Large N Strong/Weak Coupling Phase Transition and the Correspondence Principle

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We argue that the large N strong/weak phase transition is a generic phenomenon in a finite temperature supersymmetric Yang-Mills theory of maximal supersymmetry. $\mathcal{N} = 4$, $D = 4$ SYM is the canonical example, where we also argue that the large N Hawking-Page phase transition disappears for a sufficiently small coupling. The Hawking-Page transition temperature is lowered by the first $\alpha'$ correction. Physically, the strong/weak phase transition is identified with the correspondence point of Horowitz and Polchinski. We also try to construct toy models to demonstrate the large N phase transitions, with limited success.

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1. Introduction

Supersymmetric Yang-Mills theories (SYM) exhibit rich dynamics. The study of matrix theory [1] and Maldacena’s conjecture [2,3] reveals much of the unusual physics of SYM with maximal supersymmetry in the large N limit. Small energy gaps, different energy scales are only a few examples of many intriguing properties of these theories. While it is surely a great hint that both matrix theory and AdS/CFT correspondence encode gravitational physics in the large N limit, the major technical obstacle to our understanding of these models is to truly resolve the large N problem.

Another surprise the AdS/CFT correspondence brings to us is the possibility of solving the confinement problem of QCD in the large N limit. The hope might be hindered by a large N strong/weak phase transition [4]. According to [5], D4-brane and D3-brane theories when compactified on a Euclidean circle lead to QCD and 3D pure Yang-Mills theory, provided the radius of this circle is much smaller than the intrinsic scale of the lower dimensional theory. The latter can be viewed as the low energy sector of the original theory at a finite temperature. The strong/weak phase transition of the free energy reflects the same physics of this low energy sector in the $D = 4$ case, since its occurrence is independent of the temperature. One can not completely ignore the underlying 4 dimensional theory at the transition point, since the 3D YM scale $\lambda_3 = \lambda T \sim T$, the KK modes have the same energy scale as the 3D QCD string tension. While in the $D = 5$ case, although the phase transition is in terms of temperature, so long when the M-circle scale is small enough, the transition temperature can be very high, again the phase transition will be reflected by physics of the low energy sector, namely QCD. The existence of the phase transition implies that to study the weak coupling regime, one must study the weak coupling regime directly. The weak coupling physics can not be obtained by the analytic continuation of the strong coupling physics.

The study of [4] was conducted in the infinite volume limit, and is limited to $D = 4$ case. We shall generalize it to the finite volume case, and point out that the strong/weak phase transition also exists on $S^3$. The phenomenon is also a general feature of SYM of maximal supersymmetry in various dimensions. The phase transition point is identified with the correspondence point of [4]. This makes the definition of the correspondence point precise. For the $p + 1$ dimensional SYM, if $p \neq 3$, we predict the phase transition temperature $T_c \sim \lambda^{1/(3-p)}$. Our study of phase transition is rather qualitative. To understand dynamic details, one need to study both the weak coupling side and the strong coupling side to demanding precision. There is no tool available to enable us to do so.
We shall point out the connection between the strong/weak phase transition and the correspondence principle in the next section. In the subsequent section, we argue that the large N first order Hawking-Page phase transition disappears in the weak-coupling regime \((D = 4)\), thus there is a phase transition of phase transition, which is just the strong/weak phase transition. The correction to the Hawking-Page transition temperature due to the first \(\alpha'\) correction is found in sect.4, where we determine the critical coupling \(\lambda_c\) using this result. In the final section we design some toy models to explain large N phase transitions. We show it is quite hard to come up with a good model to demonstrate a first order phase transition. We point out possible missing physics in these models.

2. Strong/weak phase transition and correspondence principle

It was argued in [4] that the free energy of \(\mathcal{N} = 4, D = 4\) SYM at a finite temperature is not a smooth function of the coupling constant \(\lambda = g^2_{YM} N\). To determine the nature of this phase transition, such as its order, is beyond the power of the currently available tools.

Denote the critical value of \(\lambda\) by \(\lambda_c\). For a \(\lambda < \lambda_c\), one can trust the SYM perturbative expansion. In this regime, one does not expect the origin \(\lambda = 0\) be an essential singularity, given the fact that the underlying zero-temperature theory is finite, and the instanton effects are suppressed in the large N limit [7]. On the other hand, for a \(\lambda > \lambda_c\), one need to use the whole string theory on manifold \(AdS_5 \times S^5\). This reminds us the correspondence principle of Horowitz and Polchinski [6]. They postulate that there exists a correspondence point at which the maximal curvature invariant at the horizon becomes of the string scale. For a smaller curvature, the system is in its black hole phase, and the semi-classical geometry is reliable in string theory. One can study the closed string theory in this background. For a larger curvature, the geometry loses its meaning in string theory, now the D-brane perturbation theory takes over. Our specificatation of a phase transition defines precisely what the correspondence point is.

Here we must note that the following argument is not meant to prove the existence of a phase transition, rather, we intend to argue that if there exists a phase transition, it is natural to identify it with the correspondence point. This identification has proven to be useful in recent work [11,12].

The connection between the strong/weak phase transition and the correspondence principle looks puzzling at the first sight. For a standard Schwarzschild black hole, the
curvature invariants would be proportional to $1/r_0^2$, where $r_0$ is the horizon size. This quantity depends on the Hawking temperature. We know that in a conformally invariant theory with an infinite volume, there can be no phase transition in terms of temperature. The fact that we are applying the correspondence principle to a AdS black hole saves the day. We will show, that all the curvature invariants in such a background are independent of $r_0$, so long as $r_0 > 0$. Thus the maximal curvature is determined by the AdS size, $R = (2\lambda)^{1/4}l_s$. Now $R \sim l_s$ if $\lambda \sim O(1)$. This is consistent with the fact that $\lambda_c \sim O(1)$.

The metric of the $AdS_5$ black hole, with a boundary $R^3 \times S^1$, is [2]

$$ds^2 = \frac{r^2}{R^2}[(1 - \frac{r_0^4}{r^4})dt^2 + \sum_{i=1}^{3} dx_i^2] + \frac{R^2}{r^2}(1 - \frac{r_0^4}{r^4})^{-1}dr^2. \quad (2.1)$$

With this metric, it is not straightforward to see that all the curvature invariants are independent of $r_0$. To do so, we follow [4], to rewrite the metric in a different coordinates system. Introduce

$$r^4 = r_0^4(1 + b^2), \quad t = \frac{R^2}{2r_0}\psi, \quad (2.2)$$

the metric becomes

$$ds^2 = \frac{R^2}{4} \left( \frac{db^2}{1 + b^2} + \frac{b^2 d\psi^2}{\sqrt{1 + b^2}} + \sqrt{1 + b^2} \sum_{i=1}^{3} dx_i^2 \right), \quad (2.3)$$

where we also rescaled $x_i$. The metric in the parentheses is universal, thus its curvature invariants are universal too. We conclude that the curvature invariants of the AdS black holes are independent of $r_0$, and determined only by the AdS size $R$.

The universal metric (2.3) is applicable only when $r_0 > 0$. The AdS black hole solution is certainly different from that of AdS. The coordinates transformation (2.2) is singular in the $r_0 = 0$ limit. In other words, the metric (2.3) describes the very near-horizon region in the $r_0 \to 0$ limit.

The observation on the connection between the large N strong/weak phase transition and the correspondence point generalizes to other SYM with the maximal supersymmetry in various dimensions. For $D \neq 4$, the YM coupling constant is dimensionful, and the construction of a dimensionless constant involves both the coupling constant and the energy density. Thus the strong/weak phase transition becomes of a finite density or a finite temperature phase transition. Let $\lambda = g^2_{YM}N/p$, where $d_p$ is a numeric factor depending only on $p$. The near-horizon metric of a nonextremal Dp-brane is [3]

$$ds^2 = \alpha' \left( f dt^2 + \frac{U^{(7-p)/2}}{\sqrt{\lambda}} \sum dx_i^2 + f^{-1} dU^2 + \sqrt{\lambda} U^{(p-3)/2} d\Omega_{8-p}^2 \right), \quad (2.4)$$
where
\[ f = \frac{U^{(7-p)/2}}{\sqrt{\lambda}} \left( 1 - \frac{U_0^{7-p}}{U^{7-p}} \right). \] (2.5)

The dilaton field is given by
\[ e^\phi = (2\pi)^{2-p} g_{YM}^2 \left( \frac{\lambda}{U^{7-p}} \right)^{(3-p)/4}. \] (2.6)

For \( p < 3 \), the string coupling becomes stronger when \( U_0 \) is smaller. For \( p > 3 \), it becomes weaker when \( U_0 \) is smaller.

As in the \( p = 3 \) case, use
\[ U^{7-p} = U_0^{7-p}(1 + b^2), \quad t = \frac{2}{7-p} U_0^{-1} \sqrt{\lambda_{\text{eff}} \psi}, \] (2.7)

where \( \lambda_{\text{eff}} = \lambda U_0^{p-3} \), the metric is put into
\[ ds^2 = \frac{4\alpha' \sqrt{\lambda_{\text{eff}}}}{(7-p)^2} \left( (1 + b^2)^{\frac{3p-17}{14-2p}} db^2 + \frac{b^2 d\psi^2}{\sqrt{1+b^2}} + \frac{(7-p)^2}{4} (1 + b^2)^{\frac{p-3}{14-2p}} d\Omega_{8-p}^2 \right) \]
\[ + \frac{U_0^{(7-p)/2}}{\sqrt{\lambda}} (1 + b^2)^{(7-p)/2} \sum dx_i^2. \] (2.8)

We see that the curvature is controlled by \( \lambda_{\text{eff}} \). When this quantity becomes order 1, the maximal curvature becomes of the string scale. Since the Hawking temperature is
\[ T = (\lambda_{\text{eff}})^{-1/2} U_0 \left( \frac{7-p}{4\pi} \right), \] (2.9)
the temperature can be identified with \( U_0 \) at the correspondence point. This in turn implies that the effective coupling at the correspondence point is \( \lambda_{\text{eff}} = \lambda T^{p-3} \sim 1 \), and the phase transition occurs at
\[ T_c \sim \lambda^{\frac{1}{p-3}}. \] (2.10)

The dilation field at the horizon is
\[ e^\phi \sim \frac{1}{N} (\lambda_{\text{eff}})^{\frac{7-p}{p}}. \] (2.11)
it is of order \( 1/N \) at the phase transition point. The large N approximation becomes quite good, and on the string side one is studying free strings propagating in the black hole background.
When the above analysis applied to $p = 4$ case, we predict that the phase transition occurs at $T_c \sim \lambda^{-1}$. For higher temperatures, $\lambda_{eff}$ is larger, and we are in the supergravity phase. For lower temperatures, we need to apply the perturbation theory of SYM or the underlying M5-brane theory. To study QCD following [5], temperature $T$ must be adjusted over the induced QCD scale $T \gg \Lambda_{QCD}$. At the QCD scale, the four dimensional QCD coupling $\lambda_4 = \lambda T$ receives quantum corrections, but it is still of order 1. Thus, the situation is close to the transition point. We believe that it is located at the small coupling side. One of hints is the observation that the Regge trajectory in the strong coupling regime displays very different pattern as expected of QCD [8].

The 5 dimensional Yang-Mills coupling is given by $g_{YM}^2 = R_{11}$, where $R_{11}$ is the radius of the M-circle around which the M5-branes are wrapped to obtain D4-branes. Thus the phase transition occurs at $\beta/R_{11} \sim N$.

Another interesting case is $p = 1$. The transition temperature is given by $T_c \sim \lambda^{1/2}$. For $T \gg T_c$, the theory is strongly coupled. If we switch the role of time and space, the theory becomes the 2D SYM living on a circle of circumference $\beta$. This can be related to matrix string theory [10]. In the strong coupling regime, the theory is effectively a conformal orbifold model, and the physics is dominated by long-strings. For $1/T$ small enough, one can ignore fermions, since fermions are anti-periodic. The theory is effectively bosonic. When $T$ is lower than $T_c$, the theory is nonabelian. The phase transition separates the abelian orbifold phase of the bosonic sector and the nonabelian phase of the NS sector.

We turn to $AdS_5$ with boundary $S^3 \times S^1$ in the next two sections. For discussions on the phase diagrams with other topologies, see [11] and [12].

3. Disappearance of Hawking-Page phase transition

Hawking and Page found some time ago [13], that there are two manifolds for a given temperature on AdS space, provided the temperature is not too low. This was re-interpreted by Witten as a large N phase transition in the $\mathcal{N} = 4 \ D = 4$ SYM [5]. This is a finite volume effect, and occurs only when the SYM lives on $S^3 \times S^1$ or $S^2 \times R \times S^1$.

The AdS black hole metric with boundary $S^3 \times S^1$ is

$$ds^2 = \left(1 + \frac{\tau^2}{R^2} - \frac{r_0^2}{r^2}\right)dt^2 + \left(1 + \frac{\tau^2}{R^2} - \frac{r_0^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2.$$  \hspace{1cm} (3.1)

For a given Hawking temperature, there are two black holes, one small, one large. The smaller one can not be regarded as a thermal state in the boundary Yang-Mills theory,
since its entropy is not an extensive quantity. The horizon radii of these two black holes are
\[ r_\pm = \frac{\pi R^2 T}{2} \left( 1 \pm \frac{2}{(\pi TR)^2} \right)^{1/2}. \] (3.2)

Use the standard formula for the Bekenstein-Hawking entropy \( A/4G_5 \), the entropy reads
\[ S_\pm = \frac{\pi^2 N^2}{2} (2\pi^2 R^3) T^3 \left( \frac{1}{2} \pm \frac{1}{2} (1 - \frac{2}{(\pi TR)^2})^{1/2} \right)^3, \] (3.3)
where we used formulas \( G_5 = G_{10}/\pi^3 R^5 \), and \( G_{10} = 2^3 \pi^6 g_s^2 (\alpha')^4 \). In the large volume or high temperature limit, \( S_- \) is not an extensive quantity. Compare \( S_+ \) with the one in the infinite volume limit \([14]\), we find that \( V_3 = 2\pi^2 R^3 \), namely the radius of \( S^3 \) is also \( R \).

Note that the precise definition of the radius of \( S^3 \) depends on the choice of time, or that only \( TR \) is convention independent.

There is a finite size correction to the entropy formula. And formula (3.3) displays a branch cut at \( \pi TR = \sqrt{2} \). There is a minimal Hawking temperature. However, the Hawking-Page transition temperature is \( \pi T_c R = \frac{3}{2} \). This value is obtained by demanding the subtracted action vanishing, thus \( r_\pm^2 = R^2 \). The critical temperature is higher than the minimal temperature, thus there is no particular physics associated with the minimal temperature.

The free energy can be obtained by using the relation \( S = \beta^2 \partial_\beta F \).

One can trust the above picture only when classical supergravity is valid, thus the radius of \( S^3 \) is much larger than the string scale. In terms of SYM, this requires \( \lambda \gg 1 \). We now argue that for a small \( \lambda \), the first order phase transition disappears. Denote the phase transition temperature by \( T_c(\lambda) \), it is a function of \( \lambda \). In the large \( \lambda \) limit, \( T_c R \) tends to a constant, \( 3/(2\pi) \). If the phase transition disappears for small \( \lambda \), there must be a critical \( \lambda_c \), at which \( T_c = 0 \). This phase transition of phase transition must be the strong/weak phase transition.

Apparently, \( T_c = 0 \) at \( \lambda = 0 \). The theory is free, and the free energy can be computed by calculating determinants of free fields. The number of free fields is of order \( O(N^2) \), so the free energy scales as \( F \sim N^2 \), and we always have the high temperature phase. For a small \( \lambda \), to include effects of interaction, one employs the loop expansion in \( \lambda \), and in the large \( N \) limit, it is sufficient to count the planar diagrams. At each loop order \( \lambda^n \), the number of planar diagrams grows as \( c^n \) \([15,16]\). There is no UV divergence, since the underlying zero temperature theory is finite. There is also no IR divergence, since both
the space and the time have a finite size. Thus, the systematic loop expansion of the free energy assumes the following form

\[ F = -N^2 T^4 R^3 \sum_{n=0}^\infty \lambda^n f_n(TR), \quad (3.4) \]

where \( f_n \) depends only on the dimensionless combination \( TR \). We do not expect effects such as UV or IR renormalons, the above series ought to be convergent for a small enough \( \lambda \), and the series typically has a finite convergent radius.

By a similar argument, each perturbative series at a given genus also has a finite convergent radius. Now it is possible that a singularity is located on the positive axis of \( \lambda \), denote this by \( \lambda_c \). Beyond this point, one can no longer trust the gauge theory perturbative expansion, and one must invoke other techniques such as the closed string theory on the AdS space to compute the free energy. For \( \lambda < \lambda_c \), the free energy always scales as \( N^2 \), and there is no Hawking-Page phase transition. The critical \( \lambda_c \) is determined by the asymptotic behavior of \( f_n(TR) \). It is still possible that \( \lambda_c \) is a function of \( TR \). Consider the limit \( TR \to \infty \). In such a limit, as argued in [4], there is a finite critical \( \lambda_c \). When one lowers \( TR \), it is possible that \( \lambda_c \) is also lowered, but cannot be pushed all the way to zero. For instance, assume that the convergent radius is proportional to \( T^\alpha \) with a positive \( \alpha \), then the asymptotic form of \( f_n \sim (1/TR)^n \alpha \). Analyzing the Feynman diagrams, such a factor can come only from the propagators of the zero modes. Now on \( S^3 \), there is no zero mode at all. Put in another way, there is a mass gap in the SYM on \( S^3 \) which is proportional to the smallest scaling dimension. Still denote the minimum of \( \lambda_c(TR) \) by \( \lambda_c \). We conclude that there is no large N Hawking-Page phase transition below \( \lambda_c \).

The existence of Hawking-Page phase transition indicates that there exists a finite \( \lambda_c \), above which the perturbative SYM ceases to be valid, and the AdS picture takes over. We shall show in the next section that the Hawking-Page temperature is lowered by the first \( \alpha' \) correction.

4. \( \alpha' \) correction to the thermodynamic quantities

The correction to thermodynamics by the first \( \alpha' \) correction was done in [17], in the infinite volume limit, where it was found that the correction to the entropy is positive. There are two ways to compute this correction. The simpler way is to substitute the Riemann curvature of the metric (3.1) directly to the term \((\alpha')^3 R^4\) to compute the subtracted
action. Another is to take the corrected metric, dilaton into account. The two calculations give the same result. The full answer of the corrected metric was found in [18], again in the infinite volume limit.

We will adopt the simpler method, hoping that the corrected metric will not change the final results. The full form of the \((\alpha')^3 R^4\) is

\[
\Delta I = -\frac{1}{16\pi G_{10}} \int \frac{1}{8} \zeta(3)(\alpha')^3 W + \ldots ,
\]  

(4.1)

where the dots denote terms depending on other fields, and

\[
W = C^{hnmk} C_{pmnq} C^{rs p} C_{rs k}^q + \frac{1}{2} C^{hkmn} C_{pqmn} C^{rs p} C_{rs k}^q .
\]

(4.2)

Introducing angular variables \(\chi, \theta, \phi\) on \(S^3\), after some calculations, we obtain those nonvanishing components of the Weyl tensor

\[
C^{ab}_{\ CD} = 3X \epsilon^{ab}_{\ CD}, \quad a,b = t,r ,
\]

\[
C^{ai}_{\ bj} = -X \delta^a_b \delta^i_j, \quad i,j = \chi, \theta, \phi ,
\]

\[
C^{ij}_{\ kl} = X (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k),
\]

(4.3)

where \(X = r_0^2/4^n\). The difference between our result and that of [17] is that \(X\) is independent of \(R^2\) in our case.

It is straightforward to substitute the above result into (4.1), the result is

\[
\Delta I = -\frac{15\pi \zeta(3)}{64} \frac{(\alpha')^3 r_0 \beta}{G_5 r_{+12}^2}.
\]

(4.4)

Using the formula \(\Delta S = \beta^2 \partial_\beta \Delta F\) we obtain

\[
\Delta S = \frac{15\pi^2 \zeta(3)}{32} N^2 (2\lambda)^{-3/2} V_3 T^3 \left[ \frac{3}{2} - \frac{1}{2} (1 - \frac{2}{(\pi TR)^2})^{1/2}\right]^3 [3 - (1 - \frac{2}{(\pi TR)^2})^{-1/2}].
\]

(4.5)

In the large volume limit we recover the result of [17]. Note that the last factor approaches minus infinity when \(T\) approaches its minimal value. At the uncorrected transition temperature \(T_c\), it becomes zero.

Demanding the subtracted action be zero, we calculate the corrected Hawking-Page temperature, up to the leading order \((\alpha')^3\)

\[
\pi R T_c = \frac{3}{2} - \frac{15}{2} \zeta(3)(\alpha')^3 / R^6 = \frac{3}{2} - \frac{15}{2} \zeta(3)(2\lambda)^{-3/2} .
\]

(4.6)
As expected, the corrected temperature is lower. The correction to the entropy \((4.3)\) at this point is weighted by \(\lambda^{-3}\). The last factor will never be too negative to render the total entropy become negative.

Although the critical temperature is lowered, we can not conclude that \(T_c\) will drop to zero at a finite \(\lambda_c\). If we are optimistic and assume the higher \(\alpha'\) terms can be ignored at \(\lambda_c\), Eq.\((4.6)\) can be used to determine \(\lambda_c\):

\[
\lambda_c = \frac{1}{2}(5\zeta(3))^{2/3} + \ldots \quad (4.7)
\]

This value is greater than 1.

As in sect.2, the connection between the strong/weak phase transition point and the correspondence point is still valid. For the larger black hole, since \(r_+ \geq R/\sqrt{2}\), the maximal curvature is always controlled by \(R\). The smaller black hole is more like a Schwarzschild black hole. Its specific heat is negative, and can be described at best as a meta-stable coherent state in the SYM theory. The horizon can be arbitrarily small, and the maximal curvature at the horizon, unlike the case discussed in sect.2, can be arbitrarily large.

5. Some toy models

We design some artificial toy models in this section to explain the occurrence of the large \(N\) phase transitions. To our knowledge, the exactly solvable models such as those studied in [21] and [22] exhibit different, although similar, phase transitions to that of Hawking and Page. Those models are all effectively one-matrix models, and the phase transition is of the third order. The Hawking-Page transition is a first order one, since the free energy scales differently in \(N\) in the low and high temperature phases.

Consider a conformal field theory defined on spacetime \(S^d \times R\). Let \(H\) denote the Hamiltonian on \(S^d\). The free energy is given by

\[
F = - \ln \left( \sum e^{-\beta H} \right). \quad (5.1)
\]

The metric on \(S^d \times R\) reads

\[
ds^2 = d\tau^2 + R^2 d\Omega^2_d, \quad (5.2)
\]

where \(R\) is the radius of \(S^d\). This metric is conformal to a flat metric on \(R^{d+1}\). Use the new coordinate \(r = \exp(\tau/R)\), the metric is mapped to

\[
ds^2 = \frac{R^2}{r^2} (dr^2 + r^2 d\Omega^2_d), \quad (5.3)
\]
the new metric in the parentheses is the flat one on $R^{d+1}$. Now the time translation $\tau \rightarrow \tau + a$ gets translated into rescaling of $r$: $r \rightarrow \exp(a/R)r$. Thus the Hamiltonian on $S^d$ is identified with $D/R$, where $D$ is the dilatation operator on $R^{d+1}$ acting at the origin. For simplicity, we still use $\beta$ to denote the combination $\beta/R$. The task of computing the free energy now becomes of counting scaling dimensions:

$$F = -\ln \left( \sum e^{-\beta D} \right). \quad (5.4)$$

A large $N$ quantum field theory is very complicated, so we will construct simplest possible large $N$ models, one matrix models or multiple-matrix models. Given a Hermitian one matrix $\phi$, there are only $N$ independent single trace operators $\text{tr}\phi^n$, $n = 1, \ldots N$. A “multi-particle” state, similar to those considered in [20], is given by the product of a string of such single trace operators,

$$\text{tr}\phi^{n_1} \ldots \text{tr}\phi^{n_l}. \quad (5.5)$$

Assign a scaling dimension $\Delta(n, \lambda)$ to operator $\text{tr}\phi^n$, where we introduced a coupling constant $\lambda$. The scaling dimension of a multi-particle operator is generally complicated. To simplify the situation, we assume the scaling dimension of the general operator be given by $\sum \Delta(n_i, \lambda)$. The partition function is then

$$\prod_{n=1}^N (1 - q^{\Delta(n, \lambda)})^{-1},$$

where $q = \exp(-\beta)$. The free energy

$$F_N = \sum_{n=1}^N \ln \left( 1 - q^{\Delta(n, \lambda)} \right). \quad (5.6)$$

The first model is specified by $\Delta(n, \lambda) = n$. All operators mimic the chiral primary operators. In such a case, we expect no phase transition when the temperature is varied. This model looks the same as the 2D QCD string on a torus [19]. The free energy is calculated by expanding the logarithmic

$$F_N = -\sum_{k=1}^{\infty} \frac{q^k}{k}(1 - q^k)^{-1} + \sum_{k=1}^{\infty} \frac{q^k}{k}(1 - q^k)^{-1} q^{Nk}. \quad (5.7)$$

There is no $1/N$ corrections. The second sum is due to the instanton contribution. Each term can be interpreted as the $k$-instanton contribution.
Our second model is given by \( \Delta(n, \lambda) = a(\lambda) \ln n + b(\lambda) \) with \( b > 0 \). If \( a(\lambda) \neq 0 \), for large \( n \), the scaling dimension receives large quantum correction. These operators mimic operators corresponding to stringy states. \( b(\lambda) > 0 \) is to ensure the positivity of the first scaling dimension.

Again, expanding the logarithmic in (5.6), we have

\[
F_N = -\sum_{k=1}^{\infty} \frac{e^{-k\beta b}}{k} \sum_{n=1}^{N} n^{-k\beta a}. \tag{5.8}
\]

Apparently, the behavior of the finite sum depends crucially on the value of \( \beta a \). The dominant contribution to \( F_N \) comes from \( k = 1 \).

\[
\sum_{n=1}^{N} n^{-\beta a} = \frac{N^{1-\beta a}}{\ln N}, \quad \beta a \neq 1 \quad \beta a = 1 \tag{5.9}
\]

The model exhibits kind of phase transition. When \( \beta a < 1 \), or \( T > a \), the free energy scales as a positive power of \( N \); it tends to a constant when \( T < a \), and grows as \( \ln N \) at \( T = a \). So \( T_c = a \) is the critical temperature. Unfortunately, there is no standard large \( N \) expansion. To see the nonanalyticity in \( N \), we write

\[
F_N = -\sum_{k=1}^{\infty} \frac{e^{-k\beta b}}{k} [\zeta(k\beta a) - \zeta(k\beta a, N + 1)] = f(1) - f(N + 1). \tag{5.10}
\]

the first term is independent of \( N \), and is regular provided \( \beta a \neq 1/k \). To examine the second term, use the Hermite formula

\[
\zeta(k\beta a, N + 1) = \frac{1}{2} (N + 1)^{-k\beta a} + \frac{1}{k\beta a - 1} (N + 1)^{1-k\beta a}
+ 2 \int_{0}^{\infty} dy ((N + 1)^2 + y^2)^{-k\beta a/2} \sin(k\beta a \theta) (e^{2\pi y} - 1)^{-1}, \tag{5.11}
\]

where \( \theta = \arctan(y/(N + 1)) \). One finds

\[
f(N + 1) = f_{\text{crit}}(N + 1) + (N + 1/2) \ln[1 - (N + 1)^{-\beta a} e^{-\beta b}]
- \int_{0}^{\infty} dy (e^{2\pi y} - 1)^{-1} \ln\left\{1 - \left(\sqrt{(N + 1)^2 + y^2 e^{-ia\theta + b}}\right)^{-\beta}\right\} \tag{5.12}
\[
\left[1 - \left(\sqrt{(N + 1)^2 + y^2 e^{ia\theta + b}}\right)^{-\beta}\right]^{-1},
\]

where the first term and the second term control the large \( N \) behavior. The first term is given by

\[
f_{\text{crit}}(N + 1) = \sum_{k=1}^{\infty} \frac{\beta a}{k\beta a - 1} e^{-k\beta b} (N + 1)^{1-k\beta a}. \tag{5.13}
\]
We are convinced that it is impossible to construct a one-matrix model to exhibit both a first order phase transition and the usual large N expansion. This leads us to examine multiple-matrix models. As soon as there are more than one matrix, the number of single-particle states at a given level starts to proliferate. For instance, consider two matrices. A general single-particle state is
\[ \text{tr} (\phi_1^{m_1} \phi_2^{n_1} \ldots \phi_1^{m_l} \phi_2^{n_l}). \] (5.14)

Define the level \( n = \sum (m_i + n_i) \). At a given level, the number of the single-particle states grows at least as the partition number \( P(n) \), since there is no longer permutation symmetry among \( m_i \) and \( n_i \), only the cyclic symmetry is retained. This growth of the single-particle states is the generic property of a string theory.

For finite \( N \), one must implement constraints on multiple-particle states. These constraints are known as the Mandelstam constraints, and as the stringy exclusion principle in the modern context \[20]. It is difficult to implement these constraints efficiently. We will introduce a cut-off in the same fashion as in (5.6). The class of our multiple-matrix models is specified by
\[ Z_N = \prod_{n=1}^{N} (1 - q^{\Delta(n, \lambda)})^{-p(n)}, \] (5.15)
where we assumed that all the single-particle states at a given level \( n \) have the same scaling dimension \( \Delta(n, \lambda) \), and the multiplicity is given by \( p(n) \). Needless to say, these models are over-simplified multiple-matrix models.

We show that some physics is missing in these toy models, so that it is impossible to come up with a first order phase transition. First, we take \( \Delta(n, \lambda) = a(\lambda) \sqrt{n} + b(\lambda) \). The factor \( \sqrt{n} \) is introduced according to \[3]. As we argued before, the growth of \( p(n) \) is at least close to the partition number \( P(n) \), thus \( p(n) \sim \exp(c\sqrt{n}) \). This introduces a problem, namely there exists a limiting temperature, the Hagedorn temperature. Above the Hagedorn temperature, the free energy will increase as an exponential of \( \sqrt{N} \). This is not allowed. So we shall assume that due to Mandelstam constraints, the growth of \( p(n) \) is weaker than \( \exp(c\sqrt{n}) \).

Take the logarithmic of (5.15), the sum can be approximated by an integral, in the large N limit. The upper bound of the integral can be pushed to infinity, since we assumed that there is no limiting temperature. Thus, the free energy is given by, in the large N limit
\[ F = \int dx p(x, N) \ln \left( 1 - e^{-\beta a \sqrt{x} - \beta b} \right), \] (5.16)
where we assumed that \( p(x, N) \) in general is a function of \( N \) too. If there were a critical \( \beta_c \) at which a first order phase transition happens, the first derivative of \( F \) would diverge. Take the first derivative

\[
\frac{dF}{d\beta} = \int dx \left( a\sqrt{x} + b \right) p(x, N) e^{\beta a \sqrt{x} + \beta b - 1}.
\]  

(5.17)

We now require that the above integral diverge for a \( \beta_c \). The divergence should not come from \( x = \infty \). If it comes from \( x = x_0 \), then \( x_0 \) must be a singularity of \( p(x, N) \), since the other factor is regular everywhere. In this case, the integral will diverge for all \( \beta \), not just for a single \( \beta_c \).

We conclude that our simple-minded toy models do not capture the physics of \( \mathcal{N} = 4, D = 4 \) SYM at a finite temperature. Many things might have gone wrong in our models. It is possible that solution to the Mandelstam constraints can not be mimicked by our ansatz. It is also possible that the assumption that the scaling dimension of a multiple-particle operator is just \( \sum \Delta(n_i \lambda) \) is badly wrong. It remains a challenging problem to construct a more realistic model to demonstrate both the first order Hawking-Page phase transition, as well as the large \( N \) strong/weak coupling phase transition.

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