Remarks on Conserved Quantities and Entropy of BTZ Black Hole Solutions. Part I: the General Setting

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Abstract. The BTZ stationary black hole solution is considered and its mass and angular momentum are calculated by means of Nöther theorem. In particular, relative conserved quantities with respect to a suitably fixed background are discussed. Entropy is then computed in a geometric and macroscopic framework, so that it satisfies the first principle of thermodynamics. In order to compare this more general framework to the prescription by Wald et al. we construct the maximal extension of the BTZ horizon by means of Kruskal–like coordinates. A discussion about the different features of the two methods for computing entropy is finally developed.
1. Introduction

Since its discovery in 1992, the so-called BTZ black hole solution has often been used in current literature as a simple but realistic model for black hole physics (see [1], [2], [3], [4], [5] and references quoted therein). Specifically, it has been assumed as a test model for both quantum gravity and for problems related to black hole entropy. Recently, the BTZ solution has been shown to be the effective solution in a dimensional reduced model and many black holes ensuing from string theory are described in terms of the BTZ solution, at least near the horizon.

Certainly, many facts about entropy remain to be understood from both a statistical and a geometrical viewpoint. In recent investigations of ours we therefore aimed to review what is known from a geometrical macroscopic viewpoint and to add some considerations which, as far as we know, are new in the literature.

The results of our investigations will be presented in two papers, whereby the material has been divided on the basis of coherence and shortness considerations. We shall refer to them as Part I (the present paper) and Part II (forthcoming). In Part I we consider the standard BTZ solution, seen as a vacuum solution for standard $(2 + 1)$-General Relativity with (negative) cosmological constant. We review and apply to the specific example some methods for defining conserved quantities (as is well known, there are several methods in literature and it is almost impossible to review all of them in a single paper; hereafter we shall follow the covariant approach to conserved quantities based on Nöther theorem - see [6], [7], [8], [9], [10], [11]). Moreover a proposal for fixing a background spacetime is suggested in order to correct so-called anomalous factors (see [12]) and to produce the expected values of conserved quantities.

Then we calculate the entropy by relying on a geometrical and global framework presented in [11]. As it was already discussed there, our general method contains the proposal of Wald et al. as a particular case (see [13], [14], [15] and references quoted therein). Here we apply on purpose also the original Wald’s recipe to show that it allows to obtain the same results but by a much longer route. We believe in fact that a serious comparison between the two methods is important also because from a theoretical viewpoint Wald’s framework requires in fact additional hypotheses with respect to ours: basically involving the surface gravity $\kappa$ which in our more general framework is not required to vanish in order to ensure the horizon to be a bifurcate Killing horizon (see [13], [14]). In the BTZ solution these additional requirements of Wald hold true (with the exception of the extremal case $r_+ = r_-$). Thence the entropy calculated by using Wald’s recipe has necessarily to agree with our computation. Other examples in which Wald’s additional hypotheses do not instead hold (such as the
Taub-NUT solutions) will be considered elsewhere (see [16]), whereby we shall show that our method works also when Wald’s recipe fails for lack of properties of the concerned solution (thus providing a geometrical recipe for the correct entropy which can be calculated on a statistical basis as in [17]).

In Part II we shall analyse a triad-affine theory with topological matter but no cosmological constant, which is called BCEA and describes the BTZ spacetimes. This theory has already appeared in the literature (see [5], [18]). It has been there shown that it exhibits an “exchange behaviour” between conserved quantities (the total mass, i.e. the Nöther conserved quantity associated to a timelike vector is the parameter \( J \) which should correspond to the angular momentum of BTZ spacetimes when described in standard General Relativity). For what concerns the entropy an exchange of inner and outer horizons has also been noticed (see [5]). In Part II we shall first obtain the same results in a geometrical and global formalism. Then we shall explicitly build a purely metric (natural) theory, which we shall call BCG theory, equivalent to the triad-affine BCEA. The BCG theory can be obtained through a generalized “dual” Legendre transformation, which may be viewed as a generalization of the Palatini first order variational approach to General Relativity. Starting from the BCG Lagrangian conserved quantities and entropy will be again calculated for BTZ spacetime. The results so obtained, in our opinion, will enlight some of the ambiguities present in the BCEA theory. In particular they allow to truly isolate the “matter” contributions to conserved quantities from the purely gravitational contribution, so to better explain the calculation previously performed by others in BCEA theory (see [5]).

2. The BTZ solution

Let us consider a spacetime manifold \( M \) (for the moment of arbitrary dimension \( n = \dim(M) \)) and the bundle \( \text{Lor}(M) \) of Lorentzian metrics over \( M \). Let us fix a trivialization and denote by \( g_{\mu\nu} \) the coefficients of the metric field (as well used as coordinates on the fibers of \( \text{Lor}(M) \)), by \( \gamma^\lambda_{\mu\nu} \) the Christoffel symbols (i.e. the coefficients of the Levi-Civita connection of the metric \( g \)), by \( r_{\mu\nu} \) the Ricci tensor and by \( r \) the scalar curvature (of the metric \( g \)).

The Hilbert-Einstein Lagrangian with negative cosmological constant \( \Lambda = -1/l^2 \) is

\[
L = \mathcal{L} \, ds = \alpha (r - 2\Lambda) \sqrt{g} \, ds
\]

where \( ds = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \) is the standard basis for \( n \)-forms over \( M \) and \( \alpha \neq 0 \) is a coupling constant. To compare with results in [5] one has to
set $\alpha = \frac{1}{2}$. Let us denote the **covariant naive momenta** by

$$\pi_{\mu\nu} = \frac{\partial L}{\partial g_{\mu\nu}} = \alpha \sqrt{g} \left( r_{\mu\nu} - \frac{1}{2} r g_{\mu\nu} + \Lambda g_{\mu\nu} \right)$$

$$p^{\mu\nu} = \frac{\partial L}{\partial r_{\mu\nu}} = \alpha \sqrt{g} g^{\mu\nu}$$

(2.2)

so that for the variation of the Lagrangian (2.1) we have

$$\delta L = \pi_{\mu\nu} \delta g^{\mu\nu} + p^{\mu\nu} \delta r_{\mu\nu}$$

(2.3)

As is well known, $\pi_{\mu\nu} = 0$ are the Euler-Lagrange equations for the the Lagrangian (2.1), i.e. Einstein field equations.

In dimension 3 there is a 2-parameter family of black hole solutions (called BTZ black holes) given by (see [1])

$$g_{\text{BTZ}} = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N_\phi dt + d\phi)^2$$

(2.4)

where we set

$$N^2 = -\mu + \frac{r^2 + J^2}{4r^2} \quad \quad \mu = \frac{r_+^2 + r_-^2}{l^2}$$

$$N_\phi = -\frac{J}{2r^2} \quad \quad J = 2 \frac{r_+ - r_-}{l}$$

(2.5)

We recall that it has been shown in [2] that $\mu$ and $J$ are respectively the ADM mass and angular momentum at infinity. One can thence apply various methods to compute the conserved quantities and the entropy via Nöther’s theorem. In the sequel we shall summarize and compare various approaches.

### 3. Nöther Theorem

The Lagrangian (2.1) is covariant with respect to the action of diffeomorphisms of spacetime $M$. Infinitesimally this is expressed by the following identity which holds for any vector field $\xi$ over $M$

$$d(i_\xi L) = \pi_{\mu\nu} \mathcal{L}_\xi g^{\mu\nu} + p^{\mu\nu} \mathcal{L}_\xi r_{\mu\nu}$$

(3.1)

where $\mathcal{L}_\xi$ denotes the Lie derivative operator and $i_\xi$ the contraction (or inner product) of forms along $\xi$. By expanding the Lie derivative of the Ricci tensor
both field equations and the Nöther conserved current can be found. In fact, by defining

\[ u^\lambda_{\mu\nu} := \gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \gamma_{\nu)} \quad \gamma_\mu := \gamma^\epsilon_{\epsilon\mu} \]  

(3.2)

the Lie derivative of the Ricci tensor can be expressed as follows

\[ \mathcal{L}_\xi r_{\mu\nu} = \nabla_\lambda \mathcal{L}_\xi (u^\lambda_{\mu\nu}) \]  

(3.3)

Thence we can recast eq. (3.1) as follows

\[ \text{Div} \, \mathcal{E}(L, \xi) = \mathcal{W}(L, \xi) \]  

(3.4)

with

\[ \mathcal{E}(L, \xi) = (p^{\mu\nu} \mathcal{L}_\xi u^\lambda_{\mu\nu} - \mathcal{L}_\xi \gamma^\lambda_{\mu\nu}) ds_\lambda \]

\[ \mathcal{W}(L, \xi) = - (\pi_{\mu\nu} \mathcal{L}_\xi g^{\mu\nu}) ds \]  

(3.5)

Here \( ds_\mu = i_{\partial_\mu} \, ds \) is the standard basis for \((n-1)\)-forms; \( \text{Div} \) denotes the formal divergence operator which acts on forms depending on \( k \) derivatives of fields according to the general rule

\[ (j^{k+1}\sigma)^* \text{Div}(\omega) = d((j^k\sigma)^* \omega) \]  

(3.6)

where \( d \) is the standard differential of forms over \( M \), \( j^k \) denotes derivation up to order \( k \) and \( \sigma \) denotes any section of the configuration bundle (in our case the bundle is \( \text{Lor}(M) \) while \( k \) is usually 1 or 2, depending on how many derivatives of \( g \) enter \( \omega \); recall that \( g \) enters the Lagrangian and the theory through its second order derivatives appearing in the curvature tensor). For functions one has \( \text{Div} F = (d_\mu F) \, dx^\mu \), where the differential operators \( d_\mu \) are called total formal derivatives.

Following the general prescription of [7] and [19], by computing (3.5) along any configuration \( \sigma \), we can define the currents \( \mathcal{E}(L, \xi, g) \) and \( \mathcal{W}(L, \xi, g) \) on \( M \). If \( g \) is a solution then \( \mathcal{W}(L, \xi, g) = 0 \) and \( \mathcal{E}(L, \xi, g) \) is conserved, i.e. it is a closed form on \( M \). Using Bianchi identities in eq. (3.4) and integrating by parts, we can (algorithmically) recast (see [7], [11], [19]) the current \( \mathcal{E}(L, \xi) \) as

\[ \mathcal{E}(L, \xi) = \tilde{\mathcal{E}}(L, \xi) + \text{Div} \, \mathcal{U}(L, \xi) \]

\[ \tilde{\mathcal{E}}(L, \xi) = 2\alpha \sqrt{g} (r_{\mu\nu} - \frac{1}{2} r g_{\mu\nu} + \Lambda g_{\mu\nu}) g^{\mu\lambda} \xi^\nu \, ds_\lambda \]  

\[ \mathcal{U}(L, \xi) = \alpha \sqrt{g} \nabla^\mu \xi^\nu \, ds_{\nu\mu} \]  

(3.7)

where \( ds_{\nu\mu} = i_{\partial_\nu} \, ds_\mu \) is the standard basis for \((n-2)\)-forms over \( M \). Again the current \( \tilde{\mathcal{E}}(L, \xi, g) = (j^{2k} g)^* \tilde{\mathcal{E}}(L, \xi) \) vanishes along solutions of field equations, while \( \mathcal{E}(L, \xi, g) - \tilde{\mathcal{E}}(L, \xi, g) = d \, \mathcal{U}(L, \xi, g) = (j^{2k} g)^* \text{Div} \, \mathcal{U}(L, \xi) \) (being
exact) is *strongly conserved*, i.e. it is conserved along any configuration \( g \) (not necessarily a solution of field equations). The current \( \mathcal{U}(L, \xi) \) is known as a *superpotential*; for the Lagrangian (2.1) (with or without cosmological constant), its value (3.7) is also known as Komar potential (see [20]). Once we specify a covariant Lagrangian \( L \) and a vector field \( \xi \) on \( M \), a conserved current \( \mathcal{E}(L, \xi) \) and a superpotential \( \mathcal{U}(L, \xi) \) are defined; moreover, once we specify a configuration \( g \), we can compute them on \( g \) obtaining \( \mathcal{E}(L, \xi, g) \) and \( \mathcal{U}(L, \xi, g) \). Now, let a region \( D \) be a compact submanifold with a boundary \( \partial D \) which is again a compact submanifold of \( M \); if \( g \) is a solution of field equations, the *conserved quantity* \( Q_D(L, \xi, g) \) is defined as the integral over a region \( D \) of codimension 1 of the current \( \mathcal{E}(L, \xi, g) \), or equivalently as the integral of \( \mathcal{U}(L, \xi, g) \) over the boundary \( \partial D \) of \( D \) because of (3.7). In applications one usually chooses \( D \) to be a spacelike slice of the ADM splitting such that \( \partial D \) is (a branch of) the spatial infinity. We stress that all quantities defined in (3.7) are \( \mathbb{R} \)-linear with respect to the vector field \( \xi \). We also stress that \( \xi \) is by no means required to be a Killing vector of the metric \( g \).

Reverting to the BTZ solution (2.4), notice that the integral over the “sphere” \( S^1_r \) of radius \( r \) of the superpotential (3.7) for the vector fields \( \partial_t \) and \( \partial_\phi \) gives respectively:

\[
Q(L, \partial_t, g_{\text{BTZ}}) = \int_{S^1_r} \mathcal{U}(L, \partial_t, g_{\text{BTZ}}) = 4\pi \alpha \frac{r^2}{l^2} \tag{3.8}
\]

\[
Q(L, \partial_\phi, g_{\text{BTZ}}) = \int_{S^1_r} \mathcal{U}(L, \partial_\phi, g_{\text{BTZ}}) = -2\pi \alpha J
\]

The *mass* \( Q(L, \partial_t, g_{\text{BTZ}}) \) diverges as one considers the limit \( r \to \infty \); this is a problem analogous to the well known *anomalous factor problem* which Komar potential is known to be affected of (see [12]). In other words the integral of the superpotential does not give the correct mass though it gives the expected *angular momentum* \( Q(L, \partial_\phi, g_{\text{BTZ}}) \). Here the anomalous factor problem is even worse than for Kerr-Newman metrics or for other asymptotically flat stationary solutions (see [8], [9], [10]). In fact, while there it was just a matter of a wrong factor to be corrected, here it is primarily a divergence to be cured. This divergence is typically due to the fact that the BTZ solution is an asymptotically anti-de Sitter spacetime and the magnitude of the timelike vector field \( \partial_t \) diverges as it approaches infinity.

There are (at least) two different possibilities to overcome this situation. First, one may define the *total conserved quantities* without any reference to their *densities*. On the other hand, one can define the conserved quantity *in a region* \( D \) as the integral over \( D \) of its density; then the total conserved
quantity is obtained by taking the limit to the whole spacelike slice, provided that the method converges to a finite result.

4. The total conserved quantities

Let us associate to any vertical vector field $X = (\delta g^{\mu\nu}) \partial/\partial g^{\mu\nu}$ over the bundle $\text{Lor}(M)$, i.e. for any variation $\delta g^{\mu\nu}$ of the (inverse) metric, an $(n-1)$-form:

$$\mathcal{F}(L, g)[X] = \alpha (g^{\lambda\rho}g_{\mu\nu} - \delta^\lambda_{(\mu} \delta^\rho_{\nu)}) \nabla_{\rho}(\delta g^{\mu\nu}) \sqrt{g} \, ds_\lambda$$

(4.1)

where the section $g$ of $\text{Lor}(M)$, i.e. a Lorentzian metric, is not required to be a solution of field equations. The correspondence $\mathcal{F}(L, g)$ is called Poincaré-Cartan morphism. Recalling (see [21] and references quoted therein) that Lie derivatives $\mathcal{L}_\xi g$ of sections $g$ can be interpreted as vertical vectors over $\text{Lor}(M)$

$$\mathcal{L}_\xi g = (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) \frac{\partial}{\partial g^{\mu\nu}}$$

(4.2)

one can rewrite the current (3.5) as:

$$\mathcal{E}(L, \xi, g) = \mathcal{F}(L, g)[\mathcal{L}_\xi g] - i_\xi L$$

(4.3)

The form $\mathcal{F}(L, g)$ is irrelevant to field equations, because it enters a divergence once one integrates by parts equation (2.3) and uses (3.1). However, it is tightly related to conserved quantities and Nöther theorem. These are in fact general features of any field theory (see [7], [11] and references quoted therein for the general framework).

The variation of the total conserved quantity is:

$$\delta_X Q_D(L, \xi, g) = \int_D \delta_X \mathcal{E}(L, \xi, g) =$$

$$= \int_D \delta_X (\mathcal{F}(L, g)[\mathcal{L}_\xi g]) - \int_D \mathcal{L}_\xi (\mathcal{F}(L, g)[X]) +$$

$$+ \int_{\partial D} i_\xi (\mathcal{F}(L, g)[X])$$

(4.4)

which suggests to us to define the variation of the corrected conserved quantities by the prescription

$$\delta_X \hat{Q}_D(L, \xi, g) = \int_{\partial D} \left( \delta_X \mathcal{U}(L, \xi, g) - i_\xi (\mathcal{F}(L, g)[X]) \right)$$

(4.5)
For the example under investigation, using expressions (3.1) and (4.1) we get
\[
\delta_X \hat{Q}_D(L, \partial_t, g_{\text{BTZ}}) = 2\pi \alpha \delta \mu \quad \delta_X \hat{Q}_D(L, \partial_\phi, g_{\text{BTZ}}) = -2\pi \alpha \delta J
\]
which can be integrated to give the total conserved quantities
\[
\hat{Q}_D(L, \partial_t, g_{\text{BTZ}}) = 2\pi \alpha \mu \quad \hat{Q}_D(L, \partial_\phi, g_{\text{BTZ}}) = -2\pi \alpha J
\]
We see that this method provides directly the total conserved quantities (as already calculated in [1]) and no extra data are needed other than the solution and the Lagrangian.

In other words, the relevant quantity to replace the Komar potential is \( \delta_X U(L, \xi, g) - i_\xi (F(L, g)[X]) \). This quantity is unaffected by the addition of divergences to the Lagrangian. In fact, if one considers a Lagrangian \( L' = \text{Div}(\beta) \) which is a total divergence one easily obtains \( \delta_X U(L', \xi, g) - i_\xi (F(L', g)[X]) = 0 \) identically.

As a second alternative approach, one can instead look for a \((n-2)\)-form over \( M \) which can be integrated over the boundary of a region \( D \) to give directly the conserved quantity in that region. This approach relies on the formal integration of equation (4.5). To perform this task one has to specify some extra information. First, one fixes some boundary conditions, e.g. usually one requires that field variations vanish on the boundary \( \partial D \). Then one seeks for a current \( B(L, \xi) \) such that, once we set, as usual, \( B(L, \xi, g) \) for the pull-back of \( B(L, \xi) \) along a section \( g \), the following holds:
\[
\delta_X B(L, \xi, g) \bigg|_{\partial D} = i_\xi (F(L, g)[X]) \bigg|_{\partial D}
\]
(4.8)

Usually there is no such a (global and covariant) current \( B(L, \xi) \). For example, for standard General Relativity one has
\[
i_\xi (F(L, g)[X]) = \delta(p^\mu{}^\nu u_\mu^\lambda \, ds_\lambda)
\]
(4.9)
which does not lead to a possible choice for \( B(L, \xi) \) because \( p^\mu{}^\nu u_\mu^\lambda \, ds_\lambda \) is not covariant, i.e. it is not a form on the bundle since \( u_\mu^\lambda \) is not a tensor. To overcome this problem one has to fix some “coherent” background connection \( \Gamma_\mu^\lambda (\text{which is assumed to be unaffected by deformations}) \) and define
\[
B(L, \xi) = p^\mu{}^\nu w_\mu^\lambda \xi^\nu \, ds_\lambda \mu
\]
\[
\begin{cases}
w_\mu^\lambda = u_\mu^\lambda - U_\mu^\lambda \\
U_\mu^\lambda = \Gamma_\mu^\nu - \delta_\mu^\lambda \Gamma_\nu \\
\Gamma_\nu = \Gamma_\nu^\lambda
\end{cases}
\]
(4.10)
In this way the correction term $\mathcal{B}(L, \xi)$ is covariant; one can recast (4.5) as

$$
\delta_X \hat{Q}_D(L, \xi, g) = \int_{\partial D} \left( \delta_X \mathcal{U}(L, \xi, g) - i_\xi \left( \mathcal{F}(L, g)[X] \right) \right) = \int_{\partial D} \delta_X \left[ \mathcal{U}(L, \xi, g) - \mathcal{B}(L, \xi, g) \right]
$$

(4.11)

that can be formally integrated giving:

$$
\hat{Q}_D(L, \xi, g) = \int_{\partial D} \left[ \mathcal{U}(L, \xi, g) - \mathcal{B}(L, \xi, g) \right]
$$

(4.12)

We stress that in order to construct a formula like (4.12) a background connection is needed (or some other “globalizing” tool). The new conserved quantities $\hat{Q}_D(L, \xi, g)$ depend both on the solution $g_{\mu\nu}$ and on the background connection $\Gamma^\lambda_{\mu\nu}$. They have thence to be interpreted as the relative conserved quantities with respect to $\Gamma^\lambda_{\mu\nu}$. The physical importance of some background for the theory of conserved quantities was already recognized in the literature; see for example [3], [17], [22]. As an example, it has been proved (see [10], [23]) that if one analyses the (charged) Kerr-Newmann solutions there exist suitable backgrounds which provide reasonable mass densities which when integrated on the boundary of a spacelike slice (i.e. on spatial infinity) give the correct total mass.

Of course the choice of $\mathcal{B}(L, \xi)$ is far not unique. In particular and for sake of simplicity the background $\Gamma^\lambda_{\mu\nu}$ can be assumed to be the Levi-Civita connection of some background metric $h_{\mu\nu}$ (also considered to remain unchanged under deformations). Then one can add a term which depends just on the background $(h_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$, which, being unchanged under deformations, does not effect (4.8). To fix such a term one can reasonably require that, if $h_{\mu\nu}$ is also a solution of fields equations (as it seems physically and mathematically reasonable to require) the relative conserved quantities of the background with respect to itself vanish. This amounts to re-define the correction as follows

$$
\tilde{\mathcal{B}}(L, \xi) = [p^{\mu\nu} w^\lambda_{\mu\nu} \xi^p + \alpha \sqrt{h} h^{\alpha\rho} (h) \nabla_{\alpha} \xi^\lambda] \, ds_{\lambda\rho}
$$

(4.13)

where $\nabla^{(h)}$ denotes the covariant derivative induced by the background $h$. Generally speaking, the correction (4.13) may be used also in asymptotically flat solutions when, however, there is a preferred vacuum ($h = \eta = $ Minkowski metric) which, being flat, reduces (4.13) to the simpler correction (4.10).
One can also derive both the corrected superpotentials $\mathcal{U} - B$ and $\mathcal{U} - \tilde{B}$ as (uncorrected) superpotentials of some suitable Lagrangian. In particular $\mathcal{U} - B$ is the superpotential for the first order Lagrangian for standard General Relativity (see [10])

$$L_1 = \left[ \alpha (r - 2\Lambda) \sqrt{g} - d_\lambda (\mathcal{L}^{\mu\nu} w^\lambda_{\mu\nu}) \right] ds$$

(4.14)

where $d_\lambda$ denotes the formal divergence while $\mathcal{U} - \tilde{B}$ is the superpotential for the equivalent Lagrangian

$$\tilde{L}_1 = \left[ \alpha (r - 2\Lambda) \sqrt{g} - d_\lambda (\mathcal{L}^{\mu\nu} w^\lambda_{\mu\nu}) - \alpha (R - 2\Lambda) \sqrt{h} \right] ds$$

(4.15)

where $R$ is the scalar curvature of the background $h$. In both cases the background has to be considered as a parameter so that $\xi$ has to be a Killing vector of the background (see [10]). We remark that both Lagrangians (4.14) and (4.15) induce a well-defined action functional for a variational principle based on fixing the metric on the boundary. We stress that in general these actions differ for surface terms from that used in [3].

For the Lagrangian (2.1) and the solution (2.4) one can obviously choose as a background another metric of the same type (2.4) with fixed values $(\mu_0, J_0)$ as parameters. By a direct computation we find for the corrected superpotential $\mathcal{U} - \tilde{B}$ the conserved quantity

$$\hat{Q}(L, \partial_t, g_{\text{BTZ}}) = 2\pi \alpha (\mu - \mu_0) \quad \hat{Q}(L, \partial_\phi, g_{\text{BTZ}}) = -2\pi \alpha (J - J_0)$$

(4.16)

Here the Komar potential of the background in (4.13) is essential in order to cure the quadratic divergence of the Komar potential of the solution. This background fixing contains, as particular cases, the backgrounds usually adopted in literature (see, for example, [1], [2], [3]). In particular, the limit $(\mu \to 0, J \to 0)$ corresponds to the vacuum state in which the black hole disappears. Another allowed choice analysed in literature (see [3]) is the anti-de Sitter space-time which corresponds to the different limit $(\mu \to -1, J \to 0)$.

5. Entropy

The entropy of a (black hole) solution is defined to be a quantity that satisfies the first principle of thermodynamics:

$$\delta X \mu = T \delta X S + \Omega \delta X J$$

(5.1)

for any variation $X$ tangent to the space of solutions, i.e. $X$ has to satisfy linearized field equations. Here $T$ and $\Omega$ are constant quantities with respect
to variations $\delta X$ (namely they are related to the unperturbed solution). Usually, they are assumed to represent the temperature and angular velocity of the horizon of the black hole, so that $S$ can be interpreted as the physical entropy of the system. The physical value of these parameters has to be provided by physical arguments, since they are almost undetermined in the present context (see [1], [4], [5], [24], [25]). Of course, one can compute one of them out of the others by requiring that equation (5.1) is integrable so that there exists a state function $S$ to fulfill the first principle of thermodynamics. However, other parameters have to be provided by physical arguments (e.g. $T$ has to be the temperature of Hawking radiation). The ultimate meaning of the work terms in (5.1) is that, for example, $\Omega \delta J$ is the change in the total mass along an isoentropic transformation. It has been shown elsewhere (see [1], [4], [5]) that, in order to make this true, $\Omega$ has to be the angular velocity of the horizon. Further terms may in general appear in (5.1) due to gauge charges. Since the example which is here under consideration has no further gauge symmetries we do not consider these further contributions.

Of course entropy should also satisfy further requirements (e.g. the second principle of thermodynamics). However, these additional requirements are generally out of control so that the first principle is what one usually requires with the hope to check the second principle afterwards.

By solving (5.1) with respect to $\delta X S$ and by setting $\xi = \partial_t + \Omega \partial_\phi$ one finds (see [11])

$$\delta X S = \frac{1}{T} \left( \delta_X U(L, \xi, g) - i\xi (\mathcal{F}(L, g)[X]) \right) \quad (5.2)$$

where $\infty$ denotes the spatial infinity of a spacelike slice. Now one can prove under quite general hypotheses (basically just requiring $\xi$ to be a Killing vector of the solution $g$) that the integrand quantity $\delta_X U(L, \xi, g) - i\xi (\mathcal{F}(L, g)[X])$ is a closed form, so that its integral does not depend on the integration domain but just on its homology.

Thence one can define the following quantity

$$\delta X S = \frac{1}{T} \int_\Sigma \left( \delta_X U(L, \xi, g) - i\xi (\mathcal{F}(L, g)[X]) \right) \quad (5.3)$$

where $\Sigma$ is any spatial surface such that $\infty - \Sigma$ does not enclose any singularity (in homological notation it is a boundary). Then equation (5.3) can be integrated to give a quantity $S$ which (because of (5.2)) satisfies the first principle of thermodynamics and which is thence a natural candidate to be interpreted as the entropy. We remark that we do not need anything but a 1-parameter family of solutions $g_{\mu\nu}$ and a Killing vector $\xi$ for the unperturbed solution.
$g^0_{\mu\nu}$ (here and everywhere we denote by $X$ the infinitesimal generator of the family, which is a solution of the linearized field equations). In particular, differently from [13], [14] and [15], we do not require anything about maximality of the solution under consideration, anything about horizons and anything about the vanishing of $\xi$ on horizons (see [11], [13]). This latter remark is particularly important for the actual calculations because, as we shall see, it simplifies considerably (both conceptually and computationally) the expressions involved.

Let now $\kappa$ denote the surface gravity so that $T = \kappa/(2\pi)$ is the temperature of the Hawking radiation of BTZ, as shown in [4], [24] by means of Euclidean path integrals. Let us set $\Omega = -N_\phi(r_+)$ which can be shown to be the angular velocity of the BTZ horizon. We remark that because of the value of $\Omega$ the Killing vector $\xi$ becomes null on the horizon $N: (r = r_+)$. We then easily get

$$\delta_X S = 8\pi^2 \alpha \delta r_+$$

which in turn gives

$$S = 8\pi^2 \alpha r_+$$

for the entropy.

The original Wald’s recipe for the entropy needs to choose a particular $\Sigma$ (the bifurcation surface) on which the Killing vector $\xi$, and thence the whole term $i_\xi (\mathcal{F}(L, g)[X])$ in (5.3), vanishes. Of course the existence of such a surface is ensured just in the maximal extension of the solution. [For example, for Schwarzschild solution this surface $\Sigma$ corresponds to the surface $U = 0$, $V = 0$ in Kruskal coordinates. Notice that the 2-sphere $U = V = 0$ is not covered by spherical coordinates $(t, r, \theta, \phi)$ or outgoing (nor ingoing) coordinates. To be more precise if one considers any cross section $t = t_0$ and $r = 2m$ of the horizon then $\xi$ does not vanishes on none of it for any value of $t_0$. Thus Kruskal coordinates are needed in a somehow “essential” way to apply Wald’s recipe.]

In order to apply the original Wald’s recipe to BTZ solution, one should then first build Kruskal-type coordinates as shown in [26]. Once one has verified that $\kappa$ is a non-vanishing constant on the horizon (which is false in the extreme case), this can be done in two steps. First of all we define Eddington-Finkelstein coordinates $(u, \rho, \varphi)$ which are the parameters along the flows of the vector fields $(\xi, \zeta, X)$ we are going to define. The vector field $\xi = \partial_t + \Omega \partial_\varphi$ is the Killing vector whose Killing horizon $N$ has to be extended; $u$ is the parameter along its flow; $X$ is a vector field tangent to $N$ such that $X^\lambda \nabla_\lambda u = 0$. Finally $\zeta$ is a vector field such that the following conditions have to be satisfied

$$\begin{cases}
\xi^\lambda \nabla_\lambda u = 1 \\
g(\zeta, \xi) = 1 \quad |\zeta|^2 = 0 \text{ on } N \\
g(\zeta, X) = 0
\end{cases}$$

(5.6)
In these new coordinates the vector fields \((\xi, \zeta, X)\) read as

\[
\begin{align*}
\xi &= \frac{\partial}{\partial u} \\
\zeta &= \frac{\partial}{\partial \rho} \\
X &= \frac{\partial}{\partial \varphi}
\end{align*}
\] (5.7)

Notice that even in Eddington-Finkelstein coordinates the vector field \(\xi\) does not vanish anywhere in the chart domain. The BTZ metric reads as

\[
g_{\text{BTZ}} = -f(r(\rho)) du^2 + 2 du \, d\rho + \Phi(r(\rho)) du \, d\varphi + \Psi(r(\rho)) d\varphi^2
\] (5.8)

where we set

\[
\begin{align*}
f(r) &= N^2 - r^2 (N_\phi + \Omega)^2 \\
\Phi(r) &= \frac{N^2 - r^2 (N_\phi^2 - \Omega^2)}{\Omega} \\
\Psi(r) &= -\frac{N^2 - r^2 (N_\phi - \Omega)^2}{4\Omega^2}
\end{align*}
\] (5.9)

Here \(r(\rho)\) is obtained by inverting the change of coordinates, namely

\[
\begin{align*}
\rho(r) &= \int_{r_+}^{r} \frac{dr}{F(r)} \\
F(r) &= \sqrt{\frac{\Psi(r)}{r^2}}
\end{align*}
\] (5.10)

We remark that \(F(r)\) is well defined in a neighbourhood of the horizon \(N\). Now we can define Kruskal-type coordinates \((U, V, \varphi)\) as

\[
\begin{align*}
U &= e^{\kappa u} \\
V &= -\rho \, e^{-\kappa u} \exp \left(2\kappa \int_{0}^{\rho} H(\rho) d\rho \right) \\
H(\rho) &= \frac{1}{f(r(\rho))} - \frac{1}{2\kappa \rho}
\end{align*}
\] (5.11)

The product \(UV\) depends just on \(\rho\) and can be regarded as an implicit definition of \(\rho\) as a function \(\tilde{\rho}(UV)\). Then the BTZ metric reads as

\[
g_{\text{BTZ}} = G(UV) dU dV + \frac{\Phi(r(\tilde{\rho}))}{\kappa U} dU d\varphi + \Psi(r(\tilde{\rho})) d\varphi^2 \quad G(UV) = \frac{f(r(\tilde{\rho}))}{\kappa^2 UV}
\] (5.12)
In these coordinates the Killing vector $\xi$ reads as

$$\xi = -\kappa \left( V \frac{\partial}{\partial V} - U \frac{\partial}{\partial U} \right)$$

(5.13)

which finally vanishes for $\bar{\Sigma} : (U = V = 0)$.

Thus we have extended the Killing horizon $\mathcal{N}$ to a bifurcate Killing horizon and we have identified the bifurcate surface $\bar{\Sigma}$. Now one can “easily” compute the entropy:

$$\delta_X S = \frac{1}{T} \int \delta_X U[\xi]_3 d\varphi \bigg|_{U=V=0} \delta_X S = 8\pi^2 \alpha \delta r_+ \Rightarrow S = 8\pi^2 \alpha r_+$$

(5.14)

where $U[\xi]_3 d\varphi$ is the angular part of the superpotential 1-form $U[\xi]$.

We stress that the first expression in (5.14) for $\delta_X S$ is meaningful just on the bifurcate surface where we can ignore the contribution of the term $i_\xi(\mathcal{F}(L, g)[X])$. We stress moreover that the above method fails in the extreme case. In fact, in this case $\kappa = 0$ and the derivation of Kruskal-type coordinates fails at equations (5.11) and (5.12) (notice that in this case $f(r)$ identically vanishes). On the contrary, as we proved (see eq. (5.4)) the second expression in (5.14) for $\delta_X S$ is correct on any surface $\Sigma$. In this way the entropy is not related directly to a quantity computed on the horizon (see [17] for a discussion of entropy of Taub-NUT solutions).

The computations of this Section have been carried out by using MapleV and Tensor package.

6. Conclusion and Perspectives

We have determined and discussed the entropy of BTZ solutions. In our framework the entropy (5.2) is clearly related, by its own definition, to Nöther charges by the first principle of thermodynamics; actually our proposal is to determine a priori exactly the quantity that satisfies the first principle of thermodynamics, so that it can be, a posteriori, physically interpreted as the entropy.

At first, entropy is a quantity computed at spatial infinity (5.2), as all conserved quantities are. Then one can compute it also by an integral on finite regions, provided that the Nöther generator $\xi$ is a Killing vector of the solution under consideration (see equation (5.3)). Finally, if the surface gravity $\kappa$ does not vanish on the horizon, one can extend such an horizon to a bifurcate Killing horizon and compute integrals on the bifurcate Killing surface on which $\xi$ vanishes (see equation (5.14)). This latter step is completely useless and computationally boring in applications as well as in the theoretical framework, as we hope to have shown in the present paper. Furthermore, our framework,
being intrinsically and geometrically formulated in a global setting, is in fact valid for a much larger class of theories, namely all field theories with a gauge invariance, so-called gauge-natural theories (see [11]). For the same reasons no requirements on signature and/or dimension of spacetime are needed, as our framework relies only on globality and covariance of the Lagrangian. We remark that in the original Wald’s procedure, $\kappa \neq 0$ is used for two different purposes, namely to compute the temperature and to prove the existence of the bifurcation surfaces. Non-extremality is essential to the second issue. Since, in the extreme case, the construction of Kruskal-like coordinates and bifurcation surface break down (as Wald himself noticed), we believe that because of this there is little hope to treat the extreme case through any approaches based on bifurcation surfaces. We instead believe that our approach, which does not use bifurcation surface, is in a good position to treat the extremal cases, too. Clearly, the extreme cases have to be discussed separately and we hope to address the problem in a forthcoming paper. As we already announced in the Introduction, Part II will revisit the above results in the light of BCEA theories (see [5] and [18]).

Future investigations will be devoted to those cases (e.g. Taub-NUT solutions) in which Wald’s prescription for entropy cannot apply at all, as noticed by Wald himself (see [26]) and other authors (see also [17]). In these cases, Wald’s prescription fails because of various reasons, e.g. because the orbits of timelike vectors are closed and extra contributions to the entropy are due to singularities other than those enclosed in Killing horizons (in particular the Misner string). Both these reasons prevent the application of the latter prescription. Since our general prescription does not require the existence of a bifurcate Killing surface, it allows to overcome these problems, in Taub-NUT solutions. Results will be published in [16], where they will be shown to agree with those found in [17] by another formalism.

We finally remark that our formalism is hopefully in a good position to be generalized to non-stationary black holes, since extra contributions due to non-stationarity seem to be under control.

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