Some Paradoxes in Special Relativity

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The special theory of relativity is the foundation of modern physics, but its unusual postulate of invariant vacuum speed of light results in a number of plausible paradoxes. This situation leads to radical criticisms and suspicions against the theory of relativity. In this paper, from the perspective that the relativity is nothing but a geometry, we give a uniform resolution to some famous and typical paradoxes such as the ladder paradox, the Ehrenfest’s rotational disc paradox. The discussion shows that all the paradoxes are caused by misinterpretation of concepts. We misused the global simultaneity and the principle of relativity. As a geometry of Minkowski space-time, special relativity can never result in a logical contradiction.

Keywords: Clifford algebra, Minkowski space-time, paradox, simultaneity

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I. INTRODUCTION

In order to reconcile the confliction between Newtonian Mechanics and Electromagnetism, in 1905 Einstein introduced the special relativity by two postulates, that is, the vacuum light speed is independent of observer and the Galilean principle of relativity. The first postulate is so unusual that a number of plausible paradoxes are long standing without a resolution of consensus. The suspicions that the relativity may be contradictory always exist among physicists. In the non-mainstream literatures, we can see a lot of radical criticisms against the theory of relativity.

Historically, we can find the profound theoretical and experimental origin of relativity. In the late 19th century, Henri Poincaré suggested that the principle of relativity holds for all laws of nature. Joseph Larmor and Hendrik Lorentz discovered that Maxwell’s equations were invariant only under a certain change of time and length units. This left quite a bit of confusion among physicists, many of whom thought that a luminiferous aether is incompatible with the relativity principle.

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In their 1905 papers on electrodynamics, Henri Poincaré and Albert Einstein explained that with the Lorentz transformations the relativity principle holds perfectly. Einstein elevated the principle of relativity to an axiom of the theory and derived the Lorentz transformations from this principle combined with the principle of constant vacuum speed of light. These two principles were reconciled with each other by a re-examination of the fundamental meanings of space and time intervals[1].

However, the physical description of the constant light speed is ambiguous and looks contradictory in the common sense. What is the light speed? why should it be related to the vacuum? and how can a speed be independent of moving frames are the usual complains and arguments. Such ambiguous statement is not easy to be clarified in physics, which involves a number of basic concepts and theories. It even leads to misinterpretation and puzzles among sophistic experts. As a matter of fact, the speed of light c has complicated physical background and meanings involving the mystic properties of electromagnetic field, and is hidden in almost all equations but with different facets[2]. As the most fundamental concepts or principles in physics, the simpler the meanings and expressions, the better the effectiveness and universality.

In essence, the relativity is nothing but a 4-dimensional geometry. The confusion and paradoxes are caused by conceptual misinterpretation. In this paper, we give a unified resolution to some famous paradoxes according to the underlying geometry of the special relativity. The following typical paradoxes in relativity and some versions of their resolutions can be found in the Wikipedia free encyclopedia and some textbooks.

1. **Twins paradox[3]**: The twin paradox is a thought experiment in special relativity, in which a twin who makes a journey into space in a high-speed rocket will return home to find he has aged less than his identical twin who stayed on Earth. This result appears puzzling on this basis: the laws of physics should exhibit symmetry. Either twin sees the other twin as travelling; so each should see the other aging more slowly. How can an absolute effect (one twin really does age less) result from a relative motion?

2. **Bell’s spaceship paradox[4]**: It is a thought experiment in special relativity involving accelerated spaceships and strings. Two spaceships, which are initially at rest in some common inertial reference frame are connected by a taut string. At time zero in the common inertial frame, both spaceships start to accelerate, with a constant proper acceleration as measured by an on-board accelerometer. Question: does the string break? i.e., does the distance between the two spaceships increase?
3. **The ladder paradox**[5] The ladder paradox (or barn-pole paradox) is a thought experiment in special relativity. If a ladder travels horizontally it will undergo a length contraction and will therefore fit into a garage that is shorter than the ladder’s length at rest. On the other hand, from the point of view of an observer moving with the ladder, it is the garage that is moving and the garage will be contracted. The garage will therefore need to be larger than the length at rest of the ladder in order to contain it. Suppose the ladder is travelling at a fast enough speed that, from the frame of reference of the garage, its length is contracted to less than the length of the garage. Then from the frame of reference of the garage, there is a moment in time when the ladder can fit completely inside the garage, and during that moment one can close and open both front and back doors of the garage, without affecting the ladder. However, from the frame of reference of the ladder, the garage is much shorter than the length of the ladder (it is already shorter at rest, plus it is contracted because it is moving with respect to the ladder). Therefore there is no moment in time when the ladder is completely inside the garage; and there is no moment when one can close both doors without it hitting the ladder. This is an apparent paradox.

4. **Ehrenfest paradox**[6]: The Ehrenfest paradox concerns the rotation of a “rigid” disc in the theory of relativity. In its original formulation as presented by Paul Ehrenfest 1909 in the Physikalische Zeitschrift, it discusses an ideally rigid cylinder that is made to rotate about its axis of symmetry. The radius $R$ as seen in the laboratory frame is always perpendicular to its motion and should therefore be equal to its value $R_0$ when stationary. However, the circumference $2\pi R$ should appear Lorentz-contracted to a smaller value than at rest, by the usual factor $\gamma = \sqrt{1 - v^2}$. This leads to the contradiction that $R = R_0$ and $R < R_0$.

II. GEOMETRY OF THE MINKOWSKI SPACE-TIME

A. Basic Concepts and Settings

Like the Euclidean geometry, if we directly express the special relativity as geometry of the Minkowski space-time at the beginning, and then explain the corresponding rules between the geometrical concepts and the reality, the problem becomes clearer and simpler. Mathematically special relativity is a simple theory[7]. In some sense, the postulate of constant light speed is a restriction on the metric of the space-time, and the principle of relativity explains the rules of transformation between coordinate systems. The following contents are some well known knowledge,
but we organize them consistently in logic.

The events or points in the space-time can be described by 4 independent coordinates. This is a basic fact and can be only accepted as a fundamental assumption\cite{8, 9}. A more profound philosophical reason for the 1+3 dimensions is that, the world is related to the elegant mathematical structure of the quaternion or Clifford algebra as shown below. Assume $S(T, X, Y, Z)$ is a static coordinate system, where the physical meanings of the coordinates $(T, X, Y, Z)$ is that, $T$ is the period counting of an idealized oscillator without any interaction with other matter, and $(X, Y, Z)$ is measured by an idealized rigid unit box. We imply that such measurement can be performed everywhere, and the values assign a unique coordinate to each event in the space-time. However, this operation is not enough to describe the property of the space-time, because the space-time is measurable, in which the distance between two events is independent of measurement. Similar to the Euclidean geometry, the space-time manifold has a consistent distance $ds$ between events. Of course, it is just an assumption in logic and should be tested by experiments. According to the electrodynamics, which is mature theory proved by numerous experiments, we find that in the Nature the consistent distance should be quadratic form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In special relativity, which implies the space-time is uniform and flat, then we have

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2, \quad \forall(T, X, Y, Z), \tag{2.1}$$

where and hereafter we set $c = 1$ as unit if it is not confused. Sometimes we use $\vec{X} = (X^1, X^2, X^3)$ and so on to denote the 3-dimensional vector.

(2.1) is nothing but a kind of Pythagorean theorem, which defines the Minkowski space-time. (2.1) is the starting point of the following discussions. Although (2.1) is compatible with the postulate of constant light speed, it gets rid of the definition of light speed $c$, and becomes clear and simple in logic. The explanation of $c$ is related with the complicated properties of electromagnetic field which should be explained according to electrodynamics.

The distance formula (2.1) and the dynamical equations in the space-time all have the elegant structure of quaternion or the Clifford algebra $Cl_{1,3}[10, 11, 12, 13, 14]$. Assume $\gamma^\mu$ is a set of basis satisfying the fundamental relation of the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad \mu, \nu \in \{0, 1, 2, 3\}, \tag{2.2}$$

which can be realized by the Dirac matrices, then the vector in the Minkowski space-time can be expressed by $d\mathbf{X} = dX^\mu \gamma_\mu \in Cl_{1,3}(\mathbb{R})$, and (2.1) simply becomes the geometric product $ds^2 = d\mathbf{X}^2$. The geometric product of any vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by $\mathbf{a} \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$. And the dynamical
equation of all realistic fields $\Psi = (\psi_1, \psi_2, \cdots, \psi_n)$ becomes $\partial \Psi = f(\Psi)$, where $\partial = \gamma^\mu \partial_\mu$ is quaternion differential operator, and $f(\Psi)$ consists of some tensorial products of $\Psi$ such that the dynamical equation is covariant[9, 14].

From (2.1), we can derive a series of equivalent coordinate system according to the principle of relativity, in which the distance of the space-time also takes the form (2.1). Obviously, such coordinate transformation must be linear, which forms the Poincaré group. This means the new coordinate systems should be constructed by the similar operational procedure, which give the definition of inertial coordinate systems. The transformation is generally given by

$$dx \equiv dx^\mu \gamma_\mu = \gamma^0 \Omega dX \Omega \gamma^0,$$  

(2.3)

where $\Omega = \omega^\mu \gamma_\mu$ are any given vector satisfying $\Omega^2 = 1$. By (2.3), we can easily check the following results: (I). $dx^2 = dX^2$; (II). $\Omega$ corresponds to the boost transformation if $\vec{\omega} \in \mathbb{R}^3$, and to the rotating transformation if $\vec{\omega} \in \mathbb{C}^3$.

The rotating transformation has clear geometric meaning, and is well understood. The paradoxes only involves the boost transformation, so we take

$$\Omega = \cosh \frac{\xi}{2} \gamma_0 + \sinh \frac{\xi}{2} \gamma_1$$  

(2.4)

as example to show the problem. In this case, expanding (2.3) we get the usual Lorentz transformation

$$T = t \cosh \xi + x \sinh \xi, \quad X = t \sinh \xi + x \cosh \xi, \quad Y = y, \quad Z = z,$$  

(2.5)

$$t = T \cosh \xi - X \sinh \xi, \quad x = X \cosh \xi - T \sinh \xi, \quad y = Y, \quad z = Z,$$  

(2.6)

where $O(t, x, y, z)$ stands for the new coordinate system. The two coordinate systems describe the same space-time, and each one assign a unique coordinate to every event or point of the space-time. (2.5) and (2.6) is the 1-1 mapping between two coordinate systems for each point.

Now we examine the relation between two coordinate system, and physical meaning of parameter $\xi$. For the origin of $O(t, x, y, z)$, namely, the point $x = y = z = 0$, by (2.6) we get relations

$$X = VT, \quad Y = 0, \quad Z = 0,$$  

(2.7)

$$t = \sqrt{1 - V^2} T,$$  

(2.8)

where $V = \tanh \xi$, and then we have

$$\cosh \xi = \frac{1}{\sqrt{1 - V^2}}, \quad \sinh \xi = \frac{V}{\sqrt{1 - V^2}},$$  

(2.9)
The kinematical meaning of (2.7) is obvious, that is, the origin \((0, 0, 0)\) of \(O(t, x, y, z)\) moves along \(X\) at speed \(V\) with respect to \(S(T, X, Y, Z)\). So the new coordinate system \(O(t, x, y, z)\) is a reference frame moves at relative speed \(V\) with respect to \(S(T, X, Y, Z)\).

However, the relation (2.8) is unusual and counterintuitive, which means the moving oscillator slows down with respect to the static one. On the other hand, the static clock also moves with respect to \(O(t, x, y, z)\), and then the static clock should also be slower than the clock in \(O(t, x, y, z)\). This is clock paradox, which is the origin of the twin paradox. From the derivation of (2.8) we find that two times \(T\) and \(t\) have different physical position. \(T\) is the global time in \(S(T, X, Y, Z)\), but \(t\) is just a local time in the viewpoint of the observer at the static coordinate system \(S(T, X, Y, Z)\). The hyperplane of simultaneity does not generally exist for all moving frame[15]. Now we give more discussions on this problem.

![Figure 1: The real space of the world is an evolving hyperplane. Only in one class of special coordinate system we have global simultaneity.](image)

From FIG.(1) we find that, only in one special coordinate system, the real world can define the concept of global simultaneity, otherwise it will result in contradiction. We show this by detailed discussion. Assume that the global simultaneity holds in coordinate system \(S\), that is the world evolves from \(T = 0\) to \(T = T_0\) as shown in FIG.(1). In this case, according to the Lorentz transformation (2.6), the real space evolves from the hyperplane \(AB\) to \(CD\) in \(O\), which is a tilting line without simultaneity. If we toughly define the simultaneity \(t = t_0\) in system \(O\), we find that we
actually erase the history in the region \( x < 0 \), and fill up the future in the region \( x > 0 \). Of course such treatment is absurd, because the evolution of the world is even not uniquely determined, and we can neither exactly forecast the future, nor change the history of the past. In this sense, the hyperplanes \( T = T_0 \) and \( t = t_0 \) describe two different worlds. This evolving characteristic of Lorentz manifold is essentially different from that of Riemann manifold. This distinction is similar to the characteristics of the solutions to the hyperbolic partial differential equation and elliptic one. The speciality of the temporal coordinate can also be identified from the form of the Clifford algebra (2.3).

The above descriptions briefly explain the intrinsic property of the Minkowski space-time and define the inertial reference frames, which get rid of the complicated explanation of constant light speed. The constant light speed can be derived from electrodynamics compatible with the Minkowski space-time. The equations (2.1)-(2.6) are the fundamental relations in Minkowski space-time, and all other properties can be derived from these relations. The following discussions are only based on these relations and the principle of relativity.

### B. Kinematics of a Particle

The motion of one point (i.e. a particle) is described by \((T(p), X(p), Y(p), Z(p))\) in coordinate system \(S(T, X, Y, Z)\), where \(p\) is the time-like parameter of the world line of the particle. Usually we set \(p = T\) or \(p = \tau\) (the proper time defined below). In the following discussion, we assume that in the reference frame \(S(T, X, Y, Z)\) we realistically have the global simultaneity \(T = T_0\). The velocity of the particle \(v\) with respect to frame \(S(T, X, Y, Z)\) is defined by

\[
\vec{v} = \left( \frac{dX}{dT}, \frac{dY}{dT}, \frac{dZ}{dT} \right) = \frac{d\vec{R}}{dT},
\]

where \((d\vec{R},dT)\) means the coordinate elements on the world line. We define the proper time of the particle as \(d\tau \equiv ds\). Substituting (2.10) into (2.1), we get

\[
d\tau = \sqrt{1 - v^2}dT,
\]

where \(\vec{v} = \vec{v}(T) < 1\) is an arbitrary continuous function.

In a moving reference frame \(O(t, x, y, z)\), the speed with respect to \(O(t, x, y, z)\) is defined by \(\vec{u} = \frac{d}{dt}\vec{r}\). According to the coordinate transformation rule (2.6), we get

\[
dt(T) = (\cosh \xi - v_x \sinh \xi) dT = \frac{1 - v_x V}{\sqrt{1 - V^2}}dT,
\]
then the transformation law of speed between two coordinate system becomes

\begin{align*}
  u_x &= \frac{v_x \cosh \xi - \sinh \xi}{\cosh \xi - v_x \sinh \xi} = \frac{v_x - V}{1 - v_x V}, \quad (2.13) \\
  u_y &= \frac{v_y}{\cosh \xi - v_x \sinh \xi} = \frac{\sqrt{1 - V^2}}{1 - v_x V} v_y, \quad (2.14) \\
  u_z &= \frac{v_z}{\cosh \xi - v_x \sinh \xi} = \frac{\sqrt{1 - V^2}}{1 - v_x V} v_z. \quad (2.15)
\end{align*}

By (2.12)-(2.15), we can check

\begin{equation}
  \sqrt{1 - u^2} = \frac{\sqrt{1 - v^2}}{\cosh \xi - v_x \sinh \xi} = \frac{\sqrt{1 - v^2} \sqrt{1 - V^2}}{1 - v_x V} = \sqrt{1 - v^2} \frac{dT}{dt}, \quad (2.16)
\end{equation}

or simply,

\begin{equation}
  \sqrt{1 - u^2} dt = \sqrt{1 - v^2} dT. \quad (2.17)
\end{equation}

(2.17) can be also derived from (2.1) by the definition of speed, which implies the proper time between two definite events (2.11) is an invariant quantity independent of coordinate system and the moving state of the particle. It is the intrinsic time homogenously recorded by the idealized oscillator carried by the particle. So the time dilation is independent of whether the particle is accelerating or moving at constant speed. More formally, we have

**Theorem 1.** The proper time \( \tau \) of a particle moves at speed \( \vec{u}(t) \) with respect to an inertial coordinate system \( O(t, x, y, z) \) can be generally calculated by

\begin{equation}
  \tau = \int_0^t \sqrt{1 - u^2(t)} dt. \quad (2.18)
\end{equation}

\( \tau \) is an invariant scalar independent of whether \( u(t) \) is a constant or not, and what inertial coordinate system is referred to. It is the intrinsic time of the particle.

If we take the proper time \( \tau \) as the parameter of the world line of the particle, we can define the 4-dimensional speed \( U^\mu \) and acceleration \( a^\mu \) of the particle as follows,

\begin{equation}
  U^\mu \equiv \dot{X}^\mu = \frac{1}{\sqrt{1 - v^2}} (1, \vec{v}), \quad a^\mu \equiv \dot{U}^\mu, \quad (2.19)
\end{equation}

where the over dot stands for \( \frac{d}{d\tau} \). Or in the form of Clifford algebra

\begin{equation}
  U = \dot{X}^\mu \gamma_\mu, \quad a = \dot{U}^\mu \gamma_\mu. \quad (2.20)
\end{equation}

By the invariance of \( \tau \) and (2.6), we get

**Theorem 2.** Under coordinate transformation (2.5) and (2.6), the transformation law for the 4-dimensional speed is given by

\begin{equation}
  u^0 = U^0 \cosh \xi - U^1 \sinh \xi, \quad u^1 = U^1 \cosh \xi - U^0 \sinh \xi, \quad u^2 = U^2, \quad u^3 = U^3, \quad (2.21)
\end{equation}
where $U^\mu$ and $u^\mu$ are respectively 4-dimensional speed of a particle relatively to the coordinate system $S(T, X, Y, Z)$ and $O(t, x, y, z)$. Or in the general form of Clifford algebra under (2.3)

$$u \equiv u^\mu \gamma_\mu = \gamma^0 \Omega U \Omega \gamma^0.$$ (2.22)

By the principle of relativity, (2.21) or (2.22) actually holds for any 4-dimensional (contravariant) vector such as $a^\mu$. It should be pointed out that, the coordinate system and observer are different concepts which are sometimes confused. A concrete coordinate system is the language for the observer to express his laws and results of measurement, and it is a mathematical setting of the observer for the space-time rather than the observer himself.

III. THE PARADOXES FOR PARTICLES

A. the Twins Paradox

The twins paradox involves the computation of the proper time of two particles in the Minkowski space-time. Starting with Paul Langevin in 1911, there have been numerous explanations of this paradox, all based upon there being no contradiction because there is no symmetry: only one twin has undergone acceleration and deceleration, thus differentiating the two cases. This is the standard explanation and consensus\[3\]. There are also some other explanations such as using the radar time to define the hypersurface of simultaneity for a observer, where the hypersurface depends on the future world line of the observer, so it seems quite ambiguous and complicated. We can not confuse the coordinate system with the observer himself or the object observed.

In fact, if we measure the proper time of the twins in any moving inertial coordinate system $O(t, x, y, z)$, and the result is the same as in $S(T, X, Y, Z)$, then the paradox vanishes. In mathematics, the theorems 1 and 2 actually imply the results should be the same. For clearness, in what follows, we make a detailed computation in a moving coordinate system and compare the results. Such calculation can clearly show how the subtle concepts of coordinates and principle of relativity works, and the comparison will dismiss any ambiguity and suspicion.

Assume in $S(T, X, Y, Z)$ the coordinates of the world lines of the static and the travelling twins are respectively given by

$$\bar{X}^\mu = (T, 0, 0, 0), \quad X^\mu = (T, X(T), Y(T), Z(T)).$$ (3.1)

The travelling twin takes off at $T = 0$, and return home at $T = T_0$. According to the definition of
In order to calculate the proper time in moving \(O(t, x, y, z)\), we must measure the coordinates of the world lines of the twins in \(O(t, x, y, z)\). So the relation between the coordinates in two coordinate systems is the key of the paradox. Of course the coordinates are connected by the Lorentz transformation (2.5) and (2.6). No matter whether we obtain the coordinates of world lines by the practical measurement or by transformation, the results should be the same. This is the essential implication of the principle of relativity. Substituting (3.1) into (2.6), we get the coordinates of the two twins measured by observer in \(O(t, x, y, z)\) respectively by

\[
\begin{align*}
\tilde{t}(T) &= T \cosh \xi, \quad \tilde{x}(T) = -T \sinh \xi, \quad \tilde{y} = \tilde{z} = 0, \quad T \in (0, T_0) \quad (3.4) \\
t(T) &= T \cosh \xi - X(T) \sinh \xi, \quad x(T) = X(T) \cosh \xi - T \sinh \xi, \quad y = z = 0, \quad (3.5)
\end{align*}
\]

where \(T\) acts as parameter. Then the proper time of the twins \(\tilde{\tau}\) and \(\tau\) measured in \(O(t, x, y, z)\) should respectively be the following

\[
\begin{align*}
\tilde{\tau} &= \int_0^{T_0} \sqrt{d\tilde{t}^2 - d\tilde{x}^2} = \int_0^{T_0} \sqrt{\cosh^2 \xi - \sinh^2 \xi} dT = T_0, \quad (3.6) \\
\tau &= \int_0^{T_0} \sqrt{dt^2 - dx^2} \\
&= \int_0^{T_0} \sqrt{(\cosh \xi - v(T) \sinh \xi)^2 - (v(T) \cosh \xi - \sinh \xi)^2} dT \\
&= \int_0^{T_0} \sqrt{1 - v^2(T)} dT. \quad (3.7)
\end{align*}
\]

Comparing (3.6)-(3.7) with (3.2)-(3.3), we learn that the proper time is indeed independent of coordinate system, and we still have \(\tilde{\tau} > \tau\). The clock of the travelling twin slows down. Here we can not confuse the proper time \(\tau\) with the time coordinate \(t\).

In the above integrals, the events that “The travelling twin takes off at \(T = 0\), and return home at \(T = T_0\)” is important. Only under this condition, the region of integration \((0, T_0)\) has definite meaning independent of coordinate system. This condition implies the travelling twin moving along a closed spacial curve at varying speed. This paradox has nothing to do with the curved space-time. However, if we accept the fact that only in one coordinate system (strictly speaking, it is a class of coordinate systems connected by transformation of translation and rotation) we have global simultaneity, which defines the cosmic time acting as an absolute standard, then such condition can be cancelled.
The uniqueness of simultaneity means the World is just one solution of the physical laws, so it does not contradict the principle of relativity. In what follows, we have some other examples to deal with the unique global simultaneity and covariance consistently. The resolution of the Rietdijk-Putnam-Penrose’s Andromeda paradox also needs the uniqueness of simultaneity in a world.

B. the Spaceship Paradox

The Bell’s spaceship paradox involves the computation of the synchronous distance between two particles in different coordinate system. Assume two spaceships $A$ and $B$ have the same structure, and they are at rest in the static coordinate system $S(T,X,Y,Z)$ when $T \leq 0$. They are linked by an elastic string. Each spaceship carries a standard clock adjusted to be synchronous before taking off and have the same operational schedule of engine. Each spaceship can be treated as a point in $S(T,X,Y,Z)$. The following calculations basically agree with that of Dewan & Beran and also Bell.

According to their operation, both spaceships should have the same proper acceleration (i.e. 4-dimensional form) $a$ along $X$. Then the equation of motion for any one spaceship is given by

$$\frac{d}{dT}X = v(T), \quad (3.8)$$

$$\frac{d}{d\tau}u^1 = \frac{1}{\sqrt{1 - v^2}} \frac{d}{dT} \frac{v}{\sqrt{1 - v^2}} = a(\tau), \quad (3.9)$$

in which $u^1$ is the 4-vector speed, $a(\tau)$ is a given function determined by operational schedule of engine. Theoretically, we can solve the solution $(v(T), X(T))$ from the equations with initial values. Usually the solutions can not be expressed by elementary functions, but they certainly take the following form

$$v = f(T), \quad (f(0) = 0), \quad (3.10)$$

$$X = \int_0^T v(T)dT + C_1, \quad (3.11)$$

where $C_1$ is a constant determined by the initial coordinate of the particle. Then we get the coordinate difference between $A$ and $B$ at time $T$ from (3.11) as

$$X_A(T) - X_B(T) = X_A(0) - X_B(0) \equiv L. \quad (3.12)$$

So the coordinate difference is a constant in $S(T,X,Y,Z)$. 
When the speed becomes constant, we can define a comoving coordinate system \( O(t,x,y,z) \) for spaceships \( A \) and \( B \). Then we can measure the proper distance between them, namely the rigid length \( L_0 = \overline{AB} \) with local simultaneity \( dt = 0 \). By the first equation of Lorentz transformation (2.6) and \( dt = t_A - t_B = 0 \), we get

\[
T_A - T_B = [X_A(T_A) - X_B(T_B)] \tanh \xi = [X_A(T_A) - X_B(T_B)]v
\]

and then

\[
T_A - T_B = \frac{vL}{1 - v^2}, \quad X_A(T_A) - X_B(T_B) = \frac{L}{1 - v^2}. \tag{3.14}
\]

Again by the second equation of Lorentz transformation (2.6), we get the proper distance,

\[
L_0 = (x_A - x_B)_{t_A = t_B}
\]

\[
= [X_A(T_A) - X_B(T_B)] \cosh \xi - (T_A - T_B) \sinh \xi = \frac{L}{\sqrt{1 - v^2}} \tag{3.15}
\]

(3.15) means the string stretches.

The above calculation can be regarded as a standard derivation for the Lorentz-Fitzgerald contraction of a moving rod. More formally, we have

**Theorem 3.** For a rod \( AB \) has proper length \( L_0 = \overline{AB} \), assume the rod is parallel to the \( X \)-axis of the coordinate system \( S(T,X,Y,Z) \), and moves along \( X \) at constant speed \( v \), then in \( S(T,X,Y,Z) \), the simultaneous distance (actually the coordinate difference) \( L \) between \( A \) and \( B \) is given by

\[
L = L_0 \sqrt{1 - v^2}, \tag{3.16}
\]

which is less than the proper length by a factor \( \sqrt{1 - v^2} \).

In the case of the constant acceleration \( a \), by (3.9) we have

\[
\frac{1}{2} \ln \frac{1 + v}{1 - v} + \frac{v}{1 - v^2} = 2aT. \tag{3.17}
\]

The solution \( X(T) \) is not integrable, so the trajectory with a constant acceleration may be not \( X = \sqrt{T^2 + a^{-2} + C_0} \) as suggested in some textbooks. This is caused by the invalid definition of constant acceleration.

Now we discuss a variation of the Bell’s paradox. That is, what speed relation between two spaceships can keep the proper length \( L_0 = L \). Similar to the derivation of (3.14), substituting

\[
X_A(T) = \int_0^T v_A dT + L, \quad X_B(T) = \int_0^T v_B dT \tag{3.18}
\]
into the first equation of Lorentz transformation (2.6), we have the coordinate relations under \( t_A = t_B \)

\[
T_A - T_B = \frac{[X_A(T_B) - X_B(T_B)]v}{1 - v^2} = \frac{v}{1 - v^2} \left( \int_0^{T_B} (v_A - v_B) dT + L \right),
\]

(3.19)

\[
X_A(T_A) - X_B(T_B) = \frac{1}{1 - v^2} \left( \int_0^{T_B} (v_A - v_B) dT + L \right),
\]

(3.20)

Again substituting the above relations into the second equation of (2.6), we get the proper length \( L_0 \) when two spaceships reach the same speed \( v_A = v_B = v = \tanh \xi \) as

\[
L_0 = \frac{[X_A(T_A) - X_B(T_B)] \cosh \xi - (T_A - T_B) \sinh \xi}{\sqrt{1 - v^2} \left( \int_0^{T_B} (v_A - v_B) dT + L \right)}.
\]

(3.21)

By \( L = L_0 \), we get the constraint on the speed of the two spaceships

\[
\int_0^{T_B} (v_B - v_A) dT = (1 - \sqrt{1 - v^2})L_0.
\]

(3.22)

IV. THE PARADOXES FOR RIGID BODY

A. the Ladder Paradox

It is well known, we can not define a rigid body in the framework of relativity. The consistent matter model should be 4-dimensional fields\([9, 19]\). However rigid model can be discussed in an alternative method similar to the discussion of the Bell’s spaceships. That is, we analyze the motion of the front and back points of a rod with some suitable elasticity.

The principle of relativity implies a useful manipulation which is easily overlooked, that is, the solution to a physical problem in one coordinate system can be transformed into the solution in other coordinate system via simultaneous transformation of coordinates and variables. To analyze the coordinate relation between the moving ladder and the garage, we only need to calculate the trajectories of the front and back points in the coordinate system with global simultaneity, then we can get the corresponding results in other coordinate system by Lorentz transformation, and the results should be the same obtained by practical measurement in the moving reference frame.

Assume in the coordinate system \( S(T, X, Y, Z) \) the garage is static and we have the global simultaneity \( dT = 0 \). The ladder moves in \( X \) direction at speed \( u > 0 \). We label the front and back points of the ladder respectively by \( A \) and \( B \). Then the above results for the linked Bell’s
spaceships can be employed. By (3.11), we have the coordinates of A and B in $S(T,X,Y,Z)$

$$X_A(T) = uT + L, \quad X_B(T) = uT. \tag{4.1}$$

The proper length of the ladder is $L_0 = L(1-u^2)^{-\frac{1}{2}}$. Denote the front and back door of the garage by C and D, then we have coordinates of the door

$$X_C(T) = L_1, \quad X_D(T) = 0. \tag{4.2}$$

In the point of view of the observer in $S(T,X,Y,Z)$, if $L_1 > L$, we can close the door C at $T \leq \frac{1}{u}(L_1 - L)$, and close the door D at $T \geq 0$. So the garage contains the ladder within the time

$$0 \leq T \leq \frac{1}{u}(L_1 - L), \tag{4.3}$$

no matter whether $L_1 < L_0$ or not.

Now we examine the two events of closing doors C and D in the comoving reference frame $O(t,x,y,z)$ moving at speed $V = u$. Substituting (4.1) and (4.2) into the Lorentz transformation (2.6), we get coordinates in $O(t,x,y,z)$ as follows

$$x_A(T) = \frac{L}{\sqrt{1-u^2}}, \quad x_C(T) = \frac{L_1 - uT}{\sqrt{1-u^2}},$$

$$t_A(T) = \sqrt{1-u^2}T - \frac{uL}{\sqrt{1-u^2}}, \quad t_C(T) = \frac{T - uL_1}{\sqrt{1-u^2}}, \tag{4.4}$$

and

$$x_B(T) = 0, \quad x_D(T) = \frac{-uT}{\sqrt{1-u^2}}, \tag{4.6}$$

$$t_B(T) = \sqrt{1-u^2}T, \quad t_D(T) = \frac{T}{\sqrt{1-u^2}}. \tag{4.7}$$

By (4.4), we find if $T \leq \frac{1}{u}(L_1 - L)$, we have $x_A \leq x_C$, so we can also close the door C. Again by (4.6), we find if $T \geq 0$, we have $x_B \geq x_D$, then we can close the door D. The combination of the conditions is just (4.3). However, $t_C(T) \neq t_D(T)$, which means the doors are closed at different moment in $O(t,x,y,z)$. If we want to close the doors synchronously in $O(t,x,y,z)$, namely $t_C(T_C) = t_D(T_D)$, then we get $T_C = T_D + uL_1$. Substituting it into (4.3) we find if

$$L_1 \geq \max(L, \frac{uL}{1-u^2}), \quad 0 \leq T_D < T_D + uL_1 = T_C \leq \frac{1}{u}(L_1 - L), \tag{4.8}$$

the garage contains the ladder simultaneously in $O(t,x,y,z)$.

The above example shows how the kinematics in the Minkowski space-time is logically consistent with different coordinate system. The Minkowski space-time is an evolving manifold[15,
for which only in one special inertial reference frame we have the global simultaneity. For any physical process, we can solve the dynamical and kinematical equations in one convenient coordinate system, and then transform the solution into the one in the other coordinate system by the principle of relativity. Since the transformation between coordinates and variables are 1-1 linear mapping, it certainly does not lead to contradiction.

The Supplee’s submarine paradox is also a paradox for rigid body\[21\]. But it is a dynamical problem involving the buoyant force and hydrodynamics, so the detailed calculation will be much complicated. However, we can also remove the paradox by the principle of relativity. If we get a solution in one coordinate system, we can transform the solution into the one described in other coordinate system via linear coordinate and variable transformations. The transformed solution is identical to the one solved in the new coordinate system.

\[\text{B. the Ehrenfest Paradox}\]

The Ehrenfest paradox may be the most basic phenomenon in relativity that has a long history marked by controversy and which still gets different interpretations published in peer-reviewed journals\[22\]. The paradox involves the curvilinear coordinate system and global simultaneity. Most of the following calculation and analysis were given in [15], but we introduce them here for the integrality and clearness of the illustration.

The coordinate transformation between the rotating cylindrical coordinate system (the Born chart) \(B(t, r, \phi, z)\) and the static Cartesian chart \(S(T, X, Y, Z)\) is given by\[6, 23, 24\]

\[
T = t, \quad X = r \cos(\phi + \omega t), \quad Y = r \sin(\phi + \omega t), \quad Z = z. \tag{4.9}
\]

The line element \((2.1)\) in \(B(t, r, \phi, z)\) becomes

\[
ds^2 = (1 - r^2 \omega^2)dt^2 - 2r^2 \omega dtd\phi - dr^2 - r^2 d\phi^2 - dz^2. \tag{4.10}
\]

It includes the ‘cross-terms’ \(dtd\phi\), so the time-like vector \(\partial_t\) is not orthogonal to the spatial one \(\partial_\phi\). This is the key of the paradox. In the perspective of the observer in static coordinate system \(S(T, X, Y, Z)\), the simultaneity means \(dT = 0\), which leads to \(dt = 0\), and then the spatial length element is given by

\[
dl = |ds|_{dT=0} = \sqrt{dr^2 + r^2 d\phi^2 + dz^2}. \tag{4.11}
\]

\(dl\) is identical to the length in the static cylindrical coordinate system, so the circumference at radius \(R\) is still \(2\pi R\) rather than \(2\pi R\sqrt{1 - R^2 \omega^2}\). We can not analyze the kinematics in the
curvilinear coordinate system in the point of view for inertial Cartesian coordinate system. We must use more general concepts and relations.

However, for an idealized clock attached to the rotational disc, the proper time is different from that at rest. In this case \( r|\omega| < 1 \), by \( dr = d\phi = dz = 0 \), we get the proper time element of the moving clock as

\[
d\tau = |ds|_{dl=0} = \sqrt{1 - r^2 \omega^2} dt. 
\tag{4.12}
\]

(4.12) shows the relativistic effect, the moving clock slows down. In the region \( r|\omega| \geq 1 \), the Born coordinate is still a 1-1 smooth transformation in the region \( |\phi| < \pi \), so it is still valid. However we can not fix a clock in the rotational disc \( B(t, r, \phi, z) \), because in this case \( ds \) is imaginary, which means \( dt \) becomes space-like element. This is one difference between mathematical coordinate and physical process.

In the perspective of an observer attached at \((r_0, \phi_0, z_0)\) in the rotational coordinate system, the spatial vector bases \( (\partial_r, \partial_\phi, \partial_z) \) are also orthogonal to each other, and then the line element between two local events should take the form \( \delta s^2 = \delta t^2 - \delta r^2 - g_{\phi\phi}\delta \phi^2 - \delta z^2 \), where \( \delta t \) is his local time element. By the universal expression of line element (4.10), we have

\[
ds^2 = \delta t^2 - dr^2 - \frac{r^2}{1 - r^2 \omega^2} d\phi^2 - dz^2, \tag{4.13}
\]

where \( r = r_0 \) for this specified observer and \( \delta t \) is given by

\[
\delta t = \sqrt{1 - r^2 \omega^2} dt - \frac{r^2 \omega}{\sqrt{1 - r^2 \omega^2}} d\phi. \tag{4.14}
\]

The simultaneity of this observer means \( \delta t|_{r=r_0} = 0 \). For all observer attached in the rotational coordinate system, the induced Riemannian line element in the quotient spatial manifold \((r, \phi, z)\) is generally given by

\[
dl^2 = dr^2 + \frac{r^2}{1 - r^2 \omega^2} d\phi^2 + dz^2, \quad (r|\omega| < 1), \tag{4.15}
\]

which corresponds to the so called Langevin-Landau-Lifschitz metric.

Since the 1-form (4.14) is not integrable, so we can not define a global time for all observers attached in the rotational coordinate system. The physical reason is that the underlying manifold of the Born chart is still the original Minkowski space-time. If we toughly redefine a homogenous time \( \tilde{t} \) orthogonal to the hypersurface (4.15), then we get a new curved space-time equipped with metric

\[
ds^2 = d\tilde{t}^2 - dr^2 - \frac{r^2}{1 - r^2 \omega^2} d\phi^2 - dz^2, \quad (r|\omega| < 1). \tag{4.16}
\]
Needless to say, (4.16) and (4.13) describe different space-time, because they are not globally equivalent to each other. However, they are equivalent locally.

Generally speaking, the Lorentz transformation is only a local manipulation in the tangent space-time of a manifold, and the global Lorentz transformation only holds between Cartesian charts in the Minkowski space-time[8]. In the curved space-time, by introducing the vierbein and separating the temporal component[25], the vector and transformation can also be locally expressed in the form of Clifford algebra (2.3) or (2.22) with varying $\Omega$.

V. SOME REMARKS

The above analysis shows how a subtle concept could be misinterpreted. Such situation also exists in quantum theory. The Einstein’s relativity is essentially a geometrical theory of the space-time, so it will be much understandable and effective to directly treat it as Minkowski geometry similar to the Euclidean one. The postulate of constant light velocity involves a number of physical concepts and processes without explicit definitions and explanations, which leads to unnecessary confusions for the one who is unfamiliar with the details of electrodynamics and wave equations.

Except for the requirement that the physical equations have the same form in all admissible frames of reference, some of the profound implications of the principle of relativity are usually overlooked in textbooks. One is that, if we get the solution to a physical process in one coordinate system, then we actually get the solution in all coordinate system, because the solution can be uniquely transformed from one coordinate system to the other though two fold transformation of coordinates and variables. The coordinates are just artificial labels. The coordinate transformation can never lead to contradictions if the transformation is invertible and smooth enough.

The compatible matter model in the Minkowski space-time should be fields. The classical concepts such as particles and rigid body are only mathematical idealization for convenience of treatments. In fundamental level, it is much natural to take a particle as a special state of a field[8, 9, 19]. It is also more reliable and natural to treat the space-time and fields as two different physical systems with mutual interaction.

There are two special coordinate systems for a concrete physical system. They are the Gaussian normal coordinate system with global simultaneity, and the comoving ones with respect to the particles. All other coordinate systems are mainly helpful and convenient in mathematics. From this perspective, some concepts such as the Lorentz transformation of temperature may be meaningless in physics, because we can only well define temperature in the comoving coordinate system.
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