Projectively invariant symbol map and cohomology of vector fields Lie algebras intervening in quantization

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Abstract

We define the unique (up to normalization) symbol map from the space of linear differential operators on $\mathbb{R}^n$ to the space of polynomial on fibers functions on $T^*\mathbb{R}^n$, equivariant with respect to the Lie algebra of projective transformations $sl_{n+1} \subset \text{Vect}(\mathbb{R}^n)$. We apply the constructed $sl_{n+1}$-invariant symbol to studying of the natural one-parameter family of $\text{Vect}(M)$-modules on the space of linear differential operators on an arbitrary manifold $M$. Each of the $\text{Vect}(M)$-action from this family can be interpreted as a deformation of the standard $\text{Vect}(M)$-module $S(M)$ of symmetric contravariant tensor fields on $M$. We define (and calculate in the case: $M = \mathbb{R}^n$) the corresponding cohomology of $\text{Vect}(M)$ related with this deformation. This cohomology realize the obstruction for existence of equivariant symbol and quantization maps. The projective Lie algebra $sl_{n+1}$ naturally appears as the algebra of symmetries on which the involved $\text{Vect}(M)$-cohomology is trivial.

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Introduction

The general problem of quantization is to associate linear operators on a Hilbert space with Hamiltonian functions on a symplectic manifold. In the simplest situation when the symplectic manifold is a cotangent bundle $T^*M$ over a manifold $M$, the usual quantization procedure consists of establishing a correspondence between the space $\text{Pol}(T^*M)$ of functions on $T^*M$ polynomial on fiber and the space $\mathcal{D}(M)$ of linear differential operators on $M$:

$$\sigma : \mathcal{D}(M) \rightarrow \text{Pol}(T^*M) \quad \text{symbol map.}$$

and the inverse

$$\sigma^{-1} : \text{Pol}(T^*M) \rightarrow \mathcal{D}(M) \quad \text{quantization map}$$

(see e.g. [2]).

Let $\text{Diff}(M)$ be the group of diffeomorphisms of $M$ and $\text{Vect}(M)$ be the Lie algebra of vector fields on $M$. Space $\mathcal{D}(M)$ has a natural 1-parameter family of $\text{Diff}(M)$- and $\text{Vect}(M)$-modules. To define it, one considers arguments of differential operators as tensor-densities of an arbitrary degree $\lambda$ (see [8], [15], [12]).

One of the difficulties of quantization can be formulated as follows. There is no quantization map (and symbol map) equivariant with respect to the action of $\text{Diff}(M)$. Two spaces: $\text{Pol}(T^*M)$ and $\mathcal{D}(M)$ are not isomorphic as modules over $\text{Diff}(M)$ or $\text{Vect}(M)$. To avoid this difficulty, some of methods of quantization are based on a choice of Darboux coordinates in $T^*M$ (e.i. the well-known Weyl quantization and the Moyal-Weyl deformation quantization). Other ones (as the modern approach of B. Fedosov [9]) fix a linear (symplectic) connection on $T^*M$. Another way is to use the representation theory (as in the case of geometric quantization).

This paper contains two main parts.

(a) We consider a natural embedding $sl_{n+1} \subset \text{Vect}(\mathbb{R}^n)$ ($sl_{n+1}$ acts on $\mathbb{R}^n$ by infinitesimal projective transformations).

The first main result of this paper is the existence of the unique (up to normalization) $sl_{n+1}$-equivariant symbol map on $\mathbb{R}^n$. This implies that $\mathcal{D}(\mathbb{R}^n)$ is isomorphic to $\text{Pol}(T^*\mathbb{R}^n)$ as a module over $sl_{n+1}$.

The same result holds for manifolds endowed with a projective structure (an atlas with linear-fractional coordinate transformations). The main examples are: $M = \mathbb{P}^n, S^n, T^n$.

Remarks. 1. In the one-dimensional case ($n = 1$), $sl_2$-equivariant symbol map and quantization map was obtained (in a more general situation of pseudodifferential operators)
in a recent work by P.B. Cohen, Yu. I. Manin and D. Zagier [5]. We check that in the case $n = 1$, our formulæ coincide with those of [5]. We were not aware of this paper while the calculation of the $sl_{n+1}$-equivariant symbol has been done.

2. In the (algebraic) case of global differential operators on $\mathbb{C}P^n$, existence and uniqueness of the $sl_{n+1}$-equivariant symbol is a corollary of Borho–Brylinski’s results [3]. Space $\mathcal{D}(\mathbb{C}P^n)$ as a module over $sl_{n+1}$ has a decomposition in a sum of irreducible submodules with *multiplicities one*. This implies the uniqueness result. Our explicit formulæ are valid in the holomorphic case and define an isomorphism between $\mathcal{D}(\mathbb{C}P^n)$ and the space of polynomial on fibers functions on $T^*\mathbb{C}P^n$.

(b) In the second part of this paper, we study the one-parameter family of $\text{Vect}(M)$-modules on $\mathcal{D}(M)$ for an arbitrary smooth manifold $M$.

Space $\text{Pol}(T^*M)$ of fiber-wise polynomials on $T^*M$ is naturally isomorphic (as a $\text{Diff}(M)$-module) to the space of symmetric contravariant tensor fields on $M$: $\Gamma(S(TM))$. Action of $\text{Diff}(M)$ and $\text{Vect}(M)$ on $\mathcal{D}(M)$ can be realized as a *deformation* of the standard action on $\text{Pol}(T^*M)$. This viewpoint makes possible to apply the cohomology technique: the action of $\text{Vect}(M)$ on $\mathcal{D}(M)$ is distinguished from the action of $\text{Vect}(M)$ on $\text{Pol}(T^*M)$ by certain cohomology classes of $\text{Vect}(M)$. More precisely, this approach leads to the first group of $\text{Vect}(M)$-cohomology with coefficients in the space of linear operators on symmetric contravariant tensor fields. This cohomology realizes obstructions for existence of equivariant quantization (and symbol) map.

The cohomological approach to studying of $\text{Vect}(M)$-module structure on the space of differential operators was proposed in [8] in the case of second order operators.

The second main result of this paper is calculation of the first group of $\text{Vect}(\mathbb{R}^n)$-cohomology vanishing on $sl_{n+1}$, with coefficients in the space of differentiable linear operators on symmetric contravariant tensor fields. This result includes classification of bilinear $sl_{n+1}$-equivariant maps $\text{Vect}(\mathbb{R}^n) \otimes \Gamma(S^k(TM)) \to \Gamma(S^l(TM))$ vanishing on subalgebra $sl_{n+1} \subset \text{Vect}(\mathbb{R}^n)$. Such operations are multi-dimensional analogues of so-called Gordan transvectants [3] (bilinear $sl_2$-equivariant operators on tensor-densities on $S^1$).

Lie algebra $sl_{n+1}$ appears as the maximal subalgebra of $\text{Vect}(\mathbb{R}^n)$ on which all the involved cohomology classes vanish. The restriction to $sl_{n+1}$ of the $\text{Vect}(\mathbb{R}^n)$-modules on $\mathcal{D}(\mathbb{R}^n)$, is trivial: all the $sl_{n+1}$-modules on $\mathcal{D}(\mathbb{R}^n)$ are isomorphic to the module of symmetric contravariant tensor fields.

The problem of isomorphism of the $\text{Vect}(M)$-modules on $\mathcal{D}(M)$ for different values of the parameter $\lambda$ was solved in a series of recent papers [8], [13], [12]. We will classify the factor-modules $\mathcal{D}_\lambda^k(M)/\mathcal{D}_\lambda^l(M)$ (of the module of $k$-th order operators over the module of $l$-th order operators).

(c) We will apply the $sl_{n+1}$-equivariant symbol and quantization maps to define $sl_{n+1}$-equivariant quantization of $T^*S^n$. Namely, we will define a $sl_{n+1}$-equivariant star-product
on $T^*S^n$ and give a $sl_{n+1}$-equivariant version of quantization of the geodesic flow on $T^*S^n$.

We hope, that the appearance of projective symmetries in the context of quantization is natural. We do not know any works on this subject except the special one-dimensional case (cf. [3],[17]).

Relations of differential operators with projective geometry have already been studied by classics (see e.g. [21],[4]). The most known example is the Sturm-Liouville operator $d^2/dx^2 + u(x)$ describing a projective structure on $\mathbb{R}$ (or on $S^1$ if $u(x)$ is periodic). This example is related to the well-known Virasoro algebra (see [14]).

1 Projective Lie algebra $sl_{n+1}$

We call the projective Lie algebra, the Lie algebra of vector fields on $\mathbb{R}^n$ generated by the following vector fields:

$$\frac{\partial}{\partial x^i}, \quad x^i \frac{\partial}{\partial x^j}, \quad x^i E,$$

where

$$E = x^j \frac{\partial}{\partial x^j}$$

(we omit the sign of a sum over repeated indices) is the Euler field. This Lie algebra is isomorphic to $sl_{n+1}$. The vector fields (1) are infinitesimal generators of (locally defined on $\mathbb{R}^n$) linear-fractional transformations.

The constant and linear vector fields (first two terms in (1)) generate a subalgebra of $sl_{n+1}$ called the affine Lie algebra.

The projective Lie algebra is well-defined globally on $\mathbb{P}^n \supset \mathbb{R}^n$. The space of vector fields generated by the vector fields (1) is invariant under linear-fractional coordinate changes.

Remark 1.1. Lie subalgebra $sl_{n+1} \subset \text{Vect}(\mathbb{R}^n)$ is maximal in the following sense. Given an arbitrary polynomial vector field $X \notin sl_{n+1}$, then, a Lie algebra generated by $X$ together with the vector fields (1), coincides with the whole Lie algebra of polynomial vector fields on $\mathbb{R}^n$ (see [16] for a similar result).

Lie algebra $sl_{n+1}$ can be defined as the kernel of certain cohomology classes of $\text{Vect}(\mathbb{R}^n)$. This interesting relation between the cohomology of $\text{Vect}(\mathbb{R}^n)$ and the projective Lie algebra $sl_{n+1}$ is the main tool of this paper.

1.1 $sl_{n+1}$ as a cohomology kernel

Consider the space $\mathcal{S}_2^1$ of symmetric $(1,2)$-tensor fields on $\mathbb{R}^n$:

$$T = T^k_{ij}(x) \cdot dx^i dx^j \otimes \partial_k,$$
here and below $\partial_k = \frac{\partial}{\partial x^k}$. There exist a non-trivial cocycle $\gamma: \text{Vect}(\mathbb{R}^n) \to S_2^1$:

$$\gamma(X) = \partial_i \partial_j (X^k) \cdot dx^i dx^j \otimes \partial_k,$$

where $X = X^k \frac{\partial}{\partial x^k}$.

Cocycle $\gamma$ can be written in the following form. Let $D$ be the differential, acting on symmetric tensor fields: $D(a) = \partial_i (a) \cdot dx^i$ (operator $D$ depends on the choice of coordinates). Then, $\gamma(X) = D^2(X)$.

**Remark 1.2.** Note, that cocycle $\gamma$ is just the non-linear term in the standard (infinitesimal) coordinate transformations of the Christoffel symbols of a linear connection.

The important property of the projective subalgebra $sl_{n+1} \subset \text{Vect}(\mathbb{R}^n)$ is that the restriction to $sl_{n+1}$ of the cohomology class $[\gamma]$ vanishes.

**Proposition 1.3.** (i) In the multi-dimensional case ($n \geq 2$), there exists a unique (up to a constant) cocycle on $\text{Vect}(\mathbb{R}^n)$ with values in $S_2^1$ vanishing on $sl_{n+1}$:

$$\bar{\gamma}(X) = \gamma(X) - \frac{2}{n+1} \partial_i \partial_k (X^k) \cdot dx^i dx^j \otimes \partial_j$$

(ii) The vector fields (1) generate the space of solutions of the equation $\bar{\gamma}(X) = 0$.

**Proof:** The uniqueness of $\bar{\gamma}$ is a corollary of Theorem 5.2 below.

The cocycle $\bar{\gamma}$ is a projection of $\gamma$ to the space of zero-trace tensor fields (such that $T^i_{ij} \equiv 0$). To rewrite $\bar{\gamma}$ in an intrinsic way, introduce a $(1,1)$-tensor $\Delta = dx^j \otimes \partial_j$. Then,

$$\bar{\gamma}(X) = \gamma(X) - \frac{2}{n+1} D(\text{div}X) \Delta.$$

The cocycle (2) was considered in [18].

### 1.2 Projectively equivariant cocycles

A general property of 1-cocycles is that a 1-cocycle vanishing on a Lie subalgebra is equivariant with respect to this subalgebra. In particular, the cocycle $\bar{\gamma}$ on $\text{Vect}(\mathbb{R}^n)$ defines a $sl_{n+1}$-equivariant map from $\text{Vect}(\mathbb{R}^n)$ to $S_2^1$.

Indeed, the 1-cocycle relation: $L_X(\bar{\gamma}(Y)) - L_Y(\bar{\gamma}(X)) = \bar{\gamma}([X,Y])$ implies the property of $sl_{n+1}$-equivariance: $L_X(\bar{\gamma}(Y)) = \bar{\gamma}([X,Y])$, for every $X \in sl_{n+1}$.

### 1.3 Gelfand-Fuchs cocycle

In the one-dimensional case, $\bar{\gamma} \equiv 0$ and and Proposition 1.2 does not hold. The cocycle $\gamma$ in this case is trivial.
However, there exists a non-trivial cocycle on \( \text{Vect}(\mathbb{R}) \) with values in the quadratic differentials vanishing on the subalgebra \( \mathfrak{sl}_2 \):

\[
X \frac{d}{dx} \mapsto X'' \ (dx)^2.
\]

This is (a version of) so-called Gelfand-Fuchs cocycle (see e.g. [11]) related to the Virasoro algebra.

We will define a multi-dimensional analogue of the cocycle (3) in Section 5.4.

2 Equivariant symbol

Let \( \mathcal{D} \) be the space of scalar linear differential operators on \( \mathbb{R}^n \):

\[
A = a^{i_1\ldots i_k}_{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} + \cdots + a^i_i \frac{\partial}{\partial x^i} + a_0
\]

with coefficients \( a^{i_1\ldots i_j}_{i_1 \ldots i_j} = a^{i_1\ldots i_j}(x^1, \ldots, x^n) \in C^\infty(\mathbb{R}^n) \), number \( k \) is the order of \( A \).

Denote \( \mathcal{D}^k \subset \mathcal{D} \) the space of all \( k \)-th order linear differential operators.

2.1 Diff(\( \mathbb{R}^n \))- and Vect(\( \mathbb{R}^n \))-module structures

Let us recall the definition of the natural 1-parameter family of Diff(\( \mathbb{R}^n \))- and Vect(\( \mathbb{R}^n \))-module structures on \( \mathcal{D} \) (see [8], [15], [12]).

Consider a 1-parameter family of Diff(\( \mathbb{R}^n \))-actions on \( C^\infty(\mathbb{R}^n) \):

\[
g^\lambda_*(\phi) := \phi \circ g^{-1} \cdot \left| \frac{Dg^{-1}}{Dx} \right|^{\lambda},
\]

where \( g \in \text{Diff}(\mathbb{R}^n) \), \( \phi \in C^\infty(\mathbb{R}^n) \), \( |Dg^{-1}/Dx| \) is the Jacobian of \( g^{-1} \) and \( \lambda \in \mathbb{R} \) (or \( \mathbb{C} \)).

The 1-parameter family of Diff(\( \mathbb{R}^n \))-actions on \( \mathcal{D} \) is given by:

\[
g^\lambda(A) = g^\lambda_* A(g^\lambda_*)^{-1}.
\]

**Remark 2.1.** Operator \( g^\lambda_* \) is the natural action of Diff(\( \mathbb{R}^n \)) on the space of tensor-densities of degree \( \lambda \) on \( \mathbb{R}^n \): \( \phi = \phi(x^1, \ldots, x^n)(dx^1 \wedge \cdots \wedge dx^n)^\lambda \). For example, in the case \( \lambda = 0 \), one has the standard Diff(\( M \))-action on functions, the value \( \lambda = 1 \) corresponds to the Diff(\( M \))-action on differential \( n \)-forms.

Consider the operator of Lie derivative on tensor-densities of degree \( \lambda \):

\[
L^\lambda_X(\phi) = X^i \frac{\partial \phi}{\partial x^i} + \lambda \partial_i(X^i)\phi,
\]
where $X \in \text{Vect}(\mathbb{R}^n)$. (Note, that this formula does not depend on the choice of local coordinates.) The 1-parameter family of $\text{Vect}(\mathbb{R}^n)$-modules on space $\mathcal{D}$ is defined by the commutator:

$$\text{ad}L^X_A := L^X_A \circ A - A \circ L^X_A.$$ (6)

**Notation 2.2.** Let us denote:
(a) $\mathcal{F}_\lambda$ the $\text{Diff}(\mathbb{R}^n)$- and $\text{Vect}(\mathbb{R}^n)$-modules of tensor-densities on $\mathbb{R}^n$ of degree $\lambda$.
(b) $\mathcal{D}_\lambda$ and $\mathcal{D}_\lambda^k$ the $\text{Diff}(\mathbb{R}^n)$- and $\text{Vect}(\mathbb{R}^n)$-modules of differential operators defined by (5) and (6).

2.2 Main definition

It is well known (see Sections 4.2 – 4.4 for the precise statements), that the notion of a symbol of a differential operator is not intrinsic. Only the principal symbol has a geometrical meaning. In other words, there is no symbol map equivariant with respect to $\text{Diff}(\mathbb{R}^n)$ (or $\text{Vect}(\mathbb{R}^n)$).

Consider the restriction of the $\text{Vect}(\mathbb{R}^n)$-modules $\mathcal{D}_\lambda$ to the projective Lie algebra $sl_{n+1}$. Introduce the coordinates $(x^i, \xi_i)$, $i = 1, \ldots, n$ on $T^*\mathbb{R}^n$, then $\text{Pol}(T^*\mathbb{R}^n)$ is the space of polynomials in $\xi$ with coefficients in $C^\infty(\mathbb{R}^n)$. We are looking for a symbol map on $\mathbb{R}^n$:

$$\sigma^\lambda : \mathcal{D}_\lambda \to \text{Pol}(T^*\mathbb{R}^n)$$
equivariant with respect to the action of $sl_{n+1}$.

Let us formulate the first main result of this paper.

**Theorem I.** (i) For every $\lambda$, there exists a unique $sl_{n+1}$-equivariant symbol map $\sigma^\lambda$, such that for every $A \in \mathcal{D}$, the higher order term of polynomial $\sigma^\lambda_A(\xi)$ coincides with the principal symbol of $A$.
(ii) $\sigma^\lambda$ maps a differential operator $A = a^{i_1 \cdots i_k}_{i_{k-m} \cdots i_m} \partial_{i_1} \cdots \partial_{i_k}$ to the polynomial $\sigma^\lambda_A(\xi) = \sum_{m=0}^k \bar{a}_{k-m}^{i_1 \cdots i_{k-m}} \xi_{i_1} \cdots \xi_{i_{k-m}}$ with the coefficients:

$$\bar{a}_{k-m} = \sum_{m=0}^k c_{k-m}^{(m)} a_k^{(m)}$$

where $a_k^{(m)}$ is the “divergence”: $a_k^{(m)} = \sum_{j} \partial_{j_{i_1 \cdots j_{k-m} j_{1 \cdots j_{m}}}}$ and $c_{k-m}$ are constants:

$$c_{k-m} = (-1)^m \frac{\binom{k}{m} \binom{(n+1)\lambda+k-1}{m}}{\binom{2k+n-m}{m}}$$ (7)

**Remark 2.3.** The condition of $sl_{n+1}$-equivariance is already sufficient to determine each term $\bar{a}_{w-m}$ up to a constant. The supplementary condition, that the higher order term of polynomial $P^\lambda_A(\xi)$ coincides with the principal symbol, fixes the normalization.
Theorem I'. The unique (up to normalization) $sl_{n+1}$-equivariant quantization map is the map inverse to the $sl_{n+1}$-equivariant symbol map. It associates to a monomial $\bar{a}_k = \bar{a}_k^{i_1...i_k}\xi_{i_1}...\xi_{i_k}$ the differential operator:

$$(\sigma^\lambda)^{-1}(\bar{a}_k) = \sum_{m=0}^{k} c_{k-m}^m (\pi_k^{(m)})^{i_1...i_k-m} \partial_{i_1}...\partial_{i_k-m}$$

where

$$c_{k-m}^m = \binom{k}{m} \frac{(n+1)\lambda+k-1}{m} \binom{2k+n-1}{m} \quad (8)$$

Remark 2.4. In the one-dimensional case ($n = 1$) the formulæ (7), (8) are valid also for the space of pseudodifferential operators. In this case these formulæ coincide with (4.11) and (4.10) of [5].

2.3 Example: second order operators and quadratic Hamiltonians

Let us apply the general formulæ (8), (9) in the case of a quadratic polynomials on $T^*\mathbb{R}^n$ and second order differential operators.

(a). The $sl_{n+1}$-equivariant symbol of second order differential operator $A = a_{ij}^2 \partial_i \partial_j + a_1^i \partial_i + a_0$ is: $\sigma^\lambda_A(\xi) = \bar{a}_{ij}^2 \xi_i \xi_j + \bar{a}_{1}^i \xi_i + \bar{a}_0$, where

$$\bar{a}_{ij}^2 = a_{ij}^2$$

$$\bar{a}_{1}^i = a_1^i - 2 \frac{(n+1)\lambda+1}{n+3} \partial_j (a_{ij}^2)$$

$$\bar{a}_0 = a_0 - \lambda \partial_i (a_1^i) + \lambda \frac{(n+1)\lambda+1}{n+2} \partial_i \partial_j (a_{ij}^2)$$

(b). The $sl_{n+1}$-equivariant quantization map $(\sigma^\lambda)^{-1}$ associates the second order differential operator $A = a_{ij}^2 \partial_i \partial_j + a_1^i \partial_i + a_0$:

$$a_{ij}^2 = \bar{a}_{ij}^2$$

$$a_1^i = \bar{a}_1^i + 2 \frac{(n+1)\lambda+1}{n+3} \partial_j (\bar{a}_{ij}^2)$$

$$a_0 = \bar{a}_0 + \lambda \partial_i (\bar{a}_1^i) + \lambda \frac{(n+1)((n+1)\lambda+1)}{(n+2)(n+3)} \partial_i \partial_j (\bar{a}_{ij}^2)$$

with a polynomial $\mathcal{P}(\xi) = \bar{a}_{ij}^2 \xi_i \xi_j + \bar{a}_{1}^i \xi_i + \bar{a}_0$ on $T^*M$.

We will consider in Section 7.3 the special case $\lambda = 1/2$ and take a non-degenerate quadratic form $H$. Then, one has a metric $H^{-1} = g_{ij} \, dx^i dx^j$ on $\mathbb{R}^n$. The corresponding
volume form $\sqrt{g}$ identifies 1/2-densities with functions: $F_{1/2} \cong C^\infty(\mathbb{R}^n)$. The operator $(\sigma^\lambda)^{-1}(H)$ in this case is interpreted as a Laplace-Beltrami operator.

### 2.4 Structure of $sl_{n+1}$-module

Space of polynomial on fibers functions $\text{Pol}(T^*\mathbb{R}^n) = C^\infty(\mathbb{R}^n)[\xi_1, \ldots, \xi_n]$ has a very simple $\text{Diff}(\mathbb{R}^n)$- and $\text{Vect}(\mathbb{R}^n)$-module structures. It is isomorphic to the direct sum of modules of symmetric contravariant tensor fields:

$$\text{Pol}^k(T^*\mathbb{R}^n) \cong S^0 \oplus \ldots \oplus S^k,$$

where $S^i := \Gamma(S^i(T^*\mathbb{R}^n))$ is the space of symmetric $(l,0)$-tensor fields.

The structure of the $\text{Diff}(\mathbb{R}^n)$- and $\text{Vect}(\mathbb{R}^n)$-modules of differential operators is much more complicated. However, Theorem I implies that the restriction of these modules to the projective Lie algebra $sl_{n+1}$ is, in some sense, trivial:

**Corollary 2.5.** For every $\lambda$, $D^k_\lambda$ is isomorphic as a $sl_{n+1}$-module to $S^0 \oplus \ldots \oplus S^k$. In particular, $sl_{n+1}$-modules $D_\lambda$ are isomorphic to each other for different values of $\lambda$.

Indeed, the symbol map $\sigma^\lambda$ is an isomorphism of $sl_{n+1}$-modules.

### 2.5 Diagonalization of intertwining operators

The uniqueness of $\sigma^\lambda$ implies that in terms of the $sl_{n+1}$-equivariant symbol, every isomorphisms of modules of differential operators has a diagonal form:

**Corollary 2.6.** A linear map

$$T : D^k_\lambda \to D^k_\mu$$

is $sl_{n+1}$-equivariant if and only if the symbols $\sigma^\lambda_\lambda$ and $\sigma^\mu_{T(A)}$ are proportional:

$$(\sigma^\lambda_0(A), \ldots, \sigma^\lambda_k(A)) = (\alpha_0 \sigma^\mu_0(T(A)), \ldots, \alpha_k \sigma^\mu_k(T(A)))$$

where $\sigma^\lambda_i(A)$ is the homogeneous component of order $i$ of $\sigma^\lambda_\lambda$, $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ are arbitrary constants.

In other words, the map $\sigma T \sigma^{-1}$ on $S^0 \oplus \ldots \oplus S^k$ multiplies each term of the direct sum by a constant.

### 3 Polynomial language, proof of Theorem I

In this section we prove Theorems I and $\Gamma$. 


Using the “standard” symbol, we represent every differential operator on \( \mathbb{R}^n \) as polynomials on \( T^*\mathbb{R}^n \). The operator (4) corresponds to the polynomial

\[
P_A = \sum_{l=0}^{k} a^{i_1...i_l} \xi_{i_1} \cdots \xi_{i_l},
\]

where \( \xi_i \) are coordinates on the fiber dual to \( x^i \).

Following [15], we rewrite the conditions of equivariance in terms of differential equations.

### 3.1 Operator of Lie derivative

The operator \( L^\lambda_X(f) \) (Lie derivative of tensor-densities of degree \( \lambda \)) is given by a bilinear map \( L^\lambda : \text{Vect}(M) \otimes \mathcal{F}_\lambda(M) \rightarrow \mathcal{F}_\lambda(M) \). Thus, \( L^\lambda \) is a differential operator on sections of a certain vector bundle over \( M \times M \). Introducing coordinates \( \xi, \eta \) on fibers of \( T^*\mathbb{R}^n \times T^*\mathbb{R}^n \), one obtains the corresponding polynomial on \( T^*\mathbb{R}^n \times T^*\mathbb{R}^n \):

\[
P_{L^\lambda_X(f)} = X^i \eta_i f + \lambda X^i \xi_i f = \langle X, \eta \rangle f + \lambda \langle X, \xi \rangle f
\]

(the first term corresponds to the derivatives of \( f \) and the second one corresponds to the derivatives of \( X \)).

Let us give the polynomial realization for the operator \( \text{ad} L^\lambda \) (of commutator with Lie derivative): \( \text{ad} L^\lambda : \text{Vect}(M) \otimes \mathcal{D} \rightarrow \mathcal{D} \). The polynomial \( P_{[L^\lambda_X,A]} \) is defined on \( T^*\mathbb{R}^n \times T^*\mathbb{R}^n \) (with coordinates \( \xi, \eta, \zeta \) on fibers). The explicit formula is as follows (see [15]):

\[
P_{[L^\lambda_X,A]} = \langle X, \eta \rangle \cdot P_A - \tau_\xi P_A \cdot P_X - \lambda \tau_\xi P_A \cdot \langle X, \xi \rangle,
\]

where \( P_A = \sum a^{i_1...i_l} \xi_{i_1} \cdots \xi_{i_l} \) and \( P_X = X^i \xi_i \), coordinates \( \xi \) and \( \eta \) correspond to derivatives of \( X \) and \( A \) respectively, \( \tau_\xi \) acts on a polynomial \( P(\zeta) \) to give the polynomial:

\[
\tau_\xi P = P(\zeta + \xi) - P(\zeta).
\]

This formula is a result of easy direct computations, the term \( \tau_\xi (P_A) \) appears from the Leibnitz rule by application of the differential operator \( A \).

### 3.2 Equivariance condition

We are looking for a differentiable linear map \( \sigma^\lambda : \mathcal{D} \rightarrow \text{Pol}(T^*\mathbb{R}^n) \) commuting with the \( sl_{n+1} \)-action. Let us represent \( \sigma^\lambda \) in a polynomial form.

Take \( A \in \mathcal{D} \), denote \( P = P_A(\zeta) \) the corresponding polynomial. Then, the polynomial associated with \( \sigma^\lambda(A) \) is defined on \( T^*\mathbb{R}^n \times T^*\mathbb{R}^n \):

\[
P_{\sigma^\lambda(A)} = P_{\sigma^\lambda(A)}(\eta, P),
\]
where the coordinates $\eta$ represent derivatives of $P$. It is linear in $P$. Denote this polynomial $\mathcal{P}$.

A simple direct calculation gives the $sl_{n+1}$-equivariance condition in terms of polynomials. For every $X \in sl_{n+1}$:

$$
\mathcal{P}(\xi + \eta, \langle X, \eta \rangle P - (\lambda \langle X, \xi \rangle + P_x)\tau_x P) - \langle X, \eta \rangle \mathcal{P}(\eta, P) \\
= (X.\mathcal{P})(\eta, P) - P_x \langle \xi, \partial \rangle \mathcal{P}(\eta, P)
$$

(9)

where $(X.\mathcal{P})$ means the derivative of coefficients of $\mathcal{P}$ in the direction of the vector field $X$.

### 3.3 Proof of Theorem I

The Taylor expansion of the equation (9) with respect to $\xi$ is of the form:

$$(X.\mathcal{P})(\eta, P) - (\rho(X \otimes \xi)\mathcal{P})(\eta, P) \\
- \frac{1}{2} \langle X, \eta \rangle (\xi \partial \eta)^2 \mathcal{P}(\eta, P) + (\xi, \partial) \mathcal{P}(\eta, X(\xi \partial \xi)P) + \frac{1}{2} \mathcal{P}(\eta, X(\xi \partial \xi)^2 P^k) \\
+ \lambda \langle X, \xi \rangle \mathcal{P}(\eta, (\xi \partial \xi)P) + o(|\xi|^3) = 0
$$

where $\rho(X \otimes \xi)$ is the canonical expansion of the natural action $(X \otimes \xi)(Y) = \langle Y, \xi \rangle X$ of the matrix $X \otimes \xi$ on the space of polynomials such as $\mathcal{P}$.

Vanishing of the terms of degree 0 and 1 in $\xi$ is equivalent to the invariance with respect to the constant vector fields and the linear vector fields respectively. Therefore, $\mathcal{P}$ has constant coefficients and it is invariant under the natural action of $gl(n, \mathbb{R})$. Hense $\mathcal{P}(\eta, P^k_Y)$ is a polynomial in the variables $u := \langle Y, \xi \rangle$ and $v := \langle Y, \eta \rangle$. We denote it $\mathcal{P}^k(u, v)$. It is homogeneous of degree $k$ in $u, v$.

It remains to impose the invariance under the action of the vector fields (10) which are of degree 2 or, equivalently, under the vector fields $X^\alpha = \alpha(x)E$, where $\alpha \in \mathbb{R}^{n*}$ is arbitrary. We now consider the terms of order 2 in $\xi$. Since $X^\alpha$ is of degree 2 and since the terms of order $\leq 1$ have previously been killed, it can be easily achieved if we note that, for every $\theta \in \mathbb{R}^{n*}$, $(X^\alpha, \theta)$ is the function $x \mapsto \alpha(x)\theta(x)$. For instance, a term like $\langle X, \eta \rangle (\xi \partial \eta)^2 \mathcal{P}(\eta, P)$ equals

$$2\alpha_i \eta_j \partial \eta_i \mathcal{P}(\eta, P)
$$

(where $\alpha_i, \eta_j$ are the components of $\alpha$ and $\eta$ in the canonical basis of $\mathbb{R}^n$). Taking $P = P^k_Y$ and noticing that

$$\partial \eta_i \mathcal{P}(\eta, P^k_Y) = Y^i \partial v \mathcal{P}^k,
$$

this may be further rewritten as

$$2\langle Y, \alpha \rangle v \partial v \mathcal{P}^k.
$$

In this way, one obtains the equation:

$$u \partial v \partial u \mathcal{P}^k + (n + k) \partial \mathcal{P}^k + k((n + 1)\lambda + k - 1)\mathcal{P}^{k-1} = 0.
$$
This leads to the relations:

\[ c^k_l = \frac{k((n+1)\lambda + k - 1)}{(k-l)(k+l+n)} c^{k-1}_{l+1}. \]

where \( c^k_l \) is the coefficient of the monomial \( u^l v^{k-l} \) in \( P^k \). Together with the normalization condition \( c^l_l = 1 \), this is equivalent to the formula (7).

Hence the theorem.

### 3.4 Proof of Theorem I’

We prove Theorem I’ in the same way as Theorem I. The uniqueness of the symbol map shows that it is sufficient to find a \( \mathfrak{sl}_{n+1} \)-equivariant map from \( \text{Pol}(T^*\mathbb{R}^n) \) to \( \mathcal{D} \) (satisfying the normalization condition). Such a map is necessarily equal to \( (\sigma^\lambda)^{-1} \). Let \( \mathcal{P}(\eta, P) \), where \( P \in \text{Pol}(T^*\mathbb{R}^n) \) be the corresponding polynomial.

As above, one gets that the polynomial \( \mathcal{P}(\eta, P) \) is homogeneous:

\[ \mathcal{P}_l(\eta, (Y, \zeta)^k) = \bar{c}^k(Y, \eta)^{k-l} (Y, \zeta)^l \]

(the condition of equivariance with respect to the affine Lie algebra).

The equivariance with respect to the quadratic part of \( \mathfrak{sl}_{n+1} \) leads to the relations:

\[ \bar{c}^k_l = \frac{(l+1)((n+1)\lambda + l)}{(k-l)(k+l+n)} \bar{c}^k_{l+1}. \]

The formula (8) follows.

### 4 Space of differential operators on a manifold as a module over the Lie algebra of vector fields

Let \( M \) be a smooth orientable manifold of dimension \( n \geq 2 \) and \( \mathcal{D}(M) \) be the space of scalar linear differential operators on \( M \). Space \( \mathcal{D}(M) \) has a natural 1-parameter family of modules over the Lie algebra of vector fields \( \text{Vect}(M) \) (see [8], [15], [12]). The definition is analogous to those of Section 2.1: \( \text{Vect}(M) \) acts on \( \mathcal{D}(M) \) by the commutator with the operator of Lie derivative on \( \lambda \)-densities on \( M \).

In local coordinates, the invariant formulæ (4)–(6) are valid.

In this section we will study the factor-modules \( \mathcal{D}^k_\lambda(M)/\mathcal{D}^l_\lambda(M) \). Our objective is to solve the problem of isomorphism between these modules for different values of \( \lambda \) and to compare them with the module of tensor fields \( S^k \oplus \ldots \oplus S^{l+1} \).
4.1 Action of Vect(M) on $\mathcal{D}_\lambda^k(M)$ in terms of equivariant symbols

Let us fix an arbitrary local coordinates on $M$ and apply the $sl_{n+1}$-equivariant symbol $\sigma^\lambda$. The operator:

$$\sigma^\lambda \circ \text{ad} L_X^\lambda \circ (\sigma^\lambda)^{-1} : S^k \oplus \ldots \oplus S^0 \rightarrow S^k \oplus \ldots \oplus S^0$$

expresses the action of a vector field $X$ on $\mathcal{D}_\lambda^k(M)$ in terms of the equivariant symbol. This action is clearly of the form:

$$\bar{a}_k^X = L_X(\bar{a}_k)$$
$$\bar{a}_{k-1}^X = L_X(\bar{a}_{k-1}) + \gamma_1^\lambda(X, \bar{a}_k)$$
$$\bar{a}_{k-2}^X = L_X(\bar{a}_{k-2}) + \gamma_1^\lambda(X, \bar{a}_{k-1}) + \gamma_2^\lambda(X, \bar{a}_k)$$

......

(10)

where $L_X(\bar{a}_i)$ is the Lie derivative on the space of symmetric contravariant tensor fields $S^i$, $\gamma_i^\lambda$ is a bilinear map:

$$\gamma_i^\lambda : \text{Vect}(M) \otimes S^k \rightarrow S^{k-p}.$$  \hfill (11)

Indeed, the higher order term $\bar{a}_k$ is the principal symbol and it transforms as a tensor field, the transformation law for $\bar{a}_{k-1}$ has a correction term depending on $\bar{a}_k$ etc.

**Lemma 4.1.** Operations $\gamma_i^\lambda$ satisfy the following properties:

(a) $sl_{n+1}$-equivariance:

$$L_X(\gamma(Y,a)) = \gamma([X,Y],a) + \gamma(Y, L_X(a)), \quad X \in sl_{n+1}, a \in S^j;$$  \hfill (12)

(b) vanishing on $sl_{n+1}$:

$$\gamma(X,a) \equiv 0, \quad X \in sl_{n+1}.$$  \hfill (13)

**Proof:** follows from the $sl_{n+1}$-equivariance of the symbol map $\sigma^\lambda$.

4.2 Modules $\mathcal{D}_\lambda^k/\mathcal{D}_\lambda^{k-2}$

Consider the factor-modules $\mathcal{D}_\lambda^k(M)/\mathcal{D}_\lambda^{k-2}(M)$, where $k \geq 2$ and $\dim M \geq 2$.

**Theorem 4.2.** (i) All the Vect($M$)-modules $\mathcal{D}_\lambda^k(M)/\mathcal{D}_\lambda^{k-2}(M)$ with $\lambda \neq 1/2$, are isomorphic to each other and nonisomorphic to the direct sum of modules of tensor fields.

(ii) The module of differential operators on half-densities is exceptional:

$$\mathcal{D}_\lambda^{k/2}(M)/\mathcal{D}_\lambda^{k/2-2}(M) \cong S^k \oplus S^{k-1}.$$  

**Proof.** The action of Vect($M$) on the module $\mathcal{D}_\lambda^k(M)/\mathcal{D}_\lambda^{k-2}(M)$ is given by the two first terms $(\bar{a}_k^X, \bar{a}_{k-1}^X)$ of the formula (10). The structure of the Vect($M$)-module is, therefore, determined by bilinear map $\gamma_1^\lambda(X, \bar{a}_k)$. 

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Proposition 4.3. Bilinear map \( \gamma_1^\lambda(X, \bar{a}_k) \) is given by:

\[
\gamma_1^\lambda(X, \bar{a}_k) = (2\lambda - 1) \frac{k(k - 1)(n + 1)}{2(2k + n - 1)} \langle \bar{\gamma}(X), a_k \rangle
\]  

(14)

where \( \bar{\gamma} \) is the \( sl_{n+1} \)-equivariant 1-cocycle \([3]\) on \( \text{Vect}(R) \) and \( \langle , \rangle : \mathcal{S}_2^1 \otimes \mathcal{S}^k \rightarrow \mathcal{S}^{k-1} \) is the contraction of tensors:

\[
\langle T, a \rangle_{\lambda_1 \cdots \lambda_{n-1}} = \frac{1}{k-1} \sum_{s=1}^{k-1} T^{ij}_{\lambda \cdots \mu} a_{i_1 \cdots i_{n-1} j}. 
\]

**Proof.** First, remark that \( \gamma_1^\lambda(X, \bar{a}_k) \) is a second order differential operator. Since \( \gamma_1^\lambda \) vanishes on \( sl_{n+1} \), it is of second order in \( X \) (and of zero order in \( a \)):

\[
\gamma_1^\lambda(X, \bar{a}_k) = \alpha \sum_{s=1}^{k-1} \partial_i \partial_j (X^{i_s}) \bar{a}_{i_1 \cdots i_{k-1} i j} + \beta \partial_i \partial_j (X^j) \bar{a}_{i_1 \cdots i_{k-1} i},
\]

where \( \alpha, \beta \) are some constants.

Consider the operator \( A = (\sigma^\lambda)^{-1}(\bar{a}_k) = \bar{a}_{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k} + \cdots \). In the commutator \([L_X^\lambda, A] \) it is sufficient to only the terms of order zero in \( \bar{a}_k \) and of order \( \leq 2 \) in \( X \). One has:

\[
[L_X^\lambda, A] = -k \bar{a}_{i_1 \cdots i_{k-1}} \partial_i (X^{i_k}) \partial_{i_1} \cdots \partial_{i_k} - \left( \frac{k(k-1)}{2} \bar{a}_{i_1 \cdots i_{k-2} i j} \partial_i \partial_j (X^{i_k}) + \lambda k \bar{a}_{i_1 \cdots i_{k-3} i j} \partial_i \partial_j (X^j) \right) \partial_{i_1} \cdots \partial_{i_{k-1}} + \cdots.
\]

Now, to determine the constants \( \alpha, \beta \), let us apply the symbol map \( \sigma^\lambda \) and consider only the terms of second order in \( X \) and zero order in \( \bar{a}_k \). One easily gets:

\[
\alpha = -\frac{k}{2} - c_{k-1}^k
\]

and

\[
\beta = -\lambda k - c_{k-1}^k,
\]

where \( c_{k-1}^k \) is the first nontrivial constant in the formula \([5]\) of the symbol \( \sigma^\lambda \). Finally,

\[
\alpha = \frac{k(n + 1)}{2(2k + n - 1)} (2\lambda - 1) \quad \text{and} \quad \beta = -\frac{k(k - 1)}{2k + n - 1} (2\lambda - 1).
\]

The formula \([14]\) follows.

Let us prove Theorem 4.2, Part (i).

For every \( \lambda, \mu \neq 1/2 \), modules \( D^\lambda_\chi(M)/D^{k-2}_\chi(M) \) and \( D^\mu_\mu(M)/D^{k-2}_\mu(M) \) are isomorphic. The isomorphism is unique (up to a constant). The formula \([14]\) implies that this isomorphism \( \Lambda^\lambda \mapsto \Lambda^\mu \) is as follows:

\[
(\bar{a}_k^\mu, \bar{a}_{k-1}^\mu) = (\bar{a}_k^\lambda, \frac{2\mu - 1}{2\lambda - 1} \bar{a}_{k-1}^\lambda).
\]  

(15)
An important fact is that this isomorphism is well defined (does not depend on the choice of coordinates). Indeed, the locally defined map commutes with Vect(M)-action and with Diff(M)-action. Therefore, the formula (15) does not change under the coordinate transformations.

It follows from the uniqueness of the sl\(_{n+1}\)-equivariant symbol that \( D^k_\lambda(M)/D^{k-2}_\lambda(M) \not\cong \mathcal{S}^k \oplus \mathcal{S}^{k-1} \), if \( \lambda \neq 1/2 \). It is also a corollary of the cohomological interpretation of \( \gamma_1 \) (cf. Section 5.4).

Theorem 4.2, Part (i) is proven.

4.3 Exceptional case \( \lambda = 1/2 \)

In the case \( \lambda = 1/2 \), the term \( \gamma_1(X, \bar{a}_k) \) vanishes. The Vect(M)-action (10) in this case is just the standard action on \( \mathcal{S}^k \oplus \mathcal{S}^{k-1} \).

Theorem 4.2 is proven.

It follows from this theorem that for \( \lambda \neq 1/2 \), already the second term of a symbol of a differential operator is not intrinsically defined. However, in the exceptional case of differential operators on 1/2-densities, the two first terms of the sl\(_{n+1}\)-equivariant symbol have geometric sense. One obtains the following amazing remark:

**Corollary 4.4.** The sl\(_{n+1}\)-equivariant symbol defines an equivariant map:

\[
(\sigma^{1/2}_k, \sigma^{1/2}_{k-1}) : D^k \to \mathcal{S}^k \oplus \mathcal{S}^{k-1}.
\]

The geometrical reason is as follows. The Vect(M)-module \( D^k_{1/2} \) has a symmetry: the operator of conjugation. Every operator \( A \in D^k_{1/2} \) can be decomposed in a sum: \( A = A_0 + A_1 \), where \( A_0^* = (-1)^k A_0 \) and \( A_1^* = (-1)^{k-1} A_1 \). Then, \( \sigma^{1/2}_k \) is the principal symbol of \( A_0 \) and \( \sigma^{1/2}_{k-1} \) is the principal symbol of \( A_1 \).

4.4 Modules \( D^k_\lambda/D^l_\mu \) in multi-dimensional case

Consider the Vect(M)-modules \( D^k_\lambda(M)/D^l_\mu(M) \) with \( k - l \geq 3 \) in the multi-dimensional case (\( \text{dim} \ M \geq 2 \)). The following result shows that there is no nontrivial diffeomorphisms between the Vect(M)-modules in this case.

**Theorem 4.5.** (i) Modules \( D^k_\lambda(M)/D^l_\lambda(M) \) and \( D^k_\mu(M)/D^l_\mu(M) \), where \( k - l \geq 3 \) are isomorphic if and only if \( \lambda + \mu = 1 \).

(ii) There is no isomorphism between the modules \( D^k_\lambda(M)/D^l_\lambda(M) \) and the module of tensor fields \( \mathcal{S}^k \oplus \ldots \oplus \mathcal{S}^{l+1} \).

**Proof.** The isomorphism in Part (i) is given by the standard conjugation of differential operators. This map defines a general isomorphism

\[
* : D_\lambda \to D_{1-\lambda}
\]
(cf. [3],[13]).

To prove that there is no other isomorphisms, it is sufficient to consider the case \( k-l = 3 \).

Indeed, module \( D^k_X(M)/D^l_Y(M) \) with \( k-l \geq 3 \) projects on \( D^k_X(M)/D^{k-3}_X(M) \).

Suppose, that modules \( D^k_X(M)/D^{k-3}_X(M) \) and \( D^k_Y(M)/D^{k-3}_Y(M) \), are isomorphic. It follows from the formula (15), that in terms of \( sl_{n+1} \)-equivariant symbol, the isomorphism is unique (up to a constant) and given in local coordinates by:

\[
(\bar{a}_k^\mu, \bar{a}_{k-1}^\mu, \bar{a}_{k-2}^\mu) = (\tilde{a}_k^\lambda, \frac{2\mu - 1}{2\lambda - 1} \tilde{a}_{k-1}^\lambda, \frac{(2\mu - 1)^2}{(2\lambda - 1)^2} \tilde{a}_{k-2}^\lambda).
\]

Indeed, its restriction to the submodule \( D^{k-1}_X(M)/D^{k-3}_X(M) \) and the projection on the factor \( D^k_X(M)/D^{k-2}_X(M) \) must be isomorphism.

Now, it follows from the formula (15) that the last formula defines an isomorphism if and only if \( \gamma_2^\mu(X, \bar{a}_k) = \frac{(2\mu - 1)^2}{(2\lambda - 1)^2} \gamma_2^\lambda(X, \bar{a}_k) \). A direct calculation shows that if \( \lambda + \mu \neq 1 \), then this equality is not satisfied (cf. the formula (22) below).

Theorem 4.5 is proven.

Part (ii) of Theorem 4.5 confirms the fact (well-known “in practice”) that a symbol of a differential operator can not be defined in an intrinsic way (even in the case of differential operators on \( 1/2 \)-densities).

### 4.5 Modules of second order differential operators

Consider the modules of second order differential operators \( D^2_X \).

These modules have been classified in [3]. The result is: for every \( \lambda \neq 0, \frac{1}{2}, 1 \), all the \( \text{Vect}(M) \)-modules \( D^2_X(M) \) are isomorphic to each other, modules \( D^2_0(M) \cong D^2_1(M) \) and \( D^2_2(M) \) are particular.

For \( \lambda, \mu \neq 0, \frac{1}{2}, 1 \), there exists a unique (up to a constant) intertwining operator

\[
\mathcal{L}^2_{\lambda, \mu} : D^2_X(M) \to D^2_Y(M)
\]

(see [3]).

(a). Let us express \( \mathcal{L}^2_{\lambda, \mu} \) in terms of the \( sl_{n+1} \)-equivariant symbol (7). It follows from Corollary 2.6, that the map \( (\sigma^\mu)^{-1} \circ \mathcal{L}^2_{\lambda, \mu} \circ \sigma^\lambda \) is diagonal.

**Proposition 4.6.** The \( \text{Vect}(M) \)-module isomorphism \( \mathcal{L}^2_{\lambda, \mu}(A) \) is defined in terms of \( sl_{n+1} \)-equivariant symbol by:

\[
(\sigma^2_0(B), \sigma^4_1(B), \sigma^6_0(B)) = (\sigma^2_0(A), \sigma^4_1(A), \frac{\lambda(\lambda - 1)}{\mu(\mu - 1)} \sigma^4_0(A)),
\]

where \( B = \mathcal{L}^2_{\lambda, \mu}(A) \) and \( \sigma_i \) are the homogeneous components of \( \sigma \).

**Proof.** This formula follows from the explicit expression for the \( \text{Vect}(M) \)-action (10) in the case of second order operators. By straightforward calculations (cf. Section 5.5) one has:

\[
\begin{align*}
\bar{a}_2^X &= L_X(\bar{a}_2) \\
\bar{a}_1^X &= L_X(\bar{a}_1) + \gamma_1(X, \bar{a}_1) \\
\bar{a}_0^X &= L_X(\bar{a}_0) + \frac{\lambda(\lambda - 1)}{n+2} \gamma_2(X, \bar{a}_2)
\end{align*}
\]

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where
\[ \gamma_2(X, \bar{a}_2) = 2\partial_i \partial_j \partial_k (X^k) \bar{a}^{ij}_2 + 2\partial_j \partial_k (X^k) \partial_i (\bar{a}^{ij}_2) - (n + 1) \partial_i \partial_j (X^k) \partial_k (\bar{a}^{ij}_2) \] (16)
and \( \gamma_1(X, \bar{a}_1) \) is given by (14).

Proposition 4.6 is proven.

(b). Let us give an intrinsic expression for the map \( L_{\lambda,\mu}^2 \).

Every second order differential operator can be expressed (not in a unique way) as a linear combination of:
- zero-order operator \( \phi \mapsto f\phi \) (of multiplication by a function),
- first order operator \( L_\lambda^X \) (of Lie derivative),
- a symmetric expression \[ [L_\lambda^X, L_\lambda^Y]_+ = L^X_\lambda \circ L^Y_\lambda + L^Y_\lambda \circ L^X_\lambda, \]
where \( f \in C^\infty(M) \), \( X, Y, Z \in \text{Vect}(M) \).

Proposition 4.7. The isomorphism \( L_{\lambda,\mu}^2 \) is:
\[
L_{\lambda,\mu}^2([L_\lambda^X, L_\lambda^Y]_+) = [L_\mu^X, L_\mu^Y]_+ \\
L_{\lambda,\mu}^2(L^X_\lambda) = \frac{2\lambda - 1}{2\mu - 1} L^Y_{\mu} \\
L_{\lambda,\mu}^2(f) = \frac{\lambda(\lambda - 1)}{\mu(\mu - 1)} f
\]

Proof. The \( sl_{n+1} \)-equivariant symbol of the operator of Lie derivative is: \( \sigma^\lambda_{L_\lambda^X} = X^i \xi_i \).

In the same way, \( \sigma^\lambda_{[L_\lambda^X, L_\lambda^Y]_+} = \bar{a}^{ij}_2 \xi_i \xi_j + \bar{a}^i_1 \xi_i + \bar{a}_0 \), where
\[
\bar{a}^{ij}_2 = X^i Y^j + Y^i X^j \\
\bar{a}^i_1 = \frac{2\lambda - 1}{n + 3} \left( 2(X^i \partial_j (Y^j) + Y^i \partial_j (X^j)) - (n + 1)(X^j \partial_j (Y^i) + Y^j \partial_j (X^i)) \right) \\
\bar{a}_0 = -\frac{2\lambda(\lambda - 1)}{n + 2} \left( X^i \partial_i \partial_j (Y^j) + X^i \partial_i \partial_j (Y^j) + \partial_i (X^i) \partial_j (Y^j) \right) - (n + 1) \partial_i (X^i) \partial_j (Y^j)
\] (17)

The result follows.

The explicit formula for \( L_{\lambda,\mu}^2 \) in terms of coefficients of differential operators was obtained in \[ \text{[8]} \].

Remark. The expression for \( L_{\lambda,\mu}^2 \) in terms of Lie derivatives is intrinsic, but it is a nontrivial fact that it does not depend on the choice of \( X, Y \) and \( f \) representing the same differential operator. The expression for \( L_{\lambda,\mu}^2 \) in terms of symbols is well-defined locally, but it is a nontrivial fact that it is invariant with respect to coordinate changes. The two facts are corollaries of the third one: the two formulæ represent the same map.
5 Cohomology of Vect(R^n) with coefficients in operators on tensor fields

The relation between the Vect(M)-modules of differential operators \( D_\lambda \) and the cohomology of Vect(M) with coefficients in Hom(\( S^k, S^l \)) was noticed (in the particular case of second order differential operators) in [8]. The measure of the difference between Vect(M)-module \( D_k \lambda \) and the module of symmetric contravariant tensor fields \( S^0 \oplus \cdots \oplus S^k \) is represented by a class of the first cohomology group: \( H^1(\text{Vect}(M); \text{Hom}(S^k(M), S^l(M))) \).

In this section we calculate the first group of differentiable cohomology of Vect(R^n) vanishing on the subalgebra sl_{n+1}:

\[
H^1(\text{Vect}(R^n), sl_{n+1}; \text{Hom}_{diff}(S^k, S^l)),
\]

where Hom_{diff}(S^k, S^l)) is the space of differential operators from \( S^k \) to \( S^l \). This cohomology group is defined using the cochains on Vect(R^n) vanishing on sl_{n+1} (cf.[11]). We calculate explicitly the cocycles representing nontrivial cohomology classes and interpret them as the obstruction for existence of equivariant symbol map.

5.1 Modules of differential operators and cohomology

Each term \( \gamma^\lambda_i \) in [11] defines a linear map \( c_i^\lambda : \text{Vect}(R^n) \rightarrow \text{Hom}(S^k, S^{k-i}) \) by:

\[
c_i^\lambda(X) := \gamma^\lambda_i(X,.).
\]

All the maps \( c_i^\lambda \) are differentiable and vanish on the projective subalgebra sl_{n+1}.

The following two remarks follows from the fact that the formula [11] is a Vect(R^n)-action.

(a) Operator \( c_1^\lambda \) is a 1-cocycle:

\[
[L_X, c_1(Y)] - [L_Y, c_1(X)] - c_1([X,Y]) = 0,
\]

for every \( X,Y \in \text{Vect}(R^n) \).

(b) If \( \gamma_1^\lambda \equiv 0 \), that means in the case \( \lambda = \frac{1}{2} \), it is clear that the two maps \( c_{1/2}^2 \) and \( c_{1/2}^3 \) are 1-cocycles. We will show that \( c_{1/2}^3 \equiv 0 \).

Note, that for general values of \( \lambda \), operator \( c_2^\lambda \) satisfies the relation: \( dc_2 = [c_1, c_1]_+ \), where \([ , ]_+ \) is the Massey product.

Important remark. The cocycles \( c_1^\lambda \) and \( c_{1/2}^2 \) are nonzero and represent nontrivial classes of cohomology. This fact follows immediately from the uniqueness of sl_{n+1}-equivariant symbol (Theorem I). It follows also directly from the fact that these cocycles are sl_{n+1}-equivariant. Indeed, if \( c = db \), then \( b \) is sl_{n+1}-equivariant, but there is no sl_{n+1}-equivariant operators from \( S^k \) to \( S^l \).
5.2 Bilinear $sl_{n+1}$-equivariant operators

In this section we classify the bilinear $sl_{n+1}$-equivariant differential operators (11) vanishing on the projective subalgebra $sl_{n+1} \subset \text{Vect}(\mathbb{R}^n)$.

This means, we consider operations $\gamma_p$ satisfying the conditions (12) and (13) (In the contrast with the one-dimensional case, the vanishing condition does not follow from $sl_{n+1}$-equivariance.)

**Example 5.1: The transvectants.** Let us first recall the situation in the one-dimensional case ($M = S^1, \mathbb{R}$).

Bilinear $sl_2$-equivariant maps on tensor-densities on $\mathbb{R}$ or $S^1$, $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+m}$ were classified by Gordan [13]. For every $\lambda, \mu$ and $m = 0, 1, 2, \ldots$, there exists a remarkable operation called the transvectant of order $m$:

$$J_m(\phi, \psi) = \sum_{i+j=m} (-1)^i m! \binom{2\lambda + m - 1}{i} \binom{2\mu + m - 1}{j} \phi^{(i)} \psi^{(j)}$$

(18)

where $\phi^{(i)} = d^i \phi / dx^i$. The operation (18) is unique (up to a constant) for almost all values of $\lambda$ and $\mu$.

The transvectant $J_k$ with $k \geq 2$ vanishes on the projective Lie algebra $sl_2$.

Let us consider the multi-dimensional case ($n \geq 2$). The following answer is rather unexpected and completely different from the classification in the one-dimensional case.

**Theorem 5.2.** The space of $sl_{n+1}$-equivariant operations (17) vanishing on $sl_{n+1}$ is as follows:

(i) $k > p \geq 2$, there exist 2 independent operations.

(ii) $k = p \geq 2$, there exists a unique (up to a constant) operation.

(iii) $p = 1, k \geq 2$, there exists a unique (up to a constant) operation.

(iv) There is no such operations if $k = p = 1$.

**Proof.** Let us first consider only the equivariance and vanishing conditions with respect to the affine Lie algebra (the subalgebra of $sl_{n+1}$ generated by the constant and linear vector fields in (1)). A bilinear map $\gamma_p : \text{Vect}(\mathbb{R}^n) \otimes \mathcal{S}^k \rightarrow \mathcal{S}^{k-p}$, equivariant with respect to the affine subalgebra of $sl_{n+1}$, is given by a homogeneous differential operator of order $p+1$ (cf. Section 3.3). The condition, that $\gamma_p$ vanish on the affine subalgebra means that the expression $\gamma_p(X, a)$ does not contain terms of order $\leq 1$ in $X$.

Therefore, $\gamma_p$ in the form:

$$\gamma_p(X, a) = \sum_{u=1}^p \left( \sum_{s=1}^{k-p} \alpha_u \partial_{j_1} \cdots \partial_{j_{u+1}}(X^{i_s}) \partial_{j_{u+2}} \cdots \partial_{j_{p+1}}(a^{i_k \cdots i_{k-p} j_1 \cdots j_{p+1}}) + \beta_u \partial_{j_1} \cdots \partial_{j_u} \partial_k(X^k) \partial_{j_{u+1}} \cdots \partial_{j_{p}}(a^{i_1 \cdots i_{k-p} j_1 \cdots j_{p}}) \right) \partial_{i_1} \otimes \cdots \otimes \partial_{i_{k-p}}$$

(19)
where $\alpha_u, \beta_u, \delta_u$ are constants, $u = 1, \ldots, p$, and $\delta_1 = 0$.

In invariant terms (cf. Section 1.2) this formula reads as follows:

$$
\gamma_p(X, a) = \sum_{u=1}^{p} \left( \alpha_u \langle D^{u+1}(X), a^{(p-u-1)} \rangle + \beta_u \langle D^u(X'), a^{(p-u)} \rangle + \delta_u \langle D^u(X), D(a^{(p-u-1)}) \rangle \right)
$$

where $D$ is the differential acting on symmetric tensor fields (cf. Section 1.2), $X' = \text{div} X$ and $a^{(l)}$ is the divergence of order $l$ of $a$ (cf. Section 2.2).

Now, it is sufficient to impose the equivariance ad vanishing conditions with respect to the quadratic vector fields $X = x^i x^j \partial_j$ in $sl_{n+1}$.

**Lemma 5.3.** The equivariance condition: $L_X(\gamma_p(Y, a)) - \gamma_p(Y, L_X(a)) = \gamma_p([X,Y], a)$ is equivalent to the following recurrent system of linear equations:

$$
\begin{aligned}
-u(u+2)\alpha_{u+1} + (k-p)\delta_{u+1} + (p-u)(2k+n-p+u)\alpha_u &= 0 \\
-u(u+1)\beta_{u+1} + (u+1)\delta_{u+1} + (p-u)(2k+n-p+u)\beta_u &= 0 \\
-(u^2-1)\delta_{u+1} + (p-u)(2k+n-p+u)\delta_u &= 0 \\
(u+1)\delta_{u+1} + (u+1)\alpha_u + (k-p+u)\delta_u + (n+1)\beta_u &= 0
\end{aligned}
$$

(20)

where $u = 1, \ldots, p$ (and $\delta_1 = 0$).

Note, that the condition of equivariance with respect to $X$ implies: $\gamma_p(X, .) \equiv 0$. Indeed, the vanishing condition reads:

$$
2\alpha_1 + (n+1)\beta_1 + 2\delta_2 = 0,
$$

which coincides with the last equation of the system for $u = 1$.

**Proof of the lemma.** It can be obtained by a quite complicated straightforward calculation. Let us give here a proof based on the polynomial representation of differential operators (cf. Section 3).

Take (without loosing generality) $a = Y^k$, where $Y$ is a vector field. The polynomial corresponding to $\gamma_p(X,Y^k)$ is:

$$
P_{\gamma} = P_X P_Y^{k-(p+1)} \sum_{u=1}^{p} \alpha_u \langle Y, \xi \rangle^{u+1} \langle Y, \eta \rangle^{p-u} + P_Y^{k-p} \langle X, \xi \rangle \sum_{u=1}^{p} \beta_u \langle Y, \xi \rangle^{u} \langle Y, \eta \rangle^{p-u} + P_Y^{k-p} \langle X, \eta \rangle \sum_{u=1}^{p} \delta_u \langle Y, \xi \rangle^{u} \langle Y, \eta \rangle^{p-u}
$$
where the variables $\xi$ and $\eta$ correspond to derivatives of $X$ and $Y$ respectively.

Let us consider $v_1 = \langle X, \xi \rangle$, $w_1 = \langle Y, \xi \rangle$, $v_2 = \langle X, \eta \rangle$, $w_2 = \langle Y, \eta \rangle$, $x = P_X$ and $y = P_Y$ as independent variables. One can check that the equivariance condition leads to the following differential equations on $P_\gamma$:

\[
\left((w_2 + w_1)\partial_{w_1}v_2 + y\partial_{w_1x} + y\partial_{w_2} + (n + 1)\partial_{v_1}\right)P_\gamma = 0
\]

and

\[
(-w_1\partial_{w_1}^2 + v_2\partial_{w_1}v_2 + x\partial_{xw_1} + w_2\partial_{w_2}^2 + v_1\partial_{w_1}v_2 + x\partial_{w_2y} + 2w_1\partial_{w_1}w_2 + 2y\partial_{w_2y} + (n + 1)\partial_{w_2})P_\gamma = 0
\]

where $\partial$ means a partial derivative.

One readily obtains the system (20).

Lemma 5.3 is proven.

Proof of Theorem 5.2. (i) It is easy to see, that the system given by first three equations of (20) with the condition $2\alpha_1 + (n + 1)\beta_1 + 2\delta_2 = 0$, is of dimension 2: every solution is defined by $\alpha_1, \beta_1$. The last equation is a linear combination of the other ones. This proves Part (i) of the theorem.

(ii) In the case $k = p$, one has the following supplementary conditions: $\alpha_i = 0$, since in this case $\gamma_p(X, a)$ is a function. In this case the system has a unique (up to a multiple) solution defined by the value of $\beta_1$.

Remark, that in the particular case $k = p = 2$, the operation (16) is a solution of the considered problem.

(iii) For $p = 1, k \geq 2$ every solution is proportional to the operation (14).

(iv) For $k = p = 1$, the system (20) has no solutions.

Theorem 5.2 is proven.

5.3 Calculation of the first cohomology vanishing on $sl_{n+1}$, multi-dimensional analogue of the Gelfand-Fuchs cocycle

The following theorem is the second main result of this paper.

Theorem II. In the multi-dimensional case ($n \geq 2$),

\[
H^1(Vect(\mathbb{R}^n), sl_{n+1}; \text{Hom}_{diff}(S^k, S^l)) = \begin{cases} 
\mathbb{R}, & k - l = 1, l \neq 0 \\
\mathbb{R}, & k - l = 2 \\
0, & \text{otherwise}
\end{cases}
\]

Remark 5.4. Theorem II essentially reduces the problem of calculation of the cohomology group $H^1(Vect(\mathbb{R}^n); \text{Hom}_{diff}(S^k, S^l))$ to calculation of the cohomology group $H^1(sl_{n+1}; \text{Hom}_{diff}(S^k, S^l))$. 
**Proof of the theorem.** A 1-cocycle on $\text{Vect}(\mathbb{R}^n)$ vanishing on a subalgebra of $\text{Vect}(\mathbb{R}^n)$, is necessarily equivariant with respect to this subalgebra (cf. Section 1.2). Let us use Theorem 5.2. We will calculate explicitly all the $\mathfrak{sl}_{n+1}$-equivariant cocycles on $\text{Vect}(\mathbb{R}^n)$ with values in $\text{Hom}_{\text{diff}}(S^k, S^{k-1})$. The condition of $\mathfrak{sl}_{n+1}$-equivariance guarantees, that these cocycles are nontrivial (cf. Section 5.1).

(a). If $k - l = p = 1$ and $k \geq 2$, there exists a unique $\mathfrak{sl}_{n+1}$-equivariant operation $\gamma_1$ proportional to (14). The corresponding linear map $c_1$ is a nontrivial 1-cocycle.

(b). If $k = p = 1$ there is no $\mathfrak{sl}_{n+1}$-equivariant operations.

(ii). Consider the case $p = 2$. There exists a two-dimensional space of $\mathfrak{sl}_{n+1}$ operations $c_2 : \text{Vect}(\mathbb{R}^n) \to \text{Hom}_{\text{diff}}(S^k, S^{k-2})$ vanishing on $\mathfrak{sl}_{n+1}$.

**Proposition 5.5.** There exists a unique (up to a constant) cocycle $c_2$ on $\text{Vect}(\mathbb{R}^n)$ with values in $\text{Hom}_{\text{diff}}(S^k, S^{k-2})$ vanishing on $\mathfrak{sl}_{n+1}$. It is defined by the bilinear map $\gamma_2$ given by the formula (19) with the coefficients:

$$
\begin{align*}
\alpha_1 &= k - 2 \\
\alpha_2 &= \frac{1}{6}(k - 2)(2k + n + 1) \\
\beta_1 &= 1 \\
\beta_2 &= 1 \\
\delta_2 &= -\frac{1}{2}(2k + n - 3)
\end{align*}
$$

Cocycle $c_2$ is nontrivial.

**Proof.** It follows from Theorem 5.2, that the space of $\mathfrak{sl}_{n+1}$-equivariant operations (14) vanishing on $\mathfrak{sl}_{n+1}$ is of dimension 2 for $p = 2, k > p$. The cocycle condition reads:

$$
L_X(\gamma_2(Y, a)) + \gamma_2(Y, L_X(a)) - L_Y(\gamma_2(X, a)) - \gamma_2(X, L_Y(a)) = \gamma_2([X, Y], a).
$$

This relation adds one more equation: $\beta_1 = \beta_2$ to the general system (20). To obtain this equation, it is sufficient to collect the terms with $\partial_{i_1}\partial_{i_2}(X^j)\partial_{j_m}(Y^m)\alpha_{i_1i_2...i_k}$. The unique (up to a constant) solution of the completed system of linear equations is given by the formula (21).

The corresponding map $c_2$ is indeed a 1-cocycle. One can verify this fact by a direct calculation.

Proposition 5.5 is proven.

**Remark 5.6.** The cocycle $c_2$ is a multi-dimensional analogue of the Gelfand-Fuchs cocycle (13) in the following sense. In the one-dimensional case ($n = 1$), the corresponding cocycle on $\text{Vect}(\mathbb{R})$ is given as multiplication by the Gelfand-Fuchs cocycle.

We will show in Section 5.4, that the operation $\gamma_2$ defined by (21) is proportional to $\gamma_1^{1/2}$.

3. Consider the case $p \geq 3$. It is easy to see that there is no solutions of the system (20) satisfying the cocycle condition. Indeed, collecting the terms with $\partial_{i_1}\partial_{i_2}(X^j)\partial_{j_m}(Y^m)\partial_{i_3}...\partial_{i_p}(a^{i_1i_2...i_k})$ in the cocycle relation, one has $\beta_1 = \beta_2$. Collecting the terms with
\[ \partial_{i_1} \cdots \partial_{i_p}(X^j) \partial_j(\partial_m(Y^m)) a^{i_1 \cdots i_k}, \] 

one has \( \beta_1 = \beta_p \). The system (21) together with the two new equations has no solutions.

Theorem II is proven.

**Corollary 5.7.** All the bilinear maps \( \gamma_{2p+1}^{1/2} \) vanish.

Indeed, since \( \gamma_1^{1/2} = 0 \), the first nonzero map \( \gamma_{2p+1}^{1/2} \), where \( p \geq 1 \), defines a Vect(\( \mathbb{R}^n \))-cocycle, which is nontrivial because of sl\(_{n+1}\)-equivariance.

### 5.4 Calculation of \( \gamma_2^\lambda \)

Let us now give the explicit formula for the bilinear map \( \gamma_2^\lambda(X, \bar{a}_k) \) from the formula (10) of the Vect(\( \mathbb{R}^n \))-action on linear differential operators.

**Proposition 5.8.** The operation

\[ \gamma_2^\lambda(X, \bar{a}_k) = \frac{k(k-1)}{2(2k+n-2)} \tilde{\gamma}_2^\lambda(X, \bar{a}_k), \]

where \( \tilde{\gamma}_2^\lambda(X, \bar{a}_k) \) is the operation (19) with the coefficients:

\[
\begin{align*}
\alpha_1 &= -\frac{(k-2) \left(2(n+1)^2 \lambda(\lambda-1) + 2k^2 + 2kn - 4k + n^2 - n^2 + 2 \right)}{2k+n-1} \\
\alpha_2 &= -(k-2) \left( n+1 \right) \lambda(\lambda-1) + \frac{1}{3} (k^2 + kn + n^2 - n + k + n) \\
\beta_1 &= \frac{(4k+n-5)(n+1)\lambda(\lambda-1) + (k-2)(k-1)}{2k+n-1} \\
\beta_2 &= (4k-6)(n+1)\lambda(\lambda-1) + (k-2)n \\
\delta_2 &= -(n+1)^2 \lambda(\lambda-1) - (k-2)(k+n+1)
\end{align*}
\]

**Proof.** Proof is analogous to those of Proposition 4.3.

Indeed, the operator \( A = (\sigma^\lambda)^{-1}(\bar{a}_k) \) is as follows:

\[ A = \bar{a}_k^{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k} + \tilde{c}_{k-1} \partial_{i_1} \bar{a}_k^{i_1 \cdots i_{k-1}} \partial_{i_1} \cdots \partial_{i_{k-1}} + \cdots, \]

where \( \tilde{c}_{k-1} \) is the first coefficient in the quantization map (8):

\[ \tilde{c}_{k-1} = \frac{k((n+1)\lambda + k-1)}{2k+n-1} = -c_{k-1}^k. \]
Then,
\[
[L_X, A] = (X^i \partial_j (\bar{a}^{i_1 \ldots i_k}) - k \bar{a}^{i_1 \ldots i_{k-1} i} \partial_i (X^{i_k})) \partial_{i_1} \ldots \partial_{i_k}
\]
\[
- \left( \frac{k(k-1)}{2} \bar{a}^{i_1 \ldots i_{k-2} ij} \right) \partial_i \partial_j (X^{i_k}) + \lambda k \bar{a}^{i_1 \ldots i_{k-1} i} \partial_i \partial_j (X^j)
\]
\[
+ c_{k-1}^j \partial_j (\bar{a}^{i_1 \ldots i_{k-2} ij}) \partial_j (X^{i_{k-1}}) \partial_{i_1} \ldots \partial_{i_{k-1}}
\]
\[
- \left( \frac{k(k-1)(k-2)}{6} \bar{a}^{i_1 \ldots i_{k-3} ij l} \right) \partial_i \partial_j \partial_l (X^{i_{k-2}}) + \frac{(k-1)(k-2)}{2} \lambda \bar{a}^{i_1 \ldots i_{k-2} ij} \partial_i \partial_j \partial_l (X^l)
\]
\[
+ \frac{(k-1)(k-2)}{2} c_{k-1}^j \partial_l (\bar{a}^{i_1 \ldots i_{k-3} ij l}) \partial_j \partial_l (X^{i_{k-2}})
\]
\[
+(k-1)c_{k-1}^j \partial_l (\bar{a}^{i_1 \ldots i_{k-2} ij}) \partial_j \partial_l (X^l) \partial_{i_1} \ldots \partial_{i_{k-2}} + \ldots
\]

where \ldots means the terms of order \( \geq 2 \) in \( \bar{a}_k \), or \( \geq 4 \) in \( X \).

Now, one has to apply the symbol map \( \sigma^a \) and collect third order terms to get:

\[
\alpha_1 = -\left( \frac{k(k-2)}{2} c_{k-2}^{\frac{1}{2}} + 2(k-2) c_{k-2}^k + \frac{(k-1)(k-2)}{2} c_{k-1}^k + (k-2) c_{k-1}^k c_{k-2}^{\frac{1}{2}} \right)
\]
\[
\alpha_2 = -\left( \frac{k(k-1)(k-2)}{6} + \frac{k}{2} c_{k-2}^{k-1} + 2 c_{k-2}^k \right)
\]
\[
\beta_1 = -\left( 2 c_{k-2}^k + k \lambda c_{k-2}^{k-1} + (k-1) \lambda c_{k-1}^k + c_{k-1}^k c_{k-2}^{k-1} \right)
\]
\[
\beta_2 = -\left( \frac{k(k-2)}{2} \lambda + k c_{k-2}^{k-1} \lambda + \frac{k}{2} c_{k-2}^{k-1} + 2 c_{k-2}^k \right)
\]
\[
\delta_2 = -\left( c_{k-2}^k + \frac{k}{2} c_{k-2}^{k-1} \right)
\]

where \( c_{k-2}^k \) is the second coefficient in the formula (7).

Substituting the explicit expressions for \( c_{k-1}^k, c_{k-2}^k, c_{k-2}^{k-1} \) and \( c_{k-2}^k \), one obtains the formula (23).

Proposition 5.8 is proven.

**Remark 5.9.** Operation \( \bar{\gamma}_2^{1/2} \) coincides with \( \gamma_2 \) from Proposition 5.4.
6 Modules of (pseudo)differential operators on $S^1$

Consider the space of pseudodifferential operators on the circle $S^1$:

$$A = \sum_{i=0}^{\infty} a_{k-i} \left( \frac{d}{dx} \right)^{k-i}$$

(23)

where $a_{k-i} \in C^\infty(S^1)$, $k \in \mathbb{R}$.

Group $\text{Diff}(S^1)$ and Lie algebra $\text{Vect}(S^1)$ act on the space of pseudodifferential operators in the same was as on the space of differential operators. Denote $\Psi^\text{D}_k$ the $\text{Diff}(S^1)$- and $\text{Vect}(S^1)$- modules defined by the formulæ on the space of operators (23) (5) and (6).

We will study $\text{Diff}(S^1)$-modules $\Psi^\text{D}_k/\Psi^\text{D}_{k-l}$. In the particular case: $k \in \mathbb{Z}_+, l = k + 1$, this module is just the module of differential operators on $S^1$.

It follows from the uniqueness of transvectants (cf. Section 3.1) that in terms of $sl_2$-equivariant symbol, the action of a vector field on a (pseudo)differential operator is written via the transvectants (18):

$$\bar{a}_k^X = L_k^X(\bar{a}_k)$$
$$\bar{a}_{k_1}^X = L_{k-1}^X(\bar{a}_{k_1})$$
$$\bar{a}_{k-2}^X = L_{k-2}^X(\bar{a}_{k-2}) + t_k^2(\lambda) J_3(X, \bar{a}_k)$$
$$\bar{a}_{k-3}^X = L_{k-3}^X(\bar{a}_{k-3}) + t_{k-1}^2(\lambda) J_3(X, \bar{a}_{k-1}) + t_k^3(\lambda) J_4(X, \bar{a}_k)$$
$$\ldots$$
$$\bar{a}_s^X = L_s^X(\bar{a}_s) + \sum_{i=s+2}^{k} ti^{i-s}(\lambda) J_{i-s+1}(X, \bar{a}_i)$$

(24)

where $ti^{i-s}(\lambda)$ are some polynomials.

6.1 Cohomology of $\text{Vect}(\mathbb{R})$ vanishing on $sl_2$

In the one-dimensional case, symmetric tensor fields are just tensor-densities: $\mathcal{S}^k \cong \mathcal{F}_{-k}$. The result below follows from the classification of $sl_2$-equivariant bilinear maps on tensor-densities.

Proposition 6.1. $H^1(\text{Vect}(\mathbb{R}), sl_2; \text{Hom}_{\text{diff}}(\mathcal{F}_{-k}, \mathcal{F}_{-l})) = \begin{cases} \mathbb{R}, & k - l = 2 \\ \mathbb{R}, & k - l = 3 \\ 0, & \text{otherwise} \end{cases}$

Proof. In the one-dimensional case, all the operations (14) are proportional to the transvectants (18). One has: $J_1(X, a) = L_1^X(a)$ and $J_2(X, a) = X''a$. The corresponding linear map $c_2 : \text{Vect}(S^1) \to \text{Hom}(\mathcal{F}_\lambda, \mathcal{F}_{\lambda})$ is a coboundary: $c_2 = db$, where $b \in \text{Hom}(\mathcal{F}_{-k}, \mathcal{F}_{-k+1})$, $b(a) = a'$.

1the results of this section hold also in the complex case, e.g. when $M$ is the upper half-plane.
The transvectants \( J_3(X,a) = X'''a \) (multiplication by the Gelfand-Fuchs cocycle (3)) and
\[ J_4(X,a) = sX^{(IV)}a + 2X'''a' \]
correspond to nontrivial cocycles \( c_3 \) and \( c_4 \).

For \( J_p \) with \( p \geq 5 \), the corresponding linear maps are no more cocycles.

### 6.2 Classification of \( \text{Diff}(S^1) \)-modules \( \Psi D^k_\lambda/\Psi D^{k-l}_\lambda \)

The classification of \( \text{Diff}(S^1) \)-modules \( \Psi D^k_\lambda/\Psi D^{k-l}_\lambda \) follows from the formula (24). As in the multi-dimensional case, zeroes of the polynomials \( t^j_k(\lambda) \) corresponds to exceptional modules.

We will formulate the result for general values of \( k \).

**Proposition 6.2.** There exists an isomorphism of \( \text{Diff}(S^1) \)-modules
\[
\Psi D^k_\lambda/\Psi D^{k-l}_\lambda \cong \Psi D^k_\mu/\Psi D^{k-l}_\mu
\]
where \( k \neq 0, 1/2, 1, 3/2, \ldots \) in the following cases:

1. \( l \geq 2 \);
2. \( l = 3 \), if \( t^2_k(\lambda), t^2_k(\mu) \neq 0 \);
3. \( l = 4 \), if \( \lambda, \mu \) are not the roots of polynomials \( t^2_k, t^2_{k-1}, t^3_k \);
4. \( l \geq 4 \), if and only if \( \lambda + \mu = 1 \).

The proof is analogous to the proof of Theorems 4.2 and 4.5. We will give the explicit formulæ for the polynomials \( t^2_k, t^2_{k-1} \) and \( t^3_k \) in Section 6.3.

**Remark: the duality.** There exists a nondegenerate natural pairing between spaces \( \Psi D^k/\Psi D^l \) and \( \Psi D^{l-2}/\Psi D^{k-2} \). It is given by so-called Adler trace \([1]\): if \( A \in \Psi D^k \), where \( k \in \mathbb{Z} \), then
\[
\text{tr}(A) = \int_{S^1} a_1(x) \, dx.
\]

Let now \( A \in \Psi D^k/\Psi DO^l \) and \( B \in \Psi D^{l-2}/\Psi D^{k-2} \). Put
\[
(A, B) := \text{tr}(\tilde{A}\tilde{B}),
\]
where \( \tilde{A} \in \Psi D^k, \tilde{B} \in \Psi D^{l-2} \) are arbitrary liftings of \( A \) and \( B \).

Adler’s trace is equivariant with respect to the action (3).

This means that the pairing ( , ) is well-defined on \( \text{Vect}(S^1) \)-modules. Indeed, \( ([L^*_\lambda, A], B) + (A, [L^*_\lambda, B]) = 0 \) for every \( X \in \text{Vect}(S^1) \) (see [3] for the details and interesting properties of the transvectants).
6.3 Relation to the Bernoulli polynomials

The polynomials $t^j_k(\lambda)$ are particular cases of the coefficients in the $SL_2$-equivariant $\star$-product considered in [3]. We will not give the explicit formula here (see the formula (4.3) of [3]).

Let us give here first examples of polynomials $t^j_k(\lambda)$.

\[
t^2_k(\lambda) = \frac{k(k-1)}{2k-1}\left(\lambda^2 - \lambda - \frac{(k+1)(k-2)}{12}\right)
\]
\[
t^3_k(\lambda) = \frac{k}{6}\lambda(2\lambda - 1)(\lambda - 1)
\]

Already these examples evoke an idea about the relation of polynomials $t^j_k(\lambda)$ to the well-known Bernoulli polynomials. Indeed,

\[
t^2_k(\lambda) = \frac{k(k-1)}{2k-1}\left(B_2(\lambda) - \frac{k(k-1)}{12}\right), \quad t^3_k(\lambda) = \frac{k}{12}B_3(\lambda)
\]

where $B_s$ is the Bernoulli polynomial of degree $s$, e.g.:

- $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$,
- $B_3(x) = x^3 - 3x^2/2 + x/2$, $B_4(x) = x^4 - 2x^3 + x^2 - 1/30$,
- $B_5(x) = x^5 - 5x^4/2 + 5x^3/3 - x/6$.

The next examples are:

\[
t^4_k(\lambda) = \frac{k(k-1)(k-2)}{2(2k-3)(2k-5)}\left(B_4(\lambda) + \frac{2k^2 - 6k + 3}{24}B_2(\lambda)\right)
\]
\[
- \frac{3k^4 + 18k^3 - 35k^2 + 8k + 2}{480}
\]
\[
t^5_k(\lambda) = \frac{k(k-1)}{15(2k-7)}\left(B_5(\lambda) + \frac{5(k-1)(k-3)}{24}B_3(\lambda)\right)
\]

**Proposition 6.3.** Polynomials $t^{2j}_k(\lambda)$ are combinations of $B_{2s}$ with $s = 0, 1, \ldots, j$ and polynomials $t^{2j+1}_k(\lambda)$ are combinations of $B_{2s+1}$ with $s = 0, 1, \ldots, j$.

**Proof.** This statement is a simple corollary of the isomorphism $\mathcal{D}_\lambda \cong \mathcal{D}_{1-\lambda}$. Indeed, with respect to the involution $\lambda' = 1/2 - \lambda$, the Bernoulli polynomials verify the condition: $B_s(\lambda') = (-1)^sB_s(\lambda)$.

7 Some generalizations and applications of the $sl_{n+1}$-equivariant symbol

Equivariance of the symbol map $\sigma^{\lambda}$ and the inverse quantization map with respect to the action of $sl_{n+1}$ leads to their natural generalization. The formulae (7) and (8) are invariant
under the linear-fractional coordinate changes. Therefore, these formulae are well-defined globally on the cotangent bundle of a locally projective manifold.

Let us recall here the definition.

7.1 Projective structures

A projective structure on a manifold $M$ is defined by a projective atlas: an atlas with linear-fractional coordinate changes.

More precisely, a covering $(U_i)$ with a family of local diffeomorphisms $\phi_i : U_i \to \mathbb{P}^n$ is called a projective atlas if the local transformations $\phi_j \circ \phi_i^{-1} : \mathbb{P}^n \to \mathbb{P}^n$ are projective (i.e., are given by the action of the group $PGL_{n+1}$ on $\mathbb{P}^n$).

Examples of locally projective manifolds are: $\mathbb{R}^n, S^n, T^n, S^l \times T^m$ etc.

A projective structure defines locally on $M$ an action of the Lie group $SL_{n+1}$ by linear-fractional transformations and a (locally defined) action of the Lie algebra $sl_{n+1}$ generated by vector fields $[\mathfrak{L}]$, for every system of local coordinates of a projective atlas. This action is stable with respect to linear-fractional transformations (the space of vector fields $[\mathfrak{L}]$ is well-defined globally on $\mathbb{P}^n$).

Remarks 7.1. (a) In the case of simply connected manifold $M$ endowed with a projective structure, the local action of Lie algebra $sl_{n+1}$ is defined globally on $M$.

(b) All projective structures on a simply connected manifold are diffeomorphic to each other.

(c) Any surface admits a projective structure, the problem of existence of projective structures for 3-dimensional manifolds is open.

One has the following simple corollary of $sl_{n+1}$-equivariance of the symbol $\sigma^\lambda$.

Corollary 7.2. Given a manifold $M$ endowed with a projective structure, the symbol map $\sigma^\lambda$, given in arbitrary projective atlas by the formula (7), is well defined globally on $M$.

7.2 $SL_{n+1}$-equivariant star-products on $T^*M$

Let us show that for every $\lambda$, the $sl_{n+1}$-equivariant quantization map $[\mathfrak{L}]$ defines a star-product on $T^*M$. One obtains, therefore, a 1-parameter family of $sl_{n+1}$-equivariant star-products. All of them are equivalent to each other.

Given a quantization map $\sigma^{-1} : \text{Pol}(T^*M) \to \mathcal{D}(M)$, let us introduce a new parameter $\hbar$. For a homogeneous polynomial $P$ of degree $k$ put:

$$Q_\hbar(P) = \hbar^k \sigma^{-1}(P).$$
Define a new associative but non-commutative operation of multiplication on \( \text{Pol}(T^*M) \):

\[
F \ast_h G := Q_h^{-1}(Q_h(F) \cdot Q_h(G)).
\] (25)

The corresponding algebra is isomorphic to the associative algebra of differential operators on \( M \).

The result of the operation (25) is a formal series in \( \hbar \). It has the following form:

\[
F \ast_h G = FG + \sum_{k \geq 1} \hbar^k C_k(F, G),
\]

where the higher order terms \( C_k(F, G) \) are some differential operators.

Recall, that such an operation of is called a *star-products* if \( C_1(F, G) \) coincides with the standard Poisson bracket on \( \text{Pol}(T^*M) \), modulo symmetric in \( F \) and \( G \) terms:

\[
C_1(F, G) = \{ F, G \} + \text{terms symmetric in } (F, G)
\]

An elementary calculation shows that the associative operation corresponding to the quantization map (8) satisfies this property.

### 7.3 \( SL_{n+1} \)-equivariant quantization of geodesic flow on \( T^*S^n \)

Consider a nondegenerate quadratic form \( H = g^{ij} \xi_i \xi_j \) on \( T^*S^n \). We will apply the \( sl_{n+1} \)-equivariant quantization map (8) in the special case of \( \lambda = 1/2 \). This can be considered as a version of quantization of the geodesic flow for the corresponding metric \( g = H^{-1} = g^{ij} dx^i dx^j \). Note, that different approaches to this problem have already been considered (see \([7],[19],[22]\)).

The quantization map (8) associates to \( H \) a symmetric second order differential operator on \( \mathcal{F}_{1/2} \). In coordinates of projective structure this operator is given by the formula:

\[
A_H = g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \partial_j (g^{ij}) \frac{\partial}{\partial x^i} + \frac{(n + 1)}{4(n + 2)} \partial_i \partial_j (g^{ij})
\]

It follows from the symmetricity, that \( A_H \) is a Laplace–Beltrami operator. This means, \( A_H = \Delta + \Phi \), where \( \Delta \) is the Laplace operator corresponding to the metric \( g \) and \( \Phi \) is a function.

Let us give the explicit formula for the potential \( \Phi \) in the case when the coordinates of the projective structure are normal coordinates for the metric \( g \).

Recall, that for every point, there exist so-called normal coordinates in some neighborhood, such that

\[
\Gamma^k_{ij} = 0, \quad \text{and} \quad \partial_i(\Gamma^k_{ij}) = \frac{1}{3} (R^k_{li,j} + R^k_{lj,i})
\]
The condition that projective coordinates are normal coordinates for a metric is a sort of “compatibility condition” for the projective structure and metric. An example of such situation is the standard metric and the standard projective structure on $S^n$.

**Proposition 7.3.** In normal coordinates,

$$A_H = \Delta - \frac{(n+1)}{12(n+2)} R$$

where $R$ is the scalar curvature.

**Proof.** In normal coordinates, $\Delta = g^{ij} \partial_i \partial_j$ (see [7]). The formula (26) is a corollary of the expression: $\partial_i \partial_j (g^{ij}) = -\frac{1}{3} R$ (which can be verified by simple calculations).

**Remark 7.4.** Different methods of quantization leads to the formula $A_H = \Delta + cR$ with various values of the constant $c$. The formula (26) gives a new value of $c$ different from those of [7], [19], [22].

### 8 Discussion: two questions

(a) The definition of the $sl_{n+1}$-equivariant symbol is purely geometric. It would be very interesting to understand its relations with the algebraic structure of the space of differential operators.

The first simple observation in this direction is a remarkable (and a-priori unexpected) coincidence between the coefficients in the two formulæ for the intertwining operator $L^2_{\lambda,\mu}$ (cf. Propositions 4.6 and 4.7).

The second observation is the formula (17) for the $sl_{n+1}$-equivariant symbol of the anticommutator of two Lie derivatives: $[L^\lambda_X, L^\mu_Y]_+ = L^\lambda_X \circ L^\mu_Y + L^\mu_Y \circ L^\lambda_X$. In the right hand side of the formula (17) one obtains two interesting operations in $X$ and $Y$.

The precise problem is as follows. Calculate the $sl_{n+1}$-equivariant symbol of a symmetrized product of $k$ Lie derivatives:

$$[L^\lambda_{X_1}, \ldots, L^\lambda_{X_k}]_+ := Sym_{1,\ldots,k}(L^\lambda_{X_1} \circ \ldots \circ L^\lambda_{X_k})$$

(b) The second question concerns the cocycle $c_2$ on $\text{Vect}(\mathbb{R}^n)$ (cf Proposition 5.4). Cocycle $c_2$ is related with the module of differential operators on half-densities $D^{1/2}_{i/2}$ (cf. Remark 5.7). Note, that $c_2$ is well defined globally on $\text{Vect}(\mathbb{R}^n)$ and $\text{Vect}(S^n)$ (cf. Section 7.1).

Cocycle $c_2$ is a multi-dimensional analogue of the Gelfand-Fuchs cocycle in the form (8). Indeed, in the one-dimensional case it is proportional to $c_2(X)(a) = X^m a$.

The formula (21) is a result of calculations. It would be interesting to study the algebraic and geometric properties of cocycle $c_2$. Namely, one can specify it for the Lie algebras of Hamiltonian, contact or unimodular vector fields.
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