POLYNOMIAL INVARIANTS ON MATRICES AND PARTITION, BRAUER ALGEBRAS

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Abstract. We identify the dimension of the centralizer of the symmetric group $S_d$ in the partition algebra $A_d(\delta)$ and in the Brauer algebra $B_d(\delta)$ with the number of multidigraphs with $d$ arrows and the number of disjoint union of directed cycles with $d$ arrows, respectively. Schur-Weyl duality as a fundamental theory, we conclude that each centralizers are related with the $G$-invariant space $P^d(M_n(k))^G$ of degree $d$ homogeneous polynomials on $n \times n$ matrices, where $G$ is the orthogonal group and the group of permutation matrices, respectively. Our approach gives a uniform way to show that the dimensions of $P^d(M_n(k))^G$ are stable for sufficiently large $n$.

1. Introduction

Through the paper, $k$ is an infinite field. The characteristic of $k$ is denoted by $\text{char}(k)$. For a group $G$ and a $G$-set $X$, $X^G$ denotes the set of fixed points in $X$ under the action of $G$. Let $P(M_n(k))$ be the space of polynomial functions on the set $M_n(k)$ of $n \times n$ matrices over a field $k$ and let $P^d(M_n(k))$ be the space of homogeneous polynomial functions of degree $d$. Our primary interest in this paper is the dimensions of $P^d(M_n(k))^G$ of polynomial invariants under the conjugation action of a subgroup $G$ of the general linear group $GL_n(k)$. An interesting result of Willenbring ([15]) along this line is that the dimensions of the space $P^d(M_n(\mathbb{C}))^{O_n(\mathbb{C})}$, where $O_n(\mathbb{C})$ is the orthogonal group over $\mathbb{C}$, stabilize for $n$ which are larger than or equal to $d$. In particular, there exists a stable limit of the Hilbert series of the rings $\{P(M_n(\mathbb{C}))^{O_n(\mathbb{C})}\}_{n \geq 1}$:

$$\lim_{n \to \infty} \dim_{\mathbb{C}} P^d(M_n(\mathbb{C}))^{O_n(\mathbb{C})} = \sum_{d=0}^{\infty} h_d t^d = \sum_{d=0}^{\infty} \left[ \lim_{n \to \infty} \dim_{\mathbb{C}} P^d(M_n(\mathbb{C}))^{O_n(\mathbb{C})} \right] t^d.$$ 

It is also shown in [15] that the coefficient $h_d$ in $\lim_{n \to \infty} H(t)$ is equal to the size of the set $U^G_d$ of directed graphs with $d$ arrows, whose connected components are directed cycles.

In [16], this stability phenomenon is extended to a large family of invariant rings arising from classical symmetric pairs (for the precise definition, see [16, Section 1.1]). Among those, the invariant ring $P(M_{2m}(k))^{Sp_{2m}(\mathbb{C})}$ appears, where $Sp_{2m}(\mathbb{C})$ denotes the symplectic group. Moreover, the stable Hilbert series for the family $\{P(M_{2m}(\mathbb{C}))^{Sp_{2m}(\mathbb{C})}\}_{m \geq 1}$ is equal to

\[ \lim_{n \to \infty} \dim_{\mathbb{C}} P^d(M_n(\mathbb{C}))^{Sp_{2m}(\mathbb{C})} = \sum_{d=0}^{\infty} h_d t^d = \sum_{d=0}^{\infty} \left[ \lim_{n \to \infty} \dim_{\mathbb{C}} P^d(M_n(\mathbb{C}))^{Sp_{2m}(\mathbb{C})} \right] t^d. \]
the Hilbert series for those cases, too. Note that the orthogonal group

Interesting enough, during the identification of dim

$\tilde{\beta}_d$ there is a bijection between $\tilde{\beta}_d$ and the set $\tilde{\mathcal{I}}_{2d}$ of fixed point free involutions on $\{1, 2, \ldots, 2d\}$. Interesting enough, during the identification of $\dim_k P^d(M_n(k))^{O(n,q)}$ with the number of
directed graphs in $\mathcal{U}_d^O$, Willenbring considered an action of $\mathfrak{S}_d$ on the set $\mathcal{I}_d$ and it turned out that there is a bijection from the set of orbits to the set $\mathcal{U}_d^O$ ([15, Section 3]). Because the action of $\mathfrak{S}_d$ coincides with the conjugation action on $\tilde{\beta}$, Willenbring’s bijection can be understood as a bijection between the natural basis of $\mathcal{C}_{\beta_d}^\mathfrak{S}_d(k\mathfrak{S}_d)$ and the set $\mathcal{U}_d^O$ (See Remark 4.19). On the other hand, in [13], Shalile studied the basis of the centralizer $\mathcal{U}_d$ of directed graphs in $\mathcal{U}_d$. It is parametrized by the generalized cycle types, which are multisets of certain equivalence classes of sequences in the letters $\{U, L, T\}$. Observing a similarity between the construction of generalized cycle types and Willenbring’s bijection, we give a concrete bijection between the set $\mathcal{U}_d^O$ and the set $\mathcal{E}_d$ of generalized cycle types (Section 4.5).

In the perspective of the equality (1.1), one may consider other families $\{G_n\}_{n \geq 1}$ different from the orthogonal groups and the symplectic groups. We studied the cases $G_n = \Sigma_n$ where $\Sigma_n$ denotes the group of permutation matrices in $GL_n(k)$ provided $\text{char}(k) = 0$. By the Schur-Weyl duality for $\Sigma_n$ ([8, 9]), if $n \geq 2d$, then there is an isomorphism $\Psi_{n,d}$ from the Partition algebra $\mathcal{A}_d(n)$ to $\text{End}_{k\Sigma_n}(V^\otimes d)$, which is an extension of the action of symmetric group $\mathfrak{S}_d$ on $V^\otimes d$. Recall that the partition algebra $\mathcal{A}_d(\delta)$ has the diagram basis $\beta_d$ parametrized by the set $\Pi_{2d}$ of set partitions of $\{1, 2, \ldots, 2d\}$. Since the symmetric group $\mathfrak{S}_d$ acts on the set $\beta_d$ by conjugation, the centralizer $\mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathfrak{S}_d)$ has a natural basis consisting of the orbit sums. Hence one can deduce the stability of the Hilbert series of the ring $P(M_n(k))^{\Sigma_n}$ similarly as the case of the Brauer algebra.

Actually, the partition algebra case is better than the Brauer algebra case in the following sense: there is a nice basis $\{x_\pi \mid \pi \in \Pi_{2d}\}$ of $\mathcal{A}_d(n)$ which is compatible with the image and the kernel of the homomorphism $\Psi_{n,d}$ ([8, 11]). Hence the dimensions in (1.1) is calculable even in the case $n < 2d$. A key result in this direction is to extend the Willenbring’s bijection between $\tilde{\beta}_d$ and $\mathcal{U}_d^O$ in the case of partition algebra. We find a bijection between the set $\Pi_{2d}$ of set partitions and the set $\mathcal{L}_d$ of multidigraphs with $d$ arrows labeled by $\{1, \ldots, d\}$ bijectively. This bijection very is convenient, because the conjugation action of $\mathfrak{S}_d$ on $\Pi_{2d}$ is just the permuting the edge labels of multidigraphs in $\mathcal{L}_d$ (Theorem 3.5). In conclusion, we have a bijection between the natural basis of the centralizer $\mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathfrak{S}_d)$ and the set $\mathcal{U}_d$ of (unlabeled) multidigraphs with $d$ arrows. Then by the compatibility of the basis $\{x_\pi \mid \pi \in \Pi_{2d}\}$ with $\Psi_{n,d}$, we show that the dimension of $P^d(M_n(k))^{\Sigma_n}$ is equal to the cardinality of $\mathcal{U}_{d, \leq n}$, where $\mathcal{U}_{d, \leq n}$ denotes the set of multidigraphs with $d$ arrows whose number of non-isolated vertices is less than or equal to $n$ (Corollary 3.11). A similar result appeared in [10] (see Remark 3.12).

We present the results on partition algebra at first and the results on Brauer algebra next, because it is natural to explain the Brauer algebra cases as a restriction of the partition algebra case.

2. Polynomial invariants on matrices and Schur algebras

2.1. Homogeneous polynomial functions on matrices. For a finite dimensional vector space $W$ over $k$, let $P(W)$ be the subalgebra of the algebra $k^W$ of functions from $W$ to $k$ generated by the dual space $W^*$ of $W$. We call $P(W)$ the algebra of polynomial functions on $W$. Let $\{x_i\}_{i \in I}$ be a basis of $W^*$. Then the algebra $P(W)$ is equal to the subalgebra
k[x_i; i ∈ I] of k^W generated by the functions x_i. Since k is an infinite field, the algebra
P(W) = k[x_i; i ∈ I] can be identified with the polynomial algebra with #I indeterminates.

A polynomial function f ∈ P(W) is called homogeneous of degree d if f(cw) = c^d f(w) for all w ∈ W and c ∈ k.

We have a vector space decomposition

\[ P(W) = \bigoplus_{d \geq 0} P^d(W), \]

where P^d(W) denotes the space of homogeneous polynomials of degree d. Note that P^d(−) is a contravariant endofunctor on the category of finite-dimensional vector spaces over k: for a linear map φ : W → V, P^d(φ) : P^d(V) → P^d(W) is the linear map given by f → P^d(φ)(f) := f ◦ φ for f ∈ P^d(W).

Note that d! ≠ 0 in a field k if and only if either char(k) = 0 or char(k) > d. The following lemma is well-known.

**Lemma 2.1.** (see, for example, [12, Section 1.5]) Assume that either char(k) = 0 or char(k) > d. Then there is an isomorphism

\[ η : P^d(−) \cong P^d(−)^* \]

between the contravariant endofunctors on the category of finite-dimensional vector spaces over k.

Our main object is the space P^d(M_n(k)) of homogeneous polynomial functions on the space M_n(k) of n × n-matrices. There is a left action of the group GL_n(k) of invertible n × n matrices on P^d(M_n(k)) by conjugation:

\[ (g.f)(X) := f(g^{-1} X g) \quad (g ∈ GL_n(k), \ f ∈ P^d(M_n(k)), \ X ∈ M_n(k)). \]

The dual space P^d(M_n(k))^* is also a left GL_n(k)-module with the action given by

\[ (g.Ψ)(f) := Ψ(g^{-1}.f) \quad (g ∈ GL_n(k), \ Ψ ∈ P^d(M_n(k))^*, \ f ∈ P^d(M_n(k))). \]

**Proposition 2.2.** Let n, d ∈ Z_{≥1}. Assume that either char(k) = 0 or char(k) > d. Then the GL_n(k)-module P^d(M_n(k)) is isomorphic to its dual P^d(M_n(k))^*.

**Proof.** Let Φ : M_n(k) → M_n(k)^* be the linear isomorphism given by

\[ A ↦ (B ↦ tr(AB)) \quad \text{for } A, B ∈ M_n(k). \]

Let c_g : M_n(k) → M_n(k) be the map given by X ↦ gXg^{-1}. Then we have a commutative diagram below for every g ∈ GL_n(k):

\[
\begin{array}{ccc}
M_n(k)^* & \xrightarrow{Φ^{-1}} & M_n(k) \\
(c_g)^* \downarrow & & \downarrow c_g \\
M_n(k)^* & \xrightarrow{Φ^{-1}} & M_n(k)
\end{array}
\]
The following diagram is commutative

$$
\begin{array}{ccc}
P^d(M_n(k)) & \xrightarrow{\eta_{M_n(k)}} & P^d(M_n(k))^* \\
\downarrow & & \downarrow \\
P^d(c_{g-1}) & \xrightarrow{P^d((c_{g-1})^*)} & P^d(M_n(k))^* \\
\downarrow & & \downarrow \\
P^d(M_n(k)) & \xrightarrow{\eta_{M_n(k)}} & P^d(M_n(k))^* \\
\end{array}
$$

where the left square comes from the isomorphism in Lemma 2.1 and the right square is obtained by applying the covariant functor $P^d(-)^*$ to the square in (2.1).

Since the actions of $GL_n(k)$ on $P^d(M_n(k))$ and $P^d(M_n(k))^*$ satisfy that $g.f = P^d(c_{g-1})(f)$ and $g.\Psi = P^d(c_{g})^*(\Psi)$, respectively, the composition

$$P^d(\Phi^{-1})^* \circ \eta_{M_n(k)} : P^d(M_n(k)) \to P^d(M_n(k))^*$$

is an isomorphism of $GL_n(k)$-modules, as desired. \hfill \Box

2.2. Schur algebras and polynomial invariants on matrices. Let $V = V_k = k^n$ be the natural representation of $GL_n(k)$. Then $GL_n(k)$ acts diagonally on the $d$-th tensor power $V^\otimes d$ from the left. For a group $G$, the group algebra over $k$ is denoted by $kG$. Let us denote by

$$\Phi_{n,d} : kGL_n(k) \to \text{End}_k(V^\otimes d)$$

the algebra homomorphism induced by the action.

On the other hand the symmetric group $\mathfrak{S}_d$ acts on $V^\otimes d$ from the right by place permutation:

$$\text{(2.2)} \ (w_1 \otimes \cdots \otimes w_d).\sigma := (w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(d)}) \quad \text{for } w_1, \ldots, w_d \in V, \text{ and } \sigma \in \mathfrak{S}_d.
$$

It induces an algebra homomorphism $\Psi_{n,d} : k\mathfrak{S}_d^{\text{op}} \to \text{End}(V^\otimes d)$, where $A^{\text{op}}$ denotes the opposite ring of a ring $A$. Then we have

$$\text{(2.3)} \quad \Psi_{n,d}(k\mathfrak{S}_d^{\text{op}}) = \text{End}_{kGL_n(k)}(V^\otimes d).
$$

If $n \geq d$, then $\Psi_{n,d}$ is an isomorphism of $k$-algebras.

We call the image $\Phi_{n,d}(kGL_n(k))$ the Schur algebra and denote it $S_{n,d}(k)$. Then the centralizer of the symmetric group action on $V^\otimes d$ is the same with the Schur algebra $S_{n,d}(k)$ (for example, see [7, Theorem 2.13]):

$$S_{n,d}(k) := \Phi_{n,d}(kGL_n(k)) = \text{End}_{k\mathfrak{S}_d^{\text{op}}}(V^\otimes d).
$$

For each function $f$ on $GL_n(k)$, define a function $\hat{f}$ on $kGL_n(k)$ as follows: if $\phi = \sum_{g \in GL_n(k)} a_g g$, then $\hat{f}(\phi) := \sum_{g \in GL_n(k)} a_g f(g)$.

An important relation between $P^d(M_n(k))^*$ and $S_{n,d}(k)$ can be summarized as in the following theorem.
Example 2.6. Let \( \mathcal{P} \) be a subgroup of \( G \) that satisfies the property that the algebra \( \Psi_{n,d} \) of \( \mathcal{P} \) has a basis formed by the elements \( \Phi_{n,d}(\alpha) \in S_{n,d}(k) \) for some \( \alpha \in kGL_n(k) \).

Define a left \( GL_n(k) \)-module structure on \( S_{n,d}(k) \) by

\[
\langle f, g \xi \rangle := \langle g^{-1} f, \xi \rangle \quad \text{for } f \in P^d(M_n(k)), \quad \xi = \Phi_{n,d}(\phi) \in S_{n,d}(k) \text{ for some } \phi \in kGL_n(k).
\]

Theorem 2.5. Theorem 2.3. There is a nondegenerate bilinear form

\[
\langle \ , \ \rangle : P^d(M_n(k)) \times S_{n,d}(k) \to k
\]
given by

\[
\langle f, \xi \rangle := \tilde{f}(\phi), \quad \text{where } f \in P^d(M_n(k)), \quad \xi = \Phi_{n,d}(\phi) \in S_{n,d}(k) \text{ for some } \phi \in kGL_n(k).
\]

Theorem 2.5. Let \( k \) be an infinite field. Assume that \( \text{char}(k) = 0 \) or \( \text{char}(k) > d \). Then for any subgroup \( G \) of \( GL_n(k) \), we have

\[
S_{n,d}(k)^G = \mathcal{E}_{\text{End}_{\Phi_{n,d}(kGL_n(V^\otimes d))}}(\Psi_{n,d}(k\mathcal{E}^\text{op}_d)).
\]

Proof. By the linearity we may assume that \( \xi = \Phi_{n,d}(\phi) \) for some \( \phi \in GL_n(k) \). Then we have

\[
\langle f, g \xi \rangle = \langle g^{-1} f, \xi \rangle = (g^{-1} f)(\phi) = (g^{-1} f)(\phi) = f(g\phi g^{-1}) = \langle f, \Phi_{n,d}(g\phi g^{-1}) \rangle
\]
so that

\[
g \xi = \Phi_{n,d}(g) \circ \Phi_{n,d}(\phi) \circ \Phi_{n,d}(g^{-1}) = \Phi_{n,d}(g) \circ \xi \circ \Phi_{n,d}(g^{-1}),
\]
as desired. \( \square \)

Combining Proposition 2.4 and Proposition 2.5, we obtain the following theorem.

Theorem 2.5. Let \( k \) be an infinite field. Assume that \( \text{char}(k) = 0 \) or \( \text{char}(k) > d \). Then for any subgroup \( G \) of \( GL_n(k) \), we have

\[
\dim_k P^d(M_n(k))^G = \dim_k \mathcal{E}_{\text{End}_{\Phi_{n,d}(kGL_n(V^\otimes d))}}(\Psi_{n,d}(k\mathcal{E}^\text{op}_d)).
\]

Example 2.6. Let \( G = GL_n(k) \) in Theorem 2.5. Assume that \( \text{char}(k) = 0 \). It is well-known that the algebra \( P(M_n(k))^{GL_n(k)} \) is isomorphic to the \( k \)-algebra of symmetric polynomials in \( n \) variables as graded algebras (see, for example, [3] Example 1.2). It follows that

\[
\dim_k (P^d(M_n(k))^{GL_n(k)}) = \# \{ \lambda \vdash d \mid \ell(\lambda) \leq n \},
\]
where \( \lambda \vdash d \) denotes a partition \( \lambda \) of \( d \), and \( \ell(\lambda) \) denotes the length of a partition \( \lambda \).

On the other hand, the dimension of \( P^d(M_n(k))^{GL_n(k)} \) is equal to the dimension of the center of the algebra \( \Psi_{n,d}(k\mathcal{E}^\text{op}_d) \) by Theorem 2.5 and (2.3). Thus one can recover the above equality in this case from a result in [6] Theorem 7: the set \( \{ \Psi_{n,d}(c_\lambda) \mid \lambda \vdash d, \ell(\lambda) \leq n \} \) forms a basis of the center of \( \Psi_{n,d}(k\mathcal{E}^\text{op}_d) \), where \( c_\lambda \) denotes the sum of all permutations whose cycle type is \( \lambda \) for a partition \( \lambda \) of \( d \).
3. Group of permutation matrices and Partition algebras

3.1. Partition algebra and Schur-Weyl duality. We recall the partition algebra and its bases following [2] Section 2.1, Section 2.2.

For \( d \in \mathbb{Z}_{\geq 1} \), we consider the set partitions of \([1, 2d] := \{1, 2, \ldots, 2d\}\) into disjoint nonempty subsets, called blocks, and define

\[
\Pi_{2d} := \{\text{set partitions of } [1, 2d]\}.
\]

For each \( \pi \in \Pi_{2d} \) let \(|\pi|\) be the number of blocks of \( \pi \). We associate a diagram \( D_\pi \) to \( \pi \) as follows: It has two rows of \( d \) vertices each, with the bottom vertices indexed by \( 1, 2, \ldots, d \) and the top vertices indexed by \( d+1, d+2, \ldots, 2d \) from left to right. Vertices are connected by an edge if they lie in the same block of \( \pi \). Note that the way the edges are drawn is immaterial, what matters is that the connected components of the diagram \( D_\pi \) correspond to the blocks of the set partition \( \pi \). Thus, \( D_\pi \) represents the equivalence class of all diagrams with connected components equal to the blocks of \( \pi \).

We write \( i' := i + d \) if \( 1 \leq i \leq d \) and \( i' := i - d \) if \( d + 1 \leq i \leq 2d \) briefly. We denote \( \beta_d \) as the set \( \{D_\pi | \pi \in \Pi_{2d}\} \). Sometimes we will confuse \( \pi \) and \( D_\pi \) for simplicity.

**Definition 3.1.** Let \( k \) be a field and \( \delta \in k\setminus\{0\} \). The Partition algebra \( A_d(\delta) \) is an associative \( k \)-algebra with \( \beta_d \) as a basis where the multiplication \( D_\pi D_\sigma \) for any two diagrams \( D_\pi, D_\sigma \) is defined as the following:

(i) Draw two diagrams vertically, \( D_\pi \) on the top and \( D_\sigma \) at the bottom.

(ii) Identify the vertices in the bottom row of \( D_\pi \) with those in the top row of \( D_\sigma \).

(iii) Delete all connected components that entirely lie in the middle row of the joined diagrams. We denote by \( \pi_1 \ast \pi_2 \) the set partition represented by thus obtained diagram.

(iv) The product is given by \( D_\pi D_\sigma = \delta^{[\pi_1 \ast \pi_2]} D_{\pi_1 \ast \pi_2} \), where \([\pi_1 \ast \pi_2]\) denote the number of blocks removed from the middle row.

We call \( \beta_d \) the diagram basis of \( A_d(\delta) \).

The set partition \( \pi_1 \ast \pi_2 \) and the nonnegative integer \([\pi_1 \ast \pi_2]\) depend only on the underlying set partitions \( \pi_1, \pi_2 \) and are independent of the diagrams chosen to represent them. In particular, the product \( D_{\pi_1} D_{\pi_2} \) depends only on the set partitions \( \pi_1 \) and \( \pi_2 \).

Note that a set partition of \([1, 2d]\) each of whose blocks is of the form \( \{i, j\} \) for some \( 1 \leq i, j \leq d \) can be identified with the permutation \( \sigma \) in \( \mathfrak{S}_d \) given by \( \sigma(i) = j \). Under this correspondence, the group algebra \( k\mathfrak{S}_d \) is embedded into \( A_d(\delta) \). Note that we have

\[
D_\sigma D_\pi D_{\sigma'} = D_{\sigma \ast \pi \ast \sigma'}
\]

for \( \pi \in \Pi_{2d} \) and \( \sigma, \sigma' \in \mathfrak{S}_d \). In particular, the symmetric group \( \mathfrak{S}_d \) acts on \( \Pi_{2d} \) by conjugation:

\[
\sigma \ast \pi := \sigma \ast \pi \ast \sigma^{-1}.
\]

Let \( k \) be a field and \( V \) be a finite-dimensional \( k \)-vector space. We fix a basis \( \{v_i \in V | i = 1, 2, \ldots, n\} \) and identify \( GL_n(k) \) with the group of automorphisms on \( V \). Let \( \Sigma_n \) be the subgroup of \( GL_n(k) \) consisting of all the permutation matrices. Then the group \( \Sigma_n \) is isomorphic to the symmetric group \( \mathfrak{S}_n \).
For each sequence \( i = (i_1, \ldots, i_d) \in [1, n]^d \), set
\[
v_i := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_d}
\]
so that \( \{v_i \mid i \in [1, n]^d\} \) forms a basis of the tensor power \( V^\otimes d \).

For \( s, r \in [1, n]^d \), let \( E^r_s \) be the element in \( \text{End}(V^\otimes d) \) given by
\[
E^r_s v_t = \delta_{s,t} v_r
\]
for all \( t \in [1, n]^d \). For each \( \pi \in \Pi_{2d} \), define
\[
N_d(\pi) = \{(u_1, \ldots, u_{2d}) \in [1, n]^{2d} \mid \text{such that if } i, j \text{ belong to the same block of } \pi \text{ then } u_i = u_j\}.
\]

Define a linear map \( \Psi_{n,d} : \mathcal{A}_d(n)^{\text{op}} \to \text{End}(V^\otimes d) \) by
\[
\Psi_{n,d}(D_\pi) = \sum_{(r,r') \in N_d(\pi)} E^r_{s'}.
\]

**Theorem 3.2.** ([8, Theorem 3.6], [9]) Assume that \( k \) is a field with \( \text{char}(k) = 0 \). The map \( \Psi_{n,d} \) is an algebra homomorphism whose image \( \Psi_{n,d}(\mathcal{A}_d(n)^{\text{op}}) \) is equal to \( \text{End}_{k\Sigma_n}(k)(V^\otimes d) \).

Moreover if \( n \geq 2d \), then \( \Psi_{n,d} \) is injective.

Note that the restriction of \( \Psi_{n,d} \) onto \( \mathcal{G}_d^{\text{op}} \) is equal to \( \mathcal{G}_d \).

Hence combining the Theorem 3.2 with Theorem 2.5 we obtain

**Theorem 3.3.** Assume that \( k \) is a field with \( \text{char}(k) = 0 \).

If \( n \geq 2d \), then
\[
\dim_k P^d(M_n(k))^{\Sigma_n} = \dim_k \mathcal{G}_d(k\mathcal{A}_d(n))\mathcal{G}_d.
\]

### 3.2. Multidigraphs and the Centralizer of \( \mathcal{G}_d \) in \( \mathcal{A}_d(\delta) \).

**Definition 3.4.** A directed multigraph, shortly a multidigraph, is an unlabeled graph that is made of countably many unlabeled vertices and a set of edges with directions, called arrows. We call a multidigraph is vertex-labeled (respectively, edge-labeled) if there is a function from the set of vertices (respectively, the set of edges) to a set of labels.

We will associate three multidigraphs \( \tilde{\psi}_d(\pi), \psi_d(\pi) \) and \( \phi_d(\pi) \) to a set partition \( \pi \) as follows:
Let \( \tilde{\psi}_d(\pi) \) be the vertex-labeled and edge-labeled multidigraph such that (1) the vertices are labeled by the blocks of \( \pi \), and (2) it has \( d \) arrows such that for each \( 1 \leq i \leq d \), there is exactly one arrow with the label \( i \) which starts from the vertex containing \( i \) and ends at the vertex containing \( i' \). Let \( \psi_d(\pi) \) be the edge-labeled multidigraph obtained by removing the labels of the vertices of \( \tilde{\psi}_d(\pi) \). Finally, removing the edge labels of \( \psi_d(\pi) \), we obtain a unlabeled multidigraph \( \phi_d(\pi) \).

Let \( \mathcal{L}_d \) be the set of edge-labeled multidigraphs of \( d \) arrows with bijective labelings by the set \( \{1,2,\ldots,d\} \) and \( \mathcal{U}_d \) be the set of multidigraphs with \( d \)-arrows.

Then, the map \( \psi_d : \Pi_{2d} \to \mathcal{L}_d \) is bijective and its inverse is given as follows:
(i) For a given multidigraph in \( \mathcal{L}_d \), label all the vertices with the empty sets.
(ii) For each \( 1 \leq i \leq d \), add \( i \) to the source and \( i' \) to the target of the arrow with the label \( i \). The label of a vertex of the graph forms a block of \( \psi_d^{-1}(g) \).
Theorem 3.5. For $\sigma \in \mathcal{G}_d$ and $\pi \in \Pi_{2d}$, the multidigraph $\psi_d(\sigma \ast \pi \ast \sigma^{-1})$ is obtained from $\psi_d(\pi)$ by permuting the arrow labels by $\sigma$.

Proof. We may assume that $\sigma$ is a simple transposition $(i\ i+1)$ for some $1 \leq i \leq d-1$. Let $b_1,b_1',b_2$ and $b_2'$ be the blocks of $\pi$ containing $i,i',i+1$ and $(i+1)'$ respectively. Note that the set partition $\sigma \ast \pi$ is obtained from $\pi$ by changing the blocks $b_1$ and $b_2$ into $c_1 := (b_1 \setminus \{i\}) \cup \{i+1\}$ and $c_2 := (b_2 \setminus \{i+1\}) \cup \{i\}$, respectively. Similarly the set partition $\pi \ast \sigma^{-1}$ is obtained from $\pi$ by changing the blocks $b_1'$ and $b_2'$ into $c_1' := (b_1' \setminus \{i'\}) \cup \{(i+1)\}'$ and $c_2' := (b_2' \setminus \{(i+1)\}' \cup \{i'\}$, respectively. Thus the set partition $\sigma \ast \pi \ast \sigma^{-1}$ is obtained from $\pi$ by changing the blocks $b_1,b_1',b_2$ and $b_2'$ into the subsets $c_1,c_1',c_2$ and $c_2'$, respectively, and all the other blocks remain the same. It follows that the multidigraph $\psi_d(\sigma \ast \pi \ast \sigma^{-1})$ is obtained from $\psi_d(\pi)$ by exchanging the arrows with labels $i$ and $i+1$, as desired. \qed

The following is an immediate consequence of Theorem 3.5.

Corollary 3.6. Two set partitions $\pi_1, \pi_2$ are $\mathcal{G}_d$-conjugate if and only if $\phi_d(\pi_1) = \phi_d(\pi_2)$.

For a multidigraph $G \in \mathcal{U}_d$, set

$$E_G := \{ \pi \in \Pi_{2d} | \phi_d(\pi) = G \}, \quad \text{and} \quad \gamma_G := \sum_{\pi \in E_G} D_\pi.$$

That is, each $E_G$ is an orbit in $\Pi_{2d}$ of the conjugation action of $\mathcal{G}_d$, and $\gamma_G$ is the sum of the diagram basis elements in the orbit $E_G$.

Corollary 3.7. The set

$$\Gamma_d := \{ \gamma_G \in \mathcal{A}_d(\delta) | G \in \mathcal{U}_d \}$$

forms a basis for $\mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathcal{G}_d)$, the centralizer of $k\mathcal{G}_d$ in $\mathcal{A}_d(\delta)$. In particular, the dimension of $\mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathcal{G}_d)$ is equal to $\# \mathcal{U}_d$, the number of unlabeled multidigraph with $d$-arrows. The dimension $\dim_k \mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathcal{G}_d)$ is independent from the choice of $\delta$.

Proof. Since $\{D_\pi\}_{\pi \in \Pi_{2d}}$ is a basis for $\mathcal{A}_d(\delta)$, the set $\Gamma_d$ is linearly independent over $k$ by the definition.

Write an element $v$ in $\mathcal{A}_d(\delta)$ as $v = \sum_{\pi \in \Pi_{2d}} a_\pi D_\pi$ for some $a_\pi \in k$. For any element $\sigma \in \mathcal{G}_d$, we have

$$D_\sigma v D_{\sigma^{-1}} = \sum_{\pi \in \Pi_{2d}} a_\pi D_\sigma D_\pi D_{\sigma^{-1}} = \sum_{\pi \in \Pi_{2d}} a_\pi D_{\sigma \ast \pi \ast \sigma^{-1}} = \sum_{\pi \in \Pi_{2d}} \sum a_{\sigma^{-1} \ast \pi \ast \sigma} D_\pi.$$

Hence $v = D_\sigma v D_{\sigma^{-1}}$ if and only if $a_{\sigma^{-1} \ast \pi \ast \sigma} = a_\pi$ for all $\pi \in \Pi_{2d}$. Thus for $v \in \mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathcal{G}_d)$, we have

$$v = \sum_{G \in \mathcal{U}_d} a_G \gamma_G$$

for some $a_G \in k$, as desired. \qed

Corollary 3.8. Let $k$ be an infinite field. If $n \geq 2d$, then the dimension of the space $P^d(M_n(k))^{\Sigma_n}$ is equal to $\# \mathcal{U}_d$, the number of multidigraphs with $d$ arrows. In particular, the sequence $(\dim_k P^d(M_n(k))^{\Sigma_n})_n$ is stable for $n \geq 2d$. 

Proof. If char\((k) = 0\), then we have
\[
\dim_k P^d(M_n(k))^\Sigma_n = \dim_k \mathcal{C}_{\text{End}_k \mathfrak{S}_n(V^\otimes d)}(\Psi_{n,d}(k \mathfrak{S}_d)) = \dim_k \mathcal{C}_{\mathcal{A}_d(n)}(k \mathfrak{S}_d)
\]
by Theorem 2.5 and Theorem 3.2. Hence we get the desired result by Corollary 3.7. Since
\[
P^d(M_n(k))^\Sigma_n
\]
is the invariant ring associated with the permutation representation induced from the conjugation action of \(\Sigma_n\) on the standard basis of \(M_n(k)\), its Hilbert series is independent of the ground field \(k\) (see, for example [11, Corollary 3.2.2]). □

3.3. The Orbit basis and the Centralizer of \(\mathfrak{S}_d\) in \(\mathcal{A}_d(\delta)\) when \(n < 2d\). In this subsection, we recall the orbit basis of \(\mathcal{A}_d(\delta)\) following [1, Section 2.3]. The set \(\Pi_{2d}\) of set partitions of \([1, 2d]\) forms a lattice under the partial order given by
\[
\pi \preceq \rho \text{ if every block of } \pi \text{ is contained in a block of } \rho.
\]
In this case we say that \(\pi\) is a refinement of \(\rho\) and that \(\rho\) is a coarsening of \(\pi\).

The orbit basis \(\{x_\pi | \pi \in \Pi_{2d}\}\) of \(\mathcal{A}_d(\delta)\) is defined by the following coarsening relation with respect to the diagram basis :
\[
D_\pi := \sum_{\pi \preceq \rho} x_\rho.
\]
Then
\[
x_\pi = \sum_{\pi \preceq \rho} \mu_{2d}(\pi, \rho) D_\rho
\]
for some integers \(\mu_{2d}(\pi, \rho)\) each of which satisfies that
\[
\mu_{2d}(\pi, \rho) = \mu_{2d}(\sigma * \pi * \sigma', \sigma * \rho * \sigma') \quad \text{for } \sigma, \sigma' \in \mathfrak{S}_d.
\]
See, for example, an explicit formula for \(\mu_{2d}(\pi, \rho)\) in [1, (2.18)].

It follows by (3.1), (3.2) and (3.3) that
\[
D_\sigma x_\pi D_{\sigma'} = x_{\sigma * \pi * \sigma'}.
\]
for \(\sigma, \sigma' \in \mathfrak{S}_d\) and \(\pi \in \Pi_d\) ([1, Section 4.1]).

The orbit basis is interesting, since it is compatible with the homomorphism \(\Psi_{n,d}\) in the following sense.

**Theorem 3.9.** ([1, Theorem 3.8(a)], [8, Theorem 3.6])

(i) The set \(\{\Psi_{n,d}(x_\pi) | \pi \in \Pi_{2d}, |\pi| \leq n\}\) forms a basis of \(\text{Im}(\Psi_{n,d}) = \text{End}_{k \mathfrak{S}_d}(V^\otimes d)\).

(ii) The set \(\{x_\pi | \pi \in \Pi_{2d}, |\pi| > n\}\) forms a basis of \(\text{Ker}(\Psi_{n,d})\).

Note that the set of multidigraphs with \(d\) arrows is partitioned into
\[
U_d = U_{d,1} \sqcup \cdots \sqcup U_{d,2d},
\]
where \(U_{d,k}\) consists of the multidigraphs with \(d\) arrows and \(k\) non-isolated vertices. For each \(1 \leq k \leq 2d\), let
\[
U_{d,k} := \bigsqcup_{t \leq k} U_{d,t}.
\]
Theorem 3.10. The set \( \{ \sum_{\pi \in E_G} x_{\pi} \mid G \in \mathcal{U}_d \} \) forms a basis of the centralizer \( \mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathcal{S}_d) \)
and the set \( \{ \Psi_{n,d}(\sum_{\pi \in E_G} x_{\pi}) \mid G \in \mathcal{U}_{d,\leq n} \} \) forms a basis of the centralizer \( \mathcal{C}_{\Psi_{n,d}(\mathcal{A}_d(n)^{op})}(k\mathcal{S}_d^{op}) \).
Moreover, we have
\[
\Psi_{n,d}(\mathcal{C}_{\mathcal{A}_d(\delta)}(k\mathcal{S}_d^{op})) = \mathcal{C}_{\Psi_{n,d}(\mathcal{A}_d(n)^{op})}(k\mathcal{S}_d^{op}).
\]

Proof. Note that \( \{ \Psi_{n,d}(\sum_{\pi \in E_G} x_{\pi}) \mid G \in \mathcal{U}_{d,\leq t} \} \) is linearly independent over \( k \) for any \( 1 \leq t \leq 2d \) by Theorem 3.9 (i). Write an element \( v \) in \( \mathcal{A}_d(\delta) \) as \( v = \sum_{\pi \in \Pi_{2d}} a_{\pi} x_{\pi} \) for some \( a_{\pi} \in k \).

For any element \( \sigma \in \mathcal{S}_d \), we have
\[
D_{\sigma}vD_{\sigma^{-1}} = \sum_{\pi \in \Pi_{2d}} a_{\pi} D_{\sigma}x_{\pi}D_{\sigma^{-1}} = \sum_{\pi \in \Pi_{2d}} a_{\pi} x_{\sigma^{-1}\pi\sigma^{-1}} = \sum_{\pi \in \Pi_{2d}} a_{\sigma^{-1}\pi\sigma} x_{\pi}.
\]
Taking \( \delta = n \) and applying \( \Psi_{n,d} \), we obtain
\[
\Psi_{n,d}(D_{\sigma^{-1}})\Psi_{n,d}(v)\Psi_{n,d}(D_{\sigma}) = \sum_{\pi \in \Pi_{2d}} a_{\sigma^{-1}\pi\sigma} \Psi_{n,d}(x_{\pi}) = \sum_{\pi \in \Pi_{2d}, |\pi| \leq n} a_{\sigma^{-1}\pi\sigma} \Psi_{n,d}(x_{\pi}).
\]

Because of Theorem 3.9 (i), we conclude that \( \Psi_{n,d}(v) = \Psi_{n,d}(D_{\sigma^{-1}})\Psi_{n,d}(v)\Psi_{n,d}(D_{\sigma}) \) if and only if \( a_{\sigma^{-1}\pi\sigma} = a_{\pi} \) for all \( \pi \in \Pi_{2d} \) with \(|\pi| \leq n\).

Thus for \( \Psi_{n,d}(v) \in \mathcal{C}_{\Psi_{n,d}(\mathcal{A}_d(\delta)^{op})}(k\mathcal{S}_d^{op}) \), we have
\[
(3.5) \quad \Psi_{n,d}(v) = \sum_{G \in \mathcal{U}_{d,\leq n}} a_{G} \Psi_{n,d}\left(\sum_{\pi \in E_G} x_{\pi}\right)
\]
for some \( a_{G} \in k \). Hence we obtain the second assertion. The first can be shown in a similar way. Because
\[
\Psi_{n,d}(\mathcal{C}_{\mathcal{A}_d(\delta)^{op}}(k\mathcal{S}_d^{op})) \subset \mathcal{C}_{\Psi_{n,d}(\mathcal{A}_d(n)^{op})}(k\mathcal{S}_d^{op})
\]
the equation (3.5) proves the last assertion, too. \( \square \)

By the same reasoning as in Corollary 3.8, we obtain

Corollary 3.11. Let \( k \) be an infinite field. The dimension of the space \( P^d(M_n(k))^{\Sigma_n} \) is equal to \( \#\mathcal{U}_{d,\leq n} \), the number of multidigraphs with \( d \) arrows whose number of non-isolated vertices is less than or equal to \( n \).

Remark 3.12. The above result appeared in [10] in the following way. In [10, Theorem 3], the number of non-isomorphic multigraphs (without loops) with \( d \) edges and \( n \) vertices is identified with the dimension of the space \( SF(n,d) \) of certain polynomial invariants of degree \( d \). And the authors remarked that the above corollary can be obtained by the same way. See [10, 4.Concluding remarks 2].

4. Orthogonal groups, symplectic groups and Brauer algebras

4.1. Brauer algebras. Let \( k \) be a field and \( \delta \in k \setminus \{0\} \). For \( d \in \mathbb{Z}_{\geq 1} \), set
\[
\tilde{\Pi}_{2d} := \{ \pi \in \Pi_{2d} \mid \text{each block of } \pi \text{ is of size 2} \} \quad \text{and} \quad \tilde{\beta}_d := \{ D_{\pi} \mid \pi \in \tilde{\Pi}_{2d} \} \subset \beta_d.
\]
Then the subspace $B_d(\delta)$ of the partition algebra $A_d(\delta)$ spanned by $\tilde{\beta}_d$ is stable under the multiplication. We call it the Brauer algebra with parameter $\delta$.

Note that the subset $\{s_j, e_j \mid j = 1, 2, \ldots, d - 1\}$ of $\tilde{\beta}_d$ is a generating set of the algebra $B_d(\delta)$, where

$$
\begin{align*}
1 & \quad 2 & j-1 & j & j+1 & j+2 & d-1 & d \\
\vdots & & \vdots & \times & \vdots & \vdots & \vdots & \vdots \\
1 & \quad 2 & j-1 & j & j+1 & j+2 & d-1 & d \\
\end{align*}
$$

We have a tower of algebras

$$
\mathfrak{k} \mathfrak{S}_d \subset B_d(\delta) \subset A_d(\delta).
$$

In particular, the set $\tilde{\Pi}_{2d}$ is stable under the conjugation action of $\mathfrak{S}_d$. By the same reasoning in the proof of Corollary 3.7, we obtain

**Proposition 4.1.** The set

$$
\left\{ \sum_{\pi \in E(G)} D_\pi \mid G \in \phi_d \left( \tilde{\Pi}_{2d} \right) \right\}
$$

forms a basis for $C_{B_d(\delta)}(\mathfrak{k} \mathfrak{S}_d)$, the centralizer of $\mathfrak{k} \mathfrak{S}_d$ in $B_d(\delta)$. In particular, the dimension of $C_{B_d(\delta)}(\mathfrak{k} \mathfrak{S}_d)$ is independent from the choice of $\delta$.

We will study the set $\phi_d \left( \tilde{\Pi}_{2d} \right)$ further in the last subsection.

4.2. **Orthogonal groups.** Let $k$ be an infinite field and let $V$ be a $k$-vector space with a fixed basis $\{v_i \in V \mid i = 1, 2, \ldots, n\}$.

Let $(\cdot, \cdot)_q$ and $(\cdot, \cdot)_{q'}$ be symmetric bilinear forms on $V$ given by

$$(v_i, v_j)_q = \delta_{i,j} \quad \text{and} \quad (v_i, v_j)_{q'} = \delta_{i,j},$$

respectively, where $\overline{j} = n + 1 - j$. Then the orthogonal groups $O(n, q)$ and $O(n, q')$ are given by

$$O(n, q) := \{ f \in GL_n(k) \mid (fv, fw)_q = (v, w)_q \text{ for all } v, w \in V \} = \{ f \in GL_n(k) \mid f^T f = I_n \}$$

and

$$O(n, q') := \{ g \in GL_n(k) \mid (gv, gw)_{q'} = (v, w)_{q'} \text{ for all } v, w \in V \} = \{ g \in GL_n(k) \mid g^T I_n g = I'_n \}$$
respectively, where $I_n$ is the identity matrix in $GL_n(k)$ and $I'_n$ is the $n \times n$ permutation matrix

$$I'_n = \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & 1 & & \\ \end{pmatrix}.$$

There is an algebra homomorphism $\Psi_{q,n,d}' : B_d(n)^{op} \to \text{End}(V^d)$ given by

$$\Psi_{q,n,d}'(s_j)(v_{i_1} \otimes \cdots \otimes v_{i_d}) = v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_d},$$
and

$$\Psi_{q,n,d}'(e_j)(v_{i_1} \otimes \cdots \otimes v_{i_d}) = \delta_{i_j,i_{j+1}} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \left( \sum_{k=1}^{n} v_k \otimes v_k \right) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_d}.$$

**Theorem 4.2.** (\cite[Theorem 1.2 (b))L] Let $k$ be an infinite field with $\text{char}(k) \neq 2$. Then the image $\Psi_{q,n,d}'(B_d(n)^{op})$ is equal to the space $\text{End}_k(O(n,q'))(V^d)$. Moreover if $n \geq d$, then the homomorphism $\Psi_{q,n,d}'$ is injective.

From Theorem 2.5 and Theorem 4.2 we have

**Theorem 4.3.** Let $k$ be an infinite field with $\text{char}(k) \neq 2$. Assume that either $\text{char}(k) = 0$ or $\text{char}(k) > d$. If $n \geq d$, then

$$\dim_k P^d(M_n(k))^{O(n,q')} = \dim_k \mathcal{O}_{B_d(n)}(kS_d).$$

In particular, if $\text{char}(k) = 0$, then the sequence $\left( \dim_k P^d(M_n(k))^{O(n,q')} \right)_{n=1}^{\infty}$ is stable for $n \geq d$.

Assume that $\text{char}(k) \neq 2$ and $\sqrt{-1} \in k$. Then the map $\varphi_O : V \to V$ given by

$$\varphi_O(v_i) = \begin{cases} v_i + \frac{\sqrt{-1}v_i}{2} & \text{if } 1 \leq i \leq \frac{n}{2} \\ -\frac{\sqrt{-1}}{2}v_i + \frac{1}{2}v_i & \text{if } i > \frac{n+1}{2} \\ v_i & \text{if } i = \frac{n+1}{2} \end{cases}$$

satisfies that

$$\langle \varphi_O(v), \varphi_O(w) \rangle_q' = \langle v, w \rangle_q$$

for $v, w \in V$.

In other words, $\varphi_O$ is an isometry between the quadratic spaces $(V, \langle \cdot, \cdot \rangle_q)$ and $(V, \langle \cdot, \cdot \rangle_q')$. Actually the assumption $\sqrt{-1} \in k$ is necessary if we want to have isometries between $(V, \langle \cdot, \cdot \rangle_q)$ and $(V, \langle \cdot, \cdot \rangle_q')$ for sufficiently many different dimensions.

**Proposition 4.4.** Let $k$ be a field with $\text{char}(k) \neq 2$. If $\dim_k V \equiv 2$ or $3$ (mod 4), and the quadratic spaces $(V, \langle \cdot, \cdot \rangle_q)$ and $(V, \langle \cdot, \cdot \rangle_q')$ are isometric to each other, than $\sqrt{-1} \in k$. 
Proof. Assume that $n = 4k + 2$ (respectively, $4k + 3$) for some $k \in \mathbb{Z}_{\geq 0}$. One can show that matrix $I_n'$ is congruent to a diagonal matrix $D_n$ whose entries are $2k+1$ many $-1$'s and $2k+1$ many (respectively $2k+2$ many) 1's, by a basis change similar to $\phi_O$. Hence $I_n$ is congruent to $I_n'$, then $I_n$ is congruent to $D_n$. In particular, $1 = \det(I_n)$ and $-1 = \det(D_n)$ belongs to the same square class of $k$ and hence $k$ contains $\sqrt{-1}$, as desired. □

Assume that $k$ is an infinite field such that $\sqrt{-1} \in k$ and $\text{char}(k) \neq 0$. Set

$$(4.2) \quad \Psi_{n,d}^q(D_n) = (\varphi_O^{-1})^\otimes d \circ \Psi_{n,d}^\prime(D_n) \circ \varphi^\otimes d \quad \text{for } D_n \in \mathcal{B}_d(n)^{\text{op}}.$$  

Then we have an analogue of Theorem 4.2: the image $\Psi_n^q$ of the algebra homomorphism $\Psi_{n,d}^q$ is equal to the space $\text{End}_{kO(n,q)}(V^\otimes d)$, and the homomorphism $\Psi_{n,d}^\prime$ is injective, provided $n \geq d$ (Cf. [5, Proposition 2.8]).

From Theorem 2.5 we obtain

**Theorem 4.5.** Let $k$ be an infinite field with char($k$) $\neq 2$ and $\sqrt{-1} \in k$. Assume that either char($k$) $= 0$ or char($k$) $> d$. If $n \geq d$, then

$$\dim_k P^d(M_n(k))^{O(n,q)} = \dim_k \mathcal{E}_{\mathcal{B}_d(n)}(k \mathcal{E}_d).$$

In particular, if char($k$) $= 0$, then the sequence $(\dim_k P^d(M_n(k))^{O(n,q)})_{n=1}^\infty$ is stable for $n \geq d$.

**Remark 4.6.** One can check that

$$\Psi_{n,d}^q(s_j)(v_1 \otimes \cdots \otimes v_id) = v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_j \otimes v_{i+2} \otimes \cdots \otimes v_id,$$

and

$$\Psi_{n,d}^q(e_j)(v_1 \otimes \cdots \otimes v_id) = \delta_{i,i+1} v_1 \otimes \cdots \otimes v_{i-1} \otimes \left( \sum_{k=1}^n v_k \otimes v_k \right) \otimes v_{i+2} \otimes \cdots \otimes v_id.$$  

4.3. Symplectic groups. Let $k$ be an infinite field and let $V$ be a $k$-vector space with a fixed basis $\{v_i \in V \mid i = 1, 2, \ldots, n\}$. Through this subsection, we assume that $n = 2m$ for some $m \in \mathbb{Z}_{\geq 1}$.

The *symplectic group* $Sp_n(k)$ is the subgroup of $GL_n(k)$ given by

$$Sp_n(k) := \{ f \in GL_n(k) \mid f^T J_n f = J_n \},$$

where

$$J_n = \left( \begin{array}{c|c} O & I_m \\ \hline -I_m & O \end{array} \right).$$

There is an algebra homomorphisms $\Psi_{n,d}^s : \mathcal{B}_d(-n)^{\text{op}} \to \text{End}(V^\otimes d)$ given by

$$\Psi_{n,d}^s(s_j)(v_1 \otimes \cdots \otimes v_id) = -v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_j \otimes v_{i+2} \otimes \cdots \otimes v_id,$$

and

$$\Psi_{n,d}^s(e_j)(v_1 \otimes \cdots \otimes v_id) = v_1 \otimes \cdots \otimes v_{i-1} \otimes \left( \sum_{k=1}^m v_{m+k} \otimes v_k - v_k \otimes v_{m+k} \right) \otimes v_{i+2} \otimes \cdots \otimes v_id.$$
Theorem 4.7. ([2] Proposition 1.3, Theorem 1.4]) Let $k$ be an infinite field. Then the image $\Psi_{n,d}^s(B_d(-n)^{op})$ is equal to the space $\text{End}_{kS_{pn}(k)}(V^{\otimes d})$. Moreover, if $n \geq 2d$, then the homomorphism $\Psi_{n,d}^s$ is injective.

From Theorem 2.5 and Theorem 4.7 we obtain the following theorem.

Theorem 4.8. Let $k$ be an infinite field and assume that either $\text{char}(k) = 0$ or $\text{char}(k) > d$. If $n \geq 2d$, then

$$\dim_k P^d(M_n(k))^{S_{pn}(k)} = \dim_k \mathcal{C}_{B_d(-n)}(kS_d).$$

In particular, if $\text{char}(k) = 0$, then the sequence $(\dim_k P^d(M_n(k))^{S_{pn}(k)})_{n=1}^{\infty}$ is stable for $n \geq 2d$.

Remark 4.9. Note that $S_{pn}(k)$ is the group of linear transformations preserving the bilinear form $(\cdot, \cdot)_s$ on $V$ which is represented by the matrix $J_n$ with respect to the basis we fixed. In [2], a different bilinear form $(\cdot, \cdot)_d$ is considered but it is isometric to $(\cdot, \cdot)_s$ over an arbitrary field. One may take an isometry given by a permutation of the elements in the basis. The algebra homomorphism $\Psi_{n,d}^s$ is the conjugation of that of [2] by the isometry.

4.4. Centralizer of $\mathcal{S}_d$ in $B_d(\delta)$ and disjoint union of directed cycles. A directed cycle is a directed graph whose underlying undirected graph is a cycle. Let $L_d^O$ be the subset of $L_d$ consisting of directed graphs in which each connected component is a directed cycle. In other words, $L_d^O$ is the set of disjoint union of directed cycles whose total number of arrows is $d$.

Lemma 4.10. The map $\psi_d$ induces a bijection from $\tilde{\Pi}_{2d}$ to $L_d^O$.

Proof. Let $\pi \in \tilde{\Pi}_{2d}$. Fix $a_1 \in [1, 2d]$. Set $a_i := b'_{i-1}$ unless $b'_{i-1} = a_j$ for some $j < i$, and let $b_i$ be the other element in the block of $a_i$. If $b'_{i-1} = a_j$ for some $j < i_0$, then $j = 1$. Indeed, if $j > 1$, then $b_{j-1} = a_{i_0-1} = a_j = b'_{j-1}$ so that $i_0 - 1 = j - 1$ which is a contradiction. It follows that the full subgraph of $\tilde{\psi}_d(\pi)$ with vertices $\{a_1, b_1|a_2, b_2|\ldots|a_{i_0-1}, b_{i_0-1}\}$ forms a directed cycle, because the degree of each vertex of $\tilde{\psi}_d(\pi)$ equals two. Repeating the procedure, we conclude that $\tilde{\psi}_d(\pi)$ is a multidigraph in which every connected component is a directed cycle.

It is clear that the restriction of $\tilde{\psi}_d^{-1}$ on $L_d^O$ is a map into $\tilde{\Pi}_{2d}$. \qed

Let $U_d^O$ be a subset of $U_d$ consisting of the multidigraphs in which every connected component is a directed cycle. Then, since $\sigma$-conjugation on $\psi_d(\pi)$ is permuting the arrow labels, $\mathcal{S}_d$-conjugacy classes of $\tilde{\Pi}_{2d}$ is in bijection with $U_d^O$ under $\phi_d$.

Hence, by the same proof as in Corollary 3.7 we obtain the following theorem.

Theorem 4.11. The dimension of the centralizer $\mathcal{C}_{B_d(\delta)}(k\mathcal{S}_d)$ is equal to $\#U_d^O$, the number of disjoint union of directed cycle in which the total number of arrow is $d$.

Corollary 4.12. Let $k$ be an infinite field with $\text{char}(k) = 0$ or $\text{char}(k) > d$.

(1) If $n \geq d$, then

$$\dim_k P^d(M_n(k))^{O(n,q')} = \#U_d^O.$$
(2) If \( n \geq d \), and \( \sqrt{-1} \in k \), then
\[
\dim_k P^d(M_n(k))^{O(n,q)} = \# U_d^O.
\]

(3) If \( n \geq 2d \), then
\[
\dim_k P^d(M_n(k))^{Sp_n(k)} = \# U_d^O.
\]

Remark 4.13. If \( k = \mathbb{C} \), then it recovers the stable behavior of Hilbert series by Willenbring in the cases of orthogonal groups \( O(n,q) = O_n(\mathbb{C}) \) and symplectic groups \( Sp_n(\mathbb{C}) \). More precisely, (2) recovers [15, Theorem 4.1], and (2) together with (3) recover the equality
\[
\lim \text{HS}(GL_n(\mathbb{R})) = \lim \text{HS}(GL_m(\mathbb{H}))
\]
in [16, Section 1.8]. Here the notation \( \text{HS}(G_0) \) stands for the Hilbert series of the ring \( S(g)^K \), where \((G,K)\) is the symmetric pair corresponding to the real form \( G_0 \), \( g \) is the Lie algebra of the complex reductive group \( G \), and \( S(g) \) denotes the symmetric algebra of \( g \). When \( G_0 = GL_n(\mathbb{R}) \) (respectively, \( G_0 = GL_m(\mathbb{H}) \)), the corresponding symmetric pair is \((GL_n(\mathbb{C}), O(n,q))\) (respectively, \((GL_{2m}(\mathbb{C}), Sp_{2m}(\mathbb{C}))\)) so that \( g \) is isomorphic to \( M_n(\mathbb{C}) \) (respectively, \( M_{2m}(\mathbb{C}) \)) and the ring \( S(g)^K \) is isomorphic to \( P(M_n(\mathbb{C}))^{O(n,q)} \) (respectively, \( P(M_{2m}(\mathbb{C}))^{Sp_{2m}(\mathbb{C})} \)). The stable limit \( \lim \text{HS}(G_0) \) is defined as the formal power series whose coefficients are given by the limits of the coefficients of the Hilbert series \( \text{HS}(G_0) \) as \( n \to \infty \).

4.5. Generalized Cycle Types of Brauer diagrams. In this subsection, we recall Shalile’s description of the \( \sigma \)-conjugate classes of Brauer diagrams (called the generalized cycle types in [13, 14]) and compare it with the description \( \phi_d(\bar{\Pi}_{2d}) \) by giving an explicit isomorphism between of them.

Definition 4.14. ([13, Definition 2.1])
For a diagram \( D \in \beta_d \), we get a string, sequence of letters \( \{U, L, T\} \), as the following process:

(i) Starting from a dot in a diagram, move to the other dot which is connected to the original dot by an edge.

(ii) If this edge is connected to a dot in the other row which we have started, then mark this edge as “T”.

If the edge is connecting two dots in the same row, then mark it as “U” if it was in the first row and “L” if it was in the second row.

(iii) From the dot we arrived by (i), move to the other dot which is in the same column.

(iv) Continue (i)∼(iii) until we reach to the dot which we started. Here, we get a string of letters, composed with \( U \), \( L \), or \( T \).

From the given process, we get a multiset of strings in \( \{U, L, T\} \).

From choosing another dot that was not counted in the above process and continuing (i)∼(iv) until there are no remaining dots, one derives a multiset of strings.

Note that one can have different multiset of strings by starting from a different dot.

Remark 4.15. (Cf. [13, Remark 2.4] ) It is clear that \( U \) cannot come right after \( U \) in a string of a diagram. If \( T \) comes after \( U \), the edge starts from the bottom to the top row. and hence the next edge starts from the bottom row. Therefore, after \( U \), another \( U \) cannot
appear again until \( L \) comes after. Similarly, after \( L \), another \( L \) cannot appear again until \( U \) comes after.

**Definition 4.16.** ([13] Definition 2.5])
Let \( C = l_1l_2 \cdots l_\alpha \) be a string in \( \{ U, L, T \} \) for some \( \alpha \in \mathbb{Z}_{\geq 1} \). Let the reversing \( r \) and shifting \( s \) be functions of switching the string of letters \( C \) as
\[
\begin{align*}
r(C) &= l_\alpha l_{\alpha-1} \cdots l_1 \\
s(C) &= l_2l_3 \cdots l_\alpha l_1
\end{align*}
\]
For strings \( C_1 = l_1l_2 \cdots l_\alpha \) and \( C_2 = k_1k_2 \cdots k_\alpha \) having the same length, define the relation \( \sim \) as
\[
C_1 \sim C_2 \quad \text{if and only if} \quad C_2 = rt_1s^{t_2}(C_1) \quad \text{for some} \ t_1, t_2 \in \mathbb{Z}_{\geq 0}
\]
Then, \( \sim \) is an equivalence relation because \( sr = rs^{\alpha-1}, r^2 = \text{id}, \) and \( s^\alpha = \text{id} \).

For each \( \pi \in \tilde{\Pi}_{2d} \), we call the multiset of equivalence classes of strings obtained by the procedure in Definition 4.14 the generalized cycle type (GCT, in short) of \( \pi \).

**Example 4.17.** Consider when \( d = 4 \). Let \( \delta \in k \setminus \{0\} \) and two diagrams \( D_1, D_2 \in B_d(\delta) \) be
\[
D_1 := \begin{array}{cccc}
1' & 2' & 3' & 4' \\
\cdot & \cdot & \cdot & \cdot \\
1 & 2 & 3 & 4
\end{array}, \quad D_2 := \begin{array}{cccc}
1' & 2' & 3' & 4' \\
\cdot & \cdot & \cdot & \cdot \\
1 & 2 & 3 & 4
\end{array}
\]
Starting from 1' of \( D_1 \), we proceed as 1' \( \xrightarrow{U} \) 3' \( \xrightarrow{3} \) 4 \( \xrightarrow{T} \) 1 \( \xrightarrow{4} \) 4. We did not pass through 2 in the previous process, so starting again from 2' to obtain 2' \( \xrightarrow{T} \) 2 \( \xrightarrow{2} \) 2'. The resulting generalized cycle type of \( D_1 \) is \( \{ ULT, T \} \).

Similarly, start from 1' of \( D_2 \), we have 1' \( \xrightarrow{T} \) 2 \( \xrightarrow{2} \) 2' \( \xrightarrow{U} \) 3' \( \xrightarrow{3} \) 4 \( \xrightarrow{T} \) 1 \( \xrightarrow{4} \) 1', and start again from 4 to obtain 4 \( \xrightarrow{T} \) 4' \( \xrightarrow{4} \) 4. Then the generalized cycle type of \( D_2 \) is \( \{ TUL, T \} \).

Since \( ULT = s^{-1}(TUL) \), \( D_1 \) and \( D_2 \) have the same generalized cycle type.

**Definition 4.18.** Let \( \mathcal{E}_d \) be the set of multisets of equivalence classes of strings in \( U, L, T \) of length \( d \) such that in each string of the multiset
\begin{itemize}
  \item[(i)] no string of the form \( UT^iU \) or \( LT^iL \) \( (i \geq 0) \) appears in any representative, and
  \item[(ii)] the numbers of occurrence of \( U \) and that of \( L \) are the same.
\end{itemize}
By Remark 4.15 every generalized cycle type of an element \( \pi \in \tilde{\Pi}_{2d} \) belongs to the set \( \mathcal{E}_d \).

Define \( \rho_d : \mathcal{U}_d^O \rightarrow \mathcal{E}_d \) as the following :
\begin{itemize}
  \item[(i)] For a digraph in \( \mathcal{U}_d^O \), label each vertex as \( U \) if \( \rightarrow \cdot \leftarrow \), \( L \) if \( \leftarrow \cdot \rightarrow \), and \( T \) if \( \rightarrow \cdot \rightarrow \) or \( \leftarrow \cdot \leftarrow \).
  \item[(ii)] Start from a vertex and make a string of letters of \( U, L, T \)’s by following the edges in one orientation.
\end{itemize}
Let \( f : \mathcal{L}_d^O \rightarrow \mathcal{U}_d^O \) be the function of forgetting the labels on arrows and set \( \overline{\rho}_d := \rho_d \circ f \).
**Remark 4.19.** One can check easily that the map $\rho_d \circ \psi_d$ is nothing but the map in Definition 4.14 which Shalile defined. Also, note that $\psi_d|\tilde{\Pi}_{2d}$ is the map Willenbring considered in [15, Section 3] under the identification $\tilde{\beta}_d$ with the set of fixed point free involutions on $[1, 2d]$.

Hence we obtain a commutative diagram below:

\[
\begin{array}{ccc}
\tilde{\Pi}_{2d} & \xrightarrow{\psi_d} & \mathcal{E}_d^O \\
& & \searrow \rho_d \\
\ E_d & \xrightarrow{\rho_d} & \mathcal{U}_d^O
\end{array}
\]

We define $\nu_d : \mathcal{E}_d \rightarrow \mathcal{U}_d^O$ as follows:

It is enough to define $\nu_d(C)$ for an element in $\mathcal{E}_d$ of the form $C = l_1l_2\cdots l_d$.

1. If $l_i = T$ for all $1 \leq i \leq d$, then $\nu_d(C)$ is the directed cycle in Figure 1.

2. We may assume that $l_1 = U$ by Definition 4.18 (ii). Construct a planar directed graph $G_i$ inductively as follows:
   - Let $G_1 := \bullet^1 \leftarrow \bullet^2$.
   - For $2 \leq i \leq d - 1$,
     - (i) If $l_i = U$, then $G_i$ is the graph obtained from $G_{i-1}$ by adding one incoming arrow to the right most extreme vertex of $G_{i-1}$. That is, $G_i = G_{i-1} \leftarrow \bullet^{i+1}$ where $\bullet^i$ is the lately added vertex in $G_{i-1}$.
     - (ii) If $l_i = L$, then $G_i$ is the graph obtained from $G_{i-1}$ by adding one outgoing arrow to the right most extreme vertex of $G_{i-1}$. That is, $G_i = G_{i-1} \rightarrow \bullet^{i+1}$.
     - (iii) If $l_i = T$, then $G_i$ is the graph obtained from $G_{i-1}$ by adding one arrow to the right most extreme vertex of $G_{i-1}$ with the same direction to the adjacent one.
   - For $i = d$, $\nu_d(C) = G_d$ is the graph obtained from $G_{d-1}$ adding an arrow connecting the two extreme vertices from $\bullet^d$ to $\bullet^1$.

**Theorem 4.20.** The maps $\rho_d$ and $\nu_d$ are inverses to each other.

**Proof.** The followings are consequences of the definition of $\nu_d$. 
(1) In the case of (b)-(i), since \( l_1 = U \), there exists \( j < i \) such that \( l_j = L \) and \( l_{j+1} = l_{j+2} = \cdots = l_{i-1} = T \). Then, \( G_{i-1} = G_{j-1} \rightarrow \bullet^{i+1} \rightarrow \bullet^{i+2} \rightarrow \cdots \rightarrow \bullet^i \). So, the local configuration in this step is always \( \bullet^{i-1} \rightarrow \bullet^i \leftarrow \bullet^{i+1} \).

(2) In the case of (b)-(ii), since \( U \) has to appear first before \( L \), there exists \( j < i \) such that \( l_j = U \) and \( l_{j+1} = l_{j+2} = \cdots = l_{i-1} = T \). Then, \( G_{i-1} = G_{j-1} \leftarrow \bullet^{i+1} \leftarrow \bullet^{i+2} \leftarrow \cdots \leftarrow \bullet^i \). So, the local configuration in this step is always \( \bullet^{i-1} \leftarrow \bullet^i \rightarrow \bullet^{i+1} \).

(3) In the case of (c), \( l_d \) is either \( L \) or \( T \) by Definition 4.18 (i) and the assumption \( l_1 = U \). By these local characterizations of \( \nu_d(C) \), it is straightforward to see that the compositions \( \rho_d \) and \( \nu_d \) are the identities. \( \square \)

Corollary 4.21. The set of generalized cycle types of elements in \( \widetilde{\Pi}_{2d} \) is equal to \( \mathcal{E}_d \).

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