Some Remarks on Strichartz Estimates for Homogeneous Wave Equation *

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Abstract

We give several remarks on Strichartz estimates for homogeneous wave equation with special attention to the cases of $L^\infty_t$ estimates, radial solutions and initial data from the inhomogeneous Sobolev spaces. In particular, we give the failure of the endpoint estimate $L^4_t L^\infty_x$ for $n=2$.

1 Introduction

As is well-known, Strichartz-type estimate is of particular importance in the low regularity well-posedness theory for semilinear wave and Schrödinger equations, e.g. in [PoSi93], [LdSo95] and [CaWe90]. Recently, there are many diverse advances in extending Strichartz-type estimates for both wave equations and Schrödinger equations, such as in [Tao00], [Fc03p], [MaNaNaOz03p], [Stbz04p] etc. Since the appearance of [KeTa98], it is generally believed that the Strichartz estimates of homogeneous equation has been totally solved and the only remained problem is to extend it to the inhomogeneous equation. However, as we know, there are still some gaps for homogeneous estimate of wave equation and it seems that it causes some confusions (many authors have different statement concerning such estimate).

In this paper, we’ll concern solely on Strichartz estimate and its variants for homogeneous wave equation. For the counterpart of Schrödinger equation, one may consult [KeTa98] and [Tao00]. As usual, we denote the space of Schwartz class by $S$ and use $f = \mathcal{F}(f)$ denote the Fourier transform of $f \in S'$ and let $p(D)f = \mathcal{F}^{-1}(p(|\xi|)\hat{f}(\xi))$. Also, we use $H^s$ to denote the usual homogeneous Sobolev space $D^{-s}L^2(\mathbb{R}^n)$ for $s < n/2$(note that for $s \geq n/2$, one would interpret such spaces as the subspace of $S'$ modulo polynomials of degree less than or equals $|s - n/2|$). Moreover, we define the homogeneous Besov space $\dot{B}^s_{p,q}$ for $s < n/p$ or $s = n/p$ with $q = 1$ as follows. Let $\Delta_j f := \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\hat{f}(\xi))$ be the usual homogeneous Littlewood-Paley projection, $\|f\|_{\dot{B}^s_{p,q}} = \|2^{js}\Delta_j f\|_{L^p L^q}$. Then $\dot{B}^s_{p,q} = \{f \in S', \|f\|_{\dot{B}^s_{p,q}} < \infty, \text{ and } \sum_{j \in \mathbb{Z}} \Delta_j f = f \text{ in } S'\}$. For general $s$, one should introduce such space in $S'$ modulo finite degree polynomials. Note also that $u = \cos(tD)f + D^{-1}\sin(tD)g$ solves homogeneous wave equation $\Box u = 0$ with data $(f, g)$, so we only need to state the estimate for operator $\exp(itD)$.

At first, we give a definition.

Definition. Let $n \geq 2$ and $2 \leq q, r \leq \infty$, we say that the triple $(q, r, n)$ is admissible if

$$
\frac{1}{q} \leq \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r}).
$$

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And we say the triple is radial-admissible if $q < (n - 1)(\frac{1}{2} - \frac{1}{r})$. \hfill (1.2)

The classical Strichartz-type estimates are essentially the following single frequency estimate:

**Theorem 1** (Essential Strichartz Estimate) Let $n \geq 2$, then the following two statements are equivalent,

(I) the single frequency estimate

$$\|\exp(itD)f(x)\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2},$$ \hfill (1.3)

valid for all $f \in L^2$ with supp$(\hat{f}) \subset \{1/2 < \lambda < 2\}$;

(II) $(q, r, n)$ is admissible and $(q, r, n) \neq (2, \infty, 3)$.

The positive results are given in [GVe95] and [KeTa98]. The necessary condition $q \geq 2$ is given by time-translation invariant argument, and (1.1) follows from A.W.Knapp’s counterexample $\hat{f} = \chi_A$ with $A = \{\xi_1/2 < \xi_1 < 3/2, |\xi_j| < \epsilon, 2 \leq j \leq n\}$ by letting $\epsilon \to 0$ (note that this example is non-radial). The forbidden triple $(q, r, n) = (2, \infty, 3)$ is given in [Tao] (with previous results in [KlMa93] and [Mo98]).

By applying homogeneous Littlewood-Paley decomposition and scaling, one would get immediately the following (throughout this paper, $b = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$).

**Corollary 2** (Classical Strichartz Estimate) Let $b = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$, then for all admissible $(q, r, n)$ with $r < \infty$,

$$\|\exp(itD)f(x)\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^b},$$ \hfill (1.4)

Moreover, for all admissible $(q, r, n)$ except that $(q, r, n) = (2, \infty, 3)$,

$$\|\exp(itD)f(x)\|_{L_t^q B_x^0} \lesssim \|f\|_{\dot{H}^b},$$ \hfill (1.5)

$$\|\exp(itD)f(x)\|_{L_t^q L_x^r} \lesssim \|f\|_{B_{x,1}^0}.$$ \hfill (1.6)

Then a question arises naturally: Can (1.3) valid with $r = \infty$? Or if it fails, how can it be improved to restricted cases such as spherically symmetry or angular regularity?

In fact, we have the following results by supplement the known results.

**Theorem 3** Let $b = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$, then we have (1.3) for all admissible $(q, r, n)$ except that $(q, r) = (\text{max}(2, \frac{4}{n-1}), \infty)$ and $(q, r) = (\infty, \infty)$. On the other hand, in order for (1.3) to be valid for all $f \in S$, we need $(q, r, n)$ admissible, $(q, r) \neq (\infty, \infty)$ and $q > \frac{1}{n-1}$.

**Remark 1** As stated in Theorem 3, the only remained open problem for homogeneous estimate now is the endpoint $(2, \infty, n)$ with $n \geq 4$. Alternatively, we have (ref Proposition 2)

$$\|\exp(itD)f(x)\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^b}, \|f\|_{\dot{H}^b}^{1-\theta}.$$ \hfill (1.7)

For the valid region of $(1/q, 1/r)$ for (1.4), see Figure 7.

**Remark 2** For $r = \infty$, it seems that there are some confusions for the validness of estimate (1.4), except for $(2, \infty, 3)$ and $(\infty, \infty, n)$. Many authors have different statements for such estimate. So the partial purpose of Theorem 3 is to clarify this confusion.

**Remark 3** Note that the embeddings $\dot{H}^{n/2} \subset L^\infty$ and $\dot{H}^{n/2} \subset L^\infty$ are both fail to valid even for radial function, then (1.4) fails for $(q, r) = (\infty, \infty)$ for any $n$. 

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Remark 4 As a complement to the failure of some $r = \infty$ estimate in (1.4), we have a simple but somewhat interesting result (Proposition 4). Let $2 \leq q < \infty$, we have

$$\|\exp(itD)f(x)\|_{L^\infty_x L^q_t} \lesssim \|f\|_{H^b}.$$  

For radial function, the region of “admissible” triple can be vastly improved (the angular improvement will be given in Theorem 7 of Section 3).

Theorem 4 Let $(q, r, n)$ be radial-admissible and $(q, r) \neq (\infty, \infty)$, then (1.4) valid for all radial $f$.

For the completeness of exposition, we also state here the correspond estimate in the Sobolev space $H^s$. Here $b+$ denotes the $b + \epsilon$ with $\epsilon > 0$ arbitrary small.

Theorem 5 Let $n \geq 2$, and $u(t, x)$ be the solution to $\Box u = 0$ with data $(f, g)$. Then the estimate

$$\|u\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$  

valid with $s = b+$ if $n \geq 3$, $b \geq 1$, $(q, r, n)$ admissible and $(q, r, n) \neq (2, \infty, 3)$, on the other hand, (1.7) valid with $s = b+$ only if $b \geq 1$, $(q, r, n)$ admissible and $(q, r, n) \neq (2, \infty, 3)$. Moreover, if $n \geq 3$, $b \geq 1$, $(q, r, n)$ admissible and $(q, r) \neq (2, \infty, (\infty, \infty)$, then (1.7) valid with $s = b$. And if (1.7) valid with $s = b$, we need $n \geq 3$, $b \geq 1$, $(q, r, n)$ admissible, $(q, r, n) \neq (2, \infty, 3)$ and $(q, r) \neq (\infty, \infty)$.

Note that (1.7) valid for $s$ implies it’s validness for $s+$, also the $s+$ failure implies the $s$ failure.

From the figure 3 of Theorem 5, we see that there is a new limitation for $H^s$ estimate, due to the fact that $b-1$ may less than 0. However, if one substitute $du$ for $u$ in (1.7) with $du = (\partial_t u, \nabla u)$, one can eliminate such limitation. In fact,

$$\|\exp(itD)f\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s} \Rightarrow \|du\|_{L^q_t L^r_x} \lesssim \|f\|_{H^{s-1}} + \|g\|_{H^s} \Rightarrow \|\cos(tD)f\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s}.$$
Hence

**Theorem 6** Let \( n \geq 2 \), and \( u(t, x) \) be the solution to \( \Box u = 0 \) with data \((f, g)\). Then we have

\[
\|du\|_{L^q_t L^r_x} \lesssim \|f\|_{H^{s+1}} + \|g\|_{H^s}
\]

with \( s = b+ \) if and only if \((q, r, n)\) admissible and \((q, r, n) \neq (2, \infty, 3)\). Moreover, for admissible \((q, r, n)\) except that \((q, r) = (\max(2, \frac{4}{n-1}), \infty)\) and \((q, r) = (\infty, \infty)\), we have valid with \( s = b \). On the other hand, if valid with \( s = b \), then \((q, r, n)\) is admissible and \((q, r, n) \neq (2, \infty, 3), (\infty, \infty, n)\).

2 General Case with \( r = \infty \)

In this section, we concern on Theorem 6. Firstly, we give some remarks on a particular triple. The forbidden triple \((q, r, n) = (2, \infty, 3)\) achieves special attention in the study of Strichartz estimate.

- This forbidden triple for \((1.4)\) is found in [KlMa93], they also find that the triple is "admissible" if the data is radial symmetric.
- In [Mo98], the author gives this triple's inadmissible with the \( L^\infty \) norm substitute by \( BMO \) in \((1.4)\). The correspond assertion in Theorem 1 may be found in [Tao], which implies the result in [Mo98]. (Since the Littlewood-Paley projection of \( BMO \) is in \( L^\infty \)).
- However recently, in [MaNaNaOz03P], the authors show that if one substitutes the \( L^2 L^\infty \) norm by \( L^2_t L^\infty_x \) for any \( p < \infty \), then \((1.4)\) will be valid with the operator
bound like $\sqrt{p}$ as $p \to \infty$. (Instead, as noted there, one may get the $L^2L^\infty$ bound for the data with some angular regularity by Sobolev embedding $H_\theta^{s,p} \subset L^\infty$, $p > 2$. This fact will be used in Theorem 7 below.)

As stated in Theorem 3 there are many $q$ such that (2.1) valid with $r = \infty$. In fact, Klainerman and Machedon has stated this and explicitly prove the $n = 3$ case in [KlMa98]. We restate it here.

**Proposition 1** If $\max(\frac{4}{n-1}, 2) < q < \infty$ and $r = \infty$, then (1.3) valid.

The proof of this result in [KlMa98] is subtle. However it follows easily from classical Strichartz estimate in Corollary 2 and the following generalized Gagliardo-Nirenberg inequality (Lemma 1.4 in [Ev97]).

**Lemma.** Let $a, c \in (1, \infty)$, $\alpha, \beta \in (0, n)$ and $\alpha a < n < \beta c$, then

$$\|f\|_{L^\infty} \lesssim \|D^a f\|_{L^q}^\theta \|D^b f\|_{L^\infty}^{1-\theta},$$

where $\theta = \left(\frac{a}{n} - \frac{1}{c}\right)/\left(\frac{a}{n} - \frac{1}{c} + \frac{\alpha}{\beta} - \frac{\alpha}{\beta} \frac{n}{n-1}\right)$.

Hence for $\max(\frac{4}{n-1}, 2) < q < \infty$, let $a = 2, \alpha = b < n/2$, one may choose $2 \leq c \leq \infty$ and $\beta$ such that $((1-\theta)q, c, n)$ admissible. In fact, one only needs to choose $\beta = \min\left(\frac{a}{c} + \frac{1}{2} - \frac{1}{q}, \frac{n}{2} + \frac{n}{n-1} - \frac{1}{q}\right) > \frac{2}{q}$. Then we can apply Lemma to yield

$$\|\exp(itD)f\|_{L^q_t L^\infty_x} \lesssim (\|D^b \exp(itD)f\|_{L^q_t} \|D^\beta \exp(itD)f\|_{L^\infty_x}^{1-\theta})_{L^q_t} \lesssim \|f\|_{H^{\beta - \epsilon}} \|D^\beta \exp(itD)f\|_{L^{\infty, (1-\theta)}_x}^{1-\theta} \lesssim \|f\|_{H^{\beta - \epsilon}}.$$

So is Proposition 1. Similar argument yields the following

**Proposition 2** Let $n \geq 4$ and $b = \frac{n-1}{2}$, then

$$\|\exp(itD)f(x)\|_{L^2_t L^\infty_x} \lesssim \|f\|_{H^{\beta - \epsilon}} \|f\|_{H^{\beta + \delta}}^{1-\theta}$$

with $\epsilon \in (0, \frac{n(n-3)}{2(n-1)}), \\delta \in (0, n)$ and $\delta = (\delta + \epsilon)\theta$.

Now let’s restate the stated necessary condition here.

**Proposition 3** If (1.4) valid, then $q > \frac{4}{n-1}$.

**Proof** Note that Knapp’s counterexamples and time-translation invariant argument give a necessary condition of $\frac{1}{q} \leq \min\left(\frac{1}{2}, \frac{2}{n-1} \left(\frac{1}{2} - \frac{1}{q}\right)\right)$, then the only remained triple is $(\frac{4}{n-1}, \infty, n)$ with $n = 2, 3$. We use the contradiction argument. Assume that (1.4) valid for such triple, then for any $g \in S(\mathbb{R})$ and $f \in S(\mathbb{R}^n)$, we have

$$| \int g(t)\delta(x - te_1), D^{-b}e^{itD}f(x)\delta dt | \leq \|D^{-b}e^{itD}f\|_{L^q_t} \|g\|_{L^\infty_t} \lesssim \|f\|_{L^q} \|g\|_{L^q_t}$$

with $e_1 = (1, 0, \cdots, 0)$. Apply Plancherel theorem to the left hand side with respect to $x$ yield that

$$\langle \hat{f}(-\xi), \int |\xi|^{-b}e^{-it\langle\xi + \xi_1\rangle}g(t)\delta \xi \rangle \lesssim \|f\|_{L^q} \|g\|_{L^q_t} \|g\|_{L^q_t}.$$

From this, one has

$$\|\|\xi|^{-b}\hat{g}(\xi + \xi_1)\| \lesssim \|g\|_{L^q_t}. \quad (2.1)$$
Make coordinate transformation (denote \( \xi = (\xi_1, \xi') \) and \( \lambda = (\lambda_1, \lambda') \)) \( \lambda_1 = |\xi| + \xi_1 \geq 0 \), \( \lambda' = \xi' \), we have \( \xi_1 = \frac{\lambda^2 - |\lambda'|^2}{2\lambda_1} \), \( |\xi| = \frac{|\lambda|^2}{2\lambda_1} \), and \( d\xi = \frac{|\lambda|^2}{2\lambda_1^2} \) \( d\lambda \). Note that \( 2(1 - 2b) = 1 - n \) and set \( \lambda' = \lambda_1 y \), then

\[
\left\| |\xi|^{-b}\hat{g}(|\xi| + \xi_1) \right\|^2_{L^2_{\xi}} = \int_{R^n_+} \left( \frac{|\lambda|^2}{2\lambda_1} \right)^{-2b} |\hat{g}(\lambda_1)|^2 \left( \frac{|\lambda|^2}{2\lambda_1^2} \right)^{2} d\lambda
\]

\[
= C \int_0^\infty \lambda'^{2b-2} |\hat{g}(\lambda_1)|^2 \int_{R^{n-1}} (\lambda^2 + |\lambda'|^2)^{1-2b} d\lambda' d\lambda_1
\]

\[
= C \int_0^\infty \lambda'^{2b-2} |\hat{g}(\lambda_1)|^2 d\lambda_1 \int_{R^{n-1}} (1 + |y|^2)^{\frac{1-n}{2}} dy = \infty
\]

provided that \( g \neq 0 \) in \( S \), which contradicts to (2.1). □

As a complement to the failure of some \( r = \infty \) estimate in (1.4), we give here a simple but somewhat interesting result.

**Proposition 4** Let \( 2 \leq q < \infty \) and \( b = \frac{4}{q} - \frac{1}{q} \), we have for any \( n \),

\[
\| \exp(itD) f(x) \|_{L^q_x} \lesssim \| f \|_{H^s_x}
\]

**Proof** Let \( \alpha = 1/2 - 1/q \), \( M \) denote the space of finite measure and \( S \) denote the usual spherical measure. Note that if set \( x = r\hat{\omega}, x = \lambda\omega \),

\[
\| F^{-1}_\xi(f)(x) \|_{L^q_x} \simeq \| \int e^{i\lambda \omega \cdot x} f(\lambda \omega) \lambda^{n-1} d\lambda dS(\omega) \|_{L^q_x}
\]

\[
\lesssim \| \int \| F^{-1}_r(f(\lambda \omega) \lambda^{n-1}) (\omega \cdot x) \|_{L^q_x} dS(\omega)
\]

\[
\lesssim \| \int f(\lambda \omega) \lambda^{n-1} \|_{M_x} dS(\omega)
\]

\[
\lesssim \| f(\lambda \omega) \lambda^{n-1} \|_{L^q_x(\omega)} ,
\]

then

\[
\| \exp(itD) f(x) \|_{L^q_x} \lesssim \| \exp(itD) f(x) \|_{L^q_x L^q_x}
\]

\[
\simeq \| F^{-1}_\xi \left( \delta(\tau - |\xi|) |\tau|^{\alpha} \hat{f}(\xi) \right) \|_{L^q_x L^q_x}
\]

\[
\lesssim \| F^{-1}_\xi \left( \delta(\tau - |\xi|) |\tau|^{\alpha} \hat{f}(\xi) \right) \|_{L^q_x L^q_x}
\]

\[
\lesssim \| \delta(\tau - \lambda) |\tau|^{\alpha} \hat{f}(\lambda \omega) \lambda^{n-1} \|_{L^q_x L^q_x(\omega)}
\]

\[
= \| \| \tau |^{\alpha} \hat{f}(\tau \omega) \|_{L^q_x L^q_x(\omega)}
\]

\[
= \| \xi |^{\alpha} \hat{f}(\xi) \|_{L^q_x} \simeq \| f \|_{H^s_x}
\]

**3 Radial Improvement**

Now we turn to the radial or angular improvements of (1.4). Recently, Sterbenz [Stbz04p] gets some improvement for \( n \geq 4 \) in condition that the data with some addition angular regularity. As remarked there, the \( n = 3 \) counterpart follows directly as a combination of result in [MaNaNaOz03p] with the Proposition 3.4 in [Stbz04p]. We summarize the complete results here.
Theorem 7 Let $n \geq 3$ be the number of spatial dimensions, $\sigma_\Omega = n - 1$, $\sigma = \frac{n - 1}{2}$. Then for every $\epsilon > 0$, there is a $C_\epsilon$ depending only on $\epsilon$ such that the following set of estimates hold for any $f \in \mathcal{S}$:

$$
\|e^{itD}f(x)\|_{L^q_tL^p_x} \lesssim C_\epsilon (\|\Omega\|^s f(x)\|_{\dot{H}^3}) ,
$$

(3.1)

where we have that $r < \infty$, $s = (1 + \epsilon) \left( \frac{n - 1}{q} + \frac{n - 3}{2} \right)$, $\frac{1}{q} + \frac{3}{r} = \frac{2}{q} - b$, and $\frac{1}{q} + \frac{n}{r} < \frac{n - 1}{2}$. All of the implicit constants in the above inequality depend on $n$, $q$, and $r$. Here $\Omega_{ij} := x_i \partial_j - x_j \partial_i$, $\Delta_{sph} := \sum_{i<j} \Omega_{ij}$, $|\Omega|^s = (-\Delta_{sph})^{\frac{s}{2}}$, and

$$
\|\Omega\|^s f(x)\|_{\dot{H}^3} = \|f\|_{\dot{H}^3} + \|\Omega\|^s f(x)\|_{\dot{H}^3} .
$$

Note that this result is only proved for the case $n \geq 3$, and it seems that the argument in [Stbz04p] can only work for $n \geq 3$. It would be interesting to extend this result to $n = 2$ (compare with Theorem 4).

Now we give the proof of Theorem 4.

**Proof of Theorem 4** In view of Theorem 7 and the usual interpolation, the theorem is reduced to the proof of $r = \infty$ case. Let $|x| = r$, $|\xi| = \lambda$, $x \cdot \xi = r \lambda \cos \theta = r \lambda y$ and $\hat{g}(\lambda) := \lambda^{n-1} \hat{f}(\lambda) H(\lambda) \in L^3$ with $H$ the usual Heaviside function. Then

$$
e^{itD}f = \int e^{i(x-\xi+t|\xi|^2)} \hat{f}(\xi) d\xi
\approx \int_0^\infty \int_0^{\infty} e^{i \lambda (t + r \cos \theta)} \hat{f}(\lambda) \lambda^{n-1} (\sin \theta)^{n-2} d\lambda d\theta
\approx \int_0^\lambda \int_0^{r \cos \theta} \hat{f}(\lambda) (\sin \theta)^{n-2} d\theta
= \int_0^1 g(ry + t)(1 - y^2)^{\frac{n-2}{2}} dy := I(r,t) ,
$$

Now if $n \geq 3$, $(1 - y^2)^{\frac{n-2}{2}} \leq 1$, $|I(r,t)| \leq \int_0^1 |g(ry + t)|dy = \frac{1}{t} \int_{t-r}^{t+r} |g(z)|dz$, then for any $2 \leq q < \infty$,

$$
\|e^{itD}f\|_{L^q_tL^p_x} \lesssim \|I\|_{L^q_tL^p_x} \lesssim \|Mg(t)\|_{L^q_x} \lesssim \|g\|_{L^p} \lesssim \|g\|_{\dot{H}^{\frac{n}{2}}-1/q} \simeq \|f\|_{\dot{H}^{\frac{n}{2}}-1/q} ,
$$

here $M$ is the usual maximal operator. If $n = 2$, note that $(1 - y^2)^{-\frac{1}{2}} \in L^q_2$ with $J = [-1,1]$ and $p \in (2, \infty)$, $|I(r,t)| \lesssim \|g(t + ry)\|_{L^p_\nu}$, then for any $q \in (2, \infty)$, let $p = 1 + q/2$,

$$
\|e^{itD}f\|_{L^q_tL^p_x} \lesssim \|g(t + ry)\|_{L^q_\nu L^p_\nu} \lesssim \|M(|g|)(t)\|_{L^q_\nu L^p_\nu} \lesssim \|g\|_{\dot{H}^{\frac{n}{2}}-1/q} \simeq \|f\|_{\dot{H}^{\frac{n}{2}}-1/q} .
$$

4 Inhomogeneous Space $H^s$

In this section, based on Theorem 3, we use scaling argument and Sobolev embedding to derive estimate in $H^s = (1 - \Delta)^{-s/2} L^2$ instead of $\dot{H}^s$, i.e., we give the proof of Theorem 4 and Theorem 5. Note that $u = \cos(tD)f + D^{-1} \sin(tD)g = v + w$ solve the homogeneous wave equation $\Box u = 0$ with data $(f,g)$, we’ll deal with the two parts separately below. In this section, we use $b^+$ to denote $b + \epsilon$ with $\epsilon > 0$ arbitrary small.
Note that for any \((q, r, n)\) admissible, we have \(b \geq 0\). Then for all admissible \((q, r, n)\) except that \((q, r) = (\max(2, \frac{4}{n-1}), \infty)\) and \((q, r) = (\infty, \infty)\), we have the \(H^b\) estimate for 
\[
\|v\|_{L^1_t L^q_x} \lesssim \|(1 + D)^\frac{p}{p} v\|_{L^1_t L^q_x} \lesssim \|f\|_{H^{b+}} .
\]
The argument in Proposition 3 yields that one need \(s > \frac{4 \pm 1}{n-1}\) for the \(H^s\) estimate of \(\hat{v}\) with 
\[r = \infty.\]
Note that

\(H^s\) estimate valid for some \(s \Rightarrow\) single frequency estimate

then no \(H^s\) estimate valid for \((2, \infty, 3)\). By duality argument and decay estimate, we get the \((4, \infty, 2)\) estimate for \(f\) with \(\hat{f}\) support in unit ball, then we get \((4, \infty, 2)\) estimate with \(s = b+\). In conclusion, we have proved and the following (both see Figure 4)

**Proposition 5** Let \(n \geq 2\). We have

\[
\|\cos(tD) f\|_{L^1_t L^q_x} \lesssim \|f\|_{H^{b+}}
\]  
(4.1)

with \(s = b+\) iff \((q, r, n)\) admissible and \((q, r, n) \neq (2, \infty, 3)\). Moreover, for admissible \((q, r, n)\) except that \((q, r) = (\max(2, \frac{4}{n-1}), \infty)\) and \((q, r) = (\infty, \infty)\), we have \((4.1)\) valid with \(s = b\). On the other hand, if \((1.3)\) valid with \(s = b\), then \((q, r, n)\) is admissible and \((q, r, n) \neq (2, \infty, 3), (\infty, \infty, n)\).

**Proposition 6** The same result as in Proposition 5 valid for the estimate

\[
\|\exp(itD) f\|_{L^1_t L^q_x} \lesssim \|f\|_{H^{b+}}
\]  
(4.2)

Moreover, \((1.2)\) fail to be valid if \((q, r) = (\frac{4}{n-1}, \infty)\) and \(s = b\).

And hence we have the Theorem 3.

For \(w = D^{-1} \sin(tD)g\), the situation is different. A scaling argument yields

**Proposition 7** If \(b < 1\), then for any \(s \in \mathbb{R}\) the estimate

\[
\|w\|_{L^1_t L^q_x} \lesssim \|g\|_{H^{b-1}}
\]  
(4.3)

fails. If \(b = 1\), then \((4.3)\) valid for \(s = b+\) if and only if \((1.3)\) with \(s = b\) valid.

**Proof** For the \(b < 1\) part, we need only to show the failure for large \(s\). For such \(s\), let 
\[w_{\lambda}(t, x) = \lambda^{\frac{1}{4} + \frac{q}{p}} w(\lambda t, \lambda x),\]
and hence 
\[g_{\lambda}(x) = \lambda^{\frac{1}{4} + \frac{q}{p} - 1} g(\lambda x).\]
Then if \((4.3)\) valids, we have for \(s \geq 1\),

\[
\|w\|_{L^1_t L^q_x} = \|w_{\lambda}\|_{L^1_t L^q_x} \lesssim \|g_{\lambda}\|_{H^{b-1}} + \|g_{\lambda}\|_{L^2} = \lambda^{\frac{1}{4} + \frac{q}{p} - 1} \|g\|_{H^{b-1}} + \lambda^{1 - \frac{q}{p}} \|g\|_{L^2} \rightarrow 0
\]

by letting \(\lambda \rightarrow 0\), so is contradicted.

For \(b = 1\), if \((1.3)\) valid for \(s = b+\), then

\[
\|w\|_{L^1_t L^q_x} = \|w_{\lambda}\|_{L^1_t L^q_x} \lesssim \|g_{\lambda}\|_{H^{b}} \lesssim \|g_{\lambda}\|_{H^{b}} + \|g_{\lambda}\|_{L^2} = \lambda^{0} \|g\|_{H^{b}} + \|g\|_{L^2} \rightarrow \|g\|_{L^2}
\]

with \(\lambda \rightarrow 0\).

If \(b > 1\), then we have \(H^{b-1}\) estimate once we have \(\dot{H}^{b-1}\) estimate. On the other hand, for \((q, \infty, n)\) with \(q > \frac{4}{n-1}\) and \(q \geq 2\) such that \(b > 1\), there exists \(p\) such that \((1.3)\) valid for \((q, p, n)\) with \(\lambda_1 = \frac{3}{2} - \frac{b}{q} - \frac{1}{q} - 1 \geq 0\),

\[
\|w\|_{L^1_t L^q_x} \lesssim \|(1 + D)^{\frac{1}{p} + \frac{q}{p}} w\|_{L^1_t L^q_x} \lesssim \|(1 + D)^{\frac{1}{p} + D^{\lambda_1}} g\|_{L^2} \lesssim \|g\|_{H^{-1+b+}} .
\]

Combining Proposition 4 with the previous observation, we get (for its figure, see Figure 5).
Proposition 8 The estimate (4.3) valid with \( s = b^+ \) if \((q, r, n)\) admissible, \(b \geq 1, n \geq 3\) and \((q, r, n) \neq (2, \infty, 3)\). On the other hand, \(b < 1\) or \((q, r, n) = (2, \infty, 3)\) imply that (4.3) fail with \(s = b^+\). If \((q, r, n)\) admissible, \(b \geq 1, n \geq 3\) and \((q, r) \neq (2, \infty), (\infty, \infty)\), we have (4.3) valid with \(s = b\).

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