Matroid and Tutte-connectivity in infinite graphs

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Abstract

We relate matroid connectivity to Tutte-connectivity in an infinite graph. Moreover, we show that the two cycle matroids, the finite-cycle matroid and the cycle matroid, in which also infinite cycles are taken into account, have the same connectivity function. As an application we re-prove that, also for infinite graphs, Tutte-connectivity is invariant under taking dual graphs.

1 Introduction

This work is part of a project to develop a theory for infinite matroids that is analogous to its finite counterpart. In the initial paper of this project [11], we extended extended previous work of Higgs [14, 13] and Oxley [15] by giving equivalent definitions of (finite or infinite) matroids in terms of independence, bases, circuits, closure and (relative) rank, just as one is used to for finite matroids. Since then, in a series of papers [3, 5, 2, 4, 1], several other aspects of infinite matroids have been explored, among them graphic matroids [7] and matroid connectivity [9].

These two last aspects are the focus of the current work: Connectivity in graphic matroids. For cycle matroids of finite graphs matroid connectivity translates into a purely graph theoretic notion. A graph \( G \) is \( k \)-Tutte-connected if for every \( \ell \leq k \) and every partition \( X, Y \) of its edge set into sets of at least \( \ell \) edges each, the number of vertices incident with both an edge in \( X \) and an edge in \( Y \) is greater than \( \ell \). Tutte [17] proved that a finite graph is \( k \)-Tutte-connected if and only if its cycle matroid is \( k \)-connected.

The main result of this work is an extension of this fact to infinite graphs and matroids. For this, let us call a graph \( G \) finitely separable if any two vertices may be separated by the deletion finitely many edges, and let us define its finite-cycle matroid, by declaring any edge set not containing the edge set of a finite cycle to be independent.

**Theorem 1.** Let \( k \geq 2 \) be an integer. A finitely separable graph is \( k \)-Tutte-connected if and only if its finite-cycle matroid is \( k \)-connected.

If the graph in the theorem is infinite, the finite-cycle matroid clearly will be infinite as well. But what does it mean for an infinite matroid to be \( k \)-connected? A finite matroid \( M \) is \( k \)-connected if for any \( \ell \leq k \) and any partition of its ground set into two sets \( X, Y \) of at least \( \ell \) elements each it follows that \( r(X) + r(Y) - r(M) \geq \ell \). Clearly, this definition is useless for infinite matroids as the involved ranks will usually be infinite. In [9] we therefore gave a rank-free definition that carries over to infinite matroids. To argue that our definition is
the right one, we showed that this notion of connectivity has the same properties as in finite matroids and we, furthermore, extended Tutte’s linking theorem to at least a large subclass of infinite matroids. Theorem 1 confirms our claim further.

In [7], we observed that any finitely separable graph has not one but two cycle matroids: The finite-cycle matroid and the cycle matroid, in which any edge set containing a finite or infinite cycle is said to be dependent. Here, an infinite cycle in the graph is the homeomorphic image of the unit circle in a natural topological space obtained from the graph (often by compactifying it). This definition was proposed by Diestel and Kühn in a completely graph-theoretical context and was subsequently seen to be extremely fruitful as it allows to extend virtually any result about cycles in a finite graph to at least a large class of infinite graphs; see Diestel [10] for an introduction.

The cycle matroid and the finite-cycle matroid coincide in a finite graph but will usually be different in infinite graphs. However, as we shall observe in Theorem 10, they always have the same connectivity and even the same connectivity function.

Finally, as an application of our argumentation, we get another extension of a result known for finite graphs: Tutte-connectivity is invariant under taking duals.

Theorem 2. [8] Let $G$ and $G^*$ be a pair of dual graphs, and let $k \geq 2$. Then $G$ is $k$-Tutte-connected if and only if $G^*$ is $k$-Tutte-connected.

We remark that this is not a new result. In [8] we gave a graph-theoretical proof. Here, we will see a matroidal variant.

2 Infinite cycles

A graph is finitely separable if any two vertices can be separated by finitely many edges. Let us fix a finitely separable graph $G = (V, E)$ in this section.

A ray of $G$ is a one-way infinite path. Two rays are edge-equivalent if for every finite set of edges $F$ there is a component of $G - F$ that contains subrays of both rays. The equivalence classes of this relation are the edge-ends $\mathcal{E}(G)$ of $G$.

We view the edges of $G$ as disjoint homeomorphic images of the unit interval $[0, 1]$, and define the quotient space $X_G$ by identifying these copies of $[0, 1]$ at their common endvertices. Let us define a topological space $||G||$ on $X_G \cup \mathcal{E}(G)$ by specifying the basic open sets: These are all sets of the form $C$, which consists of a topological component of $X_G - Z$ for some finite set $Z$ of inner points of edges together with all edge-ends that have a ray lying entirely in $C$. We remark that normally this space will not be Hausdorff: No edge-end can be separated from a vertex that sends infinitely many edge-disjoint paths to one of its rays. However, and this is the reason for imposing finite separability, two vertices may always be topologically distinguished. For a locally finite $G$, that is, a graph in which every vertex has finite degree, the space $||G||$ coincides with the Freudenthal compactification.

For us a cycle of $||G||$ is a homeomorphic image of the unit circle $S^1$ in $||G||$. This definition of cycles includes the traditional finite cycles but allows
also other cycles, which then contain necessarily infinitely many vertices and edges. An arc in \( |G| \) is the homeomorphic image of the unit interval \([0, 1]\). A standard subspace of \(|G|\) is the closure of a subgraph of \( G \) in \(|G|\). The set of edges that are completely contained in a standard subspace \( X \) are denoted by \( E(X) \). Cycles as well as arcs that have their endpoints in \( V \cup E(G) \) are standard subspaces \([10]\). A topological spanning tree of \(|G|\) is a standard subspace that is path-connected in \(|G|\) and which contains every vertex of \( G \) but no cycle. For more details see \([7]\).

In Figure 1 some of the introduced concepts are illustrated. The graph there, the double ladder, has two edge-ends, one to the left and one to the right. The infinite cycle \( C \) in bold lines goes through these two edge-ends. Moreover, while \( C + f \) is a spanning tree of the graph it is not (even including the two edge-ends) a topological spanning tree, simply because it contains the infinite cycle \( C \). On the other hand, \( C - e \) can be seen to be one. Its connectivity is ensured by the edge-ends.

3 Infinite matroids

As finite matroids, infinite matroids come with a number of different axiom systems. We only describe here the independence axioms. Let \( E \) be a set, let \( \mathcal{I} \subseteq 2^E \) be a set of subsets of \( E \), and denote by \( \mathcal{I}^{\text{max}} \) the sets in \( \mathcal{I} \) that are maximal under inclusion. We say that \( M = (E, \mathcal{I}) \) is a matroid with independent sets \( \mathcal{I} \) if the following axioms are satisfied:

(I1) \( \emptyset \in \mathcal{I} \).

(I2) \( \mathcal{I} \) is closed under taking subsets, that is if \( I \in \mathcal{I} \) and \( J \subseteq I \) then \( J \in \mathcal{I} \).

(I3) For all \( I \in \mathcal{I} \setminus \mathcal{I}^{\text{max}} \) and \( I' \in \mathcal{I}^{\text{max}} \) there is an \( x \in I' \setminus I \) such that \( I \cup \{x\} \in \mathcal{I} \).

(IM) The set \( \{ I' \in \mathcal{I} : I \subseteq I' \subseteq X \} \) has a maximal element, whenever \( I \subseteq X \subseteq E \) and \( I \in \mathcal{I} \).

Infinite matroids show the same properties as finite matroids. In particular, they possess bases (\( \subseteq \)-maximal independent sets), circuits (minimal dependent sets) and a natural notion of duality, in much of the same way as finite matroids, see \([11]\). We will use the normal matroid terminology. For instance, for any subset \( X \) of the ground set \( E \) of a matroid \( M \) we will write \( M|X \) for the restriction of \( M \) to \( X \), and we write \( M - X = M|(E \setminus T) \) for the matroid obtained by deleting the elements in \( X \) from \( M \).

In \([9]\), the connectivity function \( \kappa \) is extended to infinite matroids. For any \( X \subseteq E(M) \) in a matroid \( M \), choose a basis \( B \) of \( M|X \) and a basis \( B' \) of \( M - X \),
and pick a set $F \subseteq B \cup B'$ so that $(B \cup B') \setminus F$ is a basis of $M$. Then we set $\kappa_M(X) := |F| \in \mathbb{N} \cup \{\infty\}$ (we do not distinguish between different infinite cardinalities). We remark that the value $\kappa_M(X)$ is independent of the choice of the bases and of the choice of $F$. Moreover, $F$ may be chosen to be a subset of $B$ or of $B'$, if necessary. This definition of the connectivity function has similar properties as the traditional connectivity function of a finite matroid.

For finite matroids, the two notions coincide. For more details and a proof that $\kappa$ is well-defined, see [9].

We call a partition $(X, Y)$ of $E$ a $\ell$-separation if $\kappa_M(X) \leq \ell - 1$ and $|X|, |Y| \geq \ell$. The matroid $M$ is $k$-connected if there exists no $\ell$-separation with $\ell < k$.

Infinite graphs are a natural source of infinite matroids. Two dual matroids are normally associated with a finite graph, the cycle matroid and the bond matroid. These matroids can be extended verbatim to an infinite graph $G = (V, E)$, that we assume to be finitely separable. Let $I$ be the set of all edge sets $I \subseteq E$ not containing the edge set of any finite cycle of $G$. Then $I$ is the set of independent sets of a matroid $M_{FC}(G)$, the finite-cycle matroid of $G$. Its circuits are precisely the edge sets of cycles, and its bases coincide with the spanning forests, the sets that form a spanning tree on every component. In a similar fashion, we may now define a matroid whose circuits are the finite bonds, the finite-bond matroid $M_{FB}(G)$. However, $M_{FC}(G)$ and $M_{FB}(G)$ are no longer dual. Rather the dual of $M_{FB}(G)$ is the cycle matroid $M_c(G)$, whose circuits are precisely the edge sets of (finite or infinite) cycles of $|G|$. If $G$ is connected then the bases of $M_c(G)$ are the edge sets of topological spanning trees of $|G|$ and vice versa; see [7].

If the graph $G$ is infinite and 2-connected then the two matroids $M_{FC}(G)$ and $M_c(G)$ will differ. As an illustration, consider again the double ladder in Figure 1. The set of edges in bold will be independent in $M_{FC}(G)$ but not in $M_c(G)$.

### 4 Matroid connectivity in infinite graphs

In a graph $G$, denote for $X \subseteq E(G)$ by $V[X]$ the set of vertices that are incident with an edge in $X$. Let $c(X)$ be the number of components of the subgraph $(V[X], X)$ of $G$.

Our first aim is the following theorem:

**Theorem 3.** Let $G$ be a 2-connected finitely separable graph, and let $X \subseteq E(G)$, and $Y := E(G) \setminus X$. Then the following statements hold:

(i) $\kappa_{M_{FC}(G)}(X) = \infty$ if and only if $|V[X] \cap V[Y]| = \infty$; and

(ii) if $\kappa_{M_{FC}(G)}(X) < \infty$ then

$$\kappa_{M_{FC}(G)}(X) = |V[X] \cap V[Y]| - c(X) - c(Y) + 1.$$ 

Statement (ii) is exactly as for finite graphs when the traditional connectivity function is used, see Tutte [17]. We shall need two lemmas for the proof of Theorem 3.
Lemma 4. Let \( G \) be a finitely separable graph, and let \( D \) be an infinite set of edge-disjoint finite cycles. Then there exists an infinite subset \( D' \) of \( D \) and a vertex \( v \) of \( G \) so that any two distinct cycles in \( D' \) are disjoint outside \( v \).

Proof. Let \( C_1, C_2, \ldots \) be an enumeration of (countably many of) the cycles in \( D \). Inductively we will delete certain cycles from \( D \) while ensuring in each step that we keep infinitely many cycles. In step \( i \), assuming \( C_i \) has not been deleted, we go through the finitely many vertices of \( C_i \), one by one. Then for a vertex \( w \) of \( C_i \), unless \( w \) lies in all but finitely many of the remaining \( C_j \), we delete from \( D \) all those \( C_j \) that contain \( w \). If \( w \) lies in all but finitely many of the remaining \( C_j \), we skip to the next vertex of \( C_i \) without deleting any cycles. Denote the resulting infinite subset of \( D' \) by \( D \).

Now, if the cycles in \( D' \) are pairwise disjoint, choose any vertex of \( G \) for \( v \) and observe that this choice of \( D' \) and \( v \) is as desired. So, assume that there is a vertex \( v \) shared by two cycles in \( D' \). Pick the smallest index \( i \) for which there is a \( j \neq i \) so that \( C_i \) and \( C_j \) both contain \( v \) and so that \( C_i, C_j \in D' \). Note that \( v \), as well as any other vertex that lies in two cycles of \( D' \), is contained in infinitely many cycles in \( D' \); otherwise we would have deleted all but one of those cycles incident with \( v \).

Suppose there exists a second vertex \( w \) contained in two cycles of \( D' \). If \( k \) is the lowest index with \( w \in V(C_k) \) and \( C_k \in D' \) then why have we not deleted all those cycles \( C_i \) containing \( w \) with \( l > k \) from \( D' \) in step \( k \)? Precisely because all but finitely many of the cycles in \( D' \) contain \( w \). In particular, infinitely many of those cycles in \( D' \) that contain \( v \) must also contain \( w \). By picking a \( v-w \) path in each of those cycles we obtain infinitely many edge-disjoint \( v-w \) paths, which is impossible in a finitely separable graph.

The following lemma is a straightforward combination of Lemmas 4.1 and 4.2 in [8]:

Lemma 5. Let \( G \) be a 2-connected finitely separable graph, and let \( X', Y' \) be edge sets of \( G \) so that there are infinitely many vertices that are incident with both an edge in \( X' \) and an edge in \( Y' \). Then there are infinitely many edge-disjoint finite cycles in \( G \), each of which contains an edge of \( X' \) and \( Y' \).

We now prove a first part of Theorem 3:

Lemma 6. Let \( G \) be a 2-connected finitely separable graph, and let \( X \subseteq E(G) \), and \( Y := E(G) \setminus X \). If \( |V[X] \cap V[Y]| = \infty \) then \( \kappa_{MFC(G)}(X) = \kappa_{MFC(G)}(Y) = \infty \).

Proof. For each vertex in \( V[X] \cap V[Y] \) pick one incident edge in \( X \) and one in \( Y \); denote the set of these edges by \( X' \subseteq X \) and \( Y' \subseteq Y \), respectively. Applying Lemma 5 in conjunction with Lemma 4, we obtain a vertex \( v \) and an infinite set \( D \) of finite cycles, each of which contains an edge of \( X \) and of \( Y \), and so that any two cycles either meet only in \( v \) or not at all. As no cycle in \( D \) has its edge set entirely in \( X \) or entirely in \( Y \) it follows that neither \( I_X := X \cap \bigcup_{C \in D} E(C) \) nor \( I_Y := Y \cap \bigcup_{C \in D} E(C) \) contains the edge set of a finite cycle of \( G \). To see that also neither contains the edge set of an infinite cycle, observe that each of the graphs \( (V[I_X], I_X) - v \) and \( (V[I_Y], I_Y) - v \) is the union of (vertex-)disjoint (finite) paths, and therefore none contains a ray.

Thus \( I_X \) and \( I_Y \) are independent in both matroids \( MFC(G) \) and \( MC(G) \). Let \( T_X \) be a basis of \( M[X] \) containing \( I_X \), and let \( T_Y \supseteq I_Y \) be a basis of \( M[Y] \), where
Choose a set of edges $T$ any finite circuit, and is thus independent in the edge set of a spanning tree of $G$. Hence $\kappa_M(X) = |F| = \infty$. \hfill \Box

**Proof of Theorem 3.**

(i) By Lemma 6 we only need to consider the case when $|V[X] \cap V[Y]| < \infty$. Pick a basis $T_X$ of $M_{FC}(G)|X$, and let $T_Y$ be a basis of $M_{FC}(G)|Y$. Because $G$ is finitely separable, there is a finite set of edges separating $u$ from $v$ in $(V[X], X)$, for every of the finitely many pairs of vertices $u, v \in V[X] \cap V[Y]$. Denote by $F$ the union of all those edges, and observe that $F$ is a finite edge set. By the choice of $F$ the set $(T_X \cup T_Y) \setminus F$ cannot contain any finite circuit, and is thus independent in $M_{FC}(G)$. As $|F| < \infty$ is therefore an upper bound for $\kappa_{M_{FC}(G)}(X)$ the result follows.

(ii) Pick a spanning tree on every component of $(V[X], X)$ and denote the union of their edge sets by $T_X$. We define $T_Y$ for $(V[Y], Y)$ in a similar way. Choose a set of edges $F \subseteq X$ so that $(T_X \cup T_Y) \setminus F$ is a basis of $M_{FC}(G)$, i.e. the edge set of a spanning tree of $G$.

We claim that

$$\text{if } c(X) = c(Y) = 1 \text{ then } \kappa_{M_{FC}(G)}(X,Y) = |V[X] \cap V[Y]| - 1. \quad (1)$$

Let us prove the claim. Each vertex of $U := V[X] \cap V[Y]$ must lie in a distinct component of $(V[T_X], T_X \setminus F)$ since otherwise there exists a path in $(V[T_X], T_X \setminus F)$ that starts and ends in $U$ but is otherwise disjoint from $U$. This path can be extended with edges in $T_Y$ to a finite cycle that still misses $F$, which is impossible as $(T_X \cup T_Y) \setminus F$ is the edge set of a tree. As $(V[T_X], T_X)$ is connected and as each deletion of a single edge increases the number of components by exactly one, we obtain $|F| \geq |U| - 1$. Suppose, on the other hand, that $|F| > |U| - 1$. Then there exists a component of $(V[T_X], T_X) \setminus F$ that contains no vertex of $U$. Pick an edge $e \in F$ with one of its endvertices in this component. Setting $T := (T_X \setminus F) \cup T_Y$, we observe that $\{e\}$ is a cut of $(V[T], T + e)$. However, as $T$ is the edge set of a spanning tree of $G$, there has to be a cycle in $T + e$ containing $e$, a contradiction. This proves (1).

We now proceed by induction on $c(X) + c(Y)$, which is indeed a finite number as $|V[X] \cap V[Y]|$ is an upper bound for both $c(X)$ and $c(Y)$. Since the induction start is established by (1), we may assume that $(V[X], X)$ has two components $K$ and $K'$. Insert a new edge $f$ between $K$ and $K'$, and set $G' := G + f$ and $X' := X \cup \{f\}$. Clearly, $(X', Y)$ is a partition of $E(G')$. Since $c(X') = c(X) - 1$, the induction yields

$$\kappa_{M_{FC}(G')}(X', Y) = |V[X] \cap V[Y]| - (c(X) - 1) - c(Y) + 1.$$

We shall now show that $\kappa_{M_{FC}(G')}(X', Y) = \kappa_{M_{FC}(G)}(X, Y) + 1$. Observe that then $T_X + f$ is (the edge set of) a maximal spanning forest of $(V[X'], X') \subseteq G'$. Moreover, $(T_X \setminus F) \cup T_Y = ((T_X + f) \setminus (F \cup \{f\})) \cup T_Y$ is a spanning tree of $G'$, too. Thus

$$\kappa_{M_{FC}(G')}(X, Y') = |F \cup \{f\}| = |F| + 1 = \kappa_{M_{FC}(G)}(X,Y) + 1,$$

which finishes the proof. \hfill \Box
Next, let us show that the connectivity functions of $M_{FC}(G)$ and $M_{C}(G)$ coincide. For this, we should be able to modify the proof of Theorem 3 in order to make it work for $M_{C}(G)$, too. Rather then repeating the argument we will pursue a different approach, for which we will need a small lemma and a result from [6].

**Lemma 7.** Let $G$ be a finitely separable graph, and let $H$ be an induced subgraph of $G$ so that $N(G - H)$ is a finite set. Then every cycle $C \subseteq H$ of $|G|$ contains a cycle of $|H|$.

To prove the lemma, we use a theorem that is the direct consequence of Theorems 6.3 and 6.5 of Diestel and Kühn [12]:

**Theorem 8** (Diestel and Kühn [12]). Let $Z$ be a set of edges in a finitely separable graph $G$. Then $Z$ is the edge set of an edge-disjoint union of cycles of $|G|$ if and only if $Z$ meets every finite cut of $G$ in an even number of edges.

**Proof of Lemma 7.** Consider such a cycle $C$ of $|G|$ that is completely contained in $H$, and suppose that $E(C)$ is not the edge set of an edge-disjoint union of cycles of $|H|$. By Theorem 8 there is a finite cut $F$ of $H$ so that $E(C) \cap F$ is an odd set. The cut $F$ partitions $N(G - H)$ into two sets $A$ and $B$ (one of them possibly empty). Since every two vertices in $G$ can be separated by finitely many edges there is a finite subset of $E(G) \setminus E(H)$ that separates $A$ from $B$ in $G - E(H)$. Choosing a minimal such set $F'$ ensures that $F \cup F'$ is a finite cut of $G$. Then $|E(C) \cap (F \cup F')| = |E(C) \cap F|$ is odd, implying with Theorem 8 that $E(C)$ is not the edge set of an edge-disjoint union of cycles of $|G|$, in particular that $C$ is not a cycle of $|G|$, a contradiction. \( \square \)

We will make use of the fact that for a connected and finitely separable graph $G$ there is always a common basis of $M_{FC}(G)$ and $M_{C}(G)$:

**Theorem 9.** [6] Every connected finitely separable graph $G$ has a spanning tree that does not contain the edge set of any (infinite) cycle of $|G|$.

**Theorem 10.** Let $G$ be a 2-connected finitely separable graph. Then $\kappa_{M_{FC}(G)}(X) = \kappa_{M_{C}(G)}(X)$ for all $X \subseteq E(G)$.

**Proof.** Consider a set $X \subseteq E(G)$ and put $Y := E(G) \setminus X$. If $V[X] \cap V[Y]$ is an infinite set then $\kappa_{M_{FC}(G)}(X) = \kappa_{M_{C}(G)}(X)$ by Lemma 8.

So, assume $V[X] \cap V[Y]$ to be finite. By Theorem 8 there is for each component $K$ of $(V[X], X)$ a spanning tree not containing the edge set of any cycle of $|K|$. Lemma 7 ensures that also no edge set of any cycle of $|G|$ lies in this spanning tree. Consequently, the union $T_X$ of the edge sets of those spanning trees is a basis of $M_{FC}(G)|X$ as well as of $M_{C}(G)|X$. We define $T_Y$ analogously for $(V[Y], Y)$.

Next, pick $F \subseteq T_X \cup T_Y \setminus (T_X \cup T_Y) \setminus F$ is a basis of $M_{C}(G)$. Clearly, the set $(T_X \cup T_Y) \setminus F$ is independent in $M_{FC}(G)$, too. If it is even a basis in $M_{FC}(G)$ then we have $\kappa_{M_{FC}(G)}(X) = |F| = \kappa_{M_{C}(G)}(X)$ as desired. So, suppose $T := (T_X \cup T_Y) \setminus F$ fails to be a basis, which implies that $(V[T], T)$ is not (graph-theoretically) connected. As a basis of $M_{C}(G)$ for a connected graph, $T$ is the edge set of a topological spanning tree of $|G|$. In particular, the topological spanning tree is path-connected and will therefore contain an arc $A$ between two vertices of distinct (graph-theoretical) components of $(V[T], T)$. The arc
We say that a graph \( \ell < k \) has \( \ell \)-Tutte-connectivity if \( k \) vertices or \( k \)-Tutte-separation with \( \ell \)-separation for any \( \ell < k \).

The following theorem clearly includes Theorem 1:

**Theorem 11.** Let \( G \) be a finitely separable graph. Then for integers \( k \geq 2 \) the following statements are equivalent:

(i) \( G \) is \( k \)-Tutte-connected;

(ii) \( M_{\text{FC}}(G) \) is \( k \)-connected; and

(iii) \( M_{\text{C}}(G) \) is \( k \)-connected.

**Proof.** Observe that we may assume \( G \) to be 2-connected and that \( G \) is an infinite graph. (For finite graphs, see Tutte [17]—note that \( M_{\text{FC}}(G) \) and \( M_{\text{C}}(G) \) coincide in this case.) In light of Theorem 10 we only need to prove that \( G \) has a \( k \)-Tutte-separation with \( k \leq m \) if and only if \( M_{\text{FC}}(G) \) has an \( \ell \)-separation with \( \ell \leq m \).

First, let \( (X, Y) \) be a \( k \)-Tutte-separation \( (X, Y) \) of \( G \), which implies \( |V[X] \cap V[Y]| \leq k \). Since \( c(X), c(Y) \geq 1 \) this yields with Theorem 3 that \( \kappa_{M_{\text{FC}}(G)} \leq k - 1 \). Consequently, \( (X, Y) \) is a \( k \)-separation of \( M_{\text{FC}}(G) \).

Conversely, let there be an \( \ell \)-separation in \( M_{\text{FC}}(G) \), and choose an \( \ell \)-separation \( (X, Y) \) of \( M_{\text{FC}}(G) \) so that \( c(X) + c(Y) \) is minimal among all \( \ell \)-separations of \( M_{\text{FC}}(G) \). Since \( G \) is infinite, we may assume that \( Y \) is an infinite set.

First, we claim that

\[
(V[Y], Y) \text{ is connected.}
\]

(2)

If \( (V[Y], Y) \) is not connected then there is a component \( K \) of \( (V[Y], Y) \) so that \( Y' := Y \setminus E(K) \) is an infinite set. With \( X' := X \cup E(K) \) we see that both \( X' \) and \( Y' \) have at least \( \ell \) elements. Moreover, it holds that \( |V[X] \cap V[Y]| = |V[X'] \cap V[Y']| + |V[X] \cap V[K]| \) and \( c(Y) = c(Y') + 1 \). The set of components of \( (V[X'], X') \) is comprised of components of \( (V[X], X) \) and of the union of those components of \( (V[X], X) \) that have a vertex with \( K \) in common together with \( K \). Since there are at most \( |V[X] \cap V[K]| \) components of the latter kind, we

\[\begin{align*}
\text{Theorem 11.} & \quad \text{Let } G \text{ be a finitely separable graph. Then for integers } k \geq 2 \text{ the following statements are equivalent:} \\
(i) & \quad G \text{ is } k\text{-Tutte-connected;} \\
(ii) & \quad M_{\text{FC}}(G) \text{ is } k\text{-connected; and} \\
(iii) & \quad M_{\text{C}}(G) \text{ is } k\text{-connected.}
\end{align*}\]
obtain $c(X) \leq c(X') + |V[X] \cap V[K]| - 1$. It follows with Theorem 3 that $$\kappa_{Mec(G)}(X', Y') = |V[X'] \cap V[Y']| - c(X') - c(Y') + 1 \leq |V[X] \cap V[Y]| - |V[X] \cap V[K]| - c(X)$$ $$+ |V[X] \cap V[K]| - 1 - c(Y') + 1 + 1 = |V[X] \cap V[Y]| - c(X) - c(Y) + 1 + 1 \leq \ell - 1.$$ Thus, $(X', Y')$ is an $\ell$-separation with $c(X') + c(Y') < c(X) + c(Y)$, contradicting the choice of $(X, Y)$.

Second, we show that $$|V[K] \cap V[Y]| \leq \ell \text{ for every component } K \text{ of } (V[X], X). \tag{3}$$ Suppose there exists a component $M$ of $(V[X], X)$ with $|V[M] \cap V[Y]| \geq \ell + 1$. Denoting by $\mathcal{K}$ the components of $(V[X], X)$ we get $$\ell - 1 \geq |V[X] \cap V[Y]| - c(X) - c(Y) + 1 \geq \sum_{K \in \mathcal{K} \setminus \{M\}} |V[K] \cap V[Y]| + (\ell + 1) - c(X) - c(Y) + 1.$$ That $G$ is connected implies $|V[K] \cap V[Y]| \geq 1$ for every $K \in \mathcal{K}$. Hence $$\ell - 1 \geq (c(X) - 1) + (\ell + 1) - c(X) - c(Y) + 1 = \ell + 1 - c(Y).$$ This yields $c(Y) \geq 2$, which is impossible by (2). Therefore, (3) is proved.

Next, we see that there is a component $M$ of $(V[X], X)$ with $|E(M)| \geq |V[M] \cap V[Y]|$. \tag{4}

If (4) is false then we have $|V[K] \cap V[Y]| \geq |E(K)| + 1$ for all $K \in \mathcal{K}$. This, however, implies with $c(Y) = 1$ that $$\ell - 1 \geq |V[X] \cap V[Y]| - c(X) - c(Y) + 1 = \sum_{K \in \mathcal{K}} |V[K] \cap V[Y]| - c(X) \geq \sum_{K \in \mathcal{K}} (|E(K)| + 1) - c(X) = |X|.$$ As $(X, Y)$ is an $\ell$-separation, $X$ is required to have at least $\ell$ elements, which shows that (1) holds.

Finally, with the component $M$ from (4) we set $\bar{X} := E(M)$ and $\bar{Y} := E(G) \setminus E(M)$. Then $k := |V[\bar{X}] \cap V[\bar{Y}]| = |V[M] \cap V[Y]| \leq \ell$, by (3). As $|\bar{X}| \geq k$ and $|\bar{Y}| = \infty$ it follows that $(\bar{X}, \bar{Y})$ is a $k$-Tutte-separation with $k \leq \ell$, as desired.

We remark that the arguments in the proof are not new. Indeed, (2) is inspired by Tutte \cite{17} and steps (3), (4) are quite similar to the proof of Lemma 5.3 in \cite{8}. 
6 Tutte-connectivity and duality

In this final section, we deduce a matroidal proof of the fact that Tutte-connectivity is invariant under duality (Theorem 2).

Two finitely separable countable graphs $G$ and $G^*$ defined on the same edge set $E$ are a pair of duals if any edge set $F \subseteq E$ is the edge set of a cycle of $|G|$ if and only if $F$ is a bond of $G^*$. (A bond is a minimal non-empty cut.) As for finite graphs, a (countable) finitely separable graph is planar if and only if it has a dual, see [6] for a proof and more details.

We need two more results.

Lemma 12. [9] The connectivity function is invariant under duality, that is, $\kappa_M(X) = \kappa_{M^*}(X)$ for any subset $X$ of a matroid $M$.

Theorem 13. [7] Let $G$ and $G^*$ be a pair of countable dual graphs, each finitely separable, and defined on the same edge set $E$. Then $M^*_C(G) = M^*_{FC}(G^*)$.

Consider a pair of countable dual graphs $G$ and $G^*$. Then, by Theorem 11 $G$ is $k$-Tutte-connected if and only if $M^*_{FC}(G)$ is $k$-connected. Since $M^*_C(G) = (M^*_C(G^*))^*$ by Theorem 13 and since matroid connectivity is invariant under taking duals (Lemma 12) this is precisely the case when $M^*_C(G^*)$ is $k$-connected. Finally, Theorem 11 again shows that $M^*_C(G^*)$ is $k$-connected if and only if $G^*$ is $k$-Tutte-connected. This proves Theorem 2.

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