CHARGED ROTATING BLACK HOLES IN EQUILIBRIUM

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Abstract

Axially symmetric, stationary solutions of the Einstein-Maxwell equations with disconnected event horizon are studied by developing a method of explicit integration of the corresponding boundary-value problem. This problem is reduced to non-linear system of algebraic equations which gives relations between the masses, the angular momenta, the angular velocities, the charges, the distance parameters, the values of the electromagnetic field potential at the horizon and at the symmetry axis. A found solution of this system for the case of two charged non-rotating black holes shows that in general the total mass depends on the distance between black holes. Two-Killing reduction procedure of the Einstein-Maxwell equations is also discussed.
1 Introduction

In this paper we study axially symmetric, stationary solutions of the Einstein-Maxwell equations having disconnected event horizon (N black holes). The solutions of this kind have been known since the early days of the general relativity Ref. [1] and, from our point of view, are among the most interesting exact solutions of the Einstein equations. A great merit of these solutions is that they have a clear mechanical interpretation. It allows us to hope that the solutions of this class will help us to understand N-body problem which, we think, is one of the most fundamental and difficult problem of the classical general relativity. We begin from description of the main physical properties of these solutions.

In general the solutions of this family are parameterized by the mass, the charge, the angular momentum of each of the black hole and by the distances between them. Conical singularities at the non-extreme symmetry axis components prevent the fall of the black holes on each other. Interpreting these symmetry axis components as 'matter strings' we get a natural definition of the interaction force between the black holes, namely, the tension of 'string'. It is exactly equal to the deficit angle of conical singularity Ref. [1](see also Ref. [2]). The interaction force of two non-charged black holes tends to the Newtonian limit as the distance parameter becomes large Refs. [1, 3] and goes to infinity as the distance parameter approaches the value for which two components of the horizon intersect Ref. [4]. The generalization of these results for the case of charged black holes hasn’t been done yet.

It is worth mentioning that for so-called extreme black holes (the horizon is contracted to points) the interaction force can be equal to zero Ref. [5]. Unfortunately, this is the only known case when the balance between the gravity attraction force and the electromagnetic repulsion force takes place.

The total mass of N Schwarzschild black holes in equilibrium doesn’t depend on the distances. However, it is a function of the distance parameters for the rotating black holes that is stipulated by the spin-spin interaction of the black holes. The charge 'densities' of the black holes must also effect on the total mass since some part of the energy is needed for the charge redistribution in the final black hole. In the case of the static model we find that the mass of two charged black holes
depends on the distance parameter when their charge 'densities' are different. To determine the masses of the black holes one must solve very complex constraint equations for the angular momenta and the angular velocities of the black holes Ref. [6]. A partial solution of these constraints was found in Ref. [3]. We obtain the equations for the physical parameters of the charged black holes in this paper.

Despite of the presence of the conical singularity the gravity action on the stationary N black holes solutions remains finite. It allows us to construct the thermodynamics for the system of N black holes. The thermodynamic properties of this system are known only for non-rotating black holes Ref. [7]. The surface area of the horizon of two Schwarzschild black holes increases when the distance parameter decreases.

All stationary, axially symmetric solutions with disconnected event horizon satisfy a certain boundary-value problem for the Einstein-Maxwell equations. The boundary conditions of this problem are the regularity conditions for the symmetry axis and for the event horizon Ref. [8]. The condition of asymptotic flatness is a part of the event horizon definition. The uniqueness and existence theorem for this problem was proven in Refs. [2, 4, 9, 10] for non-charged black holes, and in Ref. [11] for charged ones.

The Einstein-Maxwell equations with two commuting symmetries belong to a wide class of completely integrable equations in the sense: they can be written as compatibility conditions for some auxiliary linear system of equations Refs [13, 14]. Using this fact one can derive 2N-soliton solution. The 2N-soliton solutions do not necessarily include the family of the solutions studied in this paper. However, we have showed Ref. [6] that N black holes (non-charged) family is a subclass of the 2N-soliton family of Belinskii-Zakharov Ref. [17].

The 2N-soliton solutions of the Einstein-Maxwell equations can be constructed by a slight modification of the Belinskii-Zakharov method Ref. [16]. Different approaches were applied in Refs. [13, 15, 18]. The 2N-soliton family of Refs. [13, 15] can not contain the black holes solutions (likely, the family of Ref. [18] either). It was extended in Ref. [19] with the use of the technique suggested in Ref. [20]. A key problem with the 2N-soliton solution is that this solution has extra parameters which must be excluded from it to get N black holes solution. The only way to do
this is to satisfy the regularity conditions of the horizon and the symmetry axis by
imposing some constraints on the parameters. In this paper we use these conditions
at the beginning.

In the end of this section we wish to emphasize that physical solutions for two
black holes in equilibrium are those the interaction force of which is equal to zero.

In the next section we state the boundary-value problem, which defines N
charged rotating black holes in equilibrium strictly.

2 Boundary-value problem

In this section we formulate the boundary-value problem for the Einstein-Maxwell
equations, which describes the all possible axially symmetric, stationary solutions
having disconnected event horizon. Space-time manifold is said to be stationary and
axial symmetric if there are two commuting Killing fields of which one is timelike,
k\(\mu\), and the other is spacelike, m\(\mu\). The vector fields k\(\mu\) and m\(\mu\) are generators of
the isometry groups which are isomorphic R and SO(2) respectively. We denote
these fields \(K_0 = k\) and \(K_1 = m\) either. By definition, the vector fields \(K_A\) satisfy
the Killing equation and commute,

\[ \nabla_\mu K_{A\nu} + \nabla_\nu K_{A\mu} = 0, \quad [K_A, K_B] = 0. \]  (1)

Indexes denoted by the Latin capitals run 0 and 1.

It is more convenient to state the conditions for the self-dual electromagnetic
field \(F^\dagger\),

\[ F^\dagger = F + i * F, \quad *F_{\mu\nu} = \frac{1}{2} e_{\mu\nu\alpha\beta} F^{\alpha\beta}, \]

than for the electromagnetic field \(F\). Here, \(e_{\mu\nu\alpha\beta}\) is the alternating tensor and * is
the Hodge operator. We assume that \(F^\dagger\) is invariant under the action of the
isometry groups, viz

\[ \mathcal{L}_{K_A} F^\dagger = 0, \]  (2)

where \(\mathcal{L}_{K_A}\) is the Lie derivative in the direction \(K_A\), and the field circularity condi-
tion holds,

\[ F^\dagger_{\mu\nu} K^\mu_A K^\nu_B = 0. \]  (3)
Using $F^\dagger$ we can write the Einstein-Maxwell equations in the form
\[ R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad T_{\mu\nu} = \frac{1}{8\pi} F_{\mu\alpha}^\dagger F_{\nu\alpha}^\dagger, \quad \nabla_{[\mu} F_{\nu\alpha]}^\dagger = 0. \] (4)

Let us introduce notations
\[ g_{AB} = K_{A\mu} K_{B}^\mu, \quad \Phi_{\alpha A} = F_{\alpha\beta A}^\dagger K_{\beta}^A \] (5)
and a matrix-valued twist potential
\[ \omega_{\alpha A}^B = e^{\alpha\mu\nu\beta} K_{A\mu} \nabla_\nu K_{B\beta}. \] (6)

From the Maxwell equations and the condition (2) we get that
\[ \Phi_{\alpha A} = \nabla_\alpha \Phi_A. \] (7)

A simple corollary of the definitions (5) and (6) is
\[ e_{\mu\nu\theta\beta} \omega_{\alpha A}^B K_{C}^{\alpha} K_{D}^{\nu} = g_{AC} \nabla_\mu g_{BD} - g_{AD} \nabla_\mu g_{BC}. \] (8)

Let us now recall the basic properties of the Killing fields
\[ \nabla_\mu \nabla_\nu K_{A}^\alpha = R_{\beta\mu\nu\alpha} K_{A}^{\beta}, \quad \nabla_\mu \nabla_\alpha K_{\mu A} = R_{\alpha\beta} K_{\beta A}, \]
which are consequence of Eq. (1) and the curvature tensor definition. Using them it is easy to check that
\[ \nabla_{[\alpha} \omega_{\beta]}^A_{AB} + 2i \nabla_{[\alpha} (\bar{\Phi}_B \nabla_\beta \Phi_A) = 0. \] (9)

Furthermore, from the Einstein-Maxwell equations we see that
\[ e_{\alpha\beta\mu\nu} K_{A}^{\mu} R_{\nu\gamma} \gamma_{B} = e_{\alpha\beta\mu\nu} K_{A}^{\mu} F_{\gamma\delta}^{\dagger} F_{\gamma\delta}^{\dagger} K_{B}^{\gamma} = -i e_{\alpha\beta\mu\nu} K_{A}^{\mu} * F_{\gamma\delta}^{\dagger} F_{\gamma\delta}^{\dagger} K_{B}^{\gamma} \]
\[ = i (\Phi_{\alpha B} \Phi_{\beta A} - \Phi_{\beta B} \Phi_{\alpha A}) = 2i \nabla_{[\alpha} (\bar{\Phi}_B \nabla_\beta \Phi_A). \]

Hence, the identity (9) can be written in the following form
\[ \nabla_{[\alpha} \omega_{\beta]}^A_{AB} + 2i \nabla_{[\alpha} (\bar{\Phi}_B \nabla_\beta \Phi_A) = 0. \] (10)

From Eq. (10) we conclude that there exists a matrix potential $Y$ such that
\[ \nabla_\alpha Y_{AB} = \omega_{\alpha AB} + 2i \bar{\Phi}_B \nabla_\alpha \Phi_A. \]
The last equation gives
\[
e_{\mu \nu \theta \alpha} \nabla^\alpha Y_{AB} K^\theta_C K^\nu_D = e_{\mu \nu \theta \alpha} \omega^\alpha_{AB} K^\theta_C K^\nu_D + 2i \Phi_B e_{\mu \theta \alpha} \nabla^\alpha \Phi_A K^\theta_C K^\nu_D. \tag{11}
\]

It can be shown that the condition (3) is equivalent to the existence of a two-surface which is orthogonal to both Killing fields. From now on we chose a coordinate system adopted to the Killing fields, \( K_0 = \frac{\partial}{\partial x_0}, \ K_1 = \frac{\partial}{\partial x_1}, \ x_0 = t, \ x_1 = \phi. \) Then
\[
d s^2 = \gamma_{ab} dx^a dx^b + g_{AB} dx^A dx^B, \tag{12}
\]
where \( \gamma_{ab} \) and \( g_{AB} \) don’t depend on \( t, \phi. \) From the condition (3) we derive that \( F^\dagger \) can be represented in the form
\[
F^\dagger_{\alpha \beta} = 2g^{AB} \Phi_{A[\alpha} K_{\beta]B}. \tag{13}
\]
Here \( g^{AB} \) denotes \((g^{-1})_{AB}.\) The self-dual property of \( F^\dagger \) leads to the identity
\[
g^{AB} e_{\alpha \beta \mu \nu} \Phi^\mu_{A} K^\nu_B K^\beta_C = i \Phi_{\alpha C}. \tag{14}
\]
Restricting the equation (11) to the two-surface that is orthogonal \( K_A \) and using Eq. (8) we obtain
\[
\rho \epsilon_{CD} * dY_{AB} = g_{AD} dg_{BC} - g_{AC} dg_{BD} + 2i \rho \epsilon_{CD} \Phi_B * d\Phi_A. \tag{15}
\]
Here \( * \) is the Hodge operator with respect to \( \gamma_{ab} \) and
\[
-\rho^2 = \det g, \ \epsilon_{01} = 1, \ \epsilon_{AB} = -\epsilon_{BA}.
\]
Projection of the identity (14) to the same surface gives
\[
\rho g^{AB} \epsilon_{BC} * d\Phi_A = i d\Phi_C. \tag{16}
\]
In the matrix notations, Eqs. (15) and (16) can be written as
\[
\rho * dY = -gcdg + 2i \rho * d\Phi \Phi^* \tag{17}
\]
and
\[
d\Phi = i \rho \epsilon g^{-1} * d\Phi. \tag{18}
\]
Here \( \Phi \) is a column vector and \( \Phi^* \) is its Hermitian conjugation; \( \Phi^* = (\bar{\Phi}_0, \bar{\Phi}_1). \)
With the help of Eq. (17) and Eq. (18) we can show that the Einstein-Maxwell equations take the form

\[ d(\rho G^{-1} \ast dG) = 0, \]  

(19)

where

\[
G = \begin{pmatrix} g + 2\Phi\Phi^* & \Phi \\ \Phi^* & 1/2 \end{pmatrix}; \quad G^{-1} = \begin{pmatrix} g^{-1} & -2g^{-1}\Phi \\ -2\Phi^*g^{-1} & 4\Phi^*g^{-1}\Phi + 2 \end{pmatrix}.
\]

(20)

Notice that

\[ G^* = G, \quad \det G = -\frac{\rho^2}{2}. \]

(21)

To prove (19), we note first that

\[
\rho G^{-1} \ast dG = \begin{pmatrix} \epsilon dY & i\epsilon d\Phi \\ -2\Phi^*\epsilon dH + 2id(\Phi^*\epsilon g) & -2i\Phi^*\epsilon d\Phi \end{pmatrix}
\]

(22)

where \( H \) is Kinnersley’s potential, \( dH = dY + idg \). So, we need only to check the validity of the following identities

\[ d\Phi^* \wedge \epsilon d\Phi = d\Phi^* \wedge \epsilon dH = 0. \]

From Eq. (17) and Eq. (18), it is easy to see that

\[ dH = i\rho \epsilon g^{-1} \ast dH. \]

(23)

Then, we have

\[ d\Phi^* \wedge \epsilon d\Phi = *d\Phi^* \wedge \epsilon \ast d\Phi = -d\Phi^* \wedge \epsilon d\Phi = 0 \]

and

\[ d\Phi^* \wedge \epsilon dH = *d\Phi^* \wedge \epsilon \ast dH = -d\Phi^* \wedge \epsilon dH = 0. \]

Here we use the standard property of the Hodge operator with respect to \( \gamma_{ab} \) and the equations (18), (23).

It is worth mentioning that the same equation (19) was derived in Ref. [14] but the matrices \( G \) defined there and here are different.

We now turn to discussion of the boundary conditions. From Eq. (17) one sees that \( d \ast d\rho = 0 \). Let \( \rho \) and \( z \), where \( z \) defined by \( dz = \ast d\rho \), be a coordinate system of the two-surface. Then

\[ \gamma_{ab}dx^a dx^b = f(z, \rho)(d\rho^2 + dz^2), \quad dz = \ast d\rho, \quad d\rho = -\ast dz. \]
One can prove that the function \( f(z, \rho) \) is uniquely determined by the matrix \( G \). We won’t write these equations since they aren’t need us in this paper. Virtually, the factor \( f \) is irrelevant to the problem of N black holes since we can compute the interaction force using the properties of the metric coefficient \( g_{11} \) Ref. \[3\]. Let us remark that the Hodge operator is conformally invariant and independent of \( f \).

Define

\[
g_{00} = -V, \quad g_{11} = X, \quad g_{01} = W.
\]

The event horizon and the axis of symmetry can be described as follows. The set of points with \( \rho = 0 \) and \( X = 0 \) is the symmetry axis while the set of points with \( \rho = 0 \) and \( X > 0 \) is the event horizon. Denote through \( z_k \) \((k = 1, \ldots, 2N)\) \( z \)-coordinates of the intersection points of the horizon and the symmetry axis. Then an interval \( I_i = [z_{2i}, z_{2i-1}] \) corresponds to the horizon of \( i \)th black hole. The symmetry axis components we denote by \( \Gamma_i \) for \( i = 1, \ldots, N+1 \) where \( \Gamma_i = (z_{2i-1}, z_{2i-2}) \). It is assumed that \( z_0 = -\infty \) and \( z_{2N+1} = +\infty \).

We pass to a new coordinate system in a neighborhood of \( i \)th black hole,

\[
\rho^2 = (\lambda^2 - m_i^2)(1 - \mu^2), \quad m_i = \frac{z_{2i} - z_{2i-1}}{2}, \quad z = \frac{z_{2i} + z_{2i-1}}{2} = \lambda\mu.
\]

For this coordinate system the horizon and the symmetry axis regularity conditions can be written in the form \[8\]

\[
\left( \begin{array}{cc} 1 & \Omega_i \\ 0 & 1 \end{array} \right) g \left( \begin{array}{cc} 1 & 0 \\ \Omega_i & 1 \end{array} \right) = \left( \begin{array}{cc} (\lambda^2 - m_i^2)\hat{V}(\lambda, \mu) & \rho^2\hat{W}(\lambda, \mu) \\ \rho^2\hat{W}(\lambda, \mu) & (1 - \mu^2)\hat{X}(\lambda, \mu) \end{array} \right),
\]

\[
(\Phi_0 + \Omega_i\Phi_1) = \Phi_i^H + (\lambda^2 - m_i^2)\hat{\Phi}_0(\lambda, \mu), \quad \Phi_1 = \Phi_i^A + (1 - \mu^2)\hat{\Phi}_1(\lambda, \mu),
\]

where \( \Phi_0, \hat{\Phi}_0, \hat{\Phi}_1, \hat{V}, \hat{W} \) are smooth functions nowhere equal to zero, and \( \Omega_i, \Phi_i^H \) \( \Phi_i^A \) are some constants. The second condition of \(25\) is understood as the regularity condition for the electromagnetic field in the vicinity of the symmetry axis components, \( \Phi_1 = \Phi_i^A \) in \( i \)th component \((i = 1, \ldots, N + 1)\) of the symmetry axis. The constant \( \Omega_i \) is the angular velocity of \( i \)th black hole. Note that one can derive the conditions for the imaginary part of \( \Phi \) from the results of Ref. \[8\] using the equation \(18\).
In addition, we require that

\begin{align*}
X &= \rho^2(1 + O(1/r)), \quad W = \rho^2O(1/r^3), \quad V = 1 + O(1/r), \\
\Phi_0 &= O(1/r), \quad \Phi_1 = O(1),
\end{align*}

as \( r \to \infty \) where \( r = \sqrt{z^2 + \rho^2} \) and the asymptotics \([26]\) and \([27]\) are differentiable.

In particular, \( \Phi_{1,z} = O(1/r) \) and \( \Phi_{1,\rho} = O(1/r) \). The function \( \Phi_1 \) is given up to an additive constant which we fix requiring that

\begin{align*}
\lim_{z \to -\infty} \Phi_1(z, \rho) &= \Phi_{A_1} = i\bar{q}, \\
\lim_{z \to +\infty} \Phi_1(z, \rho) &= \Phi_{A_N+1} = -i\bar{q},
\end{align*}

where \( q \) is the total charge of the system (see below). The last condition is compatible with the equations \([19]\).

The charge of \( j \)th black hole is defined by an integral

\begin{align*}
\bar{q}_j &= Q_j - iP_j = -\frac{1}{4\pi} \int_{S_j} \bar{F}^{\mu\nu} dS_{\mu\nu},
\end{align*}

where \( Q_j \) is the electric charge, \( P_j \) is the magnet monopole charge and \( S_j \) is a two-surface surrounding \( j \)th black hole. We assume that the two-surface belongs to a spacelike hypersurface which is invariant under the action of the rotational isometry and its normal orientation is chosen in the direction of the space infinity. Let \( S_j \) be a surface of revolution of a curve \( C_j \). Then

\begin{align*}
\bar{q}_j &= \frac{i}{2} \int_{C_j} d\Phi_1 = \frac{i}{2} (\Phi_{A_j} - \Phi_{A_j}^A).
\end{align*}

Taking this into account we see that the constant \( q \) in the normalization condition \([28]\) is the sum of the charges of all black holes \( q = \sum q_i \).

The mass and the angular momentum of the system are defined by integrals

\begin{align*}
M &= -\frac{1}{4\pi} \int_{S_{\infty}} k^{\mu\nu} dS_{\mu\nu}, \quad L = \frac{1}{8\pi} \int_{S_{\infty}} m^{\mu\nu} dS_{\mu\nu},
\end{align*}

where \( S_{\infty} \) is the space infinity which is a two-sphere. Using Stokes’ theorem and the Einstein-Maxwell equations we obtain

\begin{align*}
L &= \sum_i L_i, \quad L_i = \frac{1}{8\pi} \int_{H_i} \mathcal{L}^{\mu\nu} dS_{\mu\nu}, \quad \mathcal{L}_{\mu\nu} = m_{\mu\nu} - \Phi_1 \bar{F}^\dagger_{\mu\nu},
\end{align*}

and

\begin{align*}
M &= \sum_i M_i, \quad M_i = -\frac{1}{4\pi} \int_{H_i} \mathcal{M}^{\mu\nu} dS_{\mu\nu}, \quad \mathcal{M}_{\mu\nu} = k_{\mu\nu} - \Phi_0 \bar{F}^\dagger_{\mu\nu}.
\end{align*}
Here $H_i$ is the horizon component. Since the horizon is a two-surface of revolution we get

$$L_i = \frac{1}{8}(\bar{Y}_{11}(z_{2i}) - \bar{Y}_{11}(z_{2i-1})), \quad M_i = \frac{1}{4}(\bar{Y}_{10}(z_{2i}) - \bar{Y}_{10}(z_{2i-1})).$$  \hspace{1cm} (34)$$

A simple corollary of the boundary conditions (24) and (25) is the identity $Y_{10,z} + \Omega_j Y_{11,z}|_{I_j} = 2 + 2i\Phi^H_1 \Phi_{1,z}$. Using it we find that

$$M_i = m_i + 2\Omega_i L_i + \Phi^H_1 q_i.$$  \hspace{1cm} (35)$$

Recall $m_i = (z_{2i} - z_{2i-1})/2$.

Under the gauge transformation $\Phi_1 \rightarrow \Phi_1 + a$ the quantities $M_i$ and $L_i$ change as $M_i \rightarrow M_i - a\Omega_i q_i$ and $L_i \rightarrow L_i - \frac{1}{2}aq_i$, respectively. We define the physical mass and the angular momentum of one black hole by formulas

$$M_i^{ph} = M_i + \Omega_i q_i, \quad L_i^{ph} = L_i + \frac{\Phi^A_{i+1} + \Phi^A_i}{4},$$ \hspace{1cm} (36)$$

as $\rho = 0$ and $z \in \Gamma_i$ while

$$\hat{\Omega}_i \rho G^{-1} \rho G^{-1} = \begin{pmatrix} 0 & -\bar{Y}_{00,z} & 2\Phi^A_1 \bar{Y}_{00,z} \\ 0 & -\bar{Y}_{01,z} & 2\Phi^A_1 \bar{Y}_{01,z} \\ i\Phi_{0,z} & -2i\Phi^A_1 \Phi_{0,z} \end{pmatrix},$$ \hspace{1cm} (37)$$

as $\rho = 0$ and $z \in I_i$ and $\rho G_{zz} G^{-1} = O(1)$ as $\rho \rightarrow 0$. 

10
3 Constraint equations

Not all of the parameters introduced in the previous section are independent. In this section we discuss a way of deriving the relations between the masses, the angular momenta, the angular velocities, the charges, the distance parameters, the values of the electromagnetic field potential at the horizon and at the symmetry axis.

The system of equations (19) is a compatibility condition of the following pair of matrix linear differential equations

\[ D_1 \psi = \frac{\rho^2 G_{zz} G^{-1} - \omega \rho G_{\rho} G^{-1}}{\omega^2 + \rho^2} \psi, \quad D_2 \psi = \frac{\rho^2 G_{\rho} G^{-1} + \omega \rho G_{zz} G^{-1}}{\omega^2 + \rho^2} \psi, \]

(39)

where \( D_1, D_2 \) are commuting differential operators,

\[ D_1 = \partial_z - \frac{2\omega^2}{\omega^2 + \rho^2} \partial_{\omega}, \quad D_2 = \partial_{\rho} + \frac{2\omega \rho}{\omega^2 + \rho^2} \partial_{\omega}, \]

and \( \omega \) is a complex parameter that doesn’t depend on the coordinates. Studying the non-charged black holes in Ref. [6] we have showed that well-known methods of investigation of completely integrable equations [24] can be applied to the problem of N black holes. Moreover, this problem occurs to be explicitly solvable provided that the relations between the parameters are found. We proceed discussing the principle steps of Ref. [6].

Let \( \omega \) be a root of the equation

\[ \omega^2 - 2\omega(k - z) - \rho^2 = 0, \]

(40)

where, now, \( k \) is an independent spectral parameter. From Eq. (40) it follows that

\[ \partial_z \omega = -\frac{2\omega^2}{\omega^2 + \rho^2}, \quad \partial_{\rho} \omega = \frac{2\omega \rho}{\omega^2 + \rho^2}. \]

Hence, \( \psi'(k) = \psi(\omega(k)) \) is a solution of the linear equation

\[ \partial_z \psi'(k) = \frac{\rho^2 G_{zz} G^{-1} - \omega \rho G_{\rho} G^{-1}}{\omega^2 + \rho^2} \psi'(k). \]

(41)

Invariance of Eq. (40) with respect to the transformation \( \omega \rightarrow -\rho^2/\omega \) allows us to fix a branch of the function \( \omega(k) \) through the inequality \( |\omega| > \rho \). Then, from Eq. (40) we see that

\[ \omega \rightarrow 2(k - z), \quad \rho \rightarrow 0, \quad \omega \rightarrow 2(k - z), \quad z \rightarrow \infty. \]
After choosing a branch of $\omega$, we can define the monodromy matrix $T(z,y)$, which, by definition, is a solution to Eq. (11) such that $T(y,y) = I$. Using the boundary conditions (27) and (28), $T(z,y)$ can be explicitly evaluated at $\rho = 0$. More precisely, for $\rho = 0$ and $z, y \in \Gamma$, one obtains

$$T(z,y) = \left( \begin{array}{ccc} 1 & \frac{\gamma_0(z) - \gamma_0(y)}{2(k-y)} & -\Phi_i^A \frac{\gamma_0(z) - \gamma_0(y)}{k-y} \\ 0 & k-z & i\Phi_i^A \frac{\Phi_0(z) - \Phi_0(y)}{k-y} + 2\Phi_i^A \frac{z-y}{k-y} \\ 0 & -i\Phi_0(z) - \Phi_0(y) & 1 + i\Phi_i^A \frac{\Phi_0(z) - \Phi_0(y)}{k-y} \end{array} \right),$$

while for $\rho = 0$ and $z, y \in I_i$ one gets

$$T(z,y) = \hat{\Omega}_i^{-1} \left( \begin{array}{ccc} \frac{k-z}{k-y} & i\Phi_i^H \Phi_1(z) - \Phi_1(y) & 0 \\ \frac{\gamma_1(z) - \gamma_1(y)}{2(k-y)} & 1 \\ \frac{\Phi_1(z) - \Phi_1(y)}{k-y} & 0 \end{array} \right) \hat{\Omega}_i.$$

The reduced monodromy matrix $T(k)$ is defined by

$$T(k) = \lim_{y \to -\infty, z \to +\infty} e_+^{-1}(k,z)T(z,y)e_-(k,y),$$

where

$$e_\pm(k,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega(k,z) & \mp 2i\bar{q} \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix $T(k)$ doesn’t depend on $\rho$. It is a consequence of the conditions (26), (27) and (28). Furthermore, as $z \in \Gamma_{N+1}$ and $y \in \Gamma_1$ the monodromy matrix $T(z,y)$ can be presented in the form

$$T(z,y) = T(z,z_{2N})T(z_{2N},z_{2N-1}) \ldots T(z_1,y).$$

Taking the last formula and the known expressions for $T(z,y)$ at $\rho = 0$ into account one finds the limit (42) easily. We write the result in the following form

$$T(k) = K_{N+1}T_{N}K_{N}T_{N-1} \ldots K_1 T_1 K_1.$$

Here

$$T_j = \begin{pmatrix} 1 - \frac{2M_j}{k-z_{2j-1}} & -4\Omega_j M_j & \frac{4M_j \phi_{2j-1}}{k-z_{2j-1}} \\ \frac{2L_j^{ph} - i|q_j|^2}{(k-z_{2j})(k-z_{2j-1})} & k-z_{2j} & \frac{2L_j^{ph} - i|q_j|^2 \phi_{2j}}{k-z_{2j}} \\ -\frac{q_j}{k-z_{2j-1}} & -2q_j \Omega_j & 1 + \frac{2q_j \phi_{2j}}{k-z_{2j-1}} \end{pmatrix}.$$
and

\[ K_j = \begin{pmatrix} 1 & D_j & 0 \\ 0 & 1 & 0 \\ 0 & -i(\bar{\phi}_{2j-1} - \bar{\phi}_{2j-2}) & 1 \end{pmatrix}, \]

where \( D_j = \bar{Y}_{00}(z_{2j-1}) - \bar{Y}_{00}(z_{2j-2}) \) and \( \phi_k = \Phi_0(z_k) \). Recall \( z_0 = -\infty \) and \( z_{2N+1} = +\infty \). We note also that

\[ \phi_{2j} = \Phi_H - \Omega_j \Phi^A_{j+1}, \quad \phi_{2j-1} = \Phi_H - \Omega_j \Phi^A_j \]

and

\[ 2L_j + \Phi^A_{j+1}q_j = 2L_j^{ph} - i|q_j|^2. \]

The equations (39) are invariant with respect to the transformations

\[ \psi(\omega) \to G(\psi^*(-\frac{\rho^2}{\bar{\omega}}))^{-1}, \quad \psi(\omega) \to \Phi_g(\omega)\psi^*(\bar{\omega})^{-1}, \]

where

\[ \Phi_g(\omega) = \begin{pmatrix} 2|\Phi_0|^2 & 2\Phi_0\bar{\Phi}_1 - i\omega & \Phi_0 \\ 2\Phi_1\bar{\Phi}_0 + i\omega & 2|\Phi_1|^2 & \Phi_1 \\ \bar{\Phi}_0 & \bar{\Phi}_1 & 1/2 \end{pmatrix}. \]

Thus, the reduced monodromy matrix \( T(k) \) has to satisfy

\[ T(k) = \begin{pmatrix} -I & 0 \\ 0 & 1/2 \end{pmatrix} T^*(\bar{k}) \begin{pmatrix} -I & 0 \\ 0 & 2 \end{pmatrix} \] (44)

and

\[ T(k) = \begin{pmatrix} \sigma_y & 0 \\ 0 & 1/2 \end{pmatrix} T^*(\bar{k})^{-1} \begin{pmatrix} \sigma_y & 0 \\ 0 & 2 \end{pmatrix}. \] (45)

Here \( I \) is the unit 2x2 matrix, \( \sigma_y \) is the Pauli matrix and \( T^* \) is the Hermitian conjugation of \( T \). The matrix \( T(k) \) can not satisfy Eq. (44) if the physical parameters are arbitrary. This equation (44) defines the relations between the parameters. Unfortunately, these relations are quite complicated and we can study them in particular cases only.

Let us consider one black hole first. From Eq. (44) we obtain

\[ D_1 = 2\Omega_1 M_1 - i|q_1|^2 M_1 \Omega_1 \frac{L_1}{L_1}, \quad D_2 = 2\Omega_1 M_1 + i|q_1|^2 M_1 \Omega_1 \frac{L_1}{L_1} \]

and

\[ \phi_1 = (\phi^H - i\Omega_1)q_1, \quad \phi_2 = (\phi^H + i\Omega_1)q_1, \quad \phi^H = \frac{L_1 + M_1 \Omega_1 |q_1|^2}{2L_1 M_1}. \]
Then, the equations (44) and (35) are eventually reduced to the familiar formulas for the mass and the angular velocity of one black hole, viz

\[ \Omega_1 = \frac{a}{(M_1 + m_1)^2 + a^2}, \quad M_1^2 = m_1^2 + a^2 + |q_1|^2, \quad a = \frac{L_1}{M_1}. \]

We can solve Eqs. (44) for the case of two non-rotating black holes \( (\Omega_1 = \Omega_2 = 0) \) with the charges of the magnet monopoles equal to zero \( (P_1 = P_2 = 0) \) as well. In this case, \( L_1^{ph} = L_2^{ph} = 0, \) \( M_1^{ph} = M_1 \) and \( M_2^{ph} = M_2, \) while Eqs. (44) and (35) are eventually reduced to a system of two equations

\[ M_1 = m_1 + \phi_1^H Q_1^2, \quad M_2 = m_2 + \phi_2^H Q_2^2, \quad (46) \]

where

\[ \phi_1^H = \frac{1}{M_1 + m_1} \left( 1 + \frac{2M_1 Q_2 - 2M_2 Q_1}{Q_1(M_1 + M_2 + R)} \right), \]

\[ \phi_2^H = \frac{1}{M_2 + m_2} \left( 1 - \frac{2M_1 Q_2 - 2M_2 Q_1}{Q_2(M_1 + M_2 + R)} \right), \]

and \( R \) is a distance parameter chosen such a way that \( z_1 = -m_1, z_2 = m_1, z_3 = R - m_2 \) and \( z_4 = R + m_2. \) For completeness, we note that

\[ D_1 = -i\Phi_1^H Q_1, \quad D_2 = -i(\Phi_2^H Q_2 - \Phi_1^H Q_1), \quad D_3 = i\Phi_2^H Q_2 \]

and

\[ \Phi_1^H = \phi_1^H Q_1, \quad \Phi_2^H = \phi_2^H Q_2. \]

The system of Eqs. (46) is one of the main results of this paper. It gives the relations between parameters \( M_1, M_2 \) and \( m_1, m_2. \) It is preferred to consider \( m_i, \) which are the irreducible masses, as independent parameters of the problem. First, because they describe the event horizon. Secondly, in the general relativity, there is no notion of the energy density hence, we think, there is no sense to fix the mass of \( i \)th black hole. Moreover, the total mass \( M = M_1 + M_2 \) contains the energy of the electromagnetic field surrounding the black holes as well. Let \( m_i \) be independent parameters then in the general case the masses \( M_i^{ph} \) become functions of the distance parameter. These functions must be derived from Eqs. (44) and (35).

In the particular case of Eqs. (46) we see that \( M_1 \) and \( M_2 \) don’t depend on \( R \) only if \( M_1 Q_2 = M_2 Q_1. \) This is true if \( m_1 Q_2 = m_2 Q_1. \) The quantities \( Q_i/2m_i \) have the sense of charge per unit length. Hence, if the charge density of two black holes treated as one dimensional objects, namely, the intervals \( I_i \) are different the total mass of the system depends on the distance between the black holes.
Acknowledgements

This work was partly supported by RFBR grant No. 00-01-00480.

References

[1] R. Bach and H. Weyl. Math. Z., 13, 132 (1921).
[2] G. Weinstein, Comm. Pure Appl. Math., 43, 903 (1990).
[3] G. G. Varzugin, Theor. Math. Phys., 116, 1024 (1998); gr-qc/0005033.
[4] G. Weinstein, Trans. Am. Math. Soc., 343, 899 (1994).
[5] J. B. Hartle and S. W. Hawking, Comm. Math. Phys., 26, 87, (1972).
[6] G. G. Varzugin, Theor. Math. Phys., 111, 667 (1997); gr-qc/0004073.
[7] M. S. Costa and M. J. Perry, Nucl. Phys. B, 521, 469; hep-th/0008109 (2000).
[8] B. Carter. Black hole equilibrium states. In: Black Hole. Eds C. De Witt and B. S. De Witt. N. Y.: Gordon and Breach, (1973).
[9] G. Weinstein, Comm. Pure Appl. Math., 45, 1183 (1992).
[10] Y. Li and G. Tian, Manuscripta Math., 73, 83 (1991).
[11] G. Weinstein, Comm. Part. Diff. Eqs., 21, 1389 (1996).
[12] W. Kinnersley and D. Chitre, J. Math. Phys., 19, 1926; J. Math. Phys., 19, 2037 (1978).
[13] G. A. Aleksejev, Pis’ma Zh. Eksp. Teor. Phys., 32, 301 (1980).
[14] M. Gurses and B. C. Xanthopoulos, Phys. Rev. D, 26, 1912 (1982).
[15] G. Neugebauer and D. Kramer, J. Phys. A, 16, 1927 (1983).
[16] A. Eris, M. Gurses and A. Karasu, J. Math. Phys., 25, 1489 (1984).
[17] V. A. Belinskii and V. E. Zakharov, JETP, 50, 1 (1979).
[18] I. Hauser and F. J. Ernst, Phys. Rev. D, 20, 1783 (1979).
[19] E. Ruiz, V. S. Manko and J. Martin, Phys. Rev. D, 51, 4192 (1995).
[20] N. R. Sibgatullin, Oscillations and Waves in Strong Gravitational and Electromagnetic Fields, Springer-Verlag, Berlin (1991).
[21] G. A. Aleksejev, In: Trudy MIAN, 176, 211 (1987).

[22] W. Kinnersley, J. Math. Phys., 19, 1529 (1977).

[23] V. A. Belinskii and V. E. Zakharov, JETP, 48, 985 (1978).

[24] L. A. Takhtajan and L. D. Faddeev, Hamiltonian Approach in the Theory of Soliton, Berlin-Heidelberg-New York (1987).