RELATIVE DEHN FUNCTIONS, HYPERBOLICALLY EMBEDDED SUBGROUPS AND COMBINATION THEOREMS

HADI BIGDELY AND EDUARDO MARTÍNEZ-PEDROZA

Abstract. Consider the following classes of pairs consisting of a group and a finite collection of subgroups:
• \( C = \{ (G, \mathcal{H}) \mid \mathcal{H} \text{ is hyperbolically embedded in } G \} \)
• \( D = \{ (G, \mathcal{H}) \mid \text{the relative Dehn function of } (G, \mathcal{H}) \text{ is well-defined} \} \).
Let \( G \) be a group that splits as a finite graph of groups such that each
vertex group \( G_v \) is assigned a finite collection of subgroups \( \mathcal{H}_v \), and
each edge group \( G_e \) is conjugate to a subgroup of some \( H \in \mathcal{H}_v \) if \( e \) is
adjacent to \( v \). Then there is a finite collection of subgroups \( \mathcal{H} \) of \( G \) such
that
(1) If each \( (G_v, \mathcal{H}_v) \) is in \( C \), then \( (G, \mathcal{H}) \) is in \( C \).
(2) If each \( (G_v, \mathcal{H}_v) \) is in \( D \), then \( (G, \mathcal{H}) \) is in \( D \).
(3) For any vertex \( v \) and for any \( g \in G_v \), the element \( g \) is conjugate
   to an element in some \( Q \in \mathcal{H}_v \) if and only if \( g \) is conjugate to an
   element in some \( H \in \mathcal{H} \).
That edge groups are not assumed to be finitely generated and that they
do not necessarily belong to a peripheral collection of subgroups of an
adjacent vertex are the main differences between this work and previous
results in the literature.

1. Introduction

Consider the following classes of pairs consisting of a group and a finite
collection of subgroups:
• \( C = \{ (G, \mathcal{H}) \mid \mathcal{H} \text{ is hyperbolically embedded in } G \} \)
• \( D = \{ (G, \mathcal{H}) \mid \text{the relative Dehn function of } (G, \mathcal{H}) \text{ is well-defined} \} \).

Theorem 1.1. Let \( G \) be a group that splits as a finite graph of groups such
that each vertex group \( G_v \) is assigned a finite collection of subgroups \( \mathcal{H}_v \), and
each edge group \( G_e \) is conjugate to a subgroup of some \( H \in \mathcal{H}_v \) if \( e \) is
adjacent to \( v \). Then there is a finite collection of subgroups \( \mathcal{H} \) of \( G \) such
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(1) If each \( (G_v, \mathcal{H}_v) \) is in \( C \), then \( (G, \mathcal{H}) \) is in \( C \).
(2) If each \( (G_v, \mathcal{H}_v) \) is in \( D \), then \( (G, \mathcal{H}) \) is in \( D \).
(3) For any vertex \( v \) and for any \( g \in G_v \), the element \( g \) is conjugate in
   \( G_v \) to an element of some \( Q \in \mathcal{H}_v \) if and only if \( g \) is conjugate in \( G \)
to an element of some \( H \in \mathcal{H} \).

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The theorem is trivial without the third item in the conclusion; indeed, the pair \((G, \{G\})\) belongs to both \(C\) and \(D\). In comparison with previous results in the literature, our main contribution is that our combination results do not assume that edge groups are finitely generated or contained in \(H\).

The notion of a hyperbolically embedded collection of subgroups was introduced by Dahmani, Guirardel and Osin \cite{DGO17}. A pair \((G, H)\) in \(C\) is called a \textit{hyperbolically embedded pair} and we write \(H \hookrightarrow_h G\). Our combination results for hyperbolically embedded pairs \((G, H)\) generalize analogous results for relatively hyperbolic pairs in \cite{Dah03, Ali05, Osi06a, MR08, BW13} and for hyperbolically embedded pairs \cite{DGO17, MO15}.

The notions of finite relative presentation and relative Dehn function \(\Delta_{G,H}\) of a group \(G\) with respect to a collection of subgroups \(H\) were introduced by Osin \cite{Osi06b} generalizing the notions of finite presentation and Dehn function of a group. A pair \((G, H)\) is called \textit{finitely presented} if \(G\) is finitely presented relative to \(H\), and \(\Delta_{G,H}\) is called the \textit{Dehn function of the pair} \((G, H)\). While a finitely presented group has a well-defined Dehn function, in contrast, the Dehn function of a finitely presented pair \((G, H)\) is not always well-defined, for a characterization see \cite[Thm. E(2)]{HMPS21}. Our result generalizes combination results for pairs \((G, H)\) with well-defined Dehn function by Osin \cite[Thms. 1.2 and 1.3]{Osi06a}.

We prove Theorem \ref{thm:main} for the case of graphs of groups with a single edge, since then the general case follows directly by induction on the number of edges of the graph. This particular case splits into three subcases corresponding to the three results stated below. The proofs of these subcases use characterizations of pairs \((G, H)\) being hyperbolically embedded \cite[Thm. 5.9]{MPR22}, and having a well-defined Dehn function \cite[Thm. 4.7]{HMPS21} in terms of existence of \(G\)-graphs with certain properties that relate to Bowditch’s fineness \cite{Bow12}. These characterizations are discussed in Section 2. The proof of Theorem \ref{thm:main} for the case of a graph of groups with a single edge entails the construction of graphs satisfying the conditions of those characterizations for the fundamental group of the graph of groups.

We use the existing graphs for the vertex groups as building blocks. Our main result reduces to the following statements.

\textbf{Theorem 1.2 (Amalgamated Product).} For \(i \in \{1, 2\}\), let \((G_i, H_i \cup \{K_i\})\) be a pair and \(\partial_i \colon C \to K_i\) a group monomorphism. Let \(G_1 \ast_C G_2\) denote the amalgamated product determined by \(G_1 \xleftarrow{\partial_1} C \xrightarrow{\partial_2} G_2\), and let \(H = H_1 \cup H_2\). Then:

\begin{enumerate}
  \item If \(H_i \cup \{K_i\} \hookrightarrow_h G_i\) for each \(i\), then \(H \cup \{(K_1, K_2)\} \hookrightarrow_h G_1 \ast_C G_2\).
  \item If \((G_1, H_i \cup \{K_i\}) \in D\) for each \(i\), then \((G_1 \ast_C G_2, H \cup \{(K_1, K_2)\}) \in D\).
  \item For any \(g \in G_i\), the element \(g\) is conjugate in \(G_i\) to an element of some \(Q \in H_i \cup \{K_i\}\) if and only if \(g\) is conjugate in \(G\) to an element of some \(H \in H \cup \{(K_1, K_2)\}\).
\end{enumerate}
In the following statements, for a subgroup $K$ of a group $G$ and an element $g \in G$, the conjugate subgroup $gKg^{-1}$ is denoted by $K^g$.

**Theorem 1.3** (HNN-extension I). Let $(G, \mathcal{H} \cup \{K\})$ be a pair with $K \neq L$, $C$ a subgroup of $K$, and $\varphi: C \to L$ a group monomorphism. Let $G*\varphi$ denote the HNN-extension $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$. Then:

1. If $\mathcal{H} \cup \{K, L\} \hookrightarrow_h G$ then $\mathcal{H} \cup \{(K^t, L)\} \hookrightarrow_h G*\varphi$.
2. If $(G, \mathcal{H} \cup \{K, L\}) \in \mathcal{D}$, then $(G*\varphi, \mathcal{H} \cup \{(K^t, L)\}) \in \mathcal{D}$.
3. For any $g \in G$, the element $g$ is conjugate in $G$ to an element of some $Q \in \mathcal{H} \cup \{K, L\}$ if and only if $g$ is conjugate in $G*\varphi$ to an element of some $H \in \mathcal{H} \cup \{(K^t, L)\}$.

Note that the third items of Theorems 1.2 and 1.3 follow directly from standard arguments in combinatorial group theory. This article focuses on proving the other statements.

**Corollary 1.4** (HNN-extension II). Let $(G, \mathcal{H} \cup \{K\})$ be a pair, $C$ a subgroup of $K$, $s \in G$, and $\varphi: C \to K^s$ a group monomorphism. Let $G*\varphi$ denote the HNN-extension $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$. Then:

1. If $\mathcal{H} \cup \{K\} \hookrightarrow_h G$ then $\mathcal{H} \cup \{(K, s^{-1}t)\} \hookrightarrow_h G*\varphi$.
2. If $(G, \mathcal{H} \cup \{K\}) \in \mathcal{D}$, then $(G*\varphi, \mathcal{H} \cup \{(K, s^{-1}t)\}) \in \mathcal{D}$.
3. For any $g \in G$, the element $g$ is conjugate in $G$ to an element of some $Q \in \mathcal{H} \cup \{K\}$ if and only if $g$ is conjugate in $G*\varphi$ to an element of some $H \in \mathcal{H} \cup \{(K, s^{-1}t)\}$.

**Proof.** First we prove the statement in the case that $s$ is the identity element of $G$. Let $L$ be the HNN-extension $L = K*\varphi$. Observe that there is a natural isomorphism between $G*\varphi$ and the amalgamated product $G*_{K,L} L$. In this case, the conclusion of the corollary is obtained directly by invoking Theorem 1.2, since the pair $(L, \{L\})$ is in both classes $\mathcal{C}$ and $\mathcal{D}$.

Now we argue in the case that $s \in G$ is arbitrary. Let $\psi: C \to K$ the composition $I_s \circ \varphi$ where $I_s$ is the inner automorphism $I_s(x) = s^{-1}xs$. Since

$$G*\varphi = \langle G, t \mid c^{-1}t = \varphi(c)s^{-1} \text{ for all } c \in C \rangle,$$

there is a natural isomorphism $G*\varphi \to G*\psi$ which restricts to the identity on the base group $G$, and the stable letter of $G*\psi$ corresponds to $s^{-1}t$ in $G*\varphi$. Since $\psi$ maps $C \leq K$ into $K$, we have reduced the case of arbitrary $s \in G$ to the case that $s$ is the identity in $G$ and the statement of the corollary follows. \qed

Let us describe the argument proving our main result using the three previous statements. The argument relies on the following observation.

**Remark 1.5.** If a pair $(G, \mathcal{H} \cup \{L\})$ belongs to $\mathcal{C}$ (respectively $\mathcal{D}$) and $g \in G$ then $(G, \mathcal{H} \cup \{L^g\})$ belongs to $\mathcal{C}$ (respectively $\mathcal{D}$). This statement can be seen directly from the original definitions of hyperbolically embedded collection of subgroups $\text{DG017}$, and relative Dehn function $\text{Os06b}$. It can be also
deduced directly from Theorems 2.2 and 2.9 respectively in the main body of the article.

Proof of Theorem 1.1. The case of a tree of groups satisfying the hypothesis of the theorem follows from Theorem 1.2 and Remark 1.5. Then the general case reduces to the case of a graph of groups with a single vertex, where the vertex group corresponds to the fundamental group of a maximal tree of groups. In the case of a graph of groups with a single vertex, each edge corresponds to applying either Theorem 1.3 or Corollary 1.4 together with Remark 1.5.

We conclude the introduction with a more detailed comparison of our results with previous results in the literature.

1. Dahmani, Guirardel and Osin proved Theorem 1.2(1) in the case that \( \partial_1 : C \to K_1 \) is an isomorphism and \( K_1 \) is finitely generated [DGO17, Thm 6.20]; and Theorem 1.3(1) in the case that \( C = K \) and \( K \) is finitely generated [DGO17, Thm 6.19].

2. Osin proved Theorem 1.2(2) in the case that \( \partial_1 : C \to K_1 \) is an isomorphism and \( K_1 \) is finitely generated, see [Osi06a, Thm 1.3]; and Theorem 1.3(2) in the case that \( C = K \) and \( K \) is finitely generated, see [Osi06a, Thm 1.2].

3. Under the assumptions of Theorem 1.1 if each \( (G_v, H_v) \in C \) for every vertex \( v \), and there is at least one \( v \) such that \( H_v \) is nontrivial in \( G_v \), the existence of a nontrivial collection \( \mathcal{H} \) such that \( (G, \mathcal{H}) \in C \) follows from results of Minasyan and Osin [MO15, Cor. 2.2 and 2.3] and the characterization of acylindrical hyperbolicity in terms of existence of proper infinite hyperbolically embedded subgroups by Osin [Osi16]; by a nontrivial collection we mean that it contains a proper infinite subgroup. This alternative approach does not guarantee that the collection \( \mathcal{H} \) satisfies the third condition of Theorem 1.1.

4. Theorems 1.2(1) and 1.3(1), in the case that \( G_i \) is hyperbolic relative to \( H_i \) for \( i = 1, 2 \), follow from results of Wise and the first author [BW13, Thm. A].

Organization. The rest of the article consists of four sections. In Section 2 we review characterizations of pairs \((G, \mathcal{H})\) being hyperbolically embedded and having well-defined Dehn functions in terms of actions on graphs. In Section 3 we reduce the proof of Theorems 1.2 and 1.3 to prove two technical results, Theorems 3.1 and 3.2. Their proofs are the content of Sections 4 and 5 respectively.

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2. Characterizations using Fineness

In this section, we describe a characterization of pairs \((G, \mathcal{H})\) being hyperbolically embedded, Theorem 2.2; and a characterization of the pairs having a well-defined Dehn function, Theorem 2.9. These characterizations are in terms of existence of \(G\)-graphs with certain properties that relate to Bowditch’s fineness \cite{Bow12}, a notion that is defined below. The characterizations are re-statements of previous results in the literature \cite[Thm. 5.9]{MPR22} and \cite[Thm. 4.7]{HMPS21}. This section also includes a couple of lemmas that will be of use in later sections.

All graphs \(\Gamma = (V, E)\) considered in this section are simplicial, so we consider the set of edges \(E\) to be a collection of subsets of cardinality two of the vertex set \(V\).

Let \(\Gamma\) be a simplicial graph, let \(v\) be a vertex of \(\Gamma\), and let \(T_v \Gamma\) denote the set of the vertices adjacent to \(v\). For \(x, y \in T_v \Gamma\), the angle metric \(\angle_v(x, y)\) is the combinatorial length of the shortest path in the graph \(\Gamma - \{v\}\) between \(x\) and \(y\), with \(\angle_v(x, y) = \infty\) if there is no such path. The graph \(\Gamma\) is fine at \(v\) if \((T_v \Gamma, \angle_v)\) is a locally finite metric space. A graph is fine if it is fine at every vertex.

It is an observation that a graph \(\Gamma\) is fine if and only if for every pair of vertices \(x, y\) and every positive integer \(n\), there are finitely many embedded paths between \(x\) and \(y\) of length at most \(n\); for a proof see \cite{Bow12}.

2.1. Hyperbolically embedded pairs. In \cite[Definition 2.9]{Osi16}, Osin defines the notion of a collection of subgroups \(\mathcal{H}\) being hyperbolically embedded into a group \(G\). This relation is denoted as \(\mathcal{H} \hookrightarrow_h G\) and, in this case, we say that the pair \((G, \mathcal{H})\) is a hyperbolically embedded pair. In this article we use the following characterization of hyperbolically embedded collection proved in \cite{MPR22} as our working definition.

**Definition 2.1** (Proper pair). A pair \((G, \mathcal{H})\) is proper if \(\mathcal{H}\) is a finite collection of subgroups such that no two distinct infinite subgroups are conjugate in \(G\).

**Theorem 2.2** (Criterion for hyperbolically embedded pairs). \cite[Theorem 5.9]{MPR22} A proper pair \((G, \mathcal{H})\) is a hyperbolically embedded pair if and only if there is a connected \(G\)-graph \(\Gamma\) such that

1. There are finitely many \(G\)-orbits of vertices.
2. Edge \(G\)-stabilizers are finite.
3. Vertex \(G\)-stabilizers are either finite or conjugates of subgroups in \(\mathcal{H}\).
4. Every \(H \in \mathcal{H}\) is the \(G\)-stabilizer of a vertex of \(\Gamma\).
5. \(\Gamma\) is hyperbolic.
6. \(\Gamma\) is fine at \(V_\infty(\Gamma) = \{v \in V(\Gamma) \mid v\) has infinite stabilizer\}.

**Definition 2.3.** We refer to a graph \(\Gamma\) satisfying the conditions of Theorem 2.2 as a \((G, \mathcal{H})\)-graph.
Let us observe that in [MPR22, Theorem 2.2] is proved for the case that \( H \) consists of a single infinite subgroup, and the authors observe that the argument in the case that \( H \) is a finite collection of infinite subgroups (such that no pair of distinct infinite subgroups in \( H \) are conjugate in \( G \)) follows by the same argument. Then the general case in which \( H \) is a finite collection of subgroups follows from the following statement: If \( H \) is a collection of subgroups and \( K \) a finite subgroup of a group \( G \) then:

1. \( H \to_h G \) if and only if \( H \cup \{K\} \to_h G \).
2. There is \((G,H)\)-graph if and only if there is a \((G,H \cup \{K\})\)-graph.

The first statement is a direct consequence of the definition of hyperbolically embedded collection by Osin [Osi16]. The if part of the second statement is trivial, and the only if part follows directly from [AMP22, Thm. 3.4].

2.2. Relative presentations. In [Osi06b, Chapter 2], Osin introduces the notions of relative presentation of a group with respect to a collection of subgroups, and relative Dehn functions. We briefly recall these notions below.

Let \( G \) be a group and let \( H \) be a collection of subgroups. A subset \( S \) of \( G \) is a relative generating set of \( G \) with respect to \( H \) if the natural homomorphism

\[
F(S,H) = F(S)* \bigast_{H \in H} H \to G
\]

is surjective, where \( F(S) \) denotes the free group with free generating set \( S \). A relative generating set of \( G \) with respect to \( H \) is called a generating set of the pair \((G,H)\). Let \( R \subseteq F(S,H) \) be a subset that normally generates the kernel of the above homomorphism. In this case, we have a short exact sequence of groups

\[
1 \to \langle\langle R\rangle\rangle \to F(S,H) \to G \to 1,
\]

and the triple

\[
\langle S,H \mid R \rangle
\]

is called a relative presentation of \( G \) with respect to \( H \), or just a presentation of the pair \((G,H)\). Abusing notation, we write \( G = \langle S,H \mid R \rangle \). If both \( S \) and \( R \) are finite we say that the pair \((G,H)\) is finitely presented.

**Lemma 2.4.** Let \( G \) be a group and let \( H_0 \sqcup H \) be a collection of subgroups. Let \( P \) denote the subgroup of \( G \) generated by \( S_0 \) and the subgroups in \( H_0 \). If

\[
G = \langle S_0 \cup S, H_0 \cup H \mid R_0 \cup R \rangle \quad \text{and} \quad P = \langle S_0, H_0 \mid R_0 \rangle
\]

then

\[
G = \langle S, H \cup \{P\} \mid R' \rangle,
\]

where \( R' \) is the image of \( R \) under the natural epimorphism \( \varphi: F(S_0 \cup S, H_0 \cup H) \to F(S, H \cup \{P\}) \).
Proof. Let $A = F(S, \mathcal{H})$, $B = F(S_0, \mathcal{H}_0)$, $K$ the normal subgroup of $B$ generated by $R_0$ and $N$ the normal subgroup of $A \ast B = F(S_0 \cup S, \mathcal{H}_0 \cup \mathcal{H})$ generated by $R$. Our hypotheses imply that the natural epimorphisms $A \ast B \to G$ and $B \to P$ induce short exact sequences

$$1 \to \langle \langle N, K \rangle \rangle \to A \ast B \to G \to 1, \quad \text{and} \quad 1 \to K \to B \to P \to 1.$$ 

Let us identify $P = B/K$. The natural epimorphism of the statement of the lemma

$$\varphi: A \ast B \to A \ast (B/K)$$

induces an isomorphism

$$\hat{\varphi}: \frac{A \ast B}{\langle \langle N, K \rangle \rangle} \to \frac{A \ast (B/K)}{\varphi(N)} = \frac{A \ast P}{\varphi(N)}.$$ 

By the definition of $N$, we have that $\varphi(N)$ is the normal subgroup of $A \ast P$ generated by $R' = \varphi(R)$. Therefore the natural epimorphism $A \ast P \to G$ induces a short exact sequence

$$1 \to \langle \langle R' \rangle \rangle \to A \ast P \to G \to 1$$

which concludes the proof. \qed

The following pair of lemmas allow us to conclude that certain amalgamated products and HNN-extensions preserve relative finite presentability.

**Lemma 2.5** (Amalgamated Products). For $i \in \{1, 2\}$, let $(G_i, \mathcal{H}_i \cup \{K_i\})$ be a pair, $\partial_i: C \to K_i$ a group monomorphism. Let $G_1 \ast_C G_2$ denote the amalgamated product determined by $G_1 \xrightarrow{\partial_1} C \xleftarrow{\partial_2} G_2$, and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. If

$$G_i = \langle S_i, \mathcal{H}_i \cup \{K_i\} | R_i \rangle$$ 

then

$$G_1 \ast_C G_2 = \langle S_1 \cup S_2, \mathcal{H} \cup \{(K_1, K_2)\} | R_1 \cup R_2 \rangle.$$ 

**Proof.** Observe that $\langle S_1 \cup S_2, \mathcal{H} \cup \{(K_1, K_2)\} | R_1 \cup R_2, \partial_1(c) = \partial_2(c) \text{ for all } c \in C \rangle$ is a relative presentation of $G_1 \ast_C G_2$. Since the subgroup $\langle K_1, K_2 \rangle \leq G_1 \ast_C G_2$ is isomorphic to the amalgamated product $K_1 \ast_C K_2$, we have that $\langle K_1, K_2 | \partial_1(c) = \partial_2(c) \text{ for all } c \in C \rangle$ is a relative presentation of $\langle K_1, K_2 \rangle$. The proof concludes by invoking Lemma 2.4. \qed

**Lemma 2.6** (HNN-extension). Let $(G, \mathcal{H} \cup \{K, L\})$ be a pair with $K \neq L$, $C$ a subgroup of $K$, $\varphi: C \to L$ a group monomorphism, and let $G \ast_\varphi$ denote the HNN-extension $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$. If

$$G = \langle S, \mathcal{H} \cup \{K, L\} | R \rangle$$

then

$$G \ast_\varphi = \langle S, t, \mathcal{H} \cup \{(K, L)^t\} | R' \rangle,$$

where $R'$ is the set of relations obtained by taking each element of $R$ and replacing all occurrences of elements $k \in K$ by words $t^{-1}k^tt$. In particular, $R$ and $R'$ have the same cardinality.
Proof. Let $J$ denote the subgroup $K^t$, and let $\psi: K \to J$ be the isomorphism $\psi(k) = tkt^{-1}$. Observe that $(S, t, \mathcal{H} \cup \{K, L\} \mid R, tct^{-1} = \varphi(c)$ for all $c \in C$) is a presentation for the pair $(G*_{\varphi}, \mathcal{H} \cup \{K, L\})$. Therefore

$$G*_{\varphi} = \langle S, t, \mathcal{H} \cup \{J, L\} \mid R', \psi(c) = \varphi(c) \text{ for all } c \in C \rangle.$$ 

A consequence of Britton’s lemma is that the subgroup $\langle J, L \rangle \leq G*_{\varphi}$ is isomorphic to the amalgamated product $J*_{\varphi(C)} L$. Hence,

$$\langle J, L \rangle = \langle \{J, L\} \mid \psi(c) = \varphi(c) \text{ for all } c \in C \rangle.$$ 

The proof concludes by invoking Lemma 2.4. □

2.3. Relative Dehn Functions. Suppose that $\langle S, \mathcal{H} \mid R \rangle$ is a finite relative presentation of the pair $(G, \mathcal{H})$. For a word $W$ over the alphabet $S = S \cup \bigsqcup_{H \in \mathcal{H}} (H - \{1\})$ representing the trivial element in $G$, there is an expression

$$W = \prod_{i=1}^{k} f_i^{-1} R_i f_i$$

where $R_i \in R$ and $f_i \in F(S)$. We say a function $f: \mathbb{N} \to \mathbb{N}$ is a relative isoperimetric function of the relative presentation $\langle S, \mathcal{H} \mid R \rangle$ if, for any $n \in \mathbb{N}$, and any word $W$ over the alphabet $S$ of length $\leq n$ representing the trivial element in $G$, one can write $W$ as in (3) with $k \leq f(n)$. The smallest relative isoperimetric function of a finite relative presentation $\langle S, \mathcal{H} \mid R \rangle$ is called the relative Dehn function of $G$ with respect to $\mathcal{H}$, or the Dehn function of the pair $(G, \mathcal{H})$. This function is denoted by $\Delta_{G,H}$. Theorem 2.7 below justifies the notation $\Delta_{G,H}$ for the Dehn function of a finitely presented pair $(G, \mathcal{H})$.

For functions $f, g: \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there exist constants $C, K, L \in \mathbb{N}$ such that $f(n) \leq Cg(Kn) + Ln$ for every $n$. We say $f$ and $g$ are asymptotically equivalent, denoted as $f \asymp g$, if $f \preceq g$ and $g \preceq f$.

Theorem 2.7. [Osi06b, Theorem 2.34] Let $G$ be a finitely presented group relative to the collection of subgroups $\mathcal{H}$. Let $\Delta_1$ and $\Delta_2$ be the relative Dehn functions associated to two finite relative presentations. If $\Delta_1$ takes only finite values, then $\Delta_2$ takes only finite values, and $\Delta_1 \asymp \Delta_2$.

The Dehn function of a pair $(G, \mathcal{H})$ is well-defined if it takes only finite values. This can be characterized in terms of fine graphs as follows.

Definition 2.8 (Cayley-Abels graph for pairs). A Cayley-Abels graph of the pair $(G, \mathcal{H})$ is a connected cocompact simplicial $G$-graph $\Gamma$ such that:

1. edge $G$-stabilizers are finite,
2. vertex $G$-stabilizers are either finite or conjugates of subgroups in $\mathcal{H}$,
3. every $H \in \mathcal{H}$ is the $G$-stabilizer of a vertex of $\Gamma$, and
4. any pair of vertices of $\Gamma$ with the same $G$-stabilizer $H \in \mathcal{H}$ are in the same $G$-orbit if $H$ is infinite.
Theorem 2.9. Let $(G, \mathcal{H})$ be a proper pair. The following statements are equivalent.

1. The Dehn function $\Delta_{G, \mathcal{H}}$ is well-defined.
2. $(G, \mathcal{H})$ is finitely presented and there is a fine Cayley-Abels graph of $(G, \mathcal{H})$.
3. $(G, \mathcal{H})$ is finitely presented and every Cayley-Abels graph of $(G, \mathcal{H})$ is fine.

Theorem 2.9 is essentially [HMPS21, Theorem E] together with a result on Cayley-Abels graphs from [AMP22, Theorem H]. This is described below.

Concrete examples of Cayley-Abels graphs can be exhibited using the following construction introduced by Farb [Far98], see also [Hru10].

Definition 2.10 (Coned-off Cayley graph). Let $(G, H)$ be a pair, and let $S$ be a finite relative generating set of $G$ with respect to $H$. Denote by $G/H$ the set of all cosets $gH$ with $g \in G$ and $P \in \mathcal{H}$. The coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{H}, S)$ is the graph with vertex set $G \cup G/H$ and edges of the following type

- $\{g, gs\}$ for $s \in S$ and $g \in G$,
- $\{x, gH\}$ for $g \in G$, $H \in \mathcal{H}$ and $x \in gH$.

That a pair $(G, \mathcal{H})$ has a well-defined function is characterized in terms of fineness of coned-off Cayley graphs.

Theorem 2.11. [HMPS21, Theorem E] Let $(G, \mathcal{H})$ be finitely presented pair with finite generating set $S$. The Dehn function $\Delta_{G, \mathcal{H}}$ is well-defined if and only if the coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{H}, S)$ is fine.

Every coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{H}, S)$ with $S$ a finite relative generating set is a Cayley-Abels graph. Observe now that Theorem 2.9 follows from the following result.

Theorem 2.12. [AMP22, Theorem H] If $\Gamma$ and $\Delta$ are Cayley-Abels graphs of the proper pair $(G, \mathcal{H})$, then:

1. $\Gamma$ and $\Delta$ are quasi-isometric, and
2. $\Gamma$ is fine if and only if $\Delta$ is fine.

3. Combination Theorems for Graphs

In this section, we state two technical results, Theorems 3.1 and 3.2, which will be proven in the subsequent sections. The section includes how to deduce the main results of the article, Theorems 1.2 and 1.3, from these technical results.

Theorem 3.1. For $i \in \{1, 2\}$, let $(G_i, \mathcal{H}_i \cup \{K_i\})$ be a pair and $\partial_i: C \to K_i$ a group monomorphism. Let $G = G_1 *_{C} G_2$ denote the amalgamated product determined by $G_1 \overset{\delta_1}{\leftarrow} C \overset{\delta_2}{\to} G_2$, and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. Let $\Gamma_i$ be a $G_i$-graph that has a vertex $x_i$ with $G_i$-stabilizer $K_i$. Then there is a $G$-graph $\Gamma$ with the following properties:
(1) \( \Gamma \) has a vertex \( z \) such that \( G_2 = \langle K_1, K_2 \rangle \), and there is a \( G_i \)-equivariant inclusion \( \Gamma_i \hookrightarrow \Gamma \) that maps \( x_i \) to \( z \).

(2) If \( \Gamma_i \) is connected for \( i = 1, 2 \), then \( \Gamma \) is connected.

(3) If every \( H \in \mathcal{H}_i \cup \{ K_i \} \) is the \( G_i \)-stabilizer of a vertex of \( \Gamma_i \) for \( i = 1, 2 \), then every \( H \in \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \} \) is the \( G \)-stabilizer of a vertex of \( \Gamma \).

(4) If vertex \( G_i \)-stabilizers in \( \Gamma_i \) are finite or conjugates of subgroups in \( \mathcal{H}_i \cup \{ K_i \} \) for \( i = 1, 2 \), then vertex \( G \)-stabilizers in \( \Gamma \) are finite or conjugates of subgroups in \( \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \} \).

(5) If \( \Gamma_i \) has finite edge \( G_i \)-stabilizers for \( i = 1, 2 \), then \( \Gamma \) has finite edge \( G \)-stabilizers.

(6) If \( \Gamma_i \) has finitely many \( G_i \)-orbits of vertices (edges) for \( i = 1, 2 \), then \( \Gamma \) has finitely many \( G \)-orbits of vertices (resp. edges).

(7) If \( \Gamma_i \) is fine for \( i = 1, 2 \), then \( \Gamma \) is fine.

(8) If \( \Gamma_i \) is fine at \( V_\infty (\Gamma_i) \) for \( i = 1, 2 \), then \( \Gamma \) is fine at \( V_\infty (\Gamma) \).

(9) If \( \Gamma_i \) is hyperbolic for \( i = 1, 2 \), then \( \Gamma \) is hyperbolic.

Let us explain how Theorem 1.2 follows from the above result.

Proof of Theorem 1.2. For the first statement, suppose \( \mathcal{H}_i \cup \{ K_i \} \) is hyperbolically embedded in \( G_i \). Then \( \mathcal{H}_i \cup \{ K_i \} \) is an almost malnormal collection of subgroups of \( G_i \) by [DG01] Prop. 4.33. In particular, \( (G_i, \mathcal{H}_i \cup \{ K_i \}) \) is a proper pair. By Theorem 2.2 there is a \( (G_i, \mathcal{H}_i \cup \{ K_i \}) \)-graph \( \Gamma_i \). Let \( x_i \) be a vertex of \( \Gamma_i \) with \( G_i \)-stabilizer \( K_i \). Applying Theorem 3.1 to \( \Gamma_1, \Gamma_2, x_1 \) and \( x_2 \), we obtain a \( (G_1 \ast_C G_2, \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \}) \)-graph. Note that \( (G_1 \ast_C G_2, \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \}) \) is a proper pair by an standard argument using normal forms. Then invoke Theorem 2.2 to obtain that \( \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \} \) is hyperbolically embedded in \( G_1 \ast_C G_2 \).

The second statement is proved analogously. Suppose the relative Dehn function of \( (G_i, \mathcal{H}_i \cup \{ K_i \}) \) is well-defined. By [Osi06a] Prop. 2.36, the pair \( (G_i, \mathcal{H}_i \cup \{ K_i \}) \) is proper. It follows that \( (G_1 \ast_C G_2, \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \}) \) is also a proper pair by an standard argument using normal forms. By Theorem 2.9 \( (G_i, \mathcal{H}_i \cup \{ K_i \}) \) is finitely presented and admits a fine Cayley-Abels graph \( \Gamma_i \). In particular, there is a vertex \( x_i \in \Gamma_i \) with \( G_i \)-stabilizer equal to \( K_i \). Apply Theorem 3.1 to \( \Gamma_1, \Gamma_2 \) and the vertices \( x_1, x_2 \) to obtain a fine Cayley-Abels graph \( \Gamma \) for the pair \( (G_1 \ast_C G_2, \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \}) \). Since \( (G_1 \ast_C G_2, \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \}) \) is finitely presented by Lemma 2.5 then Theorem 2.9 implies that the relative Dehn function of \( (G_1 \ast_C G_2, \mathcal{H} \cup \{ \langle K_1, K_2 \rangle \}) \) is well-defined. 

**Theorem 3.2.** Let \( (G, \mathcal{H} \cup \{ K, L \}) \) be a pair with \( K \neq L \), \( C \leq K \), and \( \varphi : C \to L \) a group monomorphism. Let \( G \ast \varphi \) denote the HNN-extension \( \langle G, t \mid tct^{-1} = \varphi (c) \text{ for all } c \in C \rangle \). Let \( \Delta \) be a \( G \)-graph that has vertices \( x \) and \( y \) such that their \( G \)-stabilizers are \( K \) and \( L \) respectively, and their \( G \)-orbits are disjoint. Then there is a \( G \ast \varphi \)-graph \( \Gamma \) with the following properties:
1. Γ has a vertex z such that \( G_z = \langle K^t, L \rangle \), and there is a \( G \)-equivariant inclusion \( \Delta \hookrightarrow \Gamma \) such that \( x \mapsto t^{-1}z \) and \( y \mapsto z \).

2. If \( \Delta \) is connected, then \( \Gamma \) is connected.

3. If every \( H \in \mathcal{H} \cup \{K, L\} \) is the \( G \)-stabilizer of a vertex of \( \Delta \), then every \( H \in \mathcal{H} \cup \{\langle K^t, L \rangle \} \) is the \( G \)-stabilizer of a vertex of \( \Gamma \).

4. If vertex \( G \)-stabilizers in \( \Delta \) are finite or conjugates of subgroups in \( \mathcal{H} \cup \{K, L\} \), then vertex \( G \)-stabilizers in \( \Gamma \) are finite or conjugates of subgroups in \( \mathcal{H} \cup \{\langle K^t, L \rangle \} \).

5. If \( \Delta \) has finite edge \( G \)-stabilizers, then \( \Gamma \) has finite edge \( G \)-stabilizers.

6. If \( \Delta \) has finitely many \( G \)-orbits of vertices (edges), then \( \Gamma \) has finitely many \( G \)-orbits of vertices (resp. edges).

7. If \( \Delta \) is fine, then \( \Gamma \) is fine.

8. If \( \Delta \) is fine at \( V_\infty(\Delta) \), then \( \Gamma \) is fine at \( V_\infty(\Gamma) \).

9. If \( \Delta \) is hyperbolic, then \( \Gamma \) is hyperbolic.

**Proof of Theorem 1.3.** This proof is completely analogous to the proof of Theorem 1.2: invoke Theorem 3.2 and Lemma 2.6 instead of Theorem 3.1 and Lemma 2.5 respectively.

4. **Amalgamated Products and Graphs**

This section describes an argument proving Theorem 3.1. While the statement of this result seems intuitive, we are not aware of a full account of those techniques in a common framework, so this section provides a detailed construction.

4.1. **Pushouts in the category of \( G \)-sets.** Let \( \phi: R \to S \) and \( \psi: R \to T \) be \( G \)-maps. The pushout of \( \phi \) and \( \psi \) is defined as follows. Let \( Z \) be the \( G \)-set obtained as the disjoint union of \( S \sqcup T \) identifying \( s \in S \) with \( t \in T \) if there is \( r \in R \) such that \( \phi(r) = s \) and \( \psi(r) = t \). There are canonical \( G \)-maps \( i: S \to Z \) and \( j: T \to Z \) such that \( i \circ \phi = j \circ \psi \). This construction satisfies the universal property of pushouts in the category of \( G \)-sets.

**Proposition 4.1.** Let \( \phi: R \to S \) and \( \psi: R \to T \) be \( G \)-maps. Consider the pushout

\[
\begin{array}{ccc}
S & \xrightarrow{i} & Z \\
\downarrow{\phi} & & \downarrow{j} \\
R & \xleftarrow{\psi} & T
\end{array}
\]

of \( \phi \) and \( \psi \). Suppose there is \( r \in R \) such that \( R = G.r \). If \( s = \phi(r) \), \( t = \psi(r) \) and \( z = i(s) \) then \( G_z = \langle G_s, G_t \rangle \).

**Proof.** Since \( i \) and \( j \) are \( G \)-maps, \( \langle G_s, G_t \rangle \leq G_z \). Conversely, let \( g \in G_z \). If \( g \in G_s \) then \( g \in \langle G_s, G_t \rangle \). Suppose \( g \notin G_s \). Then there is a sequence

\[
s = s_0, r = r_0, t = t_0, r_0', s_1, r_1, t_1, r_1', \ldots, r_k', s_{k+1} = g.s
\]
such that
  • $s_i = \phi(r_i)$ and $\psi(r_i) = t_i$, for $0 \leq i \leq k$;
  • $t_i = \psi(r'_i)$ and $\phi(r'_i) = s_{i+1}$, for $0 \leq i \leq k$.

Since $R = G.r_0$, there are elements $a_0, a_1, \ldots, a_k$ and $b_0, b_1, \ldots, b_{k-1}$ of $G$
such that

$$a_i.r_i = r'_i \quad \text{and} \quad b_j.r'_j = r_{j+1}$$

for $0 \leq i \leq k$ and $0 \leq j < k$. Then

$$g.s = \phi(r'_k) = \phi(a_kb_ka_{k-1}\ldots b_0a_0.r) = a_kb_{k-1}a_{k-2}\ldots b_0a_0.s$$

and hence $a_kb_{k-1}a_{k-2}\ldots b_0a_0 \in gG_s$. Since $G_s \leq \langle G_s, G_t \rangle$, to prove that

$g \in \langle G_s, G_t \rangle$ is enough to show that $a_i, b_j \in \langle G_s, G_t \rangle$. We will argue by

induction.

First note that since $\phi$ and $\psi$ are $G$-maps

$$a_i.s_i = s_{i+1} \quad \text{and} \quad b_j.t_j = t_{j+1},$$

and hence

$$G_{s_{i+1}} = a_iG_s.a_i^{-1} \quad \text{and} \quad G_{t_{j+1}} = b_jG_t.b_j^{-1}.$$ 

Moreover, $t_i = \psi(r_i) = \psi(r'_i) = \psi(a_i.r_i) = a_i.t_i$ implies

$$a_i \in G_{t_i},$$

and analogously $s_{j+1} = \phi(r_{j+1}) = \phi(b_j.r'_j) = b_j.s_{j+1}$ implies

$$b_j \in G_{s_{j+1}}.$$

Since $t_0 = t$ and $s_0 = s$, we have that

$$a_0 \in G_{t_0} \leq \langle G_s, G_t \rangle, \quad \text{and} \quad b_0 \in G_{s_0} = a_0G_{s_0}a_0^{-1} \leq \langle G_s, G_t \rangle.$$ 

Suppose $i < k$, $a_i, b_i \in \langle G_s, G_t \rangle$, $G_{s_i} \leq \langle G_s, G_t \rangle$ and $G_{t_i} \leq \langle G_s, G_t \rangle$. Then

$$a_{i+1} \in G_{t_{i+1}} = b_iG_{t_i}b_i^{-1} \leq \langle G_s, G_t \rangle,$$

and hence

$$G_{s_{i+1}} = a_iG_s.a_i^{-1} \leq \langle G_s, G_t \rangle.$$ 

In the case that $i + 1 < k$,

$$b_{i+1} \in G_{s_{i+2}} = a_{i+1}G_{s_{i+1}}a_{i+1}^{-1} \leq \langle G_s, G_t \rangle.$$ 

Therefore, by induction, $a_i, b_j \in \langle G_s, G_t \rangle$ for $0 \leq i \leq k$ and $0 \leq j < k$. \qed

4.2. Extending actions on sets. In the case that $K$ is a subgroup of $G$
and $S$ is a $K$-set, one can extend the $K$-action on $S$ to a $G$-set $G \times_K S$ that
we now describe. Up to isomorphism of $K$-sets, we can assume that $S$ is a
disjoint union of $K$-sets

$$S = \bigsqcup_{i \in I} K/K_i$$

where $K/K_i$ is the $K$-set consisting of left cosets in of a subgroup $K_i$ of $K$.

Then the $G$-set $G \times_K S$ is defined as a disjoint union of $G$-sets

$$G \times_K S := \bigsqcup_{i \in I} G/K_i.$$
Observe that the canonical $K$-map
\[ \iota: S \to G \times_K S, \quad K_i \mapsto K_i \]
is injective. This construction satisfies a number of useful properties that we summarize in the following proposition.

For $n$ a natural number and a set $X$, let $[X]^n$ denote the collection of a subsets of $X$ of cardinality $n$. If $X$ is a $G$-set, then $[X]^n$ is a $G$-set with action defined as $g.\{x_1, \ldots, x_n\} = \{g.x_1, \ldots, g.x_n\}$.

**Proposition 4.2.** Let $K \leq G$ and $S$ a $K$-set.

1. The $K$-map $\iota: S \to G \times_K S$ induces a bijection of orbit spaces $S/K \to (G \times_K S)/G$.
2. For each $s \in S$, the $K$-stabilizer $K_s$ equals the $G$-stabilizer $G_{\iota(s)}$.
3. If $T$ is a $G$-set and $f: S \to T$ is $K$-equivariant, then there is a unique $G$-map $\hat{f}: G \times_K S \to T$ such that $\hat{f} \circ \iota = f$.
4. If $\iota(s) \cap g.\iota(s) \neq \emptyset$ for $g \in G$, then $g \in K$ and $\iota(s) = g.\iota(s)$.
5. If $f$ induces an injective map $S/K \to T/G$ and $K_s = G_{f(s)}$ for every $s \in S$, then $\hat{f}$ is injective.
6. If $j: [S]^n \to G \times_K [S]^n$ is a canonical map, then for every $n \in \mathbb{N}$, there is a $G$-equivariant injection $\hat{i}: G \times_K [S]^n \to [G \times_K S]^n$ such that $\hat{i} \circ j = \hat{i}$ where $\hat{i}: [S]^n \to [G \times_K S]^n$ is the natural $K$-map induced by $\iota: S \to G \times_K S$.

**Proof.** The first four statements are observations. For the fifth statement, suppose $\hat{f}(\iota(s_1)) = \hat{f}(g.\iota(s_2))$. Then $f(s_1) = g.f(s_2)$. Since the map $S/K \to T/G$ induced by $f$ is injective, we have that $s_1$ and $s_2$ are in the same $K$-orbit in $S$, say $s_2 = k.s_1$ for $k \in K$. It follows that $f(s_1) = g.k.f(s_1)$, and since $K_{s_1} = G_{f(s_1)}$, we have that $gk \in K_{s_1}$. Therefore $\iota(s_1) = \iota(gk.s_1) = g.\iota(k.s_1) = g.\iota(s_2)$.

The sixth statement is proved as follows. The $K$-map $\iota: S \to G \times_K S$ naturally induces a $K$-map $\iota: [S]^n \to [G \times_K S]^n$. By the third statement, there is a unique $G$-map $\hat{i}: G \times_K [S]^n \to [G \times_K S]^n$ such that $\hat{i} \circ j = \hat{i}$ where $j: [S]^n \to G \times_K [S]^n$. As a consequence of the fourth statement, $\iota: [S]^n \to [G \times_K S]^n$ induces an injective map $[S]^n/K \to [G \times_K S]^n/G$ and $K_A = G_{\iota(A)}$ for every $A \in [S]^n$; therefore $\hat{i}$ is injective.

As the reader might have noticed, this construction is an instance of general categorical phenomena; that formulation will have no use in this article so we will not discuss it.

### 4.3. Graphs as 1-dimensional complexes.

While the objectives of this section only require us to consider simplicial graphs, the category of simplicial graphs does not have pushouts [Sta3]. For this reason, it is convenient to work within the framework of 1-dimensional complexes or equivalently graphs in the sense that we describe below. We will only consider a particular class of pushouts of graphs that behaves well over simplicial graphs.
A graph is an triple \((V, E, r)\), where \(V\) and \(E\) are sets, and \(r: E \to [V]^1 \cup [V]^2\) is a function where \([V]^n\) is the collection of subsets of \(V\) of cardinality \(n\). Elements of the set \(V\) and \(E\) are called vertices and edges respectively; the function \(r\) is called the attaching map. For a graph \(\Gamma\), we denote \(V(\Gamma)\) and \(E(\Gamma)\) its vertex and edge set, respectively. If \(v \in V(\Gamma)\), \(e \in E(\Gamma)\) and \(v \in r(e)\), then \(v\) is incident to \(e\). Vertices incident to the same edge are called adjacent. A morphism of \(\phi: (V, E, r) \to (V', E', r')\) of G-graphs is a pair of G-maps \(\phi_0: V \to V'\) and \(\phi_1: E \to E'\) such that there is a commutative diagram of G-maps

\[
\begin{array}{ccc}
E & \xrightarrow{\phi_1} & E' \\
\downarrow r & & \downarrow r' \\
[V]^1 \cup [V]^2 & \xrightarrow{\phi_0} & [V']^1 \cup [V']^2
\end{array}
\]

where the horizontal bottom arrow is the natural G-map induced by \(\phi_0: V \to V'\). A morphism \((\phi_0, \phi_1)\) of G-graphs is a monomorphism if both G-maps are injective. The graph \((V, E, r)\) is simplicial if every edge has two distinct endpoints and \(r\) is injective, or equivalently, \(E\) can be regarded as a subset of \([V]^2\).

Let \(G\) be a group. A \(G\)-graph is a graph \((V, E, r)\) where \(V\) and \(E\) are \(G\)-sets, and \(r\) is a \(G\)-map with respect to the natural \(G\)-action on \([V]^1 \cup [V]^2\) induced by the \(G\)-set \(V\). A \(G\)-action on a graph \(\Gamma\) has no inversions if for every \(e \in E\) and \(g \in G\) such that \(g.e = e\), \(g.v = v\) for every \(v \in r(e)\). For a \(G\)-action without inversions on a graph \(\Gamma\) and a \(K \leq G\), let \(\Gamma^K\) denote the maximal subgraph of \(\Gamma\) invariant by \(K\), that is, \(V(\Gamma^K) = V(\Gamma)^K\) and \(E(\Gamma^K) = E(\Gamma)^K\).

4.4. Extending group actions on graphs. Let \(K\) be a subgroup of \(G\), and let \(\Lambda = (V, E, r)\) be a \(K\)-graph. Define

\[
G \times_K \Lambda = (G \times_K V, G \times_K E, \tilde{r})
\]

where \(\tilde{r}\) is unique G-map induced by the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{j} & G \times_K E \\
\downarrow r & \downarrow \iota & \downarrow \tilde{r} \\
[V]^2 & \xrightarrow{\iota} & [G \times_K V]^2
\end{array}
\]

where \(\iota: V \hookrightarrow G \times_K V\) and \(j: E \hookrightarrow G \times_K E\) are the canonical \(K\)-maps, see Lemma 4.2[3]. Note that there is a canonical \(K\)-equivariant embedding

\[
\Lambda \hookrightarrow G \times_K \Lambda
\]

induced by \(\iota\) and \(j\). We consider \(\Lambda\) a \(K\)-subgraph of \(G \times_K \Lambda\).

**Remark 4.3.** If \(\Lambda\) is a simplicial \(K\)-graph without inversions, then \(G \times_K \Lambda\) is simplicial \(G\)-graph without inversions.
4.5. **Pushouts of graphs.** Let $X$ and $Y$ be $G$-graphs, let $C \leq G$ be a subgroup and suppose $X^C$ and $Y^C$ are non-empty. Let $x \in X^C$ and $y \in Y^C$ be vertices. The $C$-pushout $Z$ of $X$ and $Y$ with respect to the pair $(x, y)$ is the $G$-graph $Z$ obtained by taking the disjoint union of $X$ and $Y$ and then identifying the vertex $g.x$ with the vertex $g.y$ for every $g \in G$.

Equivalently, the $C$-pushout $Z$ of $X$ and $Y$ with respect to the pair $(x, y)$ is the $G$-graph $Z$ whose vertex set $V(Z)$ is the pushout of the $G$-maps $\kappa_1: G/C \to V(X)$ and $\kappa_2: G/C \to V(Y)$ given by $C \mapsto x$ and $C \mapsto y$; and edge set the disjoint union of the $G$-sets $E(X)$ and $E(Y)$, and attaching map $E(Z) \to V(Z)^2$ given by the union of the attaching maps for $X$ and $Y$ postcomposed with the maps $V(X) \to V(Z)$ and $V(Y) \to V(Z)$ defining the pushout.

The standard universal property of pushouts holds for this construction: if $j_1: X \to W$ and $j_2: Y \to W$ are morphisms of $G$-graphs such that $j_1 \circ \kappa_1 = j_2 \circ \kappa_2$, then there is a unique morphism of $G$-graphs $Z \to W$ such that above diagram commutes.

**Remark 4.4.** Let $Z$ be the $C$-pushout of $X$ and $Y$ with respect to a pair $(x, y)$.

1. For any vertex $x$ in $X$, $G_x = G_{\kappa_1(x)}$ or $x$ is in the image of $\kappa_1$.
2. For any edge $e$ of $X$, $G_e = G_{\kappa_1(e)}$.
3. If $X/A$ and $Y/B$ both have finitely many vertices (resp. edges), then $Z/G$ has finitely many vertices (resp. edges).

**Example 4.1.** Let $G = A \ast_C B$ where $A$ and $B$ are free abelian groups of rank two, and $C$ is maximal cyclic subgroup of $A$ and $B$. Let $X$ be the $A$-graph consisting of a single vertex with the trivial $A$-action, and define $Y$ analogously for $B$. Then the graph $G \times_A X$ is the edgeless $G$-graph with vertex set the collection of left cosets of $G/A$; and analogously $G \times_B Y$ is the edgeless graph with vertex set $G/B$. Let $Z$ be the $C$-pushout of $X$ and $Y$. By Proposition 4.3(4), $Z$ is a connected edgeless $G$-graph and hence a single vertex. (Note that the algebraic nature of $A$, $B$ and $C$ was not used in the argument).

**Example 4.2.** Let $A = \langle a_1, a_2, a_3 \mid [a_1, a_2] \rangle$ and $B = \langle b_1, b_2, b_3 \mid [b_1, b_2] \rangle$, and let $X = \hat{\Gamma}(A, \langle a_1, a_2 \rangle, a_3)$ and $Y = \hat{\Gamma}(B, \langle b_1, b_2 \rangle, b_3)$ be the coned-off Cayley graphs. Note that $X$ is the Bass-Serre tree of the splitting of $A$ as
the graph of groups

\[
\begin{array}{c}
\circ \quad 1 \quad \circ \\
\circ \quad 1 \quad \circ \\
\end{array}
\]

with two vertices and two edges with trivial edge group.

Let \( G = A \ast_C B \) be the amalgamated product where \( C \) corresponds to the cyclic subgroup \( \langle a_1 \rangle \leq A \) and \( \langle b_1 \rangle \leq B \). Consider the \( C \)-pushout \( Z \) of \( G \times_A X \) and \( G \times_B Y \). By the fourth, fifth and sixth statements of Proposition 4.5 below, \( Z \) is a tree, it contains three distinct \( G \)-orbits of vertices, two of these \( G \)-orbits have all representatives with trivial stabilizer, and there is a vertex \( z \) with stabilizer \( \langle a_1, a_2, b_2 \rangle = \langle a_1, a_2 \rangle \ast_{\langle a_1 \rangle = \langle b_1 \rangle} \langle b_1, b_2 \rangle \), and there distinct orbits of edges all with representatives having trivial stabilizer. Hence \( Z \) is the Bass-Serre tree of a splitting of \( G \) given by the graph of groups

\[
\begin{array}{c}
\circ \quad 1 \quad \circ \\
\circ \quad 1 \quad \circ \\
\end{array}
\]

with three vertices and four edges. In particular, \( Z \) is the coned-off Cayley graph of \( \hat{\Gamma}(G, G_y, \{a_3, b_3\}) \).

**Proposition 4.5.** Let \( G \) be the amalgamated free product group \( A \ast_C B \), let \( X \) be a \( A \)-graph, and let \( Y \) be a \( B \)-graph. Let \( x \in X_G \) and \( y \in Y_G \) be vertices. Let \( Z \) be the \( C \)-pushout of \( G \times_A X \) and \( G \times_B Y \) with respect to \( (x, y) \). The following properties hold:

1. If \( z = \iota_1(x) = \iota_2(y) \), then the homomorphism \( A_x \ast_C B_y \rightarrow G \) induced by the inclusions \( A_x \leq G \) and \( B_y \leq G \) is injective and has image \( G_z \). In particular \( G_z = \langle A_x, B_y \rangle \) is isomorphic to \( A_x \ast_C B_y \).
2. The morphism \( X \hookrightarrow G \times_A X \xrightarrow{\iota_1} Z \) is an \( A \)-equivariant embedding. Analogously, \( Y \hookrightarrow G \times_B Y \xrightarrow{\iota_2} Z \) is a \( B \)-equivariant embedding. From here on, we consider \( X \) and \( Y \) as subgraphs of \( Z \) via these canonical embeddings.
3. For every vertex \( v \) (resp. edge \( e \)) of \( Z \), there is \( g \in G \) such that \( g.v \) is a vertex (resp. is an edge \( g.e \)) of the subgraph \( X \cup Y \).
4. For every vertex \( v \) of \( X \) which is not in the \( A \)-orbit of \( x \), \( A_v = G_v \) where \( G_v \) is the \( G \)-stabilizer of \( v \) in \( Z \). Analogously for every vertex \( v \) of \( Y \) not in the \( B \)-orbit of \( y \), \( B_v = G_v \).
5. For every edge \( e \) of \( X \) (resp. \( Y \)), \( A_e = G_e \) (resp. \( B_e = G_e \)) where \( G_e \) is the \( G \)-stabilizer of \( e \) in \( Z \).
6. If the complexes \( X/A \) and \( Y/B \) both have finitely many vertices (resp. edges), then \( Z/G \) has finitely many vertices (resp. edges).
7. If \( X \) and \( Y \) are connected, then \( Z \) is connected.
8. Suppose \( X \) and \( Y \) are connected. Then the vertex \( z \) is a cut vertex of \( Z \). Moreover, the closure of any connected component of \( Z \setminus G.z \), equals \( g.X \) or \( g.Y \) for some \( g \in G \).
If $X$ and $Y$ are simplicial connected graphs, then $Z$ is simplicial connected graph.

Proof. The first item is a direct consequence of Proposition 4.1. For the second item, to show that $\kappa_1 : G \times_A X \rightarrow Z$ is an $A$-equivariant embedding is enough to only consider vertices of $X$ that are in the $A$-orbit of $x$. Suppose that $a.x$ and $x$ with $a \in A$ both map to $z \in Z$. Then $a \in G_z = A_x \ast_C B_y$ and therefore $a \in A_x$ and hence $a.x = x$. An analogous argument shows that $\kappa_2 : G \times_B Y \rightarrow Z$ is a $B$-equivariant embedding. Item three follows directly from the definition of $Z$, and items four to six are consequences of Proposition 4.2.

To prove the seventh statement suppose that $X$ and $Y$ are connected graphs. The subgraph $X \cup Y$ of $Z$ is connected since both $X$ and $Y$ contain the vertex $z$. On the other hand, any vertex of $Z$ belongs to a translate of $X \cup Y$ by an element of $G$. Therefore to prove that $Z$ is connected, it is enough to show that for any $g \in G$ there is a path in $Z$ from $z$ to $g.z$. For any $g \in G$ and $a \in A$, there is a path from $g.z$ to $ga.z$ in $Z$: indeed, there is a path from $z$ to $a.z$ in the connected $A$-subgraph $X$ of $Z$, and hence there is a path from $g.z$ to $ga.z$ in $Z$. Analogously, for any $g \in G$ and $b \in B$, there is a path from $g.z$ to $gb.z$. Since any element of $G$ is of the form $a_1 b_1 \ldots a_n b_n$ with $a_i \in A$ and $b_i \in B$, there is a path from $z$ to $g.z$ for any $g \in G$.

We conclude proving the fifth statement. Observe that $G$ splits as $G = A \ast_{A_x} (A_x \ast_C B_y) \ast_{B_y} B$ where the subgroups $A_x$, $B_y$ and $A_x \ast_C B_y$ are naturally identified with the $G$-stabilizers of $x \in G \times_A X$, $y \in G \times_B Y$, and $z \in Z$. Let $T$ denote the Bass-Serre tree of this splitting. The vertex and edge sets of $T$ can be described as

$$V(T) = \frac{G}{A} \sqcup \frac{G}{(A_x \ast_C B_y)} \sqcup \frac{G}{B}$$

and

$$E(T) = \{\{gA, g(A_x \ast_C B_y)\} \mid g \in G\} \sqcup \{\{g(A_x \ast_C B_y), gB\} \mid g \in G\}$$

respectively. Consider an $A$-map from $X$ to $T$ that maps every vertex of $X$ not in the $A$-orbit of $x$ to the vertex $A$, and $x \mapsto A_x \ast_C B_y$. This induces a unique $G$-map $j_1 : G \times_A X \rightarrow T$. Analogously, there is $B$-map $Y \rightarrow T$ that maps every vertex not in $B$-orbit of $y$ to the vertex $B$ and $y \mapsto A_x \ast_C B_y$; this induces a unique $G$-map $j_2 : G \times_B Y \rightarrow T$. 

![Diagram](attachment:diagram.png)
Consider the $G$-maps $\kappa_1: G/C \to G \times_A X$ and $\kappa_2: G/C \to G \times_B Y$ given by $C \mapsto x$ and $C \mapsto y$ respectively. Since $j_1 \circ \kappa_1 = j_2 \circ \kappa_2$, the universal property implies that there is a surjective $G$-map $\xi: Z \to T$.

To show that $z$ is a cut vertex of $Z$, it suffices to prove that $\xi^{-1}(\xi(z)) = \{z\}$. Note that $\xi^{-1}(\xi(z))$ is contained in the orbit $G.z$. Observe that if $g.z \in \xi^{-1}(\xi(z))$ then $g(A_x \ast_C B_y) = A_x \ast_C B_y$ and hence $g \in A_x \ast_C B_y$. Since $A_x \ast_C B_y$ is the $G$-stabilizer of $z$, we have that $g.z = z$. This shows that $\xi^{-1}(\xi(z)) = \{z\}$.

The last item is a direct consequence of eighth item. □

4.6. Proof of Theorem 3.1

**Lemma 4.6.** Let $U$ be a collection of cut vertices of a connected graph $\Gamma$. Let $\Omega$ be the set whose elements are the closures of the connected components of $\Gamma \setminus U$.

1. $\Gamma$ is a hyperbolic graph if and only if there is $\delta > 0$ such that every $\Delta \in \Omega$ is a $\delta$-hyperbolic graph.
2. For any vertex $v$ of $\Gamma$, the following statements are equivalent:
   - $\Gamma$ is fine at $v$.
   - For every $\Delta \in \Omega$, if $v$ is a vertex of $\Delta$, then $\Delta$ is fine at $v$.

**Proof.** The first statement is an observation. For the second statement, if $\Gamma$ is fine at $v$, then any subgraph containing $v$ is fine at $v$. Conversely, let $v$ be a vertex of $\Gamma$ such that any $\Delta$ containing $v$ is fine at $v$.

Suppose that $v \notin U$. Then there is a unique $\Delta$ that contains $v$ as a vertex. Since every element of $U$ disconnects $\Gamma$, the metric spaces $(T_v \Delta, \angle_v)$ and $(T_v \Gamma, \angle_v)$ coincide. Since $\Delta$ is fine at $v$, then $\Gamma$ is fine at $v$.

Suppose that $v \in U$. Observe that if $x, y \in T_v \Gamma$ and $x$ and $y$ belong to different subgraphs in $\Omega$, then $\angle_v(x, y) = \infty$. Therefore, every ball of finite radius in $T_v \Gamma$ centered at $x$ is a ball of finite radius in $T_v \Delta$ centered at $x$ for some $\Delta$. Since by assumption, $\Delta$ is fine at $v$, every ball of finite radius in $T_v \Gamma$ centered at $x$ is finite.

**Proof of Theorem 3.1.** Let $\Gamma$ be the $C$-pushout of the $G$-graphs $G \times_{G_1} \Gamma_1$ and $G \times_{G_2} \Gamma_2$ with respect to $(x_1, x_2)$, and let $z$ be the image of $x_1$ in $\Gamma$. The first six properties of $\Gamma$ are direct corollaries of Proposition 4.5. Observe that Proposition 4.5 implies that the vertex $z$ disconnects $\Gamma$ and has the property that the closure of each connected component of $\Gamma \setminus G.z$ is a subgraph isomorphic to $\Gamma_1$ and $\Gamma_2$. Then Lemma 4.6 implies the last three properties. □

5. HNN-Extensions

This section describes a proof of Theorem 3.2. The argument is analogous to the one proving Theorem 3.1. In this case, we need to construct a $G \ast \varphi$-graph from a given $G$-graph that we call the $\varphi$-coalescence.
**Definition 5.1.** (Coalescence in sets) Let $H$ be a subgroup of a group $A$, let $\varphi: H \to A$ be a monomorphism and let $G = A*_{\varphi}$. Let $X$ be an $A$-set, $x \in X^H$ and $y \in X^{\varphi(H)}$. The $\varphi$-**Coalescence of $X$ with respect to $(x, y)$** is the quotient $Z$ of $G \times_A X$ by the equivalence relation generated by $gt.x \sim g.y$ for all $g \in G$. Note that the quotient map
\[ \rho: G \times_A X \to Z \]
is $G$-equivariant.

**Example 5.1.** Let $\varphi: A \to A$ be a group automorphism and consider the HNN-extension $G = A*_{\varphi}$. Let $X$ be the $A$-set consisting of a single point. Then $G \times_A X$ is the $G$-space $G/A$, and then the $\varphi$-Coalescence of $X$ is again a single point.

**Example 5.2.** Consider a free product $A = H_1 * H_2$. Let $\varphi: H_1 \to H_2$ be an isomorphism, and $G = A*_{\varphi}$. Let $X$ be the $A$-set $A/H_1 \cup A/H_2$ of all left cosets of $H_1$ and $H_2$ in $A$. Then $G \times_A X$ is the $G$-set of left cosets $G/H_1 \cup G/H_2$. The $\varphi$-coalescence $Z$ of $X$ with respect to the pair $(H_1, H_2)$ is the quotient $G \times_A X$ by identifying $gH_1$ and $gH_2$ for every $g \in G$. Hence $Z$ is isomorphic as a $G$-set to $G/H_1$.

**Lemma 5.2.** Let $H$ be a subgroup of a group $A$, let $\varphi: H \to A$ be a monomorphism and let $G = A*_{\varphi}$. Let $X$ be an $A$-set, let $x, y \in X$ be in different $A$-orbits such that $A_x = H$, $A_y = \varphi(H)$. If $Z$ is the $\varphi$-Coalescence of $X$ with respect to $(x, y)$, and $z = \rho(y)$, then:

1. the $G$-stabilizer $G_z$ equals $\varphi(H)$, and
2. the $A$-map $j: X \to Z$ defined by the commutative diagram
\[
\begin{array}{ccc}
G \times_A X & \xrightarrow{\rho} & Z \\
\downarrow z & & \\
X & \xrightarrow{j} & Z,
\end{array}
\]
is injective.

**Proof.** Since $\rho$ is $G$-equivariant, $\varphi(H) = A_y$ and $\rho(y) = z$, we have $\varphi(H) \subseteq G_z$. Suppose $g \in G$ and $g.z = z$. Then $g.y \sim y$ in $G \times_A X$. Thus there is a chain of basic relations in $G \times_A X$
\[ g.y \sim g_2.x \sim g_3.y \sim g_4.x \sim \cdots \sim g_n.x \sim y, \]
that means $tg^{-1}_2 \in \varphi(H)$, $tg^{-1}_3 \in \varphi(H)$, $tg^{-1}_4 \in \varphi(H)$, $tg^{-1}_5 \in \varphi(H)$ and so on until $tg^{-1}_n \in \varphi(H)$. Therefore
\[ g^{-1} = (tg_2^{-1}g_3)(tg_4^{-1}g_3)(tg_5^{-1}g_5) \cdots (tg_n^{-1}) \in \varphi(H), \]
which implies $g \in \varphi(H)$. This shows that $G_z = \varphi(H)$.

Now we prove the second statement. By Lemma 5.2, the natural $A$-map $X \to G \times_A X$ is injective. Observe that $Z$ is obtained as a quotient of $G \times_A X$ by the $G$-equivariant equivalence relation generated by the basic relation $t.x \sim y$. Hence to prove injectivity of $X \to G \times_A X \to Z$, it is enough to show
that the restriction to $A.x \cup A.y$ is injective. Assume there are $a_1, a_2 \in A$ such that $a_1.x$ and $a_2.x$ map to the same element in $Z$. Then letting $a = a_2^{-1}a_1$, both $a.x$ and $x$ map to the same element in $Z$. Hence $a.x \sim x$ which implies that $a t^{-1}.y \sim t^{-1}.y$. Therefore $(a t^{-1}).y \sim y$ and thus by the first statement, $t a t^{-1} \in \varphi(H)$, and hence $a^{-1} \in t^{-1}\varphi(H)t = H$. This results in $a \in H$. Therefore $a_2^{-1}a_1 \in H$ and $a_1.x = a_2.x$. We have shown that the restriction $A.x \to Z$ is injective. With a similar argument one can show that $A.y \to Z$ is also injective. □

**Definition 5.3.** (Coalescence in graphs) Let $H$ be a subgroup of a group $A$, let $\varphi: H \to A$ be a monomorphism and let $G = A* \varphi$. Let $X$ be an $A$-graph, let $x, y \in V(X)$ such that $x \in X^H$ and $y \in X^{\varphi(H)}$. The $\varphi$-Coalescence of $X$ with respect to $(x, y)$ is the quotient $\mathbb{Z}$ of the $G$-graph $G \times_A X$ by the equivalence relation generated by $gt.x \sim g.y$ for all $g \in G$. Let $\rho: G \times_A X \to \mathbb{Z}$ denote the quotient map.

**Remark 5.4.** Equivalently, the $\varphi$-Coalescence $\mathbb{Z}$ of the $A$-graph $X$ with respect to $(x, y)$ is the $G$-graph with vertex set the $\varphi$-Coalescence of the $A$-set $V(X)$, edge set $G \times_A E(X)$, and attaching map $E(\mathbb{Z}) \to V(\mathbb{Z})^2$ is defined as a composition

$$
\begin{array}{ccc}
E(X) & \longrightarrow & V(X)^2 \\
\downarrow & & \downarrow \\
E(\mathbb{Z}) = G \times_A E(X) & \longrightarrow & G \times_A V(X)^2 \\
\downarrow & & \downarrow \\
V(\mathbb{Z})^2 & \end{array}
$$

where the horizontal middle arrows is induced by the attaching map $E(X) \to V(X)^2$ (see Lemma 4.2(3)) and the the bottom vertical map is induced by the quotient map $G \times_A V(X) \to V(\mathbb{Z})$.

**Lemma 5.5.** $\mathbb{Z}$ is a simplicial graph and, in particular, the quotient map $\rho: G \times_A X \to \mathbb{Z}$ is a simplicial $G$-map.

**Proof.** Since $V(X) \to V(\mathbb{Z})$ is injective (Lemma 5.2(2)) and the image of $E(X) \to V(X)^2$ does not intersect the diagonal, it follows that the image $E(X) \to V(\mathbb{Z})^2$ does not intersect the diagonal. Since the $G$-translates of $E(X)$ cover $E(\mathbb{Z}) = G \times_Z E(X)$, it follows that the image of $E(\mathbb{Z}) \to V(\mathbb{Z})^2$ does not intersect the diagonal. □

**Proposition 5.6.** Let $H$ be a subgroup of a group $A$, let $\varphi: H \to A$ be a monomorphism and let $G = A* \varphi$. Let $X$ be an $A$-graph, let $x, y \in X$ in different $A$-orbits such that $A_x = H$, $A_y = \varphi(H)$. If $\mathbb{Z}$ is the $\varphi$-Coalescence of $X$ with respect to $(x, y)$, and $z = \rho(y)$, then the following properties hold:

1. $G_z = \varphi(H)$.
2. The map $X \hookrightarrow G \times_A X \to \mathbb{Z}$ is an $A$-equivariant embedding.
From here on, we consider $X$ as a subgraph of $Z$ via this canonical embedding.

(3) For every vertex $v$ (resp. edge $e$) of $Z$, there is $g \in G$ such that $g.v$ is a vertex (resp. is an edge $g.e$) of $X$.

(4) For every vertex $v$ of $X$ which is not in the $A$-orbit of $x$, $A_v = G_v$ where $G_v$ is the $G$-stabilizer of $v$ in $Z$.

(5) For every edge $e$ of $X$ $A_e = G_e$ where $G_e$ is the $G$-stabilizer of $e$ in $Z$.

(6) If the complex $X/A$ has finitely many vertices (resp. edges), then $Z/G$ has finitely many vertices (resp. edges).

(7) If $X$ is connected, then $Z$ is connected.

(8) Suppose $X$ is connected. Then the vertex $z$ is a cut vertex of $Z$. Moreover, the closure of any connected component of $Z \setminus G.z$, equals $g.X$ for some $g \in G$.

(9) If $X$ is a connected simplicial graph, then $Z$ is connected and simplicial.

The following argument is analogous to the proof of Proposition 4.5.

Proof. The first and second statements are direct consequences of Lemma 5.2 when considering $V(X)$ and $E(X)$ as $A$-sets. Items three to six follow directly from the definition of $Z$ and Proposition 4.2.

Suppose $X$ is connected. By Proposition 4.2, the graph $G \times_A X$ is a disjoint union of copies of the connected subgraph $X$, and hence any element in $Z$ belongs to a translate of $X$ by an element of $G$. Therefore, to prove that $Z$ is connected, it is sufficient to show that for any $g \in G$ there is a path in $Z$ from $z$ to $g.z$.

First observe that if there is a path from $z$ to $g.z$, then there is path from $z$ to $gt.z$. Indeed, there is a path from $x$ to $y$ in the connected subgraph $X$ of $G \times_A X$, and hence there is path from $z = \rho(t.x)$ to $t.z = \rho(t.y)$ in $Z$. Therefore, there is a path from $g.z$ to $gt.z$ in $Z$, and in particular a path from $z$ to $g.t.z$.

Now observe that if there is path from $z$ to $g.z$, then there is path from $z$ to $ga.z$ for any $a \in A$. Indeed, there is a path from $z$ to $a.z$ in the connected $A$-subgraph $X$ of $Z$. Hence, there is a path from $g.z$ to $ga.z$ in $Z$, and in particular a path from $z$ to $ga.z$.

To conclude, any $g \in G$ is a product of the form $g = a_1t^{\epsilon_1}a_2t^{\epsilon_2} \cdots a_n t^{\epsilon_n}a_{n+1}$ with $a_i \in A$ and $\epsilon_i = \pm 1$. Therefore, an induction argument using the two previous statements shows that there is a path from $z$ to $g.z$ in $Z$ for every $g \in G$.

Now we prove the last statement. Consider the barycentric subdivision $T$ of the Bass-Serre tree of the splitting $G*_{q}$. Specifically, $T$ is the tree with vertex set

$$V(T) = G/A \sqcup G/H$$
and edge set
\[ E(T) = \{\{gA, g\varphi(H)\} \mid g \in G\} \sqcup \{\{gA, gH\} \mid g \in G\}. \]
Note that all the edges of \( T \) are \( G \)-translates of the following two edges attached at the vertex \( tH \),
\[
\begin{array}{ccc}
A & \{A, \varphi(H)\} & tH \\
\hline
 & tA & \{tA, tH\} & tA
\end{array}
\]
Suppose that the \( A \)-set \( V(X) = \bigcup_{i \in I} A/A_i \sqcup A/H \sqcup G/\varphi(H) \). Then
\[ V(G \times_A X) = \bigcup_{i \in I} G/A_i \sqcup G/H \sqcup G/\varphi(H), \]
and hence there is an induced \( G \)-equivariant simplicial map
\[
\psi: G \times_A X \to T
\]
defined on vertices by
\[
A_i \mapsto H, \quad H \mapsto H, \quad \varphi(H) \mapsto tH.
\]
Note that any edge in \( G \times_A X \) of the form \( \{gA_i, gaA_j\} \) for \( g \in G \) and \( a \in A \) is mapped to the vertex \( gA \) in \( T \); and edges of the form \( \{gA_i, gH\} \) and \( \{gA_i, g\varphi(H)\} \) are mapped to the edges \( \{gA, gH\} \) and \( \{gA, g\varphi(H)\} \) of \( T \) respectively. This map induces a simplicial \( G \)-equivariant map \( \pi: Z \to T \) such that the following diagram commutes,
\[
\begin{array}{ccc}
G \times_A X & \xrightarrow{\rho} & Z \\
\downarrow{\psi} & & \downarrow{\pi} \\
 & & T
\end{array}
\]
Indeed this diagram commutes since \( \psi \) is \( G \)-equivariant and \( \psi(tH) = \psi(\varphi(H)) \).
To show that \( z \) is a cutpoint of \( Z \) it is enough to show that \( z \) is the preimage of the cutpoint \( tH \) of \( T \) by the simplicial map \( \pi \). Observe that \( \pi(z) = tH \) and \( \psi^{-1}(tH) = \{tH, \varphi(H)\} \), and therefore \( \pi^{-1}(tH) = \rho(tH) = \rho(\varphi(H)) = \{z\} \).

The last item is a consequence of the eighth item. \( \square \)

**Proof of Theorem 3.2.** The argument is the same as the one used to prove Theorem 3.1, the only difference is the use of Proposition 5.6 instead of Proposition 4.5. \( \square \)

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