EXCHANGEABLE RANDOM PARTITIONS FROM
MAX-INFINITELY-DIVISIBLE DISTRIBUTIONS

STILIAN STOEV AND YIZAO WANG

Abstract. The hitting partitions are random partitions that arise from the
investigation of so-called hitting scenarios of max-infinitely-divisible distribu-
tions. We study a large class of max-infinitely-divisible distributions with
exchangeable hitting partitions obtained by size-biased sampling from jumps
of a subordinator. The Lévy measure of the subordinator governs the exponent
measure of the max-infinitely-divisible distribution, which is also expressed as
a scale-mixture of a multivariate max-stable law.

1. Introduction

Recently there has been a renewed interest in the study of multivariate records
in extreme value theory (e.g. 8–10 and references therein), motivated especially
by the latest advances on the so-called hitting scenarios for extremal events. The
notion of a hitting scenario originated from the investigation of the conditional
laws of max-stable processes [3, 20]. They also arise naturally in the expression
of the likelihood for max-stable models [16, 19], and play a crucial role in random
tessellations determined by max-stable processes [4]. The latest advances on hitting
scenarios are motivated by their connections to concurrence probabilities, and the
framework can be naturally stated in the language of random partitions [5, 6].

Our focus here is on the probabilistic aspects of the hitting partitions of multi-
variate max-stable distributions, recently introduced in [5]. We begin with recalling
the definition. First, let \( \{ \xi_\ell \}_{\ell \in \mathbb{N}} \) be a measurable enumeration of points of a Pois-
son point process on \( \mathbb{R}_+ \) with intensity \( \alpha x^{-\alpha - 1} dx \) for some \( \alpha > 0 \), and let \( \{ Y_\ell \}_{\ell \in \mathbb{N}} \) be i.i.d. copies of certain non-negative random vector \( Y = (Y_1, \ldots, Y_n) \) with finite
\( \alpha \)-moments. Then, it is well known, that the vector

\[
Z \equiv (Z_1, \ldots, Z_n) \equiv \left( \bigvee_{\ell=1}^\infty \xi_\ell Y_{\ell,1}, \ldots, \bigvee_{\ell=1}^\infty \xi_\ell Y_{\ell,n} \right)
\]

has a multivariate max-stable \( \alpha \)-Fréchet distribution [2]. Throughout, we write
\( \vee \equiv \max \). Recall that a random variable \( Z \) is said to be \( \alpha \)-Fréchet if
\( \mathbb{P}(Z \leq x) = \exp\{-\sigma^\alpha / x^\alpha\} \), for \( x \in (0, \infty) \), with scale parameter \( \sigma > 0 \). The vector \( Z \) is
max-stable with \( \alpha \)-Fréchet marginals if all its non-negative max-linear combinations
\( \bigvee_{k=1}^n a_k Z_k \) are \( \alpha \)-Fréchet distributed, for all \( a_k \geq 0 \), \( k = 1, \ldots, n \).

Date: June 15, 2018.
2010 Mathematics Subject Classification. Primary, 60G70, Secondary, 60C05.
Key words and phrases. exchangeable random partition, multivariate max-infinitely-divisible
distribution, Poisson–Dirichlet distribution, paintbox partition.
Given a max-stable Fréchet random vector with the representation (1.1), the induced hitting partition is determined as follows. Set

$$\ell^*(k) := \text{argmax}_{\ell \in \mathbb{N}} \{\xi_{\ell}Y_{\ell,k}\}, k = 1, \ldots, n.$$ 

Note that $\{\xi_{\ell}Y_{\ell,k}, \ell \in \mathbb{N}\}$ is a simple Poisson process on $(0, \infty)$ and hence with probability one, $\ell^*(k)$ is uniquely determined for every $k$; we restrict ourselves to this event from now on.

The hitting partition of $[n] \equiv \{1, \ldots, n\}$, for $n \in \mathbb{N}$, denoted by $\Pi_n$, is the random partition of equivalence classes induced by the relation

$$i \sim j \text{ if and only if } \ell^*(i) = \ell^*(j), \text{ for all } i, j \in [n].$$

Here, $i \sim j$ reads as $i$ and $j$ are in the same block of the partition. Recall that a partition of $[n]$ is a collection of disjoint sets, the union of which is $[n]$.

Thus far, most of the research on the hitting partitions has focused on the so-called concurrence probability, that is, the probability of the event that the hitting partition $\Pi_n$ consists of a single block [5, 6]. The concurrence probability has the following expression

$$p(n) \equiv P(\Pi_n = \{[n]\}) = \mathbb{E}\left(\frac{1}{\mathbb{E}(\bigvee_{k=1}^n Y_k^*/Y_k \mid Y)}\right),$$

where $Y^*$ is an independent copy of $Y$. This result was established in [5, Theorem 2.2] by using the Slyvniak–Mecke formula, and the same method in principle could yield formulae for the probability of the entire distribution of the hitting partition (see, e.g., [3]). The general expressions are however neither explicit nor intuitive.

The motivation of this paper is to study specific choices of $Y$, where the induced hitting partition has an explicit probability mass function. Our starting point is an example from a very recent paper [5, Example 3.1], where $\alpha \in (0, 1)$ and $Y = (Y_1, \ldots, Y_n)$ has i.i.d. 1-Fréchet components. Then, the distribution of $Z = (Z_1, \ldots, Z_n)$ in (1.1) becomes multivariate $\alpha$-logistic (see Example 3.2 below). In this case, the concurrence probability has a simple-looking formula

$$p(n) = \prod_{k=1}^{n-1} \left(1 - \frac{\alpha}{k}\right) = \frac{\prod_{k=1}^{n-1}(k-\alpha)}{(n-1)!}, n \in \mathbb{N}.$$ 

In this paper, first we explain this formula by showing that the hitting partition in this case is the exchangeable random partition induced by the Poisson–Dirichlet distribution with parameters $(\alpha, 0)$. Poisson–Dirichlet distributions and exchangeable random partitions are fundamental objects in combinatorial stochastic process, with numerous applications, notably in nonparametric inference and population genetics [1, 14]. An outstanding family of exchangeable random partitions are the ones induced by the Poisson–Dirichlet (PD($\alpha, \theta$)) distribution, referred to as the PD($\alpha, \theta$) partitions for short. The legitimate values of the parameters are $\alpha < 0$, $\theta = -m\alpha$ for some $m \in \mathbb{N}$ or $\alpha \in [0, 1], \theta > -\alpha$. For any selected partition of $[n]$ with block sizes $n_1, \ldots, n_k$ (such that $n_1 + \cdots + n_k = n, n_1, \ldots, n_k \geq 1$), the probability of a PD($\alpha, \theta$) taking the value of this partition equals

$$p_{\alpha, \theta}(n_1, \ldots, n_k) = \frac{(\theta + \alpha)_{k-1\alpha} \prod_{i=1}^{k}(1 - \alpha)_{n_i-1\alpha i\alpha}}{(\theta + 1)_n^{1\alpha}},$$

where $\prod$ is the product of all terms $1 - \alpha$ for all $i$ with $n_i \geq 1$. This formula is obtained by using the Slyvniak–Mecke formula. The general expressions are however neither explicit nor intuitive.
where \((x)_{m \uparrow} := \prod_{k=0}^{m-1} (x + k\alpha)\). (See [14, Theorem 3.2, Definition 3.3].) The Poisson–Dirichlet random partitions are actually exchangeable random partitions of \(\mathbb{N}\), although we focus on their restriction to \([n]\) most of the time.

**Proposition 1.1.** The hitting partition \(\Pi_n\) of the \(\alpha\)-logistic max-stable model \(\alpha \in (0,1)\), is a \(\text{PD}(\alpha, 0)\) partition.

The result follows essentially from the *paintbox representation* of exchangeable random partitions, to be reviewed in Section 2, and the max-stability property of Fréchet distributions. By recognizing the random weights in the paintbox representation as the (normalized) jumps of an \(\alpha\)-stable subordinator, we obtain that the hitting partition is in fact the \(\text{PD}(\alpha, 0)\) partition. Thus, our method is completely different from the one applied in [5].

Moreover, it turns out that the same idea of the proof can be applied to a larger family of hitting partitions, associated with a class of max-infinitely-divisible (max-i.d.) distributions. The hitting partitions of this family of max-i.d. distributions can be viewed as exchangeable random partitions obtained via *size-biased sampling* of jumps from a subordinator [14, Chapter 4.1], and we hence name the corresponding class of distributions *subordinated max-i.d. distributions*. Our main result Theorem 2.3 characterizes the connection between subordinated max-i.d. distributions and the corresponding exchangeable random partitions, and implies Proposition 1.1 as a special case. In addition, the more general framework beyond max-stable distributions allows us to identify another class of max-i.d. distributions, of which the hitting partitions have the \(\text{PD}(0, \theta)\) laws, for \(\theta > 0\).

The paper is organized as follows. In Section 2 we introduce the multivariate subordinated max-i.d. distributions and prove that their hitting partitions are exchangeable. In Section 3 we compute the cumulative distribution function of the subordinated max-i.d. distributions, and explain briefly how to extend the results to max-i.d. random sup-measures.

## 2. Hitting Partitions of Subordinated Max-i.d. Distributions

We shall consider a multivariate max-i.d. distribution with the following representation

\[
(\zeta_1, \ldots, \zeta_n) \equiv \left( \bigvee_{\ell=1}^{\infty} J_{\ell} Y_{\ell,1}, \ldots, \bigvee_{\ell=1}^{\infty} J_{\ell} Y_{\ell,n} \right),
\]

where \(J \equiv \{J_{\ell}\}_{\ell \in \mathbb{N}}\) is a Poisson point process on \(\mathbb{R}_+\) with intensity measure \(\nu\), and \(\{Y_\ell\}_{\ell \in \mathbb{N}}\) are i.i.d. random vectors independent from \(J\), each \(Y_\ell = (Y_{\ell,1}, \ldots, Y_{\ell,n})\) is a collection of independent 1-Fréchet random variables, with scale parameters \(\sigma = (\sigma_1, \ldots, \sigma_n) \in (0, \infty)^n\). The values of \(\sigma\) shall not have any impact in most of the discussions, until Remark 3.3 at the very end.

We assume throughout that

\[
\nu(\mathbb{R}_+) = \infty \quad \text{and} \quad \int_0^{\infty} (1 \wedge x) \nu(dx) < \infty.
\]

This ensures that the Poisson process \(J\) has infinitely many points and

\[
J_* := \sum_{\ell=1}^{\infty} J_{\ell} < \infty \ \text{a.s.}
\]
We interpret \( \{J_\ell\}_{\ell \in \mathbb{N}} \) as the jump sizes of a subordinator with Lévy measure \( \nu \), and \( J_\ell \) has the same law of such a subordinator at time 1. We name \( (\zeta_1, \ldots, \zeta_n) \) as a subordinated max-i.d. random vector with Lévy measure \( \nu \).

**Definition 2.1.** For all \( n \in \mathbb{N} \), set
\[
\ell^*(k) := \arg\max_{\ell \in \mathbb{N}} \{J_\ell Y_{\ell, k}\}, \quad k = 1, \ldots, n,
\]
and define now the random partition \( \Pi_n \) of \([n]\) for \( n \in \mathbb{N} \) by \([12]\) as before. We refer to the so-defined random partition as the *hitting partition* of the max-i.d. distribution in \([21]\).

As before, \( \ell^*(k) \) in \((2.2)\) is uniquely defined with probability one.

**Remark 2.2.** Given a max-i.d. vector, say \( \zeta = (\zeta_1, \ldots, \zeta_n) \), it is known that its law uniquely determines a Poisson representation. Namely, there exists a unique measure \( \mu \) on \((\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n))\), known as the Lévy (or exponent) measure of \( \zeta \), such that for all \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \),
\[
\mathbb{P}(\zeta_k \leq x_k, \quad k = 1, \ldots, n) = \exp \left( -\mu([0, x]^n) \right),
\]
and, for a Poisson point process \( \Psi = \{\Psi_\ell\}_{\ell \in \mathbb{N}} \) on \( \mathbb{R}_+^n \) with mean measure \( \mu \),
\[
(\zeta_1, \ldots, \zeta_n) \overset{d}{=} \left( \sup_{\ell \in \mathbb{N}} \Psi_{\ell, 1}, \ldots, \sup_{\ell \in \mathbb{N}} \Psi_{\ell, n} \right)
\]
(see \([13]\) Chapter 5)). Our starting point is the assumption that the Poisson point process \( \Psi \) has the disintegration representation (not unique in general)
\[
(\zeta_1, \ldots, \zeta_n) \overset{d}{=} (J_\ell Y_{\ell, 1}, \ldots, J_\ell Y_{\ell, n}).
\]
Clearly, the hitting partition does not depend on this specific representation, but only on \( \Psi \) (as done in \([3]\) and hence uniquely by the law of \( \zeta \). Our focus here is on the special case where the representation \((2.3)\) holds with \((Y_{\ell, 1}, \ldots, Y_{\ell, n})\) being independent and Fréchet.

Our main result is to show that in this framework, the random partition \( \Pi_n \) is exchangeable, and in particular it has the same law as a paintbox partition (a.k.a. partition generated by random sampling) with random weights
\[
P_\ell := \frac{J_\ell}{\sum_{\ell'=1}^\infty J_{\ell'}}, \quad \ell \in \mathbb{N} \quad \text{and} \quad P_0 := 0.
\]
We first review the background of the paintbox construction. Recall that a paintbox partition with respect to weight \( s = (s_0, s_1, \ldots) \) such that \( s_0 \geq 0, s_1 \geq s_2 \geq \cdots \geq s_n \geq \cdots \geq 0 \) and \( \sum_{k=0}^\infty s_k = 1 \), is a canonical way to obtain exchangeable random partitions of \( \mathbb{N} \) as follows. Let \( \{X_n\}_{n \in \mathbb{N}} \) be i.i.d. sampling from \( \mathbb{N}_0 \) with distribution \( \mathbb{P}(X_1 = \ell) = s_\ell, \ell \in \mathbb{N}_0 \). Color the set of natural numbers \( \mathbb{N} = \{1, 2, \ldots\} \) as follows. If \( X_\ell > 0 \), we paint \( \ell \) in color \( \mathbb{C}_i \), otherwise, all \( i \)'s with \( X_i = 0 \) are colored in different colors that are also different from all other colors used in the paintbox. Thus \( \mathbb{N} \) is partitioned into disjoint blocks by different colors. Formally, this partition is induced by the equivalence relation \( i \sim j \) if \( X_i = X_j > 0 \) for all \( i, j \in \mathbb{N} \). Notice that \( i \in \mathbb{N} \) such that \( X_i = 0 \) forms a singleton block by itself.

It is well known and easy to see that the so-obtained partition of \( \mathbb{N} \) is exchangeable. Moreover, Kingman’s representation theorem \([14]\) Theorem 2.2) says that every exchangeable random partition of \( \mathbb{N} \) can be obtained by a paintbox partition
with possibly random weights $s$. In this case, conditionally on $s$, $\{X_n\}_{n\in\mathbb{N}}$ are i.i.d. with distribution $P(X_1 = \ell \mid s) = s_\ell, \ell \in \mathbb{N}$.

Therefore, a convenient way to characterize the law of an infinite exchangeable partition is to identify the random weights of the corresponding paintbox partition. Our discussions focus on finite partitions: if a finite partition can be obtained by a paintbox partition with a finite number of i.i.d. samplings, it is clearly exchangeable, with the law determined by the weights, and we still refer to it as a paintbox partition.

**Theorem 2.3.** For all $n \in \mathbb{N}$, the hitting partition $\Pi_n$ associated to (2.1) is an exchangeable random partition of $[n]$, which has the same law as a paintbox partition with random weights $\{P_\ell\}_{\ell \in \mathbb{N}_0}$ given by (2.4).

**Proof.** To see this, we first observe that conditioning on $J$, for each $k = 1, \ldots, n$, the distribution of $\ell^*(k)$ is determined by $\tilde{P}_\ell(k) \equiv P(\ell^*(k) = \ell \mid J), \ell \in \mathbb{N}$, and conditioning on $J$ we have that $\{\ell^*(k)\}_{k=1,\ldots,n}$ are independent, since so are the $Y_{\ell,k}$'s. The probability $\tilde{P}_\ell(k)$ turns out to be independent from $k$. Indeed, we have

$$\tilde{P}_\ell(k) = P(J_\ell Y_{\ell,k} > \max_{\ell' \neq \ell} J_{\ell'} Y_{\ell',k} \mid J)$$

$$= P(J_\ell Y_{1,k} > \left(\sum_{\ell' \neq \ell} J_{\ell'}\right) Y_{2,k} \mid J)$$

$$= \frac{J_\ell}{\sum_{\ell'=1}^\infty J_{\ell'}} = P_\ell, \ell \in \mathbb{N}.$$  

Relation (2.5) follows from the max-stability of the Fréchet distribution. Namely, the property that

$$\bigvee_{\ell=1}^\infty a_\ell Y_\ell \overset{d}{=} \left(\sum_{\ell=1}^\infty a_\ell\right) Y_1$$

for i.i.d. 1-Fréchet random variables $\{Y_\ell\}_{\ell \in \mathbb{N}}$. Relation (2.6), on the other hand follows from the property

$$P(a Y_1 > b Y_2) = \frac{a}{a + b},$$

valid for all $a, b > 0$ where $Y_1$ and $Y_2$ are i.i.d. 1-Fréchet random variables. This completes the proof.  

The aforementioned framework of exchangeable random partitions based on jump sizes of subordinators (2.4) is well known. In fact, if the Lévy measure $\nu$ has a density $\rho$, then under mild conditions explicit formula for the random partition generalizing (1.4) is available in terms of $\rho$. See [13] and [14, Exercise 4.1.2]. To keep the presentation short, we shall instead explain only two special examples in full detail here. Recall that the Poisson–Dirichlet distribution refers to a two-parameter family of ranked frequencies of $\{P_\ell\}_{\ell \in \mathbb{N}_0}$, indexed by $(\alpha, \theta)$ with either $\alpha < 0, \theta = -m\alpha$ for some $m \in \mathbb{N}$, or $\alpha \in [0,1], \theta > -\alpha$. When the frequencies are ordered in size-biased order, the corresponding law of the same two-parameter
family is known as the Griffiths–Engen–McCloskey (GEM) distribution. The size-biased frequencies, denoted by \( \{ \tilde{P}_\ell \}_{\ell \in \mathbb{N}} \), have the representation
\[
(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \ldots) \overset{d}{=} (W_1, (1 - W_1)W_2, (1 - W_1)(1 - W_2)W_3, \ldots),
\]
where \( \{ W_\ell \}_{\ell \in \mathbb{N}} \) are independent beta random variables with parameters \((1 - \alpha, \theta + \ell \alpha)\) respectively.

In particular, explicit examples relating random weights from subordinators via \((2.4)\) to Poisson–Dirichlet distributions have been well known, and we shall make use the following two from them (see [14, Chapter 4.2]):

(i) If \( \nu(dx) = \alpha x^{\alpha - 1} dx, \alpha \in (0, 1) \), then the subordinator is an \( \alpha \)-stable subordinator. The ranked frequencies have the law of PD\((\alpha, 0)\).

(ii) If \( \nu(dx) = \theta x^{\alpha - 1} e^{-x} dx, \theta > 0 \), then the subordinator is a Gamma process.

The ranked frequencies have the law of PD\((0, \theta)\).

The following corollary follows immediately, including Proposition 1.1 as the first case.

**Corollary 2.4.** For the subordinated max-i.d. distribution \((2.1)\) with Lévy measure \( \nu \), the induced hitting partition \( \Pi_n \) is:

(i) PD\((\alpha, 0)\), \( \alpha \in (0, 1) \) if \( \nu(dw) = \alpha w^{\alpha - 1} dw \).

(ii) PD\((0, \theta)\), \( \theta > 0 \) if \( \nu(dw) = \theta w^{\alpha - 1} e^{-w} dw \).

**Remark 2.5.** The class of hitting partitions arising from \((2.1)\) does not contain all exchangeable random partitions. By allowing dependence among, and/or other types of distributions of, \( Y_{\ell,1}, \ldots, Y_{\ell,n} \), one could obtain other exchangeable random partitions by the same mechanism. In particular, we do not see how the PD\((\alpha, \theta)\) partition arises from the subordinated max-i.d. distribution for other choices of the parameters.

**Remark 2.6.** The paintbox argument in the proof of Theorem 2.3 applies without change to the case where \((2.1)\) is an infinite max-i.d. sequence indexed by \( \mathbb{N} \). In this case, one obtains exchangeable partitions of \( \mathbb{N} \). We stated Theorem 2.3 in the finite-dimensional setting of partitions on \([n]\) for simplicity and in order to draw connections to the exiting results on the concurrence probability (e.g. formula \((1.3)\)).

3. **Cumulative distribution functions of max-i.d. distributions**

We have proposed a family of max-i.d. distributions, motivated by their connection to exchangeable random partitions. We now examine their finite-dimensional distributions, which can be expressed through the Laplace transform of the subordinator random variable \( J_\ast \).

**Proposition 3.1.** For \((\zeta_1, \ldots, \zeta_n)\) as in \((2.1)\),
\[
(3.1) \quad \mathbb{P}(\zeta_k \leq x_k, k = 1, \ldots, n) = L_{J_\ast} \left( \sum_{k=1}^{n} \frac{\sigma_k}{x_k} \right), x_k \geq 0, k = 1, \ldots, n,
\]
where
\[
L_{J_\ast}(t) := \mathbb{E} \exp(-tJ_\ast) = \exp \left( - \int_{0}^{\infty} (1 - e^{-tx}) \nu(dx) \right), t > 0
\]
is the Laplace transform of \( J_\ast \). In particular,
\[
(3.2) \quad (\zeta_1, \ldots, \zeta_n) \overset{d}{=} J_\ast(Y_{1,1}, \ldots, Y_{1,n}).
\]
exchangeable hitting partitions

Proof. Observe that
\[
\{\zeta_k \leq x, \; k = 1, \ldots, n\} = \left\{ \bigvee_{\ell=1}^\infty J_{\ell} \bigvee_{k=1}^n \frac{Y_{\ell,k}}{x_k} \leq 1 \right\}.
\]
Furthermore, note that
\[
Y_{\ell,k}/d = \sigma_k \mathbf{Z}_{\ell,k},
\]
where the \(\mathbf{Z}_{\ell,k}\) are i.i.d. standard 1-Fréchet (with scale parameter 1). Therefore, by conditioning on \(\mathbf{J}\) and applying the max-stability property, we obtain
\[
\left(\bigvee_{\ell=1}^\infty J_{\ell} \bigvee_{k=1}^n \frac{\sigma_k \mathbf{Z}_{\ell,k}}{x_k} = \bigvee_{\ell=1}^\infty J_{\ell} \bigvee_{k=1}^n \frac{\sigma_k \mathbf{Z}_{\ell,k}}{x_k} \right) = \left(\sum_{k=1}^n \frac{\sigma_k \mathbf{Z}_{1,1}}{x_k} \right) J_{\ast} \left(\sum_{k=1}^n \frac{\sigma_k \mathbf{Z}_{1,1}}{x_k} \right).\]
This proves (3.1). For the second part, it suffices to check the finite-dimensional distributions of the right-hand side of (3.2) are the same as in (3.1), which follows from a similar calculation as above. \(\square\)

In particular, the PD(\(\alpha, 0\)) and PD(0, \(\theta\)) partitions are the hitting partitions of the following two families of max-i.d. distributions, respectively.

**Example 3.2** (\(\alpha\)-Logistic max-stable distribution). Suppose that \(J_{\ast}\) is obtained from the standard \(\alpha\)-stable subordinator for some \(\alpha \in (0, 1)\) with \(\nu(dx) = \Gamma(1 - \alpha) \alpha x^{-\alpha - 1}dx\) (this is different from our earlier choice by a multiplicative constant, which does not effect the law of the hitting partition, but \(L_{J_{\ast}}\)). Then, \(L_{J_{\ast}}(t) = e^{-t^\alpha}, \; t > 0\), and (3.1) implies that
\[
\mathbb{P}(\zeta_k \leq x, \; k = 1, \ldots, n) = \exp\left(-\left(\sum_{k=1}^n \frac{\sigma_k}{{x_k}}\right)^\alpha\right).
\]
Moreover, this max-i.d. distribution is in particular a max-stable Fréchet distribution. This is essentially the \(\alpha\)-logistic max-stable distribution, which is conventionally standardized to have 1-Fréchet marginals with scale parameter 1 (corresponding to \((\zeta_1^\alpha, \ldots, \zeta_n^\alpha)\) here). See \([\ref{7}]\) and the references therein for more details and extensions of the logistic family of distributions.

**Example 3.3** (Gamma-subordinated max-i.d. distribution). Suppose \(\nu(dx) = \theta x^{-1}e^{-x}dx\) for some \(\theta > 0\). In this case, \(J_{\ast}\) is well known to have the Gamma (\(\theta, 1\)) distribution, and
\[
L_{J_{\ast}}(t) = \frac{1}{\Gamma(\theta)} \int_0^\infty e^{-tx}x^{\theta-1}e^{-x}dx = \frac{1}{(1+t)^\theta}.
\]
Relation (3.1) reads
\[
\mathbb{P}(\zeta_k \leq x, \; k = 1, \ldots, n) = \left(1 + \sum_{k=1}^n \frac{\sigma_k}{{x_k}}\right)^{-\theta}, \; x_k \geq 0, k = 1, \ldots, n.
\]

**Remark 3.4.** The subordinated max-i.d. distributions can be extended immediately to max-i.d. random sup-measures, a topic which has raised some recent interest in the literature \([\ref{11}, \ref{12}]\). Indeed, the law of the corresponding random sup-measures \(\mathcal{M}\) (as a random element in the space of sup-measures equipped with the
sup-vague topology) is uniquely determined by its finite-dimensional distributions \cite[Theorem 11.5]{18}, which essentially we have already computed in Proposition 3.1.

Namely, we can define a family of max-i.d. random sup-measures on a generic measurable space \((E, \mathcal{E})\) equipped with a \(\sigma\)-finite measure \(\mu\), in the form of

\[ M(\cdot) \equiv M_{\mu, \nu}(\cdot) := \bigvee_{\ell=1}^{\infty} J_\ell N_\ell(\cdot), \]

where \(J \equiv \{J_\ell\}_{\ell \in \mathbb{N}}\) a Poisson point process with mean measure \(\nu\) as before and \(\{N_\ell\}_{\ell \in \mathbb{N}}\) are i.i.d. independently scattered 1-Fréchet random sup-measures on \((E, \mathcal{E})\) with control measure \(\mu\) \cite{17}, independent from \(J\). Proposition 3.1 then yields

\[ \mathbb{P}(M(A_{k}) \leq x_k, \ k = 1, \ldots, n) = L_{J_*} \left( \sum_{k=1}^{n} \frac{\mu(A_k)}{x_k} \right), \quad x_k \geq 0, \ k = 1, \ldots, n, \]

for disjoint \(\{A_k\}_{k=1, \ldots, n}\), which, as mentioned earlier, determines the law of \(M\).

In the case that \((E, \mathcal{E}) \equiv (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\), this has the following immediate consequences: if \(\mu\) is translation-invariant \((\mu(x + \cdot) = \mu(\cdot) \text{ for all } x \in \mathbb{R}^d)\), then \(M\) is translation-invariant in the sense that \(M(x + \cdot) \overset{d}{=} M(\cdot)\); if \(\mu\) is homogeneous \((\mu(\lambda \cdot) = \lambda^H \mu(\cdot) \text{ for all } \lambda > 0 \text{ for some } H > 0)\), then \(M\) is self-similar in the sense that \(M(\lambda \cdot) \overset{d}{=} \lambda^H M(\cdot)\), for all \(\lambda > 0\). The corresponding logistic random sup-measure with \((E, \mathcal{E}, \mu) \equiv (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \text{Leb})\) and \(J\) corresponding to Example 3.2 seems the most interesting. It is the only example within the class of subordinated max-i.d. random sup-measures that is max-stable, translation-invariant, and self-similar.

**Acknowledgement.** SS’s research was partially supported by the NSF FRG grant DMS-1462368. YW’s research was partially supported by NSA grant H98230-16-1-0322 and Army Research Laboratory grant W911NF-17-1-0006.

**References**

[1] Berestycki, N. (2009). *Recent progress in coalescent theory*, volume 16 of *Ensaios Matemáticos [Mathematical Surveys*]. Sociedade Brasileira de Matemática, Rio de Janeiro.

[2] de Haan, L. (1984). A spectral representation for max-stable processes. *Ann. Probab.*, 12(4):1194–1204.

[3] Dombry, C. and Eyi-Minko, F. (2013). Regular conditional distributions of continuous max-infinitely divisible random fields. *Electron. J. Probab*, 18(7):1–21.

[4] Dombry, C. and Kabluchko, Z. (2018). Random tessellations associated with max-stable random fields. *Bernoulli*, 24(1):30–52.

[5] Dombry, C., Ribatet, M., and Stoew, S. (2017). Probabilities of concurrent extremes. *Journal of the American Statistical Association*, to appear.

[6] Dombry, C. and Zott, M. (2018). Multivariate records and hitting scenarios. *Extremes*, 21(2):343–361.

[7] Fougères, A.-L., Nolan, J. P., and Rootzén, H. (2009). Models for dependent extremes using stable mixtures. *Scand. J. Stat.*, 36(1):42–59.

[8] Gnedin, A. V. (2007). The chain records. *Electron. J. Probab.*, 12:no. 26, 767–786.

[9] Goldie, C. M. and Resnick, S. (1989). Records in a partially ordered set. *Ann. Probab.*, 17(2):678–699.
[10] Hashorva, E. and Hüsler, J. (2005). Multiple maxima in multivariate samples. *Statist. Probab. Lett.*, 75(1):11–17.
[11] Molchanov, I. and Strokorb, K. (2016). Max-stable random sup-measures with comonotonic tail dependence. *Stochastic Process. Appl.*, 126(9):2835–2859.
[12] O’Brien, G. L., Torfs, P. J. J. F., and Vervaat, W. (1990). Stationary self-similar extremal processes. *Probab. Theory Related Fields*, 87(1):97–119.
[13] Pitman, J. (2003). Poisson-Kingman partitions. In *Statistics and science: a Festschrift for Terry Speed*, volume 40 of *IMS Lecture Notes Monogr. Ser.*, pages 1–34. Inst. Math. Statist., Beachwood, OH.
[14] Pitman, J. (2006). *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.
[15] Resnick, S. I. (1987). *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability*. A Series of the *Applied Probability Trust*. Springer-Verlag, New York.
[16] Stephenson, A. and Tawn, J. (2005). Exploiting occurrence times in likelihood inference for componentwise maxima. *Biometrika*, 92(1):213–227.
[17] Stoev, S. A. and Taqqu, M. S. (2005). Extremal stochastic integrals: a parallel between max-stable processes and $\alpha$-stable processes. *Extremes*, 8(4):237–266 (2006).
[18] Vervaat, W. (1997). Random upper semicontinuous functions and extremal processes. In *Probability and lattices*, volume 110 of *CWI Tract*, pages 1–56. Math. Centrum, Centrum Wisk. Inform., Amsterdam.
[19] Wadsworth, J. L. and Tawn, J. A. (2013). A new representation for multivariate tail probabilities. *Bernoulli*, 19(5B):2689–2714.
[20] Wang, Y. and Stoev, S. A. (2011). Conditional sampling for spectrally-discrete max-stable random fields. *Adv. in Appl. Probab.*, 43(2):463–481.

Stilian Stoev, Department of Statistics, University of Michigan, Ann Arbor., 439 W. Hall, 1085 S. University, Ann Arbor, MI 48109-1107, USA.  
E-mail address: stoev@umich.edu

Yizao Wang, Department of Mathematical Sciences, University of Cincinnati, 2815 Commons Way, Cincinnati, OH, 45221-0025, USA.  
E-mail address: yizao.wang@uc.edu