Composite Weak Bosons in a Confining Gauge Theory without Goldstone Bosons: a Strong Coupling Expansion Analysis

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Abstract

We consider a confining Yang-Mills theory without Goldstone Bosons which could describe the bosonic sector of the weak interactions. This model can be gauge invarantly regularized on a lattice. A strong coupling analysis of the low lying bound state spectrum indicates that the vector isor triplet bound state (the right quantum number to represent the W-boson) could be the lightest state if the mass of the pseudoscalar isosinglet is raised sufficiently by the effect of the chiral anomaly (in analogy to the \( \eta' \) mechanism of QCD).

This work is a preliminary study in support to an intensive lattice Monte Carlo analysis of the model.
1 Introduction

Today, all the known fundamental interactions of elementary particles (strong, weak and electromagnetic) are described by the Standard Model (SM) [1]. The matter fields of the SM (quarks and leptons) and the Higgs boson are considered to be elementary. They interact with each other by the exchange of gauge bosons which are also considered to be elementary. The structure of the theory is essentially determined once the matter fields and their transformation laws under the local gauge transformations are specified. The SM has received a great deal of phenomenological support and the structure of the model has been confirmed to a high degree of accuracy.

In spite of the beautiful corroboration of the SM by experiments, we may naturally ask the questions: how elementary are quarks and leptons, the gauge fields and the Higgs boson? The possibility of discovering further substructure within the particles of the SM remains a viable option for the physics which lies beyond that model.

The aim of introducing substructures is to construct a simple fundamental theory with a few degrees of freedom which should be able to reproduce the SM as an effective theory. In this way one hopes to be able to solve three important theoretical problems of the SM: 1) the family replication of the matter fields, 2) the existence of too many parameters and 3) the fine tuning problem. Several models treat the quarks and lepton, the gauge bosons and the Higgs boson as composite systems. Today, a conspicuous number of theorems exists which have ruled most of the existing models [2] and radically restricted the possibilities to construct realistic composite models. Two models have survived: the "Strongly coupled SM" (SCSM) [3] and the Yang-Mills theories without Goldstone bosons [4]. The aim of the SCSM is not to construct a simple fundamental theory beyond the SM, but to propose an alternative to the usual Higgs mechanism. However, the Yang-Mills theory without Goldstone bosons proposes to describe composite weak bosons starting from simple fundamental principles.

To be precise this last model considers the photon to remain elementary and switched off. The weak gauge bosons $W^\pm$ and $Z$ then form a mass degenerate triplet. This model is a usual Yang-Mills theory, with gauge fields and fermion fields, but with a special choice of the degrees of freedom. The degrees of freedom are characterized by the local gauge group, by the global isospin group and by the choice of the fermion fields (Dirac or Majorana spinors). To be plausible a composite model of the weak bosons has to reproduce the known weak boson spectrum: the lightest bound states have to be the W-bosons and heavier bound states have to lie in an experimentally unexplored energy range. The only possibility to have a Yang-Mills theory which reproduces the weak boson spectrum is to choose the degrees of freedom in a way that they naturally avoid bound states lighter than the vector isotriplet of the theory (which characterizes the W-boson triplet). This is possible if the unwanted light bound states which naturally show up as Goldstone bosons or pseudo Goldstone bosons in many models (like, for example, a pseudoscalar iso-nonsinglet, which would
be the pion analogue of QCD) vanish by the Pauli principle, i.e. they are symmetric combinations of Grassmann variables. To obtain such a theory it is sufficient to choose the hypercolor gauge group to be $SU(2)$, the isospin group to be $SU(2)$ and the fermion fields to be Majorana spinors.

In our work we will investigate the spectrum of this theory for two reasons: first, it is a very interesting project by itself to investigate the spectrum of a confining theory which exhibits a spectrum which is completely different from the QCD state spectrum and, second, this theory contains fundamental ingredients needed to describe a realistic model of composite weak bosons. Because of the non-perturbative nature of the problem the model must be treated on a lattice in Euclidean space. We emphasize that the spectrum in Euclidean space is equal to the spectrum in Minkowski space. A lattice regularization à la Wilson [5] is possible because the choice of the isospin group $SU(2)$ allows us to replace the Dirac mass term and the Dirac-type Wilson term by a hypercolor gauge invariant Majorana expression.

As a first step of our investigation we present a strong coupling expansion. We will explain in a formal way the technic that we have introduced to perform the strong coupling expansion. In particular we have developed an algorithmic way to easily identify and evaluate the diagrams of the expansion. This algorithm is very useful because the mass splitting between the lightest bound states show up only at higher order of the strong coupling expansion and to obtain a significant result one has to evaluate many hundreds of diagrams. From the result of this calculation we show that the vector isotriplet bound state of this theory is lighter than all other bound states provided that the chiral anomaly rises sufficiently the pseudoscalar isosinglet mass, in analogy to the $\eta'$ mechanism in QCD [6].

This work is a preliminary analysis in support to an intensive lattice Monte Carlo simulation where the mass of the pseudoscalar isotriplet is evaluated explicitely [7]. Outline of the paper: In section 2 we describe the model, in section 3 we explain the technic of our strong coupling expansion, in section 4 we classify the composite operators according to the CP eigenvalues and in section 5 we evaluate the spectrum of these CP eigenstates.
The Confining Gauge Theory without Goldstone Bosons

We start with a definition of our model\cite{footnote}. For pedagogical reasons, in this section, we formulate our theory using Weyl spinors in a second step constructing the Majorana spinors and rewriting the action in a QCD-like form.

2.1 Definition of the fermion fields

We consider a gauge theory whose fermion content is represented by a Weyl spinor $F^A_{\alpha,a}(x)$. Here $\alpha$ denotes the (undotted) spinor index ($\alpha = 1, 2$), $A$ denotes the fundamental representation index of a global SU(2) isospin group ($A = 1, 2$) and $a$ denotes the fundamental representation index of the local SU(2) hypercolor gauge group ($a = 1, 2$). Starting from this fermion field $F$ the following hypercolor singlet operators for the low lying bound states can be formed:

- **Lorentz scalars**: with fermion number +2 resp. -2
  \[
  S(x) = F(x)QF(x) \\
  \bar{S}(x) = F^\dagger(x)QF^\dagger(x)
  \]  
  \[(1)\]

- **Lorentz vectors**: isosinglet resp. isotriplet
  \[
  V_\mu(x) = F^\dagger(x)\sigma_\mu F(x) \\
  V^I_\mu(x) = F^\dagger(x)\sigma_\mu T^I F(x)
  \]  
  \[(2)\]

- **Lorentz tensor**:
  \[
  B^I_{\mu\nu}(x) = F(x)Q\tilde{\sigma}_\mu\sigma_\nu T^I F(x) \\
  \bar{B}^I_{\mu\nu}(x) = F^\dagger(x)\sigma_\mu\tilde{\sigma}_\nu T^I QF^\dagger(x)
  \]  
  \[(3)\]

where $\{\sigma_\mu\} = (i, \sigma_j)$ ($\mu = 0, \ldots, 3$) denotes the four component generalization of the Pauli matrices acting on the spin indices of the Weyl spinors in the Euclidean space-time and $\{T^I\}$ ($I = 1, 2, 3$) are the Pauli matrices of the global isospin group. The matrix $Q$ represents the antisymmetric matrix in spin, hypercolor and isospin space, which correspond to the Kronecker product of $i\sigma_2, i\tau_2, iT_2$ (the antisymmetric matrices in spin, hypercolor and isospin space, respectively)

\[
Q = i\sigma_2 \otimes i\tau_2 \otimes iT_2
\]

\footnote{In \cite{footnote} an unfortunate sign error has entered the calculations due to a wrong factor $i$ in the fourth and fifth term of eq. (3). To interpret the result correctly one has to interchange isosinglet with isotriplet. I thank H.Schleret for a discussion of this point.}
The notation \( F(x)QF(x) \), e.g., is a shorthand for the contraction of all the spin, isospin and hypercolor indices of the fermion fields with the matrix \( Q \). Obviously among the composite fields (1), (2) and (3) there is only one candidate for a Goldstone boson, namely \( S \), associated with a spontaneous breakdown of the fermion number transformation group \( F \to e^{i\alpha}F \) by the condensate \( \langle \bar{S} \rangle = \langle S \rangle \neq 0 \), which should arise in the confining phase. It is expected that the CP odd part of \( S \) (namely, \( S - \bar{S} \)) acquires a mass by the chiral anomaly in analogy to the \( \eta' \) in QCD. Notice that a Lorentz scalar isotriplet operator vanishes by the Pauli principle. As discussed above this last property is fundamental for any realistic electroweak composite model.

### 2.2 The Wilson lattice action

There is a SU(2) hypercolor matrix \( U(b) \) (in the fundamental representation) defined on each oriented lattice bond \( b \). Our convention is that

\[
U(-b) = U^\dagger(b) \tag{4}
\]

An oriented path \( \omega \) on the lattice is a set of bonds

\[
\omega = b_1 \cup b_2 \cup \ldots \cup b_n \tag{5}
\]

such that the endpoint of \( b_i \) is the startpoint of \( b_{i+1} \) for \( 1 \leq i \leq n - 1 \). We can associate a SU(2) hypercolor matrix with \( \omega \) by defining the path ordered product

\[
U(\omega) = U(b_1)U(b_2)\ldots U(b_n) \tag{6}
\]

The \( \sigma \)-matrices are defined as follows:

\[
\sigma(b) = \begin{cases} 
\sigma_\mu & \text{if } b \text{ in } +\mu \text{ direction} \\
-\sigma_\mu & \text{if } b \text{ in } -\mu \text{ direction}
\end{cases} \tag{7}
\]

We will denote by \( p \subset \Lambda \) an elementary plaquette of the lattice \( \Lambda \) and with \( \partial p \) the boundary of \( p \). The complete Wilson fermion action on a lattice \( \Lambda \) written in the Weyl spinor notation is given by

\[
S = \beta \sum_{p \subset \Lambda} TrU(\partial p) - 2k \sum_{b = (x,y) \subset \Lambda} \left\{ \begin{array}{c}
F^+(x)\sigma(b)U(b)F(y) + \frac{r}{2} \left( F(x)QU(b)F(y) + F^+(x)U(b)QF^+(y) \right) \\
-\frac{1}{2} \sum_{x \in \Lambda} \{ F(x)QF(x) + F^+(x)QF^+(x) \}
\end{array} \right\} \tag{8}
\]

The parameter \( \beta \) is related to the bare coupling constant by \( \beta = \frac{4}{g^2} \), \( k \) is the hopping parameter and \( r (=1) \) multiplies the Majorana version of the Wilson term. The Weyl fermions are represented by the Grassmann variables \( F(x) \) and \( F^+(x) \). Lattice points are denotes by \( x \in \Lambda \). \( (xy) \subset \Lambda \) represent the oriented bond between nearest neighbour pairs of lattice points.
2.3 Compact notation

Defining $\tilde{\sigma}^\mu = (-i, \sigma^\mu)$ one can easily check the following three relations

\begin{align*}
  i\tau_2 U(b)\tau_2 &= -U(b)^T \\
  i\sigma_2 \tilde{\sigma}^\mu i\sigma_2 &= \sigma^\mu T \\
  QU(b)\tilde{\sigma}^\mu Q &= U(b)^T \sigma^\mu T
\end{align*}

They can be used to rewrite the first part of the fermionic kinetic term in the action as a sum of two equivalent parts

\begin{equation}
  \sum_{b=xy} F_1^T(x)\sigma(b)U(b)F(y) = \frac{1}{2} \sum_{b=xy} F_1^T(x)\sigma(b)U(b)F(y) + \frac{1}{2} \sum_{b=xy} F(x)Q\tilde{\sigma}(b)U(b)QF_1(y)
\end{equation}

This observation allows us to define the following four-spinor fermion field and to rewrite the action in a compact form, which is similar to the Wilson action of QCD:

\begin{equation}
  \varphi^A_{\tilde{\alpha}a}(x) = \begin{pmatrix} F^A_{\tilde{\alpha}a}(x) \\ F_\dagger^A_{\tilde{\alpha}a}(x) \end{pmatrix}
\end{equation}

where the index $\tilde{\alpha}$ runs from 1 to 4 and indicates the two upper spinor components $F^A_{1,a}$ and $F^A_{2,a}$ for $\tilde{\alpha} = 1, 2$ and the two lower spinor components $F_\dagger^A_{1,a}$ and $F_\dagger^A_{2,a}$ for $\tilde{\alpha} = 3, 4$. Its transpose is simply defined by $\varphi^T = (F^T, F_\dagger)$.

In the notation (13) the isospin transformation $F \rightarrow e^{i\tilde{\alpha}\tilde{T}F}$ is

\[ \varphi \rightarrow \exp\left\{ i\tilde{\alpha} \begin{pmatrix} \tilde{T} & 0 \\ 0 & Q\tilde{T}Q^T \end{pmatrix} \right\} \varphi \]

The spin matrices are defined in terms of ordinary $\gamma$-matrices by

\begin{equation}
  \Gamma(b) = \begin{cases} 
  \Gamma^\mu = r + \gamma^\mu & \text{if } b \text{ in } +\mu \text{ direction} \\
  \Gamma^\mu = r - \gamma^\mu & \text{if } b \text{ in } -\mu \text{ direction}
  \end{cases}
\end{equation}

The $\gamma$-matrices are hermitian 4x4 matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ which we choose as follows:

\begin{equation}
  \gamma^\mu = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}
\end{equation}

and we also define

\begin{equation}
  \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{equation}
With this choice of $\varphi$ and $\Gamma(b)$ and with eq. (12) we can rewrite the action (8) in the following compact form

$$S = \beta \sum_{p \subset \Lambda} TrU(\partial p) \left. \right. -k \sum_{b=(xy)} \varphi^T(x) \left( \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right) \Gamma(b)U(b) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right) \varphi(y) \left. \right. -\frac{1}{2} \sum_x \varphi^T(x)A\varphi(x) \tag{17}$$

where the matrix $A$ is defined by

$$A = \left( \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right) = \left( \begin{array}{cc} Q & 0 \\ 0 & Q \end{array} \right) \tag{18}$$

We note that the action (17) is invariant under the isospin transformation. In section 4 we will go one step further and use fermion fields of the form

$$\psi = \left( \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right) \varphi \text{ and } \bar{\psi} = \varphi^T \left( \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right) \tag{19}$$

in terms of which the action closely resembles the Wilson action of QCD.

$$S = \beta \sum_{p \subset \Lambda} TrU(\partial p) -k \sum_{b=(xy)} \bar{\psi}(x)\Gamma(b)U(b)\psi(y) -\frac{1}{2} \sum_x \bar{\psi}(x)\psi(x) \equiv S_{Gauge}(U) - \bar{\psi} \frac{M(U)}{2} \psi \equiv S_{Gauge}(U) - \bar{\psi} \frac{M(U)}{2} \psi \tag{17}$$

We emphasize that one has to be careful since, unlike in QCD, $\bar{\psi}$ and $\psi$ are not independent fields. They are Majorana fields with hypercolor and isospin.

### 3 Strong coupling expansion rules

We will use functional techniques to give a simple derivation of Wilson’s Feynman rules for the strong coupling expansion that we will formulate on the lattice. By coupling the various degrees of freedom to sources, we will obtain a compact formal expression for the Green’s functions of the theory. Strong coupling expansions have been discussed in [7,8]. Here we analyse the changes needed for the Majorana case for an expansion requiring high order.

#### 3.1 Functional integration

Quantization of the fermion degrees of freedom can be carried out by expressing the transition amplitude directly as a sum over all possible paths in field space connecting
the initial and the final states. The partition sum is then

\[ Z = \int [d\varphi][dU] \exp \{-S(\varphi, U)\} \]

The sum over the paths for the boson degrees of freedom is a functional integral over the element of the gauge group U(b):

\[ [dU] = \prod_{b \subset \Lambda} dU(b) \quad (20) \]

The sum over the paths for the fermion degrees of freedom is a functional integral over the anticommuting fermion fields \( \varphi(x) \):

\[ [d\varphi] = \prod_{x \in \Lambda} d\varphi(x) \quad (21) \]

Thus they are elements of a Grassmann algebra. Notice that in contrast to QCD here there are no independent conjugate variables \( \bar{\varphi} \). The integration (21) is defined as follows

- **Grassmann integration**: In a n-dimensional Grassmann algebra, the n Grassmann generators \( \theta_1, \theta_2, \ldots, \theta_n \) satisfy \( \{ \theta_i, \theta_j \} = 0 \) (for \( i, j = 1, 2, \ldots, n \)). The symbols \( d\theta_1, d\theta_2, \ldots, d\theta_n \) are defined by their algebraic properties \( \{ d\theta_i, d\theta_j \} = \{ d\theta_i, \theta_j \} = 0 \) and

\[
\int d\theta_i = 0 \\
\int d\theta_i \theta_j = \delta_{ij}
\]

From these rules one derives that for any *totally antisymmetric* matrix \( A \)

\[
\int d\theta_1 d\theta_2 \ldots d\theta_n \exp(\frac{1}{2}(\theta, A\theta)) = \sqrt{\det A} \quad \text{(22)}
\]

which is the essence of the Grassmann integration. We obtain also a formula with shifted variables

\[
\int d\theta_1 d\theta_2 \ldots d\theta_n \exp(\frac{1}{2}(\theta, A\theta) + (\eta, \theta)) = \sqrt{\det A} \exp(-\frac{1}{2}(\eta, A^{-1}\eta)) \quad \text{(23)}
\]

where \( \eta = (\eta_1, ..., \eta_n) \) are Grassmann variables.

The strong coupling expansion is most easily derived and compactly formulated through the introduction of external sources coupled to the degrees of freedom of the theory. In this approach, Green’s functions are obtained by differentiating with
respective to the various sources. Corresponding to the degrees of freedom \( U^{aa'}(b) \) and \( \varphi_{a,a}'(x) \) we introduce the sources \( K^{aa'}(b) \) and \( \lambda_{a,a}'(x) \), respectively.

Perturbation theory begins by breaking the action \( S \) into two parts: the unperturbed action \( S_0 \) and the perturbation action \( S_I \). For our strong coupling expansion we choose

\[
S_0 = -\frac{1}{2} \sum_x \varphi^T(x) A \varphi(x) \tag{24}
\]

\[
S_I = \beta \sum_p Tr U(\partial p) - k \sum_{b=\langle xy\rangle} \varphi^T(x) \left( \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right) \Gamma(b) U(b) \left( \begin{array}{cc} 0 & Q \\ 1 & 0 \end{array} \right) \varphi(y) \tag{25}
\]

A \( n \)-points Green’s function can then be written as

\[
\langle \varphi_{\alpha_1 a_1}^A(x_1) \ldots \varphi_{\alpha_n a_n}^A(x_n) \rangle = \\
= \frac{1}{Z} \int [d\varphi][dU] \varphi_{\alpha_1 a_1}^A(x_1) \ldots \varphi_{\alpha_n a_n}^A(x_n) \exp \{-S(\varphi, U)\} = \\
= \frac{1}{Z} \delta \frac{\partial}{\partial \lambda_{\alpha_1 a_1}^A(x_1)} \ldots \delta \frac{\partial}{\partial \lambda_{\alpha_n a_n}^A(x_n)} \exp \left\{ -S_I \left( \frac{\delta}{\delta \lambda} \frac{\delta}{\delta K} \right) \right\} \times \\
\times \int [d\varphi][dU] \exp \left\{ -S_0(\varphi, U) + \sum_x \lambda^T(x) \varphi(x) + \sum_b K(b) U(b) \right\} \right\}_{\lambda, K=0}
\]

where \( Z = \int [d\varphi][dU] \exp \{-S(\varphi, U)\} \). The notation \( \lambda^T \varphi \), for example, is a shorthand for the contraction of the hypercolor, spin and isospin indices. The division by \( Z \) is equivalent to the omission of disconnected diagrams with at least one factor being a vacuum diagram (no external legs).

We note that the unperturbed action depends on the fermion fields only. The functional integration over the fermion fields yields (using eq. (22-23))

\[
\int [d\varphi] \exp \left\{ -S_0(\varphi) + \sum_x \lambda^T(x) \varphi(x) \right\} = \\
= \int [d\varphi] \exp \left\{ \sum_x \left[ \frac{1}{2} \varphi^T(x) A \varphi(x) + \lambda^T(x) \varphi(x) \right] \right\} = \\
= \exp \left\{ -\sum_x \lambda^T(x) \frac{A^{-1}}{2} \lambda(x) \right\} \tag{27}
\]

The functional integral over the gauge group gives us

\[
\int [dU] \exp \left\{ \sum_b K^{aa'}(b) U^{aa'}(b) \right\} = \prod_b D(K(b), K(-b)) \tag{28}
\]

\( \lambda(x) \) can be expressed by two sources associated to the two Weyl fermion fields \( F(x) \) and \( F^\dagger(x) \), respectively: \( \lambda_{a,a}^A = \left( \begin{array}{c} \eta_{\alpha,a}(x) \\ \bar{\eta}_{\dot{\alpha},a}(x) \end{array} \right) \). Notice that the bar over \( \eta \) simply denotes that \( \eta \) and \( \bar{\eta} \) are two different sources.
where $\prod_{b=(xy)}$ denotes a product over all nearest neighbour pairs of lattice points and

$$D(K(b), K(-b)) = \int dg \exp \left\{ K^{aa'}(b) g^{aa'} + K^{aa'}(-b) (g^{-1})^{aa'} \right\}$$

(29)

Here $dg$ denotes the Haar measure over the gauge group. Putting all this together, we obtain the expression for the n-point Green’s function (we omit the hypercolor, isospin and spin indices for simplicity)

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle = \frac{1}{Z} \delta\delta \lambda(x_1) \ldots \delta\delta \lambda(x_n) \exp \left\{ -S_I \left( \frac{\delta}{\delta \lambda}, \frac{\delta}{\delta K} \right) \right\} \times$$

$$\times \exp \left\{ -\sum_x \lambda^T(x) \frac{A^{-1}}{2} \lambda(x) \right\} \times D(K(b), K(-b)) \bigg|_{\lambda,K=0}$$

$$= \frac{\delta}{\delta \lambda(x_1)} \ldots \frac{\delta}{\delta \lambda(x_n)} \exp \left\{ -S_I \left( \frac{\delta}{\delta \lambda}, \frac{\delta}{\delta K} \right) \right\} \times \tilde{Z}[\lambda, K] \bigg|_{\lambda,K=0}$$

(30)

where the last factor $\tilde{Z}$ denotes the generating functional

$$\tilde{Z}[\lambda, K] = \frac{1}{Z} \exp \left\{ -\sum_x \lambda^T(x) \frac{A^{-1}}{2} \lambda(x) \right\} \times \prod_b D(K(b), K(-b))$$

(31)

The expansion of the exponential of the perturbation part of the action $\exp\{-S_I\}$ in a power series generates the strong coupling series. The terms of this series may be represented by graphs similar to the Feynman graphs in ordinary perturbation theory. The graphical rules for calculating the Green’s function can be read off from the equation (30). In the next subsections we will present simple algorithmic Feynman rules for the two point Green’s function of the form $\langle \varphi(x) H \varphi(y) \rangle$ or $\langle \varphi(x) M_1 \varphi(x) \varphi(y) M_2 \varphi(y) \rangle$ where $H$, $M_1$ and $M_2$ denote some general matrices in spin, isospin and hypercolor space which produce hypercolor singlets and respect the covariance of the bilinear forms $\varphi M_j \varphi$ (for example $\varphi \varphi$ is not covariant\footnote{One can’t contract dotted spinor indices with undotted ones.}). For the fermion string through the graphs along the path $\omega$ we only need the familiar Dirac-fermion rules. But for the vertices at the external points $x$ and $y$ the rules are different from the Dirac-fermion ones, due to the Majorana nature of the fermion.

### 3.2 Group integrals

In equation (30) we are left with the integral $D(K(b), K(-b))$. This integral is a generating functional for integrals of polynomials of group matrices through the relation

$$\int dg g^{a_1 a'_1} \ldots g^{a_n a'_n} (g^{-1})^{b_1 b'_1} \ldots (g^{-1})^{b_m b'_m} =$$

$$= \frac{\delta}{\delta K^{a_1 a'_1}(b)} \ldots \frac{\delta}{\delta K^{a_n a'_n}(b)} \frac{\delta}{\delta K^{b_1 b'_1}(-b)} \ldots \frac{\delta}{\delta K^{b_m b'_m}(-b)} D(K(b), K(-b)) \bigg|_{K=0}$$

(32)
The structure of the expansion is revealed considering the integrals over the group elements on the path $\omega$. A general discussion of integration over SU(N) group elements is given by Creutz [8]. For our purposes it is useful to know the following two results (together with the Haar measure normalization $\int dg = 1$):

\begin{align}
\int dg g^{aa'} &= 0 \quad (33) \\
\int dg g^{aa'}(g^{-1})^{bb'} &= \frac{1}{N}\delta_{ab}\delta_{a'b} \quad (34)
\end{align}

We consider a closed path $\omega$ which forms a loop from a point $x \in \Lambda$ to a point $y \in \Lambda$ and returns to $x$

\begin{align}
\omega &= \omega^+ \cup \omega^- \\
\omega^+ : x &\mapsto y \\
\omega^- : y &\mapsto x \quad (35)
\end{align}

(the path can have any other closed form, it can be, for instance, the union of two disconnected closed paths: a loop from $x$ to $x$ and another loop from $y$ to $y$). The graphs with non trivial group integral are given by the following algorithm:

- **The path**: We draw the path $\omega$ on the lattice $\Lambda$. We choose a direction of the path. Each bond $b \in \omega$ defines a link between two points. The direction of this link is defined by the direction of the path.

- **Plaquettes addition**: In order to have a non vanishing group integral over a hypercolor singlet polynomial of group matrices we have to add plaquettes in the following way. A plaquette is bounded by four directed links. For each given path $\omega$ we add plaquettes on the lattice $\Lambda$ in such a way that there are $2n$ ($n$ is some integer number) or zero links for each bond $b \in \Lambda$ and if there are $2n$ links at a bond $b$, half of the links are directed along $+b$ and the other half along $-b$.

- **Vertices and bonds**: We consider only the bonds with $2n$ links. This set of bonds is called the *bond set* and the respective set of start and end points of these bonds is called *vertex set*. The bond set forms a *general graph*.

- **Connected graph**: We call *graph* $G(\omega)$ a connected general graph constructed from a path $\omega$ with the previous three steps of this algorithm. The graph $G(\omega)$ is not unique, but the group integral of two different $G(\omega)$ of the same path $\omega$ and with the same number of vertices is unique.

- **Group integral**: We denote with $\Omega_{ab',a'b}(G(\omega))$ the group integral of the path $\omega$ for graph $G(\omega)$. The hypercolor indices $aa'$ and $bb'$ indicate the hypercolor indices at the begin $(a,b)$ and at the end $(a',b')$ of the paths $\omega^+$ and $\omega^-$, respectively.
Graph of type I: We call graphs of type I constructed from a path $\omega$ with the first three steps of this algorithm with $n=1$. All graphs $G(\omega)$ of type I of the same path $\omega$ and with the same number of plaquettes have the same number $V$ of vertices and $B$ of bonds. The group integral for a graph of type I yields a very simple result:

$$\Omega_{ab'a'b}(G(\omega)) = N^{(V-B-1)} \delta_{ab} \delta_{a'b}$$

(36)

A term $N$ arises from contracting the hypercolor indices of eq. (34) at each vertex and a term $1/N$ arises from each bond with two links (see eq. (34)).

Graph of type II: We call graph of type II a graph which is not of type I. The group integral for a graph of type II has not a simple form and a general result like for the previous case. The group integral has to be evaluated separately following the methods explained in [8].

3.3 Strong coupling Feynman diagram

A strong coupling Feynman diagram $D(G,\omega)$ is formed by a path $\omega$ and one of its graphs $G(\omega)$. The order of a term in the strong coupling expansion which is characterized by a diagram $D(G,\omega)$ is given by the number $P$ of plaquettes of the graph $G(\omega)$ and by the length of the path $|\omega|$. A diagram will give a contribution to the Green’s function proportional to $(-\beta)^P k^{||\omega||}$. The amplitude $A(D(G,\omega))$ of the diagram is given by:

$$A(D(G,\omega)) = (-1)^L \Omega_{ab'a'b}(G(\omega)) \Sigma_{aa'bb'}(\omega) (-\beta)^P k^{||\omega||}$$

(37)

where $L$ is the number of internally closed loops formed by the path and $\Sigma(\omega)$ is the trace term arising from the trace over the matrices of spin and isospin. The trace over the hypercolor is obtained by contracting the indices $aa'$ and $bb'$. This last trace term $\Sigma$ is different from the analogous term that one expects from a strong coupling expansion with Dirac fermions and its evaluation requires care. Therefore we will discuss it in a separate subsection.

3.4 The trace term

In this subsection we consider only two-point Green’s functions of the form:

$$\langle \varphi(x)M\varphi(x)\varphi(y)M\varphi(y) \rangle$$

The normalized Green’s function is

$$\langle \varphi(x)M\varphi(x)\varphi(y)M\varphi(y) \rangle = \frac{1}{Z} \int [d\varphi][dU] \varphi(x)M\varphi(x)\varphi(y)M\varphi(y) \exp\{-S(\varphi, U)\}$$

See Fig. 1a

See Fig. 1b
When we consider only the fermion integration, the normalized fermion expectation value will be

$$\langle \varphi(x)M\varphi(x)\varphi(y)M\varphi(y)\rangle_{\text{fermion}} = \frac{1}{Z} \int [d\varphi] \varphi(x)M\varphi(x)\varphi(y)M\varphi(y) \exp\{-S(\varphi, U)\}$$

This can be written as a sum over paths forming fermion loops connecting x and y. There are two possibilities:

- **Isomultiplet**: If the matrix $M$ contains some isospin matrix $T^I$ there is no contribution from separate loops at x and y, since $T^I$ is traceless. Only loops connecting x with y are allowed.

- **Isosinglet**: If the matrix $M$ does not contain a $T^I$ term the contributions from separate loops at x and y are allowed. If two loops are connected by plaquettes then they correspond to a chiral anomaly contributions. If the two loops are not connected by plaquettes (i.e. they do not form a connected diagram) then the truncated Green’s function has to be defined as follows\(^6\)

$$\langle \varphi(x)M\varphi(x)\varphi(y)M\varphi(y)\rangle_{\text{truncated}} = \langle \varphi(x)M\varphi(x)\varphi(y)M\varphi(y) \rangle - \langle \varphi(x)M\varphi(x) \rangle \langle \varphi(y)M\varphi(y) \rangle$$  \hspace{1cm} (38)

For the moment we concentrate ourselves on the non-anomaly terms. We will discuss the anomaly later in section 5. We have defined the closed loop path $\omega$ from a point x to a point y as the union of two distinct paths $\omega^+$ and $\omega^-$. Each of these two paths is the union of the $|\omega^\pm|$ bonds which compose them

$$\omega^+ = \bigcup_{j=1}^{|\omega^+|} \{x_{j-1}, x_j\}$$

$$\omega^- = \bigcup_{j=1}^{|\omega^-|} \{y_{j-1}, y_j\}$$

(39)

where $x_0 = x$, $x_{|\omega^+|} = y$; $y_0 = y$ and $y_{|\omega^-|} = x$.

We use the shorthand $\delta\lambda(x) \equiv \frac{\delta}{\delta \lambda(x)}$ for the differentiation of the fermion sources. The contribution of the fermion along the path is expressed by the derivative of the fermionic sources at each lattice point on the path. The kinetic term, proportional to the hopping parameter $k$, yields terms of the following form

$$\Theta_{aa'}(\omega^+) \tilde{Z} = \delta\lambda^T(x) \left( \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right) \Gamma(\omega^+) \times \delta_{aa'} \times \left( \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right) \delta\lambda(y) \mid_{\lambda(z) = 0 \text{ for } x \neq z \neq y}$$

\(^6\)Candidates here are the CP eigenstates $S_{\pm} = S \pm \bar{S}$. However, one can easily show that $\langle S_- \rangle = 0$ (the contribution of a graph to $\langle S_- \rangle$ cancels with the contribution of the same graph with all the space direction reversed) which avoids spontaneous CP violation and $\langle S_+ \rangle \neq 0$ due to a spontaneous breaking of the U(1) fermion number transformation. The vacuum expectation value of the vector fields is zero, because a spontaneous breaking of the Lorentz invariance should not occur.
\[ \Theta_{bb'}(\omega^-) \tilde{Z} = \delta \lambda^T(y) \left[ \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right] \Gamma(\omega^-) \times \delta_{bb'} \times \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right] \delta \lambda(x) \tilde{Z} \bigg|_{\lambda(z)=0 \text{ for } x\neq z \neq y} \]

where

\[ \Theta_{aa'}(\omega^+) = \delta_{aa'} \prod_{j=1}^{\omega^+} \delta \lambda^T(x_{j-1}) \left[ \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right] \Gamma(\langle x_{j-1}, x_j \rangle) \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right] \delta \lambda(x_j) \]

\[ \Theta_{bb'}(\omega^-) = \delta_{bb'} \prod_{j=1}^{\omega^-} \delta \lambda^T(y_{j-1}) \left[ \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right] \Gamma(\langle y_{j-1}, y_j \rangle) \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right] \delta \lambda(y_j) \]

When we differentiate the exponential of the quadratic form \( \lambda^T A^{-1} \lambda \) in \( \tilde{Z} \) twice with respect to \( \lambda \) we have

\[
\delta \lambda^A_{\alpha,a}(x) \delta \lambda^B_{\beta,b}(x) \times \exp \{-\lambda^T(x) \frac{A^{-1}}{2} \lambda(x)\} = \\
= (A_{AB,\tilde{\alpha}\tilde{\beta},ab} + (A^{-1} \lambda(x)) A_{\alpha,a}(A^{-1} \lambda(x))_{B,\beta,b} \times \exp \{-\lambda^T(x) \frac{A^{-1}}{2} \lambda(x)\} \]

where

\[
A_{AB,\tilde{\alpha}\tilde{\beta},ab} = \left[ \begin{array}{cc} Q & 0 \\ 0 & Q \end{array} \right]
\]

denotes the *totally antisymmetric* matrix in isospin, spin and hypercolor space. Eq. (40) is obtained using eq. (42) in (41)\[\text{\textsuperscript{7}}\]. The external source derivative \( \frac{\delta}{\delta K_{cc'}(\langle x_{j-1}, x_j \rangle)} \) associated to the gauge field can be viewed as an identity matrix \( \delta_{cc'} \) because of eq. (34) and the algorithm of sect. 3.2. The remaining terms \( \delta_{aa'} \) and \( \delta_{bb'} \) arise from contracting all the hypercolor indices of the \( \delta_{cc'} \) for each bond of the path. The remaining \( \delta \lambda(x) \) and \( \delta \lambda(y) \) in eq. (40) will contract their indices with the indices of the derivatives \( \delta \lambda(x) \) and \( \delta \lambda(y) \) arising from the propagator \( \langle \varphi(x) M \varphi(x) \varphi(y) M \varphi(y) \rangle \) when we substitute the fermion fields \( \varphi \) with the derivative \( \delta \lambda \). Applying these derivatives to the quadratic form in \( \tilde{Z} \) we obtain

\[
\Sigma_{aa' bb'}(\omega) = Tr \left\{ \delta \lambda(x) M \delta \lambda(x) \times \Theta_{aa'}(\omega^+) \times \delta \lambda(y) M \delta \lambda(y) \times \Theta_{bb'}(\omega^-) \tilde{Z} \right\} \bigg|_{\lambda=0} \]

Using eq. (40) in (44) we obtain

\[
\Sigma_{aa' bb'}(\omega) = \delta_{aa'} \delta_{bb'} \times \]

\[
\times Tr \left\{ \delta \lambda(x) M \delta \lambda(x) \left[ \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right] \Gamma(\omega^+) \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right] \delta \lambda(y) \times \right. \]

\[
\times \delta \lambda(y) M \delta \lambda(y) \times \delta \lambda(y) \left[ \begin{array}{cc} Q & 0 \\ 0 & 1 \end{array} \right] \Gamma(\omega^-) \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q \end{array} \right] \delta \lambda(x) \times \tilde{Z} \bigg|_{\lambda=0} \]

\[\text{\textsuperscript{7}}\text{All terms involving the second term in the bracket on the right hand side of eq. (42) will vanish because of the backtracking (one obtains gamma matrix multiplications of the form } \Gamma(b) \Gamma(-b) = 0 \text{ or because they are proportional to a } \lambda(x_j) \text{ which will be set to zero in eq. (40).}\]
The result one obtains from eq. (45) may be very different from the analogous result for Dirac fermions. The difference is due to the fact that in the Dirac case there are two independent fermionic external sources (say $\lambda$ and $\bar{\lambda}$) while in our case there is only one. In the Dirac analogue to eq. (45) there are two $\delta \lambda$ and two $\delta \bar{\lambda}$, and the contraction of the indices is evident. In our case we have four $\delta \lambda$ and also four times the same functional derivative which leads to a more tricky contraction of the indices.

In eq. (45) one can contract the indices of the derivatives $\delta \lambda(x)$ and $\delta \lambda(y)$ in many different ways, because each $\delta \lambda$ appears four times. The proper contraction of the $\lambda$ using an explicit notation leads to a sum of four terms

$$\Sigma_{aa'bb'}(\omega) = \delta_{aa'} \delta_{bb'} \times \times \text{Tr} \left\{ \Gamma(\omega^+) \tilde{M} \Gamma(\omega^-) \tilde{M} - \Gamma(\omega^+) \tilde{M} \Gamma(\omega^-) \tilde{M} \Gamma(\omega^+) \tilde{M} \Gamma(\omega^-) \tilde{M} \Gamma(\omega^+) \tilde{M} \right\}$$

(46)

where

$$\tilde{M} = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$$

(47)

4 CP eigenstates

In weak interactions CP is a good quantum number but not C and P separately. Therefore, we perform a classification of the composite operators (1-3) according to the CP eigenvalues. The CP transformation of the Weyl spinor $F$ is defined by

$$F^{\text{CP}}(x) = i\tau_2 i\sigma_2 F^\dagger(x_P)$$

(where $x_P = (t, -\vec{x})$) which fixes the phase. In the compact notation we can rewrite the CP transformation in the following form:

$$\psi^{\text{CP}}(x) = T_2 \gamma_0 \psi(x_P)$$
$$\bar{\psi}^{\text{CP}}(x) = \bar{\psi}(x_P) \gamma_0 T_2$$

(48)

We can build CP odd and CP even combinations of the composite fields (1), (2) and (3). These eigenstates can be expressed in a very compact form with the definitions (19).

- **Scalar CP eigenstates**: The scalar CP even combination of scalars is defined by

$$S_+(x) = \bar{\psi}(x) \psi(x)$$

(49)

The pseudoscalar CP odd combination is defined by

$$S_-(x) = i\bar{\psi}(x) \gamma_5 \psi(x)$$

(50)

---

8 Lorentz scalars $S$, vectors $V^\mu$ and tensors $T^{\mu\nu}$ are CP even if $CP S = S$, $CP V^\mu = V^\mu$ and $CP T^{\mu\nu} = T^{\mu\nu}$ and are CP odd if $CP S = -S$, $CP V^\mu = -V^\mu$ and $CP T^{\mu\nu} = -T^{\mu\nu}$, respectively.
• Vector CP eigenstates: The vector isosinglet CP even combination is defined by

\[ V^\mu(x) = \frac{i}{2} \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x) \] (51)

The isotriplet vector states is defined by

\[ V^{\mu I}(x) = \frac{1}{2} \bar{\psi}(x) \gamma^\mu T^I \psi(x) \] (52)

This state is CP even for I=1,3 and CP odd for I=2.

• Tensor CP eigenstates: The tensors are defined as follows:

\[ \tilde{B}^{\mu\nu I} = \bar{\psi}(x) \sigma^{\mu\nu} T^I \psi(x) \] (53)

and its dual

\[ \tilde{B}^{*}_{\mu\nu I} = \epsilon^{\mu\nu\sigma\rho} \tilde{B}^I_{\sigma\rho} \] (54)

where \( \sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \). \( \tilde{B}^{\mu\nu I} \) has CP properties opposite to the vector isotriplet \( V^I \) ones. Its dual \( \tilde{B}^{*}_{\mu\nu I} \) has CP properties opposite to \( \tilde{B}^{\mu\nu I} \).

The propagators of the CP eigenstates going from the point \( x \) to the point \( y \) along some path \( \omega \) can be evaluated with the help of eq. (37) and choosing the appropriate representation of the matrix \( M \) in eq. (46). In the CP formalism we can write a propagator in the form

\[ \langle (\bar{\psi}(x)\tilde{M}\psi(x))^\dagger \bar{\psi}(y)\tilde{M}\psi(y) \rangle = \sum_{\omega:x\to y} \sum_{G(\omega)} A(D(G,\omega)) = \sum_{\omega:x\to y} \sum_{G(\omega)} C_M \times Tr(\tilde{M}^I G(\omega^+) \tilde{M}^I G(\omega^-)) \times (-1)^L \Omega(G(\omega)(-\beta)^P k^{\omega}) \] (56)

where \( C_M \) is some constant and \( \Omega(G(\omega)) = \Omega_{ab} \delta_{ab} \delta_{b\omega} \). Notice that for a graph of type I we obtain \( \Omega(G) = N^{(V-B)} \) (see eq. (36)).

9Notice that using eq. (10) we obtain

\[ V^\mu = F^\dagger(x)\sigma^\mu F(x) - F^T(x)Q\sigma^\mu QF^\dagger(x) = F^\dagger(x)\sigma^\mu F(x) + F^\dagger(x)\sigma^\mu F(x) = 2F^\dagger(x)\sigma^\mu F(x) \]

10Notice that the charged and neutral weak bosons are defined by

\[
\begin{pmatrix}
W^+ \\
W_3 \\
W^-
\end{pmatrix} = 
\begin{pmatrix}
W_1 + iW_2 \\
W_3 \\
W_1 - iW_2
\end{pmatrix}
\]

The CP properties are: \( CP W_3(x) = W_3(x_P) \) and \( CP W^+(x) = W^-(x_P) \).
5 Strong coupling expansion masses

In order to compute the mass of the CP eigenstates we will consider the static propagators of these states.

\[ G_{\text{static}}(t) = \sum_{\vec{x}} \langle \bar{\psi}(0, \vec{x}) \tilde{M} \psi((0, \vec{x})) \rangle \langle \bar{\psi}(t, \vec{x}) \tilde{M} \psi((t, \vec{x})) \rangle \]  

(57)

First we assume that \( x_0 = 0 \) and \( y_0 = t \) and \( \vec{x} = \vec{y} \) (summed over all \( \vec{x} \)) to interpret eq. (56) as a static propagator. The lowest order diagram representing a static propagator is a double fermion line in the time direction as shown in Fig. 2a. Since there is a non-zero probability for a transition from the static state to a more complicated state, the full propagator is given by a sequence of excited states connected by static propagators as shown in Fig. 2b. Each diagram \( D(G(\omega), \omega) \) of a general path \( \omega \) can be viewed as a space-time process contributing to the excitation of the static propagation. These excitations renormalize the mass of the static propagator.

5.1 The unrenormalized static propagator

We denote with \( G_{\tilde{M}}(t) \) a static propagator. For the CP eigenstates we will consider the operators given by the following set of \( \tilde{M} \)'s.

| \( \tilde{M} \) | flavor |
|----------------|--------|
| \( \gamma_5 + \gamma_5 \gamma_0 \) | \( S_- \) |
| \( \gamma_k \tilde{T}^I + \sigma^{0k} \tilde{T}^I \) | \( V^{kl} \) |
| \( \gamma_5 \gamma_k \) | \( V^k \) |
| \( \sigma^{kj} \tilde{T}^I \) | \( B^{kji} \) |
| 1 | \( S_+ \) |

We will use the shorthands

\( \tilde{S}_-, \tilde{V}^{kl}, V^k, \tilde{B}^{kji}, S_+ \)

to denote the flavor of the bound states and its corresponding matrix \( \tilde{M} \). The non renormalized mass of a \( \tilde{M} \) state can be found from the property of the unrenormalized static propagator (for \( t \to \infty \))

\[ G_0^{\tilde{M}}(t) = C_{\tilde{M}} \times \exp(-m_0 t) \]  

(58)

where \( C_{\tilde{M}} \) is a constant. The path \( \omega_0 \) denotes the path of Fig. 2a. The diagram of this path contains no plaquettes. Computing the amplitude (37) of this diagram we

11Because of the relations \( \partial_{\mu} V^\mu = C \times S_- \) (when the anomaly is neglected) and \( \partial_{\mu} B^{\mu I} = C' \times V^{\mu I} \) (where \( C \) and \( C' \) are some constants) and because of the Fourier transformation with zero momenta \( \vec{p} \) in eq. (57) the static propagator mixes \( S_- \) with \( V^0 \) and \( V^{kl} \) with \( B^{kji} \).
have to evaluate the corresponding $\Gamma$ matrices:

\[
\begin{align*}
\Gamma(\omega^+) &= (\Gamma^0)^t = 2^{(t-1)}\Gamma^0 \\
\Gamma(\omega^-) &= (\Gamma^0)^t = 2^{(t-1)}\tilde{\Gamma}^0
\end{align*}
\]

(59)

The group integral of this diagram is $\Omega(\omega) = 2$ and so we find from eq. (56) that

\[
G_0^\tilde{M}(t) = C_\tilde{M} \times (4k^2)^t
\]

(60)

The static unrenormalized propagator $G_0^\tilde{M}(t)$ is different from zero only for two CP eigenstates:

\[
\tilde{M} = \tilde{S}_-, \tilde{V}^{kJ}
\]

(61)

From eq. (56) we find that for these bound states the non renormalized mass is

\[
m_0 = -\log(4k^2)
\]

(62)

The static propagator of the other CP eigenstates is zero. Their mass is $m_0 = \infty$ and remain very large even after renormalization. This result is a pathologic default of the strong coupling expansion. It does not describes any physics. The only information that one can obtain after renormalization is the spectrum of the states for which $G_0(t) \neq 0$. In the next subsections we will compare the masses of these two states.

5.2 Renormalization of the static propagator

Having computed the static propagator, we will now compute the renormalization produced by intermediate exited states. Fig. 3 shows us such an excitation, with initial point $z$ and final point $w$, where the static unrenormalized propagator arrives and departs. For fixed $z$ and $w$, we will sum over all intermediate exited states to obtain a total weight for the event. We denote this weight by $D^\tilde{M}(z, w)$. The full propagator takes the form:

\[
G^\tilde{M}(x, y) = \sum_{z_i, w_i (i=1\ldots n)} G_0^\tilde{M}(x, z_1)D^\tilde{M}(z_1, w_1)G_0^\tilde{M}(w_1, z_2)D^\tilde{M}(z_2, w_2) \times \\
\cdots \times D^\tilde{M}(z_n, w_n)G_0^\tilde{M}(w_n, y)
\]

(63)

where the points $z_i$ and $w_i$ are required to lie between $x$ and $y$ and to be ordered so that each $w_i$ is "later" than $z_i$, which is itself "later" than $w_{i-1}$. This last eq. (63) represents the picture in Fig. 2b.

In order to find the renormalized mass of the propagator, we consider the full static propagator

\[
G^\tilde{M}(t) = \sum_{y_0 - x_0 = t} G(x, y)
\]

(64)

12For the notation see eq. (14). We have fixed $r=1$ which correspond to the case $r=1$ in QCD.
Similarly we define

\[ D^\hat{M}(t) = (G_0^\hat{M}(t))^{-1} \sum_{y_0 - x_0 = t} D^\hat{M}(x, y) \] (65)

and then (63) gives

\[ G^\hat{M}(t) = e^{-m_0 t} \sum_{s_i, t_i (i=1...n)} D^\hat{M}(t_1 - s_1) \ldots D^\hat{M}(t_n - s_n) = e^{-m_0 t} e^{P_M t} \] (66)

where \( t_i \) and \( s_i \) are the corresponding time coordinates of \( w_i \) and \( z_i \). So the renormalized mass \( m_M \) is

\[ m_M = m_0 - p_M \] (67)

The leading order of \( p_M \) can just be written in the form

\[ p_M = \sum_{w = \{t' \geq 0, w\}} (G_0^\hat{M}(t'))^{-1} D^\hat{M}(0, w) = \sum_{w = \{t' \geq 0, w\}} (4k^2)^{-t'} D^\hat{M}(0, w) \] (68)

To prove this last equation we have to insert it into the last term of (66) and to expand the exponential function containing \( p_M \), then we have to compare the result with the first term of (66). The excitation term \( D^\hat{M}(z, w) \) is defined starting from eq. (37) in the following way:

\[ D^\hat{M}(z, w) = \sum_{\omega: z \rightarrow w} \sum_{G(\omega)} A(D(G, \omega)) C^\hat{M} \] (69)

where the sum is over all closed paths \( \omega \) from \( z \) to \( w \) and return and over all graphs \( G(\omega) \). \( A(D(G, \omega)) \) denotes the amplitude (37) of an intermediate excitation diagram.

#### 5.3 Results

In order to calculate the renormalized mass of the two states \( \vec{V}^{kl} \) and \( \vec{S}_- \) to next leading order we have collected all the excitation diagram for all \( t' \geq 0 \) and \( \vec{w} \) in eq. (68) corresponding to the \( k^4 \) and \( \beta^4 \) perturbation order. Our results are valid in the asymptotic sense in a neighborhood of the origin in the \((k, \beta)\) plane.

- **Vector isotriplet mass**: First we write the vector isotriplet mass \( m_{\vec{V}^{kl}} \) up to fourth order in \( \beta \) and \( k \). Our result is

\[ m_{\vec{V}^{kl}} = -log(4k^2) - 3k^2 + \frac{3}{2} k^2 \beta - \frac{3}{4} k^2 \beta^2 + \frac{7}{8} k^2 \beta^3 - \frac{5}{16} k^2 \beta^4 \]

\[ -16k^4 + 15k^4 \beta - \frac{107}{4} k^4 \beta^2 + 14k^4 \beta^3 - \frac{272}{3} k^4 \beta^4 \] (70)

\(^{13}\)Notice that \( (G_0^\hat{M}(t))^{-1} = \frac{1}{C_M} e^{m_0 t} = \frac{1}{C_M} (4k^2)^{-t} \)
• **Pseudoscalar mass:** We calculate the pseudoscalar mass neglecting the anomaly terms, since they arise only at order $k^{14}$ and are generally severely suppressed in strong coupling expansions \([10]\). These terms will be discussed later. In this calculation the pseudoscalar mass comes out to be less than the vector mass. This is not surprising in view of the suppression of the anomaly. In the next subsection the consideration of the anomaly terms will show that the pseudoscalar bound state behaves like the $\eta'$ bound state in QCD and in particular that it acquires a positive contribution to its mass from the anomaly terms. Our result is

$$m_{\tilde{S}_-} = -\log(4k^2) - 3k^2 + \frac{3}{2}k^2 \beta - \frac{3}{4}k^2 \beta^2 + \frac{7}{8}k^2 \beta^3 - \frac{5}{16}k^2 \beta^4 - 36k^4 + 12k^4 \beta - \frac{153}{4}k^4 \beta^2 + \frac{65}{4}k^4 \beta^3 - \frac{237}{2}k^4 \beta^4$$  \(71)\)

### 5.4 The anomaly terms

Here we will calculate the leading order anomaly contribution to the mass splitting $\Delta m_{\tilde{S}_-}^{\text{anomaly}} = m_{\tilde{S}_-}^{\text{tot}} - m_{\tilde{S}_-}$ where $m_{\tilde{S}_-}^{\text{tot}}$ denotes the pseudoscalar mass when the anomalies are turned on. We use the methods derived in section 3 to determine the contributions of the anomaly terms. The mass splitting is given by excitations which are characterized by two disconnected closed loops:

$$\omega_{\text{anomaly}} = \omega^1 \cup \omega^2$$

$$\omega^1 : x \rightarrow x$$

$$\omega^2 : y \rightarrow y$$

The diagram which corresponds to these two paths can be found by the algorithm of section 3.2. The amplitude of this diagram is given by the formula (37). The trace term can be evaluated by modifying eq. (45) to the new paths type. The result is

$$\Sigma(\omega_{\text{anomaly}}) =$$

$$= Tr \left\{ \delta \lambda(x) M_1 \delta \lambda(x) \times \delta \lambda(x) \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \Gamma(\omega^+) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \delta \lambda(x) \times \right.$$  

$$\times \delta \lambda(y) M_2 \delta \lambda(y) \times \delta \lambda(y) \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \Gamma(\omega^-) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \delta \lambda(y) \times \tilde{Z} \right\} |_{\lambda=0}$$  \(72)\)

For the pseudoscalar bound state $\tilde{S}_-$ the result takes the form

$$\Sigma_{\tilde{S}_-}(\omega_{\text{anomaly}}) = 4 \left[ Tr \left( \gamma_5 \Gamma^0 \Gamma(\omega^1) \right) \right] \times Tr \left( \gamma_5 \Gamma^0 \Gamma(\omega^2) \right]$$  \(73)\)

To find all paths which yield a non trivial contribution to the trace term (75) one has to find all the loops $\omega^i (i=1,2)$ for which

$$Tr[\gamma_5 \Gamma^0 \Gamma(\omega^i)] \neq 0$$  \(74)\)
Clearly this last equation is true if $\Gamma(\omega^i)$ contains a $\gamma_5$. This condition requires that $\omega^i$ covers bonds in any direction, since $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. The reader can convince himself that the shortest loops must have at least eight bonds. The allowed lowest order loops are plotted in Fig. 4. The spin trace (74) is the same for all these loops, up to a sign: $Tr[\gamma_5\Gamma^0\Gamma(\omega^i)] = \pm 32$.

An excitation diagram which contributes to the anomaly mass splitting is composed by two of the loops in Fig. 4 (one for each end of the excitation) connected by plaquettes to make a hypercolor singlet as required by the algorithm of section 3.2. The smallest excitations that can be made in this way are superpositions of two copies of the same loop, going in opposite direction. The lowest order of $\Delta m_{\tilde{S}_-}$ is given by the sum of all such diagrams. It is proportional to $k^{14}\beta_0$ with a positive coefficient. The leading order contributions to $\Delta m_{\tilde{S}_-}$ involving $\beta$ are proportional to $k^{14}\beta^n$ ($n=1,\ldots,4$). These terms are generated by excitations made out of two of the loops shown in Fig. 4, linked by $n$ plaquettes. It is easy to check from these excitations that when $\beta$ is turned on the effect is to increase the mass of the pseudoscalar bound state by

$$\Delta m_{\tilde{S}_-}^{\text{anomaly}} = 2^{12}k^{14}\left(405 - 216\beta + \frac{189}{2}\beta^2 - \frac{189}{8}\beta^3 + \frac{189}{32}\beta^4\right) \quad (75)$$

### 5.5 Conclusion

We have obtained two classes of particles in this theory:

- **Finite mass**: There are two bound states which have finite mass in the strong coupling expansion: The pseudoscalar bound state $\tilde{S}_-$ and the vector isotriplet bound state $\tilde{V}_{kI}$. The pseudoscalar bound state $\tilde{S}_-$ turns out to be lighter than the vector isotriplet bound state in the strong coupling expansion. It behaves like the $\eta'$ bound state in QCD and can acquire a larger mass by the effect of the chiral anomaly.

- **Heavy mass**: The composite operators $S_+,$ $\tilde{B}^{ijI}$ and $V^k$ (see sect. 5.1) correspond in the strong coupling expansion to states with infinite mass. Such a behavior is known in QCD, where for example the axial vector meson gets an infinite mass in the strong coupling expansion [11]. In fact, in nature, axial vector mesons are much heavier than the vector mesons.

The interesting feature of the use of Majorana fermions as basic constituents is that the spin one isotriplet bound states (having the correct assignment to be candidates for the weak bosons) can be the lightest bound state, provided that the chiral anomaly rises the mass of the pseudoscalar isosinglet sufficiently.

We have used the results of this work as the starting point of a detailed investigation of the low lying spectrum of this theory by a lattice Monte Carlo simulation [11]. In particular we have numerically evaluated the contribution of the chiral anomaly to the pseudoscalar isosinglet mass.
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Figure Caption

1. a) A possible graph of type I.  b) A possible graph of type II.

2. a) A static unrenormalized propagator.  b) An excitation of the static unrenormalized propagator.

3. Close up of an intermediate excitation.

4. The four geometrically distinct loops which contribute to the trace (74).
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