Multiplier Theorem for Fourier Series in continuous-discrete Sobolev orthogonal polynomials.

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Abstract. In paper we study the multipliers of Fourier series in polynomials orthogonal in continuous-discrete Sobolev’s spaces. Multiplier Theorem for Fourier-Sobolev series is obtained. This result based on the representation of the Fejér kernel, on the construction of the “lumpbacked majorant” and weighted estimates of maximal functions.

Key words. Orthogonal polynomials, Fourier series, multipliers, partial sums, Fejér’s averages, Dirichlet’s kernel, Fejér kernel, Sobolev’s polynomials, continuous-discrete spaces, Lebesgue’s points, continuously-discrete Sobolev spaces, multipliers of convergence

Let \( \theta(x) \) be a positive Borel measure in \([-1,1] \), with infinitely many points of increasing and let the masspoints \( a_k, \ -1 \leq a_k \leq 1, \ k = 1,2,\ldots,m \). For \( f \) and \( g \) in \( L^2_{\theta}[-1,1] \) such that there exist the derivatives in \( a_k \), one can introduce the inner product

\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)d\theta(x) + \sum_{k=1}^{m} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)g^{(i)}(a_k),
\]

where \( M_{k,i} \geq 0 \) (\( i = 0,1,2,\ldots,N_k - 1 \); \( M_{k,N_k} > 0, k = 1,2,\ldots,m \)) and \( \theta(\{a_k\}) = 0 \) (\( k = 1,2,\ldots,m \)).

Linear spaces with this inner product is called «continuous-discrete Sobolev spaces».

Let \( \{\tilde{q}_n(x), n \in \mathbb{Z}_+, \mathbb{Z}_+ = \{0,1,2,\ldots\}; x \in [-1,1] \} \) be the sequence of polynomials of degree \( n \) with a positive leading coefficients orthonormal with respect to the inner product (continuous-discrete Sobolev orthonormal polynomials)

\[
\langle \tilde{q}_n, \tilde{q}_m \rangle = \int_{-1}^{1} \tilde{q}_n(x)\tilde{q}_m(x)d\theta(x) + \sum_{k=1}^{m} \sum_{i=0}^{N_k} M_{k,i} \tilde{q}_n^{(i)}(a_k)\tilde{q}_m^{(i)}(a_k) = \delta_{n,m}.
\]

Denote by \( \mathcal{R}_p(1 \leq p < \infty) \) the set of functions

\[
\mathcal{R}_p = \left\{ f, \ f, \int_{-1}^{1} |f(x)|^p d\theta(x) < \infty; \ f^{(i)}(a_k) - \text{ exist; } \ i = 0,1,2,\ldots,N_k; \ -1 \leq a_k \leq 1(k = 1,2,\ldots,m) \right\}.
\]

To each \( f \in \mathcal{R}_p \) we assign Fourier-Sobolev series

\[
f(x) \sim \sum_{k=0}^{\infty} c_k(f)\tilde{q}_k(x) \ (x \in [-1,1]),
\]

with Fourier coefficients

\[
c_k(f) = \langle f, \tilde{q}_k \rangle = \int_{-1}^{1} f(x)\tilde{q}_k(x)d\theta(x) + \sum_{s=1}^{m} \sum_{i=0}^{N_s} M_{s,i} f^{(i)}(a_s)\tilde{q}_k^{(i)}(a_s).
\]

We consider the following sequence of the real numbers

\[
\Phi = \{\phi_k, \ k = 0,1,2,\ldots; \ \phi_0 = 1; \ \{\phi_k\}_{k=0}^{\infty} \in 1^\infty \}.
\]

For any function \( f \in \mathcal{R}_p \) by their Fourier-Sobolev series we introduce the linear transformation \( T \) defined by relation

\[
T(f; x; \Phi) \sim \sum_{k=0}^{\infty} \phi_k c_k(f)\tilde{q}_k(x). \tag{1}
\]

Transformation \( T \) is called the multiplier operator, the sequence \( \{\phi_k\}_{k=0}^{\infty} \) is called the multiplier of convergence and series \( (1) \) is called the multiplier of convergence and series \( (1) \) is called the multiplier series.
We investigate some problems of pointwise and uniform multipliers of convergence for Fourier-Sobolev series. Multiplier Theorem for the Fourier-Sobolev series is obtained. There are many papers have been devoted to continuous-discrete Sobolev orthonormal polynomials and Fourier series (see, for example [1]–[27]).

Some results about multipliers of the Fourier series in polynomials orthonormal in continuous-discrete Sobolev spaces were announced in [20].

Let $N_k^*$ be-the-positive integer number defined by

$$N_k^* = \begin{cases} N_k + 1, & \text{if } N_k \text{ is odd}, \\ N_k + 2, & \text{if } N_k \text{ is even}, \end{cases}$$

$$w_N(x) = \prod_{k=1}^{m}(x - a_k)^{N_k^*}, \quad N = \sum_{k=1}^{m} N_k^*, \quad \pi_{N+1}(x) = \int_{-1}^{1} w_N(t)dt.$$ 

Orthonormal polynomials $\tilde{q}_n(x)$ satisfy the following recurrence relation

$$\pi_{N+1}\tilde{q}_n(x) = \sum_{j=0}^{N+1} d_{n+j,j}\tilde{q}_{n+j}(x) + \sum_{j=1}^{N+1} d_{n,j}\tilde{q}_{n-j}(x) (n \in \mathbb{Z}_+; \tilde{q}_{-j} = 0, j = 1, 2, \ldots; d_{n,s} = 0, n = 0, 1, \ldots, s-1).$$

Define by

$$\varepsilon_m = (-1, 1) \cup_{s=1}^{m} \{a_s\}.$$ 

The sequence $\Phi = \{\phi_n, n = 0, 1, 2, \ldots; \phi_0 = 1\}$ is called quasiconvex if

$$\sum_{k=0}^{\infty} (k+1)|\Delta^2 \phi_k| < \infty,$$

where $\Delta \phi_k = \phi_k - \phi_{k+1}, \Delta^2 \phi_k = \Delta(\Delta \phi) = \phi_k - 2\phi_{k+1} + \phi_{k+2} (k = 0, 1, \ldots, n)$.

**Theorem 1.** Let the orthonormal polynomial system $\{\tilde{q}_k(x)\}_{k=0}^{\infty}$ be satisfy the following condition

$$|\tilde{q}_k(t)| \leq h(t) (t \in \varepsilon_m) \quad (2)$$

and for the recurrence coefficients the estimate

$$\sum_{j=1}^{N+1} \sum_{l=0}^{N+1} \sum_{s=0}^{\infty} |d_{s+j,j} - d_{s+j+l,j}| + |d_{s+j,l} - d_{s+j+l,l}| < \infty \quad (3)$$

holds. If for quasiconvex sequence $\Phi$ the relation

$$\phi_k = O\left(\frac{1}{\ln k}\right) (k \to \infty) \quad (4)$$

holds, then the following statements are valid:

(i) let for each function $f \in \mathfrak{R}_p (1 \leq p < \infty)$ be fulfilled

$$\int_{-1}^{1} |f(t)|^p h^p(t)\,d\theta(t) < \infty, \quad \int_{-1}^{1} h^p(t)\,d\theta(t) < \infty, \quad (5)$$

then at every Lebesgue’s point $x \in \varepsilon_m$ (and, consequently, a.e.) the series $[\mathbf{1}]$ converges

$$T(f; x; \Phi) = \sum_{k=0}^{\infty} \phi_k \tilde{c}_k(f) \tilde{q}_k(x);$$

(ii) in addition, suppose function $f$ is continuous in $[-1, 1]$ and the measure $d\theta(x)$ is absolutely continuous and

$$d\theta(x) = \omega(x)\,dx, \quad \omega(x) \text{ is continuous in } \varepsilon_m; \quad (6)$$

then the series $[\mathbf{1}]$ is uniformly converges on compact subsets $K \subset \varepsilon_m$. 
We define for $f \in \mathcal{R}_p$ the space $W^p_\theta(F)(1 \leq p < \infty)$ for subset $F \subseteq [-1, 1]$:  

$$W^p_\theta(F) = \{ f, \|f\|_{W^p_\theta(F)} < +\infty, \|f\|_{L^p_\theta(F)} = \|f\|_{W^p_\theta(F)}^p + \sum_{k=1}^{N_k} \sum_{i=0}^{N_k} M_k, |f^{(i)}(a_k)|^p \}. $$

The space $W^p_\theta([-1,1])(1 \leq p < \infty)$ is not complete.

**Theorem 2.** Let the orthonormal polynomial system $\{\tilde{q}_k(x)\}_{k=0}^\infty$ be satisfy the following condition (2), (3), (6) and

$$\sup_{n \in \mathbb{Z}_+} \sum_{j=0}^{n} |q_j^{(i)}(a_n)| < \infty (i = 0, 1, \ldots, N_k; s = 1, 2, \ldots, m),$$

$$|g||h||_{L^p_\theta([-1,1])} < \infty, \|h||L^p_\theta([-1,1]) < \infty \left(1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1\right).$$

If the sequence $\Phi$ is quasiconvex and satisfy (1), then for $f \in W^p_\theta([-1,1])(1 < p < \infty)$, satisfying (5), on any compact subsets $K \subseteq \varepsilon_m$ the following estimate

$$||T(f;x;\Phi)||_{W^p_\theta(K)} \leq C_p ||f||_{W^p_\theta([-1,1])},$$

holds, where the constant $C_p > 0$ indepent on function $f$ and the sequence $\Phi$.

**Remark.** Symmetric Gegenbauer-Sobolev orthonormal polynomials $\{\tilde{B}_n^{(\alpha)}(x)\}_{n \in \mathbb{Z}_+; x \in [-1,1]}$ orthonormal in an inner product

$$\langle f, g \rangle _a = \int_{-1}^1 f(x)g(x)w_\alpha(x)dx + M [f(1)g(1) + f(-1)g(-1)] + N [f'(1)g'(1) + f'(-1)g'(-1)] \quad (M \geq 0; N \geq 0),$$

where

$$w_\alpha(x) = \frac{\Gamma(2\alpha + 2)}{2^{2\alpha+1}\Gamma(\alpha + 1)}(1 - x^2)^\alpha \left(\alpha > \frac{1}{2}\right),$$

satisfying the conditions (2), (3), (5), (7), (8).

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