Boundary Wess-Zumino-Novikov-Witten Model from the Pairing Hamiltonian

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(Dated: November 16, 2010)

Correlation functions in the Wess-Zumino-Novikov-Witten (WZNW) theory satisfy a system of Knizhnik-Zamolodchikov (KZ) equations, which involve constants of motion of an exactly solvable model, known as Gaudin magnet. We show that modified KZ equations, where the Gaudin operators are replaced by constants of motion of the closely related pairing Hamiltonian, give rise to a deformed WZNW model that contains terms breaking translational symmetry. This boundary WZNW model is identified and solved. The solution establishes a connection between the WZNW model and the pairing Hamiltonian in the theory of superconductivity. We also argue and demonstrate on an explicit example that our general approach can be used to derive exact solutions to a variety of dynamical systems.

PACS numbers: 71.10.Pm, 74.20.Fg, 02.30.Ik

I. INTRODUCTION

The Wess-Zumino-Novikov-Witten (WZNW) model plays an important role in physics. Historically, the $SU(2)$ version of the model with topological coupling $k = 1$ was used to describe low-energy dynamics of a one-dimensional spin-$1/2$ Heisenberg antiferromagnet. At higher integer couplings, $k$, it describes quantum critical points in the parameter space of quantum antiferromagnetic spin-$S$ chains, with $S = k/2$. Many other applications of the WZNW model have emerged in various contexts lately (see, e.g., Ref. [2]). The Lagrangian formulation of the theory is given by a non-linear sigma-model defined in the Euclidean space by the action

$$S_{WZNW}(g) = \frac{k}{16\pi} \int d^2z \bar{z} \text{tr} \left[ \partial_\alpha g^1 \partial^\alpha g \right] - \frac{i k}{24\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{tr} \left[ g^1 \partial_\mu g g^1 \partial_\nu g g^1 \partial_\rho g \right],$$

where integration in the second topological Wess-Zumino term is over a three-dimensional ball, $x = (z, \bar{z}, \xi) \in B^3$, whose boundary at $\xi = 0$ is the two-dimensional sphere, $S^2 = \partial B^3$, which corresponds to a compactified complex plane parametrized by $(z, \bar{z})$ and $g(z, \bar{z}) \in SU(2)$. The integer parameter $k$ in Eq. (1) is the level of the corresponding conformal field theory (CFT). The WZNW action is invariant under conformal and non-Abelian current algebras. The current algebra transformations have a chiral structure, i.e. they act on the group element $g(z, \bar{z})$ as $g'(z, \bar{z}) = U(z) g(z, \bar{z}) U(\bar{z})$. Here $U(z)$ and $\bar{U}(\bar{z})$ are independent elements of the group $SU(2)$. This property allows to study the holomorphic ($z$-dependent) and antiholomorphic ($\bar{z}$-dependent) sectors of the model separately (below, we focus on the holomorphic sector).

The $N$-point correlators of primary fields, $G(z_1 \ldots z_N) = \langle \phi(z_1, \bar{z}_1) \ldots \phi(z_N, \bar{z}_N) \rangle_{S_{WZNW}}$, satisfy the Knizhnik-Zamolodchikov (KZ) equations,

$$\left( (k + 2) \partial_{z_l} - \hat{H}_l^G \right) G(\{z_i\}) = 0,$$

with

$$\hat{H}_l^G = \sum_{l \neq l'} w(z_l, z_{l'}) \hat{S}_l \cdot \hat{S}_{l'}$$

$l, l' = 1, 2, \ldots N$, and $w_{l,l'} = (z_l - z_{l'})^{-1}$. Here $\hat{S}_l = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ and $\hat{S}_l^g = SU(2)$ generators. Amazingly, operators $\hat{H}_l^G$ in Eq. (2) are formally equivalent to the integrals of motion of a seemingly unrelated Gaudin magnet model[2a]. The Gaudin magnet represents a quantum spin Hamiltonian, with effective long-range interactions between spins, which is exactly solvable “by design.” Its Hamiltonian can be represented as a linear combination of the mutually commuting integrals of motion, $\hat{H}_l^G, \hat{H}_l^{G}$ as follows:

$$\hat{H} = 2 \sum_l z_l \hat{H}_l^G.$$

One general question that we formulate in this paper is whether it is possible to derive deformed WZNW models, whose correlators satisfy modified KZ equations [2] with a different set of operators $\hat{H}_l$. Below, we answer this question in the affirmative by providing an example of this “reverse engineering approach” and finding a boundary WZNW model, which corresponds to the operators $\hat{H}_l$ representing the integrals of motion of the discrete pairing Hamiltonian (Richardson model)[2, 13, 15] closely related to the Gaudin magnet. It descends from the familiar BCS Hamiltonian

$$\hat{H}_{\text{RBCS}} = \sum_{l,s = \pm} z_l \hat{c}_{l,s}^\dagger \hat{c}_{l,s} - \lambda \sum_{l, l' S} \hat{c}_{l+}^\dagger \hat{c}_{l-} \hat{c}_{l'-} \hat{c}_{l'+},$$

where $\hat{c}_{l,s}^\dagger$ and $\hat{c}_{l,s}$ are fermion creation/annihilation operators corresponding to a single-particle state $|l\rangle$ with
energies $z_i$ and spin $s = \pm$. If $\lambda > 0$, the ground state is a superconductor with all fermions paired. Then, the operators $\hat{c}_i^{\dagger} \hat{c}_i$, $\hat{c}_i^{\dagger} \hat{c}_{i+1}$, and $\sum_s (\hat{c}_i^{\dagger} \hat{c}_i - 1/2)/2$ become algebraically equivalent to the Pauli matrices $\hat{\sigma}^i$, $\hat{\sigma}_i^{\dagger}$, and $\hat{\sigma}_i^3$ (Anderson pseudospins). The corresponding spin Hamiltonian is the integrable Richardson model, which can be presented in two identical ways

$$\hat{H}_{\text{Rich}} = \sum_i z_i (1 + \hat{\sigma}_i^3) - \frac{\lambda}{4} \sum_{i,j} \hat{\sigma}_i^+ \hat{\sigma}_j^- \quad (6)$$

$$\equiv -\sum_i \left(2z_i \hat{H}_R^z - z_i + \frac{\lambda}{4}\right) + \lambda \left(\sum_i \hat{H}_R^z\right)^2,$$

where $\hat{\sigma}_i^{3,\pm}$ are Pauli matrices, and the operators $\hat{H}_R^z = -\hat{\sigma}_i^3/2 + \lambda \hat{H}_G^z$ represent $N$ mutually commuting $[\hat{H}_R^z, \hat{H}_G^z] = 0$ conserved "currents." Note that $\sum_i \hat{H}_G^z = 0$, and hence the second term in Eq. (6) can be simplified as $\sum_i \hat{H}_G^z = -\sum_i \hat{\sigma}_i^3/2$ to give the "total pseudo-spin magnetization," which separates the Hilbert space into sectors with different numbers of Cooper pairs, which were actually studied in Ref. 7.

Note that the Gaudin model is closely related to the Richardson model (6,7) and corresponds to its infinite space into sectors with different numbers of Cooper pairs, $\sigma = 0, 3$.

We first present the main result for the boundary WZNW action:

$$G(z_1, \cdots, z_N) = \langle \phi_{s_1}(z_1) \cdots \phi_{s_N}(z_N) \rangle_{S_{\text{WZNW}}}$$

$$\equiv \langle \Phi \rangle \langle \phi_{s_1}(z_1) \cdots \phi_{s_N}(z_N) \rangle_{S_{\text{WZNW}}},$$

which is the "left" boundary term, and the "right" boundary term, $S_{\text{bound}}^R[C]$ is given by $[9]$ with $z \to \bar{z}$ and $J_3^S(z) \to J_3^S(\bar{z})$. In Eq. (10), $J_3^3(z)$ is a component of the "left" current in the $SU(2)$ WZNW theory, defined in a standard way: $J^a(z) = (k/2) \text{tr} \left[ \hat{S}^a g(z, \bar{z}) \partial_z \bar{g}^A(z, \bar{z}) \right]$, $a = \pm, 3$. Note that due to conformal invariance the "left" currents do not depend on $\bar{z}$ and likewise the "right" currents, $J^a(\bar{z})$ do not depend on $z$. Note that the term $[10]$ breaks translational invariance of the model and hence can be interpreted as a generalized impurity $z$. Below we prove that the boundary action gives rise to generalized KZ Eqs. (2) and present exact results for the corresponding correlation functions.

II. DERIVATION OF THE BOUNDARY WZNW ACTION

We are seeking to prove that a correlation function of arbitrary primary fields in the $SU(2)$ boundary WZNW model $[8]$,

$$G(z_1, \cdots, z_N) = \langle \phi_{s_1}(z_1) \cdots \phi_{s_N}(z_N) \rangle_{S_{\text{WZNW}}}$$

$$\equiv \langle \Phi \rangle \langle \phi_{s_1}(z_1) \cdots \phi_{s_N}(z_N) \rangle_{S_{\text{WZNW}}},(10)$$

satisfies the generalized KZ equations (2) with operators $[11]$. Here, $\Phi [C] = e^{-S_{\text{bound}}(C)}$ and $s_i$ stands for the spin, $0 \leq s_i \leq (k/2)$, $i = 1, \ldots N$.

To solve the KZ equations (2) we look for $\Phi [C]$ in the form $[12]$.

$$\Phi [C] = e^{\int_C dzq(z)J^3(z)}$$

where $q(z)$ is an analytic and differentiable function in $C$, and utilize the two standard key ingredients of the $SU(2)$ WZNW theory and CFTs $[13]$. (i) The crux here is the operator product expansion satisfied by the currents with the same chiralities (currents with different chiralities commute); (ii) Action of the Virasoro generators on primary fields. Then the expression for the correlation function $G(z_1, \cdots, z_N)$ can be simplified by contracting $J^3(z)$ in $\Phi [C] = \sum_p (1/p!) \left( \int_C dzq(z)J^3(z) \right)^p$ with all primary fields

$$G(z_1, \cdots, z_N) = \left\langle e^{\sum_p q(z_p)S^3\Theta [C, z_p]} \phi_{s_1}(z_1) \cdots \phi_{s_N}(z_N) \rightangle,$$

where the functional averaging with respect to the $S_{\text{WZNW}}$ is understood. Using the standard technique of Ref. 12 and taking into account the boundary operator we find that indeed the following identity holds

$$\partial_{z_i} - q'(z_i) \hat{S}_i^3 \Theta [C, z_i] - \hat{H}_G^z [C] G(z_1, \cdots, z_N) = 0,$$

where $\hat{H}_G^z [C]$ is the rotated Gaudin Hamiltonian defined in (7). Note that if $q(z) = -z/k$ and all $z_i (i = 1, \ldots N)$ are inside $C$, Eq. (11) precisely reproduces modified KZ equations (2) with the Gaudin integrals of motion replaced with those of the Richardson model with the interaction parameter $\lambda$. Hence, we recover the amazing fact that the correlation functions of the boundary
where operators, look for eigenstates of a set of commuting Hamiltonian operators, $\chi \in C$. For example if $k = 1$ and all primary fields in $G(z_1 \cdots z_N)$ are from $s_i = 1/2$ representation space of SU(2), then $M = N/2$ and $|0\rangle = \left(\begin{array}{c}0 \\ 1 \\ \vdots \\ 0 \end{array}\right)$ in the basis where the primary fields are defined by spin $s$ and its $z$-projection, $m = -s \ldots s$, their correlation function $\langle \phi_{s_i}^{m_i}(z_1) \cdots \phi_{s_N}^{m_N}(z_N) \rangle$ is connected to the general expression (12) as follows

$$
\langle \phi_{s_i}^{m_i}(z_1) \cdots \phi_{s_N}^{m_N}(z_N) \rangle = \langle s_{N}, m_{N} \rangle \cdots \langle s_{1}, m_{1} \rangle G(z_1 \cdots z_N),
$$

where $\langle s_{i}, m_{i} \rangle$ is the known solution to the KZ Eqs. for the canonical SU(2) WZNW model. In general these solutions can be expressed analytically in terms of multi-variable confluent hypergeometric functions. It is noted when $\lambda \to \infty$, the Richardson pseudospin model reduces to the Gaudin magnet, and consequently $\chi$ in Eq. (13) reduces to $\chi_0$. Moreover, analysis of Eqs. (12) at $q(z) = -z/k\lambda$ suggests that the integral over $u_1 \cdots u_M$ in Eq. (12) has a saddle point defined by the condition

$$
\frac{1}{\lambda} + \sum_{\alpha \neq \beta} \frac{1}{u_\alpha - u_\beta} + \sum_{i=1}^{N} s_i z_i = 0. \tag{18}
$$

Interestingly, this condition coincides with Richardson equations for the eigenvalues of the reduced BCS Hamiltonian (9).

**IV. BOSONIZED ACTION AT k = 1: IMPLICATIONS**

The WZNW model at $k = 1$ can also be realized as a free boson theory with central charge $c = 1$. Following the standard bosonization technique we introduce a scalar field, $\phi = \bar{\phi}(z) + \bar{\phi}(\bar{z})$, and rewrite the $z$-component of the current in the form $J_z = \frac{1}{\sqrt{2\pi}} \partial_z \phi$. Note that this representation of $J_z^2$ is correct only locally. The full action (11) at $k = 1$ reads

$$
S = \frac{1}{4\pi} \int dz dz' \partial_z \phi \partial_z \bar{\phi} + \frac{i}{\lambda \sqrt{2\pi}} \int_C dz \partial_z \phi + a. c., \tag{19}
$$

where $a. c.$ stands for an antiholomorphic contribution. We note that this bosonized version of the $k = 1$ action was discussed earlier in Refs. [13,14] by Sierra. Here we emphasize that due to the presence of the boundary term defined by an arbitrary contour $C$ and
conformal invariance of the WZNW model make the boundary WZNW model a very useful tool to classify and study low-energy, strong coupling disordered quantum systems as well as various systems driven out of equilibrium. Let us use bosonic version of the $k = 1$ theory as an illustrative example. If all $z_i$ are real, we have a standard physical Richardson pseudo-spin model. Here contour $C$ can be chosen as boundary of a narrow strip encompassing all $z_i$, which explicitly shows that we have an equilibrium system. On contrary, if some of $z_i$ have nonzero imaginary part, the contour can be a circle with radius $R$. In this case parametrization $z = it + x$, where $t$ is the dimensionless time and $x$ is the dimensionless coordinate, is inconvenient, as we generate a complicated time dependent term in our bosonic Hamiltonian. Interestingly enough, this, from a first sight abstract problem is closely related to another, well defined and physically motivated system. Consider the conformal map, $z \to w$, where $w = it + x$, which transforms a disc with radius $R$ to an infinite strip, see Fig. 1. The bulk action is invariant under such transformations, while the boundary term will transform into $\sim \int_{\delta C} dw \sinh^{-2} \left( \frac{w-B_2-B_3}{2R_3} \right) \varphi [z(w)]$. Here contour $\delta C$ is the boundary of a strip of width $\pi R_3$ centered at $t_0 = R_1$. This term contributes to the Hamiltonian and makes it time dependent. It describes a single instantaneous perturbation on the system at $t = t_0$, which however does not brake integrability. Remarkably, we can extract enormous information about physical properties of such systems by analyzing exact correlation functions.

V. PRACTICAL APPLICATIONS OF THE BOUNDARY WZNW MODEL

Boundary action Eq. (9) together with the expression (16) for the correlation functions represent our main mathematical result. As argued, it has important consequences for a variety of seemingly unrelated physical models, notably dynamical systems. We provide here an explicit example of such correspondence between Maxwell-Bloch (MB) theory of a two-level laser, which is shown to map onto the BWZNW model. Below, we derive for the first time an exact solution to the system of MB equations with damping

$$\partial_\xi \mathcal{E} + \gamma \mathcal{E} = \mathcal{P},$$
$$\partial_\xi \mathcal{P} + \gamma \mathcal{P} = N \mathcal{E},$$
$$\partial_\xi N + \frac{1}{2}(\mathcal{P}^* + \mathcal{E}^\ast \mathcal{P}) = -\gamma_N N + N_0,$$

where $\mathcal{E}(\eta, \xi)$ is the complex electrical field amplitude, $\mathcal{P}(\eta, \xi)$ is the polarization of the medium, $\mathcal{N}(\eta, \xi)$ is the population inversion, and $\eta = \Omega x/c$ and $\xi = \Omega (t - x/c)$ are given in terms of real space, $x$, and time, $t$, with $\Omega$ being a physical constant that depends on material and cavity medium, and $c$ is the speed of light. In Eq. (20), $\gamma \geq 0$ is a decay rate of energy losses inside the laser medium and the constants $\gamma_\perp$ and $\gamma_\parallel$ are damping coefficients of medium polarization and population inversion. Dissipation in the population inversion equation tends to return $N$ to $N_0/\gamma_N$, which is determined by the pumping.

Amazingly, the Hamiltonian formulation of (20) for $\gamma_\parallel = \gamma_\perp = \gamma = 0$ reduces to a set of KZ equations with linear $q(z)$ of $\alpha \xi^2$. We show now that the BWZNW model with $q(z) = \alpha x^2 + \beta z^2$ and the corresponding generalized KZ Eqs. (11) describe the system (20) with finite damping parameters $\gamma, \gamma_\parallel \geq 0$. First, we observe by analogy with Ref. [27] that the set of MB equations (20) with damping and pumping can be obtained from the compatibility condition of the following system of linear differential equations with complex spectral parameter $z \in \mathbb{C}$

$$\partial_\xi \psi = \left( z - \frac{\gamma_\perp}{2} \right) \sigma_3 + U_0 \psi,$$

and

$$\left( \partial_\eta + \frac{N_0}{z} \partial_\xi \right) \psi = \left( \frac{\hat{\rho}}{4z} - \frac{\gamma}{2} \sigma_3 \right) \psi,$$

where

$$U_0 = \frac{1}{2} \left( \begin{array}{cc} 0 & -\varepsilon \xi \\ -\varepsilon^* \xi & 0 \end{array} \right), \quad \hat{\rho} = \frac{1}{2} \left( \begin{array}{cc} N & -\mathcal{P} \\ -\mathcal{P}^* & -N \end{array} \right).$$

According to the method of isomonodromy solutions of differential equations, a variety of solutions of the MB equations can be obtained by classifying solutions of an auxiliary equation, $\partial_\xi \psi(z) = A(z, \xi, \eta) \psi(z)$, that are consistent with the original MB equations. We found that consistent with (21) choice of $A(z, \xi, \eta)$, which produces $N$-soliton solutions of MB equations in the presence of pumping and damping, reads

$$A(z, \xi, \eta) = (\xi - \xi_0)\sigma_3 + \sum_{j=1}^{N} \frac{A_j}{z - z_j},$$

$$z_j = \sqrt{2N_0\eta - k_j^2},$$

with parameters $k_j^2, \xi_0 \in \mathbb{R}$. Substituting expression for $\partial_\xi \psi(z)$ together with Eq. (24) into Eqs. (21), writing compatibility conditions and equating the residues of the poles at $z = z_i, i = 1 \ldots N$, one will obtain for functions $A_j$:

$$\partial_\xi A_j = \left( z_j - \frac{\gamma_\parallel}{2} \right) \sigma_3 + U_0, A_j,$$

$$\partial_\eta A_j = \left( \frac{\hat{\rho}}{4z_j} - \frac{\gamma}{2} \right) \sigma_3 + U_0, A_j$$

where

$$U_0 = \frac{1}{\xi - \xi_0} \left( \sum_{i=1}^{N} A_i - \text{diag} \sum_{i=1}^{N} A_i \right),$$

$$\hat{\rho} = 4N_0 \left( \xi - \xi_0 \right) \sigma_3 - \sum_{i=1}^{N} \frac{A_i}{z_i}.$$
Eqs. (24) admit a Hamiltonian structure with the Poisson brackets \[ \{ (A_m)_{ab}, (A_n)_{cd} \} = \delta_{mn} \delta_{bc} - \delta_{bd} \delta_{ac}, \] which corresponds to the \( sl(2) \) algebra on a chain. Therefore, Eqs. (24) acquire the form \[ \partial_t A_j = \{ A_j, H_\xi \}, \quad \partial_{\eta} A_j = \{ A_j, H_\eta \}, \] with Hamiltonian operators
\[ H_\xi = \sum_{k=1}^{N} \left( z_k - \frac{\gamma_k}{2} \right) \text{tr}(A_k \sigma_3) + S^- S^+ + S^+ S^- \frac{1}{\xi - \xi_0}, \] \[ H_k = \sum_{k=1}^{N} \left( \frac{\xi - \xi_0}{z_k} - 2 \gamma \right) \text{tr}(A_k \sigma_3) + \sum_{j=1}^{N} \frac{\text{tr}(A_k A_j)}{z_k (z_k - z_j)}, \] and \( S^{+,-} = (\sum_{k=1}^{N} A_k)_{12,21}. \)

Quantization of the MB system implies replacement of Poisson brackets by commutators, \( \{ \xi, \eta \} \rightarrow [\xi, \eta] \), and introduction of a quantum wave function, \( \Psi(\xi, \eta) \). Then the \( sl(2) \) algebra (24) acquires the matrix realization
\[ A_k = i \left( S_k^3 S_k^+ - S_k^- S_k^3 \right), \] which, together with transformation \( \xi \rightarrow i \xi \), leads to the set of Hamiltonian operators corresponding to (20):
\[ \hat{h}_\xi = \frac{i}{2} \sum_{k=1}^{N} \left( z_k - \frac{\gamma_k}{2} \right) \hat{S}_k^3 + \frac{i}{2} \hat{S}_k^+ \hat{S}_k^- + \hat{S}_k^- \hat{S}_k^+ \frac{1}{\xi - \xi_0}, \]
\[ \hat{h}_k = i \frac{\gamma_k}{2} \left( \frac{\xi - \xi_0}{z_k} - 2 \right) \hat{S}_k^3 - i \frac{\gamma_k}{2} \sum_{j \neq k} \hat{S}_k \hat{S}_j + \frac{i}{2} \sum_{j \neq k} \hat{S}_k \hat{S}_j, \] where \( k = 1 \ldots N \), and \( \hat{S}_k^\pm = \sum_{j \neq k} \hat{S}_j^\pm \). On a quantum level, the “wave function,” \( \Psi(\xi, z_1 \ldots z_N) \), satisfying the set of “multitime,” \( t \rightarrow (\xi, z_1 \ldots z_N) \), Schrödinger equations,
\[ i \partial_t \Psi = \hat{h}_\xi \Psi, \]
\[ i \partial_{z_k} \Psi = \hat{h}_k \Psi, \] unambiguously determine the solution of MB equations. As we see, the second set of Schrödinger equations coincides with generalized KZ equations (14) with the following parameters: \( k + 2 = 1, i(\xi - \xi_0) = 1/\lambda, q(z) = -z/\lambda - \gamma z^2, \) and \( C = C_\infty \), which encompasses all \( z_j = (2N_0 \eta - k_j^2)^{1/2} \), where \( k_j, \xi_0 \in \mathbb{R} \) are free parameters. The first equation is formally an ordinary Schrödinger equation with “time,” \( \xi = \Omega(t - x/c) \). Therefore, this maps the problem onto a dynamical boundary WZNW model, with the boundary action \( S_{MB} = \int_{C_\infty} dz q(z) J^2(z) \), correlation functions of which depend on an additional parameter, \( \xi \), playing the role of time.

By analogy with Eq. (12), solution for \( \Psi(\xi, z_1 \ldots z_N) \) is then found to be
\[ \Psi(\xi, z_1 \ldots z_N) = \int \prod_{k=1}^{M} du_k \chi_{MB} (\{ u_\alpha \} | \{ z_i \} ) \times \mathcal{N}(\{ u_\alpha \} | \{ z_i \}), \]
with
\[ \chi_{MB} (\{ u_\alpha \} | \{ z_i \}) = (\xi - \xi_0)^i (\sum_{m=-M}^{M} c_{2i(\xi - \xi_0)} \rho_{m}) \chi_0 (\{ u_\alpha \} | \{ z_i \}) \times \exp \left[ \frac{1}{\lambda} \sum_{i=1}^{N} \left( \frac{\gamma_i}{2} - \lambda q(z_i) \right) + \sum_{\alpha} u_\alpha \right]. \]

Eqs. (33) and (32) determine the form of the wave function of quantum states in the quantized MB system. Importantly, this wave function is cardinaly different from the correlation function of primary fields in the bulk WZNW model. This is because the expression in the right-hand-side of Eq. (33) must be integrated in Eq. (32) together with \( \chi_0 \), while only the later appears in the WZNW model.

In order to find the solution of the system of classical MB equations (34), one should average the angular momentum operators, \( S^\pm, S^3 \), with respect to the wave functions (33). Then the solution of classical MB equations for physical quantities \( \mathcal{E}, \mathcal{P}, \) and \( \mathcal{N} \) can be found from Eqs. (33), (35), and (36) as follows:
\[ \mathcal{E}^2 = \left\langle \Psi^* \left( \frac{\kappa^2}{\xi - \xi_0} \sum_{j=1}^{N} \hat{S}_j^3 \right) \Psi \right\rangle, \]
\[ \mathcal{P}^2 = 64N_0^2 \kappa \left\langle \Psi^* \left( \sum_{j=1}^{N} \hat{S}_j^\pm \right) \Psi \right\rangle, \]
\[ \mathcal{N} = 8N_0 \left\langle \Psi^* \left( \xi - \xi_0 \kappa \sum_{j=1}^{N} \hat{S}_j^3 \right) \Psi \right\rangle. \]

These equations provide \( N \)-soliton solutions to the MB system. In general these solutions have compact integral representations which can be evaluated and compared with other numerical and experimental data. In Appendix we evaluate this integral for \( N = 2 \) soliton case and express the solution for \( \mathcal{E}, \mathcal{P}, \) and \( \mathcal{N} \) in terms of known Kummer confluent hypergeometric functions.

VI. CONCLUSION

In conclusion, this work has introduced a method of reverse construction of boundary WZNW models from the generalized Knizhnik-Zamolodchikov equations satisfied by the exact correlation functions and demonstrated the application of this method on the explicit example of KZ equations with conserved currents of the Richardson model. Thereby, we established a direct connection between the discrete pairing model of superconductivity and the boundary WZNW model, which we identified and solved. We have established that the solutions of modified KZ equations are defined by the off-shell states of the Richardson model. Our construction is close in spirit but technically different from the BCS/CFT correspondence discussed earlier by Sierra. Our main moti-
mutation has been to precisely identify the boundary operator in the WZNW model, which is related to the Richardson-type models. Our other motivation has been to outline a range of practical applications of the discovered correspondence, which is argued to be very wide and includes a variety of dynamical systems that can be mapped on the BWZNW theories and solved exactly in many cases. One such mapping and solution for the dynamical system describing radiation of a two-level laser with pumping and damping was presented. Dynamic properties of the laser were computed exactly exploiting the integrability of this latter system.

Acknowledgements - This research was supported by the NSF CAREER award, DMR-0847224.

VII. APPENDIX

Here we derive the analytical expression of the two-soliton solution of MB equations, when \( N = 2 \) and \( M = 1 \). For simplicity we consider the case with finite pumping, \( N_0 \), and medium polarization damping, \( \gamma_{\perp} \), coefficients, but with \( \gamma = 0 \). In this case we have one integration parameter \( u \) and two parameters, \( z_1, z_2 \). Then from Eqs. (13) and (14) it follows that

\[
\mathcal{V}(u) = \sum_{i=1}^{2} \frac{S^i_1}{u - z_i}\ket{\downarrow, \downarrow} + \frac{1}{u - z_1}\ket{\uparrow, \downarrow} + \frac{1}{u - z_2}\ket{\downarrow, \uparrow}. \tag{35}
\]

In order to construct the wave function \( \Psi \), we should integrate \( \mathcal{V}(u) \) together with the term corresponding to \( \chi_0, \((-i\lambda \xi)(z_1 - z_2))^{-1/2(k + 2)} \), over \( u \) along a contour surrounding the branch-cut at \((z_1, z_2)\) [see Eq. (12) and the discussion]. To perform the resulting integration, we make use of the identity

\[
\int_C du(u - z_1)^{-a}(u - z_2)^{-1+a} e^{-bu} = B(a, 1 - a) F_1[1 - a; 1; b; (z_2 - z_1)] \tag{36}
\]

where \( F_1 \) is the Kummer confluent hypergeometric function and \( B(a, 1 - a) \) is the Beta-function. Then for \( a = 1/2 \) and \( b = 1/\lambda \) we have

\[
\frac{1}{\lambda} F_1 \left[ \frac{1}{2}; 1; \frac{(z_1 - z_2)}{\lambda} \right] = e^\frac{\pi i}{2} I_0 \left[ \frac{(z_1 - z_2)}{2\lambda} \right] \tag{37}
\]

with \( I_0 \) being the modified Bessel function of zero order. Now, returning to the real time, \( \xi \rightarrow -i\xi \), by analytic continuation, and keeping causal behavior of the wave function, \( \Psi \), we will have

\[
\Psi = \pi (\xi - \xi_0)^{-i} e^{-|\xi - \xi_0|^{2} - |(\xi - \xi_0)(z_1 - z_2)|} \left( z_1 - z_2 \right)^{-\frac{1}{4} i e\frac{i(\xi - \xi_0)(z_2 - z_1)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_2 - z_1)}{2} \right] \ket{\uparrow, \downarrow} + (z_2 - z_1)^{-\frac{1}{4} i e\frac{i(\xi - \xi_0)(z_1 - z_2)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_1 - z_2)}{2} \right] \ket{\downarrow, \uparrow}. \tag{38}
\]

According to Eqs. (31), to find \( \mathcal{E} \) one should act by the operator \( S_1^+ + S_2^+ \) on the expression (35) for \( \Psi \) and calculate the norm. By doing so and after some simple algebra we obtain

\[
\mathcal{E} = \frac{\pi}{|\xi - \xi_0|} e^{-|\xi - \xi_0|^{2} - |(\xi - \xi_0)(z_1 - z_2)|} \Abs \left( z_1 - z_2 \right)^{-\frac{1}{4} i e\frac{i(\xi - \xi_0)(z_2 - z_1)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_2 - z_1)}{2} \right] + (z_2 - z_1)^{-\frac{1}{4} i e\frac{i(\xi - \xi_0)(z_1 - z_2)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_1 - z_2)}{2} \right]. \tag{39}
\]

where the notation \( \Abs \) means absolute value. Similarly, the expressions for the polarization of the medium \( P \) and the population inversion \( N \) read:

\[
P = 8\pi N_0 e^{-|\xi - \xi_0|^{2} - |(\xi - \xi_0)(z_1 - z_2)|} \Abs \left[ \frac{1}{z_2} (z_1 - z_2)^{-\frac{1}{4} i e\frac{i(\xi - \xi_0)(z_2 - z_1)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_2 - z_1)}{2} \right] + \frac{1}{z_1} (z_2 - z_1)^{-\frac{1}{4} i e\frac{i(\xi - \xi_0)(z_1 - z_2)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_1 - z_2)}{2} \right] \right]. \tag{40}
\]
\begin{align}
\mathcal{N} &= 8\pi N_0 e^{-(k_1 z_1 + k_2 z_2)} \left[ \left( \xi - \xi_0 + \frac{1}{2z_2} \right) (z_1 - z_2) - \frac{1}{2z_1} \right]^2 e^{\frac{i(\xi - \xi_0)(z_2 - z_1)}{2}} I_0 \left[ \frac{i(\xi - \xi_0)(z_2 - z_1)}{2} \right] \\
&+ \left( \xi - \xi_0 + \frac{1}{2z_2} - \frac{1}{2z_1} \right) (z_2 - z_1)^2 \left( \frac{i(\xi - \xi_0)(z_2 - z_1)}{2} \right) I_0 \left[ \frac{i(\xi - \xi_0)(z_2 - z_1)}{2} \right].
\end{align}

We remind the reader that in all expressions above \( z_j = \sqrt{2N_0\eta - k_j^2}, \) \( j = 1, 2, \) with constant parameters \( k_1 \) and \( k_2. \) Here, \( \eta = \Omega x/c, \xi = \Omega (t - x/c) \) with \( \Omega \) being a physical constant that characterizes the material and the cavity medium. To the best of our knowledge, Eqs. (39), (40), and (41) have never been derived in the literature before.

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