REMARKS ON NONLINEAR EQUATIONS WITH MEASURES

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To the memory of I. V. Skrypnik

Abstract. We study the Dirichlet boundary value problem for equations with absorption of the form $-\Delta u + g \circ u = \mu$ in a bounded domain $\Omega \subset \mathbb{R}^N$ where $g$ is a continuous odd monotone increasing function. Under some additional assumptions on $g$, we present necessary and sufficient conditions for existence when $\mu$ is a finite measure. We also discuss the notion of solution when the measure $\mu$ is positive and blows up on a compact subset of $\Omega$.

1. Introduction

In this paper we discuss some aspects of the boundary value problem

$$
-\Delta u + g \circ u = \mu \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega,
$$

where $\mu \in \mathcal{M}_\rho(\Omega)$, i.e. $\mu$ is a Borel measure such that

$$
\int_\Omega \rho d|\mu| < \infty, \quad \rho(x) = \text{dist}(x, \partial \Omega).
$$

In addition we define a notion of solution in the case that $\mu$ is a positive Borel measure which may explode on a compact subset of the domain and discuss the question of existence and uniqueness in this case. We always assume that $g \in C(\mathbb{R})$ is a monotone increasing function such that $g(0) = 0$. To simplify the presentation we also assume that $g$ is odd.

A function $u \in L^1(\Omega)$ is a weak solution of the boundary value problem $(1.1)$, $\mu \in \mathcal{M}_\rho$, if $u \in L^p_\rho(\Omega)$, i.e.

$$
\int_\Omega g(u)\rho dx < \infty
$$

and

$$
\int_\Omega (-\nu \Delta \phi + g \circ \nu \phi)dx = \int_\Omega \phi d\mu
$$

for every $\phi \in C^2_0(\Omega)$ (= space of functions in $C^2(\Omega)$ vanishing on $\partial \Omega$).

We say that $u$ is a solution of the equation

$$
-\Delta u + g \circ u = \mu \quad \text{in } \Omega
$$

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if \( u \) and \( g \circ u \) are in \( L^1_{\text{loc}}(\Omega) \) and (1.2) holds for every \( \phi \in C_c^2(\Omega) \).

Brezis and Strauss [6] proved that, if \( \mu \) is an \( L^1 \) function the problem possesses a unique solution. This result does not extend to arbitrary measures in \( \mathcal{M}_\rho(\Omega) \).

Denote by \( \mathcal{M}_\rho^0 \) the set of measures \( \mu \in \mathcal{M}_\rho \) for which (1.1) is solvable. A measure in \( \mathcal{M}_\rho^0 \) is called a \( g \)-good measure. It is known that, if a solution exists then it is unique.

We say that \( g \) is subcritical if \( \mathcal{M}_\rho^g = \mathcal{M}_\rho \). Benilan and Brezis, [5] and [4] proved that the following condition is sufficient for \( g \) to be subcritical:

\[
\int_0^1 \frac{g(r^{2-N})}{r^{N-1}} dr < \infty.
\]

In the case that \( g \) is a power non-linearity, i.e., \( g = g_q \) where \( g_q(t) = |t|^q \text{sign } t \) in \( \mathbb{R} \), \( q > 1 \), this condition means that \( q < q_c := N/(N - 2) \). Benilan and Brezis also proved that, if \( g = g_q \) and \( q \geq q_c \), problem (1.1) has no solution when \( \mu \) is a Dirac measure.

Later Baras and Pierre [3] gave a complete characterization of \( \mathcal{M}_\rho^g \) in the case that \( g = g_q \) with \( q \geq q_c \). They proved that a finite measure \( \mu \) is \( g_q \)-good if and only if \( |\mu| \) does not charge sets of \( \bar{C}_{2,q'} \) capacity zero, \( q' = q/(q - 1) \). Here \( \bar{C}_{\alpha,p} \) denotes Bessel capacity with the indicated indices.

In the present paper we extend the result of Baras and Pierre to a large class of non-linearities and also discuss the notion of solution in the case that \( \mu \) is a positive measure which explodes on a compact subset of \( \Omega \).

2. Statement of results

Denote by \( \mathcal{H} \) the set of even functions \( h \) such that

\[
\begin{align*}
&h \in C^1(\mathbb{R}), \quad h(0) = 0, \quad \text{\( h \) is strictly convex}, \\
&h'(0) = 0, \quad h'(t) > 0 \quad \forall t > 0, \quad \lim_{t \to \infty} h'(t) = \infty.
\end{align*}
\]

For \( h \in \mathcal{H} \) denote by \( L^h(\Omega) \) the corresponding Orlicz space in a domain \( \Omega \subset \mathbb{R}^N \):

\[
L^h(\Omega) = \{ f \in L^1_{\text{loc}}(\Omega) \mid \exists k > 0 : h \circ (f/k) \leq 1 \}
\]

with the norm

\[
\|f\|_{L^h} = \inf\{k > 0 \mid h \circ (f/k) < \infty\}.
\]

Further denote by \( h^* \) the conjugate of \( h \). Since, by assumption, \( h \) is strictly convex, \( h' \) is strictly increasing so that

\[
h^*(t) = \int_0^t (h')^{-1}(s)ds.
\]

Let \( G \) be the Green kernel for \( -\Delta \) in \( \Omega \) and denote

\[
\mathcal{G}_\mu(x) = \int_{\Omega} G(x,y)d\mu(y) \quad \forall x \in \Omega, \quad \mu \in \mathcal{M}_\rho(\Omega).
\]
For every $h \in H$, the capacity $C_{2,h}$ in $\Omega$ is defined as follows. For every compact set $E \subset \Omega$ put:

\begin{equation}
C_{2,h}(E) = \sup\{\mu(\Omega) : \mu \in \mathcal{M}(\Omega), \mu \geq 0, \mu(E^c) = 0, \|\mathcal{G}\mu\|_{L^h^*} \leq 1\}.
\end{equation}

If $O$ is an open set:

\[C_{2,h}(O) = \sup\{C_{2,h}(E) : E \subset O, E \text{ compact}\}.\]

For an arbitrary set $A \subset \Omega$ put

\[C_{2,h}(A) = \inf\{C_{2,h}(O) : A \subset O \subset \Omega, O \text{ open}\}.\]

This definition is compatible with (2.2) : when $E$ is compact the value of $C_{2,h}(E)$ given by the above formula coincides with the value given by (2.2), (see [2]).

We say that $h$ satisfies the $\Delta_2$ condition if there exists $C > 0$ such that $h(a + b) \leq c(h(a) + h(b)) \forall a, b > 0$.

If $h \in H$ satisfies this condition then, $L^h$ is separable (see [8]) and the capacity $C_{2,h}$ has the following additional properties (see [2]).

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. For every $A \subset \Omega$,

\begin{equation}
C_{2,h}(A) = \sup\{C_{2,h}(E) : E \subset A, E \text{ compact}\}
\end{equation}

and for every increasing sequence of sets $\{A_n\}$

\begin{equation}
\lim C_{2,h}(A_n) = C_{2,h}(\bigcup A_n).
\end{equation}

Furthermore, for every $A \subset \Omega$

\begin{equation}
C_{2,h}(A) = \inf\{\|f\|_{L^h} : f \in L^h(\Omega), \mathcal{G}f \geq 1 \text{ on } A\}.
\end{equation}

If $h \in H$ and both $h$ and $h^*$ satisfy the $\Delta_2$ condition then $L^h$ is reflexive [8].

Finally we denote by $\mathcal{G}$ the space of odd functions in $C(\mathbb{R})$ such that $h := |g| \in H$ and by $\mathcal{G}_2$ the set of functions $g \in \mathcal{G}$ such that $h$ and $h^*$ satisfy the $\Delta_2$ condition. For $g \in \mathcal{G}$ put

\[L^g := L^{|g|}, \quad C_{2,g} := C_{2,|g|}, \quad g^*(t) = |g|^*(t)\text{sign } t \forall t \in \mathbb{R}.\]

In the sequel we assume that $\Omega$ is a bounded domain of class $C^2$. The first theorem provides a necessary and sufficient condition for the existence of a solution of (1.1) in the spirit of [3].

**Theorem 2.1.** Let $g \in \mathcal{G}_2$ and let $\mu$ be a measure in $\mathcal{M}_\nu(\Omega)$. Then problem (1.1) possesses a solution if and only if $\mu$ vanishes on every compact set $E \subset \Omega$ such that $C_{2,g^*}(E) = 0$. This condition will be indicated by the notation $\mu \prec C_{2,g^*}$.

Next we consider problem (1.1) when $\mu$ is a positive Borel measure which may explode on a compact set $F \subset \Omega$. In this part of the paper we assume
that \( g \in \mathcal{G}_2 \) and that \( g \) satisfies the Keller – Osserman condition \([9]\) and \([12]\).

This condition ensures that the set of solutions of

\[
- \Delta u + g \circ u = 0
\]

in \( \Omega \) is uniformly bounded in compact subsets of \( \Omega \). Therefore, if \( E \subset \Omega \) and \( E \) is compact then there exists a maximal solution of

\[
- \Delta u + g \circ u = 0 \quad \text{in } \Omega \setminus E, \quad u = 0 \quad \text{on } \partial \Omega.
\]

This solution will be denoted by \( U_E \).

**Notation.** Consider the family of positive Borel measures \( \mu \) in \( \Omega \) such that:

1. There exists a compact set \( F \subset \Omega \) such that, for every open set \( O \supset F \),
\[
\mu(\Omega \setminus \bar{O}) < \infty
\]
2. \( \mu(A) = \infty \) for every non-empty Borel set \( A \subset F \).

The set \( F \) will be called the singular set of \( \mu \). The family of measures \( \mu \) of this type will be denoted by \( \mathcal{B}_\infty(\Omega) \).

**Definition 2.2.** Assume that \( g \in \mathcal{G}_2 \) and that \( g \) satisfies the Keller – Osserman condition. If \( \nu \in \mathcal{M}^g_\rho(\Omega) \) denote by \( v_\nu \) the solution of \((1.1)\) with \( \mu \) replaced by \( \nu \).

Let \( \mu \in \mathcal{B}_\infty(\Omega) \) and let \( F \) be the singular set of \( \mu \). A function \( u \in L^1_{\text{loc}}(\bar{\Omega} \setminus F) \) (i.e., \( u \in L^1(\Omega \setminus \bar{O}) \) for every neighborhood \( O \) of \( F \)) is a generalized solution of \((1.1)\) if:

1. \( u \) satisfies \((1.2)\) for every \( \phi \in C^2_\infty(\bar{\Omega}) \) such that \( \text{supp } \phi \subset \Omega \setminus F \).
2. \( u \geq V_F := \sup \{ v_\nu : \nu \in \mathcal{M}^g_\rho(\Omega), \nu \geq 0, \text{supp } \nu \subset F \} \).

**Theorem 2.3.** Assume that \( g \in \mathcal{G}_2 \) and that \( g \) satisfies the Keller – Osserman condition. Let \( \mu \in \mathcal{B}_\infty \) with singular set \( F \). Then:

1. If \( \mu \) vanishes on every compact set \( E \subset \Omega \setminus F \) such that \( C_{2,g^*}(E) = 0 \) then the generalized solution is unique.

2. If \( g \) satisfies the subcriticality condition \((1.4)\) then problem \((1.1)\) possesses a unique generalized solution for every \( \mu \in \mathcal{B}_\infty \).

3. Let \( g = g_q, q \geq q_c \). If \( \mu \prec C_{2,g^*} \) in \( \Omega \setminus F \) then \((1.1)\) possesses a unique solution.

### 3. Proof of Theorem 2.1

The proof is based on several lemmas. We assume throughout that the conditions of the theorem are satisfied.

Denote by \( L^1_\rho(\Omega) \) the Lebesgue space with weight \( \rho \) and by \( L^g_\rho(\Omega) \) the Orlicz space with weight \( \rho \).

Further denote by \( W^k L^g(\Omega) \), \( k \in \mathbb{N} \), the Orlicz-Sobolev space consisting of functions \( v \in L^g(\Omega) \) such that \( D^\alpha v \in L^g(\Omega) \) for \( |\alpha| \leq k \).

Under our assumptions the set of bounded functions in \( L^g \) is dense in this space (see \([8]\)). Consequently, by \([7]\), \( C^\infty(\Omega) \) is dense in \( W^k L^g(\Omega) \). As
a consequence of the reflexivity of \( L^g \) the space \( W^k L^g(\Omega) \) is reflexive. Let \( W^k L^g(\Omega) \) denote the closure of \( C^\infty_c(\Omega) \) in \( W^k L^g(\Omega) \). The dual of this space, denoted by \( W^{-k} L^{g^*}(\Omega) \) is the linear hull of \( \{ D^\alpha f : f \in L^g(\Omega), \ |\alpha| \leq k \} \). The standard norm in \( W^k L^g(\Omega) \) is given by

\[
\|v\|_{W^k L^g} = \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^g}
\]

and the norm in \( W^{-k} L^{g^*} \) is defined as the norm of the dual space of \( W^k_0 L^g \).

The spaces \( W^k L^g_0 \) and \( W^{-k} L^{g^*}_0 \) are defined in the same way.

**Lemma 3.1.** If \( \mu \in \mathcal{M}_g(\Omega) \) is a g-good measure then (1.1) has a unique solution, which we denote by \( v_\mu \). The solution satisfies the inequality

\[
(3.1) \quad \|v_\mu\|_{L^1(\Omega)} + \|v_\mu\|_{L^g(\Omega)} \leq C \|\mu\|_{\mathcal{M}_g(\Omega)}
\]

where \( C \) is a constant depending only on \( g \) and \( \Omega \).

If \( \mu_j \in \mathcal{M}_g(\Omega), \ j = 1, 2 \) are g-good measures and \( \mu_1 \leq \mu_2 \) then \( v_{\mu_1} \leq v_{\mu_2} \).

These results are well-known (see e.g. [13]).

**Lemma 3.2.** Let \( \mu \in \mathcal{M}_g(\Omega) \) be a positive measure such that \( \mathbb{G}_\mu \in L^1_{\text{loc}}(\Omega) \). Then \( \mu \) is g good.

**Proof.** Let \( \{ \Omega_n \} \) be a \( C^2 \) uniform exhaustion of \( \Omega \). Then \( \mathbb{G}_\mu \in L^g(\Omega_n) \) is a positive supersolution of problem (1.1) in \( \Omega_n \). Therefore – as the zero function is a subsolution – there exists a solution, say \( u_n \), of (1.1) in \( \Omega_n \) and, by Lemma 3.1

\[
\int_{\Omega_n} u_n dx + \int_{\Omega_n} g \circ u_n \rho_n dx \leq C \int_{\Omega_n} \rho_n d\mu,
\]

where \( \rho_n(x) = \text{dist}(x, \partial \Omega_n) \) and \( C \) is a constant depending only on \( g \) and the \( C^2 \) character of \( \Omega_n \). Since \( \Omega_n \) is uniformly \( C^2 \), the constant may be chosen to be independent of \( n \). Moreover \( \{ u_n \} \) is increasing. Therefore \( u = \lim u_n \in L^1(\Omega) \cap L^g_0(\Omega) \) is the solution of (1.1). \( \square \)

**Lemma 3.3.** (a) If \( \mu \in \mathcal{M}_g \) and \( |\mu| \) is g-good then \( \mu \) is g-good. (b) \( T \in W^{-2} L^g(\Omega) \) if and only if \( T = \Delta h \) for some \( h \in L^g(\Omega) \). (c) If \( \mu \) is a positive measure in \( W^{-2} L^g_{\text{loc}}(\Omega) \) then \( \mathbb{G}_\mu \in L^g(\Omega) \). If, in addition, \( \mu \in \mathcal{M}_g(\Omega) \) then \( \mu \) is g-good.

**Proof.** (a) Assuming that \( |\mu| \) is g-good, let \( v \) be the solution of (1.1) with \( \mu \) replaced by \( |\mu| \). Then \( v \) is a supersolution and \( -v \) is a subsolution of (1.1). Therefore (1.1) has a solution.

(b) If \( T = \Delta h \) then, for every \( \phi \in C^\infty_c(\Omega) \),

\[
T(\phi) = \int_{\Omega} h \Delta \phi dx, \quad |T(\phi)| \leq \|h\|_{L^g} \|\phi\|_{W^2 L^g^*}.
\]

As \( C^\infty_c \) is dense in \( W^2_0 L^g \), \( T \) defines a continuous linear functional on this space; consequently \( T \in W^{-2} L^g(\Omega) \).
On the other hand if $T \in W^{-2}L^g(\Omega)$, put
\[ S(\Delta \phi) := T(\phi) \quad \forall \phi \in W^2_L^g. \]
Note that for $\phi$ in this space we have $\phi = G_{-\Delta \phi}$. Therefore $S$ is well defined on the subspace of $L^g$ given by $\{ \Delta \phi : \phi \in W^2_L^g \}$. Therefore there exists $h \in L^g(\Omega)$ such that
\[ T(\phi) = \int_{\Omega} h \Delta \phi \, dx \quad \forall \phi \in W^2_L^g. \]
It follows that $T = \Delta h$.

(c) Let $\mu$ be a positive measure in $W^{-2}L^g_{loc}(\Omega)$. By part (b), if $\Omega' \subseteq \Omega$ is a subdomain of class $C^2$ there exists $h \in L^g(\Omega')$ such that $\mu = \Delta h$. Then $h + G_{\mu}$ is an harmonic function in $\Omega'$; consequently $G_{\mu} \in L^g_{loc}(\Omega')$ and finally $G_{\mu} \in L^g(\Omega)$. If, in addition, $\mu \in \mathcal{M}_p(\Omega)$ then, by Lemma 3.2 $\mu$ is $g$ good.

**Lemma 3.4.** Assume that $\mu \in \mathcal{M}_p(\Omega)$ is $g$ good. Then:

(i) There exists $f \in L^1_p(\Omega)$ and $\mu_0 \in W^{-2}L^g_{loc}(\Omega) \cap \mathcal{M}_p(\Omega)$ such that $\mu = f + \mu_0$.

(ii) $\mu \prec C_{2,g^*}$.

**Proof.** Assume that $\mu$ is $g$-good and let $u$ be the solution of (1.1). Then $\mu = f + \mu_0$ where $f := g \circ u \in L^1_p$, $\mu_0 := \mu - g \circ u$ and $u = G_{\mu_0} \in L^g(\Omega)$.

This implies that
\[ \phi \mapsto \int_{\Omega} \phi \, d\mu_0 = \int_{\Omega} \Delta \phi u \, dx \quad \forall \phi \in C^\infty(\Omega) \]
is continuous on $C^2_0(\Omega)$ with respect to the norm of $W^2L^g_{loc}(\Omega)$. Therefore, the functional can be extended to a continuous linear functional on $W^2L^g_{loc}(\Omega')$ for every $\Omega' \subseteq \Omega$. Thus $\mu_0 \in W^{-2}L^g_{loc}(\Omega) \cap \mathcal{M}_p(\Omega)$.

(ii) In view of (2.3) it is sufficient to prove that $\mu$ vanishes on compact sets $E$ such that $C_{2,g^*}(E) = 0$.

**Assertion.** If $\nu \in W^{-2}L^g_{loc}(\Omega)$ then $\nu(E) = 0$ for every compact set $E$ such that $C_{2,g^*}(E) = 0$.

This assertion and part (i) imply part (ii).

Suppose that there exists a set $E$ such that $C_{2,g^*}(E) = 0$ and $\nu(E) \neq 0$. Then there exists a compact subset of $E$ on which $\nu$ has constant sign. Therefore we may assume that $E$ is compact and that $\nu$ is positive on $E$. We may assume that $\nu \in W^{-2}L^g(\Omega)$; otherwise we replace $\Omega$ by a $C^2$ domain $\Omega' \subseteq \Omega$.

Let $\{ V_n \}$ be a sequence of open neighborhoods of $E$ such that $V_{n+1} \subset V_n$ and $V_n \downarrow E$. Then there exists a sequence $\{ \varphi_n \}$ in $C^\infty_c(\Omega)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ in $V_{n+1}$, $\text{supp} \varphi_n \subset V_n$ and $\| \varphi_n \|_{g^*} \rightarrow 0$.

This is proved in the same way as in the case of Bessel capacities. We use (2.5) and the fact that $C^\infty(\Omega)$ is dense in $W^2L^g_p(\Omega)$ (7). Furthermore we
use an extension of the lemma on smooth truncation [1, Theorem 3.3.3] to Sobolev-Orlicz spaces with an integral number of derivatives. The extension is straightforward.

Hence,

\[(3.2)\quad \int_{\Omega} \varphi_n \, d\nu \to 0.\]

On the other hand,

\[\int_{\Omega} \varphi_n \, d\nu \geq \nu(V_n +1) - |\nu|(V_n \setminus \bar{V}_{n+1}) \to \nu(E) > 0.\]

This contradiction proves the assertion. □

**Lemma 3.5.** Let \(\mu\) be a positive measure in \(\mathcal{M}_\rho(\Omega)\). If \(\mu\) vanishes on every compact set \(E \subset \Omega\) such that \(C_{2,g^*}(E) = 0\) then \(\mu\) is the limit of an increasing sequence of positive measures \(\{\mu_n\} \subset W^{-2}L^g(\Omega)\).

**Proof.** Since \(\mu\) is the limit of an increasing sequence of measures in \(\mathcal{M}(\Omega)\) it is sufficient to prove the lemma for \(\mu \in \mathcal{M}(\Omega)\). Let \(\varphi \in W^2_0L^g(\Omega)\) and denote \(\tilde{\varphi} = G\Delta \varphi\).

Then \(\tilde{\varphi}\) is equivalent to \(\varphi\).

Suppose that \(\{\varphi_n\}\) converges to \(\varphi\) in \(W^2_0L^g(\Omega)\). Then \(\Delta \varphi_n \to \Delta \varphi\) in \(L^g\). Consequently, by [2, Theorem 4], there exists a subsequence such that \(\tilde{\varphi}_n' \to \tilde{\varphi}\) \(C_{2,g^*}\)-a.e. (i.e., everywhere with the possible exception of a set of \(C_{2,g^*}\)-capacity zero). As \(\mu\) vanishes on sets of capacity zero, it follows that \(\tilde{\varphi}_n' \to \tilde{\varphi}\) \(\mu\)-a.e..

Every \(\varphi \in W^2_0L^g(\Omega)\) is the limit of a sequence \(\{\varphi_n\} \subset C^\infty_c(\Omega)\). Hence \(\varphi_n \to \tilde{\varphi}\) \(\mu\)-a.e. and consequently \(\tilde{\varphi}\) is \(\mu\)-measurable.

Therefore the functional \(p : W^2_0L^g(\Omega) \hookrightarrow [0, \infty]\) given by

\[p(\varphi) := \int_{\Omega} (\tilde{\varphi})_+ \, d\mu\]

is well defined. The functional is sublinear, convex and l.s.c.: if \(\varphi_n \to \varphi\) in \(W^2_0L^g(\Omega)\) then (by Fatou’s lemma)

\[p(\varphi) \leq \lim \inf p(\varphi_n).\]

Furthermore,

\[p(a\varphi) = ap(\varphi) \quad \forall a > 0.\]

Therefore the result follows by an application of the Hahn-Banach theorem, in the same way as in [3, Lemma 4.2]. □

**Proof of Theorem 2.1.** By Lemma 3.4 the condition \(\mu \prec C_{2,g^*}\) is necessary for the existence of a solution. We show that the condition is sufficient.

If \(\mu \prec C_{2,g^*}\) then \(|\mu| \prec C_{2,g^*}\). By Lemma 3.3 if \(|\mu|\) is \(g\)-good then \(\mu\) is \(g\)-good. Therefore it remains to prove the sufficiency of the condition for positive \(\mu\). In this case, by Lemma 3.5 there exists an increasing sequence
of positive measures \( \{ \mu_n \} \subset W^{-2}L^0(\Omega) \) such that \( \mu_n \uparrow \mu \). By Lemma 3.3 the measures \( \mu_n \) are \( g \)-good. Denote by \( u_n \) the solution of (4.1) with \( \mu \) replaced by \( \mu_n \). By Lemma 3.1 \( u_n \geq 0 \), \( \{ u_n \} \) increases and \( \{ u_n \} \) is bounded in \( L^1(\Omega) \cap L^0_\mu(\Omega) \). Therefore \( u = \lim u_n \in L^1(\Omega) \cap L^0_\mu(\Omega) \) and \( u_n \to u \) in this space. Consequently \( u \) is the solution of (1.1).

4. Proof of Theorem 2.3

(i) Let \( \{ O_n \} \) be a decreasing sequence of open sets such that \( \tilde{O}_{n+1} \subset O_n \), \( \tilde{O}_n \subset \Omega \) and \( O_n \downarrow F \) and \( O_n \) is of class \( C^2 \). By Theorem 2.1 the condition \( \mu < C_{2,g}^* \) in \( \Omega \setminus F \) is necessary and sufficient for the existence of a solution of the equation

\[
-\Delta u + g \circ u = \mu \quad \text{in} \quad \Omega_n := \Omega \setminus \tilde{O}_n
\]

such that \( u = 0 \) on the boundary. By a standard argument, it follows that, under this condition: for every \( f \in L^1(\partial \Omega \cup \partial O_n) \), (4.1) has a solution such that \( u = f \) on the boundary. As \( g \) satisfies the Keller – Osserman condition, it also follows that (4.1) has a solution \( u_n \) such that \( u_n = 0 \) on \( \partial \Omega \) and \( u_n = \infty \) on \( \partial O_n \). Denote by \( v_n \) the solution of (4.1) vanishing on \( \partial \Omega \cup \partial O_n \) and put

\[
v_{0,\mu} = \lim v_n, \quad \bar{u}_\mu = \lim u_n.
\]

Then \( v_{0,\mu} \) is the smallest positive solution of (4.1) vanishing on \( \partial \Omega \) while \( \bar{u}_\mu \) is the largest such solution. In particular \( \bar{u}_\mu \geq v_\nu \) for every \( \nu \in \mathcal{M}_2^* \) such that \( \operatorname{supp} \nu \subset F \). Thus \( \bar{u}_\mu \) is the largest generalized solution of (1.1).

Next we construct the minimal generalized solution of (1.1). The function \( u_{0,\mu} + V_F \) is a supersolution and \( \max(u_{0,\mu}, V_F) \) is a subsolution of (4.1), both vanishing on the boundary. Let \( w_n \) denote the solution of (4.1) such that \( w_n = 0 \) on \( \partial \Omega \) and \( w_n = \max(u_{0,\mu}, V_F) \) on \( \partial O_n \). Then

\[
w_{n+1} \leq w_n \leq u_{0,\mu} + V_F
\]

and consequently, \( w = \lim w_n \) is the smallest solution of (1.1) such that

\[
\max(u_{0,\mu}, V_F) \leq w \leq u_{0,\mu} + V_F.
\]

It follows that \( w \) is a generalized solution of (1.1). Since any such solution dominates \( \max(u_{0,\mu}, V_F) \) it follows that \( w \) is the smallest generalized solution of the problem. It is easy to see that \( w = \bar{u}_\mu \), as given by (2.7).

Since \( g \) is convex, monotone increasing and \( g(0) = 0 \) we have

\[
g(a) + g(b) \leq g(a + b) \quad \forall a, b \in \mathbb{R}_+.
\]

Therefore \( \bar{u}_\mu - u_{0,\mu} \) is a subsolution of (2.6) in \( \Omega \setminus F \). Consequently \( \bar{u}_\mu - u_{0,\mu} \leq U_F \) and

\[
(4.2) \quad \max(u_{0,\mu}, U_F) \leq \bar{u}_\mu \leq u_{0,\mu} + U_F.
\]

Put \( \Omega_n = \Omega \setminus \tilde{O}_n \). Let \( u_{n,\mu} \) be the solution of the problem

\[
-\Delta u + g \circ u = \mu \quad \text{in} \quad \Omega_n, \\
u = V_F \quad \text{on} \quad \partial O_n, \quad u = 0 \quad \text{on} \quad \partial \Omega.
\]
Then \( \{ \vec{u}_n \} \) increases and \( u = \lim \vec{u}_n \).

Similarly, if \( \bar{u}_n \) is the solution of the problem

\[
-\Delta u + g \circ u = \mu \quad \text{in } \Omega_n, \\
\bar{u} = U_F \quad \text{on } \partial \Omega_n, \\
u = 0 \quad \text{on } \partial \Omega.
\]

then \( \{ \bar{u}_n \} \) increases and, in view of (4.2), \( \bar{u} = \lim \bar{u}_n \). Therefore, if \( \mu = \mu \) then \( \mu = \mu \).

(ii) We assume that in addition to the other conditions of the theorem, \( g \) satisfies the subcriticality condition. In this case, for every point \( z \in \Omega \) and \( k \in \mathbb{R} \), there exists a solution \( u_{k,z} \) of the problem

\[
-\Delta u + g \circ u = k \delta_z \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\]

Put \( w_z = \lim_{k \to \infty} u_{k,z} \). By definition \( w_z = V_{\{z\}} \). We also have \( w_z = U_{\{z\}} \).

This follows from the fact that \( g \) satisfies the Keller – Osserman condition. This condition implies that there exists a decreasing function \( \psi \in C(0, \infty) \) such that \( \psi(t) \to \infty \) as \( t \to 0 \) and every positive solution \( u \) of (4.3) satisfies

\[
C_2 \psi(|x-z|) \leq u(x) \leq C_1 \psi(|x-z|).
\]

The constant \( C_1 \) depends only on \( g, N \). Because of the boundary condition the constant \( C_2 \) depends on \( z \). However for \( z \) in a compact subset of \( \Omega \) one can choose \( C_2 \) to be independent of \( z \).

This inequality implies that

\[
w_z \leq U_{\{z\}} \leq C_1/C_2 w_z.
\]

If \( F \) is a compact subset of \( \Omega \) put

\[
F' = \{ x \in \Omega : \text{dist}(x, F) \leq \frac{1}{2} \text{dist}(F, \partial \Omega) \}.
\]

Let \( x \in F' \setminus F \) and let \( z \) be a point in \( F \) such that \( |x-z| = \text{dist}(x, F) \). Then there exists a positive constant \( C(F) \) such that

\[
C(F)\psi(|x-z|) \leq u_z(x) \leq V_F(x) \leq U_F(x) \leq C_1 \psi(|x-z|).
\]

It follows that there exists a constant \( c \) such that

\[
\text{(4.4) } U_F(x) \leq cV_F(x)
\]

for every \( x \in F' \). Since \( U_F \) and \( V_F \) vanish on \( \partial \Omega \) it follows that (4.4) (with possibly a larger constant) remains valid in \( \Omega \setminus F' \). This is verified by a standard argument using Harnack’s inequality and the fact that \( g \) satisfies the Keller – Osserman condition. Thus (4.4) is valid in \( \Omega \setminus F \). By an argument similar to the one introduced in [10 Theorem 5.4], this inequality implies that \( U_F = V_F \).

(iii) For the case considered here, it was proved in [11] that \( U_F = V_F \). Therefore uniqueness follows from part (i). \( \Box \)
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