String Field Theory for $d \leq 0$ Matrix Models via Marinari-Parisi

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ABSTRACT

We generalize the Marinari-Parisi definition for pure two dimensional quantum gravity ($k = 2$) to all non unitary minimal multicritical points ($k \geq 3$). The resulting interacting Fermi gas theory is treated in the collective field framework. Making use of the fact that the matrices evolve in Langevin time, the Jacobian from matrix coordinates to collective modes is similar to the corresponding Jacobian in $d = 1$ matrix models. The collective field theory is analyzed in the planar limit. The saddle point eigenvalue distribution is the one that defines the original multicritical point and therefore exhibits the appropriate scaling behaviour. Some comments on the nonperturbative properties of the collective field theory as well as comments on the Virasoro constraints associated with the loop equations are made at the end of this letter. There we also make some remarks on the fermionic formulation of the model and its integrability, as a nonlocal version of the nonlinear Schrödinger model.

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Hermitian one matrix models defined at a single point describe at their double scaling multicritical points minimal nonunitary conformal matter coupled to two dimensional quantum gravity.

Pure gravity \((k = 2)\) as well as all higher multicritical points of even order \(k\) \((k = 4, 6, ...)\) exhibit non perturbative ambiguities, instabilities and violations of the Schwinger-Dyson (loop) equations.

These problems can be traced back to the fact that the critical matrix potentials of even order \(k\) are bottomless. Thus a sensible definition of these models should correspond to well-defined stabilized matrix potentials.

Bottomless matrix potentials occur also in certain multimatrix models describing unitary matter coupled to two dimensional quantum gravity such as the two matrix model corresponding to the Ising case. Thus, the problem of stabilization is associated not only with one matrix models.

Marinari and Parisi have suggested a possible way out of this difficulty in the case of pure gravity by supersymmetrizing the model.

The bosonic sector hamiltonian in is also the forward Fokker-Planck hamiltonian associated with the Langevin equation whose force term is minus the gradient of the original zero dimensional matrix action.

Therefore, the definition of pure two dimensional quantum gravity in is equivalent to the stabilization procedure developed in for bottomless actions, as far as the bosonic sector of the former is concerned.

The stabilized pure gravity model of was further analyzed by , where the one eigenvalue double scaled hamiltonian was extracted, and a nonperturbative ambiguity free analysis of the density of particles and density of state was made.

The forward Fokker-Planck hamiltonian used in for the pure gravity case, reads

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3i.e. - matrix models defined over a zero dimensional space time. We will refer to them as zero dimensional models. The stabilized models, in which matrices depend upon the Langevin time coordinate will be referred to as the one dimensional models. One should not, of course, mix this terminology with “dimension” counting in target space - i.e. values of the central charge of matter coupled to gravity.

4In the existence of such an \(H\) was postulated, without specifying its details to define “time ordered” correlators in the zero dimensional theory.
for a general matrix potential $U(\Phi)$

$$H = \frac{1}{2} Tr \left[ -\frac{\partial}{\partial \Phi} + \frac{\beta}{2} U'(\Phi) \left( \frac{\partial}{\partial \Phi} + \frac{\beta}{2} U'(\Phi) \right) \right]$$ (1)

Here $\Phi$ is an hermitian matrix, depending on the Langevin time coordinate (i.e. the bosonic coordinate of superspace used in [11]).

$H$ in eq. (1) is a well defined Schrödinger operator. Its potential is clearly bounded from below and grows to plus infinity as the matrix eigenvalues become infinite.

Therefore, a well defined unique normalizable ground state vector $\Psi_0(\Phi)$ exists. Indeed, if $U(\Phi)$ is bounded from below such that the Boltzman weight of the zero dimensional matrix model is normalizable this ground state is given by

$$\Psi_0(\Phi) = \frac{1}{\sqrt{Z}} e^{-\frac{\beta}{2} Tr U(\Phi)}, \quad Z = \int dN^2 \Phi e^{-\beta Tr U(\Phi)}$$ (2)

In this case the vacuum energy is strictly zero and supersymmetry is not broken. Moreover, expectation values of operators, all at infinite Langevin time project only onto the ground state $\Psi_0$, and are identical to the corresponding correlators in the original zero dimensional matrix model:

$$\langle \Psi_0 | \mathcal{O}_1(\Phi) \cdots \mathcal{O}_n(\Phi) | \Psi_0 \rangle = \frac{1}{Z} \int dN^2 \Phi e^{-\beta Tr U(\Phi)} \mathcal{O}_1(\Phi) \cdots \mathcal{O}_n(\Phi)$$ (3)

If $U(\Phi)$ is unbounded from below, the zero dimensional Boltzman weight is unnormalizable and the corresponding matrix model exists only at a saddle point level. Supersymmetry is broken and the vacuum energy $E_0$ is positive. Alternatively - the appropriate Langevin equation has only runaway solutions and the Fokker-Planck probability density at any finite portion of matrix eigenvalue space, decays asymptotically in Langevin time $t$ as $e^{-E_0 t}$ [15] [16].

However, Fokker-Planck averages of operators normalized by the Fokker-Planck average of the unit operator are well defined as the Langevin time goes to infinity, and correspond to

$$\langle \mathcal{O}_1(\Phi) \cdots \mathcal{O}_n(\Phi) \rangle_{t \to \infty} = \int dN^2 \Phi \left| \Psi_0(\Phi) \right|^2 \mathcal{O}_1(\Phi) \cdots \mathcal{O}_n(\Phi) |_{t=\infty}$$ (4)

\footnote{For this property to hold also in the double scaling limit, we must ensure that the vacuum remains nondegenerate even as $N \to \infty$, i.e. - that the energy eigenvalue $E_1$ of the first excited state of $H$ does not coalesce with $E_0$. As was shown in [11] the mass gap $E_1 - E_0$ double scales in the WKB approximation for $k = 2$. Moreover, we will show that it double scales for any value of $k$. Thus the vacuum state remains nondegenerate. This was tacitly assumed in [12].}
Here $\Psi_0(\Phi)$ is the normalizable ground state of $\mathcal{H}$, and all the operators on the r.h.s. of Eq. (4) are at $t = \infty$. We thus consider Eq. (4) as the stabilized definition for correlators in case of bottomless matrix potentials.

As is well known, the laplacian over hermitean matrices acquires the form

$$- Tr \frac{\partial^2}{\partial \Phi^2} = - \frac{1}{\Delta(x)} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \Delta(x) + \left( \text{nonsinglet } U(N) \text{ angular momentum terms} \right)$$

(5)

where $x_i$ are the matrix eigenvalues and $\Delta(x_i)$ is the Vandermonde determinant. This leads to the mapping of eigenvalue dynamics onto that of a one dimensional Fermi gas \cite{15}. Clearly, the ground state $\Psi_0(\Phi)$ mentioned above is a $U(N)$ singlet.

For a generic potential $U(\Phi)$, the hamiltonian in Eq. (1) contains a two body interaction term among eigenvalues\footnote{Note that for $U'(\Phi) = \sum_{0 \leq n \leq N} C_n Tr \Phi^n$ we obtain $\mathcal{H}_{int} = \sum_{0 \leq n} C_n \sum_{\ell=0}^{n-1} Tr \Phi^\ell Tr \Phi^{n-1-\ell}$, therefore from the point of view of the (noncritical) one dimensional matrix theory $\mathcal{H}_0 = \frac{1}{2} Tr(-\frac{\partial^2}{\partial x_i^2} + \frac{\beta}{4} U'(\Phi)^2)$, whose eigenvalues form a noninteracting Fermi gas in the singlet sector, $\mathcal{H}_{int}$ may be interpreted as higher curvature terms \cite{20} that push the system to its multicritical point.}

$$\mathcal{H}_{int} = \frac{\beta}{4} Tr \frac{\partial}{\partial \Phi} U'(\Phi) = \frac{\beta}{4} \sum_{i,j} U'(x_i) - U'(x_j) \frac{x_i - x_j}{x_i - x_j}$$

(6)

Thus, generally – the one dimensional gas of eigenvalues is an interacting Fermi gas.

The matrix potential of minimal degree that leads to the $k = 2$ multicritical (pure gravity) point is $U_2(\Phi) = -\frac{\lambda}{6} \Phi^3 + \frac{1}{2} \Phi^2 + \Phi$ where we have followed the normalizations of \cite{19}. The critical coupling constant corresponding to the $k = 2$ point is $\lambda_c = 1$. The matrix potential $U_2(\Phi)$ is bottomless. Therefore the stabilized theory will exhibit spontaneous supersymmetry breaking. This issue was analyzed in \cite{21}. In this case, the interaction term in Eq. (6) reduces to an interaction of the eigenvalues with a constant background field, proportional to the number of eigenvalues, $N$, namely, $\mathcal{H}_{int} = \frac{\beta N}{4} Tr(\lambda \Phi - 1)$.

Therefore, the singlet sector of Eq. (1) reduces effectively to a system of $N$ non interacting Fermions in one dimension\footnote{In \cite{11} $N/\beta$ was set to 1 and $\lambda$ was varied, while in \cite{13} $\lambda$ was set to its critical value $\lambda_c = 1$ and $N/\beta$ was varied around its critical value $(N/\beta)_c = 1.$}

$$\mathcal{H}_{singlet} = \beta^2 \sum_{i=1}^{N} \left( -\frac{1}{2\beta^2} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right) = \beta^2 \sum_{i=1}^{N} h(x_i, p_i)$$

(7)
where the external one body potential is

\[ V(x) = \frac{1}{8}(U'_2(x))^2 + \frac{1}{4} \frac{N}{\beta}(\lambda x - 1) \]  

(8)

The ground state of the Fermi gas described in Eqs. (7) and (8) is obtained by filling the first \( N \) one particle levels of \( V(x) \). The associated Fermi energy must be evaluated self-consistently from the \( N \)-dependent potential \( V(x) \). Therefore, unlike the case of the \( d=1 \) model, the Fermi energy is not a free parameter that can be used to define the double scaling limit.

In order to study the higher stabilized multicritical points \( (k \geq 3) \), one has to cope with the interaction term in Eq. (6). Since only the \( U(N) \)-singlet ground state \( \Psi_0(\Phi) \) of \( \mathcal{H} \) is involved – it is natural to analyze the interacting gas in terms of the Fermion density operator – i.e., the collective field \( \phi(x) \) associated with the matrix \( \Phi \).

Following [22] [23] we define the collective field as

\[ \phi(x) = \frac{1}{\beta} T_r \delta(1 \cdot x - \Phi) = \frac{1}{\beta} \sum_{i=1}^{N} \delta(x - x_i) \]  

(9)

which implies that \( \phi(x) \) is a non negative operator and obeys the normalization condition

\[ \int \phi(x) dx = \frac{N}{\beta} \]  

(10)

Since \( \Phi \) depends on the Langevin time \( t \) and its dynamics is fixed by \( \mathcal{H} \) in Eq. (1), the Jacobian of the transformation from the matrix eigenvalue variables to the collective field is exactly the same Jacobian as in the \( d = 1 \) case [23] [24].

In this letter, we concentrate on the planar approximation to \( \mathcal{H} \) in order to establish the fact that the \( k - th \) order multicritical behaviour is respected by our formalism.

Discussing the exact loop equations and the associated non perturbative effects is deferred to a subsequent publication [24].

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8We have used the normalization of [24].

9The use of precisely this Jacobian is dictated by the very definition of the supersymmetrized model in accordance with [24] [25], bypassing the need to invoke arguments of the type used by [24], or postulates about the form of the zero dimensional partition function as in [23], that are needed if one makes the transformation to collective modes directly in the zero dimensional matrix model.
The planar collective field action $S_0[\phi] = \beta^3 \int \mathcal{L}_0 dxdt$ for the matrix Hamiltonian $\mathcal{H}$ in Eq. (1) is given by \cite{22, 23}.

\[
\mathcal{L}_0 = -\left[\frac{\pi^2}{6} \phi^3(x) + \left(\frac{U'(x)}{8} - \mu_F\right)\phi(x)\right] + \frac{1}{4} \int dy \frac{U'(x) - U'(y)}{x - y} \phi(x)\phi(y)
\]  

(11)

Here $\mu_F$ is a Lagrange multiplier (the chemical potential) that enforces the constraint of Eq. (10). Unlike the case of $d = 1$ matter \cite{23}, $\mu_F$ is not a free parameter whose deviation from a critical value is used to define the scaling behaviour. Eq. (11) yields the planar contribution to the genus expansion. The term $\frac{1}{8}(U'(x))^2$ is the external effective potential in which the eigenvalues move. Its contribution to the action clearly produces the $\frac{\beta^2}{8} T_c(U'(\Phi))^2$ term in Eq. (1). Similarly, the nonlocal bilinear interaction term in Eq. (11) is the collective field version for $\mathcal{H}_{int}$ in Eq. (6). The cubic term in Eq. (11) arises due to the transformation from matrix eigenvalues to collective modes. If one considers the first few terms in the $1/\beta$ expansion of the collective field action, including the kinetic term that is identical to the one used in $d = 1$ matrix models, one obtains (up to ambiguities known in this expansion) a nonlocal collective field theory analogous to the one developed in \cite{23}.

However, unlike the latter, fluctuations of our collective field theory around the WKB solution do not correspond to genuine string field components due to the fact that the target space dimension is negative. Nevertheless, it might include minor fractions of string field components – namely, discrete states.

The planar collective field equation of motion is readily found to be

\[-\frac{\delta S[\phi]}{\delta \phi(x)}|_{\text{planar}} = \frac{\pi^2}{2} \phi(x)^2 + \frac{U'(x)^2}{8} - \mu_F
\]

\[-\frac{1}{2} \int dy \frac{U'(x) - U'(y)}{x - y} \phi(y) = 0
\]

where $\phi(x)$ is subjected to the constraint of Eq. (10), and that by definition, $\phi(x)$ is nonnegative.

A crucial observation is that for $\frac{N}{\beta} = 1$ and $\mu_F = 0$, this nonlinear nontrivial integral equation is identical to the planar limit of the Schwinger-Dyson equation obeyed by the loop
operator in the original zero dimensional matrix model \[19\] \[18\] \[17\]

\[ \begin{align*}
F(z)^2 - U'(z)F(z) + \eta(z) &= 0 \\
F(z) &= \lim_{\beta \approx N \to \infty} \frac{1}{\beta} \left[ \frac{1}{z - \Phi} \right] \\
\eta(z) &= \lim_{\beta \approx N \to \infty} \frac{1}{\beta} \left[ \frac{U'(z) - U'({\Phi})}{z - \Phi} \right]
\end{align*} \] (13)

when \( z \) approaches the real axis. \[9\]

The fact that Eq. (12) is identical to the planar loop equation of the original matrix model is not surprising and conforms with the postulates of \[12\]. Moreover, it seems that the WKB expansion of Eqs. (3) and (4) should correspond term by term to the genus expansion of the corresponding Schwinger-Dyson (i.e. loop) equations in the original model\[14\]

Therefore, under the conditions \( N = 1 \) and \( \mu_F = 0 \), \( \phi(x) \) that solves Eq. (12) is just the planar limit eigenvalue density of the original Dyson gas in an external potential \( U(x) \).

Thus, for a matrix potential \( U(\Phi) \) in the universality class of the \( k \)-th multicritical point, \( \phi(x) \) will exhibit \( k \)-th order multicritical behaviour. In particular, if we chose \[19\] \[14\]

\[ U'_k(x) = \frac{k!(k+1)!}{(2k)!} [(2 - x)^k - 1 - (4 - x^2)^{\frac{k}{2}}]_+ \] (14)

then \( \phi(x) \) is supported only along the closed segment [-2,2] on the real axis and is given by

\[ \phi_k(x) = \begin{cases} 
\frac{k!(k+1)!}{2\pi(2k)!}[4 - x^2]^\frac{k}{2} (2 - x)^{-k-1}, & |x| \leq 2 \\
0, & \text{otherwise}
\end{cases} \] (15)

and satisfies

\[ \int_{-2}^{2} \phi_k(x) dx = 1 . \] (16)

\[10\] As \( z = x - i\epsilon , \ \epsilon \to 0^+ \) we have \[19\] \[18\] \[17\]

\[ F(z) = \frac{1}{2} U'(x) + i\pi \phi(x) \]

thus making the imaginary part of Eq. (13) vanishing and its real part proportional to Eq. (12).

\[11\] It can be shown \[27\], for example, that an Ehrenfest theorem, associated with Eq. (4), given by \( \langle \frac{\partial}{\partial x} \frac{1}{2} \left( \frac{1}{z - \Phi} \right) \rangle_+ \geq 0 \) leads order by order in the WKB expansion (in the \( N < \beta \) phase) to the genus expansion of the corresponding loop equation in the original matrix model, of which Eq. (13) is the leading (planar) term.

\[12\] The (+) subscript means that in an expansion of the r.h.s. of Eq. (14) around \( x = \infty \) we keep only nonnegative powers of \( x \).
Eqs. (14)-(16) are the solution to Eqs. (10) and (12) precisely at the k-th multicritical point.

In order to have a complete solution of the stabilized model on the sphere one has to show that these equations are properly deformed by turning on the cosmological constant. Namely, that under a deformation of $TrU_k(\Phi)$ by the picture operator $Tr\Phi$, $TrU_k(\Phi) \rightarrow TrU_k(\Phi) + \mu_B Tr\Phi$, where $\mu_B$ is the bare cosmological constant, there exists a solution to Eqs. (10) and (12), supported along a single segment on the real eigenvalue axis, such that $1 - \frac{N}{\beta}$ scales as $\beta^{-2k/2k+1}$. Such a deformation of the normalization condition in Eq. (16) may be obtained by allowing the multicritical end-point (i.e. $x = 2$) of the support of $\phi_k(x)$ in Eq. (15) to vary, on a proper scale.\footnote{In the d=1 matrix model \cite{23}, variation of the chemical potential $\mu_F$ changes the location of the classical turning points which are the end points of supp$\{\phi(x)\}$.} This scale must be that of the double scaled fluctuations of the matrix near its critical point $\Phi_c = 2$, i.e., $\beta^{-2/2k+1}$.

Therefore, the required eigenvalue distribution should be supported along a segment [-2,b] where

$$b = 2 - \epsilon \beta^{-2/2k+1}$$

in which $\epsilon$ is a finite real parameter.

Such a deformation of $\phi_k(x)$ alone is not enough to obtain the desired scaling behaviour of $1 - \frac{N}{\beta}$, since it generically induces all k-1 relevant deformations \cite{1} \cite{19} present at the $k^{th}$ multicritical point. The desired solution to Eqs. (10) and (12) must therefore include counterterms that will cancel these unwanted scaling contributions to $1 - \frac{N}{\beta}$. Thus, it must have the general form

$$\phi(x) = \frac{C_k}{\pi} (2 + x)^{\frac{1}{2}} (b - x)^{\frac{k-2}{2}} + \sum_{n=1}^{k-1} \beta^{-2(k-n)} \alpha_n^{(k)} \frac{C_n}{\pi} (2 + x)^{\frac{1}{2}} (b - x)^{n-\frac{1}{2}}$$

on its support [-2,b].

Here $C_n = \frac{n!(n+1)!}{2(2n)!}$ normalizes $\int_{-2}^{b} \phi_n(x)$ to unity when $b \rightarrow 2$ as in Eqs. (15) and (16).

$\beta^{-2(k-n)/2k+1}$ ($n \leq k - 1$) are the scaling dimensions of the relevant perturbations at the k-th multicritical point \cite{1} \cite{19} ensuring that $\phi(x)$ scales as a whole as $\beta^{-2k/2k+1}$ when $b - x \sim y \beta^{-2/2k+1}$.
Finally, $\alpha_n^{(k)} \ (n \leq k - 1)$ are the double scaled couplings of the relevant scaling operators, that will be uniquely fixed by the scaling behaviour of $1 - \frac{N}{\beta}$.

Indeed, using the elementary integral $\int_{-2}^{b} \frac{C_n}{\pi} (2 + x)^\frac{1}{2} (b - x)^{n - \frac{1}{2}} dx = \left(\frac{b + 2}{4}\right)^{n + 1}$ Eqs. (10), (17) and (18) yield

$$\frac{N}{\beta} = (1 - \frac{\epsilon}{4}\lambda)^{k+1} + \sum_{n=1}^{k-1} \alpha_n^{(k)} \lambda^{k-n} (1 - \frac{\epsilon}{4}\lambda)^{n+1}$$

(19)

where we have set $\lambda = \beta^{-\frac{2}{2k+1}}$.

In the vicinity of the $k$-th multicritical point, $\frac{N}{\beta}$ scales as

$$\frac{N}{\beta} = 1 - t \beta^{-\frac{2}{2k+1}}$$

(20)

where $t$ is the renormalized cosmological constant. Therefore, expanding the r.h.s. of Eq. (19) in powers of $\lambda$, the coefficients of $\lambda, \lambda^2, \ldots, \lambda^{k-1}$ must vanish, providing $k - 1$ equations that uniquely fix the $k - 1$ unknowns ($\alpha_n$) in Eq. (18). The latter are readily found to be given in closed form by

$$\alpha_n^{(k)} = \binom{k+1}{n+1} \left(\frac{\epsilon}{4}\right)^{k-n}$$

(21)

whence $\frac{N}{\beta} = 1 - (k + 1) \left(\frac{\epsilon}{4}\right)^{k} \beta^{-\frac{2}{2k+1}} \cdot (1 + \mathcal{O}(\beta^{-\frac{2}{2k+1}}))$. Therefore, the renormalized cosmological constant is given by$^{14}$ $t = (k + 1) \left(\frac{\epsilon}{4}\right)^k$, or equivalently

$$\epsilon = 4 \left(\frac{t}{k+1}\right)^{1/k}$$

(22)

Eqs. (17), (18), (21) and (22) comprise together the desired deformed solution $\phi(x)$ to Eq. (12).

To this eigenvalue distribution we may add relevant scaling deformations with arbitrary couplings, that will not alter the normalization condition in Eq. (10) $^{19}$.

Eq. (22) exhibits the well known scaling relations $^{1}$ on the sphere between the specific heat $\epsilon$ of the original matrix model and the cosmological constant $t$. That is, Eq. (22) implies that the string susceptibility is given by $\gamma_{str} = -1/k$. Equivalently, it implies that it is impossible

$^{14}$Eqs. (21) and (22) were obtained in closed form by M. Moshe.
to obtain a negative cosmological constant in our solution for even values of \( k \), keeping \( \epsilon \) real – i.e., that it holds only in the \( N < \beta \) phase of the theory.

General solutions of the planar loop equation (Eq. (12)) in the \( N > \beta \) phase of multicritical points of even order \( k \) would probably have branch point singularities in the complex eigenvalue plane (arranged in complex conjugate pairs) other than the two branch points on the real axis \( \Re \) \( \Im \), giving rise to multicut (planar) macroscopic loops which may be supported at negative loop lengths where they are also oscillatory and nonpositive definite). The latter fact, should however, in our opinion, be considered as a sickness of the planar loop equation, or equivalently of the saddle point (WKB) solution of the stabilized matrix model – rather than as an \( a \ priori \) signal of non-perturbative instabilities of the Marinari-Parisi procedure.

\( \beta \phi(x) \) given by Eqs. (18) and (21) clearly equals the WKB approximation to the particle density of the interacting Fermi gas of eigenvalues (integrated up to the Fermi energy).

By construction, \( \phi(x) \) scales as \( \beta^{-\frac{2k-1}{2k+1}} \) in terms of the scaling variable \( y = \beta^{\frac{2}{2k+1}}(b - x) \). Therefore, according to well known arguments, the WKB density of one-particle states at the Fermi energy, corresponding to one-particle excitations of the ground state of the gas is given by\(^{15}\)

\[
\frac{1}{\beta} \left( \frac{\partial N}{\partial E} \right)_{E=E_F} \sim \int_{-2}^{b} \frac{dx}{\phi(x, b)} \approx \beta^{\frac{2k-3}{2k+1}} \int_{0}^{\infty} \frac{dy}{\phi(y, \epsilon)} = \beta^{\frac{2k-3}{2k+1}} \rho(\epsilon) \tag{23}
\]

This integral is generically well behaved, and diverges only as \( \epsilon \to 0 \), i.e., as one approaches the \( k \)-th multicritical point.\(^{16}\)

Thus, the energy-gap of these excitations double-scales as

\[
\Delta(\epsilon) = \left[ \frac{1}{\beta} \left( \frac{\partial N}{\partial E} \right)_{E=E_F} \right]^{-1} \sim \beta^{-\frac{2k-3}{2k+1}} \frac{1}{\rho(\epsilon)} \sim 1 - \frac{N}{\beta} \left| \frac{2k-3}{2k} \right|
\]

where the last proportionality is expected on general scaling arguments. For odd values of \( k \), our solution \( \phi(x) \) is well defined either in the \( N < \beta \) phase or in the \( N > \beta \) phase. It has the same singularity structure either for positive \( \epsilon \) and \( t \) or for negative ones. Thus, in such cases \( \Delta(\epsilon) \) in Eq. (24) is well defined in both phases.

\(^{15}\)Such one-particle states should exist at least in a framework of Hartree-Fock analysis of the gas.

\(^{16}\)\( \phi(x) \) given by Eqs. (18) and (21) obviously vanishes only at \( x = b, -2 \), where it has generically Wigner semicircle singularities as long as \( \epsilon \neq 0 \).
In case of even values of \( k \), \( \phi(x) \) we have found exists only in the \( N < \beta \) phase where \( \Delta(\epsilon) \) is real and positive. Recall that a positive real double scaled excitation gap is essential for the stabilization of the model by Eqs. (3) and (4) to work.

For \( k = 2 \) our results reproduce the WKB analysis of [11] [13] in the \( N < \beta \) phase of the matrix model. Indeed, in this case, Eqs. (17), (18), (21), (22) and (24) read

\[
\phi(x) = \begin{cases} 
\frac{1}{4\pi}[(2 + x)(2 - \epsilon\beta^{-2/5} - x)]^{\frac{1}{2}}(2 + \frac{x}{2}\beta^{-2/5} - x) & -2 \leq x \leq 2 - \epsilon\beta^{-2/5} \\
0 & \text{otherwise}
\end{cases}
\]

as well as \( \frac{N}{\beta} = 1 - 3(\frac{x}{4})^2\beta^{-4/5} \) and \( \Delta(\epsilon) = \frac{\sqrt{6\pi}}{4\pi^2} \beta^{-1/5} = \frac{1}{\pi^2}(\frac{3}{4}t)^{\frac{1}{2}}\beta^{-1/5} \sim (1 - \frac{N}{\beta})^{\frac{1}{4}} \).

This single segment supported \( \phi(x) \) corresponds to the fact that the Fermi energy of the eigenvalue gas described by Eq. (7), coincides with the shallow local minimum of \( V(x) \) in Eq. (8) near \( x = +2 \). This eigenvalue distribution leads to a single cut macroscopic (planar) loop operator [13] whose Laplace transform is supported only at non-negative loop lengths [9] [28]. Analysis of Eqs. (7) and (8) in the \( N > \beta \) phase was made in [9]. We have briefly described it in our comments following Eq. (22), where we have also stated our interpretation of the difficulties pointed out in [9] [17].

We have remarked above\(^7\) that the analysis of the \( k = 2 \) point in [13] were performed by keeping the original cubic matrix potential critical while varying \( N/\beta \) around its critical value 1. At this point of our discussion it is clear that [13] has been successful in doing so because precisely for that cubic potential does \( N/\beta \) couple to its appropriate scaling perturbation in \( H_{int} \) – namely, the puncture operator \( \text{Tr}\Phi \).

Up to this point we have established that all stabilized multicritical one matrix models are equivalent on the sphere to the corresponding original (zero dimensional) models. This is expected also to hold to all orders in the genus expansion [12] [27] (at least in the \( N < \beta \) phase in the case of even \( k \)). This conclusion is a good starting point and a motivation to study the non-perturbative nature of the stabilized multicritical models.

It is well known that the set of all multicritical points of (zero dimensional) one matrix models form an integrable hierarchy – namely that of the KdV equation [1] [2] [29] [14].\

\(^{17}\)We have also constructed two solutions to Eq. (12) for \( k = 2 \) and \( \frac{N}{\beta} > 1 \) in which \( \mu_F \neq 0 \). One of these solutions with a single segment support exhibits scaling behaviour of the \( k = 3 \) point. The other solution with two real and two complex conjugate branch points seems to correspond to the one discussed in [9] .
The latter may be represented by a set of Virasoro constraints that annihilate the exact partition function of the matrix model \[30\].

Thus, in order to establish quantitative results concerning nonperturbative differences between the original and stabilized matrix models one must first try to reformulate the latter as an integrable hierarchy.

Such a construction, if possible, may be carried either by attempting to formulate the loop equations of the stabilized models in a manner analogous to \[31\] or by checking whether the interacting Fermi gas itself forms an integrable system.

In the first case, one makes an explicit use of the basic observables of the matrix model – namely the loop operators – hence an immediate comparison of the two types of models may be done. It may well be that using the exact collective field theory in this respect, without expanding its Jacobian in powers of \(1/\beta\), turns out to be quite valuable \[25\], especially due to the fact that this Jacobian (for the stabilized model) is identical to the one used in \(d=1\) matrix models \[24\]. The enormous symmetry possessed by the collective field theory in the latter case \[31\] or by its Fermionic counterpart \[32\] might have counterparts in our case as well (even though surely in our case we have more complicated Fermi “Fluid dynamics”).

The other method proposed above will yield, if it turns out to be successful, the entire spectrum of the interacting Fermi gas and its exact S-matrix. This may be by itself an interesting result in the theory of \(1 + 1\) dimensional integrable models – extending the local nonlinear Schrödinger model into a nonlocal version (with special one and two body interactions – derived from multicritical potentials) \[18\].

**Summary and Conclusions**

We have shown that the Marinari-Parisi definition of pure gravity (\(k=2\)) may be extended to stabilize all higher one matrix multicritical points. This was done by demonstrating the equivalence...
lence on the sphere of the original and stabilized models. It seems that this equivalence should hold to all orders in the genus expansion. The collective field of the stabilized matrix model turned out to be useful in coping with eigenvalue interactions in our semiclassical treatment of the Fermi gas.

We have pointed out that, since the whole structure of multicritical points of one matrix models may be transferred to the Marinari-Parisi arena, the most important questions are whether they form there an integrable hierarchy as the original models do, and if the latter is answered on the affirmative – how does it differ from the original KdV hierarchy?

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