Estimating the number of communities by Stepwise Goodness-of-fit

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Abstract

Given a symmetric network with \( n \) nodes, how to estimate the number of communities \( K \) is a fundamental problem. We propose Stepwise Goodness-of-Fit (StGoF) as a new approach to estimating \( K \). For \( m = 1, 2, \ldots \), StGoF alternately uses a community detection step (pretending \( m \) is the correct number of communities) and a goodness-of-fit step. We use SCORE \cite{SCORE} for community detection, and propose a new goodness-of-fit measure. Denote the goodness-of-fit statistic in step \( m \) by \( \psi_n^{(m)} \). We show that as \( n \to \infty \), \( \psi_n^{(m)} \to N(0,1) \) when \( m = K \) and \( \psi_n^{(m)} \to \infty \) in probability when \( m < K \). Therefore, with a proper threshold, StGoF terminates at \( m = K \) as desired.

We consider a broad setting where we allow severe degree heterogeneity, a wide range of sparsity, and especially weak signals. In particular, we propose a measure for signal-to-noise ratio (SNR) and show that there is a phase transition: when SNR \( \to 0 \) as \( n \to \infty \), consistent estimates for \( K \) do not exist, and when SNR \( \to \infty \), StGoF is consistent, uniformly for a broad class of settings. In this sense, StGoF achieves the optimal phase transition. Stepwise testing algorithms of similar kind (e.g., \cite{1,20}) are known to face analytical challenges. We overcome the challenges by using a different design in the stepwise algorithm and by deriving sharp results in the under-fitting case (\( m < K \)) and the null case (\( m = K \)). The key to our analysis is to show that SCORE has the Non-Splitting Property (NSP). The NSP is non-obvious, so additional to rigorous proofs, we also provide an intuitive explanation.

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1 Introduction

In network analysis, how to estimate the number of communities \( K \) is a fundamental problem. In many recent approaches, \( K \) is assumed as known a priori (see for example \cite{2,7,19,30,39,37} on community detection, \cite{13,38} on mixed-membership estimation, and \cite{20} on dynamic community detection). Unfortunately, \( K \) is rarely known in applications, so the performance of these approaches hinges on how well we can estimate \( K \).

The primary interest of this paper is how to estimate \( K \). Given a symmetric and connected social network with \( n \) nodes and \( K \) communities, let \( A \) be the adjacency matrix:

\[
A_{ij} = \begin{cases} 
1, & \text{if node } i \text{ and node } j \text{ have an edge,} \\
0, & \text{otherwise,}
\end{cases} \quad 1 \leq i \neq j \leq n. \quad (1.1)
\]
As a convention, self-edges are not allowed so all the diagonal entries of $A$ are 0. Denote the $K$ perceivable communities by $\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_K$. We model the network by the widely-used degree-corrected block model (DCBM) \cite{19}. For each $1 \leq i \leq n$, we encode the community label of node $i$ by a vector $\pi_i \in \mathbb{R}^K$ where
\begin{equation}
    i \in \mathcal{N}_k \iff \pi_i(k) = 1 \text{ and } \pi_i(m) = 0 \text{ for } m \neq k. \tag{1.2}
\end{equation}
Moreover, for a $K \times K$ symmetric nonnegative matrix $P$ which models the community structure and positive parameters $\theta_1, \theta_2, \ldots, \theta_n$ which model the degree heterogeneity, we assume the upper triangular entries of $A$ are independent Bernoulli variables satisfying
\begin{equation}
    \mathbb{P}(A_{ij} = 1) = \theta_i \theta_j \cdot \pi_i' P \pi_j \equiv \Omega_{ij}, \quad 1 \leq i < j \leq n, \tag{1.3}
\end{equation}
where $\Omega$ denotes the matrix $\Theta \Pi P \Theta'$, with $\Theta$ being the $n \times n$ diagonal matrix $\text{diag}(\theta_1, \ldots, \theta_n)$ and $\Pi$ being the $n \times K$ matrix $[\pi_1, \pi_2, \ldots, \pi_n]'$. For identifiability, we assume
\begin{equation}
    \text{all diagonal entries of } P \text{ are } 1. \tag{1.4}
\end{equation}
Write for short $\text{diag}(\Omega) = \text{diag}(\Omega_{11}, \Omega_{22}, \ldots, \Omega_{nn})$, and let $W$ be the matrix where for $1 \leq i, j \leq n$, $W_{ij} = A_{ij} - \Omega_{ij}$ if $i \neq j$ and $W_{ij} = 0$ otherwise. In matrix form, we have
\begin{equation}
    A = \Omega - \text{diag}(\Omega) + W, \quad \text{where we recall } \Omega = \Theta \Pi P \Theta'. \tag{1.5}
\end{equation}
In the special case of $\theta_1 = \theta_2 = \ldots = \theta_n$, DCBM reduces to the stochastic block model (SBM) \cite{8}. In this paper, we focus on DCBM, but the idea is extendable to the degree-corrected mixed-membership (DCMM) model \cite{38,15}, where mixed membership is allowed; see Remark 3 below.

Real world networks have a few interesting features that we frequently observe.

- **Severe degree heterogeneity.** The distribution of the node degrees has a power-law tail, implying severe degree heterogeneity. Therefore, the sparsity level for individual nodes (measured by the number of edges) may vary significantly from one to another.

- **Network sparsity.** The overall network sparsity may range significantly from one network to another.

- **Weak signal.** The community structure is masked by strong noise, and the signal-to-noise ratio (SNR) is usually relatively small.

For analysis, we let $n$ be the driving asymptotic parameter, and allow $(\Theta, P, \Pi)$ to depend on $n$, so that DCBM is broad enough to cover all interesting range of these metrics. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)'$, $\theta_{\text{max}} = \max\{\theta_1, \ldots, \theta_n\}$, and $\theta_{\text{min}} = \min\{\theta_1, \theta_2, \ldots, \theta_n\}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_K$ be the $K$ nonzero eigenvalues of $\Omega$, arranged in the descending order of magnitudes. The following were suggested by existing literature (e.g., \cite{17,14}). First, a reasonable metric for network sparsity is $\|\theta\|$ and a reasonable metric for the degree heterogeneity is $\theta_{\text{max}} / \theta_{\text{min}}$. Second, the range of interest for $\|\theta\|$ is
\begin{equation}
    C \sqrt{\log(n)} \leq \|\theta\| \leq C \sqrt{n}, \tag{1.6}
\end{equation}
where $C > 0$ is a generic constant. Third, the signal strength and noise level are captured by $|\lambda_K|$ and $\|W\|$, respectively. When $\theta_{\text{max}} \leq C \theta_{\text{min}}$ and some mild conditions hold (e.g., $\|P\| \leq C$),
\begin{equation}
    \lambda_1 \asymp \|\theta\|^2, \quad \text{and} \quad \|W\| = \text{a multi-log}(n) \text{ term } \cdot \sqrt{\lambda_1} \text{ with high probability}, \tag{1.7}
\end{equation}
(examples for multi-log($n$)-terms are $\sqrt{\log(n)}, \log \log(n)$, etc.), so a reasonable metric for the signal to noise ratio (SNR) is $|\lambda_K| / \sqrt{\lambda_1}$. When $\theta_{\text{max}} / \theta_{\text{min}} \to \infty$, we need an adjusted SNR; see Section 2. We consider two extreme cases.
• **Strong signal case.** $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_K|$ are at the same magnitude, and so $\text{SNR} \approx \sqrt{\lambda_1}$.

• **Weak signal case.** $|\lambda_K|/\sqrt{\lambda_1}$ is much smaller than $\sqrt{\lambda_1}$ and grows to $\infty$ slowly as $n \to \infty$ (in our range of interest, $\lambda_1$ may grow to $\infty$ rapidly as $n \to \infty$, so for example, we may have $\text{SNR} = \log \log(n)$ and $\lambda_1 = \sqrt{n}$).

Section 2.3 suggests that when $\text{SNR} = o(1)$, consistent estimate for $K$ does not exist, so the weak signal case is a very challenging case. Motivated by the above observations, it is desirable to find a consistent estimate for $K$ that satisfies the following requirements.

• (R1). Allow severe degree heterogeneity (i.e., $\theta_{\text{max}}/\theta_{\text{min}}$ may tend to $\infty$).

• (R2). Optimally adaptive to network sparsity, where $\|\theta\|$ may be as small as $O(\sqrt{\log(n)})$ or be as large as $O(\sqrt{n})$.

• (R3). **Attain the information lower bound.** Consistent for both the strong signal case where $\text{SNR}$ is large and the weak signal case where $\text{SNR}$ may be as small as $\log \log(n)$ (say).

**Example 1.** Recently, a frequently considered DCBM is to assume $P = P_0$ and $\theta_i \sim \sqrt{\alpha_n}$ for all $1 \leq i \leq n$, where $\alpha_n > 0$ is a scaling parameter and $P_0$ is a fixed matrix. It is seen that $\lambda_1, \ldots, \lambda_K$ are at the same order, so the model only considers the strong signal case.

**Example 2 (A special DCBM).** Let $e_1, \ldots, e_K$ be the standard basis vectors of $\mathbb{R}^K$. Fixing a positive vector $\theta \in \mathbb{R}^n$ and a scalar $b_n \in (0, 1)$, we consider a DCBM where $K$ is fixed, each community has $n/K$ nodes, and $P = (1-b_n)I_K + b_n1_K1_K'$. In this model, $(1-b_n)$ measures the “dis-similarity” of different communities and is small in the more challenging case when different communities are similar. By basic algebra, $\lambda_1 \approx \|\theta\|^2$, $\lambda_2 = \ldots = \lambda_K \approx \|\theta\|^2(1-b_n)$, and $\text{SNR} \approx \|\theta\|/(1-b_n)$. In the very sparse case, $\|\theta\| = O(\sqrt{\log(n)})$. In the dense case, $\|\theta\| = O(\sqrt{n})$. When $b_n \leq c_0$ for a constant $c_0 < 1$, $|\lambda_K| \geq C(1+\|\theta\|)$ and $\text{SNR} \approx \|\theta\|$; we are in the strong signal case if $\|\theta\| \geq n^a$ for a constant $a > 0$. When $b_n = 1 + o(1)$ and $\|\theta\|(1-b_n) = \log \log(n)$ (say), $\text{SNR} \approx \log \log(n)$ and we are in the weak signal case.

### 1.1 Literature review and our contributions

In recent years, many interesting approaches for estimating $K$ have been proposed, which can be roughly divided into the spectral approaches, the cross validation approaches, the penalization approaches, and the likelihood ratio approaches.

Among the spectral approaches, Le and Levina [21] proposed to estimate $K$ using the eigenvalues of the non-backtracking matrix or Bethe Hessian matrix. The approach uses ideas from mathematical graph theory, and is quite interesting for it is different from most statistical approaches. Unfortunately, the approach requires relatively strong conditions for consistency. For example, their Theorem 4.1 only considers the idealized SBM model in the very sparse case, where $\theta_1 = \theta_2 = \ldots = \theta_n = 1/\sqrt{n}$ and $P = P_0$ for a fixed matrix $P_0$. Liu et al. [28] proposed to estimate $K$ by using the classical scree plot approach with careful theoretical justification, but the approach is known to be unsatisfactory in the presence of severe degree heterogeneity, for it is hard to derive a sharp bound for the spectral norm of the noise matrix $W$ (e.g., [14]). Therefore, their approach requires the condition of $\theta_{\text{max}} \leq C\theta_{\text{min}}$. The paper also imposed the condition of $\|\theta\| = O(\sqrt{n})$ so it did not address the settings of sparse networks (see [16] for the interesting range of $\|\theta\|$). Among the cross-validation approaches, we have [11, 25], and among the penalization approaches, we have [33, 3] [20], where $K$ is estimated by the integer that optimizes some objective functions. For example, Salda et al. [33] used a BIC-type objective function and [3] [20] used an objective function of the Bayesian model selection flavor. However, these methods did not provide explicit theoretical guarantee on consistency (though a partial
result was established in [25], which stated that under SBM, the proposed estimator \( \hat{K} \) is no greater than \( K \) with high probability.

For likelihood ratio approaches, Wang and Bickel [36] proposed to estimate \( K \) by solving a BIC type optimization problem, where the objective function is the sum of the log-likelihood and the model complexity. The major challenge here is that the likelihood is the sum of exponentially many terms and is hard to compute. In a remarkable paper, Ma et al. [29] extended the idea of [36] by proposing a new approach that is computationally more feasible.

On a high level, we can recast their methods as a stepwise testing or sequential testing algorithm. Consider a stepwise testing scheme where for \( m = 1, 2, \ldots \), we construct a test statistic \( \ell_n^{(m)} \) (e.g. log-likelihood) assuming \( m \) is the correct number of communities. We estimate \( K \) as the smallest \( m \) such that the pairwise log-likelihood ratio \( (\ell_n^{(m+1)} - \ell_n^{(m)}) \) falls below a threshold. As mentioned in [36, 29], such an approach faces challenges. Call the cases \( m < K \), \( m = K \), and \( m > K \) the under-fitting, null, and over-fitting cases, respectively.

- We have to analyze \( \ell_n^{(m)} \) for both the under-fitting case and the over-fitting case, but we do not have efficient technical tools to address either case.
- It is hard to derive sharp results on the limiting distribution of \( (\ell_n^{(m+1)} - \ell_n^{(m)}) \) in the null case, and so it is unclear how to pin down the threshold.

Ma et al. [29] (see also [36]) made interesting progress but unfortunately the problems are not resolved satisfactorily. For example, they require hard-to-check strong conditions on both the under-fitting and over-fitting cases. Also, in the over-fitting case, it is unclear whether their results are sharp, and in the under-fitting case, it is unclear how to standardize \( \ell_n^{(m+1)} - \ell_n^{(m)} \) as the variance term is unknown; as a result, how to pin down the threshold remains unclear. Most importantly, both papers focus on the setting in Example 1 (see above), where severe degree heterogeneity is not allowed and they only consider the strong signal case.

In this paper, we propose Stepwise Goodness-of-Fit (StGoF) as a new approach to estimating \( K \). Our approach follows a different vein, so it is different not only by the particular procedures we use, but also in the design of the stepwise testing. In detail, for \( m = 1, 2, \ldots \), StGoF alternately uses two sub-steps, a community detection sub-step where we apply SCORE [14] assuming \( m \) is the correct number of communities, and a Goodness-of-Fit (GoF) sub-step. We propose a new GoF approach and let \( \psi_n^{(m)} \) be the GoF test statistic in step \( m \). Assuming SNR \( \rightarrow \infty \), we show that

\[
\psi_n^{(m)} \begin{cases} 
\rightarrow N(0, 1), & \text{when } m = K \text{ (null case),} \\
\rightarrow \infty \text{ in probability,} & \text{when } 1 \leq m < K \text{ (under-fitting case).} 
\end{cases}
\]  

(1.8)

This gives rise to a consistent estimate for \( K \). Note that we have derived \( N(0, 1) \) as the explicit limiting null distribution which is crucial in our study. To prove (1.8), the key is to show that in the under-fitting case, SCORE has the so-called Non-Splitting Property (NSP), meaning that all nodes in each (true) community are always clustered together. See Section 1.3 for what the analytical challenges are and how the NSP helps overcome the challenges. In the over-fitting case, \( m > K \). The NSP does not hold and so the analytical challenge remains, but the design of StGoF and the sharp results in (1.8) help avoid the analysis in this case.

For the stepwise testing algorithms in [36, 29], analysis in the over-fitting case can not be avoided, as we need to analyze \( \ell_n^{(m+1)} - \ell_n^{(m)} \) for \( m = 1, 2, \ldots, K \); see details therein.

To assess the optimality, we use the phase transition, a well-known optimality framework. It is related to the minimax framework but can be frequently more informative [5, 11, 31, 32]. We show that when SNR \( \rightarrow \infty \), (1.8) gives rise to an estimator that is consistent in a broad setting. We also obtain an information lower bound by showing that when SNR \( \rightarrow 0 \), consistent estimates for \( K \) do not exist. This suggests that our consistency result is sharp in terms of the
rate of SNR, so we say that StGoF achieves the optimal phase transition; see Section 2.3. As far as we know, such a phase transition result on estimating $K$ is new.

In order to achieve the optimal phase transition, a procedure needs to work well in the weak signal case. Since most existing methods have been focused on the strong signal case, it is unclear whether they achieve the optimal phase transition. Our contributions are as follows.

- We propose StGoF as a new approach to estimating $K$, where we use both a different design for stepwise testing and a new GoF test.
- We derive $N(0, 1)$ as the explicit limiting null distribution, and use the NSP of SCORE to derive tight bounds in the under-fitting case. These sharp results and the design of StGoF allow us to avoid the analysis in the over-fitting case and so to overcome the technical challenges faced by stepwise testing of this kind. Such an analytical strategy is extendable to other settings (e.g., the study of directed or bipartite graphs).
- We show that StGoF achieves the optimal phase transition when $\theta_{\text{max}} \leq C \theta_{\text{min}}$ and consistent in broad settings (e.g., weak signals, severe degree heterogeneity, and a wide range of sparsity). In particular, StGoF satisfies all requirements (R1)-(R3) as desired.

Compared to [14], both papers study SCORE, but the goal of [14] is community detection where $K$ is assumed as known, and the analysis were focused on the null case ($m = K$). Here, the goal is to estimate $K$: SCORE is only used as part of our stepwise algorithm, and the analysis of SCORE is focused on the under-fitting case ($m < K$), where the property of SCORE is largely unknown, and our results on the NSP of SCORE are new.

The proof of NSP is non-trivial when $m < K$. It depends on the row-wise distances of the matrix $\Xi$ consisting of the first $m$ columns of $[\xi_1, \ldots, \xi_K]\Gamma$, where $\xi_k$ is the $k$-th eigenvector of $\Omega$ and $\Gamma$ is an orthogonal matrix dictated by the Davis Kahan $\sin(\theta)$ theorem [4]. $\Gamma$ is hard to track without a strong eigen-gap assumption, and when it ranges, the row-wise distances of $\Xi$ are the same when $m = K$ but may vary significantly when $m < K$. This is why the study on SCORE is much harder in the under-fitting case than in the null case. See Section 3.

1.2 The stepwise Goodness-of-Fit (StGoF) algorithm

StGoF is a stepwise algorithm where for $m = 1, 2, \ldots$, we alternately use a community detection step and a Goodness-of-Fit (GoF) step. In principle, we can view StGoF as a general framework, and for both steps, we may use different algorithms. However, for most existing community detection algorithms (e.g., [2, 7, 38]), it is unclear whether they have the desired theoretical properties (especially the NSP), so we may face analytical challenges. For this reason, we choose to use SCORE [14], which we prove to have the NSP. For GoF, existing algorithms (e.g., [10, 22]; see Remark 2 for more discussion) do not apply to the current setting, so we propose a new GoF measure called the Refitted Quadrilateral (RQ).

In detail, fixing a tolerance parameter $0 < \alpha < 1$ and letting $z_\alpha$ be the $\alpha$ upper-quantile of $N(0, 1)$, StGoF runs as follows. Input the adjacency matrix $A$ and initialize $m = 1$. 

![Figure 1: The flow chart of StGoF.](image)
• (a) Community detection. If \( m = 1 \), let \( \hat{\Pi}^{(m)} \) be the \( n \)-dimensional vector of 1’s. If \( m > 1 \), apply SCORE to \( A \) assuming \( m \) is the correct number of communities and obtain an \( n \times m \) matrix \( \hat{\Pi}^{(m)} \) for the estimated community labels.

• (b) Goodness-of-Fit. Assuming \( \hat{\Pi}^{(m)} \) is the matrix of true community labels, we obtain an estimate \( \hat{\Omega}^{(m)} \) for \( \Omega \) by refitting the DCBM, following (1.10)-(1.11) below. Obtain the Refitted Quadrilateral test score \( \psi_n^{(m)} \) as in (1.13)-(1.16).

• (c) Termination. If \( \psi_n^{(m)} \geq z_\alpha \), repeat (a)-(b) with \( m = m + 1 \). Otherwise, output \( m \) as the estimate for \( K \). Denote the final estimate by \( \hat{K}^*_n \).

We recommend \( \alpha = 1\% \) or 5\%. See Figure 1 for the flow chart of the algorithm.

We now fill in the details for steps (a)-(b). Consider (a) first. The case of \( m = 1 \) is trivial so we only consider the case of \( m > 1 \). Let \( \lambda_k \) be the \( k \)-th largest (in magnitude) eigenvalue of \( A \), and let \( \hat{\xi}_k \) be the corresponding eigenvector. For each \( m > 1 \), we apply SCORE as follows. Input \( A \) and \( m \). Output: the estimated \( n \times m \) matrix of community labels \( \hat{\Pi}^{(m)} \).

- Obtain the first \( m \) eigenvectors \( \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_m \) of \( A \). Define the \( n \times (m - 1) \) matrix of entry-wise ratios \( \hat{R}^{(m)} \) by \( \hat{R}^{(m)}(i,k) = \hat{\xi}_{k+1}(i)/\hat{\xi}_1(i), 1 \leq i \leq n, 1 \leq k \leq m - 1 \).

- Cluster the rows of \( \hat{R}^{(m)} \) by the classical \( k \)-means assuming we have \( m \) clusters. Output \( \hat{\Pi}^{(m)} = [\hat{\pi}_1^{(m)}, \ldots, \hat{\pi}_n^{(m)}] \) (\( \hat{\pi}_i^{(m)}(k) = 1 \) if node \( i \) is clustered to cluster \( k \) and 0 otherwise).

Existing study of SCORE has been focused on the null case of \( m = K \). Our interest here is on the under-fitting case \((1 < m < K)\), where the property of SCORE is largely unknown.

Consider (b). The idea is to pretend that the SCORE estimate \( \hat{\Pi}^{(m)} \) is accurate. We then use it to estimate \( \Omega \) by re-fitting, and check how well the estimated \( \hat{\Omega} \) fits with the adjacency matrix \( A \). In detail, let \( d_i \) be the degree of node \( i \), \( 1 \leq i \leq n \), and let \( \hat{\mathcal{N}}^{(m)}_k \) be the set of nodes that SCORE assigns to group \( k \), \( 1 \leq k \leq m \). We decompose \( \mathbf{1}_n \) as follows

\[
\mathbf{1}_n = \sum_{k=1}^{m} \hat{\mathbf{1}}_k^{(m)}, \quad \text{where } \hat{\mathbf{1}}_k^{(m)}(j) = 1 \text{ if } j \in \hat{\mathcal{N}}^{(m)}_k \text{ and 0 otherwise.} \quad (1.9)
\]

For most quantities that have superscript \((m)\), we may only include the superscript when introducing these quantities for the first time, and omit it later for notational simplicity when there is no confusion. Introduce a vector \( \hat{\theta}^{(m)} = (\hat{\theta}_1^{(m)}, \hat{\theta}_2^{(m)}, \ldots, \hat{\theta}_n^{(m)})' \in \mathbb{R}^n \) and a matrix \( \hat{B}^{(m)} \in \mathbb{R}^{m,m} \) where for all \( 1 \leq i \leq n \) and \( 1 \leq k, \ell \leq m \),

\[
\hat{\theta}_i^{(m)} = [d_i/(\hat{\mathbf{1}}_i'A\mathbf{1}_n)] \cdot \sqrt{\hat{\mathbf{1}}_i'A\hat{\mathbf{1}}_i}, \quad \hat{\theta}_k^{(m)} = (\hat{\mathbf{1}}_k'A\hat{\mathbf{1}}_k)/\sqrt{(\hat{\mathbf{1}}_k'A\mathbf{1}_n)(\hat{\mathbf{1}}_k'A\hat{\mathbf{1}}_k)}. \quad (1.10)
\]

Let \( \hat{\Theta}^{(m)} = \text{diag}(\hat{\theta}) \). We refit \( \Omega \) by

\[
\hat{\Omega}^{(m)} = \hat{\Theta}^{(m)}\hat{\Pi}^{(m)}\hat{\Theta}^{(m)}(\hat{\Pi}^{(m)})'\hat{\Theta}^{(m)}. \quad (1.11)
\]

Recall that \( \Theta = \Theta \Pi \Pi' \Theta \) and \( P \) has unit diagonal entries. In the ideal case where \( m = K \), \( \hat{\Pi}^{(m)} = \Pi \), and \( A = \Omega \), we can verify that \( (\hat{\Theta}^{(m)}, \hat{\Pi}^{(m)}, \hat{\Omega}^{(m)}) = (\Theta, P, \Omega) \). This suggests that the refitting in (1.11) is reasonable. The Refitted Quadrilateral (RQ) test statistic is then

\[
Q_n^{(m)} = \sum_{i_1,i_2,i_3,i_4 \text{ (dist)}} (A_{i_1i_2} - \hat{\Omega}_{i_1i_2}^{(m)}) (A_{i_2i_3} - \hat{\Omega}_{i_2i_3}^{(m)}) (A_{i_3i_4} - \hat{\Omega}_{i_3i_4}^{(m)}) (A_{i_4i_1} - \hat{\Omega}_{i_4i_1}^{(m)}), \quad (1.12)
\]

("dist" means the indices are distinct). Without the refitted matrix \( \hat{\Omega}^{(m)} \), \( Q_n^{(m)} \) reduces to

\[
C_n = \sum_{i_1,i_2,i_3,i_4 \text{ (dist)}} A_{i_1i_2}A_{i_2i_3}A_{i_3i_4}A_{i_4i_1}, \quad (1.13)
\]

\footnote{As the network is connected, \( \hat{\xi}_1 \) is uniquely defined with all positive entries, by Perron’s theorem \[14\].}
which is the total number of quadrilaterals in the networks \([10]\). This is why we call \(Q_n^{(m)}\) the refitted quadrilaterals.

We now discuss the mean and variance of \(Q_n^{(m)}\) in the null case of \(m = K\). In this case, first, it turns out that the variance can be well-approximated by \(8C_n\). Second, while that \(\mathbb{E}[Q_n^{(K)}] = 0\) in the ideal case of \(\bar{\Omega}^{(K)} = \Omega\), in the real case, \(\bar{\Omega}^{(K)} \neq \Omega\) and \(\mathbb{E}[Q_n^{(K)}]\) is comparable to the standard deviation of \(Q_n^{(K)}\). Therefore, the mean is not negligible in the null case, and we need bias correction.

Motivated by these, for any \(m \geq 1\), we introduce two vectors \(\hat{g}^{(m)}, \hat{h}^{(m)} \in \mathbb{R}^m\) where

\[
\hat{g}_k^{(m)} = (1_k^I \hat{\theta})/\|\hat{\theta}\|_1, \quad \hat{h}_k^{(m)} = (1_k^I \hat{\theta}^2 1_k^I)^{1/2}/\|\hat{\theta}\|, \quad 1 \leq k \leq m.
\]

Write for short \(\hat{V}^{(m)} = \text{diag}(\hat{P} \hat{h})\) and \(\hat{H}^{(m)} = \text{diag}(\hat{h})\). We estimate the mean of \(Q_n^{(m)}\) by

\[
\hat{B}_n^{(m)} = 2\|\hat{\theta}\|^4 \cdot \|\hat{g}^{(m)}\|_2 \hat{V}^{(m)} \hat{H}^{(m)} \hat{H}^{(m)} \hat{V}^{(m)} \hat{g}^{(m)}
\]

where for matrices \(A\) and \(B\), \(A \circ B\) is their Hadamard product \([9]\). We show that in the null case, \(\hat{B}_n^{(m)}\) is a good estimate for \(\mathbb{E}[Q_n^{(m)}]\) and in the under-fitting case, it is much smaller than the leading term of \(Q_n^{(m)}\) and so is negligible. Finally, the StGoF statistic is defined by

\[
\psi_n^{(m)} = [Q_n^{(m)} - \hat{B}_n^{(m)}]/\sqrt{8C_n}.
\]

The computational cost of the StGoF algorithm is determined by (i) the number of iterations, (ii) the cost of SCORE, and (iii) the cost of computing \(\psi_n^{(m)}\) in \((1.16)\). For (i), we show in Section \(2\) that, under mild conditions, StGoF terminates in exactly \(K\) steps with high probability. For (ii), the costs are from implementing PCA and \(k\)-means \([14]\). PCA is manageable even for very large networks, and the complexity is \(O(n^2d)\) for each \(m\) if we use the power method, where \(\bar{d}\) is the average degree. In practice, the \(k\)-means is usually implemented with the Lloyd’s algorithm which is fast (e.g., only a few seconds when \(n\) is a few thousands). In theory, the computational cost of \(k\)-means for our setting is polynomial-time, since the dimension of each row of \(\hat{R}^{(m)}\) is \((m - 1)\). For (iii), the following lemma shows the complexity is polynomial time.

\begin{lemma}
For each \(m = 1, 2, \ldots, K\), the complexity for computing \(\psi_n^{(m)}\) is \(O(n^2\bar{d})\), where \(\bar{d}\) is average degree of the network.
\end{lemma}

\begin{remark}
The RQ test has some connections to the SgnQ test in \([17]\), but is for different problem and is more sophisticated. The RQ test is for goodness-of-fit. It depends on the matrix \(\Omega^{(m)}\), refitted for each \(m\) using the community detection results by SCORE. The SgnQ test is for global testing, where the goal is to test \(K = 1\) vs. \(K > 1\). The SgnQ test is not stepwise, and does not depend on any results of community detection. In particular, to analyze RQ, we need new technical tools, where the NSP of SCORE plays a key role.
\end{remark}

\begin{remark}
Existing GoF algorithms include \([10, 22]\), but they only address the much narrower settings (e.g., dense networks with stochastic block model and strong signals). As mentioned in \([10]\), it remains unclear how to generalize these approaches to the DCBM setting here. In principle, a GoF approach only focuses on the null case, and can not be used for estimating \(K\) without sharp results in the under-fitting case, or the over-fitting case, or both.
\end{remark}

\begin{remark}
In this paper, we are primarily interested in DCBM, but the idea can be extended to the broader DCMM \([35, 15]\), where mixed-memberships are allowed. To this end, we need to replace SCORE by Mixed-SCORE \([15]\) (an adapted version of SCORE for networks with mixed memberships), and modify the refitting step accordingly. The analysis of the resultant procedure is much more challenging so we leave it to the future.
\end{remark}
1.3 Why StGoF works and how it overcomes the challenges

We briefly explain why StGoF achieves the optimal phase transition. Recall that a reasonable measure for the SNR is $|\lambda_k|/\sqrt{\lambda_1}$, where $\lambda_k$ is the $k$-th largest (in magnitude) eigenvalue of $\Omega$; see (1.7). In Section 4 we show that if $|\lambda_k|/\sqrt{\lambda_1} \to \infty$, then

$$
\begin{cases}
\psi_n^{(m)} \to N(0,1), & \text{if } m = K, \\
E[\psi_n^{(m)}] \asymp (\sum_{k=m+1}^K \lambda_k^4)/\lambda_1^2 & \text{and so } \psi_n^{(m)} \to \infty \text{ in probability, }\text{if } 1 \leq m < K,
\end{cases}
$$

(1.17)

where we note that $(\sum_{k=m+1}^K \lambda_k^4)/\lambda_1^2 \geq (\lambda_k/\sqrt{\lambda_1})^4$ when $m < K$. Combining these with the definition of $\hat{K}_\alpha^*$ gives $P(\hat{K}^*_\alpha \neq K) \leq \alpha + o(1)$. Hence, $\hat{K}^*_\alpha$ is consistent if we let $\alpha$ tend to 0 slowly enough. When SNR $\to 0$, Section 2.2 shows that consistent estimation for $K$ is impossible in the minimax sense. Therefore, StGoF achieves the optimal phase transition.

The main technical challenge is how to analyze $\psi_n^{(m)}$ in the under-fitting case, where we not only need sharp row-wise large deviation bounds for the matrix $\hat{R}^{(m)}$, but also need to establish the NSP of SCORE. To see why NSP is important, note that $Q_n^{(m)}$ depends on $\hat{Q}^{(m)}$ (see (1.12)), where $\hat{Q}^{(m)}$ is obtained by refitting using the SCORE estimate $\hat{\Pi}^{(m)}$, and depends on $A$ in a complicate way. The dependence poses challenges for analyzing $Q_n^{(m)}$, to overcome which, a conventional approach is to use concentrations. However, $\hat{\Pi}^{(m)}$ has $\exp(O(n))$ possible realizations, and how to characterize the concentration of $\hat{\Pi}^{(m)}$ is known to be a challenging problem (e.g., [36, 29]).

However, if we are able to show that SCORE has the NSP (meaning that for each $1 \leq m \leq K$, nodes in each (true) community are always clustered together), then $\hat{\Pi}^{(m)}$ only has $\binom{K}{m}$ possible realizations, because we only have $K$ true communities. In fact, $\hat{\Pi}^{(m)}$ may have even fewer possible realizations if we impose some mild conditions. This means that for each $1 \leq m \leq K$, $\hat{\Pi}^{(m)}$ only concentrates on a few non-stochastic matrices. Using such a concentration result and conventional union bound, we can therefore remove the technical hurdle for analyzing $\psi_n^{(m)}$ in the under-fitting case; see Section 4 for details.

For the over-fitting case of $m > K$, SCORE produces $m$ clusters but we only have $K$ true communities. In this case, NSP won’t hold, and it is unclear how to derive sharp results for $\psi_n^{(m)}$. For stepwise testing procedures of similar kind, such a challenge was noted in [36, 29].

Our approach avoids the analysis of the over-fitting case by a different design in stepwise testing and sharp results in the null and under-fitting cases.

1.4 Efron’s empirical null for real applications

In theory, a good approximation for the null distribution of $\psi_n^{(m)}$ is $N(0,1)$ (see (1.17) and Theorem 2.1 where we show $\psi_n^{(m)} \to N(0,1)$ in the null case). Such a result requires some model assumptions, which may be violated in real applications (e.g., outliers, artifacts). When this happens, a good approximation for the null distribution of $\psi_n^{(m)}$ is no longer $N(0,1)$ (i.e., theoretical null), but $N(u, \sigma^2)$ (i.e., empirical null) for some $(u, \sigma) \neq (0, 1)$. Such a phenomenon has been repeatedly noted in the literature. For example, Efron [5] argued that due to artifacts or model misspecification, the empirical null frequently works better for real data than the theoretical null. The problem is then how to estimate the parameters $(u, \sigma^2)$ of the empirical null.

---

2This explains why in StGoF we do not use the refitted triangle (RT) $T_n = \sum_{i_1 \neq i_2 \neq i_3} A_{i_1 i_3} \hat{\Omega}_{i_1 i_2} A_{i_2 i_3}$, which is comparably easier to analyze. The power of RT depends on $(\sum_{k=m+1}^K \lambda_k^4)/\lambda_1^2$, where $\lambda_{m+1}, \ldots, \lambda_K$ may have different signs and so may cancel with each other.

3To shed light on why $\hat{\Pi}^{(m)}$ has so many possible realizations, suppose we wish to group $n$ iid samples from $N(0,1)$ into two clusters with the same size. It is seen that we have $\exp(O(n))$ possible clustering results.
We propose a bootstrap approach to estimating \((u, \sigma^2)\). Recall that \(\hat{\lambda}_k\) is the \(k\)-th largest eigenvalue of \(A\) and \(\xi_k\) is the corresponding eigenvector. Fixing \(N > 1\) and \(m > 1\), letting \(\hat{M}^{(m)} = \sum_{k=1}^{m} \hat{\lambda}_k \xi_k \xi_k^T\) and let \(\hat{S}^{(m)} = A - \hat{M}^{(m)}\). For \(b = 1, 2, \ldots, N\), we simultaneously permute the rows and columns of \(\hat{S}^{(m)}\) and denote the resultant matrix by \(\hat{S}^{(m,b)}\). Truncating all entries of \((\hat{M}^{(m)} + \hat{S}^{(m,b)})\) at 1 at the top and 0 at the bottom, and denote the resultant matrix by \(\hat{\Omega}^{(b)}\). Generate an adjacency matrix \(A^{(b)}\) such that for all \(1 \leq i < j \leq n\), \(A^{(b)}_{ij}\) are independent Bernoulli samples with parameters \(\hat{\Omega}^{(b)}_{ij}\) (we may need to repeat this step until the network is connected). Apply StGoF to \(A^{(b)}\) and denote the resultant statistic by \(Q_n^{(b)}\). We estimate \(u\) and \(\sigma\) by the empirical mean and standard deviation of \(\{Q_n^{(b)}\}_{b=1}^{N}\), respectively. Denote the estimates by \(\hat{u}^{(m)}\) and \(\hat{\sigma}^{(m)}\), respectively. The bootstrap StGoF statistic is then
\[
\psi_n^{(m,*)} = \frac{Q_n^{(m)} - \hat{u}^{(m)}}{\hat{\sigma}^{(m)}}, \quad m = 1, 2, \ldots,
\]
where \(Q_n^{(m)}\) is the same as in (1.16). Similarly, we estimate \(K\) as the smallest integer \(m\) such that \(\psi_n^{(m,*)} \leq z_n\), for the same \(z_n\) in StGoF. We recommend \(N = 25\), as it usually gives stable estimates for \(\hat{u}^{(m)}\) and \(\hat{\sigma}^{(m)}\). See Section 3 for details.

The original StGoF works well for real data where the DCBM is reasonable, but for data sets where DCBM is significantly violated, bootstrap StGoF may help. For the 6 data sets considered in Section 4, two methods perform similarly for all but one data set. This particular data set is suspected to have many outliers, and bootstrap StGoF performs significantly better. For theoretical analysis, we focus on the original StGoF statistics \(\psi_n^{(m)}\) as in (1.16).

### 1.5 Content

Sections 2-3 contain main theoretical results. In Section 2, we show that StGoF is consistent for \(K\) uniformly in a broad class of settings. We also present the information lower bound and show that StGoF achieves the optimal phase transition. In Section 3, we show that SCORE has the Non-Splitting Property (NSP) for \(1 \leq m \leq K\). We also shed light on why SCORE has the NSP and what the technical challenges are. In Section 4, we prove the main results. Section 5 presents numerical results with real and simulated data. The appendix contains the proofs for secondary theorems and lemmas.

In this paper, \(C > 0\) denotes a generic constant which may vary from case to case. For any numbers \(\theta_1, \ldots, \theta_n\), \(\theta_{\text{max}} = \max\{\theta_1, \ldots, \theta_n\}\), and \(\theta_{\text{min}} = \min\{\theta_1, \ldots, \theta_n\}\). For any vectors \(\theta = (\theta_1, \ldots, \theta_n)^T\), both \(\text{diag}(\theta)\) and \(\text{diag}(\theta_1, \ldots, \theta_n)\) denote the \(n \times n\) diagonal matrix with \(\theta_i\) being the \(i\)-th diagonal entry, \(1 \leq i \leq n\). For any vector \(a \in \mathbb{R}^n\), \(\|a\|_q\) denotes the Euclidean \(\ell^q\)-norm (we write \(\|a\|\) for short when \(q = 2\)). For any matrix \(P \in \mathbb{R}^{m,n}\), \(\|P\|\) denotes the matrix spectral norm, and \(\|P\|_{\text{max}}\) denotes the entry-wise maximum norm. For two positive sequences \(\{a_n\}\) and \(\{b_n\}\), we say \(a_n \sim b_n\) if \(\lim_{n \to \infty} \{a_n/b_n\} = 1\) and \(a_n \asymp b_n\) if there are constants \(c_2 > c_1 > 0\) such that \(c_1 a_n \leq b_n \leq c_2 a_n\) for sufficiently large \(n\).

### 2 Optimal phase transition

This section contains the first part of our main results, where we discuss the consistency and optimality of StGoF. Section 3 contains the second part of our main results, where we discuss the NSP of SCORE [13].

Consider a DCBM with \(K\) communities as in (1.5). We assume
\[
\|P\| \leq C, \quad \|\theta\| \to \infty, \quad \text{and} \quad \theta_{\text{max}} \sqrt{\log(n)} \to 0.
\]
(2.1)
The first one is a mild regularity condition on the \(K \times K\) community structure matrix \(P\). The other two are mild conditions on sparsity. See (1.6) for the interesting range of \(\|\theta\|\). We exclude
the case where $\theta_i = O(1)$ for all $1 \leq i \leq n$ for convenience, but our results continue to hold in this case provided that we make some small changes in our proofs. Moreover, for $1 \leq k \leq K$, let $\mathcal{N}_k$ be the set of nodes belonging to community $k$, let $n_k$ be the cardinality of $\mathcal{N}_k$, and let $\theta^{(k)}$ be the $n$-dimensional vector where $\theta_i^{(k)} = \theta_i$ if $i \in \mathcal{N}_k$ and $\theta_i^{(k)} = 0$ otherwise. We assume the $K$ communities are balanced in the sense that

$$
\min_{1 \leq k \leq K} \left\{ \frac{n_k}{n}, \|\theta^{(k)}\|_1/\|\theta\|_1, \|\theta^{(k)}\|/\|\theta\| \right\} \geq C. \tag{2.2}
$$

In the presence of severe degree heterogeneity, the valid SNR for SCORE is

$$
s_n = a_0(\theta)(|\lambda_K|/\sqrt{\lambda_1}), \quad \text{where } a_0(\theta) = (\theta_{\min}/\theta_{\max}) \cdot (\|\theta\|/\sqrt{\theta_{\max}}\|\theta\|_1) \leq 1.
$$

In the special case of $\theta_{\max} \leq C\theta_{\min}$, it is true that $a_0(\theta) \approx 1$ and $s_n \propto |\lambda_K|/\sqrt{\lambda_1}$. In this case, $s_n$ is the SNR introduced [1.7]. We assume

$$
s_n \geq C_0 \sqrt{\log(n)}, \quad \text{for a sufficiently large constant } C_0 > 0. \tag{2.3}
$$

In the special case of $\theta_{\max} \leq C\theta_{\min}$, (2.3) is equivalent to $|\lambda_K|/\sqrt{\lambda_1} \geq C \sqrt{\log(n)}$, which is mild. See Remark 6 for more discussion. Define a $K \times K$ diagonal matrix $H$ by $H_{kk} = \|\theta^{(k)}\|/\|\theta\|$, $1 \leq k \leq K$. For the matrix $HPH$ and $1 \leq k \leq K$, let (largest means largest in magnitude)

$$
\mu_k \text{ be the } k\text{-th largest eigenvalue and } \eta_k \text{ be the corresponding eigenvector}.
$$

By Perron’s theorem [9], if $P$ is irreducible, then the multiplicity of $\mu_1$ is 1, and all entries of $\eta_1$ are all strictly positive. Note also the size of the matrix $P$ is small. It is therefore only a mild condition to assume that for a constant $0 < c_0 < 1$,

$$
\min_{2 \leq k \leq K} |\mu_1 - \mu_k| \geq c_0 |\mu_1|, \quad \text{and } \frac{\max_{1 \leq k \leq K} \{\eta_1(k)\}}{\min_{1 \leq k \leq K} \{\eta_1(k)\}} \leq C. \tag{2.4}
$$

In fact, (2.4) holds if all entries of $P$ are lower bounded by a positive constant or $P \to P_0$ for a fixed irreducible matrix $P_0$. We also note that the most challenging case for network analysis is when the matrix $P$ is close to the matrix of 1’s (where it is hard to distinguish one community from another), and (2.4) always holds in such a case. In this paper, we implicitly assume $K$ is fixed. This is mostly for simplicity, as there is really no technical hurdle for the case of diverging $K$. See Remark 5 for more discussion.

### 2.1 The null case and a confidence lower bound for $K$

In the null case, $m = K$. In this case, if we apply SCORE to the rows of $\hat{R}^{(m)}$ assuming $m$ clusters, then we have perfect community recovery. As a result, StGoF provides a confidence lower bound for $K$.

**Theorem 2.1.** Fix $0 < \alpha < 1$. Suppose we apply StGoF to a DCBM model where [2.1]-[2.4] hold. As $n \to \infty$, up to a permutation of the columns of $\bar{\Pi}^{(K)}$, $P(\bar{\Pi}^{(K)} \neq \Pi) \leq Cn^{-3}$, $\psi_n^{(K)} \to N(0,1)$ in law, and $P(\hat{K}^*_n \leq K) \geq (1 - \alpha) + o(1)$.

Theorem 2.1 is proved in Section 4. Theorem 2.1 allows for severe degree heterogeneity. If the degree heterogeneity is moderate, $s_n \approx |\lambda_K|/\sqrt{\lambda_1}$, and we have the following corollary.

**Corollary 2.1.** Fix $0 < \alpha < 1$. Suppose we apply StGoF to a DCBM model where [2.1]-[2.4] hold. Suppose $\theta_{\max} \leq C\theta_{\min}$ and $|\lambda_K|/\sqrt{\lambda_1} \geq C_0 \sqrt{\log(n)}$ for a sufficiently large constant $C_0 > 0$. As $n \to \infty$, up to a permutation of the columns of $\bar{\Pi}^{(K)}$, $P(\bar{\Pi}^{(K)} \neq \Pi) \leq Cn^{-3}$, $\psi_n^{(K)} \to N(0,1)$ in law, and $P(\hat{K}^*_n \leq K) \geq (1 - \alpha) + o(1)$. 

Theorem 2.1 and Corollary 2.1 show that $\hat{K}_n^*$ provides a level-$(1 - \alpha)$ confidence lower bound for $K$. If $\alpha$ depends on $n$ and tends to 0 slowly enough, these results continue to hold. In this case, $P(\hat{K}_n^* \leq K) \geq 1 + o(1)$. In cases (e.g., when the SNR is slightly smaller than those above) where perfect community recovery is impossible but the fraction of misclassified nodes is small, the asymptotic normality continues to hold. Same comments apply to Theorem 2.3 and Corollary 2.2.

2.2 The under-fitting case and consistency of StGoF

In the under-fitting case, $m < K$. We focus on the case of $1 < m < K$ as the case of $m = 1$ is trivial. Suppose we apply SCORE to the rows of $R^{(m)}$ assuming $m$ is the correct number of communities and let $\hat{\Pi}^{(m)}$ be the matrix of estimated community labels as before. When $1 < m < K$, we underestimate the number of clusters, so perfect community recovery is impossible. However, SCORE satisfies the Non-Splitting Property (NSP). Recall that $\Pi$ is the matrix of true community labels.

**Definition 2.1.** Fix $K > 1$ and $m \leq K$. We say that a realization of the $n \times m$ matrix of estimated labels $\hat{\Pi}^{(m)}$ satisfies the NSP if for any pair of nodes in the same (true) community, the estimated community labels are the same. When this happens, we write $\Pi \succeq \hat{\Pi}^{(m)}$, meaning the partition (into clusters) on the left is finer than that on the right.

**Theorem 2.2.** Consider a DCBM where (2.1)-(2.4) hold. With probability at least $1 - O(n^{-3})$, for each $1 < m \leq K$, $\Pi \succeq \hat{\Pi}^{(m)}$ up to a permutation in the columns.

**Theorem 2.3.** Fix $0 < \alpha < 1$. Suppose we apply StGoF to a DCBM model where (2.1)-(2.4) hold. As $n \rightarrow \infty$, $\min_{1 \leq m < K} \{\psi_n^{(m)}\} \rightarrow \infty$ in probability and $P(\hat{K}_n^* \neq K) \leq \alpha + o(1)$.

**Corollary 2.2.** Fix $0 < \alpha < 1$. Suppose we apply StGoF to a DCBM model where (2.1)-(2.2) and (2.4) hold. Suppose $\theta_{\max} \leq C\theta_{\min}$ and $|\lambda_K|/\sqrt{\lambda_1} \geq C_0\sqrt{\log(n)}$ for a sufficiently large constant $C_0 > 0$. As $n \rightarrow \infty$, $\min_{1 \leq m < K} \{\psi_n^{(m)}\} \rightarrow \infty$ in probability and $P(\hat{K}_n^* \neq K) \leq \alpha + o(1)$.

Note that in Theorem 2.3 and Corollary 2.2 if we let $\alpha$ depend on $n$ and tend to 0 slowly enough, then we have $P(\hat{K}_n^* = K) \rightarrow 1$.

**Remark 4.** While the NSP of SCORE largely facilitates the analysis, it does not mean that StGoF ceases to work well once NSP does not hold; it is just harder to analyze in such cases. Numerical experiments confirm that StGoF continues to behave well even when NSP does not hold exactly. How to analyze StGoF in such cases is an interesting problem for the future.

**Remark 5.** In this paper, we assume $K$ is fixed. For diverging $K$, the main idea of our paper continues to be valid, but we need to revise several things (e.g., definition of consistency and SNR, some regularity conditions, phase transition) to reflect the role of $K$. The proof for the case of diverging $K$ can be much more tedious, but aside from that, we do not see a major technical hurdle. Especially, the NSP of SCORE continues to hold for a diverging $K$. Then, with some mild conditions, we can show that $\hat{\Pi}^{(m)}$ has very few realizations, so the analysis of StGoF is readily extendable. That we assume $K$ as fixed is not only for simplicity but also for practical
relevance. For example, real networks may have hierarchical tree structure, and in each layer, the number of leaves (i.e., clusters) is small (e.g., [12, 13, 23, 24]). Therefore, we have small $K$ in each layer when we perform hierarchical network analysis. Also, the goal of real applications is to have interpretable results. For example, for community detection, results with a large $K$ is hard to interpret, so we may prefer a DCBM with a small $K$ to an SBM with a large $K$. In this sense, a small $K$ is practically more relevant.

Remark 6. Conditions (2.3) is the main condition that ensures (a) SCORE yields exact community recovery when $m = K$, and (b) SCORE has the NSP when $1 \leq m < K$. The condition is much weaker than those in existing works (e.g., [36], [29]), and can not be significantly improved in the case of $\theta_{\text{max}} \leq C\theta_{\text{min}}$ (see phase transition results in Section 2.3). The more difficult case where $\theta_{\text{max}}/\theta_{\text{min}}$ tends to $\infty$ rapidly has never been studied before, at least for estimating $K$, and it is unclear whether we can find an alternative algorithm that satisfies (a)-(b) under a significantly weaker condition than (2.3). On the other hand, we can view StGoF as a general framework for estimating $K$, where SCORE may be improved or replaced by some other procedures satisfying (a)-(b) in the future as researchers continue to make advancements in this area, so whether (2.3) can be further improved does not affect our main contributions (see Section 1.1 for our contributions).

2.3 Information lower bound and phase transition

In Theorem 2.2 and Corollary 2.2 we require the SNR, $|\lambda_k|/\sqrt{\lambda_1}$, to tend to $\infty$ at a speed of at least $\sqrt{\log(n)}$. Such a condition cannot be significantly relaxed. For example, if SNR $\to 0$, then it is impossible to have a consistent estimate for $K$. The exact meaning of this is described below.

We say two DCBM models are asymptotically indistinguishable if for any test that tries to decide which model is true, the sum of Type I and Type II errors is no smaller than $1 + o(1)$, as $n \to \infty$. Given a DCBM with $K$ communities, our idea is to construct a DCBM with $(K + m)$ communities for any $m \geq 1$, and show that two DCBM are asymptotically indistinguishable, provided that the SNR of the latter is $o(1)$.

In detail, fixing $K_0 \geq 1$, consider a DCBM with $K_0$ communities that satisfies (1.1)-(1.4). Let $(\Theta, \Pi, \bar{P})$ be the parameters of this DCBM, and let $\bar{\Omega} = \Theta \Pi \Pi^T \Theta$. When $K_0 > 1$, let $(\beta', 1)'$ be the last column of $\bar{P}$, and let $S$ be the sub-matrix of $\bar{P}$ excluding the last row and the last column. Given $m \geq 1$ and $b_n \in (0, 1)$, we construct a DCBM model with $(K_0 + m)$ communities as follows. We define a $(K_0 + m) \times (K_0 + m)$ matrix $P$:

$$P = \begin{bmatrix} S & \beta' 1_{m+1}' \beta M \\ 1_{m+1}\beta' & \beta 1_{m+1} M \\ \end{bmatrix}, \quad \text{where} \quad M = (1 - b_n)I_{m+1} + b_n 1_{m+1}1_{m+1}'. \quad (2.5)$$

When $K_0 = 1$, we simply let $P = \frac{m+1}{1+mb_n} M$. Let $\bar{\ell}_i \in \{1, \ldots, K_0\}$ be the community label of node $i$ defined by $\Pi$. We generate labels $\ell_i \in \{1, \ldots, K_0 + m\}$ by

$$\ell_i = \begin{cases} \bar{\ell}_i, & \text{if } \bar{\ell}_i \in \{1, \ldots, K_0 - 1\}; \\ \text{uniformly drawn from } \{K_0, K_0 + 1, \ldots, K_0 + m\}, & \text{if } \bar{\ell}_i = K_0. \end{cases} \quad (2.6)$$

Let $\Pi$ be the corresponding matrix of community labels. This gives rise to a DCBM model with $(K_0 + m)$ communities, where $\Omega = \Theta \Pi \Pi^T \Theta$. Note that $P$ does not have unit diagonals, but we can re-parametrize so that it has unit diagonals. In detail, introduce a $(K_0 + m) \times (K_0 + m)$ diagonal matrix $D$ where $D_{kk} = \sqrt{T_{kk}}$, $1 \leq k \leq K_0 + m$. Now, if we let $P^* = D^{-1}PD^{-1}$, $\theta^*_i = \theta_i ||D\pi_i||_1$, and $\Theta^* = \text{diag}(\theta^*_1, \ldots, \theta^*_n)$, then $P^*$ has unit-diagonals and $\Omega = \Theta^* \Pi P^* \Pi^T \Theta^*$.

Here some rows of $\Pi$ are random (so we may call the corresponding model the random-label DCBM), but this is conventional in the study of lower bounds. Let $\lambda_k$ be the $k$th largest
The case of \( C \leq a_n < C_0 \sqrt{\log(n)} \) is more delicate. Sharp results are possible if we consider more specific models (e.g., for a scaling parameter \( \alpha_n > 0 \), \( \theta_i/\alpha_n \) are iid from a fixed distribution \( F \), and the off-diagonals of \( P \) are the same). We leave this to the future.

3 The non-splitting property (NSP) of SCORE

This section contains the second part of our main theoretical results. We first present the main technical tools for proving Theorem 2.2 (i.e., the NSP of SCORE), and then prove Theorem 2.2.
Why NSP holds is non-obvious, so in Section 3.3 we also shed light by providing an intuitive explanation and several examples. The NSP may hold in many other unsupervised learning settings, and the gained insight in Section 3.3 may serve as a good starting point for studying NSP in these settings.

Here, the primary focus of our study on SCORE is on the under-fitting case of \( m < K \), while existing study on SCORE (e.g., [14]) has been focused on the null case of \( m = K \). In the last two paragraphs of Section 1.1, we have briefly explained why the study in the under-fitting case is much harder. This section will further explain this with details.

Recall that in the SCORE step, for each \( 1 < m \leq K \), we apply the \( k \)-means to the rows of an \( n \times (m - 1) \) matrix \( \hat{R}^{(m)} \), where \( \hat{R}^{(m)}(i, k) = \frac{\xi_{k+1}(i)}{\xi_1(i)} \), \( 1 \leq i \leq n \), \( 1 \leq k \leq m - 1 \), and \( \xi_k \) is the \( k \)-th eigenvector (eigenvectors are arranged in the descending order in magnitudes of corresponding eigenvalues) of the adjacency matrix \( A \). Viewing each row of \( \hat{R}^{(m)} \) as a point in \( \mathbb{R}^{m-1} \), we will show that there is a polytope in \( \mathbb{R}^{m-1} \) with vertices \( v_1, v_2, \ldots, v_K \) such that with large probability, row \( i \) of \( \hat{R}^{(m)} \) falls close to \( v_k \) if node \( i \) belongs to the true community \( k \), for all \( 1 \leq i \leq n \). Therefore, the \( n \) rows form \( K \) clusters (but \( K \) and true cluster labels are unknown), each being a true community. To show that SCORE satisfies the NSP, the goal is to show that the \( k \)-means algorithm will not split any of these \( K \) clusters. See Figure 2 where we illustrate the NSP with an example with \( (K, m) = (4, 3) \).

**Figure 2:** Illustration for what NSP means \((K, m) = (4, 3)\). The rows of \( \hat{R}^{(m)} \) (blue crosses) form \( K \) clusters (red: cluster centers) each of which is a true community \((K \text{ and true cluster labels are unknown})\). SCORE aims to cluster all rows of \( \hat{R}^{(m)} \) into \( m \) clusters. Left: Voronoi diagram of \( k \)-means when the NSP does not hold (which will not happen according to our proof). Right: Voronoi diagram when the NSP holds.

**Definition 3.1** (Bottom up pruning and minimum pairwise distances). Fixing \( K > 1 \) and \( 1 < m \leq K \), consider a \( K \times (m - 1) \) matrix \( U = [u_1, u_2, \ldots, u_K]' \). First, let \( d_K(U) \) be the minimum pairwise distance of all \( K \) rows. Second, let \( u_k \) and \( u_\ell \) (\( k < \ell \)) be the pair that satisfies \( \|u_k - u_\ell\| = d_K(U) \) (if this holds for multiple pairs, pick the first pair in the lexicographical order).

Remove row \( \ell \) from the matrix \( U \) and let \( d_{K-1}(U) \) be the minimum pairwise distance for the remaining \((K - 1)\) rows. Repeat this step and define \( d_{K-2}(U), d_{K-3}(U), \ldots, d_2(U) \) recursively.

Note that \( d_K(U) \leq d_{K-1}(U) \leq \ldots \leq d_2(U) \).

For example, if \((K, m) = (4, 3)\), and the rows of \( U \) are \((1, 0), (1, 0), (0, 1)\) and \((1, 1)\), then \( d_4(U) = 0, d_3(U) = 1, \) and \( d_2(U) = \sqrt{2} \). The following theorem is the key to prove the NSP of SCORE, and is proved in the appendix.

**Theorem 3.1.** Fix \( 1 < m \leq K \) and let \( n \) be sufficiently large. Suppose \( x_1, x_2, \ldots, x_n \in \mathbb{R}^{m-1} \) take \( K \) distinct values \( u_1, u_2, \ldots, u_K \). Letting \( U = [u_1, u_2, \ldots, u_K]' \) and \( F_k = \{1 \leq i \leq n : x_i = u_k\} \), for \( 1 \leq k \leq K \), suppose \( \min_{1 \leq k \leq K} |F_k| \geq a_0 n \) and \( \max_{1 \leq k \leq K} \|u_k\| \leq C_0 \cdot d_m(U) \), for constants \( 0 < a_0 < 1, C_0 > 0 \). Suppose we apply \( k \)-means to a set of \( n \) points \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \) assuming \( m \) clusters. Let \( \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_m \) be the resultant clusters (which are not necessarily...
unique). There is a number \( c = c(\alpha_0, C_0, m) > 0 \) such that if \( \max_{1 \leq i \leq n} \| \hat{x}_i - x_i \| \leq c \cdot d_m(U) \), then \( \# \{1 \leq j \leq m : \hat{S}_j \cap F_k \neq \emptyset \} = 1 \), for each \( 1 \leq k \leq K \).

When we apply Theorem 3.1 to prove Theorem 2.2, all conditions required in Theorem 3.1 can be deduced from those in Theorem 2.2, so we do not need any additional conditions. See Lemma 3.3 and Section 3.2. Theorem 3.1 is a general result on \( k \)-means and may be useful in many other unsupervised settings. The proof is non-trivial for the following reasons.

- The objective function of the \( k \)-means is complicate, and the \( k \)-means solution is not necessarily unique. See Example 3.
- Theorem 3.1 only requires that there are at least \( m \) true cluster centers the minimum pairwise distance of which is large. If we assume a stronger condition, say, the minimum pairwise distance of all \( K \) cluster centers is large (i.e., \( \max_{1 \leq k \leq K} \| u_k \| \leq C_0 \cdot d_K(U) \)), the proof is much easier, but unfortunately, such a condition does not always hold in our settings. See Example 4 below.

Example 3. Suppose \( (K, m) = (4, 3) \) and \( F_1, F_2, F_3, F_4 \) have equal sizes. We view \( u_1, u_2, \ldots, u_K \) as the vertices of a quadrilateral in \( \mathbb{R}^2 \). Suppose we apply the \( k \)-means to \( x_1, x_2, \ldots, x_n \) and let \( C_1, C_2, C_3 \) be the resultant clusters. Suppose that among the 6 different pairs of vertices, \( (u_1, u_2) \) is the pair with the smallest distance. In this case, the three clusters are \( C_1 = F_1 \cup F_2, C_2 = F_3, \) and \( C_3 = F_4 \), and the cluster centers are \( (u_1 + u_2)/2, u_3, \) and \( u_4 \). If the quadrilateral is a square or rectangle, then among the 6 pairs of indices, more than one pairs have the smallest pairwise distance, so the \( k \)-means solutions are not unique.

Now, to prove Theorem 2.2, the idea is to apply Theorem 3.1 with \( \hat{x}_i \) being row \( i \) of \( \hat{R}^{(m)} \).

To do this, we study the geometrical structure underlying \( \hat{R}^{(m)} \) in the under-fitting case, where the ideal polytope and tight row-wise large deviation bounds for \( \hat{R}^{(m)} \) play a key role.

3.1 Geometric structure, ideal polytopes, and row-wise bounds

Fix \( 1 \leq k \leq K \). Let \( \lambda_k \) be the \( k \)-th largest (in magnitude) eigenvalue of the \( n \times n \) matrix \( \Omega \) and let \( \xi_k \) be the corresponding unit-\( \ell^2 \)-norm eigenvector. By Davis-Kahan \( \sin(\theta) \)-theorem [4], the two matrices \( [\xi_1, \ldots, \xi_K] \) and \( [\xi_1, \ldots, \xi_K] \) only match well with each other by a rotation matrix \( \Gamma : [\xi_1, \ldots, \xi_K] \approx [\xi_1, \ldots, \xi_K] \Gamma \). Let \( \Xi \) be the matrix consisting of the first \( m \) columns of \( [\xi_1, \ldots, \xi_K] \Gamma \). The geometrical structure underlying \( \Xi \) is the key to our study.

In the null case of \( m = K \), the geometric structure was studied in [14] [15]. For the under-fitting case of \( 1 < m < K \), the study is much harder. The reason is that, \( \Gamma \) is hard to track without a strong condition on the eigen-gap of \( \Omega \), as well as \( \Gamma \) ranges, the row-wise distances of \( \Xi \) remain the same when \( m = K \), but may vary significantly when \( m < K \). To deal with this, we need relatively tedious notations and harder proofs, compared to those in [14] [15].

Recall that \( \mu_k \) is the \( k \)-th largest (in magnitude) eigenvalue of the \( K \times K \) matrix \( HPH \), and \( \eta_k \) is the corresponding unit-\( \ell^2 \)-norm eigenvector. We now relate \( (\mu_k, \eta_k) \) to \( (\lambda_k, \xi_k) \) above. The following lemma is proved in the appendix.

Lemma 3.1. Consider a DCBM where [2.2] holds and let \( \lambda_k, \mu_k, \eta_k, \xi_k \) be as above. We have the following claims. First, \( \lambda_k = \| \theta \|^2 \mu_k \) for \( 1 \leq k \leq K \). Second, the multiplicity of \( \mu_1 \) is 1 and all entries of \( \eta_1 \) have the same sign, and the same holds for \( \lambda_1 \) and \( \xi_1 \). Last, if \( \eta_k \) is an eigenvector of \( HPH \) corresponding to \( \mu_k \), then \( \| \theta \|^{-1} \Theta \Pi H^{-1} \eta_k \) is an eigenvector of \( \Omega \) corresponding to \( \lambda_k \), and conversely, if \( \xi_k \) is an eigenvector of \( \Omega \) corresponding to \( \lambda_k \), then \( \| \theta \|^{-1} H^{-1} \Pi \Theta \xi_k \) is an eigenvector of \( HPH \) corresponding to \( \mu_k \).
From now on, let \( \eta_1 \) be the unique unit-\( \ell^2 \)-norm eigenvector of \( HPH \) corresponding to \( \lambda_1 \) that have all positive entries. Note that \( \eta_2, \ldots, \eta_K \) may not be unique. Fix a particular candidate for \( \eta_2, \ldots, \eta_K \), say, \( \eta^*_2, \ldots, \eta^*_K \). Let
\[
[\xi_1, \xi_2, \ldots, \xi_K] = \|\theta\|^{-1} \Theta \Pi H^{-1} [\eta_1, \eta^*_2, \ldots, \eta^*_K].
\]

**Definition 3.2.** Given any \((K-1) \times (K-1)\) orthogonal matrix \( \Gamma \) and \( 2 \leq k \leq K \), let \( \eta_k(\Gamma) \) be the \((k-1)\)-th column of \([\eta^*_2, \eta^*_3, \ldots, \eta^*_K] \Gamma \), with \( \eta_k(i, \Gamma) \) being the \( i \)-th entry, \( 1 \leq i \leq K \), and let \( \xi_k(\Gamma) \) be the \((k-1)\)-th column of \([\xi^*_2, \xi^*_3, \ldots, \xi^*_K] \Gamma \), with \( \xi_k(j, \Gamma) \) being the \( j \)-th entry, \( 1 \leq j \leq n \).

Note that \( (\eta_1, \xi_1) \) are uniquely defined (up to a factor of \( \pm 1 \)), but \{\( (\eta_k, \xi_k) \)\}_{2 \leq k \leq K} are not necessarily unique. However, by Lemma 3.1 and basic linear algebra, there is a collection of \{\( (\eta^*_2, \eta^*_3, \ldots, \eta^*_K) \)\} for all possible candidates of \{\( \{\eta_2(\Gamma), \ldots, \eta_K(\Gamma)\} \)\} give all possible candidates of \{\( \xi_2, \ldots, \xi_K \)\}. In the special case where \( \mu_2, \ldots, \mu_K \) are distinct, \( \mathcal{A} \) is the set of all \((K-1) \times (K-1)\) diagonal orthogonal matrices, and in the special case where \( \mu_2 = \ldots = \mu_K \), \( \mathcal{A} \) is the set of all \((K-1) \times (K-1)\) orthogonal matrices.

Fix \( 1 \leq m \leq K \) and a \((K-1) \times (K-1)\) orthogonal matrix \( \Gamma \) (which is not necessarily in \( \mathcal{A} \)). We define a \( K \times (m-1) \) matrix \( V^{(m)}(\Gamma) \) and an \( n \times (m-1) \) matrix \( R^{(m)}(\Gamma) \) by
\[
V^{(m)}(k, \ell; \Gamma) = \eta_{k+1}(k; \Gamma)/\eta_1(k), \quad 1 \leq k \leq K, \ 1 \leq \ell \leq m-1,
\]
and
\[
R^{(m)}(i, \ell; \Gamma) = \xi_{\ell+1}(i; \Gamma)/\xi_1(i), \quad 1 \leq i \leq n, \ 1 \leq \ell \leq m-1.
\]

We note that \( V^{(m)}(\Gamma) \) is the sub-matrix of \( V^{(K)}(\Gamma) \) consisting the first \((m-1)\) columns; same comments for \( R^{(m)}(\Gamma) \). Write \( V^{(m)}(\Gamma) = [v^{(m)}_1(\Gamma), \ldots, v^{(m)}_K(\Gamma)]^T \) and \( R^{(m)}(\Gamma) = [r^{(m)}_1(\Gamma), \ldots, r^{(m)}_n(\Gamma)]^T \), so that \( (v^{(m)}_k(\Gamma))^T \) is the \( k \)-th row of \( V^{(m)}(\Gamma) \) and \( (r^{(m)}_i(\Gamma))^T \) is the \( i \)-th row of \( R^{(m)}(\Gamma) \), \( 1 \leq k \leq K, 1 \leq i \leq n \). For notational simplicity, we may drop “\( \Gamma \)” when there is no confusion. Recall that for \( 1 \leq k \leq K \), \( \mathcal{N}_k \) denotes the \( k \)-th true community. The following lemma is proved in the appendix.

**Lemma 3.2 (The ideal polytope).** Consider a DCBM model where \( (2.4) \) holds. For any \( 1 \leq m \leq K \) and fixed \((K-1) \times (K-1)\) orthogonal matrix \( \Gamma \), \( r^{(m)}_i(\Gamma) = v^{(m)}_k(\Gamma) \) for any \( i \in \mathcal{N}_k \) and \( 1 \leq k \leq K \).

Therefore, the \( m \) rows of \( R^{(m)}(\Gamma) \) have at most \( K \) distinct values, \( (v^{(m)}_1(\Gamma))^T, (v^{(m)}_2(\Gamma))^T, \ldots, (v^{(m)}_K(\Gamma))^T \). For an “easy” setting, \( d_K(V^{(m)}(\Gamma)) \geq C \), so the minimum pairwise distance of these \( K \) rows are large. In a more “difficult” case, we may have \( d_K(V^{(m)}(\Gamma)) = 0 \). However, we can always find \( m \) rows of \( V^{(m)}(\Gamma) \) so that the minimum pairwise distance of which is no smaller than a constant \( C \). This is the following lemma, which is proved in the appendix.

**Lemma 3.3.** Consider a DCBM model where \( (2.2) \) and \( (2.4) \) hold. Fix \( 1 \leq m \leq K \) and an \((K-1) \times (K-1)\) orthogonal matrix \( \Gamma \), we have \( d_m(V^{(m)}(\Gamma)) \geq \sqrt{2} \) when \( m = K \), and \( d_m(V^{(m)}(\Gamma)) \geq C \) when \( 1 < m < K \), where the constant \( C > 0 \) does not depend on \( \Gamma \).

We should not expect that \( d_K(V^{(m)}(\Gamma)) \geq C \) holds for all rotation \( \Gamma \). We can only show a weaker claim of \( d_m(V^{(m)}(\Gamma)) \geq C \) as in Lemma 3.3. Below, we use a special example to illustrate how \( \Gamma \) affect \( d_K(V^{(m)}(\Gamma)) \).

**Example 4.** Consider a special case of Example 2 where \( P = (1 - b_n)I_K + b_n 1_K 1_K^T \), \( 0 < b_n < 1 \), and \( \|\theta^{(k)}\| = \|\theta\|/\sqrt{K} \) for \( 1 \leq k \leq K \) (as a result, \( HPH = (1/K)P \)). Note that the eigenvectors of \( HPH \), denoted by \( \eta_1, \eta_2, \ldots, \eta_K \), do not depend on \( b_n \). We take the case of \( (K, m) = (3, 2) \) for example. In this case, \( \eta_1 = (1/\sqrt{3})[1, 1, 1]^T \), and a candidate for \( \{\eta_2, \eta_3\} \) is
\[\eta_2^* = (1/\sqrt{2})[1, -1, 0]^t, \text{ and } \eta_3^* = (1/\sqrt{6})[1, 1, -2]^t, \] and all possible candidates for \{\eta_2, \eta_3\} are given by

\[
[\eta_2^*, \eta_3^*] \Gamma, \quad \Gamma = \Gamma(\theta) = \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix}, \quad 0 \leq \theta < 2\pi.
\]

Now, \(d_2(V(\tau^2))\) changes continuously in \(\theta\) and takes values in \([0, \sqrt{3}/\sqrt{2}]\), and hits 0 when \(\theta \in \{\pi/6, \pi/2, 5\pi/6, 7\pi/6, 3\pi/2, 11\pi/6\}\). However, \(d_2(V(\tau^2)) \geq \sqrt{3}/\sqrt{2}\) for all \(\theta\).

Similarly, we write \(\hat{R}^{(m)} = [\hat{r}_1^{(m)}, \hat{r}_2^{(m)}, \ldots, \hat{r}_n^{(m)}]^t\), so that \((\hat{r}_i^{(m)})^t\) is the \(i\)-th row of \(\hat{R}^{(m)}\). The following lemma provides a tight row-wise large-deviation bound for \(\hat{R}^{(m)}\) and is proved in the appendix.

**Lemma 3.4.** Consider a DCBM model where \(\{2.1\}, \{2.4\}\) hold. With probability \(1-O(n^{-3})\), there exists a \((K-1) \times (K-1)\) orthogonal matrix \(\Gamma\) (which may depend on \(n\) and \(\hat{R}^{(K)}\)) such that as \(n \to \infty\), \(\|\hat{r}_i^{(m)} - \hat{r}_i^{(m)}(\Gamma)\| \leq \|\hat{r}_i^{(K)} - \hat{r}_i^{(K)}(\Gamma)\| \leq Cs_n^{-1/\sqrt{\log(n)}}, \) for all \(1 < m < K\) and \(1 \leq i \leq n\).

For illustration, we assume \(d_K(V^{(m)}) \geq C\) for all \(1 < m \leq K\) (we have dropped \(\Gamma\) to simplify notations) so the minimum pairwise distance of the \(K\) rows of \(V^{(m)}\) is no smaller than \(C\). In this case, Lemmas \(3.2, 3.4\) say that the \(n\) rows of \(R^{(m)}\) have \(K\) distinct values, \((v_1^{(m)})^t, \ldots, (v_K^{(m)})^t\), and partitioning the rows with respect to different values gives exactly \(K\) true communities. Note that we can view \(v_1^{(m)}, v_2^{(m)}, \ldots, v_K^{(m)}\) as the vertices of a polytope in \(\mathbb{R}^{m-1}\). See Figure 3 for an illustration of \(K=4\). In this case, \(v_1^{(m)}, v_2^{(m)}, v_3^{(m)}, v_4^{(m)}\) are the vertices of a tetrahedron when \(m=4\), the vertices of a quadrilateral when \(m=3\), and \(K\) scalars when \(K=2\). By Lemma 3.4 and the condition \(\{2.3\}\), for all \(1 \leq i \leq n\), \(\|v_i^{(m)} - v_i^{(m)}\|\) is much smaller than \(d_K(V^{(m)})\). Therefore, the \(n\) rows of \(\hat{R}^{(m)}\) also form \(K\) clusters, each being a true community. If we apply k-means assuming \(K\) clusters, then we can fully recover the true communities. Unfortunately, \(K\) is unknown. In the under-fitting case, \(m < K\) and we under-estimate the number of clusters. However, Theorem 3.1 guarantees that, although we are not able to recover all true communities, the NSP holds.

![Figure 3: An example (K = 4). From left to right: m = 4, 3, 2. Red dots: the 4 distinct rows of R(m), v_1^{(m)}, v_2^{(m)}, v_3^{(m)}, v_4^{(m)}. Blue crosses: rows of \(\hat{R}^{(m)}\). The red dots are the vertices of a tetrahedron when m = 4, vertices of a quadrilateral when m = 3, and scalars when m = 2. For each m, the n rows of \(\hat{R}^{(m)}\) are seen to have K clusters, each of which is a true community.](image)

### 3.2 Proof of Theorem 2.2

By Lemma 3.4 there is an event \(E\), where \(P(E^c) = O(n^{-3})\), and on this event there exists a \((K-1) \times (K-1)\) orthogonal matrix \(\Gamma\) (which may depend on \(n\) and \(\hat{R}^{(K)}\)) such that

\[
\max_{1 \leq i \leq n} \|\hat{r}_i^{(m)} - \hat{r}_i^{(m)}(\Gamma)\| \leq Cs_n^{-1/\sqrt{\log(n)}}, \quad \text{for all } 1 < m \leq K.
\]

Fix \(1 < m \leq K\). By Lemma 3.2, \(\hat{r}_i^{(m)}(\Gamma) = v_i^{(m)}(\Gamma)\) for each \(i \in \mathcal{N}_k\) and \(1 \leq k \leq K\). Suppose \(v_1^{(m)}(\Gamma), \ldots, v_K^{(m)}(\Gamma)\) have \(L\) distinct values, where \(L\) may depend on \(m\) and \(\Gamma\) and \(L \geq m\) by
Lemma 3.3 Note that whenever two vectors (say) $v_1^{(m)}(\Gamma)$ and $v_2^{(m)}(\Gamma)$ are identical, we can always treat $N_1$ and $N_2$ as the same cluster before we apply Theorem 3.1. Therefore, without loss of generality, we assume $L = K$, so $v_1^{(m)}(\Gamma), \ldots, v_K^{(m)}(\Gamma)$ are distinct. It suffices to show that, on the event $E$, none of $N_1, N_2, \ldots, N_K$ is split by the k-means.

We now apply Theorem 3.1 with $\hat{x}_i = r_i^{(m)}$, $x_i = r_i^{(m)}(\Gamma)$, $F_k = N_k$, and $U = V^{(m)}(\Gamma)$. Note that by Lemma 3.3, $d_m(U) \geq C$. Also, in the proof of Lemma 3.3, we have shown that $\max_{1 \leq i < K} \|v_k^{(m)}(\Gamma)\| \leq C$. It follows that the $\ell^2$-norm of each row of $U$ is bounded by $C \cdot d_m(U)$. Additionally, on the event $E$, $\max_{1 \leq i < n} \|\hat{x}_i - x_i\| \leq C s_n^{-1} \sqrt{\log(n)}$. As long as $s_n \geq C_0 \sqrt{\log(n)}$ for a sufficiently large constant $C_0$, we have $\max_{1 \leq i < n} \|\hat{x}_i - x_i\| \leq c \cdot d_m(U)$ for a sufficiently small constant $c$. The claim now follows by applying Theorem 3.1.

3.3 Why NSP holds: intuitive explanations and examples

Why NSP holds is non-obvious, so we provide an intuitive explanation and some examples. The NSP may hold for many other unsupervised learning settings, and this section may be especially helpful if we wish to extend our ideas to other settings. Since the NSP in general settings is already proved above and the purpose here is to provide some insight, we consider settings where

$$d_K(V^{(m)}(\Gamma)) \geq C. \quad (3.10)$$

This condition is stronger than the condition $d_m(V^{(m)}(\Gamma)) \geq C$ needed in Theorem 3.1 (e.g., see Example 4). Also, for notational simplicity, we drop “$\Gamma$” below.

We start by introducing the minimum gap as a measure for the stability of the clustering results by k-means. Fixing $1 < m < K$, consider $n$ points $u_1, u_2, \ldots, u_n \in \mathbb{R}^{m-1}$ and let $U = [u_1, u_2, \ldots, u_n]^T$. Suppose we cluster $u_1, u_2, \ldots, u_n$ into $m$ clusters using the k-means.

Definition 3.3. Let $c_1, c_2, \ldots, c_m$ be any possible cluster centers from k-means (the set is not necessarily unique). Let $d_1(u_i; c_1, \ldots, c_m)$ and $d_2(u_i; c_1, \ldots, c_m)$ be the distances between $u_i$ and its closest cluster center and the distance between $u_i$ and its second closest cluster center, respectively. The minimum gap for the clustering results is defined by

$$g_m(U) = \min \{\text{all possible } c_1, c_2, \ldots, c_m\} \min_{1 \leq i \leq n} \{d_2(u_i; c_1, \ldots, c_m) - d_1(u_i; c_1, \ldots, c_m)\}.$$ 

We now explain why NSP holds for the under-fitting case. We start by considering the oracle case where we apply k-means to the $n$ rows of the non-stochastic matrix $R^{(m)}(\Gamma)$.

Theorem 3.2. Consider a DCBM model where (2.2) holds. Fix $1 < m < K$ and any $(K - 1) \times (K - 1)$ orthogonal matrix $\Gamma$. Let $V^{(m)}(\Gamma)$ and $R^{(m)}(\Gamma)$ be as in (3.8) and (3.9), respectively. If $d_K(V^{(m)}(\Gamma)) > 0$ and we apply the k-means to rows of $R^{(m)}(\Gamma)$, then NSP holds and $g_m(R^{(m)}(\Gamma)) \geq C d_K(V^{(m)}(\Gamma))$, where $C$ only depends on the constant in (2.2).

Theorem 3.2 is proved in the appendix. In the oracle case, since $r_i^{(m)} = r_j^{(m)}$ when $i$ and $j$ are in the same community, the NSP must hold once we have $g_m(R^{(m)}) > 0$ (otherwise we can easily find a contradiction). At the same time, it is less obvious why $g_m(R^{(m)}) \geq C d_K(V^{(m)})$ holds. Below, we use two examples for further illustration. In these examples, we assume $K = 4$, and let $N_1, N_2, N_3, N_4$ be the true communities. We assume these communities have equal sizes. We consider the cases of $m = 2$ and $m = 3$, separately.

Example 5a. When $m = 3$, the four points $v_1^{(m)}, \ldots, v_4^{(m)}$ are the vertices of a quadrilateral in $\mathbb{R}^2$. Following Example 3, it is seen $g_m(R^{(m)}) \geq (1/2)\|v_1^{(m)} - v_2^{(m)}\| \equiv (1/2)d_K(V^{(m)})$.

Example 5b. When $m = 2$, $v_1^{(m)}, \ldots, v_4^{(m)}$ are scalars. Without loss of generality, we assume $v_1^{(m)} < v_2^{(m)} < v_3^{(m)} < v_4^{(m)}$. In Section B.7, we show that $g_m(R^{(m)}) \geq [(3 - \sqrt{3})/2] \cdot d_K(V^{(m)})$. 

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In the real case, we take an intuitive approach to explain why NSP holds for the k-means (see Theorem 3.1 for a rigorous proof). Recall that $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_K$ are the true communities. Suppose we apply the k-means to the rows of $\hat{R}^{(m)}$ and obtain $m$ clusters with centers $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_m$. Suppose we also apply the k-means to the rows of $R^{(m)}$ and obtain $m$ clusters $c_1, c_2, \ldots, c_m$. Under some regularity conditions, we expect to see that

$$\max_{1 \leq k \leq m} \|\hat{c}_k - c_k\| \leq C \max_{1 \leq k \leq m} \|\hat{r}_i - r_i\|, \quad \text{up to a permutation of } c_1, c_2, \ldots, c_m. \quad (3.11)$$

By Lemma 3.4, the right hand side is $\leq CS_n^{-1}\sqrt{\log(n)}$ with large probability. In the k-means on rows of $R^{(m)}$, it follows from Theorem 3.2 that every row $i$ for $i \in \mathcal{N}_k$ is clustered into a cluster with center $c_j$, for some $1 \leq j \leq m$. By Definition 3.3

$$\|r_i - c_j\| + g_m(R^{(m)}) \leq \|r_i - c_j\|, \quad \text{for any } \ell \neq j.$$

Combining it with (3.11), except for a small probability, for all $i \in \mathcal{N}_k$ and $\ell \neq j$,

$$\|\hat{r}_i - \hat{c}_j\| \leq \|r_i - c_j\| + \|\hat{r}_i - r_i\| + \|\hat{c}_j - c_j\| \leq \|r_i - c_j\| + CS_n^{-1}\sqrt{\log(n)},$$

$$\|\hat{r}_i - \hat{c}_j\| \geq \|r_i - c_j\| - \|\hat{r}_i - r_i\| - \|\hat{c}_j - c_j\| \geq \|r_i - c_j\| - CS_n^{-1}\sqrt{\log(n)}.$$

It follows that

$$\|\hat{r}_i - \hat{c}_j\| \leq \|\hat{r}_i - \hat{c}_\ell\| + \left[2CS_n^{-1}\sqrt{\log(n)} - g_m(R^{(m)})\right].$$

Therefore, as long as $2CS_n^{-1}\sqrt{\log(n)} < g_m(R^{(m)})$, $\hat{c}_j$ is the closest cluster center to $\hat{r}_i$, for every $i \in \mathcal{N}_k$. This shows that except for a small probability, the whole set $\mathcal{N}_k$ is assigned to the cluster with center $\hat{c}_j$, i.e., NSP holds.

While the above explanation is intuitive and easy to understand, quite strong conditions are needed when we try to solidify each step. For example, while (3.11) sounds correct intuitively, it may not hold in some cases when the k-means solutions are not unique. Condition (3.10) may not hold in some cases either, due to the rotation aforementioned. To show NSP in the general settings as in our paper, we need Theorem 3.1 and Lemmas 3.2-3.4. On the other hand, the intuitive explanation here is easy-to-understand, and may provide a starting point for proving NSP in other unsupervised learning settings.

Remark 7. A simpler version of Theorem 3.1 was proved in [29], under stronger conditions of (a) when we apply the k-means to $\{x_1, x_2, \ldots, x_n\}$, the k-means solution is unique, and (b) $d_K(U) \geq C$ (with the same notations as in Theorem 3.1). Unfortunately, [29] only proved their claim for the special case of $(K, m) = (3, 2)$ (for general $(K, m)$, the proof is non-trivial due to complex combinatorics). Also, conditions (a)-(b) are hard to check especially as we need them to hold for $U = V^{(m)}(\Gamma)$ with all $\Gamma$ and all $m$; see Examples 3-4. For example, as illustrated in Example 4, when $\Gamma$ ranges continuously, (b) tends to fail for some $m$. To make sure (b) holds, [29] assumes a relatively strong condition (b1): $P \rightarrow P_0$ for a fixed matrix $P_0$ with distinct eigenvalues. This is a strong signal case where $\lambda_1, \lambda_2, \ldots, \lambda_K$ (eigenvalues of $\Omega$) are at the same magnitude, and the eigen-gaps are also at the same magnitude; see Example 1. In this case, the $\Gamma$ in David-Kahan $\sin(\theta)$ theorem is uniquely determined, so (b) holds. However, our primary interest is in the more challenging weak signal case, where typically $|\lambda_2|/\lambda_1 \rightarrow 0$. In this case, (b1) won’t hold, because the only $P_0$ that can be the limit of $P$ is the $K \times K$ matrix of all ones, where the $K$ eigenvalues are not distinct.

4 The behavior of the RQ test statistic

In this section, we prove Theorems 2.1 and 2.3. Corollaries 2.1 and 2.2 follow directly from Theorems 2.1 and 2.3, respectively, so the proofs are omitted. All other theorems and lemmas are proved in the appendix.
4.1 Proof of Theorem 2.1 (the null case of \( m = K \))

First, it is seen that the first item is a direct result of Theorem 2.2. By definitions,

\[
\mathbb{P}(\hat{K}_n^* \leq K) \geq \mathbb{P}(\psi_n^{(K)} \leq z_\alpha),
\]

and so the last item follows once the second item is proved. Therefore, we only need to show the second item. Recall that when \( m = K \),

\[
\psi_n^{(K)} = \left[ Q_n^{(K)} - B_n^{(K)} \right] / \sqrt{SC_n},
\]

where \( Q_n^{(K)} \), \( B_n^{(K)} \), and \( C_n \) are defined in (4.13), (1.12) and (1.15), respectively, which we reiterate below:

\[
Q_n^{(K)} = \sum_{i_1, i_2, i_3, i_4 (\text{dist})} (A_{i_1 i_2} - \hat{A}_{i_1 i_2})(A_{i_2 i_3} - \hat{A}_{i_2 i_3})(A_{i_3 i_4} - \hat{A}_{i_3 i_4})(A_{i_4 i_1} - \hat{A}_{i_4 i_1}),
\]

\[
C_n = \sum_{i_1, i_2, i_3, i_4 (\text{dist})} A_{i_1 i_2}A_{i_2 i_3}A_{i_3 i_4}A_{i_4 i_1}, \quad B_n^{(K)} = 2\parallel \hat{\theta} \parallel^4 \cdot [g' V^{-1} (\hat{P} \hat{H}^2 \hat{P} \circ \hat{P} \hat{H}^2 \hat{P}) V^{-1} g].
\]

In the first equation here, \( \hat{\Omega}^{(K)} \) depends on the estimated community label matrix \( \hat{\Pi}^{(K)} \). To facilitate the analysis, it’s desirable to replace \( \hat{\Pi}^{(K)} \) by the true membership matrix \( \Pi \). By the first claim of the current theorem, this replacement only has a negligible effect.

Formally, we introduce \( \hat{\Omega}^{(K,0)} \) to be the proxy of \( \hat{\Omega}^{(K)} \) with \( \hat{\Pi}^{(K)} \) in its definition replaced by \( \Pi \). Moreover, define \( Q_n^{(K,0)} \) to be the proxy of \( Q_n^{(K)} \) with \( \hat{\Omega}^{(K)} \) replaced by \( \hat{\Omega}^{(K,0)} \) in its definition, and define the corresponding counterpart of \( \psi_n^{(K)} \) as

\[
\psi_n^{(K,0)} = [Q_n^{(K,0)} - B_n^{(K)}] / \sqrt{SC_n}.
\]

Then, for any fixed number \( t \in \mathbb{R} \) we have

\[
\left| \mathbb{P}(\psi_n^{(K)} \leq t) - \mathbb{P}(\psi_n^{(K,0)} \leq t) \right| \leq \mathbb{P}(\hat{\Pi}^{(K)} \neq \Pi) \to 0, \quad \text{as } n \to \infty,
\]

where the last step follows from the first claim in the current theorem. Hence by elementary probability, to prove \( \psi_n^{(K)} \to N(0,1) \) in law, if suffices to show \( \psi_n^{(K,0)} \to N(0,1) \) in law.

Recall that if we neglect the difference in the main diagonal entries, then \( A - \Omega = W \). By definition, we expect that \( \hat{\Omega}^{(K,0)} \approx \Omega \), and so \( (A - \hat{\Omega}^{(K,0)}) \approx W \). This motivates us to define

\[
\tilde{Q}_n = \sum_{i_1, i_2, i_3, i_4 (\text{dist})} W_{i_1 i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1}, \tag{4.12}
\]

At the same time, for short, let \( b_n \) and \( c_n \) be the oracle counterparts of \( B_n^{(K)} \) and \( C_n \)

\[
c_n = \sum_{i_1, i_2, i_3, i_4 (\text{dist})} \Omega_{i_1 i_2} \Omega_{i_2 i_3} \Omega_{i_3 i_4} \Omega_{i_4 i_1}, \quad b_n = 2\parallel \theta \parallel^4 \cdot [g' V^{-1} (PH^2 \circ P \circ H^2 \circ P)V^{-1} g]. \tag{4.13}
\]

Here, two vectors \( g, h \in \mathbb{R}^K \) are defined as \( g_k = (1_k^T \theta) / \parallel \theta \parallel_1 \) and \( h_k = (1_k^T \Theta 1_k)^{1/2} / \parallel \theta \parallel \), where \( 1_k \) is for short of \( 1_k^{(K)} \), which is defined as

\[
1_k^{(K)}(i) = 1 \text{ if } i \in N_k \text{ and } 0 \text{ otherwise.}
\]

Moreover, \( V = \text{diag}(Pg) \), and \( H = \text{diag}(h) \). The following lemmas are proved in the appendix.

**Lemma 4.1.** Under the conditions of Theorem 2.4, we have \( \mathbb{E}[C_n] = c_n \asymp \parallel \theta \parallel^8 \) and \( \text{Var}(C_n) \leq C\parallel \theta \parallel^8 \cdot (1 + \parallel \theta \parallel^6 | \parallel \theta \parallel |) \), and so \( C_n / c_n \to 1 \) in probability for \( c_n \) defined in (4.13).
Lemma 4.2. Under the conditions of Theorem 2.1, \( \tilde{Q}_n / \sqrt{8c_n} \rightarrow N(0, 1) \) in law.

Lemma 4.3. Under the conditions of Theorem 2.1, \( \mathbb{E}(Q_n^{(K,0)} - \tilde{Q}_n - b_n)^2 = o(||\theta||^8) \).

Lemma 4.4. Under the conditions of Theorem 2.1, we have \( b_n \asymp ||\theta||^4 \) and \( B_n^{(K)}/b_n \rightarrow 1 \) in probability for \( b_n \) defined in (4.13).

Among these lemmas, the proof of Lemma 4.3 is the most complicated one, as it requires computing the bias in \( Q_n^{(m)} \) caused by the refitting step; see Section C.8 in the appendix for details.

We now prove Theorem 2.1. Rewrite \( \psi_n^{(K,0)} \) as

\[
\sqrt{\frac{c_n}{C_n}} \left[ \tilde{Q}_n / \sqrt{8c_n} + \frac{(Q_n^{(K,0)} - \tilde{Q}_n - b_n)}{\sqrt{8c_n}} + \frac{(b_n - B_n^{(K)})}{\sqrt{8c_n}} \right] = \sqrt{\frac{c_n}{C_n}} \cdot [I + (II) + (III)],
\]

(4.14)

where \((I) = \tilde{Q}_n / \sqrt{8c_n}, (II) = (Q_n^{(K,0)} - \tilde{Q}_n - b_n) / \sqrt{8c_n}, \) and \((III) = (b_n - B_n^{(K)}) / \sqrt{8c_n}) \). Now, first by Lemmas 4.1, 4.2,

\[
c_n/C_n \rightarrow 1 \ \text{in probability, and } (I) \rightarrow N(0, 1) \ \text{in law.} \quad (4.15)
\]

Second, by Lemma 4.2,

\[
\mathbb{E}[(II)^2] \leq (8c_n)^{-1} \cdot \mathbb{E}[(Q_n^{(K,0)} - \tilde{Q}_n^{(K,0)} - b_n)^2] \leq c_n^{-1} \cdot o(||\theta||^8), \quad (4.16)
\]

where the right hand side is \( o(1) \) for \( c_n \asymp ||\theta||^8 \) by Lemma 4.1. Last, by Lemma 4.1-4.4, we have \( b_n \asymp \sqrt{c_n} \asymp ||\theta||^4 \) and \( B_n^{(K)}/b_n \rightarrow 1 \), and so

\[
(III) = \left( \frac{b_n}{\sqrt{8c_n}} \right) \cdot \left( \frac{B_n^{(K)}}{b_n} - 1 \right) \rightarrow 0. \quad (4.17)
\]

Inserting (4.15)-(4.17) into (4.14) gives the claim and concludes the proof of Theorem 2.1.

### 4.2 Proof of Theorem 2.3 (the under-fitting case of \( m < K \))

In the proof of Theorem 2.1, we start from replacing \( \hat{\Pi}^{(K)} \) with the true community label matrix \( \Pi \). However, when \( m < K \), \( \hat{\Pi}^{(m)} \) does not concentrate on one particular label matrix. Below, we introduce a collection of label matrices, \( \mathcal{G}_m \), consisting of all possible realizations of \( \hat{\Pi}^{(m)} \) when NSP holds. We then study the GoF statistic on the event that \( \hat{\Pi}^{(m)} = \Pi_0 \) for a fixed \( \Pi_0 \in \mathcal{G}_m \).

Recall that \( \Pi \) is the true community label matrix. Fix \( 1 \leq m < K \). Let \( \mathcal{G}_m \) be the class of \( n \times m \) matrices \( \Pi_0 \), where each \( \Pi_0 \) is formed as follows: let \( \{1, 2, \ldots, K\} = S_1 \cup S_2 \cup \cdots \cup S_m \) be a partition, column \( \ell \) of \( \Pi_0 \) is the sum of all columns of \( \Pi \) in \( S_\ell \), \( 1 \leq \ell \leq m \). Let \( L_0 \) be the \( K \times m \) matrix of 0 and 1 where

\[
L_0(k, \ell) = 1 \text{ if and only if } k \in S_\ell, \quad 1 \leq k \leq K, \ 1 \leq \ell \leq m. \quad (4.18)
\]

Therefore, for each \( \Pi_0 \in \mathcal{G}_m \), we can find an \( L_0 \) such that \( \Pi_0 = \Pi L_0 \). Note that each \( \Pi_0 \) is the community label matrix where each community implied by it (i.e., “pseudo community”) is formed by merging one or more (true) communities of the original network.

Fix a \( \Pi_0 \) and let \( \mathcal{N}_1^{(m,0)}, \mathcal{N}_2^{(m,0)}, \ldots, \mathcal{N}_m^{(m,0)} \) be the \( m \) “pseudo communities” associated with \( \Pi_0 \). Recall that \( \hat{\theta}^{(m)}, \hat{\Theta}^{(m)} \) and \( \hat{P}^{(m)} \) are refitted quantities obtained by using the adjacency matrix \( A \) and \( \hat{\Pi}^{(m)} \); see (1.9)-(1.10). To misuse the notations a little bit, let \( \hat{\theta}^{(m)}, \hat{\Theta}^{(m)} \) and \( \hat{P}^{(m)} \) be the proxy of \( \hat{\theta}^{(m)}, \hat{\Theta}^{(m)} \) and \( \hat{P}^{(m)} \) respectively, constructed similarly by (1.9)-(1.10), but with \( \hat{\Pi}^{(m)} \) replaced by \( \Pi_0 \). Introduce

\[
\hat{\Theta}^{(m,0)} = \hat{\Theta}^{(m,0)} \Pi_0 \hat{P}^{(m,0)} \Pi_0', \quad (4.19)
\]
where the rank of $P$.

The following lemma gives an equivalent expression of $\Omega^{(m,0)}$ and is proved in the appendix.

**Lemma 4.5.** Fix $K > 1$ and $1 \leq m \leq K$. Let $\Pi_{0} = \Pi L_{0} \in \mathcal{G}_{m}$ and $\Omega^{(m,0)}$ be as above. Write $D = \Pi' \Theta \Pi \in \mathbb{R}^{K,K}$ and $D_{0} = \Pi_{0}' \Theta \Pi_{0} \in \mathbb{R}^{m,K}$. Let $P_{0}$ be the $K \times K$ matrix given by

$$P_{0} = \text{diag}(PD_{1}K) \cdot L_{0} \cdot \text{diag}(D_{0}PD_{1}K)^{-1}(D_{0}PD_{0}'\text{diag}(D_{0}PD_{1}K))^{-1} \cdot L_{0}' \cdot \text{diag}(PD_{1}K),$$

where the rank of $P_{0}$ is $m$. Then, $\Omega^{(m,0)} = \Theta \Pi P_{0} \Pi' \Theta$.

This lemma says that $\Omega^{(m,0)}$ has a similar expression as $\Omega$, with $P$ replaced by a rank-$m$ matrix $P_{0}$. When $m = K$, $\mathcal{G}_{m}$ has only one element $\Pi$; then $(P_{0}, \Omega^{(m,0)})$ reduces to $(P, \Omega)$.

We expect $\hat{\Omega}^{(m,0)}$ to concentrate at $\Omega^{(m,0)}$. This motivates the following proxy of $Q^{(m,0)}_{n}$.

$$\hat{\tilde{Q}}^{(m,0)}_{n} = \sum_{i, i_{2}, i_{3}, i_{4}(\text{dist})} (A_{i_{1}i_{2}} - \hat{\Omega}^{(m,0)}_{i_{1}i_{2}})(A_{i_{2}i_{3}} - \hat{\Omega}^{(m,0)}_{i_{2}i_{3}})(A_{i_{3}i_{4}} - \Omega^{(m,0)}_{i_{3}i_{4}})(A_{i_{4}i_{1}} - \hat{\Omega}^{(m,0)}_{i_{4}i_{1}}).$$

Introduce

$$\hat{\tilde{\Omega}}^{(m,0)} = \Omega - \Omega^{(m,0)}.$$  \hspace{1cm} (4.23)

Recall that $A = (\Omega - \text{diag}(\Omega)) + W$, we rewrite $\hat{\tilde{Q}}^{(m,0)}_{n}$ as

$$\hat{\tilde{Q}}^{(m,0)}_{n} = \sum_{i, i_{2}, i_{3}, i_{4}(\text{dist})} (W_{i_{1}i_{2}} + \hat{\Omega}^{(m,0)}_{i_{1}i_{2}})(W_{i_{2}i_{3}} + \hat{\Omega}^{(m,0)}_{i_{2}i_{3}})(W_{i_{3}i_{4}} + \Omega^{(m,0)}_{i_{3}i_{4}})(W_{i_{4}i_{1}} + \hat{\Omega}^{(m,0)}_{i_{4}i_{1}}).$$ \hspace{1cm} (4.24)

Note that when $m = K$ and $\Pi_{0} = \Pi$, the statistic $\hat{\tilde{Q}}^{(m,0)}_{n}$ reduces to $\hat{\tilde{Q}}_{n}$ defined in (4.12).

The matrix $\hat{\tilde{\Omega}}^{(m,0)}$ captures the signal strength in $\hat{\tilde{Q}}^{(m,0)}_{n}$. From now on, for notation simplicity, we write $\hat{\tilde{\Omega}}^{(m,0)} = \tilde{\Omega}$ in the rest of the proof. Let $\hat{\lambda}_{k}$ be the $k$-th largest (in magnitude) eigenvalue of $\tilde{\Omega}$ and recall that $\hat{\lambda}_{k}$ is the $k$-th largest (in magnitude) eigenvalue of $\Omega$. In light of (4.23), we write $\tilde{\Omega} = \Omega^{(m,0)} + \tilde{\Omega}$ and apply Weyl’s theorem for singular values (see equation (7.3.13) of [9]). Note that $\Omega^{(m,0)}$ has a rank $m$ and $\Omega$ has a rank $K$. By Weyl’s theorem, for all $1 \leq k \leq K - m$, $|\lambda_{m+k}| \leq |\lambda_{m+1}(\Omega^{(m,0)})| + |\hat{\lambda}_{k}| = |\lambda_{k}|$. It follows that

$$\text{tr}(\tilde{\Omega}^{4}) \geq \sum_{k=1}^{K-m} |\hat{\lambda}_{k}|^{4} \geq \sum_{k=m+1}^{K} |\lambda_{k}|^{4}.$$  \hspace{1cm} (4.25)

As we will see in Lemma 4.7 below, $\text{tr}(\tilde{\Omega}^{4})$ is the dominating term of $\mathbb{E}[\hat{\tilde{Q}}^{(m,0)}_{n}]$. Define

$$\tau^{(m,0)} = |\hat{\lambda}_{1}|/\lambda_{1}.$$ \hspace{1cm} (4.25)

For notation simplicity, we write $\tau^{(m,0)} = \tau$, but keep in mind both $\tilde{\Omega}$ and $\tau$ actually depend on $m$ and $\Pi_{0} \in \mathcal{G}_{m}$. The following lemmas are proved in the appendix.
Lemma 4.6. Under the conditions of Theorem 2.3, for each 1 \leq m \leq K, let \( \tilde{\Omega} \) and \( \tau \) be defined as in (4.21) and (4.25). The following statements are true:

- There exists a constant \( C > 0 \) such that \( |\tilde{\Omega}_{ij}| \leq C\tau \theta_i \theta_j \), for all \( 1 \leq i, j \leq n \).
- \( c_n \preceq \| \theta \|^8, \lambda_1 \preceq \| \theta \|^2 \), and \( \tau = O(1) \).
- \( \text{tr}(\tilde{\Omega}^4) \geq C\tau^4 \| \theta \|^8 \), and \( \tau \| \theta \| \to \infty \).

Lemma 4.7. Under the condition of Theorem 2.3, for 1 \leq m < K,

\[
\mathbb{E}[\tilde{Q}_n^{(m,0)}] = \text{tr}(\tilde{\Omega}^4) + o(\| \theta \|^4), \quad \text{Var}(\tilde{Q}_n^{(m,0)}) \leq C(\| \theta \|^8 + \tau^6 \| \theta \|^8 \| \theta \|^4).
\]

Lemma 4.8. Under the condition of Theorem 2.3, for 1 \leq m < K,

\[
\mathbb{E}[Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)}] = o(\| \theta \|^8), \quad \text{Var}(Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)}) \leq o(\| \theta \|^8) + C\tau^6 \| \theta \|^8 \| \theta \|^4.
\]

Lemma 4.9. Under the conditions of Theorem 2.3 for 1 \leq m < K, there exists a constant \( C > 0 \), such that \( \mathbb{P}(B_n^{(m)} \leq C\| \theta \|^4) \geq 1 + o(1) \).

We now prove Theorem 2.3. Note that by Theorem 2.1, the second item of Theorem 2.3 follows once the first item is proved. Therefore we only consider the first item, where it is sufficient to show that for all \( 1 < m < K \),

\[
\psi_n^{(m)} \to \infty, \quad \text{in probability}.
\]

By the NSP of the solutions produced by SCORE, which is shown in Theorem 2.2, there exists an event \( A_n \) with \( \mathbb{P}(A_n^c) \leq Cn^{-3} \) as \( n \to \infty \), such that on event \( A_n \) we have \( \Pi_n^{(m)} \in \mathcal{G}_m \). This further indicates that on event \( A_n \) we have

\[
\psi_n^{(m)} \geq \min_{\Pi_0 \in \mathcal{G}_m} \psi_n^{(m,0)}, \quad (4.26)
\]

where \( \psi_n^{(m,0)} \) is defined in (4.20). The LHS is hard to analyze, but the RHS is relatively easy to analyze. Then further notice that the cardinality of \( \mathcal{G}_m \) is \( |\mathcal{G}_m| = m^K \), which is of constant order as long as \( K \) is constant. Therefore to prove \( \psi_n^{(m)} \to \infty \) in probability, it suffices to show that for any fixed \( \Pi_0 \in \mathcal{G}_m \),

\[
\psi_n^{(m,0)} \to \infty, \quad \text{in probability}. \quad (4.27)
\]

We now show (4.27). Rewrite \( \psi_n^{(m,0)} \) as

\[
\sqrt{\frac{c_n}{C_n}} \cdot \left[ \frac{Q_n^{(m,0)}}{\sqrt{8c_n}} - \frac{B_n^{(m)}}{\sqrt{8c_n}} \right] = \sqrt{\frac{c_n}{C_n}} \cdot [(I) - (II)], \quad (4.28)
\]

where \( (I) = Q_n^{(m,0)}/\sqrt{8c_n} \), and \( (II) = B_n^{(m)}/\sqrt{8c_n} \). First, by Lemma 4.1 (since \( C_n \) and \( c_n \) do not depend on \( m \), this lemma applies to both the null case and the under-fitting case),

\[
c_n/C_n \to 1 \quad \text{in probability}. \quad (4.29)
\]

Second, by Lemma 4.6 \( c_n \preceq \| \theta \|^8 \). Combining it with Lemma 4.9 gives that there is a constant \( C > 0 \) such that

\[
\mathbb{P}((II) \leq C) \geq 1 + o(1). \quad (4.30)
\]

Last, by Lemma 4.6, 4.8

\[
\mathbb{E}((I)) \geq C\tau^4 \| \theta \|^4 \cdot [1 + o(1)] \to \infty, \quad \text{Var}((I)) \leq C(1 + \tau^6 \| \theta \|^6).
\]
Therefore, by Chebyshev’s inequality, for any constant $M > 0$,

$$
P((I) < M) \leq \left( \frac{1}{C \sqrt{\mathbb{E}(\mathbb{E}[I] - M)^2}} \right) \leq \left( \frac{1 + \tau^6 \|\theta\|^6_3}{(\tau^4 \|\theta\|^4_3 (1 + o(1)) - M)^2} \right),$$

(4.31)

where on the denominator, $\tau \|\theta\| \to \infty$ by Lemma 4.6. Note that under our conditions, $\|\theta\|^3 = o(\|\theta\|^2)$ and $\|\theta\| \to \infty$. Combining these, the RHS of (4.31) tends to 0 as $n \to \infty$. Inserting (4.29)-(4.31) into (4.28) proves the claim, and concludes the proof of Theorem 2.3.

5 Real data analysis and simulation study

5.1 Real data analysis

For real data analysis, we consider 6 different data sets as in Table 5.1, which can be downloaded from http://www-personal.umich.edu/~mejn/netdata/. We now discuss the true $K$. For the dolphin network, it was argued in [27] that both $K = 2$ or $K = 4$ are reasonable. For UKfaculty network, we symmetrize the network by ignoring the directions of the edges. There are 4 school affiliations for the faculty members so we take $K = 4$. For the football network, we take $K = 11$. The network was manually labelled as 12 groups, but the 12th group only consists of the 5 “independent” teams that do not belong to any conference and do not form a conference themselves. For polblogs network, Le and Levina [21] suggest that $K = 3$, but it was argued by [15] that a more appropriate model for the network is a degree corrected mixed-membership (DCMM) model with two communities, so $K = 2$ is also appropriate.

We compare StGoF and bootstrap StGoF (StGoF*) with the BIC approach by Wang and Bickel [36], BH approach by Le and Levina [21], ECV approach by Li et al. [25], and NCV approach by Chen and Lei [1]. For all these methods, we use the R package “randnet” to implement them. Note that among these approaches, ECV and NCV are cross validation (CV) approaches and the results vary from one repetition to the other. Therefore, we run each method for 25 times and report the mean and SD. The StGoF* uses bootstrapping mean and standard deviation and is also random, but the SDs are 0 for five data sets. Most methods require a feasible range of $K$ as a priori. We take $\{1, 2, \ldots, 15\}$ as the range in this section.

| Name     | $n$ | $K$ | BIC   | BH   | ECV   | NCV   | StGoF | StGoF* |
|----------|-----|-----|-------|------|-------|-------|-------|--------|
| Dolphins | 62  | 2, 4| 2     | 2    | 3.08(0.91) [2, 5] | 2.20(2.71) [1, 15] | 2     | 3      |
| Football | 115 | 11  | 10    | 10   | 11.28(0.61) [11, 13] | 12.36(1.15) [11, 15] | 10    | 10     |
| Karate   | 34  | 2   | 2     | 2    | 2.60(1.00) [1, 6] | 2.56(0.58) [2, 4] | 2     | 2      |
| UKfaculty| 81  | 4   | 4     | 3    | 5.56(1.61) [3, 11] | 2.40(0.28) [2, 3] | 4     | 3      |
| Polblogs | 1222| 2   | 6     | 8    | 4.88(1.13) [4, 8] | 2(0.00) [2, 2] | 2*    | 2      |
| Polbooks | 105 | 2, 3| 3     | 4    | 7.56(2.66) [2, 15] | 2.08(0.71) [2, 5] | 5     | 2.4(0.25) [2, 3] |

Table 1: Comparison of estimated $K$. Take ECV for Dolphins for example: for 25 independent repetitions, the estimated $K$ have a mean of 3.08 and a SD of 0.91, ranging from 2 to 5 (SD of StGoF* are 0 for the first 5 data sets).

The polblogs network is suspected to have outliers, so most of the methods do not work well. For this particular network, the mean of StGoF is much larger than expected, so we choose to estimate $K$ by the $m$ that minimizes $\psi_n^m(m)$ for $1 \leq m \leq 15$ (for this reason, we put a * next to 2 in the table). Note that StGoF* correctly estimates $K$ as 2. The polblogs network is suspected have a significant faction of mixing nodes (e.g., [15]), which explains why StGoF overestimates $K$. Fortunately, for both data sets, StGoF* estimates $K$ correctly, suggesting the bootstrapping means and standard deviations help standardize $Q_n^m$. 
5.2 Simulations

We now study StGoF with simulated data. We compare StGoF with BIC, ECV, NCV via a small scale simulations (for StGoF, $\alpha = 0.05$). We do not include StGoF* since there is no model specification. We do not include ECV and NCV are relatively close to that of StGOF when the communities are relatively small.

Given $(n, K)$, a scalar $\beta_n > 0$ that controls the sparsity, a symmetric non-negative matrix $P \in \mathbb{R}^{K \times K}$, a distribution $f(\theta)$ on $(0, \infty)$, and a distribution $g(\pi)$ on the standard simplex of $\mathbb{R}^n$, we generate the adjacency matrix $A \in \mathbb{R}^{n \times n}$ as follows:

1. Generate $\tilde{\theta}_1, \tilde{\theta}_2, ..., \tilde{\theta}_n$ iid from $f(\theta)$. Let $\theta_i = \beta_n \cdot \tilde{\theta}_i / \|\tilde{\theta}\|$ and $\Theta = \text{diag}(\theta_1, ..., \theta_n)$.
2. Generate $\pi_1, \pi_2, ..., \pi_n$ iid from $g(\pi)$, and let $\Pi = [\pi_1, \pi_2, ..., \pi_n]'$.
3. Let $\Omega = \Theta \Pi \Pi' \Theta$. For each experiment below, once $\Omega$ is generated, we keep it fixed, and use it to generate $A$ according to the DCBM, for 100 times independently.

For all algorithms, we measure the performance by the fraction of times the algorithm correctly estimates the true number of communities $K$ (i.e., accuracy). Note that $\|\theta\| = \beta_n$, and $\text{SNR} \propto \|\theta\|/(1 - b_n)$. For the experiments, we let $\beta_n$ range so to cover many different sparsity levels, but let $\|\theta\|(1 - b_n)$ be at a more or less the same level, so the problem of estimating $K$ is not too difficult or too easy; see details below. We consider three experiments, and each experiment has some sub-experiments.

**Experiment 1.** In this experiment, we study how degree heterogeneity affect the results and comparisons. Fixing $(n, K) = (600, 4)$, we let $P$ be the $4 \times 4$ matrix with unit diagonals and off-diagonals $P(k, \ell) = 1 - [(1 - b_n)(|k - \ell| + 1)]/K$, where $1 \leq k, \ell \leq 4$ and $k \neq \ell$. Such matrix is called a Toeplitz matrix. Let $g(\pi)$ be the uniform distribution over $e_1, e_2, e_3, e_4$ (the standard basis vectors of $\mathbb{R}^4$).

We consider three sub-experiments, Exp 1a-1c. In these sub-experiments, we keep $(1 - b_n)\|\theta\|$ fixed at 9.5 so the SNR’s are roughly at the same level. We let $\beta_n$ range from 10 to 14 so to cover both the more sparse and the more dense cases. Moreover, for the three sub-experiments, we take $f(\theta)$ to be $U(2, 3)$ (uniform distribution), Pareto($8, .375$) (8 is the shape parameter and .375 is the scale parameter), and two point mixture $0.95\delta_1 + 0.05\delta_3$ ($\delta_n$ is a point mass at $a_n$), respectively. Note that from Exp 1a to Exp 1c, the degree heterogeneity is increasingly more severe on average.

The estimation accuracy is presented in Figure 3, where StGoF is seen to consistently outperform other approaches. Also, from Exp 1a to Exp 1c, the estimation accuracy for all algorithms get consistently lower, suggesting that when the degree heterogeneity gets more severe, the problem of estimating $K$ gets more challenging.

**Experiment 2.** In this experiment, we study how the relative sizes of different communities affect the results and comparisons. For $b_n > 0$ to be determined, we set $(n, K) = (1200, 3)$, $f(\theta)$ as Pareto($10, 0.375$), and let $P$ be the $3 \times 3$ matrix satisfying $P(k, \ell) = 1 - |k - \ell|(1 - b_n)/2$, $1 \leq k, \ell \leq 3$. We let $\beta_n$ range in $\{12, 13, ..., 17\}$ and keep $(1 - b_n)\|\theta\|$ fixed at 10 so the SNR’s are roughly at the same level. We take $g(\pi)$ as the distribution with weights $a, b$, and $(1 - a - b)$ on vectors $e_1, e_2, e_3$ (the standard basis vectors of $\mathbb{R}^3$), respectively. Consider three sub-experiments, Exp 2a-2c, where we take $(a, b) = (.30, .35), (.25, .375), and (.20, .40)$, respectively, so the three communities in the network are slightly unbalanced, moderately unbalanced, and slightly unbalanced, respectively.

Figure 4 presents the estimation accuracy. First, StGoF consistently outperforms NCV, ECV and BIC. Second, when the three communities get increasingly unbalanced, all methods become less accurate, suggesting that estimating $K$ gets increasingly harder. Last, the performance of ECV and NCV are relatively close to that of StGOF when the communities are relatively
Figure 4: Left to right: Experiment 1a, 1b, and 1c, where the degree heterogeneity are increasingly more severe (x-axis: sparsity. y-axis: accuracy). Results are based on 100 repetitions.

Figure 5: Left to right: Experiment 2a, 2b, and 2c (x-axis: ||θ|| (sparsity level); y-axis: estimation accuracy. The results are based on 100 repetitions.

balanced (e.g., Exp 2a), but are comparably more unsatisfactorily when the models are more unbalanced (e.g., Exp 2b-2c).

**Experiment 3.** We study how robust these algorithms are in cases of model misspecification. Fix \((n, K) = (600, 4)\). We let \(f(\theta)\) be the uniform distribution \(U(2, 3)\), and let \(P\) be the 4 × 4 matrix with unit diagonals and where for 1 ≤ \(k, \ell\) ≤ 4 and \(k \neq \ell\), \(P(k, \ell) = 1 - (1 - b_n)(|k - \ell| + 1)/K\). We consider two sub-experiments, Exp 3a-3b. For sparsity, we let \(\beta_n\) range from 11 to 16 in Exp 3a and range from 11 to 18 in Exp 3b. For different \(\beta_n\), we choose \(b_n\) so that \((1 - b_n)||\theta||\) is fixed at 10.5. Moreover, in Exp 3a, we allow mixed-memberships. We take \(g(\pi)\) to be the mixing distribution which puts probability \(0.2\) on \(e_1, e_2, e_3, e_4\) (standard basis vectors of \(\mathbb{R}^4\)), respectively, and let \(\pi\) be the symmetric \(K\)-dimensional Dirichlet distribution for the remaining probability of \(0.2\). Once we have \(\theta_i, \pi_i, \) and \(P\), we let \(\Omega_{ij} = \theta_i \theta_j \pi_i' P \pi_j\), 1 ≤ \(i, j\) ≤ \(n\), similar to that in DCBM. In Exp 3b, we allow outliers. First, we let \(g(\pi)\) be the mixing distribution that puts masses \(0.25\) on \(e_1, e_2, e_3, e_4\), and obtain \(\Omega\) as in DCBM. We then randomly select 10% of the nodes and re-define \(\Omega_{ij}\) as \(\rho_n\) if either \(i\) or \(j\) is selected, where \(\rho_n = n^{-2} \sum_{1 \leq i, j \leq n} \Omega_{ij}\).

Figure [6] presents the estimation accuracy. The two cross-validation methods (ECV and NCV) are not model-based algorithms and are expected to be less affected by model misspecification, so we can use their results as a benchmark to evaluate the performances of StGoF and the likelihood-based approach BIC. Figure [6] shows that StGoF continues to perform well in all settings, suggesting that it is not sensitive to model misspecification. The performance of BIC, if compared to those in Experiments 1-2, is less satisfactory, suggesting that the method is more sensitive to the model misspecification.
Figure 6: Experiment 3a (left) and 3b (right) (x-axis: $\|\theta\|$ (sparsity level). y-axis: estimation accuracy). The results are based on 100 repetitions.

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Appendix for “Estimating the number of communities by Stepwise Goodness-of-fit”

The appendix contains the proof of theorems and lemmas in the main article. Section A proves Lemma 1.1 and Theorems 2.4, 2.5. Section B proves Theorems 3.1, 3.2 and Lemmas 3.3, 3.4. Section C proves Lemmas 4.1, 4.9. Section D proves the technical lemmas needed in Sections A-C.

A Proof of results in Sections 1, 2

A.1 Proof of Lemma 1.1

For the goodness-of-fit test, it contains calculation of (a) $\hat{\Omega}^{(m)}$ as the refitted $\Omega$, (b) $Q_n^{(m)}$ as the main term, (c) $B_n^{(m)}$ as the bias correction term and (d) $C_n$ as the variance estimator.

For (a), it requires calculation of $d_i$ for $1 \leq i \leq n$, and $\hat{1}_k^i A \hat{1}_f$ and $\hat{1}_k^i A \hat{1}_n$ for $1 \leq k, \ell \leq m$ with $m \leq K$. Since $d_i$ needs $O(d_i)$ operations, it takes $O(n d)$ for calculating $d_i$, $1 \leq i \leq n$. Similarly, it takes $O(1_k^i A \hat{1}_f)$ to calculate $1_k^i A \hat{1}_f$ and $O(1_k^i A \hat{1}_n)$ to calculate $1_k^i A \hat{1}_n$, $1 \leq k, \ell \leq m$. The total complexity is then $O(n d)$. By (1.12),

$$\hat{\Omega}^{(m)}(i, j) = \hat{g}^{(m)}(i) \hat{g}^{(m)}(j) (\hat{\pi}^{(m)})^T \hat{V}^{(m)} \hat{\pi}^{(m)},$$

whose calculation takes $O(m^2)$ operations. Hence, calculation of $\hat{\Omega}^{(m)}$ needs $O(m^2 n^2)$ operations. Combining together, we conclude that step (a) costs $O(m^2 n^2)$.

For (b), $Q_n^{(m)}$ can be calculated using the same form in Theorem 1.1 of [17]. As is shown there, this step requires $O(n^2 d)$ operations.

For (c), given $\hat{\Omega}^{(m)}$ and $\hat{P}^{(m)}$, the calculation of $\hat{g}^{(m)}$, $\hat{V}^{(m)}$ and $\hat{H}^{(m)}$ only takes $O(n)$. By (1.15), calculation of $B_n^{(m)}$ only involves calculate $||\hat{\theta}||$ and $\hat{g}^T \hat{V}^{-1} (\hat{P} \hat{H}^2 \hat{P} \hat{V})^{-1} \hat{g}$. The first part needs $O(n)$ operations. The second part only involves vectors in $\mathbb{R}^m$ and matrices in $\mathbb{R}^{m,m}$. Moreover since $m \leq K$ and $K$ is fixed, it takes at most $o(n)$ operations. Combining above, step (c) costs $O(n)$.

For (d), the calculation follows from Proposition A.1 of [16]. It should be noted $C_n$ is denoted as $\hat{C}_n$ there, and it requires calculation of (i) trace of a matrix, (ii) $A^4$ for matrix $A$ and (iii) quadratic form of matrix $A$ and $A^2$. For (i), it only takes $O(n)$. For (ii), it takes at most $O(n^2)$. For (ii), we can compute $A^k$ recursively from $A^k = A^{k-1} A$. It suffices to consider the complexity of computing $BA$, for an arbitrary $n \times n$ matrix $B$. The $(i, j)$-th entry of $BA$ is $\sum_{\ell, A_{\ell j} \neq 0} B_{i \ell} A_{\ell j}$, where the total number of nonzero $A_{\ell j}$ equals to $d_j$, the degree of node $j$. Hence, the complexity of computing the $(i, j)$-th entry of $BA$ is $O(d_j)$. It follows that the complexity of computing $BA$ is $O(n^2 d)$.

Combining above, the goodness-of-fit test needs $O(n^2 d)$ operations.

A.2 Proof of Theorem 2.4

First, we show the claims on $|\lambda_{k}| / \sqrt{\lambda_1}$. Define a diagonal matrix $H$ by $H_{kk} = ||\theta||^{-1} \sqrt{\sum_{i, \ell = k} \theta_i^2}$, for $1 \leq k \leq K$. Note that $H$ is also stochastic. By Lemma 3.1, the eigenvalues of $\Omega$ are equal to the eigenvalues of $||\theta||^2 HPH$, i.e.,

$$\lambda_k = ||\theta||^2 \cdot \lambda_k (HPH), \quad 1 \leq k \leq K.$$
It follows that
\[ |\lambda_K|/\sqrt{\lambda_1} = \|\theta\| \cdot |\lambda_K(HPH)|/\sqrt{\lambda_1(HPH)}. \tag{A.32} \]

Below, we first study the matrix \( H \) and then show the claims.

Consider the matrix \( H \). Let \( \tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_{K_0} \) be the (non-stochastic) communities of the DCBM with \( K_0 \) communities. For each \( 1 \leq k \leq K_0 \), let \( \theta^{(k)} \in \mathbb{R}^n \) be such that \( \theta_i^{(k)} = \theta_{i,0} \cdot 1 \{ i \in \tilde{N}_k \} \). By definition,
\[ H_{kk}^2 = \|\theta\|^{-2} \left\{ \begin{array}{ll} \|\theta^{(k)}\|^2, & \text{for } 1 \leq k \leq K_0 - 1, \\ \sum_{i \in \tilde{N}_{K_0}} \theta_i^2 \cdot 1 \{ i = k \}, & \text{for } K_0 \leq k \leq K_0 + m. \end{array} \right. \]

Since (2.2) is satisfied, \( \|\theta\|^2 \geq \|\theta^{(k)}\|^2 \geq C\|\theta\|^2 \), for \( 1 \leq k \leq K_0 \). It implies that
\[ C^{-1} \leq H_{kk} \leq C, \quad \text{for } 1 \leq k \leq K_0 - 1. \tag{A.33} \]

Fix \( k \geq K_0 \). The \( n \) indicators \( 1 \{ \ell_i = k \} \) are iid Bernoulli variables with a success probability of \( 1/m+1 \). Therefore, \( \mathbb{E}H_{kk}^2 = \frac{1}{m+1}\|\theta\|^{-2}\|\theta^{(K_0)}\|^2 \). Furthermore, by Hoeffding’s inequality,
\[ \mathbb{P}\left( \|\theta\|^2(H_{kk}^2 - \mathbb{E}H_{kk}^2) > t \right) \leq 2 \exp\left( -\frac{t^2}{2\sum_{i \in \tilde{N}_{K_0}} \theta_i^4} \right). \]

By (2.1), \( \theta_{max} = \sqrt{\log(n)} \rightarrow 0 \). Hence, \( \sum_{i \in \tilde{N}_{K_0}} \theta_i^4 \leq \theta_{max}^2 \|\theta^{(K_0)}\|^2 \ll \|\theta\|^2 / \log(n) \). Taking \( t = \|\theta\| \) in the above equation yields \( H_{kk}^2 - \mathbb{E}H_{kk}^2 \leq \|\theta\|^{-1} \) with probability \( 1 - o(n^{-1}) \). We have seen that \( \mathbb{E}H_{kk}^2 = \frac{1}{m+1}\|\theta\|^{-2}\|\theta^{(K_0)}\|^2 \), which is bounded above and below by constants. Additionally, \( \|\theta\|^{-1} = o(1) \). Combining these results gives
\[ C^{-1} \leq H_{kk} \leq C, \quad \text{with probability } 1 - o(n^{-1}), \quad \text{for any } k \geq K_0. \tag{A.34} \]

It follows from (A.33) and (A.34) that
\[ \|H\| \leq C, \quad \|H^{-1}\| \leq C, \quad \text{with probability } 1 - o(n^{-1}). \tag{A.35} \]

Consider the the upper bound for \( |\lambda_K|/\sqrt{\lambda_1} \). It suffices to get an upper bound for \( |\lambda_K(HPH)| \) and a lower bound for \( \lambda_1(HPH) \). Note that \( |\lambda_K(HPH)| \) is the smallest singular value of \( HPH \), which can be different from the absolute value of the smallest eigenvalue. Therefore, we cannot use Cauchy’s interlacing theorem [9] to relate \( |\lambda_K(HPH)| \) to the smallest eigenvalue of \( M \). We need a slightly longer proof. Write
\[ P = \begin{bmatrix} S & \beta \beta_{m+1} \beta_{m+1} \\ 1_{m+1} \beta_{m+1} & 1_{m+1} \beta_{m+1} \end{bmatrix} + \begin{bmatrix} 0_{(K_0-1) \times (K_0-1)} & 0_{(K_0-1) \times (K_0-1)} \\ 1_{1 \times (K_0-1)} & M - 1_{m+1} \end{bmatrix} = P^* + \Delta. \]

The matrix \( P^* \) can be re-expressed as ((\( e_{K_0} \) is the \( K_0 \)th standard basis of \( \mathbb{R}^{K_0} \))
\[ P^* = \begin{bmatrix} I_{K_0} & S \beta \beta_{m+1} \\ 1_{m+1} \beta_{m+1} & 1_{m+1} \beta_{m+1} \end{bmatrix}. \]

Therefore, the rank of \( P^* \) is only \( K_0 \). Then, \( H P^* H \) is also a rank-\( K_0 \) matrix. Consequently, for \( K = K_0 + m \),
\[ \lambda_K(H P^* H) = 0. \]

By Weyl’s inequality [9], \( |\lambda_K(HPH) - \lambda_K(H P^* H)| \leq \|H \Delta H\| \). Combining these results gives
\[ |\lambda_K(HPH)| \leq \|H \Delta H\|. \tag{A.36} \]
Note that $\|\Delta\| = \| \frac{m+1}{1+mb_n} M - 1_{m+1} 1'_{m+1} \|$. $M$ is a matrix whose diagonals are 1 and off-diagonals are equal to $b_n$. As a result, $\Delta$ is a matrix whose diagonals are equal to $\frac{m(1-b_n)}{1+mb_n}$ and off-diagonals are equal to $\frac{-(1-b_n)}{1+mb_n}$. It follows immediately that

$$\|\Delta\| \leq C(1-b_n).$$

We plug it into (A.36) and apply (A.35). It yields that

$$|\lambda_K(HPH)| \leq C(1-b_n). \quad (A.37)$$

Furthermore, $\lambda_1(P) \geq P_{11} = 1$ and $\lambda_1(P) \leq \|H^{-1}\|^2 \lambda_1(HPH)$. Combining it with (A.35) gives

$$\lambda_1(HPH) \geq C^{-1}. \quad (A.38)$$

Note that (A.37)-(A.38) hold with probability $1-o(n^{-1})$, because their derivation utilizes (A.35). We plug (A.37)-(A.38) into (A.32) to get $|\lambda_K/\sqrt{\lambda_1}| \leq C/\|\theta\|(1-b_n)$, with probability $1-o(n^{-1})$. This proves the upper bound of $|\lambda_K/\sqrt{\lambda_1}|$.

Consider the the lower bound for $|\lambda_K|/\sqrt{\lambda_1}$. Using (A.35), we have

$$|\lambda_K(HPH)|^{-1} = \|(HPH)^{-1}\| \leq \|H^{-1}\|^2 \cdot \|P^{-1}\| \leq C\|P^{-1}\|. \quad (A.39)$$

We then bound $\|P^{-1}\|$. Write

$$P = A + B, \quad \text{where} \quad A = \begin{bmatrix} S & \frac{m+1}{1+mb_n} M \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & \beta 1'_{m+1} \\ 1_{m+1}' \beta' & 0 \end{bmatrix}.$$ 

The matrix $B$ is a rank-2 matrix, which can be re-expressed as

$$B = XD^{-1}X', \quad \text{where} \quad X = \begin{bmatrix} \beta & \beta' \\ \beta' & 1_{m+1}' \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$ 

We use the matrix inversion formula to get

$$\|P^{-1}\| = \|(A + XD^{-1}X')^{-1}\| = \|A^{-1} - A^{-1}X(D + X' A^{-1}X)^{-1}X' A^{-1}\| \leq \|A^{-1}\| \cdot \left(1 + \|X(D + X' A^{-1}X)^{-1}X' A^{-1}\|\right) \|A^{-1}\| \cdot \left(1 + \|D + X' A^{-1}X\| \|X' A^{-1}X\|\right). \quad (A.40)$$

By direct calculations, writing $M_0 = \frac{1+mb_n}{m+1} M$ and $1 = 1_{m+1}$ for short, we have

$$X' A^{-1}X = \begin{bmatrix} \beta' S^{-1}\beta + 1 & \beta' S^{-1}\beta - 1 \\ \beta' S^{-1}\beta - 1 & \beta' S^{-1}\beta + 1 \\ \beta' S^{-1}\beta + 1 & \beta' S^{-1}\beta - 1 \end{bmatrix}.$$ 

Note that $M1 = (1+mb_n)1$. It implies that $M^{-1}1 = \frac{1}{1+mb_n} 1$. As a result, $1'M_0^{-1}1 = \frac{1+mb_n}{m+1} 1'M_0^{-1}1 = \frac{1+mb_n}{m+1} 1' \left( \frac{1}{1+mb_n} 1 \right) = 1$.

Plugging it into the expression of $X' A^{-1}X$ gives

$$X' A^{-1}X = \begin{bmatrix} \beta' S^{-1}\beta + 1 & \beta' S^{-1}\beta - 1 \\ \beta' S^{-1}\beta - 1 & \beta' S^{-1}\beta + 1 \end{bmatrix}.$$ 

It follows from direct calculations that

$$\left( D + X' A^{-1}X\right)^{-1} \left( X' A^{-1}X\right) = \frac{1}{2} \left[ \begin{array}{cc} 1 & -1 \\ 1 & \frac{3 \beta' S^{-1}\beta + 1}{\beta' S^{-1}\beta - 1} \end{array} \right]. \quad (A.41)$$
Under the condition $|\beta S^{-1}\beta - 1| \geq C$, the absolute value of $\frac{3\beta S^{-1}\beta + 1}{\beta S^{-1}\beta - 1}$ is bounded by a constant. Therefore, the spectral norm of the matrix in (A.40) is bounded by a constant. We plug it into (A.41) to get
\[ \|P^{-1}\| \leq C \|A^{-1}\| \leq C \max \{|\lambda_{\min}(S)|^{-1}, |\lambda_{\min}(M)|^{-1}\}. \]

The minimum eigenvalue of $M$ is $(1 - b_n)$. Hence, under the condition of $|\lambda_{\min}(S)| \gg 1 - b_n$, we immediately have $\|P^{-1}\| \leq C(1 - b_n)^{-1}$. We plug it into (A.39) to get
\[ |\lambda_K(HPH)| \geq C^{-1}(1 - b_n). \] (A.42)

Additionally, $\|\tilde{P}\| \leq C$ by (2.1). It follows from the connection between $P$ and $\tilde{P}$ in (2.5) that $\|P\| \leq C$. Combining it with (A.35) gives $\|HPH\| \leq C$, i.e.,
\[ \lambda_1(HPH) \leq C. \] (A.43)

Here (A.42) and (A.43) are satisfied with probability $1 - o(1)$, because their derivation uses (A.35). We plug (A.42) and (A.43) into (A.32). It yields that $|\lambda_K/\sqrt{\lambda_1} \geq C^{-1}\|\theta\|(1 - b_n)$, with probability $1 - o(1)$. This proves the lower bound of $|\lambda_K/\sqrt{\lambda_1}$.

Next, we show that, if $\|\theta\|(1 - b_n) \to 0$, the two random-label DCBM models associated with $m_1$ and $m_2$ are asymptotically indistinguishable. It is sufficient to show that each random-label DCBM is asymptotically indistinguishable from the (fixed-label) DCBM with $K_0$ communities.

Fix $m \geq 1$. Let $f_0(A)$ and $f_1(A)$ be the respective likelihood of the (fixed-label) DCBM and the random-label DCBM. Write $\tilde{\Omega} = \Theta \Pi \tilde{P} \Theta$ and $\Omega = \Theta P \Pi \Theta$. It is seen that
\[ f_0(A) = \prod_{1 \leq i < j \leq n} \tilde{\Omega}_ij^{A_{ij}}(1 - \tilde{\Omega}_ij)^{1 - A_{ij}}, \quad f_1(A) = \int \prod_{1 \leq i < j \leq n} \Omega_\alpha^{A_{ij}}(1 - \Omega_\alpha)^{1 - A_{ij}} d\mathbb{P}(\Pi). \]

Recall that $\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_{K_0}$ are the (non-stochastic) communities in the first DCBM. We observe that $\tilde{\Omega}_ij \neq \Omega_\alpha$ only when both $i$ and $j$ are in $\tilde{N}_{K_0}$. Therefore, the likelihood ratio is
\[ L(A) = \frac{f_1(A)}{f_0(A)} = \prod_{\{i,j\} \subseteq \tilde{N}_{K_0}, i < j} \left( \frac{\Omega_\alpha}{\tilde{\Omega}_ij} \right)^{A_{ij}} \left( \frac{1 - \Omega_\alpha}{1 - \tilde{\Omega}_ij} \right)^{1 - A_{ij}} d\mathbb{P}(\Pi). \] (A.44)

When $i, j$ are both in $\tilde{N}_{K_0}$, it is seen that
\[ \tilde{\Omega}_ij = \theta_i \theta_j, \quad \Omega_\alpha = \theta_i \theta_j \cdot \pi_i \cdot (m + 1) \cdot (1 + m b_n) \cdot \pi_j, \]
where $\pi_i = e_k$ if and only if $\ell_i = K_0 - 1 + k$, $1 \leq k \leq m + 1$, and $e_1, e_2, \ldots, e_{m+1}$ are the standard bases of $\mathbb{R}^{m+1}$. Here we have mis-used the notation $\pi_i$; previously, we use $\pi'_i$ to denote the $i$-th row of $\Pi$, but currently, the $i$-th row of $\Pi$ is $(0^{K_0 - 1}, \pi'_i)$. Define
\[ z_i = \pi_i - \frac{1}{m + 1} \mathbf{1}_{m+1}, \quad \text{for all } i \in \tilde{N}_{K_0}. \]

The random vectors $\{z_i\}_{i \in \tilde{N}_{K_0}}$ are independently and identically distributed, satisfying $\mathbb{E}z_i = \mathbf{0}$ and $\|z_i\| \leq 1$. In the paragraph below (A.36), we have seen that
\[ \begin{bmatrix} m & -1 & \cdots & -1 \\ -1 & m & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & m \end{bmatrix} \equiv 1_{m+1}^t 1_{m+1} + G. \]
The matrix $G$ satisfies that $G1_{m+1} = 0$ and $\|G\| \leq C(1 - b_n)$. It follows that

$$
\begin{align*}
\Omega_{ij} &= \theta_i \theta_j \cdot \pi'_i \left(1 + 1_{m+1}^T + G\right) \pi_j \\
&= \theta_i \theta_j + \theta_i \theta_j (\pi'_i G \pi_j) \\
&= \theta_i \theta_j + \theta_i \theta_j \left( \frac{1}{m+1} 1_{m+1} + z_i \right) \left( \frac{1}{m+1} 1_{m+1} + z_j \right) \\
&= \theta_i \theta_j (1 + z'_i G z_j).
\end{align*}
$$

We plug it into \((A.44)\) to get

$$
L(A) \equiv \frac{f_2(A)}{f_1(A)} = \mathbb{E}_z \left\{ \prod_{i,j \in \mathcal{N}_{\kappa_0}, i < j} \left( 1 + z'_i G z_j \right)^{A_{ij}} \left[ \frac{1 - \theta_i \theta_j (1 + z'_i G z_j)}{1 - \theta_i \theta_j} \right]^{1 - A_{ij}} \right\}. \tag{A.46}
$$

The $\chi^2$-distance between two models is $\mathbb{E}_{A \sim f_0}[(L(A) - 1)^2]$. To show that the two models are asymptotically indistinguishable, it suffices to show that the $\chi^2$-distance is $o(1)$ \[33\]. Using the property that $\mathbb{E}_{A \sim f_0}[(L(A) - 1)^2] = \mathbb{E}_{A \sim f_0}[L^2(A)] - 1$, we only need to show

$$
\mathbb{E}_{A \sim f_0}[L^2(A)] \leq 1 + o(1). \tag{A.47}
$$

We now show \((A.47)\). Write $L(A) = \mathbb{E}_z [g(A, z)]$, where $g(A, z)$ is the term inside the expectation in \((A.46)\). Let $\{\tilde{z}_i\}_{i \in \mathcal{N}_{\kappa_0}}$ be an independent copy of $\{z_i\}_{i \in \mathcal{N}_{\kappa_0}}$. Then,

$$
\mathbb{E}_{A \sim f_0}[L^2(A)] = \mathbb{E}_{A \sim f_0} \left[ \mathbb{E}_z [g(A, z)] \cdot \mathbb{E}_{\tilde{z}} [g(A, \tilde{z})] \right] = \mathbb{E}_{z, \tilde{z}} \left\{ \mathbb{E}_{A \sim f_0}[g(A, z)g(A, \tilde{z})] \right\}. \tag{A.48}
$$

Using the expression of $g(A, z)$ in \((A.46)\), we have

$$
g(A, z) g(A, \tilde{z}) = \prod_{i,j \in \mathcal{N}_{\kappa_0}, i < j} \left[ (1 + z'_i G z_j)(1 + \tilde{z}'_i G \tilde{z}_j) \right]^{A_{ij}} \left[ \frac{1 - \theta_i \theta_j (1 + z'_i G z_j)(1 + \tilde{z}'_i G \tilde{z}_j)}{(1 + \theta_i \theta_j)^2} \right]^{1 - A_{ij}}.
$$

Here $A_{ij}$’s are independent Bernoulli variables, where $\mathbb{P}(A_{ij} = 1) = \theta_i \theta_j$. If we take expectation with respect to $A_{ij}$ in each term of the product, it gives

$$
\begin{align*}
\prod_{i,j \in \mathcal{N}_{\kappa_0}, i < j} \left( 1 + z'_i G z_j \right) \cdot \mathbb{P}(A_{ij} = 1) &= \prod_{i,j \in \mathcal{N}_{\kappa_0}, i < j} \left( 1 + z'_i G z_j \right) \cdot \mathbb{P}(A_{ij} = 1) \\
&= \frac{1 - \theta_i \theta_j (1 + z'_i G z_j)(1 + \tilde{z}'_i G \tilde{z}_j)}{(1 - \theta_i \theta_j)^2}.
\end{align*}
$$

As a result,

$$
\mathbb{E}_{A \sim f_0}[g(A, z)g(A, \tilde{z})] = \prod_{i,j \in \mathcal{N}_{\kappa_0}, i < j} \left[ 1 + \frac{\theta_i \theta_j}{1 - \theta_i \theta_j} (z'_i G z_j)(\tilde{z}'_i G \tilde{z}_j) \right] \\
\leq \exp \left( \sum_{i,j \in \mathcal{N}_{\kappa_0}, i < j} \frac{\theta_i \theta_j}{1 - \theta_i \theta_j} (z'_i G z_j)(\tilde{z}'_i G \tilde{z}_j) \right),
$$

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where the second line is from the inequality that $1 + x \leq e^x$ for all $x \in \mathbb{R}$. We plug it into (A.48). Then, to show (A.47), it suffices to show that
\[
E_{z, \bar{z}}[\exp(Y)] \leq 1 + o(1), \quad \text{where} \quad Y = \sum_{i, j} \frac{\theta_i \theta_j}{1 - \theta_i \theta_j} (z_i' G z_j)(\bar{z}_i' G \bar{z}_j). \quad (A.49)
\]

We now show (A.49). We drop the subscript $\{i, j\} \subseteq \tilde{N}_K$ in most places to make notations simpler. The matrix $G$ can be re-written as
\[
G = \frac{1 - b_n}{1 + mb_n} \left[(m + 1)I_{m+1} - (m + 1)'_{m+1} \right].
\]
Additionally, $z_i' 1_{m+1} \equiv 0$. It follows that $z_i' G z_j = \frac{(m + 1)(1 - b_n)}{1 + mb_n} (z_i' \bar{z}_j)$. As a result,
\[
Y = \frac{(m + 1)^2(1 - b_n)^2}{(1 + mb_n)^2} \sum_{i, j} \frac{\theta_i \theta_j}{1 - \theta_i \theta_j} (z_i' \bar{z}_j)(\bar{z}_i' z_j)
\]
\[
= \frac{1}{(m + 1)^2} \sum_{1 \leq k, \ell \leq m + 1} \frac{(m + 1)^4(1 - b_n)^2}{(1 + mb_n)^2} \sum_{i, j} \frac{\theta_i \theta_j}{1 - \theta_i \theta_j} z_i(k) z_i(\ell) \bar{z}_i(k) \bar{z}_i(\ell).
\]

By Jensen’s inequality, $\exp(Y) = \exp\left(\frac{1}{(m + 1)^2} \sum_{k, \ell} Y_{k\ell} \right) \leq \frac{1}{(m + 1)^2} \sum_{k, \ell} \exp(Y_{k\ell})$. It follows that
\[
E_{z, \bar{z}}[\exp(Y)] \leq \frac{1}{(m + 1)^2} \sum_{1 \leq k, \ell \leq m + 1} E_{z, \bar{z}}[\exp(Y_{k\ell})] \leq \max_{1 \leq k, \ell \leq m + 1} E_{z, \bar{z}}[\exp(Y_{k\ell})].
\]
Therefore, to show (A.49), it suffices to show that, for each $1 \leq k, \ell \leq m + 1$,
\[
E_{z, \bar{z}}[\exp(Y_{k\ell})] \leq 1 + o(1). \quad (A.50)
\]

Fix $(k, \ell)$. We now show (A.50). Define $\sigma_i = z_i(k) \bar{z}_i(\ell)$, for all $i \in \tilde{N}_K$. Then,
\[
Y_{k\ell} = \frac{(m + 1)^4(1 - b_n)^2}{(1 + mb_n)^2} \sum_{i, j} \frac{\theta_i \theta_j}{1 - \theta_i \theta_j} \sigma_i \sigma_j
\]
\[
= \frac{(m + 1)^4(1 - b_n)^2}{(1 + mb_n)^2} \sum_{i, j} \left(\theta_i^2 \theta_j^2 \sigma_i \sigma_j \right)
\]
\[
= \sum_{s = 1}^{\infty} (1 - \theta_{\max}^2)^2 \exp\left(\sum_{s = 1}^{\infty} w_s X_s \right) \leq \sum_{s = 1}^{\infty} w_s \exp(X_s).
\]

In the second line above, we used the Taylor expansion $\theta_i^2 \theta_j^2 = \sum_{s = 1}^{\infty} \theta_i^s \theta_j^s$. It is valid because $|\theta_i \theta_j| \leq \theta_{\max}^2 = o(1)$. In the third line, we have switched the order of summation. It is valid because the double sum is finite if we take the absolute value of each summand. The numbers $\{w_s\}_{s = 1}^{\infty}$ satisfy that $\sum_{s = 1}^{\infty} w_s = 1$. By Jensen’s inequality,
\[
\exp(Y_{k\ell}) = \exp\left(\sum_{s = 1}^{\infty} w_s X_s \right) \leq \sum_{s = 1}^{\infty} w_s \exp(X_s).
\]

By Fatou’s lemma,
\[
E_{\sigma}[\exp(Y_{k\ell})] \leq \sum_{s = 1}^{\infty} w_s \cdot E_{\sigma}[\exp(X_s)] \leq \max_{s \geq 1} E_{\sigma}[\exp(X_s)] \quad (A.51)
\]
It remains to study $X_s$. Note that

$$X_s = \frac{(m+1)^4(1-b_n)^2}{(1+mb_n)^2(1-\theta_{\text{max}}^2)\theta_{\text{max}}^2} \sum_{i<j} \theta_i^4 \theta_j^4 \sigma_i \sigma_j$$

$$= \frac{(m+1)^4(1-b_n)^2}{(1+mb_n)^2(1-\theta_{\text{max}}^2)\theta_{\text{max}}^2} \left[ \frac{1}{2} \sum_{i,j} \theta_i^4 \theta_j^4 \sigma_i \sigma_j - \sum_i \theta_i^2 \sigma_i^2 \right]$$

$$\leq \frac{(m+1)^4(1-b_n)^2}{2(1+mb_n)^2(1-\theta_{\text{max}}^2)\theta_{\text{max}}^2} \left( \sum_i \theta_i^4 \sigma_i \right)^2.$$

Note that the summation is over $i \in \hat{N}_{K_0}$. Let $\theta^* \in \mathbb{R}^n$ be defined by $\theta_i^* = \theta_i \cdot 1\{i \in \hat{N}_{K_0}\}$. Since $1 - \theta_{\text{max}}^2 \geq 1/2$ and $\|\theta^*\|^2 \leq \theta_{\text{max}}^2 \|\theta^*\|^2 \leq \theta_{\text{max}}^2 \|\theta\|^2$, we have

$$X_s \leq \frac{a_0(1-b_n)^2\|\theta\|^2}{\|\theta^*\|^2} \left( \sum_i \theta_i^4 \sigma_i \right), \quad (A.52)$$

for a constant $a_0 > 0$. We apply Hoeffding’s inequality to get that, for all $t > 0$,

$$\mathbb{P} \left( \left| \sum_i \theta_i^4 \sigma_i \right| > t \right) \leq 2 \exp \left( -\frac{t^2}{2 \sum_i \theta_i^2} \right) = 2 \exp \left( -\frac{t^2}{2 \|\theta\|^2} \right). \quad (A.53)$$

For any nonnegative variable $X$, using the formula of integration by part, we can derive that $\mathbb{E} [\exp(aX)] = 1 + a \int_0^\infty \exp(at) \mathbb{P}(X > t)dt$. As a result,

$$\mathbb{E}_\sigma [\exp(X_s)] \leq \mathbb{E}_\sigma \left\{ \exp \left[ \frac{a_0(1-b_n)^2\|\theta\|^2}{\|\theta^*\|^2} \left( \sum_i \theta_i^4 \sigma_i \right) \right] \right\}$$

$$= 1 + \frac{a_0\|\theta\|^2(1-b_n)^2}{\|\theta^*\|^2} \int_0^\infty \exp \left( -\frac{a_0\|\theta\|^2(1-b_n)^2}{\|\theta^*\|^2} t \right) \mathbb{P} \left\{ \left( \sum_i \theta_i^4 \sigma_i \right)^2 > t \right\} dt$$

$$\leq 1 + \frac{a_0\|\theta\|^2(1-b_n)^2}{\|\theta^*\|^2} \int_0^\infty \exp \left( -\frac{a_0\|\theta\|^2(1-b_n)^2}{\|\theta^*\|^2} t \right) \exp \left( -\frac{t}{2 \|\theta\|^2} \right) dt$$

$$= 1 + \frac{a_0\|\theta\|^2(1-b_n)^2}{\|\theta^*\|^2} \int_0^\infty \exp \left( -1 - 2a_0\|\theta\|^2(1-b_n)^2 \frac{t}{2 \|\theta\|^2} \right) dt$$

$$= 1 + \frac{2a_0\|\theta\|^2(1-b_n)^2}{1 - 2a_0\|\theta\|^2(1-b_n)^2}.$$ 

The right hand side does not depend on $s$, so the same bound holds for $\max_{s \geq 1} \{ \mathbb{E}_\sigma [\exp(X_s)] \}$. When $\|\theta\|^2(1-b)^2 \to 0$, this upper bound is $1 + o(1)$. Plugging it into (A.51) gives (A.50). Then, the second claim follows. \hfill \Box

### A.3 Proof of Theorem 2.5

We show a slightly stronger argument. Given $1 \leq K_1 < K_2 \leq m_0$, let $\mathcal{M}_n(K_1, K_2, a_n)$ be the sub-collection of $\mathcal{M}_n(m_0, a_n)$ corresponding to $K_1 \leq K \leq K_2$. Note that

$$\inf \left\{ \sup_{\mathcal{M}_n(m_0, a_n)} \mathbb{P}(\hat{K} \neq K) \right\} \geq \inf \left\{ \sup_{\mathcal{M}_n(K_1, K_2, a_n)} \mathbb{P}(\hat{K} \neq K) \right\}.$$ 

It suffices to lower bound the right hand side.
Fix an arbitrary DCBM model with \((K_1 - 1)\) communities. For each \(1 \leq m \leq K_2 - K_1 + 1\), we use (2.5) and (2.6) to construct a random-label DCBM with \((K_1 - 1 + m)\) communities, where \(b_n = 1 - c\|\theta\|^2 - a_n\), for a constant \(c\) to be decided. Let \(P_k\) denote the probability measure associated with the \(k\)-community random-label DCBM, for \(K_1 \leq k \leq K_2\). By Theorem 2.4, we can choose an appropriately small constant \(c\) such that \(|\lambda_K|/\sqrt{K_1} \geq a_n\) with probability \(1 - o(n^{-1})\), under each \(P_k\). Additionally, using a proof similar to that of (A.34), we can show that (A.55)-(A.56) are satisfied with probability \(1 - o(n^{-1})\). Therefore, under each \(P_k\), the realization of \((\Theta, H, P)\) belongs to \(M_n(K_1, K_2, a_n)\) with probability \(1 - o(n^{-1})\). Then, for any \(K\),

\[
\sup_{M_n(K_1, K_2, a_n)} P(\hat{K} \neq K) \geq \max_{K_1 \leq k \leq K_2} P_k(\hat{K} \neq K) + o(n^{-1}). \tag{A.54}
\]

To bound the right hand side of (A.54), consider a multi-hypothesis testing problem: Given an adjacency matrix \(A\), choose one out of the models \(\{P_k\}_{K_1 \leq k \leq K_2}\). For any test \(\psi\), define

\[
\bar{p}(\psi) = \frac{1}{K_2 - K_1 + 1} \sum_{k=K_1}^{K_2} P_k(\psi \neq k).
\]

We apply [35, Proposition 2.4]. It yields that

\[
\frac{1}{K_2 - K_1} \sum_{k=K_1+1}^{K_2} \chi^2(P_k, P_{K_1}) \leq \alpha^* \quad \Rightarrow \quad \inf_{\psi} \bar{p}(\psi) \geq \sup_{\alpha < \tau < 1} \left\{ \frac{\tau(K_2 - K_1)}{1 + \tau(K_2 - K_1)} [1 - \tau(\alpha^* + 1)] \right\}.
\]

We have shown in Theorem 2.4 that \(\alpha^* = o(1)\). By letting \(\tau = 1/2\) in the above, we immediately find that

\[
\inf_{\psi} \bar{p}(\psi) \geq \frac{K_2 - K_1}{2 + (K_2 - K_1)} \left( 1 - \frac{1 + o(1)}{2} \right) \geq 1/6 + o(1). \tag{A.55}
\]

Now, given any estimator \(\hat{K}\), it defines a test \(\psi_{\hat{K}}\), where \(\psi_{\hat{K}} = \hat{K}\) if \(K_1 \leq \hat{K} \leq K_2\) and \(\psi_{\hat{K}} = K_1\) otherwise. It is easy to see that

\[
\bar{p}(\psi_{\hat{K}}) \leq \max_{K_1 \leq k \leq K_2} P_k(\hat{K} \neq k). \tag{A.56}
\]

Combining (A.55)-(A.56) gives that \(\max_{K_1 \leq k \leq K_2} P_k(\hat{K} \neq k) \geq 1/6 + o(1)\). We plug it into (A.54) to get the claim.

\[\square\]

**B Proof of results in Section 3**

**B.1 Proof of Lemma 3.1**

By definition of \(H\), we have \(\Pi^2 \Theta = \|\theta\|^2 \cdot H^2\). As a result, the matrix \(U = \|\theta\|^{-1} \Pi H^{-1}\) satisfies that \(U'U = I_K\). We now write

\[
\Omega = \Theta \Pi H \Pi' \Theta = \|\theta\|^2 \cdot U' \cdot (H \Pi H') \cdot U, \quad \text{where} \quad U'U = I_K.
\]

Since \(U\) contains orthonormal columns, the nonzero eigenvalues of \(\Omega\) are the nonzero eigenvalues of \(\|\theta\|^2 (H \Pi H')\). This proves that \(\lambda_k = \|\theta\|^2 \mu_k\). Furthermore, there is a one-to-one correspondence between the eigenvectors of \(\Omega\) and the eigenvectors of \(H \Pi H'\) through

\[
[\xi_1, \xi_2, \ldots, \xi_k] = U[\eta_1, \eta_2, \ldots, \eta_K].
\]

It follows that \(\xi_k = U \eta_k = \|\theta\|^{-1} \Pi H^{-1} \eta_k\). This proves the claim about \(\xi_k\). We can multiply both sides of the equation \(\xi_k = U \eta_k\) by \(\|\theta\|^{-1} H^{-1} \Pi' \Theta\) from the left. It yields that

\[
\|\theta\|^{-1} H^{-1} \Pi' \Theta \xi_k = (\|\theta\|^{-1} H^{-1} \Pi' \Theta)(\|\theta\|^{-1} \Theta H^{-1} \eta_k)
\]

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\[ = \|\theta\|^{-2} H^{-1} (\Pi' \Theta^2 \Pi) H^{-1} \eta_k = \eta_k. \]

This proves the claim about \( \eta_k \). Last, the condition \([2.4]\) ensures that the multiplicity of \( \mu_1 \) is 1 and that \( \mu_1 \) is a strictly positive vector. It follows that \( \lambda_1 \) has a multiplicity of 1. Note that \( \xi_k = U \eta_k \) implies
\[
\xi_1(i) = \|\theta\|^{-1} \theta_i \sum_{k=1}^{K} \sum_{1 \leq i \leq K} H_{kk}^{-1} \pi_i(k) \eta_1(k) \geq \|\theta\|^{-1} \theta_i \min_{1 \leq k \leq K} \{ H_{kk}^{-1} \eta_1(k) \}.
\]

Since \( \eta_1 \) is a positive vector and \( H \) is a positive diagonal matrix, we conclude that all entries of \( \xi_1 \) are positive.

**B.2 Proof of Lemma 3.2**

We fix an arbitrary \((K - 1) \times (K - 1)\) orthogonal matrix \( \Gamma \) and drop “\( \Gamma \)” in the notations \( \eta_k, \xi_k, r_i, v_k \). By Definition 3.2
\[
[\eta_1, \eta_2, \ldots, \eta_K] = [\eta_1, \eta_2^*, \ldots, \eta_K^*] [1 \quad \Gamma], \quad [\xi_1, \xi_2, \ldots, \xi_K] = [\xi_1, \xi_2^*, \ldots, \xi_K^*] [1 \quad \Gamma].
\]

Here, \( \eta_1, \eta_2^*, \ldots, \eta_K^* \) is a particular candidate of the eigenvectors of \( HPH \) and \( \xi_1, \xi_2^*, \ldots, \xi_K^* \) is linked to \( \eta_1, \eta_2^*, \ldots, \eta_K^* \) through
\[
[\xi_1, \xi_2^*, \ldots, \xi_K^*] = \|\theta\|^{-1} \Theta \Pi H^{-1}[\eta_1, \eta_2^*, \ldots, \eta_K^*].
\]

It follows immediately that
\[
[\xi_1, \xi_2, \ldots, \xi_K] = \|\theta\|^{-1} \Theta \Pi H^{-1}[\eta_1, \eta_2, \ldots, \eta_K]. \quad \tag{B.57}
\]

As a result, for any true community \( \mathcal{N}_k \),
\[
\xi_1(i) = \theta_i / (\|\theta\| H_{kk}) \cdot \eta_1(k), \quad \text{for all } i \in \mathcal{N}_k.
\]

We plug it into the definition of \( R^{(m)} \) to get that for each \( i \in \mathcal{N}_k \) and \( 1 \leq \ell \leq m - 1 \),
\[
R^{(m)}(i, \ell) = \frac{\xi_{\ell+1}(i)}{\xi_1(i)} = \frac{\theta_i / (\|\theta\| H_{kk}) \cdot \eta_{\ell+1}(k)}{\theta_i / (\|\theta\| H_{kk}) \cdot \eta_1(k)} = \frac{\eta_{\ell+1}(k)}{\eta_1(k)} = V^{(m)}(k, \ell).
\]

It follows that \( v_{i}^{(m)} = v_{k}^{(m)} \) for each \( i \in \mathcal{N}_k \). \( \square \)

**B.3 Proof of Lemma 3.3**

The matrix \( V^{(K)}(\Gamma) \) was studied in \([14] [15]\). Since the pairwise distances for rows in \( V^{(K)}(\Gamma) \) are invariant of \( \Gamma \), the quantity \( d_K(V^{(K)}(\Gamma)) \) does not change with \( \Gamma \) either. Using Lemma B.3 of \([14]\), we immediately know that \( d_K(V^{(K)}(\Gamma)) \geq \sqrt{2} \).

Below, we fix \( 1 < m < K \) and a \((K - 1) \times (K - 1)\) orthogonal matrix \( \Gamma \), and study \( d_m(V^{(m)}(\Gamma)) \). For notation simplicity, we drop “\( \Gamma \)” when there is no confusion.

We apply a bottom up pruning procedure (same as in Definition 3.1) to \( V^{(m)}(\Gamma) \). First, we find two rows \( v_k^{(m)} \) and \( v_{\ell}^{(m)} \) that attain the minimum pairwise distance (if there is a tie, pick the first pair in the lexicographical order) and change the \( \ell \)th row to \( v_{k}^{(m)} \) (suppose \( k < \ell \)). Denote the resulting matrix by \( V^{(m,K-1)}(\Gamma) \). Next, we consider the rows of \( V^{(m,K-1)}(\Gamma) \) and similarly find two rows attaining the minimum pairwise distance and replace one row by the other. Denote the resulting matrix by \( V^{(m,K-2)}(\Gamma) \). We repeat these steps to get a sequence of matrices:
\[
V^{(m,K)}, V^{(m,K-1)}, V^{(m,K-2)}, \ldots, V^{(m,2)}, V^{(m,1)}.
\]
where \( V^{(m,K)} = V^{(m)} \) and for each \( 1 \leq k \leq K \), \( V^{(m,k)} \) has at most \( k \) distinct rows. Comparing it with the definition of \( d_k(V^{(m)}) \) (see Definition 3.1), we find that \( V^{(m,k-1)} \) differs from \( V^{(m,k)} \) in only 1 row, and the difference on this row is a vector whose Euclidean norm is exactly \( d_k(V^{(m)}) \). As a result,

\[
\|V^{(m,k)} - V^{(m,k-1)}\| = d_k(V^{(m)}), \quad 2 \leq k \leq K.
\]  

(B.58)

By triangle inequality and the fact that \( d_k(V^{(m)}) \leq d_{k-1}(V^{(m)}) \), we have

\[
\|V^{(m,K)} - V^{(m,m-1)}\| \leq \sum_{k=m}^{K} d_k(V^{(m)}) \leq (K - m + 1) \cdot d_m(V^{(m)}).
\]

To show the claim, it suffices to show that

\[
\|V^{(m,K)} - V^{(m,m-1)}\| \geq C.
\]  

(B.59)

We now show (B.59). Introduce two matrices

\[
V^{(m,K)}_* = [1_K, V^{(m,K)}], \quad V^{(m,m-1)}_* = [1_K, V^{(m,m-1)}],
\]

where \( 1_K \) is the \( K \)-dimensional vector of 1s. Adding the vector \( 1_K \) as the first column changes neither the number of distinct rows nor pairwise distances among rows. Additionally,

\[
\|V^{(m,K)} - V^{(m,m-1)}\| = \|V^{(m,K)}_* - V^{(m,m-1)}_*\|.
\]  

(B.60)

Let \( \sigma_m(U) \) denote the \( m \)-th singular value of a matrix \( U \). Since \( V^{(m,m-1)}_* \) has at most \((m-1)\) distinct rows, its rank is at most \((m-1)\). As a result,

\[
\sigma_m(V^{(m,m-1)}_*) = 0.
\]  

(B.61)

We then study \( \sigma_m(V^{(m,K)}_*) \). Note that

\[
V^{(m,K)}_* = [1_K, V^{(m)}] = \begin{bmatrix} 1 & v_1^{(m)} \\ \vdots & \vdots \\ 1 & v_K^{(m)} \end{bmatrix} = [\text{diag}(\eta_1)]^{-1} \cdot [\eta_1, \eta_2(\Gamma), \ldots, \eta_m(\Gamma)],
\]  

(B.62)

where \( \eta_1, \eta_2(\Gamma), \ldots, \eta_K(\Gamma) \) is one choice of eigenvectors of \( HPH \) indexed by \( \Gamma \) (see Definition 3.2) and \( \text{diag}(\eta_1) \) is the diagonal matrix whose diagonal entries are from \( \eta_1 \). Write for short \( Q = [\eta_1, \eta_2(\Gamma), \ldots, \eta_m(\Gamma)] \). We have

\[
(V^{(m,K)}_*)^t V^{(m,K)}_* = Q^t [\text{diag}(\eta_1)]^{-2} Q.
\]

(B.63)

By the last item of (2.4) and that \( \|\eta_1\| = 1 \), we conclude that \( \eta_1(k) \approx 1/\sqrt{K} \) for all \( 1 \leq k \leq K \). In particular, there exists a constant \( c > 0 \) such that \( (\text{diag}(\eta_1)]^{-2} - cI_K \) is a positive semi-definite matrix. It follows that \( (Q^t [\text{diag}(\eta_1)]^{-2} Q - cQ^t Q) \) is a positive semi-definite matrix. Therefore,

\[
\lambda_m((V^{(m,K)}_*)^t V^{(m,K)}_*) \geq \lambda_m(cQ^t Q) = c \cdot \lambda_m(Q^t Q),
\]  

(B.64)

where \( \lambda_m(\cdot) \) denotes the \( m \)-th largest eigenvalue of a symmetric matrix. By (3.7), for some pre-specified choice of eigenvectors, \( \eta_1^*, \eta_2^*, \ldots, \eta_K^* \), of \( HPH \),

\[
Q^t Q = I_m.
\]

Note that \([\eta_1^*, \eta_2^*, \ldots, \eta_K^*] \) and \( \text{diag}(1, \Gamma) \) are both \( K \times K \) orthogonal matrices. Then, their product is also an orthogonal matrix, and the columns in \( Q \) are orthonormal. It follows that

\[
Q^t Q = I_m.
\]
This shows that the right hand side of (B.63) is equal to $c$. The left hand side of (B.63) is equal to $\sigma_m^2(V_{s(m,K)})$. It follows that

$$\sigma_m(V_{s(m,K)}) \ge C.$$  \hspace{1cm} (B.64)

We now combine (B.61) and (B.64), and apply Weyl’s inequality for singular values \cite[Corollary 7.3.5]{horn}. It gives

$$C \le \sigma_m(V_{s(m,K)}) - \sigma_m(V_{s(m,m-1)}) \le \|V_{s(m,K)} - V_{s(m,m-1)}\|.$$  

Combining it with (B.60) gives (B.59). The claim follows immediately. \hfill $\square$

**Remark.** The proof of Theorem 2.2 uses $\max_{1 \le k \le K} \|v_{k}(m)\| \le C$, and we prove this claim here. Note that $v_{k}(m)(\Gamma)$ is a sub-vector of the $k$th row of $V_{s(m,K)}$. In light of (B.62), the row-wise $\ell_2$-norms of $V_{s(m,K)}$ are uniformly bounded by $C\|\text{diag}^{-1}(\eta_1)\|$. We have argued that $\eta_1(k) \approx 1/\sqrt{K} \le C$ for all $1 \le k \le K$. As a result, $\max_{1 \le k \le K} \|v_{k}(m)(\Gamma)\| \le C$.

**B.4 Proof of Lemma 3.4**

Since $\|\hat{r}_i - \tilde{r}_i\| \le \|\hat{r}_i - \hat{r}_i\|$, we only need to show the claim for $m = K$. Write $\hat{r}_i(K)(\Gamma) = \hat{r}_i(\Gamma)$ for short. In the special case of $\Gamma = I_{K-1}$ (i.e., $\eta_k(\Gamma) = \eta_k$ for $2 \le k \le K$, by Definition 3.2), we further write $\hat{r}_i = \hat{r}_i(I_{K-1})$ for short. It is easy to see that

$$\hat{r}_i(\Gamma) = \Gamma \cdot r_i, \quad \text{for any orthogonal matrix } \Gamma \in \mathbb{R}^{(K-1) \times (K-1)}.$$  

It suffices to show that with probability $1 - O(n^{-3})$ there exists a $(K-1) \times (K-1)$ orthogonal matrix $\Gamma$, which may depend on $n$ and $\hat{R}(K)$, such that

$$\max_{1 \le i \le n} \|\hat{r}_i - \Gamma \cdot r_i\| \le C s_n^{-1} \sqrt{\log(n)}.$$  

Such a bound was given by Theorem 4.1 of \cite{15} (also, see Lemma 2.1 of \cite{15} for a special case where $\lambda_2, \ldots, \lambda_K$ are at the same order). \hfill $\square$

**B.5 Proof of Theorem 3.2**

The key of proof is the following lemma, which characterizes the change of the k-means objective under perturbation of cluster assignment. Consider the problem of clustering points $y_1, y_2, \ldots, y_n$ into two disjoint clusters $A$ and $B$. The k-means objective is the residual sum of squares by setting the two cluster centers as the within-cluster means. Now, we move a subset $C$ from cluster $A$ to cluster $B$. The new clusters are $\bar{A} = A \setminus C$ and $\bar{B} = B \cup C$, and the cluster centers are updated accordingly. There is an explicit formula for the change of the k-means objective:

**Lemma B.1.** For any $y_1, y_2, \ldots, y_n \in \mathbb{R}^d$ and subset $M \subset \{1, 2, \ldots, n\}$, define $\bar{y}_M = \frac{1}{|M|} \sum_{i \in M} y_i$. Let $\{1, 2, \ldots, n\} = A \cup B$ be a partition, and let $C$ be a strict subset of $A$. Write $\hat{A} = A \setminus C$ and $\hat{B} = B \cup C$. Define

$$\text{RSS} = \sum_{i \in A} \|y_i - \bar{y}_A\|^2 + \sum_{i \in B} \|y_i - \bar{y}_B\|^2, \quad \text{RSS} = \sum_{i \in \hat{A}} \|y_i - \bar{y}_A\|^2 + \sum_{i \in \hat{B}} \|y_j - \bar{y}_B\|^2.$$  

Then,

$$\text{RSS} - \text{RSS} = \frac{|B||C|}{|B| + |C|} \|\bar{y}_C - \bar{y}_B\|^2 - \frac{|A||C|}{|A| - |C|} \|\bar{y}_C - \bar{y}_A\|^2.$$  

This lemma is proved by elementary calculation, which is relegated to Section D.1. It shows that the change of k-means objective depends on the distances from $\bar{y}_C$ to two previous cluster centers.
We now apply Lemma B.1 to prove the claim. For notation simplicity, we drop “$\Gamma$” and omit the superscript $m$, i.e., we write $r_i^{(m)}(\Gamma) = r_i$ and $v_k^{(m)}(\Gamma) = v_k$. By Lemma 3.2 and the condition (2.2),

- The $n$ points $r_1, r_2, \ldots, r_n$ take $K$ distinct values, $v_1, \ldots, v_K$.
- The minimum pairwise distance of $v_1, v_2, \ldots, v_K$ is defined as $d_K(V) > 0$.
- For each $v_k$, there are at least $a_0n$ points, corresponding to nodes in community $\mathcal{N}_k$, that are equal to $v_k$, where $a_0 > 0$ is a constant determined by condition (2.2).

First, we show that any optimal solution of the k-means clustering on $\{r_1, r_2, \ldots, r_n\}$ satisfies NSP. We prove by contradiction. If this is not true, there must exist a community $\mathcal{N}_k$ and two clusters, say, $S_1$ and $S_2$, such that $\mathcal{N}_k \cap S_1 \neq \emptyset$ and $\mathcal{N}_k \cap S_2 \neq \emptyset$. Note that we have either $S_1 \setminus \mathcal{N}_k \neq \emptyset$ or $S_2 \setminus \mathcal{N}_k \neq \emptyset$ (if both $S_1$ and $S_2$ are contained in $\mathcal{N}_k$, then we can combine these two clusters and construct another cluster assignment with a smaller residual sum of squares, which conflicts with the optimality of the solution). Without loss of generality, we assume $S_1 \setminus \mathcal{N}_k \neq \emptyset$. We now move an arbitrary $r_i \in \mathcal{N}_k \setminus S_1$ to $S_2$ and update the cluster centers (i.e., within-cluster means) accordingly. Let $RSS$ and $\tilde{RSS}$ be the respective k-means objective before and after the change. We apply Lemma B.1 to get that

$$\tilde{RSS} - RSS = \frac{|S_2|}{|S_2| + 1} \|r_i - c_2\|^2 - \frac{|S_1|}{|S_1| - 1} \|r_i - c_1\|^2. \quad (B.65)$$

Since $i$ is clustered to $S_1$ in the optimal solution, it must be true that $\|r_i - c_1\| \leq \|r_i - c_2\|$, which further implies that $\|v_k - c_1\| \leq \|v_k - c_2\|$. At the same time, if we take any $j \in \mathcal{N}_k \setminus S_2$, we can similarly derive that $\|v_k - c_2\| \leq \|v_k - c_1\|$. Combining the above gives $\|v_k - c_1\| = \|v_k - c_2\|$. It follows that $\|r_i - c_1\| = \|r_i - c_2\|$.

We immediately see that

$$\tilde{RSS} - RSS = \left( \frac{|S_2|}{|S_2| + 1} - \frac{|S_1|}{|S_1| - 1} \right) \|r_i - c_1\|^2 = -\frac{|S_1| + |S_2|}{(|S_2| + 1)(|S_1| - 1)} \|r_i - c_1\|^2.$$

The optimality of k-means solutions ensures that $\tilde{RSS} - RSS \geq 0$. Therefore, the above equality is possible only if $\|r_i - c_1\| = 0$. However, $\|r_i - c_1\| = 0$ implies $c_1 = c_2$, which is impossible.

Second, define $g(r_i; c_1, c_2, \ldots, c_m) \equiv d_2(r_i; c_1, \ldots, c_m) - d_1(r_i; c_1, \ldots, c_m)$, which is the gap between the distances from $r_i$ to the closest and second closest cluster centers. We aim to show that $g(r_i; c_1, c_2, \ldots, c_m)$ has a uniform lower bound for all $1 \leq i \leq n$. Fix $i$. Without loss of generality, we assume $c_1$ and $c_2$ are the cluster centers closest and second closest to $r_i$. Then, $i$ is clustered to $S_1$. Suppose $i \in \mathcal{N}_k$. The NSP we proved above implies that

$\mathcal{N}_k \subset S_1$.

Again, by NSP, there are only two possible cases: (a) $S_1 = \mathcal{N}_k$, and (b) $S_1$ is the union of $\mathcal{N}_k$ and some other true communities.

In case (a), we immediately have $c_1 = v_k$. It follows that $\|r_i - c_1\| = \|v_k - c_1\| = 0$.

Furthermore, for any $j \in S_2$, $r_j$ equals to some $v_{\ell}$ that is distinct from $v_k$. Therefore,

$$\|r_i - c_2\| = \|v_k - c_2\| \geq \min_{j \in S_2} \|v_k - r_j\| \geq \min_{\ell \neq k} \|v_k - v_{\ell}\| = d_K(V).$$
As a result,

\[ g(r_i; c_1, c_2, \ldots, c_m) = \| r_i - c_2 \| - \| r_i - c_1 \| = \| r_i - c_2 \| \geq d_K(V). \]

The proves the claim in case (a).

In case (b), we consider moving \( N_k \) from \( S_1 \) to \( S_2 \), and let \( RSS \) and \( \tilde{RSS} \) denote the respective k-means objective before and after the change. Applying Lemma [B.1], we obtain

\[
\tilde{RSS} - RSS = \frac{|S_2| |N_k|}{|S_2| + |N_k|} \| v_k - c_2 \|^2 - \frac{|S_1| |N_k|}{|S_1| - |N_k|} \| v_k - c_1 \|^2.
\]

Let \( \Delta = \| v_k - c_2 \|^2 - \| v_k - c_1 \|^2 \). By direct calculations,

\[
\tilde{RSS} - RSS = \frac{|S_2| |N_k|}{|S_2| + |N_k|} \Delta - \frac{|N_k|^2 (|S_1| + |S_2|)}{(|S_2| + |N_k|)(|S_1| - |N_k|)} \| v_k - c_1 \|^2.
\]

The optimality of k-means solutions implies that \( \tilde{RSS} \geq RSS \). It follows that

\[
\Delta \geq \frac{|N_k|(|S_1| + |S_2|)}{|S_2||S_1| - |N_k|} \| v_k - c_1 \|^2.
\]

Note that \( |N_k| \geq a_0 n \), \( |S_1| - |N_k| \leq n \), and \( \frac{|S_1| + |S_2|}{|S_2|} \geq 1 \). It is seen that \( \frac{|N_k|(|S_1| + |S_2|)}{|S_2||S_1| - |N_k|} \geq a_0 \). As a result,

\[
\| v_k - c_2 \|^2 - \| v_k - c_1 \|^2 = \Delta \geq a_0 \| v_k - c_1 \|^2.
\]

It implies that \( \| v_k - c_2 \|^2 \geq (1 + a_0) \| v_k - c_1 \|^2 \), i.e.,

\[
\| v_k - c_2 \| - \| v_k - c_1 \| \geq (\sqrt{1 + a_0} - 1) \| v_k - c_1 \|.
\]

We then derive a lower bound on \( \| v_k - c_1 \| \). Here, \( c_j \) is the mean of \( r_i \)'s in \( S_1 \). For any \( j \in S_1 \setminus N_k \), \( r_j \) equals to some \( v_\ell \) that is distinct from \( v_k \). As a result, \( \| v_k - r_j \| = \min_{\ell \neq k} \| v_k - v_\ell \| \geq d_K(V) \), for all \( j \in S_1 \setminus N_k \). It follows that

\[
\| v_k - c_1 \| = \| v_k - \left( \frac{|N_k|}{|S_1|} v_k + \frac{1}{|S_1|} \sum_{j \in S_1 \setminus N_k} r_j \right) \|
\]

\[
= \left\| \frac{1}{|S_1|} \sum_{j \in S_1 \setminus N_k} (r_j - v_k) \right\|
\]

\[
= \frac{|S_1 \setminus N_k|}{|S_1|} \left\| \left( \frac{1}{|S_1 \setminus N_k|} \sum_{j \in S_1 \setminus N_k} r_j \right) - v_k \right\|
\]

\[
\geq \frac{|S_1 \setminus N_k|}{|S_1|} \min_{j \in S_1 \setminus N_k} \| r_j - v_k \|
\]

\[
\geq a_0 \cdot d_K(V), \quad \text{(B.68)}
\]

where in the last inequality we have used \( |S_1| \leq n \) and \( |S_1 \setminus N_k| \geq a_0 n \) (because \( S_1 \) is the union of \( N_k \) and at least one other community). Combining (B.67) and (B.68) gives

\[
g(r_i; c_1, c_2, \ldots, c_m) \geq a_0 (\sqrt{1 + a_0} - 1) d_K(V).
\]

This proves the claim in case (b). \qed
B.6 Proof of Theorem 3.1

Write for short $d_m = d_m(U)$ and $\delta = \max_{1 \leq i \leq n} \|\hat{x}_i - x_i\|$. Given any partition $\{1, 2, \ldots, n\} = \bigcup_{k=1}^{m} B_k$ and vectors $b_1, b_2, \ldots, b_m \in \mathbb{R}^d$, define

$$R(B_1, \ldots, B_m; b_1, \ldots, b_m) = n^{-1} \sum_{k=1}^{m} \sum_{i \in B_k} \|x_i - b_k\|^2. \quad (B.69)$$

Fixing $B_1, \ldots, B_m$, the value of $R(B_1, \ldots, B_m; b_1, \ldots, b_m)$ is minimized when $b_k$ is the average of $x_i$'s within each $B_k$. When $b_1, \ldots, b_m$ take these special values, we skip them in the notation. Namely, define

$$R(B_1, \ldots, B_m) = R(B_1, \ldots, B_m; \bar{x}_1, \ldots, \bar{x}_m), \quad \text{where } \bar{x}_k = |B_k|^{-1} \sum_{i \in B_k} x_i. \quad (B.70)$$

We define $\hat{R}(B_1, \ldots, B_m; b_1, \ldots, b_m)$ and $\hat{R}(B_1, \ldots, B_m)$ similarly but replace $x_i$ by $\hat{x}_i$. We shall prove the claim by contradiction. Suppose there is $1 \leq k \leq K$ such that $F_k$ intersects with more than one $\hat{S}_j$. By pigeonhole principle, there exists $j_1$, such that $|F_k \cap \hat{S}_{j_1}| \geq m^{-1} |F_k|$. Let $\hat{S}_{j_2}$ be another cluster that intersects with $F_k$. We have

$$|F_k \cap \hat{S}_{j_1}| \geq m^{-1} \alpha_0 n, \quad F_k \cap \hat{S}_{j_2} \neq \emptyset,$$

Below, we aim to show: There exists $C_1 = C_1(\alpha_0, C_0, m)$ such that

$$\min_{\hat{S}_1, \ldots, \hat{S}_m} R(\hat{S}_1, \ldots, \hat{S}_m) \geq R(\hat{S}_1, \ldots, \hat{S}_m) - C_1 \delta \cdot d_m, \quad (B.71)$$

where the minimum on the left hand side is taken over possible partitions of $\{1, 2, \ldots, n\}$ into $m$ clusters. We also aim to show that there exists $C_2 = C_2(\alpha_0, C_0, m)$ such that we can construct a clustering structure $\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_m$ satisfying that

$$R(\hat{S}_1, \ldots, \hat{S}_m) \leq R(\hat{S}_1, \ldots, \hat{S}_m) - C_2 \cdot d_m^2. \quad (B.72)$$

Combining $(B.71)$ and $(B.72)$ gives

$$R(\hat{S}_1, \ldots, \hat{S}_m) - C_1 \delta \cdot d_m \leq R(\hat{S}_1, \ldots, \hat{S}_m) - C_2 \cdot d_m^2$$

This is impossible if $C_1 \delta \cdot d_m < C_2 \cdot d_m^2$. Hence, we can take

$$c(\alpha_0, C_0, m) < C_2/C_1.$$

There is a contradiction between $(B.71)$ and $(B.72)$ whenever $\delta \leq c \cdot d_m$. The claim follows.

It remains to prove $(B.71)$ and $(B.72)$. Consider $(B.71)$. For an arbitrary cluster structure $B_1, B_2, \ldots, B_m$, let $\hat{R}(B_1, \ldots, B_m), R(B_1, \ldots, B_m), \hat{x}_k$ and $x_k$ be defined as in $(B.70)$. By direct calculations,

$$\hat{x}_i - \hat{x}_k - (x_i - x_k) = \frac{|B_k| - 1}{|B_k|} (\hat{x}_i - x_i) - \frac{1}{|B_k|} \sum_{j \in B_k, j \neq i} (\hat{x}_j - x_j).$$

Since $\|\hat{x}_j - x_j\| \leq \delta$ for all $1 \leq j \leq n$, the above equality implies that $\|\hat{x}_i - \hat{x}_k - (x_i - x_k)\| \leq \delta$. As a result, $\|\hat{x}_i - \hat{x}_k\|^2 \leq \|x_i - x_k\|^2 + 2\delta \|x_i - x_k\| + \delta^2$. It follows that

$$\hat{R}(B_1, \ldots, B_m) \leq R(B_1, \ldots, B_m) + 2\delta n^{-1} \sum_{k=1}^{m} \sum_{i \in B_k} \|x_i - x_k\| + \delta^2$$

$$\leq R(B_1, \ldots, B_m) + 2\delta \sqrt{R(B_1, \ldots, B_m)} + \delta^2$$
where the second line is from the Cauchy-Schwarz inequality. It follows that \( \sqrt{R(B_1, \ldots, B_m)} \leq \sqrt{R(B_1, \ldots, B_m) + \delta} \). We can switch \( R(B_1, \ldots, B_m) \) and \( R(B_1, \ldots, B_m) \) to get a similar inequality. Combining them gives

\[
\sqrt{R(B_1, \ldots, B_m)} - \sqrt{R(B_1, \ldots, B_m) + \delta} \leq \sqrt{R(B_1, \ldots, B_m)} - \sqrt{R(B_1, \ldots, B_m) + \delta}.
\]

This inequality holds for an arbitrary partition \((B_1, B_2, \ldots, B_m)\). We now apply it to \((\hat{S}_1, \ldots, \hat{S}_m)\), which are the clusters obtained from applying k-means on \(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\). We also consider applying k-means on \(x_1, x_2, \ldots, x_n\) and let \(S_1, S_2, \ldots, S_m\) denote the resultant clusters. By optimality of the k-means solutions,

\[
\hat{R}(\hat{S}_1, \ldots, \hat{S}_m) \leq \hat{R}(S_1, \ldots, S_m).
\]

Combining it with (B.73) gives

\[
\sqrt{R(\hat{S}_1, \ldots, \hat{S}_m)} \leq \sqrt{\hat{R}(\hat{S}_1, \ldots, \hat{S}_m) + \delta} \leq \sqrt{R(S_1, \ldots, S_m) + \delta} \leq \sqrt{R(S_1, \ldots, S_m) + 2\delta}.
\]

Since \(\max_{1 \leq i \leq n} \|x_i\| \leq C_0 \cdot d_m\), we can easily see that \(R(S_1, \ldots, S_m) \leq C_0^2 \cdot d_m^2\). It follows that, as long as \(\delta \leq d_m/4\),

\[
R(\hat{S}_1, \ldots, \hat{S}_m) \leq R(S_1, \ldots, S_m) + 4\delta \sqrt{R(S_1, \ldots, S_m)} + 4\delta^2 \leq R(S_1, \ldots, S_m) + 4C_0\delta \cdot d_m + \delta \cdot d_m \leq R(S_1, \ldots, S_m) + (4C_0 + 1)\delta \cdot d_m.
\]

As a result,

\[
\min_{\hat{S}_1, \ldots, \hat{S}_m} R(\hat{S}_1, \ldots, \hat{S}_m) = R(S_1, \ldots, S_m) \geq R(\hat{S}_1, \ldots, \hat{S}_m) - (4C_0 + 1)\delta \cdot d_m.
\]

This proves (B.71) for \(C_1 = 4(C_0 + 1)\).

Consider (B.72). Define

\[
w_j = |\hat{S}_j|^{-1} \sum_{i \in \hat{S}_j} x_i, \quad \text{for each} \ 1 \leq j \leq m.
\]

Using the notations in (B.69), (B.70), we write \(R(\hat{S}_1, \ldots, \hat{S}_m) = R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m)\). We aim to construct \(\{(\hat{S}_j, \hat{w}_j)\}_{1 \leq j \leq m}\) such that

\[
R(\hat{S}_1, \ldots, \hat{S}_m, \hat{w}_1, \ldots, \hat{w}_m) \leq R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m) - C_2 \cdot d_m^2.
\]

Since \(R(\hat{S}_1, \ldots, \hat{S}_m) = \min_{b_1, \ldots, b_m} R(\hat{S}_1, \ldots, \hat{S}_m, b_1, \ldots, b_m)\), we immediately have

\[
R(\hat{S}_1, \ldots, \hat{S}_m) \leq R(\hat{S}_1, \ldots, \hat{S}_m, \hat{w}_1, \ldots, \hat{w}_m) \leq R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m) - C_2 \cdot d_m^2.
\]

This proves (B.72).

What remains is to construct \(\{(\hat{S}_j, \hat{w}_j)\}_{j=1}^m\) so that (B.70) is satisfied. Let \(\hat{w}_j = |\hat{S}_j|^{-1} \sum_{i \in \hat{S}_j} \hat{x}_i\), for \(1 \leq j \leq m\). Then, \(\{(\hat{S}_j, \hat{w}_j)\}_{1 \leq j \leq m}\) are the clusters and cluster centers obtained by applying the k-means algorithm on \(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\). The k-means solution guarantees to assign each point to the closest center. Take \(i \in F_k \cap \hat{S}_{j_1}\) and \(i' \in F_k \cap \hat{S}_{j_2}\). It follows that

\[
\|\hat{x}_i - \hat{w}_{j_1}\| \leq \|\hat{x}_i - \hat{w}_{j_2}\|, \quad \|\hat{x}_{i'} - \hat{w}_{j_2}\| \leq \|\hat{x}_{i'} - \hat{w}_{j_1}\|.
\]
Since \( x_i = x_i' = u_k \) and \( \max\{\|\hat{x}_i - x_i\|, \|\hat{x}_i' - x_i\|, \|\hat{w}_{j_1} - w_{j_1}\|, \|\hat{w}_{j_2} - w_{j_2}\|\} \leq \delta \), we have
\[
\|u_k - w_{j_1}\| \leq \|\hat{x}_i - \hat{w}_{j_1}\| + 2\delta \leq \|\hat{x}_i - \hat{w}_{j_2}\| + 2\delta \leq \|u_k - w_{j_2}\| + 4\delta.
\]
Similarly, we can derive that \( \|u_k - w_{j_1}\| \leq \|u_k - w_{j_1}\| + 4\delta \). Combining them gives
\[
\|\|u_k - w_{j_1}\| - \|u_k - w_{j_2}\|\| \leq 4\delta.
\]
This inequality tells us that \( \|u_k - w_{j_1}\| \) and \( \|u_k - w_{j_2}\| \) are sufficiently close. Introduce
\[
C_3 = \frac{m^{-1}\alpha_0}{36 \times 12C_0}.
\]
Below, we consider two cases: \( \|u_k - w_{j_1}\| < C_3 \cdot d_m \) and \( \|u_k - w_{j_1}\| \geq C_3 \cdot d_m \).

In the first case, \( \|u_k - w_{j_1}\| < C_3 \cdot d_m \). The definition of \( d_m \) guarantees that there are \( m \) points from \( \{u_1, u_2, \ldots, u_K\} \) such that their minimum pairwise distance is \( d_m \). Without loss of generality, assume these \( m \) points are \( u_1, u_2, \ldots, u_m \). If \( k \in \{1, 2, \ldots, m\} \), then the distance from \( u_k \) to any of the other \( (m-1) \) points is at least \( d_m \). If \( k \notin \{1, 2, \ldots, m\} \), then \( u_k \) cannot be simultaneously within a distance of \( d_m/2 \) to two or more points of \( u_1, u_2, \ldots, u_m \). In other words, there exists at least \( (m-1) \) points from \( u_1, u_2, \ldots, u_m \) whose distance to \( u_k \) is at least \( d_m/2 \). Combining the above situations, we conclude that there exist \( (m-1) \) points from \( \{u_1, u_2, \ldots, u_K\} \), which we assume to be \( u_1, u_2, \ldots, u_m \) without loss of generality, such that
\[
\min_{1 \leq \ell \neq s \leq m-1} \|u_\ell - u_s\| \geq d_m, \quad \min_{1 \leq \ell \leq m-1} \|u_\ell - u_k\| \geq d_m/2.
\]
We then consider two sub-cases. In the first sub-case, there exists \( \ell \in \{1, 2, \ldots, m-1\} \) such that \( |F_\ell \cap (\tilde{S}_{j_1} \cup \tilde{S}_{j_2})| \geq m^{-1}\alpha_0 n \). Then, at least one of \( \tilde{S}_{j_1} \) and \( \tilde{S}_{j_2} \) contains more than \( (m^{-1}\alpha_0/2)n \) nodes from \( F_\ell \). We only study the situation of \( |F_\ell \cap \tilde{S}_{j_1}| \geq (m^{-1}\alpha_0/2)n \). The proof for the situation of \( |F_\ell \cap \tilde{S}_{j_2}| \geq (m^{-1}\alpha_0/2)n \) is similar and omitted. We modify the clusters and cluster centers \( \{\tilde{S}_j, w_j\}\) as follows:

(i) Combine \( \tilde{S}_{j_2} \setminus F_\ell \) and \( \tilde{S}_{j_1} \) into one cluster and set the cluster center to be \( w_{j_1} \).

(ii) Create a new cluster as \( \tilde{S}_{j_2} \cap F_\ell \) and set the cluster center to be \( u_\ell \).

The other clusters and cluster centers remain unchanged. Namely, we let
\[
\tilde{S}_j = \begin{cases} 
\tilde{S}_{j_1} \cup (\tilde{S}_{j_2} \setminus F_\ell), & \text{if } j = j_1, \\
\tilde{S}_{j_2} \cap F_\ell, & \text{if } j = j_2, \\
\tilde{S}_{j}, & \text{if } j \notin \{j_1, j_2\}, 
\end{cases}
\]
\[
\tilde{w}_j = \begin{cases} 
u_\ell, & \text{if } j = j_2, \\
w_j, & \text{otherwise}. 
\end{cases}
\]
Recall that \( n \cdot R(B_1, \ldots, B_m, b_1, \ldots, b_m) = \sum_{j=1}^m \sum_{i \in B_j} \|x_i - b_j\|^2 \). By direct calculations,
\[
\Delta = n \cdot R(\tilde{S}_1, \ldots, \tilde{S}_m, w_1, \ldots, w_m) - n \cdot R(\tilde{S}_1, \ldots, \tilde{S}_m, \tilde{w}_1, \ldots, \tilde{w}_m)
\]
\[
= \sum_{i \in (\tilde{S}_{j_2} \setminus F_\ell)} (\|x_i - w_{j_2}\|^2 - \|x_i - u_\ell\|^2) - \sum_{i \in (\tilde{S}_{j_2} \setminus F_\ell)} (\|x_i - w_{j_1}\|^2 - \|x_i - w_{j_2}\|^2)
\]
\[
= \Delta_2 - \Delta_1.
\]
Here \( \Delta_1 \) is the increase of the residual sum of squares (RSS) caused by the operation (i) and \( \Delta_2 \) is the decrease of RSS caused by the operation (ii).
\[
\Delta_1 = \sum_{i \in (\tilde{S}_{j_2} \setminus F_\ell)} (\|x_i - w_{j_1}\| - \|x_i - w_{j_2}\|)(\|x_i - w_{j_1}\| + \|x_i - w_{j_2}\|)
\]
\[
\leq \sum_{i \in (\hat{S}_j \setminus F_\ell)} \|w_{j_1} - w_{j_2}\| \cdot (\|x_i - w_{j_1}\| + \|x_i - w_{j_2}\|)
\leq |\hat{S}_{j_2} \setminus F_\ell| \cdot \|w_{j_1} - w_{j_2}\| \cdot 4C_0 \cdot d_m,
\]

where the third line is from the triangle inequality and the last line is because \(\max_{1 \leq j \leq m} \|w_j\| \leq \max_{1 \leq j \leq n} \|x_i\| \leq C_0 \cdot d_m\). Note that \(\|w_{j_1} - w_{j_2}\| \leq \|u_k - w_{j_1}\| + \|u_k - w_{j_2}\|\). We have assumed \(\|u_k - w_{j_1}\| < C_3 \cdot d_m\) in this case. Combining it with \((\text{B.77})\), as long as \(\delta < (C_3/4) \cdot d_m\),

\[
\|w_{j_1} - w_{j_2}\| \leq 2\|u_k - w_{j_1}\| + 4\delta \leq 3C_3 \cdot d_m.
\]

It follows that

\[
\Delta_1 \leq 12C_0C_3 \cdot nd_m^2.
\]

Since \(x_i = u_\ell\) for \(i \in F_\ell\), we immediately have

\[
\Delta_2 = |\hat{S}_{j_2} \cap F_\ell| \cdot \|u_\ell - w_{j_2}\|^2.
\]

We have assumed \(\|u_k - w_{j_1}\| \leq C_3 \cdot d_m\) in this case. Combining it with \((\text{B.77})\) and \((\text{B.78})\) gives

\[
\|u_\ell - w_{j_2}\| \geq \|u_\ell - u_k\| - \|u_k - w_{j_2}\|
\geq \|u_\ell - u_k\| - (\|u_k - w_{j_1}\| + 4\delta)
\geq d_m/2 - (C_3 \cdot d_m + 4\delta).
\]

Recall that \(C_3 = \frac{m^{-1} \alpha_0}{36 \times 12C_0} < 1/12\). Then, as long as \(\delta < (1/48)d_m\), we have \(\|u_\ell - w_{j_2}\| \geq d_m/3\).

It follows that

\[
\Delta_2 \geq (m^{-1} \alpha_0/2)n \cdot (d_m/3)^2 \geq \frac{m^{-1} \alpha_0}{18} \cdot nd_m^2.
\]

As a result,

\[
\Delta = \Delta_2 - \Delta_1 \geq \left(\frac{m^{-1} \alpha_0}{18} - 12C_0C_3\right) \cdot nd_m^2.
\]

We plug in the expression of \(C_3\), the right hand side is \((m^{-1} \alpha_0/36) \cdot nd_m^2\). It follows that

\[
R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m) - R(\hat{S}_1, \ldots, \hat{S}_m, \hat{w}_1, \ldots, \hat{w}_m) \geq \frac{m^{-1} \alpha_0}{36} \cdot d_m^2.
\]

This gives \((\text{B.76})\) in the first sub-case.

In the second sub-case, \(|F_\ell \cap (\hat{S}_j \cup \hat{S}_j^*)| < m^{-1} \alpha_0 n\) for all \(1 \leq \ell \leq m - 1\). For each \(F_\ell\), by pigeonhole principle, there exists at least one \(j \in \{1, 2, \ldots, m\}\) such that \(|F_\ell \cap \hat{S}_j| \geq m^{-1}|F_\ell| \geq m^{-1} \alpha_0 n\). Denote such a \(j\) by \(j_\ast\); if there are multiple indices satisfying the requirement, we pick one of them. This gives

\[
j_1^\ast, j_2^\ast, \ldots, j_{m-1}^\ast \in \{1, 2, \ldots, m\} \setminus \{j_1, j_2\}.
\]

These \((m - 1)\) indices take at most \((m - 2)\) distinct values. By pigeonhole principle, there exist \(1 \leq \ell_1 \neq \ell_2 \leq m - 1\) such that \(j_{\ell_1}^\ast = j_{\ell_2}^\ast = j^\ast\), for some \(j^\ast \notin \{j_1, j_2\}\). Recalling \((\text{B.75})\), we let \(w_{j^\ast}\), denote the average of \(x_{i}\)’s in \(\hat{S}_j^\ast\). Since \(\|u_{\ell_1} - u_{\ell_2}\| \geq d_m\), the point \(w_{j^\ast}\) cannot be simultaneously within a distance of \(d_m/2\) to both \(u_{\ell_1}\) and \(u_{\ell_2}\). Without loss of generality, suppose

\[
\|u_{\ell_1} - w_{j^\ast}\| \geq d_m/2.
\]

We modify the clusters and cluster centers \(\{(\hat{S}_j, w_j)\}_{1 \leq j \leq m}\) as follows:

(i) Combine \(\hat{S}_{j_1}\) and \(\hat{S}_{j_2}\) into one cluster and set the cluster center to be \(w_{j_1}\).  

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(ii) Split \( \hat{S}_j \) into two clusters, where one is \( (\hat{S}_j \cap F_{\ell_1}) \), and the other is \( (\hat{S}_j \setminus F_{\ell_1}) \); the two cluster centers are set as \( u_{\ell_1} \) and \( w_{j^*} \), respectively.

The other clusters and cluster centers remain unchanged. Namely, we let

\[
\tilde{S}_j = \begin{cases} 
\hat{S}_j \cup \hat{S}_j, & \text{if } j = j_1, \\
\hat{S}_j \cap F_{\ell_1}, & \text{if } j = j_2, \\
\hat{S}_j \setminus F_{\ell_1}, & \text{if } j = j^*, \\
\hat{S}_j, & \text{if } j \notin \{j_1, j_2, j^*\},
\end{cases}
\]

\[
\tilde{w}_j = \begin{cases} 
u_{\ell_1}, & \text{if } j = j_2, \\
w_j, & \text{otherwise}.
\end{cases}
\]

By direct calculations,

\[
\Delta \equiv n \cdot R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m) - n \cdot R(\tilde{S}_1, \ldots, \tilde{S}_m, \tilde{w}_1, \ldots, \tilde{w}_m)
\]

\[
= \sum_{i \in (\hat{S}_j \cap F_{\ell_1})} (\|x_i - w_j\|^2 - \|x_i - u_{\ell_1}\|^2) - \sum_{i \in \hat{S}_j} (\|x_i - w_j\|^2 - \|x_i - w_{j^*}\|^2)
\]

\[
\equiv \Delta_2 - \Delta_1,
\]

where \( \Delta_1 \) is the increase of RSS caused by (i) and \( \Delta_2 \) is the decrease of RSS caused by (ii). We can bound \( \Delta_1 \) in a similar way as in the previous sub-case, and the details are omitted. It gives

\[
\Delta_1 \leq 12C_0C_3 \cdot nd_m^2.
\]

Since \( x_i = u_{\ell_1} \) for all \( i \in F_{\ell_1} \), we immediately have

\[
\Delta_2 = |\hat{S}_j \cap F_{\ell_1}| \cdot \|u_{\ell_1} - w_j\|^2 \geq (m^{-1}a_0n) \cdot (d_m/2)^2 \geq \frac{m^{-1}a_0}{4} \cdot nd_m^2.
\]

As a result, \( \Delta \geq \left( \frac{m^{-1}a_0}{4} - 12C_0C_3 \right) m^{-1}a_0 \cdot nd_m^2. \) If we plug in the expression of \( C_3 \), it becomes \( \geq \left( \frac{2}{9} m^{-1}a_0 \right) \cdot nd_m^2. \) This gives

\[
R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m) - R(\tilde{S}_1, \ldots, \tilde{S}_m, \tilde{w}_1, \ldots, \tilde{w}_m) \geq \frac{2m^{-1}a_0}{9} \cdot d_m^2.
\]

This gives (B.76) in the second sub-case.

In the second case, \( \|x_k - w_j\|^2 \geq C_3 \cdot d_m. \) We recall that \( |F_k \cap \hat{S}_j| \geq m^{-1}a_0n. \) Let \( E \) be a subset of \( F_k \cap \hat{S}_j \) such that \( |E| = \frac{|F_k \cap \hat{S}_j|}{2}. \) Note that \( |\hat{S}_j \setminus E| \leq n. \) We have

\[
\hat{S}_j \setminus E \neq \emptyset, \quad \text{and} \quad \frac{|E|}{|\hat{S}_j \setminus E|} \geq m^{-1}a_0/2.
\]

We now modify the clusters and cluster centers \( \{(\hat{S}_j, w_j)\}_{1 \leq j \leq m} \) as follows:

- Move the subset \( E \) from \( \hat{S}_j_1 \) to \( \hat{S}_j_2 \), and update each cluster center to be the within cluster average of \( x_i \)’s.

The other clusters and cluster centers remain unchanged. Namely, we let

\[
\begin{align*}
\tilde{S}_j = \begin{cases} 
\hat{S}_j \setminus E, & \text{if } j = j_1, \\
\hat{S}_j_2 \cup E, & \text{if } j = j_2, \\
\hat{S}_j, & \text{if } j \notin \{j_1, j_2\},
\end{cases}
\end{align*}
\]

\[
\tilde{w}_j = \begin{cases} 
u_{\ell_1}, & \text{if } j = j_2, \\
w_j, & \text{otherwise}.
\end{cases}
\]

We apply Lemma B.1 to \( A = \hat{S}_j_1, B = \hat{S}_j_2, \) and \( C = E, \) and note that \( x_i = u_k \) for all \( i \in E. \) It follows that

\[
\Delta \equiv n \cdot R(\hat{S}_1, \ldots, \hat{S}_m, w_1, \ldots, w_m) - n \cdot R(\tilde{S}_1, \ldots, \tilde{S}_m, \tilde{w}_1, \ldots, \tilde{w}_m)
\]
This gives (B.76) in the second case. We combine (B.81), (B.82) and (B.84), and take the assumed

\[ \parallel \]

Let \( B.7 \) Proof of the claim in Example 4b of Section 3

Without loss of generality, we assume 4 all communities have equal sizes, then

\[ \parallel \]

It follows that \( C \)

communities have equal sizes, then

\[ \parallel \]

Choose \( C \)

Then, (B.76) is satisfied for all cases. This completes the proof of (B.72).

By (B.77), \( \parallel \)

It follows that, as long as \( \delta < (C_3/16) \cdot d_m, \)

\[ \parallel \]

where the last line is because \( 16\delta^2 \leq C_3 \delta \cdot d_m \leq \delta \cdot \parallel u_k - w_{j_1} \parallel. \) We plug it into (B.83) to get

\[ \Delta \geq \frac{|E|^2}{|S_{j_1} \setminus E|} \parallel u_k - w_{j_1} \parallel - \frac{|\tilde{S}_{j_1}| \cdot |E|}{|S_{j_1} \setminus E|} \cdot 9\delta \cdot \parallel u_k - w_{j_1} \parallel \]

\[ \geq |E| \cdot (m^{-1}a_0/2) \cdot \parallel u_k - w_{j_1} \parallel - |E| \cdot 9\delta \cdot \parallel u_k - w_{j_1} \parallel \]

\[ \geq |E| \cdot \parallel u_k - w_{j_1} \parallel \cdot \left( \frac{C_3 m^{-1}a_0}{2} d_m - 9\delta \right), \]

where the second line is because \( |E| \geq (m^{-1}a_0/2) \cdot |\tilde{S}_{j_1} \setminus E| \) and the last line is because we have assumed \( \parallel u_k - w_{j_1} \parallel \geq C_3 \cdot d_m \) in the current case. As long as \( \delta < \frac{C_3 m^{-1}a_0}{27} \cdot d_m, \) the number in brackets is \( \geq \frac{C_3 m^{-1}a_0}{6} d_m. \) We also plug in \( |E| = \left[ m^{-1}a_0 / 2 \right] n \) and \( \parallel u_k - w_{j_1} \parallel \geq C_3 \cdot d_m \) to get

\[ \Delta \geq \frac{m^{-1}a_0 n}{2} \cdot C_3 d_m \cdot \frac{C_3 m^{-1}a_0}{6} d_m \geq \frac{C_3^2 m^{-2}a_0^2}{12} \cdot n d_m^2. \]

It follows that

\[ R(\tilde{S}_1, \ldots, \tilde{S}_m, w_1, \ldots, w_m) - R(\tilde{S}_1, \ldots, \tilde{S}_m, \tilde{w}_1, \ldots, \tilde{w}_m) \geq \frac{C_3^2 m^{-2}a_0^2}{12} \cdot n d_m^2. \]  

(B.84)

This gives (B.76) in the second case. We combine (B.81), (B.82) and (B.84), and take the minimum of the right hand sides of three inequalities. Since \( m^{-1}a_0 < 1 \) and \( C_3^2 < 1/3, \) we choose

\[ C_2 = (1/12)C_3^2 m^{-2}a_0^2. \]

Then, (B.76) is satisfied for all cases. This completes the proof of (B.72).

We remark that the scalar \( c = c(a_0, C_0, m) \) is not exactly \( C_2/C_1. \) In the derivation of (B.71) and (B.72), we have imposed other restrictions on \( \delta, \) which can be expressed as \( \delta \leq C_4 \cdot d_m, \) where \( C_4 \) is determined by \( (C_0, \alpha, d_m) \) and \( (C_1, C_2, C_3). \) Since \( (C_1, C_2, C_3) \) only depend on \( (\alpha, C_0, m), \) \( C_4 \) is a function of \( (\alpha, C_0, m) \) only. We take \( c = \min\{C_2/C_1, C_4\}. \)

\[ \square \]

B.7 Proof of the claim in Example 4b of Section 3

In Example 4b of Section 3 we have the following claim.

**Lemma B.2.** Let \( R(m) \) and \( V(m) \) be as in \( \{ 3.9 \} \) and \( \{ 3.8 \}, \) respectively. If \( (K, m) = (4, 2) \) and all 4 communities have equal sizes, then

\[ g_m(R(m)) \geq ((3 - \sqrt{3})/2) d_k(V(m)). \]

We now show the claim. For short, let \( x_k = v_k^{(m)} \) for all \( 1 \leq k \leq 4 \) and let \( d_* = g_m(R(m)). \)

Without loss of generality, we assume \( x_1 = 0, x_2 = 1, x_3 = x, \) and \( x_4 = y, \) where \( y > x > 1. \) Let \( z = y - x. \) It is seen \( d_k(V(m)) = \min\{1, x - 1, z\}. \) To show the claim, it is sufficient to show

\[ d_* \geq \frac{3 - \sqrt{3}}{2} \min\{1, x - 1, z\}. \]  

(B.85)
By definitions,

\[ d_\ast = \min_{\{\text{all possible } c_1, c_2\}} \min_{1 \leq i \leq 4} \{d_i(c_1, c_2)\}, \quad (B.86) \]

where for \( 1 \leq i \leq 4, d_i(c_1, c_2) \geq 0 \) is the difference between the distance from \( x_i \) to the center of the cluster to which \( x_i \) does not belong and the distance from \( x_i \) to the center of the cluster to which \( x_i \) belongs. For simplicity, we write \( d_i = d_i(c_1, c_2) \) when there is no confusion.

For the four points \( x_1, x_2, x_3, x_4 \), we have three possible candidates (a)-(c) for the clustering results (which of them is the actual clustering result depends on the values of \( x, y \)):

- (a). The left most point forms one cluster, the other three form the other cluster.
- (b). The left two points form one cluster, the other two points form the other cluster.
- (c). The left three points form one cluster, the right most point forms the other cluster.

Recall that for any \( n \) points \( x_1, x_2, \ldots, x_n \), the RSS for the k-means solution with \( K \) clusters is

\[ RSS = \sum_{k=1}^{K} \sum_{i \in \text{cluster } k} (x_i - c_k)^2, \]

where \( c_1, c_2, \ldots, c_K \) are the cluster centers. For (a), the two cluster centers are \( c_1 = 0 \) and \( c_2 = (1 + x + y)/3 \). In this case, the RSS is \( S_1 = x^2 + y^2 + 1 - (1/3)(x + y + 1)^2 \). For (b), the two cluster centers are \( c_1 = 1/2 \) and \( c_2 = (x+y)/2 \), and the RSS is \( S_2 = (1/2) + (1/2)(x-y)^2 \). For (c), the two cluster centers are \( c_1 = (1+x)/3 \) and \( x_2 = y \), and the RSS is \( S_3 = x^2 + 1 - (1/3)(x+1)^2 \).

It is seen that the actual clustering result is as in (a) if and only if \( S_1 \leq S_2 \) and \( S_1 \leq S_3 \); similar for (b) and (c).

Recall that \( z = y - x \). Consider the two-dimensional space with \( x \) and \( z \) being the two axes. As in Figure 7, we partition the region \( \{(x, z) : x > 1, z > 0\} \) into three sub-regions as follows.

- Region (I). \( \{(x, z) : 2x + z < 2 + \sqrt{3}, z < 1\} \).
- Region (II). \( \{(x, z) : z < (2x - 1)/\sqrt{3}, 2x + z > 2 + \sqrt{3}\} \).
- Region (III). \( \{(x, z) : z > 1, z > (2x - 1)/\sqrt{3}\} \).

![Figure 7: In the two dimensional space with x and z being the two axes, the whole region \( \{(x, z) : x > 1, z > 0\} \) partitions into three sub-regions (I), (II), and (III), respectively.](image-url)
Note that any point \((x, z)\) in our range of interest either belongs to one of the three regions, or falls on one of the boundaries of these regions. We now show the claim by consider the three regions in Figure 7 separately. The discussions for the case where \((x, z)\) fall on the boundaries of these regions are similar so are omitted.

Consider Region (I). In this region, by elementary algebra, we have \(S_1 < S_2\) and \(S_1 < S_3\). Therefore, case (a) is the final clustering result, where the two clusters are \(\{x_1\}\) and \(\{x_2, x_3, x_4\}\), respectively, with cluster centers \(c_1 = 0\) and \(c_2 = (x + y + 1)/3\). By definitions, for \((x, z)\) in Region (I), \(d_1 = |c_2 - 0| - |c_1 - 0| = (1 + x + y)/3\), \(d_2 = |c_1 - 1| - |c_2 - 1| = (5 - x - y)/3\), \(d_3 = |c_1 - x| - |c_2 - x| = (x + y + 1)/3\) if \(2x > y + 1\) and \(d_3 = (5x - y - 1)/3\) otherwise, and \(d_4 = |c_1 - y| - |c_2 - y| = (x + y + 1)/3\). By elementary algebra, it is seen that \(d_2\) is the smallest among \(\{d_1, d_2, d_3, d_4\}\). Combining this with (B.86) gives that for \((x, z)\) in Region (I), \(d_* = (5 - x - y)/3 = (5 - 2x - z)/3\). Note that for \((x, z)\) in Region (I), \(2x + z < 2 + \sqrt{3}\). It follows \(2(x - 1) + z < 2\sqrt{3}\), and \(\min\{1, x - 1, z\} \leq \sqrt{3}/3\). Combining these, \(d_*/\min\{1, x - 1, z\} \geq (\sqrt{3}/3 - 1)\).

Consider Region (II). In this region, by elementary algebra, \(S_2 \leq S_1\) and \(S_2 < S_3\). Therefore, case (b) is the actual clustering result, so the two cluster centers are \(c_1 = 1/2\) and \(c_2 = (x + y)/2\), respectively. By definitions, \(d_1 = |c_2 - 0| - |c_1 - 0| = (x + y - 1)/2\), \(d_2 = |c_2 - 1| - |c_1 - 1| = (x + y - 3)/2\), \(d_3 = |c_1 - x| - |c_2 - x| = (3x - y - 1)/2\), and \(d_4 = |c_1 - y| - |c_2 - y| = (x + y - 1)/2\). By elementary algebra, among the four numbers \(\{d_1, d_2, d_3, d_4\}\), \(d_2\) is the smallest when \(z < 1\) and \(d_3\) is smallest when \(z > 1\). Combining this with (B.86) gives that for \((x, z)\) in Region (II),
\[
d_* = \begin{cases} (x + y - 3)/2 = (2x + z - 3)/2, & \text{if } z < 1, \\ (3x - y - 1)/2 = (2x - z - 1)/2, & \text{if } z \geq 1. \end{cases}
\]

Consider the case of \(z < 1\) first. In this case, \(\min\{1, x - 1, z\} = \min\{x - 1, z\} > 0\), and \(2x + z - 3 \geq (2 - 2/\sqrt{3})(x - 1) + (1 - 1/\sqrt{3})z\) since \(2x + z > (2 + \sqrt{3})\) in Region (II). Therefore,
\[
d_* = \frac{2x + z - 3}{2 \min\{x - 1, z\}} \geq \frac{(2 - 2/\sqrt{3})(x - 1) + (1 - 1/\sqrt{3})z}{2 \min\{x - 1, z\}},
\]
where the right hand side is no smaller than
\[(2 - 2/\sqrt{3}) + (1 - 1/\sqrt{3})/2 = (3 - \sqrt{3})/2.\]

Consider the case \(z \geq 1\). In this case, \(\min\{1, x - 1, z\} = \min\{x - 1, 1\} > 0\), and \((2x - z - 1) \geq (2 - 2/\sqrt{3})(x - 1) + (1 - 1/\sqrt{3})\) since \(z \leq (2x - 1)/\sqrt{3}\). Therefore,
\[
d_* = \frac{2x - z - 1}{2 \min\{x - 1, 1\}} \geq \frac{(2 - 2/\sqrt{3})(x - 1) + (1 - 1/\sqrt{3})}{2 \min\{x - 1, 1\}},
\]
where the right hand side is no smaller than
\[(2 - 2/\sqrt{3}) + (1 - 1/\sqrt{3})/2 = (3 - \sqrt{3})/2.\]

Combining the above, we have that in Region (II),
\[
d_* \geq \frac{(3 - \sqrt{3})}{2} \min\{1, x - 1, z\}. \tag{B.88}
\]

Consider Region (III). By elementary algebra, it is seen \(S_3 < S_1\) and \(S_3 < S_2\) in this case. Therefore, case (c) is the actual clustering result, so the two cluster centers are \(c_1 = (1 + x)/3\) and
\( c_2 = y \), respectively. By definitions, \( d_1 = |c_2 - 0| - |c_1 - 0| = (3y - x - 1)/3, d_2 = |c_2 - 1| - |c_1 - 1| = (-1 - x + 3y)/3 \) if \( x > 2 \) and \( d_2 = (x + 3y - 5)/3 \) elsewise, \( d_3 = |c_1 - 1| - |c_2 - x| = (1 - 5x + 3y)/3, \) and \( d_4 = |c_1 - y| - |c_2 - y| = (-1 - x + 3y)/3 \). By elementary algebra, \( d_3 \) is the smallest in \( \{d_1, d_2, d_3, d_4\} \). Combining these with (B.86) gives that for \((x, z)\) in Region (III),
\[
d_* = (1 - 5x + 3y)/3 = (1 - 2x + 3z)/3, \quad \min\{1, x - 1, z\} = \min\{1, x - 1\}.
\]
When \( x > 2, \min\{1, x - 1\} = 1 \), and the minimum of \( d_* \) in Region (III) is \((\sqrt{3} - 1)\) attained at \((x, z) = (2, \sqrt{3})\). When \( x < 2, \min\{1, x - 1\} = x - 1 \). Therefore, \( d_*/\min\{1, x - 1\} = (z - 1/3)/(x - 1)/2/3 \), where the minimum in Region (III) is \(2/\sqrt{3}\), attained at \((x, z) = ((\sqrt{3}+1)/2, 1)\). Combining these, we have that for \((x, z)\) in Region (III),
\[
d_* \geq (\sqrt{3} - 1) \min\{1, x - 1, z\}.
\]
Combining (B.87)-(B.89) gives the claim. \(\square\)

**C Proof of results in Section 4**

**C.1 Proof of Lemma 4.1**

Consider the first two claims. It is easy to see that \( E[C_n] = c_n \). In the proof of Theorem 3.1 of [16], it has been shown that
\[
c_n = \text{tr}(\Omega^4) + O(\|\theta\|^4) = \text{tr}(\Omega^4) + o(\|\theta\|^8).
\]
Moreover, \( \lambda_1^4 \leq \text{tr}(\Omega^4) \leq K\lambda_1^4 \). In the proof of Theorem 2.4 we have seen that \( \lambda_1 = \|\theta\|^2 \cdot \lambda_1(HPH') \). Using the condition (2.2) and the fact that \( P \) has unit diagonals, we have \( \lambda_1(HPH') \geq C\lambda_1(P) \geq C \). Similarly, since we have assumed \( \|P\| \leq C \) in (2.1), \( \lambda_1(HPH') \leq C\lambda_1(P) \leq C \). Here, \( C \) is a generic constant. We have proved that
\[
E[C_n] = c_n = \|\theta\|^8.
\]
To compute the variance of \( C_n \), write
\[
C_n = \bar{Q}_n + \Delta, \quad \text{where} \quad \bar{Q}_n = \sum_{i_1, i_2, i_3, i_4, (d_{i_1})} W_{i_1 i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1}.
\]
The variance of \( \Delta \) is computed in the proof of Lemma B.2 of [16]. Using the upper bound of the variance of \((\sum_{CC(I_n)} \Delta_{i_1 i_2 i_3 i_4}^{(k)})\) for \( k = 1, 2, 3 \) there, we have
\[
\text{Var}(\Delta) \leq C\|\theta\|^6 \|\theta\|^8.
\]
Furthermore, we show in the proof of Lemma 4.2 that \( \text{Var}(\bar{Q}_n) = 8c_n \cdot [1 + o(1)] \). It follows that \( \text{Var}(\bar{Q}_n) \approx c_n \times \|\theta\|^8 \). Combining these results gives
\[
\text{Var}(C_n) \leq C\|\theta\|^8 \cdot [1 + \|\theta\|^6].
\]
Consider the last claim. For any \( \epsilon > 0 \), using Chebyshev’s inequality, we have
\[
P(|C_n/c_n - 1| \geq \epsilon) \leq (c_n \epsilon)^{-2}\text{Var}(C_n) \leq \frac{C(1 + \|\theta\|^6)}{\epsilon^2 \|\theta\|^8}.
\]
Here we have used the first two claims. Since \( \|\theta\|^6 \leq \theta_{\max} \|\theta\|^2 = o(\|\theta\|^8), \) the rightmost term is \( o(1) \) as \( n \to \infty \). This proves that \( C_n/c_n \to 1 \) in probability. \(\square\)
C.2 Proof of Lemma 4.2

In the proof of Theorem 3.2 of [16], it was shown that $\tilde{Q}_n/\sqrt{\text{Var}(\tilde{Q}_n)} \to N(0,1)$ in law (in the proof there, $\tilde{Q}_n/\sqrt{\text{Var}(\tilde{Q}_n)}$ is denoted as $S_{n,n}$). It remains to prove Var($\tilde{Q}_n$) = $8c_n \cdot [1 + o(1)]$.

Note that for each ordered quadruple $(i, j, k, \ell)$ with four distinct indices, there are 8 summands in the definition of $\tilde{Q}_n$ whose values are exactly the same; these summands correspond to $(i_1, i_2, i_3, i_4) \in \{(i, j, k, \ell), (j, k, i, \ell), (k, i, j, \ell), (\ell, i, j, k), (j, i, k, \ell), (i, k, j, \ell), (k, j, i, \ell), (\ell, k, i, j)\}$. We treat these 8 summands as in an equivalent class. Denote by $CC_4$ the collection of all such equivalent classes. Then, for any doubly indexed sequence $\{x_{ij}\}_{1 \leq i \leq j \leq n}$ such that $x_{ij} = x_{ji}$, it is true that $\sum_{i_1, i_2, i_3, i_4 \in (dist)} x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1} = 8 \sum_{CC_4} x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1}$. In particular,

$$\tilde{Q}_n = 8 \sum_{CC_4} W_{i_1 i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1}.$$

The summands are independent of each other, and the variance of $W_{i_1 i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1}$ is equal to $\Omega_{i_1 i_2}^{*} \Omega_{i_2 i_3}^{*} \Omega_{i_3 i_4}^{*} \Omega_{i_4 i_1}^{*}$, where $\Omega_{ij} = \Omega_{ij}(1 - \Omega_{ij})$. As a result,

$$\text{Var}(\tilde{Q}_n) = 64 \sum_{CC_4} \Omega_{i_1 i_2}^{*} \Omega_{i_2 i_3}^{*} \Omega_{i_3 i_4}^{*} \Omega_{i_4 i_1}^{*} = 8 \sum_{i_1, i_2, i_3, i_4 \in (dist)} \Omega_{i_1 i_2}^{*} \Omega_{i_2 i_3}^{*} \Omega_{i_3 i_4}^{*} \Omega_{i_4 i_1}^{*}.$$

Recall that $c_n = \sum_{i_1, i_2, i_3, i_4 \in (dist)} \Omega_{i_1 i_2} \Omega_{i_2 i_3} \Omega_{i_3 i_4} \Omega_{i_4 i_1}$. Then,

$$|\text{Var}(\tilde{Q}_n) - 8c_n| \leq 8 \sum_{i_1, i_2, i_3, i_4 \in (dist)} |\Omega_{i_1 i_2} \Omega_{i_2 i_3} \Omega_{i_3 i_4} \Omega_{i_4 i_1} - \Omega_{i_1 i_2}^{*} \Omega_{i_2 i_3}^{*} \Omega_{i_3 i_4}^{*} \Omega_{i_4 i_1}^{*}|$$

$$\leq 8 \sum_{i_1, i_2, i_3, i_4 \in (dist)} \Omega_{i_1 i_2} \Omega_{i_2 i_3} \Omega_{i_3 i_4} \Omega_{i_4 i_1} \cdot C \cdot \|\theta\|_{\text{max}}$$

$$= 8c_n \cdot O(\theta_{\text{max}}^2).$$

Since $\theta_{\text{max}} = o(1)$ by the condition (2.1), we immediately have $\text{Var}(\tilde{Q}_n) = 8c_n \cdot [1 + o(1)]$. $\Box$

C.3 Proof of Lemma 4.3

The proof is combined with the proof of Lemma 4.8 see below.

C.4 Proof of Lemma 4.4

Consider the first claim. Since $b_n = 2\|\theta\|^{4} \cdot [g'V^{-1}(PH^2P \circ PH^2P)V^{-1}g]$ (see (4.13)), it suffices to show that

$$g'V^{-1}(PH^2P \circ PH^2P)V^{-1}g \succeq 1.$$

The vectors $g, h \in \mathbb{R}^K$ are defined by $g_k = (1_k^t \theta)/\|\theta\|_1$ and $h_k = (1_k^t \Omega^2 1_k)^{1/2}/\|\theta\|$, where $1_k$ is for short of $1_k^{(K)}$. By condition (2.2), $c_1 \leq g_k \leq 1$ and $c_1 \leq h_k^{2} \leq 1$ for $1 \leq k \leq K$, and $\|P\| \leq c_2$, for some constants $c_1, c_2 \in (0, 1)$.

For the upper bound, by $h_k^{2} \leq 1$ and $\|P\| \leq c_2$, we have $\|(PH^2P \circ (PH^2P))\| \leq C$. Since $P$ has unit diagonals and $g_k \geq c_1$, the diagonal elements of $V = \text{diag}(Pg)$ is no less than $c_1$. Hence

$$g'V^{-1}(PH^2P \circ PH^2P)V^{-1}g \leq \|g'V^{-1}\|^2 \cdot \|PH^2P \circ PH^2P\| \leq C.$$

(C.90)

For the lower bound, since $P$ has unit diagonals and $h_k^2 \geq c_1$, we can lower bound diagonal elements of $PH^2P \circ PH^2P$ by $c_2$. Since $g \in \mathbb{R}^K$ is a non-negative vector with entries summing to 1, the diagonal elements of $V = \text{diag}(Pg)$ is no more than $\max_{k,l} P_{k,l} \leq \|P\| \leq c_2$. Therefore
each entry of vector \(gV^{-1}\) is at least \(c_1/c_2\). Since \(PH^2P \circ PH^2P \in \mathbb{R}^{(K,K)}\) is non-negative matrix and \(g'V^{-1} \in \mathbb{R}^{|K|}\) is non-negative vector, we can lower bound

\[
g'V^{-1} (PH^2P \circ PH^2P)V^{-1}g \geq c_1^2 \|g'V^{-1}\|^2 \geq C, \tag{C.91}
\]

Combining (C.90)-(C.91), we completes the proof of the first claim.

Consider the second claim. Introduce the following event

\[
A_n = \{\hat{\Pi}^{(K)} = \Pi, \text{ up to a permutation in the columns of } \hat{\Pi}^{(K)}\}. \tag{C.92}
\]

By Theorem 2.4 when \(m = K\), SCORE exactly recovers \(\Pi\) with probability \(1 - o(n^{-3})\), i.e.,

\[
P(A_n^c) \leq Cn^{-3} = o(1).
\]

This means if we replace every \(\hat{\Pi}^{(K)}\) in the definition of \(B_n^{(K)}\) with \(\Pi\), and denote the resulting quantity as \(B_n^{(K,0)}\), the above inequality immediately implies that \(B_n^{(K)}/B_n^{(K,0)} \overset{p}{\rightarrow} 1\). So we only need to prove \(B_n^{(K,0)}/b_n \overset{p}{\rightarrow} 1\). Since we will never use the original definition of \(B_n^{(K)}\) in the rest of the proof, without causing any confusion we will suspend the original definitions of \(B_n^{(K)}\) and the quantities used to define \(B_n^{(K)}\), including \((\hat{\theta}, \hat{g}, \hat{V}, \hat{P}, \hat{H})\), and use them to actually denote the correspondents with every \(\hat{\Pi}^{(K)}\) replaced by \(\Pi\).

Recall the formulas for \(B_n^{(K)}\) and \(b_n\) in (1.15) and (4.13), we have

\[
\frac{B_n^{(K)}}{b_n} = \frac{\|\hat{\theta}\|^4}{\|\theta\|^4} \cdot \frac{\hat{g}'V^{-1}(\hat{P}\hat{H}^2\hat{P} \circ \hat{P}\hat{H}^2\hat{P})V^{-1}\hat{g}}{g'V^{-1}(PH^2P \circ PH^2P)V^{-1}g}. \tag{C.93}
\]

To show that \(B_n^{(K)}/b_n \rightarrow 1\), we need the follow lemma, which is proved in Section D.2.

**Lemma C.1.** Suppose the conditions of Theorem 2.4 hold. Let \(1_n \in \mathbb{R}^n\) be the vector of 1’s, and let \(1_k \in \mathbb{R}^n\) be the vector such that \(1_k(i) = 1\{i \in N_k\}\), for \(1 \leq i \leq n\) and \(1 \leq k \leq K\). As \(n \rightarrow \infty\), for all \(1 \leq k \leq K\),

\[
\frac{1_n' \Omega_{1_k} 1_n}{\|1_n\|^2} \overset{p}{\rightarrow} 1, \quad \frac{1_k' \Omega_{1_k} 1_k}{\|1_k\|^2} \overset{p}{\rightarrow} 1, \quad \frac{1_k' \Omega_{1_k} 1_k}{\|1_k\|^2} \overset{p}{\rightarrow} 1.
\]

Moreover, let \(d_i\) be the degree of node \(i\) and let \(d_i^* = (\Omega_{1_k})_{ii}\), for \(1 \leq i \leq n\). Write \(D = \text{diag}(d) \in \mathbb{R}^{n \times n}\) and \(D^* = \text{diag}(d^*) \in \mathbb{R}^{n \times n}\). As \(n \rightarrow \infty\), for all \(1 \leq k \leq K\),

\[
\|\hat{\theta}\|^2/\|\theta\|^2 \overset{p}{\rightarrow} 1, \quad \|\hat{\theta}\|/\|\theta\| \overset{p}{\rightarrow} 1, \quad \frac{1_k' D^21_k}{1_k' (D^*)^2 1_k} \overset{p}{\rightarrow} 1.
\]

First, by Lemma C.1 \(\|\hat{\theta}\|/\|\theta\| \overset{p}{\rightarrow} 1\). It follows from the continuous mapping theorem that

\[
\|\hat{\theta}\|^4/\|\theta\|^4 \overset{p}{\rightarrow} 1. \tag{C.94}
\]

Second, recall that \(g_k = (1_k' \hat{\theta})/\|\theta\|_1\) and \(\hat{g}_k = (1_k' \hat{\theta})/\|\hat{\theta}\|_1\), where by (1.10), we have the equality \(1_k' \hat{\theta} = (1_k' \theta) \cdot \sqrt{1_k' A_k 1_k}/(1_k' A_1 1_n)\). Here, keep in mind that we have replaced \(\hat{\Pi}^{(K)}\) with \(\Pi\), which implies that \(1_k = 1_k\). The vector \(d\) is such that \(d = A_{1_n}\). It follows that \(1_k' \theta = \sqrt{1_k' A_k 1_k}\). Furthermore, \(1_k' \Omega_{1_k} = (1_k' \theta)^2\), because \(P\) has unit diagonal. Combining the above gives

\[
\frac{\hat{g}_k}{g_k} = \frac{1_k' \hat{\theta}}{1_k' \theta} \cdot \frac{\|\theta\|_1}{\|\hat{\theta}\|_1} = \frac{\sqrt{1_k' A_k 1_k}}{1_k' \Omega_{1_k}} \cdot \frac{\|\theta\|_1}{\|\hat{\theta}\|_1} \overset{p}{\rightarrow} 1, \quad 1 \leq k \leq K. \tag{C.95}
\]

Third, note that by definition and basic algebra, both \(P\) and \(\hat{P}\) have unit diagonals. We compare their off-diagonals. By (1.10), \(\hat{P}_{kl} = 1_l' A_{1_l}/\sqrt{(1_k' A_{1_k})(1_l' A_{1_l})}\). At the same time, it can be easily verified that \(P_{kl} = 1_k' \Omega_{1_k}/\sqrt{(1_k' A_{1_k})(1_l' A_{1_l})}\).

Introduce

\[
X = \frac{\sqrt{(1_k' \Omega_{1_k})(1_k' A_k)}}{\sqrt{(1_k' A_{1_k})(1_l' A_{1_l})}}
\]
By Lemma C.1, $X \overset{p}{\rightarrow} 1$. We re-write

$$P_{kl} - P_{kt} = \frac{1_k' A 1_k - 1_k' \Omega 1_k}{\sqrt{(1_k' A 1_k)(1_k' A 1_k)}} + P_{kt}(X - 1) = \frac{1_k' W 1_k}{(1_k' \theta)(1_k' \theta)} X + P_{kt}(X - 1),$$

where in the last inequality we have used the fact that $1_k' \Omega 1_k = (1_k' \theta)^2$ for all $1 \leq k \leq K$. Note that $\mathbb{E}[1_k' W 1_k] = 0$. Moreover, $\text{Var}(W_{ij}) \leq \|P\|_{\max} \theta_i \theta_j \leq C \theta_i \theta_j$. It follows that $\text{Var}(1_k' W 1_k) \leq C(1_k' \theta)(1_k' \theta)$. Therefore,

$$\mathbb{E} \left[ \frac{1_k' W 1_k}{(1_k' \theta)(1_k' \theta)} \right]^2 \leq \frac{C}{(1_k' \theta)(1_k' \theta)} = O(\|\theta\|_1^{-2}) = o(1).$$

Hence, $\frac{1_k' W 1_k}{(1_k' \theta)(1_k' \theta)} \overset{p}{\rightarrow} 0$. Combining the above results, we have

$$P_{kl} - P_{kt} \overset{p}{\rightarrow} 0, \quad 1 \leq k, \ell \leq K. \tag{C.96}$$

Fourth, since $V = \text{diag}(Pg)$ and $\hat{V} = \text{diag}(\hat{P}g)$, it follows from (C.95) and (C.96) that

$$\hat{V}_{kk}/V_{kk} \overset{p}{\rightarrow} 1, \quad 1 \leq k \leq K. \tag{C.97}$$

Last, note that $H^2, \hat{H}^2 \in \mathbb{R}^{K \times K}$ are diagonal matrices, with $k$-th diagonal elements being $h_k^2$ and $\hat{h}_k^2$, respectively. By (1.14), $\hat{h}_k^2 = (1_k' \Theta^2 1_k)/\|\theta\|^2$. In addition, by (1.10), for any $i \in N_k^*$, we have $\theta_i^2 = d_i^2(1_k' A 1_k)/(1_k' A 1_k)^2$. We thus re-write

$$\hat{H}_{kk} \equiv \hat{h}_k^2 = \frac{(1_k' D^2 1_k) \cdot (1_k' A 1_k)}{(1_k' A 1_k)^2 \cdot \|\theta\|^2}.$$

Additionally, $h_k = (1_k' \Theta^2 1_k)/\|\theta\|^2$, as defined in the paragraph below (4.13). By direct calculations, $(1_k' \Omega 1_k) / \sqrt{1_k' \Omega 1_k} = [(1_k' \theta) \sum_l P_{kl}(1_l' \theta)] / (1_k' \theta) = \sum_l P_{kl}(1_l' \theta)$. Also, for any $i \in N_k$, we have $d_i^2 = (\Omega_i 1) = \theta_i \sum_l P_{kl}(1_l' \theta)$. It implies that $1_k' (D^2) 1_k = (1_k' \Theta^2 1_k) \sum_l P_{kl}(1_l' \theta)^2$. We can use these expressions to verify that

$$H_{kk} \equiv h_k^2 = \frac{(1_k' (D^2) 1_k) \cdot (1_k' \Omega 1_k)}{(1_k' \Omega 1_k)^2 \cdot \|\theta\|^2}.$$

We apply Lemma C.1 to obtain that

$$\hat{H}_{kk}/H_{kk} \overset{p}{\rightarrow} 1, \quad 1 \leq k \leq K. \tag{C.98}$$

We plug (C.94), (C.95), (C.96), (C.97) and (C.98) into (C.93). It follows from elementary probability that $B_n^{K^*}/b_n \rightarrow 1$. This gives the second claim. \hfill $\square$

### C.5 Proof of Lemma 4.5

Recall $N_1^{(m,0)}, N_2^{(m,0)}, \ldots, N_m^{(m,0)}$ are “fake” communities associated with $\Pi_0$, and we decompose the vector $1_n \in \mathbb{R}^n$ as follows

$$1_n = \sum_{k=1}^m 1_{(m,0)}^k, \quad \text{where } 1_{k}^{(m,0)}(j) = 1 \text{ if } j \in N_{(m,0)}^{(m,0)} \text{ and } 0 \text{ otherwise.} \tag{C.99}$$

Notice for $\Pi_0 \in G_m$ defined in (4.18), there exists an $K \times m$ matrix $L_0$ such that $\Pi_0 = \Pi L_0$.

By definitions, $\Omega^{(m,0)} = \Theta^{(m,0)} \Pi_0 P^{(m,0)} \Pi_0^T \Theta^{(m,0)}$. Here $\Theta^{(m,0)}$ and $P^{(m,0)}$ are obtained by replacing $(d, A, (I_k, \Pi_k))$ by $(d_i^*, 1_k^{(m,0)}, \Omega)$ in the definition (4.10). It yields that, for $1 \leq k, \ell \leq m$ and $i \in N_{(m,0)}^{(m,0)}$,

$$\theta_i^{(m,0)} = \frac{d_i^*}{(1_k^{(m,0)}) \Omega_n} \cdot \sqrt{(1_k^{(m,0)}) \Omega 1_k^{(m,0)}}, \quad P_{kt}^{(m,0)} = \frac{(1_k^{(m,0)}) \Omega_k^{(m,0)} (1_t^{(m,0)})}{\sqrt{(1_k^{(m,0)}) \Omega 1_k^{(m,0)}} \sqrt{(1_t^{(m,0)}) \Omega 1_t^{(m,0)}}}.$$
As a result, for \( i \in \mathcal{N}_k^{(m,0)} \) and \( j \in \mathcal{N}_k^{(m,0)} \),

\[
\Omega_{ij}^{(m,0)} = \theta_i^{(m,0)} \theta_j^{(m,0)} P_{kl}^{(m,0)} = d_i^* d_j^* \cdot \frac{(1_k^{(m,0)})' \Omega_1^{(m,0)}}{[(1_k^{(m,0)})' \Omega_1^{(m,0)}] \cdot [(1_k^{(m,0)})' \Omega_1^{(m,0)}]}, \tag{C.100}
\]

Note that \((1_k^{(m,0)})' \Omega_1^{(m,0)} = (\Pi_0' \Omega_0)_{k\ell}\). Since \( \Omega = \Theta \Pi P' \Theta \) and \( D_0 = \Pi_0' \Theta \Pi \), we immediately have \( \Pi_0' \Omega_0 = \Pi_0' \Theta \Pi P' \Theta \Pi_0 = D_0 PD_0' \). It follows that

\[
(1_k^{(m,0)})' \Omega_1^{(m,0)} = (D_0 PD_0')_{k\ell}, \quad 1 \leq k, \ell \leq m.
\]

Similarly, \((1_k^{(m,0)})' \Omega_1 = (\epsilon_k' \Pi_0' \Omega (\Pi_1 K)) = \epsilon_k' \Pi_0' \Theta \Pi P' \Theta \Pi_1 K = \epsilon_k' D_0 PD_1 K \). This gives

\[
(1_k^{(m,0)})' \Omega_1 = \text{diag}(D_0 PD_1 K)_{kk}, \quad 1 \leq k, \ell \leq m.
\]

We plug the above equalities into \((C.100)\). It follows that, for \( i \in \mathcal{N}_k^{(m,0)} \) and \( j \in \mathcal{N}_k^{(m,0)} \),

\[
\Omega_{ij}^{(m,0)} = d_i^* d_j^* \cdot \left[ \left( \text{diag}(D_0 PD_1 K) \right)^{-1}(D_0 PD_0' \text{diag}(D_0 PD_1 K))^{-1} \right]_{k\ell}. \tag{C.101}
\]

Write for short

\[
M = \left[ \text{diag}(D_0 PD_1 K) \right]^{-1}(D_0 PD_0' \text{diag}(D_0 PD_1 K))^{-1}. \tag{C.102}
\]

Then, \((C.101)\) can be written equivalently as

\[
\Omega_{ij}^{(m,0)} = d_i^* d_j^* \cdot \sum_{k=1}^{m} M_{k\ell} \cdot 1\{i \in \mathcal{N}_k^{(m,0)}\} \cdot 1\{j \in \mathcal{N}_k^{(m,0)}\}. \tag{C.103}
\]

By definition, \( L_0(u, k) = 1\{u \in \mathcal{N}_k^{(m,0)}\} \), for \( 1 \leq u \leq K \) and \( 1 \leq k \leq m \). Therefore, we have the equalities: \( 1\{i \in \mathcal{N}_k^{(m,0)}\} = \sum_{u=1}^{K} L_0(u, k) \cdot 1\{i \in \mathcal{N}_u\} \) and \( 1\{j \in \mathcal{N}_k^{(m,0)}\} = \sum_{v=1}^{K} L_0(v, \ell) \cdot 1\{j \in \mathcal{N}_v\} \). Combining them with the above equation gives

\[
\Omega_{ij}^{(m,0)} = d_i^* d_j^* \cdot \sum_{u=1}^{K} 1\{i \in \mathcal{N}_u\} \cdot 1\{j \in \mathcal{N}_u\} \sum_{k=1}^{m} L_0(u, k) L_0(v, \ell) M_{k\ell} = d_i^* d_j^* \cdot \sum_{u=1}^{K} 1\{i \in \mathcal{N}_u\} \cdot 1\{j \in \mathcal{N}_v\} \cdot (L_0 ML_0')_{uv}. \tag{C.104}
\]

By definition, \( d_i^* = \Omega_1 = \Omega(\Pi_1 K) \). Since \( \Omega = \Theta \Pi P' \Theta \), we immediately have

\[
d_i^* = \theta_i \cdot \pi_i' \Pi P' \Theta \Pi_1 K. \tag{C.105}
\]

Similarly, we have \( d_j^* = \theta_j \cdot \sum_{u=1}^{K} \text{diag}(PD_1 K)_{uv} \cdot 1\{j \in \mathcal{N}_v\} \). Plugging the expressions of \( (d_i^*, d_j^*) \) into \((C.103)\) gives

\[
\Omega_{ij}^{(m,0)} = \theta_i \theta_j \sum_{u,v=1}^{K} 1\{i \in \mathcal{N}_u\} 1\{j \in \mathcal{N}_v\} \text{diag}(PD_1 K)_{uv} (L_0 ML_0')_{uv} \cdot \text{diag}(PD_1 K)_{vv} = \theta_i \theta_j \cdot \pi_i' \left[ \text{diag}(PD_1 K) L_0 ML_0' \text{diag}(PD_1 K) \right] \pi_j. \tag{C.104}
\]

Combining it with the expression of \( M \) in \((C.102)\) gives the claim.
C.6 Proof of Lemma 4.6

The claim of \( c_n \asymp \|\theta\|^8 \) is proved in Lemma 4.1. To prove the claim of \( \lambda_1 \asymp \|\theta\|^2 \), we note that by Lemma 3.3, \( \lambda_k = \|\theta\|^2 \cdot \lambda_k(HPH) \), where \( H \) is the diagonal matrix such that \( H_{kk} = \|\theta(k)\|^2 / \|\theta\|^2 \).

By the condition (2.2), all the diagonal entries of \( H \) are between \([c, 1]\), for a constant \( c \in (0, 1) \).

It follows that \( \lambda_1(HPH) \asymp \lambda_1(P) \). Since \( \lambda_1 \geq P_{11} = 1 \) and \( \lambda_1 \leq \|P\| \leq C \), we have \( \lambda_1(P) \asymp 1 \).

Combining the above gives

\[
\lambda_1 \asymp \|\theta\|^2 \lambda_1(P) \asymp \|\theta\|^2.
\]

We then prove the claims related to the matrix \( \tilde{\Omega} \). First, we show the upper bound of \( \|\tilde{\Omega}_{ij}\| \) and the lower bound of \( \text{tr}(\tilde{\Omega}^4) \). Recall that \( \tilde{\Omega} = \Omega - \Omega(m, 0) \). By Lemma 4.5, \( \Omega(m, 0) = \Theta \Pi P_0 \Pi' \Theta \) for a rank-\( m \) matrix \( P_0 \). It follows that

\[
\tilde{\Omega} = \Theta \Pi (P - P_0) \Pi' \Theta.
\]

(\text{C.105})

Let \( H \) be the same diagonal matrix as above. It can be easily verified that \( \|\theta\|^2 \cdot H^2 = \Pi' \Theta^2 \Pi \).

This means that the matrix \( U = \|\theta\|^{-1} \Pi H^{-1} \) satisfies the equality \( U'U = I_K \). As a result, we can write \( \tilde{\Omega} = U \cdot (\|\theta\|^2 \cdot H(P - P_0)H) \cdot U' \).

Since \( U \) contains orthonormal columns, the nonzero eigenvalues of \( \tilde{\Omega} \) are the same as the nonzero eigenvalues of \( \|\theta\|^2 \cdot H(P - P_0)H \), i.e.,

\[
\tilde{\lambda}_k = \|\theta\|^2 \cdot \lambda_k(H(P - P_0)H), \quad 1 \leq k \leq m.
\]

In particular, \( |\tilde{\lambda}_1| = \|\theta\|^2 \cdot \|H(P - P_0)H\| \asymp \|\theta\|^2 \cdot \|P - P_0\| \asymp \lambda_1 \|P - P_0\| \), where we have \( \|H\| \asymp \|H^{-1}\| \asymp 1 \), and \( \lambda_1 \asymp \|\theta\|^2 \). Combining it with the definition of \( \tau \) gives

\[
\tau \asymp \|P - P_0\|.
\]

(\text{C.106})

Consider \( |\tilde{\Omega}_{ij}| \). By \( |\tilde{\Omega}_{ij}| = |\theta_i \theta_j| |\pi_i(P - P_0)\pi_j| \leq \|\theta\| C \|P - P_0\| \), we plug in \( \|\theta\| \) to get \( |\tilde{\Omega}_{ij}| \leq C \tau \|\theta\|, \) for \( 1 \leq i, j \leq n \). Consider \( \text{tr}(\tilde{\Omega}^4) \). We have seen that \( |\tilde{\lambda}_1| \asymp \|\theta\|^2 \cdot \|P - P_0\| \asymp \tau \|\theta\|^2 \).

As a result, \( \text{tr}(\tilde{\Omega}^4) \asymp \tilde{\lambda}_1^4 \asymp \tau^4 \|\theta\|^8 \).

Next, we study the order of \( \tau \). Note that \( \Omega = \Omega(m, 0) + \tilde{\Omega} \). We aim to apply Weyl's inequality. In our notation, \( \lambda_k(\cdot) \) refers to the \( k \)th largest eigenvalue (in magnitude) of a symmetric matrix. As a result, \( |\lambda_k(\cdot)| \) is the \( k \)th singular value. By Weyl's inequality for singular values (equation (7.3.13) of [9]), we have

\[
|\lambda_{r+s-1}(\Omega)| \leq \max\{|\lambda_{r}(\Omega(m, 0))| + |\lambda_{s}(\tilde{\Omega})|, \quad \text{for } 1 \leq r, s \leq n - 1.
\]

Since \( \Omega(m, 0) \) only has \( m \) nonzero eigenvalues, by taking \( r = m + 1 \) and \( s = k \) in the above, we immediately have

\[
|\lambda_{m+k}(\Omega)| \leq |\lambda_k(\tilde{\Omega})| = |\tilde{\lambda}_k|, \quad 1 \leq k \leq K - m.
\]

(\text{C.107})

In particular, \( |\tilde{\lambda}_1| \geq |\lambda_{m+1}| \geq |\lambda_K| \). At the same time, \( \lambda_1 \asymp \|\theta\|^2 \) and by definition, \( \tau = |\tilde{\lambda}_1| / \lambda_1 \).

It follows that

\[
\tau \|\theta\| \asymp (|\lambda_K| / \lambda_1) \|\theta\| \geq C(|\lambda_K| / \sqrt{\lambda_1}) \rightarrow \infty.
\]

This gives \( \tau \|\theta\| \rightarrow \infty \). We then prove \( \tau \leq C \). In light of (\text{C.106}), it suffices to show \( \|P_0\| \leq C \).

Consider the expression of \( P_0 \) in Lemma 4.5. It is easy to see that \( \|L_0\| \leq C \), \( \|D_0PD_0^T\| \leq C\|\theta\|^2 \), and \( \|\text{diag}(PD_0^T)\| \leq C\|\theta\|_1 \). As a result,

\[
\|P_0\| \leq C\|\theta\|_1 \cdot \|\text{diag}(D_0PD_0^T)^{-1}\|^2.
\]

(\text{C.108})

Since \( D_0 = \Pi_0' \Theta \Pi \) and \( D = \Pi' \Theta \Pi \), it is true that \( D_0PD_0^T = \Pi_0' \Theta \Pi P_0 \Pi' \Theta \Pi = \Pi_0' \Theta \Pi P \Pi' \Theta P_0 \Pi' \Theta = \Pi_0' \Theta \Pi \).

Next, for each \( 1 \leq k \leq m \),

\[
[\text{diag}(D_0PD_0^T)]_{kk} = (\Pi_0' \Omega_{(n)})_k = \sum_{i \in \mathcal{X}_n} d_i^*, \quad \text{where } d^* = \Omega_{(n)}.
\]
Here $\mathcal{N}_{1}(m,0), \mathcal{N}_{2}(m,0), \ldots, \mathcal{N}_{m}(m,0)$ are the pseudo-communities defined by $\Pi_{0}$. Suppose $i \in \mathcal{N}_{t}$ for some true community $\mathcal{N}_{t}$. Then, $d_{t}^{i} \geq \sum_{j \in \mathcal{N}_{t}} \theta_{i} \theta_{j} P_{ij} = \theta_{i} \|\theta\|_{1} \geq C \theta_{i} \|\theta\|_{1}$. Moreover, for any $\Pi_{0} \in \mathcal{G}_{m}$, each pseudo-community $\mathcal{N}_{k}(m,0)$ is the union of one or more true communities. It yields that $\sum_{i \in \mathcal{N}_{k}(m,0)} \theta_{i} \geq \min_{1 \leq t \leq K} \{\|\theta\|_{1}\} \geq C \|\theta\|_{1}$. Combining these results gives $\sum_{i \in \mathcal{N}_{k}(m,0)} d_{t}^{i} \geq C \|\theta\|_{1}^{2}$. This shows that each diagonal entry of $\text{diag}(D_{0} PD_{1} K)$ is lower bounded by $C \|\theta\|_{1}^{2}$. We immediately have

$$\|\text{diag}(D_{0} PD_{1} K)^{-1}\| \leq C \|\theta\|_{1}^{-2}. \quad (C.109)$$

Combining (C.108) and (C.109) gives $\|P_{b}\| \leq C$. The claim $\tau \leq C$ then follows from (C.106).

### C.7 Proof of Lemma 4.7

Recall that $W = A - \Omega$. Given any $n \times n$ symmetric matrix $T$, we can define a random variable as follows:

$$Q_{W}(T) = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} (W_{i_{1}i_{2}} + T_{i_{1}i_{2}})(W_{i_{2}i_{3}} + T_{i_{2}i_{3}})(W_{i_{3}i_{4}} + T_{i_{3}i_{4}})(W_{i_{4}i_{1}} + T_{i_{4}i_{1}}). \quad (C.110)$$

Then, $\tilde{Q}_{n}(m,0)$ is a special case with $T = \tilde{\Omega}(m,0)$, where $\tilde{\Omega}(m,0)$ is defined in (4.23). We study the general form of $Q_{W}(T)$. By an expansion of each summand, we can write $Q_{W}(T)$ as the sum of $2^{4}$ post-expansion sums. Each post-expansion sum takes a form

$$X = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} a_{i_{1}i_{2}} b_{i_{2}i_{3}} c_{i_{3}i_{4}} d_{i_{4}i_{1}}, \quad (C.111)$$

where each of $a_{i_{j}}, b_{i_{j}}, c_{i_{j}}, d_{i_{j}}$ may take value in $\{W_{ij}, T_{ij}\}$. We divide the post-expansion sums into 6 common types and compute the mean and variance of each of them (see Table 2 for the special case of $T = \tilde{\Omega}(m,0)$). For example, the post-expansion sum $\sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} T_{i_{1}i_{2}} T_{i_{2}i_{3}} T_{i_{3}i_{4}} T_{i_{4}i_{1}}$ is non-stochastic and has a zero variance. Its mean equals to $\text{tr}(T^{4}) - \Delta$, where $\Delta$ contains the sum of $T_{i_{1}i_{2}} T_{i_{2}i_{3}} T_{i_{3}i_{4}} T_{i_{4}i_{1}}$ when some of the indices $(i_{1}, i_{2}, i_{3}, i_{4})$ are equal. As another example, the post-expansion sum $\sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} W_{i_{1}i_{2}} W_{i_{2}i_{3}} W_{i_{3}i_{4}} W_{i_{4}i_{1}}$ has a zero mean, and since the summands are mutually uncorrelated, its variance is $\sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} \Omega_{i_{1}i_{2}} \Omega_{i_{2}i_{3}} \Omega_{i_{3}i_{4}} \Omega_{i_{4}i_{1}}$, where $\Omega_{ij} = \Omega_{ij}(1 - \Omega_{ij})$.

| Type | # | $(N_{0}, N_{W})$ | Examples | Mean | Variance |
|------|---|-----------------|----------|------|----------|
| I    | 1 | (0, 4)          | $X_{1} = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} W_{i_{1}i_{2}} W_{i_{2}i_{3}} W_{i_{3}i_{4}} W_{i_{4}i_{1}}$ | 0 | $\propto \|\theta\|^{8}$ |
| II   | 4 | (1, 3)          | $X_{2} = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} \Omega_{i_{1}i_{2}} \Omega_{i_{2}i_{3}} \Omega_{i_{3}i_{4}} \Omega_{i_{4}i_{1}}$ | 0 | $\leq C \tau^{2} \|\theta\|^{4} \|\theta\|^{4} = o(\|\theta\|^{4})$ |
| IIIa | 4 | (2, 2)          | $X_{3} = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} \Omega_{i_{1}i_{2}} \Omega_{i_{2}i_{3}} \Omega_{i_{3}i_{4}} \Omega_{i_{4}i_{1}}$ | 0 | $\leq C \tau^{4} \|\theta\|^{6} \|\theta\|^{2} = o(\tau^{4} \|\theta\|^{6} \|\theta\|^{2})$ |
| IIIb | 2 | (2, 2)          | $X_{4} = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} \tilde{\Omega}_{i_{1}i_{2}} \tilde{\Omega}_{i_{2}i_{3}} \tilde{\Omega}_{i_{3}i_{4}} \tilde{\Omega}_{i_{4}i_{1}}$ | 0 | $\leq C \tau^{4} \|\theta\|^{2} = o(\|\theta\|^{2})$ |
| IV   | 4 | (3, 1)          | $X_{5} = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} \tilde{\Omega}_{i_{1}i_{2}} \tilde{\Omega}_{i_{2}i_{3}} \tilde{\Omega}_{i_{3}i_{4}} \tilde{\Omega}_{i_{4}i_{1}}$ | 0 | $\leq \tau^{6} \|\theta\|^{2} \|\theta\|^{2}$ |
| V    | 1 | (4, 0)          | $X_{6} = \sum_{i_{1},i_{2},i_{3},i_{4}(\text{dist})} \tilde{\Omega}_{i_{1}i_{2}} \tilde{\Omega}_{i_{2}i_{3}} \tilde{\Omega}_{i_{3}i_{4}} \tilde{\Omega}_{i_{4}i_{1}} \sim \text{tr}(\tilde{\Omega}^{4})$ | 0 | $\leq \tau^{6} \|\theta\|^{4} \|\theta\|^{4}$ |

Here we omit the calculation details, because similar calculations were done in [17]. In their Theorem 4.4, they analyzed $Q_{W}(T)$ for $T$ equal to a rank-1 matrix (denoted by $\tilde{\Omega}$ there). However, their proof does not rely on the condition that $\tilde{\Omega}$ is rank-1 and applies to any symmetric matrix. They actually proved the following lemma:

**Lemma C.2.** Consider a DBCM model where (2.3) and [2.3] hold. Let $W = A - \Omega$ and let $Q_{W}(T)$ be the random variable defined in (C.110). As $n \to \infty$, suppose there is a constant $C > 0$ and a scalar $\alpha_{n} > 0$ such that $\alpha_{n} \leq C$, $\alpha_{n} \|\theta\| \to \infty$, and $|T_{ij}| \leq C \alpha_{n} \theta_{i} \theta_{j}$ for all $1 \leq i, j \leq n$. Then, $E[Q_{W}(T)] = \text{tr}(T^{4}) + o(\|\theta\|^{4})$ and $\text{Var}(Q_{W}(T)) \leq C(\|\theta\|^{8} + \alpha_{n}^{6} \|\theta\|^{8} \|\theta\|^{6})$. 

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We now set \( T = \tilde{\Omega}^{(m,0)} \) and verify the conditions of Lemma C.2. Recall that \( \tau = \hat{\lambda}_1/\lambda_1 \), where \( \hat{\lambda}_1 \) and \( \lambda_1 \) are the respective largest (in magnitude) eigenvalue of \( \tilde{\Omega}^{(m,0)} \) and \( \Omega \). By Lemma 4.6
\[
\tau \leq C, \quad \tau \|\theta\| \to \infty, \quad |\tilde{\Omega}_{ij}^{(m,0)}| \leq C\tau \theta_i \theta_j, \quad \text{for all } 1 \leq i, j \leq n.
\]
Therefore, we can apply Lemma C.2 with \( \alpha_n = \tau \). The claim follows immediately. \( \square \)

**C.8 Proof of Lemma 4.8**

Before proceed, recall [4.24] that
\[
\tilde{Q}^{(m,0)} = \sum_{i_1,i_2,i_3,i_4(\text{dist})} (W_{i_1,i_2} + \tilde{\Omega}_{i_1i_2}^{(m,0)})(W_{i_2,i_3} + \tilde{\Omega}_{i_2i_3}^{(m,0)})(W_{i_3,i_4} + \tilde{\Omega}_{i_3i_4}^{(m,0)})(W_{i_4,i_1} + \tilde{\Omega}_{i_4i_1}^{(m,0)}).
\]

Here \( \tilde{\Omega}^{(m,0)} = \Omega - \Omega^{(m,0)} \) and \( \Omega^{(m,0)} \) is as in [4.21]. By Lemma 4.5, \( \Omega^{(m,0)} = \Theta \Pi P_0 \Pi' \Theta \), for a rank-\( m \) matrix \( P_0 \). If \( m = K \) and \( \Pi_0 = \Pi \), it can be verified that \( P_0 = P \). Therefore, \( \Omega^{(m,0)} = \Omega \), and \( \tilde{\Omega}^{(m,0)} \) reduces to a zero matrix. In this case, \( \tilde{Q}^{(m,0)} \) reduces to \( Q_n \) in [4.12]. It means that we can treat Lemma 4.3 as a “special case” of Lemma 4.8 with \( \tilde{\Omega}^{(m,0)} \) being a zero matrix. We thus combine the proofs of two lemmas.

We now show the claim. First, we introduce two proxies of \( Q^{(m,0)} \). By definition,
\[
Q^{(m,0)} = \sum_{i_1,i_2,i_3,i_4(\text{dist})} (A_{i_1,i_2} - \tilde{\Omega}_{i_1i_2}^{(m,0)})(A_{i_2,i_3} - \tilde{\Omega}_{i_2i_3}^{(m,0)})(A_{i_3,i_4} - \tilde{\Omega}_{i_3i_4}^{(m,0)})(A_{i_4,i_1} - \tilde{\Omega}_{i_4i_1}^{(m,0)}).
\]

By [4.19], \( \tilde{\Omega}^{(m,0)} \) is defined by \( \hat{\theta}, \Pi_0 \), and \( \hat{P} \). For \( 1 \leq k \leq m, \) let \( \lambda_k^{(m,0)} \) and \( \psi_k^{(m,0)} \) be the same as in [C.96]. Then, \( (\hat{\theta}, \hat{P}) \) are obtained by replacing \( \psi_k^{(m,0)} \) in [4.10]. For the rest of the proof, we write \( \psi_k = 1_k^{(m,0)} \) for short. It follows that, for \( 1 \leq k, \ell \leq K \) and \( i \in \mathcal{N}_k \),
\[
\hat{\theta}_i^{(m,0)} = \frac{1}{\sqrt{1_k^{(m,0)} A_k}} \sqrt{1_k^{(m,0)} A_k}, \quad \hat{P}_{k\ell}^{(m,0)} = \frac{1_k^{(m,0)} A_{k\ell}}{\sqrt{(1_k^{(m,0)} A_k)(1_{k\ell}^{(m,0)})}}, \quad \text{with } 1_k = 1_k^{(m,0)} \text{ (for short)}.
\]

We plug it into [4.19] and note that \( d = A_1 \). It yields that, for \( i \in \mathcal{N}_k^{(m,0)} \) and \( j \in \mathcal{N}_k^{(m,0)} \),
\[
\tilde{\Omega}_{ij}^{(m,0)} = d_{ij} \cdot \hat{U}_{kl}^{(m,0)}, \quad \text{where } \hat{U}_{kl}^{(m,0)} = \frac{1_k^{(m,0)} A_{kl}}{(1_k^{(m,0)} d)(1_{kl}^{(m,0)})}.
\]

At the same time, in [C.100], we have seen that (recall: \( d^* = \Omega_1 \))
\[
\Omega_{ij}^{(m,0)} = d_{ij}^* \cdot U_{kl}^{(m,0)}, \quad \text{where } U_{kl}^{(m,0)} = \frac{1_k \Omega_{kl}}{1_k d^*(1_{kl} d^*)}.
\]

Note that \( \Omega, d^* \) are approximately \( \mathbb{E}[A], \mathbb{E}[d] \) but there is subtle difference. We thus introduce an intermediate quantity:
\[
U_{k\ell}^{(m,0)} = \frac{1_k \mathbb{E}[A] 1_{\ell}}{(1_k \mathbb{E}[d])(1_{\ell} \mathbb{E}[d])}.
\]

We now use [C.112]–[C.114] to decompose \( A_{ij} - \tilde{\Omega}_{ij}^{(m,0)} \). Recall that \( \tilde{\Omega}_{ij}^{(m,0)} = \Omega_{ij} - \Omega_{ij}^{(m,0)} \). We immediately have
\[
A_{ij} - \tilde{\Omega}_{ij}^{(m,0)} = W_{ij} + \tilde{\Omega}_{ij}^{(m,0)} + \Omega_{ij}^{(m,0)} - \tilde{\Omega}_{ij}^{(m,0)}.
\]
Note that and so we introduce a proxy to this term as 

\[
\Omega_{ij}^{(m, 0)} = U_{k\ell} \left[ (\mathbb{E}_d_i)(\mathbb{E}_d_j - d_j) + (\mathbb{E}_d_j)(\mathbb{E}_d_i - d_i) \right].
\]

(C.116)

By comparing it with (4.24), we can see that the above expression of 

\[
Q_m(\hat{E}^i_{(m, k\ell)} = \left[ (m, k\ell) \right] (\hat{E}^i_{(m, k\ell)} + d^*_{ij}U_{k\ell} - (\mathbb{E}_d_i)(\mathbb{E}_d_j)U_{k\ell} + U_{k\ell}(d_i - \mathbb{E}_d_i)(d_j - \mathbb{E}_d_j) + (U_{k\ell} - \tilde{U}_{k\ell})(\mathbb{E}_d_i)(\mathbb{E}_d_j))
\]

where \( \tilde{r}_{ij}^{(m, 0)} = \tilde{U}_{k\ell}(d_i - \mathbb{E}_d_i)(d_j - \mathbb{E}_d_j) \)

(C.117)

and

\[
\epsilon_{ij}^{(m, 0)} = [d^*_{ij}d^*_{ij}U_{k\ell} - (\mathbb{E}_d_i)(\mathbb{E}_d_j)U_{k\ell} + (U_{k\ell} - \tilde{U}_{k\ell})(\mathbb{E}_d_i)(\mathbb{E}_d_j) + (U_{k\ell} - \tilde{U}_{k\ell})(\mathbb{E}_d_i)(d_j - \mathbb{E}_d_j) + (\mathbb{E}_d_j)(d_i - \mathbb{E}_d_i)]
\]

We plug the above results into (C.115) to get

\[
A_{ij} - \tilde{\Omega}_{ij}^{(m, 0)} = \tilde{\Omega}_{ij}^{(m, 0)} + W_{ij} + \delta_{ij}^{(m, 0)} + \epsilon_{ij}^{(m, 0)} + \tilde{r}_{ij}^{(m, 0)}.
\]

(C.119)

We then use (C.119) to define two proxies of \( Q_n^{(m, 0)} \). For any \( 1 \leq i \neq j \leq n \), let

\[
X_{ij} = \tilde{\Omega}_{ij}^{(m, 0)} + W_{ij} + \delta_{ij}^{(m, 0)} + \tilde{r}_{ij}^{(m, 0)} + \epsilon_{ij}^{(m, 0)},
\]

\[
X_{ij}^* = \tilde{\Omega}_{ij}^{(m, 0)} + W_{ij} + \delta_{ij}^{(m, 0)} + \tilde{r}_{ij}^{(m, 0)} + \epsilon_{ij}^{(m, 0)},
\]

\[
X_n = \tilde{\Omega}_{ij}^{(m, 0)} + W_{ij} + \delta_{ij}^{(m, 0)} + \tilde{r}_{ij}^{(m, 0)} + \epsilon_{ij}^{(m, 0)},
\]

\[
\tilde{X}_{ij} = \tilde{\Omega}_{ij}^{(m, 0)} + W_{ij}.
\]

(C.120)

Correspondingly, we introduce

\[
Q_n^{(m, 0)} = \sum_{i, j, k, \ell} X_{i \ell, i, j, k, \ell}(\mathbb{E}_d_i)(\mathbb{E}_d_j) - d_{ij}
\]

\[
\tilde{Q}_n^{(m, 0)} = \sum_{i, j, k, \ell} \tilde{X}_{i \ell, i, j, k, \ell}(\mathbb{E}_d_i)(\mathbb{E}_d_j) - d_{ij}
\]

\[
Q_n^{(s, m, 0)} = \sum_{i, j, k, \ell} X_{i \ell, i, j, k, \ell}(\mathbb{E}_d_i)(\mathbb{E}_d_j) - d_{ij}
\]

\[
\tilde{Q}_n^{(s, m, 0)} = \sum_{i, j, k, \ell} \tilde{X}_{i \ell, i, j, k, \ell}(\mathbb{E}_d_i)(\mathbb{E}_d_j) - d_{ij}
\]

(C.121)

By comparing it with (4.24), we can see that the above expression of \( \tilde{Q}_n^{(m, 0)} \) is the same as before. Additionally, by (C.119), the above expression of \( Q_n^{(m, 0)} \) is also equivalent to the definition. The other two quantities, \( Q_n^{(s, m, 0)} \) and \( \tilde{Q}_n^{(s, m, 0)} \), are the two proxies we introduce here.

Next, we decompose

\[
Q_n^{(m, 0)} - \tilde{Q}_n^{(m, 0)} = (Q_n^{(s, m, 0)} - \tilde{Q}_n^{(m, 0)}) + (\tilde{Q}_n^{(s, m, 0)} - Q_n^{(m, 0)}) + (Q_n^{(m, 0)} - \tilde{Q}_n^{(s, m, 0)}).
\]

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For any random variables $X, Y, Z$, we know that $\mathbb{E}[X + Y + Z] = \mathbb{E}X + \mathbb{E}Y + \mathbb{E}Z$ and $\text{Var}(X + Y + Z) \leq 3\text{Var}(X) + 3\text{Var}(Y) + 3\text{Var}(Z)$. Therefore, to show the claim, we only need to study the mean and variance of each term in the above equation. The next three lemmas are proved in Sections D.3–D.5.

Lemma C.3. Let $b_n = 2\|\theta\|^4 \cdot [g'V^{-1}(PH^2P \circ PH^2P)V^{-1}g]$ be the same as in (4.13). Under conditions of Lemma 4.3, it is true that

$$\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}] = b_n + o(\|\theta\|^4), \quad \text{and} \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) = o(\|\theta\|^8),$$

Let $\tau = \tilde{\lambda}_1/\lambda_1$ be the same as in (4.25). Under conditions of Lemma 4.8, it is true that

$$\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}] = o(\tau^4\|\theta\|^8), \quad \text{and} \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) \leq C\tau^6\|\theta\|^8\|\theta\|^8 + o(\|\theta\|^8).$$

Lemma C.4. Under conditions of Lemma 4.3, it is true that

$$\mathbb{E}[\bar{Q}_n^{(m,0)} - Q_n^{(m,0)}] = o(\|\theta\|^4), \quad \text{and} \quad \text{Var}(\bar{Q}_n^{(m,0)} - Q_n^{(m,0)} = o(\|\theta\|^8).$$

Under conditions of Lemma 4.8, it is true that

$$\mathbb{E}[\bar{Q}_n^{(m,0)} - Q_n^{(m,0)}] = o(\|\theta\|^4 + \tau^4\|\theta\|^8), \quad \text{and} \quad \text{Var}(\bar{Q}_n^{(m,0)} - Q_n^{(m,0)}) = o(\|\theta\|^8 + \tau^6\|\theta\|^8\|\theta\|^\delta).$$

Lemma C.5. Under conditions of Lemma 4.3, it is true that

$$\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}] = o(\|\theta\|^4), \quad \text{and} \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) = o(\|\theta\|^8).$$

Under conditions of Lemma 4.8, it is true that

$$\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}] = o(\|\theta\|^4 + \tau^4\|\theta\|^8), \quad \text{and} \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) = o(\|\theta\|^8 + \tau^6\|\theta\|^8\|\theta\|^\delta).$$

We now prove Lemma 4.3 and Lemma 4.8. By Lemma C.3–Lemma C.5 under the conditions of Lemma 4.3,

$$\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}] = b_n = o(\|\theta\|^4), \quad \text{and} \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) = o(\|\theta\|^8),$$

which implies $\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)} - b_n] = o(\|\theta\|^8)$ and completes the proof of Lemma 4.3. Under the conditions of Lemma 4.8, it follows from Lemma C.3–Lemma C.5 that

$$\mathbb{E}[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}] = o(\tau^4\|\theta\|^8) \quad \text{and} \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) \leq C\tau^6\|\theta\|^8\|\theta\|^\delta + o(\|\theta\|^8),$$

which completes the proof of Lemma 4.8. \qed

C.9 Proof of Lemma 4.9

Let $G_m$ be the class of $n \times m$ membership matrices that satisfy NSP (the definition of $G_m$ is in Section 4.2). By Theorem 2.2, $\Pi^{(m)} \in G_m$ with probability $1 - O(n^{-3})$. Given any $\Pi_0 \in G_m$, let $B_n^{(m)}(\Pi_0)$ be defined in the same way as in (1.15), except that $(\theta, \hat{g}, \hat{V}, \hat{P}, \hat{H})$ are defined based on $\Pi_0$ instead of $\Pi^{(m)}$. Then, with probability $1 - O(n^{-3})$,

$$B_n^{(m)} \leq \max_{\Pi_0 \in G_m} B_n(\Pi_0).$$

It follows from the probability union bound that

$$\mathbb{P}(B_n^{(m)} > C\|\theta\|^4) \leq \sum_{\Pi_0 \in G_m} \mathbb{P}(B_n(\Pi_0) > C\|\theta\|^4) + O(n^{-3}).$$
Since \( m < K \) and \( K \) is finite, \( \mathcal{G}_m \) has only a bounded number of elements. Therefore, it suffices to show that
\[
P(B_n(\Pi_0) > C\|\theta\|^4) = o(1), \quad \text{for each } \Pi_0 \in \mathcal{G}_m. \quad \text{(C.122)}
\]
We now show (C.122). From now on, we fix \( \Pi_0 \in \mathcal{G}_m \) and write \( B_n(\Pi_0) = B_n \) for short. By \[1.15\] and direct calculations,
\[
B_n = 2\|\hat{\theta}\|^4 \cdot \hat{g}\hat{V}^{-1}(\hat{P}\hat{H}^2\hat{P} \circ \hat{P}\hat{H}^2\hat{P})\hat{V}^{-1}\hat{g} = 2\|\hat{\theta}\|^4 \cdot \sum_{1 \leq k, \ell \leq m} \hat{g}_k\hat{g}_\ell |(\hat{P}\hat{H}^2\hat{P})_{k,\ell}|^2 / \left( (\hat{P}\hat{g}) \cdot (\hat{P}\hat{g}) \right),
\]

where \( \hat{P}_k \) denotes the \( k \)th column of \( \hat{P} \). We have mis-used the notations \((\hat{\theta}, \hat{g}, \hat{V}, \hat{P}, \hat{H})\), using them to refer to the counterparts of original definitions with \( \Pi^{(m)} \) replaced by \( \Pi_0 \). Denote by \( N'(m,0), N'(m,0), \ldots, N'(m,0) \) the pseudo-communities defined by \( \Pi_0 \). Let \( 1_k^{(m,0)} \in \mathbb{R}^n \) be such that \( 1_k^{(m,0)}(i) = 1 \{ i \in N'(m,0) \} \). We write \( 1_k = 1_k^{(m,0)} \) when there is no confusion. By \[1.14\],
\[
\hat{g} = (1_k'\hat{\theta}) / \|\hat{\theta}\|_1, \quad \hat{h}_k = (1_k'\hat{\theta}^21_k) / \|\hat{\theta}\|^2, \quad 1 \leq k \leq m.
\]
Note that \( \hat{g}, \hat{h} \) and \( \hat{P} \) all have non-negative entries, with all entries of \( \hat{g} \) and \( \hat{h} \) further bounded by 1. Moreover, the diagonals of \( \hat{P} \) are all equal to 1. It follows that, for all \( 1 \leq k, \ell \leq m, \)
\[
0 \leq \hat{g}_k \leq \hat{P}_k \hat{g}, \quad \text{and} \quad 0 \leq (\hat{P}\hat{H}^2\hat{P})_{k,\ell} \leq (\hat{P}^2)_{k,\ell}.
\]
As a result,
\[
B_n \leq 2\|\hat{\theta}\|^4 \sum_{k,\ell=1}^m |(\hat{P}^2)_{k,\ell}|^2 \leq 2\|\hat{\theta}\|^4 \cdot m^4 \|\hat{P}\|_{\text{max}}^4,
\]
where \( \|\cdot\|_{\text{max}} \) is the element-wise maximum norm. Below, we study \( \|\hat{P}\|_{\text{max}} \) and \( \|\hat{\theta}\| \) separately.

First, we bound \( \|\hat{P}\|_{\text{max}} \). By \[1.10\],
\[
\hat{P}_{k,\ell} = (1_k'A1\ell)/(1_k'A1_k)(1\ell'A1\ell).
\]
Write \( 1_k'A1\ell = \sum_{i \in N'(m,0), j \in N'(m,0)} A_{ij} \), where \( E[A_{ij}] = \Omega_{ij} \), and \( \sum_{i \in N'(m,0), j \in N'(m,0)} \text{Var}(A_{ij}) \leq \sum_{i \in N'(m,0), j \in N'(m,0)} C\theta_i\theta_j \leq C(1_k'\theta)(1\ell'\theta) \). We apply the Bernstein’s inequality \[5.4\] to get
\[
\mathbb{P}(\|1_k'A1\ell - 1_k'\Omega_1\ell\| > t) \leq 2 \exp\left(-\frac{t^2/2}{C(1_k'\theta)(1\ell'\theta) + t/3}\right), \quad \text{for all } t > 0.
\]
By NSP, each pseudo-community \( N'(m,0) \) contains at least one true community, say, \( N_k \). Combining it with the condition \[2.2\] gives \( 1_k'\theta \geq \sum_{i \in N_k} \theta_i \geq C\|\theta\|_1 \). At the same time, \( 1_k'\theta \leq \|\theta\|_1 \). We thus have \( 1_k'\theta \asymp \|\theta\|_1 \gg \sqrt{\log(n)} \). Similarly, we can show that \( 1\ell'\Omega_1 \gg \|\theta\|_1^2 \). In the above equation, if we choose \( t = C_1\|\theta\|_1 \sqrt{\log(n)} \) for a properly large constant \( C_1 > 0 \), then the right hand side is \( O(n^{-3}) \). In other words, with probability \( 1 - O(n^{-3}) \),
\[
\|1_k'A1\ell - 1_k'\Omega_1\ell\| \leq C\|\theta\|_1 \sqrt{\log(n)}.
\]
Since \( 1_k'\Omega_1 \gg \|\theta\|_1^2 \gg \|\theta\|_1\sqrt{\log(n)} \), the above implies \( 1_k'A1\ell \gg \|\theta\|_1^2 \). We combine this result with the probability union bound. It follows that there exists a constant \( C_2 > 1 \) such that with probability \( 1 - O(n^{-3}) \),
\[
C_2^{-1}\|\theta\|_1^2 \leq \min_{1 \leq k, \ell \leq m} \{1_k'A1\ell\} \leq \max_{1 \leq k, \ell \leq m} \{1_k'A1\ell\} \leq C_2\|\theta\|_1^2 \quad \text{(C.124)}
\]
We plug it into the expression of \( \|\hat{P}\|_{\text{max}} \) above and can easily see that
\[
\|\hat{P}\|_{\text{max}} \leq C, \quad \text{with probability } 1 - O(n^{-3}). \quad \text{(C.125)}
\]
Second, we bound $\|\hat{\theta}\|^2$. By (1.10), $\hat{\theta}_i = d_i \sqrt{\frac{1}{n_k} \sum_k} (1_k \theta_k) / (1_k \theta_k A_1)$ for $i \in \mathcal{N}^{(m,0)}$. It follows that

$$\|\hat{\theta}\|^2 = \sum_{k=1}^m \frac{(1_k \theta_k^2)}{(1_k \theta_k A_1)^2}, \text{ where } D = \text{diag}(d_1, d_2, \ldots, d_n).$$

Note that $1_k \theta_k A_1 = \sum_{i=1}^m 1_k \theta_k$. It follows from (C.124) that $1_k \theta_k A_1 \approx \|\theta\|^2$ and $1_k \theta_k A_1 \approx \|\theta\|^2$. As a result, $\|\hat{\theta}\|^2 \leq C\|\theta\|^2 + \sum_{i=1}^m \frac{(1_k \theta_k^2)}{(1_k \theta_k A_1)^2}$. Since $\sum_{k=1}^m (1_k \theta_k^2) = \|\theta\|^2$, we immediately have $\|\hat{\theta}\|^2 \leq C\|\theta\|^2 \|\theta\|^2$, with probability $1 - O(n^{-3})$. (C.126)

Recall that $d_i = \sum_{j \neq i} A_{ij} = \sum_{j \neq i} \Omega_{ij} + W_{ij}$. Then,

$$\|d\|^2 = \sum_{i=1}^n \sum_{j \neq i, s \neq i} \Omega_{ij} + W_{ij} (\Omega_{is} + W_{is}) = \sum_{i,j,s \neq i} \Omega_{ij} + 2 \sum_{i \neq j} \left( \sum_{i \neq i} \Omega_{is} \right) W_{ij} + \sum_{i,j,s \neq i} W_{ij}^2 + \sum_{i,j,s \neq i} W_{ij} W_{is}.$$

Since $\sum_{s \notin \{i,j\}} \Omega_{is} \leq C\theta_1 \|\theta\|_1$, we have $\mathbb{E}[X_1^2] \leq \sum_{i \neq j} C\theta_1^2 \|\theta\|_1^2$. Moreover, $X_2 \geq 0$ and $\mathbb{E}[X_2] = \sum_{i \neq j} \mathbb{E}[W_{ij}^2] \leq C\|\theta\|_1^2$. Last, $\mathbb{E}[X_3^2] = 2 \sum_{i,j,s \neq i} \text{Var}(W_{ij} W_{is}) \leq C \sum_{i,j,s \neq i} \theta_1^2 \theta_1 \theta_1 \leq C\|\theta\|^2 \|\theta\|^2$. By Markov’s inequality, for any sequence $\epsilon_n \to 0$,

$$|X_1| \leq C\epsilon^{-1}\|\theta\|^2 \|\theta\|^2, \quad |X_2| \leq C\epsilon^{-1}\|\theta\|^2, \quad |X_3| \leq C\epsilon^{-1}\|\theta\|^2 \|\theta\|^2.$$

It is not hard to see that we can choose a property $\epsilon_n \to 0$ so that all the right hand sides are $o(\|\theta\|^2 \|\theta\|^2)$. Then, with probability $1 - \epsilon_n$,

$$\|d\|^2 = \sum_{i,j,s \neq i, s \neq i} \Omega_{ij} \Omega_{is} + o(\|\theta\|^2 \|\theta\|^2) \leq C\|\theta\|^2 \|\theta\|^2.$$

We plug it into (C.126) to get

$$\|\hat{\theta}\|^2 \leq C\|\theta\|^2, \quad \text{with probability } 1 - O(1).$$

Then, (C.122) follows from plugging (C.125) and (C.127) into (C.123). This proves the claim. 

### D Proof of secondary lemmas

#### D.1 Proof of Lemma B.1

Note that for any set $M \subset \{1, 2, \ldots, n\}$ and $z \in \mathbb{R}^d$,

$$\sum_{i \in M} \|y_i - z\|^2 = \sum_{i \in M} \|(y_i - \bar{y}_M) + (\bar{y}_M - z)\|^2$$

$$= \sum_{i \in M} \|y_i - \bar{y}_M\|^2 + 2(\bar{y}_M - z)' \sum_{i \in M} (y_i - \bar{y}_M) + |M| \|\bar{y}_M - z\|^2$$

$$= \sum_{i \in M} \|y_i - \bar{y}_M\|^2 + |M| \|\bar{y}_M - z\|^2.$$

The clusters associated with $RSS$ are $A = \tilde{A} \cup C$ and $B$, and the clusters associated with $\tilde{RSS}$ are $\tilde{A}$ and $\tilde{B} = C \cup B$. By direct calculations,

$$RSS = \sum_{i \in A} \|y_i - \bar{y}_A\|^2 + \sum_{i \in C} \|y_i - \bar{y}_A\|^2 + \sum_{i \in B} \|y_i - \bar{y}_B\|^2$$
\[
\widehat{RSS} = \sum_{i \in A} \|y_i - \bar{y}_A\|^2 + \sum_{i \in C} \|y_i - \bar{y}_C\|^2 + \sum_{i \in B} \|y_i - \bar{y}_B\|^2
\]

\[
\widehat{RSS} - RSS = (|B|\|y_B - \bar{y}_B\|^2 + |C|\|y_C - \bar{y}_B\|^2) - (|A|\|y_A - \bar{y}_A\|^2 + |C|\|y_C - \bar{y}_A\|^2). \quad \text{(D.128)}
\]

By definition,
\[
\bar{y}_A = \frac{|A| - |C|}{|A|} \bar{y}_A + \frac{|C|}{|A|} \bar{y}_C, \quad \bar{y}_B = \frac{|B|}{|B| + |C|} \bar{y}_B + \frac{|C|}{|B| + |C|} \bar{y}_C.
\]

Re-arranging the terms, we have
\[
\bar{y}_A - \bar{y}_A = \frac{|C|}{|A| - |C|} (\bar{y}_A - \bar{y}_C), \quad \bar{y}_B - \bar{y}_B = \frac{|C|}{|B| + |C|} (\bar{y}_C - \bar{y}_B), \quad \bar{y}_C - \bar{y}_B = \frac{|B|}{|B| + |C|} (\bar{y}_C - \bar{y}_B).
\]

We plug (D.129) into (D.128) to get
\[
\widehat{RSS} - RSS = \left(\frac{|B|}{(|B| + |C|)^2} + \frac{|C|}{(|B| + |C|)^2}\right) \|\bar{y}_C - \bar{y}_B\|^2
\]

\[
- \left(\frac{|A| - |C|}{|A| - |C|} \frac{|C|^2}{|B| + |C|} + \frac{|C|}{|B| + |C|}\right) \|\bar{y}_C - \bar{y}_A\|^2
\]

\[
= \frac{|B|}{|B| + |C|} \|\bar{y}_C - \bar{y}_B\|^2 - \frac{|A| |C|}{|A| - |C|} \|\bar{y}_C - \bar{y}_A\|^2.
\]

This proves the claim. \[\square\]

### D.2 Proof of Lemma C.1

Recall that \(1_k \in \mathbb{R}^n\) is such that \(1_k(i) = \{i \in N_k\}\), \(D = \text{diag}(d_1, d_2, \ldots, d_n)\), and \(d^* = \Omega_1\). We re-state the claims as

\[
\frac{1'_k A_1 n}{1'_k \Omega_1 n} \xrightarrow{p} 1, \quad \frac{1'_k A_1 n}{1'_k \Omega_1 n} \xrightarrow{p} 1, \quad \frac{1'_k A_11_k}{1'_k \Omega_1 k} \xrightarrow{p} 1. \quad \text{(D.130)}
\]

and

\[
\frac{\|\hat{\theta}\|_1}{\|\theta\|_1} \xrightarrow{p} 1, \quad \frac{\|\hat{\theta}\|_1}{\|\theta\|_1} \xrightarrow{p} 1, \quad \frac{1'_k D^2 1_k}{1'_k (D^*)^2 1_k} \xrightarrow{p} 1. \quad \text{(D.131)}
\]

We note that convergence in \(\ell^2\)-norm implies convergence in probability. Hence, to show \(X \xrightarrow{p} 1\) for a random variable \(X\), it is sufficient to show \(\mathbb{E}[(X - 1)^2] \to 0\). Using the equality \(\mathbb{E}[(X - 1)^2] = (\mathbb{E}X - 1)^2 + \text{Var}(X)\), we only need to prove that \(\mathbb{E}[X] \to 1\) and \(\text{Var}(X) \to 0\), for each variable \(X\) on the left hand sides of (D.130)-(D.131).

First, we prove the three claims in (D.130). Since the proofs are similar, we only show the proof of the first claim. Note that \(1'_k \Omega_1 n = \sum_{\ell} (1'_k \theta)(1'_k \theta) P_{k\ell}\). Under the conditions (2.1)-(2.2), \(1'_k \Omega_1 n \propto \|\theta\|^2\). Additionally, \(1'_n \text{diag}(\Omega) 1_n = \|\theta\|^2\). It follows that

\[
\frac{|\mathbb{E}[1'_n A_1 n]|}{1'_n \Omega_1 n} - 1 = \frac{1'_n \text{diag}(\Omega) 1_n}{1'_n \Omega_1 n} \propto \frac{\|\theta\|^2}{\|\theta\|_1} = o(1),
\]

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where the last inequality is because \( \|\theta\|^2 \leq \theta_{\max} \|\theta\|_1 \leq C \|\theta\|_1 \) and \( \|\theta\|_1 \rightarrow \infty \). Also, since the upper triangular entries of \( A \) are independent, \( \text{Var}(I_{n}^{\prime}A_1) = 4\text{Var}(\sum_{i<j} A_{ij}) \leq 4\sum_{i<j} \Omega_{ij} \leq C\|\theta\|_{1}^{2} \). It follows that

\[
\frac{\text{Var}(I_{n}^{\prime}A_1)}{(I_{n}^{\prime}N_1)^2} \leq \frac{C\|\theta\|_{1}^{2}}{\|\theta\|_{1}^{2}} = o(1).
\]

Combining the above gives \((1_{k}^{\prime}A_{1n})/(1_{k}^{\prime}N_{1n}) \rightarrow 1\).

Second, we show the first claim in \((D.131)\). By Theorem 2.2 \( \hat{\Pi}^{(K)} = \Pi \), with a probability of \( 1 - O(n^{-3}) \). It is sufficient to consider the re-defined \( \hat{\theta} \) where \( \Pi^{(K)} \) is replaced with \( \Pi \). Combining it with the definition in \((1.10)\), we have \( \hat{\theta}_i = d_i \sqrt{1_{k}^{\prime}A_1k}/(1_{k}^{\prime}A_1n) \). It follows that

\[
\|\hat{\theta}\|_1 = \sum_{k=1}^{K} (1_{k}^{\prime}d) \sqrt{1_{k}^{\prime}A_1k} \leq \sum_{k=1}^{K} \sqrt{1_{k}^{\prime}A_1k},
\]

where the last equality is because of \( d = A_1n \). At the same time, it is easy to see that \( 1_{k}^{\prime}\Omega_{1k} = (1_{k}^{\prime}\theta)P_{kk}(1_{k}^{\prime}\theta) = (1_{k}^{\prime}\theta)^2 \), which implies \( \|\theta\|_1 = \sum_{k=1}^{K} \sqrt{1_{k}^{\prime}\Omega_{kk}} \). We thus have

\[
\|\hat{\theta}\|_1 = \sum_{k=1}^{K} \delta_k X_k, \quad \text{where} \quad \delta_k = \frac{\sqrt{1_{k}^{\prime}\Omega_{kk}}}{\sum_{t=1}^{K} \sqrt{1_{t}^{\prime}\Omega_{tt}}}, \quad X_k = \sqrt{1_{k}^{\prime}A_1k}. \]

By the last claim in \((D.130)\) and the continuous mapping theorem, \( X_k \xrightarrow{p} 1 \) for each \( 1 \leq k \leq K \). Also, \( \sum_{k=1}^{K} \delta_k = 1 \). It follows immediately that \( \sum_{k=1}^{K} \delta_k X_k \xrightarrow{p} 1 \). This proves \( \|\hat{\theta}\|_1/\|\theta\|_1 \xrightarrow{p} 1 \).

Next, we show the last claim in \((D.131)\). Recall that \( d^* = \Omega_{1n} \) and \( D^* = \text{diag}(d^*) \). Then, for \( i \in N_k \), \( \sum_{t \in N_k}(d^*_t)^2 \leq C\sum_{t \in N_k} \theta_i \|\theta\|_1^2 \leq C\|\theta\|_1^2 \|\theta\|_1^2 \). At the same time, \( d^*_i \geq \theta_i P_{kk}(1_{k}^{\prime}\theta) \geq C\|\theta\|_1 \theta_i \), where we have used the condition \((2.2)\). As a result, \( \sum_{t \in N_k}(d^*_t)^2 \geq C\|\theta\|_1^2 \sum_{t \in N_k} \theta_i^2 \geq C\|\theta\|_1^2 \|\theta\|_1^2 \), where we have used \((2.2)\) again. Combining the above gives

\[
1+k(D^*)^2 1_k \geq \|\theta\|^2 \|\theta\|^2. \tag{D.132}
\]

Note that \( 1_{k}^{\prime}D^2 1_k = \sum_{i \in N_k} \sum_{i+j \neq 0} A_{it}^2 = \sum_{i,j} \sum_{t \in N_k \setminus \{i,j\}} A_{it}A_{jt} \). Similarly, \( 1_{k}^{\prime}(D^*)^2 1_k = \sum_{i,j} \sum_{t \in N_k \setminus \{i,j\}} \Omega_{it} \Omega_{jt} \). We now write

\[
1_{k}^{\prime}D^2 1_k = \sum_{i \in N_k} \sum_{t \in N_k \setminus \{i\}} A_{it}^2 + 2 \sum_{i<j} \sum_{t \in N_k \setminus \{i,j\}} A_{it}A_{jt},
\]

\[
1_{k}^{\prime}(D^*)^2 1_k = \sum_{i \in N_k} \Omega_{it}^2 + 2 \sum_{i<j} \sum_{t \in N_k \setminus \{i,j\}} \Omega_{it} \Omega_{jt}.
\]

Note that \( \mathbb{E}[A_{it}^2] = \mathbb{E}[A_{it}] = \Omega_{it} \) and \( \mathbb{E}[A_{it}A_{jt}] = \Omega_{it} \Omega_{jt} \). As a result,

\[
|\mathbb{E}[1_{k}^{\prime}D^2 1_k] - 1_{k}^{\prime}(D^*)^2 1_k| \leq \sum_{i \in N_k} (\Omega_{it} - \Omega_{it}^2) + \sum_{i} \Omega_{it}^2 + 2 \sum_{i<j} (\Omega_{it} \Omega_{ji} + \Omega_{ij} \Omega_{jj})
\]

\[
\leq C \sum_{i \in N_k} \theta_i \theta_i + \|\theta\|^2 + C \sum_{i,j} \theta_i^2 \theta_j
\]

\[
\leq C \|\theta\|_1^2 + \|\theta\|^2 + \|\theta\|_1^3 \|\theta\|_1
\]

\[
\leq C \|\theta\|_1^3,
\]

where the last line is because \( \|\theta\|^3 \leq \theta_{\max} \|\theta\|_1 \leq C \|\theta\|_1 \). Combining it with \((D.132)\) gives

\[
\left| \frac{\mathbb{E}[1_{k}^{\prime}D^2 1_k]}{1_{k}^{\prime}(D^*)^2 1_k} - 1 \right| \leq \frac{C \|\theta\|_1^2}{\|\theta\|^2} = o(1). \tag{D.133}
\]

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We then compute the variance. Write for short $X = \sum_{i<j} \sum_{t \in \mathcal{N}_k \setminus \{i,j\}} A_{it}A_{jt}$. Note that

$$\text{Var}(\mathbf{1}'_k D^2 \mathbf{1}_k) \leq 2 \text{Var}\left(\sum_i \sum_{t \in \mathcal{N}_k \setminus \{i\}} A_{it}^2\right) + 2 \text{Var}(2X) \leq C \sum_i \sum_{t \in \mathcal{N}_k} \Omega_{it} + 8 \text{Var}(X) \leq C \|\theta\|^2_1 + 8 \text{Var}(X).$$

Since $A_{it}A_{jt} = (\Omega_{it} + W_{it})(\Omega_{jt} + W_{jt})$, we write

$$X = \sum_{i<j} \sum_{t \in \mathcal{N}_k \setminus \{i,j\}} \Omega_{it} \Omega_{jt} + 2 \sum_j \sum_{t \in \mathcal{N}_k \setminus \{j\}} \left( \sum_i \Omega_{it} \right) W_{jt} + \sum_{i<j} \sum_{t \in \mathcal{N}_k \setminus \{i,j\}} W_{it} W_{jt} \equiv X_0 + 2X_1 + X_2.$$

Here, $X_0$ is non-stochastic. Therefore, $\text{Var}(X) = \text{Var}(2X_1 + X_2) \leq 8 \text{Var}(X_1) + 2 \text{Var}(X_2)$. It is seen that $\text{Var}(X_1) \leq \sum_j \sum_{t \in \mathcal{N}_k} (\sum_i \Omega_{it})^2 \cdot \Omega_{jt} \leq C \sum_j \sum_{t \in \mathcal{N}_k} (\theta_1 \|\theta\|_1)^2 \cdot \Omega_{jt} \leq C \|\theta\|^3_3 \|\theta\|^2_1$. Additionally, the summands in $X_2$ are mutually uncorrelated, so $\text{Var}(X_3) \leq C \sum_{i<j} \sum_{t \in \mathcal{N}_k} \Omega_{it} \Omega_{jt} \leq C \sum_{i,j,t} \Omega_{it} \Omega_{jt} \leq C \|\theta\|^2_1 \|\theta\|^2_1$. Combining the above gives

$$\text{Var}(X) \leq C \left(\|\theta\|^3_3 \|\theta\|^2_1 + \|\theta\|^2_2 \|\theta\|^2_1\right) \leq C \|\theta\|^3_3 \|\theta\|^3_3,$$

where in the second inequality we have used $\|\theta\|^2 \leq \|\theta\|_1 \|\theta\|_3$, which is a direct consequence of the Cauchy-Schwarz inequality. We combine the above to get

$$\text{Var}(\mathbf{1}'_k D^2 \mathbf{1}_k) \leq C \left(\|\theta\|^2_1 + \|\theta\|_3 \|\theta\|_3\right) \leq C \left(\|\theta\|^2_1 + \theta_{\text{max}} \|\theta\|^2_1 \|\theta\|^3_3\right),$$

where in the second inequality we have used $\|\theta\|_3 \leq \theta_{\text{max}} \|\theta\|^2_1$. Combining it with (D.132) gives

$$\frac{\text{Var}(\mathbf{1}'_k D^2 \mathbf{1}_k)}{(\mathbf{1}'_k (D^*)^2 \mathbf{1}_k)^2} \leq \frac{C \|\theta\|^4_1}{\|\theta\|^4_1 + C \theta_{\text{max}} \|\theta\|^2_1 \|\theta\|^3_3} = o(1). \quad (D.134)$$

By (D.133) and (D.134), we have $(\mathbf{1}'_k D^2 \mathbf{1}_k)/(\mathbf{1}'_k (D^*)^2 \mathbf{1}_k) \xrightarrow{P} 1$.

Last, we show the second claim in (D.131). Since $\theta_1 = d_1 \sqrt{\mathbf{1}'_k A \mathbf{1}_k / (\mathbf{1}'_k A \mathbf{1}_k)}$, we have

$$\|\theta\|^2 = \sum_{k=1}^\infty \left(\mathbf{1}'_k D^2 \mathbf{1}_k \mathbf{1}'_k A \mathbf{1}_k \right) / \left(\mathbf{1}'_k A \mathbf{1}_k\right)^2.$$

At the same time, $\mathbf{1}'_k \Omega \mathbf{1}_k = (\mathbf{1}'_k \theta)^2$ and $\mathbf{1}'_k \Omega \mathbf{1}_n = (\mathbf{1}'_k \theta) \sum_{\ell=1}^K P_{k\ell}(\mathbf{1}'_k \theta)$. Furthermore, for $i \in \mathcal{N}_k$, $d_i^* = (\mathbf{1}'_k \mathbf{1}_k)_i = \theta_i \sum_{k=1}^K P_{k\ell}(\mathbf{1}'_k \theta)$, and so $\mathbf{1}'_k (D^*)^2 \mathbf{1}_n = (\mathbf{1}'_k \theta^2 \mathbf{1}_k) \sum_{\ell=1}^K P_{k\ell}(\mathbf{1}'_k \theta)^2$. Combining these equalities gives

$$\|\theta\|^2 = \sum_{k=1}^\infty \mathbf{1}'_k \Theta^2 \mathbf{1}_k = \sum_{k=1}^\infty \left(\mathbf{1}'_k (D^*)^2 \mathbf{1}_k \mathbf{1}'_k \Omega \mathbf{1}_k \right) / \left(\mathbf{1}'_k \Omega \mathbf{1}_n\right)^2.$$

It follows that

$$\frac{\|\theta\|^2}{\|\theta\|^2} = \sum_{k=1}^\infty \delta_k X_k, \quad \text{where} \quad \delta_k = \frac{|\mathbf{1}'_k (D^*)^2 \mathbf{1}_k \mathbf{1}'_k \Omega \mathbf{1}_k|}{\sum_{\ell=1}^K |\mathbf{1}'_k (D^*)^2 \mathbf{1}_k \mathbf{1}'_k \Omega \mathbf{1}_k| / \left(\mathbf{1}'_k \Omega \mathbf{1}_n\right)^2}; \quad X_k = \frac{\mathbf{1}'_k D^2 \mathbf{1}_k \mathbf{1}'_k A \mathbf{1}_k \mathbf{1}'_k \Omega \mathbf{1}_k}{\mathbf{1}'_k (D^*)^2 \mathbf{1}_k \mathbf{1}'_k \Omega \mathbf{1}_k / \left(\mathbf{1}'_k A \mathbf{1}_k\right)^2}.$$

By the claims in (D.130) and the last claim in (D.131), as well as the continuous mapping theorem, we have $X_k \xrightarrow{P} 1$ for each $1 \leq k \leq K$. Since $\sum_{k=1}^K \delta_k = 1$, it follows that $\sum_{k=1}^K \delta_k X_k \xrightarrow{P} 1$. This proves that $\|\theta\|^2 / \|\theta\|^2 \xrightarrow{P} 1$. By the continuous mapping theorem again, $\|\theta\| / \|\theta\| \xrightarrow{P} 1$. \qed
D.3 Proof of Lemma C.3

We introduce a notation $M_{ijk\ell}(X) = X_{ij}X_{jk}X_{\ell i}$, for any symmetric $n \times n$ matrix $X$ and distinct indices $(i, j, k, \ell)$. Using the definition in (C.121), we can write

$$Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)} = \sum_{i_1, i_2, i_3, i_4 (\text{dist})} [M_{i_1 i_2 i_3 i_4}(X^*) - M_{i_1 i_2 i_3 i_4}(\tilde{X})], \quad \text{where} \quad \begin{cases} X^*_{ij} = \Omega_{ij}^{(m,0)} + W_{ij} + \delta_{ij}^{(m,0)}, \\ \tilde{X}_{ij} = \Omega_{ij}^{(m,0)} + W_{i\ell}. \end{cases}$$

For the rest of the proof, we omit superscripts in $\tilde{Q}_n^{(m,0)}$ and $\delta_{ij}^{(m,0)}$ to simplify notations. From the expression of $X^*_{ij}$ and $\tilde{X}_{ij}$, we notice that $[M_{i_1 i_2 i_3 i_4}(X^*) - M_{i_1 i_2 i_3 i_4}(\tilde{X})]$ expands to $3^4 - 2^4 = 65$ terms. Consequently, there are 65 post-expansion sums in $Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)}$, each with the form

$$\sum_{i_1, i_2, i_3, i_4 (\text{dist})} a_{i_1 i_2} b_{i_3 i_4} c_{i_3 i_4} d_{i_4 i_1}, \quad \text{where} \quad a, b, c, d \in \{\Omega, W, \delta\}.$$

In the first 4 columns of Table 3, we group these post-expansion sums into 15 distinct terms, where the second column shows the counts of each distinct term. For example, in the setting of Lemma 4.3, $\Omega$ reduces to a zero matrix. Therefore, any post-expansion sum that involves $\Omega$ is zero. Then, it follows from Table 3 that

$$Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)} = 4Y_1 + 4Z_1 + 2Z_2 + 4T_1 + F,$$

(D.35)

where the expression of $(Y_1, Z_1, Z_2, T_1, F)$ are given in the fourth column of Table 3. Similarly, in the setting of Lemma 4.8, we have $Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)} = 4Y_1 + 8Y_2 + 4Y_3 + \ldots + 4T_2 + F$. These are elementary calculations.

To show the claim, we need to study the mean and variance of each post-expansion sum. We take $Y_1$ for example. Let $N_1^{(m,0)}, N_2^{(m,0)}, \ldots, N_m^{(m,0)}$ be the pseudo-communities defined by $\Pi_0$.

For each $1 \leq i \leq n$, let $\tau(i) \in \{1, 2, \ldots, m\}$ be the index of the pseudo-community that contains node $i$. By (C.116),

$$\delta_{i_1 i_2} = U_{\tau(i_1) \tau(i_2)} [(\mathbb{E}d_{i_1})(\mathbb{E}d_{i_2} - d_{i_2}) + (\mathbb{E}d_{i_1} - d_{i_1})]$$

$$= U_{\tau(i_1) \tau(i_2)} : \mathbb{E}d_{i_1} : \left( - \sum_{j : j \neq i_2} W_{j i_2} \right) + U_{\tau(i_1) \tau(i_2)} : \mathbb{E}d_{i_1} : \left( - \sum_{\ell : \ell \neq i_2} W_{\ell i_2} \right)$$

$$= -2 \sum_{j : j \neq i_2} U_{\tau(i_1) \tau(i_2)} : \mathbb{E}d_{i_1} : W_{j i_2}. \quad \text{(D.36)}$$

It follows that

$$Y_1 = -2 \sum_{i_2, i_3, i_4, j} \left( \sum_{i_1} U_{\tau(i_1) \tau(i_2)} : \mathbb{E}d_{i_1} \right) : W_{j i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1},$$

where we note that the indices $\{i_1, i_2, i_3, i_4, j\}$ have to satisfy the constraint that $i_1, i_2, i_3, i_4$ are distinct and that $j \neq i_2$. We can see that $Y_1$ is a weighted sum of $W_{j i_2} W_{i_2 i_3} W_{i_3 i_4} W_{i_4 i_1}$, where the summmands have zero mean and are mutually uncorrelated. The mean and variance of $Y_1$ can be calculated easily. We will use the same strategy to analyze each term in Table 3— we use the expansion of $\delta_{i j}$ in (D.36) to write each post-expansion sum as a weighted sum of monomials of $W$, and then we calculate the mean and variance. The calculations can become very tedious for some terms (e.g., $T_1, T_2$ and $F$), because of combinatorics. Fortunately, similar calculations were done in the proof of Theorem 4.4 in [17], where they analyzed a special case with $U_{k \ell} \equiv 1/v$ for all $1 \leq k, \ell \leq m$. However, their proof does not rely on that $U_{k \ell}$’s are equal but only require that $U_{k \ell}$’s have a uniform upper bound. Essentially, they have proved the following lemma:
Lemma D.1. Consider a DCBM model where (2.1)–(2.2) and (2.4) hold. Let $W = A - \Omega$ and 
$\Delta = \sum_{i,j \in \bar{N}_m \setminus \bar{N}_k} [M_{i,j} \Omega + W + \delta - M_{i,j} \Omega - W]$, where $\Omega$ is a non-stochastic symmetric matrix, $\delta_{ij} = n \cdot \sum_{l=1}^{N_m} \delta_{i,l} \cdot \sum_{l=1}^{N_m} \delta_{j,l}$. Let $\Omega = \{\Omega_{ij}\}$ be an $N_m \times N_m$ matrix, with entries $\Omega_{ij}$. Then, $\|\Omega\|_2 < \infty$ and $\vartheta = \|\Omega\|_2$. Furthermore, if $\Omega$ is a zero matrix, then $\|\Delta\|_2 \leq 0$. Then, $|\Omega|_{1,\infty} \leq C\vartheta^2 + o(\vartheta^2)$.

We check the conditions of Lemma D.1. By Lemma 4.8, $\sigma_\geq C$, $\tau, \vartheta \to \infty$, and $\|\bar{\Omega}\|_2 \leq C\tau, \vartheta$. We now verify that $U_{kl}$ has a uniform upper bound for all $1 \leq k, \ell \leq m$. By (C.141).

$$U_{kl} = \frac{1_k^2 E[A] 1_k}{|1_k^2 E[d]| (1_k^2 E[d])},$$

where $1_k = 1_k^{(m,0)}$ is the same as in (C.99). Since $E[A_{ij}] = \Omega_{ij} \leq C\vartheta, \vartheta$, we have $0 \leq 1_k^2 E[A] 1_k \leq C\|\vartheta\|_2^2$. At the same time, by the NSP of SCORE, for each $1 \leq k, \ell \leq m$, there is at least one true community $N_k$, such that $N_k \subset N_k^{(m,0)}$. It follows that $1_k^2 E[d] = \sum_{i,j \in N_k \setminus \bar{N}_k} \Omega_{ij} \geq \Omega_{ij} \geq \sum_{k,\ell \in N_k \setminus \bar{N}_k} \vartheta, \vartheta, \vartheta, \vartheta$. We plug these results into $U_{kl}$ to get

$$0 \leq U_{kl} \leq C\|\vartheta\|_1^{-2}. \tag{D.137}$$

Then, the conditions of Lemma D.1 are satisfied. We apply this lemma with $\alpha_n = \tau$ and $\lambda_{ij} = U_{kl}$ for $i \in N_k^{(m,0)}$ and $j \in N_k^{(m,0)}$. It yields that, under the conditions of Lemma 4.8,

$$|E[Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}]| = o(\tau^4\|\vartheta\|_8), \quad \text{Var}(Q_n^{(m,0)} - \bar{Q}_n^{(m,0)}) \leq C\tau^6\|\vartheta\|_8^4 + o(\|\vartheta\|_8^4),$$
and that under the conditions of Lemma 4.3 (where Ω is a zero matrix)
\[ |E[Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)}]| \leq C\|\theta\|^4, \quad \text{Var}(Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)}) \leq o(\|\theta\|^8). \]

This proves all the desirable claims except for the following one: Under conditions of Lemma 4.3.

It remains to show that, under the conditions of Lemma 4.3,
\[ E[Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)}] = b_n + o(\|\theta\|^4). \]  
(D.138)

We now show (D.138). By (D.135), we only need to calculate the expectations of \( Y_1, Z_1, Z_2, T_1 \) and \( F \). From Table 3, \( E[Y_1] = 0 \). We now study \( E[Z_1] \). Recall that \( \delta_{ij} = U_{\tau(i)\tau(j)}(E_{d_i}) - (E_{d_j}) \), where \( \tau(i) \) is the index of pseudo-community defined by \( \Pi_0 \) that contains node \( i \). We plug \( \delta_{ij} \) into \( Z_1 \), by elementary calculations,
\[
Z_1 = \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)\tau(i_3)\tau(i_4)}(E_{d_i})(E_{d_j} - d_j)^2(E_{d_k} - d_k)W_{i_3i_4}W_{i_4i_1} + 2 \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)\tau(i_3)\tau(i_4)}(E_{d_i})(E_{d_j} - d_j)(E_{d_k} - d_k)W_{i_3i_4}W_{i_4i_1} + \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)\tau(i_3)\tau(i_4)}(E_{d_i} - d_i)(E_{d_j} - d_j)^2(W_{i_3i_4} - d_i)W_{i_4i_1}.
\]

We write it as \( Z_1 = Z_{11} + 2Z_{12} + Z_{13} \). For \( Z_{1k} \), we can further replace \( E_{d_i} - d_i \) by \( \sum_j W_{ji} \) and write \( Z_{1k} \) as a weighted sum of monomials of \( W \). Then, \( E[Z_{1k}] \neq 0 \) if some of the monomials are \( W_{i_3i_4}^2W_{i_4i_1} \). This will not happen in \( Z_{11} \) and \( Z_{12} \), and so only \( Z_{13} \) has a nonzero mean. It is seen that
\[
E[Z_{13}] = E \left[ \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)\tau(i_3)\tau(i_4)}(E_{d_i})(E_{d_j} - d_j)^2 \left( \sum_{k \neq i_3} W_{i_3k}W_{i_4i_1} \right) \right].
\]
\[
= E \left[ \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)\tau(i_3)\tau(i_4)}(W_{i_3i_4})(E_{d_i})(E_{d_j} - d_j)^2 \right] \cdot E \left[ W_{i_3i_4}^2W_{i_4i_1} \right].
\]

Here, in the second line, we only keep \( (j, k) = (i_4, i_4) \), because other \( (j, k) \) only contribute zero means. Recall that we are considering the setting of Lemma 4.3 where \( m = K \) and \( \Pi_0 = \Pi \). In C.113, we introduce a proxy of \( U_{k\ell} \) as \( U_{k\ell}^* = (1_k^T\Omega\Omega_k)/(1_k^T\Omega_k\Omega_k) \), for all \( 1 \leq k, \ell \leq K \). Note that \( \Omega_{ij} = \theta_i\theta_j \) for \( i \in N_k \) and \( j \in N_k \). At the same time, by (4.13), \( g_k = (1_k^T\theta)/\|\theta\|_1 \), and \( V_{kk} = (\text{diag}(Pg))_{kk} = \sum_{\ell} P_{k\ell}(1_{\ell}^T\theta)/\|\theta\|_1 \). It follows that
\[
U_{k\ell}^* = \frac{P_{k\ell}(1_{\ell}^T\theta)/\|\theta\|_1}{(1_k^T\theta)(1_k^T\theta)/\|\theta\|_1} \cdot P_{k\ell} = \frac{P_{k\ell}}{V_{kk}V_{\ell\ell}\|\theta\|_1^2}. \]

Comparing \( U_{k\ell} \) with \( U_{k\ell}^* \) (see (C.113)-(C.114)), the difference is negligible. (We can rigorously justify this by directly computing the difference caused by replacing \( U_{k\ell} \) with \( U_{k\ell}^* \), similarly as in the proof of \( c_n = \text{tr}(\Omega^4) + o(\|\theta\|^8) \) in Section C.1; see details therein. Such calculations are too elementary and so omitted.) We thus have
\[
U_{k\ell} = [1 + o(1)] \cdot \frac{P_{k\ell}}{V_{kk}V_{\ell\ell}\|\theta\|_1^2}. \]  
(D.140)
Furthermore, for \( i \in \mathcal{N}_k \),
\[
E[d_i] = [1 + o(1)] \sum_{j=1}^{n} \Omega_{ij} = [1 + o(1)] \cdot \theta_i \left[ \sum_{\ell=1}^{K} P_{k\ell}(1_i^\ell) \right] = [1 + o(1)] \cdot \theta_i \|\theta\|_1 V_{kk}.
\]  \hspace{1cm} (D.141)

Also, \( E[W^2_{ij}] = \Omega_{ij}(1 - \Omega_{ij}) = \Omega_{ij}[1 + o(1)]. \) We plug these results into (D.139) to get
\[
E[Z_{13}] = [1 + o(1)] \sum_{k_1, k_2, k_3, k_4} \frac{P_{k_1 k_2} P_{k_3 k_4} P_{k_4 k_1}}{V_{k_1 k_2} V_{k_3 k_4} \|\theta\|_1^2} \cdot (\theta^2_2 \|\theta\|_1^2) V_{kk}^2 \cdot \Omega_{ii} \Omega_{ij},
\]  \hspace{1cm} (D.142)

where in the third line we have used the definition of \( H \) which gives \( H_{kk} = (1_k^T \Theta^2 1_k)^{1/2}/\|\theta\| \). It follows that
\[
E[Z_1] = E[Z_{13}] = [1 + o(1)] \cdot b_n/2.
\]  \hspace{1cm} (D.143)

We then study \( E[Z_2] \). Similarly, we first plug in \( \delta_{ij} = U_{\tau(i)\tau(j)}(\langle Ed_i \rangle(\langle Ed_j \rangle - d_j) + (\langle Ed_j \rangle(\langle Ed_i \rangle - d_i)) \) and then plug in \( d_i - Ed_i = \sum_{j \neq i} W_{ij} \). This allows us to write \( Z_2 \) as a weighted sum of monomials of \( W \). When calculating \( E[Z_2] \), we only keep monomials of the form \( W^2_{i_1 i_4} W^2_{i_2 i_3} \). It follows that
\[
E[Z_2] = E \left[ \sum_{i_1, i_2, i_3, i_4 (\text{dist})} U_{\tau(i_1)\tau(i_2)}(\langle Ed_{i_1} \rangle(\langle Ed_{i_2} \rangle - d_{i_2}) W_{i_2 i_3} U_{\tau(i_3)\tau(i_4)}(\langle Ed_{i_3} \rangle(\langle Ed_{i_4} \rangle - d_{i_4}) W_{i_4 i_1} \right]
\]  \hspace{1cm} (D.144)

\[
= E \left[ \sum_{i_1, i_2, i_3, i_4 (\text{dist})} U_{\tau(i_1)\tau(i_2)}(\langle Ed_{i_1} \rangle W^2_{i_2 i_3} U_{\tau(i_3)\tau(i_4)}(\langle Ed_{i_3} \rangle W^2_{i_4 i_1} \right]
\]  \hspace{1cm} (D.145)

\[
= 2 \sum_{k_1, k_2, j=1}^{k_3, k_4} \sum_{i \in \mathcal{N}_j} P_{k_1 k_2} P_{k_3 k_4} \left( \frac{\theta_i^2 \|\theta\|_1^2}{V_{k_1 k_2} V_{k_3 k_4} \|\theta\|_1^2} \right) \Omega_{ii} \Omega_{ij} \cdot \Omega_{ii} \Omega_{ij},
\]  \hspace{1cm} (D.146)

\[
= [1 + o(1)] \cdot \|\theta\|^2 \cdot \Theta^2 \|\theta\|^{-1} \cdot \langle (PH^2 P) \circ (PH^2 P) \rangle V^{-1} g.
\]  \hspace{1cm} (D.147)

Here, the first two lines come from discarding terms with mean zero, the fourth line is because of (D.140) and (D.141), and the last line is obtained similarly as in the equation above (D.142). Hence,
\[
E[Z_2] = b_n \cdot [1 + o(1)].
\]  \hspace{1cm} (D.148)
We then study $\mathbb{E}[T_1]$. We plug in $\delta_{ij} = U_{\tau(i)\tau(j)}(\mathbb{E}d_i)(\mathbb{E}d_j - d_j) + \left(\mathbb{E}d_j\right)(\mathbb{E}d_i - d_i)$ to get

$$
T_1 = 2 \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)} U_{\tau(i_2)\tau(i_3)} U_{\tau(i_3)\tau(i_4)} \times \left(\mathbb{E}d_{i_1}\right)(\mathbb{E}d_{i_2} - d_{i_2})^2 \left(\mathbb{E}d_{i_3}\right)^2 \left(\mathbb{E}d_{i_4} - d_{i_4}\right) W_{i_4 i_1} + \text{rem}
$$

$$
\equiv 2T_{11} + \text{rem}.
$$

We claim that

$$
|\mathbb{E}[\text{rem}]| = o(\|\theta\|^4).
$$

The calculations here are similar to those in Equation (E.176) of [17], where $T_1$ there (with a slightly different meaning) is decomposed into $2T_{1a} + 2T_{1b} + 2T_{1c} + 2T_{1d}$. Here, $T_{11}$ is analogous to $T_{1d}$, and the remainder term is analogous to $2T_{1a} + 2T_{1b} + 2T_{1c}$. In [17], it was shown that $|\mathbb{E}[T_{1a}]| + |\mathbb{E}[T_{1b}]| + |\mathbb{E}[T_{1c}]| = o(\|\theta\|^4);$ see Equations (E.179)-(E.181) in [17]. We can adapt their proof to show $|\mathbb{E}[\text{rem}]| = o(\|\theta\|^4).$ Since the calculations are elementary, we omit the details to save space. We then compute $\mathbb{E}[T_{11}]$. Since $\mathbb{E}d_i - d_i = -\sum_{j: i \neq j} W_{ij}$, it follows that

$$
\mathbb{E}[T_{11}] = -E\left[ \sum_{k_1, k_2} \sum_{k_3, k_4} U_{k_1 k_2} U_{k_2 k_3} U_{k_3 k_4} (\mathbb{E}d_{i_1}) \left( \sum_{i_5, i_6, i_7, i_8} W_{i_5 i_6} \right)^2 (\mathbb{E}d_{i_3})^2 \left( \sum_{i_9} W_{i_9 i_6} \right) W_{i_4 i_1} \right]
$$

$$
= -E\left[ \sum_{k_1, k_2} \sum_{k_3, k_4} U_{k_1 k_2} U_{k_2 k_3} U_{k_3 k_4} (\mathbb{E}d_{i_1}) \left( \sum_{i_5, i_6, i_7, i_8} W_{i_5 i_6} \right)^2 (\mathbb{E}d_{i_3})^2 \mathbb{E}[W_{i_4 i_1}] \right]
$$

$$
= -E\left[ \sum_{k_1, k_2} \sum_{k_3, k_4} U_{k_1 k_2} U_{k_2 k_3} U_{k_3 k_4} (\mathbb{E}d_{i_1}) (\mathbb{E}d_{i_3})^2 \mathbb{E}[W_{i_4 i_1}] \right] \cdot [1 + o(1)] \left( \theta_{i_1} \|\theta\| \sum_{k_5} P_{k_2 k_5 g_{k_5}} \right)
$$

$$
= -[1 + o(1)] \sum_{k_1, k_2, k_3, k_4} P_{k_1 k_2} P_{k_2 k_3} P_{k_3 k_4} P_{k_4 k_1} \sum_{k_5} \frac{\theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4}}{V_{k_2 k_5}} \left( \sum_{j=1}^{4} \sum_{i_j \in N_{i_j}} \theta_{i_j}^2 \theta_{i_3}^2 \theta_{i_4} \right)
$$

$$
= -[1 + o(1)] \cdot \|\theta\|^4 \delta^{V-1} \cdot \left( (PH^2 P) \circ (PH^2 P) \right) V^{-1} g,
$$

where we have plugged in (D.140)-(D.141) in the second last line, and the last line can be derived similarly as in the equation above (D.142). We have proved $\mathbb{E}[T_{11}] = -[1 + o(1)] \cdot b_n / 2$. Then,

$$
\mathbb{E}[T_1] = 2\mathbb{E}[T_{11}] + o(\|\theta\|^4) = -b_n \cdot [1 + o(1)].
$$

We then study $\mathbb{E}[F]$. Similar to the analysis of $T_1$, after plugging in $\delta_{ij} = U_{\tau(i)\tau(j)}(\mathbb{E}d_i)(\mathbb{E}d_j - d_j) + \left(\mathbb{E}d_j\right)(\mathbb{E}d_i - d_i)$, we can obtain that

$$
F = \text{rem} + 2 \sum_{i_1, i_2, i_3, i_4} U_{\tau(i_1)\tau(i_2)} U_{\tau(i_2)\tau(i_3)} U_{\tau(i_3)\tau(i_4)} \times \left(\mathbb{E}d_{i_1}\right)(\mathbb{E}d_{i_2} - d_{i_2})^2 \left(\mathbb{E}d_{i_3}\right)^2 \left(\mathbb{E}d_{i_4} - d_{i_4}\right)^2 \left(\mathbb{E}d_{i_1}\right),
$$

$$
\equiv \text{rem} + 2F_1,
$$

where $|\mathbb{E}[\text{rem}]| = o(\|\theta\|^4)$.

The proof of $|\mathbb{E}[\text{rem}]| = o(\|\theta\|^4)$ is similar to the proof of (E.188)-(E.189) in [17]. There they analyzed a quantity $F$, which bears some similarity to the $F$ here, and decomposed $F = 2F_a +$
We now study the case $b \parallel 4$, $o \parallel E$. We can mimic their proof to show $\|b\|_2 = 1$ and that

$$\left| \sum_{i \in \mathbb{N}} \left( \sum_{j} c_{ij} \sum_{k} \frac{b_{ik} P_{km}}{V_{k2} V_{k4}} \right) \theta_{ij} \theta_{ik} \theta_{ik} \right| = \frac{1}{2} \| \theta \|_4 \cdot \| \theta \|_1 \cdot \frac{1}{2} \sum_{i \in \mathbb{N}} \sum_{j} \left( \sum_{k} \frac{b_{ik} P_{km}}{V_{k2} V_{k4}} \right) \theta_{ij} \theta_{ik} \theta_{ik}$$

where in the second line we discard terms with mean zero, in the third line we plug in (D.140)-(D.141), and in the last line we use elementary calculations similar to those in the equation above (D.142). It follows that $E[F_1] = [1 + o(1)] \cdot b_n/2$ and that

$$E[F] = 2E[F_1] + o(\|\theta\|^4) = [1 + o(1)] \cdot b_n.$$  \hfill (D.145)

We now plug (D.142), (D.143), (D.144), and (D.145) into (D.135) to get

$$E[Q_n^{s(m,0)} - \tilde{Q}_n^{s(m,0)}] = 4E[Z_1] + 2E[Z_2] + 4E[T_1] + E[F] = [1 + o(1)] \cdot \left( 4(b_n/2) + 2b_n - 4b_n + b_n \right) = [1 + o(1)] \cdot b_n.$$

Since $b_n \asymp \|\theta\|^4$, (D.138) follows immediately. \hfill \Box

### D.4 Proof of Lemma C.4

Similar to the proof of Lemma C.3, we use the notation $M_{ijki}(X) = X_{ij} X_{jk} X_{ki} X_{li}$. By (C.121),

$$\tilde{Q}_n^{s(m,0)} - Q_n^{s(m,0)} = \sum_{i_1, i_2, i_3, i_4(\text{dist})} [M_{i_1 i_2 i_3 i_4}(\tilde{X}^*) - M_{i_1 i_2 i_3 i_4}(X^*)],$$

where

$$\begin{align*}
\tilde{X}_{i_1}^{c_{ij}} &= \tilde{\tilde{X}}_{i_1}^{c_{ij}} + W_{i_1} + \delta_{ij}^{c_{ij}} + \tilde{r}_{ij}^{c_{ij}}, \\
X_{i_1}^{c_{ij}} &= \tilde{\tilde{X}}_{i_1}^{c_{ij}} + W_{i_1} + \delta_{ij}^{c_{ij}}.
\end{align*}$$

For the rest of the proof, we omit the superscripts $(m,0)$ in $(\tilde{\tilde{X}}, \tilde{\delta}, \tilde{r})$. There are $4^4 - 3^4 = 175$ post-expansion sums in $Q_n^{s(m,0)} - Q_n^{s(m,0)}$, each with the form

$$S \equiv \sum_{i_1, i_2, i_3, i_4(\text{dist})} a_{i_1 i_2 i_3 i_4} d_{i_1 i_2} c_{i_3 i_4} d_{i_3 i_4}, \quad \text{where} \quad a, b, c, d \in \{ \tilde{\tilde{X}}, W, \delta, \tilde{r} \}.$$

Here we use $S$ as a generic notation for any post-expansion sum. To show the claim, it suffices to bound $E[S]$ and $\text{Var}(S)$ for each post-expansion sum $S$.

We now study $S$. Let $N_1^{s(m,0)}, N_2^{s(m,0)}, \ldots, N_k^{s(m,0)}$ be the pseudo-communities defined by $\Pi_0$. By (C.116) and (C.117), for $i \in N_1^{s(m,0)}$ and $j \in N_k^{s(m,0)},$

$$\delta_{ij} = U_{ikl} [(\mathbb{E}d_i)(d_j - \mathbb{E}d_j) + (\mathbb{E}d_j)(d_i - \mathbb{E}d_i)], \quad \tilde{r}_{ij} = -\tilde{U}_{ikl}(d_i - \mathbb{E}d_i)(d_j - \mathbb{E}d_j).$$
The term $\tilde{U}_{k\ell}$ has a complicated correlation with each summand, so we want to “replace” it with $U_{k\ell}$. Introduce a proxy of $\tilde{r}_{ij}$ as

$$r_{ij} = -U_{k\ell}(d_i - \mathbb{E}d_i)(d_j - \mathbb{E}d_j) \quad (D.147)$$

We define a proxy of $S$ as

$$T = \sum_{i_1,i_2,i_3,i_4(\text{dist})} a_{i_1i_2} b_{i_3i_4} c_{i_1i_2} d_{i_3i_4}, \quad \text{where} \quad a, b, c, d \in \{\tilde{\Omega}, W, \delta, r\}. \quad (D.148)$$

We note that $T$ is also a generic notation, and it has a one-to-one correspondence with $S$. For example, if $S = \sum_{i_1,i_2,i_3,i_4(\text{dist})} \delta_{i_1i_2} W_{i_2i_3} \tilde{\Omega}_{i_3i_4} \tilde{r}_{i_4i_1}$, then $T = \sum_{i_1,i_2,i_3,i_4(\text{dist})} \delta_{i_1i_2} W_{i_1i_3} \tilde{\Omega}_{i_3i_4} \tilde{r}_{i_4i_1}$; if $S = \sum_{i_1,i_2,i_3,i_4(\text{dist})} \delta_{i_1i_2} \tilde{r}_{i_1i_2} \tilde{r}_{i_3i_4} W_{i_4i_1}$, then $T = \sum_{i_1,i_2,i_3,i_4(\text{dist})} \delta_{i_1i_2} \tilde{r}_{i_1i_2} \tilde{r}_{i_3i_4} W_{i_4i_1}$. Therefore, to bound the mean and variance of $S$, we only need to study $T$ and $S - T$ separately.

First, we study the mean and variance of $T$. Since $d_i - \mathbb{E}d_i = \sum_{j:j \neq i} W_{ij}$, we can write $\delta_{ij}$ as a linear form of $W$ and $r_{ij}$ as a quadratic form of $W$. We then plug them into the expression of $T$ and write $T$ as a weighted sum of monomials of $W$. Take $T = \sum_{i_1,i_2,i_3,i_4(\text{dist})} r_{i_1i_2} W_{i_1i_3} W_{i_2i_4} W_{i_3i_1}$, for example. It can be re-written as (note: $\tau(i)$ is the index of pseudo-community that contains node $i$)

$$T = -\sum_{i_1,i_2,i_3,i_4(\text{dist})} U_{\tau(i_1)\tau(i_2)} \left( \sum_{j_1,j_1 \neq i_1} W_{i_1j_1} \right) \left( \sum_{j_2,j_2 \neq i_2} W_{i_2j_2} \right) W_{i_3i_4} W_{i_4i_1}$$

$$= -\sum_{i_1,i_2,i_3,i_4(\text{dist})} U_{\tau(i_1)\tau(i_2)} W_{i_1j_1} W_{i_2j_2} W_{i_3i_4} W_{i_4i_1}.$$  

Then, we can compute the mean and variance of $T$ directly. We use the same strategy to analyze each of the 175 post-expansion sums of the form $T$. Similar calculations were conducted in the proof of Lemma E.11 of [17]. The setting of Lemma E.11 is a special case where $U_{k\ell} \equiv 1/v$ for a scalar $v$. However, their proof does not rely on that $U_{k\ell}$’s are equal to each other. Instead, their proof only requires a universal upper bound on $U_{k\ell}$. In fact, they have proved the following lemma:

**Lemma D.2.** Consider a DCBM model where $[2.4]$ and $[2.6]$ hold. Let $W = A - \Omega$ and $\Delta = \sum_{i_1,i_2,i_3,i_4(\text{dist})} \left[ M_{i_1i_2i_3i_4}(\tilde{\Omega} + W + \delta + r) - M_{i_1i_2i_3i_4}(\Omega + W + \delta) \right]$, where $\tilde{\Omega}$ is a non-stochastic symmetric matrix, $\delta_{ij} = d_i - \mathbb{E}d_i$, $\{v_{ij}, u_{ij}\} \subseteq \mathbb{R}$ are non-stochastic scalars, $d_i$ is the degree of node $i$, and $M_{i_1i_2i_3i_4}$ is as defined above. As $n \to \infty$, suppose there is a constant $C > 0$ and a vector $\alpha_n > 0$ such that $\alpha_n \leq C$, $\alpha_n\|\theta\| \to \infty$, $|\tilde{\Omega}_{ij}| \leq C\alpha_n\theta_i \theta_j$, $|v_{ij}| \leq C\|\theta\|^{-1}$, and $|u_{ij}| \leq C\|\theta\|^{-1}$ for $1 \leq i, j \leq n$. Let $T$ be an arbitrary post-expansion sum of $\Delta$. Then, $|E[T]| \leq C\alpha_n^2\|\theta\|^6 + o(\|\theta\|^4)$ and $\text{Var}(T) = o(\alpha_n^6\|\theta\|^8\|\theta\|^6 + \|\theta\|^8)$.

We apply Lemma D.2 for $\alpha_n = \tau$ and $v_{ij} = u_{ij} = U_{\tau(i)\tau(j)}$. By Lemma 4.6, $\tau \leq C$, $\tau\|\theta\| \to \infty$, and $|\tilde{\Omega}_{ij}| \leq C\tau\theta_i \theta_j$. In (D.137), we have seen that $|U_{k\ell}| \leq C\|\theta\|^{-1}$. The conditions of Lemma D.2 are satisfied. We immediately have: Under the conditions of Lemma 4.8 (note: $\tau\|\theta\| \to \infty$)

$$|E[T]| \leq C\tau^2\|\theta\|^6 + o(\|\theta\|^4) = o(\tau^4\|\theta\|^8), \quad \text{Var}(T) = o(\tau^6\|\theta\|^6\|\theta\|^6 + \|\theta\|^8). \quad (D.149)$$

and under the conditions of Lemma 4.3 (i.e., $\tilde{\Omega}$ is a zero matrix and $\tau = 0$),

$$|E[T]| = o(\|\theta\|^4), \quad \text{Var}(T) = o(\|\theta\|^8). \quad (D.150)$$

Next, we study the variable $(S - T)$. In (D.146) and (D.148), if we group the summands based on pseudo-communities $(i_1, i_2, i_3, i_4)$, then we have

$$S = \sum_{1 \leq k_1, k_2, k_3, k_4 \leq m} S_{k_1k_2k_3k_4} \quad \text{and} \quad T = \sum_{1 \leq k_1, k_2, k_3, k_4 \leq m} T_{k_1k_2k_3k_4}. $$

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where \(S_{k_1 k_2 k_3 k_4}\) contains all the summands such that \(i_s \in \mathcal{N}_{k_s}^{(m, 0)}\) for \(s = 1, 2, 3, 4\). By straightforward calculations and definitions of \((\hat{r}_{ij}, \tilde{r}_{ij})\), we have

\[
S_{k_1 k_2 k_3 k_4} = \hat{U}_{k_1 k_2}^s \hat{U}_{k_2 k_3}^s \hat{U}_{k_3 k_4}^s \hat{U}_{k_4 k_1}^s \sum_{i_s \in \mathcal{N}_{k_s}^{(m, 0)}} \hat{a}_{i_1 i_2} \hat{b}_{i_2 i_3} \hat{c}_{i_3 i_4} \hat{d}_{i_4 i_1},
\]

\[
T_{k_1 k_2 k_3 k_4} = U_{k_1 k_2}^s U_{k_2 k_3}^s U_{k_3 k_4}^s U_{k_4 k_1}^s \sum_{i_s \in \mathcal{N}_{k_s}^{(m, 0)}} \tilde{a}_{i_1 i_2} \tilde{b}_{i_2 i_3} \tilde{c}_{i_3 i_4} \tilde{d}_{i_4 i_1},
\]

where \(\hat{a}_{ij}, \hat{b}_{ij}, \hat{c}_{ij}, \hat{d}_{ij} \in \{ \Omega_{ij}, W_{ij}, \delta_{ij}, -(d_i - E d_i)(d_j - E d_j) \}\).

Here \(\ell_a \in \{0, 1\}\) is an indicator about whether \(a_{ij}\) takes the value of \(\hat{r}_{ij}\) in \(S\), and \((\ell_b, \ell_c, \ell_d)\) are similar. For example, if \(S = \sum_{i_1, i_2, i_3, i_4(dist)} \delta_{i_1 i_2} W_{i_2 i_4} \hat{\Omega}_{i_3 i_4} \hat{r}_{i_4 i_1}\), then \((\ell_a, \ell_b, \ell_c, \ell_d) = (0, 0, 0, 1)\); if \(S = \sum_{i_1, i_2, i_3, i_4(dist)} \delta_{i_1 i_2} \hat{r}_{i_2 i_3} \hat{r}_{i_3 i_4} W_{i_4 i_1}\), then \((\ell_a, \ell_b, \ell_c, \ell_d) = (0, 1, 1, 0)\). For any post-expansion sum \(S\) considered here, \(1 \leq \ell_a + \ell_b + \ell_c + \ell_d \leq 4\). To study the difference between \(S_{k_1 k_2 k_3 k_4}\) and \(T_{k_1 k_2 k_3 k_4}\), we introduce an intermediate term

\[
R_{k_1 k_2 k_3 k_4} = \left( \frac{1}{\|\theta\|^4} \right) \|\theta\|^{\ell_a + \ell_b + \ell_c + \ell_d} \sum_{i_s \in \mathcal{N}_{k_s}^{(m, 0)}} \tilde{a}_{i_1 i_2} \tilde{b}_{i_2 i_3} \tilde{c}_{i_3 i_4} \tilde{d}_{i_4 i_1}.
\]

In fact, \(R_{k_1 k_2 k_3 k_4}\) has a similar form as \(T_{k_1 k_2 k_3 k_4}\) except that the scalar \(U_{k_\ell}\) in the definition of \(r_{ij}\) (see \(\text{(D.147)}\)) is replaced by \(1/\|\theta\|^2\). We apply Lemma \(\text{D.2}\) with \(u_{ij} = 1/\|\theta\|^2\). It yields that, under conditions of Lemma \(\text{4.3}\)

\[
\mathbb{E}[R_{k_1 k_2 k_3 k_4}] = o(\|\theta\|^4), \quad \text{Var}(R_{k_1 k_2 k_3 k_4}) = o(\|\theta\|^8),
\]

and under conditions of Lemma \(\text{4.8}\)

\[
\mathbb{E}[R_{k_1 k_2 k_3 k_4}] \leq C \tau^2 \|\theta\|^6 + o(\|\theta\|^4), \quad \text{Var}(R_{k_1 k_2 k_3 k_4}) = o(\|\theta\|^8 + \tau^6 \|\theta\|^{8} \|\theta\|^{6}).
\]

Particularly, since \(\mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \text{Var}(X)\) for any variable \(X\), we have

\[
\|\theta\|^{-4} \mathbb{E}[R_{k_1 k_2 k_3 k_4}] \leq \begin{cases} 
\begin{aligned}
o(\|\theta\|^4), & \text{for setting of Lemma } \text{4.3} \\
C \tau^4 \|\theta\|^{8} + o(\|\theta\|^4 + \tau^6 \|\theta\|^{4} \|\theta\|^{6}), & \text{for setting of Lemma } \text{4.8}
\end{aligned}
\end{cases}
\]

\[
= \begin{cases} 
\begin{aligned}
o(\|\theta\|^4), & \text{for setting of Lemma } \text{4.3} \\
C \|\theta\|^8, & \text{for setting of Lemma } \text{4.8}
\end{aligned}
\end{cases}
\]

Note that in deriving \(\text{(D.151)}\) we have used \(\tau \leq C\) and \(\tau^6 \|\theta\|^4 \|\theta\|^6 \leq \tau^6 \|\theta\|^4 \theta_{\max}^2 \|\theta\|^4 \leq C \|\theta\|^{8}\).

We now investigate \((S_{k_1 k_2 k_3 k_4} - T_{k_1 k_2 k_3 k_4})\). By condition \(\text{2.1}\), \(\sqrt{\log(n)} \ll \|\theta\|^1/\|\theta\|^2\). Hence, we can take a sequence of \(x_n\), such that \(\sqrt{\log(n)} \ll x_n \ll \|\theta\|^1/\|\theta\|^2\), and define the event \(E_n:\)

\[
E_n = \left\{ \left| U_{k\ell} - \hat{U}_{k\ell} \right| \leq \frac{C_0 x_n}{\|\theta\|^1}, \text{ for all } 1 \leq k, \ell \leq m \right\},
\]

where \(C_0 > 0\) is a constant to be decided. To bound the probability of \(E_n^c\), we recall that (by definitions in \(\text{(C.112)}\) and \(\text{(C.114)}\))

\[
\hat{U}_{k\ell} = \frac{1^s_k \mathbb{E}[A]_{k\ell}}{(1^s_k d)(1^s d)} \quad \text{and} \quad U_{k\ell} = \frac{1^s_k \mathbb{E}[A]_{k\ell}}{(1^s_k \mathbb{E}[d])(1^s d \mathbb{E}[d])},
\]

where \(1^s_k\) is a shorthand notation for \(1_k^{(m, 0)}\) in \(\text{(C.99)}\). Using Bernstein’s inequality and mimicking the argument from (E.299)-(E.300) of [17], we can easily show that, there is a constant \(C_1 > 0\) such that, for any \(1 \leq k, \ell \leq m\),

\[
\mathbb{P}\left( 1^s_k \mathbb{E}[A]_{k\ell} - 1^s_k \mathbb{E}[A]_{k\ell} > x_n \|\theta\|^1 \right) \leq 2 \exp(-C_1 x_n^2).
\]
By probability union bound, with probability $1 - 2m^2 \exp(-C_1 x_n^2)$,

$$\max_{1 \leq k, t \leq m} \{|1_k^t A 1_t - 1_k^t \mathbb{E}[A] 1_t|\} \leq x_n \|\theta\|_1.$$  

Furthermore, $1_k^t d - 1_k^t \mathbb{E}[d] = \sum_{t=1}^m \{1_k^t \mathbb{E}[A] 1_t - 1_k^t \mathbb{E}[A] 1_t\}$. So, with probability $1 - 2m^2 \exp(-C_1 x_n^2)$,

$$\max_{1 \leq k, t \leq m} \{|1_k^t d - 1_k^t \mathbb{E}[d]|\} \leq m \cdot x_n \|\theta\|_1.$$  

At the same time, we know that $1_k^t \mathbb{E}[A] 1_t = \|\theta\|_1^2$ and $1_k^t \mathbb{E}[d] = \|\theta\|_1^2$. We plug the above results into the expressions of $U_{kl}$ and $\hat{U}_{kl}$ and can easily find that, with probability $1 - 2m^2 \exp(-C_1 x_n^2)$,

$$\max_{1 \leq k, t \leq m} |\hat{U}_{kl} - U_{kl}| \leq C_0 x_n / \|\theta\|_1^3,$$

for some constant $C_0 > 0$ ($C_0$ still depends on $m$, but $m$ is bounded here). We use the same $C_0$ to define $E_n$. Then,

$$\mathbb{P}(E_n^c) \leq 2m^2 \exp(-C_1 x_n^2) = o(n^{-L}),$$

for any fixed $L > 0$,  \hspace{1cm} (D.154)

where the last equality is due to $x_n^2 \gg \log(n)$. We aim to use \[ (D.154) \] to bound $\mathbb{E}[|S_{k_1 k_2 k_3 k_4} - T_{k_1 k_2 k_3 k_4}| \cdot I_{E_n^c}]$. It is easy to see the trivial bound $|\hat{U}_{kl}| \leq 1$ and $|U_{kl}| \leq 1$. Also, recall that $\tilde{a}_{ij}$ takes value in $\{\bar{\Omega}_{ij}, W_{ij}, \hat{a}_{ij} - (d_i - \mathbb{E}d_i)(d_j - \mathbb{E}d_j)\}$, and so $|a_{ij}| \leq n^2$; we have the same bound for $|\hat{b}_{ij}|, |\tilde{c}_{ij}|, |\hat{d}_{ij}|$. This gives a trivial bound

$$(S_{k_1 k_2 k_3 k_4} - T_{k_1 k_2 k_3 k_4})^2 \leq 2S_{k_1 k_2 k_3 k_4}^2 + 2T_{k_1 k_2 k_3 k_4}^2 \leq 2(n^4 \cdot n^8)^2 + 2(n^4 \cdot n^8)^2 = 4n^{24}.$$  

Combining it with \[ (D.154) \], we have

$$\mathbb{E}[|T_{k_1 k_2 k_3 k_4} - S_{k_1 k_2 k_3 k_4}|^2 \cdot I_{E_n^c}] \leq 4n^{24} \cdot 2m^2 \exp(-C_1 x_n^2) = o(1).$$  \hspace{1cm} (D.155)

At the same time, on the event $E_n$,

$$|S_{k_1 k_2 k_3 k_4} - T_{k_1 k_2 k_3 k_4}| = \left| \hat{U}_{k_1 k_2} \hat{U}_{k_2 k_3} \hat{U}_{k_3 k_4} \hat{U}_{k_4 k_1} - U_{k_1 k_2} U_{k_2 k_3} U_{k_3 k_4} U_{k_4 k_1} \right| \cdot \|\theta\|_1^2 \max_{1 \leq k, t \leq m} |\hat{U}_{kl} / U_{kl} - 1| \cdot \|\theta\|_1^2,$$

$\leq C \|\theta\|_1^2 \cdot \max_{1 \leq k, t \leq m} \left| \hat{U}_{kl} - U_{kl} \right| \cdot \left| R_{k_1 k_2 k_3 k_4} \right|,$

$= C x_n \|\theta\|_1^{-1} \cdot \left| R_{k_1 k_2 k_3 k_4} \right|,$

where the fourth line is because $\|\theta\|_1^{-1} \leq \|U_{kl}\| \leq C \|\theta\|_1^{-2}$ (e.g., see \[ (D.137) \]) and the last line is because $x_n \ll \|\theta\|_1 / \|\theta\|_2^2$. It follows that

$$\mathbb{E}[|T_{k_1 k_2 k_3 k_4} - S_{k_1 k_2 k_3 k_4}| \cdot I_{E_n}] = o(\|\theta\|^{-4}) \cdot \mathbb{E}[R_{k_1 k_2 k_3 k_4}^2].$$  \hspace{1cm} (D.156)

We combine \[ (D.155) \] and \[ (D.156) \] and plug in \[ (D.151) \]. It follows that

$$\mathbb{E}[|T_{k_1 k_2 k_3 k_4} - S_{k_1 k_2 k_3 k_4}|^2] = o(\|\theta\|^{-4}) \cdot \mathbb{E}[R_{k_1 k_2 k_3 k_4}^2] + o(1) = \left\{ \begin{array}{ll} o(\|\theta\|^{-4}), & \text{under conditions of Lemma 4.3} \\ o(\|\theta\|^{-8}), & \text{under conditions of Lemma 4.8} \end{array} \right\}$$

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Since \( m \) is bound, we immediately know that
\[
\mathbb{E}[(S - T)^2] = \begin{cases} o(\|\theta\|^4), & \text{under conditions of Lemma } 4.3 \\ o(\|\theta\|^8), & \text{under conditions of Lemma } 4.8 \end{cases} \tag{D.157}
\]

Last, we combine the results on \( T \) and the results on \( (S - T) \). By (D.149)-(D.150) and (D.157),
\[
|\mathbb{E}[S]| \leq |\mathbb{E}[T]| + |\mathbb{E}[S - T]| \\
\leq |\mathbb{E}[T]| + \sqrt{\mathbb{E}[(S - T)^2]} \\
= \begin{cases} o(\|\theta\|^4) + o(\|\theta\|^2) = o(\|\theta\|^4), & \text{for setting of Lemma } 4.3 \\ o(\|\theta\|^8) + o(\|\theta\|^4) = o(\|\theta\|^8), & \text{for setting of Lemma } 4.8 \end{cases}
\]

Additionally,
\[
\text{Var}(S) \leq 2\text{Var}(T) + 2\text{Var}(S - T) \\
\leq 2\text{Var}(T) + 2\mathbb{E}[(S - T)^2] \\
\leq \begin{cases} o(\|\theta\|^8) + o(\|\theta\|^4) = o(\|\theta\|^8), & \text{for setting of Lemma } 4.3 \\ o(\|\theta\|^8 + \tau^6\|\theta\|^8\|\theta\|^\delta) + o(\|\theta\|^8) = o(\|\theta\|^8 + \tau^6\|\theta\|^8\|\theta\|^\delta), & \text{for setting of Lemma } 4.8 \end{cases}
\]

This gives the desirable claim. \( \square \)

### D.5 Proof of Lemma C.5

Similar to the proof of Lemma C.3, we use the notation \( M_{ijkt}(X) = X_{ij}X_{jk}X_{kt}X_{ti} \). By (C.121),
\[
Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)} = \sum_{i_1,i_2,i_3,i_4(\text{dist})} [M_{i_1,i_2,i_3,i_4}(X) - M_{i_1,i_2,i_3,i_4}(\tilde{X}^*)],
\]
where
\[
\begin{align*}
X_{ij} &= \tilde{\Omega}_{ij}^{(m,0)} + W_{ij} + \delta_{ij}^{(m,0)} + \hat{\epsilon}_{ij}^{(m,0)} + \epsilon_{ij}^{(m,0)}, \\
\hat{X}_{ij}^{*} &= \tilde{\Omega}_{ij}^{(m,0)} + W_{ij} + \delta_{ij}^{(m,0)} + \hat{\epsilon}_{ij}^{(m,0)}.
\end{align*}
\]

We shall omit the superscripts \((m,0)\) in \((\tilde{\Omega}, \delta, \tilde{\tau}, \epsilon)\). Let \(N_1^{(m,0)}, N_2^{(m,0)}, \ldots, N_m^{(m,0)}\) be the pseudo-communities defined by \(\Pi_0\). By (C.118), \(\epsilon_{ij} = \alpha_{ij} + \beta_{ij} + \gamma_{ij}\), where for \(i \in N_k^{(m,0)}\) and \(j \in N_l^{(m,0)}\),
\[
\begin{align*}
\alpha_{ij} &= d_i d_j U_{ij}^* - (\mathcal{E}d_i) (\mathcal{E}d_j) U_{ij}, \\
\beta_{ij} &= (U_{ij} - \bar{U}_{ij}) (\mathcal{E}d_i), \\
\gamma_{ij} &= (U_{ij} - \bar{U}_{ij}) [(\mathcal{E}d_i) (d_j - \mathcal{E}d_j) + (\mathcal{E}d_j) (d_i - \mathcal{E}d_i)]. \tag{D.158}
\end{align*}
\]

Therefore, we can write
\[
Q_n^{(m,0)} - \tilde{Q}_n^{(m,0)} = \sum_{i_1,i_2,i_3,i_4(\text{dist})} [M_{i_1,i_2,i_3,i_4}(\tilde{\Omega} + W + \delta + \tilde{\tau} + \alpha + \beta + \gamma) - M_{i_1,i_2,i_3,i_4}(\tilde{\Omega} + W + \delta + \tilde{\tau})].
\]

There are \( 7^4 - 4^4 = 2145 \) post-expansion sums. Let \( S \) be the generic notation for any such post-expansion sum. Similarly as in the proof of Lemma C.3, we group the summands according to which pseudo-communities \((i_1, i_2, i_3, i_4)\) belong to, i.e., we write \( S = \sum_{1 \leq k_1,k_2,k_3,k_4 \leq m} S_{k_1,k_2,k_3,k_4} \), where
\[
S_{k_1,k_2,k_3,k_4} = \sum_{j=1}^{4} \sum_{i_j \in N_{k_j}^{(m,0)}} a_{i_1,i_2,i_3,i_4} d_{i_4,i_1}, \quad \text{where } a, b, c, d \in \{ \tilde{\Omega}, W, \delta, \tilde{\tau}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \}. \tag{D.159}
\]
It suffices to study the mean and variance of each $S_{k_1 k_2 k_3 k_4}$.

Let $\tau$ and $r_{ij}$ be the same as in (1.25) and (D.147). Define

$$
\alpha_{ij} = \frac{\tau\|\theta\|_1}{\theta_{\max}} \left[ d_1^t d_2^t U_{kt} - (Ed_i)(Ed_j)U_{kt} \right],
\beta_{ij} = \frac{\tau U_{kt}(Ed_i)(Ed_j)},
\gamma_{ij} = U_{kt}(Ed_i)(d_j - Ed_j) + (Ed_j)(d_i - Ed_i).
$$

We introduce a proxy of $S_{k_1 k_2 k_3 k_4}$ as

$$
S_{k_1 k_2 k_3 k_4}^* = \frac{4}{j=1} \sum_{i,j \in N^L_{k_1 k_2 k_3 k_4}} a_{ij} b_{ij} c_{ij} d_{ij}, \text{ where } a, b, c, d \in \{\Theta, W, \delta, r, \alpha, \beta, \gamma\}. \tag{D.160}
$$

Reviewing the expressions of $(\Theta, W, \delta, r, \alpha, \beta, \gamma)$, we know that $S_{k_1 k_2 k_3 k_4}^*$ can always be written as a weighted sum of monomials of $W$, and so we can calculate the mean and variance of $S_{k_1 k_2 k_3 k_4}^*$ (the straightforward calculations are still tedious, but later we will introduce a simple trick to do that). Comparing (D.160) with (D.158) and $r_{ij}$ with $\tilde{r}_{ij}$, we observe that, for $i \in N^L_{k_1 k_2 k_3 k_4}$ and $j \in N^{L}_{k_1 k_2 k_3 k_4}$,

$$
\tilde{r}_{ij} = \frac{\tilde{U}_{kt}}{U_{kt}} r_{ij}, \quad \tilde{\alpha}_{ij} = \frac{\tau_{\max}}{\tau_{\max} \|\theta\|_1} \alpha_{ij}, \quad \tilde{\beta}_{ij} = \frac{U_{kt} - \tilde{U}_{kt}}{\tau U_{kt}} \beta_{ij}, \quad \tilde{\gamma}_{ij} = \frac{U_{kt} - \tilde{U}_{kt}}{U_{kt}} \gamma_{ij}.
$$

We plug them into (D.159) to get

$$
S_{k_1 k_2 k_3 k_4}^* = \left( \frac{\tilde{U}_{kt}}{U_{kt}} \right)^{N_{\bar{S}}} \left( \frac{\theta_{\max}}{\tau_{\max} \|\theta\|_1} \right)^{N_{\bar{\alpha}}} \left( \frac{U_{kt} - \tilde{U}_{kt}}{U_{kt}} \right)^{N_{\bar{\beta}}} \left( \frac{U_{kt} - \tilde{U}_{kt}}{U_{kt}} \right)^{N_{\bar{\gamma}}} \left( \frac{U_{kt} - \tilde{U}_{kt}}{U_{kt}} \right)^{N_{\bar{\gamma}}} S_{k_1 k_2 k_3 k_4}^*, \tag{D.162}
$$

where $N_{\bar{S}}$ is the count of $\{a, b, c, \|\theta\|_1\}$ in (D.159). Taking the value of $\bar{r}$, and $(N_{\bar{\alpha}}, N_{\bar{\beta}}, N_{\bar{\gamma}})$ are similar.

For any post-expansion sum considered here, $1 \leq N_{\bar{\alpha}} + N_{\bar{\beta}} + N_{\bar{\gamma}} \leq 4$. The notation $(\frac{\tilde{U}_{kt}}{U_{kt}})^{N_{\bar{S}}}$ is interpreted in this way: For example, if in (D.159) only $a$ takes the value of $\bar{r}$, then $N_{\bar{S}} = 1$ and $(\frac{\tilde{U}_{kt}}{U_{kt}})^{N_{\bar{S}}} = \frac{\tilde{U}_{kt}}{U_{kt}}$. If $(a, b, c)$ take the value of $\bar{r}$, then $N_{\bar{S}} = 3$ and $(\frac{\tilde{U}_{kt}}{U_{kt}})^{N_{\bar{S}}} = \frac{\tilde{U}_{kt}}{U_{kt}}$. In (D.162), $S_{k_1 k_2 k_3 k_4}$ is a random variable whose mean and variance are relatively easy to calculate. The factor in front of $S_{k_1 k_2 k_3 k_4}^*$ has a complicated correlation with the summands in $S_{k_1 k_2 k_3 k_4}$, but fortunately we can apply a simple bound on this factor. Consider the event $E_n$ as in (D.152).

We have shown in (D.151) that $P(E_n^c) = o(n^{-L})$ for any fixed $L > 0$. Therefore, the event $E_n^c$ has a negligible effect on the mean and variance of $S_{k_1 k_2 k_3 k_4}$, i.e.,

$$
E[S_{k_1 k_2 k_3 k_4}^2] \cdot I_{E_n^c} = o(1).
$$

On the event $E_n$, we have $\max_{k, \ell, r} \{||\tilde{U}_{kt} - U_{kt}/U_{kt}|| \leq C_0 x_n/\|\theta\|_1$. It follows that

$$
|S_{k_1 k_2 k_3 k_4}| \leq \max_{k, \ell, r} \left| \frac{\tilde{U}_{kt}}{U_{kt}} \right|^{N_{\bar{S}}} \left( \frac{\theta_{\max}}{\tau_{\max} \|\theta\|_1} \right)^{N_{\bar{\alpha}}} \left( \frac{U_{kt} - \tilde{U}_{kt}}{U_{kt}} \right)^{N_{\bar{\beta}}} \left( \frac{\max_{k, \ell, r}}{\tau U_{kt}} \right)^{N_{\bar{\gamma}}} \left| S_{k_1 k_2 k_3 k_4}^* \right|
$$

$$
\leq C \left( \frac{\tau_{\max}}{\|\theta\|_1} \right)^{N_{\bar{S}}} \left( \frac{x_n}{\|\theta\|_1} \right)^{N_{\bar{\alpha}}} \left( \frac{\tau}{\|\theta\|_1} \right)^{N_{\bar{\beta}}} \left( \frac{x_n}{\|\theta\|_1} \right)^{N_{\bar{\gamma}}} \left| S_{k_1 k_2 k_3 k_4}^* \right|.
$$

Since $x_n \ll \|\theta\|_1$ and $\tau \|\theta\|_1 \rightarrow \infty$, we immediately have $\frac{x_n}{\|\theta\|_1} = o(\left( \frac{1}{\|\theta\|_1} \right))$, $\frac{x_n}{\|\theta\|_1} = o\left( \frac{1}{\|\theta\|_1} \right)$ and $\frac{\theta_{\max}}{\tau_{\max}} = o\left( \frac{1}{\|\theta\|_1} \right)$. It follows that

$$
|S_{k_1 k_2 k_3 k_4}| = o(1) \cdot \|\theta\|_1^{-N_{\bar{S}} + N_{\bar{\alpha}} + 2N_{\bar{\gamma}}}, \left| S_{k_1 k_2 k_3 k_4}^* \right|, \text{ on the event } E_n.
$$
Combining the above gives

\[
E[S^2_{k_1k_2k_3k_4}] = E[S^2_{k_1k_2k_3k_4} \cdot I_{E_i}] + E[S^2_{k_1k_2k_3k_4} \cdot I_{E_f}]
= o(1) \cdot \|\theta\|^{-(2N_k + 2N_j + 4N_\gamma)} \cdot E[(S^*_{k_1k_2k_3k_4})^2] + o(1). \tag{D.163}
\]

It remains to bound \(E[(S^*_{k_1k_2k_3k_4})^2]\). As we mentioned, we can write \(S^*_{k_1k_2k_3k_4}\) as a weighted sum of monomials of \(W\) and calculate its mean and variance directly. However, given that there are \(2145\) types of \(S^*_{k_1k_2k_3k_4}\), the calculation is still very tedious. We now use a simple trick to relate the \(S^*_{k_1k_2k_3k_4}\) to the post-expansion sums we have analyzed in Lemmas \(C.3\) and \(C.4\). We first bound \(|\alpha_{ij}|\) in (D.160). Since \(d^*_{i} = E[d_i] + \Omega_{ij},\)

\[
|\alpha_{ij}| \leq \frac{\tau\|\theta\|_1}{\theta_{\text{max}}} \left( E[d_i]E[d_j]|U^*_{kl} - U_{kl}| + (\Omega_{ij}E[d_j] + \Omega_{jj}E[d_i])U^*_{kl} + \Omega_{ii}\Omega_{jj}U^*_{kl} \right).
\]

By basic algebra, \(|(x_1 + x_2)/(y_1 + y_2) - x_1/y_1| \leq |x_2/y_1 + y_2|/(y_1 + y_2) + |x_1y_2|/|(y_1 + y_2)y_1|\). We apply it on (C.113)-(C.114) and note that \(1_k(\Omega - E[A])1_k = 1_k \diag(\Omega)1_k = O(\|\theta\|^2)\) and \(1_k(d^* - E[d]) = 1_k \diag(\Omega)1_n = O(\|\theta\|^2)\). It yields

\[
|U^*_{kl} - U_{kl}| \leq \frac{|1_k(\Omega_{ii} - 1_kE[A])1_k|}{(1_k' d^*)(1_k' d^*)} + \frac{|(1_k' E[A]1_k)(1_k' d^*) - (1_k' E[d])(1_k' E[d])|}{(1_k' d^*)(1_k' d^*)} \leq C\|\theta\|^{-3} \cdot 1_k \diag(\Omega)1_n + C\|\theta\|^{-6} \cdot (1_k' d^*)(1_k' d^*) - (1_k' E[d])(1_k' E[d]) \leq C\|\theta\|^{-3} \cdot \|\theta\|^2 + C\|\theta\|^{-6} \cdot \|\theta\|^2 \leq C\|\theta\|^{-3} \theta_{\text{max}},
\]

where in the last line we have used \(\|\theta\|^2 \leq \theta_{\text{max}}\|\theta\|_1\). Combining the above gives

\[
|\alpha_{ij}| \leq C\frac{\tau\|\theta\|_1}{\theta_{\text{max}}} \left[ \theta_1 \theta_2 \|\theta\|_1 \theta_3 \theta_4 \|\theta\|_1 \|\theta\|_1^{-3} \theta_{\text{max}} + (\theta_1^2 \theta_2 \|\theta\|_1 + \theta_3^2 \theta_4 \|\theta\|_1) \cdot \|\theta\|_1^{-2} \right] \leq C\tau \theta_1 \theta_2.
\]

Additionally, in (D.160), we observe that \(\gamma_{ij} = \delta_{ij}\). Since \(|U_{kl}| \leq C\|\theta\|^{-1}\) and \(E[d_i] \leq C\|\theta\|_1\), it is true that \(|\beta_{ij}| \leq C\tau \theta_1 \theta_2\). We summarize the results as

\[
|\alpha_{ij}| \leq C\tau \theta_1 \theta_2, \quad |\beta_{ij}| \leq C\tau \theta_1 \theta_2, \quad \gamma_{ij} = \delta_{ij}. \tag{D.164}
\]

It says that \(\gamma\) is the same as \(\delta\), and \((\alpha, \beta)\) behave similarly as \(\tilde{\Omega}\). Consequently, the calculation of mean and variance of \(S^*_{k_1k_2k_3k_4}\) in (D.161) can be carried out by replacing \((\alpha, \beta, \gamma)\) with \((\tilde{\Omega}, \tilde{\Omega}, \delta)\). In other words, we only need to study a sum like

\[
S^*_{k_1k_2k_3k_4} = \sum_{j=1}^{4} \sum_{i_j \in N_{k_j}(\emptyset,0)} a_{i_1i_2i_3i_4}d_{i_1i_2i_3i_4}, \quad \text{where} \ a,b,c,d \in \{\Omega, W, \delta, r\}.
\]

Let \(N_{\Omega}, N_{W}, N_{\delta}, N_{r}, N_{\alpha}, N_{\beta}, N_{\gamma}\) be the count of different terms in \(\{a,b,c,d\}\) determined by \(S^*_{k_1k_2k_3k_4}\), where these counts sum to 4. In \(S^*_{k_1k_2k_3k_4}\), the counts become \(N_{\Omega} = N_{\Omega} + N_{\alpha} + N_{\beta} + N_{\gamma}\), \(N_{W} = N_{W}\), \(N_{\delta} = N_{\delta} + N_{\gamma}\) and \(N_{r} = N_{r}\). Luckily, anything like \(S^*_{k_1k_2k_3k_4}\) has been analyzed in Lemmas \(C.3\) and \(C.4\). Especially, in light of (D.163), the mean and variance contributed by any post-expansion sum considered here must be dominated by the mean and variance of some post-expansion sum considered in Lemmas \(C.3\) and \(C.4\). We thus immediately obtain the claim, without any extra calculation.