Quantum theory of magnetic quadrupole lenses for spin-$\frac{1}{2}$ particles

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Abstract

General guidelines for constructing a quantum theory of charged-particle beam optics starting \textit{ab initio} from the basic equations of quantum mechanics, appropriate to the situation under study. In the context of spin-$\frac{1}{2}$ particles, these guidelines are used starting with the Dirac equation. The spinor theory just constructed is used to obtain the transfer maps for normal and skew magnetic quadrupoles respectively. As expected the traditional transfer maps get modified by the quantum contributions. The classical limit of the quantum formalism presented, reproduces the well-known Lie algebraic formalism of charged-particle beam optics.

Keywords: Beam physics, Beam optics, Accelerator optics, Spin-$\frac{1}{2}$ particle, Anomalous magnetic moment, Quantum mechanics, Dirac equation, Foldy-Wouthuysen transformation, Polarization, Thomas-Bargmann-Michel-Telegdi equation, Magnetic quadrupole lenses, Stern-Gerlach kicks, Quantum corrections to the classical theory.

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1 Introduction and Formalism

Charged-particle beam optics, or the theory of transport of charged-particle beams through electromagnetic systems, is traditionally dealt with using classical mechanics. This is the case in ion optics, electron microscopy, accelerator physics etc \([1]-[3]\). The classical treatment of charged-particle beam optics has been extremely successful, in the designing and working of numerous optical devices from electron microscopes to very large particle accelerators. It is natural to look for a prescription based on the quantum theory, since any physical system is quantum at the fundamental level! Such a prescription is sure to explain the grand success of the classical theories and may also help towards a deeper understanding and designing of certain charged-particle beam devices. To date the curiosity to justify the success of the classical theories as a limit of a quantum theory has been the main motivation to look for a quantum prescription. But, with ever increasing demand for higher luminosities and the need for polarized beam accelerators in basic physics, we strongly believe that the quantum theories, which up till now were an isolated academic curiosity will have a significant role to play in designing and working of such devices.

It is historically very curious that the quantum approaches to the charged-particle beam optics have been very modest and have a very brief history as pointed out in the third volume of the three-volume encyclopaedic text book of Hawkes and Kasper \([5]\). In the context of accelerator physics the grand success of the classical theories originates from the fact that the de Broglie wavelength of the high energy beam particle is very small compared to the typical apertures of the cavities in accelerators. This and related details have been pointed out in the recent article of Chen \([6]\).

A beginning of a quantum formalism starting \textit{ab initio} with the Dirac equation was made only recently \([7]-[8]\). The formalism of Jagannathan \textit{et al} was the first one to use the Dirac equation to derive the focusing theory of electron lenses, in particular for magnetic and electrostatic axially symmetric and quadrupole lenses respectively. The formalism of Jagannathan \textit{et al} further outlined the recipe to obtain a quantum theory of aberrations. Details of these and some of the related developments in the quantum theory of charged-particle beam optics can be found in the references \([7]-[14]\). I shall briefly state the central theme of the quantum formalism with some of the more recent results relevant to accelerator optics.

The starting point to obtain a quantum prescription is to build a theory based on the basic equations of quantum mechanics appropriate to the situation under study. For situations when either there is no spin or spinor effects are believed to be small and ignorable we start with the scalar Klein-Gordon and Schrödinger equations for relativistic and nonrelativistic cases respectively. For electrons, protons and other spin-$\frac{1}{2}$ particles it is natural to start with the Dirac equation, the equation for spin-$\frac{1}{2}$ particles. In practice we do not have to care about the other (higher spin) equations.

In many situations the electromagnetic fields are static or can reasonably assumed to be static. In many such devices one can further ignore the times of flights which are negligible or of not direct interest as the emphasis is more on the profiles of the trajectories. The idea is to analyze the evolution of the beam parameters of the various individual charged-particle beam optical elements (quadrupoles, bending
magnets, · · ·) along the optic axis of the system. This in the language of the quantum formalism would require to know the evolution of the wavefunction of the beam particles as a function of ‘s’, the coordinate along the optic axis. Irrespective of the starting basic time-dependent equation (Schrödinger, Klein-Gordon, Dirac, · · ·) the first step is to obtain an equation of the form

\[ i\hbar \frac{\partial}{\partial s} \psi(x, y; s) = \hat{\mathcal{H}}(x, y; s) \psi(x, y; s) , \]

where \((x, y; s)\) constitute a curvilinear coordinate system, adapted to the geometry system. For systems with straight optic axis, as it is customary we shall choose the optic axis to lie along the \(Z\)-axis and consequently we have \(s = z\) and \((x, y; z)\) constitutes a rectilinear coordinate system. Eq. (1) is the basic equation in the quantum formalism and we call it as the beam-optical equation; \(\mathcal{H}\) and \(\psi\) as the beam-optical Hamiltonian and the beam wavefunction respectively. The second step requires to obtain a relationship for any relevant observable \(\langle \mathcal{O} \rangle(s)\) at the transverse-plane at \(s\) to the observable \(\{\langle \mathcal{O} \rangle(s_{in})\}\) at the transverse plane at \(s_{in}\), where \(s_{in}\) is some input reference point. This is achieved by the integration of the beam-optical equation in (2)

\[ \psi(x, y; s) = \hat{U}(s, s_{in}) \psi(x, y; s_{in}) , \]

which gives the required transfer maps

\[ \langle \mathcal{O} \rangle(s_{in}) \rightarrow \langle \mathcal{O} \rangle(s) = \langle \psi(x, y; s)|\mathcal{O}|\psi(x, y; s) \rangle , \]

\[ = \langle \psi(x, y; s_{in})|\hat{U}^{\dagger}\mathcal{O}\hat{U}|\psi(x, y; s_{in}) \rangle . \]

The two-step algorithm stated above may give an over-simplified picture of the quantum formalism than, it actually is. There are several crucial points to be noted. The first-step in the algorithm of obtaining the beam-optical equation is not to be treated as a mere transformation which eliminates \(t\) in preference to a variable \(s\) along the optic axis. A clever set of transforms are required which not only eliminate the variable \(t\) in preference to \(s\) but also gives us the \(s\)-dependent equation which has a close physical and mathematical analogy with the original \(t\)-dependent equation of standard time-dependent quantum mechanics. The imposition of this stringent requirement on the construction of the beam-optical equation ensures the execution of the second-step of the algorithm. The beam-optical equation is such, that all the required rich machinery of quantum mechanics becomes applicable to compute the transfer maps characterizing the optical system. This describes the essential scheme of obtaining the quantum formalism. Rest is mostly a mathematical detail which is built in the powerful algebraic machinery of the algorithm, accompanied with some reasonable assumptions and approximations dictated by the physical considerations. For instance, a straight optic axis is a reasonable assumption and paraxial approximation constitute a justifiable approximation to describe the ideal behaviour.

Before explicitly looking at the execution of the algorithm leading to the quantum formalism in the spinor case, we further make note of certain other features.
Step-one of the algorithm is achieved by a set of clever transformations and an exact expression for the beam-optical Hamiltonian is obtained in the case of Schrödinger, Klein-Gordon and Dirac equations respectively, without resorting to any approximations! We expect this to be true even in the case of higher-spin equations. The approximations are made only at step-two of the algorithm, while integrating the beam-optical equation and computing the transfer maps for averages of the beam parameters. Existence of approximations in the description of nonlinear behaviour is not uncommon and should come as no surprise, afterall the beam optics constitutes a nonlinear system. The nature of these approximations can be best summarized in the optical terminology as; a systematic procedure of expanding the beam optical Hamiltonian in a power series of \(|\hat{\pi}_\perp/p_0|\) where \(p_0\) is the design (or average) momentum of beam particles moving predominantly along the direction of the optic axis and \(\hat{\pi}_\perp\) is the small transverse kinetic momentum. The leading order approximation along with \(|\hat{\pi}_\perp/p_0| \ll 1\) constitutes the paraxial or ideal behaviour and higher order terms in the expansion give rise to the nonlinear or aberrating behaviour. It is seen that the paraxial and aberrating behaviour get modified by the quantum contributions which are in powers of the de Broglie wavelength (\(\lambda_0 = 2\pi\hbar/p_0\)). Lastly, and importantly the question of the classical limit of the quantum formalism; it reproduces the well known Lie algebraic formalism \([20]\) of charged-particle beam optics pioneered by Dragt et al.

Let us start with the Dirac equation for a beam made up of particle of charge \(q\), mass \(m_0\) and anomalous magnetic moment \(\mu_a\) in a static electromagnetic field with potentials \((\phi(r), A(r))\)

\[
\hat{H}_D |\psi_D\rangle = E |\psi_D\rangle,
\]

where \(|\psi_D\rangle\) is the time-independent 4-component Dirac spinor, \(E\) is the energy of the beam particle and the Hamiltonian \(\hat{H}_D\), including the Pauli term is

\[
\hat{H}_D = \beta m_0c^2 + \frac{e}{c} \alpha \cdot (-i\hbar \nabla - qA) - \mu_a \beta \Sigma \cdot B,
\]

\[
\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix},
\]

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We multiply \(\hat{H}_D\) (on the left) by \(\alpha_z/c\) and rearrange the terms to get

\[
\hat{H}_D = -p_0\beta \chi \alpha_z - qA_z I + \alpha_z \alpha_\perp \cdot \hat{\pi}_\perp + (\mu_a/c)\beta \alpha_z \Sigma \cdot B,
\]

\[
\chi = \begin{pmatrix} \xi \mathbb{I} & 0 \\ 0 & -\xi^{-1} \mathbb{I} \end{pmatrix}, \quad \xi = \sqrt{\frac{E + m_0c^2}{E - m_0c^2}}.
\]

Eq. (6) is not in the standard form. We need to get \(\hat{H}_D\) into a beam-optical form in close analogy with the standard Dirac equation, as required by step-one of the
algorithm. Define,
\[
M = \frac{1}{\sqrt{2}}(I + \chi\alpha_z), \quad M^{-1} = \frac{1}{\sqrt{2}}(I - \chi\alpha_z),
\]
and
\[
|\psi_D\rangle \rightarrow |\psi'\rangle = M|\psi_D\rangle.
\]
Then
\[
i\hbar \frac{\partial}{\partial z} |\psi'\rangle = \hat{H}' |\psi'\rangle, \quad \hat{H}' = M\hat{H}_D M^{-1} = -p_0\beta + \hat{E} + \hat{O},
\]
with the matrix elements of \(\hat{E}\) and \(\hat{O}\) given by
\[
\hat{E}_{11} = -qA_z \bar{1} - (\mu_a/2c) \left\{ (\xi + \xi^{-1}) \sigma_\perp \cdot B_\perp + (\xi - \xi^{-1}) \sigma_z B_z \right\},
\]
\[
\hat{E}_{12} = \hat{E}_{21} = 0,
\]
\[
\hat{E}_{22} = -qA_z \bar{1} - (\mu_a/2c) \left\{ (\xi + \xi^{-1}) \sigma_\perp \cdot B_\perp - (\xi - \xi^{-1}) \sigma_z B_z \right\},
\]
and
\[
\hat{O}_{11} = \hat{O}_{22} = 0,
\]
\[
\hat{O}_{12} = \xi \left[ \sigma_\perp \cdot \hat{\pi}_\perp - (\mu_a/2c) \left\{ i \left( \xi - \xi^{-1} \right) (B_x\sigma_y - B_y\sigma_x) \right. \right.
\]
\[
- \left. \left( \xi + \xi^{-1} \right) B_z \right\} \left. \bar{1} \right],
\]
\[
\hat{O}_{21} = -\xi^{-1} \left[ \sigma_\perp \cdot \hat{\pi}_\perp + (\mu_a/2c) \left\{ i \left( \xi - \xi^{-1} \right) (B_x\sigma_y - B_y\sigma_x) \right. \right.
\]
\[
+ \left. \left( \xi + \xi^{-1} \right) B_z \right\} \left. \bar{1} \right].
\]

These transformations give us the beam-optical Hamiltonian as required by step-one of the algorithm. In the Dirac theory, the lower spinor components are much smaller than the upper spinor components. This is also true for the beam-optical equation derived above. The analogy between the standard Dirac equation and the beam-optical equation, thus constructed can be summarized in the following table

| Standard Dirac Equation | Beam Optical Form |
|-------------------------|-------------------|
| \(m_0c^2\beta + \hat{E}_D + \hat{O}_D\) | \(-p_0\beta + \hat{E} + \hat{O}\) |
| \(m_0c^2\) | \(-p_0 = i\hbar \frac{\partial}{\partial z}\) |
| Positive Energy | Forward Propagation |
| Nonrelativistic, \(|\pi| \ll m_0c\) | Paraxial Beam, \(|\pi_\perp| \ll \hbar k\) |
| Non relativistic Motion | Paraxial Behavior |
| + Relativistic Corrections | + Aberration Corrections |

From the above table it is clear that there is a very close analogy between the derived beam-optical form and the standard Dirac equation. Having established this analogy we can now execute step-two of the algorithm. As is well known the Foldy-Wouthuysen machinery [21] enables one to take the proper nonrelativistic limit of the Dirac equation and provides a systematic procedure for obtaining the relativistic corrections to a desired degree of accuracy in powers of \(1/m_0c^2\). With the analogy in the table above, we are now in position to adopt the Foldy-Wouthuysen machinery to
the beam-optical situation. The procedure of the Foldy-Wouthuysen-like machinery
gives us the paraxial behaviour and a systematic procedure to compute aberrations
to all orders in powers of \(1/p_0\). To leading order we get

\[
|\psi'\rangle = \exp\left(\frac{1}{2p_0}\beta \hat{O}\right) |\psi^{(1)}\rangle .
\]  

(12)

Then

\[
\begin{align*}
i\hbar \frac{\partial}{\partial z} |\psi^{(1)}\rangle &= \hat{\mathcal{H}}^{(1)} |\psi^{(1)}\rangle , \\
\hat{\mathcal{H}}^{(1)} &= \exp\left(-\frac{1}{2p_0} \beta \hat{O}\right) \hat{\mathcal{H}}' \exp\left(\frac{1}{2p_0} \beta \hat{O}\right) \\
&\quad - i\hbar \exp\left(-\frac{1}{2p_0} \beta \hat{O}\right) \frac{\partial}{\partial z} \left\{ \exp\left(\frac{1}{2p_0} \beta \hat{O}\right) \right\} \\
&= -p_0 \beta + \hat{\mathcal{E}}^{(1)} + \hat{\mathcal{O}}^{(1)}, \\
\hat{\mathcal{E}}^{(1)} &= \hat{E} - \frac{1}{2p_0} \beta \hat{O}^2 + \cdots , \\
\hat{\mathcal{O}}^{(1)} &= -\frac{1}{2p_0} \beta \left\{ [\hat{\mathcal{O}}, \hat{\mathcal{E}}] + i\hbar \frac{\partial}{\partial z} \hat{\mathcal{O}} \right\} + \cdots .
\end{align*}
\]  

(13)

The lower pair of components of \(|\psi^{(1)}\rangle\) are almost vanishing compared to the upper pair and the odd part of \(\hat{\mathcal{H}}^{(1)}\) is negligible compared to its even part we can effectively introduce a Pauli-like two-component spinor formalism based on the representation. Calling \(\hat{\mathcal{H}}_{11}^{(1)}\) as \(\hat{\mathcal{H}}\)

\[
i\hbar \frac{\partial}{\partial z} |\tilde{\psi}\rangle = \hat{\mathcal{H}} |\tilde{\psi}\rangle ,
\]

\[
\hat{\mathcal{H}} \approx \left( -p_0 - qA_z + \frac{1}{2p_0} \hat{\pi}_+^2 \right) - \frac{1}{p_0} \left\{ (q + \epsilon) B_z S_z + \gamma \epsilon \mathbf{B}_\perp \cdot \mathbf{S}_\perp \right\} ,
\]

with \(\hat{\pi}_\perp^2 = \hat{\pi}_x^2 + \hat{\pi}_y^2\), \(\epsilon = 2m_0 \mu_a / \hbar\), \(\gamma = E/m_0 c^2\), \(\mathbf{S} = \frac{1}{2} \hbar \sigma\).  

(14)

Up to now, all the observables, the field components, time etc., have been defined
in the laboratory frame. The covariant description of the spin of the Dirac particle has
simple operator representation in the rest frame of the particle. In the analysis of the
spin-dynamics in accelerator physics it is customary to use such a representation [18].
So we define the following transform which takes us from the beam-optical form to
the accelerator optical form

\[
|\tilde{\psi}\rangle = \exp\left\{ \frac{i}{2p_0} (\hat{\pi}_x \sigma_y - \hat{\pi}_y \sigma_x) \right\} |\psi^{(A)}\rangle .
\]  

(15)

Details of the accelerator optical transform are found elsewhere [13, 16, 17]. Up to
the paraxial approximation the accelerator-optical Hamiltonian is

\[
i\hbar \frac{\partial}{\partial z} |\psi^{(A)}\rangle = \hat{H}^{(A)} |\psi^{(A)}\rangle ,
\]
\[
\hat{H}^{(A)} \approx \left(-p_0 - qA_z + \frac{1}{2p_0} \hat{\pi}_z^2 \right) + \frac{\gamma m_0}{p_0} \mathbf{\Omega}_s \cdot \mathbf{S},
\]
with \( \mathbf{\Omega}_s = -\frac{1}{\gamma m_0} \left\{ qB + \epsilon \left( B_{||} + \gamma B_\perp \right) \right\}. \) (16)

Only at this stage we need to know the specific geometry of the magnetic lens, i.e., the form of the fields characterizing the lens, for computing the transfer maps. We shall briefly consider the normal and skew quadrupoles respectively in the next sections and see how the traditional transfer maps get modified by the quantum contributions.

2 Example-I: Normal Magnetic Quadrupole

An ideal normal magnetic quadrupole of length \( \ell \) is characterized by the field

\[
\mathbf{B} = (-Gy, -Gx, 0), \tag{17}
\]
corresponding to the vector potential

\[
\mathbf{A} = \left(0, 0, \frac{1}{2} G (x^2 - y^2)\right), \tag{18}
\]
situated in the transverse-planes \( z = z_{in} \) and \( z = z_{out} + \ell \), with \( G \) constant inside the lens and zero outside. The accelerator-optical Hamiltonian is

\[
\hat{H}(z) = \begin{cases} 
\hat{H}_F = -p_0 + \frac{1}{2p_0} \hat{\pi}_z^2, & \text{for } z < z_{in} \text{ and } z > z_{out}, \\
\hat{H}_{L}(z) = -p_0 + \frac{1}{2p_0} \hat{\pi}_z^2 - \frac{1}{2} qG (x^2 - y^2) + \frac{p_0}{\ell} (y \sigma_x + x \sigma_y), & \text{for } z_{in} \leq z \leq z_{out}, \text{ with } \eta = (q + \gamma \epsilon)G\ell \hbar/2p_0^2.
\end{cases} \tag{19}
\]

The subscripts \( F \) and \( L \) indicate, the field-free and the lens region respectively.

Best way to compute \( \hat{U} \) is via the interaction picture, used in the Lie algebraic formulation \([20]\) of classical beam optics. Using the transfer operator thus derived (details found elsewhere \([13, 17]\)) we get the transfer maps for averages with the subscripts \( in \) and \( out \) standing for \( (z_{in}) \) and \( (z_{out}) \) respectively

\[
M_q = \begin{pmatrix}
\cosh(\sqrt{K\ell}) & -\frac{1}{\sqrt{K}} \sinh(\sqrt{K\ell}) & 0 & 0 \\
\sqrt{K} \sinh(\sqrt{K\ell}) & \cosh(\sqrt{K\ell}) & 0 & 0 \\
0 & 0 & \cos(\sqrt{K\ell}) & \frac{1}{\sqrt{K}} \sin(\sqrt{K\ell}) \\
0 & 0 & -\sqrt{K} \sin(\sqrt{K\ell}) & \cos(\sqrt{K\ell})
\end{pmatrix},
\]

\[
T_q = \begin{pmatrix}
\langle x \rangle \\
\langle \hat{p}_x \rangle / p_0 \\
\langle y \rangle \\
\langle \hat{p}_y \rangle / p_0
\end{pmatrix}_{out} \approx T_q
\begin{pmatrix}
\langle x \rangle \\
\langle \hat{p}_x \rangle / p_0 \\
\langle y \rangle \\
\langle \hat{p}_y \rangle / p_0
\end{pmatrix}_{in} + \eta
\begin{pmatrix}
\left( \frac{\cosh(\sqrt{K\ell}) - 1}{K\ell} \right) \langle \sigma_y \rangle \\
\left( \frac{\sinh(\sqrt{K\ell})}{\sqrt{K\ell}} \right) \langle \sigma_y \rangle \\
\left( \frac{\cos(\sqrt{K\ell}) - 1}{K\ell} \right) \langle \sigma_x \rangle \\
\left( \frac{\sin(\sqrt{K\ell})}{\sqrt{K\ell}} \right) \langle \sigma_x \rangle
\end{pmatrix}.
\]
For spin, the transfer map reads

\[ \langle S_x \rangle_{\text{out}} \approx \langle S_x \rangle_{\text{in}} + 4\pi \eta \left( \left( \frac{\sinh(\sqrt{K\ell})}{\sqrt{K\ell}} \right) \langle xS_x \rangle_{\text{in}} + \left( \frac{\cosh(\sqrt{K\ell}) - 1}{K\ell p_0} \right) \langle \hat{p}_x S_x \rangle_{\text{in}} \right) , \]

\[ \langle S_y \rangle_{\text{out}} \approx \langle S_y \rangle_{\text{in}} - 4\pi \eta \left( \left( \frac{\sin(\sqrt{K\ell})}{\sqrt{K\ell}} \right) \langle yS_y \rangle_{\text{in}} - \left( \frac{\cos(\sqrt{K\ell}) - 1}{K\ell p_0} \right) \langle \hat{p}_y S_y \rangle_{\text{in}} \right) , \]

\[ \langle S_z \rangle_{\text{out}} \approx \langle S_z \rangle_{\text{in}} - 4\pi \eta \left\{ \left( \frac{\sinh(\sqrt{K\ell})}{\sqrt{K\ell}} \right) \langle xS_x \rangle_{\text{in}} - \left( \frac{\sin(\sqrt{K\ell})}{\sqrt{K\ell}} \right) \langle yS_y \rangle_{\text{in}} \right. \]
\[ + \left. \left( \frac{\cosh(\sqrt{K\ell}) - 1}{K\ell p_0} \right) \langle \hat{p}_x S_x \rangle_{\text{in}} + \left( \frac{\cos(\sqrt{K\ell}) - 1}{K\ell p_0} \right) \langle \hat{p}_y S_y \rangle_{\text{in}} \right\} . \]  

(21)

Thus, we get the a fully quantum mechanical derivation for the traditional transfer maps (transfer matrices) \[4\]. In addition we also get the spinor contributions, the Stern-Gerlach kicks as they are called. In recent years there has been a campaign to make a spin-splitter \[23\] device to produce polarized beams using the Stern-Gerlach kicks. The spinor theory of charged-particle beam optics, in principle supports the spin-splitter devices.

### 3 Example-II: Skew Magnetic Quadrupole

For a skew-magnetic-quadrupole lens the field is given by

\[ B = (-G_s y, G_s x, 0) , \]

(22)

corresponding to the vector potential

\[ A = (0, 0, -G_s x y) . \]

(23)

The accelerator-optical Hamiltonian is

\[ \hat{H}(z) = \begin{cases} 
\hat{H}_F = -p_0 + \frac{1}{2p_0} \hat{p}_\perp^2 , & \text{for } z < z_{\text{in}} \text{ and } z > z_{\text{out}} , \\
\hat{H}_L(z) = -p_0 + \frac{1}{2p_0} \hat{p}_\perp^2 + \frac{1}{2} q G_s x y - \frac{q}{v_0} \eta_s (x \sigma_y - y \sigma_x) , & \text{for } z_{\text{in}} \leq z \leq z_{\text{out}} , \end{cases} \]

(24)

and the corresponding transfer operator is

\[ \hat{U}_{i,L}(z_{\text{out}}, z_{\text{in}}) \]
Then we get the transfer maps straightforward to see that the transfer maps of the skew quadrupole can be obtained along the optic axis. For spin, the transfer map reads

\[
\begin{align*}
M_{\text{sq}} &= \frac{1}{2} \begin{pmatrix}
C^+ & \frac{S^+}{\sqrt{K_s}} & C^- & \frac{S^-}{\sqrt{K_s}} \\
-\sqrt{K_s}S^- & C^+ & -\sqrt{K_s}S^+ & \frac{C^-}{\sqrt{K_s}} \\
C^- & S^- & C^- & \frac{S^+}{\sqrt{K_s}} \\
-\sqrt{K_s}S^+ & C^- & -\sqrt{K_s}S^- & C^+
\end{pmatrix}, \quad T_{\text{sq}} = M_{\text{sq}} M_{\text{sq}}^<. \quad (26)
\end{align*}
\]

Again we get a quantum derivation of the traditional transfer maps \[4\] with some spinor contributions. Due to the algebraic machinery of the quantum formalism it is straightforward to see that the transfer maps of the skew quadrupole can be obtained by rotation of the transfer maps for the normal quadrupole by \(\pi/4\) along the optic axis. For spin, the transfer map reads

\[
\begin{align*}
\langle S_x \rangle_{\text{out}} &\approx \langle S_x \rangle_{\text{in}} + \frac{2\pi \eta_s}{\lambda_0} \left\{ -\left( \frac{S^+}{\sqrt{K_s}} \right) \langle x \rangle_{\text{in}} + \left( \frac{C^-}{K_s \ell p_0} \right) \langle \hat{p}_x \rangle_{\text{in}} + \left( \frac{C^- - 2}{K_s \ell p_0} \right) \langle p_x \rangle_{\text{in}} \right\}, \\
\langle S_y \rangle_{\text{out}} &\approx \langle S_y \rangle_{\text{in}} + \frac{2\pi \eta_s}{\lambda_0} \left\{ -\left( \frac{S^-}{\sqrt{K_s}} \right) \langle y \rangle_{\text{in}} + \left( \frac{C^+ - 2}{K_s \ell p_0} \right) \langle \hat{p}_y \rangle_{\text{in}} + \left( \frac{C^+ - 2}{K_s \ell p_0} \right) \langle p_y \rangle_{\text{in}} \right\}, \\
\langle S_z \rangle_{\text{out}} &\approx \langle S_z \rangle_{\text{in}} + \frac{2\pi \eta_s}{\lambda_0} \left\{ \left( \frac{S^+}{\sqrt{K_s}} \right) \langle x \rangle_{\text{in}} + \left( \frac{C^-}{K_s \ell p_0} \right) \langle \hat{p}_x \rangle_{\text{in}} + \left( \frac{C^- - 2}{K_s \ell p_0} \right) \langle p_x \rangle_{\text{in}} \right\} + \left\{ \left( \frac{S^-}{\sqrt{K_s}} \right) \langle y \rangle_{\text{in}} + \left( \frac{C^+ - 2}{K_s \ell p_0} \right) \langle \hat{p}_y \rangle_{\text{in}} + \left( \frac{C^+ - 2}{K_s \ell p_0} \right) \langle p_y \rangle_{\text{in}} \right\}. \quad (27)
\end{align*}
\]
4 Concluding Remarks and Directions for Future Research

To summarize, an algorithm is presented for constructing a quantum theory of charged-particle beam optics, starting ab initio from the basic equations of quantum mechanics appropriate to the situation. This algorithm is used to construct a spinor theory of accelerator optics starting ab initio from the Dirac equation, taking into account the anomalous magnetic moment. The formalism is demonstrated by working out the examples of the normal and skew magnetic quadrupoles respectively and as expected there are small quantum contributions, which are proportional to the powers of the de Broglie wavelength. It is further shown that the quantum formalism presented, in the classical limit reproduces the Lie algebraic formalism of charged-particle beam optics.

The present algorithm is suited for constructing a quantum theory of charged-particle beam optics at the single-particle level. The next logical step would be to extend it to accommodate the realistic multiparticle dynamics. We feel that clues to such a formalism can come from the experience gained from the so-called quantum-like models. These phenomenological models have been extensively developed in recent years to explain the classical collective behavior of a charged-particle beam, by associating with the classical beam a quantum-like wavefunction obeying a Schrödinger-like equation with the role of \( \hbar \) being played by the beam emittance \( \varepsilon \).

One practical application of the quantum formalism would be to get a deeper understanding of the polarized beams. A proposal to produce polarized beams using the proposed spin-splitter devices based on the classical Stern-Gerlach kicks has been presented recently.

Lastly it is speculated that the quantum theory of charged-particle beam optics will be able to resolve the choice of the position operator in the Dirac theory and the related question of the form of the force experienced by a charged-particle in external electromagnetic fields. This will be possible provided one can do an extremely high precision experiment to detect the small differences arising in the transfer maps from the different choices of the position operators. These differences shall be very small, i.e., proportional to powers of the de Broglie wavelength. It is the extremely small magnitude of these minute differences which makes the exercise so challenging and speculative!

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