Abstract. This paper considers a class of space fractional partial differential equations (FPDEs) that describe gas pressures in fractured media. First, the well-posedness, uniqueness, and the stability in \( L_\infty(\mathbb{R}) \) of the considered FPDEs are investigated. Then, the reference tracking problem is studied to track the pressure gradient at a downstream location of a channel. This requires manipulation of gas pressure at the downstream location and the use of pressure measurements at an upstream location. To achieve this, the backstepping approach is adapted to the space FPDEs. The key challenge in this adaptation is the non-applicability of the Lyapunov theory which is typically used to prove the stability of the target system as, the obtained target system is fractional in space. In addition, a backstepping adaptive observer is designed to jointly estimate both the system’s state and the disturbance. The stability of the closed loop (reference tracking controller/observer) is also investigated. Finally, numerical simulations are given to evaluate the efficiency of the proposed method.

Key words. Reference Tracking, Observer Design, Fractional systems, Well-posedness, Uniqueness.

AMS subject classifications. 35R11, 93C20, 93C40, 35A08.

1. INTRODUCTION. Asymptotic output regulation, which allows a system’s output to follow a desired reference, is a challenging problem that has received considerable research attention, particularly with regard to the presence of disturbances. This problem has been widely studied and solved for different perturbed systems described either using ordinary differential equations (ODEs) or partial differential equations (PDEs), including parabolic PDEs, which describe a wide variety of time-dependent phenomena, such as heat conduction and particle diffusion (e.g., [32], [11]). For example, in [23], the output regulation problem was investigated for lumped-parameter linear systems. An extension of this result was discussed in [9] and [24] for nonlinear systems. Subsequent studies have investigated the robust output regulation problem for linear distributed parameter systems (e.g., [8], [21], [19]). More recently, this approach has been extended to different PDE systems [34], [33]. In particular, in [11], the author considered the output regulation problem for boundary-controlled parabolic PDEs in the presence of disturbances under the assumption that these disturbances and the desired reference can be modeled using a known finite dimensional signal.

On the other hand, fractional PDEs (FPDEs) have been used for accurately modeling and analyzing numerous phenomena in different scientific and engineering fields (e.g., [31], [44]) owing to their suitability to describe the dynamics of several problems. Both time and space FPDEs (fractional derivatives in time and space, respectively) can capture multi scale features of complex physical phenomena due to the non locality of the fractional operators [4], [6], [7], [12] and [30]. Furthermore, time FPDEs have been more widely investigated in the literature than space FPDEs, predominantly in the field of control. For example, [29] proposed to solve boundary feedback stabilization and disturbance rejection problems for time fractional PDE. whereas [45] solves...
the output feedback stabilization problem for an unstable time fractional PDE. [28] and [41] addresses the observer-based robust stabilisation problem and dynamic output feedback control for non-linear fractional uncertain systems and fractional order systems respectively. [14] considers approximate controllability problem for fractional diffusion equations.

In [38], the authors concluded that particle transport behavior may be parsimoniously described using a fractional advection dispersion equation (FADE) by phenomenological discussion of the arbitrary average velocity and non-zero dispersion coefficient. Furthermore, in [10], an unconditionally stable second-order difference method for two-dimensional FADE was proposed.

Moreover, very few work investigates the tracking problem for FPDEs. [17] investigates the reference tracking problem for time fractional advection dispersion equation in presence of disturbances. Less investigations have been done on space fractional PDEs despite their importance in describing a large variety of real life phenomena such as gas production in fractured media [3], solute transport in heterogeneous porous media [38], [37], Plumes spread in laboratories in [6] [7], and Transport affected by hydraulic conditions at a distance on the Earth surface in [10]. To the best of the authors knowledge, there is no paper dealing with stability, reference tracking and observer design problems for space FPDE. Furthermore, some work exists on studying the well-posedness of space FDEs. For instance, in [20], the authors solved analytically and in terms of Green function the homogeneous time and space FADE. and in [2], the non homogeneous Riesz-Feller space FADE has been solved analytically.

In this paper, we consider a class of Caputo space FPDE, which describes for instance the gas production in fractured media. such mechanisms is done by drilling wells through gas saturated rock to force the gas to flow through the drilled well into the production pipelines. The underground layers burden pressure causes a chaotic explosion of the saturated rocks, which yields to unequal spatial distribution of the pressure and which is modeled by a spatial fractional derivative. In [3], the authors solves the non-Newtonian non-Darcy fractional-derivatives flux equations using physics-preserving averaging schemes that incorporates both, original and shifted, Grunwald-Letnikov (GL) approximation formulas preserving the physics, by reducing the shifting effects, while maintaining the stability of the system. They derived the system’s equations and discussed the discretization schemes. Then, they illustrated the physics-preserving averaging scheme. Some authors believe that a minimum pressure gradient (called threshold pressure gradient (TPG)) is required before a liquid starts to flow in a porous medium. It has been proven as well in [39] that the pressure gradient has a much greater effect on gas mobility and oil recovery than pressure-decline rate has. That is why, in this paper, we aim at tracking the pressure gradient at a final position to follow a desired trajectory. This tracking process can be seen as a decreasing process of the gradient pressure as well by taking a desired trajectory which is as small as possible.

The main objectives of this paper are to study the reference tracking problem for boundary controlled space FPDE modelling the gas production in fractured media in the presence of disturbances. This is done by adapting the backstepping approach to the space FPDEs. The key challenge here is the non-applicability of the Lyapunov theory which is commonly used to prove the stability of the target system as the obtained target system is fractional in space. The contribution of this paper is to prove the asymptotic stability of the space fractional target system to ensure the reference tracking process. This is done by deriving the analytical solution of the target system in terms of Green functions. Then, taking the advantage of the fractional differen-
tiation order which will allow the green function to converge asymptotically to zero.
With a similar reasoning another contribution of the paper would be to extract some
conditions on the system’s parameters that ensure the stability in $L_{\infty}(\mathbb{R})$ of the
considered model. Moreover, an additional contribution of this paper is the design of a
backstepping adaptive state observer which is required by the output feedback control.
In addition, the stability of the closed loop (reference tracking controller-observer) is
proved. Moreover, a fundamental solution for the non homogeneous Caputo space
FPDE is proposed and its uniqueness is studied.

This paper is organized as follows, in section 2, a review of some preliminaries on
fractional calculus is presented. In section 3, the considered problem is formulated.
The fundamental solution of the considered FPDE using Duhamel’s principle and
the generalized Leibnitz differentiation rule is introduced in section 4. Section 5
aims to derive some sufficient conditions to guarantee the stability in $L_{\infty}(\mathbb{R})$ of the
considered problem. Section 6 studies the reference tracking problem for boundary
controlled space FPDE in the presence of disturbances by proving the asymptotic
stability in $L_{\infty}(\mathbb{R})$ of the tracking error. The backstepping adaptive observer design
for the considered problem is given in section 7. Section 8 studies the stability of the
closed loop (reference tracking controller-observer). Simulation examples are given as
well to support the proposed results. Finally, a general conclusion will summarize the
obtained results.

2. Fractional Calculus Preliminaries. This section presents useful definitions
and results on fractional PDE.

**Definition 2.1 (Gamma function).** [35, 16, 15] Gamma function is defined by:

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt, \quad z \in \mathbb{R}^+,$$

**Definition 2.2.** [35, 16, 15]
Riemann-Liouville fractional integral of order $\alpha$ with $n-1 < \alpha \leq n, n \in \mathbb{N}$ is defined by:

$$aJ_\alpha^t f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau,$$

where, $t \in [a, b], f(.) \in C[a, b]$. We denote: $J_\alpha^t f(t) := aJ_\alpha^t f(t)$.

**Definition 2.3.** [35, 16, 15]
Caputo fractional derivative of order $\alpha$ with $n-1 < \alpha \leq n, n \in \mathbb{N}$ is given by:

$$C_\alpha^t D_\alpha^t f(t) = \begin{cases} \frac{\partial^n f(t)}{\partial t^n}, & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(\tau)}{\partial \tau^n}d\tau, \end{cases}$$

where, $t \in [a, b], f(.) \in C^n[a, b]$. We denote:

$$C_0^t D_\alpha^t f(t) := C_0^t D_0^t f(t).$$

For more information on the Caputo fractional derivative, we refer the readers to
[35].
Remark 2.4. Caputo time fractional derivative with a negative non integer order is defined in [35] by:

\[ C^\alpha_t f(t) = J^{-\alpha}_t f(t), \]

for all \( \alpha \in \mathbb{R}^- / \mathbb{Z}^- \).

**Theorem 2.5.** Let \( f(t) \in C^n[a, b], t \in [a, b], \) we have:

\[ C^\alpha_t J^\alpha f(t) = f(t), \]

where \( n - 1 < \alpha \leq n, n \in \mathbb{N} \), see [35] for the proof.

**Definition 2.6.** [36] The right-sided Caputo fractional derivative of order \( \alpha \) with \( n - 1 < \alpha \leq n, n \in \mathbb{N} \) is defined by:

\[ D^\alpha_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^b (\tau - t)^{n-\alpha-1} \frac{\partial^n f(\tau)}{\partial \tau^n} d\tau, \]

where, \( t \in [a, b], f(.) \in C^n[a, b]. \)

**Definition 2.7.** [36] The right-sided Caputo time fractional derivative with a negative non integer order \( \alpha \in \mathbb{R}^- / \mathbb{Z}^- \) is defined by:

\[ C^\alpha_{t,b} f(t) = J^{-\alpha}_{t,b} f(t), \]

where, \( J^{-\alpha}_{t,b} f(\cdot) = J^{-\alpha}_{t,b} g(\cdot), \alpha \in \mathbb{R}^- / \mathbb{Z}^- \) is the right-sided Riemann-Liouville fractional integral of order \( -\alpha \) defined in [36] by:

\[ J^\beta_{t,b} f(t) = \frac{(-1)^n}{\Gamma(\beta)} \int_t^b (t - \tau)^{\beta-1} f(\tau) d\tau, \]

where, \( n - 1 < \beta \leq n, t \in [a, b], f(.) \in C[a, b] \)

**Definition 2.8.** Fourier transform definition, its inverse and the Fourier transform of the classical derivative are given by:

Let \( f(t) \in C^m(\mathbb{R}), F\{f(t)\}(s) \in L^1(\mathbb{R}) \) then, we have:

\[
\begin{align*}
F\{f(t)\}(s) &= \int_{-\infty}^{+\infty} e^{ist} f(t) dt, \\
F^{-1}\{\hat{f}(s)\}(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ist} \hat{f}(s) ds, \\
F\{D^m_t f(t)\}(s) &= (-is)^m F\{f(t)\}(s),
\end{align*}
\]

where, \( F\{f(t)\}(s) := \hat{f}(s), s \in \mathbb{R}, m \in \mathbb{N}, i^2 = -1. \)

**Lemma 2.9.** Fourier transform of Caputo fractional derivative is given in [20] by:

Let \( f(t) \in C^m(\mathbb{R}) \), then:

\[ F\{C^\alpha_t f(t)\}(s) = (-is)^\alpha F\{f(t)\}(s) \]

where, \( n - 1 < \alpha < n, n \in \mathbb{N} \).

**Lemma 2.10.** Fractional integration by parts was given in [36] by:

Let \( f(t), g(t) \in C^n(\mathbb{R}) \), then:
A solution \( u(t) \) of this system is said to be stable if, given any \( \| x \| \) with \( \| x \|_X \leq \delta \), there exists a \( \delta = \delta(\epsilon) > 0 \) such that, for all \( u_0 \in X \) with \( \| u_0 \|_X \leq \delta \), the corresponding solution satisfies \( \| u(t) \|_X \leq \epsilon \) for all \( t \geq 0 \). If in addition there exists a \( \delta^* \) such that for all initial conditions with \( \| u_0 \|_X \leq \delta^* \) the corresponding solution satisfies
\[
\lim_{t \to \infty} \| u(t) \|_X = 0,
\]
then this solution is said to be asymptotically stable.
Remark 2.14. In what follows, we will be considering the stability and the asymptotic stability in $L_\infty(\mathbb{R})$. That is, taking $X = L_\infty(\mathbb{R})$ in definition 2.13.

Remark 2.15. In what follows, we will be using the following norms:

$$
\| P(x,t) \|_\infty = \sup_{x \in [0,L]} |P(x,t)|, \quad \text{and} \quad \| P(x,t) \|_2 = \left( \int_0^L |P(x,t)|^2 dx \right)^{1/2}.
$$

which are defined over $L_\infty(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively.

**Theorem 2.16.** Let $f(t) \in L^1(\mathbb{R})$. Then the Fourier transform of $f(t)$ is bounded in $L^1(\mathbb{R})$ by:

$$
|\mathcal{F}\{f(t)\}(s)| \leq \int_{-\infty}^{+\infty} |f(t)| dt = \| f(t) \|_{L^1(\mathbb{R})}.
$$

**Theorem 2.17.** Fourier transform of fractional integral is given in [35] by:

$$
\mathcal{F}\{J_t^\alpha f(t)\}(s) = \frac{1}{(is)^\alpha} \mathcal{F}\{f(t)\}(s), \quad s \neq 0
$$

where, $n - 1 < \alpha < n, n \in \mathbb{N}$.

### 3. Problem Statement

This section presents the model’s equations. We consider the gas production mechanism which is done by drilling wells. When a vertical well is drilled through a gas saturated rock, the underground layers burden pressure causes the gas to flow through the drilled well into the production pipelines as shown in figure 1. Table 1 summarizes the system’s parameters.

![Fig. 1: Gas production process](image)

#### 3.1. Gas Production Fractional Model

The standard modeling approach for gas pressure is recalled in the appendix A. However, due to the rocks explosion caused by the gas pressure, the constructed pores don’t have the same shapes and dimensions. Thus, the variation of the pressure with respect to space is not equally distributed which means that the Darcy law equation doesn’t describe fully the considered phenomena. Therefore, the authors in [3] suggested to use the non-Darcy law, where the first spatial derivative of the pressure is replaced with a fractional spatial derivative of order $\alpha$, where $0 < \alpha \leq 1$ represents the diffusion coefficient. Therefore the FPDE that fully describes the considered phenomena is given with the boundary
Table 1: Parameters Description

| Symbol | Description |
|--------|-------------|
| \( t \) | Time        |
| \( x \) | Space       |
| \( P(x, t) \) | Gas pressure |
| \( \rho(x, t) \) | Density     |
| \( \varphi(x, t) \) | Porosity    |
| \( u(x, t) \) | Velocity    |
| \( k(t) \) | Permeability |
| \( \mu \) | Viscosity   |
| \( \alpha \) | Diffusion coefficient |
| \( Q(x, t) \) | Gas production flow |
| \( C \) | Variation of the porosity with respect to the pressure |
| \( u(t) \) | Control input |

conditions and the system control by:

\[
\begin{aligned}
\frac{\partial P(x, t)}{\partial t} - \frac{1}{C} \frac{k(t)}{\mu} \frac{\partial}{\partial x} \mathcal{C}D_x^\alpha P(x, t) &= \frac{1}{C\rho} Q(x, t), \\
P(x, 0) &= g_0(x), \\
P_x(0, t) &= 0, P(L, t) = u(t),
\end{aligned}
\]

(3.1)

where \( t > 0, x \in [0, L], u(t) \) is the control input, \( P(x, t) \) is the gas pressure distributed in space and in time. The permeability \( k(t) \) and the viscosity \( \mu \) are both positive. \( \mathcal{C}D_x^\alpha P(x, t) \) is the Caputo space fractional derivative of order \( \alpha \). \( u(t) \) is the control input.

Remark 3.1. It has been proved in [35] that if we consider the Caputo fractional derivative then the operator \( \frac{\partial}{\partial x} \mathcal{C}D_x^\alpha P(x, t) \) in (3.1) cannot be written as \( \mathcal{C}D_x^\beta P(x, t) \) with \( 1 < \beta \leq 2 \).

The main objective of this paper is to track the pressure gradient at final position in the presence of disturbances using some measurements and a boundary control. However, before that, we first solve the problem analytically. Then, we study the well-posedness and the stability of the considered system.

4. Fundamental Solution of the Space FPDE. In this section, we first derive the fundamental solution for the non homogeneous Caputo space fractional PDE. For this purpose, we use the Fourier transform of Caputo fractional derivative (2.2). Then, we investigate the uniqueness of the obtained solution.

4.1. Analytic Solution.

Theorem 4.1. Assuming that \( Q(x, \cdot), g_0(x) \in \mathcal{L}^1(\mathbb{R}) \) and \( u(t) \in C^1(\mathbb{R}^+) \), system (3.1) admits at least one solution and it is given by:

\[
P(x, t) = u(t) + \int_{-\infty}^{+\infty} \int_0^t G_\alpha(\mid x - y \mid, t - \tau) Q(y, \tau) d\tau dy + \int_{-\infty}^{+\infty} G_\alpha(\mid x - y \mid, t) \tilde{g}_0(y) dy.
\]
where, \( \bar{g}_0(x) := g_0(x) - u(0) \), \( \bar{Q}(x, t) := \frac{1}{C_\rho} Q(x, t) - \frac{\partial}{\partial t} u(t) \) and \( G_\alpha(x, t) \) is the Green function defined by:

\[
G_\alpha(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\alpha s} e^{\frac{1}{\mu} \phi^{\alpha+1}(s)} ds,
\]

with \( \phi^{\alpha+1}(s) = (-is)^{\alpha+1} \).

**Proof.** Starting from the system (3.1), we define the new coordinate \( \bar{P}(x, t) \),

\[
(4.1) \quad \bar{P}(x, t) := P(x, t) - u(t),
\]

where, \( u(t) \) is the boundary condition in system (3.1). The system for the new coordinate becomes:

\[
(4.2) \quad \begin{cases}
\frac{\partial \bar{P}(x, t)}{\partial t} - \frac{1}{C_\mu} \frac{k(t)}{\mu} D^\alpha_x \bar{P}(x, t) = \bar{Q}(x, t), \\
\bar{P}(x, 0) = \bar{g}_0(x), \\
\bar{P}_x(0, t) = 0, \bar{P}(L, t) = 0,
\end{cases}
\]

where, \( \bar{g}_0(x) \) is the new initial condition and \( \bar{Q}(x, t) \) is the new source term given respectively by:

\[
\bar{g}_0(x) := g_0(x) - u(0) \quad \bar{Q}(x, t) := \frac{1}{C_\rho} Q(x, t) - \frac{\partial}{\partial t} u(t).
\]

Then, we multiply the gas pressure \( \bar{P}(x, t) \) in (3.1) by the indicator function of \([0, L]\) to have the FPDE defined for \( x \in \mathbb{R} \). Then we apply the superposition principle to write the solution of (3.1) as \( \bar{P} = h + v \), where, for every \( t > 0, x \in \mathbb{R} \), \( u \) and \( v \) are the solution of the following problems respectively:

\[
(4.3) \quad \begin{cases}
\frac{\partial h(x, t)}{\partial t} - \frac{1}{C_\mu} \frac{k(t)}{\mu} D^\alpha_x h(x, t) = 0, \\
h(x, 0) = \bar{g}_0(x),
\end{cases}
\]

\[
(4.4) \quad \begin{cases}
\frac{\partial v(x, t)}{\partial t} - \frac{1}{C_\mu} \frac{k(t)}{\mu} D^\alpha_x v(x, t) = \bar{Q}(x, t), \\
v(x, 0) = 0,
\end{cases}
\]

because, both the Caputo fractional derivative and classical derivative are linear. We start by solving system (4.4), we apply the Duhamel’s principle [25] ( which allows to solve first the homogeneous linear, PDE, and then superposing to find the solution of the original PDE), the solution of (4.4) is given by:

\[
(4.5) \quad v(x, t) = \int_0^t \mathbb{V}(x, t, \tau)d\tau,
\]
where, for all $\tau \in [0,t]$, for every $t > 0$, $x \in \mathbb{R}$. We start by computing the time first derivative of $v(x, t)$ in (4.5):

\[
\frac{\partial}{\partial t} v(x, t) = \frac{\partial}{\partial t} \int_0^t V(x, t, \tau) d\tau,
\]

(4.6)

\[
= \int_0^t \frac{\partial}{\partial t} V(x, t, \tau) d\tau + V(x, t = \tau),
\]

using the classical Leibnitz differentiation rule. Now, we compute the space fractional derivative of order $\alpha$ of $v(x, t)$ in (4.5):

\[
C^D_\alpha x v(x, t) = \int \frac{\partial}{\partial x} C^D_\alpha x V(x, t, \tau) d\tau,
\]

(4.7)

\[
\Rightarrow \frac{\partial}{\partial x} C^D_\alpha x v(x, t) = \int \frac{\partial}{\partial x} C^D_\alpha x V(x, t, \tau) d\tau.
\]

Thus, $V(., ., \tau)$ satisfies the following system:

(4.8)

\[
\begin{cases}
\frac{\partial V(x, t)}{\partial t} - \frac{1}{C} \frac{k(t)}{\mu} \frac{\partial}{\partial x} C^D_\alpha x V(x, t) = 0, \\
\lim_{t \to \tau} V(x, t) = \bar{Q}(x, \tau).
\end{cases}
\]

Then, by applying the Fourier transform to the FPDE with respect to the variable $x$, we have:

\[
\begin{cases}
\mathcal{F}\left\{ \frac{\partial V(x, t)}{\partial t} \right\} - \frac{1}{C} \frac{k(t)}{\mu} \mathcal{F}\left\{ \frac{\partial}{\partial x} C^D_\alpha x V(x, t) \right\} = 0, \\
\mathcal{F}\{V(x, t = \tau)\} = \mathcal{F}\{\bar{Q}(x, \tau)\},
\end{cases}
\]

using the Fourier transform of the first spatial derivative (definition 2.8) and Caputo fractional derivative (2.2), we get:

(4.9)

\[
\begin{cases}
\frac{\partial \tilde{V}(s, t)}{\partial t} - \frac{1}{C} \frac{k(t)}{\mu} (-is)^{\alpha+1} \tilde{V}(s, t) = 0, \\
\tilde{V}(s, t = \tau) = \tilde{\bar{Q}}(s, \tau),
\end{cases}
\]

where, $\tilde{\bar{Q}}(s, \tau) = \mathcal{F}\{\bar{Q}(x, \tau)\}(s, \tau)$. Using the initial condition in (4.8), the solution of (4.9) is given by:

(4.10)

\[
\tilde{V}(s, t) = e^{\frac{1}{C} \frac{k(t)}{\mu} (-is)^{\alpha+1}(t-\tau)} \tilde{\bar{Q}}(s, \tau),
\]

where, $\tilde{\bar{Q}}(s, \tau) = \mathcal{F}\{\bar{Q}(x, \tau)\}(s, \tau)$. Applying the inverse Fourier transform (2.8), (4.10) becomes:

\[
V(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixs} e^{\frac{1}{C} \frac{k(t)}{\mu} (-is)^{\alpha+1}(t-\tau)} \tilde{\bar{Q}}(s, \tau) ds,
\]

which can be written as:

(4.11)

\[
V(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixs} \mathcal{G}_\alpha(s, t-\tau) \tilde{\bar{Q}}(s, \tau) ds,
\]
where, $G_\alpha(x,t)$ is the Green function defined in [20] by:

$$G_\alpha(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isx} e^{\frac{i}{\mu} \phi^{\alpha+1}(s)t} ds$$

and characterized by:

$$\tilde{G}_\alpha = \mathcal{F}\{G_\alpha\}(s,t) = e^{\frac{i}{\mu} \phi^{\alpha+1}(s)t},$$

where,

$$\phi^{\alpha+1}(s) = (-is)^{\alpha+1}.$$  

$\mathcal{F}$ is the Fourier Transform, more details in the green function can be found in [2], [20]. Then, using the inverse Fourier transform (2.8), (4.11) becomes:

$$V(x,t) = \int_{-\infty}^{+\infty} G_\alpha(x-y,t-\tau) \bar{Q}(y,\tau) dy,$$

Thus, using (4.5), the solution of (4.4) is given by:

$$v(x,t) = \int_{-\infty}^{+\infty} \int_{0}^{t} G_\alpha(x-y,t-\tau) \bar{Q}(y,\tau) d\tau dy$$

Similarly, the solution of (4.3) is given by:

$$h(x,t) = \int_{-\infty}^{+\infty} G_\alpha(x-y,t) \bar{g}_0(y) dy.$$  

Finally, the fundamental solution of (3.1) is:

$$P(x,t) = u(t) + \int_{-\infty}^{+\infty} \int_{0}^{t} G_\alpha(x-y,t-\tau) \bar{Q}(y,\tau) d\tau dy + \int_{-\infty}^{+\infty} G_\alpha(x-y,t) \bar{g}_0(y) dy.$$  

### 4.2. Uniqueness.

The uniqueness of the solution is a direct result from the linearity of the integral operator.

**Theorem 4.2.** Assuming that $g_0(x) \in L^1(\mathbb{R})$ and $u(t) \in C^1(\mathbb{R}^+)$, Let $P_1(x,t)$ and $P_2(x,t)$ be two solutions of system (3.1) with different source terms $Q_1(x,\cdot), Q_2(x,\cdot) \in L^1(\mathbb{R})$ respectively. Then, the condition $Q_1(x,t) = Q_2(x,t)$ implies that $P_1(x,t) = P_2(x,t)$.

**Proof.** Suppose that system (3.1) admits two solutions $P_1(x,t)$ and $P_2(x,t)$ with different source terms $Q_1(x,\cdot), Q_2(x,\cdot) \in L^1(\mathbb{R})$. Then, by Theorem 4.1, $P_1(x,t)$ and $P_2(x,t)$ are both given by:

$$P_1(x,t) = u(t) + \int_{-\infty}^{+\infty} \int_{0}^{t} G_\alpha(x-y,t-\tau) Q_1(y,\tau) d\tau dy + \int_{-\infty}^{+\infty} G_\alpha(x-y,t) \bar{g}_0(y) dy + u(t).$$

and

$$P_2(x,t) = \int_{-\infty}^{+\infty} \int_{0}^{t} G_\alpha(x-y,t-\tau) Q_2(y,\tau) d\tau dy + \int_{-\infty}^{+\infty} G_\alpha(x-y,t) \bar{g}_0(y) dy + u(t).$$
Then, by (4.16) and (4.17):
\[ P_1(x,t) - P_2(x,t) = \int_{-\infty}^{+\infty} \int_0^t G_\alpha(x-y,t-\tau) [Q_1(y,\tau) - Q_2(y,\tau)] d\tau dy \]
completes the proof.

5. Stability Conditions. In this section, we aim to derive some sufficient conditions to guarantee the stability of the problem given in (3.1).

**Theorem 5.1.** Suppose that \( \bar{Q}(x,t) \) is separable on its variables \( \bar{Q}(x,t) = T(t)q(x) \) such that \( q(x) \in L^1(\mathbb{R}) \) and \( T(t) \) is bounded on time, this means that
\[ \bar{Q}(x,t) \leq rq(x). \]
Suppose as well that \( \bar{g}_0(x) \in L^1(\mathbb{R}) \) and \( k(t) \) is bounded from below \( (k_0 \leq k(t)) \) and that the control \( u(t) \) is bounded as well. Then, the solution of system (3.1) is stable in \( L_\infty(\mathbb{R}) \). That is:
\[ \lim_{t \to +\infty} \|P(x,t)\|_\infty < \infty \]

**Proof.** Using Theorem 4.1, the solution of (3.1) is given by (4.15). \( \bar{Q}(x,t) \) and \( u(t) \) are bounded in time which means that:
\[ \bar{Q}(x,t) \leq rq(x). \]
\[ u(t) \leq u_{max}. \]
Thus, (3.1) becomes:
\[ P(x,t) = u(t) + \int_{-\infty}^{+\infty} \int_0^t G_\alpha(x-y,t-\tau) \bar{Q}(y,\tau) d\tau dy + \int_{-\infty}^{+\infty} G_\alpha(x-y,t) \bar{g}_0(y) dy. \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_0^t \tilde{G}_\alpha(s,t-\tau) \bar{Q}(s,t) e^{-isx} d\tau ds + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{G}_\alpha(s,t) \bar{g}_0(s) e^{-isx} ds + u(t), \]
because both \( \bar{Q}(x,t) \) and \( \bar{g}_0(x) \) are in \( L^1(\mathbb{R}) \) which means that their Fourier transforms \( \tilde{Q}(x,t) \) and \( \tilde{g}_0(x) \) exist. Then, we get:

\[ |P(x,t)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_0^t \tilde{G}_\alpha(s,t-\tau) e^{-isx} \tilde{Q}(s,t) d\tau ds + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{G}_\alpha(s,t) e^{-isx} \bar{g}_0(s) ds + |u_{max}|, \]
\[ \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_0^t \tilde{G}_\alpha(s,t-\tau) e^{-isx} \tilde{Q}(s,t) d\tau ds + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{G}_\alpha(s,t) e^{-isx} \bar{g}_0(s) ds + |u_{max}|, \]
\[ \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|r\bar{q}(s)e^{-isx}\| \int_0^t \tilde{G}_\alpha(s,t-\tau) d\tau ds + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|\tilde{G}_\alpha(s,t)\| \bar{g}_0(s) ds + |u_{max}|. \]

By taking the sup on \( x \) over \([0,L]\) of (5.1), we get:

\[ \|P(x,t)\|_\infty \leq \frac{|r|}{2\pi} \int_{-\infty}^{+\infty} \|\bar{q}(s)\| \int_0^t \|\tilde{G}_\alpha(s,t-\tau)\| d\tau ds + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|\tilde{G}_\alpha(s,t)\| \bar{g}_0(s) ds + |u_{max}|. \]
On the other hand, we have that, for all $0 < \alpha \leq 1$:

$$
\phi^{\alpha+1}(s) = (-is)^{\alpha+1} = s^{\alpha+1}(-i)^{\alpha+1} = s^{\alpha+1}e^{-i\frac{\pi}{2}(\alpha+1)}
$$

$$
= s^{\alpha+1}\{\cos\left(\frac{\pi}{2}(\alpha + 1)\right) - i \sin\left(\frac{\pi}{2}(\alpha + 1)\right)\}
= s^{\alpha+1}\{-\sin\left(\frac{\pi}{2}\alpha\right) - i \cos\left(\frac{\pi}{2}\alpha\right)\}.
$$

(5.4)

$$
\tilde{G}_{\alpha}(s, t) = e^{\frac{i}{\pi^2}k(t)\phi^{\alpha+1}(s)t}
$$

and

$$
\phi^{\alpha+1}(s) = (-is)^{\alpha+1}.
$$

On the other hand, we have that, for all $0 < \alpha \leq 1$:

$$
\phi^{\alpha+1}(s) = (-is)^{\alpha+1} = s^{\alpha+1}(-i)^{\alpha+1} = s^{\alpha+1}e^{-i\frac{\pi}{2}(\alpha+1)}
$$

$$
= s^{\alpha+1}\{\cos\left(\frac{\pi}{2}(\alpha + 1)\right) - i \sin\left(\frac{\pi}{2}(\alpha + 1)\right)\}
= s^{\alpha+1}\{-\sin\left(\frac{\pi}{2}\alpha\right) - i \cos\left(\frac{\pi}{2}\alpha\right)\}.
$$

(5.5)

if $s < 0$ we have that, for all $0 < \alpha \leq 1$:

$$
\phi^{\alpha+1}(s) = (-is)^{\alpha+1} = s^{\alpha+1}(-i)^{\alpha+1} = s^{\alpha+1}e^{-i\frac{\pi}{2}(\alpha+1)}
$$

$$
= s^{\alpha+1}\{\cos\left(\frac{\pi}{2}(\alpha + 1)\right) - i \sin\left(\frac{\pi}{2}(\alpha + 1)\right)\}
= s^{\alpha+1}\{-\sin\left(\frac{\pi}{2}\alpha\right) - i \cos\left(\frac{\pi}{2}\alpha\right)\}.
$$

(5.6)

Thus from (5.5) and (5.6) we have that:

$$
\text{Re}\{\phi^{\alpha+1}(s)\} \leq 0, \quad 0 < \alpha \leq 1, \forall s \in \mathbb{R},
$$

using the fact that both the permeability and the viscosity are both positive and the linear-compressibility with $C > 0$, the sign of the real part of $\phi^{\alpha+1}(s)$ is negative. Using (5.4) we have that:

$$
|\tilde{G}_{\alpha}(s, t)| = e^{\frac{i}{\pi^2}k(t)\text{Re}\{\phi^{\alpha+1}(s)\}t}
$$

(5.7)

$$
\Rightarrow \lim_{t \to +\infty} |\tilde{G}_{\alpha}(s, t)| = \lim_{t \to +\infty} e^{\frac{i}{\pi^2}k(t)\text{Re}\{\phi^{\alpha+1}(s)\}t}
$$

(5.8)
Thus, by (5.7):

\[
(5.9) \quad \lim_{t \to +\infty} |\mathcal{G}_\alpha(s, t)| = 0.
\]

Using the fact that \(k(t)\) is bounded from below, (5.4) and (5.7), we have:

\[
\int_0^t \mathcal{G}_\alpha(s, t - \tau) d\tau = \int_0^t e^{\frac{1}{\tau} \int_0^t \frac{k(t)}{\nu} \phi^{\alpha+1}(s)(t-\tau) d\tau} d\tau
\]

\[
\Rightarrow |\int_0^t \mathcal{G}_\alpha(s, t - \tau) d\tau| \leq \int_0^t |e^{\frac{1}{\tau} \int_0^t \frac{k(t)}{\nu} \phi^{\alpha+1}(s)(t-\tau) d\tau}| d\tau
\]

\[
\leq \int_0^t e^{\frac{1}{\nu} \frac{|k(t)|}{\mu} \text{Re}\{\phi^{\alpha+1}(s)\}(t-\tau)} d\tau
\]

\[
(5.10) \quad \Rightarrow \lim_{t \to +\infty} |\int_0^t \mathcal{G}_\alpha(s, t - \tau) d\tau| \leq \lim_{t \to +\infty} \int_0^t e^{\frac{1}{\nu} \frac{|k(t)|}{\mu} \text{Re}\{\phi^{\alpha+1}(s)\}(t-\tau)} d\tau
\]

\[
= \lim_{t \to +\infty} \frac{-1}{e^{\frac{1}{\nu} \frac{|k(t)|}{\mu} \text{Re}\{\phi^{\alpha+1}(s)\}} - 1} \frac{1}{\frac{|k(t)|}{\mu} \text{Re}\{\phi^{\alpha+1}(s)\}} = \frac{1}{\frac{|k(t)|}{\mu} |s|^{\alpha+1} \sin(\frac{\pi}{2} \alpha)}
\]

Thus, using (5.9) and (5.10), (5.3) becomes:

\[
(5.11) \quad \lim_{t \to +\infty} \|P(x, t)\|_\infty \leq \frac{|r|}{2\pi} \frac{1}{\frac{|k(t)|}{\mu} \sin(\frac{\pi}{2} \alpha)} \int_{-\infty}^{+\infty} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds + |u_{\text{max}}|
\]

Now, we focus on computing the integral in (5.11). Because the term \(1/|s|^{\alpha+1}\) is undefined around zero, the integral was divided to study the convergence of the integral in (5.11) using Theorem 2.16 as follows:

\[
(5.12) \quad \int_{-\infty}^{+\infty} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds = \int_{-1}^{1} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds + \int_{-\infty}^{-1} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds + \int_{1}^{+\infty} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds
\]

We start by computing the following integral:

\[
\int_{-\infty}^{-1} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds \leq \|q(x)\|_{L^1(\mathbb{R})} \int_{-\infty}^{-1} \frac{1}{|s|^{\alpha+1}} ds
\]

\[
(5.13) \quad = \|q(x)\|_{L^1(\mathbb{R})} \int_{1}^{+\infty} \frac{1}{s^{\alpha+1}} ds
\]

\[
\Rightarrow \int_{-\infty}^{-1} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds \leq \frac{1}{\alpha} \|q(x)\|_{L^1(\mathbb{R})}
\]

using Theorem 2.16, following the same reasoning as in (5.13), we have that:

\[
(5.14) \quad \int_{1}^{+\infty} |\tilde{q}(s)| \frac{1}{|s|^{\alpha+1}} ds \leq \frac{1}{\alpha} \|q(x)\|_{L^1(\mathbb{R})}
\]
Now, we compute the remaining integral in (5.12):

\[
\int_{-1}^{t} |q(s)| \frac{1}{|s|^{\alpha+1}} ds = \int_{-1}^{0} |q(s)| \frac{1}{s^{\alpha+1}} |ds + \int_{0}^{1} |q(s)| \frac{1}{s^{\alpha+1}} |ds
\]

[5.15]

\[
= \lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} |q(s)| \frac{1}{s^{\alpha+1}} |ds + \lim_{\epsilon \to 0} \int_{\epsilon}^{1} |q(s)| \frac{1}{s^{\alpha+1}} |ds
\]

\[
= \lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} |F(J^{\alpha+1} y)(s)| ds + \lim_{\epsilon \to 0} \int_{\epsilon}^{1} |F(J^{\alpha+1} y)(s)| ds
\]

\[
\leq \lim_{\epsilon \to 0} (2 - 2\epsilon) \|J^{\alpha+1} y\|_{L^1(\mathbb{R})} = 2 \|J^{\alpha+1} y\|_{L^1(\mathbb{R})}
\]

\[
\Rightarrow \int_{-1}^{1} |\hat{q}(s)| \frac{1}{|s|^{\alpha+1}} ds \leq 2 \|J^{\alpha+1} y\|_{L^1(\mathbb{R})}
\]

using Theorems 2.16 and 2.17. Then, using (5.12)-(5.15), (5.11) becomes

\[
(5.16) \quad \lim_{t \to +\infty} \|P(x, t)\|_{L^\infty} \leq \frac{|r|}{2\pi} \frac{1}{C^{|k_0|}} \sin \left(\frac{\pi}{2} \alpha \right) \left(2 \|J^{\alpha+1} y\|_{L^1(\mathbb{R})} + \|y\|_{L^1(\mathbb{R})}\right)
\]

\]

6. Reference Tracking process. In this section, we study the reference tracking problem of the system given in (3.1). We aim to track the pressure gradient at final position for the boundary controlled space FPDE in the presence of some disturbances using some measurements.

Consider the space FPDE (3.1) with \(L = 1, g_0(x) = 0, Q(x, t) = d_1(t)f(x)\), where \(d_1(t)\) is a disturbance. We also consider a disturbance \(d_2(t)\) on the boundary, leading to the following system:

\[
(6.1) \quad \begin{cases}
\frac{\partial P(x, t)}{\partial t} - \frac{k(t)}{C \mu} \frac{\partial}{\partial x} D_x^\alpha P(x, t) = \frac{1}{C \rho} d_1(t)f(x), \\
P(x, 0) = 0, \\
P(1, t) = u(t), \quad P_x(0, t) = d_2(t),
\end{cases}
\]

where \(t > 0, x \in [0, 1]\), \(P(x, t)\) is the gas pressure distributed in space and in time. The permeability \(k(t)\) and the viscosity \(\mu\) are both positive, \(f(x)\) is the source term. \(u(t)\) represents the control input. \(C D_x^\alpha P(x, t)\) is the Caputo space fractional derivative of order \(\alpha\) with \(0 < \alpha \leq 1\). It is important to emphasize that we are dealing with steplike and sinusoidal disturbances. For the purpose of tracking, we consider the measurements \(P(x, t)|_{x=0}\) and the output to be tracked \(y(t)\) such that: \(P_x(x, t)|_{x=1} = y(t)\).

**Remark 6.1.** In the tracking process we will not use the measurements \((P(x, t)|_{x=0})\). Instead, we will use its fractional in space derivative of order \(\alpha\). This is possible thanks to the results in [42] and [43], where it has been proven that the fractional derivative of a signal can be estimated using the measurements of the signal even if this signal is noisy. We propose then to use the measurement \(y_m(t)\) where, \((C D_x^\alpha P(x, t))|_{x=0} = y_m(t)\).
The objective is to track asymptotically the output \( y(t) \) to a desired trajectory \( y_d(t) \), in other words we want to ensure the following:

\[
\lim_{t \to +\infty} e(t) := \lim_{t \to +\infty} (y(t) - y_d(t)) = 0.
\]

### 6.1. Output regulation using Volterra integral transformation

In this part, we extend the well-known backstepping approach to the space FPDE in order to design a controller that guarantees the state feedback output regulation for the considered problem. Using the same analogy as in [11], the disturbances \( d_1(\cdot), d_2(t), \) the measurements \( y_m(t), \) and the reference \( y_d(t) \) can be written in the space spanned by the finite dimensional signal \( V(t) \) which satisfies:

\[
\begin{cases}
    V'(t) = SV(t), \\
    V(0) = V_0,
\end{cases}
\]

where, \( t > 0, S \) is a known matrix having distinct and negative eigenvalues. Thus we can write:

\[
\begin{cases}
    V'(t) = SV(t), t > 0 \quad V(0) = V_0, \\
    d_1(t) = a^T V(t), \quad d_2(t) = b^T V(t), \\
    y_d(t) = c^T V(t), y_m(t) = q^T V(t)
\end{cases}
\]

where, \( V_0, a, b, c, q \in \mathbb{C}^{n_v} \) with \( n_v \) an arbitrary chosen order. System (6.4) allows the modelling of unknown steplike and sinusoidal exogenous signals. We start by introducing the Volterra coordinates transformation [45], [27] and [40]:

\[
w(x, t) = \mathcal{V}\{P(\cdot, t)\}(x) := P(x, t) - \int_0^x K(x, y) P(y, t) dy.
\]

This transformation is invertible, the formula for the inverse can be found in [27]. We consider the following space FPDE target system:

\[
\begin{cases}
\frac{\partial w(x, t)}{\partial t} - \frac{k(t)}{C_{\mu}} \frac{\partial}{\partial x} C^\alpha_D K(x, t) w(x, t) = r^T(x)V(t), \\
(C^\alpha_D w(x, t))|_{x=0} = q^T V(t), \quad w(x, t) = m^T V(t),
\end{cases}
\]

where,

\[
r^T(x) = a^T \mathcal{V}\{f\}(x)
\]

and \( u(t) = \int_0^t K(1, y) P(y, t) dy + m^T V(t), \)

with the kernel system given by:

\[
\begin{cases}
    \frac{\partial}{\partial x} \left( C^\alpha_D K(x, y) \right) = C^\alpha_D K_{y,x} \left( \frac{\partial}{\partial y} K(x, y) \right), \\
    C^\alpha_D K_{x,x} |_{y=x} = 0, \\
    C^\alpha_D K_{x,y} |_{y=x} = 0, \\
    k(x, x) = 0, \\
    K(x, 0) \neq 0
\end{cases}
\]

where, \( C^\alpha_D K_{y,x} \) is the right sided Caputo derivative defined in (2.1)
Lemma 6.2. Using transformation (6.5), if there exists a twice continuously differentiable kernel function $K(x, y)$ satisfying (6.7) then, system (6.1) is equivalent to (6.6).

Proof. We start by computing the time classical derivative of the new coordinate $w(x, t)$:

$$
\frac{\partial}{\partial t} w(x, t) = \frac{\partial}{\partial t} P(x, t) - \int_0^x k(x, y) \frac{\partial}{\partial t} P(y, t) dy
$$

$$
= \frac{k(t)}{C\mu} \frac{\partial}{\partial x} D_x^\alpha P(x, t) - \frac{k(t)}{C\mu} \int_0^x K(x, y) \frac{\partial}{\partial y} D_y^\alpha P(y, t) dy
$$

$$
+ \frac{1}{C\rho} a^T V(t) f(x) - \frac{1}{C\rho} \int_0^x k(x, y) a^T V(t) f(y) dy
$$

$$
= \frac{k(t)}{C\mu} \frac{\partial}{\partial x} D_x^\alpha P(x, t) + \frac{1}{C\rho} a^T V(t) \mathcal{V}\{f\}(x)
$$

$$
+ \frac{k(t)}{C\mu} \int_0^x \frac{\partial}{\partial y} K(x, y) C D_y^\alpha P(y, t) dy
$$

$$
- \frac{k(t)}{C\mu} [K(x, y) C D_y^\alpha P(y, t)]_{y=0}^{y=x}.
$$

(6.8)

where, (6.8) is obtained by first applying a classical integration by parts then the fractional integration by parts (2.3). Using the generalized Leibnitz differentiation rule (2.12), we obtain the Caputo spatial fractional derivative of the new coordinate $w(x, t)$:

$$
C D_x^\alpha w(x, t) = C D_x^\alpha P(x, t) - \int_0^x \frac{C}{y} D_x^\alpha K(x, y) P(y, t) dy
$$

$$
- [C D_x^{\alpha-1} K(x, y) P(y, t)]|_{y=x},
$$

(6.9)

then, by the classical Leibnitz differentiation rule, we obtain:

$$
\frac{\partial}{\partial x} C D_x^\alpha w(x, t) = \frac{\partial}{\partial x} C D_x^\alpha P(x, t)
$$

$$
- \int_0^x \frac{\partial}{\partial x} \frac{C}{y} D_x^\alpha K(x, y) P(y, t) dy
$$

$$
- [\frac{C}{y} D_x^{\alpha-1} K(x, y) P(y, t)]|_{y=x}
$$

$$
- \frac{\partial}{\partial x} ([\frac{C}{y} D_x^{\alpha-1} K(x, y) P(y, t)]|_{y=x}).
$$

(6.10)
Thus, by (6.1), (6.4), (6.5), (6.7), (6.8), (6.9) and (6.10) we have:

$$\frac{\partial w(x,t)}{\partial t} k(t) C_{\mu} \frac{\partial}{\partial x} C_{D}^{\alpha} w(x,t)$$

(6.11) $$= \left( \frac{1}{C_{\rho}} \alpha^{T} V \{f\}(x) - b^{T} \frac{k(t)}{C_{\mu}} \left[C_{D}^{\alpha-1}_{y,x} \frac{\partial}{\partial y} K(x,y) \right]_{y=0} + q^{T} \frac{k(t)}{C_{\mu}} K(x,0) \right) V(t)$$

By looking to (6.11), we conclude that we need an extra condition on the kernel and which is $K(x,0) \neq 0$ in order not to loss the measurements. We set $w(1,t) = w_{1}(t)$, where by (6.4), $w_{1}(t)$ can be written as: $w_{1}(t) = m^{T} V(t)$

Using (6.1) and transformation (6.5), we obtain :

(6.12) $$u(t) = P(1,t) = \int_{0}^{1} K(1,y) P(y,t) dy + m^{T} V(t).$$

Which completes the proof.

LEMMA 6.3. The kernel system (6.7) admits at least one family of twice continuously differentiable solutions $K(x,y)$ in the triangle $0 \leq y \leq x \leq 1$ and which is given by:

(6.13) $$K(x,y) = (x-y)^{2m+1}, \quad m \in \mathbb{N}$$

Proof. Consider system (6.7) in $0 \leq y \leq x \leq 1$. To check that the kernel function given in (6.13) is valid, we start by computing both sides of the kernel PDE using Theorem 5.1:

(6.14) $$\frac{\partial}{\partial x} C_{x} D_{x}^{\alpha} (x-y)^{2m+1} = \frac{2m+2}{2m+1-\alpha} (x-y)^{2m-\alpha},$$

and

(6.15) $$C_{D}^{\alpha} \frac{\partial}{\partial y} (x-y)^{2m+1} = \frac{2m+2}{2m+1-\alpha} (x-y)^{2m-\alpha},$$

By (6.14) and (6.15) the proposed kernel function satisfies the kernel PDE. Let’s check the boundary conditions: we have:

$$C_{y} D_{x}^{\alpha-1} K(x,y) |_{y=x} = \frac{2m+2}{2m+3-\alpha} (x-y)^{2m+2-\alpha} |_{y=x} = 0,$$

which means that:

$$C_{D}^{\alpha} K(x,y) |_{y=x} = \frac{2m+2}{2m+2-\alpha} (x-y)^{2m+1-\alpha} |_{y=x} = 0,$$

and

$$K(x,x) = 0.$$

Finally, $K(x,0) = (x)^{2m+1} \neq 0$, which completes the proof.

The objective now is to determine $m_T$ which guarantees that $w(x, t)$ will achieve
the output regulation at steady state. We define the tracking error:

$$e(x, t) = w(x, t) - M^T(x)V(t),$$

where $M^T$ has to be determined. Let’s now define the following systems:

$$
\begin{cases}
\frac{\partial e(x, t)}{\partial t} = \frac{k(t)}{C^\mu} \frac{\partial}{\partial x} D^\alpha_x e(x, t) \\
e_x(0, t) = 0, \quad e(1, t) = 0, \\
(V^{-1}\{e_x(x, t)\})|_{x=1} = e(t),
\end{cases}
$$

and the system:

$$
\begin{cases}
M^T(x)S - \frac{k(t)}{C^\mu} \frac{\partial}{\partial x} D^\alpha_x M^T(x) = r^T(x), \\
M^T_x(x)|_{x=0} = b^T, \\
(V^{-1}\{M^T(x)\})|_{x=1} = c^T,
\end{cases}
$$

**Theorem 6.4.** Using (6.3), (6.5) and (6.16) if there exists a twice continuously
differentiable function $M^T(x)$ solution of (6.18) then, the tracking error $e(x, t)$ sat-
\isatises (6.17), where, $m_T$ is chosen as follows:

$$[M^T(x)]|_{x=1} = m^T,$$

**Proof.** We start by computing the time classical derivative and the space frac-
tional derivative of the tracking error (6.16), and replace them in (6.1) we get:

$$
\begin{split}
\frac{\partial e(x, t)}{\partial t} - \frac{k(t)}{C^\mu} \frac{\partial}{\partial x} D^\alpha_x e(x, t) = \\
r^T(x)V(t) - M^T(x)S V(t) + \frac{k(t)}{C^\mu} \frac{\partial}{\partial x} D^\alpha_x M^T(x)V(t).
\end{split}
$$

We take $M^T(x)$ to be solution of (6.18). Thus, (6.19) becomes (6.17). Furthermore
if we chose $m^T$ that satisfies:

$$[M^T(x)]|_{x=1} = m^T,$$

which is in (6.17) equivalent to the condition:

$$e(1, t) = 0.$$

The control in (6.12) becomes:

$$u(t) = P(1, t) = \int_0^1 K(1, y) P(y, t) dy + [M^T(x)]|_{x=1} V(t).$$

**Theorem 6.5.** The tracking error system (6.17) is asymptotically stable in $L^\infty(\mathbb{R})$.}\]
Proof. The tracking error function given by (6.17) is a particular case of (3.1) for a source term equal to zero. Thus, the solution of (6.17) is given in Theorem 4.1 by (4.15):

\[
e(x, t) = \mathcal{G}_\alpha(\lvert x - y \rvert, t)e_0(y)dy.
\]

(6.21)

\[
e(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isx} \tilde{G}_\alpha(s, t)e_0(s)ds.
\]

where,

\[
e_0(x) = e(x, 0) = g_0(x) - \int_0^x K(x, y)g_0(y)dy - M^T(x)V_0
\]

(6.22)

\[M^T(x) \text{ is twice continuously differentiable, thus, } e_0(x) \text{ is in } L^1(\mathbb{R}) \text{ which means that its Fourier transform } \tilde{e}_0(s) \text{ exist. Then, we get:}
\]

\[
\lvert e(x, t) \rvert \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert \tilde{G}_\alpha(s, t)e^{-isx}\tilde{e}_0(s) \rvert ds
\]

(6.23)

\[
\lvert e(x, t) \rvert \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert \tilde{G}_\alpha(s, t)\rvert \lvert \tilde{e}_0(s) \rvert ds.
\]

By taking the sup on \(x\) over \([0, L]\) of (6.23), we get:

\[
\|e(x, t)\|_\infty \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert \tilde{G}_\alpha(s, t)\rvert \lvert \tilde{e}_0(s) \rvert ds,
\]

(6.24)

taking the limit of (6.24), we get:

\[
\lim_{t \to +\infty} \|e(x, t)\|_\infty \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{t \to +\infty} \lvert \tilde{G}_\alpha(s, t)\rvert \lvert \tilde{e}_0(s) \rvert ds,
\]

(6.25)

where

\[
\tilde{G}_\alpha(s, t) = e^{\frac{s^2}{2}\frac{\phi^{\alpha+1}(s)}{\alpha}}
\]

and

\[
\phi^{\alpha+1}(s) = (-is)^{\alpha+1}.
\]

On the other hand, we have from equation (5.9) that:

\[
\lim_{t \to +\infty} \lvert \tilde{G}_\alpha(s, t) \rvert = 0,
\]

(6.26)

using the fact that both the permeability and the viscosity are both positive and the linear-compressibility with \(C > 0\). Thus, using (6.26), (6.25) becomes:

\[
\lim_{t \to +\infty} \|e(x, t)\|_\infty = 0
\]

(6.27)

which completes the proof. \(\Box\)
Remark 6.6. Notice that the kernel system in (6.7) is non-integer order PDE with very complex boundary conditions and with right sided fractional derivative $C^\alpha_{D_x^g}$ and shifted fractional derivative $C^\alpha_{D_x^s}$ which makes it hard to solve. In (6.13), we propose one possible family of solutions which satisfies (6.7). To avoid having this non-integer kernel PDE, in the next part, we propose an alternative method which simplifies the Kernel PDE into an integer PDE which has a unique solution.

6.2. Novel coordinates transformation for output regulation. In this part, we propose an alternative coordinates transformation to the Voltera transformation (6.5) which allows to achieve a stable target system similar to (6.6) but with an integer kernel system simpler than the one in (6.7) and with less complex derivations. This method can be used for PDEs with a more complex space operator than the time one (which is the case for our system (6.1)).

We assume that the disturbances $d_1(\cdot)$, $d_2(t)$, the measurements $y_m(t)$, and the reference $y_d(t)$ in (6.1) can be written in the space spanned by the finite dimensional signal $V(t)$ which satisfies (6.3) and (6.4). We start by introducing the new coordinates transformation:

$$w(x,t) = V\{P(\cdot,t)\}(x) := P(x,t) - \int_0^t P(x,\tau)l(\tau,t)d\tau.$$  

(6.28)

This transformation is invertible, the formula for the inverse will be discussed later. We consider the following space fractional target system:

$$\frac{\partial w(x,t)}{\partial t} - \frac{k(t)}{c_p} \frac{\partial}{\partial x} D^\alpha_x w(x,t) = \frac{1}{C_p} a^T V\{f\}(x)V(t),$$

$$w(0,t) = n^T V(t),$$

$$w(1,t) = m^T V(t),$$

(6.29)

where,

$$u(t) = \int_0^t P(1,\tau)l(\tau,t)d\tau + m^T V(t).$$

and the kernel system:

$$\frac{\partial}{\partial t} l(\tau,t) = - \frac{\partial}{\partial \tau} l(\tau,t),$$

$$l(0,t) = \hat{l}(t),$$

(6.30)

where, $\hat{l}(\cdot)$ is a known differentiable function.

Theorem 6.7. Using transformation (6.28), if there exist a continuously differentiable kernel function $l(t,\tau)$ satisfying (6.30) then, system (6.1) is equivalent to (6.29).

Proof. We start by computing the time classical derivative of the new coordinate $w(x,t)$:

$$\frac{\partial}{\partial t} w(x,t) = \frac{\partial}{\partial t} P(x,t)$$

(6.31)

$$- \int_0^t P(x,\tau) \frac{\partial}{\partial t} l(\tau,t)d\tau - P(x,t)l(t,t),$$
using the non-integer Leibnitz rule. The Caputo spatial fractional derivative of the new coordinate \( w(x,t) \) is given by:

\[
(CD_x^\alpha) w(x,t) = CD_x^\alpha P(x,t) - \int_0^t CD_x^\alpha P(x,\tau) l(\tau,t)d\tau
\]

Thus, we have:

\[
\frac{k(t)}{C\mu} \frac{\partial}{\partial x} CD_x^\alpha w(x,t) = \frac{k(t)}{C\mu} \frac{\partial}{\partial x} CD_x^\alpha P(x,t)
\]

\[
- \int_0^t \frac{k(t)}{C\mu} \frac{\partial}{\partial x} CD_x^\alpha P(x,\tau) l(\tau,t)d\tau
\]

\[
= \frac{\partial}{\partial t} P(x,t) - \int_0^t \frac{\partial}{\partial \tau} P(x,\tau) l(\tau,t)d\tau
\]

\[
- \frac{1}{C\rho} a^T V(t)f(x) + \frac{1}{C\rho} \int_0^t l(\tau,t)a^T V(\tau)f(x)d\tau
\]

\[
= \frac{\partial}{\partial t} P(x,t) + \int_0^t P(x,\tau) \frac{\partial}{\partial \tau} l(\tau,t)d\tau - [P(x,\tau)l(\tau,t)]|_{\tau=0}^{\tau=t}
\]

\[
- \frac{1}{C\rho} a^T V(t)f(x) + \frac{1}{C\rho} \int_0^t l(\tau,t)a^T V(\tau)f(x)d\tau
\]

where, (6.33) is obtained by applying a classical integration by parts. Thus, by (6.1), (6.4), (6.28), (6.31), and (6.33) we have:

\[
\frac{\partial w(x,t)}{\partial t} \frac{k(t)}{C\mu} \frac{\partial}{\partial x} CD_x^\alpha w(x,t) = \frac{1}{C\rho} a^T V\{f\}(x)V(t)
\]

We set \( w(1,t) = w_1(t) \), where by (6.4), \( w_1(t) \) can be written as: \( w_1(t) = m^T V(t) \)

Using (6.1) and transformation (6.28), we obtain :

\[
u(t) = P(1,t) = \int_0^t P(1,\tau) l(\tau,t)d\tau + m^T V(t).
\]

Which completes the proof.

**Theorem 6.8.** The kernel system (6.30) admits a unique continuously differentiable solution \( l(t,\tau) \) in the triangle \( 0 \leq \tau \leq t \) and which is given by:

\[
l(t,\tau) = l(t-\tau),
\]

The objective now is to determine \( m^T \) which guarantees that \( w(x,t) \) will achieve the output regulation at steady state. We define the tracking error:

\[
E(x,t) = w(x,t) - M^T(x)V(t),
\]

where \( M^T \) has to be determined. Let’s now define the following systems:

\[
\begin{cases}
\frac{\partial E(x,t)}{\partial t} = \frac{k(t)}{C\mu} \frac{\partial}{\partial x} CD_x^\alpha E(x,t) \\
E(0,t) = 0, & E(1,t) = 0,
\end{cases}
\]

\[
\left[ V^{-1}\left\{ E_x(x,t) \right\} \right]_{x=1} = E(t),
\]
and the system:

\[
\begin{align*}
M^T(x)S - \frac{k(t)}{C\mu} \frac{\partial}{\partial x} \alpha \frac{D_x^n M^T(x)}{\partial x} &= r^T(x), \\
M^T(x)|_{x=0} &= n^T, \\
(V^{-1}\{M^T(x)\})|_{x=1} &= c^T,
\end{align*}
\]

(6.39)

**Theorem 6.9.** Using (6.3), (6.28) and (6.37) if there exists a solution of (6.39) then, the tracking error \( e(x,t) \) satisfies (6.38), where, \( m^T \) is chosen as follows:

\[
[M^T(x)]|_{x=1} = m^T,
\]

Proof. We start by computing the time classical derivative and the space fractional derivative of the tracking error (6.37), and replace them in (6.1) we get:

\[
\frac{\partial E(x,t)}{\partial t} - \frac{k(t)}{C\mu} \frac{\partial}{\partial x} \alpha \frac{D_x^n E(x,t)}{\partial x} =
\]

\[
r^T(x)V(t) - M^T(x)Sv(t) + \frac{k(t)}{C\mu} \frac{\partial}{\partial x} \alpha \frac{D_x^n M^T(x)}{\partial x}V(t).
\]

(6.40)

We take \( M^T(x) \) to be solution of (6.39). Thus, (6.40) becomes (6.38). Furthermore if we chose \( m^T \) that satisfies:

\[
[M^T(x)]|_{x=1} = m^T,
\]

which is in (6.38) equivalent to the condition:

\[
E(1,t) = 0.
\]

The control in (6.35) becomes:

\[
u(t) = P(1,t) = \int_0^t P(1,\tau)v(\tau,t) d\tau + [M^T(x)]|_{x=1}V(t).
\]

(6.41)

**Theorem 6.10.** The tracking error system (6.38) is asymptotically stable in \( L_\infty(\mathbb{R}) \).

The proof is similar to the proof of Theorem 6.5

**Theorem 6.11.** The inverse of transformation (6.28) is given by:

\[
P(x,t) = V^{-1}\{w(.,t)\}(x) := w(x,t) + \int_0^t w(x,\tau)L(\tau,t) d\tau.
\]

(6.42)

with:

\[
L(\tau,t) = \hat{L}(t - \tau),
\]

(6.43)

where,

\[
\hat{L}(t) = L(0,t).
\]
Proof. The proof is similar to the proof of Theorem 6.7. using transformation (6.43), we compute the first temporal derivative and the fractional spatial derivative of \( P(x,t) \) and inject it in (6.1). Using (6.1), we conclude the following system for the kernel \( L(\tau,t) \):

\[
\begin{aligned}
\frac{\partial}{\partial \tau} L(\tau,t) &= -\frac{\partial}{\partial \tau} L(\tau,t), \\
L(0,t) &= \dot{L}(t),
\end{aligned}
\]

System (6.44) admits a unique solution given in [13] by:

\[
L(\tau,t) = \dot{L}(t - \tau),
\]

where, \( \dot{L}(t) = L(0,t) \).

which completes the proof.

7. Adaptive boundary observer for space fractional PDEs subject to domain and boundary disturbances

To design the tracking controllers in (6.20) and (6.41), we need to recover the state \( P(x,t) \) of system (6.1). We recall that, it has been mentioned in Remark 6.1 that in the tracking process, we did not use the measurements \( (z_m(t) = P(x,t)|_{x=0}) \). Instead, we used its fractional in space derivative of order \( \alpha \) \( (y_m(t) = C D_2^{\alpha} P(x,t)|_{x=0}) \). This was possible thanks to the results in [41, 42], where it has been proven that the fractional derivative of a signal can be estimated using the measurements of the signal even if this signal is noisy. In the adaptive observer design, we propose then to use both the measurements \( z_m(t) \) and \( y_m(t) \) where, \( P(x,t)|_{x=0} = z_m(t) \) and \( (C D_2^{\alpha} P(x,t))|_{x=0} = y_m(t) \). which will prove to be judicious later. Thus, we design the following observer for system (6.1):

\[
\begin{aligned}
\frac{\partial \hat{P}(x,t)}{\partial t} - \frac{k(t)}{C_{\mu}} \frac{\partial}{\partial \tau} C D_2^{\alpha} \hat{P}(x,t) &= \frac{1}{C_{\rho}} \hat{d}_1(t) f(x) + u(x,t) \\
- \frac{k(t)}{C_{\mu}} H_1(x)(\hat{P}(0,t) - z_m(t)) - \frac{k(t)}{C_{\mu}} H_2(x)(C D_2^{\alpha} \hat{P}(0,t) - y_m(t)), \\
\hat{P}(x,0) &= \hat{P}_0(x), \hat{P}(1,t) = u(t), \\
\hat{P}_2(0,0) &= \hat{d}_2(t).
\end{aligned}
\]

This observer is designed similarly as in [15-17] for a parabolic systems. Where, \( H_1(x) \) and \( H_2(x) \) are a space-dependent observer gains, \( \hat{P}_0(x) \) is an arbitrary initial condition which satisfies some conditions that will be determined later. \( \hat{d}_1(t) \) and \( \hat{d}_2(t) \) are the disturbances estimates and \( u(x,t) \) is an additional feedback term that will be determined latter. We introduce the following usual estimation errors:

\[
\text{State estimation error: } \hat{P}(x,t) = \hat{P}(x,t) - P(x,t).
\]

\[
\text{Disturbances estimation error: } \hat{\theta}(t) = \hat{\theta}(t) - \theta(t) := \begin{bmatrix} \hat{d}_1(t) \\ \hat{d}_2(t) \end{bmatrix},
\]

where,

\[
\hat{\theta}(t) := \begin{bmatrix} \hat{d}_1(t) \\ \hat{d}_2(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{d}_1(t) \\ \hat{d}_2(t) \end{bmatrix} := \begin{bmatrix} \hat{d}_1(t) - d_1(t) \\ \hat{d}_2(t) - d_2(t) \end{bmatrix}.
\]
Then, using (7.1), it follows that the state estimation error in (7.2) satisfies the following system:

\[
\begin{align*}
\frac{\partial \tilde{P}(x,t)}{\partial t} - \frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a \tilde{P}(x,t) &= \frac{1}{C_\rho} \tilde{d}_1(t) f(x) + u(x,t) \\
t &= \frac{k(t)}{C_\mu} H_1(x) \tilde{P}(0,t) - \frac{k(t)}{C_\mu} H_2(x) C^D_2 \tilde{P}(0,t), \\
\end{align*}
\]

(7.5)

\[
\begin{align*}
\tilde{P}(x,0) &= \tilde{P}_0(x), \tilde{P}(1,t) = 0, \\
\tilde{P}_x(0,t) &= \tilde{d}_2(t),
\end{align*}
\]

7.1. Finite-dimensional backstepping-like transformation.

The objective now is to cancel the terms on \( \tilde{d}_1(t) \) and \( \tilde{d}_2(t) \) in the observer error system (7.5). To this end, we consider the finite-dimensional backstepping-like transformation defined in [15-17] as:

\[
w(x,t) = \tilde{P}(x,t) - \lambda_1(x,t) \tilde{d}_1(t) - \lambda_2(x,t) \tilde{d}_2(t) := \tilde{P}(x,t) - \Lambda(x,t) \tilde{\theta}(t),
\]

where,

\[
\Lambda(x,t) = \begin{bmatrix} \lambda_1(x,t) & \lambda_2(x,t) \end{bmatrix}
\]

Using the PDE in (7.5), \( w(x,t) \) satisfies the following PDE:

\[
\begin{align*}
\frac{\partial w(x,t)}{\partial t} &= \frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a \tilde{P}(x,t) + \frac{1}{C_\rho} \tilde{d}_1(t) f(x) - H_1(x) \tilde{P}_x(1,t) \\
&\quad + u(x,t) - \frac{\partial}{\partial t} (\Lambda(x,t) \tilde{\theta}(t)), \\
&= \frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a w(x,t) - H_1(x) w_x(1,t) + u(x,t) - \Lambda(x,t) \dot{\tilde{\theta}}(t) \\
&\quad + \left[ \frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a \lambda_1(x,t) - \frac{\partial}{\partial t} \lambda_1(x,t) - H_1(x) \frac{\partial}{\partial x} \lambda_1(x,t) + \frac{1}{C_\rho} f(x) \right] \tilde{d}_1(t) \\
&\quad + \left[ \frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a \lambda_2(x,t) - \frac{\partial}{\partial t} \lambda_2(x,t) - H_1(x) \frac{\partial}{\partial x} \lambda_2(x,t) \right] \tilde{d}_2(t),
\end{align*}
\]

This suggests the following choice of the feedback expression for \( u(x,t) \):

\[
u(x,t) = \Lambda(x,t) \dot{\tilde{\theta}}(t)
\]

and the following trajectory of the auxiliary states:

\[
\begin{align*}
\frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a \lambda_1(x,t) - \frac{\partial}{\partial t} \lambda_1(x,t) - H_1(x) \frac{\partial}{\partial x} \lambda_1(x,t) + \frac{1}{C_\rho} f(x) &= 0 \\
\frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a \lambda_2(x,t) - \frac{\partial}{\partial t} \lambda_2(x,t) - H_1(x) \frac{\partial}{\partial x} \lambda_2(x,t) &= 0
\end{align*}
\]

Thus, (7.7) becomes:

\[
\begin{align*}
\frac{\partial w(x,t)}{\partial t} &= \frac{k(t)}{C_\mu} \frac{\partial}{\partial x} D_x^a w(x,t) - H_1(x) w_x(1,t)
\end{align*}
\]
This is completed by the following initial and boundary conditions:

\[(7.10)\]
\[
\begin{cases}
\lambda_1(x,0) = 0 \\
\frac{\partial}{\partial x}\lambda_1(x,t)|_{x=0} = \lambda_1(1,t) = 0
\end{cases}
\begin{cases}
\lambda_2(x,0) = 0 \\
\lambda_2(1,t) = 0 \\
\frac{\partial}{\partial x}\lambda_2(x,t)|_{x=0} = 1.
\end{cases}
\]

**Theorem 7.1.** There exist a unique auxiliary states \(\lambda_1(x,t)\) and \(\lambda_2(x,t)\) that satisfies the PDEs in (7.8) and the initial and boundary conditions in (7.10). Furthermore, Theses auxiliary states are stable. That is:

\[
\lim_{t \to +\infty} \|\Lambda(x,t)\|_\infty < \infty
\]

**Proof.** The proof is given in appendix B.

The choice of initial and boundary conditions in (7.10) is efficient because they allow the following initial and boundary condition for the system \(w(x,t)\):

\[(7.11)\]
\[
\begin{cases}
w(x,0) = \tilde{P}_0(x) \\\nw_x(0,t) = w(1,t) = 0
\end{cases}
\]

Thus the system on \(w(x,t)\) become:

\[(7.12)\]
\[
\begin{cases}
\frac{\partial w(x,t)}{\partial t} = k(t) \frac{\partial}{\partial x} C^D_\alpha x w(x,t) - H_1(x)w(0,t) - H_2(x) C^D_\alpha x w(0,t) \\
w(x,0) = \tilde{P}_0(x) \\
w_x(0,t) = w(1,t) = 0
\end{cases}
\]

### 7.2. Infinite-dimensional backstepping-like transformation and observer gains selection

we propose to extend the backstepping approach to the space FPDE in (7.5) for the determination of the space dependent gains \(H_1(x)\) and \(H_2(x)\). We start by introducing the Voltera coordinates transformation [45], [27] and [40]:

\[(7.13)\]
\[
Z(x,t) = V\{\tilde{P}(.,t)\}(x) := w(x,t) - \int_0^x R(x,y)w(y,t)dy.
\]

This transformation is invertible, the formula for the inverse can be found in [27]. We consider the following space FPDE target system:

\[(7.14)\]
\[
\begin{cases}
\frac{\partial Z(x,t)}{\partial t} = \frac{k(t)}{C^\mu} \frac{\partial}{\partial x} D^\alpha_x Z(x,t) = 0, \\
Z(x,0) = V\{\tilde{P}_0(x)\}(x) := Z_0(x) \\
Z(1,t) = Z_x(0,t) = 0,
\end{cases}
\]

and the kernel system:

\[(7.15)\]
\[
\begin{cases}
\frac{\partial}{\partial x} D^\alpha_{x,y} R(x,y) = C^D_{\alpha,x} \frac{\partial}{\partial y} R(x,y), \\
C^D_{\alpha,x} D^\alpha_{x-1,y} R(x,y)|_{y=x} = 0, \\
C^D_{\alpha,x} D^\alpha_{x,y} R(x,y)|_{y=x} = 0, \\
R(x,x) = 0,
\end{cases}
\]
where, $^C D_{y,x}^\alpha$ is the right sided Caputo derivative defined in (2.1). With the extra conditions that allow the choice of the observer gains:

\begin{align}
  \begin{cases}
    ^C D_{y,x}^\alpha R(x,y)|_{y=0} = V\{H_1\}(x), \\
    R(x,y)|_{y=0} = -V\{H_2\}(x),
  \end{cases}
\end{align}

Theorem 7.2. Using transformation (7.13), if there exist a twice continuously differentiable kernel function $R(x,y)$ satisfying (7.15) then, system (7.5) is equivalent to (7.14).

Proof. We start by computing the time classical derivative of the new coordinate $Z(x,t)$ using the same derivations as in (6.8):

\begin{align}
  \frac{\partial}{\partial t} Z(x,t) = \frac{\partial}{\partial t} w(x,t) - \int_0^x R(x,y) \frac{\partial}{\partial t} w(y,t) dy \\
  = \frac{k(t)}{C\mu} \frac{\partial}{\partial x} \left[ D_x^\alpha w(x,t) + w(0,t)V\{H_1\}(x) + D_x^\alpha w(0,t)V\{H_2\}(x) \right]
\end{align}

\begin{align}
  &- \frac{k(t)}{C\mu} \int_0^x \frac{\partial}{\partial x} D_x^\alpha R(x,y) w(y,t) dy \\
  &+ \left[ \frac{C}{\mu} D_x^\alpha R(x,y) w(y,t) \right]_{y=x} \\
  &- \frac{k(t)}{C\mu} \left[ R(x,y) D_y^\alpha w(y,t) \right]_{y=x}
\end{align}

where, (7.17) is obtained by first applying a classical integration by parts then the fractional integration by parts (2.3). Using the generalized Leibnitz differentiation rule (2.12) and the same derivations as in (6.9), we obtain the Caputo spatial fractional derivative of the new coordinate $Z(x,t)$:

\begin{align}
  ^C D_x^\alpha Z(x,t) = &^C D_x^\alpha w(x,t) \\
  - &\int_0^x \left[ \frac{C}{\mu} D_x^\alpha R(x,y) w(y,t) \right]_{y=x} \\
  - &\left[ \frac{C}{\mu} D_x^\alpha R(x,y) w(y,t) \right]_{y=x}
\end{align}

then, by the classical Leibnitz differentiation rule, we obtain:

\begin{align}
  \frac{\partial}{\partial x} \left[ ^C D_x^\alpha Z(x,t) \right] = &\frac{\partial}{\partial x} \left[ ^C D_x^\alpha w(x,t) \right] \\
  - &\int_0^x \frac{\partial}{\partial x} \left[ ^C D_x^\alpha R(x,y) w(y,t) \right]_{y=x} \\
  - &\left[ \frac{C}{\mu} D_x^\alpha R(x,y) w(y,t) \right]_{y=x}
\end{align}

Thus, by (3.1), (7.13), (7.15), (7.17), (7.18) and (7.19) we have:

\begin{align}
  \frac{\partial Z(x,t)}{\partial t} \frac{k(t)}{C\mu} \frac{\partial}{\partial x} \left[ ^C D_x^\alpha Z(x,t) \right] = 0
\end{align}

Which completes the proof. □
Remark 7.3. The kernel system (7.15) admits at least one family of twice continuously differentiable solutions $K(x, y)$ in the triangle $0 \leq y \leq x \leq 1$ and which is given by (6.13).

From the conditions in (7.16) we have that, the gains $H_1(x)$ and $H_2(x)$ should be chosen as follows:

$$H_1(x) = \mathcal{V}^{-1}\{CD^\alpha_{y,x}R(x, y)|_{y=0}\}$$

and

$$H_2(x) = -\mathcal{V}^{-1}\{R(x, y)|_{y=0}\}$$

Theorem 7.4. System (7.15) admits a unique solution given by:

$$Z(x, t) = \int_{-\infty}^{+\infty} G_\alpha(|x-y|, t-\tau)Z_0(y)dy$$

if $Z_0(x) \in L^1(\mathbb{R})$ (which means that $\hat{P}_0(x) \in L^1(\mathbb{R})$) and $k(t)$ is bounded from below ($k_0 \leq k(t)$). Then, the observer target system given in (7.15) is asymptotically stable in $L_{\infty}(\mathbb{R})$. Furthermore, $w(x, t)$ is asymptotically stable in $L_{\infty}(\mathbb{R})$ as well:

$$\lim_{t \to +\infty} \|w(x, t)\|_{\infty} = 0$$

Proof. Using Theorem 4.1, system (7.15) admits a unique solution given by:

$$Z(x, t) = \int_{-\infty}^{+\infty} G_\alpha(|x-y|, t-\tau)Z_0(y)dy$$

Using Theorem 5.1, because $Z_0(x) \in L^1(\mathbb{R})$ and $k(t)$ is bounded from below ($k_0 \leq k(t)$). Then, the observer target system given in (7.15) is asymptotically stable. Thus:

$$\lim_{t \to +\infty} \|Z(x, t)\|_{\infty} = 0$$

Thus using the inverse of the transformation (7.13) given in [ ] by:

$$(7.22) \quad w(x, t) = Z(x, t) + \int_0^x \hat{R}(x, y)Z(y, t)dy$$

taking the sup over $[0, x]$ of the absolute value of (7.22) and using Schwarz inequality, we get:

$$(7.23) \quad \|w(x, t)\|_{\infty} \leq \|Z(x, t)\|_{\infty} + \|\hat{R}(x, y)\|_2 \|Z(y, t)\|_2$$

Then, we take the limit of (7.23) as $t \to +\infty$:

$$(7.24) \quad \lim_{t \to +\infty} \|w(x, t)\|_{\infty} \leq \lim_{t \to +\infty} \|Z(x, t)\|_{\infty} + \|\hat{R}(x, y)\|_2 \lim_{t \to +\infty} \|Z(y, t)\|_2$$

using (7.21) and the equivalence of the $L^2$ and the $L^\infty$ norms we get that:

$$\lim_{t \to +\infty} \|w(x, t)\|_{\infty} = 0$$

The result of Theorem 7.4 is quite interesting but, we still need to prove that the observer error $\hat{P}(x, t)$ is asymptotically stable as well. Actually, in view of (7.6), one has to show that $\hat{\theta}(t)$ is also exponentially vanishing and $|\Lambda(x, t)|$ is bounded. Before that, we will first investigate the selection of the estimate of the disturbances $\hat{\theta}(t)$.
7.3. Disturbances adaptive law selection. The choice of the estimate of the disturbances is model free thanks to the choice of \( \Lambda(x,t) \) which allowed the rejection of the effect of the disturbances from the state estimation process. That is why, we propose the parameter adaptive law in [1] which is enough to guarantee the exponential stability of the disturbances estimation error \( \tilde{\theta}(t) \) independently of the state estimation problem.

**Theorem 7.5.** Using the following parameter adaptive law parameter adaptive law in [1]:

\[
\dot{\hat{\theta}}(t) = \frac{R(t)\Lambda^T(0,t)}{1 + \Lambda(0,t)^T\Lambda(0,t)} \hat{P}(0,t)
\]

\[
\dot{\hat{R}}(t) = R(t) - \frac{R(t)\Lambda^T(0,t)\Lambda(0,t)R(t)}{1 + \Lambda(0,t)^T\Lambda(0,t)}
\]

where, \( R(t) \in \mathbb{R}^{2 \times 2}, \ \hat{\theta}(0) \) and \( R(0) = R_0 \) are arbitrarily chosen with \( R_0 = R_0^T > 0 \). the disturbance estimation error \( \tilde{\theta}(t) \) is exponentially stable.

**Proof.** consider the following parameter adaptive law parameter adaptive law in [1]:

\[
\dot{\hat{\theta}}(t) = \frac{R(t)\Lambda^T(0,t)}{1 + \Lambda(0,t)^T\Lambda(0,t)} \hat{P}(0,t)
\]

\[
\dot{\hat{R}}(t) = R(t) - \frac{R(t)\Lambda^T(0,t)\Lambda(0,t)R(t)}{1 + \Lambda(0,t)^T\Lambda(0,t)}
\]

The choice of using only the measurements \( z_m(t) \) for the disturbances estimation will prove its efficiency later. This parameter adaptive law is a variant of the least squares estimator, commonly referred to forgetting factor least squares [22]. Thus (7.25) is equivalent to:

\[
\frac{dR^{-1}}{dt} = -R^{-1}(t) + \frac{\Lambda^T(0,t)\Lambda(0,t)}{1 + \Lambda(0,t)^T\Lambda(0,t)}
\]

using transformation (7.6), (7.25) equivalent to:

\[
\dot{\tilde{\theta}}(t) = -\frac{R(t)\Lambda^T(0,t)\Lambda(0,t)}{1 + \Lambda(0,t)^T\Lambda(0,t)} \hat{\theta}(t) + \frac{R(t)\Lambda^T(0,t)}{1 + \Lambda(0,t)^T\Lambda(0,t)} w(0,t)
\]

Using the following Lyapunov function:

\[
V_1(t) = \tilde{\theta}^T(t)R^{-1}(t)\hat{\theta}(t)
\]

using (7.25),(7.26) and (7.27), it has been proved in [1] that:

\[
\dot{V}_1(t) = \tilde{\theta}^T(t)R^{-1}(t)\hat{\theta}(t) + 2\tilde{\theta}^T(t)R^{-1}(t)\hat{\theta}(t) \leq -V_1(t) + w^2(0,t)
\]

using Theorem 7.4, \( w^2(0,t) \) is asymptotically vanishing. Thus, from (7.29) and by the comparison lemma [26], \( V_1(t) \) is exponentially vanishing. In view of (7.28) so is \( \tilde{\theta}(t) \).

**Theorem 7.6.** the observer error \( \tilde{P}(x,t) \) in (7.5) is asymptotically stable in \( L_\infty(\mathbb{R}) \). That is,

\[
\lim_{t \to +\infty} \| \tilde{P}(x,t) \|_\infty = 0
\]
Proof. We start by the transformation in (7.6)

\[ \dot{P}(x,t) = w(x,t) + \Lambda(x,t)\dot{\theta}(t), \]

taking the sup over \([0,x]\) of the absolute value of (7.30) and using Cauchy-schwarz inequality, we get:

\[ \|\dot{P}(x,t)\|_{\infty} \leq \|w(x,t)\|_{\infty} + |\dot{\theta}(t)|\|\Lambda(x,t)\|_{\infty}, \]

by taking the limit of (7.31) as \(t \to +\infty\) and using the fact that \(\Lambda(x,t)\) is stable (Theorem 7.1) and that \(\dot{\theta}(t)\) vanishes exponentially (Theorem 7.5) we get the result \(\square\)

8. Observer Based Output Regulation. Since we obtain an approximated state \(\hat{P}(x,t)\) from the output by observer (7.1), it follows from the output controllers (6.20) and (6.41) that an observer-based output regulation controller should be designed as:

\[ u(t) = P(1,t) = \int_{0}^{1} k(1,y)\hat{P}(y,t)dy + [M^{T}(x)]|_{x=1}V(t). \]

and

\[ u(t) = P(1,t) = \int_{0}^{t} \hat{P}(1,\tau)l(\tau,t)d\tau + [M^{T}(x)]|_{x=1}V(t). \]

respectively.

Theorem 8.1. Under feedback controllers (8.1) (respectively (8.2)), if \(\dot{P}_0(x) \in L^{1}(\mathbb{R})\) and \(k(t)\) is bounded from below \((k_0 \leq k(t))\), we have that the closed loop system:

\[
\begin{cases}
\frac{\partial P(x,t)}{\partial t} - k(t) \frac{\partial}{\partial x} C D_{x}^a P(x,t) = \frac{1}{C_{\mu}} \hat{d}_1(t)f(x), \\
P(x,0) = 0, \\
P(1,t) = \int_{0}^{1} K(1,y)\hat{P}(y,t)dy + [M^{T}(x)]|_{x=1}V(t), \\
P_{z}(0,t) = \hat{d}_2(t), \\
P(0,t) = z_m(t), C D_{x}^a P(0,t) = y_m(t), \\
\frac{\partial P(x,t)}{\partial t} - k(t) \frac{\partial}{\partial x} C D_{x}^a P(x,t) = \frac{1}{C_{\mu}} \hat{d}_1(t)f(x) + u(x,t) \\
- \frac{k(t)}{C_{\mu}} H_1(x)(\hat{P}(0,t) - z_m(t)) - \frac{k(t)}{C_{\mu}} H_2(x) C D_{x}^a \hat{P}(0,t) - y_m(t), \\
\dot{\hat{P}}(x,0) = \hat{P}_0(x), \\
\dot{\hat{P}}(1,t) = \int_{0}^{1} K(1,y)\hat{P}(y,t)dy + [M^{T}(x)]|_{x=1}V(t), \\
\dot{\hat{P}}_{z}(0,t) = \hat{d}_2(t),
\end{cases}
\]

is asymptotically stable. Where, \(\hat{d}_1(t)\) and \(\hat{d}_2(t)\) have to satisfy the adaptive law in (7.25)
Proof. Using the observer error variable \( \tilde{P} = \tilde{P}(x, t) - P(x, t) \), (8.3) becomes:

\[
\begin{aligned}
\frac{\partial P(x, t)}{\partial t} - \frac{k(t)}{\alpha C} D_x^\alpha P(x, t) &= \frac{1}{\alpha \rho} d_1(t) f(x), \\
P(x, 0) &= 0, \\
P(1, t) &= \int_0^1 K(1, y) \hat{P}(y, t) dy + [M^T(x)]|_{x=1} V(t), \\
P_2(0, t) &= d_2(t), \\
(8.4)
\end{aligned}
\]

Using the transformations in (6.5) and (6.16), system (8.4) is equivalent to:

\[
\begin{aligned}
\frac{\partial e(x, t)}{\partial t} &= \frac{k(t)}{\alpha C} D_x^\alpha e(x, t) \\
e(x, 0) &= e_0(x), \quad e_x(0, t) = 0, \\
(8.5)
\end{aligned}
\]

where, \([V^{-1}\{Z(y, t)\}] = \tilde{P}(x, t)\). Using the transformations in (7.6) and (7.13), system (8.5) is equivalent to:

\[
\begin{aligned}
\frac{\partial e(x, t)}{\partial t} &= \frac{k(t)}{\alpha C} D_x^\alpha e(x, t) \\
e(x, 0) &= e_0(x), \quad e_x(0, t) = 0, \\
\frac{\partial Z(x, t)}{\partial t} - \frac{k(t)}{\alpha \rho} D_x^\alpha Z(x, t) &= 0, \\
Z(x, 0) &= Z_0(x), \\
Z(1, t) &= Z_2(0, t) = 0
\end{aligned}
\]

It is clear from (8.6) that the observer error \( Z(x, t) \) is asymptotically stable using Theorem 7.4 (because, \( \tilde{P}_0(x) \in \mathcal{L}^1(\mathbb{R}) \) and \( k(t) \) is bounded from bellow \( (k_0 \leq k(t)) \)). Thus, the output tracking error \( e(x, t) \) is asymptotically stable as well using Theorem 6.5 (no conditions required on \( e_0(x) \)).

9. Numerical results. In this section, we present some numerical results to show the efficiency of the presented method to solve the reference tracking problem and for the adapted observer design.

We consider the following state \( P(x, t) = 4e^{-t} \sin(2\pi x) \) and the following system parameters: \( k = 5, \mu = C = L = 1, \rho = 1.0726 \). Signal \( V(t) \) in system (6.3) is chosen such that \( S = -25, V_0 = 1 \). The reference is chosen as follows: \( y_d(t) = \sin(2\pi t) \). The choice of the kernel functions \( K(x, y) \) and \( R(x, y) \) given by (6.13) is important since it affects the efficiency of the algorithm. In this regard, we propose using polynomial kernel functions that satisfy systems (6.7) and (7.15) for \( m = 1 \), and for which the fractional derivatives are easy to calculate, of the following form:

\[ K(x, y) = R(x, y) = (x - y)^3, \]
whose the fractional derivative is known analytically and given by:
\[ C_y D_y^\alpha K(x, y) = C_y D_y^\alpha R(x, y) = \frac{\Gamma(4)}{\Gamma(4 - \alpha)} (x - y)^{3-\alpha}, \]
using Theorem 2.11. Figures 2 and 3 show the resulting tracking behaviour when using a high order compensator and the corresponding tracking error behavior after adding Gaussian noise to the measurements with mean equal to zero and standard deviation \( \sigma \). Clearly, the estimates get very close to their true variables after a transient period. The above observations confirm the theoretical asymptotic performance described in Theorem 6.5.

![Fig. 2: Reference tracking process (left) and reference tracking error (right) for \( \sigma = 0.01 \).](image1)

![Fig. 3: Reference tracking process (left) and reference tracking error (right) for \( \sigma = 0.1 \).](image2)

Figure 4 (left) shows the time evolution of the state estimate at a particular position in the spatial domain, Figure 4 (right) represents the gain \( H_1(x) \) behavior for different values of \( \alpha \). Figure 5 shows the state estimation error for the whole space and time domains without noise (left) and with added noise (right). Figure 6 shows the accuracy of the disturbances estimation after a transient period. The above observations confirm the theoretical asymptotic performance described in Theorems 7.4 and 7.5.

**Conclusion.** This paper dealt with a class of boundary controlled space FPDE in the presence of disturbances describing the gas pressure in fractured media. The main contributions of this paper were first to study the stability of the considered problem. Then, to track the pressure gradient at final position in the presence of disturbances using backstepping approach. Moreover, an adaptive observer has been
designed to estimate the system’s state. A fundamental solution for the considered model was given in this paper its uniqueness has been also studied. A future direction of this work would be to validate the obtained results by numerical simulations, and to generalize these results for the space and time FPDE. A further direction could be the adaptation of the adaptive observer for a time fractional PDE and to compare to the results in [18] (which is a asymptotic and robust method for the state estimation for PDEs which can be applied for time fraction PDEs as well).

10. Appendix A. We recall the conventional PDE model’s derivation for the gas pressure [3]:
1. Mass conservation law:
\[
\frac{\partial (\rho \phi)(P(x,t))}{\partial t} + \nabla \cdot (\rho u)(P(x,t)) = Q(x,t),
\]

2. Assume the non-compressible fluid, which means that the density does not change with respect to the pressure:
\[
\frac{\partial \phi}{\partial P}(P(x,t)) \frac{\partial P(x,t)}{\partial t} + \rho \nabla \cdot u(P(x,t)) = Q(x,t),
\]

3. Assume a constant-compressible rock (a particular case of the linear-compressibility), which means that the porosity is constant with respect to the pressure:
\[
\frac{\partial P(x,t)}{\partial t} + \frac{1}{C} \nabla \cdot u(P(x,t)) = \frac{1}{C \rho} Q(x,t),
\]

4. Using the Darcy flow equation:
\[
\frac{\partial P(x,t)}{\partial t} + \frac{1}{C} \nabla \cdot \left(-\frac{k(t)}{\mu} \frac{\partial P(x,t)}{\partial x}\right) = \frac{1}{C \rho} Q(x,t).
\]

11. Appendix B. Proof of Theorem (7.1):
Consider the auxiliary states in (7.8) with the initial and boundary conditions in (7.10):
\[
\begin{cases}
\frac{\partial \lambda_1(x,t)}{\partial t} = \frac{k(t)}{C \mu} \frac{\partial}{\partial x} C D_x^\alpha \lambda_1(x,t) - H_1(x) \frac{\partial}{\partial x} \lambda_1(x,t) + \frac{1}{C \rho} f(x) \\
\lambda_1(x,0) = 0 \\
\frac{\partial \lambda_1(x,t)}{\partial x}|_{x=0} = \lambda_1(1,t) = 0
\end{cases}
\]
and
\[
\begin{cases}
\frac{\partial \lambda_2(x,t)}{\partial t} = \frac{k(t)}{C \mu} \frac{\partial}{\partial x} C D_x^\alpha \lambda_2(x,t) - H_1(x) \frac{\partial}{\partial x} \lambda_2(x,t) \\
\lambda_2(x,0) = 0 \\
\lambda_2(1,t) = 0 \\
\frac{\partial \lambda_2(x,t)}{\partial x}|_{x=0} = 1.
\end{cases}
\]
Notice that systems (11.1) and (11.2) are not very different from system (3.1), the only differences are the advection terms and the homogeneity of the boundary conditions. That is why, we propose to introduce the change of variables in [32] which allows the cancellation of the advection term.
To do so, we transform the boundary conditions in (11.2) to homogeneous Bcs to be able to use the transformation in [32], we introduce the change of variables:
\[
\tilde{\lambda}_2(x,t) = \lambda_2(x,t) - (x - 1)
\]
which allows the homogeneity of the boundary conditions. Thus \(\tilde{\lambda}_2(x,t)\) satisfies:
\[
\begin{cases}
\frac{\partial}{\partial t} \tilde{\lambda}_2(x,t) = \frac{k(t)}{C \mu} \frac{\partial}{\partial x} C D_x^\alpha \tilde{\lambda}_2(x,t) - H_1(x) \frac{\partial}{\partial x} \tilde{\lambda}_2(x,t) + H(x) \\
\tilde{\lambda}_2(x,0) = \lambda_2,0(x) \\
\tilde{\lambda}_2(1,t) = 0 \\
\frac{\partial}{\partial x} \tilde{\lambda}_2(x,t)|_{x=0} = 0.
\end{cases}
\]
where, $H(x) = \frac{k(t)}{C\mu} \frac{\partial}{\partial x} D_x^\alpha x - H_1(x)$ and $\lambda_{2,0}(x) = -(x - 1)$.

We introduce now the transformation in [32]:

\begin{equation}
\bar{x} = \frac{1}{\sqrt{\frac{k(t)}{C\mu}}} x
\end{equation}

the transformation in (11.3) is well defined thanks to the fact that $k(t), c$ and $\mu$ are all positive. Thus, using transformation (11.3), we have:

\begin{equation}
\lambda_1(x, t) = \lambda_1(\bar{x}, t) e^{-\bar{g}_1(\bar{x})}
\end{equation}

and

\begin{equation}
\lambda_2(x, t) = \lambda_2(\bar{x}, t) e^{-\bar{g}_2(\bar{x})}
\end{equation}

where,

\begin{equation}
\bar{g}_1(\bar{x}) = \frac{1}{2} \int_0^\infty \left( \frac{k(t)}{C\mu} \frac{\partial}{\partial x} D_x^\alpha x \lambda_1(x, t) - H_1(x) \frac{\partial}{\partial x} \lambda_1(x, t) \right) |_{x=\bar{x}-1(s)} ds
\end{equation}

and

\begin{equation}
\bar{g}_2(\bar{x}) = \frac{1}{2} \int_0^\infty \left( \frac{k(t)}{C\mu} \frac{\partial}{\partial x} D_x^\alpha x \lambda_2(x, t) - H_1(x) \frac{\partial}{\partial x} \lambda_2(x, t) \right) |_{x=\bar{x}-1(s)} ds
\end{equation}

Then, systems (11.1) and (11.2) become:

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} \lambda_1(\bar{x}, t) = \frac{\partial}{\partial \bar{x}} D_x^\alpha \lambda_1(\bar{x}, t) + \frac{1}{e^\rho f(x)} \\
\lambda_1(\bar{x}, 0) = 0 \\
\lambda_1(1, t) = 0
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} \lambda_2(\bar{x}, t) = \frac{\partial}{\partial \bar{x}} D_x^\alpha \lambda_2(\bar{x}, t) + H(\bar{x}) \\
\lambda_2(\bar{x}, 0) = \lambda_{2,0}(\bar{x}) \\
\lambda_2(1, t) = 0
\end{cases}
\end{equation}

Then, systems (11.8) and (11.9) can be solved following the same reasoning as the proof of Theorem 4.1. Thus:

\begin{equation}
\lambda_1(\bar{x}, t) = \frac{1}{e^{\rho}} \int_{-\infty}^{+\infty} \int_{0}^{t} \mathcal{G}_\alpha(\bar{x} - y, t - \tau) f(y)) d\tau dy
\end{equation}

where, $\mathcal{G}_\alpha(\bar{x}, t)$ is the Green function defined by:

\begin{equation}
\mathcal{G}_\alpha(\bar{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik\bar{x}} e^{i\alpha^{\alpha+1}(s)} t dk
\end{equation}

Similarly, the solution of (11.9):

\begin{equation}
\lambda_2(\bar{x}, t) = \int_{-\infty}^{+\infty} \mathcal{G}_\alpha(\bar{x} - y, t) \lambda_{2,0}(y) dy + (\bar{x} - 1) + \int_{-\infty}^{+\infty} \int_{0}^{t} \mathcal{G}_\alpha(\bar{x} - y, t - \tau) H(y) d\tau dy.
\end{equation}
The uniqueness of (11.10) and (11.12) is a direct result from the linearity of the integral and the stability comes from Theorem 4.1 as $f(x) \in L^1(\mathbb{R})$. Thus:

(11.13) \[ \lim_{t \to +\infty} \| \lambda_1(\vec{x}, t) \|_\infty < \infty \]

and

(11.14) \[ \lim_{t \to +\infty} \| \lambda_2(\vec{x}, t) \|_\infty < \infty \]

which is equivalent to

\[ \lim_{t \to +\infty} \| \lambda_1(x, t) \|_\infty < \infty \]

and

\[ \lim_{t \to +\infty} \| \lambda_2(x, t) \|_\infty < \infty \]

by taking the norm of (11.4) and (11.5) and using Cauchy–Schwarz inequality and using (11.13), (11.14) and the fact that: $H_1(x) = V^{-1} \{ C D_{y,x}^\alpha R(x, y) \} \in L^1(\mathbb{R})$.

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