THE CALABI-YAU EQUATION ON THE KODAIRA-THURSTON
MANIFOLD, VIEWED AS A $S^1$-BUNDLE OVER A 3-TORUS

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Abstract. We prove that the Calabi-Yau equation on the Kodaira-Thurston
manifold has a unique solution for every $S^1$-invariant initial datum.

1. Introduction

The celebrated Calabi-Yau theorem affirms that given a compact Kähler man-
ifold $(M, \Omega, J)$ with first Chern class $c_1(M)$, every $(1, 1)$-form $\tilde{\rho} \in 2\pi c_1(M)$ is the
Ricci form of a unique Kähler metric whose Kähler form belongs to the cohomo-
logy class $[\Omega]$. This theorem was conjectured by Calabi in [4] and subse-
quently proved by Yau in [15]. The Calabi-Yau theorem can be alternatively reformulated in terms
of symplectic geometry by saying that, given a compact Kähler manifold $(M^n, \Omega, J)$ and
a volume form $\sigma$ satisfying the normalizing condition
\[
\int_M \sigma = \int_M \Omega^n,
\]
then there exists a unique Kähler form $\tilde{\Omega}$ on $(M, J)$ solving
\begin{align*}
\tilde{\Omega}^n &= \sigma, \\
[\tilde{\Omega}] &= [\Omega].
\end{align*}
Equation (1) still makes sense in the almost-Kähler case, when $J$ is merely an
almost-complex structure. In this more general context (1) is usually called the
Calabi-Yau equation.

In [5] Donaldson described a project about compact symplectic 4-manifolds in-
volving the Calabi-Yau equation and showed the uniqueness of the solutions. Don-
alson’s project is principally based on a conjecture stated in [5] whose confirm-
ation would lead to new fundamental results in symplectic geometry. Donaldson’s project
was partially confirmed by Taubes in [9] and strongly motivates the study of the
Calabi-Yau equation on non-Kähler 4-manifolds.

In [10] Weinkove proved that the Calabi-Yau equation can be solved if the torsion
of $J$ is sufficiently small and in [13] Tosatti, Weinkove and Yau proved the Don-
alson conjecture assuming an extra condition on the curvature and the torsion of
the almost-Kähler metric. Furthermore, Tosatti and Weinkove studied in [12] the
Calabi-Yau equation on the Kodaira-Thurston manifold assuming the initial datum $\sigma$
invariant under the action of a 2-dimensional torus $T^2$. The Kodaira-Thurston
is historically the first example of symplectic manifold without Kähler structures
(see[11, 1]) and it is defined as the direct product of a compact quotient of the
3-dimensional Heisenberg group by a lattice with the circle $S^1$. In [6] it is proved
that when $\sigma$ is $T^2$-invariant, the Calabi-Yau equation on the Kodaira-Thurston
manifold can be reduced to a Monge-Ampère equation on a torus which has always

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solution. Moreover in [3] the same equation is studied in every $T^2$-fibration over a 2-torus.

The *Kodaira-Thurston manifold* is defined as the compact 4-manifold $M = \text{Nil}^3/\Gamma \times S^1$, where $\text{Nil}^3$ is the 3-dimensional real Heisenberg group

$$\text{Nil}^3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

and $\Gamma$ is the lattice in $\text{Nil}^3$ of matrices having integer entries.

Therefore $M$ is parallelizable and has the global left-invariant co-frame

$$e^1 = dy, \quad e^2 = dx, \quad e^3 = dt, \quad e^4 = dz - xdy$$

inducing the standard metric $g = \sum(e^k \otimes e^k)$ and the triple of non-degenerate 2-forms

$$\Omega_1 = e^1 \wedge e^2 + e^3 \wedge e^4, \quad \Omega_2 = e^1 \wedge e^3 + e^4 \wedge e^2, \quad \Omega_3 = e^1 \wedge e^4 + e^2 \wedge e^3.$$ 

Every pair $(\Omega_k, g)$ specifies an almost complex structure $J_k$ making $(g, J_k)$ an almost Hermitian structure. It turns out that $J_1$ is integrable, whereas $(\Omega_2, J_2), (\Omega_3, J_3)$ are almost-Kähler, since $\Omega_2$ and $\Omega_3$ are both symplectic. This fact actually implies that $\Omega_2 + i\Omega_3$ is a *holomorphic-symplectic* structure on $(M, J_1)$.

Moreover, $M$ can be viewed as the total space of an $S^1$-bundle over the 3-dimensional torus $T^3 = T_{xy}^2 \times S^1_1$ therefore it is rather natural to extend the analysis in [12] [6] when $\sigma$ is $S^1$-invariant instead of $T^2$-invariant. We show that the Calabi-Yau equation on the Kodaira-Thurston manifold has a unique solution assuming the initial datum $\sigma$ invariant under the action of the fiber $S^1$. The result holds for the canonical almost-Kähler structures defined by the symplectic forms $\Omega_2$ and $\Omega_3$ and more in general for a family of almost-Kähler structures $(\Omega, J)$ such that $e^4$ is orthogonal to $e^1, e^2, e^3$ with respect to the Riemannian metric induced by $(\Omega, J)$.

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### 2. THE MAIN RESULTS

Since $\text{Nil}^3/\Gamma \times S^1 = (\text{Nil}^3 \times \mathbb{R})/(\Gamma \times \mathbb{Z})$, the Kodaira-Thurston manifold $M$ is a 2-step nilmanifold and every left-invariant almost-Kähler structure on $\text{Nil}^3 \times \mathbb{R}$ projects to an almost-Kähler structure on $M$. Moreover, the compact 3-dimensional manifold $N = \text{Nil}^3/\Gamma$ is the total space of an $S^1$-bundle over a 2-dimensional torus $T^2$ with projection $\pi_{xy} : N \rightarrow T^2_{xy}$ and $M$ inherits a structure of principal $S^1$-bundle over the 3-dimensional torus $T^3 = T_{xy}^2 \times S^1_1$, i.e.

$$S^1 \longrightarrow N \times S^1 = M \quad \longrightarrow \quad T^2 \times S^1 = T^3.$$ 

Furthermore $M$ is parallelizable and has the global left-invariant co-frame

$$e^1 = dy, \quad e^2 = dx, \quad e^3 = dt, \quad e^4 = dz - xdy,$$

satisfying the structure equations

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12},$$

where $e^{ij} = e^i \wedge e^j$.

Notice that a differential form $\gamma$ on $M$ is invariant by the action of the fiber $S^1_z$ if and only if its coefficients with respect to the global basis $e^{j_1 \cdots j_k} = e^{j_1} \wedge \cdots \wedge e^{j_k}$ do not depend on the variable $z$. 
In this paper we mainly focus on the almost-Kähler structure \((\Omega_2, J_2)\) which we denote by \((\Omega, J)\) in order to simplify the notation. The Calabi-Yau equation on \((M, \Omega, J)\) can be written as

\[(\Omega + d\alpha)^2 = e^F \Omega^2,\]

where the unknown \(\alpha\) is a smooth 1-form on \(M\) such that

\[J(d\alpha) = d\alpha,\]

and the datum \(F\) is a smooth function on \(M\) such that

\[\int_M e^F \Omega^2 = \int_M \Omega^2.\]

Our main result is the following

**Theorem 1.** For every smooth \(S^1\)-invariant volume form \(e^F \Omega^2\) satisfying condition (5), equation (3) has a unique solution satisfying (4).

Since uniqueness follows from a general result in [5], then we need only to prove existence. This will be done in two steps. Firstly in Section 3 we reduce equation (3) to a fully nonlinear PDE on the 3-dimensional base torus \(T^3\). Then in Section 5 we show that such an equation is solvable. Section 4 concerns the a-priori estimates needed in Section 5.

With some minor changes in the proof, it is possible to generalize Theorem 1 to a large class of invariant almost-Kähler structures on the Kodaira-Thurston manifold

**Theorem 2.** Let \((\Omega, J)\) be an invariant almost-Kähler structure on \(M\) with induced metric \(g\). Assume that \(e^4\) is orthogonal to \(e^1, e^2, e^3\) with respect to \(g\) and that

\[Je^4 \in \text{span}_Q \{e^1, e^2, e^3\}.\]

Then the Calabi-Yau equation (3) has a unique solution satisfying (4) for every \(S^1\)-invariant volume form \(e^F \Omega^2\) satisfying (5).

It is clear that the almost-Kähler structure \((\Omega_2, J_2)\) occurs as a case considered in Theorem 2. Moreover Theorem 2 works for \((\Omega_3, J_3)\) and, more generally, for the family of almost-Kähler structures

\[\omega_\theta = \cos \theta \Omega_2 + \sin \theta \Omega_3, \quad g = \sum_{i=1}^4 e^i \otimes e^i\]

when \(\tan \theta \in \mathbb{Q}\).

### 3. Reduction to a single equation

Denote by \(dV\) the volume form \(dx \wedge dy \wedge dt\) on \(T^3\).

**Proposition 3.** Let \(F\) a smooth function on \(T^3\) satisfying the following condition

\[\int_{T^3} e^F dV = 1\]

and let \(u : T^3 \to \mathbb{R}\) be a smooth function such that

\[\int_{T^3} u dV = 0.\]

Consider the 1-form

\[\alpha = d' u - u e^1.\]

Then \(d\alpha\) satisfies (4). Moreover \(\alpha\) solves (3) if and only if \(u\) solves the following PDE

\[(u_{xx} + 1)(u_{yy} + u_{tt} + u_t + 1) - u_{xy}^2 - u_{xt}^2 = e^F.\]
Proof. Let
\[ e_1 = \partial_y + x\partial_z, \quad e_2 = \partial_x, \quad e_3 = \partial_t, \quad e_4 = \partial_z \]
be the dual frame to \( \{e^1, e^2, e^3, e^4\} \). Then we have
\[
\dd\bar{u} = -dJdu = -\sum_{i,k=1}^{3} u_{ik} e^i \wedge J e^k - u_2 de^4
\]
\[
= -\sum_{i,k=1}^{3} u_{ik} e^i \wedge J e^k - u_2 e^{12},
\]
where
\[
(11) \quad u_1 = e_1 u = u_y, \quad u_2 = e_2 u = u_x, \quad u_3 = e_3 u = u_t.
\]
Observe that \( e_4 u = 0 \), since \( u \) does not depend on \( z \).

We have
\[
d(ue^1) = -u_2 e^{12} - u_3 e^{13}.
\]
Therefore
\[
\dd\bar{u} = -\sum_{i,k=1}^{3} u_{ik} e^i \wedge J e^k + d(ue^1) + u_3 e^{13}.
\]
It follows that \( da \) is of type \((1, 1)\) and that
\[
(\Omega + da)^2 = \left((1 + u_{22})(1 + u_{11} + u_{33} + u_3) - (u_{23})^2 - (u_{12})^2\right)\Omega^2,
\]
and equation (3) becomes
\[
(12) \quad (u_{22} + 1)(u_{11} + u_{33} + u_3 + 1) - (u_{23})^2 - (u_{12})^2 = e^F.
\]
Thanks to (11), equations (12) and (10) coincide. \(\square\)

4. A priori estimates

We begin by fixing some notation. Functions on the 3-torus can be identified with functions \( u: \mathbb{R}^3 \to \mathbb{R} \) which are 1-periodic in each variable.

For any non-negative integer \( n \), we denote by \( C^n(T^3) \) the Banach space of \( C^n \) functions \( u: T^3 \to \mathbb{R} \) equipped with norm
\[
\|u\|_{C^n} = \max_{m \leq n} |u|_{C^m},
\]
where
\[
|u|_{C^m} = \max_{i+j+k=m} \sup_{T^3} |\partial^i_x \partial^j_y \partial^k_t u(x, y, t)|.
\]

Given \( 0 < \rho < 1 \) and \( u \in C^0(T^3) \), we set
\[
\left[u(x, y, t)\right]_{\rho} = \sup_{0 < |h| \leq 1} |u(x + h_1, y + h_2, t + h_3) - u(x, y, t)| |h|^{-\rho}.
\]
For every non-negative integer \( n \) and real number \( 0 < \rho < 1 \), define the space \( C^{n+\rho}(T^3) \) of functions \( u \in C^n(T^3) \) such that
\[
|u|_{C^{n+\rho}} = \max_{i+j+k=n} \sup_{T^3} \left[\partial^i_x \partial^j_y \partial^k_t u(x, y, t)\right]_{\rho} < \infty.
\]
\( C^{n+\rho}(T^3) \) is a Banach space with respect to the norm
\[
\|u\|_{C^{n+\rho}} = \max\left\{\|u\|_{C^n}, |u|_{C^{n+\rho}}\right\}.
\]

In conclusion we have defined \( C^\sigma(T^3) \) for every non-negative real number \( \sigma \).

Finally, we denote by \( \check{C}^\sigma(T^3) \) the closed subspace of all \( u \in C^\sigma(T^3) \) satisfying
\[
\int_{T^3} u \, dV = 0,
\]
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where \( dV \) denotes as usual the volume form \( dx \wedge dy \wedge dt \) on \( T^3 \).

4.1. \( C^0 \)-estimate.

**Proposition 4.** Let \( u \in C^2(T^3) \) be a solution to (10). Then we have

\[
\begin{align*}
\text{(13)} & \quad u_{xx} > -1 \\
\text{and} & \\
\text{(14)} & \quad u_{yy} + u_{tt} + u_t > -1.
\end{align*}
\]

**Proof.** Indeed, from equation (10), we have

\[
(u_{yy} + u_{tt} + u_t + 1)(u_{xx} + 1) \geq e^F > 0.
\]

This implies that \( u_{yy} + u_{tt} + u_t + 1 \) and \( u_{xx} + 1 \) have always the same sign. But at a point where \( u \) attains its minimum, the second derivatives are non-negative. In particular \( u_{xx} + 1 \) must be positive. \( \square \)

**Proposition 5.** Let \( u \in C^2(T^3) \) be a solution to (10). Then we have

\[
\begin{align*}
\text{(15)} & \quad \lvert u_x \rvert \leq 1.
\end{align*}
\]

**Proof.** Fix \( y, t \in [0, 1] \) and let \( x_0 \) be a point in \( [0, 1] \) where \( x \mapsto u(x, y, t) \) attains its minimum. In particular, we have \( u_t(x_0, y, t) = 0 \). Integrating (13) yields

\[
\begin{align*}
\quad & u_x(x, y, t) \geq -(x - x_0) \geq -1, \quad \text{for } x_0 \leq x \leq x_0 + 1 \\
\text{and} & \\
\quad & -u(x, y, t) \geq -(x_0 - x) \geq -1, \quad \text{for } x_0 - 1 \leq x \leq x_0.
\end{align*}
\]

By periodicity, these estimates hold for all \( x \). This proves estimate (15). \( \square \)

**Theorem 6.** For all \( u \in \tilde{C}^2(T^3) \) satisfying equation (10), and all \( 1 < p < \infty \), we have

\[
\begin{align*}
\text{(16)} & \quad \lVert \nabla |u|^{p/2} \rVert_{L^2} \leq \frac{p^2}{16} \lVert u \rVert_{L^p}^p + \frac{p^2}{16} \left( p + \frac{8}{p-1} \lVert 1 - e^F \rVert_{C^0} \right) \lVert u \rVert_{L^p}^{p-1}.
\end{align*}
\]

**Proof.** Set

\[
\alpha = -(u_t + u)e^4 + u_y e^3 - u_x e^4.
\]

By Proposition 5, \( \alpha \) solves (3). Then

\[
\begin{align*}
\text{(17)} & \quad (e^F - 1) \Omega^2 = d\alpha \wedge (\Omega + \tilde{\Omega}),
\end{align*}
\]

where

\[
\tilde{\Omega} = \Omega + d\alpha.
\]

Since

\[
\begin{align*}
d(u |u|^{p-2}) & = |u|^{p-2} du + u(p-2) |u|^{p-3} \frac{u}{|u|} du \\
& = (p-1) |u|^{p-2} du,
\end{align*}
\]

then

\[
\begin{align*}
\text{(18)} & \quad \int_{T^3} d\left( (u |u|^{p-2} \alpha) \wedge (\Omega + \tilde{\Omega}) \right) = \\
& = (p-1) \int_{T^3} |u|^{p-2} du \wedge \alpha \wedge (\Omega + \tilde{\Omega}) + \int_{T^3} |u|^{p-2} u (e^F - 1) \Omega^2
\end{align*}
\]

and Stokes’ theorem implies

\[
\begin{align*}
\text{(19)} & \quad \int_{T^3} |u|^{p-2} du \wedge \alpha \wedge (\Omega + \tilde{\Omega}) = \frac{1}{p-1} \int_{T^3} (1 - e^F) |u|^{p-2} u \Omega^2.
\end{align*}
\]
Taking into account that
\[ du = u_y e^1 + u_x e^2 + u_t e^3, \]
\[ da = u_x(e^{12} - e^{24}) + (u_{yy} + u_{tt} + u_{t})e^{13} - u_{xy}(e^{14} - e^{23}) - u_{xx} e^{24}, \]
(20)
\[ \Omega = u_x(e^{12} - e^{24}) + (1 + u_{yy} + u_{tt} + u_{t})e^{13} - u_{xy}(e^{14} - e^{23}) - (1 + u_{xx}) e^{24}, \]
we have
(21)
\[ du \wedge \alpha \wedge \Omega = \frac{1}{2} (u_x^2 + u_y^2 + u_t (u_t + u)) \Omega^2, \]
and
(22)
\[ du \wedge \alpha \wedge \tilde{\Omega} = \frac{1}{2} \left( (u_x^2 + (u_t + u/2)^2)(1 + u_{xx}) + u_x^2 + u_y^2 + u_t + u_t \right) \Omega^2 \\
- (u_x u_y u_{xy} + u_x (u_t + u/2) u_{xt}) \Omega^2 - \frac{1}{8} u^2 (1 + u_{xx}) \Omega^2. \]
But from (10) and (13) we conclude that the quadratic form
\[ Q(\theta) = (1 + u_{xx})(\theta_1^2 + \theta_2^2) + (1 + u_{yy} + u_{tt} + u_t)\theta_2^2 - 2u_{xy}\theta_1 \theta_2 - 2u_{xt}\theta_2 \theta_3 \]
is positive-definite. Then from (19), (21) and (22) we obtain
(23)
\[ \int_{T^3} |u|^{p-2} \left( u_x^2 + u_y^2 + u_t (u_t + u) \right) dV - \frac{1}{4} \int_{T^3} |u|^p (1 + u_{xx}) dV \leq \\
\leq \frac{2}{p-1} \int_{T^3} (1 - e^F) |u|^{p-2} u dV. \]
An integration by parts gives
\[ \int_{T^3} |u|^{p-2} uu_t = -\int_{T^3} \partial_t \left( |u|^{p-2} u \right) u dV = (1 - p) \int_{T^3} |u|^{p-2} uu_t dV, \]
therefore we have
\[ \int_{T^3} |u|^{p-2} uu_t dV = 0. \]
Since, moreover
\[ \int_{T^3} |u|^p u_{xx} dV = -p \int_{T^3} |u|^{p-2} u u_{xt}^2 dV, \]
estimates (15) and (23) imply
(24)
\[ \int_{T^3} |u|^{p-2} |\nabla u|^2 dV \leq \frac{1}{4} \int_{T^3} |u|^p dV + \frac{p}{4} \left( \frac{2}{p-1} ||1 - e^F||_{C^0} \right) \int_{T^3} |u|^{p-1} dV. \]
But the left-hand side can be rewritten as
\[ \int_{T^3} |u|^{p-2} |\nabla u|^2 dV = \frac{4}{p^2} \int_{T^3} |\nabla |u|^{p/2}|^2 dV. \]
Then (24) becomes
(25)
\[ \int_{T^3} |\nabla |u|^{p/2}|^2 dV \leq \\
\leq \frac{p^2}{16} \int_{T^3} |u|^p dV + \frac{p^2}{4} \left( \frac{2}{p-1} ||1 - e^F||_{C^0} \right) \int_{T^3} |u|^{p-1} dV. \]
Since \( T^3 \) has measure 1, we have
(26)
\[ ||u||_{L^{p-1}} \leq ||u||_{L^p}. \]
Estimate (16) follows from (25) and (26).
It is rather natural to compare estimate (16) with the classical a priori Yau’s estimate

\[ \| \nabla \varphi \|^2_{L^2} \leq \frac{mp^2}{4p-1} \left( \| 1 - e^F \|_{C^0} \right) \| \varphi \|_{L^p} \]

involving the solutions \( \varphi \) to the complex Monge-Ampère equation \( (\omega + dd^c \varphi)^m = e^F \omega^m \) in 2m-dimensional Kähler manifolds (see for instance [8, Proposition 5.4.1]). The right-end side of (16) contains the extra term \( \| u \|_{L^p}^p \) due to the presence of \( -ue^F \) in (9). This is a problem in the first step of \( C^0 \)-estimate, that is with \( p = 2 \). We take care of this in the next two propositions.

**Proposition 7.** We have

(27) \[ 2\pi \| u \|_{L^2} \leq \| \nabla u \|_{L^2}, \quad \text{for all } u \in \tilde{C}^1(T^3). \]

*Proof.* Consider the Fourier series expansion

\[ \| u \|_{L^2}^2 = \sum_{(k,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |\hat{u}_{k,m,0}|^2, \]

where

\[ \hat{u}_{k,m,0} = \int_{T^3} e^{-2\pi i(kx + my + nt)} u(x, y, t) \, dV. \]

We have

\[ |\hat{\nabla \hat{u}_{k,m,0}|^2 = 4\pi^2(k^2 + m^2 + n^2)|\hat{u}_{k,m,0}|^2 \geq 4\pi^2|\hat{u}_{k,m,0}|^2, \]

for all \( (k, m, n) \in \mathbb{Z}^3 \setminus \{(0,0,0)\} \).

It follows that

\[ \| \nabla u \|_{L^2}^2 = \sum_{(k,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |\nabla \hat{u}_{k,m,0}|^2 \geq 4\pi^2 \sum_{(k,m) \in \mathbb{Z}^2 \setminus \{(0,0,0)\}} |\hat{u}_{k,m,0}|^2 \]

\[ = 4\pi^2 \| u \|_{L^2}^2. \]

Since

\[ \| \nabla |u| \|_{L^2} = \| \nabla u \|_{L^2}, \]

we obtain (27). \( \square \)

**Proposition 8.** For all \( u \in \tilde{C}^2(T^3) \) satisfying equation (10) we have

(28) \[ \| \nabla u \|_{L^2} \leq \frac{16\pi}{16\pi^2 - 1} \| 1 + e^F \|_{C^0}. \]

*Proof.* From (16) with \( p = 2 \) and (27) we obtain

\[ \| \nabla u \|_{L^2}^2 \leq \frac{1}{16\pi^2} \| \nabla u \|_{L^2}^2 + \frac{1}{4\pi} \left( 1 + \| 1 - e^F \|_{C^0} \right) \| \nabla u \|_{L^2}, \]

which readily implies the statement. \( \square \)

Now we are ready to prove an a priori \( C^0 \) estimate for the solutions to (10):

**Theorem 9.** There exists a positive constant \( C \) such that

(29) \[ \| u \|_{C^0} \leq C \| 1 + e^F \|_{C^0}, \]

for all \( u \in \tilde{C}^2(T^3) \) satisfying equation (10) and all \( F \in C^0(T^3) \) satisfying condition (7).
Proof. From Sobolev Imbedding Theorem (see for instance [2, Theorem 5.4]), there exists a positive constant $C'$ such that

$$(30) \quad \|w\|_{L^6}^2 \leq C'(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2),$$

for all $w$ in the Sobolev space $W^{1,2}(T^3)$.

Then from (16) and (30) we have

$$(31) \quad \|u\|_{L^3}^p \leq C'\left(1 + \frac{p^2}{16}\right) \|u\|_{L^p}^p + C'\frac{p^2}{16} \left(p + \frac{8}{p-1}\right) \|1 + e^F\|_{C^0}) \|u\|_{L^p}^{-1},$$

for all $1 < p < \infty$. If $p \geq 2$ and $u \neq 0$,

(31) implies

$$\left(\frac{\|u\|_{L^3}}{\|u\|_{L^p}}\right)^p \leq C'\frac{1}{2} p^3 \left(1 + \|1 + e^F\|_{C^0} \|u\|_{L^2}^{-1}\right).$$

It follows that

$$\frac{\|u\|_{L^{3p_k}}}{\|u\|_{L^p}} \leq (M_{p_k}^3)^{1/p_k}, \quad \text{for all } k \in \mathbb{Z}_+, \quad \text{with} \quad (32) \quad M = C'\frac{1}{2} \left(1 + \|1 + e^F\|_{C^0} \|u\|_{L^2}^{-1}\right)$$

and

$$p_k = 2 \cdot 3^k.$$

Then

$$\frac{\|u\|_{L^{3p_k}}}{\|u\|_{L^2}} \leq \prod_{k=0}^n (M_{p_k}^3)^{1/p_k}, \quad \text{for all } n \in \mathbb{Z}.$$ 

But

$$\prod_{k=0}^\infty (M_{p_k}^3)^{1/p_k} = \exp\left(\sum_{k=0}^\infty \frac{1}{2} \frac{1}{3^k} (\log(8M) + 3k \log 3)\right) = (8M)^3/4 \cdot 3^{\mu/2},$$

with

$$\mu = \sum_{k=1}^\infty \frac{k}{3^k} < \infty.$$

Then

$$(33) \quad \|u\|_{C^0} = \sup_{n \in \mathbb{N}} \|u\|_{L^{pn}} \leq (8M)^{3/4} \cdot 3^{\mu/2} \|u\|_{L^2}.$$ 

Now from (32), (27) and (28) we have

$$M^{3/4} \|u\|_{L^2} = \left(C'\frac{1}{2}\right)^{3/4} \left(\|u\|_{L^2} + \|1 + e^F\|_{C^0}\right)^{3/4} \|u\|_{L^2}^{1/4}$$

$$\leq \left(C'\frac{1}{2}\right)^{3/4} \left(\frac{16\pi^2}{16\pi^2 - 7} \|1 + e^F\|_{C^0}\right)^{1/4},$$

and (29) follows from (33). \qed
4.2. $C^0$-estimate of $\Delta u$. Let

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_t^2,$$

be the Laplacian on $\mathbb{R}^3$.

**Proposition 10.** Let $u \in \tilde{C}^2(T^3)$ be a solution to equation (10). Then we have

$$0 < 2e^{F/2} \leq 2 + \Delta u + u_t,$$

and

$$0 < \lambda_- I \leq H \leq \lambda_+ I$$

where $I$ is the identity matrix,

$$H = \begin{bmatrix} 1 + u_{yy} + u_{tt} + u_t & u_{xy} & u_{xt} \\ u_{xy} & 1 + u_{xx} & 0 \\ u_{xt} & 0 & 1 + u_{xx} \end{bmatrix}$$

and

$$\lambda_{\pm} = \frac{1}{2} \left( 2 + \Delta u + u_t \pm \sqrt{(2 + \Delta u + u_t)^2 - 4e^F} \right).$$

**Proof.** Inequality (34) follows from (13), (14) and (10). A simple computation shows that the characteristic polynomial of $H$ is

$$(\lambda - (1 + u_{xx}))(\lambda^2 - (2 + \Delta u + u_t)\lambda + e^F).$$

Then the eigenvalues of $H$ are $\lambda_{\pm}$ and $1 + u_{xx}$. Since

$$(2 + \Delta u + u_t)^2 - 4e^F = ((1 + u_{yy} + u_{tt} + u_t) - (1 + u_{xx}))^2 + u_{xy}^2 + u_{xt}^2$$

$$\geq ((2 + \Delta + u_t) - 2(1 + u_{xx}))^2,$$

we have

$$\lambda_- \leq 1 + u_{xx} \leq \lambda_+$$

and the proof is complete. $\Box$

**Theorem 11.** Given $F \in C^2(T^3)$ satisfying condition (7), there exists a positive constant $C$, depending only on $\|F\|_{C^2}$, such that

$$|\Delta u|_{C^0} \leq C(1 + |u|_{C^1}),$$

for all $u \in \tilde{C}^4(T^3)$ solution to equation (10).

**Proof.** From equation (10) we obtain

$$\left( \Delta F + |\nabla F|^2 + F_t \right) e^F =$$

$$= (1 + u_{yy} + u_{tt} + u_t)(\Delta u_{xx} + u_{xxt}) + (1 + u_{xx})(\Delta u_{yy} + u_{yyt} + \Delta u_{tt} + u_{ttt}) +$$

$$+ (1 + u_{xx})(\Delta u_t + u_{tt}) + 2\nabla u_{xx} \cdot \nabla(u_{yy} + u_{tt} + u_t) -$$

$$- 2u_{xy}(\Delta u_{xy} + u_{xyt}) - 2|\nabla u_{xy}|^2 - 2u_{xt}(\Delta u_{xt} + u_{xtt}) - 2|\nabla u_{xt}|^2.$$

Consider

$$G = (2 + \Delta u + u_t)e^{-\mu u},$$

where

$$\mu = \frac{\epsilon}{\max(2 + \Delta u + u_t)}$$

and $\epsilon$ is a positive constant to be chosen later. Differentiating (39) yields

$$\nabla G = e^{-\mu u} \left( \nabla(\Delta u + u_t) - \mu(2 + \Delta u + u_t)\nabla u \right),$$
\[(\nabla \otimes \nabla)G = -\mu e^{-\mu u} (\nabla u \otimes \nabla (\Delta u + u_t) + \nabla (\Delta u + u_t) \otimes \nabla u) + \mu^2 e^{-\mu u} (2 + \Delta u + u_t) \nabla u \otimes \nabla u + e^{-\mu u} \left( (\nabla \otimes \nabla)(\Delta u + u_t) - \mu (2 + \Delta u + u_t)(\nabla \otimes \nabla) u \right). \]

At a point where \(G\) attains its maximum value we have \(\nabla G = 0\) and \((\nabla \otimes \nabla)G \leq 0\), that is
\[(\nabla \otimes \nabla)(\Delta u + u_t) = \mu (2 + \Delta u + u_t) \nabla u, \]
and
\[(\nabla \otimes \nabla)(\Delta u + u_t) \leq \mu (2 + \Delta u + u_t) \left( (\nabla \otimes \nabla) u + \mu \nabla u \otimes \nabla u \right). \]

In particular, we obtain
\[
\left( \mu (2 + \Delta u + u_t) (u_{xy} + \mu u_x u_y) - (\Delta u_{xy} + u_{xyt}) \right)^2 \leq \left( \mu (2 + \Delta u + u_t) (u_{xx} + \mu u_x^2) - (\Delta u_{xx} + u_{xxt}) \right) \cdot \left( \mu (2 + \Delta u + u_t) (u_{yy} + \mu u_y^2) - (\Delta u_{yy} + u_{yyt}) \right).
\]

and
\[
\left( \mu (2 + \Delta u + u_t) (u_{xt} + \mu u_x u_t) - (\Delta u_{xt} + u_{xtt}) \right)^2 \leq \left( \mu (2 + \Delta u + u_t) (u_{xx} + \mu u_x^2) - (\Delta u_{xx} + u_{xxt}) \right) \cdot \left( \mu (2 + \Delta u + u_t) (u_{tt} + \mu u_t^2) - (\Delta u_{tt} + u_{ttt}) \right).
\]

Then \( (10) \) implies
\[(1 + u_{yy} + u_{tt} + u_t)(\Delta u_{xx} + u_{xxt}) + (1 + u_{xx})(\Delta u_{yy} + u_{yyt} + \Delta u_{tt} + u_{ttt}) - 2 u_{xy}(\Delta u_{xy} + u_{xyt}) - 2 u_{xt}(\Delta u_{xt} + u_{xtt}) \leq \mu (2 + \Delta u + u_t) (1 + u_{yy} + u_{tt} + u_t) (u_{xx} + \mu u_x^2) + \mu (2 + \Delta u + u_t) (1 + u_{xx}) (u_{yy} + \mu u_y^2 + u_{yt}) - 2 \mu (2 + \Delta u + u_t) (u_{xy}(u_{xy} + \mu u_x u_y) + u_{xt}(u_{xt} + \mu u_x u_t)). \]

Substituting \( (41) \) and \( (42) \) into \( (40) \), and using \( (44) \), we obtain
\[(\Delta F + |\nabla F|^2 + F_1) e^F \leq \mu (2 + \Delta u + u_t) (1 + u_{yy} + u_{tt} + u_t) (u_{xx} + \mu u_x^2) + \mu (2 + \Delta u + u_t) (1 + u_{xx}) (u_{yy} + u_{tt} + u_t) (u_{xx} + \mu u_x^2) + \mu (2 + \Delta u + u_t) (1 + u_{xx}) u_t + 2 \nabla u_{xx} \cdot \nabla (u_{yy} + u_{tt} + u_t) + - 2 \mu (2 + \Delta u + u_t) (u_{xy}(u_{xy} + \mu u_x u_y) + u_{xt}(u_{xt} + \mu u_x u_t)). \]

\footnote{Here we make use of the tensor product notation:
\[A \otimes B = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}, \text{ with } A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \]

In particular \((\nabla \otimes \nabla) u\) is the Hessian matrix of \(u\), and \((\text{tr}(\nabla \otimes \nabla)) u = \Delta u\).}
On the other side, from (11) we have

\begin{equation}
\mu^2 (2 + \Delta u + u_t)^2 |\nabla u|^2 = |\nabla (\Delta u + u_t)|^2 = \\
= |\nabla u_{xx}|^2 + |\nabla (u_{yy} + u_{tt} + u_t)|^2 + 2 \nabla u_{xx} \cdot \nabla (u_{yy} + u_{tt} + u_t) \\
\geq 2 \nabla u_{xx} \cdot \nabla (u_{yy} + u_{tt} + u_t).
\end{equation}

Eventually form (13), and (14) we obtain

\begin{equation}
(\Delta F + |\nabla F|^2 + F_t) e^F \leq \\
\leq \mu (2 + \Delta u + u_t) \left( (1 + u_{yy} + u_{tt} + u_t) u_{xx} + (1 + u_{xx}) (u_{yy} + u_{tt} + u_t) \right) \\
- 2 \mu (2 + \Delta u + u_t) (u_{yy}^2 + u_{tt}^2) \\
+ 2 \mu^2 (2 + \Delta u + u_t) \left( (1 + u_{yy} + u_{tt} + u_t) u_x^2 + (1 + u_{xx}) (u_y^2 + u_t^2) \right) \\
+ \mu^2 (2 + \Delta u + u_t)^2 |\nabla u|^2 \\
\leq 2 \mu (2 + \Delta u + u_t) e^F - \mu (2 + \Delta u + u_t)^2 + 3 \mu^2 (2 + \Delta u + u_t)^2 |\nabla u|^2.
\end{equation}

at any point where $G$ attains its maximum value. Let $(x_0, y_0, t_0)$ be a point where $G$ attains its maximum value. Set

\[ M = 2 + \Delta u(x_0, y_0, t_0) + u_t(x_0, y_0, t_0) \]

and

\[ u_0 = u(x_0, y_0, t_0), \]

so that

\[ \max G = Me^{-\mu u_0}. \]

From (45) we get

\begin{equation}
\mu M^2 \leq |(\Delta F + F_t) e^F|_{C^0} + 2 \mu M \|e^F\|_{C^0} + 3 \mu^2 M^2 |u|_{C^1}^2.
\end{equation}

Denote by $\tilde{u}$ the value of $u$ at a point where $2 + \Delta u + u_t$ attains its maximum. Then we have

\begin{equation}
M \leq \max (2 + \Delta u + u_t) \leq Me^{\mu (\tilde{u} - u_0)} \leq Me^{2\mu |u|_{C^0}}.
\end{equation}

Moreover, (10) and (11) imply

\[ 2\mu = \frac{2\epsilon}{\max (2 + \Delta u + u_t)} \leq \epsilon e^{-\min F/2} \leq e^{-\min F/2}, \]

then, (47) yields

\begin{equation}
\epsilon \exp \left( -e^{-\min F/2} |u|_{C^0} \right) \leq \mu M \leq \epsilon
\end{equation}

and

\begin{equation}
\exp \left( -e^{-\min F/2} |u|_{C^0} \right) \max (2 + \Delta u + u_t) \leq M.
\end{equation}

Eventually from (46), (48), and (49) we obtain

\[ \epsilon \exp \left( -2e^{-\min F/2} |u|_{C^0} \right) \max (2 + \Delta u + u_t) \leq \\
\leq |(\Delta F + F_t) e^F|_{C^0} + 3 \epsilon |u|_{C^1}^2, \]

that is

\begin{equation}
\max (2 + \Delta u + u_t) \leq \\
\leq \exp \left( 2e^{-\min F/2} |u|_{C^0} \right) \left( \frac{1}{\epsilon} |(\Delta F + F_t) e^F|_{C^0} + 3 |u|_{C^1}^2 \right).
\end{equation}
Since

$$|\Delta u|_{C^0} \leq \max(2 + \Delta u + u_t) + 2 + |u_t|_{C^0},$$

estimate (27) follows from (24), (25) and (30), with

$$\epsilon = \frac{1}{1 + |u|_{C^1}}.$$ 

Now we prove a simple interpolation inequality.

**Theorem 12.** For all $\epsilon > 0$ there exists a positive constant $M_\epsilon$ such that

$$|u|_{C^1} \leq M_\epsilon |u|_{C^0} + \epsilon |\Delta u|_{C^0}, \quad \text{for all } u \in C^2(T^3).$$

In order to prove the last theorem we need the following elementary lemma.

**Lemma 13.** For all $R > 0$ and all $0 < \mu \leq 1/4$ we have

$$M_{\mu,R} = \sup_{|h| \leq 1/2} |h|^{-\mu} \int_{|z| \leq R} \left| \frac{z-h}{|z-h|^3} - \frac{z}{|z|^3} \right| dz < \infty,$$

where $z, h \in \mathbb{R}^3$.

**Proof.** Consider the following cases:

1. $|z| > |h|^\mu$,
2. $|z-h| > |h|^\mu$,
3. $|z| \leq |h|^\mu$ and $|z-h| \leq |h|^\mu$.

We have the elementary estimate

$$\left| \frac{z-h}{|z-h|^3} - \frac{z}{|z|^3} \right| = \left| \int_0^1 \frac{d}{ds} \frac{z-sh}{|z-sh|^3} ds \right| = \left| \int_0^1 \left( \frac{-1}{|z-sh|^3} \frac{h}{|z-sh|^5} \cdot (z-sh) \right) ds \right| \leq 4|h| \int_0^1 \frac{ds}{|z-sh|^3}.$$

Then in case (1) we have

$$\left| \frac{z-h}{|z-h|^3} - \frac{z}{|z|^3} \right| \leq \frac{4|h|}{(|h|^\mu - |h|)^3} = \frac{4|h|^{1-3\mu}}{(1 - |h|^{1-\mu})^3} \leq \frac{4|h|^{\mu}}{(1 - 2(\mu-1)/\mu)^3},$$

while in case (2) we have

$$\left| \frac{z-h}{|z-h|^3} - \frac{z}{|z|^3} \right| \leq 4|h| \int_0^1 \frac{ds}{|z-h + (1-s)h|^3} \leq \frac{4|h|^\mu}{(|h|^\mu - |h|)^3} \leq \frac{4|h|}{(1 - 2(\mu-1)/\mu)^3}.$$

It remains to consider case (3). But then we obtain

$$|h|^{-\mu} \int_{|z| \leq R} \left| \frac{z-h}{|z-h|^3} - \frac{z}{|z|^3} \right| dz \leq |h|^{-\mu} \int_{|z| \leq R} \frac{dz}{|z-h|^2} + |h|^{-\mu} \int_{|z| \leq R} \frac{dz}{|z|^2} \leq \int_{|z| \leq R} \frac{dz}{|z-h|^{2+\mu}} + \int_{|z| \leq R} \frac{dz}{|z|^{2+\mu}} \leq 2 \int_{|z| \leq R+1} \frac{dz}{|z|^{2+\mu}} < \infty.$$
Proof of Theorem 12. Green’s Representation Formula (see e.g. [7, formula (2.16)]) implies

\[ u(p) = \int_{|q|=R} \left( \frac{u(q)}{4\pi |q-p|^2} + \frac{\nabla u(q) \cdot q}{4\pi R |q-p|} \right) d\sigma(q) - \int_{|q| \leq R} \frac{\Delta u(q)}{4\pi |q-p|} dq, \]

where

\[ d\sigma(q) = R^2 \sin \theta \, d\phi \wedge d\theta \]

and

\[
\begin{cases}
q_1 = R \sin \phi \cos \theta, \\
q_2 = R \cos \phi \cos \theta, \\
q_3 = R \sin \theta,
\end{cases}
\]

are coordinates on the sphere with radius \( R \) and centered at the origin. Set

\[ f(p) = \int_{|q|=R} \left( \frac{u(q)}{4\pi |q-p|^2} + \frac{\nabla u(q) \cdot q}{4\pi R |q-p|} \right) d\sigma(q), \]

and

\[ g(p) = \int_{|q| \leq R} \frac{\Delta u(q)}{4\pi |q-p|} dq. \]

Now

\[(\nabla \otimes \nabla) f(p) = \frac{1}{4\pi} \int_{|q|=R} \frac{6}{|q-p|^4} (q-p) \otimes (q-p) d\sigma(q) \]

\[- \frac{1}{4\pi} \int_{|q|=R} \frac{2}{|q-p|^3} d\sigma(q) \]

\[+ \frac{1}{4\pi} \int_{|q|=R} \frac{2}{R |q-p|^2} (q-p) \otimes (q-p) d\sigma(q) \]

\[- \frac{1}{4\pi} \int_{|q|=R} \frac{\nabla u(q) \cdot q}{R |q-p|} d\sigma(q), \]

then, for all \( 0 < r < R \) we have

\[ \sup_{|p| \leq r} |(\nabla \otimes \nabla) f(p)| \leq 8R^2 \frac{|u|_{C^0}}{(R - r)^3} + 9R^2 \frac{|u|_{C^1}}{(R - r)^3}. \]

This estimate shows that \( f \) belongs to \( C^{1+\mu}(B_r) \), where \( B_r \) is the ball centered at the origin and with radius \( r \), and satisfies the estimate:

\[ |f|_{C^{1+\mu}(B_r)} \leq \frac{9R^2}{(R - r)^3} \left( |u|_{C^0} + (R - r) |u|_{C^1} \right) |h|^{1-\mu}. \]

On the other hand we have

\[ \nabla g(p) = \nabla \int_{|q| \leq R} \frac{\Delta u(q)}{4\pi |q-p|} dq = \int_{|q| \leq R} \frac{\Delta u(q)}{4\pi |q-p|^3} (q-p) dq. \]

Then

\[ |\nabla g(p + h) - \nabla g(p)| \leq \frac{|u|_{C^0}}{4\pi} \int_{|q| \leq R} \frac{|q - p - h|}{|q - p|^3} \left| \frac{q - p}{|q-p|^3} \right| dq \]

\[\leq \frac{|u|_{C^0}}{4\pi} \int_{|q| \leq R + r} \frac{|q - h|}{|q - h|^3} - \frac{q}{|q|^3} dq, \quad \text{for all } p \in B_r, \]

\[2 \text{ Observe that } |\nabla u(q) \cdot q| \leq 3R |u|_{C^1}. \]
and from Lemma 13 we obtain
\[ |h|^{-\mu} |\nabla g(p + h) - \nabla g(p)| \leq (4\pi)^{-1} M_{\mu,2R} |\Delta u|_{C^0}, \]
for $|p| \leq r$ and $|h| \leq 1/2$.

Consider now $0 < r < R$ such that $Q \subset B_r$. Then from (51), (52), (53), (54), and (55) we obtain that there exists a positive constant $M$ such that
\[ |u|_{C^{1+\mu}} \leq M \left( |u|_{C^0} + |u|_{C^1} + |\Delta u|_{C^0} \right), \quad \text{for all } u \in C^2(T^3). \]

Thanks to standard interpolation inequality theory (see [7, section 6.8]), for all $\epsilon > 0$ there exists a positive constant $M'_{\epsilon}$ such that
\[ |u|_{C^1} \leq M'_{\epsilon} |u|_{C^0} + \epsilon |u|_{C^{1+\mu}}, \quad \text{for all } u \in C^{1+\mu}(T^3). \]

From (56) and (57) with $\epsilon = \epsilon_{\mu} = 1/(2M)$ we obtain
\[ |u|_{C^1} \leq 2(M + M'_{\epsilon}) |u|_{C^0} + 2M |\Delta u|_{C^0}, \quad \text{for all } u \in C^2(T^3). \]

Using (57) a second time with $\epsilon = \epsilon' > 0$, we get in the end
\[ |u|_{C^1} \leq \left( M'_{\epsilon'} + 2\epsilon (M + M'_{\epsilon'}) \right) |u|_{C^0} + 2M \epsilon' |\Delta u|_{C^0}, \quad \text{for all } u \in C^2(T^3). \]

**Theorem 14.** Given $F \in C^2(T^3)$ satisfying condition (7), there exists a positive constant $C$, depending only on $\|F\|_{C^2}$, such that
\[ |\Delta u|_{C^0} \leq C, \]
for all $u \in \tilde{C}^4(T^3)$ solution to equation (10).

**Proof.** From Theorems 9, 11 and 12 we obtain that there exists a positive constant $C'$ such that for all $\epsilon > 0$ we have:
\[ |\Delta u|_{C^0} \leq C' \left( 1 + M'_{\epsilon} + \epsilon |\Delta u|_{C^0} \right), \]
for all $u$ satisfying the hypotheses of the theorem. \hfill \Box

**Corollary 15.** Given $F \in C^2(T^3)$ satisfying condition (7), there exists a positive constant $C$, depending only on $\|F\|_{C^2}$, such that
\[ |u|_{C^1} \leq C, \]
for all $u \in \tilde{C}^4(T^3)$ solution to equation (10).

**Proof.** It follows from Theorems 12 and 14. \hfill \Box

**Corollary 16.** Equation (10) is strictly uniformly elliptic, in the sense that there exists a positive constant $C$, depending only on $\|F\|_{C^2}$, such that for all $u \in \tilde{C}^4(T^3)$ solution to equation (10), we have
\[ \Lambda^{-1} \leq \lambda_- \leq \lambda_+ \leq \Lambda, \]
where $\lambda_{\pm}$ are defined in formula (59).
4.3. $C^{2+\rho}$-estimate. We begin by recalling a theorem of [14], which greatly simplifies the estimate of second derivatives. In [14] the theorem has been stated locally, but on compact manifolds it holds globally.

**Theorem 17 (14 Theorem 5.1).** Let $\tilde{\Omega}$ be the solution of the Calabi-Yau equation

$$\tilde{\Omega} = e^F \Omega,$$

on a compact almost-Kähler manifold $(M^{2n}, \Omega, J)$.

Assume there are two constants $C_0 > 0$ and $0 < \rho_0 < 1$ such that $F \in C^{\rho_0}(M^{2n})$ and

$$\text{tr} \tilde{\tilde{g}} \leq C_0,$$

where $\tilde{\tilde{g}}$ is the Riemannian metric associated to $\tilde{\Omega}$.

Then there exist two constants $C > 0$ and $0 < \rho < 1$, depending only on $M^{2n}$, $\Omega$, $J$, $C_0$ and $\|F\|_{C^{\rho_0}}$, such that

$$\|\tilde{\tilde{g}}\|_{C^0} \leq C.$$

Using this Theorem we easily prove the following estimate.

**Theorem 18.** Given $F \in C^2(T^3)$ satisfying condition (7), there exist constants $C > 0$ and $\rho > 0$, both depending only on $\|F\|_{C^2}$, such that

$$\|u\|_{C^{2+\rho}} \leq C,$$

for all $u \in \tilde{\mathcal{C}}^4(T^3)$ solution to equation (10).

**Proof.** From (20) we obtain that the Riemannian metric $\tilde{\tilde{g}}$ is represented by the matrix

$$\begin{bmatrix}
1 + u_{yy} + u_{tt} + u_t & u_{xy} & 0 & u_{xt} \\
0 & 1 + u_{xx} & u_{xt} & 0 \\
u_{xt} & 0 & 1 + u_{yy} + u_{tt} + u_t & -u_{xy} \\
0 & u_{xt} & -u_{xt} & 1 + u_{xx}
\end{bmatrix}.$$

Then

$$\text{tr} \tilde{\tilde{g}} = 2(2 + \Delta u + u_t).$$

Thanks to Theorem 14 and Corollary 16 we can apply Theorem 17 and get that

$$\max\{1 + u_{xx}\}_{C^\rho}, \ 1 + u_{yy} + u_{tt} + u_t\}_{C^\rho}, \|u_{xy}\|_{C^\rho}, \|u_{xt}\|_{C^\rho} \leq C,$$

where $C$ depends only on $\|F\|_{C^2}$.

Now the estimates of the second derivatives can be obtained as follows. Given a solution $u$ of equation (10), we have that $u$ can be viewed as a solution of the equation

$$a_1 u_{xx} + a_2(u_{yy} + u_{tt}) + 2a_3 u_{xy} + 2a_4 u_{xt} + bu_t = f,$$

with

$$a_1 = 1 + u_{yy} + u_{tt} + u_t, \quad a_2 = 1 + u_{xx}, \quad a_3 = -u_{xy}, \quad a_4 = -u_{xt}, \quad b = 1 + u_{xx}, \quad f = 2e^F - (2 + \Delta u + u_t).$$

Equation (60) is uniformly elliptic, with coefficients in $C^\rho(T^3)$. Thanks to (59) and Theorem 9 standard Schauder theory gives the estimate (58). □
5. Proof of Theorem 19

Proposition 19. Assume \( u \in \tilde{C}^{2+\rho}(T^3) \) is a solution to equation (10) with \( \rho > 0 \). If \( F \in C^{\infty}(T^3) \) then \( u \in \tilde{C}^{\infty}(T^3) \).

Proof. By Proposition 10 the equation (10) is elliptic. Then from [10, Theorem 4.8, Chapter 14], it follows that \( u \) belongs to the Sobolev space \( W^{n,2}(T^3) \), for all \( n \in \mathbb{Z}_+ \). But this implies that \( u \in C^{\infty}(T^3) \).

Thanks to Proposition 8, Theorem 1 is an immediate consequence of the following

Theorem 20. Let \( F \in C^{\infty}(T^3) \) satisfy (7). Then equation (10) has a solution \( u \in \tilde{C}^{\infty}(T^3) \).

Proof. We apply the continuity method (see [7 Section 17.2]). For \( 0 \leq \theta \leq 1 \), let

\[
\mathcal{S}_\theta = \left\{ u \in \tilde{C}^{\infty}(T^3) : (1 + u_{yy} + u_{tt} + u_t)(1 + u_{xx}) - u_{xy}^2 - u_{xt}^2 = e^{F_\theta} \right\}
\]

where

\[
F_\theta = \log(1 - \theta + \theta e^{F}).
\]

Note that \( 0 \in \mathcal{S}_0 \) and that \( \mathcal{S}_1 \) consists in the solutions to (10) lying in \( \tilde{C}^{\infty}(T^3) \). Since

\[
\max_{0 \leq \theta \leq 1} \|F_\theta\|_{C^2} < \infty,
\]

and

\[
\int_{T^3} e^{F_\theta} \, dV = \int_{T^3} (1 - \theta + \theta e^{F}) \, dV = 1,
\]

by Theorem 18 there exists a real number \( \rho > 0 \) such that

\[
\sup_{u \in \mathcal{S}} ||u||_{C^{2+\rho}} < \infty,
\]

with

\[
\mathcal{S} = \bigcup_{0 \leq \theta \leq 1} \mathcal{S}_\theta \neq \emptyset.
\]

Since \( 0 \in \mathcal{S}_0 \), the set \( \{ \theta \in [0, 1] : \mathcal{S}_\theta \neq \emptyset \} \) is not empty and we can define

\[
\mu = \sup \{ \theta \in [0, 1] : \mathcal{S}_\theta \neq \emptyset \}.
\]

In order to complete the proof we have to show that \( \mathcal{S}_\mu \neq \emptyset \) and \( \mu = 1 \).

- \( \mathcal{S}_\mu \neq \emptyset \). By the definition of \( \mu \) there exist two sequences \( (\theta_k) \subset [0, 1] \) and \( (u_k) \subset \tilde{C}^{\infty}(T^3) \) such that \( (\mu_k) \) is increasing and \( u_k \in \mathcal{S}_{\theta_k} \) for all \( k \). Thanks to (52), the sequence \( (u_k) \) is bounded in \( \tilde{C}^\rho(T^3) \), then by Ascoli-Arzelà Theorem there exists a subsequence \( (u_{k_j}) \) convergent in \( \tilde{C}^{2+\rho/2}(T^3) \). Let \( v = \lim_{j} u_{k_j} \), Then \( v \) belongs to \( \tilde{C}^{2+\rho/2}(T^3) \) and satisfies the equation

\[
(1 + v_{yy} + v_{tt} + v_t)(1 + v_{xx}) - v_{xy}^2 - v_{xt}^2 = e^{F_v}.
\]

By Proposition 10 the equation (10) is elliptic and the standard theory of elliptic equations implies that \( v \) belongs to \( \tilde{C}^{\infty}(T^3) \) (see e.g. [10 Chapter 14]). In particular, \( v \) belongs to \( \mathcal{S}_\mu \), which turns out to be not empty.

- \( \mu = 1 \). Assume by contradiction \( \mu < 1 \) and define the non-linear \( C^\infty \) operator

\[
\begin{align*}
T & : \tilde{C}^\rho(T^3) \times [0, 1] \to C^{\rho-2}(T^3), \\
T(u, \theta) & = (1 + u_{yy} + u_{tt} + u_t)(1 + u_{xx}) - u_{xy}^2 - u_{xt}^2 - e^{F_v}.
\end{align*}
\]

Since \( \mathcal{S}_\mu \) is not empty, there exists \( v \in \mathcal{S}_\mu \) such that \( T(v, \mu) = 0 \). Compute

\[
\partial_\mu T(v, \mu) w = Lw,
\]
Then we have
\[ Lw = (1 + v_{yy} + v_{tt} + v_1)w_{xx} + (1 + v_{xx}) (w_{yy} + w_{tt} + w_t) - 2v_{xy} w_{xy} - 2v_{xt} w_{xt}. \]

Proposition 16 implies that \( L : \mathcal{C}^{2+p}(T^3) \to \tilde{\mathcal{C}}^p(T^3) \) is elliptic. Then by Strong Maximum Principle \( L = 0 \) implies that \( u \) is constant. This shows that \( L \) is one-to-one on \( \mathcal{C}^{2+p} \). Moreover, by ellipticity, \( L \) has closed range, thus Schauder Theory and Continuity Method (see [7, Theorem 5.2]) show that \( L \) is onto. Therefore by Implicit Function Theorem there exists an \( \epsilon > 0 \) such that
\[ T(u, \theta) = 0 \]
is solvable with respect to \( u \) for every \( \theta \in (\mu - \epsilon, \mu + \epsilon) \). Thanks to Proposition 19 these solutions belong to \( \tilde{\mathcal{C}}^\infty(T^3) \). Then \( \mathcal{G}_y \neq \emptyset \) for all \( \mu < \theta < \mu + \epsilon \), in contradiction with the definition of \( \mu \). \( \square \)

6. Proof of Theorem 2

Consider on \( M \) an almost-Kähler structure \((\Omega, J)\) such that \( e^4 \) is orthogonal to \( e^1, e^2, e^3 \) with respect to the Riemannian metric induced by \((\Omega, J)\). Then we can find an invariant orthonormal co-frame \( \{f^1, f^2, f^3, f^4\} \) such that
\[ \text{span}_g \{f^1, f^2, f^3\} = \text{span}_g \{e^1, e^2, e^3\}, \quad \text{span}_g \{f^4\} = \text{span}_g \{e^4\}, \]
and
\[ \Omega = f^{13} - f^{24}. \]

Let \( u \) be an \( S^1 \)-invariant function on \( M \), that is not depending on \( z \), and set
\[ u_i = f_i u, \]
where \( \{f_1, f_2, f_3, f_4\} \) is the frame dual to \( \{f^1, f^2, f^3, f^4\} \). Observe that
\[ u_4 = 0, \]
because \( \text{span}_g \{f_4\} = \text{span}_g \{e_4\} \). From (63) we know that there exists a non-singular matrix \( A \) such that
\[ e^1 = \sum_{j=1}^3 A_j^i f^j, \quad \text{for } i \in \{1, 2, 3\}, \quad e^4 = A_4^i f^4. \]

Then we have
\[ df^i = \frac{1}{A_i^4} de^4 = \frac{1}{A_i^4} e^{12} = \frac{1}{A_i^4} \sum_{j,k=1}^3 A_j^i A_k^j f^{jk} = \lambda^1 f^{23} + \lambda^2 f^{13} + \lambda^3 f^{12}, \]
with
\[ \lambda^1 = \frac{A_1^j A_2^j - A_1^j A_3^j}{A_4^j}, \quad \lambda^2 = \frac{A_1^j A_2^j - A_2^j A_3^j}{A_4^j}, \quad \lambda^3 = \frac{A_1^j A_3^j - A_2^j A_3^j}{A_4^j}. \]

It follows that
\[ dd^c u = - \sum_{i,j=1}^3 u_{ij} f^i \wedge J f^j + u_2 (\lambda^1 f^{23} + \lambda^2 f^{13} + \lambda^3 f^{12}) \]
\[ = - \sum_{i,j=1}^3 u_{ij} f^i \wedge J f^j + d(u(\lambda^1 f^3 - \lambda^3 f^1) - \sum_{i=1}^3 \lambda^i u_i) f^{13} \]

Therefore, if we set
\[ \alpha = d^c v - u(\lambda^1 f^3 - \lambda^3 f^1), \]
(65)
we have that \( d\alpha \) is of type \((1, 1)\) and

\[
d\alpha = -\sum_{i,j=1}^{3} u_{ij} f^i \wedge J f^j + \left( \sum_{i=1}^{3} \lambda^i u_i \right) f^{13}.
\]

At this point, a simple computation shows that \( \alpha \) satisfies (4) if and only if \( u \) satisfies the following PDE

\[
(1 + u_{22}) \left( 1 + u_{11} + u_{33} + \sum_{i=1}^{3} \lambda^i u_i \right) - (u_{12})^2 - (u_{13})^2 = e^F.
\]

In conclusion we have proven the following

**Proposition 21.** Consider on \( M \) an almost-Kähler structure \((\Omega, J)\) such that \( e^4 \) is orthogonal to \( e^1, e^2, e^3 \) with respect to the Riemannian metric induced by \((\Omega, J)\). Let \( F : T^3 \to \mathbb{R} \) be a smooth function satisfying (4). Let \( \{f^1, f^2, f^3, f^4\} \) be an invariant orthonormal co-frame for which (63) and (64) are satisfied. Let \( u : T^3 \to \mathbb{R} \) be a smooth function satisfying (8). Then the 1-form (65) solves (3) if and only if \( u \) is a solution to the non-linear PDE (66).

Equation (66) is very similar to equation (12). As a matter of facts the only really new feature is that the unknown \( u \) is periodic with respect to the lattice generated by \( \{e_1, e_2, e_3\} \) and not with respect to the one generated by the new frame \( \{f_1, f_2, f_3\} \). Then the proof of Proposition 5 fails unless \( u \) is periodic in the direction of \( f_2 \). However from assumption (6) it follows that there exist 4 integers \( n, j, k, m \), with \( n > 0 \), such that

\[
nf_2 = je_1 + ke_2 + me_3.
\]

Now we can estimate \( u_2 \).

**Proposition 22.** Let \( \pi : \mathbb{R}^3 \to T^3 \) be the quotient map. Assume (67) is satisfied. Then the map

\[
T : \mathbb{R} \to T^3,
\]

\[
T(s) = \pi \left( \frac{j}{n} s, \frac{k}{n} s, \frac{m}{n} s \right),
\]

is periodic. Let \( \nu \) be the minimum positive period of \( T \). Then

\[
|u_2| \leq \nu.
\]

for all solution \( u \) to equation (66).

**Proof.** Fix \((x, y, t) \in \mathbb{R}^3\), and consider the periodic function

\[
v(s) = u(x + js/n, y + ks/n, t + ms/n).
\]

We have

\[
v''(s) = u_{22} u(x + js/n, y + ks/n, t + ms/n) \geq -1.
\]

Let \( s_0 \in [0, \nu] \) be a critical point of \( v \). Then we have

\[
v'(s) = \int_{s_0}^{s} v''(r) \, dr \begin{cases} \geq -(s - s_0) \geq -\nu, & s_0 \leq s \leq s_0 + \nu, \\ \leq -(s - s_0) \geq -\nu, & s_0 - \nu \leq s \leq s_0. \end{cases}
\]

By periodicity we get that these estimates hold everywhere, in particular we get

\[
|u_2(x, y, t)| = |v'(0)| \leq \nu.
\]

The rest of the proof of Theorem 2 can be obtained by a slight modification of the argument used to prove Theorem 1 and it is left to the reader.
References

[1] E. Abbena, An example of an almost Kähler manifold which is not Kählerian, *Boll. Un. Mat. Ital. A* (6) 3 (1984), no. 3, 383–392.

[2] R.A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, Orlando, 1975.

[3] E. Buzano, A. Fino and L. Vezzoni, The Calabi-Yau equation for $T^2$-bundles over the non-Lagrangian case, *Rend. Semin. Mat. Univ. Politec. Torino* 69 (2011), no. 3, 281–298.

[4] E. Calabi, On Kähler manifolds with vanishing canonical class, in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pp. 78–89. Princeton University Press, Princeton, N.J., 1957.

[5] S.K. Donaldson, Two-forms on four-manifolds and elliptic equations. *Inscribed by S.S. Chern*, 153–172, Nankai Tracts Math. 11, World Scientific, Hackensack N.J., 2006.

[6] A. Fino, Y.Y. Li, S. Salamon and L. Vezzoni, The Calabi-Yau equation on 4-manifolds over 2-tori, *Trans. Amer. Math. Soc.* 365 (2013), no. 3, 1551–1575.

[7] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.

[8] D.D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. xii+436 pp.

[9] C.H. Taubes, Tamed to compatible: symplectic forms via moduli space integration. *J. Symplectic Geom.* 9 (2011), no. 2, 161–250.

[10] M.E. Taylor, *Partial Differential Equations III, Nonlinear Equations*, Applied Mathematical Sciences, vol.V 117, Springer, New York, NY, 1996.

[11] W.P. Thurston, Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* 55 (1976), 467–468.

[12] V. Tosatti and B. Weinkove, The Calabi-Yau equation on the Kodaira-Thurston manifold, *J. Inst. Math. Jussieu* 10 (2011), no. 2, 437–447.

[13] V. Tosatti, B. Weinkove and S.T. Yau, Taming symplectic forms and the Calabi-Yau equation, *Proc. London Math. Soc.* 97 (2008), no. 2, 401–424.

[14] V. Tosatti, Y. Wang, B. Weinkove and X. Yang, $C^{2,\alpha}$ estimates for nonlinear elliptic equations in complex and almost complex geometry. *arXiv:1402.0554*

[15] S.T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, *Comm. Pure Appl. Math.* 31 (1978), no. 3, 339–411.

[16] B. Weinkove, The Calabi-Yau equation on almost-Kähler four-manifolds, *J. Differential Geom.* 76 (2007), no. 2, 317–349.

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