KREIN-TYPE THEOREMS AND ORDERED STRUCTURE
FOR CAUCHY–DE BRANGES SPACES

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ABSTRACT. We extend some results of M.G. Krein to the class of entire functions which
can be represented as ratios of discrete Cauchy transforms in the plane. As an application
we obtain new versions of de Branges’ Ordering Theorem for nearly invariant subspaces
in a class of Hilbert spaces of entire functions. Examples illustrating sharpness of the
obtained results are given.

1. Introduction

1.1. Krein’s theorem. M.G. Krein’s theorem about the Cartwright class functions plays
a seminal role in entire function theory and its applications to spectral theory of linear
operators. Recall that an entire function $F$ is said to be of Cartwright class if it is of finite
exponential type and

$$\int_{\mathbb{R}} \frac{\log^+ |F(x)|}{1 + x^2} dx < \infty.$$ 

Krein’s theorem can be stated as follows (for the necessary background see Section 2):

Theorem (M.G. Krein). Let $F$ be an entire function. If $F$ is a function of bounded
type both in the upper half-plane $\mathbb{C}^+$ and the lower half-plane $\mathbb{C}^-$, then $F$ is a function
of Cartwright class. In particular, $F$ is of finite exponential type and its type is equal to
max($\text{mt}_+(F)$, $\text{mt}_-(F)$), where $\text{mt}_+(F)$ and $\text{mt}_-(F)$ denote the mean type of $F$ in $\mathbb{C}^+$ and
in $\mathbb{C}^-$, respectively.

For different approaches to this result see [10, Part II, Chapter 1] or [16, Lecture 16]; its
applications to the spectral theory of non-dissipative operators can be found, e.g., in [9, Section IV.8].

A typical situation when Krein’s theorem is applicable is when $F$ can be represented
as a ratio of two Cauchy transforms of some discrete measures on $\mathbb{R}$. Namely, for $T =$
\( \{t_n\}_{n=1}^\infty \subset \mathbb{R} \) and \( a = \{a_n\}_{n=1}^\infty \in \ell^1 \) consider the discrete Cauchy transform
\[
\mathcal{C}_a(z) = \sum_n \frac{a_n}{z - t_n}.
\]
Condition \( a \in \ell^1 \) can be relaxed. Assume that \( t_n \neq 0 \) and, for some \( k \in \mathbb{N} \),
\[
\sum_n |t_n|^{-k-1}|a_n| < \infty.
\]
Then we define the regularized Cauchy transform as
\[
(1.1) \quad \mathcal{C}_{a,k}(z) = \sum_n a_n \left( \frac{1}{z - t_n} + \frac{1}{t_n} + \cdots + \frac{z^{k-1}}{t_n^k} \right) = z^k \sum_n \frac{a_n}{t_n(z - t_n)}
\]
(we do not need to regularize the Cauchy kernel at infinity when \( k = 0 \)). The functions of the form \( \mathcal{C}_{a,k} \) are of bounded type in \( \mathbb{C}^+ \) and in \( \mathbb{C}^- \). Therefore, if an entire function \( F \) can be represented as \( \mathcal{C}_{a,k}/\mathcal{C}_{b,m} \), then \( F \) is of finite exponential type.

A special case of the above statement is the following theorem, also due to Krein (see [14, Theorem 4] or [16, Lecture 16]): Assume that \( F \) is an entire function, which is real on \( \mathbb{R} \), with simple real zeros \( t_n \neq 0 \) and such that, for some integer \( k \geq 0 \), we have
\[
\sum_n \frac{1}{|t_n|^{k+1}|F'(t_n)|} < \infty
\]
and
\[
(1.2) \quad \frac{1}{F(z)} = R(z) + \sum_n \frac{1}{F'(t_n)} \cdot \left( \frac{1}{z - t_n} + \frac{1}{t_n} + \cdots + \frac{z^{k-1}}{t_n^k} \right),
\]
where \( R \) is some polynomial. Then \( F \) is of Cartwright class. Krein [14, Theorem 5] showed also that the condition \( t_n \in \mathbb{R} \) can be relaxed to the Blaschke condition \( \sum_n |t_n|^{-2} \text{Im } t_n| < \infty \). Some further refinements of this result are due to A.G. Bakan and V.B. Sherstyukov (see, e.g., [21] and references therein).

One of the goals of this paper is to extend the above results to the case of Cauchy transforms of measures which are supported by some discrete set \( \{t_n\} \) in \( \mathbb{C} \) where \( t_n \) are no longer real. In what follows we assume that \( T = \{t_n\} \subset \mathbb{C} \), \( t_n \) are pairwise distinct, and \( |t_n| \to \infty \) as \( n \to \infty \). To simplify the formulas we assume, as above, that \( 0 \notin T \) (if \( 0 \in T \) then an obvious modification of the formulas is required, but all results remain true).

**Question.** Let \( F \) be an entire function such that \( F = \mathcal{C}_{a,k}/\mathcal{C}_{b,m} \) for some \( a, b, k, m \). Under what conditions on \( T \) can we conclude that \( F \) is a function of finite exponential type?

We find several conditions on \( T \) ensuring that this (and even more) is true. We also provide some examples which show the sharpness of these conditions.
1.2. **The spaces of Cauchy transforms.** One motivation for the study of the above question is related to spectral theory of rank one perturbations of compact normal operators. Recently, D.V. Yakubovich and the second author [4] applied a functional model to the study of rank one perturbations of compact selfadjoint operators. Using this model, a number of results was obtained about completeness and spectral synthesis for such perturbations. The functional model in question acts in the so-called de Branges spaces of entire functions which can be identified with the spaces of Cauchy transforms of discrete measures supported by \( \mathbb{R} \). To extend the results of [4] to the case of rank one perturbations of *normal* (non-selfadjoint) compact operators one needs to have analogues of Krein’s theorems for the case of non-real \( t_n \). A similar functional model for perturbations of normal operators was constructed in [5]; it also acts in some space of discrete Cauchy transforms, which we introduce now.

Let \( T = \{ t_n \}_{n=1}^{\infty} \) be a set as above and let \( \mu = \sum_n \mu_n \delta_{t_n} \) be a positive measure such that \( \sum_n \frac{\mu_n}{|t_n|^2+1} < \infty \). Also let \( A \) be an entire function which has only simple zeros and whose zero set \( \mathcal{Z}_A \) coincides with \( T \). With any such \( T, A \) and \( \mu \) we associate the space \( \mathcal{H}(T, A, \mu) \) of entire functions,

\[
\mathcal{H}(T, A, \mu) := \left\{ f : f(z) = A(z) \sum_n a_n \mu_n^{1/2} \frac{1}{z-t_n}, \ a = \{a_n\} \in \ell^2 \right\}
\]

equipped with the norm \( \|f\|_{\mathcal{H}(T, A, \mu)} := \|a\|_{\ell^2} \). In what follows, the spaces \( \mathcal{H}(T, A, \mu) \) will be called *Cauchy–de Branges spaces*.

The spaces \( \mathcal{H}(T, A, \mu) \) were introduced in full generality by Yu. Belov, T. Mengestie and K. Seip [6]. Essentially, they are spaces of Cauchy transforms. We multiply them by the entire function \( A \) to get rid of poles and make the elements entire, but essentially the space does not depend on the choice of \( A \). The spaces with the same \( T, \mu \) and different \( A \)-s are isomorphic. In what follows we will always assume that \( T \) has a finite convergence exponent (i.e., \( \sum_n |t_n|^{-K} < \infty \) for some \( K > 0 \)) and \( A \) is some canonical product of the corresponding order. We call the pair \( (T, \mu) \) the *spectral data* for \( \mathcal{H}(T, A, \mu) \).

It is noted in [6] that each space \( \mathcal{H}(T, A, \mu) \) is a reproducing kernel Hilbert space and, moreover, if \( \mathcal{H} \) is a reproducing kernel Hilbert space of entire functions such that

(i) \( \mathcal{H} \) has the *division property*, that is, \( \frac{f(z)}{z-w} \in \mathcal{H} \) whenever \( f \in \mathcal{H} \) and \( f(w) = 0 \),

(ii) there exists a Riesz basis of reproducing kernels in \( \mathcal{H} \),

then \( \mathcal{H} = \mathcal{H}(T, A, \mu) \) (as sets with equivalence of norms) for some choice of the parameters.

Note that the functions \( A'(t_n) \mu_n \cdot \frac{A(z)}{z-t_n} \) form an orthogonal basis in \( \mathcal{H}(T, A, \mu) \) and are the reproducing kernels at the points \( t_n \).
In the case when $T \subset \mathbb{R}$ and $A$ is real on $\mathbb{R}$, the space $\mathcal{H}(T, A, \mu)$ is a de Branges space. De Branges spaces’ theory is a deep and important field which has numerous applications to operator theory, spectral theory of differential operators and even to number theory. For the basics of de Branges theory we refer to de Branges’ monograph [7] and to [20]; some further results and applications can be found in [1, 4, 11, 17, 18, 19].

Since the spaces $\mathcal{H}(T, A, \mu)$ are defined in terms of Cauchy transforms and also are a generalization of de Branges spaces, the term a Cauchy–de Branges space seems to be appropriate.

We believe it is a noteworthy problem to extend certain aspects of de Branges theory to the more general setting of Cauchy–de Branges spaces. One of the most striking features of de Branges spaces is the ordered structure of their subspaces which are themselves de Branges spaces: if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two de Branges subspaces of a de Branges space $\mathcal{H}$, then either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$ [7, Theorem 35]. Here we study this problem for Cauchy–de Branges spaces and, as an application of the Krein-type theorems obtained in the first part of the paper, establish the ordering property for a class of these spaces.

1.3. Main results. We will develop the Krein-type theory for measures with non-real supports in the following three cases:

(i) $\mathbf{Z}$: $T$ is the zero set of some entire function of zero exponential type;

(ii) $\mathbf{\Pi}$: $T$ lies in some strip and has finite convergence exponent;

(iii) $\mathbf{A}_\gamma$: $T$ lies in some angle of size $\pi \gamma$, $0 < \gamma < 1$, and the convergence exponent of $T$ is strictly less than $\gamma^{-1}$.

**Theorem 1.1.** Let $T$ satisfy one of the conditions $\mathbf{Z}$, $\mathbf{\Pi}$ or $\mathbf{A}_\gamma$. Assume that $F$ is an entire function such that $F = \frac{C_{a,k}}{C_{b,m}}$, where $C_{a,k}$ and $C_{b,m}$ are regularized Cauchy transforms with poles in $T$ defined in (1.1). Then $F$ is a function of finite exponential type.

In cases $\mathbf{Z}$ and $\mathbf{A}_\gamma$, the function $F$ is of zero exponential type. In cases $\mathbf{\Pi}$ and $\mathbf{A}_\gamma$, $F$ is of Cartwright class with respect to some line.

As a corollary of Theorem 1.1 we see that if a finite order function $F$ with zeros in a strip or a function of order less than $\gamma^{-1}$ with zeros in the angle of size $\pi \gamma$ admits the representation (1.2), then $F$ is a function of exponential type. The first of these observations was proved in [21] where Krein-type theorems for functions with zeros in a strip were studied.

The relation between the size of the angle and the order in the case $\mathbf{A}_\gamma$ is optimal.
Theorem 1.2. For any $\gamma \in (0, 1)$ there exists an entire function $F$ of order precisely $\gamma^{-1}$ such that all its zeros $t_n$ are simple, lie in an angle of size $\pi \gamma$ and

$$\sum_n \frac{1}{|F'(t_n)|} < \infty, \quad \frac{1}{F(z)} = \sum_n \frac{1}{F'(t_n)(z-t_n)}.$$

We use Theorem 1.1 to establish Ordering Theorems for the Cauchy–de Branges spaces $\mathcal{H}(T, A, \mu)$. In fact, we consider a more general and, in a sense, more natural class of subspaces: these are nearly invariant (or division-invariant) subspaces. A closed subspace $\mathcal{H}_0$ of a Cauchy–de Branges space $\mathcal{H}$ is said to be nearly invariant if there is $w_0 \in \mathbb{C}$ such that $\frac{f(z)}{z-w_0} \in \mathcal{H}_0$ whenever $f \in \mathcal{H}_0$ and $f(w_0) = 0$. It is known that this property is equivalent to a stronger division invariance property: for any $w \in \mathbb{C}$ such that there exists $f \in \mathcal{H}_0$ with $f(w) \neq 0$ ($w$ is not a common zero for $\mathcal{H}_0$),

$$f \in \mathcal{H}_0, \quad f(w) = 0 \implies \frac{f(z)}{z-w} \in \mathcal{H}_0.$$

In the context of Hardy spaces in general domains the equivalence of nearly invariance and division invariance is shown in [2, Proposition 5.1]; a similar argument works for general spaces of analytic functions (see Proposition 5.1 below).

While the de Branges theory guarantees a rich structure of de Branges subspaces in a de Branges space, it is not clear whether there always exist many subspaces of $\mathcal{H}(T, A, \mu)$ which have a Riesz basis of reproducing kernels (i.e., are Cauchy–de Branges spaces themselves). However, there exist many nearly invariant subspaces. A natural construction of a nearly invariant subspace is as follows. Given a function $G \in \mathcal{H}(T, A, \mu)$, consider the subspace of $\mathcal{H}$ defined as

$$\mathcal{H}_G = \text{span} \left\{ \frac{G}{z-\lambda} : G(\lambda) = 0 \right\}.$$

We can also define $\mathcal{H}_G$ in a slightly more general situation when $G$ possibly is not in $\mathcal{H}(T, A, \mu)$, but $\frac{G}{z-\lambda} \in \mathcal{H}(T, A, \mu)$ whenever $G(\lambda) = 0$. It is easy to see that if $G$ has simple zeros then $\mathcal{H}_G$ is nearly invariant (and, thus, division-invariant). Clearly, any subspace $\mathcal{H}$ which is itself a Cauchy–de Branges space is of the form (1.3) (indeed, if $\mathcal{H} = \mathcal{H}(T_1, A_1, \mu_1)$, then $\mathcal{H} = \mathcal{H}_{A_1}$). We do not know at present whether any division-invariant subspace of $\mathcal{H}(T, A, \mu)$ is of the form $\mathcal{H}_G$.

We will consider mostly nearly invariant subspaces $\mathcal{H}_0$ without common zeros, that is, such that $\mathcal{Z}(\mathcal{H}_0) = \emptyset$, where $\mathcal{Z}(\mathcal{H}_0) = \{ w \in \mathbb{C} : f(w) = 0 \text{ for any } f \in \mathcal{H}_0 \}$. Note that subspaces of the form $\mathcal{H}_G$ have no common zeros when zeros of $G$ are simple.

Now we formulate two theorems which extend de Branges’ Ordering Theorem to Cauchy–de Branges spaces.
Theorem 1.3. Let $T$ satisfy one of the conditions $Z$ or $A_{\gamma}$ and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two nearly invariant subspaces of $\mathcal{H}(T, A, \mu)$ without common zeros. Then either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$.

To state a similar result for the strip case $\Pi$ we need to impose some conditions. Otherwise the statement is no longer true even in the case of real zeros.

Theorem 1.4. Let $T \subset \{-h \leq \text{Im} z \leq h\}$, $h > 0$, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two nearly invariant subspaces of $\mathcal{H}(T, A, \mu)$ without common zeros. Assume, moreover, that $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed under the $*$-transform $f \mapsto f^*$, where $f^*(z) = \overline{f(\overline{z})}$. Then either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$.

In other words, in the above cases the set of all nearly invariant subspaces without common zeros (and, in particular, the set of all Cauchy–de Branges subspaces) of $\mathcal{H}(T, A, \mu)$ is totally ordered by inclusion. However, without any restrictions on the growth or spectrum localization the ordered structure for nearly invariant subspaces fails. This is illustrated by the following

Theorem 1.5. There exists a space $\mathcal{H}(T, A, \mu)$ of order 2 and two nearly invariant and $*$-closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ without common zeros such that neither $\mathcal{H}_1 \subset \mathcal{H}_2$ nor $\mathcal{H}_2 \subset \mathcal{H}_1$. Moreover, these subspaces can be chosen to be of the form (1.3).

The paper is organized as follows. In Section 2 we discuss our main tools from function theory. Theorem 1.1 is proved in Section 3, while in Section 4 counterexamples are given illustrating its sharpness. Ordering theorems 1.4 and 1.5 are proved in Section 5. Construction of two nearly invariant subspaces which do not contain each other is presented in Section 6.

2. Preliminaries

In what follows we write $U(x) \lesssim V(x)$ if there is a constant $C$ such that $U(x) \leq CV(x)$ holds for all $x$ in the set in question. We write $U(x) \asymp V(x)$ if both $U(x) \lesssim V(x)$ and $V(x) \lesssim U(x)$. The standard Landau notations $O$ and $o$ also will be used.

For the basic notions (such as order and type) of entire function theory see, e.g., [15, 16]. The order of an entire function $f$ will be denoted by $\rho(f)$ and its zero set by $\mathbb{Z}_f$. We denote by $D(z, R)$ the disc with center $z$ of radius $R$. The symbol $m_2$ will denote the area Lebesgue measure in $\mathbb{C}$, while, for a measurable set $E \subset \mathbb{R}$, we denote its one-dimensional Lebesgue measure by $|E|$.
2.1. Functions of bounded type. In this subsection we recall some definitions and basic facts about functions of bounded type.

Denote by \( H^p = H^p(\mathbb{C}^+) \), \( 1 \leq p \leq \infty \), the standard Hardy spaces in the upper half-plane. For the inner-outer factorization and other basic properties of the Hardy spaces see, e.g., [12]. Recall that if \( m \) is a non-negative function on \( \mathbb{R} \) such that \( \log m \in L^1(\mathbb{R}, \frac{dt}{t^2+1}) \), then we can define the outer function \( O_m \) as

\[
O_m(z) = \exp \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) \log m(t) \, dt \right).
\]

We will use the following well-known estimates for outer functions. By a very rough estimate \( \frac{y}{(t-x)^2+y^2} \lesssim (y + \frac{x^2+1}{y})\frac{1}{t^2+1} \) we have for \( z = x + iy = re^{i\theta} \in \mathbb{C}^+, r \geq 1 \),

\[
\| \log |O_m(z)| \| \lesssim \frac{y}{\pi} \int_{\mathbb{R}} |\log m(t)| \frac{1}{(t-x)^2 + y^2} \lesssim y + \frac{x^2+1}{y} \lesssim \frac{r}{\sin \theta}.
\]

In particular, for any \( \delta > 0 \),

\[
(2.2) \quad \| \log |O_m(z)| \| \lesssim |z|, \quad \delta < \arg z < \pi - \delta, \quad |z| \geq 1.
\]

A function \( f \) analytic in \( \mathbb{C}^+ \) is said to be of bounded type, if \( f = g/h \) for some functions \( g, h \in H^\infty \). If, moreover, \( h \) can be taken to be outer, we say that \( f \) is in the Smirnov class \( N_+ = N_+(\mathbb{C}^+) \). Analogously, we can define functions of bounded type and Smirnov class functions in any given half-plane.

If \( f \) is a function of bounded type in \( \mathbb{C}^+ \), it has the canonical factorization \( f = OBS_1/S_2 \), where \( O \) is the outer factor for \( f \), \( B \) is a Blaschke product and \( S_1, S_2 \) are some (mutually prime) singular inner functions. We define the mean type of \( f \) as

\[
\text{mt}(f) = \limsup_{y \to \infty} \frac{\log |f(iy)|}{y}.
\]

The mean type is equal to \( a \) if and only if there is a factor \( e^{-iaz} \) in the canonical factorization of \( f \). If we assume additionally that \( f \) is continuous up to \( \mathbb{R} \) then the singular inner functions can not have singularities on \( \mathbb{R} \) and so \( S_1/S_2 = e^{iaz} \) for some \( a \in \mathbb{R} \). Thus, in this case \( f \in N_+(\mathbb{C}^+) \) if and only if \( \text{mt}(f) \leq 0 \).

Estimate (2.2), a similar estimate for the singular factor and the Hayman theorem [16, Lecture 15] which gives an estimate from below for the Blaschke product outside a union of angles of arbitrarily small total size imply the following estimates:

**Lemma 2.1.** If \( f \) is a function of bounded type in \( \mathbb{C}^+ \) and \( \text{mt}(f) = a \), then, for any \( \varepsilon, \delta > 0 \), there exists \( R > 0 \) such that

\[
\log |f(z)| \leq (a \sin \delta + \varepsilon)|z|, \quad \delta < \arg z < \pi - \delta, \quad |z| \geq R.
\]
More generally, if \( f = O B_1 S_1 e^{iaz} \) where \( O \) is the outer factor, \( B_1 \) and \( B_2 \) Blaschke products, \( S_1, S_2 \) singular inner functions and \( a = \text{mt}(f) \), then for any \( \varepsilon, \delta > 0 \), there exist \( R \) and a set \( E \subset [\delta, \pi - \delta] \) which is a union of intervals of total length less than \( \varepsilon \) such that

\[
(a \sin \delta - \varepsilon)|z| \leq \log |f(z)| \leq (a \sin \delta + \varepsilon)|z|, \quad \arg z \notin E, \quad |z| \geq R.
\]

For the theory of the Cartwright class we refer to [10, 13, 16].

The following lemma will be often useful.

**Lemma 2.2.** Let \( \mathbb{H}_+ \) and \( \mathbb{H}_- \) be two complement half-planes and assume that \( T \subset \mathbb{H}_- \). Then any regularized Cauchy transform \( C_{a,k} \) given by \((1.1)\) is a function from the Smirnov class in \( \mathbb{H}_+ \).

**Proof.** Without loss of generality, let \( \mathbb{H}_+ = \mathbb{C}^+ \). It is well known that if \( f \) is analytic in \( \mathbb{C}^+ \) and \( \text{Im} f > 0 \), then \( f \) is in the Smirnov class \([10, \text{Part 2, Ch. 1, Sect. 5}]\). Thus, if \( u_n > 0 \) and \( \sum_n u_n < \infty \), then the function \( \sum_n \frac{u_n}{t_n - z} \) is in the Smirnov class \( \mathcal{N}_+ \). Consequently, \( \sum_n \frac{u_n}{t_n - z} \in \mathcal{N}_+ \) for any \( \{v_n\} \in \ell^1 \). Finally, \( f(z) = z \) also is in \( \mathcal{N}_+ \) and the result follows immediately from formula \((1.1)\).

\[ \Box \]

### 2.2. Estimates of Cauchy transform in the complex plane

The following two results from [3] about the asymptotic behaviour of Cauchy transforms of measures in the plane will be useful. We say that \( \Omega \subset \mathbb{C} \) is a set of zero area density if

\[
\lim_{R \to \infty} \frac{m_2(\Omega \cap D(0, R))}{R^2} = 0.
\]

**Lemma 2.3.** [3, Proof of Lemma 4.3] Let \( \nu \) be a finite complex Borel measure in \( \mathbb{C} \). Then, for any \( \varepsilon > 0 \), there exists a set \( \Omega \) of zero area density such that

\[
\left| \int_{\mathbb{C}} \frac{\nu(\xi)}{z - \xi} - \frac{\nu(\mathbb{C})}{z} \right| < \frac{\varepsilon}{|z|}, \quad z \in \mathbb{C} \setminus \Omega.
\]

The following result from [3], which is due to A. Borichev, can be considered as an extension of the classical Liouville theorem.

**Theorem 2.4.** [3, Lemma 4.2] If an entire function \( f \) of finite order is bounded on \( \mathbb{C} \setminus \Omega \) for some set \( \Omega \) of zero area density, then \( f \) is a constant.

Next we discuss growth properties of functions in the spaces \( \mathcal{H}(T, A, \mu) \).

**Lemma 2.5.** Let \( A \) be an entire function with the zero set \( T \) and let \( A \) be of order \( \rho \). Then for any \( \varepsilon > 0 \), there exists a set \( E \subset (0, \infty) \) of zero linear density (i.e., \( |E \cap (0, R)| = o(R), R \to \infty \)) such that for any entire function \( f \) of the form \( f = AC_{a,k} \),

\[
|f(z)| \lesssim |z|^{\rho + k + 1 + \varepsilon} |A(z)|, \quad |z| \notin E.
\]
In particular, if \( A \) is of order \( \rho \) and type \( \sigma \), then any element of \( \mathcal{H}(T, A, \mu) \) is of order at most \( \rho \) and of type at most \( \sigma \) with respect to this order.

Proof. Let \( a = (a_n) \) be such that \( \sum_n |t_n|^{-k-1}|a_n| < \infty \). In view of the representation

\[
A(z)C_{a,k}(z) = A(z)P(z) + A(z)z^{k+1} \sum_n \frac{a_n}{t_n^{k+1}(z-t_n)},
\]

where \( P \) is a polynomial of degree at most \( k \), it suffices to prove the statement for \( C_{a,k} \) with \( a \in \ell^1 \).

For a fixed sufficiently large \( n \in \mathbb{N} \) let \( \mathcal{R} = \{ z : 2^n \leq |z| \leq 2^{n+1} \} \) and \( \mathcal{R}' = \{ z : 2^{n-1} \leq |z| \leq 2^{n+2} \} \). Let \( \{t_{n_1}, t_{n_2}, \ldots, t_{n_p}\} = T \cap \mathcal{R}' \). Since \( A \) is of order \( \rho \), we have \( p \lesssim 2^{(\rho+\varepsilon)n} \).

Let \( f = A \sum_n \frac{a_n}{z-t_n} \). Then, for \( z \in \mathcal{R} \),

\[
\left| \sum_{t_n \not\in \mathcal{R}'} \frac{a_n}{z-t_n} \right| \lesssim \sum_{t_n \not\in \mathcal{R}'} \left| \frac{a_n}{t_n} \right| \lesssim 1.
\]

By the classical Cartan’s lemma [15, Chapter 1, §7], there exist discs \( D_j, j = 1, \ldots, p \), of radii \( r_j \) such that \( \sum_{j=1}^p r_j < 2 \) and

\[
\min_{z \in \mathcal{R} \setminus \bigcup_j D_j} \text{dist} (z, T \cap \mathcal{R}') \geq \frac{1}{p}.
\]

Hence, for \( z \in \mathcal{R} \setminus \bigcup_j D_j \), we have

\[
\left| \sum_{t_n \in \mathcal{R}'} \frac{a_n}{z-t_n} \right| \lesssim p \sum_{t_n \in \mathcal{R}'} |a_n| \lesssim p \lesssim |z|^\rho+\varepsilon.
\]

Now we repeat this procedure for any \( n \in \mathbb{N} \) and define \( E \) as the set of \( r \) such that \( \{ |z| = r \} \cap (\bigcup D_j) \neq \emptyset \). Then (2.4) holds for any \( z \) with \( |z| \notin E \). \( \square \)

Note that we have the following criterion for the inclusion of \( f \) into \( \mathcal{H}(T, A, \mu) \).

**Theorem 2.6.** Let \( \mathcal{H}(T, A, \mu) \) be a Cauchy–de Branges space and let \( A \) be of finite order. Then an entire function \( f \) is in \( \mathcal{H}(T, A, \mu) \) if and only if the following three conditions hold:

(i) \( \sum_n |f(t_n)|^2 |A(t_n)|^2 \mu_n < \infty \);

(ii) there exists a set \( E \subset (0, \infty) \) of zero linear density and \( N > 0 \) such that \( |f(z)| \leq |z|^N |A(z)|, |z| \notin E \);

(iii) there exists a set \( \Omega \) of positive area density such that \( |f(z)| = o(|A(z)|), |z| \to \infty, z \in \Omega \).
Proof. The necessity of (i) is obvious since for \( f = A \sum_n \frac{c_n \mu_n^{1/2}}{z-t_n} \in \mathcal{H}(T, A, \mu) \) we have \( f(t_n) = A'(t_n) c_n \mu_n^{1/2} \) and \( \{c_n\} \in l^2 \). The necessity of (ii) is proved in Lemma 2.5. Finally, the representation
\[
\frac{f(z)}{A(z)} = z \sum_n \frac{c_n \mu_n^{1/2}}{t_n(z-t_n)} - \sum_n \frac{c_n \mu_n^{1/2}}{t_n}
\]
and Lemma 2.3 imply that \( f(z)/A(z) = o(1) \) as \( |z| \to \infty \) outside a set of zero density.

To prove the sufficiency consider the function
\[
H(z) = \frac{f(z)}{A(z)} - \sum_n \frac{f(t_n)}{A'(t_n)(z-t_n)}
\]
which is well defined by (i) and entire. Condition (ii) and Lemma 2.5 imply that \( H \) is a polynomial. Finally, note that, by the same argument as above, \( \sum_n \frac{f(t_n)}{A'(t_n)(z-t_n)} \) tends to zero as \( |z| \to \infty \) on some set \( \Omega_1 \) whose complement has zero density. Hence, by (iii), \( |H(z)| \to 0, |z| \to \infty, z \in \Omega \cap \Omega_1 \). Since the set \( \Omega \cap \Omega_1 \) is obviously unbounded, we conclude that \( H \equiv 0 \). Thus, \( f \) has the required representation with \( c_n = \frac{f(t_n)}{A'(t_n)\mu_n^{1/2}} \). \( \square \)

In many cases one can relax the conditions (ii)–(iii) and require the estimates on a smaller set.

Note that, for \( f \in \mathcal{H}(T, A, \mu) \),
\[
\|f\|^2_{\mathcal{H}(T, A, \mu)} = \|\{c_n\}\|^2_{l^2} = \sum_n \frac{|f(t_n)|^2}{|A'(t_n)|^2 \mu_n}.
\]
Thus, the space \( \mathcal{H}(T, A, \mu) \) is isometrically embedded into the space \( L^2(\nu) \), where
\[
(2.5) \quad \nu = \sum_n |A'(t_n)|^{-2} \mu_n^{-1} \delta_{t_n}.
\]

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In what follows we assume that \( F = C_{a,k}/C_{b,m} \) and \( T \) satisfies one of the conditions \( \mathbf{Z}, \Pi, \) or \( \mathbf{A}_\gamma \).

In the Case \( \mathbf{Z} \) the result follows directly from Lemma 2.5. Let \( A \) be a function of zero exponential type with zero set \( T \). Then \( H = AC_{a,k} \) is an entire function and by Lemma 2.5 we have \( |H(z)| \lessgtr |z|^N |A(z)| \) for \( |z| \) outside some small exceptional set. We conclude that \( H \) is of zero exponential type. Hence, if \( F = C_{a,k}/C_{b,m} \), then we can write \( F = H_1/H_2 \) for two functions \( H_1, H_2 \) of minimal type. By the standard estimates of the minimum of modulus for entire functions [16, Chapter 1, §8], \( F \) is of minimal type.
Case II. Without loss of generality, let \( T \subset \{-h \leq \text{Im } z \leq h\} \). Then, by Lemma 2.2, \( F \) is of bounded type in the half-planes \( \mathbb{C}^+ + ih \) and \( \mathbb{C}^- - ih \).

Since \( T \) has a finite convergence exponent, there exists an entire function \( A \) of finite order such that \( Z_A = T \). Then, as above, we can write \( F = H_1/H_2 \) where \( H_1 = AC_{a,k} \), \( H_2 = AC_{b,m} \). By Lemma 2.5, \( \rho(H_j) \leq \rho(A) \) whence \( \rho(F) \leq \rho(A) \). Choose \( \varepsilon \in (0,1) \) such that \( \rho(F) < \pi/(2\varepsilon) \). By Lemma 2.1, there exists \( R > 0 \) such that

\[
\log |F(z)| \lesssim |z|, \quad \arg z \in [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon], \quad |z| > R.
\]

Since \( \rho(F) < \pi/(2\varepsilon) \), we can apply the standard Phragmén–Lindelöf principle to the angles \(-\varepsilon < \arg z < \varepsilon \) and \( \pi - \varepsilon < \arg z < \pi + \varepsilon \) to conclude that \( F \) is of exponential type. Since \( F \) is of bounded type in \( \mathbb{C}^+ih \), we have \( \log |F(t + ih)| \in L^1(\frac{dt}{t^2+1}) \). Therefore \( F(z + ih) \) is of Cartwright class and, finally, \( F \) is of Cartwright class.

Case A. Here we follow essentially the method of de Branges [7, Theorem 11]. Put \( A(\gamma_1, \gamma_2) = \{z : \gamma_1 < \arg z < \gamma_2\} \). Choose \( \gamma' \in (\gamma, 1) \) such that \( \rho(A) \leq 1/\gamma' \). Without loss of generality we can assume that \( T \subset A(\delta, \pi \gamma + \delta) \) where \( \delta \) is so small that \( \pi \gamma + \delta < \pi \gamma' \).

By Lemma 2.2, \( F \) is of bounded type in \( \{ -\pi + \delta < \arg z < \delta \} \) and \( \{ \pi \gamma + \delta < \arg z < \pi \gamma + \delta + \pi \} \). Then, by Lemma 2.1, we have

\[
\log |F(z)| \lesssim |z|, \quad \pi \gamma' \leq \arg z \leq 2\pi + \delta/2.
\]

It remains to estimate \( |F| \) in the angle \( A(\delta/2, \pi \gamma') \).

By Lemma 2.2, \( F \) is of bounded type in \( \mathbb{C}^- \). Then \( \log |F| \in L^1(\frac{dt}{t^2+1}) \). Let \( G \) be an outer function in \( \mathbb{C}^+ \) such that \( |G| = |F|^{-1} \) on \( \mathbb{R} \). Since \( F \) is of bounded type in the half-plane \( \{ \pi \gamma + \delta < \arg z < \pi \gamma + \delta + \pi \} \), we have

\[
\limsup_{r \to \infty} \frac{\log |F(re^{i\gamma'})|}{r} < \infty.
\]

Choose sufficiently large \( h > 0 \) so that \( \widetilde{F} = FGe^{ihz} \) is bounded on the ray \( \{ \arg z = \gamma' \} \). Also, \( \widetilde{F} \) is bounded on \( \mathbb{R} \). Let us show that \( \widetilde{F} \) is bounded in \( A(0, \pi \gamma') \). By Lemma 2.5, \( \rho(F) \leq \rho(A) \). Choose \( \varepsilon > 0 \) such that \( \rho(F) + \varepsilon < \frac{1}{\gamma'} \). Then we have

\[
\log |\widetilde{F}(z)| \lesssim |z| + |z|^\rho(F) + \varepsilon + \log |G(z)|, \quad z \in A(0, \pi \gamma'), \quad |z| \geq 1.
\]

Consider the function \( F_1(z) = \widetilde{F}(z^{\gamma'}) \). Then \( F_1 \) is analytic in \( \mathbb{C}^+ \), continuous up to \( \mathbb{R} \), and bounded on \( \mathbb{R} \). Using the estimate (2.1) we get

\[
\log |F_1(z)| \lesssim r^{\gamma'} + r^{\gamma'(\rho(F)+\varepsilon)} + \frac{r^{\gamma'}}{\sin \gamma' \theta}, \quad z = re^{i\theta} \in \mathbb{C}^+, \quad r \geq 1.
\]
We conclude that
\[
\lim_{r \to \infty} \frac{1}{r} \int_0^\pi \log^+ |F_1(r^i\theta)| \sin \theta \, d\theta = 0.
\]
By the de Branges version of the Phragmén–Lindelöf principle [7, Theorem 1], \( F_1 \) is bounded in \( \mathbb{C}^+ \). Thus, \( \tilde{F} \) is bounded in \( A(0, \pi \gamma') \). Using the fact that \( |\log |G(z)|| \lesssim |z| \), \( z \in A(\delta/2, \pi - \delta/2) \), we conclude that \( \log |F(z)| \lesssim |z| \) for \( z \in A(\delta/2, \pi \gamma') \), \( |z| \geq 1 \). Thus, \( F \) is of finite exponential type.

It remains to show the \( F \) is of zero type. Since \( \log |F| \in L^1\left(\frac{dt}{t^2+1}\right) \), \( F \) is of Cartwright class and so we have \( \lim \inf_{|x| \to \infty} \frac{\log |F(x)|}{|x|} \leq 0 \). Hence, by Lemma 2.1, \( F \) is of non-positive mean type in the half-planes \( \{ \pi \gamma + \delta < \arg z < \pi \gamma + \delta + \pi \} \) and \( \{ \pi + \delta < \arg z < 2\pi + \delta \} \). Therefore, for any \( \varepsilon > 0 \), \( \log |F(z)| \leq \varepsilon |z| \) when \( \pi \gamma' < \arg z < 2\pi + \delta/2 \) and \( |z| \) is sufficiently large. The standard Phragmén–Lindelöf principle now implies that \( F \) is of zero exponential type.

Given \( T \), denote by \( \mathcal{C} \) the class of all regularized Cauchy transforms \( \mathcal{C}_{a,k} \) with poles on \( T \), by \( \mathcal{C}/\mathcal{C} \) the class of functions of the form \( \mathcal{C}_{a,k}/\mathcal{C}_{b,m} \), etc. Then, by the same arguments as above one easily obtains the following result that we will use in what follows:

**Corollary 3.1.** Let \( T \) satisfy one of the conditions \( Z \), \( \Pi \) or \( A_\gamma \). If \( F \in \mathcal{C} + \mathcal{C} \cdot \mathcal{C}/\mathcal{C} \), then the conclusions of Theorem 1.1 hold.

**Remark 3.2.** We do not know whether in the case \( \Pi \) the condition that \( T \) has finite convergence exponent can be omitted.

### 4. Counterexamples to Krein-type theorem

In this section we prove Theorem 1.2. However, we start with a simpler example for the special case of the half-plane. Namely, we show that there exists a function \( F \) with zeros in the lower half-plane which admits the representation (1.2) and \( F \) is of order 1, but of **maximal** (i.e., infinite) type. This shows the sharpness of our results in case \( A_{\gamma} \) in the limit case \( \gamma = 1 \). This example will play an important role in the construction in Theorem 1.5.

**Example 4.1.** There exists an entire function \( F \) of order 1 with simple zeros \( t_n \) in \( \mathbb{C}^- \) such that \( \sum_n |F''(t_n)|^{-1} < \infty \) and

\[
\frac{1}{F(z)} = \sum_n \frac{1}{F''(t_n) (z - t_n)}.
\]

but \( F \) is of maximal type with respect to order 1.
Let \( n_k \) be an increasing sequence such that \( n_{k+1} - n_k \geq 1 \). Put
\[
G(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{e^{2\pi iz}}{e^{2\pi n_k}} \right).
\]
Then \( G \) is an entire function with zeros \( z_{m,k} = m - in_k, m \in \mathbb{Z}, k \in \mathbb{N} \). By simple estimates of lacunary canonical products, for any \( N > 0 \) there exists \( C > 0 \) such that
\[
\prod_{k=1}^{\infty} \left| 1 - \frac{w}{e^{2\pi n_k}} \right| \geq C|w|^N \text{dist}(w, \{e^{2\pi n_k}\}).
\]
Hence, \( |G(z)| \asymp 1, \text{Im} \, z \geq 0, |G(z)| \gtrsim \text{dist}(z, \{m - in_k\}_{m,k}) \) and so \( |G'(m - in_k)| \gtrsim 1 \). Put \( F = PG \) where \( P \) is a polynomial of degree at least 3 whose zeros are not in the set \( \{m - in_k\} \). Then \( \sum_{t_n \in \mathbb{Z}_F} |F'(t_n)|^{-1} < \infty \) since \( \sum_{m,k} |m - in_k|^{-3} < \infty \). Let us show that the entire function
\[
H(z) = \frac{1}{F(z)} - \sum_n \frac{1}{F'(t_n)(z - t_n)}
\]
is identically zero. From the estimates on \( G \) it follows that \( |H(z)| \lesssim 1 \) when \( \text{dist}(z, \{t_n\}) \geq 1/2 \), and \( H(iy) \to 0, y \to \infty \). Hence, \( H \equiv 0 \).

If \( n_k = k \), the function \( F \) is of order 2. However, taking \( n_k \) to be sufficiently sparse (e.g., \( n_k = 2^k \)) we obtain an example of a function of order 1 and maximal type which has expansion (4.1).

Now we pass to the proof of Theorem 1.2. In what follows let \( D = A(0, \pi \gamma) \) be the angle of size \( \pi \gamma \) and \( \Gamma = \partial D \) be its boundary (oriented from \( e^{i\pi \gamma} \infty \) to \( +\infty \)).

**Lemma 4.2.** Let \( g \) be a function analytic in a slightly larger angle \( A(-\varepsilon, \pi \gamma + \varepsilon) \) for some \( \varepsilon > 0 \) and assume that, for some \( C > 0 \), we have
\[
|g(z)| + |g'(z)| \leq \frac{C}{1 + |z|}, \quad \text{dist}(z, \Gamma) \leq (|z| + 1)^{-1}.
\]

Then for the Cauchy integral of \( g \) over \( \Gamma \) we have
\[
\int_{\Gamma} \frac{g(w)}{z - w} \, dw = \frac{1}{z} \int_{\Gamma} g(w) \, dw + o\left(\frac{1}{z}\right), \quad |z| \to \infty.
\]
Proof. We split the integral into three parts:
\[
\int_{\Gamma} \frac{g(w)}{z - w} dw - \frac{1}{z} \int_{\Gamma} g(w) dw = \int_{\{|w|<|z|/2\}} \frac{wg(w)}{z-w} dw + \int_{\{|w-z|<((|z|+1)^{-1}\}} \frac{wg(w)}{z-w} dw \nonumber \]
\[
+ \int_{\{|w|\geq|z|/2,|w-z|\geq(|z|+1)^{-1}\}} \frac{wg(w)}{z-w} dw = I_1 + I_2 + I_3. \nonumber \]

Clearly, \(|I_1| \lesssim |z|^{-2}, \ |z| \geq 2, \) and
\[
|I_3| \lesssim \int_{\{|w| \geq |z|/2\}} |wg(w)| |dw| \lesssim |z|^{-2}. \nonumber \]

Finally, to estimate the integral \(I_2\) for \(z\) close to \(\Gamma\) note that
\[
\left| \int_{\{|w-z|<((|z|+1)^{-1}\}} \frac{g(w)}{w-z} dw \right| \leq \left| \int_{\{|w-z|<((|z|+1)^{-1}\}} \frac{g(z)}{w-z} dw \right| \nonumber \]
\[
+ \left| \int_{\{|w-z|<((|z|+1)^{-1}\}} \frac{g(w)-g(z)}{w-z} dw \right| \lesssim |g(z)| + \max_{|z-\zeta|<((|z|+1)^{-1}\}} |g'(\zeta)| \lesssim |z|^{-4}. \nonumber \]

Combining these inequalities we obtain the estimate of the lemma. \(\square\)

Proof of Theorem 1.2. Let \(D\) and \(\Gamma\) be as above. Let \(f(z) = z^{-4} \sin^4 z\). Then \(f\) is an entire function of finite exponential type. Put \(g(z) = f(z^{1/\gamma}), \ z \in D, \) and define the function \(F\) for \(z \in \mathbb{C} \setminus \overline{D}\) by the contour integral
\[
F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(w)}{z-w} dw, \quad z \in \mathbb{C} \setminus \overline{D} \nonumber \]
(note that \(|g(w)| \lesssim |w|^{-4/\gamma}\) and so there is no problem with convergence).

Now we use a well-known trick to show that \(F\) admits a continuation to an entire function. Let \(R > 0\) and \(\tilde{\Gamma}\) be the contour \(\{te^{i\pi \gamma} : t \geq R\} \cup \{re^{i\theta} : 0 \leq \theta \leq \pi \gamma\} \cup \{t \geq R\}\) (also oriented from \(e^{i\pi \gamma} \infty\) to \(+\infty\) and let \(\tilde{D} \subset D\) be the domain such that \(\partial \tilde{D} = \tilde{\Gamma}\). Put
\[
\tilde{F}(z) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{g(w)}{z-w} dw, \quad z \in \mathbb{C} \setminus \overline{\tilde{D}} \nonumber \]
Then, for \(z \in \mathbb{C} \setminus \overline{D}\),
\[
F(z) - \tilde{F}(z) = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{g(w)}{z-w} dw = 0 \nonumber \]
where \(\Gamma_0\) is the counterclockwise oriented boundary of the sector \(S = \{re^{i\theta} : 0 < r < R, 0 < \theta < \pi \gamma\}\) and the integral is zero since \(g\) is analytic inside \(S\) and continuous up
to the boundary. Thus, $\tilde{F}$ is a continuation of $F$ to a larger domain $\mathbb{C} \setminus \overline{D}$. Since $R$ is arbitrary, we conclude that $F$ has an entire extension. Moreover, by the same argument we have a representation for $F$ inside $D$:

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(w)}{z - w} dw + g(z), \quad z \in D.$$ 

Note that $g$ satisfies the hypotheses of Lemma 4.2. Indeed, $|f(z)| + |f'(z)| \lesssim (|z| + 1)^{-4}$ for $\text{dist}(z, \mathbb{R}) \geq 1$ which implies estimate (4.2). Also, since $f$ is even and non-negative, it follows that $\alpha := (2\pi i)^{-1} \int_{\Gamma} g(w) dw \neq 0$. Thus,

$$(4.3) \quad F(z) = \frac{\alpha}{z} + o(1), \quad z \in \mathbb{C} \setminus D,$$

and we conclude that $F$ has at most finite number of zeros in $\mathbb{C} \setminus D$. Let us analyse the zeros of $F$ inside $D$. We have

$$(4.4) \quad F(z) = g(z) + \frac{\alpha}{z} + o\left(\frac{1}{z}\right), \quad z \in D, \ |z| \to \infty.$$ 

Equivalently, this means that for $G(z) = F(z^\gamma)$ we have

$$G(z) = \frac{\sin^4 z}{z^4} + \frac{\alpha}{z^\gamma} + o\left(\frac{1}{|z|^\gamma}\right), \quad z \in \mathbb{C}^+.$$ 

The unperturbed equation

$$\frac{\sin^4 z}{z^4} + \frac{\alpha}{z^\gamma} = 0$$

has zeros in $\mathbb{C}^+$ whose asymptotics can be easily computed. Namely, if we write $-8\alpha = re^{i\beta}$, then the solutions $z_k = x_k + iy_k$ of the equation $(e^{-iz} - e^{iz})^4 = re^{i\beta} z^{4-\gamma}$ in $\mathbb{C}^+$ will have the asymptotics

$$\begin{cases} 
x_k = \frac{\pi k}{2} + \frac{\beta}{4} + o(1), \\
y_k = (1 - \frac{\gamma}{4}) \ln k + (1 - \frac{\gamma}{4}) \ln \frac{\pi}{2} + o(1),
\end{cases}$$

as $k \to \infty$. It is easy to see that

$$\left| \frac{\sin^4 z}{z^4} + \frac{\alpha}{z^\gamma} \right| \gtrsim \frac{1}{|z|^\gamma}$$

when $z \in \mathbb{C}^+$, $\text{dist}(z, \{z_k\}) \geq \frac{1}{10}$ and $|z|$ is sufficiently large. By the Rouché theorem, for sufficiently large $k$ the disc $D(z_k, 1/10)$ contains exactly one zero (say, $s_k$) of $G$ and these are all zeros of $G$ except a finite number.

Moreover,

$$|G'(s_k)| \asymp \frac{\sin^3 s_k \cos s_k}{|s_k|^4} \asymp \frac{1}{|s_k|^\gamma}.$$
It follows from formulas (4.3) and (4.4) that \( F \) is an entire function of order \( \gamma^{-1} \) and of finite type. The zeros of \( F \) are given by \( t_k = s_k^\gamma \). Dividing and multiplying by a polynomial we can assume without loss of generality that all zeros of \( F \) are simple and lie in \( D \). Since 
\[
|F'(t_k)| \asymp |s_k|^{1-2\gamma} \gtrsim |t_k|^{-1} \asymp |k|^{-\gamma},
\]
we can multiply \( F \) by a polynomial \( P \) of sufficiently large degree (with zeros in \( D \)) to achieve
\[
\sum_n \frac{1}{|F'(t_n)P(t_n)|} < \infty.
\]
Slightly abusing the notation we now denote by \( \{t_n\} \) the zero set of \( FP \). It remains to show that \( FP \) has the required simple fraction expansion. We have
\[
\frac{1}{F(z)P(z)} = \sum_n \frac{1}{(FP)'(t_n)(z-t_n)} + H(z)
\]
for some entire function \( H \). Since \( FP \) is of order \( \gamma^{-1} \), we conclude that \( H \) is of order at most \( \gamma^{-1} \). However, \( |F(z)P(z)| \gtrsim 1 \), \( z \in \mathbb{C} \setminus D \) (since \( |F(z)| \asymp |z|^{-1} \) there). Also, for any \( \varepsilon > 0 \) we have \( |F(z)P(z)| \asymp |g(z)| \to \infty \) when \( |z| \to \infty \) and \( z \in A(\varepsilon, \pi\gamma - \varepsilon) \). From this we conclude that \( H \equiv 0 \). \( \square \)

5. Proof of Theorems 1.3 and 1.4

We first state a simple proposition which shows that nearly invariance implies division-invariance. The proof is similar to [2, Proposition 5.1] and we omit it. Let \( \mathcal{H} \) be a reproducing kernel Hilbert space which consists of analytic functions in some domain \( D \) and has the division property. Recall that, for a closed subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \), we denote by \( \mathcal{Z}(\mathcal{H}_0) \) the set of its common zeros.

**Proposition 5.1.** Assume that there exists \( w_0 \in D \) such that \( \frac{f(z)}{z-w_0} \in \mathcal{H}_0 \) whenever \( f \in \mathcal{H}_0 \) and \( f(w_0) = 0 \). Then, for any \( w \in D \setminus \mathcal{Z}(\mathcal{H}_0) \) and any \( f \in \mathcal{H}_0 \) such that \( f(w) = 0 \), we have \( \frac{f(z)}{z-w} \in \mathcal{H}_0 \).

We pass to the proofs of Theorems 1.3 and 1.4. The key idea of the proof is due to L. de Branges [7, Theorem 35]. Assume that neither \( \mathcal{H}_1 \subset \mathcal{H}_2 \) nor \( \mathcal{H}_2 \subset \mathcal{H}_1 \) and choose nonzero functions \( F_1, F_2 \in \mathcal{H} \) such that \( F_1 \perp \mathcal{H}_2 \) but \( F_1 \) is not orthogonal to \( \mathcal{H}_1 \), while \( F_2 \perp \mathcal{H}_1 \) but \( F_2 \) is not orthogonal to \( \mathcal{H}_2 \).
Let $F \in \mathcal{H}_1$ and $G \in \mathcal{H}_2$. Define two functions

$$f(w) = \left( F - \frac{F(w)G}{G(w)}, F_1 \right)_{\mathcal{H}(T,A,\mu)} = \int \frac{F(z) - \frac{F(w)G(z)}{G(w)}F_1(z)}{z - w} d\nu(z),$$

$$g(w) = \left( \frac{G}{F(w)}F - g, F_2 \right)_{\mathcal{H}(T,A,\mu)} = \int \frac{G(z) - \frac{G(w)F(z)}{F(w)}F_2(z)}{z - w} d\nu(z),$$

where $\nu$ is the measure defined by (2.5) such that the embedding $\mathcal{H}(T, A, \mu) \subset L^2(\nu)$ is isometric. The functions $f$ and $g$ are well-defined and analytic on the sets $\{w : G(w) \neq 0\}$ and $\{w : F(w) \neq 0\}$, respectively.

**Step 1:** $f$ and $g$ are entire functions, $f$ does not depend on the choice of $G$ and $g$ does not depend on the choice of $F$.

Let $f_1$ be a function associated in a similar way to $G_1 \in \mathcal{H}_1$,

$$f_1(w) = \int \frac{F(z) - \frac{F(w)G_1(z)}{G_1(w)}F_1(z)}{z - w} d\nu(z).$$

Then, for $G(w) \neq 0$ and $G_1(w) \neq 0$, we have

$$f_1(w) - f(w) = F(w) \int \frac{G_1(z) - G(w)G_1(z)}{G(w)G_1(z)} \frac{F_1(z)}{z - w} d\nu(z) = 0,$$

since $\frac{G_1(z)}{G(w)G_1(z)} \in \mathcal{H}_2$.

Now choosing $G$ such that $G(w) \neq 0$ we can extend $f$ analytically to a neighborhood of the point $w$. Thus, $f$ and $g$ are entire functions.

**Step 2:** $f$ and $g$ are of zero exponential type.

Recall that we denote by $\mathcal{C}$ the class of all regularized Cauchy transforms with poles in $T$. Since $F$ and $G$ are in $\mathcal{H}(T, A, \mu)$, we have $F/G, G/F \in \mathcal{C}$ and so $F/G, G/F \in \mathcal{C}/\mathcal{C}$.

Hence,

$$f, g \in \mathcal{C} + \mathcal{C} \cdot \frac{\mathcal{C}}{\mathcal{C}}.$$

By Corollary 3.1 $f$ and $g$ are of zero exponential type.

**Step 3:** Either $f$ or $g$ is identically zero.

Given $w$ such that $F(w) \neq 0, G(w) \neq 0$, we have

$$|f(w)| \leq \left| \int \frac{F(z)F_1(z)}{z - w} d\nu(z) \right| + \frac{|F(w)|}{|G(w)|} \cdot \left| \int \frac{G(z)F_1(z)}{z - w} d\nu(z) \right|,$$

$$|g(w)| \leq \left| \int \frac{G(z)F_2(z)}{z - w} d\nu(z) \right| + \frac{|G(w)|}{|F(w)|} \cdot \left| \int \frac{F(z)F_2(z)}{z - w} d\nu(z) \right|.$$ 

(5.1)
By Lemma 2.5, there exist \( M > 0 \) and a set \( E \subset (0, \infty) \) of zero linear density such that
\[
|f(w)| \lesssim |w|^M \left( 1 + \left| \frac{F(w)}{G(w)} \right| \right), \quad |g(w)| \lesssim |w|^M \left( 1 + \left| \frac{G(w)}{F(w)} \right| \right), \quad |w| \notin E.
\]
We conclude that
\[
\min \left( |f(w)|, |g(w)| \right) \lesssim |w|^M, \quad |w| \notin E.
\]
Since \( E \) has zero linear density, we can choose a sequence \( R_j \to \infty \) such that \( R_j \notin E \) and \( R_{j+1}/R_j \leq 2 \). Applying the maximum principle to the annuli \( R_j \leq |z| \leq R_{j+1} \), we conclude that
\[
\min \left( |f(w)|, |g(w)| \right) \lesssim |w|^M, \quad |w| \geq 1.
\]
Since both \( f \) and \( g \) are of zero exponential type, a small variation of a well-known deep result by de Branges [7, Lemma 8] gives that either \( f \) or \( g \) is a polynomial.

Assume that \( f \) is a nonzero polynomial. By Lemma 2.3, there exists a set \( \Omega \) of zero area density such that
\[
\left| \int \frac{F(z)F_1(z)}{z-w} d\nu(z) \right| + \left| \int \frac{G(z)F_1(z)}{z-w} d\nu(z) \right| = O\left( \frac{1}{|w|} \right), \quad w \notin \tilde{\Omega}.
\]
Hence, \( |F(w)/G(w)| \to \infty \) as \( |w| \to \infty \), \( w \notin \Omega \), and so
\[
|g(w)| \leq \left| \int \frac{G(z)F_2(z)}{z-w} d\nu(z) \right| + \frac{|G(w)|}{|F(w)|} \left| \int \frac{F(z)F_2(z)}{z-w} d\nu(z) \right| = O\left( \frac{1}{|w|} \right), \quad w \notin \Omega \cup \tilde{\Omega},
\]
where \( \tilde{\Omega} \) is another set of zero area density (here we again applied Lemma 2.3). Thus, \( g \) tends to zero outside a set of zero density and so \( g \equiv 0 \) by Theorem 2.4.

**Step 4: End of the proof.**

Without loss of generality, let \( f \equiv 0 \). Then
\[
\frac{F(w)}{G(w)} \int \frac{G(z)F_1(z)}{z-w} d\nu(z) = \int \frac{F(z)F_1(z)}{z-w} d\nu(z)
\]
for any \( F \in \mathcal{H}_1 \), \( G \in \mathcal{H}_2 \).

Recall that \( F_1 \) is not orthogonal to \( \mathcal{H}_1 \) and so we can choose \( F \in \mathcal{H}_1 \) such that \( \langle F, F_1 \rangle = \int FF_1 d\nu \neq 0 \). Then, by Lemma 2.3,
\[
\left| \int \frac{F(z)F_1(z)}{z-w} d\nu(z) \right| \gtrsim \frac{1}{|w|}, \quad w \notin \Omega,
\]
for some set \( \Omega \) of zero density. Since \( G \perp F_1 \) for any \( G \in \mathcal{H}_2 \), we have (again by Lemma 2.3)
\[
\left| \int \frac{G(z)F_1(z)}{z-w} d\nu(z) \right| = o\left( \frac{1}{|w|} \right), \quad |w| \to \infty, \ w \notin \tilde{\Omega},
\]
where \( \tilde{\Omega} \) is another set of zero density. We conclude that \(|F(w)/G(w)| \to \infty \) when \(|w| \to \infty \) outside the set of zero density \( \Omega \cup \tilde{\Omega} \) (for any \( G \in \mathcal{H}_2 \)). Applying this fact and Lemma 2.3 to \( g \) we conclude that \(|g(w)| \to 0 \) outside a set of zero density and so \( g \equiv 0 \) by Theorem 2.4.

Thus, we have

\[
G(w) = \frac{F(z)F_2(z)}{z - w} \nu(z) = \int \frac{G(z)F_2(z)}{z - w} \nu(z)
\]

and we may repeat the above argument. Choose \( G \in \mathcal{H}_2 \) such that \( \langle G, F_2 \rangle = \int GF_2 \nu \neq 0 \). Then, by Lemma 2.3, the modulus of the right-hand side in (5.2) is \( \gtrsim |w|^{-1} \), while the left-hand side is \( o(|w|^{-1}) \) when \(|w| \to \infty \) outside a set of zero density. This contradiction proves Theorem 1.3. \( \square \)

**Proof of Theorem 1.4.** The proof essentially coincides with the proof of Theorem 1.3. Let \( f \) and \( g \) be defined as above. By Step 1, \( f \) and \( g \) are entire functions. Since \( f, g \in \mathcal{C} + \mathcal{C} \cdot \mathcal{C}/\mathcal{C} \), we conclude by Corollary 3.1 that \( f \) and \( g \) are of finite exponential type.

Now we need to show that \( f \) and \( g \) are of zero type. For this we can use the property that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are closed under the \(*\)-transform. Therefore, we can also take \( F^* \) and \( G^* \) in place of \( F \) and \( G \) in the definition of \( f \) and \( g \). Since \( T \subset \{-h \leq \text{Im} \ z \leq h\} \), we have

\[
\int \frac{\alpha(z)}{z - w} \nu(z) \lesssim |\text{Im} \ w|^{-1}, \quad |\text{Im} \ w| \geq 2h,
\]

for any \( \alpha \in L^1(\nu) \). Thus,

\[
|f(w)| \lesssim \left( 1 + \min \left\{ \left\| \frac{F(w)}{G(w)} \right\|, \left\| \frac{F(w)}{F_2(w)} \right\| \right\} \right) \frac{1}{|\text{Im} \ w|}, \quad |\text{Im} \ w| \geq 2h.
\]

Note that \( F/G \in \mathcal{C}/\mathcal{C} \). Therefore, \( F/G \) is a function of bounded type in \( \mathbb{C}^+ + ih \) and in \( \mathbb{C}^- - ih \). Then we can write for \( w \in \mathbb{C}^+ \),

\[
\frac{F(w + ih)}{G(w + ih)} = O \frac{B_1S_1}{B_2S_2} e^{jaw}, \quad \frac{F(w - ih)}{G(w - ih)} = \tilde{O} \frac{\tilde{B}_1\tilde{S}_1}{\tilde{B}_2\tilde{S}_2} e^{ibw},
\]

where \( O, \tilde{O} \) are the corresponding outer factors, \( B_1, B_2, \tilde{B}_1, \tilde{B}_2 \) are Blaschke products in \( \mathbb{C}^+ \), \( S_1, S_2, \tilde{S}_1, \tilde{S}_2 \) are singular inner functions in \( \mathbb{C}^+ \) (without factors of the form \( e^{icw} \)) and \( a, b \in \mathbb{R} \). If at least one of the numbers \( a \) or \( b \) is non-negative we have, by Lemma 2.1,

\[
\log |f(w)| = o(|w|), \quad |\text{Im} \ w| \geq 2h, \quad \arg w \notin E,
\]

where \( E \subset [0, 2\pi] \) is a union of interval of arbitrarily small total length. Since \( f \) is an entire function of exponential type, the classical Phragmén–Lindelöf principle implies that \( f \) is
of zero type. Assume that both $a < 0$ and $b < 0$. Then for $g$ we have a similar estimate

$$|g(w)| \lesssim \left(1 + \min \left\{ \left| \frac{G(w)}{F(w)} \right|, \left| \frac{G(\overline{w})}{F(\overline{w})} \right| \right\} \right) \frac{1}{|\text{Im} w|}, \quad |\text{Im} w| \geq 2h.$$  

We conclude from factorizations of $G/F$ in $\mathbb{C}^+ + ih$ and in $\mathbb{C}^- - ih$ that $g$ tends to zero outside the union of angles of arbitrarily small total size whence $g \equiv 0$.

Thus, we have seen that

(i) either both $f$ and $g$ are of zero type, and we can proceed to Step 3 as in the proof of Theorem 1.3;

(ii) or one of the functions $f$ or $g$ is identically zero, and we can go directly to Step 4.

The end of the proof is the same as in Theorem 1.3. 

\square

Remark 5.2. Let us mention that Lemma 2.3 and Theorem 2.4 are essential in the case $Z$.

In the cases $\Pi$ and $A$, it is sufficient to consider the asymptotics of the Cauchy transforms along the rays lying outside the strip or the angle in question.

Remark 5.3. Theorems 1.3 and 1.4 can be extended with essentially the same proofs to the case of nearly invariant subspaces having the same sets of common zeros (counting with multiplicities).

6. COUNTEREXAMPLE TO THE ORDERING THEOREM

In this section we prove Theorem 1.5. Our construction is similar to Example 4.1. To have the symmetry with respect to $\mathbb{R}$ we rotate the function $G$ from Example 4.1.

Proof of Theorem 1.5. Step 1. Define the entire functions $A_1$ and $A_2$ by

$$A_1(z) = \prod_{k=1}^{\infty} \left(1 - \frac{e^{2\pi z}}{e^{2\pi k}} \right), \quad A_2(z) = \prod_{k=1}^{\infty} \left(1 - \frac{e^{-2\pi z}}{e^{2\pi k}} \right),$$

and put $A = A_1A_2$. Then the zero set of $A$ is given by $T = \{t_n\} = \{k + im : k \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{Z}\}$. Put $\mu_n = |t_n|^{-3}$.

Now we construct the functions $G_1$ and $G_2$ as follows. Let $P_1$ be some polynomial of degree 4 such that $\mathcal{Z}_{P_1} \subset \mathcal{Z}_{A_2}$ and let $G_1$ be an entire function with simple zeros $\tilde{t}_n = t_n + \delta_n$, $t_n \in \mathcal{Z}_{A_2} \setminus \mathcal{Z}_{P_1}$, $|\delta_n| < 1/100$, such that:

(i) $\mathcal{Z}_{G_1}$ is so close to the set $\mathcal{Z}_{A_2} \setminus \mathcal{Z}_{P_1}$ that

$$|G_1(z)P_1(z)| \asymp |A_2(z)|, \quad \text{dist} (z, \mathcal{Z}_{A_2}) > 1/10;$$

(ii) $\mathcal{Z}_{G_1} \cap T = \emptyset.$
Note that condition (6.1) implies (by the maximum modulus principle applied to the function \((z - t_n)G_1P_1/A_2\) in \(|z - t_n| \leq 1/10\)) that
\[
|G_1(t_n)P_1(t_n)| \lesssim |A'_2(t_n)|, \quad t_n \in \mathcal{Z}_{A_2}.
\]

Analogously, we define a polynomial \(P_2\) and the function \(G_2\) such that \(|G_2(z)P_2(z)| \approx |A_1(z)|\) when \(\text{dist} (z, \mathcal{Z}_{A_1}) > 1/10\).

**Step 2.** Let us show that \(G_1, G_2\) belong to the corresponding space \(\mathcal{H}(T, A, \mu)\). Similarly to Example 4.1 we have
\[
|A_1(z)| \approx 1, \quad \Re z \leq 0,
\]
\[
|A'_1(t_n)| \gtrsim 1, \quad t_n \in \mathcal{Z}_{A_1},
\]
\[
|A_1(z)| \gtrsim 1, \quad \text{dist} (z, \mathcal{Z}_{A_1}) > 1/10.
\]
Hence,
\[
\left| \frac{G_1(z)}{A(z)} \right| \lesssim \left| \frac{1}{P_1(z)A_1(z)} \right| \lesssim \frac{1}{|P_1(z)|}, \quad \text{dist} (z, T) > 1/10.
\]

Also, by (6.1) and (6.2),
\[
\sum_{t_n \in T \setminus \mathcal{Z}_{P_1}} \frac{|G_1(t_n)|^2}{|A'(t_n)|^2|A_1(t_n)|^2} \lesssim \sum_{t_n \in \mathcal{Z}_{A_1}} \frac{|A_1(t_n)|^2}{|P_1(t_n)|^2|A_2(t_n)|^2|A'_1(t_n)|^2|A'_2(t_n)|^2} + \sum_{t_n \in \mathcal{Z}_{A_1} \cap \mathcal{Z}_{P_1}} \frac{|G_1(t_n)|^2}{|A_1(t_n)|^2|A_2(t_n)|^2|A'_2(t_n)|^2|A'_1(t_n)|^2}.
\]

Since \(\mu_n = |t_n|^{-3} \) and \(|P_1(t_n)| \approx |t_n|^4\), \(t_n \in T \setminus \mathcal{Z}_{P_1}\), we conclude that \(\sum_{t_n \in T} |G_1(t_n)|^2/(|A'(t_n)|^2|A_1(t_n)|^2|A'_1(t_n)|^2|A'_2(t_n)|^2) < \infty\). By Theorem 2.6, \(G_1 \in \mathcal{H}(T, A, \mu)\).

Clearly, we can construct \(G_1\) and \(G_2\) so that \(G_1 = G_1, G_2^* = G_2\), whence the spaces \(\mathcal{H}_{G_1}\) and \(\mathcal{H}_{G_2}\) are \(\ast\)-closed.

**Step 3.** Now we construct a function \(f \in \mathcal{H}(T, A, \mu)\) such that \(f \perp \mathcal{H}_{G_1}\), but \(\left< \frac{G_1}{z - \lambda}, f \right> \neq 0\) for some \(\lambda \in \mathcal{Z}_{G_2}\). Thus, \(\mathcal{H}_{G_2}\) is not contained in \(\mathcal{H}_{G_1}\). By the symmetry of the construction, also \(\mathcal{H}_{G_1}\) is not contained in \(\mathcal{H}_{G_2}\).

Since \(|A_1(z)| \approx 1, \Re z \leq 0\), we have \(\sum |A_1(t_n)|^2|t_n|^{-3} < \infty\). Put
\[
f(z) = A(z) \sum_n \frac{A_1(t_n)}{|t_n|^3(z - t_n)} = A(z) \sum_n \frac{A_1(t_n)\mu_n^{1/2}}{|t_n|^{3/2}(z - t_n)}.
\]
Then \(f \in \mathcal{H}(T, A, \mu)\). For \(\lambda \in \mathcal{Z}_{G_1}\) we have
\[
\left< \frac{G_1}{z - \lambda}, f \right> = \sum_n \frac{G_1(t_n)}{(t_n - \lambda)A'(t_n)\mu_n^{1/2}} \cdot \frac{A_1(t_n)}{|t_n|^{3/2}} = \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_1(t_n)}{A'_2(t_n)(t_n - \lambda)}.
\]
To show that the last expression is 0, we prove the interpolation formula
\[ \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_1(t_n)}{A'_2(t_n)(z - t_n)} = \frac{G_1(z)}{A_2(z)}. \]

Once it is proved, taking \( z = \lambda \in \mathcal{Z}_{G_1} \) we obtain that \( \langle \frac{G_1}{z - \lambda}, f \rangle \). As usual, consider the entire function
\[ H(z) = \frac{G_1(z)}{A_2(z)} - \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_1(t_n)}{A'_2(t_n)(z - t_n)}. \]

By (6.2), \( |G_1(t_n)/A'(t_n)| \lesssim |P_1(t_n)|^{-1} \) for \( t_n \in \mathcal{Z}_{A_2} \setminus \mathcal{Z}_P \), and it is easy to see that
\[ \left| \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_1(t_n)}{A'_2(t_n)(z - t_n)} \right| \to 0, \quad |z| \to \infty, \quad \text{dist} (z, \mathcal{Z}_{A_2}) > 1/10. \]

Since, by (6.1), \( |G_1(z)/A_2(z)| \asymp |P_1(z)|^{-1} \) when \( \text{dist} (z, \mathcal{Z}_{A_2}) > 1/10 \), we conclude that \( |H(z)| \to 0 \), when \( |z| \to \infty \) and \( \text{dist} (z, \mathcal{Z}_{A_2}) > 1/10 \). Hence, \( H \equiv 0 \).

**Step 4.** It remains to show that \( \langle \frac{G_2}{z - \lambda}, f \rangle \neq 0 \) for some \( \lambda \in \mathcal{Z}_{G_2} \). As above we have
\[ \left\langle \frac{G_2}{z - \lambda}, f \right\rangle = \sum_{n} \frac{G_2(t_n)}{(t_n - \lambda)A'(t_n)|t_n|^{1/2}} \cdot \frac{A_1(t_n)}{|t_n|^{3/2}} = \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_2(t_n)}{A'_2(t_n)(t_n - \lambda)}. \]

Assume that \( \langle \frac{G_2}{z - \lambda}, f \rangle = 0 \) for any \( \lambda \in \mathcal{Z}_{G_2} \). Then the entire function \( A_2(z) \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_2(t_n)}{A'_2(t_n)(z - t_n)} \) vanishes on \( \mathcal{Z}_{G_2} \) and we can write
\[ A_2(z) \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_2(t_n)}{A'_2(t_n)(z - t_n)} = G_2(z)U(z) \]

for some entire function \( U \). Since \( G_2 \neq 0 \) on \( T \), comparing the values at \( t_n \in \mathcal{Z}_{A_2} \), we obtain \( U(t_n) = 1, \; t_n \in \mathcal{Z}_{A_2} \). Hence, we can write \( U = 1 + A_2V \) for some entire function \( V \). Dividing by \( G_2A_2 \) we get
\[ V(z) = \frac{1}{G_2(z)} \sum_{t_n \in \mathcal{Z}_{A_2}} \frac{G_2(t_n)}{A'_2(t_n)(z - t_n)} - \frac{1}{A_2(z)}. \]

We know that \( |A_2(z)| \gtrsim 1 \) and \( |G_2(z)| \gtrsim |P_2(z)|^{-1} \) when \( \text{dist} (z, T) > 1/10 \). Since \( |G_2(t_n)| \asymp |A_1(t_n)|/|P_2(t_n)| \asymp 1/|P_2(t_n)|, \; t_n \in \mathcal{Z}_{A_2} \), we also see that the Cauchy transform in the above formula tends to zero when \( |z| \to \infty, \; \text{dist} (z, T) > 1/10 \). We conclude that \( V \) is at most a polynomial. However, we also have \( \lim_{x \to +\infty} |G_2(x)| = \infty \), whence \( V \) is at most a constant. Since \( \lim_{x \to +\infty} A_2(x) = 1 \), we conclude that this constant is \(-1\).
We have arrived to the following representation:

\[(6.3) \quad \sum_{t_n \in \mathbb{Z}} \frac{G_2(t_n)}{A_2'(t_n)(z - t_n)} = \frac{G_2(z)}{A_2(z)} - G_2(z)\]

and we need to show that it is impossible. Note that (6.3) implies that

\[|1 - A_2(x)| \lesssim |G_2(x)|^{-1}, \quad x > 0.\]

However,

\[1 - A_2(x) = 1 - \prod_{k=1}^{\infty} (1 - e^{-2\pi x - 2\pi k}) > 1 - (1 - e^{-2\pi x - 2\pi}) = e^{-2\pi x - 2\pi}.\]

On the other hand, it is clear that

\[\frac{\log |G_2(x)|}{x} \to \infty, \quad x \to +\infty, \quad \text{dist}(x, \mathbb{Z}G_2) > 1/10.\]

Indeed, for the lacunary product \(L(z) = \prod_{k=1}^{\infty} (1 - e^{-2\pi x - 2\pi k}z)\) and for any \(N, \delta > 0\), we have \(|L(z)| \gtrsim |z|^N, \text{dist}(z, \{e^{2\pi k}\}) > \delta\). By the construction, \(|G_2(x)| \asymp |L(e^x)|/|P_2(x)| \gtrsim e^{Nz}/|P_2(x)|\) when \(\text{dist}(x, \mathbb{Z}G_2) > 1/10\). This contradiction proves Theorem 1.5. \(\Box\)

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