THE DAVEY STEWARTSON SYSTEM IN WEAK $L^p$ SPACES

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Abstract. We study the global Cauchy problem associated to the Davey-Stewartson system in $\mathbb{R}^n$, $n=2,3$. Existence and uniqueness of solution are established for small data in some weak $L^p$ space. We apply an interpolation theorem and the generalization of the Strichartz estimates for the Schrödinger equation derivated in [CVeV]. As a consequence we obtain self-similar solutions.

1. Introduction

This paper is concerned with the initial value problem (IVP) associated to the Davey-Stewartson system

$$
\begin{aligned}
&i\partial_t u + \delta \partial^2_{x_1} u + \sum_{j=2}^n \partial^2_{x_j} u = \chi |u|^\alpha u + bu \partial_{x_1} \varphi, \\
&\partial_{x_1} \varphi + m \partial^2_{x_2} \varphi + \sum_{j=3}^n \partial^2_{x_j} \varphi = \partial_{x_1} (|u|^\alpha), \\
&u(x,0) = u_0(x)
\end{aligned}
$$

where $u = u(x,t)$ is a complex-valued function and $\varphi = \varphi(x,t)$ is a real-valued function.

The exponent $\alpha$ is such that $\frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2}$, the parameters $\chi$ and $b$ are constants in $\mathbb{R}$, $\delta$ and $m$ are real positive and we can consider $\delta, \chi$ normalized in such a way that $|\delta| = |\chi| = 1$.

The Davey-Stewartson systems are 2D generalization of the cubic 1D Schrödinger equation,

$$
i\partial_t u + \Delta u = |u|^2 u
$$

and model the evolution of weakly nonlinear water waves that travel predominantly in one direction but which the amplitude is modulated slowly in two horizontal directions.

System (P), $n = 2$, $\alpha = 2$, was first derived for Davey and Stewartson [DS] in the context of water waves, but its analysis did not take account of the effect of surface tension (or capillarity). This effect was later included by Djordjevic and Redekopp [DR] who have shown that the parameter $m$ can become negative when capillary effects are important. Independently, Ablowitz and Haberman [AH] obtained a particular form of (P), $n = 2$, as an example of completely integrable model also generalizing the two-dimensional nonlinear Schrödinger equation.

When $(\delta, \chi, b, m) = (1, -1, 2, -1), (-1 - 2, 1, 1), (-1, 2, -1, 1)$ the system (P), $n = 2$, is referred as $DSI$, $DSII$ defocusing and $DSII$ focusing respectively in the inverse scattering literature. In these cases several results concerning the existence of solutions or lump solutions.

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have been established ([AF], [AnFr], [AS], [C], [FS], [FSu], [Su]) by the inverse scattering techniques.

In [GS], Ghidaglia and Saut studied the existence of solutions of IVP (1), $n = 2$, $\alpha = 2$. They classified the system as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic, according to respective sign of $(\delta, m) : (+, +), (+, -), (-, +), (-, -)$.

For the elliptic-elliptic and hyperbolic-elliptic cases, Ghidaglia and Saut [GS] reduced the system (1), $n = 2$, to the nonlinear cubic Schrödinger equation with a nonlocal nonlinear term, i.e.

$$i\partial_t u + \delta \partial^2_{x_1} u + \partial^2_{x_2} u = \chi |u|^2 u + H(u),$$

where $H(u) = (\Delta^{-1} \partial^2_{x_1} |u|^2)u$. They showed local well-posedness for data in $L^2, H^1$ and $H^2$ using Strichartz estimates and the continuity properties of the operator $\Delta^{-1}$.

The remaining cases, elliptic-hyperbolic and hyperbolic-hyperbolic, were treated by Linares and Ponce [LP1], Hayashi [H1], [H2], Chihara [Ch], Hayashi and Hirata [HH1], [HH2], Hayashi and Saut [HS] (see [LP2] for further references).

Here we will concentrate in the elliptic-elliptic and hyperbolic-elliptic cases. We start with the motivation for this work:

From the condition $m > 0$ we are allowed to reduce the Davey-Stewartson system (1) to the Schrödinger equation

$$\begin{cases}
i\partial_t u + \delta \partial^2_{x_1} u + \sum_{j=2}^n \partial^2_{x_j} u = \chi |u|^\alpha u + buE(|u|^\alpha), & \forall x \in \mathbb{R}^n, n = 2, 3, t \in \mathbb{R}, \\
u(x, 0) = u_0(x),
\end{cases}$$

where

$$\hat{E}(\hat{f})(\xi) = \frac{\xi_1^2}{\xi_1^2 + m\xi_2^2 + \sum_{j=3}^n \xi_j^2} \hat{f}(\xi) = p(\xi) \hat{f}(\xi).$$

Now observe that if $u(x, t)$ satisfies

$$i\partial_t u + \delta \partial^2_{x_1} u + \sum_{j=2}^n \partial^2_{x_j} u = \chi |u|^\alpha u + buE(|u|^\alpha),$$

then also does $u_\beta(x, t) = \beta^{2/\alpha} u(\beta x, \beta^2 t)$, for all $\beta > 0$.

Therefore it is natural to ask whether solutions $u(x, t)$ of (1) exist and satisfy, for $\beta > 0$:

$$u(x, t) = \beta^{2/\alpha} u(\beta x, \beta^2 t).$$

Such solutions are called self-similar solutions of the equation (2).

Therefore supposing local well posedness and $u$ a self-similar solution we must have

$$u(x, 0) = u_\beta(x, 0), \forall \beta > 0,$$

i.e.,

$$u_0(x) = \beta^{2/\alpha} u_0(\beta x).$$

In other words, $u_0(x)$ is homogeneous with degree $-2/\alpha$ and every initial data that gives a self-similar solution must verify this property. Unfortunately, those functions do not belong to the usual spaces where strong solutions exists, such as the Sobolev spaces $H^s(\mathbb{R}^n)$. We shall therefore replace them by other functional spaces that allow homogeneous functions.

There are many motivations to find self-similar solutions. One of them is that they can give a good description of the large time behaviour for solutions of dispersive equations.
The idea of constructing self-similar solutions by solving the initial value problem for homogeneous data was first used by Giga and Miyakawa [GM] for the Navier Stokes equation in vorticity form. The idea of [GM] was used latter by Cannone and Planchon [CP], Planchon [P] (for the Navier-Stokes equation); Kwak [K], Snoussi, Tayachi and Weissler [STW] (for nonlinear parabolic problems); Kavian and Weissler [KW], Pecher [Pe], Ribaud and Youssfi [RY2] (for the nonlinear wave equation); Cazenave and Weissler [CW1], [CW2], Ribaud and Youssfi [RY1], Furioli [F] (for the nonlinear Schrödinger equation).

In [CVVeVi] Cazenave, Vega and Vilela studied the global Cauchy problem for the Schrödinger equation

\[ i\partial_t u + \Delta u = \gamma|u|^{\alpha}u, \quad \alpha > 0, \gamma \in \mathbb{R}, \ (x,t) \in \mathbb{R}^n \times [0, \infty). \] (4)

Using a generalization of the Strichartz estimates for the Schrödinger equation they showed that, under some restrictions on \( \alpha \), if the initial value is sufficiently small in some weak \( L^p \) space then there exists a global solution. This result provided a common framework to the classical \( H^s \) solutions and to the self-similar solutions. We follow their ideas in our work.

From the condition \( m > 0 \) we are allowed to reduce the Davey-Stewartson system (1) to the Schrödinger equation (2). Now comparing Schrödinger equations (2) and (4) we observe that we have the nonlocal term \( uE(|u|^2) \) to treat. The main ingredient to do that will be an interpolation theorem and the generalization of the Strichartz estimates for the Schrödinger equation derivated in [CVVeVi]. As a consequence, we prove existence and uniqueness (in the sense of distributions) to the IVP problem (2). As a consequence we find self-similar solutions for the problem (2) in the case \( \delta > 0 \).

To study the IVP (2) we use its integral equivalent formulation

\[ u(t) = U(t)u_0 + i \int_0^t U(t-s) \left( \chi|u|^\alpha u + buE(|u|^\alpha) \right)(s)ds, \] (5)

where \( U(t)u_0 \) defined as

\[ \hat{U}(t)\hat{u}_0(\xi) = e^{-it\psi(\xi)}\hat{u}_0(\xi), \] (6)

\[ \psi(\xi) = 4\pi^2 \delta \xi_1^2 + 4\pi^2 \sum_{j=2}^{n} \xi_j^2, \]

is the solution of the linear problem associated to (2).

We also define the subspace \( Y \subset S'(\mathbb{R}^n) \) where:

\[ Y = \{ \varphi \in S'(\mathbb{R}^n) : U(t)\varphi \in L_{-\frac{\alpha(n+2)}{2}}^{\alpha(n+2)}(\mathbb{R}^{n+1}) \}, \]

\[ \| \varphi \|_Y = \| U(t)\varphi \|_{L_{-\frac{\alpha(n+2)}{2}}^{\alpha(n+2)}(\mathbb{R}^{n+1})}, \]

and

\[ L_{-\frac{\alpha(n+2)}{2}}^{\alpha(n+2)}(\mathbb{R}^{n+1}) \]

are weak \( L^p \) spaces that we define latter.

Our main result in this paper reads as follows:

**Theorem 1.** There exists \( \delta_1 > 0 \) such that given \( \frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2} \) and \( u_0 \in Y \) with \( \|u_0\|_Y < \delta_1 \) then there exists a unique solution \( u \in L_{-\frac{\alpha(n+2)}{2}}^{\alpha(n+2)}(\mathbb{R}^{n+1}) \) of (5) such that \( \|u\|_{L_{-\frac{\alpha(n+2)}{2}}^{\alpha(n+2)}(\mathbb{R}^{n+1})} < 3\delta_1 \).
To obtain this result we will use the contraction mapping theorem and some estimates for the nonlocal operator \( E \), defined in (3).

As a consequence of Theorem 1 we show that giving any initial data in \( Y \) and assuming the existence of a solution \( u \) to the integral equation (5) we have that \( u \) is the solution (in the weak sense) of the differential equation (2). We emphasize that Theorem 1 provides the existence of solutions to the equation (5) under the assumption of small initial data.

**Proposition 2.** Given \( \frac{4(n+1)}{n(n+2)} < \alpha < \frac{4(n+1)}{n^2} \), \( u_0 \in Y \) and let \( u \in L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1}) \) be the solution of (5). It follows that \( t \in \mathbb{R} \rightarrow u(t) \in S'(\mathbb{R}^n) \) is continuous and \( u(0) = u_0 \). In particular, \( u \) is a solution of (2). Moreover \( u(t_0) \in Y \) for all \( t_0 \in \mathbb{R} \). In addition, there exist \( u_\pm \) such that \( \|U(t)u_\pm\|_L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1}) < \infty \) and \( U(-t)u(t) \rightarrow u_\pm \) in \( S'(\mathbb{R}^n) \) as \( t \rightarrow \pm \infty \).

This paper is organized as follows: in section 2 we show the main theorem. In preparation for that we will establish some needed estimates for the integral operator.

Last section will be devoted to find self-similar solutions.

2. Global existence in weak \( L^p \) spaces

Let us define the weak \( L^p \) spaces we will use in the following:

**Definition 3.** \( L^{p,\infty}(\mathbb{R}^n) = \{ f : \mathbb{R}^n \rightarrow \mathbb{C} \text{ measurable} ; \|f\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda \alpha(\lambda, f)^{1/p} < \infty \} \)

where

\[
\alpha(\lambda, f) = \mu(\{ x \in \mathbb{R}^n ; |f(x)| > \lambda \}),
\]

and

\[
\mu = \text{Lebesgue measure}.
\]

The reader should refer to [BeL] for details.

**Remark 4.** Using a change of variables it is easy to see that for any \( \varphi \in S'(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \) and \( \tau \in \mathbb{R} \):

\[
\|U(t)\varphi\|_{L^{p,\infty}(\mathbb{R}^{n+1})} = \|U(t + \tau)\varphi\|_{L^{p,\infty}(\mathbb{R}^{n+1})},
\]

where \( U(t) \) is the unitary group defined in (3).

The next theorem establishes a relationship between Lorentz Spaces \( L^{p,\infty} \) and \( L^\theta \) spaces:

**Theorem 5** (Interpolation’s theorem). Given \( 0 < p_0 < p_1 \leq \infty \), then for all \( p \) and \( \theta \) such that \( \frac{1}{p} = \frac{1-p_0}{p_0} + \frac{p}{p_1} \) and \( 0 < \theta < 1 \) we have :

\[
(L^{p_0}, L^{p_1})_{\theta, \infty} = L^{p,\infty} \quad \text{with} \quad \|f\|_{(L^{p_0}, L^{p_1})_{\theta, \infty}} = \|f\|_{L^{p,\infty}},
\]

where

\[
(L^{p_0}, L^{p_1})_{\theta, \infty} = \{ a \text{ Lebesgue measurable}; \|a\|_{(L^{p_0}, L^{p_1})_{\theta, \infty}} := \sup_{t > 0} t^{-\theta} k(t, a) < \infty \}
\]

and

\[
k(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|_{L^{p_0}} + t \|a_1\|_{L^{p_1}}).
\]

**Proof.** We refer to [BeL] for a proof of this theorem. \( \square \)
Remark 6. Another relationship between Lorentz Spaces and $L^p$ spaces is given by the following decomposition:

Let $1 \leq p_1 < p < p_2 < \infty$. Then

$$L^{p\infty} = L^{p_1} + L^{p_2}.$$  

The next result is a generalization of the classical Strichartz estimates for the Schrödinger equation. This was proved by Vilela in [McV].

Theorem 7. Consider $r, \tilde{r}, q$ and $\tilde{q}$ such that

$$2 < r, \tilde{r} \leq \infty, \quad \frac{1}{r} - \frac{1}{\tilde{r}} < \frac{2}{n},$$

$$\frac{1}{q} - \frac{2}{n} \frac{1}{r} + \frac{n}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right) = 1, \quad \left\{ \begin{array}{ll}
\frac{n-2}{n} \frac{r}{r} & \neq \infty \\
\frac{1}{r} & \leq \frac{n}{n-2} \left( 1 - \frac{1}{\tilde{r}} \right) \end{array} \right. \quad \text{if } n = 2,$$

and

$$0 < \frac{1}{q} < 1 - \frac{n}{2} \left( \frac{1}{r} + \frac{1}{r} - 1 \right) \quad \text{if } 1 \frac{1}{r} + \frac{1}{r} \geq 1,$$

$$- \frac{n}{2} \left( \frac{1}{r} + \frac{1}{r} - 1 \right) < \frac{1}{q} < 1 \quad \text{if } \frac{1}{r} + \frac{1}{r} < 1.$$

Then we have the following inequalities:

$$\| \int_0^t e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \|_{L^q_t L^r_x} \leq c \| F \|_{L^q_\tilde{r}_t L^\tilde{r}_x}, (9)$$

$$\| \int_{-\infty}^t e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \|_{L^q_t L^r_x} \leq c \| F \|_{L^q_\tilde{r}_t L^\tilde{r}_x},$$

$$\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \|_{L^q_t L^r_x} \leq c \| F \|_{L^q_\tilde{r}_t L^\tilde{r}_x}.$$  

Proof. We refer to [McV] for a proof of this theorem. \hfill \Box

Remark 8. Theorem 7 also holds for $U(t)$.

To prove Theorem 1 we need some results:

Proposition 9. Consider $F : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{C}$. Then for $1 < p < \infty$:

$$\| E(F) \|_{L^{p\infty}(\mathbb{R}^{n+1})} \leq \| F \|_{L^{p\infty}(\mathbb{R}^{n+1})}.$$  

Instead of proving Proposition 9 we establish a more general result:

Lemma 10. Let $A$ be a linear injective operator and suppose that for each $1 \leq p < \infty$ there exists $1 \leq q = q(p) < \infty$ such that $A : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded. Then $A$ is bounded from $L^{p\infty}(\mathbb{R}^n)$ to $L^{q\infty}(\mathbb{R}^n)$. 
Proof. In fact, fix $1 \leq p < \infty$. Take $1 \leq p_0$, $p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$.

By Theorem 5 we have $\|A(f)\|_{L^p(\mathbb{R}^n)} = \|A(f)\|_{(L^{p_0}, L^{p_1})\theta}$.

If

$$f = f_0 + f_1 \in L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n),$$

then

$$A(f) = A(f_0) + A(f_1) \in L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n),$$

and

$$\|A(f_j)\|_{L^{p_j}(\mathbb{R}^n)} \leq \|f_j\|_{L^{p_j}(\mathbb{R}^n)}, \ j = 0, 1.$$

So

$$K(t, A(f)) = \inf_{A(f) = A(f_0) + A(f_1)} (\|F_0\|_{L^{p_0}(\mathbb{R}^n)} + t\|F_1\|_{L^{p_1}(\mathbb{R}^n)})$$

$$\leq \inf_{A(f) = A(f_0) + A(f_1)} (\|A(f_0)\|_{L^{p_0}(\mathbb{R}^n)} + t\|A(f_1)\|_{L^{p_1}(\mathbb{R}^n)})$$

$$\leq \inf_{A(f) = A(f_0) + A(f_1)} (\|f_0\|_{L^{p_0}(\mathbb{R}^n)} + t\|f_1\|_{L^{p_1}(\mathbb{R}^n)}).$$

Since $A$ is injective, $A(f) = A(f_0) + A(f_1)$ implies $f = f_0 + f_1$ Lebesgue almost everywhere. Then

$$K(t, A(f)) \leq \inf_{f_0 + f_1} (\|f_0\|_{L^{p_0}(\mathbb{R}^n)} + t\|f_1\|_{L^{p_1}(\mathbb{R}^n)}) = K(t, f).$$

Using Theorem 5 once more we obtain the result. \qed

Observe that since the linear operator $E$ defined in (3) is injective and satisfies

$$\|E(F)\|_{L^q(\mathbb{R}^{n+1})} \leq \|F\|_{L^q(\mathbb{R}^{n+1})}$$

for all $1 < p < \infty$ (see [X]), the Proposition 9 will be a consequence of Lemma 10.

Now we define two integral operators:

$$G(F)(x, t) = \int_0^t U(t-s) F(\cdot, s)(x) ds, \quad (10)$$

and

$$(TT^*F)(x, t) = \int_{-\infty}^{+\infty} U(t-\tau) F(x, \tau) d\tau, \quad (11)$$

where $U(t)$ is the group defined in (6). We prove the following properties about them:

**Proposition 11.** Let $1 \leq p$, $r < \infty$ such that

$$\frac{1}{p} - \frac{1}{r} = \frac{2}{n + 2},$$

and

$$\frac{2(n + 1)}{n} < r < \frac{2(n + 1)(n + 2)}{n^2}.$$ 

Then

$$\|G(F)\|_{L^r(\mathbb{R}^{n+1})} \leq c\|F\|_{L^p(\mathbb{R}^{n+1})}, \quad (12)$$

and

$$\|TT^*(F)\|_{L^r(\mathbb{R}^{n+1})} \leq c\|F\|_{L^p(\mathbb{R}^{n+1})}, \quad (13)$$
Proof. To prove Property (12) we need Theorem 7 (with $U(t)$ instead of $e^{it\Delta}$) and the interpolation theorem. In fact taking $r = q$ and $r' = q' = p$ in Theorem 10 the hypothesis (1) becomes
\[
\frac{1}{p} - \frac{1}{r} = \frac{2}{n + 2},
\]
and the inequality (10) becomes
\[
\|G(F)\|_{L^r(\mathbb{R}^{n+1})} \leq c\|F\|_{L^p(\mathbb{R}^{n+1})},
\]
(14)
The restriction $\frac{2(n+1)}{n} < r < \frac{2(n+1)(n+2)}{n^2}$ comes from hypothesis (8).
The result follows applying Lemma 10 to inequality (14). Property (13) is proved exactly the same way.

Now we are ready to prove our main result:

Proof of Theorem 7: Consider the following operator
\[
(\Phi u)(t) = U(t)u_0 - iG(\chi|u|^\alpha u + buE(|u|^\alpha))(t),
\]
(15)
$G$ as in (10). We want to use the Picard fixed point theorem to find a solution of $u = \Phi(u)$ in $B(0,3\delta_1) = \{f \in L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1}); \|f\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} \leq 3\delta_1\}$.

To prove $\Phi(B(0,3\delta_1) \subset B(0,3\delta_1)$ take $u \in B(0,3\delta_1)$.
Using the hypothesis $\|U(t)u_0\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} < \delta_1$ and Proposition 11 combined with the definition $\Phi(\cdot)$ in (13), we obtain
\[
\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} \leq 2\left(\delta_1 + \|u|^{\alpha}u\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} + \|buE(|u|^\alpha)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})}\right).
\]
Applying Proposition 9 and Holder’s inequality we get
\[
\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} \leq 2\left(\delta_1 + \|u|^{\alpha+1}\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} + |b|\|u\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})}\|u(t)\|^\alpha_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})}\right).
\]
Using that $u \in B(0,3\delta_1)$ and choosing $0 < \delta_1 \ll 1$ we have
\[
\|\Phi(u)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} \leq 2c\delta_1 + 4c(3\delta_1)^{\alpha+1} + 4c|b|(3\delta_1)^{\alpha+1} < 3\delta_1.
\]
Now we prove the contraction in $B(0,3\delta_1)$. Take $u,v \in B(0,3\delta_1)$:
\[
\Phi(u) - \Phi(v) = iG(\chi(|v|^\alpha v - |u|^\alpha u)) + iG(b(vE(|v|^\alpha) - uE(|u|^\alpha))).
\]
By Proposition 11 we get
\[
\|\Phi(u) - \Phi(v)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} \leq 2c\left(\|v|^{\alpha} - |u|^{\alpha}\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} + \|u|^{\alpha}(u - v)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})}\right) + 2c|b|\left(\|E(|v|^\alpha)(v - u)\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})} + \|u(E(|v|^\alpha) - E(|u|^\alpha))\|_{L^{\frac{\alpha(n+2)}{2}}(\mathbb{R}^{n+1})}\right).
\]
Applying Holder’s inequality and Proposition 9 we obtain
\[ \|\Phi(u) - \Phi(v)\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} \]
\[ \leq 2c\left(\|v\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} ||u|\alpha - |u|\alpha\|_{L^{\frac{n+2}{2}}}(\mathbb{R}^{n+1}) + \|u\|^{\alpha}_{L^{\frac{n+2}{2}}}(\mathbb{R}^{n+1})\right)\]
\[ + 2c|\beta|\left(\|v\|^{\alpha-1}_{L^{\frac{n+2}{2}}}(\mathbb{R}^{n+1}) ||u - v\|_{L^{\frac{n+2}{2}}}(\mathbb{R}^{n+1}) + \|u\|^{\alpha-1}_{L^{\frac{n+2}{2}}}(\mathbb{R}^{n+1})\right). \]

Now we set
\[ g(u) = |u|\alpha. \]
It follows by the Mean Value Theorem that
\[ |g(u) - g(v)| \leq c(\alpha)(|u|^{\alpha-1} + |v|^{\alpha-1})|u - v|. \]
This property and Holder’s inequality imply that
\[ \|v|\alpha - |u|\alpha\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} \]
\[ \leq c(\alpha) \left(\|u\|^{\alpha-1}_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} ||u - v\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} + \|v\|^{\alpha-1}_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} ||u - v\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})}\right). \]

Finally by the last inequality and the hypothesis \( u, v \in B(0, 3\delta_1) \) we get
\[ \|\Phi(u) - \Phi(v)\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})} \leq \delta_1^{\alpha}(c_1 + c_2|\beta|)||u - v\|_{L^{\frac{n+2}{2}}(\mathbb{R}^{n+1})}. \]
Taking 0 < \( \delta_1 \ll 1 \) we get the contraction. \( \square \)

**Proof of Proposition 2:** By hypothesis \( u \in L^{\frac{n+2}{2}}(\mathbb{R}^{n+1}) \). So by Holder’s inequality and Proposition 9
\[ |u|\alpha u \quad \text{and} \quad uE(|u|\alpha) \in L^{\frac{n+2}{2}}(\mathbb{R}^{n+1}). \]
Now, by Remark 6 we can write
\[ |u|\alpha u = f_1 + f_2 \quad \text{and} \quad uE(|u|\alpha) = f_3 + f_4, \]
where \( f_j \in L^{p_j}(\mathbb{R}^{n+1}) \) for some \( 1 \leq p_1 < \frac{\alpha(n+2)}{2(\alpha+1)} < p_2 < \infty \) and \( 1 \leq p_3 < \frac{\alpha(n+2)}{2(\alpha+1)} < p_4 < \infty \).
Replacing (16) in (5) we get
\[ u(t) = U(t)u_0 + i\chi G(f_1)(t) + i\chi G(f_2)(t) + ibG(f_3)(t) + ibG(f_4)(t). \]
Observe that from the decomposition (17) we have that \( u(t) \in S'(\mathbb{R}^n) \).
Now, if we take \( \phi \in S(\mathbb{R}^n) \) then \( U(t)\phi \in C(\mathbb{R} : S(\mathbb{R}^n)) \) and also \( G(\phi)(t) \in C(\mathbb{R} : S(\mathbb{R}^n)) \).
By duality we can extend \( U(t) \) to \( S'(\mathbb{R}^n) \) and get \( U(t)\phi \in C(\mathbb{R} : S'(\mathbb{R}^n)) \) for \( \phi \in S(\mathbb{R}^n) \).
Using dominated convergence theorem we have \( G(\phi)(t) \in C(\mathbb{R} : S'(\mathbb{R}^n)) \) for \( \phi \in S'(\mathbb{R}^n) \) and by (17)
\[ u(t) \in C(\mathbb{R} : S'(\mathbb{R}^n)). \]
Letting \( t \to 0 \) in (17) we get \( u(0) = u_0 \).
Now we prove that \( u(t) \) satisfies the equation
\[ iu_t + \vec{d}u_{x_1} + \sum_{j=2}^n u_{x_jx_j} = \chi |u|^\alpha u + buE(|u|^\alpha), \]
in \( S'(\mathbb{R}^n) \) for all \( t \in \mathbb{R} \).
Define $F(u) := \chi |u|^\alpha u + buE(|u|^\alpha)$.

Note that by (10) and (18) we have

$$F(u)(t) \in C(\mathbb{R}, S'(\mathbb{R}^n)).$$

Using the integral equation (3) and the definition of the operator $G$ in (10) we have the following expression for $u(t)$

$$u(t) = U(t)u_0 + iG(Fu)(t).$$

(20)

Using group properties, Lebesgue dominated convergence theorem and the Lebesgue differentiation theorem combined with the expression of $u(t)$ in (20) we obtain that for any $\phi \in S(\mathbb{R}^n)$

$$i \lim_{h \to 0} \left( \frac{u(t+h) - u(t)}{h} \right) \phi = (-\delta \partial_{x_1} + \sum_{j=2}^n \partial_{x_j})u(t) + F(u)(t), \phi,$$

where $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$, which proves (19).

To prove $\|u(t_0)\|_{\mathcal{Y}} < \infty$, take $r = \frac{\alpha(n+2)}{2}$ on the Inequality (13) of Proposition 11.

Then we have

$$\|TT^*F\|_{L^\frac{\alpha(n+2)}{2}(\mathbb{R}^{n+1})} \leq c\|F\|_{L^\frac{\alpha(n+2)}{2}(\mathbb{R}^{n+1})}.$$ (21)

From the last property and identity (12), $\forall t_0 \in \mathbb{R}$ we get

$$\|U(t)\|_{L^\frac{\alpha(n+2)}{2}} \leq \|F\|_{L^\frac{\alpha(n+2)}{2}}.$$ (22)

Now taking $\chi(t_0, t)F$ instead of $F$ in the last inequality we have

$$\|U(t)G(F)(t_0)\|_{L^\frac{\alpha(n+2)}{2}} \leq \|F\|_{L^\frac{\alpha(n+2)}{2}}.$$ (23)

Now taking $t = t_0$ in the integral equation (3) and applying $U(t)$ we have

$$U(t)u_0 = U(t + t_0)u_0 + iU(t)G(\chi |u|^\alpha u + buE(|u|^\alpha))(t_0).$$

Combining property (4), inequality (22) and the same arguments as in Theorem 4 we obtain

$$\|U(t)u(t_0)\|_{L^\frac{\alpha(n+2)}{2}} \leq 2\|U(t)u_0\|_{L^\frac{\alpha(n+2)}{2}} + 4\|u\|_{L^\frac{\alpha(n+2)}{2}} + 4|b|\|u\|_{L^\frac{\alpha(n+2)}{2}} \|u\|_{L^\frac{\alpha(n+2)}{2}} < \infty.$$

Finally, to prove the last statement of the theorem we set

$$u_+ = u_0 + i \int_0^\infty U(-\tau)(\chi |u|^\alpha u + buE(|u|^\alpha))(\tau)d\tau.$$

(24)

It follows from Inequalities (21) that:

$$\|U(t)u_+\|_{L^\frac{\alpha(n+2)}{2}} \leq 2 \left( \|U(t)u_0\|_{L^\frac{\alpha(n+2)}{2}} + \|\chi |u|^\alpha u + buE(|u|^\alpha)\|_{L^\frac{\alpha(n+2)}{2}} \right) < \infty.$$ (25)
We deduce from the decompositions in \([16]\) that
\[
U(-t)u(t) - u_+ = \int_t^\infty U(-\tau)(\chi|u|^{\alpha}u + buE(|u|^{\alpha}))(\tau)d\tau \to 0 \text{ in } S'(\mathbb{R}^n) \text{ as } t \to \infty.
\]
The result for \(t \to -\infty\) is proved similarly.

3. Self-similar solutions

In this section we find self-similar solutions to \((2)\) for \(\delta > 0\). Without lost of generality we can suppose \(\delta = 1\), so our equation becomes:

\[
\begin{cases}
   iu_t + \Delta u = \chi|u|^{\alpha}u + buE(|u|^{\alpha}), \\
   u(x, 0) = u_0(x),
\end{cases}
\]
\[\forall x \in \mathbb{R}^n, n = 2, 3, t \in \mathbb{R}, \quad (23)\]

We will need the following proposition:

**Proposition 12.** Let \(\varphi(x) = |x|^{-p}\) where \(0 < \text{Re } p < n\). Then \(e^{t\Delta} \varphi\) is given the explicit formula below for \(x \neq 0\) and \(t > 0\):

\[
e^{t\Delta} \varphi(x) = |x|^{-p} \sum_{k=0}^{m} A_k(a, b) e^{k\pi i / 2} \left(\frac{|x|^2}{4t}\right)^{-k} + |x|^{-p} A_{m+1}(a, b) \left(\frac{|x|^2}{4t}\right)^{-m-1} \frac{(m + 1)e^{ak/2}}{\Gamma(m + 2 - b)}
\]
\[\times \int_0^\infty \int_0^1 (1 - s)^m \left(-i - \frac{4ts}{|x|^2}\right)^{a-m-1} e^{-\tau^{m+1-b}dsd\tau}\]
\[+ e^{i|x|^2/4t} |x|^{-n+p}(4t)^{\frac{n-p}{2}} \sum_{k=0}^{l} B_k(b, a) e^{-(n+2k)p/4} \left(\frac{|x|^2}{4t}\right)^{-k}\]
\[\times \int_0^\infty \int_0^1 (1 - s)^l \left(-i - \frac{4ts}{|x|^2}\right)^{-b-l-1} e^{-\tau^{l+1-a}dsd\tau},\]

where \(a = p/2, b = (n-p)/2, m, l \in \mathbb{N}\) such that \(m + 2 > \text{Re } b\) and \(l + 2 > \text{Re } a\) and

\[
A_k(a, b) = \frac{\Gamma(a + k)\Gamma(k + 1 - b)}{\Gamma(a)\Gamma(1 - b)k!}, \quad B_k(b, a) = \frac{\Gamma(b + k)\Gamma(k + 1 - a)}{\Gamma(a)\Gamma(1 - a)k!}
\]

where \(\Gamma\) denotes the gamma function.

**Proof.** We refer to \([CW1]\) for a proof of this proposition. \(\square\)

We already know that a self-similar solution must have an homogeneous initial condition with degree \(-2/\alpha\). So the idea is to prove that \(u_0(x) = \epsilon|x|^{-2/\alpha} \in Y\) where \(0 < \epsilon \ll 1\). Then by Theorem \([1]\) and Proposition \([2]\) we have existence and uniqueness for equation \((23)\) in \(Y\). Since \(u(x, t)\) and \(\beta^{2/\alpha}u(\beta x, \beta^2 t)\) are both solutions, we must have \(u = u_\beta\) and therefore self-similar solutions in \(Y\).

To prove that \(u_0 \in Y\), we consider the homogeneous problem with initial condition
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\[ u_0(x) = |x|^{-2/\alpha}; \]
\[
\begin{align*}
    iu_t + \Delta u &= 0, \\
    u(x, 0) &= |x|^{-2/\alpha}. \\
\end{align*}
\forall x \in \mathbb{R}^n, n = 2, 3, t \in \mathbb{R}. \tag{24}
\]

We know that the solution to the equation (24) is given by

\[ u(x, t) = U(t)u_0(x), \]

where \( U(t) = e^{it\Delta}. \)

Since \( u_\beta(x, t) = \beta^{2/\alpha}u(\beta x, \beta^2 t), \beta > 0 \) is also a solution, we must have

\[ \beta^{2/\alpha}u(\beta x, \beta^2 t) = U(t)u_0(x) = u(x, t). \]

Taking \( \beta = 1/\sqrt{t} \) we get

\[ u(x, t) = t^{-1/\alpha}f(x/\sqrt{t}), \tag{25} \]

where \( f(x) = u(x, 1). \)

By Proposition [12] we have that for \( \alpha > 2/n \)

\[ |f(x)| \leq c(1 + |x|)^{-\sigma} \text{ where } \sigma = \begin{cases} 
\frac{2}{\alpha}; & \alpha \geq 4/n \\
\frac{n - 2}{2\alpha}; & \alpha < 4/n.
\end{cases} \tag{26} \]

Next, we calculate \( \alpha(\lambda, u) = |\{(x, t); |u(x, t)| > \lambda\}|. \)

By (25) and (26)

\[
\alpha(\lambda, u) \leq \int \left\{(x, t); t^{-1/\alpha} \left(1 + \frac{|x|}{\sqrt{t}}\right)^{-\sigma} > \lambda \right\} d(x, t) \leq \int \left\{(x, t); 0 \leq t < \lambda^{-\alpha} \text{ and } |x| < t^{1/2}((t\lambda^\alpha)^{-1/\alpha} - 1) \right\} d(x, t) \\
\leq c\lambda^{-n/2} \int_0^{\lambda^{-\alpha}} t^\frac{n}{2\alpha \sigma} \left[1 - (t\lambda^\alpha)^{\frac{1}{\alpha \sigma}}\right]^n dt \leq \lambda^{-\frac{\alpha(n+2)}{2}}.
\]

Therefore \( \|U(\cdot)u_0\|_{L^\alpha(R^{n+1})} \leq c. \)

Choosing \( 0 < \epsilon \ll 1 \) and taking the initial condition \( u_0(x) = \epsilon |x|^{-2/\alpha} \) we conclude the result.

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