Stability and second-order lateral stiffness of embedded piles with generalized end-boundary conditions on non-homogeneous soil

Carlos A. Vega-Posada\textsuperscript{1,*}, Jeisson A. Higuita-Villa\textsuperscript{2} and Julio C. Saldarriaga-Molina\textsuperscript{1}

\textsuperscript{1} Dept. of Civil and Environ. Eng., Univ. de Antioquia, UdeA, Calle 67 # 53-108. A. A. 1226 Medellin, Colombia
\textsuperscript{2} Undergrad student, Dept. of Civil and Environ. Eng., Univ. de Antioquia, Calle 67 # 53-108. A. A. 1226 Medellin, Colombia

E-mail: (*) carlosa.vega@udea.edu.co, jeisson.higuita@udea.edu.co, julio.saldarriaga@udea.edu.co

Abstract. This paper presents a simplified analytical method to conduct elastic stability and second-order lateral stiffness analysis of piles embedded in a two-parameter elastic soil. This work is an extension to a work recently presented by the first author, but here, the emphasis is given to study the effect of end-boundary conditions and soil non-homogeneity on the pile’s buckling load and lateral stiffness. The proposed formulation includes the effect of i) semi-rigid connections and lateral transverse springs at the ends of the pile, ii) an external transverse load acting along the pile, iii) soil non-homogeneity, and iv) the second-parameter of the elastic soil. The influence of the modulus of subgrade reaction, degrees of non-homogeneity, and intermediate end-boundary conditions on the pile response are investigated via a parametric study. The proposed solution can be employed to perform either lateral deformation or elastic buckling analysis.

1. Introduction

The elastic stability and lateral behavior of embedded piles have been a topic of great interest among the geotechnical community. Extensive research has been conducted on this subject due to the wide range of practical applications in civil engineering and other related engineering disciplines. Different approaches ranging from simple elastic analytical methods to very sophisticated finite element analyses have been used to investigate the mechanisms behind the soil-pile interaction phenomenon [1–5]. From a practical perspective, the simplifications of some methods are so that they may not capture the basic mechanisms that influence the pile response, and others are so elaborated that they become difficult to implement for practical applications.

Most of the available methods found in the literature are derived to conduct either elastic stability or static analysis of piles, no both, and they are limited to homogeneous soils and classical end-boundary conditions [6–11]. When conventional methods are used to solve the Governing Differential Equation (GDE), the mathematical development becomes complex to tackle from an analytical perspective. The solution becomes even more difficult to find when the soil’s inhomogeneity is introduced in the formulation. Therefore, it is still pertinent the search of alternative methods that could provide accurate results while maintaining a degree of simplicity and practicability.
The aim of this paper is to present a ready-to-use semi-analytical approach to conduct both elastic stability and second-order lateral stiffness analysis of piles. This work is an extension to a work recently presented by the first author [12], but here the emphasis is given to investigate the effects of semi-rigid connections and soil non-homogeneity on the stability and lateral response of a pile element. The GDE is solved using the well-known Differential Transformation Method (DTM). The proposed solution can be applicable to piles with different slenderness ratios and various end-rotational and -lateral transverse connections. The effect of different lateral soil pressures, degrees of non-homogeneity, and intermediate end-boundary conditions on the pile response are investigated via a parametric study.

2. Pile’s Structural Model

2.1. Assumptions

Figure 1 shows the structural model of the proposed element. The pile is connected to ends A and B by semi-rigid connections and linear transverse springs with stiffness $k_a$ and $S_a$, and $k_b$ and $S_b$, respectively. The pile has stiffness $EI$, length $L$, and is embedded in a two-parameter elastic soil, where $k_s$ is the modulus of subgrade reaction (i.e., represented by a Winkler spring) and $k_G$ is the shear layer linking the individual springs at the top ends. $k_s$ varies with depth/length in a linear fashion following the expression $k_s(x) = k_0 + cx$. $k_s(x) = k_0$ when $x = 0$ and $k_s(x) = k_1$ when $x = L$. Here, $k_0$ and $k_1$ are the subgrade reaction at the surface and at the depth/length $L$. The pile is loaded along its span with an external transverse load $q(x)$, and at each end with axial and horizontal loads $(P, V, M)$.

![Figure 1. Structural model of the proposed pile element.](image)

2.2. Governing Differential Equation (GDE)

The GDE of the pile element described in Figure 1 is expressed as [12]:

$$\frac{d^4\ddot{y}}{d\bar{x}^4} + F \frac{d^2\ddot{y}}{d\bar{x}^2} + \Lambda(\bar{x})\ddot{y} = \Omega(\bar{x})$$

(1)

where $F = (L^2/EI)(P - k_G)$, $\Lambda(\bar{x}) = (L^4/EI)(k_0 + cL\ddot{x})$, and $\Omega(\bar{x}) = (L^3/EI)(a_0 + a_1L\ddot{x} + a_2(L\ddot{x})^2)$.

The boundary conditions (B.Cs) at ends A and B are given as: At $\bar{x} = 0$

$$M_a = \frac{3EI\rho_a}{(1 - \rho_a) L} \frac{d\ddot{y}}{d\bar{x}} + \frac{EI}{L} \frac{d^2\ddot{y}}{d\bar{x}^2} = 0$$

(2a)

$$V_a - S_a L \ddot{y} - (P - k_G) \frac{d\ddot{y}}{d\bar{x}} - \frac{EI}{L^2} \frac{d^3\ddot{y}}{d\bar{x}^3} = 0$$

(2b)

At $\bar{x} = 1$

$$M_b = \frac{3EI\rho_b}{(1 - \rho_b) L} \frac{d\ddot{y}}{d\bar{x}} - \frac{EI}{L} \frac{d^2\ddot{y}}{d\bar{x}^2} = 0$$

(2c)
\[ V_b - S_b L \ddot{y} + (P + k_G) \frac{d\ddot{y}}{dt} + EI \frac{d^3\ddot{y}}{dt^3} = 0 \]  \hspace{1cm} (2d)

The normalized transverse deflection \( \ddot{y} \) of the element in terms of the DTM is expressed as:

\[ y(\xi) = \ddot{Y}(0) + \ddot{Y}(1) \xi + \ddot{Y}(2) \xi^2 + \ddot{Y}(3) \xi^3 \ldots + \ddot{Y}(m) \xi^m = \sum_{k=0}^{m} \ddot{Y}(k) \xi^k \]  \hspace{1cm} (3)

where \( \dddot{Y}(k) \) are the transformed coefficients of the polynomial function. Using the DTM, the GDE of the proposed structural element can be converted into the following recurrence equation.

\[
\ddot{Y}(k + 4) = \frac{1}{(k + 4)(k + 3)(k + 2)(k + 1)} \left[ \frac{L^3}{EI} (a_0 \delta(k) + a_1 \delta(k - 1)L)
+ a_2 \delta(k - 2)L^2 - (k + 2)(k + 1)F \dddot{Y}(k + 2) - \sum_{r=0}^{k} \dddot{Y}(k - r) \dddot{Y}(r) \right]
\]  \hspace{1cm} (4)

Equation (4) is the GDE that controls the mechanical response of the proposed pile element.

### 2.3. Parametric analysis

A parametric analysis is conducted to investigate the influence of rotational and lateral constraints and soil non-homogeneity on the elastic stability and second-order lateral stiffness response of embedded piles. Initially, the proposed formulation is validated with other available, but limited in scope, analytical approaches.

Figure 2 shows \( P_{cri}/P_E \) for a free-free pile and values of \( F = 0, 0.2, 0.4, 0.6, 0.8 \) and 1. Here, \( F = k_o/k_i \) is defined as the soil non-homogeneity index. \( F = 1, F = 0 \) and \( F \neq 0 \) represent a uniform, triangular linear, and trapezoidal distribution of \( k_o \), respectively. \( \lambda = (k_i L^3/EI)^{0.5} \) is the soil-pile stiffness ratio. As expected, it is observed that the elastic stability capacity of the pile increases as \( \lambda \) increases (i.e., soil stiffness increases). Also, \( P_{cri} \) increases as \( F \) increases (i.e., \( k_o \) moves from a triangular linear variation towards a uniform distribution), highlighting the contribution of the upper portion of the soil on the elastic stability. For this classical end-boundary condition case, the results show an excellent agreement between the proposed formulation and those presented by [13].

**Figure 2.** Variation in \( P_{cri}/P_E \) with \( F \) for a free-free pile. \( F=0 \) (triangular linear distribution of \( k_o \)) and \( F=1 \) (uniform distribution of \( k_o \))
Figure 3. $P_{crit}$ values for a pile in a soil with (a) $F = 0$ and (b) $F = 1$ and various end-boundary conditions. (△) fixed; (○) hinged; and (□) free.

Figure 4. Effect of a (a) rotational ($\rho = \rho_a$ and $S_a = 0$) and (b) translational restraint ($S = S_a$ and $\rho = 0$) at the pile head on $P_{crit}$. Pile is free at the bottom and embedded in a soil with $F = 0$.

Figure 3 shows $P_{crit}/P_E$ values for a pile in a soil with $F = 0$ and $F = 1$, and with various classical B.C.s. at the top and bottom ends. For the case when $F = 0$, it is observed that the end-boundary condition appreciably influences $P_{crit}$. The greatest values are achieved for a pile with a clamped head, and begin to decrease as the B.C. moves towards a free condition. A similar trend is observed for $F = 1$, but for this case the critical load values are significantly greater than for $F = 0$. As predicted by Euler, and as corroborated with the analysis, when $\lambda = 0$ (i.e., soil is neglected), $P_{crit}/P_E = 4$ for the clamped-clamped condition, 1.0 for the clamped with sidesway uninhibited-clamped condition, and 0.25 for a free-fixed condition. Again, the results agree well with those presented by [13]. The effect of intermediate B.C.s. (i.e., rotational and translational restraints) is shown in Figure 4. It is observed that $P_{crit}$ increases as the stiffness of both rotational and translational restraints increases. For the lateral restraint, it is interesting to note that, for a given $\lambda$, there exists a value of $S$ where the pile is fully braced and an increase in $P_{crit}$ is no longer appreciable. For instance, when $\lambda = 25$ (i.e., soft or loose deposit) a value of $S_a = 100$ is enough to achieve the maximum value of $P_{crit}$. This threshold value increases as $\lambda$ increases.

Next, a parametric study is conducted to evaluate the second-order lateral stiffness response of an
Figure 5. Variation of $S_{\delta}$ with $R$ when (a) $F = 0$ and (a) $F = 1$

embedded pile. The following definitions are adopted herein [14]: $S_{\delta} = \frac{S_{L}L^3}{EI}$, $R = \frac{PL^2}{2EI}$, and $D_{z} = \frac{K_{z}L^3}{EI}$. Where $S_{\delta}$ is the lateral stiffness index, $R$ is the axial load index, and $D_{z}$ is the soil lateral stiffness index. The analysis is presented for a pile free at the bottom ($S = 0$ and $\rho = 0$) and partially inhibited against rotation at its head ($S = 0$ and $\rho = \rho_{\delta}$), and subjected to an axial load and a lateral transverse force at its head (i.e., one unit lateral load). Figure 5 shows the variation of $S_{\delta}$ as a function of $R$ and $D_{z}$ when $F = 0$, and for different magnitudes of semi-rigid connections at the pile-head. Figure 5 shows the variation of the lateral stiffness index with the axial index load for three values of the soil lateral index. Figure 5 (a), (c), and (e) show the values for $F = 0$, and Figure 5 (b), (d), and (f) for $F = 1$. For $F = 0$, three general characteristics are clearly identified: i) $S_{\delta}$ decreases in an almost linear fashion as the applied axial load increases, and until the critical load is reached; ii) both, the magnitude of $S_{\delta}$ and $R$ increase
as the soil lateral index increases, and iii) the pile lateral stiffness gradually increases as the magnitude of the rotational constraint ($\rho$) increases. For $F = 1$, the same general trends as those of $F = 0$ are observed; but here, the values of $P_{cri}$, for a given $D_x$, are significantly greater than those of $F = 0$. This response highlights the contribution of the upper soil on $P_{cri}$. Also, $S_\delta$ smoothly decreases as the axial load increases but, unlike the case when $F = 0$, there is an abrupt loss in the pile lateral capacity when the applied axial load reaches the critical load.

3. Summary and Conclusions
The governing differential equation of a pile reacting against a non-homogeneous soil was presented. The proposed solution was used to investigate the effect of generalized end-boundary conditions ($\rho$ and $S$) and soil non-homogeneity ($F$) on the pile’s critical load ($P_{cri}$) and second-order lateral stiffness ($S_\lambda$). For classical end-boundary conditions, the results from the proposed model were validated against results from already available analytical approaches. The agreement was excellent. The parametric study showed that both $P_{cri}$ and $S_\lambda$ were significantly influenced by the end-boundary conditions and soil non-homogeneity. $P_{cri}$ increased as the magnitude of the rotational and transverse lateral restraints increased, and as the soil non-homogeneity moved from a linear distribution towards a uniform distribution. For both $F = 1$ and $F = 0$, (i) $S_\delta$ and $R$ increased as $D_x$ increased, and (ii) $S_\delta$ decreased in an almost linear and smooth fashion until the critical load was reached. For $F = 1$, there was an abrupt dropped in $S_\delta$ when $P_{cri}$ was reached.

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