Thermodynamic quantities of independent harmonic oscillators in microcanonical and canonical ensembles in the Tsallis statistics

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Received 5 September 2022 / Accepted 8 January 2023 / Published online 27 January 2023
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Abstract. We study the energy and the entropies of \( N \) independent harmonic oscillators in the microcanonical and the canonical ensembles in the Tsallis classical and the Tsallis quantum statistics of entropic parameter \( q \), where \( N \) is the number of the oscillators and the value of \( q \) is larger than one. The energy and the entropies are represented with the physical temperature, and the well-known expressions are obtained for the energy and the Rényi entropy. The difference between the microcanonical and the canonical ensembles is the existence of the condition for \( N \) and \( q \) in the canonical ensemble: \( N(q-1) < 1 \). The condition does not appear in the microcanonical ensemble. The entropies are \( q \)-dependent in the canonical ensemble, and are not \( q \)-dependent in the microcanonical ensemble. For \( N(q-1) < 1 \), this difference in entropy is quite small, and the entropy in the canonical ensemble does not differ from the entropy in the microcanonical ensemble substantially.

1 Introduction

Power-like distributions appear in various fields of science. The Tsallis statistics is a possible extension of the Boltzmann–Gibbs statistics, and the statistics gives a power-like distribution. The Tsallis statistics has a parameter \( q \), and the value \( |1 - q| \) is the measure of the deviation from the Boltzmann–Gibbs statistics. This statistics has been applied to various phenomena [1]. Various systems should be examined to clarify the effects of the Tsallis statistics. One of the systems is the system composed of harmonic oscillators.

The system composed of harmonic oscillators is basic to describe a large number of different phenomena. In the studies of phenomena, the potential is often approximated with a harmonic potential. A field is often decomposed into harmonic oscillators in field theories to proceed with calculations. The system of harmonic oscillators is a good base to calculate physical quantities. Therefore, the study of independent harmonic oscillators is required in the Tsallis statistics [2], in addition to the study of a harmonic oscillator [3]. Different ensembles can be applied to calculate quantities in the Tsallis statistics, as in the Boltzmann–Gibbs statistics.

It is worth to study simple models in different ensembles to gain some insights. In the Boltzmann–Gibbs statistics, the quantities are calculated in several ensembles such as microcanonical and canonical ensembles, and the value of a thermodynamic quantity in the microcanonical ensemble is the same as that in the canonical ensemble. In the unconventional statistics, the equivalence between the ensembles is not always clear. For example, the quantity in the microcanonical ensemble is equal to that in the canonical ensemble in the presence of short range force. However this conjecture may not be guaranteed in the presence of long range force. Long range force may alter the properties of the statistics, and may be an origin of the Tsallis statistics. Calculations in different ensembles are required when the equivalence between the ensembles is not guaranteed.

It is not always easy to calculate the quantities of harmonic oscillators in the Tsallis statistics because of the power-like distribution. A self-consistent equation which is often called norm equation [4] should be satisfied in the canonical ensemble in the Tsallis statistics. The physical temperature [5–14], which characterizes the equilibrium, was introduced and has been used to describe quantities in the Tsallis statistics. The Tsallis entropy is related to the Rényi entropy, and the Rényi entropy is related directly to the physical temperature.

The purpose of this paper is to calculate the quantities, the energy and the entropies, and to clarify the differences between ensembles in the system of independent harmonic oscillators with the same frequency in the Tsallis statistics with escort average [1,3]. We
attempt to derive the energy and the entropies in the microcanonical and the canonical ensembles in the Tsallis classical and the Tsallis quantum statistics of the entropic parameter \( q \). The condition for the number of the oscillators \( N \) and the deviation \( q - 1 \) in the canonical ensemble is derived for \( q > 1 \). The expression of the energy with the physical temperature is derived, and the difference between the entropy in the microcanonical ensemble and that in the canonical ensemble is given for \( N(q - 1) < 1 \). The difference between the ensembles in the Tsallis statistics is clarified for \( N \) independent harmonic oscillators.

In this paper, the energy and the entropies are calculated in the microcanonical and the canonical ensembles. The energy and the entropies are obtained in the Tsallis classical statistics in Sect. 2 and in the Tsallis quantum statistics in Sect. 3. The Tsallis entropy in the microcanonical ensemble, the microcanonical temperature \( T_{ph} \), the Tsallis entropy in the canonical ensemble, and the Tsallis entropy in the microcanonical ensemble are shown numerically in Sect. 4. The last section is assigned for conclusions.

\[ \Omega_0(U^{MC}) = \frac{1}{\Gamma(N + 1)} \left( \frac{U^{MC}}{\hbar \omega} \right)^N, \tag{2} \]

where \( \hbar \) is the Dirac’s constant and \( \Gamma(x) \) is the Gamma function. The number of states \( W(U^{MC}, \delta U^{MC}) \) between \( U^{MC} \) and \( U^{MC} + \delta U^{MC} \) is

\[ W(U^{MC}, \delta U^{MC}) = \frac{1}{(N - 1)!} \left( \frac{U^{MC}}{\hbar \omega} \right)^{N-1} \left( \frac{\delta U^{MC}}{\hbar \omega} \right). \tag{3} \]

The Tsallis entropy \( S_{T_q}^{MC} \) is given \([1,15,16]\) by

\[ S_{T_q}^{MC} = \ln_q W = \frac{W^{1-q} - 1}{1-q}, \tag{4} \]

where \( \ln_q x \) is the \( q \)-logarithm function. The Rényi entropy \( S_{R_q}^{MC} \) is related to the Tsallis entropy \([7,17]\):

\[ S_{R_q}^{MC} = \frac{1}{1-q} \ln(1 + (1-q)S_{T_q}^{MC}) = \ln W. \tag{5} \]

This equation indicates that the Rényi entropy \( S_{R_q}^{MC} \) is equal to the Boltzmann–Gibbs entropy in the microcanonical ensemble in the equilibrium. Using the Stirling’s formula, we have

\[ S_{R_q}^{MC} \sim (N - 1) \left[ 1 + \ln \left( \frac{U^{MC}}{(N-1)\hbar \omega} \right) \right] + \ln \left( \frac{\delta U^{MC}}{\hbar \omega} \right). \tag{6} \]

The Tsallis entropy is also obtained by substituting \( W(U^{MC}, \delta U^{MC}) \) into Eq. (4).

We introduce the microcanonical temperature \( T^{MC} \) and microcanonical physical temperature \( T_{ph}^{MC} \) by

\[ \frac{1}{T^{MC}} = \frac{\partial S^{MC}_{T_q}}{\partial U^{MC}} = W^{-q} \frac{\partial W}{\partial U^{MC}}, \tag{7a} \]

\[ \frac{1}{T_{ph}^{MC}} = \frac{\partial S_{R_q}^{MC}}{\partial U^{MC}} = \frac{1}{W} \frac{\partial W}{\partial U^{MC}}. \tag{7b} \]

Therefore, we have the relation

\[ \frac{1}{T^{MC}} = W^{1-q} \frac{1}{T_{ph}^{MC}}. \tag{8} \]

It is possible to replace \( N - 1 \) with \( N \) for large \( N \), and we have

\[ \frac{1}{T_{ph}^{MC}} = \frac{\partial S_{R_q}^{MC}}{\partial U^{MC}} \sim \frac{N}{U^{MC}}. \tag{9} \]
Therefore, we have the equation:

\[ U_{\text{MC}}^{\text{MC}} \sim N T_{\text{ph}}^{\text{MC}}. \]  \hspace{1cm} (10)

The expression of the energy represented with \( T_{\text{ph}}^{\text{MC}} \) is the same as that given in the Boltzmann–Gibbs statistics.

The Rényi entropy, Eq. (6), is represented with \( T_{\text{ph}}^{\text{MC}} \) using Eq. (11) for \( N \gg 1 \):

\[ S_{R q}^{\text{MC}} \sim N \ln \left( \frac{T_{\text{ph}}^{\text{MC}}}{\hbar \omega} \right) + N. \]  \hspace{1cm} (11)

Equation (5) is rewritten as

\[ S_{T q}^{\text{MC}} = \frac{1 - e^{-(q-1) S_{R q}^{\text{MC}}}}{q - 1}. \]  \hspace{1cm} (12)

Using Eq. (11), the Tsallis entropy in the microcanonical ensemble is given by

\[ S_{T q}^{\text{MC}} \sim \frac{1}{q-1} \left[ 1 - \left( \frac{\hbar \omega}{e T_{\text{ph}}^{\text{MC}}} \right)^{N(q-1)} \right]. \]  \hspace{1cm} (13)

It is possible to obtain the relation between \( T_{\text{ph}}^{\text{MC}} \) and \( T_{\text{MC}}^{\text{MC}} \). By differentiating \( S_{T q}^{\text{MC}} \) with respect to \( U_{\text{MC}}^{\text{MC}} \), we have

\[ \frac{1}{T_{\text{MC}}^{\text{MC}}} = e^{-(q-1) S_{R q}^{\text{MC}}} \frac{1}{T_{\text{ph}}^{\text{MC}}}. \]  \hspace{1cm} (14)

Substituting the expression of \( S_{R q}^{\text{MC}} \), Eq. (11), into Eq. (14), we have the relation between \( T_{\text{ph}}^{\text{MC}} \) and \( T_{\text{MC}}^{\text{MC}} \):

\[ T_{\text{ph}}^{\text{MC}} \sim \left( \frac{\hbar \omega}{e T_{\text{MC}}^{\text{MC}}} \right)^{\frac{N(q-1)}{1+q(q-1)}} T_{\text{MC}}^{\text{MC}}. \]  \hspace{1cm} (15)

Finally, we have the expression of \( U_{\text{MC}}^{\text{MC}} \) with \( T_{\text{MC}}^{\text{MC}} \):

\[ U_{\text{MC}}^{\text{MC}} \sim \left( \frac{\hbar \omega}{e T_{\text{MC}}^{\text{MC}}} \right)^{\frac{N(q-1)}{1+q(q-1)}} N T_{\text{MC}}^{\text{MC}}. \]  \hspace{1cm} (16)

It is possible to expand \( U_{\text{MC}}^{\text{MC}} \) with respect to \( N(q-1) \). We obtain

\[ U_{\text{MC}}^{\text{MC}} \sim N T_{\text{MC}}^{\text{MC}} \left\{ 1 - N(q-1) \left[ \ln \left( \frac{T_{\text{MC}}^{\text{MC}}}{\hbar \omega} \right) + 1 \right] \right\}. \]  \hspace{1cm} (17)

This approximate equation is valid for \( |N(q-1) \ln((\hbar \omega)/(e T_{\text{MC}}^{\text{MC}}))| \ll 1 \).

### 2.2 Canonical ensemble in the Tsallis classical statistics

We adopt the escort average in this study and calculate the partition function \( Z \). The escort average \( \langle A \rangle_q \) of a quantity \( A \) is given by

\[ \langle A \rangle_q = \frac{\int dx_1 dp_1 \cdots dx_N dp_N \ A (p(E))^q}{\int dx_1 dp_1 \cdots dx_N dp_N (p(E))^q}, \]  \hspace{1cm} (18)

where \( p(E) \) is the probability and \( h \) is the Planck's constant. The partition function \( Z \) is given by

\[ Z = \int \frac{dx_1 dp_1 \cdots dx_N dp_N}{h^N} \left[ 1 - (1-q) \frac{\beta^C}{Z_{1-q}^C} (E_1 + E_2 + \cdots + E_N - U^C) \right]^{\frac{1}{1-q}}, \]  \hspace{1cm} (19)

where \( E_j \) is given by Eq. (1b), \( \beta^C \) is the inverse canonical temperature, \( U^C \) is the energy in the canonical ensemble. To calculate \( Z \), we define \( J_N^x(x) \) by

\[ J_N^x(x) = \int_0^\infty dx_1 dx_2 \cdots dx_N \ [xU^C - x(E_1 + E_2 + \cdots + E_N)]^\gamma. \]  \hspace{1cm} (20)

The recurrence relation of \( J_N^x(x) \) is

\[ J_N^x(x) = \frac{1}{x(\gamma+1)} J_{N-1}^{x+1}(x), \quad \gamma + 1 < 0, x \neq 0. \]  \hspace{1cm} (21)

Equation (21) is used recursively, and we have

\[ J_N^x(x) = \frac{1}{x^N} \frac{(1 + \mu U^C)^\gamma + N}{(\gamma+1)(\gamma+2)\cdots(\gamma+N)} \]  \hspace{1cm} (22)

The partition function is rewritten as

\[ Z = \frac{1}{(\hbar \omega)^N} J_N^x(\mu), \]  \hspace{1cm} (23)

where \( \mu \) and \( \gamma \) are set as follows:

\[ \mu = (1-q) \beta^C / Z^{1-q}, \]  \hspace{1cm} (24a)

\[ \gamma = 1/(1-q). \]  \hspace{1cm} (24b)
The energy $U^C$ under the escort average is given by

$$
U^C = \frac{1}{(\hbar \omega)^N Z^q} \int_0^{\infty} dE_1 dE_2 \ldots dE_N \left( (1 + x U^C) - x (E_1 + E_2 + \ldots + E_N) \right)^{q\gamma} - \frac{1}{(q\gamma + 1)} \frac{\partial}{\partial x} \left( (1 + x U^C) - x (E_1 + E_2 + \ldots + E_N) \right)^{q\gamma+1}.
$$

Therefore, we have

$$
N_U = \frac{1}{(\hbar \omega)^N Z^q} \int_0^{\infty} dE_1 dE_2 \ldots dE_N \left( (1 + x U^C) - x (E_1 + E_2 + \ldots + E_N) \right)^{q\gamma} \bigg|_{x=\mu}.
$$

We use the following relation:

$$
\begin{align*}
(E_1 + E_2 + \ldots + E_N) [1 + x U^C] - x (E_1 + E_2 + \ldots + E_N) \right)^{q\gamma} & = U^C \right)^{q\gamma} + \frac{1}{(q\gamma + 1)} \frac{\partial}{\partial x} \left( (1 + x U^C) - x (E_1 + E_2 + \ldots + E_N) \right)^{q\gamma+1}.
\end{align*}
$$

The numerator of Eq. (25), $N_U$, is

$$
N_U = \frac{1}{(\hbar \omega)^N Z^q} \int_0^{\infty} dE_1 dE_2 \ldots dE_N \left( (1 + x U^C) - x (E_1 + E_2 + \ldots + E_N) \right)^{q\gamma} \bigg|_{x=\mu}.
$$

We obtain the expression of $U^C$ with $J_N^\gamma(\mu)$:

$$
U^C = \frac{N_U}{D_U} = \frac{1}{(\hbar \omega)^N Z^q} \left( U^C J_N^\gamma(\mu) - \frac{1}{(q\gamma + 1)} \left[ \frac{\partial}{\partial x} J_N^{\gamma+1}(x) \right]_{x=\mu} \right).
$$

This equation is reduced to

$$
\frac{\partial}{\partial x} \left( 1 + x U^C \right)^{q\gamma+1} \bigg|_{x=\mu} = 0.
$$

The above equation gives

$$
U^C = \frac{N}{\mu(q\gamma + 1)}.
$$

With Eqs. (24a) and (24b), we have

$$
U^C = \frac{N}{Z^{1-q}} = NT^C Z^{1-q}.
$$

The canonical physical temperature is given by

$$
\frac{1}{T_{ph}^C} = \frac{\beta^C}{Z^{1-q}} = \frac{1}{T_{ph}^C} Z^{1-q}.
$$

The energy $U^C$ represented with $T_{ph}^C$ is

$$
U^C = NT_{ph}^C.
$$

Equation (36) in the canonical ensemble is equivalent to Eq. (10) in the microcanonical ensemble when $T_{ph}^C$ equals $T_{ph}$. The number of the oscillators $N$ is restricted from the above: $N < 1/(q-1)$.

We attempt to obtain the entropies by calculating the partition function. The partition function $Z$ with Eqs. (24a) and (24b) is explicitly given by

$$
Z = \left( \frac{T_{ph}^C}{\hbar \omega} \right)^N \left[ \frac{1}{(2-q)(3-2q) \ldots ((N+1) - Nq)} \right]^{1/\gamma+1} \prod_{j=1}^N \left[ \frac{1}{((j+1) - jq)} \right], \quad 1 - q + N < 0.
$$

The Rényi entropy $S_{Rq}^C$ is given by

$$
S_{Rq}^C = \ln Z = N \ln \left( \frac{U^C}{N \hbar \omega} \right).
$$
when ensemble is given by

\[
S_{R} \sim \frac{1}{q - 1} \ln \left( \frac{T_{ph}^{C}}{\hbar \omega} \right) + N + \frac{1}{2} N(q - 1).
\]  

(39)

Using Eq. (39), the Tsallis entropy in the canonical ensemble is given by

\[
S_{R}^{C} \sim \frac{1}{q - 1} \ln \left( \frac{T_{ph}^{C}}{\hbar \omega} \right) + N + \frac{1}{2} N(q - 1).
\]  

(40)

The Rényi entropy \( S_{R}^{C} \) in the canonical ensemble resembles the entropy \( S_{R}^{MC} \) in the microcanonical ensemble. The difference between \( S_{R}^{C} \) and \( S_{R}^{MC} \), Eq. (11), for large \( N \) is

\[
S_{R}^{C} - S_{R}^{MC} \sim \frac{1}{2} N(q - 1),
\]  

(41)

when \( T_{ph}^{MC} \) equals \( T_{ph}^{C} \). We note that \( N(q - 1) \) is positive and less than one in the canonical ensemble. Therefore, \( S_{R}^{C} \) equals \( S_{R}^{MC} \) substantially for \( 0 < N(q - 1) < 1 \). It is found from Eq. (41) that \( S_{R}^{C} \) equals \( S_{R}^{MC} \) in the Boltzmann–Gibbs limit (\( q \) approaches one).

We also calculate the difference between \( S_{T}^{MC} \) and \( S_{T}^{C} \) for \( T_{ph}^{MC} = T_{ph}^{C} \):

\[
S_{T}^{C} - S_{T}^{MC} \sim \frac{1}{q - 1} \left( \frac{\hbar \omega}{e T_{ph}^{C}} \right)^{N(q - 1)} \ln \left( \frac{T_{ph}^{C}}{\hbar \omega} \right) + N \left( 1 - \exp \left( -\frac{1}{2N} (N(q - 1))^2 \right) \right).
\]  

(42)

We have the relation, \( 0 < (N(q - 1))^2 < N(q - 1) < 1 \), for \( q > 1 \). We obtain the following expression for \( T_{ph}^{MC} = T_{ph}^{C} \) by expanding the exponential term in the square brackets of Eq. (42):

\[
S_{T}^{C} - S_{T}^{MC} \sim \frac{1}{2} N(q - 1) \left( \frac{\hbar \omega}{e T_{ph}^{C}} \right)^{N(q - 1)}.
\]  

(43)

It is easily seen from the above expression that \( S_{T}^{C} \) equals \( S_{T}^{MC} \) in the Boltzmann–Gibbs limit.

### 3 Harmonic oscillators in the Tsallis quantum statistics

In this section, we calculate the quantities of \( N \) independent harmonic oscillators in the microcanonical ensemble and the canonical ensemble in the Tsallis quantum statistics.

The quantized Hamiltonian is given by

\[
\hat{H} = \sum_{j=1}^{N} \left( \frac{1}{2m} \dot{p}_{j}^{2} + \frac{1}{2} m \omega^{2} x_{j}^{2} \right) = \sum_{j=1}^{N} \hbar \omega \left( \hat{n}_{j} + \frac{1}{2} \right),
\]  

(44)

where \( \hat{n}_{j} \) is the number operator.

#### 3.1 Microcanonical ensemble in the Tsallis quantum statistics

We attempt to obtain the expression of \( U^{MC} \) by applying the standard procedure. The energy \( U^{MC} \) of \( N \) independent harmonic oscillators is given by

\[
U^{MC}(n_{1}, n_{2}, \ldots, n_{N}) = \left( M + \frac{N}{2} \right) \hbar \omega, \quad M = \sum_{j=1}^{N} n_{j},
\]  

(45)

where \( n_{j} \) is an integer which is larger than or equal to zero. The number of states \( W(M, N) \) is

\[
W(M, N) = \left( \frac{M + N - 1}{M} \right) = \frac{(M + N - 1)!}{M!(N - 1)!}.
\]  

(46)

The Rényi entropy \( S_{R}^{MC} \) and the Tsallis entropy \( S_{T}^{MC} \) are given by \( \ln W \) and \( \ln W \) respectively. Using the Stirling’s formula, we have the approximated value of \( W \):

\[
W(M, N) \sim \frac{(M + N - 1)^{M + N - 1}}{M^{M}(N - 1)^{N - 1}}.
\]  

(47)

The Rényi entropy \( S_{R}^{MC} \) is

\[
S_{R}^{MC} \sim (M + N - 1) \ln (M + N - 1) - M \ln M - (N - 1) \ln (N - 1).
\]  

(48)

For fixed \( N \), the inverse of the microcanonical physical temperature (Eq. (7b)) is given by

\[
\frac{1}{T_{ph}^{MC}} = \frac{\partial S_{R}^{MC}}{\partial U^{MC}} = \frac{1}{\hbar \omega} \frac{\partial S_{R}^{MC}}{\partial M} = \frac{1}{\hbar \omega} \ln \left( \frac{M + N - 1}{M} \right),
\]  

(49)
For large \( N \), we have
\[
\frac{M + N - 1}{M} \sim \frac{U^{MC} + \frac{N \hbar}{2}}{U^{MC} - \frac{N \hbar}{2}}.
\]
(50)

From Eqs. (49) and (50), the well-known expression of \( U^{MC} \) is obtained:
\[
U^{MC} = N \hbar \omega \left( f_B(T_{ph}^{MC}) + \frac{1}{2} \right),
\]
(51)
where \( f_B(T) \) is given by
\[
f_B(T) = \frac{1}{\exp \left( \frac{\hbar \omega}{T} \right) - 1}.
\]
(52)

The Tsallis entropy \( S_{Tq}^{MC} \) is given by
\[
S_{Tq}^{MC} = W^{1-q} - 1, \quad \frac{1}{1-q}.
\]
(53)

The microcanonical temperature \( T^{MC} \) (Eq. (7a)) is
\[
1 - \frac{\partial S_{Tq}^{MC}}{\partial U^{MC}} = W^{-q} \frac{\partial W}{\partial U^{MC}} = \frac{1}{\hbar \omega} W^{-q} \frac{\partial W}{\partial M}.
\]
(54)
This leads to
\[
\frac{\hbar \omega}{T^{MC}} = W^{1-q} \frac{\partial}{\partial M} \ln W.
\]
(55)

The relation between \( T_{ph}^{MC} \) and \( W \) is already given in Eq. (49) with \( S_{Rq}^{MC} = \ln W \):
\[
1 - \frac{\partial S_{Tq}^{MC}}{\partial U^{MC}} = \frac{1}{\hbar \omega} \frac{\partial W}{\partial M} \ln W.
\]
(56)

Therefore, the relation between \( T_{ph}^{MC} \) and \( T^{MC} \) is
\[
1 - \frac{\partial S_{Tq}^{MC}}{\partial U^{MC}} = W^{1-q} \frac{1}{T_{ph}^{MC}}.
\]
(57)

This relation was already given as Eq. (8).

The expression of \( U^{MC} \) with Eq. (57) is given by
\[
U^{MC} = N \hbar \omega \left( \frac{1}{\exp \left( \frac{W^{1-q} \hbar \omega}{T^{MC}} \right) - 1} + \frac{1}{2} \right).
\]
(58)

From Eq. (58), we have the energy \( U^{MC} \) represented with \( T^{MC} \):
\[
U^{MC} \sim N \hbar \omega \left( f_B(T^{MC}) + \frac{1}{2} \right) + N(1-q) \left[ (f_B(T^{MC}) + 1) \ln (f_B(T^{MC}) + 1) \right. \\
\left. - f_B(T^{MC}) \ln f_B(T^{MC}) \right]
\times f_B(T^{MC}) \left( f_B(T^{MC}) + 1 \right) \left( \frac{\hbar \omega}{T^{MC}} \right).
\]
(59)

Equation (59) at high \( T^{MC} \) is
\[
U^{MC} \sim N T^{MC} \left\{ 1 - N(q-1) \left[ \ln \left( \frac{T^{MC}}{T^{MC}} \right) + 1 \right] \right\},
\]
(60)
where the condition \( |N(q-1) \ln((\hbar \omega)/(eT^{MC}))| \ll 1 \) should be satisfied in Eq. (60). The expression of \( U^{MC} \) in the quantum case, Eq. (51), coincides with that in the classical case, Eq. (10), at high \( T^{MC} \). The expression of \( U^{MC} \) in the quantum case, Eq. (60), also coincides with that in the classical case, Eq. (17), at high \( T^{MC} \).

The Rényi entropy \( S_{Rq}^{MC} \) is also given by
\[
S_{Rq}^{MC} \sim N \left\{ (1 + f_B(T_{ph}^{MC})) \ln (1 + f_B(T_{ph}^{MC})) \\
- f_B(T_{ph}^{MC}) \ln (f_B(T_{ph}^{MC})) \right\}.
\]
(61)
At high \( T_{ph}^{MC} \), we have
\[
S_{Rq}^{MC} \sim N \ln \left( \frac{T_{ph}^{MC}}{\hbar \omega} \right) + N.
\]
(62)
Equation (62) is the same as Eq. (11). Therefore, the expression of the Tsallis entropy is given by Eq. (13).

### 3.2 Canonical ensemble in the Tsallis quantum statistics

The density operator \( \hat{\rho} \) under the escort average in the canonical ensemble is given by
\[
\hat{\rho} = Z^{-1} \left( 1 - (1-q) \frac{\beta^C}{c_q} (\hat{H} - U^C) \right)^{1-\frac{1}{q}},
\]
(63a)
\[
c_q = \text{Tr} [\hat{\rho}^q],
\]
(63b)
\[
Z = \text{Tr} \left[ \left( 1 - (1-q) \frac{\beta^C}{c_q} (\hat{H} - U^C) \right)^{1-\frac{1}{q}} \right],
\]
(63c)
where the escort average \( \langle \hat{A} \rangle_q \) of an operator \( \hat{A} \) is defined by
\[
\langle \hat{A} \rangle_q = \frac{\text{Tr} [\hat{\rho}^q \hat{A}]}{\text{Tr} [\hat{\rho}^q]}.
\]
(64)
The following relation between \( c_q \) and \( Z \) is used to find the expressions of physical quantities:
\[
c_q = Z^{1-q}.
\]
(65)
The partition function $Z$ is calculated as

$$Z = \sum_{n_1, \ldots, n_N=0}^{\infty} (ab + a(n_1 + \cdots + n_N))^{\frac{1}{q}}\tau_q,
$$

(66a)

where

$$a = (q-1) \left( \frac{\beta C \hbar \omega}{c_q} \right),
$$

(66b)

$$ab = 1 + (q-1) \left( \frac{\beta C}{c_q} \right) \left( \frac{1}{2} N \hbar \omega - U^C \right). \quad (66c)$$

The partition function $Z$ is represented with the Barnes zeta function $\zeta_B$:

$$Z = a^{\frac{1}{q-1}} \zeta_B \left( \frac{1}{q-1}, b; N \right), \quad (67)$$

where $\zeta_B$ (see also Appendix A) is given by

$$\zeta_B(s, \alpha; N) = \sum_{n_1, \ldots, n_N=0}^{\infty} \frac{1}{(\alpha + n_1 + n_2 + \cdots + n_N)^s}. \quad (68)$$

In the same way, $c_q$ is calculated directly using the definition $c_q = \text{Tr} [\hat{\rho}^q]$:

$$c_q = \frac{1}{Z} a^{\frac{q}{q-1}} \zeta_B \left( \frac{q}{q-1}, b; N \right). \quad (69)$$

We obtain the following self-consistent equation (norm equation) from Eq. (65):

$$a\zeta_B \left( \frac{1}{q-1}, b; N \right) = \zeta_B \left( \frac{q}{q-1}, b; N \right). \quad (70)$$

We obtain the energy $U^C$ approximately from the self-consistent equation (norm equation), Eq. (70). For large $\alpha$, we have

$$\zeta_B(1 + z, \alpha; N) \sim \frac{1}{\prod_{j=0}^{N-1}(z-j)} \frac{1}{\alpha^{z-(N-1)}} + O \left( \frac{1}{\alpha^{1+z-(N-1)}} \right), \quad (1 + z > N). \quad (71)$$

Applying Eq. (71) to Eq. (70), we obtain the product $ab$ approximately for large $b$ (high physical temperature):

$$ab = \frac{\prod_{j=0}^{N-1} ((2q - j(q-1)))}{\prod_{j=0}^{N-1} (1 - (j+1)(q-1))} = \frac{\prod_{j=0}^{N-1} (1 - (j+1)(q-1))}{\prod_{j=0}^{N-1} (1 - j(q-1))} = 1 - N(q-1), \quad N < (q-1)^{-1}. \quad (72)$$

Equation (66c) gives the energy $U^C$:

$$U^C = N T_{ph} + \frac{1}{2} \hbar \omega. \quad (73)$$

We also calculate the escort average of the Hamiltonian directly with Eq. (65):

$$U^C = \frac{\text{Tr} [\hat{\rho}^q \hat{H}]}{\text{Tr} [\hat{\rho}^q]} = \frac{1}{2} \hbar \omega + \frac{1}{Z} \frac{\alpha q}{\zeta_B \left( \frac{q}{q-1}, b; N \right)} q_{\zeta B} \left( \frac{q}{q-1}, b; N \right). \quad (74)$$

Another form of the energy $U^C$ is also derived:

$$U^C = \frac{\hbar \omega}{Z} a^{\frac{q}{q-1}} \zeta_B \left( \frac{1}{q-1}, b; N \right) + \left( \frac{N}{2} - b \right) \zeta_B \left( \frac{q}{q-1}, b; N \right). \quad (75)$$

It is possible to calculate Eq. (74) approximately with Eq. (67) using the approximate expression of Barnes zeta function, Eq. (71). For large $b$, we obtain

$$\frac{\zeta_B \left( \frac{q}{q-1}, b; N \right) \sim b^{-1}[1 - N(q-1)], \quad N < (q-1)^{-1}. \quad (76)$$

We have Eq. (73) again from Eq. (74).

The Rényi entropy $S^C_{Rq}$ can be calculated with the partition function $Z$: $S^C_{Rq} = \ln Z$. For large $b$, the partition function $Z$ is approximated as

$$Z = a^{\frac{1}{q-1}} \zeta_B \left( \frac{1}{q-1}, b; N \right) \sim \frac{\prod_{j=0}^{N-1} (q-1)^{N-j}}{\prod_{j=0}^{N-1} (1 - (j+1)(q-1))} = \frac{b^N}{(ab)^{(q-1)-N}}. \quad (77)$$
We have the following expression of $S_{Rq}^C$ for large $N$ by expanding $\ln Z$ with respect to $N(q-1)$:

$$S_{Rq}^C \sim N \ln \left( \frac{T_{ph}^C}{\hbar \omega} \right) + N + \frac{1}{2} N(q-1). \quad (78)$$

Equation (78) is the same as Eq. (39). Therefore, the expression of the Tsallis entropy is given by Eq. (40).

We obtain the difference between $S_{Rq}^C$, Eq. (78), and $S_{Rq}^{MC}$, Eq. (62), at high physical temperature in the quantum statistics. The difference is

$$S_{Rq}^C - S_{Rq}^{MC} = \frac{1}{2} N(q-1), \quad (79)$$

when $T_{ph}^C$ equals $T_{ph}^{MC}$. We note that the quantity $N(q-1)$ is restricted below one in the canonical ensemble. Therefore, $S_{Rq}^C$ equals $S_{Rq}^{MC}$ substantially for $0 < N(q-1) < 1$, as shown in the classical statistics. These results in the quantum statistics coincide with those in the classical statistics. At high physical temperature, the difference between $S_{Tq}^C$ and $S_{Tq}^{MC}$ in the quantum statistics is the same as the difference in the classical statistics. Therefore, the difference between $S_{Tq}^C$ and $S_{Tq}^{MC}$ is given by Eq. (42).

### 4 Numerical values of Tsallis entropies

In this section, we calculate the Tsallis entropies numerically to show the difference between the Tsallis entropy in the microcanonical ensemble and the Tsallis entropy in the canonical ensemble. The physical temperature $T_{ph}^{MC}$ in the microcanonical ensemble and the physical temperature $T_{ph}^C$ in the canonical ensemble are set to the physical temperature $T_{ph}$, and the scaled physical temperature $T_{ph}^S$ is defined by $T_{ph}/(\hbar \omega)$. The expressions of the Rényi entropies were obtained in the previous sections. The expression of the Rényi entropy $S_{Rq}^{MC}$ in the microcanonical ensemble is $N \ln(T_{ph}^{MC}) + N$ and that of the Rényi entropy $S_{Rq}^C$ in the canonical ensemble is $N \ln(T_{ph}^C) + N + N(q-1)/2$. The difference between $S_{Rq}^C$ and $S_{Rq}^{MC}$ is $N(q-1)/2$. We use these expressions of the Rényi entropies. The Tsallis entropies are calculated numerically using the relation between the Rényi entropy $S_{Rq}$ and the Tsallis entropy $S_{Tq}$ in this section: $S_{Tq} = (1 - \exp(- (q-1) S_{Rq}))/ (q-1)$. The Tsallis entropy $S_{Tq}$ goes to $1/(q-1)$ as $T_{ph}^C$ goes to infinity for $q > 1$. Therefore, the difference between $S_{Tq}^C$ and $S_{Tq}^{MC}$ goes to zero as $T_{ph}^C$ goes to infinity, as found from Eq. (43).

Figure 1a shows the Tsallis entropies $S_{Tq}^{MC}$ as functions of $T_{ph}^{MC}$ at $N(q-1) = 0.9$ for $N = 10, 30$, and 50. Figure 1b shows the Tsallis entropies $S_{Tq}^C$ as functions of $T_{ph}^C$ at $N(q-1) = 0.9$ for $N = 10, 30$, and 50. Figure 1c shows the differences $S_{Tq}^C - S_{Tq}^{MC}$ as functions of $T_{ph}^C$ at $N(q-1) = 0.9$ for $N = 10, 30$, and 50. The range of $T_{ph}^C$ is $1 \leq T_{ph}^C \leq 30$ in these calculations, because it is considered that the expressions of the Rényi entropies are valid for high physical temperature. As shown in Fig. 1a, b, the values of $S_{Tq}^{MC}$ and $S_{Tq}^C$ at $T_{ph}^C = 30$ are approximately $N/0.9$. The differences are quite small compared with the values of the Tsallis entropies, as shown in Fig. 1c. The behaviors in Fig. 1(c) are consistent with Eq. (43): the approximate expression of the difference, Eq. (43), is a function of $N(q-1)$.

Figure 2 shows the differences between the Tsallis entropy $S_{Tq}^C$ in the canonical ensemble and the Tsallis entropy $S_{Tq}^{MC}$ in the microcanonical ensemble, $S_{Tq}^C - S_{Tq}^{MC}$, in the range of $1 \leq T_{ph}^{MC} \leq 30$. Figure 2a shows the differences between $S_{Tq}^C$ and $S_{Tq}^{MC}$ at $T_{ph}^C = 1.018$ for $N = 10, 30$, and 50. As is expected, the differences go to zero as $T_{ph}^{MC}$ goes to infinity. The approximate equation, Eq. (43), depends on $N(q-1)$. Therefore, the difference $S_{Tq}^C - S_{Tq}^{MC}$ decreases as $N$ increases at high $T_{ph}^C$ when $q$ is fixed, as shown in Fig. 2a. Figure 2b shows the differences between $S_{Tq}^C$ and $S_{Tq}^{MC}$ at $N = 30$ for $q = 1.01, 1.02, 1.03$. The difference $S_{Tq}^C - S_{Tq}^{MC}$ decreases as $q$ increases at high $T_{ph}^{MC}$ when $N$ is fixed, as shown in Fig. 2b. The behavior of the difference at fixed $N$ is similar to that at fixed $q$, because the approximate equation, Eq. (43), depends on $N(q-1)$.

It is shown numerically that the differences, $S_{Tq}^C - S_{Tq}^{MC}$, are quite small compared with the values of the Tsallis entropies in the case of $N(q-1) < 1$. These results indicate that the Tsallis entropy in the canonical ensemble is not different from the Tsallis entropy in the microcanonical ensemble substantially for $N(q-1) < 1$. It seems that the difference $S_{Tq}^C - S_{Tq}^{MC}$ goes to zero as $T_{ph}^{MC}$ goes to infinity. This behavior can be explained from the fact that the Rényi entropy in the microcanonical ensemble is approximately $N \ln(T_{ph}^{MC}) + N$ and the fact that the difference $S_{Rq}^C - S_{Rq}^{MC}$ is approximately $N(q-1)$.

### 5 Conclusions

We studied the thermodynamic quantities in the Tsallis classical and the Tsallis quantum statistics of the entropic parameter $q$, where the value of $q$ is larger than one. We treated the $N$ independent harmonic oscillators with the same frequency, where $N$ is the number of the oscillators. We introduced the physical temperature which characterizes the equilibrium. The energy was represented with the physical temperature, and the Rényi entropy in the microcanonical ensemble was compared with the Rényi entropy in the canonical ensemble. The Tsallis entropy was calculated with the relation between the Tsallis entropy $S_{Tq}$ and the Rényi entropy $S_{Rq}$: $S_{Rq} = (1 - q)^{-1} \ln(1 + (1 - q) S_{Tq})$, and the Tsallis...
Fig. 1 The Tsallis entropies $S^{MC}_{Tq}$ as functions of $T^{sc}_{ph}$ in the microcanonical ensemble, the Tsallis entropies $S^{C}_{Tq}$ as functions of $T^{sc}_{ph}$ in the canonical ensemble, and the differences between these two entropies, $S^{C}_{Tq} - S^{MC}_{Tq}$, in the range of $1 \leq T^{sc}_{ph} \leq 30$ at $N(q-1) = 0.9$ for $N = 10, 30,$ and $50$.

Fig. 2 The differences between the Tsallis entropy $S^{C}_{Tq}$ in the canonical ensemble and the Tsallis entropy $S^{MC}_{Tq}$ in the microcanonical ensemble, $S^{C}_{Tq} - S^{MC}_{Tq}$, in the range of $1 \leq T^{sc}_{ph} \leq 30$. 

The differences $S^{C}_{Tq} - S^{MC}_{Tq}$ at $q = 1.018$ for $N = 10, 30,$ and $50$. 

The differences $S^{C}_{Tq} - S^{MC}_{Tq}$ at $N = 30$ for $q = 1.01, 1.02,$ and $1.03$. 

The differences between $S^{C}_{Tq}$ and $S^{MC}_{Tq}$.
entropy in the microcanonical ensemble was compared with the Tsallis entropy in the canonical ensemble. The physical temperature $T_{ph}$ is less than or equal to the temperature $T$ for $q > 1$. The microcanonical physical temperature $T_{ph}^{MC}$ and the microcanonical temperature $T^{MC}$ has the relation $T_{ph}^{MC} = W^{1-q}T^{MC}$, where $W$ is the number of states. The quantity $W^{-1}$ is larger than or equal to one for $W \geq 1$ and $q > 1$. Therefore, the physical temperature $T_{ph}^{MC}$ is less than or equal to the temperature $T^{MC}$. The canonical physical temperature $T_{ph}^{C}$ and the canonical temperature $T^{C}$ has the relation $T_{ph}^{C} = Z^{1-q}T^{C}$, where $Z$ is the partition function. The partition function $Z$ is larger than or equal to one for $S_{q}^{C} \geq 0$ because of the relation $Z = \exp(S_{q}^{C})$, where $S_{q}^{C}$ is the Rényi entropy in the canonical ensemble. The canonical physical temperature $T_{ph}^{C}$ is less than or equal to the temperature $T^{C}$ for $q > 1$.

The condition $N(q-1) < 1$ appears in the canonical ensemble, while this condition does not appear in the microcanonical ensemble. Such conditions were already obtained in the previous studies [2, 6]. As expected, the maximum value of $N$ which satisfies $N(q-1) < 1$ goes to infinity, as $q$ approaches one. This condition is explained by the power-law behavior of the distribution for $q > 1$ in the Tsallis statistics. This condition does not appear for $q = 1$ (the Boltzmann–Gibbs statistics), because the distribution decreases exponentially.

The existence of the condition is the apparent difference between the microcanonical ensemble and the canonical ensemble in the Tsallis statistics.

It was shown that the energy $U$ represented with the physical temperature $T_{ph}$ is not $q$-dependent at high $T_{ph}$ in both the ensembles: the energy has the well-known relation $U = NT_{ph}$ at high physical temperature. At high physical temperature, the expression of $U$ represented with $T_{ph}$ in the Tsallis statistics is the same as that in the Boltzmann–Gibbs statistics.

The $q$-dependence of the Rényi entropy in the canonical ensemble slightly differs from that in the microcanonical ensemble. The Rényi entropy represented with the physical temperature is not $q$-dependent in the microcanonical ensemble. Therefore, the Tsallis entropy represented with the physical temperature is not $q$-dependent in the microcanonical ensemble. This result is consistent with the result within Tsallis formalism in the microcanonical ensemble [10]. In contrast, the Rényi entropy has the $q$-dependent term, $N(q-1)/2$, in the canonical ensemble. The Rényi entropy in the canonical ensemble is not different from the Rényi entropy in the microcanonical ensemble substantially, because $N(q-1)$ is less than one. In the same way, the Tsallis entropy in the canonical ensemble is not different from the Tsallis entropy in the microcanonical ensemble substantially, because $N(q-1)$ is less than one.

A self-consistent equation (norm equation) appears generally in the Tsallis-3 statistics which is the Tsallis statistics with escort average, where the Tsallis-3 statistics is the statistics employed in this study. In contrast, such a self-consistent equation does not appear in the Tsallis-2 statistics which is the Tsallis statistics with $q$-average: the $q$-average is defined by $\sum_{j} A_{j}(p_{j}(\beta^{(2)}))^{q}$, where $\beta^{(2)}$ is the inverse temperature in the Tsallis-2 statistics. The inverse temperature $\beta^{(2)}$ and the inverse temperature $\beta^{(3)}$ are related each other [3], where $\beta^{(3)}$ is the inverse temperature in the Tsallis-3 statistics. The escort average of Hamiltonian, $U^{(3)}$, can be represented as a function of $\beta^{(2)}$ without solving the self-consistent equation. The physical temperature is also represented as a function of $\beta^{(2)}$. Therefore, we can plot the escort average of a quantity as a function of the physical temperature numerically [19]. However, the self-consistent equation appears when $U^{(3)}$ is represented with $\beta^{(3)}$, because the relation between $\beta^{(2)}$ and $\beta^{(3)}$ contains the energy $U^{(3)}$.

We comment on the relation between the Tsallis statistics and the other statistics in the equilibrium. In the microcanonical ensemble, it is considered that an entropy $S^{MC}$ is calculated with the number of states $W$: $S^{MC} = f(W)$, where $f$ is a function. The Tsallis entropy $S_{q}^{MC}$ is also given by $S_{q}^{MC}(W) = \ln_{q} W$. The number of states $W$ is represented as $W = f^{inv}(S^{MC})$ when the inverse function $f^{inv}$ of $f$ exists. In that case, the entropies are related each other: $f^{inv}(S^{MC}) = \ln_{q}(f^{inv}(S^{MC}))$. Evidently, this relation can be generalized to the relation between two entropies. In the canonical ensemble, the condition $N(q-1) < 1$ appears in the Tsallis statistics of entropic parameter $q$. Such conditions may appear in other statistics. The difference of the conditions is the difference between statistics even when the entropies are related.

In this paper, we discussed the difference between the quantity in the microcanonical ensemble and the quantity in the canonical ensemble in the Tsallis statistics of the entropic parameter $q$ for $N$ independent harmonic oscillators. The condition $N < (q-1)^{-1}$ appears in the canonical ensemble. The various properties of the unconventional statistics will be studied in the future.

**Funding** This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

**Data availability statement** This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This study is theoretical, and the graphs were drawn with the equations given in this paper.]

**Conflict of interest** The author declares no competing interest.
Appendix A: Hurwitz zeta function and Barnes zeta function

In this appendix, we give the approximate expressions of Hurwitz and Barnes zeta functions. The derivation is also given in the appendices A and B of the reference [2].

The Hurwitz zeta function \( \zeta_H \) is defined by

\[
\zeta_H(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)^s}.
\]

(A.1)

In this appendix, we treat the case of \( s > 1 \) and \( \alpha > 0 \). Applying the Euler–Maclaurin formula, we have

\[
\zeta_H(1+z, \alpha) = \frac{1}{z\alpha^2} + \frac{1}{2z\alpha^{1+z}} + \sum_{k=1}^{M-1} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \Gamma(z+k+1) \frac{1}{\alpha^{z+k+1}}
\]

\[
- \frac{(-1)^{M}}{M!} \int_0^\infty \text{d}x B_M(x-[x]) f^{(M)}(x)
\]

(A.2)

\( (z > 0, \alpha > 0) \),

where \( B_k \) is Bernoulli number. The Hurwitz zeta function can be expressed in the following forms [18]. From Eq. (A.2), we find that \( \zeta_H(1+z, \alpha) \) for \( \alpha \gg 1 \) behaves as

\[
\zeta_H(1+z, \alpha) \sim \frac{1}{z\alpha^2}.
\]

(A.3)

The Barnes zeta function [20,21] is defined by

\[
\zeta_B(s, \alpha|\omega_N) = \sum_{n_1, \ldots, n_N=0}^{\infty} \frac{1}{(\alpha + \omega_1 n_1 + \ldots + \omega_N n_N)^s}
\]

\( \omega_N = (\omega_1, \omega_2, \ldots, \omega_N) \),

(A.4)

where \( s > N, \alpha > 0, \) and \( \omega_j > 0 \). The Barnes zeta function for sufficiently large \( \alpha \) has the following relation

\[
\zeta_B(1+z, \alpha|\omega_{N-1}) \sim \frac{1}{\omega_N} \zeta_B(z, \alpha|\omega_N).
\]

(A.5)

This relation is derived with Eq. (A.3). Using the recurrence relation, Eq. (A.5), we have the following approximate expression of \( \zeta_B \) for \( \alpha \gg 1 \):

\[
\zeta_B(1+z, \alpha|\omega_N) \sim \frac{1}{\omega_N} \left( \prod_{j=0}^{N-1} (z-j) \prod_{j=1}^{N} \omega_j \right) \alpha^{-z(N-1)}
\]

\( (z - (N-1) > 0) \).

(A.6)

In the present case, \( \omega_N \) is set to \( \omega_N = (1, 1, \ldots, 1) \). For simplicity, we use the following notation for the Barnes zeta function \( \zeta_B \) with \( \omega_N = (1, 1, \ldots, 1) \):

\[
\zeta_B(s, \alpha; N) = \sum_{n_1, \ldots, n_N=0}^{\infty} \frac{1}{(\alpha + n_1 + n_2 + \cdots + n_N)^s}
\]

(A.7)

The approximate expression of \( \zeta_B(1+z, \alpha; N) \) for \( \alpha \gg 1 \) is

\[
\zeta_B(1+z, \alpha; N) \sim \frac{1}{\left( \prod_{j=0}^{N-1} (z-j) \right) \alpha^{-z(N-1)}}
\]

\( (z - (N-1) > 0) \).

(A.8)

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