Lower bounds of fractional Dehn twist coefficients and modular invariants

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Abstract In this paper, we aim to give explicit lower bounds of modular invariants of families of curves of genus 2 and 3. According to the relation between fractional Dehn twists and modular invariants, we give the sharp lower bounds of fractional Dehn twist coefficients and classify pseudo-periodic maps with minimal coefficients firstly. Then we give explicit lower bounds of modular invariants, which are sharp for genus 2. We also obtain equivalent conditions for families of curves with these bounds. As an application, we give a better uniform lower bounds for the effective Bogomolov conjecture for genus 2 and 3.

Keywords Lower bounds, fractional Dehn twists, modular invariants

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1 Introduction

1.1 Modular invariants

Without mention we always work on complex number field $\mathbb{C}$. A family of projective curves of genus $g$ is a surjective holomorphic morphism $f : S \to C$ whose general fiber is a smooth curve of genus $g$, where $S$ is a smooth projective surface, and $C$ is a smooth projective curve of genus $b$. Let $M_g$ be the moduli space of smooth curves of genus $g$, and $\overline{M}_g$ be the Deligne-Mumford compactification of $M_g$.

The intersection theory of divisors of $\overline{M}_g$ is very beautiful. The intersection of rational divisor class $\gamma$ of $\overline{M}_g$ with curves $D \subset \overline{M}_g$ is also interested in the theory of birational geometry of $\overline{M}_g$. Numerically, the intersection of $\gamma$ with $D$ can be regarded as the degree of $\gamma$ on $D$. The modular invariant of $f$ corresponding to $\gamma$ is defined as the degree $\gamma(f) = \deg J_f^*(\gamma)$ ([Ta10]), where $J_f : C \to \overline{M}_g$ is the induced moduli map. The modular invariant $\gamma(f)$ satisfies the base change property, i.e., if $\tilde{f} : \tilde{X} \to \tilde{C}$ is the pullback fibration of $f$ under a base change $\pi : \tilde{C} \to C$ of degree $d$, then $\gamma(f) = d \cdot \gamma(f)$. Let $\lambda$ be the Hodge divisor class of $\overline{M}_g$, $\delta$ be the boundary divisor class, and their corresponding modular invariants be $\lambda(f)$ and $\delta(f)$. We also denote by $\kappa(f)$ the modular invariant corresponding to $\kappa = 12\lambda - \delta$. These kinds of modular invariants are called Arakelov invariants ([Ja14]) in number field case, and the modular invariant $\lambda(f)$ is Faltings height in particular.

The minimal uniform lower bounds for these invariants are interesting. In 1991, Mazur raised a question on the minimized Faltings heights of varieties in $\mathbb{P}^N$, which was studied by Zhang ([Zh96]) partly. Some uniform lower bounds for Faltings $\delta$-invariant of curves...
are also obtained by Faltings et al ([Fa84] [Wi16]...). But all these known bounds are not sharp.

In this paper, we consider the above uniform lower bounds problem in the case of curves over function fields. Our goal is to get sharp lower bounds depending only on g and characterize families with these minimal lower bounds.

When \( g = 1 \), then the best lower bounds are \( \kappa(f) = 0 \), \( \lambda(f) \geq \frac{1}{12} \) and \( \delta(f) \geq 1 \). We believe that these bounds of modular invariants are known to experts. Hence we only considered \( g \geq 2 \).

For \( g = 2 \), we have the following sharp lower bounds of modular invariants.

**Theorem 1.1.** Let \( f : S \to C \) be a non-isotrivial fibration of genus 2, then

\[
\lambda(f) \geq \frac{1}{60}, \quad \kappa(f) \geq \frac{1}{15}, \quad \delta(f) \geq \frac{1}{12}
\]

and each equality can be reached. Furthermore,

1) \( \lambda(f) = \frac{1}{60} \) if and only if \( \delta(f) = \frac{1}{12} \) if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is either Figure (2-1a) or Figure (2-1b).

\[
\begin{align*}
(2-1a) & \quad & (2-1b) \\
2 & \quad & 4 \\
6 & \quad & 5 \\
3 & \quad & 2 \\
C_{v_1} & & C_{v_2} \\
C_{v_1} & & C_{v_2} \\
2 & \quad & 1
\end{align*}
\]

2) \( \kappa(f) = \frac{1}{15} \) if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is either Figure (2-0a) or Figure (2-0b).

\[
\begin{align*}
(2-0a) & \quad & (2-0b) \\
1 & \quad & 2 \quad 3 \quad 2 \\
C_v & & 1 \\
2 & \quad & 6 \\
C_v & & 3 \\
1 & \quad & 3
\end{align*}
\]

We prove that the lower bounds in Theorem 1.1 are optimum by giving examples in Section 5.

**Theorem 1.2.** There exists a family of fibrations \((f_{\lambda,n} : S_n \to \mathbb{P}^1)_{n \in \mathbb{N}}\) (resp. \((f_{\kappa,n} : S_n \to \mathbb{P}^1)_{n \in \mathbb{N}}\)) of genus 2 with \( \lambda(f_{\lambda,n}) = \frac{1}{60} \), \( \delta(f_{\lambda,n}) = \frac{1}{12} \) (resp. \( \kappa(f_{\kappa,n}) = \frac{1}{15} \)).

For \( g = 3 \), we have the following results.

**Theorem 1.3.** Let \( f : S \to C \) be a family of curves of genus \( g = 3 \) with \( \delta(f) \neq 0 \), then

\[
\kappa(f) \geq \frac{2}{35}, \quad \delta(f) \geq \frac{1}{30}, \quad \lambda(f) \geq \frac{1}{105}
\]

Moreover, \( \delta(f) = \frac{1}{30} \) if and only if all singular fiber of \( f \) have periodic monodromy except one whose dual graph is one of figures (3-1a), (3-1b) and (3-1c) as follows.
For $g \geq 4$, we have the following lower bounds depending only on $g$.

**Theorem 1.4.** Suppose $f : S \to C$ is a fibration of genus $g \geq 3$, and $\delta(f) \neq 0$, then

$$\lambda(f) \geq \frac{1}{16g(2g+1)}, \quad \delta(f) \geq \frac{1}{4(2g+1)^2}, \quad \kappa(f) \geq \frac{g-1}{4g^2(2g+1)}.$$ 

Let $\lambda(g)$ be the sharp lower bound of $\lambda(f)$ for non-isotrivial families of curves $f$ of genus $g$. From the above, we know that $\lambda(1) = \frac{1}{12}$, $\lambda(2) = \frac{1}{60}$. Since modular invariants are heights in arithmetic algebraic geometry, we raise the following effective question which relates to finiteness of points on curves.

**Question 1.5.** Is there a positive real number $r_0 > 0$ with

$$\inf_{g \geq 1} \lambda(g) \geq r_0?$$

### 1.2 Fractional Dehn twist coefficients

It is proved that the modular invariant $\delta(f)$ is a summation of fractional Dehn twist coefficients ([Liu]). So to get results in Section 1.1, we need sharp lower bounds of fractional Dehn twist coefficients, which is an interesting problem in low-dimensional topology.

It is known that Dehn twists are the generators of the mapping class group, and fractional Dehn twist coefficients are also important in 3-manifolds. These coefficients were first studied by Gabai and Oertel in [GO89], and then applied in many aspects ([HKM07, HM18]...). The bounds of these coefficients are studied in many different contexts, see [HM18 Theorem 1], [IK17 Section 7], [KR13 Theorem 2.16], [Liu, Theorem 1.5].

For our purpose, we consider fractional Dehn twists coefficients in pseudo-periodic maps, and try to give their sharp uniform lower bounds which depends only on $g$. Before we state our results, we will introduce some notations first.

Let $\Sigma_g$ be a closed connected Riemann surface of genus $g \geq 2$. The mapping class group $\text{Mod}(\Sigma_g)$ of $\Sigma_g$ is the group of isotopy classes of orientation preserving homeomorphism of $\Sigma_g$. The Nielsen-Thurston classification theorem says that any mapping class $\phi \in \text{Mod}(\Sigma_g)$ is either periodic, pseudo-Anosov, or reducible. The homeomorphism $\phi$ is reducible if there exists finite simple closed curves $\mathcal{C} = \{\gamma_1, \ldots, \gamma_r\}$ on $\Sigma_g$ such that the restriction of $\phi$ on $\Sigma_g - \mathcal{C}$ is either periodic or pseudo-Anosov. If $\phi \in \text{Mod}(\Sigma_g)$ is periodic, or $\phi$ is reducible and the restriction is periodic, then $\phi$ is said to be pseudo-periodic. We may assume $\mathcal{C}$ satisfies the following additional conditions: (i) $\gamma_i$ does not bound a disk on $\Sigma_g$, and (ii) $\gamma_i$ is not parallel to $\gamma_j$ if $i \neq j$ ([MM11, Lemma 1.1]). Such $\mathcal{C}$ is called an admissible system of cut curves.

Given a pseudo-periodic map $\phi$, a sufficiently high power $\phi^m$ preserves each cut curve $\gamma_1, \ldots, \gamma_r$. Denote by $T_{\gamma_i}$ the (right-hand) Dehn twist of $\Sigma_g$ along $\gamma_i$, then there is a factorization of $\phi$ into a commutative product $\phi^m = T_{\gamma_1}^k \cdots T_{\gamma_r}^k$. The fractional Dehn twist coefficient of $\phi$ along $\gamma_i$ is defined to be $c(\phi, \gamma_i) = k_i/m$ ([Liu, Section 2.2.2]).
If $\phi \in \text{Mod}(\Sigma_g)$ is a pseudo-periodic map of negative twist, that is, $c(\phi, \gamma) < 0$ for each $\gamma \in \mathcal{C}$, then there exists a unique local family $f_\phi : S \rightarrow \Delta$ whose monodromy homeomorphism around its central fiber is equal (up to isotopy and conjugation) to $\phi$ ([Im09, Theorem 4.1]). Here, the local family $f_\phi : S \rightarrow \Delta$ means a proper surjective holomorphic map from a complex surface $S$ to the unit disk of the complex plane $\Delta$, and only the central fiber $F_\phi = f_\phi^{-1}(0)$ over the origin is singular. We also call $F_\phi$ the singular fiber of $\phi$.

It is known that the topological types of local families $f$ of genus $g \geq 2$ are 1-1 correspondent to the conjugacy classes of pseudo-periodic maps $\phi$ of negative twist [MM11, Theorem 0.2]. Almost all the topological types of local families can be determined by dual graphs of their central fibers. (It is easy to check that the dual graphs in this paper determine topological types of the corresponding local families.) Hence we denote the conjugacy class of $\phi$ by the dual graph $G(F_\phi)$ for simplicity.

Now we give the sharp lower bounds of fractional Dehn twist coefficients ($\lvert c(\phi, \gamma) \rvert$ in fact), and classify the pseudo-periodic maps with these bounds.

**Theorem 1.6.** Let $\phi \in \text{Mod}(\Sigma_g)$ be a pseudo-periodic map of negative twist, and $\gamma \in \mathcal{C}$.

1. If $g = 2$, then
   \[ \lvert c(\phi, \gamma) \rvert \geq \frac{1}{12}, \]
   and the equality holds if and only if $(G(F_\phi), \gamma)$ is either Figure (2-1a) or Figure (2-1b). In each figure, we use thick edges to denote $\gamma$ using the correspondence in [Liu, Theorem 4.2].

2. If $g = 3$, then
   \[ \lvert c(\phi, \gamma) \rvert \geq \frac{1}{30}, \]
   and the equality holds if and only if $(G(F_\phi), \gamma)$ is one of figures (3-1a), (3-1b) and (3-1c).

Moreover, we can obtain more precise results.

If $\gamma$ is a non-separated cut curve, then $\gamma$ is said to be of type 0. If $\gamma$ is separated, and the least genus of the two connected components is $i \geq 1$, then $\gamma$ is said to be of type $i$.

**Theorem 1.7.** Let $\phi \in \text{Mod}(\Sigma_g)$ be a pseudo-periodic map of negative twist, and $\gamma \in \mathcal{C}$. Then the sharp lower bounds $c$ of $\lvert c(\phi, \gamma) \rvert$ are as follows.

| $g$ | type 0 | type 1 |
|-----|--------|--------|
| 2   | $\frac{1}{3}$ | $\frac{1}{12}$ |
| 3   | $\frac{1}{12}$ | $\frac{1}{12}$ |

Furthermore,

1. (2-0) if $g = 2$, $\gamma$ is of type 0, then $\lvert c(\phi, \gamma) \rvert = \frac{1}{3}$ if and only if $(G(F_\phi), \gamma)$ is either Figure (2-0a) or Figure (2-0b).

2. (2-1) if $g = 2$, $\gamma$ is of type 1, then $\lvert c(\phi, \gamma) \rvert = \frac{1}{12}$ if and only if $(G(F_\phi), \gamma)$ is either Figure (2-1a) or Figure (2-1b).

3. (3-0) if $g = 3$, $\gamma$ is of type 0, then $\lvert c(\phi, \gamma) \rvert = \frac{1}{12}$ if and only if $(G(F_\phi), \gamma)$ is one of the following figures
\[(3-0a)\]
\[(3-0b)\]
\[(3-0c)\]

(3-1) if \(g = 3\), \(\gamma\) is of type 1, then \(|c(\phi, \gamma)| = \frac{1}{30}\) if and only if \((G(F_0), \gamma)\) is one of figures (3-1a), (3-1b) and (3-1c).

For general \(g \geq 4\), there is a uniform lower bound of fractional Dehn twist coefficients in \([\text{Liu}]\) Theorem 1.5 which is not sharp.

### 1.3 Effective Bogomolov conjecture

Though we have given a uniform lower bound of the effective Bogomolov conjecture for general \(g\), see \([\text{LT17}]\). We now give a better bound for \(g = 2, 3\).

Fix an algebraically closed field \(k\) of characteristic zero and a smooth proper connected curve \(Y/k\). Define \(K\) to be the field of rational functions on \(Y\). Let \(C\) be a smooth proper geometrically connected curve of genus at least 2 over the function field \(K\). Denote by \(f : X \rightarrow Y\) the minimal regular model of the curve \(C\) over \(Y\), where \(X\) is a smooth projective surface over \(k\). Choose a divisor \(D\) of degree 1 on \(\bar{C} = C \times_k \bar{K}\) and consider the embedding of \(C\) into its Jacobian \(\text{Jac}(C) = \text{Pic}^0(C)\) given on geometric points by \(j_D(x) = [x] - D\). Define

\[
a'(D) = \lim \inf_{x \in C(\bar{K})} \hat{h}(j_D(x)).
\]

where \(\hat{h}\) is the canonical Néron-Tate height on the Jacobian associated to the symmetric ample divisor \(\Theta + [-1]^*\Theta\). As \(C(\bar{K})\) may not be countable, the \(\lim \inf\) is taken to mean the limit over the directed set of all cofinite subsets of \(C(\bar{K})\) of the infimum of the heights of points in such a subset.

**Theorem 1.8.** Let \(C/K\) be a smooth proper geometrically connected curve of genus \(g \geq 2\), if the semistable reduction of \(C\) is not smooth, then

\[
\inf_{D \in \text{Div}^1(C)} a'(D) \geq \begin{cases} 
\frac{1}{2280}, & g = 2; \\
\frac{1}{3276}, & g = 3.
\end{cases}
\]

Remark that the bounds given in \([\text{LT17}]\) are \(\frac{1}{12160}\) for \(g = 2\) and \(\frac{1}{19656}\) for \(g = 3\).

The organization of this paper is as follows.

In Section 2 we give notations of modular invariants \(\delta_i(f)\) and valencies of periodic maps, which will be used in our proofs. In Section 3 we prove Theorem 1.6 and Theorem 1.7 using the theory of classification of singular fibers in \([\text{AI02}]\). We divide Theorem 1.7 into four parts in Section 3. Theorem 1.6 is in fact a corollary of Theorem 1.7. By the correspondence in \([\text{Liu}]\) we obtain sharp lower bounds of \(\delta_i(f)\). As an immediate application, we prove Theorem 1.8 at the end of Section 3. Modular invariants of families of curves of genus 2 are then easy to get, so we prove Theorem 1.1 first in Section 4. Similarly, we prove Theorem 1.3 and Theorem 1.4 using slope inequalities. Theorem 1.2 is proved in Section 5.
2 Preliminaries

2.1 Modular invariants

Let $\Delta_0, \Delta_1, \ldots, \Delta_{[g/2]}$ be the boundary divisors of $\overline{M}_g$, $\delta_i$ be the $\mathbb{Q}$-divisor classes corresponding to $\Delta_i$, and $\delta_i(f)$ be the modular invariants corresponding to $\delta_i$. Then

$$\delta(f) = \delta_0(f) + \delta_1(f) + \cdots + \delta_{[g/2]}(f).$$ (2.1)

If $g = 1$, then $\delta(f)$ is the number of poles of the $J$-function of the family (see [Li16] for generalization). When $g \geq 2$, it is shown ([Ta94] [Ta96]) that $\lambda(f) = 0$ if and only if $\kappa(f) = 0$ if and only if $f$ is an isotrivial family. In this paper, we always assume that $f$ is non-isotrivial, then $\lambda(f)$ and $\kappa(f)$ are positive rational number.

Let $F$ be a singular fiber of $f$, and $\bar{F}$ be its $d$-th semistable model ([LT17], P207). Let $p$ be a node of $\bar{F}$. We say $p$ is of type $\theta$ if the normalization of $\bar{F}$ at $p$ is connected. Otherwise, the normalization at $p$ has two connected components, and we say $p$ is of type $i$, where $i$ is the minimum of the arithmetic genera of the two components. Denote by $\delta_i(\bar{F})$ the number of nodes of type $i$ in $\bar{F}$, then we define

$$\delta_i(F) := \frac{\delta_i(\bar{F})}{d}, \quad (i = 0, 1, \ldots, [g/2]),$$ (2.2)

which is independent of the choice of the semistable model $\bar{F}$ of $F$. Let $F_1, \ldots, F_s$ be all singular fibers of $f$. It is shown that, in [LT17],

$$\delta_i(f) = \delta_i(F_1) + \cdots + \delta_i(F_s), \quad i = 0, 1, \ldots, [g/2].$$ (2.3)

If we restrict $f$ to a neighborhood of $f(F) \in C$, we can get a local family $f_\delta$ whose dual graph is $F$, and we denote by $\phi_\delta$ the pseudo-periodic map determined by $f_\delta$. On the other hand, let $\phi \in \text{Mod}(\Sigma_g)$ be a pseudo-periodic map of negative twist. Then, for each $i \geq 0$, we have ([Li16] Theorem 1.2))

$$\delta_i(F_\phi) = \delta_i(f_\phi) = \sum_{\gamma \in \mathcal{C}_i} |c(\phi, \gamma)|,$$ (2.4)

where $\mathcal{C}_i = \{ \gamma \in \mathcal{C} : \gamma \text{ is of type } i \}$. So, if $\delta(F_\phi) = 0$, then $F_\phi$ has smooth reduction, and $\phi$ has periodic monodromy.

2.2 Valencies of periodic maps

Let $\Sigma$ be a connected real 2-dimensional manifold with or without boundary. When we emphasize its complex structure, we call $\Sigma$ a Riemann surface.

Let $\phi : \Sigma \to \Sigma$ be a periodic homeomorphism of order $n \geq 2$, and $p$ be a point on $\Sigma$. There is a positive integer $m_p$ such that the points $p, \phi(p), \ldots, \phi^{m_p-1}(p)$ are mutually distinct and $\phi^{m_p}(p) = p$. If $m_p = n$, we call the point $p$ a simple point of $\phi$, while if $m_p < n$, we call $p$ a multiple point of $\phi$.

Let $\gamma$ be a cut curve in $\mathcal{C}$, and $m = m_\gamma$ be the smallest positive integer such that $\phi^m(\gamma) = \gamma$, i.e., $\phi^m(\gamma) = \gamma$ as a set and $\phi^m$ preserving the orientation of $\gamma$. The restriction of $\phi^m$ to $\gamma$ is a periodic map of order, say, $\lambda \geq 1$. Let $q$ be any point on $\gamma$, and suppose that the images of $q$ under the iteration of $\phi^m$ are ordered $(q, \phi^m(q), \phi^{2m}(q), \ldots, \phi^{(\lambda-1)m}(q))$.
viewed in the direction of $\vec{\gamma}$, where $\sigma$ is an integer with $0 \leq \sigma \leq \lambda - 1$, $\gcd(\sigma, \lambda) = 1$, and $\sigma = 0$ iff $\lambda = 1$. The triple $(m, \lambda, \sigma)$ is called the valency of $\vec{\gamma}$ with respect to $\phi$.

We define the valency of a boundary curve (i.e., a connected component of the boundary $\partial \Sigma$) as its valency with respect to $\phi$, assuming it has the orientation induced by the surface $\Sigma$. The valency of a multiple point $p$ is defined to be the valency of the boundary curve $\partial D_p$, oriented from the outside of a disk neighborhood $D_p$ of $p$.

Let $\Sigma$ be a surface of genus $g$ with $k$ boundary curves $\partial_1, \ldots, \partial_k$. Let $\phi : \Sigma \to \Sigma$ be an orientation-preserving homeomorphism which satisfies:

1. there is a disjoint union of simple closed curves $\mathcal{C} = \bigcup_{j=1}^{r} \gamma_j$ such that $\mathcal{C}$ and $\partial \Sigma = \bigcup_{j=1}^{k} \partial_j$ do not intersect each other,
2. $\Sigma - \mathcal{C}$ is connected,
3. $\phi(\mathcal{C}) = \mathcal{C}$ and $\phi|_{\Sigma - \mathcal{C}}$ is periodic.

Then we can extend $\phi$ to a periodic map on a closed surface $\tilde{\Sigma}$ easily ([AI02, Lemma 1.2]), and classify the valency data for each $(g, r, k)$ similarly to Lemma 2.1 (see [AI02, Section 2.2]).

Suppose $\Pi : \Sigma \to \Sigma'$ is the $n$-fold cyclic covering induced by $\phi$, where $\Sigma'$ is the quotient surface of $\Sigma$ with respect to $\phi$. Let $\{q_1, \ldots, q_l\} \subseteq \Sigma'$ be the set of branch points. If $\tilde{q}_i$ is a point of the pre-image $\Pi^{-1}(q_i)$ of $q_i$, and let the valency of $\tilde{q}_i$ be $(m_i, \lambda_i, \sigma_i)$. Then we know that $m_i$ is the number of points in $\Pi^{-1}(q_i)$ and $\lambda_i = n/m_i$. Since the valencies of points in $\Pi^{-1}(q_i)$ are the same, we can define the valency of $q_i$ to be the valency of $\tilde{q}_i$.

For brevity’s sake, if we have the data of valencies $(n/\lambda_i, \lambda_i, \sigma_i)$ ($1 \leq i \leq l$), we symbolically write $\sigma_1/\lambda_1 + \cdots + \sigma_l/\lambda_l$ which is called the total valency. We also write the order $n$ of the map and the genus $g'$ of $\Sigma'$. However if $g' = 0$, the genus is omitted.

A periodic map can be represented by its total valency. For the reader’s convenience, we list the classification of periodic maps in [AI02, Lemma 1.4] here.

**Lemma 2.1.** Non-identical conjugacy classes of periodic maps of closed surfaces of genus $1 \leq g \leq 2$ are classified as follows:

(i) $g = 1$

1. $n = 6$: $1/6 + 1/3 + 1/2, 5/6 + 2/3 + 1/2$.
2. $n = 4$: $1/4 + 1/4 + 1/2, 3/4 + 3/4 + 1/2$.
3. $n = 3$: $1/3 + 1/3 + 1/3, 2/3 + 2/3 + 2/3$.
4. $n = 2$: $1/2 + 1/2 + 1/2 + 1/2$.
5. $g' = 1$, $n$ is arbitrary and $\Pi : \Sigma \to \Sigma'$ is an unramified covering.

(ii) $g = 2$

1. $n = 10$: $1/10 + 2/5 + 1/2, 3/10 + 1/5 + 1/2, 7/10 + 4/5 + 1/2, 9/10 + 3/5 + 1/2$.
2. $n = 8$: $1/8 + 3/8 + 1/2, 5/8 + 7/8 + 1/2$.
3. $n = 6$: $1/6 + 1/6 + 2/3, 5/6 + 5/6 + 1/3, 1/3 + 2/3 + 1/2 + 1/2$.
4. $n = 5$: $1/5 + 1/5 + 3/5, 1/5 + 2/5 + 2/5, 2/5 + 4/5 + 4/5, 3/5 + 3/5 + 4/5$.
5. $n = 4$: $1/4 + 3/4 + 1/2 + 1/2$.
6. $n = 3$: $1/3 + 1/3 + 2/3 + 2/3$.
8. $g' = 1$, $n = 2, 1/2 + 1/2$.

2.3 **Representation of a pseudo-periodic map**

Let $\phi : \Sigma_g \to \Sigma_g$ be a pseudo-periodic map, and $\mathcal{C} = \{\gamma_1, \ldots, \gamma_r\}$ be the admissible system of cut curves. Then the restriction of $\phi$ on $\mathcal{B} = \Sigma_g - \mathcal{C}$ is isotopic to a periodic
Now we use a weighted graph \( G_\mathcal{C} \) to denote the decomposition \( \Sigma = \mathcal{B} \cup \mathcal{C} \). A vertex \( v \) in \( G_\mathcal{C} \) corresponds to a connected component \( B_v \) of \( \mathcal{B} \), and an edge \( e \) corresponds to a separated cut curve \( \gamma_e \) in \( \mathcal{C} \), where \( \gamma_e \) is adjacent to two connected components of \( \mathcal{B} \). We define the weight of a vertex \( v \) to be \( g(B_v) + \rho(v) \), where \( g(B_v) \) is the genus of \( B_v \), and \( \rho(v) \) is the number of cut curves only adjacent to \( B_v \). We use a small circle to denote a vertex, and the number inside the small circle means \( g(B_v) + \rho(v) \). We omit the number when it is zero.

Note that a weighted graph may represent different decompositions. For example, the graph (II) in Lemma 3.1 represents four types of decompositions, that is, the component corresponding to \( v_1 \) has genus \( i_1 \) and is adjacent to \( 1 - i_1 \) non-separating curves in \( \mathcal{C} \), and the component corresponding to \( v_2 \) has genus \( i_2 \) and is adjacent to \( 1 - i_2 \) non-separating curves in \( \mathcal{C} \) (\( 0 \leq i_1 \leq 1, \ 0 \leq i_2 \leq 1 \)).

The map \( \phi : \Sigma_g = \mathcal{B} \cup \mathcal{C} \rightarrow \Sigma_g = \mathcal{B} \cup \mathcal{C} \) induces an automorphism \( \sigma_\phi \) on the weighted graph \( G_\mathcal{C} \). Here an automorphism of \( G_\mathcal{C} \) means an automorphism of the graph such that the weight \( (g(B_v), \rho(v)) \) coincides with \( (g(B_{\sigma(v)}), \rho(\sigma(v))) \) for each vertex \( v \) of \( G_\mathcal{C} \), see [AI02, Section 3.3] for an example.

For each cut curve \( \gamma \in \mathcal{C} \), there exists a minimal integer \( \alpha \) such that \( \phi^\alpha(\gamma) = \bar{\gamma} \). The curve \( \gamma \) is said to be amphidrome if \( \alpha \) is even and \( \phi^{\alpha/2}(\gamma) = -\bar{\gamma} \) (where \( \bar{\gamma} \) and \( -\bar{\gamma} \) denote the same \( \gamma \) with the opposite directions assigned) and non-amphidrome otherwise. There exists a minimal integer \( L \) such that \( \phi^L \) restricts to an annulus of \( \gamma \) isotopic to a Dehn twist of \( e \) times \( (e \in \mathbb{Z}) \). The rational number \( ea/L \) is called the screw number of \( \phi \) about \( \gamma \), and is denoted by \( s(\gamma) \). We may always assume that \( s(\gamma) \neq 0 \) for each \( \gamma \in \mathcal{C} \) (see [MM11, P5]).

For each \( \gamma \in \mathcal{C} \), denote by \( m_\gamma \) the length of the cyclic orbit of \( \gamma \) under the permutation caused by \( \phi \), that is,

\[
m_\gamma = \#\{\phi^k(\gamma) : k \in \mathbb{N}\}.
\]

Our classification is based on the following theorem in [MM11].

**Theorem 2.2.** The conjugacy class of a pseudo-periodic map \( \phi : \Sigma_g \rightarrow \Sigma_g \) of negative twist is determined by the following data: an admissible system \( \mathcal{C} \) of cut curves, the induced automorphism \( \sigma_\phi \) of \( G_\mathcal{C} \), the screw numbers \( s(\gamma) \) for each \( \gamma \in \mathcal{C} \), and the valency data of the periodic maps which stabilize the connected components of \( \Sigma_g - \mathcal{C} \).

Now we give the formula of fractional Dehn twist coefficients in [Liu, Theorem 4.5] and the formula of screw number (see [AI02, Section 2.1]). Let \( \mathcal{A}_\gamma \) be the annular neighborhood of \( \gamma \). Let \( (m_1, \lambda_1, \sigma_1) \) and \( (m_2, \lambda_2, \sigma_2) \) be the valencies of the two boundary curves of \( \mathcal{A}_\gamma \). If \( \gamma \) is non-amphidrome, then \( m_1 = m_2 = m_\gamma \), and

\[
|c(\phi, \gamma)| = \frac{|s(\gamma)|}{m_\gamma} = \frac{1}{m_\gamma} \left( \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + K \right),
\]

(2.5)

where \( K \geq -1 \) is an integer, and \( \mu_i \) are integers with

\[
\sigma_i \mu_i \equiv 1 \mod \lambda_i, \quad 0 \leq \mu_i \leq \lambda_i - 1.
\]

If \( \gamma \) is amphidrome, then \( \gamma \) is of type 0, and the two boundary curves have the same valency \( (2m_\gamma, \lambda, \sigma) \) where \( 2m_\gamma = \alpha \), and

\[
|c(\phi, \gamma)| = \frac{|s(\gamma)|}{2m_\gamma} = \frac{1}{m_\gamma} \left( \frac{\mu}{\lambda} + K \right),
\]

(2.6)
where \( K \geq -1 \) is an integer, and \( \mu \) is an integer with
\[
\sigma \mu \equiv 1 \mod \lambda, \quad 0 \leq \mu \leq \lambda - 1.
\]

In the following, we will use more precise notations if necessary. For example, let \( \gamma \) be the cut curve \( e \) in the graph (II) in Lemma 3.1. Then we denote the valencies of the two boundary curves of \( \mathcal{A}_\gamma \) by \((m_1, \lambda_{v_1, \gamma}, \sigma_{v_1, \gamma})\) and \((m_2, \lambda_{v_2, \gamma}, \sigma_{v_2, \gamma})\), and denote by \( n(B_{v_i}) \) the order of the periodic homeomorphism of \( B_{v_i} \) induced by \( \phi \). Thus \( n(B_{v_i}) = m_\gamma \lambda_{v_i, \gamma} \).

3 Bounds of fractional Dehn twist coefficients

3.1 Genus 2 case

First we give the classification of decompositions of Riemann surfaces of genus two.

**Lemma 3.1.** The decompositions of a Riemann surface of genus two by an admissible system of cut curves can be classified in terms of weighted graphs (I)-(III) as follows.

\[
\begin{align*}
&\text{(I)} \quad \begin{array}{c}
2 \\
v_1
\end{array} \\
&\text{(II)} \quad \begin{array}{c}
1 \\
v_1 \\
\gamma
\end{array} \\
&\text{(III)} \quad \begin{array}{c}
1 \\
v_1 \\
\gamma
\end{array}
\end{align*}
\]

**Proof.** This problem is equivalent to classifying stable curves of genus two, which is trivial.

**Theorem 3.2.** Let \( \phi \in \text{Mod}(\Sigma_2) \) be a pseudo-periodic map of negative twist, and \( \gamma \) be a cut curve of type 1, then
\[
|c(\phi, \gamma)| \geq \frac{1}{12},
\]
and the equality holds if and only if \((G(F_\phi), \gamma)\) is either Figure (2-1a) or Figure (2-1b).

**Proof.** The possible cut curve \( \gamma \) is \( e \) in (II). In this case, the automorphism \( \sigma_\phi \) of the graph (II) induced by \( \phi \) satisfies that \( \sigma_\phi(e) = e, \sigma_\phi(v_i) = v_i \) for \( i = 1, 2 \), and \( m_\gamma = 1 \).

Claim 1: if \( \rho(v_i) = 1, g(B_{v_i}) = 0 \) for some \( i \), then \( |c(\phi, \gamma)| \geq \frac{1}{12} \).

Proof of Claim 1: We may assume that \( \rho(v_1) = 1, g(B_{v_1}) = 0 \). Let \( \gamma' \) be the cut curve only adjacent to \( B_{v_1} \), and \( \mathcal{A}_{\gamma'} \) be the annular neighbourhood of \( \gamma' \). Then \( \phi \) maps boundary curves of \( \mathcal{A}_{\gamma'} \) (resp. \( \mathcal{A}_{\gamma} \)) to boundary curves of \( \mathcal{A}_{\gamma'} \) (resp. \( \mathcal{A}_{\gamma} \)). Thus \( n(B_{v_1}) \leq 2 \), since the automorphism of Riemann sphere with three fixed points is identity. Hence \( \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 2 \). Now consider \( B_{v_2} \), we have \( n(B_{v_2}) \leq 6 \): if \( \rho(v_2) = 0 \), then \( g(B_{v_2}) = 1 \) and \( n(B_{v_2}) \leq 6 \) by Lemma 2.1 (i); if \( \rho(v_2) = 1 \), then \( g(B_{v_2}) = 0 \) and \( n(B_{v_2}) \leq 2 \) as above. Thus, by (2.5),
\[
|c(\phi, \gamma)| = \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K \geq \frac{1}{\text{lcm}(\lambda_{v_1, \gamma}, \lambda_{v_2, \gamma})} > \frac{1}{12},
\]
and we finish the proof of Claim 1.

Now we only need to prove the case that \( \rho(v_i) = 0, g(B_{v_i}) = 1, \ i = 1, 2 \). Since there is one edge \( e \) adjacent to \( v_i \ (i = 1, 2) \), we know that the restriction of \( \phi \) on \( B_{v_i} \) can not
induce an unramified covering of degree $n \geq 2$. Thus $n(B_v) \leq 6$ and the valency data are classified in Lemma 2.1 (i). By (2.5) and Lemma 2.1 (i),
\[ |c(\phi, \gamma)| = \frac{\mu_{v_1,\gamma}}{\lambda_{v_1,\gamma}} + \frac{\mu_{v_2,\gamma}}{\lambda_{v_2,\gamma}} + K \geq \frac{1}{12}. \]

If the equality holds, then $K = -1$. Furthermore, by Lemma 2.1 and Theorem 2.2, the cut curves and the pseudo-periodic maps with the lowest bound are classified as follows:

\begin{align*}
(2.1a) & \quad B_{v_1} : \frac{3}{2} + \frac{2}{3} + \frac{1}{2}, \quad B_{v_2} : \frac{4}{3} + \frac{1}{2} + \frac{1}{2}, \quad K = -1; \\
(2.1b) & \quad B_{v_1} : \frac{3}{2} + \frac{2}{3} + \frac{1}{2}, \quad B_{v_2} : \frac{3}{2} + \frac{3}{4} + \frac{1}{2}, \quad K = -1.
\end{align*}

Here we write valency data of $\gamma$ by bold face characters. For the reader’s convenience, we take the case (2-1a) as an example, the valencies of the two boundary curves of $A_{\gamma}$ are $(m_{v_1,\gamma}, \lambda_{v_1,\gamma}, \sigma_{v_1,\gamma}) = (1, 6, 5)$ and $(m_{v_2,\gamma}, \lambda_{v_2,\gamma}, \sigma_{v_2,\gamma}) = (1, 4, 1)$. Thus $\mu_{v_1,\gamma} = 5, \mu_{v_2,\gamma} = 1$, and
\[ |c(\phi, \gamma)| = \frac{5}{6} + \frac{1}{4} + (-1) = \frac{1}{12}. \]

Using the correspondence in [Liu, Section 4], it is easy to check that the dual graphs of the above two pseudo-periodic maps are Figure (2-1a) and Figure (2-1b) respectively. By Theorem 2.2, there is no other pseudo-periodic maps $(\phi, \gamma)$ with $|c(\phi, \gamma)| = \frac{1}{12}$.

**Corollary 3.3.** Let $f$ be a family of curves of genus 2 with $\delta_1(f) \neq 0$, then
\[ \delta_1(f) \geq \frac{1}{12}, \]
and the equality holds if and only if all the singular fibers of $f$ have smooth reduction except one whose dual graph is either Figure (2-1a) or Figure (2-1b).

**Proof.** Let $F$ be a singular fiber of $f$, then

Claim 2: $\delta_1(F) \geq \frac{1}{12}$, and $\delta_1(F) = \frac{1}{12}$ if and only if the dual graph of $F$ is either Figure (2-1a) or Figure (2-1b).

Proof of Claim 2: By (2.4) and Theorem 3.2 we have
\[ \delta_1(F) = \sum_{\gamma \in \mathcal{G}_{\phi_F, 1}} |c(\phi_F, \gamma)| \geq \frac{1}{12}. \]

If $\delta_1(F) = \frac{1}{12}$, then $\phi_F$ has only one cut curve $\gamma$ of type 1, and the possible dual graphs of $(\phi_F, \gamma)$ are Figure (2-1a) and Figure (2-1b). Hence we obtain the claim.

Since $\delta_1(f) \neq 0$, there is a singular fiber of $f$, say $F_1$, with $\delta_1(F_1) \neq 0$ by (2.3). So
\[ \delta_1(f) \geq \delta_1(F_1) \geq \frac{1}{12}, \]
and $\delta_1(f) = \delta_1(F_1) = \frac{1}{12}$ if and only $\delta_1(F_1) = \frac{1}{12}$, and $\delta_1(F_i) = 0, i \geq 2$. Then we complete the proof by Claim 2.

**Theorem 3.4.** Let $\phi \in \text{Mod}(\Sigma_2)$ be a pseudo-periodic map of negative twist, and $\gamma$ be a cut curve of type 0, then
\[ |c(\phi, \gamma)| \geq \frac{1}{3}, \]
and the equality holds if and only if $(G(F_\phi), \gamma)$ is either Figure (2-0a) or Figure (2-0b).
In this case, we may assume that $(m_1, \lambda_1, \sigma_1)$ and $(m_2, \lambda_2, \sigma_2)$ be the valencies of the boundary curves of $\mathcal{A}_\gamma$.

(II). First consider $\rho(v_1) = 1$. Then $g(B_{v_1}) = 1$ and $m_\gamma = 1$. If $\gamma$ is non-amphidrome, then $\lambda_1 = \lambda_2 = n(B_{v_1})$. By (2.5) and Lemma 2.1, we have that

$$|c(\phi, \gamma)| = \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + K \geq \frac{1}{3}.$$ 

Furthermore, the equality holds if and only if $K = -1$ and the valency data of the boundary curves of $\mathcal{A}_\gamma$ is

$$(2-0a) \ v_1 : \frac{2}{3} + \frac{2}{3} + \frac{1}{3}, \ K = -1.$$ 

If $\gamma$ is amphidrome, then $\lambda = n(B_{v_1})/2 \leq 3$. Thus, by (2.6),

$$|c(\phi, \gamma)| = \frac{\mu}{\lambda} + K \geq \frac{1}{3}.$$ 

Furthermore, the equality holds if and only if $K = 0$, and the valency data of the boundary curves of $\mathcal{A}_\gamma$ is

$$(2-0b) \ v_1 : \frac{1}{3} + \frac{1}{3} + \frac{1}{3}, \ K = 0.$$ 

(II). Now we consider $\rho(v_1) = 2$, then $g(B_{v_1}) = 0$. Let the two cut curves be $\gamma = \gamma_1$ and $\gamma_2$. Then $m_\gamma \leq 2$. Note that $\phi|_{B_{v_1}}$ maps $\mathcal{A}_{\gamma_i}$ to $\mathcal{A}_{\gamma_j}$, and maps boundary curves of $\mathcal{A}_{\gamma_i}$ to those of $\mathcal{A}_{\gamma_j}$, where $1 \leq i, j \leq 2$. If $m_\gamma = 1$, then $\lambda \leq n(B_{v_1}) \leq 2$, and thus $|c(\phi, \gamma)| \geq \frac{1}{2}$. If $m_\gamma = 2$, then $\lambda = 1$ which is independent of the action of $\phi$ on boundary curves (note that if $\gamma$ is amphidrome, then $\lambda = n(B_{v_1})/2m_\gamma$). Thus $|c(\phi, \gamma)| \geq 1$, by (2.5) and (2.6).

Case (II) In this case, we may assume $\gamma$ is adjacent to $B_{v_1}$ only and $m_\gamma \leq 2$, then we have $\rho(v_1) = 1, g(B_{v_1}) = 0$. We have that $\lambda = 1$ which is independent of the action of $\phi$ on boundary curves, and thus $|c(\phi, \gamma)| \geq \frac{1}{2}$.

Case (III) We may assume that $\gamma = e_1$.

(III1). If $m_\gamma \geq 2$, then $\lambda_{v_1, \gamma} = \lambda_{v_2, \gamma} = 1$. So $|c(\phi, \gamma)| \geq \frac{1}{2}$.

(III2). If $m_\gamma = 1$, then $\lambda_{v_1, \gamma} = \lambda_{v_2, \gamma} \leq 2$. So $|c(\phi, \gamma)| \geq \frac{1}{2}$.

\[\square\]

Corollary 3.5. Let $f$ be a family of curves of genus 2 with $\delta_0(f) \neq 0$, then

$$\delta_0(f) \geq \frac{1}{3},$$ 

and the equality holds if and only if all the singular fibers of $f$ have smooth reduction except one whose dual graph is either Figure (2-0a) or Figure (2-0b).

Proof. Similar to the proof of Corollary 3.3. \[\square\]

3.2 Genus 3 case

In this subsection, we use the same method as Section 3.1 to discuss lower bounds of fractional Dehn twist coefficients in genus 3 case.
Lemma 3.6. The decompositions of a Riemann surface of genus three by an admissible system of cut curves can be classified in terms of weighted graphs (A)-(O) as follows.

(A) \[ \begin{array}{c} 3 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 2 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \]

(B) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_2 \end{array} \]

(C) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_2 \end{array} \]

(D) \[ \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_1 \end{array} \]

(E) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \]

(F) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_2 \end{array} \]

(G) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \]

(H) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_3 \end{array} \]

(I) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \]

(J) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \]

(K) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_5 \end{array} \]

(L) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_5 \end{array} \xrightarrow{e_5} \begin{array}{c} 1 \\ v_6 \end{array} \]

(M) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_5 \end{array} \xrightarrow{e_5} \begin{array}{c} 1 \\ v_6 \end{array} \]

(N) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_5 \end{array} \xrightarrow{e_5} \begin{array}{c} 1 \\ v_6 \end{array} \]

(O) \[ \begin{array}{c} 1 \\ v_1 \end{array} \xrightarrow{e_1} \begin{array}{c} 1 \\ v_2 \end{array} \xrightarrow{e_2} \begin{array}{c} 1 \\ v_3 \end{array} \xrightarrow{e_3} \begin{array}{c} 1 \\ v_4 \end{array} \xrightarrow{e_4} \begin{array}{c} 1 \\ v_5 \end{array} \xrightarrow{e_5} \begin{array}{c} 1 \\ v_6 \end{array} \]

Proof. See [AI02, Lemma 3.2]. □

Theorem 3.7. Let \( \phi \in \text{Mod}(\Sigma_3) \) be a pseudo-periodic map of negative twist, and \( \gamma \) be a cut curve of type 1, then

\[ |c(\phi, \gamma)| \geq \frac{1}{30}, \]

and the equality holds if and only if \((G(F_\phi), \gamma)\) is one of figures (3-1a), (3-1b) and (3-1c).

Proof. The possible cut curves are in (B), (C), (D), (F), (G), (I), and (K).

Case (B) In this case, \( \gamma \) is \( e_1 \), and \( m_\gamma = 1 \). Similar to Claim 1 in the proof of Theorem 3.2 we may assume that \( g(B_{v_1}) \geq 2, g(B_{v_2}) \geq 1 \), and thus \( n(B_{v_1}) \leq 10 \) and \( n(B_{v_2}) \leq 6 \)
classified in Lemma 2.1. By (2.5),

\[ |c(\phi, \gamma)| = \frac{\mu_{\psi_1, \gamma}}{\lambda_{\psi_1, \gamma}} + \frac{\mu_{\psi_2, \gamma}}{\lambda_{\psi_2, \gamma}} + K \geq \frac{1}{30}. \]

If the equality holds, then \( K = -1 \). Furthermore, by Lemma 2.1 the cut curves and the pseudo-periodic maps with the lowest bound are classified as follows:

(3-1a) \( B_{v_1} : \frac{3}{10} + \frac{1}{5} + \frac{1}{7}, \quad B_{v_2} : \frac{1}{1} + \frac{1}{5} + \frac{1}{3}, \ K = -1; \)

(3-1b) \( B_{v_1} : \frac{1}{10} + \frac{2}{5} + \frac{2}{3}, \quad B_{v_2} : \frac{5}{10} + \frac{2}{3} + \frac{1}{2}, \ K = -1; \)

(3-1c) \( B_{v_1} : \frac{1}{5} + \frac{1}{5} + \frac{2}{3}, \quad B_{v_2} : \frac{5}{6} + \frac{2}{3} + \frac{1}{2}, \ K = -1. \)

The dual graphs of the above three pseudo-periodic maps are Figure (3-1a), Figure (3-1b), and Figure (3-1c) respectively.

Case (C) We may assume that \( \gamma = e_1 \). Then \( m_\gamma \leq 2 \). We may assume that \( g(B_{v_i}) = 1 \) and \( n(B_{v_i}) \leq 6 \), where \( i = 1, 3 \). Then

\[ |c(\phi, \gamma)| \geq \frac{1}{2} \left( \frac{\mu_{\psi_1, \gamma}}{\lambda_{\psi_1, \gamma}} + \frac{\mu_{\psi_3, \gamma}}{\lambda_{\psi_3, \gamma}} + K \right) \geq \frac{1}{24}. \]

Case (D) In this case, we may assume that \( \gamma = e_1 \). Then \( \lambda_{v_4, \gamma} = 1, \ \lambda_{v_1, \gamma} \leq 6, \) and \( m_\gamma \leq 3 \). Thus

\[ |c(\phi, \gamma)| \geq \frac{1}{3} \left( \frac{\mu_{\psi_4, \gamma}}{\lambda_{\psi_4, \gamma}} + \frac{\mu_{\psi_1, \gamma}}{\lambda_{\psi_1, \gamma}} + K \right) \geq \frac{1}{18}. \]

The rest cases are similar, and we omit their proofs.

\[ \square \]

**Corollary 3.8.** Let \( f \) be a family of curves of genus 3 with \( \delta_1(f) \neq 0 \), then

\[ \delta_1(f) \geq \frac{1}{30}, \]

and the equality holds if and only if all the singular fibers of \( f \) have smooth reduction except one whose dual graph is one of figures (3-1a), (3-1b) and (3-1c).

**Proof.** Similar to the proof of Corollary 3.3. \( \square \)

**Lemma 3.9.** Let \( \phi \in \text{Mod}(\Sigma_3) \) be a pseudo-periodic map of negative twist, and \( \gamma \) be a cut curve of type 0 which is adjacent to one component only. Then

\[ |c(\phi, \gamma)| \geq \frac{1}{5}, \]

and the equality holds if and only if \( (G(F_\phi), \gamma) \) is one of the following figures (3-0-0a), (3-0-0b) and (3-0-0c).

\[ \begin{align*}
\text{(3-0-0a)} & \qquad \text{(3-0-0b)} & \qquad \text{(3-0-0c)} \\
1 & 3 & 5 & 4 & 1 & 5 & 2 & 2 & 1 \\
& c_\gamma & & & 1 & 2 & 10 & 3 & 2 & 1 \\
\end{align*} \]
Proof. The proof is similar to that of Case (I) in Theorem 3.4.

Assume $\gamma$ is adjacent to $B_v$ with $g(B_v) + \rho(v) \leq 3$ and $\rho(v) \geq 1$. Let $(m_1, \lambda_1, \sigma_1)$ and $(m_2, \lambda_2, \sigma_2)$ be the valencies of the boundary curves of $\mathcal{A}_\gamma$.

If $\gamma$ is non-amphidrome, then $\lambda_1 = \lambda_2 = n_B(v)/m_\gamma$. Since $\rho(v) \geq 1$ and $g = 3$, by Lemma 2.1, we have that

$$|c(\phi, \gamma)| = \frac{1}{m_\gamma}(\mu_1 \lambda_1 + \mu_2 \lambda_2 + K) \geq \frac{1}{5}.$$

Furthermore, the equality holds if and only if $g(B_v) = 2$, $m_\gamma = \rho(v) = 1$, and the valencies of the boundary curves of $\mathcal{A}_\gamma$ are one of the following cases:

- (3-0-0a) $B_v : \frac{4}{5} + \frac{3}{5} + \frac{2}{5}$, $K = -1$.
- (3-0-0b) $B_v : \frac{2}{5} + \frac{2}{5} + \frac{1}{5}$, $K = -1$.

The dual graphs of the above pseudo-periodic maps $\phi$ are Figure (3-0-0a) and Figure (3-0-0b) respectively.

If $\gamma$ is amphidrome, then the valencies of the boundary curves of $A_\gamma$ are $(2m_\gamma, \lambda, \sigma)$. Here $\lambda = n_B(v)/2m_\gamma$. Since $\rho(v) \geq 1$, $g(B_v) \leq 2$, we have $n_B(v) \leq 10$. Thus

$$|c(\phi, \gamma)| = \frac{1}{m_\gamma}(\mu \lambda + K) \geq \frac{2}{n_B(v)} \geq \frac{1}{5}.$$

Furthermore, the equality holds if and only if $g(B_v) = 2$, $m_\gamma = \rho(v) = 1$, and the valencies of the boundary curves of $\mathcal{A}_\gamma$ are

- (3-0-0c) $v : \frac{3}{10} + \frac{1}{5} + \frac{1}{2}$, $K = 0$.

The dual graph of the above pseudo-periodic map $\phi$ is Figure (3-0-0c).

\[\Box\]

**Theorem 3.10.** Let $\phi \in \text{Mod}(\Sigma_3)$ be a pseudo-periodic map of negative twist, and $\gamma$ be a cut curve of type 0. Then

$$|c(\phi, \gamma)| \geq \frac{1}{12},$$

and the equality holds if and only if $(G(F_\phi), \gamma)$ is one of figures (3-0a), (3-0b) and (3-0c).

**Proof.** By Lemma 3.9, we may assume that $\gamma$ is adjacent to two connected components. Thus $\gamma$ is in (E)-(O).

**Case (E)** Without loss of generality, we may assume $\gamma = e_1$.

**Case (E)1.** If the automorphism $\sigma_\phi$ induced by $\phi$ on the graph (E) satisfies that $\sigma_\phi(e_1) = e_1, \sigma_\phi(e_2) = e_2$, then $m_\gamma = 1$, and $\lambda_{v_j, \gamma} = n(B_{v_j})$ for $j = 1, 2$. Hence $\lambda_{v_j, \gamma} = n(B_{v_j}) \leq 4$ by Lemma 2.1. Thus

$$|c(\phi, \gamma)| = \frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K \geq \frac{1}{12}.$$

Furthermore, the equality holds if and only if the valencies of the boundary curves of $\mathcal{A}_\gamma$ are

- (3-0a) $B_{v_1} : \frac{3}{5} + \frac{3}{5} + \frac{1}{2}$, $B_{v_2} : \frac{1}{5} + \frac{1}{5} + \frac{1}{2}$, $K = -1$.

The dual graph of the above pseudo-periodic map $\phi$ is Figure (3-0a).
(E2). If the automorphism $\sigma_\phi$ of the graph (E) satisfies that $\sigma_\phi(e_1) = e_2, \sigma_\phi(e_2) = e_1$, then $m_\gamma = 2$, and $\lambda_{v_1, \gamma} = n(B_{v_1})/2$ for $j = 1, 2$. By Lemma 2.1 we have that $n(B_{v_1}) \leq 6$, and thus $\lambda_{v_1, \gamma} \leq 3$. Thus

$$|c(\phi, \gamma)| = \frac{1}{2}\left(\frac{\mu_{v_1, \gamma}}{\lambda_{v_1, \gamma}} + \frac{\mu_{v_2, \gamma}}{\lambda_{v_2, \gamma}} + K\right) \geq \frac{1}{12}.$$ 

Furthermore, the equality holds if and only if the valencies of the boundary curves of $\partial_a$ are one of the following two cases:

(3-0b) $B_{v_1} : \frac{1}{6} + \frac{2}{3} + \frac{1}{2}$, $B_{v_2} : \frac{1}{4} + \frac{1}{4} + \frac{1}{1} + K = -1$;

(3-0c) $B_{v_1} : \frac{5}{6} + \frac{2}{3} + \frac{1}{2}$, $B_{v_2} : \frac{5}{4} + \frac{2}{4} + \frac{1}{1}, K = -1$.

The corresponding dual graphs are Figure (3-0b) and Figure (3-0c) respectively.

In the following, we prove that $|c(\phi, \gamma)| > \frac{1}{12}$ in all the rest cases. Though the method is the same as above, we give the detail proof here for the reader’s convenience.

Case (F) We may assume that $\gamma = e_1$. Since the rational component $B_{v_1}$ ($i = 3, 4$) is adjacent to three edges, and $\sigma_\phi(e_1)$ is either $e_1$ or $e_2$, we know that $n(B_{v_1}) \leq 2$. Thus $m_\gamma = 2$, $\lambda_{v_1, \gamma} = 2$, and

$$|c(\phi, \gamma)| \geq \frac{1}{4}.$$ 

Case (G) We may assume that $\gamma = e_1$.

(G1). If the automorphism $\sigma_\phi$ of the graph (G) such that $\sigma_\phi(e_1) = e_1, \sigma_\phi(e_2) = e_2$, then $n(B_{v_2}) = 1$, $m_\gamma = 1$, $\lambda_{v_1, \gamma} = \lambda_{v_1, e_2} = n(B_{v_1})$, and $\lambda_{v_2, \gamma} = n(B_{v_2}) = 1$. By Lemma 2.1 we know that $\lambda_{v_j, \gamma} = n(B_{v_j}) \leq 4$ ($j = 1, 2$). Thus

$$|c(\phi, \gamma)| \geq \frac{1}{4}.$$ 

(G2). If $\sigma_\phi(e_1) = e_2, \sigma_\phi(e_2) = e_1$, then $m_\gamma = 2$, $\lambda_{v_2, \gamma} = 1$ and $\lambda_{v_1, \gamma} = \lambda_{v_1, e_2} = \frac{1}{2} n(B_{v_1}) \leq 3$. Thus

$$|c(\phi, \gamma)| \geq \frac{1}{6}.$$ 

Case (H) We may assume that $\gamma = e_1$.

(H1). If $m_\gamma = 3$, that is, $\sigma_\phi(e_1) = e_2, \sigma_\phi(e_2) = e_3, \sigma_\phi(e_3) = e_1$, then $\lambda_{v_2, \gamma} = 1, \lambda_{v_1, \gamma} = \frac{1}{3} n(B_{v_1}) \leq 2$, so

$$|c(\phi, \gamma)| \geq \frac{1}{6}.$$ 

(H2). If $m_\gamma = 2$, then we may assume that $\sigma_\phi(e_1) = e_2, \sigma_\phi(e_2) = e_1, \sigma_\phi(e_3) = e_3$, and $n(B_{v_2}) = 2$. Thus $\lambda_{v_2, \gamma} = \frac{1}{2} n(B_{v_2}) = 1, \lambda_{v_1, \gamma} = \frac{1}{2} n(B_{v_1}) \leq 3$, so

$$|c(\phi, \gamma)| \geq \frac{1}{6}.$$ 

(H3). If $m_\gamma = 1$, and $\sigma_\phi(e_i) = e_i$ ($i = 2, 3$), then $\lambda_{v_2, \gamma} = 1, \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 6$, so

$$|c(\phi, \gamma)| \geq \frac{1}{6}.$$ 

(H4). If $m_\gamma = 1$, and $\sigma_\phi(e_2) = e_3, \sigma_\phi(e_3) = e_2$, then $\lambda_{v_2, \gamma} = n(B_{v_2}) = 2, \lambda_{v_1, \gamma} = n(B_{v_1}) \leq 6$, so

$$|c(\phi, \gamma)| \geq \frac{1}{6}.$$
Case (I) We may assume that $\gamma = e_1$. In this case, we always have $\lambda_{v_1,\gamma} = \lambda_{v_2,\gamma} \leq 2$, and $m_\gamma \leq 3$. So
\[ |c(\phi, \gamma)| \geq \frac{1}{6}. \]

Case (J) If $\gamma$ is either $e_1$ or $e_2$, then $|c(\phi, \gamma)| \geq \frac{1}{4}$, similarly as Case (I).
Otherwise, we may assume $\gamma$ is $e_3$. If $\sigma_\phi(e_3) = e_3, \sigma_\phi(e_4) = e_4$, then $m_\gamma = 1, \lambda_{v_1,\gamma} \leq 2$ and \( \lambda_{v_2,\gamma} \leq 6 \). If $\sigma_\phi(e_3) = e_4, \sigma_\phi(e_4) = e_3$, then $\sigma_\phi(v_1) = v_2, \sigma_\phi(v_2) = v_1, m_\gamma = 2, \lambda_{v_1,\gamma} = 1$ and $\lambda_{v_2,\gamma} \leq 3$. In both cases, we have
\[ |c(\phi, \gamma)| \geq \frac{1}{6}. \]

Case (K) Let $\gamma$ be any cut curve $e_j$ of type 0. The two end components are both rational. Since the period of each rational component $B_{v_i}$ has $n(B_{v_i}) \leq 2$, $m_\gamma \leq 2$, we have that
\[ |c(\phi, \gamma)| \geq \frac{1}{4}. \]
Similarly, we can obtain the same results for Cases (M) and (N), and $|c(\phi, \gamma)| \geq \frac{1}{8}$ for Case (O). Note that in Case (O), $m_\gamma \leq 4$.

Case (L) We may assume that $\gamma = e_1$.

(L1) If $m_\gamma \geq 3$, then $\lambda_{v_1,\gamma} = \lambda_{v_2,\gamma} = 1$. So
\[ |c(\phi, \gamma)| \geq \frac{1}{4}. \]

(L2) If $m_\gamma \leq 2$, then $\lambda_{v_1,\gamma} = \lambda_{v_2,\gamma} \leq 4$. So
\[ |c(\phi, \gamma)| \geq \frac{1}{6}. \]

\[ \text{Corollary 3.11. Let } f \text{ be a family of curves of genus } 3 \text{ with } \delta_0(f) \neq 0, \text{ then} \]
\[ \delta_0(f) \geq \frac{1}{6}, \]
and the equality holds if and only if all the singular fibers of $f$ have smooth reduction except one whose dual graph is one of figures (3-0a), (3-0b) and (3-0c).

\textit{Proof.} Let $F$ be singular fiber of $f$, then we claim that:

Claim 3: $\delta_0(F) \geq \frac{1}{6}$, and $\delta_0(F) = \frac{1}{6}$ if and only if the dual graph of $F$ is one of figures (3-0a),(3-0b) and (3-0c).

Proof of Claim 3: By Lemma 3.9 and (2.4), we may assume that the dual graph of the stable model $\tilde{F}$ of $F$ is one of (E)--(O). For each of these graphs, there are at least two cut curves of type 0 adjacent to two connected components. So, by (2.4) and Theorem 3.10, we have
\[ \delta_0(F) = \sum_{\gamma \in \Theta_{F,0}} |c(\phi, \gamma)| \geq 2 \cdot \frac{1}{12} = \frac{1}{6}. \]
If $\delta_1(F) = \frac{1}{6}$, then $F$ has exactly two cut curve $\gamma$ of type 0 with $|c(\phi, \gamma)| = \frac{1}{12}$. Thus we obtain Claim 3.

The rest of the proof is similar to that of Corollary 3.3. \qed
Remark 3.12. From the proof of Theorem 3.10 we know that if \( \gamma \) is a cut curve of type 0 in \((F)-(O)\), and \( \gamma \) is adjacent to two connected components, then \(|e(\phi, \gamma)| \geq \frac{1}{6}\). Similarly to the proof of Corollary 3.11 we have that:

If the dual graph of the stable model \( \tilde{F} \) of \( F \) is one of \((F)-(O)\), then \( \delta_0(F) \geq \frac{1}{3} \).

\[ \text{Proof of Theorem 1.8.} \]

Denote by \( f \) the family of curves corresponding to \( C/K \). By the assumption, we know that the semistable model of \( f \) is not smooth. So either \( \delta_0(f) \neq 0 \) or \( \delta_1(f) \neq 0 \). By \cite{Ci11, Theorem 2.4},

\[ \inf_{D \in \text{Div}^1(\bar{C})} a'(D) \geq \frac{1}{2(2g+1)} \left( \frac{(g-1)^2}{2g(7g+5)} \delta_0(f) + \sum_{i \in \{0, g/2, \ldots, g\}} \frac{2i(g-i)}{g} \delta_i(f) \right). \]

If \( \delta_1(f) \neq 0 \), then

\[ \inf_{D \in \text{Div}^1(\bar{C})} a'(D) \geq \begin{cases} \frac{1}{120}, & \text{if } g = 2, \\ \frac{1}{315}, & \text{if } g = 3. \end{cases} \]

If \( \delta_0(f) \neq 0 \), then

\[ \inf_{D \in \text{Div}^1(\bar{C})} a'(D) \geq \begin{cases} \frac{1}{2280}, & \text{if } g = 2, \\ \frac{1}{3276}, & \text{if } g = 3. \end{cases} \]

Comparing these two cases, we get our result.

\[ \square \]

4 Proof of Theorems 1.1, 1.3 and 1.4

Now we can use results in Section 3 to prove Theorem 1.1, Theorem 1.3, and Theorem 1.4.

\[ \text{Proof of Theorem 1.1} \]

Since \( f \) is a family of curves of genus 2, we know that (\cite{Ta10, (4.4)})

\[ \lambda(f) = \frac{1}{10} \delta_0(f) + \frac{1}{5} \delta_1(f), \quad \kappa(f) = \frac{1}{5} \delta_0(f) + \frac{7}{5} \delta_1(f). \]  

(4.1)

Because \( f \) is non-isotrivial, either \( \delta_0(f) \neq 0 \) or \( \delta_1(f) \neq 0 \). So there are the following two cases.

Case 1: \( \delta_0(f) \neq 0 \). By (4.1) and Corollary 3.5, we know that

\[ \lambda(f) \geq \frac{1}{10} \delta_0(f) \geq \frac{1}{10} \times \frac{1}{3} = \frac{1}{30}, \quad \delta(f) \geq \delta_0(f) \geq \frac{1}{3}, \quad \kappa(f) \geq \frac{1}{5} \delta_0(f) \geq \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}. \]  

(4.2)

Case 2: \( \delta_1(f) \neq 0 \). Similarly, we have that

\[ \lambda(f) \geq \frac{1}{5} \times \frac{1}{12} = \frac{1}{60}, \quad \delta(f) \geq \frac{1}{12}, \quad \kappa(f) \geq \frac{7}{5} \times \frac{1}{12} = \frac{7}{60}. \]

From the above two cases, it is easy to see that

\[ \lambda(f) \geq \min\{\frac{1}{30}, \frac{1}{60}\} = \frac{1}{60}, \quad \delta(f) \geq \min\{\frac{1}{3}, \frac{1}{12}\} = \frac{1}{12}, \quad \kappa(f) \geq \min\{\frac{1}{15}, \frac{7}{60}\} = \frac{1}{15}. \]

Moreover, \( \lambda(f) = \frac{1}{60} \) if and only if \( \delta(f) = \frac{1}{12} \) if and only if \( \delta_1(f) = \frac{1}{12} \). So we obtain Theorem 1.1 (1) by Corollary 3.3. Similarly, we can get Theorem 1.1 (2).

Now we have completed the proof except that each equality of (1.1) can be reached, which will be proved by examples in Section 5.

\[ \square \]
Remark 4.1. From Theorem 1.1 we know that if $f$ has $\lambda(f) = \frac{1}{160}$, then $\delta(f) = \frac{1}{112}$, and $\kappa(f) = \frac{7}{60} \neq \frac{1}{15}$ by Noether equality. Hence there does not exist a non-isotrivial family $f$ such that both $\lambda(f)$ and $\kappa(f)$ are minimal.

In order to see our method for $g \geq 3$ clearly, we prove Theorem 1.4 first.

Proof of Theorem 1.4 Since $\delta(f) \neq 0$, there exists $0 \leq i \leq \lfloor g/2 \rfloor$ such that $\delta_i(f) \neq 0$. From [LT17, Theorem 1.4], we know that if $\delta_i(f) \neq 0$, then

$$\delta_i(f) \geq \begin{cases} \frac{1}{4g^2}, & \text{if } i = 0, \\ \frac{1}{(4i+2)(4(g-i)+2)}, & \text{if } i \geq 1. \end{cases}$$

In this proof, we use the following Moriwaki’s inequality (see [Mo98, Theorem D])

$$(8g + 4)\lambda(f) \geq g\delta_0(f) + \sum_{i=1}^{\lfloor g/2 \rfloor} 4i(g-i)\delta_i(f). \quad (4.3)$$

If $\delta_0(f) \neq 0$, then, by (4.3),

$$\lambda(f) \geq \frac{1}{8g + 4} \cdot g \cdot \frac{1}{4g^2}.$$

If $\delta_i(f) \neq 0$ for some $i \geq 1$, then

$$\lambda(f) \geq \frac{1}{8g + 4} \cdot 4i(g-i) \cdot \frac{1}{(4i+2)(4(g-i)+2)}.$$

Hence, combining all these cases, we have that

$$\lambda(f) \geq \frac{1}{8g + 4} \cdot \min \left\{ g \cdot \frac{1}{4g^2}, \frac{4(g-1)}{(4+2)(4(g-1)+2)}, \ldots, \frac{4\lfloor \frac{g}{2} \rfloor (g-\lfloor \frac{g}{2} \rfloor)}{(4\lfloor \frac{g}{2} \rfloor + 2)(4(g-\lfloor \frac{g}{2} \rfloor)+2)} \right\} \geq \frac{1}{16g(2g+1)}.$$

By Cornalba-Harris-Xiao’s slope inequality ([Xi87, Theorem 6.1.5]), we have that

$$\kappa(f) \geq \frac{4g - 4}{g} \lambda(f) \geq \frac{g-1}{4g^2(2g+1)}.$$

\[ \square \]

Applied the proof of Theorem 1.4 to the case $g = 3$, we get Theorem 1.3 by Corollary 3.8 and Corollary 3.11 directly. So we omit the proof of Theorem 1.3.

5 Proof of Theorem 1.2

Before the proof of Theorem 1.2 we introduce the notation of ramification index.

A reduced divisor $D$ of $S$ is called vertical, if $f(D)$ is a point. If $D$ contains no vertical component, then $f$ induces a morphism $\phi : D \to C$. Let

$$\rho = \rho_1 \circ \cdots \circ \rho_r : (\tilde{S}, \tilde{D}) \rightarrow (S_r, D_r) \rightarrow \cdots \rightarrow (S_1, D_1) \rightarrow (S_0, D_0) = (S, D)$$
be the resolution of $D$, where $D_i$ is the strict transform of $D_{i-1}$, $\tilde{D}$ is smooth, and $\rho_i$ is a blow-up at a singularity of $D_{i-1}$ with multiplicity $m_i$. Then the relative ramification index of $\phi$ is defined to be

$$r(D) := \deg \tilde{R} + \sum_{i=1}^{r} m_i(m_i - 1),$$  \hspace{1cm} (5.1)$$

where $\deg \tilde{R}$ is the ramification index of the induced morphism $\tilde{\phi} : \tilde{D} \to C$. Then

$$r(D) = K_{S/C} + D^2,$$  \hspace{1cm} (5.2)$$

which is a generalized Riemann-Hurwitz formula (see [Xi92, Lemma 2.4.8]).

Now we give our examples.

**Proposition 5.1.** There is a family of fibrations $(f_{\lambda,n} : S_n \to \mathbb{P}^1)_{n \in \mathbb{N}}$ of genus 2 with

$$\lambda(f_{\lambda,n}) = \frac{1}{60}, \quad \delta(f_{\lambda,n}) = \frac{1}{12}.$$  

**Proof.** Let $\Gamma_t$ be the fiber over $t \in \mathbb{P}^1$ of the second projection $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, $p_2((x,t)) = t$.

Let $R_h$ be the divisor on $\mathbb{P}^1$, whose affine equation is

$$h(x,t) = x^6 + (15x^4 + 40x^3)t - (45x^2 + 24x)t^2 + 5t^3.$$  \hspace{1cm} (5.3)$$

Let $R_{\lambda,n} = R_h + \Gamma_{\infty} + \sum_{i=1}^{2n} \Gamma_i$, where $n \geq 0$ and $\Gamma_i$'s are generic fibers of $p_2$. Here, when $n = 0$, the sum means that there is no generic fiber. Then there is an invertible sheaf $\delta R_{\lambda,n}$ with $O_P(R_{\lambda,n}) \cong \delta^2 R_{\lambda,n}$, and there is a double cover $\pi_n : S'_n \to P$ whose branch locus is $R_{\lambda,n}$, see [BPV84, §I17]. Taking birational transforms, we can obtain a relative minimal fibration $f_{\lambda,n} : S_n \to \mathbb{P}^1$ induced by the second projection $p_2$.

**Case 1:** $n = 0$. Denote $f_{\lambda,0}$ (resp. $R_{\lambda,0}$) by $f_{\lambda}$ (resp. $R_{\lambda}$) for brief.

**Claim A:** There are exactly three singular fibers $F_0 = f_{\lambda}^{-1}(0)$, $F_{-1} = f_{\lambda}^{-1}(-1)$ and $F_{\infty} = f_{\lambda}^{-1}(\infty)$ in $f_{\lambda}$. Moreover, $F_0$ is the fiber in Theorem 1.1, $F_{-1}$ is of type [VIII-1] and $F_{\infty}$ is of type [II], see [NU73] for the notations.

Assume Claim A firstly, then we have that

$$\lambda(f_{\lambda}) = \frac{1}{60}, \quad \delta(f_{\lambda}) = \frac{1}{12},$$

by Theorem 1.1 for both $F_{-1}$ and $F_{\infty}$ have smooth reduction. Furthermore, $\kappa(f_{\lambda}) = \frac{7}{60}$ by the Noether equality $12\lambda(f_{\lambda}) = \kappa(f_{\lambda}) + \delta(f_{\lambda})$.

**Case 2:** $n \geq 1$.

Comparing $R_{\lambda,n}$ with $R_{\lambda}$, it is easy to see that $f_{\lambda,n}$ has $2n + 3$ singular fibers, three of them are the same as singular fibers of $f_{\lambda}$ and the rest are all of type $[I_{0-0-0}]$. Hence

$$\lambda(f_{\lambda,n}) = \frac{1}{60}, \quad \kappa(f_{\lambda,n}) = \frac{7}{60}, \quad \delta(f_{\lambda,n}) = \frac{1}{12},$$

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by Theorem 1.1.

For each integer $n > 0$, the family $f_{\lambda,n}$ is the same as $f_{\lambda}$ except a finite fibers. So the image of $f_{\lambda,n}$ in $\M_g$ induced by the moduli map $J_{f_{\lambda,n}} : \P^1 \to \M_g$ is the same as that of $f_{\lambda}$. Hence we will complete our proof after proving Claim A.

Proof of Claim A: See Figure 1 for the branch locus $R_{\lambda}$ in $P$.

Denote by $r_i(R_h)$ the contribution of the point $(x, t) = (i, i)$ ($i = -1, 0, \infty$) to the relative ramification index $r(R_h)$.

$F_0$: Let $p$ be the point $(x, t) = (0, 0)$. The local equation of $R_h$ near $p$ is $h(x, t)$ in (5.3), thus

1. The root $x = 0$ of $h(x, 0) = 0$ is with multiplicity 6.
2. The point $p$ is a singularity of $R_h$ with multiplicity $m_1 = 3$, and the vertical direction is a tangent line of $R_h$ with multiplicity 2.
3. From the following figure of resolution of $p$, we have that $r_0(R_h) = m_1(m_1 - 1) + m_2(m_2 - 1) + 3 = 11$, where 3 comes from the contribution of smooth ramification points (see (5.1)).

Hence the dual graph of $F_0$ is Figure 2-1b.

$F_{-1}$: The local equation of $R_h$ near $(x, t) = (-1, -1)$ is

$$h_{-1}(u, s) := h(u - 1, s - 1) = u^6 - 6u^5 + (15u^4 - 20u^3 + 60u^2 - 72u + 32)s + (-45u^2 + 66u - 36)s^2 + 5s^3.$$ So $R_h$ is smooth near $(u, s) = (0, 0)$, and $u = 0$ is a root of $f(u, 0) = u^6 - 6u^5$ with multiplicity 5. Thus $r_{-1}(R_h) = 4$. 

Figure 1: Branch locus of $f_{\lambda}$
The local equation of $R_h$ near $(x, t) = (-1, -1)$ is the same as $y^2 = x^5 + t$, and $F_{-1}$ is of type [VIII-1] whose dual graph is Figure 2(a). (See Figure 2 where $\bullet$ denotes a smooth elliptic curve.)

Figure 2: Singular fibers in fibrations with minimal modular invariants

$F_{\infty}$: Let $q$ be the point $(x, t) = (\infty, \infty)$. The local equation of $R_h$ near $q$ is

$$h_{\infty}(w, r) := w^6 r^3 (w, r) = 5w^6 - (24w^5 + 45w^4)r + (40w^3 + 15w^2)r^2 + r^3. \quad (5.4)$$

Then we know that

1. The root $w = 0$ of $h_{\infty}(w, 0) = 0$ is of multiplicity 6.
2. The point $q$ is a singularity of $R_h$ with multiplicity $m_1 = 3$, and the vertical direction is a tangent line of $R_h$ with multiplicity 3.
3. From the following figure of resolution of $q$, we have that $r_{\infty}(R_h) = m_1(m_1 - 1) + m_2(m_2 - 1) + 3 = 15.$

Hence the local equation of $R_h$ near $q$ is the same as $y^2 = t\prod_{i=1}^{3}(x^2 + \alpha_it)$, $F_{\infty}$ is of type [II] and the dual graph of $F_{\infty}$ is Figure 2(b).

Now we know that the relative ramification of $R_h$ is

$$r(R_h) = K_{P/P^1} R_h + R_{h}^2 = 30 \geq r_{-1}(R_h) + r_{0}(R_h) + r_{\infty}(R_h) = 30.$$ 

So $f_\lambda$ has no other singular fibers.

Proposition 5.2. There is a family of fibrations $(f_{\kappa,n} : X_n \to \mathbb{P}^1)_{n \in \mathbb{N}}$ of genus 2 with $\kappa(f_{\kappa,n}) = \frac{1}{15}$, $\lambda(f_{\kappa,n}) = \frac{1}{30}$, $\delta(f_{\kappa,n}) = \frac{1}{3}$.

Proof. This proof is similar to that of Proposition 5.1.

Let $\Gamma_t$ be the fiber over $t \in \mathbb{P}^1$ of the second projection

$$p_2 : P = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad p_2((x, t)) = t.$$ 

Let $R_g$ be the divisor on $P$, whose affine equation is

$$g(x, t) = 5x^6 - 18x^5 + (15x^4 + 20x^3)t + (-45x^2 + 30x - 16)t^2 + 9t^3. \quad (5.5)$$
Let $R_{\kappa,n} = R_g + \Gamma_{\infty} + \sum_{i=1}^{2n} \Gamma_i$, where $n \geq 0$ and $\Gamma_i$'s are generic fibers of $p_2$. Combining with the second projection $p_2$, let $f_{\kappa,n} : X_n \to \mathbb{P}^1$ be the relative minimal fibration determined by the double cover over $P$ whose branch locus is $R_{\kappa,n}$.

Case 1: $n = 0$. Denote $f_{\kappa,0}$ (resp. $R_{\kappa,0}$) by $f_{\kappa}$ (resp. $R_{\kappa}$) for brief.

Claim B: There are exactly three singular fibers $F_1 = f_{\kappa}^{-1}(1)$, $F_0 = f_{\kappa}^{-1}(0)$ and $F_{\infty} = f_{\kappa}^{-1}(\infty)$ in $f_{\kappa}$. Moreover, $F_1$ is the fiber in Theorem 1.1, $F_0$ is of type [IX-1] and $F_{\infty}$ is of type [II], see [NU73] for the notations.

Assume Claim B firstly, then we have that

$$\kappa(f_{\kappa}) = \frac{1}{15}, \quad \delta(f_{\kappa}) = \frac{1}{3},$$

by Theorem 1.1 for both $F_0$ and $F_{\infty}$ have smooth reduction. Furthermore, $\lambda(f_{\kappa}) = \frac{1}{30}$ by the Noether equality.

Case 2: $n \geq 1$.

Comparing $R_{\kappa,n}$ with $R_{\kappa}$, it is easy to see that $f_{\kappa,n}$ has $2n + 3$ singular fibers, three of them are the same as singular fibers of $f_{\kappa}$ and the others are all of type $[I_0 - 0 - 0]$. Hence

$$\lambda(f_{\kappa,n}) = \frac{1}{30}, \quad \kappa(f_{\kappa,n}) = \frac{1}{15}, \quad \delta(f_{\kappa,n}) = \frac{1}{3},$$

by Theorem 1.1.

For each integer $n > 0$, the family $f_{\kappa,n}$ is the same as $f_{\kappa}$ except a finite fibers. So the image of $f_{\kappa,n}$ in $\mathcal{M}_g$ induced by the moduli map is the same as that of $f_{\kappa}$. Hence we will complete the proof after proving Claim B.

Proof of Claim B: See Figure 3 for the branch locus $R_{\kappa}$ in $P$.

![Figure 3: Branch locus of $f_{\kappa}$](image)

Denote by $r_i(R_g)$ the contribution of the point $(x, t) = (i, i)$ $(i = 0, 1, \infty)$ to the relative ramification index $r(R_g)$.

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\textbf{F}_0: \text{ Let } p \text{ be the point } (x, t) = (0, 0). \text{ The local equation of } R_g \text{ near } p \text{ is } g(x, t) \text{ in (5.5)}, \text{ then we know that }

1. The root } x = 0 \text{ of } g(x, 0) = 0 \text{ is with multiplicity } 5.

2. The point } p \text{ is a singularity of } R_g \text{ with multiplicity } m_1 = 2, \text{ and the vertical direction is a tangent line of } R_g \text{ with multiplicity } 2.

3. From the following figure of resolution of } p, \text{ we have that } r_0(R_g) = m_1(m_1 - 1) + m_2(m_2 - 1) = 8, \text{ where } 4 \text{ comes from the contribution of smooth ramification points.}

Hence the local equation of } R_g \text{ near } p \text{ is the same as } y^2 = x^5 + t^2, \text{ } F_0 \text{ is of type } [\text{IX} - 1] \text{ and the dual graph of } F_0 \text{ is Figure 2(c).}

\textbf{F}_1: \text{ Let } q \text{ be the point } (x, t) = (1, 1). \text{ The local equation of } R_g \text{ near } q \text{ is }

\[ g_1(u, s) := g(u + 1, s + 1) = 5u^6 + 12u^5 + (15u^4 + 80u^3 + 60u^2)r - (45u^2 + 60u + 4)r^2 + 9r^3. \]

Then we know that

1. The root } u = 0 \text{ of } g_1(u, 0) = 0 \text{ is of multiplicity } 5.

2. The point } q \text{ is a singularity of } R_g \text{ with multiplicity } m_1 = 2, \text{ and the vertical direction is a tangent line of } R_g \text{ with multiplicity } 2.

3. From the following figure of resolution of } q, \text{ we have that } r_1(R_g) = m_1(m_1 - 1) + m_2(m_2 - 1) + 3 = 7, \text{ where } 3 \text{ comes from the contribution of smooth ramification points.}

Hence the local equation of } R_g \text{ near } q \text{ is the same as } y^2 = (x^2 + t)(x^3 + t), \text{ and the dual graph of } F_1 \text{ is Figure (2-0a).}

\textbf{F}_\infty: \text{ The local equation of } R_g \text{ near } (x, t) = (\infty, \infty) \text{ is }

\[ g_\infty(w, r) = w^6r^3g(1/w, 1/r) = 9w^6 + (-16w^6 + 30w^5 - 45w^4)r + (20w^3 + 15w^2)r^2 + (-18w + 5)r^3. \] \hfill (5.6)

It is easy to see that this is the same as } F_\infty \text{ in Proposition 5.1. In particular, } r_\infty(R_g) = 15.

Now we know that the relative ramification of } R_g \text{ is }

\[ r(R_g) = K_P/F_1 R_g + R_g^2 = 30 \geq r_0(R_g) + r_1(R_g) + r_\infty(R_g) = 30. \]

So } f_\kappa \text{ has no other singular fibers.}
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References

[AI02] T. Ashikaga, M. Ishizaka: Classification of degenerations of curves of genus three via Matsumoto-Montesinos' theorem, Tohoku Math. J. 54 (2) (2002), 195-226.

[BPV84] W. Bath, C. Peters, A. Van de Ven: Compact complex surfaces, Springer-Verlag, 1984.

[CI11] Z. Cinkir: Zhang’s conjecture and the effective Bogomolov conjecture over function fields, Invent. Math. 183 (3) (2011), 517-562.

[Fa84] G. Faltings: Calculus on arithmetic surfaces, Ann. Math. 119 (1984), 387-424.

[GO89] D. Gabai, U. Oertel: Essential laminations in 3-manifolds, Ann. Math. 130 (1) (1989), 41-73.

[HM18] M. Hedden, T. Mark: Floer homology and fractional Dehn twists, Adv. Math. 324 (2018), 1-39.

[HKM07] K. Honda, W. Kazez and G. Matic: Right-veering diffeomorphisms of compact surfaces with boundary, Invent. Math. 169 (2007), 427-449.

[Im09] Y. Imayoshi: A construction of holomorphic families of Riemann surfaces over the punctured disk with given monodromy, In Handbook of Teichmüller theory, Vol II, vol 13 of IRMA Lect. Math. Theor. Phys., 93-130, Eur. Math. Soc., Zürich, 2009.

[IK17] T. Ito, K. Kawamuro: Essential open book foliation and fractional Dehn twist coefficient, Geom. Dedicata 187 (2017), 17-67.

[Ja14] A. Javanpeykar: Polynomial bounds for Arakelov invariants of Belyi curves. With an appendix by Peter Bruin, Algebra Number Theory 8 (1) (2014), 89-140.

[KR13] W. Kazez, R. Roberts: Fractional Dehn twists in knot theory and contact topology, Algebr. Geom. Topol. 13 (6) (2013), 3603-3637.

[Liu] X.-L. Liu: Fractional Dehn twists and modular invariants, http://arxiv.org/abs/1912.13236.

[Li16] X.-L. Liu: Modular invariants and singularity indices of hyperelliptic fibrations, Chin. Ann. Math. Ser. B 37 (2016), 875-890.

[LT17] X.-L. Liu, S.-L. Tan: Uniform bound for the effective Bogomolov conjecture, C. R. Acad. Sci. Paris, Ser. I 355 (2) (2017), 205-210.
[Li17] Y. Liu: *A characterization of virtually embedded subsurfaces in 3-manifolds*, Trans. Amer. Math. Soc. **369** (2) (2017), 1237-1264.

[MM11] Y. Matsumoto, J. M. Montesinos-Amilibia: *Pseudo-periodic maps and degeneration of Riemann surfaces*, Lecture Notes in Mathematics **2030**, Springer-Verlag, 2011.

[Mo98] A. Moriwaki: *Relative Bogomolov’s inequality and the cone of positive divisors on the moduli space of stable curves*, J. Amer. Math. Soc. **11** (1998), 569-600.

[NU73] Y. Namikawa, K. Ueno: *The complete classification of fibres in pencils of curves of genus two*, Manus. Math. **9** (1973), 143-186.

[Ta94] S.-L. Tan: *On the base changes of pencils of curves, I*, Manus. Math. **84** (1994), 225–244.

[Ta96] S.-L. Tan: *On the base changes of pencils of curves, II*, Math. Z. **222** (1996), 655–676.

[Ta10] S.-L. Tan: *Chern numbers of a singular fiber, modular invariants and isotrivial families of curves*, Acta Math. Viet. **35** (1) (2010), 159-172.

[Xi87] G. Xiao: *Fibered algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449-466.

[Xi92] G. Xiao: *The fibrations of algebraic surfaces*, Shanghai Scientific & Technical Publishers, 1992 (in Chinese).

[Wi16] R. Wilms: *New explicit formulas for Faltings delta-invariant*, Invent. Math. **209** (2) (2017), 481-539.

[Zh96] S.-W. Zhang: *Heights and reductions of semi-stable varieties*, Compositio Math. **104** (1996), 77-105.

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