RELATIVE ZETA DETERMINANTS AND THE QUILLEN METRIC

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Abstract. We compute the relation between the Quillen metric and the canonical metric on the determinant line bundle for a family of elliptic boundary value problems of Dirac-type. To do this we present a general formula relating the \(\zeta\)-determinant and the canonical determinant for a class of higher-order elliptic boundary value problems.

1. Introduction

The determinant line bundle for a family of first-order elliptic operators over a closed manifold was first studied in remarkable papers by Quillen [10], Atiyah and Singer [1], Bismut [2], Bismut and Freed [3]. Using spectral \(\zeta\)-functions it was shown that the regularized geometry of the determinant line bundle encodes a subtle relation between the Bismut local family’s index theorem and fundamental non-local spectral invariants. A great deal of subsequent work has been directed towards a generalization of this theory to manifolds with boundary. The corresponding object of interest is the determinant bundle for a family of elliptic boundary value problems (EBVPs). The choice of boundary conditions for the family introduces a new degree of complexity into the problem, or, more optimistically, a new degree of freedom. Indeed, explicit computations of \(\zeta\)-determinants on closed manifolds are generally achieved by an identification with a spectrally equivalent EBVP. The crucial fact behind this is a canonical identification of the space of interior solutions to the Dirac operator with their boundary traces defined by the Poisson operator, enabling one to resolve the EBVP by solving an equivalent problem for pseudodifferential operators on the space of boundary sections.

In [12] (math/9812124) it was explained how these facts lead to a regularization procedure for determinants of Dirac type operators and a corresponding regularized geometry of the determinant line bundle for families of EBVPs. In this note simple formulas are given relating the canonical metric and the Quillen \(\zeta\)-function metric. Details of the constructions here along with the relation between the \(\zeta\)-function and canonical curvature will be presented in [13].
Both the canonical regularization and the $\zeta$-function regularization depend on choices. The former depends on an admissible choice of a basepoint boundary condition, the corresponding canonical determinant is essentially the quotient of the (unregularized) determinant of the EBVP with that defined by the basepoint condition. The later depends on a choice of spectral cut. Up to the choice of basepoint the construction of the canonical regularization is purely topological, being achieved via natural isomorphisms of the determinant line bundle. Furthermore, it is a completely algebraic operator-theoretic honest Fredholm determinant. The first fact is encouraging that this ‘god-given’ regularization may be closely related to the $\zeta$-determinant. The second fact is rather less encouraging, the $\zeta$-determinant is defined by an analytic regularization and its meromorphic continuation relies on delicate asymptotic properties. Both points, however, are correct. The $\zeta$-determinant really is a subtle analytic object not accessible via operator determinants, however the relative $\zeta$-determinant with respect to boundary conditions $P^1, P^2$ is actually not mysterious and really is given by a Fredholm determinant over the boundary.

To compute the relation between the metrics we have to study the Dirac Laplacian EBVP, which \textit{a priori} is a quite complicated operator since one must additionally restrict the range of the EBVP for the Dirac operator to lie in the domain of the adjoint EBVP. However, two key ideas reduce this to a computable problem. First, we use a useful trick which identifies a certain class of higher-order EBVPs with an equivalent first-order system (see Grubb \cite{Grubb1} for a recent similar method used to compute the asymptotic expansion of the heat kernel via ‘doubling-up’). Second, we use an explicit formula for the relative parametrix of the Schwartz kernel of an EBVP relative to different choices of boundary condition which generalizes the constructions in \cite{Wojciechowski}. One is therefore reduced to computing only a relative determinant, and this serendipitously depends only on boundary data.

The results in this paper are closely connected with those of Mueller \cite{Mueller} on relative determinants on non-compact manifolds. This will be discussed in \cite{Scott}. The general results presented here are a generalization of the formulas in the paper \cite{Wojciechowski} of K.Wojciechowski and the author in which the special case of first-order self-adjoint EBVPs over an odd-dimensional manifold was treated via variational methods. Here we remove those three conditions: first-order, self-adjoint, odd-dimensions. I refer to \cite{Wojciechowski} and \cite{Scott} for background material and to \cite{Grubb2} for a detailed presentation of the 1-dimensional case. I thank Gerd Grubb for helpful conversations.
2. The Determinant Line Bundle for Families of Higher-order EBVPs

We consider a smooth fibration of manifolds \( \pi : Z \to B \) with fibre diffeomorphic to a compact manifold \( X \) with boundary \( \partial X = Y \), and endowed possibly with a vertical Hermitian coefficient bundle \( \xi \to Z/B \) with compatible connection, and such that the tangent bundle along the fibres \( T(Z/B) \) is oriented, spin and endowed with a Riemannian metric \( g_{T(Z/B)} \). This data defines a corresponding family of Dirac operators \( \mathbb{D}_0 = \{ D_b : b \in B \} : \mathcal{F}^0 \to \mathcal{F}^1 \). Here \( \mathcal{F}^i \) are the infinite-dimensional bundles on \( B \) with fibre at \( b \) the Frechet space of smooth sections \( C^\infty(M_b, S^i_b) \), where \( S^i_b \) are the appropriate Clifford bundles. Since the dimension of \( X \) is unrestricted here, we shall not specify whether \( \mathbb{D} \) is the family of total Dirac operators or, in even dimensions, a family of chiral Dirac operators acting between the bundles of positive and negative chirality fields.

The corresponding structures are inherited on the boundary fibration \( \partial \pi : \partial Z \to B \) of closed manifolds with fibre \( Y \). We assume a collar neighbourhood \( V = [0, 1] \times \partial Z \) of \( \partial Z \) in which \( g_{T(Z/B)} \) has the product form \( du^2 + g_{Y/B} \), where \( u \) is a normal coordinate to \( \partial Z \) and \( g_{Y/B} \) the induced metric on the boundary fibration, and that the Hermitian metrics, connections on the bundles \( S(Z/B) \), \( \xi \) split similarly. In \( V_b = [0, 1] \times \partial X_b \) the operators \( D_b \) have the form \( \sigma_b(y)(\partial/\partial u + A_b) \) where \( \sigma_b : S_{V_b} \to S_{V_b} \) is a unitary bundle isomorphism and \( A_b \) is an elliptic self-adjoint operator identified with the Dirac operator over \( Y_b \). Each fibre of the bundle \( \mathcal{H}_Y \) is endowed with a \( \mathbb{Z}_2 \)-grading \( \mathcal{H}_{Y,b} = H^+_b \oplus H^-_b \) where \( H^+_b \) (resp. \( H^-_b \)) is the direct sum of the eigenspaces of \( A_b \) with non-negative (resp. negative) eigenvalues. Associated to the gradings we have a (smooth) Grassmann bundle \( \mathcal{G}_{r \infty}(\mathcal{H}_Y) \to B \) with fibre the infinite-dimensional smooth Grassmannian \( \mathcal{G}_{r \infty}(\mathcal{H}_{Y,b}) \) parameterising projections \( P \in \text{End}(H_b) \) which differ by a smoothing operator from the projection \( \Pi_{\geq b} \) with image \( H^+_b \), where projection means self-adjoint indempotent. \( \mathcal{G}_{r \infty}(\mathcal{H}_{Y,b}) \) is a dense submanifold of the larger Grassmannian \( \mathcal{G}_{r \infty}(\mathcal{H}_{Y,b}) \) parameterising projections such that \( P - \Pi_{\geq b} \) is a pseudodifferential operator of order -1. A Grassmann section for the family \( \mathbb{D}_0 \) is defined to be a smooth section \( \mathbb{F} \) of the fibration \( \mathcal{G}_{r \infty}(\mathcal{H}_Y) \), and we denote the space of Grassmann sections by \( \mathcal{G}_{r \infty}(0)(Y/B) \). Such sections always exist.

More generally, for \( i = 0, \ldots, r - 1 \), let \( \mathbb{D}_i = \{ D_{i,b} : b \in B \} : \mathcal{F}^i \to \mathcal{F}^{i+1} \) be families of compatible Dirac operators over \( X \), defined as above. The bundle \( \mathcal{F}^1 \) has fibre \( \mathcal{F}^1_b = C^\infty(M_b, S^i_b) \), with \( S^i_b \) the corresponding bundles of Clifford-modules. We then consider the family of \( r \)th order elliptic differential operators

\[
(1) \quad \mathbb{D}^{(r)} = \mathbb{D}_{r-1} \cdot \mathbb{D}_{r-2} \cdot \ldots \cdot \mathbb{D}_0 : \mathcal{F}^0 \to \mathcal{F}^r.
\]
Thus $\mathbb{D}^{(r)}$ parameterizes the operators $D_{b} = D_{r,b} \cdot D_{r-1,b} \cdot \ldots \cdot D_{1,b} : F_{b}^{0} \to F_{b}^{r}$. In $V_{b} = [0,1] \times \partial X_{b}$ the operators $D_{i,b}$ have the product form $\sigma_{b}(y)(\partial / \partial u + A_{i,b})$. For odd $r = 2k + 1$ we assume that $F^{k} = F^{k+1}$.

For each $D_{i}$ we have an associated space $\mathcal{G}_{r}^{(i)}(Z/B)$ of Grassmann sections, and to an $r$-tuple $P^{r} = (P_{0}, \ldots, P_{r-1})$ of Grassmann sections $P_{i} \in \mathcal{G}_{r}^{(i)}(Z/B)$ we have a family $(\mathbb{D}^{(r)}, P^{r})$ of EBVPs parameterizing the operators

$$D_{P^{r}} = D_{b} : \text{dom}(D_{P^{r}}) \to L^{2}(X_{b}; S_{b}^{0}),$$

dom$(D_{P^{r}}) = \{s \in H^{r}(X_{b}; S_{b}^{0}) : P_{b,r-1}\gamma_{r-1}(D_{b,r-1} \ldots D_{b,1}s) = 0, \ldots, P_{b,0}\gamma_{0}s = 0\}$

where $P_{b} = (P_{b,0}, \ldots, P_{b,r-1})$, and $\gamma_{i} : H^{i+1}(X_{b}; S_{b}^{1}) \to H^{i+1/2}(Y_{b}, S_{b}^{1,y})$ is the restriction operator.

To analyze $(\mathbb{D}^{(r)}, P^{r})$ we consider the ‘equivalent’ family of first-order Dirac type operators

$$\hat{D}_{b} = \begin{pmatrix} 0 & 0 & \ldots & 0 & \mathbb{D}_{r-1} \\ 0 & 0 & \ldots & \mathbb{D}_{r-2} & -I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{0} & -I & 0 & \ldots & 0 \end{pmatrix} : F_{0} \oplus F_{1} \oplus \ldots \oplus F_{r-1} \to F_{r} \oplus F_{r-1} \oplus \ldots \oplus F_{1}$$

In $V_{b} = [0,1] \times \partial X_{b}$ the operators $\hat{D}_{b}$ parameterized by $\hat{D}$ are of Dirac-type with the product form $\hat{\sigma}_{b}(y)(\partial / \partial u + \hat{A}_{b})$, where $\hat{\sigma}_{b} : \bigoplus_{i=1}^{r-1} S_{b}^{i} \to \bigoplus_{i=0}^{r-1} S_{b}^{r-j+1}$ is a unitary bundle isomorphism, and $\hat{A}_{b} = (\bigoplus_{i=0}^{r-1} A_{i,b}) + R$, where $R$ is an operator of order 0 on the boundary fields built from the $\sigma_{i,b}(y)$. We therefore have for each operator $\hat{D}_{b}$ a Grassmannian $\mathcal{G}_{r-1}(\mathcal{H}_{Y_{b}})$ of projections associated to the spectral grading $\mathcal{H}_{Y_{b}} = H_{b}^{0} \oplus H_{b}^{-}$ defined by the elliptic self-adjoint operator $\bigoplus_{i=0}^{r-1} A_{i,b}$, where $\mathcal{H}_{Y_{b}} = \bigoplus_{i=0}^{r-1} L^{2}(Y_{b}; S_{b}^{i,y})$. Globally, we obtain a Grassmann bundle, a space of Grassmann sections $\mathcal{G}_{r-1}(Z/B)$, and for each $\hat{P} \in \mathcal{G}_{r-1}(Z/B)$ a family of first-order EBVPs $(\hat{D}, \hat{P})$ and a family of $r^{th}$ order EBVPs $(D^{(r)}, \hat{P})$ with dom $(D_{P_{b}}) = \{s \in H^{r}(X_{b}; S_{b}^{0}) : \hat{P}_{b}\gamma_{D^{r}s} = 0\}$, where $\gamma_{D^{r}s} = (\gamma_{0}s, \gamma_{1}(D_{b,1}s), \ldots, \gamma_{r-1}(D_{b,r-1} \ldots D_{b,1}s))$.

A Grassmann section $\mathbb{P} \in \mathcal{G}_{r-1}(Z/B)$ is equivalent to a smooth ungraded Frechet subbundle $\mathcal{W} \to B$ of $\mathcal{H}_{Y}$, and for each such pair of Grassmann sections $\mathbb{P}^{0}, \mathbb{P}^{1}$ we have the smooth family of Fredholm operators

$$(\mathbb{P}^{0}, \mathbb{P}^{1}) \in C^{\infty}(B; \text{Hom}(\mathcal{W}^{0}, \mathcal{W}^{1})), \quad (\mathbb{P}^{0}, \mathbb{P}^{1})_{b} = P_{b}^{1}P_{b}^{0} : W_{b,0} \to W_{b,1},$$

where $\mathcal{W}^{i}$ are the bundles defined by $\mathbb{P}^{i}$, and hence a (Segal) determinant line bundle $\text{DET}(\mathbb{P}^{0}, \mathbb{P}^{1})$ with its canonical determinant section $b \mapsto \det(P_{b}^{1}P_{b}^{0})$. Associated to $\hat{D}$ we
have a canonical Grassmann section $P(\mathcal{D})$, equal at $b$ to the Calderon projection $P(\mathcal{D}_b)$ with range equal to the space of boundary traces $K_b = K(\mathcal{D}_b) = \gamma \text{Ker}(\mathcal{D}_b) \subset H_{Y_b}$ of solutions to the Dirac operator, where $\gamma$ is the restriction operator to the boundary of $X_b$.

For each $P \in \mathcal{G}_{r-1}(Y/B)$ the pair $(\mathcal{D}, P)$ therefore has a canonically associated Fredholm family

$$S(P) := (P(\mathcal{D}), P) : K(\mathcal{D}) \to W,$$

where $K(\mathcal{D})$ has fibre $K_b$, with determinant line bundle $\text{DET}(S(P))$ with determinant section $b \mapsto \det(S(P_b))$, where $S(P_b) := P_b P(\mathcal{D}_b)$. On the other hand, we have the smooth Fredholm families $(\mathbb{D}(r), P)$ and $(\mathcal{D}, P)$, with associated determinant line bundles $\text{DET}(\mathbb{D}(r), P), \text{DET}(\mathcal{D}, P)$ over $B$ endowed with their respective determinant sections $b \mapsto \det(\mathbb{D}(P_b))$ and $b \mapsto \det(\mathcal{D}(P_b))$.

**Theorem 2.1.** For $P_b \in \mathcal{G}_{r-1}(H_{Y_b})$, $\mathbb{D}_{P_b}, \mathcal{D}_{P_b}$ are Fredholm operators with kernel and cokernel consisting of smooth sections. One has

$$\text{index}(\mathbb{D}(P_b)) = \text{index}(\mathcal{D}(P_b)) = \text{index}(S(P_b)).$$

For $P, P_1, P_2 \in \mathcal{G}_{r-1}(Z/B)$ there are canonical isomorphisms of determinant line bundles

$$\text{DET}(\mathbb{D}(r), P) \cong \text{DET}(\mathcal{D}, P) \cong \text{DET}(S(P)),$$

preserving the determinant sections $\det(\mathbb{D}(P_b)) \longleftrightarrow \det(\mathcal{D}(P_b)) \longleftrightarrow \det(S(P_b))$, and

$$\text{DET}(\mathbb{D}(r), P_1) \cong \text{DET}(\mathbb{D}(r), P_2) \otimes \text{DET}(P_1, P_2).$$

With the identifications of Theorem 2.1 at hand, we obtain a commutative diagram of canonical isomorphisms

$$\text{DET}(\mathbb{D}(r), P_1) \xrightarrow{\cong} \text{DET}(\mathbb{D}(r), P_2) \otimes \text{DET}(P_1, P_2)$$

$$\downarrow \cong \quad \quad \downarrow \cong$$

$$\text{DET}(S(P_1)) \xrightarrow{\cong} \text{DET}(S(P_2)) \otimes \text{DET}(P_1, P_2)$$

in which, by (3), the vertical maps take the determinant sections to each other, while in the bottom map

$$\det(P_1^1 S(P_2^2)) \longleftrightarrow \det(S(P_2^2)) \otimes \det(P_1^1, P_2^2).$$

If we assume the EBVPs $\mathbb{D}_{P_b}$ are invertible at $b$, then all the above operators are invertible, and we obtain two non-zero canonical elements $\det(S(P_1))$ and $\det(P_1^1 S(P_2^2))$ in the complex line $\text{Det}(S(P_1))$, where for brevity we have dropped the $b$ subscript. The first element is identified with $\det(\mathbb{D}(P_1)) \in \text{Det}(\mathbb{D}(P_1))$ by the isomorphism (3), while the
first maps to an element of \( \text{Det}(D_{P_1}) \) we shall denote by \( \det(D_{(P_1, P_2)}) \). The two elements therefore differ by the complex number \( \det_{C(P_2)} D_{P_1} \) where

\[
\det(D_{P_1}) = \det_{C(P_2)}(D_{P_1}), \det(D_{(P_1, P_2)}).
\]

\( \det_{C(P_2)} D_{P_1} \) is called the \textit{canonical regularization of the determinant of the} \( r \text{th order EBVP} \) \( D_{P_1} \) \textit{relative to the basepoint} \( P_2 \). Notice that this is a purely \textit{topological} regularization. We compute:

\textbf{Lemma 2.2.}

\[\det_{C(P_2)} D_{P_1} = \det_{F_{r,K}} \left( \frac{S(P_1)}{P_1 S(P_2)} \right). \tag{7}\]

Here \( \det_{F_{r,K}} \) means the Fredholm determinant taken on the Calderon subspace \( K = K(\widehat{D}) \) defined by \( \widehat{D} \), and the operator quotient means \( (P_1 S(P_2))^{-1} S(P_1) : K(\widehat{D}) \to K(\widehat{D}) \).

3. 

**Computation of the relative zeta-function determinant**

In order to compare the canonical regularization (7) with the \( \zeta \)-function regularization we study the relative \( \zeta \)-determinant of the \( r \text{th order EBVPs} \) \( D_{P_1}, D_{P_2} \) with \( P_1, P_2 \in \widehat{G}_{r,\infty}(\mathcal{H}_b) \). We assume the operators have a common ray of minimal growth \( l_{\theta} = \{ \arg \lambda = \theta : \lambda \in \mathbb{C} \} \). This means that there is an open neighbourhood of \( l_{\theta} \) disjoint from the spectrum of the operators \( D_{P_1} \) and \( D_{P_2} \). We further assume that there exists \( r > 0 \) such that for \( |\lambda| > r \) the \( L^2 \) operator norms along this ray \( \|(D_{P_1} - \lambda)^{-1}\| \) are \( O(1/|\lambda|) \). For \( \Re(s) > 0 \) we can therefore define

\[D_{P_1}^{-s} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-s}(D_{P_1} - \lambda)^{-1} d\lambda, \tag{8}\]

where \( \Gamma \) is the contour beginning at \( \infty \) traversing \( l_{\theta} \) to a small circle around the origin, anti-clockwise around the circle, then back along the ray to \( \infty \). For \( \Re(s) > n/r \), \( n = \dim X \), the operator \( D_{P_1}^{-s} \) has a continuous kernel and the \textit{spectral} \( \zeta \)-function

\[\zeta_{\theta}(s, D_{P_1}) = \text{Tr}(D_{P_1}^{-s}) \tag{9}\]

is well-defined and holomorphic.

The analytic continuation of the \( \zeta \)-function to the whole of \( \mathbb{C} \) depends (as does the construction of the canonical regularization) on properties, first, of the \textit{Poisson operator}

\[\mathcal{K}_b = C^\infty(Y_b; \hat{S}_b|_Y) \to C^\infty(X_b; \hat{S}_b), \]

which defines an isomorphism \( \mathcal{K}_b : K(\widehat{D}_b) \to \text{Ker}({\mathcal{F}}_{\theta}), \) and thus a canonical identification of interior solutions with boundary traces. See [11, 4, 15] for details of the construction. Second, we need the \textit{relative parametrix}, given in Proposition 3.1 below, which tells us how the Schwartz kernel of the inverse
operator changes under a change of boundary condition. For a trace-class operator $A : F_{r,b} \oplus F_{r-1,b} \oplus \ldots F_{1,b} \to F_{0,b} \oplus F_{1,b} \oplus \ldots F_{r-1,b}$ written as an $r \times r$ block matrix of operators relative to the direct sum decomposition, we use the notation $[A]_{(1,1)}$ for the component in the top-left $(1,1)$ position. We have:

**Proposition 3.1.** Let $P^1, P^2 \in \hat{G}_{r,\infty}(H_Y)$. If $D_{p_1}, D_{p_2}$ are invertible, then

$$D_{p_1}^{-1} = D_{p_2}^{-1} - \left[K_1(P^1)P_1^1 \gamma \hat{D}_{p_2}^{-1}\right]_{(1,1)}$$

Let $\lambda$ be a complex number such that $D_{p_1} - \lambda$ and $D_{p_2} - \lambda$ are both invertible and assume that $F_{0,b} = F_{r,b}$. Then

$$\left(D_{p_1} - \lambda\right)^{-1} = \left(D_{p_2} - \lambda\right)^{-1} - \left[K_1(P^1)P_1^1 \gamma \hat{D}_{p_2}^{-1}\right]_{(1,1)}.$$

Here $K_1(P) = K_1 S_1(P)^{-1} P$, where $K_1$ is the Poisson operator of $D_1$, $S_1(P) = PP(D_1)$, and

$$\hat{D}_1 = \begin{pmatrix} -\lambda & 0 & \ldots & 0 & D_{r-1} \\ 0 & 0 & \ldots & D_{r-2} & -I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_0 & -I & 0 & \ldots & 0 \end{pmatrix} : F_{b,0} \oplus F_{b,1} \oplus \ldots F_{b,r-1} \to F_{b,0} \oplus F_{b,r-1} \oplus \ldots F_{b,1}.$$

The remainder term on the right-side of (10) and (11) is a smoothing operator.

**Proof.** We find that $D_{p_i}^{-1} = \left[\hat{D}_{p_i}^{-1}\right]_{(1,1)}$ and hence to prove (10) it is enough to prove that $\hat{D}_{p_1}^{-1} = \hat{D}_{p_2}^{-1} - K_1(P^1)P_1^1 \gamma \hat{D}_{p_2}^{-1}$, and this follows in a similar way to (11). If $P^1, P^2 \in \hat{G}_{r,\infty}(Z/B)$ then $P^1(I - P^2)$ is a smoothing operator, and so the last statement is immediate. \qed

In a recent papers of Grubb [3, 4], generalizing joint work with Seeley [7], it was shown that for $(m+1)r > n = \dim X$ there is as asymptotic expansion as $\lambda \to \infty$

$$\text{Tr} \left(\partial^2_{\lambda}(D_{p} - \lambda)^{-1}\right) \sim \sum_{j=0}^{\infty} (a_j + b_j)(-\lambda)^{(n-j)/r-m-1} + \sum_{k=0}^{\infty} (c_k \log(-\lambda) + \bar{c}_k)(-\lambda)^{-k/r-m-1},$$

where the $a_j$ are integrals of densities locally determined by the symbol of $D$ and the $b_j, c_k$ similarly with densities determined locally by the symbol of $D$ and $P$. The $\bar{c}_k$ are in general globally determined. From this one obtains the meromorphic continuation of
\( \zeta_\theta(s, D_P) \) to \( \mathbb{C} \) and the full pole structure. If the coefficients \( c_0, \tilde{c}_0 \) vanish there is no pole at \( s = 0 \) and so we can then define the \( \zeta \)-determinant

\[
\det_{\zeta, \theta}(D_{P_1}) := \exp\left(-\frac{d}{ds}_{|s=0} \zeta_\theta(s, D_{P_1})\right).
\]

From \([5, 6]\) this applies to the case of the Dirac Laplacian for \( P^i \in \hat{G}_{\infty}(HY) \) (see Section 4) and also to the first-order self-adjoint problems considered in \([5]\). See also \([17]\).

On the other hand \([K_\lambda(P^1)P^1\gamma \tilde{D}_{\lambda, P^2}^{-1}]_{(1,1)} \) is a smoothing and hence trace-class operator and its trace it determined explicitly by only boundary data. Hence the formula \((11) \) tells us that the natural object to compute is the relative determinant

\[
\det_{\zeta, \theta}(D_{P_1}, D_{P_2}) := \exp\left(-\frac{d}{ds}_{|s=0} \zeta_\theta(s, D_{P_1}, D_{P_2})\right),
\]

where the relative spectral \( \zeta \)-function \( \zeta_\theta(s, D_{P_1}, D_{P_2}) = \text{Tr} \left(D_{P_1}^{-1} - D_{P_2}^{-1}\right) \) is well-defined and holomorphic for \( \text{Re}(s) > 0 \) and has a meromorphic continuation with simple poles at \( s = -k/r, k = 1, 2, \ldots \).

For a function with an asymptotic expansion \( f(x) \xrightarrow{x \to \infty} \sum_{j=0}^{\infty} a_j x^{\alpha_j} \log^k x \), where \( k = 0, 1 \) and \( \infty > \alpha_0 > \alpha_1 > \ldots \) and \( \alpha_k \to -\infty \), the regularized limit \( \text{LIM}_{x \to \infty} f(x) \) is defined to be equal to the constant \( a_0 \) term in the expansion. We obtain the following fundamental relation between the \( \zeta \)-function regularization and the canonical regularization:

**Theorem 3.2.** One has:

\[
\frac{\det_{\zeta, \theta}(D_{P_1})}{\det_{\zeta, \theta}(D_{P_2})} = \frac{\det_C(P_2)}{\det_C(P_1)} \cdot e^{-\text{LIM}_{x \to \infty} \log \det_C(P_2)/(D_{\lambda, P_1})}.
\]

where \( \det_C(P_2)/(D_{\lambda, P_1}) = \det_{F_{r, K}}((P^1 S_{\lambda}(P^2))^{-1} S_{\lambda}(P^1)) \). If \( \zeta_\theta(0, D_{P_1}) = \zeta_\theta(0, D_{P_2}) \), then \( \text{LIM} \) can be replaced by (the unregularized) lim.

**Proof.** We have that \( \det_{\zeta, \theta}(D_{P_1}, D_{P_2}) = \det_{\zeta, \theta}(D_{P_1}) / \det_{\zeta, \theta}(D_{P_2}) \) while a direct computation yields

\[
\text{Tr} \left\{ (D_{P_1} - \lambda)^{-1} - (D_{P_2} - \lambda)^{-1} \right\} = \frac{\partial}{\partial \lambda} \log \det_{F_{r, K}} \left( \frac{S_{\lambda}(P^1)}{P_{P_1 S_{\lambda}(P^2)}^{P_1}} \right)
\]

from which the equality follows.
Formula (14) may be regarded as a formula for determinants in the spirit of the Atiyah-Singer index formula for elliptic operators: the left-side of (14) is the (relative) analytical-determinant constructed from \( \zeta \)-function analysis, while \( \text{det}_{C}(P_{2})(D_{P_{1}}) \) is the (relative) topological-determinant constructed purely from natural topological properties of the determinant line bundle.

4. Applications to the geometry of \( \text{DET}(D, P) \)

Let \( (D, P) \) be a family of first-order EBVPs with \( P \in \text{Gr}_{\infty}(Z/B) \). There is a metric and connection on the determinant line bundle \( \text{DET}(D, P) \) associated to the canonical regularization. Let \( \Delta_{P} \) be the Dirac Laplacian \( \Delta = D^{*}D \) with domain

\[
\text{dom}(\Delta_{P}) = \{ s \in H^{2}(X, S) \to L^{2}(X, S) : P\gamma s = 0, \ P^{*}\gamma Ds = 0 \},
\]

where \( P^{*} = \sigma(I - P)\sigma^{*} \) is the adjoint boundary condition for \( D^{*} \) (not to be confused with the adjoint of the (self-adjoint) operator \( P \)) and \( S \) the Clifford bundle of (chiral) spinors. Over the open \( U \) subset of \( B \) where the EBVPs are invertible the canonical metric is defined by

\[
\|\text{det}(D_{P})\|_{\zeta}^{2} = \text{det}_{\zeta}(\Delta_{P}) := \text{det}_{Fr,K}(S(P)^{*}S(P)),
\]

where \( S(P) := PP(D) : K(D) \to \text{range}(P) \) and \( P(D) \) is the Calderon projection of the (first-order) operator \( D \). On the other hand, \( \text{DET}(D, P) \) has a Quillen metric, defined by \( \zeta \)-function regularization \([10, 3]\) over \( U \) by

\[
\|\text{det}(D_{P})\|_{\zeta}^{2} = \text{det}_{\zeta}(\Delta_{P}) := e^{-\zeta_{\Delta_{P}}(0)}
\]

where \( \zeta_{\Delta_{P}}(s) = \text{Tr} (\Delta_{P}^{s}) \) is defined around 0 by analytic continuation. For the global construction of these objects see \([10, 3, 12]\). From Theorem 3.2 we compute:

**Theorem 4.1.** Over \( U \subset B \)

\[
\frac{\|\text{det}(D_{P_{1}})\|_{\zeta}}{\|\text{det}(D_{P_{2}})\|_{\zeta}} = \frac{\|\text{det}(D_{P_{1}})\|_{C}}{\|\text{det}(D_{P_{2}})\|_{C}}.
\]

A holomorphic line bundle \( L \) endowed with a metric \( g \) and local holomorphic section \( s \), has a canonical compatible connection with curvature 2-form \( R^{g} = \partial\bar{\partial} \log \|s\|^{2}_{g} \). If we assume that \( \mathcal{H}_{Y} \) is a trivial bundle and that \( D \) depends holomorphically on \( b \), then for two choices of constant Grassmann sections \( \mathbb{P}_{1} := P^{1}, \mathbb{P}_{2} := P^{2} \) the determinant line bundles \( \text{DET}(D, P_{i}) \) the corresponding \( \zeta \) and \( C \) metrics have respective canonical curvature 2-forms \( R_{\zeta}^{i}, R_{C}^{i} \in \Omega^{2}(B) \) and we have:
Corollary 4.2. With the above assumptions,

\[ R_1^2 - R_2^2 = R_1^2 - R_2^2. \]

This would apply, for example, to a family of \( \bar{\partial} \)-operators coupled to a Hermitian vector bundle over a Riemann surface with boundary. The extension of (19) to more general families of EBVPs will be presented in [13].

Finally, we remark the \( \zeta \)-determinant formulae recently published in [8] for a \( r \)th order EBVP in dimension one follow from Theorem 3.2, while for self-adjoint first-order EBVPs over odd-dimensional manifolds (here the condition \( \zeta_\theta(0, D_{P_1}) = \zeta_\theta(0, D_{P_2}) \) applies) Theorem 3.2 reproves the result of [15].

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