Point-free theories of space and time

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Abstract

The paper is in the field of Region Based Theory of Space (RBTS), sometimes called mereotopology. RBTS is a kind of point-free theory of space based on the notion of region. Its origin goes back to some ideas of Whitehead, De Laguna and Tarski to build the theory of space without the use of the notion of point. More information on RBTS and mereotopology can be found, for instance, in [73]. Contact algebras present an algebraic formulation of RBTS and in fact give axiomatizations of the Boolean algebras of regular closed sets of some classes of topological spaces with an additional relation of contact. An exhaustive study of this theory is given in [22]. Dynamic contact algebra (DCA) [76] (see also [74, 75]) introduced by the present author, is a generalization of contact algebra studying regions changing in time and presents a formal explication of Whitehead’s ideas of integrated point-free theory of space and time. DCA is an abstraction of a special dynamic model of space, called also snapshot or cinematographic model and the paper [76] contains the expected representation theorem with respect to such models. In the present paper we introduce a new version of DCA which is a simplified version of the definition from [76] and similar to that of [75]. The aim is to use this version as a representative example of a DCA and to develop for this example not only the

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snapshot models but also topological models and the expected topological duality theory, generalizing in a certain sense the well known Stone duality for Boolean algebras. Abstract topological models of DCAs present a new view on the nature of space and time and show what happens if we are abstracting from their metric properties.

**Keywords**: Boolean algebra, clan, cluster, (pre)-contact relation, dynamic mereotopology, (Stone-type)-duality, regular-closed set, space-time, temporal relation, ultrafilter.

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**Preface**

The present work can be considered as a continuation of the essay "Region-Based Theory of Space: Algebras of Regions, Representation Theory and Logics" ([73]). The essay contains a short history of the Region-Based Theory of Space (RBTS) and a survey of the corresponding literature (till 2006), an exposition of the mathematical apparatus of this approach based on contact algebras and a description of some propositional spatial logics related to RBTS. In this approach "region-based" means that the notion of region, taken as an abstraction of material or geometric body, is considered as one of the base notions of the theory. The theory is also "point-free" in a sense that the typical geometric notion of "point" is not considered as a primitive (undefinable) notion of the theory and should be defined in a later stage of the theory. Later on we consider RBTS and "point-free theory of space" as synonyms.

The motivation of the point-free approach to the theory of space was formulated for the first time by Alfred North Whitehead in 1915 in his lecture Space, Time, and Relativity (published as chapter VIII of [83]). In the same lecture Whitehead also claims that the same approach should also be applied to the theory of time, and, motivated by the relativity theory, that the theory of time should not be developed separately from the theory of space and they both should be developed in one integrated point-free theory of space and time. In this context "point-free" means that neither space points, nor time points (instances of time, moments) are considered as primitive notions of the theory.

The present essay is devoted mainly to the point-free theories of space and time and so the title. Point-free theories of space and time are also "region-
based” because they consider changing or moving regions. So, we consider also another equivalent name: Region-Based Theory of Space and Time - RBTST.

The text of the paper is structured as follows. Section 1 is the Introduction. We start with some discussion about point free theory of space and time and present with more details the discussions about the nature of space and time between Leibnitz and Newton, Leibnitz’ ”relational” view on space and time and Newton’s ”absolute space” and ”absolute time”. We consider the Whitehead’s viewpoint on this subject and his motivations why the theory of space and time should be ”point-free” and ”region-based”. We describe shortly Whitehead’s contributions to this idea and some other sources and finally we present our concrete strategy of how to build an integrated point-free theory of space and time. In Section 2 we summarize some facts of contact algebras and precontact algebras taken from [21, 73, 26] to be used later on. In Section 3 we introduce a concrete point-based model of dynamic space called snapshot model or cinematographic model. This model is used as a source of motivated axioms for a various versions of the abstract notion of dynamic contact algebra. Section 4 is devoted to the abstract notion of one special version of dynamic contact algebra (DCA), considered as a representative example of DCA. In Section 5 we introduce topological point-based models called dynamic mereotopological spaces (DMS) and develop the intended topological representation theory. Section 6 is devoted to the expected topological duality theory for DCAs and DMSes, generalizing the famous Stone Duality Theorem for Boolean algebras. Section 7 is for some conclusions, discussions and open problems. In a separate Appendix we present a short survey of results on RBTS obtained after 2007 making in this way a more close connection with the present essay [73].

We consider [66], [32] and [53] as standard reference books correspondingly for Boolean algebras, topology and category theory.

1 Introduction

1.1 Point-based and point-free theories of space and time

In mathematics the theory of space is identified with geometry which includes various geometrical disciplines. Well known example is the classical Euclidean geometry. Typical for all axiomatically presented geometries is that they follow the standard Euclidean approach to consider the notion of
"point" as one of the basic undefinable notions of the theory and similarly for the notions "straight line" and "plain". Sometimes straight lines and plains are considered as certain sets of points satisfying some additional axioms, so, point in geometry is always a primitive notion. But neither points, nor straight lines and plains have a separate existence in reality, so the truths for these notions do not correspond to some observational truths for the real things. In a sense "points", "straight lines" and "plains" are some kind of suitable fictions and it is not good to put fictions on the base of the so respectable mathematical theory as geometry, considered as a certain theory of reality. This issue gives rise to serious discussions, which we will comment on below.

So, what is a point-free theory of space? Contemporary example is the point-free topology [45]. Standardly topology is considered as an abstract theory of space formalizing the notion of continuity and is considered as a set of points with some distinguished subsets called open sets. Instead, point-free topology is based on lattice theory considering the members of the lattice representing open sets. In general by a point-free theory of space we mean an axiomatic theory of space in which the notion of point is not assumed as a primitive notion. For a given (point-based) geometry, for instance Euclidean geometry, its point-free reformulation means it to be reaxiomatized equivalently on a point-free basis of primitive notions. This means that points are not disregarded at all but are given by certain definitions in the new axiomatization. Among the first authors who criticized the standard Euclidean point-based approach to the theory of space and appealing to a point-free bases for the theory I can mention Whitehead [83, 84, 85, 86, 87], De Laguna [48, 49, 50] and Tarski [69].

According to time we can say that there is no specific pure mathematical area like geometry, which is devoted exclusively to the theory of time. Only some investigations on temporal logic (TL) (see, for instance, [5]) introduced the so called time structures devoted to a separate study of time. Time structures are systems in the form \((T, \prec)\), where \(T\) is a nonempty set whose elements are called "time points" or "moments of time" and \(\prec\) is a binary relation between time points called "before-after" relation, reading: \(i \prec j\) - \(i\) is before \(j\), or equivalently \(j\) is after \(i\) (other relations between time points are also possible). Such structures are studied to be used as a semantics of TL. The before-after relation may satisfy various sets of some meaningful conditions which fact makes possible to have various different time structures and hence different TL systems. If, for instance, \(T\) is the set of real numbers and \(\prec\) is the strong inequality \(<\), then \((T, \prec)\) is called "real time structure",
and similarly for "rational" or "integer (discrete) time structure". Thus, by definition all temporal structures of the above kind are point-based. But moments of time, like space points, also are some abstract fictions - we can not see in reality a time moment. So the problem to avoid time points in TL also exists. And indeed there are TL systems with a more realistic semantics based on time intervals and some relations between them according to their possible positions to each other. However, the intuition of time intervals and their interrelations is based on their representation as ordered pairs of time points \((x, y)\) such that \(x \prec y\) and \(x \neq y\), and \(x, y\) taken from some linearly ordered time structure (for instance real numbers). So, time intervals and their interrelations again are reduced to time points. There is also a point of view to consider interval structures as intuitively more clear and extract from their structure the notion of time point and a kind of before-after relation. But time intervals are also "suitable fictions", abstract fictions having no separate existing in reality, so the above criticism also holds.

Both time and space are central notions in physics, but physics takes his mathematical apparatus from mathematics (unless we can treat mathematical physics just as a part of mathematics). Newtonian physics adopts, for instance, Newtonian notions of absolute space and absolute time considered them independent from the material things, independent from each other and having a separate existence in reality (see for this view, for instance [31, 44]). In relativistic physics space and time are not independent and are considered as one spacetime system. In special relativity, this is the Minkowski spacetime in which points are called events and are identified with tuples of real numbers \((x_1, x_2, x_3, x_4)\) where \(x_1, x_2, x_3\) are meant as space coordinates of the event and \(x_4\) is meant as its time coordinate. So in Minkowski spacetime time is the fourth coordinate, which makes the system to be four dimensional with 3 spatial dimensions and one time dimension. Minkowski spacetime differs from the 4-dimensional Euclidean space because it has a different metrics convenient for describing special relativity in which gravitation is not considered.

An axiomatic presentation of Minkowskian spacetime geometry is given by A. A. Robb in [63]. Robb’s system has only two primitive notions: "instant" intuitively meant as a spacetime point and the "before-after" relation between spacetime points interpreted intuitively as a causal ordering of things. Robb named his relation "after" and its converse "before" and presented for it an appealing illustration by means of the Euclidean conic model of 3-dimensional Minkowski spacetime, which motivated him to call the relation "after" conic order. Because "after" is a temporal relation and space fea-
tures (as well as all other notions of the system) are definable by it, this fact motivates Robb to state that time is more fundamental than space and to call his system "geometry of time and space" putting time on the first place. Probably this shows in a certain sense that both time and space are based on a more deep concept like causality. Spacetime systems based on before-after relation interpreted as a causality relation are called causality theories of spacetime (see, for instance [89]).

A readable axiomatic treatment of Minkowski spacetime and some related spacetimes based on a more natural and classically oriented basis of primitive concepts is given by R. Goldblatt in [34]. Modal logics with a relational semantics based on some versions of Minkowski spacetime relation "after" are also studied - see Goldblatt [33] and Shehtmann [65].

General relativity theory is a generalization of special relativity by assuming the effects of gravitation. An intensive research on axiomatic foundations of relativity theories is initiated by a Hungarian group of logicians organized by I. Nemeti and H. Andreka [2]. But, let us note again, both Newtonian and relativistic spacetime theories are not point-free and the problem of their point-free reformulation is still open (the situation in quantum physics is still unclear).

Spacetime systems in which space and time are considered together like in relativity theory are used in applied mathematics for describing certain systems of dynamically changing spatial objects. Such spacetime systems are combinations of some spatial structure (geometry) and some temporal structure (theory of time). For one such construction of concrete spacetime system see, for instance, [47]. It was based on the so called snapshot construction and it is natural to be named snapshot spacetime. As a rule such spatio-temporal systems are also point-based, so their point-free re axiomatization is an open problem. Later on we will discuss such systems with more details and will consider them as a starting point for various versions of an integrated point-free theory of space and time.

1.2 "Relational theory of space and time", a discussion between Newton and Leibniz, Whitehead’s view and his program for building a "point-free theories of space and time"

The question of whether points of space and time have to be considered as real things, raises hot philosophical discussions and puts the more serious
question whether space and time itself are also "suitable fictions". A typical example is the discussion between Leibnitz and Newton about the nature of space and time. Leibnitz’ position is known now as the "relational view of space and time": space and time are mathematical fictions and the things in reality are connected by some spacetime relations and the mathematical theories of space and time just describe the properties of these relations. Space expresses the coexistence of things, while time expresses an order of successive things. Newton’s position advocates the view of "absolute space" and "absolute time" discussed in the previous section (for more details about the discussion between Leibniz and Newton see, for instance, [31, 44]).

At the beginning of 20 Century probably the first who adopted in some form Leibnitz’ relational view of space and time and formulated the problem of its correct mathematical reinterpretation as a point-free theory of space and time was Alfred North Whitehead.

Whitehead is well known among logicians as a co-author with Bernard Russell in their famous book Pricipia Mathematica, published in three volumes in 1910-1913 and dedicated to the foundation of mathematics [88]. It is said in the preface of volume III of the book that geometry is reserved for the final volume IV. But probably due to some disagreements between the authors about the nature of space (and probably of time), volume IV had not been written.

The best articulation of the original Whitehead’s view about space and time is given in the following quote (pages 194,195 of [83]) of Whitehead’s lecture *Space, Time, and Reality*:

"...We may conceive of the points of space as self-subsistent entities which have the indefinable relation of being occupied by the ultimate stuff (matter, I will call it) which is there. Thus, to say that the sun is there (wherever it is) is to affirm the relation of occupation between the set of positive and negative electrons which we call the sun and a certain set of points, the points having an existence essentially independent of the sun. This is the absolute theory of space. The absolute theory is not popular just now, but it has very respectable authority on its side Newton, for one so treat it tenderly. The other theory is associated with Leibnitz.

Our spare concepts are concepts of relations between things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between the matter which is, in common language, said to be in space."
It follows from the relativity theory that a point should be definable in terms of the relations between material things. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics, in much the same state as was the theory of parallels in the eighteenth century."

It can be concluded from this quote that Whitehead accepted Leibnitz' "relational theory of space and time" in a more relaxed form: we have to build the theory of space staring from more realistic primitive notions avoiding points, lines and plains and introducing them by suitable definitions. From his other writings, for instance from his main philosophical book Process and Reality [87] (which we will discuss with more details after words) such more realistic notions are regions as abstractions of material bodies and some natural relations between them. In contemporary terminology the above quote is nothing but a program for building of a point-free theory of space, and also for building of an integrated point-free theory of space and time as it is considered in relativity theory. From the phrase

"This is an unwritten chapter of mathematics, in much the same state as was the theory of parallels in the eighteenth century"

we may conclude that Whitehead considered this as a difficult and a serious problem. This problem has two forms, first, concerning only space, and second, concerning both space and time taken together. Since geometry as a theory of space exists as a branch of mathematics separately from the theory of time, this is the problem to build the point-free theory of space. And since the theory of time appeared mostly in mathematical physics as an integrated theory of space and time - this is just the related problem to build an integrated point-free theory of space and time.
1.3 Whitehead’s contribution and other roots of building of point-free theories of space and time

In the lecture *The Anatomy of Some Scientific Ideas* (Chapter VII in the same book cited above [84]) Whitehead describes, among others, how such a "point-free theory" should be build. First he considers as a base notion the notion of "event" a feature existing in space and in time. Second, the theory should be based on the theory of "whole and a part" (named by other authors mereology - see, for instance [67] and more recently [61]) and definitions of the "points of time" and "points of space" to be done by his "principle of convergence", renamed in his later publications by "the method of extensive abstraction".

An attempt to build such a theory is given in the Whitehead’s books [84] and [85]. This attempt was criticized from philosophical and from methodological points of view by De Laguna in the papers [48, 49, 50], where he presented his own approach for point-free theory of space based on mereology. De Laguna’s system has the primitives "solid" as an abstraction of physical body and a ternary relation between solids named "can connect". Intuitively the solids $a$, $b$ and $c$ are in the relation "can connect" if $a$ can be "moved" so that "to connect" $b$ and $c$. Here "to connect" means to touch or to overlap. De Laguna showed how to define points, lines and surfaces using a modification of Whitehead’s method of extensive abstraction. We will not comment De Laguna’s critical remarks, but it have to be mentioned that Whitehead considered them seriously and changed radically his system, published in Process and Reality (P&R) [87] (see page 440 of P&R [87] where Whitehead correctly gives credits to De Laguna’s criticism and comments how to avoid the defects of his approach to the definition of point presented in [84] and [85]. Instead of De Laguna’s notion of "solid" Whitehead uses the term "region" with the same intuitive meaning, and instead of the De Laguna’s ternary relation "can connect" he used the simplified binary relation of connection (called in the recent literature contact). The main idea of Whitehead’s new approach is described in Part IV of the book - "The theory of extension" and the mathematical details are presented in Chapters II and III of P&R. The exposition is almost mathematical and consists of a series of enumerated definitions and assumptions without any attempt "to reduce these enumerated characteristics to a logical minimum from which the remainder can be deduced by strict deduction" (p. 449).

By means of the connection relation, Whitehead defines in Chap. II some other relations between regions: part-of, overlap, external connection, and
tangential inclusion. Chapter II ends with the definition of a point (Def. 16). Chapter III contains all preliminary formal definitions and assumptions needed in the definitions of a straight line (Def. 6) and definition of of a plane (Def. 8) as certain sets of regions using the method of extensive abstraction. Because the text is sketchy these two chapters of P&R have to be considered as an extended program containing all needed details in order to develop Whitehead’s new theory of space in a strictly mathematical manner. Namely, this is what is called now the root of ”region-based theory of space” (RBTS), or equivalently - point-free theory of space. Another root is, of course, De Laguna’s papers [48, 49, 50], but still De Laguna’s system has no precise contemporary interpretation with adequate models and representation theory. As another root it have to be mentioned Tarski [69], who developed a point-free version of Euclidean geometry called ”Foundations of the geometry of solids”. It is based on mereology extended with the primitive notion of ball which is used in the definition of point. Also we owe to Tarski the reinterpretation of mereology (the mereological system of Lesniewski) to the notion of Boolean algebra (BA) (namely complete BA with deleted zero) and also the good topological model of complete BA as algebra of regular open (or regular closed) subsets of a topological space. In an algebra of regular closed sets solids (or regions) are just the regular closed sets and the relation of ”contact” has a very natural definition - having a common point. These facts can be considered as the roots of the first definitions of the notion of contact algebra (CA) as an extension of BA with the contact relation (for the history of CA see [73]). Now the version of CA from [22] is commonly considered as the simplest point-free formulation of RBTS with standard models the algebras of regular closed sets of topological spaces. This fact motivates some authors to use another name of RBTS - mereotopology - a combination of mereology with topology: the BA represents mereological component and the contact relation which has a topological nature represents the topological component of the system.

Let us mention that RBTS as a point-free approach to the theory of space can be considered now as a well established branch of mathematics with applications in computer science which is open for further research. For the results of RBTS till 2006 see our essay [73] as well as the survey papers [8, 62], and [39] which contains also information of applications of RBTS in computer science. Some possibly incomplete information on the further development of RBTS and some related topics after 2007 is given in the Appendix of this paper.

Let us return to the integrated point-free theory of space and time. As we
have mentioned spacetime systems from mathematical physics are not point- free and the Whitehead’s early program formulated in his lecture *Space, Time, and Relativity* can be considered as a kind of program or a wish to build such a theory. Whitehead’s view on the nature of time developed in his books [83, 84, 86, 87] is mainly philosophical and changed over years. For instance in [83, 84] he uses a more common time terminology: instances of time, moments, but in [86, 87] he renamed his theory of time as ”epochal theory of time” (ETT) considering *epochs* as certain atomic instances of time. Probably the reason for this new terminology is that the Whitehead’s notion of epoch is one of the central notions of his later theory of time. Whitehead did not propose how ETT can be formalized and integrated with the point-free theory of space. Unlike his quite detailed program for building point-free mathematical theory of space, presented in P&R Whitehead did not describe analogous program for his ETT. He introduced and analyzed many notions related to ETT but mainly in an informal way using his own quite complicated philosophical terminology which makes extremely difficult to obtain clear mathematical theory corresponding to ETT.

An attempt to build a theory incorporated both space and time was recently made in [28, 29], but the system is not point-free with respect to time: time points are presented directly in the system.

1.4 The first attempts in building of an integrated point-free theories of space and time and a possible strategy how to realize such a task

Having in mind the situation about building an integrated point-free theory of space and time discussed at the end of the previous section, the present author decided to make the first steps in building such a theory (or examples of such theories). The results till now appeared in the series of papers started from 2010: [74, 75, 76], and (jointly with P. Dimitrov) in [13]. Because the notions of space and time are so rich, our aim in this project was to start with a simple system describing in a point free manner (some aspects of) both space and time and their mutual relationships, and then to refine the system step by step removing some defects and extending its expressive power. First we had to find a strategy how to build such systems and what requirements they should satisfy in order to treat them as point-free axiomatic systems of space and time.

We found that the following requirements will be useful.
1. **In order to follow Whitehead style the system should be region-based and should be based on mereology.** Regions will correspond to changing or moving objects and following Tarski the regions should form a Boolean algebra.

2. **The regions should be equipped with a number of basic spatio-temporal relations with well motivated meaning.** The relations are called basic because they have to be used in the definitions of some other meaningful relations. The meaning of basic relations should be determined by an appropriate set of axioms. What does this mean? - see the next two requirements:

3. **The system should have a meaningful standard adequate set-theoretical point based spacetime model describing the change of regions and the meaning of the spatio-temporal relations.** "Meaningful" means that the model is in accordance with our point-based spatial and temporal intuition which we obtained during our basic education in mathematics and physics. "Standard" means that we consider that this model give the intended point-based intuition of the basic relations.

4. **"Adequate" in 3. means that we can extract from the system in a canonical way a standard model, called the canonical model of the system, and to define an isomorphism mapping of the system into its canonical model.** Here "to extract" means to define both space points and time points within the system and also all other ingredients needed to construct the model. "To construct the model" means to use only the axioms of the system and standard set-theoretical constructions. So the theory should have the form of ordinary axiomatical mathematical theory.

5. **The main problem in realization of 2 and 4 is how to find the needed axioms.** This is the most difficult part of the realization of the program. One way, which we follow, is to start with the standard model and to proof for it enough statements considered further as possible axioms. But which true sentences to accept as axioms? Practically this is the following informal task: make an initial hypothesis of the possible steps of the construction of the canonical model and look what axiom (or a set of axioms) are needed to prove the correctness of a given step. If the required axiom (or axioms) is not in the list, see if it is true in the standard model and add it to the list. This is a long experimental mathematical procedure which is not always successful, and, as Whitehead commented in the quote from section 1.2, many attempts have to be done in order to obtain a satisfying result.
If we succeed in the realization of the above five requirements then obviously the resulting system will be point-free, the standard models indeed will be models of the system and the isomorphism of the system into its canonical model will show that the choice of the axioms is successful and that the standard point-based model and the point-free axiomatic systems are in certain sense equivalent. The expressivity power of the system will depend on the choice of the basic spatio-temporal relations between regions, so further steps of improving the system is to consider larger and a richer system of basic relations.

As we have seen, the realization of such a strategy is to start with the standard point-based model of spacetime and to find a successful construction of space points, time points and other ingredients of the model. Whitehead do this by his method of "extensive abstraction" which results to a complicated constructions. In contemporary mathematics, for instance in the Stone representation theory of Boolean algebras [68] and the theory of proximity spaces [56, 70] there are more good methods for defining abstract points: ultrafilters, clans, clusters and others. The success of the realization of the above scheme depends also of what kind of concrete point-based model is chosen to start with. Because standard point-based models are concrete constructions involving space points and time points, we adopted a special construction called "snapshot construction" and the resulting models - called "snapshot spacetime models". This is a very simple and intuitive construction which we mentioned in Section 1.1 [1]. Intuitively the snapshot construction is a formalization and generalization of the real method of describing an area of changing objects by making a video: for each moment of time the video camera makes a snapshot of the current spatial configurations of the objects and the series of the snapshots can be used to construct the point based spacetime model of change (see Remark 3.2 about the limitation of the analogy of the method of "snapshot construction with making video).

The first paper [74] from the above mentioned series of papers was experimental - we just wanted to see if the above described strategy works. That is why we included only two spatio-temporal relations between changing objects which do not suppose that time flaws: \( aC^sb \) - stable contact \((a \text{ and } b \text{ are always in a contact})\) and \( aC^ub \) - unstable contact \((a \text{ and } b \text{ are sometimes in a contact})\). The paper [75] makes the next step assuming that time flaws and in the point based model the moments of time are equipped with "before-after" relation. It contains two relations which do not depend on before-after relation: space contact \( aC^s b \) - there is a moment of time in which \( a \text{ and } b \) are in a space contact, time contact \( aC^t b \) - there is a moment
of time in which $a$ and $b$ exist simultaneously. The third relation, called *preceding* just uses the before-after relation: there is a moment $s$ in which $a$ exists and a later moment $t$, $s \prec t$, in which $b$ exists. This is a quite rich system for space and time, but it was not able to describe *past*, *present* and *future*. This was possible in the system from [76] in which we added the notion of the so called time representative, a region existing only at a given moment of time, or epoch in Whitehead’s terminology, which is using as name of the corresponding epoch, for instance ”the epoch of Leonardo”.

The paper [13] studies some new spacetime systems extending the system from [76] with new axioms and some propositional (quantifier-free) logics based on these systems. Other results in this direction are included in the papers [59] and [57, 58] which generalize [74] putting the system on pure relational base and without operations on regions.

In this paper, starting from Section 3, we will present with some details one not very complicated spacetime system just in order to show how the method works. The new thing is that we will supply the system not only with snapshot models, but also with topological models which will give more information on the nature of space points and time points.

## 2 Contact and precontact algebras

In this section we summarize some facts about contact and precontact algebras which are needed later on. We assume a familiarity of the reader with the basic theory of Boolean algebras, filters, ideals, ultrafilters and the Stone representation of Boolean algebra by ultrafilters.

### 2.1 Definitions of contact and precontact algebras

**Definition 2.1. Contact algebra** [22]. Let $(B, 0, 1, \leq, +, \cdot, *)$ be a non-degenerate Boolean algebra with complement denoted by $*$ and let $C$ be a binary relation in $B$. $C$ is called a contact relation in $B$ if the following axioms are satisfied:

1. **(C1)** If $aCb$ then $a \neq 0$ and $b \neq 0$,  
2. **(C2)** If $aCb$ and $a \leq a'$ and $b \leq b'$ then $a'Cb'$,  
3. **(C3')** If $aC(b + c)$ then $aCb$ or $aCc$,  
4. **(C3'')** If $(a + b)Cc$ then $aCc$ or $bCc$,  
5. **(C4)** If $aCb$ then $bCa$.  

(C5) If $a.b \neq 0$ then $aCb$.

We write $C'$ for the complement of $C$. If $C$ is a contact relation in $B$, then the algebra $A = (B, C)$ is called a contact algebra.

If we do not assume axioms (C4) and (C5), then $C$ is called a precontact relation in $B$ and the pair $(B, C)$ is called a precontact algebra.

If $A = (B, C)$ is a precontact (contact) algebra then we will write also $A = (B_A, C_A)$, where $B_A = (B, 0, 1, \leq, +, \cdot, *)$ and $C_A = C$.

In this paper we will consider also Boolean algebras with several precontact and contact relations satisfying some interacting axioms. Examples will be the dynamic contact algebras to be introduced later on.

Let us mention that if we assume (C4) only one of the axioms (C3') and (C3") is needed. Note also that (C5) is equivalent (on the base of the precontact axioms) to the following more simple axiom

(C5') If $a \neq 0$ then $aCa$.

From (C5') and (C1) it follows that $a \neq 0$ iff $aCa$.

In the present context we treat the Boolean part of the contact algebra as its mereological component and the contact relation - as its mereotopological component. In our treating of mereology we consider the zero element 0 as a non-existing region and this can be used to define the ontological predicate of existence $E(a)$: "$a$ ontologically exists", in the following way:

$E(a)$ iff $a \neq 0$.

For simplicity, instead of "ontologically exists" we will say simply "exists" and from the context it will be clear that this is not the existential quantifier.

The definitions of mereological relations "part-of" and "overlap" are the following:

- $a$ is part of $b$ iff $a \leq b$, i.e. part-of is just the Boolean ordering,
- $a$ overlaps $b$ (in symbols $aOb$) iff there exists a region $c \neq 0$ such that $c \leq a$ and $c \leq b$ iff $a.b \neq 0$.

Note that by the definition of overlap the axiom (C5) can be presented in this way: $aOb$ implies $aCb$.

**Remark 2.2.** It is easy to see that the relation $O$ of overlap satisfies all axioms of contact relation and by axiom (C5) it can be considered as the smallest contact in $B$. Non-degenerate Boolean algebras have also another
contact $C_{\text{max}}$ definable by "$a \neq 0$ and $b \neq 0$". It follows by axiom (C1) that this is the largest contact in $B$.

By means of the contact relation we may reproduce the definitions of some mereotopological relations considered by Whitehead:

- **external contact**: $aC^{E}b \iff \text{def } aCb$ and $a.b = 0$, the common points of $a$ and $b$ are on their boundaries.
- **non-tangential inclusion**: $a \ll b \iff \text{def } a\overline{C}b^{*}$, called also deep inclusion - $a$ is included in $b$ not touching the boundary of $b$.
- **tangential inclusion**: $a \leq^{T} b \iff \text{def } a \leq b$ and $a \not\ll b$, $a$ is included in $b$ and touches the boundary of $b$.

**Intuitive examples**: A cup on a table is in an external contact with the table. If a nail is driven into the table then it is tangentially included into the table. If the nail is deeply embedded into the table so that his head is not seen, then the nail is non-tangentially included in the table.

Contact relation has the following interesting property, stated in the next lemma.

**Lemma 2.3.** ([74], Lemma 1.1. (vi)) For any $a, b, p, q \in B$: if $pCq$ and $a\overline{C}b$ then either $(p.a^{*})C(q.a^{*})$ or $(p.b^{*})C(q.b^{*})$.

Precontact algebras were considered under the name of proximity algebras in [26]. We will be interested later on contact and precontact algebras satisfying the following additional axiom:

(CE) If $a\overline{C}b$ then $(\exists c)(a\overline{C}c$ and $(c^{*}\overline{C}b)$.

This axiom is called sometimes Efremovich axiom, because it is used in the definition of Efremovich proximity spaces [56]. Let us note that the largest contact $C_{\text{max}}$ satisfies the Efremovich axiom.

### 2.2 Examples of contact and precontact algebras

**Topological example of contact algebra.** The intended example of contact algebra is a topological one and can be defined in the following way. Let $X$ be a topological space and $Cl$ and $Int$ be the operations of closure and interior of a subset of $X$. A set $a \subseteq X$ is called regular closed if $a = Cl(Int(a))$. The set $RC(X)$ of regular closed subsets of $X$ is a Boolean algebra with respect to the following operations and constants: $0 = \emptyset$, $1 = \ldots$
Relational examples of precontact and contact algebras. Let $X$ be a nonempty set, whose elements are considered as points and $R$ be a reflexive and symmetric relation in $X$. Pairs $(X, R)$ with reflexive and symmetric $R$ are called by Galton adjacency spaces (see [26]).

One can construct a contact algebra from an adjacency space as follows: take a class $B$ of subsets of $X$ which form a Boolean algebra under the set-theoretical operations of union $a + b = a \cup b$, intersection $a.b = a \cap b$ and complement $a^* = X \setminus a$ and define contact $C_R$ between two members of $B$ as follows: $aC_R b$ iff there exist $x \in a$ and $y \in b$ such that $xRy$. It can easily be verified that all axioms of contact are satisfied.

Let us note that there are more general adjacency spaces in which neither reflexivity nor symmetry for the relation $R$ are assumed (see [26]). We reserve the name ”adjacency space” for such more general spaces and for the special case where $R$ is a reflexive and symmetric relation we will say ”adjacency spaces in the sense of Galton”. If we repeat the above construction then the axioms (C1), (C2), (C3') and (C3'') will be true but in general the axioms (C4) and (C5) will not be satisfied and in this way we obtain examples of precontact algebras which are not contact algebras. The relational models of contact and precontact algebras are called also discrete models.

The following lemma will be of later use:

**Lemma 2.4. Characterization of reflexivity, symmetry and transitivity.** [26] Let $(X, R)$ be an adjacency space and $(B(X), C_R)$ be the precontact algebra over all subsets of $X$. Then the following conditions hold:

(i) $R$ is a symmetric relation in $X$ iff $(B(X), C_R)$ satisfies the axiom (C4) If $aC_R b$ then $bC_{Ra}$,

(ii) $R$ is reflexive relation in $X$ iff $(B(X), C_R)$ satisfies the axiom (C5) If $a.b \neq \emptyset$ then $aCb$,
(iii) R is a transitive relation in X iff \((B(X), C_R)\) satisfies the axiom

\((CE)\) If \(a \subseteq b\) then \((\exists c)(a \subseteq c \text{ and } c \subseteq b)\).

In the proof of the above lemma the following equivalent definition of the precontact relation \(a \subseteq b\) will be helpful. For a subset \(a \subseteq X\) define \((R)a \stackrel{\text{def}}{=} \{x \in X : (\exists y \in a)(xRy)\}\). Then obviously we have: \(a \subseteq b\) iff \(a \cap \langle R \rangle b \neq \emptyset\).

The operation \(\langle R \rangle a\) comes from the relational semantics of modal logic and represents the operation of possibility (for more information for this connection see [4]). The following property of the operation \(\langle R \rangle a\) can be proved: \(R\) is transitive relation on \(X\) iff for all \(a \subseteq X\): \(\langle R \rangle \langle R \rangle a \subseteq \langle R \rangle a\). Then by pure set-theoretical transformations one can show that the Efremovic axiom (CE) is equivalent to this property, which proves (iii).

### 2.3 Algebras with several precontact relations

In this section we will introduce Boolean algebras with two precontact relations satisfying two special interacting axioms which will be used in the definition of dynamic contact algebra. First we will present their relational examples.

Let \((W, R, S)\) be a relational system with two relations. We consider the following two first-order conditions for \(R\) and \(S\):

\((R \circ S \subseteq S)\) If \(xRy\) and \(ySz\), then \(xSz\) (The composition of \(R\) with \(S\) is included in \(S\)).

\((S \circ R \subseteq S)\) If \(xSy\) and \(yRz\), then \(xSz\) (The composition of \(S\) with \(R\) is included in \(S\)).

The system \((W, R, S)\) defines in an obvious way set-theoretical Boolean algebra with two precontact relations \(C_R\) and \(C_S\).

Consider the following two conditions for the precontact relations \(C_R\) and \(C_S\) which are similar to the Efremovich axiom (CE):

\((C_R C_S)\) If \(a \subseteq b\) then there exists \(c \subseteq W\) such that \(a \subseteq c\) and \(c \subseteq \langle S \rangle b\), and

\((C_S C_R)\) If \(a \subseteq b\) then there exists \(c \subseteq W\) such that \(a \subseteq c\) and \(c \subseteq \langle R \rangle b\).

We call the conditions \((C_R C_S)\) and \((C_S C_R)\) compositional axioms for \(C_R\) and \(C_S\).
**Lemma 2.5.** (i) The condition \((C_RC_S)\) is fulfilled between precontact relations \(C_R\) and \(C_S\) iff the condition \((R \circ S \subseteq S)\) is satisfied.

(ii) The condition \((C_SC_R)\) is fulfilled between precontacts relations \(C_R\) and \(C_S\) iff the condition \((S \circ R \subseteq S)\) is satisfied.

The proof is similar to the proof of Lemma 2.4 (iii). In the proof of (i) use the following equivalences: \((R \circ S \subseteq S)\) iff for all \(a \subseteq X\) \(\langle R \rangle \langle S \rangle a \subseteq \langle S \rangle a\) iff \((C_RC_S)\) and similarly for (ii) by exchanging the places of \(R\) and \(S\).

**2.4 Discrete (relational) representation of contact and precontact algebras.**

One way to obtain a representation theory of precontact algebras with relational representation of precontact is to consider ultra filters as the set of abstract points of a given precontact algebra \(A = (B, C)\) (as in the Stone representation theory of Boolean algebras) and to define the relation \(R\) in the set of ultrafilters \(\text{Ult}(A)\) of \(A\) as follows. For \(U, V \in \text{Ult}(A)\):

\[ URV \leftrightarrow_{def} (\forall a, b \in B)(a \in U \text{ and } b \in V \Rightarrow aCb). \]

For \(a \in B\) define also the Stone embedding: \(s(a) = \{U \in \text{Ult}(A) : a \in U\}\).

**Definition 2.6.** The relational system \((\text{Ult}(A), R)\) with just defined \(R\) is called a canonical adjacency space over \(A\) and \(R\) is called the canonical adjacency relation on \(\text{Ult}(A)\).

Note that the definition of the canonical relation \(R\) is meaningful for arbitrary filters. In order to prove some facts for the canonical relation some constructions of filters and ideals will be needed and some technical lemmas have to be introduced.

First we remind the well known Separation Lemma for filters and ideals in Boolean algebra and the Extension Lemma for proper filters.

**Lemma 2.7.** (i) **Separation Lemma.** If \(F\) is a filter and \(I\) is an ideal in a Boolean algebra such that \(F \cap I = \emptyset\), then there exists an ultrafilter \(U\) such that \(F \subseteq U\) and \(U \cap I = \emptyset\).

(ii) **Extension Lemma.** Every proper filter can be extended into an ultrafilter.
The sum of two filters: If \( F \) and \( G \) are filters, then \( F \oplus G = \{ a.b : a \in F, b \in G \} \) is the smallest filter containing both \( F \) and \( G \). \( 0 \in F \oplus G \) iff there exists \( a \in F \) and \( a^* \in G \).

**Lemma 2.8. Technical lemma for the canonical relation.** Let \( A = (B, C) \) be a precontact algebra, \( F \) and \( G \) be filters in \( A \) and \( FRG \) be the canonical relation between them corresponding to \( C \). Define the following sets:

\[
\begin{align*}
I_{C}^{1}(F) &= \{ b : (\exists a \in F)(aCb) \}, & I_{C}^{1}(G) &= \{ a : (\exists b \in G)(aC^*b) \}, \\
F_{C}^{1}(F) &= \{ b : (\exists a \in F)(aC^*b^*) \}, & F_{C}^{1}(G) &= \{ a : (\exists b \in G)(a^*Cb) \}.
\end{align*}
\]

Then the following equivalencies are true:

(i) \( FRG \iff I_{C}^{1}(F) \cap G = \emptyset \), and \( I_{C}^{1}(F) \) is an ideal.

(ii) \( FRG \iff F \cap I_{C}^{1}(G) = \emptyset \), and \( I_{C}^{1}(G) \) is an ideal.

(i') If \( G \) is an ultrafilter then \( FRG \iff F_{C}^{1}(F) \subseteq G \), and \( F_{C}^{1}(F) \) is a filter.

(ii') If \( F \) is an ultrafilter, then \( FRG \iff F_{C}^{1}(G) \subseteq F \), and \( F_{C}^{1}(G) \) is a filter.

**Proof.** The proof follows by a direct verification of the corresponding definitions. \( \square \)

**Lemma 2.9. [26] R-extension Lemma.** Let \( U_0 \) and \( V_0 \) be filters in a precontact algebra \( (B, C) \) and let \( U_0 RV_0 \). Then there exist ultrafilters \( U \) and \( V \) such that \( U_0 \subseteq U, V_0 \subseteq V \) and \( URV \).

**Proof.** By Lemma 2.8 \( U_0 RV_0 \iff I_{C}^{1}(U_0) \cap V_0 = \emptyset \). Then by the Separation Lemma for filters and ideals 2.7 there exists an ultrafilter \( V \) such that \( V_0 \subseteq V \) and \( I_{C}^{1}(U_0) \cap V = \emptyset \). From \( I_{C}^{1}(U_0) \cap V = \emptyset \) again by Lemma 2.8 we obtain \( U_0 RV \). So we have extended \( U_0 \) into the ultrafilter \( U \). Similarly repeating this procedure for \( V_0 \) we can extend it into an ultrafilter \( V \). \( \square \)

**Lemma 2.10. [26] Canonical Lemma 1.**

(i) \( aCb \iff \) there exist ultrafilters \( U, V \) such that \( URV \), \( a \in U \) and \( b \in V \).

(ii) \( aCb \iff s(a)C_{R}b \).

**Proof.** For (i) define first the filters generated by \( a \) and \( b \): \( [a] = \{ c : a \leq c \} \) and \( [b] = \{ c : b \leq c \} \). Second, \( aCb \) implies \( [a]R[b] \) and then apply the \( R \)-extension Lemma 2.9. Condition (ii) follows from (i). \( \square \)
Lemma 2.11. [26] Canonical Lemma 2. Let $A = (B, C)$ be a precontact algebra. Then:

(i) $R$ is a symmetric relation in $\text{Ult}(A)$ iff $C$ satisfies the axiom $(C4)$.

(ii) $R$ is a reflexive relation in $\text{Ult}(A)$ iff $C$ satisfies the axiom $(C5)$.

(iii) $R$ is transitive relation in $\text{Ult}(A)$ iff $C$ satisfies the Efremovich axiom $(CE)$ $a \overset{c}{\rightarrow} b \Rightarrow (\exists c)(a \overset{c}{\rightarrow} c \text{ and } c \overset{*}{\rightarrow} b)$.

Proof. We will demonstrate only the proof of (iii).

Proof of $(\Rightarrow)$. Suppose that $R$ is a transitive relation. We will prove $(CE)$. Suppose $a \overset{c}{\rightarrow} b$ and in order to obtain a contradiction suppose that $(\exists c)(a \overset{c}{\rightarrow} c \text{ and } c \overset{*}{\rightarrow} b)$ is not true. We will show that there are ultrafilters $U, V$ and $W$ such that $URV$, $VRW$, but $URW$ which contradicts the assumption on transitivity of $R$.

Let $[a] =_{def} \{ c : a \leq c \}$ and $[b] =_{def} \{ b : b \leq c \}$ and define (see Lemma 2.8): $\Gamma = F^1_C([a]) \oplus F^2_C([b])$. $\Gamma$ is a proper filter containing $F^1_C([a])$ and $F^2_C([b])$. If we assume that $0 \in \Gamma$, then there is a $c$ such $c \in F^1_C([a])$ and $c \in F^2_C([b])$. This implies that $a \overset{c}{\rightarrow} c \text{ and } c \overset{*}{\rightarrow} b$ contrary to the assumption that there is no such $c$. So $\Gamma$ is a proper filter and can be extended into an ultrafilter $V$ such that $F^1_C([a]) \subseteq V$ and $F^2_C([b]) \subseteq V$. By Lemma 2.8) (i') and (ii') we obtain $[a]RV$ and $VR[b]$. By Lemma 2.9 extend $[a]$ and $[b]$ to ultrafilters $U$ and $W$ such that $URV$ and $VRW$, $a \in U$ and $b \in W$. But by assumption we have $a \overset{b}{\rightarrow} c$ which shows that $URW$ - the desired contradiction.

Proof of $(\Leftarrow)$. Suppose that $(CE)$ holds and for the sake of contradiction that $R$ is not transitive. Then there exist ultrafilters $U, V$ and $W$ such that $URV$, $VRW$, but $URW$. So, there exist $a \in U$ and $b \in W$ such that $a \overset{b}{\rightarrow} c$. By $(CE)$ there exists $c$ such that $a \overset{c}{\rightarrow} c \text{ and } c \overset{*}{\rightarrow} b$. We have two cases for $c$:

Case 1: $c \in V$. But $a \in U$ and $URV$, so $a \overset{c}{\rightarrow} c - a$ contradiction with $a \overset{c}{\rightarrow} c$.

Case 2: $c \not\in V$, so $c \overset{*}{\rightarrow} c \in V$. But $b \in W$ and $VRW$ imply $c \overset{*}{\rightarrow} b$ - a contradiction with $c \overset{*}{\rightarrow} b$.

The following lemma will be used later on. It is the canonical analog of Lemma 2.5 concerning algebras with several precontact relations.

Lemma 2.12. Canonical Lemma 3. Let $A = (B, C_1, C_2)$ be Boolean algebra with two precontact relations $C_1$ and $C_2$ and let $R_1$ and $R_2$ be their canonical relations in the canonical structure $(\text{Ult}(A), R_1, R_2)$. Then the following conditions are true:
(i) $A$ satisfies the condition
\[(C_1, C_2) \ a \bar{C}_1 b \Rightarrow (\exists c)(a \bar{C}_1 c \text{ and } c^* \bar{C}_2 b) \iff (Ult(A), R_1, R_2) \text{ satisfies the condition.}
\[(R_1 \circ R_2 \subseteq R_1) \ U R_1 V \text{ and } V R_2 W \Rightarrow U R_1 W.
\]
(ii) $A$ satisfies the condition
\[(C_2, C_1) \ a \bar{C}_1 b \Rightarrow (\exists c)(a \bar{C}_2 c \text{ and } c^* \bar{C}_2 b) \iff (Ult(A), R_1, R_2) \text{ satisfies the condition.}
\[(R_2 \circ R_1 \subseteq R_1) \ U R_2 V \text{ and } V R_1 W \Rightarrow U R_1 W.
\]

Proof. The proof is similar to the proof of condition (iii) of 2.11.

**Theorem 2.13. Relational representation theorem for precontact and contact algebras [26].** Let $A = (B, C)$ be a precontact algebra, $(Ult(A), R)$ be the canonical adjacency space of $A$ and $s$ be the stone embedding. Then:

(i) $s$ is an embedding of $(B, C)$ into the precontact algebra over the canonical adjacency space $(Ult(A), R)$.

(ii) If $(B, C)$ is a contact algebra then the precontact algebra over the canonical adjacency space over $(B, C)$ is a contact algebra.

Proof. The proof follows from Lemma 2.10 and Lemma 2.11 and the fact that $s$ is an isomorphic embedding of the Boolean algebra $B$ into the algebra of all subsets of $Ult(A)$.

The above representation theorem for the case of contact algebras is not the intended one because the contact is not of Whiteheadian type, namely sharing a common point. In the next section we will describe another representation of contact algebras using topology, which presents an Whiteheadian type contact between regions. As we shall see, the reason is that ultrafilters as abstract points are not enough to model the Whiteheadean contact and we need to introduce another kind of abstract points.
2.5 Topological representation of contact algebras. Clans.

First we will introduce another kind of abstract points in contact algebras called clans.

Definition 2.14. Definition of clan. [22] Let \( A = (B, C) \) be a contact algebra. A subset \( \Gamma \subseteq B \) is called a clan in \( (B, C) \) if it satisfies the following conditions:

(i) \( 1 \in \Gamma \) and \( 0 \notin \Gamma \),
(ii) If \( a \in \Gamma \) and \( a \leq b \) then \( b \in \Gamma \),
(iii) If \( a + b \in \Gamma \) then \( a \in \Gamma \) or \( b \in \Gamma \)
(iv) If \( a, b \in \Gamma \) then \( aCb \).

\( \Gamma \) is a maximal clan if it is a maximal set under the set inclusion. We denote by \( \text{Ult}(\Gamma) \) the set of all ultrafilters contained in \( \Gamma \) and by \( \text{Clans}(A) \) - the set of all clans of \( A \).

Subsets of \( B \) satisfying (i), (ii) and (iii) are called grills. So clans are grills satisfying (iv).

The above definition is an algebraic abstraction from an analogous notion in the proximity theory (see, for instance, [70], from where we adopt the name clan).

Let us note that ultrafilters are clans, but there are other clans and they can be obtained by the following construction.

Let \( \sum \) be a nonempty set of ultrafilters of \( (B, C) \) such that if \( U, V \in \sum \), then \( URV \), where \( R \) is the canonical adjacency relation of \( C \) on the set of ultrafilters of \( (B, C) \). Such sets of ultrafilters are called \( R \)-cliques. An \( R \)-clique is maximal, if it is a maximal set under the set-inclusion. By the axiom of choice every \( R \)-clique is contained in a maximal \( R \)-clique. Let \( \Gamma \) be the union of all ultrafilters from \( \sum \). Then it can be verified that \( \Gamma \) is a clan. Moreover, every clan can be obtained by this construction from an \( R \)-clique and there is an obvious correspondence between maximal cliques and maximal clans. All these facts about clans are contained in the following technical lemma:

Lemma 2.15. [22] Clan Lemma. (i) Every ultrafilter is a clan.
(ii) The complement of a clan is an ideal.
(iii) Every clan is contained in a maximal clan (by the Zorn Lemma),
(iv) Let $\sum$ be an $R$-clique and $\Gamma(\sum) = \bigcup_{\Gamma \in \sum} \Gamma$. Then $\Gamma(\sum)$ is a clan.

(v) If $U, V \in \text{Ult}(\Gamma)$ then $URV$, so $\text{Ult}(\Gamma)$ is an $R$-clique,

(vi) If $\Gamma$ is a clan and $a \in \Gamma$ then there is an ultrafilter $U \in \text{Ult}(\Gamma)$ such that $a \in U$,

(vii) Let $\Gamma$ be a clan and $\sum$ be the $R$-clique $\text{Ult}(\Gamma)$. Then $\Gamma = \Gamma(\sum)$, so every clan can be defined by an $R$-clique as in (iv),

(viii) If $\sum$ is a maximal $R$-clique then $\Gamma(\sum)$ is a maximal clan,

(ix) If $\Gamma$ is a maximal clan then $\text{Ult}(\Gamma)$ is a maximal $R$-clique,

(x) For all ultrafilters $U, V$: $URV$ iff there exists a (maximal) clan $\Gamma$ such that $U, V \in \text{Ult}(\Gamma)$,

(xi) For all $a, b \in B$: $aCb$ iff there exists a (maximal) clan $\Gamma$ such that $a, b \in \Gamma$,

(xii) For all $a, b \in B$: $a \not\leq b$ iff there exists clan (ultrafilter) $\Gamma$ such that $a \in \Gamma$ and $b \notin \Gamma$.

Proof. We invite the reader to prove the lemma by himself or to consult [22]. As an example we will give proofs only of some parts of the lemma in order to connect it with the discrete representation of contact algebras.

(vi) Let $\Gamma$ be a clan and $a \in \Gamma$. Then obviously $[a] \subseteq \Gamma$ and consequently $[a] \cap \Gamma = \emptyset$. But $[a]$ is a filter, $\overline{\Gamma}$ is an ideal (by (ii)) and by the Separation Theorem for filters and ideals there exists an ultrafilter $U$ such that $[a] \subseteq U$ and $U \cap \overline{\Gamma} = \emptyset$. This implies that $a \in U$ and $U \subseteq \Gamma$.

(ix) ($\Rightarrow$). Let $aCb$. Then by Lemma 2.10 there exist ultrafilters $U, V$ such that $URV$, $a \in U$ and $b \in V$. Since $R$ is a reflexive and symmetric relation, then $\sum = \{U, V\}$ is a clique and by (iv) $\Gamma = U \cup V$ is a clan such that $a, b \in \Gamma$.

(ix) ($\Leftarrow$). This direction follows by the definition of clan.

Lemma 2.16. [22] Let $\Gamma$ be a clan in a contact algebra $A = (B, C)$. Then the following holds for any $a \in B$:

$a^* \in \Gamma$ iff $(\forall b \in B)(a + b = 1 \Rightarrow b \in \Gamma)$.

Proof. By a direct verification.
The topological representation theory of contact algebras is based on the following construction taken from [22]. Let \( A = (B, C) \) be a contact algebra and let \( X = \text{Clans}(A) \) and for \( a \in B \), define \( g(a) = \{ \Gamma \in \text{Clans}(B) : a \in \Gamma \} \). We introduce a topology in \( X \) taking the set \( B = \{ g(a) : a \in B \} \) as the base of closed sets in \( X \). The obtained topological space \( X \) is called the canonical topological space of \( (B, C) \).

**Lemma 2.17.** [22]

(i) \( g(0) = \emptyset, \ g(1) = X \),

(ii) \( g(a + b) = g(a) \cup g(b) \),

(iii) \( a \preceq b \iff g(a) \subseteq g(b) \).

(iv) \( a = 1 \iff g(a) = X \).

(v) \( g(a^*) = \text{Cl}_X(X \setminus g(a)) = \text{Cl}_X - g(a) \)

(vi) \( g(a) \) is a regular closed subset of \( X \).

**Proof.** (i) and (ii) follow directly from the definition of clan, (iii) follows from Lemma 2.10 (xii) and (iv) follows from (iii). (v) follows from the following sequence of equivalencies:

for any clan \( \Gamma : \Gamma \in g(a^*) \iff a^* \in \Gamma \iff (\forall b \in B)(a + b = 1 \implies b \in \Gamma) \iff (\forall b \in B)(g(a) \cup g(b) = X \implies \Gamma \in g(b)) \iff (\forall b \in B)(X \setminus g(a) \subseteq g(b) \implies \Gamma \in g(b)) \iff \text{Cl}_X(X \setminus g(a)) = \text{Cl}_X - g(a) \).

For (vi) By (v) \( g((a^*)^*) = \text{Cl}_X - \text{Cl}_X - g(a) = \text{Cl}_X(\text{Int}_X(a)) \). \( \square \)

**Theorem 2.18.** Topological representation theorem for contact algebras [22] (see also [73]). (i) The mapping \( g \) is an embedding from \( (B, C) \) into the canonical contact algebra \( \text{RC}(X) \) of \( (B, C) \).

(ii) The canonical space of \( (B, C) \) is \( T_0 \), compact and semiregular.

Note that a topological space is semiregular if it has a base of regular-closed sets.

**Proof.** We will give a proof only of (i). By Lemma 2.17 we see that \( g \) isomorphically embeds \( B \) into \( \text{RC}(X) \) where \( X = \text{Clans}(A) \) and the topology is determined by the closed basis \( \{ g(a) : a \in B \} \). It remains to show that \( g \) preserves contact:

\( aCb \iff (\text{by Lemma 2.15 (ix)}) \) there exists a clan \( \Gamma \) such that \( a \in \Gamma \) and \( b \in \Gamma \) iff there exists a clan \( \Gamma \) such that \( \Gamma \in g(a) \) and \( \Gamma \in g(b) \iff g(a) \cap g(b) \neq \emptyset \), i.e. \( g(a) \) and \( g(b) \) have a common point. \( \square \)
Let us note that in the above representation theorem two kinds of abstract points have been used: ultrafilters and clans which are not ultrafilters (ultrafilters as clans are used in the Clan Lemma (xii). Note that in the relational representation (Theorem 2.13) contact is characterized by the adjacency relations between ultrafilters. It is possible that two regions are in a relational contact and not share an ultrafilter. By adding more points (namely clans) this situation is excluded because we can find a clan-like point in both regions. We may consider ultrafilter points as simple atoms. Since clans are unions of adjacent ultrafilters, this suggests to consider clans as molecules composed by atoms. It is interesting to know how these two kinds of points are distributed in the set $g(a)$ of points associated with a given region $a$. For instance it can be proved that the set $BP(a) = g(a) \setminus \text{Int}(g(a)$ of boundary points of $g(a)$ do not contain any ultrafilter point. In some sense the above facts throw a new light on the ancient atomistic view of space.

**Remark 2.19.** Let us note that the clans corresponding to the largest contact $C_{\text{max}}$ (which can be named $C_{\text{max}}$-clans) are just the gills and that there is only one maximal grill - just the union of all ultrafilters. Analogously the clans and maximal clans corresponding to the smallest contact, the overlap relation $O$ in a Boolean algebra (O-clans) are ultrafilters (see Example 3.1 in [22]).

### 2.6 Factor contact algebras determined by sets of clans.

The following is a construction of a contact algebra from a given contact algebra $A$ and given set of clans of $A$. The construction is taken from [74] and the reader is invited to consult the paper for the details.

Let $\Delta$ be an ideal in a Boolean algebra $B$. It is known from the theory of Boolean algebras that the relation $a \equiv_\Delta b$ iff $a.b^* + a^*.b \in \Delta$ is a congruence relation in $B$ and the factor algebra $B/\equiv_\Delta$ under this congruence (called also factor algebra under $\Delta$ and denoted by $B/\Delta$) is a Boolean algebra. Denote the congruence class determined by an element $a$ of $B$ by $|a|_\Delta$ (or simply by $|a|$). Boolean operations in $B/\Delta$ are defined as follows: $|a| + |b| = |a + b|, \ |a|.|b| = |a.b|, \ |a|^* = |a^*|, \ 0 = |0|, \ 1 = |1|$. Recall that Boolean ordering in $B/\Delta$ is defined by $|a| \leq |b|$ iff $a.b^* \in \Delta$ (see [66] for details).

Let $A$ be a contact algebra and $\alpha \subseteq \text{Clans}(A), \ \alpha \neq \emptyset$. Now we will define a construction of a contact algebra $B_\alpha$ corresponding to $\alpha$. Define $I(\alpha) = \{a \in B : \alpha \cap g(a) = \emptyset\}$. It is easy to see that $I(\alpha)$ is a proper ideal in $B$, i.e. $1 \notin I(\alpha)$. The congruence defined by $I(\alpha)$ is denoted by $\equiv_\alpha$. 
So we have $a \equiv_\alpha b$ iff $a^*b + a.b^* \in I(\alpha)$ iff $a^*b \in I(\alpha)$ and $a.b^* \in I(\alpha)$. Now define $B_\alpha$ to be the Boolean algebra $B/I(\alpha)$. We define a contact relation $C_\alpha$ in $B_\alpha$ as follows: $|a|_\alpha |b|_\alpha$ iff $\alpha \cap g(a) \cap g(b) \neq \emptyset$, where $g(a) = \{ \Gamma \in \text{Clans}(B) : a \in \Gamma \}$ (see the topological representation theorem of contact algebras).

**Lemma 2.20.** $(B_\alpha, C_\alpha)$ is a contact algebra.

Let us note that in the Boolean algebra $B_\alpha$ the following conditions are true: $|a|_\alpha \neq |0|_\alpha$ iff $a \notin I(\alpha)$ iff there exists a clan $\Gamma \in \alpha$ such that $a \in \Gamma$.

### 2.7 Contact algebras satisfying the Efremovich axiom (CE).

**Clusters.**

We will show in this section that in contact algebras satisfying the Efremovich axiom (CE) we can introduce a new kind of abstract points called clusters. Our definition is an algebraic abstraction of the analogous notion used in the compactification theory of proximity spaces (see for instance [56]). Clusters will be used later on to define time points in dynamic contact algebras.

**Definition 2.21. Clusters.** [22] Let $(B, C)$ be a contact algebra. A subset $\Gamma \subseteq B$ is called a cluster in $(B, C)$ if it is a clan satisfying the following condition:

(Cluster) If $a \notin \Gamma$ then there exists $b \in \Gamma$ such that $a \mathrel{C} b$.

The set of clusters of $A = (B, C)$ is denoted by Clusters$(A)$.

**Lemma 2.22.** Let $A = (B, C)$ be a contact algebra satisfying the Efremovich axiom (CE). Then:

(i) $\Gamma$ is a cluster in $(B, C)$ iff $\Gamma$ is a maximal clan in $(B, C)$.

(ii) Every clan is contained in a unique cluster.

**Proof.** Let us note that the above lemma is a lattice-theoretic version of a result of Leader about clusters in proximity spaces mentioned in [70]. One can prove this lemma having in mind the following facts. First, it follows from Lemma 2.11 that if $C$ is a contact relation satisfying the Efremovich axiom (CE), then the canonical relation for $C$ is an equivalence relation. Second, the maximal $R$-cliques of an equivalence relation are the equivalence classes of $R$. And third, clusters in the presence of (CE) are unions of such $R$-equivalence classes (by 2.15 ).
Lemma 2.23. Let \((B, C)\) be a contact algebra satisfying the Efremovich axiom \((CE)\). Then for any \(a, b \in B\): \(aCb\) iff there is a cluster \(\Gamma\) containing \(a\) and \(b\).

Proof. \(aCb\) iff (by Lemma 2.15) there exists a maximal clan \(\Gamma\) containing \(a\) and \(b\). By Lemma 2.22 \(\Gamma\) is a cluster. \(\square\)

Note that we cannot prove a representation theorem for contact algebras satisfying the Efremovich axiom as subalgebras of regular closed sets using only clusters as abstract points, because we cannot distinguish in general different regions by means of clusters. Ultrafilters can distinguish different regions, but in general they are not clusters.

The following lemma states how we can distinguish clusters.

Lemma 2.24. Let \(A = (B, C)\) be a contact algebra satisfying the Efremovich axiom and let \(\Gamma, \Delta\) be clusters. Then the following conditions are equivalent:

(i) \(\Gamma \neq \Delta\),
(ii) there exist \(a \in \Gamma\) and \(b \in \Delta\) such that \(aCb\),
(iii) there exists \(c \in B\) such that \(c \notin \Gamma\) and \(c^* \notin \Delta\).

Proof. (i) \(\Rightarrow\) (ii) Suppose \(\Gamma \neq \Delta\), then, since they are maximal clans, there exists \(a \in \Delta\) and \(a \notin \Gamma\). Consequently, there exists \(b \in \Gamma\) such that \(aCb\), so (ii) is fulfilled.

(ii) \(\Rightarrow\) (iii) Suppose that there exist \(a \in \Gamma\) and \(b \in \Delta\) such that \(aCb\). From \(aCb\) we obtain by the Efremovich axiom that there exists \(c\) such that \(aCc\) and \(c^*Cb\). Conditions \(a \in \Gamma\) and \(aCb\) imply \(c \notin \Gamma\). Similarly \(b \in \Delta\) and \(c^*Cb\) imply \(c^* \notin \Delta\).

(iii) \(\Rightarrow\) (i) Suppose that there exists \(c \in B\) such that \(c \notin \Gamma\) and \(c^* \notin \Delta\) and for the sake of contradiction that \(\Gamma = \Delta\). Since \(c + c^* = 1\) then ether \(c \in \Gamma\) or \(c^* \in \Delta\) - a contradiction. \(\square\)

Remark 2.25. We have mentioned in Remark 2.19 that \(C_{\text{max}}\)-clans are grills and that there is only one maximal \(C_{\text{max}}\)-clan just the union of all ultrafilters. Because \(C_{\text{max}}\) satisfies the Efremovich axiom, then there is only one \(C_{\text{max}}\)-cluster - the maximal grill.
3 A dynamic model of space and time based on snapshot construction

In this section, following mainly [76, 13] we will give a specific point-based spacetime structure called dynamic model of space and time (DMST) built by a special construction mentioned in Section 1 and called snapshot construction. Because the notion of time structure is one of the base ingredients of the construction we start with this notion.

3.1 Time structures

Time structures of the forma $T = (T, \prec)$ were introduced in Section 1.1 as relational systems used as a semantic basis of temporal logic. Let us remind that $T$ is a non-empty set whose elements are called "time points" (moments, Whitehead’s epochs). The binary relation $\prec$ is called "before-after" relation (or "time order") with the standard intuitive meaning of $i \prec j$: the moment $i$ is before the moment $j$, or equivalently, $j$ is after $i$. We also suppose that $T$ is supplied with the standard notion of equality denoted as usual by $=$. We do not presuppose in advance any fixed set of conditions for the relation $\prec$. One possible list of first-order conditions for $\prec$ which are typical for some systems of temporal logic, are the following. We describe them with their specific names and notations which will be used in this paper.

- (RS) Right seriality $(\forall m)(\exists n)(m \prec n)$,
- (LS) Left seriality $(\forall m)(\exists n)(n \prec m)$,
- (Up Dir) Updirectedness $(\forall i,j)(\exists k)(i \prec k$ and $j \prec k)$,
- (Down Dir) Downdirectedness $(\forall i,j)(\exists k)(k \prec i$ and $k \prec j)$,
- (Circ) Circularity $(\forall i,j)(i \prec j \rightarrow (\exists k)(j \prec k$ and $k \prec i))$
- (Dens) Density $i \prec j \rightarrow (\exists k)(i \prec k$ and $k \prec j)$,
- (Ref) Reflexivity $(\forall m)(m \prec m)$,
- (Irr) Irreflexivity $(\forall m)(\text{not } m \prec m)$,
- (Lin) Linearity $(\forall m,n)(m \prec n$ or $n \prec m)$,
- (Tri) Trichotomy $(\forall m,n)(m = n$ or $m \prec n$ or $n \prec m)$,
- (Tr) Transitivity $(\forall i,j,k)(i \prec j$ and $j \prec k \rightarrow i \prec k)$.
We call the set of formulas (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) time conditions. If the relation \( \prec \) satisfies the condition (Irr) it will be called ”strict”. If \( \prec \) satisfies (Ref) the reading of \( i \prec j \) should be more precise: ”\( i \) is equal or before \( j \)”.

Note that the above listed conditions for time ordering are not independent. Taking some meaningful subsets of them we obtain various notions of time order. Of course this list is not absolute and is open for extensions but in this paper we will consider only these 11 conditions.

3.2 The snapshot construction and the dynamic model of space and time

The snapshot construction is a specific method of constructing a dynamic model of space. It is a formalization of the following intuitive idea. Suppose we are observing an area of changing regions, called ”dynamic regions” and we want to describe this area. In our everyday life such a description can be realized by a video camera making a video. In this way the camera can be interpreted as a fixed observer. The description is realized by making a snapshot of the observed area for each moment of the camera’s time. Namely the series of these snapshots can be considered as a realization of the description of the area of changing or moving regions and each snapshot can be considered as a static spatial description of the area for the corresponding time moment. This procedure can be formalized and generalized as follows. First we start with certain time structure \( \underline{T} = (T, \prec) \), described in the previous section. The formalization of the action ”making snapshots” is the following. To each moment \( i \in T \) we associate a contact algebra \( A_i = (B_i, 0_i, 1_i, \leq_i, +_i, \cdot_i, \ast_i, C_i) = (B_i, C_i) \), called ”coordinate contact algebra”. We assume that the algebra \( (B_i, C_i) \) realizes the static description of the dynamic regions at the moment \( i \in T \) and can be considered as the corresponding ”snapshot” of the area at the moment \( i \in T \). In this way each dynamic region \( a \) is represented by a series \( \langle a_i \rangle_{i \in T} \) such that for each \( i \in T, a_i \in B_i \). The series \( \langle a_i \rangle_{i \in T} \) is considered also as a life history of \( a \). We identify \( a \) with the series \( \langle a_i \rangle_{i \in T} \) and will write \( a = \langle a_i \rangle_{i \in T} \). The set of all dynamic regions is denoted by \( B \). We consider \( B \) as a Boolean algebra with Boolean operations defined coordinate-wise. For instance:

\[ a + b = \langle a_i +_ib_i \rangle_{i \in T}, \quad 0 = \langle 0_i \rangle_{i \in T}, \quad 1 = \langle 1_i \rangle_{i \in T}, \quad \text{etc.} \]

Let us define the Cartesian product (direct product) \( \mathbb{B} \) of the coordinate
Boolean algebras $B_i$, $i \in T$, namely $B = \prod_{i \in T} B_i$. Obviously $B$ is a subalgebra of $\mathbb{B}$. Now we introduce the following important definition

**Definition 3.1.** By a dynamic model of space and time (DMST) we understand the system $M = \langle (T, \prec), \{ (B_i, C_i) : i \in T \}, \mathcal{B}, \mathbb{B} \rangle$. We say that $M$ is a full model if $B = \mathbb{B}$, and that $M$ is a rich model if $B$ contains all regions $a = \langle a_i \rangle_{i \in T}$ such that for all $i \in T$ either $a_i = 0_i$ or $a_i = 1_i$. (obviously every full model is a rich model).

Dynamic model of space and time will be called sometimes ”snapshot model” or ”cinematographic model”.

Let us note that DMST is a very expressive model with the main component the Boolean algebra $\mathcal{B}$ of dynamic regions which can be supplied with additional structure by various ways using the other components of the model. Before doing this let us make some observations and introduce some terminology.

Let $a = \langle a_i \rangle_{i \in T}$ and $b = \langle b_i \rangle_{i \in T}$ be two dynamic regions. Then $a \leq b$ (in the Boolean algebra $\mathcal{B}$ or in $\mathbb{B}$) iff $(\forall i \in T) (a_i \leq i b_i)$. If $a_i \neq 0_i$ for some $i \in T$ we say that $a$ exists at the moment $i$. It is possible for some dynamic region $a \neq 0$ to have many successive (with respect to $\prec$) moments of time in which it is alternatively existing and non-existing (for example viruses in biology). Also it is quite possible for two different regions $a$ and $b$ that there exists a moment of time $i$ (possibly not only one) such that $a_i = b_i$.

Example: before the World War II we have one Germany, after that for some time - two Germanies, West Germany and East Germany, now again one Germany, and what will be in the future we do not know. Note that in DMST coordinate contact algebras are presented as point-free spatial systems, but they can equivalently be presented by their point-based representative copies according to the representation theory of contact algebras. So, in DMST we do not have one space, but for each $i \in T$ a concrete local space $X_i$ with his own set of points. Of course all such observations put some ontological questions about the meaning of ”existence”, ”equality” and other abstract metaphysical concepts which we will not discuss in this paper.

**Remark 3.2.** Let us note that the analogy of ”snapshot construction” with making a video have to be considered more carefully and not literally, because video is based on visual observation. Normally what we (or camera) see is considered as existing at the moment of observation. But this is true only for objects which are not far from the observer. For instance seeing a star on the sky does not mean that this star is existing at the moment of observation.
it is quite possible that this star had ceased to exist a billion years before and this fact is based on the finite velocity of light. So, if we use a video (or some optic devices) for obtaining information for dynamically changing area of regions, for some of them which are far from the observer we need additional information for their status of existing and spatial configuration at the moment of observing. For instance, if I observe the Sun from which the light travels to the Earth several minutes I can conclude that it exists at the moment of observation, just because it is not possible for it to stop existing for such a short time. Having in mind the above, the phrase "snapshot at the moment t of the area of dynamic regions" has to be considered just as attaching to t the contact algebra \((B_t, C_t)\) considered as the real (actual) static description of spatial configurations of regions of the area at the moment t no matter how we can obtain this information. The analogy with video film is considered only as a way to illustrate the snapshot construction.

3.3 Standard dynamic contact algebras

Let \(M =<(T, \prec), \{ (B_i, C_i) : i \in T \}, B, B>\) be a given DMST. As we mentioned in the previous section, the Boolean algebra \(B\) of dynamic regions can be supplied with some additional relational structure in different ways. In this section we will give the first step introducing three spatio-temporal relations in \(B\).

- **Space contact** \(aC^s b\) iff \((\exists m \in T)(a_mC_mb_m)\).

  Intuitively space contact between \(a\) and \(b\) means that there is a time point \(i \in T\) in which \(a\) and \(b\) are in a contact \(C_i\) in the corresponding coordinate contact algebra \((B_i, C_i)\).

- **Time contact** \(aC^t b\) iff \((\exists m \in T)(a_m \neq 0_m \text{ and } b_m \neq 0_m)\).

  Intuitively time contact between \(a\) and \(b\) means that there exists a time point in which \(a\) and \(b\) exist simultaneously. Note that \(a_m \neq 0_m\) and \(b_m \neq 0_m\) means just that \(a\) and \(b\) exist at the time point \(m\). This relation can be considered also as a kind of **simultaneity relation** or **contemporaneity relation** studied in Whitehead’s works and special relativity.

- **Local precedence** or simply **Precedence** \(aBb\) iff \((\exists m, n \in T)(m < n \text{ and } a_m \neq 0_m \text{ and } b_n \neq 0_n)\).

  Intuitively \(a\) is in a local precedence relation with \(b\) (in words \(a \text{ precedes } b\)) means that there is a time point in which \(a\) exists which is before a time point
in which $b$ exists, which motivates the name of $B$ as a (local) precedence relation. Note the following similarity between the relations $C^t$ and $B$: if in the definition of $B$ we replace the relation $\prec$ with $=$, then we obtain just the definition of $C^t$.

**Lemma 3.3.** Let $M = (T, \prec), \{(B_i, C_i) : i \in T\}, B, B$ be a rich DMST. Then the relations $C^s$, $C^t$ and $B$ satisfy the following abstract conditions:

(i) $C^s$ is a contact relation,

(ii) $C^t$ is a contact relation satisfying the following additional conditions:

$(C^s \subseteq C^t)$ $aC^sb \rightarrow aC^tb$.

$(C^tE)$ $aC^tb \rightarrow (\exists c \in B)(aC^tc$ and $c^C b)$ - the Efremovich axiom for $C^t$.

(iii) $B$ is a precontact relation satisfying the following additional conditions (see for these conditions Section 2.3):

$(C^tB)$ $a\overline{B}b \Rightarrow (\exists c \in B)(aC^tc$ and $c^C b)$,

$(B C^t)$ $a\overline{B}b \Rightarrow (\exists c \in B)(aC^tc$ and $c^C b)$.

**Proof.** Let us note that the requirement that the model $M$ is rich is needed only in the verifications of the conditions $(C^tE)$, $(C^tB)$ and $(B C^t)$ which required constructions of new regions. As an example we shall verify only the condition $(B C^t)$. The proof for the other conditions is similar.

Suppose $a\overline{B}b$ and define $c$ coordinate-wise:

$$c_k = \begin{cases} 0_k, & \text{if } a_k \neq 0_k \\ 1_k, & \text{if } a_k = 0_k. \end{cases}$$

Since the model is rich then $c$ certainly belongs to $B$. The verification of the conclusion $aC^t c$ and $c^C b$ is straightforward.

\[ \square \]

**Definition 3.4.** Standard Dynamic Contact Algebra. Let $M = (T, \prec), \{(B_i, C_i) : i \in T\}, B, B$ be a DMST and let us suppose that the algebra $B$ of dynamic regions enriched with the relations $C^s$, $C^t$ and $B$ satisfies the conclusions of Lemma 3.3. Then the system $(B, C^s, C^t, B)$ is called standard dynamic contact algebra (standard DCA) over DMST.
Let us note that Lemma 3.3 ensures that standard DCAs exist. We call them "standard", because they are concrete and will be considered as standard models of abstract DCA (to be introduced and study later on). Shortly speaking the definition of abstract DCA is to rephrase the present definition in an abstract way. Let us remaind that the aim to start with concrete point-based model for spcetime is to use it as a source of motivated axioms.

3.4 A characterization of the abstract properties of time structures with some time axioms

We do not presuppose in the formal definition of DMST that the time structure \((T, \prec)\) satisfies some abstract properties of the precedence relation. In this section we shall see that all abstract properties of the precedence relation mentioned in Section 3.1 are in an exact correlation with some special conditions of time contact \(C^t\) and precedence relation \(B\) called time axioms. The correlation is given in the next table:

| Property | Formal Expression |
|----------|-------------------|
| **Right seriality** | \((\forall m)(\exists n)(m \prec n) \iff (rs) a \neq 0 \rightarrow aB1,\) |
| **Left seriality** | \((\forall m)(\exists n)(n \prec m) \iff (ls) a \neq 0 \rightarrow 1Ba,\) |
| **Updirectedness** | \((\forall i, j)(\exists k)(i \prec k \land j \prec k) \iff (up dir) a \neq 0 \land b \neq 0 \rightarrow aBp \lor bBp^*,\) |
| **Downdirectedness** | \((\forall i, j)(\exists k)(k \prec i \land k \prec j) \iff (down dir) a \neq 0 \land b \neq 0 \rightarrow pBb \lor p^*Bb,\) |
| **Closure** | \((\forall i, j)(\exists k)(i \prec j \rightarrow (\exists k)(k \prec i \land j \prec k)) \iff (cirk) aBb \rightarrow bBp \lor p^*Bb\) |
| **Density** | \((\forall i, j)(\exists k)(i \prec k \land k \prec j) \iff (dens) aBb \rightarrow aBp \lor p^*Bb,\) |
| **Reflexivity** | \((\forall m)(m \prec m) \iff (ref) aC^t b \rightarrow aBb,\) |
| **Irreflexivity** | \((\forall m)(m \neq m) \iff (irr) aBb \rightarrow (\exists c, d)(aC^t c \land bC^t d \land cC^t d),\) |
From here we obtain \( c \) regions \( a \in C \) such that \( i, j \). From this and \( b \neq 0 \) we get that \( aBb \) or \( bBa \).

**Lemma 3.5. Correspondence Lemma 1.** Let \( M = \langle \langle T, \prec \rangle, \{B_i, C_i \mid i \in T \}, B, B \rangle \) be a rich DMST and let \( B \) be enriched with the relations \( C_t \) and \( B \). Then all the correspondences in the above table are true in the following sense: the left site of a given equivalence is true in \( (T, \prec) \) iff the right site is true in \( B \).

**Proof.** We will show the proof for two cases: (Irr) and (Circ).

**Case 1:** (Irr) \( \iff \) (irr).

(Irr) \( \iff \) (irr). Suppose \( \text{Irr} \). This condition is equivalent also to the following one: \( m \prec n \implies m \neq n \). To prove (irr) suppose \( aBb \). Then there exist \( i, j \) such that \( a_i \neq 0 \), \( b_j \neq 0 \) and \( i \prec j \) which implies \( i \neq j \). Define the regions \( c \) and \( d \) coordinate-wise as follows:

\[
c_k = \begin{cases} 1_k, & \text{if } k = i \\ 0_k, & \text{if } k \neq i \end{cases}
\]

\[
d_k = \begin{cases} 1_k, & \text{if } k = j \\ 0_k, & \text{if } k \neq j \end{cases}
\]

From here we obtain \( c_i = 1_i \neq 0_i \) and \( d_j = 1_j \neq 0_j \). Since \( a_i \neq 0 \), we get \( aC^t c \). Since \( b_j \neq 0 \), we get \( bC^t d \). In order to show that \( cC^t d \) suppose the contrary: \( cC^t d \). This implies that there is \( k \in T \) such that \( c_k \neq 0_k \) and \( d_k \neq 0_k \). By the definitions of \( c \) and \( d \) we get that \( c_k = 1_k \) (and hence \( k = i \)) and \( d_k = 1_k \) (and hence \( k = j \)) and consequently \( i = j \) - a contradiction. Thus \( cC^t d \) which has to be proved.

(irr) \( \iff \) (Irr). Suppose (irr) and that (Irr) is not true. Then there exists \( i \) such that \( i \prec i \). Define \( a \) coordinate-wise as follows:

\[
a_k = \begin{cases} 1_k, & \text{if } k = i \\ 0_k, & \text{if } k \neq i \end{cases}
\]

From here we get that \( a_i = 1_i \neq 0_i \) and since \( i \prec i \) we obtain \( aBa \). By (irr) there are \( c \) and \( d \) such that \( aC^t c, aC^t d \) and \( cC^t d \). From the definition of \( a \) we have that \( a_k \neq 0_k \) only for \( k = i \). From this and \( aC^t c \) we get that \( c_i \neq 0_i \) and from \( aC^t d \) that \( d_i \neq 0_i \). Consequently \( cC^t d \) a contradiction with \( cC^t d \), which ends the proof.
Case 2: $(\text{Circ}) \iff (\text{circ})$.

$(\text{Circ}) \implies (\text{circ})$. Suppose that $(\text{Circ})$ is true. To prove $(\text{circ})$ suppose $a \mathcal{B} b$. Then there are $i, j \in T$ such that $a_i \neq 0_i$, $b_j \neq 0_j$ and $i \prec j$. By Circ there is a $k \in T$ such that $j \prec k$ and $k \prec i$. Let $p$ be arbitrary dynamic region. There are two cases: Case a: $p_k \neq 0_k$ which implies $p \mathcal{B} a$.

Case b: $p_k = 0_k$. Then $p_k^* = 1_k \neq 0_k$ which implies $b \mathcal{B} p^*$.

$(\text{circ}) \implies (\text{Circ})$. Suppose $(\text{circ})$ holds. In order to prove $(\text{Circ})$ suppose $i \prec j$. Define $a, b$ and $p$ as follows:

$$a_m = \begin{cases} 1_m, & \text{if } m = i \\ 0_m, & \text{if } m \neq i \end{cases}, \quad b_n = \begin{cases} 1_n, & \text{if } n = j \\ 0_n, & \text{if } n \neq j \end{cases}, \quad p_k = \begin{cases} 1_k, & \text{if } k \prec i \\ 0_k, & \text{if } k \nprec i \end{cases}.$$ 

By the definitions of $a$ and $b$ we obtain that $a_i \neq 0_i$ and $b_j \neq 0_j$. Since $i \prec j$ we get $a \mathcal{B} b$. By $(\text{Circ})$ we obtain $b \mathcal{B} p$ or $p^* \mathcal{B} a$. Consider the two cases separately.

Case I: $b \mathcal{B} p$. This implies that there exist $m, k \in T$ such that $n \prec k$, $b_n \neq 0_m$ (hence $b_n = 1_n$ and $n = j$) and $p_k \neq 0_k$ (and hence $p_k = 1_k$ and $k \prec i$). From here we get $j \prec k$ and $k \prec i$ - just what have to be proved.

Case II: $p^* \mathcal{B} a$. This implies that there exist $k, m \in T$ such that $k \prec m$, $p_k^* \neq 0_k$ (and hence $p_k^* = 1_k$, $p_k = 0_k$ and $k \nprec i$) and $a_m \neq 0_m$ (and hence $a_m = 1_m$ and $m = i$). From here we get $k \prec i$ which contradicts $k \nprec i$. So this case is impossible and the previous case implied what is needed.

Definition 3.6. The formulas $(\text{rs})$, $(\text{ls})$, $(\text{up dir})$, $(\text{down dir})$, $(\text{circ})$, $(\text{dens})$, $(\text{ref})$, $(\text{irr})$, $(\text{lin})$, $(\text{tri})$, $(\text{tr})$, included in the above table are called "time axioms" and will be considered as additional axioms for abstract DCAs.

The above lemma is very important because it states that the abstract properties of the time structure of a given rich model of space are determined by the time axioms which contain only variables for dynamic regions and time points are not mention. This correlation suggests to consider (abstract) DCAs satisfying some of the time axioms.

3.5 Time representatives and NOW

In this section, following [76] we present another enrichment of the expressive power of standard DCA by new constructs called time representatives, universal time representatives and NOW. Since this material will not be
used later on in this paper, the presentation is sketchy and without proofs. For more details the reader is invited to consult [76].

First about the intuitions behind these notions. Consider the phrases: ”the epoch of Leonardo”, ”the epoch of Renaissance”, ”the geological age of the dinosaurs”, ”the time of the First World War”, etc. All these phrases indicate a concrete unit of time named by something which happened or existed at that time and not in some other moment (epoch) of time. These examples suggest to introduce in DMST a special set of dynamic regions called time representatives, which are regions existing at a unique time point. The formal definition is the following:

**Definition 3.7.** A region $c$ in a DMST is called a time representative if there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i$, $c_j = 0_j$. We say also that $c$ is a representative of the time point $i$ and indicate this by writing $c = c(i)$. In the case when $c_i = 1_i$, $c$ is called universal time representative. We denote by $TR$ the set of universal time representatives and by $UTR$ the set of universal time representatives.

Time representatives and universal time representatives always exist in rich models. Let $i \in T$, then the following region $c = c(i)$ is the universal time representative corresponding to the time point $i$:

$$c_k = \begin{cases} 1_k, & \text{if } k = i \\ 0_k, & \text{if } k \neq i. \end{cases}$$

If for a given $i \in T$ there exists $a$ such that $a_i \neq 0_i$ and $a_i \neq 1_i$ then $c.a$ is time representative of $i$ which is not universal time representative.

The existence of universal time representatives for each $i \in T$ suggests to consider enriched time structures $(T, \prec, \text{now})$, where now is a fixed element of $T$ corresponding to the present epoch. We denote by NOW the universal time representative of now. Let us note that the extension of the language of standard DCA with time representatives and NOW enriches considerably its expressive power and makes possible to consider Past, Present and Future. Examples:

- $a$ exists now - $a \text{NOW}$,
- $a$ will exist in the future - $\text{NOW} \triangleright a$,
• a will always exist in the future - \( (\forall c \in TR)(NOW \exists bc \rightarrow aC^t c) \),
• a was existing in the past - \( a \exists NOW \),
• a is in a contact with b now - \( a \exists NOWCs b \),
• a will be in a contact with b - \( (\exists c \in UTR)(NOW \exists bc \text{ and } a.eC^t sb) \),
• a and b are always in a contact - \( (\forall c \in UTR)(a.eC^t sb) \).

For more information about time representatives see [76]

4 Dynamic contact algebra (DCA)

We adopt in this paper the following definition of abstract dynamic contact algebra.

**Definition 4.1.** The algebraic system \( A = (B_A, C^s_A, C^t_A, B_A) \) is called dynamic contact algebra (DCA) provided the following conditions are satisfied:

\( (BA) \) \( B_A = (B_A, \leq, 0, 1, +, *, \cdot) \) is a nondegenerate Boolean algebra.
\( (CC^s) \) \( C^s_A \) is a contact relation in \( B_A \), called space contact,
\( (CC^t) \) \( C^t_A \) is a contact relation in \( B_A \), called time contact and satisfying the following two axioms:
\( (C^s \subseteq C^t) \) \( aC^s_A b \Rightarrow aC^t_A b \).
\( (C^tE) \) \( aC^t_A b \Rightarrow (\exists c)(aC^t_A c \text{ and } c^*C^t_A) \), the Efremovich axiom for \( C^t_A \).
\( (PreC^\exists B) \) \( B_A \) is a precontact relation in \( B_A \), called local precedence and satisfying the following two axioms:
\( (C^tB) \) \( a\overline{B}A b \Rightarrow (\exists c)(a\overline{C}A c \text{ and } c^*\overline{B}A b) \).
\( (BC^t) \) \( a\overline{C}A B \Rightarrow (\exists c)(a\overline{B}A c \text{ and } c^*\overline{C}A b) \).

We consider also DCA satisfying additionally some of the time axioms \( (rs), (ls), (up \ dir), (down \ dir), (circ), (dens), (ref), (lin), (tri), (tr) \) (see Definition 3.6). (Note that here the axiom \( (irr) \) is excluded for reasons which will be explained later, see Remark 4.10).
Since DCAs are algebraic systems we adopt the standard algebraic notions of isomorphism between two DCAs $A_1$ and $A_2$ and isomorphic embedding of $A_1$ into $A_2$. If $A_1$ and $A_2$ are isomorphic we will denote this by $A_1 \cong A_2$.

Note that the name "dynamic contact algebra" is used in the papers [74, 75, 76, 13] as an integral name for point-free theories of space and time with different definitions in different papers. This is just for economy of names. The definition used in [76] incorporates also time representatives but for the purposes of this paper we decided to adopt more simple definition which is based only on the relations $C^s$, $C^t$ and $B$. It is similar to the definition of DCA from [75], but the present definition is based on a more strong axioms, so it has a different theory. Note also that the just introduced DCA has models - these are the standard DCAs from Definition 3.4 and they will be considered as standard models of the present definition of DCA. Our first aim is to show that DCAs are representable by means of models.

**Lemma 4.2.** DCA is a generalization of CA.

*Proof.* Let $A = (B_A, C^s_A, C^t_A, B_A)$ be a contact algebra. Set $C^s_A = C_A$, $aC^t_A b$ iff $a \neq 0$ and $b \neq 0$ (the maximal contact of $A$) and $B_A = C^t_A$. Then it is easy to see that $A$ with thus defined relations is a DCA.

**Remark 4.3.** One note to the Lemma 4.2. If we interpret contact algebras as dynamic contact algebras as in Lemma 4.2 the obtained reinterpretation of contact algebra has topological models which are different from the standard topological models of contact algebras (see section 5.5). So the stated equivalence in the Lemma 4.2 is only about the corresponding algebraic structures.

It is true if we consider CA with an additional contact - the definable maximal contact $(C_{max})_A$ with $a(C_{max})_A b \iff a \neq 0$ and $b \neq 0$. Such extended contact algebras have topological models which are different from the standard topological models of contact algebras (see section 5.5).

**4.1 Facts about ultrafilters, clans and clusters in DCA**

Let $A = (B_A, C^s_A, C^t_A, B_A)$ be a DCA. We denote by $Ult(A)$ the set of ultrafilters of $A$ and by $R^s_A$, $R^t_A$ and $\prec_A$ we denote correspondingly the canonical relations of $C^s_A, C^t_A$ and $B_A$ (for the definition of canonical relation see Definition 2.6). Since $C^s_A$ and $C^t_A$ are contact relations, then $R^s_A$ and $R^t_A$ are reflexive and symmetric relations (Lemma 2.11). Since $C^t_A$ satisfies the
Efremovich axiom \((C^tE)\), the relation \(R^t_A\) is transitive (Lemma 2.11), which implies the following statement:

The relation \(R^t_A\) is an equivalence relation. \hfill (1)

By the axioms \((C^tB)\) and \((BC^s)\) the relation \(<_A\) satisfies the following conditions (see Lemma 2.12) for arbitrary \(U, V, W \in \text{Ult}(A)\):

\[
(R^t \circ<_A) UR^t_A V \text{ and } V<_A W \Rightarrow U<_A W, \hfill (2)
\]

\[
(_A \circ R^t \subseteq_A) U<_A V \text{ and } VR^t_A W \Rightarrow U<_A W. \hfill (3)
\]

Conditions (2) and (3) imply the following more general condition

\[
UR^t_A U_0 \text{ and } U_0<_A V_0 \text{ and } V_0 R^t_A V \Rightarrow U<_A V. \hfill (4)
\]

The clans determined by the contact \(C^t_A\) are called s-clans and their set is denoted by \(s\text{-Clans}(A)\). The clans determined by \(C^s_A\) are called t-clans and their set is denoted by \(t\text{-Clans}(A)\). By axiom \((C^s \subseteq C^t)\) every s-clan is a t-clan. Note that every ultrafilter is both an s-clan and a t-clan. So we have the inclusions:

\[
\text{Ult}(A) \subseteq s\text{-Clans}(A) \subseteq t\text{-Clans}(A). \hfill (5)
\]

If \(\Gamma\) is a t-clan we denote by \(\text{Ult}(\Gamma)\) the set of ultrafilters included in \(\Gamma\). \hfill (6)

By axiom \((C^tE)\) maximal t-clans are clusters and by Lemma 2.23 they are unions of the equivalence classes of ultrafilters determined by the equivalence relation \(R^t_A\). The set of clusters is denoted by \(\text{Clust}(A)\). Note that (see Lemma 2.22)

Every t-clan (s-clan) is contained in a unique cluster. \hfill (7)

So there is a function \(\gamma_A: t\text{-Clans}(A) \rightarrow \text{Clusters}(A)\) with the following properties:

\[
(\gamma_1) \text{ If } \Gamma \in t\text{-Clans}(A), \text{ then } \gamma_A(\Gamma) \in \text{Clust}(A),
\]

\[
(\gamma_2) \text{ If } \Gamma \in \text{Clust}(A), \text{ then } \gamma_A(\Gamma) = \Gamma. \hfill (8)
\]

Now we extend the relation \(<\) to hold between t-clans (and hence between clusters) by the same definition used for ultrafilters: for \(\Gamma, \Delta \in t\text{-Clans}(A)\)

\[
\Gamma<_A \Delta \iff_{def} \forall a, b \in B_A)(a \in \Gamma \text{ and } b \in \Delta \Rightarrow aB_A b). \hfill (9)
\]
Lemma 4.4. The following conditions are equivalent for any \( \Gamma, \Delta \in t\text{-}\text{Clans}(A) \):

(i) \( \Gamma \prec_A \Delta \),
(ii) For all \( U \in \text{Ult}(\Gamma) \) and \( V \in \text{Ult}(\Delta) \): \( U \prec_A V \),
(iii) There exist \( U_0 \in U\text{LT}(\Gamma) \) and \( V_0 \in U\text{lt}(\Delta) \) such that: \( U_0 \prec_A V_0 \).

Proof. (i)\( \Rightarrow \) (ii). Suppose (i) holds and to prove (ii) suppose \( a \in U \in \text{Ult}(\Gamma) \) and \( b \in V \in \text{Ult}(\Delta) \). Then \( a \in \Gamma \) and \( b \in \Delta \) and by (i) and (9) we get \( aB_b \) which proves (ii).

(ii)\( \Rightarrow \) (iii) is obvious.

(iii)\( \Rightarrow \) (i). Suppose (iii): \( U_0 \prec_A V_0 \) for some \( U_0 \in U\text{LT}(\Gamma) \) and \( V_0 \in U\text{LT}(\Delta) \). In order to show (i) suppose \( a \in \Gamma \) and \( b \in \Delta \) and proceed to show that \( aB_b \).

Since \( a \in \Gamma \), then there exist an ultrafilter \( U \) such that \( a \in U \in \text{Clans}(\Gamma) \) and an ultrafilter \( V \) such that \( b \in V \in \text{Clans}(\Delta) \) (see Lemma 2.15). Then \( U \text{R}_A U_0 \) and \( V \text{R}_A V_0 \). Since \( U_0 \prec_A V_0 \), then by (4) we get \( U \prec_A V \). But \( a \in U, b \in V \) and \( U \prec_A V \) imply \( aB_b \).

Lemma 4.5. For all t-clans \( \Gamma, \Delta \) if \( \Gamma \prec_A \Delta \), then there exists a cluster \( \Gamma' \) and a cluster \( \Delta' \) such that \( \Gamma \subseteq \Gamma' \) and \( \Delta \subseteq \Delta' \) and \( \Gamma' \prec_A \Delta' \).

Proof. The proof follows from the fact that every t-clan can be extended into unique cluster and the relation \( \prec_A \) between extensions is preserved by the properties of this relation stated in Lemma 4.4.

The next three definitions will be used later on. For \( a \in B_A \) set:

\[
g_A(a) = \text{def} \{ \Gamma \in t\text{-}\text{Clans}(A) : a \in \Gamma \}; \tag{10}
g_A^s(a) = \text{def} \{ \Gamma \in s\text{-}\text{Clans}(A) : a \in \Gamma \} = g_A(a) \cap s\text{-}\text{Clans}(A), \tag{11}
g_A^{\text{clust}}(a) = \text{def} \{ \Gamma \in \text{Clusters}(A) : a \in \Gamma \} = g_A(a) \cap \text{Clusters}(A). \tag{12}
\]

Lemma 4.6. The following equivalencies are true for arbitrary \( a, b \in B_A \):

(i) \( aC^t_A b \text{ iff there exists a } t\text{-} \text{clan (cluster) } \Gamma \text{ containing } a \text{ and } b \text{ iff } g_A(a) \cap g_A(b) \neq \emptyset \text{ (see (10) and (12)).} \)

(ii) \( aC^s_A b \text{ iff there exists an } s\text{-} \text{clan } \Gamma \text{ containing } a \text{ and } b \text{ iff } g_A^s(a) \cap g_A^s(b) \neq \emptyset \text{ (see (11))} \).
(iii) $a \mathcal{B}_A b$ iff there exist $t$-clans (clusters) $\Gamma, \Delta$ such that $\Gamma \prec \Delta$, $a \in \Gamma$ and $b \in \Delta$ iff there exist $t$-clans (clusters) $\Gamma, \Delta$ such that $\Gamma \prec \Delta$ and $g_A(a) \neq \emptyset, g_A(b) \neq \emptyset$ (see (10) and (12)).

Proof. (i) and (ii) follow from Lemma 2.15 and definitions (10), (11) and (12). For (iii) suppose $a \mathcal{B}_A b$. Then by Lemma 2.10 there are ultrafilters $U, V$ such that $U \prec_A V$. Then there are clusters $\Gamma, \Delta$ such that $U \subseteq \Gamma$ and $V \subseteq \Delta$, so $a \in \Gamma$ and $b \in \Delta$. By Lemma 4.4 we obtain that $\Gamma \prec_A \Delta$. The converse implication follows from the definition of $\prec$.

The next lemma is a more detailed reformulation of Lemma 4.6 which will be used in Section 4.3.

**Lemma 4.7.**

(i) $a \mathcal{C}_A^s b$ iff there exists a cluster $\Gamma$ and an $s$-clan $\Delta$ containing $a$ and $b$ such that $\Delta \subseteq \Gamma$.

(ii) $a \mathcal{C}_A^t b$ iff there exist a cluster $\Gamma$ and $s$-clans $\Delta, \Theta$ such that $a \in \Delta$, $b \in \Theta$ and $\Delta, \Theta \subseteq \Gamma$.

(iii) $a \mathcal{B}_A b$ iff there exist clusters $\Gamma, \Delta$, such that $\Gamma \prec \Delta$ and there exist $s$-clans $\Theta \subseteq \Gamma$ and $\Lambda \subseteq \Delta$, $a \in \Theta$ and $b \in \Lambda$.

(iv) $a \not\leq b$ iff $a.b^* \neq 0$ iff there exists a cluster $\Gamma$ and an $s$-clan $\Delta \subseteq \Gamma$ such that $a.b^* \in \Delta$.

Proof. The proof follows from Lemma 4.6 and the fact that every $s$-clan and $t$-clan is contained in a cluster.

The system $(s\text{-Clans}(A), t\text{-Clans}(a), \text{Clusters}(A), \gamma_A, \prec_A)$ is called the clan structure of $A$.

Since any contact algebra is a DCA (Lemma 4.2) it is interesting to know which are $s$-clans, $t$-clans and clusters of $A$. Obviously $s$-clans are just the clans of $A$ with (respect to $C$), $t$-clans are just the grills of $A$ (they are unions of ultrafilters). There is only one maximal grill in $A$ - the union of all ultrafilters and this is the unique cluster in $A$ (with respect to $C_A^t$). The relation $\prec$ is just the universal relation in the set of all grills.

### 4.2 Extracting the time structure of DCA

Let $A = (B_A, C_A^s, C_A^t, \mathcal{B}_A)$ be a DCA. The first step to represent $A$ in some DMSP by the snapshot construction is to extract the time structure of $A$. 

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This means to define the time points of $A$ and the corresponding "before-after" relation. From Lemma 4.6 we see that the relations $C_A^t$ and $B_A$ which have a temporal nature can be characterized by means of clusters. This suggests the time points of $A$ to be identified with the clusters of $A$ and the before-after relation to be identified with the relation $\prec$ defined by (9) and restricted to the set of clusters. So we have the following

**Definition 4.8. Canonical time structure.** The system

$$T_A = (\text{Clusters}(A), \prec_A)$$

where $\prec_A$ is restricted to $\text{Clusters}(A)$ is considered as the canonical time structure of $A$.

It is interesting to see if there is a correspondence between time properties of $T_A$ and the corresponding time axioms like in Lemma 3.5. This is possible for all time conditions except (Irr). First we will present ultrafilter characterization of time axioms by means of conditions on the set $\text{Ult}(A)$ expressible by the canonical relations $R^t_A$ and $\prec_A$, considered as a relation between ultrafilters (so these conditions will be for the structure $(\text{Ult}(A), \prec_A, R^t_A)$).

The corresponding table is the following. Note that the names of ultrafilter conditions are the same for the names for the corresponding time conditions from Section 3.1. enclosed by curly brackets. $U, V, W$ below are considered as variables ranging on ultrafilters.

| Condition | Symbol | Description |
|-----------|--------|-------------|
| RS        | $\forall U \exists V (U \prec_A V)$ | $a \neq 0 \rightarrow aB1,$ |
| RS        | $\forall U \exists V (U \prec_A V)$ | $a \neq 0 \rightarrow aB1,$ |
| LS        | $\forall U \exists V (V \prec_A U)$ | $a \neq 0 \rightarrow 1B_a,$ |
| LS        | $\forall U \exists V (V \prec_A U)$ | $a \neq 0 \rightarrow 1B_a,$ |
| Up Dir    | $\forall U \exists V (W \prec_A U$ and $V \prec_A W)$ | $a \neq 0 \wedge b \neq 0 \Rightarrow aBp$ or $bBp^*,$ |
| Up dir    | $\forall U \exists V (W \prec_A U$ and $V \prec_A W)$ | $a \neq 0 \wedge b \neq 0 \Rightarrow aBp$ or $bBp^*,$ |
| Down Dir  | $\forall U, V (\exists W) (W \prec_A U$ and $W \prec_A V)$ | $a \neq 0 \wedge b \neq 0 \Rightarrow pB_a$ or $p^*B_b,$ |
| Down dir  | $\forall U, V (\exists W) (W \prec_A U$ and $W \prec_A V)$ | $a \neq 0 \wedge b \neq 0 \Rightarrow pB_a$ or $p^*B_b,$ |
| Circ      | $aBb \Rightarrow \exists W (W \prec_A U$ and $V \prec_A W)$ | $aBb \Rightarrow \exists W (W \prec_A U$ and $V \prec_A W)$ |
| Circ      | $aBb \Rightarrow \exists W (W \prec_A U$ and $V \prec_A W)$ | $aBb \Rightarrow \exists W (W \prec_A U$ and $V \prec_A W)$ |
| Dens      | $aBb \Rightarrow \exists W (U \prec_A W$ and $W \prec_A V)$ | $aBb \Rightarrow \exists W (U \prec_A W$ and $W \prec_A V)$ |
| Dens      | $aBb \Rightarrow \exists W (U \prec_A W$ and $W \prec_A V)$ | $aBb \Rightarrow \exists W (U \prec_A W$ and $W \prec_A V)$ |
| Ref       | $\forall U (U \prec_A U)$ | $\forall U (U \prec_A U)$ |
\(aCt b \Rightarrow aBb,\)
\(\langle \text{Lin} \rangle (\forall U, V)(U \prec V \text{ or } V \prec U) \iff\)
\(\langle \text{Tri} \rangle (\forall U, V)(UR_A V \text{ or } U \prec_A V \text{ or } V \prec_A U) \iff\)
\(\langle \text{Tr} \rangle U \prec_A V \text{ and } V \prec_A W \Rightarrow U \prec_A W \iff\)
\((\text{ref})\ aCt b \Rightarrow aBb,\)
\(\langle \text{Lin} \rangle (\forall U, V)(U \prec V \text{ or } V \prec U) \iff\)
\(\langle \text{Tri} \rangle (\forall U, V)(UR_A V \text{ or } U \prec_A V \text{ or } V \prec_A U) \iff\)
\(\langle \text{Tr} \rangle U \prec_A V \text{ and } V \prec_A W \Rightarrow U \prec_A W \iff\)
\((\text{lin})\ a \neq 0 \text{ and } b \neq 0 \Rightarrow aBb \text{ or } bBa,\)
\(\langle \text{Tri} \rangle (\forall U, V)(UR_A V \text{ or } U \prec_A V \text{ or } V \prec_A U) \iff\)
\(\langle \text{Tr} \rangle U \prec_A V \text{ and } V \prec_A W \Rightarrow U \prec_A W \iff\)
\((\text{tri})\ a \neq 0 \text{ and } b \neq 0 \Rightarrow aCt_A b \Rightarrow aB_A b \text{ or } bB_A a,\)
\(\langle \text{Tri} \rangle (\forall U, V)(UR_A V \text{ or } U \prec_A V \text{ or } V \prec_A U) \iff\)
\(\langle \text{Tr} \rangle U \prec_A V \text{ and } V \prec_A W \Rightarrow U \prec_A W \iff\)

The table with clusters can be obtained from the above one replacing ultrafilter variables \(U, V, W\) with cluster variables \(\Gamma, \Delta, \Theta\) and \(R^t_A\) (which occurs only in the condition \(\langle \text{Tr} \rangle\)) with equality =.

**Lemma 4.9. Correspondence Lemma 2.** The following equivalencies are true for each raw of the above table:

(i) The left-side condition is true in the structure \((Ult(A), \prec_A, R^t_A)\).

(ii) The left-side condition in its cluster interpretation is true in the canonical time structure \((Clusters(A), \prec_A)\).

(iii) The right-side condition is true in \(DCA A\).

Proof. We illustrate the proof checking three examples. Let us start with the easiest case - (ref). We will prove the following implications:

(i) \(\forall U \in Ult(A)(U \prec_A U) \Rightarrow (\forall \Gamma \in Clusters(A))(\Gamma \prec_A \Gamma) \Rightarrow\)

(ii) \(\forall a, b \in B_A (aCt_A b \Rightarrow aB_A b) \Rightarrow (i).\)

(i) \(\Rightarrow\) (ii). Suppose (i) and to prove (ii) suppose that \(\Gamma \in Clusters(A)\) and that an ultrafilter \(U_0 \subseteq \Gamma\). By (i) \(U_0 \prec_A U_0\) and by Lemma 4.4 we get that \(\Gamma \prec_A \Gamma.\)

(ii) \(\Rightarrow\) (iii). Suppose (ii) and in order to show (iii) suppose \(aCt_A b\) and proceed to show \(aB_A b\). Condition \(aCt_A b\) implies that there is a cluster \(\Gamma\) containing \(a\) and \(b\). By (ii) we have \(\Gamma \prec_A \Gamma\). But \(a \in \Gamma\) and \(b \in \Gamma\) implies (by the definition of \(\prec\)) that \(aB_A b\).

(iii) \(\Rightarrow\) (i). Suppose (iii) and in order to prove (i) suppose that \(U \in Ult(B)\) and \(a, b \in U\). Then \(a.b \neq 0\) which implies \(aCt_A b\) (\(Ct_A\) is a contact relation) and hence by (iii) we get that \(aB_A b\). By the definition of the canonical relation \(\prec_A\) for ultrafilters, this shows that \(U \prec_A U\).

The next example is (tri). We will prove the following implications:
(i) \( UR_A^t V \) or \( U \prec_A V \) or \( V \prec_A U \) \( \implies \) (ii) \( \Gamma = \Delta \) or \( \Gamma \prec_A \Delta \) or \( \Delta \prec_A \Gamma \) \( \implies \) (iii) \( aCt \) and \( bCt \) and \( cCt \) \( \implies \) \( aBb \) or \( bBa \) \( \implies \) (i).

(i) \( \implies \) (ii). Suppose (i) and let \( \Gamma, \Delta \in \text{Clusters}(A) \). To show (ii) suppose that \( \Gamma, \Delta \in \text{Clusters}(A) \). If \( \Gamma = \Delta \), then (ii) is OK. Suppose \( \Gamma \neq \Delta \). Then by Lemma 2.24 there exist \( a \notin \Gamma \) and \( b \notin \Delta \) such that \( aCt \) and \( bCt \). Consequently there are ultrafilters \( U, V \) such that \( a \in U \in \text{Ult}(\Gamma) \) and \( b \in V \in \text{Ult}(\Delta) \). Since \( aCt \) and \( bCt \), then \( U \prec A V \). This implies by (i) that \( U \prec A V \) or \( V \prec A U \). Since \( U \subseteq \Gamma \) and \( V \subseteq \Delta \), then by Lemma 4.4 we get \( \Gamma \prec_A \Delta \) or \( \Delta \prec_A \Gamma \).

(ii) \( \implies \) (iii). Suppose (ii) and in order to show (iii) suppose \( a \neq 0 \) and \( b \neq 0 \). Then there are \( \Gamma, \Delta \in \text{Clusters}(A) \) such that \( a \in \Gamma \) and \( b \in \Delta \). By (ii) there are three cases:

**Case I:** \( \Gamma = \Delta \). Then \( aCt \).

**Case II:** \( \Gamma \prec_A \Delta \). Then \( aBt \).

**Case III:** \( \Delta \prec_A \Gamma \). Then \( bBt \).

(iii) \( \implies \) (i). Suppose (iii) and for the sake of contradiction assume that (i) is not true. Then there are ultrafilters \( U, V \) such that \( U \prec A V \) and \( V \prec A U \). Then there are \( a_1, b_1 \) such that \( a_1 \in U \), \( b_1 \in V \) and \( a_1Ct \), there are \( a_2, b_2 \) such that \( a_2 \in U \), \( b_2 \in V \) and \( a_2Ct \), and there are \( a_3, b_3 \) such that \( a_3 \in U \), \( b_3 \in V \) and \( a_3Ct \). Let \( a = a_1a_2a_3 \) and \( b = b_1b_2b_3 \). Since \( U \), \( V \) are ultrafilters then \( a \in U \) and \( b \in V \), so \( a \neq 0 \) and \( b \neq 0 \). It can be shown also that \( aCt \), \( aBt \) and \( bBt \) which contradicts (iii).

Let us consider as a last example (tr). By Lemma 2.11 we already know that (i) \( \iff \) (iii). It remains to show (i) \( \iff \) (ii).

(i) \( \implies \) (ii). Suppose (i) and in order to prove (ii) suppose that \( \Gamma \prec_A \Delta \) and \( \Delta \prec_A \Theta \). Suppose for the contrary that \( \Gamma \not\prec_A \Theta \). Then by Lemma 4.4 there are ultrafilters \( U \in \text{Ult}(\Gamma) \) and \( W \in \text{Ult}(\Theta) \) such that \( \Gamma \not\prec_A W \). Then by (i) \( U \not\prec_A V \) or \( V \not\prec_A W \) for any \( V \in \text{Ult}(B) \). Take some \( V \in \text{Ult}(\Delta) \).

**Case I:** \( U \not\prec_A V \). Then \( U \in \text{Ult}(\Gamma) \), \( V \in \text{Ult}(\Delta) \) and \( \Delta \prec_A \Delta \) implies \( U \prec_A V \) - a contradiction.

**Case II:** \( V \not\prec_A W \). Then \( V \in \text{Ult}(\Delta) \), \( W \in \text{Ult}(\Theta) \) and \( \Delta \prec_A \Theta \) implies \( V \prec_W \) - a contradiction.

(ii) \( \implies \) (i). Suppose (ii) and in order to show (i) suppose \( U \prec_A V \) and \( V \prec_A W \). Take some \( U \subseteq \Gamma \), \( V \subseteq \Delta \) and \( W \subseteq \Theta \). By Lemma 4.4 we get \( \Gamma \prec_A \Delta \) and \( \Delta \prec_A \Theta \). By (ii) this implies \( \Gamma \prec_A \Theta \). But \( U \subseteq \Gamma \) and \( W \subseteq \Theta \) which implies \( U \prec_A W \).
One remark for the proofs of the remaining cases of this lemma is to show first the equivalence \((i) \implies (iii)\) follow in the style of the proof of Lemma 2.11 and Lemma 2.12. Then the proof of \((i) \implies (ii)\) is more easy by application of Lemma 4.4.

**Remark 4.10.** Let us explain why we excluded the axiom \((\text{irr})\) from the list of time axioms and the Correspondence Lemma. The reason is that we can not prove the equivalence \(\langle \text{Irr} \rangle \iff (\text{irr})\). One can easily proof the implication \(\langle \text{Irr} \rangle \implies (\text{irr})\), but we do not know if the converse has a proof (we believe not) or if there is a stronger first-order sentence like \((\text{irr})\) for which the equivalence holds. This equivalence is true in rich standard DCA and the reason is the possibility to define special regions due to richness. The language of the abstract version of DCA can not express a property similar to richness but in a DCA enriched with time representatives discussed in Section 3.5 the treatment of this case is possible because the language is more expressive (see [76]).

Since any contact algebra \(A\) is a DCA which is the canonical time structure of \(A\)? The set \(T\) of time points is the singleton set \(\{\Gamma\}\) where \(\Gamma\) is the maximal grill in \(A\) (the union of all ultrafilters) and \(<\) is just the equality. So the time of \(A\) has only one moment and the clock of \(A\) is not ticking - the time is "stopped" or degenerated. That is why contact algebras can be considered as static (no time is hidden in them) and the RBTS based on contact algebras - as a static mereotopology.

### 4.3 Extracting canonical coordinate contact algebras and the canonical standard DCA

Let \(A = (B_A, C_A^r, C_A^l, B_A)\) be a DCA and let \(T_A = (\text{Clusters}(A), \prec_A)\) be the canonical time structure of \(A\). The next step in the snapshot construction is for each \(\Gamma \in \text{Clusters}(A)\) to define in a canonical way the coordinate contact algebra \(A_\Gamma = (B_\Gamma, C_\Gamma)\).

Because \(\Gamma\) is a cluster, consider the set
\[
\hat{\Gamma} = \{\Delta \in s-\text{Clans}(A) : \Delta \subseteq \Gamma\}.
\]

We will consider the construction of factor contact algebra determined by sets of clans described in Section 2.6. So we adopt the following definition.

**Definition 4.11.** Canonical coordinate contact algebra. We define \((B_\Gamma, C_\Gamma)\), denoted for simplicity by \(B_\Gamma = (B_\Gamma, C_\Gamma)\) to be the contact algebra
defined by the factor construction from Sections 2.6 applied to the contact algebra \((B_A, C_A^s)\) and the set of s-clans \(\widehat{\Gamma}\). The algebra \((B_\Gamma, C_\Gamma)\) is called the **canonical coordinate contact algebra** corresponding to the time point \(\Gamma\).

Remaind that the elements of \(B_\Gamma\) are now of the form \(|a|_\Gamma\) defined by the congruence \(\equiv_{\widehat{\Gamma}}\) (see Section 2.6) and \(|a|_\Gamma |C_\Gamma| b|_\Gamma\) iff \(\widehat{\Gamma} \cap g(a) \cap g(b) \neq \emptyset\), where 

\[g(a) = \{\Gamma \in s\text{-Clans}(A); a \in \Gamma\} \]

**Definition 4.12. Canonical standard DCA.** Having the canonical time structure \(T_B = (\text{Clusters}(A), \prec_A)\) and the set of canonical contact algebras \(A_\Gamma = (B_\Gamma, C_\Gamma), \Gamma \in \text{Clusters}(A)\) we define by the snapshot construction described in Sections 3.2 and 3.3 the full **canonical standard DCA** \(A^{\text{can}} = (B, C^s, C^t, B)\), where 

\[B = \prod_{\Gamma \in \text{Clusters}(A)} B_\Gamma\]

is the Cartesian product of the coordinate Boolean algebras.

We define an embedding function \(h\) from \(A\) into \(A^{\text{can}}\) coordinatewise as follows: for \(a \in B_A\) and for each \(\Gamma \in \text{Clusters}(A)\), 

\[h_\Gamma(a) = |a|_\Gamma\]

The next lemma is important because it shows that the time axioms are preserved by the construction of the full canonical standard DCA.

**Lemma 4.13.** Let \(A\) be a DCA and \(A^{\text{can}}\) be the full canonical standard dynamic contact algebra associated to \(A\). Then for each time axiom \(\alpha\) from the list of time axioms \((\text{rs}), (\text{ls}), (\text{up dir}), (\text{down dir}), (\text{circ}), (\text{dens}), (\text{ref}), (\text{lin}), (\text{tri}), (\text{tr})\) the following equivalence is true: \(\alpha\) holds in \(A\) iff \(\alpha\) holds in \(A^{\text{can}}\).

**Proof.** By Lemma 4.9 \(\alpha\) is true in \(A\) iff the corresponding condition \(\hat{\alpha}\) is true in the canonical time structure \(T_B = (\text{Clusters}(A), \prec_A)\) iff (by Lemma 3.5) \(\alpha\) is true in the full standard DCA \(A^{\text{can}}\). \(\square\)

**Lemma 4.14. Embedding Lemma.** Let \(A\) be a DCA and \(h\) be the mapping defined in Definition 4.12. Then:

(i) \(h\) preserves Boolean operations.

(ii) \(aC^s b\) in \(A\) iff there exists \(\Gamma \in \text{Clusters}(a)\) such that \(|a|_\Gamma C_\Gamma |b|_\Gamma\) iff \(h(a)C^s_{A^{\text{can}}} h(b)\) in \(A^{\text{can}}\).

(iii) \(aC^t b\) in \(A\) iff there exists \(\Gamma \in \text{Clusters}(A)\) such that \(|a|_\Gamma \neq |0|_\Gamma\) and \(|b|_\Gamma \neq |0|_\Gamma\) iff \(h(a)C^t_{A^{\text{can}}} h(b)\) in \(A^{\text{can}}\).
(iv) \( a \mathbb{B}_A b \) in \( A \) iff there exist \( \Gamma, \Delta \in \text{Clusters}(A) \) such that \( \Gamma \prec \Delta \) and \( |a|_\Gamma \neq |0|_\Gamma \) and \( |b|_\Delta \neq |0|_\Delta \) iff \( h(a) \mathbb{B}(A)_{\text{A}_{\text{can}}} h(b) \) in \( \text{A}_{\text{can}} \).

(v) \( a \nleq b \) in \( A \) iff there exist \( \Gamma \in \text{Clusters}(A) \) such that \( |a|_\Gamma \nleq |b|_\Gamma \) iff \( h(a) \nleq h(b) \) in \( \text{A}_{\text{can}} \).

(vi) \( a = b \) iff \( h(a) = h(b) \), i.e. \( h \) is an embedding.

\textbf{Proof.} (i) The statement is obvious, because the elements of the coordinate algebras are equivalence classes determined by a congruence relations in \( A \) and that Boolean operations in \( \text{A}_{\text{can}} \) are defined coordinatewise.

(ii) \( aC^*_A b \) in \( A \) iff (by Lemma 4.7) there exist a cluster \( \Gamma \) and s-clans \( \Delta, \Theta \) such that \( a \in \Delta, b \in \Theta \) and \( \Delta, \Theta \subseteq \Gamma \) iff (by the definition of \( \hat{\Gamma} \) and \( g \), see (11), (12)) there exists \( \Gamma \in \text{Clusters}(A) \) such that \( \hat{\Gamma} \cap g(a) \cap g(b) \neq \emptyset \) iff (by the factorization construction) there exist \( \Gamma \in \text{Clusters}(A) \) such that \( |a|_\Gamma C \cap |b|_\Gamma \) iff \( h(a)C^s_{\text{A}_{\text{can}}} h(b) \) in \( \text{A}_{\text{can}} \).

(iii) \( aC'_A b \) in \( A \) iff (by Lemma 4.7) there exist clusters \( \Gamma, \Delta, \) such that \( \Gamma \prec \Delta \) and there exist s-clans \( \Theta \subseteq \Gamma \) and \( \Lambda \subseteq \Delta \), \( a \in \Theta \) and \( b \in \Lambda \) iff there exist \( \Gamma \in \text{Clusters}(A) \) such that \( \hat{\Gamma} \cap g(a) \neq \emptyset \) and \( \hat{\Gamma} \cap g(b) \neq \emptyset \) iff (by the factorization construction) there exist \( \Gamma \in \text{Clusters}(A) \) such that \( |a|_\Gamma \neq |0|_\Gamma \) and \( |b|_\Gamma \neq |0|_\Gamma \) iff \( h(a)C^t_{\text{A}_{\text{can}}} h(b) \) in \( \text{A}_{\text{can}} \).

(iv) \( a \mathbb{B}_A b \) in \( A \) iff (by Lemma 4.7) there exist clusters \( \Gamma, \Delta, \) such that \( \Gamma \prec \Delta \) and there exist s-clans \( \Theta \subseteq \Gamma \) and \( \Lambda \subseteq \Delta \), \( a \in \Theta \) and \( b \in \Lambda \) iff there exist \( \Gamma, \Delta \in \text{Clusters}(A) \) such that \( \hat{\Gamma} \cap g(a) \neq \emptyset \) and \( \hat{\Delta} \cap g(b) \neq \emptyset \) iff (by the factorization construction) there exist clusters \( \Gamma, \Delta, \) such that \( \Gamma \prec \Delta \), \( |a|_\Gamma \neq |0|_\Gamma \) and \( |b|_\Delta \neq |0|_\Delta \) iff \( h(a) \mathbb{B}_{\text{A}_{\text{can}}} h(b) \) in \( \text{A}_{\text{can}} \).

(v) \( a \nleq b \) in \( A \) iff \( a.b^* \neq 0 \) iff there exists a cluster \( \Gamma \) and an s-clan \( \Delta \subseteq \Gamma \) such that \( a.b^* \in \Delta \) iff there exists \( \Gamma \in \text{Clusters}(A) \) such that \( \hat{\Gamma} \cap g(a.b^*) \neq \emptyset \) iff (by the factorization construction) \( |a|_\Gamma \nleq |b|_\Gamma \) iff \( h(a) \nleq h(b) \) in \( \text{A}_{\text{can}} \).

(vi) \( a = b \) iff \( h(a) = h(b) \) - by (v) and the fact that \( a = b \) iff \( a \leq b \) and \( b \leq a \). \( \square \)

\section{4.4 Representation Theorem for DCAs by means of snapshot models}

\textbf{Theorem 4.15. Representation Theorem for DCA by means of snapshot models.} Let \( A \) be a DCA. Then there exists a full standard DCA \( \mathbb{B} \) and an isomorphic embedding \( h \) of \( A \) into \( \mathbb{B} \). Moreover, \( A \) satisfies some of the time axioms iff the same axioms are satisfied in \( \mathbb{B} \).
Proof. The proof is a direct corollary of Lemma 4.14 and Lemma 4.13 by taking $B = A^{can}$.

This Theorem shows that the meaning of the (point-based) standard DCA built by the snapshot construction is coded by the axioms of the abstract DCA which is point-free. Note, however, that this representation theorem is of embedding type, like the representation theorem for Boolean algebras as algebras of sets: every Boolean algebra can be isomorphically embedded into the Boolean algebra of subsets of some universe. The theorem does not guarantee one-one correspondence between set models and algebras via some isomorphism. The same situation is with DCAs and standard (point-based) DCAs. But adding topology we may characterize more deeply point models and like in the Stone topological representation theorem for Boolean algebras to establish a one-one correspondence between algebras and topological models. That is why we introduce and develop in the next Section topological models for DCAs.

5 Topological models for dynamic contact algebras

5.1 What kind of topological models for DCA we need?

What kind of topological models for DCA we need? We need topological spaces $X$ such that their algebra $RC(X)$ of regular closed subsets to model the algebra of regions. Note that regions in this algebra are related between each other by three differen relations - space contact $C^s$, time contact $C^t$ and precedence $B$, the first two acting as contact relations and the third - as precontact relation. This means that the realization of the contact $aC^sb$ should be $a$ and $b$ to have a common point and for $aC^tb$ also $a$ and $b$ to have a common point and these common points should be of different kind - points characterized space contact - space points, and points characterized time contact - time points. So regions should contain at least two kinds of points - space and time points and $aC^sb$ should hold if they share a space point, and $aC^tb$ should hold if $a$ and $b$ share time point. According to the third relation $B$, it should act as a precontact by means of some binary relation between time points. Also, in order to characterize $C^t$ as a simultaneity relation we need a special subclass of ”bigger” time points to be interpreted as ”moments of time” and the other time points to be considered as
parts of the bigger time points, such that simultaneous time points to form different disjoint classes. So space should have different classes of points similar to the clan structure of DCA. The topology in this space, as in the representation theory for contact algebras, should be generated by a subalgebra of the Boolean algebra of regular closed subsets of the space taken as a closed base for the topology. And finally, in order to prove topological representation theorem for DCA, we should be able to extract in a canonical way the same type of topological space from the structure of DCA. Obviously the abstract points of such a topology should be the different kinds of clans in DCA and their interrelations. So, this is the intuition which we will put in the definition of the special topological spaces introduced in Section 5.3 called Dynamic Mereotopological Spaces (DMS). Since DCA is a generalizations of contact algebra, we follow some terminology and ideas from the representation and duality theory for contact algebras given recently by Goldblatt and Grice in [35]. Since we will represent a given DCA $A$ as a subalgebra of the regular closed subsets $RC(S)$ of certain DMS $S$, we need some "lifting" conditions guaranteeing that $A$ satisfies some abstract conditions (for instance the time axioms and some others) iff $RC(S)$ satisfies the same axioms. This will be subject of the next section.

5.2 Lifting conditions

Let $A_i = (B_{A_i}, C^s_{A_i}, C^t_{A_i}, B_{A_i}), i = 1, 2$ be two algebras with a signature of DCA such that $C^s_{A_i}$ and $C^t_{A_i}$ be contact relations and $B_{A_i}$ be a precontact relation. We assume also that $A_1$ is a subalgebra of $A_2$. This means that $B_{A_1}$ is a Boolean subalgebra of $B_{A_2}$ and that the relations from the list $C^s_{A_1}, C^t_{A_1}, B_{A_1}$ are restrictions of the corresponding relations from the list $C^s_{A_2}, C^t_{A_2}, B_{A_2}$ to $B_{A_1}$. We need some abstract "lifting" conditions guaranteeing that $A_1$ satisfies the remaining axioms of DCA and possibly some time axioms from the list time axioms (rs), (ls), (up dir), (down dir), (circ), (dens), (ref), (lin), (tri), (tr) iff $A_2$ satisfies the same axioms. The conditions are given in the next definition and extend similar conditions considered in [73](pages 283-4 ) only for contact algebras. For convenience the elements from the set $B_{A_i}$ are denoted correspondingly by $a_i, b_i, c_i, ...$ etc.

**Definition 5.1. Lifting conditions.** Having in mind the above notations we say that the Boolean subalgebra $A_1$ is said to be a Boolean dense subalgebra of $A_2$ if

$$(Dense) \forall a_2(a_2 \neq 0 \Rightarrow \exists a_1)(a_1 \neq 0 \text{ and } a_1 \leq a_2),$$

50
and to be a co-dense subalgebra of $A_2$ if

\((\forall a_2)(a_2 \neq 1 \Rightarrow (\exists a_1)(a_1 \neq 1 \text{ and } a_2 \leq a_1))\).

It is easy to see that (Dense) is equivalent to (Co-dense).

Let $C$ be any of the relations $C_{A_2}^s, C_{A_2}^t, B_{A_2}$ and its restriction to $B_{A_1}$ to be denoted also by $C$. We say that $A_1$ is a $C$-separable subalgebra of $A_2$ if the following condition is satisfied:

\((C-\text{separation}) (\forall a_2, b_2)(a_2 C b_2 \Rightarrow (\exists a_1, b_1)(a_1 C b_1 \text{ and } a_2 \leq a_1 \text{ and } b_2 \leq b_1))\).

Conditions (Dense), (Co-dense) and (C-separable) for all $C$ from the set \{${C_{A_2}^s, C_{A_2}^t, B_{A_2}}$\} are called lifting conditions. If all lifting conditions are satisfied then $A_1$ is said to be a stable subalgebra of $A_2$.

If $g$ is an isomorphic embedding of $A_1$ into $A_2$, then $g$ is said to be a dense (co-dense) embedding provided that $g(A_1)$ is a dense (co-dense) subalgebra of $A_2$. We say that $g$ is a $C$-separable embedding if $g(A_1)$ is a $C$-separable subalgebra of $A_2$. If all lifting conditions are satisfied, then $g$ is called a stable embedding of $A_1$ into $A_2$.

**Lemma 5.2. Lifting Lemma.** Let $A_i = (B_{A_i}, C_{A_i}^s, C_{A_i}^t, B_{A_i})$, $i = 1, 2$ be two algebras with a signature of DCA such that $C_{A_1}^s$ and $C_{A_2}^s$ be contact relations and $B_{A_i}$ be a precontact relation and let $A_1$ be a stable subalgebra of $A_2$. Let $Ax$ be any of the following list of axioms of DCA : $(C^s \subseteq C^t)$, $(C^t E)$, $(C^t B)$, $(B C^t)$, or any from the list of time axioms. Then $Ax$ is true in $A_1$ iff $Ax$ is true in $A_2$.

**Proof.** Let us start with the case when $Ax$ is the axiom $(C^s \subseteq C^t) a C^s b \Rightarrow a C^t b$. Suppose first that $(C^s \subseteq C^t)$ is true in $A_1$ and for the sake of contradiction that it is not true in $A_2$. Then for some $a_2, b_2$ we have: $a_2 C^s b_2$ and $a_2 \overline{C^t}_b b_2$. Then by the condition $(C^t\text{-separation})$ we obtain: there exist $a_1, b_1$, such that $a_2 \leq a_1$, $b_2 \leq b_1$ and $a_1 \overline{C^t}_b b_1$. From here and $a_2 C^s b_2$ we get $a_1 C^s b_1$ which by $a_1 \overline{C^t}_b b_1$ shows that the axiom $(C^s \subseteq C^t)$ is not true in $A_1$ - a contradiction. Suppose now that the axiom is true in $A_2$. Since it is an universal formula, then it is trivially true in $A_1$.

Consider now that $Ax$ is the axiom $(C^t E) a C^t b \Rightarrow (\exists c)(a C^t c$ and $c \overline{C^t}_b b)$. Suppose first that $(C^t E)$ is true in $A_1$. In order to show that it is true in $A_2$ suppose $a_2 \overline{C^t}_b b$. Then by the condition $(C^t\text{-separation})$ there exist $a_1, b_1$ such that $a_1 \overline{C^t}_b b_1$, $a_2 \leq a_1$ and $b_2 \leq b_1$. By the assumption that $(C^t E)$ is true in $A_1$, $a_1 \overline{C^t}_b b_1$ implies that $(\exists c_1)(a_1 C^t c_1$ and $c_1 \overline{C^t}_b b_1)$. From here we
obtain \( a_2C^t c_1 \) and \( c_1^*C^t b_2 \). Obviously \( c_1 \) and \( c_1^* \) are in \( B_{A_2} \) which shows that \((C^t{E})\) is true in \( A_2 \).

Suppose now that \((C^t{E})\) is true in \( A_2 \) and in order to prove it in \( A_1 \) suppose \( a_1C^t b_1 \). Since \( a_1, b_1 \) are also in \( B_{A_2} \), then by the assumption there is \( c_2 \) such that \( a_1C^t c_2 \) and \( c_2C^t b_1 \). Then by the condition \((C^t{E})\) applied to \( a_1C^t c_2 \) there exist \( a_1', c_1' \) such that \( a_1 \leq a_1' \), \( c_2 \leq c_1' \) and \( a_1'C^t c_1' \). Analogously from \( c_2C^t b_1 \) we infer that there exist \( c_1'', b_1' \) such that \( b_1 \leq b_1' \), \( c_1'' \leq c_1' \) and \( c_1'C^t b_1' \). Manipulating with inequalities and monotonicity conditions for \( C^t \) we finally obtain \( a_1C^t c_1' \) and \( c_1'C^t b_1 \) which shows that \((C^t{E})\) holds in \( A_1 \).

In a similar way one can treat the case for the axioms \((C^t{B})\) and \((B^t{C})\).

As an example we will treat one case for time axioms just to show that the things go in a similar way. We consider the axiom \((\text{lin})\) \( a \neq 0 \) and \( b \neq 0 \Rightarrow aBb \) or \( bBa \). Suppose first that \((\text{lin})\) is true in \( A_1 \) and in order to show that it is true in \( A_2 \) suppose \( a_2 \neq 0 \) and \( b_2 \neq 0 \). Then by the condition \((\text{dence})\) there exists \( a_1 \neq 0 \) such that \( a_1 \leq a_2 \) and there exists \( b_1 \neq 0 \) such that \( b_1 \leq b_2 \). By the assumption \( a_1 \neq 0 \) and \( b_1 \neq 0 \) imply \( a_1Bb_1 \) or \( b_1Ba_1 \). By monotonicity conditions for \( B \) we get \( a_2Bb_2 \) or \( b_2Ba_2 \) which finishes the proof for this direction. For the converse direction suppose that \((\text{lin})\) is true in \( A_2 \). Since \((\text{lin})\) is an universal sentence it trivially holds in the subalgebra \( A_1 \).

5.3 Dynamic Mereotopological Spaces (DMS)

**Definition 5.3. Dynamic Mereotopological Space.** A system \( S = (X^t_S, X^s_S, T^t_S, \prec_S, M^t_S) \) is called Dynamic Mereotopological Space (DMS, DM-space) if the next axioms are satisfied.

**The axioms of DMS:**

- (S1) \( X^t_S \) is a nonempty topological space, the elements of \( X^t_S \) are called partial time points of \( S \).
- (S2) \( M^t_S \) is a subalgebra of the algebra \( RC(X^t_S) \) of regular closed sets of \( X^t_S \) and \( M^t_S \) is a closed base of the topology of \( X^t_S \).
- (S3) The sets \( X^t_S, X^s_S \) and \( T^t_S \) are non-empty sets satisfying the following inclusions:

\[
X^t_S \subseteq X^s_S, \quad T^t_S \subseteq X^t_S.
\]
The elements of $X_S^t$ are called **space points of** $S$, hence every space point is a partial time point. The elements of $T_S$ are called **time points of** $S$.

- **(S4)** For $a \in RC(X_S^t)$: if $a \neq \emptyset$, then $a \cap X_S^t \neq \emptyset$ and
  - **(S5)** $\prec_S$ is a binary relation in $X_S^t$ called **before-after relation**. The subsystem $(T_S, \prec_S)$ is called the **time structure of** $S$.

**Definitions:** For $a, b \in RC(X_S^t)$ define:

\[
\begin{align*}
  aC^t_S b & \text{ iff } a \cap b \neq \emptyset, \text{ time contact}, \\
  aC^s_S b & \text{ iff } a \cap b \cap X_S^t \neq \emptyset, \text{ space contact}, \\
  aB_S b & \text{ iff there exist } x, y \in X_S^t \text{ such that } x \prec_S y, x \in a \text{ and } y \in b, \text{ precedence}, \\
  RC(S) & = \text{def } (RC(X_S^t), C_S^t, C_S^s, B_S), \text{ regular-sets algebra of } S, \\
  S^+ & = \text{def } (M_S, C_S^t, C_S^s, B_S) \text{ with the above defined relations restricted to } M_S.
\end{align*}
\]

It can easily be seen that $C_S^s$ and $C_S^t$ are contact relations in $RC(X_S^t)$ and that $B$ is a precontact relation (for $C_S^s$ use axiom (S4)).

- **(S6)** The system $S^+$ is a DCA. $S^+$ is called the **canonical DCA of** $S$ or the **dual of** $S$.
- **(S7)** For $x, y \in X_S^t$, $x \prec_S y$ iff $(\forall a, b \in M_S)(x \in a, y \in b \Rightarrow aB_S b)$.
- **(S8)** If $x \in T_S$ then $\rho_S(x)$ is a cluster in $S^+$.

**Lemma 5.4.** Let $S = (X_S^t, X_S^s, T_S, \prec_S, M_S)$ be a DMS. Then:

(i) If $x \in X_S^t$, then $\rho_S(x)$ is a $t$-clan in $S^+$.
(ii) If $x \in X_S^s$, then $\rho_S(x)$ is an $s$-clan in $S^+$.
(iii) If \( x \in T_S \), then \( \rho_S(x) \) is a cluster in \( S^+ \).

(iv) Let \( \prec_S \) be the canonical relation of \( B \) between \( t \)-clans of \( S^+ \) (see (9) for the definition). Then Axiom (S7) of DMS is equivalent to the following statement: for all \( x, y \in X^t_S \), \( x \prec_S y \) iff \( \rho_S(x) \prec_S \rho_S(y) \).

(v) \( S \) is \( T_0 \) space iff \( (\forall x, y \in X^t_S)(\rho_S(x) = \rho_S(y) \Rightarrow x = y) \). (or, equivalently, \( S \) is \( T_0 \) iff \( \rho_S \) is an injective mapping from \( X^t_S \) into the \( t \)-clans of \( S^+ \)).

Proof. For (i) and (ii) - by an easy verification of the corresponding definitions. For (iii) this is just the axiom (S8) for DMS. (iv) is trivial on the base of the definition of the relation \( \prec_M \). (v) is easy if we take in consideration the definition \( T_0 \) property, the definition of \( \rho_S \) and the fact that \( M_S \) is a closed base of the topology of \( X^t_S \). \( \square \)

Definition 5.5. (1) A \( t \)-clan (\( s \)-clan, \( t \)-cluster) \( \Gamma \) of \( S^+ \) is called a point \( t \)-clan (\( s \)-clan, \( t \)-cluster) if there is a point \( x \in X^t_S \) (\( x \in X^s_S \), \( x \in T_S \)) such that \( \Gamma = \rho_S(x) \).

(2) \( S \) is \textbf{DM-compact} (dynamic mereocompact) space if every \( t \)-clan, \( s \)-clan and \( t \)-cluster of \( S^+ \) is respectively a point \( t \)-clan, \( s \)-clan and a \( t \)-cluster.

The following Lemma is obvious.

Lemma 5.6. Let \( S \) be a DMS. Then the following two conditions are equivalent:

(i) \( S \) is DM-compact,

(ii) \( \rho_S \) is a surjective mapping from \( X^t_S \) onto the set of all \( t \)-clans of \( S^+ \). More over \( \rho_S \) maps \( X^s_S \) onto the set of all \( s \)-clans of \( S^+ \) and it maps \( T_S \) onto the set of all clusters of \( S^+ \).

Corollary 5.7. Let \( S \) be a \( T_0 \) and DM-compact DMS. Then \( \rho_S \) is a one-one mapping from \( X^t_S \) onto the set of all \( t \)-clans of \( S^+ \).

Proof. By Lemma 5.4 (v) and Lemma 5.6 \( \square \)

Remark 5.8. The notions of DM-space and DM-compactness are analogous to the notions of mereotopological space and mereocompactness introduced by Goldblatt and Grice in [35]. Their definitions are the following. A mereotopological space is a pair \( S = (X_S, M_S) \) where \( X \) is a topological space and \( M_S \) is a subalgebra of the Boolean algebra \( RC(X_S) \) of regular
closed sets of $X_S$ considered as closed base of the topology of $X$. Let $S^+$ be the contact algebra $(M_S, C_S)$ where $C_S$ is the standard topological contact between regular closed sets. $S$ is mereocompact if every clan of the contact algebra $S^+$ is a point clan in the sense of Definition 5.5 (in fact the definition of mereocompactness in [35] is slightly different but equivalent to the given here). So, if $S = (X^t_S, X^*_S, T_S, \prec_S, M_S)$ is a DM-space then the pair $(X^t, M_S)$ is mereotopological space and if $S$ is DM-compact then $(X^t, M_S)$ is mereocompact. Mereotopological spaces have been introduced by Goldblatt and Grice in order to develop a topological duality theory for contact algebras. Similarly, we introduce the notion of DM-space to be used in the topological representation theory and duality theory of DCAs. Let us note that mereotopological space is not a special case of DM-spaces by the following reasons (see Remark 4.3). In our case contact algebras are used to obtain the notion of dynamic contact algebra. On the other hand contact algebras can be considered as a special case of dynamic contact algebras (see Lemma 4.2) by adding to their signature some definable relations. In this interpretation of contact algebras they have corresponding topological spaces which are not the same as mereotopological spaces considered by Goldblatt and Grice (for the DMS spaces corresponding to contact algebras see Section 5.6). Because our exposition is quite similar to that of Goldblatt and Grice and in some sense is an adaptation of their method to the case of DCAs, we recommend the paper [35] to the reader of the present text. For convenience we even use similar and compatible notations with [35].

**Lemma 5.9.** Let $S$ be a DM-compact space. Then the topological space $X^t_S$ is compact.

**Proof.** According to Remark 5.8 DM-compactness of $S$ implies that the pair $(X^t_S, M_S)$ is a mereocompact space and then the statement follows from Theorem 4.2.(3) of [35]. We present below the proof illustrating our definition of DM-compactness.

In order to prove the compactness of $X^t_S$, it suffices to prove the following. Let $I \subseteq M_S$ be a nonempty set and let $A = \bigcap\{a \in M_S : a \in I\}$. If for every finite $I_0 \subseteq I$ the set $\bigcap\{a \in M_S : a \in I_0\} \neq \emptyset$, then $A \neq \emptyset$. The fact that $\bigcap\{a \in M_S : a \in I_0\} \neq \emptyset$ for every finite subset $I_0$ of $I$ guarantees the existence of an ultrafilter $U$ in the subset of all subsets of $X^t_S$ such that $\{a \in M_S : a \in I\} \subseteq U$. Let $\Gamma = \{a \in M_S : a \in U\}$. Then it is easy to see that $\Gamma$ is a t-clan. Then by DM-compactness there exists $x \in X^t_S$ such that $\Gamma = \rho_S(x)$. Hence for every $a \in I$ we have the following:
\[ a \in I \implies a \in U \implies a \in \Gamma \implies a \in \rho_S(x) \implies x \in a \implies x \in A \implies A \neq \emptyset \]

**Lemma 5.10.** Let \( S \) be a DM-compact space. Then the following equivalences are true:

(i) \( aC_S^ib \) iff \( a \cap b \cap T_S \neq \emptyset \).

(ii) \( aB_Sb \) iff \( (\exists x \in a \cap T_S)(\exists y \in a \cap T_S)(x \prec y) \).

**Proof.** (i) \( (\Rightarrow) \) Suppose \( aC_S^ib \). Then there exists \( xX_S^b \) such that \( x \in a \) and \( x \in b \), so \( a, b \in \rho_S(x) \). By Lemma 5.4 (i) \( \rho_S(x) \) is a t-clan in \( S^+ \). Then extend \( \rho_S(x) \) into a cluster \( \Gamma \). By DM-compactness there exists a point \( y \in T_S \) such that \( \Gamma = \rho_S(y) \). This implies \( \rho_S(x) \subseteq \rho_S(y) \) and hence \( a, b \in \rho_S(y) \). So, \( y \in a, y \in b, y \in T_S \) and consequently \( a \cap b \cap T_S \neq \emptyset \). The converse implication \( (\Leftarrow) \) is trivial because \( T_S \subseteq X_S \).

(ii) \( (\Rightarrow) \) Suppose \( aB_Sb \). Then there exist \( x', y' \in X_S^b \), such that \( x' \in a \) (hence \( a \in \rho_S(x') \)), \( y' \in b \) (hence \( b \in \rho_S(y') \)) and \( x' \prec y' \) and hence by Lemma 5.4 (v) \( \rho(x') \prec_{S^+} \rho(y') \), where \( \prec_{S^+} \) is the canonical relation of \( B_{S^+} \) in the set of t-clans of \( S^+ \) (see (9) for the definition \( \prec_{S^+} \)). Then by Lemma 4.5 there exist clusters \( \Gamma, \Delta \) in \( S^1(+) \) such that \( \rho_S(x') \subseteq \Gamma, \rho_S(y') \subseteq \Delta \) and \( \Gamma \prec_{S^+} \Delta \). By DM-compactness there exist \( x, y \in T_S \), such that \( \Gamma = \rho_S(x) \) and \( \Delta = \rho_S(y) \), which implies \( \rho_S(x) \prec_{S^+} \rho_S(y) \). Then again by Lemma 5.4 (v) we get \( x \prec_S y \). Also we have \( \rho_S(x') \subseteq \rho_S(x) \) and \( \rho_S(x') \subseteq \rho_S(x) \) which imply \( x \in a \), and hence \( x \in (a \cap T_S), y \in b \) and hence \( y \in (b \cap T_S) \). The converse implication \( (\Leftarrow) \) is trivial because \( T_S \subseteq X_S \). \( \square \)

**Lemma 5.11.** Let \( S = (X_S^b, X_S^s, T_S, \gamma_S, \prec_S, M_S) \) be a DM-compact DMS. Then the set \( X_S^b \) of space points of \( S \) with a subset topology is a dense subset of \( X_S^b \).

**Proof.** Let \( Cl \) denote be the closure operation of \( X_S^b \). We have to show that \( ClX_S^b = X_S^b \). Suppose that this is not true, i.e. there exists \( x \in X_S^b \) such that \( x \notin ClX_S^b \). Since \( M_S \) is a closed base of the topology of \( X_S^b \) then there exits \( a \in M_S \) such that \( X_S^b \subseteq a \) and \( x \notin a \). Then \( a \notin \rho_S(x) \), which is a t-clan in \( S^+ \). Then for all ultrafilters \( U \subseteq \rho_S(x) \) we have that \( a \notin U \), and let \( U \) be such one. But \( U \) is an s-clan, so by DM-compactness there is a point \( y \in X_S^b \) such that \( U = \rho_S(y) \). Because \( U \subseteq \rho_S(x) \) we obtain \( \rho_S(y) \subseteq \rho_S(x) \). From here we obtain that \( a \notin \rho_S(y) \) and consequently \( y \notin a \). But \( y \in X_S^b \subseteq a \), so \( y \in a - a \) contradiction. \( \square \)
Lemma 5.12. ([12], page 271) Let $X$ be a dense subspace of a topological space $Y$ and let $RC(X)$ and $RC(Y)$ be the corresponding Boolean algebras of regular closed sets of $X$ and $Y$. Let for $a \in RC(X)$, $h(a) = Cl_Y(a)$. Then $h : RC(X) \rightarrow RC(Y)$ is an isomorphism from $RC(X)$ onto $RC(Y)$. For $b \in RC(Y)$ converse mapping $h^{-1}$ acts as follows: $h^{-1}(b) = b \cap X$.

Corollary 5.13. The Boolean algebra $RC(X^S)$ of regular closed subsets of $X^S$ is isomorphic to the Boolean algebra $RC(X^T)$.

Proof. The lemma is a corollary of Lemma 5.11 and Lemma 5.12.

In the next section we study some other consequences of DM-compactness.

5.4 Canonical filters in DM-compact spaces

We assume in this section that $S$ is a DM-compact space. The aim of the section is to introduce a technical notion - canonical filter, generalizing a similar notion from [73]. By means of canonical filters and the assumption of DM-compactness of a given $S$ we will establish that the algebra $S^+$ is a stable subalgebra of $RC(S)$ in the sense of Definition 5.1 which fact implies several important consequences.

Definition 5.14. Let $A \in RC(X^T)$. Then the set $F_A = \{a \in M_S : A \subseteq a\}$ is called canonical filter of $S^+$.

Lemma 5.15. Let $A, B \in RC(X^S)$. Then:

(i) $F_A$ is a filter in $S^+$.

(ii) $\forall x \in X^S \downarrow : x \in A$ iff $F_A \subseteq \rho_S(x)$.

(iii) $A \neq X^S \downarrow$ iff there exists $a \in M_S$ such that $A \subseteq a$ and $a \neq X^S \downarrow$.

Let $R^t, R^s, \prec$ be the canonical relations between filters corresponding to the relations $C^t_S, C^s_S, B_S$ from the DCA algebra $S^+$. Then:

(iv) $AC^t_S B$ iff $F_A R^t F_B$.

(v) $AC^s_S B$ iff $F_A R^s F_B$.

(vi) $AB_S B$ iff $F_A \prec F_B$.

Proof. (i) The proof is by a direct checking the corresponding definitions.
(ii) The implication from left to right is by straightforward checking. For the converse direction we will reason by contraposition. Suppose $x \notin A$. Now we will apply the fact that $M_S$ is a closed base of the topology of $X$. Because $A$ is a regular closed set then $A$ is a closed set and then there exists $a \in M_S$ such that $A \subseteq a$ and $x \notin a$. Then $a \in F_A$ and $a \notin \rho_S(x)$, so $F_A \nsubseteq \rho_S(x)$.

(iii) can be derived by direct application of (ii).

(iv) ($\Rightarrow$). Suppose $AC^t_S B$. Then there is a point $x \in X^t_S$ such that $x \in A$ and $x \in B$. By (ii) this implies

1. $F_A \subseteq \rho_S(x)$ and
2. $F_B \subseteq \rho_S(x)$.

In order to show $F_A \prec F_B$ suppose $a \in F_A$ and $b \in F_B$ and proceed to show $F_A R^t F_B$. Then by (1) and (2) we get $a \in \rho_S(x)$ and hence $x \in a$, and $b \in \rho_S(x)$ and hence $x \in b$, which shows $a \cap b \neq \emptyset$. So, $a \prec b$.

($\Leftarrow$). Suppose $F_A R^t F_B$. By Lemma 2.9 there exist ultrafilters $U, V$ such that $F_A \subseteq U$, $F_B \subseteq V$ and $UR^t V$. Let $\Gamma = U \cup V$. Obviously $F_A \subseteq \Gamma$ and $F_B \subseteq \Gamma$. By Lemma 2.15 $\Gamma$ as a union of $R^t$-related ultrafilters is a $t$-clan in $S^+$ and by DM-compactness there is $x \in X^t_S$ such that $\Gamma = \rho_S(x)$. Hence $F_A \subseteq \rho_S(x)$ and $F_B \subseteq \rho_S(x)$. By (ii) $x \in A$ and $x \in B$ hence $A \cap B \neq \emptyset$, so $AC^t_S B$.

(v) the proof is similar to (iv)- it is used that if $\Gamma$ is an $s$-clan in $S^+$ then by the DM-compactness there is point $x \in X^t_S$ such that $\Gamma = \rho_S(x)$.

(vi)($\Rightarrow$). The proof is similar to the proof of (iv) ($\Rightarrow$) (DM-compactness is not needed).

(vi) ($\Leftarrow$). Suppose $F_A \prec F_B$. Then by the Lemma 2.9 there exist ultrafilters $U, V$ in $S^+$ such that $F_A \subseteq U$, $F_B \subseteq V$ and $U \prec V$. Since ultrafilters are $t$-clans by DM-compactness there exist points $x, y \in X^t_S$ such that $U = \rho_S(x)$ and $V = \rho_S(y)$ and hence $\rho_S(x) \prec \rho_S(y)$. By Lemma 5.14 (iv) we get $x \prec_S y$. Thus we have: $F_A \subseteq \rho_S(x)$ (hence by (ii) $x \in A$) and $F_B \subseteq \rho_S(y)$ (hence $y \in B$) and $x \prec_S y$. By the definition of $\mathcal{B}_S$ we obtain $A B_S B$.

Lemma 5.16. The following conditions are true for $S$:

(i) The algebra $S^+$ is a stable Boolean sub-algebra of $RC(S)$.

(ii) $RC(S)$ is a DCA.
Proof. (i) We first show that $S^+$ satisfies the lifting conditions (see Definition 5.1) and then (i) is a corollary of Lemma 5.2. First we verify the lifting condition (co-dense). Suppose $A \in RC(X^+_S)$ and $A \neq X^+_S$. Then by Lemma 5.15 (iii) there exists $a \neq M_S$ such that $a \neq X^+_S$ and $A \subseteq a$. We do not treat (dense) because it is equivalent to (co-dense).

To verify the condition (C-separation) for $C \in \{C^t_S, C^g_S, B_S\}$ we proceed as follows. Looking at the conditions (iv), (v), (vi) we see that they have the following common form. Let $R$ be the canonical relation between filters corresponding to the relation $C$. Then for any $A, B \in RC(X^+_S)$: $ACB$ iff $F_ARF_B$. Taking the negation in both sides we obtain: $A \nleq B$ iff $F_ARF_B$ iff there exists $a, b \in M_S$ such that $A \subseteq a, B \subseteq b$ and $a \nleq b$. Thus: $F_ARF_B$ implies that for some $a, b \in M_S$, $A \subseteq a, B \subseteq b$ and $a \nleq b$ which is the (C-separation) condition. Note that just this implication needed DM-compactness in Lemma 5.15.

(ii) is a corollary of (i) and the fact that $S^+$ is a DCA, so by Lemma 5.2 the axioms $(C^g \subseteq C^t), (C^t E), (C^t B)$ and $(BC^t)$ are lifted from $S^+$ to $RC(S)$.

Lemma 5.17. Let $(\varphi)$ be any of the time axioms: (rs), (ls), (up dir), (down dir), (circ), (dens), (ref), (lin), (tri), (tr) Then the following conditions are equivalent:

(i) $(\varphi)$ is true in the algebra $S^+$.

(ii) $(\varphi)$ is true in the algebra $RC(S)$.

Proof. The proof follows from Lemma 5.16 (i) and Lemma 5.2.

Lemma 5.18. Let $S$ be DM-compact DMS, $RC(S)$ be its regular-sets algebra, $(T_S, \prec_S)$ be its time structure and let $(T^+_S, \prec_{S^+})$ be the canonical time structure of $S^+$ (see Definition 4.8). Let $(\Phi)$ be the time condition from the list (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Irr), (Lin), (Tr) (condition (Tri) is excluded). Then the following conditions are true:

(i) $(\varphi)$ is true in $(T_S, \prec_S)$ iff $(\varphi)$ is true in $(T^+_S, \prec_{S^+})$.

(ii) If $S$ is $T0$ DMS, then: (Tri) is true in $(T_S, \prec_S)$ iff (Tri) is true in $(T^+_S, \prec_{S^+})$.

Proof. (i) Let us remind that the members of $T^+_S$ are clusters of $S^+$, which we will denote by $\Gamma, \Delta, \Theta, \ldots$. We will demonstrate the proof considering the case (Dense) the proofs for the other cases go in the same manner.
Lemma 5.17: \( \forall i,j \) \((i < j \Rightarrow \exists k)(i < k \text{ and } k < j)\).

\( \Rightarrow \) Suppose \((\text{Dense})\) is true in \((T_S, \prec_S)\) and let \(\Gamma, \Delta \in T_S^+\) and \(\Gamma \prec_S^+ \Delta\). Then by DM-compactness there exist \(x, y \in T_S\) such that \(\Gamma = \rho_S(x)\), and \(\Delta = \rho_S(y)\), so \(\rho_S(x) \prec_S^+ \rho_S(y)\). By Lemma 5.4 (iv) we obtain \(x \prec_S y\) and by (Dence) there exists \(z \in T_S\) such that \(x \prec_S z \prec_S y\). Again by Lemma 5.4 (iv) we obtain \(\rho_S(x) \prec_S^+ \rho_S(z) \prec_S^+ \rho_S(y)\). Because \(\rho_S(z)\) is a cluster in \(S_+\), we put \(\Theta = \rho_S(z)\) and obtain \(\prec_S^+ \Theta \prec_S^+ \Delta\) which shows that \((\text{Dense})\) is true in \((T_S^+, \prec_S^+)\).

\( \Leftarrow \) Suppose \((\text{Dense})\) is true in \((T_S^+, \prec_S^+)\), \(x, y \in T_S\) and \(x \prec_S y\). Then \(\rho_S(x) \prec_S^+ \rho_S(y)\). By (Dence) there exists a cluster \(\Theta\) (hence there exists \(z \in T_S\) with \(\rho_S(z) = \Theta\)) such that \(\rho_S(x) \prec_S^+ \rho_S(z) \prec_S^+ \rho_S(y)\). This implies \(x \prec_S z \prec_S y\) which shows that \((\text{Dense})\) is true in \((T_S, \prec_S)\).

(ii) The case of \((\text{Tri})\) \((\forall i,j)\) \((i = j \text{ or } i < j \text{ or } j < i)\).

(\(\Rightarrow\)) The proof of this implication is straightforward and requires neither DM-compactness nor \(T_0\) property.

(\(\Leftarrow\)) Suppose \((\text{Tri})\) is true in \((T_S^+, \prec_S^+)\) and let \(x, y \in T_S\). Then \(\rho_S(x), \rho_S(y)\) are clusters in \(S^+\). Then by (Tri) we have \(\rho_S(x) = \rho_S(y)\) or \(\rho_S(x) \prec_S^+ \rho_S(y)\) or \(\rho_S(y) \prec_S^+ \rho_S(x)\). \textbf{Case 1:} \(\rho_S(x) = \rho_S(y)\). Since \(\rho_S(x)\) and \(\rho_S(y)\) are also \(t\)-clans then by the assumption that \(S\) is a \(T_0\) space case 1 implies \(x = y\) (by Lemma 5.4 (v)).

\textbf{Case 2:} \(\rho_S(x) \prec_S^+ \rho_S(y)\). By Lemma 5.4 (iv) this implies \(x \prec_S y\).

\textbf{Case 3:} \(\rho_S(y) \prec_S^+ \rho_S(x)\). Again by Lemma 5.4 (iv) this implies \(y \prec_S x\). Thus, (Tri) is fulfilled in the time structure \((T_S, \prec_S)\).

Lemma 5.19. \textbf{Topological definability.} Let \((T_S, \prec_S)\) be the time structure of \(S\), \((\Phi)\) be the time condition from the list \((RS), (LS), (Up \ Dir), (Down \ Dir), (Circ), (Dens), (Ref), (Lin), (Tri) (Tr)\) and \((\Phi)\) be the corresponding time axiom from the list \((rs), (ls), (up \ dir), (down \ dir), (circ), (dens), (ref), (lin), (tri), (tr)\). Then the following conditions are equivalent (for the case of \((Tri)\) we assume also that \(S\) is \(T_0\)):

(i) \((\Phi)\) is true in \((T_S, \prec_S)\)

(ii) \((\varphi)\) is true in \((RC)(S))\).

\textbf{Proof.} \((\Phi)\) is true in \((T_S, \prec_S)\) iff (by Lemma 5.18) \((\Phi)\) is true in the canonical time structure of \(S^+, (T_S^+, \prec_S^+)\) iff (by Lemma 4.9 \((\varphi)\) is true in \(S^+\) iff (by Lemma 5.17) \((\varphi)\) is true in the algebra \(RC(S)\). \(\square\)
5.5 Canonical DMS for DCA and topological representation theorem for DCA

Let $A = (B_A, C^t_A, C^s_A, B_A)$ be a DCA. We associate to DCA in a canonical way a DM-space denoted by $A_+$ and called the **canonical DMS of $A$** or the **dual DMS of $A$** as follows:

- $A_+ = \text{def} (X^t_A, X^s_A, T_A, \prec_A, M_A)$, where:
  - $X^t_A = \text{t-Clans}(A)$,
  - $X^s_A = \text{s-Clans}(A)$ and
  - $T_A = \text{Clusters}(A)$.
- $\prec_A$ is the before-after relation in the set $X^t_A$ defined by (9). The structure $(T_A, \prec_A)$ - the time structure of $A$ is now the time structure of $A_+$.

$M_A$ is defined as follows and is used to introduce a topology in the set $X^t_A$ considering it as a basis of the closed sets in the topology:

- For $a \in B_A$ let $g_A(a) = \{\Gamma \in \text{t-Clans}(A) : a \in \Gamma\}$ and put

- $M_A = \{g_A(a) : a \in B_A\}$.

By the topological representation theory of contact algebras (see Section 2.5) the set $\{g_A(a) : a \in B_A\}$ defines a topology in the set $X^t_A$ and $g_A$ is an isomorphic embedding of $B_A$ into the algebra $RC(X^t_A)$ and $M_A$ is a Boolean subalgebra of $RC(X^t_A)$ isomorphic to $B_A$.

We define the algebra $(A_+)^+$ - the dual of $A_+$ as follows.

- $(A_+)^+ = \text{def} (M_A, C^t_{A_+}, C^s_{A_+}, B_{A_+})$.

Having in mind the topological representation theory of contact algebras (see Section 2.5 and Lemma 4.6 it can be seen that $g_A$ is also an isomorphism from $A$ onto $(A_+)^+$, so $(A_+)^+ = g_A(B_A)$ which proves the following lemma.

**Lemma 5.20.** $A$ is isomorphic to $(A_+)^+$ and hence $(A_+)^+$ is a DCA.

By definition we have $\rho_{A_+} = \text{def} \{g_A(a) \in M_A : \Gamma \in g_A(a)\} = \{g_A(a) \in M_A : a \in \Gamma\}$.

**Lemma 5.21.** (i) For any $\Gamma \in X^t_A \rho_{A_+}(\Gamma)$ is a $C^t_{A_+}$-clan in $(A_+)^+$.

(ii) For any $\Gamma \in X^s_A \rho_{A_+}(\Gamma)$ is a $C^s_{A_+}$-clan in $(A_+)^+$.

(iii) For any $\Gamma \in T_A \rho_{A_+}(\Gamma)$ is a $C^t_{A_+}$-cluster in $(A_+)^+$. 

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Proof. The proof is by a routine verification of the corresponding definitions and using the results about the clan structure of DCA developed in Section 4.1. As an example we will demonstrate the proof of (iii)

Let $\Gamma \in T_A$. Then $\Gamma$ is a cluster in $A$, so $\Gamma$ is a $t$-clan in $A$. By (i) $\rho_{A_+}(\Gamma)$ is a $C'_{A_+}$-clan in $(A_+)^+$. We will show that $\rho_{A_+}(\Gamma)$ is a $C'_{A_+}$-cluster in $(A_+)^+$. Suppose that for some $a \in B_A$, $g_A(a) \notin \rho_{A_+}(\Gamma)$. Then $\Gamma \notin g_A(a)$, so $a \notin \Gamma$. Then there exists $b \in B_A$ such that $b \notin \Gamma$ and $a \overline{C_A} b$. Then $g_A(b) \notin \rho_{A_+}(\Gamma)$ and $g_A(a) \cap g_A(b) = \emptyset$, so $g_A(a) \overline{B}_A g_A(b)$. Note that (iii) verifies the DMS axiom (S7) for $A_+$. □

Lemma 5.22. (i) If $\alpha \in RC(X_A)$ and $\alpha \neq \emptyset$, then $\alpha \cap X_A^* \neq \emptyset$.

(ii) Let $\Gamma, \Delta$ be t-clans in $A$. Then: $\Gamma \prec_A \Delta$ iff for all $g_A(a), g_A(b) \in M_A(\Gamma \in g_A(a)$ and $\Delta \in g_A(b)$ implies $g_A(a) \overline{B}(A_+)^+ g_A(b)$.

Proof. (i) The proof is based on the fact that every t-clan in $A$ is an s-clan in $A$. This lemma verifies axiom S4 for DMS for the system $A_+$. This verifies the DMS axiom (S4) for $A_+$.

(ii) Let $\Gamma, \Delta$ be t-clans in $A$. Having in mind the relevant definitions the implication from left to the right is obvious. For the converse implication suppose that

($\sharp$) for all $g_A(a), g_A(b) \in M_A$, the conditions $\Gamma \in g_A(a)$ and $\Delta \in g_A(b)$ imply $g_A(a) \overline{B}(A_+)^+ g_A(b)$

and proceed to show $\Gamma \prec_A \Delta$. By (9) this means that for some $a, b \in B_A$ we have $a \overline{B}_A b$. To this end suppose $a \in \Gamma$ and $b \in \Delta$. Then $\Gamma \in g_A(a)$ and $\Delta \in g_A(b)$. By ($\sharp$) we get $g_A(a) \overline{B}(A_+)^+ g_A(b)$ which by the definition of $B(\overline{A_+})^+$ means that for some t-clans $\Gamma', \Delta'$ in $A$ we have $\Gamma' \in g_A(a)$, $\Delta' \in g_A(b)$ and $\Gamma' \prec_A \Delta'$. This implies $a \in \Gamma'$ and $b \in \Delta'$ and by the definition of $\Gamma' \prec_A \Delta'$ (see (9)) that $a \overline{B}_A b$ - end of the proof. Note that (ii) verifies the DMS axiom (S7) for $A_+$. □

Lemma 5.23. $A_+$ is a DMS.

Proof. The proof follows from Lemma 5.21, Lemma 5.22 and Lemma 5.20 which establish the DMS axioms (S4),(S6), (S7) and (S8). The other axioms are obviously true. □
Lemma 5.24. Let $A$ be a DCA. Then:

(i) If $\Gamma$ is a t-clan in $(A_+)^+$, then $\Gamma^t = {a \in B_A : g_A(a) \in \Gamma}$ is a t-clan in $A$ and $\rho_{A_+}(\Gamma^t) = \Gamma$.

(ii) If $\Gamma$ is an s-clan in $(A_+)^+$, then $\Gamma^s = {a \in B_A : g_A(a) \cap s\text{-}Clans(A) \in \Gamma}$ is an s-clan in $A$ and $\rho_{A_+}(\Gamma^s) = \Gamma$.

(iii) If $\Gamma$ is a cluster in $(A_+)^+$, then $\Gamma^{\text{clust}} = {a \in B_A : g_A(a) \cap \text{Clusters}(A) \in \Gamma}$ is a cluster in $A$ and $\rho_{A_+}(\Gamma^{\text{clust}}) = \Gamma$.

Proof. (i) Let $\Gamma$ be a t-clan in $(A_+)^+$. Note that the elements of $\Gamma$ are in the form $g_A(a)$, $a \in B_A$. It is easy to verify the grill properties of $\Gamma^t$. In order to verify the t-clan property let $a, b \in \Gamma^t$, then $g_A(a), g_A(b) \in \Gamma$, so $g_A(a) \cap g_A(b) \neq \emptyset$ which by Lemma 4.6 is equivalent to $aC_b^t b$ which finally shows that $\Gamma^t$ is a t-clan in $A$.

Let us show that $\rho_{A_+}(\Gamma^t) = \Gamma$. Indeed, using Lemma 5.21 we obtain: $\rho_{A_+}(\Gamma^t) = \{g_A(a) : a \in M_A : a \in \Gamma^t\} = \{g_A(a) : a \in M_A : g_A(a) \in \Gamma\} = \Gamma$.

(ii) Let $\Gamma$ be an s-clan in $(A_+)^+$. In this case the elements of $\Gamma$ are in the form $g_A(a) \cap s - Clans(A)$. We will verify only the s-clan property of $\Gamma^s$. Let $a, b \in \Gamma^s$, then $g_A(a) \cap s - Clans(A), g_A(b) \cap s - Clans(A) \in \Gamma$. Since $\Gamma$ is an s-clan, then $(g_A(a) \cap s - Clans(A)) \cap (g_A(b) \cap s - Clans(A)) = g_A(a) \cap (g_A(b) \cap s - Clans(A)) \neq \emptyset$. This by (11) is equivalent to $g_A^s(a) \cap g_A^s(b) \neq \emptyset$ which by Lemma 4.6 is equivalent to $aC^s b$.

The proof of the equality $\rho_{A_+}(\Gamma^s) = \Gamma$ is as follows: $\rho_{A_+}(\Gamma^s) = \{g_A(a) : a \in M_A : a \in \Gamma^s\} = \{g_A(a) : a \in M_A : g_A(a) \cap s\text{-}Cans(A) \in \Gamma\} = \Gamma$.

(iii) Let $\Gamma$ be a cluster in $(A_+)^+$. In this case the elements of $\Gamma$ are of the form $g_A(a) \cap \text{Clusters}(A)$. The proof that $\Gamma^{\text{clust}}$ is a t-clan is similar to that of (i). Let us show that $\Gamma^{\text{clust}}$ is a cluster in $A$. Let $a \not\in \Gamma^{\text{clust}}$, then $g_A(a) \cap \text{Clusters}(A) \not\in \Gamma$, hence (because $\Gamma$ is a cluster in $(A_+)^+$) there is $g_A(b) \cap \text{Clusters}(A)$ such that $g_A(b) \cap \text{Clusters}(A) \not\in \Gamma$ and $(g_A(a) \cap \text{Clusters}(A)) \cap (g_A(b) \cap \text{Clusters}(A)) = g_A(a) \cap (g_A(b) \cap \text{Clusters}(A)) \neq \emptyset$. This by (12) is equivalent to $g_A^{\text{clust}}(a) \cap g_A^{\text{clust}}(b) \neq \emptyset$ which by Lemma 4.6 is equivalent to $aCO^t b$. This shows that $\Gamma^{\text{clust}}$ is a cluster in $A$.

The proof of the equality $\rho_{A_+}(\Gamma^{\text{clust}}) = \Gamma$ is similar to the corresponding proof of the above two cases.

The following theorem is important.

Theorem 5.25. $A_+$ is T0 and DM-compact.
Proof. By Lemma 5.4(v) $A_+$ has T0 property iff for every two members $\Gamma, \Delta$ of $X_A^t (=t$-Clans$(A))$ the following holds: if $\rho_{A_+}(\Gamma) = \rho_{A_+}(\Delta)$, then $\Gamma = \Delta$.

Suppose $\rho_{A_+}(\Gamma) = \rho_{A_+}(\Delta)$ and for the sake of contradiction that $\Gamma \neq \Delta$, so $\Gamma \not\subseteq \Delta$ or $\Delta \not\subseteq \Gamma$. Considering the first case this means that there exists $a$ such that $a \in \Gamma$ and $a \notin \Delta$. Then by Lemma 5.21 $g_A(a) \in \rho_{A_+}(\Gamma)$ and $g_A(a) \notin \rho_{A_+}(\Delta)$ which shows that $\rho_{A_+}(\Gamma) \neq \rho_{A_+}(\Delta)$ - a contradiction. In a similar way the second case also implies a contradiction.

For DM-compactness we have to show the following three things:

(i) Every t-clan $\Gamma$ of $(A_+)^+$ is a point t-clan,

(ii) Every s-clan of $(A_+)^+$ is a point s-clan,

(iii) Every cluster of $(A_+)^+$ is a point cluster.

Proof of (i). Let $\Gamma$ be a t-clan of $(A_+)^+$. To show that $\Gamma$ is a point t-clan we have to find $\Delta \in X^t_A (= t$-Clans$(A))$ such that $\Gamma = \rho_{A_+}(\Delta)$. Let $\Delta = \Gamma^t$. By Lemma 5.24 (i) $\Gamma^t$ is a t-clan in $A$ and hence it is in $X^t_A$. More over we have $\rho_{A_+}(\Gamma^t) = \Gamma$.

The proofs of (ii) and (iii) are similar by using Lemma 5.24 (ii) and (iii).

Theorem 5.26. Topological representation theorem for DCA. Let $A$ be a DCA. Then the following conditions for $A$ are true:

(i) $(A_+)^+$ is a stable subalgebra of the algebra $RC(A_+)$.

(ii) The algebra $RC(A_+)$ is a DCA.

(iii) The function $g_A$ is a stable isomorphic embedding of $A$ into $RC(A_+)$.

(iv) If $Ax$ is a time axiom, then $Ax$ is true in $A$ iff $Ax$ is true in $RC(A_+)$.

Proof. (i) By Theorem 5.25 $A_+$ is a DM-compact DMS and hence by Lemma 5.16 (i) $(A_+)^+$ is a stable Boolean subalgebra of $RC(A_+)$. (ii) follows from (i) and Lemma 5.16 (ii). (iii) By Lemma 5.20 $g_A$ is an isomorphism from $A$ onto $(A_+)^+$ and hence by (i) $g_A$ is a stable isomorphic embedding of $A$ into $RC(A_+)$. (iv) follows from Lemma 5.25 and Lemma 5.17. □
5.6 Contact algebra as a special case of dynamic contact algebras

Let $A = (B_A, C_A)$ be a contact algebra. By Lemma 4.2 $A$ can be considered a DCA algebra on the base of the following definable relations: $a, b \in B_A$:

1. $aC_A^t b \Leftrightarrow aC_A^{\max} b \Leftrightarrow a \neq 0$ and $b \neq 0$.
2. $aB_A b \Leftrightarrow aC_A^t b$.
3. $aC_A^s b \Leftrightarrow aC_A^t b$.

Let $A = (B_A, C_A^s, C_A^t, B_A)$ be a DCA which satisfies the above conditions. It is obviously equivalent to the contact algebra $(B_A, C_A)$. Condition (3) is just giving another name of $C_A^s$, and conditions (1) and (2) can be relaxed correspondingly to the following:

1'. If $a \neq 0$ and $b \neq 0$, then $aC_A^t b$,
2'. If $a \neq 0$ and $b \neq 0$, then $aB_A b$.

Obviously (1') implies (1) and (2') implies (2). Hence if a DCA satisfies (1') and (2'), then it is equivalent to the contact algebra $(B_A, C_A)$. Condition (1) then makes t-clans to coincide with grills, and to have only one cluster, denote it by $t_0$ (the only time point of $A$) which is just the union of all ultrafilters in $A$. Condition (2) implies that $B_A = C_A^t$ which makes the relation $\prec_A$ to be the universal relation between grills and especially for $t_0$ to have $t_0 \prec_A t_0$. This suggests the following formal definition.

**Definition 5.27.** We say that $A$ is a **trivial DCA** if it satisfies the conditions (1') and (2').

Thus for the dual space $A_+$ of a trivial DCA we have that $T_A = \{t_0\}$ is a singleton set and that $t_0$ is the only point of $A$. This suggests to consider this as a characteristic property of a DMS corresponding in some sense to a trivial DCA and to adopt the following formal definition.

**Definition 5.28.** We say that $S$ is a trivial DMS if the set $T_S = \{t_0\}$ is a singleton with a single time point $t_0$ and $t_0 \prec_S t_0$.

**Lemma 5.29.** Let $S$ be a T0 and DM-compact space. Then the following two conditions are equivalent:

(i) $S$ is trivial DMS.

(ii) The dual algebra $S^+$ is a trivial DCA.
Proof. (i)⇒(ii). Suppose that $S$ is trivial DMS. First we will show that the DCA algebra $S^+$ has at most one cluster. Note that it has clusters. Let $\Gamma, \Delta$ be two clusters. By DM-compactness there is $x \in T_S$ such that $\rho_S(x) = \Gamma$ and $y \in T_S$ such that $\rho_S(y) = \Delta$. But $T_S$ is a singleton, so $x = y$ which implies $\Gamma = \rho_S(x) = \rho_S(y) = \Delta$. So we have only one cluster, say $\Gamma_0$.

In order show (ii) it is sufficient that the following is true for arbitrary regular closed sets $\alpha, \beta \in RC(X^+_S)$:

If $\alpha \neq \emptyset$ and $\beta \neq \emptyset$, then $\alpha C^t_S \beta$ and then $\alpha B_S \beta$.

Suppose $\alpha \neq \emptyset$ and $\beta \neq \emptyset$, then there exist $x \in \alpha$ and $y \in \beta$. Now we will apply the properties of canonical filters (see Lemma 5.15 from Section 5.4). Conditions $x \in \alpha$ and $y \in \beta$ imply $F_\alpha \subseteq \rho_S(x)$ and $F_\beta \subseteq \rho_S(y)$. $\rho_S(x)$ and $\rho_S(y)$ are t-clans in $S^+$ and can be extended into clusters. But there is only one cluster $\Gamma_0 = \rho_S(z)$ for some $z \in T_S$. Hence $F_\alpha \subseteq \rho_S(z)$ and $F_\beta \subseteq \rho_S(z)$. Then by the properties of canonical filters we get $z \in \alpha$ and $z \in \beta$, so $\alpha \cap \beta \neq \emptyset$ which shows $\alpha C^t_S \beta$. Because $z$ is the only element of $T_S$ we have $z \prec_S z$ which also shows that $\alpha B_S \beta$.

(ii)⇒(i) Let $S^+$ be a trivial DCA. We mentioned that the condition (1) makes t-clans to coincide with grills. Because there exists only one maximal grill - the union of all ultrafilters, then there exists only one cluster, say $\Gamma_0$. By DM-compactness there exists $x \in T_S$ such that $\rho_S(x) = \Gamma_0$. We will show that $T_S$ is a singleton. Suppose that $y \in T_S$. By axiom S8 of DMS $\rho_S(y)$ is a cluster an because we have only one cluster $\Gamma_0$ we have $\rho_S(y) = \Gamma_0$. So $\rho_S(x) = \rho_S(y)$. Because $S$ is a $T0$ space this equality implies $x = y$. 

Theorem 5.30. New topological representation theorem for contact algebras. Let $A = (B_A, C_A)$ be a contact algebra. Consider it as a trivial DCA. Then the following conditions are true.

(i) The regular set-algebra $RC(A_+)$ is a trivial DCA.

(ii) The function $g_A$ is a stable isomorphic embedding of $A$ into $RC(A_+)$. 

Proof. The Theorem is a consequence of of Theorem 5.26 - Topological representation theorem for DCA. Condition (iii) of the theorem says that the function $g_A$ is a stable isomorphic embedding of $A$ into $RC(A_+)$. This proves our condition (ii). Let us note that it is easy to see that the lifting Lemma 5.2 is true for the formulas $(1')$ and $(2')$. This implies that the conditions $(1')$ and $(2')$ are true in $RC(A_+)$, so $RC(A_+)$ is a trivial DCA and this proves our condition (i).
6 Topological duality theory for DCA

In this section we extend the topological representation of DCAs to a topological duality theory of DCAs in terms of DMSes. We assume basic knowledge of category theory: categories, morphisms, functors and natural isomorphisms (see, for instance, Chapter I from [53]). Since DCA is a generalization of contact algebra, and DMS is a generalization of mereotopological space, the developed duality theory in this section will generalize the duality theory for contact algebras and mereotopological spaces presented by Goldblatt and Greice in [35] and some proofs below will be the same as in [35]. Other topological duality theories for contact and precontact algebras are presented in [20] and it is possible to generalize them for DCAs, but in this paper we follow the scheme of [35] for two purposes: first, because the corresponding notion of DMS fits quite well to the topological representation theory for DCS-s, and second, because the proofs in this case are more short.

6.1 The categories DCA and DMS

**Definition 6.1.** The category $\text{DCA}$ consists of the class of all DCAs supplied with the following morphisms, called DCA-morphisms.

Let $A_i = (B_{A_i}, C_{A_i}, C_{t_{A_i}}, B_{A_i})$, $i = 1, 2$ be two DCAs. Then $f : A_1 \rightarrow A_2$ is a DCA-morphism if it is a mapping $f : B_{A_1} \rightarrow B_{A_2}$ which satisfies the following conditions:

(f 1) $f$ is a Boolean homomorphism from $B_{A_1}$ into $B_{A_2}$.

For all $a, b \in B_{A_1}$:

(f 2) if $f(a)C_{A_2}^s f(b)$, then $aC_{A_1}^s b$,

(f 3) if $f(a)C_{A_2}^t f(b)$, then $aC_{A_1}^t b$,

(f 4) if $f(a)B_{A_2}^s f(b)$, then $aB_{A_1}^s b$.

$A_1$ is the domain of $f$ and $A_2$ the codomain of $f$.

We define $f_+ =_{\text{def}} f^{-1}$ acting on $t$-clans of $A_2$ as follows: for $\Gamma \in t$-Clans$(A_2)$, $f^{-1}(\Gamma) =_{\text{def}} \{a \in B_{A_1} : f(a) \in \Gamma\}$.

A DCA-morphism $f : A_1 \rightarrow A_2$ is a DCA-isomorphism (in the sense of category theory) if there is a DCA-morphism $g : A_2 \rightarrow A_1$ such that the compositions $f \circ g$ and $g \circ f$ are the identity morphism of their domains. It is
a well known fact that this definition is equivalent to the standard algebraic
definition of isomorphism in universal algebra: \( f \) is bijection and preserves
contact.

**Definition 6.2.** The category \( \text{DMS} \) consists of the class of all DMSes
equipped with suitable morphisms called DMS morphism. The definition
is as follows. Let \( S_i = (X^t_{S_i}, X^s_{S_i}, T_{S_i}, \preceq_{S_i}, M_{S_i}), i = 1, 2 \) be two DMSes. A
DMS-morphism is a mapping
\[
\theta: X_{S_1} \rightarrow X_{S_2}
\]
such that:

1. \( \theta(1) \) if \( x \in X^s_{S_1} \), then \( \theta(x) \in X^s_{S_2} \).
2. \( \theta(2) \) If \( x \preceq_{S_1} y \), then \( \theta(x) \preceq_{S_2} \theta(y) \).

Let \( a \subseteq X^t_{S_2} \) and \( \theta^{-1}(a) = \{ x \in X^t_{S_1} : \theta(x) \in a \} \). We define \( \theta^+ = \theta^{-1} \).

The next two requirements for \( \theta \) are the following:

1. \( \theta(3) \) If \( a \in M_{S_2} \) then \( \theta^{-1}(a) \in M_{S_1} \) and
2. \( \theta(4) \) the map \( \theta^{-1}: M_{S_2} \rightarrow M_{S_1} \) is a Boolean algebra homomorphism from
   \( (M_2) \) into \( (M_1) \).

Note that in \( M_S \) the join operation is a set theoretical union of regular
closed sets. Since meets in Boolean algebra is definable by the join and the
complement *, for the condition \( \theta(4) \) it is sufficient to assume that \( \theta^{-1} \)
preserves complement.

A DMS-morphism \( \theta : S_1 \rightarrow S_2 \) is a DMS-isomorphism if there exists a
converse DMS-morphism \( \eta : S_2 \rightarrow S_1 \) such that the compositions \( \theta \circ \eta \) and
\( \eta \circ \theta \) are identity morphisms in the corresponding domains.

The following lemma states an equivalent definition of DMS-isomorphism.
Similar statement for mereotopological isomorphism is Theorem 2.2 from
[35].

**Lemma 6.3.** Let \( S, S' \) be DM-spaces and \( \theta : S \rightarrow S' \) be DMS-morphism
from \( S \) into \( S' \). Let \( a \subseteq X^t_S \) and define \( \theta[a] = \{ \theta(x) : x \in a \} \). Then the
following two conditions are equivalent:

1. \( \theta(a) \) is a DMS-isomorphism from \( S \) onto \( S' \).
2. \( \theta(a) \) is a bijection from \( X^t_S \) onto \( X^t_{S'} \), satisfying the following conditions:
   1. \( \theta(1) \) If \( \theta(x) \in X^s_{S'}, \) then \( x \in X^s_S \).
   2. \( \theta(2) \) If \( \theta(x) \prec_{S'} \theta(y) \), then \( x \prec_S y \).
   3. \( \theta(3) \) If \( a \in M_S \), then \( \theta[a] \in M_{S'} \).
Proof. (i)⇒(ii) Suppose that $\theta$ is a DMS isomorphism from $S$ onto $S'$. Then obviously $\theta$ is a bijection with converse $\eta$ such that $\theta$ is a DMS-morphisms from $S$ onto $S'$ and $\eta$ is a DMS-morphism from $S'$ onto $S$ such that the composition $\theta \circ \eta$ is the identity in $S'$ and $\eta \circ \theta$ is the identity in $S$. To show (1) let $\theta(x) \in X^s_{S'}$. Then $x = \eta(\theta(x)) \in X^s_S$, because $\eta$ is a DMS-morphism from $S'$ onto $S$. In a similar way we show (2). To show (3) let $a \in M_S$.

Let $f : A_1 \rightarrow A_2$ and $g : A_2 \rightarrow A_3$ be two DCA-morphisms. The composition $h = f \circ g$ is a mapping $h : B_{A_1} \rightarrow B_{A_3}$ acting as follows; for $a \in B_{A_1}$: $h(a) = g(f(a))$. In a similar way we define composition for DMS morphisms.

The following lemma has an easy proof.

**Lemma 6.4.** (i) The composition of two DCA-morphisms is a DCA-morphism. The identity mapping $1_A$ on each DCA $A$ is a DCA-morphism. Hence $\text{DCA}$ is indeed a category.

(ii) The composition of two DMS-morphisms is a DMS-morphism. The identity mapping $1_S$ on each DMS $S$ is a DMS-morphism. Hence $\text{DMS}$ is indeed a category.

It follows from Lemma 6.4 that $\text{DCA}$ and $\text{DMS}$ are indeed categories. We denote by $\text{DMS}^*$ the full subcategory of $\text{DMS}$ of all $\text{T0}$ and DM-compact DMSes.

We introduce two contravariant functors.
Φ: \textbf{DCA}→\textbf{DMS}, and Ψ: \textbf{DMS}→\textbf{DCA} as follows:

(I) For a given DCA $A$ we put $\Phi(A) = A_+$ and for a DCA-morphism $f : A \rightarrow A'$ we put $\Phi(f) = f_+$ and prove that $f_+$ is a DMS-morphism from $(A')_+$ into $A$.

(II) For a given DMS $S$ we put $\Psi(S) = S^+$ and for a DMS-morphism $\theta : S \rightarrow S'$ we put $\Psi(\theta) = \theta^+$ and prove that $\theta^+$ is a DMS morphism from $(S')^+$ into $S$.

(III) We show that for each DCA $A$ the mapping $g_A(a) = \{ \Gamma \in t\text{-Clans}(A) : a \in \Gamma \}, a \in B_A$ is a natural isomorphism (in the sense of category theory (see [53] Chapter I, 4.)) from $A$ to $\Psi(\Phi(A)) = (A_+)^+$.

(IV) We show that for each T0 and DM-compact DMS $S$ the mapping $\rho_S(x) = \{ a \in M_S : x \in a \}, x \in X^t_S$, is a natural isomorphism from $S$ to $\Phi(\Psi(S) = (S^+)^+$. All this shows that the category \textbf{DCA} is dually equivalent to the category \textbf{DMS} of T0 and DM-compact DMS. The realization of (I)-(IV) is given in the next subsection.

6.2 Facts for DCA-morphisms and DMS-morphisms

\textbf{Lemma 6.5.} Every DMS-morphism is a continuous mapping.

\textit{Proof.} Let $\theta : S \rightarrow S'$ be a DMS-morphism. Since $\theta^{-1}$ maps $M_{S'}$ (which is the closed basis of the topology of $S'$) into $M_S$, then $\theta$ is continuous. □

\textbf{Lemma 6.6.} Let $f : A \rightarrow A'$ be a DCA-morphism. Then:

(i) If $\Gamma$ is a t-clan in $A'$ then $f^{-1}(\Gamma) = \{ a \in B_A : f(a) \in \Gamma \}$ is a t-clan in $A$.

(ii) If $\Gamma$ is an s-clan in $A'$ then $f^{-1}(\Gamma) = \{ a \in B_A : f(a) \in \Gamma \}$ is an s-clan in $A$.

\textit{Proof.} The proof consists of a routine check of the corresponding definitions of t-clan and s-clan. □

\textbf{Lemma 6.7.} (i) Let $A, A'$ be two DCAs and $f : A \rightarrow A'$ be a DCA-morphism. Then $f_+$ is a DMS-morphism from $(A')_+$ to $A_+$. 

(ii) The mapping $g_A(a) = \{ \Gamma \in t\text{-Clans}(A) : a \in \Gamma \}, a \in B_A$ is a natural DCA-isomorphism of $A$ onto $\Psi(\Phi(A)) = (A_+)^+$. 

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Proof. (i) Remind that \((A')_+ = (t\text{-Clans}(A'), s\text{-Clans}(A'), Clusters(A'), \prec_{A'}, M_{A'})\). If \(\Gamma \in t\text{-Clans}(A')\), then by Lemma 6.6 \(f^{-1}(\Gamma)\) is a t-clan of \(A\) and similarly for the case when \(\Gamma\) is an s-clan. This shows that the condition \((\theta 1)\) for DMS-morphisms is fulfilled. For the condition \((\theta 2)\) let \(\Gamma \prec_{A'} \Delta, \Gamma, \Delta \in t\text{-Clans}(A')\). We have to show that \(f^{-1}(\Gamma) \prec_{A} f^{-1}(\Delta)\).

By the definition of \(\prec_{A}\) for clans (see (9)) this means the following. Let \(a \in f^{-1}(\Gamma)\), \(b \in f^{-1}(\Delta)\). Then \(f(a) \in \Gamma\) and \(f(b) \in \Delta\). But \(\Gamma \prec_{A'} \Delta\), so \(f(a) \mathcal{B}_{A'} f(b)\), which by (f 4) implies \(a \mathcal{B}_A b\). This by (9) sows that \(\Gamma \prec_{A} \Delta\).

The next step is to verify the condition \((\theta 3)\) of DMS-morphisms, namely that \((f_+)^+\) maps the members of \(M_{A'}\) into the members of \(M_A\). Note that the members of \(M_A\) are of the form \(g_A(a)\) for \(a \in B_A\) and that \(g_A(a) = \{\Gamma \in t\text{-Clans}(A) : a \in \Gamma\}\) and similarly for the members of \(M_{A'}\). In order to verify \((\theta 3)\) we will show that for any \(a \in B_A\) the following equality holds which indeed shows that \((f_+)^+\) maps \(M_A\) into \(M_{A'}\):

\[
(f_+)^+ (g_A(a)) = g_{A'}(f(a))
\]

(13)

To show (13) note that \((f_+)^+ (g_A(a))\) is a subset of \(t\text{-Clans}(A')\). So let \(\Gamma \in t\text{-Clans}(A')\). Then the following sequence of equivalences proves (13):

\[
\Gamma \in (f_+)^+ (g_A(a)) \iff \Gamma \in (f^{-1})^{-1}(g_A(a)) \iff f^{-1}(\Gamma) \in g_A(a) \iff a \in f^{-1}(\Gamma) \iff f(a) \in \Gamma \iff \Gamma \in g_{A'}(f(a)).
\]

Now we verify the condition \((\theta 4)\) of DMS-morphisms: \((f_+)^+\) preserves the Boolean complement. We show this by applying (13) and the facts that \(f\) and \(g_{A'}\) acts as Boolean homomorphisms:

\[
((f_+)^+_+)(g_A(a)^*) = (f_+)^+(g_A(a^*)) = g_{A'}f(a^*) = (f_+)^+(g_A(a^*)) = ((f_+)^+_+(g_A(a^*))^*)
\]

(ii) The statement that \(g_A\) is a natural isomorphism in the sense of category theory means the following: first, that \(g_A\) is indeed an isomorphism from \(A\) onto \(A_+\) (this is the Theorem 5.20) and second, that for any DCA-morphism \(f : A \rightarrow A'\), the following equality should be true: \(g_{A'} \circ f = (f_+)^+ \circ g_A\).

By the definition of the composition \(\circ\) for DCA-morphisms this equality is equivalent to the following: for any \(a \in B_A\) the following holds:

\[
g_{A'}(f(a)) = (f_+)^+(g_A(a)),\text{ which is just (13).}
\]

\[\Box\]

Lemma 6.8. Let \(S, S'\) be two DMS-s and \(\theta : S \rightarrow S'\) be a DMS-morphism from \(S\) to \(S'\). Then \(\theta^+\) is a DCA-morphism from \((S')^+\) to \(S^+\).

Proof. We have to verify that \(\theta^+ = \theta^{-1}\) satisfies the conditions (f1)-(f4) for DCA-morphism. Condition (f1) is fulfilled by the condition \((\theta 4)\) for
DMS-morphisms. For condition (f2) suppose that for some \( a, b \in M_S \), \( \theta^{-1}(a)C \theta^{-1}(b) \) and proceed to show \( aC_S b \). This implies that there exists \( x \in X'_S \) such that \( x \in \theta^{-1}(a) \) and \( x \in \theta^{-1}(b) \). From here we obtain \( \theta(x) \in a \), \( \theta(x) \in b \) and \( \theta(x) \in X'_S \) (by condition (\( \theta1 \)) for DMS morphism) which yields \( aC_S b \). In a similar way one can verify condition (f3).

For (f4) suppose \( \theta^{-1}(a)B_S \theta^{-1}(b) \) and proceed to show that \( aB_S b \). Then there exist \( x, y \in X'_S \) such that \( x <_S y, x \in \theta^{-1}(a), y \in \theta^{-1}(b) \). This implies \( \theta(x) \in a, \theta(y) \in b \), and by (\( \theta1 \)) and (\( \theta2 \)) that \( \theta(x), \theta(y) \in X'_S \) and \( \theta(x) <_S \theta(y) \). This implies \( aB_S b \). \( \square \)

Before the formulation of the next statement let us see what is \( (S^+)_+ \) for a DMS \( S \). \( S^+ \) is the dual of \( S \) which is the DCA algebra \((M_S,C^+_S,C^{+_S}_S,B_S)\)
(see Definition 5.3). Then \((S^+)_+ \) is the dual space of the algebra \( S^+ \) which is \((S^+)_+ = (X'_S,X^+_S,T_{S^+},<_S,M_{S^+}), \) where \( X'_S \) is the set of t-clans of \( S^+ \), \( X^+_S \) is the set of s-clans of \( S^+ \), \( T_{S^+} \) is the set of clusters of \( S^+ \), \( <_S \) is the relation defined by (9) between t-clans, and \( M_{S^+} \) is the set \((g_{S^+}(a) : a \in M_S), \) where \( g_{S^+}(a) =_{def} \{ \Gamma \in t-clans(S^+) : a \in \Gamma \} \) (see Section 5.5).

**Lemma 6.9.** (i) Let \( S \) be a DMS. Then \( \rho_S \) is a DMS-morphism from \( S \) to \((S^+)_.\)

(ii) Let \( S \) be a DM-compact DMS and let for \( a \subseteq X'_S \), \( \rho_S[a] =_{def} \{ \rho_S(x) : x \in a \} \). Then for \( a \in M_S \): \( \rho_S[a] = g_{S^+}(a) \) (for the function \( g_A \) for a DCA A see Section 5.5).

(iii) If \( S \) is T0 and DM-compact, then \( \rho_S \) is a DMS-isomorphism from \( S \) onto \((S^+)_.\).

(iv) If \( S \) is a T0 and DM-compact DMS, then \( \rho_S \) is a natural isomorphism from \( S \) to \( \Phi(\Psi(S)) = (S^+)_.\)

**Proof.** (i) We have to verify whether \( \rho_S \) satisfies the conditions (\( \theta1 \))-\( \theta4 \)) for DMS-morphisms. By Lemma 5.4 \( \rho_S(x) \) is a t-clan in \( S^+ \) for \( x \in X'_S \) and an s-clan in \( S^+ \) for \( x \in X^+_S \). This verifies the conditions (\( \theta1 \)) and (\( \theta2 \)) for DMS-morphisms. Condition (\( \theta2 \)) is guaranteed by axiom (7) for DMS and Lemma 5.4 (iv). For condition (\( \theta3 \)) we have to show that \( (\rho_S)^{-1} \) transforms the members from \( M_{S^+} \) into the members from \( M_S \) (recall that the members of \( M_{S^+} \) are of the form \( g_{S^+}(a), a \in M_S \), see the text before the lemma). This can be seen from the following equality
\[
(\rho_S)^{-1}(g_{S^+}(a)) = a
\]
Indeed, for $x \in X_S^t$ we have:

$$x \in (\rho_S)^{-1}(g_S(x)) \iff \rho_S(x) = g_S(x) \iff \rho_S(x) \iff x \in a.$$  

For condition (74) we have to show that $(\rho_S)^{-1}$ preserves Boolean complement. The following sequence of equalities proves this: $(\rho_S)^{-1}(g_S(x)) = a^* = ((\rho_S)^{-1}(g_S(x)))^*$, which is true on the base of (14).

(ii) Suppose $a \in M_S$ and let us show first $\rho_S[a] \subseteq g_S(a)$:

$$\rho_S(x) \in \rho_S[a] \Rightarrow x \in a \Rightarrow a \in \rho_S(x) \Rightarrow \rho_S(x) \in g_S(a) \quad \text{(because $\rho_S(x)$ is a t-clan in the DCA algebra $S^+$)}.$ For the converse inclusion, let $\Gamma$ be a t-clan in $S^+$. Then by DM-compactness there exists $x \in X_S^t$ such that $\Gamma = \rho_S(x)$. Then for $a \in M_S$:

$$\Gamma \in g_S(a) \Rightarrow a \in \Gamma \Rightarrow a \in \rho_S(x) \quad \text{and} \quad x \in a \Rightarrow \rho_S(x) \in \rho_S[a] \Rightarrow \Gamma \in \rho_S[a].$$

(iii) Let $S$ be T0 and DM-compact. Then by Lemma 5.7 then $\rho_S$ is a one-one mapping from $X_S^t$ onto the set of all t-clans of $S^+$, which are the points of $(S^+)$. By (i) $\rho_S$ is a DMS-morphism from $S$ to $(S^+)$. So in order to show that $\rho_S$ is a DMS-isomorphism from $S$ onto $(S^+)$, we have to see if $\rho_S$ satisfies the conditions (1), (2) and (3) of Lemma 6.3 (ii).

For condition (1) suppose $\rho_S(x) \in X_S^t$. Then $\rho_S(x)$ is a t-clan in $M_S$. By DM-compactness there exists $y \in X_S^t$ such that $\rho_S(x) = \rho_S(y)$. By T0 condition this implies $x = y$, so $x \in X_S^t$.

For condition (2) suppose $\rho_S(x) \not<_{S^+} \rho_S(y)$. Then by Lemma 5.4 and axiom (S7) for DMS we obtain $x \not<_{S} y$.

For condition (3) suppose $a \in M_S$ and proceed to show that $\theta[a] \in M_{(S^+)\Gamma}$. By (ii) $\theta[a] = g_S(a)$ and since $g_S(a) \in M_{(S^+)\Gamma}$ we get $\theta[a] \in M_{(S^+)\Gamma}$.

Thus the conditions of (1), (2) and (3) are fulfilled which proves that $\rho_S$ is a DMS-isomorphism from $S$ onto $(S^+)$. 

(iv) Let $S$ be a T0 and DM-compact DMS. In order $\rho_S$ to be a natural isomorphism from $S$ to $(S^+)\Gamma$ it has to satisfy the following two conditions: first, $\rho_S$ have to be a DMS-isomorphism - this is guaranteed by (iii), and second, for every DMS morphism $\theta : S \Rightarrow S'$ the following equality should be true: $\theta \circ \rho_S = \rho_S \circ (\theta^+)$. This equality is equivalent to the following condition: for $x \in X_S^t$

$$(\theta^+) \circ (\rho_S(x)) = \rho_{S'}(\theta(x)) \quad (15)$$

The following sequence of equivalencies proves (15). For $a \in M_{S'}$

$a \in (\theta^+) \circ (\rho_S(x)) \iff a \in (\theta^+) \circ (\rho_S(x)) \iff \Theta^+(a) \in \rho_S(x) \iff x \in \theta^+(a) \iff x \in \theta^{-1}(a) \theta(x) \in a \iff a \in \rho_{S'}(\theta(x)).$
As applications of the developed theory we establish some isomorphism correspondences between the objects of the two categories. The isomorphism between two objects will be denoted by the symbol $\cong$.

**Lemma 6.10.** Let $A, A'$ be two DCAs. Then the following conditions are equivalent:

(i) $A \cong A'$,

(ii) $A_+ \cong (A')_+$,

(iii) $(A_+)^+ \cong ((A')_+)^+$

*Proof.*\((i) \Leftrightarrow (iii)\). By Lemma 5.20 we have $A \cong (A_+)^+$ and $A' \cong (A'_+)^+$. This makes obvious the equivalence $(i) \Leftrightarrow (iii)$.

\((i) \Rightarrow (ii)\) Suppose $A \cong A'$, then there exists a one-one mapping $f$ from $A$ onto $A'$ with a converse mapping $h$ such that $f : A \to A'$ is a DCA morphism from $A$ onto $A'$ and $h : A' \to A$ is a DCA morphism from $A'$ onto $A$ such that the composition $f \circ h$ is the identity mapping in $A'$ and the composition $h \circ f$ is the identity mapping in $A$. Then by Lemma 6.7 $f_+$ is a DMS-morphism from $A'_+$ onto $A_+$ and $h_+$ is a DMS-morphism from $A_+$ onto $A'_+$.

We shall show the following:

1. The composition $f_+ \circ h_+$ is the identity in $A'_+$, and
2. The composition $h_+ \circ f_+$ is the identity in $A_+$.

Then, by the definition of DMS-isomorphism this will imply that both $f_+$ and $h_+$ are DMS-isomorphisms in the corresponding directions.

Note that the members of $A_+$ are the t-clans of $A$ and similarly for $A'_+$.

To show (1) let $\Gamma$ be a point of the space $A'_+$, i.e. $\Gamma$ is a t-clan in $A'$. We shall show that $(f_+ \circ h_+)(\Gamma) = \Gamma$ which will prove (1). This is seen from the following sequence of equivalencies where $a$ is an arbitrary element of $B_{A'}$:

\[
a \in (f_+ \circ h_+)(\Gamma) \iff a \in (f_+)(h_+)(\Gamma) \iff a \in f^{-1}(h_+(\Gamma)) \iff f(a) \in h_+(\Gamma) \iff h(f(a)) \in \Gamma \iff a \in \Gamma.
\]

Here we use that $h(f(a)) = a$ for $a \in B_{A'}$ because $h$ is the converse of the one-one mapping $f$ from $B_A$ onto $B_{A'}$.

In a similar way we show (2).

\((ii) \Rightarrow (iii)\) The proof is similar to the above one. Suppose $A_+ \cong (A'_+)_+$, then there exists a one-one mapping $\theta$ and its converse $\eta$ such that $\theta$ is a DMS-morphism from $A_+$ onto $(A'_+)_+$ and $\eta$ is a DMS-morphism from $(A'_+)_+$ onto
Then by Lemma 6.8 \( \theta^+ \) is a DCA-morphism from \( (A'_+)^+ \) into \( (A_+)^+ \) and \( \eta^+ \) is a DCA-morphism from \( (A'_+)^+ \) into \( (A_+)^+ \). We shall show that both \( \theta^+ \) and \( \eta^+ \) are DCA-isomorphisms in the corresponding directions by showing that their compositions are identities in the corresponding domains. Let us note that the domain of \( \theta^+ \) are the members of the algebra \( (A'_+)^+ \) which are of the form \( g_{A'}(a) \), \( a \in B_{A'} \), and similarly for the members of \( (A_+)^+ \). Namely we will show the following two things:

3) \( \theta^+ \circ \eta^+ \)(\( g_{A'}(a) \)) = \( g_{A'}(a) \) for any \( a \in B_{A'} \),
4) \( \eta^+ \circ \theta^+ \)(\( g_{A'}(a) \)) = \( g_{A'}(a) \) for any \( a \in B_{A} \).

To show (3) note that \( g_{A'}(a) = \{ \Gamma \in t - clans(A') : a \in \Gamma \} \). So let \( \Gamma \in t - clans(A') \). Then the following sequence of equivalents proves (3):

\[ \Gamma \in (\theta^+ \circ \eta^+)(g_{A'}(a)) \iff \Gamma \in (\theta^+ (eta^+(g_{A'}(a)))) \iff \Gamma \in (\theta^{-1}(eta^+(g_{A'}(a)))) \iff \theta(\Gamma) \in (eta^+(g_{A'}(a))) \iff \eta(\theta(\Gamma)) \in g_{A'}(a) \iff \Gamma \in g_{A'}(a). \]

We have just used that \( \eta(\theta(\Gamma)) = \Gamma \), because \( \eta \) is the converse of the one-one mapping \( \theta \) from \( X^t_{A'} = t - clans(A) \) onto \( X^t_{(A')_+} = t - clans(A') \). The proof of (4) is similar.

**Lemma 6.11.** Let \( S, S' \) be two DMSes. Then the following conditions are equivalent:

1. \( S \cong S' \),
2. \( S^+ \cong (S')^+ \),
3. \( (S^+)_+ \cong ((S')^+)_+ \).

**Proof.** The proof is analogous to the proof of Lemma 6.10

As a corollary from Lemma 6.10 and Lemma 6.11 we obtain the following addition to the topological representation theorem for DCAs.

**Corollary 6.12.** There exists a bijective correspondence between the class of all, up to DCA-isomorphism DCAs, and the class of all, up to DMS-isomorphism DMSes; namely, for every DCA-algebra \( A \) the corresponding DMS of \( A \) is \( A_+ \) - the canonical algebra of \( A \); and for every DMS \( S \) the corresponding DCA of \( S \) is \( S^+ \) - The canonical algebra of \( S \).
6.3 Topological duality theorem for DCAs

In this section we prove the third important theorem of this paper.

**Theorem 6.13. Topological duality theorem for DCAs.** The category \( DCA \) of all dynamic contact algebras is dually equivalent to the category \( DMS^* \) of all \( T_0 \) and DM-compact DMSes.

**Proof.** The proof follows from Lemma 6.7, Lemma 6.8 and Lemma 6.9.

The above theorem has several consequences to some important subcategories of \( DCA \) and \( DMS \). The first example is the following. Let \( Ax \) be a subset of the set of temporal axioms (rs), (ls), (up dir), (down dir), (circ), (dens), (ref), (lin), (tri), (tr). Consider the class of all DCAs satisfying the axioms from \( Ax \). It is easy to see that this class forms a full subcategory of the category of all DCAs under the DCA-morphism. Denote this subcategory by \( DCA(Ax) \). Let \( \hat{Ax} \) be the subset of the corresponding to \( Ax \) time condition from the list (RS), (LS), (Up Dir), (Down Dir), (Circ), (Dens), (Ref), (Lin), (Tri), (Tr). Consider the class of all \( T_0 \) and DM-compact DMSes which satisfy the axioms \( \hat{Ax} \). It is easy to see that this class is a full subcategory of the category \( DMS^* \) of all \( T_0 \) and DM-compact dynamic mereotopological spaces. Denote this subcategory by \( DMS(\hat{Ax})^* \)

**Theorem 6.14.** The category \( DCA(Ax) \) of all dynamic contact algebras satisfying \( Ax \) is dually equivalent to the category \( DMS(\hat{Ax})^* \) of all \( T_0 \) and DM-compact DMSes satisfying \( \hat{Ax} \).

**Proof.** Let \( S \) be a \( T_0 \) and DM-compact DMS. It follows by Lemma 5.19 that \( S \) satisfies \( \hat{Ax} \) iff \( S^+ \) satisfies \( Ax \). Now the theorem is a corollary of Theorem 6.13.

Another subcategory of \( DCA \) is the class of all trivial DCAs with the same morphisms. Denote it by \( DCA_{trivial} \). The corresponding subcategory of \( DMS^* \) with the same morphisms is the class of all trivial \( T_0 \) and DM-compact DMSes. Denote it by \( DMS^*_{trivial} \). The following theorem is also an obvious consequence of Theorem 6.13.

**Theorem 6.15.** The category \( DCA_{trivial} \) is dually isomorphic to the category \( DMS^*_{trivial} \).
In [35] Goldblatt and Grice proved that the category of contact algebras $\text{CA}$ is dually isomorphic to the category $\text{MS}^*$ of mereocompact and $T_0$ mereotopological spaces (for the relevant definitions see Remark 4.3). Although trivial DCAs can be identified with CAs, the morphisms considered between them in [35] are different. Also the corresponding trivial $T_0$ DM-compact spaces are different from mereotopological spaces. So, the result obtained by Goldblatt and Grice is different from Theorem 6.15. We hope that it can be derived from 6.15 by proving that the category of $\text{CA}$ is equivalent to the category $\text{DCA}_{\text{trivial}}$ and that the category $\text{MS}^*$ is equivalent to the category $\text{DMS}^*_{\text{trivial}}$. We left this problem for a future work.

7 Concluding remarks

Overview. The aim of this paper is to present with some details a version of point-free theory of space and time based on a special representative example of a dynamic contact algebra (DCA). The axioms of the algebra are true sentences from a concrete point-based model, the snapshot model, developed in Section 3. Theorem 4.15 - the Representation theorem for DCA by snapshot models shows that the chosen axioms are enough to code the intuition based on snapshot construction which can be considered as the cinematographic model of spacetime. In Section 4 we introduced topological models of DCAs giving them another intuition based on topology. These models are based on the notion of Dynamic mereotopological space (DMS). Let us note that the abstract definition of DCA can be considered as a "dynamic generalization" of contact algebra, which in a sense is a certain point-free theory of space called also mereotopology. In this relation contact algebras can be considered as a "static mereotopology" while dynamic contact algebras can be considered as a "dynamic mereotopology". Let us note that topological models of contact algebras, which are considered as the standard models of this notion, contain one type of points, which are just the "space points" while dynamic mereotopological spaces contain three kinds of points: partial time points, time points and space points, which are also partial time points. Time points realize the time contact, while space points realize the space contact. The fact that each space point is a partial time point says that space in this model is reduced to time, a feature quite similar to the Robb’s axiomatic treating of Minkowskian spacetime geometry in which space is reduced to time (see [63] and the discussion in Section 1.1). Another common feature of both snapshot and topological models is that the properties of the underline time structure corresponds to the validity of time axioms which
are point-free conditions for dynamic regions formulated by the relations of
time contact $C^t$ and precedence relation $B$. Because regions are observable
things, then recognizing which time axioms they satisfy we may conclude
which abstract properties satisfies the corresponding time structure.

**Discussions and some open problems.** Time contact relation $aC^t b$, and
precedence relation $aB b$ between two dynamic regions $a$ and $b$ in snapshot
models are defined by the predicate "existence" defined in Boolean algebras
as follows: $E(a)$ iff $a \neq 0$. One may ask if this predicate is a good one.
It has the following disadvantage - there are too many existing regions and
only one non-existing - the zero region. For instance we can not see the zero
region, but we can see on the sky a non-existing star - see Remark 3.2. What
we see is different from 0 but this does not mean that it is existing at the mo-
moment of observation. So the adopted in this paper definition for "existence"
is approximate one and we need a more exact definition corresponding to
what we intuitively mean by "actual existence". This is a serious problem
discussed in our papers [78, 79] in which we introduce an axiomatic defi-
nition and corresponding models of predicate "actual existence" (denoted
by $AE(a)$) and a corresponding relation between regions called "actual con-
tact". The predicate $E(a)$ satisfies the axioms of $AE(a)$ and is the simplest
one, but $AE$ is more general - it is possible for some region $a$ to have $a \neq 0$
but not $AE(a)$ like "non-existing stars" discussed in Remark 3.2. One of
our future plans is to reconstruct the theory of the present paper on the
base of the more realistic predicates of actual existence and actual contact.

Another subject of discussion is the relation $aC^t b$ called "time contact"
which is a kind of simultaneity relation. Special relativity theory (SR) also
studies a kind of simultaneity relation and states (and proves) that it is not
absolute and is relative to a given observer. Is it possible to relate these two
notions? In general these two relations are different because in our system
this is a relation between regions and in SR it is between events, which are
not regions but space-time points. Nevertheless we will try to find some cor-
respondence. By event in SR one normally assume a space point, taken from
our ordinary space, with attached time-point (a date), taken from a clock at-
tached to the space point with the assumption that all attached clocks work
synchronously (the possibility to have synchronized clocks in all points of
our space is explained by Einstein in [30] by a special synchronization pro-
cedure). So events are pairs $(A, t)$, where $A$ is a space point and $t$ is a real
number interpreted as a date. According to Einstein’s natural definition,
two events $(A_1, t_1), (A_2, t_2)$ are simultaneous if $t_1 = t_2$ which shows that si-
multaneity is an equivalence relation. Note that Einstein did not give formal
definition of "event", but in the terminology of Minkowski spacetime, which is a formal explication of SR spacetime, events are just spacetime points and two spacetime points are simultaneous if they have equal time coordinates. In our system we do not introduce the notion of event but in the abstract DCA an (approximate) analog of event can be identified with a pair \((U, \Gamma)\) where \(U\) is an ultrafilter and \(\Gamma\) is a cluster containing \(U - U\) is a space point and \(\Gamma\) is a time point (see Section 5.5). Let \((U_i, \Gamma_i), i=1,2\) be two events in DCA. Then, according to the simultaneity relation between events it can be easily seen that \((U_1, \Gamma_1)\) is simultaneous with \((U_2, \Gamma_2)\) iff \(U_1 R^t U_2\) which is just the canonical relation between ultrafilters corresponding to the contact relation \(C^t\). Note that \(R^t\) is also an equivalence relation as the simultaneity relation in SR is. So an analog of SR simultaneity relation in our theory is the relation \(R^t\) considered between "events" in the sense of DCA.

An analog of our before-after relation \(<\) between events in SR is \((A_1, t_1) \prec (A_2, t_2)\) iff \(t_1 < t_2\). This relation, like simultaneity, is not absolute and is relative to the observer. Note also that it is different from the Robb’s causal relation "before” taken as the unique basic relation between events in the axiomatic presentation of Minkowski geometry [63]). The natural analog of the above relation between DCAs ”events” is \((U_1, \Gamma_1) \prec (U_2, \Gamma_2) \iff def \Gamma_1 \prec \Gamma_2\). But we have \(\Gamma_1 \prec \Gamma_2\) iff \(U_1 \prec U_2\) which shows that the relation coincides with the canonical relation \(<\) between ultrafilters corresponding to the precontact relation \(\mathcal{B}\). This shows that the canonical relation \(<\) between ultrafilters which is used to characterize \(\mathcal{B}\) is not an analog of the Robb’s causal relation (let us denote it by \(\prec_{Robb}\) which has a special definition in Minkowski spacetime by means of its metric. An analog of this definition in Einstein’s SR is the following: \((A_1, t_1) \prec_{Robb} (A_2, t_2)\) iff \(|AB| \leq |t_1 - t_2|\) and \(t_1 < t_2\). This relation is stronger than the relation \(<\). It will be nice to have an abstract version of DCA containing stronger than \(\mathcal{B}\) precontact relation corresponding to causality. We put this problem to the list of our future plans.

Comparing the presented in this paper theory with SR we see that there is another feature which differs the corresponding theories: RS considers many observers and can prove that some relations between events like simultaneity are relative to corresponding observer, while a given DCA \(A\) is based on only one observer, denote it by \(O(A)\) (this observer can be identified with an abstract person describing the standard dynamic model of space which is isomorphic to \(A\)). So, because we have only one observer in our formalism, we can not give formal proofs whether the basic relations between regions are relative or not to the observer. Hence, building a theory like DCA
incorporating many observers is the next open problem.

One possibility for a theory with many observers describing one and the same reality is to consider a family of DCAs with some relations between them. Let $A$ and $A'$ be two DCAs from such a set. Examples of possible relations between them are, for instance, the following:

1. The observers $O(A)$ and $O(A')$ are at rest to each other, they have synchronous clocks, and have possibilities to exchange information.

2. The observers $O(A)$ and $O(A')$ are not at rest to each other but are not accelerated.

3. The case of (2) with accelerated observers.

How to characterize abstractly such relations? Maybe by establishing some morphism-like relations between $A$ and $A'$. An example of a set of DCAs with some morphisms between them is the category $\text{DCA}$ considered as a small category (the class of DCAs is a set). Then a natural question is what are saying the DCA-morphisms between the algebras considered as algebras produced by observers describing one and the same reality. For instance, what is the meaning of the condition on DCA-morphism $f : A \rightarrow A'$:

(\sharp) If $f(a)C_A f(b)$, then $aC_{A'} b$

If we interpret $f$ as a way for the observer $O(A)$ to point out some regions to the observer $O(A')$, then (\sharp) says that if $O(A')$ sees that the pointed regions are in a time contact, then the same has been seen by $A$. Similar interpretation have the other conditions on DCA-morphisms concerned $C^s$ and $C^b$. This means that $O(A')$ is seen the reality in the same way as $O(A)$.

Let us finish this section by formulating one more open problem. The axiomatization of Minkowski geometry presented by Robb [63] is point-based: the primitive concepts are points and the binary relation ”before” on points satisfying some axioms. The problem is to present a point-free characterization of Minkowskian geometry similar to DCA eventually with more spatio-temporal primitive relations between regions.

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### Appendix: Short review of papers on RBTS

In this appendix we present a short, probably incomplete review of papers on RBTS appeared after 1977 and not discussed in [73]. The papers are classified in several groups.

**I) Results on mereology.** First I want to mention here some papers devoted to a detailed analysis of results obtained by Polish logicians in the field of mereology and RBTS. The book Metamereology [61] extends some results on mereology, the paper [37] is devoted to a detailed analysis of Grzegorczyk point-free theory of space [36] and the paper [38] - to a full analysis of Tarski geometry of solids ([69]).

**II) Further results on contact and precontact algebras.** The papers [25, 11] contain some technical results on contact algebras. The paper [19] transfers the notion of dimension from topology to the corresponding notion of some classes of contact algebras and the paper [71] extends contact algebras with connectedness predicates and studies the corresponding quantifier-free logics. The paper [18] characterizes contact algebras on Euclidean spaces. The papers [23, 24] presented topological representation theorem for precontact algebras and new representation theorems for some classes of contact algebras.
(III) Duality theory of contact and precontact algebras and some related systems. There are many papers generalizing De Vries duality theorem [80] mainly with applications to topology: [6], [7], [14], [15], [16], [17], [9] - for Boolean algebras with quasi-modal operators which are equivalent to precontact algebras, [10] - for subordination Tarski algebras with application to De Vries duality. A paper about duality theory for contact and precontact algebras is [20] which include also some generalizations of the Stone Duality Theorem. Another duality theorem for contact algebras is based on mereotopological spaced is presented in [35].

(III) Generalizations of contact algebras. The paper [60] contains a generalization of contact algebra based only on the standard mereological relations part-of, overlap and underlap plus standard mereotopological relations of contact, dual contact and non-tangential inclusion and studies also a modal logic based on these relations. The paper [43] studies generalizations of contact algebras based on distributive lattices with three basic mereotopological relations of contact, dual contact and non-tangential inclusion taken as primitive relations. Representation theorems for extended contact algebras based on equivalence relations is in the paper [3]. Generalization of contact algebra based on non-distributive lattices is presented in [40, 81, 82].

Another generalization of contact algebra is the notion of sequent algebra which presents Tarski and Scott consequence relations as mereotopological relations - see [77] and [42]. In standard models with regular closed subsets of a topological space Tarski consequence relation $a_1, \ldots, a_n \vdash b$ is defined as $a_1 \cap, \ldots, \cap a_n \subseteq b$, which makes possible to define n-ary contact by $C_n(a_1, \ldots, a_n) \iff a_1, \ldots, a_n \not\vdash 0$ and ordinary contact as $aCb \iff a, b \not\vdash 0$. Generalizations of contact algebras with predicates of actual existence and actual contact are subject of [78, 79]. In standard contact algebras the predicate of existence is defined as follows: $E(a) \iff a \neq 0$. This is a quite strong predicate, because the only non-existing region is 0. The generalization is to relax this definition as follos: take a fixed grill $\Gamma$ (see Definition 2.14) and define $E(a) \iff a \in \Gamma$. Another line of generalizations is to consider Boolean algebras with contact relation and measure - see [51] and [52].

(IV) Modal and Quantifier-free logics based on contact and precontact algebras. Modal logics based on mereological and mereotopological relations arising from contact algebras or topology are presented in [55] and [60]. Papers on quantifier-free logics in the style of [4] related to contact algebras and their extensions and generalizations are [71] for log-
ics with connectedness predicates, [47] - studying them form computational point of view, [43],[41], [42] - for logics based on extended contact algebras Quantifier-free logics related to contact algebras with measure are [51] and [52].