Ground state properties and excitation spectra of non-Galilean invariant interacting Bose systems

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We study the ground state properties and the excitation spectrum of bosons which, in addition to a short-range repulsive two body potential, interact through the exchange of some dispersionless bosonic modes. The latter induces a time dependent (retarded) boson-boson interaction which is attractive in the static limit. Moreover the coupling with dispersionless modes introduces a reference frame for the moving boson system and hence breaks the Galilean invariance of this system. The ground state of such a system is depleted linearly in the boson density due to the zero point fluctuations driven by the retarded part of the interaction. Both quasiparticle (microscopic) and compressional (macroscopic) sound velocities of the system are studied. The microscopic sound velocity is calculated up the second order in the effective two body interaction in a perturbative treatment, similar to that of Beliaev for the dilute weakly interacting Bose gas. The hydrodynamic equations are used to obtain the macroscopic sound velocity. We show that these velocities are identical within our perturbative approach. We present analytical results for them in terms of two dimensional parameters – an effective interaction strength and an adiabaticity parameter – which characterize the system. We find that due the presence of several competing effects, which determine the speed of the sound of the system, three qualitatively different regimes can be in principle realized in the parameter space and discuss them on physical grounds.

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The superfluid properties of Bose-systems have been studied and greatly understood since the early fifties. While an ideal Bose gas condenses but does not exhibit superfluidity, an interacting Bose gas with repulsive interaction does.

The microscopic description of excitation spectrum of weakly interacting dilute Bose gas (WIDBG) has been first considered by Bogoliubov. It has been shown that the interaction among the particles modifies the character of low lying elementary excitation in the condensed state of the system and as a result long-wavelength excitations obey the linear dispersion law, thus having a sound-wave-like character rather then particle-like as in the normal state of the system. Within this lowest order treatment, referred to nowadays as Bogoliubov approximation, the microscopic sound velocity was found to be identical to the macroscopic, compressional, sound velocity of the system. Going beyond this lowest order theory, Beliaev has developed a perturbation expansion of the WIDBG model discussed in Ref. and considered the second order corrections to the Bogoliubov results. His results confirmed the equivalence of microscopic and macroscopic sound velocities. This equivalence has been generalized to all orders of perturbation theory by Gavoret and Nozières in Ref. invoking explicitly the Galilean invariance of the system.

In a recent paper we have addressed ourselves to a class of systems composed of two coupled bosonic subsystems: propagating bosons, interacting through a short-range repulsive interaction, and local dispersionless bosonic degree’s of freedom (for brevity hereafter termed as phonons) to which they are coupled. The coupling to the latter not only supplements the short range two body potential of bosons by a time dependent counterpart (attractive at low energies) but also introduces a reference frame for the moving boson system and hence breaks the Galilean invariance of this system. The aim of the present work is to study the ground state properties and the excitation spectrum of such a system, focusing on the renormalization of long-wavelength sound-like excitation due to the presence of the competing interactions in the system mentioned above.

We first determine the microscopic sound velocity from the single-particle excitation spectrum. The latter has been calculated in Ref. within the Beliaev-Popov second-order perturbation theory generalized to our system. Here we consider the continuum version of the model discussed in Ref. and show that the leading order terms in a density expansion can be treated analytically. We discuss the modification of the perturbation expansion due to the presence of retarded interaction and the range of validity of our treatment. To calculate the macroscopic sound velocity we base ourself on the hydrodynamic equations, explicitly incorporating the non-Galilean invariance of the system. We find that both velocities are identical up to the order considered in the present approximation.

The Hamiltonian which describes the above discussed scenario is given by

\[ H = \sum_q (\varepsilon_q - \mu) b_q^\dagger b_q + \frac{g}{2N} \sum_{k,k',p} b_{k'}^\dagger b_{k'} b_k^\dagger b_{k+p} \]
\[ + \omega_0 \sum_q (a_q^\dagger a_q + \frac{1}{2}) - \frac{\alpha \omega_0}{\sqrt{N}} \sum_{k,q} b_k^\dagger b_{k+q} [a_q^\dagger + a_{-q}] , \]

where \( b_k^{(1)} \) and \( a_q^{(1)} \) denote the boson and phonon annihilation (creation) operators, \( \epsilon_q = q^2/(2m) \), \( m \) and \( \mu \) is the boson mass and chemical potential, respectively. The interaction among bosons is assumed to be properly renormalized and is given by \( g = 4\pi a/m \). We note, that working with a renormalized interaction, one has to be careful to avoid double counting of the diagrams which have already been included in the ladder sum for the renormalized interaction, that defines the \( s \)-wave scattering length \( a \) of two particles in vacuum. The strength of the boson-phonon coupling is given by \( \alpha \omega_0 \), \( \alpha \) being dimensionless parameter and \( \omega_0 \) denotes the frequency of the dispersionless phonon mode.

In our treatment we assume both, the dilute limit of the boson system and weak boson-phonon coupling in order to set up a perturbation theory for the above introduced Hamiltonian at \( T = 0 \). The phonon mode can be effectively integrated out leading to the effective two body potential \( \Gamma_{q,\omega} = g + \alpha^2 \omega_0^2 D_{q,\omega}^{(0)} \), where \( D_{q,\omega}^{(0)} = 2\omega_0/\left(\omega^2 - \omega_0^2\right) \) is the bare phonon Green’s function. The phonon mediated interaction vanishes at high frequencies as \( 1/\omega^2 \) and hence no special renormalization or ultraviolet cut-off is required in order to perform calculations to a given order.

The lowest order approach to the above Hamiltonian is constructed within the Bogoliubov scheme by separating out in Eq. (1) the condensate part of the Bose field \( b_k = b_k + n^2 \delta_{k,0} \) (\( n_c \) being the condensate fraction) and keeping only the bilinear part of the resulting Hamiltonian. To this order of approximation one obtains two branches of the excitation spectrum of the system:

\[ \omega_{\pm,q} = \frac{E_q^2 + \omega_0^2}{2} \pm \frac{1}{2} \sqrt{\left(E_q^2 - \omega_0^2\right)^2 + 16\alpha^2 \omega_0^2 \epsilon_q n_c} \]

with \( E_q = |\epsilon_q^2 + 2gn_c\epsilon_q|^{1/2} \) denoting the spectrum of Bogoliubov quasi-particles in the absence of the coupling with phonons.

When the bosons condense, the phonons, being initially coupled to the boson density (symmetry restoring variable in the case of gauge symmetry breaking), get hybridized with the Goldstone mode. The resulting two normal modes describe Bogoliubov type excitations and Einstein phonons. For momenta close to where the level crossing of bare excitation spectra occurs neither mode is predominantly a phonon nor a Bogoliubov quasi-particle. In the long-wavelength limit the lower branch reduces to a sound-like dispersion \( \omega_{-q} \approx v_0 q \), where \( v_0 = \left(\frac{g - 2\alpha^2 \omega_0}{\omega_0}\right)^{1/2} \) is a characteristic sound velocity with the repulsive interaction having been reduced by the static part of the attractive phonon mediated interaction.

The Bogoliubov scheme amounts to taking into account collisions between condensate quasi-particles, between condensate and out-of-condensate quasi-particles, while neglecting scattering among out-of-condensate quasi-particles. The interaction among the particles depletes the ground state condensate leading to a finite density \( \bar{n} \) of non condensed particles. For the weakly interacting dilute Bose gas, the depletion is given by \( \bar{n} = n - n_c \approx n^{3/2} \) and is small in the low density limit \( n \ll 1 \). Beliaev has considered the second order corrections to the Bogoliubov result and has shown that the expansion parameter of the theory is the so called gas parameter \( an^{1/3} \). The latter measures the effective range of the interaction \( a \) (the \( s \)-wave scattering length) relative to the inter-particle distance \( d \sim n^{1/3} \). All the second-order corrections to the Bogoliubov results (for example corrections to the chemical potential \( \mu \), squared sound velocity \( v_s^2 \) ) appear with higher powers in density \( \sim n^{3/2} \) and are small in the dilute limit even if the interaction is strong. As it follows, these arguments no longer apply to the case we consider. The reason is the following: even in the vanishing density limit the system does not reduce to the non interacting one and one ends up with the well known polaron problem. In other words, due to the retarded character of phonon mediated interaction, one particle can interact with itself, by creating the lattice deformation at given time and absorbing it later on.

A generalization of the Beliaev-Popov second-order perturbation theory to the presently studied problem has been discussed by us in Ref. 8. As we have already mentioned, here we restrict ourselves to the leading order in density contributions, which allows us to obtain analytical results. We calculate the corrections to the chemical potential, microscopic sound velocity and discuss the expansion parameter and the limit of the validity of the perturbation theory.

To begin with we calculate the depletion of the ground state depletion, b) normal component of the boson self-energy due to the coupling with phonons, c) anomalous component of the boson self-energy, d) renormalized four point vertex.

\[ \text{FIG. 1. a) Leading order in density contribution to ground state depletion, b) normal component of the boson self-energy due to the coupling with phonons, c) anomalous component of the boson self-energy, d) renormalized four point vertex.} \]
the zig-zag line the condensate (given by a factor \( n_0^{1/2} \)). This contribution is exclusively due to the phonon mediated retarded interaction. It vanishes for the instantaneous interaction since in this case the end points (see Fig. 1a) would correspond to the same time, which involves both “particle” and “hole” type excitations. Only the former one is non-vanishing for \( T = 0 \).

The corresponding analytical expression of the diagram shown in Fig. 1a is as follows

\[
\hat{n} = \alpha^2 \omega_0^2 n_c \int \frac{d\omega dq}{2\pi (2\pi)^3} \left[ \mathcal{G}_{\omega}^{(0)} \right]^2 \mathcal{G}_{\omega}^{(0)} \frac{q^2 dq}{2\pi^2} \frac{1}{(\epsilon_q + \omega_0)^2} = \frac{\alpha^2 (2m\omega_0)^2 n_c}{8\pi}, \tag{3}
\]

In Eq. (3) \( \mathcal{G}_{\omega}^{(0)} = (\omega - \epsilon_q + i\eta)^{-1} \) denotes the bare boson Green’s function. As a result, to the leading order in density, the ground state depletion and the condensate fraction takes the form:

\[
\hat{n} = \frac{\gamma \lambda}{2} n_c + O \left( n_c^2 \right), \quad n_c \simeq \frac{n}{1 + \gamma \lambda/2}, \tag{4}
\]

where we have introduced two dimensionless parameters: \( \lambda = 2\alpha^2 \omega_0/\gamma \) which is the strength of phonon mediated interaction relative to the direct boson-boson repulsion and \( \gamma = a/q_0 \) being the ratio of two characteristic lengths – the scattering length \( a \) and \( q_0 = \sqrt{2m\omega_0} \) being the momentum at which crossing of the boson and the phonon bear spectrum occurs. We note that the parameter \( \gamma \) always appear in the combination \( \gamma \lambda \) which does not explicitly involve the boson-boson repulsion and is equivalent to the so-called Migdal parameter in the electron-phonon problem. However, as it follows, the interpretation of our results in terms of \( \gamma \) and \( \lambda \) is more transparent and we keep this notations, rather using the conventional one.

The renormalization of the spectrum given by Eq. (3), due to the second order corrections can be obtained from the poles of the dressed boson propagator. The diagonal \( (\mathcal{G}_{\omega}^{(0)}\mathcal{G}_{\omega}^{(0)}) \) and off-diagonal \( (\mathcal{G}_{\omega}^{(0)}\mathcal{G}_{\omega}) \) elements of the boson Green’s function are expressed through the normal \( (\Sigma_{\omega}) \) and anomalous \( (\Sigma_{\omega}) \) components of the self-energy

\[
\mathcal{G}_{\omega} = \left[ \omega + \Sigma_{\omega} \right]/\mathcal{D}_{\omega}, \quad \hat{\mathcal{G}}_{\omega} = \left[ -\hat{\Sigma}_{\omega} \right]/\mathcal{D}_{\omega}, \quad \mathcal{D}_{\omega} = \left[ \omega - \mathcal{A}_{\omega} \right]^2 - [\epsilon_q + \Sigma_{\omega}]^2 + 2\Sigma_{\omega}, \tag{5}
\]

where \( \epsilon_q = \omega - \mu \), \( \Sigma_{\omega} = [\Sigma_{\omega} + \Sigma_{-\omega} - 2\Sigma_{\omega}]/2 \) and \( \mathcal{A}_{\omega} = [\Sigma_{\omega} - \Sigma_{-\omega}]/2 \) are symmetric and antisymmetric functions in \( \omega \), respectively. Invoking the Hugenholtz-Pines theorem we have the relation \( \mu = \Sigma_{0,0} - \Sigma_{0,0} = \bar{S}_0 \).

Expanding the self-energies in the long-wavelength, low frequency limit as follows

\[
\mathcal{A}_{\omega} = \bar{A}_{\omega}, \quad \mathcal{S}_{\omega} - \mu = \bar{S}_1 \omega^2 + \bar{S}_2 q^2 \tag{6}
\]

we obtain the microscopic sound velocity from Eq. (3) as

\[
e^2 = \frac{\Sigma_{0,0}(1 + \bar{S}_2)}{m(1 + |\bar{A}|^2 + 2|\bar{S}_1|\Sigma_{0,0})}. \tag{7}
\]

Let us first discuss the self-energy expansion parameters defined in Eq. (4). As we have already mentioned, we are interested in the leading order (linear in density) contributions. Since the anomalous self-energy \( \Sigma_{0,0} \) entering into Eq. (3) vanishes linearly with density, there will be no contribution to this order coming from \( \bar{S}_1 \) and only those from the remaining \( |\bar{A}| \) and \( \bar{S}_2 \) (which has nonzero value in the limit of vanishing density) will contribute to this order. The first non-zero contribution to \( \bar{A} \) comes from the diagram presented in Fig. 1b. Substituting the bare boson and phonon Green’s functions in the corresponding formula of this diagram, and expanding the result of the integration in powers of the external frequency \( \omega \), after straightforward algebra one obtains \( \bar{A} = \gamma \lambda/2 \). One also verifies, that for the dispersionless Einstein phonons considered here, the phonon mediated self-energy (see Fig. 1b) being local in space is momentum independent. Hence, for the momentum expansion parameter \( \bar{S}_2 (\bar{A}) \) one has \( \bar{S}_2 = 0 \) within the same order of approximation.

At this point, it is interesting to check whether the exact relations

\[
|\bar{A}| = \partial \hat{n}/\partial n_c, \quad \bar{S}_2 q^2 = n_c^{-1}[\epsilon_q' - \epsilon_0'] \tag{8}
\]

derived in Ref. 3 still holds, given the above obtained perturbative results. In Eq. (3) \( \epsilon_0' \) and \( \epsilon_q' \) stand for the energy per unit volume of the system at rest and the system moving uniformly with a speed \( v = q/m \), respectively (prime denotes that the contribution from the condensate phase is eliminated). With the help of Eq. (3), one can easily verify that the first relation Eq. (8) is recovered and not violated by the perturbative theory. As for the second relation, invoking the Galilean invariance of the system it has been linked in Ref. 4 to the change in kinetic energy of the moving system leading to \( \bar{S}_2 = n_c^{-1} \bar{n}/(2m) \). For the present case [see Eq. (3)] this gives a finite value \( \bar{S}_2 = \gamma \lambda/(4m) \) instead of zero as has been discussed above. However, as it will be shown in details below, due to the non-Galilean invariance in the present case there is an additional contribution to the energy of the moving system, stemming from the retarded boson phonon interaction. Considering both these contributions, it turns out that to within this order of approximation of our perturbative treatment they cancel each other and one recovers \( \bar{S}_2 = 0 \).

To the leading order in a density expansion the anomalous self-energy is given by the diagram shown in Fig. 1c, corresponding to \( \Sigma_{0,0} = \Gamma n_c \) with \( \Gamma \) denoting a four point vertex dressed by the phonon mediated interaction (see Fig. 1d) with all external momenta and frequencies taken to be zero. The renormalization of the four point vertex
In the second order is shown in Fig. 1d. The diagrams $d_3$ and $d_4$ from Fig. 1d represent the renormalization of the boson-boson and the boson-phonon vertices, respectively. The diagrams $d_5$ and $d_6$ describe the two-body $t$-matrix renormalization arising from the phonon-mediated interaction. The last two diagrams ($d_7$ and $d_8$) represent the exchange processes and, like the diagrams for vertex renormalization, are nonzero due to the time dependence of the phonon mediated interaction. We note that there are no equivalent diagrams with both interaction lines being the boson-boson renormalized interaction $g$, since either such contributions involve backward propagators of the bosons in the vacuum and hence vanish or they have already been included in $g$ and do not have to be double counted. After substituting the bare boson and phonon Green’s functions in the corresponding standard analytical formulas of the above discussed diagrams, all the integration can be done analytically and one arrives to the following expression for $\Gamma$ (see Ref. [1]):

$$\Gamma = g[1 - \lambda + 4\gamma\lambda - 17\gamma\lambda^2/8]$$ (9)

Based on the above obtained result the sound velocity from Eq. (7) can be written as:

$$v^2 \simeq \frac{\partial e}{\partial n} \left[ 1 + \frac{\lambda}{8}( -8 + 20\gamma - 5\gamma\lambda) + O(\gamma^2\lambda^2) \right].$$ (10)

This expression implies the perturbation expansion in powers of $\lambda$ and $\gamma\lambda$. The dilute limit does not necessary imply the convergence of the perturbation expansion and we have to assume moreover $\lambda << 1$ and $\gamma << 1/\lambda$.

Before providing a physical interpretation of these results, let us examine the macroscopic sound velocity of the system to within the same order of perturbation theory. For that purpose we start with the well known equations of motion for the two conjugate dynamical variables: the number operator $n$ and the macroscopic phase of the condensate $\phi$. It reads as:

$$\frac{\partial n}{\partial t} = \frac{\partial e_{n,\phi}}{\partial \phi}, \quad \frac{\partial \phi}{\partial t} = -\frac{\partial e_{n,\phi}}{\partial n},$$ (11)

where $e_{n,\phi}$ denotes the energy of the system per unit volume. The next step is to consider a hydrodynamic potential flow of the boson system with the superfluid velocity defined as $\mathbf{v}_s = \nabla\phi/m$. Then the change in energy of the system at finite but small $v_s$ can be written, without loss of generality, as $\delta e_{n,v_s} = e_{n,v_s} - e_{n,0} = \Lambda n v_s^2/2$, where $\Lambda$ is the phase stiffness of the superfluid system. Taking the grad from both sides of the equation (11) and using the thermodynamic relation $\partial e_{n,\phi}/\partial n = \mu$ one arrives to the Josephson relation for the superfluid velocity field $m\mathbf{v}_s/\partial t = -\nabla \mu$. The two equations of motion (11) can be combined into:

$$\frac{\partial^2 n}{\partial t^2} = c^2\nabla^2 n,$$ (12)

with $c^2 = \Lambda/\chi$ being the speed of macroscopic sound, and $1/\chi = \partial \mu/\partial n$ the compressibility of the system.

In the case of Galilean invariant system the change of the system’s energy is exclusively due to the kinetic energy term in the Hamiltonian and is given by $\delta e_{n,v_s} = \delta e_{n,v_s}^\text{kin} = nmv_s^2/2$. The potential energy, being only a function of the relative positions of the particles, does not change when the boson system moves with finite velocity. Then the phase stiffness is given by $\Lambda = n/m$ and one recovers the well known expression for the speed of compressional sound $c^2 = n/(m\chi)$. However, in the case of coupling with phonons, the system is no longer Galilean invariant since then this coupling introduces the reference frame for the moving system of bosons. In other words, the phonon mediated interaction, being retarded, depends on the frequency and hence on the velocity of the particles. Therefore, there will be an additional contribution to $\delta e_{n,v_s}$ due to the boson-phonon coupling $\delta e_{n,v_s}^{\text{B-P}}$.

$$\delta e_{n,v_s}^{\text{B-P}} = \Lambda^s n^2 v_s^2/2,$$ (13)

The next step is to calculate compressibility of the system. To this end we evaluate the chemical potential by using the Hugenholtz-Pines relation $\mu = \Sigma_{0,0} - \Sigma_{0,1}$ and using the various diagrams for $\Sigma_{0,0}$ and $\Sigma_{0,1}$ one finds that most of them cancel pairwise, and to leading order (linear in density) we have $\mu = const + (g - 2\alpha\omega_0)n + \mu^{(2)}$. The constant term describes the negative density independent shift of the chemical potential due to the coupling with phonons and does not contribute to the compressibility. The second term is the first order contribution given by the reduced boson repulsion arising from the static attractive part of phonon mediated interaction. The last term describes the second order corrections and can be

![FIG. 2. Ground state energy correction due to the boson-phonon coupling. The quantities in the parentheses correspond to the frame moving together with the boson system.](image-url)
written as $\mu^{(2)} = [2d_5 + d_6 + 2d_7 + d_8]n_c$ (see Ref. [1]) where $d_i$ denotes the diagrams shown in Fig. 1d and the corresponding values are given in Ref. [10]. The final result for compressibility is written as

$$\chi^{-1} = 1 - \lambda - 3\gamma\lambda - 9\gamma\lambda^2/8. \quad (14)$$

Finally, upon using Eqs. (12), (13), and (14) one obtains the macroscopic sound velocity:

$$c^2 \simeq \frac{gn}{m} \left[1 + \frac{\lambda}{8}(-8 + 20\gamma - 5\gamma\lambda) + O(\gamma^2\lambda^2)\right], \quad (15)$$

which is thus identical to the microscopic one (11).

Let us now discuss our main findings. Our perturbative expansion assumes $\lambda \ll 1$ and $\gamma\lambda \ll 1$. The first condition is necessary to insure the stability of the condensate against the collapse when the attractive interaction overcompensates the repulsive one, the second is equivalent to the one of the Migdal approximation in the electron-phonon problem. For small $\lambda \ll 1$ one can neglect the last, higher order, term in Eq. (13) and obtains the following expression for the sound velocity $c^2 \simeq c_0^2[1 + \lambda(-1 + 5\gamma/2)]$, with $c_0^2 = gn/m$ being the speed the sound in the absence of any boson-phonon coupling. Analyzing this expression as a function of $\gamma$ one finds three different regimes of behavior:

i) for $\gamma < 2/5$, corresponding to a small phonon mode frequency, the sound velocity decreases when the boson-phonon coupling is switched on,

ii) for $\gamma = 2/5$ the linear in $\lambda$ contribution is exactly zero, and the sound velocity does not depend on $\lambda$, provided the latter is small as we have already assumed, and

iii) if $\gamma > 2/5$, corresponding to a high phonon mode frequency, the coefficient in front of $\lambda$ becomes positive and the sound becomes stiffer when the boson-phonon coupling is switched on.

While intuitively the softening of the sound would be naturally expected, the increase of the sound is a rather surprising result. It can be understood as follows. The softening of the sound is due to two effects: First the phonon mediated interaction, being attractive, reduces the boson-boson repulsion, and second it enhances the boson mass. However, if the reduction of the repulsive interaction in the lowest order is not strong enough, it could be overcompensated by the enhancing of the boson density vertex due to the coupling with phonons. Moreover, this enhancement could also overcompensate the reduction of sound velocity due to the enhancement of the boson mass. Both effects, mass enhancement and vertex renormalization have the same origin, being only driven by the time dependence of the phonon mediated interaction. Hence these effects have the same order of magnitude and, acting in different directions, strongly compete. This is manifest in the three different regimes found in our perturbative approach. We would like to point out, that all regimes can in principle be realized within the range of validity of the present treatment.

In our previous paper analyzing numerically Beliaev-Popov self energies for different values of $\alpha$ (boson-phonon coupling strength), without restricting ourself to the leading order terms in density expansion, discussed here, we came to the conclusion that the sound velocity was practically unaffected by the coupling with the phonons. This situation which corresponds to the second regime discussed above.

In conclusion, we have studied the ground state properties and the excitation spectrum of the system of bosons which, in addition to short-range repulsive two body potential, interact through the exchange of the other dispersionless bosonic modes, for brevity termed as phonons. We have assumed the dilute limit of the boson system and weak boson-phonon coupling and considered the second order perturbation theory, equivalent of the Beliaev-Popov treatment of the weakly interacting dilute Bose gas. We have shown that the ground state of such a system is depleted linearly in the boson density due to the zero point fluctuations driven by retarded part of the interaction. We have derived analytical expressions for the microscopic and macroscopic sound velocities of the system and shown that up to the order of our perturbative approach they are identical. We found that three physically different regimes can be realized in the parameter space of the system and discussed them on physical grounds. While one can think of a number of phenomena which might be described in terms of the model we considered, we leave the application of our study to realistic systems for some future work.

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deriving these equations no Galilean invariance has been invoked in Ref. 7 and hence they still hold in present case. First we point out, that the diagram given in Fig. 1a that has been already evaluated Eq. (3) is the basic element of the diagrams $d_3$, $d_4$ and $d_7$ shown in Fig. 1d. Based on this observation one finds $d_3 = d_7 = g\gamma\lambda/2$ and $d_4 = -2\alpha^2\omega_0\gamma\lambda/2$. One farther verifies, that diagram $d_5$, which can be obtained from $d_7$ by reversing an arrow of one of the boson propagators, is twice bigger. This leads to $d_5 = 2d_7 = g\gamma\lambda$ since in the former one ($d_5$) both two poles (in lower and upper half of the complex plane) of the phonon propagator contribute. The same holds for $d_6$ and $d_8$ - and it is straightforward to show $d_6 = 2d_8 = -3\alpha^2\omega_0\gamma\lambda/2$. Putting all these contributions together and noting that the diagrams $d_3$, $d_4$, $d_5$, and $d_7$ enters with the symmetric factor 2, one obtains $\tilde{\Gamma} = d_1 + d_2 + 8d_3 + 2d_4 + 3d_8$. Substituting in the last expression those corresponding values of the diagrams, discussed above, one recovers Eq. (11).

The second order corrections to the chemical potential $\mu^{(2)}$ have been discussed in details in Ref. 8. The diagrammatic representation of the contribution to $\mu^{(2)}$ due to the boson phonon coupling is shown in Fig. 7 of that paper. The contributions due to the boson-boson interaction is obtained by replacing the phonon propagator by the boson-boson interaction line in these diagrams. The leading order in density terms, can then be obtained by replacing the dressed boson propagators in these diagrams by corresponding perturbation series and keeping only the leading, linear in density terms. In this way the same diagrams as those shown in Fig. 1d of the present paper are generated and one recovers the analytical formula for $\mu^{(2)}$ presented in the text.