On the stability of the open-string QED neutron and dark matter

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Abstract We study the stability of a hypothetical QED neutron, which consists of a color-singlet system of two $d$ quarks and a $u$ quark interacting with the QED interaction. As a quark cannot be isolated, the intrinsic motion of the three quarks in the lowest-energy state may lie predominantly in 1+1 dimensions, as in a $d$-$u$-$d$ open string. The attractive $d$-$u$ and $u$-$d$ QED interactions may overcome the weaker repulsive $d$-$d$ QED interaction to bind the three quarks together. We examine the QED neutron in a phenomenological three-body problem in 1+1 dimensions with an effective interaction extracted from Schwinger’s exact QED solution in 1+1 dimensions. The phenomenological model in a variational calculation yields a stable QED neutron at 44.5 MeV. The analogous QED proton with two $u$ quarks and a $d$ quark has been found to be too repulsive to be stable and does not have a bound or continuum state, onto which the QED neutron can decay via the weak interaction. Consequently, the QED neutron is stable against the weak decay, has a long lifetime, and is in fact a QED dark neutron. It may be produced following the deconfinement-to-confinement phase transition of the quark gluon plasma in high-energy heavy-ion collisions. Because of the long lifetime of the QED dark neutron, self-gravitating assemblies of QED dark neutrons or dark antineutrons may be good candidates for a part of the primordial dark matter produced during the phase transition of the quark gluon plasma in the evolution of the early Universe.

Keywords Anomalous soft photons · X17 · E38 · Schwinger QED2 · Open string · Dark matter

1 Introduction

Recent experimental observations of (i) the anomalous soft photons [1,2,3,4,5,6,7,8,9], (ii) the X17 particle at about 17 MeV [10,11,12], and (iii) the E38 particle at about 38 MeV [13,14,15,16] have generated a great deal of interests [17]-[52]. With a mass in the region of many tens of MeV, the produced anomalous particles appear to lie outside the domain of the Standard Model. Many speculations have been proposed for these objects, including the cold quark-gluon plasma, QED mesons, the fifth force of Nature, the extension of the Standard Model, the QCD axion, dark matter and many others [53].

Among the many proposed explanations for the above three anomalies, the description of the quantized QED mesons [28,29,30,31,32,33,34] has the prospect of linking them together in a consistent framework. We note that the anomalous soft photons are consistently produced as observed excess $e^+e^-$ pairs when hadrons are produced, and they are not produced when hadrons are not produced [12,34,54]. The correlated production of anomalous soft photons alongside with hadrons suggests that a parent particle of the anomalous soft photons is likely to contain some elements of the hadrons, such as a light quark-antiquark pair. The transverse momentum range of the anomalous soft photons of many tens of MeV/c suggests further that the parent particle is likely the QED excitation of such a
quark-antiquark pair. For, it was shown by Schwinger previously that massless fermions (and their antiparticles) interacting in a gauge field with a coupling constant $g_{\alpha \beta}$ in 1+1 dimensions lead to stable bosons with a mass $m = g_{\alpha \beta}/\sqrt{\alpha}$ [31,32,33,34,35]. A Schwinger boson can be viewed as the stable collective excitation of the fermions and their gauge fields in the vacuum. It can be alternatively viewed as a confined open-string state of a fermion-antifermion pair bound together by their mutual gauge field interaction. We shall review Schwinger’s boson in (1+1)D QED in Appendix A and apply it to quarks interacting in QED and QCD in Appendix B. Thus, if we treat the light quarks as Schwinger’s massless fermions subject to QED and QCD gauge interactions as discussed in Appendix A and B, the ratio of the mass of a $q\bar{q}$ composite of the QED excitation (a QED meson) to the mass of a $q\bar{q}$ composite of the QCD excitation (a QCD meson) as given by Eq. (B.16) of Appendix B would be of order

$$m_{\text{QCD meson}}/m_{\text{QED meson}} \sim g_{\text{QED}}/g_{\text{QCD}} \sim \sqrt{\alpha_{\text{QED}}/\alpha_{\text{QCD}}} \sim \sqrt{1/137}/0.6 \sim 1/5,$$  

placement the QED meson within the domain of the anomalous soft photons. It was therefore proposed in [30,31] that QED excitations of quarks may lead to confined, open-string QED mesons with masses of many tens of MeV. These QED mesons may be produced simultaneously along with the QCD mesons when the quark system is excited in high-energy collisions [1,2,3,4,5,6,7,8,9], and the excess $e^+e^-$ pairs may arise from the decays of these QED mesons. By examining both the QCD string and the QED string excitations, using the method of bosonization [30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49], and extrapolating from the QCD string excitations to the QED string excitations, the mass of the $I(J^P)=0(0^-)$ isoscalar QED meson was predicted to be $17.9\pm1.8$ MeV and the mass of the isovector $(I(J^P)=1(0^-), I_3=0)$ QED meson to be $36.4\pm3.8$ MeV [31]. These QED meson masses match those of the X17 particle, the E38 particle, and the possible parent particles of the anomalous soft photons, indicating that these anomalous particles are consistent with their description as QED excitations of the quark vacuum.

It is commonly argued that quarks experience QCD and QED interactions simultaneously and that the quark confinement is a property of quarks in QCD alone. It may appear at first sight that the above suggestion of stable and QED excitations of confining quarks may appear impossible. Is it ever possible for quarks (and antiquarks) to interact with the QED interaction alone without the QCD interaction? Can there be stable collective QED excitations of quarks and antiquarks, similar to the stable collective QCD quark-antiquark excitations? If quarks can interact with the QED interaction alone, can there be additionally other similarly favorable neutral multiquark systems stabilized by the QED interaction?

With regard to the first question whether it is ever possible for quarks (and antiquarks) to interact with the QED interaction alone, we can consider as an example the production of a quark and an antiquark pair by $e^+ + e^- \to \gamma^* \to q + \bar{q}$ or $e^+ + e^- \to \gamma^* \to q + \bar{q}$ with a center-of-mass energy $\sqrt{s}$ in the range $(m_q + m_{\bar{q}}) < \sqrt{s} < m_\pi$, where the sum of the rest masses of the light quark and light antiquark is of order a few MeV and $m_\pi \sim 135$ MeV [29]. The incident $e^+ + e^-$ pair is in a colorless color-singlet state, and thus the produced $q\bar{q}$ pair is also a color-singlet pair. The produced $q\bar{q}$ pair, if they can ever be produced in this range of $\sqrt{s}$, can only interact with the QED interaction but not with the QCD interaction, because the QCD interaction would result in a composite $q\bar{q}$ hadron state with a center-of-mass energy $\sqrt{s}$ beyond this energy range, in a contradictory manner. It is therefore possible for a quark and an antiquark to interact with QED interaction alone, if they are produced in the range of $\sqrt{s}$ below the pion mass. If the produced quark and the antiquark are not confined by the QED interaction, the produced quark-antiquark pair will appear as a continuum state with separated fractional charges of a quark and an antiquark, at energies away from the bound state energies. The non-observation of fractional charges for the $e^+ + e^- \to q + \bar{q}$ reaction in this $\sqrt{s}$ range away from the bound state energies, is consistent with the confinement of quarks in QED interactions in (3+1)D.

In other reactions, there are circumstances in which a $q\bar{q}$ pair can be produced with a center-of-mass energy $\sqrt{s}$ lower than the pion mass and can interact in the QED interaction alone without the QCD interaction. For example, we mentioned earlier the production of a $q\bar{q}$ pair interacting in QED interactions as a possible description for the anomalous soft photons, the X17 particle, and the E38 particle with masses in the range of many tens of MeV, as described in [31]. In other circumstances in the deconfinement-to-confinement phase transition of the quark-gluon plasma in high-energy heavy-ion collisions, a deconfined quarks and a deconfined antiquark in close spatial proximity can coalesce to become a $q\bar{q}$ pair with a pair energy below the pion mass, and they can interact in QED interactions alone.

In the dynamics of quarks and antiquarks in a general reaction, the quark currents and the gauge fields are not single-element functions. They are in fact $3\times3$ color matrices. Quarks in triplet $\bar{3}$ representation and
antiquarks in 3 representation form the product group of $3 \otimes 3 = 1 \oplus 8$, which contains the color-singlet 1 subgroup and the color-octet 8 subgroup. Quark currents $j^\mu$ and gauge fields $A^\mu$ in the color-singlet and color-octet subgroups execute collective dynamics within their respective subgroups, as discussed in Appendix A and B. Quarks and antiquarks can therefore interact with the QED interaction alone within the color-singlet subgroup, if their center-of-mass energy is below the pion mass.

To describe the quark-QCD-QED dynamics in detail, we envisage the vacuum to comprise of quarks filling up the negative-energy Dirac sea, with transient quark-antiquark pairs, gluons, and photons lurking here, occupying states above the Dirac sea and valence antiquarks as unoccupied hole states below the Dirac sea. The interaction Lagrangian is

\[
\mathcal{L}^\text{int} = \mathcal{L}^\text{int}_\text{QED} + \mathcal{L}^\text{int}_\text{QCD},
\]

\[
\mathcal{L}^\text{int}_\text{QED} = \sum_{f,a} g^\text{QED} \overline{Q}^f_{\mu} \gamma_\mu (A^\mu_{\text{QED}})_{a} \psi^f_{a}(x),
\]

\[
\mathcal{L}^\text{int}_\text{QCD} = \sum_{f,a} g^\text{QCD} O^f_{\mu} \gamma_\mu (A^\mu_{\text{QCD}})_{a} \psi^f_{a}(x),
\]

where $\mu = 0, 1, 2, 3$ are the space-time index, $a = 1, 2, 3$ is the color label, $f$ is the flavor label, $g^\text{QED,QCD}$ are the coupling constants, and $O^f_{\mu}$ is the charge number of $f$-flavor quark in QED and QCD respectively. In particular, the QED gauge field $A^\mu_{\text{QED}}(x)$ consists of the component $A^\mu_0(x)$ associated with the generator $t^0$ of the QED U(1) gauge group,

\[
A^\mu_{\text{QED}}(x) = A^\mu_0(x) t^0, \quad \text{where} \ t^0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The QCD gauge field $A^\mu_{\text{QCD}}(x)$ consists of eight components $A^i_{\text{QCD}}(x)$ associated with the eight Gell-Mann matrices generators, $i = 1, 2, 3, 8$ of the QCD color SU(3) gauge group,

\[
A^\mu_{\text{QCD}}(x) = \sum_{i=1,2,3,8} A^i_{\text{QCD}}(x) t^i, \quad 2 \text{Tr} \{ t^a t^b \} = 3 \delta^{ab}.
\]

The initial QED and QCD gauge field disturbances will act on the quark field through the Dirac equation

\[
\gamma_\mu \left[ i \partial^\mu + \sum_a^{8} g^\text{QED}_f g^\text{QCD}_f \gamma_\mu (A^\mu_{\text{QED}})_{a} \right] \psi^f_a(x) = 0
\]

where $g^\text{QED}_f = g^\text{QED} Q_f^\text{QED}$ and $g^\text{QCD}_f = g^\text{QCD} Q_f^\text{QCD}$, $Q_f^\text{QED} = 2/3$, $Q_f^\text{QCD} = -1/3$, and $Q_{f(a,d,s)}^\text{QCD} = -1$. The initial QED and QCD gauge field disturbances will induce a change of the state vector $\psi^f_i(x)$ of all quark states and subsequently a change of the quark currents $j^\mu_i(x)$. Just as the gauge fields, the induced quark currents $j^\mu_i(x)$ are likewise a $3 \times 3$ matrices in color space which can be similarly described in terms of 9 independent generators,

\[
j^\mu_i(x) = \sum_{i=0,1,2,8} j^\mu_i(x) t^i.
\]

In particular, they can be divided into the QED quark current $j^\mu_i(x) = j^\mu_0(x) t^0$ where

\[
j^\mu_0(x) = \sum_{f,a} g^\text{QED} Q_f^\text{QED} \overline{Q}^f_{\mu} \gamma_\mu (A^\mu_{\text{QED}})_{a} \psi^f_a(x) \big|_{x' \to x}
\]

and the QCD quark currents $j^\mu_i(x) = \sum_{f,a} g^\text{QCD} Q_f^\text{QCD} \overline{Q}^f_{\mu} \gamma_\mu (A^\mu_{\text{QCD}})_{a} \psi^f_a(x) \big|_{x' \to x}.

Through the Maxwell equations for QCD and QED, the color-singlet QCD current $j^\mu_{\text{QCD}}$ will generate a color-singlet QED gauge fields $A^\mu_{\text{QED}}$, while the color-octet QCD currents $j^\mu_{\text{QCD}}$ will generate a color-octet QCD gauge fields $A^\mu_{\text{QCD}}$. Along the evolution loop $A^\mu \to j^\mu \to A^\mu$, the requirement that the generated gauge fields $A^\mu$ at the end of loop be the same as the gauge field disturbance $A^\mu$ initially introduced provides the condition for stable QED and QCD collective excitations of the quark vacuum as described in Appendix B.

For quarks in QCD, the idealization of a flux tube in $(3+1)D(x_1,x_2,x_3,x_4)$ as a one-dimensional string in $(1+1)D(x_1,x_2)$ is well known since the works of the dual resonance model \[74\] the Nambu-Goto string model \[71,72\], the t’Hoof two-dimensional model \[86,73,74\], the inside-outside-cascade model \[69\], the yo-yo string model \[75\] and the Lund model \[76\]. QCD lattice gauge calculations exhibit explicitly the structure of a flux tube in 3+1 dimensions \[71,73,77\].

For quarks in QED, whether quarks are confined in the U(1) QED gauge interaction in (3+1)D has not been unequivocally demonstrated, although there have been strong hints pointing to the confinement of quarks in QED if they can interact in the QED interaction alone. The principal cause of uncertainty arise from the fact that the QED U(1) gauge field admits two different versions of gauge theories with different confinement properties. As emphasized by Polyakov \[80,81\] and Drell et al. \[82\], there is the compact U(1) QED gauge theory containing the gauge fields $A^\mu$ as angular variables with periodic properties. Defined on a lattice, the compact U(1) QED gauge theory has the gauge field
was subsequently confirmed by Drell and collaborators

dealing with the coupling constants, no matter how weak. Such a result

transverse confinement persists for all non-vanishing

gauge fields. This is carried out in [34]. Specifically, Polyakov [80, 81]
result to infer transverse confinement of quarks interact-
acting in compact QED interaction in the transverse

which has self-interacting photons. There is also the

non-compact U(1) QED gauge field theory with the
gauge field action

which has non-interacting photons. Even though the

compact and the non-compact theories have the same

continuum limit, they have different confinement prop-
erties. A pair of opposite charges in non-compact QED
interaction are not confined whereas a pair of opposite
charges in compact QED interaction are confined under
appropriate conditions [80][81][82].

As pointed out by Yang [33], the quantization and the

commensurate property of the electric charges of the

interacting particles imply the compact property of the

underlying QED gauge group. Because quarks

are confined and quark electric charges are quantized and

commensurate, it is reasonable to propose that

quarks and antiquarks interact in the compact QED
U(1) interaction. Accordingly, we can apply the results

of Polyakov [80][81], Drell et al. [82], and Schwinger [53]

[55] to simplify the dynamics of quarks and the QED
gauge field from (3+1)D(x1,x2,x3,x0) to (1+1)D(x2,x0),
as is carried out in [54]. Specifically, Polyakov [80][81]

showed that a pair of opposite electric charges interact-

ing in compact QED interaction in the transverse

(2+1)D(x2,x0) space-time are confined, and that the

transverse confinement persists for all non-vanishing

confining constants, no matter how weak. Such a result

was subsequently confirmed by Drell and collaborators
[82]. As explained in [82], the transverse confinement

of the opposite charges in compact QED gauge field in

(2+1)D(x2,x0) arises from the periodicity of the
gauge field as a function of the spatial angular vari-

able, leading to transverse gauge photons that are self-

interacting through a periodic and bounded potential in

the neighborhood of the electric charge. These trans-

verse gauge photons interact among themselves, they do

not radiate away, and they join the two opposite charges

by a confining force [82]. Upon applying Polyakov’s re-

sult to infer transverse confinement of quarks interact-

ing in compact QED in the transverse space of \{x1,x2\},
the dynamics in the (3+1)D(x2,x0) space-time can

be approximated as the dynamics in an idealized

(1+1)D(x2,x0) space-time, with the information in the

transverse degrees of freedom on the \{x1\} transverse

plane stored as input parameter properties in the ide-
alized (1+1)D(x2,x0) space-time. It is necessary to ex-
amine in this idealized (1+1)D(x2,x0) space-time arena

whether quarks and gauge fields are longitudinally con-

fined. If we next consider further that the light quarks

can be approximated to be massless, then according
to Schwinger’s exact solution for massless fermions in

such a (1+1)D(x2,x0) QED [54][55], a light quark and

its antiparticle interacting in the QED interaction will

be longitudinally confined just as well and will form a

stable QED quark-antiquark system [34].

From such considerations, as reviewed in Appendix
A and B and in [28][31][34], we follow Schwinger and we
find theoretically that there is a stable QED quark

current j_{QED}(x) = j_{QED}^{0}(x) which obeys the Klein-Gordon

equation with a QED meson mass, m_{QED \text{ meson}}, in addi-
tion to a stable QCD quark current j_{QCD}^{u}(x) along a unit vector in the t^{1,2,...,8} generator space which

obeys the Klein-Gordon equation with a different QCD

meson mass, m_{QCD \text{ meson}} [28][31][32][33][34]. In 1+1 di-

mensional space-time, stable collective dynamics of the

QED and QCD currents and their associated gauge

fields can be independently excited in their respective

color subgroup spaces. The corresponding confined QED

and QCD mesons are string-like excitations occurring

diffent energies in such 1+1 dimensional space-

It is interesting to point out that in previous lattice
gauge calculations, Wilson, t’Hooft, Polyakov, Kogut,
Susskind, Mandelstam, Banks, Jaffe, Peskin, Drell, Guth,
Kondo and many others [35][31][30][27][26][30][31][32][81][82]
[28][34] showed that fermions and antifermions in the

compact QED U(1) gauge interaction in 3+1 dimen-
sions have a confining phase for strong coupling and a

non-confining phase for weak coupling. Recent lattice
gauge calculations using tensor networks with dynamical
fermions for compact QED in 3+1 dimensions con-
firm such a result [94]. As quarks are fermions, we expect
therefore that under appropriate conditions, quarks
interacting in QED in 3+1 dimensions can be confined.

The observation and the proposed interpretation of the
anomalous particles as confined composite qq states
provide a special impetus to examine in future lattice
gauge calculations whether quarks in color SU(3) inter-
acting in compact QED U(1) gauge fields are confined
or not, given the quark QED coupling constant and
light quark masses as they are and assuming the QCD
gauge fields to be non-participating spectators. In this
regard, efficient methods of lattice gauge calculations
with dynamical quarks using the tensor network [94],

\begin{equation}
S = \frac{1}{2g^2} \sum_{x,\alpha\beta} (1 - \cos F_{x,\alpha\beta}), \tag{11}
\end{equation}

where g is the coupling constant and F_{x,\alpha\beta} is

\begin{align}
F_{x,\alpha\beta} &= A_{x,\alpha} + A_{x+\alpha,\beta} - A_{x+\beta,\alpha} - A_{x,\beta}, \tag{12a}
\end{align}

with \(-\pi \leq A_{x,\alpha} \leq \pi,

\begin{align}
F_{x,\alpha\beta} &= A_{x,\alpha} + A_{x+\alpha,\beta} - A_{x+\beta,\alpha} - A_{x,\beta}, \tag{12b}
\end{align}

with \(-\pi \leq A_{x,\alpha} \leq \pi,

\begin{align}
S = \frac{1}{4g^2} \sum_{x,\alpha\beta} F_{x,\alpha\beta}^2, \tag{13a}
\end{align}

with \(-\infty \leq A_{x,\alpha} \leq +\infty,

\begin{align}
S = \frac{1}{4g^2} \sum_{x,\alpha\beta} F_{x,\alpha\beta}^2, \tag{13b}
\end{align}

with \(-\infty \leq A_{x,\alpha} \leq +\infty,
dual presentation [93], magnetic-field digitization [96], regulating magnetic fluctuations [97], or other efficient methods will be of great interest. Clearly, whatever the theoretical predictions there may be, the confining property of quarks interacting in QED interactions in 3+1 dimensions must be tested by experiments. The present investigation on QED mesons and the stable quark systems in QED interactions serves to facilitate the examination of such an important question.

If quarks and antiquarks can interact with the QED interaction alone under appropriate conditions, can there be other similarly favorable neutral quark systems stabilized by the QED interaction between the constituents in the color-singlet subgroup, with the color-octet QCD gauge interaction as a spectator field? Of particular interest is the QED neutron with the gauge interaction as a spectator field? Of particular interest is the QED neutron with the $d$, $u$, and $d$ quarks. The three quarks form a color product group of $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$, which contains a color-singlet subgroup $1$ where the color-singlet current and the color-singlet QED gauge field reside. In the color-singlet system of the three quarks at energies below the QCD nucleon mass, the three quarks can interact with the QED interaction alone. The QED interaction is attractive between quark electric charges of opposite signs, and is repulsive between quark electric charges of the same sign. We depict in Fig. 1 the color-singlet $d$-$u$-$d$ system with three different colors, and we display the different QED forces acting on the three quarks, with the magnitude proportional to $|Q_i Q_j|$ for the force between quark $i$ and quark $j$, an attractive force for negative $Q_i Q_j$, and a repulsive force for positive $Q_i Q_j$. The attractive $d$-$u$ and $u$-$d$ QED interactions between the two $d$ quarks and the center $u$ quark in Fig. 1(a) may overcome the weaker repulsive $d$-$d$ QED interaction between the two $d$ quarks, to bind the three quarks in the QED neutron. The lowest-energy state is likely to reside in 1+1 dimensions because they are confined, and there is an attraction between the $u$ quark and the $d$ quarks, but a repulsion between the two $d$ quarks. However, the analogous configuration of the QED proton in Fig. 1(b) with two $u$ quarks and one $d$ quark in the $u$-$d$-$u$ configuration may likely be unstable because of the stronger repulsion between the two $u$ quarks in comparison with the weaker attractive interaction between the $d$ and the $u$ quarks. We would like to show quantitatively in section 2 that this may indeed be the case. The conclusions we will reach concerning the stability of the QED neutron and proton will also be applicable to the stability of QED antinucleons.

It is worth pointing out that theoretical and experimental investigations on the QED neutron are interesting on many accounts. First is the possibility of its being a new exotic member in the families of particles of the Standard Model. Its properties probably set it apart from other particles because it is a special combination of the light quark fields and the QED gauge fields that has not yet been known up to now. It also calls for a better understanding on the role of the interplay between different fields in the confinement process, whether confinement is limited only to constituents in the presence of the QCD gauge fields, or it suffices to involve also the light quark fields and the QED gauge fields by themselves without the QCD interactions. Clearly, this is a fundamental question that can be answered only by experiments. The observation of the anomalous soft photons, the X17, and the E38 particles indicate that the confined QED meson states can arise from QED interactions of quarks and antiquarks. The investigations presented here and elsewhere [28,29,30,31] will provide additional experimental tests on such a basic property.

There is also the additional interest in the QED neutron, along with the QED mesons and their corresponding antiparticle counterparts, as candidate particles for the primordial dark matter because they are massive, they may be produced during the deconfinement-to-confinement stage of the quark gluon plasma phase transition, and the quark-gluon plasma phase may occur in the early history of the Universe or in high-energy heavy-ion collisions.

To study the stability of the QED neutron and proton with three light quarks, we need to develop the tools for the relativistic three-body problem by cali-
brating the effective two-body QED interaction using Schwinger’s exact QED solution for massless fermions in 1+1 dimensions in the Appendix. We generalize the relativistic two-body problem to the three-body problem and evaluate in section 2 the ground state energy for the $d$-$u$-$d$ configuration in a variational calculation, which yields a stable QED neutron. Section 3 deals with the property of the lowest state of the QED neutron. We further examine the potential energy as a function of the separations between the quarks. In section 4, we study the stability of the analogous QED $u$-$d$-$u$ proton. We find that there is no stable energy minimum for the QED proton. The total potential between quarks interacting in QED alone exhibits repulsive behavior that does not allow the binding of the $u$-$d$-$u$ system.

In section 5, we discuss other favorable quark configurations for which attractive QED interactions may be present to stabilize the composite multiquark QED system. In section 6, we examine the decay and the detection of the QED neutron to facilitate its detection. We discuss the implication of the instability of the QED proton and the possibility of the QED neutron to be a dark neutron with a very long lifetime. In Section 7, we conclude our discussions. In Appendix A, we review Schwinger’s exact solution of massless fermions in QED in 1+1 dimensions and see how it may be transcribed into a two-body problem. In Appendix B, we study how the color-singlet and the color-octet excitations arise in the quark-QCD-QED medium by applying Schwinger’s model to massless quarks in QCD and QED in (1+1)D. In Appendix C we examine the two-body problem from which the effective interaction is determined. Appendix D gives the two-body solution. Appendix E shows that for the lowest energy two-body bound state a variational calculation gives the same result as solving the two-body problem in a wave equation. Appendix F examines the decays and the detection of QED mesons.

2 Stability of the QED neutron with three quarks interacting in QED interactions

To study the QED neutron, we construct a composite system of two $d$ quarks and one $u$ quark by selecting quarks of three different colors to form a color-singlet state and search for its lowest energy bound state. In the formulation of Dirac, Todorov, Crater, Van Alf"{a}tine, and Sazdjian, and many others, relativistic many-body treatments of bound states have been carried out in QCD and QED with a high degree of successes. The basic ingredients consist of treating particles and antiparticles as independent positive-energy entities with effective interactions between them. Each particle obeys a mass-shell constraint on: (i) the momentum, (ii) the particle mass, and (iii) the effective interactions from the other particles. The effective interactions can be obtained by matching with the perturbative or non-perturbative counterparts of the field theory or by phenomenological considerations. In accordance with Dirac’s constraint dynamics, relativistic many-body interactions must however be compatible with each other, resulting in additional functional requirements or additional terms in the equivalent Schrödinger-type equations whose eigenvalues lead to the eigenstates and the masses of the composite particle in question.

We consider a three-quark system with a two-body effective interaction $\Phi_{ij}(x_{ij})$ arising from the particle $j$ at $x_j$ acting on the particle $i$ at $x_i$, depending on the relative coordinate $x_{ij} = x_i - x_j$. The lowest-energy state is likely to reside in 1+1 dimensions because quarks are confined, and there is an attraction between the $u$ quark and the $d$ quarks but a repulsion between the two $d$ quarks. We place the $d$, $u$, and $d$ quarks on the $x$-axis with coordinate labels $x_1$ and $x_3$ for the two $d$ quarks and $x_2$ for the $u$ quark. By allowing all $x_i$ coordinates to assume both positive and negative values, while fixing the center of mass position (see Eq. (23) below), we allow all possible arrangement of the ordering of the positions of the three quarks in the variations. We wish to find out quantitatively whether the attractive QED interactions between the $d$ quarks and the $u$ quark can overcome the repulsive and weaker QED interaction between the two $d$ quarks so as to stabilize the QED neutron, as discussed schematically in Fig. 1(a).

For simplicity in our first survey, we neglect particle spins whose effects on the confining effective interaction are expected to be small\footnote{Schwinger’s exact solution for massless fermions in QED in 1+1 dimensions indicates that the boson mass of the composite fermion-antifermion system depends only on the gauge coupling constant and is independent of the total spin of the fermion-antifermion pair. Consequently, it can be inferred that spin effects on the effective confining interaction are small.}. The spin will lead to a fine-structure splitting of the QED neutron states which can be studied when more data become available. We work in the three-quark center-of-mass system in which $P = p_1 + p_1 + p_3 - (P^0, P) = (M, 0)$, and the relative coordinate $x_{ij\perp} = (x_i - x_j)$ transverse to $P$ involves only spatial coordinates $x_i$ and $x_j$. The three particle momenta in the CM system are

$$p_i = (\epsilon_i, q_i), \quad i = 1, 2, 3, \quad (14)$$

where $\epsilon_i = \frac{p_i \cdot P}{\sqrt{P^2}}, \quad q_i = p_i - \frac{p_i \cdot P}{P^2} P, \quad (15)$

A space-time vector $x_{\perp}$ is transverse to another space-time vector $P$ if $x_{\perp} \cdot P = 0$. 

$$\text{if } x_{\perp} \cdot P = 0.$$
and we consider the particles to be of positive energy only, with \( \epsilon_i > 0 \). The rest mass \( M \) of the composite particle is

\[
M = p^0 = \epsilon_1 + \epsilon_2 + \epsilon_3.
\]

(16)

We generalize the two-body equations of (C.17a) and (C.17b) to the three-body problem by imposing three mass-shell constraints relating the momenta, the masses, and their interactions in the form

\[
\mathcal{H}_1|\Psi\rangle = \left\{ p_1^2 - m_1^2 - \left[ \Phi_{12}(x_{12}) + \Phi_{13}(x_{13}) \right] \right\} |\Psi\rangle = 0,
\]

(17a)

\[
\mathcal{H}_2|\Psi\rangle = \left\{ p_2^2 - m_2^2 - \left[ \Phi_{21}(x_{21}) + \Phi_{23}(x_{23}) \right] \right\} |\Psi\rangle = 0,
\]

(17b)

\[
\mathcal{H}_3|\Psi\rangle = \left\{ p_3^2 - m_3^2 - \left[ \Phi_{31}(x_{31}) + \Phi_{32}(x_{32}) \right] \right\} |\Psi\rangle = 0.
\]

(17c)

The compatibility conditions on the above mass-shell constraints lead to the requirement that \( \Phi_{ij}(x_{ij})- \Phi_{ji}(x_{ij}) \) be \( x_{ij} = x'_{ij\perp} \) which is transverse to the combined momentum \( P_{ij}=p_i + p_j \). This \( x'_{ij\perp} \) coordinate should be the relative spatial coordinate in the frame in which the center-of-mass of the system of constituents \( i \) and \( j \) is at rest. For the three-body problem, the center-of-mass motion of any two constituents \( i \) and \( j \) has a velocity \( V_{ij}=(q_i + q_j)/(\epsilon_i + \epsilon_j) \) and may not be at rest. Even though \( V_{ij} \) may not be zero, it will be constrained and limited in a bound state. It is reasonable to neglect such velocities so that we can approximate the \( x'_{ij\perp} \) that is transverse to momentum \( P_{ij} \) to be the relative coordinate \( x_{ij\perp} \) that is transverse to the total center-of-mass momentum \( P \) instead. In such an approximation, the relative coordinate \( x_{ij\perp} \) in the effective interaction \( \Phi_{ij}(x'_{ij\perp}) \) becomes just the relative coordinate \( x_{ij\perp} \) of \( i \) and \( j \) in the three-quark-center-of-mass system. Corrections for such an approximation will be of order \( V_{ij} \), which can be taken as velocity-dependent corrections in future refinements.

In the wave equations \[17a\]-\[17c\], the phenomenological effective QED interaction extracted from Schwinger’s exact QED solution in Eqs. (D.30) and (D.40) in Appendix D is

\[
\Phi_{ij}(x_{ij\perp}) = \frac{2\epsilon_i \epsilon_j}{\epsilon_i + \epsilon_j} (-Q_i Q_j) \kappa | x_i - x_j | \text{ with } \kappa = e^2/4\pi,
\]

(18)

where \( e = e_{2D} = g_{2D}^{QED} \) is the QED coupling constant\(^3\) in \((1+1)D\).

\(^3\)We adopt here the notations that \( e \) is actually \( e_{2D} \), the QED coupling constant in \( 1+1 \) dimensions, and \( e_{3D} \) is the QED coupling constants in \( 3+1 \) dimensions, with the fine structure constant defined by

\[
\alpha = \frac{\alpha_{\text{EM}}}{e^{2D}} = \frac{\alpha_{\text{EM}}}{4\pi} = 1/137.
\]

Equations \[17\] and \[18\] constitute the system of relativistic three-body equations for three quarks interaction in the effective QED interactions. We would like to investigate whether there is a lowest-energy equilibrium state of the QED neutron by using a variational wave function. It is convenient to choose a Gaussian variational wave function of the spatial dimensionless spatial variables \( y_1, y_2, y_3 \) with standard deviations \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) as variational parameters,

\[
\Psi(y_1, y_2, y_3) = N \exp \left\{ \frac{-y_1^2}{4\sigma_1^2} - \frac{-y_2^2}{4\sigma_2^2} - \frac{-y_3^2}{4\sigma_3^2} \right\},
\]

(19)

where \( y_i = \sqrt{\kappa} x_i \). The charge numbers of the quarks are \( Q_1 = Q_3 = -1/3 \), and \( Q_2 = 2/3 \). The expectation values of \[17a\]-\[17c\] using the variational wave function \( \Psi \) are

\[
\left\langle \Phi_{ij} \right| \frac{e^2}{\kappa} \left| \Phi_{ij} \right\rangle = \left\langle \frac{1}{2\sigma_1^2} - \frac{y_1^2}{4\sigma_1^2} \right\rangle + \frac{m_1^2}{\kappa}
\]

(20a)

\[
\left\langle \Phi_{ij} \right| \frac{e^2}{\kappa} \left| \Phi_{ij} \right\rangle = \left\langle \frac{1}{2\sigma_2^2} - \frac{y_2^2}{4\sigma_2^2} \right\rangle + \frac{m_2^2}{\kappa}
\]

(20b)

\[
\left\langle \Phi_{ij} \right| \frac{e^2}{\kappa} \left| \Phi_{ij} \right\rangle = \left\langle \frac{1}{2\sigma_3^2} - \frac{y_3^2}{4\sigma_3^2} \right\rangle + \frac{m_3^2}{\kappa}
\]

(20c)

Because of the symmetry of the two \( d \) quarks we can assume for the lowest-energy state

\[
\sigma_1 = \sigma_3,
\]

(21)

so that the variational parameters consist only of \( \sigma_1 \) and \( \sigma_2 \). We look for the state with the lowest composite mass \( M \) in the variations of \( \sigma_1 \) and \( \sigma_2 \),

\[
\delta^2 M(\sigma_1, \sigma_2) \frac{\delta \sigma_1 \delta \sigma_2}{\delta \sigma_1 \delta \sigma_2} = 0.
\]

(22)

The motion of the three quarks should maintain a fixed center of mass for the composite system. It is necessary for the coordinates of the three quarks to satisfy the center-of-mass condition on the spatial coordinates,

\[
\sum_{i=1}^{3} \epsilon_i y_i = 0.
\]

The variational wave function \( \Psi \) of Eq. (19) is normalized according to

\[
\int dy_1 dy_2 dy_3 |\Psi(y_1, y_2, y_3)|^2 \delta(\epsilon_1 y_1 + \epsilon_2 y_2 + \epsilon_3 y_3) = 1.
\]

(24)
Because of the CM condition, there are actually only two independent spatial variables which can be chosen to be \( y_1 \) and \( y_2 \). However, we need to treat all three spatial variables as independent in the beginning, and impose the CM constraint [25] only when we evaluate the expectation values in [20] to calculate \( \epsilon_i \) and \( M \) at the end.

![Fig. 2. The mass \( M \) of the QED neutron as a function of the variation parameters \( \sigma_1, \sigma_2 \) in units of \( h/\sqrt{\kappa} = 8.29 \text{ fm} \). The QED neutron has an energy minimum at \( M = 44.5 \text{ MeV} \) at \( \sigma_1/\sqrt{\kappa} = 2.40 \) and \( \sigma_2/\sqrt{\kappa} = 1.09 \).](image)

In the evaluation of the QED neutron mass \( M \), the unknown quantities \( \epsilon_i \) are needed to defined the effective interactions. They can be obtained self-consistently and iteratively with initial guesses. Knowing the effective interactions and the variational parameters \( \sigma_1 \) and \( \sigma_2 \), we evaluate the expectation values on the right hand sides of (20a)-(20c) numerically. The calculated values of \( \epsilon_i \) on the right hand sides of (20a)-(20c) can form the basis of the next iteration until convergence is achieved. In the numerical calculations, we have used quark masses \( m_u = 2.16 \text{ MeV} \) and \( m_d = 4.67 \text{ MeV} \).

By such variational calculations, we find that the mass \( M \) as a function of \( \sigma_1 \) and \( \sigma_2 \) has an energy minimum, \( M = 44.5 \text{ MeV} \), at \( \sigma_1 = 2.40h/\sqrt{\kappa} = 19.9 \text{ fm} \) and \( \sigma_2 = 1.05h/\sqrt{\kappa} = 8.71 \text{ fm} \) as shown in Fig. 2.

### Table 1 Properties of the lowest-energy QED neutron

| Quantity | Value |
|----------|-------|
| \( M \) (mass of the QED neutron) | 44.5 MeV |
| \( \sqrt{\epsilon_1^2} = \sqrt{\epsilon_2^2} \) (the \( d \) quark) | 11.3 MeV |
| \( \sqrt{\epsilon_1^2} \) (the \( u \) quark) | 21.8 MeV |
| \( \sqrt{((x_1-x_2)^2)^2} \) (between \( d \) quark and \( u \) quark) | 20.4 fm |
| \( \sqrt{((x_3-x_1)^2)^2} \) (between two \( d \) quarks) | 28.2 fm |
| \( \sigma_1 \) (of wave function for the \( d \) quarks) | 19.9 fm |
| \( \sigma_2 \) (of wave function for the \( u \) quark) | 8.71 fm |

The force vectors in Fig. 1d give a qualitative description of the various forces leading to the binding of the three quarks in a QED neutron. The variational calculations demonstrate the stability of the QED neutron in a quantitative analysis. It is illuminating to see how the effective interactions between the three quarks can bind them together into a QED neutron from a more quantitative viewpoint as an illustration. For such a purpose, we add the wave equations in (17) and get the total mass-shell condition

\[
\left\{ \sum_{i=1}^{3} (\epsilon_i^2 - m_i^2) - \sum_{i=1}^{3} q_i^2 \right\} \Psi(x_1, x_2, x_3) = 0.
\]

This is just a three body system with a total effective interaction

\[
\Phi_{\text{tot}}(x_1, x_2, x_3) = 2[\Phi_{12}(x_{12}) + \Phi_{13}(x_{13}) + \Phi_{23}(x_{23})].
\]

We can acquire a better understanding how the three quarks can bind together in the QED neutron when we examine various components of the effective interactions between different pairs of quarks as a function of a set of representative spatial coordinates. We can choose a sample set of representative coordinates such that the first \( d \) quark coordinate is \( y_1 \), the \( u \) quark coordinate is at \( y_2 = 0 \), and the second \( d \) quark coordinate is at \( y_3 = -y_1 \) because of the CM constraint. For such a sample set of representative coordinates, we can study the behavior of various effective interactions which can be expressed as functions of a single variable \( y_1 \). The various effective interactions \( 2\Phi_{ij}/\sqrt{\kappa}e_1 \) between quark
Effective interactions as a function of $y_1$ at $y_2=0$ and $y_3=-y_1$

$\Phi_{tot} = 2\Phi_{12} + 2\Phi_{23} + 2\Phi_{31}$, and $\Phi_{tot}$ is the total effective interaction for a selected sample $d-u-d$ arrangement of the three quarks shown at the upper left corner. The potentials are obtained as a function of $y_1$, at $y_2 = 0$, $y_3 = -y_1$, where $y_1$, $y_2$ and $y_3$ are the positions of the $d$, $u$, and $d$ quarks, respectively.

At the minimum energy point, the values of $\epsilon_1$ and $\epsilon_2$ are $\epsilon_1 = 0.4767\sqrt{\kappa}$, $\epsilon_2 = 0.9163\sqrt{\kappa}$, and so $\epsilon_2/\epsilon_1 = 1.922$ and the above dependencies can be evaluated. We show the effective interactions $\Phi_{tot}$ and $\Phi_{ij}$ between different quarks, as a function of $y_1$ for the QED neutron at $y_2 = 0$, $y_3 = -y_1$ in Fig. 3. The attractive $u-d$ interactions $2\Phi_{12}/\sqrt{\kappa}\epsilon_1$ and $2\Phi_{23}/\sqrt{\kappa}\epsilon_1$ are shown as the dashed curve. The repulsive $d-d$ interaction $2\Phi_{13}/\sqrt{\kappa}\epsilon_1$ is shown as the dashed-dot curve in Fig. 3. The total effective interaction $\Phi_{tot}$ is displayed as the solid curve which is a confining interaction that binds the three quarks together. Hence there is a stable QED neutron arising from the balances of the mutual electrostatic forces between the quarks.

With the above confining potential in Fig. 3 as an illustrative example, we can understand the energy minimum of the QED neutron in two intuitive ways. In the description of the classical string $[75,76]$, the two $d$ quarks execute yo-yo motion shuttling about the $u$ quark back and forth after reaching the longitudinal turning points in the confining potential of Fig. 3. In the quantum mechanical description, the attractive net confining QED interaction as shown in Fig. 3 between the quarks is counterbalanced by the quantum stress pressure [113] that arises from the derivatives of the single-particle wave function, reaching the lowest-energy equilibrium between the attractive QED interaction and the quantum stress pressure.

4 The stability of the QED proton and the QED neutron weak decay

We would like to study next whether QED color-singlet proton with two $u$ quarks and a $d$ quark can be stable. For such a calculation, we carry out the variational calculations as in the above QED neutron case, with the $u$ and $d$ quarks in the QED neutron replaced by $d$ and $u$ quarks respectively. That is, we consider the $u$, $d$, and $u$ quarks to be placed on the $x$-axis with coordinate labels $x_1$ and $x_3$ for the two $u$ quarks and $x_2$ for the $d$ quark. By allowing all $x_i$ coordinates to assume both positive and negative values, while fixing the center of mass position (Eq. (23)), we allow all possible arrangement of the ordering of the positions of the three quarks in the variations, including both the linear $u-d-u$ configuration as in Fig. 4(b) and the $u-u-d$. In this case of QED proton, we have $Q_1 = 2/3$, $Q_2 = -1/3$, and $Q_3 = 2/3$.

Our variations over a very large range of $\sigma_1$ and $\sigma_2$ values fail to find an energy minimum. Extending the range of $\sigma$ will only drive the total energy of the system lower with the $u$ quarks farther and farther apart without the energy turning to a minimum. The condition of (22) cannot be satisfied for this case. We can understand the failure by looking at the total potential $2\Phi_{tot}$ at a sample arrangement shown in Fig. 4. The effective potentials at $y_2 = 0$, $y_3 = -y_1$ for the sample case with $\sigma_1 = \sigma_3 = 2.09\sqrt{\kappa}$ and $\sigma_2 = 2.80\sqrt{\kappa}$, which give $\epsilon_2/\epsilon_1 = 3.71$ are shown in Fig. 4. The magnitude of the sum of the attractive effective interactions $2\Phi_{12} - 2\Phi_{23}$ between the $d$ quark and the two $u$ quarks is smaller than the magnitude of the repulsive interaction $2\Phi_{13}$ between the two $u$ quarks. The total effective interaction $\Phi_{tot}$ is repulsive; it decreases as $|y_1|$ increases. Hence, a QED proton does not possess a stable bound state. The QED proton also does not possess a continuum state with isolated quarks because the isolation of quarks as color-triplet quarks is forbidden. Therefore, the QED proton does not exist either as a stable bound state nor a continuum state with isolated quarks. There is no QED proton state.
Effective interactions as a function of $y_1$

\[ \Phi_{ij} = 2\Phi_{12} + 2\Phi_{13} + 2\Phi_{23} \text{ in units of } \sqrt{\kappa \epsilon} \text{ for a QED proton, where } \Phi_{ij} \text{ is the effective interaction between quarks at } y_1 \text{ and } y_2 \text{ and } \Phi_{\text{tot}} \text{ is the total effective interaction, for a selected sample } u-d-u \text{ arrangement of the three quarks shown at the left upper corner. The potentials are obtained as a function of } y_1, y_2 = 0, y_3 = -y_1, \text{ where } y_1, y_2 \text{ and } y_3 \text{ are the positions of the } u, d, \text{ and } u \text{ quarks respectively. The total effective interaction } \Phi_{\text{tot}} \text{ shown as the solid curve is a linear repulsive interaction, indicating that the QED proton is not stable.}

5 Other favorable QED quark configurations and QED quark matter

Although the QED proton with two up quarks and a down quark cannot have a stable bound or continuum state, other neutral QED quark and antiquark systems and their corresponding charge-conjugate counterparts may have stable configurations arising from quarks and antiquarks interacting in QED interactions in the color-singlet $1$ subgroup of the direct product color group, with the QCD gauge interactions as spectator fields. Whether heavy quarks can also interact in QED interactions in this sector cannot be excluded, and it is of interest to examine such a possibility theoretically so as to facilitate future experimental assessments on the occurrence probabilities. By including heavy quarks also among the quarks interacting in QED interactions with the QCD gauge fields as non-participating spectators in Fig. 5, there can be a large number of favorable configurations of neutral open-string systems in which the forces acting on the quarks and antiquarks are subject to a linear QED force that is attractive between two charges of opposite signs and repulsive for two charges of the same sign. The QED attractive forces in these configurations may overwhelm the repulsive forces to lead to stable color-singlet QED composite particles with different flavor contents and quark numbers.

Fig. 5 Some favorable neutral open-string configurations for the lowest energy states involving (a) a quark and an antiquark, (b) three quarks, and (c) three quarks plus a quark-antiquark pair, for quarks and antiquarks interacting in QED forces in the color-singlet $1$ subgroup of the product group. Their electric field lines of force are indicated by arrows and the quark electric charge numbers are listed. These open-string configurations may be stabilized by the linear QED forces which are attractive between charges of opposite signs and repulsive between charges of the same sign.

In Fig. 5, we list different choices of flavors for each of the quarks. The quark and the antiquark in Fig. 5(a) have electric charges of opposite signs, and the attractive QED forces between them can stabilize the system. There can be additional flavor mixing considerations if one further assumes flavor SU(2) or SU(3) symmetries as discussed in [31]. Fig. 5(b) shows three quarks, with two quarks of electric charges of $(-1/3)$ and a quark of
charge $(2/3)$. They are the analogue of the QED neutron and are likely to be stable. Fig. 5(c) shows a linear chain of five quarks. An example of such a configuration is $d$-$u$-$\bar{u}$-$u$-$d$ which can be built even longer as $d$-$u$-$\bar{u}$-$u$-$\bar{u}$-$d$-$u$-$\bar{u}$-$d$-$u$-$\bar{u}$-$d$-$u$, with $n = 0, 1, 2...$. It have electric charges with alternating signs such that the QED forces between them may be attractive and balanced to stabilize the system.

In the configurations in Fig. 5, one can construct color-singlet states by choosing color-anticolor combinations for the quark-antiquark combinations in Fig. 5(a), three different colors for the three-quark combination in Fig. 5(b), and three different colors for the three-quark combination, and a color and an anticolor for the additional quark-antiquark pair in Fig. 5(c). It will be of interest to study theoretically and experimentally whether these neutral color-singlet quark systems are stable.

Stable QED mesons and QED neutrons, if found to occur, can lead to the QED composite matter, a new type of matter that is a collection of QED mesons and QED neutrons interacting with each other by weak Van der Waal-type interactions between electric multipoles. The QED composite matter can make a transition to become a deconfined quark-QED plasma at the phase transition temperatures $T_{\text{QED}}^{\text{trans}}$. By dimensional analysis following Eq. (17), the ratio of the phase-transition temperatures, $T_{\text{QED}}^{\text{trans}}/T_{\text{QCD}}^{\text{trans}}$, at which QED and QCD mesons become deconfined, would be approximately proportional to their gauge field coupling constants. This places $T_{\text{trans}}^{\text{QED}}/T_{\text{trans}}^{\text{QCD}} \sim 20$ MeV. In the finite baryon density regime, the deconfinement phase transition occurs when the degenerate fermion pressure of the dense QED neutrons overwhelming the QED confinement interaction $\Phi$.

The quark-QED plasma with varying net baryon densities will have its equilibrium density and its own equation of state. They may be produced in high-energy heavy ion collisions, in the core of neutron stars, or in some stages of neutron star mergers. It has been suggested that the deconfined phase of quark matter may be present in the core of massive neutron stars. In this case, QED quark photon plasma and QED neutrons may also be present in the hadron-quark matter transition region close to the deconfined quark matter core. Future studies on the quark-QED plasma properties and its possible phase transition to QED neutrons will be of great interest.

6 The detection of a QED neutron

A QED neutron is a composite object containing two $d$ quarks and one $u$ quark interacting in QED interactions. The relativistic three-body equations of $(17a)$, $(17b)$, $(17c)$ with a confining potential such as shown in Fig. 8 are expected to have many eigenstates and eigenvalues. The inclusion of spin-spin, spin-orbit, and other interactions between the quarks in a future fully three-dimensional calculation will add a greater degree of complexity to the spectrum of the QED neutron.

As the detection of a QED neutron will depend on its decay products, we would like to examine how a QED neutron decays from an excited state $n_{\text{QED}}^{\text{initial}}$ to a final state $n_{\text{QED}}^{\text{final}}$. We note first of all what it will not do. It will not dissociate itself into isolated quark constituents because of the non-isolation nature of the quarks. It will not decay by weak interactions into a positively charged entity because there is no bound or continuum final QED proton state.

![](image)

**Fig. 6** The decay modes of a QED neutron from an excited eigenstate $n_{\text{QED}}^{\text{initial}}$ to the final eigenstate $n_{\text{QED}}^{\text{final}}$: (a) by the emission of a photon, and (b) by the emission of a QED meson which subsequently can decay into real or virtual photons, or a dilepton pair as described in Fig. 9(a), 9(b), and 9(c) in Appendix F.

In 1+1 dimensions, an excited state of QED neutron cannot decay as the photon is represented in effect by an effective potential $\Phi$ as discussed in Appendix C, and the quarks do not radiate photon. In the physical 3+1 dimensions, the transverse structure of the flux tube must be taken into account and the photon emission channel from the quark opens up. A quark can make a sharp change of its trajectory turning to the transverse direction with the emission of a photon from the excited $n_{\text{QED}}$. By such an emission process at the vertex $V_1$ as depicted in diagrams in Fig. 6(a), a quark in an excited QED neutron at the initial excited eigenstate $n_{\text{QED}}^{\text{initial}}$ can de-excite to reach the final eigenstate $n_{\text{QED}}^{\text{final}}$. The multipolarity of the photon transition will depend on the spins and the parities of the initial and final states in question. Alternatively, a valence quark in an initial excited eigenstate $n_{\text{QED}}^{\text{initial}}$ can de-excite by the emission of a photon at the vertex $V_1$ in Fig. 6(b) leading to the production of a quark-
antiquark pair at the vertex $V_2$. The produced quark can join up with the remaining two quarks of the initial QED neutron $n_{QED}^*(\text{initial})$ to become the final QED neutron eigenstate $n_{QED}^* (\text{final})$, while the produced antiquark can combine with the valence quark to form a QED meson. In such a decay with the emission of a QED meson as depicted in Fig. 6(b), the flavor of the produced pair at $V_2$ must agree with the flavor of the valence quark emitting the virtual photon at $V_1$ so that the flavor and the charge of the final QED neutron remains unchanged.

It is easy to envisage that successive emissions of photons and QED mesons will allow an excited QED neutron $n_{QED}^*$ to de-excite, and eventually reach the lowest energy QED neutron state. Through out the de-excitation process, the three quarks constituents remain bound to each other as an entity, in the conservation of the QED neutron number, while the emitted QED mesons will decay into real photons, virtual photons in the form of dileptons, or a dilepton pair as described by diagrams Figs. 9(a), 9(b), and 9(c) in Appendix F. At the end of the de-excitation, only the ground state QED neutron remains and it does not radiate because it is the lowest energy QED neutron state. Because the lowest-energy QED neutron ground state does not decay or radiate, it is therefore a QED dark neutron.

With the exception of the QED dark neutron which has no decay products, the detection of a QED neutron can be carried out by searching for their decay products of photons and QED mesons arising from the de-excitation of the excited QED neutron states with the emitted QED mesons detected as diphoton resonances. We envisage that by the coalescence of the quarks of different colors, QED neutrons at the lowest-energy state as well as the excited states may be produced during the deconfinement-to-confinement phase transition of the quark gluon plasma. The de-excitation of the excited QED neutron states will yield photons and QED mesons of various energies exhibiting the spectrum of the QED neutron system. The de-excitation energies provide information on the QED neutron structure. We may rely on the presence of these emitted photons and QED mesons to reconstruct the complete spectrum of the QED neutron. The de-excitation may also go through many steps with sequential emissions of QED mesons and/or photons. Accordingly, we can look for unknown photons and QED mesons that accompany the production of other photons and QED mesons.

We note with keen interest that QED neutrons may be produced by the coalescence of deconfined quarks in the core of a dense neutron star or in high-energy heavy-ion collisions during the phase transition of the quark gluon plasma. The production of the QED neutron or its excited states can be used as a signature for the quark-gluon plasma production, if the QED neutron can be so identified. It is necessary to identify the QED neutron if it is produced. A QED neutron can be identified by its QED meson emission spectral lines which exhibit its own characteristic structure. With the mass of the lowest energy QED neutron state predicted to be 44.5 MeV, and if a harmonic oscillator spectrum for the QED neutron can be an order-of magnitude guide, we expect the masses of the emitted QED mesons in the decay of an excited QED neutron state to be of order 50 MeV. We can therefore estimate the production of diphoton resonances with an invariant mass of order 50 MeV to accompany the production of the QED neutron. A way to distinguish those QED mesons as arising from the QED neutron decay or from a $q\bar{q}$ pair production is to study the differences of the QED meson emission spectral lines as a function of the probability for the occurrence of deconfinement, which is correlated with the mass, energy, and centrality of the high-energy collision process. For those collisions with a low probability leading to deconfinement, the detected QED mesons emission spectral lines will not contain those lines emitted from the excited QED neutrons. On the other hand, for collisions with a high likelihood of attaining deconfinement, there will be excited QED neutrons with the accompaniment of the emission of QED mesons with these spectral lines from their de-excitation. By judicial correlations with the probability of such collisions, those QED meson spectral lines associated with the de-excitation of the excited QED neutrons may be separated. The on-set of these extra QED meson lines arising from the decay of excited QED neutrons may be a good signature of the on-set of deconfinement and a signature of the quark-gluon plasma formation.

The search of the QED neutron may also be carried out by elastic and inelastic collisions of the QED neutron with electrons in semiconductors and superconductors, using detectors designed to search for dark matter particles with a mass below 1 GeV/$c^2$. (See Ref. [115] for a review and a list of references for such detectors). The proposed QED neutron has a predicted mass of 44.5 MeV. It is a linear electric quadrupole and it interacts with an electron by the electromagnetic quadrupole interaction $e^2 a^2 \cos^2 \theta / 3|r|^3$ where $r$ is the radius vector from the center of mass of the QED neutron to the electron, $\theta$ is the opening angle between $r$ and the QED neutron linear axis, and $a$ is the separation between the $u$ and the $d$ quark. The separation $a$ has been estimated in Table I to be 20.4 fm. In a charged-coupled device (CCD) detector, the collision between the QED neutron and the electrons of the device may promote an electron from the valence band to
the conduction band. In a transition-edge sensor (TES) detector, which is a superconducting thin film held very close to its transition temperature, the collision may deposit energy onto the thin film. As a result of the energy deposition, the superconducting film will make a transition to the normal state signaled by an increase in its resistance. Another method makes use of a superconducting nanowire single-photon detector (SNSPD) of a thin film of superconducting material carrying a constant applied current. The collision between the QED neutron and the electrons of the detector may disrupt the superconductivity and create a measurable signal. Future studies of the feasibility on the search for the QED neutron using semiconductor and superconductor devices will be of great interest.

7 Summary, Conclusions, and discussions

The recent observations of the anomalous soft photons, the X17 particle, and the E38 particle have been consistently interpreted as the production of QED mesons in [21,22,23,24] arising from the QED excitation of quarks and antiquarks in the QCD+QED gauge field vacuum. Heretofore our usual experience of quarks interacting with gauge fields have been confined mainly to the situation with quarks interacting with both QCD and QED gauge interactions simultaneously, and they are invariably accompanied by gluon exchange interactions. How is it possible for a quark and an antiquark to interact with the QED interaction alone without the QCD interaction?

We have presented arguments in the Introduction to show that a quark and an antiquark can interact with the QED interaction alone if they are produced in the range \( m_q + m_{ar{q}} < \sqrt{s} < m_x \). We note that the QED and QCD excitations of the quark vacuum can be independent excitations. The important ingredient in resolving the conceptual difficulties rests on the fact that the quark currents and gauge fields are not single-element quantities in QED and QCD dynamics. On the contrary, they are \( 3 \times 3 \) matrices in color space, forming a colors-singlet \( 1 \) subgroup and a color-octet \( 8 \) subgroup. The quark QCD and QED currents and gauge fields can be independently excited in their respective subgroups. Hence, there can be QED excitations at the lower energies of many tens of MeV, leading to the production of the QED mesons, with the QCD gauge fields as non-participating background spectators.

If light quarks can interact with QED interactions in the color-singlet sector with the QCD interaction as non-participating background fields, a new frontier will be opened up for exploration because quarks carry electric charges, electric charges of opposite signs attract, and attractive interactions result in stable and confined composite quark states. There will be many composite systems of light quarks and antiquarks in which the attractive QED forces allow the composite system to form bound and confined states. The many quantum numbers that characterize the quarks will also add complexities to the spectrum of these composite particles. For example, for light quarks with two flavors and \( S=0 \) moving in phase and out of phases with each other, we show earlier that there can be the \((I=0,J_3=0)\) state at 17.9±1.8 MeV and the \((I=1,J_3=0)\) state at 36.4±3.8 MeV. We do not know whether QED mesons formed by a light quark and a light antiquark of the same flavor are allowed or not. If they are allowed, then by using the semi-empirical formula for the QED meson state energy developed in [31], one locates theoretically the \( S=0\) single-flavor excitation \( dd \) QED meson state at 21.2±2.1 MeV and the \( uu \) QED meson state at 34.7±3.5 MeV.

The possible occurrence of the QED mesons can be tested by searching for the decay products of two real photons, two virtual photons in the form of double dilepton pairs, or a single dilepton pair. The X17 particle observed in the decays of the \(^{3}He^*\) and \(^{8}Be^*\) [10,11] with an \( e^+e^-\) invariant mass of 17 MeV and the state at 19±1 MeV in emulsion studies [12] match the predicted mass of the isoscalar \( 0(0^-) \) QED meson [31]. The E38 MeV particle, observed in high-energy \( pC, dC, dCu \) collisions at Dubna [13,14] with a \( \gamma \gamma \) invariant mass of about 38 MeV, matches the predicted mass of the isovector QED meson [31]. These are encouraging experimental observations. The QED \((uu)\) and QED \((dd)\) states have yet to be located and identified experimentally. Further experimental measurements in the low invariant mass region will be of great interest.

The QED mesons are not the only color-singlet composite states arising from quarks interacting in QED interactions in the color-singlet subgroup, with the QCD gauge fields as non-participating spectators. The QED neutron with two \( d \) quarks and one \( u \) quark with three different colors can form a color-singlet composite system. The QED neutron can be stable because the attractive QED interactions between two \( d \) quarks and the \( u \) quark overcome the weaker repulsion between the two \( d \) quarks. With a phenomenological three-body model in 1+1 dimensions with an effective interaction between electric charges extracted from Schwinger’s exact QED solution, we find quantitatively in a variational calculation that there is a QED neutron energy minimum at a mass of 44.5 MeV. The analogous QED proton with two \( u \) quarks and a \( d \) quark has been found to be too repulsive to be stable and does not have a bound or continuum state.
Because of its composite nature, there will likely be many excited states in the QED neutron. The excited states are expected to decay by emitting photons and/or QED mesons to make transitions to lower QED neutron states. One of the two $d$ quarks may decay into an $u$ quark by way of the weak interaction. However, because the QED proton does not possess a stable bound state nor a continuum state of isolated quarks, the rate of the QED neutron weak decay into a QED proton is zero.

Among all QED neutron states, the ground QED neutron state located at 44.5 MeV distinguishes itself from higher excited QED neutron states as a stable particle without decay products. It can only decay by a baryon-number non-conserving transition, which presumably has a very long lifetime. As a consequence, the lowest state QED neutron is a dark neutron. The QED antineutron ground states is likewise a dark antineutron. The only mode of destruction for a QED dark neutron and a QED dark antineutron is their mutual annihilation, with the production of photons and QED mesons.

It is of interest to discuss the relation between the QED neutron at 44.5 MeV and the QCD neutron at 939.6 MeV that is confined by QCD gauge interactions. We envisage that the QED neutron is an energy minimum in the color-singlet subgroup of quark current and QED gauge field, while the QCD neutron is an energy minimum in the color-octet subgroup of quarks interacting with the exchanges of gluons between quarks. They are similar to the multiple energy minima which occur in the collective energy surface of nuclear systems in \[115,116\]. As members of different subgroups, they do not mix in the lowest order but can be mixed in second-order perturbation theory by photon interactions. The mixing of QCD neutron in the QED neutron may be small because the QED neutron can be connected to the QCD neutron by electromagnetic interactions but the difference in their longitudinal extensions of wave functions and the large energy denominator in second-order perturbation theory may make the mixing quite small. The sea quarks arise predominantly from the splitting of the gluons into quark-antiquark pairs which appear in the color-octet subgroup of the product group. Because the QED neutron resides in the color-singlet subgroup, the presence of the sea quarks in the QCD color-octet subgroup will not likely affect the stability of the QED neutron.

On account of their being predicted to be stable particles with a very long lifetime without decay products, the QED dark neutrons and QED dark antineutrons may be good candidate particles for a part of the dark matter. We envisage that in the early evolution of the Universe after the big bang, the Universe will go through the quark-gluon plasma phase with deconfined quarks and gluons. As the primordial matter expands and cools down the quark-gluon plasma undergoes a phase transition from the deconfined phase to the confined phase, deconfined quarks of three different colors may coalesce to form color-singlet states. While many of the color-singlet systems of three quarks have sufficient energy to form hadrons, there may be some produced color-singlet three-quark systems in which their total energy is below the QCD neutron energy of about 1 GeV. These three-quark systems may form QED neutron states that are bound by QED interactions. The de-excitation of the excited QED neutron state will find its way down to the lowest energy state of the QED dark neutron. Such QED dark neutrons and its excited states may occur at the deconfinement-to-confinement phase transition of the quark-gluon plasma and may be a signature of the deconfinement-to-confinement transition of the quark gluon plasma in high-energy heavy-ion collisions. Self-gravitating assemblies of QED dark neutrons may be stable astrophysical objects. Because of its long lifetime, self-gravitating QED dark neutron assemblies (and similarly QED dark antineutron assemblies) of various sizes may be good candidates for a part of the primordial dark matter produced during the deconfinement-to-confinement phase transition of the quark gluon plasma in the evolution of the early Universe.

In another matter, LIGO recently observed the merger of two neutron stars through the detection of their gravitational waves in 2017 \[118\]. Such a merger will likely lead to the production of a quark matter with deconfined quarks \[119,120,121\]. The authors of \[119,120\] proposed a new signature for a first-order hadron-quark phase transition in merging neutron stars, which may provide the opportunity to study the properties of the post-merger quark matter. As the quark matter cools and undergoes the deconfinement-to-confinement phase transition during the merging process, the coalescence of deconfined quarks to become confined quarks will produce QED neutrons in the post-merger environment.

In another astrophysical frontier, it has been suggested that deconfined quark matter may be present in the core of a massive neutron star \[114\]. In such a neutron star, the transition region close to the core may contain QED neutron matter arising from the coalescence of deconfined quarks. In matter of modeling massive neutron star mergers of such a type, it may be necessary to take into account the presence of such a QED neutron region. Therefore, the study of the equation of states and the thermodynamical and phase transition properties of the QED neutron matter will be of
great interest. QED neutrons in the neutron star environment will experience a strong magnetic field which will affect the QED neutron in a significant way. While the confinement of QED neutron may continue to be operative, we expect that the additional magnetic interaction will set the QED neutron into a rotational motion so that the quarks in the QED neutron will go from a 1+1 dimensional dynamics to a full three-dimensional dynamics. It will be of interest to study how the magnetic field will affect the QED neutron and its stability.

As it is suggested here that the confinement to deconfinement phase transition at the early history of the Universe in the quark-gluon plasma phase may generate the QED dark neutron assemblies as seeds for the primordial dark matter, it will be of great interest to study whether QED dark neutrons and/or its excited states may be produced in high-energy heavy-ion collisions where quark gluon plasma may be produced. The detection of the QED dark neutrons may be made by searching for the photons and/or QED mesons during the de-excitation from its excited states. The de-excitation of the excited QED neutron states will yield photons and QED mesons with its own characteristic QED meson emission spectrum. The capability of precise dilepton measurements in high-energy heavy-ion collisions may make it possible to study the spectrum of the produced QED mesons through their decays into two virtual photons, as discussed in Appendix F. We may rely on the presence of these emitted photons or produced QED mesons to reconstruct the spectrum of the QED neutron.

For simplicity in the present first survey of the QED neutron, we have neglected the spin degree of freedom. While the spin will not likely affect the stability, the quark confinement, and the gross structure of the QED neutron, it will play a significant role in the fine structure and the spectrum of the QED neutron. The spin degree of freedom, along with the orbital angular momentum, the collective rotation, and the collective vibration should be taken into account in future studies. Theoretical investigations on the internal structure and the energy spectrum of the QED neutron will be valuable to assist the detection of the produced QED neutrons. How the QED mesons and QED neutron interact with themselves and with hadrons will open up another avenue to explore the interplay between species from the same branch and from different branches of the quark family of the Standard Model. Furthermore, the possibility of the many-body interaction between QED dark neutrons forming a bound multi-QED-neutron system and the interaction between the QED neutron matter and other standard QCD matter will add other dimensions to the complexity of matter associated with the QED dark neutron. It will be of great interest to extend the frontier of QED neutrons both theoretically and experimentally.

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Appendix A: Schwinger’s boson and massless fermions in 1+1 dimensional QED

Our goal is to study the stability of color-singlet states involving three light quarks in QED interactions. To pave the way for such an investigation, we would like to sharpen our theoretical tools by examining the analogous two-body problem of a massless fermion-antifermion pair in QED, for which the exact solution from Schwinger is already known [54,55].

Schwinger showed previously that in 1+1 dimensions, massless fermions and antifermions interacting with the QED gauge interaction with a coupling constant \( e = e_A = g_{\mu\nu} \) give rise to a bound boson with a mass \( m \), given by [54,55] \( m = \frac{e}{\sqrt{2}} \). (A.1)

where the QED coupling constant \( e \) has the dimension of a mass in 1+1 dimensions. In terms of the description of \( e \) as a unit of charge and \( Q \) as the charge number, the fermion can be described as possessing a charge \( e \) and a charge number \( Q = 1 \) and the antifermion a charge \( (-e) \) and a charge number \( Q = (-1) \) in the Schwinger model.

A derivation of Schwinger’s exact solution of (A.1) can be found in [55] and explained in details in Chapter 6 of [98]. Recent generalizations and extensions of the Schwinger model have been presented in [99,100,101]. It is illuminating to review its salient points here to see in what way we may treat Schwinger’s boson as a relativistic two-body problem. Schwinger’s exact solution can be obtained self-consistently as a many-body field theory problem involving the response of all fermions in the presence of a perturbing gauge field \( A^\mu \). We start by considering a vacuum state in which all the negative-energy states of the massless fermions in the
Dirac sea are occupied. A disturbance in the fermion density and/or fermion current $j^\mu$ will generate a perturbing gauge field $A^\mu$. The presence of the perturbing $A^\mu$ induces a change of the gauge phases of fermion field operator $\psi$ through the Dirac equation

$$\gamma_\mu (i\partial_\mu - eA^\mu)\psi (x) = 0,$$

(A.2)

where $\gamma^\mu$ are the gamma matrices in 1+1 dimensions. The change of the gauge phases of fermion field operator $\psi$ in turn lead to a change of the fermion current $j^\mu$. By imposing the Schwinger modification factor, $e^{ie\int_\xi \gamma^\mu A^\mu (x)dx}$, to ensure the gauge invariance of the fermion Green’s function, the induced fermion current $j^\mu$ is given implicitly as a function of the perturbing gauge field $A^\mu$ by

$$j^\mu (x) = -\frac{e}{2} \left\{ \lim_{x^0 \rightarrow x^0 +} + \lim_{x^0 \rightarrow x^0 -} \right\} tr \left[ e^{ie\int_\xi \gamma^\mu A^\mu (x)dx} \times (T(\psi^\dagger (x')\gamma^\mu \psi (x)) \right].$$

(A.3)

where $T$ is the time-order operator. Upon evaluating the above limits from the left and from the right at the space-time point $x = (x^0, x^1)$, the fermion current singularities from the left and from the right cancel and the induced gauge-Invariant fermion current $j^\mu$ is found to be related explicitly to the perturbing gauge field $A^\mu$ by

$$j^\mu = -\frac{e}{\pi} \left[ A^\mu - \partial^\mu \frac{1}{\partial x^\nu} \partial_\nu A^\nu \right].$$

(A.4)

The resultant fermion current $j^\mu$ in turn leads to a new gauge field $\tilde{A}^\mu$, through the Maxwell equation,

$$\partial_\nu F^{\mu\nu} = \partial_\nu (\partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu) = ej^\mu.$$  

(A.5)

From Eqs. (A.4) and (A.5), the self-consistency of the resultant gauge field $\tilde{A}^\mu$ matching the initial perturbing gauge field $A^\mu$ leads to the gauge field satisfying the Klein-Gordon equation

$$-\Box A^\mu + \frac{e^2}{\pi} A^\mu = 0.$$  

(A.6)

Therefore, massless fermions and antifermions interacting non-perturbatively in a gauge field with a coupling constant $e$ in 1+1 dimensions result in a boson field with a quanta of mass $m = e/\sqrt{\pi}$.

It is clear from the above review that the exact solution of the boson state does not lend itself readily to a simple quantum mechanical two-body problem of valence fermion (quark) and valence antifermion (antiquark) involving a simple fundamental two-body interaction, because it involves concepts of massless fermions, gauge invariance, gauge field self-consistency, and the cancellation of the fermion current singularities that are beyond the conventional two-body problems with a simple two-body interaction. In spite of this being the case, it is desirable to construct a phenomenological two-body model for the valence fermion and antifermion with an effective interaction $\Phi$ that can be calibrated to contain the basic properties of the theory and to yield Schwinger’s exact result. Examples of such an approach can be found in the successes of relativistic and non-relativistic hadron spectroscopy where the non-perturbative QCD solution involving the lattice gauge theory is approximated by a two-body theory with phenomenological effective interactions (see for example, [122, 123, 124, 125, 126, 104, 105, 106, 107, 108, 110]).

Being a phenomenological two-body theory, such a theory and its generalizations will need to be worked out with considerable theoretical support and persistent confrontation with experiments so that it can be refined and readjusted, should new experimental data and new theoretical predictions become available. The present investigation on the stability of the QED neutron represents an exploration along such lines.

An additional advantage of a successful phenomenological two-body problem treatment rests on it ability to simplify the calculations, to retain the essential features, to provide an intuitive understanding, and to help solve problems that may not be solvable in a full treatment of the field theory, paving the way for our analysis on the stability of the three-quark system in section 2.

Appendix B: The separation and the independence of the color-singlet and color-octet $q\bar{q}$ excitations

It is instructive to review how the color-singlet and color-octet $q\bar{q}$ excitations can arise in a quark-QCD-QED system. We start with the quark-QCD-QED vacuum which is the lowest energy state with quarks filling up the negative-energy Dirac sea and interacting in QCD and QED interactions. We introduce a disturbance which creates one or many $q\bar{q}$ pairs, as for example, (i) during the de-excitation of a highly-excited nuclear state with a proton pulling outside an alpha particle core, (ii) in a high-energy nuclear collision, or (iii) in an excited system of a quark and an antiquark stretching out from each other at high energies after the annihilation of a high energy $e^+e^-$ pair, as represented schematically by Figs. (1a), (1b), and (1c) of [33]. Final state interactions allow the creation of only those final $q\bar{q}$ states at eigenenergies of the QCD or QED mesons because the density of final states away from the meson eigenenergies is zero on account of the confinement of quarks.

We focus our attention on one of the lowest energy produced $q\bar{q}$ pairs. For the created $q\bar{q}$ pair to be in
a QCD or QED meson state, there must be a directi
dominate direction of the quark and the anti-
direction, which can be taken to be the longitudinal di-
we can infer the occurrence of transverse con-
finement of the created q ̅q pair and the created QCD and QED gauge fields at the moment of the their cre-
ation, from Polyakov’s results of the confinement of op-
ne of opposite charges in compact Abelian and non-Abelian gauge theories in (2+1)D [50,51]. The cylindrical flux tube arising from the subsequent longitudinal stretching of the initial quark and antiquark will remain trans-
versely confined, as discussed in [29,33]. We can thus
study the longitudinal dynamics of the quark and the
antiquark system by idealizing the cylindrical flux tube as a one-dimensional string in (1+1)D, with its informa-
tion on the transverse flux tube radius RT stored in the
new gauge field coupling constant g = g_{\mu\nu} / \sqrt{RT}
in (1+1)D, and its transverse quark mass m obtained from
the eigenvalue of a transverse equation of motion of the
quark in the flux tube [28,31,34]. Because the
masses of light quarks are small, it is reasonable to
assume light quarks to be massless so that Schwinger’s model of massless charges can be applied.

We are now in a position to examine the color de-
grees of freedom in the longitudinal dynamics of quarks and antiquarks in a general quark-QCD-QED system in the idealized (1+1)D. As we mentioned in the Introduction, the quark current \( j^\mu \) and the gauge fields \( A^\mu \) are not single-element functions. They are in fact 3\times3 color matrices. Quarks in color-triplet 3 representation and antiquarks in color 3* representation form the product group of 3 \( \otimes \) 3* = \{1\} \( \otimes \) 8, which contains the color-singlet 1 subgroup and the color-octet 8 subgroup of generators.

In the general case, the color-octet QCD gauge fields
\( A^\mu(x) = \sum_{\lambda=1}^{8} A_\lambda^\mu(x) t^\lambda \) contains eight non-Abelian degrees of freedom and their couplings will lead to color excitations, the majority of which will not lead to stable collective excitations. To get stable collective excitations, we represent the color-octet degrees of freedom by a single unit vector \( \tau^1 \) oriented randomly in the eight-
dimensional SU(3) color generator space [28,31,34],

\[
\tau^1 = \sum_{i=1}^{8} n_i t^i, \quad \text{with} \quad n_i^2 + n_2^2 + \ldots + n_8^2 = 1, \quad (B.7)
\]

where \( n_\lambda = \text{2Tr}\{\tau^1 t^\mu\} \), and we restrict the dynamics of the QCD gauge fields only to the gauge field am-
plitude \( A^\mu_\lambda(x) \) along the \( \tau^1 \) direction, while keeping the \( \tau^1 \) orientation fixed. The gauge fields of the quark-QCD-
QED system are then described by the color-singlet am-
plitudes \( A^\mu_0(x) \) for QED dynamics and the color-octet amplitude \( A^\mu_\lambda(x) \) for QCD dynamics as [28,31,34]

\[
A^\mu(x) = A^\mu_0(x) \tau^0 + A^\mu_\lambda(x) \tau^\lambda = \sum_{\lambda=0}^{8} A^\mu_\lambda(x) \tau^\lambda, \quad (B.8)
\]

where \( \tau^0 = t^0, \quad 2\text{Tr}(\tau^\lambda \tau^\lambda) = 2\delta^\lambda_0, \) and \( \lambda, \lambda' = 0, 1. \) The quark currents \( j^\mu(x) \) can be likewise represented by the
color-singlet and color-octet current components as

\[
j^\mu(x) = j^\mu_0(x) \tau^0 + j^\mu_\lambda(x) \tau^\lambda = \sum_{\lambda=0}^{8} j^\mu_\lambda(x) \tau^\lambda. \quad (B.9)
\]

Because \( \tau^0 \) and \( \tau^1 \) commute, the gauge fields in this
restricted subspace are Abelian.

We can now examine the dynamics of the quarks
fields and the gauge fields. We start with initial gauge
fields \( A_\lambda^\mu \) which affect the quark field. The Dirac equation (A.2)
for the quark field is

\[
\gamma^\mu \left( i\partial^\mu + \sum_{\lambda=0}^{8} g^\lambda A^\mu_\lambda(x) \tau^\lambda \right) \psi(x) = 0. \quad (B.10)
\]

After obtaining the solutions to the above Dirac equation for the quark field \( \psi(x) \), we evaluate the quark current \( j^\mu_\lambda(x) \) as a function of the applied initial gauge field \( A_\lambda^\mu(x) \),

\[
\begin{align*}
\frac{1}{2} \left\{ \lim_{x' = x + x''} x'' \rightarrow x' - x'' \quad \text{and} \quad \lim_{x'' = x + x'} x'' \rightarrow x' - x'' \right\} \\
\text{tr} \left[ e^{i \int_{x''}^{x'} \sum_{\lambda} (-g^\lambda A^\mu_\lambda(t)) \lambda^\mu d\nu(T(\bar{\psi}(x') \gamma^\mu \tau^\lambda \psi(x))) \right] 
\end{align*}
\]

The induced quark current \( j^\mu_\lambda(x) \) as a function of the initial applied gauge field \( A_\lambda^\mu \) is found to be [34]

\[
\begin{align*}
\frac{g^\lambda}{\pi} \left( A^\mu_\lambda(x) - \frac{\partial \psi}{\partial x^\mu} \right) \\
\text{where} \lambda = 0 \text{for QED}, \quad \lambda = 1 \text{for QCD}. \quad (B.12)
\end{align*}
\]
remain colorless and cannot become colored, when the color-singlet quark density oscillation absorbs a gauge boson. Hence, it can only absorb a QED gauge boson \( A_\mu^\alpha \) but not a QCD gauge boson \( A_\mu^\mu \) to make the color-singlet quark density oscillation observable. On the other hand, a color-octet quark density oscillation cannot become observable until it bleaches its octet color to become colorless. Hence it can only absorb a QCD gauge boson \( A_\mu^\mu \) but not a QED gauge boson \( A_\mu^\nu \) to make the color-octet quark density oscillation observable. Therefore, in the collective dynamics of the quark-QCD-QED medium, quark currents of the type \( j_\mu^\nu(x) \), is affected only by gauge fields of the same type \( \lambda \), \( A_\mu^\lambda(x) \), as shown in Eq. (B.12).

The induced quark current \( j_\mu^\nu(x) \) generates a new gauge fields \( \tilde{A}_\nu^\lambda(x) \) through the Maxwell equation, which in the Abelian approximation of Eq. (B.13) is

\[
-\partial_\nu \partial^\nu \tilde{A}_\nu^\lambda(x) + \partial_\lambda \partial^\lambda \tilde{A}_\nu^\lambda(x) = g^\lambda j_\mu^\nu(x). 
\]

(B.13)

Stable self-consistent collective dynamics of the quark field and the gauge fields can be obtained when the newly generated gauge fields \( \tilde{A}_\nu^\lambda(x) \) are the same as the initial applied gauge fields \( A_\mu^\lambda(x) \). Such self-consistency can be achieved by setting \( \tilde{A}_\nu^\lambda(x) = A_\mu^\lambda(x) \) and substituting Eq. (B.13) into Eq. (B.12). We get the Klein-Gordon equation for the currents

\[
-\partial^\nu \partial_\nu \lambda j_\lambda^\mu = m_\lambda^2 j_\lambda^\mu, 
\]

which corresponds to the occurrence of a boson of a stable and independent collective excitation of the quark-QCD-QED medium, with a mass \( m_\lambda \) given by

\[
m_\lambda^2 = \frac{(g^\lambda)^2}{\pi}, \quad \begin{cases} \lambda = 0 & \text{for QED} \\ \lambda = 1 & \text{for QCD} \end{cases}
\]

(B.15)

From another perspective, the separation and the independence of the color-singlet and the color-octet excitations are possible because the gauge-invariant relations between the charge currents \( j^\mu \) and the gauge fields \( A^\mu \) in (1+1)D in Eqs. (A.4) and (A.5) (and similarly in Eqs. (B.12) and (B.13)) are each a linear function of \( j^\mu \) and \( A^\mu \). As a consequence, there is a principle of superposition of currents and gauge fields of different color components in color-space. Thus, the quark-QCD-QED medium possesses stable and independent collective QCD and QED excitations with different bound states masses \( m_\lambda \), depending on the coupling constants \( g^\lambda \). From Eq. (B.15), the ratio of the QED meson mass to the QCD meson mass is of order

\[
\frac{m_{\text{QED} \text{ meson}}}{m_{\text{QCD} \text{ meson}}} \sim \frac{g^\text{QED}}{g^\text{QCD}} \sim \sqrt{\frac{\alpha_{\text{QED}}}{\alpha_{\text{QCD}}}} \sim \sqrt{\frac{1/137}{0.6}} \sim 1/9, 
\]

which is Eq. (1).

**Appendix C: Relativistic two-body problem**

The relativistic two-body wave equations for the wave function \( \Psi \) for QED interactions in 1+1 dimensions consist of two mass-shell constraints on each of the interacting particles \([104,105,106,107,108,109,110,111,112]\).

\[
\mathcal{H}_1|\Psi\rangle = \left\{ p_1^2 - m_1^2 - \Phi_{12}(x_{12}) \right\} |\Psi\rangle = 0, \quad (C.17a)
\]

\[
\mathcal{H}_2|\Psi\rangle = \left\{ p_2^2 - m_2^2 - \Phi_{21}(x_{21}) \right\} |\Psi\rangle = 0. \quad (C.17b)
\]

As emphasized in Appendix A, Schwinger’s exact solution in field theory resulting in longitudinal confinement is in fact the solution of the collective motion for a many-body problem of great complexity. It is not a simple two-body problem of a fundamental interaction. In transcribing the Schwinger field theory problem as a phenomenological two-body problem, there are choices of different forms of the two-body effective interaction. An alternative form uses the QED confining linear interaction in the Coulomb gauge as the time-like vector potential \( V_{12}(r) \) in a minimum substitution form, and it leads to \(|(p_1^0 - V_{12}(x_{12}))^2 - p_2^2 - m_3^2|\Psi = 0\). However, such an effective interaction has the unpleasant feature that the \( V_{12}^2 \) term in the above Schrödinger-type equation leads to solutions whose behavior do not match the behavior of the Schwinger solution at large separations. It becomes necessary to introduce an additional scalar confining interaction \( S_{12}(x_{12}) \) so that the wave equation becomes \(|(p_1^0 - V_{12}(x_{12}))^2 - p_2^2 - (m_3^2 + S_{12}(x_{12}))|\Psi = 0\). The choice of \( V_{12} = S_{12} \) used in [109] leads to a cancellation of the \( V_{12}^2 \) term with the \( S_{12}^2 \) term at large separations and a total effective interaction similar to the \( \Phi \) used here. In view of the phenomenological nature of the confining interaction, the present effective confining interaction \( \Phi_{12} \) in Eq. (C.17b) serves well as a description that can match the confinement property of the Schwinger solution.

We would like to calibrate the above effective interaction \( \Phi_{1j} \) by comparing the solution of the above two-body problem with Schwinger’s exact QED solution in 1+1 dimensions. We construct the total Hamiltonian \( \mathcal{H} \) from these constraints by

\[
\mathcal{H} = \sum_{i=1}^{2} \mathcal{H}_i. \quad (C.18)
\]
In order that each of these constraints be conserved in time we must have
\[ [\mathcal{H}_i, \mathcal{H}_j] \psi = i \frac{d\mathcal{H}_i}{dt} \psi = 0. \] (C.19)

As a consequence, the above equation leads to the compatibility condition between the two constraints [103, 105, 106, 107, 108, 109, 110].

\[ [\mathcal{H}_i, \mathcal{H}_j] \psi = 0. \] (C.20)

Since the masses \( m_1 \) and \( m_2 \) commute with the operators, this implies
\[ \left( [p^2_1 \Phi_{21}(x_{21})] - [p^2_2 \Phi_{12}(x_{12})] \right) \psi = 0. \] (C.21)

The above equation cannot be satisfied if \( \Phi_{12}(x_{12}) \neq \Phi_{21}(x_{21}) \). The simplest way to satisfy the above equation is to take
\[ \Phi_{12}(x_{12}) = \Phi_{21}(x_{21}) = \chi(x_{12}), \] (C.22)

which is the relativistic analogue of Newton’s third law. The compatibility condition (C.21) then requires the effective interaction \( \Phi(x_{12}) \) to depend only on the coordinate \( x_{12} \) transverse to the total momentum \( P = p_1 + p_2 \),
\[ \Phi(x_{12}) = \Phi(x_{12} \perp), \] (C.23)

where \( x_{12\perp} = (x_1 - x_2) - \frac{(x_1 - x_2) \cdot P}{P^2} P \). (C.24)

We shall work in the CM system where the total momentum \( P = (P^0, P^1) = (M, 0) \), \( M \) is the invariant mass of the composite system, and the relative coordinate \( x_{i\perp} = (x_i - x_2) \) involves only spatial coordinates \( x_1 \) and \( x_2 \). The particle momentum \( p_i \) can be separated out into a component \( \epsilon_i \parallel P \) and a component \( q_i \perp P \) as
\[ p_i = (\epsilon_i, q_i) = \epsilon_i \frac{P}{\sqrt{P^2}} + q_i, \quad i = 1, 2, \] (C.25)

where \( \epsilon_i = \frac{p_i \cdot P}{\sqrt{P^2}} \), and \( q_1 + q_2 = 0. \) (C.26)

In terms of \( \epsilon_i \), the invariant mass of the composite system \( M \) is given by
\[ M = P^0 = \epsilon_1 + \epsilon_2. \] (C.27)

The two-body wave equations (C.17a) and (C.17b) in the CM system becomes
\[ \epsilon_1^2 |\Psi\rangle = \{ q_1^2 + m_1^2 + \Phi(x_{12\perp}) \} |\Psi\rangle, \] (C.28a)
\[ \epsilon_2^2 |\Psi\rangle = \{ q_2^2 + m_2^2 + \Phi(x_{12\perp}) \} |\Psi\rangle. \] (C.28b)

Because \( q_2 = (-q_1) \), the second equation of the above is simply
\[ (\epsilon_2^2 - \epsilon_1^2) < |\Psi\rangle = (m_1^2 - m_2^2) |\Psi\rangle. \] (C.29)

It is only necessary to solve for the eigenstate of the first Schrödinger-type equation (C.28a) to obtain \( \epsilon_1 \), and the quantity \( \epsilon_2 \) can be obtained as an algebraic equation from (C.29). The knowledge of \( \epsilon_1 \) and \( \epsilon_2 \) (taken to be positive) then gives the invariant mass \( M \) of the interacting two-body system.

### Appendix D: Schwinger’s QED boson as a relativistic two-body problem in 1+1 dimensions

The brief summary presented in Appendix A makes it plain that the Schwinger boson that is confined and bound in 1+1 dimensions is in fact a non-linear self-consistent solution of a many-body system of great complexity. It is desirable to construct a phenomenological two-body problem for QED in 1+1 dimensions involving a valence fermion and a valence antifermion with an effective phenomenological interaction \( \Phi(x_{12\perp}) \) that can be calibrated to contain the basic properties of the theory and to yield Schwinger’s exact QED result in 1+1 dimensions.

Accordingly, we consider the two-body wave equations (C.28a) and (C.28b) (or (C.29)) in the CM system for two charge particles with charge numbers \( Q_1 \) and \( Q_2 \), interacting with a phenomenological effective QED interaction \( \Phi(x_{12\perp}) \)
\[ \Phi(x_{12\perp}) = \frac{2e^2}{\epsilon_1 + \epsilon_2} (-Q_1 Q_2) \kappa |x_1 - x_2|. \] (D.30)

The effective QED interaction \( \Phi(x_{12\perp}) \) in 1+1 dimensions has been chosen such that:

1. In the Coulomb gauge, the interaction energy between charges \( Q_1 e \) and \( Q_2 e \) in 1+1 dimensional QED is \( (-Q_1 Q_2 e^2/2) |x_1 - x_2| \) [47], which is indeed confining for the attractive interaction between unlike charges. It is reasonable to use such a spatially linear interaction in the phenomenological two-body problem. The value of the coefficient parameter \( \kappa \) of such a linear potential in (D.30) will be affected by the use of the phenomenological reduced mass factor, Schwinger’s self-consistency condition on the gauge field, and the gauge invariance constraint. It is therefore appropriate to extract \( \kappa \) phenomenologically from Schwinger’s exact solution.

2. The quantity \( \kappa \) is proportional to the square of the coupling constant \( e^2 \). The exact value of \( \kappa \) will be chosen to give Schwinger’s solution of \( m = e^2/\sqrt{\pi} \) for a massless fermion-antifermion pair interacting in the QED interaction.

3. The effective interaction contains the charge factor \( (-Q_1 Q_2) \), which leads to an attractive interaction if \( Q_1 Q_2 < 0 \), and a repulsive interaction if \( Q_1 Q_2 > 0 \), as in standard QED interaction in quantum electrodynamics.

4. The reduced mass factor \( 2(\epsilon_1 \epsilon_2)/(\epsilon_1 + \epsilon_2) \) has been chosen to give the proper reduced mass in the non-relativistic two-body wave equation. We have however used the particle energy \( \epsilon_i \) in lieu of the rest mass \( m_i \), to make it applicable also to the massless limit. The reduced mass factor depends on the
Our present task is to obtain the eigenstates for the
wave equation (C.28a) with the effective potential of (D.30). Schwinger’s case of massless fermion and antifermion corresponds to \( Q_1 = 1, Q_2 = -1, m_1 = m_2 = 0, \epsilon_1 = \epsilon_2 = \epsilon, \) and \( M = 2\epsilon. \) Using the dimensionless variable 
\[ y = \sqrt{\kappa}(x_1 - x_2), \]  
the effective interaction \( \Phi(y)/\kappa \) is then
\[ \Phi(y)/\kappa = \frac{\epsilon}{\sqrt{\kappa}}|y|, \]  
and the wave equation (C.28a) becomes
\[ \left\{ \frac{\partial^2}{\partial y^2} + \frac{\epsilon}{\sqrt{\kappa}}|y| - \frac{\epsilon^2}{\kappa} \right\}\Phi(y) = 0. \]  
We show in Fig. 7 the dimensionless effective potential, \( \Phi(y)/\sqrt{\kappa}, \) expressed in units of \( \sqrt{\kappa}\epsilon. \) It is a linearly rising function of the dimensionless spatial separation \( |y| \) between the charges, expressed units of \( 1/\sqrt{\kappa}. \) The solution of the wave equation with a linearly rising interaction is the Airy function. The wave equation (D.33) becomes the Airy equation
\[ \left\{ \frac{\partial^2}{\partial z^2} - \left( z - \frac{\epsilon}{\sqrt{\kappa}}\right)^{3/2} \right\}\Psi(z) = 0, \]  
where \( z = (\frac{\epsilon}{\sqrt{\kappa}})^{1/2}|y|. \)
\[ \Psi(z) = \text{Ai}(|z| - (\frac{\epsilon}{\sqrt{\kappa}})^{3/2}). \]  
The eigenstate is obtained by matching the wave function and its derivative at \( z = 0. \) There are two types of eigenstates with even and odd parities:
\[ \text{even parity: } \Psi'(z)|_{z=0} = \text{Ai}'(-(\frac{\epsilon}{\sqrt{\kappa}})^{3/2}) = 0, \]  
\[ \text{odd parity: } \Psi(z)|_{z=0} = \text{Ai}(-(\frac{\epsilon}{\sqrt{\kappa}})^{3/2}) = 0. \]  
We label the locations where the wave function or its derivative are zero as \((-a_s)\) with \( \text{Ai}(-a_s) = 0 \) or \( \text{Ai}'(-a_s) = 0. \) Then the eigenvalues of the wave equations \( \epsilon \) are given by
\[ \left( \frac{\epsilon}{\sqrt{\kappa}} \right)^{3/2} = a_s, \quad \text{or} \quad \epsilon = a_s^{3/4}\sqrt{\kappa}. \]  

Table 2 Solution of the two-body problem with the effective interaction \( \Phi(y)/\sqrt{\kappa}, \) for the lowest states. Here \( n \) is the number of nodes of the two-body wave function, and \( M/\sqrt{\kappa} \) is the dimensionless measure of the composite particle mass.

| n | Parity | \( a_s \) | \( \epsilon/\sqrt{\kappa} \) | \( M/\sqrt{\kappa} \) | \( \sqrt{|q|^2}/\sqrt{\kappa} \) | \( \sqrt{|q|^4}/\sqrt{\kappa} \) |
|---|---|---|---|---|---|---|
| 0 | even | 1.02 | 1.01 | 2.02 | 0.862 | 0.60 |
| 1 | odd | 2.34 | 1.89 | 3.98 | 1.38 | 1.09 |
| 2 | even | 3.25 | 2.42 | 4.84 | 1.78 | 1.41 |
| 3 | odd | 4.09 | 2.87 | 5.74 | 2.10 | 1.66 |
| 4 | even | 4.82 | 3.26 | 6.50 | 2.38 | 1.89 |

Table 2 gives the values of \( a_s, \) energy \( \epsilon/\sqrt{\kappa}, \) (mass \( M/\sqrt{\kappa}, \) \( \sqrt{|q|^2}/\sqrt{\kappa} \), and \( \sqrt{|q|^4}/\sqrt{\kappa} \) of the lowest five states. Fig. 7 displays the energies as the horizontal lines, with their corresponding wave functions \( \Psi(y) \) exhibiting different number of nodes. In the two-body problem with the phenomenological two-body interaction, the mass of the lowest state for the phenomenological potential is \( M = 2.02\sqrt{\kappa}. \) On the other hand, the Schwinger’s exact solution (A.1) in field theory gives \( M = \epsilon/\sqrt{\pi}. \) Therefore, by matching the mass of the lowest eigenstate from the phenomenological two-body theory with the mass from Schwinger’s field theory, we find \( \kappa \) to be
\[ \kappa = \frac{\epsilon^2}{4.08\pi} \sim \frac{\epsilon^2}{4\pi}, \]  
where for simplicity, we shall approximate the denominator \( 4.08\pi \) to be \( 4\pi. \) We have thus obtained the phenomenological interaction \( \Phi \) in Eq. (D.30) between two electric charges interacting in QED in \( 1+1 \) dimensions. Such a knowledge of the effective QED interaction between two electric charges will enable us to study the states of three quarks interacting in \( 1+1 \) dimensional QED.

We need the value of the coupling constant \( \epsilon = e_{2D} \) which can be obtained from \( e_{1D}. \) In the physical world of
3+1 dimensions, the one-dimensional open string without a structure is in fact an idealization of a flux tube with a transverse radius $R_T$. The masses calculated in 1+1 dimensions can represent physical mesonic masses, when the structure of the flux tube is properly taken into account. Upon considering the structure of the flux tube in the physical 3+1 dimensions, we find that the coupling constant $e_{2D}$ in 1+1 dimensions is related to the physical coupling constants $e_{3D}$ in 3+1 dimensions by the flux tube radius $R_T$, \[ 2531 \] \[ 2534 \] \[ 2535 \]

\[ \alpha_{3D} = 1/137 \] and \[ R_T = 0.4 \text{ fm} \] \[ 2531 \], which yields \[ \sqrt{\kappa} = 23.8 \text{ MeV} \] and \[ h/\sqrt{\kappa} = 8.3 \text{ fm} \].

As a final check of a faithful representation of the phenomenological two-body model for the Schwinger exact field theory in 1+1 dimensions, we note that they share the distinct property that the mass of the system increases with the increase in the magnitude of the coupling constant, in contrast to a non-confining interaction such as the positronium where the mass of the composite system decreases with the increase in the magnitude of the coupling constant.

We note from the above solutions of the two-body problem that a fermion-antifermion composite system with the phenomenological QED interaction possesses excited states with a higher number of nodes $n$, in addition to the lowest state with $n = 0$. They represent high string vibrational excitations of the fermion-antifermion system as an open string. As a rough guide for future searches of QED meson states in $(q\bar{q})$ composite systems, we can treat the observed X17 state at \[ E = 16.70 \text{ MeV} \] \[ 10 \] to be the lowest band heads of the $(I = 0, n = 0)$ state, and the E38 state at \[ E = 37.38 \text{ MeV} \] \[ 14 \] to be the lowest band heads of the $(I = 1, I_3 = 0, n = 0)$ state respectively. We can then use the QED meson solutions as displayed in Fig. 7 and Table I by assuming the observed X17 state at 16.70 MeV \[ 10 \], as the band heads of the $(I = 1, n = 0)$ and the E38 state at \[ E = 37.38 \text{ MeV} \] \[ 14 \] as the band head of the $(I = 1, I_3 = 0, n = 0)$ states respectively. The right panel gives the transition energy \[ E(I=0,n+1) - E(I=0,n) \] as dashed lines.

and the deviations from the linear potential shape may modify the spectrum.

App. E: Variational Calculation for the lowest two-body bound state energy

Before we apply the effective interaction $\Phi_{12}(x_{12})$ to the three-quark problem, we wish to test here whether a variational calculation using the effective interaction \[ (D.30) \] will also lead to the same two-body bound state mass for the lowest-energy two-body state. The success of the variational calculations will pave the way in a similar variational calculation for the three-body problem in section 2. We therefore evaluate the lowest two-body bound state energy by using a Gaussian variational wave function. For massless quarks with $Q_1 = 1$, $Q_2 = -1$, we rewrite \[ (D.33) \] as

\[ \{ \mathcal{H}_0 - E^2 \} \psi = 0, \] \[ \text{where } \mathcal{H}_0 = -\frac{\partial^2}{\partial y^2} + E|y|, \text{ and } E = \frac{e}{\sqrt{\kappa}}. \] \[ (E.42) \] \[ (E.43) \]

We introduce a Gaussian variational wave function with the variational parameter $\sigma$,

\[ \psi(y) = \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^{1/2} \exp\left\{ -\frac{y^2}{4\sigma^2} \right\} = N \exp\left\{ -\frac{y^2}{4\sigma^2} \right\}. \] \[ (E.44) \]

We obtain

\[ \langle \mathcal{H}_0 \rangle (\sigma) = \frac{1}{4\sigma^2} + \frac{2\sigma E}{\sqrt{2\pi}}. \] \[ (E.45) \]
From the requirement of $\delta \langle H_0 \rangle(\sigma)/\delta \sigma = 0$, we get
\begin{equation}
\sigma = \left(\frac{\sqrt{2\pi}}{4E}\right)^{1/3}, \tag{E.46}
\end{equation}
at which
\begin{equation}
\langle H_0 \rangle = \frac{3E}{\sqrt{2\pi}}. \tag{E.47}
\end{equation}
From Eq. (E.43), the value of $E^2$ at the equilibrium value of $\sigma$ becomes
\begin{equation}
E^2 = \langle H_0 \rangle = \frac{3E}{\sqrt{2\pi}}. \tag{E.48}
\end{equation}
Eliminating $\sigma$ from Eqs. (E.46) and (E.48) we get
\begin{equation}
E = \left(\frac{3}{(2\sqrt{\pi})^{3/2}}\right)^{1/3} = 1.034 \tag{E.49}
\end{equation}
\begin{equation}
\sigma = \left(\frac{\sqrt{2\pi}}{4E}\right)^{1/3} = 0.876. \tag{E.50}
\end{equation}
The above variational calculation gives $E = \epsilon/\sqrt{\kappa} \sim 1$ as given in (E.49), and thus
\begin{equation}
\epsilon = \sqrt{\kappa}, \quad M = 2\epsilon = \frac{e}{\sqrt{\pi}}, \tag{E.51}
\end{equation}
which agrees with the lowest eigenenergy obtained by solving the wave equation directly. We find indeed that the variational calculation can give the correct lowest-energy bound state mass. This justifies the use of the variational calculations in the three-body problem to obtain the lowest-energy state of a QED neutron in section 2.

Appendix F: The decay and detection of composite $q\bar{q}$ QED mesons

Fig. 9 Fig. 9(a) depicts the diagram for the decay of the QED meson $X$ into two real photons $(\gamma_1\gamma_2)$, Fig. 9(b) the decay of the QED meson $X$ into two virtual photons $(\gamma_1^\star\gamma_2^\star)$, and Fig. 9(c) the decay of the QED meson $X$ into a dilepton $(e^+e^-)$ pair.

We would like to review and extend here our knowledge on the decay of the QED meson $\Psi$ to facilitate the experimental detection of QED mesons and the QED neutron. We consider a $q\bar{q}$ composite system $X$ formed by a valence quark $q_1$ and a valence antiquark $\bar{q}_1$ interacting with the effective QED interaction $\Phi$. As shown in Appendix D, there can be many eigenstate solutions of the relativistic two-body equations for the composite system $X$. The additional intrinsic degrees of freedom of the quarks will add further to the complexity of the spectrum and the number of the composite particles $X$. In the $1+1$ dimensional description, the composite particle $X$ cannot decay, as the quarks execute a yo-yo motion along the string in an idealization of the flux tube, and the photon is represented by an effective interaction binding the quarks. In the physical $3+1$ dimensions where the structure of the flux tube is taken into account and the photon decay channel opens up, the quark and the antiquark at different transverse coordinates in the tube traveling from opposing longitudinal directions can make a sharp change of their trajectories turning to the transverse direction to annihilate, leading to the emission of photons as depicted in Fig. 9(a), 9(b), and 9(c). The number of emitted photons depends on the spin and parity of the decaying system $X[127,128]$. We have illustrated the case of two photon decay in Fig. 9(a), 9(b), and 9(c). The emitted photon can be two real photons $(\gamma_1^\star\gamma_2^\star)$ in Fig. 9(a), two virtual photons $(\gamma_1^\star\gamma_2^\star)$ in Fig. 9(b), or a dilepton $(e^+e^-)$ pair in Fig. 9(c).

There can be excited QED meson states $X^\ast$ with nodal number $n$ which can de-excite to the lower QED meson state $X$ with a smaller nodal number $n' < n$, with the emission of $\gamma$, $\gamma\gamma$, and $\gamma^\ast\gamma^\ast$ as shown in Fig. 10.

![Diagrams for the de-excitation of the excited QED meson $X^\ast$ to the QED meson $X'$ with particle emissions: Fig. 10(a) $X^\ast \rightarrow X' + \gamma$, Fig. 10(b) $X^\ast \rightarrow X' + \gamma + \gamma$, and Fig. 10(c) $X^\ast \rightarrow X' + \gamma^\ast + \gamma^\ast$.](image)

For the decay from a bound state $X$, the decay amplitude needs to be folded in with the bound state momentum wave function. For $X \rightarrow k_1 + k_2$ where the final states $(k_1, k_2)$ are $(\gamma_1, \gamma_2)$, $(\gamma_1^\ast, \gamma_2^\ast)$, or $(e^+, e^-)$ as in diagrams 9(b), 9(c), and 9(d), the decay amplitudes are $[106,107,$]

\begin{equation}
M(X \rightarrow k_1 + k_2) = \int d^3q_1 \Psi(q_1) \Gamma(q_1 + \bar{q}_2 \rightarrow k_1 + k_2). \tag{F.52}
\end{equation}
where $\tilde{\Psi}(q_1)$ is the bound state momentum wave function of a constituent, $q_1 = q_{\gamma_1}q_{\gamma_2}/2$, and $\Gamma(q_1 + q_2 \rightarrow k_1 + k_2)$ is the Feynman amplitude for the diagram of $q_1 + q_2 \rightarrow k_1 + k_2$. (see e.g. Eqs. (3.6) of [106] or Eq. (2.2) of [107] for the case of two-photon decay). The two-body spatial wave function $\Psi(y)$ for the QED meson X in Eqs. (D.33) has been expressed in terms of the dimensionless relative spatial coordinate $y = \sqrt{\kappa}(x_{q_1} - x_{q_2})$.

The corresponding wave function in the momentum space, $\tilde{\Psi}(q/2\sqrt{\kappa})$ is the Fourier transform of $\Psi(y)$ of Eqs. (D.32) and (D.36).

$$\tilde{\Psi}(q/2\sqrt{\kappa}) = \tilde{\Psi}(q/\sqrt{\kappa}) = \frac{1}{\sqrt{2\pi}}e^{-i(q_2/\sqrt{\kappa})y}d\gamma_dq$$ (F.53)

where $q = q_1 - q_2 = 2q_1$, and the constituent momentum $q_1$ is expressed in units of $\sqrt{\kappa}$. The momentum distribution of the composite $q_1q_2$ system $X$ with a mass $M$ is

$$\frac{dN(P; q)}{DPdq} = \delta(P^2 - M^2)|\tilde{\Psi}(q/2\sqrt{\kappa})|^2.$$ (F.54)

We show in Figs. 11(a) the momentum wave functions $\tilde{\Psi}(q/2\sqrt{\kappa})$, and in Fig. 11(b) the corresponding momentum probability function $|\tilde{\Psi}(q/2\sqrt{\kappa})|^2$, for the lowest five states of the composite system. We can understand the contents of Eq. (F.54) intuitively as follows: the invariant square of the momentum sum, $P^2$, probes the invariant mass $M^2$ of the composite QED meson, whereas the invariant square of the momentum difference, $q^2$, measures the inverse spatial size of the decaying QED meson as $h/\sqrt{|q^2|}$, and the number of nodes of $|\tilde{\Psi}(q/2\sqrt{\kappa})|^2$ in Eq. (F.54) and Fig. 11(b) reveals the internal structure of the composite $q_1q_2$ system.

$$|\tilde{\Psi}(q/2\sqrt{\kappa})|^2$$ (b)

$$\Psi(q/2\sqrt{\kappa})$$ (a)

**Fig. 11** Fig. 11(a) gives the two-body wave function in momentum space $\tilde{\Psi}(q/2\sqrt{\kappa})$ of the composite $q\bar{q}$ system as a function of the relative momentum $q = q_1 - q_2 = 2q_1$ in units of $2\sqrt{\kappa}$, and Fig. 11(b) gives $|\tilde{\Psi}(q/2\sqrt{\kappa})|^2$.

The decay via two virtual photons in diagram Fig. 9(b) may provide an interesting probe to yield additional information on the decaying QED meson parent particle. One can construct the invariant momenta square of the sum and differences of the 4-momenta of the two virtual photons,

$$P_{\gamma_1\gamma_2}^2 = (p_{\gamma_1} + p_{\gamma_2})^2,$$

$$Q_{\gamma_1\gamma_2}^2 = -(p_{\gamma_1} - p_{\gamma_2})^2,$$ (F.55)

and $P \equiv P_{\gamma_1\gamma_2} = \sqrt{P_{\gamma_1\gamma_2}^2}$, $Q \equiv Q_{\gamma_1\gamma_2} = \sqrt{-Q_{\gamma_1\gamma_2}^2}$. Experimental measurement of the virtual diphoton pair distribution

$$\frac{dN(P; q)}{dPdq} \frac{dN(Q; q)}{dPdq} = \frac{dN(P; Q)}{dPdq}$$ (F.56)

will provide useful information on the composite $q\bar{q}$ particles of massless light quarks from virtual diphoton decay measurements. Specifically, one makes a scatter plot of diphoton events on the two dimensional plane of $P$ and $Q$. In actual experiments with the presence of cuts and windows in various kinematical regions, one may resort to the use of the event mixing method to normalize the distribution as

$$\frac{dN(P; Q)}{dPdq} \mid_{\text{norm}} \equiv \frac{\frac{dN(P; Q)}{dPdq}}{\frac{dN(P; Q)}{dPdq} \mid_{\text{mixed events}}}.$$ (F.57)

The above normalized distribution $dN(P; Q)/dPdq \mid_{\text{norm}}$ is expected to cluster sharply around the invariant mass.
M = (P + Q)/2 of the composite particles and spread out in width in the direction of Q. One can alternative display the distribution in terms of the rotated coordinates (P + Q)/2 and a small stripe of P − Q > 0.

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