MULTIPLICATION OPERATOR AND EXCEPTIONAL JACOBI POLYNOMIALS

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Abstract. Below the normalized zero-counting measure based on the regular zeros of exceptional Jacobi polynomials, and the normalized weighted reciprocal of the Christoffel function with respect to exceptional Jacobi polynomials are investigated. It is proved that both measures tend to the equilibrium measure of the interval of orthogonality in weak-star sense. The main tool of this study is the multiplication operator and examination of the behavior of zeros of the corresponding average characteristic polynomial. Finally, as an application of multiplication operator, the zeros of certain self-inversive polynomials are examined.

1. Introduction

First we sketch some problems which are strongly related to each other. These problems have an extended literature, below only some examples are cited. Let \( w \) be a weight function on a real interval \( I \), that is \( w > 0 \) in the interior of \( I \), and \( w \) has finite moments. Let \( \{ q_k \}_{k=0}^{\infty} \) be the standard orthonormal polynomials on \( I \) with respect to \( w \). Let \( \xi_i := \xi_{i,k} \), \( i = 1, \ldots, k \) the zeros of \( q_k \). In several cases (also in more general circumstances) it is proved that the normalized counting measure based on the zeros of standard orthogonal polynomials tends to the equilibrium measure of the interval of orthogonality is weak-star sense, that is

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{\xi_i} = \mu_{e,I},
\]

see e.g. [17] and the references therein.

On the other hand the measure, defined as the weighted reciprocal of the Christoffel function, tends to the equilibrium measure of \( I \) again:

\[
\lim_{n \to \infty} d\mu_n(x) = \lim_{n \to \infty} \frac{1}{n} K_n(x,x)d\mu(x) = \mu_{e,I},
\]

see e.g. [20], [11].

The three-term recurrence relation fulfilled by standard orthogonal polynomials, implies that the multiplication operator, \( M : f(x) \to xf(x) \), acting on the weighted \( L^2_w \) space has a close relation to the previous two measures. As \( M \) can be represented by a tridiagonal (Jacobi) matrix, denoting by \( \pi_n \) the projection operator to the \( n \)-dimensional subspace, the eigenvalues of \( \pi_nM\pi_n \) are just the zeros of \( q_n \) and limits like (1) and (2) can be proved, see e.g. [29], [28].

On the other hand, considering \( x_1, \ldots, x_N \) as random variables, the joint probability

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distribution on $\mathbb{R}^N$ is $g_{N,n}(x_1,\ldots,x_N) = c(n,N) \det |K_n(x_i, x_j)|_{i,j=1}^N \prod_{i=1}^N W(x_i)$, where $c(n,N)$ is a normalization factor. The expectation $\mathbb{E}$ refers to $\rho$. The average characteristic polynomial is $\chi_n(z) := \mathbb{E} \left( \prod_{i=1}^n (z - x_i) \right)$. According to [30] (2.2.10), the zeros of the average characteristic polynomial are just the zeros of $q_n$ again. Thus (1) and (2) can be proved by this technique as well, see e.g. [11], [12].

Below we deal with exceptional Jacobi polynomials. Exceptional orthogonal polynomials are complete systems of polynomials with respect to a positive measure. They are different from the standard or from the classical orthogonal polynomials, since exceptional families have finite codimension in the space of polynomials. Similarly to the classical ones exceptional polynomials are eigenfunctions of Sturm-Liouville-type differential operators but unlike the classical cases, the coefficients of these operators are rational functions. Exceptional orthogonal polynomials also possess a Bochner-type characterization as each family can be derived from one of the classical families applying finitely many Darboux transformations, see [6]. Exceptional orthogonal polynomials were introduced recently by Gómez-Ullate, Kamran and Milson, cf. e.g. [8], [9] and the references therein. These families of polynomials play a fundamental role for instance in the construction of bound-state solutions to exactly solvable potentials in quantum mechanics. In the last few years there has been a great deal of activity in this area both by mathematicians and physicists, cf. e.g. [11], [25], [13], [7], etc. The location of zeros of exceptional orthogonal polynomials is also examined, cf. e.g. [8], [10], [18], [15], [16].

The results on zero-distribution summarized above can be derived considering the three-term recurrence relation fulfilled by standard orthogonal polynomials. Since exceptional orthogonal polynomials fulfil $2L + 1$ recurrence formulae with $L \geq 2$, the situation here is different. In [14] by combinatorial methods it is proved that in one codimensional Jacobi case $\mu_n$ tends to $\mu_{c,1}$ in weak-star sense. The relation between the transformed, normalized zero-counting measure based on the zeros of the modified average characteristic polynomial and $\mu_n$ is also studied there.

The aim of this investigation is to extend (1) and (2) to exceptional Jacobi polynomials of any codimension. To examine the relation between the (modified) average characteristic polynomial and the exceptional orthogonal polynomials the main tool is the multiplication operator. The location of zeros of exceptional Jacobi polynomials is investigated by outer ratio asymptotics. Finally the multiplication operator method is applied to determine the zeros on the unite circle of certain self-inversive polynomials.

2. Notation and the main result

2.1. General construction of exceptional orthogonal polynomials. Subsequently we use the Bochner-type characterization of exceptional polynomials given in [6]. Classical orthogonal polynomials $\{P_n^{[0]}\}_{n=0}^\infty$ are eigenfunctions of the second order linear differential operator with polynomial coefficients

$$T[y] = py'' + qy' + ry,$$

and its eigenvalues are denoted by $\lambda_n$. $T$ can be decomposed as

$$T = BA + \lambda, \quad \text{with } A[y] = b(y' - wy), \quad B[y] = \hat{b}(y' - \hat{w}y), \quad \text{(3)}$$
where \( b, w \) are rational functions and

\[
\hat{b} = \frac{p}{b}, \quad \hat{w} = -w - \frac{q}{p} + \frac{b'}{b}.
\]

Then the exceptional polynomials are the eigenfunctions of \( \hat{T} \), that is the partner operator of \( T \), which is

\[
\hat{T}[y] = (AB + \tilde{\lambda})[y] = py'' + \hat{q}y' + \hat{r}y,
\]

where

\[
\hat{q} = q + p' - 2b'b'p, \quad \hat{r} = r + q' + wp' - \frac{b'}{b}(q + p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p,
\]

and \( w \) fulfills the Riccati equation

\[
p(w' + w^2) + qw + r = \tilde{\lambda},
\]

cf. [6 Propositions 3.5 and 3.6].

(3) and (5) ensure that

\[
\hat{T}AP_n[0] = \lambda_nAP_n[0],
\]

so exceptional polynomials can be obtained from the classical ones by application of (finite) appropriate first order differential operator(s) to the classical polynomials.

This observation motivates the notation below

\[
AP_n[0] = b\left(P_n[0]\right)' - bwP_n[0]' =: P_n[1],
\]

(and recursively \( A_sP_n[s-1] =: P_n[s] \) in case of \( s \) Darboux transformations.) The degree of \( P_n[1] \) is usually greater than \( n \). \( \left\{P_n[1]\right\}_n=0^\infty \) is an orthogonal system on \( I \) with respect to the weight

\[
W := \frac{pw_0}{b^2},
\]

where \( w_0 \) is one of the classical weights.

**Remark.** - Since at the endpoints of \( I \) (if there is any) \( p \) may possesses zeros, \( b \) can be zero here as well, but \( b \) does not have zeros inside \( I \), otherwise the moments of \( W \) would not be finite.

- As each \( P_n[1] \) is a polynomial \( (n = 0, 1, \ldots) \), applying the operator \( A \) to \( P_0[0] \) it can be seen that \( bw \) must be a polynomial and to \( P_1[0] \) shows that \( b \) itself is also a polynomial.

- Let us recall that \( r = 0 \) in the classical differential operators. Considering (7) and comparing degrees, \( w \) itself can not be a polynomial. By the same reasons \( pw \) can not be a polynomial unless it is of degree one. \( w \) is a rational function and it has no poles in \((-1, 1)\).

- Expressing (3) as

\[
\hat{b}BP_n[1] = p\left(P_n[1]\right)' + \left(pw + q - \frac{b'}{b}\right)P_n[1] = (\lambda_n - \tilde{\lambda})bP_n[0],
\]

it can be easily seen that if \( P_n[1] \) had got a double zero at \( x_0 \in \text{int} I \), then \( P_0[0](x_0) = 0 \), and by (9) \( \left(P_n[0]\right)'(x_0) = 0 \), which is impossible, that is that zeros of \( P_n[1] \) which are in the interior of the interval of orthogonality are simple.
Let $a$ be (one of) the finite endpoint(s) of the interval of orthogonality. Again by (11) one can derive that if $b(a) \neq 0$, then $P_n^{[1]}(a) \neq 0$ as well.

If there was an $n \in \mathbb{N}$ such that $P_n^{[1]}(a) = 0$, then $b(a) = 0$ and by (13) $(bw)(a) = 0$ and so $P_n^{[1]}(a) = 0$ for all $n$. That is, taking $\tilde{P}_n^{[1]} := b_1 \left( P_n^{[0]} \right)' - b_1 w P_n^{[0]}$, where $b_1(x) = \frac{b(x)}{x - a}$, we arrive to the exceptional system orthogonal with respect to $W = \frac{p_{\text{max}}}{b_1}$. Thus we can assume that $\forall \ n\ P_n^{[1]}(a) \neq 0$, and if $b(a) = 0$, then $(bw)(a) \neq 0$.

### 2.2. Exceptional Jacobi polynomials

Let the $n^{th}$ Jacobi polynomial defined by Rodrigues’ formula:

$$
(1 - x)^\alpha (1 + x)^\beta P_n^{\alpha, \beta}(x) = \frac{(-1)^n}{2^n n!} \left( (1 - x)^{\alpha+n} (1 + x)^{\beta+n} \right)^{(n)},
$$

where $\alpha, \beta > -1$.

$$
p_k := p_k^{\alpha, \beta} = \frac{P_k^{\alpha, \beta}}{\varrho_k^{\alpha, \beta}},
$$

(12) $\varrho_k^{\alpha, \beta} := \left( \varrho_k^{\alpha, \beta} \right)^2 = \frac{2^{\alpha+\beta+1} \Gamma(k + \alpha + 1) \Gamma(k + \beta + 1)}{(2k + \alpha + \beta + 1) \Gamma(k + 1) \Gamma(k + \alpha + \beta + 1)}$.

The orthonormal Jacobi polynomials which fulfil the following differential equation (cf. (4.2.1))

$$
(1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0.
$$

With

$$
P_n^{[0]} = p_n^{(\alpha, \beta)} = p_n
$$

and

$$
P_n^{[1]} = AP_n^{[0]} = b(P_n^{[0]})' - bwP_n^{[0]}.
$$

The next examples of $X_m$ Jacobi polynomials, that is exceptional Jacobi polynomials given by one Darboux transformation and of codimension $m$ can be found in [10]. These are as follows.

$$
w_0 = w^{(\alpha, \beta)} = (1 - x)^\alpha (1 + x)^\beta,
$$

where $\alpha, \beta$ are defined appropriately, see [10] Proposition 5.1.

$$
T[y] = (1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' = BA - (m - \alpha)(m + \beta + 1),
$$

where $A$ and $B$ are defined in (3), and

$$
b(x) = (1 - x)P_m^{(-\alpha, -\beta)}(x), \quad w(x) = (\alpha - m) \frac{P_m^{(-\alpha-1, -\beta-1)}(x)}{(1 - x)P_m^{(-\alpha, -\beta)}(x)}.
$$

So the defined exceptional Jacobi polynomials, $P_n^{[1]} := AP_n^{[0]}$, are orthogonal with respect to

$$
W(x) = \frac{(1 - x^2)w^{(\alpha, \beta)}(x)}{(1 - x)^2 \left( P_m^{(-\alpha, -\beta)}(x) \right)^2} = \frac{w^{(\alpha-1, \beta+1)}(x)}{\left( P_m^{(-\alpha, -\beta)}(x) \right)^2}.
$$
and the space spanned by these classes are $m$-codimensional in the space of polynomials.

We restrict our investigations to one-step Darboux transformation case. As it is mentioned above, $b$ can be zero at $\pm 1$. Subsequently we assume that $b$ has got at most simple zeros at the endpoints of the interval of orthogonality, that is

\[(17) \quad \frac{p}{b} = \tilde{p} \quad \text{is bounded on} \quad [-1, 1].\]

Introducing the notation

\[(18) \quad b(x) = (1 - x)^{1-\varepsilon_1} (1 + x)^{1-\varepsilon_2} \tilde{p}(x),\]

where $\varepsilon_i = \pm 1$, $i = 1, 2$; the exceptional Jacobi polynomial system, $\{\hat{P}_n^{[1]}\}_{n=0}^{\infty}$, is orthogonal on $(-1, 1)$ with respect to

\[(19) \quad W = \frac{w^{(\alpha + \varepsilon_1, \beta + \varepsilon_2)}}{b^2}.\]

The orthonormal system is denoted by $\{\hat{P}_n\}_{n=0}^{\infty}$. The codimension is given by the degree of $\tilde{b}$ cf. [6].

2.3. Formulation of the main theorem.

**Theorem 1.** Let $\{\hat{P}_n\}_{n=0}^{\infty}$ be the orthonormal system of exceptional Jacobi polynomials generated by one Darboux transformation from the original Jacobi polynomials and of arbitrary codimension. If $\alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2}$, then

\[(20) \quad \nu_n \rightarrow \nu_e\]

in weak-star sense, where

\[d\nu_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \hat{P}_k^2(x) W(x) dx\]

and $\nu_e$ is the equilibrium measure of $[-1, 1]$.

If $n$ is large enough, $\hat{P}_n^{[1]}$ has $m$ exceptional zeros in $\mathbb{C} \setminus [-1, 1]$, $n$ simple regular zeros in $(-1, 1)$. The exceptional zeros tend to the zeros of $\tilde{b}$. If $\alpha, \beta \geq -\frac{1}{2}$, then

\[(21) \quad \tilde{\nu}_n \rightarrow \tilde{\nu}_e,\]

where

\[(22) \quad \tilde{\nu}_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_{kn}},\]

and $x_{kn}, k = 1, \ldots, n$ are the regular zeros of $\hat{P}_n$. The convergence is meant in weak-star sense.
3. Multiplication operator

In this section we investigate the location of zeros of the modified characteristic polynomial introduced in [14]. In our case it is as follows. Let us denote by \( \{ \hat{P}_n \}_{n=0}^{\infty} \) the orthonormal system of exceptional Jacobi polynomials, that is

\[
\hat{P}_n := \frac{P_n^{[1]}}{\sigma_n}, \quad \text{where} \quad \sigma_n := \|P_n^{[1]}\|_{W,2}.
\]

Let

\[
K_N(x, y) := \sum_{k=0}^{N-1} \hat{P}_k(x) \hat{P}_k(y).
\]

\( x_1, \ldots, x_N \) are random variables, the joint probability distribution on \( \mathbb{R}^N \) is

\[
\rho_{N,n}(x_1, \ldots, x_N) = c(n, N) \det |K_n(x_i, x_j)|_{i,j=1}^{N} \prod_{i=1}^{N} W(x_i),
\]

where \( c(n, N) \) is a normalization factor. The expectation \( \mathbb{E} \) refers to \( \rho \).

Let \( b \) be as in (15). Recalling that \( b \) is a polynomial, let

\[
Q(x) := \int x \tilde{b},
\]

where \( \tilde{b}(x) \) is defined in (17). Since the definition of \( Q \) let the constant term be chosen, we choose it to be zero, say.

Considering \( Q \), the modified average characteristic polynomial is defined as

\[
\chi_N(z) := \chi_N^Q(z) = \mathbb{E} \left( \prod_{i=1}^{N} (z - Q(x_i)) \right),
\]

cf. [14]. Denote by \( z_i \) the zeros of \( \chi_N(z) \). Define the normalized zero-counting measure by \( \nu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i} \), and the modified empirical distribution, \( \hat{\nu}_N^Q = \frac{1}{N} \sum_{i=1}^{N} \delta_{Q(x_i)} \). It is proved that

\[
\lim_{N \to \infty} \left| \mathbb{E} \left( \int x' d\hat{\nu}_N^Q(x) \right) - \int x' d\nu_N(x) \right| = 0,
\]

see [14] Theorem 5.1]. That is the examination of \( \nu_N \) leads to the description of the behavior of the normalized “Christoffel function measure”, that is

\[
d\mu_n(x) = \frac{1}{n} K_n(x, x) W(x) dx.
\]

Denoting by \( z_i = Q(y_i), i = 1, \ldots, n \), we define

\[
\tilde{\nu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{y_i}.
\]

The main theorem of this section is as follows.
Theorem 2. In exceptional Jacobi case, if \( \alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2} \), then for all \( l \in \mathbb{N} \)

\[
\lim_{n \to \infty} \left( \int_{-1}^{1} Q^l d\nu_n - \int_{-1}^{1} Q^l d\mu_e \right) = 0.
\]

where \( \mu_e \) is the equilibrium measure of \([-1,1]\).

For this study our main tool is the multiplication operator generated by \( Q \), \( M : f \to Qf \), cf. [14]. The exceptional orthogonal polynomials fulfill the next recurrence formula with constant coefficient: \( Q\hat{P}_n = \sum_{k=-L}^{L} \hat{u}_{n,k} \hat{P}_{n+k} \) (see \([21]\) and \([14] \) (3.4))). After the normalization above the recurrence relation is modified as

\[
Q\hat{P}_n = \sum_{k=-L}^{L} u_{n,k} \hat{P}_{n+k},
\]

where \( u_{n,k} = \frac{e_{n+k}}{\sigma_{n+k}} \hat{u}_{n,k} \). That is \( M \) can be represented by an infinite matrix, \( M_e \), in the orthonormal basis \( \{\hat{P}_n\}_{n=0}^{\infty} \). This operator, acting on \( L^2_W[-1,1] \) and simultaneously on \( l^2 \) is denoted by \( M_e \) as well. By (30) the matrix of \( M_e \) is \( 2L+1 \)-diagonal:

\[
M_e = \begin{bmatrix}
  u_{0,0} & u_{0,1} & \cdots & u_{0,L} & 0 & 0 & \cdots \\
  u_{1,-1} & u_{1,0} & \cdots & u_{1,L-1} & u_{1,L} & 0 & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots \\
  u_{L,-L} & u_{L,-L+1} & \cdots & u_{L,0} & \cdots & u_{L,L} & 0 \\
  0 & u_{L+1,-L} & \cdots & \vdots & \vdots & \vdots & \cdots & u_{L+1,L} \\
  \vdots & 0 & u_{L+j,-L} & \cdots & \vdots & \vdots & \cdots & \cdots & \cdots
\end{bmatrix}.
\]

It can be easily seen that \( M_e \) is symmetric since

\[
u_{k,j} = \int_{-1}^{1} Q\hat{P}_k \hat{P}_{k+j} W^2 = \int_{-1}^{1} Q\hat{P}_{k+j} \hat{P}_{(k+j)-j} W^2 = u_{k+j,-j}.
\]

It is proved (see [14] and the references therein) that the characteristic polynomial of the truncated multiplication matrix coincides with the average characteristic polynomial, that is

\[
\det(zI_N - \pi_N M \pi_N) = E \left( \prod_{i=1}^{N} (z - Q(x_i)) \right).
\]

Let us recall that \( \{\hat{P}_n\}_{n=0}^{\infty} \) is an orthonormal system on \([-1,1]\) with respect to \( W = \frac{w_0}{\sigma_n} \), where \( w_0 \) is a classical Jacobi weight function. Besides this exceptional orthonormal polynomial system, there is a standard orthonormal polynomial system, \( \{g_n\}_{n=0}^{\infty} \), on \([-1,1]\) with respect to \( W \). These standard orthonormal polynomials fulfill the three-term recurrence relation

\[
x g_n = a_{n+1} g_{n+1} + b_n g_n + a_n g_{n-1}
\]

(see e.g. [30] (3.2.1)). According to (34) the multiplication operator on \( L^2_W[-1,1] \), \( A : f \to xf \) can be
represented in the (Schauder) basis \( \{ q_n \} \) as an infinite tridiagonal matrix (denoted by \( A \) again)

\[
A = \begin{bmatrix}
    b_0 & a_1 & 0 & 0 & \ldots \\
    a_1 & b_1 & a_2 & 0 & \ldots \\
    0 & a_2 & b_2 & a_3 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

and so \( A \) acts on \( L^2_0 \) and on \( l^2 \). Let \( Q(x) \) be as above. The the multiplication operator, \( M : f \mapsto Qf \) can be represented as \( M = Q(A) \), and in the basis \( \{ q_n \} \) it has a matrix \( M_q = Q(A) \). Let \( \Pi_n^q \) and \( \Pi_n^c \) be the projections to \( \text{span}\{q_0, \ldots, q_{n-1}\} \), and to \( \text{span}\{\bar{P}_0, \ldots, \bar{P}_{n-1}\} \), respectively. Let \( A_{n \times n} := \Pi_n^c A \Pi_n^q \), \( M_{c, n \times n} := \Pi_n^c M_n^c \Pi_n^q \), \( M_{q, n \times n} := \Pi_n^q M_n^c \Pi_n^q \). To prove Theorem 2 we compare the trace of the truncated operator in the different bases.

**Proposition 1.** With the notation \( (19) \) let us assume that \( \alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2} \). Then for each \( l \in \mathbb{N} \)

\[
\lim_{n \to \infty} \frac{1}{n} (\text{Tr}((Q(A_{n \times n}))^l)) - \text{Tr}((M_{c, n \times n})^l)) = 0.
\]

First we need some technical lemmas. Let \( C = [c_{ij}]_{i,j=0}^\infty \) be an infinite matrix. \( c_{ij} = i_j \) stands for the elements of \( C \). As above its \( n \)th principal minor matrix is \( C_{n \times n} := [c_{ij}]_{i,j=0}^{n-1} \). Its \( i \)th row is \( i \) and \( j \)th column is \( j \). With this notation we can state the next lemma.

**Lemma 1.** Let \( C = [c_{ij}]_{i,j=0}^\infty \) be a \( 2k + 1 \)-diagonal, infinite matrix. Let \( P = \sum_{i=0}^M p_i x^l \) be a fixed polynomial. If the entries of \( C \) are bounded, that is there is a \( K \) such that \( |c_{ij}| < K \) for all \( 0 \leq i, j < \infty \), then

\[
\lim_{n \to \infty} \frac{1}{n} (\text{Tr}(P(C_{n \times n})) - \text{Tr}((P(C))_{n \times n})) = 0.
\]

**Proof.** It is obvious that for each \( l \in \mathbb{N} \) \( C^l \) is \( 2(kl) + 1 \) diagonal:

Indeed, \( c_{ij} = 0 \) if \( |i-j| > k \). Assuming that \( (C^l)_{ij} = 0 \) if \( |i-j| > kl \) we compute \( C_{ij}^{l+1} = (iC, C_j) = \sum_{|p-j|\leq k} C_{ip} C_{pj} \). That is \( C_{ij}^{l+1} = 0 \) if \( |i-j| > (l+1)k \).

By induction on \( l \) we show that the elements of the principal diagonal are coincide in \( C_{n \times n}^l \) and \( (C^{l})_{n \times n} \) except at most the last \( k(l-1) \) ones.

Let us assume that \( (C_{n \times n}^{l})_{ij} = (iC_{n \times n})_{ij} \) if \( i \leq n-(l-1)k \) or \( j \leq n-(l-1)k \). (For \( l = 1 \) it is obviously fulfilled.) Let \( i \leq n - kl \). By the assumption and by \( 2k + 1 \)-diagonality \( i((C_{n \times n})^l)_{ij} = i(C^{l})_{ij} \), and similarly if \( j \leq n - k \), then \( (C_{n \times n})_{ij} = C_{ij} \). Thus \((C_{n \times n})^{l+1})_{ij} = (iC^{l+1})_{ij} \) if \( i \leq n - kl \) and \( j \leq n - k \). If \( i \leq n - kl \) and \( j > n - k \) then \( C_{ij} - (C_{n \times n})_{ij} \) starts with \( n \) zeros, thus if \( i \leq n - kl \) and \( j > n - k \), \( i((C_{n \times n})^{l+1})_{ij} = C^{l+1})_{ij} \) again. For \( j \leq n - kl \) we have the same by symmetry.

Finally considering that the rows and columns contain finitely many non-zero elements, these imply that \( \frac{1}{n} (\text{Tr}((C_{n \times n})^l) - \text{Tr}((C^{l})_{n \times n})) \leq \frac{1}{n} k(l-1)(2kl + 1)^l K^l \), that is

\[
\lim_{n \to \infty} \frac{1}{n} (\text{Tr}((C_{n \times n})^l) - \text{Tr}((C^{l})_{n \times n})) = 0.
\]
Since $\text{Tr}(P(C_{n\times n})) = \sum_{i=0}^{M} P_i \text{Tr}((C_{n\times n})^i)$ and $\text{Tr}((P(C))_{n\times n}) = \sum_{i=0}^{M} P_i \text{Tr}((C^i)_{n\times n})$, (36) implies (35).

To ensure the boundedness of the entries of the matrices in question we recall the asymptotic behavior of recurrence coefficients. Since $W > 0$ on $(-1, 1)$, by [23 Theorem 4.5.7] (see also [26]) in formula (34) the recurrence coefficients fulfill the asymptotics

$$\lim_{n \to \infty} a_n = \frac{1}{2} \quad \text{and} \quad \lim_{n \to \infty} b_n = 0.$$  

Similarly to (37), the coefficients in (30) fulfill the symmetric limit relation

$$\lim_{n \to \infty} u_{n,j} =: U_{|j|},$$

where $U_{|j|}$ depends on the polynomial $\tilde{b}$ (cf. (3), (17)) as follows. Let

$$\tilde{b}(x) = \sum_{k=0}^{L-1} d_k x^k.$$  

Then

$$U_{|j|} = \begin{cases} \sum_{p=\text{max}(1,1)}^{\left\lceil \frac{j}{2} \right\rceil} \frac{d_{2p-1}}{2p} \left(\frac{2p}{2p-1}\right)^{\frac{1}{2p}}, & \text{if } |j| = 2l \\ \sum_{p=\text{max}(1,1)}^{\left\lceil \frac{j}{2} \right\rceil} \frac{d_{2p}}{2p+1} \left(\frac{2p+1}{2p}\right)^{\frac{1}{2p+1}}, & \text{if } |j| = 2l + 1, \end{cases}$$

see [14 Proposition 1] and [12] below.

Let us recall $\frac{\tilde{b}}{x}$ is bounded on $[-1, 1]$ and $W(x) = \frac{(1-x)^{1+\varepsilon_1}(1+x)^{1+\varepsilon_2}}{y^2} (\varepsilon_i = \pm 1)$, where $b(x) = \tilde{b}(x)(1-x)^{1-\varepsilon_1}(1+x)^{1-\varepsilon_2}$ and $0 < c < \tilde{b} < C$ on $[-1, 1]$.

Subsequently the next lemma proved by Badkov (see [1]) is useful. Here we cite the formulation given in [22].

**Lemma A.** [22 Lemma 2.E] Let $\{q_k\}_{k=0}^{\infty}$ be the standard orthonormal system with respect to $W$. For each $j \geq 0$ integer

$$\left| q_k^{(j)}(x) \right| \leq c \left( \frac{k}{\sqrt{1-x^2}} \right)^j \frac{1}{\left(\sqrt{1-x} + \frac{1}{x}\right)^{\alpha + \varepsilon_1} \left(\sqrt{1+x} + \frac{1}{x}\right)^{\beta + \varepsilon_2} \sqrt{1-x^2} + \frac{1}{x}}.$$  

We need the next estimations on norms of exceptional Jacobi polynomials.

**Lemma 2.** With the notation [23], if $\alpha + \frac{\varepsilon_1}{2}, \beta + \frac{\varepsilon_2}{2} > -\frac{1}{2}$,

$$\sigma_k = \sqrt{k(\alpha + \beta + 1) + \lambda}.$$  

If $\alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2}$, $0 \leq \delta \leq \min \left\{ \frac{1}{4}, \frac{\alpha + \varepsilon_1}{2} + \frac{1}{4}, \frac{\beta + \varepsilon_2}{2} + \frac{1}{4} \right\}$

$$\left\| \tilde{P}_k(x)\sqrt{W(x)(1-x^2)^{\frac{1}{2}-\delta}} \right\|_{\infty} \leq c k^{\max\{\varepsilon_1, \varepsilon_2\} - 1 + 2\delta},$$

where $c$ is a constant (independent of $k$).
Proof.\footnote{12}: With the notation $p_k^{(\alpha,\beta)} = p_k$, cf. \footnote{14}

\[
\sigma_k^2 \hat{P}_k^2 W = (bp'_k - bwp_k) \frac{p_{\alpha,\beta}}{b^2}
\]

\[
= (p'_k)^2 w_{(\alpha+1,\beta+1)} + (bw)^2 \frac{p_{\alpha,\beta}}{b^2} - 2b^2 wp'_k p_k \frac{p_{\alpha,\beta}}{b^2}.
\]

By \footnote{7} $pw^2 = \tilde{\lambda} - qw - pu'$ and considering \footnote{13} $(w_{(\alpha+1,\beta+1)})' = qw_{(\alpha,\beta)}$

\[
\sigma_k^2 \hat{P}_k^2 W = (p'_k)^2 w_{(\alpha+1,\beta+1)} + \tilde{\lambda} p_k^2 w_{(\alpha,\beta)} - \left(p_k^2 w_{(\alpha+1,\beta+1)}\right)'.
\]

Thus

\[
\sigma_k^2 = \int_{-1}^{1} (p'_k)^2 w_{(\alpha+1,\beta+1)} + \tilde{\lambda} p_k^2 w_{(\alpha,\beta)} - \left(p_k^2 w_{(\alpha+1,\beta+1)}\right)' = I_1 + I_2 + I_3.
\]

According to \footnote{30} (4.21.7) $p'_k = \frac{k+\alpha+\beta+1}{2} p_{k-1} \frac{p_{\alpha+1,\beta+1}}{\epsilon_{k,\beta}}$

\[
I_1 = \left(\frac{k+\alpha+\beta+1}{2} \frac{p_{\alpha+1,\beta+1}}{\epsilon_{k,\beta}}\right)^2.
\]

$I_2 = \tilde{\lambda}$. By the assumption on $\alpha$ and $\beta$ one can see that $I_3 = 0$. Substituting the values of the corresponding $\epsilon_{k,\beta}$, \footnote{12} is proved.

\footnote{13}:

\[
\left|\hat{P}_k(x)\sqrt{W(x)}(1-x^2)^{\frac{\delta}{2}}\right| \leq \frac{c}{\sigma_k} \left(|b(x)p'_k(x)| + |(bw)(x)p_k(x)|\right)(1-x)^{\frac{\alpha+1}{2}} + \frac{1}{2}(1+x)^{\frac{\beta+2}{2}} + \frac{1}{4}
\]

\[
= K_1 + K_2.
\]

\[
K_1 \leq |p_{\alpha+1,\beta+1,0}(1-x)^{\frac{\alpha+1}{2}} + \frac{1}{2}(1+x)^{\frac{\beta+2}{2}} + \frac{1}{4},
\]

which is bounded, see e.g. \footnote{30} (8.21.10), or \footnote{11}. Since $bw$ is bounded on $[-1,1]$ (it is a polynomial), by \footnote{11}

\[
K_2 \leq \frac{c}{k} \left|p_k(x)\right|(1-x)^{\frac{\alpha+1}{2}} + \frac{1}{2}(1+x)^{\frac{\beta+2}{2}} + \frac{1}{4} - \delta
\]

\[
\leq \frac{c}{k} \frac{1}{(\sqrt{1-x} + \frac{1}{k})^\alpha (\sqrt{1+x} + \frac{1}{k})^\beta \sqrt{1-x^2 + \frac{1}{k}}} \leq \frac{c k^{max\{-\epsilon_1,-\epsilon_2\}-1+2\delta}}{1-\epsilon_1 - \frac{\alpha+1}{2} + \frac{1}{4} - \delta},
\]

To prove Proposition \footnote{11} we introduce the operator $O$ which changes the basis in the Hilbert space in question:

\footnote{44}

\[
O^{-1} M_\delta O = M_\varepsilon,
\]

where

\[
O = \left[a_{ij}\right]_{i,j=0}^{\infty},
\]

and

\footnote{45}

\[
\hat{P}_j = \sum_{i=0}^{j+m} a_{ij} q_i.
\]
Proof: (of Proposition 11)

Recalling that $A$ is tridiagonal and $M_c$ is $2L + 1$ diagonal and by \(47\) and \(48\) their entries are bounded, by Lemma 11 it is enough to show that

\[
\lim_{n \to \infty} \frac{1}{n} \left( \text{Tr}(\pi_n^p Q^l(A)\pi_n^q) - \text{Tr}(\pi_n^p M_c \pi_n^q) \right) = 0.
\]

Considering \(44\) it is enough to prove that

\[
\lim_{n \to \infty} \frac{1}{n} \left( \text{Tr}(\pi_n^p Q^l(A)\pi_n^q) - \text{Tr}(\pi_n^p O^{-1} Q^l(A)O \pi_n^q) \right) = 0.
\]

To prove \(47\) we compute the diagonal elements of the second term. According to \(45\), for $0 \leq i \leq n - 1$

\[
i(O^{-1} Q^l(A)O)_i = \sum_{k = \max\{0, i - L\}}^{\infty} i(O^{-1} Q^l(A))_k k O_i
\]

and considering the $2L + 1$-diagonality of $Q^l(A)$

\[
i(O^{-1} Q^l(A))_k = \sum_{j=0}^{i+L} O_{ij}^{-1} j(Q^l(A))_k = \sum_{\max\{0, k - i\} \leq j \leq \min\{i + L, k + L\}} j O_i j(Q^l(A))_k.
\]

Thus

\[
\sum_{i=0}^{n-1} i(O^{-1} Q^l(A)O)_i = \sum_{i=0}^{n-1} \sum_{k=0}^{i+L} \sum_{\max\{0, k-i\} \leq j \leq \min\{i+L, k+i\}} O_{ij} O_{kj} j(Q^l(A))_k
\]

\[
= \sum_{i=0}^{n-1} \sum_{k=0}^{i+L} \sum_{\max\{0, k-i\} \leq j \leq \min\{i+L, k+i\}} O_{ij} O_{kj} j(Q^l(A))_k
\]

\[
= M_n + H_n.
\]

By reversing the order of summation

\[
M_n = \sum_{k=0}^{L} \sum_{i=0}^{n-1} O_{ki}^2 k(Q^l(A))_k + \sum_{k=L+1}^{n-1} k(Q^l(A))_k \sum_{i=k-L}^{n-1} O_{ki}^2 + \sum_{k=n}^{n-1+L} k(Q^l(A))_k \sum_{i=k-L}^{n-1} O_{ki}^2
\]

\[
= E_{1,n} + M_n + E_{2,n}.
\]

Thus the sequence in \(47\) becomes

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left( k(Q^l(A))_k - k(O^{-1} Q^l(A)O)_k \right)
\]

\[
= \frac{1}{n} \left( \sum_{k=0}^{L} k(Q^l(A))_k + E_{1,n} + E_{2,n} \right) + \frac{1}{n} \sum_{k=L+1}^{n-1} k(Q^l(A))_k \left( 1 - \sum_{i=k-L}^{n-1} O_{ki}^2 \right) + \frac{1}{n} H_n.
\]

According to \(40\) the entries $i(Q^l(A))_k$ are bounded, and by orthogonality

\[
\sum_{i=k-L}^{n-1} O_{ki}^2 = 1,
\]
thus the first term tends to zero, when \( n \) tends to infinity. By the same reasons the second term can be estimated as

\[
\frac{1}{n} \sum_{k=L+1}^{n-1} k(Q^l(A))_k \left( 1 - \sum_{i=k-L}^{n-1} o^2_{ki} \right) \leq C \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=n}^{\infty} o^2_{ki}.
\]

Let \( i > k \). Referring to (15)

\[
o_{ki} = \int_{-1}^{1} \hat{P}_i q_k W = \frac{1}{\lambda_i} \int_{-1}^{1} \left( \hat{P}_i'' + i \hat{P}_i' + \hat{P}_i \right) q_k W,
\]

where the differential equation of \( \hat{P}_i \) is given by (41) and (43), integrating by parts the first term we have

\[
\int_{-1}^{1} p \hat{P}_i'' q_k W = - \int_{-1}^{1} \hat{P}_i'(pq_k W)' = - \int_{-1}^{1} \hat{P}_i'(q q_k + p q_k) W,
\]

because \( pW' = (\hat{q} - p')W \), cf. (6).

So (51) and (52) imply that

\[
o_{ki} = \frac{1}{\lambda_i} \int_{-1}^{1} -p \hat{P}_i'' q_k W + i \hat{P}_i q_k W = I_{ki} + \Pi_{ki}.
\]

Considering (3) (9) and (4),

\[
p \hat{P}_i' = \frac{\lambda_i - \tilde{\lambda}_i}{\sigma_i} b p_i - \left( pw + q - \frac{b'}{b} \right) \hat{P}_i.
\]

By (54)

\[
I_{ki} = - \frac{\lambda_i - \tilde{\lambda}_i}{\sigma_i \lambda_i} \int_{-1}^{1} b p_i q_k W + \frac{1}{\lambda_i} \int_{-1}^{1} \left( pw + q - \frac{b'}{b} \right) \hat{P}_i q_k W = I_{ki}^{(1)} + I_{ki}^{(2)}.
\]

By (17) \( |pw + q - \frac{b'}{b}| \) is bounded on \([-1, 1]\). Indeed, only the first term needs some investigation; recalling that \( bw \) is a polynomial, notice that \( pw = \frac{b'}{b} bw \).

Thus, according to (51) and (53)

\[
|I_{ki}^{(2)}| \leq \frac{ck}{t^2} \left\| \hat{P}_k(x) \sqrt{W(x)}(1 - x^2)^{\frac{1}{2}} \right\| \infty
\]

\[
\times \int_{-1}^{1} \frac{\sqrt{1 - x^{\alpha + \epsilon_1}}}{(1 - x^2)^{\frac{1}{2}} (\sqrt{1 - x^2} + \frac{1}{k})^{\frac{1}{2}} (\sqrt{1 - x} + \frac{1}{k})^{\alpha + \epsilon_1}} \left( \sqrt{1 + x} + \frac{1}{k} \right)^{\beta + \epsilon_2} dx
\]

\[
\leq c \frac{k}{t^2} J_k.
\]

\[
J_k \leq c \int_{-1}^{0} \frac{\left( \sqrt{1 + x} \right)^{\beta + \epsilon_2 + \frac{3}{2} - 2\delta}}{(\sqrt{1 + x} + \frac{1}{k})^{\beta + \epsilon_2 + \frac{3}{2} + 2\delta} (1 + x)^{1 - \delta}} dx
\]

\[
+ c \int_{0}^{1} \frac{\left( \sqrt{1 - x} \right)^{\alpha + \epsilon_1 + \frac{3}{2} - 2\delta}}{(\sqrt{1 - x} + \frac{1}{k})^{\alpha + \epsilon_1 + \frac{3}{2} + 2\delta} (1 - x)^{1 - \delta}} \leq c \frac{k^{2\delta}}{\delta},
\]

where the last inequality fulfills if \( \alpha + \epsilon_1 + \frac{3}{2} - 2\delta \geq 0 \) and \( \beta + \epsilon_2 + \frac{3}{2} - 2\delta \geq 0 \). Let \( \delta = ck^{-\frac{3}{2}} \). Then

\[
|I_{ki}^{(2)}| \leq c \frac{k^{\frac{3}{2}}}{t^2},
\]
where the norm of \( q \) can be estimated by a constant independently of \( p \). For sake of simplicity let us denote by \( c(i) := \frac{1}{\sigma_i \lambda_i} \). After simplification

\[
I^{(1)}_{ki} = c(i) \int_{-1}^{1} \frac{\hat{p}_k p_i w^{(\alpha, \beta)}}{b}.
\]

Recalling that \( \frac{1}{b} \) is bounded on \([-1, 1]\), let \( S_{i-k-2} \) be its uniformly best approximating polynomial on \([-1, 1]\) of degree \( i - k - 2 \). Then, as its degree is less than \( i \), \( S_{i-k-2} \hat{q}_k' \) is orthogonal to \( p_i \) with respect to \( w^{(\alpha, \beta)} \). Thus

\[
I^{(1)}_{ki} = c(i) \int_{-1}^{1} \left( \frac{1}{b} - S_{i-k-2} \right) \hat{p}_k p_i w^{(\alpha, \beta)}.
\]

Taking into account (13), (42) and (3) \( c(i) \leq \frac{1}{i} \). Since \( \left( \frac{1}{b} \right)' \) is bounded on \([-1, 1]\] too, according to the classical Jackson’s theorem

\[
|I^{(1)}_{ki}| \leq c \frac{k}{i(i-k)} \int_{-1}^{1} |\hat{p}_k p_i w^{(\alpha, \beta)}| \leq \frac{c}{i(i-k)} \int_{-1}^{1} \frac{|q_k'(x)|}{\sqrt{W(x)}} \|q_k'(x)\|_2 \sqrt{W(x)} \frac{b(x)}{(1-x^2)^{\frac{3}{4}}} dx
\]

\[
\leq \frac{k}{i(i-k)} \int_{-1}^{1} |p_i(x)|(1-x)^{\frac{3}{4}}(1+x)^{\frac{3}{4}} dx = O\left( \frac{1}{i^{\frac{3}{4}}} \right),
\]

where the norm of \( q_k' \) is estimated by \( [11] \). Finally by \( [30] \ (7.34.1) \) the last integral can be estimated by a constant independently of \( i \), that is

\[
|I^{(1)}_{ki}| \leq c \frac{k}{i(i-k)}.
\]

To estimate \( II_{ki} \) first we remark that \( \hat{r} \) is bounded on \([-1, 1]\). Indeed, it is clear that we have to deal with only the endpoints. Recalling that \( \frac{P}{b} \) is bounded on \([-1, 1]\) and considering \([10] \), \( \hat{q} \) is bounded there too. As it is mentioned in the Remark of section 2 \( \hat{P}_n(1) \neq 0 \), that is due to (5) and (8) \( \hat{r} \) must be bounded in 1. In \(-1\) it is the same. Thus, by Cauchy-Schwarz inequality

\[
|II_{ki}| \leq c \frac{1}{i} \int_{-1}^{1} |\hat{P}_i| \sqrt{W} |q_k| \sqrt{W} \leq c \frac{1}{i^2}.
\]

Since \( \sum_{i=0}^{\infty} a_{ki}^2 \leq 1 \), according to \( [30] \), \( [10] \) and \( [58] \)

\[
\frac{1}{n} \sum_{k=0}^{n-1} a_{ki}^2 \leq c \frac{1}{n} \sum_{k=n}^{\infty} \sum_{i=n}^{\infty} a_{ki}^2 + O\left( \frac{1}{n^{\frac{3}{2}}} \right)
\]

\[
\leq \frac{c}{n} \sum_{k=n}^{\infty} \sum_{i=n}^{\infty} \left( \frac{1}{i^4} + \frac{1}{i^2 n^{\frac{3}{4}}} \right) + O\left( \frac{1}{n^{\frac{3}{2}}} \right) = O\left( \frac{1}{\sqrt{n}} \right) + O\left( \frac{1}{n^{\frac{3}{2}}} \right).
\]

Finally we estimate the error term \( \frac{1}{n} H_n \), cf \([3] \) and \([49] \). Using the properties of \( |j(Q^l(A))_k| \) and \( o_{ki} \) again, it can be estimated as

\[
\frac{1}{n} |H_n| \leq c \frac{1}{n} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\max(0, k-i) \leq j \leq i+L} \sum_{j \neq k} |o_{ji}| o_{ki}.
\]
Proof. (of Theorem 2) Let \( \sum \) \( (Q^j(A))_k \), \( \max_{0 \leq j \leq k+L} \). That is

\[
+ \frac{1}{n} \sum_{i=(l-1)L+1}^{n-1} \sum_{k=0}^{L-1} \sum_{0 \leq j \leq k+L, j \neq k} a_{ji} o_{ki} (Q^j(A))_k = \Sigma_1(n) + \Sigma_2(n).
\]

Certainly, \( \lim_{n \to \infty} \Sigma_1(n) = 0 \). Splitting \( \Sigma_2(n) \) to two parts and changing the ordering of summation,

\[
\begin{align*}
\Sigma_2(n) \leq & \, \frac{1}{n} \sum_{i=(l-1)L+1}^{n-1} \sum_{k=0}^{L-1} \sum_{0 \leq j \leq k+L, j \neq k} a_{ji} o_{ki} (Q^j(A))_k \\
+ & \, \frac{1}{n} \sum_{k=L}^{n-1+L} \sum_{k-L \leq j \leq k+L, j \neq k} (Q^j(A))_k \sum_{i=(l-1)L+1}^{n-1} o_{ji} o_{ki} \\
\leq & \, \frac{1}{n} \sum_{k=0}^{L-1} \sum_{0 \leq j \leq k+L, j \neq k} |j(Q^j(A))_k| \sum_{i=(l-1)L+1}^{n-1} |o_{ji} o_{ki}| \\
+ & \, \frac{1}{n} \sum_{k=L}^{n-1+L} \sum_{k-L \leq j \leq k+L, j \neq k} (Q^j(A))_k \sum_{i=(l-1)L+1}^{n-1} o_{ji} o_{ki} = \Sigma_{21}(n) + \Sigma_{22}(n).
\end{align*}
\]

Again, \( \lim_{n \to \infty} \Sigma_{21}(n) = 0 \). According to (45) \( o_{ki} = 0 \) if \( i \leq k - L \). Thus, by orthogonality \( \sum_{i=(l-1)L+1}^{n-1} o_{ji} o_{ki} = \sum_{i=n}^{n-1} o_{ji} o_{ki} \). That is

\[
\Sigma_{22}(n) \leq c \frac{1}{n} \sum_{k=L}^{n-1+L} \sum_{i=n}^{\infty} |o_{ji} o_{ki}| \leq c \frac{1}{n} \sum_{k=L}^{n-1+L} \max_{k-L \leq j \leq k+L} \sum_{i=n}^{\infty} o_{ji}^2.
\]

So we can proceed as in (59) again, and so \( \lim_{n \to \infty} \Sigma_{22}(n) = 0 \).

\[\]
4. OUTER RATIO ASYMPTOTICS

In [10] a family of exceptional Jacobi polynomials is given. Among other properties the location of their zeros is described there. In this section we describe the behavior of zeros and give outer ratio asymptotics and a Heine-Mehler type formula by the general formulation of exceptional Jacobi polynomials.

Recalling the general construction of exceptional orthogonal polynomials with one-step Darboux transform, cf. (9) and (3),

\[
\frac{P^{[1]}_{n-1}}{P^{[1]}_n} = \frac{b(p^{[0]}_{n-1})'}{b(p^{[0]}_n)'} - bw \frac{p^{[0]}_{n-1}}{p^{[0]}_n},
\]

Classical orthonormal polynomials satisfy the following relation (cf. e.g. [30, (4.5.5), (5.1.14), (5.5.10)].

\[
 p \left( p^{[0]}_n \right)' = A_n p^{[0]}_{n+1} + B_n p^{[0]}_n + C_n p^{[0]}_{n-1},
\]

where \( A_n, B_n \) and \( C_n \) are real numbers and \( p \) is the coefficient of the second derivative in the differential equation of the classical orthogonal polynomials. Thus

\[
\frac{P^{[1]}_{n-1}}{P^{[1]}_n} = \frac{P^{[0]}_{n-1}}{P^{[0]}_n} \frac{b \left( A_n p^{[0]}_{n+1} + B_n p^{[0]}_n + C_n p^{[0]}_{n-1} \right) - bw p^{[0]}_n}{\left( A_n p^{[0]}_{n+1} + B_n p^{[0]}_n + C_n p^{[0]}_{n-1} \right) - bw p^{[0]}_n}.
\]

Now we return to exceptional Jacobi polynomials.

Notation. Denote by \( Z_b \) the zeros of the polynomial \( \tilde{b} \), cf. [15], [17]. Similarly to [10] Proposition 5.6 the next asymptotics fulfills.

Proposition 2. For exceptional Jacobi polynomials given by [15]

\[
\lim_{n \to \infty} \frac{P^{[1]}_{n-1}(z)}{P^{[1]}_n(z)} = z - \sqrt{z^2 - 1}, \quad z \in \mathbb{C} \setminus [-1, 1] \setminus Z_b
\]

locally uniformly, where \( \sqrt{z^2 - 1} \) means that branch of the function for which \( |z - \sqrt{z^2 - 1}| < 1 \) on \( \mathbb{C} \setminus [-1, 1] \).

Proof. In Jacobi case [60] becomes

\[
\frac{P^{[1]}_{n-1}}{P^{[1]}_n} = \frac{P^{[0]}_{n-1}}{P^{[0]}_n} \frac{b \left( A_n p^{[0]}_{n+1} + B_n p^{[0]}_n + C_n p^{[0]}_{n-1} \right) - bw p^{[0]}_n}{\left( A_n p^{[0]}_{n+1} + B_n p^{[0]}_n + C_n p^{[0]}_{n-1} \right) - bw p^{[0]}_n}.
\]

Classical Jacobi polynomials fulfill the asymptotics below.

\[
\lim_{n \to \infty} \frac{P^{[0]}_{n-1}(z)}{P^{[0]}_n(z)} = z - \sqrt{z^2 - 1}
\]

uniformly on the compact subsets of \( \mathbb{C} \setminus [-1, 1] \) (cf. e.g. [21]). Taking into consideration that \( \lim_{n \to \infty} \frac{\varrho_n^{-1}}{\varrho_n} = 1, \) etc. (for \( \varrho_n \) see [12]), according to [30] (4.5.5)]

\[
\lim_{n \to \infty} \frac{A_n}{n} = \frac{1}{2}, \quad \lim_{n \to \infty} \frac{B_n}{n} = \frac{\alpha - \beta}{2}, \quad \lim_{n \to \infty} \frac{C_n}{n} = -\frac{1}{2}.
\]
Since \( bwp \) is a polynomial (and usually \( pw \) is not) \( \frac{bwp}{n} \) tends to zero locally uniformly on \( \mathbb{C} \), considering (64) and (65) we arrive to the statement.

**Notation.** Let us denote by
\[
Z := \{ z \in \mathbb{C} : \forall U \text{ neighborhood of } z \exists N \in \mathbb{N} \text{ such that } \forall P_n^{[1]} \exists w \in U, P_n^{[1]}(w) = 0 \text{ if } n > N \}.
\]

**Corollary 1.** \([-1,1] \subset Z \).

**Proof.** For sake of self-containedness we repeat the proof of [21, Theorem 5].

\( Z \) is closed. If there was a point \( t \in (-1,1) \) which is not in \( Z \), there would be a compact neighborhood, \( E \subset \mathbb{C} \), of \( t \) and a subsequence of polynomials such that
\[
\frac{P_n^{[1]}(z)}{P_n^{[0]}(z)} \rightarrow z - \sqrt{z^2 - 1} \text{ on } E \text{ uniformly, which is impossible.}
\]

Let us recall that \( m = \deg \tilde{b} \) is the codimension of the exceptional system. With this notation we have

**Proposition 3.** If \( n \) is large enough, \( P_n^{[1]} \) has \( m \) exceptional zeros (with multiplicity), that is \( m \) zeros out of the interval of orthogonality. Moreover the exceptional zeros tend to the zeros of \( \tilde{b} \), when \( n \) tends to infinity.

**Proof.** Similarly to the computation above
\[
\frac{P_n^{[1]}}{n P_n^{[0]}} = \frac{p b(P_n^{[0]})'}{p n P_n^{[0]}} - \frac{b w}{n} = \frac{b}{p} \left( \frac{A_n P_n^{[0]}}{n} + B_n + \frac{C_n P_n^{[0]}}{n} - \frac{b w}{n} \right),
\]
that is by (64) and (65)
\[
\lim_{n \to \infty} \frac{P_n^{[1]}}{n P_n^{[0]}} = -\frac{b(z)}{\sqrt{z^2 - 1}},
\]
where the convergence is locally uniform on \( \mathbb{C} \setminus [-1, 1] \). Applying Hurwitz’s theorem the statement is proved.

Let \( j_{\alpha}(z) = \Gamma(\alpha + 1) (\frac{2}{z})^\alpha J_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1) \Gamma(k+\alpha+1)} (\frac{2}{z})^{2k} \) be the Bessel function. Classical Jacobi polynomials fulfil the next Mehler-Heine formula (see [30, Theorem 8.1.1])
\[
\lim_{n \to \infty} \frac{P_n^{\alpha,\beta}(\cos \frac{z}{n})}{n^\alpha} = \frac{1}{\Gamma(\alpha+1)} j_{\alpha}(z),
\]
where the convergence is locally uniform on the complex plane.

If \( b(1) = 0 \), that is \( \varepsilon_1 = -1 \) cf. [18], we introduce the notation \( b(x) = (1-x)b_1(x) \). Similarly to [10, Proposition 5.7] the next limit is valid.

**Proposition 4.** If \( \alpha \geq -\frac{\varepsilon_1}{2} \)
\[
\lim_{n \to \infty} \frac{\partial_n}{n^{\alpha+1+\varepsilon_1}} P_n^{[1]}(\cos \frac{z}{n}) = c j_{\alpha+\varepsilon_1}(z),
\]
where the convergence is locally uniform on the complex plane and \( c \) is a constant depending on \( \alpha \) and \( b \).
Proof. $\varepsilon_1 = 1$:
By the classical formula
\[(P_n^{\alpha, \beta})' = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{\alpha, \beta+1},\]
see [30] (4.21.7) and considering [14]
\[
\frac{\varrho_n}{n^{\alpha+2}} P_n^{[1]} \left( \cos \frac{z}{n} \right)
\]
\[= \left( \frac{\alpha}{\alpha+2} \right) P_n^{[1]} \left( \cos \frac{z}{n} \right) - \left( \frac{1}{\alpha+2} \right) \frac{n^{\alpha+1}}{n^2} P_n^{[1]} \left( \cos \frac{z}{n} \right).
\]
Since $||j_n|| \leq 1$ (see [2] Ch. 7 7.3 (4)), applying (66) the statement is proved in the first case with $c = \frac{b(1)}{2(\alpha+2)}$.
$\varepsilon_1 = -1$:
Taking into consideration the next classical formula (see eg. [10] (73)):
\[(1-x)(P_n^{\alpha, \beta})' = \alpha P_n^{\alpha, \beta} - (n+\alpha)P_n^{\alpha-1, \beta+1} \]
we have
\[
\varrho_n^{[1]} P_n^{[1]} = b_1 (\alpha P_n^{\alpha, \beta} - (n+\alpha)P_n^{\alpha-1, \beta+1}) - bw P_n^{\alpha, \beta}.
\]
Recalling the Remark in Section 2 ($bw(1) \neq 0$). We show that the polynomial $\alpha b_1 - bw$ vanishes at 1. Indeed, by [15] and [3]
\[
bBP_n^{[1]}(1) = \left( p \left( P_n^{[1]} \right)' + p - \frac{b'}{b} \right) P_n^{[1]}(1) = \left( wp + q - \frac{b'}{b} \right) (1)(-bw p_1(1) = (\lambda_n - \tilde{\lambda}) b p_1(1) = 0.
\]
Thus referring to [13]
\[
0 = \left( wp + q - \frac{b'}{b} \right) (1) = \left( p \frac{bw + q - \frac{b'}{b}}{b} \right) (1)
\]
\[= \frac{1 + 1}{b_1(1)} (bw)(1) - 2\alpha - 2 - \frac{1 + 1}{b_1(1)} (-b_1(1) + (1 - 1)b_1'(1)) = 2 \left( \frac{bw(1)}{b_1(1)} - \alpha \right).
\]
That is
\[
\varrho_n P_n^{[1]}(x) = (\alpha b_1 - bw)(x) P_n^{\alpha, \beta}(x) - b_1(x)(n+\alpha)P_n^{\alpha-1, \beta+1}(x)
\]
\[= (1-x)s(x) P_n^{\alpha, \beta}(x) - b_1(x)(n+\alpha)P_n^{\alpha-1, \beta+1}(x),
\]
where $s(x)$ is a polynomial. Applying [30] (4.5.4)
\[(n+\alpha)P_n^{\alpha-1, \beta} - (n+1)P_n^{\alpha-1, \beta} - b_1(n+\alpha)P_n^{\alpha-1, \beta+1}.
\]
So again by (66) and the uniform boundedness of the corresponding Bessel functions
\[
\lim_{n \to \infty} \frac{\varrho_n}{n^{\alpha}} P_n^{[1]} \left( \cos \frac{z}{n} \right) = -\frac{b_1(1)}{\Gamma(\alpha)} J_{\alpha-1}(z).
\]
According to Proposition [3] $P_n^{[1]}$ has $m$ exceptional zeros out the interval of orthogonality, and by the Remark of Section 2 $P_n^{[1]}(-1) \neq 0$, $P_n^{[1]}(1) \neq 0$. Because $m$ is the number of gaps in the sequence of degrees, if $n$ is large enough, $P_n^{[1]}$ has to possess $n$ zeros in $(-1,1)$. These are the regular zeros, and as it is noted these zeros are simple. The distribution of regular zeros can be derived from a theorem of Erdős and Turán, see [5].
Theorem B. Let \( 1 \geq \zeta_{1,n} > \cdots > \zeta_{n,n} \geq -1, n \in \mathbb{N}_+ \) any system of points, and let \( \eta_{i,n} \in [0, \pi] \) be defined by \( \zeta_{i,n} = \cos \eta_{i,n} \). Let \( \omega_n(\zeta) = \prod_{i=1}^n (\zeta - \zeta_{i,n}) \). If for all \( \zeta \in [-1, 1] \)

\[
|\omega_n(\zeta)| \leq \frac{A(n)}{2^n}
\]

holds, then for every subinterval \([\gamma, \delta] \subset [0, \pi]\) we have

\[
(67) \quad \left| \sum_{\gamma \leq i_n \leq \delta} 1 - \frac{\delta - \gamma}{\pi} \right| < \frac{8}{\log 3} \sqrt{n \log A(n)}.
\]

A simple application of the previous theorem is the next one.

Proposition 5. Let \( x_{1,n}, \ldots, x_{m,n} \) be the regular zeros of the exceptional Jacobi polynomials, \( P_n^{[1]} = P_n^{[1],\alpha,\beta} \), \( \alpha, \beta \geq -\frac{1}{2} \). Let \( x_{in} = \cos \varphi_{in} \). For every \([\gamma, \delta] \subset [0, \pi]\)

\[
\left| \frac{1}{n} \sum_{\gamma \leq i_n \leq \delta} 1 - \frac{\delta - \gamma}{\pi} \right| \leq c \sqrt{\frac{\log n}{n}},
\]

where \( c \) is a constant, depends on \( \alpha, \beta, b, \nu \), but is independent of \( n \).

Proof. Let \( P_n^{[1]} = l_n q_{m,n} s_n \), where \( q_{m,n} \) and \( s_n \) are monic polynomials of degree \( m \) and \( n \), respectively, the zeros of \( q_{m,n} \) are the exceptional zeros of \( P_n^{[1]} \), and \( s_n \) possesses the regular ones. \( l_n \) is the leading coefficient of \( P_n^{[1]} \). Since the zeros of \( q_{m,n} \) tends to the zeros of \( \tilde{b} \), if \( n \) is large enough, there are constants \( k \) and \( K \) independent of \( n \) such that \( 0 < k < |q_{m,n}| < K \) on \([-1, 1]\). Thus for \( x \in [-1, 1] \)

\[
|s_n(x)| \leq \frac{1}{k|l_n|} |P_n^{[1]}(x)| \leq \frac{1}{k|l_n|} (\|bP_n\| + \|bwP_n\|),
\]

where the norm sign refers to the sup-norm on \([-1, 1]\), and \( P_n = P_n^{(\alpha,\beta)} \). By \( \|P_n\| = n^{\max\{\alpha,\beta\}} \) and by \( 4.21.6 \) the leading coefficient \( L_n^{(\alpha,\beta)} \) of \( P_n^{(\alpha,\beta)} \) is \( L_n^{(\alpha,\beta)} = \frac{\Gamma(2n+\alpha+\beta)}{2^{2n+\alpha+\beta} \Gamma(n+1)^2} \). Since \( |l_n| \geq c \min\{nL_n^{(\alpha,\beta+1)}, L_n^{(\alpha,\beta)}\} \geq cL_n^{(\alpha,\beta)} \),

\[
|s_n(x)| \leq c2^n \frac{(n+\max\{\alpha+1,\beta+1\})}{(2n+\alpha+\beta)} \leq c2^n \frac{\Gamma(n+\max\{\alpha+1,\beta+1\})}{\Gamma(n+\alpha+\beta+1)} \leq c2^n \frac{\Gamma(2n+\alpha+\beta+2n)}{\Gamma(n+\alpha+\beta+1+2n)} \leq c2^n \frac{\Gamma(\alpha+\beta+2n+2)}{\Gamma(n+\alpha+\beta+2n)}.
\]

According to (67) the estimation above ensures the result.

5. Proof of Theorem 1

Corollary 5.2 states that in the \( X_m \) Jacobi case that is if the exceptional Jacobi polynomials are derived by one Darboux transformation and are of codimension \( m \), for all \( l \geq 0 \)

\[
\lim_{n \to \infty} \left( \int_{-1}^1 Q^l d\mu_n - \int Q^l d\nu_n \right) = 0.
\]
According to Theorem 2 if \( \alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2} \), then for all \( l \in \mathbb{N} \)

\[
\lim_{n \to \infty} \left( \int Q^l \tilde{\nu}_n - \int_{-1}^1 Q^l d\mu_e \right) = 0.
\]

That is

\[
\lim_{n \to \infty} \left( \int_{-1}^1 Q^l d\mu_n - \int_{-1}^1 Q^l d\mu_e \right) = 0
\]

for all \( l \in \mathbb{N} \). As it is pointed out in the proof of [14 Theorem 4.1], \( \text{span}\{Q^l : l \in \mathbb{N}\} \) is dense in \( C[-1, 1] \), which implies the result.

[21]:

The location of regular and exceptional zeros is explained above.

Then Proposition [5] ensures that the normalized counting measure based on \( \varphi_{kn}, k = 1, \ldots, n \) tends to the normalized arc-measure on \([0, \pi]\). Recalling that \( x_n = \cos \varphi_{kn} \), a substitution implies the statement.

**Remark.** - [20] is the extension of [14 Theorem 4.1] to any codimension.

- In the standard case the \( n \)th average characteristic polynomial coincides with the \( n \)th orthogonal polynomial. In exceptional case these two polynomials are different moreover are of different degrees but as it is pointed out for all \( l \in \mathbb{N} \)

\[
\lim_{n \to \infty} \left( \int Q^l \tilde{\nu}_n - \int Q^l d\tilde{\mu}_n \right) = 0,
\]

where \( \tilde{\nu}_n \) and \( \tilde{\mu}_n \) are defined in [29] and [22], respectively.

### 6. Certain Self-inversive Polynomials

In this section we use the multiplication operator and the infinite matrix \( M_e \) introduced in Section 3 to investigate certain self-inversive polynomials.

The whole investigation of Section 3 was independent of the constant term of the polynomial \( Q = \int x b \). Recalling (35) and (40), in this section we define

\[
Q(x) := \int x b - U_0,
\]

where \( \int x b \) means the primitive function without any constant term. The operator \( M_e \) refers to this \( Q \). \( M_e \) can be decomposed to a bounded symmetric and a compact symmetric part cf. [32], that is

\[
M_e = M_{e,s} + M_{e,c},
\]

where

\[
M_e = \begin{bmatrix}
-u_{0,0} + U_0 & u_{0,1} & \cdots & \cdots & u_{0,L} & 0 & 0 & \cdots \\
u_{1,-1} & u_{1,0} - U_0 & \cdots & \cdots & u_{1,L-1} & u_{1,L} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{L,-L} & u_{L,-L+1} & \cdots & u_{L,0} - U_0 & \cdots & u_{L,L} & 0 & \vdots \\
0 & u_{L+1,-L} & \cdots & \vdots & \vdots & \vdots & \cdots & u_{L+1,L} \\
\vdots & \vdots & \cdots & u_{L+j,-L} & \cdots & \vdots & \cdots & \vdots
\end{bmatrix}
\]
and

\begin{equation}
M_{e,s} = \begin{bmatrix}
0 & U_1 & \ldots & U_L & 0 & 0 & \ldots \\
U_1 & 0 & \ldots & U_{L-1} & U_L & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
U_L & \ldots & 0 & \ldots & \ldots & \ldots & U_L \\
0 & U_L & \vdots & \ldots & \vdots & \ldots & \ldots \\
\vdots & \vdots & U_L & \ldots & \vdots & \ldots & \ldots \\
\end{bmatrix}.
\end{equation}

Investigation of $M_{e,s}$ leads to the so-called self-inversive or palindrome polynomials, cf. [21, Lemma 12]. A polynomial $P$ of degree $n$ with real coefficients is self-inversive if $z^n P \left( \frac{1}{z} \right) = P(z)$. The location of zeros of self-inversive polynomials has been extensively studied, see e.g. [19], [31], etc. Of course, the zeros of a self-inversive polynomial are symmetric with respect to the unite circle. One of the statements on location of zeros is the next one (see eg. [31]): if $P_{2m}(z) = \sum_{k=0}^{2m} a_k z^k$ is self-inversive and $|a_m| > \sum_{k \leq m} |a_k|$, then $P_{2m}$ has no zeros on the unite circle.

In our special case we get something similar. Let us recall that $b(x) = \sum_{k=0}^{m} d_k x^k$, and the requirements on $b$ are as follows. Let $b(x) > 0$ if $x \in (-1, 1)$, and it has at least simple zeros at $-1$ and $1$, $bw$ is a polynomial, where $w$ is a rational solution of (7). Now define

$$P_{2L, \lambda}(z) = \sum_{k=1}^{L} U_k (z^{L+k} + z^{L-k}) - \lambda z^L = \tilde{P}_{2L}(z) - \lambda z^L,$$

where $U_k$ depends on $b$ see (40).

**Statement 1.** $P_{2L, \lambda}(z)$ has no zeros on the unite circle if and only if

$$\lambda \notin \left\{ 2 \sum_{k=1}^{L} (-1)^k U_k, 2 \sum_{k=1}^{L} U_k \right\} = (-1)^L \left[ \tilde{P}_{2L}(z)(-1), \tilde{P}_{2L}(1) \right].$$

Let us consider $M_{e,s}$ as an operator on $l^2$ and on the Hardy space

$$H^2 := \left\{ f(z) = \sum_{k=0}^{\infty} c_k z^k : \text{is holomorphic on } |z| < 1, \lim_{r \to 1} f(re^{i\varphi}) = f(e^{i\varphi}) \text{ a.e. } \varphi \in (0, 2\pi) \right\}.$$

It is a Hilbert space under the norm $\|f\|^2 = \sum_{k=0}^{\infty} |c_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\varphi})|^2 d\varphi$.

**Lemma 3.** $M_{e,s} - \lambda I$ has a bounded inverse if and only if $P_{2L, \lambda}(z)$ has no zeros on the unite circle.

**Proof.** With this interpretation if $f \in H^2$,

$$(M_{e,s}f)(z) = \sum_{k=1}^{L} U_k (z^k + z^{-k}) f(z) - \sum_{k=1}^{L} U_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{j+k}.$$

Let $g \in H^2$ be arbitrary. Then

$$f(z) = \frac{z^L g(z) + \sum_{k=1}^{L} U_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{L+j-k}}{P_{2L, \lambda}(z)}.$$
Proof. Let us recall the information on \( \sigma \) isolated point of \( M \) solution it must be of rank of the coefficient matrix coincides with its less size. That is to get a unique \((70) =\)

\[
\begin{aligned}
P_{2L,\lambda}(z) = \frac{z^L g(z) + \sum_{j=0}^{L-1} \frac{f^{(j)}(0)}{j!} \sum_{k=j}^{L-1} U_{L-k+j} z^k}{P_{2L,\lambda}(z)}.
\end{aligned}
\]

By symmetry, counting with multiplicity, \( P_{2L,\lambda} \) has \( l \) zeros in the unite disc and \( 2(L - l) \) on the unite circle. \( f \in H^2 \) if and only if these zeros can be compensated. That is if \( \xi_m \) are the zeros of the denominator (in the closed unite disc) of multiplicity \( k_m \) with \( \sum_m k_m = 2L - l \), then it means a system of linear equations in the numerator. According to \( (70) \) this system looks like

\[
\begin{bmatrix}
C_0(\xi_1) & C_1(\xi_1) & \cdots & C_{L-1}(\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
C_0(\xi_2) & C_1(\xi_2) & \cdots & C_{L-1}(\xi_2) \\
C_0'(\xi_2) & C_1'(\xi_2) & \cdots & C_{L-1}'(\xi_2) \\
\vdots & \vdots & \ddots & \vdots \\
C_0((k-1)\xi_2) & C_1((k-1)\xi_2) & \cdots & C_{L-1}((k-1)\xi_2) \\
\vdots & \vdots & \ddots & \vdots \\
c_{2L-1,0} & \cdots & \cdots & c_{2L-1,L-1}
\end{bmatrix}
\begin{bmatrix}
f(0) \\
f'(0) \\
f''(0) \\
\vdots \\
f^{(L-1)}(0)
\end{bmatrix}
= \begin{bmatrix}
b(\xi_1) \\
b(\xi_2) \\
b'(\xi_2) \\
\vdots \\
b^{(k-1)}(\xi_2)
\end{bmatrix},
\]

where \( C_j(x) = \frac{1}{L} \sum_{k=0}^{L-1} U_{L-k+j} x^k, j = 0, \ldots, L-1; b(x) = -x^L g(x) \), the coefficient matrix is of \( 2L - l \times L \) and the lenght of the "unknown" vector is \( L \).

We deal with the rank of the coefficient matrix. Multiplying by an appropriate constant \( k \)th column and subtracting it from the \( k-1 \)th one \( (0 < k \leq L - 1) \) - starting the process with the last column as \( C_{L-2} - (L - 1) \frac{U_{L-2}}{U_{L-1}} C_{L-1}, C_{L-3} - (L - 2) \frac{U_{L-3}}{U_{L-2}} C_{L-2}, \ldots \), and repeating this procedure starting with the new \( L - 2 \)th column, etc. it can be easily seen that we get a similar matrix which can be described on the same way as above by the modified function \( \tilde{C}_j(x) = x^j \). Thus the rank of the coefficient matrix coincides with its less size. That is to get a unique solution it must be of \( L \times L \) wich means that \( 2L - l = L \), that is \( l = L \) which means that \( P_{2L} \) has \( L \) zeros inside the unite disc. The unique solution of this linear system is equivalent with the existence of a (well-defined) bounded inverse of the operator and it can be characterized by the above mentioned location of the zeros of \( P_{2L,\lambda} \).

**Lemma 4.** The spectrum of \( M_{e,s} \) is \( Q([-1,1]) \).

**Proof.** Let us recall the information on \( M_{e,s} \). According to Weyl’s theorem (see e.g. [27] sec. 134]) the essential spectrum of \( M_{e,s} \) agrees with the essential spectrum of \( M_e \). Taking into consideration that the spectrum of the multiplication operator is the closure the range of \( Q \) on \([-1,1] \) (see e.g. [27] sec. 150]) the essential spectrum and the spectrum of \( M_e \) are the same. As the spectrum of \( M_{e,s} \) does not contains any isolated points as well, it also coincides with \( Q([-1,1]) \).

The last statement can be proved in this setup as follows. Let us consider \( P_{2L,\lambda}(r, \varphi) \) \((z = (r \cos \varphi, r \sin \varphi)) \) as a function from \( \mathbb{R}^{1+2} \) to \( \mathbb{R}^2 \). If there was an isolated point of \( \sigma(M_{e,s}) \), then there would be a \((\lambda_0, r_0, \varphi_0), r_0 = 1 \), such that \( P_{2L,\lambda_0}(r_0, \varphi_0) = 0 \) and there would be a neighborhood of \( \lambda_0 \) such that for any \( \lambda \) from this neighborhood the zeros of \( P_{2L,\lambda} \) are not on the unite circle.

So \( P_{2L,\lambda}(r, \varphi) = (p_1(\lambda, r, \varphi), p_2(\lambda, r, \varphi)) \). Let us consider \( \partial_2 P_{2L,\lambda} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \),
where $a_{11} = \frac{\partial}{\partial r} p_i$, $a_{12} = \frac{\partial}{\partial r} p_i$, $i = 1, 2$. As det $\frac{\partial^2 P_{2L, \lambda}}{\partial r^2} = \frac{1}{4} (a_{12}^2 + a_{12}^2)$, we can apply the implicit function theorem at $(\lambda_0, r_0, \varphi_0)$ that is there is a neighborhood of $\lambda_0$ denoted by $U$ and an arc $g$ in $U$ such that if $\lambda \in U$ then $P_{2L}(\lambda, g(\lambda)) = P_{2L, \lambda}(r(\lambda), \varphi(\lambda)) = 0$. If $\lambda \in U$ $g'(\lambda) = \begin{bmatrix} r'(\lambda) \\ \varphi'(\lambda) \end{bmatrix}$, where $r'(\lambda) = \frac{\lambda^2}{a_{12}^2 + a_{12}^2} \sum_{k=1}^{L} kU_k \cos k\varphi(r^k - r^{-k})$ (that is $r'(\lambda_0) = 0$), and $\varphi'(\lambda) = \frac{-\lambda^2}{a_{12}^2 + a_{12}^2} \sum_{k=1}^{L} kU_k \sin k\varphi(r^k + r^{-k})$. Thus the slope of the tangent line at $\lambda_0$ to $g$ is $-\cot(\varphi(\lambda_0))$, which is just the the slope of the tangent line to the unit circle at the same point, and the curvature of $g$ at $\lambda_0$ is 1. Considering the symmetry of the zeros with respect to the unite circle $g$ has to coincide locally with the unite circle, which means that $\lambda_0$ cannot be isolated.

**Proof.** (of the Statement) Notice, that $b \geq 0$ on $[-1, 1]$, thus $Q$ is increasing here and $Q(-1) < Q(1)$. That is to prove the statement it is enough to compute these two values. In view of (40)

\[
2 \sum_{k=1}^{L} (-1)^k U_k
\]

Changing the order of summation and by the definition of $Q$

\[
S = \sum_{p=l}^{2p} \frac{d_{p-1}}{2p} \left( 1 - \frac{1}{2^{2p}} \binom{2p}{p} \right) \pm 2 \sum_{p=l}^{L} \frac{d_{2p}}{2p+1} \binom{2p}{p} + 1 \frac{1}{2^{2p+1}} =: S.
\]

Thus the spectrum of the operator $[M]_{e,s}$ is $\sum_{k=1}^{L} (-1)^k U_k$, $2 \sum_{k=1}^{L} U_k$, which proves the statement.

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