We give a new formula for the antipode of the algebra of rooted trees, directly in terms of the bialgebra structure. The equivalence, proved in this paper, among the three available formulae for the antipode, reflects the equivalence among the Bogoliubov–Parasiuk–Hepp, Zimmermann, and Dyson–Salam renormalization schemes.

Keywords: Antipode, rooted tree, renormalization.

1. Introduction

More than two years ago, Kreimer [9] discovered that there is a Hopf algebra structure encoding Zimmermann’s forest formula [14] in perturbative renormalization theory. Shortly afterwards, an essential coincidence was found between Kreimer’s algebra and the Hopf algebras introduced by Connes and Moscovici in connection with the index problem for $K$-cycles on foliations [5].

A unified treatment in terms of the algebra of rooted trees $H_R$ was developed in [2]: there to each (superficially divergent) Feynman diagram a sum of rooted trees is assigned; the assignment is straightforward when the diagram contains only disjoint or nested subdivergences (as it then leads to a single tree), but it does not work smoothly for overlapping divergences [13].

The central role in the application of Kreimer–Connes–Moscovici algebras is played by the antipode. In [2] two equivalent definitions of the antipode in $H_R$ were given, representing respectively —in the framework of the algebra of rooted trees— the recursive Bogoliubov–Parasiuk–Hepp procedure for renormalizing Feynman integrals with subdivergences, and Zimmermann’s forest formula which solves that recursion; that indeed they correspond to the antipode of the Hopf algebra of rooted trees is implied rather than proven.

Here we construct the antipode for $H_R$, giving a new formula for computing it in terms of the coproduct; and then we show its equivalence to each of the formulae by Connes and Kreimer. It turns out that this new formula corresponds to the Dyson–Salam procedure for renormalization.

2. The antipode of the Hopf algebra of rooted trees

To establish the notation, we briefly recall some basic facts concerning the antipode of a Hopf algebra (consult [1,6,8,12] for proofs), and then the algebra of rooted trees.

Given a unital algebra $(A, m, u)$ and a counital coalgebra $(C, \Delta, \varepsilon)$ over a field $\mathbb{F}$, the convolution of two elements $f, g$ of the vector space of $\mathbb{F}$-linear maps $\text{Hom}(C, A)$ is defined as the map $f \ast g \in \text{Hom}(C, A)$ given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A.$$
This product turns $\text{Hom}(C, A)$ into a unital algebra, where the unit is the map $u \circ \varepsilon$. In this paper $F$ is the field of real numbers $\mathbb{R}$.

A bialgebra $H = (A, m, u, \Delta, \varepsilon)$ in which the identity map $id_H$ is invertible under convolution is a Hopf algebra, and its (necessarily unique) convolution inverse $S$ is called the antipode. The property $id_H * S = S * id_H = u \circ \varepsilon$ boils down to the commutativity of the diagram:

\[
\begin{array}{c}
H \otimes H & \xrightarrow{\Delta} & H & \xrightarrow{\Delta} & H \otimes H \\
\downarrow \scriptstyle{\text{id} \otimes S} & & \downarrow \scriptstyle{u \circ \varepsilon} & & \downarrow \scriptstyle{S \otimes \text{id}} \\
H \otimes H & \xrightarrow{m} & H & \xleftarrow{m} & H \otimes H.
\end{array}
\] (2.1)

In particular, if $\Delta(a) = \sum_j a'_j \otimes a''_j$, then

\[
\varepsilon(a) 1_H = u \circ \varepsilon(a) = m \circ (\text{id} \otimes S) \circ \Delta(a) = \sum_j a'_j S(a''_j),
\] (2.2)

and likewise $\varepsilon(a) 1_H = \sum_j S(a'_j)a''_j$.

The antipode is always a unital algebra and a counital coalgebra antihomomorphism. When $H$ is either commutative or cocommutative, $S^2 = \text{id}$; in particular, $S$ is bijective in such cases. Another important property is that a bialgebra morphism $\ell: H \rightarrow H'$ between Hopf algebras is automatically compatible with the antipodes: $\ell \circ S = S' \circ \ell$ [7,12].

A rooted tree is a finite set of points, called vertices, joined by oriented lines that do not intersect, so that all the vertices have exactly one incoming line, except the root which has only outgoing lines. In particular, there is a unique branch that joins the root with any other vertex. One actually works with isomorphic classes of trees. Two rooted trees are isomorphic if the number of vertices with given length and fertility is the same for all possible choices of lengths and fertilities, where the fertility of a vertex is the number of its outgoing lines and its length is the number of lines that make up the unique branch joining it to the root.

For concrete examples, it will be convenient to have a list of a few isomorphic classes of rooted trees, say with four vertices or fewer:

\[\diamond t_1 \diamond t_2 \diamond t_31 \diamond t_32 \diamond t_41 \diamond t_42 \diamond t_43 \diamond t_44\]

A simple cut $c$ of a tree $T$ is a subset of its lines such that the path from the branch to any other vertex includes at most one line of $c$. Deleting the cut branches produces several subtrees; the component containing the original root (the trunk) is denoted $R_c(T)$. The remaining branches also form rooted trees, where in each case the new root is the vertex immediately below the deleted line; $P_c(T)$ denotes the set of these pruned branches. Here, for instance, are the possible simple cuts of $t_{42}$:
The set of nontrivial simple cuts of a tree $T$ will be denoted by $C(T)$; we consider also the “empty cut” $c = \emptyset$, for which $R_{\emptyset}(T) = T$ and $P_{\emptyset}(T) = \emptyset$.

The algebra of rooted trees $H_R$ is the commutative algebra generated by symbols $T$, one for each isomorphism class of rooted trees, plus a unit 1 corresponding to the empty tree; the product of trees is written as the juxtaposition of their symbols. The counit $\varepsilon : H_R \to \mathbb{R}$ is the linear map defined by $\varepsilon(1) := 1\mathbb{R}$ and $\varepsilon(T_1T_2\ldots T_n) = 0$ if $T_1, \ldots, T_n$ are trees. Kreimer defined a map $\Delta : H_R \to H_R \otimes H_R$ on the generators, extending it as an algebra homomorphism, as follows:

$$\Delta 1 := 1 \otimes 1; \quad \Delta T := T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T). \quad (2.3)$$

Notice that $P_c(T)$ is the product of the several subtrees pruned by the cut $c$. For instance,

$$\begin{align*}
\Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1, \\
\Delta(t_2) &= t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1, \\
\Delta(t_{31}) &= t_{31} \otimes 1 + 1 \otimes t_{31} + t_2 \otimes t_1 + t_1 \otimes t_2, \\
\Delta(t_{32}) &= t_{32} \otimes 1 + 1 \otimes t_{32} + 2t_1 \otimes t_2 + t_1^2 \otimes t_1, \\
\Delta(t_{41}) &= t_{41} \otimes 1 + 1 \otimes t_{41} + t_{31} \otimes t_1 + t_2 \otimes t_2 + t_1 \otimes t_{31}, \\
\Delta(t_{42}) &= t_{42} \otimes 1 + 1 \otimes t_{42} + t_1 \otimes t_{32} + t_2 \otimes t_2 + t_1 \otimes t_{31} + t_2t_1 \otimes t_1 + t_1^2 \otimes t_2, \\
\Delta(t_{43}) &= t_{43} \otimes 1 + 1 \otimes t_{43} + 3t_1 \otimes t_{32} + 3t_1^2 \otimes t_2 + t_1^3 \otimes t_1, \\
\Delta(t_{44}) &= t_{44} \otimes 1 + 1 \otimes t_{44} + t_{32} \otimes t_1 + 2t_1 \otimes t_{31} + t_1^2 \otimes t_2.
\end{align*} \quad (2.4)$$

A most useful tool is the sprouting of a new root; namely the morphism $L : H_R \to H_R$ given by the linear map defined by

$$L(T_1 \ldots T_k) := T,$$

where $T$ is the rooted tree obtained by conjuring up a new vertex as its root and extending lines from this vertex to each root of $T_1, \ldots, T_k$. For instance,

$$L(\textcircled{1} \textcircled{2}) = \textcircled{1} \textcircled{2} \quad \text{and} \quad L(\textcircled{1} \textcircled{2}) = \textcircled{1} \textcircled{2} \quad (2.5)$$

The proof that $\Delta$ is indeed a coproduct is based on the formula

$$\Delta \circ L = L \otimes 1 + (\text{id} \otimes L) \circ \Delta. \quad (2.6)$$

For details see [2] or [6].

When dealing with particular Hopf algebras, the antipode is often determined by specific properties of the algebras in question, and the defining property of the antipode is scarcely used. The latter turns out to be extremely useful in our context, however. We compute the antipode $S : H_R \to H_R$ by exploiting its very definition as the convolution inverse of the identity in $H_R$, via a geometric series:

$$S := (\text{id})^{*1} = (u \circ \varepsilon - (u \circ \varepsilon - \text{id}))^{*1} = u \circ \varepsilon + (u \circ \varepsilon - \text{id}) + (u \circ \varepsilon - \text{id})^2 + \cdots$$
Lemma 2.1. If \( T \) is a rooted tree with \( n \) vertices, the geometric series expansion of \( S(T) \) has at most \( n + 1 \) terms.

Proof. The claim is certainly true for \( t_1 \). Assume that it holds for all trees with \( n \) vertices. Let \( T \) be a rooted tree with \( n + 1 \) vertices; then

\[
(u \circ \varepsilon - \text{id})^{*(n+2)}(T) = (u \circ \varepsilon - \text{id}) * (u \circ \varepsilon - \text{id})^{*(n+1)}(T)
\]

\[
= m \circ [(u \circ \varepsilon - \text{id}) \otimes (u \circ \varepsilon - \text{id})^{*(n+1)}] \circ \Delta(T)
\]

\[
= m \circ [(u \circ \varepsilon - \text{id}) \otimes (u \circ \varepsilon - \text{id})^{*(n+1)}] \left( T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T) \right).
\]

The first and second term vanish because \((u \circ \varepsilon - \text{id})1 = 0\). By the induction hypothesis the third term is zero. \(\square\)

As an immediate corollary we obtain that \( S \) so defined is indeed the antipode.

One of the advantages of this formulation is that we obtain a fully explicit formula for \( S \) from the coproduct table. If \( a \in H^n, \Delta(a) = \sum_{i_1} a'_{i_1} \otimes a''_{i_1}, \Delta(a'_{i_1}) = \sum_{i_2} a'_1 i_2 \otimes a''_{i_1 i_2} \) and in general \( \Delta(a''_{i_1,...,i_k}) = \sum_{i_{k+1}} a'_{i_1,...,i_{k+1}} \otimes a''_{i_1,...,i_{k+1}}, \) then

\[
(u \circ \varepsilon - \text{id})^{k+1}(a) = (-1)^{k+1} \sum_{i_1,...,i_k} b'_{i_1} b''_{i_1 i_2} \cdots b'_{i_1,...,i_k} b''_{i_1,...,i_k},
\]

where

\[
b'_{i_1,...,i_j} := \begin{cases} 0 & \text{if } a'_{i_1,...,i_j} = 1 \text{ or } a''_{i_1,...,i_j} = 1, \\ a'_{i_1,...,i_j} & \text{otherwise}, \end{cases}
\]

and

\[
b''_{i_1,...,i_j} := \begin{cases} 0 & \text{if } a''_{i_1,...,i_j} = 1, \\ a''_{i_1,...,i_j} & \text{otherwise}. \end{cases}
\]

For instance, using (2.4),

\[
S(t_{42}) = -t_{42} + (t_1 t_{32} + t_2^2 + t_1 t_{31} + 2t_1^2 t_2) - (5t_1^2 t_2 + 2t_1^4) + 3t_1^3
\]

\[
= -t_{42} + t_1 t_{32} + t_2^2 + t_1 t_{31} - 3t_1^2 t_2 + t_1^4.
\]

Similarly, if we denote by \( t' \) the rooted tree in (2.5) with 5 vertices, then

\[
S(t') = -t' + (2t_1 t_{42} + 2t_2 t_{31} + t_1^2 t_{32} + 3t_1 t_2^2)
\]

\[
- (2t_1^2 t_{32} + 6t_1 t_2^2 + 2t_1^2 t_{31} + 8t_1^3 t_2 + t_1^5) + (12t_1^3 t_2 + 6t_1^5) - 6t_1^5
\]

\[
= -t' + 2t_1 t_{42} + 2t_2 t_{31} - t_1^2 t_{32} - 3t_1 t_2^2 - 2t_1^2 t_{31} + 4t_1^3 t_2 - t_1^5.
\]

The reader will find without difficulty that these formulae correspond to the original Dyson–Salam procedure for renormalizing Feynman graphs with subdivergences: see, for instance, [11].
In correspondence with Bogoliubov’s recursive formula for renormalization, equations
\( m \circ (S \otimes \text{id}) \circ \Delta(T) = 0 \) and (2.4) suggest to define the antipode recursively, as indeed done
by Connes and Kreimer:

\[
S_B(T) := -T - \sum_{c \in C(T)} S_B(P_c(T)) R_c(T).
\]

For instance,

\[
S_B(t_{42}) = -t_{42} - S_B(t_1)t_{32} - S_B(t_2)t_2 - S_B(t_1)t_{31} - S_B(t_2t_1)t_1 - S_B(t_1^2)t_2,
\]

which gives again (2.7). We next check that \( S_B \) is indeed the antipode.

**Proposition 2.2.** If \( T \) is any rooted tree, then \( S(T) = S_B(T) \).

**Proof.** For convenience, we abbreviate \( \eta := u \circ \varepsilon - \text{id} \). The statement holds, by a direct
check, if \( T \) has 1, 2 or 3 vertices. If it holds for all rooted trees with at most \( n \) vertices and
if \( T \) is a rooted tree with \( n + 1 \) vertices, then

\[
S(T) = \eta(T) + \sum_{j=1}^{n} \eta^* T \eta(T) = -T + m \circ \left( \sum_{j=1}^{n} \eta^* \otimes \eta \right) \circ \Delta(T)
\]

\[
= -T + m \circ \sum_{j=1}^{n} \eta^* \otimes \eta \left( T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T) \right)
\]

\[
= -T - \sum_{c \in C(T)} \sum_{j=1}^{n} \eta^* \left( P_c(T) \right) R_c(T)
\]

\[
= -T - \sum_{c \in C(T)} S_B(P_c(T)) R_c(T) = S_B(T),
\]

where the penultimate equality uses the inductive hypothesis. \( \square \)

Zimmermann’s forest formula corresponds to the following nonrecursive formula for
the antipode:

\[
S_Z(1) := 1, \quad S_Z(T) := - \sum_{d \in D(T)} (-1)^{\#d} P_d(T) R_d(T),
\]

where \( D(T) \) is the set of all cuts, not necessarily simple, including the empty cut, and \( \#d \)
is the cardinality of \( d \).

**Proposition 2.3.** If \( T \) is any rooted tree, then \( S(T) = S_Z(T) \).

**Proof.** First we prove that, for an arbitrary rooted tree \( T \),

\[
S(L(T)) = -L(T) - S(T) t_1 - \sum_{c \in C(T)} S(P_c(T)) L(R_c(T)). \tag{2.8}
\]
Indeed, if \( T \) has \( n \) vertices, then, by Lemma 2.1 and (2.6),

\[
S(L(T)) = -L(T) + m \circ \left( \sum_{j=1}^{n} \eta^{*j} \otimes \eta \right) \circ \Delta(L(T))
\]

\[
= -L(T) + m \circ \sum_{j=1}^{n} \eta^{*j} \otimes \eta (L(T) \otimes 1 + (\text{id} \otimes L) \circ \Delta(T))
\]

\[
= -L(T) + m \circ \sum_{j=1}^{n} \eta^{*j} \otimes \eta \\
\left( L(T) \otimes 1 + T \otimes t_1 + 1 \otimes L(T) + \sum_{c \in C(T)} P_c(T) \otimes L(R_c(T)) \right)
\]

\[
= -L(T) - S(T) t_1 - \sum_{c \in C(T)} S(P_c(T)) L(R_c(T)).
\]

Suppose that \( S_Z \) were also to satisfy (2.8). Since any rooted tree can be written as an image of \( L \), and on the right side \( S \) is applied only to rooted trees of strictly fewer vertices, the proposition will follow by induction on the number of the vertices. It remains, therefore, to prove that (2.8) holds for \( S_Z \).

For a given rooted tree \( T \), let \( \ell_0 \) be the new line in \( L(T) \), and \( v_1, \ldots, v_k \) the vertices of length one with respect to the root of \( T \). Now \( D(L(T)) = A \sqcup B \) where \( d \in A \) or \( B \) according as \( \ell_0 \in d \) or not. Thus,

\[
S_Z(L(T)) = - \left( \sum_{d \in A} \sum_{d \in B} \right) (-1)^{#d} P_d(L(T)) R_d(L(T)). \tag{2.9}
\]

If \( d \in A \), then \( e = d \setminus \{\ell_0\} \) is a cut of \( T \); moreover, \( R_d(L(T)) = t_1 \), \( P_d(L(T)) = P_e(T) R_e(T) \) and \( #d = #e + 1 \), so that the first sum of (2.9) equals \( -S(T) t_1 \).

For a given \( d \in B \setminus \{\emptyset\} \) and each \( j \in K := \{1, \ldots, k\} \), let \( \ell_j \) be the line in \( d \) closer to the root that is linked to \( v_j \) (if any). Then \( c' := \{\ell_j : j \in K\} \) is a simple cut of \( T \). If \( #c' \) is odd, we set \( c := c' \), whereas if \( #c' \) is even, we set \( c := c' \setminus \{\ell_s\} \), where \( s \) is the smallest integer in \( K \) for which there is a line with the required property. In either case, we take \( e := d \setminus c \). Clearly \( R_d(L(T)) = L(R_c(T)) \), \( (-1)^{#d+1} = (-1)^{#e} \), and \( P_d(L(T)) = P_e(T) R_e(T) \), where we use the temporary notation \( T_e := P_e(T) \). It follows that the second sum of (2.9) equals

\[
\sum_{c \in C(T)} \sum_{e \in D(T_e)} (-1)^{#e} P_e(T_e) R_e(T_e) L(R_c(T)) = - \sum_{c \in C(T)} S_Z(P_c(T)) L(R_c(T)).
\]

Finally, since the summand for the empty cut is \(-L(T)\), the proposition is proved. \( \square \)

In summary, modulo the distiction between the antipode and the “twisted” or “renormalized” antipode [3,4,10], Kreimer’s algebraic approach reflects, within the framework of the algebra of rooted trees, the equivalence of the Dyson–Salam, the Bogoliubov–Parasiuk–Hepp and the Zimmermann procedures for renormalizing Feynman diagrams.
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