Seiberg–Witten monopoles and flat \( \text{PSL}(2, \mathbb{R}) \)-connections

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Abstract

I show that flat \( \text{PSL}(2, \mathbb{R}) \)-connections on three-manifolds satisfying certain ‘stability condition’ can be interpreted as solutions of the Seiberg–Witten equations with two spinors. This is used to construct explicit examples of the Seiberg–Witten moduli spaces. Also, I show that in this setting blow up sets satisfy certain non-trivial topological restrictions.

1 Introduction

Studies of the boundary points of the moduli space of solutions to the Seiberg–Witten equations with multiple spinors are concerned with some interesting phenomena, which are new and of importance beyond the Seiberg–Witten theory. Similar phenomena occur for example in the studies of the Vafa–Witten, Kapustin–Witten, complex anti-self-duality, Hermitian Yang–Mills, \( G_2 \)- and Spin(7)-instanton equations and many other interesting geometric PDEs.

To set the stage, let \( M \) be a closed oriented Riemannian three-manifold. Pick a spin structure and denote by \( \mathcal{S} \) the corresponding spinor bundle. Recall that \( \mathcal{S} \) is a Hermitian rank 2 bundle such that \( \text{End}_0(\mathcal{S}) \) is isomorphic to \( T^*_C M = T^* M \otimes \mathbb{C} \), where the subscript 0 indicates the subbundle of trace-free endomorphisms.

Let \( E \) be any fixed Hermitian vector bundle of rank 2 such that \( \Lambda^2 E \) is trivial, so that the structure group of \( E \) is SU(2). Fix also an SU(2)-connection \( b \) on \( E \).

For a Hermitian line bundle \( \mathcal{L} \) the bundle \( \text{Hom}(E; \mathcal{S} \otimes \mathcal{L}) \) will be referred to as the twisted spinor bundle. If \( \Psi \) is a section of the twisted spinor bundle, then

\[
\mu(\Psi) = \Psi \Psi^* - \frac{1}{2} |\Psi|^2
\]

(1)

is a trace-free Hermitian endomorphism of \( \mathcal{S} \) (the twist by \( \mathcal{L} \) is immaterial here). Using the isomorphisms \( \text{End}_0(\mathcal{S}) \cong T^*_CM \cong \Lambda^2 T^*_CM \), \( \mu(\Psi) \) can be identified with a purely imaginary 2-form on \( M \). With this at hand, the Seiberg–Witten equations with two spinors read

\[
\mathcal{D}_{a \otimes b} \Psi = 0 \quad \text{and} \quad F_a = \mu(\Psi),
\]

(2)

where \( a \) is a Hermitian connection on \( \mathcal{L} \), see [HW15] for more details. Let me just point out that while (2) looks just like the classical Seiberg–Witten equations, an essential difference lies in the
structure of the quadratic map $\mu$, which in a local trivialization of $E$ can be written as follows

$$(\psi_1, \psi_2) \mapsto \psi_1 \psi_1^* - \frac{1}{2} |\psi_1|^2 + \psi_2 \psi_2^* - \frac{1}{2} |\psi_2|^2.$$ 

In particular, $\mu$ is no longer proper, which, in some sense, is a source of potential non-compactness of the moduli space of solutions. The failure of the compactness is discussed in detail in [HW15]. In fact, I construct below fairly explicit examples of such moduli spaces, in particular some non-compact ones, see the discussion at the end of Section 3.

According to [HW15], if $(a_k, \Psi_k)$ is a non-convergent sequence of solutions of (2), then a subsequence of $(a_k, \|\Psi_k\|_{L^2}^{-1} \Psi_k)$ converges to some $(a, \Psi)$ over $M \setminus Z$, where $Z$ is a closed subset of $M$ of Hausdorff dimension at most one. Moreover, the pointwise norm of $\Psi$ extends as a continuous function to all of $M$, $|\Psi|^{-1}(0) = Z$, $\|\Psi\|_{L^2(M \setminus Z)} = 1$, and $a$ is flat with the monodromy in $\{\pm 1\}$. Thus, in a certain sense solutions of (3) describe the topological boundary of the moduli space of solutions of the Seiberg–Witten equations with two spinors. Notice that a solution of (3) does not need to be defined along $Z$.

It follows from the proof of [HW15, Thm. 1.1] that if the sequence $(a_k, \|\Psi_k\|^{-1} \Psi_k)$ converges to $(a, \Psi)$ in the sense described above, then the set

$$\left\{ m \in M \mid \exists r_k \to 0 \text{ s.t. } r_k \int_{B_{r_k}(m)} |F_{A_k}|^2 \to \infty \right\}$$

is contained in $Z$, where $B_r(m)$ is the geodesic ball of radius $r$ centered at $m$. This motivates the following.

**Definition 4 ([Hay19])**. A closed nowhere dense set $Z \subset M$ is called a blow-up set for the Seiberg–Witten equation with two spinors, if there is a solution $(a, \Psi)$ of (3) defined over $M \setminus Z$ such that the following holds:

(i) $|\Psi|$ extends as a Hölder-continuous function to all of $M$ and $Z = |\Psi|^{-1}(0)$;

(ii) $\int_{M \setminus Z} |\nabla A\Psi|^2 < \infty$.

Condition (ii) in the above definition is of technical nature and will not be used directly in the discussion below.

It is easy to see that if the determinant line bundle $L^2$ is non-trivial, then $Z \neq \emptyset$. Indeed, assume that for a non-trivial $L^2$ there is a solution $(A, \Psi)$ of (3) such that $\Psi$ vanishes nowhere, i.e., $Z = \emptyset$. The equation $\mu(\Psi) = \Psi \Psi^* - \frac{1}{2} |\Psi|^2 = 0$ implies that $\operatorname{ker} \Psi^* = \{0\}$ pointwise. Hence, $\Psi$ is surjective everywhere, which in turn yields that $\Psi$ is an isomorphism, because $\operatorname{rk} E = \operatorname{rk} (\mathcal{S} \otimes \mathcal{L})$. Therefore, $\Lambda^2 E \cong \Lambda^2 (\mathcal{S} \otimes \mathcal{L}) \cong L^2$, which yields that $L^2$ is trivial thus providing a contradiction.

Thus, in order to construct a compactification of the moduli space of the Seiberg–Witten monopoles with two spinors one needs to understand properties of blow up sets $Z$. In particular, one can ask whether there are any other restrictions on $Z$ apart from being closed and of Hausdorff dimension at most one.

A topological restriction for $Z$ has been established in [Hay19]. Namely, it has been shown that $Z$ supports a homology class, which is Poincaré dual to $c_1(L^2)$. 


In this manuscript a topological restriction of another type is obtained. Before explaining this, let me note that in general $Z$ does not need to be smooth, see however [Zha17] for the most general regularity statement currently known. Recent results by Taubes–Wu [TW20] yield a strong evidence that $Z$ may be singular in general. Nevertheless, it is conceivable that after a suitable perturbation (of the background metric, say) the blow up set will become a smoothly embedded 1-submanifold, i.e., a link in $M$. In any case, to formulate the next result, let me assume that $Z$ is a link indeed.

Thus, put $E = \$ $ with $b$ being the Levi–Civita connection and assume that $Z$ is a link in $M$. Denote by $Z_1$ the union of all components of $Z$ such that the monodromy of $a$ along the meridian is non-trivial.

**Theorem 5.** Let $(a, \Psi, Z)$ be a solution of (3) on an integral homology sphere $M$ with $E = \$ $ and $b$ being the Levi–Civita connection. If $Z$ is smooth and $\Delta_{Z_1}(t)$ denotes the Alexander polynomial of $Z_1$, then $\Delta_{Z_1}(-1) = 0$. In particular, $Z$ consists of at least two connected components if it is smooth.

The proof of this theorem can be found on Page 13. Concerning the very last claim about the disconnectedness of $Z$, it is well-known that the Alexander polynomial of a 1-component link does not vanish at the point $-1$, see for example [Lic97, Cor. 6.11] in the case $M = S^3$. In any case, this will be also clear from the proof of Theorem 5.

While a very particular choice of the twist is required for the proof of Theorem 5, the topological restrictions obtained are stable under small deformations. To explain, notice first that solutions of (3) correspond [HW15, App. A] to certain $\mathbb{Z}/2$ harmonic 1-forms [Tau13, Tau15], for which $Z$ is a part of the data. In any case, if $Z$ is an embedded link, then the space of all $\mathbb{Z}/2$ harmonic 1-forms in some neighborhood is cut out by a Fredholm map [Tak15]. Hence, it is reasonable to expect that if $(a, \Psi, Z)$ is a non-trivial solution of (3) for $(g, b) = (g, \nabla^{LC})$, then for any other choice of parameters $(\hat{g}, \hat{b})$, which are sufficiently close to $(g, \nabla^{LC})$ and admit a non-trivial solution $(\hat{a}, \hat{\Psi}, \hat{Z})$, the set $\hat{Z}$ is also an embedded link, which is isotopic to $Z$. In this case, $\Delta_{\hat{Z}_1}(-1)$ must vanish too. In this sense, the conclusion of Theorem 5 is a manifestation of generic properties of blow up sets.

A crucial step in the proof of Theorem 5 is a correspondence between solutions of (2) and (3) with flat $\text{PSL}(2, \mathbb{R})$-connections on $M$ and $M \setminus Z$ respectively satisfying certain stability condition, see Definition 10 and Lemmas 12 and 17 below. Combining this with known topological and analytic results, the proof of the above theorem follows quite easily.

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## 2 Basic constructions

To fix notations, for a principal $G$-bundle $P \to M$, denote by $\mathcal{A}(P)$ the space of all connections on $P$ and recall that $\mathcal{A}(P)$ is an affine space modeled on $\Omega^1(ad P)$, where $ad P := P \times_{G, ad} \mathfrak{g}$ with $\mathfrak{g}$ being the Lie algebra of $G$. The gauge group

\[ \mathcal{G}(P) = \{ \psi: P \to P \mid \pi \circ \psi = \pi, \quad \psi(p \cdot g) = \psi(p) \cdot g \} \]
acts naturally on \( \mathcal{A}(P) \) by pull-backs.

Recall also that the group \( \text{PSL}(2, \mathbb{C}) = \text{SU}(2, \mathbb{C})/\pm 1 \) acts transitively on the hyperbolic space

\[
\mathbb{H}^3 := \{ H \in M_2(\mathbb{C}) \mid H^* = H, \, \det H = 1 \} \quad \text{by} \quad g \cdot H = gHg^*.
\]

Since the stabilizer of the identity matrix is \( \text{PSU}(2) = \text{SU}(2)/\pm 1 \cong \text{SO}(3) \), we have \( \mathbb{H}^3 = \text{PSL}(2, \mathbb{C})/\text{PSU}(2) \).

Let \( \pi: Q^c \to M \) be a principal \( \text{PSL}(2, \mathbb{C}) \)-bundle, where \( M \) is a closed oriented Riemannian three manifold just as in the preceding section. Choose a reduction of the structure group to \( \text{PSL}(2, \mathbb{C}) \), that is a \( \text{PSL}(2, \mathbb{C}) \)-subbundle \( Q \subset Q^c \). Recall that such reduction corresponds to a section \( s \) of the bundle

\[
Q^c \times_{\text{PSL}(2, \mathbb{C})} \text{PSL}(2, \mathbb{C})/\text{PSU}(2) = Q^c \times_{\text{PSL}(2, \mathbb{C})} \mathbb{H}^3
\]
or, equivalently, an equivariant map \( \hat{s}: Q^c \to \mathbb{H}^3 \). Explicitly, \( \hat{s} \) and \( s \) are related by \( Q = \{ q \in Q^c \mid \hat{s}(q) = 1 \} \). Notice that \( Q^c \) can be recovered from \( Q \), since \( Q^c = Q \times_{\text{PSL}(2, \mathbb{C})} \text{PSL}(2, \mathbb{C}) \).

For any connection \( A \) on \( Q^c \), which is an equivariant 1-form on \( Q^c \) with values in \( \mathfrak{psl}(2, \mathbb{C}) \), write \( A = a + (\pi^*b)i \), where \( a \) is a connection on \( Q \) while \( b \) is a 1-form on \( M \) with values in \( \mathfrak{ad} Q \). Then \( 0 = F_A = F_a + (d_a)b - \frac{1}{2}[b \land b] \), that is \( A \) is flat if and only if

\[
d_a b = 0 \quad \text{and} \quad F_a = \frac{1}{2}[b \land b], \tag{6}
\]
cf. [Hit87]. Notice that the space of solutions of these equations is invariant under the action of \( \mathcal{G}(Q^c) \), which is referred to as the complex gauge group below.

**Remark 7.** Sometimes I identify \( b \in \Omega^1(\mathfrak{ad} Q) \) with \( \pi^*b \in \Omega^1(Q; \mathfrak{su}(2)) \) dropping the pull-back from the notations if this is unlikely to lead to a confusion.

**Definition 8.** I say that \( A = a + (\pi^*b)i \) is a **stable flat** \( \text{PSL}(2, \mathbb{C}) \)-connection, if \( A \) is flat and the following condition holds:

\[
d_a^* b = -* d_a * b = 0 \tag{9}
\]

The above stability condition has been studied at least starting from [Hit87, Don87, Cor88] and can be understood as follows. By writing connections on \( Q^c \) as pairs just like above, we have an isomorphisms \( \mathcal{A}(Q^c) \cong \mathcal{A}(Q) \times \Omega^1(\mathfrak{ad} Q) \cong T^* \mathcal{A}(Q) \). In particular, \( \mathcal{A}(Q^c) \) has a natural symplectic structure. The gauge group \( \mathcal{G}(Q) \), which is referred to as a `real gauge group’ in the sequel, acts in a Hamiltonian fashion, and the corresponding moment map can be identified with

\[
\mathcal{A}(Q) \times \Omega^1(\mathfrak{ad} Q) \to \text{Lie}(\mathcal{G}(Q)) \cong \Omega^0(\mathfrak{ad} Q), \quad (a, b) \mapsto d_a^* b.
\]

Hence, (9) demands that \( (a, b) \) lies in the zero locus of this moment map. This is the familiar `stability condition’ from algebraic/symplectic geometry.

Notice also that (9) is preserved by the **real gauge group** \( \mathcal{G}(Q) \), but not by the complex one.

**Definition 10.** I say that a pair \( (A, Q^r) \) is a **flat stable** \( \text{PSL}(2, \mathbb{R}) \)-connection, if the following holds:

- \( A = a + (\pi^*b)i \) is a solution of (6) and (9);
- \( Q^r \subset Q^c \) is a \( \text{PSL}(2, \mathbb{R}) \)-subbundle such that \( A \) reduces to \( Q^r \), i.e., the restriction of \( A \) to \( Q^r \) takes values in \( \mathfrak{psl}(2, \mathbb{R}) \).
Remark 11. For any PSL(2, ℂ)-connection $A$ and any $q_0 \in Q^c$, let $\text{Hol}(A, q_0)$ be the holonomy group of $A$ relative to $q_0$. Let $Q_{\text{hol}}(A, q_0)$ denote the holonomy bundle of $A$, that is $Q_{\text{hol}}(A, q_0)$ consists of all those $q \in Q^c$ which can be connected with $q_0$ by a horizontal curve. Then $Q_{\text{hol}}(A, q_0)$ is a principal $\text{Hol}(A, q_0)$-bundle and it is well known that $A$ reduces to $Q_{\text{hol}}(A, q_0)$, see for example [Nom55, Prop. 2]. Then for any $g \in \text{PSL}(2, \mathbb{C})$ we have

$$\text{Hol}(A, q_0 g) = g^{-1} \text{Hol}(A, q_0) g \quad \text{and} \quad Q_{\text{hol}}(A, q_0 g) = Q_{\text{hol}}(A, q_0) g.$$ 

If $\text{Hol}(A, q_0) \subset \text{PSL}(2; \mathbb{R})$, then there exists a unique PSL(2, ℛ)-bundle $Q^r(A, q_0) \supset \text{Hol}(A, q_0)$ such that $A$ restricts to $Q^r(A, q_0)$. Any other choice of the basepoint yields

$$Q_{\text{hol}}(A, q_0 \cdot g) = Q_{\text{hol}}(A, q_0) g \quad \implies \quad Q^r(A, q_0 \cdot g) = Q^r(A, q_0) \cdot g,$$

where $g \in \text{PSL}(2, \mathbb{C})$.

Hence, if $q_0$ lies in the normalizer of PSL(2, ℛ) and $A$ reduces to $Q^r$, then $A$ also reduces to $Q^r \cdot g_0$. Notice that in this case $\text{Hol}(A, q_0)$ and $\text{Hol}(A, q_0 \cdot g_0)$ are related by an outer automorphism of PSL(2, ℛ). I shall come back to this point in Section 3 again.

Notice that in the above definition I fix the standard embedding PSL(2, ℛ) $\subset$ PSL(2, ℂ). Since PSL(2, ℛ) $\cap$ PSU(2) = U(1), this yields a distinguished copy of U(1) both in PSU(2) and PSL(2, ℛ). In the latter case, this copy of U(1) is a preferred maximal compact subgroup.

The reduction of the structure group of $Q^r$ to PSU(2) induces a reduction of the structure group of $Q^r$ to U(1). Indeed, the corresponding U(1)-subbundle is simply

$$P := \{ p \in Q^r \mid \hat{s}(p) = 1 \} = Q^r \cap Q.$$

Let $L$ be the complex Hermitian line bundle associated with $P$ and the standard U(1)-representation. Denote by $\chi$ the fiberwise symplectic form on $L$. A combination of the wedge-product and $\chi$ yields a fiberwise quadratic map $\chi(\cdot \wedge \cdot) : \text{Sym}^2(T^*M \otimes L) \to \Lambda^2 T^* M$.

Since $P$ can be also viewed as a reduction of the structure group of $Q$ to U(1), the bundle $adQ$ splits into the trivial bundle of rank 1 and a bundle of rank 2. The latter is naturally isomorphic to $L$.

**Lemma 12.** For each flat stable PSL(2, ℛ)-connection there exists a unique triple $(L, a, b)$, where $L$ is a Hermitian line bundle, $a$ is a Hermitian connection on $L$, and $b$ is a 1-form with values in $L$, such that $(a, b)$ satisfies

$$(d_a + d^*_a) b = 0 \quad \text{and} \quad F_a = \chi(b \wedge b)i. \quad (13)$$

Conversely, for any triple $(L, a, b)$ as above such that $(a, b)$ satisfies (13) there is a unique principal PSU(2)-bundle $Q := P \times_{U(1)} \text{PSL}(2, \mathbb{C})$ and a unique flat stable PSL(2, ℛ)-connection with

$$Q^r := P \times_{U(1)} \text{SL}(2, \mathbb{R}) \subset P \times_{U(1)} \text{PSL}(2, \mathbb{C}) =: Q^r. \quad (14)$$

**Proof.** Let $u(1) \subset \text{psl}(2; \mathbb{R})$ be the Lie algebra of the maximal compact subgroup U(1) $\subset$ PSL(2, ℛ). Notice that with respect to the decomposition $\text{sl}(2, \mathbb{C}) = \text{su}(2) \oplus \text{su}(2)i$ we have $\text{psl}(2; \mathbb{R}) = u(1) \oplus u(1)^{-i}$, where $u(1)^{-i}$ is the orthogonal complement of $u(1)$ in su(2). Choose an orthonormal oriented basis $(\eta_1, \eta_2, \eta_3)$ of su(2) such that $u(1) = \mathbb{R} \eta_1$ and $\text{span}(\eta_2, \eta_3) = u(1)^{-i}$, say

$$\eta_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \text{and} \quad \eta_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (15)$$
Notice that there is a unique symplectic form $\chi$ on $u(1)^\perp$ such that $\chi(\eta_2, \eta_3) = 1$, that is $(\eta_2, \eta_3)$ is a symplectic basis of $u(1)^\perp$.

If $Q^r \subset Q^r$ is a principal $\text{PSL}(2, \mathbb{R})$-subbundle as above, then $A = a + (\pi^* b)i$ reduces to $Q^r$ if and only if the real part takes values in the one-dimensional subspace $u(1) \subset \mathfrak{su}(2)$, while the imaginary part takes values in $u(1)^{\perp}i$. In other words, $a$ is a connection on $P = Q^r \cap Q$ and $b$ is a one-form on $M$ with values in $\mathcal{L} = \mathbb{C} \times \mathbb{U}(1) \subset \mathbb{C}$.

Furthermore, the splitting $\mathfrak{su}(2) = u(1) \oplus u(1)^{\perp}$ yields the decomposition $\text{ad} Q^r = \mathbb{R} \oplus \mathcal{L}$. Since $[\eta_2, \eta_3] = 2\eta_1$, the restriction of the Lie-brackets to $\mathcal{L}$ takes values in $\mathbb{R}$ and equals $2\chi$. Hence, the real part of $F_A$ vanishes if and only if $F_a = \chi(b \wedge b)i$. Thus, a $\text{PSL}(2, \mathbb{R})$-connection $A = a + (\pi^* b)i$ is flat and stable if and only if (13) holds.

Conversely, given a Hermitian line bundle $\mathcal{L}$ together with a solution $(a, b)$ of (13), essentially the same computation (reading backwards) yields that $A = a + (\pi^* b)i$ is a framed flat stable $\text{PSL}(2, \mathbb{R})$-connection with $Q^r$ given by (14). \hfill $\square$

**Example 16.** Let $\mathcal{L}$ be the product line bundle $\mathbb{C} := M \times \mathbb{C}$ and $a = \vartheta \equiv \text{the product connection}$. Assume furthermore that $b$ is a real-valued harmonic 1-form. Then for any $w \in \mathbb{C}$ the pair $(\vartheta, wb)$ is clearly a solution of (13). In this case the holonomy of $A$ is contained in a one-parameter subgroup generated by an element of $u(1)^{\perp} \otimes i \subset \mathfrak{sl}(2, \mathbb{R})$. In particular, $\text{Hol}(A)$ is abelian.

Starting from a different perspective, consider the Seiberg–Witten equations (2) with $E = \mathcal{S}$, which is equipped with the Levi–Civita connection. From now on I will assume this particular twist throughout even if this is not mentioned explicitly. In this case we have a well-defined trace map

$$\text{tr}: \text{Hom}(\mathcal{S}, \mathcal{S} \otimes \mathcal{L}) \cong \text{End}(\mathcal{S}) \otimes \mathcal{L} \to \mathcal{L}.\,$$

Denote by $\text{Hom}_0(\mathcal{S}, \mathcal{S} \otimes \mathcal{L}) \cong \text{End}_0(\mathcal{S}) \otimes \mathcal{L}$ the subbundle of traceless homomorphisms. The Clifford multiplication (twisted by the identity map on $\mathcal{L}$) provides an isomorphism

$$\text{Cl}: T_{\mathbb{C}}^* M \otimes \mathcal{L} \to \text{End}_0(\mathcal{S}) \otimes \mathcal{L},$$

which in turn yields an isomorphism

$$\Upsilon: \text{Hom}(\mathcal{S}, \mathcal{S} \otimes \mathcal{L}) \longrightarrow T_{\mathbb{C}}^* M \otimes \mathcal{L} \oplus \mathcal{L},$$

where $\Upsilon = (\text{Cl}^{-1}, \text{tr})$ and $\text{Cl}^{-1}$ is extended trivially to the trace-component.

**Lemma 17.** A pair $(a, \Psi)$ such that $\text{tr} \Psi = 0$ is a solution of (2) with $E = \mathcal{S}$ and $b$ being the Levi-Civita connection if and only if $(a, b) = (a, \text{Cl}^{-1}(\Psi))$ solves (13). Moreover, the following holds:

(i) If $\mathcal{L}$ is non-trivial, then for any solution of (2) we have $\text{tr} \Psi = 0$;

(ii) Any solution $(a, \Psi)$ of (2) with $\text{tr} \Psi \neq 0$ is gauge-equivalent to a pair $(\vartheta, \omega)$, where $\vartheta$ is the product connection on $\mathcal{L} = \mathbb{C}$ and $\omega$ is a purely imaginary harmonic 1-form. In particular, in this case $(a, \Psi)$ corresponds to a flat $\text{PSL}(2, \mathbb{R})$-connection with an abelian holonomy.
Proof. Let \((a, \Psi)\) be a solution of (2) such that \(\text{tr } \Psi = 0\). A straightforward computation, whose details can be found in Appendix A, yields that \(\mu(\Psi) = \text{Cl} (\chi(b \wedge b)i)\) so that \((a, b)\) solves (13) indeed.

Assume now that \(s := \text{tr } \Psi \neq 0\). It follows from (2) that \(s\) is a \(\nabla^a\)-covariantly constant section. Since \(a\) is Hermitian, \(s\) vanishes nowhere, hence proving \((i)\). Furthermore, \(a\) is the product connection with respect to the trivialization given by \(s\). Just like above, the traceless component \(\Psi_0\) of \(\Psi\) can be identified with some complex-valued 1-form \(b_1\) and (2) translates into

\[
(d + d^*) b_1 = 0 \quad \text{and} \quad F_a = \chi(b_1 \wedge b_1) + 2 * \text{Re } b_1,
\]

where we also have \(F_a = F_\varphi = 0\).

Furthermore, writing \(b_1 = b_{10} + b_{11}i\) we obtain \(b_{10} \wedge b_{11} + 2 * b_{10} = 0\), which yields in turn

\[
2 |b_{10}|^2 = 2 b_{10} \wedge * b_{10} = -b_{10} \wedge b_{10} \wedge b_{11} = 0,
\]

i.e., \(\text{Re } b_1 = 0\).

Thus, in the case \(s = \text{tr } \Psi \neq 0\), a solution \((a, \Psi)\) of (2) yields a trivialisation of \(\mathcal{L}\); Moreover, \(a\) is the product connection with respect to this trivialisation, and \(b_1\) is a purely imaginary harmonic 1-form.

Let \((a, \Psi)\) be a solution of (2) with \(E = \mathcal{C}\) and \(b\) being the Levi–Civita connection. Regarding \((a, \Psi)\) as a pair \((a, b)\) just like in Lemma 17 and recalling Lemma 12, we obtain that \(A = a + (\pi^* b)i\) is a flat stable \(\text{PSL}(2, \mathbb{R})\)-connection. Assign to \((a, \Psi)\) the holonomy representation

\[
\rho_{a, \Psi} : \pi_1(M) \to \text{PSL}(2, \mathbb{R})
\]

of \(A\). Of course, \(\rho_{a, \Psi}\) is well-defined up to the conjugation in \(\text{PSL}(2, \mathbb{R})\) only.

### 3 Involutions and a homeomorphism between moduli spaces

Pick \(\alpha \in H^2(M; \mathbb{Z})\) and let

\[
\mathcal{M}_{SW2}(\alpha) := \{(a, \Psi) \mid (a, \Psi) \text{ solves } (2) \text{ and } c_1(\mathcal{L}) = \alpha\}/C^\infty(M; \text{U}(1))
\]

be the moduli space of solutions of (2) with a fixed line bundle \(\mathcal{L}\). Denote also

\[
\mathcal{M}_{SW2} := \bigsqcup_{\alpha \in H^2(M; \mathbb{Z})} \mathcal{M}_{SW2}(\alpha).
\]

We shall see below that the above union is in fact finite.

It is well-known that the moduli space of solutions of the classical Seiberg–Witten equations (just one spinor) is equipped with an involution [Mor96, Sect. 6.7]. This construction carries over essentially verbatim to the present setting and yields an involution \(\sigma : \mathcal{M}_{SW2} \to \mathcal{M}_{SW2}\) such that \(\sigma : \mathcal{M}_{SW2}(\alpha) \to \mathcal{M}_{SW2}(-\alpha)\) is a homeomorphism for each \(\alpha \in H^2(M; \mathbb{Z})\).

However, it is easier and more convenient to describe this involution in terms of solutions of (13). Thus, set

\[
\mathcal{M}(\alpha) := \{(a, b) \text{ solves } (13) \text{ and } c_1(\mathcal{L}) = \alpha\}/C^\infty(M; \text{U}(1)) \quad \text{and} \quad \mathcal{M} := \bigsqcup_{\alpha} \mathcal{M}(\alpha).
\]
Letting $\mathcal{L}^\vee$ denote the dual of $\mathcal{L}$, define $\sigma: \mathcal{A}(\mathcal{L}) \times \Omega^1(M; \mathcal{L}) \to \mathcal{A}(\mathcal{L}^\vee) \times \Omega^1(M; \mathcal{L}^\vee)$ by

$$
\sigma(a, b) = (a^\vee, b^\vee),
$$

where $a^\vee$ is the connection dual to $a$ and if $b$ equals $\omega \otimes s$ locally, then $b^\vee = \bar{\omega} \otimes \langle \cdot, s \rangle$. Then $\sigma$ preserves the space of solutions of (13) and yields a homeomorphism $\sigma: \mathcal{M}(\alpha) \to \mathcal{M}(-a)$. This defines implicitly an involution, still denoted by the same letter, on $\mathcal{M}_{SW2}$.

**Lemma 18.** Let $Q^r$ be a principal $\text{PSL}(2, \mathbb{C})$-bundle equipped with a principal $\text{PSU}(2)$-subbundle $Q$. Let $g_0 \in \text{PSL}(2, \mathbb{C})$ denote a non-trivial representative in $\mathcal{N}(\text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, where $\mathcal{N}(\text{PSL}(2, \mathbb{R}))$ is the normalizer of $\text{PSL}(2, \mathbb{R})$ in $\text{PSL}(2, \mathbb{C})$. Let $A$ be a flat stable $\text{PSL}(2, \mathbb{C})$-connection which reduces to a principal $\text{PSL}(2, \mathbb{R})$-bundle $Q^r \subset Q^c$. If $P = Q^r \cap Q$ and $A|_P = a + (\pi^*b)i$, then $A$ also reduces to $Q^r \cdot g_0$ and the corresponding $\text{U}(1)$-bundle $P^\vee := (Q^r \cdot g_0) \cap Q$ is naturally isomorphic to the $\text{U}(1)$-bundle of $\mathcal{L}^\vee$. Moreover, the restriction of $A|_{P^\vee}$ corresponds to the pair $\sigma(a, b)$.

**Proof.** Choose

$$
g_0 = \begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix}.
$$

We have $P^\vee = (Q^r \cdot g_0) \cap Q = P \cdot g_0$, since $g_0 \in \text{PSU}(2)$. Moreover, the natural map $P \to P^\vee$, $p \mapsto p \cdot g_0$ is bijective and fiber-preserving, however maps $p \cdot z$ to $p \cdot zg_0 = (p \cdot g_0) \cdot \bar{z}$, where $z \in \text{U}(1) = \text{PSL}(2, \mathbb{R}) \cap \text{PSU}(2)$. Therefore, $P^\vee$ is isomorphic to the principal $\text{U}(1)$-bundle corresponding to $\mathcal{L}^\vee$. Moreover, if $A|_P = a + (\pi^*b)i$, then $A|_{P \cdot g_0} = g_0^{-1} (a + (\pi^*b)i)g_0 = -a + (\pi^*b)i$, where $b_2\xi_2 + b_3\xi_3 = b_2\xi_2 - b_3\xi_3$. This finishes the proof of this lemma.

Furthermore, let

$$
\mathcal{R}(M) := \text{Hom} \left( \pi_1(M), \text{PSL}(2, \mathbb{R}) \right)/\text{conj}.
$$

be the $\text{PSL}(2, \mathbb{R})$-representation variety of $M$, where the quotient is taken by the conjugation in $\text{PSL}(2, \mathbb{R})$. Observe that $\mathcal{R}(M)$ is also equipped with an involution, which I also denote by $\sigma$. Indeed, keeping to the notations of Lemma 18, $\sigma$ is defined by $\sigma(\rho) = g_0\rho g_0^{-1}$. Moreover, the natural map

$$
\hat{H}: \mathcal{M}_{SW2} \to \mathcal{R}(M), \quad \hat{H}([a, \Psi]) = [\rho_a, \Psi]
$$

is equivariant, that is $\hat{H} \circ \sigma = \sigma \circ \hat{H}$. Thus, $\hat{H}$ yields a map

$$
H: \mathcal{M}_{SW2}/\langle \sigma \rangle \longrightarrow \mathcal{R}(M)/\langle \sigma \rangle.
$$

A representation $\pi_1(M) \to \text{PSL}(2, \mathbb{R})$ is said to be irreducible, if no point in $\mathbb{C}P^1$ is fixed by all elements of $\pi_1(M)$. Here $\text{PSL}(2, \mathbb{R})$ acts on $\mathbb{C}P^1$ via the standard embedding $\text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$, where the latter group acts via Möbius transformations. Denote by $\mathcal{R}^{\text{irr}}(M)$ the subspace of classes of irreducible representations and set

$$
\mathcal{M}^{\text{irr}}_{SW2} := \{(a, \Psi) \mid (a, \Psi) \text{ solves } (2) \text{ and } \rho_{a, \Psi} \text{ is irreducible} \}/C^\infty(M; \text{U}(1)).
$$

Notice that if $\Psi \equiv 0$, the corresponding $\text{PSL}(2, \mathbb{R})$-representation cannot be irreducible so that $\mathcal{M}^{\text{irr}}_{SW2}$ consists of irreducible solutions in the sense of the Seiberg–Witten theory.
Since $\text{PSL}(2, \mathbb{C})$ acts freely on the space of irreducible $\text{PSL}(2, \mathbb{C})$-representations, $\sigma$ acts freely on $\mathcal{R}^{\text{irr}}(M)$ and, therefore, on $\mathcal{M}^{\text{irr}}_{SW2}$ too. Clearly, by restriction we obtain a map

$$H : \mathcal{M}^{\text{irr}}_{SW2}/\langle \sigma \rangle \rightarrow \mathcal{R}^{\text{irr}}(M)/\langle \sigma \rangle$$

still denoted by the same letter.

**Proposition 20.** $H$ is a homeomorphism.

**Proof.** By a result of Donaldson\(^1\) [Don87], an irreducible representation $\rho : \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$ yields a flat stable $\text{PSL}(2, \mathbb{C})$-connection $A_\rho$ on a $\text{PSL}(2, \mathbb{C})$-bundle $Q^c$. Pick a reduction of the structure group of this bundle to $\text{PSU}(2)$.

Since the holonomy of $A_\rho$ is contained in $\text{PSL}(2, \mathbb{R})$, there is a $\text{PSL}(2, \mathbb{R})$-subbundle $Q^r \subset Q^c$ such that $A_\rho$ reduces to $Q^r$, cf. Remark 11. Let $q_0 \in Q^r$ be a base point such that the holonomy representation of $A_\rho$ equals $\rho$. If the holonomy representation of $A_\rho$ with respect to any other base point $q_0 \cdot g, g \in \text{PSL}(2, \mathbb{C})$, is also contained in $\text{PSL}(2, \mathbb{R})$, then $g \in N(\text{PSL}(2, \mathbb{R}))$. Hence, $A_\rho$ reduces to exactly two $\text{PSL}(2, \mathbb{R})$-subbundles, namely $Q^r$ and $Q^r \cdot g_0$. If $A_\rho|_{Q^r} = a_\rho + (\pi^*b_\rho)i$, then $[a_\rho, b_\rho] \in \mathcal{M}/\langle \sigma \rangle$ is well defined. By Lemma 17 we obtain a unique class $[a, \Psi] \in \mathcal{M}_{SW2}/\langle \sigma \rangle$ such that $H([a, \Psi]) = [\rho]$. This finishes the proof of this proposition. \(\square\)

It is a well-known fact from the Seiberg–Witten theory, that the moduli space of the classical Seiberg–Witten monopoles is non-empty for finitely many spin$^c$-structures only, see for example [Mor96, Thm. 5.2.4]. To the best of my knowledge, it is not known whether this holds for the Seiberg–Witten equations with two or more spinors. However, Proposition 20 can be used to show that for $E = \$ this finiteness property still holds.

**Corollary 21.** If $E = \$ is equipped with the Levi–Civita connection, then the set

$$\{ \alpha \in H^2(M; \mathbb{Z}) \mid \mathcal{M}_{SW2}(\alpha) \neq \varnothing \}$$

is finite.

**Proof.** Choose a basis $(\sigma_1, \ldots, \sigma_{b_2})$ of $H_2(M)/\text{Tor}$ and represent each $\sigma_j$ by an embedded surface $\Sigma_j$. Pick $\alpha \in H^2(M; \mathbb{Z})$ and a line bundle $\mathcal{L}$ such that $c_1(\mathcal{L}) = \alpha$. If $\mathcal{M}_{SW2}(\alpha) \neq \varnothing$, by the Milnor–Wood inequality [Mil58] we obtain

$$|c_1(\mathcal{L}|_{\Sigma_j})| = |\langle \sigma_j, c_1(\mathcal{L}) \rangle| \leq \text{genus}(\Sigma_j) - 1.$$ 

This implies the statement of this corollary. \(\square\)

Let me finish this section with some examples of $\text{PSL}(2; \mathbb{R})$ representation varieties—hence, also of the moduli space of solutions to (2)—paying a particular attention to the case of integral homology spheres, since this will be of interest in the next section.

For the Brieskorn homology sphere

$$\Sigma(p, q, r) := \{ z \in \mathbb{C}^3 \mid z^p_1 + z^q_2 + z^r_3 = 0 \} \cap S^5,$$

(22)

\(^1\)In [Don87] the base manifold is of dimension two, however this is not really used in the proof.
where \(p, q, r\) are coprime positive integers, the \(\text{PSL}(2, \mathbb{R})\)-representation variety is finite [KY16]. Moreover, all non-trivial representations are irreducible. In particular, for \(M = \Sigma(p, q, r)\) the space \(\mathcal{M}_{SW2}\) is compact.

The Brieskorn homology spheres can be also used to construct examples of homology spheres, for which the \(\text{PSL}(2, \mathbb{R})\) representation varieties are non-compact as follows. Assume \(M_1\) and \(M_2\) are two arbitrary closed three-manifolds each admitting an irreducible representation \(\rho_i: \pi_1(M_i) \rightarrow \text{PSL}(2, \mathbb{R})\). Assume also for the sake of simplicity that each \(\rho_i\) is rigid, i.e., that \([\rho_i]\) is an isolated point in \(\mathcal{R}(M_i)\). By the van Kampen theorem, the fundamental group of the connected sum \(M := M_1 \# M_2\) is the free product \(\pi_1(M_1) \ast \pi_1(M_2)\) so that we obtain a non-trivial family of representations \(\rho_A: \pi_1(M_1) \ast \pi_1(M_2) \rightarrow \text{PSL}(2, \mathbb{R})\) corresponding to \((\rho_1, A \rho_2 A^{-1})\), where \(A \in \text{PSL}(2, \mathbb{R})\) is a parameter. Notice that \(\rho_A\) is conjugate to \(\rho_B\) if and only if \(A = B\). It is easy to see that \(\rho_{A_k}\) converges in \(\mathcal{R}(M)\) if and only if \(A_k\) converges in \(\text{PSL}(2, \mathbb{R})\). Hence, \(\mathcal{R}(M)\) is non-compact. In particular, this yields the following: If \(M\) is the connected sum of two Brieskorn homology spheres — and, hence, also a homology sphere — then the moduli space of solutions of (2) on \(M\) is non-compact. Hence [HW15], (3) admits non-trivial solutions on such \(M\) for any background metric, even though \(M\) does not support any non-trivial honest harmonic 1-form.

### 4 Blow up sets

Let \(Z\) be a blow up set and \((a, \Psi)\) a solution of (3) just as in Definition 4. In view of Lemma 17, the case \(\text{tr } \Psi \neq 0\) is easy to analyse, hence from now on I will assume that \(\text{tr } \Psi = 0\). Just like in Section 2, \(\text{Cl}^{-1}(\Psi) := b \in \Omega^1(M \setminus Z; \mathcal{L})\) satisfies

\[
(d_a + d_a^*) b = 0, \quad \sigma(b \wedge b) = 0 \quad \text{over } M \setminus Z. \tag{23}
\]

Of course, \(|b|\) also extends to \(M\) as a continuous function and \(|b|^{-1}(0) = Z\).

If we consider \(b\) as a section of \(\text{Hom}(TM; \mathcal{L})\), then the condition \(\sigma(b \wedge b)\) implies that the image of \(b\) in \(\mathcal{L}\) is one-dimensional over \(M \setminus Z\). Hence, we can define the real line bundle \(\mathcal{I} := \text{Im } b \subset \mathcal{L}\) over \(M \setminus Z\) and consider \(b\) as a 1-form with values in \(\mathcal{I}\). Hence, a solution of (23) can be thought of a \(\mathbb{Z}/2\) harmonic 1-form in the following sense.

**Definition 24 ([Tau13]).** Let \((\omega, Z, \mathcal{I})\) be a triple such that
- \(Z\) is a closed subset of \(M\) of Hausdorff dimension at most one;
- \(\mathcal{I}\) is a real Euclidean line bundle over \(M \setminus Z\) equipped with a canonical Euclidean connection \(\nabla\);
- \(\omega \in \Gamma(M \setminus Z; T^* M \otimes \mathcal{I})\) satisfies \(d \omega = 0 = d^* \omega\);
- \(\int_{M \setminus Z} |\nabla \omega|^2 < \infty\);
- \(\omega\) extends as a Hölder-continuous function to all of \(M\) and \(|\omega|^{-1}(0) = Z\).

Under these circumstances \((\omega, Z, \mathcal{I})\) is called a \(\mathbb{Z}/2\) harmonic 1-form.

From now on I assume that \(M\) is an integral homology sphere and that the blow up set \(Z\) is a 1-dimensional submanifold of \(M\), that is a link in \(M\). Let \(Z_1\) denote the union of all components of \(Z\) such that \(\mathcal{I}\) is non-trivial on the meridian of any component. Notice that \(Z_1 \neq \emptyset\), since otherwise \(\mathcal{I}\) would be trivial so that \(\omega\) would extend as a non-trivial harmonic 1-form to all of \(M\).
The double covering branched along a link plays an important rôle in what follows below, therefore let me pause for a while to recall this construction. Let me pick \( Z_1 \) as a branching set and for this reason it is convenient to choose an orientation of each component of \( Z_1 \), albeit this will play a minor rôle below. Denoting by \( N_{Z_1} \) a (small) tubular neighborhood of \( Z_1 \) so that the closure \( \tilde{N}_{Z_1} \) consists of \( k \) solid tori, where \( k \) is the number of connected components of \( Z_1 \). The Mayer–Vietoris sequence applied to \( M \setminus Z_1 \) and \( N_{Z_1} \) yields
\[
H_1(M \setminus Z_1) = \mathbb{Z} \mu_1 \oplus \ldots \mathbb{Z} \mu_k,
\]
where each \( \mu_j \) is a meridian of a connected component of \( Z_1 \), see for example [Pra07, PP. 367–368].

Denote \( \hat{Y} := M \setminus N_{Z_1} \) and consider the homomorphism
\[
\pi_1(Y) \to H_1(Y) \cong H_1(M \setminus Z_1) \to \mathbb{Z} \to \mathbb{Z}/2,
\]
where the first arrow stands for the abelianization homomorphism, the second one is given by \( \sum a_j \mu_j \mapsto \sum a_j \), and the last one is the canonical projection. The kernel of this homomorphism is an index 2 subgroup of \( \pi_1(M \setminus N_{Z_1}) \). Let \( \hat{Y} \) denote the corresponding (unbranched) double covering of \( Y \). This is an oriented manifold with boundary consisting of \( k \) disjoint solid tori. We glue in \( \tilde{N}_{Z_1} \) via a homomorphism \( h: \partial \hat{Y} \to \partial \tilde{N}_{Z_1} \) such that \( h(\tilde{\mu}_j) = \mu_j \), where \( \tilde{\mu}_j \) is the preimage in \( \hat{Y} \) of \( \mu_j \). The result of this is a smooth oriented manifold \( M \), which is called the double covering of \( M \) branched along \( \tilde{Z}_1 \).

Notice that the construction yields naturally a smooth map \( \pi: \hat{Y} \to Y \), which can be extended as a map \( \hat{\pi}: \hat{M} \to M \) such that \( \pi \) is 2-to-1 over \( M \setminus Z_1 \) and 1-to-1 over the branching set \( Z_1 \). One way to fix such extension is as follows. Chose an identification \( \tilde{N}_{Z_1} \simeq \mathbb{C} \cup (S^1 \times D) \), where \( D = \{|z| \leq 1\} \) is the unit disc in \( \mathbb{C} \). This identification can be chosen so that the canonical involution on \( \hat{Y} \) extends as a map \((\theta, z) \mapsto (\theta, -z)\) on each connected component of \( \tilde{N}_{Z_1} \), and we can set \( \pi(\theta, z) = (\theta, z^2) \). Albeit this particular extension is very common in the literature, notice that this is by no means unique. I shall come back to this point in the proof of Proposition 26 below.

Recall that a representation \( \rho: G \to \text{PSL}(2; \mathbb{R}) \) of a group \( G \) is called metabelian, if \( \rho([G, G]) \) is an abelian subgroup.

**Proposition 26.** Let \( M \) be an integral homology sphere. For a solution \((a, b)\) of (23) denote \( A := a + bi \), which is a flat \( \text{PSL}(2; \mathbb{R}) \)-connection on \( M \setminus Z \). Then the following holds:

(i) \( \pi^* \mathcal{I} \) is trivial over \( \hat{M} \setminus \hat{Z}_1 \);
(ii) The holonomy representation of \( \pi^* A \) is abelian and non-trivial.
(iii) The holonomy representation of \( A \) is metabelian.
(iv) The first Betti number of \( \hat{M} \) is positive.

**Proof.** The flat bundle \( \mathcal{I} \) corresponds to a homomorphism \( \pi_1(M \setminus Z_1) \to \mathbb{Z}/2 \), which factors through \( H_1(M \setminus Z_1) \), since \( \mathbb{Z}/2 \) is abelian:
\[
\tau: \pi_1(M \setminus Z_1) \to H_1(M \setminus Z_1) \to \mathbb{Z}/2.
\]

By inspection, this coincides with (25) taking into account \( \pi_1(Y) \cong \pi_1(M \setminus Z_1) \). Then the pull-back \( \pi^* \mathcal{I} \) is a flat bundle corresponding to the homomorphism
\[
\pi_1(M_2 \setminus \hat{Z}_1) \xrightarrow{\pi^*} \pi_1(M \setminus Z_1) \xrightarrow{\tau} \mathbb{Z}/2,
\]
which is trivial, since the image of $\pi_+$ is in the kernel of $\tau$. This proves \((i)\).

To prove \((ii)\), consider the following basis of $\mathfrak{psl}(2, \mathbb{R})$:

\[
\xi_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \xi_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

cf. (15). Notice that the 1-parameter group generated by $\xi_1$ is isomorphic to $U(1)$, whereas the one parameter group corresponding to any non-trivial $\xi \in \text{span}\{\xi_2, \xi_3\}$ is isomorphic to $\mathbb{R}$.

By \((i)\), $\pi^*a$ is a flat connection with trivial holonomy on a trivial line bundle. Therefore, after applying a gauge transformation we can assume that $a$ is the product connection on the product line bundle.

Furthermore, since $\mathcal{I}$ is a subbundle of $\mathcal{L}$, $\pi^*\mathcal{I} = \mathbb{R}$ is a subbundle of $\pi^*\mathcal{L}$, i.e., we have a trivialization of $\pi^*\mathcal{L}$ over $\hat{M} \setminus \hat{Z}_1$. Hence, $\pi^*b$ can be viewed as an $\mathbb{R}^2 = \text{span}\{\xi_2, \xi_3\}$-valued 1-form and $\pi^*A = d + (\pi^*b)i$ can be viewed as a connection on the product $\text{PSL}(2; \mathbb{R})$-bundle. Then $F_{\pi^*A} = 0$ together with $(d + d^*) \pi^*b = 0$ imply that

\[
\pi^*b = \omega \xi \tag{27}
\]

for some fixed $\xi \in \text{span}\{\xi_1, \xi_2\}$, where $\omega$ is a 1-form. Moreover, $\omega$ is closed, and therefore the holonomy of $\pi^*A$ along a loop $\gamma$ is given by

\[
\text{Hol}_\gamma(\pi^*A) = \exp\left(\int_\gamma \omega \xi\right).
\]

In other words, the holonomy of $\pi^*A$ is determined by the periods of $\omega$. Even though $\omega$ is only continuous along $\hat{Z}_1$, I claim that the de Rham cohomology class of $\omega$ is well-defined and non-trivial. The proof of this claim, however, requires some background material, which is introduced first. I follow the line of argument of [Wan93, Sect. 1.2] in this part.

Thus, pick a component of $\hat{Z}_1$ and identify its neighbourhood with $S^1 \times D$ so that the canonical involution acts by multiplication by $-1$ on $D$ just as above. It is convenient to extend $\pi$: $\hat{Y} \to Y$ to $\hat{M}$ so that on each component of $\hat{N}_{Z_1}$ it has the following form

\[
\hat{\pi}: S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}, \quad (\theta, z) \mapsto (\theta, z^2/|z|).
\]

Notice that $\hat{\pi}$ is smooth away from $\hat{Z}_1$ but is only Lipschitz near $\hat{Z}_1$. Nevertheless, $\hat{\pi}$ has the following important advantage over $\pi$: For any smooth metric $g_0$ on $\hat{M}$ there exist positive constants $c_1$ and $c_2$ such that the inequalities

\[
c_1 g_0 \leq \hat{\pi}^* g \leq c_2 g_0
\]

hold on $\hat{M} \setminus \hat{Z}_1$ in the sense of quadratic forms. Hence, $\hat{\pi}^* g$ is a Lipschitz metric on $\hat{M}$ in the sense of [Tel83, Sec. 3]. In particular, the spaces of $L^2$-functions (forms) on $\hat{M}$ with respect to $\hat{\pi}^* g$ and $g_0$ coincide and the corresponding norms are equivalent.

Furthermore, denote by $\mathcal{H}^1_1$ the space of all 1-forms $\omega$ on $\hat{M}$ such that the following holds: $\omega$ is smooth on $\hat{M} \setminus \hat{Z}_1$, $\omega \in L^2(T^*\hat{M})$, and $d\omega = 0 = d(\hat{\pi}^* \omega)$ pointwise on $\hat{M} \setminus \hat{Z}_1$, cf. [Wan93, Def. 6]. We have a version of Hodge theorem in this setting [Tel83, Thm. 4.1], i.e., a natural isomorphism $\mathcal{H}^1_1 \to H^1(\hat{M}; \mathbb{R})$. 

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With this understood, \( \hat{\pi}^* \omega \) is clearly a non-trivial harmonic 1-form with respect to \( \hat{\pi}^* g \) taking values in \( \hat{\pi}^* \mathcal{I} \cong \mathbb{R} \). In particular, by the version of the Hodge theorem mentioned above, \( [\hat{\pi}^* \omega] = [\pi^* \omega] \) represents a non-trivial class in \( H^1(\hat{\mathcal{M}}; \mathbb{R}) \). This finishes the proof of (ii) and proves (iv) as well.

To prove (iii), notice that we have the following short exact sequence
\[
1 \rightarrow \pi_1(\hat{\mathcal{M}}) \rightarrow \pi_1(M \setminus Z_1) \xrightarrow{\tau} \{\pm 1\} \rightarrow 1
\]
where \( \tau \) sends meridians of each components of \( Z_1 \) to \(-1\). Combining this with (ii), we obtain that the holonomy of \( A \) lies in the subgroup \( H \) generated by matrices of the form \( \exp(t \xi) \), where \( \xi \in \text{span}\{\xi_1, \xi_2\} \) as in (27), and a matrix \( B \in \{\exp(t \xi_1)\} \) such that \( B^2 = 1 \). Concretely,
\[
B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Here I think of \( \text{PSL}(2; \mathbb{R}) \) as \( \text{SL}(2; \mathbb{R})/\pm 1 \) so that \( B^2 \) is the identity element indeed.

It is easy to check directly that \( H \) is a metabelian subgroup of \( \text{PSL}(2, \mathbb{R}) \), i.e., \( [H, H] = \{\exp(t \xi)\} \) is abelian. Thus, the holonomy representation of \( A \) is metabelian as claimed. \( \square \)

**Proof of Theorem 5.** It is well-known that if the Alexander polynomial of a link in an integral homology sphere does not vanish at \(-1\), then the corresponding double branched covering is a rational homology sphere [Lic97, Cor. 9.2] (see also [Kaw96, Sect. 5.5] and [PS17, Prop. 3.1]). For a one-component link, i.e., a knot, the double branched covering is always a rational homology sphere [Lic97, P. 95]. \( \square \)

Let me note in passing that examples of \( \mathbb{Z}/2 \) harmonic 1-forms on homology 3-spheres such that the corresponding branching set \( Z \) is a link will appear elsewhere.

## A An algebraic property of \( \mu \)

The only purpose of this appendix to provide some details of the calculation concerning the quadratic map \( \mu \) used in the proof of Lemma 17. More precisely, I claim that the diagram
\[
\begin{array}{ccc}
T^*_{\mathbb{C}}M & \xrightarrow{\text{Cl}} & \text{End}_0(\mathcal{I}) \\
\chi(\cdot \wedge \cdot) \downarrow & & \downarrow \mu \\
\Lambda^2 T^* M \otimes \mathbb{R} i & \xrightarrow{\text{Cl}} & \text{End}_0(\mathcal{I})
\end{array}
\]
commutes, where the horizontal arrows represent the Clifford multiplication, the left vertical arrow represents a combination of the standard symplectic product on \( \mathbb{C} \cong \mathbb{R}^2 \) and the wedge-product, and the right vertical arrow is given by (1).

It suffices to show that the following diagram
\[
\begin{array}{ccc}
(\mathbb{R}^3)^* \otimes \mathbb{C} & \xrightarrow{\text{Cl}} & \text{End}_0(\mathbb{C}^2) \\
\chi(\cdot \wedge \cdot) \downarrow & & \downarrow \mu \\
\Lambda^2 (\mathbb{R}^3)^* \otimes \mathbb{R} i & \xrightarrow{\text{Cl}} & \text{End}_0(\mathbb{C}^2)
\end{array}
\]
commutes. To this end, let \((e^*_1, e^*_2, e^*_3)\) be the standard basis of \((\mathbb{R}^3)^*\). Since the Clifford multiplication maps \(e^*_j\) to \(\eta_j\), which is defined by (15), we have

\[ b := \sum_{j=1}^{3} z_j e^*_j \longrightarrow \begin{pmatrix} z_3 i & -z_1 - i z_2 \\ z_1 - z_2 i & -z_3 i \end{pmatrix} =: \Psi. \]

Then a straightforward computation yields

\[ \mu(\Psi) = \begin{pmatrix} -\overline{(z_1 \bar{z}_2 - z_2 \bar{z}_1)} i & \overline{(z_3 \bar{z}_1 - z_1 \bar{z}_3)} i - z_3 \bar{z}_2 + z_2 \bar{z}_3 \\ (\overline{z_3 \bar{z}_1 - z_1 \bar{z}_3}) i + z_3 \bar{z}_2 - z_2 \bar{z}_3 & \overline{(z_1 \bar{z}_2 - z_2 \bar{z}_1)} i \end{pmatrix}. \]

Furthermore, writing \(z_j = x_j + y_j i\), we obtain

\[ \chi(b \wedge b) = (x_1 y_2 - x_2 y_1) e^*_1 \wedge e^*_2 + (x_1 y_3 - x_3 y_1) e^*_1 \wedge e^*_3 + (x_2 y_3 - x_3 y_2) e^*_2 \wedge e^*_3. \]

Hence,

\[ \text{Cl} \left( \chi(b \wedge b) i \right) = 2 \begin{pmatrix} -(x_1 y_2 - x_2 y_1) & x_1 y_3 - x_3 y_1 - (x_2 y_3 - x_3 y_2) i \\ x_1 y_3 - x_3 y_1 + (x_2 y_3 - x_3 y_2) i & x_1 y_2 - x_2 y_1 \end{pmatrix}. \]

Therefore, we obtain \(\mu(\Psi) = \text{Cl} \left( \chi(b \wedge b) i \right)\) by inspection. This establishes the commutativity of (28).

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