Central Limit Theorems for Moving Average Random Fields with Non-Random and Random Sampling

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Abstract. For a Lévy basis \( L \) on \( \mathbb{R}^d \) and a suitable kernel function \( f : \mathbb{R}^d \to \mathbb{R} \), consider the continuous spatial moving average field \( X = (X_t)_{t \in \mathbb{R}^d} \) defined by \( X_t = \int_{\mathbb{R}^d} f(t - s) \, dL(s) \). Based on observations on finite subsets \( \Gamma_n \) of \( \mathbb{Z}^d \), we obtain central limit theorems for the sample mean and the sample autocovariance function of this process. We allow sequences \( (\Gamma_n) \) of deterministic subsets of \( \mathbb{Z}^d \) and of random subsets of \( \mathbb{Z}^d \). The results generalise existing results for time indexed stochastic processes (i.e. \( d = 1 \)) to random fields with arbitrary spatial dimension \( d \), and additionally allow for random sampling. The results are applied to obtain a consistent and asymptotically normal estimator of \( \mu > 0 \) in the stochastic partial differential equation \((\mu - \Delta)X = dL\) in dimension 3, where \( L \) is Lévy noise.

1. Introduction

Many statistical models with more than one spatial dimension are described by a linear stochastic partial differential equation with some additive noise, which means that we have a random field \( X \) on \( \mathbb{R}^d \) satisfying

\[
\mathcal{L}(\mu)X = dL, \tag{1.1}
\]

where \( \mathcal{L}(\mu) \) is a linear partial differential operator depending on some parameter \( \mu \) and \( dL \) denotes some noise, for example Gaussian or stable noise. If \( \mathcal{L}(\mu) \) has an integrable fundamental solution \( G_\mu \), the mild solution of (1.1) can be written as

\[
X_t = \int_{\mathbb{R}^d} G_\mu(t - s) \, dL(s), \tag{1.2}
\]

where \( dL \) denotes the additive noise, see for example [1], [2], [18] and [24]. The solution (1.2) is a so called continuous moving average random field. The additive noise \( dL \) studied in this paper will be a Lévy white noise, where the Gaussian white noise and stable noise are included. A detailed study of Lévy white noise can be found in [12], where it is also shown that a Lévy white noise defines a Lévy basis in the sense of Rajput and Rosinski [19]. Random fields of the form

\[
X_t = \int_{\mathbb{R}^d} f(t - s) \, dL(s), \tag{1.3}
\]

with a suitable kernel function \( f : \mathbb{R}^d \to \mathbb{R} \) and a Lévy basis \( L \) on \( \mathbb{R}^d \) (as in (1.2) with \( f = G_\mu \)) can be seen as a continuous and spatial extension of the discrete time moving average processes \( Z = (Z_t)_{t \in \mathbb{Z}} \), defined by

\[
Z_t = \sum_{k \in \mathbb{Z}} a_{t-k} W_k, \tag{1.4}
\]
where \((W_k)_{k \in \mathbb{Z}}\) is an independent and identically distributed sequence and \(a_k, k \in \mathbb{Z}\) are real coefficients.

In many cases one is interested in estimating the parameter \(\mu\) of the equation (1.1). If we know how the fundamental solution \(G_\mu\) depends on the parameter \(\mu\), it is sometimes possible to give moment estimators for \(\mu\). Of particular interest are estimators of the mean \(\mathbb{E}(X_t)\) and the autocovariance \(\text{cov}(X_t, X_{t+h})\) for \(t, h \in \mathbb{R}^d\).

In most applications only discrete spatial data is available, for example observations based on a finite subset \(\Gamma_n\) of the regular grid \(\mathbb{Z}^d\). A natural estimator for \(\mathbb{E}X_t\) is then the sample mean \(\frac{1}{|\Gamma_n|} \sum_{s \in \Gamma_n} X_s\), while a natural estimator for the autocovariance \(\text{cov}(X_t, X_{t+h})\) is the (adjusted) sample autocovariance

\[
\gamma_n^*(h) := \frac{1}{|\Gamma_n|} \sum_{s \in \Gamma_n} X_s X_{s+h}, \quad h \in \mathbb{Z}^d
\]

(assuming that the Lévy basis and hence \(X\) have mean zero and that for each \(s \in \Gamma_n\), both \(X_s\) and \(X_{s+h}\) are observed). Motivated by this, in this paper we will provide central limit theorems for the sample mean and sample autocovariance function as defined in (1.5) for continuous spatial moving average random fields as defined in (1.3) (equivalently, (1.2)), when the kernel function \(f\) decays sufficiently fast and the Lévy basis has finite variance or finite fourth moment and mean zero, respectively.

The sampling sequence \((\Gamma_n)_{n \in \mathbb{N}}\) will be a nested sequence of finite subsets of \(\mathbb{Z}^d\) satisfying \(|\Gamma_n| \to \infty\) and some extra conditions, and it will be either a sequence of deterministic subsets (referred to as non-random sampling) or a sequence of random subsets (referred to as random-sampling), more precisely of the form \(\Gamma_n = \{t \in [-n, n]^d \cap \mathbb{Z}^d | Y_t = 1\}\), where \((Y_t)_{t \in \mathbb{Z}^d}\) is a \(\{0, 1\}\)-valued stationary ergodic random field on \(\mathbb{Z}^d\). In the case of non-random sampling, we will need slightly higher moment conditions on the Lévy basis.

Central limit theorems for the sample mean and the sample autocovariance of (1.4) are classic and can be found e.g. in Chapter 7 of the book [6] (for \(d = 1\)). On the other hand, central limit theorems for Lévy driven moving average processes based on discrete low-frequency observations have only recently attracted attention, and this also only in dimension \(d = 1\), i.e. for continuous time series and not spatial data. In [7], the asymptotics of the sample mean and sample autocovariance are studied when \(f\) decays sufficiently fast and \(L\) has finite second or fourth moment, respectively. [22] studies the situation when \(f\) decays slowly leading to a long-memory process \(X\), while [11] considers the heavy tailed situation when the Lévy process \(L\) is in the domain of attraction of a stable non-normal distribution, and in [3] the case of random sampling when the process \(X\) is sampled at a renewal sequence is treated. Observe that all these results are in dimension \(d = 1\) only. The results of this paper can be seen as a generalization of the results of [7], who have \(d = 1\) and \(\Gamma_n = \{1, 2, \ldots, n\}\), to arbitrary spatial dimensions \(d \in \mathbb{N}\) and more general sets \(\Gamma_n\), and additionally allowing random sampling as described above.

The paper is organized as follows. In the next section, we fix notation and recall the notion of Lévy bases. Then, in Section 3 we state the main results of the present paper. These are central limit theorems for the sample mean as described above for
non-random and random sampling (Theorems 3.1 and 3.6 respectively), and central limit theorems for the sample autocovariance as described above for non-random and random sampling (Theorems 3.8 and 3.9 respectively). In Section 4 we apply the results to a random field given as a solution as in (1.1), more specifically, we consider the stochastic partial differential equation
\[(\mu - \Delta)X = dL\]
in dimension \(d = 3\), where \(\Delta\) denotes the Laplace operator, and obtain a consistent and asymptotically normal estimator of \(\mu > 0\) based on the sample mean. Finally, Sections 5 and 6 contain the proofs of the main theorems for the sample mean and the sample autocovariance, respectively.

2. Notation and Preliminaries

To fix notation, by a distribution on \(\mathbb{R}\) we mean a probability measure on \((\mathbb{R}, B(\mathbb{R}))\) with \(B(\mathbb{R})\) being the Borel \(\sigma\)-algebra on \(\mathbb{R}\). By a measure on \(\mathbb{R}^d\), \(d\) a natural number, we always mean a positive measure on \((\mathbb{R}^d, B(\mathbb{R}^d))\). The set \(B_b(\mathbb{R}^d)\) is the set of all bounded Borel measurable sets. The Dirac measure at a point \(b \in \mathbb{R}\) will be denoted by \(\delta_b\), the Gaussian distribution with mean \(a \in \mathbb{R}\) and variance \(b \geq 0\) by \(N(a, b)\) and the Lebesgue measure by \(\lambda^d\) on \(\mathbb{R}^d\). If a random vector \(X\) has law \(\mathcal{L}\) we write \(X \sim \mathcal{L}\). Weak convergence of measures will be denoted by “\(\Rightarrow\)”. We write \(\mathbb{N} = \{1, 2, \ldots\}\), \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) and \(\mathbb{Z}, \mathbb{R}\) for the set of integers and real numbers respectively. The indicator function of a set \(A \subset \mathbb{R}\) is denoted by \(1_A\). By \(L^p(\mathbb{R}^d, A)\) for \(1 \leq p < \infty\) and \(A \subset \mathbb{C}\) we denote the set of all Borel-measurable functions \(f : \mathbb{R}^d \to A\) such that \(\int_{\mathbb{R}^d} |f(x)|^p \lambda^d(dx) < \infty\). If \(A = \mathbb{R}\) we simply write \(L^p(\mathbb{R}^d)\). For two different sets \(A, B \subset \mathbb{R}^d\), we denote \(\text{dist}(A, B) := \inf\{\|x - y\| : x \in A \text{ and } y \in B\}\), where \(\|\cdot\|\) is the euclidean norm. We write ‘a.e.’ to denote almost everywhere and ‘a.s.’ to denote almost surely. \(|A|\) denotes the number of elements of the set \(A\).

We are interested in integrals of the form \(\int_{\mathbb{R}^d} f(u) dL(u)\), where \(dL\) denotes the integration over a Lévy basis. A Lévy basis can be understood in the following way:

**Definition 2.1** (see [19] p. 455). A Lévy basis is family \((L(A))_{A \in B_b(\mathbb{R}^d)}\) of real valued random variables such that

i) \(L(\bigcup_{n=0}^\infty A_n) = \sum_{n=0}^\infty L(A_n)\) a.s. for pairwise disjoint sets \((A_n)_{n \in \mathbb{N}_0} \subset B_b(\mathbb{R}^d)\) with \(\bigcup_{n \in \mathbb{N}_0} A_n \in B_b(\mathbb{R}^d)\),

ii) \(L(A_1)\) are independent for pairwise disjoint sets \(A_1, \ldots, A_n \in B_b(\mathbb{R}^d)\) for every \(n \in \mathbb{N}\),

iii) there exist \(a \in [0, \infty), \gamma \in \mathbb{R}\) and a Lévy measure \(\nu\) on \(\mathbb{R}\) (i.e. a measure \(\nu\) on \(\mathbb{R}\) such that \(\nu(\{0\}) = 0\) and \(\int_\mathbb{R} \min\{1, x^2\} \nu(dx) < \infty\)) such that

\(\mathbb{E}e^{iazL(A)} = \exp\left(\psi(z)\lambda^d(A)\right)\)

for every \(A \in B_b(\mathbb{R}^d)\), where

\(\psi(z) := iz + \frac{1}{2}az^2 + \int_\mathbb{R} \left(e^{ixz} - 1 - ixz1_{[-1,1]}(x)\right)\nu(dx), \quad z \in \mathbb{R}\).
The triplet \((a, \nu, \gamma)\) is called the characteristic triplet of \(L\) and \(\psi\) its characteristic exponent. By the Lévy-Khintchine formula, \(L(A)\) is then infinitely divisible.

It can be shown that the characteristic triplet is unique; conversely, to every \(a \in [0, \infty)\), \(\gamma \in \mathbb{R}\) and Lévy measure \(\nu\) there exists a Lévy basis with \((a, \nu, \gamma)\) as characteristic triplet. It follows from the general theory of infinitely divisible distributions that for a Lévy basis \(L\) with characteristic triplet \((a, \nu, \gamma)\) and \(p \in [1, \infty)\), we have

\[
\int |x|^p \nu(dx) < \infty \quad \text{if and only if} \quad \mathbb{E}|L(A)|^p < \infty \quad \text{for some (equivalently, all)} \quad A \in B_b(\mathbb{R}^d) \quad \text{with} \quad \lambda^d(A) > 0.
\]

In that case,

\[
\mathbb{E}L(A) = \lambda^d(A) \mathbb{E}L([0, 1]^d).
\]

Integration of deterministic functions with respect to Lévy bases is described by Rajput and Rosinski \([19]\); in particular for simple functions \(f\) of the form \(f = \sum_{j=1}^{n} x_j 1_{A_j}\) with \(x_j \in \mathbb{R}\) and \(A_j \in B_b(\mathbb{R}^d)\), the integral \(\int f(u) dL(u)\) for \(A \in B(\mathbb{R}^d)\) is defined as \(\sum_{j=1}^{n} x_j L(A_j \cap A)\). A general Borel-measurable function \(f : \mathbb{R}^d \to \mathbb{R}\) is called integrable with respect to \(L\), if there exists a sequence of simple functions \((f_n)_{n \in \mathbb{N}}\) such that \(f_n \to f \ \lambda^d\text{-a.e.} \) and such that \(\int f_n(u) dL(u)\) converges in probability as \(n \to \infty\) for every \(A \in B(\mathbb{R}^d)\), in which case this limit is denoted by \(\int_A f(u) dL(u)\), see \([19\) p.460]. Rajput and Rosinski also characterize integrability of functions. In particular, if \(f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\) and \(\mathbb{E}L([0, 1]^d)^2 < \infty\), or if \(f \in L^2(\mathbb{R}^d), \mathbb{E}L([0, 1]^d)^2 < \infty\) and \(\mathbb{E}L([0, 1]^d) = 0\), then the integral \(\int_{\mathbb{R}^d} f(u) dL(u)\) is well-defined and satisfies \(\mathbb{E} \left(\int_{\mathbb{R}^d} f(u) dL(u)\right)^2 < \infty\). This follows by standard calculations. Moreover, for two such functions \(f, g\) we have

\[
\text{cov} \left( \int_{\mathbb{R}^d} f(u) dL(u), \int_{\mathbb{R}^d} g(u) dL(u) \right) = \sigma^2 \int_{\mathbb{R}^d} f(u) g(u) \lambda^d(du),
\]

where \(\sigma^2 = \mathbb{E}L([0, 1]^d)^2\). For a stationary random field \(X = (X_t)_{t \in \mathbb{R}^d}\) with finite second moment we write \(\gamma_X(t) := \text{cov}(X_t, X_0)\).

### 3. Main results

In this section, we formulate our main results. Our sampling grid will always be \(\mathbb{Z}^d\), but observe that every result can be extended to the sampling set \(\Delta A \mathbb{Z}^d = \{\Delta v : v \in \mathbb{Z}^d\}\), where \(A\) is an orthogonal \(d \times d\)-matrix and \(\Delta > 0\), because the Lévy basis is invariant (in distribution) under orthogonal transformations and any scale transformation can be applied to the Lévy basis instead to the lattice by transporting the scaling parameter to the triplet \((a, \gamma, \nu)\). Our sampling sets \(\Gamma_n\) will then be subsets of \(\mathbb{Z}^d\). The process under consideration is given by \(X_t = \int_{\mathbb{R}^d} f(t-s) dL(s)\),
where \( f : \mathbb{R}^d \to \mathbb{R} \) is integrable with respect to the Lévy basis \( L \). By homogeneity of the Lévy basis, it is easy to see that \((X_t)_{t \in \mathbb{R}^d}\) is a strictly stationary random field, meaning that its finite dimensional distributions are shift invariant.

The proof of Theorem 3.1 and Theorem 3.6 are in Section 5 and the proofs of Theorem 3.8 and Theorem 3.9 in Section 6.

3.1. Central limit theorems for the sample mean. In this and the next section, we give central limit theorems (CLTs) for the sample mean.

**Theorem 3.1.** Let \( L \) be a Lévy basis with \( \mathbb{E}(L([0, 1]^d)^2 < \infty \) and \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), and let

\[
X_t := \int_{\mathbb{R}^d} f(t - u) dL(u), \quad t \in \mathbb{R}^d.
\]

Let \((\Gamma_n)_{n \in \mathbb{N}}\) be a sequence of finite subsets of \( \mathbb{Z}^d \) such that

a) \( \Gamma_n \subset \Gamma_{n+1} \) for every \( n \in \mathbb{N} \),

b) \( |\Gamma_n| \to \infty \) as \( n \to \infty \), and

c) \( a^n_l := \frac{|\{(t,s) \in \Gamma_n \times \Gamma_n : t-s = l\}|}{|\Gamma_n|} \) converges as \( n \to \infty \) to some \( a_l \) for each \( l \in \mathbb{Z}^d \).

Assume that

\[
\sum_{n \in \mathbb{N}} \sup_{t \in \mathbb{Z}^d} a^n_l \int_{\mathbb{R}^d} |f(-u)f(t-u)| \lambda^d(du) < \infty.
\]

Then

\[
\sum_{t \in \mathbb{Z}^d} a_t |\text{cov}(X_t, X_0)| < \infty,
\]

and

\[
\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t \in \Gamma_n} \left( X_t - \mathbb{E}L([0, 1]^d) \int_{\mathbb{R}^d} f(u) \lambda^d(du) \right) \overset{d}{\to} N \left( 0, \sum_{t \in \mathbb{Z}^d} a_t \text{cov}(X_t, X_0) \right).
\]

**Remark 3.2.** From the definition of \( a^n_l \) it is obvious that \( 0 \leq a^n_l \leq 1 \), hence necessarily also \( a_l \in [0, 1] \) for each \( l \in \mathbb{N} \).

A sufficient condition for (3.1) to hold is hence that

\[
\sum_{t \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |f(-u)f(t-u)| \lambda^d(du) < \infty.
\]

Denoting

\[
F(u) := \sum_{t \in \mathbb{Z}^d} |f(u + t)|, \quad u \in \mathbb{R}^d,
\]

it is easy to see that \( F \) is periodic and that

\[
\sum_{t \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |f(-u)f(t-u)| \lambda^d(du) = \int_{\mathbb{R}^d} |f(u)|F(u) \lambda^d(du)
\]
\[ \int_{[0,1]^d} \sum_{t \in \mathbb{Z}^d} |f(u + t)| F(u) \lambda^d(du) \]

\[ = \int_{[0,1]^d} F(u)^2 \lambda^d(du), \]

so that \( F \in L^2([0,1]^d) \) is a sufficient condition for (3.1) to hold. Observe however that there are also other cases when (3.1) holds but \( F \notin L^2([0,1]^d) \). For example, when the sets \( \Gamma_n \) are contained in some hyperplane of \( \mathbb{R}^d \), then many of the \( a^n_l \) will be 0.

**Example 3.3.** Let \( \Gamma_n = (-n, n]^d \cap \mathbb{Z}^d \). Then it is clear that \( a^n_l \) in Theorem 3.1 will converge to 1 as \( n \to \infty \) for each \( l \in \mathbb{Z}^d \). Sequences that satisfy \( \lim_{n \to \infty} a^n_l = 1 \) for each \( l \) are called *Følner*. They play an important role in ergodic theorems in the theory of amenable groups, see [17].

Another example of sequences \( (\Gamma_n) \) satisfying the assumptions of Theorem 3.1 can be obtained as realisations of certain random subsets, in which also the limits \( a^n_l \) may be non-trivial (i.e. different from 0 or 1). This follows from the next lemma, where we use the concept of ergodicity on \( \mathbb{Z}^d \), see [23, Definition 1.1, p. 52].

**Lemma 3.4.** Let \( (Y_t)_{t \in \mathbb{Z}^d} \) be a \( \{0, 1\} \)-valued stationary ergodic random field such that \( \mathbb{E}Y_0 \neq 0 \) (i.e. \( P(Y_0 = 0) < 1 \)). We define

\[ \Gamma_n := \{ t \in [-n, n]^d \cap \mathbb{Z}^d : Y_t = 1 \}. \]

Then \( (\Gamma_n)_{n \in \mathbb{N}} \) satisfies

\[ \frac{|\{(t, s) \in \Gamma_n \times \Gamma_n : t - s = l\}|}{|\Gamma_n|} \to \frac{\mathbb{E}Y_0^2}{\mathbb{E}Y_0^2} \quad \text{a.s. for } n \to \infty. \]

Especially, \( (\Gamma_n)_{n \in \mathbb{N}} \) satisfies almost surely the assumptions of Theorem 3.1.

**Proof.** This is an easy application of the ergodic properties of \( Z_t \). We write

\[ \frac{|\{(t, s) \in \Gamma_n \times \Gamma_n : t - s = l\}|}{|\Gamma_n|} \sum_{t \in [-n, n]^d \cap [-n - l, n - l]^d \cap \mathbb{Z}^d} Y_t Y_{t+l} \]

\[ = \frac{|[-n, n]^d \cap [-n - l, n - l]^d \cap \mathbb{Z}^d|}{|[-n, n]^d \cap \mathbb{Z}^d|} \cdot \frac{|[-n, n]^d \cap [-n - l, n - l]^d \cap \mathbb{Z}^d|}{|[-n, n]^d \cap \mathbb{Z}^d|} \cdot \sum_{t \in [-n, n]^d \cap \mathbb{Z}^d} Y_t \]

Letting \( n \) go to infinity we obtain the assertion from the ergodic theorem for random fields (e.g. Lindenstrauss [17, Theorem 1.3]).

**Example 3.5.** Let \( (Z_t)_{t \in \mathbb{Z}^d} \) be a random field of independent and identically distributed random variables. A typical example of an ergodic random field is the moving average random field \( M_t := \sum_{l \in \mathbb{Z}^d} a_l Z_{t-l} \), where \( (a_l)_{l \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} \) such that the
sum is well-defined, i.e. the sum of the absolute values is almost surely finite. Let \( \varphi : \mathbb{R} \to \{0,1\} \) be a measurable function, then the random field \( \varphi(M_t) \) is an ergodic and stationary random field. Assuming that \( \varphi(M_t) > 0 \) with probability greater than 0, \( \varphi(M_t) \) satisfies the assumption of Lemma 3.4.

3.2. From Non-Random Sampling to Random Sampling. We obtain a CLT on sequences \((\Gamma_n)_{n \in \mathbb{N}}\) similar to the construction as in Lemma 3.4 under the assumption that \((Y_t)_{t \in \mathbb{Z}^d}\) is \( \alpha \)-mixing, which means that

\[
\alpha_Y(k; u, v) := \sup\{\alpha(\sigma(Y_t, t \in A), \sigma(Y_t, t \in B)) : \text{dist}(A, B) \geq k, |A| \leq u, |B| \leq v\} \to 0
\]

for \( k \to \infty \) for every \( u, v \in \mathbb{N} \), where for two \( \sigma \)-fields \( \mathcal{F} \) and \( \mathcal{G} \), \( \alpha(\mathcal{F}, \mathcal{G}) \) is defined by

\[
\sup\{|P(A)P(B) - P(A \cap B)| : A \in \mathcal{F}, B \in \mathcal{G}\}.
\]

A related but much stronger condition is \( h \)-dependence. A stationary random field \( Y = (Y_t)_{t \in \mathbb{Z}^d} \) or \( Y = (Y_t)_{t \in \mathbb{R}^d} \) is \( h \)-dependent (\( h > 0 \)), if for every two finite subsets \( A, B \subset \mathbb{Z}^d \) (\( \subset \mathbb{R}^d \), resp.) the two \( \sigma \)-fields \( \sigma(Y_s : s \in A) \) and \( \sigma(Y_s : s \in B) \) are independent if \( \text{dist}(A, B) > h \).

**Theorem 3.6.** Let \((Y_t)_{t \in \mathbb{Z}^d}\) be a \( \{0,1\} \)-valued \( \alpha \)-mixing random field, which is independent of the Lévy basis \( L \) and satisfies \( P(Y_0 = 1) > 0 \). Moreover, assume there exists a \( \delta > 0 \) such that \( Y \) satisfies

i) for every \( u, v \in \mathbb{N} \) it holds \( \alpha_Y(k; u, v)k^d \to 0 \) for \( k \to \infty \),

ii) for every \( u, v \in \mathbb{N} \) such that \( u + v \leq 4 \) it holds \( \sum_{k=0}^{\infty} k^{d-1} \alpha_Y(k; u, v) < \infty \) and especially \( \sum_{k=0}^{\infty} k^{d-1} \alpha_Y(k; 1, 1)^{\delta/(2+\delta)} < \infty \).

Let

\[
\Gamma_n := \{t \in [-n, n]^d \cap \mathbb{Z}^d : Y_t = 1\},
\]

and \( X = (X_t)_{t \in \mathbb{R}^d} \) be a moving average random field with \( X_t = \int_{\mathbb{R}^d} f(t-u) dL(u) \) with \( \mathbb{E}[L([0,1]^d)]^{2+\delta} < \infty \) and \( f \in L^1(\mathbb{R}^d) \cap L^{2+\delta}(\mathbb{R}^d) \). If

\[
\sum_{t \in \mathbb{Z}^d} \mathbb{E}Y_0Y_t \int_{\mathbb{R}^d} |f(-u)||f(t-u)| \lambda^d(du) < \infty,
\]

then we have that

\[
\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t \in \Gamma_n} (X_t - \beta) \xrightarrow{d} \mathcal{N} \left( 0, \sum_{t \in \mathbb{Z}^d} \mathbb{E}Y_0 \text{cov} \left( Y_t(X_t - \beta), Y_0(X_0 - \beta) \right) \right),
\]

where \( \beta = \mathbb{E}L([0,1]^d) \int_{\mathbb{R}^d} f(u) \lambda^d(du) \). In the special case that \( Y \) is \( h \)-dependent for some finite \( h > 0 \), it is enough to assume that \( \mathbb{E}[L([0,1]^d)]^2 < \infty \) and \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \).

**Example 3.7.** Every \( h \)-dependent random field \( Y \) is \( \alpha \)-mixing with \( \alpha_Y(k; u, v) = 0 \) for \( |k| > h \). Other examples of (non-\( h \)-dependent) random fields \( Y \) with suitable mixing rates can be constructed by [10], Theorem 2, p. 58.
3.3. Non-Random Sampling of the Autocovariance. Our object of interest is the estimator

$$\gamma_n^*(t) := \frac{1}{|\Gamma_n|} \sum_{s \in \Gamma_n} X_s X_{s+t}$$

for some $\Gamma_n \subset \mathbb{Z}^d$ of the autocovariance $\gamma_X(t) = \text{cov}(X_0, X_t)$. We assume that $\Gamma_n$ satisfies the same conditions as in Theorem 3.1. We state a central limit theorem for the sample autocovariance which can be proven similar to Theorem 3.1. Nevertheless, the calculations are a little bit longer.

We assume that

$$E L([0, 1]^d)^4 < \infty, \ E L([0, 1]^d) = 0, \ \sigma^2 := E L([0, 1]^d)^2 > 0$$

and denote

$$\eta := \sigma^{-4} E L([0, 1]^d)^4.$$

Theorem 3.8. Let $m \in \mathbb{N}$ and $\Delta_1, \ldots, \Delta_m \in \mathbb{Z}^d$, $\Gamma_n$ as in Theorem 3.1, and let $(X_t)_{t \in \mathbb{R}^d} = \left(\int_{\mathbb{R}^d} f(t-s) dL(s)\right)_{t \in \mathbb{R}^d}$ be a moving average random field such that it satisfies the assumptions (3.2), $f \in L^2(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$ and

$$\sum_{l \in \mathbb{Z}^d} \sup_{n \in \mathbb{N}} a_n|f(u)f(u + l)f(u + \Delta_p)f(u + l + \Delta_d)|\lambda^d(du) < \infty$$

for every $p, d \in \{1, \ldots, m\}$ and

$$\sum_{l \in \mathbb{Z}^d} \sup_{n \in \mathbb{N}} a_n^2 \gamma_X(l)^2 < \infty.$$

Then

$$\sqrt{|\Gamma_n|} (\gamma_n^*(\Delta_1) - \gamma_X(\Delta_1), \ldots, \gamma_n^*(\Delta_m) - \gamma_X(\Delta_m)) \xrightarrow{d} N(0, V),$$

the multivariate normal distribution with mean 0 and covariance matrix $V = (v_{pq})_{p,q \in \{1, \ldots, m\}}$ given by

$$v_{pq} = \sum_{l \in \mathbb{Z}^d} a_l \left(\eta - 3\right)\sigma^4 \int_{\mathbb{R}^d} f(u)f(u + \Delta_p)f(u + l + \Delta_q)|\lambda^d(du)$$

$$+ \gamma_X(l)\gamma_X(l + \Delta_q - \Delta_p) + \gamma_X(l + \Delta_q)\gamma_X(l - \Delta_p).$$

3.4. Random Sampling of the Autocovariance. Now we present a theorem similar to Theorem 3.6.

Theorem 3.9. Let $(Y_t)_{t \in \mathbb{Z}^d}$ be a $\{0, 1\}$-valued $\alpha-$mixing random field with mixing rates as in Theorem 3.1 ($\delta > 0$), which is independent of the Lévy basis $L$. Let $X = (X_t)_{t \in \mathbb{R}^d}$ be a moving average random field with $X_t = \int_{\mathbb{R}^d} f(t-u) dL(u)$ such that (3.2)
holds with $|\mathbb{E}|L([0, 1]^d)|^{4+\delta} < \infty$ and $f \in L^2(\mathbb{R}^d) \cap L^{4+\delta}(\mathbb{R}^d)$. Let $\Delta_1, \ldots, \Delta_m \in \mathbb{Z}^d$ and for every $p, d \in \{1, \ldots, m\}$ assume that
\[
\sum_{t \in \mathbb{Z}^d} \mathbb{E}Y_0Y_t \int_{\mathbb{R}^d} |f(u) f(u+t)f(u+\Delta_p)f(u+t+\Delta_d)| \lambda^d(du) < \infty
\]
and
\[
\sum_{t \in \mathbb{Z}^d} \mathbb{E}Y_0Y_t \gamma_X(l)^2 < \infty.
\]
Then for $\Gamma_n := \{t \in [-n, n]^d \cap \mathbb{Z}^d : Y_t = 1\}$ we have
\begin{equation}
\sqrt{|\Gamma_n|} \left( \gamma_n^\ast(\Delta_1) - \gamma_X(\Delta_1), \ldots, \gamma_n^\ast(\Delta_m) - \gamma_X(\Delta_m) \right) \overset{d}{\to} N(0, V),
\end{equation}
with covariance matrix $V = (v_{pq})_{p,q \in \{1, \ldots, m\}}$ given by
\[
v_{pq} = \sum_{l \in \mathbb{Z}^d} \mathbb{E}Y_0Y_l \left( \frac{(\eta - 3)\sigma^4}{2\pi} \int_{\mathbb{R}^d} f(u)f(u+\Delta_p)f(u+l)f(u+l+\Delta_q)\lambda^d(du) \right. \\
+ \gamma_X(l)\gamma_X(l+\Delta_p) + \gamma_X(l+\Delta_p)\gamma_X(l+\Delta_q) \bigg).
\]

4. Applications

In this section we present an application of the previously stated theorems. We fix the dimension $d = 3$ and estimate the parameter $\mu > 0$ of the equation
\begin{equation}
(\mu - \Delta)X = dL,
\end{equation}
where $L$ is a Lévy basis with $\mathbb{E}L([0, 1]^3)^2 < \infty$. The mild solution of (4.1) can be written as
\begin{equation}
X(x) = \int_{\mathbb{R}^d} G_\mu(x-z)dL(z),
\end{equation}
where $G_\mu(x) := \frac{\exp(-\sqrt{\mu}||x||)}{||x||}$ for $x \neq 0$, see [8, Definition 3.5] for the notion of the mild solution. That $G_\mu$ is a fundamental solution of $(\mu - \Delta)X = \delta_0$ follows e.g. from [16, Section 2.1, Equation (21)]. We see that $G_\mu \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, so $X$ exists since $\mathbb{E}L([0, 1]^3)^2 < \infty$.
Calculating the mean we obtain
\[
\mathbb{E}X(x) = \mathbb{E}X(0) = \mathbb{E}L([0, 1]^3) \int_{\mathbb{R}^3} \frac{\exp(-\sqrt{\mu}||x||)}{||x||} dx = \frac{4\pi \mathbb{E}L([0, 1]^3)}{\mu},
\]
where the last equality follows by using spherical coordinates. Our moment estimator is then given by
\begin{equation}
\hat{\mu}_n = 4\pi \mathbb{E}L([0, 1]^3) \frac{|\Gamma_n|}{\sum_{k \in \Gamma_n} X(k)}.
\end{equation}
Corollary 4.1. Let \( \hat{\mu}_n \) be defined as in (4.3), \( \mathbb{E}L([0,1]^3) \neq 0 \) and \( \Gamma_n \subset \mathbb{Z}^3 \) satisfying the assumptions of Theorem 3.1. Then \( \hat{\mu}_n \) defines a consistent and asymptotically normal estimator.

Proof. By Theorem 3.1 we conclude that \( \hat{\mu}_n^{-1} \) is asymptotically normal, as

\[
\sum_{t \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |G_\mu(-u)G_\mu(t-u)| \lambda^d(du) = 0
\]

which is finite for \( 0 < \varepsilon < \sqrt{\bar{\mu}} \). Asymptotic normality and consistency of \( \hat{\mu}_n^{-1} \) implies consistency of \( \hat{\mu}_n \), and from both we obtain asymptotical normality of \( \hat{\mu}_n \). □

If in the situation above, additionally \( \Gamma_n \) is a tempered Følner sequence, which means that

\[
\lim_{n \to \infty} \frac{(\{(k + \Gamma_n) \setminus \Gamma_n\} \cup (\Gamma_n \setminus (\Gamma_n + k)))}{|\Gamma_n|} = 0 \text{ for all } k \in \mathbb{Z}^3 \quad \text{and}
\]

\[
\left| \bigcup_{k<n} (-\Gamma_k + \Gamma_n) \right| \leq C|\Gamma_n| \text{ for some constant } C > 0,
\]

then the estimator \( \hat{\mu}_n \) is strongly consistent by [17, Theorem 1.2, p. 260]. A simple example of a tempered Følner sequence is \( (-n, n) \cap \mathbb{Z}^d \).

5. Proof of Theorems 3.1 and 3.6

Since

\[
X_t = \int_{\mathbb{R}^d} f(t-u) dL'(u) + \mathbb{E}(L([0,1]^d)) \int_{\mathbb{R}^d} f(u) \lambda^d(du),
\]

where the mean zero Lévy basis \( L' \) is defined by

\[
L'(A) := L(A) - \mathbb{E}L([0,1]^d) \lambda^d(A), \quad A \in \mathcal{B}_b(\mathbb{R}^d),
\]

and since

\[
\text{cov} (Y_t X_t, Y_0 X_0) = \text{cov} (Y_t (X_t - \mathbb{E}X_t), Y_0 (X_0 - \mathbb{E}X_0))
\]

in Theorem 3.6 by independence of \( X \) and \( Y \), we may and do assume for rest of this section that \( \mathbb{E}L([0,1]^d) = 0 \).

Proof of Theorem 3.1. For every \( h \in \mathbb{N} \) we define a new random field \( (X_t^{(h)})_{t \in \Delta A \mathbb{Z}^d} \) by

\[
X_t^{(h)} := \int_{\mathbb{R}^d} f(t-u) 1_{[-h,h]^d}(t-u) dL(u).
\]
It is obvious that \( (X_t^{(h)})_{t \in \Delta A \mathbb{Z}^d} \) is \( 2 \sqrt{dh} + 1 \)-dependent.

We want to use [15, Theorem 2, p. 135], which states the following: If we have a sequence \( \{X_{n_z}, z \in V_n \subset \mathbb{Z}^d\}, n \in \mathbb{N}, \) of \( m_n \)-dependent random fields \( (m_n \geq 1) \) with \( |V_n| \to \infty, \) \( \mathbb{E}X_{n_z} = 0 \) for all \( z \in V_n, \mathbb{E}\left(\sum_{z \in V_n} X_{n_z}\right)^2 = 1 \) and satisfying the conditions

\[
\sup_{n \in \mathbb{N}} \sum_{z \in V_n} \mathbb{E}X_{n_z}^2 < \infty \quad \text{and} \quad m_n^{2d} \sum_{z \in V_n} \mathbb{E}X_{n_z}^2 1_{|X_{n_z}| \geq m_n^{-2d}} \to 0 \quad \text{as} \quad n \to \infty
\]

for every \( \varepsilon > 0, \) then \( \sum_{z \in V_n} X_{n_z} \xrightarrow{d} \mathcal{N}(0, 1) \) as \( n \to \infty. \) In our case \( m_n \) is constant, so the conditions are simpler. We set \( U_t^{(n, h)} := \frac{1}{\sqrt{|\Gamma_n|}} X_t^{(h)}. \) We calculate that

\[
\mathbb{E}\left(\sum_{t \in \Gamma_n} U_t^{(n, h)}\right)^2 = \frac{1}{|\Gamma_n|} \sum_{t, s \in \Gamma_n} \mathbb{E}X_t^{(h)} X_s^{(h)} = \frac{1}{|\Gamma_n|} \sum_{t, s \in \Gamma_n} \gamma_{X^{(h)}}(t - s) = \sum_{l \in \mathbb{Z}^d} a_l^n \gamma_l^{X^{(h)}}(l).
\]

Letting \( n \) go to infinity, we obtain by Lebesgue’s dominated convergence theorem

\[
\mathbb{E}\left(\sum_{t \in \Gamma_n} U_t^{(n, h)}\right)^2 \to \sum_{t \in \mathbb{Z}^d} a_l \gamma_l^{X^{(h)}}(t).
\]

Furthermore, we immediately see that

\[
\sum_{t \in \Gamma_n} \mathbb{E}(U_t^{(n, h)})^2 = \frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} \mathbb{E}(X_t^{(h)})^2 = \gamma_{X^{(h)}}(0) < \infty
\]

and

\[
\sum_{t \in \Gamma_n} \mathbb{E}\left(\left(U_t^{(n, h)}\right)^2 1_{|U_t^{(n, h)}| \geq \varepsilon}\right) = \frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} \mathbb{E}(X_t^{(h)})^2 1_{|X_t^{(h)}| \geq \varepsilon \sqrt{|\Gamma_n|}}
\]

\[
= \mathbb{E}(X_0^{(h)})^2 1_{|X_0^{(h)}| \geq \varepsilon \sqrt{|\Gamma_n|}} \to 0 \quad \text{for} \quad n \to \infty.
\]

Hence all conditions of [15, Theorem 2, p. 135] as stated above are satisfied and we conclude that

\[
\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t \in \Gamma_n} X_t^{(h)} \xrightarrow{d} Y^{(h)}
\]

for \( n \to \infty \) with \( Y^{(h)} \sim \mathcal{N}(0, \sum_{t \in \mathbb{Z}^d} a_t \gamma_l^{X^{(h)}}(t)). \)

Observe that \( \lim_{h \to \infty} \gamma_{X^{(h)}}(t) = \gamma_X(t) \) for all \( t \in \mathbb{Z}^d \) by (2.1) and dominated convergence and \( |\gamma_{X^{(h)}}(t)| \leq \sigma^2 \int_{\mathbb{R}^d} |f(-u)||f(t-u)| \lambda^d(du), \) hence we conclude by dominated convergence that

\[
\lim_{h \to \infty} \sum_{t \in \mathbb{Z}^d} a_t \gamma_{X^{(h)}}(t) = \sum_{t \in \mathbb{Z}^d} a_t \gamma_X(t)
\]
and hence

\[ Y^{(h)} \xrightarrow{d} Y \sim N(0, \sum_{t \in \mathbb{Z}^d} a_t \gamma_X(t)) \text{ for } h \to \infty. \]

As in (5.1), we obtain

\[
\mathbb{E} \left( \frac{1}{\sqrt{|\Gamma_n|}} \sum_{t \in \Gamma_n} (X_t - X_t^{(h)}) \right)^2 = \sum_{l \in \mathbb{Z}^d} a_l^n \gamma_{X-X(n)}(l)
\]

\[
= \sum_{l \in \mathbb{Z}^d} a_l^n \int_{\mathbb{R}^d} f(l - u) 1_{\mathbb{R}^d \setminus [-h, h)^d}(t - u) f(-u) 1_{\mathbb{R}^d \setminus [-h, h)^d}(-u) \lambda^d(du),
\]

hence

\[
\lim_{h \to \infty} \lim_{n \to \infty} \mathbb{E} \left( \frac{1}{\sqrt{|\Gamma_n|}} \left( \sum_{t \in \Gamma_n} X_t - X_t^{(h)} \right) \right)^2 = 0
\]

from Lebesgue’s dominated convergence theorem for series. An application of Chebyshev’s inequality gives for \( \varepsilon > 0 \),

\[
\lim_{h \to \infty} \lim_{n \to \infty} P \left( \frac{1}{\sqrt{|\Gamma_n|}} \left| \sum_{t \in \Gamma_n} X_t - X_t^{(h)} \right| > \varepsilon \right) = 0.
\]

The claim then follows by a variant of Slutsky’s theorem, e.g. [6, Proposition 6.3.9, pp. 207-208]. \( \square \)

**Proof of Theorem 3.6.** The proof is very similar to the proof of Theorem 3.1. Let us start by approximating \( X_t \) by \( X_t^{(h)} \) as above. Observe that

\[
\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t \in \Gamma_n} X_t^{(h)} = \frac{(2n)^{d/2}}{\sqrt{|\Gamma_n|}} \frac{1}{(2n)^{d/2}} \sum_{t \in (-n,n]^d \cap \mathbb{Z}^d} X_t^{(h)} Y_t.
\]

We know that \( \frac{(2n)^{d/2}}{\sqrt{|\Gamma_n|}} \to \left( \sqrt{\mathbb{E} Y_0} \right)^{-1} \), which follows from the ergodic theorem. Furthermore, as \( (X_t^{(h)}) \) is \((2\sqrt{d}h + 1)-dependent and Y is \( \alpha\)-mixing, we obtain that \( (X_t^{(h)} Y_t)_{t \in \mathbb{Z}} \) is \( \alpha\)-mixing with the same rate as \( Y \). From this and conditions i) and ii) of Theorem 3.6 we conclude by [10, Theorem 3, p. 48] that

\[
\frac{1}{(2n)^{d/2}} \sum_{t \in \Gamma_n} X_t^{(h)} \xrightarrow{d} N \left( 0, \sum_{t \in \mathbb{Z}^d} \frac{1}{\mathbb{E} Y_0} \text{cov} (X_t^{(h)} Y_t, X_0^{(h)} Y_0) \right) \text{ for } n \to \infty.
\]

Now by the same arguments as above we conclude that this theorem holds true when \( Y \) is \( \alpha\)-mixing. When \( Y \) is even \( h'\)-dependent for some \( h' \), then \( (X_t^{(h')} Y_t)_{t \in \mathbb{Z}^d} \) is \( \max\{h', 2\sqrt{d}h + 1\}\)-dependent and we can use [15, Theorem 2, p. 135] instead of [10, Theorem 3, p. 48] and hence need weaker moment conditions. \( \square \)
6. Proof of Theorems 3.8 and 3.9

Proposition 6.1. Let $f_1, \ldots, f_4 \in L^4(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. It holds true that

\[
\mathbb{E} \prod_{i=1}^{4} f_i(t) dL(t) = (\eta - 3)\sigma^4 \int_{\mathbb{R}^d} f_1(u) f_2(u) f_3(u) f_4(u) \lambda^d(du) \\
+ \sigma^4 \int_{\mathbb{R}^d} \prod_{i=1,2} f_i(u) \lambda^d(du) \int_{\mathbb{R}^d} \prod_{i=3,4} f_i(u) \lambda^d(du) \\
+ \sigma^4 \int_{\mathbb{R}^d} \prod_{i=1,3} f_i(u) \lambda^d(du) \int_{\mathbb{R}^d} \prod_{i=2,4} f_i(u) \lambda^d(du) \\
+ \sigma^4 \int_{\mathbb{R}^d} \prod_{i=1,4} f_i(u) \lambda^d(du) \int_{\mathbb{R}^d} \prod_{i=2,3} f_i(u) \lambda^d(du).
\]

Proof. Follows directly from the proof of [3, Lemma 4.1].

Proposition 6.2. Under the assumptions of Theorem 3.8, for $\Delta p, \Delta q \in \mathbb{Z}^d$, we have

\[
|\Gamma_n| \text{cov} (\gamma_n^*(\Delta p), \gamma_n^*(\Delta q)) \to \sum_{l \in \mathbb{Z}^d} a_l T_l \quad \text{for } n \to \infty,
\]

where

\[
T_l := (\eta - 3)\sigma^4 \int_{\mathbb{R}^d} f(u) f(u + l) f(u + \Delta p) f(u + l + \Delta q) \lambda^d(du) \\
+ \gamma_X(l) \gamma_X(l + \Delta q - \Delta p) + \gamma_X(l + \Delta q) \gamma_X(l - \Delta p).
\]

Proof. A direct calculation gives us

\[
|\Gamma_n| \text{cov} (\gamma_n^*(\Delta p), \gamma_n^*(\Delta q)) = \frac{1}{|\Gamma_n|} \sum_{s,t \in \Gamma_n} \text{cov} (X_s X_{t+\Delta p}, X_s X_{s+\Delta q}) \\
= \frac{1}{|\Gamma_n|} \sum_{s,t \in \Gamma_n} \mathbb{E} (X_s X_{t+\Delta p} X_{s+\Delta q}) - \gamma_X(\Delta p) \gamma_X(\Delta q) \\
= \frac{1}{|\Gamma_n|} \sum_{s,t \in \Gamma_n} \mathbb{E} (X_0 X_{s-t} X_{s+t+\Delta p} X_{s-\Delta q}) - \gamma_X(\Delta p) \gamma_X(\Delta q) \\
= \frac{1}{|\Gamma_n|} \sum_{s,t \in \Gamma_n} T_{s-t},
\]

which follows from Proposition 6.1 and we get that

\[
\frac{1}{|\Gamma_n|} \sum_{s,t \in \Gamma_n} T_{s-t} = \sum_{l \in \mathbb{Z}^d} a_l^T l.
\]
By our assumptions and Lebesgue’s dominated convergence theorem for series we conclude that
\[
|\Gamma_n| \text{cov} (\gamma_n^*(\Delta_p), \gamma_n^*(\Delta_q)) \to \sum_{t \in \mathbb{Z}^d} a_t T_t \quad \text{for } n \to \infty.
\]

\[
\square
\]

\textit{Proof of Theorem 3.8.} Let \( h \in \mathbb{N} \) and \( X_t^{(h)} \) be given by
\[
X_t^{(h)} := \int_{\mathbb{R}^d} f^{(h)}(t-u) \, dL(u),
\]
where \( f^{(h)}(u) := f(u) \mathbf{1}_{[-h,h]^d}(u). \) We define
\[
U_t^{(h)} := (X_t^{(h)} X_{t+\Delta_1}^{(h)}, \ldots, X_t^{(h)} X_{t+\Delta_m}^{(h)}).
\]
Now observe that \((U_t^{(h)})_{t \in \mathbb{Z}^d}\) is \((2\sqrt{dh} + 2 \sup_{i=1,\ldots,m} \|\Delta_i\| + 1)-dependent. We want to show that
\[
(6.1) \quad \frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} (U_t^{(h)} - (\gamma_{X_t^{(h)}}(\Delta_1), \ldots, \gamma_{X_t^{(h)}}(\Delta_m))) \rightarrow^{d} Y^{(h)} = N(0, V^{(h)})
\]
as \( n \to \infty, \) where \( V^{(h)} = (v_{pq}^{(h)})_{p,q \in \{1,\ldots,n\}} \) is defined by \((3.5)\) with \( f \) replaced by \( f^{(h)}.\)
Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m \setminus \{0\} \). Define \( K_t^{(h)} := \alpha(U_t^{(h)} - (\gamma_{X_t^{(h)}}(\Delta_1), \ldots, \gamma_{X_t^{(h)}}(\Delta_m)))^T, \)
which is also \((2\sqrt{dh} + 2 \sup_{i=1,\ldots,m} \|\Delta_i\| + 1)-dependent. Then we see that \( \mathbb{E} K_t^{(h)} = 0 \) and

\[
\frac{1}{|\Gamma_n|} \mathbb{E} \left( \sum_{t \in \Gamma_n} K_t^{(h)} \right)^2 = \frac{1}{|\Gamma_n|} \sum_{t,s \in \Gamma_n} \mathbb{E} K_t^{(h)} K_s^{(h)}
\]

\[
= \frac{1}{|\Gamma_n|} \sum_{t,s \in \Gamma_n} \mathbb{E}(\alpha(U_t^{(h)} - (\gamma_{X_t^{(h)}}(\Delta_1), \ldots, \gamma_{X_t^{(h)}}(\Delta_m)))^T \alpha((U_s^{(h)} - (\gamma_{X_s^{(h)}}(\Delta_1), \ldots, \gamma_{X_s^{(h)}}(\Delta_m)))^T)
\]

\[
= \frac{1}{|\Gamma_n|} \sum_{t,s \in \Gamma_n} \mathbb{E} \sum_{i,j=1}^m \alpha_i \alpha_j (X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X_t^{(h)}}(\Delta_i))(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_{X_s^{(h)}}(\Delta_j))
\]

\[
= \frac{1}{|\Gamma_n|} \sum_{t,s \in \Gamma_n} \sum_{i,j=1}^m \alpha_i \alpha_j \text{cov}(X_t^{(h)} X_{t+\Delta_i}^{(h)}, X_s^{(h)} X_{s+\Delta_j}^{(h)}).
\]

By Proposition 6.2 we conclude that

\[
\frac{1}{|\Gamma_n|} \mathbb{E} \left( \sum_{t \in \Gamma_n} K_t^{(h)} \right)^2 \rightarrow \sum_{i,j=1}^m \alpha_i \alpha_j v_{ij}^{(h)}
\]

for \( n \to \infty. \) Furthermore, for every \( \varepsilon > 0 \) we have

\[
\lim_{n \to \infty} \frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} \mathbb{E}(K_t^{(h)})^2 1_{|K_t^{(h)}| \geq |\Gamma_n|\varepsilon}
\]
\[
= \lim_{n \to \infty} \mathbb{E}(K_0^{(h)})^2 \mathbf{1}_{|K_0^{(h)}| \geq |\Gamma_n| \varepsilon} = 0
\]
and
\[
\frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} \mathbb{E}(K_t^{(h)})^2 = \mathbb{E}(K_0^{(h)})^2 < \infty.
\]

By [15, Theorem 2, p. 135] we conclude that
\[
\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t \in \Gamma_n} K_t^{(h)} \xrightarrow{d} N(0, \sum_{i,j=1}^{m} \alpha_i \alpha_j v_{ij}^{(h)}), n \to \infty.
\]

By the Crámer-Wold Theorem we see that (6.1) holds true. Next we have to show that \(V^{(h)} \to V\) for \(h \to \infty\). But this follows from dominated convergence, since \(f^{(h)} \to f\) in \(L^4(\mathbb{R}^d)\) and in \(L^2(\mathbb{R}^d)\) as \(h \to \infty\), since \(|f^{(h)}| \leq |g|\) and by (2.1). Hence we get
\[
Y^{(h)} \xrightarrow{d} Y \sim N(0, V) \text{ as } h \to \infty.
\]

The claim will now follow by [6, Proposition 6.3.9, pp. 207-208] if we can show that for any \(\varepsilon > 0\),
\[
(6.2)
\lim_{h \to \infty} \lim_{n \to \infty} P \left( \sqrt{|\Gamma_n|} \left| \gamma_n^*(\Delta_i) - \gamma_X(\Delta_i) - \frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} X_t^{(h)} X_{t+\Delta_i}^{(h)} + \gamma_X^{(h)}(\Delta_i) \right| > \varepsilon \right) = 0.
\]

This follows by showing that
\[
(6.3)
\lim_{h \to \infty} \lim_{n \to \infty} \mathbb{E}|\Gamma_n| \left| \gamma_n^*(\Delta_i) - \gamma_X(\Delta_i) - \frac{1}{|\Gamma_n|} \sum_{t \in \Gamma_n} X_t^{(h)} X_{t+\Delta_i}^{(h)} + \gamma_X^{(h)}(\Delta_i) \right|^2 = 0.
\]

as an application of the Dominated convergence Theorem similar to the end of the proof of [7, Theorem 3.5, p. 1302] and therefore we obtain our desired result. \(\square\)

**Proof of Theorem 3.9.** We observe that
\[
\sum_{t \in \Gamma_n} (X_t X_{t+\Delta_i} - \gamma_X(\Delta_i)) = \sum_{t \in [-n,n) \cap \mathbb{Z}^d} Y_t (X_t X_{t+\Delta_i} - \gamma_X(\Delta_i))
\]

and
\[
\text{cov} \left( Y_t (X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_X^{(h)}(\Delta_i)), Y_s (X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_X^{(h)}(\Delta_j)) \right)
= \mathbb{E}Y_t (X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_X^{(h)}(\Delta_i)) Y_s (X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_X^{(h)}(\Delta_j))
- \mathbb{E}Y_t (X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_X^{(h)}(\Delta_i)) \mathbb{E}Y_s (X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_X^{(h)}(\Delta_j))
= \mathbb{E}Y_t \mathbb{E}(X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_X^{(h)}(\Delta_i)) \mathbb{E}(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_X^{(h)}(\Delta_j)).
\]

Repeating the same steps as in the proof of Theorem 3.6 gives the claim. \(\square\)
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References

[1] O.E. Barndorff-Nielsen, F. Benth and A. Veraart, Modelling electricity futures by ambit fields, Adv. Appl. Prob., 46(3), 719-745 (2014).
[2] D. Berger, Lévy driven CARMA generalized processes and stochastic partial differential equations, arXiv: 1904.02928 (2019).
[3] D. Brandes, I.V. Curator, On the sample autocovariance of a Lévy driven moving average process when sampled at a renewal sequence, arXiv:1804.02254 (2018).
[4] E. Bolthausen, On the central limit theorem for stationary mixing random fields, Ann. of Prob., Vol. 10, No. 4, 1047-1050 (1982).
[5] P.J. Brockwell and Y. Matsuda, Continuous auto-regressive moving average random fields on $\mathbb{R}^d$, J. R. Stat. Soc. Ser. B. Stat. Methodol. 79, no. 3, 833-857 (2017).
[6] P.J. Brockwell and R.A. David, Time Series: Theory and Methods, 2nd edition, Springer (1990).
[7] S. Cohen and A. Lindner, A central limit theorem for the sample autocorrelations of a Lévy driven continuous time moving average process, J. Stat. Plan. Inference 143, 1295-1306 (2013).
[8] R.C. Dalang and T. Humeau, Random field solutions to linear SPDEs driven by symmetric pure jump Lévy space-time white noise, arXiv:1809.09999 (2018).
[9] J. Dedecker, A central limit theorem for stationary random fields, Prob. Th. Rel. Fields, 110 (3), 397-426 (1998).
[10] P. Doukhan, Mixing: Properties and Examples, Lecture Notes in Statistics, Springer (1994).
[11] M. Drapatz, Limit theorems for the sample mean and sample autocovariances of continuous time moving averages driven by heavy-tailed Lévy noise, ALEA, Lat. Am. J. Probab. Math. Stat. 14, 403-426 (2017).
[12] J. Fageot and T. Humeau, Unified view on Lévy white noise: general integrability conditions and applications to linear SPDE, arXiv:1708.02500 (2017).
[13] J. Fageot and M. Unser, Scaling Limits of Solutions of Linear Stochastic Differential Equations Driven by Lévy White Noises, J. Theor. Probab. https://doi.org/10.1007/s10959-018-0809-1 (2018).
[14] L. Grafakos, Classical Fourier Analysis, 2nd edition, Springer (2008).
[15] L. Heinrich, Asymptotic behaviour of an empirical nearest-neighbour distance function for stationary Poisson cluster processes, Math. Nachr. 136 (1), 131-148 (1988).
[16] R.M. Höfer and J.J.L. Velázquez, The Method of Reflections, Homogenization and Screening for Poisson and Stokes Equations in Perforated Domains, Arch Rational Mech Anal 227, 1165–1221, (2018).
[17] E. Lindenstrauss, Pointwise Theorems for Amenable Groups, Invent. Math. 146, 259-295 (2001).
[18] V.S. Pham, Lévy driven causal CARMA random fields, arXiv:1805.08807 (2018).
[19] B.S. Rajput and J. Rosinski, Spectral Representations of Infinitely Divisible Processes, J. Probab. Th. Rel. Fields 82, 451-487 (1989).
[20] M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier analysis, self-adjointness*, Academic Press, New York (1975).

[21] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge studies in advanced mathematics (2007).

[22] F. Spangenberg, *Limit theorems for the sample autocovariance of a continuous time moving average process with long memory*, arXiv:1502.04851 (2015).

[23] A. Tempelman, *Ergodic Theorems for Group Actions: Informational and Thermodynamical Aspects*, Mathematics and Its Applications, Springer-Science+Business Media (1992).

[24] J.B. Walsh, *An introduction to stochastic partial differential equations*, École d’Été de Probabilités de Saint Flour XIV-1984, pages 265-439. Springer (1986).

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