 Disorder resistivity of solid neutron-star matter

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Lower limits are found for the disorder electrical resistivity of solid neutron-star matter in the neutron-drip region which is amorphous and heterogeneous in nuclear charge. This temperature-independent resistivity, large compared with that produced by phonon scattering, has direct consequences for theories of neutron-star magnetic field generation and evolution.

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Following the discovery of neutron stars in 1967, very many papers have been published containing theories of the nature and origin of their magnetic flux distributions. The original view was that the distributions are fossil fields amplified during gravitational collapse. A recent variant of this idea is field generation by rapid dynamo action during a brief interval of convective motion immediately after that process. But some authors have doubted that amplified fossil fields can be of the order of magnitude inferred from measured pulsar periods and their time-derivatives, and have considered thermoelectric field generation in the solid crust and liquid surface region of the star during the early years after its formation. This Letter shows that disorder electrical resistivity is present in solid neutron-star matter. It is temperature-independent and so large that, with the impurity resistivity, it severely constrains other than fossil-field theories.

At times when thermoelectric processes have become negligible, and with no movement of matter, the evolution of the magnetic flux density $B$ in the solid crust is given by the Maxwell induction equation with Ohm’s law and neglect of displacement currents,

$$\frac{\partial B}{\partial t} = -\nabla \times \left( \frac{c^2}{4\pi\sigma} \nabla \times B \right) - \nabla \times \left( \frac{c}{4\pi ne} (\nabla \times B) \times B \right),$$

where $\sigma$ is the zero-field conductivity and $n$ the electron number density. Eq. (1) assumes that $B$ is not so large in relation to $n$ that classical conductivity fails (see, for example, Ref. [5]). The existence of disorder resistivity leads to small values of $\sigma$, renders negligible the term in (1) bilinear in $B$ representing Hall drift, and allows unambiguous calculation of field evolution within the crust.

Previously, values of $\sigma$ have been derived from calculations of electron-phonon scattering in a homogeneous $bcc$ lattice [6], and from scattering [6, 8, 9] by fractional concentrations $c_i$ of impurity charges $Z_i$ with mean $\bar{Z}$ and mean square deviation $Q = \sum_i c_i (Z_i - \bar{Z})^2$. The latter has been treated as a free parameter, usually assumed to satisfy $Q \ll 1$. For these $\sigma$, the Hall term in (1) is, in many cases, larger than the ohmic dissipative linear term [10, 11]. Eq. (1) has some similarity with the vorticity equation for a viscous incompressible fluid (see, for example, Ref. [12]) and a number of authors [11, 13, 14] have investigated the existence of a cascade to higher-wavenumber components analogous with the Kolmogorov cascade in fluid turbulence. This would have a significant effect on evolution because, although Hall drift is not in itself dissipative, the ohmic dissipation rate for any flux-distribution component is dependent on the square of its wavenumber.

It must be emphasized that the references cited so far are not complete but provide an introduction to a very large literature on a subject of some complexity. Solutions of Eq. (1) are not simple if the Hall term is appreciable. Also, if $\sigma$ is derived from electron-phonon resistivity, it is so highly temperature-dependent that the cooling of the star and the evolution of its magnetic flux distribution must, in principle, be calculated in parallel. The present work shows that disorder becomes the dominant source of resistivity as the interior cools below $10^9$ K and so obviates these problems.

There is now evidence that the assumption of a $bcc$ lattice, on which calculations of $\sigma$ have been based, is not correct. Weak-interaction equilibrium is not reached during the formation and cooling of the neutron star so that, as a consequence of shell structure and of the small differences in formation enthalpy between nuclei of different $Z$, the solid is both heterogeneous and amorphous [15, 16]. (This work applies to neutron stars that have not undergone significant mass-accretion. The high mass-transfer rates in certain binary systems can completely replace the natural solid crust with accreted matter of heterogeneous $Z$.) It is well known that amorphous systems have high and substantially temperature-independent resistivities (see, for example, Ref. [18]). In solid neutron-star matter, disorder is the most important source of resistivity and does not seem to have been considered previously.
If the weak-interaction strength were (hypothetically) adjusted so as to allow the attainment of equilibrium, an amorphous and $Z$-heterogeneous solid would transform to a $bcc$ structure with homogeneous nuclear charge and a population of lattice defects. Thus an amorphous structure with minimum disorder can be obtained by the reverse transformation, starting from a defect-free $bcc$ lattice and sequentially changing each nuclear charge $Z \rightarrow Z_i$ randomly so as to give fractional concentrations $c_i$. This procedure has been carried out for a system of 24357 $bcc$ lattice sites contained within a spherical volume using distributions $c_i$ of $Z_i$ found by the methods of Ref. [12, 13]. These are based on equilibrium $bcc$ lattices with charge $Z_{CLDM}$ whose parameters have been obtained at temperature $T = 0$ for each of the matter densities in Table I using the compressible liquid-drop model of Lattimer et al [19]. The densities selected are in the neutron-drip region and represent an interval equivalent to most of the depth of the crust. Calculations of formation enthalpies for nuclei with $20 < Z_i < 50$ in equilibrium with the uniform continuum [12, 13] give values of the $c_i$ for thermal equilibrium, at an initial temperature $T_0 = 5 \times 10^9$ K, which is near the melting temperature $T_m$ of the solid. There is movement away from weak-interaction equilibrium during the early stages of cooling and the fractional concentrations $c_i$ evolve to frozen-in values giving the parameters $Q$ of Table I. This produces a small shift $6Z$ in nuclear charge so that the mean charge of Table I is $\bar{Z} = Z_{CLDM} + \delta Z$. Because there are considerable uncertainties in obtaining the shell-corrected nuclear formation enthalpies, a range of possibilities is considered [10]. Rows labelled sp in Table I are based on values of the $c_i$ found from formation enthalpies given by the Strutinski procedure for shell-correction (see, for example, Ref. [20]) with proton pairing. Rows labelled p are for $c_i$ given by formation enthalpies with proton pairing but no shell correction.

Each change of nuclear charge $Z \rightarrow Z_i$ produces displacements of all other lattice sites in the system considered. Our assumption here is that these are identical with those found in Ref. [17] for an isolated point-defect in a homogeneous $bcc$ lattice. (This, of course, is not strictly correct because as the sequence of transformations $Z \rightarrow Z_i$ is made, the neighbours of a given nucleus become heterogeneous in charge and their positions deviate from those of the $bcc$ lattice.) These displacements were calculated for each of the neighbour sets $\alpha = 1 - 3$, but for $\alpha \geq 3$, they are assumed equal to those produced by a point defect in an isotropic incompressible elastic medium. The volume change represented by these displacements is determined, principally, by the properties of the Coulomb-electron stress tensor in the neutron-drip region whose isotropic components are between one and two orders of magnitude larger than the off-diagonal, and by the fact that $\kappa \approx 1$, where $\kappa$ is the electron screening wavenumber [13, 16]. Thus the displacements and volume change are such that the electron density, averaged over a volume of the order of a Wigner-Seitz cell, adjusts to values almost exactly equal to those of the undisturbed lattice with Fermi wavenumber $k_{Fe}$. The result of calculating the total displacement for each lattice site in the system considered is that the radial density distribution of nuclear positions relative to the origin site changes from a sum of $\delta$-functions in the undisturbed $bcc$ system to a sum of gaussian functions of approximately constant variance $w^2$. There is also a change in the mean spatial density of sites, but it is statistically consistent with zero. Values of $w$ have been calculated for each matter density and are given in Table I.

Our calculation of disorder resistivity follows closely the classic paper of Ziman [21] on liquid metals. Comparison with the exact electron scattering matrix element [22] as a function of the momentum transfer $q$ indicates that for the size of momentum transfer allowed in the neutron-drip region, $0 < q < 2k_{Fe}$, the Born approximation is adequate for scattering by the potential inside an electrically neutral Wigner-Seitz cell. The Fourier transform of the charge density is $ZeF(q)$. It assumes the uniform proton density of a CLDM nucleus [19] and uniform electron density inside a charge-neutral spherical cell whose radius is defined by $\bar{Z}$ and $k_{Fe}$. The scattering amplitude for the whole system is then a sum of two terms; one dependent on $\bar{Z}$ and the other on the differences $(Z_i - \bar{Z})$. The static structure function derived from the first term,

$$S_d(q) = \frac{1}{N} \left| \sum_{i=1}^{N} e^{iq \cdot r_i} \right|^2,$$

(2)

which vanishes in the absence of disorder, has been obtained directly, to terms of order $q^2 w^2$, from the Fourier transform of the radial density distribution of the nuclear positions $r_i$. Nuclei of any specific charge $Z_i$ are randomly distributed and the set of such nuclei has static structure function $S_z = 1$.

There is no interference between the two amplitude terms and so the resistivity $\mathcal{R}$ is

$$\mathcal{R} = \mathcal{R}_z + \mathcal{R}_d = \frac{4\pi e^2}{Z c \mu_e} \left( q \Lambda_z + \bar{Z}^2 \Lambda_d \right),$$

(3)

where $\mu_e$ is the electron chemical potential. The modified Coulomb integrals are,

$$\Lambda_{dz} = \int_0^{2k_{Fe}} dq \left( 1 - \frac{q^2}{4k_{Fe}^2} \right) F^2(q) S_d(z)(q)e^{-2W},$$

(4)

in which the final term, the Debye-Waller factor, can be set equal to unity at all $q < 2k_{Fe}$ for the relevant $T \ll T_D$, where $T_D$ is the Debye temperature of a neutron-drip $bcc$ lattice. The first term in Eq. (3) is identical with the standard expression for impurity resistivity but the Table II values of $\mathcal{R}_z$ obtained from it are smaller than those
TABLE I: Disordered systems at matter density $\rho$ have mean nuclear charge $\bar{Z}$ and electron Fermi wavenumber $k_F$, giving an equivalent bcc lattice constant $a$. The parameter $Q$ is the mean square nuclear charge deviation and $w^2$ is the variance of the gaussian radial distributions of nuclear positions relative to the origin site. Rows labelled sp are for the cases in which nuclear concentrations $c_i$ have been obtained by the procedures of Ref. [16] with shell corrections and proton pairing. Rows labelled p are for $c_i$ given by proton pairing alone.

| $\rho$ (10$^{13}$ g cm$^{-3}$) | $\bar{Z}$ | $k_F$ (10$^{15}$ cm$^{-1}$) | $a$ (10$^{-13}$ cm) | $Q$ | $w^2$ |
|-----------------|--------|-----------------|-------------|-----|------|
| sp              | 1.6    | 37.8            | 0.231       | 56.6| 11.9 | 1.47 |
| sp              | 3.7    | 35.3            | 0.286       | 44.7| 6.0  | 1.80 |
| sp              | 8.8    | 39.0            | 0.363       | 36.4| 19.0 | 1.90 |
| p               | 1.6    | 34.6            | 0.231       | 55.0| 5.0  | 1.40 |
| p               | 3.7    | 33.8            | 0.286       | 44.1| 17.4 | 2.46 |
| p               | 8.8    | 34.4            | 0.363       | 34.9| 24.3 | 3.19 |

of Ref. [8] owing to the different assumption in the present work concerning $F(q)$. (The assumption of [8], that $F(q)$ is the nuclear form factor divided by the small-q limit of the longitudinal static dielectric constant, gives the bracketed $R_e$ values.) The second is the disorder resistivity and, even for the minimum disorder case, it is the larger term given the $F(q)$ of this Letter, as shown in Table II.

TABLE II: Resistivities are listed for systems with the properties given in Table I, $R_d^{\min}$ from the effect of minimum disorder and $R_e$ from charge heterogeneity; $R = R_d^{\min} + R_e$. The bracketed values of $R_e$ are for the form factor assumed in [8]. The final column gives an estimate of the resistivity arising from maximum disorder as defined here.

| $\rho$ (10$^{13}$ g cm$^{-3}$) | $R_d^{\min}$ (10$^{-24}$s) | $R_e$ (10$^{-24}$s) | $R$ (10$^{-24}$s) | $R_d^{\max}$ (10$^{-24}$s) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| sp              | 1.6             | 0.49            | 0.31 (0.73)     | 0.80            | 35               |
| sp              | 3.7             | 0.39            | 0.11 (0.29)     | 0.50            | 21               |
| sp              | 8.8             | 0.19            | 0.15 (0.55)     | 0.34            | 10               |
| p               | 1.6             | 0.40            | 0.14 (0.33)     | 0.54            | 35               |
| p               | 3.7             | 0.70            | 0.33 (0.88)     | 1.03            | 21               |
| p               | 8.8             | 0.47            | 0.22 (0.80)     | 0.69            | 10               |

If component separation were too extensive, our assumption about electron wave functions would fail and the solid would become ordered over those length scales which determine resistivity. However, neutron star crusts differ from white dwarf interiors in which component separation is possible (see, for example Ref. [22]) because cooling to $T \approx 0.5T_m$ occurs rapidly, in 10$^2$–3s. Vacancy mobility (or its analogue in an amorphous system) is the dominant diffusion mechanism enabling component separation to occur [24]. Under the assumption that sufficient sinks exist in the solid structure to maintain a thermal equilibrium vacancy concentration during cooling, the jump frequency for a given nucleus is of the order of $\nu_D \exp(-\beta(H_{F_v} + \bar{z}))$, where $\nu_D$ is the Debye frequency and $\beta^{-1} = k_B T$. In a bcc lattice, the jump activation energy given in Ref. [24] can be expressed, approximately, as $\bar{z} \approx 15\mu V_{WS} \delta^2/4$, in which $\mu$ is the shear modulus, $V_{WS}$ is the mean volume per nucleus and $\delta^2 = 0.08$. Evaluation of this expression, as an estimate of $\bar{z}$ for an amorphous system, gives values much smaller than calculated vacancy formation enthalpies $H_{F_v}$ [24] which are therefore the principal factor determining diffusion rates. For these formation enthalpies, jump frequencies become negligible at $T < T_m$ relative to cooling rates. A further consideration is the formation of clusters of like nuclei which might form the initial stages of component separation. Approximate calculations [12] of the formation enthalpy of some specific forms of cluster show that any decrease in enthalpy attained is smaller than the free energy term derived from configurational entropy at $T_m$. On these grounds, we believe that component separation in neutron-star matter is insignificant and that the amorphous nature of the solid persists during cooling.

The consequences of disorder resistivity are unambiguous. From Eq. (1), the exponential decay time for a mode of wavelength $\lambda$ is $\tau = \sigma \lambda^2/\pi c^2$. Under the assumption that $\sigma$ is derived entirely from the high-density $R$ of Table II, the decay time is $\tau = \tau_0 = 0.7 - 1.3 \times 10^6$ yr for a mode with $\lambda/2 = 10^5$ cm, the approximate depth of the whole crust in a 1.4$M_{\odot}$ neutron star. This limits the ap-
plicability of models in which the field is confined to the crust or has been submerged within it by post core-collapse accretion. Non-fossil field models, in which considerable amplification occurs after formation of the neutron star, rely on processes which are optimized at small electron density $n$. This is true of thermoelectric amplification and of amplification arising from movement of flux by Hall drift to regions of higher resistivity. The fields of $10^{14}-15$ G thought to be present in the anomalous X-ray pulsars (AXP) may evolve by Hall drift. There are several possible mechanisms for the persistent emission of these objects involving departures from hydromagnetic equilibrium in the star. But the resistivities of Table II mean that, unless the field is almost irrotational in the solid, ohmic dissipation must be a source of strong thermal emission with a lifetime $\sim 10^{5-6}$ yr whether or not hydromagnetic equilibrium exists.

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