Stochastic Stackelberg games

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Abstract

In this paper, we consider a discrete-time stochastic Stackelberg game where there is a defender (also called leader) who has to defend a target and an attacker (also called follower). Both attacker and defender have conditionally independent private types, conditioned on action and previous state, that evolve as controlled Markov processes. The objective is to compute the stochastic Stackelberg equilibrium of the game where defender commits to a strategy. The attacker’s strategy is the best response to the defender strategy and defender’s strategy is optimum given the attacker plays the best response. In general, computing such equilibrium involves solving a fixed-point equation for the whole game. In this paper, we present an algorithm that computes such strategies by solving smaller fixed-point equations for each time $t$. This reduces the computational complexity of the problem from double exponential in time to linear in time. Based on this algorithm, we compute stochastic Stackelberg equilibrium of a security example.

I. INTRODUCTION

In the past decade, Stackelberg games have been used extensively in the security of real world systems such as to protect ports, airports and wildlife [2], [3], [8], [11]. A Bayesian Stackelberg game is played between two players: a defender and an attacker. The attacker has a private type that only she observes, however, the defender knows the prior on that state. The defender commits to a strategy that is observable to the attacker. The attacker then plays a best response to attacker’s strategy to maximize its utility. Knowing that the attacker will play a best response, the defender commits to and plays a strategy that maximizes its utility. Such pair of strategies is called a Stackelberg equilibrium. It is known that such strategies can provide higher utility to the defender than obtained in a Nash equilibrium of the game. Such games have been used in the real world by security agencies such as the US Coast Guard, the Federal Air Marshals Service, Department of Electrical and Computer Engineering, University of Texas, Austin (dvasal@utexas.edu, https://sites.google.com/view/dvasal/home).
and the Los Angeles Airport Police [12]. Similar algorithms are used in wildlife protection in Uganda and Malaysia [16].

Most of the above real world applications of Stackelberg equilibrium are based on single-shot Bayesian game models. However, in many practical scenarios, the attacker and defender interact periodically, thus reducing the applicability of such models. Solving a dynamic stochastic Stackelberg game when both the attacker and the defender have a private Markovian states is computationally challenging. This is because unlike other games, in such dynamic games of asymmetric information, there is coupling of players’ strategies across time. Since strategy of a player is a map from each history of the game which grows exponentially with time, the space of strategies of the players is double exponential in, rendering such problems intractable. Recently, there has been results on sequential decomposition of certain classes of games of asymmetric information [13]–[15]. In repeated Stackelberg security games, there have been other approaches to mitigate this issue. For instance Kar et al in [7] consider a repeated Stackelberg game and use a new human behavior model to study such games. Mareki et.al. in [9] study a Bayesian repeated Stackelberg game where they assume defenders are myopic, thus significantly simplifying the analysis of finding the equilibrium. Balcan et al in [1] consider a learning theoretic approach to study a repeated Stackelberg game between attacker and defender where they use regret analysis to learn attacker’s types, and show sub-linear regret for both complete and partial information models.

In this paper, we provide a dynamic programming like sequential decomposition algorithm for stochastic dynamic Stackelberg games to compute equilibria with fully rational attacker and defender. Our algorithm consists of a backward recursive step which, for each time t, involves solving a fixed-point equation for the attacker and an optimization problem for the defender. This reduces the complexity of finding Markovian equilibria of such games from double exponential to linear in time. Based on this algorithm, we study a security game where we numerically find its Stackelberg equilibria.

The paper is structured as follows. We present our model in Section II. We discuss background material and solution concept in Section III. In Section IV, we present our main result of providing an algorithm to compute Markovian equilibrium strategies. In Section V, we discuss an infinite horizon version of the problem. In Section VI, we discuss the complexity of the proposed algorithm. In Section VII, we present a numerical example. We conclude in Section VII. All proofs are presented in the Appendices.
A. Notation

We use uppercase letters for random variables and lowercase for their realizations. For any variable, subscripts represent time indices and superscripts represent player indices. We use notation \( A_{t,t'} \) to represent the vector \( (A_t, A_{t+1}, \ldots A_{t'}) \) when \( t' \geq t \) or an empty vector if \( t' < t \).

We remove superscripts or subscripts if we want to represent the whole vector, for example \( A_t \) represents \( (A_1, \ldots, A_N) \). In a similar vein, for any collection of sets \( (X^i)_{i \in N} \), we denote \( \times_{i \in N} X^i \) by \( X \). For any finite set \( S \), \( \Delta(S) \) represents the space of probability measures on \( S \) and \( |S| \) represents its cardinality. We denote by \( P^g \) (or \( E^g \)) the probability measure generated by \( g \). We denote the set of real numbers by \( \mathbb{R} \). For a probabilistic strategy profile of players \( (\sigma^i)_{i \in N} \) where the probability of action \( a^i_t \) conditioned on \( a_{1:t-1}, x^i_{1:t} \) is given by \( \sigma^i (a^i_t | a_{1:t-1}, x^i_{1:t}) \), we use the notation \( \sigma^i_t (a^i_t | a_{1:t-1}, x^i_{1:t}) \) to represent \( \prod_{j \neq i} \sigma^j (a^j_t | a_{1:t-1}, x^j_{1:t}) \). All equalities/inequalities involving random variables are to be interpreted in the \( a.s. \) sense. For mappings with range function sets \( f : A \to (B \to C) \) we use square brackets \( f[a] \in B \to C \) to denote the image of \( a \in A \) through \( f \) and parentheses \( f[a](b) \in C \) to denote the image of \( b \in B \) through \( f[a] \). A controlled Markov process with state \( X_t \), action \( A_t \), and horizon \( [T] \) is denoted by \( (X_t, A_t)_{t \in [T]} \).

II. Model

We consider a stochastic Stackelberg game over a time horizon \( [T] \triangleq \{1, 2, \ldots, T\} \) with perfect recall as follows. Suppose there are two kinds of players: a leader and a follower. Both the leader and the follower have private types, \( x^l_t \in \mathcal{X}^l, x^f_t \in \mathcal{X}^f \), respectively, at time \( t \), where \( x^l_t, x^f_t \) evolve as a conditionally independent controlled Markov processes in the following way,

\[
P(x^l_t, x^f_t | a_{1:t-1}, x^l_{1:t-1}) = Q(x^l_t | a_{t-1}, x^l_{t-1}) Q(x^f_t | a_{t-1}, x^l_{t-1}), \tag{1a}
\]

where \( a_t = (a^l_t, a^f_t), x_t = (x^l_t, x^f_t) \) and \( Q \) are known kernels. Leader takes action \( a^l_t \in \mathcal{A}^l \) at time \( t \) on observing \( a_{1:t-1}, x^l_{1:t} \), and the follower takes action \( a^f_t \in \mathcal{A}^f \) at time \( t \) on observing \( a_{1:t-1} \) and \( x^l_{1:t} \), where \( a_{1:t-1} \) is the common information among players, and \( x^l_{1:t} (x^f_{1:t-1}) \) is the private information of the leader (and the follower, respectively). The sets \( \mathcal{A}^l, \mathcal{A}^f, \mathcal{X}^l, \mathcal{X}^f \) are assumed to be finite. Let \( \sigma^i = (\sigma^i_t)_{t \in [T]} \) be a probabilistic strategy of player \( i \in \{l, f\} \) where \( \sigma^i_t : \mathcal{A}^{i-1} \times (\mathcal{X}^i)^t \to \mathcal{P}(\mathcal{A}^i) \) such that player \( i \) plays action \( A^i_t \) according to \( A^i_t \sim \sigma^i_t (\cdot | a_{1:t-1}, x^l_{1:t}) \).

Let \( \sigma \triangleq (\sigma^i)_{i \in \{l,f\}} \) be a strategy profile of all players. At the end of interval \( t \), the leader receives
an instantaneous reward $R_l^t(x_t, a_t)$ and the follower receives an instantaneous reward $R_f^t(x_t, a_t)$.

Suppose players discount their rewards by a discount factor $\delta \leq 1$.

III. PRELIMINARIES

In this section, we present some preliminaries.

A. Stackelberg Equilibrium

The Stackelberg equilibrium is defined for a game as follows. For a given strategy profile of the leader, $\sigma^l$, the follower maximizes its total discounted expected utility over finite horizon $T$,

$$\max_{\sigma^f} \mathbb{E}^{\sigma^l, \sigma^f} \left\{ \sum_{t=1}^{T} \delta^{t-1} R_f^t(X_t, A_t) \right\}. \quad (2)$$

Let $BR_f^l(\sigma^l)$ be the set of optimizing strategies of the follower given a strategy $\sigma^l$ of the leader, i.e.

$$BR_f^l(\sigma^l) = \arg \max_{\sigma^f} \mathbb{E}^{\sigma^l, \sigma^f} \left\{ \sum_{t=1}^{T} \delta^{t-1} R_f^t(X_t, A_t) \right\} \quad (3)$$

The leader finds its optimal strategy that maximizes its total expected discounted reward given that the follower will use its best response to it,

$$\tilde{\sigma}^l \in \max_{\sigma^l} \mathbb{E}^{\sigma^l, BR_f^l(\sigma^l)} \left\{ \sum_{t=1}^{T} \delta^{t-1} R_l^t(X_t, A_t) \right\}, \quad (4)$$

Then $(\tilde{\sigma}^l, \tilde{\sigma}^f)$ constitute a Stackelberg equilibrium where $\tilde{\sigma}^f \in BR_f^l(\tilde{\sigma}^l)$.

B. Perfect Stackelberg equilibrium

In this paper, we will consider both players’ equilibrium policies that only depend on their current states $x_t^l, x_t^f$ and action history, i.e. at equilibrium, $a_t^i \sim \tilde{\sigma}_i^l(\cdot|a_{1:t-1}, x_t^i), i \in \{l, f\}$.\footnote{Note, however, that for the purpose of equilibrium, the optimization will be performed in the space of all possible strategies that may depend on the entire history of state.}

For the game considered, we introduce a notion of Perfect Stackelberg Equilibrium (PSE), inspired by perfect Bayesian equilibrium [6] as follows.

Let $(\tilde{\sigma}, \mu)$ be a PSE of the game, where $\mu = (\mu_t)_{t \in [T]}$, and for any $t, a_{1:t-1}$, $\mu_t[a_{1:t-1}] = (\mu^l_t[a_{1:t-1}], \mu^f_t[a_{1:t-1}]) \in \mathcal{P}(\mathcal{X}^l) \times \mathcal{P}(\mathcal{X}^f)$ is the equilibrium belief on the current state $(x_t^l, x_t^f)$, given the action history $a_{1:t-1}$, i.e. $\mu_t^i[a_{1:t-1}](x_t^i) = P(x_t^i|a_{1:t-1}), i \in \{l, f\}$. Then for all $t \in [T]$,
C. Common agent approach

We recall that the leader and the follower generate their actions at time $t$ as follows, $a^l_t \sim \sigma^l_t(\cdot|a_{1:t-1}, x^l_{1:t})$ and $a^f_t \sim \sigma^f_t(\cdot|a_{1:t-1}, x^f_{1:t})$. An alternative way to view the problem is as follows. As is done in the common information approach [10], at time $t$, a fictitious common agent observes the common information $a_{1:t-1}$ and generates prescription functions $\gamma_t = (\gamma^l_t, \gamma^f_t) = \psi_t[a_{1:t-1}]$. Player $i$ uses its prescription function $\gamma^i_t$ to operate on its private information to produce its action $a^i_t$, i.e. $\gamma^i_t : (\mathcal{X}^i)^t \rightarrow \mathcal{P}(\mathcal{A}^i)$ and $a^i_t \sim \gamma^i_t(\cdot|x^i_{1:t})$. It is easy to see that for any $\sigma$ policy profile of the players, there exists an equivalent $\psi$ profile of the common agent (and vice versa) that generates the same control actions for every realization of the information of the players.

Here, we will consider Markovian common agent’s policy as follows. We call a common agent’s policy be of “type $\theta$” if the common agent observes the common belief $\pi_t$ derived from the common observation $a_{1:t-1}$, and generates prescription functions $\gamma_t = (\gamma^l_t, \gamma^f_t) = \theta_t[\pi_t]$, where $\pi_t = (\pi^l_t, \pi^f_t)$ and $\pi^l_t$ is a belief on the current state $x^l_t$ defined as, $\pi^l_t(x^l_t) = P^\theta(x^l_t|a_{1:t-1})$ for $i \in \{l, f\}$. The player $i$ uses prescription function $\gamma^i_t$ to operate on its current private type $x^i_t$ to produce its action $a^i_t$, i.e. $\gamma^i_t : \mathcal{X}^i \rightarrow \mathcal{P}(\mathcal{A}^i)$ and $a^i_t \sim \gamma^i_t(\cdot|x^i_t)$.

In the next lemma we show that for any given $\theta$ policy, the belief states $\pi^i_t$ can be updated recursively as follows. Let $\pi^i_t(x^i_t) := Q^i(x^i_t)$.

**Lemma 1**: $\pi_t$ can be factorized as $\pi_t(x_t) = \prod_{i=l,f} \pi^i_t(x^i_t)$ where each $\pi^i_t$ can be updated through an update function $\pi^i_{t+1} = F^i(\pi^i_t, \gamma^i_t, a_t)$ and $F^i$ is independent of common agent’s policy $\psi$. We also say $\pi^i_{t+1} = F_i(\pi^i_t, \gamma^i_t, a_t)$.

**Proof**: Please see Appendix A.
**Definition 1:** We call a strategy profile \( \sigma \) Markov PSE (MPSE), if it is a PSE of type \( \theta \).

In the next section, we design an algorithm to compute MPSE of the game.

**IV. Algorithm for MPSE Computation**

**A. Backward Recursion**

In this section, we define an equilibrium generating function \( \theta = (\theta_i^t)_{i \in \{l,f\}, t \in \{T\}} \), where \( \theta_i^t : \mathcal{P}(\mathcal{X}^l) \times \mathcal{P}(\mathcal{X}^f) \rightarrow \{\mathcal{X}^i \rightarrow \mathcal{P}(\mathcal{A}^i)\} \) and a sequence of functions \((V^l_t, V^f_t)_{t \in \{1,2,...,T+1\}}\), where \( V^i_t : \mathcal{P}(\mathcal{X}^l) \times \mathcal{P}(\mathcal{X}^f) \times \mathcal{X}^i \rightarrow \mathbb{R} \), in a backward recursive way, as follows.

1. Initialize \( \forall \pi_{T+1} \in \mathcal{P}(\mathcal{X}^l) \times \mathcal{P}(\mathcal{X}^f) \), \( x_{T+1}^l \in \mathcal{X}^l, i = l, f \),

\[
V^l_{T+1}(\pi_{T+1}, x_{T+1}^l) \triangleq 0.
\]

\[
V^f_{T+1}(\pi_{T+1}, x_{T+1}^f) \triangleq 0
\]

2. For \( t = T, T-1, \ldots, 1 \), \( \forall \pi_t \in \mathcal{P}(\mathcal{X}) \), let \( \theta_t[\pi_t] \) be generated as follows. Set \( \tilde{\gamma}_t = \theta_t[\pi_t] \), where \( \tilde{\gamma}_t = (\tilde{\gamma}_t^l, \tilde{\gamma}_t^f) \) is the solution of the following fixed-point equation. For a given \( \pi_t, \gamma_t^i \), define \( BR_t(\pi_t, \gamma_t^i) \) as follows,

\[
BR_t^f(\pi_t, \gamma_t^f) = \left\{ \tilde{\gamma}_t^f : \forall x_t^l \in \mathcal{X}^l, \tilde{\gamma}_t^f(\cdot|x_t^l) \in \arg \max_{\gamma_t^f(\cdot|x_t^l)} \mathbb{E}^f(\cdot|x_t^l) \gamma_t^f, \pi_t \right\}
\]

\[
BR_t^l(\pi_t, \gamma_t^l) = \left\{ R_t^l(X_t, A_t) + \delta V_{t+1}^l(\mathbb{E}(\pi_t, \tilde{\gamma}_t^l, A_t, X_{t+1}^l)|\pi_t, x_t^l) \right\}
\]

(9)

where expectation in (9) is with respect to random variables \((X_t^l, A_t, X_{t+1}^l)\) through the measure \( \pi_t(x_t^l) \gamma_t^f(a_t^f|x_t^l) \gamma_t^l(a_t^l|x_t^l)Q(x_{t+1}^l|x_t^l, a_t^l) \) and \( \mathbb{E} \) is defined in (2).

Then let for all \( \pi_t, \theta[\pi_t] = (\tilde{\gamma}_t^l, \tilde{\gamma}_t^f) \) is a solution of the following fixed-point equation (if it exists),

\[
\tilde{\gamma}_t^f \in BR_t^f(\pi_t, \tilde{\gamma}_t^l)
\]

(10)

and

\[
\tilde{\gamma}_t^l \in \arg \max_{\gamma_t^l} \mathbb{E}^{BR_t^l(\gamma_t^f, \gamma_t^l, \pi_t)} \left\{ R_t^l(X_t, A_t) + \delta V_{t+1}^l(\mathbb{E}(\pi_t, \gamma_t^l, BR_t^l(\gamma_t^f, \gamma_t^l), X_{t+1}^l)|\pi_t, x_t^l) \right\}
\]

(11)

where the above expectation is defined with respect to random variables \((X_t^l, A_t, X_{t+1}^l)\) through the measure \( \pi_t(x_t^l) \tilde{\gamma}_t^f(a_t^f|x_t^l) \gamma_t^l(a_t^l|x_t^l)Q(x_{t+1}^l|x_t^l, a_t^l) \), and \( \tilde{\gamma}_t^f \in BR_t^f(\gamma_t^f) \).
Let \((\tilde{\gamma}_t^l, \tilde{\gamma}_t^f)\) be a pair of solution of the above operation. Then set \(\forall x_t^f \in \mathcal{X}^f\),

\[
V_t^f(\pi_t, x_t^f) \triangleq \mathbb{E}_{\tilde{\gamma}_t^f(\cdot|x_t^f), \pi_t} \left\{ R_t^f(X_t, A_t) + \delta V_{t+1}^f(E(\tilde{\pi}_t, \tilde{\gamma}_t^f, A_t), X_{t+1}^f) \big| x_t^f \right\}
\]

\[
V_t^l(\pi_t, x_t^l) \triangleq \mathbb{E}_{\tilde{\gamma}_t^l, \pi_t} \left\{ R_t^l(X_t, A_t) + \delta V_{t+1}^l(E(\tilde{\pi}_t, \tilde{\gamma}_t^l, A_t), X_{t+1}^l) \big| x_t^l \right\}.
\]

(12) \hspace{10cm} (13)

B. Forward Recursion

Based on \(\theta\) defined in the backward recursion above, we now construct a set of strategies \(\tilde{\sigma}\) (through beliefs \(\mu\)) in a forward recursive way as follows.

1. Initialize at time \(t = 1, i = l, f\),

\[
\mu_t^i[\phi](x_1^i) := Q^i(x_1^i).
\]

(14)

2. For \(t = 1, 2 \ldots T, a_{1:t-1} \in \mathcal{H}_{t+1}^l, x_{1:t}^l \in (\mathcal{X}^l)^t, x_{1:t}^f \in (\mathcal{X}^f)^t, i = l, f\)

\[
\tilde{\sigma}_t^i(a_t^i|a_{1:t-1}, x_{1:t}^i) := \theta_t^i[\mu_t^i[\eta_t^i]](a_t^i|x_t^i)
\]

\[
\mu_{t+1}^i[h_{t+1}^c] := F^i(\mu_t^i[h_t^c], \theta_t^i[\mu_t^i[\eta_t^i]], a_t)
\]

(15) \hspace{10cm} (16)

where \(F^i\) is defined in (2).

**Theorem 1:** A strategy profile \(\tilde{\sigma}\), as constructed through backward/forward recursion algorithm above is an MPSE of the game

**Proof:** We will prove this theorem in two parts. In Part 1 for the follower, we prove that \(\tilde{\sigma}^f \in BR_t^f(h_t^f, \tilde{\sigma}^f)\) i.e. \(\forall t \in [T], \forall \sigma^f, h_t^f = (a_{1:t-1}, x_{1:t}^f)\)

\[
\mathbb{E}_{\tilde{\sigma}_t^f, \tilde{\sigma}_t^f, \mu} \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n)|a_{1:t-1}, x_{1:t}^f \right\} \geq \mathbb{E}_{\tilde{\sigma}_t^f, \sigma_t^f, \mu} \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n)|a_{1:t-1}, x_{1:t}^f \right\}.
\]

(17)

In Part 2 for the leader, we show that

\[
\mathbb{E}_{\tilde{\sigma}_t^l, \tilde{\sigma}_t^l, \mu[a_{1:t-1}]} \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^l(X_n, A_n)|a_{1:t-1}, x_{1:t}^l \right\} \geq \mathbb{E}_{\sigma_t^l, BR_t^l(\sigma_t^l), \mu[a_{1:t-1}]} \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^l(X_n, A_n)|a_{1:t-1}, x_{1:t}^l \right\},
\]

(18)

where \(\tilde{\sigma}^f \in BR_t^f(h_t^c, \tilde{\sigma}^f)\), as shown in Part 1.

Combining both the parts prove the above result. The proof is presented in Appendix C. ■
V. INFINITE HORIZON

The above results can be extended to infinite horizon case when the reward functions $R^t, R^f$ are time homogenous and are absolutely bounded, and $\delta < 1$. In the following, we present the algorithm, where its proof is presented in the supplementary file.

A. Backward Recursion

Define an equilibrium generating function $\theta = (\theta^i)_{i \in \{l, f\}}$, where $\theta^i : \mathcal{P}(\mathcal{X}^i) \times \mathcal{P}(\mathcal{X}^f) \to \{\mathcal{X}^i \to \mathcal{P}(\mathcal{A}^i)\}$ and a vector of functions $(V^l, V^f)$, where $V^i : \mathcal{P}(\mathcal{X}^i) \times \mathcal{P}(\mathcal{X}^f) \times \mathcal{X}^i \to \mathbb{R}$ through the following one-shot fixed-point equation. For a given $\gamma$, set

$$\theta^i = \mathbb{E}^{\gamma^i} \left(V^i(x) + \delta V^i(\mathbb{F}(\pi^i, \gamma^i, A), X') | \pi^i, x^i \right).$$

where expectation in (9) is with respect to random variables $(X^i, A, X')$ through the measure $\pi^i(x^i)\gamma^i(a^i|x^i)\gamma^f(a^f|x')Q(x', a)$. And $\mathbb{F}$ is defined in (2).

Then let $(\tilde{\gamma}^l, \tilde{\gamma}^f, V^l, V^f)$ be a solution of the following fixed-point equation,

$$\tilde{\gamma}^l \in BR^l(\pi, \tilde{\gamma}^l)$$

$$\tilde{\gamma}^l \in \arg \max_{\gamma^l} \mathbb{E}^{BR^l(\pi^l, \gamma^l)} \pi \left\{ R^l(x, A) + \delta V^l(\mathbb{F}(\pi^l, \gamma^l, A), X') | \pi^l, x^l \right\}$$

where the above expectation is defined with respect to random variables $(X, A, X')$ through the measure $\pi(x)\gamma^l(a^l|x)\gamma^f(a^f|x')Q(x', x, A)$. And $\tilde{\gamma}^l \in BR^l(\pi^l, \gamma^l)$. And $\forall x^l \in \mathcal{X}^l$,

$$V^l(\pi^l, x^l) = \mathbb{E}^{\gamma^l} \pi \left\{ R^l(x, A) + \delta V^l(\mathbb{F}(\pi^l, \tilde{\gamma}^l, A), X') | x^l \right\}.$$  

$$V^l(\pi^l, x^l) = \mathbb{E}^{\gamma^l} \pi \left\{ R^l(x, A) + \delta V^l(\mathbb{F}(\pi^l, \tilde{\gamma}^l, A), X') | x^l \right\}.$$  

B. Forward Recursion

Based on $\theta$ defined above, we now construct a set of strategies $\sigma$ (through beliefs $\mu$) in a forward recursive way as follows.

1. Initialize at time $t = 1, i = l, f$,

$$\mu^i_t[\phi](x^i_1) := Q^i(x^i_1).$$
2. For \( t = 1, 2 \ldots T \), \( a_{1:t} \in \mathcal{H}_{t+1}^c, x_{1:t}^i \in (\mathcal{X}^l)^t, x_{1:t}^f \in (\mathcal{X}^f)^t \),

\[
\tilde{\sigma}_i^t(a_i^t|a_{1:t-1}, x_{1:t}^i) := \theta_i^t[\mu_i[h_i^f]](a_i^t|x_i^t) \tag{25}
\]

\[
\mu_{t+1}^i[h_{t+1}^c] := F_i(\mu_t^i[h_t^f], \theta_i^t[\mu_t^i[h_t^f]], a_t) \tag{26}
\]

where \( F_i \) is defined in (2).

**Proof:** Please see the supplementary file.

### VI. Complexity

In general, computing a Stackelberg equilibrium involves solving a fixed-point equation in the space of strategies of both the players for all histories of the game i.e. of the form \( \sigma = f(\sigma) \) where \( f \) is appropriately defined from (4). For any time \( t \), since \( \sigma_i^t : \mathcal{A}_i^{t-1} \times (\mathcal{X}_i^l)^t \rightarrow \mathcal{P}(\mathcal{A}_i^l) \), there exist \( |\mathcal{P}(\mathcal{A}_i^l)||\mathcal{A}_i^{t-1}||\mathcal{X}_i^l|^t \) number of possible strategies of the player \( i = l, f \). Since the complexity at the last time \( t \) dominates, solving a Stackelberg equilibrium reduces to solving a fixed-point equation in the space of \( \times_{i=1,2}|\mathcal{P}(\mathcal{A}_i^l)||\mathcal{A}_i^{T-1}||\mathcal{X}_i^l|^T \) number of strategies.

In our algorithm, each time \( t \) involves solving a fixed-point equation (10),(11), for every \( \pi_t \), where \( \pi_t \in \mathcal{P}(\mathcal{X}_i^l) \times \mathcal{P}(\mathcal{X}_f^f) \). Thus computing a Stackelberg equilibrium involves solving \( T|\mathcal{P}(\mathcal{X}_i^l) \times \mathcal{P}(\mathcal{X}_f^f) | \) smaller fixed-point equations (10) (11). Therefore, our algorithm reduces the computational dependence on \( T \) from double exponential to linear. The complexity of solving each smaller fixed-point equation depends on the specific model parameters and is an important direction for future research.

### VII. Security Example

In this section, we consider a repeated Stackelberg game as a security example. We assume that \( \mathcal{X}_f = \mathcal{A}_f = \{0, 1\}, \mathcal{X}_l = \phi \) and type of the defender is static i.e. \( Q(x_{t+1}|x_t, a_t) = \mathbb{1}(x_{t+1} = x_t) \). We assume \( \delta = 0.6 \). Let \( p^f = \gamma^f(1), p^{f,0} = \gamma^f(1|0) \) and \( p^{f,1} = \gamma^f(1|1) \) and the rewards of the players are given in Table I below.

The equilibrium strategies and value functions are provided in Figures 1–3. Interestingly, the equilibrium strategies of the players are pure strategies that exhibit “complementary discontinuities” [4], [5].

In this paper, we study a general leader/defender, follower/attacker security game where both the attacker and the defender have private types that evolves as conditionally independent controlled Markov process, conditioned on action history. We present a novel dynamic...
TABLE I: Game matrix for X=0

| X=0     | Attacker A1 | Attacker A2 |
|---------|-------------|-------------|
| Defender D1 | (2, 1)     | (4, 0)     |
| Defender D2 | (1, 0)     | (3, 2)     |

TABLE II: Game matrix for X=1

| X=1     | Attacker A1 | Attacker A2 |
|---------|-------------|-------------|
| Defender D1 | (3, 2)     | (2, 0)     |
| Defender D2 | (0, 1)     | (1, 1)     |

Fig. 1: Probability of follower taking action 1 when its state is low and high

(a) Low

(b) High

Fig. 2: Utility of the follower when its state is low and high

(a) Low

(b) High
programing like methodology to sequentially decompose the problem of computing Markov perfect Stackelberg equilibrium for these games. Based on this algorithm we study a repeated security game where we numerically compute the equilibrium policies. In general, this algorithm can further increase the applicability of Stackelberg security games in dynamic security settings and in dynamic mechanism design where a leader commits to a policy and the follower best responds to it.

VIII. ACKNOWLEDGEMENT

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APPENDIX A

Lemma 2: $\pi_t$ can be factorized as $\pi_t(x_t) = \prod_{i=l,f} \pi^i_t(x^i_t)$ where each $\pi^i_t$ can be updated through an update function $\pi^i_{t+1} = F^i(\pi^i_t, \gamma^i_t, a_t)$ and $F^i$ is independent of common agent’s policy $\psi$. We also say $\pi_{t+1} = F(\pi_t, \gamma_t, a_t)$.

Proof: We prove this by induction. Since $\pi_1(x_1) = \prod_{i=l,r} Q^i_1(x^i_1)$, the base case is verified.
Now suppose \( \pi_t(x_t) = \prod_{i=l,r} \pi_t^i(x_t^i) \). Then,

\[
\pi_{t+1}(x_{t+1}) = P^\psi(x_{t+1}|a_{1:t}, \gamma_{1:t+1})
= P^\psi(x_{t+1}|a_{1:t}, \gamma_{1:t})
= \sum_{\tilde{x}_{t+1}} \frac{P^\psi(x_t, a_t, x_{t+1}|a_{1:t-1}, \gamma_{1:t})}{P^\psi(\tilde{x}_t, x_{t+1}, a_t|a_{1:t-1}, \gamma_{1:t})}
= \frac{\sum_{x_t} \pi_t(x_t) \prod_{i=1}^N \gamma_t^i(a_t^i | x_t^i) Q_t^i(x_{t+1}^i | x_t^i, a_t)}{\sum_{\tilde{x}_t \tilde{x}_{t+1}} \pi_t(\tilde{x}_t) \prod_{i=1}^N \gamma_t^i(a_t^i | \tilde{x}_t^i) Q_t^i(\tilde{x}_{t+1}^i | \tilde{x}_t^i, a_t)}
= \prod_{i=l,r} \frac{\sum_{x_t^i} \pi_t^i(x_t^i) \gamma_t^i(a_t^i | x_t^i) Q_t^i(x_{t+1}^i | x_t^i, a_t)}{\sum_{\tilde{x}_t^i} \pi_t^i(\tilde{x}_t^i) \gamma_t^i(a_t^i | \tilde{x}_t^i)}
= \prod_{i=l,r} \pi_{t+1}^i(x_{t+1}^i),
\]

where (27e) follows from induction hypothesis. It is assumed in (27c)-(27e) that the denominator is not 0. If denominator corresponding to any \( \gamma_t^i \) is zero, we define

\[
\pi_{t+1}^i(x_{t+1}^i) = \sum_{x_t^i} \pi_t^i(x_t^i) Q_t^i(x_{t+1}^i | x_t^i, a_t),
\]

where \( \pi_{t+1} \) still satisfies (27f). Thus \( \pi_{t+1}^i = F^i(\pi_t^i, \gamma_t^i, a_t) \) and \( \pi_{t+1} = F(\pi_t, \gamma_t, a_t) \) where \( F^i \) and \( F \) are appropriately defined from above.

**APPENDIX B**

**Claim 1:** For any policy profile \( g \) and \( \forall t \),

\[
P^g(x_{1:t}|a_{1:t-1}) = \prod_{i=1}^N P^g(x_t^i|a_{1:t-1})
\]

**Proof:**

\[
P^g(x_{1:t}|a_{1:t-1}) = \frac{P^g(x_{1:t}, a_{1:t-1})}{\sum_{x_{1:t}} P^g(\tilde{x}_{1:t}, a_{1:t-1})}
= \frac{\Pi_{i=1}^N (Q_1^i(x_1^i)g_1^i(a_1^i | x_1^i)) \Pi_{i=2}^N (Q_n^i(x_{i|n-1}^i, a_{n-1})g_n^i(a_{n-1}, x_{1:n}^i))}{\sum_{\tilde{x}_{1:t}} \Pi_{i=1}^N (Q_1^i(\tilde{x}_1^i)g_1^i(a_1^i | \tilde{x}_1^i)) \Pi_{i=2}^N (Q_n^i(\tilde{x}_{i|n-1}^i, a_{n-1})g_n^i(a_{n-1}, \tilde{x}_{1:n}^i))}
= \frac{\Pi_{i=1}^N (Q_1^i(x_1^i)g_1^i(a_1^i | x_1^i)) \Pi_{i=2}^N (Q_n^i(x_{i|n-1}^i, a_{n-1})g_n^i(a_{n-1}, x_{1:n}^i))}{\Pi_{i=1}^N (\sum_{\tilde{x}_{1:t}} Q_1^i(\tilde{x}_1^i)g_1^i(a_1^i | \tilde{x}_1^i)) \Pi_{i=2}^N (Q_n^i(\tilde{x}_{i|n-1}^i, a_{n-1})g_n^i(a_{n-1}, \tilde{x}_{1:n}^i))}
= \prod_{i=1}^N \frac{Q_1^i(x_1^i)g_1^i(a_1^i | x_1^i)}{\sum_{\tilde{x}_1^i} Q_1^i(\tilde{x}_1^i)g_1^i(a_1^i | \tilde{x}_1^i)} \frac{\Pi_{i=2}^N (Q_n^i(x_{i|n-1}^i, a_{n-1})g_n^i(a_{n-1}, x_{1:n}^i))}{\Pi_{i=2}^N (Q_n^i(\tilde{x}_{i|n-1}^i, a_{n-1})g_n^i(a_{n-1}, \tilde{x}_{1:n}^i))}
\]
where (32a) follows from Lemma 5 and (32b) follows from Lemma 3 in Appendix D. This proof is identical to the proof of Theorem 1 in [15]. We reproduce the proof here for completeness. We prove this using induction and the results in Lemma 3, 4 and 5 proved in Appendix D. For base case at \( t = T \), \( \forall (a_{1:T-1}, x_{1:T}^f) \in \mathcal{H}_T^{f}, \sigma^f \)

\[
\mathbb{E}_{\tilde{\sigma}^f, \tilde{\sigma}^f, \mu_t}[\sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n)|a_{1:t-1}, x_{1:t}^f] \geq \mathbb{E}_{\sigma^f, \sigma^f, \mu_t}[\sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n)|a_{1:t-1}, x_{1:t}^f] \tag{31}
\]

This proof is identical to the proof of Theorem 1 in [15]. We reproduce the proof here for completeness. We prove this using induction and the results in Lemma 3, 4 and 5 proved in Appendix D. For base case at \( t = T \), \( \forall (a_{1:T-1}, x_{1:T}^f) \in \mathcal{H}_T^{f}, \sigma^f \)

\[
\mathbb{E}_{\tilde{\sigma}^f, \tilde{\sigma}^f, \mu_T[a_{1:T-1}]} \left\{ R_T^f(X_T, A_T)|a_{1:T-1}, x_{1:T}^f \right\} = V_T^f(\mu_T[a_{1:T-1}], x_T^f) \tag{32a}
\]

\[
\geq \mathbb{E}_{\sigma^f, \sigma^f, \mu_T[a_{1:T-1}]} \left\{ R_T^f(X_T, A_T)|a_{1:T-1}, x_{1:T}^f \right\}, \tag{32b}
\]

where (32a) follows from Lemma 5 and (32b) follows from Lemma 3 in Appendix D.

Let the induction hypothesis be that for \( t + 1 \), \( \forall a_{1:t} \in \mathcal{H}_{t+1}^c, x_{1:t+1}^f \in (X^f)^{t+1}, \sigma^f \),

\[
\mathbb{E}_{\tilde{\sigma}^f, \tilde{\sigma}^f, \mu_{t+1}[a_{1:t}]} \left\{ \sum_{n=t+1}^{T} R_n^f(X_n, A_n)|a_{1:t}, x_{1:t+1}^f \right\} \tag{33a}
\]

\[
\geq \mathbb{E}_{\sigma^f, \sigma^f, \mu_{t+1}[a_{1:t}]} \left\{ \sum_{n=t+1}^{T} R_n^f(X_n, A_n)|a_{1:t}, x_{1:t+1}^f \right\}. \tag{33b}
\]

Then \( \forall (a_{1:t-1}, x_{1:t}^f) \in \mathcal{H}_t^{f}, \sigma^f \), we have

\[
\mathbb{E}_{\tilde{\sigma}^f, \tilde{\sigma}^f, \mu_t[a_{1:t-1}]} \left\{ \sum_{n=t}^{T} R_n^f(X_n, A_n)|a_{1:t-1}, x_{1:t}^f \right\} = V_t^f(\mu_t[a_{1:t-1}], x_t^f) \tag{34a}
\]

\[
\geq \mathbb{E}_{\sigma^f, \mu_t[a_{1:t-1}]} \left\{ R_t^f(X_t, A_t) + V_{t+1}^f(\mu_{t+1}[a_{1:t-1}A_t], x_{t+1}^f)|a_{1:t-1}, x_{1:t}^f \right\} \tag{34b}
\]
where (34a) follows from Lemma 5, (34b) follows from Lemma 3, (34c) follows from Lemma 5, (34d) follows from induction hypothesis in (33b) and (34e) follows from Lemma 4. Moreover, construction of $\theta$ in (9), and consequently definition of $\tilde{\sigma}$ in (15) are pivotal for (34e) to follow from (34d).

\section*{Appendix D}

\begin{lemma}
$\forall t \in \mathcal{T}, (a_{1:t-1}, x_{1:t}^f) \in \mathcal{H}_f, \sigma_f^t$

$$V_t^f(\mu[a_{1:t-1}], x_{1:t}^f) \geq \mathbb{E}^\sigma_t^f \{ R_t^f(X_t, A_t) + V_{t+1}^f(\mathbb{E}(\mu[a_{1:t-1}], \tilde{\sigma}_t(\cdot|a_{1:t-1}, \cdot), A_t), X_{t+1}^f) |a_{1:t-1}, x_{1:t}^f \}.$$ \tag{35}
\end{lemma}

\begin{proof}
We prove this lemma by contradiction.

Suppose the claim is not true for $t$. This implies $\exists \tilde{t}, \tilde{\sigma}_t^f, \tilde{a}_{1:t-1}, \tilde{x}_{1:t}^f$ such that

$$\mathbb{E}^\tilde{\sigma}_t^f \{ R_t^f(X_t, A_t) + V_{t+1}^f(\mathbb{E}(\mu[a_{1:t-1}], \tilde{\sigma}_t(\cdot|a_{1:t-1}, \cdot), A_t), X_{t+1}^f) |a_{1:t-1}, x_{1:t}^f \} > V_t^f(\mu[a_{1:t-1}], \tilde{x}_{1:t}^f).$$ \tag{36}

We will show that this leads to a contradiction.

Construct

\begin{align*}
\hat{\gamma}_t^f(a_t^f | x_t^f) = \begin{cases} 
\tilde{\sigma}_t^f(a_t^f | \tilde{a}_{1:t-1}, \tilde{x}_{1:t}^f) & x_t^f = \tilde{x}_t^f \\
\text{arbitrary} & \text{otherwise}.
\end{cases}
\end{align*} \tag{37}
Then for \( \hat{a}_{1:t-1}, \hat{x}_{1:t} \), we have

\[
V_t^f(\mu_t[\hat{a}_{1:t-1}], \hat{x}_t^f)
= \max_{\gamma_t} \mathbb{E} \gamma_t(\cdot|x_t^f) \sigma_t[\hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \left\{ R_t^f(\hat{x}_t x_t^f, a_t) + V_{t+1}^f(\mu_t[\hat{a}_{1:t-1}], \hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \right\}
\]

\[
V_{t+1}^f(\mu_t[\hat{a}_{1:t-1}], \hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \left\{ R_t^f(X_t, a_t) + V_{t+1}^f(\mu_t[\hat{a}_{1:t-1}], \hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \right\}
\]

\[
= \sum_{x_{t+1}, \mu_t[\hat{a}_{1:t-1}]} \left\{ R_t^f(\hat{x}_t x_t^f, a_t) + V_{t+1}^f(\mu_t[\hat{a}_{1:t-1}], \hat{x}_{t+1}^f) \right\}
\]

\[
\times \mu_t[\hat{a}_{1:t-1}](x_t^f) \gamma_t(\cdot|x_t^f) \sigma_t(\cdot|x_t^f) Q_t^f(x_{t+1}^f) \right\}
\]

\[
= \sum_{x_{t+1}, \mu_t[\hat{a}_{1:t-1}]} \left\{ R_t^f(\hat{x}_t x_t^f, a_t) + V_{t+1}^f(\mu_t[\hat{a}_{1:t-1}], \hat{x}_{t+1}^f) \right\}
\]

\[
\times \mu_t[\hat{a}_{1:t-1}](x_t^f) \gamma_t(\cdot|x_t^f) \sigma_t(\cdot|x_t^f) Q_t^f(x_{t+1}^f) \right\}
\]

\[
= \mathbb{E} \gamma_t(\cdot|x_t^f) \sigma_t[\hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \left\{ R_t^f(\hat{x}_t x_t^f, a_t) + V_{t+1}^f(\mu_t[\hat{a}_{1:t-1}], \hat{x}_{t+1}^f) \right\}
\]

\[
> V_t^f(\mu_t[\hat{a}_{1:t-1}], \hat{x}_t^f)
\]

where (38a) follows from definition of \( V_t^f \) in (12), (38d) follows from definition of \( \gamma_t^f \) and (38f) follows from (36). However this leads to a contradiction. \( \blacksquare \)

**Lemma 4:** \( \forall t \in T, (a_{1:t}, x_{1:t+1}) \in \mathcal{H}_{t+1}^f \) and \( \sigma_t^f \)

\[
\mathbb{E} \sigma_t^f[\hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \sum_{n=t+1}^T R_n^f(X_n, A_n) \left| a_{1:t}, x_{1:t+1}^f \right\}
\]

\[
= \mathbb{E} \sigma_t^f[\hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}]) \sum_{n=t+1}^T R_n^f(X_n, A_n) \left| a_{1:t}, x_{1:t+1}^f \right\}
\]

Thus the above quantities do not depend on \( \sigma_t^f \).

**Proof:** Essentially this claim stands on the fact that \( \mu_{t+1}^f[a_{1:t}] \) can be updated from \( \mu_t^f[a_{1:t-1}], \hat{a}_t \)

and \( a_t \), as \( \mu_{t+1}^f[a_{1:t}] = F(\mu_t^f[a_{1:t-1}], \hat{a}_t(a_{1:t-1}, \cdot), a_t) \) as in Claim 2. Since the above expectations involve random variables \( X_{t+1}^f, A_{t+1:T}, X_{t+2:T} \), we consider the probability

\[
P\sigma_t^f[\hat{a}_{1:t-1}, \cdot, \mu_t[\hat{a}_{1:t-1}](x_{t+1}^f, a_{t+1:T}, x_{t+2:T}^f) | a_{1:t}, x_{1:t+1}^f) = \frac{N_r}{D_r}
\]

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where

\[ N_t = \sum_{x_t} \mu_t^{[a_{1:t-1}]}(x_t) \sigma^f_t(x_t | a_{1:t-1}, x^f_{1:t}) \]  

(41a)

\[
= \sum_{x_t} \mu_t^{[a_{1:t-1}]}(x_t) \sigma^f_t(x_t | a_{1:t-1}, x^f_t) \sigma^f_t(a_t | a_{1:t-1}, x^f_t) \tilde{\sigma}^f_t(a_t | a_{1:t-1}, x^f_t)
\]  

(41b)

\[
Q(x_{t+1} | x_t, a_t) \mathbb{P}^{f}_{x_{1:T}} \mathbb{P}^{f}_{x_{1:T}}(a_{1:T} | a_{1:t-1}, x^f_t, x^f_{t+1})
\]  

(41c)

\[
= \sum_{x_t} \mu_t^{[a_{1:t-1}]}(x_t) \sigma^f_t(a_t | a_{1:t-1}, x^f_t) \tilde{\sigma}^f_t(a_t | a_{1:t-1}, x^f_t) Q^f(x^f_{t+1} | x_t, a_t)
\]  

(41d)

\[
P^{f}_{x_{1:T}} w^{f}_{x_{1:T}}(a_{1:t-1}) (a_{t+1:T}, x^f_{t+1} | a_{1:t-1}, x^f_{1:t}, x^f_{t+1})
\]  

(41e)

By canceling the terms \( \sigma_t^f(\cdot) \) and \( Q_t^f(\cdot) \) in the numerator and the denominator, (40) is given by

\[
\frac{\sum_{x_t} \mu_t^{[a_{1:t-1}]}(x_t) \tilde{\sigma}^f_t(a_t | a_{1:t-1}, x^f_t) Q^f(x^f_{t+1} | x_t, a_t)}{\sum_{x_t} \mu_t^{[a_{1:t-1}]}(x_t) \tilde{\sigma}^f_t(a_t | a_{1:t-1}, x^f_t)}
\]  

(41f)

\[
\times P^{f}_{x_{1:T}} w^{f}_{x_{1:T}}(a_{1:t-1}) (a_{t+1:T}, x^f_{t+1} | a_{1:t}, x^f_{1:t}, x^f_{t+1})
\]  

(41g)

\[
= \mu_t^{[a_{1:t-1}]}(x^f_{t+1}) P^{f}_{x_{1:T}} w^{f}_{x_{1:T}}(a_{1:t-1}) (a_{t+1:T}, x^f_{t+1} | a_{1:t}, x^f_{1:t}, x^f_{t+1})
\]  

(41h)

where (41g) follows from using the definition of \( \mu_{t+1}^{[a_{1:t}]}(x^f_t) \) in the forward recursive step in (16) and the definition of the belief update in (27).

**Lemma 5:** \( \forall t \in \mathcal{T}, (a_{1:t-1}, x^f_{1:t}) \in \mathcal{H}^f_t \),

\[
V_t^f(\mu_t^{[a_{1:t-1}]}(x_t^f) = E^{f}_{x_{1:T}}(a_{1:t-1}) \left\{ \sum_{n=t}^T R_n^f(X_n, A_n) \right\}.
\]  

(42)

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Proof: We prove the lemma by induction. For \( t = T \),

\[
\mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:T-1}]} \left\{ R^f_T(X_T, A_T) \big| a_{1:T-1}, x^f_{1:T} \right\} \\
= \sum_{x^f_T, a_T} R^f_T(x_T, a_T) \mu_T[a_{1:T-1}] (x^f_T, a_T) \tilde{\sigma}^f_T(a_T|a_{1:T-1}, x^f_{1:T}) \tilde{\sigma}_T(a_T|a_{1:T-1}, x^f_{1:T}) \tag{43a}
\]

\[
= V^f_T(\mu_T[a_{1:T-1}], x^f_T), \tag{43b}
\]

where (43b) follows from the definition of \( V^f_t \) in (12) and the definition of \( \tilde{\sigma}_T \) in the forward recursion in (15).

Suppose the claim is true for \( t + 1 \), i.e., \( \forall t \in \mathcal{T}, (a_{1:t}, x^f_{1:t+1}) \in \mathcal{H}^{f}_{t+1} \)

\[
V^f_{t+1}(\mu_{t+1}, a_{1:t}, x^f_{1:t+1}) = \mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t}], a_{1:t}} \left\{ R^f_t(X_t, A_t) \big| a_{1:t-1}, x^f_{1:t} \right\} \sum_{n=t+1}^T R^f_n(X_n, A_n) \big| a_{1:t}, x^f_{1:t+1} \right\} \tag{44}
\]

Then \( \forall t \in \mathcal{T}, (a_{1:t-1}, x^f_{1:t}) \in \mathcal{H}^{f}_{t}, \) we have

\[
\mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t-1}]} \left\{ \sum_{n=t+1}^T R^f_n(X_n, A_n) \big| a_{1:t-1}, x^f_{1:t} \right\} \\
= \mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t-1}]} \left\{ R^f_t(X_t, A_t) + \mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t-1}]} \left\{ \sum_{n=t+1}^T R^f_n(X_n, A_n) \big| a_{1:t-1}, x^f_{1:t} \right\} \right\} \tag{45a}
\]

\[
= \mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t-1}]} \left\{ R^f_t(X_t, A_t) + \sum_{n=t+1}^T R^f_n(X_n, A_n) \big| a_{1:t-1}, x^f_{1:t} \right\} \tag{45b}
\]

\[
= \mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t-1}]} \left\{ R^f_t(X_t, A_t) + V^f_{t+1}(\mu_{t+1}[a_{1:t-1} A_t], X^f_{t+1}) \big| a_{1:t-1}, x^f_{1:t} \right\} \tag{45c}
\]

\[
= \mathbb{E}_{\hat{\theta}^f_{i:T}, \mu_T[a_{1:t-1}]} \left\{ R^f_t(X_t, A_t) + V^f_{t+1}(\mu_{t+1}[a_{1:t-1} A_t], X^f_{t+1}) \big| a_{1:t-1}, x^f_{1:t} \right\} \tag{45d}
\]

\[
= V^f_t(\mu_t[a_{1:t-1}], x^f_t), \tag{45e}
\]

where (45b) follows from Lemma 4 in Appendix D, (45c) follows from the induction hypothesis in (44), (45d) follows because the random variables involved in expectation, \( X^f_{t}, A_t, X^f_{t+1} \) do not depend on \( \hat{\theta}^f_{i:T}, \hat{\theta}^f_{i:T+1} \) and (45e) follows from the definition of \( \hat{\theta}^f_t \) in the forward recursion in (15), the definition of \( \mu_{t+1} \) in (16) and the definition of \( V^f_t \) in (12). □
APPENDIX E
PART 2: LEADER

In the following we will show that, $\forall \sigma^l$

\[
\mathbb{E}^{\tilde{\sigma}^l, \sigma^l, \mu} \left\{ \sum_{n=t}^{T} \delta^{n-t} R^l_n(X_n, A_n) | h^l_t \right\} 
\geq \mathbb{E}^{\sigma^l, BR^l(\sigma^l), \mu} \left\{ \sum_{n=t}^{T} \delta^{n-t} R^l_n(X_n, A_n) | h^l_t \right\},
\]

(46)

where $\tilde{\sigma}^l \in BR^l(\sigma^l)$, as shown in Part 1.

Proof: We prove the above result using induction and from results in Lemma 13 and 7 proved in Appendix F.

For base case at $t = T$, $\forall (a_{1:T-1}, x^l_{1:T}) \in H^l_T, \sigma^l$

\[
\mathbb{E}^{\tilde{\sigma}^l_T, \sigma^l_T, \mu_T[a_{1:T-1}]} \left\{ R^l_T(X_T, A_T) | a_{1:T-1}, x^l_{1:T} \right\} 
= V^l_T(\mu_T[a_{1:T-1}], x^l_T) \quad (47a)
\]

\[
\geq \mathbb{E}^{BR^l(\sigma^l_T), \mu_T[a_{1:T-1}]} \left\{ R^l_T(X_T, A_T) | a_{1:T-1}, x^l_{1:T} \right\}. \quad (47b)
\]

where (47a) follows from Lemma 7 and (47b) follows from Lemma 13 in Appendix F. Let the induction hypothesis be that for $t + 1$, $\forall (a_{1:t}, x^l_{1:t}) \in H^l_{t+1}, \sigma^l$,

\[
\mathbb{E}^{\tilde{\sigma}^l_{t+1:T}, x^l_{1:t} a_{1:t-1, t}} \left\{ \sum_{n=t+1}^{T} R^l_n(X_n, A_n) | a_{1:n-1}, x^l_{1:n} \right\} 
\geq \mathbb{E}^{BR^l(\sigma^l_{t+1:T}), x^l_{1:t} a_{1:t-1, t}} \left\{ \sum_{n=t+1}^{T} R^l_n(X_n, A_n) | a_{1:n-1}, x^l_{1:n} \right\} \quad (48a)
\]

Then $\forall (a_{1:t-1}, x^l_{1:t}) \in H^l_{t}, \sigma^l$, we have

\[
\mathbb{E}^{\tilde{\sigma}^l_{1:T}, \sigma^l_{1:T}, \mu[a_{1:t-1}]} \left\{ \sum_{n=t}^{T} R^l_n(X_n, A_n) | a_{1:t-1}, x^l_{1:t} \right\} 
= V^l_t(\mu[a_{1:t-1}], x^l_t) \quad (49a)
\]

\[
\geq \mathbb{E}^{BR^l(\mu[a_{1:t-1}], \gamma^l), \mu[a_{1:t-1}]} \left\{ R^l_t(X_t, A_t) + V^l_{t+1}(\mathbb{E}^{\mu[a_{1:t-1}], \gamma^l, BR^l(\gamma^l), X^l_{t+1}} | x^l_t) \right\} \quad (49b)
\]

\[
= \mathbb{E}^{BR^l(\sigma^l), \mu[a_{1:t-1}]} \left\{ R^l_t(X_t, A_t) + \mathbb{E}^{BR^l(\tilde{\sigma}^l_{t+1:T}), BR^l(\tilde{\gamma}^l), F(\mu[a_{1:t-1}], \gamma^l, BR^l(\gamma^l), A_t)} \right\} \quad (49c)
\]

\[
\left\{ \sum_{n=t+1}^{T} R^l_n(X_n, A_n) | a_{1:t-1}, A_t, x^l_{1:t}, X^l_{t+1} | a_{1:t-1}, x^l_{1:t} \right\} \quad (49c)
\]
\[ \geq \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}^{l}) \sigma_{t}^{l}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ R_{t}^{l} (X_{t}, A_{t}) + \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t+1:T}^{l}) \sigma_{t+1:T}^{l}, F(\mu_{[a_{1:t-1}]} \mid \gamma_{t}^{l}, A_{t}) \right\} \]

\[
\left\{ \sum_{n=t+1}^{T} R_{n}^{l} (X_{n}, A_{n}) \mid a_{1:t-1}, A_{t}, x_{1:t}, X_{1:t} \right\} \mid a_{1:t-1}, x_{1:t} \right\} (49d)
\]

\[
= \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}^{l}) \sigma_{t}^{l}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ R_{t}^{l} (X_{t}, A_{t}) + \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}^{l}) \sigma_{t}^{l}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ \sum_{n=t+1}^{T} R_{n}^{l} (X_{n}, A_{n}) \mid a_{1:t-1}, A_{t}, x_{1:t}, X_{1:t} \right\} \right\} \right\} \right\} (49e)
\]

\[
\mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}^{l}) \sigma_{t}^{l}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ \sum_{n=t}^{T} R_{n}^{l} (X_{n}, A_{n}) \mid a_{1:t-1}, A_{t}, x_{1:t}, X_{1:t} \right\} \right\} (49f)
\]

where (49a) follows from Lemma 7, (49b) follows from Lemma 13, (49c) follows from Lemma 7 and \( \gamma_{t}^{l} \) in (49c) satisfy \( \forall x_{t}, x_{t+1} \) (such a \( \gamma_{t}^{l} \) exists from Lemma 9),

\[
\gamma_{t}^{l}(a_{t}^{l} \mid x_{t}^{l}) = \sigma_{t}^{l}(a_{t}^{l} \mid a_{1:t-1}, x_{1:t}^{l})
\]

(50)

\[
F(\mu_{[a_{1:t-1}]} \mid a_{1:t-1}, \gamma_{t}^{l} \mid \mathcal{R}^{l}, \pi_{t}, \gamma_{t}^{l}, A_{t})(x_{t+1}) = P^{\mathcal{BR}^{l} (\sigma_{t+1:T}^{l}) \sigma_{t+1:T}, \sigma_{t+1:T}, \sigma_{t+1:T}, \sigma_{t+1:T} \mid a_{1:t}(x_{t+1} \mid a_{1:t})}
\]

(51)

(49d) follows from induction hypothesis in (48a) and (49e) follows from (27).

\[\]

\[\]

**APPENDIX F**

**Lemma 6:** \( \forall t \in [T], (a_{1:t-1}, x_{1:t}^{l}) \in \mathcal{H}_{t}, \gamma_{t}^{l} \)

\[
V_{t}^{f} (\mu_{[a_{1:t-1}]} \mid x_{t}^{l}) \geq \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}^{l}) \sigma_{t}^{l}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ R_{t}^{l} (X_{t}, A_{t}) + \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}^{l}) \sigma_{t}^{l}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ \sum_{n=t+1}^{T} R_{n}^{l} (X_{n}, A_{n}) \mid a_{1:t-1}, A_{t}, x_{1:t}, X_{1:t} \right\} \right\} (52)
\]

where \( \gamma_{t}^{l} \) satisfies \( \forall x_{t+1} \) (such a \( \gamma_{t}^{l} \) exists from Lemma 9),

\[
\gamma_{t}^{l}(a_{t}^{l} \mid x_{t}^{l}) = \sigma_{t}^{l}(a_{t}^{l} \mid a_{1:t-1}, x_{1:t}^{l})
\]

(53)

\[
F(\mu_{[a_{1:t-1}]} \mid a_{1:t-1}, \gamma_{t}^{l} \mid \mathcal{R}^{l}, \pi_{t}, \gamma_{t}^{l}, A_{t})(x_{t+1}) = P^{\mathcal{BR}^{l} (\sigma_{t+1:T}^{l}) \sigma_{t+1:T}, \sigma_{t+1:T}, \sigma_{t+1:T} \mid a_{1:t}(x_{t+1} \mid a_{1:t})}
\]

(54)

**Proof:** We prove this lemma by contradiction. Suppose the claim is not true for \( t \). This implies \( \exists \sigma_{t}, \hat{a}_{1:t-1}, \hat{x}_{1:t} \) such that

\[
\mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}) \sigma_{t}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ R_{t}^{l} (X_{t}, A_{t}) + \mathbb{E}^{\mathcal{BR}^{l} (\sigma_{t}) \sigma_{t}, \mu_{[a_{1:t-1}]} \mid X_{t}, A_{t}} \left\{ \sum_{n=t+1}^{T} R_{n}^{l} (X_{n}, A_{n}) \mid \hat{a}_{1:t-1}, \hat{x}_{1:t} \right\} \right\} > V_{t}^{f} (\mu_{[\hat{a}_{1:t-1}]} \mid \hat{x}_{t}^{l}),
\]

(55)
where $\hat{\gamma}_t^l$ satisfies (such a $\hat{\gamma}_t^l$ exists from Lemma 9)

$$\hat{\gamma}_t^l(a_t^l|x_t^l) = \hat{\sigma}_t^l(a_t|\hat{a}_{1:t-1},\hat{x}_{1:t}^l)$$  \hspace{1cm} (56)

$$F(\mu_t[a_{1:t-1}], \hat{\gamma}_t^l, BR^f(\mu_t[a_{1:t-1}], \hat{\gamma}_t^l), a_t)(x_{t+1})$$

$$= p^{BR^f}(\hat{\sigma}_{1:t-1}^{l}, \hat{\sigma}_{t}^{l}, \hat{\sigma}_{t+1}^{l}: \mu_{a_{1:t-1}, \hat{a}_{1:t-1}, \overrightarrow{a}})(x_{t+1}|\hat{a}_{1:t-1}, a_t)$$  \hspace{1cm} (57)

Then for $\hat{a}_{1:t-1}$, we have

$$V_t^l(\mu_t[\hat{a}_{1:t-1}], \hat{x}_{1:t}^l)$$

$$= \max_{\gamma_t^l} E^{\gamma_t^l BR^f(\pi_t, \gamma_t^l), \mu_t[\hat{a}_{1:t-1}]} \{ R_t^l(X_t, A_t) + V_{t+1}^l(\mu_t[\hat{a}_{1:t-1}], \hat{\gamma}_t^l, BR^f(\mu_t[\hat{a}_{1:t-1}], \hat{\gamma}_t^l, A_t), X_{t+1}^l)|\hat{x}_{1:t}^l) \}$$  \hspace{1cm} (58a)

$$= E^{\hat{\sigma}_t^l \mu_t[\hat{a}_{1:t-1}]} \{ R_t^l(X_t, A_t) + V_{t+1}^l(\mu_t[\hat{a}_{1:t-1}], \hat{\gamma}_t^l, BR^f(\mu_t[\hat{a}_{1:t-1}], \hat{\gamma}_t^l), A_t), X_{t+1}^l)|\hat{a}_{1:t-1}, \hat{x}_{1:t}^l) \}$$  \hspace{1cm} (58b)

where (58b) follows from definition of $V_t^l$ in (12), (58d) follows from definition of $\hat{\gamma}_t^l$ and (58e) follows from (55). However this leads to a contradiction.  \hspace{1cm} $\blacksquare$

**Lemma 7**: $\forall t \in [T], (a_{1:t-1}, x_{1:t}^l) \in \mathcal{H}_T^l$

$$V_t^l(\mu_t[a_{1:t-1}], x_{1:t}^l) = E^{\hat{\sigma}_t^l \pi_t \mu_t[a_{1:t-1}]} \left\{ \sum_{n=t}^{T} R_n^l(X_n, A_n)|a_{1:t-1}, x_{1:t}^l) \right\}$$  \hspace{1cm} (59)

**Proof**: We prove the lemma by induction. For $t = T$,

$$E^{\hat{\sigma}_T^l \pi_T \mu_T[a_{1:T-1}]} \{ R_T^l(X_T, A_T)|a_{1:T-1}, x_{1:T}^l) \}$$

$$= \sum_{x_T, a_T} \mu_T^l[a_{1:t-1}](x_T^l) R_T^l(x_T, a_T) \hat{\sigma}_T^l(a_T|a_{1:T-1}, x_{1:T}^l) \hat{\sigma}_T^l(a_T|a_{1:T-1}, x_{1:T}^l)$$  \hspace{1cm} (60a)

$$= V_T^l(\mu_T[a_{1:T-1}], x_T^l),$$  \hspace{1cm} (60b)

where (60b) follows from the definition of $V_T^l$ in (12) and the definition of $\hat{\sigma}_T$ in the forward recursion in (15).
Suppose the claim is true for $t + 1$, i.e., $\forall t \in [T], (a_{1:t}, x_{1:t+1}) \in H_{t+1}^f$

$$V_{t+1}^f(\mu_{t+1}[a_{1:t}], x_{t+1}) = \mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{t+1}[a_{1:t}]} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) | a_{1:t}, x_{1:t+1} \right\}. \tag{61}$$

Then $\forall t \in [T], (a_{1:t-1}, x_{1:t}) \in H_t^f$, we have

$$\mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}} \left\{ \sum_{n=t}^{T} R_n(X_n, A_n) | a_{1:t-1}, x_{1:t} \right\} = \mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}} \left\{ R_t(X_t, A_t) + \right.$$

\begin{align}
\mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) | a_{1:t-1}, A_t, x_{1:t}, X_{t+1} \right\} | a_{1:t-1}, x_{1:t} \right\} & \tag{62a}
\mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}} \left\{ \sum_{n=t+1}^{T} R_n(X_n, A_n) | a_{1:t-1}, A_t, x_{1:t}, X_{t+1} \right\} | a_{1:t-1}, x_{1:t} \right\} \tag{62b}
\mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}} \left\{ R_t(X_t, A_t) + V_{t+1}^f(\mu_{a_{1:t-1}}[a_{1:t}], X_{t+1}) | a_{1:t-1}, x_{1:t} \right\} \tag{62c}
\mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}} \left\{ R_t(X_t, A_t) + V_{t+1}^f(\mu_{a_{1:t-1}}[a_{1:t}], X_{t+1}) | a_{1:t-1}, x_{1:t} \right\} \tag{62d}

\mathbb{E}_{t+1:T}^f \sigma_{t+1:T, \mu_{a_{1:t}}}, x_{t} \right\}, \tag{62e}
\end{align}

where (62b) follows from an identical argument for leader as Lemma 4 in Appendix D, (62c) follows from the induction hypothesis in (61), (62d) follows because the random variables involved in expectation, $X_t^f, A_t, X_{t+1}$ do not depend on $\sigma_{t+1:T}^f \sigma_{t+1:T}$ and (62e) follows from the definition of $\sigma_t$ in the forward recursion in (15), the definition of $\mu_{t+1}$ in (16) and the definition of $V_t^f$ in (12).

**Lemma 8:** Let $\tilde{\sigma}_t^f = BR_t^f(\sigma_t^f)$ and $\tilde{\gamma}_t^f = BR_t^f(\gamma_t^f)$. Then $\forall \alpha_f, a_{1:t-1}, x_{1:t}, \gamma_t^f(a_f | x_t) = \tilde{\gamma}_t^f \alpha_f(a_f | a_{1:t-1}, x_{1:t})$, where,

$$BR_t^f(\sigma_t^f) := \cap_t \arg \max_{\sigma_f} \mathbb{E}_{t}^f \sigma_f \left\{ \sum_{n=t}^{T} \tilde{\sigma}_t^f \sigma_f | a_{1:t}, x_{1:t} \right\} \tag{63}$$

$$BR_t^f(\pi_t[a_{1:t-1}], \gamma_t^f) = \left\{ \tilde{\gamma}_t^f : \forall x_t^f \in X_f, \tilde{\gamma}_t^f (\cdot | x_t^f) \in \arg \max_{\gamma_t^f (\cdot | x_t^f)} \mathbb{E}_{t}^f (\gamma_t^f, x_t^f, \pi_t) \right\} \tag{64}$$

where

$$\pi_t[a_{1:t-1}] = P_{t+1}^f \sigma_t^f \sigma_{t+1:T}^f (x_t | a_{1:t-1}). \tag{65}$$

May 6, 2020
\textbf{Proof:}

\[
\max_{\gamma_t^f(\cdot|x_t^f)} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ R_t^f(X_t, A_t) + \delta V_t^{f_f} (E(\pi_t, \gamma_t^f, A_t), X_t^f) \mid \pi_t[a_{1:t-1}], x_t^f \right\} \\
= \max_{\gamma_t^f(\cdot|x_t^f)} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ R_t^f(X_t, A_t) + \right. \\
\mathbb{E}[\tilde{\sigma}_t^{1:T} \tilde{\gamma}_t^{1:T}, F(\pi_t, \gamma_t^f, A_t) \left\{ \sum_{n=t+1}^{T} \delta^{n-t} R_n^f(X_n, A_n) \mid h_t^f \right\} \mid \pi_t[a_{1:t-1}], x_t^f \right\} \tag{66a} \\
= \max_{\gamma_t^f(\cdot|x_t^f)} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ R_t^f(X_t, A_t) + \right. \\
\mathbb{E}[\tilde{\sigma}_t^{1:T} \tilde{\gamma}_t^{1:T}, \pi_t \left\{ \sum_{n=t+1}^{T} R_n^f(X_n, A_n) \mid a_{1:t-1}, A_t, x_{1:t}^f, X_{1:t}^f \right\} \mid a_{1:t-1}, x_{1:t}^f \right\} \tag{66b} \\
= \max_{\sigma_t^f} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ R_t^f(X_t, A_t) + \right. \\
\mathbb{E}[\tilde{\sigma}_t^{1:T} \tilde{\gamma}_t^{1:T}, \tilde{\sigma}_t^{1:T}, \tilde{\sigma}_t^{1:T}, \pi_t \left\{ \sum_{n=t+1}^{T} R_n^f(X_n, A_n) \mid a_{1:t-1}, A_t, x_{1:t}^f, X_{1:t}^f \right\} \mid a_{1:t-1}, x_{1:t}^f \right\} \tag{66c} \\
= \max_{\sigma_t^f} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ R_t^f(X_t, A_t) + \right. \\
\mathbb{E}[\tilde{\sigma}_t^{1:T} \tilde{\gamma}_t^{1:T}, \tilde{\sigma}_t^{1:T}, \tilde{\sigma}_t^{1:T}, \tilde{\sigma}_t^{1:T}, \pi_t \left\{ \sum_{n=t+1}^{T} \delta^{n-t} R_n^f(X_n, A_n) \mid h_t^f \right\} \right\} \tag{66d} \\
= \max_{\tilde{\sigma}_t^{1:T}} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n) \mid h_t^f \right\} \tag{66e} \\
= \max_{\tilde{\sigma}_t^{1:T}} \mathbb{E}[\gamma_t^f(\cdot|x_t^f)] \gamma_t^f, \pi_t \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n) \mid h_t^f \right\} \tag{66f} \\
= \mathbb{E}[\tilde{\sigma}_t^{1:T} \tilde{\gamma}_t^{1:T}, \tilde{\sigma}_t^{1:T}, \pi_t] \left\{ \sum_{n=t}^{T} \delta^{n-t} R_n^f(X_n, A_n) \mid h_t^f \right\} \tag{66g}
\]

where (66a) follows from Lemma 5 in Appendix D, (66b) follows from Lemma 4 in Appendix D, (66c) follows from the definition of \(\pi_t\) in (65), (66f) follows from (17), and (66g) follows from the definition of \(\tilde{\sigma}\).

\textbf{Lemma 9:} Let \(\tilde{\sigma}\) be an MPSE. For any given \(\sigma_t^f, a_{1:t-1}, x_{1:t}^f\), let

\[
\gamma_t^f(\sigma_t^f | a_{1:t-1}, x_{1:t}^f) = \sigma_t^f(a_t^f | a_{1:t-1}, x_{1:t}^f) \tag{67}
\]
Then

\[ P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}, \sigma^{f}_{1:t+1:T}) \sigma^{l}_{1:t-1} \sigma^{f}_{t+1:T}(x_{t+1} | a_{1:t}) = F(\mu_t[a_{1:t-1}], \gamma^{f}_t, \tilde{B}R_i^{f}(\mu_t[a_{1:t-1}], \gamma^{f}_t), a_t)(x_{t+1}) \]  

(68)

**Proof:** Let \( \sigma^{f}_t = [BR_i^{f}(a_{1:t-1}, \sigma^{i}_{1:t-1}, \sigma^{l}_{1:t-1}, \sigma^{f}_{1:t+1:T})]_t \) and \( \gamma^{f}_t = \tilde{B}R_i^{f}(\mu_t[a_{1:t-1}], \gamma^{f}_t) \). Then from Lemma 8, \( \forall a^{f}_t, a_{1:t-1}, x^{f}_{1:t}, \gamma^{f}_t(a^{f}_t | x^{f}_t) = \sigma^{f}_t(a^{f}_t | a_{1:t-1}, x^{f}_{1:t}) \), \( \forall a^{f}_t, x^{f}_t \). Thus

\[ P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}) \sigma^{i}_{1:t-1} \sigma^{l}_{t}(x_{t+1} | a_{1:t}) \]

\[ = \frac{\sum_{x^{f}_{1:t}} P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}) \sigma^{l}_{1:t-1} \sigma^{i}_{t}(x_{1:t}, a_t, x_{t+1} | a_{1:t-1})}{\sum_{x_{1:t}} P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}) \sigma^{l}_{1:t-1} \sigma^{i}_{t}(x_{1:t}, a_t | a_{1:t-1})} \]

(69)

\[ \sum_{x^{f}_{1:t}} P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}) \sigma^{l}_{1:t-1} \sigma^{i}_{t}(x_{1:t}) \sigma^{l}_{t}(a^{f}_t | a_{1:t-1}, x^{f}_{1:t}) \sigma^{f}_t(a^{f}_t | a_{1:t-1}, x^{f}_{1:t}) \]

\[ = \frac{Q(x_{t+1} | x_t, a_t)}{\sum_{x^{f}_{1:t}} P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}) \sigma^{i}_{1:t-1} \sigma^{l}_{t}(x_{1:t}, a_t | a_{1:t-1}, x^{f}_{t})} \]

(70)

\[ \sum_{x^{f}_{1:t}} P^{BR_i}(a_{1:t-1}, \hat{\sigma}^{i}_{1:t-1}, \sigma^{l}_{1:t-1}) \sigma^{i}_{1:t-1} \sigma^{l}_{t}(x_{1:t}) \gamma^{f}_t(a^{f}_t | x^{f}_t) \gamma^{f}_t(a^{f}_t | x^{f}_t) Q(x_{t+1} | x_t, a_t) \]

\[ = \frac{\sum_{x^{f}_{1:t}} \mu_t[a_{1:t-1} | x_t] \gamma^{f}_t(a^{f}_t | x^{f}_t) \gamma^{f}_t(a^{f}_t | x^{f}_t) Q(x_{t+1} | x_t, a_t)}{\sum_{x^{f}_{1:t}} \mu_t[a_{1:t-1} | x_t] \gamma^{f}_t(a^{f}_t | x^{f}_t) \gamma^{f}_t(a^{f}_t | x^{f}_t)} \]

(71)

\[ = F(\mu_t[a_{1:t-1}], \gamma^{f}_t, \tilde{B}R_i^{f}(\mu_t[a_{1:t-1}], \gamma^{f}_t), a_t)(x_{t+1}) \]

(72)

**APPENDIX G**

Part of the following proof (corresponding to the follower) is identical to the proof of Theorem 3 in [15]. It is reproduced for the sake of completion. We divide the proof into two parts: first we show that for \( i = l, f \) the value function \( V^i \) is at least as big as any reward-to-go function; secondly we show that under the strategy \( \tilde{\sigma}^i \), reward-to-go is \( V^i \).

**Part 1:** For any \( \sigma^f \) define the following reward-to-go functions

\[ W^{f, \sigma^f}(h^f_t) = E^{\sigma^f, \tilde{\sigma}^f, \mu_t[b^f_t]} \left\{ \sum_{n=t}^{\infty} \delta^{n-t} R^f(X_n, A_n) \mid h^f_t \right\} \]

(74a)
Since $X^i, A^i$ are finite sets the reward $R^i$ is absolutely bounded, the reward-to-go $W^{i, \sigma^i}_t(h^i_t)$ is finite $\forall \, i, t, \sigma^i, h^i_t$.

For any $h^i_t \in \mathcal{H}_t^i$,

$$V^i(\underline{\mu}[h^i_t], x^i_t) - W^{i, \sigma^i}_t(h^i_t) = \left[ V^i(\underline{\mu}[h^i_t], x^i_t) - W^{i, \sigma^i, T}_t(h^i_t) \right]$$

$$+ \left[ W^{i, \sigma^i, T}_t(h^i_t) - W^{i, \sigma^i}_t(h^i_t) \right]$$

(75)

(76)

Combining results from Lemmas 10 and 11 the term in the first bracket in RHS of (75) is non-negative. Using (74), the term in the second bracket is

$$\left( \delta^{T+1-t} \right) \mathbb{E}^{\sigma^i, \tilde{\sigma}^i, \mu[h^i_t]} \left\{ \sum_{n=T+1}^{\infty} \delta^{n-(T+1)} R^i(X_n, A_n) \right\}$$

(77)

+ $V^i(\Pi_{T+1}^t, X^i_{T+1}) \mid h^i_t \right\}$. 

(78)

The summation in the expression above is bounded by a convergent geometric series. Also, $V^i$ is bounded. Hence the above quantity can be made arbitrarily small by choosing $T$ appropriately large. Since the LHS of (75) does not depend on $T$, this results in

$$V^i(\underline{\mu}[h^i_t], x^i_t) \geq W^{i, \sigma^i}_t(h^i_t).$$

(79)

**Part 2:** Since the strategy $\tilde{\sigma}$ generated in (25) is such that $\tilde{\sigma}^i_t$ depends on $h^i_t$ only through $\underline{\mu}[h^i_t]$ and $x^i_t$, the reward-to-go $W^{i, \tilde{\sigma}^i}_t$, at strategy $\tilde{\sigma}$, can be written (with abuse of notation) as

$$W^{i, \tilde{\sigma}^i}_t(h^i_t) = W^{i, \tilde{\sigma}^i}_t(h^i_t)$$

(80)

$$= \mathbb{E}^{\tilde{\sigma}, \mu[h^i_t]} \left\{ \sum_{n=t}^{\infty} \delta^{n-t} R^i(X_n, A_n) \mid \underline{\mu}[h^i_t], x^i_t \right\}. 

(81)
For any \( h^i_t \in \mathcal{H}^i_t \),
\[
W^{i,\hat{\sigma}^i}_t(\mu[h^c_t], x^i_t) = \mathbb{E}^{\hat{\sigma},\mu[h^c_t]} \left\{ R^i(X_t, A_t) + \delta W^{i,\hat{\sigma}^i}_{t+1} \right. \\
\left( F(\mu[h^c_t], \theta[\mu[h^c_t]], A_{t+1}), X^i_{t+1} \right) \mid \mu[h^c_t], x^i_t \right\} \tag{82a}
\]
\[
V^i(\mu[h^c_t], x^i_t) = \mathbb{E}^{\hat{\sigma},\mu[h^c_t]} \left\{ R^i(X_t, A_t) + \delta V^i \right. \\
\left( F(\mu[h^c_t], \theta[\mu[h^c_t]], A_{t+1}), X^i_{t+1} \right) \mid \mu[h^c_t], x^i_t \right\} \tag{82b}
\]

Repeated application of the above for the first \( n \) time periods gives
\[
W^{i,\hat{\sigma}^i}_t(\mu[h^c_t], x^i_t) = \mathbb{E}^{\hat{\sigma}_t,\mu[h^c_t]} \left\{ \sum_{m=t}^{t+n-1} \delta^{m-t} R^i(X_t, A_t) \\
+ \delta^n W^{i,\hat{\sigma}^i}_{t+n}(\Pi_{t+n}, X^i_{t+n}) \mid \mu[h^c_t], x^i_t \right\} \tag{83a}
\]
\[
V^i(\mu[h^c_t], x^i_t) = \mathbb{E}^{\hat{\sigma}_t,\mu[h^c_t]} \left\{ \sum_{m=t}^{t+n-1} \delta^{m-t} R^i(X_t, A_t) \\
+ \delta^n V^i(\Pi_{t+n}, X^i_{t+n}) \mid \mu[h^c_t], x^i_t \right\} \tag{83b}
\]

Here \( \Pi_{t+n} \) is the \( n \)-step belief update under strategy and belief prescribed by \( \hat{\sigma}, \mu \).

Taking differences results in
\[
W^{i,\hat{\sigma}^i}_t(\mu[h^c_t], x^i_t) - V^i(\mu[h^c_t], x^i_t) = \delta^n \mathbb{E}^{\hat{\sigma},\mu[h^c_t]} \left\{ W^{i,\hat{\sigma}^i}_{t+n}(\Pi_{t+n}, X^i_{t+n}) - V^i(\Pi_{t+n}, X^i_{t+n}) \mid \mu[h^c_t], x^i_t \right\} \tag{84}
\]
\[
\sup_{h^i_t} \left\{ W^{i,\hat{\sigma}^i}_t(\mu[h^c_t], x^i_t) - V^i(\mu[h^c_t], x^i_t) \right\} \leq \delta^n \sup_{h^i_t} \mathbb{E}^{\hat{\sigma},\mu[h^c_t]} \tag{86}
\]
\[
\left\{ \left| W^{i,\hat{\sigma}^i}_{t+n}(\Pi_{t+n}, X^i_{t+n}) - V^i(\mu[h^c_t], x^i_t) \right| \mid \mu[h^c_t], x^i_t \right\}. \tag{87}
\]

Now using the fact that \( W^{i}_{t+n}, V^i \) are bounded and that we can choose \( n \) arbitrarily large, we get
\[
\sup_{h^i_t} \left| W^{i,\hat{\sigma}^i}_t(\mu[h^c_t], x^i_t) - V^i(\mu[h^c_t], x^i_t) \right| = 0.
\]
In this section, we present four lemmas. Lemma 10 and 11 are intermediate technical results needed in the proof of Lemma 12. Then the results in Lemma 12 and 15 are used in the previous section for the proof of Theorem 3. The proofs for Lemma 10 and 11 below aren’t stated as they are analogous (the only difference being a non-zero terminal reward in the finite horizon model) to the proofs of Lemma 3 and 4 from Appendix D, used in the proof of Theorem 1.

Define the reward-to-go $W_{f,\sigma_f^f,T}(h_f^f)$ for the follower and strategy $\sigma_f^f$ as

$$W_{f,\sigma_f^f,T}(h_f^f) = \mathbb{E}_{\sigma_f^f, \tilde{\sigma}_l^f, \mu_t}[\sum_{n=t}^{T} \delta^{n-t} R_f^f(X_n, A_n) + \delta^{T+1-t} G_f^f(\Pi_{T+1}^f, X_{T+1}^f) | h_f^f].$$  

(88)

Here agent $i$’s strategy is $\sigma_f^f$ whereas leader uses strategy $\tilde{\sigma}_l^f$ defined above. Since $\mathcal{X}_f^f, \mathcal{A}_f^f$ are assumed to be finite and $G_f^f$ absolutely bounded, the reward-to-go is finite $\forall i, t, \sigma_f^f, h_f^f$.

In the following, any quantity with a $T$ in the superscript refers the finite horizon model with terminal reward $G_f^f$.

**Lemma 10:** For any $t \in \mathcal{T}$, $h_f^f$ and $\sigma_f^f$,

$$V_{f,T}^f(\mu_t[h_c^f], x_f^f) \geq \mathbb{E}_{\sigma_f^f, \tilde{\sigma}_l^f, \mu_t}[R_f^f(X_t, A_t) + \delta V_{t+1}^{f,T}(\mathbb{E}(\mu_t[h_c^f], \tilde{\sigma}_t(\cdot | \mu_t[h_c^f], \cdot), A_t), X_{t+1}^f) | h_f^f].$$  

(89)

**Lemma 11:**

$$\mathbb{E}_{\sigma_l^f, \tilde{\sigma}_l^f, \mu_t[h_c^f], a_t}[\sum_{n=t+1}^{T} \delta^{n-(t+1)} R_f^f(X_n, A_n) + \delta^{T+1-t} G_f^f(\Pi_{T+1}^f, X_{T+1}^f) | h_f^f, a_t, x_{t+1}^f].$$  

(90)

$$= \mathbb{E}_{\sigma_l^f, \tilde{\sigma}_l^f, \mu_t[h_c^f]}[\sum_{n=t+1}^{T} \delta^{n-(t+1)} R_f^f(X_n, A_n) + \delta^{T+1-t} G_f^f(\Pi_{T+1}^f, X_{T+1}^f) | h_f^f, a_t, x_{t+1}^f].$$  

(91)

The result below shows that the value function from the backwards recursive algorithm is higher than any reward-to-go.

**Lemma 12:** For any $t \in \mathcal{T}$, $h_f^f$ and $\sigma_f^f$,

$$V_{f,T}^f(\mu_t[h_c^f], x_f^f) \geq W_{f,\sigma_f^f,T}(h_f^f).$$  

(92)
Proof 1: We use backward induction for this. At time $T$, using the maximization property from (9) (modified with terminal reward $G^f$),

$$V_{T}^{f,T}(\mu_r[h_T^c], x_T^f)$$

$$= \mathbb{E}^{\tilde{\gamma}^{f,T}(x_T^f), \gamma^{f,T}, \mu_T[h_T^c]}[R^f(X_T, A_T) + \delta G^f(\sum_{t=1}^{T} \mathbb{E}^{\mu_T[h_T^c], \gamma^T, A_T}(X_T^f) | \mu_r[h_T^c], x_T^f)$$

$$\geq \mathbb{E}^{\sigma^f, \tilde{\gamma}^{f,T}, \mu_T[h_T^c]}[R^f(X_T, A_T) + \delta G^f(\sum_{t=1}^{T} \mathbb{E}^{\mu_T[h_T^c], \gamma^T, A_T}(X_T^f) | \mu_r[h_T^c], x_T^f)$$

$$= W_{T}^{f,T}(h_T^f)$$

Here the second inequality follows from (9) and (12) and the final equality is by definition in (88).

Assume that the result holds for all $n \in \{t + 1, \ldots, T\}$, then at time $t$ we have

$$V_{t}^{f,T}(\mu_r[h_t^c], x_t^f)$$

$$\geq \mathbb{E}^{\sigma^f, \tilde{\gamma}^{f,T}, \mu_T[h_T^c]}[R^f(X_T, A_t) + \delta V_{t+1}^{f,T}(\sum_{t=1}^{T} \mathbb{E}^{\mu_T[h_T^c], \gamma^T, A_t}(X_T^f) | \mu_r[h_T^c], x_T^f)$$

$$+ \delta^{T-t} G^f(\sum_{t=1}^{T} \mathbb{E}^{\mu_T[h_T^c], \gamma^T, A_t}(X_T^f) | \mu_r[h_T^c], x_T^f)$$

$$= W_{t}^{f,T}(h_t^f)$$

Here the first inequality follows from Lemma 10, the second inequality from the induction hypothesis, the third equality follows from Lemma 11 and the final equality by definition (88).

APPENDIX I

PART 2: LEADER

In this section, we present three lemmas. Lemma 13 is an intermediate technical result needed in the proof of Lemma 14. Then the results in Lemma 14 are used in the previous section for the proof of Theorem 3. The proof for Lemma 13 below isn't stated as it is analogous (the only difference being a non-zero terminal reward in the finite horizon model) to the proofs of Lemma 3 from Appendix F, used in the proof of Theorem 1.
Define the reward-to-go $W_{t,\sigma_t}^{T,\gamma}$ for the leader and strategy $\sigma_t$ as

$$W_{t,\sigma_t}^{T,\gamma}(h_t) = \mathbb{E}_{\sigma_t}^{\mathcal{H}}[\prod_{n=t}^{T} \delta^{n-t} R_t(X_n, A_n) + \delta^{T+1-t} G_{T+1}(\Pi_{T+1}, X_{T+1}^{l}) | h_t].$$  \hspace{1cm} (95)

where $\Pi_{T+1}(x_{T+1}) = P_{\sigma_t}^{\mathcal{H}}(x_{T+1} | h_t), \forall x_{T+1}$. Since $\mathcal{X}_t, \mathcal{A}_t$ are assumed to be finite and $G_{T+1}$ absolutely bounded, the reward-to-go is finite $\forall t, \sigma_t, h_t$.

In the following, any quantity with a $T$ in the superscript refers the finite horizon model with terminal reward $G_{T+1}$.

**Lemma 13**: $\forall t \in [T], (a_{1,t-1}, x_{1,t}) \in \mathcal{H}_t, \sigma_t$

$$V_{t}^{l}((\mu_t[a_{1,t-1}], x_{1,t}^l) \geq \mathbb{E}^{\mathcal{H}}(\alpha_t | a_{1,t-1} \in \mathcal{A}_t) \{ R_t(X_t, A_t) + \delta \Pi_{t+1}(a_{1:t-1} | \gamma_t^l, X_{t+1}^{l}) | (a_{1:t-1}, x_{1:t}) \}$$  \hspace{1cm} (96)

where $\gamma_t^l$ satisfies $\forall x_{t+1}$ (such a $\gamma_t^l$ exists from Lemma 9),

$$\gamma_t^l(a_{1,t-1} | x_{1,t}) = \sigma_t^l(a_{1,t-1} | x_{1,t})$$  \hspace{1cm} (97)

$$F(\mu_t[a_{1,t-1}], \gamma_t^l, BR_t(\pi_t, \gamma_t^l), a_t)(x_{t+1}) = P_{\sigma_t}^{\mathcal{H}}(\sigma_t^l, \sigma_t^l, \sigma_t^l, \sigma_t^l, x_{1:t})$$  \hspace{1cm} (98)

The result below shows that the value function from the backwards recursive algorithm is higher than any reward-to-go.

**Lemma 14**: For any $t \in \mathcal{T}$, $h_t^l$ and $\sigma_t^l$,

$$V_{t}^{l,\sigma_t}(\mu_t[h_t^l], x_{1,t}) \geq W_{t}^{l,\sigma_t}(h_t^l).$$  \hspace{1cm} (99)

**Proof 2**: We use backward induction for this. At time $T$, using the maximization property from (9) (modified with terminal reward $G_{T+1}$),

$$V_{T}^{l}(\mu_T[h_T^l], x_{T})$$

\hspace{1cm} :\hspace{1cm} \mathbb{E}^{\gamma_T^l}(|x_T^l), \gamma_T, BR_T(\gamma_T), h_T^l) [R_T(X_T, A_T) + \delta G_{T+1}(E(\mu_T[h_T^l], \gamma_T, X_{T+1}^l) | \mu_T[h_T^l], x_T^l)]$$  \hspace{1cm} (100a)

\hspace{1cm} \geq \mathbb{E}_{\gamma_T^l}(|x_T^l), BR_T(\gamma_T^l), h_T^l) [R_T(X_T, A_T) + \delta G_{T+1}(E(\mu_T[h_T^l], \gamma_T^l, X_{T+1}^l) | \mu_T[h_T^l], x_T^l)]$$  \hspace{1cm} (100b)

$$= W_{T}^{l,\sigma_T}(h_T^l)$$  \hspace{1cm} (100d)
Here the second inequality follows from (9) and (12) and the final equality is by definition in (95). Let \( \gamma^t_i \) satisfies \( \forall x_{t+1} \) (such a \( \gamma^t_i \) exists from Lemma 9),

\[
\gamma^t_i(a^t_i|x^t_i) = \sigma^t_i(a^t_i|a_{1:t-1}, x^t_{1:t})
\]

(101)

\[
F(\mu[a_{1:t-1}], \gamma^t_i, BR^f_t(\pi_t, \gamma^t_i), a_t)(x_{t+1}) = P^{BR^f_t(\delta^t_{1:t-1} \sigma^t_{1:t+1:T}) \delta^t_{1:t-1} \sigma^t_{1:t+1:T}}(x_{t+1}|a_{1:t})
\]

(102)

Assume that the result holds for all \( n \in \{t+1, \ldots, T\} \), then at time \( t \) we have

\[
V^{i,T}_t(\mu_t[h^t_i], x^t_i) \geq \mathbb{E}^{BR^f(\sigma^t_i)\mu_t[a_{1:t-1}]} \{ R^t(X_t, A_t) + V^{i+1}_t(F(\mu[a_{1:t-1}], \gamma^t_i, BR^f_t(\pi_t, \gamma^t_i), A_t), x^t_{1:t}) \}
\]

(103b)

\[
\geq \mathbb{E}^{\sigma^t_i, BR^f(\mu_t[h^t_i], a_{1:t-1}), \mu_t[(\gamma^t_i)^T]} [R^t(X_t, A_t) + \delta \mathbb{E}^{\sigma^t_i, BR^f(\mu_t[h^t_i], a_{1:t-1}), \mu_t[(\gamma^t_i)^T]} [\sum_{n=t+1}^{T} \delta^{n-(t+1)} G^d(I_{t+1}, X^{l}_{T+1}) | h^t_i, A_t, X^{l}_{t+1}] | h^t_i]
\]

(103c)

\[
= \mathbb{E}^{\sigma^t_i, BR^f(\mu_t[h^t_i], a_{1:t-1}), \mu_t[(\gamma^t_i)^T]} [\sum_{n=t}^{T} \delta^{n-t} R^t(X_n, A_n) + \delta^{T+1-t} G^d(I_{t+1}, X^{l}_{T+1}) | h^t_i]
\]

(103d)

\[
= W^{i,T}_t(h^t_i)
\]

(103e)

Here the first inequality follows from Lemma 13, the second inequality from the induction hypothesis and the fourth equality by definition (95).

**APPENDIX J**

The following result highlights the similarities between the fixed-point equation in infinite horizon and the backwards recursion in the finite horizon.

**Lemma 15:** Consider the finite horizon game with for \( i = l, f \), \( G^i \equiv V^i \). Then \( V^{i,T}_t = V^i \), \( t \in \{1, \ldots, T\} \) satisfies the backwards recursive construction stated above (adapted from (9) and (12)).

**Proof:** Use backward induction for this. Consider the finite horizon algorithm at time \( t = T \), noting that \( V^{i,T}_T = G^i \equiv V^i \),

\[
BR^f_T(\pi, \gamma^i) = \left\{ \bar{\gamma}^i_T : \bar{\gamma}^i_T \in \arg \max_{\gamma^i_T} \mathbb{E}^\gamma^i_T(X, A) + \delta V^i(F(\pi_T, \gamma^i_T, A_T), x^{l}_{T+1}) \right\}_T
\]

(104a)
where expectation in (9) is with respect to random variables \((X_T^l, A_T, X_T')\) through the measure 
\[
\pi_T(x_T^l)\gamma_T(a_T^l|x_T^l)\gamma_T(a_T'|x_T^l)Q(x_T'|x_T, a_T) \quad \text{and} \quad \mathbb{E}
\]
and 
\[
\mathbb{E}^{\bar{B}R_T^l}(\pi_T\gamma_T)\gamma_{\bar{T}}^l, \pi_T \left\{ R_T^l(X, A) + \delta V^l(\mathbb{E}(\pi_T^l, \gamma_T^l, \bar{B}R_T^l(\pi_T\gamma_T^l), A_T^l), X_T'|\pi_T, x_T^l) \right\} \quad (104c)
\]
where the above expectation is defined with respect to random variables \((X_T, A_T, X_T')\) through the measure \(\pi_T(x)\gamma_T^l(a_T^l)\gamma_T(a_T'|x_T)Q(x_T'|x_T-1, a_T)\), and \(\tilde{\gamma}_T \in \bar{B}R_T^l(\pi_T\gamma_T)\). And \(\forall x_T^l \in \mathcal{X}^l\),
\[
V_{T}^{l,T}(\pi_T^l, x_T^l) = \mathbb{E}^{\tilde{\gamma}_T^l(\cdot|x_T^l)}\gamma_{\bar{T}}^l, \pi_T \left\{ R_T^l(x, A) + \delta V^l(\mathbb{E}(\pi_T^l, \tilde{\gamma}_T^l, A_T^l), X_T'|\pi_T, x_T^l) \right\} \quad (104d)
\]
\[
V_{T}^{l,T}(\pi_T^l, x_T^l) = \mathbb{E}^{\tilde{\gamma}_T^l, \pi_T} \left\{ R_T^l(X, A) + \delta V^l(\mathbb{E}(\pi_T^l, \tilde{\gamma}_T^l, A_T^l), X_T'|\pi_T, x_T^l) \right\} \quad (104e)
\]
Comparing the above set of equations with (19), we can see that the pair \((V, \tilde{\gamma})\) arising out of (19) satisfies the above. Now assume that \(V_{n}^{l,T} \equiv V^l\) for all \(n \in \{t + 1, \ldots, T\}\). At time \(t\), in the finite horizon construction from (104), (12), substituting \(V^l\) in place of \(V_{t+1}^{l,T}\) from the induction hypothesis, we get the same set of equations as (104). Thus \(V_{t}^{l,T} \equiv V^l\) satisfies it.

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