RICCI FALL-OFF IN STATIC, GLOBALLY HYPERBOLIC, NON-SINGULAR SPACETIMES

DAVID GARFINKLE
STEVEN G. HARRIS

Abstract. What restrictions are there on a spacetime for which the Ricci curvature is such as to produce convergence of geodesics (such as the preconditions for the Singularity Theorems) but for which there are no singularities? We answer this question for a restricted class of spacetimes: static, geodesically complete, and globally hyperbolic. The answer is that, in at least one spacelike direction, the Ricci curvature must fall off at a rate inversely quadratic in a naturally-occurring Riemannian metric on the space of static observers. Along the way, we establish some global results on the static observer space, regarding its completeness and its behavior with respect to universal covering spaces.

1. Introduction

The Ricci curvature of a spacetime is what is used to drive the Singularity Theorems of Hawking and Penrose [1,2]. Essentially, if $\text{Ric}$ along a geodesic has a positive lower bound, then a conjugate point must occur on the geodesic within a certain length [3,4]. However, in a globally hyperbolic spacetime timelike-related points must be joined by maximal geodesics (no conjugate points). One can then obtain a contradiction if Ricci curvature is sufficiently large and all geodesics are complete.

So suppose we disallow singularities—assume geodesic completeness—but keep global hyperbolicity: What is forced to happen? Just how must the Ricci curvature behave?

To simplify matters, we shall also assume a static spacetime. The energy condition we shall use is the null energy condition: $\text{Ric}(N,N) \geq 0$ for all null vectors $N$.

In section 2 we prove some results about the space of static observers. Section 3 contains our result on the behavior of the Ricci tensor. In section 4 we discuss the nature of this result and consider two examples of the behavior of the Ricci tensor in spacetimes satisfying our conditions.

2. The space of static observers

Our spacetime $M$ comes equipped with a field of distinguished observers, the static observers (i.e., the integral curves of the timelike Killing field); this amounts to a one-dimensional foliation of $M$. Any foliation $\mathcal{F}$ of a manifold $M$ gives rise to the leaf space $Q$, the quotient of $M$ by the equivalence relation of two points being equivalent if they lie on the same leaf of the foliation. For us, $\mathcal{F}$ is the set of static
observer worldlines; the space $Q$ of static observers is the most natural object to use for analyzing the geometry of $M$.

Geroch [5], in the context of a stationary spacetime, introduced the notion of examining the observer space $Q$. In the general stationary spacetime, there is no guarantee that $Q$ is even remotely well-behaved topologically (it can be totally indiscrete!). However, it is shown in Harris [6] that a chronological stationary spacetime must have an observer space that is almost a manifold: locally Euclidean, but possibly not Hausdorff.

In order to show that $Q$ is fully a manifold (i.e., that it is Hausdorff), all we need do is show that the Killing field $U$ is a complete vector field [6]. It is shown in Proposition 9.30 of O’Neill [7] that geodesic completeness implies completeness of the Killing field. However, we want to use the weaker condition of either timelike or null geodesic completeness. We now show

**Lemma 1.** Let $(M, g)$ be a stationary spacetime that is timelike or null geodesically complete. Then the Killing field is complete.

**Proof.** For any point $p$ let $\gamma_p$ be the integral curve of the Killing field $U$ for which $\gamma_p(0) = p$. Let $(a_p, b_p)$ be the maximum interval on which this curve is defined. Let $q$ be a point not on $\gamma_p$ such that there is a timelike (respectively, null) geodesic from $p$ to $q$ and also from $q$ to a point $p' \neq p$ on $\gamma_p$. We will show first that $a_q \leq a_p$ and $b_q \geq b_p$.

We proceed by constructing a two-surface $\Sigma$ with boundary $\gamma_p$ and $\gamma_q$ and ruled by timelike (respectively, null) geodesics. Let $\zeta_0$ be the geodesic from $p$ to $q$ with $\zeta_0(0) = p$ and $\zeta_0(1) = q$. Let $X$ be the tangent vector to $\zeta_0$ at $p$ and extend $X$ along $\gamma_p$ by $[X, U] = 0$; note that since $U$ is Killing, its flow is an isometry, so $X$ is timelike (respectively, null) at all points. Now let $\zeta_t$ be the geodesic with $\zeta_t(0) = \gamma_p(t)$ and $\zeta_t'(0) = X$ (by timelike or null geodesic completeness, $\zeta_t(s)$ is defined for all $s$); let $\Sigma = \{\zeta_t(s) | t \in (a_p, b_p), s \in [0, 1]\}$. Note that $\zeta_t$ is the image of $\zeta_0$ under the isometry generated by $U$ within $\Sigma$; in particular, $\zeta_t(1) = \gamma_q(t)$. In other words, $\gamma_q$ is defined for at least the same parameter values as is $\gamma_p$: $a_q \leq a_p$ and $b_q \geq b_p$.

By the same argument, $a_{p'} \leq a_q$ and $b_{p'} \geq b_q$: $(a_{p'}, b_{p'}) \supset (a_p, b_p)$; but, since $p$ and $p'$ are on the same integral curve of $U$, this can happen only if $a_p = -\infty$ and $b_p = \infty$. It follows that $U$ is complete. □

Now using the results of [6] we have

**Theorem 2.** Let $M$ be a timelike or null geodesically complete, chronological, stationary spacetime. Then the space $Q$ of stationary observers is a (Hausdorff) manifold. □

The observer space $Q$ tells quite a lot about the topology and causal structure of $M$: It contains all of the information on the “spacelike topology” of $M$. This is so in a rather strong sense:

**Theorem 3.** Let $M$ and $Q$ be as in the previous theorem. Then $M$ is diffeomorphic to $Q \times \mathbb{R}$. Furthermore, for any edgeless, achronal, embedded spacelike hypersurface $N$ in $M$, $N$ must be diffeomorphic to $Q$.

**Proof.** Let $\pi : M \to Q$ be the projection to the orbit space, making $M$ a smooth line bundle over $Q$ (the differentiable structure on $Q$ comes from local identification
with a cross-section in a flow-box for the Killing field). Since the basespace $Q$ is a (Hausdorff) manifold, the bundle has global cross-sections, and $M$ is diffeomorphic to $Q \times \mathbb{R}$ (see, for instance, [10], Theorem I.5.7). Let $N$ be a spacelike hypersurface in $M$. We will see that $\pi$, restricted to $N$, provides a diffeomorphism with $Q$, under the right circumstances.

Since $N$ is spacelike and of the same dimension as $Q$ and is a manifold without boundary, and $\pi_*$ kills only timelike vectors, $\pi(N)$ must be an open subset of $Q$. With $N$ achronal, $\pi$ must be injective on $N$ ($\pi(x) = \pi(y)$ implies there is a Killing orbit from $x$ to $y$, which is a timelike curve between the two points). It follows that $\pi$ provides a diffeomorphism from $N$ to its image in $Q$; we need only see how $N$ being edgeless makes this image all of $Q$. Since $N$ is assumed embedded and achronal, the easiest way to formulate “edgeless” is to take $N$ to be closed in $M$.

Since $\pi(N)$ is open, we need only prove it closed ($Q$ is connected, being the continuous image of the connected space $M$). Let $q$ be a point in $Q$, the limit of points $\{\pi(x_n) = q_n\}$ with $x_n \in N$. Pick a point $p$ in $\gamma = \pi^{-1}(q)$, the Killing orbit corresponding to $q$. Let $P$ be a cross-section through $p$ of the Killing-field foliation. Let $\gamma_n$ be the foliate corresponding to $q_n$; then each $\gamma_n$ intersects $P$ in precisely one point $p_n$. Let each $\gamma_n$ be parametrized as an integral curve of the Killing field with $\gamma_n(0) = p_n$, and similarly parametrize $\gamma$ with $\gamma(0) = p$. We then have each $x_n = \gamma_n(t_n)$ for some $t_n$. Another way to formulate that is in terms of the $R$-action on $M$ induced by the Killing action: $x_n = t_n \cdot p_n$, where $s \cdot y$ denotes movement along the Killing orbit through $y$ by parameter-value $s$.

All we need do now is show that the $\{t_n\}$ have an accumulation point $t$, for then $x = \gamma(t) = t \cdot p$ will be a limit point of the $x_n$, so $x \in N$ and $q = \pi(x) \in \pi(N)$. But the only way for the $\{t_n\}$ to avoid having an accumulation point is if they go off to plus or minus infinity. Consider any small neighborhood $V$ around $q$ in $Q$, and let $W$ be the tubular neighborhood $\pi^{-1}(V)$ of $\gamma$ in $M$; we pick $V$ sufficiently small that $W$ is a standard static spacetime. For $x_n \in W$, the past and future null cones from $x_n$ strike the side of $W$, so that the portion $W_0$ of $W$ not timelike-related to $x_n$ is relatively compact. For all $m$ with $q_m \in V$, $x_m = t_m \cdot p_m$ must lie in $W_0$ (since $N$ is achronal). But for $|s|$ sufficiently large, $s \cdot p_m$ will lie outside $W_0$; thus, the $\{t_m\}$ must be bounded. □

Thus, in particular, when $M$ is globally hyperbolic, $Q$ has the topology of a Cauchy surface. Note, however, that when $M$ is static, it does not follow that the restspaces—the hypersurfaces perpendicular to the Killing field—have the topology of $Q$. This is because the restspaces need not, in general, be achronal (although any restspace is necessarily a covering space of $Q$). An example would be the Minkowski cylinder $S^1 \times \mathbb{L}^1$ ($\mathbb{L}^n$ denotes Minkowski $n$-space), with Killing field $U = d/dt + k(d/d\theta)$ for some small non-zero constant $k$; the restspaces are spacelike helices (topologically lines), while $Q$ is a circle.

The stationary observer space $Q$ comes equipped with a natural Riemannian metric $h$, as shown by Geroch [5]: The spacetime metric $g$ can be represented (locally, in general; globally, if $Q$ is a Hausdorff manifold) as $g = -(\Omega \circ \pi)\alpha^2 + \pi^*h$, where $\Omega$ is a function on $Q$, $\pi$ is projection to $Q$, $\alpha$ is the one-form obeying $\alpha U = 1$ and $\text{Ker}(\alpha) = U^\perp$, and $h$ is a Riemannian metric on $Q$; $|U|^2 = \Omega \circ \pi$. The spacetime is static if and only if $d\alpha = 0$.

When $M$ is static, $h$ is the metric coming from the projection of the restspaces to $Q$. It turns out, however, that this is not quite the appropriate metric to use. For
many purposes, the appropriate metric on $Q$ is $\tilde{h} = \Omega^{-1} h$; call this the conformal metric on the static observer space $Q$. This is the metric useful for calculating the causal properties of $M$ in terms of curves on $Q$, and it also has this nice property:

**Theorem 4.** Let $M$ be a timelike or null geodesically complete, static, globally hyperbolic spacetime. Then the static observer space $Q$ is complete in the conformal metric.

**Proof.** We first examine the case in which $M$ is simply connected: The spacetime metric is $g = (\Omega \circ \pi) [-\alpha^2 + \pi^* \tilde{h}]$. Since $(M, g)$ is globally hyperbolic it follows that $(M, \tilde{g})$ is globally hyperbolic where $\tilde{g} = -\alpha^2 + \pi^* \tilde{h}$. Since $g$ is static, $d\alpha = 0$. Since $M$ is simply connected it follows that there is a globally defined scalar function $\tau$ on $M$ such that $d\tau = \alpha$. The function $\tau$ measures parameter-value along the integral curves of the Killing field $U$; since this is complete (Lemma 1), the map $\tau : M \to \mathbb{R}$ is onto. Then $\pi$ and $\tau$ provide the global identification $(M, \tilde{g}) \cong (Q, \tilde{h}) \times \mathbb{L}^1$. It then follows from proposition 2.54 of reference [8] that $(Q, \tilde{h})$ is complete.

We next consider the general case: Let $\tilde{M}$ be the universal covering space for $M$, with $p_M : \tilde{M} \to M$ the canonical projection. We need to know that we can apply the previous paragraph to $(\tilde{M}, \tilde{g})$, where $\tilde{g} = p_M^* g$. A lemma to this effect is in order.

**Lemma 4.1.** Let $M$ be a spacetime and $\tilde{M}$ its universal covering space, with metric induced from $M$. Then $\tilde{M}$ inherits all these properties from $M$:

1) static or stationary
2) energy conditions
3) geodesic completeness of any type
4) global hyperbolicity.

**Proof of 4.1.** Any vector field on $M$ induces a corresponding field on $\tilde{M}$; if the first is Killing, so is the second, and the same goes for the integrability of the perpendicular distributions. Any energy conditions are clearly inherited (the projection being a local isometry), as is geodesic completeness of any type (any appropriate geodesic segment projects to an extendible segment, and the extension lifts to an extension).

Suppose $M$ is globally hyperbolic. There is a Cauchy surface $\Sigma$ in $M$. Let $\Sigma = p^{-1}(\Sigma)$, where $p : \tilde{M} \to M$ is the standard projection. Then $\tilde{\Sigma}$ is a Cauchy surface for $\tilde{M}$: For any inextendible timelike curve $\tilde{c}$ in $\tilde{M}$, $c = p \circ \tilde{c}$ is an inextendible timelike curve in $M$. For any parameter-value $t$, $c(t) \in \Sigma$ if and only if $\tilde{c}(t) \in \tilde{\Sigma}$. Thus, $\tilde{c}$ intersects $\tilde{\Sigma}$ in exactly one point, since that is true for $c$ and $\Sigma$. This shows that $\tilde{M}$ is also globally hyperbolic. $\square$

(Although not needed for our purposes here, it is perhaps worth noting that the causality, chronology, and strong causality properties are also inherited by universal covering spaces.)

By Lemma 4.1, we can apply the results of the first paragraph to $(\tilde{M}, \tilde{g})$: Let $\tilde{Q}$ be the space of static observers in $\tilde{M}$, with $\tilde{\pi} : \tilde{M} \to \tilde{Q}$ the projection; then $\tilde{Q}$ is complete in the conformal metric $\tilde{h}$ (notation here is that $\tilde{g} = - (\Omega \circ \tilde{\pi}) \tilde{\alpha}^2 + \tilde{\pi}^* \tilde{h} = (\tilde{\Omega} \circ \tilde{\pi}) \tilde{\alpha}^2 + \tilde{\pi}^* \tilde{h}$). All we need to do is compare this with $Q$ and its conformal metric $h$.

First note that $P_M$ induces a map $P_Q : \tilde{Q} \to Q$, since if $\tilde{x}$ and $\tilde{y}$ in $\tilde{M}$ lie on the same Killing orbit $\tilde{z}$, then $x$ and $y$ lie on the same Killing orbit $z$, where $z = p_{\tilde{M}}^* \tilde{z}$.
y = p_M \bar{y}, \text{ and } \gamma = p_M \circ \bar{\gamma}; \text{ we have } p_Q \circ \bar{\pi} = \pi \circ p_M. \text{ Also note that the metric induced on } \bar{Q} \text{ via } p_Q \text{ from that on } Q \text{ is the same as that which } \bar{Q} \text{ inherits from the static structure } \bar{\pi} : \bar{M} \to \bar{Q} \text{ (because } p_M \text{ is a local isometry)—} p_Q^* h = \bar{h}. \text{ This applies also to the respective conformal metrics (since the conformal factor is the same in both cases)—} p_Q^* \bar{h} = \bar{h}. \text{ Thus, if we can show that } p_Q : \bar{Q} \to Q \text{ is the standard projection (universal covering map) of the universal covering space for } Q, \text{ then the completeness of } (\bar{Q}, \bar{h}) \text{ will imply, by standard covering space arguments, the completeness of } (Q, \bar{h}). \text{ Thus, we need}

**Lemma 4.2.** Let } M \text{ be a chronological stationary spacetime with a complete Killing field; let } \pi : M \to Q \text{ be the projection to the stationary orbit space. Let } p_M : M \to \bar{M} \text{ be the universal covering map from the universal covering space for } M, \text{ and let } \bar{\pi} : \bar{M} \to \bar{Q} \text{ be the projection to the stationary orbit space. Let } p_Q : \bar{Q} \to Q \text{ be the map induced by } p_M \text{ (i.e., } p_Q \circ \bar{\pi} = \pi \circ p_M). \text{ Then } p_Q \text{ is the universal covering map from the universal covering space for } Q.

**Proof of 4.2.** By the results in [6], } Q \text{ is a (Hausdorff) manifold; since } \bar{M} \text{ inherits chronology, stationarity, and completeness of the Killing field from } M, \bar{Q} \text{ is also a manifold. The map } p_Q \text{ is onto (an orbit in } M \text{ is mapped onto by the corresponding orbit in } \bar{M} \text{) and is locally a homeomorphism (since } p_M \text{ is, and the orbit spaces are formed analogously in the two spacetimes). Therefore, to show that } p_Q \text{ is a covering map, all that we need is that it evenly covers small sets in } Q, \text{ i.e., that for any } q \in Q, \text{ there is a neighborhood } U \text{ of } q \text{ such that } p_Q^{-1}(U) \text{ is the disjoint union of sets } \{V_\alpha\} \text{ in } Q \text{ with each restriction } p_Q \mid_{V_\alpha} : V_\alpha \to U \text{ being a homeomorphism.}

Given } q \in Q, \text{ pick any simply connected neighborhood } U \text{ of } q \text{ such that } \pi : M \to Q \text{ is locally trivial over } U, \text{ i.e., } \pi^{-1}(U) \cong U \times \mathbb{R}. \text{ Then } \pi^{-1}(U) \text{ is simply connected also, so it is evenly covered by } p_M \text{ (a universal covering map evenly covers an open set if and only if it is simply connected). Thus, } p_M^{-1}(\pi^{-1}(U)) \text{ is a disjoint union of open sets } \{W_\alpha\} \text{ in } M, \text{ each one carried homeomorphically to } \pi^{-1}(U) \text{ by } p_M. \text{ Let } V_\alpha = \bar{\pi}(W_\alpha) \text{ in } \bar{Q}.

In order to show each } V_\alpha \text{ is open, we need to show } W_\alpha \text{ is full, i.e., contains only entire equivalence classes of points; that means that } W_\alpha \text{ must contain the entire orbit of any point it contains. But if } \bar{\gamma} \text{ is an orbit in } \bar{M} \text{ intersecting } W_\alpha, \text{ then } \gamma = p_M \circ \bar{\gamma} \text{ is an orbit in } M \text{ intersecting } \pi^{-1}(U). \text{ It follows that } \gamma \text{ wholly lies in } \pi^{-1}(U), \text{ so } \bar{\gamma} \text{ lies in the disjoint union of the } \{W_\beta\}; \text{ since these are disjoint open sets, } \bar{\gamma} \text{ must lie entirely in } W_\alpha. \text{ Thus, } V_\alpha \text{ is open. Also, the collection } \{V_\beta\} \text{ is disjoint: } \{W_\beta\} \text{ is disjoint and made up of full sets.}

It remains to be shown that, restricted to any } V_\alpha, \text{ } p_Q \text{ is a homeomorphism to } U; \text{ this is straight-forward: First, } p_Q \mid_{V_\alpha} \text{ is onto } U, \text{ because } p_M(W_\alpha) = \pi^{-1}(U) \text{ and } \pi(\pi^{-1}(U)) = U. \text{ Next, it is injective: If } p_Q(\bar{q}_1) = p_Q(\bar{q}_2), \text{ then for any points } \bar{x}_1 \text{ in the corresponding orbits } \bar{\gamma}_1 \text{ in } \bar{M}, \pi(p_M(\bar{x}_1)) = \pi(p_M(\bar{x}_2)). \text{ Since } \pi \text{ is trivial over } U, \text{ this means that } p_M(\bar{x}_1) \text{ and } p_M(\bar{x}_2) \text{ lie on the same orbit in } M; \text{ since } p_M \text{ is injective on } W_\alpha, \text{ that means the orbits } \bar{\gamma}_1 \text{ and } \bar{\gamma}_2 \text{ must coincide, so } \bar{q}_1 = \bar{q}_2. \text{ That } p_Q \text{ is continuous follows automatically from the commuting diagram } p_Q \circ \bar{\pi} = \pi \circ p_M. \text{ Finally, } p_Q \text{ is open: An open set in } \bar{Q} \text{ is precisely } \bar{\pi}(Z), \text{ where } Z \text{ is a full open set in } \bar{M}; \text{ then } p_Q(\bar{\pi}(Z)) = \pi(p_M(Z)). \text{ Since } p_M \text{ is an open map (being a covering projection), } p_M(Z) \text{ is open in } M. \text{ Since } Z \text{ is made up of } \bar{M}\text{-orbits, } p_M(Z) \text{ is made up of } M\text{-orbits, i.e., it is full; therefore, } \pi(p_M(Z)) \text{ is open in } Q.
Thus, \( p_q \) is a covering projection. Since \( \tilde{M} \) is simply connected and \( \tilde{M} \cong \tilde{Q} \times \mathbb{R} \), it follows that \( \tilde{Q} \) is simply connected: \( p_q \) is the universal covering map from the universal covering space for \( Q \). \( \square \)

This completes the proof of Theorem 4. \( \square \)

It is the complete Riemannian manifold \((Q, \tilde{h})\) that provides the proper context to discuss the geometry of \( M \).

3. Results

Let us now apply the energy condition. For a static spacetime the Killing vector \( U \) is an eigenvector of \( R^a_b \) (the linear map corresponding to \( \text{Ric} \)) (see [11]). Call the corresponding eigenvalue \(-S\), i.e., \( S = \text{Ric}(U, U)/|U|^2 \). Then we have

**Proposition 5.** In a static spacetime obeying the null energy condition, for any unit timelike vector \( T \), \( \text{Ric}(T, T) \geq S \).

**Proof.** Let \( V = U/|U| \), the unit vector in the direction of \( U \). Let \( X \) be any unit spacelike vector orthogonal to \( V \); then \( V + X \) is a null vector. The null energy condition then yields \( S + \text{Ric}(X, X) \geq 0 \).

Now let \( T \) be any unit timelike vector. Then \( T \) has the form \( T = aV + bX \) where \( X \) is a unit spacelike vector orthogonal to \( V \) and \( a \) and \( b \) satisfy \( a^2 - b^2 = 1 \). We then find

\[
\text{Ric}(T, T) = a^2 S + b^2 \text{Ric}(X, X) = (a^2 - b^2) S + b^2 [S + \text{Ric}(X, X)] \geq S.
\]

\( \square \)

For any point \( q_0 \) in \( Q \) and any \( r > 0 \), let \( B_r(q_0) = \{ q \in Q \mid d(q, q_0) \leq r \} \), where \( d \) is the distance function from the conformal metric \( \tilde{h} \). Note that since \((Q, \tilde{h})\) is complete, \( B_r(q_0) \) is always compact. Let \( S_r(q_0) = \min_{B_r(q_0)} S \) (\( S \) can be thought of as a function on \( Q \)). The main result of this paper is

**Theorem 6.** Let \((M, g)\) be a static, globally hyperbolic, timelike or null geodesically complete spacetime satisfying the null energy condition. Then for each point \( q_0 \in Q \), for all \( r > 0 \), \( S_r(q_0) \leq K/r^2 \), where \( K = 3\pi^2/(4|U|^2_{q_0}) \).

**Proof.** Again, we first treat the case where \( M \) is simply connected. Given \( q_0 \in Q \), let \( \gamma \) be the integral curve of \( U \) corresponding to \( q_0 \). For each \( r > 0 \) let \( \zeta \) be a unit-speed maximal geodesic joining \( \gamma(0) \) to \( \gamma(2r) \). The curve \( \zeta \) is timelike in the metric \( \tilde{g} = -d\tau^2 + \pi^*\tilde{h} \) (with notation the same as in the proof of Theorem 4). Since \( \tau \) measures the parameter along \( \gamma \), it then follows that the projection of \( \zeta \) to \( Q \) must remain in \( B_r(q_0) \). Going back to the spacetime metric \( g \) and using the result of Proposition 5, we then find \( \text{Ric}(\tilde{\zeta}, \tilde{\zeta}) \geq S_r(q_0) \). Since \( \zeta \) is maximal it has no conjugate points, and it has length at least \( 2r|U|_{q_0} \). Then applying the Lorentzian analogue of Myers’ theorem [4] we find that \( S_r(q_0) \leq 3\pi^2/(4|U|^2_{q_0} r^2) \).

Now we consider the case where \( M \) is not simply connected. As before, let \( M \) be the universal covering space of \( M \) with projection \( p_M : \tilde{M} \to M \) and \( \tilde{g} \) the induced metric. Then, by Lemma 4.1, \((M, \tilde{g})\) is static, globally hyperbolic, and timelike or null geodesically complete.
null geodesically complete, and satisfies the null energy condition. With \( \tilde{B}_r \) and \( \tilde{S}_r \) defined on the static observer space \( \tilde{Q} \), we have \( \tilde{S}_r(\tilde{q}) \leq 3\pi^2/\left(4|\tilde{U}|^2_{\tilde{q}} r^2\right) \) for any \( \tilde{q} \in \tilde{Q} \).

By Lemma 4.2, the induced map \( p_Q : \tilde{Q} \to Q \) is a covering projection, and \( p_Q^* \tilde{h} = \tilde{h} \). It follows that for \( p_Q(\tilde{q}) = q \) and \( p_Q(\tilde{q}_0) = q_0 \), \( \tilde{d}(\tilde{q}, \tilde{q}_0) \geq d(q, q_0) \), where \( \tilde{d} \) and \( d \) are the respective conformal distance functions (since projection by \( p_Q \) preserves conformal lengths of curves). For any \( q_0 \in Q \), pick a \( \tilde{q}_0 \in p_Q^{-1}(q_0) \); then for any \( r > 0 \), there will be some point \( \tilde{q} \in \tilde{B}_r(\tilde{q}_0) \) with \( \tilde{S}(\tilde{q}) \leq K/r^2 \) (where \( K = 3\pi^2/\left(4|\tilde{U}|^2_{\tilde{q}_0}\right) \); since \( |\tilde{U}|_{\tilde{q}_0} = |U|_{q_0} \), this is the correct \( K \)). Let \( q = p_Q(\tilde{q}) \); then \( S(q) = \tilde{S}(\tilde{q}) \), and \( d(q, q_0) \leq r \). Therefore, \( S_r(q_0) \leq K/r^2 \). \( \square \)

Since \( Q \) is complete in the conformal metric, we can always find a direction—that is to say, a geodesic emanating from \( q_0 \)—such that, along that direction, \( S \) actually goes down at least as fast as \( K/r^2 \).

None of what we have done presupposes that \( S \) is actually positive, though the conclusion is vacuous if \( S \) is anywhere non-positive, once \( r \) is large enough for \( B_r \) to contain any such point. If we assume positivity of \( S \), we obtain a restriction on the spacelike topology:

**Corollary 7.** Let \( M \) be a static, globally hyperbolic, timelike or null geodesically complete spacetime satisfying the energy condition that \( \text{Ric}(T, T) > 0 \) for all timelike vectors \( T \). Then the static observer space \( Q \) is not compact.

**Proof.** This energy condition implies the null energy condition, so Theorem 6 yields \( S_r \leq K/r^2 \) for all \( r \); this implies \( \inf \inf Q S \leq 0 \). On the other hand, \( S = \text{Ric}(U, U)/|U|^2 \) is positive at all points of \( Q \), so if \( Q \) were compact, \( S \) would have to have a positive infimum. \( \square \)

### 4. Discussion

We expect some result along the lines that in order to avoid gravitational collapse, spacetime must not contain “too much matter.” This naive expectation is made more precise by noting that global hyperbolicity and geodesic completeness put constraints on the Ricci tensor, specifically, on \( \text{Ric}(T, T) \) where \( T \) is the unit tangent vector to a maximal timelike geodesic segment: This component of \( \text{Ric} \) is obliged, somewhere along any such geodesic segment, to be less than an inverse-quadratic function of the length of the segment.

The specific form of the results we have obtained depends upon the spacetime being foliated by a system of “canonical” observers, with measurement being made with respect to this canonical observer space. For any such foliated spacetime, if there are sufficient completeness conditions to assure that there is a maximal geodesic between any pair of points on each observer orbit, and that the observer orbits are infinitely long, then we will be able to conclude that there will be points of small Ricci-value associated with each observer. The difficulty comes in finding a geometric framework for these points. In physical terms, what is wanted is this: Given two events on the worldline of a base-point canonical observer, consider an arbitrary observer moving from one event to the other one; we need to be able to say how far that observer can wander in the canonical observer space. If we can place a bound on that wandering, in terms of the proper time between these
two events, then we can say that the place of small Ricci-value associated to the maximal geodesic between those events occurs (in the canonical observer space) within that bound from the base-point.

In this paper we have considered static spacetimes, largely because we can then easily bound the wandering of an arbitrary observer between events on the worldline of a static observer. Even a generalization to stationary spacetimes becomes problematical due to the difficulty of finding such a bound.

From the conditions used in theorem 6 one might have hoped for a stronger bound on the behavior of the Ricci tensor. We have placed bounds on only one component of $\text{Ric}$. Furthermore we have shown only that that component must fall off in one direction rather than in all directions. We now consider two examples that show that stronger bounds of this sort do not apply. First consider the Einstein Static Universe [1] whose metric is given by $g = -dt^2 + a^2 h_{S^3}$, where $a$ is a constant and $h_{S^3}$ is the metric of the unit three-sphere. This spacetime is a homogeneous, isotropic, and static universe with Killing field $U = d/dt$. The matter in this universe is dust and a cosmological constant. The Einstein Static Universe satisfies all the hypotheses of Theorem 6; it also satisfies $\text{Ric}(U, U) = 0$, i.e., $S = 0$. However, $\text{Ric}(X, X) = 2/a^2$ for any unit vector $X$ orthogonal to $U$. Thus $S$ vanishes, but the other components of $\text{Ric}$ do not fall off at all.

Our second example is Melvin’s Magnetic Universe [9] whose metric is given by

$$ds^2 = F^2 (-dt^2 + d\rho^2 + dz^2) + F^{-2} \rho^2 d\phi^2 .$$

Here $F = 1 + (b_0 \rho/2)^2$ where $b_0$ is a constant. This spacetime is a static, cylindrically symmetric solution of the Einstein-Maxwell equations. It represents an infinitely long tube of magnetic flux held together by its own gravity. The constant $b_0$ is the value of the magnetic field on the axis. Melvin’s Magnetic Universe satisfies all of the hypotheses of Theorem 6. The static observer space with the conformal metric is given by $(Q, h) = (\mathbb{R}^3, dp^2 + dz^2 + F^{-4} \rho^2 d\phi^2)$. Straightforward calculation shows that $S = b_0^2/F^4$ and that, with $q_0$ given by $\rho = z = 0$, $S_r = b_0^2 / \left[1 + (b_0 r/2)^2\right]^4$.

We see that $S_r$ falls off at an appropriate rate in $r$, but that $S$ falls off only as $\rho$ goes to infinity and is independent of position along the cylindrical axis (the $z$ direction). Thus spacetimes satisfying the hypotheses of Theorem 6 need not have $S$ falling off in all directions.

Acknowledgements

We would like to thank Gary Horowitz and Mael Melvin for helpful discussions. We would also like to thank the Aspen Center for Physics for hospitality. This research was supported in part by NSF grant DMS9310477 to St. Louis University, NSF Grant PHY9408439 to Oakland University and by a Cottrell College Science Award of Research Corporation to Oakland University.

References

[1] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge University, Cambridge, 1973.
[2] R. M. Wald, General Relativity, University of Chicago, Chicago, 1984.
[3] J. Cheeger and D. G. Ebin, Comparison Theorems in Riemannian Geometry, North-Holland, Amsterdam, 1975.
[4] S. G. Harris, Indiana U. Math. J. 31 (1982), 289–308.
[5] R. Geroch, J. Math. Phys. 12 (1971), 918–924.
[6] S. G. Harris, Class. Quantum Grav. 9 (1992), 1823–1827.
[7] B. O’Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.
[8] J. K. Beem and P. E. Ehrlich, Global Lorentzian Geometry, Marcel Dekker, New York, 1981.
[9] M. A. Melvin, Phys. Lett. 8 (1964), 65–68.
[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. I, Interscience, New York, 1963.
[11] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, Exact Solutions of Einstein’s Field Equations, Cambridge University Press, Cambridge, 1980.

Department of Physics, Oakland University, Rochester, MI 48309
E-mail address: garfinkl@oakland.edu

Department of Mathematics, St. Louis University, St. Louis, Mo 63103, USA
E-mail address: harrissg@sluca.slu.edu