COOL PIONS MOVE AT LESS THAN THE SPEED OF LIGHT

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At nonzero temperature, pions propagate through a thermal medium at less than the speed of light. About low temperature, this effect begins not at $\sim T^2$, but at next to leading order, $\sim T^4$. We also derive the generalization of the relation of Gell-Mann, Oakes, and Renner to nonzero temperature.

1 General analysis

In this proceeding we give a pedagogical explanation of recent work of ours. In this section we show how the velocity of pions, or more generally Goldstone bosons, changes with temperature in a thermal distribution.

This in itself is pretty trivial, and familiar from other contexts. For examples, the appropriate analogy to Goldstone bosons in a theory which is relativistically invariant at zero temperature are spin waves in an antiferromagnet. This is because the dispersion relation for such spin waves is $\omega =vp$, where $\omega$ is the frequency, $p$ the magnitude of the spatial momentum, and so $v$ is the velocity. (Spin waves in a ferromagnet behave as $\omega \sim p^2$.) In a relativistically invariant system, of course $v = c$, but in antiferromagnets, the velocity $v$ is just some parameter that depends upon details of the couplings, etc. Thus for antiferromagnets, it is completely unremarkable that $v$ changes with temperature, since the couplings change with temperature as well.

This is in agreement with Goldstone’s theorem, which tells us that for the Goldstone modes, the inverse pion propagator vanishes at zero spatial momentum, $p = 0$. Thus it is fine if $\omega$ vanishes like some constant $v$ times $p$, where $v$ changes with temperature.

Less trivially, one can also understand the damping of pions or spin waves. Even without calculation, we expect that the damping rate vanishes at zero spatial momentum, since by Goldstone's theorem the inverse pion propagator must vanish at zero momentum, for both the real and imaginary parts. How precisely the imaginary parts, and so the damping rate, vanishes is an interesting question which we face when we get into the gory details below.

A similar problem is the propagation of light in a medium with an index of refraction greater than one. The analogy is imprecise, though, because in a thermal bath Debye screening generates a finite correlation length for photons, whereas in the broken symmetric phase, Goldstone bosons always have an infinite correlation length.
A better example is the propagation of light in a background magnetic field, where at two loop order the velocity of light is less than that in vacuum by an amount $\sim \alpha^2 B^2/m^4$, where $m$ is the mass of the electron. There is a technical similarity here as well. The deviations from the speed of light are proportional to the energy density $\sim B^2$ for photons in a background $B$ field, and to the free energy density $\sim T^4$ for thermal pions. For the former, this is unsurprising, since one wouldn’t expect anything to depend upon $B$, only on $B^2$. But for thermal pions, that there aren’t terms of $\sim T^2$ is at first sight unexpected. However old arguments in a different guise tell us that to leading order, $\sim T^2$, lorentz covariance is manifest in a pion gas, so $v = 1$. Indeed, it is amusing that for thermal pions, the deviation of $v^2$ from unity is proportional to $8\pi^2/45 T^4$, Eqs. (21) and (23), which is 16/3 times the free energy density of a relativistic pion gas.

Quite remarkably, this is analogous to the behavior of spin waves in ferromagnets at low temperature, where the dispersion relation is $\omega(k) = c(T)k^2$. As first shown by Dyson, $c(T)$ changes with temperature but only at next to leading order in an expansion about $T = 0$ and the correction is proportional to the free energy density of the gas of spin waves. Presumably, the same is true for other systems with Goldstone modes at low temperature. Also, perhaps this a hint of the existence of a more general relation, which is valid nonperturbatively; analogous, say, to the expression for the speed of sound in a medium in terms of the pressure and density.

Consequently, the basic physics which we consider is hardly revolutionary. Nevertheless, we think that we have a novel way of looking at these phenomena, for instead of considering inverse (pion) propagators, we use PCAC, and speak of pion decay constants.

At zero temperature, the pion decay constant $f_\pi \sim 93\, MeV$ is defined by

$$\langle 0 | A_\mu^a | \pi^b(P) \rangle = i f_\pi \delta^{ab} P^\mu, \quad (1)$$

where $A_\mu^a$ is the axial vector current, and the pion has momentum $P^\mu = (p^0, \vec{p})$.

Once stated, our basic point is obvious. At nonzero temperature, because of the presence of the medium, there are, in general, two distinct pion decay constants. The timelike component of the current has one,

$$\langle 0 | A_\mu^0 | \pi^b(P) \rangle_T = i f_\pi \delta^{0b} p^\mu, \quad (2)$$

and the spatial part of the current has another,

$$\langle 0 | A_\mu^i | \pi^b(P) \rangle_T = i f_\pi \delta^{ib} p^\mu.$$

At this point we should be careful and qualify exactly in which regime we are computing. First, all matrix elements are computed at a temperature $T$ in the imaginary time formalism; then, the timelike component of the
momentum, $p^0$, is analytically continued from euclidean to Minkowski values, $p^0 = -i\omega + 0^+$. The two pion decay constants, $f^t_\pi$ and $f^s_\pi$, are defined about zero momentum, $\omega$ and $p \to 0$; typically, each has both a real and an imaginary part.

In a colloquial fashion, we only deal with soft, “cool” pions. Soft means that all components of the momenta are small relative to (the real parts) of $f^t_\pi$ and $f^s_\pi$. “Cool” means that the temperature is well below that for the restoration of chiral symmetry. This is certainly a well defined regime, where the sigma meson is heavy and its abundance is Boltzmann suppressed. Once the sigma becomes light, with a mass less than the $f_\pi$’s, then one enters the regime of the critical point, and our analysis does not apply.

We certainly aren’t the first people to realize that there are two pion decay constants at nonzero temperature: that’s an old story in nonrelativistic systems; for pions, it was pointed out before by Kirchbach and Riska, by Thorsson and Wirzba, and probably by others as well.

What we want to emphasize is that whenever there are two, distinct pion decay constants, then the propagation of pions must move off of the light cone. The argument is elementary: whether or not the vacuum respects the chiral symmetry, the symmetry is still good on the pion mass shell. Thus if we take the derivative of the axial current, we can use it to define the pion mass shell:

$$f^t_\pi p^2_0 + f^s_\pi p^2 = 0|_\pi \text{ mass shell}. \tag{3}$$

This expression has both a real and an imaginary part. The mass shell condition for pions is then,

$$p^0 = -i\omega - \gamma, \tag{4}$$

where the real part gives

$$\omega^2 = v^2 p^2 \approx \frac{\text{Re} f^s_\pi}{\text{Re} f^t_\pi} p^2. \tag{5}$$

and the imaginary part fixes

$$\gamma \approx \frac{1}{2\omega \text{Re} f^t_\pi} \left( +\text{Im} f^t_\pi \omega^2 - \text{Im} f^s_\pi p^2 \right) \geq 0, \tag{6}$$

In doing so, we assume that the imaginary parts are small relative to the real parts,

$$\text{Im} f^{t,s}_\pi \ll \text{Re} f^{t,s}_\pi. \tag{7}$$

One could probably gain some general understanding of what happens when this condition doesn’t hold, and pions are strongly damped, but we haven’t tried.
With Eq. (6) in hand, we can understand the possible behavior of the damping rate about zero momentum. The simplest possibility is that the imaginary parts of $f^t_π$ and $f^s_π$ are nonzero at zero momentum; then the damping rate vanishes linearly as $p → 0$. One can show that in this case, the imaginary part of the pion self energy vanishes quadratically about zero momentum, so Goldstone’s theorem is evidently obeyed. However, in some cases in which calculations have been done, the damping rate vanishes faster than linearly. For example, in the linear sigma model discussed in the next section, it vanishes exponentially. This is special to the linear sigma model at the order treated, since the only intermediate state through which the pion can scatter involves a virtual sigma meson, which is heavy, and so Boltzmann suppressed. Also, for spin waves in an antiferromagnet, the damping rate vanishes quadratically about zero momentum.

We quote, without derivation, our results on the generalization of the relation of Gell-Mann, Oakes, and Renner to nonzero temperature. Using methods of Shore and Veneziano, we find that the dynamic pion mass, which is the pole for complex $p^0$ at $p = 0$, is

$$m^2_π = \frac{2m_π \langle q\bar{q} \rangle_T}{(\text{Re} f^t_π)^2}.$$  

This is the same expression as at zero temperature, except that instead of $f_π$ entering, it is the real part of $f^t_π$ which appears. Of course, both $\text{Re} f^t_π$ and the value of the quark chiral condensate, $\langle q\bar{q} \rangle_T$, are functions of temperature. A relation like this was first obtained by Thorsson and Wirzba, although they just wrote $f^t_π$ instead of its real part.

It is worth emphasizing that this is the dynamic pion mass, and not the static pion mass. The latter is given by the pole for zero frequency, $p^0 = 0$, in the complex $p$ plane. This mass is just

$$m^{\text{static}}_π = \frac{m_π}{v},$$

and so is less than $m_π$ when $v < 1$. This is interesting, because what is typically measured in numerical simulations of lattice gauge theory are the static and not the dynamic masses; the static mass is then an upper bound on the dynamic mass.

2 Lowest order

We now consider where these effects first appear in an expansion about zero temperature. Using either a nonlinear or a linear sigma model, to leading
order in $T^2/f_\pi^2$, $f_\pi^t(T) = f_\pi^s(T)$, so pions move undamped at the speed of light. The reason for this was given by Dey, Eletsky, and Ioffe,\textsuperscript{7} who considered the thermal average of the two point function of either vector or axial vector currents. For completeness, we redo their analysis in terms of the thermal average of the one point function:

\[ \langle 0 | A_a^\mu | \pi^b \rangle_T \equiv \sum_n \frac{\langle n | e^{-H/T} A_a^\mu | n; \pi^b \rangle}{\sum_n \langle n | e^{-H/T} | n \rangle}. \]

(10)

At low temperature, $T \ll f_\pi$, the states $|n\rangle$ contain only pions. Further, to lowest order in $T^2/f_\pi^2$ we can truncate states with the fewest number of pions to obtain

\[ \langle 0 | A_a^\mu | \pi^b \rangle_T = \langle 0 | A_a^\mu | \pi^b \rangle_{T=0} + \sum_c \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} n(k) \langle \pi^c(k) | A_a^\mu | \pi^c(k) ; \pi^b \rangle_{T=0} + \ldots \]

(11)

where $k = |k|$ and $n(k)$ is the Bose-Einstein statistical distribution function. The Bose-Einstein distribution function is built up by summing over states: in the bra state, first one pion with momentum $k$, weighted as $e^{-k/T}$, then two pions with the same momentum, weighted as $e^{-2k/T}$, etc. In bra states with more than one pion, all pions beyond the first are completely disconnected from the matrix element, and only contribute through their statistical weighting. Summing over all such pions then generates $e^{-k/T}/(1-e^{-k/T}) = n(k)$. Using $A_0^a$ as an interpolating field for $\pi^a$, and the canonical commutation relations of current algebra, the thermal pions $\pi^c(k)$ are eliminated to obtain

\[ \sum_c \langle \pi^c(k) | A_a^\mu | \pi^c(k) ; \pi^b \rangle_{T=0} \sim -\frac{2}{f_\pi} \langle 0 | A_a^\mu | \pi^b \rangle_{T=0}. \]

(12)

To leading order in $T^2/f_\pi^2$, then,

\[ \langle 0 | A_a^\mu | \pi^b \rangle_T = \left( 1 - \frac{T^2}{12 f_\pi^2} \right) \langle 0 | A_a^\mu | \pi^b \rangle_{T=0}. \]

(13)

Since the matrix element is lorentz covariant at zero temperature, it remains so to $\sim T^2/f_\pi^2$, and $f_\pi^t = f_\pi^s$.\textsuperscript{a}

\textsuperscript{a}Strictly speaking, the Gibbs average is only defined for diagonal matrix elements but, to leading order in $T^2$ at least, (14) reproduces the known answer. It would be interesting to know what is the definition valid to any order.
3 Nonlinear sigma model

It would of great interest to perform calculations in the nonlinear sigma model, to ask: how does the damping rate vanish about zero momentum? and, is \( v < 1 \)? Both effects can first appear at two loop order, \( \sim T^4/f_\pi^4 \). Calculations of the damping rate have been performed by Gavin \textit{et al.};\(^{10,11,12}\) however, they used various (physically reasonable) approximations to the \( \pi\pi \) scattering amplitude, and so it is not apparent how \( \gamma(p) \) vanishes as \( p \to 0 \) in the chiral limit.

We note, however, that there is indirect support for \( v < 1 \) in the results of Schenk.\(^{11,12}\) He also computed not in the strict chiral limit, but using various approximations to the \( \pi\pi \) scattering amplitude. In fig. 5 of Ref. [11] and fig. 7 of Ref. [12], Schenk graphs \( f(p) = \omega(p)/\sqrt{p^2 + (m_\pi^0)^2} \), where \( m_\pi^0 \) is the mass at zero temperature, and \( \omega(p) \) his computed quasiparticle pion mass.

At leading order, \( \omega_{\text{lo}}(0) > m_\pi^0 \), and so \( f_{\text{lo}}(0) > 1 \). As \( p \) increases, \( f_{\text{lo}}(p) \) decreases monotonically to one. This is what happens when \( \omega_{\text{lo}}(p) = \sqrt{p^2 + \omega_{\text{lo}}(0)^2} \), or \( v_{\text{lo}} = 1 \).

At next to leading order, \( \omega_{\text{nlo}}(0) < m_\pi^0 \). This is not particularly suprising. However, as \( p \) increases, \( f_{\text{nlo}}(p) \) first decreases, and then increases, approaching one from below. That is, there is a “dip” in \( f_{\text{nlo}}(p) \). Assume that the dispersion relation is

\[
\omega_{\text{nlo}}(p)^2 = v_{\text{nlo}}(p)^2 p^2 + \omega_{\text{nlo}}(0)^2 .
\]

(14)

Of course \( v_{\text{nlo}}(p) \to 1 \) for \( p \gg \omega_{\text{nlo}}(0) \). For the dip to occur, however, at zero momentum the velocity must satisfy

\[
v_{\text{nlo}}(0) < \frac{\omega_{\text{nlo}}(0)}{m_\pi^0} < 1 .
\]

(15)

The first inequality guarantees that a dip occurs in \( f_{\text{nlo}}(p) \); the second is because Schenk’s results show that \( \omega_{\text{nlo}}(0) < m_\pi^0 \). These are both strict inequalities, and so \( v_{\text{nlo}}(0) < 1 \). This is in accord with our analysis.

4 Linear sigma model

In this section we describe, hopefully in a more comprehensible way than in Ref. [1], where these effects first show up in a linear sigma model in an expansion about zero temperature. The modifications of pion propagation were already computed long ago by Itoyama and Mueller;\(^{14}\) the only thing which we are doing is to turn their results for the pion propagator into results for \( f_\pi^\pm \) and \( f_\pi^\tau \).

\(^{b}\)To leading order in a virial expansion, as eq. (2.4) of ref. [11], we estimate \( \gamma \sim p(T^4/f_\pi^4) \) about zero momentum in the chiral limit.
The lagrangian of the linear sigma model is given by

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} (\phi^2)^2 - h \sigma ,$$  \hspace{1cm} (16)

where $\phi = (\sigma, \vec{\pi})$ is an $O(4)$ isovector field. We introduce a background magnetic field $h$ which is proportional to the current quark mass $m$. For $h = 0$, the vacuum expectation value of the $\sigma$ is $\sigma_0 = \sqrt{\mu^2 / \lambda}$, where we then shift $\sigma \rightarrow \sigma_0 + \sigma$; for two flavors, $f_\pi = \sigma_0$.

We want to compute terms of order $\sim T^4$. There are many such terms, both at one and at two loop order. To avoid being drowned in calculational details, we extract a small subset of terms which are dominant in weak coupling, those $\sim T^4/(f_\pi^2 m_\pi^2)$ at one loop order. In weak coupling, $m_\pi^2 = 2 \lambda f_\pi^2$, and so these terms are larger by $1/\lambda$ than terms $\sim T^4/f_\pi^4$. Indeed, we are lucky not to have to go to two loop order in the first place, as is required in the nonlinear model.

We first review the calculations of Itoyama and Mueller. There are several diagrams which contribute to the pion self energy, fig. 5 of Ref. [13]. However, the only one which can produce the effects we are looking for is where a pion splits into a pion and a sigma, and then recombines. This is because we need a nontrivial momentum dependence in the pion self energy, and this is the only diagram that has it. The integral for this particular diagram is proportional to

$$I(P) = tr_K \frac{1}{K^2((P - K)^2 + m_\pi^2)} ,$$  \hspace{1cm} (17)

where $tr_K = T \sum_{n=-\infty}^{+\infty} \int d^3 k / (2\pi)^3$.

For the real parts of the pion self energy, the trick is simply to expand this integral in powers of the external momentum $P$. At lowest order, this is trivial, since we just take the propagator for the sigma meson, and replace it by a constant:

$$I(P) \sim \frac{1}{m_\pi^2} tr_K \frac{1}{K^2} \sim \frac{1}{m_\pi^2} \frac{T^2}{12} .$$  \hspace{1cm} (18)

We can forget about any ultraviolet divergences along the way, since they will be taken care of by zero temperature renormalization, as usual. (We might have to worry if we found any logarithmic ultraviolet divergences, but we’re only throwing away powers divergences, which is ok.)

To go beyond leading order, then, to $\sim T^4$, all we have to do is to expand the integral in $P$ to $\sim P^2$. Doing so, we find that there are two types of
integrals which enter. One is that of Eq. (18), while the other is
\[ \text{tr}_K \frac{K^\mu K^\nu}{K^2} \sim (\delta^{\mu\nu} - 4n^\mu n^\nu) \frac{\pi^2 T^4}{90}, \] (19)
where \( n^\mu = (1, \vec{0}) \). This appearance of the timelike vector \( n^\mu \) is entirely where \( v < c \) will enter at this order, since it produces different terms in the pion self energy \( \sim p_0^2 \) and \( p^2 \). For both integrals in Eqs. (18) and (19), evidently the scale of the loop momentum \( K \sim T \). This justifies the approximations made.

These are all the integrals which are needed to get the real part of the pion self energy. For the imaginary part, we need the integral
\[ \text{Im } \mathcal{I}(P)|_{\omega \sim p_{<m_s}} \sim \frac{1}{16\pi} e^{\frac{m_\pi^2}{4pT}}, \] (20)
The imaginary part is exponentially small because one needs to scatter off of a sigma meson in the thermal distribution at low temperature. Sigmas are heavy at low temperature, so the probability to pop one out of the distribution is Boltzmann suppressed.

Introducing the quantities
\[ t_1 = \frac{T^2}{12 f_\pi^2}, \quad t_2 = \frac{\pi^2 T^4}{45 f_\pi^2 m_\pi^2}, \quad t_3 = \frac{1}{32\pi} \frac{m_\pi^4}{f_\pi^2 p^2} e^{\frac{m_\pi^2}{4pT}}, \] (21)
to this order the inverse pion propagator is
\[ Z_\pi \Delta^{-1}(P) \sim (1 + t_1 + 6t_2)p_0^2 + (1 + t_1 - 2t_2)p^2 + n_\pi^2 (1 + 3t_1/2) - 2ip^2 t_3. \] (22)
The zero of this expression determines the pion mass shell. In the chiral limit, it is
\[ ip^0 \sim vp - ip t_3, \] (23)
where the pion velocity is given by
\[ v^2 \sim 1 - 8t_2, \] (24)
The result for the pion velocity agrees with (A.8) of Ref. [13]. There appears to be a misprint for the pion damping rate in (A.8') of Ref. [13].

To evaluate \( f_\pi^2 \) and \( f_\sigma^2 \) in the linear sigma model we need the axial current,
\[ A_\mu^a = (\sigma_0 + \sigma)\partial_\mu \pi^a - \pi^a \partial_\mu \sigma. \] (25)
Then we take the two point function of axial currents and saturate it with a single pion in the intermediate state; from Eqs. (9) and (11) of Ref. [1], this equals
\[ \langle 0| \partial^\mu A^a_\mu \partial^\nu A^b_\nu |0 \rangle_T \sim \delta^{ab} \text{Re } f_\pi^4 \left( f_\pi^2 p_0^2 + f_\sigma^2 p^2 \right), \] (26)
Generally, for small momenta this two point function is dominated by the
collection of a single pion, since the collection of other states is of higher
order in momenta, \( \sim (p^2)^2 \), etc. This remains true even if the condition of Eq. (7) is not satisfied: even for strongly damped pions, the dominant contribution
to Eq. (26) arises from single pions in the intermediate state.

At one loop order, the diagrams which contribute to \( f_t^\pi \) and \( f_s^\pi \) are given
in fig. (5) of Ref. [15]. As for the pion self energy, the only diagrams which can
produce the effects which we are looking for are those involving a virtual pion-
sigma meson loop. In the calculation of the two point function in Eq. (26),
this kind of loop enters in two ways. First, simply as self energy insertions on
the intermediate pion. Secondly, from the form of the axial current, they can
also enter through either vertex, as a type of form factor.

For the real parts of the pion decay constants, the integrals in Eqs. (18)
and (19) suffice. For the form factor diagram involving a loop with a sigma
meson and a pion, again one expands the propagator of the sigma meson in
powers of the external momentum. As a form factor, only terms linear in \( P \)
need be retained.

For the imaginary parts, we need two new integrals,

\[
tr_K \frac{k^0}{K^2((P-K)^2 + m_\sigma^2)} \sim \frac{i}{16\pi} \left( \frac{m_\sigma^2}{4p} + T \right) \exp \left( -\frac{m_\sigma^2}{4pT} \right) .
\]  

and

\[
tr_K \frac{k^i}{K^2((P-K)^2 + m_\sigma^2)} \sim \frac{p^i}{16\pi} \left( \frac{m_\sigma^2}{4p} - T \right) \exp \left( -\frac{m_\sigma^2}{4pT} \right) .
\]  

These expressions are valid near the light cone, \( \omega \sim p \ll m_\sigma \). In these integrals,
the dominant loop momentum \( k \) is very large, \( k \sim m_\sigma^2/p \gg m_\sigma \). Thus one
typically retains only those terms \( \sim k \) and not \( \sim p \).

The results for \( f_t^\pi \) and \( f_s^\pi \) are

\[
f_t^\pi \sim (1 - t_1 + 3t_2 + i t_3) f_\pi , \quad f_s^\pi \sim (1 - t_1 - 5t_2 - it_3) f_\pi .
\]  

It is then simple to verify that Eqs. (5) and (6) give the same mass shell condition as found directly from the pion propagator in Eq. (23).

Given that

\[
\langle \mathcal{P} \rangle \sim \sigma_0(T) \sim \sigma_0(0) (1 - 3t_1/2) ,
\]  

we also verify our generalization of the formula of Gell-Mann, Oakes, and
Renner in Eq. (8) for the dynamic pion mass:

\[
m_\pi^2(T) \sim m_\pi^2 (1 + t_1/2 - 6t_2) .
\]  

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From the propagator, this result is really trivial, since it follows simply by using the wave function renormalization of the pion, \( Z_\pi \sim 1 + t_1 + 6t_2 \), and finding the pole in \( p_0^2 \). In detail, what matters is that the coefficient of \( t_2 \) in \( Z_\pi \) and in \((f_\pi^0)^2\) are equal.

Acknowledgments

R. D. P. would like to thank Misha Shifman for inciteful comments made during a talk on a related subject in November, 1995. Misha began by stating that “Using PCAC at nonzero temperature...”, and then went on from there. This led us to pause and ask, exactly what does PCAC mean at nonzero temperature? This is the really the essence of our work.

M.T. would like to thank R. Brout for his comments on spin waves in ferromagnets.

This work is supported by a DOE grant at Brookhaven National Laboratory, DE-AC02-76CH00016.

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