Nonnegative Whitney Extension Problem for $\mathbb{C}^1(\mathbb{R}^n)$

Black Jiang

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Abstract

Let $f$ be a real-valued function on a compact subset in $\mathbb{R}^n$. We show how to decide if $f$ extends to a nonnegative and $\mathbb{C}^1$ function on $\mathbb{R}^n$. There has been no known result for nonnegative $\mathbb{C}^m$ extension from a general compact set $E$ when $m > 0$. The nonnegative extension problem for $m \geq 2$ remains open.

1 Introduction

For $m,n \geq 1$, we write $\mathbb{C}^m(\mathbb{R}^n)$ to denote the vector space of continuously differentiable functions on $\mathbb{R}^n$ whose derivatives up to $m$-th order are bounded and continuous. Let $\mathbb{C}^m_+(\mathbb{R}^n)$ be the convex collection of elements in $\mathbb{C}^m(\mathbb{R}^n)$ that are also nonnegative on $\mathbb{R}^n$.

In this paper, we consider the following problem.

Problem 1 (Nonnegative Whitney Extension Problem). Let $E \subseteq \mathbb{R}^n$ be compact. Let $f : E \to [0, \infty)$. How can we decide if there exists $F \in \mathbb{C}^m_+(\mathbb{R}^n)$ with $F = f$ on $E$?

When $E$ is finite, $[8,10]$ provide solutions to Problem 1 with further control on the size of the derivatives of the extension (an extension without derivative control always exists in this case). It is related to the $\mathbb{C}^m$ selection problem. However, when $E$ is infinite, the strategies employed in $[8,10]$ collapse, because they rely on a Calderón-Zygmund decomposition procedure which may not terminate when $E$ is infinite. There has been no known answer to Problem 1 when $E \subseteq \mathbb{R}^n$ is infinite.

Problem 1 is a variant the following classical problem posed by H. Whitney $[13–15]$.

Problem 2 (Whitney Extension Problem). Let $E \subseteq \mathbb{R}^n$ be compact. Let $f : E \to \mathbb{R}$. How can we decide if there exists $F \in \mathbb{C}^m(\mathbb{R}^n)$ with $F = f$ on $E$?

In a series of papers $[4–6]$, Ch. Fefferman answered Problem 2. A key ingredient in Fefferman’s solution is the notion of Glaeser refinement, inspired by $[1,9]$. We briefly discuss the main idea of $[6]$ here.

To each $x \in E$ we assign an affine subspace $H_f(x) \subseteq \mathcal{P}^m$, where $\mathcal{P}^m$ denotes the polynomial of $n$ variables of degree no greater than $m$. The subspace $H_f(x)$ satisfies the following crucial property:

\begin{equation}
\text{(1.1) If } F \in \mathbb{C}^m(\mathbb{R}^n) \text{ satisfies } F = f \text{ on } E, \text{ then } \partial^m_x F \in H_f(x).
\end{equation}

Here, $\partial^m_x F$ denotes the degree $m$ Taylor polynomial of $F$ about the point $x$. For instance, we may take $H_f(x) = \{ P \in \mathcal{P}^m : P(x) = f(x) \}$. Then solving Problem 2 then amounts to the following problem.
(1.2) Decide if there exists \( F \in C^m(\mathbb{R}^n) \) such that \( \partial^m F \in H_\ell(x) \) for all \( x \in E \).

To achieve this goal, the author uses the procedure called “Glaeser refinement” (See Definition 2.1) on each of the subspace \( H_\ell(x) \), which produce another subspace \( H_\ell(x) \subseteq H_\ell(x) \subseteq P^m_n \) that possibly excludes some jets at \( x \) that cannot arise as the jets of a \( C^m \) function that agrees with \( f \) on \( E \). The author first shows that the Glaeser refinement stabilizes (i.e. the procedure does not produce new proper subspace) after a controlled number (depending only on \( m \) and \( n \)) of times. The author then shows that if the stabilized subspace is nonempty for each \( x \in E \), then there exists \( F \in C^m(\mathbb{R}^n) \) with \( \partial^m F \in H_\ell(x) \) for all \( x \in E \), hence solving Problem 2.

In this paper, we adapt the technology described above to solve Problem 1 for \( m = 1 \) (see Theorem 2 in Section 2). To account for nonnegativity, we associate to each \( x \in E \) a subset

\[
\Gamma_\ell(x) = \left\{ P \in P^1 : \text{there exists } F \in C^1_+(\mathbb{R}^n) \text{ such that } f(x) = f(x) \text{ and } \partial^1_x F = P \right\}.
\]

Solving Problem 1 then amounts to deciding whether there exists \( F \in C^1_+(\mathbb{R}^n) \) such that \( \partial^1_x F \in \Gamma_\ell(x) \) for each \( x \in E \).

To this end, we will apply Glaeser refinement to each of the subset \( \Gamma_\ell(x) \). Following [6], we will first prove that each subset \( \Gamma_\ell(x) \) will eventually stabilize after a finite number of refinement. Next, we show that if, for each \( x \in E \), we start with \( \Gamma_\ell(x) \) and arrive at some \( \Gamma_\ell(x) \neq \emptyset \) after a certain number of refinement, and that \( \Gamma_\ell(x) \) is its own Glaeser refinement; then there exists \( F \in C^1_+(\mathbb{R}^n) \) such that \( \partial^1_x F \in \Gamma_\ell(x, \infty) \) for each \( x \in E \), hence solving Problem 1 for \( m = 1 \).

This paper is part of a literature on extension and interpolation, going back to the seminal works of H. Whitney [13–15]. We refer to the interested readers to [4–8] and references therein for the history and related problems. For further discussion on Glaeser refinement, we direct the readers to [2,3,11].

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We will start from scratch and redefine all the notions.

2 Preliminaries and Main Results

Fix integers \( m, n \geq 0 \).

- We will use Euclidean distance \( |.| \) on \( \mathbb{R}^n \). We use \( B(x, r) \) to denote the open ball of radius \( r \) centered at \( x \).
- We use \( C^m_{\text{loc}}(\mathbb{R}^n) \) to denote the vector space of \( m \)-times continuously differentiable functions on \( \mathbb{R}^n \). We use \( C^m(\mathbb{R}^n) \) to denote the subspace of \( C^m_{\text{loc}}(\mathbb{R}^n) \) consisting of elements whose derivatives up to \( m \)-th order are bounded on \( \mathbb{R}^n \). We use \( C^m(\mathbb{R}^n) \) to denote the convex subcollection of elements in \( C^m(\mathbb{R}^n) \) that are also nonnegative on \( \mathbb{R}^n \).
We use $\mathcal{P}^m$ to denote the space of polynomials of $n$ variables and degree less or equal to $m$. For $x \in \mathbb{R}^n$ and $F \in C^m_{\text{loc}}(\mathbb{R}^n)$, we use $\partial^m_x F$ to denote the $m$-jet of $F$ at $x$, which we identify with the degree $m$ Taylor polynomial of $F$ at $x$

$$\partial^m_x F(y) := \sum_{|\alpha| \leq m} \frac{\partial^\alpha F(x)}{\alpha!} (y - x)^\alpha.$$ 

We use $\mathcal{R}_x^m$ to denote the ring of $m$-jets at $x$. It is clear that $\mathcal{R}_x^m$ is isomorphic to $\mathcal{P}^m$ as vector spaces, but we will distinguish them. Let $P, P' \in \mathcal{R}_x^m$, we define the jet product of $P$ and $P'$ in $\mathcal{R}_x^m$ to be

$$P \circ_x^m P' := \partial^m_x (PP').$$

We assume that $k^\#$ is a sufficiently large integer depending only on $m$ and $n$. See [6] for an estimate of the size of $k^\#$.

We first define the notion of Glaeser refinement.

**Definition 2.1.** Let $E \subseteq \mathbb{R}^n$ be compact. For each $x \in E$, supposed we are given a subset (not necessarily affine and possibly empty) $\Phi_0(x) \subseteq \mathcal{R}_x^m$. We define each $\Phi_\ell(x)$ inductively:

Let $x_0 \in E$, $P_0 \in \mathcal{R}_{x_0}^m$, and $\ell \geq 0$, we say that $P_0 \in \Phi_{\ell+1}(x_0)$ if the following holds.

(1) Given $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x_1, \cdots, x_{k^\#} \in B(x_0, \delta)$, there exist $P_1, \cdots, P_{k^\#} \in \mathcal{P}$, with $P_j \in \Phi_\ell(x_j)$ for $j = 0, \cdots, k^\#$, such that

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq \epsilon |x_i - x_j|^{m - |\alpha|} \text{ for } |\alpha| \leq m \text{ and } 0 \leq i, j \leq k^\#.$$ 

We define the Glaeser refinement of $\Phi_\ell(x_0)$ to be $\Phi_{\ell+1}(x_0)$.

**Remark 2.1.** Without further assumption on $\Phi_0(x)$, we do not know if $\Phi_0(x)$ will stabilize after finite number of Glaeser refinement, i.e., if $\Phi_{\ell^*+1}(x) = \Phi_{\ell^*}(x)$ for some $\ell^* < \infty$.

We make a definition for a class of subsets of $\mathcal{P}^m$ that will be known to stabilize.

**Definition 2.2.** Let $E \subseteq \mathbb{R}^n$ be compact. For $x \in E$, let $\Phi(x) \subseteq \mathcal{R}_x^m$. We call $\Phi(x)$ a **Glaeser fiber** if $\Phi(x) = \emptyset$ or $\Phi(x)$ has the form

$$\Phi(x) = P^x + I(x)$$

where $P^x \in \mathcal{R}_x^m$ and $I(x) \subseteq \mathcal{R}_x^m$ is an ideal.

The main theorem in [6] that provides an answer to Problem 2 is the following.

**Theorem 1 ([6]).** Let $E \subseteq \mathbb{R}^n$ be compact. Suppose that for each $x \in E$, we are given a nonempty Glaeser fiber $\Phi_\ast(x) \subseteq \mathcal{R}_x^m$. Assume that $\Phi_\ast(x)$ is its own Glaeser refinement. Then there exists $F \in C^m(\mathbb{R}^n)$ with $\partial^m_x F \in \Phi_\ast(x)$ for all $x \in E$.

We explain how we go from Theorem 1 to answer Problem 2. We begin with

(2.2) $$\Phi_0(x) := H_x := \{ P \in \mathcal{P}_n^m : P(x) = f(x) \}. $$

Then $\Phi_0(x)$ has the form $f(x) + m_0(x)$, where $f(x)$ is the constant polynomial and

(2.3) $$m_0(x) := \{ \phi^x \in \mathcal{P}_n^m : \text{ There exists } F \in C^m(\mathbb{R}^n) \text{ such that } F(x) = 0 \text{ and } \partial^m_x F = \phi^x \}$$
is clearly an ideal in $\mathcal{R}_{x_0}^{m}$. Lemma 2.1 in [6] shows that $\Phi_\ell(x)$ is still a Glaeser fiber under Glaeser refinement if we start with (2.2). Lemma 2.2 in [6] then shows that with this choice of $\Phi_0$, we have $\Phi_\ell(x) = \Phi_\ell^*(x)$ for all $\ell \geq \ell^*$, where $\ell^* = 2\dim \mathcal{P}^m + 1$. Therefore, deciding whether $f : E \to \mathbb{R}$ extends to a $C^m$ function amounts to computing the $\ell^*$-th Glaeser refinement of $H_f$.

Now we describe the key objects in this paper that are analogous to those above but also take into consideration of nonnegativity.

**Definition 2.3.** Let $E \subseteq \mathbb{R}^n$ be a compact subset. Let $f : E \to [0, \infty)$. For $x \in E$ and $M > 0$, we define

$$\Gamma^{[m]}_\ell(x) := \{ P \in \mathcal{P}^m : \text{There exists } F \in C^m_+(\mathbb{R}^n) \text{ and } \partial_x F = P \}.$$ 

**Remark 2.2.** $\Gamma^{[m]}_\ell(x)$ is in general not a Glaeser fiber if $m \geq 2$. However, for $m = 1$, we will see in Lemma 3.3 that it is. We will also see in Lemma 3.4 that it remains Glaeser after refinement.

Our main result of the paper is the following.

**Theorem 2.** Let $m = 1$. Let $E \subseteq \mathbb{R}^n$ be compact, and let $f : E \to [0, \infty)$ be given. For each $x \in E$, let $\Phi_0(x) := \Gamma^{(1)}_\ell(x)$, and for $\ell \geq 0$, let $\Phi_{\ell+1}(x)$ be the Glaeser refinement of $\Phi_{\ell}(x)$ defined by (2.1).

If $\Phi_{2n+3}(x) \neq \emptyset$ for each $x \in E$, then there exists $F \in C^1_+(\mathbb{R}^n)$ such that $\partial_x^1 F \in \Gamma^{(1)}_\ell(x)$ for each $x \in E$. In particular, there exists $F \in C^1_+(\mathbb{R}^n)$ such that $F = f$ on $E$.

To prove Theorem 2, we will show that under its hypotheses, the hypotheses of Theorem 1 (with $m = 1$) are satisfied. Theorem 1 then produces a $C^1$ function, which is not necessarily nonnegative, and whose jet at each $x \in E$ belongs to $\Gamma^{(1)}_\ell(x, \infty)$. We will then use these jets to reconstruct a nonnegative counterpart that takes the same jet at each $x \in E$, hence solving Problem 1. The reconstruction uses a variant of the classical Whitney Extension Theorem.

## 3 Main ingredients

In this section, we prove the main ingredients.

### 3.1 Preservation of Glaeser fiber

The main result we prove in this subsection is the following lemma, which states Glaeser fiber remains Glaeser after refinement.

**Lemma 3.1.** Suppose $\Phi_\ell(x)$ is a Glaeser fiber for each $x \in E$, then $\Phi_{\ell+1}(x)$ is a Glaeser fiber for each $x \in E$.

**Proof.** We expand the argument given by Lemma 2.1 in [6].

Fix $x_0 \in E$. If $\Phi_{\ell+1}(x_0) = \emptyset$, there is nothing to prove.

Suppose $\Phi_{\ell+1}(x_0) \neq \emptyset$. Pick arbitrary $P_{\ell+1}^{x_0} \in \Phi_{\ell+1}(x_0)$. Let

$$I_{\ell+1}(x_0) := \Phi_{\ell+1}(x_0) - P_{\ell+1}^{x_0}. \tag{3.1}$$

To show that $\Phi_{\ell+1}(x_0)$ is a Glaeser fiber, it suffices to show that $I_{\ell+1}(x_0)$ is an ideal in $\mathcal{R}_{x_0}^{m}$. 


By assumption, for each \( x \in E \),

\[
\Phi_{\ell}(x) = P_{\ell}^x + I_{\ell}(x),
\]

where \( P_{\ell}^x \in R_\ell^m \), and \( I_{\ell}(x) \subseteq R_\ell^m \) is an ideal.

**Claim 3.1.** \( I_{\ell+1}(x_0) \) are defined by the following procedure:

\( (3.3) \) \( \phi_0 \in I_{\ell+1}(x_0) \) if and only if the following holds: given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( x, \ldots, x_k \in E \cap B(x_0, \delta) \), there exists \( \phi_1, \ldots, \phi_k \), with \( \phi_j \in I_\ell(x_j) \) for \( j = 0, \ldots, k^\# \), such that

\( (3.4) \) \[ |\partial^\alpha (\phi_i - \phi_j)(x_i)| \leq \epsilon |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m, 0 \leq i, j \leq k^\#. \]

**Proof of Claim 3.1.** First we show sufficiency. Suppose \( \phi_0 \in I_{\ell+1}(x_0) \). Fix \( \epsilon > 0 \). Write \( P_0 = P_{\ell+1}^{x_0} \).

Define

\( \hat{\phi}_0 := P_0 + \phi_0 \).

By (3.1),

\( \hat{\phi}_0 \in I_{\ell+1}(x_0) \).

Applying Definition 2.1 to \( P_0 \) and \( \hat{\phi}_0 \) with \( \frac{1}{2} \epsilon \) in place of \( \epsilon \), we find a \( \delta > 0 \) such that for any \( x, \ldots, x_k \in E \cap B(x_0, \delta) \), there exist \( P_j, \hat{\phi}_j \in \Phi_\ell(x_i) \) for \( j = 1, \ldots, k^\# \), such that

\( (3.5) \) \[ |\partial^\alpha (P_i - P_j)(x_i)| \leq \frac{1}{2} \epsilon_0 |x_i - x_j|^{m-|\alpha|}, \]

\[ |\partial^\alpha (\hat{\phi}_i - \hat{\phi}_j)(x_i)| \leq \frac{1}{2} \epsilon_0 |x_i - x_j|^{m-|\alpha|}. \]

Now, let

\( \phi_j := \hat{\phi}_j - P_j \) for \( j = 1, \ldots, k^\# \).

Thanks to (3.5), the \( \phi_j \)'s satisfy (3.4).

Now we need to check that \( \phi_j \in I_\ell(x_j) \) for \( j = 1, \ldots, k^\# \). Indeed, by induction hypothesis, the \( \Phi_\ell(x_i)'s \) are Glaeser fiber, so

\( \phi_j \in I_\ell(x_j) \) for \( j = 1, \ldots, k^\# \).

This proves sufficiency.

Now we show necessity. Let \( \epsilon_0 > 0 \) be given. Apply Definition 2.1 to \( P_0 \) and apply the latter condition in (3.3) to \( \phi_0 \), with \( \frac{1}{2} \epsilon_0 \) in place of \( \epsilon \), we see that there exists \( \delta > 0 \) such that for any \( x, \ldots, x_k \in E \cap B(x_0, \delta) \), there exist \( P_j \in \Phi_\ell(x_j) \) and \( \phi_j \in I_\ell(x_j) \), \( j = 1, \ldots, k^\# \), satisfying

\( (3.6) \) \[ |\partial^\alpha (P_i - P_j)(x_i)| \leq \frac{1}{2} \epsilon_0 |x_i - x_j|^{m-|\alpha|}, \]

\[ |\partial^\alpha (\phi_i - \phi_j)(x_i)| \leq \frac{1}{2} \epsilon_0 |x_i - x_j|^{m-|\alpha|}. \]

Now, let

\( \hat{\phi}_j := P_j + \phi_j \) for \( j = 1, \ldots, k^\# \).
By induction hypothesis, the $\Phi_t(x_j)$’s are Glaeser, so

$$\hat{\phi}_j \in \Phi_t(x_j) \text{ for all } j = 1, \ldots, k^\#.$$ 

Thanks to (3.6), we have

$$|\partial^\alpha (\hat{\phi}_i - \hat{\phi}_j)(x_i)| \leq \epsilon_0 |x_i - x_j|^{m-|\alpha|} \text{ for } 0 \leq i, j \leq k^\#.$$ 

Therefore,

(3.8) \hspace{1cm} \hat{\phi}_0 \in \Phi_{t+1}(x_0). 

Now, (3.1), (3.7), and (3.8) together imply $\phi_0 \in I_{t+1}(x_0)$. This proves necessity, and concludes the proof of the claim. \hfill \Box

To finish the proof of the lemma, we fix $\phi_0 \in I_{t+1}(x_0)$ and $\tau \in R^m_{x_0}$. It suffices to show that

(3.9) \hspace{1cm} \hat{\phi}_0 := \phi_0 \circ_{x_0} \tau \in I_{t+1}(x_0). 

Let $\epsilon_0 > 0$. Let $\delta, x_1, \ldots, x_{k^\#}, \phi_1, \ldots, \phi_{k^\#}$ be as in (3.3) with $A^{-1}\epsilon_0$ in place of $\epsilon$, for some $A > 0$ to be determined. Define

$$\hat{\phi}_j := \phi_j \circ_{x_j} \tau \text{ for } j = 1, \ldots, k^\#.$$ 

Since $I_t(x_j) \subseteq R^m_{x_j}$ is an ideal by assumption, we have $\hat{\phi}_j \in I_t(x_j)$ for all $j = 1, \ldots, k^\#$. Moreover, by the classical Whitney Extension Theorem for finite set (see e.g. [12]) and (3.4), for each distinct pair $x_i, x_j$, we may find $F^{ij} \in C^m(\mathbb{R}^n)$ such that

(3.10) \hspace{1cm} |\partial^\alpha F| \leq M \text{ for } |\alpha| \leq m \text{ on } \mathbb{R}^n

where $M$ is a number depending only on $m, n$, and $\epsilon_0$, and that

(3.11) \hspace{1cm} \partial^m_{x_{i}^{\nu}} F^{ij} = \phi_\nu \text{ for } \nu = i, j. 

Therefore, $\hat{\phi}_\nu = \partial^m_{x_{i}^{\nu}} (F^{ij} \cdot \tau)$ for $\nu = i, j$. Taylor’s theorem, combined with (3.10) and (3.11), implies

$$|\partial^\alpha (\hat{\phi}_i - \hat{\phi}_j)(x_i)| = \left| \partial^\alpha (F^{ij} \cdot \tau - \partial^m_{x_j^{\nu}} (F^{ij} \cdot \tau))(x_j) \right| \leq B_\tau \cdot A^{-1} \epsilon_0 |x_i - x_j|^{m-|\alpha|} \text{ for } |\alpha| \leq m. 

Here, we may take $B_\tau$ to be a number that depends only on $M$ and $\tau$. Taking $A > B_\tau$, we can conclude that

$$|\partial^\alpha (\hat{\phi}_i - \hat{\phi}_j)(x_i)| \leq \epsilon_0 |x_i - x_j|^{m-|\alpha|} \text{ for all } |\alpha| \leq m, 0 \leq i, j \leq k^\#.$$ 

Hence, we have shown (3.9). The lemma is proved. \hfill \Box

Lemma 3.2. Suppose $\Phi_t(x) \subseteq R^m_x$ is a Glaeser fiber for each $x \in E$ and $t \geq 0$. If $\ell^* = 2 \dim \mathcal{P}^m + 1$, then for each $x \in E$, $\Phi_t(x) = \Phi_{t^*}(x)$ for all $t \geq \ell^*$.

The argument is the same as the proof of Lemma 2.2 in [6], which is inspired by [1] and [9]. We direct the interested readers to those cited above as well as [11] for a discussion on stabilization of Glaeser refinement.
3.2 Nonnegative Whitney Extension Theorem

In this subsection, we sketch the proof of the nonnegative version of the classical Whitney Extension Theorem [13].

**Theorem 3.** Let $E \subseteq \mathbb{R}^n$ be compact. Let $f : E \to [0, \infty)$. Let $\{P^x : x \in E\}$ be a collection of polynomials such that $P^x \in \Gamma_f^{(m)}(x)$ for all $x \in E$. Suppose

\[ |\partial^\alpha (P^x - P^y)(x)| = o(|x - y|^{m-|\alpha|}) \quad \text{as} \quad |x - y| \to 0, \quad \text{for all} \quad x, y \in E \quad \text{and} \quad |\alpha| \leq m. \]

Then there exists $F \in C_m(\mathbb{R}^n)$ with $\partial^m_x F = P^x$ for all $x \in E$.

**Sketch of Proof.** Let $\mathcal{W}_E$ be a Whitney cover of $\mathbb{R}^n \setminus E$, namely, $\mathcal{W}_E = \{Q\}_{Q \in \mathcal{W}_E}$ such that the following hold.

- Each $Q \in \mathcal{W}_E$ is a closed cube in $\mathbb{R}^n$.
- If $Q, Q' \in \mathcal{W}_E$ and $Q \neq Q'$, then interior$(Q) \cap$ interior$(Q') = \emptyset$.
- For every $Q \in \mathcal{W}_E$,
  \[ \frac{1}{4} \text{diam}(Q) \leq \text{dist}(Q, E) \leq 4 \text{diam}(Q). \]

Let $\{\varphi_Q : Q \in \mathcal{W}_E\}$ be a $C^m$ partition of unity satisfying

- $\sum_{Q \in \mathcal{W}_E} \varphi_Q(x) = 1$ for all $x \in \mathbb{R}^n \setminus E$.
- $\text{supp}(\varphi_Q) \subseteq \frac{1}{2}Q$.
- $|\partial^\alpha \varphi_Q| \leq C(\text{diam}(Q))^{-|\alpha|}$ for all $|\alpha| \leq m$.

For the existence of such covering and partition of unity, see e.g. [12, 13].

For each $x \in E$, since $P^x \in \Gamma_f(x)$, there exists $F^x \in C^m(\mathbb{R}^n)$ such that $\partial^m_x F^x = P^x$.

For each $Q \in \mathcal{W}_E$, we pick a representative point $r_Q \in E$ (not necessarily unique) such that

\[ \text{dist}(r_Q, Q) = \text{dist}(E, Q). \]

We also let $F_Q := F^{r_Q}$.

Define

\[ F(x) := \left\{ \begin{array}{ll}
\sum_{Q \in \mathcal{W}_E} \varphi_Q(x) F_Q(x) & x \in \mathbb{R}^n \setminus E \\
0 & x \in E
\end{array} \right. . \]

We want to show that $F \in C^m(\mathbb{R}^n)$ with $F \geq 0$ on $\mathbb{R}^n$, and $\partial^m_x F = P^x$ for all $x \in E$.

It is clear that $F \geq 0$ on $\mathbb{R}^n$, since all of the $\varphi_Q$’s and the $F_Q$’s are.

It is also clear that $F$ is $C^m$ away from $E$ since each of the $F_Q$’s are and the supports of the $\varphi_Q$’s have bounded overlap. Therefore, it suffices to examine the differentiability property of $F$ near the set $E$ and the jet of $F$ on $E$. 


By Taylor’s theorem,
\[ \partial^\alpha F_Q(x) = \partial^\alpha P \tau_Q(x) + o(|x - \tau_Q|^{m-|\alpha|}) \text{ as } x \to \tau_Q \text{ for all } |\alpha| \leq m. \]

The compatibility condition (3.12) then implies that
\[ \partial^\alpha F_Q(\hat{x}) \to \partial^\alpha P(\hat{x}) \]

uniformly along any sequence of cubes \( Q \in \mathcal{W}_E \) converging to \( \hat{x} \in \mathbb{E}^1 \). Therefore, \( F \in C^m_{\text{loc}} \) near \( E \) and \( \partial^m F = P^x \) for each \( x \in E \). Since \( E \) is compact, we can conclude that \( F \in C^m(\mathbb{R}^n) \).

This completes the sketch of the proof.

\[ \square \]

### 3.3 Properties of \( \Gamma_t \)

Recall Definition 2.3 and (2.2). For the rest of this section, we fix \( m = 1 \). We write \( \mathcal{P} \) for \( \mathcal{P}^1 \), \( \partial_x \) for \( \partial^1 \), and \( \Gamma_t(x) \) for \( \Gamma_t^{(1)}(x) \).

**Lemma 3.3.** If \( f(x) > 0 \), then \( \Gamma_t(x) = H_t(x) \). If \( f(x) = 0 \), then \( \Gamma_t(x) = \{0\} \). In particular, for each \( x \in E \), \( \Gamma_t(x) \) is a Glaeser fiber (see Definition 2.2).

**Proof.** Suppose \( f(x) > 0 \). It is clear that \( \Gamma_t(x) \subseteq H_t(x) \). It suffices to show that reverse inclusion. Let \( P \in H_t(x) \). Then \( P(x) = f(x) > 0 \). Since \( P \) is continuous, there exists \( \delta > 0 \) such that \( P \geq 0 \) on \( B(x, \delta) \). Let \( \chi \) be a \( C^1 \)-cutoff function such that \( \chi \equiv 1 \) near \( x \) and \( \text{supp}(\chi) \subseteq B(x, \delta) \). Then \( F := \chi \cdot P \in C^1_{\text{loc}}(\mathbb{R}^n) \) with \( \partial_x F = P \). Therefore, \( H_t(x) \subseteq \Gamma_t(x) \).

Suppose \( f(x) = 0 \). It is clear that the zero polynomial \( 0 \in \Gamma_t(x) \). Suppose \( P \in \Gamma_t(x) \), then there exists \( F \in C^1_{\text{loc}}(\mathbb{R}^n) \) such that \( \partial_x F = P \). Since \( F \geq 0 \) on \( \mathbb{R}^n \), \( F \) has a local minimum at \( x \), so \( \nabla F(x) = 0 \). Hence, \( P \equiv 0 \).

\[ \square \]

**Lemma 3.4.** Let \( \ell \geq 0 \). For each \( x \in E \), \( \Gamma_t(x) \) is a Glaeser fiber.

**Proof.** We have shown in Lemma 3.3 that \( \Gamma_t(x) \) is a Glaeser fiber for each \( x \in E \). Therefore, the Lemma follows from Lemma 3.1.

\[ \square \]

**Remark 3.1.** A subtle difference between Lemma 3.4 and Lemma 2.1 in [6] is that \( \Gamma_t(x) \) is a translate of an ideal that possibly depends on the function \( f \).

**Lemma 3.5.** For each \( x \in E \), \( \Gamma_{t^*}(x) = \Gamma_{2n+3}(x) \) for all \( \ell^* \geq 2n + 3 \).

**Proof.** By Lemma 3.4, \( \Gamma_t(x) \) is a Glaeser fiber for each \( x \in E \) and \( \ell \geq 0 \). Therefore, by Lemma 3.2, \( \Gamma_t(x) \) is a stabilized Glaeser fiber if \( \ell^* \geq 2 \dim \mathcal{P} + 1 = 2n + 3 \).

\[ ^{1} \text{Here we define dist}(x, F) = \inf \{|x - y| : y \in F\} \text{ for } x \in \mathbb{R}^n \text{ and } F \subseteq \mathbb{R}^n \text{ closed.} \]
4 Proof of the main theorem

In this section, we fix $m = 1$. We write $P$ for $\mathcal{P}^1$, $\partial_\phi$ for $\partial_\phi^1$, and $\Gamma_\ell(x)$ for $\Gamma_\ell^{(1)}(x)$.

Proof of Theorem 2. First, we want to show that under the hypotheses of Theorem 2, the hypotheses of Theorem 1 are satisfied.

Let $\ell^* \geq 2n + 3$. Then, $\Gamma_\ell^*(x)$ is a Glaeser fiber, thanks to Lemma 3.4. By Lemma 3.5, $\Gamma_\ell^*(x)$ is its own Glaeser refinement. Hence, the hypotheses of Theorem 1 are satisfied.

By Theorem 1, there exists $F_0 \in \mathcal{C}^1(\mathbb{R}^n)$, not necessarily nonnegative, such that

$$\partial_\phi F_0 \in \Gamma_\ell^*(x) \subseteq \Gamma_0(x) = \Gamma_\ell(x).$$

Consider the family of polynomials

$$F := \{P^x = \partial_\phi F_0 : x \in E\}.$$

By Taylor’s theorem, $F$ satisfies (3.12). Thanks to (4.1), $F$ satisfies the hypothesis of Theorem 3 (with $m = 1$). Therefore, there exists $F \in \mathcal{C}_+^1(\mathbb{R}^n)$, such that $\partial_\phi F = \partial_\phi F_0$ for each $x \in E$. In particular, $F(x) = f(x)$. This concludes the proof.

Remark 4.1. In [11], the authors showed that for $\mathcal{C}^1(\mathbb{R}^n)$ without the nonnegative constraint, it suffices to take $k^# = 2$ in the first refinement and $k^# = 1$ in the subsequent refinements, and the number of refinement $\ell^*$ till stabilization can be reduced to $n \leq \ell^* \leq n + 1$. It will be interesting to see if these bounds still hold for $\mathcal{C}_+^1(\mathbb{R}^n)$.

References

[1] Edward Bierstone, Pierre D. Milman, and Wieslaw Pawlucki. Differentiable functions defined in closed sets. A problem of Whitney. Invent. Math., 151(2):329–352, 2003.

[2] Edward Bierstone, Pierre D. Milman, and Wieslaw Pawlucki. Higher-order tangents and Fefferman’s paper on Whitney’s extension problem. Ann. of Math. (2), 164(1):361–370, 2006.

[3] Edward Bierstone and Pierre D. Milman. $\mathcal{C}^m$-norms on finite sets and $\mathcal{C}^m$ extension criteria. Duke Math. J., 137(1):1–18, 2007.

[4] Charles Fefferman. A sharp form of Whitney’s extension theorem. Ann. of Math. (2), 161(1):509–577, 2005.

[5] Charles Fefferman, A Generalized Sharp Whitney Theorem for Jets. Rev. Mat. Iberoamericana, 21(2):577–688, 2005.

[6] Charles Fefferman, Whitney’s extension problem for $\mathcal{C}^m$. Ann. of Math. (2), 164(1):313–359, 2006.

[7] Charles Fefferman, Arie Israel, and Garving K. Luli, Finiteness principles for smooth selections. Geom. Funct. Anal., 26(2): 422–477, 2016.

[8] Charles Fefferman, Arie Israel, and Garving K. Luli, Interpolation of data by smooth nonnegative functions. Rev. Mat. Iberoamericana, 33(1): 305–324, 2016.
[9] Georges Glaeser. Étude de quelques algèbres tayloriennes. *J. Analyse Math.*, 6:1–124; erratum, insert to 6 (1958), no. 2, 1958.

[10] Fushuai Jiang and Garving K. Luli, Nonnegative $C^2(\mathbb{R}^2)$ interpolation. In preparation.

[11] Boažaz Klartag and Nahum Zobin, $C^1$ extension of functions and stabilization of Glaeser refinements. *Rev. Mat. Iberoamericana*, 23(2):635–669, 2007.

[12] Elias M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Monographs in Harmonic Analysis. Princeton University Press, Princeton, NJ, 1970.

[13] Hassler Whitney. Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.*, 36(1):63–89, 1934.

[14] Hassler Whitney. Differentiable Functions Defined in Closed Sets. I. *Trans. Amer. Math. Soc.*, 36(2): 369–387, 1934.

[15] Hassler Whitney. Functions differentiable on the boundaries of regions *Ann. of Math. (2)* 35(3): 482–485, 1934.