Towards the Thermodynamics of Localization Processes

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We study the entropy time evolution of a quantum mechanical model, which is frequently used as a prototype for Anderson’s localization. Recently Latora and Baranger [V. Latora, M. Baranger, Phys. Rev. Lett. 82, 520(1999)] found that there exist three entropy regimes, a transient regime of passage from dynamics to thermodynamics, a linear in time regime of entropy increase, namely a thermodynamic regime of Kolmogorov kind, and a saturation regime. We use the non-extensive entropic indicator recently advocated by Tsallis [C. Tsallis, J. Stat. Phys. 52, 479 (1988)] with a mobile entropic index $q$, and we find that with the adoption of the “magic” value $q = Q = 1/2$ the Kolmogorov regime becomes more extended and more distinct than with the traditional entropic index $q = 1$. We adopt a two-site model to explain these properties by means of an analytical treatment and we argue that $Q = 1/2$ might be a typical signature of the occurrence of Anderson’s localization.

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I. INTRODUCTION

In this paper we focus our attention on the process of localization discovered by Anderson, and we discuss the corresponding time evolution using the non-extensive thermodynamics view of Tsallis. The subject of Tsallis non-extensive thermodynamics is attracting the interest of an ever increasing number of investigators in different branches of the complexity theory (see, for instance, Ref. 5). We think, however, that the connection with dynamics, especially quantum dynamics, is not yet explored with the attention that this subject would require.

To make easier for the reader to understand the conceptual difficulty of this problem, it is convenient to make a short review of this intriguing subject. First of all, we want to remind the reader that there are two classes of entropic indicators. The first is that of the indicators expressed in terms of trajectories, which are consequently confined to the classical case. The second class is that of the entropic indicators expressed in terms of either classical or quantum distributions. The second class is directly connected to the focus of the present paper, which is in fact devoted to a quantum problem. However, we are convinced that there is a subtle equivalence between the two classes of entropy indicators, and that understanding some properties of the first class can make easier for us to make a proper balance on the results obtained in the present paper.

A. Entropies in terms of trajectories

The Kolmogorov-Sinai (KS) entropy is considered to be a dynamic property of a single classical trajectory measuring the randomness which is responsible for the thermodynamic properties of the system under study. In the recent literature there are interesting examples of adoption of this class of entropic indicators either in the ordinary extensive form or in the new non-extensive form of Tsallis. We think that the more recent work of Jin and Grigolini is of special interest for the present paper. The stimulus for this work was given by the heuristic arguments of Refs. 3-5 which provide in fact a strong motivation for the generalization of the KS entropy. The authors of Ref. 5 show that the Tsallis non-extensive generalization of the KS entropy, referred to as Kolmogorov-Sinai-Tsallis (KST) entropy, can be expressed in terms of an average over the invariant distribution $p(x)$, thereby implying the stationarity condition. The explicit form of this KST entropy is:

$$H_q(t) = [1 - k(q)] \int dx p(x)^q \xi(t, x)^{1-q} / (q - 1).$$

Note that the departure point of the theory leading to this interesting result is given by a repartition of the phase space of the system under study into cells of small size. This is mirrored by the fact that $k(q) \equiv (2/l)^q - 1$, where $2/l$ denotes the size of these cells. The function $\xi(t, x)$ is defined by:
\[ \xi(t) = \lim_{|\Delta y(0)| \to 0} \left| \frac{\Delta y(t)}{\Delta y(0)} \right|. \]  

We are now in the right position to show how the detection of the entropic index corresponding to the thermodynamic nature of the dynamics under study is carried out. Let us assume that the sensitivity to the initial condition is expressed by:

\[ \xi(t) = [1 + (1 - Q)\lambda_Q t]^{1/(1-Q)}. \]  

Let us replace Eq. (3) into Eq. (1) and let us change the mobile index \( q \). It is evident that the resulting time evolution of the KST entropy is a power law behavior faster or slower than the linear in time evolution. A linear in time entropy increase can only be found if the entropic index \( q = Q \) is adopted. This is the entropic index corresponding to the real nature of the thermodynamics of the system. The fact that \( Q \neq 1 \) means that the thermodynamics of the system is not extensive. If it happens that the magic entropic index \( Q \) has the traditional value \( Q = 1 \), then the connection between thermodynamics and dynamics is established in the manner recently pointed out, for instance, in the illuminating book by Zaslavsky [8]. This means a dynamics-thermodynamics connection, with dynamics characterized by the ordinary Lyapunov coefficients. Note in fact that \( Q = 1 \) means that the sensitivity to initial conditions given by Eq. (3) becomes the ordinary exponential sensitivity and that the thermodynamic nature of the system is expressed by the ordinary KS entropy. It is worth remarking that at \( Q = 1 \) the KST entropy of Eq. (1) adheres to the prescriptions of the Pesin theorem in its ordinary form [16].

These interesting properties have been originally pointed out in the important work of Refs. [13–15]. However, the special form for \( H_q(t) \) of Eq. (1), in addition to taking into account the heuristic arguments of Refs. [13–15], serves also the useful purpose of establishing the genuine entropic index \( Q \) even when dynamics are not exactly characterized by the form of Eq. (3), and system’s dynamics are rather characterized by a distribution of power indices [17,18]. In this case the search for the proper entropic index \( Q \) by means of the prescription of Eq. (3) is especially convenient [19].

The connection between this classical case and the quantum case under study in this paper is given by the fact that both cases imply \( Q < 1 \). Before addressing the discussion of the consequences of this property for quantum diffusion, it is convenient to let the reader know about the meaning of \( Q \neq 1 \) in the case of classical diffusion. The case \( Q > 1 \) has been discussed in Ref. [20] and it has been pointed out that it leads to a form of diffusion faster than ordinary Brownian motion. Note that in this case the sensitivity to initial condition is given by Eq. (3) with \( Q > 1 \). This means that two trajectories with very close initial conditions depart one from the other so fast as to make their distance diverge at a finite time. It is thus plausible that \( Q > 1 \) yields a diffusion faster than the ordinary diffusion.

Much more important for the conclusions of this paper is the case where \( Q < 1 \). In the classical case this means a localization process occurring within a finite time scale. As shown in Ref. [15], the first step to prove this conjecture is given by the paper by Tsallis and Bukman [21]. These authors studied a family of non-linear Fokker-Planck equation and proved that the adoption of the Tsallis non-extensive entropy yield the exact form of solution, and that the case \( Q < 1 \) corresponds to subdiffusion. On the other hand, both the KST entropy of Eq. (1) and the dynamic approach to diffusion [22] imply the stationary assumption. It is shown [15] that subdiffusion in a stationary condition yields localization with a finite time scale. This theoretical prediction has been fully supported by the numerical results of Ref. [18] on the dynamics of the logistic map at the onset of chaos. At the end of this paper we shall come back to discussing the consequences that this finding has on quantum diffusion, if the assumption is made that a sort of statistical equivalence exists between fractal dynamics in classical physics and the quantum dynamics of the Anderson Hamiltonian.

B. Entropies in terms of distributions

As earlier mentioned, the second class of entropies is expressed in terms of the classical or quantum Liouville density [23,24] by means of either the traditional Gibbs-Boltzmann form or the Tsallis [3] non-extensive form. In literature we find the implicit assumption that these two forms of entropic indicators are equivalent [25]. We take this equivalence for granted but, in principle, we leave open the choice between the extensive and non-extensive form. The assumption of equivalence between the two classes of entropy will afford a criterion to make a choice. In fact, we note that the KST entropy as well as the ordinary KS entropy has the meaning of entropy increase per unit time: Thus it seems to be natural to interpret the regime where the von Neumann entropy is found to increase linearly in time as the expression of a genuinely thermodynamical condition. This point of view is supported by the interesting results of Zurek and Paz [23], and by the more recent computer calculations of other authors [24,25].

Notice that both Ref. [24] and Ref. [25] support the view of Zurek and Paz, and that all these papers [23–25] establish a connection between dynamics and thermodynamics in the deep classical regime. The work of [25] has
established that the entropy time evolution studied by Pattanayak and Brumer \[24\] results in a power law increase if we assign to the Planck constant values corresponding to the region of transition from the classical to the quantum regime. The authors of Ref. \[24\] did not consider this condition to be relevant, probably because it is far from the condition of linear in time increase, which is correctly judged by them as the only one with a thermodynamical significance. In Ref. \[26\] also this special condition has been recognized as being thermodynamic due to the adoption of the non-extensive perspective advocated by Tsallis \[3–5\]: The authors of this paper \[26\] pointed out that \( q < 1 \) in this case reflects the occurrence of Anderson localization.

Another important paper, worth of mention, is that of Latora and Baranger \[27\]. These authors studied several maps, characterized by deterministic chaos. This is a remarkable paper since it corresponds to studying the time evolution of the Gibbs entropy expressed in terms of the probability distribution, and can be considered as the classical counterpart of the time evolution of the von Neumann entropy of Ref. \[25\]. These authors find that the time evolution of the Gibbs entropy is characterized by three time regimes: (i) an early regime of exponential increase, (ii) an intermediate-time regime of linear increase, and, finally, (iii), a saturation regime. The earlier remarks naturally lead us to conclude, in a full accordance with the view of these authors \[27\], that dynamics become compatible with thermodynamics only in the intermediate-time regime. We shall refer to this intermediate time regime as Kolmogorov regime, for reasons that are made clear by the equivalence between the two classes of entropy.

C. Purpose and outline of the paper

To substantiate the conjecture of Ref. \[26\] on the non-extensive nature of the thermodynamics of a localization process, we devote this paper to the numerical treatment of the Anderson tight-binding Hamiltonian. In other words, we study a prototype of Anderson localization processes, rather than the quantum kicked rotator. It is well known that the saturation of the energy increase of the quantum kicked rotator is due to quantum correlations, namely, the same cause as that responsible for Anderson’s localization \[28\]. However, the two systems, although equivalent, are not identical and there might be the doubt that the remarks of Ref. \[26\] do not apply to the genuine model of Anderson’s localization. Thus, we think that the direct study of the prototype model of Anderson’s localization, if we find that also in this case the mobile entropic index \( q \) must be assigned a value \( Q < 1 \), will make more convincing our conjecture about non-extensive thermodynamics and Anderson’s localization.

The outline of this paper is as follows. Section II defines the model under study in this paper and points out that the Anderson noise has the twofold role of creating statistical mechanics and localization. The former aspect, in an apparent conflict with the latter, is compatible with the existence of a thermodynamic perspective. Section III is devoted to the illustration of the numerical results of this paper. This numerical treatment has to be considered as a sort of exact treatment of the entropy time evolution triggered by Anderson randomness. An analytical treatment of the problem, shedding light into the reasons why the Tsallis entropy indicator is a so efficient indicator, is illustrated in Section IV. Finally, in Section V we aim at making a proper balance on the results of this paper.

II. MASTER EQUATION, ANDERSON LOCALIZATION AND STATISTICAL MECHANICS

The main purpose of this Section is to show that the tight-binding Hamiltonian system that we use to discuss Anderson’s localization is a remarkable example of joint action of randomness and order. This is, in other words, a system which is not equivalent to the classical condition of full chaos. Rather, as we shall see, this is system equivalent to the condition of weak classical chaos, or to the condition of sporadic randomness.

We study a system described by the following Hamiltonian

\[ H = H_0 + W, \]  

where

\[ H_0 \equiv \sum_m E_m |m\rangle\langle m| \]  

and

\[ W \equiv V \sum_m (|m\rangle\langle m+1| + |m+1\rangle\langle m|). \]

This is the Hamiltonian originally taken into account by Anderson \[14\]. We make this Hamiltonian result in a transport process different from the ballistic diffusion of a perfect crystal assuming that
\[ E_m = \epsilon + \phi_m. \] (7)

Here we are assuming that with changing site there is a fluctuation \( \phi_m \) around the common value \( \epsilon \). We assume no correlation among different sites, namely

\[ \langle \phi_m \phi_{m'} \rangle = A \delta_{mm'}. \] (8)

It has to be pointed out that according to the prescriptions of quantum statistical mechanics, any entropy indicator must be expressed in terms of the density matrix associated to the Hamiltonian of Eq.(4). The time evolution of this density matrix is unitary, and consequently any form of entropic indicator, expressed in terms of the density matrix, is time independent. From this point of view, there is no difference between the system under study and a system characterized by regular dynamic properties, and thus strongly departing from the randomness condition intuitively associated to the second principle.

This is disconcerting. To a first sight, in fact, the Anderson prescriptions of Eqs.(7) and (8) sow seed of randomness into the system dynamics and the entropy indicator should make this randomness ostensible. So the question is raised of how to make the entropy indicator sensitive to this randomness.

The source of entropy increase in the model of Zurek and Paz \[23\] is given by the deterministic chaos that the system would exhibit in the classical limit. To trigger a regime of entropy increase, these authors took into account the influence of the environment as a source of dephasing, a process that does not imply any exchange of energy between the system and its environment. This is in line with the second principle which forces entropy to increase (or to remain constant if the process is reversible) if no thermal exchange with the environment is allowed. This means that an interaction between system and environment is allowed, provided that it does not cause any energy exchange. The perspective of Zurek and Paz is not trivial: This is so because, even if the entropy increase is made possible by the key ingredient of external fluctuations, these are so weak that the time scale of the process of transition from dynamics to thermodynamics is determined by the Lyapunov coefficients, namely, a genuinely dynamic property of the Hamiltonian system under study.

Here we find that the Anderson randomness is a sort of counterpart of the deterministic chaos randomness of Zurek and Paz \[23\]. To reveal this randomness we must adopt the statistical density matrix defined by:

\[ \rho_S(t) \equiv \int d[\phi] w([\phi]) \rho([\phi], t). \] (9)

Any contribution to the integral of Eq.(9), \( \rho([\phi], t) \), is an ordinary density matrix corresponding to a given random distribution of the energy fluctuations \( \phi_m \). The symbol \([\phi]\) denotes a given Anderson realization, namely, \([\phi] \equiv \phi_1, \phi_2, \ldots, \phi_i, \ldots\). Note that, as a consequence of the assumption of Eq.(8), we have

\[ w([\phi]) = \cdots p(\phi_{m-1}) p(\phi_m) p(\phi_{m+1}) \cdots \] (10)

We make the assumption that the random distribution of the site energies follows the Cauchy prescription

\[ p(\phi) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + \phi^2}. \] (11)

where \( p(\phi) \) denotes the probability that the energy of a given site fluctuates by the quantity \( \phi \) about the common value \( \epsilon \). It is interesting to remark that the time evolution of the average density matrix \( \rho_S(t) \) of Eq.(9) becomes identical to that of a perfect lattice if \( \gamma = 0 \). If, on the contrary, the value of the parameter \( \gamma \) increases, the time evolution of the statistical density matrix \( \rho_S(t) \) increasingly departs from the prescription of unitary time evolution. Consequently, \( \gamma \), the width of the Anderson noise, can be regarded as the randomness intensity of the system.

We are therefore in a position to establish a comparison between the time evolution of \( \rho_S(t) \) and the density matrix of the quantum kicked rotor \[23][24\]. In the latter case the source of randomness is given by the deterministic chaos of the classical time evolution of the system. In both cases, randomness has a twofold role. In the early time region of the process this randomness creates a condition of transport similar to that of ordinary Brownian motion. At later times, the diffusion process is quenched by the occurrence of the Anderson localization.

The model under study rests only on the two parameters \( \gamma \) and \( V \), and it would be tempting for us to refer ourselves to the condition:

\[ \gamma < V. \] (12)

as weak chaos. On the same token, we are tempted to refer ourselves to the condition:
\(\gamma > V.\) (13)

as a condition of strong chaos. This would be incorrect, since, as we shall see through the joint use of numerical calculations (Section III) and analytical theory (Section IV), in both conditions the Anderson randomness and the quantum correlations are present, and, in a sense, the condition of Eq. (13) has the effect of realizing Anderson’s localization at earlier times. Thus, both conditions have to be considered as being equivalent to the weak chaos of classical mechanics. The case of Eq. (13) makes it possible to rest on analytical calculations. Therefore in Section IV we shall focus on the condition of Eq. (13).

The twofold role of the Anderson randomness has been studied, in the case \(V < \gamma,\) in an earlier publication by Mazza and Grigolini [29]. The authors of this paper prove that the time evolution of the site population \(p_n(t)\) is driven by the following generalized master equation:

\[
\frac{\partial}{\partial t} p_n(t) = -\sum_{m \neq n} \int_0^t dt' \Xi_{nm}(t-t')[p_n(t') - p_m(t')].
\] (14)

The authors of Ref. [29] prove that in the deep regime of strong Anderson noise (\(\gamma > V\)) the memory kernel \(\Xi_{nm}(t-t')\) becomes

\[
\Xi_{nm}(t) = 2K(t)(\delta_{n,m'+1} + \delta_{n,m'+1}),
\] (15)

where

\[
K(t) = \frac{V^2 \gamma}{\gamma^2 - V^2} \frac{1}{h^2} [\gamma \exp(-2\gamma t/h) - V \exp(-2V t/h)] + \frac{V^3}{\pi \gamma h^2} \cos(2V t/h).
\] (16)

It is worth stressing that the oscillatory term on the r.h.s. of Eq. (16) (the second term on the r.h.s. of Eq. (16)) does not play any relevant role and was introduced by the authors of Ref. [29] for the minor purpose of reproducing the weak and fast oscillations revealed by the numerical treatment. In the theoretical treatment of Section IV we shall make an approximation equivalent to disregarding the influence of this term.

The main result of Ref. [29] is that in the time region

\[
\frac{h}{\gamma} << t << \frac{h}{V}
\] (17)

the time evolution of the system is virtually indistinguishable from ordinary Brownian diffusion. This is so because the negative and slow exponential appearing in the r.h.s. of Eq. (16) is not yet strong enough as to balance the fast and strong exponential. In the case \(\gamma >> V\) Anderson’s localization occurs at the time \(h/V\). Note that the condition of ordinary statistical mechanics is recovered for \(V \to 0,\) while keeping the quantity \(V^2/\gamma\) constant. In this limiting condition the slow negative tail of Eq. (13) can be neglected, and the resulting statistical process is indistinguishable from that where the dephasing process is due to the environment. In this case, as we shall see in Section V, we recover the condition of ordinary statistical mechanics, denoted by \(Q = 1.\)

III. NUMERICAL RESULTS

The numerical calculations have been done producing, first of all, about 1,000 realizations of the Hamiltonian system of Eqs. (4), (5) and (6). We have also checked that in the time span illustrated by Figures in this letter the result is not changed if the number of realizations is increased. Each realization is obtained by using a random noise generator which assigns to any site \(|m\rangle\) a fluctuation \(\phi_m\) so as to realize the Cauchy prescription of Eq. (11). The Hamiltonian of each realization is diagonalized so as determine the corresponding time evolution and the corresponding density matrix. For any realization the initial condition is given by the wave function \(\langle \psi(0) | = | m = 0 \rangle.\) Finally an average over all the realizations is made.

The numerical results concern these two distinct cases: (i) \(\gamma >> V\) and (ii) \(\gamma << V.\) Fig. 4 refers to case (i) and Figs. 5 and 6 to case (ii). Let us examine case (i) first. Fig. 4 shows the second moment of the distribution \(M_2 = \sum_m L^2 p_m(t).\) \(L\) denotes the lattice spacing and for simplicity we assume \(L = 1.\) We see that the time evolution of \(M_2(t)\) undergoes a non linear time increase for a time interval of the order of \(h/\gamma.\) After this first time region, the increase of \(M_2(t)\) becomes linear in time. The regime of linear increase lasts for a time of the order of \(h/V.\) In the last time regime the function \(M_2(t)\) tends to become time independent, thereby signalling the occurrence of Anderson’s localization.
Fig. 2 illustrates the time evolution of the two entropy indicators, \( S_1(t) \) and \( S_{1/2}(t) \) as well as that of \( M_2(t) \). In fact, as earlier discussed in detail, we think that some interesting information can be derived from the observation of the entropy indicator:

\[
S_q(t) \equiv \frac{1}{q-1} \left( 1 - T\rho_S(t)^q \right).
\]

(18)

It is well known that the traditional von Neumann entropy is obtained from Eq. (18) setting \( q = 1 \). Notice that the choice of \( q = Q = 1/2 \) is the result of a search that will be discussed in Section IV. Here we limit ourselves to noticing that the transition to the linear regime with \( q = 0.5 \) is faster than in the case \( q = 1 \). We also notice that the slope of this entropy increase per time unit is \( 2V/h \), corresponding to the theoretical predictions of Section V. In conclusion, the time behavior of both \( S_1(t) \) and \( S_{1/2}(t) \) is a reflection of the three time regimes revealed by the time evolution of \( M_2(t) \). With the help of Fig. 2 we see that with \( q = 1/2 \) the time duration of the Kolmogorov regime is slightly more extended than in the case \( q = 1 \). With the help of Fig. 3 we also see that with \( S_1(t) \) the linear increase is not clearly separated from the regime of logarithmic dependence in time, which finally is changed into the saturation regime corresponding to the occurrence of Anderson’s localization. Thus, \( S_{1/2}(t) \) is an indicator of the three regimes slightly more efficient than \( S_1(t) \). This aspect, as we shall see with the help of Fig. 2 and 3, is more pronounced in case (ii).

Fig. 2 and 3 illustrate the same properties as those of Fig. 1 referred to case (ii). We see that the function \( M_2(t) \) signals a transition to the statistical regime of Brownian diffusion at a time of the order \( \hbar/V \). In the time scale explored by Fig. 3 there is no sign of the occurrence of Anderson’s localization, which takes place at a much later time. Even in this case the entropic analysis reveals the existence of the three regimes of Latora and Baranger. Even in this case the Kolmogorov regime lasts for a time of the order of \( \hbar/V \) and even in this case the entropy \( S_{1/2}(t) \) is a much more accurate indicator of the Kolmogorov regime. Even in this case the slope of \( S_{1/2}(t) \) in the regime of linear increase is given by \( 2V/\hbar \). We note that at a time of the order of \( \hbar/V \) the entropy \( S_{1/2}(t) \) makes a transition to a regime of logarithmic dependence on time. This aspect is made clear by Fig. 3 whose abscissas are expressed in a logarithmic scale for that purpose. Again the adoption of \( S_1(t) \) as entropy indicator does not establish a clear distinction between the regime of linear increase and that of logarithmic dependence on time.

Of some interest is also Fig. 4, which shows the time evolution of \( S_q(t) \) for several values of the mobile entropic index \( q \). We see that all these entropic indicators signal a transition to the regime of logarithmic dependence on time at times of the order \( \hbar/V \). We also note in the first and second time regime a pattern of curves reminding the form of a leaf. This leaf effect will be discussed again in Section IV. We also note that this leaf effect is reminiscent of that revealed by the study of the kicked quantum rotor of Ref. [26]. The authors of Ref. [26] made the conjecture that this leaf effect might be related to the occurrence of Anderson’s localization. With the help of the numerical results of this Section and of the theoretical analysis of Section IV, in the concluding remarks of Section V we shall address again the discussion of this interesting issue.

IV. THE NON-EXTENSIVE INTERPRETATION: AN ANALYTICAL TREATMENT

The numerical analysis of Ref. [29], supported also by the numerical treatment of this paper, proves that the deep regime of strong Anderson randomness is characterized by the important fact that the transition from \( m \) to \( m + 1 \) is statistically independent of that from \( m \) to \( m - 1 \). This means that it is possible to carry out an analytical treatment based on the study of only two sites. The adoption of the distribution of Eq. (11) yields the following values for the four elements of the statistical density matrix of Eq. (3):

\[
(p_{S(t)})_{11} = \int_{-\infty}^{+\infty} dE \frac{1}{\pi} \frac{2\gamma}{4\gamma^2 + E^2} \left[ 1 - \frac{4V^2}{4V^2 + E^2} \sin^2(\sqrt{E^2 + 4V^2} \frac{t}{2\hbar}) \right],
\]

(19)

and

\[
(p_{S(t)})_{12} = \int_{-\infty}^{+\infty} dE \frac{2\gamma V}{\pi} \frac{1}{4\gamma^2 + E^2} \sqrt{\frac{1}{E^2 + 4V^2}} \sin(\sqrt{E^2 + 4V^2} \frac{t}{\hbar}).
\]

(20)

Of course \((p_{S(t)})_{21} = (p_{S(t)})_{12}^*\) and \((p_{S(t)})_{22} = 1 - (p_{S(t)})_{11}\). By diagonalizing the two by two density matrix, we find the eigenvalues

\[
\Lambda_1(t) = \frac{1}{2} + \sqrt{[(p_{S(t)})_{11} - (p_{S(t)})_{22}]^2 + 4(p_{S(t)})_{12}(p_{S(t)})_{21}}.
\]

(21)
and
\[ \Lambda_2(t) = \frac{1}{2} - \frac{\sqrt{[(\rho S(t))_{11} - (\rho S(t))_{22}]^2 + 4(\rho S(t))_{12}(\rho S(t))_{21}}}{2}. \]  

The time evolution of the Tsallis entropy corresponding to the mobile entropic index \( q \) is given by
\[ S_q(t) = \frac{1 - \Lambda_1(t)^q - \Lambda_2(t)^q}{q - 1}. \]  

The expression of Eq. (23) is not yet suitable for an analytical discussion of the problem under study, since it depends on integrals defining the terms of Eqs. (19) and (20). These integrals can be easily solved if we make the approximation of neglecting \( V^2 \) compared to \( E^2 \). This approximation is equivalent to that of disregarding the r.h.s. of Eq. (16)). We thus obtain:
\[ \rho_S(t)_{11} = 1 - \frac{V}{2(\gamma^2 - 4V^2)}[\gamma(1 - \exp(-2Vt/\hbar)) - V(1 - \exp(-2\gamma t/\hbar))] \]  
and
\[ \rho_S(t)_{12} = \frac{iV}{2\gamma}[1 - \exp(-2\gamma t/\hbar)]. \]  

With help of Eqs. (24) and (25) the time evolution of the Tsallis entropy becomes analytical. It is interesting to note that if the entropy \( S_q(t) \) is plotted for different values of the entropic index \( q \) it results in the same kind of leaf-shape effect as that given by the numerical results of Section IV (see Fig. 4). It is possible to prove analytically that the magic value of the mobile entropic index \( q, Q \), is \( Q = 1/2 \). The numerical leaf-shape effect means that the adoption of the magic value of \( q \) results in a linear increase of entropy as a function of time after a transient process of time duration \( 1/\gamma \). Thus, we make an expansion of \( S_q(t) \) supplementing the condition \( V >> \gamma \) (which is made necessary to give credibility to the two-site model) with additional conditions \( Vt << \gamma t \approx 1 \). We thus obtain from Eq. (23), with Eqs. (21), (22), (24) and (25):
\[ S_q(t) \approx \frac{qV^2t^2 - (V^2t^2)^q}{q - 1}. \]  

It is easy to prove that the linear dependence on \( t \) is obtained by assigning to \( q \) the magic value \( Q = 1/2 \). In fact, in this case we derive from Eq. (26):
\[ S_{1/2}(t) \approx -\frac{qV^2t^2}{\hbar^2} + \frac{2Vt}{\hbar} \approx \frac{2Vt}{\hbar}. \]  

This means that the rate of entropy increase is:
\[ \frac{d}{dt}S_{1/2}(t) \approx \frac{2V}{\hbar}. \]  

It is interesting to notice that the case where the conditions for statistical mechanics are realized by environmental fluctuations, the two-state model discussed in this Section would lead to a master equation identical to that of Eq. (24) with the memory kernel of Eq. (16) replaced by:
\[ K(t) = \frac{4V^2}{\hbar^2}\exp(-\frac{\sigma}{\hbar}t). \]  

In this specific case an analytical treatment of the same kind as that illustrated above yields \( Q = 1 \). In Fig. 5 and 6 we compare the entropy time evolution produced by Anderson’s randomness to the externally induced entropy increase. In fact in Fig. 5 we study the leaf effect associated to Anderson’s randomness \( \gamma = 100 \), changing the mobile entropic index \( q \) from \( q = 1 \) to \( q = 0.45 \). Comparing Fig. 5 to Fig. 6 we see that the typical leaf effect, induced by Anderson’s randomness, is lost if the entropy increase is only of external origin. We also note that the leaf effect in this case is similar to that of Fig. 4, the only remarkable difference being the fact that the large \( \gamma \) condition has the effect of strongly reducing the time duration of the first time regime.

With the help of Figs. 5 and 6 we note that in the case of merely external randomness the rate of entropy increase is proportional to \( V^2/\sigma \). In fact from Figs. 5 and 6 we see that the entropy rate is proportional to \( 1/\sigma \) and to \( V^2 \),
respectively. In other words, we find that in this case the entropy increase corresponds to the rate of the environment induced dephasing process. This has to be contrasted with the earlier discovery of Eq. \( (28) \). We see, in other words, that in the case of merely external randomness the “magic” entropic index is given by the conventional entropic index \( Q = 1 \) and that the adoption of this magic entropic index reveals an ordinary source of randomness. In the case where the only source of randomness is internal, namely, is Anderson’s randomness, the “magic” entropic index is given by value \( Q = 0.5 \). The corresponding non-extensive entropy is the proper entropic indicator signalling that a thermodynamic view is still possible in spite of dynamics dominated by strong correlations.

V. CONCLUDING REMARKS

The first indisputable result of this paper is given by Figs. 1, 2, and 3. These figures prove that the the tight-binding Hamiltonian of Eq. (1) is a source of entropy time evolution clearly showing the three regimes recently discovered by Latora and Baranger (27). Furthermore, from Fig. 1 we see that the regime of Brownian diffusion discussed by Mazza and Grigolini (29) is a regime of constant KS entropy.

On the basis of this observation, we would be tempted to conclude that this Kolmogorov regime is characterized by ordinary statistical mechanics, but this interpretation would not be not totally satisfactory. In fact, it is well known (29) that this regime of apparently ordinary statistical mechanics is compatible with the silent action of quantum correlations. In the case \( \gamma >> V \) this has to do with the silent action of the negative tail of Eq. (16), whose time scale is \( \hbar / (2V) \). At the end of this transient process, Anderson’s localization takes place. Thus, we find to be to some extent embarrassing to interpret this regime as a manifestation of ordinary statistical mechanics. According to the illuminating picture illustrated by Zaslavsky in his recent book (8), ordinary statistical mechanics are closely related to a deterministic motion of the type of the Bernouilli shift, map, namely, a case of dynamics with no memory. Here, on the contrary, the ensuing process of Anderson’s localization is a consequence of the action of quantum correlations, even if this remained silent through the whole Kolmogorov regime of Fig. 1. For this reason, we find it to be extremely interesting that the adoption of the Tsallis entropy makes a different interpretation emerge. This is so because the regime of steady entropy increase per unit time becomes well distinct only if we use the magic value \( Q = 1/2 \), which implies that the Brownian diffusion regime studied by Mazza and Grigolini (29) is actually a form of non-extensive rather than extensive thermodynamics. This is compatible with the fact that this process is characterized by the silent action of quantum correlations which are responsible for the occurrence of Anderson’s localization. We note that this interpretation is also supported by the observation, numerical and analytical, that the rate of entropy increase is \( 2V/\hbar \). In the case \( \gamma >> V \) this is also the rate of establishment of Anderson’s localization.

The results illustrated by Fig. 2 are an exciting confirmation of the fact that the magic value \( Q = 0.5 \) results in in a marked regime of linear entropy increase. However, in this case the onset of Anderson localization takes place at a time scale much larger than \( \hbar / V \), and this makes it difficult to maintain the claim about a direct connection between \( Q < 1 \) and Anderson’s localization. We think that in this specific case the frequency \( V/\hbar \) has to do with the coherent motion of a regular lattice. The memory kernel \( K(t) \) of Eq. (14) undergoes many oscillations with frequencies of this order of magnitude before producing localization. Thus, in this specific case the connection between \( Q < 1 \) and Anderson’s localization seems to be much less direct.

Is it possible that \( Q < 1 \) might signal quantum coherent motion, without necessarily implying Anderson’s localization? We think that some more research work must be devoted to this intriguing issue. The results of this paper give some more support to the conjecture of Ref. (29) that \( Q < 1 \) is a signature of localization. However, we want to point out that the arguments of Section IA cannot yet be used as a compelling evidence that this is true in general, especially because the arguments of Section IA refer to the classical case of fractal dynamics. We are convinced that there exists a statistical equivalence between fractal dynamics and Anderson’s randomness. We hope that the results of this paper might trigger further research to prove, or disprove, this interesting conjecture.

Although this conjecture is not yet proved, it cannot be easily dismissed either. From the theory of Section IV we see that if Anderson’s randomness vanishes, and entropy increase only rests on external fluctuations, the magic entropic index \( Q \) is given the ordinary value \( Q = 1 \), again. On top of that, the rate of entropy increase is \( V^2/\gamma \), in accordance with the fact that entropy measures randomness, and that the rate of decoherence is a proper measure of system’s randomness. Thus, we can conclude that the adoption of Tsallis entropy makes it possible to adopt a thermodynamics perspective, even when the entropy rate signals a coherent property, \( 2V/\hbar \), rather than an incoherent dephasing process, and that the miracle is possible due to the adoption of the entropic index \( Q = 0.5 \).
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\[
\frac{1}{1-Q} = \frac{1}{\alpha_{\text{min}}} - \frac{1}{\alpha_{\text{max}}}
\]
(30)
to the value $\beta = 0$ thereby always implying the condition $Q < 1$ to be fulfilled.
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FIG. 1. The entropic indicators and the second moment of diffusion as a function of time. $\gamma/h = 4$ and $V/h = 1$. The curve denoted by the $+$'s is the Tsallis entropy with $Q = 0.5$; the curve denoted by the $X$'s is the Gibbs entropy (the Tsallis entropy with $Q = 1$); the curve denoted by the $*$'s is the second moment of the distribution, $M_2(t)$. Time is expressed in units of $h/V$.

FIG. 2. The entropic indicators and the diffusion second moment as a function of time. $\gamma/h = 0.3$ and $V/h = 1$. The curve denoted by the $+$'s is the Tsallis entropy with $Q = 0.5$; the curve denoted by the $X$'s is the Gibbs entropy (the Tsallis entropy with $Q = 1$); the curve denoted by the $*$'s is the second moment of the distribution, $M_2(t)$. Time is expressed in units of $h/V$. 
FIG. 3. The Tsallis entropy with $q = 0.5$ expressed with respect to $\log(t)$ with the same parameters as those of Fig. 2.

FIG. 4. The entropic indicators $S_q(t)$ as a function of time. The values of the parameters are: $\gamma/\hbar = 0.3$ and $V/\hbar = 1$. The plotted curves from the bottom to the top refer to: $q = 0.3$, $q = 0.4$, $q = 0.5$, $q = 0.6$ and $q = 0.7$. 
FIG. 5. The entropic indicators $S_q(t)$ as a function of time with Anderson’s randomness, $\gamma/\hbar = 100$ and $V/\hbar = 1$. The plotted curves from the bottom to the top refer to: $q = 1$, $q = 0.55$, $q = 0.5$, $q = 0.45$.

FIG. 6. The entropic indicators $S_q(t)$ in the presence of only external randomness as a function of time. The value of system’s parameters are $\sigma = 100$ and $V/\hbar = 1$. The plotted curves from the bottom to the top refer to: $q = 1$, $q = 0.55$, $q = 0.5$, $q = 0.45$. 

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FIG. 7. Externally generated time increase of $S_1(t)$. $V/h = 1$. From the bottom to the top: $\sigma/h = 400, \sigma/h = 200, \sigma/h = 100$.

FIG. 8. Externally generated time increase of $S_1(t)$. $\sigma/h = 400$. From the bottom to the top: $V/h = 0.5, V/h = 1, V/h = 2$. 