Type I on (Generalized) Voisin-Borcea Orbifolds and Non-perturbative Orientifolds

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Abstract

We consider non-perturbative four dimensional $\mathcal{N} = 1$ space-time supersymmetric orientifolds corresponding to Type I compactifications on (generalized) Voisin-Borcea orbifolds. Some states in such compactifications arise in “twisted” open string sectors which lack world-sheet description in terms of D-branes. Using Type I-heterotic duality as well as the map between Type IIB orientifolds and F-theory we are able to obtain the massless spectra of such orientifolds. The four dimensional compactifications we discuss in this context are examples of chiral $\mathcal{N} = 1$ supersymmetric string vacua which are non-perturbative from both orientifold and heterotic points of view. In particular, they contain both D9- and D5-branes as well as non-perturbative “twisted” open string sector states. We also explain the origins of various inconsistencies arising in such compactifications for certain choices of the gauge bundle.

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I. INTRODUCTION

In the recent years much progress has been made in understanding six and four dimensional orientifold compactifications. Various six dimensional orientifold vacua were constructed, for instance, in \[11\]. Generalizations of these constructions to four dimensional orientifold vacua have also been discussed in detail in \[5\]–\[13\]. In many cases the perturbative world-sheet approach to orientifolds gives rise to consistent anomaly free vacua in six and four dimensions. However, as was pointed out in \[11\], there are cases where the perturbative orientifold description is inadequate as it misses certain non-perturbative sectors giving rise to massless states. In certain cases this inadequacy results in obvious inconsistencies such as lack of tadpole and anomaly cancellation. Examples of such cases were discussed in \[9\]–\[11\]. In other cases, however, the issue is more subtle as the non-perturbative states arise in anomaly free combinations, so that they are easier to miss.

Thus, let us consider Abelian $T^6/\Gamma$ orbifold compactifications of Type I with $\mathcal{N} = 1$ supersymmetry in four dimensions. The requirement that the orbifold group $\Gamma$ act crystallographically on $T^6$ restricts the allowed choices of $\Gamma$ to $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_{12}$ and $\mathbb{Z}_{12}'$ for $\Gamma \approx \mathbb{Z}_N$, and to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, $\mathbb{Z}_2 \otimes \mathbb{Z}_4$, $\mathbb{Z}_2 \otimes \mathbb{Z}_6$, $\mathbb{Z}_2 \otimes \mathbb{Z}_6^*$, $\mathbb{Z}_3 \otimes \mathbb{Z}_3$, $\mathbb{Z}_3 \otimes \mathbb{Z}_6$, and $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ for $\Gamma \approx \mathbb{Z}_N \otimes \mathbb{Z}_M$. Here we use star and prime to distinguish different cyclic groups of the same order which act differently on $T^6$ (the precise actions of these orbifold groups on $T^6$ are given in the subsequent sections). In some of the above cases Type I-heterotic duality enables one to argue along the lines of \[7\]–\[11\] that the non-perturbative states become heavy and decouple once the corresponding orbifold singularities are appropriately blown up. In particular, these are the $\mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Z}_7 \otimes \mathbb{Z}_3$ and $\mathbb{Z}_6^*$ \[8\], and $\mathbb{Z}_2 \otimes \mathbb{Z}_6^*$ \[13\] orbifold cases. As was argued in \[11\], in other perturbatively tadpole free orientifolds, such as the $\mathbb{Z}_6'$, $\mathbb{Z}_2 \otimes \mathbb{Z}_6$, $\mathbb{Z}_3 \otimes \mathbb{Z}_6$ and $\mathbb{Z}_6 \otimes \mathbb{Z}_6$ \[9\], and $\mathbb{Z}_{12}$ \[12\] orbifold cases, the non-perturbative open string sectors do give rise to massless states which do not decouple even for blown-up orbifolds. Recently some examples of non-perturbative orientifolds were explicitly constructed in \[14\]. Thus, the four dimensional example discussed in \[14\] is based on the $\mathbb{Z}_6$ orbifold. The purpose of this paper, which is the follow-up of \[14\], is to extend the discussions of \[14\] to understand all other orientifold cases as well. In particular, we will discuss the $\mathbb{Z}_2 \otimes \mathbb{Z}_6$, $\mathbb{Z}_3 \otimes \mathbb{Z}_6$, $\mathbb{Z}_6 \otimes \mathbb{Z}_6$ and $\mathbb{Z}_{12}$ cases. The first three cases lead to anomaly free non-perturbative orientifolds. The latter case, however, turns out to suffer from a non-perturbative anomaly whose origins we explain in detail in section V. Moreover, we elaborate on the rest of the orbifold compactifications mentioned above, namely, the $\mathbb{Z}_2 \otimes \mathbb{Z}_4$ and $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ \[3\], and $\mathbb{Z}_8$, $\mathbb{Z}_8'$ and $\mathbb{Z}_{12}$ \[10\] cases, which where shown to suffer from perturbative tadpoles in \[3\] and \[10\], respectively. In \[11\] this was explained by considering F-theory duals of these compactifications. In this paper we elaborate the arguments of \[11\]. In particular, we point out that the corresponding inconsistencies arise due to the particular choices of the gauge bundle in the models of \[3\]–\[10\] (these gauge bundles are perturbative from the orientifold viewpoint).

The origin of non-perturbative states in orientifold compactifications can already be understood in six dimensions. Thus, in the K3 orbifold examples of \[3\] the orientifold projection is not $\Omega$, which we will use to denote that in the smooth K3 case, but rather $\Omega J'$, where $J'$ maps the $g$ twisted sector to its conjugate $g^{-1}$ twisted sector (assuming $g^2 \neq 1$) \[13\]. Geometrically this can be viewed as a permutation of two $\mathbb{P}^1$’s associated with each fixed point of the orbifold \[11\]. (More precisely, these $\mathbb{P}^1$’s correspond to the orbifold...
blow-ups.) This is different from the orientifold projection in the smooth case where (after blowing up) the orientifold projection does not permute the two \( \mathbb{P}^1 \)'s. In the case of the \( \Omega J' \) projection the “twisted” open string sectors corresponding to the orientifold elements \( \Omega J'g \) are absent \([16,11]\). However, if the orientifold projection is \( \Omega \), then the “twisted” open string sectors corresponding to the orientifold elements \( \Omega g \) are present \([11]\). In fact, these states are non-perturbative from the orientifold viewpoint and are required for gravitational anomaly cancellation in six dimensions. In certain cases Type I-heterotic duality allows one to understand such sectors and construct the corresponding models explicitly \([14]\).

In four dimensional orientifolds with \( \mathcal{N} = 1 \) supersymmetry there are always sectors (except in the \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) model of \([1]\) which is completely perturbative from the orientifold viewpoint) such that there is only one \( \mathbb{P}^1 \) per fixed point, so only the \( \Omega \) orientifold projection is allowed. This results in non-perturbative “twisted” open string sectors, which, as we have already mentioned, decouple in certain cases once the appropriate blow-ups are performed. In other cases we must include these states to obtain the complete description of a given orientifold.

Some of the non-perturbative orientifolds have perturbative heterotic duals. However, non-perturbative orientifolds with, say, D5-branes are non-perturbative from the heterotic viewpoint. In this paper we are therefore exploring some vacua in the region \( \mathcal{D} \) in Fig.1 (which has been borrowed from the second reference in \([13]\)). Thus, the \( \mathbb{Z}_6' \) orbifold compactification discussed in \([4]\), as well as the \( \mathbb{Z}_2 \otimes \mathbb{Z}_6 \) orbifold compactification we discuss in section V of this paper, are examples of four dimensional chiral \( \mathcal{N} = 1 \) supersymmetric string vacua which are non-perturbative from both orientifold and heterotic points of view.

The key point which allows us to understand four dimensional non-perturbative orientifolds discussed in this paper is the fact that these models correspond to Type I compactifications on generalized Voisin-Borcea orbifolds of the form \( (T^2 \otimes K3)/\mathbb{Z}_N \) \((N = 2, 3, 4, 6)\). One of the simplifying features of these compactifications is that once non-perturbative K3 orientifolds are understood (and this is partially facilitated by the fact that in six dimensions the gravitational anomaly cancellation condition is rather constraining), it is possible to extend the corresponding results to generalized Voisin-Borcea compactifications. Another nice feature is that, by T-dualizing along \( T^2 \), we can map these modes to F-theory compactifications on the corresponding Calabi-Yau four-folds, and the (at least partially) geometric picture arising in the F-theory context is very helpful in understanding such vacua.

The rest of this paper is organized as follows. In section II we use the map between Type I compactifications on K3 and F-theory on Voisin-Borcea orbifolds to determine which gauge bundles are perturbative from the orientifold viewpoint. In section III we give explicit models corresponding to K3 compactifications of Type I with these perturbative gauge bundles. In section IV we move down to four dimensions and consider Type I compactifications on Voisin-Borcea orbifolds. In particular, using the map between such Type I vacua and F-theory on Calabi-Yau four-folds we give the constraints necessary for tadpole cancellation in the cases where the gauge bundle is perturbative. In particular, this discussion relies on the results of section II. In section V we discuss Type I compactifications on generalized Voisin-Borcea orbifolds \( (T^2 \otimes K3)/\mathbb{Z}_N \) with \( N = 3, 4, 6 \). In particular, we argue that the \( \mathbb{Z}_{12} \) case suffers from a non-perturbative anomaly. In section VI we discuss some directions for extending our results. In particular, we consider Type I on K3 with non-zero NS-NS \( B \)-field and the gauge bundles with vector structure.
II. TYPE I ON K3 (ORBIFOLDS)

In this section we discuss six dimensional $\mathcal{N} = 1$ supersymmetric Type I compactifications on K3. In particular, what we would like to understand here is which choices of the gauge bundle are perturbative from the orientifold viewpoint. Our discussion here applies to both Type I and Spin(32)/$\mathbb{Z}_2$ heterotic compactifications on K3, which is due to Type I-heterotic duality. The dictionary between the two descriptions includes mapping 5-branes on the Type I side, which are made of some number of D5-branes, to NS 5-branes (or, equivalently, small instantons) on the heterotic side.

Thus, consider Type I on a K3 surface. In order to determine a particular ground state of the theory, we must specify not only the compactification space, which in this case is K3, but also the choice of the gauge bundle. Thus, in the orientifold description of Type I non-perturbative effects can arise from two different sources: (1) the perturbative orientifold description may fail to capture all states arising due to, say, various singularities in K3 itself; (2) a given choice of the gauge bundle may not have perturbative orientifold description. The second source of non-perturbative physics is generally more non-trivial to understand. Here, however, we will be interested in determining which choices of the gauge bundle do possess perturbative orientifold description, and once we understand this class of gauge bundles, we can then focus on the first source of non-perturbative effects which are easier to handle due to various hints from geometry.

The case of K3 is particularly convenient as (unlike in the case of Calabi-Yau three-folds) topologically we are dealing with one surface. This enables us to understand perturbative gauge bundles in the K3 case by going to orbifold limits which have a simple description. Thus, let us consider orbifold K3’s of the form $K3 = T^4/\mathbb{Z}_N$, where $N = 2, 3, 4, 6$ are the choices allowed by the requirement that the orbifold act crystallographically on the four-torus $T^4$. Let $z_1, z_2$ be the complex coordinates parametrizing $T^4$. Then the action of the generator $g$ of $\mathbb{Z}_N$ is given by

$$gz_1 = \omega z_1, \quad gz_2 = \omega^{-1} z_2,$$

where $\omega \equiv \exp(2\pi i/N)$.

Next, consider Type I on $T^4/\mathbb{Z}_N$. We will assume that the (untwisted) NS-NS $B$-field is trivial. Then we can view Type I on $T^4/\mathbb{Z}_N$ as the $\Omega$ orientifold of Type IIB on $T^4/\mathbb{Z}_N$. (More precisely, the $\Omega$ orientifold is well defined after the appropriate blow-ups of the orbifold singularities other than $\mathbb{Z}_2$.) The theory has $\mathcal{N} = 1$ supersymmetry in six dimensions. The massless spectrum of the closed string sector consists of the gravity supermultiplet, one tensor supermultiplet, and 20 hypermultiplets (which are neutral under the non-Abelian gauge symmetries in the open string sector). In the open string sector we have 32 D9-branes, which is required by the untwisted tadpole cancellation conditions. In the $\mathbb{Z}_3$ case we do not have any D5-branes. In the $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6$ cases tadpole cancellation conditions also require introduction of 32 D5-branes.

To understand the gauge bundles in all of the above cases, recall that in the case of K3 anomaly cancellation requires that the number $n_I$ of instantons embedded in the gauge bundle and the number $n_5$ of 5-branes (that is, small instantons) must add up to 24: $n_I + n_5 = 24$. In the $\mathbb{Z}_3$ all 24 instantons are embedded in the gauge bundle, so the number of 5-branes is zero. This gauge bundle is therefore perturbative from the heterotic viewpoint. On the
orientifold side it is defined by the action of the orbifold group on the Chan-Paton charges which is given by the following $16 \times 16$ matrix \[3\]:

$$Z_3 : \quad \gamma_g = \text{diag}(\alpha I_4, \alpha^{-1} I_4, I_8),$$

where $\alpha \equiv \exp(2\pi i/3)$, and $I_n$ denotes the $n \times n$ identity matrix.

Next, consider the $Z_2$ case. Here we have 32 D5-branes which form $n_5 = 8$ dynamical 5-branes. Here two pairings take place - one due to the orientifold projection, and the other one due the $Z_2$ orbifold projection. We therefore conclude that $n_I = 16$ in this case. This gauge bundle is non-perturbative from the heterotic viewpoint. On the orientifold side the corresponding Chan-Paton matrix is given by \[2\]:

$$Z_2 : \quad \gamma_g = \text{diag}(i I_8, -i I_8).$$

As to the $Z_4$ and $Z_6$ cases, the corresponding models are on the same moduli as the $Z_2$ model. Indeed, in all three of these cases the gauge group can be Higgsed completely. In fact, in the $Z_4$ and $Z_6$ cases this can already be done just with the matter arising in the sectors which are perturbative from the orientifold viewpoint \[1\]. Then the gravitational anomaly cancellation condition implies that after Higgsing the number of hypermultiplets in the open string sector is the same in all three models (the closed string spectra of these models are identical), hence they are on the same moduli. This implies that in the $Z_4$ and $Z_6$ cases we also have $n_5 = 8$ and $n_I = 16$. In the orientifold language the corresponding Chan-Paton matrices are given by \[4\]:

$$Z_4 : \quad \gamma_g = \text{diag}(\beta I_4, -\beta I_4, \beta^{-1} I_4, -\beta^{-1} I_4),$$

$$Z_6 : \quad \gamma_g = \text{diag}(i \alpha I_2, -i \alpha I_2, i \alpha^{-1} I_2, -i \alpha^{-1} I_2, i I_4, -i I_4),$$

where $\beta \equiv \exp(\pi i/4)$.

Thus, the gauge bundles that are perturbative from the orientifold viewpoint have $(n_5, n_I) = (0, 24)$ and $(n_5, n_I) = (8, 16)$. It is instructive to understand these gauge bundles a bit better. In particular, here we would like to discuss them in the heterotic and F-theory pictures. On the heterotic side there are three topologically distinct gauge bundles with $(n_5, n_I) = (0, 24)$. These are characterized by the generalized second Stiefel-Whitney class $\tilde{w}_2$ (which is an element of $H^2(K3, \mathbb{Z})$, or, more precisely, of $H^2(K3, \mathbb{Z}_2)$ as $\tilde{w}_2$ is defined modulo a shift by twice a lattice vector of $H^2(K3, \mathbb{Z})$) \[18\]. The three distinct gauge bundles correspond to: (1) $\tilde{w}_2 = 0$; (2) $\tilde{w}_2 \cdot \tilde{w}_2 = 0 \pmod{4}$; (3) $\tilde{w}_2 \cdot \tilde{w}_2 = 2 \pmod{4}$. In the first case we have vector structure, while (2) and (3) are the cases without vector structure.

To understand the above three cases better, it is useful to think about the corresponding gauge bundles in terms of the $Z_2$ orbifold action on the $\text{Spin}(32)/\mathbb{Z}_2$ degrees of freedom in the heterotic context. This action can be viewed in terms of a $\mathbb{Z}_2$ valued shift $V$ of the $\text{Spin}(32)/\mathbb{Z}_2$ lattice. Let us use the Cartan basis of $SO(32)$ to write $V$. If $2V$ belongs to

\[1\] Throughout this paper (unless specified otherwise) we work with $16 \times 16$ (rather than $32 \times 32$) Chan-Paton matrices for we choose not to count the orientifold images of the corresponding D-branes.
the $SO(32)$ lattice, then we have vector structure, and $\tilde{w}_2 = 0$. Two inequivalent gauge bundles of this type which are perturbative from the heterotic viewpoint are given by $V = ((1/2)^2 \ 0^{14})$ and $V = ((1/2)^6 \ 0^{10})$ (here we have chosen the surviving spinor of Spin$(32)/\mathbb{Z}_2$ to be given by $((1/2)^{16})$ modulo $SO(32)$ shifts). In particular, they satisfy the modular invariance constraints such as the level matching condition. (Thus, the gauge bundle given by $V = ((1/2)^2 \ 0^{14})$ corresponds to the standard embedding for the $\mathbb{Z}_2$ orbifold.) These gauge bundles are non-perturbative from the orientifold viewpoint.

However, we can have $V$ such that $2V$ does not belong to the $SO(32)$ lattice, albeit it does belong to the Spin$(32)/\mathbb{Z}_2$ lattice. Two inequivalent gauge bundles of this type are given by $V = ((1/4)^{16})$ and $V = ((1/4)^{15}(3/4))$. Note that for such gauge bundles $\tilde{w}_2 \neq 0$, and $\tilde{w}_2 \cdot \tilde{w}_2 = (2V)^2$, where by $(2V)^2$ we mean the length squared of the 16-vector $2V$. The gauge bundle given by $V = ((1/4)^{16})$ is non-perturbative from the heterotic viewpoint - it does not satisfy the level matching condition. In fact, this is precisely the gauge bundle in the Type IIB orientifold of $T^4/\mathbb{Z}_2$ \[2\]. As we have already mentioned, this gauge bundle corresponds to embedding 16 instantons accompanied by 8 5-branes to cancel the anomalies. On the other hand, the gauge bundle $V = ((1/4)^{15}(3/4))$ is perturbative from the heterotic viewpoint - it satisfies the level matching requirement, and corresponds to embedding all 24 instantons. It is, however, non-perturbative from the orientifold viewpoint.

Finally, let us discuss the F-theory duals of the above cases. In \[2\] it was shown that in the $\tilde{w}_2 = 0$ case the dual F-theory compactification is on the Calabi-Yau threefold given by the $T^2$ fibration over the base $\mathbf{F}_4$. (Here $\mathbf{F}_n$ are Hirzebruch surfaces.) In the case of the $\mathbb{Z}_3$ orbifold this can be seen in two ways. First, we can Higgs the $U(8) \otimes SO(16)$ gauge group at the orbifold point (see Table I) down to $SO(8) \otimes SO(8)$ with 10 adjoint hypermultiplets in the first $SO(8)$ and no matter in the second $SO(8)$. This is precisely the spectrum of the F-theory dual on the Calabi-Yau three-fold with the base $\mathbf{F}_4$ (at the Voisin-Borcea orbifold point - see section IV). On the other hand, we can explicitly construct the $SO(8) \otimes SO(8)$ point in the string language. Let K3 be $(T^2 \otimes T^2)/\mathbb{Z}_3$, where $\mathbb{Z}_3$ simultaneously rotates the two $T^2$'s by $2\pi/3$. Then consider the following choice of the gauge bundle. First, let us turn on a $\mathbb{Z}_2$ valued Wilson line on the $a$-cycle of, say, the first $T^2$ such that it breaks $SO(32)$ down to $SO(16) \otimes SO(16)$. Next, let us turn on the second $\mathbb{Z}_2$ valued Wilson line on the $b$-cycle of the same $T^2$ such that it further breaks $SO(16) \otimes SO(16)$ down to $SO(8)^4$. It is not difficult to see that this theory has $\mathbb{Z}_3$ symmetry under which, say, the first three $SO(8)$'s are cyclically permuted. In fact, since the $T^2$'s are hexagonal, under the $\mathbb{Z}_3$ rotation the $a$-cycles maps to the $b$-cycle, and so the first Wilson line maps to the second Wilson line. The choice of the gauge bundle corresponding to the $\mathbb{Z}_3$ orbifold then precisely consists of the twist permuting the first three $SO(8)$'s, while leaving the last $SO(8)$ untouched. The resulting model has $SO(8)_3 \otimes SO(8)_1$ gauge group, where the subscript indicates the current algebra level via which the corresponding subgroup is realized. The untwisted sector contains the gravity supermultiplet, one tensor supermultiplet, 2 neutral hypermultiplets, and one hypermultiplet in the adjoint of $SO(8)_3$. The twisted sector contains 18 neutral hypermultiplets (corresponding to the blow-up modes of the orbifold), and 9 hypermultiplets in the adjoint of $SO(8)_3$. This spectrum precisely matches that of the F-theory dual corresponding to the base $\mathbf{F}_4$.

As to the cases with $\tilde{w}_2 \cdot \tilde{w}_2 = 0 \pmod{4}$ and $\tilde{w}_2 \cdot \tilde{w}_2 = 2 \pmod{4}$, the corresponding dual F-theory compactifications are on Calabi-Yau three-folds given by $T^2$ fibrations over
\( F_0 \) respectively \( F_1 \) \[20\]. Here we note that these cases also correspond to Voisin-Borcea orbifolds, but at the latter points in the respective moduli spaces the corresponding bases are singular leading to additional matter \([21]\) precisely such that the resulting massless spectra match those of the heterotic/Type I duals. Also, as was pointed out in \([11]\), in the \( \bar{w}_2 \cdot \bar{w}_2 = 0 \) (mod 4) case one can also view the dual F-theory compactification as on a singular Calabi-Yau three-fold (with Hodge numbers \((h^{1,1}, h^{2,1}) = (3,51)\)) given by the “non-geometric” \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) orbifold of \( T^2 \otimes T^2 \otimes T^2 \) with discrete torsion \([22]\). This F-theory compactification is in turn on the same moduli as that on a smooth Calabi-Yau three-fold (with Hodge numbers \((h^{1,1}, h^{2,1}) = (3,243)\)) given by the \( T^2 \) fibration over the base \( \mathbb{P}^1 \otimes \mathbb{P}^1 \) (that is, the smooth \( F_0 \) surface) \([11]\).

To summarize, the gauge bundles which are perturbative from the orientifold viewpoint are those with \((n_5, n_I) = (0, 24), \bar{w}_2 = 0 \) (this is the case for the gauge bundle in the \( \mathbb{Z}_3 \) case \(1\)), and \((n_5, n_I) = (8, 16), \bar{w}_2 \cdot \bar{w}_2 = 0 \) (mod 4) (this is the case for the gauge bundles in the \( \mathbb{Z}_2 \) \([3]\), \( \mathbb{Z}_4 \) \([4]\), and \( \mathbb{Z}_6 \) \([5]\) cases).

### III. NON-PERTURBATIVE K3 ORIENTIFOLDS

In the previous section we have discussed the gauge bundles in K3 compactifications of Type I which are perturbative from the orientifold viewpoint. In this section we would like to discuss the explicit Type I vacua arising upon compactifications on K3 with these gauge bundles. These compactifications can be viewed as the \( \Omega \) orientifolds of Type IIB on (the corresponding orbifold limits of) K3. Thus, the \( \mathbb{Z}_2 \) case with the gauge bundle given by \( \mathbb{Z}_2 \) \([6]\) (the action of the orbifold group on both D9- and D5-branes is the same) was discussed in \([1,2]\). This model is completely perturbative from the orientifold viewpoint. Thus, the “\( \Omega \)-twisted” sector of the orientifold can be viewed as the usual (“untwisted”) 99 and 55 open string sectors. The “\( \Omega \)-twisted” sector (with \( g^2 = 1 \) in this case) can be viewed as the usual (“untwisted”) 59 open string sector. These sectors are perturbative from the orientifold point of view. The massless open spectrum of this model is given in Table I.

In the \( \mathbb{Z}_3 \) case we do have non-perturbative sectors, however. Thus, the “\( \Omega \)-twisted” sector of the orientifold corresponds to the untwisted 99 open string sector (there are no D5-branes in this case). The “\( \Omega g \)- and \( \Omega g^2 \)-twisted” sectors (with \( g^2 = 1 \) in this case) correspond to the \( \mathbb{Z}_3 \) “twisted” 99 open string sector. This model has a perturbative heterotic dual which enables one to obtain its massless spectrum \([12]\), and the latter is given\(^2\) in Table I.

\(^2\)For the sake of simplicity, throughout this paper we omit the \( U(1) \) charges. In the untwisted open string sectors they are straightforward to determine from the standard Chan-Paton charge assignments. In the twisted open string sectors, however, these \( U(1) \) charges are generically fractional (in the normalization where the fundamental of \( SU(N) \subset U(N) \) in the untwisted open string sectors has the \( U(1) \) charge +1). In certain cases one can determine these charges using Type I-heterotic duality (as on the heterotic side these charges can be computed perturbatively in the corresponding twisted sectors). In other cases determining the twisted sector \( U(1) \) charges can be more involved. Also, some of the twisted closed string states can transform non-trivially under gauge transformations of some of the \( U(1) \)’s. The corresponding \( U(1) \) charges (in the appropriate
In the \( \mathbb{Z}_6 \) case we do not have a perturbative heterotic dual as the Type I model contains D5-branes. Nonetheless, it is still possible to determine the spectrum of the model. Thus, the \( \mathbb{Z}_3 \) twisted 99 open string sector is the same as in the \( \mathbb{Z}_3 \) model with the projection onto the \( \mathbb{Z}_2 \) invariant states (note that \( \mathbb{Z}_6 \approx \mathbb{Z}_3 \otimes \mathbb{Z}_2 \)). In particular, the 9 fixed points in the original \( \mathbb{Z}_3 \) twisted sector are combined into 5 linear combinations invariant under \( \mathbb{Z}_2 \) plus 4 linear combinations which pick up a minus sign under the \( \mathbb{Z}_2 \) action. Taking this into account, it is straightforward to determine the \( \mathbb{Z}_2 \) invariant states in the \( \mathbb{Z}_3 \) twisted 99 open string sector (see [14] for details). The \( \mathbb{Z}_3 \) twisted 55 open string sector is a bit more subtle - here we assume that all D5-branes are sitting on top of each other at the orientifold 5-plane located at the origin of K3 (\( z_1 = z_2 = 0 \)). This implies that the D5-branes only feel the singularity in K3 located at the origin. That is, the other 8 of the original 9 fixed points in the \( \mathbb{Z}_3 \) twisted sector play no role in this discussion as the twisted 55 states arise due to the local geometry near the origin. This results in the fact that the corresponding multiplicity in the \( \mathbb{Z}_3 \) twisted 55 open string sectors is 1. Finally, the “\( \Omega g \)- and \( \Omega g^3 \)-twisted” sectors (with \( g^6 = 1 \) in this case) correspond to the \( \mathbb{Z}_3 \) twisted 59 open string sector. In this case local geometry once again determines the multiplicity of states to be 1 (and also fixes their quantum numbers) [14]. The massless spectrum of the \( \mathbb{Z}_6 \) model is summarized in Table I. Note that the twisted 99 and 55 open string sectors no longer exhibit the naive “T-duality”. This is due to the fact that these sectors do not arise via a straightforward orbifold reduction of the corresponding (\( SO(32) \)) gauge theory (with \( N = 2 \) supersymmetry in six dimensions). More concretely, the T-duality transformation involves \( \mathbb{Z}_2 \) reflection of the complex coordinates \( z_1, z_2 \). The \( \mathbb{Z}_3 \) singularities, however, are not invariant under this \( \mathbb{Z}_2 \) action, hence the lack of T-duality symmetry.

Finally, let us consider the \( \mathbb{Z}_4 \) case which was not discussed in [14]. The main difficulty in this case is that (unlike in the \( \mathbb{Z}_3 \) and \( \mathbb{Z}_6 \) cases) this model has no perturbative heterotic dual, nor do we have any sectors of the theory which can be understood by performing a perturbative computation on the heterotic side. Fortunately, however, anomaly cancellation requirements (which are quite constraining in six dimensions) along with the relevant geometric picture allow us to determine the massless spectrum in this case as well. Thus, the gravitational anomaly cancellation condition reads:

\[
n_H - n_V = 273 - 29n_T ,
\]

where \( n_H, n_V, n_T \) are the numbers of hypermultiplets, vector multiplets and tensor multiplets, respectively. In this case we have \( n_T = 1 \) tensor multiplets and \( n_H = 20 \) hypermultiplets from the closed string sector. All the vector multiplets arise in the open string sector. In particular, \( n_V = 256 \) as the gauge group is \([U(8) \otimes U(8)]_{99} \otimes [U(8) \otimes U(8)]_{55}\). The anomaly cancellation condition [1] then implies that the total number of open string hypermultiplets must be: \( n_H^0 = n_H - n_H^c = 480 \). The perturbative sectors, namely, the untwisted 99 plus 55 as well as 59 sectors give rise to 368 hypermultiplets in this model (see Table I). Thus, 112 = 4 \times 28 hypermultiplets must come from the twisted open string sectors. These are the \( \mathbb{Z}_4 \) twisted 99 and 55 open string sectors corresponding to the “\( \Omega g \)- and \( \Omega g^3 \)-twisted” sectors (with \( g^4 = 1 \) in this case). Note that there is no “twisted” 59 sector in this model.
From the anomaly cancellation we get a hint that the twisted 99 and 55 open string sector hypermultiplets must transform in the antisymmetric representation $28$ of the $SU(8)$ subgroups. In fact, since the two $SU(8)$’s within, say, the 55 open string sector are on the equal footing, we conclude that the only way we can cancel anomalies is by assuming that in the twisted 55 sector we have one copy of twisted hypermultiplets in $(28,1)_{55}$ and $(1,28)_{55}$, and similarly for the twisted 99 sector. In the 55 sector this is actually what one expects from the geometric picture: just as in the $\mathbb{Z}_6$ case here we have all D5-branes on top of each other at the orientifold 5-plane located at the origin of K3, so we expect only one copy of non-perturbative twisted hypermultiplets arising due to the local geometry at the vicinity of the origin (the latter being the relevant singularity in K3). However, for the 99 sector the conclusion that we have only one copy of twisted hypermultiplets might appear a bit puzzling - the D9-branes wrap the entire K3, and naively one expects that they should “feel” all 4 $\mathbb{Z}_4$ fixed points.

Let us try to understand this point a bit better. An important observation here is that the naive T-duality is expected to be a symmetry of this background - indeed, the $\mathbb{Z}_2$ reflection of the complex coordinates $z_1, z_2$ involved in the T-duality transformation leaves the corresponding $\mathbb{Z}_4$ singularities invariant. Thus, the $\mathbb{Z}_4$ twisted 55 and 99 open string sectors should have T-duality symmetry. Then we must explain how come in the 99 sector we do not get 4 copies of the corresponding matter representations but only one. The point here is that we expect massless non-perturbative states to arise only for true geometric singularities. However, if there is a non-zero twisted $B$-field turned on inside of the $\mathbb{P}^1$ associated with a given blow-up mode (that is, with a given fixed point), then non-perturbative states do not arise [24]. Since we are dealing with the $\mathbb{Z}_4$ twisted sectors, the corresponding twisted $B$-field can only take values $0, \pm 1/4, 1/2$ (here we normalize the $B$-field so that it is defined modulo a unit shift). The $B$-field must be zero at the origin. At the other three fixed points, however, it must be non-zero. Note that it cannot be $\pm 1/4$ as this would not be invariant under the orientifold action $\Omega$ (recall that the $B$-field is odd under the action of $\Omega$). However, the $B$-field at these three fixed points can take a half-integer value which (taking into account that it is only defined modulo an integer shift) is invariant under the action of $\Omega$. Thus, to obtain a consistent $\mathbb{Z}_4$ orbifold background (with all D5-branes located at the origin of K3), we must turn on half-integer twisted $B$-field at the other three fixed points. Then the massless spectrum of the $\mathbb{Z}_4$ model, which is summarized in Table I, is free of anomalies. Here we should note that in the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ models the twisted $B$-field (in the $\mathbb{Z}_3$ twisted sectors) must be zero as non-zero values such as $\pm 1/3$ would not be invariant under the action of $\Omega$. This is why in these models all singularities (that is, fixed points) are relevant in the twisted 99 sector.

Here we would like to point out that all three $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6$ models are on the same moduli. As we have already mentioned, at generic points with completely broken gauge symmetry these models correspond to F-theory compactifications on the Calabi-Yau three-fold (with the Hodge numbers $(h^{1,1}, h^{2,1}) = (3, 243)$) given by the $T^2$ fibration over the base $\mathbb{P}^1 \otimes \mathbb{P}^1$.

**IV. TYPE I ON VOISIN-BORCEA ORBIFOLDS**

In this section we consider four dimensional $\mathcal{N} = 1$ supersymmetric Type I compactifications on Calabi-Yau three-folds known as Voisin-Borcea orbifolds [25,26]. To aid the
presentation, in the next subsection we review various basic facts about Voisin-Borcea orbifolds following [27, 21, 11]. To clarify some points discussed in the previous sections, we also briefly discuss F-theory compactifications on these spaces. Having reviewed the relevant background material, we then discuss Type I compactifications on these Calabi-Yau three-folds as well as their dual F-theory compactifications on the corresponding Calabi-Yau four-folds known as Borcea four-folds.

A. Voisin-Borcea Orbifolds

Let \( W_2 \) be a K3 surface (which is not necessarily an orbifold of \( T^4 \)) which admits an involution \( J \) such that it reverses the sign of the holomorphic two-form \( dz_1 \wedge dz_2 \) on \( W_2 \). Consider the following quotient:

\[
Y_3 = (T^2 \otimes W_2) / J ,
\]

where \( Y = \{1, S\} \approx \mathbb{Z}_2 \), and \( S \) acts as \( Sz_0 = -z_0 \) on \( T^2 \) (\( z_0 \) being a complex coordinate on \( T^2 \)), and as \( J \) on \( W_2 \). This quotient is a Calabi-Yau three-fold with \( SU(3) \) holonomy which is elliptically fibered over the base \( B_2 = W_2 / B \), where \( B = \{1, J\} \approx \mathbb{Z}_2 \).

Nikulin gave a classification [28] of possible involutions of K3 surfaces in terms of three invariants \((r, a, \delta)\) (for a physicist’s discussion, see, e.g., [27, 21]). The result of this classification is plotted in Fig.2 (which has been borrowed from [11]) according to the values of \( r \) and \( a \). The open and closed circles correspond to the cases with \( \delta = 0 \) and \( \delta = 1 \), respectively. (The cases denoted by “\( \otimes \)” are outside of Nikulin’s classification, and we will discuss them shortly.) In the case \((r, a, \delta) = (10, 10, 0)\) the base \( B_2 \) is an Enriques surface, and the corresponding \( Y_3 \) has Hodge numbers \((h^{1,1}, h^{2,1}) = (11, 11)\). In all the other cases the Hodge numbers are given by:

\[
\begin{align*}
h^{1,1} &= 5 + 3r - 2a , \\
h^{2,1} &= 65 - 3r - 2a .
\end{align*}
\]

(8) (9)

For \((r, a, \delta) = (10, 10, 0)\) the \( \mathbb{Z}_2 \) twist \( S \) is freely acting (that is, it has no fixed points). For \((r, a, \delta) = (10, 8, 0)\) the fixed point set of \( S \) consists of two curves of genus 1. The base \( B_2 \) in this case is \( \mathbb{P}^2 \) blown up at 9 points. In all the other cases the fixed point set of \( S \) consists of one curve of genus \( g \) plus \( k \) rational curves, where

\[
\begin{align*}
g &= \frac{1}{2} (22 - r - a) , \\
k &= \frac{1}{2} (r - a) .
\end{align*}
\]

(10) (11)

It is sometimes useful to separate the above Hodge numbers into the contribution \((h_0^{1,1}, h_0^{2,1})\) of the untwisted sector and the contribution \((h_s^{1,1}, h_s^{2,1})\) of the \( \mathbb{Z}_2 \) twisted (that is, \( S \)-twisted) sector. These are given by

\[
\begin{align*}
h_0^{1,1} &= r + 1 , \\
h_0^{2,1} &= 21 - r .
\end{align*}
\]

(12) (13)
and
\[ h^{1,1}_s = 4(k + 1) , \]  
\[ h^{2,1}_s = 4g , \]
respectively.

Note that except for the cases with \( a = 22 - r, r = 11, \ldots, 20 \), the mirror pair of \( Y_3 \) is given by the Voisin-Borcea orbifold \( \bar{Y}_3 \) with \( \bar{r} = 20 - r, \bar{a} = a \). Under the mirror transform we have: \( \bar{g} = f, \bar{f} = g \), where \( f = k + 1 \).

In the cases \( a = 22 - r, r = 11, \ldots, 20 \), the mirror would have to have \( \bar{r} = 20 - r \) and \( \bar{a} = a = \bar{r} + 2 \), where \( \bar{r} = 0, \ldots, 9 \). We have depicted these cases in Fig.2 using the “⊗” symbol. In particular, we have plotted cases with \( a = r + 2, r = 0, \ldots, 9 \). Extrapolation to \( r = 10 \) is motivated by the fact that in this case we get \( (h^{1,1}, h^{2,1}) = (11, 11) \) which is the same as for \( (r, a, \delta) = (10, 10, 0) \). In [11] it was argued that these Voisin-Borcea orbifolds also exist, albeit they are singular. In fact, some of them can be constructed explicitly (see [11] for details).

As we already mentioned, here we would like to briefly review F-theory compactifications on Voisin-Borcea orbifolds (which correspond to \( N = 1 \) supersymmetric vacua in six dimensions). Thus, consider F-theory on \( Y_3 \) with \( (r, a, \delta) \neq (10, 10, 0) \) or \( (10, 8, 0) \). This gives rise to the following massless spectrum in six dimensions. The number of tensor multiplets is \( T = r - 1 \). The number of neutral hypermultiplets is \( H = 22 - r \). The gauge group is \( SO(8) \otimes SO(8)^k \). There are \( g \) adjoint hypermultiplets of the first \( SO(8) \). There are no hypermultiplets charged under the other \( k SO(8) \)'s. Under mirror symmetry \( g \) and \( f = k + 1 \) are interchanged. Thus, the vector multiplets in the adjoint of \( SO(8)^k \) are traded for \( g - 1 \) hypermultiplets in the adjoint of the first \( SO(8) \). That is, gauge symmetry turns into global symmetry and vice-versa. This can be pushed further to understand F-theory compactifications on Calabi-Yau three-folds with \( a = r + 2, r = 1, \ldots, 10 \), which give the following spectra. The number of tensor multiplets is \( T = r - 1 \). There are \( H = 22 - r \) neutral hypermultiplets. In addition there are \( g = 10 - r \) hypermultiplets transforming as adjoints under a global \( SO(8) \) symmetry. There are no gauge bosons, however. It is not

\[ ^3 \text{In the case } (r, a, \delta) = (10, 10, 0) \text{ we have } T = 9 \text{ tensor multiplets, } H = 12 \text{ hypermultiplets, and no gauge vector multiplets. In the case } (r, a, \delta) = (10, 8, 0) \text{ we have } T = 9 \text{ tensor multiplets, } H = 12 \text{ neutral hypermultiplets, gauge vector multiplets corresponding to } SO(8) \otimes SO(8), \text{ one hypermultiplet in the adjoint of the first } SO(8), \text{ and one hypermultiplet in the adjoint of the second } SO(8). \]

\[ ^4 \text{Here we must exclude the case with } r = 0, a = 2 \text{ for the F-theory prediction would be } T = -1 \text{ tensor multiplets. This Calabi-Yau three-fold does exist [11], but it is singular and F-theory compactification on this space does not appear to have a local Lagrangian description. However, an extremal transition [21] between this Calabi-Yau three-fold and another Voisin-Borcea orbifold could lead to a phase transition into a well defined vacuum.} \]
difficult to verify that this massless spectrum is free of gravitational anomalies in six dimensions. In fact, F-theory compactifications on these singular spaces are equivalent to F-theory compactifications on smooth Calabi-Yau three-folds according to the following relation \[11\]:

\[
\begin{align*}
\text{F-theory on } \mathcal{Y}_3 \text{ with } (h^{1,1}, h^{2,1}) &= (r + 1, 61 - 5r) \text{ is equivalent to } \\
\text{F-theory on } \hat{\mathcal{Y}}_3 \text{ with } (\hat{h}^{1,1}, \hat{h}^{2,1}) &= (r + 1, 301 - 29r) \quad (r = 1, \ldots, 9) .
\end{align*}
\]

Thus, for instance, for \( r = 2 \) we get \((\hat{h}^{1,1}, \hat{h}^{2,1}) = (3, 243)\), which is the elliptic Calabi-Yau given by the \( T^2 \) fibration over the base \( \mathbb{P}^1 \otimes \mathbb{P}^1 \). We have already encountered this Calabi-Yau three-fold in the previous sections. Note that F-theory compactifications on Voisin-Borcea orbifolds with \((r, a, \delta) = (2, 2, 0), (2, 2, 1)\) are on the same moduli as the background corresponding to \((\hat{h}^{1,1}, \hat{h}^{2,1}) = (3, 243)\). The reason is that the corresponding bases in the former two cases (namely, the Hirzebruch surfaces \( \mathbf{F}_0 \) and \( \mathbf{F}_1 \)) are singular at the Voisin-Borcea points \[21\]. After the appropriate blow-ups additional matter arises precisely such that the moduli count matches that for the \((\hat{h}^{1,1}, \hat{h}^{2,1}) = (3, 243)\) compactification.

### B. Type I on Voisin-Borcea Orbifolds

In this subsection we consider some general aspects of Type I compactifications on Voisin-Borcea orbifolds reviewed in the previous subsection. In discussing such a compactification, we must specify the gauge bundle. We discussed perturbative (from the orientifold viewpoint) gauge bundles for K3 compactifications in section II. Our goal here will be to determine which gauge bundles are perturbative from the orientifold viewpoint for the Voisin-Borcea compactifications. It turns out that one can answer this question using the map between Type IIB orientifolds and F-theory. In fact, as we will see in a moment, the requirement that the gauge bundle be perturbative greatly restricts the possible choices of the compactification space itself.

Thus, consider Type I compactification on the Voisin-Borcea orbifold \( \mathcal{Y}_3 \) labeled by \((r, a, \delta) \neq (10, 10, 0)\). We can view this as the \( \Omega \) orientifold of Type IIB on \( \hat{\mathcal{Y}}_3 \). This theory contains 32 D9-branes as well as D5-branes. Thus, there are D5-branes filling \( \mathbb{R}^{3,1} \otimes \mathcal{C} \), where \( \mathbb{R}^{3,1} \) is the non-compact four dimensional Minkowski space-time, whereas \( \mathcal{C} \) is the set of points (of real dimension 2) in \( \mathcal{B}_2 \) fixed under the action of the \( \mathbb{Z}_2 \) twist \( S \) (see the previous subsection for notations). There may be other 5-branes depending upon the choice of the space \( \mathcal{Y}_3 \). Thus, if the K3 surface \( \mathcal{W}_2 \) (recall that the base of the elliptic fibration is \( \mathcal{B}_2 = \mathcal{W}_2 / J \)) can be written as the quotient \( \mathcal{W}_2 = \mathcal{W}'_2 / \mathbb{Z}_2 \) (here \( \mathcal{W}'_2 \) can be either \( T^4 \) or another K3 surface), where the generator \( R \) of \( \mathbb{Z}_2 \) reflects both of the complex coordinates \( z_1, z_2 \) on \( \mathcal{W}'_2 \), then we have other 5-branes in the theory. First, we have D5-branes filling \( \mathbb{R}^{3,1} \otimes \mathcal{C}' \), where \( \mathcal{C}' \) is the set of points (of real dimension 2) in \( \mathcal{B}_2 \) fixed under the action of the twist \( SR \). Also, there are 5-branes filling \( \mathbb{R}^{3,1} \otimes T^2 \) (here \( T^2 \) is the fibre \( T^2 \)). More precisely, as we will see in a moment, this is the case for perturbative gauge bundles (while for other choices of the gauge bundle these 5-branes may be absent).

To understand which gauge bundles are perturbative from the orientifold viewpoint in the case of Voisin-Borcea compactifications, it is convenient to T-dualize the corresponding Type I background in the direction of the fibre \( T^2 \). This way we arrive at the \( \Omega \hat{\mathcal{J}}(-1)^{F_L} \) orientifold of Type IIB on \( \mathcal{Y}_3 \) (which contains various 7-branes plus, in certain cases, 3-branes). Here \( \hat{\mathcal{J}} \) reflects the complex coordinate \( z_0 \) on the fibre \( T^2 \) (while acting trivially
on the base $\mathcal{B}_2$), and $F_L$ is the left-moving (space-time) fermion number. This orientifold can now be mapped to the F-theory compactification on an elliptically fibered Calabi-Yau four-fold via the map of [29] which we review next.

Consider an $\Omega J(-1)^{F_L}$ orientifold of Type IIB on $\mathcal{Y}_3$, where $\bar{J}$ reverses the sign of the holomorphic 3-form $dz_1 \wedge dz_2 \wedge dz_3$ on $\mathcal{Y}_3$, and the set of points fixed under $\bar{J}$ in $\mathcal{Y}_3$ has real dimension 4. Then following [29] we can map this orientifold to (a limit of) F-theory on a Calabi-Yau four-fold $X_4$ defined as

$$X_4 = (\bar{T}^2 \otimes Y_3)/X,$$  

(17)

where $X = \{1, \bar{S}\} \approx \mathbb{Z}_2$, and $\bar{S}$ acts as $\bar{S}z_0 = -z_0$ on $\bar{T}^2$ ($z_0$ is a complex coordinate on $\bar{T}^2$), and as $\bar{J}$ on $Y_3$. To avoid confusion, here we use tilde to distinguish between the fibre $\bar{T}^2$ in the elliptic fibration over the base $B_3 = Y_3/B$, where $B \equiv \{1, \bar{J}\} \approx \mathbb{Z}_2$, and the fibre $T^2$ in the elliptic fibration over the base $B_2 = \mathcal{W}_2/Y$.

Note that in the case where $Y_3$ is a Voisin-Borcea orbifold, the above four-fold $X_4$ can be viewed as a quotient $(\mathcal{W}_2 \otimes \mathcal{W}_2)/Y$. Here $\mathcal{W}_2 \equiv (T^2 \otimes T^2)/X$ (recall that $X = \{1, \bar{S}\} \approx \mathbb{Z}_2$, and $\bar{S}$ acts as $\bar{S}z_0 = -z_0$ and $\bar{S}z_0 = -z_0$ on the complex coordinates parametrizing $T^2$ and $T^2$, respectively). Thus, $\mathcal{W}_2$ is (the $\mathbb{Z}_2$ orbifold limit of) a K3 surface. The action of $Y = \{1, S\}$ on the two K3 surfaces $\mathcal{W}_2$ and $\mathcal{W}_2$ is given by the corresponding Nikulin’s involutions labeled by the integers $(\bar{r}, \bar{a}, \bar{\delta})$ and $(r, a, \delta)$ respectively. (In fact, it is not difficult to show that in the present case $\bar{r} = 18$ and $\bar{a} = 4$.) These Calabi-Yau four-folds (which have $SU(4)$ holonomy) generically are singular spaces. The Euler number $\chi$ of such a four-fold is given by [26]

$$\frac{1}{24} \chi = 12 + \frac{1}{4}(\bar{r} - 10)(r - 10).$$  

(18)

Taking into account that in the case at hand we have $\bar{r} = 18$, we obtain

$$\frac{1}{24} \chi = 12 + 2(r - 10) = 2(r - 4).$$  

(19)

Now consider F-theory compactification on such a four-fold. To cancel space-time anomaly we need to introduce $\chi/24$ 3-branes, which implies that this number must be non-negative or else supersymmetry appears to be broken [30]. It then follows that

Type I on Voisin-Borcea orbifolds can be tadpole free only for $r \geq 4$.  

(20)

This is a non-trivial constraint with interesting implications for Type I model building. For instance, if we take Type I on a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold of $T^2 \otimes T^2 \otimes T^2$ with discrete torsion, the Hodge numbers are $(h^{1,1}, h^{2,1}) = (3, 51)$ (note that this is the mirror Calabi-Yau of a usual $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold without discrete torsion whose Hodge numbers are $(h^{1,1}, h^{2,1}) = (51, 3)$), and we have $r = 2$ and $a = 4$. Thus, Type I compactification on this space is always anomalous (that is, there is no choice of the gauge bundle consistent with tadpole cancellation).

Thus, let us consider Type I compactifications with $r \geq 4$. In the dual F-theory picture the number of 3-branes required for anomaly cancellation is $2(r - 4)$. If this number is non-zero, we have a choice of where to place these 3-branes: (i) we can keep them in the bulk; from the Type I point of view these correspond to dynamical 5-branes (made of some
number of D5-branes); (ii) alternatively, we can “dissolve” them into the 7-branes; from the Type I viewpoint this corresponds to embedding a certain Spin(32)/Z₂ gauge bundle. The corresponding instantons are no longer point-like (at generic points). Thus, we see that we need to specify additional data, which is the choice of the gauge bundle. What we are interested in understanding here is which choices are perturbative from the orientifold viewpoint.

To answer this question recall that in K3 compactifications the total number of instantons must be 24. By these we actually mean the number $n_I$ of instantons embedded in the gauge bundle plus the number $n_5$ of 5-branes transverse to K3. In the case of the Voisin-Borcea compactifications this number 24 is reduced to 12 - the corresponding pairing takes place due to the additional $Z_2$ projection by the orbifold group $Y$. Thus, $\chi/24$ must be at least 12. This implies that the statement (20) can be made even stronger: Type I on Voisin-Borcea orbifolds can be tadpole free only for $r \geq 10$. (21)

In fact, since we have a $Z_2$ twist generated by $S$, we must include 32 D5-branes (filling $R^{3,1} \otimes C$ - see above). These correspond to 8 dynamical 5-branes: here two pairings take place - one due to the $\Omega$ orientifold projection, and the other one due to the $Z_2$ orbifold projection. Note that here we are making an assumption that the corresponding background is perturbative from the orientifold viewpoint - it is $a priori$ not obvious at all that in other cases we must include D5-branes. In fact, here we can give an example where this is not the case. Thus, consider Spin(32)/Z₂ heterotic on $T^4/Z_2$ with the standard embedding of the gauge bundle (which we discussed in section II). This theory is perturbative from the heterotic viewpoint, hence no 5-branes are present. That is, no D5-branes would be present on the dual Type I side either. This compactification, however, is non-perturbative from the orientifold viewpoint.

Thus, we must have at least 20 3-branes (12 from K3, and 8 from the D5-branes filling $R^{3,1} \otimes C$). This implies that, for the gauge bundle to be perturbative, we must have $r \geq 14$. In fact we will now show that there are only two choices of $r$ for which we can have perturbative gauge bundle: $r = 14$ and $r = 18$. First, suppose $r > 14$. Then we must have additional 5-branes whose number is $2(r - 14)$. However, the only other way we can obtain 5-branes perturbatively in the orientifold language is to have 8 dynamical 5-branes (arising from 32 D5-branes) filling $R^{3,1} \otimes C'$ (these are perpendicular to those filling $R^{3,1} \otimes C$). This then implies that we must have $r = 18$. On the other hand, if $r = 14$ we must make sure that there are no 5-branes filling $R^{3,1} \otimes C'$. This implies that in this case the K3 surface $W_2$ cannot be written as the quotient $W'/Z_2$. It then follows that there are no 5-branes transverse to K3 (that is, $W_2$) either (that is, $n_5 = 0$ and $n_I = 24$ in the corresponding K3 compactification).

The above discussion shows that the gauge bundle can be perturbative if and only if $r = 14$ or $r = 18$. Let us apply these constraints to the cases where the K3 surface $W_2$ is a toroidal orbifold, that is, $W_2 = T^4/Z_N$ with $N = 2, 3, 4, 6$. In the $N = 2$ case (without discrete torsion between $S$ and the generator $g$ of the K3 orbifold group) the Hodge numbers of the corresponding Voisin-Borcea orbifold are $(h^{1,1}, h^{2,1}) = (51, 3)$, and $r = 18, a = 4$. We then arrive at a consistent Type I compactification on the $Z_2 \otimes Z_2$ orbifold discussed in [3]. In the $N = 3$ case the Hodge numbers are $(h^{1,1}, h^{2,1}) = (35, 11)$, and we arrive at the non-perturbative orientifold corresponding to Type on $Z_6'$ orbifold recently constructed in [14]. For later convenience we give the massless spectrum of this model in Table II.
Next, consider the $N = 4$ case. Suppose there is no discrete torsion between $S$ and $g^2$ (where $g$ is the generator of the K3 orbifold group). Then the Hodge numbers are $(h^{1,1}, h^{2,1}) = (61, 1)$, and, therefore, $r = 20$ and $a = 2$. On the other hand, if we include discrete torsion between $S$ and $g^2$, we obtain $(h^{1,1}, h^{2,1}) = (3, 51)$ just as in the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ case with discrete torsion. Thus, in Type I on the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold we cannot cancel tadpoles if we choose the perturbative gauge bundle. (In fact, in the case with discrete torsion it is impossible to cancel tadpoles for any choice of the gauge bundle. In the case without discrete torsion tadpole cancellation might be possible at the expense of choosing a gauge bundle which is non-perturbative from the orientifold viewpoint.) This explains why the naive perturbative tadpole cancellation conditions were found in Ref. [9] to have no solution. Here the map to F-theory provides a simple geometric interpretation of this fact.

The above conclusion about perturbative inconsistency of Type I on $\mathbb{Z}_2 \otimes \mathbb{Z}_4$ can be used to understand other similar models. Note that the sectors of the orientifold labeled by $\Omega$, $\Omega S$, $\Omega g^k$, $k = 1, 2, 3$, and $\Omega Sg^2$ are perturbatively well defined (as the corresponding K3 orientifolds are well defined). This implies that the “troublesome” sectors are the $\Omega Sg$ and $\Omega Sg^3$ sectors. This is precisely the conclusion that was reached in Ref. [11]. Note that $Sg$ is the generator of an orbifold group $\mathbb{Z}_4$, which, to avoid confusion, we will refer to as $\mathbb{Z}_4'$. In particular, $T^6/\mathbb{Z}_4'$ is a Calabi-Yau three-fold with $SU(3)$ holonomy. Thus, we conclude that Type I on the $\mathbb{Z}_4'$ orbifold is not consistent for perturbative gauge bundles. In fact, we can further deduce that perturbative gauge bundles would lead to inconsistencies in the $\mathbb{Z}_4 \otimes \mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_8'$ orbifold models as well since all of these orbifold groups contain $\mathbb{Z}_4'$ as a subgroup. This is in accord with the discussion in Ref. [11], where it was also argued why in the $\mathbb{Z}_8$ model choosing the perturbative gauge bundle leads to similar inconsistencies as well. Thus, the discussion in Ref. [11] explains the results of Ref. [10] (where it was shown that the naive perturbative tadpole cancellation conditions in the $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_8'$ and $\mathbb{Z}_8''$ cases have no solution) using the geometric picture via the map to F-theory. Here the $\mathbb{Z}_2 \otimes \mathbb{Z}_4$ example serves as an illustration of the discussion in Ref. [11].

Finally, let us discuss the case $N = 6$ (without discrete torsion between $S$ and $g^2$, where $g$ is the generator of the K3 orbifold group). The Hodge numbers in this case are $(h^{1,1}, h^{2,1}) = (51, 3)$ just as in the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ case without discrete torsion. This implies that $r = 18$ and $a = 4$. This compactification, therefore, should be consistent for the appropriate perturbative gauge bundle. We will explicitly construct this model (via the corresponding non-perturbative orientifold) in the next subsection.

### C. The $\mathbb{Z}_2 \otimes \mathbb{Z}_6$ Model

In this subsection we are going to give the massless spectrum of the non-perturbative orientifold based on the $\mathbb{Z}_2 \otimes \mathbb{Z}_6$ orbifold compactification of Type I theory. This background has $\mathcal{N} = 1$ supersymmetry in four dimensions. Here we should emphasize that the choice of the gauge bundle in the model we discuss in this subsection is perturbative from the orientifold viewpoint. The non-perturbative sectors arise due to singularities in the Calabi-Yau itself as discussed in Introduction. That is, non-perturbative states in this theory arise along the lines of our discussion in section III for K3 compactifications of Type I.

Thus, consider Type I on the Voisin-Borcea orbifold $Y_3 = (T^2 \otimes T^2 \otimes T^2)/(\mathbb{Z}_2 \otimes \mathbb{Z}_6)$, where the generators $R_3$ and $g$ of $\mathbb{Z}_2$ respectively $\mathbb{Z}_6$ have the following action on the complex
coordinates $z_1, z_2, z_3$ parametrizing the three two-tori:

$$R_iz_j = -(-1)^{δ_{ij}}z_j, \quad i, j = 1, 2, 3,$$

$$θz_1 = z_1, \quad θz_2 = αz_2, \quad θz_3 = α^{-1}z_3,$$

where $R_1 ≡ g^3$, $R_2 ≡ R_1R_3$, $θ ≡ g^2$, and $α ≡ \exp(2πi/3)$. The Calabi-Yau three-fold $Y_3$ (which has $SU(3)$ holonomy) has the Hodge numbers $(h^{1,1}, h^{2,1}) = (51, 3)$, so that there are 54 chiral supermultiplets in the closed string sector.

In the open string sector we have 32 D9-branes, and also three sets of D5-branes with 32 D5-branes in each set. Thus, the world-volumes of the D5$_i$-branes are four non-compact space-time dimensions plus the two-torus parametrized by the complex coordinate $z_i$. For the sake of definiteness, in the following we will concentrate on the brane configuration where all D5$_i$-branes are placed on top of each other at the orientifold 5$_i$-plane located at the origin in the transverse dimensions (that is, $z_j = 0, j \neq i$).

Up to equivalent representations the Chan-Paton matrices are given by:

$$γ_{θ, 9} = \text{diag}(W \otimes I_2, I_8),$$

$$γ_{R_i, 9} = iσ_i \otimes I_8.$$

Here $W ≡ \text{diag}(α, α, α^{-1}, α^{-1})$, and $σ_i$ are the usual $2 \times 2$ Pauli matrices. (The action on the D5$_i$-branes is similar.) The perturbative (from the orientifold viewpoint) massless open string spectrum of this model can be obtained using the standard orientifold techniques, and is given in Table III. Thus, the $Ω$-twisted sector corresponds to the untwisted 99 plus 5$_1$5 open string sectors, while $ΩR_i$ twisted sector corresponds to the untwisted 95$_i$ plus 5$_j$5$_k$ ($j \neq k \neq i$) open string sectors.

The non-perturbative sectors in this orientifold are the following. The $Ωθ$- and $Ωθ^2$-twisted sectors correspond to the twisted 99 plus 5$_1$5 open string sectors, while the $ΩR_iθ$- and $ΩR_iθ^2$-twisted sectors correspond to the twisted 95$_i$ plus 5$_j$5$_k$ ($j \neq k \neq i$) open string sectors. The twisted 95$_1$ and 95$_2, 3$ sectors are straightforward to work out starting from the twisted 95 sector in the six dimensional $Z_6$ model in Table I respectively the twisted 95 sector in the four dimensional $Z_6'$ model in Table II and projecting onto the corresponding $Z_2$ invariant states. The twisted 5$_2$5$_3$ sector is the same as the twisted 95$_1$ sector (with the obvious substitutions of the gauge quantum numbers). The twisted 5$_1$5$_2, 3$ sectors are the same as the twisted 95$_3, 2$ sectors except that the multiplicity of the states in the latter is 1, while it is 3 in the former. This is due to the fact that the the corresponding 5-branes only feel the local geometry in the vicinity of the corresponding fixed points at which they are placed. (This is in complete parallel with the corresponding discussion for the six dimensional cases in [14] and section III.) As to the twisted 99, 5$_2$5$_2$ and 5$_3$5$_3$ sectors, they can be worked out by starting from the twisted 99 and 55 sectors in the $Z_6'$ model in Table II and projecting onto the corresponding $Z_2$ invariant states. Similarly, the twisted 5$_1$5$_1$ sector (as well as the twisted 99 sector) can be worked out by starting from the twisted 55 sector in the $Z_6$ model in Table I and projecting onto the corresponding $Z_2$ invariant states. The complete

\[5\] Here we note that the perturbative spectrum of this model was discussed in [3].
massless spectrum of the $\mathbb{Z}_2 \otimes \mathbb{Z}_6$ model including both perturbative and non-perturbative (from the orientifold viewpoint) states is given in Table III. Note that all non-Abelian gauge anomalies cancel in this model.

V. TYPE I ON GENERALIZED VOISIN-BORCEA ORBIFOLDS

In the previous section we considered Type I compactifications on Voisin-Borcea orbifolds of the form $(T^2 \otimes \text{K3})/\mathbb{Z}_2$. Here we are going to discuss Type I compactifications on generalized Voisin-Borcea orbifolds of the form

$$\mathcal{Y}_3 = (T^2 \otimes \mathcal{W}_2)/\mathbb{Z}_N,$$  \hspace{1cm} (26)

where $\mathcal{W}_2$ is a K3 surface, and the generator $\eta$ of $\mathbb{Z}_N$ acts as $\eta z_0 = \omega z_0$ on the fibre $T^2$ (parametrized by the complex coordinate $z_0$), and as $\eta \Omega_2 = \omega^{-1} \Omega_2$ on $\mathcal{W}_2$ (parametrized by the complex coordinates $z_1, z_2$). Here $\omega \equiv \exp(2\pi i/N)$, and $\Omega_2 \equiv dz_1 \wedge dz_2$ is the holomorphic two-form on $\mathcal{W}_2$. Note that the holomorphic three-form $\Omega_3 \equiv dz_0 \wedge dz_1 \wedge dz_2$ on $\mathcal{Y}_3$ is invariant under the action of $\eta$, which implies that $\mathcal{Y}_3$ is a Calabi-Yau three-fold with $SU(3)$ holonomy (which is actually elliptically fibered over the base $\mathcal{W}_2/\mathbb{Z}_N$) provided that the action of the orbifold group $\mathbb{Z}_N$ is a symmetry of $T^2 \otimes \mathcal{W}_2$. The latter requirement implies that $N$ is restricted to the values $N = 2, 3, 4, 6$ so that the action of $\mathbb{Z}_N$ on $T^2$ is crystallographic. Note that in the $N = 2$ case we have Voisin-Borcea orbifolds discussed in the previous section. In this section we will be interested in the cases $N = 3, 4, 6$.

Let us start with the $N = 4$ case. Suppose $\mathcal{W}_2$ is an orbifold K3. Then the three-fold $\mathcal{Y}_3$ can be either the $\mathbb{Z}_4 \otimes \mathbb{Z}_2$ or $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ orbifold of $T^2 \otimes T^4$. We have already discussed these cases in the previous section where we saw that for perturbative gauge bundles tadpoles cannot be canceled in these models.

Next, let us consider the $N = 6$ case with $\mathcal{W}_2$ an orbifold K3. Then $\mathcal{Y}_3$ can be the $\mathbb{Z}_6 \otimes \mathbb{Z}_2$, $\mathbb{Z}_6 \otimes \mathbb{Z}_3$, $\mathbb{Z}_6 \otimes \mathbb{Z}_6$ or $\mathbb{Z}_6^* \otimes \mathbb{Z}_2$ orbifold of $T^2 \otimes T^4$. The first case was explicitly constructed in the previous section. We will discuss the last case, which was explicitly constructed in [13], in a moment. The other two cases are straightforward to work out: the perturbative part of the spectrum is obtained using the standard orientifold techniques [8], whereas the non-perturbative twisted open string sector states can be easily worked out by starting with the corresponding twisted states in the $\mathbb{Z}_6$ model in Table I, the $\mathbb{Z}_6'$ model in Table II, and the $\mathbb{Z}_2 \otimes \mathbb{Z}_6$ model in Table III, and performing the appropriate ($\mathbb{Z}_2$ and/or $\mathbb{Z}_3$) orbifold projections. Here the following remark is in order. Note that in the $\mathbb{Z}_6 \otimes \mathbb{Z}_3$ and $\mathbb{Z}_6 \otimes \mathbb{Z}_6$ models a priori there are two types of twisted open string sectors. First, we have open string sectors corresponding to the orientifold group elements labeled by $\Omega \rho$, where $\rho$ is an orbifold group element such that for some choice of $k \in 2\mathbb{Z}$ the set of points in $T^2 \otimes \mathcal{W}_2$ fixed under $\rho^k$ is of real dimension 2. These are precisely the twisted open string sectors which can be deduced via the appropriate orbifold projections of the “parent” models mentioned above. This is due to the fact that such twisted open string sectors are either projections of those in the corresponding $T^2 \otimes \text{K3}$ compactifications, or of twisted 59 open string sectors which can be read off by appropriately projecting that in the $\mathbb{Z}_6'$ model in Table II. The second type of twisted open string sectors correspond to the orientifold group elements labeled by $\Omega \rho$ such that for some choice of $k \in 2\mathbb{Z}$ the set of points fixed under
\( \rho^k \) is zero dimensional. Such twisted open string sectors were argued in \([11]\) to give rise to states that become heavy and decouple upon appropriately blowing up the corresponding singularities in the compactification space \([7,8,11]\). We will discuss this point in more detail shortly. Here, however, an important observation is that the second type of twisted open string sectors do not give rise to massless states once we consider blown-up orbifolds.

For the sake of brevity here we will not give the spectra of the \( \mathbb{Z}_6 \otimes \mathbb{Z}_3 \) and \( \mathbb{Z}_6 \otimes \mathbb{Z}_6 \) models. Let us mention, however, that the gauge groups in these two cases are \( [U(2)^6 \otimes U(4)]_{99} \otimes [U(2)^6 \otimes U(4)]_{55} \) respectively \( [U(2)^3 \otimes Sp(4)]_{99} \otimes \bigotimes_{i=1}^{2} [U(2)^3 \otimes Sp(4)]_{5,5} \). Thus, the \( \mathbb{Z}_6 \otimes \mathbb{Z}_6 \) model is non-chiral if we ignore the \( U(1) \) charges. The \( \mathbb{Z}_6 \otimes \mathbb{Z}_3 \) model is chiral as in the perturbative open string sectors there are states transforming under \( SU(3) \times SU(3) \) and \( SU(2) \times SU(2) \) sectors. This turns out to be the case in the \( \mathbb{Z}_6 \) model as well. The fact that non-perturbative states to decouple as well.

Let us point out that the \( \mathbb{Z}_6^* \otimes \mathbb{Z}_2 \) model was explicitly constructed in \([13]\). The discussion of twisted open string sectors in this model is completely analogous to that in the \( \mathbb{Z}_6^* \otimes \mathbb{Z}_2 \) case. Let us therefore consider the generalized Voisin-Borcea orbifolds with \( N = 3 \) (and the \( \mathbb{Z}_6 \) case is a particular example of this type).

For \( N = 3 \) we have the following possibilities: \( \mathbb{Z}_3^* \otimes \mathbb{Z}_2 \approx \mathbb{Z}_3^*, \mathbb{Z}_3^* \otimes \mathbb{Z}_4 \approx \mathbb{Z}_{12}, \mathbb{Z}_3 \otimes \mathbb{Z}_3, \) and \( \mathbb{Z}_3 \otimes \mathbb{Z}_6 \). The last case we have already considered. The \( \mathbb{Z}_3 \otimes \mathbb{Z}_3 \) case was explicitly constructed in \([8]\). Just as in the \( \mathbb{Z}_6 \otimes \mathbb{Z}_3 \) and \( \mathbb{Z}_6 \otimes \mathbb{Z}_6 \) cases, here we also have twisted open string sectors of two types. As was shown in \([8]\), in this model twisted states of both types decouple upon appropriately blowing up the orbifold singularities, so that the remaining massless spectrum coincides with that obtained in the perturbative orientifold approach. This turns out not to be the case in the \( \mathbb{Z}_{12} \) model which we will discuss in detail in a moment. However, some of the sectors in the \( \mathbb{Z}_{12} \) model are similar to those in the \( \mathbb{Z}_6^* \) model, which was explicitly constructed in \([8]\), so let us discuss the latter in a bit more detail.

Thus, consider Type I on the generalized Voisin-Borcea orbifold \( (T^2 \otimes T^2 \otimes T^2)/\mathbb{Z}_6^* \), where the action of the generator \( g \) of \( Z_6^* \) on the complex coordinates \( z_1, z_2, z_3 \) parametrizing the three two-tori is given by:

\[
gz_1 = \alpha z_1 , \quad gz_2,3 = -\alpha z_2,3 ,
\]

(27)

where \( \alpha \equiv \exp(2\pi i/3) \). In this case we have 32 D9-branes and 32 D5-branes, the latter wrapping the first \( T^2 \). The non-perturbative open string sectors correspond to the orientifold group elements labeled by \( \Omega g \) plus \( \Omega g^5 \) and \( \Omega g^2 \) plus \( \Omega g^4 \). Note that the singularities in the \( \mathbb{Z}_6^* \) twisted sectors are a subset of singularities in the \( \mathbb{Z}_3^* \) twisted sectors. This implies that if the blow-ups in the \( \mathbb{Z}_3^* \) model lead to the decoupling of non-perturbative open string sector states, the same should hold in the \( \mathbb{Z}_6^* \) model as well. The fact that non-perturbative twisted open string sector states indeed decouple in the blown-up \( \mathbb{Z}_6^* \) model was shown in \([8]\) using Type-I heterotic duality. Thus, in the corresponding \( \mathbb{Z}_6^* \) model we expect all non-perturbative states to decouple as well.
A. The $\mathbb{Z}_{12}$ Model: A Non-perturbative Anomaly

We are now ready to discuss the $\mathbb{Z}_{12}$ model. The generator $g$ of $\mathbb{Z}_{12}$ has the following action on the three complex coordinates in the compact directions:

$$gz_1 = \alpha z_1, \quad gz_2 = i\alpha z_2, \quad gz_3 = -i\alpha z_3.$$  \hspace{1cm} (28)

Note that $\mathbb{Z}_{12} \supset \mathbb{Z}^*_6$. The corresponding Calabi-Yau four-fold has the Hodge numbers $(h^{1,1}, h^{2,1}) = (29, 5)$, so that the closed string spectrum contains 34 chiral supermultiplets. Here we note that the Hodge numbers in the $\mathbb{Z}^*_6$ case are also $(h^{1,1}, h^{2,1}) = (29, 5)$. That is, in both the $\mathbb{Z}^*_6$ and $\mathbb{Z}_{12}$ models the compactification space is the same, but the gauge bundles are different.

In the open string sector we have 32 D9-branes as well as 32 D5-branes (in the following we will focus on the brane configuration where all D5-branes are sitting on top of each other at the orientifold 5-plane located at $z_2 = z_3 = 0$). The perturbative open string sector of this model can be obtained using the standard orientifold techniques (once the action of the orbifold group on the Chan-Paton factors is specified - see below). The non-perturbative sectors corresponding to the orientifold group elements labeled by $\Omega g$ plus $\Omega g^3$, $\Omega g^4$ plus $\Omega g^8$, and $\Omega g^5$ plus $\Omega g^7$ do not give rise to massless states (after the appropriate blow-ups) for the same reasons as in the $\mathbb{Z}^*_6$ case. However, the $\mathbb{Z}_4$ twisted open string sectors corresponding to the orientifold group elements labeled by $\Omega g^3$ plus $\Omega g^9$ do give rise to non-perturbative massless states. These states can be obtained by starting from the $\mathbb{Z}_4$ model in Table I and projecting onto the $\mathbb{Z}^*_3$ invariant states. There is, however, a subtlety in this projection, so let us discuss the action of the orbifold group on the Chan-Paton factors in a bit more detail.

The point here is that to have a perturbative (from the orientifold viewpoint) gauge bundle, we must make sure that all the twisted tadpole cancellation conditions (derived in the perturbative orientifold approach) cancel. In particular, the twisted tadpole cancellation condition for the $\mathbb{Z}_{12}$, $\mathbb{Z}^*_6$, $\mathbb{Z}_4$ and $\mathbb{Z}_2$ twisted Chan-Paton matrices (for both D9- and D5-branes) read $[8,10]$:

$$\text{Tr}(\gamma_g k) = 0, \quad k = 1, 2, 3, 5, 6, 7, 9, 10, 11.$$  \hspace{1cm} (29)

On the other hand, the $\mathbb{Z}_3^*$ twisted Chan-Paton matrices are fixed by the corresponding tadpole cancellation conditions as in the $\mathbb{Z}_3^*$ model of $[8]$ (see the next subsection for details). The only way to satisfy all of these twisted tadpole cancellation conditions is to consider $32 \times 32$ Chan-Paton matrices which (up to equivalent representations) are given by

$$\gamma_{g^4} = \text{diag}(\alpha I_{12}, \alpha^{-1} I_{12}, I_8), \quad \gamma_{g^3} = \text{diag}(U \otimes I_3, U \otimes I_3, U \otimes I_2).$$  \hspace{1cm} (30)

where $U \equiv \text{diag}(\beta, -\beta, \beta^{-1}, -\beta^{-1})$, and $\beta \equiv \exp(\pi i/4)$. Note that the same would not be possible if we considered $16 \times 16$ matrices. That is, the tadpole cancellation conditions require that both 16 D-branes and their orientifold images participate in canceling the tadpoles at the same time, while separate cancellations of the corresponding tadpoles for the 16 D-branes and their orientifold images cannot be achieved. This, in particular, implies that (since the 16 D-branes and their orientifold images are mapped to each other by the orientifold
projection) the Chan-Paton matrix $\gamma_\Omega$ interchanges the twisted Chan-Paton matrices with their conjugates: $\gamma_\Omega : \gamma_{g^k} \to \gamma_{g^{-k}}$. Such an action is perfectly well defined in the six dimensional $\Omega J'$ orientifolds of $\mathbb{Z}_3$ as in those cases the orientifold projection $\Omega J'$ maps the $g$-twisted sector to its conjugate $g^{-1}$-twisted sector. However, as we have already pointed out in Introduction, in four dimensions the orientifold projection is $\Omega$ which maps the $g$-twisted sector to itself (here we assume that the appropriate blow-ups have been performed). It is then a priori far from being obvious whether it is consistent to have $\gamma_\Omega : \gamma_{g^k} \to \gamma_{g^{-k}}$ while $\Omega : g^k \to g^k$. In fact, we are going to argue that such an action leads to inconsistencies. Here we should point out that such an inconsistency cannot be seen perturbatively in the orientifold picture if we just focus on the gauge (that is, the open string) sector of the theory. Indeed, the perturbative open string sector of this model by itself is perfectly sensible (see below). However, if one couples open and closed strings, then one expects an inconsistency to show up - thus, the twisted closed string sector states (which couple to the corresponding D-branes) feel the orientifold projection in a way which is not compatible with the action of the orientifold projection on the Chan-Paton factors. That is, the orientifold group actions on the closed and open string degrees of freedom are not compatible. It is, however, difficult to see this open-closed coupling inconsistency directly as the $\Omega$ projection in the closed string sector makes sense only after the appropriate blow-ups have been performed, so we are no longer at the orbifold point in the corresponding Calabi-Yau moduli space, which makes world-sheet computations rather involved (and practically useless, at least for our purposes here).

There are, however, other ways to see this inconsistency. First, we can try to understand it using Type I-heterotic duality. Here we should emphasize that quantifying the following statements in this context is difficult as the heterotic dual of the $\mathbb{Z}_{12}$ model would be non-perturbative - we have 5-branes in this case. Nonetheless, the following somewhat hand-waving argument might still be useful. Thus, on the heterotic side the embedding of the gauge bundle (that is, of the orbifold action on the gauge degrees of freedom) can be understood in terms of the appropriate $\mathbb{Z}_{12}$ valued Spin(32)/$\mathbb{Z}_2$ lattice shifts. (These shifts are $\mathbb{Z}_{12}$ valued w.r.t. the Spin(32)/$\mathbb{Z}_2$ lattice, but are $\mathbb{Z}_{24}$ valued w.r.t. the $SO(32)$ lattice.) The $\Omega$ projection on the heterotic side can be thought of as pairing 32 real Majorana fermions (in the real fermion representation of the Spin(32)/$\mathbb{Z}_2$ degrees of freedom) into 16 complex world-sheet fermions. The $\mathbb{Z}_{12}$ valued Spin(32)/$\mathbb{Z}_2$ lattice shifts in this language translate into $U(1)^{16}$ phase rotations of the complex fermions. For these rotations to be consistent, we must make sure that they can be written in the complex $16 \times 16$ basis. If this cannot be achieved, that is, if we can only do this in the $32 \times 32$ bases, then the model is inconsistent. In particular, the scattering amplitudes would be inconsistent. In the following subsection we will relate this argument to the analogous statement in the dual F-theory picture, where the above inconsistency can be seen in a geometric way. However, for now let us take the above gauge bundle and discuss the corresponding massless spectrum arising in the $\mathbb{Z}_{12}$ model. As we will see in a moment, the non-perturbative spectrum turns

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$^6$On the Type I side this would amount to an inconsistency arising in the world-sheet theory of a probe D-string placed in this background. I would like to thank E. Gimon for a discussion in a related but somewhat different context.
out to be anomalous.

Once the twisted Chan-Paton matrices are fixed as above, the perturbative (from the orientifold viewpoint) states in the $\mathbb{Z}_{12}$ model are straightforward to work out. The resulting perturbative massless open string sector is given\(^7\) in Table IV. Note that the non-Abelian gauge anomalies cancel in the untwisted open string sector.

Next, let us discuss the non-perturbative open string sectors, that is, the $\mathbb{Z}_4$ twisted open string sectors. These are straightforward to work out starting from the $\mathbb{Z}_4$ twisted open string sectors in the $\mathbb{Z}_{12}$ model in Table I and projecting onto the $\mathbb{Z}_3^*$ invariant states. Note that the resulting twisted 99 and 55 open string sectors are apparently T-dual for the same reason as in the $\mathbb{Z}_4$ model in Table I - the $\mathbb{Z}_4$ fixed points are invariant under the $\mathbb{Z}_2$ reflections involved in the T-duality transformation (recall that the other twisted sectors such as $\mathbb{Z}_3^*$, $\mathbb{Z}_6^*$ and $\mathbb{Z}_{12}$ do not give rise to non-perturbative states once the appropriate blow-ups are performed). In fact, it is important here that the $\mathbb{Z}_4$ sector fixed point located at the origin contains no twisted $B$-field, whereas the other three fixed points have half-integer valued $B$-field stuck inside of the corresponding $\mathbb{P}^1$’s (see section III for details). In particular, this configuration possesses the required $\mathbb{Z}_3$ symmetry.

Next, we list the resulting non-perturbative states arising in the $\mathbb{Z}_4$ twisted open string sectors (these states are $\mathcal{N} = 1$ chiral supermultiplets - see Table IV for notations):

\[
(\bar{3}, 1, 1, 1, 1, 1)_{99(T)/55(T)}; \\
(1, 1, \bar{3}, 1, 2, 1)_{99(T)/55(T)}; \\
(1, 1, \bar{3}, 1, 1, 1)_{99(T)/55(T)}; \\
(3, 1, 1, 1, \bar{3}, 1)_{99(T)/55(T)}; \\
(1, 3, 1, 1, 1, 1)_{99(T)/55(T)}; \\
(1, 1, 1, \bar{3}, 1, 2)_{99(T)/55(T)}; \\
(1, 1, 1, 3, 1, 1)_{99(T)/55(T)}; \\
(1, \bar{3}, 1, 1, 1, 2)_{99(T)/55(T)};
\]

where the first four of these states come from the hypermultiplet transforming in (28, 1), and the last four come from the hypermultiplet transforming in (1, 28) upon $\mathbb{Z}_3^*$ projecting the $[U(8) \otimes U(8)]_{99/55}$ quantum numbers in the $\mathbb{Z}_4$ model in Table I. Note that the above twisted spectrum suffers from $SU(3)^4$ non-Abelian gauge anomalies. That is, non-perturbatively we see an inconsistency in this model. In the next subsection we will give additional arguments clarifying the origins of this non-perturbative anomaly using the map between Type IIB orientifolds and F-theory.

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\(^7\)Here we note that the perturbative open string spectrum of the $\mathbb{Z}_{12}$ model was discussed in [10]. The untwisted open string spectrum given in Table IV, however, differs from that in [10]. In particular, note that the spectrum should be invariant under permuting the two $U(2)$ subgroups (accompanied by an appropriate interchange/conjugation of the corresponding $U(3)$ subgroups) in, say, the 99 open string sector, which follows from the action of the orbifold group on the Chan-Paton factors. This is the case for the spectrum in Table IV, but not for the spectrum given in [10].
B. Map to F-theory

In the previous section we gave the map between Type I on Voisin-Borcea orbifolds and F-theory. In this subsection we would like to do the same in the case of generalized Voisin-Borcea orbifolds with \( N = 3, 4, 6 \). There is a subtlety in this map (as was pointed out in [11]) in the \( N = 3, 6 \) cases, and here we will give make this map precise.

Naively, we can start from \( \mathcal{Y}_3 = (T^2 \otimes \mathcal{W}_2)/\mathbb{Z}_2 \), T-dualize along the fibre \( T^2 \), and map the resulting model to F-theory via \([29]\) to arrive at the F-theory compactification on the following Calabi-Yau four-fold:

\[
\mathcal{X}_4 = (\bar{T}^2 \otimes T^2 \otimes \mathcal{W}_2)/(\mathbb{Z}_2 \otimes \mathbb{Z}_N),
\]

where the action of the generator \( \bar{S} \) of \( \mathbb{Z}_2 \) is given by \( \bar{S}z_0 = -\bar{z}_0, \bar{S}z_0 = -z_0 \), and it acts trivially on \( \mathcal{W}_2 \); the action of the generator \( \eta \) of \( \mathbb{Z}_N \) is given by \( \eta\bar{z}_0 = \bar{z}_0, \eta z_0 = \omega z_0, \eta \Omega_2 = \omega^{-1}\Omega_2 \). Here \( \bar{z}_0 \) and \( z_0 \) parametrize \( T^2 \) respectively \( T^2 \), \( \omega \equiv \exp(2\pi i/N) \), and \( \Omega_2 \) is the holomorphic two-form on \( \mathcal{W}_2 \) (which is a K3 surface). Note, however, that the \( \mathbb{Z}_N \) singularities in the \( \mathcal{Y}_3 \) fibre \( T^2 \) are invariant under \( \bar{S} \) only in the \( N = 2, 4 \) cases. Thus, there is no subtlety in the above map in these cases. In the \( N = 3, 6 \) cases, however, the \( \mathbb{Z}_N \) singularities are not invariant under \( \bar{S} \), so that we must be a bit more careful. In fact, the issue in the \( N = 6 \) case is the same as in the \( N = 3 \) case, so we can focus on the latter. Moreover, instead of considering the generalized Voisin-Borcea orbifolds \( \mathcal{Y}_3 \), we will discuss this point for a simpler example, namely, the \( \mathbb{Z}_3^3 \) orbifold. The discussion for the other cases is completely analogous. Thus, let us start from Type I on \( \mathcal{Y}_3 = (T^2 \otimes T^2 \otimes T^2)/\mathbb{Z}_3^3 \), where the generator \( g \) of \( \mathbb{Z}_3^3 \) acts on the complex coordinates \( z_1, z_2, z_3 \) parametrizing the three two-tori as \( g z_i = \alpha z_i \) \((i = 1, 2, 3, \alpha \equiv \exp(2\pi i/3))\). The Hodge numbers of this Calabi-Yau three-fold are given by \((h^{1,1}, h^{2,1}) = (36, 0)\).

In this Type I compactification, which was studied in detail in [1,2,3,22], we have 32 D9-branes and no D5-branes. The twisted tadpole cancellation conditions read \( \text{Tr}(\gamma_g) = -2 \) (in the \( 16 \times 16 \) basis). This implies that (up to equivalent representations) we have

\[
\gamma_g = \text{diag}(\alpha I_6, \alpha^{-1}I_6, I_4),
\]

where \( \alpha \equiv \exp(2\pi i/3) \). The gauge group of this model is \( U(12) \otimes SO(8) \), and the (untwisted) 99 open string sector contains the following chiral supermultiplets: \( 3 \times (66, 1) \) plus \( 3 \times (\mathbb{T}_2, 8_v) \).

Here we would like to ask what is the T-dual (where we T-dualize along both directions in the first \( T^2 \)) of this Type I background. That is, we would like to understand the dual Type IIB orientifold with D7-branes. In fact, to T-dualize we must first turn on Wilson lines which break the original \( SO(32) \) gauge group (before \( \mathbb{Z}_3^* \) orbifolding) to \( SO(8)^4 \). Indeed, the corresponding T-dual of Type I on \( T^2 \otimes T^2 \otimes T^2 \) is the \( \Omega \bar{J}(-1)^F_L \) orientifold of Type IIB on \( T^2 \otimes T^2 \otimes T^2 \), where \( Jz_1 = -z_1 \), and \( Jz_{2,3} = z_{2,3} \). Note that \( T^2/J \) has four fixed points at which are located the corresponding orientifold 7-planes. We must place 8 D7-branes at each of these orientifold 7-planes to properly cancel tadpoles (this statement is precise in the perturbative orientifold context). Thus, the gauge group is \( SO(8)^4 \). In fact, let us see what are the locations of these fixed points. In order for \( T^2 \)'s to have \( \mathbb{Z}_3^* \) symmetry (by which we are ultimately going to mod out), they must be hexagonal. Let us focus on the first \( T^2 \) whose
volume we will denote by $v_1$. Then the metric on $T^2$ is given by $g_{ab} = \sqrt{v_1/3} e_a \cdot e_b$, $a, b = 1, 2$, where $e_a$ are the vectors spanning the $SU(3)$ lattice. That is, $e_1^2 = e_2^2 = -2e_1 \cdot e_2 = 2$. The $\tilde{J}$ fixed points are located at $\xi_0 = 0$, $\xi_a = e_a/2$, $a = 1, 2, 3$, where $e_3 \equiv -(e_1 + e_2)$.

The T-dual Type I configuration of the above Type IIB orientifold is given by the following. Start from Type I with 32 D9-branes on $T^2 \otimes T^2 \otimes T^2$. Turn on two Wilson lines - one on the $a$-cycle and the other one on the $b$-cycle of the first $T^2$ - such that they break the $SO(32)$ gauge symmetry down to $SO(8)^4$. This is precisely the configuration we are looking for. Note that this setup is symmetric under $Z_3$ valued permutations of any three of the four $SO(8)$ subgroups accompanied by $2\pi/3$ rotations of the first $T^2$. Under these rotations the first Wilson line maps to the second Wilson line (as the $a$-cycle maps to the $b$-cycle on $T^2$). Therefore, we can mod out by the $Z_3^*$ symmetry. The corresponding gauge bundle is given by the following Chan-Paton matrix:

$$\gamma_g = \text{diag}(P_3 \otimes I_4, W),$$

where $P_3$ is a $3 \times 3$ matrix of cyclic permutations, and $W \equiv \text{diag}(\alpha I_2, \alpha^{-1} I_2)$. Note that we still have $\text{Tr}(\gamma_g) = -2$, so that the twisted tadpole cancellation conditions are satisfied. The resulting gauge group is $SO(8) \otimes U(4)$, and the matter consists of chiral supermultiplets in $3 \times (28, 1)$ plus $3 \times (1, 6)$. The $SO(8)$ factor arises as the diagonal subgroup of the original $SO(8)^3$ subgroup on which the permutation matrix $P_3$ is acting. The $U(4)$ factor is the corresponding subgroup of the fourth $SO(8)$.

The above $SO(8) \otimes U(4)$ model actually is on the same moduli as the original $U(12) \otimes SO(8)$ model \[7]. The former point in the moduli space is T-dual of the corresponding Type IIB orientifold with D7-branes, and the latter can now be readily mapped to F-theory. The corresponding compactification space is given by the four-fold $\mathcal{X}_4 = (T^2 \otimes T^2 \otimes T^2)/(Z_2 \otimes Z_3^*)$, which has $SU(4)$ holonomy, and is an elliptic fibration of $T^2$ over the base $(T^2 \otimes T^2 \otimes T^2)/(Z_2 \otimes Z_3^*)$.

We can generalize the above discussion to other examples of this type. In particular, Type I on $\mathcal{Y}_3 = (T^2 \otimes \mathcal{W}_2)/\mathbb{Z}_N$ (where $\mathcal{W}_2$ is either a K3 surface or $T^4$) with the appropriately turned on Wilson lines is dual to F-theory on the elliptic four-fold \[32]. The Euler number of a four-fold is expressed in terms of the corresponding Hodge numbers via:

$$\chi = 4 + 2h^{1,1} - 4h^{2,1} + 2h^{3,1} + h^{2,2}. \quad (35)$$

However, for elliptic four-folds we have

$$h^{2,2} = 44 + 4h^{1,1} - 2h^{2,1} + 4h^{3,1}, \quad (36)$$

so it suffices to give the Hodge numbers $(h^{1,1}, h^{2,1}, h^{3,1})$ to specify such a four-fold. In particular, we have

$$\chi/6 = 8 + h^{1,1} - h^{2,1} + h^{3,1}. \quad (37)$$

For illustrative purposes, here we give the Hodge numbers of the four-folds corresponding to Type I on the $Z_3^*, Z_6^*, Z_{12}^*$, and $Z_6^* \otimes Z_2$ orbifolds (other cases are straightforward to work out):
\[ Z_3^* : \quad (h_1^{1,1}, h_2^{2,1}, h_3^{3,1}) = (32, 21, 9), \quad \chi/24 = 7, \]
\[ Z_6^*, Z_{12} : \quad (h_1^{1,1}, h_2^{2,1}, h_3^{3,1}) = (32, 6, 14), \quad \chi/24 = 12, \]
\[ Z_6^* \otimes Z_2 : \quad (h_1^{1,1}, h_2^{2,1}, h_3^{3,1}) = (60, 1, 1), \quad \chi/24 = 17. \]

Note that in the \( Z_6^* \) and \( Z_{12} \) cases we have the same four-fold. This is not surprising as the corresponding three-folds were also the same.

Finally, we would like to apply the above map to F-theory to better understand the non-perturbative inconsistency we have encountered in the \( Z_{12} \) case with the particular gauge bundle discussed in the previous subsection. The point here is that in the F-theory language the action of the orientifold projection \( \Omega \) is geometrized - it is mapped to the action of \( \tilde{J} \). Now, in the case of the four-fold compactifications of F-theory we must specify the action of \( \tilde{J} \) not only on the four complex coordinates, but also on the gauge bundle. More precisely, in, say, the \( Z_{12} \) case we have to specify the action of the orbifold group \( Z_{12} \) on various gauge degrees of freedom, that is, we have to specify the gauge bundle. In the orientifold language this is described by the Chan-Paton matrices, while in the heterotic language this is done in terms of the \( \text{Spin}(32)/\mathbb{Z}_2 \) lattice shifts. In the F-theory context we must also specify the orbifold action of the corresponding gauge degrees of freedom. Thus, in the \( Z_3^* \) example discussed above the \( Z_3^* \) orbifold acts geometrically on the three \( SO(8) \)'s (corresponding to \( D_4 \) singularities in the fibre \( \tilde{T}^2 \)) by permuting them - this is precisely what happens to the three \( \tilde{J} \) fixed points \( \xi_a = e_a/2, \quad a = 1, 2, 3 \), under the action of \( Z_3^* \). However, the action of \( Z_3^* \) on the fourth \( SO(8) \) (corresponding to the fixed point \( \xi_0 = 0 \)) is no longer purely geometric - it breaks \( SO(8) \) down to \( U(4) \), which implies that there is non-trivial gauge bundle associated with the embedding of the \( Z_3^* \) orbifold action on the corresponding gauge degrees of freedom. The situation here is similar to what happens in the heterotic compactifications - we must embed some number of instantons which break the gauge group. Now, the action of \( \tilde{J} \) on the \( Z_{12} \) twisted sectors as well as the corresponding gauge bundles must be one and the same representation of the orbifold group. In the gauge bundle we discussed in the previous subsection the orbifold group (in the four-fold context) is \( Z_2 \otimes Z_{12} \), whereas its embedding in terms of the gauge bundle would be the non-Abelian dihedral group \( D_{12} \). These two actions are clearly incompatible, hence the non-perturbative inconsistency (which, in particular, manifests itself via a non-perturbative anomaly) in the \( Z_{12} \) model.

**VI. EXTENSIONS**

In the previous sections we have discussed various non-perturbative orientifolds corresponding to Type I compactifications on K3 and Calabi-Yau three-folds. Our discussion so far has been confined to the cases with trivial NS-NS antisymmetric \( B \)-field. It would be interesting to understand non-perturbative orientifolds with non-zero \( B \)-field, and one example of such a compactification was recently discussed in [14]. We will not consider these models in detail in this paper as we are planning to discuss such compactifications elsewhere [33]. However, here we would like to point out an additional set of models arising in the perturbative K3 orientifolds (which complement those discussed in [1]) as their generalizations to non-perturbative orientifolds might lead to interesting new models in six and four dimensions.
The key point here is the following. Consider the Ω orientifold of Type IIB on $T^4/\mathbb{Z}_2$ without the $B$-field. This is the model of $[1,2]$. Note that the action of the orbifold group on the Chan-Paton factors is given by the Chan-Paton matrix

$$\gamma_g = \text{diag}(iI_8, -iI_8),$$

where $g$ is the generator of $\mathbb{Z}_2$. This corresponds to the gauge bundle without vector structure (see section II for details). Here we can ask whether we can take the Chan-Paton matrix to be given by

$$\gamma_g = \text{diag}(I_8, -I_8),$$

which would also satisfy the twisted tadpole cancellation conditions. In this case we would have the gauge bundle with vector structure. However, this choice can be seen to be inconsistent. The point is that the gauge group in this case is $[Sp(16) \otimes Sp(16)]_{99} \otimes [Sp(16) \otimes Sp(16)]_{55}$ (in our conventions $Sp(2N)$ has rank $N$), and the matter consists of hypermultiplets in

$$(16, 16; 1, 1)_{99}, \quad (1, 1; 16, 16)_{55},$$

$$\frac{1}{2}(16, 1; 16, 1)_{59}, \quad \frac{1}{2}(1, 16; 1, 16)_{59}.$$

Note that in the 59 sectors we have half-hypermultiplets in real representations, which is inconsistent, albeit the gravitational anomaly cancellation condition is satisfied in this case (note that the number of extra tensor multiplets is zero).

Note, however, that, as was pointed out in [4], if we turn on the $B$-field of rank $b$ ($b = 2, 4$ in the case of K3 compactifications), the 59 sector states come with multiplicity $2^{b/2}$ (while the rank of the gauge group is reduced by $2^{b/2}$). This implies that the 59 sector states no longer need to come in half-hypermultiplets. Thus, consider the $b = 2$ case. The Chan-Paton matrix is given by

$$\gamma_g = \text{diag}(I_4, -I_4).$$

The gauge group now is $[Sp(8) \otimes Sp(8)]_{99} \otimes [Sp(8) \otimes Sp(8)]_{55}$, and the matter consists of hypermultiplets in

$$(8, 8; 1, 1)_{99}, \quad (1, 1; 8, 8)_{55},$$

$$(8, 1; 8, 1)_{59}, \quad (1, 8; 1, 8)_{59}.$$

This spectrum is completely consistent, and, in particular, the gravitational anomaly cancels (the number of extra tensor multiplets in this model is 4, which follows from the results of [4]). Note that this model was originally discussed in [4].

In fact, we can generalize the above discussion to other perturbative K3 orientifolds, that is, the $\Omega J'$ orientifolds of Type IIB on $T^4/\mathbb{Z}_N$, $N = 2, 3, 4, 6$. (Note that in the $\mathbb{Z}_2$ case the action of $J'$ is trivial.) Actually, in the $\mathbb{Z}_3$ case nothing changes as we do not have a $\mathbb{Z}_2$ subgroup, but in other cases we do obtain new models with non-zero $B$-field. Here we give the twisted Chan-Paton matrices for these models, which we label $[N, b]$ ($g$ is the generator of $\mathbb{Z}_N$):
\[ \gamma_g = \text{diag}(I_4, -I_4), \]
\[ \gamma_g = \text{diag}(I_2, -I_2), \]
\[ \gamma_g = \text{diag}(\alpha I_4, \alpha^{-1} I_4), \]
\[ \gamma_g = I_4, \]
\[ \gamma_g = \text{diag}(i I_2, -i I_2, I_2, -I_2), \]
\[ \gamma_g = \text{diag}(i, -i, 1, -1), \]
\[ \gamma_g = \text{diag}(\alpha I_2, -\alpha I_2, \alpha^{-1} I_2, -\alpha^{-1} I_2), \]
\[ \gamma_g = \text{diag}(I_2, -I_2), \]

where \( \alpha \equiv \exp(2\pi i/3) \). These models, which were worked out in [35], are summarized in Table V. Note that in all of these models the massless spectra satisfy the gravitational anomaly cancellation condition (6). Here we note that the \([6, 2]\) model in Table V is the same as the corresponding \([6, 2]\) model without vector structure discussed in [4]. Also, the \([2, 4]\) and \([6, 4]\) models in Table V are the same. Moreover, it is not difficult to show that each of the \([N, b]\) models in Table V is on the same moduli as the corresponding \([N, b]\) model without vector structure discussed in [4].

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FIG. 1. A schematic picture of the space of four dimensional $\mathcal{N} = 1$ Type I and heterotic vacua. The region $\mathcal{A} \cup \mathcal{B}$ corresponds to perturbative Type I vacua. The region $\mathcal{A} \cup \mathcal{C}$ corresponds to perturbative heterotic vacua. The vacua in the region $\mathcal{A}$ are perturbative from both the Type I and heterotic viewpoints. The region $\mathcal{D}$ contains both non-perturbative Type I and heterotic vacua.
FIG. 2. Open circles and dots represent the original Voisin–Borcea orbifolds. The line of $\otimes$’s corresponds to the extension discussed in section IV.
### TABLE I

The massless spectra of the six dimensional $\mathcal{N} = 1$ supersymmetric Type IIB orientifolds on $T^4/Z_N$ for $N = 2, 3, 4, 6$. The semi-colon in the column “Charged Hypermultiplets” separates 99 and 55 representations. The subscript “$U$” indicates that the corresponding (“untwisted”) state is perturbative from the orientifold viewpoint. The subscript “$T$” indicates that the corresponding (“twisted”) state is non-perturbative from the orientifold viewpoint. The $U(1)$ charges are not shown, and by “neutral” hypermultiplets we mean that the corresponding states are not charged under the non-Abelian subgroups.

| Model | Gauge Group | Charged Hypermultiplets | Neutral Hypermultiplets | Extra Tensor Multiplets |
|-------|-------------|-------------------------|-------------------------|-------------------------|
| $Z_2$ | $U(16)_{99} \otimes U(16)_{55}$ | $2 \times (120; 1)_U$  $2 \times (1; 120)_U$  $(16; 16)_U$ | 20 | 0 |
| $Z_3$ | $[U(8) \otimes SO(16)]_{99}$ | $(28, 1)_U$  $(8, 16)_U$  $9 \times (28, 1)_T$ | 20 | 0 |
| $Z_4$ | $[U(8) \otimes U(8)]_{99} \otimes [U(8) \otimes U(8)]_{55}$ | $(28, 1; 1, 1)_U$  $(1, 28; 1, 1)_U$  $(8, 8; 1, 1)_U$  same as above with 99 ↔ 55  $(8, 1; 8, 1)_U$  $(1, 8; 1, 8)_U$  $(28, 1; 1, 1)_T$  $(1, 28; 1, 1)_T$  $(1, 1; 28, 1)_T$  $(1, 1; 28)_T$ | 20 | 0 |
| $Z_6$ | $[U(4) \otimes U(4) \otimes U(8)]_{99} \otimes [U(4) \otimes U(4) \otimes U(8)]_{55}$ | $(6, 1; 1, 1, 1)_U$  $(1, 6, 1; 1, 1, 1)_U$  $(4, 1, 8; 1, 1, 1)_U$  $(1, 4, 8; 1, 1, 1)_U$  same as above with 99 ↔ 55  $(4, 1, 1; 4, 1, 1)_U$  $(1, 4, 1; 1, 4, 1)_U$  $(1, 1, 8; 1, 1, 8)_U$  $5 \times (6, 1; 1, 1, 1)_T$  $5 \times (1, 6, 1; 1, 1, 1)_T$  $4 \times (4, 4, 1; 1, 1, 1)_T$  $(1, 1, 1; 6, 1, 1)_T$  $(1, 1, 1; 1, 6, 1)_T$  $(4, 4, 1; 1, 1, 1)_T$  $(1, 4, 1; 1, 4, 1)_T$ | 20 | 0 |
| Model | Gauge Group | Charged Chiral Multiplets | Neutral Chiral Multiplets |
|-------|-------------|----------------------------|--------------------------|
| \(Z'_6\) | \([U(4) \otimes U(4) \otimes U(8)]_{99} \otimes [U(4) \otimes U(4) \otimes U(8)]_{55}\) | \((1, 1, 28; 1, 1, 1)_{U}\) | 46 |
|       |             | \((1, 1, 28; 1, 1, 1)_{U}\) |                          |
|       |             | \((4, 4, 1; 1, 1, 1)_{U}\)  |                          |
|       |             | \((\bar{4}, \bar{4}, 1; 1, 1, 1)_{U}\) |                          |
|       |             | \((4, 1, 8; 1, 1, 1)_{U}\)  |                          |
|       |             | \((1, \bar{4}, \bar{8}; 1, 1, 1)_{U}\) |                          |
|       |             | \((\bar{6}, 1, 1; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, 6, 1; 1, 1, 1)_{U}\)  |                          |
|       |             | \((\bar{4}, 1, 8; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, 4, \bar{8}; 1, 1, 1)_{U}\) |                          |
|       |             | \((4, 4, 1; 1, 1, 1)_{U}\)  |                          |
|       |             | \((4, 1, 1; 1, 1, 1)_{U}\)  |                          |
|       |             | \((1, 4, 1; 1, 1, 8)_{U}\)  |                          |
|       |             | \((1, 1, 8; 1, 4, 1)_{U}\)  |                          |
|       |             | \((\bar{4}, 1, 1; 1, 1, 1)_{U}\) |                          |
|       |             | \((4, 4, 1; 1, 1, 1)_{U}\)  |                          |
|       |             | \((1, 1, \bar{8}; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, 4, 1; 1, 1, 1)_{U}\)  |                          |
|       |             | \((1, \bar{4}, 1; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, \bar{4}, 1; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, 1, \bar{4}; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, 1, \bar{4}; 1, 1, 1)_{U}\) |                          |
|       |             | \((1, 1, 1; 4, 1, 1)_{U}\)  |                          |
|       |             | \((1, 1, 1; 6, 1, 1)_{U}\)  |                          |
|       |             | \((1, 4, 1; 1, 4, 1)_{U}\)  |                          |
|       |             | \((1, 1, 4; 1, 4, 1)_{U}\)  |                          |
|       |             | \((6, 1, 1; 1, 1, 1)_{T}\)  |                          |
|       |             | \((3, \bar{6}, 1; 1, 1, 1)_{T}\) |                          |
|       |             | \((6, \bar{1}, 1; 1, 1, 1)_{T}\) |                          |
|       |             | \((3, 6, 1; 1, 1, 1)_{T}\)  |                          |
|       |             | \((3, 1, \bar{6}, 1; 1, 1, 1)_{T}\) |                          |
|       |             | \((3, 1, 1, 1; 4, 1, 1)_{T}\) |                          |
|       |             | \((3, 1, 1, 1; 6, 1, 1)_{T}\) |                          |
|       |             | \((3, 1, 1, 1; \bar{4}, 1, 1)_{T}\) |                          |
|       |             | \((3, 1, 1, 1; \bar{4}, 1, 1)_{T}\) |                          |
|       |             | \(3 \times (1, 1, 4; 1, 4, 1)_{T}\) |                          |
|       |             | \(3 \times (1, 1, 4; 1, 4, 1)_{T}\) |                          |

**TABLE II.** The massless spectrum of the four dimensional \(\mathcal{N} = 1\) supersymmetric Type IIB orientifold on \(T^6/Z'_6\). The semi-colon in the column “Charged Chiral Multiplets” separates 99 and 55 representations. The subscript “\(U\)” indicates that the corresponding ("untwisted") state is perturbative from the orientifold viewpoint. The subscript “\(T\)” indicates that the corresponding ("twisted") state is non-perturbative from the orientifold viewpoint. The \(U(1)\) charges are not shown, and by “neutral” chiral multiplets we mean that the corresponding states are not charged under the non-Abelian subgroups.
TABLE III. The massless spectrum of the four dimensional $\mathcal{N} = 1$ supersymmetric Type IIB orientifold on $T^6/(\mathbb{Z}_2 \otimes \mathbb{Z}_6)$. The semi-colon in the column “Charged Chiral Multiplets” separates 99 and the corresponding 5,5i representations. The subscript “U” indicates that the corresponding (“untwisted”) state is perturbative from the orientifold viewpoint. The subscript “T” indicates that the corresponding (“twisted”) state is non-perturbative from the orientifold viewpoint. The $U(1)$ charges are not shown, and by “neutral” chiral multiplets we mean that the corresponding states are not charged under the non-Abelian subgroups. Note that 16 and 28 are reducible representations of $SU(4)$ and $Sp(8)$, respectively (in our conventions $Sp(2N)$ has rank $N$).
TABLE IV. The perturbative massless spectrum of the four dimensional $\mathcal{N} = 1$ supersymmetric Type IIB orientifold on $T^6/\mathbb{Z}_{12}$. The semi-colon in the column “Charged Chiral Multiplets” separates 99 and 55 representations. The subscript “$U$” indicates that the corresponding (“untwisted”) state is perturbative from the orientifold viewpoint. The $U(1)$ charges are not shown, and by “neutral” chiral multiplets we mean that the corresponding states are not charged under the non-Abelian subgroups.
| Model | $b$ | Gauge Group | Charged Hypermultiplets | Neutral Hypermultiplets | Extra Tensor Multiplets |
|-------|-----|-------------|------------------------|------------------------|-----------------------|
| $Z_2$ | 2   | $[Sp(8) \otimes Sp(8)]_{99}$ $[Sp(8) \otimes Sp(8)]_{55}$ | (8, 8; 1, 1) (1, 1; 8, 8) (8, 1; 8, 1) (1, 8; 1, 8) | 16 | 4 |
|       | 4   | $[Sp(4) \otimes Sp(4)]_{99}$ $[Sp(4) \otimes Sp(4)]_{55}$ | (4, 4; 1, 1) (1, 1; 4, 4) 2 × (4, 1; 4, 1) 2 × (1, 4; 1, 4) | 14 | 6 |
| $Z_4$ | 2   | $[U(4) \otimes Sp(4) \otimes Sp(4)]_{99}$ $[U(4) \otimes Sp(4) \otimes Sp(4)]_{55}$ | (4, 4, 1; 1, 1, 1) (4, 1, 4; 1, 1, 1) (1, 1, 1; 4, 4, 1) (1, 1, 1; 4, 1, 4) (4, 1, 1; 1, 4, 1) (1, 4, 1; 4, 1, 1) (1, 1, 4; 4, 1, 1) | 14 | 6 |
|       | 4   | $[U(2) \otimes Sp(2) \otimes Sp(2)]_{99}$ $[U(2) \otimes Sp(2) \otimes Sp(2)]_{55}$ | (2, 2, 1; 1, 1, 1) (2, 1, 2; 1, 1, 1) (1, 1, 1; 2, 2, 1) (1, 1, 1; 2, 1, 2) 2 × (2, 1, 1; 1, 2, 1) 2 × (2, 1, 1; 1, 1, 2) 2 × (1, 2, 1; 2, 1, 1) 2 × (1, 1, 2; 2, 1, 1) | 13 | 7 |
| $Z_6$ | 2   | $[U(4) \otimes U(4)]_{99}$ $[U(4) \otimes U(4)]_{55}$ | (6, 1; 1, 1) (1, 6; 1, 1) (4, 4; 1, 1) (1, 1; 6, 1) (1, 1; 1, 6) (1, 1; 4, 4) 2 × (4, 1; 4, 1) 2 × (1, 4; 1, 4) | 14 | 6 |
|       | 4   | $[Sp(4) \otimes Sp(4)]_{99}$ $[Sp(4) \otimes Sp(4)]_{55}$ | (4, 4; 1, 1) (1, 1; 4, 4) 2 × (4, 1; 4, 1) 2 × (1, 4; 1, 4) | 14 | 6 |

TABLE V. The massless spectra of the six dimensional Type IIB orientifolds on $T^4/Z_N$ for $N = 2, 4, 6$, and various values of $b$ (the rank of the $B$-field) worked out in [35]. The semi-colon in the column “Charged Hypermultiplets” separates 99 and 55 representations.
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