Can a pure state remain pure in the Unruh effect?

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Introduction. – The indistinguishability between the quantum vacuum fluctuation and the thermal fluctuation is one of the fundamental discoveries in the last century [1, 2]. Several authors introduced such a connection between the thermal bath and the quantum vacuum as the thermalization theorem [3–5]. The theorem tells that the inertial vacuum behaves as a thermal bath for a uniformly accelerated observer. The temperature of the bath is proportional to the acceleration of the detector. Later Unruh put this theorem in the context of particle creation, which after him is known as the renowned Unruh effect [1, 6–8]. It is also closely connected with the Hawking radiation in the near horizon region [9]. A uniformly accelerated two-level system (TLS) perceives the vacuum state of a free scalar field as a thermal bath in which it undergoes a non-equilibrium evolution. As a result, the system will reach a thermal state at equilibrium [2, 10]. Therefore, one can map this type of system to an open quantum system (OQS), where the effective temperature of the effective thermal bath proportional to the acceleration (α) of the TLS [11, 12].

Thermalization is an irreversible process. Boltzmann first introduced the concept of irreversibility using the classical collision model [13]. The quantum analog of this problem is first described by Von Neumann [14]. Since then, the quantum thermalization problem is one of the prestigious topics in modern physics [17, 18]. The thermal state can be identified as a mixed state whose entropy is maximum. The initial memory is lost during this irreversible evolution. No local conserved quantities survive at thermal equilibrium. Hence, it is a non-integrable phase [18]. Anderson first investigated non-thermal features in an isolated random lattice due to impurity. There exist several localized states, which skip the thermalization. It is known as Anderson-localization [19, 20]. These phases are also integrable as various local conserved quantities persist at equilibrium [18]. Many-body localization, quantum scars are the other examples of the non-thermal phases in the closed many quantum systems [17]. Also, in the case of many-particle OQS, thermal to non-thermal phase transition occurs for a critical value of a particular system parameter [21]. The steady-state configuration suddenly changes at the critical point. It is further known as the dissipative phase transition (DPT) in the literature [22–28]. Several experimental verifications support the existence of DPT, e.g., Dicke quantum phase transition in optical cavity [29].

As the temperature goes down, the quantum effect dominates the thermal dissipation. So, at very low temperature, the spatially correlated thermal bath behaves as a source of long-range quantum correlations, and the cooperativity between the atoms arises [21, 27]. This behavior of the thermal bath is called the common-environment effect [30]. As a result, a persistent entanglement occurs in the system of non-interacting spins in a thermal bath. So, the final steady-state is a non-thermal state [31, 32]. The whole dynamics is protected by several symmetry operators in the non-thermal limit [21, 24, 25]. Therefore the eigenstates of the symmetry operator have the possibility to remain unchanged throughout the dynamics. It also predicts the preservation of the purity of particular initial states. These dissipationless states are called dark states [21, 24]. Even for the markovian systems, they can carry the initial memories in the equilibrium, which has huge implications in quantum computation, and quantum optics [35]. The steady-state solution has a degeneracy.
at this limit. When the temperature increases, the degeneracy is explicitly broken, and the cooperativity between the atoms is lost, and the steady-state becomes thermal [21].

Motivated from the above discussions of the temperature-driven phase transition, we show that the acceleration-driven phase transition is possible for a uniformly accelerated non-interacting two spin system in an inertial vacuum. It is well known that, for a single atom ensemble, the final steady-state is always thermal. Non-thermality arises when we have more than one atom in the ensemble [11]. An inter-atoms distance-dependent thermal-nonthermal signature of the Unruh effect in resonance Casimir-Polder interaction (RCPI) was reported earlier [36]. Moreover, the energy shift due to Casimir-Polder interaction (CPI) is very small compared to the Zeeman levels, so it is very hard to observe experimentally [27].

We observe that for such a system with fixed interatomic distance, the low acceleration regime is entirely given a notion of non-thermality. The dynamics are invariant under a weak symmetry transformation. The emergence of the two-atomic dark state is the most important feature of this non-thermal phase, and beyond the critical limit of the acceleration, the steady-state becomes thermal. The system acts as a collection of single atom ensembles in this phase. Although, this effect is completely different from the non-markovian Unruh effect, where the non-thermality arises due to non-markovian effects [33, 34]. We also use the exact experimental values estimated by Bell, and Leinass [8] to give a proper notion of the non-thermal regime, which can be proven by using the appropriate instrumental setup in the future.

Throughout the paper, we follow the given notation. The four-vector is denoted by ‘x,’ the time is represented by ‘t’ and the proper time by ‘τ.’ We denote the space co-ordinate by ‘X,’ (X = X, Y, Z).

Dynamics of the accelerated atoms through a massless scalar field.—Here, we consider a system of two spin half particles uniformly accelerated in Minkowski spacetime and interacts with a free massless scalar field. The atoms are following the hyperbolic trajectory with respect to the inertial observer. The positions of the atoms in terms of proper time are written by,

\[ t_m(\tau) = \frac{1}{\alpha} \sinh \alpha \tau, \quad X_m = \frac{1}{\alpha} \cosh \alpha \tau, \]

\[ Y_m = 0, \quad Z_m(\tau) = Z_a, \quad Z_b(\tau) = Z_b. \quad (m = \{a, b\}). \quad (1) \]

We define L to be the proper distance between the two atoms located at the positions \((t_a(\tau), x_a(\tau))\) and \((t_b(\tau), x_b(\tau))\). The Hamiltonian of the system without field is expressed as [11],

\[ \mathcal{H} = \mathcal{H}_3^a + \mathcal{H}_3^b + \mathcal{H}_{\text{sl}}. \quad (2) \]

Throughout the paper, we use the natural units, \(h = c = 1\). Here we assume the two spins have the same Zeeman levels, and the free Hamiltonian of the system is

\[ \mathcal{H}_3^a = \mathcal{H}_3^b = \frac{1}{2} \omega_3^{(1)} \sigma_3^a + \frac{1}{2} \omega_3^{(2)} \sigma_3^b (\text{The superscript in the Pauli matrices represent the atom number}). \]

\(\sigma_i\) are the Pauli matrices, and \(\omega_i\) is the frequency of Zeeman levels. The free Hamiltonian of the scalar field \(\mathcal{H}_S^a\) can be expressed as,

\[ \mathcal{H}_S^a = \int \frac{d^3k}{(2\pi)^3} \omega_3^a a^\dagger(k)a(k). \]

\(\mathcal{H}_{\text{sl}} = \mathcal{H}_{\text{sl}}^a + \mathcal{H}_{\text{sl}}^b \) where \(\mathcal{H}_{\text{sl}}^a\) is the coupling Hamiltonian between the atom and free massless scalar field. The coupling Hamiltonian between the atom-scalar field is assumed to be [11],

\[ \mathcal{H}_{\text{sl}} = \lambda \sum_{\mu=0}^{3} \left[ \sigma_\mu^1 \otimes \phi_\mu(x_1) + \sigma_\mu^2 \otimes \phi_\mu(x_2) \right], \]

(3)

where \(\lambda\) is the coupling constant, \(\phi\) represents the scalar field and \(x_1, x_2\) are the individual trajectories of the two atoms. Here, \(\phi_\mu(x) = \sum_{a=1}^{N} \left[ \chi_\mu^a \phi^-(x) + (\chi_\mu^a)^\dagger \phi^+(x) \right] \).

\(\phi^\pm(x)\) is the positive and negative field operator of the free scalar field and \(\chi_\mu^a\) are the corresponding complex coefficients. The two-spin system and the scalar field are initially uncorrelated, so the total initial density matrix can be written as \(\rho(0) = \rho_a(0) \otimes |0\rangle \langle 0|\). Here, \(|0\rangle\) is the vacuum state of the scalar field and \(\rho_a(0)\) is the initial density matrix of the system. Under the weak-coupling approximation, using the Von-Neumann-Liouville equation in the interaction frame and taking the trace over the field variable, we get the dynamical equation of the system, which is written below [11],

\[ \frac{d\rho_a(\tau)}{d\tau} = -i \left[ \mathcal{H}_{\text{amb}}, \rho_a(\tau) \right] + \mathcal{L}(\rho_a(\tau)) . \]

(4)

\(\mathcal{L}(\rho_a(\tau))\) is the dissipator of the master-equation and \(\mathcal{H}_{\text{amb}}\) is known as the Lamb-shift Hamiltonian that leads to the renormalization of Zeeman Hamiltonian. This Lamb shift gives rise to the CPI between the spins [33-41]. The forms are given by,

\[ \mathcal{L}(\rho_a) = \sum_{a,b=1}^{3} S_{ab}^{jk} \sigma_a^k \sigma_b^j \rho_a \left( \sigma_a^k \sigma_b^j - \frac{1}{2} \left( \sigma_a^k \sigma_b^k \right) \right), \]

(5)

\[ \mathcal{H}_{\text{amb}} = - \frac{i}{2} \sum_{a,b=1}^{3} S_{ab}^{jk} \sigma_a^k \sigma_b^j . \]

(6)

The field correlation time-scale is taken to be smaller than the relaxation time-scale of the system [14]. The dynamics are completely positive and trace preservation holds. Here, we assume \(\chi_\mu^a\) satisfies \(\sum_{a=1}^{N} \chi_\mu^a (\chi_\mu^a)^\dagger = \delta_{\mu \nu}\). So the field correlation functions are diagonal i.e. \(G_{\nu}^{ab}(x - y) = \delta_{\nu} G^{ab}(x - y)\). The expressions of the field-correlation function is obtained by,

\[ G^{ab}(\Delta \tau) = \langle 0 | \hat{\Phi}(\tau, x_a) \hat{\Phi}(\tau', x_b) | 0 \rangle . \]

(7)
Here $\Delta \tau = (\tau - \tau')$. The correlation function is assumed to be stationary. For a massless scalar field, in 4-dimensional Minkowski spacetime, the standard expression of this function is written as \[ G(x, x') = -\frac{1}{4\pi^2 R^2} , \] where $R = \sqrt{(t-t' - i\epsilon)^2 - | \mathbf{x} - \mathbf{x}' |^2}$. This is called the positive-frequency Wightman function. Here $i\epsilon$ is a small constant in the complex plane and acts as a regulator to avoid the divergence and make sure about the function’s analytic property over the chosen complex plane. The Fourier transform along the trajectory of the atoms is given below [44],\[ G^{ab}(\omega) = \frac{1}{2\pi} \frac{\omega}{1 - e^{-2\pi\omega/\alpha}} f_{ab}(\alpha) . \] Here, $f_{ab}(\alpha) = \sin \left( \frac{\omega}{2\pi} \sinh^{-1}(\alpha L/2) \right) / L \omega \sqrt{1 + L^2 \alpha^2/4}$ for $a \neq b$, and $f_{ab}(\alpha) = 1$ for $a = b$. The explicit form of Kossakowski matrix, which is an essential component of the Lindbladian, is obtained by [11],\[ \delta^{ab}_{jk} = A^{ab} \delta_{jk} - i B^{ab} \epsilon_{jkl} \delta_{3l} - A^{ab} \delta_{3k} \delta_{3l} . \] $A^{ab}(B^{ab})$ are the even (odd) combination of the Fourier transform $G^{ab}$ for the positive and negative Zeeman frequency. For a massless scalar field, the explicit forms of $A^{ab}, B^{ab}$ are written as [14],\[ B^{ab} = \frac{\chi^2 \omega}{8\pi} f_{ab}(\alpha), A^{ab} = B^{ab} \times \coth(\pi\omega/\alpha) . \] Here $a, b \in \{1, 2\}$. We use $f_{ab}(\alpha)$ as $f$ in the remaining part.

Dissipative phase transition (DPT) and Unruh effect. The steady-state solution of the QME gives the notion of the thermal equilibrium of the system. In the Liouville space, the QME can be written as, $\frac{d}{d\tau} \rho_s = \hat{L} \rho_s$. Here, $\hat{L}$ is a column vector of dimension $N^2 \times 1$ and $\hat{L}$ is the Lindbladian superoperator of dimension $N^2 \times N^2$ ($N$ is the dimension of the Hilbert space). $\hat{L}$ is a non-Hermitian matrix [25]. The eigenvector corresponding to the zero eigenvalues is the steady-state solution of the QME. DPT is defined as, for a continuous changing of any system parameter, when the degeneracy of the eigenstate of zero eigenvalues of $\hat{L}$ is abruptly broken, the system makes a transition between two different phases. This description is quite similar to the quantum phase transition, where the ground state degeneracy is broken at a critical limit \[ \tau_c \]. In this case, the steady-state solutions depend on the behavior of $f$. For a single spin-1/2 system and $2 \times 2$ matrix, the observables are $\{ \sigma_x, \sigma_y, \sigma_z \}$ (one constraint-trace preservation) and the corresponding equations are known as the Bloch equations [40]. Following the above argument for two spin systems, there exist fifteen observables. Still, as we have symmetry in the system (the Zeeman levels of the spins are the same), the number of observables is reduced to nine, which can be written as [21],\[ M_i = \frac{1}{2} \mathrm{Tr}_s \left( \sigma^{(1)}_i \otimes I + I \otimes \sigma^{(2)}_i \right) \rho_s , \] \[ M_{ii} = \frac{1}{4} \mathrm{Tr}_s \left( \sigma^{(2)}_i \right) \rho_s , \] \[ M_{ij} = \frac{1}{4} \mathrm{Tr}_s \left( \sigma^{(1)}_i \otimes \sigma^{(2)}_j \right) \rho_s . \] (12) Although the choice is not unique. The Bloch-type equations for our case is written as,\[ \begin{pmatrix} \dot{M}_x \\ \dot{M}_{zz} \\ \dot{M}_c \end{pmatrix} = \begin{pmatrix} -A^{11} & 0 & 2B^{12} \\ -B^{12} & -A^{11} & A^{12} \\ -B^{12} & 2A^{12} & -A^{11} \end{pmatrix} \begin{pmatrix} M_x \\ M_{zz} \\ M_c \end{pmatrix} + \begin{pmatrix} B^{11} \\ 0 \\ 0 \end{pmatrix} . \] (13) Here $M_c = M_{xx} + M_{yy}$. Expectation value of the other observables goes to zero at $\tau \rightarrow \infty$. Here, $f$ varies from 0 \leq \alpha = \alpha_c \leq 1$. There exist a critical value of $\alpha = \alpha_c$ for fixed $L$, when $f \rightarrow 1$. In this limit, $A^{11} = A^{12}, B^{11} = B^{12}$, the Lindbladian super-operator $\hat{L}$ is invariant under the following unitary transformation, $\hat{U}(\kappa) \hat{L} \hat{U}^\dagger(\kappa) = \hat{L}$ where $\kappa$ is real. We define, $\hat{\tilde{U}}(\kappa) = \exp[-iD\kappa]$. Here, $D = D_1 \otimes I - I \otimes D^F$. $D$ is precisely the weak symmetry super-operator, as it commutes with the full $\hat{L}$, unlike the strong symmetry, where the operator commute with all the individual components of $\hat{L}$ [25]. As a consequence of the Noether theorem, there exists a local conserved quantity, which is written as,\[ \frac{d}{d\tau} (M_{xx} + M_{yy} + M_{zz}) = 0 . \] (14) As we increase the number of spins, several two quantum local quantities will be conserved (i.e. $\mathrm{Tr}_s \left( \{ \sum_i \sigma^{(i)} \otimes \sigma^{(i+1)} \right) \rho_s \}$). Here, $n$ is the spin-number and $i = (x, y, z)$. It is called the localization [18]. Due to this symmetry the final steady-state has an initial value dependence. In FIG. [1] (b), we numerically plot of the distribution of the eigenvalues of $\hat{L}$ in the increasing order. $A^{11}, B^{11}$ is chosen to be fixed. The contribution from the Lamb shift is neglected, so, all values are either zero or real negative. For $f = 1$, the Lindbladian superoperator has two zero eigenvalue so the steady-state is degenerate, and as $f$ decreases, the degeneracy is broken. The steady-state solutions for $f = 1$ is given by,\[ M^{eq}_z = \frac{M_x (3 + 4(M_x + M_3))}{3 + M_3^2} , \] \[ M^{eq}_c = \frac{M_z^2 - 4(M_2 + M_3)}{2(3 + M_3^2)} . \] (15) Here, $M_c(\tau = 0) = M_2, M_{zz}(\tau = 0) = M_3$ and $M_0 = \tanh(\pi\omega_c/\alpha)$. The final steady-state is a non-thermal one. This steady-state solutions are exactly matches with
Increasing the value of $\alpha$, we get $0 \leq f < 1$, and for a large value of $\alpha$, $f = 0$. At the transition point $\alpha \approx 2 \times 10^{22}$ ms$^{-2}$, the value of $f(\alpha)$ starts to fall from 1. (b) shows the plot of the distribution of the eigenvalues of $\hat{L}$ for two different values of $f$ (effect of Lamb shift is neglected and $A^{11}, B^{11}$ is fixed, $A^{11} = 4, B^{11} = 1$). Here $p$ is the position of the eigenvalues [1 to 16] in the increasing order. For $f = 1$, there is a degeneracy of the zero eigenvalues (denoted by the red “+” sign), as $f$ decreases ($f = 0.8$), the degeneracy is broken (denoted by the blue “+” sign). (c), (d), (e) show the plot of the first-order derivative of $\{M_z, M_{zz}, M_c\}$ w.r.t. $\alpha$. At the transition point of $\alpha = \alpha_c$, the function diverges. Therefore, for $f = 1$, we get a localized phase, and $f < 1$, it is a thermal phase.

The Lindbladian is temperature. Bell and Leinaas showed in a 3.1 km circular accelerator in SPEAR at Stanford, and nearly 1200 k temperature can be produced by circulating the ultra-relativistic electrons at a centripetal acceleration of $\alpha = 2.9 \times 10^{23}$ m/s$^{-2}$ [5]. So we, vary $\alpha$ from $(10^{21}$ ms$^{-2} \rightarrow 10^{25}$ ms$^{-2}$). Distance between the atoms is chosen to be $L = 6 \times 10^{-7} m$, as the Lamb shift or the Casimir-Polder interaction can be observed at a distance of $\mu m$ range [50] and Zeeman frequency is chosen as $\omega_0 = 10^{14}$ Hz. In principle, localization in a thermal bath occurs at zero temperature. Several experimental techniques like reservoir engineering, laser
cooling ion-trap, etc., are developed in the last few decades, which are used to enhance the bath-correlation length at $T > 0$ to trap the atoms [51–54]. Therefore, one can produce such a localized phase even at non-zero temperatures. Similarly, the non-thermality will arise at $\alpha = 0$. Lowering the inter-atomic distance value and the increasing value of Zeeman frequency will increase the field correlation length. As a result, we can get a localized phase at $\alpha \neq 0$. These limits, we are using here, are also experimentally justifiable [8].

For this choice of value, we numerically plot $f(\alpha)$ vs. $\alpha$. In FIG. 1 (a), for $\alpha < 2 \times 10^{22}ms^{-2}$, we get $f = 1$, which signifies a localized phase and increasing $\alpha$, $f$ falls below 1, which denotes the existence of thermal phase for higher acceleration. So we get two different regions of $f$, which signifies two separate phases.

There exists an explicit symmetry breaking in the system. The first-order derivative with respect to $\alpha$, of the set of observables $\mathcal{O} = (M_x, M_{zz}, M_z)$ diverge at $\alpha = \alpha_c$ ($i.e.$, $\lim_{\alpha \rightarrow \alpha_c} \frac{d\mathcal{O}}{d\alpha} = \infty$). Beyond the critical limit, the steady-state configuration abruptly changes. Therefore, a first-order phase transition (from a localized to a thermal phase) happens in the system. In FIG. 1 (c), (d), (e), we numerically show that, the first-order derivative of $(M_x, M_{zz}, M_z)$ diverges at $\alpha_c \approx 2 \times 10^{22}ms^{-2}$, which also predicts the existence of two different phases. Non zero value of $M_z$ in the non-thermal phase indicates the existence of field-induced entanglement in the system [11]. Concurrence is a well-known measure of the entanglement between the two spin systems [55]. The expression for the concurrence $(C(\rho_s))$ is given by, $C(\rho_s) = \max\{0, 4|\rho_s| - \sqrt{(1 + 4M_{zz})^2 - 4M_z^2}\}$. On the other hand, in the thermal phase, the entanglement vanishes ($M_z = 0$), and the spins behave as a separate system.

**Conclusion.**– For a two-atom system accelerating through a massless scalar field, the vacuum state of the field acts like a correlated bath. At the lower values of acceleration ($f = 1, \alpha < \alpha_c$), the system goes to a localized phase, and the inertial vacuum acts as a common environment. There exist a conserved quantity. Increase the value of acceleration, the symmetry of the system is explicitly broken ($f < 1, \alpha < \alpha_c$), the system goes to a thermal phase. In this two limit ($f = 1, f < 1$), the steady-state solution of the observables $(M_x, M_{zz}, M_z)$ are different. Hence the first-order derivative of the observables diverges at the transition point. The Von-Neumann entropy $(-Tr_\alpha(\rho_s\ln\rho_s))$ in the non-thermal phase is not an extensive quantity [21]. In this phase, the dark states do not evolve. Therefore, this kind of state is putting a limitation on the thermalization theorem. When we increase the number of atoms in the ensemble, the vacuum state from the point of view of the inertial atomic detector still behaves as a canonical ensemble as a point of view of a uniformly accelerated observer. But at the lower value of acceleration, the system is at a frozen state, which is protected by several symmetry operators, so it cannot feel any temperature. The principle of equal a priori probabilities do not hold here. So, a pure dark state remains pure.

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