A class of punctured simplex codes which are proper for error detection

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Abstract—Binary linear \([n,k]\) codes that are proper for error detection are known for many combinations of \(n\) and \(k\). For the remaining combinations, existence of proper codes is conjectured. In this paper, a particular class of \([n,k]\) codes is studied in detail. In particular, it is shown that these codes are proper for many combinations of \(n\) and \(k\) which were previously unsettled.

Index Terms—Error detection, proper codes, satisfactory codes, simplex codes, punctured codes, ugly codes.

I. INTRODUCTION

In this paper, we study binary linear \([n,k]\) codes (codes of length \(n\) and dimension \(k\)) used for error detection on the binary symmetric channel. A comprehensive introduction to the field is given in [I]. The basic definitions are given in Section II. A main quantity is the probability of undetected error of a code. If the probability of undetected error is an increasing function on the interval \([0,1/2]\), the code is known as proper for error detection.

It is believed that proper codes exist for all lengths \(n\) and dimensions \(k\). However, this has been shown only for some cases. In particular, proper \([n,k]\) codes are known to exist for any given \(k\) when \(n\) is sufficiently large. The best known result in this direction was given by Kløve and Yari [2] who showed that proper codes exist for

\[
n \geq 2^{k-1} (2^{k-5} + 2^{(k-5)/2}) \quad \text{when} \quad k \geq 5.
\]

In this paper, we study a particular class of \([n,k]\) codes where \(n > 2^{k-1}\). One of our results is that these codes are proper for many values of \(n\) and \(k\) where the existence of proper codes was previously unknown. In particular, we improve the bound (I).

We first consider \(n\) in the range \(2^{k-1} < n < 2^k\). The Hamming bound proves that the dual of an \([n,k]\) code in this case has minimum distance at most 3. Moreover, an \([n,n-k]\) code with minimum distance 3 can be obtained by shortening the \([2^k - 1,2^k - 1-k]\) Hamming code.

Two \([n,k]\) codes \(C\) and \(C'\) are equivalent if there exists a permutation \(\pi\) of \(\{1,2,\ldots,n\}\) such that

\[
C' = \{c_{\pi(1)},c_{\pi(2)},\ldots,c_{\pi(n)}\} \mid (c_1,c_2,\ldots,c_n) \in C\}.
\]

If two codes are equivalent, then it may happen that the corresponding (repeatedly) punctured codes are not equivalent.

Let \(H_k\) be some \(k \times (2^k - 1)\) matrix having as columns all possible nonzero vectors of length \(k\). The code generated by \(H_k\) is the simplex code \(S_k\), and the code having \(H_k\) as parity check matrix is the well-known Hamming code. Note that the order of the columns is not specified; all the equivalent codes are named Hamming codes.

II. ERROR DETECTION

We start by defining \(P_{ue}(C,p)\), the undetected error probability for an \([n,k]\) code \(C\) when used on the binary symmetric
channel with error probability $p$:

$$P_{\text{ue}}(C, p) = \sum_{w=1}^{n} A_w p^w (1 - p)^{n-w}, \quad (3)$$

where $A_w$ is the number of codewords having Hamming weight $w$, see e.g. [1, p. 38].

One can also express this polynomial in terms of the weight distribution of the dual code, see e.g. [1, Theorem 2.4]. If $A_w^\perp$ is the number of codewords having Hamming weight $w$ in the dual code $C^\perp$, we have:

$$P_{\text{ue}}(C, p) = 2^{k-n} \sum_{w=0}^{n} A_w^\perp (1 - 2p)^w - (1 - p)^n. \quad (4)$$

As mentioned in the introduction, if $P_{\text{ue}}(C, p)$ is an increasing function on $[0, \frac{1}{2}]$, the code $C$ is called proper for error detection. If

$$P_{\text{ue}}(C, p) \leq P_{\text{ue}}(C, 1/2)$$

for every $p \in [0, \frac{1}{2}]$, $C$ is called good for error detection. If

$$P_{\text{ue}}(C, p) \leq 2^{k-n}$$

for every $p \in [0, \frac{1}{2}]$, $C$ is called satisfactory for error detection, see [1, p. 38]. A code that is not satisfactory is called ugly. When a code is proper then it is satisfactory; so, if it is ugly it is clearly not proper (nor good).

### III. The Code Construction

We first describe a particular parity check matrix $H_k$ for the Hamming code. For $0 \leq m \leq k - 1$, let $H_k^{(m)}$ be the $k \times 2^m$ matrix constructed as follows:

- The first $k - m - 1$ rows are all-zero vectors.
- Row $k - m$ is the all-one vector.
- In the $m \times 2^m$ matrix consisting of the last $m$ rows, the columns are ordered lexicographically.

Then

$$H_k = \left[ H_k^{(k-1)} | H_k^{(k-2)} | \ldots | H_k^{(0)} \right].$$

We illustrate this with an example. For $k = 4$, we get

$$H_4^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, H_4^{(1)} = \begin{bmatrix} 00 \\ 11 \\ 01 \end{bmatrix}, H_4^{(2)} = \begin{bmatrix} 0000 \\ 1111 \\ 0011 \\ 0101 \end{bmatrix},$$

$$H_4^{(3)} = \begin{bmatrix} 11111111 \\ 00001111 \\ 00110011 \\ 01010101 \end{bmatrix},$$

and so

$$H_4 = \left[ H_4^{(3)} | H_4^{(2)} | H_4^{(1)} | H_4^{(0)} \right] = \begin{bmatrix} 1111111100000000 \\ 0000111111110000 \\ 0011001100111100 \\ 0101010101010101 \end{bmatrix}. \quad (5)$$

We let $H_k(n)$ denote the $k \times n$ matrix containing the first $n$ columns of $H_k$. For example

$$H_4(11) = \begin{bmatrix} 11111111000 \\ 00001111111 \\ 0011001100110 \end{bmatrix}. \quad (6)$$

We let $S_{n,k}$ denote the code generated by $H_k(n)$. We see that $S_{2^k-1,k}$ is the first order Reed-Muller code and $S_{2^k-1,k}$ is the simplex code. Both of these codes are known to be proper (and this is easy to show). The Hamming code is $S_{2^{k-1},k}$. The code having $H_k(n)$ as parity check matrix is a shortened Hamming code which we denote by $C_{n,n-k}$. We note that $C_{n,n-k} = S_{n,k}^\perp$. In the rest of the paper (except Section VII) we will assume that $2^{k-1} < n \leq 2^k - 1$.

**Theorem 1:** The codes $S_{n,k}$ and $D_{n,k}$ are equivalent.

**Proof:** We first illustrate by the example $k = 4$ and $n = 11$, that is, the matrices (7) and (9). Adding the second row in (6) to the third and forth rows, we get

$$H_4(11) = \begin{bmatrix} 11111111000 \\ 00001111111 \\ 0011001100110 \end{bmatrix}. \quad (6)$$

This is an alternative generator matrix for $S_{11,4}$. The last three columns are the same in (7) and (9), and the first eight columns of (7) are a permutation of the first eight columns in (9). Hence, $S_{11,4}$ and $D_{11,4}$ are equivalent.

In the general case, if $n \in [2^k - 2^{k-m} + 1, 2^k - 2^{k-m-1} - 1]$ for some $m$, $1 \leq m \leq k - 1$, we add row $m + 1$ in $H_k(n)$ to all the rows below. This gives an alternative generator matrix $H_k(n)$ for $S_{n,k}$. The first $2^{k-1}$ columns of $H_k(n)$ are a permutation of the binary representations of $i \in [2^{k-1}, 2^k]$, the next $2^{k-2}$ columns of $H_k(n)$ are a permutation of the binary representations of $i \in [2^{k-2}, 2^{k-1}]$, etc. The final $n - 2^k + 2^{k-m}$ columns in $H_k(n)$ and $M_{n,k}$ are the same. Hence, $S_{n,k}$ and $D_{n,k}$ are equivalent.

If $n = 2^k - 2^{k-m}$ for some $m$, $1 \leq m \leq k$, the same argument shows that the columns of $H_k(n)$ are a permutation of the columns of $M_{n,k}$, and so again $S_{n,k}$ and $D_{n,k}$ are equivalent.

### IV. Weight Distribution of $S_{n,k}$

The main question we consider is: for which $n$ and $k$ is $S_{n,k}$ proper for error detection? We will also in some cases consider the simpler question: for which $n$ and $k$ is $S_{n,k}$ satisfactory for error detection?

We note that this is equivalent to the question: for which $n$ and $k$ is $C_{n,n-k}$ satisfactory for error detection? The reason is the following known lemma.

**Lemma 1:** [1, Theorem 2.8]. A code is satisfactory if and only if the dual code is satisfactory.

To determine the probability of undetected error for $S_{n,k}$, we have to determine its weight distribution. This is done in this section. We break the argument down into a number of lemmas.
We first give some further notations. We observe that the matrix
\[
\begin{bmatrix}
H^{(k-1)}_k & H^{(k-2)}_k & \cdots & H^{(k-m)}_k
\end{bmatrix}
\]
has length
\[
\sum_{j=1}^{m} 2^{k-j} = 2^k - 2^{k-m}.
\]
For a given \(n\), let \(m\) be determined by
\[
2^k - 2^{k-m} < n \leq 2^k - 2^{k-m-1}.
\]
Since \(2^k - 2^{k-m} < n < 2^k\), we have \(1 \leq m \leq k - 1\). Let \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) denote the last column of \(H_k(n)\).

**Lemma 2:** Let \(2^k - 2^{k-m} < n \leq 2^k - 2^{k-m-1}\). Then \(\alpha_1 = \ldots = \alpha_m = 0\), \(\alpha_{m+1} = 1\),
\(\alpha_{m+2}, \ldots, \alpha_k\) are determined by
\[
\sum_{i=0}^{k-m-2} \alpha_{k-i} 2^i = n - 1 - 2^k + 2^{k-m}.
\]

**Proof:** The last column in \(H_k(n)\) is a column in \(H^{(k-m-1)}_k\). Hence (7) follows. Moreover, its number in \(H^{(k-m-1)}_k\) is \(n - 1 - (2^k - 2^{k-m})\) when we count the first column as number zero. The columns in \(H^{(k-m-1)}_k\) are ordered lexicographically and so (8) follows.

Let \(w_i\) denote the weight of the \(i\)-th row in \(H_k(n)\). As usual, \([x]\) denotes the largest integer less than or equal to \(x\).

**Lemma 3:** Let \(2^k - 2^{k-m} < n \leq 2^k - 2^{k-m-1}\). Then
\[
w_1 = \cdots = w_m = 2^k \tag{9}
\]
and
\[
w_{m+1} = n - 2^k - 2^{k-m-1}. \tag{10}
\]
If \(m + 2 \leq i \leq k\) and \(\alpha_i = 0\), then
\[
w_i = 2^{k-i} \left[ \frac{n - 1}{2^{k-i+1}} \right]. \tag{11}
\]
If \(m + 2 \leq i \leq k\) and \(\alpha_i = 1\), then
\[
w_i = n - 2^k - \left[ \frac{n - 1}{2^{k-i+1}} \right] - 2^{k-i}. \tag{12}
\]

**Proof:** All the rows of \(H_k\) have weight \(2^{k-1}\). The first \(m\) rows of \(H_k(n)\) are obtained from rows in \(H_k\) removing some zeros. Hence \(w_i = 2^k - 1\) for \(1 \leq i \leq m\).

Before we go on with the proof, let us take a closer look at \(H^{(m)}_k\). Row \(i > m\) consists of consecutive blocks of zeros and ones, each block of length \(2^k-i\). We use the term double block for a zero-block combined with the following one-block; it has length \(2^k-i+1\). Now, let
\[
n = \left\lfloor \frac{n - 1}{2^{k-i+1}} \right\rfloor 2^{k-i+1} + \nu \quad \text{where} \quad 1 \leq \nu \leq 2^{k-i+1}. \tag{13}
\]
Then row \(i\) in \(H_k(n)\) consists of \(\left\lfloor \frac{n - 1}{2^{k-i+1}} \right\rfloor\) double blocks of length \(2^{k-i+1}\), each of weight \(2^k-i\), followed by an incomplete double block of length \(\nu\) that has to be considered further (when \(\nu = 2^{k-i+1}\), the incomplete double block is, of course, a full double block).

If \(\alpha_i = 0\), the incomplete double block is all zero, and so (11) follows.

If \(\alpha_i = 1\), the incomplete double block consists of a full block (of length \(2^{k-i}\)) of zeros followed by an incomplete block of ones of length \(\nu - 2^{k-i}\). Hence
\[
w_i = \left. 2^{k-i} \left[ \frac{n - 1}{2^{k-i+1}} \right] + \nu - 2^{k-i} \right.
\]
\[
= \left. 2^{k-i} \left[ \frac{n - 1}{2^{k-i+1}} \right] + n - \left[ \frac{n - 1}{2^{k-i+1}} \right] 2^{k-i+1} - 2^{k-i} \right.
\]
\[
= n - 2^k - \left[ \frac{n - 1}{2^{k-i+1}} \right] - 2^{k-i}.
\]
This proves (12). The proof of (10) is similar (and even simpler).

**Lemma 4:** Consider sums of rows from \(H_k(n)\).

a) Any of the \(2^m - 1\) non-zero sums of some of the first \(m\) rows have weight \(2^{k-1}\).

b) Any of the \(2^m\) sums containing row \(m + 1\) and zero or more previous rows have weight \(w_{m+1}\).

c) For \(m + 2 \leq i \leq k\), \(2^{i-2}\) sums containing row \(i\) and some previous rows have weight \(w_i\) and the other \(2^{i-2}\) sums have weight \(n - w_i\).

**Proof:** For each sum of rows from the first \(m\), the corresponding sum of rows in \(H_k\) are codewords in the simplex code \(S_k\). These always have weight \(2^{k-1}\). Since only zeros have been removed to get the corresponding rows in \(H_k(n)\), their sum also has weight \(2^{k-1}\). This proves a).

Let \(i \geq m + 2\). We note that in the set of positions of a double block in row \(i \geq m + 2\) in \(H^{(k-j)}_k\) for \(j < m + 1\), the elements of any previous row are all zero or all one. Therefore, the weight of these positions in any sum of row \(i\) and a combination of previous rows is \(2^{k-i}\).

It remains to consider the contribution to the weight from the last \(\nu\) positions (where \(\nu\) is defined by (13)).

Case I) \(\alpha_i = 0\). In this case, all the last \(\nu\) elements of row \(i\) are zeros. Any previous row has all zeros or all ones in these positions, and so the weight of the elements in these positions in any sum is either 0 or \(\nu\). Hence, the weight of the sum is either \(w_i\) or \(n - w_i\), where \(w_i\) is given by (11). Moreover, row \(m + 1\) has all ones in the last \(\nu\) positions. Hence, half of the \(2^{i-2}\) sums has weight \(w_i\) and the other half has weight \(n - w_i\).

Case II) \(\alpha_i = 1\). In this case, all the last \(\nu\) elements of row \(i\) are \(2^{k-i}\) zeros followed by \(\nu - 2^{k-i}\) ones. The weight of the last \(\nu\) elements in a sum is therefore \(\nu - 2^{k-i} + 2^{k-i}\). Hence, the weight of a sum is \(w_i\) or \(n - w_i\), where now \(w_i\) is given by (12). As done above, considering sums containing row \(m + 1\), we can see that the multiplicities of these two weights are the same. This proves c).

Finally, consider row \(m + 1\). Any previous row has all zeros in the last \(\nu\) positions. Hence, the weight of any sum involving row \(m + 1\) and previous rows is \(w_{m+1}\). This proves b).

We next give an alternative expression for \(w_i\).

**Lemma 5:** Let \(2^k - 2^{k-m} < n \leq 2^k - 2^{k-m-1}\).

a) If \(m + 2 \leq i \leq k\) and \(\alpha_i = 0\), then
\[
w_i = 2^{k-1} - 2^{k-m-1} + \sum_{j=m+2}^{i-1} \alpha_j 2^{k-1-j}.
\]
b) If \( m + 2 \leq i \leq k \) and \( \alpha_i = 1 \), then

\[
    w_i = 2^{k-i} - 2^{k-m-i} + 1 + \sum_{j=m+2}^{i-1} \alpha_j 2^{k-j} + \sum_{j=m+1}^{k} \alpha_j 2^{k-j}.
\]

c) Further,

\[
    w_{m+1} = 2^{k-1} - 2^{k-m-1} + 1 + \sum_{j=m+2}^{k} \alpha_j 2^{k-j}.
\]

**Proof:** From (8) we get

\[
\begin{align*}
    n - 1 &= 2^k - 2^m + \sum_{j=m+2}^{k} \alpha_j 2^{k-j}.
\end{align*}
\]

Hence

\[
    \frac{n - 1}{2^{k-i+1}} = 2^{i-1} - 2^{i-m-1} + \sum_{j=m+2}^{i-1} \alpha_j 2^{i-j} + r
\]

where

\[
    r = \sum_{j=1}^{k} \alpha_j 2^{i-j} \leq \sum_{j=1}^{k} 2^{i-j} = 1 - 2^{i-1} - k < 1.
\]

Hence

\[
    \left| \frac{n - 1}{2^{k-i+1}} \right| = 2^{i-1} - 2^{i-m-1} + \sum_{j=m+2}^{i-1} \alpha_j 2^{i-j}
\]

and so, by (10),

\[
    w_i = 2^{k-i} \left| \frac{n - 1}{2^{k-i+1}} \right| = 2^{k-1} - 2^{k-m-1} + \sum_{j=m+2}^{i-1} \alpha_j 2^{k-j}.
\]

This proves a).

Similarly, if \( \alpha_i = 1 \), (12) gives

\[
\begin{align*}
    w_i &= n - 2^{k-i} \left| \frac{n - 1}{2^{k-i+1}} \right| - 2^{k-i} \\
    &= 1 + 2^k - 2^{k-m} + \sum_{j=m+2}^{k} \alpha_j 2^{k-j} - 2^{k-i} \\
    &\quad - 2^{k-1} + 2^{k-m-1} - \sum_{j=m+2}^{i-1} \alpha_j 2^{k-1-j} \\
    &= 1 + 2^{k-1} - 2^{k-m-1} + \sum_{j=m+2}^{i-1} \alpha_j 2^{k-1-j} \\
    &\quad + \alpha_i 2^{k-i} + \sum_{j=i+1}^{k} \alpha_j 2^{k-j} - 2^{k-i} \\
    &= 1 + 2^{k-1} - 2^{k-m-1} + \sum_{j=m+2}^{i-1} \alpha_j 2^{k-1-j} \\
    &\quad + \sum_{j=i+1}^{k} \alpha_j 2^{k-j},
\end{align*}
\]

since \( \alpha_i = 1 \). This proves b). Finally, c) follows directly by substituting the expression for \( n \) in the expression for \( w_{m+1} \) in (10).

**Example 1:** Consider \( n = 2^k - 2^{k-m-1} \), where \( 1 \leq m < k \).

We have

\[
    \alpha_i = 0 \text{ for } 1 \leq i \leq m \text{ and } \alpha_i = 1 \text{ for } m + 1 \leq i \leq k.
\]

Using Lemma (5) we see that \( S_{n,k} \) has \( 2^m - 1 \) codewords of weight \( w_1 = 2^{k-1} \) and \( 2^{k-2} - 2^m \) codewords of weight

\[
    2^{k-1} - 2^{k-m-2} = n/2.
\]

In particular, the minimum distance is \( n/2 \). Hence, the code \( S_{n,k} \) is proper (see [1] Theorem 2.2).

**Lemma 6:** Let \( 2^{k-2} - 2^m < n < 2^{k-2} - 2^{k-m-1} \)

and \( m + 2 \leq i \leq k \).

a) If \( \alpha_i = \alpha_{i+1} \), then \( w_{i+1} = w_i \).

b) If \( \alpha_i = 0 \) and \( \alpha_{i+1} = 1 \), then \( w_{i+1} > w_i \).

c) If \( \alpha_i = 1 \), \( \alpha_{i+1} = 0 \), and \( \alpha_j = 1 \) for all \( j \) such that \( i + 2 \leq j \leq k \), then \( w_{i+1} = w_i \).

d) In all cases,

\[
    w_1 \geq w_{m+1} > w_k \geq w_{k-1} \geq w_{k-2} \geq \cdots \geq w_{m+2}.
\]

In particular, the minimum distance \( d \) of \( S_{n,k} \) is \( m + 2 \).

**Proof:** a) If \( \alpha_i = \alpha_{i+1} = 1 \), then Lemma (5) gives

\[
    w_{i+1} - w_i = \alpha_i 2^{k-1-i} = 0.
\]

e) \( w_{m+1} > n/2 \).

**Proof:** If \( \alpha_i = \alpha_{i+1} = 0 \), then Lemma (5) gives

\[
    w_{i+1} - w_i = \alpha_i 2^{k-1-i} - \alpha_{i+1} 2^{k-(i+1)} = 0.
\]

b) If \( \alpha_i = 0 \) and \( \alpha_{i+1} = 1 \), then Lemma (5) gives

\[
    w_{i+1} - w_i = 1 + \sum_{j=i+2}^{k} \alpha_j 2^{k-j} > 0.
\]

c) If \( \alpha_i = 1 \) and \( \alpha_{i+1} = 0 \), then Lemma (5) gives

\[
    w_{i+1} - w_i = 2^{k-1-i} - 1 - \sum_{j=i+2}^{k} \alpha_j 2^{k-j}.
\]

We have

\[
\sum_{j=i+2}^{k} \alpha_j 2^{k-j} \leq \sum_{j=i+2}^{k} 2^{k-j} = 2^{k-1-i} - 1
\]

with equality if and only if \( \alpha_j = 1 \) for \( i + 2 \leq j \leq k \).

d) We have

\[
\sum_{j=m+2}^{k} \alpha_j 2^{k-j} \leq \sum_{j=m+2}^{k} 2^{k-j} = 2^{k-m-1} - 1,
\]

and so

\[
    w_{m+1} \leq 2^{k-1} - 2^{k-m-1} + 1 + 2^{k-m-1} - 1 = w_1.
\]
Further, both for \( \alpha_k = 0 \) and \( \alpha_k = 1 \), Lemma [5] gives
\[
w_{m+1} - w_k = 1 + \sum_{j=m+2}^{k-1} \alpha_j 2^{k-j-1} > 0.
\]

For \( m + 2 \leq i \leq k \), a), b), c1), and c2) show that \( w_i \geq w_{i-1} \).
e) Equation \[(10)\] implies that
\[n - 2w_{m+1} = 2^k - 2^{k-m} - n < 0.
\]
\[\]
Let
\[n(k,m) = 2^k - 2^{k-m} + 2^{k-m-2} = 2^{k-1} - 2^{k-m-1},\]
the midpoint of the interval \([2^k - 2^{k-m}, 2^k - 2^{k-m-1}]\).

**Lemma 7:** Let \( d \) be the minimum distance of \( S_{n,k} \).
a) If \( 2^k - 2^{k-m} \leq n \leq n(k,m) \), then
\[d = 2^{k-1} - 2^{k-m-1}.
\]
b) If \( n(k,m) \leq n \leq 2^k - 2^{k-m-1} \), then
\[n - d = 2^{k-1} - 2^{k-m-2}.
\]

**Proof:** a) We have
\[n - 1 - 2^k + 2^{k-m} \leq 2^{k-2} - 1.
\]
Hence \( \alpha_{m+2} = 0 \). By Lemma [5]
\[w_{m+2} = 2^{k-1} - 2^{k-m-1},\]
b) We have \( \alpha_{m+2} = 1 \). From [8] and Lemma [5],
\[n - w_{m+2} = 2^{k-1} - 2^{k-m-1} + \alpha_{m+2} 2^{k-m-2} = 2^{k-1} - 2^{k-m-2}.
\]

**Lemma 8:** a) If \( w_{m+2} = w_k \), then
\[A_d = 2^{k-1} - 2^m.
\]
b) If \( w_{m+2} < w_k \) and \( i \geq m + 2 \) is given by
\[w_{m+2} = w_i < w_{i+1},
\]
then
\[A_d = 2^{i-1} - 2^m.
\]
In particular, \( A_d \geq 2^m \) in all cases.

**Proof:** The conditions imply that \( w_j \) has value \( d \) exactly for \( m + 2 \leq j \leq i \) (where \( i = k \) for case a). Hence
\[A_d = \sum_{j=m+2}^{i} 2^{j-2} = 2^{i-2} - 2^m.
\]

\[\]

### V. Probability of Undetected Error of \( S_{n,k} \) and \( C_{n,k} \)

From [3], [4], and Lemma [3] we get the following theorems.

**Theorem 2:** Let \( 2^k - 2^{k-m} < n \leq 2^k - 2^{k-m-1} \). Then
\[
P_{ue}(S_{n,k}, p) = (2^m - 1)p^{w_1}(1 - p)^{n-w_1} + 2^m p^{w_{m+1}}(1 - p)^{n-w_{m+1}} + \sum_{i=m+2}^{k} 2^i - 2^{i-2} \left\{ (1 - 2p)^{w_i} + (1 - 2p)^{n-w_i} \right\}.
\]

**Theorem 3:** Let \( 2^k - 2^{k-m} < n \leq 2^k - 2^{k-m-1} \). Then
\[
P_{ue}(C_{n,k}, p) = (2^m - 1)(1 - 2p)^{w_1} + 2^m (1 - 2p)^{w_{m+1}} + \sum_{i=m+2}^{k} 2^i - 2^{i-2} \left\{ (1 - 2p)^{w_i} + (1 - 2p)^{n-w_i} \right\}.
\]

**Example 2:** Consider \( n = n(k,m) = 2^k - 3 \cdot 2^{k-m-2} \) where \( k \geq m + 3 \). Then
\[\alpha_i = 0, \text{ for } 1 \leq i \leq m, \quad \alpha_{m+1} = 1, \quad \alpha_{m+2} = 0, \quad \alpha_i = 1 \text{ for } m + 3 \leq i \leq k.
\]

Using Lemmas [3] and [8] we get
\[w_i = 2^{k-1} \text{ for } 1 \leq i \leq m, \quad w_{m+1} = 2^{k-1} - 2^{k-m-2}, \quad w_{m+2} = 2^{k-1} - 2^{k-m-1}, \quad n - w_{m+2} = 2^{k-1} - 2^{k-m-2},
\]
\[n - w_i = w_i = 2^{k-1} - 3 \cdot 2^{k-m-3} \text{ for } m + 3 \leq i \leq k.
\]

Hence, the weight distribution of \( S_{n,k} \) is given by Table I.

**Example 3:** For \( k = 9 \) and \( m = 1 \) in Example 2 we get \( n = 320 \) and
\[
P_{ue}(S_{320,9}, p) = 2 p^{128}(1 - p)^{192} + 504 p^{160}(1 - p)^{160} + 4 p^{192}(1 - p)^{128} + p^{256}(1 - p)^{64}.
\]

In Fig. I we give the graphs of \( P_{ue}(S_{320,9}, p) \) and the terms \( 2 p^{128}(1 - p)^{192} \) and \( 504 p^{160}(1 - p)^{160} \). The contributions from the last two terms, \( 4 p^{192}(1 - p)^{128} \) and \( p^{256}(1 - p)^{64} \), are so small that they are not visible on the graph. The graph illustrates that \( S_{320,9} \) is ugly. For small \( p \) (up to

\[\]
approximately 0.42), $2p^{128}(1-p)^{192}$ is the dominating term; in this region the difference

$$P_{ue}(S_{320,9},p) - 2p^{128}(1-p)^{192}$$

is so small that it is not visible on the graph. For $p$ close to 0.5, the term $504p^{160}(1-p)^{160}$ dominates.

Theorem 2 can be used to determine if the code $S_{n,k}$ is proper and Theorem 3 if the code $C_{n,n-k}$ is proper. We just compute $\frac{dP_{ue}}{dp}$ and check the presence or absence of real roots in $(0,\frac{1}{2})$. For moderate values of $n$ and $k$ (e.g. $k \leq 20$), this is feasible in a reasonable time.

Before we give a main general result, we quote two lemmas from [1].

**Lemma 9:** a) [1] Theorem 2.2: if $w \geq n/2$, then

$$p^w(1-p)^{n-w}$$

is increasing on $[0,1/2]$.

b) [1] Lemma 3.5: if $(n-\sqrt{n})/2 \leq w \leq n/2$, then

$$p^w(1-p)^{n-w} + p^{n-w}(1-p)^w$$

is increasing on $[0,1/2]$.

Let

$$\tau_{k,m} = \min\left\{2k-m-2, \frac{1 + \sqrt{1 + 2k+2 - 2k-m+2}}{2}\right\},$$

$$\zeta_{k,m} = \min\left\{2k-m-1, \frac{1 - \sqrt{1 + 2k+2 - 2k-m+1}}{2}\right\}.$$

**Theorem 4:** For $k > m \geq 1$, if

$$2k - 2k-m \leq n \leq 2k - 2k-m + \tau_{k,m},$$

(15)

or

$$2k - 2k-m-1 - \zeta_{k,m} \leq n \leq 2k - 2k-m-1,$$

(16)

then $S_{n,k}$ is proper.

**Proof:** For $n = 2^k - 2k-m$ and $n = 2^k - 2k-m-1$, $S_{n,k}$ is proper by Example 1. For $2^k - 2k-m < n < 2^k - 2k-m-1$,

(9), Lemma 6a), and Lemma 6e) imply that

$$w_1 = \cdots = w_m \geq w_{m+1} > n/2$$

and so $P^{w_i}(1-p)^{n-w_i}$ is increasing on $[0,1/2]$ for $1 \leq i \leq m+1$ by Lemma 9a).

Now, consider $n = 2^k - 2k-m + \eta$, where $0 < \eta \leq 2k-m-2$.

By Lemma 7a),

$$d = 2k-1 - 2k-m-1 < \frac{n}{2}$$

By Lemmas 8d) and 9b),

$$p^{w_i}(1-p)^{n-w_i} + p^{n-w_i}(1-p)^{w_i}$$

is increasing for $m + 2 \leq i \leq k$ if $2d \geq n - \sqrt{n}$, that is, if

$$2k - 2k-m = 2d \geq 2^k - 2k-m + \eta - \sqrt{2^k - 2k-m + \eta}.$$  

This is equivalent to

$$\sqrt{2^k - 2k-m + \eta} \geq \eta$$

and

$$2k - 2k-m + \eta \geq \eta^2.$$  

Solving this for $\eta$, we get

$$\eta \leq 1 + \sqrt{1 + 4(2^k - 2k-m)}.$$  

We see that if (15) is satisfied, then all the terms in

$$P_{ue}(S_{n,k},p)$$

are increasing on $[0,1/2]$. Consequently, $S_{n,k}$ is proper.

Next, let $n = 2^k - 2k-m-1 - \eta$ where $0 < \eta \leq 2k-m-2 - 1$.

Then, by Lemma 7b),

$$d = 2k-1 - 2k-m-2 - \eta.$$  

We want

$$2^k - 2k-m-1 - 2\eta = 2d \geq 2^k - 2k-m-1 - \eta - \sqrt{2^k - 2k-m-1 - \eta}.$$  

Solving for $\eta$, we get $\eta^2 + \eta \leq 2^{2k-m-1} - 1$ and so

$$\eta \leq -1 + \sqrt{1 + 4(2^k - 2k-m-1)}.$$  

As above, if (16) is satisfied, then $S_{n,k}$ is proper.

When $S_{n,k}$ is proper, then it is satisfactory, and so, by Lemma 1, $C_{n,n-k}$ is satisfactory. Hence we get the following corollary.

**Corollary 1:** If $n$ is in the range defined by (15) or (16) for some $m \geq 1$, then $C_{n,n-k}$ is satisfactory.

**Theorem 5:** a) If

$$m \geq \left\lceil \frac{k-3}{2}\right\rceil,$$  

(17)

then $S_{n,k}$ is proper for all $n \in \left[2^k - 2k-m, 2^k - 2k-m-1\right]$.

b) $S_{n,k}$ is proper for all $n \in \left[2^k - 2k-\left\lceil \frac{k-3}{2}\right\rceil, 2^k - 1\right]$. 

Fig. 1. Plot of $P_{ue}(S_{320,9},p)$ (solid line), $2p^{128}(1-p)^{192}$ (dashed line), $504p^{160}(1-p)^{160}$ (dotted line), and $2^9-320$ (long dashed).
Proof: We have $S_{k,m} = 2^{k-m-2}$ if and only if
\[
1 + \sqrt{1 + 2^{k+2} - 2^{k+m+2}} \geq 2^{k-m-2}. \tag{18}
\]
We observe that $2^{k-m-2}$ decreases with increasing $m$ and $1 + \sqrt{1 + 2^{k+2} - 2^{k+m+2}}$ increases with increasing $m$.

Let $x = 2^{k-m}$. Then (18) is equivalent to the following sequence of inequalities
\[
1 + \sqrt{1 + 2^{k+2} - 8x} \geq x,
1 + 2^{k+2} - 8x \geq (x-1)^2 = x^2 - 2x + 1,
2x^2 - 2x + 1 \leq 2^{k+2}.
\]
(19)

For $k = 2m + 3$ we get $x = 2^{m+2}$ and so
\[
x^2 + 6x = 2^{2m+4} + 6 \cdot 2^{m+2} = 2^{2m+5} = 2^{k+2}
\]
for all $m \geq 1$. However, for $k = 2m + 4$ we get $x = 2^{m+3}$ and so
\[
x^2 + 6x = 2^{2m+6} + 6 \cdot 2^{m+3} > 2^{2m+6} = 2^{k+2}.
\]
Hence, (18) is satisfied if and only if $k \leq 2m + 3$, that is when (17) is satisfied. Therefore, if (17) is satisfied, then $S_{n,k}$ is proper for all $n \in [2^k - 2^{k-m}, 2k - 2^{k-m} + 2^{k-m-2}]$.

Next, since $-2^{k-m+1} > -2^{k-m+2}$, we see that if (18) is satisfied, then
\[
-1 + \sqrt{1 + 2^{k+2} - 2^{k+m+1}} \geq \sqrt{1 + 2^{k+2} - 2^{k+m+2}} - 1 \geq 2^{k-m-1}.
\]
Hence, $S_{k,m} = 2^{k-m-2} - 1$, and so $S_{n,k}$ is proper also for all $n \in [2^k - 2^{k-m} + 2^{k-m-2} + 1, 2k - 2^{k-m-1}]$.

This, combined with the result above, proves a).

Since $S_{n,k}$ is proper for $n \in [2^k - 2^{k-m}, 2k - 2^{k-m-1}]$ for all $m \geq \left\lceil \frac{k}{2} \right\rceil$, b) follows.

Based on the previous theorems, we have found a set of values of $n$ for which $S_{n,k}$ is proper and, hence, satisfactory. For other values of $n$, the existence of real roots of $\frac{dp}{dn}$ in $(0, \frac{1}{2})$ must be checked. However, for large values of $k$ and $n$ in general it may be difficult to numerically compute the polynomial's real roots, or even just to determine the existence of real roots (e.g. using Sturm’s chain). However, in many cases we can decide that the code $S_{n,k}$ (and hence $C_{n,n-k}$) is not satisfactory (i.e. ugly) by showing that $P_{ue}(C,p) > 2^{k-n}$ for some value of $p$. How should the value of $p$ be chosen? There is no theory that can give an exact answer to this question. However, it is known that if the minimum distance of the code is $d$, then $A_d p^d(1-p)^{n-d}$ is often the dominating term of $P_{ue}(C,p)$, except for large $p$. This is well illustrated by the example of $S_{320,9}$ given in Fig. I. Since $p^d(1-p)^{n-d}$ has its maximum for $p = d/n$, a good choice for $p$ may be $p = d/n$. This gives the following sufficient condition for $S_{n,k}$ to be ugly:

\[
2^{k-n} < A_d \left(\frac{d}{n}\right)^d \left(1 - \frac{d}{n}\right)^{n-d} = A_d 2^{-n} h(d/n) \tag{20}
\]
where
\[
h(x) = -x \log_2 x - (1-x) \log_2 (1-x)
\]
is the binary entropy function.

We can reformulate this to the following well-known sufficient condition for a code to be ugly (see e.g. [1, Theorem 2.11] or [5]):
\[
A_d > 2^{k-n} n h(d/n). \tag{21}
\]

We showed in Example II that $S_{2k-2k-m-1,k}$ is proper for all $m < k$. In general, $S_{n,k}$ and $C_{n,n-k}$ may be ugly for some values of $n$ when $2^k - 2^{k-m} < n < 2^k - 2^{k-m-1}$.

We have checked that $S_{n,k}$ is proper for all $n \geq 2^{k-1}$ when $k \leq 8$. When $k \geq 9$, $S_{n,k}$ is ugly for some values of $n$. An example is $S_{320,9}$ in Fig. I.

Lemma 10: For a given $k$, let
\[
g(n) = k - n + n h\left(\frac{d}{n}\right).
\]
Then $g(n)$ is increasing with $n$ on $[2^k - 2^{k-m}, n(k,m)]$ and decreasing with increasing $n$ on $[n(k,m), 2^k - 2^{k-m-1}]$, where $n(k,m)$ was given in (14).

Proof: From the definitions of $h(x)$ and $g(n)$, we get
\[
g(n) = k - n + n \log_2 n - d \log_2 d - (n - d) \log_2 (n - d).
\]
By Lemma 7b), $d$ is constant for $n \in [2^k - 2^{k-m}, n(k,m)]$.
Considering $n$ as a real variable for the moment, direct calculations give
\[
\frac{dg(n)}{dn} = -1 + \log_2 \left(\frac{n}{n - d}\right).
\]
Since $d < n/2$, we get $\frac{dg(n)}{dn} < 0$.

Similarly, for $n \in [n(k,m), 2^k - 2^{k-m-1}]$, $n - d$ is constant by Lemma 7b), and so
\[
\frac{dg(n)}{dn} = -1 + \log_2 \left(\frac{n}{d}\right) > 0.
\]

Note: The weight distribution of $S_{n(k,m),k}$ was given in Table II. In particular, $A_2 = 2^m$ for $n = n(k,m)$. Moreover, $A_d \geq 2^m$ for all $n \in [2^k - 2^{k-m}, 2^k - 2^{k-m-1}]$. Hence, (21) is satisfied for some such $n$ if and only if it is satisfied for $n = n(k,m)$.

For $n = n(k,m) = 2^{k-m-2}(2m^2 + 3)$ we have, from Table II that
\[
d = 2^{k-m-1}(2m^2 - 3),
\]
and so
\[
\frac{d}{n} = \frac{2m^2 - 2}{2m^2 + 3}.
\]

For a fixed $m$, let
\[
G(k) = k - m - 2^{k-m-2} U_m \tag{22}
\]
where we consider $k$ a real variable, and where
\[
U_m = (2m^2 - 3)\left\{1 - h\left(\frac{2m^2 - 2}{2m^2 + 3}\right)\right\}. \tag{23}
\]
Then (21), for $n = n(k, m)$, can be rewritten as

$$G(k) < 0. \quad (24)$$

We get

$$G'(k) = 1 - 2^{k-m-2} U_m \ln 2, \quad (25)$$

$$G''(k) = -2^{k-m-2} U_m (\ln 2)^2. \quad (26)$$

To analyze $G(k)$ further, we first give some relations for $h(x)$.

**Lemma 11:** For $0 < x < 1/2$, we have

$$h(x) < 1 - \frac{(1 - 2x)^2}{2 \ln 2} \quad (27)$$

and

$$h(x) > 1 - \frac{(1 - 2x)^2}{2 \ln 2} - \frac{1}{\ln 2} \sum_{i=1}^{\infty} \frac{(1 - 2x)^{2i}}{2i(2i - 1)}. \quad (28)$$

**Proof:** Using Taylor’s theorem, we get

$$h(x) = 1 - \frac{1}{\ln 2} \sum_{i=1}^{\infty} \frac{(1 - 2x)^{2i}}{2i(2i - 1)}, \quad (29)$$

for $0 \leq x \leq 1$.

The upper bound (27) follows immediately since

$$h(x) = 1 - \frac{(1 - 2x)^2}{2 \ln 2} - \frac{1}{\ln 2} \sum_{i=1}^{\infty} \frac{(1 - 2x)^{2i}}{2i(2i - 1)}.$$

Next, from (29), we get

$$0 = h(0) = 1 - \frac{1}{\ln 2} \sum_{i=1}^{\infty} \frac{1}{2i(2i - 1)}.$$

Since $(1 - 2x)^{2i} \leq (1 - 2x)^4$ for $i \geq 2$ and $x \in (0, 1/2)$, we get

$$h(x) > 1 - \frac{(1 - 2x)^2}{2 \ln 2} - \frac{1}{\ln 2} \sum_{i=2}^{\infty} \frac{(1 - 2x)^4}{2i(2i - 1)}$$

$$= 1 - \frac{(1 - 2x)^2}{2 \ln 2} - \frac{1}{\ln 2} \sum_{i=2}^{\infty} \frac{1}{2i(2i - 1)}$$

$$= 1 - \frac{(1 - 2x)^2}{2 \ln 2} - \frac{(1 - 2x)^4}{\ln 2} \left( \ln 2 - \frac{1}{2} \right). \quad (30)$$

**Lemma 12:** For $m \geq 1$, we have

$$2^{m+3} U_m \ln 2 = 1 + u_m,$$

where

$$\frac{3}{2^{m+2} - 3} < u_m < \frac{3}{2^{m+2} - 3} + \frac{(2 \ln 2 - 1)2^{m+2}}{(2^{m+2} - 3)^3}. \quad (31)$$

**Proof:** We have

$$1 - \frac{2^{m+1} - 2}{2^{m+2} - 3} = \frac{1}{2^{m+2} - 3}.$$

Hence, from (25) and (27) we get

$$2^{m+3} U_m \ln 2 > 2^{m+3} \ln 2 (2^{m+2} - 3) \frac{1}{2 \ln 2} \left( \frac{1}{2^{m+2} - 3} \right)^2$$

$$= 1 + \frac{3}{2^{m+2} - 3}.$$
TABLE III

VALUES OF $n$ FOR WHICH (21) IS SATISFIED, AND THEREFORE, BOTH $S_{n,k}$ AND $C_{n,n-k}$ ARE UGLY

| $k$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ |
|-----|---------|---------|---------|---------|
| 9   | [315, 324] |         |         |         |
| 10  | [599, 676] |         |         |         |
| 11  | [1140, 1396] |        |         |         |
| 12  | 2219, 2878 | [3280, 3367] |         |         |
| 13  | 4351, 6858 | [6438, 6844] |         |         |
| 14  | 8540, 11878 | [12717, 13888] | [14812, 14833] |         |
| 15  | 16870, 23966 | [25620, 28006] | [28671, 30013] |         |
| 16  | 33486, 48290 | [50834, 56408] | [58305, 58368] | [58370, 60396] |
| 17  | [66546, 66560] | [66595, 97028] | [99602, 113304] | [116102, 121434] |
| 18  | [132560, 194804] | [198432, 227423] | [231354, 231424] | [231451, 243631] |

a) If
\[
(2^c - 1)\mu + 2^c \mu u_m > c + \log_2 \mu,
\]
then
\[
\kappa(m) < \rho(m) + c.
\]

b) If
\[
(2^c - 1)\mu + 2^c \mu u_m < c + \log_2 \mu,
\]
then
\[
\kappa(m) > \rho(m) + c.
\]

**Proof:** Let $k = \rho(m) + c = m + \mu + \log_2 \mu + c$. Since $\mu = m + 5 + \log_2(\ln 2)$, we have
\[
G(k) = k - m - 2^{k-m-2} U_m
= \mu + \log_2 \mu + c - 2^{m+3} \mu \ln 2 U_m.
\]

By Lemma 12
\[
G(k) = \mu + \log_2 \mu + c - 2^c \mu (1 + u_m).
\]
If (32) is satisfied, then $G(k) < 0$ and so $k > \kappa(m)$. This proves a) and the proof of b) is similar.

Let
\[
\bar{\omega}(m) = \rho(m) + \frac{\log_2 \mu}{\mu \ln 2},
\]
\[
\omega(m) = \rho(m) + \frac{\log_2 \mu}{\mu \ln 2} - \frac{(\log_2 \mu)^2}{2 \mu \ln 2}.
\]

**Corollary 2:** If $m \geq 2$, then
\[
\bar{\omega}(m) < \kappa(m) < \omega(m).
\]

**Proof:** First we consider the upper bound. Let $c = \frac{\log_2 \mu}{\mu \ln 2}$. Then
\[
(2^c - 1)\mu = (e^{\ln 2} - 1)\mu = \mu \sum_{i=1}^{\infty} \frac{(\log_2 \mu)^i}{i! \mu^i}
\]
\[
> \log_2 \mu + \frac{(\log_2 \mu)^2}{2 \mu}
\]
\[
> \log_2 \mu + \frac{\log_2 \mu}{\mu \ln 2} = c + \log_2 \mu
\]
for $\log_2 \mu \geq \frac{2}{\ln 2} \approx 2.885$, that is, $\mu \geq 7.389$, i.e. $m \geq 3$. For $m = 2$ we get $(2^c - 1)\mu - (c + \log_2 \mu) \approx 0.0468 > 0$ also. In particular, (32) is satisfied for all $m \geq 2$. The upper bound therefore follows from Lemma 14.

The proof of the lower bound is similar. For $m = 2, 3, 4$ we can show it by direct computation. Some calculus shows that $(2^c - 1)\mu < \log_2 \mu$ for all $m \geq 2$ and $\mu u_m < c$ for $m \geq 5$. We skip the details. The lower bound therefore follows from Lemma 14b).

From Corollary 2 we immediately get the following result.

**Corollary 3:** We have $\kappa(m) - \rho(m) \to 0$ when $m \to \infty$.

**Theorem 6:** For all $m \geq 2$, we have
\[
K(m) = \left\lceil \omega(m) \right\rceil
\]
or
\[
K(m) = \left\lceil \omega(m) \right\rceil + 1.
\]
In particular, if there is no integer between $\omega(m)$ and $\bar{\omega}(m)$, then
\[
K(m) = \left\lceil \bar{\omega}(m) \right\rceil.
\]

**Proof:** Since $\omega(m) < \omega(m) + 1$, (31) and Corollary 2 imply that
\[
\left\lfloor \omega(m) \right\rfloor \leq K(m) = \left\lceil \kappa(m) \right\rceil \leq \left\lceil \bar{\omega}(m) \right\rceil \leq \left\lceil \omega(m) \right\rceil + 1.
\]
Further, if there is no integer between $\omega(m)$ and $\bar{\omega}(m)$, then $\left\lfloor \omega(m) \right\rfloor = \left\lceil \bar{\omega}(m) \right\rceil$.

The difference
\[
\omega(m) - \bar{\omega}(m) = \frac{(\log_2 \mu)^2}{2 \mu^2 \ln 2}
\]
is small, except for small $m$. Hence, to have an integer between $\omega(m)$ and $\bar{\omega}(m)$, $\omega(m)$ must be close to and above an integer. If we denote this integer by $2m + 5 + u$, then $m$ must be close to $\frac{u}{\ln 2} - u - 5$. We will make this statement more precise in the following lemma.

**Lemma 15:** Let $u$ be a positive integer.

a) If
\[
m \leq \frac{2^u}{\ln 2} - u - 5 - \frac{u^2}{2^u},
\]
then $\omega(m) < 2m + 5 + u$.

b) If
\[
m \geq \frac{2^u}{\ln 2} - u - 5 + \frac{u^2}{2^u},
\]
then $\omega(m) > 2m + 5 + u$.

**Proof:** In this proof, we let $m$ be a positive real variable. We note that $\omega(m)$ and $\bar{\omega}(m)$ are still well defined. Moreover,
simple calculus shows that $\frac{d\omega(m)}{dm} > 0$ and $\frac{d\omega(m)}{dm} > 0$ for all $m > 0$. Therefore, a) is equivalent to

$$\omega\left(\frac{2u}{\ln 2} - u - 5 - \frac{y^2}{2u}\right) < 2m + 5 + u$$

and similarly for b).

Proof of a). Let $m = \frac{2^n}{m_2} - u - 5 - \frac{y^2}{2u}$. Then $\mu = \frac{2^n}{m_2} - y$, where

$$y = u - \lambda + \frac{u^2}{2u}$$

and so

$$\log_2 \mu = \log_2 \left(\frac{2^n}{\ln 2}\right) + \log_2 \left(1 - \frac{y \ln 2}{2u}\right) = u - \lambda + \frac{1}{\ln 2} \cdot \ln \left(1 - \frac{y \ln 2}{2u}\right) < u - \lambda - \frac{y}{2u} < u - \lambda.$$

Hence

$$\omega(m) = 2m + 5 + \lambda + \log_2 \mu + \log_2 \left(1 - \frac{y \ln 2}{2u}\right) < 2m + 5 + u - \frac{y}{2u} + \frac{u - \lambda}{2u} = 2m + 5 + u$$

if

$$\frac{y}{2u} \geq \frac{u - \lambda}{2u} - \frac{y \ln 2}{2u}.$$

The inequality (35) is equivalent to

$$\left(u - \lambda + \frac{u^2}{2u}\right) \left(2u - y \ln 2\right) \geq 2^u(u - \lambda)$$

which in turn is equivalent to

$$u^2 \geq \left(u - \lambda + \frac{u^2}{2u}\right)^2 \ln 2.$$

This is satisfied for all $u \geq 6$. For $1 \leq u \leq 5$, we can show a) directly by numerical computation. (This completes the proof of a).

The proof of b) is similar. We give a sketch, leaving out some details. Let $m = \frac{2^n}{m_2} - u - 5 + \frac{y^2}{2u}$. Then $\mu = \frac{2^n}{m_2} - y$, where

$$y = u - \lambda - \frac{u^2}{2u}$$

and so

$$\log_2 \mu = u - \lambda + \frac{1}{\ln 2} \cdot \ln \left(1 - \frac{y \ln 2}{2u}\right) > u - \lambda - \frac{y}{2u} - \frac{y \ln 2}{2u}.$$
66593 ≤ n ≤ 97028, we get \( b_1(17, 1) = 66593 \). For \( n \) in the range \([66561, 66593]\) we have \( A_d = 30 \). Since \( 2^{k-n+n} h(d/n) \) is decreasing with increasing \( n \), \( 21 \) is not satisfied for \( n \) in the range \([66561, 66592]\). However, we see that for \( n = 66560 \), we have a jump in the value of \( A_d \) compared to \( n = 66561 \), and \( 21 \) is again satisfied for all \( n \) in the range \( 66546 ≤ n ≤ 66560 \). For \( n = 66545 \), \( 21 \) is again not satisfied.

Example 5: An interesting example occurs when \( k = 16 \) and \( m = 3 \); \( n = 58369 = b_1(16, 3) - 1 \) is an isolated value of \( n \) for which \( 21 \) is not satisfied. We have

\[
n = 2^{16} - 2^{13} + 2^{10} + 1 \quad \text{and} \quad d = 2^{15} - 2^{12}.
\]

We have \( A_d = 8 \) and

\[
A_d (d/n)^d (1 - d/n)^{-d} \approx 0.989
\]

and so \( 21 \) is not satisfied. However, \( A_{d+1} = 16 \) and

\[
A_{d+1} (d/n)^{d+1} (1 - d/n)^{-d-1} \approx 1.9106,
\]

so the contribution from this term alone is sufficient to conclude that \( S_{n,k} \) is not satisfactory after all. This shows that if \( 21 \) is not satisfied, but the second lowest weight of \( S_{n,k} \) is close the minimum weight \( d \), it may be a good idea to consider the contribution from this weight also.

VI. APPROXIMATE ANALYSIS

In Table III we see that for a given \( m \), an increasing fraction of the codes are ugly when \( k \) increases. Define \( \beta_1(k, m) \) and \( \beta_2(k, m) \) by

\[
b_1(k, m) = 2^k - 2^{k-m} + \beta_1(k, m)
\]

and

\[
b_2(k, m) = 2^k - 2^{k-m} + \beta_2(k, m).
\]

Clearly, \( 0 < \beta_1(k, m) < \beta_2(k, m) < 2^{k-m-1} \). Let

\[
\gamma_1(k, m) = (k-m) \ln 2 + \sqrt{(k-m)^2 \ln 2^2 + 2(k-m)(2^k - 2^{k-m}) \ln 2}.
\]

Theorem 7: We have

\[
\beta_1(k, m) \leq \lceil \gamma_1(k, m) \rceil.
\]

Proof: Let \( n = 2^k - 2^{k-m} + \sigma \), where \( 0 ≤ \sigma ≤ 2^{k-m-2} \).

By Lemma 7, \( d = 2^k - 2^{k-m+1} \), and by Lemma 8 \( A_d ≥ 2^m \). By (21) we see that if

\[
m > k - n + n h(d/n),
\]

then \( A_d ≥ 2^m > 2^{k-n+n h(d/n)} \), and so \( S_{n,k} \) is ugly. We have

\[
1 - 2 \frac{d}{n} = \frac{n - 2d}{n} = \frac{\sigma}{n}.
\]

By (27),

\[
h\left(\frac{d}{n}\right) < 1 - \frac{1}{2 \ln 2} \frac{\sigma^2}{n^2}.
\]

Hence

\[
k - n + n h\left(\frac{d}{n}\right) < k - \frac{\sigma^2}{2 n \ln 2} = k - \frac{\sigma^2}{2 (2^k - 2^{k-m} + \sigma) \ln 2},
\]

By (37), if

\[
m ≥ k - \frac{\sigma^2}{2 (2^k - 2^{k-m} + \sigma) \ln 2},
\]

then \( S_{n,k} \) is ugly. Since (38) is equivalent to \( \sigma ≥ \gamma_1(k, m) \), and \( \sigma \) is an integer, that is \( \sigma ≥ \lceil \gamma_1(k, m) \rceil \), we can conclude that \( \beta_1(k, m) ≤ \lceil \gamma_1(k, m) \rceil \). This proves the theorem. ■

Remark. We see that \( \beta_1(k, m) - 1 \) is an upper bound on the number of \( n \) in \([2^k - 2^{k-m+1}, 2^k - 2^{k-m} + 2^{k-m-2}]\) such that \( S_{n,k} \) is satisfactory. Therefore, a main corollary of Theorem 7 is that, for any fixed \( m \), \( \beta_1(k, m) / 2^{k-m-2} \) converges to 0 exponentially fast when \( k \) increases.

For \( 2^k - 2^{k-m} + 2^{k-m-2} < n < 2^k - 2^{k-m-1} \), we have a similar result. Let

\[
\gamma_2(k, m) = -(k-m) \ln 2 + \sqrt{(k-m)^2 \ln 2^2 + 2(k-m)(2^k - 2^{k-m-1}) \ln 2}.
\]

Theorem 8: We have

\[
\beta_2(k, m) ≥ 2^{k-m-1} - \lceil \gamma_2(k, m) \rceil.
\]

Proof: The proof is similar to the proof of Theorem 7.
where \(0 < \zeta < 2^{k-m-2}\). From Lemma 7 we get
\[
d = 2^{k-1} - 2^{k-m-2} - \zeta,
\]
and so
\[
1 - 2 \frac{d}{n} = \frac{n - 2d}{n} = \frac{\zeta}{n}.
\]
Therefore, analogously to (38) we get
\[
m \geq k - \frac{\zeta^2}{2(2^k - 2^{k-m-1} - \zeta) \ln 2}
\]
is a sufficient condition for \(S_{n,k}\) to be ugly. Since this is equivalent to \(\zeta \geq \lceil \gamma_2(k,m) \rceil\), Theorem 8 follows. ■

**Theorem 9:** Let \(N_k\) be the number of \(n \in [2^{k-1}, 2^k - 1]\) such that \(S_{n,k}\) is satisfactory. Then
\[
N_k < k + \frac{2(k+5)/2}{3} \sqrt{k^3 \ln 2}.
\]
**Proof:** The number \(N_{k,m}\) of satisfactory codes for \(n\) in the interval \([2^k - 2^{k-m} - 1, 2^k - 2^{k-m-1}]\) is at most those for \(n \in [2^k - 2^{k-m} + 1, b_1(k,m) - 1]\) plus those for \(n \in [b_2(k,m) + 1, 2^k - 2^{k-m-1}]\). The number of \(n\) in the first interval is
\[
b_1(k,m) - 1 - (2^k - 2^{k-m}) = \beta_1(k,m) - 1 \leq \lceil \gamma_1(k,m) \rceil - 1 < \gamma_1(k,m).
\]
The number of \(n\) in the second interval is
\[
2^k - 2^{k-m-1} - b_2(k,m) = 2^{k-m} - \beta_2(k,m) \leq \lceil \gamma_2(k,m) \rceil < \gamma_2(k,m) + 1.
\]
We see that, for \(1 \leq m \leq k - 1\), we have
\[
(k - m) \ln 2 + 2(2^k - 2^{k-m}) < (k - m) \ln 2 + 2(2^k - 2^{k-m-1}) < 2^{k+1}.
\]
Hence
\[
N_{k,m} < \gamma_1(k,m) + \gamma_2(k,m) + 1 < 1 + 2 \sqrt{(k - m)2^{k+1} \ln 2},
\]
and so
\[
N_k = 1 + \sum_{m=1}^{k-1} N_{k,m}
< 1 + \sum_{m=1}^{k-1} \left( 1 + 2 \sqrt{(k - m)2^{k+1} \ln 2} \right)
= k + 2 \sum_{m=1}^{k-1} \sqrt{(k - m)2^{k+1} \ln 2}
= k + 2^{(k+3)/2} \ln 2 \sum_{m=1}^{k-1} \sqrt{k - m}
< k + 2^{(k+3)/2} \ln 2 \cdot \frac{2}{3} \sqrt{k}.
\]
**Corollary 4:**
\[
\frac{N_k}{2^{k+1}} \to 0,
\]
when \(k \to \infty\).

Theorem 9 shows that \(C_{n,n-k}\) is ugly for most values of \(n\). On the other hand, \(C_{n,k-n}\) is satisfactory for many values of \(n\) as shown by Corollary 1.

**VII. A GENERALIZATION OF THE CONSTRUCTION**

The matrix \(H_k\) was defined by concatenating \(H^t(k-m)\) for \(m = 1, 2, \ldots, k\). We can generalize this by concatenating \(t_1\) copies of \(H^t(k-1)\), followed by \(t_2\) copies of \(H^t(k-2)\), \(t_3\) copies of \(H^t(k-3)\), etc. for any sequence \((t_1, t_2, \ldots, t_k)\) of positive integers. Most of the previous results carries over, with obvious modifications. For now, we only consider the construction with \(t_1 = 1\) for \(i \geq 2\), and we write \(t_1 = t\). As before, we use the notation \(S_{n,k}\) for the codes generated by the first \(n\) columns of the matrix. For large \(t\), these codes have low rate. The dual codes will have very high rate and minimum distance \(2\).

Consider \(S_{n',k}\) generated by \(H_k(n')\), and let \(S_{n,k}\), where
\[
n = 2^{k-1}(t - 1) + n' \in [2^{k-1}t, 2^{k-1}(t + 1) - 1],
\]
be the code generated by the matrix
\[
H_k(n) = H^{t(k-1)}_k | H^{t(k-2)}_k | \cdots | H^{t(1)}_k | H_k(n').
\]
We see that we get a code \(S_{n,k}\) for each \(n \geq 2^{k-1}\). Also, given \(n\), the values of \(t\) and \(n'\) are uniquely determined by (39).

From its definition, we immediately get the following lemma.

**Lemma 16:** a) The weight of the first row of \(H_k(n)\) is \(2^{k-1}(t - 1)\) larger than the weight of the first row of \(H_k(n')\).

b) For any other non-zero codeword in \(S_{n,k}\), the weight is \(2^{k-2}(t - 1)\) larger than the weight of the corresponding codeword in \(H_k(n')\).

In particular, we see that

- The minimum distance of \(S_{n,k}\) is \(2^{k-2}(t - 1)\) larger than the minimum distance of \(H_k(n')\).
- For a non-zero codeword of \(S_{n,k}\) of weight \(w\), either \(w \geq n/2\) or there is a unique other codeword in the code of weight \(n - w\).

This last property was used to prove Theorem 8. Therefore, this theorem can be directly generalized by a similar proof. Let
\[
\tau_{t,k,m} = \min \left\{ \frac{2^{k-m-2}}{2} + \sqrt{1 + \frac{2^{k+1}(t + 1) - 2^{k-m-2}}{2}} \right\},
\]
\[
\tau_{t,k,m} = \min \left\{ \frac{2^{k-m-2} - 1}{2} + \sqrt{1 + \frac{2^{k+1}(t + 1) - 2^{k-m-1}}{2}} \right\}.
\]
Note that \(\tau_{1,k,m} = \tau_{k,m}\) and \(\tau_{1,k,m} = \tau_{k,m}\).

**Theorem 10:** For \(t \geq 1\) and \(k > m \geq 1\), if
\[
2^{k-1}(t + 1) - 2^{k-m} \leq n \leq 2^{k-1}(t + 1) - 2^{k-m} + \tau_{t,k,m}
\]
or
\[
2^{k-1}(t + 1) - 2^{k-m-1} - \tau_{t,k,m} \leq n \leq 2^{k-1}(t + 1) - 2^{k-m-1},
\]
then \(S_{n,k}\) is proper.

The proof is similar to the proof of Theorem 8 and is omitted.
Theorem 11: a) If \( m \geq 1 \) and
\[
m \geq \left\lfloor \frac{k - 3 - \log_2 t}{2} \right\rfloor,
\]
then \( S_{n,k} \) is proper for all
\[
n \in [2^{k-1}(t+1) - 2^{k-m}, 2^{k-1}(t+1) - 2^{k-m-1}].
\]
b) \( S_{n,k} \) is proper for all
\[
n \in \left[ 2^{k-1}(t+1) - 2^k - k - 3, 2^{k-1}(t+1) - 1 \right].
\]
Proof: Similarly to Theorem \( 5 \) we see that if
\[
x^2 + 6x \leq 2^{k+1}(t+1),
\]
where \( x = 2^{k-m} - 1 \), then \( S_{n,k} \) is proper for all
\[
n \in [2^{k-1}(t+1) - 2^{k-m}, 2^{k-1}(t+1) - 2^{k-m-1}].
\]
We see that if \( 2^{k-2m-3} \geq t+1 \), then \( x^2 \geq 2^{k+1}(t+1) \) and so \( 43 \) is not satisfied. However, if
\[
2^{k-2m-3} \leq t,
\]
then \( x^2 \leq 2^{k+1}t \) and so
\[
x^2 + 6x \leq 2^{k+1}t + 6 \cdot 2^{k-m-1} \leq 2^{k+1}(t+1)
\]
for \( 6 \cdot 2^{m-2} \leq 1 \), that is, all \( m \geq 1 \). Since \( 43 \) is equivalent to \( k - 2m - 3 \leq \log_2 t \), we get the theorem. ■

Theorem 12: For \( k \geq 6 \) there exists an integer \( \theta(k) \leq 2^{k-5} \) such that \( S_{n,k} \) is proper for all \( n \geq 2^{k-1} \theta(k) \).
Proof: For \( k \geq 6 \) and \( t \geq 2^{k-5} \), we get
\[
\frac{k - 3 - \log_2 t}{2} \leq k - 3 - (k - 5) = 1.
\]
By Theorem \( 11b \), \( S_{n,k} \) is proper for all
\[
n \in [2^{k-1}(t+1) - 2^{k-1}, 2^{k-1}(t+1) - 1].
\]
This implies that \( S_{n,k} \) is proper for all \( n \geq 2^{k-1} - 3 \cdot 2^{k-3} + 2 \).

Corollary 5: If \( k \geq 6 \), then \( S_{n,k} \) is proper for all
\[
n \geq 2^{k-6} - 3 \cdot 2^{k-3} + 2.
\]
Proof: By Theorem \( 12 \) \( S_{n,k} \) is proper for all \( n \geq 2^{k-6} - 6 \). Next, Theorem \( 11b \) for \( t = 2^{k-5} - 1 \) shows that \( S_{n,k} \) is proper for
\[
n \in [2^{k-6} - 2^{k-2}, 2^{k-6} - 1].
\]
Finally, Theorem \( 10 \) for \( t = 2^{k-5} - 1, k \geq 6 \), and \( m = 1 \) implies that \( S_{n,k} \) is proper for
\[
n \in [2^{k-6} - 2^{k-2}, \lceil \frac{2^{k-5} - 1}{k} \rceil, 2^{k-6} - 2^{k-2}].
\]
It remains to show that
\[
\lceil \frac{2^{k-5} - 1}{k} \rceil = 2^{k-3} - 2.
\]
We have
\[
\frac{2^{k-5} - 1}{k} = \frac{-1 + \sqrt{1 + 2^{2k-4} - 2^k}}{2}
\]
\[
= \frac{-1 + \sqrt{(2^{k-2} - 2)^2 - 3}}{2}
\]
\[
= \frac{-1 + \sqrt{(2^{k-2} - 3)^2 + 2^{k-1} - 8}}{2},
\]
and so, for \( k \geq 6 \),
\[
-1 + (2^{k-2} - 3) \leq \frac{2^{k-5} - 1}{k} < -1 + (2^{k-2} - 2).
\]
This proves \( 45 \).

Until recently, the best general result of this kind was \( 11 \) Theorem 2.64: If \( k \geq 5 \) and
\[
n \geq (2^{k-1})^2 - 3,
\]
then there exists a proper \([n,k]\) code.

This bound was recently improved in \( 22 \) to the following:
If \( k \geq 5 \) and
\[
n \geq 2^{k-1} (2^{k-5} + 2^{(k-5)/2}),
\]
then there exists a proper (and self complementary) \([n,k]\) code.

Clearly, Theorem \( 12 \) above gives a further improvement and is now the best known such bound.

We can also find lower bounds on \( \theta(k) \). We consider \( S_{n,k} \) for \( n \) in the middle of the interval with \( m = 1 \), that is
\[
n = n(k, 1) + 2^{k-1}(t - 1) = 2^{k-3}(4t + 1),
\]
where \( n(k, 1) \) was defined in \( 14 \). When we consider only the term in \( S_{n,k} \) of lowest degree, we know that the case \( n = n(k, 1) + 2^{k-1}(t - 1) \) is the worst case (cfr. Lemma \( 10 \)). Moreover, this term of lowest degree is the dominating one in \( S_{n,k} \). Therefore, it is reasonable to consider these values of \( n \) when we look for non-proper \( S_{n,k} \).

Theorem 13: For \( k \geq 6 \) we have
a) \( \theta(k) \geq \theta_1(k) = \min \{ t | S_{2^{k-3}(4t+1),k} \text{ is proper} \} \).

b) \( \theta(k) \geq \theta_2(k) = \left\lceil \frac{2^{k-6}}{(k-1) \ln 2 - \frac{1}{4}} \right\rceil \).

Proof: Proof a). We see that if \( S_{2^{k-3}(4t+1),k} \) is not proper, then by the definition of \( \theta(k) \), \( \theta(k) > t \). Therefore, \( 46 \) follows.

Proof b). Again, consider \( S_{n,k} \) for \( n = 2^{k-3}(4t + 1) \). Then, by Table \( 22 \) and Lemma \( 16 \)
\[
d = 2^{k-2} + 2^{k-1} - 2^{k-2} t.
\]
Since \( A_d = 2 \), \( 21 \) implies that the code is ugly if
\[
1 > k - n + nh \left( \frac{d}{n} \right).
\]
We have
\[
\frac{d}{n} = \frac{2t}{4t + 1} \quad \text{and} \quad 1 - 2 \frac{d}{n} = \frac{1}{4t + 1},
\]
and so, by \( 27 \),
\[
-n + nh \left( \frac{d}{n} \right) < n - \left( \frac{1}{2 \ln 2} \right)^2 = - \frac{2^{k-4}}{(4t + 1) \ln 2}.
\]
Hence, if
\[
k \geq k - \frac{2^{k-4}}{(4t + 1) \ln 2},
\]
that is

\[ t \leq \frac{2^{k-6}}{(k-1) \ln 2} - \frac{1}{4}, \]

then \( S_{n,k} \) is ugly. Therefore, \( \theta(k) > t \), and the theorem follows.

We now give a lemma that is useful for studying when \( S_{n,k} \) codes in general are proper for a given \( k \).

**Lemma 17:** Let \( n \geq 2^{k-1} + 1 \) and let \( w \) be the weight of the first row of \( H_k(n) \). If \( P_{\text{ue}}(S_{n,k},p) - p^w(1-p)^{n-w} \) is increasing on \([0,1/2] \), then \( S_{n+2^k-1},u,k \) is proper for all integers \( u \geq 0 \).

**Proof:** The weight of the first row of \( H_k(n+2^{k-1}u) \) is \( 2^{k-1}u + w \). The weight of any other non-zero codeword in \( S_{n+2^k-1},u,k \) is \( 2^{k-2}u \) larger than the corresponding codeword in \( S_{n,k} \). Hence

\[
P_{\text{ue}}(S_{n+2^k-1},u,k,p) = p^{2^{k-1}u+w}(1-p)^{n-w} + p^{2^{k-2}u}(1-p)^{2^{k-2}u}\left\{ P_{\text{ue}}(S_{n,k},p) - p^w(1-p)^{n-w} \right\}.
\]

By assumption, \( P_{\text{ue}}(S_{n,k},p) - p^w(1-p)^{n-w} \) is increasing on \([0,1/2] \). Since \( p^{2^{k-2}u}(1-p)^{2^{k-2}u} \) and \( p^w(1-p)^{n-w} \) are increasing on \([0,1/2] \), we can conclude that \( P_{\text{ue}}(S_{n+2^k-1},u,k,p) \) is increasing on \([0,1/2] \), that is, \( S_{n+2^k-1},u,k \) is proper. 

For the use of this lemma, it is useful to observe that the conclusion of Theorem 10 can be improved: if \( 40 \) or \( 41 \) hold, then \( P_{\text{ue}}(S_{n,k},p) - p^w(1-p)^{n-w} \) is increasing on \([0,1/2] \). The proof carries over immediately.

Using Lemma 17 and computations, we have determined \( \theta(k) \) for \( 6 \leq k \leq 17 \). These values are given in Table VI together with the lower and upper bounds on \( \theta(k) \) in Theorem 13. We have included the bounds for \( 18 \leq k \leq 20 \).

We see that the upper bound is very loose, but \( \theta(k) \) equals the implicit lower bound \( \theta_2(k) \) for all \( k \leq 17 \). We conjecture that this may be the case for all \( k \). Table VI shows that the explicit lower bound \( \theta_2(k) \) is also loose (but substantially better than the upper bound) and the ratio \( \theta_1(k)/\theta_2(k) \) is increasing slowly with \( k \). For \( k = 13 \) the ratio is 1.375, for \( k = 16 \) it is 1.485, and for \( k = 20 \) it is 1.555. Theorem 14 below shows that the ratio is always less than 2. The lower bound \( \theta_2(k) \) has the advantage that it is explicit and it shows that \( \theta(k) \) grows exponentially with \( k \). In Appendix 1 we prove the following theorem.

**Theorem 14:** Let \( k \geq 6 \) and \( R = 2^{k-4} \). Let \( \vartheta = \vartheta(k) \) be the positive real number defined by

\[
2(4R-2)(4\vartheta+1)\left(R+\sqrt{(R-1)^2-8R\vartheta}\right)^\pi R = 1, \quad (49)
\]

where

\[
\pi = \frac{8R\vartheta+R-1-\sqrt{(R-1)^2-8R\vartheta}}{8R\vartheta+3R-1+\sqrt{(R-1)^2-8R\vartheta}} \quad (50)
\]

Then

\[ a) \quad \theta_1(k) \geq \lceil \vartheta(k) \rceil, \]
\[ b) \quad \vartheta(k) \leq \frac{2^{k-5}}{(k-2) \ln 2 + \ln(k-3) - 1/(2^{k-3} - 1)} + \frac{1}{2}, \]
\[ c) \quad \theta_1(k) > \theta(k) - \frac{k-2}{k-3} \left(4\vartheta(k) + 1\right)2^{-k}. \]

Combining Theorem 14a) and Theorem 14b), we get the following corollary.

**Corollary 6:** We have

\[ \theta_1(k) = \lceil \vartheta(k) \rceil \text{ or } \theta_1(k) = \lceil \vartheta(k) \rceil - 1. \]

Moreover, the first alternative is the most likely.

We have included \( \lceil \vartheta(k) \rceil \) in Table VI. For the range of values we have computed, i.e. \( k \leq 20 \), we have \( \theta_1(k) = \lceil \vartheta(k) \rceil \).

If the conjecture that \( \theta(k) = \theta_1(k) \) is true, then \( \lceil \vartheta(k) \rceil \) is a sharp upper bound on \( \theta(k) \). Further, if the conjecture is true, then \( S_{n,k} \) is proper for all \( n \geq 2^{k-6}/(k \ln 2) \), a substantially stronger result than what we have been able to show in Corollary 5.

When \( k \geq 6 \), let \( \Phi_k \) be the number of \( n \in [2^{k-1}+1, 2^{k-6}-1] \) such that \( S_{n,k} \) is proper.

We have shown by direct computation and the use of Lemma 17 that for \( 6 \leq k \leq 8 \), \( S_{n,k} \) is proper for all \( n \in [2^{k-1}+1, 2 \cdot 2^{k-1}] \). Hence, \( \Phi_k = 2^{k-6} - 2^{k-1} + 1 \) for \( 6 \leq k \leq 8 \).

For \( 9 \leq k \leq 12 \) we know that there are values of \( n \) where \( S_{n,k} \) is not proper. For given \( t, k, m \), let \( X_{t,k,m} \) be the set of \( n \in [2^{k-1}(t+1)-2^{k-m}, 2^{k-1}(t+1)-2^{k-m}] \) for which \( S_{n,k} \) is not proper. The set \( X_{t,k,m} \) may be empty. In particular, Theorem 11b) shows that \( X_{t,k,m} = \emptyset \) if (38) is satisfied. On the other hand, Table V shows that \( X_{1,k,1} \neq \emptyset \) for \( 9 \leq k \leq 18 \). By direct computation and the use of Lemma 17 we have shown that the values given in Table VII are the only values \( n \geq 2^{k-1}+1 \) for which \( S_{n,k} \) is not proper. The computations have been extended up to \( k = 17 \). In general, the values in \( X_{t,k,m} \) are not necessarily consecutive. For example

\[ X_{11,14,1} = [91124, 93142] \cup [93181, 93184]. \]

This is similar to what we have in Table III and the underlying reason is the same.
have shown that \( S_{n,k} \) is proper for all \( n \geq 2^{2k-6} - 3 \cdot 2^{k-3} + 2 \). Moreover, \( S_{n,k} \) is proper for at least \( 17/21 \) of the shorter lengths. A plausible conjecture \( (\theta(k) = \theta_1(k)) \) implies that \( S_{n,k} \) probably is proper for all \( n \geq 2^{2k-6}/(k \ln 2) \).

An open question for future work is the following: is it the case that if there is an \( n \in [2^{k-1} - 1, 2^k - 1] \) for which \( S_{n,k} \) is not proper, then \( n = 2^{k-3}(4t + 1) \) is such an \( n \)? If the answer is yes (which we believe it is), then this would in particular imply the conjecture referred to above.

Further work may concern searching for modifications of the construction that will extend the range of lengths where the codes are proper or satisfactory. In particular, one line of investigation could be to consider lengths less than \( 2^{k-1} \) by looking at the duals of the best known codes of minimum distance 4.

**APPENDIX I**

**Proof of Theorem 1.4**

Let

\[
P_t(p) = P_{we}(S_{2^k-1+k, k}, p).
\]

The expression for \( P_t(p) \) that was used to determine the bound \( \theta_1(k) \) is easily obtained by combining Table II (with \( m = 1 \)) and Lemma 16 since \( R = 2^{k-4} \), we have:

\[
P_t(p) = 2 p^{ARt}(1 - p)^{ARt+2R} + (16R - 8) [p(1 - p)]^{ARt+R} + 4 p^{ARt+2R}(1 - p)^{ARt} + p^{SRt}(1 - p)^{2R}.
\]

We note that \( P_t(p) \) is well defined for any positive real number \( t \). For large values of \( t \), \( P_t(p) \) is increasing on \([0, 1/2]\). For small values of \( t \), \( P_t(p) \) is first increasing, then decreasing, then again increasing. There is a limiting \( t \) such that, for this \( t \), there is a \( p_0 \) such that \( P_t'(p_0) = 0 \) and \( P_t''(p) > 0 \) for all other \( p \) in \([0, 1/2]\). In particular, this implies that \( P_t''(p_0) = 0 \). In principle, the two equations \( P_t'(p_0) = 0 \) and \( P_t''(p_0) = 0 \) can be used to determine \( p_0 \) and \( t = \theta_0 \). However, the equations are complicated, and we will consider an approximation which is easier to handle. We remark at this point that for \( t \) close to \( R \), \( P_t''(p_0) = 0 \) is not possible.

In this appendix we often drop \( k \) from \( \theta(k) \) and write just \( \theta \) when the value of \( k \) should be clear from the context. Similarly we write \( \overline{\theta} \) for \( \overline{\theta}(k) \), \( \theta_0 \) for \( \theta_0(k) \), etc.

In \( P_t(p) \), the last three terms are increasing on \([0, 1/2]\) whereas the first term is increasing on \([0, d/n]\) and decreasing on \([d/n, 1/2]\). For all \( p \in [0, 1/2] \), the first two terms are dominating. Therefore, we first consider the sum of these two terms:

\[
f_t(p) = 2 p^{ARt}(1 - p)^{ARt+2R} + (16R - 8) [p(1 - p)]^{ARt+R}
\]

and determine the \( \overline{\theta} \) and \( p_1 \) such that \( f_0'(p_1) = f_0''(p_1) = 0 \). We expect \( \overline{\theta} \) to be a good approximation to \( \theta_0 \) (and \( p_1 \) to be a good approximation to \( p_0 \)).

The remaining two terms in \( P_0(p) \) are much smaller. Moreover, they are increasing on \([0, 1/2]\). Therefore, \( \overline{\theta} \geq \theta_0 \). In particular, \( \theta_1 = [\theta_0] \leq \overline{\theta} \). This proves Theorem 1.4.
Since Combining (53) and (54), we get where Similarly, where property of $p$ solution will be closer to This reflects the fact that Solving this for $f$ we see that We have $(49)$ occurs because we have neglected the two smallest terms in $\Delta_{R,t}$ as explained above. Therefore, it is not relevant for our analysis of $P_\vartheta(p)$. Since $f'_\vartheta(p_1) = 0$, $(52)$ implies $[2\vartheta - (4\vartheta + 1)p_1](1 - p_1)^R + (4R - 2)(4\vartheta + 1)(1 - 2p_1)p_1^R = 0$. (55) Substituting the value of $p_1$ into $(55)$ and simplifying, we get (49).

We cannot find a closed expression for $\vartheta$, but for a given $k$, we can determine the value numerically. We note, however, that $\lfloor \vartheta \rfloor$ (which is the quantity we want) actually is the least integer $t$ such that

$$\frac{2(4R - 2)(4t + 1)(R + \Delta)}{4tR + R - 1 - (4t + 1)\Delta} \frac{8Rt + R - 1 - \Delta}{8Rt + 3R - 1 + \Delta} R \geq 1,$$

where $\Delta = \Delta_{R,t}$ and this observation simplifies the numeric determination of $\lfloor \vartheta \rfloor$ since we do not have to solve the equation (49), but only search for $\lfloor \vartheta \rfloor$.

To prove Theorem 14b), we first give a couple of lemmas.

Lemma 18: If $0 < 8Rt \leq (R - 1)^2$, then

$$\frac{8Rt + R - 1 - \Delta}{8Rt + 3R - 1 + \Delta} > 1 - \frac{1}{2t}.$$

Proof: The function is decreasing when $x$ is increasing. We have

$$\Delta^2 = (R - 1)^2 - 8Rt = (R - 4t)^2 - 2R - 16t^2 + 1 < (R - 4t)^2$$

and so $\Delta < R - 4t$. Hence,

$$\frac{8Rt + R - 1 - \Delta}{8Rt + 3R - 1 + \Delta} > \frac{8Rt + R - 1 - (R - 4t)}{8Rt + 3R - 1 + (R - 4t)}$$

$$= 1 - \frac{1}{2t} + \frac{4t(4t - 1) + AR - 1}{2t[(8R - 4)t + 4R - 1]} > 1 - \frac{1}{2t}.$$

Lemma 19: If $0 < 8Rt \leq (R - 1)^2$, then

$$\frac{2(4R - 2)(4t + 1)(R + \Delta)}{4tR + R - 1 - (4t + 1)\Delta} > 8R - 4.$$

Proof: We have

$$\frac{(4t + 1)(R + \Delta)}{4tR + R - 1 - (4t + 1)\Delta} > \frac{(4t + 1)(R + \Delta)}{4tR + R - (4t + 1)\Delta}$$

$$= \frac{R + \Delta}{R - \Delta} \geq 1.$$

From Lemmas 18 and 19 we see that if

$$8R \left(1 - \frac{1}{2R}\right) \left(1 - \frac{1}{2t}\right)^R \geq 1,$$

then (56) is satisfied and so $t \geq \vartheta$. Taking logarithms of (58), we get the equivalent expression

$$\ln(8R) + \ln \left(1 - \frac{1}{2R}\right) + R \ln \left(1 - \frac{1}{2t}\right) \geq 0.$$

Since $\ln(1 - x) > -x/(1 - x)$ for $x \in (0, 1)$, we see that if

$$\ln(8R) - \frac{1}{2R - 1} - \frac{R}{2t - 1} = 0,$$
then \[58\] is satisfied, and so \(t \geq \vartheta\).

Since \(R = 2^{k-4}\), we have \(\ln(8R) = (k - 1) \ln 2\).

Solving \[59\] for \(t\), we get the following relation:

\[
\vartheta \leq \frac{R}{2(k-1) \ln 2 - 2/(2R - 1)} + \frac{1}{2} \tag{60}
\]

In the proof of Lemma \[19\] we used that \(\Delta \geq 0\). However, using \[60\] we get a better bound on \(\Delta\) and hence a stronger version of Lemma \[19\] and a better bound on \(\vartheta\).

Lemma 20: If

\[
0 < t \leq \frac{R}{2(k-1) \ln 2 - 2/(2R - 1)} + \frac{1}{2} \tag{61}
\]

then

\[
\frac{2(4R - 2)(4t + 1)(R + \Delta)}{4tR + R - 1 - (4t + 1)\Delta} > (4R - 2)(k - 3).
\]

Proof: By \[61\]

\[
\Delta_{R,t}^2 = (R - 1)^2 - 8Rt
\]

\[
\geq (R - 1)^2 - 8R \cdot \frac{R}{2(k-1) \ln 2 - 2/(2R - 1)} - \frac{8R}{2}
\]

\[
> R^2 \left(1 - \frac{4}{k-1}\right)^2
\]

(62)

for \(k \geq 11\). Hence,

\[
\frac{R + \Delta}{R - \Delta} \geq \frac{R + R \left(1 - \frac{1}{k-1}\right)}{R - R \left(1 - \frac{1}{k-1}\right)} = \frac{k - 3}{2}.
\]

Therefore, if

\[
4R \left(1 - \frac{1}{2R}\right)(k - 3) \left(1 - \frac{1}{2t}\right) \geq 1,
\]

then \(t \geq \vartheta\). Taking logarithms and solving as above, we get Theorem \[14\b\] exactly as we obtained \[60\] from \[58\].

For \(6 \leq k \leq 10\), we can show that Theorem \[14\b\] is true by direct computation.

To prove Theorem \[14\c\], we first give another lemma.

Lemma 21:

a) For \(k \geq 6\) we have \(\pi < 1 - \frac{k - 3}{2(k-1)\vartheta(k) + (k - 2)}\).

b) For \(k \geq 11\) we have \(\pi^R < 2^{-k}\).

Proof: Using \[62\], we get

\[
\pi = \frac{8R\vartheta + R - 1 - \Delta_{R,\vartheta}}{8R\vartheta + 3R - 1 + \Delta_{R,\vartheta}}
\]

\[
< \frac{8R\vartheta + R - \Delta_{R,\vartheta}}{8R\vartheta + 3R + \Delta_{R,\vartheta}}
\]

\[
< \frac{8R\vartheta + R - [1 - 4/(k-1)]}{8R\vartheta + 3R + [1 - 4/(k-1)]}
\]

\[
= 1 - \frac{k - 3}{2(k-1)\vartheta(k) + (k - 2)}.
\]

This is Lemma \[21\b\]. Moreover, this implies that

\[
\pi^R < e^{-\frac{R(k-3)}{2(k-1)\vartheta(k) + (k - 2)}}.
\]

By Theorem \[14\b\]

\[
R(k-3)
\]

\[
\frac{2(k-1)\vartheta + k - 2}{2(k-1)\vartheta + k - 2}
\]

\[
> \frac{R(k-3)}{(k-2)\ln 2 + \ln(k-3) - 1/(2R - 1) + 2k - 3}
\]

\[
> k \ln 2
\]

for \(k \geq 23\), and so \(\pi^R < e^{-k \ln 2} = 2^{-k}\). Direct computations show that \(\pi^R < 2^{-k}\) also for \(11 \leq k \leq 22\). Hence Lemma \[21\b\] is proved.

To prove Theorem \[14\c\], let \(\varepsilon > 0\) and \(t > \varepsilon\). We get

\[
f'_{t-\varepsilon}(p) = 8R(t - \varepsilon) p^{4R(t-\varepsilon)-1}(1-p)^{4R(t-\varepsilon)+2R-1}
\]

\[
- [8R(t - \varepsilon) + 4R] p^{4R(t-\varepsilon)} (1-p)^{4R(t-\varepsilon)+2R-1} + (16R - 8)[4R(t - \varepsilon) + R]
\]

\[
\cdot [p(1-p)]^{4R(t-\varepsilon)+R-1}(1 - 2p)
\]

\[
= f'_t(p) - 8R \varepsilon p^{4R(t-\varepsilon)-1}(1-p)^{4R(t-\varepsilon)+2R-1}
\]

\[
+ 8R \varepsilon p^{4R(t-\varepsilon)} (1-p)^{4R(t-\varepsilon)+2R-1} - 4R\varepsilon(16R - 8)[p(1-p)]^{4R(t-\varepsilon)+R-1}(1 - 2p).
\]

Since \(f'_t(p_1) = 0\) and \((1 - 2p_1)/(1 - p_1) = 1 - \pi\), we get

\[
f'_{\vartheta-\varepsilon}(p_1) = -8R\varepsilon(1-p_1)^{4R(\vartheta-\varepsilon)+2R-1} \pi^{4R(\vartheta-\varepsilon)-1}
\]

\[
\cdot (1 - \pi) \left[1 + (8R - 4)\pi^R\right].
\]

Let

\[
g_t(p) = 4 p^{4Rt+2R}(1-p)^{4Rt} + p^{8Rt} (1-p)^2R.
\]

Then \(P_t(p) = f_t(p) + g_t(p)\). We get

\[
g'_{\vartheta-\varepsilon}(p_1) = (1 - p_1)^{8R(\vartheta-\varepsilon)+2R-1} \pi^{4R(\vartheta-\varepsilon)-1}
\]

\[
\cdot \left\{16R(\vartheta - \varepsilon) + 8R\pi^{2R} - 16R(\vartheta - \varepsilon) \pi^{2R+1}
\]

\[
+ 8R(\vartheta - \varepsilon)\pi^{4R(\vartheta-\varepsilon)-2} - 2R\pi^{4R(\vartheta-\varepsilon)+1}\right\}
\]

\[
< 8R(1-p_1)^{8R(\vartheta-\varepsilon)+2R-1} \pi^{4R(\vartheta-\varepsilon)-1}
\]

\[
\cdot \left\{2\vartheta(1 - \pi) + 1\right\} \pi^{2R} + \vartheta \pi^{4R(\vartheta-\varepsilon)-1}\right\}.
\]

Clearly, \(P'_{\vartheta-\varepsilon}(p_1) = f'_{\vartheta-\varepsilon}(p_1) + g'_{\vartheta-\varepsilon}(p_1) < 0\) if

\[
\left[2\vartheta(1 - \pi) + 1\right] \pi^{2R} \leq \varepsilon(1 - \pi),
\]

and

\[
\vartheta \pi^{4R(\vartheta-\varepsilon)-3R} \leq \varepsilon(1 - \pi)^2R.
\]

Equation \[63\] is equivalent to

\[
\left(2\vartheta + \frac{1}{1 - \pi}\right) \pi^{2R} \leq \varepsilon.
\]

We choose the \(\varepsilon\) which gives equality in \[65\], that is

\[
\varepsilon = \left(2\vartheta + \frac{1}{1 - \pi}\right) \pi^{2R}.
\]

Equation \[64\] is equivalent to

\[
\frac{1}{1 - \pi} \vartheta \pi^{4R(\vartheta-\varepsilon)-3R} \leq \frac{\varepsilon}{\pi^{2R}}(8R - 4).
\]

(67)
By (66),
\[
\frac{1}{1 - \pi} < \frac{\varepsilon}{\pi^2 R}.
\]
Further, by Theorem [14],
\[\theta < (8R - 4),\]
and finally, \(\pi < 1\) and so
\[
\pi^4 R (\theta - \varepsilon) - 3R < 1.
\]
Hence, (64) is also satisfied. Therefore
\[\theta_1 \geq \theta - \varepsilon,\]
where \(\varepsilon\) is given by (66). By Lemma 21
\[
\varepsilon = \left(2\theta + \frac{1}{1 - \pi}\right)\pi^2 R < (2\theta + \frac{2(k - 1)\theta + k - 2}{k - 3})4^{-k} = \frac{k - 2}{k - 3}(4\theta + 1)4^{-k}.
\]
Combining (68) and (69) we get Theorem 14c) for \(\varepsilon\).

Further, by Theorem 14b),
\[
\Phi = \frac{1}{2^{\frac{k - 3}{2}} - 2^{k - m - 3}} - 2^{k - m - 3} - 1.
\]

Now, consider \(1 \leq m \leq \left\lfloor \frac{k - 3}{2} \right\rfloor\) and \(1 \leq t \leq 2^{k - 2m - 3} - 1\). Then \(2^{k - m - 2} \leq 2^{k + 1}\) and so
\[
|\tau_{t, k, m}| > \tau_{t, k, m} - 1 = -\frac{1 + \sqrt{1 + 2^{k + 1}(t + 1) - 2^{k - m + 2}}}{2} \geq -\frac{1 + \sqrt{2^{k + 1}t}}{2}.
\]
Similarly,
\[
|\tau_{t, k, m}| > \frac{-3 + \sqrt{2^{k + 1}t}}{2}.
\]
Combining these two inequalities with (71), a) follows. ■

Lemma 23: For \(k \geq 6\) we have
\[
\sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k - 2m - 3} - 1 \leq \sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k + 1}\sqrt{t}.
\]

Hence
\[
\sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k - 2m - 3} - 1 \leq \sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} \left(2^{k - 3} - 2^{k - 3 - 2\left\lfloor \frac{k - 3}{2} \right\rfloor} + 2^{k - 1 - \left\lfloor \frac{k - 3}{2} \right\rfloor}\right).
\]

Similarly,
\[
\sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k - 2m - 3} - 1 = \left(2^{k - 5} - 2^{k - 3 - 2\left\lfloor \frac{k - 3}{2} \right\rfloor} - \left\lfloor \frac{k - 3}{2} \right\rfloor\right).
\]

The lemma follows from these results and Lemma 22b). ■

Lemma 24: For \(k \geq 6\) we have
\[
\sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k - 5 - 1} \sum_{t=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} \Phi_{t, k, m} = 2^{k - 5} - 2^{k - 5} - 2^{k - 1 - \left\lfloor \frac{k - 3}{2} \right\rfloor} + 1
\]
and
\[
\sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k - 5 - 1} \sum_{t=\left\lfloor \frac{k - 3}{2} \right\rfloor}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} \Phi_{t, k, m} = 2^{k - 5} - 2^{k - 5} - 2^{k - 1 - \left\lfloor \frac{k - 3}{2} \right\rfloor} + 1
\]
and
\[
\sum_{m=1}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} 2^{k - 5 - 1} \sum_{t=\left\lfloor \frac{k - 3}{2} \right\rfloor}^{\left\lfloor \frac{k - 3}{2} \right\rfloor} \Phi_{t, k, m} = 2^{k - 5} - 2^{k - 5} - 2^{k - 1 - \left\lfloor \frac{k - 3}{2} \right\rfloor} + 1
\]
Proof: The result follows directly from Lemmas 22c) and 22b) respectively.

We can now combine these results into a proof of Theorem 15.

Proof: From (70) and Lemmas 22–24, we get

\[
\Phi_k = \sum_{m=1}^{\lfloor k/2 \rfloor - 3} 2^{k-2m-3-1} \sum_{t=1}^{k-1} \Phi_{t,k,m} \\
+ \sum_{m=\lfloor k/2 \rfloor + 1}^{\lfloor k/2 \rfloor - 3} 2^{k-5-1} \sum_{t=1}^{k-1} \Phi_{t,k,m} \\
+ \sum_{m=1}^{\lfloor k/2 \rfloor - 3} \sum_{t=2^{k-2m-3}}^{2^{k-5-1}} \Phi_{t,k,m} \\
> 2^{2k-6-\lfloor k/2 \rfloor} - 2^{k-5} - 2^{k-1-\lfloor k/2 \rfloor} + 1 \\
+ \frac{3}{7} \cdot 2^{2k-6} - 2^{2k-6-\lfloor k/2 \rfloor} + \frac{2^{2k-4-3\lfloor k/2 \rfloor}}{7} \\
+ \frac{8}{21} \cdot 2^{2k-6} - 2^{k-1} - \frac{2^{2k-3-3\lfloor k/2 \rfloor}}{21} + 2^{k-1-\lfloor k/2 \rfloor} \\
- \frac{2^{k-3}}{3} - \frac{2^{k-3-2\lfloor k/2 \rfloor}}{3} + \lfloor \frac{k-3}{2} \rfloor \\
= \frac{17}{21} \cdot 2^{2k-6} - \frac{55}{3} \cdot 2^{k-5} \\
+ \frac{2^{2k-4-3\lfloor k/2 \rfloor}}{21} + \frac{2^{k-3-2\lfloor k/2 \rfloor}}{3} + \lfloor \frac{k-3}{2} \rfloor + 1.
\]

Theorem 15 follows from this expression.

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REFERENCES

[1] T. Kløve, Codes for Error Detection, World Scientific Publishing Company, 2007.
[2] T. Kløve and S. Yari, “Proper self-complementary codes”, Proceedings, 2010 Int. Symposium on Information Theory and its Applications, Tachung, Taiwan, October 17-20, 2010, pp. 118-122.
[3] A. A. Davydov, L. N. Kaplan, Yu. B. Smerkis, and G. L. Tauglikh, “Optimization of shortened Hamming codes”, Problems of Inform. Transm., vol. 17, no. 4, pp. 261–267, Oct.-Dec. 1981 (transl. from Russian).
[4] S. Xia and F. Fu, “Error detection capability of shortened Hamming codes and their dual codes”, Acta Mathematicae Applicatae Sinica, vol. 16 no. 3, pp. 292–298, July 2000.
[5] P. Perry, “Necessary conditions for good error detection”, IEEE Trans. Inform. Theory, vol. 37, no. 2, pp. 375–377, March 1991.