Spectral Analysis of the Semi-relativistic Pauli-Fierz Hamiltonian

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Abstract

We consider a charged particle, spin $\frac{1}{2}$, with relativistic kinetic energy and minimally coupled to the quantized Maxwell field. Since the total momentum is conserved, the Hamiltonian admits a fiber decomposition as $H(P), P \in \mathbb{R}^3$. We study the spectrum of $H(P)$. In particular we prove that, for non-zero photon mass, the ground state is exactly two-fold degenerate and separated by a gap, uniformly in $P$, from the rest of the spectrum.

1 Introduction and Main Results

Let us consider a classical point charge, charge $e$, mass $M$, position $q$, velocity $\dot{q}$, coupled to the field with electric field $E$ and magnetic field $B$. The coupling to the field is through a rigid charge distribution $\varphi : \mathbb{R}^3 \to \mathbb{R}_+$ normalized as $\int d\mathbf{x} \varphi(\mathbf{x}) = 1$. Then the equations of motion for the coupled system read, in units where $c = 1$,

$$\frac{\partial}{\partial t} B = -\nabla \wedge E, \quad \frac{\partial}{\partial t} B = \nabla \wedge E - e\varphi(\cdot - q)\dot{q},$$

$$\nabla \cdot E = e\varphi(\cdot - q), \quad \nabla \cdot B = 0,$$

$${\frac{d}{dt}}(M(1 - \dot{q}^2)^{-1/2}\dot{q}) = e(E \ast \varphi(q) + \dot{q} \wedge (B \ast \varphi)(q)) \quad (1)$$

with $\ast$ denoting convolution. The uncoupled system, $e = 0$, is Lorentz invariant. But the choice of the rigid charge distribution singles out a specific reference frame and hence makes the model semi-relativistic, only.

The canonical quantization of (1) results in a quantum evolution governed by the semi-relativistic Pauli-Fierz hamiltonian. Our goal is to study spectral properties of this operator. While the nonrelativistic counterpart has been investigated in considerable detail, no spectral results seem to be available for the semi-relativistic case.
The quantization procedure for (1) is described, e.g., in [21]. One writes (1) in Lagrangian form and Legendre transforms to Hamiltonian structure in using the Coulomb gauge. Since our prime example will be an electron (charge $-e$), we want to include spin $\frac{1}{2}$. As for the nonrelativistic Hamiltonian this amounts to replacing $(p + eA)^2$ by $(\sigma \cdot (p + eA))^2$ with $\sigma$ the 3-vector of Pauli spin matrices. As a result one obtains the semi-relativistic Pauli-Fierz Hamiltonian, which is given by

$$H = \gamma \sqrt{\left(\sigma \cdot (-i\nabla + eA(x))\right)^2 + M^2 + H_f}, \quad (2)$$

We introduced here by hand the factor $\gamma$, $0 < \gamma \leq 1$. Physically, as discussed above, one has $\gamma = 1$. Our units are chosen such that $\hbar = 1$. Without restriction of generality we set $e \geq 0$. $H$ acts in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}$, where $\mathcal{F}$ is the photon Fock space

$$\mathcal{F} = \sum_{n \geq 0} \otimes L^2(\mathbb{R}^3 \times \{1, 2\}) \otimes \mathbb{C}^2.$$  

$A(x)$ is the quantized vector potential defined through

$$A(x) = \sum_{\lambda=1,2} \int_{|k| \leq \Lambda} \frac{dk}{2(2\pi)^3} \varepsilon(k, \lambda) \left(e^{ikx}a(k, \lambda) + e^{-ikx}a(k, \lambda)^*\right),$$

where $\varepsilon(k, \lambda), \lambda = 1, 2,$ is the pair of polarization vectors. $k/|k|, \varepsilon(k, 1), \varepsilon(k, 2)$ are a dreibein depending measurably on $k$. For convenience we use the sharp ultraviolet cutoff $\Lambda$ which corresponds to setting $\hat{\varphi}(k) = (2\pi)^{-3/2}$ for $|k| \leq \Lambda$ and $\hat{\varphi}(k) = 0$ otherwise, $\hat{}$ denoting Fourier transform. Our results are equally valid for a smooth cutoff. $a(k, \lambda), a(k, \lambda)^*$ are the annihilation and creation operators which satisfy the standard commutation relations

$$[a(k, \lambda), a(k', \lambda')^*] = \delta_{\lambda\lambda'}\delta(k - k'), \quad [a(k, \lambda), a(k', \lambda')] = 0 = [a(k, \lambda)^*, a(k', \lambda')^*].$$

$H_f$ is the field energy,

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \omega(k)a(k, \lambda)^*a(k, \lambda). \quad (3)$$

For the Maxwell field the dispersion relation is

$$\omega(k) = |k|.$$  

Mathematically it is convenient to introduce the photon mass $m_{ph}$ through the choice

$$\omega(k) = \sqrt{k^2 + m_{ph}^2}.$$  

Readers will find more precise definitions of $A(x)$ and $H_f$ in the Appendix A.
Remark 1.1 For a fixed configuration of the vector potential the classical hamiltonian function is
\[ H_{cl}(p, q) = \sqrt{(p - eA(q))^2 + M^2}. \]
We picked here the “naive” quantization \( p \sim -i\nabla_x, \; q \sim x \), which is fairly common in the physics community \[2\]. Alternatives would be either Weyl or magnetic Weyl quantization \[14\].

By translation invariance the total momentum, i.e., the sum of the momentum of the charge and the field momentum, is conserved. The generator of translations is the total momentum operator \( P_{\text{tot}} = -i\nabla_x + P_f \) with
\[ P_f = \sum_{\lambda = 1,2} \int_{\mathbb{R}^3} dk \, k a(k, \lambda)^* a(k, \lambda). \]
It strongly commutes with the hamiltonian \( H \), namely, \( \exp[-ia \cdot P_{\text{tot}}] \exp[-itH] = \exp[-itH] \exp[-ia \cdot P_{\text{tot}}] \) for all \( a \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \). Therefore \( H \) admits the direct integral decomposition
\[ \mathcal{U} H \mathcal{U}^* = \int_{\mathbb{R}^3} H(P) \, dP, \]
acting in \( \mathbb{C}^2 \otimes \mathfrak{f}, \; B(0) = \nabla \wedge A(0) \). The unitary \( \mathcal{U} \) is defined by \( \mathcal{U} = \mathcal{F}_x \exp[i x \cdot P_f] \) where \( \mathcal{F}_x \) is the Fourier transformation with respect to \( x \). We will provide a mathematically rigorous definition of \( H \) and \( H(P) \) in section \[2\]. Our interest is mostly in the low lying spectrum of \( H(P) \) in dependence of \( P \). To get started we have to ensure the self-adjointness of \( H \) and of \( H(P) \), see section \[2\] for details.

Proposition 1.2 For any \( 0 < \gamma \leq 1, \Lambda < \infty \) and \( 0 \leq m_{\text{ph}} \), there exists \( e_* > 0 \) such that, for all \( e < e_* \), \( H \) is self-adjoint on \( \text{dom}(| - i\nabla_x |) \cap \text{dom}(H_f) \). Moreover \( H \) is essentially self-adjoint on any core of the free Hamiltonian \( H_0 = \gamma \sqrt{-\Delta_x + M^2} + H_f \).

Proposition 1.3 Choose \( \gamma, \Lambda, m_{\text{ph}} \) arbitrarily as Proposition \[1.2\]. Let \( e_* \) be given by Proposition \[1.2\]. Then, for all \( e < e_* \) and \( P \in \mathbb{R}^3 \), \( H(P) \) is self-adjoint on \( \text{dom}(|P_f|) \cap \text{dom}(H_f) \). Moreover \( H(P) \) is essentially self-adjoint on any core of the operator \( H_0(P) = \gamma \sqrt{(P - P_f)^2 + M^2} + H_f \).

Remark 1.4 There are further parts of our proof which will require small \( e_* \). Therefore we did not attempt to optimize \( e_* \) in every step. 

The spectral analysis of the nonrelativistic Pauli-Fierz hamiltonian was initiated by J. Fröhlich in his Ph.D. thesis \[15\]. Our first main result is the extension of his methods to the semi-relativistic case. While the result could be anticipated from \[5\] \[14\] \[20\], the actual proof is surprisingly technical, since the minimal coupling is under the square root, see section \[8\].
Theorem 1.5  Set $\Lambda, \gamma, m_{\text{ph}}$ arbitrarily as in Proposition 1.3. Choose $e$ as $e < e_*$. Let

$$\Delta(P) = \inf_{k \in \mathbb{R}^3} (E(P - k) + \omega(k) - E(P))$$

where

$$E(P) = \inf \text{spec}(H(P)), \quad \Sigma(P) = \inf \text{ess.spec}(H(P)).$$

Then one has

$$\Sigma(P) - E(P) = \Delta(P)$$

for all $P \in \mathbb{R}^3$.

A problem of general interest is to derive from (2) the effective dynamics of a charge subject to slowly varying external potentials and coupled to the radiation field. Very crudely, one considers the subspace of $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}$ spanned by the ground states of $H(P)$ with $P \in \mathbb{R}^3$ and constructs the effective dynamics as an approximate solution to the full dynamics lying close to that subspace. In principle this problem can be handled by space-adiabatic perturbation theory [17, 23], which as one basic input uses that $H(P)$ has a uniform spectral gap, i.e., for all $P \in \mathbb{R}^3$,

$$\inf \{ \text{spec}(H(P)) \setminus \{E(P)\} \} - E(P) =: C_g(P) \geq C_0 > 0.$$  \hspace{1cm} (6)

This can be achieved by having a maximal velocity which is strictly less than 1, which means to choose $\gamma < 1$. This choice amounts to a small modification for low energies which is anyhow the domain of validity of our model.

If $m_{\text{ph}} > 0$ and $\gamma < 1$ it is easily seen that $\Delta(P) \geq C_0 > 0$ uniformly. However this does not yet establish a spectral gap in the sense of (6), because beyond the ground state there could be other eigenvalues in the interval $[E(P), \Sigma(P)]$. In the literature there are two methods to count the number of eigenvalues. One is through positive commutator, Mourre type estimates and the other uses a pull through in order to estimate the overlap between the Fock vacuum and the ground state. For sufficiently small $P$ both methods yield the desired result. However, a uniform bound on the spectral gap seems to be difficult to achieve by such techniques. Therefore we introduce a novel method based on operator monotonicity, which we learned from the masterly works of Lieb and Loss [11, 12], together with the min-max principle.

Progressed so far, one still has to determine the degeneracy of the ground state. For the non-relativistic Pauli-Fierz model this is discussed in [10]. Later on we learned a very simple and general argument from M. Loss. We reproduce his result and show that it is applicable to the semi-relativistic Pauli-Fierz hamiltonian.

We summarize our main result in
Theorem 1.6 Fix $0 < \gamma < 1$ and $0 < m_{\text{ph}}$. Then there exists $e_* > 0$ independent of $P$, such that, for all $e < e_*$ and $P \in \mathbb{R}^3$, the following properties hold.

(i) One has
\[ \Sigma(P) - E(P) \geq (1 - \gamma)m_{\text{ph}} - ec_1 - O(e^2) > 0 \]
for all $P \in \mathbb{R}^3$, where $c_1$ and $O(e^2)$ are independent of $P$. In particular $E(P)$ is an eigenvalue.

(ii) One has
\[ \inf \{ \text{spec}(H(P)) \setminus \{E(P)\} \} - E(P) \geq (1 - ec_2 - \gamma)m_{\text{ph}} - ec_3 - O(e^2) \quad (7) \]
for all $P$, where $c_2, c_3$ and $O(e^2)$ are independent of $P$.

(iii) $E(P)$ is exactly doubly degenerate.

Remark 1.7 We remark that the lowest energy $E(P)$ and a possible spectral gap are also of importance, e.g., in scattering theory. We refer to [6, 7, 9] for the investigation of related models and to [1] for $E(P)$ when the infrared cutoff is removed. ♦

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2 Self-adjointness

2.1 Dirac operators

As a preliminary, we introduce two Dirac operators which will simplify our study.

Let us define a Dirac operator $D$ by
\[ D = \alpha \cdot \left( -i \nabla_x + eA(x) \right) + M\beta \]
living in $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}$. This is essentially self-adjoint on $C^\infty_0(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\text{fin}}$ by the Nelson’s commutator theorem [18] with the test operator $-\Delta_x + H_t$. Here
\[ \mathfrak{F}_{\text{fin}} = \text{Lin}\{ a(f_1)^* \cdots a(f_n)^* \Omega, \Omega \mid f_1(\cdot, \lambda_1), \ldots, f_n(\cdot, \lambda_n) \in C^\infty_0(\mathbb{R}^3) \]
for all $\lambda_1, \ldots, \lambda_n \in \{1, 2\}$ and $n \in \mathbb{N}$.
where \( \text{Lin}\{\cdots\} \) means the linear span of the set \( \{\cdots\} \) and \( \Omega \) is the Fock vacuum defined by \( \Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \). We denote the closure of \( D \) by the same symbol. We note that

\[
D^2 = T + M^2, \\
|D| = \sqrt{T + M^2},
\]

where the self-adjoint operator \( T \) is expressed as

\[
T = \begin{pmatrix}
(\sigma \cdot (-i\nabla_x + eA(x)))^2 & 0 \\
0 & (\sigma \cdot (-i\nabla_x + eA(x)))^2
\end{pmatrix}
\]
on \( C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\text{fin}} \).

Next let us define the following Dirac operator

\[
D(P) = \alpha \cdot (P - P_f + eA(0)) + M\beta
\]
acting in \( \mathbb{C}^4 \otimes \mathfrak{F} \). Again this is essentially self-adjoint on \( \mathbb{C}^4 \otimes \mathfrak{F}_{\text{fin}} \) by the Nelson’s commutator theorem with a test operator \( P_f^2 + H_f \). We denote its closure by the same symbol. Then one can easily observe that

\[
\mathcal{U}D\mathcal{U}^* = \int_{\mathbb{R}^3} D(P) \, dP, \\
D(P)^2 = T(P) + M^2, \\
|D(P)| = \sqrt{T(P) + M^2},
\]

where the action of the self-adjoint operator \( T(P) \) is concretely given as

\[
T(P) = \begin{pmatrix}
(\sigma \cdot (P - P_f + eA(0)))^2 & 0 \\
0 & (\sigma \cdot (P - P_f + eA(0)))^2
\end{pmatrix}
\]
on \( \mathbb{C}^4 \otimes \mathfrak{F}_{\text{fin}} \).

### 2.2 Definition of the Hamiltonians

Our definition of \( H \) and \( H(P) \) are as follow:

\[
H = \gamma|D| + H_f, \\
H(P) = \gamma|D(P)| + H_f.
\]

In this paper we occasionaneously identify a direct sum operator \( A \oplus A \) with \( A \) if no confusion occurs. Hence the above definitions mean that \( H \oplus H = \gamma|D| + H_f \oplus H_f \) and \( H(P) \oplus H(P) = \gamma|D(P)| + H_f \oplus H_f \).
2.3 Proof of Proposition 1.2

For each \( \varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\text{fin}} \), one has

\[
\| |D| \varphi \|^2 = \langle \varphi, D^2 \varphi \rangle \leq \text{const} \| (H_0 + 1) \varphi \|^2.
\]

Since \( C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\text{fin}} \) is a core of \( H_0 \), one concludes that \( \text{dom}(H_0) \subseteq \text{dom}(|D|) \).

Also note that, for \( D_0 := \alpha \cdot (-i\nabla_x) + M\beta \), one has \( \text{dom}(H_0) \subseteq \text{dom}(|D_0|) \). Let \( H_I \) be the interaction term given by

\[
H_I = |D| - |D_0|.
\]

By the above arguments, \( \text{dom}(H_0) \subseteq \text{dom}(H_I) \) holds. Using the formula

\[
|a| = \frac{1}{\pi} \int_0^\infty \frac{dt}{\sqrt{t}} \frac{a^2}{t + a^2},
\]

one has

\[
|D| - |D_0| = \frac{1}{\pi} \int_0^\infty dt \sqrt{t} (t + D^2)^{-1} \left\{ 2eA(x) \cdot (-i\nabla_x) + e^2 A(x)^2 + e\sigma \cdot B(x) \right\} (t + D_0^2)^{-1},
\]

where \( B(x) = \nabla_x \wedge A(x) \). Observe that

\[
\| A_j(x)(-i\partial_j)(t + D_0^2)^{-1}(H_0 + 1)^{-1} \|
\]

\[
\leq \| A_j(x)(-i\partial_j) - i\partial_j \|^{-1/2}(H_0 + 1)^{-1} \| | - i\partial_j \|^{-1/2}(t + D_0^2)^{-1} \|
\]

\[
\leq \text{const} t^{-3/4}
\]

for \( j = 1, 2, 3 \), and

\[
\| A(x)^2(H_0 + 1)^{-1} \| \leq \text{const},
\]

\[
\| \sigma \cdot B(x)(H_0 + 1)^{-1} \| \leq \text{const}.
\]

Combining these with (8), one obtains

\[
\| H_I(H_0 + 1)^{-1} \| \leq \text{const} (e + e^2) \int_0^\infty dt \sqrt{t} (t + M^2)^{-1} \{ t^{-3/4} + (t + M^2)^{-1} \}
\]

\[
\leq \mathcal{O}(e).
\]

Hence there exists \( e_* \) such that \( \| H_I(H_0 + 1)^{-1} \| < 1 \) for all \( e < e_* \). Now we can apply the Kato-Rellich theorem \[18\] to obtain the assertion in Proposition 1.2.
2.4 Proof of Proposition 1.3

By (10), one has

\[ \|H_1\psi\| \leq \mathcal{O}(e)\| (H_0 + \mathbb{1})\psi\|. \]  

(11)

For each \( k_0 \in \mathbb{R}^3 \), choose \( \psi \) as \( U\psi = |B_{\varepsilon,k_0}|^{-1/2} \chi_{B_{\varepsilon,k_0}} \otimes \varphi \) where \( \varphi \in \mathbb{C}^2 \otimes \mathcal{F}_{\text{fin}} \), \( \chi_S \) is the characteristic function of the set \( S \), \( B_{\varepsilon,k_0} = \{ k \in \mathbb{R}^3 | |k - k_0| < \varepsilon \} \) and \( |B_{\varepsilon,k_0}| = 4\pi\varepsilon^3/3 \). It follows from (11) that

\[ |B_{\varepsilon,k_0}|^{-1} \int_{B_{\varepsilon,k_0}} dk \| H_1(k)\varphi \|^2 \leq \mathcal{O}(e^2)|B_{\varepsilon,k_0}|^{-1} \int_{B_{\varepsilon,k_0}} dk \| (H_0(k) + \mathbb{1})\varphi \|^2, \]  

(12)

where \( H_1(P) = |D(P)| - |D_0(P)| \). Since \( H_1(P)\varphi \) and \( (H_0(P) + \mathbb{1})\varphi \) are strongly continuous in \( P \), we can take the limit as \( \varepsilon \downarrow 0 \) and obtain that

\[ \|H_1(k_0)\varphi\| \leq \mathcal{O}(e)\| (H_0(k_0) + \mathbb{1})\varphi\|. \]

Since \( k_0 \) is arbitrary and \( \mathbb{C}^2 \otimes \mathcal{F}_{\text{fin}} \) is a core of \( H_0(P) \), we have that \( \|H_1(P)(H_0(P) + \mathbb{1})^{-1}\| \leq \mathcal{O}(e) \) for all \( P \). Now we can apply the Kato-Rellich theorem [18] and obtain the assertion in the proposition. \( \square \)

3 Spectral properties

3.1 Preliminaries

In this section, we will prove Theorem 1.3. To this end, we need some preliminaries.

Let \( j_1 \) and \( j_2 \) be two localization functions on \( \mathbb{R}^3 \) so that \( j_1^2 + j_2^2 = 1 \) and \( j_1 \) is supported in a ball of radius \( R \). For each vector \( f = f(k, \lambda) \) in \( L^2(\mathbb{R}^3 \times \{1,2\}) \), we define an operator \( \mathcal{J}_i \) \( (i = 1, 2) \) by

\[ (\mathcal{J}_i f)(k, \lambda) = j_i(-i\nabla_k) f(k, \lambda). \]

Now we define a linear operator \( \mathcal{J} : L^2(\mathbb{R}^3 \times \{1,2\}) \rightarrow L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\}) \) by

\[ \mathcal{J} f = \mathcal{J}_1 f \oplus \mathcal{J}_2 f \]

for each \( f \in L^2(\mathbb{R}^3 \times \{1,2\}) \).

Let \( U \) be the natural isometry from \( \mathfrak{F}(L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\})) \) to \( \mathfrak{F} \otimes \mathfrak{F} \) where \( \mathfrak{F}(L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\})) \) is the Fock space over \( L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\}) \), see Appendix A. Concrete action of \( U \) is given by

\[ Ua(f_1 \oplus g_1)^* \ldots a(f_n \oplus g_n)^* \Omega^\oplus = [a(f_1)^* \otimes \mathbb{1} + \mathbb{1} \otimes a(g_1)^*] \ldots [a(f_n)^* \otimes \mathbb{1} + \mathbb{1} \otimes a(g_n)^*] \Omega \otimes \Omega \]
where $\Omega^\oplus$ is the Fock vacuum in $\mathfrak{F}(L^2(\mathbb{R}^3 \times \{1, 2\}) \oplus L^2(\mathbb{R}^3 \times \{1, 2\}))$. The following operator

$$\hat{\Gamma}(\mathcal{J}) := U\Gamma(\mathcal{J})$$

plays an important role in our proof. The importance of $\hat{\Gamma}(\mathcal{J})$ was discovered by Dereziński and Gérard [2].

In Appendix C we show the following formula.

**Lemma 3.1** (Localization formula) Let

$$H^\oplus(P) = \gamma \sqrt{\{\sigma \cdot (P - P_1 \otimes I - I \otimes P_1 + eA(0) \otimes I)\}^2 + M^2 + H_\ell \otimes I + I \otimes H_\ell}$$

acting in $C^2 \otimes \mathfrak{f} \otimes \mathfrak{f}$. Choose $e$ as $e < e_*$, where $e_*$ is given in Proposition 1.3. Then, for all $\varphi \in C^2 \otimes \mathfrak{f}_{\text{fin}} \otimes \mathfrak{f}_{\text{fin}}$, one obtains

$$|\langle \varphi, (H(P) - \hat{\Gamma}(\mathcal{J})^* H^\oplus(P) \hat{\Gamma}(\mathcal{J}))\varphi \rangle| \leq O(R^0)\|H(P) + I\varphi\|^2,$$

where $O(R^0)$ is a function of $R$ vanishing as $R \to \infty$.

Finally we note the following lemma.

**Lemma 3.2** One has

$$H^\oplus(P) \geq E(P) + \Delta(P)(I - P_\Omega),$$

where $P_\Omega$ is the orthogonal projection onto $C^2 \otimes \mathfrak{f} \otimes \Omega$.

**Proof.** Remark the following natural identification,

$$C^2 \otimes \mathfrak{f} \otimes \mathfrak{f} = \sum_{n \geq 0}^{\oplus} C^2 \otimes \mathfrak{f} \otimes L^2(\mathbb{R}^3 \times \{1, 2\})^{\oplus_n}.$$

Set $\mathcal{H}_n = C^2 \otimes \mathfrak{f} \otimes L^2(\mathbb{R}^3 \times \{1, 2\})^{\oplus_n}$. Each vector in $\varphi \in \mathcal{H}_n$ can be expressed as a $C^2 \otimes \mathfrak{f}$-valued symmetric function on $(\mathbb{R}^3 \times \{1, 2\})^{\times_n}$:

$$\varphi = \varphi(k_1, \lambda_1, \ldots, k_n, \lambda_n).$$

Under this identification, the action of $H^\oplus(P)$ is given by

$$(H^\oplus(P)\varphi)(k_1, \lambda_1, \ldots, k_n, \lambda_n)
= \left(H\left(P - \sum_{i=1}^n k_i \right) + \sum_{i=1}^n \omega(k_i)\right)\varphi(k_1, \lambda_1, \ldots, k_n, \lambda_n)$$

for a suitable $\varphi \in \mathcal{H}_n$. Thus, using the triangle inequality $\omega(k_1 + k_2) \leq \omega(k_1) + \omega(k_2)$, one has

$$\langle \varphi, H^\oplus(P)\varphi \rangle = \sum_{\lambda_1, \ldots, \lambda_n = \{1, 2\}} \int dk_1 \ldots dk_n \left\langle \varphi(k_1, \lambda_1, \ldots, k_n, \lambda_n), \left(H\left(P - \sum_{i=1}^n k_i \right) + \sum_{i=1}^n \omega(k_i)\right)\varphi(k_1, \lambda_1, \ldots, k_n, \lambda_n)\right\rangle \geq (\Delta(P) + E(P))\|\varphi\|^2.$$

For $n = 0$, we have $H^\oplus(P) | \mathcal{H}_0 = H(P)$. Combining the results, one reaches the assertion in the lemma. \qed
3.2 Proof of Theorem 1.5

3.2.1 Lower bound of $\Delta(P)$

In this subsubsection, we will show the following lower bound.

**Proposition 3.3** Choose $e < e_*$. Then one has that $\Sigma(P) - E(P) \geq \Delta(P)$.

**Proof.** For any $\lambda \in \text{ess.spec}(H(P))$, we can find a sequence $\{\varphi_n\}_n$ such that $||\varphi_n|| = 1$, w- $\lim \varphi = 0$ and $\lim \|(H(P) - \lambda)\varphi_n\| = 0$. For each $n \in \mathbb{N}$, one has

$$\langle \varphi_n, H(P)\varphi_n \rangle \geq \langle \varphi_n, \tilde{\Gamma}(\mathcal{J})^* H^\otimes(P) \tilde{\Gamma}(\mathcal{J}) \varphi_n \rangle - \mathcal{O}(R^0)\|(H(P) + \mathbb{I})\varphi\|^2$$

by Lemma 3.1. Thus using Lemma 3.2 one gets

$$\langle \varphi_n, H(P)\varphi_n \rangle \geq E(P) + \Delta(P) - \Delta(P)\|P_\Omega \tilde{\Gamma}(\mathcal{J})\varphi_n\|^2 - \mathcal{O}(R^0)\|(H(P) + \mathbb{I})\varphi_n\|^2.$$  \hfill (13)

First we will show that $\lim_{n \to \infty} \|P_\Omega \tilde{\Gamma}(\mathcal{J})\varphi_n\| = 0$. Remark that $\|P_\Omega \tilde{\Gamma}(\mathcal{J})\varphi_n\| = \|\Gamma(\mathcal{J})\varphi_n\|$. With $N_\Omega$ the number operator given by

$$N_\Omega = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \, a(k, \lambda)^* a(k, \lambda),$$

we also remark that $\langle \varphi_n, N_\Omega \varphi_n \rangle$ is uniformly bounded in $n$ because

$$\langle \varphi_n, N_\Omega \varphi_n \rangle \leq m^{-1}_{\text{ph}} \langle \varphi_n, H_\Omega \varphi_n \rangle \leq \langle \varphi_n, H(P)\varphi_n \rangle.$$  

Thus $\|(\mathbb{1} - \chi_N(N_\Omega)) \Gamma(\mathcal{J})\varphi\| \leq \|(\mathbb{1} - \chi_N(N_\Omega))\varphi\| = \mathcal{O}(N^0)$ holds where $\mathcal{O}(N^0)$ is a function of $N$, independent of $n$, vanishing as $N \to \infty$. Here $\chi_N(s) = 1$ if $0 \leq s \leq N$ and $\chi(s) = 0$ otherwise, moreover $\chi_N(N_\Omega)$ is defined by the functional calculus. On the other hand, $\chi_N(N_\Omega)(H_\Omega + \mathbb{I})^{-1/2} \Gamma(\mathcal{J})$ is compact for all $N$. Thus one finds that

$$\|P_\Omega \tilde{\Gamma}(\mathcal{J})\varphi_n\|^2 \leq 2\chi_N(N_\Omega) \Gamma(\mathcal{J})\varphi_n\|^2 + 2\|(\mathbb{1} - \chi_N(N_\Omega))\varphi_n\|^2 = 2\chi_N(N_\Omega)(H_\Omega + \mathbb{I})^{-1/2} \Gamma(\mathcal{J}) \varphi_n\|(H_\Omega + \mathbb{I})^{1/2} \varphi_n\| + \mathcal{O}(N^0)$$

$$= 2\chi_N(N_\Omega)(H_\Omega + \mathbb{I})^{-1/2} \Gamma(\mathcal{J}) \varphi_n\|(H_\Omega + \mathbb{I})^{1/2} \varphi_n\| \|H_\Omega + \mathbb{I}\|^{1/2} \|H(P) + \mathbb{I}\|^{-1/2} \varphi_n\| + \mathcal{O}(N^0).$$

First we take the limit $n \to \infty$. Then, by the compactness of the linear operator $\chi_N(N_\Omega)(H_\Omega + \mathbb{I})^{-1/2} \Gamma(\mathcal{J})$, the vector $\chi_N(N_\Omega)(H_\Omega + \mathbb{I})^{-1/2} \Gamma(\mathcal{J}) \varphi_n$ converges to 0 strongly which implies that $\limsup \|P_\Omega \tilde{\Gamma}(\mathcal{J})\varphi_n\| \leq \mathcal{O}(N^0)$. Then taking $N \to \infty$, one concludes that $\lim_{n \to \infty} \|P_\Omega \tilde{\Gamma}(\mathcal{J})\varphi_n\| = 0$.

Taking the limit $n \to \infty$ in both side of (13), one finds

$$\lambda \geq E(P) + \Delta(P) - \mathcal{O}(R^0)(\lambda + 1)^2.$$

Finally taking $R \to \infty$, one obtains the desired assertion. $\square$
3.2.2 Upper bound of $\Delta(P)$
We will complete our proof of Theorem 1.5 by showing the following upper bound.

**Proposition 3.4** Choose $e$ as $e < e^*$. Then we have that $\Sigma(P) - E(P) \leq \Delta(P)$.

**Proof.** For notational simplicity we set $\gamma = 1$ in this proof. For each $k_0 \in \mathbb{R}^3$, let us define

$$f_{\varepsilon,k_0} = |B_{\varepsilon,k_0}|^{-1/2} \chi_{B_{\varepsilon,k_0}},$$
$$B_{\varepsilon,k_0} = \{ k \in \mathbb{R}^3 \mid |k - k_0| \leq \varepsilon \},$$

where $\chi_A$ is the characteristic function of the measurable set $A$ and $|A|$ means the Lebesgue measure of $A$. Choose a normalized vector $\varphi_{\varepsilon} \in \text{ran}E_\Delta(H(P - k_0))$ with $\Delta = [-\varepsilon + z, z + \varepsilon]$, $z = E(P - k_0) + \omega(k_0), \varepsilon > 0$. Here for a self-adjoint operator $A$, $E_\Delta(A)$ stands for the spectral measure of $A$ for the interval $\Delta$. Let $\alpha_{\lambda}(f) = \int_{\mathbb{R}^3} dk f(k)^* a(k, \lambda)$. We will show that $\alpha_{\lambda}(f_{\varepsilon,k_0})^* \varphi_{\varepsilon} / \| \alpha_{\lambda}(f_{\varepsilon,k_0})^* \varphi_{\varepsilon} \|$ is a Weyl sequence for $z$ as $\varepsilon \downarrow 0$. Applying the pull-through formula, one has

$$\left\langle (H(P) - z)\alpha_{\lambda}(f_{\varepsilon,k_0})^* \varphi_{\varepsilon}, \psi \right\rangle$$
$$= \int_{\mathbb{R}^3} dk f_{\varepsilon,k_0}(k) \left\{ \left\langle (H(P - k) + \omega(k) - z)\varphi_{\varepsilon}, a(k, \lambda)\psi \right\rangle - \left\langle S_{k,\lambda}(P)^* \varphi_{\varepsilon}, \psi \right\rangle \right\}$$

(14)

for each normalized $\psi \in C^2 \otimes \mathfrak{S}_{\text{fin}}$, where

$$S_{k,\lambda}(P) = |D(P - k)| a(k, \lambda) - a(k, \lambda)|D(P)|.$$

As to the second term in the right hand side of (14), observe that

$$\left| \int_{\mathbb{R}^3} dk f_{\varepsilon,k_0}(k) \left\langle S_{k,\lambda}(P)^* \varphi_{\varepsilon}, \psi \right\rangle \right|$$
$$\leq \int_{\mathbb{R}^3} dk f_{\varepsilon,k_0}(k) \| S_{k,\lambda}(P)^* \varphi_{\varepsilon} \| \| \psi \|$$
$$\leq \int_{\mathbb{R}^3} dk f_{\varepsilon,k_0}(k) \| S_{k,\lambda}(P)^* (H(P - k) + \mathbb{1})^{-1} \| (H(P - k) + \mathbb{1}) \varphi_{\varepsilon} \|$$
$$\leq C \int_{\mathbb{R}^3} dk f_{\varepsilon,k_0}(k)(1 + |k|) |F_0(k, \lambda)| (E(P - k_0) + 1 + \omega(k_0) + \mathcal{O}(|k - k_0|) + \varepsilon)$$

by Lemma D.6 and (23) below, where

$$F_z(k, \lambda) = e^{\frac{\chi_A(k)\varepsilon(k, \lambda)}{\sqrt{2(2\pi)^3}}} \omega(k) e^{-ik \cdot z}.$$  (15)
Clearly the right hand side of the above inequality converges to 0 as \( \varepsilon \downarrow 0 \) because \( f_{\varepsilon,k_0} \) weakly converges to 0 in \( L^2(\mathbb{R}^3) \). Next we will estimate the first term in the right hand side of (13). One has

\[
\left| \int_{\mathbb{R}^3} dk \, f_{\varepsilon,k_0}(k) \langle (H(P-k) + \omega(k) - z) \varphi_\varepsilon, a(k,\lambda) \psi \rangle \right|
\]

\[
\leq \left[ \int_{\mathbb{R}^3} dk \, f_{\varepsilon,k_0}(k)^2 \| (N_\varepsilon + 2)^{1/2}(H(P-k) + \omega(k) - z) \varphi_\varepsilon \|^2 \right]^{1/2}
\]

\[
\times \left[ \int_{\mathbb{R}^3} dk \, \| (N_\varepsilon + 2)^{-1/2}a(k,\lambda) \psi \|^2 \right]^{1/2}
\] (16)

The term (17) is less than \( \| \psi \|^2 (= 1) \) because

\[
\int_{\mathbb{R}^3} dk \, \| (N_\varepsilon + 2)^{-1/2}a(k,\lambda) \psi \|^2 \leq \int_{\mathbb{R}^3} dk \, \langle a(k,\lambda) \psi, a(k,\lambda)(N_\varepsilon + 1)^{-1}\psi \rangle
\]

\[
= \langle \psi, N_\varepsilon(N_\varepsilon + 1)^{-1}\psi \rangle
\]

\[
\leq \| \psi \|^2.
\] (17)

As to the term (16) we need a lengthy calculation below. Note that, since \( N_\varepsilon + 2 \leq m_{\text{ph}}^{-1}H(P-k) + 2 \), one has

\[
\| (N_\varepsilon + 2)^{1/2}(H(P-k) + \omega(k) - z) \varphi_\varepsilon \|
\]

\[
\leq C\| (H(P-k) + 2)^{1/2}(H(P-k) + \omega(k) - z) \varphi_\varepsilon \|
\]

\[
\leq C\| (H(P-k) + \omega(k) - z)^{3/2} \varphi_\varepsilon \|
\]

\[
+ C|2 - z + \omega(k)|^{1/2}\| (H(P-k) + \omega(k) - z) \varphi_\varepsilon \|.
\] (18)

Note that

\[
H(P-k) + \omega(k) - z
\]

\[
= (H(P-k_0) + \omega(k_0) - z) + (H(P-k) - H(P-k_0) + \omega(k) - \omega(k_0)).
\]

Thus one has

\[
\| (H(P-k) + \omega(k) - z) \varphi_\varepsilon \|
\]

\[
\leq \| (H(P-k_0) + \omega(k_0) - z) \varphi_\varepsilon \| + \| (H(P-k) - H(P-k_0)) \varphi_\varepsilon \|
\]

\[
+ |\omega(k) - \omega(k_0)|
\]

\[
\leq \varepsilon + \| (|D(P-k)| - |D(P-k_0)|) \varphi_\varepsilon \| + |\omega(k) - \omega(k_0)|.
\] (20)

We will show that

\[
\| (|D(P-k)| - |D(P-k_0)|)(H(P-k_0) + 1)^{-1} \| \leq O(|k-k_0|).
\] (21)

To see this, we just note that, by (5),

\[
|D(P-k)| - |D(P-k_0)|
\]

\[
= \frac{1}{\pi} \int_{M_2^2} dt \, \sqrt{t + \hat{D}(P-k)^2}^{-1}\left\{ 2(-k+k_0) \cdot P_t + k^2 - k_0^2 + 2\varepsilon(-k+k_0) \cdot A(0) \right\}
\]

\[
\times (t + \hat{D}(P-k_0)^2)^{-1}.
\]
Hence, by Lemma D.3 one obtains
\[
\left\| (|D(P - k)| - |D(P - k_0)|)(H(P - k) + 1)^{-1} \right\|
\leq \frac{1}{\pi} \int_{M^2} dt \sqrt{t}^{-1} \left\| \left\{ 2(-k + k_0) \cdot P_t + k^2 - k_0^2 + 2\epsilon(-k + k_0) \cdot A(0) \right\} (H + 1)^{-1} \right\|
\times \left\| (H + 1)(t + \hat{D}(P - k_0)^{(2)^{-1}}(H(P - k_0) + 1)^{-1}) \right\|
\leq C(t^{-1} + t^{-3/2} + t^{-2}) \text{ by Lemma D.3}
\leq O(|k - k_0|)
\] (22)

which implies
\[
\left\| (H(P - k) + \omega(k) - z)\varphi_\epsilon \right\| \leq O(|k - k_0|) + \epsilon
\] (23)

by (20). As a consequence, the term (19) is estimated as
\[
(19) \leq (O(|k - k_0|) + \epsilon)|2 - \omega(k)|^{1/2}.
\] (24)

To estimate (18), observe that
\[
\left\| (H(P - k) + \omega(k) - z)^{3/2} \varphi_\epsilon \right\|^2
\leq \left\| (H(P - k) + \omega(k) - z)^2 \varphi_\epsilon \right\| \left\| (H(P - k) + \omega(k) - z)\varphi_\epsilon \right\|
\leq O(|k - k_0|) + \epsilon \text{ by (23)}
\] (25)

Since \((H(P - k) + \omega(k) - z)^2(H(P - k_0) + 1)^{-2}\) is uniformly bounded for \(k \in B_{\epsilon,k_0}\) by Lemma D.3, we obtain
\[
(18) \leq O(|k - k_0|) + C\epsilon.
\] (26)

Collecting (24) and (26), we arrive at
\[
\int_{\mathbb{R}^3} dk f_{\epsilon,k_0}(k)^2 \left\| (N_t + 1)^{1/2}(H(P - k) + \omega(k) - z)\varphi_\epsilon \right\|^2
\leq \int_{\mathbb{R}^3} dk f_{\epsilon,k_0}(k)^2 (O(|k - k_0|) + \epsilon)
= O(\epsilon^0).
\]

This means that
\[
\left\| (H(P) - z)a_{\lambda}(f_{\epsilon,k_0})^* \varphi_\epsilon \right\| \leq O(\epsilon^0).
\] (27)

Set \(\Psi_\epsilon = a_{\lambda}(f_{\epsilon,k_0})^* \varphi_\epsilon / \left\| a_{\lambda}(f_{\epsilon,k_0})^* \varphi_\epsilon \right\|\). (Note that, by the CCRs, \(\left\| a_{\lambda}(f_{\epsilon,k_0})^* \varphi_\epsilon \right\|^2 = \left\| f_{\epsilon,k_0} \right\|^2 \left\| \varphi_\epsilon \right\|^2 + \left\| a_{\lambda}(f_{\epsilon,k_0})^* \varphi_\epsilon \right\|^2 \geq 1\).) Then one can easily see that \(\Psi_\epsilon\) weakly converges to 0 as \(\epsilon \downarrow 0\) and, by (27), \(\lim_{\epsilon \to 0} \left\| (H(P) - z)\Psi_\epsilon \right\| = 0\). Hence \(\{\Psi_\epsilon\}\) is a Weyl sequence. Thus \(z = E(P - k_0) + \omega(k_0) \in \text{ess.spec}(H(P))\). Since \(k_0\) is arbitrary, one has the desired assertion in the proposition. □
4 Degenerate eigenvalues

4.1 Abstract Kramers’ degeneracy theorem

The following lemma is well-known as the Kramers’ degeneracy theorem which plays a central role in this section.

**Lemma 4.1** (Abstract Kramers’ degeneracy theorem) Let \( \vartheta \) be an antiunitary operator with \( \vartheta^2 = -1 \). (In applications \( \vartheta \) is mostly the time reversal operator.) Let \( H \) be a self-adjoint operator. Assume that \( H \) commutes with \( \vartheta \). Then each eigenvalue of \( H \) is at least doubly degenerate.

**Proof.** Let \( \mu \) be an eigenvalue of \( H \) and \( x \) be a corresponding eigenvector. Then, by the polarization identity, one has \( \langle x, \vartheta x \rangle = \langle \vartheta \vartheta x, \vartheta x \rangle = -\langle x, \vartheta x \rangle \) which means \( x \) and \( \vartheta x \) are orthogonal to each other. On the other hand, \( \vartheta x \) is an eigenvector with eigenvalue \( \mu \) too. Hence \( \mu \) is degenerate. \( \Box \)

4.2 Reality preserving operators and degenerate eigenvalues

Recall that the Hamiltonian \( H(P) \) is living in \( \mathbb{C}^2 \otimes \mathfrak{f} \). Each vector \( \varphi \in \mathbb{C}^2 \otimes \mathfrak{f} \) has the following expression:

\[
\varphi = \varphi_1 \oplus \varphi_2,
\]

\[
\varphi_i = \sum_{n \geq 0} \varphi_i^{(n)}(k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad i = 1, 2,
\]

under the identification \( \mathbb{C}^2 \otimes \mathfrak{f} = \mathfrak{f} \oplus \mathfrak{f} \). For each \( \varphi \in \mathbb{C}^2 \otimes \mathfrak{f} \), set

\[
J\varphi = j\varphi_1 \oplus j\varphi_2,
\]

\[
j\varphi_i = \sum_{n \geq 0} \varphi_i^{(n)}(k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad i = 1, 2.
\]

We say that a linear operator \( A \) on \( \mathfrak{f} \) preserves the reality w.r.t. \( j \) if \( A \) commutes with \( j \).

Since \( a(k, \lambda) \) acts by

\[
a(k, \lambda)\varphi_i = \sum_{n \geq 0} \sqrt{n+1} \varphi_i^{(n+1)}(k, \lambda, k_1, \lambda_1, \ldots, k_n, \lambda_n),
\]

one has \( ja(k, \lambda) = a(k, \lambda)j \) and \( ja(k, \lambda)^* = a(k, \lambda)^*j \) which imply

\[
jP_0 = P_0j, \quad \text{(28)}
\]

\[
jA(0) = A(0)j, \quad \text{(29)}
\]

\[
jB(0) = -B(0)j, \quad \text{(30)}
\]

\[
jH_f = H_fj, \quad \text{(31)}
\]

that is, \( P_0, A(0), iB(0) \) and \( H_f \) preserve the reality w.r.t. \( j \). (Here \( B(0) = \nabla \wedge A(0) \).)
Proposition 4.2 Let the time reversal operator $\vartheta$ be given by

$$\vartheta = \sigma_2 J.$$ 

For all $P$ and $e$, we obtain that

$$\vartheta H(P) = H(P)\vartheta.$$ 

Thus, by Lemma 4.1, each eigenvalue of $H(P)$ is at least doubly degenerate.

Proof. Since $H_f$ commutes with $\vartheta$ by (31), it suffices to show that $|D(P)|$ commutes with $\vartheta$. Our basic idea is simple. Noting the fact $\vartheta \sigma_i = -\sigma_i \vartheta$ for $i = 1, 2, 3$, we can easily see that $\vartheta$ commutes with $|D(P)|^2(= (P - P_t + e A(0))^2 + e \sigma \cdot B(0) + M^2)$ on $\mathbb{C}^2 \otimes \mathfrak{F}_{\text{fin}}$ by (28)-(30). As a consequence we could expect that $|D(P)|^2(= \sqrt{(P - P_t + e A(0))^2 + e \sigma \cdot B(0) + M^2})$ also commutes with $\vartheta$.

Unfortunately since we do not know whether the subspace $\mathbb{C}^2 \otimes \mathfrak{F}_{\text{fin}}$ is a core of $D(P)^2$ or not, the above arguments are somehow formal. However we can rigorize the arguments as follow. To clarify the $M$ dependence, we write $D(P)$ as $D_M(P)$ in this proof. Let $\tilde{\vartheta} = \vartheta \odot \vartheta$ acting in $\mathbb{C}^4 \otimes \mathfrak{F}$. Then we see that $\alpha_i \tilde{\vartheta} = -\tilde{\vartheta} \alpha_i$ and $\beta \tilde{\vartheta} = \tilde{\vartheta} \beta$ which imply $\tilde{\vartheta} D_M(P) \tilde{\vartheta}^{-1} = -D_{-M}(P)$ on $\mathbb{C}^4 \otimes \mathfrak{F}_{\text{fin}}$. Since we have already seen that $\mathbb{C}^4 \otimes \mathfrak{F}_{\text{fin}}$ is a core of $D_M(P)$ in section 2.1, this equality holds as an operator equality. Hence, by the functional calculus, one has $\tilde{\vartheta} f(D_M(P)) \tilde{\vartheta}^{-1} = f(-D_{-M}(P))$, where $f$ is real-valued. In the case where $f(s) = \sqrt{s^2}$, we have $f(-D_{-M}(P)) = f(D_M(P))$ because $D_{-M}(P)^2 = D_M(P)^2$ by the anticommutativity between $M\beta$ and $D_{M=0}(P)$. Now one can conclude that $\tilde{\vartheta} |D_M(P)| \tilde{\vartheta}^{-1} = |D_M(P)|$ holds as an operator equality. $\square$

4.3 Comments on related models

The arguments in this section are applicable to other models, e.g.,

$$H_{\text{NR,V}} = \frac{1}{2M} (\sigma \cdot (-i \nabla_x + e A(x)))^2 + H_f + V(x),$$

$$H_{\text{NR}}(P) = \frac{1}{2M} (P - P_t + e A(0))^2 + \frac{e}{2M} \sigma \cdot B(0) + H_f,$$

$$H_V = \sqrt{(-i \nabla_x + e A(x))^2 + e \sigma \cdot B(x) + M^2} + V(x) + H_f$$

with $V(x) = V(-x)$. As regards to $H_{\text{NR}}(P)$, most of the arguments of section 4.2 are valid. However, for $H_{\text{NR,V}}$ and $H_V$, we have to change the definition of $j$. $H_V$ is acting in the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$. Each vector $\varphi$ in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$ has the form

$$\varphi = \varphi_1 \oplus \varphi_2,$$

$$\varphi_i = \sum_{n \geq 0} \varphi_i^{(n)}(x; k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad i = 1, 2.$$
In this case, we define \( j \) as
\[
j\varphi_i^{(n)} = \sum_{n \geq 0} \varphi_i^{(n)}(-x; k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad i = 1, 2.
\]

Then one can check that
\[
\begin{align*}
j(-i\nabla_x) &= (-i\nabla_x)j, \\
jA(x) &= A(x)j, \\
j(iB(x)) &= (iB(x))j, \\
jV(x) &= V(x)j, \\
jH_t &= H_tj,
\end{align*}
\]

namely, all these operators preserve the reality w.r.t. this new \( j \). Hence defining the time reversal operator as \( \vartheta = \sigma_2 J \), one can see that \( H_V \) commutes with \( \vartheta \).

Thus using the abstract Kramers’ degeneracy theorem, one concludes that each eigenvalue of \( H_V \) is at least doubly degenerate. A similar modification applies to \( H_{NR,V} \).

5 Energy inequalities

To make sure that there is no further eigenvalues close to \( E(P) \) we will find self-adjoint operators \( L_+(P) \) and \( L_-(P) \) such that
\[
L_-(P) \leq H(P) \leq L_+(P). \tag{32}
\]

\( L_-(P), L_+(P) \) are given below. They can be easily diagonalized. The min-max principle allows us to obtain bounds as, e.g.,
\[
\Sigma(P) - E(P) \geq \Sigma(L_-(P)) - E(L_+(P))
\]

and more precise information, since the spectrum of \( L_{\pm}(P) \) is available, see section ~\footnote{ } for details. (Here, for a self-adjoint operator \( T \), \( \Sigma(T) = \inf \text{ess.spec}(T) \) and \( E(T) = \inf \text{spec}(T) \).)

**Proposition 5.1** (Lower bound) For any \( 0 < \gamma < 1, 0 \leq m_{ph} \) and \( P \in \mathbb{R}^3 \), one has
\[
H(|P|u) \geq L_-(P)
\]

with
\[
L_-(P) = \gamma \sqrt{P^2 + M^2} + (1 - \gamma - eC_1)H_t - eC_2 \tag{33}
\]

for suitable constants \( C_1, C_2 > 0 \) which are independent of \( e \) and \( P \), where \( u = (1, 0, 0) \).
Proof. **STEP 1.** Let $H_{SL}(P)$ be the spinless Hamiltonian. In this step, we will show the following operator inequality by developing the method in [12]:

$$H_{SL}(|P|u) \geq \gamma \sqrt{P^2 + M^2 + (1 - \gamma - eC)H_f - eC}$$  \hspace{1cm} (34)

with a strictly positive constant $C$ independent of $e$ and $P$. Clearly

$$\left(|P|u - P_1 + eA(0)\right)^2 \geq \left(|P| - P_{11} + eA(0)\right)^2.$$

Thus by the operator monotonicity of the square root (Lemma [E.1], one has

$$H_{SL}(|P|u) \geq \gamma \sqrt{(|P| - P_{11} + eA(0))^2 + M^2 + H_f}.$$  \hspace{1cm} (35)

Let $f(s) = \sqrt{s^2 + M^2}$, $s \in \mathbb{R}$. By Taylor’s theorem, one has

$$f(|P| + s) = f(|P|) + f'(|P|)s + \int_0^1 dt \left(1 - t\right) f''(|P| + ts)s^2$$

with $f'(s) = s/\sqrt{s^2 + M^2}$ and $f''(s) = M^2/(s^2 + M^2)^{3/2}$. Applying the functional calculus, we have the following operator equality

$$\sqrt{(|P| - P_{11} + eA(0))^2 + M^2} = \sqrt{P^2 + M^2 + \frac{|P|}{\sqrt{P^2 + M^2}} P_{11} + eA(0)}$$

$$+ \int_0^1 dt \left(1 - t\right) f''(t(-P_{11} + eA(0)))(-P_{11} + eA(0))^2.$$  \hspace{1cm} (35)

Since the last term in (35) is a positive operator, one obtains

$$\sqrt{(|P| - P_{11} + eA(0))^2 + M^2} \geq \sqrt{P^2 + M^2 + \frac{|P|}{\sqrt{P^2 + M^2}} P_{11} + eA(0)}$$

$$\geq \sqrt{P^2 + M^2 - H_f - \|\omega^{-1/2}F_0\|}(H_f + \mathbb{1})$$

by the standard bounds $|P_{11}| \leq H_f$ and $eA(0) \geq -\|\omega^{-1/2}F_0\|(H_f + \mathbb{1})$. This proves (34).

**STEP 2.** We will show that

$$\pm(H_{SL}(|P|u) - H(|P|u)) \leq \frac{3\pi}{M} \|\omega^{-1/2}\|k\|F_0\|\|H_f + \mathbb{1}\|.$$  \hspace{1cm} (36)

To this end, we simply note that, by (8),

$$H_{SL}(|P|u) - H(|P|u)$$

$$= \frac{1}{\pi} \int_{M^2}^{\infty} ds \sqrt{s - M^2} \left(s + (|P|u - P_1 + eA(0))^2\right)^{-1} e\sigma \cdot B(0)(s + \hat{D}(\|P|u\|^2))^{-1},$$
where \( \hat{D}(P) = D(P) - M\beta \). Noting the facts \( \|e\sigma \cdot B(0)(H_f + \mathbb{1})^{-1/2}\| \leq 6\|(1 + \omega^{-1/2})|k||F_0||\| \) and (30) in the proof of Lemma 5.3, one can see that \( \|(H_{SL}(P) - H(P))(H_f + \mathbb{1})^{-1/2}\| \leq 3\pi\|(1 + \omega^{-1/2})|k||F_0||/M. \) Now (36) is obtained.

**STEP 3.** (Proof of Proposition 5.1) From (34) and (36) it follows that

\[
H(|P|u) = H_{SL}(|P|u) + (H(|P|u) - H_{SL}(|P|u))
\]

\[
\geq H_{SL} - eC(H_f + \mathbb{1})
\]

\[
\geq \gamma\sqrt{P^2 + M^2} + (1 - \gamma - eC_1)H_f - eC_2.
\]

This proves the desired assertion in the proposition. \( \square \)

Before we proceed, we remark the following. Let \( SO(3) \) be the rotation group. Then there exists a unitary representation \( \pi \) of \( SO(3) \) such that

\[
\pi_g H(P)\pi_g^{-1} = H(g^{-1}P)
\]

for all \( g \in SO(3) \) and \( P \in \mathbb{R}^3 \), see e.g., [21]. Thus \( E(P) \) is a radial function in \( P \).

Since \( E(P) \) is rotationally symmetric in \( P \), one has an immediate corollary.

**Corollary 5.2** Choose \( \gamma < 1 \) and \( e \) sufficiently small as \( e < e_* \). One has

\[
E(P) \geq \gamma\sqrt{P^2 + M^2} - eC_1
\]

for all \( P \in \mathbb{R}^3 \), where \( C_1 \) is independent of \( e,P \).

**Proposition 5.3** (Upper bound) One obtains

\[
H(|P|u) \leq L_+(P)
\]

with

\[
L_+(P) = \gamma\left[ (|P|u - P_l)^2 + 2|P|(H_f + \|\omega^{-1/2}|F_0||) \right]
\]

\[
+ 4(H_f + \mathbb{1})P_l^2 + \|(1 + \omega^{-1/2})|F_0||^2 + \|(1 + \omega^{-1/2})|F_0||^2(H_f + \mathbb{1})
\]

\[
+ H_f + ||k||^{1/2}|F_0||^2 + M^2\right]^{1/2} + H_f.
\]

for all \( P \).

**Proof.** Observe that

\[
D(|P|u)^2 = (|P|u - P_l)^2 + 2(|P|u - P_l) \cdot eA(0) + e^2 A(0)^2 + e\sigma \cdot B(0) + M^2.
\]

Using the fundamental inequalities in Appendix A, one has

\[
|P|eA(0)^1 \leq |P|(H_f + \|\omega^{-1/2}|F_0||),
\]

\[
eA(0) \cdot P_l \leq 2(H_f + \mathbb{1})P_l^2 + \frac{1}{2}\|(1 + \omega^{-1/2})|F_0||^2,
\]

\[
e^2 A(0)^2 \leq \|(1 + \omega^{-1/2})|F_0||^2(H_f + \mathbb{1}),
\]

\[
e\sigma \cdot B(0) \leq H_f + ||k||^{1/2}|F_0||^2.
\]
Applying the operator monotonicity of the square root (Lemma E.1), one concludes (38).

By the above operator inequality, we immediately obtain

\[ \langle \eta \otimes \Omega, H(|P|u)\eta \otimes \Omega \rangle \leq \gamma \sqrt{|P|^2 + 2|P|\omega^{-1/2}|F_0| + 2|(1 + \omega^{-1/2})|F_0|^{1/2} + \|k|^{1/2}|F_0|^{1/2}} + M^2, \]

where \( \eta \uparrow = (1, 0) \). Thus taking the rotational invariance of \( E(P) \) in \( P \) into consideration, one has a following corollary.

**Corollary 5.4** One has

\[ E(P) \leq \gamma \sqrt{(|P| + eC_3)^2 + M^2 + e^2C_4} \]  

(39)

for all \( P \), where \( C_3 \) and \( C_4 \) are independent of \( P \) and \( e \).

6 **Proof of Theorem 1.6**

6.1 **Proof of Theorem 1.6 (i)**

For \( a \in \mathbb{R}^d \), \( ||a||_{\mathbb{R}^d} \) means the standard norm in \( \mathbb{R}^d \). Then one has, for example, \( \omega(k) = \|(k, m_{ph})\|_{\mathbb{R}^4} = \||(k, m_{ph})\|_{\mathbb{R}^2} \). Applying the triangle inequality and Corollary 5.2, one gets

\[
E(P - k) + \omega(k) \\
\geq \gamma \|(P - k, M)\|_{\mathbb{R}^2} + \|(k, m_{ph})\|_{\mathbb{R}^2} - eC \\
\geq \gamma \|(P - k, M)\|_{\mathbb{R}^2} + \gamma \|(k, m_{ph})\|_{\mathbb{R}^2} + (1 - \gamma)\|(k, m_{ph})\|_{\mathbb{R}^2} - eC \\
= \gamma \|(P - k, M)\|_{\mathbb{R}^4} + \gamma \|(k, m_{ph})\|_{\mathbb{R}^4} + (1 - \gamma)\|(k, m_{ph})\|_{\mathbb{R}^4} - eC \\
\geq \gamma \|(P, M + m_{ph})\|_{\mathbb{R}^4} + (1 - \gamma)m_{ph} - eC. 
\]

On the other hand, since \( \|\omega^{-1/2}|F_0|\| \leq O(e^2) \) etc., one has, by Corollary 5.4, that

\[
E(P) \leq \gamma \|(P + eC_3, \sqrt{M^2 + O(e^2)})\|_{\mathbb{R}^2} \\
\leq \gamma \|(P + eC_3, M)\|_{\mathbb{R}^2} + \gamma \|(0, \sqrt{M^2 + O(e^2)} - M)\|_{\mathbb{R}^2} \\
\leq \gamma \|(P, M)\|_{\mathbb{R}^4} + eC_3 + O(e^2). 
\]

Thus the desired assertion in the lemma follows. □
6.2 Proof of Theorem 1.6 (ii) and (iii)

We denote the infimum of \( \text{spec}(L_-(P)) \) and \( \text{ess.spec}(L_-(P)) \) by \( \mathcal{E}_-(P) \) and \( \Sigma_-(P) \) respectively. Clearly one has

\[
\mathcal{E}_-(P) = \gamma \sqrt{P^2 + M^2} - eC, \\
\Sigma_-(P) = \gamma \sqrt{P^2 + M^2} + (1 - \gamma - eC_1)m_{ph} - eC_2.
\]

Let \( \mathcal{E}_+(P) \) be the function of \( P \) which is appearing on the right hand side of (39). Then using similar arguments as in the proof of Theorem 1.6 (i), one sees that

\[
0 \leq \mathcal{E}_-(P) - \mathcal{E}_+(P) \leq C'e + \mathcal{O}(e^2)
\]

with \( e < e_* \). (Here it should be noted that \( e_* \) is independent of \( P \).) Thus taking the fact \( \mathcal{E}_-(P) \leq E(P) \leq \mathcal{E}_+(P) \) into consideration, one has

\[
0 \leq E(P) - \mathcal{E}_-(P) \leq C'e + \mathcal{O}(e^2)
\]

for \( e < e_* \), which means \( E(P) \) is close to \( \mathcal{E}_-(P) \) uniformly in \( P \). Also we note that

\[
\Sigma_-(P) - \mathcal{E}_-(P) \geq (1 - eC - \gamma)m_{ph}.
\]

Let \( E_1(P) \) be the first excited eigenvalue of \( H(P) \) (or possibly be \( \Sigma(P) \) if there is no such excited state.) Then by the operator inequality (33) and the min-max principle, one has

\[
E_1(|P|u) \geq \Sigma_-(P).
\]

(Note that, by Proposition 4.2, \( E(P) \) is always degenerate.) With the help of (43), one sees that \( E_1(P) \) is radial and

\[
E_1(P) \geq \Sigma_-(P).
\]

for all \( P \in \mathbb{R}^3 \). Thus, combining this with (41), we arrive at

\[
E_1(P) - E(P) \geq \Sigma_-(P) - \mathcal{E}_+(P) \\
\geq \Sigma_-(P) - \mathcal{E}_-(P) + (\mathcal{E}_-(P) - \mathcal{E}_+(P)) \\
\geq (1 - eC_1 - \gamma)m_{ph} - C'e - \mathcal{O}(e^2)
\]

for \( e < e_* \). This proves (ii) in the theorem.

For a self-adjoint operator \( A \), let \( E_K(A) \) be its spectral measure for the interval \((-\infty, K)\) and let \( P_{pp}(A) \) be the projection onto the linear space spanned by all eigenstates. Since, by Proposition 5.1, one has the operator inequality \( L_-(P) \leq H(|P|u) \), the following property holds,

\[
\text{tr} P_{pp}(H(|P|u)) E_{\Sigma_-(P)}(H(|P|u)) \leq \text{tr} P_{pp}(L_-(P)) E_{\Sigma_-(P)}(L_-(P)) = 2
\]

by the min-max principle. Applying (37), one has that

\[
\text{tr} P_{pp}(H(P)) E_{\Sigma_-(P)}(H(P)) = \text{tr} P_{pp}(H(|P|u)) E_{\Sigma_-(P)}(H(|P|u)) \leq 2.
\]

Thus \( H(P) \) has at most two eigenstates with corresponding eigenvalue less than \( \Sigma_-(P) \). On the other hand, one already knows that \( E(P) < \Sigma_-(P) \leq E_1(P) \) for \( e < e_* \) by (41), (42) and (43). Therefore \( E(P) \) is at most doubly degenerate. \( \square \)
A Second quantization and basic inequalities

Let \( \mathfrak{h} \) be a complex Hilbert space. The Fock space over \( \mathfrak{h} \) is defined by

\[
\mathfrak{F}(\mathfrak{h}) = \bigoplus_{n \geq 0} \mathfrak{h}^\otimes_n,
\]

where \( \mathfrak{h}^\otimes_n \) means the \( n \)-fold symmetric tensor product of \( \mathfrak{h} \) with the convention \( \mathfrak{h}^\otimes_0 = \mathbb{C} \). The vector \( \Omega = 1 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathfrak{h}) \) is called the Fock vacuum.

We denote by \( a(f) \) the annihilation operator on \( \mathfrak{F}(\mathfrak{h}) \) with a test vector \( f \in \mathfrak{h} \), its adjoint \( a(f)^\ast \), called the creation operator, is defined by

\[
a(f)^\ast \varphi = \sum_{n \geq 0} \sqrt{n+1} f \otimes_s \varphi^{(n)}
\]

for a suitable \( \varphi = \sum_{n \geq 0} \varphi^{(n)} \in \mathfrak{F}(\mathfrak{h}) \). By definition, \( a(f) \) is densely defined, closed, and antilinear in \( f \). We frequently write \( a(f)^\# \) to denote either \( a(f) \) or \( a(f)^\ast \). Creation and annihilation operators satisfy the canonical commutation relations

\[
[a(f), a(g)^\ast] = \langle f, g \rangle_{\mathfrak{h}} \mathbb{I}, \quad [a(f), a(g)] = 0 = [a(f)^\ast, a(g)^\ast]
\]

on a suitable subspace of \( \mathfrak{F}(\mathfrak{h}) \), where \( \mathbb{I} \) denotes the identity operator. We introduce a particular subspace of \( \mathfrak{F}(\mathfrak{h}) \) which will be used frequently. Let \( s \) be a subspace of \( \mathfrak{h} \). We define

\[
\mathfrak{F}_{\text{fin}}(s) = \text{Lin}\{ a(f_1)^\ast \cdots a(f_n)^\ast \Omega, \ \Omega | f_1, \ldots, f_n \in s, \ n \in \mathbb{N} \},
\]

where \( \text{Lin}\{\cdots\} \) means the linear span of the set \( \{\cdots\} \). If \( s \) is dense in \( \mathfrak{h} \), so is \( \mathfrak{F}_{\text{fin}}(s) \) in \( \mathfrak{F}(\mathfrak{h}) \).

For a densely defined closable operator \( c \) on \( \mathfrak{h} \), \( d\Gamma(c) : \mathfrak{F}(\mathfrak{h}) \to \mathfrak{F}(\mathfrak{h}) \) is defined by

\[
d\Gamma(c) \upharpoonright \text{dom}(c)^\otimes_n = \sum_{j=1}^n \mathbb{I} \otimes \cdots \otimes c_{j\text{th}} \otimes \cdots \otimes \mathbb{I}
\]

and

\[
d\Gamma(c) \Omega = 0
\]

where \( \text{dom}(c) \) means the domain of the linear operator \( c \). Here in the \( j \)-th summand \( c \) is at the \( j \)-th entry. Clearly \( d\Gamma(c) \) is closable and we denote its closure by the same symbol. As a typical example, the number operator \( N_f \) is given by \( N_f = d\Gamma(\mathbb{I}) \).

In the case where \( \mathfrak{h} = L^2(\mathbb{R}^3 \times \{1, 2\}) \), the annihilation and creation operator can be expressed as the operator valued distributions \( a(k, \lambda), a(k, \lambda)^\ast \) by

\[
a(f) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \mathcal{F}(k, \lambda)a(k, \lambda), \quad a(f)^\ast = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \ f(k, \lambda)a(k, \lambda)^\ast.
\]
Let $F$ be a measurable function on $\mathbb{R}^3$ and let the multiplication operator associated with $F$ be denoted by the same symbol: $(Ff)(k, \lambda) = F(k)f(k, \lambda)$ for $f \in L^2(\mathbb{R}^3 \times \{1, 2\})$. Then one can formally express $d\Gamma(F)$ as

$$d\Gamma(F) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk F(k)a(k, \lambda)^*a(k, \lambda).$$

For $F(k) = \omega(k)$, one has the expression (3) of $Hf = d\Gamma(\omega)$.

Lemma A.1 One has the following.

(i) $\|a(f)\varphi\| \leq \|\omega^{-1/2} f\| \|H^{1/2}_f\varphi\|.$

(ii) $\|a(f)^*\varphi\| \leq \|(1 + \omega^{-1/2}) f\| \|(H_f + \mathbb{1})^{1/2}\varphi\|.$

(iii) $a(f) + a(f)^* \leq H_f + \|\omega^{-1/2} f\|^2.$

(iv) $\|(a(f) + a(f)^*)\varphi\| \leq 2\|(1 + \omega^{-1/2}) f\| \|(H_f + \mathbb{1})^{1/2}\varphi\|.$

(v) $a(f)^1a(g)^2 \leq \|(1 + \omega^{-1/2}) f\| \|(1 + \omega^{-1/2}) g\| (H_f + \mathbb{1}).$

B Invariant domains

Lemma B.1 Let $A$ be self-adjoint and $H$ be positive and self-adjoint. Assume the following.

(i) $(H + \mathbb{1})^{-1}\text{dom}(A) \subseteq \text{dom}(A)$.

(ii) $\left| \langle Hu, Au \rangle - \langle Au, Hu \rangle \right| \leq C \|(H + \mathbb{1})u\|^2$ for all $u \in \text{dom}(A) \cap \text{dom}(H)$.

(iii) $[H, A](H + \mathbb{1})^{-1}$ can be extended to a bounded operator.

Then one has $e^{itA}\text{dom}(H) = \text{dom}(H)$ for all $t \in \mathbb{R}$.

Proof. See [4, Lemma 2]. □

C Localization estimate

In this appendix, we will establish Lemma 3.1 which is essential for the proof of Theorem 1.5. Unfortunately the proof is technically complicated because of the square root structure. We repeat the statement which we want to prove.

Lemma 3.1 Choose $e$ as $e < e_*$. For all $\varphi \in C^2 \otimes \mathfrak{f}_{\text{fin}} \otimes \mathfrak{f}_{\text{fin}}$, one obtains

$$\left| \langle \varphi, (H(P) - \tilde{\Gamma}(\mathcal{J})^*H^0(P)\tilde{\Gamma}(\mathcal{J}))\varphi \rangle \right| \leq O(R^0)\|(H(P) + \mathbb{1})\varphi\|^2,$$

where $O(R^0)$ is a function of $R$ vanishing as $R \to \infty$. 
Proof. Let us define a Dirac operator by
\[ D^\otimes(P) = \alpha \cdot (P - P_1 \otimes \mathbb{1} - \mathbb{1} \otimes P_1 + eA(0) \otimes \mathbb{1}) + M\beta. \]

We also introduce
\[ \hat{D}(P) = D(P) - M\beta, \quad \hat{D}^\otimes(P) = D^\otimes(P) - M\beta. \]

Using the formula [8], one has
\[
|D(P)| = \frac{1}{\pi} \int_{M^2}^\infty dt \frac{\hat{D}(P)^2 + M^2}{\sqrt{t - M^2(t + \hat{D}(P)^2)}},
\]
\[
|D^\otimes(P)| = \frac{1}{\pi} \int_{M^2}^\infty dt \frac{\hat{D}^\otimes(P)^2 + M^2}{\sqrt{t - M^2(t + \hat{D}^\otimes(P)^2)}}.
\]

Hence
\[
|D(P)| - \hat{\Gamma}(\mathcal{J})^*|D^\otimes(P)|\hat{\Gamma}(\mathcal{J})
= \frac{1}{\pi} \int_{M^2}^\infty dt \frac{\hat{D}(P)^2 + M^2}{\sqrt{t - M^2(t + \hat{D}(P)^2)}}\left\{ \hat{D}(P)G(P) + G(P)\hat{D}(P) - G(P)^2 \right\}
\times (t + \hat{D}^\otimes(P)^2)^{-1},
\]
where
\[ G(P) = \hat{D}(P) - \hat{D}^\otimes(P) \]
with \( \hat{D}^\otimes(P) = \hat{\Gamma}(\mathcal{J})^*\hat{D}^\otimes(P)\hat{\Gamma}(\mathcal{J}). \) Remark the following fact
\[ \|G(P)(N_f + \mathbb{1})^{-1}\| \leq \mathcal{O}(R^0), \]
see, e.g., [8] [13]. (It should be noted that the positive photon mass is crucial here.) By Lemma [C.1] below, we estimate as
\[
\left\| (N_f + \mathbb{1})^{-1}(t + \hat{D}(P)^2)^{-1}\left\{ \hat{D}(P)G(P) + G(P)\hat{D}(P) - G(P)^2 \right\} \right\|
\times (t + \hat{D}^\otimes(P)^2)^{-1}(N_f + \mathbb{1})^{-1}
\leq 2\| (N_f + \mathbb{1})^{-1}(t + \hat{D}(P)^2)^{-1}(N_f + \mathbb{1})\|\| (N_f + \mathbb{1})^{-1}G(P)\|\| (N_f + \mathbb{1})^{-1}\hat{D}(P)\|
\times \| (N_f + \mathbb{1})(t + \hat{D}^\otimes(P)^2)^{-1}(N_f + \mathbb{1})^{-1}\|\| (N_f + \mathbb{1})^{-1}G(P)\|^2
\times \| (N_f + \mathbb{1})(t + \hat{D}^\otimes(P)^2)^{-1}(N_f + \mathbb{1})^{-1}\|
\leq \mathcal{O}(R^0)(t^{-2} + t^{-3/2} + t^{-1})^2
\]
which implies
\[
\left\| (N_f + \mathbb{1})^{-1}(|D(P)| - \hat{\Gamma}(\mathcal{J})^*|D^\otimes(P)|\hat{\Gamma}(\mathcal{J}))(N_f + \mathbb{1})^{-1}\right\| \leq \mathcal{O}(R^0). \quad (45)
\]
We also note the fact
\[ \| (N_t + \mathbb{1})^{-1} (H_t - \hat{\Gamma}(\mathcal{J})^* (H_t \otimes \mathbb{1} + \mathbb{1} \otimes H_t) \hat{\Gamma}(\mathcal{J})) (N_t + \mathbb{1})^{-1} \| \leq O(R^0) \] (46)
which is proven in [8]. Collecting (45) and (46), one sees that \( |\langle \varphi, (H(P) - \hat{\Gamma}(\mathcal{J})^* H^\otimes(P) \hat{\Gamma}(\mathcal{J})) \varphi \rangle| \leq O(R^0) \| (N_t + \mathbb{1}) \varphi \|^2 \) holds.

Finally one has to show \( \| (N_t + \mathbb{1}) \varphi \| \leq C \| (H(P) + \mathbb{1}) \varphi \| \). The positive photon mass implies \( \| (N_t + \mathbb{1}) \varphi \| \leq C \| (H_t + \mathbb{1}) \varphi \| \). Applying Lemma [D.2] yields the desired results \( \| (N_t + \mathbb{1}) \varphi \| \leq C \| (H(P) + \mathbb{1}) \varphi \| \). □

**Lemma C.1** For all \( t > 0 \), one has the following.

(i) \( \| (N_t + \mathbb{1})(t + \hat{D}(P)^2)^{-1} (N_t + \mathbb{1})^{-1} \| \leq C(t^{-1} + t^{-3/2} + t^{-2}). \)

(ii) \( \| (N_t + \mathbb{1})(t + \hat{D}^\otimes(P)^2)^{-1} (N_t + \mathbb{1})^{-1} \| \leq C(t^{-1} + t^{-3/2} + t^{-2}). \)

**Proof.** (i) The essential idea is taken from [11]. First we will show that \( e^{it\hat{D}(P)}\text{dom}(N_t) = \text{dom}(N_t) \). It suffices to check the conditions (i), (ii) and (iii) in Lemma [B.1]. Noting \( [N_t, \hat{D}(P)] = -\alpha \cdot (a(F_0) - a(F_0)^*) \) on \( \mathbb{C}^4 \otimes \mathfrak{F}_{\text{fin}} \), we can check all conditions in Lemma [B.1] by Lemma [A.1].

Using the formula
\[ (\hat{D}(P)^2 + t)^{-1} = \int_\mathbb{R} ds \, g_t(s) e^{-is\hat{D}(P)} \]
with \( g_t(s) = \sqrt{\pi/2t} e^{-\sqrt{\pi} s} \), we have
\[ \| (N_t + \mathbb{1})(\hat{D}(P)^2 + t)^{-1} \varphi \| \leq \int_\mathbb{R} ds \, g_t(s) \| (N_t + \mathbb{1})e^{-is\hat{D}(P)} \varphi \| \] (47)
for each normalized \( \varphi \in \text{dom}(N_t) \). (We already know that \( e^{is\hat{D}(P)} \varphi \in \text{dom}(N_t) \).)

Set \( I_1(s) = \| (N_t + \mathbb{1})e^{-is\hat{D}(P)} \varphi \| \) and \( I_{1/2}(s) = \| (N_t + \mathbb{1})^{1/2}e^{-is\hat{D}(P)} \varphi \| \). Then one has
\[
\frac{d}{ds} I_1(s)^2 = \langle e^{-is\hat{D}(P)} \varphi, i[\hat{D}(P), (N_t + \mathbb{1})^2]e^{-is\hat{D}(P)} \varphi \rangle
= \langle e^{-is\hat{D}(P)} \varphi, (e\alpha \cdot E(N_t + \mathbb{1}) + (N_t + \mathbb{1})e\alpha \cdot E) e^{is\hat{D}(P)} \varphi \rangle,
\]
where \( E = ia(F_0) - ia(F_0)^* \). Accordingly using the standard estimate \( \| E \varphi \| \leq C \| (N_t + \mathbb{1})^{1/2} \varphi \| \), one has
\[
\frac{d}{ds} I_1(s)^2 \leq CI_{1/2}(s) I_1(s). \]
(48)

Next we will estimate \( I_{1/2}(s) \). Observe that
\[
\frac{d}{ds} I_{1/2}(s)^2 = \langle e^{-is\hat{D}(P)} \varphi, i[\hat{D}(P), N_t]e^{-is\hat{D}(P)} \varphi \rangle
\leq e \| E \varphi \|^2 \leq C \| (N_t + \mathbb{1})^{1/2} e^{-is\hat{D}(P)} \varphi \|
= CI_{1/2}(s).
\]
Solving this inequality, we get \( I_{1/2}(s) \leq C|s| + I_{1/2}(0) \). Inserting this result into (18), one has
\[
I_1(s) \leq I_1(0) + C s^2 + C|s|I_{1/2}(0).
\]
Combining this with (17), we finally obtain the assertion (i) in the lemma.

Noting the fact \( \hat{\Gamma}(\mathcal{J}) N_t \hat{\Gamma}(\mathcal{J})^* = N_t \otimes 1 + 1 \otimes N_t \), one can apply the similar arguments in the proof of (i) to show (ii).

\[\square\]

D Auxiliary estimates

In this appendix, we always choose \( e \) as \( e < e_0 \).

**Lemma D.1** For all \( P \in \mathbb{R}^3 \) and \( e \geq 0 \), one has
\[
\|D(P)(H_f + 1)^{-1}\| \leq |P| + 3 + 3e\|\omega^{-1/2}F_0\|.
\]

**Proof.** Noting Lemma A.1 and the fundamental fact \( \|P_{i,i}(H_f + 1)^{-1}\| \leq 1 \) for \( i = 1, 2, 3 \), one observes that
\[
\|D(P)(H_f + 1)^{-1}\|
= \|D(P)(H_f + 1)^{-1}\|
\leq |P| + \sum_{i=1,2,3} \|P_{i,i}(H_f + 1)^{-1}\| + e \sum_{i=1,2,3} \|A(0)(H_f + 1)^{-1}\|
\leq |P| + 3 + 3e\|\omega^{-1/2}F_0\|.
\]
This proves the assertion. \[\square\]

**Lemma D.2** For each \( n \in \mathbb{N} \), one obtains
\[
\|H_f^{n/2}(H_f + 1)^{-n/2}\| \leq \text{const},
\]
\[
\|D(P)^{n/2}(H_f + 1)^{-n/2}\| \leq \text{const},
\]
where const is independent of \( P \).

**Proof.** In the similar way in the proof of (7), one can show that both
\[
H_f^{n/2}(H_f + 1)^{-n/2}, \quad \|D(P)^{n/2}(H_f + 1)^{-n/2}\| \leq \text{const}
\]
are bounded. Thus we conclude the assertion by the fact that \( (H_f + 1)^{n/2}\varphi \) is strongly continuous in \( P \) for \( \varphi \in \mathbb{C}^4 \otimes \mathcal{F}_\text{fin} \). \[\square\]

**Lemma D.3** For each \( n \in \mathbb{N} \), we obtain
\[
\|(H_f + 1)^n(t + \hat{D}(P)^2)^{-1}(H_f + 1)^{-n}\| \leq \text{const} (t^{-1} + t^{-3/2} + \cdots + t^{-n-1})
\]
for every \( P, Q \in \mathbb{R}^3 \), where const is independent of \( P \) and \( Q \).
Sketch of proof. By Lemma A.1, one can see that \( e^{it\hat{D}(P)} \text{dom}(H^n_t) = \text{dom}(H^n_t) \).

Let us write
\[
K_{m/2}(s) = \| (H_t + 1)^{m/2} e^{-is\hat{D}(P)} \varphi \|
\]
for a normalized \( \varphi \in \text{dom}(H^n_t) \) with \( m \leq 2n \). In the case where \( m = 1 \), one has
\[
\frac{d}{ds} K_{1/2}(s)^2 = \langle e^{-is\hat{D}(P)} \varphi, i[\hat{D}(P), H_t] e^{-is\hat{D}(P)} \varphi \rangle = \langle e^{-is\hat{D}(P)} \varphi, i\alpha \cdot (a(\omega F) - a(\omega F)^*) e^{-is\hat{D}(P)} \varphi \rangle \leq CK_{1/2}(s)
\]
by the Schwarz inequality. Thus \( K_{1/2}(s) \leq K_{1/2}(0) + C|s| \) holds. In the case where \( m = 2 \), one has, by the similar arguments in the above,
\[
\frac{d}{ds} K_1(s)^2 \leq CK_{1/2}(s)K_1(s) \leq (K_{1/2}(0) + C|s|)K_1(s)
\]
which implies \( K_1(s) \leq K_1(0) + C(K_{1/2}(0)|s| + s^2) \). Repeating this procedure, one can arrive at
\[
K_{m/2}(s) \leq K_{m/2}(0) + C(K_{(m-1)/2}(0)|s| + \cdots + K_{1/2}(0)|s|^{m-1} + |s|^m).
\]

Therefore using the formula
\[
\| (H_t + 1)^{m/2} (t + \hat{D}(P)^2)^{-1} \| \leq \int_{\mathbb{R}} ds \ g_t(s) K_{m/2}(s)
\]
with \( g_t(s) = \sqrt{\pi/2|e^{-\sqrt{t}|s|}} \), one has, by putting \( m = 2n \),
\[
\| (H_t + 1)^{n}(t + \hat{D}(P)^2)^{-1} (H_t + 1)^{-n} \| \leq C(t^{-1} + t^{-3/2} + \cdots + t^{-n-1}).
\] (50)

Finally using Lemma D.2, one concludes the desired assertion in the lemma.

Lemma D.4 One has
\[
\| [H_t, |D(P)|](H(Q) + 1)^{-2} \| \leq \text{const} \ (1 + |P|)
\]
for all \( P, Q \in \mathbb{R}^3 \).

Proof. By Lemma D.3 and the following standard formula
\[
[H_t, |D(P)|] = \frac{1}{\pi} \int_{M^2} ds \sqrt{s - M^2}(s + \hat{D}(P)^2)^{-1}[H_t, \hat{D}(P)^2](s + \hat{D}(P)^2)^{-1},
\]
one computes

\[
\begin{align*}
& \left\| [H_f, D(P)] (H_f + \mathbb{1})^{-2} \right\| \\
& \leq \frac{1}{\pi} \int_{M^2} \frac{ds}{s - M^2} \left\| (s + \hat{D}(P)^2)^{-1} \right\| \left\| [H_f, D(P)] (H_f + \mathbb{1})^{-2} \right\| \\
& \quad \times \left\| (H_f + \mathbb{1})^2 (s + \hat{D}(P)^2)^{-1} (H(Q) + \mathbb{1})^{-2} \right\| \\
& \leq \frac{C}{\pi} \int_{M^2} \frac{ds}{s - M^2 s^{-1}} (1 + \|P\|)(s^{-1} + s^{-3/2} + \cdots + s^{-3}) \\
& = \text{const} (1 + |P|).
\end{align*}
\]

This completes the proof. \(\square\)

**Lemma D.5** For all \(P, Q \in \mathbb{R}^3\), one has

\[
\begin{align*}
& \left\| H(P)^2 (H(Q) + \mathbb{1})^{-2} \right\| \\
& \leq (C + |P - Q|)^2 + C(1 + |P| + |Q| + |P||Q|)(C + |P - Q|^2)
\end{align*}
\]

**Proof.** First we will show that

\[
\left\| H(P) (H(Q) + \mathbb{1})^{-1} \right\| \leq C + |P - Q|.
\]

To this end, observe that

\[
\begin{align*}
& \left\| H(P) (H(Q) + \mathbb{1})^{-1} \right\| \\
& \leq \left\| (D(P))(H(Q) + \mathbb{1})^{-1} \right\| + \left\| H_f (H(Q) + \mathbb{1})^{-1} \right\| \\
& \leq \left\| (D(Q))(H(Q) + \mathbb{1})^{-1} \right\| + \alpha \cdot (P - Q)(H(Q) + \mathbb{1})^{-1} \right\| + \left\| H_f (H(Q) + \mathbb{1})^{-1} \right\| \\
& \leq C + |P - Q|
\end{align*}
\]

by Lemma D.2. Write

\[
\begin{align*}
& H(P)^2 (H(Q) + \mathbb{1})^{-2} \\
& = \{ H(P)(H(Q) + \mathbb{1})^{-1} \}^2 + H(P)[H(P), (H(Q) + \mathbb{1})^{-1}](H(Q) + \mathbb{1})^{-1} \\
& = \{ H(P)(H(Q) + \mathbb{1})^{-1} \}^2 + H(P)(H(Q) + \mathbb{1})^{-1}[H(Q), H(P)](H(Q) + \mathbb{1})^{-2}.
\end{align*}
\]

Hence

\[
\begin{align*}
& \left\| H(P)^2 (H(Q) + \mathbb{1})^{-2} \right\| \\
& \leq (C + |P - Q|)^2 + (C + |P - Q|)\left\| [H(Q), H(P)](H(Q) + \mathbb{1})^{-2} \right\|.
\end{align*}
\]

Accordingly what we have to show next is to estimate the operator norm

\[
\left\| [H(Q), H(P)](H(P) + \mathbb{1})^{-2} \right\|.
\]
Observe that
\[ [H(Q), H(P)](H(Q) + \mathbb{1})^{-2} = \|D(P)\|D(Q)\|H(Q) + \mathbb{1})^{-2} \]
\[ + [H_t, D(P)](H(Q) + \mathbb{1})^{-2} \]
\[ + \|D(P)\|H_t(\mathbb{1})^{-2}. \]

Then one has
\[ \|\|D(P)\|D(Q)\|\varphi\| \leq \|\|D(0)\|D(Q)\|\varphi\| + \|P\|\|D(Q)\|\varphi\| \leq C\|\|H_t + \mathbb{1}\|D(Q)\|\varphi\| + \|P\|\|D(Q)\|\varphi\| \]
\[ \leq C\left(\|\|D(Q)\|H_t + \mathbb{1}\|\varphi\| + \|H_t, D(Q)\|\varphi\| + (1 + |Q|)\|P\|\|H_t + \mathbb{1}\|\varphi\|\right) \]
\[ \leq C\left((1 + |Q|)\|\|H_t + \mathbb{1}\|2\varphi\| + \|H(Q) + \mathbb{1}\|2\varphi\| + (1 + |Q|)\|P\|\|H_t + \mathbb{1}\|\varphi\|\right) \]
\[ \leq C(1 + |P| + |Q| + \|P\|Q\|)(H(Q) + \mathbb{1})^2\varphi\| . \]

In the above we have used Lemma [D.4] from the line three to the next, and from the line four to the final line, we have used Lemma [D.1]. Collecting the results, one obtains the assertion in the lemma. \(\square\)

**Lemma D.6** Let
\[ S_{k,\lambda}(P) = |D(P - k)|a(k, \lambda) - a(k, \lambda)|D(P)|. \]
Then one has
\[ \|S_{k,\lambda}(P)^*(H(P - k) + \mathbb{1})^{-1}\| \leq C(1 + |k|)|F_0(k, \lambda)|, \]
where \(C\) is independent of \(k\) and \(P\).

**Proof.** We will show that \(\|\|H(P - k) + \mathbb{1}\|^{-1}S_{k,\lambda}\| \leq C(1 + |k|)|F_0(k, \lambda)|. \) Let
\[ S_{k,\lambda} = e^{i\mathbf{k} \cdot \mathbf{r}}e^{i P_1[|D|, a(k, \lambda)]e^{-i P_1}}. \]
Then one has
\[ \mathcal{F}_xS_{k,\lambda}\mathcal{F}_x^* = \int_{\mathbb{R}^3} S_{k,\lambda}(P) dP \]
for all \((k, \lambda) \in \mathbb{R}^3 \times \{1, 2\}\). Hence it suffices to show that
\[ \|\|H_k + \mathbb{1}\|^{-1}[|D - \alpha \cdot k|, a(k, \lambda)]\| \leq C(1 + |k|)|F_0(k, \lambda)| \]
where \(H_k = |D - \alpha \cdot k| + H_t, \) because
\[ \mathcal{F}_x\int_{\mathbb{R}^3} (H(P - k) + \mathbb{1})^{-1}S_{k,\lambda}(P) dP \mathcal{F}_x \]
\[ = e^{i P_1}(H_k + \mathbb{1})^{-1}e^{-i P_1}S_{k,\lambda} \]
\[ = e^{i P_1}(H_k + \mathbb{1})^{-1}e^{-i P_1}e^{i k \cdot \mathbf{r}}e^{i P_1[|D|, a(k, \lambda)]e^{-i P_1}} \]
\[ = e^{i P_1}(H_k + \mathbb{1})^{-1}[|D - \alpha \cdot k|, a(k, \lambda)]e^{i k \cdot \mathbf{r}}e^{-i P_1}. \]
To this end, we remark that, with \( \hat{D} = D - M\beta \),

\[
[D - \alpha \cdot k, a(k, \lambda)] = \frac{1}{\pi} \int_{M^2}^\infty ds \sqrt{s - M^2} (s + (\hat{D} - \alpha \cdot k)^2)^{-1}[(\hat{D} - \alpha \cdot k)] a(k, \lambda) \\
\times (s + (\hat{D} - \alpha \cdot k)^2)^{-1} \\
= -\frac{1}{\pi} \int_{M^2}^\infty ds \sqrt{s - M^2} (s + (\hat{D} - \alpha \cdot k)^2)^{-1} \left\{ (\hat{D} - \alpha \cdot k) e \alpha \cdot F_x(k, \lambda) + e \alpha \cdot F_x(k, \lambda) (\hat{D} - \alpha \cdot k) \right\} (s + (\hat{D} - \alpha \cdot k)^2)^{-1}.
\]

(55)

Assume, for a while, that

\[
\| (H_k + 1)(s + (\hat{D} - \alpha \cdot k)^2)^{-1} (H_k + 1) \| \leq C(s^{-1} + s^{-3/2} + s^{-2}).
\]

(56)

We note the following two estimates:

\[
\| (H_k + 1)^{-1}(\hat{D} - \alpha \cdot k) \alpha \cdot F_x(k, \lambda) \|
\]

\[
= \| (H + 1)^{-1} \hat{D} \alpha \cdot F_0(k, \lambda) \|
\]

\[
\leq C|F_0(k, \lambda)|
\]

(57)

and

\[
\| (H_k + 1)^{-1} \alpha \cdot F_x(k, \lambda) (\hat{D} - \alpha \cdot k) \|
\]

\[
= \| (H + 1)^{-1} \alpha \cdot F_0(k, \lambda) (\hat{D} + \alpha \cdot k) \|
\]

\[
\leq |F_0(k, \lambda)| \| (H + 1)^{-1} \hat{D} \| + |k| |F_0(k, \lambda)| \| (H + 1)^{-1} \|
\]

\[
\leq C(1 + |k|) |F_0(k, \lambda)|.
\]

(58)

because \( \hat{D}(H + 1)^{-1} \) is bounded and \( e^{-i k \cdot x} H_k = H e^{-i k \cdot x} e^{-i k \cdot x} (\hat{D} - \alpha \cdot k) = \hat{D} e^{-i k \cdot x} \).

Collecting (55), (56), (57) and (58), one has

\[
\| [D - \alpha \cdot k, a(k, \lambda)] (H_k + 1) \|
\]

\[
\leq \frac{1}{\pi} \int_{M^2}^\infty ds \sqrt{s - M^2} s^{-1} C(1 + |k|) |F_0(k, \lambda)| (s^{-1} + s^{-3/2} + s^{-2})
\]

\[
= C(1 + |k|) |F_0(k, \lambda)|.
\]

This is what we want to show.

Finally we will prove (56). Basic strategy is similar to the proof of Lemma D.3. Let \( J_{n/2}(s) = \| (H_k + 1)^{n/2} e^{-i s (\hat{D} - \alpha \cdot k)} \phi \|, \ n = 1, 2 \) for \( \phi \in \text{dom}(H_k) \) with \( \| \phi \| = 1 \). Then, since \( [\hat{D} - \alpha \cdot k, H_k] = [\hat{D}, H_k] = \alpha \cdot (a(\omega F_x) - a(\omega F_x)^*) \), one can easily modify the proof of Lemma D.3 to conclude that

\[
J_1(s) \leq J_1(0) + C(J_{1/2}(0)|s| + s^2), \quad J_{1/2}(s) \leq J_{1/2}(0) + C|s|.
\]
Thus, using the formula
\[(t + \hat{D}^2)^{-1} = \int_{\mathbb{R}} ds \, g_t(s)e^{-is\hat{D}}\]
with \(g_t(s) = \sqrt{\pi/2t} e^{-\sqrt{t}|s|}\) and modifying the proof of Lemma [D.3], one can arrive at (56). \(\Box\)

E  Operator monotonicity of the square root

Lemma E.1 (Operator monotonicity of the square root: Unbounded version)
Let \(S\) and \(T\) be two positive self-adjoint operators (not necessarily bounded) with \(\text{dom}(S^{1/2}) \supseteq \text{dom}(T^{1/2})\). Assume that \(S \leq T\). Then one has \(\text{dom}(S^{1/4}) \supseteq \text{dom}(T^{1/4})\) and \(\sqrt{S} \leq \sqrt{T}\).

Proof. Set \(E_n = E_S([0, n])\) where \(E_S(\cdot)\) is the spectral measure of \(S\). Define \(S_n = E_n^{1/2}SE_n^{1/2}\). Then one has \(S \geq S_n \geq 0\) for all \(n \in \mathbb{N}\). Thus \(S_n \leq T\) holds for all \(n \in \mathbb{N}\). Now one has
\[(\varepsilon T + \mathbb{1})^{-1}S_n(\varepsilon T + \mathbb{1})^{-1} \leq (\varepsilon T + \mathbb{1})^{-1}T(\varepsilon T + \mathbb{1})^{-1}\]
for every \(\varepsilon > 0\). Since both sides of (59) are positive and bounded, one can apply the operator monotonicity of the square root for bounded positive operators [16] and obtain
\[\sqrt{(\varepsilon T + \mathbb{1})^{-1}S_n(\varepsilon T + \mathbb{1})^{-1}} \leq \sqrt{(\varepsilon T + \mathbb{1})^{-1}T(\varepsilon T + \mathbb{1})^{-1}}\]
for all \(\varepsilon > 0\). Taking \(\varepsilon \downarrow 0\) first, we have \(\sqrt{S_n} \leq \sqrt{T}\) for each \(n \in \mathbb{N}\). It follows that, for \(f \in \text{dom}(T^{1/4})\), one has
\[\int_0^n \lambda^{1/2}d\|E_S(\lambda)f\|^2 \leq \langle f, \sqrt{T}f \rangle\]
by the spectral theorem. Now taking \(n \to \infty\), we conclude that \(f \in \text{dom}(S^{1/4})\) and \(\langle f, \sqrt{S}f \rangle \leq \langle f, \sqrt{T}f \rangle\) by the monotone convergence theorem. \(\Box\)

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