New conjecture related to a conjecture of McIntosh

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Abstract

We introduce a new conjecture on products of two distinct primes that would provide a partial answer to a conjecture of McIntosh. Also, \((p-1) - 1\) is written in terms of a polynomial in prime \(p\) over the integers and we discuss one way this form may be useful.

Keywords: Wolstenholme’s theorem, McIntosh conjecture

1. Introduction

In 1771, Lagrange gave the first proof of an interesting property of the prime numbers we now call Wilson’s theorem. The converse of this theorem also holds.

**Theorem 1.1** (Wilson’s theorem and its converse, theorem 5.4 [4]).

\[(p−1)! ≡ −1 \pmod{p} \iff p \text{ is prime.}\]

**Remark 1.2.** For any \(n \in \mathbb{N}\),

\[
\frac{(2n-1)!}{n!} = (n+1)(n+2) \cdots (n+n-1) \equiv (n-1)! \pmod{n},
\]

so **Theorem 1.1** may also be stated as

\[
\frac{(2p-1)!}{p!} ≡ −1 \pmod{p} \iff p \text{ is prime.}
\]

A prime \(p\) satisfying the congruence

\[(p−1)! ≡ −1 \pmod{p^2}\]

is called a Wilson prime. Infinitely many such primes are conjectured to exist but only 5, 13, and 563 have been identified so far. We also think there are no integers \(n\) such that

\[(n−1)! ≡ −1 \pmod{n^3}.\]

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For $n \in \mathbb{N}$ denote $w_n = \binom{2n-1}{n-1} = \frac{1}{2} \binom{2n}{n}$. Let $n = p$ be a prime in congruence (1.1). Since $p$ is relatively prime to $(p - 1)!$, dividing by $(p - 1)!$,

$$w_p \equiv 1 \pmod{p}.$$ 

The congruence also holds for squares of odd primes, cubes of primes $\geq 5$, and for some products of distinct primes (see section 2). In 1819, Babbage [1] further showed

$$w_p \equiv 1 \pmod{p^2}$$

for primes $p \geq 3$. See conjecture 2.3 for other solutions. Finally, in 1862 Wolstenholme [12] improved on Babbage’s result.

**Theorem 1.3 (Wolstenholme’s theorem).** If $p \geq 5$ is prime, then

$$w_p \equiv 1 \pmod{p^3}.$$

James P. Jones conjectures no other solutions exist.

**Conjecture 1.4 (Jones’ conjecture).**

$$w_p \equiv 1 \pmod{p^3} \iff p \geq 5 \text{ is prime}.$$

Jones’ conjecture is true for even integers, powers of primes $\leq 10^9$ ([11], [7]), and based on computations for integers $\leq 10^9$. McIntosh [9] gives probabilistic evidence supporting the conjecture (see conjecture 2.3). A prime $p$ satisfying the congruence

$$w_p \equiv 1 \pmod{p^4}$$

is called a *Wolstenholme prime*. McIntosh conjectures infinitely many such primes exist but only 16843 and 2124679 have been found so far. McIntosh also conjectures there are no integers $n$ such that

$$w_n \equiv 1 \pmod{n^5}.$$

A summary of results known and conjectured related to Wilson’s and Wolstenholme’s theorems shows the similarity between the two notions.

| m  | Wilson       | Wolstenholme    |
|----|--------------|-----------------|
|    | $\binom{2n-1}{n-1} \equiv -1 \pmod{n^m}$ | $\binom{2n-1}{n-1} \equiv 1 \pmod{n^{2+m}}$ |
| 1  | if and only if $n$ is prime | if $n \geq 5$ is prime, conjectured only if |
| 2  | $n = 5, 13, 563 \ldots$ conj $\propto$ many | $n = 16843, 2124679 \ldots$ conj $\propto$ many |
| 3  | conjectured none | conjectured none |

In section 2, we introduce a new conjecture that would provide a partial answer to a conjecture of McIntosh [9]. In the last section, $(w_p - 1)/p^3$ is written in terms of a polynomial in prime $p$ over the integers and we discuss one way this form may be useful. See [8] for a simple proof of Wilson’s and Wolstenholme’s theorems.
2. Product of two distinct primes

From Bertrand’s postulate and [7, proposition 5, part 4], Jones’ conjecture holds for products of two consecutive odd primes. For pairs of primes in general, the following is an equivalent criteria to the product satisfying Jones’ conjecture.

**Proposition 2.1** (corollary 4, proposition 4 [7]). Let $p$ and $q$ be distinct primes $\geq 5$. Then

$$w_{pq} \equiv 1 \pmod{(pq)^3} \iff w_p \equiv 1 \pmod{q^3} \text{ and } w_q \equiv 1 \pmod{p^3}.$$ 

A direct proof is outlined in [7] that avoids the use of [7, proposition 4]. We fill in the details below. In [9] for $n \in \mathbb{N}$, the modified binomial coefficient is defined to be

$$w'_n = \left(\frac{2n-1}{n-1}\right)' = \prod_{k=1}^{n} \frac{2n-k}{k}.$$ 

Notice $w'_1 = 1$. From [3] we also have the relation

$$\left(\frac{2n-1}{n-1}\right)' = \prod_{d|n} \left(\frac{2d-1}{d-1}\right)' \quad (2.1),$$

so $w_p = w'_p$ for any prime $p$.

**Proof.** By a generalization of Wolstenholme’s theorem [9, Theorem 1],

$$w'_p \equiv 1 \pmod{(pq)^3}.$$ 

Using relation (2.1),

$$w_{pq} = w'_1w'_pw'_qw'_pq = w_PW_Pw_PW_Pq,$$

so

$$w_{pq} \equiv w_PW_Pq \pmod{(pq)^3}.$$ 

We also have from Wolstenholme’s theorem,

$$w_p \equiv 1 \pmod{p^3} \text{ and } w_q \equiv 1 \pmod{q^3},$$

so by the previous congruence,

$$w_{pq} \equiv w_q \pmod{p^3} \text{ and } w_{pq} \equiv w_p \pmod{q^3}. \quad (2.2)$$

Now assume $w_{pq} \equiv 1 \pmod{(pq)^3}$. Then by (2.2),

$$w_p \equiv 1 \pmod{q^3} \text{ and } w_q \equiv 1 \pmod{p^3}.$$ 

Conversely, assume

$$w_p \equiv 1 \pmod{q^3} \text{ and } w_q \equiv 1 \pmod{p^3},$$

Then from (2.2),

$$w_{pq} \equiv 1 \pmod{p^3} \text{ and } w_{pq} \equiv 1 \pmod{q^3}.$$ 

Since $p$ and $q$ are distinct primes, $w_{pq} \equiv 1 \pmod{(pq)^3}$ and the proof is complete. \hfill $\square$
Remark 2.2. Starting with (2.2) we can also show

\[ w_{pq} \equiv 1 \pmod{pq} \iff w_p \equiv 1 \pmod{q} \text{ and } w_q \equiv 1 \pmod{p} \]

and

\[ w_{pq} \equiv 1 \pmod{(pq)^2} \iff w_p \equiv 1 \pmod{q^2} \text{ and } w_q \equiv 1 \pmod{p^2}. \]

The only known examples of distinct primes \( p \) and \( q \) such that

\[ w_{pq} \equiv 1 \pmod{pq} \]

are (29, 937), (787, 2543), and (69239, 231433). In [2] we ask if there are actually infinitely many examples of such pairs of primes.

Let \( n = p^2 \) with \( p \) a Wolstenholme prime. By the generalization of Wolstenholme’s theorem [3, theorem 1],

\[ w'_{n} \equiv 1 \pmod{n^2}. \]

Using relation (2.1),

\[ w_n = w_{p^2} = w'_1 w'_p w'_p = w_p w'_p w'_p, \]

so

\[ w_n \equiv w_p \pmod{n^2}. \]

Since \( p \) is a Wolstenholme prime,

\[ w_p \equiv 1 \pmod{(p^2)^2}, \]

so

\[ w_n \equiv 1 \pmod{n^2}. \]

Therefore McIntosh conjectures the following:

**Conjecture 2.3** (McIntosh’s conjecture).

\[ w_n \equiv 1 \pmod{n^2} \iff n \text{ is odd prime or } n = p^2 \text{ with } p \text{ a Wolstenholme prime.} \]

If McIntosh’s conjecture is true, the only remaining composites that could violate Jones’ conjecture are the Wolstenholme primes squared. By the definition of \( w_p \) for prime \( p \geq 5 \), for any prime \( q \geq \sqrt{2p-1} \), \( q \neq p \), such that \( \frac{2p-1}{n+1} < q \leq \frac{2p-1}{n} \), \( n \in \mathbb{N} \), we have the following:

\[ n \text{ is odd } \implies q \mid w_p, \]

\[ n \text{ is even } \implies q \nmid w_p. \]

Hence most of the large prime factors of \( w_p - 1 \) reside in one of the even \( n \) inequalities above and calculations show the primes \( q < p \leq 10^5 \) such that \( q^2 \mid (w_p - 1) \) are more than 100 times smaller than \( p \). Therefore it is increasingly unlikely for \( q^2 \mid (w_p - 1) \) for a prime \( q > 2p \) the larger \( p \) is, so we conjecture the following:
**Conjecture 2.4** (New conjecture). For all but at most finitely many pairs of distinct primes \( p \) and \( q \),
\[
w_p \equiv 1 \pmod{q^2} \implies q < p.
\]
The new conjecture implies
\[
w_p \not\equiv 1 \pmod{q^2} \text{ or } w_q \not\equiv 1 \pmod{p^2}
\]
for all but at most finitely many pairs of distinct primes \( p \) and \( q \), so by Remark 2.2,
\[
w_{pq} \not\equiv 1 \pmod{(pq)^2}.
\]
However, a prime pair \((p, q)\) exception in conjecture 2.4 only means \( w_{pq} \equiv 1 \pmod{(pq)^2} \) is possible and an effective proof of conjecture 2.4 would allow us to check the finitely many exceptions. Hence the new conjecture would provide a partial answer to McIntosh’s conjecture.

3. Wolstenholme polynomials

For a prime \( p \geq 5 \),
\[
w_p = \binom{2p - 1}{p - 1} = \prod_{k=1}^{p-1} \left(\frac{p}{k} + 1\right) = \sum_{k=0}^{p-1} p^k S_{p-1,k}
\]  \hspace{1cm} (3.1)

where \( S_{n,k} \) is the \( k \)th elementary symmetric polynomial in \( n \) variables with values \( \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\} \). Note \( S_{n,0} = 1, S_{n,n} = 1/n!, \) and \( S_{n,k} = 0 \) for \( k > n \). By [2, theorem 3] and Newton’s identities relating power sums and elementary symmetric polynomials,
\[
S_{p-1,k} \equiv 0 \pmod{p}, \quad \text{even } k \leq p - 3
\]
\[
\equiv 0 \pmod{p^2}, \quad \text{odd } k \leq p - 4.
\]  \hspace{1cm} (3.2)

The fractional congruence notation (3.2) means \( p, p^2 \) divides the numerator of the rational \( S_{p-1,k} \) but not the denominator since the denominator is a product of integers \( \leq p - 1 \). More generally, the highest power of \( p \) dividing the numerator of \( S_{p-1,k}, k = 1, 3, \ldots, p - 2, \) is one higher than for \( S_{p-1,k+1} \).

**Remark 3.1.** Since \( S_{p-1,p-1} = 1/(p - 1)! \), \( S_{p-1,p-2} \equiv 0 \pmod{p} \). Also by (3.1), \( S_{p-1,1} \equiv 0 \pmod{p^2} \) is equivalent to Wolstenholme’s theorem and \( p \) is a Wolstenholme prime if and only if \( S_{p-1,1} \equiv 0 \pmod{p^3} \).

Let \( P_{n,k} \) be the \( k \)th elementary symmetric polynomial in \( n \) variables with values \( \{1, 2, \ldots, n\} \). Since the recurrence relation [10, equation 14.3] for \( P_{n,k} \) is
\[
P_{n,k} = P_{n-1,k} + nP_{n-1,k-1}
\]
and we have

\[ S_{n,n-k} = P_{n,k}/n!, \quad k = 0, 1, 2, \ldots, n, \]  

we get the recurrence relation

\[ S_{n,n-k} = S_{n-1,n-k} + \frac{S_{n-1,n-(k+1)}}{n}. \]  

So by (3.1),

\[ \frac{(w_p - 1)/p^3}{S_{p,2}} = pS_{p,4} + p^3S_{p,6} + \cdots + p^{p-6}S_{p,p-3} + p^{p-4}S_{p,p-1}. \]  

Notice (3.2) and (3.4) imply \( S_{p,m} \equiv 0 \pmod{p} \) for \( m = 2, 4, \ldots, p - 3 \). We next seek an explicit formula for the rationals \( S_{p,2}, S_{p,4}, \ldots, S_{p,p-1} \).

The Stirling numbers of the first kind and second kind, denoted \( s(n,k) \) and \( S(n,k) \) respectively, are characterized by

\[ \sum_{k=0}^{n} s(n,k)x^k = n!(\frac{x}{n})^n, \quad x \in \mathbb{R}, \]

\[ \sum_{k=0}^{n} k!(\frac{x}{k})S(n,k) = x^n, \quad x \in \mathbb{R}, \]

and we have an explicit formula [10, equation 13.32] relating the two sets of numbers,

\[ s(n,n-k) = \sum_{j=0}^{k} (-1)^j \binom{n+j-1}{k+j} \binom{n+k}{k-j} S(j+k,j). \]  

**Remark 3.2.** Since \( S(n,0) = s(n,0) = 0 \) for \( n \geq 1 \), (3.6) and the summations that follow may start at index one.

The integers \( s(n,k) \) and \( P_{n,k} \) are related [10, equation 14.5] by

\[ P_{n,k} = (-1)^k s(n+1,n+1-k), \]

so by (3.3) and (3.6), the explicit formula for \( S_{p,2}, S_{p,4}, \ldots, S_{p,p-1} \) is

\[ S_{p,p-k} = \frac{1}{p!} \sum_{j=0}^{k} (-1)^{j+1} \binom{p+j}{k+j} \binom{p+1+k}{k-j} S(j+k,j), \quad k = 1, 3, 5, \ldots, p - 2. \]

Also

\[ S_{p,p-k} = \frac{(p+1+k)!}{p!(p-k)!} \sum_{j=0}^{k} (-1)^{j+1} \frac{1}{(k+j)!(k-j)!(p+1+j)} S(j+k,j) \]

\[ = \frac{p+1}{(2k)!(p-k)!} \sum_{j=0}^{k} (-1)^{j+1} \binom{2k}{k+j} \frac{C(p,k)}{p+1+j} S(j+k,j) \]  

\[ = \frac{p+1}{(2k)!(p-k)!} \sum_{j=0}^{k} (-1)^{j+1} \binom{2k}{k+j} \frac{C(p,k)}{p+1+j} S(j+k,j) \]  

(3.7)
where \(C(p, k) = (p + 2)(p + 3) \cdots (p + 1 + k)\), so (3.5) may be written as

\[
\frac{w_p - 1}{p^3} = \frac{p + 1}{(2p - 4)!}(a_{2p - 7}p^{2p - 7} + \cdots + a_{2p}^2 + a_1p + a_0)
\]

\[
= \frac{p + 1}{(2p - 4)!}W(p), \quad a_i \in \mathbb{Z}.
\]

The properties of some of the coefficients of the Wolstenholme polynomials \(W(p)\) may easily be determined. In particular \(a_{2p - 7}, a_{2p - 8}, a_1\), and \(a_0\) come from \(k = p - 2\) in (3.7), that is from \(S_{p, 2}\) alone. So in finding the common denominator above, we see \((p - 3)! | a_0\). We also claim \(a_{2p - 7} = (2p - 5)!\). By formula (3.6),

\[
(-1)^k s(n, n - k) = \sum_{j=0}^{k} (-1)^{j+k} \left( \begin{array}{c} n + j - 1 \\ k \\ j \\ \end{array} \right) S(j + k, j)
\]

\[
= \frac{(n + k)!}{(2k)!(n - 1 - k)!} \sum_{j=0}^{k} (-1)^{j+k} \left( \begin{array}{c} 2k \\ k + j \\ \end{array} \right) S(j + k, j)
\]

\[
= \frac{1}{(2k)!} \sum_{j=0}^{k} (-1)^{j+k} \left( \begin{array}{c} 2k \\ k + j \\ \end{array} \right) D(n, k) \frac{S(j + k, j)}{n + j}
\]

with \(D(n, k) = (n - k)(n - k + 1) \cdots (n + k)\). Also from [3, proposition 1.1], for each fixed integer \(k \geq 0\), \((-1)^k s(n, n - k)\) is a polynomial in \(n\) over the rationals with leading coefficient \((2k - 1)!/(2k)!\). Since \(D(n, k)\) is a monic polynomial in \(n\),

\[
(2k - 1)! = \sum_{j=0}^{k} (-1)^{j+k} \left( \begin{array}{c} 2k \\ k + j \\ \end{array} \right) S(j + k, j),
\]

and since \(C(p, k)\) is a monic polynomial in \(p\) for fixed \(k\), (3.7) and (3.8) imply \(a_{2p - 7} = (2p - 5)!\). Data also suggests \(a_{2p - 4}\) is the largest coefficient and the sign of the coefficients between \(a_{2p - 7}\) and \(a_{p - 4}\) alternate.

Let \(p\) and \(q\) be primes such that \(p < q\) and \(q \mid (w_p - 1)\). Since \(q \nmid w_p\), the proof of [7, proposition 5, part 4] implies \(q > 2p\). Hence a prime \(q > p\) divides \((w_p - 1)/p^3\) if and only if \(q \mid W(p)\). This makes the Wolstenholme polynomials useful in situations like conjecture 2.4. For a prime \(p\), the Taylor series expansion of \(W(p)\) about \(n \in \mathbb{Z}\) is

\[
W(p) = W(n) + W'(n)(p - n) + \cdots + \frac{W^{(2p - 6)}(n)}{(2p - 6)!}(p - n)^{2p - 6},
\]

so \((p - n) \mid W(p)\) if and only if \((p - n) \mid W(n)\). Also when \((p - n) \mid W'(n), (p - n)^2 \mid W(p)\) if and only if \((p - n)^2 \mid W(n)\). If \((p - n) \nmid W'(n)\) and \(p - n\) is prime, then by Hensel’s lemma [3, theorem 3.4.1], \((p - n)^2 \mid W(p)\) if and only if \(p\) is in the congruence class \(s\) mod \((p - n)^2\) where \(s \in \mathbb{Z}\) such that \((p - n)^2 \mid W(s)\) and \((p - n) \mid (s - n)\). In the case where \(p - n\) is a prime larger than \(p\) with \((p - n) \mid W(n)\), data collected for primes \(p \leq 200\) shows \((p - n) \nmid W'(n)\). Therefore showing this data trend continues for all primes would be an important step towards proving conjecture 2.4.
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