CONNECTIVITY AND KIRWAN SURJECTIVITY FOR ISOPARAMETRIC SUBMANIFOLDS

AUGUSTIN-LIVIU MARE

Abstract. Atiyah’s formulation of what is nowadays called the convexity theorem of Atiyah-Guillemin-Sternberg has two parts: (a) the image of the moment map arising from a Hamiltonian action of a torus on a symplectic manifold is a convex polytope, and (b) all preimages of the moment map are connected. Part (a) was generalized by Terng to the wider context of isoparametric submanifolds in euclidean space. In this paper we prove a generalization of part (b) for a certain class of isoparametric submanifolds (more precisely, for those with all multiplicities strictly greater than 1). For generalized real flag manifolds, which are an important class of isoparametric submanifolds, we give a surjectivity criterion for a certain Kirwan map (involving equivariant cohomology with coefficients in \( \mathbb{Q} \)) which arises naturally in this context. Examples are also discussed.

1. Introduction

A crucial result in symplectic geometry is the convexity theorem of Atiyah-Guillemin-Sternberg. It states that if \((M, \omega)\) is a symplectic manifold acted on by a torus \(T\) in a Hamiltonian way, then the image of the moment map \(\mu : M \rightarrow \text{Lie}(T)^*\) is a convex polytope. A closely related result says \([A]\) that all pre-images of \(\mu\) are connected (or empty). The convexity theorem has been generalized by Terng \([T1]\) to the case when \(M\) is an isoparametric submanifold in a euclidean space (see section 1 of our paper for the definition; we note that we always assume that \(M\) is compact). More precisely, she proved the following theorem.

Theorem 1.1. (Terng \([T1]\)) Let \(M \subset \mathbb{R}^{n+k}\) be an isoparametric submanifold, \(q\) a point of \(M\), \(\nu_q(M)\) the normal space at \(q\) to \(M\), and \(P : \mathbb{R}^{n+k} \rightarrow \nu_q(M)\) the orthogonal projection map. Then the image of the map
\[
\mu = P|_M : M \rightarrow \nu_q(M)
\]
is a convex polytope.

For example, \(M\) can be a generalized real flag manifold, i.e. an orbit of the isotropy representation of a compact symmetric space (for more details, see section 5 of our paper). These can be realized as real loci (i.e. fixed point sets of antisymplectic involutions) of coadjoint orbits. Convexity results for such spaces had been obtained by Kostant \([K]\) (see also Duistermaat \([D]\)). But Terng’s theorem does not fit entirely into the framework of the paper by Kostant. The reason is that there are several examples of isoparametric submanifolds which are not flag manifolds, i.e. are not homogeneous: such examples were found by Ferus, Karcher and Münzner in \([FKM]\).
Now turning to the connectedness result of Atiyah mentioned above, the following question can be raised. In the context of Theorem 1.1, is any nonempty preimage of \( \mu \) connected? One can easily see that the answer is in general negative. More specifically, the real flag manifold corresponding to the symmetric space \( SU(2)/SO(2) \) is a circle and \( \mu \) is just the height function on that circle; hence almost all preimages consist of two points. Our main result is (see section 2 for the definition of multiplicities):

**Theorem 1.2.** Let \( M \subset \mathbb{R}^{n+k} \) be an isoparametric submanifold with all multiplicities greater or equal to 2, and let \( a \in \nu_q(M) \) be in the image of \( \mu \). The following statements are true.

(i) The function \( f : M \to \mathbb{R}, f(x) = \| \mu(x) - a \|^2 \), is minimally degenerate in the sense defined by Kirwan [K] (see section 3 of our paper).

(ii) The level set \( \mu^{-1}(a) \) is connected or empty.

We note that in the case of the real flag manifold arising from \( SU(2)/SO(2) \) mentioned above, the (only) multiplicity is equal to 1. So the hypothesis on multiplicities in the previous theorem is essential.

**Remarks.**

(a) In fact, Terng [T1] proved that if \( M_\xi \) is a manifold parallel to \( M \) (see section 2), then the image of \( P|_{M_\xi} : M_\xi \to \nu_q(M) \) is a convex polytope. With the methods of our paper one can prove that any preimage of the latter map is connected as well.

(b) Here is a list of examples of isoparametric submanifolds with all multiplicities at least equal to 2, together with the corresponding parallel manifolds (see the previous remark):

- adjoint orbits of compact Lie groups
- isotropy orbits of the symmetric spaces \( SU(2n)/Sp(n), E_6/F_4, SU(m+n)/S(U(m) \times U(n)) \), where \( m > n, Sp(m+n)/Sp(m) \times Sp(n) \) (for more details, see Helgason [H])
- infinitely many of the Ferus-Karcher-Münzner [FKM] examples.

(c) The idea of the proof of Theorem 1.2 goes back to Kirwan [K]. We consider the Morse stratification of \( M \) induced by \( f \). The strata are smooth submanifolds, and all but the minimum one have codimension at least 2, due to the hypothesis on multiplicities. Hence the minimum stratum is connected. Finally, we use the fact, also proved by Kirwan [K], that the latter stratum has the same Čech cohomology module \( H^0(\cdot, \mathbb{Q}) \) as the minimum set \( f^{-1}(0) = \mu^{-1}(a) \).

(d) Terng [T2] was able to extend Theorem 1.1 to the case when \( M \) is an infinite dimensional isoparametric submanifold in a Hilbert space, or any of its parallel manifolds, like in Remark (a). We conjecture that also our Theorem 1.2 can be extended to the infinite dimensional situation. The connectivity theorem proved recently by Harada, Holm, Jeffrey, and the author in [HHJM] shows that the conjecture is true for the loop group \( \Omega(G) \), which is a parallel manifold of a certain infinite dimensional isoparametric submanifold with all multiplicities equal to 2.

**Acknowledgement.** I would like to thank Jost Eschenburg for discussions and important hints concerning the topics of the paper, especially those in the last section.
2. Isoparametric submanifolds in Euclidean space

We present some basic facts concerning the theory of isoparametric submanifolds. For more details, the reader can consult Palais and Terng’s book [PT] (especially chapter 6) and the references therein.

Let $M \subset \mathbb{R}^{n+k}$ be an $n$-dimensional embedded submanifold, which is closed, complete with respect to the induced metric, and full (i.e. not contained in any affine subspace). We say that $M$ is isoparametric if any normal vector at a point of $M$ can be extended to a parallel normal vector field $\xi$ on $M$ with the property that the eigenvalues of the shape operators $A_{\xi(p)}$ (i.e. the principal curvatures) are independent on $p \in M$, as values and multiplicities. It follows that for $p \in M$, the set $\{A_{\xi(p)} : \xi(p) \in \nu(M)_p\}$ is a commutative family of selfadjoint endomorphisms of $T_p(M)$, and so it determines a decomposition of $T_p(M)$ as a direct sum of common eigenspaces $E_1(p), E_2(p), \ldots, E_r(p)$. There exist normal vectors $\eta_1(p), \eta_2(p), \ldots, \eta_r(p)$ such that

$$A_{\xi(p)}|_{E_i(p)} = \langle \xi(p), \eta_i(p) \rangle \text{id}_{E_i(p)},$$

for all $\xi(p) \in \nu_p(M)$, $1 \leq i \leq r$. By parallel extension in the normal bundle we obtain the vector fields $\eta_1, \ldots, \eta_r$, which are the principal curvature vectors. The eigenspaces from above give rise to the distributions $E_1, \ldots, E_r$ on $M$, which are called the curvature distributions. The numbers

$$m_i = \text{rank}E_i, \quad 1 \leq i \leq r,$$

are the multiplicities of $M$.

To any parallel normal vector field $\xi$ on $M$ we assign the end-point map $\pi_\xi : M \to \mathbb{R}^{n+k}$,

$$\pi_\xi(p) = p + \xi(p),$$

$p \in M$. Its image is the “parallel” manifold $M_\xi$, which is also an embedded submanifold of $\mathbb{R}^{n+k}$. Now the differential map of $\pi_\xi$ is

$$d(\pi_\xi)_p = I - A_{\xi(p)}.$$

So the focal points of $M$ in the affine normal space $p + \nu_p(M)$ are those which are in one of the hyperplanes

$$\ell_i(p) := \{p + \xi(p) : \langle \eta_i(p), \xi(p) \rangle = 1\},$$

for some $i \in \{1, 2, \ldots, r\}$. It turns out that $\ell_1(p), \ldots, \ell_r(p)$ have a unique intersection point, call it $c_0$, independent on the point $p$. Moreover, $M$ is contained in a sphere centered at $c_0$ (here we use the assumption that $M$ is compact). We do not lose any generality if we assume that $M$ is contained in the unit sphere $S^{n+k-1}$; hence $c_0$ is just the origin $0$. One shows that the group of linear transformations of $\nu_p(M)$ generated by the reflections about $\ell_1(p), \ldots, \ell_r(p)$ is a Coxeter group, whose isomorphism type is independent on $p$. We denote it by $W$ and call it the Weyl group of $M$.

The map $\pi_\xi : M \to M_\xi$ is a submersion. If $p$ is a point in $M$ and $b := \pi_\xi(p) = p + \xi(p)$, we denote by $S_{p, b}$ the connected component of $\pi_\xi^{-1}(b)$ which contains $p$. This manifold is called the slice through $p$ corresponding to $\xi$. Consider

$$I := \{i \in \{1, 2, \ldots, r\} : p + \xi(p) \in \ell_i(p)\},$$
and also the subspace $\left(\bigcap_{i \in I} \ell_i(p)\right) \perp$ of $\nu_p(M)$, where the subscript $\perp$ indicates the orthogonal complement. An important result is the so-called slice theorem, which is stated as follows (for more details, see [PT, Theorem 6.5.9]).

**Theorem 2.1.** The slice $S_{p,b}$ is a (full) isoparametric submanifold of $\left(\bigcap_{i \in I} \ell_i(p)\right) \perp + \sum_{i \in I} E_i(p)$.

### 3. Minimally degenerate functions according to Kirwan

In this section we follow chapter 10 of Kirwan’s book [K].

**Definition.** A smooth function $f : X \to \mathbb{R}$ on a closed manifold $X$ is called **minimally degenerate** if the following conditions hold.

(a) The set of critical points of $f$ is a finite union of disjoint closed subsets $C_1, C_2, \ldots, C_N$, on each of which $f$ is constant. These subsets are called the **critical sets** of $f$.

(b) For every $j = 1, \ldots, N$, there exists a submanifold $Y_j$ of $X$, which contains $C_j$, with the property that the normal bundle$^1$ of $Y_j$ in $X$ is orientable, and such that
   
   (i) $C_j$ is the subset of $Y_j$ on which $f$ takes its minimum value
   
   (ii) for any $x \in C_j$, the space $T_x Y_j$ is maximal among the subspaces of $T_x X$ on which the Hessian $\text{Hess}_x(f)$ is positive semi-definite.

A manifold $Y_j$ with the properties (i) and (ii) is called a **minimizing manifold** around $C_j$. The codimension of $Y_j$ is called the **index** of $f$ along $C_j$.

Even though minimally degeneracy is a condition weaker than nondegeneracy in the sense of Bott, it is still sufficient to induce a Morse stratification of $M$, as the following theorem shows.

**Theorem 3.1.** (Kirwan [Ki]) Let $f : X \to \mathbb{R}$ be a minimally degenerate function like above and let $g$ be a Riemannian metric on $M$. For any $j \in \{1, 2, \ldots, N\}$ we denote by $\Sigma_j = \Sigma_j(g)$ the set of all points in $M$ with the property that the $\omega$-limit of the integral line through $x$ of the vector field $-\nabla(f)$ is contained in $C_j$. Then we have as follows.

(a) There exists a metric $g$ with the property that $\Sigma_j$ is a smooth submanifold of $M$ of codimension equal to the index along $C_j$. We also have

$$M = \bigcup_{1 \leq j \leq N} \Sigma_j, \quad \Sigma_i \cap \Sigma_j = \emptyset, \text{ for } i \neq j.$$ 

The intersection of $Y_j$ with a sufficiently small neighbourhood of $C_j$ is contained in $\Sigma_j$.

(b) For any $j \in \{1, 2, \ldots, N\}$, the inclusion map $C_j \hookrightarrow \Sigma_j$ induces an isomorphism in Čech cohomology.

### 4. Proof of Theorem 1.2

Let $M^n \subset \mathbb{R}^{n+k}$ be an isoparametric submanifold. We start with the following lemma.

$^1$More specifically, $(TX/TY_j)|_{Y_j}$
Lemma 4.1. Let \( q \) be a point of \( M \), \( b \in \nu_q(M) \), and let \( S_{q,b} \) be the corresponding slice. Consider

\[
I := \{ i \in \{1, 2, \ldots, r\} : b \in \ell_i(q) \}.
\]

If \( x \) is an arbitrary point in \( S_{q,b} \), then we have:

(a) \( S_{q,b} \) is a full isoparametric submanifold in \( (\bigcap_{i \in I} \ell_i(q))^\perp + \sum_{i \in I} E_i(q) \) whose normal space at \( q \) is \( (\bigcap_{i \in I} \ell_i(q))^\perp \)

(b) \( T_xS_{q,b} = \sum_{i \in I} E_i(x) \)

(c) \( \bigcap_{i \in I} \ell_i(x) + \sum_{i \in I} E_i(x) = \bigcap_{i \in I} \ell_i(q) + \sum_{i \in I} E_i(q) \)

(d) \( \bigcap_{i \in I} \ell_i(q) \subset \nu_x(M) \cap \nu_q(M) \)

(e) \( \sum_{j \notin I} E_j(x) \) is perpendicular to \( \bigcap_{i \in I} \ell_i(q) \).

Proof. Points (a)-(d) have been proved for instance in [PT, Chapter 6]. We will prove (e).

From (c) we deduce that \( \sum_{j \notin I} E_j(x) \) is perpendicular to \( \bigcap_{i \in I} \ell_i(q) \), and from (d), the same space is perpendicular to \( \bigcap_{i \in I} \ell_i(q) \).

We study the function \( f : M \to \mathbb{R} \), \( f(x) = \|\mu(x) - a\|^2 \). First we determine its critical points. We note that this has been done already by Terng in [T2, section 3]. Let us consider an orthonormal basis \( e_1, \ldots, e_k \) in \( \nu_q(M) \). We have

\[
\mu(x) - a = \sum_{i=1}^k \langle x - a, e_i \rangle e_i,
\]

hence

\[
f(x) = \sum_{i=1}^k \langle x - a, e_i \rangle^2,
\]

which implies

\[
df_x(v) = 2 \sum_{i=1}^k \langle x - a, e_i \rangle \langle v, e_i \rangle = 2 \langle v, \sum_{i=1}^k \langle x - a, e_i \rangle e_i \rangle = 2 \langle v, \mu(x) - a \rangle,
\]

for any \( v \in T_x(M) \). So \( x \) is a critical point of \( f \) if and only if the vector \( b := \mu(x) - a \in \nu_q(M) \) is perpendicular to \( T_x(M) \), in other words, when \( x \) is a critical point of the height function \( h_b : M \to \mathbb{R}, h_b(x) = \langle b, x \rangle \). We can express this in a more concise way as

\[
\text{Crit}(f) = \mu^{-1}(a) \cup \bigcup_{b \in \nu_q(M)} \mu^{-1}(a + b) \cap \text{Crit}(h_b).
\]

We prove that the intersection in the right hand side is nonempty only for finitely many \( b \in \nu_q(M) \). We use the fact that

\[
\text{Crit}(h_b) = \bigcup_{w \in W} S_{wq,b}
\]

(see e.g. [T2, subsection 2.9]).
Lemma 4.2. There exists finitely many \( b \in \nu_q(M) \setminus \{0\} \) with the property that \( \mu^{-1}(a+b) \cap S_{wq,b} \) is nonempty.

Proof. For simplicity we assume that \( w = 1 \). Pick \( I \) an arbitrary subset of \( \{1,2,\ldots,r\} \). In \( \nu_q(M) \) we consider the subspace

\[
\ell_I := \bigcap_{i \in I} \ell_i(q).
\]

We also consider the convex hull \( \text{cvx}[(W_I)_q] \), where \( W_I \) denotes the stabilizer if \( \ell_I \). The affine span of this convex body is perpendicular to \( \ell_I \) and its dimension is just the codimension of \( \ell_I \). To justify this, we note that this affine span is the affine normal space to a certain slice, which has codimension equal to \( \dim(\ell_I^\perp) \), and which is contained in an affine space perpendicular to \( \ell_I \) (see Lemma 4.1 (a)). Consequently the intersection \( (a + \ell_I) \cap \text{cvx}[(W_I)_q] \) has at most one point. By letting \( I \) vary among all subsets of \( \{1,2,\ldots,r\} \), we obtain a finite set of points, call it \( F \). Now, if \( \mu^{-1}(a+b) \cap S_{wq,b} \neq \phi \), then \( a+b \) must belong to \( \mu(S_{wq,b}) \), which is \( \text{cvx}[(W_b)_q] \). Consequently, we have

\[
a + b \in (a + \ell_I) \cap \text{cvx}[(W_I)_q],
\]

where \( I = \{ i \in \{1,2,\ldots,r\} : b \in \ell_i(q) \} \). This implies \( a+b \in F \). \( \square \)

We have proved that the set

\[
B := \{ b \in \nu_q(M) \setminus \{0\} : \mu^{-1}(a+b) \cap \text{Crit}(h_b) \neq \phi \}
\]

is finite. The description (2) can be refined by taking into account that for any \( b \in \nu_q(M) \) we have \( h_b(x) = \langle x,b \rangle = (\mu(x),b) \), \( x \in M \). Consequently, if \( x \in \mu^{-1}(a+b) \), then \( h_b(x) = \langle a+b,b \rangle \). So

\[
\text{Crit}(f) = \mu^{-1}(a) \cup \bigcup C_{b,w}
\]

where \( C_{b,w} := \mu^{-1}(a+b) \cap S_{wq,b} \) and the union runs over all \( b \in B \) and \( w \in W \) with the property that \( h_b(S_{wq,b}) = (a+b,b) \).

Fix \( b \) and \( w \) like above. Let \( Y_{b,w} \) denote the stable manifold of the function \( h_b \) corresponding to the critical set \( S_{wq,b} \). More specifically, this consists of all points \( x \in M \) with the property that the limit at \( \infty \) of the integral line through \( x \) of the vector field \( -\nabla(h_b) \) is in \( S_{wq,b} \). We will prove the following result.

Proposition 4.3. (a) A minimizing manifold for \( f \) around \( \mu^{-1}(a) \) is \( M \) itself.

(b) For \( b \in B \) and \( w \in W \), the space \( Y_{b,w} \) is a minimizing manifold for \( f \) around \( C_{b,w} \).

(c) The only critical set of index 0 is \( \mu^{-1}(a) \).

Proof. (b) First we show that the normal bundle to \( Y_{b,w} \) is orientable. To this end we note that \( Y_{b,w} \) is just a vector bundle over \( S_{wq,b} \), hence it is homeomorphic to the latter space. But \( S_{wq,b} \) is an isoparametric submanifold with all multiplicities at least 2, hence it is simply connected. Consequently, \( Y_{b,w} \) is also simply connected. This implies the desired conclusion (we recall that any vector bundle over a simply connected space is orientable).
Next we note that if \( x \in Y_{b,w} \), then \( h_b(x) \geq h_b(S_{wq,b}) = \langle a + b, b \rangle \). So we have \( \langle \mu(x), b \rangle \geq \langle a + b, b \rangle \), which implies

\[
\langle \mu(x) - a, b \rangle \geq \langle b, b \rangle.
\]

We deduce

\[
\| \mu(x) - a \| \cdot \| b \| \geq \langle \mu(x) - a, b \rangle \geq \| b \|^2,
\]

hence

\[
f(x) \geq f(C_{b,w}).
\]

Moreover, if \( x \in Y_{b,w} \) has the property that \( f(x) = f(C_{b,w}) \), then we must have

\[
\begin{align*}
&\bullet h_b(x) = h_b(S_{wq,b}) \text{ (from equation (1))}, \text{ hence } x \in S_{wq,b} \\
&\bullet \mu(x) - a = \lambda b, \text{ for a number } \lambda \text{ (from equation (1)); we deduce that } \lambda = 1, \text{ because } \\
&\langle \mu(x) - a, b \rangle = \langle b, b \rangle.
\end{align*}
\]

Consequently, \( x \in C_{b,w} \). We have proved that the condition (b) (i) from the definition of a minimally degenerate function is satisfied.

It remains to check condition (b) (ii). Let us consider a point \( x_0 \) in \( C_{b,w} \). We construct a subspace \( V \subset T_{x_0}(M) \) with the following properties.

1. \( V \oplus T_{x_0}(Y_{b,w}) = T_{x_0}(M) \)
2. \( \text{Hess}_{x_0}(f)|_V \text{ is negative definite.} \)

First we determine the Hessians of \( h_b \) and \( f \) at the point \( x_0 \). To this end we consider the functions \( H_b \) and \( F \) given by

\[
H_b(x) = \langle x, b \rangle, \quad F(x) = \| P(x) - a \|^2, \quad x \in \mathbb{R}^{n+k}
\]

where \( P \) denotes the orthogonal projection \( \mathbb{R}^{n+k} \to \nu_q(M) \). We know that for \( v, w \in T_{x_0}(M) \) we have

\[
\text{Hess}(f)_{x_0}(v, w) := \langle \partial_v(\nabla f)(x_0), w \rangle = \langle \partial_v(\nabla F)(x_0), w \rangle + \langle A(\nabla F)_{x_0} v, w \rangle,
\]

where \( \nabla \) stands for gradient and the superscript \( \perp \) indicates the orthogonal projection on \( \nu_{x_0}(M) \). Because \( (\nabla F)_x = 2(P(x) - a) \) (see equation (1)), and \( P(x_0) - a = b \), we deduce that

\[
(5) \quad \text{Hess}(f)_{x_0} = 2(P + A_b).
\]

Similarly, the Hessian of \( h_b \) is

\[
(6) \quad \text{Hess}(h_b)_{x_0} = A_b.
\]

The tangent space \( T_{x_0}(Y_{b,w}) \) is the subspace of \( T_{x_0}(M) \) where the hessian \( \text{Hess}(h_b)_{x_0} \) is negative semidefinite. From equation (6) we deduce that for \( i \in \{1, 2, \ldots, r\} \), the restriction of \( \text{Hess}(h_b)_{x_0} \) to \( E_i(x_0) \) is given by scalar multiplication by \( \langle b, \eta_i(x_0) \rangle \). Consequently we have

\[
T_{x_0}(Y_{b,w}) = \sum E_i(x_0)
\]

where the sum runs over all \( i \) with \( \langle b, \eta_i(x_0) \rangle \geq 0 \). From elementary Morse theoretical considerations, we know that

\[
T_{x_0}(S_{wq,b}) \subset T_{x_0}(Y_{b,w}).
\]
Set
\begin{equation}
V := \sum E_i(x_0),
\end{equation}
where the sum runs over all \(i\) with \(\langle b, \eta_i(x_0) \rangle < 0\). By Lemma 4.1 (e) (with \(q\) replaced by \(wq\)), \(P\) maps \(V\) to 0. The reason is that \(b\in \ell_i(x_0)\) exactly when \(\langle b, \eta_i(x_0) \rangle = 0\) (since \(\ell_i(x_0)\) goes through the origin). By (5) and (6), the restrictions of \(\text{Hess}(f)_{x_0}\) and \(\text{Hess}(h_b)_{x_0}\) to \(V\) are the same, up to a factor of 2. Since the latter restriction is strictly negative definite, the former is like that as well.

(c) Suppose that for \(b \in B\) and \(w \in W\) the critical set \(C_{wq,b}\) has index 0. The considerations from above show that the index of \(h_b\) at \(S_{wq,b}\) is 0. This implies that the slice \(S_{wq,b}\) is the minimum set of \(h_b\) on \(M\). Consequently, for any \(x \in M\) we must have \(h_b(x) \geq h_b(S_{wq,b})\). Like in the proof of point (b), this implies \(f(x) \geq f(C_{b,w}) = \|b\|^2\), for all \(x \in M\). This is a contradiction (because \(f\) can reach the value 0), which concludes the proof. \(\square\)

Remark. The arguments used above are approximately the same as those used by Kirwan in the proof of [K, Proposition 4.15] (in the context of Hamiltonian torus actions on symplectic manifolds).

Proof of Theorem 1.2
Only point (ii) has to be proved. By Theorem 3.1, there exists a metric \(g\) on \(M\) which induces the stratification
\[ M = \Sigma_0 \cup \bigcup_{b \in B, w \in W} \Sigma_{b,w}, \]
such that \(\Sigma_0, \Sigma_{b,w}\) are smooth submanifolds. By Proposition 4.3 and equation (7), the manifolds \(\Sigma_{b,w}\) have codimension at least 2 (recall that, by hypothesis, we have \(\dim(E_i(x_0)) = m_i \geq 2\)). Consequently, \(\Sigma_0\) is connected. Since the map \(\mu^{-1}(a) \hookrightarrow \Sigma_0\) induces the linear isomorphism \(H^0(\mu^{-1}(a), \mathbb{Q}) \cong H^0(\Sigma_0, \mathbb{Q})\) (see Theorem 4.1), we deduce that \(\mu^{-1}(a)\) is connected as well. \(\square\)

5. Kirwan surjectivity for real flag manifolds

An important class of isoparametric submanifolds are the real flag manifolds. More specifically, we start with a non-compact symmetric space \(G/K\), where \(G\) is a non-compact connected semisimple Lie group and \(K \subset G\) a maximal compact subgroup. Then \(K\) is the fixed point set of an automorphism \(\tau\) of \(G\) (see for instance [H, chapter VI]). The differential map \(d(\tau)_e\) is an automorphism of \(g = \text{Lie}(G)\) and induces the Cartan decomposition
\[ g = \mathfrak{k} \oplus \mathfrak{p}, \]
where \(\mathfrak{k} = \text{Lie}(K)\) and \(\mathfrak{p}\) are the (+1)-, respectively (−1)-eigenspaces of \((d\tau)_e\).

Now let us consider \(a \subset \mathfrak{p}\) a maximal abelian subspace. The number \(k := \dim(a)\) is the rank of the symmetric space \(G/K\). The roots of the symmetric space are linear functions \(\alpha : a \to \mathbb{R}\) with the property that the space
\[ \mathfrak{g}_\alpha := \{ z \in \mathfrak{g} : [x, z] = \alpha(x)z \text{ for all } x \in a\} \]
is non-zero. The set $\Pi$ of all roots is a root system in $(\mathfrak{a}^*, \langle , \rangle)$. Let $\Pi^+ \subset \Pi$ be the set of positive roots with respect to a simple root system. For any $\alpha \in \Pi^+$ we have

$$g_\alpha + g_{-\alpha} = \mathfrak{f}_\alpha + \mathfrak{p}_\alpha,$$

where $\mathfrak{f}_\alpha = (g_\alpha + g_{-\alpha}) \cap \mathfrak{f}$ and $\mathfrak{p}_\alpha = (g_\alpha + g_{-\alpha}) \cap \mathfrak{p}$. We have the direct decompositions

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Pi^+} \mathfrak{p}_\alpha, \quad \mathfrak{f} = \mathfrak{f}_0 + \sum_{\alpha \in \Pi^+} \mathfrak{f}_\alpha,$$

where $\mathfrak{f}_0$ denotes the commutator of $\mathfrak{a}$ in $\mathfrak{k}$.

Since $[\mathfrak{f}, \mathfrak{p}] \subset \mathfrak{p}$, the space $\mathfrak{p}$ is $Ad(G) := Ad(K)$-invariant. The orbits of the action of $Ad(K)$ on $\mathfrak{p}$ are called generalized real flag manifolds. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ is an $Ad(G)$-invariant inner product on $\mathfrak{p}$, which we denote by $\langle , \rangle$.

**Proposition 5.1.** (see e.g. [PT, Example 6.5.6]) Let $M = Ad(K).q$ be the orbit of $q \in \mathfrak{a}$.

(i) If $q$ is regular (i.e. not contained in any of the hyperplanes $\ker(\alpha), \alpha \in \Pi$), then $M$ is an isoparametric submanifold of $(\mathfrak{p}, \langle , \rangle)$. The curvature distributions at $q$ are

$$E_\alpha(q) = [q, \mathfrak{f}_\alpha + \mathfrak{f}_{2\alpha}],$$

where $\alpha$ is a positive indivisible root. Hence the multiplicities of $M$ are

$$m_\alpha = \dim(\mathfrak{f}_\alpha) + \dim(\mathfrak{f}_{2\alpha}).$$

The normal space to $M$ at the point $q$ is

$$\nu_q(M) = \mathfrak{a}.$$

(ii) If $q$ is not regular, then $M$ is a manifold parallel to an isoparametric submanifold.

In [M] we have investigated the action of

$$K_0 := Z_K(\mathfrak{a}) = \{ k \in K : Ad(k)(x) = x, \text{ for all } x \in \mathfrak{a} \}$$

on $M$. It turns out that if all $m_\alpha$ are strictly greater than 1, then $K_0$ is connected and the action of $K_0$ on $M$ is equivariantly formal. In this paper we will address the following question.

**Problem.** If $\mu : M \to \mathfrak{a}$ is the restriction to $M$ of the orthogonal projection map $P : \mathfrak{p} \to \mathfrak{a}$ and $a$ is an arbitrary point in $\mathfrak{a}$, is it true that the Kirwan type map

$$\kappa : H^*_{K_0}(M, \mathbb{Q}) \to H^*_{K_0}(\mu^{-1}(a), \mathbb{Q})$$

is surjective?

We will prove that the answer to this question is affirmative under certain restrictions.

**Proposition 5.2.** (Surjectivity criterium) Assume that all multiplicities $m_\alpha$ are strictly greater than 1 and for any $b \in \mathfrak{a}$, the set

$$Z_b(b) := \{ x \in \mathfrak{f} : [x, b] = 0 \} = \mathfrak{f}_0 + \sum_{\alpha(b)=0} \mathfrak{f}_\alpha$$

is the fixed point set of a certain torus $T_b \subset K_0$. Then the map $\kappa$ described by equation (8) is surjective, for any $a \in \mathfrak{a}$.
Proof. We consider the function $f : M \to \mathbb{R}$, $f(x) = \|\mu(x) - a\|^2$. By Theorem 1.2 (i), this is a minimally degenerate function. Moreover, $f$ is $K_0$-invariant. Let $C$ be a critical set of $f$ (by equation (3)), $C$ can be $\mu^{-1}(a)$ or $C_{b,w})$. Denote $M^\pm = f^{-1}((-\infty, f(C) \pm \epsilon))$, for $\epsilon > 0$ sufficiently small. By [K, chapter 10] (see also [BTW, section 9]) we have the commutative diagram

$$
\cdots \to H^*_K(M_+, M_-) \to H^*_K(M_+) \to H^*_K(M_-) \to \cdots
$$

(9)

$$
\xymatrix{ \cdots \ar[r] & H^*_K(M_+, M_-) \ar[r] \ar[d]^{\simeq} & H^*_K(M_+) \ar[r] \ar[d] & H^*_K(M_-) \ar[r] & \cdots \\
H^*_{K_0}(\text{index}(C)) \ar[r] \ar[r] & H^*_{K_0}(C)
}
$$

where $e_C \in H^*_{K_0}(C)$ denotes the equivariant Euler class of the normal bundle $\nu(\Sigma_C)|_C$.

We will prove that $e_C$ is not a divisor of zero. If $C = \mu^{-1}(a)$, then $e_C = 1$ and the claim is obvious. Let us consider the case when $C = C_{b,w} = \mu^{-1}(a + b) \cap S_{wq,b}$ (see equation (3)). According to a criterium of Atiyah and Bott (see [AB, Proposition 13.4]), it is sufficient to prove that there exists a torus $T \subset K_0$ with the property that the only points in $\nu(\Sigma_C)|_C$ which are fixed by $T$ are those from $C$. But $\nu(\Sigma_C)|_C$ is contained in $\nu(S_{wq,b})|_C$ (because $\Sigma_C$ contains $Y_{b,w}$ on a neighbourhood of $C$, and $\dim \Sigma_C = \dim Y_{b,w}$, see Theorem 3.1). We will show that the fixed points of the torus $T_b$ (see the statement of the proposition) on $\nu(S_{wq,b})$ are exactly those from $S_{wq,b}$. First we note that the fixed point set of $T_b$ on $M$ is

$$M \cap Z_p(b) = \text{Crit}(h_b).$$

This is because if $\alpha$ is a positive root, then the space $p_\alpha$ is fixed by $T_b$ exactly when $t_\alpha$ is fixed by $T_b$, as $t_\alpha = [x, p_\alpha]$, for $x \in a$ regular. But as we already mentioned in section 4, $\text{Crit}(h_b)$ is the disjoint union of all slices $S_{wq,b}$, where $w \in W$, and the conclusion follows.

We will give an example where the criterium applies.

Example 1. We consider the symmetric space $SU(2n)/Sp(n)$. For details concerning this, the reader can consult for instance [H, chapter X, section 2]. We will confine ourselves to saying that the symplectic group $Sp(n)$ consists of all $n \times n$ nonsingular matrices with coefficients in $H = \{a + ib + jc + kd : a, b, c, d \in \mathbb{R}\}$ which preserve the canonical symplectic product on $\mathbb{R}^n$. There is a canonical embedding of this group in $SU(2n)$, as the fixed point set of a certain involutive automorphism of the group $SU(2n)$. In this case, we can choose $a$ to be the space of all matrices of type

$$a := \begin{pmatrix} iD & 0 \\ 0 & iD \end{pmatrix},$$

where $i = \sqrt{-1}$ and

$$D = \begin{pmatrix} a_1 & 0 & \ldots & 0 & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix},$$

with $a_1, \ldots, a_n \in \mathbb{R}$, $a_1 + a_2 + \ldots + a_n = 0$. The positive roots are $\alpha_{rs} : a \to \mathbb{R}$,

$$\alpha_{rs}(a) = a_r - a_s,$$
where \( r < s \). This is an irreducible root system of type \( A_{n-1} \). We have

\[
\mathfrak{k}_{\alpha_\mathfrak{r}\mathfrak{s}} = \{ \gamma E_{\mathfrak{r}\mathfrak{s}} - \bar{\gamma} E_{\mathfrak{s}\mathfrak{r}} : \gamma \in \mathbb{H} \},
\]

where \( E_{\mathfrak{r}\mathfrak{s}} \) denotes the \( n \times n \) matrix whose entries are all 0, except the one on line \( r \) and column \( s \), which is 1. In particular, all multiplicities are equal to 4. The space \( K_0 \) is just \( Sp(1)^n \).

We can describe explicitly the action of an arbitrary element \( h = (h_1, \ldots, h_n) \in Sp(1)^n \) on \( \mathfrak{k}_{\alpha_\mathfrak{r}\mathfrak{s}} \), as follows:

\[
(10) \quad h.(\gamma E_{\mathfrak{r}\mathfrak{s}} - \bar{\gamma} E_{\mathfrak{s}\mathfrak{r}}) = h^r \gamma h^s E_{\mathfrak{r}\mathfrak{s}} - h^s \bar{\gamma} h^r E_{\mathfrak{s}\mathfrak{r}}.
\]

We show that the fixed point set of the group \( Sp(1)_{\mathfrak{r}\mathfrak{s}} := \{ (h_1, \ldots, h_{r-1}, 1, h_{r+1}, \ldots, h_{s-1}, 1, h_{s+1}, \ldots, h_n) : h_p \in \mathbb{C}, |h_p| = 1 \} \subset K_0 \) on \( \mathfrak{k} \) is \( \mathfrak{k}_0 + \mathfrak{k}_{\alpha_\mathfrak{r}\mathfrak{s}} \). This follows from equation (10) and the following technical claim.

Claim. (a) If \( \gamma \in \mathbb{H} \) such that \( h_1 \gamma h_2 = \gamma \) for any \( h_1, h_2 \in \mathbb{H} \) of length 1, then \( \gamma = 0 \).

(b) The same is true if \( h_1, h_2 \) are of the form \( a + ib \), where \( a, b \in \mathbb{R}, a^2 + b^2 = 1 \).

The proof of the claim is straightforward. In fact, point (b) shows that \( \mathfrak{k}_0 + \mathfrak{k}_{\alpha_\mathfrak{r}\mathfrak{s}} \) is the fixed point set of the torus

\[
T_{\mathfrak{r}\mathfrak{s}} := \{ (h_1, \ldots, h_{r-1}, 1, h_{r+1}, \ldots, h_{s-1}, 1, h_{s+1}, \ldots, h_n) : h_p \in \mathbb{C}, |h_p| = 1 \} \subset K_0
\]

In a similar way, we can prove that the hypothesis of Proposition 5.2 is satisfied for any \( b \in \mathfrak{a} \).

Example 2. Consider the symmetric space \( \mathbb{C}P^2 = SU(3)/U(2) \), where \( U(2) \) is embedded in \( SU(3) \) via

\[
A \mapsto \left( \frac{1}{\det(A)} 0 0 \\
0 0 \begin{array}{c} z_1 \\
z_2 \end{array} \right),
\]

\( A \in U(2) \). It turns out that \( \mathfrak{p} \) consists of all matrices of the type

\[
\begin{pmatrix} 0 & -\bar{z_1} & -\bar{z_2} \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix},
\]

where \( z_1, z_2 \in \mathbb{C} \). We choose \( \mathfrak{a} \) the space of all matrices

\[
\mathfrak{a} := \begin{pmatrix} 0 & -x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

with \( x \in \mathbb{R} \). The positive roots are \( \alpha \) and \( 2\alpha \), where \( \alpha(\mathfrak{a}) = -x \) and the corresponding root spaces in \( \mathfrak{p} \) are

\[
\mathfrak{p}_\alpha = \left\{ \begin{pmatrix} 0 & 0 & -\bar{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\}, \quad \mathfrak{p}_{2\alpha} = \left\{ \begin{pmatrix} 0 & iy & 0 \\ iy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : y \in \mathbb{R} \right\}.
\]
We deduce that $m_α = 2 + 1 = 3$. An easy calculation shows that $K_0 := Z_K(α)$ is the subgroup of $U(2)$ which consists of
\[
\begin{pmatrix}
  z & 0 & 0 \\
  0 & z & 0 \\
  0 & 0 & \frac{1}{z^2}
\end{pmatrix}
\]
for $z \in \mathbb{C} \setminus \{0\}$. One can see that $K_0$ acts trivially not only on $α$, but also on $p_{2α}$. We deduce that it acts trivially on $f_{2α}$ as well. This implies that there is no subgroup of $K_0$ whose fixed point set in $f$ is just $f_0$. So the hypothesis of Proposition 5.2 is not satisfied.

References

[A] M. F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. 14 (1982), 1–15
[AB] M. Atiyah, R. Bott, The Yang-Mills functional over a Riemann surface, Phil. Trans. Roy. Soc. A308 (1982), 523–615
[D] J. J. Duistermaat Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution, Trans. Amer. Math. Soc. 275 (1) 1982, 417-429
[BTW] R. Bott, S. Tolman, and J. Weitsman, Surjectivity for Hamiltonian loop group spaces, Invent. Math. 155 (2004), 225-251
[FKM] D. Ferus, H. Karcher, and H.F. Münzner, Cliffordalgebren und neue isoparametrische Hyperflächen, Math. Z. 177 (1981), 479-502
[HHJM] M. Harada, T. Holm, L. Jeffrey, and A.-L. Mare, Connectivity properties of moment maps on based loop groups, preprint [math. sg/0503684]
[H] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, 1978
[K] F. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Princeton University Press 1984
[M] A.-L. Mare, Equivariant cohomology of real flag manifolds, Differential Geom. Appl., to appear, preprint [math. dg/0404369]
[PT] R. Palais and C.-L. Terng, Critical Point Theory and Submanifold Geometry, Lecture Notes in Mathematics 1353, Springer Verlag 1988
[T1] C.-L. Terng, Convexity theorems for isoparametric submanifolds, Invent. Math. 85 (1986), 487-492
[T2] C.-L. Terng, Convexity theorems for infinite dimensional isoparametric submanifolds, Invent. Math. 112 (1993), 9-22