AIC, $C_p$ and estimators of loss for elliptically symmetric distributions

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Abstract

In this article, we develop a modern perspective on Akaike’s Information Criterion (AIC) and Mallows’ $C_p$ for model selection, and proposes generalizations to spherically and elliptically symmetric distributions. Despite the differences in their respective motivation, $C_p$ and AIC are equivalent in the special case of Gaussian linear regression. In this case they are also equivalent to a third criterion, an unbiased estimator of the quadratic prediction loss, derived from loss estimation theory. We then show that the form of the unbiased estimator of the quadratic prediction loss under a Gaussian assumption still holds under a more general distributional assumption, the family of spherically symmetric distributions. One of the features of our results is that our criterion does not rely on the specificity of the distribution, but only on its spherical symmetry. The same kind of criterion can be derived for a family of elliptically contoured distribution, which allows correlations, when considering the invariant loss. More specifically, the unbiasedness property is relative to a distribution associated to the original density.

Keywords: variable selection, loss estimation, unbiasedness, SURE, Stein identity, spherically and elliptically symmetric distributions.

1 Introduction

The problem of model selection has generated a lot of interest for many decades now and especially recently with the increased size of datasets. In such a context, it is important to model the observed data in a way that accounts for the sparsity of the parameter space. The principle of parsimony helps to avoid classical issues such as overfitting or computational error. At the same time, the model should capture sufficient information in order to comply with some objectives of good prediction, good estimation or good selection and thus it
should not be too sparse. This principle has been elucidated by many statisticians as a trade-off between goodness of fit to data and complexity of the model (see for instance Hastie, Tibshirani, and Friedman [2008] Chapter 7). From the practitioner’s point of view, model selection is often implemented through cross-validation (see Arlot and Celisse [2010] for a review on this topic) or the minimization of criteria whose theoretical justification relies on hypotheses made within a given framework. In this paper, we review two of the most commonly used criteria, namely Mallows’ \( C_p \) and Akaike’s AIC, together with the associated theory under Gaussian distributional assumptions, and then we propose a generalization to spherically and elliptically symmetric distributions.

We will focus initially on the linear regression model

\[
Y = X\beta + \sigma \varepsilon, \tag{1}
\]

where \( Y \) is a random vector in \( \mathbb{R}^n \), \( X \) is a fixed and known full rank design matrix containing \( p \) observed variables \( x_j \) in \( \mathbb{R}^n \), \( \beta \) is the unknown vector in \( \mathbb{R}^p \) of regression coefficients to be estimated, \( \sigma \) is the noise level and \( \varepsilon \) is a random vector in \( \mathbb{R}^n \) representing the model noise, with mean zero and covariance matrix proportional to the identity matrix (we assume the proportion coefficient to be equal to one when \( \varepsilon \) is Gaussian). One subproblem of model selection is the problem of variable selection: only a subset of the variables in \( X \) gives sufficient and nonredundant information on \( Y \) and we wish to recover this subset as well as correctly estimate the corresponding regression coefficients.

Early work treated the model selection problem from the hypothesis testing point of view. For instance the Forward Selection and Backward Elimination procedures were stopped using appropriate critical values. This practice changed with Mallows’ automated criterion known as \( C_p \) (Mallows [1973]). Mallows’ idea was to propose an unbiased estimator of the scaled expected prediction error \( \mathbb{E}_\beta[\|X\hat{\beta}_I - X\beta\|^2/\sigma^2] \), where \( \hat{\beta}_I \) is an estimator of \( \beta \) based on the selected variables set \( I \subset \{1, \ldots, p\} \); \( \mathbb{E}_\beta \) denotes the expectation with respect to the sampling distribution in model (1) and \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^n \). Assuming Gaussian i.i.d. error terms, Mallows proposed the following criterion

\[
C_p = \frac{\|Y - X\hat{\beta}_I\|^2}{\hat{\sigma}^2} + 2\hat{df} - n, \tag{2}
\]

where \( \hat{\sigma}^2 \) is an estimator of the variance \( \sigma^2 \) based on the full linear model fitted with the least-squares estimator \( \hat{\beta}^{LS} \), that is, \( \hat{\sigma}^2 = \|Y - X\hat{\beta}^{LS}\|^2/(n - p) \), and \( \hat{df} \) is an estimator of \( df \), the degrees of freedom, also called the effective dimension of the model (see Hastie and Tibshirani [1990] Meyer and Woodroofe [2000]). For the least squares estimator, \( df \) is the number \( k \) of variables in the selected subset \( I \). Mallows’ \( C_p \) relies on the assumption that, if for some subset \( I \) of explanatory variables the expected prediction error is low, then those variables to be relevant for predicting \( Y \). In practice, the rule for selecting the “best” candidate is the minimization of \( C_p \). However, Mallows argues that this rule should not be applied in all cases, and that it is better to look at the shape of the \( C_p \)-plot instead, especially when some explanatory variables are highly correlated.

In 1974, Akaike proposed different automatic criteria that would not need a subjective calibration of the significance level as in hypothesis testing based approaches. His proposal was more general with application to many problems such as variable selection, factor analysis, analysis of variance, or order selection in autoregressive models (Akaike [1974]). Akaike’s motivation however was different from Mallows. Akaike considered the problem of estimating the density \( f(\cdot|\beta) \) of an outcome variable \( Y \), where \( f \) is parameterized by
\( \beta \in \mathbb{R}^p \). His aim was to generalize the principle of maximum likelihood enabling a selection between several maximum likelihood estimators \( \hat{\beta}_I \). Akaike showed that all the information for discriminating the estimator \( f(\cdot|\beta_I) \) from the true \( f(\cdot|\beta) \) could be summed up by the Kullback-Leibler divergence \( D_{KL}(\hat{\beta}_I, \beta) = \mathbb{E}[\log f(Y_{\text{new}}|\beta)] - \mathbb{E}[\log f(Y_{\text{new}}|\hat{\beta}_I)] \) where the expectation is taken over new observations. An accurate quadratic approximation to this divergence is possible when \( \hat{\beta}_I \) is sufficiently close to \( \beta \), which actually corresponds to the distance \( \|\hat{\beta}_I - \beta\|^2 / 2 \) where \( I = -\mathbb{E}[\partial^2 \log f / \partial \beta_i \partial \beta_j]_{p,j=1} \) is the Fisher-information matrix and for a vector \( z \), its weighted norm \( \|z\|_I \) is defined by \( (z' I z)^{1/2} \). By means of asymptotic analysis and by considering the expectation of \( D_{KL} \), Akaike arrived at the following criterion

\[
\text{AIC} = -2 \sum_{i=1}^{n} \log f(y_i|\hat{\beta}_I) + 2k, \tag{3}
\]

where \( k \) is the number of parameters of \( \hat{\beta}_I \). In the special case of a Gaussian distribution, AIC and \( C_p \) are equivalent up to a constant for model (1) (see Section 2.4). Hence Akaike described his AIC as a generalization of \( C_p \) to a more general class of models. Unlike Mallows, Akaike explicitly recommends the rule of minimization of AIC to identify the best model from data. Note that Ye (1998) proposed to extend AIC to more complex settings by replacing \( k \) by the estimated degrees of freedom \( \hat{df} \).

Both \( C_p \) and AIC have been criticized in the literature, especially for the presence of the constant 2 tuning the adequacy complexity/trade-off and favoring complex models in many situations, and many authors have proposed some correction (see Schwarz, 1978; Foster and George, 1994; Shibata, 1980). Despite these critics, these criteria are still quite popular among practitioners. Also they can be very useful in deriving better criteria of the form \( \delta = \delta_0 - \gamma \), where \( \delta_0 \) is equal to \( C_p \), AIC or an equivalent and \( \gamma \) is a correction function based on data. This framework, referred to as loss estimation, has been successfully used by Johnstone (1988) and Fourdrinier and Wells (1995), among others, to propose good criteria for selecting the best submodel.

Another possible criticism of \( C_p \) and AIC regards their strong distributional assumptions. Indeed, \( C_p \)'s unbiasedness in the \( i.i.d. \) Gaussian case, while AIC requires specification of the distribution. However, in many practical cases, we might not have any prior knowledge or intuition about the form of the distribution, and we want the result to be robust with respect to a wide family of distributions.

The purpose of the present paper is threefold:

- First, we show in Section 2 that the procedures \( C_p \) and AIC are equivalent to unbiased estimators of the quadratic prediction loss when \( Y \) is assumed to be Gaussian in model (1). This result is an important initial step for deriving improved criteria as is done in Johnstone (1988) and Fourdrinier and Wells (2012). Both references consider the case of improving the unbiased estimator of loss based on the data. The derivation of better criteria will not be covered in the present article.

- Secondly, we derive the unbiased loss estimator for the wide family of spherically symmetric distributions and show that, for any spherical law, this unbiased estimator is the same as that derived under the Gaussian model. The family of spherically symmetric distribution is a large family which generalizes the multivariate standard normal law. Also, the spherical assumption frees us from the independence assumption of the error terms in (1), while not rejecting it since the Gaussian law is itself spherical. Furthermore, some members of the spherical family, like the Student law,
have heavier tails than the Gaussian density allowing a better handling of potential outliers. Finally, the results of the present work do not depend on the specific form of the distribution. The last two points provide some distributional robustness.

- Thirdly, we extend these results to a model with possible correlation structure. Indeed, for a family of elliptically symmetric distributions with unknown scale matrix Σ, we derive an unbiased estimator of the invariant loss associated to Σ. To this end, the notion of unbiasedness has to be adapted to take into account the fact that Σ is unknown. Our results cover a multivariate version of the regression model where it is possible to estimate the unknown covariance matrix from the observed data.

2 Expression of AIC and $C_p$ in the loss estimation framework

2.1 Basics of loss estimation

The idea underlying the estimation of loss is closely related to Stein’s Unbiased Risk Estimate (SURE, Stein [1981]). The theory of loss estimation was initially developed for problems of estimation of the location parameter of a multivariate distribution (see e.g. Johnstone [1988] Fourdrinier and Strawderman [2003]). The principle is classical in statistics and goes as follow: we wish to evaluate the accuracy of a decision rule $\hat{\mu}$ for estimating the unknown location parameter $\mu$ (in the linear model (1), we have $\mu = X\beta$). Therefore we define a loss function, which we write $L(\hat{\mu}, \mu)$, measuring the discrepancy between $\hat{\mu}$ and $\mu$. Typical examples are the quadratic loss $\|\hat{\mu} - \mu\|^2$ and the invariant (quadratic) loss $\|\hat{\mu} - \mu\|^2/\sigma^2$. Since $L(\hat{\mu}, \mu)$ depends on unknown parameters $\mu$ or $(\mu, \sigma^2)$, it is unknown as well and can thus be assessed through an estimation using the observations (see for instance Fourdrinier and Wells [2012] and references therein for more details on loss estimation).

In this article, we only consider unbiasedness as the notion of optimality. In such a case, unbiased estimators of the loss and unbiased estimators of the corresponding risk, defined by

$$R(\hat{\mu}, \mu) := E_\mu[L(\hat{\mu}, \mu)],$$

are actually defined in the same way (see Stein [1981], Johnstone [1988]). In the sequel, we choose to refer to them as unbiased loss estimators.

**Definition 1** (Unbiasedness). Let $Y$ be a random vector in $\mathbb{R}^n$ with mean $\mu \in \mathbb{R}^n$ and let $\hat{\mu}(Y)$ be any estimator of $\mu$. An estimator $\delta_0(Y)$ of the loss $L(\hat{\mu}(Y), \mu)$ is said to be unbiased if, for all $\mu \in \mathbb{R}^n$,

$$E_\mu[\delta_0(Y)] = E_\mu[L(\hat{\mu}(Y), \mu)],$$

where $E_\mu$ denotes the expectation with respect to the distribution of $Y$.

Obtaining an unbiased estimator of the quadratic loss requires Stein’s identity (see Stein [1981]) which states that, if $Y \sim N_n(\mu, \sigma^2 I_n)$ and $g: \mathbb{R}^n \to \mathbb{R}^n$ is a weakly differentiable function, then, assuming the expectations exist,

$$E_\mu [(Y - \mu)^t g(Y)] = \sigma^2 E_\mu [\text{div}_Y g(Y)],$$

where $\text{div}_Y g(Y) = \sum_{i=1}^n \partial g_i(Y)/\partial Y_i$ is the weak divergence of $g(Y)$ with respect to $Y$. See e.g. Section 2.3 in Fourdrinier and Wells [1995] for the definition and the justification of weak differentiability.
When dealing with the invariant loss $\|X\hat{\beta} - X\beta\|^2/\sigma$, the need for estimating $\sigma^2$ leads to the following Stein-Haff identity (Lemma 3.1 in [Fourdrinier and Wells, 2012]):

$$
\mathbb{E}_{\mu, \sigma^2} \left[ \frac{1}{\sigma^2} h(Y, S) \right] = \mathbb{E}_{\mu, \sigma^2} \left[ (n - p - 2)S^{-1} h(Y, S) + 2 \frac{\partial}{\partial S} h(Y, S) \right],
$$

where $S$ is a non negative random variable independent of $Y$ such that $S \sim \sigma^2 \chi^2_{n-p}$ and $h: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is a weakly differentiable function with respect to $S$. Note that a slightly different notion of unbiasedness will be used in the elliptical case.

### 2.2 Unbiased loss estimation for the linear regression

When considering the Gaussian model in (1), we have $\mu = X\beta$, we set $\hat{\mu} = X\hat{\beta}$ and $L(\hat{\beta}, \beta)$ is defined as the quadratic loss $\|X\hat{\beta} - X\beta\|^2$. Special focus will be given to the quadratic loss since it is the most commonly used and allows tractable calculations. In practice, it is a reasonable choice if we are interested in both good selection and good prediction at the same time. Moreover, quadratic loss allows us to link loss estimation with $C_p$ and AIC.

In the following theorem, an unbiased estimator of the quadratic loss, under a Gaussian assumption, is developed using a result of Stein (1981).

**Theorem 1.** Let $Y \sim \mathcal{N}_n(X\beta, \sigma^2 I_n)$. Let $\hat{\beta} = \hat{\beta}(Y)$ be a function of the least squares estimator of $\beta$ such that $X\hat{\beta}$ is weakly differentiable with respect to $Y$. Let $\hat{\sigma}^2 = \|Y - X\hat{\beta}^{LS}\|^2/(n - p)$. Then

$$
\delta_0(Y) = \|Y - X\hat{\beta}\|^2 + (2 \text{div}_Y (X\hat{\beta}) - n)\hat{\sigma}^2
$$

is the usual unbiased estimator of $\|X\hat{\beta} - X\beta\|^2$.

**Proof.** The risk of $X\hat{\beta}$ at $X\beta$ is

$$
\mathbb{E}_{\beta} [\|X\hat{\beta} - X\beta\|^2] = \mathbb{E}_{\beta} [\|X\hat{\beta} - Y\|^2 + \|Y - X\beta\|^2]
$$

$$
+ \mathbb{E}_{\beta} [2(Y - X\beta)^t(X\hat{\beta} - Y)].
$$

Since $Y \sim \mathcal{N}_n(X\beta, \sigma^2 I_n)$, we have $\mathbb{E}_{\beta} [\|Y - X\beta\|^2] = \mathbb{E}_{\beta} [(Y - X\beta)^t Y] = n \sigma^2$ leading to

$$
\mathbb{E}_{\beta} [\|X\hat{\beta} - X\beta\|^2] = \mathbb{E}_{\beta} [\|Y - X\hat{\beta}\|^2] - n \sigma^2 + 2 \text{tr}(\text{cov}_\beta(X\hat{\beta}, Y - X\beta)).
$$

Moreover, applying Stein’s identity for the right-most part of the expectation in (8) with $g(Y) = X\hat{\beta}$ and assuming that $X\hat{\beta}$ is weakly differentiable with respect to $Y$, we can rewrite (8) as

$$
\mathbb{E}_{\beta} [\|X\hat{\beta} - X\beta\|^2] = \mathbb{E}_{\beta} [\|Y - X\hat{\beta}\|^2] - n \sigma^2 + 2 \sigma^2 \mathbb{E}_{\beta} [\text{div}_Y X\hat{\beta}].
$$

Since $\hat{\sigma}^2$ is an unbiased estimator of $\sigma^2$ independent of $\hat{\beta}^{LS}$, the right-hand side of this last equality is also equal to the expectation of $\delta_0(Y)$ given by Equation (7). Hence, according to Definition 1, the statistic $\delta_0(Y)$ is an unbiased estimator of $\|X\hat{\beta} - X\beta\|^2$. \[ \]

**Remark 1** (Estimating the variance). The estimation of the variance is a crucial problem in model selection, as different variance estimators can lead to very different models selected. However, in this article, we do not intend to develop new model selection criteria, but only to review and extend existing ones. In particular, the restriction to unbiasedness makes the estimator $\hat{\sigma}^2 = \|Y - X\hat{\beta}^{LS}\|^2/(n - p)$ natural. For more details on the issue, we refer the interested reader to [Cherkassky and Ma, 2003].
Remark 2. It is often of interest to use robust estimators of $\beta$ and $\sigma^2$. In such a case, the hypotheses of the theorem need to be modified to insure the independence between estimators $\hat{\beta}$ and $\hat{\sigma}^2$ which were implicit in the statement of the theorem. We will see in Remark 3 of the next sections that, by the use of Stein identities for the spherical and elliptical cases, the implicit assumption of independence is actually not needed.

2.3 Unbiased estimation of the invariant loss for regression

When dealing with the invariant loss, it is natural to consider estimators of $\beta$ involving the sample variance $S = \|Y - X\hat{\beta}_{LS}\|^2$, that is, $\hat{\beta}(Y, S)$. However, by consistency with the elliptical case tackled in Section 4, we consider estimators that only depend on the least squares estimator and no longer on $S$. Therefore, we give the following adaptation of Theorem 3.1 from [Fourdrinier and Wells (2012)].

Theorem 2. Let $Y \sim N_n(X\beta, \sigma^2 I_n)$ and $n \geq 5$. Let $\hat{\beta} = \hat{\beta}(Y)$ be an estimator of $\beta$ weakly differentiable with respect to $Y$ and independent of $\|Y - X\hat{\beta}_{LS}\|^2$. Then

$$\delta_0^{\text{inv}}(Y) = \frac{n - p - 2}{\|Y - X\hat{\beta}_{LS}\|^2} \|Y - X\hat{\beta}\|^2 + 2 \text{div}_Y(X\hat{\beta}) - n$$

is an unbiased estimator of the invariant loss $\|X\hat{\beta} - X\beta\|^2/\sigma^2$.

Note that, for more general estimators of the form $\hat{\beta}(Y, S)$, a correction term has to be added to (9). Thus, if $\hat{\beta}(Y, S) = \hat{\beta}_{LS}(Y) + g(\hat{\beta}_{LS}, S)$ for some function $g$, this correction is $4(X\hat{\beta}(Y, S) - Y)^t X \frac{\partial g(\hat{\beta}_{LS}, S)}{\partial S}$.

2.4 Links between loss estimation and model selection

In order to make the following discussion clearer, we recall here the formulas of the three criteria of interest for the Gaussian assumption. First the historical criterion Mallows’ $C_p$ and the extended version of AIC proposed by Ye (1998):

$$C_p(\hat{\beta}) = \frac{\|Y - X\hat{\beta}\|^2}{\hat{\sigma}^2} + 2 \hat{df} - n$$

and

$$\text{AIC}(\hat{\beta}) = \frac{\|Y - X\hat{\beta}\|^2}{\hat{\sigma}^2} + 2 \text{div}_Y(X\hat{\beta}).$$

Secondly the unbiased estimator of loss $\delta_0(Y)$:

$$\delta_0(\hat{\beta}) = \|Y - X\hat{\beta}\|^2 + (2 \text{div}_Y(X\hat{\beta}) - n)\hat{\sigma}^2.$$

We have the following link between $\delta_0$, $C_p$ and AIC:

$$\delta_0(\hat{\beta}) = \hat{\sigma}^2 \times C_p(\hat{\beta}) = \hat{\sigma}^2 \times (\text{AIC}(\hat{\beta}) - n)$$

since the statistic $\text{div}_Y(X\hat{\beta})$ is as an unbiased estimator of the (generalized) degrees of freedom [Ye (1998)].

These links between different criteria for model selection are due to the fact that, under our working hypothesis (linear model, quadratic loss, normal distribution $Y \sim N_n(X\beta, \sigma^2 I_n)$ for a fixed design matrix $X$), they can be seen as unbiased estimators of...
related quantities of interest. Note that there is also an equivalence with other model selection criteria, such as those investigated in Li (1985), Shao (1997) and Efron (2004).

The final objective is to select the “best” model among those at hand. This can be performed by minimizing either of the three proposed criteria, that is the unbiased estimator of loss $\delta_0$, $C_p$, and AIC. The idea behind this heuristic, as shown in the previous section, is that the best model in terms of prediction is the one minimizing the estimated loss. Now, from Equation (10), it can be easily seen that the three criteria differ from each other only up to a multiplicative and/or additive constant. Hence the models selected by the three criteria will be the same.

It is important to note that Theorem 1 does not rely on the linearity of the link between $X$ and $Y$ so that this work can easily be extended to nonlinear links. Therefore $\delta_0$ generalizes $C_p$ to nonlinear models. Moreover, following its definition (3), AIC implementation requires the specification of the underlying distribution. In this sense it is considered as a generalization of $C_p$ for non-Gaussian distributions. However, in practice, we might only have a vague intuition of nature of the underlying distribution and we might not be able to give its specific form. We will see in the following section that $\delta_0$, which is equivalent to the Gaussian AIC as we have just seen, can be also derived from a more general distribution context, that of spherically symmetric distributions, with no need to specify the precise form of the distribution.

Note that $C_p$ and AIC are developed first fixing $\sigma^2$ and then estimating it by $\hat{\sigma}^2$, while, for $\delta_0$, the estimation of $\sigma^2$ is integrated to the construction process. It is then natural to gather the evaluation of $\beta$ and $\sigma^2$ estimating the invariant loss

$$\frac{\|X\beta - X\hat{\beta}\|^2}{\sigma^2},$$

for which $\delta_0^{\text{inv}}(\hat{\beta})$ in an unbiased estimator. Note that $\delta_0^{\text{inv}}(\hat{\beta})$ involves the variance estimator $\|Y - X\hat{\beta}_{LS}\|^2/(n-p-2)$ instead of $\|Y - X\hat{\beta}_{LS}\|^2/(n-p)$. This alternative variance estimator was also considered in the unknown variance setting for the construction of the modified $C_p$, which is actually equivalent to $\delta_0^{\text{inv}}(\hat{\beta})$, and the corrected AIC (see Davies, Neath, and Cavanaugh, 2006, and references therein).

## 3 Unbiased loss estimators for spherically symmetric distributions

### 3.1 Multivariate spherical distributions

In the previous section, results were given under the Gaussian assumption with covariance matrix $\sigma^2 I_n$. In this section, we extend this distributional framework.

The characterization of the normal distribution as expressed by Kariya and Sinha (1989) allows two directions for generalization. Indeed, the authors assert that a random vector is Gaussian, with covariance matrix proportional to the identity matrix, if and only if its components are independent and its law is spherically symmetric. Hence, we can generalize the Gaussian assumption by either keeping the independence property and consider other laws than the Gaussian, or by relaxing the independence assumption to the benefit of spherical symmetry. In the same spirit, Fan and Fang (1985) pointed out that there are two main generalizations of the Gaussian assumption in the literature: one generalization comes from the interesting properties of the exponential form and leads to the exponential family
of distributions, while the other is based on the invariance under orthogonal transformation and results in the family of spherically symmetric distributions (which can be generalized by elliptically contoured distributions). These generalizations are complementary but go in different directions and have lead to fruitful lines of research. Note that their only common member is the Gaussian distribution. The main interest of choosing spherical distributions is that the conjunction of invariance under orthogonal transformation together with linear regression with less variables than observations brings robustness. The interest of that property is illustrated by the fact that some statistical tests designed under a Gaussian assumption, such as the Student and Fisher tests, remain valid for spherical distributions [Wang and Wells (2002); Fang and Anderson (1990)]. This robustness property is not shared by independent non-Gaussian distributions, as mentioned in Kariya and Sinha (1989).

To be self contained, we recall the definition of spherically symmetric distributions.

**Definition 2.** A random vector \( \varepsilon \) is said to be spherically symmetric if, for any orthogonal matrix \( Q \), the law of \( Q\varepsilon \) is the same as the one of \( \varepsilon \).

Note that any random vector \( \varepsilon \) in Definition 2 is necessarily centered at zero. Then, for any fixed vector \( \mu \), the distribution of \( \varepsilon + \mu \) will be said spherically symmetric around \( \mu \).

Thus, from the model in (1), the distribution of \( Y \) is the distribution of \( \sigma \varepsilon \) translated by \( \mu = X\beta \): \( Y \) has a spherically symmetric distribution about the location parameter \( \mu \) with covariance matrix equal to \( n^{-1}\sigma^2 E[\|\varepsilon\|^2]I_n \) where \( I_n \) is the identity matrix. We will write \( \varepsilon \sim S_n(0, I_n) \) and \( Y \sim S_n(\mu, \sigma^2 I_n) \). Examples of this family besides the Gaussian distribution are the Student distribution, the Kotz distribution, or any variance mixtures of Gaussian distributions.

As we will see in the sequel, the unbiased estimator of the quadratic loss \( \delta_0 \) remains unbiased for any of these distributions with no need to specify its particular form. It thus brings distributional robustness. For more details and examples of the spherical family, we refer the interested reader to Chmielewski (1981) for a historical review and Fang and Zhang (1990) for a comprehensive presentation.

### 3.2 The canonical form of the linear regression model

An efficient way of dealing with the linear regression model under spherically symmetric distributions is to use its canonical form (see Fourdrinier and Wells (1995) for details). This form will allow us to give more straightforward proofs.

Considering model (1), the canonical form consists in an orthogonal transformation of \( Y \). Using partitioned matrices, let \( Q = (Q_1 \  Q_2) \) be an \( n \times n \) orthogonal matrix partitioned such that the first \( p \) columns of \( Q \) (i.e. the columns of \( Q_1 \)) span the column space of \( X \). For instance, this is the case of the \( Q-R \) factorization where \( X = QR \) with \( R \) an \( n \times p \) upper triangular matrix. Now, according to (1), let

\[
Q^tY = \begin{pmatrix} Q_1^t \\ Q_2^t \end{pmatrix} Y = \begin{pmatrix} Q_1^t \\ Q_2^t \end{pmatrix} X\beta + \sigma Q^t\varepsilon = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + \sigma Q^t\varepsilon
\]

(11)

with \( \theta = Q_1^tX\beta \) and \( Q_2^tX\beta = 0 \) since the columns of \( Q_2 \) are orthogonal to those of \( X \). It follows from the definition that \( (Z^t \ U)^t := Q^tY \) has a spherically symmetric distribution about \( (\theta^t \ 0)^t \). In this sense, the model

\[
\begin{pmatrix} Z \\ U \end{pmatrix} = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} Q_1^t\varepsilon \\ Q_2^t\varepsilon \end{pmatrix}
\]
is the canonical form of the linear regression model (1).

This canonical form has been considered by various authors such as Cellier, Fourdrinier, and Robert (1989); Cellier and Fourdrinier (1995); Maruyama (2003); Maruyama and Strawderman (2009); Fourdrinier and Strawderman (2010); Kubokawa and Srivastava (1999). Kubokawa and Srivastava (2001) addressed the multivariate case where $\theta$ is a mean matrix (in this case $Z$ and $U$ are matrices as well) we will introduce section 4.2.

For any estimator $\hat{\beta}$, the orthogonality of $Q$ implies that

$$\|Y - X\hat{\beta}\|^2 = \|\hat{\theta} - Z\|^2 + \|U\|^2 \tag{12}$$

where $\hat{\theta} = Q^t X\hat{\beta}$ is the corresponding estimator of $\theta$. In particular, for the least squares estimator $\hat{\beta}_{LS}$, we have

$$\|Y - X\hat{\beta}_{LS}\|^2 = \|U\|^2. \tag{13}$$

In that context, we recall the Stein-type identity given by Fourdrinier and Wells (1995).

**Theorem 3** (Stein-type identity). Given $(Z, U) \in \mathbb{R}^n$ a random vector following a spherically symmetric distribution around $(\theta, 0)$, and $g : \mathbb{R}^p \to \mathbb{R}^p$ a weakly differentiable function, we have

$$E_{\theta}[(Z - \theta)^t g(Z)] = E_{\theta}[\|U\|^2 \text{div}_Z g(Z)/(n - p)], \tag{14}$$

provided both expectations exist.

Note that the divergence in Theorem 1 is taken with respect to $Y$ while the Stein type identity (14) requires the divergence with respect to $Z$ (with the assumption of weak differentiability). Their relationship can be seen in the following lemma.

**Lemma 1.** We have

$$\text{div}_Y X\hat{\beta}(Y) = \text{div}_Z \hat{\theta}(Z, U). \tag{15}$$

**Proof.** Denoting by $\text{tr}(A)$ the trace of any matrix $A$ and by $J_f(x)$ the Jacobian matrix (when it exists) of a function $f$ at $x$, we have

$$\text{div}_Y X\hat{\beta}(Y) = \text{tr}(J_{X\hat{\beta}}(Y)) = \text{tr}(Q^t J_{X\hat{\beta}}(Y) Q)$$

by definition of the divergence and since $Q^t$ is an orthogonal matrix. Now, setting $W = Q^t Y$, i.e. $Y = Q W$, applying the chain rule to the function

$$\tilde{T}(W) = Q^t X\hat{\beta}(Q W)$$

gives rise to

$$J_{\tilde{T}}(W) = J_{Q^t X\hat{\beta}}(Y) Q = Q^t J_{X\hat{\beta}}(Y) Q, \tag{16}$$

noticing that $Q^t$ is a linear transformation. Also, as according to (11)

$$W = \begin{pmatrix} Z \\ U \end{pmatrix} \quad \text{and} \quad \tilde{T} = \begin{pmatrix} \hat{\theta} \\ 0 \end{pmatrix},$$

the following decomposition holds

$$J_{\tilde{T}}(W) = \begin{pmatrix} J_{\hat{\theta}}(Z) & J_{\hat{\theta}}(U) \\ 0 & 0 \end{pmatrix},$$
where \( J_\theta(Z) \) and \( J_\beta(U) \) are the parts of the Jacobian matrix in which the derivatives are taken with respect to the components of \( Z \) and \( U \) respectively. Thus

\[
\text{tr}(J^T(Z)) = \text{tr}(J_\theta(Z)) \tag{16}
\]

and, therefore, gathering (16) and (17), we obtain

\[
\text{tr}(J_\theta(Z)) = \text{tr}(Q^t J_{X\beta}(Y) Q) = \text{tr}(QQ^t J_{X\beta}(Y)) = \text{tr}(J_{X\beta}(Y)),
\]

which is (15) by definition of the divergence.

\[\square\]

### 3.3 Unbiased estimator of loss for the spherical case

This section develops a generalization of Theorem 1 to the class of spherically symmetric distributions \( Y \sim \mathcal{S}_n(Y_{\beta}, \sigma^2) \), given by Theorem 4. To do so we need to consider the statistic

\[
\hat{\sigma}^2(Y) = \frac{1}{n-p} \|Y - X\hat{\beta}^{LS}\|^2. \tag{18}
\]

It is an unbiased estimator of \( \sigma^2 \mathbb{E}_\beta[||\varepsilon||^2/n] \). Note that, in the normal case where \( Y \sim \mathcal{N}_n(Y_{\beta}, \sigma^2 I_n) \), we have \( \mathbb{E}_\beta[||\varepsilon||^2/n] = 1 \) so that \( \hat{\sigma}^2(Y) \) is an unbiased estimator of \( \sigma^2 \).

**Theorem 4 (Unbiased estimator of the quadratic loss under a spherical assumption).** Let \( Y \sim \mathcal{S}_n(X_{\beta}, \sigma^2 I_n) \) and let \( \hat{\beta} = \hat{\beta}(Y) \) be an estimator of \( \beta \) depending only on \( Q^tY \). If \( \hat{\beta}(Y) \) is weakly differentiable with respect to \( Y \), then the statistic

\[
\delta_0(\hat{\beta}) = ||Y - X\hat{\beta}(Y)||^2 + 2 \text{div}_Y(\hat{\beta}(Y)) - n \hat{\sigma}^2(Y), \tag{19}
\]

is an unbiased estimator of \( ||X\hat{\beta}(Y) - X\beta||^2 \).

**Proof.** The quadratic loss function of \( X\hat{\beta} \) at \( X\beta \) can be decomposed as

\[
||X\hat{\beta} - X\beta||^2 = ||Y - X\hat{\beta}||^2 + ||Y - X\beta||^2 + 2(X\hat{\beta} - Y)^t(Y - X\beta). \tag{20}
\]

An unbiased estimator of the second term in the right hand side of (20) has been considered in (18). As for the third term, by orthogonal invariance of the inner product,

\[
(X\hat{\beta} - Y)^t(Y - X\beta) = (Q^tX\hat{\beta} - Q^tY)^t(Q^tY - Q^tX\beta) = (Q^t_1X\hat{\beta} - Q^t_1Y)^t(Q^t_1Y - Q^t_1X\beta) = (\hat{\beta} - \theta)\text{tr}(Z - U) = (\theta - Z)^t(Z - \theta) - ||U||^2. \tag{21}
\]

Now, since \( \hat{\theta} = \hat{\theta}(Z, U) \) depends only on \( Z \), by Stein type identity, we have

\[
\mathbb{E}[(\theta - Z)^t(Z - \theta)] = \mathbb{E}\left[\frac{||U||^2}{n-p} \text{div}_Z(\theta - Z)\right] = \mathbb{E}\left[\frac{||U||^2}{n-p} (\text{div}_Z\hat{\theta} - p)\right] \tag{22}
\]
so that

\[
\mathbb{E}[(X\hat{\beta} - Y)'(Y - X\beta)] = \mathbb{E}\left[\frac{\|U\|^2}{n-p} \left(\text{div}_Z\hat{\theta} - \frac{n}{n-p} \|U\|^2\right)\right] = \mathbb{E}[\sigma^2(Y) \text{div}_Y X\hat{\beta} - n \hat{\sigma}^2(Y)] \tag{23}
\]

by (13) and since \(\text{div}_Z\hat{\theta} = \text{div}_Y X\hat{\beta}\) by Lemma 1. Finally, gathering the expressions in (20), (18), and (23) gives the desired result.

From the equivalence between \(\delta_0\), \(C_p\) and AIC under a Gaussian assumption, and the unbiasedness of \(\delta_0\) under the wide class of spherically symmetric distributions, we conclude that \(C_p\) and AIC derived under the Gaussian distribution is as reasonable selection criteria for spherically symmetric distributions, although their original properties may not have been verified in this context. Also, criticisms concerning on the complexity of selected models still stand.

**Remark 3.** Note that the extension of Stein’s lemma in Theorem 4 implies that \(\hat{\sigma}^2\) is also an unbiased estimator of \(\sigma^2\) under the spherical assumption. Moreover, we would like to point out that the independence of \(\hat{\sigma}^2\) used in the proof of Theorem 4 in the Gaussian case is no longer necessary. Also, to require that \(\hat{\beta}\) depends on \(Q_1^TY\) only is equivalent to say that \(\hat{\beta}\) is a function of the least squares estimator. When this hypothesis is not available, an extended Stein type identity can be derived (Fourdrinier, Strawderman, and Wells, 2003).

## 4 Unbiased loss estimators for elliptically symmetric distributions

Spherical errors clearly generalize the traditional i.i.d. Gaussian case. However, it is important to take into account possible correlations among the error components, as in the Gaussian case with a general covariance matrix. The family of elliptically symmetric distributions is then the natural extension of this Gaussian framework.

Another interesting extension also would have been to non-Gaussian errors. Unfortunately, in that case, the only context in which Stein-type identities are available is the one of exponential family (Hudson, 1978) and no transition from the linear regression model to a canonical form, as in (11), is known. In return, the framework of elliptically symmetric distributions allows such an approach. This section is devoted to deriving an unbiased estimator of loss for this case.

### 4.1 Multivariate elliptical distributions

For a random vector \(Y\) that has a spherically symmetric distribution, its components are uncorrelated (and independent in the only Gaussian case). It is important to consider possible correlations through a general covariance matrix \(\Sigma\). Regarding the linear regression model, the natural extension of spherically symmetric distributions is the notion of elliptically symmetric distributions which may be defined through \(\Sigma^{-1}\)-orthogonal transformations (playing the role of orthogonal transformations in the spherical case).
**Definition 3.** A matrix $Q$ is said to be $\Sigma^{-1}$-orthogonal if it preserves the inner product associated to $\Sigma^{-1}$, that is, for all vectors $x$ and $y$, 

$$x^t \Sigma^{-1} y = (Qx)^t \Sigma^{-1} Qy,$$

or, equivalently, if 

$$Q^t \Sigma^{-1} Q = \Sigma^{-1}. \quad (24)$$

Elliptically symmetric distributions are defined as follows.

**Definition 4.** Let $\Sigma$ be a positive definite matrix. A random vector $\varepsilon$ with covariance matrix proportional to $\Sigma$ is said to be elliptically symmetric if, for any $\Sigma^{-1}$-orthogonal matrix $Q$, the law of $Q\varepsilon$ is the same as that of $\varepsilon$. We note $\varepsilon \sim E_n(0, \Sigma)$.

Note that any random vector $\varepsilon$ in Definition 4 is necessarily centered. For any fixed vector $\mu$, the distribution of $\varepsilon + \mu$ will be said elliptically symmetric around $\mu$. Clearly, a centered Gaussian random vector with covariance matrix $\Sigma$ is elliptically symmetric distributed. More generally, for any positive matrix $\Sigma$ and any random vector $W$ with a spherically symmetric distribution, the random vector $\Sigma^{1/2} W$ is elliptically symmetric distributed with covariance matrix $\Sigma$ (provided $\text{cov}(W) = I$).

Thus, the spherically symmetric distributions given in subsection 3.1 lead to examples of elliptically symmetric distributions taking into account possible correlations.

4.2 The canonical form of the linear regression model in the elliptical case

In the elliptical case, it is worth considering the multivariate linear regression model, for instance in Kubokawa and Srivastava (1999)

$$Y = X\beta + \varepsilon, \quad (25)$$

where $Y$ is an $n \times m$ matrix of response variable, $X$ the $n \times p$ known design matrix, $\beta$ is $p \times m$ matrix of unknown parameters and $\varepsilon$ an $n \times m$ random matrix of errors having an elliptically symmetric distribution with scale matrix $\Sigma$. This approach allows to take into account the fact that $\Sigma$ is unknown. We assume also that $p + m + 1 < n$. In the line of the previous section, we now specify the ellipticity of the distribution of the matrix $\varepsilon$ through its (line based) vectorization $\text{vec}(\varepsilon^t)$ such that $\varepsilon_{ij} = \text{vec}(\varepsilon^t)_{m(i-1)+j}$. Note that considering column base vectorization would have made trouble for us in the canonical decomposition of the error in (24).

In the following, we consider a specific subclass of elliptical distributions for matrices: the class of vector-elliptical (contoured) matrix distributions (see Fang and Zhang, 1990). Thus, regarding the distribution of $\varepsilon$, we assume that, and for any $(I_n \otimes \Sigma^{-1})$-orthogonal transformation $nm \times nm$ matrix $Q$, the law of $Q\text{vec}(\varepsilon^t)$ is the same as that of $\text{vec}(\varepsilon^t)$. Note that, if $\varepsilon$ has a density, it is of the form

$$\varepsilon \mapsto |\Sigma|^{-n/2} f(\text{tr}(\varepsilon \Sigma^{-1} \varepsilon^t)), \quad (26)$$

for some function $f$. The importance of the transposition is underlined by Muirhead (1982).

When the covariance exists, it is proportional to the scale matrix $\Sigma$. In that case, let $\tau^2$ be this proportionality coefficient so that

$$\mathbb{E}[\text{tr}(\varepsilon \Sigma^{-1} \varepsilon^t)] = \mathbb{E}[\text{vec}(\varepsilon \Sigma^{-1/2} \varepsilon^t)^t \text{vec}(\varepsilon \Sigma^{-1/2} \varepsilon^t)] = nm \tau^2, \quad (27)$$
since \(\text{vec}\{\varepsilon\Sigma^{-1/2}t\}\) has spherically symmetric distribution with proportionality coefficient \(\tau^2\). See Fang and Zhang (1990) for an approach through characteristic functions and more general classes of elliptical matrix distributions.

The canonical form of (25) can be derived as in the spherical case. Let \(Q = (Q_1 \ Q_2)\) be defined as previously in Section 3.2. According to (25), let

\[
Q^tY = \begin{pmatrix} Q_1^t \\ Q_2^t \end{pmatrix} Y = \begin{pmatrix} Q_1^t \\ Q_2^t \end{pmatrix} \beta + Q^t \varepsilon = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + Q^t \varepsilon
\]  

(28)

with matrices \(\theta = Q_1^tX\beta\) and \(Q_2^tX\beta = 0\) since, by construction, the columns of \(Q_2\) are orthogonal to those of \(X\). Now, using (25) and (26), \(Y - X\beta\) has density

\[
\varepsilon \mapsto |\Sigma|^{-n/2} f(\text{tr}(Q^t \varepsilon \Sigma^{-1} \varepsilon^t Q))
\]

and hence \((Z^tU^t)^t := Q^tY\) has an elliptically symmetric distribution about the corresponding column of \((\theta^t0^t)^t\). In this sense, the model

\[
\begin{pmatrix} Z \\ U \end{pmatrix} = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + \begin{pmatrix} Q_1^t \varepsilon \\ Q_2^t \varepsilon \end{pmatrix}
\]

is the canonical form of the linear regression model (28). Note that, if it exists, the joint density of \(Z\) and \(U\) is of the form

\[
(z, u) \mapsto |\Sigma|^{-n/2} F(\text{tr}((z - \theta)\Sigma^{-1}(z - \theta)^t) + \text{tr}(u\Sigma^{-1}u^t)),
\]  

(29)

for some function \(f\). Extending the vector case considered in Fourdrinier et al. (2003) we denote by \(\mathbb{E}\) the expectation with respect to the density in (29) and \(\mathbb{E}^*\) the expectation with respect to the density

\[
(z, u) \mapsto \frac{1}{C} |\Sigma|^{-n/2} F(\text{tr}((z - \theta)\Sigma^{-1}(z - \theta)^t) + \text{tr}(u\Sigma^{-1}u^t))
\]

(30)

where

\[
F(t) = \frac{1}{2} \int_0^t f(u)du.
\]  

(31)

and where it is assumed that

\[
C = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |\Sigma|^{-n/2} F(\text{tr}((z - \theta)\Sigma^{-1}(z - \theta)^t) + \text{tr}(u\Sigma^{-1}u^t)) \, dz \, du < \infty.
\]  

(32)

Note that, in the Gaussian case, these two expectations coincide.

We consider the problem of estimating \(\beta\) with the invariant loss

\[
L(\hat{\beta}, \beta) = \text{tr}((X\hat{\beta} - X\beta)\Sigma^{-1}(X\hat{\beta} - X\beta)^t).
\]  

(33)

It is worth noting that this loss can be written through the Frobenius norm associated to \(\Sigma^{-1}\) that is, for any \(n \times m\) matrix \(M\),

\[
\|M\|_{\Sigma^{-1}}^2 = \text{tr}(M\Sigma^{-1}M^t).
\]
As in the spherical case, the nature of $Q$ implies
\[
\|Y - X\hat{\beta}\|_{\Sigma}^2 = \|\hat{\theta} - Z\|_{\Sigma}^2 + \|U\|_{\Sigma}^2,
\]
which is the analog of (12).

In this context, we can give the following Stein-type identity for the multivariate elliptical distributions.

**Theorem 5 (Stein-type identity).** Given $(Z, U)$ a $n \times m$ random matrix following a vector-elliptical distribution around $(\theta, 0)$ with scale matrix $\Sigma$ and $g : \mathbb{R}^{pm} \rightarrow \mathbb{R}^{pm}$ a weakly differentiable function, we have
\[
E_{\theta} [\text{tr}(Z - \theta)\Sigma^{-1} g(Z)^t] = E_{\theta} [\|U\|_{\Sigma}^2 (n - p)m \text{div}_Z g(Z)],
\]
provided both expectations exist.

Taking into account the fact that the scale matrix of the vectorization of $(Z^t U^t)^t$ is $I_n \otimes \Sigma$, Theorem 5 can be derived from the spherical version given in Fourdrinier and Wells (1995).

To deduce an unbiased type estimator of loss for the elliptical case, the following adaption from Lemma 1 in Fourdrinier et al. (2003) is needed.

**Theorem 6 (Stein-Haff-type identity).** Under the conditions of Theorem 5, let $S = U^t U$. For any $m \times m$ matrix function $T(Z, S)$ weakly differentiable with respect to $S$, we have
\[
E [\text{tr}(T(Z, S)\Sigma^{-1})] = C E^*[2D_{1/2}^* T(Z, S) + a \text{tr}(S^{-1} T(Z, S))]
\]
with $a = n - p - m - 1$ and where the operator $D_{1/2}^*$ is be defined by
\[
D_{1/2}^* T(Z, S) = \sum_{i=1}^m \frac{\partial T_{ii}(Z, S)}{\partial S_{ii}} + \frac{1}{2} \sum_{i \neq j} \frac{\partial T_{ij}(Z, S)}{\partial S_{ij}}.
\]

### 4.3 Unbiased type estimator of loss

In this section we provide an unbiased type estimator $\delta_0^{\text{inv}}$ of the invariant loss $\|X\hat{\beta}(Y) - X\beta\|_{\Sigma}^2/\tau^2$ in the sense of the following definition.

**Definition 5 (E*-unbiasedness).** Let $Y$ be a random matrix in $\mathbb{R}^{nm}$ with mean $\mu \in \mathbb{R}^{nm}$ and let $\hat{\mu}(Y)$ be any estimator of $\mu$. An estimator $\delta_0(Y)$ of the loss $L(\hat{\mu}(Y), \mu)$ is said to be $E^*$-unbiased if, for all $\mu \in \mathbb{R}^{nm}$,
\[
E^* [\delta_0(Y)] = E [L(\hat{\mu}(Y), \mu)],
\]
where $E$ is the expectation associated with the original density in (29) while $E^*$ is the expectation with respect to the modified density (30).

In the usual notion of unbiasedness, $E^*$ is replaced by $E$. The need of $E^*$ is due to the presence of the matrix $\Sigma$. 

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Theorem 7 (Unbiased type estimator of loss for the elliptical case). Let \( \mathbf{Y} \sim \mathbf{E}_n(X\mathbf{\beta}, \Sigma) \) and let \( \hat{\mathbf{\beta}} = \hat{\mathbf{\beta}}(\mathbf{Y}) \) be an estimator of \( \mathbf{\beta} \) depending only on \( Q_1^\top \mathbf{Y} \). If \( \hat{\mathbf{\beta}}(\mathbf{Y}) \) is weakly differentiable with respect to \( \mathbf{Y} \), then the statistic

\[
\delta_0^{nw} = (n - p - m - 1)\| \mathbf{Y} - X\hat{\mathbf{\beta}} \|_{\Sigma^{-1}}^2 + 2 \text{div}_X \mathbf{X}\hat{\mathbf{\beta}}(\mathbf{Y}) - nm
\]  

(37)
is an \( \mathbb{E}^* \)-unbiased estimator of the invariant loss.

Proof. The invariant loss (33) can be decomposed as

\[
\| X\hat{\mathbf{\beta}} - X\mathbf{\beta} \|_{\Sigma^{-1}}^2 = \| \mathbf{Y} - X\hat{\mathbf{\beta}} \|_{\Sigma^{-1}}^2 - \| \mathbf{Y} - X\mathbf{\beta} \|_{\Sigma^{-1}}^2 + 2\text{tr}(X\hat{\mathbf{\beta}} - X\mathbf{\beta})\Sigma^{-1}(\mathbf{Y} - X\mathbf{\beta})^t. \tag{38}
\]

By orthogonally invariance of the classical inner product, the third term verifies

\[
\text{tr}(X\hat{\mathbf{\beta}} - X\mathbf{\beta})\Sigma^{-1}(\mathbf{Y} - X\mathbf{\beta})^t = \text{tr}(\hat{\mathbf{\theta}} - \mathbf{\theta})\Sigma^{-1}(\mathbf{Z} - \mathbf{\theta})^t.
\]

Now, since \( \hat{\mathbf{\theta}} = \hat{\mathbf{\theta}}(\mathbf{Z}, \mathbf{U}) \) depends only on \( \mathbf{Z} \), by the Stein type identity given in Theorem 3 we have

\[
\mathbb{E}\left[\text{tr}(\hat{\mathbf{\theta}} - \mathbf{\theta})\Sigma^{-1}(\mathbf{Z} - \mathbf{\theta})^t\right] = \mathbb{E}\left[ \frac{\|\mathbf{U}\|_{\Sigma^{-1}}^2}{(n - p)m} \text{div}_\mathbf{Z}(\hat{\mathbf{\theta}}) \right],
\]

so that, thanks to (27), taking expectation in (38) gives

\[
\mathbb{E}\left[\| X\hat{\mathbf{\beta}} - X\mathbf{\beta} \|_{\Sigma^{-1}}^2 \right] = \mathbb{E}\left[ \| \mathbf{Y} - X\hat{\mathbf{\beta}} \|_{\Sigma^{-1}}^2 \right] - nm\tau^2 + 2 \mathbb{E}\left[ \frac{\|\mathbf{U}\|_{\Sigma^{-1}}^2}{(n - p)m} \text{div}_\mathbf{Z}(\hat{\mathbf{\theta}}) \right].
\]

We get rid of \( \Sigma^{-1} \) applying Stein-Haff-type identity (Theorem 6) two times. First

\[
\mathbb{E}\left[ \| \mathbf{Y} - X\hat{\mathbf{\beta}} \|_{\Sigma^{-1}}^2 \right] = C \mathbb{E}^* \left[ (n - p - m - 1)\| \mathbf{Y} - X\hat{\mathbf{\beta}} \|_{\Sigma^{-1}}^2 \right].
\]

Secondly, setting \( T(\mathbf{Z}, \mathbf{S}) = \text{div}_\mathbf{Z}(\hat{\mathbf{\theta}})/(n - p)m \mathbf{S} \), we have

\[
\mathbb{E}\left[ \frac{\|\mathbf{U}\|_{\Sigma^{-1}}^2}{(n - p)m} \text{div}_\mathbf{Z}(\hat{\mathbf{\theta}}) \right] = \mathbb{E}\left[ \text{tr}(\Sigma^{-1} \mathbf{S}) \frac{\text{div}_\mathbf{Z}(\hat{\mathbf{\theta}})}{(n - p)m} \right] = C \mathbb{E}^* \left[ \text{div}_\mathbf{Z}(\hat{\mathbf{\theta}}) \right].
\]

It is worth noticing that \( C = \tau^2 \). Indeed, thanks to (27), \( \mathbb{E}\left[\|\mathbf{U}\|_{\Sigma^{-1}}^2 \right] = (n - p)m\tau^2 \) and, using Theorem 6 \( \mathbb{E}\left[\|\mathbf{U}\|_{\Sigma^{-1}}^2 \right] = C(n - p)m \) . Finally note that Lemma 1 extends to the matrix case so that \( \text{div}_\mathbf{Z}(\hat{\mathbf{\theta}}) = \text{div}_\mathbf{Y} X\hat{\mathbf{\beta}} \). \[\square\]

It is worth noting that \( m = 1 \) reduces to the spherical case and that the associated unbiased estimator of the invariant loss, adapted from equation (37), is

\[
\delta_0^{uw} = \frac{n - p - 2}{\|\mathbf{U}\|_{\Sigma^{-1}}^2} \| \mathbf{Y} - X\hat{\mathbf{\beta}} \|_{\Sigma^{-1}}^2 + 2 \text{div}_X X\hat{\mathbf{\beta}}(\mathbf{Y}) - n,
\]

which is the same as the one provided in the Gaussian case in (9).
5 Discussion

In this article we viewed the well-known model selection criteria $C_p$ and AIC through the lens of loss estimation and related them to an unbiased estimator of the quadratic prediction loss under a Gaussian assumption. We then developed unbiased estimators of loss under considerably wider distributional settings, spherically symmetric distributions and the family of elliptically symmetric distributions. In the spherical context, the unbiased estimator of loss is actually equal to the one derived under the Gaussian law. Hence, this implies that we do not have to specify the form of the distribution, the only condition being its spherical symmetry. We also conclude from the equivalence between unbiased estimators of loss, $C_p$ and AIC that their form for the Gaussian case is able to handle any spherically symmetric distribution. The spherical family is interesting for many practical cases since it allows a dependence property between the components of random vectors whenever the distribution is not Gaussian. Some members of this family also have heavier tails than Gaussian densities, and thus the unbiased estimator derived here can be robust to outliers. We also considered a generalization with elliptically symmetric distributions for the error vector, allowing a general covariance matrix $\Sigma$. However, to get such a result, a matrix regression model which allows estimation of the covariance matrix $\Sigma$ was introduced.

It is well known that unbiased estimators of loss are not the best estimators and can be improved (Johnstone 1988; Fourdrinier and Wells 2012). It was not our intention in this article to derive such estimators, but to explain why their performances can be similar when departing from the Gaussian assumption. The improvement over these unbiased estimators requires a way to assess their quality. This can be done either using oracle inequalities or the theory of admissible estimators under a certain risk criteria. Based on a proper risk, a selection rule $\delta_0$ is inadmissible if we can find a better estimator, say $\delta_\gamma$, that has a smaller risk function for all possible values of the parameter $\beta$, with strict inequality for some $\beta$. The heuristic of loss estimation is that the closer an estimator is to the true loss, the more we expect their respective minima to be close. We are currently working on improved estimators of loss of the type $\delta_\gamma(Y) = \delta_0(Y) + \gamma(Y)$, where $\gamma(Y)$ can be thought of as a data driven penalty. The selection of such a $\gamma$ term is an important, albeit difficult, research direction.

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