Free left regular bands: irreducible algebraic sets and universal theories

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Abstract

We consider the algebraic geometry over free left regular (f.l.r.) bands and describe coordinate bands associated to irreducible algebraic sets. Meanwhile, we describe finite subsemigroups of a f.l.r. bands, universal classes and theories.

1 Introduction

Universal algebraic geometry — is a branch of mathematics, where the equations over different algebraic structures (groups, algebras, lattices, etc.) are studied. The series of the papers [1, 2, 3] by E. Daniyarova, A. Miasnikov, V. Remeslennikov allows us to develop algebraic geometry over any algebraic structure in a language with no predicates. Recently there are appeared various algebraic geometries over specific algebraic structures. Mainly equations are studied over free objects: free groups ([4, 5, 6]), free commutative monoids ([7, 8, 9]), free Lie algebras ([10, 11]), free metabelian Lie algebras ([12]), free semilattices ([13]).

In the current paper we deal with semigroups defined by the identities $x^2 = x$, $xyx = xy$ (left regular bands) and develop an algebraic geometry over the free left regular band $F$ of an infinite rank. The main result of this paper is Theorem 3.1 that describes coordinate bands of irreducible algebraic sets (irreducible coordinate bands) over $F$.

Let us give the statement of this theorem (all essential definitions can be found in Section 2).

Theorem 3.1 For any finitely generated left regular band $S$ the next conditions are equivalent:

1. $S$ is an irreducible coordinate band over $F$;
2. $S$ is embeddable into $F$;
3. $S$ is a right hereditary band;
4. $S \models \varphi$, where

$$\varphi: \forall x_1, x_2, y_1, y_2 (x_1y_1 = x_2y_2 \rightarrow [(x_1 \leq x_2) \vee (x_2 \leq x_1)])$$

Moreover, Theorem 3.1 describes finitely generated subsemigroups of $F$ and allows us to decompose an algebraic set $Y$ over $F$ into a union of its irreducible subsets (irreducible components); this process is shown in Example 3.12.

Finally, Theorem 3.13 contains model-theoretic results deduced from Theorem 3.1.
2 Bands and equations

By \( F \) we shall denote the join free semilattice of an infinite rank generated by \( \{ a_i | i \in I \} \). We always assume that \( F \) contains the minimal element \( \varepsilon \). It is well-known fact that \( F \) is isomorphic to the family of all finite subsets \( \{ a_{i_1}, a_{i_2}, \ldots, a_{i_n} \} \) equipped the operation of the set union (the element \( \varepsilon \in F \) is represented by the empty set). Clearly, the inclusion of sets \( \subseteq \) induces a partial order over \( F \).

Further we denote elements of \( F \) by bold letters \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \).

For \( \mathbf{b}, \mathbf{c} \in F \) one can define \( \mathbf{c} \setminus \mathbf{b} \in F \) as the set difference. Secondly, the length \( |a| \) of \( a \in F \) is the cardinality of its representation set.

Let us introduce a linear order \( \leq F \) over \( F \) by

1. for any \( n \) the set \( \{ a \in F | |a| = n \} \) is arbitrarily ordered;
2. put \( a \leq F b \) for all \( |a| < |b| \).

The following statement is well-known in semilattice theory.

**Theorem 2.1.** Any finitely generated join semilattice is finite, and moreover it is embedded into \( F \).

According Theorem 2.1, we shall consider elements of any finite semilattice \( L \) as elements of \( F \) (with respect to a fixed embedding). Hence, one can assume that for any elements \( \mathbf{b}, \mathbf{c} \in L \) it is defined \( |\mathbf{b}|, |\mathbf{c}|, \mathbf{b} \setminus \mathbf{c} \).

A semigroup \( S \) is a left regular band if the next identities

\[
x^2 = x, \ xyx = xy
\]

hold in \( S \).

The identities above clearly give the next result.

**Theorem 2.2.** Any finitely generated left regular band is finite.

By \( \mathcal{F}_n (n \in \{ 0, 1, \ldots \} \cup \{ \infty \}) \) we denote the free left regular of the rank \( n \). The elements of \( \mathcal{F}_n \) are all words \( w \) in the alphabet \( A = \{ a_1, a_2, \ldots, a_n \} \) such that any letter \( a_i \) occur at most one time in \( w \). The product of two elements \( w_1, w_2 \in \mathcal{F}_n \) is defined as follows:

\[
w_1w_2 = w_1 \circ (w_2) \exists,
\]

where \( \circ \) is the word concatenation and the operator \( \exists \) is the deletion of all letters which occur earlier.

For elements \( \mathbf{x}, \mathbf{y} \) of a left regular band \( S \) one can define a relation

\[
x \leq y \iff xy = y.
\]

It is easy to check that \( \leq \) is a partial order over \( S \).

For elements \( \mathbf{x}, \mathbf{y} \in \mathcal{F}_n \) the relation \( \mathbf{x} \leq \mathbf{y} \) means that the word \( \mathbf{x} \) is a prefix of \( \mathbf{y} \).

Define the another relation on elements of a left regular band \( S \) by

\[
x \preceq y \iff yx = y.
\]

This relation is reflexive and transitive, but not necessarily antisymmetric.

For the free left regular band \( \mathcal{F}_n \) the relation \( \mathbf{x} \preceq \mathbf{y} \) means that the word \( \mathbf{y} \) contains all letters of the word \( \mathbf{x} \).

One can get a poset \( L \) by identifying \( \mathbf{x} \) and \( \mathbf{y} \) if \( \mathbf{x} \preceq \mathbf{y} \) and \( \mathbf{y} \preceq \mathbf{x} \). Let \( \sigma : S \to L \) denote the quotient map. \( L \) is called the support semilattice of \( S \) and \( \sigma : S \to L \) is
called the support map. Indeed, $L$ is a join semilattice. Remark that for any $s \in F_n$ the map $\sigma(s)$ gets the set of all letters occurring in $s$.

Following [14], a left regular band $S$ with an identity element $\varepsilon$ is called right hereditary if the Hasse diagram of the order $\leq$ is a tree with the minimal element $\varepsilon$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of variables, and $F(X)$ be the free left regular band generated by the set $X$. The elements of $F(X)$ are called terms.

An equality of two terms $t(X) = s(X)$ is called an equation. For example, the next expressions are equations $x_1 x_2 = x_2 x_1$, $x_1 x_2 x_3 = x_3 x_4$.

One can naturally define the solution set $V_S(t(X) = s(X))$ of an equation $t(X) = s(X)$ in a left regular band $S$. An arbitrary set of equations is called a system of equations (system for shortness). Remark that we always consider systems depending on at most finite set of variables. The solution set of a system is the intersection of solution sets of its equations.

A set $Y \subseteq S^n$ is called algebraic if there exists a system of equations with the solution set $Y$. An algebraic set is irreducible if it is not a proper finite union of algebraic sets.

Let $Y \subseteq S^n$ be an algebraic set over a left regular band $S$. Terms $s(X), t(X)$ are $Y$-equivalent if they have the same values at any point $P \in Y$. The set of such equivalence classes forms a left regular band $\Gamma_S(Y)$ which is called the coordinate band of $Y$ (see [1] for more details).

For example, in the solution set $Y \subseteq F^3$ of a system $\{yx = zx, yz = z\}$ the terms $z, yz$ are $Y$-equivalent.

A coordinate band which corresponds to an irreducible algebraic set is called irreducible. A coordinate band determines an algebraic set up to isomorphism (the isomorphism of algebraic sets was defined in [1]). Thus, one can consider the aim of algebraic geometry as the classification of coordinate bands.

A left regular band $S$ is discriminated by a band $T$ if for any distinct $s_1, s_2, \ldots, s_n \in S$ there exists a homomorphism $\psi: S \to T$ with $\psi(s_i) \neq \psi(s_j)$ for all $i \neq j$.

Let $\varphi$ be a first-order sentence of the language $\{\cdot\}$, where $\cdot$ is a binary operation. If a formula $\varphi$ holds in a band $S$ it is denoted by $S \models \varphi$. A formula $\varphi$ is universal if it is equivalent to

$$\forall x_1 \forall x_2 \ldots \forall x_n \varphi'(x_1, x_2, \ldots, x_n),$$

where $\varphi'$ is quantifier-free.

The universal closure $\text{Ucl}(S)$ of a left regular band $S$ consists of all bands $T$ such that $T \models \varphi$ for any universal formula $\varphi$ with $S \models \varphi$.

The next theorems were proved in [1, 2] for any algebraic structure, but we formulate them for left regular bands. In all theorems below $S, T$ are left regular bands and $T$ is finitely generated.

**Theorem 2.3.** [1] $T$ is an irreducible coordinate band $S$ iff $T$ is discriminated by $S$.

**Theorem 2.4.** [1] If $T$ is an irreducible coordinate band over $S$ then $T \in \text{Ucl}(S)$.

# 3 Irreducible coordinate semilattice over $F$

Denote by $F = \{a_i | i \in I\}$ the free left regular band of on infinite rank. The aim of this section is to prove the next theorem.
Theorem 3.1. For any finitely generated left regular band $S$ the next conditions are equivalent:

1. $S$ is an irreducible coordinate band over $F$;
2. $S$ is embeddable into $F$;
3. $S$ is a right hereditary band;
4. $S \models \varphi$, where

$$\varphi: \forall x_1, x_2, y_1, y_2 \ (x_1 y_1 = x_2 y_2 \rightarrow [(x_1 \leq x_2) \lor (x_2 \leq x_1)]) \quad (1)$$

Let us divide the proof into four subsections.

3.1 $(1) \Rightarrow (4)$

The formula $\varphi$ means that the equality of two elements $x_1 y_1, x_2 y_2$ implies comparability of their origins. It obviously holds in $F$. By Theorem 2.4 $\varphi$ holds in any coordinate band of an irreducible algebraic set over $F$.

3.2 $(3) \iff (4)$

$(4) \Rightarrow (3)$. Take elements $x \leq z, y \leq z$ of the band $S$. Hence, $xz = z, yz = z$ and $x = yz$. By formula $\varphi$, we have either $x \leq y$ or $y \leq x$. Hence the Hasse diagram of the order $\leq$ is a tree.

$(3) \Rightarrow (4)$. Take $x_1, x_2, y_1, y_2 \in S$ such that $A = x_1 y_1 = x_2 y_2$. We have $x_1 \leq A, x_2 \leq A$. As the Hasse diagram of the order $\leq$ is a tree, it holds either $x_1 \leq x_2$ or $x_2 \leq x_1$. Thus, $S \models \varphi$.

3.3 $(2) \Rightarrow (1)$

As any subsemigroup $S$ of $F$ is obviously discriminated by $F$, $S$ is coordinate band of an irreducible algebraic set over $F$ (Theorem 2.3).

3.4 $(3) \Rightarrow (2)$

In this section we always assume that a finitely generated left regular band $S$ with the identity element $\varepsilon$ is right hereditary (or equivalently $S \models \varphi$). By Theorem 2.2, $S$ is finite.

The plan of the proof is the following. Firstly, for any right hereditary band $S$ we define a homomorphism $h_S: S \to F$ (Lemma 3.7). The map $h_S$ is not necessarily an embedding, hence further we define a series of right hereditary bands $S = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n$ such that the homomorphism $h_{S_n}(S_n)$ is an embedding of $S_n$ into $F$ (Theorem 3.11).

By the condition, the Hasse diagram of the order $\leq$ is a tree with the root $\varepsilon$. It allows us to define the ancestor $\alpha(s) \in S$ for any $s \neq \varepsilon$.

Let $a, b, c \in S$, define

$$a \sim_c b \iff ac = bc.$$  

Obviously, $\sim_c$ is an equivalence relation.

For any $c \neq \varepsilon$ let us define a set $S_c \subseteq S$ by

$$S_c = \{s \mid s \sim_c \alpha(s), s \sim_{\alpha(c)} \alpha(s)\} \quad (2)$$
Notice that $S_c$ is not empty, since it always contains $c$. The set $S_c$ has the next obvious property.

**Lemma 3.2.** For any $y \in S_c$, $b < c$ we have $y \sim_b \alpha(y)$.

**Proof.** Assume $y \sim_b \alpha(y)$ for some $b \leq \alpha(c) < c$. Let us multiply the equality $yb = \alpha(y)b$ by $\alpha(c)$ and obtain $ya(c) = \alpha(y)\alpha(c)$. Thus, $y \sim_{\alpha(c)} \alpha(y)$ and we come to the contradiction with the definition of the set $S_c$.

Let $\leq_S$ be an arbitrary linear order over $S$. Put $a \sqsubseteq b$ for $a, b \in S \setminus \{\varepsilon\}$ if

1. $\sigma(a) \setminus \sigma(\alpha(a)) <_{\mathcal{F}} \sigma(b) \setminus \sigma(\alpha(b))$;
2. $a \leq_S b$ if $\sigma(a) \setminus \sigma(\alpha(a)) =_{\mathcal{F}} \sigma(b) \setminus \sigma(\alpha(b))$.

For any $c \in S$ define the vector

$$\chi(c) = (\chi(\alpha(c)), y_1, y_2, \ldots, y_{n_c})$$

such that

1. the sequence $y_1, y_2, \ldots, y_{n_c}$ is a permutation of all elements of the set $S_c$ ($|S_c| = n_c$);
2. $y_i \sqsubseteq y_{i+1}$;
3. $\chi(\varepsilon)$ is an empty vector.

One can recursively apply (3) to $\alpha(c)$ and obtain a new vector

$$\chi(c) = (\chi(\alpha(c)), x_1, x_2, \ldots, x_{n(\alpha(c))}, y_1, y_2, \ldots, y_{n_c})$$

It allows us to descend to any element $a \leq c$ obtaining the vector:

$$\chi(c) = (\chi(a), \ldots, y_1, y_2, \ldots, y_{n_c}).$$

**Lemma 3.3.** For any element $x$ from the vector (4) it holds $x \sim_c \alpha(x)$.

**Proof.** By the definition of the vector (4), there exists $c' \leq c$ such that $x_i \in S_{c'}$, i.e. $x \sim_{c'} \alpha(x)$ or $xc' = \alpha(x)c'$.

Let us multiply the last equality by $c$ and obtain $xc = \alpha(x)c$. Thus, $x \sim_c \alpha(x)$.

For any $s \in S \setminus \{\varepsilon\}$ let us define the strict support $\Sigma(s) \in \mathcal{F}$ by the next process.

**$\Sigma$-process**

**Step 0.** Set $\Sigma(s) := \sigma(s) \setminus \sigma(\alpha(s))$ for all $s \in S$.

**Step 1.** For any triple $(y, z, c) \in S^3$ and an element $s = \alpha(z)\alpha(y)z$ such that

$$y \sim_c \alpha(y), \quad z \sim_c \alpha(z), \quad z \sqsubseteq y, \quad \Sigma(s) \subset \Sigma(z)$$

put $\Sigma(s) := \Sigma(z)$.

**Step 2.** Repeat step 1 until one can find such $y, z, c$.

**Lemma 3.4.** For any $s \in S$ we have

$$\sigma(s) \setminus \sigma(\alpha(s)) \subseteq \Sigma(s) \subseteq \sigma(s),$$

and moreover, the $\Sigma$-process terminates after a finite number of steps.
Lemma 3.5. Let

where

For any

Let us prove the second inclusion by the induction on the number of executions of the Step 1. After the first execution of the Step 1 we have

\[ \Sigma(\alpha(z)\alpha(y)z) = \Sigma(z) = \sigma(z) \subseteq \sigma(\alpha(z)) \subseteq \sigma(\alpha(z)\alpha(y)z) = \sigma(\alpha(y)) \cup \sigma(z) = \sigma(\alpha(z)\alpha(y)z). \]

Assume that after \( n \) executions of the Step 1 the inclusion \( \Sigma(s) \subseteq \sigma(s) \) holds and the \( n+1 \)-th Step 1 is applied for a triple \( (y, z, c) \). We have

\[ \Sigma(\alpha(z)\alpha(y)z) = \Sigma(z) \subseteq \sigma(z) \subseteq \sigma(\alpha(y)) \cup \sigma(z) = \sigma(\alpha(z)\alpha(y)z) \]

that concludes the proof.

As \( \Sigma(a) \) is bounded above by \( \sigma(s) \) and \( |\Sigma(s)| \) increases, the \( \Sigma \)-process terminates after finite number of steps. \( \square \)

Fix an arbitrary linear order of free generators of \( F \)

\[ a_1 < a_2 < \ldots \]

For any \( a \in F \) the element \( \overline{a} \in F \) is obtained from \( a \) by the ordering of its letters. For instance, \( \{a_1, a_3, a_5\} = a_1a_3a_5 \in F \).

Now we are able to define a map \( h_S: S \rightarrow F \) of a right hereditary band \( S \).

\[ h_S(c) = \begin{cases} \varepsilon, & \text{if } c = \varepsilon, \\ h_S(\alpha(c))c_1c_2\ldots c_n, & \text{otherwise} \end{cases}, \quad (6) \]

where \( \chi(c) = (\chi(\alpha(c)), y_1, y_2, \ldots, y_n) \) and \( c_i = \overline{\Sigma(y_i)} \).

Lemma 3.5. Let \( S \) be right hereditary band. Suppose \( h_S \) preserves the order \( \preceq \), and equivalence relation \( \sim_c \), i.e. for any \( a, b, c \in S \) we have:

\[ a \preceq b \Rightarrow h_S(a) \preceq h_S(b), \]

\[ a \sim_c b \Rightarrow h_S(a) \sim_{h_S(c)} h_S(b). \]

Then \( h_S \) is a homomorphism.

Proof. Let us prove \( h_S(xy) = h_S(x)h_S(y) \) for all \( x, y \in S \). Denote \( z = xy \), hence \( zy = xy \) and \( z \sim_y x \). By the condition, \( h_S(z)h_S(y) = h_S(x)h_S(y) \). As \( y \preceq z \), we have \( h_S(y) \preceq h_S(z) \) and obtain \( h_S(z) = h_S(x)h_S(y) \). \( \square \)

Lemma 3.6. For any \( c \in S \) it holds \( \sigma(h_S(c)) = \sigma(c) \).

Proof. If \( c = \varepsilon \) the equality obviously holds. Assume now that the equality holds for the ancestor \( \alpha(c) \)

\[ \sigma(h_S(\alpha(c))) = \sigma(\alpha(c)) \]

Let \( \chi(c) \) be the vector (3), and the image \( h_S(c) \) is defined by (6). By Lemma 3.4, we have

\[ \sigma(h_S(c)) = \sigma(h_S(\alpha(c))) \cup \Sigma(y_i) \supseteq \sigma(\alpha(c)) \cup (\sigma(y_i) \setminus \sigma(\alpha(y_i))). \]

As \( c \in S_c \), there exists an index \( i \) such that \( y_i = c \), hence

\[ \sigma(h_S(c)) \supseteq \sigma(\alpha(c)) \cup (\sigma(c) \setminus \sigma(\alpha(c))) = \sigma(c). \quad (7) \]
Let us prove the contrary inclusion by the induction of the executions of the Step 1 in the \( \Sigma \)-process.

Before the first execution of the Step 1 we have \( \Sigma(y_i) = \sigma(y_i) \setminus \sigma(\alpha(y_i)) \) for all \( y_i \).

By the choice of \( y_i \), we have \( y_i \in S_c \), hence \( y_ic = \alpha(y_i)c \) and \( \sigma(y_i) \cup \sigma(c) = \sigma(\alpha(y_i)) \cup \sigma(c) \). By the properties of \( F \), it follows
\[
\sigma(c) \supseteq (\sigma(y_i) \setminus \sigma(\alpha(y_i))) = \Sigma(y_i).
\]

Thus,
\[
\sigma(h_S(c)) = \sigma(h_S(\alpha(c))) \bigcup_i \Sigma(y_i) = \sigma(\alpha(c)) \bigcup_i (\sigma(y_i) \setminus \sigma(\alpha(y_i))) \subseteq \sigma(\alpha(c)) \cup \sigma(c) = \sigma(c).
\]

Assume that after the \( n \)-th launching of the Step 1 the inclusion \( \sigma(h_S(c)) \subseteq \sigma(c) \) holds. Suppose the triple \( (y, z, b) \) is chosen at the \((n + 1)\)-th execution of the Step 1. If the element \( s = \alpha(z)\alpha(y)z \) does not occur in the vector \( \chi(c) \) the assignment of the Step 1 do not change \( h_S(c) \) and the inclusion \( \sigma(h_S(c)) \subseteq \sigma(c) \) still holds. Otherwise, there exists \( y_s = \alpha(z)\alpha(y)z \in \chi(c) \). As \( y \sim_c \alpha(y) \), \( z \sim_c \alpha(z) \), the elements \( y, z \) occur in the vector \( \chi(c) \) (probably, \( y \) or \( z \) belong to the subvector \( \chi(\alpha(c)) \)). By the assumption of the induction we have
\[
\sigma(h_S(c)) = \sigma(\alpha(c)) \cup \Sigma(y_k) \cup \Sigma(z) \bigcup_{i \neq k} \Sigma(y_i) \subseteq \sigma(c).
\]

By \( \Sigma(y_k) \subseteq \Sigma(z) \), we come to
\[
\sigma(h_S(c)) = \sigma(\alpha(c)) \cup \Sigma(z) \bigcup_{i \neq k} \Sigma(y_i) \subseteq \sigma(c),
\]
and after the assignment of the Step 1 we obtain
\[
\sigma(h_S(c)) = \sigma(\alpha(c)) \cup \Sigma(z) \bigcup_{i \neq k} \Sigma(y_i) = \sigma(\alpha(c)) \cup \Sigma(z) \bigcup_{i \neq k} \Sigma(y_i) \subseteq \sigma(c)
\]
that proves the inclusion
\[
\sigma(h_S(c)) \subseteq \sigma(c) \tag{8}
\]

Finally, the inclusions \((7, 8)\) conclude the proof. \( \square \)

**Lemma 3.7.** The map \( h_S: S \to F \) is a homomorphism of a right hereditary band \( S \).

**Proof.** By Lemmas 3.5, 3.6, it is sufficient to prove the implication
\[
a \sim_c b \Rightarrow h_S(a) \sim_{h_S(c)} h_S(b).
\]

According the formula \( \varphi \), the equality \( ac = bc \) gives \( a \leq b \) (similarly, one can put \( b \leq a \)).

By (4), one can write
\[
h_S(b) = h_S(a)b_1b_2\ldots b_n, \tag{9}
\]
\[
h_S(c) = c_1c_2\ldots c_m. \tag{10}
\]
By the definition of $h_S$, for any $b_i$ (resp. $c_j$) there exist $y_i \in S \ (z_j \in S)$ such that $b_i = \Sigma(y_i)$ ($c_j = \Sigma(z_j)$). Denote

$$Y = \{y_i|1 \leq i \leq n\}, \ Z = \{z_j|1 \leq j \leq m\}.$$ 

By Lemma 3.3, $y_i \sim_b \alpha(y_i)$, $z_j \sim_c \alpha(z_j)$ for all $1 \leq i \leq n, \ 1 \leq j \leq m$. Using $y_i b = \alpha(y_i) b$ and $b \leq ac$, we have the next chain

$$\alpha(y_i) \leq y_i \leq \alpha(y_i) b \leq \alpha(y_i) ac.$$ 

As the Hasse diagram of the order $\leq$ is tree, the chain above should contain $\alpha(y_i) a$, since $\alpha(y_i) \leq \alpha(y_i) a \leq \alpha(y_i) ac$. If $y_i \leq \alpha(y_i) a$ we have $\alpha(y_i) \sim_a y_i$ and we obtain the contradiction with the definition of the set $S_b$ (Lemma 3.2). Thus, $\alpha(y_i) \leq \alpha(y_i) a < y_i$. As $\alpha(y_i)$ is the ancestor of $y_i$, we obtain $\alpha(y_i) = \alpha(y_i) a$.

Since $\alpha(y_i) \leq y_i \leq \alpha(y_i) ac = \alpha(y_i) c$, hence $\alpha(y_i) \sim_c y_i$. By the definition of $h_S$, the set $Z$ contains all $y_i$. In other words, there exists a function $f: \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, m\}$ which embeds $Y$ into $Z \ (y_i = z_{f(i)})$.

As the order the elements in the vectors $\chi(b), \chi(c)$ depends only on the order $\leq_F$, the function $f$ is monotone: $i_1 < i_2 \rightarrow f(i_1) < f(i_2)$.

Let $z_{f(k)} = y_k \in Y \subseteq Z$ and $z_j \in Z \setminus Y$ such that $j < f(k)$ and there is not any $z_{j'} \in Y \subseteq Z$ with $j < j' < f(k)$.

Consider an element $s = \alpha(z) \alpha(y_k) z_j$. The triple $(y_k, z_j, c)$ satisfies the condition (5), hence $\Sigma(s)$ after the $\Sigma$-process assignments satisfies

$$\Sigma(s) \supseteq \Sigma(z_j). \quad (11)$$ 

Let us prove the inequality

$$s \leq \alpha(z_j) \alpha(y_k) y_k$$ 

As $z_j \sim_c \alpha(z), \ y_k \sim_c \alpha(y_k)$, we have $z_j c = \alpha(z_j) c, \ y_k c = \alpha(y_k) c$. Let us multiply the both equalities by $\alpha(z_j) \alpha(y_k)$ and obtain $\alpha(z_j) \alpha(y_k) z_j c = \alpha(z_j) \alpha(y_k) \alpha(z_j) c = \alpha(z_j) \alpha(y_k) c, \ \alpha(z_j) \alpha(y_k) y_k c = \alpha(z_j) \alpha(y_k) \alpha(y_k) c = \alpha(z_j) \alpha(y_k) c$. Hence

$$\alpha(z_j) \alpha(y_k) z_j c = \alpha(z_j) \alpha(y_k) y_k c.$$ 

By the formula $\varphi$, we have either $s = \alpha(z_j) \alpha(y_k) z_j \leq \alpha(z_j) \alpha(y_k) y_k$ or $s = \alpha(z_j) \alpha(y_k) z_j > \alpha(z_j) \alpha(y_k) y_k$. The first inequality immediately gives (12), therefore we assume the second one. We have the next inclusion in $F$:

$$\sigma(s) = \sigma(\alpha(z_j)) \cup \sigma(\alpha(y_k)) \cup \sigma(z_j) \supseteq \sigma(\alpha(z_j)) \cup \sigma(\alpha(y_k)) \cup \sigma(y_k).$$

From set theory it follows $\sigma(z_j) \setminus \sigma(\alpha(z_j)) \supseteq \sigma(y_k) \setminus \sigma(\alpha(y_k))$. By the definition of the order $\sqsubseteq$, we obtain $z_j \sqsubseteq y_k$ that contradicts with the order of $z_j, z_{f(k)}$ in the vector $\chi(c)$. Thus, the inequality (12) holds, and we have the next chain of elements

$$\alpha(z_j) \alpha(y_k) \leq \alpha(z_j) \alpha(y_k) z_j \leq \alpha(z_j) \alpha(y_k) y_k = \alpha(z_j) y_k \leq \alpha(z_j) y_k b = \alpha(z_j) \alpha(y_k) b$$

(in the last equality we use $y_k \sim_b \alpha(y_k)$). It follows $s \sim_b \alpha(s)$, and $s$ should occur in $\chi(b)$.

As

$$\sigma(s) \setminus \sigma(\alpha(s)) \subseteq \sigma(s) \setminus \sigma(\alpha(z_j) \alpha(y_k)) \subseteq \sigma(z_j) \setminus \sigma(\alpha(z_j)),$$

Since the subvector $(z_j, \ldots, z_{f(k)}) \subseteq \chi(c)$ does not contain any element from $Y$, $s$ occurs earlier than $z_j$ in $\chi(c)$. 
Thus, we prove above the next statement: for any \( y_k \in Y \subseteq Z \), \( z_j \in Z \setminus Y \) such that \( y_k \) occurs in \( \chi(c) \) on the right of \( z_j \) there exists \( s = \alpha(z_j) y_k z_j \in Y \subseteq Z \) with (11) and \( s \) occurs on the left of \( z_j \) in \( \chi(c) \). Moreover, \( s \) occurs on the left of \( y_k \) in the vector \( \chi(b) \).

Let us prove the equality

\[
b_1 b_2 \ldots b_n = c_1 c_2 \ldots c_{f(n)}. \tag{13}
\]

by the induction on \( n \).

Let \( n = 1 \) and \( y_1 \) be the first element of the set \( Y \). As we proved above, \( z_{f(1)} = y_1 \). If \( f(1) = 1 \) the equality (13) obviously holds. Otherwise, \( f(1) = k > 1 \) and there exists \( z_l \notin Y, l < k \). We proved above there exists an element \( s \in Y \subseteq Z \) on the left of \( z_l \) in the vector \( \chi(c) \). Hence, \( s \) should occur on the left of \( y_1 \) in the vector \( \chi(b) \). However, it is impossible, since \( y_1 \) is the first element (after \( \chi(a) \)).

Assume (13) holds for a number \( n - 1 \). Let us prove it for \( n \).

Denote \( k = f(n - 1), l = f(n) \). If \( f(n) = k + 1 \) there is not any element between the elements \( c_{f(n-1)}, c_{f(n)} \), and (13) is obviously holds. Assume now \( l > k + 1 \), hence the set \( Z_{kl} = \{ z_j | k < j < l \} \) is nonempty.

As we proved above, for any \( z_j \in Z_{kl} \) there exists \( y_{f^{-1}(j)} \in Y \subseteq Z \) such that

1. \( f^{-1}(j) \leq k \);
2. \( \Sigma(y_{f^{-1}(j)}) \supseteq \Sigma(z_j) \).

From the second item, \( c_j \leq b_{f^{-1}(j)} \) for any \( k < j < l \). Hence all \( \{ c_j | k < j < l \} \) are eliminated by the elements \( b_{f^{-1}(j)} \), which occur earlier in the right side of the equality (13). Thus, (13) holds.

By (13), we have

\[
h_{S}(a) h_{S}(c) = h_{S}(a) c_1 c_2 \ldots c_m = h_{S}(a) (c_1 c_2 \ldots c_{f(n)}) (c_{f(n)+1} c_{f(n)+2} \ldots c_m) = h_{S}(a) b_1 b_2 \ldots b_n c_{f(n)+1} c_{f(n)+2} \ldots c_m.
\]

On the other hand,

\[
h_{S}(b) h_{S}(c) = h_{S}(a) b_1 b_2 \ldots b_n c_1 c_2 \ldots c_m = h_{S}(a) b_1 b_2 \ldots b_n (c_1 c_2 \ldots c_{f(n)}) (c_{f(n)+1} c_{f(n)+2} \ldots c_m) = h_{S}(a) b_1 b_2 \ldots b_n (c_{f(n)+1} c_{f(n)+2} \ldots c_m).
\]

Thus, we obtain \( h_{S}(a) h_{S}(c) = h_{S}(b) h_{S}(c) \), i.e. \( h_{S} \) preserves the equivalence relation \( \sim_{c} \). By Lemma 3.5, \( h_{S} \) is a homomorphism. \( \square \)

The homomorphism \( h_{S} \) is not necessarily an embedding of \( S \) into \( \mathcal{F} \). Further we define a series of right hereditary bands

\[ S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq \mathcal{F}, \]

where any \( S_{i+1} \) is obtained from \( S_i \) by the adjunction of a new element.

Let \( T \) be an arbitrary right hereditary band, \( a \in T, a \neq \varepsilon \). By \( T(a) \) we denote the subsemigroup of the Cartesian square generated by the next set of elements

\[ \{(t, t) | t \in T\} \cup \{(\alpha(a), a)\}. \]

Obviously, \( T \) is embedded into \( T(a) \). However, \( (\alpha(a), \alpha(a)) \) is not the ancestor of \( (a, a) \) in \( T(a) \), since \( (\alpha(a), \alpha(a)) < (\alpha(a), a) < (a, a) \).

**Lemma 3.8.** If \( T \) is right hereditary band, so is \( T(a) \) for any \( a \in T \setminus \{\varepsilon\} \).
Proof. As \( T(a) \) is the extension of \( T \) by the element \((\alpha(a), a)\), any \( \bar{s} \in T(a) \) is the pair
\[
\bar{s} = (t\alpha(a)t', tat')
\]for some \( t, t' \in T \).

Let
\[
\bar{s}_1 = (t_1\alpha(a)t'_1, t_1at'_1), \quad \bar{s}_2 = (t_2\alpha(a)t'_2, t_2at'_2),
\]
\[
\bar{p}_1 = (p_1\alpha(a)p'_1, p_1ap'_1), \quad \bar{p}_2 = (p_2\alpha(a)p'_2, p_2ap'_2)
\]
Suppose \( \bar{s}_1\bar{p}_1 = \bar{s}_2\bar{p}_2 \), hence
\[
t_1\alpha(a)t'_1p_1\alpha(a)p'_1 = t_2\alpha(a)t'_2p_2\alpha(a)p'_2,
\]
and
\[
t_1at'_1p_1ap'_1 = t_2at'_2p_2ap'_2.
\]
Assume neither \( \bar{s}_1 \) nor \( \bar{s}_2 \) nor \( \bar{s}_1 \) in \( T(a) \). From the equalities above, it follows one of the next cases
\[
t_1\alpha(a)t'_1 \leq t_2\alpha(a)t'_2, \quad t_1at'_1 > t_2at'_2, \tag{15}
\]
\[
t_1\alpha(a)t'_1 > t_2\alpha(a)t'_2, \quad t_1at'_1 \leq t_2at'_2,
\]
\[
t_1\alpha(a)t'_1 > t_2\alpha(a)t'_2, \quad t_1at'_1 \leq t_2at'_2,
\]
\[
t_1\alpha(a)t'_1 \leq t_2\alpha(a)t'_2, \quad t_1at'_1 > t_2at'_2.
\]
Consider the first case (similarly, one can consider the another).

We have
\[
t_1\alpha(a)t'_1 \leq t_2\alpha(a)t'_2 \iff t_1\alpha(a)t'_1t_2\alpha(a)t'_2 = t_1\alpha(a)t'_1t_2t'_2 = t_2\alpha(a)t'_2, \tag{16}
\]
\[
t_2at'_2 \leq t_1at'_1 \iff t_2at'_2t_1at'_1 = t_2at'_2t_1t'_1 = t_1at'_1 \tag{17}
\]
Using (17) and \( \alpha(a) \leq a \), obtain
\[
t_1at'_1 = t_2at'_2t_1t'_1 = t_2at'_2t_1\alpha(a)t'_1 = (t_2at'_2t_1\alpha(a)t'_1)t_2t'_2.
\]
By (16),
\[
t_2at'_2(t_1\alpha(a)t'_1t_2t'_2) = t_2at'_2t_2at'_2 = t_2at'_2.
\]
Thus, \( t_1at'_1 = t_2at'_2 \) that contradicts with the strict inequality (15).

\[\Box\]

Lemma 3.9. The element \((\alpha(a), a) \in T(a) \) is the ancestor of \((a, a) \) in the right hereditary band \( T(a) \).

Proof. Assume the converse: there exists an element \( \bar{s} \in T(a) \), \((\alpha(a), a) \leq \bar{s} \leq (a, a) \). By (14), there exist \( t, t' \in T \) such that
\[
\bar{s} = (t\alpha(a)t', tat').
\]

Thus, we have the next four inequalities:
\[
\begin{aligned}
\alpha(a) &\leq t\alpha(a)t', \\
\ a &\leq tat', \\
\ t\alpha(a)t' &\leq a, \\
\ tat' &\leq a
\end{aligned}
\]
or, equivalently,

\[
\begin{align*}
\alpha(a)t't' &= t\alpha(a)t', \\
at't' &= tat', \\
t\alpha(a)t'a &= a, \\
tat' &= a \\
\end{align*}
\]

(18)

From the first and third equalities it follows \(\alpha(a)t't'a = a\). Hence \(\alpha(a)t't' \leq a\).

As \(\alpha(a)\) is the ancestor of \(a\) in \(T\), we have \(\alpha(a) = \alpha(a)t't'\). Hence, the first equation of (18) gives us

\[
\alpha(a) = t\alpha(a)t'.
\]

(19)

From the fourth equation of (18) and (19) we obtain \(\overline{s} = (t\alpha(a)t', tat') = (\alpha(a), a)\) that concludes the lemma.

Let \(T\) be a right hereditary band. Consider \(T(a)\) for some \(a \neq \varepsilon\). Let us extend \(T(a)\) by the element \(b \in T \subseteq T(a)\). The double extension \(T(a, b)\) is a subsemigroup of \(T^4\), and it has the next properties:

1. \(T\) embeds into \(T(a, b)\) by \(t \mapsto (t, t, t, t)\) for any \(t \in T\);
2. \(T(a)\) embeds into \(T(a, b)\); more precisely, the element \((\alpha, a) \in T(a)\) corresponds to \((\alpha(a), a, \alpha(a), a) \in T(a, b)\);
3. \(T(a, b)\) contains the element \((\alpha(b), \alpha(b), b, b) \notin T(a)\).

**Lemma 3.10.** Let \(T\) be a right hereditary band, \(a, b \in T\), \(a \neq b\), \(a \preceq b\), \(b \preceq a\). The elements \(A = (\alpha(a), a, \alpha(a), a) \in T(a, b)\), \(B = (\alpha(b), \alpha(b), b, b) \in T(a, b)\) have the next properties

1. \(((\alpha(a), a, \alpha(a), a)) \leq A \leq (a, a, a, a)),
   
   \(((\alpha(b), \alpha(b), b, b)) \leq B \leq (b, b, b, b))\);
2. neither \(A \preceq B\) nor \(B \preceq A\).

**Proof.**

1. It directly follows from the multiplication in \(T^4\).
2. Assume \(B \preceq A\) (similarly, one can put \(A \preceq B\)), hence \(AB = A\). Compute

\[
AB = (\alpha(a)\alpha(b), a\alpha(b), \alpha(a)b, ab).
\]

By the condition of the lemma, \(ab = a\), \(a\alpha(b) = a\), therefore

\[
AB = (\alpha(a)\alpha(b), a, \alpha(a)b, a).
\]

As \(AB = A\), we obtain \(\alpha(a)b = \alpha(a)\). Hence, \(b \preceq \alpha(a)\) that contradicts with the condition \(a \preceq b\).

Now we are able to prove the main result.

**Theorem 3.11.** Let \(S\) be a finitely generated right hereditary band. Then \(S\) is embedded into \(\mathcal{F}\).
Example 3.12. Let $S = \{x_1 x_2 x_3 = y_1 y_2 y_3\}$ be a system of equations in six variables over $F$ and $Y = V_F(S)$. Further we find all irreducible algebraic subsets (irreducible components) $Y_i$ of $Y$ such that

$$Y = \bigcup_i Y_i.$$ 

Remark that the inequality $t(X) \leq s(X)$ means the equation $t(X)s(X) = s(X)$ below.

By the formula $\varphi$, $Y$ is firstly splitted into two parts defined by the systems

$$S_1 = \begin{cases} x_1 x_2 x_3 = y_1 y_2 y_3, \\ x_1 \leq y_1 \end{cases}, \quad S_2 = \begin{cases} x_1 x_2 x_3 = y_1 y_2 y_3, \\ y_1 \leq x_1 \end{cases}$$

Let us proceed the splitting of the systems by the compare $x_1 x_2$ and $y_1$:

$$S_1 = S_{11} \cup S_{12} = \begin{cases} x_1 x_2 x_3 = y_1 y_2 y_3, \\ x_1 \leq y_1, \\ y_1 \leq x_1 x_2 \end{cases} \cup \begin{cases} x_1 x_2 x_3 = y_1 y_2 y_3, \\ x_1 \leq y_1, \\ x_1 x_2 \leq y_1 \end{cases}$$

Similarly, the comparison of $x_1$ and $y_1 y_2$ gets

$$S_2 = S_{21} \cup S_{22} = \begin{cases} x_1 x_2 x_3 = y_1 y_2 y_3, \\ y_1 \leq x_1, \\ x_1 \leq y_1 y_2 \end{cases} \cup \begin{cases} x_1 x_2 x_3 = y_1 y_2 y_3, \\ y_1 \leq x_1, \\ y_1 y_2 \leq x_1 \end{cases}$$
Theorem 3.13. For the free left regular band $F$ of an infinite rank we have:

1. the universal closure $Ucl(F)$ consists of all right hereditary left regular bands;

2. the universal theory $Thv(F)$ is axiomatizable by the next universal formulas

   (a) $\forall x, y, z \ (xy)z = x(yz)$ (associativity);
(b) $\forall x \ x^2 = x$ (idempotency);
(c) $\forall x, y \ xyx = xy$ (left regularity);
(d) $\forall x_1, x_2, y_1, y_2 (x_1 y_1 = x_2 y_2 \rightarrow [(x_1 \leq x_2) \vee (x_2 \leq x_1)])$.

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