Maximizing Entropy Yields Spatial Scaling in Social Networks

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Abstract

In addition to the well known common properties such as small world and community structures, recent empirical investigations suggest a universal scaling law for the spatial structure of social networks. It is found that the probability density distribution of an individual to have a friend at distance \( r \) scales as \( P(r) \propto r^{-1} \). The basic principle that yields this spatial scaling property is not yet understood. Here we propose a fundamental origin for this law based on the concept of entropy. We show that this spatial scaling law can result from maximization of information entropy, which means individuals seek to maximize the diversity of their friendships. Such spatial distribution can benefit individuals significantly in optimally collecting information in a social network.

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Social networks structure is found to be important since it leads to deep insights about how people interact and how social relations evolve [1–14]. It has been found that social networks possess common properties such as small-world [14] and community structure [4]. Recently, geographical properties of social networks have attracted much attention [14–27]. Several empirical studies have analyzed the distribution of distances between friends in real social networks. Liben-Nowell et al. explored the geographic properties in decentralized search within a large, online social network [25]. They used data from the LiveJournal online community with about 500,000 members, in which their state and city of residence, as well as a list of their LiveJournal friends are available. They found that the probability density function (PDF), $P(r)$, of an individual having a friend at a geographic distance $r$ is about $P(r) \propto r^{-1}$ (see supplementary I). Almost at the same time, Adamic and Adar have also found the same phenomenon [16]. They investigated a relatively small social network, the Hewlett-Packard Labs email network. In this work, the PDF of the distance is also found to scale as $P(r) \propto r^{-1}$. More recently, Lambiotte et al. investigated a large mobile phone communication network [17]. The network consists of 2.5 million mobile phone customers that have placed 810 million communications, for whom they have the geographical home location information. Their empirical results show that the mobile phone communication network has the same scaling properties in the spatial structure. They found that the probability of two nodes ($u$ and $v$) to have a long range connection of length $r(u,v)$ is $Pr(u,v) \propto r(u,v)^{-2}$. For 2-dimensional space, the number of nodes which have distance $r$ from a given node is proportional to $r^{d-1}$. This implies that the PDF of an individual to have a friend at distance $r$ is $P(r) \propto r \cdot r^{-2} = r^{-1}$. Very recently, Goldenberg and Levy investigated several large online communities, and also detected the same spatial scaling phenomenon [18]. From the above empirical investigations, one can conclude that the PDF of having a friend at distance $r$ is

$$P(r) \propto r^{-1}. \quad (1)$$

Why does the spatial structure of our social networks possess this kind of scaling property and how does it benefit us? Kleinberg has proved that in a $d$-dimensional space, when the probability of having a long range connection of length $r$ between $u$ and $v$ is $Pr(u,v) \propto r(u,v)^{-d}$, the network is optimally navigated [26–29]. For $d$-dimensional lattice, the number of nodes that have the same distance $r$ to a given node is proportional to $r^{d-1}$. So when the network structure is optimal for navigability, the PDF of the distance from a given node is $P(r) \propto r^{d-1} \cdot r^{-d} = r^{-1}$ for all $d$. This spatial scaling property enables people to send messages efficiently in minimal number of hops to
all nodes of the system. However, social networks are usually not constructed for the purpose of sending messages between unrelated individuals. Thus, there should be a fundamental origin that governs the formation of the spatial scaling law, Eq. (1).

Here we suggest that the origin of this scaling, Eq. (1), comes from a general perspective based on the concept of entropy. We hypothesize that human social behavior is based on gathering maximum information through different activities. Making friends can be regarded as a way of collecting information. To get optimal information could be a general purpose for an individual that shapes the social network architecture. We will show that a social network based on Eq. (1) is an optimal network which can benefit people in collecting maximal information.

I. MODEL

To model a social system we use a toroidal lattice to denote the world in which each node represents an individual. We assume that each individual has a finite energy $w$ which can be represented by the sum of distances between an individual and all his or her friends,

$$\sum_{v=1}^{m} r(u, v) = w,$$

where $m$ is the number of direct links of node $u$. Eq. (2) implies that every node $u$ selects its long range acquaintances $v$, one by one, until the total distance reaches $w$.

The information that node $v$ brings to $u$ can be evaluated by considering the information of node $v$ and all its neighbors. Thus, the information that $u$ collects can be expressed by the sequence of nodes as illustrated in Fig. 1 and the entropy of the whole sequence measures the amount of information. We assume that all nodes are equivalent, so the information obtained by one node can represent the information obtained by each of the other nodes. Thus, our model for constructing a social network is

$$\text{Max} \ \varepsilon = -\sum_{i=1}^{n} q_i \log q_i,$$

subjected to Eq. (2). In Eq. (3), $q_i$ denotes the frequency of node $i$ in the information sequence (see Fig. 1) and $n$ is the size of the network. When $i$ is not a neighbor and not a next nearest neighbor of $u$, $q_i = 0$, and we define $q_i \log q_i = 0$. Here, Eq. (3) implies that the information entropy $\varepsilon$ is determined by the sequence of friends and friends of friends (For considering also friends of next nearest friends, see supplementary IIA).
FIG. 1: The friends of node 1. Node 2, 3 and 4 are the friends of node 1 which Eq. (2) yields that 
\[ d(1, 2) + d(1, 3) + d(1, 4) = w. \] The size of the network is \( n = 12 \) and the information sequence is 
{2, 3, 4, 5, 6, 7, 7, 8, 9, 9, 10} and the frequencies of all nodes are \( q_2 = q_3 = q_4 = q_5 = q_6 = q_8 = q_{10} = \frac{1}{11}, \)
\( q_7 = q_9 = \frac{2}{11}, q_1 = q_{11} = q_{12} = 0. \) If one site is reached several times when constructing the long range connections from node 1 or from its nearest neighbors, the node will appear in the node sequence and in Eq. (2) the same number of times.

II. RESULTS

Our optimization model (OM) is based on Eqs. (2) and (3) which represent two competing processes. To maximize entropy (Eq. (3)), it is preferred to have friends at long distances in order to explore new parts of the network and to obtain more information. However the farther one goes he can have less friends due to the finite energy limited by Eq. (2). Assuming the PDF of having a friend at distance \( r \) obeys

\[ P(r) \propto r^{-\alpha}, \] (4)

we can explore the value of \( \alpha \) that yields maximum entropy under the condition of Eq. (2).

The optimization model is simulated on a toroidal lattice whose size is \( L \times L \) (\( L = 10000 \) means that individuals can make friends in a population of \( 10^8 \)) and lattice (‘Manhattan’) distance is employed. Because toroidal lattice is a regular network and each node has a unique index, we can calculate the lattice distance between any pair of nodes and we do not need to construct the whole network, enabling us to simulate very large lattices.

For a large enough 2-dimensional lattice, the number of nodes that have distance \( r \) from a given node is proportional to \( r \). So if \( w \to +\infty \), that means if we consider the maximal diversity
of friendships without any constraints of energy, we expect $P(r) \propto r$ to be the optimal entropy information since each node has the same probability in the information sequence. In practice, individuals naturally have a limited energy $w$. Our numerical results shown in Fig. 2a indicate that when $\alpha \approx 1$, the information entropy $\varepsilon$ is near its maximum value for a very broad range of $w$. For the range $w \in (5 \times 10^4, 10^6)$ and $f \in (50, 1000)$, we find the optimal $\alpha$ to be $\alpha = 1 \pm 0.05$.

When the size of the lattice is $L$ and $P(r) \propto r^{-1}$, the mean distance between friends is $\frac{L}{\log L}$. Therefore, we can find the average number of friends $f$ to be

$$f = \frac{w \log L}{L}$$

which gives one to one correspondence between $f$ and $w$ at the optimal state. When $L = 10000$ and $w \in (5 \times 10^4, 10^6)$ the average number of friends is $f \in (50, 1000)$ which indeed corresponds to reality [30]. In particular, when considering the average number of friends we contact in one year, $f = 300$ [30], the optimal value of $\alpha$ is $\alpha = -0.99 \pm 0.03$ (as shown in Fig. 2).

Our results suggest that $P(r) \propto r^{-1}$ is the optimal distribution for collecting information between all power law distributions. Is $P(r) \propto r^{-1}$ the optimal distribution when considering all kinds of distributions? We will demonstrate, based on the following evolutionary model (EM), that among all kinds of distributions, $P(r) \propto r^{-1}$ is still the optimal one. In the EM, we also construct a network on a lattice of size $L \times L$. A node $u_i$ is one of the neighbors of node $u$ when there is a direct link from $u$ to $u_i$. Each node $u$ has friends at distances $r(u, u_i)$ subject to $\sum_{u \in U} r(u, u_i) \leq w$, where $U$ is the set of all neighbors of node $u$. In the initial stage of the EM, $P(r)$ is set to be a uniform distribution. Then we employ the extremal optimization method [31], to maximize the entropy through the evolution of network architecture. At each step, a node is chosen randomly. For a chosen node $u$, we make two operations, deleting and adding neighbors according to the marginal improvement of entropy. Suppose $u$ has $k$ neighbors. For the deleting execution, we first calculate the marginal entropies of each neighbor of node $u$, $\{\frac{\Delta E_{u_1}}{r(u,u_1)}, \frac{\Delta E_{u_2}}{r(u,u_2)}, \ldots, \frac{\Delta E_{u_k}}{r(u,u_k)}\}$, where $\Delta E_{u_i}$ means the change in the entropy of node $u$ when we delete node $u_i$ from the neighborhood of node $u$ with other parameters being unchanged. Then we randomly select a comparatively small $\frac{\Delta E_{u_i}}{r(u,u_i)}$ with probability $Pr(u_i)$ proportional to $(\text{rank} \frac{\Delta E_{u_i}}{r(u,u_i)})^{-1-\log(k)}$ [31] and delete $u_i$ from $u$’s neighborhood. For the adding link execution, suppose $v_1, v_2, \ldots, v_h$ are all the candidates which are currently next nearest neighbors of node $u$. We first calculate the marginal entropies of each of the candidate, $\{\frac{\Delta E_{v_1}}{r(u,v_1)}, \frac{\Delta E_{v_2}}{r(u,v_2)}, \ldots, \frac{\Delta E_{v_h}}{r(u,v_h)}\}$, then we also employ the extremal optimization method to choose a node whose marginal entropy is comparatively large among all candidates’ marginal
FIG. 2: The relationship between $\varepsilon$, $w$, $f$, $\alpha$ and $L$ in the optimization model. 

**a.** The contour map shows the relationships between $w$, $\alpha$ and $\varepsilon$, for $L = 10000$. The colors indicate the value of $\varepsilon$. In **b**, the dependence of the information entropy $\varepsilon$ on $\alpha$ for $f = 300, 500, 1000$ is shown. 

**c.** The dependence of the optimal $\alpha$ on the average number of friends $f$. The error bars denote the standard deviations. 

**d.** The relationships between optimal $\alpha$ and the edge length $L$ of the lattice. From it we can see that for large $L$ the optimal $\alpha$ approaches 1. The error bars denote the standard deviations.

entropies as a friend of node $u$. We repeat the adding execution until all the candidates are chosen or the energy limit (Eq. (2)) is satisfied.

In the evolutionary model, we have to record all friends of each node and therefore a system of size $L \times L$ with $L = 10000$ is too large to simulate. So we simulate the evolutionary model on a toroidal lattice of size $100 \times 100$. We assume that the energy scales linearly with distance as suggested by Eq. (2). Thus, when reducing $L$ from 10,000 to 100 (factor of 100) we expect the corresponding energy to be reduced from order of $10^5$ to order of $10^3$. We therefore study the EM model of $L = 100$ with $w \approx 10^3$.

In order to find the optimal distribution of the distances, we first employ the optimization model described by Eqs. (2)-(4) to analyze the above case with the system size $100 \times 100$ and $w \approx 10^3$. We find that the maximum entropy is 7.18 and the corresponding $\alpha$ is $\alpha = 0.95 \pm 0.05$ (see Fig. 3a, b). Next we simulate the evolutionary model of size of $100 \times 100$ and $w \approx 10^3$. After long
FIG. 3: The results of evolutionary model when $L = 100$ and $f = 50$. 

a. The simulation results of OM on a toroidal lattice with the preset power law distribution $P(r) \propto r^{-\alpha}$. 

b. The dependence of the information entropy $\epsilon$ or $\alpha$ for $f$ around 40 in the OM. We can see that when $f = 50$, the optimal exponent is 0.95 and it is very close to $-1$. 

c. The changes of entropy in the EM with the evolution time. The entropy is fixed and the system archives a steady state. The fixed entropy is 7.15 which is very close to the entropy 7.18 in the network of $L = 100$ which we preset the distribution is $P(r) \propto r^{-1}$. The inset denotes the difference of the time-entropy curve which implies that the difference decays exponentially. From it we can see that for a sufficient long time evolution, the entropy converges to a fixed value and the system achieves a steady state. 

d. The cumulative distribution of the distance in EM is shown in log-linear plot in the steady state. We can see that this distribution is very close to $P(r) \propto r^{-1}$ (dashed line).

term evolution from the initial uniform distribution (each node modify the neighborhood more than 40000 times), the system achieves its stationary state (Fig. 3e). The maximum entropy is 7.15 and the corresponding PDF of the distance between the friends scales as $P(r) \propto r^{-1}$ (Fig. 3d and supplementary IIIB), which are very close to the results obtained by OM. So we conclude that $P(r) \propto r^{-1}$ is the optimal PDF of distances of friendships for collecting maximal information. It implies that, the spatial structure of the real social networks is the most optimal structure which leads to the maximum diversity of the friends’ location and can help individuals to collect information efficiently. We note that, it can be proved analytically, under the assumption that the energy
scales linearly with system size, i.e. \( w = cL \), for \( L \to +\infty \), that \( P(r) \propto r^{-1} \) will be the optimal distribution for maximizing entropy among all power law distributions (see supplementary IIC for detailed analysis).

III. CONCLUSION

From the empirical results, we conclude that the probability distribution of having a friend at distance \( r \) scales as \( P(r) \propto r^{-1} \) which is a universal spatial property for social networks. It is shown here that the origin of this spatial scaling law may result from the maximization of entropy which can benefit individuals for optimally collecting information.

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IV. EXPLANATION FOR SPATIAL SCALING OF LIVEJOURNAL

In the empirical study of the LiveJournal data set \cite{1}, for each distance \( r \), \( Q(r) \) is the fraction of friendships among all pairs \( u, v \) of LiveJournal users with \( r(u, v) = r \). \( Q(r) = \frac{F(r)}{S(r)} \propto r^{-1} \). Here, \( F(r) \) denotes the total number of friendships with distance \( r \) and \( S(r) \) is the total number of pairs of nodes that have distance \( r \). The LiveJournal social network has a fractal dimension of about 0.8 (they define the fractal dimension of a network as the exponent \( d \) of the best-fit function \( \text{rank}_u(v) = c \cdot r(u, v)^d \), where \( \text{rank}_u(v) \) is the number of people who live closer to \( u \) than \( v \) and \( c \) is a constant). We know that for any \( d \)-dimensional lattice, the number of nodes that have the same distance \( r \) to a given node is proportional to \( r^{d-1} \). In fractal networks, \( d \) should be the fractal dimension. Thus the probability density function \( P(r) \) of the geographic distance \( r \) between friends is about \( P(r) \propto r^{d-1} \cdot Q(r) = r^{0.8-1} \cdot r^{-1} = r^{-1.2} \), which is close to \( r^{-1} \).

V. ABOUT THE OPTIMIZATION MODEL (OM)

A. Why We Only Consider Friends and Next Nearest Friends?

We assume that the information obtained from the social network is actually related with the influence of friendships. Indeed, in our social life, our friends always talk something about their friends. Thus, we assume that friends and next nearest friends are most important and is enough to consider them in our model. However, Christakis and Fowler have found recently that the influence is mainly within three degrees of separation and call this finding the “Three Degrees of Influence Rule” \cite{2}. It is computationally difficult to take into account more than two degrees of separation of friends to study a system of \( 10^4 \times 10^4 \). We have therefor performed the numerical experiments of the OM in \( 3000 \times 3000 \) size lattice with \( w \approx 10^4 \) (\( f = 300 \)) and found that the simulated results were similar when we took into account friends and next nearest friends, and three degrees of separation (as shown in Fig. 4).
FIG. 4: The relationship between entropy and the power law exponent in different degrees of influence. The lattice size is $3000 \times 3000$, $f = 300$. We can see that the phenomena are similar in which -1 is close to the optimal exponents.

**B. Algorithm of OM**

When the lattice size is $10000 \times 10000$, it is hard to record all nodes’ links information. Thus, we first represent each node an index running from 1 to $10^8$. This way is easy to obtain a function $r(u, v)$ to calculate the lattice distance between any pair of nodes $u$ and $v$, where $u, v$ are now the running index.

In the OM model all nodes are equivalent. Without losing generality, we can set any node as $u = 1$. To construct the spatial network on the lattice, each time we first randomly generate a distance $r$ according to the distribution $P(r) \propto r^{-\alpha}$, $r \in \{1, 2, \ldots, L\}$. Then from the set of nodes which have distance $r$ from node 1, a node is chosen randomly as a friend of node 1 and a directed link is constructed. Repeating the execution until the energy achieves the limit constraint. After the executions we can get all the friends of node 1. Employing the same approach, we can also get all the next nearest friends of node 1.

**C. Analysis on OM**

In this section we will prove that if energy hold

$$w = cL,$$  \hspace{1cm} (6)
where \( c \) is constants, for \( L \to +\infty \), \( P(r) \propto r^{-1} \) is the optimal distribution for all \( P(r) \propto r^{-\alpha} \) distributions.

1. **Symbol and Expression Descriptions**

   \( P(r) \propto r^{-\alpha} \), the distribution of distance between friendships.

   \( R_\alpha \), the expectation of the distance which holds \( P(r) \propto r^{-\alpha} \).

   \( f_\alpha = \frac{w}{R_\alpha} \) is the expectation of number of friends.

   When \( w = \frac{\ell L}{\log(L)} \), \( L \) is the edge length of the lattice, \( f \) denotes the number of friends when \( \alpha = 1 \).

   \( q_{i,j}^\alpha \) denotes the probability of the connection between node \( i \) and \( j \) for a given \( \alpha \).

   \( F^\alpha = \{ \theta_1^\alpha, \theta_2^\alpha, \ldots, \theta_f^\alpha \} \), denotes the set of friends of node 1, where \( f_\alpha = \frac{w}{R_\alpha} \).

   \( q_{F^\alpha,i}^\alpha = \frac{1}{f_\alpha} \sum_{v=1}^{f_\alpha} q_{\theta_v,i}^\alpha \), denotes the probability that node \( i \) is one of friends of \( F^\alpha \).

   \( \sum_{i=1}^{f_\alpha} \frac{f_\alpha}{f_\alpha} \log \frac{f_\alpha}{f_\alpha} C_{j_2}^{\alpha} q_{F^\alpha,i}^\alpha (1 - q_{F^\alpha,i}^\alpha) f_\alpha^{-\alpha} \) denotes the expectation of entropy of node \( i \) when the chosen probability of node \( i \) is \( q_{F^\alpha,i}^\alpha \) and the time of choosing is \( f_\alpha^2 \).

   \( \varepsilon_\alpha = \sum_{i=1}^{f_\alpha} \sum_{v=1}^{f_\alpha} \frac{f_\alpha}{f_\alpha} \log \frac{f_\alpha}{f_\alpha} C_{j_2}^{\alpha} q_{F^\alpha,i}^\alpha (1 - q_{F^\alpha,i}^\alpha) f_\alpha^{-\alpha} \), denotes the expectation of entropy for a given \( F^\alpha \).

   \( E(\varepsilon_\alpha) \), denotes the expectation \( \varepsilon_\alpha \)

2. **Case 1: \( \alpha < 1 \)**

   \[
   R_\alpha = \frac{\int_1^L x^{1-\alpha} dx + O(1)}{\int_1^L x^{-\alpha} dx + O(1)} = \frac{\frac{1}{1-\alpha}(L^{2-\alpha} - 1) + O(1)}{\frac{1}{1-\alpha}(L^{1-\alpha} - 1) + O(1)} \approx \frac{1}{2-\alpha} L. \tag{7}
   \]

   Therefore, for a given \( w = cL \), where \( c \) is a constant, we have

   \[
   \lim_{L \to \infty} f_\alpha = \lim_{L \to \infty} \frac{w}{R_\alpha} = \frac{c(2-\alpha)}{1-\alpha}. \tag{8}
   \]

   Because,

   \[
   \lim_{L \to \infty} \max_{i,j} q_{i,j}^\alpha \leq \lim_{L \to \infty} \frac{1}{\int_1^L x^{-\alpha} dx + O(1)} = \lim_{L \to \infty} \frac{1}{\frac{1}{1-\alpha}(L^{1-\alpha} - 1) + O(1)} = 0 \tag{9}
   \]

   and

   \[
   q_{F^\alpha,i}^\alpha \leq \max_{i,j} q_{i,j}^\alpha. \tag{10}
   \]
Thus, for any $F^\alpha$,

$$\lim_{L \to \infty} q_{F^\alpha, i} = 0.$$  \hspace{1cm} (11)

It implies that

$$\lim_{L \to \infty} \varepsilon_\alpha = \log(\frac{c(2-a)}{1-a} + \frac{c(2-a)}{1-a}^2).$$  \hspace{1cm} (12)

Thus

$$\lim_{L \to \infty} E(\varepsilon_\alpha) = \log(\frac{c(2-a)}{1-a} + \frac{c(2-a)}{1-a}^2),$$  \hspace{1cm} (13)

which is a monotonic increasing function with $\alpha < 1$.

3. Case 2: $\alpha > 1$

Lemma: if $q \in (0, \frac{1}{3})$, for any large enough $z$ we have

$$- q \log q > - \sum_{x=1}^{z} \log \left( \frac{x}{z} C_z q \frac{x}{z} (1-q)^{z-x} \right),$$  \hspace{1cm} (14)

where $- \sum_{x=1}^{z} \frac{x}{z} \log \left( \frac{x}{z} C_z q \frac{x}{z} (1-q)^{z-x} \right)$ denotes the expectation of entropy of a node with the probability $q$ to be chose and the total choosing time is $z$ (as shown in Fig. 5).

Proof:

According to Law of Large Numbers, $\lim_{z \to \infty} - \sum_{x=1}^{z} \frac{x}{z} \log \left( \frac{x}{z} C_z q \frac{x}{z} (1-q)^{z-x} \right) = -q \log q$.

Thus, we just need to prove

$$g(z) = - \sum_{x=1}^{z} \log \left( \frac{x}{z} C_z q \frac{x}{z} (1-q)^{z-x} \right)$$  \hspace{1cm} (15)

is a monotonic increasing function.

For large enough $z$, normal distribution is a well approximation to binomial distribution then we have

$$g(z) = \int_{1}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{x}{z} \log \frac{x}{z} dx,$$  \hspace{1cm} (16)

where $\sigma^2 = zq(1-q), \mu = zq$.

$$g'(z) = \frac{\sqrt{2}}{4z^3q(1-q) \sqrt{\pi zq(1-q)}} \int_{1}^{\infty} \left[ \log \frac{x}{z} (q^2 z^2 - 3q^2 z + 3qz - x^2) - 2zq^2 + 2zq \right] e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$  \hspace{1cm} (17)

Obviously,

$$\frac{\sqrt{2}}{4z^3q(1-q) \sqrt{\pi zq(1-q)}} > 0$$  \hspace{1cm} (18)
FIG. 5: Plot of $y = -q \log q - g(z)$. From the plot we can see that Lemma is true. More over when $z$ is small $g(z) > -q \log q$ is also correct.

and

$$e^{\frac{(x-z)^2}{2zq}} x > 0$$

(19)

More over

$$\int_{1}^{z} [\log \frac{x}{z} (q^2 z^2 - 3q^2 z + 3q z - x^2) - 2zq^2 + 2zq] dx = \left( \frac{1}{9} - q^2 \right) z^3 + \Theta(z^2 \log z) > 0$$

(20)

when $q < \frac{1}{3}$, where, $\Theta(z^2 \log z)$ denotes the same order of $z^2 \log z$.

Thus, $g'_z(z) > 0$ which implies that $g(z)$ is a monotonic increasing function and

$$- q \log q > - \sum_{x=1}^{z} \frac{x}{z} \log \frac{x}{z} C_x q^{x} (1 - q)^{z-x}.$$  

(21)

For case 2, according to Lemma and Levy stable distribution property (the distance between the next nearest neighbor and the origin is also obey $P(r) \propto r^{-\alpha}$ when $\alpha > 1$). So for large enough friends number we have:

$$E(\varepsilon_\alpha) < \sum_{r=1}^{L} \frac{4r^{-\alpha}}{4rZ(\alpha)} \log \frac{r^{-\alpha}}{4rZ(\alpha)}$$

(22)

$$= \frac{1}{Z(\alpha)} \sum_{r=1}^{L} r^{-\alpha} [(\alpha - 1) \log r - \log[4Z(\alpha)]].$$  

(23)

More over we can get:

$$\lim_{L \to \infty} E(\varepsilon_\alpha) = \frac{(a - 1)(2 \log 2 + \log Z(a)) + a + 1}{2(a - 1)^2}$$

(24)
where $Z(\alpha)$ denotes $\sum_{r=1}^{L} r^{-\alpha}$. Obviously, $\frac{(a-1)2 \log 2 + \log Z(\alpha) + a + 1}{2(a-1)}$ is a monotonic increasing function. Thus, for any fixed $c$, -1 is the optimal exponent.

VI. ABOUT THE EVOLUTIONARY MODEL (EM)

A. Why we chose new friend only from the next nearest neighbors?

There are 2 reasons. The first is that, according to our real social experience, we always make some new friends who are the friends of our friends. The second is that EM is a global optimal algorithm. Thus if we choose any node as our new friend, the result will be the same theoretically.

B. How to Measure the Power Law Exponent in EM?

To accurately measure the exponent value of power law distribution is not a easy work. Especially, when the exponent is very close to $-1$. We use the least square method to evaluate the exponent value. We are afraid the least square method is not a good way, so we plot the accumulated curve. Fortunately, it can be proved that when $P(r) \propto r^{-1}$, the accumulated function in log-linear plot will be a straight line. We can see that the distribution is about $P(r) \propto r^{-1}$.

[1] Liben-Nowell, D., Novak, J., Kumar, R., Raghavan, P. and Tomkins, A. Geograph routing in social networks. Proc. Natl. Acad. 102, 11623-11628 (2005).

[2] http://www.calit2.net/newsroom/article.php?id=1558 and the up coming book: Christakis and Fowler: Connected: The Surprising Power of Social Networks and How They Shape Our Lives, Little Brown, (2009).