Big Bang Nucleosynthesis constraints on $f(T, T_G)$ gravity

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We confront $f(T, T_G)$ gravity, with Big Bang Nucleosynthesis (BBN) requirements. The former is obtained using both the torsion scalar, as well as the teleparallel equivalent of the Gauss-Bonnet term, in the Lagrangian, resulting to modified Friedmann equations in which the extra torsional terms constitute an effective dark energy sector. We calculate the deviations of the freeze-out temperature $T_f$, caused by the extra torsion terms in comparison to $\Lambda$CDM paradigm. Then we impose five specific $f(T, T_G)$ models and we extract the constraints on the model parameters in order for the ratio $|\Delta T_f/T_f|$ to satisfy the observational BBN bound. As we find, in most of the models the involved parameters are bounded in a narrow window around their General Relativity values as expected, as in the power-law model where the exponent $n$ needs to be $n \lesssim 0.5$. Nevertheless the logarithmic model can easily satisfy the BBN constraints for large regions of the model parameters. This feature should be taken into account in future model building.

PACS numbers: 98.80.-k, 04.50.Kd, 26.35.+c, 98.80.Es

I. INTRODUCTION

There are two motivations that lead to the construction of modifications of gravity. The first is purely theoretical, namely to construct gravitational theories that do not suffer from the renormalizability problems of general relativity and thus being closer to a quantum description [1, 2]. The second is cosmological, namely to construct gravitational theories that at a cosmological framework can describe the early and late accelerating eras [3–7], as well as to alleviate various observational tensions [8].

There is a rich literature on modified and extended theories of gravity. One may start from the Einstein-Hilbert Lagrangian and add extra terms, resulting in $f(R)$ gravity [9–11], in $f(G)$ gravity [12–14], in $f(G, T)$ theories [15], in $f(P)$ gravity [16–18] in Lovelock gravity [19, 20], in Weyl gravity [21], in Horndeski/Galileon scalar-tensor theories [22, 23] etc. Nevertheless, one can follow a different approach, and add new terms to the equivalent torsional formulation of gravity, resulting to $f(T)$ gravity [24, 25], to $f(T, T_G)$ gravity [26–28], to $f(T, B)$ gravity [29, 30], to scalar-torsion theories [31] etc. Torsional gravity has been proven to exhibit interesting phenomenology, both at the cosmological framework [32–60], as well as at the level of local, spherically symmetric solutions [61–78].

One crucial test that every modification of gravity should pass, that is usually underestimated in the literature, is the confrontation with the Big Bang Nucleosynthesis (BBN) data[79–83]. Specifically, the amount of modification needed in order to fulfill the late-time cosmological requirements must not at the same time spoil the successes of early-time cosmology, and among them the BBN phase. Hence, whatever are the advantages of a specific modified theory of gravity, if it cannot satisfy the BBN constraints it must be excluded [84–87].

In the present manuscript we are interested in investigating the BBN epoch in a universe governed by $f(T, T_G)$ gravity. In particular, we desire to study various specific models that are known to lead to viable phenomenology, and extract constrains on the involved model parameters. The plan of the article is the following: In Section II we briefly present $f(T, T_G)$ gravity, extracting the field equations and applying them to a cosmological framework. In Section III we summarize the BBN formalism and we provide the difference in the freeze-out temperature caused by the extra torsion terms. Then in Section IV we investigate five specific $f(T, T_G)$ models, confronting them with the observational BBN bounds. Finally, Section V is devoted to the Conclusions.

II. $f(T, T_G)$ GRAVITY

In this section we briefly review $f(T, T_G)$ gravity [26–28]. As usual in torsional formulation of gravity we use the tetrad field as the dynamical variable, which forms an orthonormal basis at the tangent space. In a coordinate basis one can relate it with the metric through $g_{\mu\nu}(x) = \eta_{AB}e_{A}^{\mu}(x)e_{B}^{\nu}(x)$, where $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$,
and with Greek and Latin letters denoting coordinate and tangent indices respectively. Applying the Weitzenböck connection $W^\lambda_{\nu\rho} \equiv e^\lambda_A \partial^\rho e^A_\nu$ [25], the corresponding torsion tensor is

$$T^\lambda_{\mu\nu} \equiv W^\lambda_{\nu\rho} - W^\lambda_{\mu\rho} = e^\lambda_A (\partial_\rho e^A_\nu - \partial_\nu e^A_\rho) ,$$  (1)

and then the torsion scalar is obtained through the contractions

$$T \equiv \frac{1}{4} T^\rho_{\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T_{\rho\mu\nu} T^\rho_{\nu\mu\rho} - T_{\rho\mu} \rho^\rho_{\mu\rho} \ ,$$  (2)

and incorporates all information of the gravitational field. Used as a Lagrangian, the torsion scalar gives rise to exactly the same equations with General Relativity, that is why the theory was named teleparallel equivalent of general relativity (TEGR).

Similarly to curvature gravity, where one can construct higher-order invariants such as the Gauss-Bonnet one, in torsional gravity one may construct higher-order torsional invariants, too. In particular, since the curvature (Ricci) scalar and the torsion scalar differ by a total derivative, in [26] the authors followed the same recipe and extracted a higher-order torsional invariant which differs from the Gauss-Bonnet one by a boundary term, namely

$$T_G = (K^\kappa_{\pi\lambda} K^\rho_{\lambda\sigma} K^\mu_{\sigma\tau} - 2K^{\mu\lambda} K^\kappa_{\pi\rho} K^\rho_{\sigma\tau} - 2K^{\kappa\lambda} K^\mu_{\pi\rho} K^\rho_{\sigma\tau} - 2K^{\mu\lambda} K^\rho_{\pi\rho} K^\rho_{\sigma\tau} + 2K^{\kappa\lambda} K^\mu_{\pi\rho} K^\rho_{\sigma\tau} + 2K^{\mu\lambda} K^\rho_{\pi\rho} K^\rho_{\sigma\tau}) \delta^\pi_{\lambda\rho\tau} (3)$$

where $K^{\mu\nu\rho}_{\rho} = -\frac{1}{2} (T^{\mu\nu}_{\rho} - T^{\nu\mu}_{\rho} - T^{\mu\nu}_{\rho})$ is the contortion tensor and the generalized $\delta^\pi_{\lambda\rho\tau}$ denotes the determinant of the Kronecker deltas. Note that similarly to the Gauss-Bonnet term, the teleparallel equivalent of the Gauss-Bonnet term $T_G$ is also a topological invariant in four dimensions.

Using the above torsional invariants one can construct the new class of $f(T, T_G)$ gravitational modifications, characterized by the action [26]

$$S = \frac{M_P^2}{2} \int d^4x \, e f(T, T_G) ,$$  (4)

with $M_P^2$ the reduced Planck mass. The general field equations of the above action can be found in [26], where one can clearly see that the theory is different from $f(R)$, $f(R, G)$ and $f(T)$ gravitational modifications, and thus it corresponds to a novel class of modified gravity.

In this case, the torsion scalar (2) and the teleparallel equivalent of the Gauss-Bonnet term (3) become

$$T = 6H^2$$  (7)

$$T_G = 24H^2(H + H^2),$$  (8)

with $H = \frac{\dot{a}}{a}$ the Hubble parameter and where dots denoting derivatives with respect to $t$.

The general field equations for the FRW geometry are [27]

$$f - 12H^2 f_T - T_G f_{T_G} + 24H^2 f_T^2 = 2M_P^2 (\rho_r + \rho_m)$$  (9)

$$f - 4(3H^2 + H) f_T - 4H f_T - T_G f_{T_G} + \frac{2}{3} T_G f_{T_G} = 8H^2 f_T = -2M_P^2 (\rho_r + \rho_m),$$  (10)

with $f_T = f_{TT} T^2 + f_{TGG} T_G T = f_{TTC} T + f_{TTC} T_G T_G$, and $f_{T_{TTT}} = f_{TTTT} T^2 + 2 f_{TTTE} T_T T + f_{TTGG} T_G T^2 + f_{TTC} T + f_{TTC} T_G T_G$, and where $f_{TTT}, f_{TTGG}...$ denote multiple partial differentiations with respect to $T$ and $T_G$. Note that in the above equations we have also introduced the radiation and matter sectors, corresponding to perfect fluids with energy densities $\rho_r, \rho_m$ and pressures $p_r, p_m$, respectively. Lastly, we mention that the above equations for $f(T, T_G) = -T + \Lambda$ recover the TEGR and General Relativity equations, where $\Lambda$ is the cosmological constant.

As we can see, we can re-write the Friedmann equations (9) and (10) in the usual form

$$3M_P^2 H^2 = (\rho_r + \rho_m + \rho_{DE})$$  (11)

$$-2M_P^2 \dot{H} = (\rho_r + \rho_m + p_r + p_m + \rho_{DE} + \rho_{DE}) ,$$  (12)

where we have defined the effective dark energy density and pressure as

$$\rho_{DE} \equiv \frac{M_P^2}{2} \left(6H^2 - f + 12H^2 f_T + T_G f_{T_G} - 24H^3 f_{T_G} \right) ,$$  (13)

$$p_{DE} \equiv \frac{M_P^2}{2} \left[-2(2\dot{H} + 3H^2) + f - 4(\dot{H} + 3H^2) f_T - 4H f_T - T_G f_{T_G} + \frac{2}{3} T_G f_{T_G} + 8H^2 f_T \right] ,$$  (14)

of gravitational origin.

### III. BIG BANG NUCLEOSYNTHESIS CONSTRAINTS

Big Bang Nucleosynthesis (BBN) was a process that took place during radiation era. Let us first present the framework which provides the BBN constraints through standard cosmology [79–83]. The first Friedmann equation from Einstein-Hilbert action can be written as

$$3H^2 = M_P^{-2} \rho ,$$  (15)
where \( \rho = \rho_r + \rho_m \). In the radiation era the radiation sector dominates hence we can write

\[
H^2 \approx \frac{M_P^2}{3} \rho_r = H_{GR}^2.
\]  
(16)

In addition it is known that the energy density of relativistic particles is

\[
\rho_r = \frac{\pi^2}{30} g_* T^4,
\]  
(17)

where \( g_* \approx 10 \) the effective number of degrees of freedom and \( T \) the temperature. Thus, if we combine (16) with (17) we obtain

\[
H(T) \approx \left( \frac{4\pi^3 g_*}{45} \right)^{1/2} \frac{T^2}{M_{Pl}},
\]  
(18)

where \( M_{Pl} = (8\pi)^{1/2} M_P = 1.22 \times 10^{19} \) GeV is the Planck mass.

During the radiation era the scale factor evolves as \( a(t) \sim t^{1/2} \). Therefore, using the relation of Hubble parameter with scale factor we find that in the radiation era the Hubble parameter evolves as \( H(t) = \frac{1}{2t} \). Combining the last one with (18) we find the relation between temperature and time. Thus, we have

\[
\frac{1}{t} \approx \left( \frac{32\pi^3 g_*}{90} \right)^{1/2} \frac{T^2}{M_{Pl}} \quad \text{(or } T(t) \approx (t/\text{sec})^{-1/2} \text{ MeV}).
\]

During the BBN we have interactions between particles. For example we have interactions between neutrons, protons, electrons and neutrinos, namely \( n + \nu_e \rightarrow p + e^- \), and finally

\[
\lambda_{pn} \equiv \lambda_{n+\nu_e\rightarrow p+e^-} + \lambda_{n+e^-\rightarrow p+\nu_e} + \lambda_{n\rightarrow p+e^-+\bar{\nu}_e}
\]

the conversion rate from neutrons to protons is \( \lambda_{pn} \) and it is equal to the sum of the three interaction conversion rates written above. Therefore, the calculation of the neutron abundance arises from the protons-neutron conversion rate [81, 82]

\[
\lambda_{pn}(T) = \lambda_{n+\nu_e\rightarrow p+e^-} + \lambda_{n+e^-\rightarrow p+\nu_e} + \lambda_{n\rightarrow p+e^-+\bar{\nu}_e}
\]

and its inverse \( \lambda_{np}(T) \), and therefore for the total rate we have \( \lambda_{tot}(T) = \lambda_{np}(T) + \lambda_{pn}(T) \). Now, we assume that the various particles (neutrinos, electrons, photons) temperatures are the same, and low enough in order to use the Boltzmann distribution instead of the Fermi-Dirac one, and we neglect the electron mass compared to the electron and neutrino energies. The final expression for the conversion rate is [84–87]

\[
\lambda_{tot}(T) = 4A T^3(4T^2 + 2 \times 3QT + 2Q^2),
\]  
(20)

where \( Q = m_n - m_p = 1.29 \times 10^{-3} \text{GeV} \) is the mass difference between neutron and proton and \( A = 1.02 \times 10^{-11} \text{GeV}^{-4} \).

We proceed in calculating the corresponding freeze-out temperature. This will arise comparing the universe expansion rate \( \frac{1}{H} \) with \( \lambda_{tot}(T) \). In particular, if \( \frac{1}{H} \ll \lambda_{tot}(T) \), namely if the expansion time is much smaller than the interaction time, we can consider thermal equilibrium [79, 80]. On the contrary, if \( \frac{1}{H} \gg \lambda_{tot}(T) \) then particles do not have enough time to interact so they decouple. The freeze-out temperature \( T_f \), in which the decoupling takes place, corresponds to \( H(T_f) = \lambda_{tot}(T_f) \approx c_q T_f^5 \), with \( c_q \approx 4A 4! \approx 9.8 \times 10^{-10} \text{GeV}^{-4} \) [84–87]. Now if we use (18) and \( H(T_f) = \lambda_{tot}(T_f) \approx c_q T_f^5 \), we acquire

\[
T_f = \left( \frac{4\pi^3 g_*}{45 M_{Pl} c_q} \right)^{1/6} \sim 0.0006 \text{ GeV}.
\]  
(21)

Using modified theories we obtain extra terms in energy density due to the modification of gravity. The first Friedmann equation (11) during radiation era becomes

\[
3M_P^2 H^2 = \rho_r + \rho_{DE},
\]  
(22)

where \( \rho_{DE} \) must be very small compared to \( \rho_r \) in order to be in accordance with observations. Hence, we can write (22) using (16) as

\[
H = H_{GR} \sqrt{1 + \frac{\rho_{DE}}{\rho_r}} = H_{GR} + \delta H,
\]  
(23)

where \( H_{GR} \) is the Hubble parameter of standard cosmology. Thus, we have \( \Delta H = \left( \sqrt{1 + \frac{\rho_{DE}}{\rho_r}} - 1 \right) H_{GR} \), which quantifies the deviation from standard cosmology, i.e. form \( H_{GR} \). This will lead to a deviation in the freeze-out temperature \( \Delta T_f \). Since \( H_{GR} = \lambda_{tot} \approx c_q T_f^5 \) and \( 1 + \frac{\rho_{DE}}{\rho_r} \approx 1 + \frac{1}{2} \frac{\rho_{DE}}{\rho_r} \), easily find

\[
\left( \sqrt{1 + \frac{\rho_{DE}}{\rho_r}} - 1 \right) H_{GR} = 5c_q T_f^4 \Delta T_f,
\]  
(24)

and finally

\[
\frac{\Delta T_f}{T_f} \approx \frac{\rho_{DE}}{\rho_r} \frac{H_{GR}}{10c_q T_f^4},
\]  
(25)

where we used that \( \rho_{DE} \ll \rho_r \) during BBN era. This theoretically calculated \( \frac{\Delta T_f}{T_f} \) should be compared with the observational bound

\[
\left| \frac{\Delta T_f}{T_f} \right| < 4.7 \times 10^{-4},
\]  
(26)

which is obtained from the observational estimations of the baryon mass fraction converted to \( ^4 \text{He} \) [88–94].

IV. BBN CONSTRAINTS ON \( f(T, T_G) \) GRAVITY

In this section we will apply the BBN analysis in the case of \( f(T, T_G) \) gravity. Let us mention here that in general, in modified gravity, inflation is not straightforward driven by an inflaton field but the inflaton is hidden inside the gravitational modification, i.e. it is one of the
extra scalar degrees of freedom of the modified graviton. Hence, in such frameworks reheating is usually performed gravitationally, and the reheating and BBN temperatures may differ from standard ones. Nevertheless, in the present work we make the assumption that we do not deviate significantly from the successful concordance scenario, in order to examine whether \( f(T, T_G) \) gravity can at first pass BBN constraints or not. Clearly a more general analysis should be performed in a separate project, to cover more radical cases too. In the following, we will examine five specific models that are considered to be viable in the literature.

1. **Model I:** \( f = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} \)

Firstly we investigate the model \( f = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} \) \[28\]. Since in our analysis we focus on the radiation era where the Hubble parameter \( H(t) = \frac{1}{T} \), we can express the derivatives of the Hubble parameter as powers of the Hubble parameter itself, e.g. \( \dot{H} = -2H^2 \) and \( H = 8H^3 \). Additionally, in order to eliminate one model parameter we will apply the Friedmann equation at present time, requiring

\[
\Omega_{DE0} \equiv \rho_{DE0}/(3M_P^2 H_0^2),
\]

where \( \Omega_{DE} \) is the dark energy density parameter and with the subscript “0” denoting the value of a quantity at present time. Doing so, and inserting \( f = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} \) into (13) and then into (25), we finally find

\[
\frac{\Delta T}{T_f} = (10c_q T_f^3)^{-1} \zeta H_0 \Omega_{DE0} (3 - 2 \beta_2)^{-3/2} \cdot \left[ (9 - 15\beta_2 + 6\beta_2^2) \left( H_0^2 + 2\beta_2 H_0 \right) \right]^{3/2} \cdot \left[ (9 + 3\beta_2 - 2\beta_2^2) H_0^4 + 9\beta_2 H_0^2 H_0 + \beta_2^2 H_0^2 H_0 \right]^{-1},
\]

where

\[
\zeta \equiv \left( \frac{4\pi^3 g_*}{45} \right)^{\frac{1}{2}} M_{Pl}^{-1}.
\]

In this expression we insert \[95\]

\[
\Omega_{DE0} \approx 0.7, \quad H_0 = 1.4 \times 10^{-42} \text{ GeV},
\]

and the derivatives of the Hubble function at present are calculated through \( H_0 = -\dot{H}_0 (1 + q_0) \) and \( H_0 = H_0^2 (j_0 + 3g_0 + 2) \) with \( q_0 = -0.503 \) the current deceleration parameter of the Universe \[95\], and \( j_0 = 1.011 \) the current jerk parameter \[96, 97\]. Hence, \( H_0 \approx -9.7 \times 10^{-89} \text{ GeV}^2 \) and \( \tilde{H}_0 \approx 4.1 \times 10^{-126} \text{ GeV}^3 \).

Using the BBN constraint (26) we conclude that \( \beta_2 \in (-2.98, -2.93) \cup (0.99, 1.01) \), where we have used (27) to find

\[
\beta_1 = \sqrt{3} H_0 \Omega_{DE0} \left[ (3 + 2\beta_2) H_0^2 + 2\beta_2 \dot{H}_0 \right]^{3/2} \cdot \left[ (9 + 3\beta_2 - 2\beta_2^2) H_0^4 + 9\beta_2 H_0^2 \dot{H}_0 + \beta_2^2 H_0^2 \ddot{H}_0 \right]^{-1}.
\]

Using the above range of \( \beta_2 \) we find that \( \beta_1 \in (2.09 \times 10^{-20}, 0.001) \cup (1.380, 1.384) \).

In Fig. 1 we depict \( |\Delta T_f/T_f| \) appearing in (28) versus the model parameter \( \beta_2 \). As we can see the allowed range is within the vertical dashed lines.

2. **Model II:** \( f = -T + a_1 T^2 + a_2 T \sqrt{T_G} \)

Let us now study the case \( f = -T + a_1 T^2 + a_2 T \sqrt{T_G} \), where \( a_1, a_2 \) are the free parameters of the theory \[28\]. In this case we find

\[
\frac{\Delta T_f}{T_f} = \frac{3}{10 c_q T_f^3} \zeta^3 T_f \left\{ \frac{\Omega_{DE0}}{3 H_0^2} \right\}^{\frac{1}{2}} \cdot \sqrt{6 a_2} \left[ \frac{\sqrt{H_0^2 + H_0}}{6 H_0} \left( 6 - \frac{2 H_0^2 - H_0 \ddot{H}_0}{H_0^2 + \dot{H}_0} \right) - 1 \right].
\]

Using the constraint (26), ans since according to (32) \( \Delta T_f/T_f \) is linear in \( a_2 \), we deduce that (32) is valid for a small region around \( 2.7 \times 10^{83} \text{ GeV}^{-2} \), where we have used the constraint from current cosmological era (27)

\[
a_1 = \frac{\Omega_{DE0}}{18 H_0^2} \sqrt{6 a_2} \sqrt{\frac{H_0^2 + H_0}{36 H_0}} \left[ 6 - \frac{2 H_0^2 - H_0 \ddot{H}_0}{H_0^2 + \dot{H}_0} \right]^{3/2}.
\]

Using the above value of \( a_2 \) we find that \( a_1 = -1.1 \times 10^{83} \text{ GeV}^{-2} \).
3. Model III: \( f = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + a_1 T^2 + a_2 T \sqrt{|T_G|} \)

Now we analyze the model \( f = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + a_1 T^2 + a_2 T \sqrt{|T_G|} \), where we have four free parameters, namely \( \beta_1, \beta_2, a_1, a_2 \) [28]. In order to simplify the analysis we will impose the constraint \(-2.99 < \beta_2 < \frac{9}{4}\), obtained above.

In this case we find

\[
\frac{\Delta T_f}{T_f} = - (60 \epsilon_q T_f)^{-1} \left\{ \frac{3 \sqrt{2} \beta_1 (3 - 2 \beta_2)^{-1/2} (1 + \beta_2 - 2 \beta_1)}{\sqrt{18 H_0^2}} \right. \\
- 18 \sqrt{\Omega_{DE0} + \frac{3 \sqrt{2} \beta_1}{18 H_0^2}} \left[ (3 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{-1/2} \\
\cdot \left[ (3 - 6 \beta_1 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right] \\
- a_2 \sqrt{6 \dot{H}_0} \left[ (3 - 6 \beta_1 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{3/2} \\
\cdot \left[ (3 + 2 \beta_2) H_0^2 + (9 + 8 \beta_2) H_0 \dot{H}_0 + \beta_2 \left( 4 H_0^2 + \dot{H}_0 \right) \right] \right\} c^2 T_f \right\} \zeta.
\]

Observing that expression (34) is linear in \( a_2 \), and using the constraint (26) and two values for \( \beta_1 \) from the aforementioned range we extracted in model I, i.e. \( \beta_1 = 1.4 \) and \( \beta_2 = 1 \), we find that (32) is valid for a small region around the point \(-3.5 \times 10^{83} \text{ GeV}^{-2} \). Using another set of values \( (\beta_1 = 0.001, \beta_2 \approx -2.96) \) we find that (32) is valid for a small region around the point \(-5.3 \times 10^{83} \text{ GeV}^{-2}\), where we have used

\[
a_1 = \frac{\Omega_{DE0}}{18 H_0^2} + \frac{\sqrt{12} \beta_1}{108 H_0^2} \left[ (3 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{-1/2} \\
\cdot \left[ (3 - 6 \beta_1 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right] \\
- \frac{\sqrt{12} \beta_1}{108 H_0^2} \left[ (3 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{-3/2} \\
\cdot \left[ (3 + 2 \beta_2) H_0^2 + (9 + 8 \beta_2) H_0 \dot{H}_0 + \beta_2 \left( 4 H_0^2 + \dot{H}_0 \right) \right],
\]

from (27). Imposing the above range of \( a_2 \) we find that \( a_1 = 1.4 \times 10^{83} \text{ GeV}^{-2} \) for the first case and \( a_1 = 2.2 \times 10^{83} \text{ GeV}^{-2} \) for the second.

4. Model IV: \( f = -T + \beta_1 \left( T^2 + \beta_2 T_G \right)^n \)

As a next model we consider the power-law model \( f = -T + \beta_1 \left( T^2 + \beta_2 T_G \right)^n \), where the free parameters are \( \beta_1, \beta_2, n \). In this model we use values of \( \beta_1, \beta_2 \) in order to constrain the power \( n \). In this case, repeating the above steps, we find

\[
\frac{\Delta T_f}{T_f} = (10 \epsilon_q)^{-1} \Omega_{DE0} H_0^{2(1-n)} \zeta^{4n-1} T_f^{n-7} (3 - 2 \beta_2)^{n-2} \\
\cdot \left[ (3 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{-2n} \\
\cdot \left[ (9 - 12 \beta_2 + 4 \beta_2^2) H_0^2 + 2 \beta_2 \dot{H}_0 \right] \\
\cdot \{ (9 + 12 \beta_2 + 4 \beta_2^2) H_0^2 + 2 \beta_2 \dot{H}_0 \}^{2n} \\
\cdot \{ (9 + 12 \beta_2 + 4 \beta_2^2) H_0^2 + 2 \beta_2 \dot{H}_0 \}^{-1}.
\]

We use the constraint (26) and four values for \( \beta_2 \) from the range we extracted in model I above. For \( \beta_2 \approx -2.9 \) we find that the constraint (26) is valid for \( n \lesssim 0.5 \). Similarly, using the value \( \beta_2 = -2 \) we find \( n \lesssim 0.47 \), while for \( \beta_2 = -1 \) we find \( n \lesssim 0.46 \). Finally, for \( \beta_2 = 1 \) we find \( n \lesssim 0.47 \). We mention that we have used the relation

\[
\beta_1 = -6 (12)^{-n} H_0^{2(1-n)} \Omega_{DE0} \left[ (3 + 2 \beta_2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{2-n} \\
\cdot \left[ (9 + 12 \beta_2 + 4 \beta_2^2) H_0^2 + 2 \beta_2 \dot{H}_0 \right] \\
\cdot \left[ (9 + 12 \beta_2 + 4 \beta_2^2) H_0^2 + 2 \beta_2 \dot{H}_0 \right]^{2n} \\
\cdot \{ (9 + 12 \beta_2 + 4 \beta_2^2) H_0^2 + 2 \beta_2 \dot{H}_0 \}^{-1},
\]

which arises from (27).

Now taking \( \beta_2 \approx -2.9 \), \( n \lesssim 0.5 \) we find \( \beta_1 \in [-6.1 \times 10^{-82}, 0.0007] \text{ GeV}^{2(1-2n)} \). Similarly, for \( \beta_2 = -2 \), \( n \lesssim 0.47 \) we find \( \beta_1 \in [-3.5 \times 10^{-74}, 5.9 \times 10^{-6}] \text{ GeV}^{2(1-2n)} \), while using \( \beta_2 = -1 \), \( n \lesssim 0.46 \) we find \( \beta_1 \in [-4.4 \times 10^{-58}, 1.2 \times 10^{-6}] \text{ GeV}^{2(1-2n)} \). Finally, for \( \beta_2 = 1 \), \( n \lesssim 0.47 \) we find \( \beta_1 \in [-6.4 \times 10^{-8}, 9.0 \times 10^{-6}] \text{ GeV}^{2(1-2n)} \).

In order to provide the above results in a more transparent way, in Fig. 2, we present \( |\Delta T_f|/T_f | \) from (35) in terms of the model parameter \( n \). As we observe, \( n \) needs to be \( n \lesssim 0.5 \) to pass the BBN constraint (26).

5. Model V: \( f = -T + a \ln \beta_1 \left( T^2 + \beta_2 T_G \right)^n \)

The last model we examine is the logarithmic one, characterized by \( f = -T + a \ln \beta_1 \left( T^2 + \beta_2 T_G \right)^n \), where \( \beta_1, \beta_2, n \) are the free parameters. Repeating the above
As an example, in Fig. 3 we present $|\Delta T_f/T_f|$ from (37) as a function of the model parameter $n$. The model parameter $n$ is allowed to take all possible values except those values around a very small region centered at $-0.0003$, in which (37) diverges. Hence, we conclude that the logarithmic $f(T, T_G)$ model can easily satisfy the BBN bounds.

![Graph showing $|\Delta T_f/T_f|$ vs the model parameter $n$.](image)

**FIG. 2:** $|\Delta T_f/T_f|$ vs the model parameter $n$ (blue solid curve), for Model IV: $f = -T + \beta_1 \left(T^2 + \beta_2 T_G\right)^n$ with $\beta_2 \approx -2.90$, and the upper bound for $|\Delta T_f/T_f|$ from (36) (red dashed line). As we observe, constraints from BBN require $n \lesssim 0.5$.

![Graph showing $|\Delta T_f/T_f|$ vs the model parameter $n$.](image)

**FIG. 3:** $|\Delta T_f/T_f|$ vs the model parameter $n$ (blue solid curve), for Model V: $f = -T + \alpha \ln \left(T^2 + \beta_2 T_G\right)^n$, choosing $\beta_1 = 0.001 \text{ GeV}^{-4n}$, $\beta_2 \approx -2.7$. The vertical dashed line at $n = -0.0003$ denotes the point where (37) diverges.

**V. CONCLUSIONS**

Modified gravity aims to provide explanations for various epochs of the Universe evolution, and at the same time to improve the renormalizability issues of General Relativity. Nevertheless, despite the specific advantages at a given era of the cosmological evolution one should be very careful not to spoil other, well understood and significantly constrained, phases, such as the Big Bang Nucleosynthesis (BBN) one.

In particular, there are many modified gravity models, which are constructed phenomenologically in order to be able to describe the late-time universe evolution at both background and perturbation level. Typically, these models are confronted with observational data such as Supernovae Type Ia (SNIa), Baryonic Acoustic Oscillations (BAO), Cosmic Microwave Background (CMB), Cosmic Chronometers (CC), Gamma-ray Bursts (GRB), growth data, etc. The problem is that although modified gravity scenarios, through the extra terms they induce, are very efficient in describing the late-time universe, quite often they induce significant terms at early times too, and thus spoiling the early-time evolution, such as the BBN phase, in which the concordance cosmological paradigm is very successful. Hence, independently of the late-universe successes that a modified gravity model may have, one should always examine whether the model can pass the BBN constraints too.

In the present work we confronted one interesting class
of gravitational modification, namely $f(T, T_G)$ gravity, with BBN requirements. The former is obtained using both the torsion scalar, as well as the teleparallel equivalent of the Gauss-Bonnet term, in the Lagrangian. Hence, one obtains modified Friedmann equations in which the extra torsional terms constitutes an effective dark energy sector.

We started by calculating the deviations of the freeze-out temperature $T_f$, caused by the extra torsion terms, in comparison to $\Lambda$CDM paradigm. We imposed five specific $f(T, T_G)$ models that have been proposed in the literature in phenomenological grounds, i.e. in order to be able to describe the late-time evolution and lead to acceleration without an explicit cosmological constant. Hence, we extracted the constraints on the model parameters in order for the ratio $|\Delta T_f/T_f|$ to satisfy the BBN bound $|\Delta T_f/T_f| < 4.7 \times 10^{-4}$. As we found, in most of the models the involved parameters are bounded in a narrow window around their General Relativity values, as expected. However, the logarithmic model can easily satisfy the BBN constraints for large regions of the model parameters, which acts as an advantage for this scenario.

We stress here that we did not fix the cosmological parameters to their General Relativity values, on the contrary we left them completely free and we examined which parameter regions are allowed if we want the models to pass the BBN constraints. The fact that in most models the parameter regions are constrained to a narrow window around their General Relativity values was in some sense expected, but in general is not guaranteed or known a priori, since many modified gravity models are completely excluded under the BBN analysis since for all parameter regions their early-universe effect is huge.

In conclusion, $f(T, T_G)$ gravity, apart from having interesting cosmological implications both in inflationary and late-time phase, possesses particular sub-classes that can safely pass BBN bounds, nevertheless the torsional modification is constrained in narrow windows around the General Relativity values. This feature should be taken into account in future model building.

Acknowledgments

This research is co-financed by Greece and the European Union (European Social Fund-ESF) through the Operational Programme “Human Resources Development, Education and Lifelong Learning” in the context of the project “Strengthening Human Resources Research Potential via Doctorate Research” (MIS-5000432), implemented by the State Scholarships Foundation (IKY). The work of N.E.M. is supported in part by the UK Science and Technology Facilities research Council (STFC) under the research grant ST/T000759/1. S.B., N.E.M. and E.N.S. also acknowledge participation in the COST Association Action CA18108 “Quantum Gravity Phenomenology in the Multimessenger Approach (QG-MM)”.

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