PROJECTIVE MODULE DESCRIPTION OF EMBEDDED NONCOMMUTATIVE SPACES

R.B. ZHANG AND XIAO ZHANG

ABSTRACT. An algebraic formulation is given for the embedded noncommutative spaces over the Moyal algebra developed in a geometric framework in [8]. We explicitly construct the projective modules corresponding to the tangent bundles of the embedded noncommutative spaces, and recover from this algebraic formulation the metric, Levi-Civita connection and related curvatures, which were introduced geometrically in [8]. Transformation rules for connections and curvatures under general coordinate changes are given. A bar involution on the Moyal algebra is discovered, and its consequences on the noncommutative differential geometry are described.

CONTENTS

1. Introduction 1
2. Moyal algebra and projective modules 4
3. Differential geometry of noncommutative vector bundles 5
  3.1. Connections and curvatures 6
  3.2. Gauge transformations 8
  3.3. Vector bundles associated to right projective modules 9
  3.4. Canonical connections and fibre metric 11
4. Embedded noncommutative spaces 11
  4.1. Embedded noncommutative spaces 11
  4.2. Example 14
5. General coordinate transformations 18
6. Bar involution and generalised Hermitian structure 20
7. Concluding remarks 22
References 24

1. INTRODUCTION

It is a long held belief in physics that the notion of spacetime as a pseudo Riemannian manifold requires modification at the Planck scale [34, 38]. Theoretical investigations in recent times strongly supported this view. In particular, the seminal paper [16] by Doplicher, Fredenhagen and Roberts demonstrated mathematically that coordinates of spacetime became noncommutative at the Planck scale, thus some form of
noncommutative geometry \[13\] appeared to be necessary in order to describe the structure of spacetime. This prompted intensive activities in mathematical physics studying various noncommutative generalisations of Einstein’s theory of general relativity \[10, 11, 29, 5, 30, 3, 1, 8, 9, 7, 6\]. For reviews on earlier works, we refer to \[31, 35\] and references therein. For more recent developments, particularly on the study of noncommutative black holes, see \[9, 7, 37, 33, 15, 2, 27, 26, 4\].

In joint work with Chaichian and Tureanu \[8\], we investigated the noncommutative geometry \[13, 22\] of noncommutative spaces embedded in higher dimensions. We first quantised a space by deforming \[21, 28\] the algebra of functions to a noncommutative associative algebra known as the Moyal algebra. Such an algebra naturally incorporates the generalised spacetime uncertainty relations of \[16\], capturing key features expected of spacetime at the Planck scale. We then systematically investigated the noncommutative geometry of embedded noncommutative spaces. This was partially motivated by Nash’s isometric embedding theorem \[32\] and its generalisation to pseudo-Riemannian manifolds \[19, 12, 23\], which state that any (pseudo-) Riemannian manifold can be isometrically embedded in Euclidean or Minkowski spaces. Therefore, in order to study the geometry of spacetime, it suffices to investigate (pseudo-) Riemannian manifolds embedded in higher dimensions. Embedded noncommutative spaces also play a role in the study of branes embedded in \(\mathbb{R}^D\) in the context of Yang-Mills matrix models \[36\].

The theory of \[8\] was developed within a geometric framework analogous to the classical theory of embedded surfaces (see, e.g., \[14\]). The present paper further develops the differential geometry of embedded noncommutative spaces by constructing an algebraic formulation in terms of projective modules, a language commonly adopted in noncommutative geometry \[13, 22\].

We shall first describe the finitely generated projective modules over a Moyal algebra, which will be regarded as noncommutative vector bundles on a quantised spacetime. We then construct a differential geometry of the noncommutative vector bundles, developing a theory of connections and curvatures on such bundles. In doing this, we make crucial use of a unique property of the Moyal algebra, namely, it has a set of mutually commutative derivations related to the usual partial derivations of functions.

Then we apply the noncommutative differential geometry developed to study the embedded noncommutative spaces introduced in \[8\]. We explicitly construct the projective modules corresponding to the tangent bundles of the noncommutative spaces, and recover from this algebraic formulation the geometric Levi-Civita connections and related curvatures introduced in \[8\]. This way, the embedded noncommutative spaces of \[8\] acquire a natural interpretation in the algebraic formalism present here.

Morally one may regard the very definition of a projective module (a direct summand of a free module) as the geometric equivalent of embedding a low dimensional manifold isometrically in a higher dimensional one. In the commutative setting of classical (pseudo-) Riemannian geometry, we make this connection more precise and explicit by showing that the projective module description of tangent bundles studied
here is a natural consequence of the isometric embedding theorems [32, 19, 12, 23]. This is briefly discussed in Theorem 7.1.

As a concrete example of noncommutative differential geometries over the Moyal algebra, we study in detail a quantum deformation of a time slice of the Schwarzschild spacetime. The projection operator yielding the tangent bundle is given explicitly, and the corresponding metric is also worked out.

As is well known, one of the fundamental principles of general relativity is general covariance. It is important to find a noncommutative version of this principle. By analyzing the structure of the Moyal algebra, we show that the noncommutative geometry developed here (initiated in [8]) retains some notion of “general covariance”. Properties of the connection and curvature under general coordinate transformations are described explicitly (see Theorem 5.1).

The Moyal algebra (over the real numbers) admits an involution similar to the bar involution in the context of quantum groups. We introduce a particularly nice class of noncommutative vector bundles over the Moyal algebra, which are associated to bar invariant idempotents and endowed with bar hermitian connections (see Section 6). In this case the bar involution takes the left tangent bundles to right tangent bundles. We show that the tangent bundles of embedded noncommutative spaces under a middle condition belong to this class.

The organisation of the paper is as follows. In Section 2, we describe the Moyal algebras and finitely generated projective modules over them. In Section 3, we discuss the differential geometry of noncommutative vector bundles on quantum spaces corresponding to Moyal algebras. In Section 4, we develop the differential geometry of embedded noncommutative spaces using the language of projective modules. As an explicit example, we study in detail the quantum deformation of a time slice of the Schwarzschild spacetime in Section 4.2. In Section 5, we study the effect of general coordinate transformations. In Section 6, we investigate properties of noncommutative vector bundles under the bar involution of the Moyal algebra. Finally, Section 7 concludes the paper with some general comments and a discussion of the natural relationship between projective modules and isometric embeddings in classical (pseudo-) Riemannian geometry.

Before closing this section, we mention that the theory of [8] has the advantage of being explicit and easy to use for computations. Using this theory, we constructed noncommutative Schwarzschild and Schwarzschild-de Sitter spacetimes in joint work with Wang [37]. Our long term aim is to develop a theoretical framework for studying noncommutative general relativity. A variety of physically motivated methods and techniques were used in the literature to study corrections to general relativity arising from the noncommutativity of the Moyal algebra. In particular, references [3, 11] studied deformations of the diffeomorphism algebra as a means for incorporating noncommutative effects of spacetime, while in [9, 7, 6] a gauge theoretical approach was taken. These approaches differ considerably from the theory of [8, 37] at the mathematical level.
We describe the Moyal algebra of smooth functions on an open region of $\mathbb{R}^n$, and the finitely generated projective modules over the Moyal algebra. This provides the background material needed in later sections, and also serves to fix notations.

We take an open region $U$ in $\mathbb{R}^n$ for a fixed $n$, and write the coordinate of a point $t \in U$ as $(t^1, t^2, \ldots, t^n)$. Let $\hbar$ be a real indeterminate, and denote by $\mathbb{R}[[\hbar]]$ the ring of formal power series in $\hbar$. Let $A$ be the set of formal power series in $\hbar$ with coefficients being real smooth functions on $U$. Namely, every element of $A$ is of the form $\sum_{i \geq 0} f_i \hbar^i$ where $f_i$ are smooth functions on $U$. Then $A$ is an $\mathbb{R}[[\hbar]]$-module in the obvious way.

Fix a constant skew symmetric $n \times n$ matrix $\theta = (\theta_{ij})$. The Moyal product on $A$ corresponding to $\theta$ is a map $\mu : A \otimes_{\mathbb{R}[[\hbar]]} A \rightarrow A$, $f \otimes g \mapsto \mu(f, g)$, defined by

$$\mu(f, g)(t) = \lim_{t' \rightarrow t} \exp \frac{\hbar \sum_{i,j} \theta_{ij} \frac{\partial}{\partial t^i} \frac{\partial}{\partial t'^j}}{\hbar} f(t)g(t').$$

(2.1)

On the right hand side, $f(t)g(t')$ means the usual product of the numerical values of the functions $f$ and $g$ at $t$ and $t'$ respectively.

It has been known since the early days of quantum mechanics that the Moyal product is associative (see, e.g., [28] for a reference). Thus the $\mathbb{R}[[\hbar]]$-module $A$ equipped with the Moyal product forms an associative algebra over $\mathbb{R}[[\hbar]]$, which is a deformation of the algebra of smooth functions on $U$ in the sense of [21]. We shall usually denote this associative algebra by $A$, but when it is necessary to make explicit the multiplication, we shall write it as $(A, \mu)$.

The partial derivations $\partial_i := \frac{\partial}{\partial t^i}$ with respect to the coordinates $t^i$ for $U$ are $\mathbb{R}[[\hbar]]$-linear maps on $A$. Since $\theta$ is a constant matrix, the Leibniz rule is valid. Namely, for any element $f$ and $g$ of $A$, we have

$$\partial_i \mu(f, g) = \mu(\partial_i f, g) + \mu(f, \partial_i g).$$

(2.2)

Therefore, the $\partial_i$ ($i = 1, 2, \ldots, n$) are mutually commutative derivations of the Moyal algebra $(A, \mu)$ on $U$.

Remark 2.1. The usual notation in the literature for $\mu(f, g)$ is $f \ast g$. This is referred to as the star-product of $f$ and $g$. Hereafter we shall replace $\mu$ by $\ast$ and simply write $\mu(f, g)$ as $f \ast g$.

Following the general philosophy of noncommutative geometry [13], we regard the associative algebra $(A, \mu)$ as defining some quantum deformation of the region $U$, and finitely generated projective modules over $A$ as (spaces of sections of) noncommutative vector bundles on the quantum deformation of $U$ defined by the noncommutative algebra $A$. Let us now briefly describe finitely generated projective modules over $A$. 

2. MOYAL ALGEBRA AND PROJECTIVE MODULES
Given an integer \( m > n \), we let \( \mathcal{A}^m \) (resp. \( \mathcal{A}_{r}^m \)) be the set of \( m \)-tuples with entries in \( \mathcal{A} \) written as rows (respectively, columns). We shall regard \( \mathcal{A}^m \) (respectively, \( \mathcal{A}_{r}^m \)) as a left (respectively, right) \( \mathcal{A} \)-module with the action defined by multiplication from the left (respectively, right). More explicitly, for \( v = (a_1 \ a_2 \ \ldots \ a_m) \in \mathcal{A}^m \), and \( b \in \mathcal{A} \), we have \( b \ast v = (b \ast a_1 \ b \ast a_2 \ \ldots \ b \ast a_m) \). Similarly for \( w = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \in \mathcal{A}_{r}^m \), we have \( w \ast b = \begin{pmatrix} a_1 \ast b \\ a_2 \ast b \\ \vdots \\ a_m \ast b \end{pmatrix} \). Let \( \mathcal{M}_m(\mathcal{A}) \) be the set of \( m \times m \)-matrices with entries in \( \mathcal{A} \). We define matrix multiplication in the usual way but by using the Moyal product for products of matrix entries, and still denote the corresponding matrix multiplication by \( \ast \). Now for \( A = (a_{ij}) \) and \( B = (b_{ij}) \), we have \( (A \ast B) = (c_{ij}) \) with \( c_{ij} = \sum_k a_{ik} \ast b_{kj} \). Then \( \mathcal{M}_m(\mathcal{A}) \) is an \( \mathbb{R}[\hbar] \)-algebra, which has a natural left (respectively, right) action on \( \mathcal{A}_{r}^m \) (respectively, \( \mathcal{A}^m \)).

A finitely generated projective left (respectively, right) \( \mathcal{A} \)-module is isomorphic to some direct summand of \( \mathcal{A}^m \) (respectively, \( \mathcal{A}_{r}^m \)) for some \( m < \infty \). If \( e \in \mathcal{M}_m(\mathcal{A}) \) satisfies the condition \( e \ast e = e \), that is, it is an idempotent, then

\[
\mathcal{M} = \mathcal{A}^m \ast e := \{ v \ast e \mid v \in \mathcal{A}^m \}, \quad \mathcal{N} = e \ast \mathcal{A}_{r}^m := \{ e \ast w \mid \mathcal{A}_{r}^m \}
\]

are respectively projective left and right \( \mathcal{A} \)-modules. Furthermore, every projective left (right) \( \mathcal{A} \)-module is isomorphic to an \( \mathcal{M} \) (respectively, \( \mathcal{N} \)) constructed this way by using some idempotent \( e \).

In Section 4, we shall give a systematic method for constructing idempotents (see (4.1)). The corresponding noncommutative vector bundles include the tangent bundles of embedded noncommutative spaces introduced in [8], which we shall investigate in depth. An explicit example of embedded noncommutative spaces will be analyzed in detail in Section 4.2 To do this, we need to develop some generalities of the differential geometry of noncommutative vector bundles using the language of projective modules over the Moyal algebra.

3. Differential Geometry of Noncommutative Vector Bundles

In this section we investigate general aspects of the noncommutative differential geometry over the Moyal algebra. We shall focus on the abstract theory here. A large class of examples will be given in Section 4 including one which will be worked out in detail.

As we shall see, the set of mutually commutative derivations \( \partial_i \) \((i = 1, 2, \ldots, n)\) of the Moyal algebra \( \mathcal{A} \) will play a crucial role in developing the noncommutative differential geometry.
3.1. Connections and curvatures. We start by considering the action of the partial derivations \( \partial_i \) on \( \mathcal{M} \) and \( \hat{\mathcal{M}} \). We only treat the left module in detail, and present the pertinent results for the right module at the end, since the two cases are similar.

Let us first specify that \( \partial_i \) acts on rectangular matrices with entries in \( \mathcal{A} \) by componentwise differentiation. More explicitly,

\[
\partial_i B = \begin{pmatrix}
\partial_i b_{11} & \partial_i b_{12} & \ldots & \partial_i b_{1l} \\
\partial_i b_{21} & \partial_i b_{22} & \ldots & \partial_i b_{2l} \\
\vdots & \ldots & \ldots & \ldots \\
\partial_i b_{k1} & \partial_i b_{k2} & \ldots & \partial_i b_{kl}
\end{pmatrix}
\quad \text{for} \quad B = \begin{pmatrix}
b_{11} & b_{12} & \ldots & b_{1l} \\
b_{21} & b_{22} & \ldots & b_{2l} \\
\vdots & \ldots & \ldots & \ldots \\
b_{k1} & b_{k2} & \ldots & b_{kl}
\end{pmatrix}.
\]

In particular, given any \( \zeta = v \ast e \in \mathcal{M} \), where \( v \in \mathcal{A}^m \) regarded as a row matrix, we have \( \partial_i \zeta = (\partial_i v) \ast e + v \ast \partial_i(e) \) by the Leibniz rule. While the first term belongs to \( \mathcal{M} \), the second term does not in general. Therefore, \( \partial_i \) (\( i = 1, 2, \ldots, n \)) send \( \mathcal{M} \) to some subspace of \( \mathcal{A}^m \) different from \( \mathcal{M} \).

Let \( \omega_i \in \mathcal{M}_m(\mathcal{A}) \) \( (i = 1, 2, \ldots, n) \) be \( m \times m \)-matrices with entries in \( \mathcal{A} \) satisfying the following condition:

\[
(3.1) \quad e \ast \omega_i \ast (1 - e) = -e \ast \partial_i e, \quad \forall i.
\]

Define the \( \mathbb{R}[\hbar] \)-linear maps \( \nabla_i \) \( (i = 1, 2, \ldots, n) \) from \( \mathcal{M} \) to \( \mathcal{A}^m \) by

\[
\nabla_i \zeta = \partial_i \zeta + \zeta \ast \omega_i, \quad \forall \zeta \in \mathcal{M}.
\]

Then each \( \nabla_i \) is a covariant derivative on the noncommutative bundle \( \mathcal{M} \) in the sense of Theorem 3.1 below. They together define a connection on \( \mathcal{M} \).

**Theorem 3.1.** The maps \( \nabla_i \) \( (i = 1, 2, \ldots, n) \) have the following properties. For all \( \zeta \in \mathcal{M} \) and \( a \in \mathcal{A} \),

\[
\nabla_i \zeta \in \mathcal{M} \quad \text{and} \quad \nabla_i (a \ast \zeta) = \partial_i (a) \ast \zeta + a \ast \nabla_i \zeta.
\]

**Proof.** For any \( \zeta \in \mathcal{M} \), we have

\[
\nabla_i (\zeta) \ast e = \partial_i (\zeta) \ast e + \zeta \ast \omega_i \ast e \\
= \partial_i \zeta + \zeta \ast (\omega_i \ast e - \partial_i e),
\]

where we have used the Leibniz rule and also the fact that \( \zeta \ast e = \zeta \). Using this latter fact again, we have \( \zeta \ast (\omega_i \ast e - \partial_i e) = \zeta \ast (e \ast \omega_i \ast e - e \ast \partial_i e) \), and by the defining property (3.1) of \( \omega_i \), we obtain \( \zeta \ast (e \ast \omega_i \ast e - e \ast \partial_i e) = \zeta \ast \omega_i \). Hence

\[
\nabla_i (\zeta) \ast e = \partial_i \zeta + \zeta \ast \omega_i = \nabla_i \zeta,
\]

proving that \( \nabla_i \zeta \in \mathcal{M} \). The second part of the theorem immediately follows from the Leibniz rule.

We shall also say that the set of \( \omega_i \) \( (i = 1, 2, \ldots, n) \) is a connection on \( \mathcal{M} \). Since \( e \ast \partial_i e = \partial_i (e) \ast (1 - e) \), one obvious choice for \( \omega_i \) is \( \omega_i = -\partial_i e \), which we shall refer to as the *canonical connection* on \( \mathcal{M} \).
Lemma 3.2. If \( \omega_i \) (\( i = 1, 2, \ldots, n \)) define a connection on \( \mathcal{M} \), then so do also \( \omega_i + \phi_i \ast e \) (\( i = 1, 2, \ldots, n \)) for any \( m \times m \)-matrices \( \phi_i \) with entries in \( \mathcal{A} \).

For a given connection \( \omega_i \) (\( i = 1, 2, \ldots, n \)), we consider \( [\nabla_i, \nabla_j] = \nabla_i \nabla_j - \nabla_j \nabla_i \) with the right hand side understood as composition of maps on \( \mathcal{M} \). By simple calculations we can show that for all \( \zeta \in \mathcal{M} \),

\[
[\nabla_i, \nabla_j] \zeta = \zeta \ast R_{ij} \quad \text{with} \quad R_{ij} := \partial_i \omega_j - \partial_j \omega_i - [\omega_i, \omega_j],
\]

where \([\omega_i, \omega_j] = \omega_i \ast \omega_j - \omega_j \ast \omega_i \) is the commutator. We call \( R_{ij} \) the curvature of \( \mathcal{M} \) associated with the connection \( \omega_i \).

For all \( \zeta \in \mathcal{M} \),

\[
[\nabla_i, \nabla_j] \nabla_k \zeta = \partial_k (\zeta) \ast R_{ij} + \zeta \ast \omega_k \ast R_{ij},
\]

\[
\nabla_k [\nabla_i, \nabla_j] \zeta = \partial_k (\zeta) \ast R_{ij} + \zeta \ast (\partial_k R_{ij} + R_{ij} \ast \omega_k).
\]

Define the following covariant derivatives of the curvature:

\[
(3.2) \quad \nabla_k R_{ij} := \partial_k R_{ij} + R_{ij} \ast \omega_k - \omega_k \ast R_{ij},
\]

we have

\[
[\nabla_k, [\nabla_i, \nabla_j]] \zeta = \zeta \ast \nabla_k R_{ij}, \quad \forall \zeta \in \mathcal{M}.
\]

The Jacobian identity \([\nabla_k, [\nabla_i, \nabla_j]] + [\nabla_j, [\nabla_k, \nabla_i]] + [\nabla_i, [\nabla_j, \nabla_k]] = 0 \) leads to

\[
\zeta \ast (\nabla_k R_{ij} + \nabla_j R_{ki} + \nabla_i R_{jk}) = 0, \quad \forall \zeta \in \mathcal{M}.
\]

From this we immediately see that \( e \ast (\nabla_k R_{ij} + \nabla_j R_{ki} + \nabla_i R_{jk}) = 0 \). In fact, the following stronger result holds.

Theorem 3.3. The curvature satisfies the following Bianchi identity:

\[
\nabla_k R_{ij} + \nabla_j R_{ki} + \nabla_i R_{jk} = 0.
\]

Proof. The proof is entirely combinatorial. Let

\[
A_{ijk} = \partial_k \partial_i \omega_j - \partial_k \partial_j \omega_i,
\]

\[
B_{ijk} = [\partial_i \omega_j, \omega_k] - [\partial_j \omega_i, \omega_k].
\]

Then we can express \( \nabla_k R_{ij} \) as

\[
\nabla_k R_{ij} = A_{ijk} + B_{ijk} - \partial_k [\omega_i, \omega_j] - [\omega_i, [\omega_j, \omega_k]].
\]

Note that

\[
A_{ijk} + A_{jki} + A_{kij} = 0,
\]

\[
B_{ijk} + B_{jki} + B_{kij} = \partial_k [\omega_i, \omega_j] + \partial_i [\omega_j, \omega_k] + \partial_j [\omega_k, \omega_i].
\]

Using these relations together with the Jacobian identity

\[
[[\omega_i, \omega_j], \omega_k] + [[\omega_j, \omega_k], \omega_i] + [[\omega_k, \omega_i], \omega_j] = 0,
\]

we easily prove the Bianchi identity. \( \square \)
3.2. **Gauge transformations.** Let $GL_m(A)$ be the group of invertible $m \times m$-matrices with entries in $A$. Let $\mathcal{G}$ be the subgroup defined by

$$\mathcal{G} = \{ g \in GL_m(A) \mid e \ast g = g \ast e \},$$

which will be referred to as the *gauge group*. There is a right action of $\mathcal{G}$ on $M$ defined, for any $\zeta \in M$ and $g \in \mathcal{G}$, by $\zeta \times g \mapsto \zeta \cdot g := \zeta \ast g$, where the right side is defined by matrix multiplication. Clearly, $\zeta \ast g \ast e = \zeta \ast g$. Hence $\zeta \ast g \in M$, and we indeed have a $\mathcal{G}$ action on $M$.

For a given $g \in \mathcal{G}$, let

$$\omega_i^g = g^{-1} \ast \omega_i \ast g - g^{-1} \ast \partial_i g.$$

Then

$$e \ast \omega_i^g \ast (1 - e) = g^{-1} \ast e \ast \omega_i \ast (1 - e) \ast g - g^{-1} \ast e \ast \partial_i (g) \ast (1 - e).$$

By (3.1),

$$g^{-1} \ast e \ast \omega_i \ast (1 - e) \ast g = -g^{-1} \ast e \ast \partial_i (e) \ast g$$

$$= -g^{-1} \ast e \ast \partial_i (e \ast g) + g^{-1} \ast e \ast \partial_i g$$

$$= -g^{-1} \ast e \ast \partial_i (g) \ast e - e \ast \partial_i e + g^{-1} \ast e \ast \partial_i g$$

$$= -e \ast \partial_i e + g^{-1} \ast e \ast \partial_i (g) \ast (1 - e).$$

Therefore,

$$e \ast \omega_i^g \ast (1 - e) = -e \ast \partial_i e.$$ 

This shows that the $\omega_i^g$ satisfy the condition (3.1), thus form a connection on $M$.

Now for any given $g \in \mathcal{G}$, define the maps $\nabla_i^g$ on $M$ by

$$\nabla_i^g \zeta = \partial_i \zeta + \zeta \ast \omega_i^g, \quad \forall \zeta.$$

Also, let $\mathcal{R}_{ij}^g = \partial_i \omega_j^g - \partial_j \omega_i^g - [\omega_i^g, \omega_j^g]$, be the curvature corresponding to the connection $\omega_i^g$. Then we have the following result.

**Lemma 3.4.** Under a gauge transformation procured by $g \in \mathcal{G}$,

$$\nabla_i^g (\zeta \ast g) = \nabla_i (\zeta) \ast g, \quad \forall \zeta \in M;$$

$$\mathcal{R}_{ij}^g = g^{-1} \ast \mathcal{R}_{ij} \ast g.$$ 

**Proof.** Note that

$$\nabla_i^g (\zeta \ast g) = \partial_i (\zeta) \ast g + \zeta \ast \partial_i g + \zeta \ast g \ast \omega_i^g = (\partial_i \zeta + \zeta \ast \omega_i) \ast g.$$

This proves the first formula.
To prove the second claim, we use the following formulae
\[
\partial_i \omega_j^g - \partial_j \omega_i^g = g^{-1} * (\partial_i \omega_j - \partial_j \omega_i) * g - \partial_i (g^{-1}) * \partial_j g + \partial_j (g^{-1}) * \partial_i g
\]
\[
\quad + [\partial_i (g^{-1}) * g, g^{-1} * \omega_j * g]_* - [\partial_j (g^{-1}) * g, g^{-1} * \omega_i * g]_*,
\]
\[
[\omega_i^g, \omega_j^g]_* = g^{-1} * [\omega_i, \omega_j]_* * g - \partial_i (g^{-1}) * \partial_j g + \partial_j (g^{-1}) * \partial_i g
\]
\[
\quad + [\partial_i (g^{-1}) * g, g^{-1} * \omega_j * g]_* - [\partial_j (g^{-1}) * g, g^{-1} * \omega_i * g]_*.
\]
Combining these formulae together we obtain \( \mathcal{R}_{ij}^g = g^{-1} \mathcal{R}_{ij} g \). This completes the proof of the lemma.

3.3. **Vector bundles associated to right projective modules.** Connections and curvatures can be introduced for the right bundle \( \hat{\mathcal{M}} = e * \mathcal{A}_r^m \) in much the same way. Let \( \hat{\omega}_i \in \mathcal{M}_m(A) \) \( (i = 1, 2, \ldots, n) \) be matrices satisfying the condition that
\[
(1 - e) * \hat{\omega}_i * e = \partial_i (e) * e.
\]
Then we can introduce a connection consisting of the right covariant derivatives \( \hat{\nabla}_i \) \( (i = 1, 2, \ldots, n) \) on \( \hat{\mathcal{M}} \) defined by
\[
\hat{\nabla}_i : \hat{\mathcal{M}} \longrightarrow \hat{\mathcal{M}}, \quad \xi \mapsto \hat{\nabla}_i \xi = \partial_i \xi - \hat{\omega}_i * \xi.
\]
It is easy to show that \( \hat{\nabla}_i (\xi * a) = \hat{\nabla}_i (\xi) * a + \xi * \partial_i a \) for all \( a \in A \).

Note that if \( \hat{\omega}_i \) is equal to \( \partial_i e \) for each \( i \), the condition (3.5) is satisfied. We call them the *canonical connection* on \( \hat{\mathcal{M}} \).

Returning to a general connection \( \hat{\omega}_i \), we define the associated curvature by
\[
\hat{\mathcal{R}}_{ij} = \partial_i \hat{\omega}_j - \partial_j \hat{\omega}_i - [\hat{\omega}_i, \hat{\omega}_j]_*.
\]
Then for all \( \xi \in \hat{\mathcal{M}} \), we have
\[
[\hat{\nabla}_i, \hat{\nabla}_j] \xi = -\hat{\mathcal{R}}_{ij} * \xi.
\]
We further define the covariant derivatives of \( \hat{\mathcal{R}}_{ij} \) by
\[
\hat{\nabla}_k \hat{\mathcal{R}}_{ij} = \partial_k \hat{\mathcal{R}}_{ij} + \hat{\omega}_k * \hat{\mathcal{R}}_{ij} - \hat{\mathcal{R}}_{ij} * \hat{\omega}_k.
\]
Then we have the following result.

**Lemma 3.5.** The curvature on the right bundle \( \hat{\mathcal{M}} \) satisfies the Bianchi identity
\[
\hat{\nabla}_i \hat{\mathcal{R}}_{jk} + \hat{\nabla}_j \hat{\mathcal{R}}_{ki} + \hat{\nabla}_k \hat{\mathcal{R}}_{ij} = 0.
\]
By direct calculations we can also prove the following result:
\[
[\hat{\nabla}_k, [\hat{\nabla}_i, \hat{\nabla}_j]] \xi = -\hat{\mathcal{R}}_{ij} (\hat{\mathcal{R}}_{ij}) * \xi, \quad \forall \xi \in \hat{\mathcal{M}}.
\]
Consider the gauge group \( \mathcal{G} \) defined by (3.3), which has a right action on \( \hat{\mathcal{M}} \):
\[
\hat{\mathcal{M}} \times \mathcal{G} \longrightarrow \hat{\mathcal{M}}, \quad \xi * g \mapsto \xi * g := g^{-1} * \xi.
\]
Under a gauge transformation procured by $g \in \mathcal{G}$,

$$\tilde{\omega}_i \mapsto \tilde{\omega}_i^g := g^{-1} \ast \tilde{\omega}_i \ast g + \partial_i (g^{-1}) \ast g.$$ 

The connection $\tilde{\nabla}_i^g$ on $\tilde{\mathcal{M}}$ defined by

$$\tilde{\nabla}_i^g \xi = \partial_i \xi - \tilde{\omega}_i^g \ast \xi$$

satisfies the following relation for all $\xi \in \tilde{\mathcal{M}}$:

$$\tilde{\nabla}_i^g (g^{-1} \ast \xi) = g^{-1} \ast \tilde{\nabla}_i^g \xi.$$ 

Furthermore, the gauge transformed curvature

$$\tilde{\mathcal{R}}_{ij}^g := \partial_i \omega_j^g - \partial_j \omega_i^g - [\tilde{\omega}_i^g, \tilde{\omega}_j^g],$$

is related to $\tilde{\mathcal{R}}_{ij}$ by

$$\tilde{\mathcal{R}}_{ij}^g = g^{-1} \ast \tilde{\mathcal{R}}_{ij} \ast g.$$ 

Given any $\Lambda \in \mathbf{M}_m(\mathcal{A})$, we can define the $\mathcal{A}$-bimodule map

$$(3.6) \quad \langle \cdot, \cdot \rangle : \mathcal{M} \otimes_{\mathbb{R}[h]} \tilde{\mathcal{M}} \longrightarrow \mathcal{A}, \quad \zeta \otimes \xi \mapsto \langle \zeta, \xi \rangle = \zeta \ast \Lambda \ast \xi,$$

where $\zeta \ast \Lambda \ast \xi$ is defined by matrix multiplication. We shall say that the bimodule homomorphism is gauge invariant if for any element $g$ of the gauge group $\mathcal{G}$,

$$\langle \zeta \cdot g, \xi \cdot g \rangle = \langle \zeta, \xi \rangle, \quad \forall \zeta \in \mathcal{M}, \xi \in \tilde{\mathcal{M}}.$$ 

Also, the bimodule homomorphism is said to be compatible with the connections $\omega_i$ on $\mathcal{M}$ and $\tilde{\omega}_i$ on $\tilde{\mathcal{M}}$ if for all $i = 1, 2, \ldots, n$

$$\partial_i \langle \zeta, \xi \rangle = \langle \nabla_i \zeta, \xi \rangle + \langle \zeta, \nabla_i \xi \rangle, \quad \forall \zeta \in \mathcal{M}, \xi \in \tilde{\mathcal{M}}.$$ 

**Lemma 3.6.** Let $\langle \cdot, \cdot \rangle : \mathcal{M} \otimes_{\mathbb{R}[h]} \tilde{\mathcal{M}} \longrightarrow \mathcal{A}$ be an $\mathcal{A}$-bimodule homomorphism defined by (3.6) with a given $m \times m$-matrix $\Lambda$ with entries in $\mathcal{A}$. Then

1. $\langle \cdot, \cdot \rangle$ is gauge invariant if $g \ast \Lambda \ast g^{-1} = \Lambda$ for all $g \in \mathcal{G}$;
2. $\langle \cdot, \cdot \rangle$ is compatible with the connections $\omega_i$ on $\mathcal{M}$ and $\tilde{\omega}_i$ on $\tilde{\mathcal{M}}$ if for all $i$,

$$e \ast (\partial_i \Lambda - \omega_i \ast \Lambda + \Lambda \ast \tilde{\omega}_i) \ast e = 0.$$ 

**Proof.** Note that $\langle \zeta \cdot g, \xi \cdot g \rangle = \zeta \ast g \ast \Lambda \ast g^{-1} \ast \xi$ for any $g \in \mathcal{G}$, $\zeta \in \mathcal{M}$ and $\xi \in \tilde{\mathcal{M}}$. Therefore $\langle \zeta \cdot g, \xi \cdot g \rangle = \langle \zeta, \xi \rangle$ if $g \ast \Lambda \ast g^{-1} = \Lambda$. This proves part (1).

Now $\partial_i \langle \zeta, \xi \rangle = \langle \partial_i \zeta, \xi \rangle + \langle \zeta, \partial_i \xi \rangle + \zeta \ast (\partial_i \Lambda - \omega_i \ast \Lambda + \Lambda \ast \tilde{\omega}_i) \ast \xi$. Thus if $\Lambda$ satisfies the condition of part (2), then $\langle \cdot, \cdot \rangle$ is compatible with the connections. 

Q.E.D.
3.4. **Canonical connections and fibre metric.** Let us consider in detail the canonical connections on $M$ and $\tilde{M}$ given by

$$\omega_i = -\partial_i e, \quad \tilde{\omega}_i = \partial_i e.$$ 

A particularly nice feature in this case is that the corresponding curvatures on the left and right bundles coincide. We have the following formula:

$$R_{ij} = \tilde{R}_{ij} = -[\partial_i e, \partial_j e].$$

Now we consider a special case of the $\mathcal{A}$-bimodule map defined by equation (3.6).

**Definition 3.7.** Denote by $\mathbf{g} : M \otimes_{\mathcal{R}[[\hbar]]} \tilde{M} \to \mathcal{A}$ the map defined by (3.6) with $\Lambda$ being the identity matrix. We shall call $\mathbf{g}$ the *fibre metric* on $M$.

**Lemma 3.8.** The fibre metric $\mathbf{g}$ is gauge invariant and is compatible with the standard connections.

**Proof.** Since $\Lambda$ is the identity matrix in the present case, it immediately follows from Lemma 3.6 (1) that $\mathbf{g}$ is gauge invariant. Note that $e \ast \partial_i (e) \ast e = 0$ for all $i$. Using this fact in Lemma 3.6 (2), we easily see that $\mathbf{g}$ is compatible with the standard connections. $\square$

4. **EMBEDDED NONCOMMUTATIVE SPACES**

In this section we study explicit examples of idempotents and related projective modules. They correspond to the noncommutative spaces introduced in [8]. The main result here is a reformulation of the theory of embedded noncommutative spaces [8] in the framework of Section 3 in terms of projective modules.

4.1. **Embedded noncommutative spaces.** We shall consider only embedded spaces with Euclidean signature. The Minkowski case is similarly, which we shall briefly allude to in Remark 4.6 at the end of this section. Given $X = (X^1 \ X^2 \ldots \ X^m)$ in $\mathcal{A}^m$, we define an $(n \times n)$-matrix $(g_{ij})_{i,j=1,2,\ldots,n}$ with entries given by

$$g_{ij} = \sum_{\alpha=1}^{m} \partial_i X^\alpha \ast \partial_j X^\alpha.$$ 

Following [8], we shall call $X$ a *noncommutative space* embedded in $\mathcal{A}^m$ if the matrix $(g_{ij})$ is invertible.

For a given noncommutative space $X$, we denote by $(g^{ij})$ the inverse matrix of $(g_{ij})$ with $g_{ij} \ast g^{jk} = g^{k\ell} \ast g_{\ell j} = \delta^k_j$ for all $i$ and $k$. Here Einstein’s summation convention is used, and we shall continue to use this convention throughout the paper. Let

$$E_i = \partial_i X, \quad \tilde{E}^i = (E_j)^i \ast g^{ji}, \quad E^i = g^{ij} \ast E_j,$$
for $i = 1, 2, \ldots, n$, where $(E_i)^t = \begin{pmatrix} \partial_i X^1 \\ \partial_i X^2 \\ \vdots \\ \partial_i X^m \end{pmatrix}$ denotes the transpose of $E_i$. Define $e \in \mathbb{M}_m(\mathcal{A})$ by

$$e := \tilde{E}^j \ast E_i$$

(4.1)

\[
\begin{pmatrix}
\partial_i X^1 \ast g^{ij} \ast \partial_j X^1 & \partial_i X^1 \ast g^{ij} \ast \partial_j X^2 & \cdots & \partial_i X^1 \ast g^{ij} \ast \partial_j X^m \\
\partial_i X^2 \ast g^{ij} \ast \partial_j X^1 & \partial_i X^2 \ast g^{ij} \ast \partial_j X^2 & \cdots & \partial_i X^2 \ast g^{ij} \ast \partial_j X^m \\
\vdots & \vdots & \ddots & \vdots \\
\partial_i X^m \ast g^{ij} \ast \partial_j X^1 & \partial_i X^m \ast g^{ij} \ast \partial_j X^2 & \cdots & \partial_i X^m \ast g^{ij} \ast \partial_j X^m
\end{pmatrix}.
\]

We have the following results.

**Proposition 4.1.**

1. Under matrix multiplication, $E_i \ast \tilde{E}^j = \delta_i^j$ for all $i$ and $j$.
2. The $m \times m$ matrix $e$ satisfies $e \ast e = e$, that is, it is an idempotent in $\mathbb{M}_m(\mathcal{A})$.
3. The left and right projective $\mathcal{A}$-modules $\mathcal{M} = \mathcal{A}^m \ast e$ and $\tilde{\mathcal{M}} = e \ast \mathcal{A}^m_r$ are respectively spanned by $E_i$ and $\tilde{E}^i$. More precisely, we have

$$\mathcal{M} = \{ a^i \ast E_i \mid a^i \in \mathcal{A} \}, \quad \tilde{\mathcal{M}} = \{ \tilde{E}^i \ast b_i \mid b_i \in \mathcal{A} \}.$$

**Proof.** Note that $g_{ij} = E_i \ast (E_j)^t$. Thus $E_i \ast \tilde{E}^j = E_i \ast (E_k)^t \ast g^{kj} = \delta_i^j$. It then immediately follows that

$$e \ast e = \tilde{E}^i \ast \left( E_i \ast \tilde{E}^j \right) \ast E_j = \tilde{E}^i \ast \delta_i^j \ast E_j = e.$$

Obviously $\mathcal{M} \subset \{ a^i \ast E_i \mid a^i \in \mathcal{A} \}$ and $\tilde{\mathcal{M}} \subset \{ \tilde{E}^i \ast b_i \mid b_i \in \mathcal{A} \}$. By the first part of the proposition, we have

$$a^i \ast E_i \ast e = a^i \ast \left( E_i \ast \tilde{E}^j \right) \ast E_j = a^i \ast E_j, \quad e \ast \tilde{E}^j \ast b_j = \tilde{E}^i \ast \left( E_i \ast \tilde{E}^j \right) \ast b_j = \tilde{E}^i \ast b_j.$$

This proves the last claim of the proposition. \qed

It is also useful to observe that $\tilde{\mathcal{M}} = \{(E_i)^t \ast b_i \mid b_i \in \mathcal{A}\}$ since $(g_{ij})$ is invertible.

We shall denote $\mathcal{M}$ and $\tilde{\mathcal{M}}$ respectively by $TX$ and $\tilde{TX}$, and refer to them as the left and right tangent bundles of the noncommutative space $X$. Note that the definition of the tangent bundles coincides with that in [8].

**Definition 4.2.** Call the fibre metric $\mathbf{g} : TX \otimes_{\mathbb{R}[\mathcal{A}]} \tilde{TX} \longrightarrow \mathcal{A}$ defined in Definition 3.7 the **metric** of the noncommutative space $X$.

The proposition below in particular shows that $\mathbf{g}$ agrees with the metric of the embedded noncommutative space defined in [8] in a geometric setting.
Lemma 4.4. Then we have the following result.

\[ (4.2) \]

Proof. Recall from Definition 3.7 that \( g \) is defined by (3.6) with \( \Lambda \) being the identity matrix. Thus for any \( \zeta = a^i \ast E_i \in TX \) and \( \xi = (E_j)^l \ast b^j \in \tilde{TX} \) with \( a_i, b_j \in \mathcal{A} \),

\[ g : \zeta \otimes \xi \mapsto g(\zeta, \xi) = a^i \ast g_{ij} \ast b^j. \]

In particular, \( g(E_i, (E_j)^l) = g_{ij} \).

Proposition 4.3. For any \( \zeta = a^i \ast E_i \in TX \) and \( \xi = (E_j)^l \ast b^j \in \tilde{TX} \) with \( a_i, b_j \in \mathcal{A} \),

\[ \nabla_i E_j = \Gamma_{ijl}^k \ast E_k, \quad \tilde{\nabla}_i \tilde{E}^j = -\tilde{E}^k \ast \tilde{\Gamma}_{ijl}^k. \]

Proof. Consider the first formula. Write \( \partial_i e = \partial_i (\tilde{E}^k) \ast E_k + \tilde{E}^k \ast \partial_i E_k \). We have

\[ \nabla_i E_j = \partial_i E_j - E_j \partial_i \ast e \]

\[ = \partial_i E_j - \left( \partial_i (E_j \ast e) - \partial_i (E_j) \ast e \right) \]

\[ = \partial_i (E_j) \ast \tilde{E}^k \ast E_k. \]

It was shown in [8] that \( \Gamma_{ij}^k = \partial_i (E_j) \ast \tilde{E}^k \). This immediately leads to the first formula. The proof for the second formula is essentially the same. \( \square \)

Note that Lemma 4.4 can be re-stated as

\[ \nabla_i \tilde{E}^j = -\tilde{\Gamma}_{ijl}^k \ast E_k, \quad \tilde{\nabla}_i (E_j)^l = (E_k)^l \ast \tilde{\Gamma}_{ijl}^k. \]

By using Lemma 3.8 and Lemma 4.4, we can easily prove the following result, which is equivalent to [8] Proposition 2.7.
Proposition 4.5. The connections are metric compatible in the sense that

\[ \partial_i \mathbf{g}(\zeta, \xi) = \mathbf{g}(\nabla_i \xi, \zeta) + \mathbf{g}(\zeta, \hat{\nabla}_i \xi), \quad \forall \zeta \in TX, \; \xi \in \mathcal{T}X. \]

For \( \zeta = E_j \) and \( \xi = (E_k)' \), we obtain from (4.4) the following result for all \( i, j, k \):

\[ \partial_i \mathbf{g}_{jk} - \Gamma_{ijk} = 0. \]

This formula is in fact equivalent to Proposition 4.5.

Define

\[ R_{kij}^l = E_k * \mathcal{R}_{ij} * \tilde{E}_l, \quad \tilde{R}_{kij}^l = -g^{lj} * E_q * \mathcal{R}_{ij} * \tilde{E}_p * g_{pk}. \]

Using \( \mathcal{R}_{ij} = \tilde{\mathcal{R}}_{ij} = -[\partial_i e, \partial_j e]_* \), we can show by some lengthy calculations that

\[ R_{kij}^l = -\partial_j \Gamma_{ik}^l - \Gamma_{jk}^p * \Gamma_{ip}^l + \partial_i \Gamma_{jk}^l + \Gamma_{jk}^l * \Gamma_{jk}^l, \]

\[ \tilde{R}_{kij}^l = -\partial_j \Gamma_{ik}^l - \Gamma_{jk}^p * \Gamma_{jk}^l + \partial_i \Gamma_{jk}^l + \Gamma_{jk}^l * \Gamma_{jk}^l, \]

which are the Riemannian curvatures of the left and right tangent bundles of the noncommutative space \( X \) given in [8, Lemma 2.12 and §4]. Therefore,

\[ [\nabla_i, \nabla_j]E_k = R_{kij}^l * E_l, \quad [\tilde{\nabla}_i, \tilde{\nabla}_j](E_k)' = (E_i)' * \tilde{R}_{kij}^l, \]

recovering the relations [8, (2.13)] and their generalisations [8, §4] to arbitrary \( m \geq n \).

Remark 4.6. We comment briefly on noncommutative spaces with Minkowski signatures embedded in higher dimensions [8]. Let \( \eta = \text{diag}(-1, \ldots, -1, 1, \ldots, 1) \) be a diagonal \( m \times m \) matrix with \( p \) of the diagonal entries being \(-1\), and \( q = m - p \) of them being \( 1 \). Given \( X = (X^1 \; X^2 \; \ldots \; X^m) \) in \( \mathcal{A}^m \), we define an \( n \times n \) matrix \((g_{ij})_{i,j=1,2,\ldots,n}\) with entries

\[ g_{ij} = \sum_{\alpha=1}^{m} \partial_i X^\alpha * \eta_{\alpha\beta} * \partial_j X^\beta. \]

We call \( X \) a noncommutative space embedded in \( \mathcal{A}^m \) if the matrix \((g_{ij})\) is invertible. Denote its inverse matrix by \((g^{ij})\). Now the idempotent which gives rise to the left and right tangent bundles of \( X \) is given by

\[ e = \eta(E_i)' * g^{ij} * E_j, \]

which obviously satisfies \( E_i * e = E_i \) for all \( i \). The fibre metric of Definition 3.8 yields a metric on the embedded noncommutative surface \( X \).

4.2. Example. We analyze an embedded noncommutative surface of Euclidean signature arising from the quantisation of a time slice of the Schwarzschild spacetime. While the main purpose here is to illustrate how the general theory developed in previous sections works, the example is interesting in its own right.

Let us first specify the notation to be used in this section. Let \( t^1 = r, \; t^2 = \theta \) and \( t^3 = \phi \), with \( r > 2m, \; \theta \in (0, \pi), \) and \( \phi \in (0, 2\pi) \). We deform the algebra of functions...
in these variables by imposing the Moyal product defined by (2.1) with the following anti-symmetric matrix

\[
(\theta_{ij})^3_{i,j=1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

Note that the functions depending only on the variable \( r \) are central in the Moyal algebra \( \mathcal{A} \). We shall write the usual pointwise product of two functions \( f \) and \( g \) as \( fg \), but write their Moyal product as \( f \ast g \).

Consider \( X = (X^1 \ X^2 \ X^3 \ X^4) \) given by

\[
X^1 = f(r) \quad \text{with} \quad (f')^2 + 1 = \left( 1 - \frac{2m}{r} \right)^{-1},
X^2 = r \sin \theta \cos \phi, \quad X^3 = r \sin \theta \sin \phi, \quad X^4 = r \cos \theta.
\]

Simple calculations yield

\[
E_1 = \partial_r X = (f' \ \sin \phi \ \sin \phi \ \cos \phi),
E_2 = \partial_\theta X = (0 \ \cos \phi \ \cos \phi \ \sin \phi \ \sin \phi),
E_3 = \partial_\phi X = (0 \ -r \sin \phi \ \sin \phi \ \sin \phi \ \cos \phi \ 0).
\]

Using these formulae, we obtain the following expressions for the components of the metric of the noncommutative surface \( X \):

\[
g_{11} = \left( 1 - \frac{2m}{r} \right)^{-1} \left[ 1 - \left( 1 - \frac{2m}{r} \right) \cos(2\theta) \sinh^2 \tilde{h} \right],
g_{12} = g_{21} = r \sin(2\theta) \sinh^2 \tilde{h},
g_{22} = r^2 \left[ 1 + \cos(2\theta) \sinh^2 \tilde{h} \right],
g_{23} = -g_{32} = -r^2 \cos(2\theta) \sinh \tilde{h} \cosh \tilde{h},
g_{13} = -g_{31} = -r \sin(2\theta) \sinh \tilde{h} \cosh \tilde{h},
g_{33} = r^2 \left[ \sin^2 \theta - \cos(2\theta) \sinh^2 \tilde{h} \right].
\]

In the limit \( \tilde{h} \to 0 \), we recover the spatial components of the Schwarzschild metric. Observe that the noncommutative surface still reflects the characteristics of the Schwarzschild spacetime in that there is a time slice of the Schwarzschild black hole with the event horizon at \( r = 2m \).

Since the metric \( (g_{ij}) \) depends on \( \theta \) and \( r \) only, and the two variables commute, the inverse \( (g^{ij}) \) of the metric can be calculated in the usual way as in the commutative case. Now the components of the idempotent \( e = (e_{ij}) = (E_i)^t \ast g^{ij} \ast E_j \) are given by
the following formulae:

\[ e_{11} = \frac{2m}{r} + \frac{2m(2m - r)(2 + \cos 2\theta)}{r^2} \bar{h}^2 + O(\bar{h}^3), \]

\[ e_{12} = \frac{m \cos \phi \sin \theta}{r \sqrt{\frac{m}{-4m + 2r}}} - \frac{2m \cos \theta \sin \phi \bar{h}}{r \sqrt{\frac{m}{-4m + 2r}}} + \frac{m(4m + r + 2m \cos 2\theta) \cos \phi \sin \theta}{r^2 \sqrt{\frac{m}{-4m + 2r}}} \bar{h}^2 + O(\bar{h}^3) \]

\[ e_{13} = \frac{m \sin \theta \sin \phi}{r \sqrt{\frac{m}{-4m + 2r}}} + \frac{2m \cos \theta \cos \phi \bar{h}}{r \sqrt{\frac{m}{-4m + 2r}}} + \frac{m(4m + r + 2m \cos 2\theta) \sin \theta \sin \phi}{r^2 \sqrt{\frac{m}{-4m + 2r}}} \bar{h}^2 + O(\bar{h}^3) \]

\[ e_{14} = \frac{m \cos \phi}{r \sqrt{\frac{m}{-4m + 2r}}} + \frac{m \cos \theta(4m - r + 2m \cos 2\theta)}{r^2 \sqrt{\frac{m}{-4m + 2r}}} \bar{h}^2 + O(\bar{h}^3) \]

\[ e_{21} = \frac{m \cos \phi \sin \theta}{r \sqrt{\frac{m}{-4m + 2r}}} + \frac{2m \cos \theta \sin \phi \bar{h}}{r \sqrt{\frac{m}{-4m + 2r}}} + \frac{m(4m + r + 2m \cos 2\theta) \cos \phi \sin \theta}{r^2 \sqrt{\frac{m}{-4m + 2r}}} \bar{h}^2 + O(\bar{h}^3) \]

\[ e_{22} = 1 - \frac{2m \sin^2 \theta \cos^2 \phi}{r} \]

\[ + \frac{m}{2r^2} \left[ 2r + 2m \cos 4\theta \cos^2 \phi - 6m \cos^2 \phi \right. \]

\[ + 2 \cos 2\theta(m + 8r + (m - r) \cos 2\phi) \] \bar{h}^2 + O(\bar{h}^3) \]

\[ e_{23} = -\frac{m \sin^2 \theta \sin 2\phi}{r} - \frac{3m \sin 2\theta}{r} \bar{h} \]

\[ + \frac{m(2(m - r) \cos 2\theta + m(-3 + \cos 4\theta)) \sin 2\phi}{2r^2} \bar{h}^2 + O(\bar{h}^3) \]

\[ e_{24} = -\frac{2m \cos \theta \cos \phi \sin \theta}{r} - \frac{m(1 + 3 \cos 2\theta) \sin \phi}{r} \bar{h} \]

\[ - \frac{m(8m + 5r + 4m \cos 2\theta) \cos \phi \sin 2\theta}{2r^2} \bar{h}^2 + O(\bar{h}^3) \]
Let us write $\mathcal{R}_{ij} = -[\partial_i e, \partial_j e]$, which is very complicated and not terribly illuminating.
However, we mention that in [37] a quantisation of the Schwarzschild spacetime was carried out (for a particular choice of $\Theta$), and the resulting noncommutative differential geometry was studied in detail. In particular, the metric, Christoffel symbols, Riemannian and Ricci curvatures were explicitly worked out. We refer to that paper for details.

5. General coordinate transformations

We now return to the general setting of Section 3 to investigate “general coordinate transformations”. Our treatment follows closely [8, §V] and makes use of general ideas of [21, 17, 28]. We should point out that the material presented is part of an attempt of ours to develop a notion of “general covariance” in the noncommutative setting. This is an important matter which deserves a thorough investigation. We hope that the work presented here will prompt further studies.

Let $(A, \mu)$ be a Moyal algebra of smooth functions on the open region $U$ of $\mathbb{R}^n$ with coordinate $t$. This algebra is defined with respect to a constant skew symmetric matrix $\theta = (\theta_{ij})$. Let $\Phi : U \longrightarrow U$ be a diffeomorphism of $U$ in the classical sense. We denote $u^i = \Phi^i(t)$, and refer to this as a general coordinate transformation of $U$.

Denote by $A_u$ the sets of smooth functions of $u = (u^1, u^2, \ldots, u^n)$. The map $\Phi$ induces an $\mathbb{R}[\hbar]$-module isomorphism $\phi = \Phi^* : A_u \longrightarrow A$ defined for any function $f \in A_u$ by

$$\phi(f)(t) = f(\Phi(t)).$$

We define the $\mathbb{R}[\hbar]$-bilinear map

$$\mu_u : A_u \otimes A_u \longrightarrow A_u, \quad \mu_u(f, g) = \phi^{-1}(\phi(f), \phi(g)).$$

Then it is well-known [21] that $\mu_u$ is associative. Therefore, we have the associative algebra isomorphism

$$\phi : (A_u, \mu_u) \sim \longrightarrow (A_t, \mu_t).$$

We say that the two associative algebras are gauge equivalent by adopting the terminology of [17].

Following [8], we define $\mathbb{R}[\hbar]$-linear operators

$$\partial_i^\phi := \phi^{-1} \circ \partial_i \circ \phi : A_u \longrightarrow A_u,$$

which have the following properties [8 Lemma 5.5]:

$$\partial_i^\phi \circ \partial_j^\phi - \partial_j^\phi \circ \partial_i^\phi = 0,$$

$$\partial_i^\phi \mu_u(f, g) = \mu_u(\partial_i^\phi(f), g) + \mu_u(f, \partial_i^\phi(g)), \quad \forall f, g \in A_u,$$

where the second relation is the Leibniz rule for $\partial_i^\phi$. Recall that this Leibniz rule played a crucial role in the construction of noncommutative spaces over $(A_u, \mu_u)$ in [8].
We shall denote by $M_m(A_u)$ the set of $m \times m$-matrices with entries in $A_u$. The product of two such matrices will be defined with respect to the multiplication $\mu_u$ of the algebra $(A_u, \mu_u)$. Then $\phi^{-1}$ acting component wise gives rise to an algebra isomorphism from $M_m(A)$ to $M_m(A_u)$, where matrix multiplication in $M_m(A)$ is defined with respect to $\mu$.

Since we need to deal with two different algebras $(A, \mu)$ and $(A_u, \mu_u)$ simultaneously in this section, we write $\mu$ and the matrix multiplication defined with respect to it by $\star$ as before, and use $\star_u$ to denote $\mu_u$ and the matrix multiplication defined with respect to it.

Let $e \in M_m(A)$ be an idempotent. There exists the corresponding finitely generated projective left (resp. right) $A$-module $M$ (resp. $\tilde{M}$). Now $e_u := \phi^{-1}(e)$ is an idempotent in $M_m(A_u)$, that is, $\phi^{-1}(e) \star_u \phi^{-1}(e) = \phi^{-1}(e)$. Write $e_u = (e^\beta_\alpha)_{\alpha, \beta=1, \ldots, m}$. This idempotent gives rise to the left projective $A_u$-module $M_u$ and right projective $A_u$-module $\tilde{M}_u$, respectively defined by

$$M_u = \{ (a^\alpha \star_u e^1_\alpha, a^\alpha \star_u e^2_\alpha, \ldots, a^\alpha \star_u e^m_\alpha) \mid a^\alpha \in A_u \},$$

$$\tilde{M}_u = \left\{ \begin{pmatrix} e^1_\beta \star_u b_\beta \\ e^2_\beta \star_u b_\beta \\ \vdots \\ e^m_\beta \star_u b_\beta \end{pmatrix} \mid b_\beta \in A_u \right\},$$

where $a^\alpha \star_u e^\beta_\alpha = \sum_\alpha \mu_u(a^\alpha, e^\beta_\alpha)$ and $e^\beta_\alpha \star_u b_\beta = \sum_\beta \mu_u(e^\beta_\alpha, b_\beta)$. Below we consider the projective module only, as the right projective module may be treated similarly.

Assume that we have the left connection

$$\nabla_i : M \longrightarrow M, \quad \nabla_i \zeta = \frac{\partial \zeta}{\partial t^i} + \zeta \star \omega_i.$$ 

Let $\omega_i^\mu := \phi^{-1}(\omega_i)$. We have the following result.

**Theorem 5.1.**

1. The matrices $\omega_i^\mu$ satisfy the following relations in $M_m(A_u)$:

$$e_u \star_u \omega_i^\mu \star_u (1 - e_u) = -e_u \star_u \partial_i^\phi e_u.$$

2. The operators $\nabla_i^\phi (i = 1, 2, \ldots, n)$ defined for all $\eta \in M_u$ by

$$\nabla_i^\phi \eta = \partial_i^\phi \eta + \eta \star_u \omega_i^\mu$$

give rise to a connection on $M_u$.

3. The curvature of the connection $\nabla_i^\phi$ is given by

$$R_i^\mu_j = \partial_i^\phi \omega_j^\mu - \partial_j^\phi \omega_i^\mu - \omega_i^\mu \star_u \omega_j^\mu + \omega_j^\mu \star_u \omega_i^\mu,$$

which is related to the curvature $R_{ij}$ of $M$ by

$$R_i^\mu_j = \phi^{-1}(R_{ij}).$$
Proof. Note that \( e_u^* u \omega^u_v (1 - e_u) = \phi^{-1} (e \ast \omega \ast (1 - e)) \). We also have \( \partial_i^u e_u = \phi^{-1} (\frac{\partial}{\partial u}) \), which leads to \( e_u^* u \partial_i^u e_u = \phi^{-1} (e \ast \phi(\partial_i^u e_u)) = \phi^{-1} (e \ast \partial_i e) \). This proves part (1). Part (2) follows from part (1) and the Leibniz rule for \( \partial_i^u \). Straightforward calculations show that the curvature of the connection \( \nabla_i^u \) is given by \( \epsilon_{ij}^u \omega_{ij}^u - \partial_i^u \omega_{ij}^u - \partial_j^u \omega_{ij}^u + \omega_{ij}^u * \omega_{ij}^u \). Now \( \partial_i^u \omega_{ij}^u = \phi^{-1} \left( \frac{\partial \omega_{ij}^u}{\partial \phi} \right) \), and \( \omega_{ij}^u * \omega_{ij}^u - \omega_{ij}^u * \omega_{ij}^u = \phi^{-1} (\omega_i \ast \omega_i) - \phi^{-1} (\omega_i \ast \omega_i) \). Hence \( \epsilon_{ij}^u = \phi^{-1} (\epsilon_{ij}) \). \( \square \)

Remark 5.2. One can recover the usual transformation rules of tensors under the diffeomorphism group from the commutative limit of Theorem 5.1 in a way similar to that in \([8, \S 5.C] \).

6. BAR INVOLUTION AND GENERALISED HERMITIAN STRUCTURE

In this section, we study a Moyal algebra analogue of the bar map of quantum groups, and investigate its implications on noncommutative geometry. Note that the \( \text{feomorphism group from the commutative limit of Theorem 5.1} \)

Lemma 6.1. Let \( \mathcal{A} \rightarrow \mathcal{A} \) be the map defined for any \( f = \sum_i f_i \hbar^i \in \mathcal{A} \), where \( f_i \) are real functions on \( U \), by \( \bar{f} = \sum_i (-1)^i f_i \hbar^i \). Then for all \( f, g \in \mathcal{A} \),

\[ \bar{f} \ast g = \bar{g} \ast \bar{f} \]

We refer to the map as the bar involution of the Moyal algebra. It is an analogue of the well known bar map, sending \( q = \exp(\hbar) \) to \( q^{-1} \), in the theory of quantum groups, which plays an important role in the study of canonical (crystal) bases.

The lemma can be easily proven by inspecting (2.1). Given any rectangular matrix \( A = (a_{rs}) \) with entries in \( \mathcal{A} \), we let \( A^\dagger \) be the matrix obtained from \( A \) by first taking its transpose then sending every matrix element to its conjugate. For example,

\[
\begin{pmatrix}
1 & b_1 & c_1 \\
2 & b_2 & c_2
\end{pmatrix}^\dagger = \begin{pmatrix}
\bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\
\bar{b}_1 & \bar{b}_2 & \bar{b}_3
\end{pmatrix}.
\]

It is clear that if the product \( A \ast B \) of two matrices are defined, then \( (A \ast B)^\dagger = B^\dagger \ast A^\dagger \).

Let \( \mathcal{A}^m = \mathcal{A}^m \) be the \( \mathbb{R}[[\hbar]] \)-module consisting of rows matrices of length \( m \) with entries in \( \mathcal{A} \). We define the form

\[
( , ) : \mathcal{A}^m \times \mathcal{A}^m \rightarrow \mathcal{A}, \quad \xi \times \xi' \mapsto (\xi, \xi') := \xi \ast \xi^\dagger.
\]

Lemma 6.2. (1) For all \( \xi, \xi' \in \mathcal{M} \) and \( a, b \in \mathcal{A} \),

\[
(\xi, \xi') = (\xi, \xi'), \quad (a \ast \xi, b \ast \xi') = a \ast (\xi, \xi') \ast b.
\]

Thus in this sense the form (6.1) is sesquilinear.
(2) \((\zeta, \zeta) = 0 \) if and only if \(\zeta = 0\).

(3) For all \(\zeta, \xi \in \mathcal{M}\) and \(A \in \mathcal{M}_m(A)\), we have
\[
(\zeta * A, \xi) = (\zeta, \xi * A^†).
\]

(4) Let the bar-unitary group \(U_m(A)\) over \(A\) be the subgroup of \(GL_m(A)\) defined by \(U_m(A) = \{ g \in GL_m(A) \mid g^† = g^{-1}\}\). Then the form (6.1) is invariant under \(U_m(A)\) in the sense that for all \(g \in U_m(A)\) and \(\zeta, \xi \in \mathcal{M}\),
\[
(\zeta * g, \xi * g) = (\zeta, \xi).
\]

It is straightforward to prove the lemma. Note that part (2) of the lemma makes the form (6.1) as nice as a positive definite hermitian form in the commutative case.

We shall call an idempotent \(e \in \mathcal{M}_m(A)\) self-adjoint (with respect to the sesquilinear form (6.1)) if
\[
e = e^†.
\]

In this case, the corresponding left and right projective modules \(\mathcal{M} = \mathcal{A}^m * e\) and \(\tilde{\mathcal{M}} = e * \mathcal{A}_m^m\) are related by
\[
\tilde{\mathcal{M}} = \left\{ \xi^† \mid \xi \in \mathcal{M} \right\}.
\]

Furthermore, the form (6.1) restricts to a sesquilinear form on \(\mathcal{M}\), which is invariant under \(\mathfrak{G} \cap U_m(A)\).

**Lemma 6.3.** Let \(\mathcal{M} = \mathcal{A}^m * e\) and \(\tilde{\mathcal{M}} = e * \mathcal{A}_m^m\) be the left and right bundles associated with a self-adjoint idempotent \(e\). Assume that the left connection \(\omega_i\) on \(\mathcal{M}\) and the right connection \(\tilde{\omega}_i\) on \(\tilde{\mathcal{M}}\) satisfy the condition
\[
(\nabla_i \zeta)^† = \tilde{\nabla}_i (\zeta^†).
\]

Then for any \(\zeta\) in \(\mathcal{M}\),
\[
(\nabla_i \zeta)^† = \tilde{\nabla}_i (\zeta^†).
\]

Furthermore, the curvatures on the left and right bundles are related by
\[
\tilde{\mathcal{R}}_{ij} = -\mathcal{R}_{ij}^†.
\]

**Proof.** Let \(\xi = \zeta^†\). We have
\[
(\nabla_i \zeta)^† = (\partial_i \zeta + \zeta * \omega_i)^† = \partial_i \zeta + \omega_i^† * \xi = \tilde{\nabla}_i \xi.
\]

This proves the first part of the lemma. Now
\[
\mathcal{R}_{ij}^† = (\partial_i \omega_j - \partial_j \omega_i - [\omega_i, \omega_j])^† = \partial_i \omega_j^† - \partial_j \omega_i^† + [\omega_i^†, \omega_j] = -\tilde{\mathcal{R}}_{ij}.
\]

This proves the second part. \(\square\)
Hereafter we shall assume that condition (6.2) is satisfied by the left and right connections. Let $M$ be the left bundle corresponding to a self-adjoint idempotent $e$. We shall say that a connection $\omega_i$ on $M$ is *hermitian with respect to the bar map* (or bar-hermitian) if $\omega_i^\dagger = \omega_i$ for all $i$. In this case, we shall also say that the bundle $M$ is *bar-hermitian*.

Note that the canonical connections $\omega_i = -\partial_i e$ on $M$ and $\tilde{\omega}_i = \partial_i e$ on $\tilde{M}$ satisfy $\omega_i = -\omega_i^\dagger$ and $\omega_i^\dagger = \omega_i$ provided that $e$ is self-adjoint. Therefore, in this case the canonical connection is bar-hermitian. Since the left and right curvatures associated to the canonical connections are equal, it follows from Lemma 6.3 that $R_{ij}^\dagger = -R_{ij}$.

We have the following result.

**Theorem 6.4.** Let $X = (X^1 \ X^2 \ \ldots \ X^m)$ in $\mathbb{A}^m$ be an embedded noncommutative surface satisfying the condition $\overline{X} := \left(\overline{X^1} \ \overline{X^2} \ \ldots \ \overline{X^m}\right) = X$. Then $X$ has the following properties.

1. The metric has the property $g_{ij} = g_{ji}$ for all $i, j$.
2. The idempotent $e = (E_i)^\dagger * g^{ij} * E_j$ is self-adjoint.
3. Equipped with the canonical connection $\omega_i = -\partial_i e$, the tangent bundle of $X$ is bar-hermitian.
4. The curvature satisfies $R_{ij}^\dagger = -R_{ij}$.

**Proof.** The given condition on $X$ implies that all the $E_i$ satisfy $E_i^\dagger = (E_i)^\dagger$. Thus

$$g_{ij} = E_i * (E_j)^\dagger = E_i * (E_j)^\dagger, \quad e = (E_i)^\dagger * g^{ij} * E_j = (E_i)^\dagger * g^{ij} * E_j.$$

Hence we have $g^\dagger_{ij} = (E_i * (E_j)^\dagger)^\dagger = E_j * (E_i)^\dagger = g_{ji}$. It then follows that $g^\dagger_{ij} = g^{ji}$. Now the idempotent $e$ satisfies

$$e^\dagger = ((E_i)^\dagger * g^{ij} * E_j)^\dagger = (E_j)^\dagger * g^ji * E_i = (E_j)^\dagger * g^{ji} * E_i = e.$$

Part (3) and part (4) follow from part (2) and the discussion preceding the proposition. \[\Box\]

Note that the quantum spacetimes studied in [37] and the example in Section 4.2 all satisfy the conditions of Theorem 6.4.

7. Concluding Remarks

We wish to point out that in the classical commutative setting, we can recover (pseudo-) Riemannian geometry from the theory developed here by using the isometric embedding theorems of [32, 19, 12, 23]. The simplification in this case is that there is no need to distinguish the left and the right tangent bundles. To describe the situation, we let $(N, g)$ be a smooth $n$-dimensional (pseudo-) Riemannian manifold with metric $g$. Denote by $C^\infty(N)$ the set of smooth functions on $N$ endowed with the usual pointwise multiplication. Let $C^\infty(N)^m$ be the space consisting of row vectors of length $m$. 

with entries in $C^\infty(N)$. By results of [32, 19, 12, 23], there exist positive integers $p, q$ (with $p + q = m$) and a set of smooth functions $X^1, \cdots, X^p, X^{p+1}, \cdots, X^m$ on $N$ such that $g = \sum_{\alpha, \beta=1}^{m} dX^\alpha \eta_{\alpha\beta} dX^\beta$, where $\eta = \text{diag}(-1, \cdots, -1, 1, \cdots, 1)$ with $p = 0$ if $N$ is Riemannian. Let $U$ be a coordinate chart of $N$ with local coordinate $(t^1, \cdots, t^n)$. We set $E_i = \left( \frac{\partial X^1}{\partial t^i}, \frac{\partial X^2}{\partial t^i}, \cdots, \frac{\partial X^m}{\partial t^i} \right)$ and define $e = \eta(E_i)^j g^{ij} E_j$ on each coordinate chart $U$. Then we have the following result.

**Theorem 7.1.**

1. The idempotent $e$ is globally defined on $N$.
2. The space $\Gamma(TN)$ of sections of the tangent bundle of $N$ is given by $C^\infty(N)^m e$.
3. For all $\zeta, \xi \in \Gamma(TN)$, we have $g(\zeta, \xi) = \zeta e \xi^t$.
4. The standard connection (with $\omega_i = -\partial_i e$) on $C^\infty(N)^m e$ is the usual Levi-Civita connection on $TX$ with the Christoffel symbol $\Gamma^k_{ij}$ defined by (4.2) and $\Upsilon_{ijk} = 0$.
5. The Riemannian curvature tensor is given by (4.6).

Returning to the noncommutative case, we recall that one can quantise any Poisson manifold following the prescription of [28]. Then one obtains a collection of noncommutative associative algebras (analogous to the Moyal algebra), one on each coordinate patch. The algebras relative to different local coordinates are gauge equivalent [28, Theorem 2.3] as discussed in Section 5. This way, one obtains a sheaf of noncommutative algebras over the Poisson manifold. The algebraic geometry of such a quantised Poisson manifold has been extensively developed by Kashiwara and Schapira [24, 25]. In principle one may extend the local theory developed in this paper to a “global” differential geometry over the quantised Poisson manifold. Work in this direction is currently under way.

Results in this paper should be directly applicable to the development of a theory of noncommutative general relativity, which is of considerable current interest in theoretical physics. We hope that the theory presented here will provide a consistent mathematical basis for this purpose. We should also mention that one may use this theory to clarify, conceptually, aspects of the many noncommutative geometries introduced in physics in recent years based on physical intuitions. For example, general features of the noncommutative geometries in [10, 11] and [3] have considerable similarity with that of [8]. These works also have the advantage of being explicit and amenable to calculations, thus have the chance to be physically tested. Therefore, it will be useful to further develop the mathematical bases of these theories by casting them into the framework of this paper. Finally we note that a noncommutative analogue of spin geometry over the Moyal algebra within the $C^*$-algebraic framework in terms of noncompact spectral triples was studied in [20]. Our treatment is complementary to that of [20].
Acknowledgement: We wish to thank Masud Chaichian and Anca Tureanu for discussions at various stages of this work. X. Zhang thanks the School of Mathematics and Statistics, the University of Sydney for the hospitality extended to him during a visit when this work was completed. Partial financial support from the Australian Research Council, National Science Foundation of China (grants 10421001, 10725105, 10731080), NKBRC (2006CB805905) and the Chinese Academy of Sciences is gratefully acknowledged.

REFERENCES

[1] L. Álvarez-Gaumé, F. Meyer, M. A. Vazquez-Mozo, Comments on noncommutative gravity, Nucl. Phys. B 75, 392 (2006).
[2] S. Ansoldi, P. Nicolini, A. Smailagic, E. Spallucci, Non-commutative geometry inspired charged black holes, Phys. Lett. B 645, 261 (2007).
[3] P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Noncommutative geometry and gravity, Class. Quant. Grav. 23, 1883 (2006).
[4] R. Banerjee, B. R. Majhi, S. K. Modak, Noncommutative Schwarzschild black hole and area law, Class. Quant. Grav. 26, 085010 (2009).
[5] M. Buric, T. Grammatikopoulos, J. Madore, G. Zoupanos, Gravity and the structure of noncommutative algebras, JHEP 0604, 054 (2006).
[6] M. Chaichian, M. Oksanen, A. Tureanu, G. Zet, Gauging the twisted Poincare symmetry as noncommutative theory of gravitation, Phys. Rev. D 79, 044016 (2009).
[7] M. Chaichian, M. R. Setare, A. Tureanu, G. Zet, On black holes and cosmological constant in noncommutative gauge theory of gravity, JHEP 0804, 064 (2008).
[8] M. Chaichian, A. Tureanu, R.B. Zhang, Xiao Zhang, Riemannian geometry of noncommutative surfaces, J. Math. Phys. 49, 073511 (2008).
[9] M. Chaichian, A. Tureanu and G. Zet, Corrections to Schwarzschild solution in noncommutative gauge theory of gravity, Phys. Lett. B 660, 573 (2008).
[10] A. H. Chamseddine, Complexified gravity in noncommutative spaces, Commun. Math. Phys. 218, 283-292 (2001).
[11] A. H. Chamseddine, \(SL(2, \mathbb{C})\) gravity with a complex vierbein and its noncommutative extension, Phy. Rev. D 69, 024015 (2004).
[12] C.J.S. Clarke, On the global isometric embedding of pesudo-Riemannian manifolds, Proc. Roy. Soc. Lond. A. 314, 417-428 (1970).
[13] A. Connes, Noncommutative geometry, Academic Press, 1994.
[14] M. P. do Carmo, Differential geometry of curves and surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976.
[15] B. P. Dolan, K. S. Gupta, A. Stern, Noncommutative BTZ black hole and discrete time, Class. Quant. Grav. 24, 1647 (2007).
[16] S. Doplicher, K. Fredenhagen, J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields, Commun. Math. Phys. 172, 187-220 (1995).
[17] V. Drinfeld, Quasi-Hopf algebras, Leningrad Math. J. 1, 1419-1457 (1990).
[18] S. Estrada-Jimenez, H. Garcia-Compean, O. Obregon, C. Ramirez, Twisted covariant noncommutative self-dual gravity, Phys. Rev. D 78, 124008 (2008).
[19] A. Friedman, Local isometric embedding of Riemannian manifolds with indefinite metric, J. Math. Mech. 10, 625-650 (1961).
[20] V. Gayral, J.M. Gracia-Bondía, B. Iochum, T. Schücker, J.C. Várilly, Moyal planes are spectral triples, Commun. Math. Phys. 246, 569-23 (2004).
[21] M. Gerstenhaber, On the deformation of rings and algebras, Ann. Math. 79, 59-103 (1964).
[22] J. M. Gracia-Bondía, J. C. Várilly, H. Figueroa, Elements of noncommutative geometry, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Boston, Inc., Boston, MA, 2001.
[23] R. E. Greene, Isometric embedding of Riemannian and pseudo Riemannian manifolds, Memoirs Am. Math. Soc. 97, 1970.
[24] M. Kashiwara, P. Schapira, Deformation quantization modules I: Finiteness and duality, arXiv:0802.1245 [math.QA];
[25] M. Kashiwara, P. Schapira, Deformation quantization modules II. Hochschild class, arXiv:0809.4309 [math.AG].
[26] H. C. Kim, M. I. Park, C. Rim, J. H. Yee, Smeared BTZ black hole from space noncommutativity, JHEP 10, 060 (2008).
[27] A. Kobakhidze, Noncommutative corrections to classical black holes, Phys. Rev. D 79, 047701 (2009).
[28] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66, 157-216 (2003).
[29] J. Madore, J. Mourad. Quantum space-time and classical gravity, J. Math. Phys. 39, 423-442 (1998).
[30] S. Majid, Noncommutative Riemannian and spin geometry of the standard $q$-sphere, Commun. Math. Phys. 256, 255-285 (2005).
[31] F. Muller-Hoissen, Noncommutative geometries and gravity. In Recent developments in gravitation and cosmology, 12-29, AIP Conf. Proc., 977, Amer. Inst. Phys., Melville, NY, 2008.
[32] John Nash, The imbedding problem for Riemannian manifolds, Ann. Math. 63, 20-63 (1956).
[33] P. Nicolini, A. Smailagic, E. Spallucci, Noncommutative geometry inspired Schwarzschild black hole, Phys. Lett. B 632, 547 (2006).
[34] H.S. Snyder, Quantized space-time, Phys. Rev. 71, 38-41 (1947).
[35] R. J. Szabo, Symmetry, gravity and noncommutativity, Class. Quant. Grav. 23, R199 (2006).
[36] H. Steinacker, Emergent gravity and noncommutative branes from Yang-Mills matrix models, Nucl. Phys. B 810, 1-39 (2009).
[37] D. Wang, R. B. Zhang, Xiao Zhang, Quantum deformations of Schwarzschild and Schwarzschild-de Sitter spacetimes, Class. Quant. Grav. 26, 085014 (2009).
[38] C. N. Yang, On quantized space-time, Phys. Rev. 72, 874 (1947).

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, SYDNEY, AUSTRALIA
E-mail address: ruibin.zhang@sydney.edu.au

INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, CHINA
E-mail address: xzhang@amss.ac.cn