Mass singularity in $\text{QED}_3$

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Abstract

We determine the position space fermion propagator in three dimensional QED based on Ward-identity and spectral representation. There is a new type of mass singularity which governs the long distance behaviour. It leads the propagator vanish at large distance more strongly than the mass term does. This term corresponds to Dynamical mass. Momentum space propagator is compared with the analysis of Schwinger-Dyson equation and our solution contains a non-perturbative effects beyond the quenched approximation with bare vertex.
I. INTRODUCTION

In the previous paper we find that the propagator has a new type of mass singularity in QED$_3$[1]. The two kinds of self-energy are derived by Ward-Identity in evaluation of the second order spectral function. To include intermediate states with infinite numbers of photon we exponentiate the one-photon matrix element. Usually three dimensional QED is thought as the high temperature limit of QED in the imaginary time formalism at finite temperature[2]. It is assumed that the three dimensional analysis leads the leading order of high temperature expansion results in four dimension[3]. In this respect first we study the position space structure of the propagator especially in the Yennie gauge. In this gauge we can avoid linear infrared divergences. In three dimension propagator has a logarithmic singularity even in the Yennie gauge. Fourier transformation to momentum space is done numerically. Low and high energy behaviour is consistent with other analysis in Euclidean space[6]. In section II Bloch-Nordsieck approximation in three dimension is reviewed. Section III is for non-perturbative effects. In section IV mass singularity in four and three dimension is compared and numerical analysis in momentum space is given. Main results are the followings.

1 Based on Ward-Identity and dispersion-theoretic representation we find position space propagator $S_F(x)$. 
2 Gauge dependent terms are second-order corrected mass and linear infrared.
divergences which vanishes in the Yennie gauge and others are gauge invariant.

3 Second-order correction to the free propagator yields two kinds of gauge
invariant mass singularity: position dependent mass and self energy. The threshold
behaviour is determined by self energy and dynamical mass is due to position
dependent mass. Exponentiation of one-photon matrix element we obtain the full
propagator. There exists dynamical symmetry breaking at $\frac{e^2}{8\pi m} = 1$ for finite
infrared cutoff.

II. BLOCH-NORDSIECK APPROXIMATION TO QED$_3$

First we consider about the charged particle which emit and absorb massless pho-
tons. Usually this process was described by spectral function; transition probability of par-
ticle into particle and photon state. Multi-photon emitted from external line is introduced
by ladder type diagrams which satisfy Ward-Identity. Let us begin by dispersion theoretic
description of the propagator[1]

$$S_F(p) = \int d^3x \exp(ip \cdot x) \langle \Omega | T\bar{\psi}(x)\psi(0) | \Omega \rangle$$
$$= \int d\omega \frac{\gamma \cdot p\rho_1(\omega) + \omega\rho_2(\omega)}{p^2 - \omega^2 + i\epsilon}$$
$$S_F(x) = \int d\omega S_F(p, \omega).$$

The field $\psi$ is renormalized and is taken to be a spinor with mass $m$. Here we introduce
intermediate states that contribute the spectral function

$$\sigma(p^2) = (2\pi)^2 \sum_N \delta^3(p - p_N) \int d^3x \exp(i p \cdot x) \langle \Omega | \bar{\psi}(x) | N \rangle \langle N | \psi(0) | \Omega \rangle .$$

Total three-momentum of the state $|N\rangle$ is $p^\mu_N$. The only intermediates $N$ contain one spinor
and an arbitrary number of photons. Setting

$$|N\rangle = |r; k_1, ..., k_n\rangle,$$
where $r$ is the momentum of the spinor $r^2 = m^2$, and $k_i$ is the momentum of $i$th soft photon, we have

$$\sigma(p^2) = \int \frac{m^2}{p^0} \sum_{n=0}^{\infty} \frac{1}{n!} \times \left( \int \frac{d^3 k}{(2\pi)^3} \right) \theta(k_0) \delta(k^2) \sum_n \delta(p - r - \sum_{i=1}^n k_i) \times \langle \Omega | \psi(x) | r; k_1, \ldots, k_n \rangle \langle r; k_1, \ldots, k_n | \psi(0) | \Omega \rangle. \tag{5}$$

Here the notation

$$(f(k))_0 = 1,$$

$$(f(k))_n = \prod_{i=1}^n f(k_i) \tag{6}$$

has been introduced. Here we define matrix element

$$T_n = \langle \Omega | \psi | r; k_1, \ldots, k_n \rangle,$$ \tag{7}

$$T_n^\mu = -\int d^3 x \exp(ik_n \cdot x) \langle \Omega | T \psi j^\mu | r; k_1, \ldots, k_{n-1} \rangle, \tag{8}$$

provided

$$\Box_x T \psi A_\mu(x) = T \psi \Box_x A_\mu(x) = T \psi(-j_\mu(x) + \partial^\nu(\partial \cdot A(x))). \tag{9}$$

$T_n$ satisfies Ward-Identity:

$$\partial^\nu T(\bar{\psi} j_\mu(x)) = -e\psi(x), \tag{10}$$

$$\partial^\nu T(\bar{\psi} j_\mu(x)) = e\bar{\psi}(x),$$

$$k_{n\mu}T_n^\mu(r, k_1, \ldots, k_n) = cT_{n-1}(r, k_1, \ldots, k_{n-1}), r^2 = m^2. \tag{11}$$

Using LSZ the one photon matrix element is given

$$T_1 = \langle in | T(\psi_{in}(x), ie \int d^3 y \bar{\psi}_{in}(y) \gamma_\mu \psi_{in}(y) A_\mu^0(y)) | r; k, in \rangle$$

$$= ie \int d^3 y d^3 z S_F(x - y) \gamma_\mu \delta^{(3)}(y - z) \exp(i(k \cdot y + r \cdot z)) e^\mu(k, \lambda) U(r, s)$$

$$= -ie \frac{(r + k) \cdot \gamma + m}{(r + k)^2 - m^2} \gamma_\mu e^\mu(k, \lambda) \exp(i(k + r) \cdot x) U(r, s), \tag{12}$$

where $U(r, s)$ is a four-component free particle spinor with positive energy

$$\sum_s U(r, s) U(r, s) = \frac{\gamma \cdot r + m}{2m}. \tag{13}$$
The spectral function $\sigma$ is given by exponentiation of one-photon matrix element,

$$T_n = \prod_{j=1}^{n} T_1(k_j),$$ \hspace{1cm} (14)

which yields an infinite ladder approximation

$$
\sigma(x) = \int \frac{md^2r}{r^0} \exp(ir \cdot x) \exp(F),
$$ \hspace{1cm} (15)

$$F = \int \frac{d^3k}{(2\pi)^2} \delta(k^2)\theta(k_0) \sum_{\lambda,S} T_1(x) T_1^+(0)$$

$$= \int \frac{d^3k}{(2\pi)^2} \exp(ik \cdot x)\delta(k^2)\theta(k_0)$$

$$\times e^2 \text{tr} \left[ \frac{(r + k) \cdot \gamma}{(r + k)^2 - m^2} \gamma^\mu \frac{(r + k) \cdot r}{(r + k)^2 - m^2} \gamma^\nu \frac{r \cdot \gamma + m}{2m} \Pi_{\mu\nu} \right]$$

$$= \int \frac{d^3k}{(2\pi)^2} \exp(ik \cdot x)\delta(k^2)\theta(k_0)$$

$$\times e^2 \left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{r \cdot k} + (d - 1) \frac{\delta(k^2)}{k^2} \right].$$ \hspace{1cm} (16)

First we take the trace of $T_1 T_1^+$ for simplicity in the infrared. In this case we take the scalar part of them and assume $\rho_1(\omega) = \rho_2(\omega)$. In general case there are two kinds of spectral function. To avoid infrared divergences we introduce photon mass $\mu$ as an infrared cut off. It is helpful to use function $D_+(x)$

$$D_+(x) = \frac{1}{(2\pi)^2 i} \int \exp(ik \cdot x) d^3k \theta(k^0) \delta(k^2 - \mu^2)$$

$$= \frac{1}{(2\pi)^2 i} \int_0^\infty J_0(kx) \frac{\pi kdk}{2\sqrt{k^2 + \mu^2}} = \frac{\exp(-\mu x)}{8\pi ix},$$ \hspace{1cm} (17)

to determine $F$. If we use parameter trick

$$\lim_{\epsilon \to 0} \int_0^\infty d\alpha \exp(i(k + i\epsilon) \cdot (x + \alpha r)) = \frac{\exp(ik \cdot x)}{k \cdot r},$$ \hspace{1cm} (18)

$$\lim_{\epsilon \to 0} \int_0^\infty d\alpha \exp(i(k + i\epsilon) \cdot (x + \alpha r)) = \frac{\exp(ik \cdot x)}{(k \cdot r)^2},$$ \hspace{1cm} (19)

the function $F$ is written in the following form

$$F = ie^2m^2 \int_0^\infty d\alpha D_+(x + \alpha r, \mu) - e^2 \int_0^\infty d\alpha D_+(x + \alpha r, \mu) - ie^2(d - 1) \frac{\partial}{\partial \mu} D_+(x, \mu)$$

$$= \frac{e^2m^2}{8\pi r^2} \left( - \frac{\exp(-\mu x)}{\mu} + x \text{Ei}(1, \mu x) \right) + (d - 1) \frac{e^2}{8\pi \mu} \exp(-\mu x),$$ \hspace{1cm} (20)
where the function $E_i(n, \mu x)$ is defined

$$E_i(n, \mu x) = \int_1^\infty \frac{\exp(-\mu xt)}{t^n} dt.$$  \hspace{1cm} (21)

It is understood that all terms which vanishes with $\mu \to 0$ are ignored. The leading non trivial contributions to $F$ are

$$E_i(1, \mu x) = -\gamma - \ln(\mu x) + O(\mu x),$$  \hspace{1cm} (22)

$$F_1 = \frac{e^2 m^2}{8\pi r^2} \left( -\frac{1}{\mu} + x(1 - \ln(\mu x) - \gamma) \right) + O(\mu),$$

$$F_2 = \frac{e^2}{8\pi r} (\ln(\mu x) + \gamma) + O(\mu),$$

$$F_g = \frac{e^2}{8\pi} \left( \frac{1}{\mu} - x \right) (d - 1) + O(\mu),$$  \hspace{1cm} (23)

$$F = \frac{e^2}{8\pi \mu} (d - 2) + \frac{\gamma e^2}{8\pi r} + \frac{e^2}{8\pi} \ln(\mu x) - \frac{e^2}{8\pi} x \ln(\mu x) - \frac{e^2}{8\pi} x(d - 2 + \gamma),$$  \hspace{1cm} (24)

where $\gamma$ is Euler’s constant. Using integrals for intermediate state for on-shell fermion

$$\int d^3x \exp(ip \cdot x) \int d^3r \delta(r^2 - m^2) \exp(ir \cdot x) f(r) = f(m),$$  \hspace{1cm} (25)

$$\int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \int d^3r \delta(r^2 - m^2) = \frac{1}{m^2 + p^2},$$  \hspace{1cm} (26)

we set $r = m$ in the phase space integral ;

$$\sigma(p) = \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \int \frac{m}{\sqrt{r^2 + m^2}} d^2r \exp(ir \cdot x) \exp(F(m, x))$$

$$= \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \frac{\exp(-m x)}{x} \exp(F(m, x)).$$  \hspace{1cm} (27)

Since we evaluated the matrix element $F$ by the on shell limit of fermion and photon, $F$ can be understood to describes self-energy[5]. In section IV and V we discuss dynamical mass, the renormalization constant, and bare mass in connection of each terms. After angular integration we get the propagator

$$\sigma(p) = \frac{m}{2\pi p} \int_0^\infty dx \sin(px) \exp(-(m + B)x)(\mu x)^{-C_x + D},$$  \hspace{1cm} (28)
where

\[ A = \frac{e^2}{8\pi \mu} (d-2) + \frac{\gamma e^2}{8\pi}, B = \frac{e^2}{8\pi} (d-2 + \gamma), C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}. \]  

(29)

Here we show up to \( O(e^2) \) propagator \( \sigma(p) \)

\[ \sigma^{(2)}(p) = \int \frac{d^3x}{(2\pi)^3} \exp(ip \cdot x) \int \frac{md^2r}{r^6} \exp(ir \cdot x) F(x) \]

\[ = \frac{m}{2\pi p} \int_0^\infty dx \sin(px) \exp(-mx) \left[ A - Bx - C \ln(mx) + D \ln(mx) \right] \]

\[ = \left[ \frac{2m(1 + A)}{m^2 + p^2} - \frac{m^2 B}{(m^2 + p^2)^2} + 2m(DI_1 - CI_2) \right], \]  

(30)

where \( I_1, I_2 \) are the following integrals

\[ I_1 = \int_0^\infty \frac{\sin(px) \exp(-mx)}{p} \ln(mx) dx \]

\[ = -\gamma \frac{\ln((m^2 + p^2)/\mu^2)}{2(m^2 + p^2)} - \ln((m - \sqrt{-p^2})/(m + \sqrt{-p^2})), \]  

(31)

\[ I_2 = \int_0^\infty \frac{\sin(px) \exp(-mx)}{p} x \ln(mx) dx \]

\[ = \frac{m}{(m^2 + p^2)^2} \left[ \ln((m - \sqrt{-p^2})/(m + \sqrt{-p^2})) + \ln((m^2 + p^2)/\mu^2) - 2(1 - \gamma) \right]. \]  

(32)

Notice that the Yennie gauge \((d = 2)\) is free from linear infrared divergences but associates a logarithmic divergences. In four dimension logarithmic divergence disappear and we have a free propagator in this gauge. This point is a clear difference between three and four dimensional case. For definiteness hereafter we choose the Yennie gauge. In this gauge the position space propagator is

\[ S_F(x) = -\int (i\gamma \cdot \partial + \omega) \frac{\exp(-\omega x)}{4\pi x} \sigma(\omega) d\omega, \]

(33)

and it is written in momentum space

\[ S_F(p) = \int d\omega \frac{(\gamma \cdot p + \omega) \sigma(\omega)}{p^2 - \omega^2 + i\epsilon} = \frac{\gamma \cdot p}{p} \sigma(p) + \sigma(p), \]

\[ \sigma(p) = \int d^3x \exp(ip \cdot x) \sigma(x), \]

\[ \sigma(x) = \frac{m \exp(-m + B)x}{4\pi x} \mu x - Cx + D. \]  

(34)

(35)

Finally we show the general spin dependent spectral function in the \( O(e^2) \) for \( d = 2 \) gauge

\[ F = (\gamma \cdot r + m) \left[ \frac{e^2}{8\pi} (-\gamma x - x \ln(mx) + \frac{e^2}{8\pi m} (\ln(mx) + \gamma)) - \gamma \cdot \gamma \frac{e^2}{8\pi m^2 x} \right]. \]  

(36)
The first term is the same as the previous one. The second term is independent of the cutoff $\mu$ and not significant in the infrared. In Euclidean space if we exponentiate it that becomes

$$
\int \frac{mr^2 dr}{\sqrt{r^2 + m^2}} \exp(ir \cdot x) \frac{\exp(-mx)}{2\pi x} (1 + \frac{ir \cdot \gamma}{\sqrt{-r^2}}) \exp((\frac{ir \cdot \gamma}{\sqrt{-r^2}} + 1)F)
$$

$$
= \int \frac{mr^2 dr}{\sqrt{r^2 + m^2}} \exp(ir \cdot x) \frac{\exp(-mx)}{2\pi x} (1 + \frac{ir \cdot x}{\sqrt{-r^2}}) \exp(\frac{F}{2}) \exp(\frac{F}{2})(\cosh(\frac{F}{2}) + i \sigma \cdot r \sinh(\frac{F}{2})).
$$

(37)

This form corresponds to the infinite ladder graph of the fermion propagator with renormalized mass $m$ and corrected vertex. In section IV we discuss physical meanings of each terms in $\sigma(x)$ and the dynamical effects in momentum space numerically.

### III. VANISHING BARE MASS AND VACUUM EXPECTATION VALUE OF CONDENSATE

In this section we examine the renormalization constant and bare mass and study the condition of vanishing bare mass based on spectral representation. The spinor propagator in position space is expressed in the following[3]

$$
S_F(x) = -\int (i\gamma \cdot \partial + \omega) \frac{\exp(-\omega x)}{4\pi x} \sigma(\omega) d\omega
$$

(38)

$$
S_F(p) = \frac{\gamma \cdot p}{p} \sigma(p) + \sigma(p).
$$

(39)

First term is related to wave function renormalization and second term is a mass function which contains renormalized and dynamical mass. The equation for the renormalization constant in terms of the spectral functions read

$$
\lim_{p \to \infty} \frac{Z^{-1}_2(\gamma \cdot p + m_0)}{p^2 - m^2 + i\epsilon} = \lim_{p \to 0} \frac{\int (\gamma \cdot p + \omega) \sigma(\omega) d\omega}{p^2 - \omega^2 + i\epsilon}.
$$

(40)

Instead we determine them directly by taking the high energy limit of $S_F$

$$
m_0 Z^{-1}_2 = \lim_{p \to \infty} \frac{1}{4} \text{tr}[S_F(p)],
$$

(41)

$$
Z^{-1}_2 = \lim_{p \to \infty} \frac{1}{4} \text{tr}[\gamma \cdot p S_F(p)].
$$

(42)

To determine $Z^{-1}_2$, first we show free case

$$
S_F^{(0)}(x) = (i\gamma \cdot \partial) \frac{1}{4\pi \sqrt{-x^2}} = i\gamma \cdot \frac{x}{4\pi (-x^2)^{3/2}} = -\frac{i\gamma \cdot x}{\sqrt{-x^2} 4\pi x^2}.
$$

(43)

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where the dimension of the $S_F^{(0)}(x)$ is equal to $1/x^2$. In momentum space

$$\frac{1}{p} \int_0^\infty \frac{x \sin(px)}{p} x^{D-1} dx \sim p^{-3-D},$$  \hspace{1cm} (44)

and this shows the ordinary expression

$$\lim_{p \to \infty} \frac{1}{4} tr(\gamma \cdot pS_F(p)) \sim p^{-1-D}. \hspace{1cm} (45)$$

In this way we obtain

$$Z_2^{-1} \sim \lim_{p \to \infty} p^{-1-D} = \begin{cases} 0 & (0 < D) \\ 1 & (0 = D) \end{cases} \hspace{1cm} (46)$$

In the same way to evaluate $Z_2^{-1}$, bare mass reads

$$\lim_{p \to \infty} \frac{1}{4} p^2 tr(S_F(p)) = m \lim_{p \to \infty} p^2 \int_0^\infty \frac{x \sin(px)}{px} x^{D-1} dx \sim mp^{-D}, \hspace{1cm} (47)$$

in the high energy limit if we use the formula

$$m_0 Z_2^{-1} \sim m \lim_{p \to \infty} p^{-D} = 0, \hspace{1cm} (48)$$

and we obtain

$$\frac{m_0 Z_2^{-1}}{m} / Z_2^{-1} \geq 0. \hspace{1cm} (49)$$

Usually mass is a parameter which appears in the Lagrangian. For example chiral symmetry is defined for the bare quantity. In ref[4] the relation between bare mass and renormalized mass in the Schwinger-Dyson equation is discussed based on renormalization group equation and shown that the bare mass vanishes in the high energy limit even if we start from the finite bare mass in the theory. It suggests that symmetry properties must be discussed in terms of renormalized quantities. In QCD bare mass vanishes in the short distance by asymptotic freedom. And Dynamical mass vanishes too. In our approximation this problem is understood that the two kinds of mass are always generated in which the second order terms in the coupling constants is gauge dependent but another is gauge invariant. In our position space propagator we can easily see the differences between bare mass and dynamical mass. But in momentum space its relation is complicated. In this case the symmetry of massless fermion is realized in the short distance. Condition of the vanishment of the bare mass depends on
the ratio of the renormalized mass and fixed coupling constant. There is a chiral symmetry at short distance where the bare or dynamical mass vanishes but its breaking must be discussed in terms of the values of the order parameter. Therefore it is interesting to study the possibility of pair condensation in our approximation. The vacuum expectation value of pair condensate is evaluated

$$\langle \psi \psi \rangle = -2m \lim_{x \to 0^+} \frac{\exp(-mx)}{x} (\mu x)^{-Cx+D}$$

$$= \begin{cases} 0 & (1 < D) \\ \text{finite} & (1 = D) \\ \infty & (1 > D) \end{cases}$$

for fixed $\mu$. We do not see confinement ($Z_2 \neq 0$) but condensation (finite $\langle \bar{\psi} \psi \rangle$) occurs at $D = 1$. It is understood that in QCD the anomalous dimension of quark or $D$ in our model vanishes in the high energy limit thus $Z_2^{-1} = \text{finite}$. In that case the vanishment of the bare mass is automatically satisfied [4]. In the weak coupling limit we obtain $Z_2 = 1, m_0 = m$ and $\langle \bar{\psi} \psi \rangle = \infty$. If we introduce chiral symmetry as global $SU(n)_L \times SU(n)_R$, it breaks dynamically into $SU(n)_V$ as in QCD [3] for $D = 1$. Our model is also applicable to relativistic model of super fluidity in three dimension.

IV. MASS SINGULARITY IN MOMENTUM SPACE

In this section we study the effect of position dependent mass, self energy in momentum space

$$M(x) = \frac{e^2}{8\pi} \ln(\mu x),$$

$$\text{self energy} = \frac{e^2}{8\pi} \ln(\mu x).$$

The position space free propagator

$$S_F(x, m_0) = -(i\gamma \cdot \partial + m_0) \frac{\exp(-m_0 x)}{4\pi x}$$

is modified by these two terms which are related to dynamical mass and wave function renormalization. To see this let us think about the effect of spectral function in position space propagator

$$\sigma(x) = \frac{\exp(-m_0 + \frac{e^2 \gamma}{8\pi} x)}{4\pi x} \mu x)^{-\frac{2}{e^2} (\mu x)^{\frac{2}{m^2}}.$$
It is easy to see that the probability of particles which are separated with each other in the long distance is suppressed by the factor \((\mu x)^{-Cx}\), and the self energy modifies the short distance behaviour from the bare \(1/x\) to \(1/x^{1-D}\). The effect of self energy for the infrared behaviour of the free particle with mass \(m\) can be seen by its Fourier transform\([1]\)

\[
\int_0^\infty x^2 \frac{\sin(px)}{px} \frac{\exp(-mx)}{4\pi x} (\mu x)^a dx = -\Gamma(a + 1) \cos \left( \frac{\pi a}{2} \right) (p^2 + m^2)^{-1-a/2} \mu^a \\
\times \left[ (\sqrt{-p^2} + m)(\frac{\sqrt{-p^2} - m}{\sqrt{-p^2} + m})^{-a/2} + (\sqrt{-p^2} - m)(\frac{\sqrt{-p^2} + m}{\sqrt{-p^2} - m})^{-a/2} \right] \\
\sim (\sqrt{-p^2} - m)^{-1-a} \text{ near } p^2 = m^2. \quad (55)
\]

Usually constant \(D\) represents the coefficient of the leading infrared divergence for fixed mass in four dimensions. Therefore self energy has the same effects in three dimensions as in four dimensions\([5]\). In the non-linear Dyson-Schwinger equation of the fermion propagator or operator product expansion we can evaluate the dynamical mass which depends on momentum\([4]\). Here we show that \(M(x)\) satisfies the same property as the dynamical mass. Let us define Fourier transform of the scalar part of the propagator

\[
F.T(\frac{\exp(-mx)}{4\pi x} (\mu x)^{-Cx+D}) = \sigma_E(p) = \frac{m}{2\pi p} \int \sin(px) \exp(-mx)(\mu x)^{-Cx+D} dx. \quad (56)
\]

The momentum dependence of \(\sigma_E(p)\) agrees well to the numerical solution of the Dyson-Schwinger equation with vanishing bare mass in the Landau gauge. It dumps as \(p^{-2-D}\) and stays constant at small \(p\)[6]. It is easy to see that short distance behaviour is described by wave renormalization part \((\mu x)^D\) and the long distance behaviour is given by \(\exp(-M(x)x) = (\mu x)^{-Cx}\). The effect of \((\mu x)^{-Cx}\), first we check the \(o(e^2)\) contribution

\[
\sigma_M(p) = -\frac{e^2}{8\pi} I_2, \quad (57)
\]

\[
I_2 = \int_0^\infty x^2 \frac{\sin(px)}{px} \frac{\exp(-mx)}{4\pi x} x \ln(\mu x) dx \\
= \frac{-m}{(p^2 + m^2)^2} \left[ \ln \left( \frac{m - \sqrt{-p^2}}{m + \sqrt{-p^2}} \right) + \ln \left( \frac{m^2 + p^2}{\mu^2} \right) - 2(1 - \gamma) \right]. \quad (58)
\]

Thus a second-order correction of the propagator dumps as \((O(1) + \ln(p^2/\mu^2))^2 e^2/p^4\) at high energy. It is not easy to see the effect of the factor \((\mu x)^{-Cx}\) in momentum space. Therefore the scalar part of the propagator \(m/(m^2 + p^2)\) receives an additional correction by \((\mu x)^{-Cx}\). If the dynamical symmetry breaking and mass generation occur it has been discussed that the propagator will have a branch point on the real \(p^2\) axis[7]. However the analysis in Minkowski
space is difficult due to the fact that the function \((\mu x)^{-C x}\) cannot be integrated analytically. Therefore we see only the perturbative results. For this purpose it may be helpful to study the Minkowski space by replacement from Euclidean distance \(R = \sqrt{x^2} \rightarrow iT\) in \(\sigma(x)[8]\),

\[
\sigma(T) \simeq \frac{\exp(-imT)}{2\pi m} (\mu iT)^{-iCT + D} = \frac{\exp(-imT)}{2\pi m} (\mu T)^{-iCT + D} \exp\left(\frac{\pi}{2} i(-iCT + D)\right) \tag{59}
\]

provided

\[
(i\mu T)^{-iCT + D} = \exp\left(\frac{\pi}{2} (C + Di)\right)^D (\cos(CT \ln(\mu T) + mT) - i\sin(CT \ln(\mu T) + mT)), \tag{60}
\]

Here we notice that \(\sigma(T)\) becomes complex valued function. Fourier transform into energy is given

\[
\sigma(E) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos(ET) \sigma(T) dT, \tag{62}
\]

\[
\int dE \sigma(E) = -\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{\sin(ET)}{T} \sigma(T) dT = -\lim_{T \to \infty} \frac{1}{2T} \pi \int_{-T}^{T} \delta(E) \sigma(T) dT. \tag{63}
\]

In free case integration over space gives

\[
\sigma_0(t) = \int dx^2 \frac{\exp(-m\sqrt{-t^2 + x^2})}{4\pi \sqrt{-t^2 + x^2}} = \frac{\exp(-imt)}{2\pi m}. \tag{64}
\]

\[
\int \sigma_0(E) dE = \lim_{T \to \infty} \frac{1}{2T} \frac{2\pi \sin(mT)}{\pi m^2}
\]

Thus energy distribution for free fermion becomes

\[
\sigma_0(E) = \frac{1}{2T} \int_{-T}^{T} \cos(ET) \sigma_0(t) dt = \frac{m \sin(mT) \cos(ET) + E \cos(mT) \sin(ET)}{2\pi Tm(E^2 - m^2 + i\epsilon)}. \tag{65}
\]

Taking imaginary part

\[
\Im \sigma_0(E) = \frac{\sin(2mT)}{2T} \tag{66}
\]

gives the residue at \(E^2 = m^2\). In free case imaginary part is \(\delta(E^2 - m^2)\) and the continuum contribution is given by an integral of the principal part in the region given by \(\theta(E^2 - m^2)\). By numerical analysis using Maple we find that \(\sigma(E)\) oscillates and diverges in the region

\[
E \simeq m_{eff} = m_0 + \frac{e^2 \gamma}{8\pi} + C \tag{67}
\]
V. SUMMARY

Infrared behaviour of the propagator was examined in the Bolch-Nordsieck approximation to QED\textsubscript{3}. Even in the Yennie gauge there is a logarithmic infrared divergence but free from linear infrared divergences. Position dependent mass and self energy are both gauge independent in our approximation. Therefore in three dimension we examine the structure of the propagator in this gauge. In our approximation confinement property is given by position dependent mass. We evaluated the renormalization constant and bare mass based on spectral function sum rule and dynamical mass in momentum space is analysed with finite infrared cutoff. In our approximation there seems to be a critical coupling constant for vacuum expectation value $\langle \overline{\psi} \psi \rangle$ which is due to wave function renormalization. This is a clear difference between lowest ladder approximation to the Schwinger-Dyson equation with bare vertex and ours. In Euclidean space we see the momentum dependence of the propagator which has been known in the analysis of the Schwinger-Dyson equations except for the wave function renormalization[6].

VI. REFERENCES

[1] Y. Hoshino, JHEP05(2003) 075; R. Jackiw, L. Soloviev, Phys. Rev. 173 (1968) 1458.
[2] S. Deser, R. Jackiw, S. Temploton, Ann. Phys. (NY) 140 (1982) 372.
[3] T. Appelquist, R. Pisarski, Phys. Rev. D23 (1981) 2305; T. Appelquist, U. Heinz, Phys. Rev. D24 (1981) 2305.
[4] K. Nishijima, Prog. Theor. Phys. 81 (1989) 878; K. Nishijima, Prog. Theor. Phys. 83 (1990) 1200.
[5] L. S. Brown, Quantum Field Theory, Cambridge University Press (1992).
[6] T. Appelquist, D. Nash, L. C. R. Wijewardhana, Critical behaviour in (2+1)-dimensional QED, Phys. Rev. Lett. 60 (1988) 1338; Y. Hoshino, T. Matsuyama, Phys. Lett. B222 (1989) 493.
[7] D. Atkinson, D. W. E. Blatt, Nucl. Phys. 151B (1979) 342; P. Maris, Phys. Rev. D52 (1995);
Y. Hoshino, Il. Nouvo. Cim. 112A (1999) 335.
[8] J. Schwinger, Particles, Sources, and Fields, volume I (1970), Perseus Books Publishing, L, L, C.