Meeting Descartes and Klein
Somewhere in a Noncommutative Space

Vladimir V. Kisil

Abstract. We combine the coordinate method and Erlangen program in the framework of noncommutative geometry through an investigation of symmetries of noncommutative coordinate algebras. As the model we use the coherent states construction and the wavelet transform in functional spaces. New examples are a three dimensional spectrum of a non-normal matrix and a quantisation procedure from symplectomorphisms.

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1. Introduction
Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.
Vladimir Arnold

We used to think by a small number of mental images which help us to understand equally good (or bad) a variety of different processes. A “Big Bang” is one of such pet ideas which we call archetypes to excuse their overloading. K. Jaspers

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On leave from the Odessa University.
argued that modern culture appeared from a black matter as result of a big bang at the *axial period* more than two thousand years ago. The expanding Universe of human culture became split into many seemingly independent galaxies—science, religion, art, etc.—each with its own complicated structure and dynamics. The cosmological belief that all future of a Universe was determined by the first minutes after the bang may not be true but has an appeal of simplicity. Anyway it is natural to expect that after the bang all parts will run away each other. Thus the appearance of *two cultures* in the sense of C.P. Snow, which are disjoint or even in an opposition, seems to be unavoidable.

Yet there is also another persistent pattern: mathematics since *Elements of Euclid* grown enormously both in qualitative and quantitative sense but did not split into several smaller independent subjects. And the border between mathematics and physics (if ever exists at all) is as thin today as in times of Archimedes. Moreover a smuggling across that border in both direction is more rewarding today than ever before. Is there a hidden rules or forces which tie them together despite of general centrifugal tendencies?

2. Descartes Meets Klein: Symmetries of Coordinate Algebras

If you would be a real seeker after truth, you must at least once in your life doubt, as far as possible, all things.
*Descartes Discours de la Méthode, 1637*

The striking example of centripetal trends in mathematics is the Cartesian coordinate method. Before the XVII century there were two big and relatively independent mathematical subjects with different (even geographically) origins: the *synthetic geometry* of Greeks and *abstract algebra* of Arabs. It was natural to expect that these fields will diverge even further during their developments. Thus the proposition of Descartes to associate geometrical problems with algebraic equations by an *introduction of coordinates* was a great manifestation of the integrity of mathematics. Another example of unexpected links between seemingly unrelated topics was the Galois discovery that solvability of *algebraic equations* depends on certain *group-theoretic properties* of their Galois group. Just putting together these two ideas one may suspect that there is a connection between geometry and group theory. *That connection was announced in the famous Erlangen program of Felix Klein [developed under the strong influence of Sophus Lie]: synthesis of geometry as the study of the properties of a space that are invariant under a given group of transformations.*

We will not retell once again the story of coordinate approach in noncommutative spaces, see for example [11] for a balanced and concise exposition. Instead we would highlight few observations oftenly overlooked in the current literature:

(1) The rule that coordinates should form an *algebra* was not introduced by Descartes originally, it is sufficient that coordinates have *any* rich algebraic structure to reflect all geometrical properties, for example, via algebraic or differential equations. The identification

\[
\text{coordinates = an associative algebra + some topology} \tag{2.1}
\]

was fixed only after the Gelfand theory of commutative Banach algebras. While the achievements based on (2.1) are really impressive [11] that identification is not necessary (see below) and could be a needless *restriction* in general.
(2) The development of noncommutative geometry was oftenly motivated and supported by the representation theory of groups. For example, the original paper on noncommutative measure and integration theory [37] led to the Plancherel theorem for noncommutative locally compact groups. Quantum theory—the stronghold of Cartesian approach to noncommutative spaces—was able to deal with elementary particles or four basic interactions only in the Erlangen spirit through their symmetry groups.

(3) The Cartesian connection of geometrical problems with algebraic equations was not a subordination of geometry to algebra, actually Descartes used it in both ways and introduced a geometrical method for solutions of quadratic equations. Forever geometry be seen as an extremely beautiful subject with its own charm: “a geometrical proofs” usually means “an elegant proof”, and its is fashionable to say “I am doing noncommutative geometry” rather than “I am studying operator algebras and applications”.

We challenge the rigid identification (2.1) in this paper and use the assumption:

\[ (2.2) \text{ coordinates are oftenly a representation space for a group action} \]

There are to approach certain problem in noncommutative spaces. The fact that sometimes coordinates form also an algebra could be useful but is not crucial anymore. We will see in the last two sections that it is even helpful to downplay the structure of algebras by abandoning the algebraic homomorphism property. The number of applications is not limited to given in the present paper, (see in addition Example 2 in [24] with the Manin plane and quantum groups [32]) and they deserve a further investigation.

One can object [11] that homogeneous spaces which are geometries in the sense of the Klein program are too restrictive to give a good model of space-time in general relativity. There is no a hard evidence to refute that claim at the moment, but we could learn from the inspiring paper [8] that symmetries of objects are usually richer than people ordinary think. For example, even the structure of a single point could be significantly enriched by introduction of a coordinate bundle over it [8, § 6] and assigning a group action in that bundle. Therefore one could make the following conjecture, which we illustrate by examples in the present paper.

Conjecture 2.1. The combination of Cartesian and Klein-Lie approaches based on the assumption (2.2) is stronger than the original Erlangen program itself and could go beyond previous limits.

The rest of the paper is organised as follows. In the next Section we will study symmetries of functional spaces which are coordinates in commutative cases. This will be our platform for an invasion to noncommutative spaces, similarly to the Gelfand structural theorem about commutative Banach algebras in the approach based on identification (2.1). The technique is widely known as coherent states construction and wavelet transform but was rediscovered many times before and after those names were coined. We show that many fundamental notions of analysis are intimately connected to that circle of ideas, which are yet not explicitly understood and used to its full power.

Intertwining commutative and noncommutative coordinates in Section 4 we will get a new description of functional calculus and related spectrum of non-normal matrices. The calculus and the spectrum are naturally connected through an appropriately extended spectral mapping theorem.
We also approach the quantisation problem with the similar ideas in Section 5 and get a natural combination of quantum and classical mechanics within the framework of the Heisenberg group.

This paper is a survey or even an essay on the subject. The three main sections are rather connected by a common idea than technically dependent. Therefore they could be looked through almost separately. More details could be found in published papers [22, 23, 25, 27] and will also appear in [26, 28].

3. Coherent States and Wavelets in Mathematics and Physics

In the 1960’s it was said (in a certain connection) that the most important discovery of recent years in physics was the complex numbers. Yu.I. Manin [31, Preface]

We would like to present a construction which produces many important objects in analytic function theory, i.e. commutative coordinate spaces, out of symmetry groups. The scheme is well known, cf. [1, 34] and got much attention in recent decades but it is not used to its full potential yet. Our main examples are provided by the one dimensional Heisenberg group \( \mathbb{H} = \mathbb{H}^1 \) [16, 44] and the \( SL_2(\mathbb{R}) \) groups [19, 30, 44]—two groups with the utmost importance [18, 19] in both mathematics and physics.

The one dimensional Heisenberg group \( \mathbb{H} \) consists of points \((s, z) = (s, x + iy)\) parametrised by \( s \in \mathbb{R} \) and \( z = x + iy \in \mathbb{C} \). The group law is given by:

\[
g \ast g' = (s, z) \ast (s', z') = (s + s' + \frac{1}{2} \Im(\bar{z}z'), z + z'),
\]

where \( \Im w \) denotes the imaginary part of a complex number \( w \). The \( \mathbb{H} \) is the necessary component (sometimes implicit) of any quantisation scheme because its Lie algebra has the only non-trivial commutator:

\[
[[X, Y]] = S,
\]

which in the Schrödinger representation (see below (3.17)) takes a form of the celebrated Heisenberg uncertainty relation.

The group \( SL_2(\mathbb{R}) \) consists of \( 2 \times 2 \) matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with real entries and determinant \( ad - bc = 1 \). Its is also isomorphic to the following three groups [44, § 8.1] which we will use below in different contexts:

1. the Lorentz type group \( SO_+(1, 1) \) in the two-dimensional Minkowski space with the metric \( ds^2 = dt^2 - dx^2 \);
2. the group \( SU(1, 1) \) of linear transformation of \( \mathbb{C}^2 \) preserving the quadratic form \( z_1^2 - z_2^2 \);
3. the symplectic group \( Sp(1) \) of linear symplectomorphisms of the two-dimensional flat phase space in classical mechanics.

It is not surprising that \( \mathbb{H} \) and \( SL_2(\mathbb{R}) \) are intimately connected to each other as we will see and use in the Section 3.

3.1. Space-Time or Phase Space from Symmetry Groups. There is a heuristic observation [41] that a space-time is not a primary concept but appears in the approach based on the identification [2, 1] as the spectrum of a maximal commutative subalgebra of the algebra of observables invariant under the fundamental group. We show in this subsection that simplest forms of the space-time and the
phase space could be naturally obtained just from symmetry groups in the context of (23). It is interesting to note that the same group could produce several rather distinct spaces, e.g., with elliptic or hyperbolic metrics.

Abstract scheme could be described as follows. Let $G$ be a group and $H$ be its closed normal subgroup, which could be trivially just $\{e\}$. Let $X = G/H$ be the corresponding homogeneous space with an invariant measure $d\mu$ and $s : X \rightarrow G$ be a Borel section in the principal bundle $G \rightarrow G/H$ [21 § 13.2]. Then any $g \in G$ has a unique decomposition of the form $g = s(x)h$, $x \in X$ and we will write $x = s^{-1}(g)$, $h = r(g) = (s^{-1}(g))^{-1}g$. Note that $X$ is a left $G$-homogeneous space with an action defined in terms of $s$ as follow:

$$g : x \mapsto g \cdot x = s^{-1}(g^{-1} * s(x))$$

where $*$ is the multiplication on $G$.

We will illustrate our consideration by a chain of examples. Each one consists of four parts numbered from (a) to (d): two cases for $G = \mathbb{H}$ with two different subgroups $H = \mathbb{R}^2$ and $H = \mathbb{R}$, see [23] for more details; other two cases for $G = SL_2(\mathbb{R})$ with subgroup $H = K$ and $H = A$ studied in [23].

**Example 3.1.** (a) We start from $\mathbb{H}$ and its subgroup $H = \mathbb{R}^2 = \{(t, z) \mid \Im(z) = 0\}$. Then $X = G/H = \mathbb{R}$ and because the Haar measure on $\mathbb{H}$ coincides with the standard Lebesgue measure on $\mathbb{R}^3$ [44 § 1.1] then invariant measure on $X$ coincides with the Lebesgue measure on $\mathbb{R}$. Mappings $s : \mathbb{R} \rightarrow \mathbb{H}$ and $r : \mathbb{H} \rightarrow H$ are defined by the identities $s(x) = (0, ix)$, $s^{-1}(t, z) = \Im x$, $r(t, u + iv) = (t, u)$. The composition law $s^{-1}((t, z) \cdot s(x)) = x + u$ reduces to Euclidean shifts on $\mathbb{R}$. We also find $s^{-1}((s(x_1))^{-1} \cdot s(x_2)) = x_2 - x_1$ and $r((s(x_1))^{-1} \cdot s(x_2)) = 0$. This $X$ is the configuration space of a particle with one degree of freedom.

(b) As a subgroup $H = \mathbb{R}$ we select now the one dimensional centre of $\mathbb{H}$ consisting of elements $(s, 0)$. Of course $X = G/H$ isomorphic to $\mathbb{C}$ and mapping $s : \mathbb{C} \rightarrow G$ simply is defined as $s(z) = (0, z)$. The invariant measure on $X$ also coincides with the Lebesgue measure on $\mathbb{C}$. Note also that composition law $s^{-1}(g \cdot s(z))$ reduces to Euclidean shifts on $\mathbb{C}$. We also find $s^{-1}((s(z_1))^{-1} \cdot s(z_2)) = z_2 - z_1$ and $r((s(z_1))^{-1} \cdot s(z_2)) = \frac{1}{2} \Im z_1 z_2$ -- the symplectic form on $\mathbb{R}^2$. In that case we get the phase space of a particle with one degree of freedom.

(c) Here we study $SL_2(\mathbb{R})$ in the form of the group $SU(1, 1)$ of $2 \times 2$ matrices with complex entries of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ such that $|\alpha|^2 - |\beta|^2 = 1$. $SL_2(\mathbb{R})$ has the only non-trivial compact closed subgroup $K$, namely the group of matrices of the form $h_\psi = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$. Any $g \in SL_2(\mathbb{R})$ has a unique decomposition of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \frac{1}{\sqrt{1 - |\alpha|^2}} \begin{pmatrix} 1 & a \\ \bar{\alpha} & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$$

where $\psi = \Im \ln \alpha$, $a = \beta \bar{\alpha}^{-1}$, and $|\alpha| < 1$ because $|\alpha|^2 - |\beta|^2 = 1$. Thus we can identify $SL_2(\mathbb{R})/H$ with the unit disk $\mathbb{D}$, see Figure 4(a), and define mapping $s : \mathbb{D} \rightarrow SL_2(\mathbb{R})$ and $r : G \rightarrow H$ as follows

$$s : a \mapsto \frac{1}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix} \quad r : \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\bar{\alpha}}{|\alpha|} \end{pmatrix}.$$
The formula $g : a \mapsto g \cdot a = s^{-1}(g^{-1} \ast s(a))$ associates with a matrix $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ the fraction-linear conformal transformation of $\mathbb{D}$ of the form

$$g : z \mapsto g \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \quad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},$$

which also can be extended to a transformation of $\hat{\mathbb{C}}$ (the one-point compactification of $\mathbb{C}$). Here $X = \mathbb{D}$ is two dimensional conformal configuration space.

(d) Now we use $SL_2(\mathbb{R})$ in the form of a Lorentz type group $SO_e(1,1)$. It is convenient to represent its elements again as $2 \times 2$-matrices but this time with Clifford algebra values. This four real dimensional Clifford algebra $\mathcal{C}(1,1)$ generated by $1$ and two imaginary units $e_1$ and $e_2$ such that

$$e_1^2 = -e_2^2 = -1, \quad e_1 e_2 = -e_2 e_1.$$

We use Sanserif font for elements of $\mathcal{C}(1,1)$. Then $SL_2(\mathbb{R})$ is represented by matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where $a\bar{a} - b\bar{b} = 1$. We have a decomposition similar to (3.3):

$$(3.6) \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = |a| \begin{pmatrix} 1 & ba^{-1} \\ -ba^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{a}{|a|} & 0 \\ 0 & \frac{a}{|a|} \end{pmatrix}.$$  

It could be seen that $ba^{-1} \in \mathbb{R}^{1,1}$, i.e. is a two-dimensional vector in Minkowski space. But now we could not get that $|ba^{-1}| < 1$, or equivalently we could not separate:

(1) topologically the Minkowski space $\mathbb{R}^{1,1}$ into interior and exterior of the unit circle;
(2) analytically \( L_2 \) on the unit “circle” into subspaces of analytic and antianalytic functions;

(3) physically the time axis into the future and the past halves, because Möbius (linear-fractional) transformations mix both sets in each case. A way out is known and is the same both in physical and mathematical situations: we need to take a double cover of \( \mathbb{R}^{1,1} \) and chose the “unit disk” \( \mathbb{D} \) as shown on Figure 1(b) and explained in its caption.

Matrices of the form 
\[
\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} e^{e_1 e_2 \tau} & 0 \\ 0 & e^{-e_1 e_2 \tau} \end{pmatrix}, \quad a = e^{e_1 e_2 \tau} = \cosh \tau + e_1 e_2 \sinh \tau, \quad \tau \in \mathbb{R}
\]

comprise a subgroup of hyperbolic rotations in \( \mathbb{R}^{1,1} \) which we denote by \( A \). We define an embedding \( s \) of \( \mathbb{D} \) for our realization of \( SL_2(\mathbb{R}) \) by the formula:

\[
s : u \mapsto \frac{1}{\sqrt{1 + u^2}} \begin{pmatrix} 1 & u \\ -u & 1 \end{pmatrix}, \quad r : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto \begin{pmatrix} a \sqrt{\frac{|a|}{|b|}} & 0 \\ 0 & \frac{b}{|b|} \end{pmatrix}
\]

The formula \( g : u \mapsto s^{-1}(g \cdot s(u)) \) gives the linear-fraction transformation \( \mathbb{D} \to \mathbb{D} \) conformal in the hyperbolic metric:

\[
g : u \mapsto g \cdot u = \frac{au + b}{-bu + a}, \quad g^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},
\]

which is similar to (3.5). We get two dimensional relativistic space-time. The appearance of Clifford algebra in relativistic case is expectable [13, 35].

In the following we call \( X \) just space understanding that in different realisations it could be either a configuration space or phase space or space-time.

### 3.2. The Vacuum and Reduced Wavelet Transform

We are ready to explain the rôle of the subgroup \( H \) in the coherent states construction: it selects the vacuum as its eigenvector among all possible physical states. Thereafter having the chosen vacuum we can generate all possible states of the system from a representation \( \rho \) of \( G \) by isometric operators in a Banach space \( B \). Coherent states are parametrised by points of the space \( X = G/H \).

**Definition 3.2.** [23] Let \( G, H, X = G/H, s : X \to G, \rho : G \to L(B) \) be as before. We say that \( b_0 \in B \) is a vacuum vector if for all \( h \in H \)

\[
\rho(h)b_0 = \chi(h)b_0, \quad \chi(h) \in \mathbb{C}.
\]

We say that set of vectors \( b_x = \rho(x)b_0, \quad x \in X \) form a family of coherent states if there exists a continuous non-zero linear functional \( l_0 \in B^* \), called the analysing functional, such that

1. \( ||b_0|| = 1, \quad ||l_0|| = 1, \quad \langle b_0, l_0 \rangle \neq 0; \)
2. \( \rho(h)^*l_0 = \chi(h)l_0 \), where \( \rho(h)^* \) is the adjoint operator to \( \rho(h) \);
3. The following equality holds

\[
\int_X \langle \rho(x^{-1})b_0, l_0 \rangle \langle \rho(x)b_0, l_0 \rangle \, d\mu(x) = \langle b_0, l_0 \rangle.
\]

With analysing functional we are able to decompose any state as a superposition of the coherent states.
The reduced wavelet transform \( W \) from a Banach space \( B \) to a space of function \( F(X) \) on a homogeneous space \( X = G/H \) defined by a representation \( \rho \) of \( G \) on \( B \), a vacuum vector \( b_0 \) and a test functional \( l_0 \) is:

\[
W : v \mapsto \hat{v}(x) = [Wv](x) = \langle \rho(x^{-1})v, l_0 \rangle = \langle v, \rho^*(x)l_0 \rangle.
\]

The inverse wavelet transform \( \mathcal{M} \) from \( F(X) \) to \( B \) is given by the formula:

\[
\mathcal{M} : \hat{v}(x) \mapsto \mathcal{M}[\hat{v}(x)] = \int_X \hat{v}(x)b_x \, d\mu(x) = \int_X \hat{v}(x)\rho(x) \, d\mu(x)b_0.
\]

The geometric action \( \mathbb{R}^2 \) of \( G : X \to X \) defines a representation \( \lambda(g) : F(X) \to F(X) \) induced by a character \( \chi \) of \( H \) as follows

\[
[\lambda(g)f](x) = \chi(r(g^{-1} \cdot x))f(g^{-1} \cdot x).
\]

For the case of trivial \( H = \{ e \} \) the representation \( \lambda(g) \) becomes the left regular representation \( \rho_l(g) \) of \( G \) on \( L_2(G) \).

**Proposition 3.4.** We have:

1. The reduced wavelet transform \( W \) and the inverse wavelet transform \( \mathcal{M} \) intertwine \( \rho \) and the representation \( \lambda \) on \( F(X) \):

\[
W\rho(g) = \lambda(g)W \quad \text{and} \quad \mathcal{M}\lambda(g) = \rho(g)\mathcal{M} \quad \text{forall} \, g \in G.
\]

2. There is an isomorphism property:

\[
\langle \mathcal{M}, \mathcal{M}^* \rangle_{F(X)} = \langle v, l \rangle_B, \quad \forall v \in B, \quad l \in B^*
\]

3. The image \( F(X) \) of \( B \) under \( W \) is \( \lambda \)-invariant subspace of \( C(G) \).

There is a physical meaning of \( \lambda(g) \): having a representation \( \rho \) of \( G \) in an abstract space \( L(B) \) of observables we could introduce a space \( X \) and realise \( B \) as functions on \( X \) with a geometric action \( \lambda \) instead of an abstract \( \rho \). This advantage allows us to use the **intertwining property** \( \lambda(g) \) in new Definition 3.3 of functional calculus and Definition 1.1 of quantisation instead of the traditional algebraic homomorphism property.

**Theorem 3.5.** The composition of transforms \( \mathcal{M} \) and \( W \)

\[
\mathcal{P} = \mathcal{M}W : B \to B
\]

is a projection of \( B \) to its linear subspace for which \( b_0 \) is cyclic. Particularly if \( \rho \) is an irreducible representation then the inverse wavelet transform \( \mathcal{M} \) is a left inverse operator on \( B \) for the wavelet transform \( W : \mathcal{M}W = I \).

**Example 3.6.** (a) We take a representation \( \sigma_h \) of \( \mathbb{H} \) in \( L_p(\mathbb{R}) \), \( 1 < p < \infty \) by operators of shift and multiplication \( [4.4, \text{§ 1.1}] \):

\[
\sigma_h(s,z) : f(y) \mapsto [\sigma_h(s,z)f](y) = e^{i(2s-\sqrt{2h}uy+huv)}f(y-\sqrt{2hu}), \quad z = u + iv,
\]

It is the Schrödinger representation with parameter \( h \). As a character of \( H = \mathbb{R}^2 \) we take the \( \chi(s,u) = e^{2it} \). The corresponding test functional \( l_0 \) satisfying to \( 3.2(2) \) is the integration \( l_0(f) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) \, dy \). Thus the wavelet transform \( \mathcal{P} \) is

\[
\hat{f}(x) = \int_{\mathbb{R}^n} \sigma(s(x)^{-1})f(y) \, dy = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\sqrt{2hu}y} f(y) \, dy
\]
and is nothing else but the Fourier transform. There is no a vacuum vector in our space \( B \), see physical implications of an absence of vacuum in \( \mathcal{M} \). We however could proceed as in \( \mathbb{R}^n \) \( \mathcal{M} \) and take a bigger space \( B' = L_\infty(\mathbb{R}^n) \supset B \) with the vacuum vector \( b_0(y) = (2\pi)^{-n/2} e^{-iyx} \) and the inverse wavelet transform \( (3.12) \) is defined by the inverse Fourier transform

\[
\hat{f}(y) = \int_{\mathbb{R}^n} f(x) b_x(y) \, dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(y) e^{-iyx} \, dx.
\]

\( \mathcal{W} \) and \( \mathcal{M} \) intertwine the left regular representation — multiplication by \( e^{iyx} \) with operators

\[
[\chi(g)f](x) = \chi(r(g^{-1} \cdot x)) f(g^{-1} \cdot x) = e^{i\xi_x} f(x - \sqrt{x}u) = f(x - \sqrt{x}u),
\]

i.e. with Euclidean shifts. From \( (3.15) \) follows the Plancherel identity:

\[
\int_{\mathbb{R}^n} \hat{v}(y) \hat{f}(y) \, dy = \int_{\mathbb{R}^n} v(x) f(x) \, dx.
\]

Thus our construction generate all important properties of the Fourier transform.

The Schrödinger representation is irreducible on \( S(\mathbb{R}^n) \) thus \( \mathcal{M} = \mathcal{W}^{-1} \). Thereafter the operator \( (3.14) \) representing operators \( \mathcal{M} \mathcal{W} = \mathcal{W} \mathcal{M} = 1 \) correspondingly give an integral resolution of the Dirac delta \( \delta(x) \):

\[
\hat{\delta}(x - y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi(x-y)} \, d\xi.
\]

**b)** As a subgroup \( H \) we select now the center of \( \mathbb{H} \) consisting of elements \((s,0)\). As a “vacuum vector” we will select the original vacuum vector of quantum mechanics—the Gauss function \( f_0(x) = e^{-x^2/2} \) which belongs to all \( L_p(\mathbb{R}^n) \). Its transformations are as follow:

\[
\hat{f}_\rho(x) = \rho (s,z) f_0(x) = e^{2it - z/2} e^{-(z^2 + x^2)/2 + \sqrt{2}zx}.
\]

Particularly \( \rho (s,0) f_0(x) = e^{-2it} f_0(x), \) i.e., it really is a vacuum vector in the sense of our Definition \( (2.2) \) with respect to \( H \). For the same reasons we could take \( l_0(x) = e^{-x^2/2} \in L_q(\mathbb{R}^n), p^{-1} + q^{-1} = 1 \) as the analysing functional.

Coherent states \( f_{\rho}(x) = \rho (s,0) f_0(x) \) belongs to \( L_q(\mathbb{R}^n) \otimes L_p(\mathbb{C}^n) \) for all \( p > 1 \) and \( q > 1, p^{-1} + q^{-1} = 1 \). Thus transformation \( (3.11) \) with the kernel \( \rho (s,0) f_0(x) \) is an embedding \( L_p(\mathbb{R}^n) \to L_p(\mathbb{C}^n) \) and is given by the formula

\[
(3.19) \quad \hat{f}(z) = e^{-z^2/2 + \sqrt{2}zx} \int_{\mathbb{R}^n} f(x) e^{-(x^2 + z^2)/2 + \sqrt{2}zx} \, dx.
\]

Then \( \hat{f}(g) \) belongs to \( L_p(\mathbb{C}^n, dg) \) or its preferably to say that function \( \hat{f}(z) = e^{-z^2/2} \hat{f}(t_0, z) \) belongs to space \( L_p(\mathbb{C}^n, e^{-|z|^2} \, dg) \) because \( \hat{f}(z) \) is analytic in \( z \). Such functions for \( p = 2 \) form the Segal-Bargmann space \( F_2(\mathbb{C}^n, e^{-|z|^2} \, dg) \) of functions \( 8, 38 \), which are analytic by \( z \) and square-integrable with respect the Gaussian measure \( e^{-|z|^2} \, dz \).

The integral in \( (3.13) \) is the well-known Segal-Bargmann transform \( 3, 38 \). The inverse is given by a realization of \( (3.13) \):

\[
(3.20) \quad f(x) = \int_{\mathbb{C}^n} \hat{f}(z) e^{-(z^2 + x^2)/2 + \sqrt{2}zx} e^{-|z|^2} \, dz.
\]
The corresponding operator $P$ (3.16) is an identity operator $L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ and (3.16) gives another integral presentation of the Dirac delta.

Integral transformations (3.14) and (3.20) intertwines the Schrödinger representation (3.17) with the following realization of representation (3.14):

$$
(3.21) \quad \chi(s, z)f(w) = f_0(z^{-1}, w) \chi(t + r(z^{-1}, w)) = f_0(w - z) e^{H + i\lambda(z w)}
$$

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|z|^2} \, dz) \to F_2(\mathbb{C}^n, e^{-|z|^2} \, dz)$ is of a separate interest and is a principal ingredient in Berezin quantization [7, 8]. Its integral kernel is

$$
K(z, w) = \exp \left( \frac{1}{2}(-|z|^2 - |w|^2) + wz \right).
$$

(c) We continue with the case of $G = SL_2(\mathbb{R})$ and $H = K$. The compact group $K \sim T$ has a discrete set of characters $\chi_m(h_\phi) = e^{-im\phi}$, $m \in \mathbb{Z}$. We consider here only $\chi_1$, see [23] for others. Let us take $X = T$—the unit circle equipped with the standard Lebesgue measure $d\phi$ normalised in such a way that

$$
(3.22) \quad \int_T |f_0(\phi)|^2 \, d\phi = 1 \text{ with } f_0(\phi) \equiv 1.
$$

From (3.1) one can find that

$$
r(g^{-1} \ast s(e^{i\phi})) = \frac{\beta e^{i\phi} + \alpha}{|\beta e^{i\phi} + \alpha|}, \quad g^{-1} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.
$$

Then the action of $G$ on $T$ defined by (3.3), the equality $d(g \cdot \phi)/d\phi = |\beta e^{i\phi} + \alpha|^{-2}$ and the character $\chi_1$ give the following realization of the formula (3.13):

$$
(3.23) \quad [\rho_1(g)f](e^{i\phi}) = \frac{1}{\beta e^{i\phi} + \alpha} f \left( \frac{\alpha e^{i\phi} + \beta}{\beta e^{i\phi} + \alpha} \right).
$$

This is a unitary representation—the mock discrete series of $SL_2(\mathbb{R})$ [44, § 8.4]. It is easily seen that $K$ acts in a trivial way (3.3) by multiplication by $\chi(e^{i\phi})$. The function $f_0(e^{i\phi}) \equiv 1$ mentioned in (3.22) transforms $[\rho_1(g)f_0](e^{i\phi}) = (\beta e^{i\phi} + \alpha)^{-1}$ and in particular has an obvious property $[\rho_1(h_\phi)f_0](\phi) = e^{i\phi} f_0(\phi)$, i.e. it is a vacuum vector with respect to the subgroup $H$. The smallest linear subspace $F_2(X) \subset L_2(X)$ spanned by all these transformations consists of boundary values of analytic functions in the unit disk and is the Hardy space. Now the reduced wavelet transform (3.11) takes the form [23]:

$$
(3.24) \quad \hat{f}(a) = |Wf|(a) = \sqrt{1 - |a|^2} \int_T \frac{f(z)}{a + z} \, d\phi,
$$

where $z = e^{i\phi}$. Of course (3.24) is the Cauchy integral formula up to factor $2\rho \sqrt{1 - |a|^2}$. The inverse wavelet transform $M$ gives an integral expression for orthoprojection Szegö onto the Hardy space.

(d) Now we consider the same group $G = SL_2(\mathbb{R})$ but pick up another subgroup $H = A$. Let $e_{12}$ denote $e_1 e_2$. The mapping from the subgroup $A \sim \mathbb{R}$ to even Clifford numbers $\chi_\sigma : a \mapsto a e^{i\sigma} = (\exp(e_1 e_2 \sigma \ln a)) = (a p_1 + a^{-1} p_2)^\sigma$ parametrised by $\sigma \in \mathbb{R}$ is a character (in a somewhat generalised sense). It represents an isometric...
rotation of $\tilde{T}$ by the angle $\alpha$. Under the present conditions we have from (3.7):

$$r(g^{-1} \ast s(u)) = \begin{pmatrix} -bu+\alpha & 0 \\ \frac{bu-a}{|bu-a|} & -bu+\alpha \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$  

Furthermore we can construct a realization of (3.13) on the functions defined on $\tilde{T}$ by the formula:

$$\langle f_1, f_2 \rangle_T^\sigma = \int_{\tilde{T}} \tilde{f}_2(t) f_1(t) \, dt.$$  

We select function $f_0(u) \equiv 1$ as our vacuum vector, it is a singular one in the same sense as $e^{i\xi z}$ is singular for the Fourier transform in Example 3.6(a). Furthermore, we can consider the identity:

$$|\rho_\sigma(g) f_0| = |1 + u^2|^{1/2} \frac{(-vb+\bar{a})^\sigma}{(-bv+a)^{1+\sigma}}, \quad g^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$  

and in particular the identity $|\rho_\sigma(g) f_0| = \bar{a}^\sigma a^{-1-\sigma} f_0(v) = a^{-1-2\sigma} f_0(v)$ for $g^{-1} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ demonstrates that it is a vacuum vector. Thus we can define the reduced wavelet transform accordingly to (3.7) and (3.11) by the formula:

$$|W_\sigma f|(u) = |1 + u^2|^{1/2} e^{12} \int_{\tilde{T}} \frac{(-ue^z + 1)^\sigma z^\sigma}{(-e^u + z)^{1+\sigma}} \, dz.$$  

where $z = e^{i\xi z}$ is the new monogenic variable and $dz = e^{12} e^{i\xi z} \, dt$ —its differential. The integral (3.27) is a singular one, its four singular points are intersections of the light cone with the origin in $u$ with the unit circle $\tilde{T}$.

The explicit similarity between (3.24) and (3.27) allows us to consider transformation $W_\sigma$ (3.27) as an analog of the Cauchy integral formula and the linear space $H_\sigma(\mathbb{T})$ generated by the coherent states $f_u(z)$ (3.26) as the correspondence of the Hardy space $H_2(\mathbb{T})$.

### 3.3. The Dirac (Cauchy-Riemann) and Laplace Operators

Consideration of Lie groups is hardly possible without consideration of their Lie algebras, which are naturally represented by left and right invariant vector fields on groups. On a homogeneous space $X = G/H$ we have also defined a left action of $G$ and can be interested in left invariant vector fields (first order differential operators). Due to the irreducibility of $F_\mathbb{T}(x)$ under left action of $G$ every such vector field $D$ restricted to $F_\mathbb{T}(x)$ is a scalar multiplier of identity $D|_{F_\mathbb{T}(x)} = cI$. We are in particular interested in the case $c = 0$.

**Definition 3.7.** A $G$-invariant first order differential operator

$$D_\rho : C_0(X,S \otimes V_\rho) \to C_0(X,S \otimes V_\rho)."
such that \(\mathcal{W}(F_2(X)) \subset \ker D_\rho\) is called \((Cauchy\text{-}Riemann-)Dirac\ operator\) on \(X = G/H\) associated with an irreducible representation \(\rho\) of \(H\) in a space \(V_\rho\) and a spinor bundle \(\mathcal{S}\).

The Dirac operator is explicitly defined by the formula \(\] (3.1)]:

\[
D_\rho = \sum_{j=1}^{n} \rho(Y_j) \otimes c(Y_j) \otimes 1,
\]

where \(Y_j\) is an orthonormal basis of \(\mathfrak{p} = \mathfrak{h}^\perp\) — the orthogonal completion of the Lie algebra \(\mathfrak{h}\) of the subgroup \(H\) in the Lie algebra \(\mathfrak{g}\) of \(G\); \(\rho(Y_j)\) is the infinitesimal generator of the right action of \(G\) on \(X\); \(c(Y_j)\) is Clifford multiplication by \(Y_j \in \mathfrak{p}\) on the Clifford module \(\mathcal{S}\). We also define an invariant Laplacian by the formula

\[
\Delta_\rho = \sum_{j=1}^{n} \rho(Y_j)^2 \otimes \epsilon_j \otimes 1,
\]

where \(\epsilon_j = c(Y_j)^2\) is +1 or −1. Note that the Dirac operator (3.28) is not a factor of the Laplacian (3.29), unless all commutators \([Y_i, Y_j]\) vanish. Thus null-solutions of \(D_\rho\) is not necessarily the null-solutions of \(\Delta_\rho\) a priori. But this happens under our assumptions.

**Proposition 3.8.** \(\] Let all commutators of vectors of \(\mathfrak{h}^\perp\) belong to \(\mathfrak{h}\), i.e. \([\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subset \mathfrak{h}\). Let also \(f_0 \) be an eigenfunction for all vectors of \(\mathfrak{h}\) with eigenvalue 0 and let also \(\mathcal{W}f_0\) be a null solution to the Dirac operator \(D\). Then \(\Delta f(x) = 0\) for all \(f(x) \in F_2(X)\).

**Example 3.9.** (a) With \(G = \mathbb{H}, H = \mathbb{R}^2,\) and \(X = \mathbb{R}\) the \(L_2(X)\) is irreducible therefore both the Dirac and Laplace operators are identically zero.

(b) With \(G = \mathbb{H}, H = \mathbb{R},\) and \(X = \mathbb{R}^2\) we could take just \(\mathbb{C}\) as a “Clifford algebra” sufficient in that case. The orthogonal completion to the centre \((s, 0, 0)\) generates two dimensional Euclidean shifts on \(\mathbb{R}^2 = \mathbb{C}\), cf. Example 3.1(b). As an orthogonal basis in that subspace we could take the differential operators \(\partial/\partial x_1\) and \(\partial/\partial x_2\), then we got the following realisation of (3.28) and (3.29):

\[
D = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},
\]

i.e. the classic Cauchy-Riemann and Laplace operators. This is a particular case of invariant operators on nilpotent Lie groups considered in \(\] and the inclusion \([\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subset \mathfrak{h}\) needed in Proposition 3.8 follows from the nilpotency of \(\mathbb{H}\).

(c) Let \(G = SL_2(\mathbb{R})\) and \(H\) be its one-dimensional compact subgroup \(K\). Then \(\mathfrak{h}^\perp\) is spanned by two vectors \(Y_1 = A\) and \(Y_2 = B\). In such a situation we can again use \(\mathbb{C}\) instead of the Clifford algebra \(\mathcal{O}(0, 2)\). Then formula (3.28) takes a simple form \(D = \rho(A + iB)\). Infinitesimal action of this operator in the upper-half plane follows from calculation in \(\] VI.5(8), IX.5(3)], it is \([D_{\mathbb{H}}f](z) = -2iy\frac{\partial f(z)}{\partial z}, z = x + iy\). Making the Cayley transform we can find its action in the unit disk \(D_2\): again the Cauchy-Riemann operator \(\partial/\partial y\) is its principal component. An explicit calculation of \(D_{\mathbb{H}}\) was done in \(\] and gives the expected answer

\[
D_{\mathbb{H}} = i\rho(A) + \rho(B) = 2y i \partial_2 + 2y \partial_4 = 2y \frac{\partial}{\partial z}.
\]
which is just a conformal invariant variant of the Cauchy-Riemann equation. The corresponding operator $\Delta$ is an invariant Laplacian.

**d)** In $\mathbb{R}^{1,1}$ the subgroup $H = A$ and its orthogonal completion is spanned by $B$ and $Z$. Thus the associated Dirac operator has the form $D = e_1\rho(B) + e_2\rho(Z)$. We need infinitesimal generators of the right action $\rho$ on the “left” half plane $\tilde{H}$. Following to [23] we find that:

$$D_{\tilde{H}} = e_1\rho(B) + e_2\rho(A) = 2y(e_1\partial_1 + e_2\partial_2).$$

In this case the Dirac operator is not elliptic and as a consequence we have in particular a singular Cauchy integral formula (3.27). Another manifestation of the same property is that primitives in the “Taylor expansion” do not belong to $F_2(T)$ itself, see Example 3.12(d). It is known that solutions of a hyperbolic system (unlike the elliptic one) essentially depend on the behaviour of the boundary value data. Thus function theory in $\mathbb{R}^{1,1}$ is not defined only by the hyperbolic Dirac equation alone but also by an appropriate boundary condition. The operator (3.29) in this case is the wave equation $\Delta = y^2(\partial_1^2 - \partial_2^2)$.

### 3.4. The Taylor Expansion.

Both the wavelet transform and its inverse are based on the family of coherent states $f_a$. For any decomposition

$$f_a(x) = \sum_{\alpha} \psi_{\alpha}(x)V_{\alpha}(a)$$

of the coherent states $f_a(x)$ by means of functions $V_{\alpha}(a)$ (where the sum can become eventually an integral) we have the Taylor expansion

$$\tilde{f}(a) = \int_X f(x)f_a(x) dx = \int_X f(x) \sum_{\alpha} \tilde{\psi}_{\alpha}(x)V_{\alpha}(a) dx = \sum_{\alpha} \int_X f(x)\tilde{\psi}_{\alpha}(x) dx V_{\alpha}(a) = \sum_{\alpha} V_{\alpha}(a)f_a,$$

where $f_a = \int_X f(x)\tilde{\psi}_{\alpha}(x) dx$. However to be useful within the presented scheme such a decomposition should be connected with the structures of $G$, $H$, and the representation $\rho$. We will use a decomposition of $f_a(x)$ by the eigenfunctions $V_{\alpha}$ of the operators $\rho(h)$, $h \in \mathfrak{h}$.

**Definition 3.10.** Let $F_2(X) = \int_A H_\alpha d\alpha$ be a spectral decomposition with respect to the operators $\rho(h)$, $h \in \mathfrak{h}$. Then the decomposition

$$f_a(x) = \int_A V_{\alpha}(a)f_a(x) d\alpha,$$

where $f_a(x) \in H_\alpha$ and $V_{\alpha}(a) : H_\alpha \to H_\alpha$ is called the Taylor decomposition of the Cauchy kernel $f_a(x)$.

Note that the Dirac operator $D$ is defined in the terms of left invariant vector fields and therefore commutes with all $\rho(h)$. Thus it also has a spectral decomposition over spectral subspaces of $\rho(h)$:

$$D = \int_A D_\delta d\delta.$$
(1) If spectral measures $d\alpha$ and $d\delta$ from (3.32) and (3.33) have disjoint supports then the image of the Cauchy integral belongs to the kernel of the Dirac operator.

(2) We say that $V_{\alpha}$ is negative if $D V_{\alpha} \neq 0$ and $V_{\alpha}$ is positive $\psi_{\alpha} \neq 0$ in the decomposition (3.30). If the intersection of positive and negative states is void then the physical states from $F(X)$ are null solutions of the Dirac operator.

Example 3.12. (a) For $G = \mathbb{H}$ and $H = \mathbb{R}^2$ the only eigenvectors of $H$ in the Schrödinger representation (3.17) are exponents $e^{ix\xi}$ and the “Taylor decomposition” over them is in fact the Fourier integral. Note that singularity of the vacuum, like in the case (d) below, implies that the primitive monomials are also outside our space $\mathcal{F}(X)$. Here the support of $da$ is $\mathbb{R}$ and from Example 3.11(a) the support of $d\delta$ is the empty set, i.e they are disjoint.

(b) For $G = \mathbb{H}$ and $H = \mathbb{R}$ the subgroup $H$ acts trivially as multiplication by a scalar on any function thus leave us an excessive freedom in the choice of the Taylor decomposition. We may wish to use Proposition 3.11 as a guideline. Example 3.9(b) tells that the Dirac operator is the Cauchy-Riemann operator $\partial / \partial \bar{z}$. Thus to get a decomposition over a disjoint support we may chose the monomial $z^k$ for the Taylor decomposition. The same choice is dictated if we wish to obtained the minimum uncertainty states (3.34).

(c) Let $G = SL_2(\mathbb{R})$ and $H = K$ be its maximal compact subgroup and $p_1$ be described by (3.23). $H$ acts on $\mathbb{T}$ by rotations. It is one dimensional and eigenfunctions of its generator $Z$ are parametrised by odd integers (due to compactness of $K$). Moreover, on the irreducible Hardy space these are positive odd integers $n = 1, 3, 5 \ldots$ and corresponding eigenfunctions are $f_{2n+1}(\phi) = e^{in\phi}$. Negative integers span the space of anti-holomorphic function and the splitting reflects the existence of analytic structure given by the Cauchy-Riemann equation from Example 3.9(c). The decomposition of coherent states $f_{\alpha}(\phi)$ by means of this functions is well known:

$$f_{\alpha}(\phi) = \sqrt{1 - |a|^2/\alpha + \phi} = \sum_{n=1}^{\infty} \sqrt{1 - |a|^2 a^{-n-1} e^{i(n-1)\phi}} = \sum_{n=1}^{\infty} V_{n}(a) f_{n}(\phi),$$

where $V_{n}(a) = \sqrt{1 - |a|^2 a^{-n-1}}$. This is the classical Taylor expansion up to multipliers coming from the invariant measure.

(d) Let $G = SL_2(\mathbb{R})$, $H = A$, and $\rho_{a}$ be described by (3.25). Subgroup $H$ acts on $\mathbb{T}$ by hyperbolic rotations:

$$\rho : z = e^{i1e^{ix}} t \rightarrow e^{2i1e^{ix}t} z = e^{i1e^{ix}(2t+\tau)}, \quad t, \tau \in \mathbb{T}.$$ 

Then for every $p \in \mathbb{R}$ the function $f_p(z) = (z)^p = e^{i1e^{ix}t}$ is an eigenfunction in the representation (3.23) for a generator $a$ of the subgroup $H = A$ with the eigenvalue $2(p - a) - 1$, cf. with (a) above. Again, due to the analytical structure reflected in the Dirac operator, we only need negative values of $p$ to decompose the Cauchy integral kernel. Thereafter for a function $f(z) \in F_2(\mathbb{T})$ we have the following Taylor expansion of its wavelet transform (3.34):

$$|\mathcal{W}_0 f|(u) = \int_0^{\infty} \frac{(e_1 u)^{p-1}}{e_1 u - 1} f_p dp, \quad \text{where} \quad f_p = \int_{\mathbb{T}} t^{p-1} d\bar{z} f(z).$$
| Notion                  | $G = \mathbb{H}, H = \mathbb{R}^2$ | $G = \mathbb{H}, H = \mathbb{R}^2$ | $G = SL_2(\mathbb{R}), H = K$ | $G = SL_2(\mathbb{R}), H = A$ |
|------------------------|-----------------------------------|-----------------------------------|-------------------------------|-------------------------------|
| Space $X$              | Real line $\mathbb{R}$           | Complex line $\mathbb{C}$        | Unit disk $\mathbb{D}$      | Disk on Fig. 1(b)          |
| Physical meaning       | Configuration space              | Phase space                       | Conformal 2d space           | Space-time (2d)            |
| Functional space $B$   | $L_p(\mathbb{R})$                | $L_p(\mathbb{C})$                | $L_p(\mathbb{T})$           | $L_p(\mathbb{T})$         |
| Vacuum vector          | $f_0(x) \equiv 1$                | $f_0(x) = e^{-x^2/2}$            | $f_0(t) \equiv 1$           | $f_0(t) \equiv 1$         |
| Wavelet transform $\mathcal{W}$ | Fourier transform               | Segal-Bargmann (SB) tr.          | Cauchy integral             | Formula (3.27)            |
| Inverse transform $\mathcal{M}$ | Fourier transform               | Inverse SB transform             | Szegö projection            |                               |
| Image $F(X)$           | Whole $L_2(\mathbb{R})$          | Segal-Bargmann space             | Hardy space                  | Monogenic space            |
| Dirac operator         | Null operator                    | Cauchy-Riemann oper.             | Cauchy-Riemann op.          | Dirac operator            |
| Laplace operator       | Null operator                    | Laplace operator                 | Laplace operator            | Wave operator              |
| Taylor expansion       | Fourier integral                 | Taylor expansion                 | Taylor expansion            | Formula (3.34)            |
| Image of convolution   | Weyl PDO (4.1) [16, 43]          | Wick operator [7, 16]            | Toplitz operator            | Sing.Int.Op. [43]         |

Table 1. Periodic table of elements of analytic function theory. The dots at the end symbolise an absence of the table end.
The last integral is similar to the Mellin transform [30, § III.3], [14, Chap. 8, (3.12)], which naturally arises from the principal series representations of $SL_2(\mathbb{R})$. ♦

Our presentation fits into Table 1, which is an extended version of the table from paper [24]. It was called there the “periodic table” of function theory and it was guessed that it could help to discover new analytic function theories just like the periodic table of chemical elements by D. Mendeleev help to find new elements. The chain [d] of above examples was presented in [23] as a sample of such a theory. Not all theoretically possible function theory are reasonable just like not all theoretical combination of protons and neutrons gives a stable chemical atom. However there are still some more interesting function theories to discover.

We interrupt here with our overview of the coherent state construction, but much more could be derived from it. See books [1, 34] for more examples of groups and physical aspects of the theory. Paper [25] contains further connections with operator theory, symbolic calculus, PDO, and Toeplitz operators which we did not mention here at all, see the last raw of the Table 1 however.

4. Functional Calculus as an Intertwining Operator

Each problem that I solved became a rule which served afterwards to solve other problems.
Descartes Discours de la M´ethode, 1637

We saw in the preceding section that coherent states naturally generate many principal objects of analysis and function theory. Particularly the wavelet transform as an intertwining operator links different spaces with the same symmetry. Could we export this observation to other areas? There are many cases where function spaces provide a model for more complicated objects, i.e. a functional calculus of operators and quantisation procedure. Thus they are the first candidates for an application of that technique.

There are several types of functional calculi, e.g. for selfadjoint bounded [36, § VII.2] or unbounded [36, § VIII.3] operators, normal operators [15, § VII.3], several commuting selfadjoint operators [42]. All of them are defined as algebraic homomorphisms from an algebra of function to an algebra of operators with some additional properties. On the language of identification (2.1) this means that functional and operator spaces are forced to have the same system of coordinates. The only exception from that rule is the Weyl functional calculus [2] which is defined just by an integral formula and is not traditionally obliged to preserve any algebraic structure. That calculus is a generalisation of the Weyl quantisation [16, § 2.1]

\[
a_h(p, q) \mapsto a_h(D, X) = \int \int \hat{a}(x, y)\sigma_h(0, x, y) \, dx \, dy
\]

obtained by an integration of the Schrödinger representation $\sigma_h$ (3.17).

However there is no need to be so restricted in our choice: we could give a meaningful general definition [22] of functional calculus without any references to an algebraic homomorphism property at all. Moreover we can define a functional calculus not only for an operator algebra $\mathfrak{A}$ but also for any $\mathfrak{A}$-module $M$ [28]. Let there be two continuous representations $\rho_f$ and $\rho_a$ of the same topological group $G$ such that $\rho_f$ act on function space $F(X, \mathbb{C})$ with a vacuum vector $b_0 \in F(X, \mathbb{C})$ and action of $\rho_a$ on an $M$-valued function space $F(X, M)$ in a way determined by an element $a \in \mathfrak{A}$. 
DEFINITION 4.1. An analytic functional calculus of an element \( a \in \mathfrak{A} \) is a continuous linear map \( \Phi : F(X, \mathbb{C}) \to F(X, M) : f(w) \mapsto [\Phi f] \) if the following conditions fulfil

1. \( \Phi \) is an intertwining operator between \( \rho_f \) and \( \rho_a \), namely

\[
[\Phi \rho_f(g) f] = \rho_a(g)[\Phi f],
\]

for all \( g \in G \) and \( f \in F(\mathfrak{A}, \mathbb{C}) \).

2. \( \Phi \) maps the vacuum vector \( b_0 \) for the representation \( \rho_f \) to the vacuum vector \( b_a \in F(X, M) \) for the representation \( \rho_a \):

\[
[\Phi b_0] = b_a.
\]

We will illustrate advantages of this approach by a construction of functional calculus for a finite dimensional non-normal matrix. We use to this end the group \( SL_2(\mathbb{R}) \) together with its discrete series mock representation \( \rho_1 \) in \( L_2(\mathbb{T}) \) as a model for a corresponding representation related to a matrix. Let \( \mathfrak{A} \) be the algebra of complex valued \( n \times n \) matrix, \( e \) be its unit and \( a \in \mathfrak{A} \) have all its eigenvalues in the unit circle \( \mathbb{D} \). Then we could define linear-fraction transformation \( g \cdot a \) of \( a \) by the formula:

\[
g^{-1} a = (\bar{\alpha} a - \bar{\beta} e)(ae - \beta a)^{-1}, \quad \text{where} \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\alpha & \bar{\beta} \end{pmatrix},
\]

in analogy with the point transformation (3.4). The set all matrices (3.4) for \( g \in SL_2(\mathbb{R}) \) becomes a \( SL_2(\mathbb{R}) \)-homogeneous space. Similarly the resolvent \( R(g^{-1} a) = (\bar{\alpha} e - \beta a)^{-1} \) is well defined for all \( g \in SL_2(\mathbb{R}) \). Let \( M = \mathbb{C}^n \) be a natural left \( \mathfrak{A} \)-module and let us consider a space of \( M \)-valued functions on \( \mathbb{D} \). Analogously to (3.4) we define a representation \( \rho_a \) of \( SL_2(\mathbb{R}) \) as follows:

\[
(\rho_a(g)f)(z) = R(g^{-1} a)f \begin{pmatrix} \bar{\alpha}z - \bar{\beta} \\ \alpha z - \beta \end{pmatrix}, \quad \text{where} \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\alpha & \bar{\beta} \end{pmatrix}.
\]

Let \( b_0(z) = 1 \) be the vacuum vector of \( \rho_1 \) and let \( H_2(\mathbb{D}, M) \) the minimal \( \rho_a \)-invariant space of \( M \)-valued functions containing all functions \( b_z(z) = x \otimes b_0(z) \), \( x \in M \). Then the functional calculus in the sense of Definition 4.1 from \( H_2(\mathbb{D}, M) \) could be constructed [28 Prop. 2.16] with the help of intertwining properties from Proposition 3.4. Indeed if we define \( \Phi = M_{\rho_1}^{\prime} W_{\rho_1} \) then:

\[
\Phi \rho_f(g) = M_{\rho_a} W_{\rho_1} \rho_1(g) = M_{\rho_a} \lambda(g) W_{\rho_1} = \rho_a(g) M_{\rho_a} W_{\rho_1} = \rho_a(g) \Phi.
\]

The explicit integral formula for \( \Phi \) coincides with the integral formula of Dunford-Riesz analytic functional calculus [15 Thm. VII.1.10]:

\[
[[\Phi f](x, z) = \int f(t)(za - te)^{-1} \, dt.
\]

Now we can use the constructed functional calculus to get a better spectral characterisation of \( a \) than just the set of its eigenvalues. The definition used in [2] to define the Weyl spectrum of operator is suitable for the generalisation.

DEFINITION 4.2. [28] A spectrum of an operator \( a \) is the support of a functional calculus \( \Phi : f(x) \mapsto f(a) \).

Because now the functional calculus is an intertwining operator its support is a collection of indecomposable intertwining operators between \( \rho_1 \) and \( \rho_a \). The entire space \( H_2(\mathbb{D}, M) \) splits into \( \rho_a \)-invariant subspaces \( V_{\lambda,k} \) generated by functions
$b_k(z) = x \otimes b_k(z)$, where $x$ is a $k$th root vector for an eigenvalue $\lambda$, i.e. $(a - \lambda e)^k x = 0$ and $(a - \lambda e)^{k-1} x \neq 0$. Such a minimal $\rho_k$-invariant subspace $V_{\lambda,k}$ up to similarity is described by the corresponding pair $(\lambda, k)$, $\lambda \in \mathbb{D}$, $k \in \mathbb{N}$. To get their classification we need the following notion.

**Definition 4.3.** [33, Chap. 4] Two holomorphic functions have $n$th order contact in a point if their value and their first $n$ derivatives agree at that point, in other words their Taylor expansions are the same in first $n+1$ terms.

A point $(z, u(n)) = (z, u, u_1, \ldots, u_n)$ of the jet space $J^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having $n$th contact at the point $z$ with the polynomial:

$$p_n(w) = u_n \frac{(w-z)^n}{n!} + \cdots + u_1 \frac{(w-z)}{1!} + u.$$  

For a fixed $n$ each holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has $n$th prolongation (or $n$-jet) $j_n f : \mathbb{D} \to \mathbb{C}^{n+1}$:

$$j_n f(z) = (f(z), f'(z), \ldots, f^{(n)}(z)).$$

The graph $\Gamma_f^{(n)}$ of $j_n f$ is a submanifold of $J^n$ which is section of the jet bundle over $\mathbb{D}$ with a fibre $\mathbb{C}^{n+1}$. We also introduce a notation $J_n$ for the map $J_n : f \mapsto \Gamma_f^{(n)}$ of a holomorphic $f$ to the graph $\Gamma_f^{(n)}$ of its $n$-jet $j_n f(z)$ (4.7).

One can prolong any map of function $\psi : f(z) \mapsto [\psi f](z)$ to a map $\psi^{(n)}$ of $n$-jets by the formula

$$\psi^{(n)}(J_n f) = J_n(\psi f).$$

For example such a prolongation $\rho_1^{(n)}$ of the representation $\rho_1$ of the group $SL_2(\mathbb{R})$ in $H_2(\mathbb{D})$ (as any other representation of a Lie group [33]) will be again a representation of $SL_2(\mathbb{R})$. Equivalently we could say that $J_n$ intertwines $\rho_1$ and $\rho_1^{(n)}$:

$$J_n \rho_1(g) = \rho_1^{(n)}(g) J_n.$$

Of course, the representation $\rho_1^{(n)}$ is not irreducible: any jet subspace $J^k$, $0 \leq k \leq n$ is $\rho_1^{(n)}$-invariant subspace of $J^n$.

Coming back to our representation $\rho_a$ (4.5) we could characterise its minimal component as follows.

**Proposition 4.4.** [28] Restriction of $\rho_a$ to $V_{\lambda,k}$ is equivalent to the extension $\rho_1^{(k)}$ of $\rho_1$ in the $k$th jet space $J^k$. Consequently the spectrum of $a$ (defined via the functional calculus $\Phi$) consists of exactly $n$ pairs $(\lambda_i, k_i)$, $\lambda_i \in \mathbb{D}$, $k_i \in \mathbb{N}$, $1 \leq i \leq n$ some of whom could coincide.

**Example 4.5.** Let $J_k(\lambda)$ denote the Jordan block of the length $k$ for the eigenvalue $\lambda$. On the Fig. 2 there are two pictures of the spectrum for the matrix

$$a = J_3 \left( \frac{3}{4} e^{i\pi/4} \right) \oplus J_4 \left( \frac{2}{3} e^{i5\pi/6} \right) \oplus J_1 \left( \frac{2}{5} e^{-i3\pi/4} \right) \oplus J_2 \left( \frac{3}{5} e^{-i\pi/3} \right).$$

Part (a) represents the conventional two-dimensional image of the spectrum, i.e. eigenvalues of $a$, and [b] describes spectrum spa arising from the wavelet construction. The first image did not allow to distinguish $a$ from many other essentially
different matrices, e.g. the diagonal matrix
\[
\text{diag}\left(\frac{3}{4}e^{i\pi/4}, \frac{2}{3}e^{i5\pi/6}, \frac{2}{5}e^{i3\pi/4}, \frac{3}{5}e^{-i\pi/3}\right).
\]
At the same time the Fig. 2(b) completely characterise a up to a similarity. Note that each point of spa on Fig. 2(b) corresponds to a particular root vector.

The three dimensional spectrum of matrices obeys the spectral mapping theorem which is a refined version of the classic theorem about mapping of eigenvalues.

**Theorem 4.6 (Spectral mapping).** \[28\] Let \(\phi : \mathbb{D} \to \mathbb{D}\) be a holomorphic map, let us define \(\phi_* f(z) = f(\phi(z))\) and its prolongation \(\phi^{(n)}\) onto the jet space \(J^n\) by \[4.3\]. Its associated action on the pairs \((\lambda, k)\) is given by the formula:

\[
\phi^{(n)}(\lambda, k) = \left(\phi(\lambda), \left[k \mod \text{deg}_\lambda \phi\right]\right),
\]
where \(\text{deg}_\lambda \phi\) denotes the degree of zero of the function \(\phi(z) - \phi(\lambda)\) at the point \(z = \lambda\) and \([x]\) denotes the integer part of \(x\). Then

\[\text{sp} \phi(a) = \phi^{(n)} \text{spa}.\]

The explicit expression for \(\phi^{(n)}\) which involves derivatives of \(\phi\) up to \(n\)th order is known, see for example \[17\], Thm. 6.2.25, but was not understood before as a form of spectral mapping.

To finish with this topic we will note that our Definitions 4.1 and 4.2 are not restricted to the case \(SL_2(\mathbb{R})\) only. They are suitable for any group \(G\) and subgroup \(H\) from the Table 1. For example, Segal-Bargmann type calculus was outlined in \[25\] and monogenic calculus of several noncommuting operators in \[22\]. This directions of research still waits a careful exploration.

5. Quantisation from the Symplectic Invariance

Approach your problem from the right end and begin with the answer. Then one day, perhaps you will find the final question.

R. van Gulik *The Hermit Clad in Crane Feather*

It is well known that quantum mechanics is full of paradoxes. The more general is the following: *there are a lot working tools and tricks which give numerical*
predictions for almost all observable effects, but the majority of them are mathematically unsound and philosophically obscure. The basic example is the question of quantisation itself.

It is often said that quantum mechanics is superior to the classical one but unfortunately we are not able to perceive its glory directly. To obtain a quantum description we have first to describe a physical system classically and then quantise that description. A popular scheme of quantisation was given by Dirac \[14\]. It says that in order to quantise a set of classical observables, which are real valued functions on the phase space, we prescribe a linear map \(\hat{f}: f \mapsto \hat{f}\) into selfadjoint operators on a Hilbert space such that for any two classical observables \(f_1\) and \(f_2\)

\[
\hat{f}_1 \hat{f}_2 - \hat{f}_2 \hat{f}_1 = \frac{\hbar}{i} \{f_1, f_2\},
\]

where \(\{\cdot,\cdot\}\) denotes the Poisson brackets in the phase space and \([\cdot,\cdot]\) the commutator in the operator algebra.

It is also known from various “no-go” theorems \[16\] that those requirements could not be satisfied beyond the polynomials of degree \(\leq 2\). On the other hand the Weyl calculus \((4.1)\) gives a good approximation to \((5.1)\) for a small Planck constant \(\hbar\). Therefore for a physicist interested in numerical predictions of measurements the quantisation problem is already solved. But mathematicians are still unhappy with that answer and actively investigate other quantisation theories, e.g. geometric quantisation, deformation quantisation, quantum groups, etc.

The requirements of Dirac \((5.1)\) are similar to the algebra homomorphism property for the functional calculus: the first map in \((5.1)\) prescribe the image of a product for two function and the second is the homomorphism between two Lie algebra structures. Therefore mappings \((5.1)\) are silently based on the identification \((2.1)\). As we saw in the previous Section one can get a progress in functional calculus if replaces the algebraic homomorphism condition by the intertwining property in the spirit of the assumption \((2.2)\). Such a change in a definition of quantisation is also possible (see below Definition \((5.1)\)) and has the following advantages:

(1) It is based on the first physical principles;
(2) It has a well defined solution in mathematical sense, which naturally turns to be the Weyl quantisation.

We will outline briefly how to quantise an elementary classical system with a phase space \(\mathbb{R}^2\) using an intertwining condition. There is a natural candidate for a group \(G\): this is the group \(Sp(1)\) (isomorphic to our permanent companion \(SL_2(\mathbb{R})\)) of linear symplectomorphisms of \(\mathbb{R}^2\)

\[
\tau(g): (p, q) \mapsto (ap + bq, cp + dq), \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

i.e. transformations preserving the symplectic form \((v_1, v_2) = p_1 q_2 - p_2 q_1, v_i = (p_i, q_i) \ [8, \S 4.1]\). Note that the symplectic form enters to the expression \((3.1)\) of the multiplication law on \(H\) making it noncommutative and pops up at the end of Example \((3.1(b))\) as expression for \(r(s(z_1))^{-1} * s(z_2))\). It is not surprising therefore that \(Sp(1)\) acts as automorphisms of \(H\) as follows:

\[
\alpha(g): (s, x, y) \mapsto (s, ax + by, cx + dy), \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Heisenberg group

Parameter $\hbar \neq 0$

Phase space ($\hbar = 0$)

Figure 3. The appearance of both quantum and classic mechanics from the same source. Automorphisms of $\mathbb{H}$ generated by symplectic group $Sp(1)$ do not mix Schrödinger representations $\rho_\hbar$ with different $\hbar$ and act by the metaplectic representation inside each of them. In the contrast those automorphisms of $\mathbb{H}$ act transitively on the set of one dimensional representations $\rho_{(p,q)}$ joining them into the phase space $\mathbb{R}^2$.

In fact $Sp(1)$ is exactly the subgroup of non-inner automorphisms of $\mathbb{H}$ which send a Schrödinger representation $\rho_\hbar$ (3.17) with any parameter $\hbar \neq 0$ to an unitary equivalent representation [16, § 1.2].

More precisely for any automorphism $\alpha(g)$, $g \in Sp(1)$ of $\mathbb{H}$ the composition $\rho_\hbar \circ \alpha(g)$ is again a representation, which is unitary equivalent to $\rho_\hbar$. Therefore should exists a unitary operator $U(g)$ in $L_2(\mathbb{R})$ such that $\rho_\hbar \circ \alpha(g) = U(g)\rho_\hbar U^{-1}(g)$. Then the correspondence $\mu : g \mapsto U(g)$ is a double valued metaplectic representation of $Sp(1)$ [16, § 4.2], [44, § 11.3] in $L_2(\mathbb{R})$. We are ready to give our definition of quantisation.

**Definition 5.1.** Quantisation $Q$ is a linear operator from $C(\mathbb{R}^2)$ to $B(L_2(\mathbb{R}))$, which intertwines two actions of the symplectic group $Sp(1)$: by symplectomorphisms $\mathbb{R}^2$ in classical mechanics and the metaplectic representation $\mu$ in quantum mechanics on $L_2(\mathbb{R})$:

$$Q\tau(g) = \mu(g)Q \quad \text{forall} \quad g \in Sp(1).$$

The following “easy-go” theorem is just an application of several known results (e.g. [16, Thm. 4.28]) about the metaplectic representation.

**Theorem 5.2.** [26] The Weyl calculus (4.1) is unique (up to equivalence) well-defined solution for the quantisation problem 5.1.

It is interesting to note that the quantisation $Q$ which exactly intertwines symplectomorphisms also approximately intertwines any canonical transformation of $\mathbb{R}^2$ modulo smoothing operators—this is statement of the important Egorov theorem [43, § VIII.1] from the theory of PDO.

One can get even more from our study of automorphism of $\mathbb{H}$. We recall that the Heisenberg group $\mathbb{H}$ besides of the family of Schrödinger representations $\sigma_\hbar$ (5.17)
with parameter $\hbar \in \mathbb{R} \setminus \{0\}$ has only in addition the family of one dimensional representations $\rho_{(p,q)}$:

$$\rho_{(p,q)} : (s,x,y) \mapsto e^{i(p\hat{s} + q\hat{y})}, \quad \text{where} \quad (p,q) \in \mathbb{R}^2.\tag{5.4}$$

That family is usually just mentioned by accurate authors in the statement of the Stone-von Neumann theorem \[16, 44\] but almost never used in any way: what could we expect interesting from commutative representations in our age of non-commutative geometry? But let us take a second look assuming that Nature does not create anything without a purpose.

The representation $\alpha$ \[5.3\] of $Sp(1)$ could be lifted to the action $\alpha^*$ on the dual object (the set of all unitary irreducible representations) $\hat{H}$ of $\mathbb{H}$. As was mentioned before each Schrödinger representation $\sigma_\hbar$ \[3.17\] is a fixed point of $\alpha^*$. In the contrast the whole family of one dimensional representations $\rho_{(p,q)}$ is just one orbit for $\alpha^*$: it acts transitively on the set $(p,q)$ by symplectomorphisms \[5.2\]. Therefore it is natural to identify the family of representation \[5.4\] with the phase space. This situation is illustrated by Figure 3. From the topology on the dual object $\hat{H}$ \[20, \S 7.2.2\] the correct place to put the entire family \[5.4\] is the point $\hbar = 0$. Moreover representations $\sigma_\hbar$, $\hbar \neq 0$ are dense in $\mathbb{R}^2$—this is a form of the correspondence principle between quantum and classical mechanics. Reader may wish to compare our Figure 3 with Figs. 6 and 7 in \[20, \S 7.2.2\] and corresponding discussion there of topology on $\hat{H}$. It is slightly speculative but we could also assume that negative values of $\hbar$ on $\hat{H}$ correspond to anti-particles because the minus sign of $\hbar$ reverses the time flow in the Schrödinger equation.

How far could that relation between quantum and classic mechanics be extend? It turns that we can introduce an appropriate notion of dynamics \[27\] on $\mathbb{H}$ which produces quantum and classical dynamics in corresponding representations \[3.17\] and \[5.4\]. That unified dynamics is based on the following definition.

**Definition 5.3.** \[27\] The $p$-mechanical brackets of two functions $k_1(s,x,y)$, $k_2(s,x,y)$ on the Heisenberg group $\mathbb{H}$ are defined as follows:

$$\{k_1, k_2\} = A(k_1 \ast k_2 - k_2 \ast k_1),\tag{5.5}$$

where $\ast$ denotes the group convolution on $\mathbb{H}$ of two functions and $A$ acts as anti-derivative with respect of the variable $s$.

The main property of $p$-mechanical brackets is a common source for quantum and classical brackets:

**Proposition 5.4.** \[27\] The images of $p$-mechanical brackets \[5.3\] under infinite dimensional representations $\sigma_\hbar$ \[3.17\], $\hbar \neq 0$ and finite dimensional representations $\rho_{(q,p)}$ \[5.4\] are quantum commutator and Poisson brackets of functions $\hat{k}_1$ and $\hat{k}_2$ respectively:

$$\rho(\{k_1, k_2\}) = \begin{cases} \frac{1}{i\hbar}[\hat{k}_1, \hat{k}_2] = \frac{1}{i\hbar}(K_1K_2 - K_2K_1), & \rho = \sigma_\hbar, \ h \neq 0; \\ \{\hat{k}_1, \hat{k}_2\} = \frac{\partial \hat{k}_1}{\partial q} \frac{\partial \hat{k}_2}{\partial p} - \frac{\partial \hat{k}_1}{\partial p} \frac{\partial \hat{k}_2}{\partial q}, & \rho = \rho_{(q,p)}. \end{cases}\tag{5.6}$$

We refer to the paper \[27\] for further details and discussion about the $p$-mechanical brackets, dynamics generated by them and its relations to quantum and classical mechanics. It convinces that not only quantum mechanics but classical too appear from representations of the Heisenberg group.
Conclusion

A mathematical idea should not be petrified in a formalised axiomatic settings, but should be considered instead as flowing as a river.

Sylvester 1878

It is accurate to say that the idea of symmetries and invariants was dominant and the most fruitful in the physics of XX century. There is no a contradiction however to hope that its systematical use will reveal many secrets and explain many mysteries of the Nature in the future as well. That idea is among very few other which are strong enough to oppose a collapse of science into many unrelated parts. Moreover the admiration of symmetries is common to both cultures (in the sense of C.P. Snow) and could be a bridge between them.

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I am grateful to the editors who published my previous paper on that topic despite of the obvious doubts it generated.

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