THE HEART OF THE BANACH SPACES

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Abstract. Consider an exact category in the sense of Quillen. Assume that in this category every morphism has a kernel and that every kernel is an inflation. In their seminal 1982 paper, Beĭlinson, Bernstein and Deligne consider in this setting a t-structure on the derived category and remark that its heart can be described as a category of formal quotients. They further point out that the category of Banach spaces is an example, and that here a similar category of formal quotients was studied by Waelbroeck already in 1962. In the current article, we give a direct and rigorous construction of the latter category by considering first the monomorphism category. Then we localize with respect to a multiplicative system. Our approach gives rise to a heart-like category not only for the Banach spaces. In particular, the main results apply to categories in which the set of all kernel-cokernel pairs does not form an exact structure. Such categories arise frequently in functional analysis.

1. Introduction

In 1982, Beĭlinson, Bernstein, Deligne published the article [2], in which the general theory of t-structures on triangulated categories and their hearts is developed. In [2, Exemple 1.3.22], they consider an exact category in which every morphism has a kernel and every kernel is an inflation. They outline in this case the derived category, the canonical left t-structure and the corresponding heart. The latter is an abelian category which contains the initial category as a full subcategory. In [2, Example 1.3.24], Bernstein, Beĭlinson, Deligne state moreover “Les hypothèses de 1.3.22 sont vérifiées pour A la catégorie des espaces de Banach […] La catégorie ᵇ [le cœur] obtenue est une catégorie de “quotients formels” B/A (pour A → B une application linéaire continue injective entre espaces de Banach) […] Des quotients formels similaires ont été considérés par L. Waelbroeck (Les quotients de b-espaces, preprint, Bruxelles, 1962).”

Indeed, Waelbroeck published between 1962 and 2005 a large number of articles in which categories of formal quotients were studied. Amazingly, his summary [28] on the “category of quotient Banach spaces” appeared in the same year in which Beĭlinson, Bernstein, Deligne published [2]. Today, the monograph [26] by Schneiders is the state of the art reference for the construction of the heart associated with a so-called quasiabelian category. Schneiders [26, p. ix] and also Bühler [6, Section IV.2.6] mention the coincidence of Waelbroeck’s category of quotients with the heart in the case of Banach spaces.

In contrast to the abstract approach of Beĭlinson, Bernstein, Deligne and Schneiders, Waelbroeck considers particular categories, i.e., Banach spaces, Fréchet spaces, or bornological spaces (in the sense of Buchwalter [4] and Hogbe-Nlend [13]), and he does not follow the standard conventions from category theory all the time. His articles suggest, however, that a category of formal quotients can be constructed directly and rigorously also in cases where the ambient category does not have the properties required in [2, Exemple 1.3.22], but satisfies some variant of the open mapping theorem from functional analysis.

In fact, categories appearing in functional analysis are the main motivation for this work. In view of the disproof of Raĭkov’s conjecture, see Rump [25], there exists a considerable amount of non-quasiabelian categories arising from functional analytic problems. We refer, in particular, to Wengenroth [33]. These categories often carry a natural exact structure that can be explicitly described, see Dierolf, Sieg [8, 9] for several examples. The exact structure will, however, always be strictly smaller than the set of all kernel-cokernel pairs. In particular, it happens that there exist kernels which do not belong to the class of inflations. These categories fail
also the conditions of [2, Exemple 1.3.22]. The aim of this article is to adapt Waelbroeck’s method in a rigorous way for a class of categories that includes these examples. To this end we introduce in Section 2 the notion of a Waelbroeck category and localize a quotient of the monomorphism category in order to construct an abelian category. If the ambient category satisfies the assumptions of [2, Exemple 1.3.22], then the resulting category coincides with the heart. In Section 3 we show that our formal theory applies to the categories of Banach and Fréchet spaces, but also to the category of LB-spaces, which is a type of category not considered by Bernstein, Beilinson, Deligne or Schneiders. Our approach makes Waelbroeck’s construction for Banach and Fréchet spaces formal, uses the standard notions of localization theory and provides a calculus of fractions for the corresponding categories of formal quotients. For all other Waelbroeck categories we obtain the same results. In Section 4, we comment on Waelbroeck’s original papers and prove additional results for Banach, Fréchet and LB-spaces.

We point out that our notion of a Waelbroeck category a priori does not cover all categories considered by Waelbroeck [30]. In particular, for the category of bornological vector spaces, which in [30] appears to be the most general framework for Waelbroeck’s construction, it is not clear that it is a Waelbroeck category in our sense. However, as the latter category is quasiabelian, see Prosmans, Schneiders [24], here the classical approach via t-structures can be applied. At the end of Section 3 we give the details for the case of bornological vector spaces and, in addition, also for complete bornological vector spaces. It turns out that the morphisms that Waelbroeck makes invertible in both cases are precisely those that we make invertible in the setting of a Waelbroeck category. From this perspective the statements of our main results are also valid for the category of (complete) bornological vector spaces.

Our notation in this paper follows the usual practice. We use the words “map” and “morphism” synonymously although not all categories under consideration are concrete. Furthermore, we use the expressions ker, cok, im and coin sometimes for the corresponding object and sometimes for the corresponding map. Finally, we assume tacitly that locally convex spaces are Hausdorff. We refer to Meise, Vogt [18] for unexplained notation from functional analysis and to Mac Lane [17] and Weibel [32] for unexplained notation from category theory. For the basics on categories of locally convex spaces we refer to Prosmans [23]. A discussion of the different types of non-abelian categories can be found in [16] by Kopylov and the author. Concerning the localization of categories we follow Gabriel, Zisman [10, Chapter I.2.2] but refer also to Miličić [21] for a very detailed exposition. Our notion of a “pulation” we adapted from Adámek, Herrlich, Strecker [1, p. 205], other naming conventions (Doolittle diagram, push-me pull-you diagram or bicartesian square) are mentioned in the literature.

Before we start now, let us point out that monomorphism categories of abelian categories are under investigation in representation theory, see, e.g., Chen [7], Zhang [34], Gao, Psaroudakis [11]. In the abelian case, the monomorphism category is exact when conflations are defined degree-wise [7, Lemma 2.1]. The author of the current paper proved that the same is true if the ambient category is at least karoubian [31]. The natural question about what happens if only “admissible pulations” are made invertible, seems to be open and will be investigated in a forthcoming work.

2. THE MONOMORPHISM CATEGORY OF AN ADDITIVE CATEGORY

Throughout the whole paper we denote by $\mathcal{A}$ an additive category. By $\text{Mon}\mathcal{A}$ we denote the category whose objects are the monomorphisms $f: X' \to X$ of $\mathcal{A}$. Given two objects $f: X' \to X$ and $g: Y' \to Y$, then a morphism $f \to g$ in $\text{Mon}\mathcal{A}$ is a pair $(\alpha', \alpha)$ of morphisms $\alpha': X' \to Y'$ and $\alpha: X \to Y$ in $\mathcal{A}$ such that the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha'} & Y' \\
\downarrow f & & \downarrow g \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]

2
commutes. Mon\(A\) is an additive category. Considering the objects of Mon\(A\) as formal quotients, and the morphisms as maps between such quotients, suggests to identify all those morphisms \((\alpha', \alpha): f \to g\) where \(\alpha\) factors through \(g\) with zero. For this purpose we put

\[
J(f, g) := \{(\alpha', \alpha): f \to g; \exists \rho: X \to Y': \alpha = g \circ \rho\}
\]

and \(\beta := \bigcup_{f, g \in \text{Mon}A} J(f, g)\),

which constitutes an ideal in Mon\(A\) as the following lemma shows. For the convenience of the reader we include its short proof and add an equivalent description of \(J(f, g)\) for later use.

**Lemma 1.** The collection \(\beta\) of morphisms is an ideal in Mon\(A\).

**Proof.** As composition is bilinear, we get that \(J(f, g) \subseteq \text{Hom}_{\text{Mon}A}(f, g)\) is a subgroup for all \(f, g\) in Mon\(A\). Let \((\alpha', \alpha) \in J(f, g)\) and \((\beta', \beta) \in \text{Hom}_{\text{Mon}A}(g, h)\). Select \(\rho\) such that \(\alpha = g \circ \rho\) and put \(\eta := \beta' \circ \rho\). Then \(h \circ \eta = h \circ \beta' \circ \rho = \beta \circ g \circ \rho = \beta \circ \alpha\), which shows \((\beta' \circ \alpha', \beta \circ \alpha) \in J(f, g)\). Let \((\alpha', \alpha) \in \text{Hom}_{\text{Mon}A}(f, g)\) and \((\beta', \beta) \in J(g, h)\). Then we may select \(\rho\) such that \(\beta = h \circ \rho\) and put \(\eta := \rho \circ \alpha\). It follows that \(h \circ \eta = h \circ \rho \circ \alpha = \beta \circ \alpha = \beta \circ \alpha\), which shows \((\beta' \circ \alpha', \beta \circ \alpha) \in J(f, g)\). \(\square\)

**Lemma 2.** For monomorphisms \(f: X' \to X\) and \(g: Y' \to Y\) in \(A\) we have

\[
J(f, g) = \{(\alpha', \alpha): f \to g; \exists \rho: X \to Y'; \alpha = g \circ \rho & \alpha' = \rho \circ f\}.
\]

**Proof.** Let \((\alpha', \alpha) \in J(f, g)\) be given. That is, the solid part of the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha'} & Y' \\
\downarrow{f} & \ldots & \downarrow{g} \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]

commutes and \(\alpha = g \circ \rho\) holds by definition. It follows \(g \circ \rho \circ f = \alpha \circ f = g \circ \alpha'\), which implies \(\rho \circ f = \alpha'\) as \(g\) is a monomorphism. \(\square\)

If we flip the diagram (1) along the dashed arrow and regard it as a morphism of complexes

\[
\begin{array}{ccccc}
\cdots & \longrightarrow & X' & \xrightarrow{f} & X & \longrightarrow & \cdots \\
\downarrow{\alpha} & \ldots & \downarrow{\alpha'} & \ldots & \downarrow{\alpha} & \ldots & \downarrow{\alpha'} \\
\cdots & \longrightarrow & Y' & \xrightarrow{g} & Y & \longrightarrow & \cdots \\
\end{array}
\]

then the above shows that the chain map \((\alpha', \alpha)\) between the two complexes is null-homotopic in the category of chain complexes. We now define the category

\[
h\text{Mon}A := (\text{Mon}A)/\beta
\]

to be the quotient with respect to the ideal of “null-homotopic” morphisms. The objects of \(h\text{Mon}A\) are those of \(\text{Mon}A\), and for objects \(f\) and \(g\) we have

\[
\text{Hom}_{h\text{Mon}A}(f, g) = \text{Hom}_{\text{Mon}A}(f, g)/J(f, g),
\]

which defines again an additive category. Our next aim is to show that in \(h\text{Mon}A\) every morphism has a kernel and a cokernel, provided that in \(A\) every morphism has a kernel and a range in the following sense.

**Definition 3.** Let \(f: X \to Y\) be a morphism in \(A\). A monomorphism \(r: R \to Y\) is called a range of \(f\), if there exists a morphism \(q: X \to R\), such that \(f = r \circ q\), and such that for all morphisms \(g: Y \to Z\), \(h: X \to Z\) and every monomorphism \(s: S \to Z\) with \(g \circ f = s \circ h\), there exists a unique \(\tilde{g}: R \to S\) such that \(g \circ r = s \circ \tilde{g}\).

Let \(r: R \to Y\) be a range of \(f: X \to Y\) and let \(q, g, s\) and \(g'\) be as in Definition 3. Then we have \(s \circ g' \circ q = g' \circ q = g \circ f = s \circ h\), which implies \(g' \circ q = h\) as \(s\) is a monomorphism.
Consequently, the diagram

```
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{r} & & \downarrow{s} \\
X & \xrightarrow{q} & R \\
\downarrow{h} & & \downarrow{s'} \\
\end{array}
```

that visualizes Definition 3 is commutative. If we take the identity for $g: Y \to Y$ above, we see that $r: R \to Y$ is unique up to a unique isomorphism. We observe that $f$ and $r$ in turn determine $q$ uniquely as $r$ is a monomorphism by definition. In the sequel, we use the notation $r_f: R_f \to Y$ and $q_f: X \to R_f$ for the range of $f$ and the induced map. For a more detailed discussion of the range property we refer to Section 3.

**Proposition 4.** Assume that every morphism in $\mathcal{A}$ has a kernel and a range. Then every morphism in $h\text{Mon}\mathcal{A}$ has a kernel and a cokernel. That is, $\mathcal{A}$ is preabelian.

*Proof.* (i) Let $(\alpha', \alpha): f \to g$ be a morphism. As $\mathcal{A}$ has all kernels, we may form the pullback

\[
\begin{array}{ccc}
P & \xrightarrow{p_1} & X \\
\downarrow{p_2} & & \downarrow{\alpha} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

in which $p_1$ is a monomorphism as $g$ is so. Since $g \circ \alpha' = \alpha \circ f$ holds, the pullback property yields a unique map $h: X' \to P$ making the following diagram commutative.

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow{h} & & \downarrow{\alpha} \\
P & \xrightarrow{p_1} & X \\
\downarrow{p_2} & & \downarrow{\alpha} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

The map $h$ is a monomorphism as $f = p_1 \circ h$ is a monomorphism. We obtain the morphism $(\text{id}_{X'}, p_1): h \to f$ in $h\text{Mon}\mathcal{A}$ and claim that it is the kernel of $(\alpha', \alpha)$. Firstly, we consider the composition

\[
\begin{array}{ccc}
X' & \xrightarrow{\text{id}_{X'}} & X' \\
\downarrow{h} & & \downarrow{f} \\
P & \xrightarrow{p_1} & X \\
\downarrow{p_1} & & \downarrow{\alpha} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

and show that $(\alpha', \alpha) \circ (\text{id}_{X'}, \beta) = 0$ in $h\text{Mon}\mathcal{A}$ holds. Indeed, we have $g \circ p_2 = \alpha \circ p_1$ and thus $(\alpha' \circ \text{id}_{X'}, \alpha \circ \beta) \in J(h, g)$. Let now $(\beta', \beta): j \to f$ be a morphism with $(\alpha', \alpha) \circ (\beta', \beta) = 0$ in $h\text{Mon}\mathcal{A}$. That is, we have a map $\rho: Z \to Y'$ with $g \circ \rho = \alpha \circ \beta$. In view of the latter, we may use again the pullback property

\[
\begin{array}{ccc}
Z & \xrightarrow{\beta} & X \\
\downarrow{\gamma} & & \downarrow{\alpha} \\
P & \xrightarrow{p_1} & X \\
\downarrow{p_2} & & \downarrow{\alpha} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]
to obtain the map $\gamma: Z \to P$ with $p_1 \circ \gamma = \beta$ and $p_2 \circ \gamma = \rho$. We get the diagram

$$
\begin{array}{c}
\xymatrix{
Z' \ar[rr]^{eta'} \ar[dd]^j & & Y' \\
X' \ar[rr]^f & & P \\
Z \ar[rr]_\gamma \ar[rr]^{\beta} \ar[u]^h & & X, \ar[u]_p \ar[u]_{p_1}
}
\end{array}
$$

where $(\beta', \gamma): j \to h$ represents a morphism in $\text{hMon}A$. Indeed, we compute

$$p_1 \circ \gamma \circ j = \beta \circ j = f \circ \beta' = f \circ \text{id}_{X'} \circ \beta' = p_1 \circ h \circ \beta',
$$

which yields $\gamma \circ j = h \circ \beta'$ as $p_1$ is a monomorphism. In view of (2), we have $(\text{id}_{X'}, p_1) \circ (\beta', \gamma) = (\beta', \beta)$ even componentwise and thus, in particular, in $\text{hMon}A$. Finally, let $(\delta', \delta): j \to h$ be another morphism with $(\text{id}_{X'}, p_1) \circ (\delta', \delta) = (\beta', \beta)$. Then

$$0 = (\text{id}_{X'}, p_1) \circ (\beta', \gamma) - (\text{id}_{X'}, p_1) \circ (\delta', \delta) = (\beta' - \delta', p_1 \circ (\gamma - \delta))
$$

holds in $\text{hMon}A$, i.e., the composition

$$
\begin{array}{c}
\xymatrix{
Z' \ar[r]^{\beta' - \delta'} \ar[d]^j & X' \ar[d]^h & X' \ar[d]^f \\
Z \ar[r]_{\gamma - \delta} & P & X
}
\end{array}
$$

is the zero morphism in $\text{hMon}A$. That is, there exists $\rho: Z \to X'$ with $f \circ \rho = p_1 \circ (\gamma - \delta)$. This allows to compute

$$p_1 \circ (\gamma - \delta) = f \circ \rho = f \circ \text{id}_{X'} \circ \rho = p_1 \circ h \circ \rho,
$$

which implies $\gamma - \delta = h \circ \rho$ as $p_1$ is a monomorphism. Looking again at the last diagram, we see that the latter means $(\delta', \delta) = (\beta', \gamma)$ in $\text{hMon}A$. We thus showed that $(\text{id}_{X'}, \beta): h \to f$ is a kernel of $(\alpha', \alpha): f \to g$.

(ii) Let $(\alpha', \alpha): f \to g$ be a morphism. We form the range of $[\alpha \ g]$ and get the factorization

$$
\begin{array}{c}
\xymatrix{
& Y' \ar[dd]^i & \\
X \ar[ur]^{[\alpha \ g]} & & \ar[dl]^h R
}
\end{array}
$$

with a monomorphism $i$. We denote by $i_1: X \to X \oplus Y'$ and by $i_2: Y' \to X \oplus Y'$ the canonical maps and claim that $(q \circ i_2, \text{id}_Y): g \to i$ is a cokernel of $(\alpha', \alpha): f \to g$. We consider the commutative diagram which represents the composition $(q \circ i_2, \text{id}_Y) \circ (\alpha', \alpha)$.

$$
\begin{array}{c}
\xymatrix{
X' \ar[r]^{\alpha'} & Y' \ar[r]^{q \circ i_2} & R \\
X \ar[u]^{f} \ar[r]_{\alpha} \ar[ur] & Y \ar[u]^{h} \ar[r]_{\text{id}_Y} & Y
}
\end{array}
$$

This composition is zero in $\text{hMon}A$, since $q \circ i_1: X \to R$ satisfies $i \circ q \circ i_1 = [\alpha \ g] \circ i_1 = \alpha = \text{id}_Y \circ \alpha$, and thus $(q \circ i_2 \circ \alpha', \text{id}_Y \circ \alpha) \in J(f, i)$ holds. This implies $q \circ i_1 \circ f = q \circ i_2 \circ \alpha'$ as $i$ is a monomorphism. Let now $(\beta', \beta): g \to h$ be another morphism with $(\beta', \beta) \circ (\alpha', \alpha) = 0$ in
Therefore, we have a map \( \rho: X \to Z' \) with \( h \circ \rho = \beta \circ \alpha \). As 
\[
\beta \circ [\alpha \, g] = \beta \circ \alpha \circ i_1 + \beta \circ g \circ i_2 = h \circ \rho \circ i_1 + h \circ \beta' \circ i_2 = h \circ [\rho \beta']
\]
holds, the range property gives \( \gamma': R \to Z' \) which makes the diagram 
\[
\begin{array}{ccc}
X \oplus Y' & \xrightarrow{\gamma} & R \\
\downarrow{\beta} & & \downarrow{\gamma'} \\
Y & \xrightarrow{\beta} & Z
\end{array}
\]
commutative. This shows that \( (\gamma', \beta): i \to h \) is a morphism in \( \text{hMon} \). We consider the diagram 
\[
\begin{array}{ccc}
X' & \xrightarrow{i'} & Y' \\
\downarrow{f} & & \downarrow{g} \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]
and observe that we have \( \gamma' \circ q \circ i_2 = [\rho \beta'] \circ i_2 = \beta' \) by (3). Thus \( (\gamma', \beta) \circ (q \circ i_2, \text{id}_Y) = (\beta', \beta) \) holds componentwise and, therefore, in particular, in \( \text{hMon} \). Finally, let \( (\delta', \delta): i \to h \) be another map with \( (\delta', \delta) \circ (q \circ i_2, \text{id}_Y) = (\beta', \beta) \) in \( \text{hMon} \). That is, 
\[
0 = (\gamma', \beta) \circ (q \circ i_2, \text{id}_Y) - (\delta', \delta) \circ (q \circ i_2, \text{id}_Y) = ((\gamma' - \delta') \circ q \circ i_2, \beta - \delta)
\]
holds in \( \text{hMon} \). Whence, the composition 
\[
\begin{array}{ccc}
Y' & \xrightarrow{\gamma_1} & R \\
\downarrow{g} & & \downarrow{\beta - \delta} \\
Y & \xrightarrow{\text{id}_Y} & Z
\end{array}
\]
is the zero morphism in \( \text{hMon} \). Thus there exists \( \sigma: Y \to Z' \) such that \( h \circ \sigma = (\beta - \delta) \circ \text{id}_Y \) is valid. Looking at the diagram, the latter means that \( (\gamma' - \delta', \beta - \delta): i \to h \) is zero in \( \text{hMon} \). Consequently, \( (\gamma', \beta) = (\delta', \delta) \) holds in \( \text{hMon} \). This establishes that \( (q \circ i_2, \text{id}_Y): g \to i \) is a cokernel of \( (\alpha', \alpha): f \to g \).

Let \( \mathcal{A} \) be a category which has all kernels and all ranges. Let \( f: X' \to X \) and \( g: Y' \to Y \) be objects in \( \text{hMon} \mathcal{A} \) and let \( (\alpha', \alpha): f \to g \) be a morphism in \( \text{hMon} \mathcal{A} \). By Proposition 4, it would by now already be possible to get the canonical factorization 
\[
\begin{array}{ccc}
\ker(\alpha', \alpha) & \xrightarrow{f} & \text{cok}(\alpha', \alpha) \\
\downarrow & & \downarrow \\
\text{cok}(k', k) & \to & \ker(c', c)
\end{array}
\]
and to determine the dashed morphism explicitly. We postpone this to the proof of Theorem 10 and refer, in particular, to the diagram (20). As a preview, and to motivate our next definition,
we mention now that the square corresponding to the latter morphism will turn out to be always
a pullback and, at the same time, a pushout square.

**Definition 5.** Let \( f: X' \to X, g: Y' \to Y \) be monomorphisms in \( A \). We say that a morphism \((\alpha', \alpha): f \to g\) in \( h\text{Mon} \ A \) is a pulation if the corresponding diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha'} & Y' \\
\downarrow{f} & & \downarrow{g} \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]

(5)

is a pulation square in \( A \), i.e., if it is simultaneously a pullback and a pushout square.

The following lemma shows that the notion introduced above is well-defined, i.e., that either
all representatives of a morphism in \( h\text{Mon} \ A \) are pulation squares, or none of them are so.

**Lemma 6.** Let \( f: X' \to X \) and \( g: Y' \to Y \) be monomorphisms in \( A \). Let \((\alpha', \alpha), (\beta', \beta): f \to g\) be morphisms in \( \text{Mon} \ A \) with \((\alpha', \alpha) - (\beta', \beta) \in J(f, g)\). If \((\alpha', \alpha)\) is a pulation, then so is \((\beta', \beta)\).

**Proof.** We first show that \((\beta', \beta)\) represents a pullback. For this purpose, let \( \nu: P \to X \) and \( \mu: P \to Y \) satisfy \( g \circ \mu = \beta \circ \nu \). We consider the solid part of the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\mu} & X' \\
\downarrow{\nu} & & \downarrow{f} \\
X & \xrightarrow{\beta} & Y
\end{array}
\]

(6)

and have to find the dashed map. By our assumptions, we may select \( \rho: X \to Y' \) such that \( \alpha - \beta = g \circ \rho \) holds. By Lemma 2 we get that also \( \alpha' - \beta' = \rho \circ f \) holds. We compute \( \alpha \circ \nu = (g \circ \rho + \beta) \circ \nu = g \circ \rho \circ \nu + \beta \circ \nu = g \circ \rho \circ \nu + g \circ \mu = g \circ (\rho \circ \nu + \mu) \). From the pullback property of (5) we get a map \( \sigma: P \to X' \) such that

\[
\begin{array}{ccc}
P & \xrightarrow{\rho \circ \nu + \mu} & X' \\
\downarrow{\nu} & & \downarrow{f} \\
X & \xrightarrow{\pul{P}} & Y
\end{array}
\]

commutes. That is, \( f \circ \sigma = \nu \) and \( \alpha' \circ \sigma = \rho \circ \nu + \mu \) hold. Combining both equations, we get \( \mu = \alpha' \circ \sigma - \rho \circ \nu = (\rho \circ f + \beta') \circ \sigma - \rho \circ \nu = \rho \circ f \circ \sigma + \beta' \circ \sigma - \rho \circ \nu = \beta' \circ \sigma \). This shows that \( \sigma \) as the dashed map in (6) makes (6) commutative. Moreover, \( \sigma \) is unique with this property as \( f \) is a monomorphism.

It remains to show that \((\beta', \beta)\) represents a pushout. Let thus \( \psi: X \to Q \) and \( \phi: Y' \to Q \) be
given with \( \psi \circ f = \phi \circ \beta' \). That is, the solid part of the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\beta'} & Y' \\
\downarrow{f} & & \downarrow{g} \\
X & \xrightarrow{\phi} & Y
\end{array}
\]

(7)
commutes. Again we need to find the dashed map. We compute \( \varphi \circ \alpha' = \varphi \circ (\rho \circ f + \beta') = \varphi \circ \rho \circ f + \varphi \circ \beta' = \varphi \circ \rho \circ f + \psi \circ f = (\varphi \circ \rho + \psi) \circ f \). From the pushout property we obtain \( \eta: Y \to Q \) such that

\[
\begin{array}{c}
X' \xrightarrow{\alpha'} Y' \\
\downarrow f \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \subsection*{Lemma 8. Let \( A \) be a Waelbroeck category.}
If

\[
\begin{array}{ccc}
Q & \xrightarrow{\eta} & Z \\
\downarrow{\mu} & & \downarrow{\alpha} \\
P \xrightarrow{p_1} & & \xrightarrow{f} Z \\
\downarrow{p_2} & & \downarrow{g} \\
X & \xrightarrow{\rho} & Y
\end{array}
\]

is a pullback diagram in \(A\) with a monomorphism \(g\), then there exists a unique isomorphism

\[
FP \cong [F\alpha]^{-1}([Fg](FX)) \text{ such that } [F\eta](q) = [F\nu](q), \ [Fp_1](z) = z
\]

and \([Fp_2](z) = [Fg]^{-1}([F\alpha](z))\).

(ii) Let \(f: X \to Y\) be a morphism in \(A\) and \(r_f: R_f \to Y\) the range of \(f\). Let \(q_f: X \to R_f\) be the morphism that satisfies \(f = r_f \circ q_f\). Then we have a unique isomorphism

\[
FR_f \cong [Ff](FX) \text{ such that } [Fr_f](y) = y \text{ and } [Fq_f](x) = [Ff](x).
\]

Let \(g: Y \to Z\), \(h: X \to S\) and \(s: S \to Z\) be given with \(g \circ f = s \circ h\). Let \(g': R_f \to S\) be the map that exists by the range property. Then we have

\[
[Fg'](y) = [Fg]^{-1}([Fg \circ Fr_f](y)).
\]

(iii) The diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha'} & Y' \\
\downarrow{f} & & \downarrow{g} \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]

is a pullback square in \(A\) if and only if the equalities

\[
[F\alpha]^{-1}([Fg](FY')) = [Ff](FX') \text{ and } FY = [F\alpha](FX) + [Fg](FY')
\]

both hold. The inclusion “\(\subseteq\)” holds automatically in both equations.

**Proof.** Firstly, we observe that \(F\) preserves monomorphisms. Indeed, if \(f: X \to Y\) is a monomorphism in \(A\), then its kernel is the morphism \(0 \to X\). As \(F\) preserves kernels by (W1) and is additive, it follows that the kernel of \(Ff\) in \(Ab\) is \(0 \to FX\). Thus \(Ff\) is injective.

(i) By the above, the formulas for \(FP\) and \(Fp_2\) are well-defined. Furthermore, we know that \(P = \ker(\alpha - g): Z \oplus X \to Y\) holds and that \(p_1 = q_1 \circ k\), \(p_2 = q_2 \circ k\) is valid. Here \(k: P \to Z \oplus X\) is the kernel mapping and \(q_1: Z \oplus X \to Z\), \(q_2: Z \oplus X \to X\) are the canonical maps. We derive

\[
FP = \ker(F\alpha - Fg): FZ \oplus FX \to FY
\]

and show that \(k: K \to FZ \oplus FX\) with \(K = [F\alpha]^{-1}([(Fg)(FX))]\) and \(k(z) = (z, [Fg]^{-1}([F\alpha](z)))\) is a kernel of \([F\alpha - Fg]\). Firstly,

\[
([F\alpha - Fg] \circ k)(z) = [F\alpha](z) - [Fg][Fg]^{-1}([F\alpha](z)) = [F\alpha](z) - [F\alpha](z) = 0
\]

for \(z \in K\). Let \(k': K' \to FZ \oplus FX\) be a morphism with \([F\alpha - Fg] \circ k' = 0\). For \(a \in K'\) we put \((z, x) = k'(a)\) and compute

\[
0 = ([F\alpha - Fg] \circ k')(a) = [F\alpha](z) - [Fg](x).
\]
Thus $[F\alpha](z) = [Fg](x)$ holds, which means that $z \in [F\alpha]^{-1}([Fg](FX)) = K$. As $Fg$ is injective, it follows further that $x = [Fg]^{-1}([F\alpha](z))$ and consequently
\[ k'(a) = (z, x) = (z, [Fg]^{-1}([F\alpha](z))) = k(z) \]
is valid. We thus get that the map $Fq_1 \circ k': K' \to K$, which in the above notation maps $a$ to $z$, satisfies $k \circ (Fq_1 \circ k') = k'$. As $k$ is injective there can only be one such map. With the comments made at the beginning, this shows (8). Equation (9) is an immediate consequence.

(ii) By (W2), $Fr_f: FR_f \to FY$ is a range of $FF: FX \to FY$ in $Ab$. Thus there exists $q_f: FX \to FR_f$ with $Fr_f \circ q_f = FF$. As $Fr_f \circ q_f = FF$ holds, it follows that $q_f = Fq$ since $FR_f$ is injective. Taking $Z = Y$ and $g = \text{id}_Y$ in the definition, we get that $Fr_f: FR_f \to FY$ satisfies the following universal property. We have $FF = Fr_f \circ q_f$ with $Fr_f$ being a monomorphism, and given any other decomposition $FF = s \circ h$ with $h: FX \to S$ and a monomorphism $s: S \to FY$, there exists a unique $m: FR_f \to S$ with $s \circ m = Fr_f$. The universal property determines $FR_f$ uniquely and so it is enough to check that
\[ R' := [FF](FX) \text{ with } r': R' \to FY, r'(y) := y \text{ and } q': FX \to R', q'(x) := [FF](x) \]
satisfies the latter. We have $FF = r' \circ q'$ and $r'$ is a monomorphism. Let $h$ and $s$ as above be given and take $y \in R'$. That is, $y \in [FF](FX)$ holds. We can select $x \in FX$ with $[FF](x) = y$ and consider $h(x)$. If $x' \in FX$ satisfies $[FF](x') = y$, then
\[ s(h(x')) = [FF](x') = y = [FF](x) = s(h(x)) \]
holds and $h(x') = h(x)$ follows as $s$ is injective. Thus $m: R' \to S$, $y \mapsto h(x)$, defines a map which satisfies $s \circ m = r'$ in view of the computation above. This shows (10).

Let now $g: Y \to Z$, $h: X \to S$ and $s: S \to Z$ be morphisms in $A$ with $g \circ f = s \circ h$ and let $g': R_f \to S$ be the map that exists by the range property, i.e., $s \circ g' = g \circ r_f$ holds. Therefore, $Fs \circ Fg' = Fg \circ Fr_f$ holds and for $y \in FR_f$ we get
\[ (Fg \circ Fr_f)(y) = (Fs \circ Fg')(y) = [Fs][[Fg']](y). \]
As $Fs$ is injective, it follows that $[Fg'](y) = [Fs]^{-1}((Fg \circ Fr_f)(y))$ and we are done.

(iii) The square (12) is a pullulation square in $A$ if and only if
\[ X' \xrightarrow{\alpha'} X \oplus Y' \xrightarrow{[\alpha \, g]} Y \]
is a kernel-cokernel pair in $A$. In view of (W3), the latter holds if and only if
\[ 0 \rightarrow FX' \xrightarrow{[r_f]} FX \oplus FY' \xrightarrow{[F\alpha \\ Fg]} FY \rightarrow 0 \] (14)
is a short exact sequence in $Ab$. We show that this holds if and only if the equalities (13) are valid.

$\Rightarrow$ Let $x \in [FF](FX')$. That is, we find $x' \in FX'$ with $[FF](x') = x$ and thus
\[ [F\alpha](x) = [F\alpha][[FF](x')] = [Fg][[F\alpha']](x') \in [Fg](FY'). \]
Consequently, $x \in [F\alpha]^{-1}([Fg](FY'))$. For the other direction, let $x \in [F\alpha]^{-1}([Fg](FY'))$. That is, $[F\alpha](x) \in [Fg](FY')$ holds. We select $y' \in FY'$ such that $[F\alpha](x) = [Fg](y')$ is valid. We get
\[ (x, y') \in [F\alpha][Fg]^{-1}(0) = [[Ff']][FX'] \]
and may select $x' \in FX'$ such that $[FF](x') = x$ holds. Therefore, we have $x \in [FF](FX')$. The second equation follows immediately, since $[F\alpha][Fg]$ is surjective.

$\Leftarrow$ We show that (14) is a short exact sequence. Since $FF$ is injective by the remark at the beginning of this proof, we obtain that $[[Ff']]$ is injective. As $[F\alpha](FX) + [Fg](FY') = FY$
holds, \([Fa \cdot Fg]\) is surjective. From the commutativity of (12) it follows that
\[ [\alpha \cdot g][\overset{-f}{\alpha'}] = \alpha \circ (-f) + g \circ \alpha' = 0 \]
and thus also \([Fa \cdot Fg][\overset{-FF}{\alpha'}] = 0\). Consequently,
\[ [\overset{-FF}{\alpha'}](FX') \subseteq [Fa \cdot Fg]^{-1}(\{0\}). \]
Let \((x', y') \in [Fa \cdot Fg]^{-1}(0)\) be given, i.e., \([Fa]([-x]) = [Fg](y') \in Fg(FY')\). Then \(-x \in [Fa]^{-1}([Fg](FY')) = [Ff](FX')\) and we select \(x' \in FX'\) with \([Ff](x') = -x\). Now we use the commutativity of (12) again to obtain
\[ (Fg \circ F\alpha')(x') = (F\alpha \circ Ff)(x') = [F\alpha]([-x]) = [Fg](y'), \]
which implies \([F\alpha'](x') = y'\), as \(Fg\) is injective. Thus, \([\overset{-FF}{\alpha'}](x') = (-Ff(x'), [F\alpha'](x')) = (x, y')\) and we are done.

\[ \square \]

Now we are ready to prove that the pulations form a multiplicative system.

**Proposition 9.** Let \(A\) be a Waelbroeck category. Then \(\Sigma = \{(\alpha', \alpha) : f \rightarrow g\text{ is a pulation}\}\) is a multiplicative system in \(hMonA\).

**Proof.** (MS1) As the diagram
\[
\begin{array}{c}
X' \xrightarrow{id_{X'}} X' \\
\downarrow f \quad \downarrow f \\
X \xrightarrow{id_X} X
\end{array}
\]
is a pulation square, we have \(id_f \in \Sigma\) for every object \(f : X' \rightarrow X\) of \(hMonA\). Moreover, if in the diagram
\[
\begin{array}{c}
X' \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z' \\
\downarrow f \quad \downarrow g \\
X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z
\end{array}
\]
both squares are pulations, then also the outer rectangle is a pulation, see Kelly [15, Lemma 5.1(a)] resp. the dual statement.

(MS2) Let \(f : X' \rightarrow X\), \(g : Y' \rightarrow Y\) and \(h : A' \rightarrow A\) be objects in \(hMonA\). Let \((\tau', \tau) : f \rightarrow g\) be in \(\Sigma\) and let \((\alpha', \alpha) : h \rightarrow g\) be an arbitrary morphism in \(hMonA\). As \(A\) has kernels, we may form the pullback
\[
\begin{array}{c}
S \xrightarrow{s_1} A \oplus X \\
\downarrow s_2 \\
Y' \xrightarrow{g} Y
\end{array}
\]
and put \(S' := A' \oplus X'\). We define \(\gamma : S \rightarrow X\) to be the composition \(\gamma := p_2 \circ s_1\), where \(p_2 : A \oplus X \rightarrow X\) is the canonical map. Furthermore, we define \(\sigma : S \rightarrow A\) to be the composition \(\sigma := p_1 \circ s_1\), where \(p_1 : A \oplus X \rightarrow A\) is the canonical map. Let \(\gamma' : S' \rightarrow X'\) and \(\sigma' : S' \rightarrow A'\) be the canonical maps. Compute
\[
[\alpha - \tau] \circ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = [\alpha \circ h - \tau \circ f] = [g \circ \alpha' - g \circ \tau'] = g \circ [\alpha' - \tau']
\]
and use the pullback property

\[
\begin{array}{c}
S' \quad \xrightarrow{\begin{bmatrix} h & 0 \\ 0 & f \end{bmatrix}} \\
\downarrow{k} \\
S \quad \xrightarrow{s_1} A \oplus X \\
\downarrow{s_2} \\
Y' \quad \xrightarrow{g} Y \\
\end{array}
\]

(15)
to get a map \( k: S' \to S \), which makes the above diagram commutative. We claim that \( k \) is a monomorphism. Let \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}: Z \to S' \) be a morphism with \( k \circ z = 0 \). That is,

\[
0 = s_1 \circ k \circ z = \begin{bmatrix} h & 0 \\ 0 & f \end{bmatrix} \circ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h \circ z_1 \\ f \circ z_2 \end{bmatrix}
\]

is valid, which yields \( z_1 = 0 \) and \( z_2 = 0 \), as \( h \) and \( f \) are monomorphisms. Consequently, \( z = 0 \) and \( k \) is a monomorphism. From the commutativity of (15) we derive, in addition, that

\[
\sigma \circ k = p_1 \circ s_1 \circ k = p_1 \circ \begin{bmatrix} h & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} h \circ 0 \end{bmatrix} = h \circ \sigma'
\]

and

\[
\gamma \circ k = p_2 \circ s_1 \circ k = p_2 \circ \begin{bmatrix} h & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} f \circ 0 \end{bmatrix} = f \circ \gamma'
\]

hold. Thus we get the diagram

\[
\begin{array}{c}
S' \quad \xrightarrow{\begin{bmatrix} h & 0 \\ 0 & f \end{bmatrix}} \quad X' \\
\downarrow{\sigma} \\
A \quad \xrightarrow{\begin{bmatrix} p_1 & o s_1 \\ p_2 & o s_1 \end{bmatrix}} \quad Y' \\
\end{array}
\]

in which the left, right, upper and lower faces are commutative. In particular, \((\sigma', \sigma): k \to h \) and \((\gamma', \gamma): k \to f \) are morphisms in \(h\text{Mon}\mathcal{A}\). We claim now that \((\alpha' \circ \sigma', \alpha \circ \sigma) = (\tau' \circ \gamma', \tau \circ \gamma)\) holds in \(h\text{Mon}\mathcal{A}\). That is, we have to show that \((\alpha' \circ \sigma', \alpha \circ \sigma) = (\tau' \circ \gamma', \tau \circ \gamma) \in J(k, g)\) holds.

Indeed, the map \( s_2: S' \to Y' \) satisfies

\[
g \circ s_2 = [\alpha - \tau] \circ s_1 = [\alpha - \tau] \circ [p_1 \circ o s_1] = \alpha \circ p_1 \circ s_1 - \tau \circ p_2 \circ s_1 = \alpha \circ \sigma - \tau \circ \gamma,
\]

which establishes the claim.

It remains to show that \((\sigma', \sigma) \in \Sigma\) holds. In view of Lemma 8.(iii), it suffices to establish

\[
[F\sigma]^{-1}([Fh](FA')) \subseteq [Fk](FS') \quad \text{and} \quad FA \subseteq [F\sigma](FS) + [Fh](FA').
\]

As \((\tau', \tau) \in \Sigma\), we get from Lemma 8.(iii) that

\[
[Ff](FX') = [F\tau]^{-1}([Fg](FY')) \quad \text{and} \quad FY = [Fg](FY') + [F\tau](FX)
\]

(16)
are valid. Now we derive from Lemma 8.(i) that

\[
FS = \{(a, x) \in FA \oplus FX; [F\alpha](a) - [F\tau](x) \in [Fg](FY')\},
\]

\[
[F\sigma](a, x) = a, [F\gamma](a, x) = x \quad \text{and} \quad [Fs_1](a', x') = (a', x') \quad \text{hold. Since} \quad [F\sigma'](a', x') = a' \quad \text{and} \quad [F\gamma'](a', x') = x' \quad \text{are the canonical maps, we get from the last part of Lemma 8.(i) that} \quad [Fk](a', x') = ([Fh](a'), [Ff](x')) \quad \text{holds.}
\]

Now we take \((a, x) \in [F\sigma]^{-1}([Fh](FA'))\), i.e., \([F\sigma](a, x) \in [Fh](FA')\). Thus we can select \(a' \in FA'\) such that \(a = [F\sigma](a, x) = [Fh](a')\). We have \([F\alpha](a) - [F\tau](x) \in [Fg](FY')\) as
\((a, x) \in FS\). That is, we find \(y' \in FY'\) such that
\[
[F\tau](x) = [F\alpha](a) - [Fg](y') = (F\alpha \circ Fh)(a') - [Fg](y') \\
= (Fg \circ F\alpha)(a') - [Fg](y') = [Fg][F\alpha'](a') - y'
\]
holds. This shows that \(x \in [F\tau]^{-1}([Fg](FY'))\). In view of (16) we then find \(x' \in FX'\) such that \([Ff](x') = x\). But then \((a, x) = ([Fh][a'), [Ff](x')) = [Fk][a')(x') \in [Fk](FS')\).

In order to establish the second inclusion, we fix \(a \in FA\) and consider \([F\alpha](a) \in FY\). By (16) we find \(x \in FX\) and \(y' \in FY'\) such that \([F\alpha](a) = [F\tau](x) + [Fg](y')\). That is, \([F\alpha](a) - [F\tau](x) \in [Fg](FY')\), and therefore, \((a, x) \in FS\) with \([F\sigma](a, x) = a\).

Next, we have to prove the second part of (MS2). For this let \(j: B' \to B\) be an object and \((\beta', \beta): f \to j\) a morphism in hMon \(A\). We put \(T := B \oplus Y\) and form the range of the map \([j_{00}\beta': B' \oplus Y' \oplus X \to B \oplus Y]\), that is, we get the factorization
\[
\begin{array}{ccc}
B \oplus Y & \xrightarrow{j} & B \oplus Y' \\
\text{B'} & \xrightarrow{\delta} & \text{X'} & \xrightarrow{\beta'} & B'
\end{array}
\]

with a monomorphism \(\ell\). Let \(\delta: Y \to B \oplus Y\) and \(\mu: B \to B \oplus Y\) be the canonical maps. Put \(\mu': B' \to T', \mu':= q \circ i_1\) and \(\delta': Y' \to T', \delta':= q \circ i_2\), where \(i_1: B' \to B' \oplus Y' \oplus X\) and \(i_2: Y' \to B' \oplus Y' \oplus X\) are the canonical maps. Then we get the diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{\beta'} & B' \\
\text{X} & \xrightarrow{\tau} & \text{B} & \xrightarrow{\mu'} & \text{B'} \\
Y' & \xrightarrow{\delta'} & T' & \xrightarrow{\mu} & \text{B}
\end{array}
\]

in which
\[
\ell \circ \mu' = \ell \circ q \circ i_1 = [j_{00}\beta'] \circ i_1 = j \circ \mu' \quad \text{and} \quad \ell \circ \delta' = \ell \circ q \circ i_2 = [j_{00}\beta'] \circ i_2 = \delta \circ g
\]
hold. Therefore, \((\mu', \mu): j \to \ell\) and \((\delta', \delta): g \to \ell\) are morphisms in hMon \(A\). To show that the cube above represents a commutative diagram in hMon \(A\), we have to verify that \((\mu' \circ \beta', \mu \circ \beta) - (\delta' \circ \tau', \delta \circ \tau)\) belongs to \(J(f, \ell)\). We define \(\rho: X \to T\) via \(\rho := q \circ i_3\), where \(i_3: X \to B' \oplus Y' \oplus X\) is the canonical map. Then we get
\[
\ell \circ \rho = \ell \circ q \circ i_3 = [j_{00}\beta'] \circ i_3 = [\beta_{-\tau}] = \mu \circ \beta - \delta \circ \tau
\]
and are done.

Now we have to show that the right face of the cube is a pulation square. As the left face has this property, we know that
\[
[F\tau]^{-1}([Fg](FY')) = [Ff](FX') \quad \text{and} \quad FY = [F\tau](FX) + [Fg](FY') \tag{17}
\]
hold. We claim that
\[
[F\mu]^{-1}([F\ell](FT')) \subseteq [Fj](FB') \quad \text{and} \quad FT \subseteq [F\mu](FB) + [F\ell](FT')
\]
are true. We use Lemma 8.(ii) to see that
\[
FT' = \{(F[j](b') + [F\beta](x), [Fg](y') - [F\tau](x)) : b' \in FB', y' \in FY', x \in FX\},
\]
\[ [F\ell](b, y) = (b, y) \text{ and } [Fg](b', y', x) = ([F\ell](b') + [F\beta](x), [Fg](y') - [F\tau](x)) \] hold.

Let \( b \in [F\mu]^{-1}([F\ell](FT')) \), i.e., \([F\mu](b) \in [F\ell](FT')\). We select \( b' \in FB', y' \in FY' \), and \( x \in FX \) such that \([F\mu](b) = [F\ell](b', y', x)\) holds. Thus

\[
(b, 0) = [F\mu](b) = (F\ell \circ Fg)(b, y', x) = \left[ F\ell \circ Fg \right](b, y', x) = ([F\ell](b') + [F\beta](x), [Fg](y') - [F\tau](x))
\]

and consequently \([Fg](y') = [F\tau](x)\) holds. That is, \( x \in [F\tau]^{-1}([Fg](FY')) = [Ff](FX)\) and we find \( x' \in FX' \) with \([Ff](x') = x\). Finally, we compute

\[
b = [F\ell](b') + [F\beta](x) = [F\ell](b') + (F\beta \circ Ff)(x') = [F\ell](b') + (F\beta \circ Ff)(x') = [F\ell](b') + (F\beta'(x')),
\]

where we see that the last expression belongs to \([Ff](FB')\). This shows the first inclusion.

For the second one, let \((b, y) \in FT\) be given. Employing (17), we select \( x \in FX \) and \( y' \in FY' \) such that \( y = [F\tau](x) + [Fg](y') \). Then we get

\[
[F\delta](y) = [F\delta \circ F\tau](x) + [F\delta \circ Fg](y') = [F\mu \circ F\beta](x) - [F\ell \circ F\rho](x) + [F\ell \circ F\delta'](y')
\]

and therefore

\[
(b, y) = [F\mu](b) + [F\delta](y) = [F\mu](b) + [F\ell](([F\delta'](y') - [F\rho](x)) \in [F\mu](B) + [F\ell](T'),
\]

which establishes the second inclusion.

(3.3) Let \((\alpha', \alpha), (\beta', \beta)\): \( f \to g \) be morphisms in \( \text{hMon}A \). Let \((\sigma', \sigma): h \to f\) be in \( \Sigma \) and assume that \((\alpha' \circ \sigma', \alpha \circ \sigma) = (\beta' \circ \sigma', \beta \circ \sigma)\) holds. With \( \gamma := \alpha - \beta \) and \( \gamma' := \alpha' - \beta' \) this means

\[
(\gamma' \circ \sigma', \gamma \circ \sigma) = ((\alpha' - \beta') \circ \sigma', (\alpha - \beta) \circ \sigma) = (\alpha' \circ \sigma', \alpha \circ \sigma) - (\beta' \circ \sigma', \beta \circ \sigma) = 0
\]

in \( \text{hMon}A \). Whence \((\gamma' \circ \sigma', \gamma \circ \sigma) \in J(h, g)\) and we find \( \rho: Z \to Y' \), such that \( \rho \circ h = \gamma' \circ \sigma' \) and \( g \circ \rho = \gamma \circ \sigma \). We use the pushout property

\[
\begin{array}{ccc}
Z' & \xrightarrow{h} & Z \\
\sigma' \downarrow & & \sigma \downarrow \\
X' & \xrightarrow{\eta} & X \\
\gamma' \downarrow & & \gamma \downarrow \\
& Y' &
\end{array}
\]

to obtain the map \( \eta \). We have \( \gamma' = \eta \circ f \) and thus \( g \circ \eta \circ f = g \circ \gamma' = \gamma \circ f \). Moreover, \( g \circ \eta \circ \sigma = g \circ \rho = \gamma \circ \sigma \) is valid. The latter two equations yield

\[
g \circ \eta \circ [\sigma f] = [g \circ \eta \circ \sigma] \quad g \circ \eta \circ f = [\gamma \circ \sigma \circ f] = [\gamma \circ \sigma \circ \gamma \circ f] = \gamma \circ [\sigma f],
\]

which implies \( g \circ \eta = \gamma \), as \([\sigma f]\) is an epimorphism by the pulation property. Now we select \((\tau', \tau) := \text{id}_f: f \to f\) and have \((\tau' \circ \gamma', \tau \circ \gamma) = (\gamma', \gamma) \in J(f, h)\), i.e., the latter is zero in \( \text{hMon}A \). Plugging in \( \alpha - \beta = \gamma \) and \( \alpha' - \beta' = \gamma' \) again, we get \((\tau' \circ (\alpha' - \beta'), \tau \circ (\alpha - \beta)) = 0\) and thus the equality \((\tau' \circ \alpha, \tau \circ \alpha) = (\tau' \circ \beta', \tau \circ \beta)\) in \( \text{hMon}A \).

Now let \((\tau', \tau): g \to h\) be in \( \Sigma \) and assume that \((\tau' \circ \alpha', \tau \circ \alpha) = (\tau' \circ \beta', \tau \circ \beta)\) in \( \text{hMon}A \) holds. As before, we get \((\tau' \circ \gamma', \tau \circ \gamma) \in J(f, h)\), and we find \( \rho: X \to Z'\) such that \( \tau \circ \gamma = h \circ \rho \) and

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\[ \tau' \circ \gamma' = \rho \circ f \text{ hold. We may thus use the pullback property} \]

\[
\begin{array}{c}
X \\ \eta \downarrow \\
\gamma \\ \downarrow \rho \\
Y' \\ \downarrow g \\
Y
\end{array}
\]

\[
\begin{array}{c}
Z' \\ \downarrow h \\
Z
\end{array}
\]

\[
\tau \\ \downarrow \text{PB} \\
\tau
\]

to obtain the map \( \eta \). Then we have immediately \( g \circ \eta = \gamma \) and Lemma 2 implies \( \eta \circ f = \gamma' \). We can thus take \( (\sigma', \sigma) := \text{id}_g : g \to g \) and have \( (\gamma' \circ \sigma', \gamma \circ \sigma) = (\gamma', \gamma) \in J(f, h) \), i.e., the latter is zero in \( \text{hMon} \mathcal{A} \). As before, this implies \( (\alpha' \circ \sigma', \alpha \circ \sigma) = (\beta' \circ \sigma', \beta \circ \sigma) \) in \( \text{hMon} \mathcal{A} \). \( \square \)

We conclude this section with our main result. We remark, that in its proof the factorization (20) will be derived without using that \( \mathcal{A} \) is a Waelbroeck category. This assumption is only needed in the second step, where we show that the induced map is a pulation. Cf. our remarks after the proof of Proposition 4.

Looking in detail at [2, second half of p. 40], where Beilinson, Bernstein, Deligne describe the heart of the t-structure considered in [2, Exemple 1.3.22] explicitly, one can see that it coincides with the category \( (\text{hMon} \mathcal{A})[\Sigma^{-1}] \), which we consider below. For a quasiabelian category \( \mathcal{A} \), a more detailed exposition can be found in the book [26, Section 1.2] by Schneiders.

**Theorem 10.** Let \( \mathcal{A} \) be a Waelbroeck category and \( \Sigma = \{ (\alpha', \alpha) : f \to g \text{ is a pulation} \} \). Then the localization \( (\text{hMon} \mathcal{A})[\Sigma^{-1}] \) is abelian.

**Proof.** The proof of [21, Lemma 2.2.1] and the preceding part of the lecture notes by Miličić show, that localizing an additive category \( \mathcal{A} \) in which every morphism has a kernel and a cokernel with respect to a multiplicative system yields a category in which again every morphism has a kernel and a cokernel. From this, and Proposition 4, it follows that \( (\text{hMon} \mathcal{A})[\Sigma^{-1}] \) has kernels and cokernels.

For a morphism \( f \) in \( \mathcal{C} \) we denote by \( \overline{f} : \text{ker} f \to \text{cok} f \) the induced morphism in \( \mathcal{C} \). Analogously we use the notation \( \overline{\varphi} \) for a morphism in \( \mathcal{C} [\Sigma^{-1}] \). The arguments in [21, p. 38 and p. 39] show the following. Let \( \Sigma \) be a multiplicative system in \( \mathcal{C} \) and let \( Q : \mathcal{C} \to \mathcal{C} [\Sigma^{-1}] \) be the canonical functor. If for a morphism \( \varphi = Q(f) \circ Q(s)^{-1} \) in \( \mathcal{C} [\Sigma^{-1}] \), the map \( Q(\overline{f}) \) is an isomorphism, then \( \overline{\varphi} \) is also an isomorphism.

In view of the above, it is enough to construct the induced morphism for a given morphism in \( \text{hMon} \mathcal{A} \) and to show that it is a pulation. For this purpose, let \( f : X' \to X \) and \( g : Y' \to Y \) be objects of \( \text{hMon} \mathcal{A} \) and let \( (\alpha', \alpha) : f \to g \) be a morphism. We first form its kernel, i.e., we consider the pullback

\[
\begin{array}{c}
X' \\ \downarrow h \\
\alpha' \\ \downarrow p_1 \\
\text{PB} \\ \downarrow \alpha \\
Y' \\ \downarrow g \\
Y
\end{array}
\]

to obtain

\[
\begin{array}{c}
X' \\ \downarrow h \\
X' \\ \downarrow f \\
T \\ \downarrow p_1 \\
X \\ \downarrow \alpha \\
Y
\end{array}
\]
Next, we form the range of \([p_1 f]: T \oplus X' \to X\) to get the cokernel and the canonical morphism \(\text{coker}(\alpha', \alpha) \to f\). We consider

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow{c} & & \downarrow{\gamma} \\
T \oplus X' & \xrightarrow{g} & Y'
\end{array}
\]

and denote by \(i_1: T \to T \oplus X'\) and \(i_2: X' \to T \oplus X'\) the canonical maps. Thus we get

\[
\begin{array}{ccc}
X' & \xrightarrow{id_Y} & X' \\
\downarrow{h} & & \downarrow{f} \\
T & \xrightarrow{p_1} & X
\end{array}
\]

and can now form the cokernel of \((\alpha', \alpha): f \to g\). In order to do this, we consider the range of \([\alpha g]: X \oplus Y' \to Y\) to get

\[
\begin{array}{ccc}
Y' & \xrightarrow{\alpha'} & Y' \\
\downarrow{\gamma} & & \downarrow{\gamma'} \\
X & \xrightarrow{id_X} & X
\end{array}
\]

and with the canonical maps \(j_1: X \to X \oplus Y'\) and \(j_2: Y' \to X \oplus Y'\) we obtain

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha'} & Y' \\
\downarrow{\alpha g} & & \downarrow{j} \\
X & \xrightarrow{\gamma} & Y
\end{array}
\]

according to Proposition 4. Next, we need to compute the kernel of \((p \circ j_2, \text{id}_Y): g \to j\). For this we have to form the pullback of \(j\) along \(\text{id}_Y\) and thus get

\[
\begin{array}{ccc}
S & \xrightarrow{\gamma} & X \\
\downarrow{p \circ j_2} & & \downarrow{j} \\
S & \xrightarrow{\text{id}_S} & S
\end{array}
\]

from whence \(\ker \text{cok}(\alpha', \alpha) = (\text{id}_Y, j): p \circ j_2 \to q\) follows. The diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{id_Y} & Y' \\
\downarrow{p \circ j_2} & & \downarrow{j} \\
S & \xrightarrow{\gamma} & Y
\end{array}
\]

represents the composition \((\ker \text{cok}(\alpha', \alpha)) \circ \text{cok}(\alpha', \alpha)\). In order to obtain the canonical map
from the cokernel of the kernel to the kernel of the cokernel, we have to show that the composition \((p \circ j_2, \text{id}_Y) \circ (\gamma', \alpha) : c \to g \to j\) is zero. We consider

\[
\begin{array}{ccc}
A' & \xrightarrow{p_0 j_2} & S \\
c & \downarrow{j} & j \\
X & \xrightarrow{p_0 j_1} & Y
\end{array}
\]

and derive \(j \circ p \circ j_1 = \text{id}_Y \circ \alpha\) from (19). Therefore, \((p \circ j_2, \text{id}_Y) \circ (\gamma', \alpha) = 0\) holds in \(\text{hMon}A\).

Consequently, \((\gamma', \alpha) : c \to g\) factors through \((\text{id}_Y, j) : p \circ j_2 \to g\). According to Proposition 4 the factorization is given by

\[
\begin{array}{ccc}
A' & \xrightarrow{\gamma'} & S' \\
\downarrow{\gamma'} & \downarrow{\gamma'} & \downarrow{\gamma'} \\
A' & \xrightarrow{p_0 j_2} & S' \\
\downarrow{\alpha} & \downarrow{j} & \downarrow{j} \\
X & \xrightarrow{p_0 j_1} & Y \\
S & \xrightarrow{\text{id}_Y} & Y \\
S & \xrightarrow{\text{id}_Y} & Y
\end{array}
\]

since we have to take the pullback

\[
\begin{array}{ccc}
Y' & \xrightarrow{p_{0j_1}} & S \\
\downarrow{\alpha} & \downarrow{j} & \downarrow{j} \\
S & \xrightarrow{\text{id}_Y} & Y \\
S & \xrightarrow{\text{id}_Y} & Y
\end{array}
\]

along the identity. Therefore, \((\gamma', p \circ j_1) : c \to p \circ j_2\) is the map induced by \((\alpha', \alpha)\) from \(\text{coker}(\alpha', \alpha)\) to \(\text{ker cok}(\alpha', \alpha)\). We get the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\text{id}_{X'}} & X' \\
\downarrow{\gamma'} & \downarrow{\gamma'} & \downarrow{\gamma'} \\
A' & \xrightarrow{p_0 j_2} & S' \\
\downarrow{\alpha} & \downarrow{j} & \downarrow{j} \\
X & \xrightarrow{p_0 j_1} & Y \\
S & \xrightarrow{\text{id}_Y} & Y \\
S & \xrightarrow{\text{id}_Y} & Y
\end{array}
\]

that represents the factorization of \((\alpha', \alpha)\), cf. also (4) and the corresponding remarks.

Going through the above again, and using Lemma 8, we first see that

\[
FT = \{(x, y') \in FX \oplus FY' ; [Fa](x) = [Fg](y')\},
\]

\([Fp_1](x, y') = x, [Fp_2](x, y') = y'\) and \([FH](x') = ([FF](x'), [Fa']([Fg'](x'))\) hold. Next, we observe that

\[
FA' = \{[Fp_1](t) + [FF](x') ; t \in FT, x' \in FX'\},
\]

\([Fc](a') = a', [Fg](t, x') = [Fp_1](t) + [FF](x'), [Fg'](a') = [Fg]^{-1}([Fa']([Fg](a'))), [Fj_1](x) = (x, 0)\) and \([Fj_2](y') = (0, y')\). Finally, we have

\[
FS = \{[Fa](x) + [Fg](y') ; x \in FX, y' \in FY'\}
\]
and observe that \([Fp](x, y_0) = [F \alpha](x) + [F_g](y_0)\) and \([F_j](s) = s\). Using the above, we are able to derive that the map \((\gamma', p \circ j_1): c \to p \circ j_2\), represented by the front face of the cube in (20), belongs to \(\Sigma\). Indeed, by Lemma 8, for this we have to show that

\[
[F(p \circ j_2)]^{-1}([F(p \circ j_2)](FY')) \subseteq [Fc](FA') \quad \text{and} \quad FS \subseteq [F \alpha](FX) + [F_g](Y')
\]

hold. Let \(x \in FX\) with \([F(p \circ j_1)] = (Fp \circ Fj_1)(x) \in [F(p \circ j_2)](FY') = [(Fp \circ j_2)](FY')\) be given. That is, there exists \(y' \in FY'\) with

\[
[F \alpha](x) = [Fp \circ Fj_1](x) = [Fp \circ Fj_2](y') = [F_g](y').
\]

Thus \((x, y') \in FT\) holds and \(x = [Fp_\alpha](x, y') + [FF'](0) \in FA'\) follows. As \(Fc\) is the inclusion map, this means that \(x \in [Fc](FA')\). We established the first inclusion and now show the second. For \(x \in FX\) we have \((Fp \circ Fj_1)(x) = [F \alpha](x)\), and for \(y' \in FY'\) we have \((Fp \circ Fj_1)(y') = F_g(y')\). Thus \(FS \subseteq [F \alpha](FX) + [F_g](Y')\) holds.

\[\square\]

3. Old and new examples for Waelbroeck categories

Before discussing examples of Waelbroeck categories, we make the following remarks on the notion of the range. Firstly, we observe that the range \(r_f: R_f \to Y\) of a morphism \(f: X \to Y\) is, in particular, an image in the sense of Mitchell [22, Section I.10]. To see this, it is enough to take \(Z = Y\) and \(g = \text{id}_Y\) in Definition 3. If we are given an abelian category, the range is isomorphic to \(\text{im} f := \ker \text{cok} f\) and to \(\text{coim} f := \text{cok} \ker f\). The proof of Lemma 8 showed already that in a module category the range is given by the “set-wise range”. In non-abelian categories this need not be true and the next result suggests that in certain cases the range should be thought of as a coinage rather than as an image.

**Lemma 11.** Let \(f: X \to Y\) be a morphism in \(\mathcal{A}\) for which \(\text{coim} f = \text{cok} \ker f\) exists. Assume that the canonical map \(i: \text{coim} f \to Y\) is a monomorphism. Then this map is a range of \(f\).

**Proof.** We consider the diagram

\[
\begin{array}{ccc}
ker f & \xrightarrow{k} & X \\
    & \searrow^h & \downarrow^c \\
    &           & \text{cok} k \\
    & \nearrow_j & \downarrow^g' \\
    &           & Z \\
\end{array}
\]

and compute \(j \circ h \circ k = g \circ f \circ k = 0\). As \(j\) is a monomorphism, we get \(h \circ k = 0\). By the universal property of the cokernel we obtain \(g': \text{cok} k \to J\) with \(g' \circ c = h\). Now we compute \(j \circ g' \circ c = j \circ h = g \circ f = g \circ i \circ c\), which gives \(j \circ g' = g \circ i\) as \(c\) is an epimorphism. \(\square\)

If \(\mathcal{A}\) is left-semiabelian, see [16], i.e., \(\mathcal{A}\) has kernels and cokernels and for any \(f: X \to Y\) the induced map \(\overline{f}: \text{coim} f \to \text{im} f\) is a monomorphism, then the assumptions of Lemma 11 are satisfied for every morphism in \(\mathcal{A}\). This means that for any non-abelian but left-semiabelian category the image will not be a range. In the category of Hausdorff locally convex spaces, see [23, §2.1] for its basic category theory, the range of \(f: X \to Y\) is given by

\[R_f = X/f^{-1}({0})\quad \text{and} \quad r_f(y) = y.\]

Observe that we have \(X/f^{-1}({0}) \cong f(X)\) as linear spaces. So, algebraically, the range is again the set-wise range. However, the topology on \(R_f = f(X)\) will for general \(f\) not coincide with the topology induced by \(Y\). On top of that, the image of \(f\) in the sense \(\text{im} f = \ker \text{cok} f\) is given by

\[\text{im} f = \overline{f(X)}\]
endowed with the topology induced by $Y$. Here we have the intuitive topology but the space will in general be strictly bigger than the set-wise range.

Next we show that the model case of the category $\mathcal{B}$ of Banach spaces with linear and continuous maps as morphisms is indeed a Waelbroeck category. The functor in this case can be chosen to be just the forgetful functor to $\text{Ab}$.

**Proposition 12.** The category of Banach spaces is a Waelbroeck category.

**Proof.** We fix $k \in \{\mathbb{R}, \mathbb{C}\}$, consider Banach spaces over $k$, and define $F: \mathcal{B} \to \text{Ab}$ to be the forgetful functor that assigns to a Banach space its underlying abelian group. Kernels and cokernels in $\mathcal{B}$ are inherited from the category of all locally convex spaces. Given $f: X \to Y$, the kernel is given by the inclusion $f^{-1}([0]) \to Y$, where $f^{-1}([0])$ is endowed with the topology induced by $X$, and the cokernel is given by the quotient map $Y \to Y/f(X)$, where $Y/f(X)$ carries the quotient topology. The range is given by the coimage according to Lemma 11. It follows that $F$ preserves kernels and ranges. Finally, it is a classical consequence of the open mapping resp. closed graph theorem that a chain of linear and continuous maps $X \to Y \to Z$ between Banach spaces is short exact when considered in Ab if and only if this holds in $\mathcal{B}$, cf. [18, Chapter 26]. We refer, in addition, to the proof of Proposition 14 below, which is an adaption of the Banach space proof.

In precisely the same way one gets that the category $\mathcal{F}$ of Fréchet spaces, with linear and continuous maps as morphisms, is a Waelbroeck category.

**Proposition 13.** The category of Fréchet spaces is a Waelbroeck category.

The categories $\mathcal{B}$ and $\mathcal{F}$ are both quasiabelian. Therefore, here one can also use the t-structure method to get the categories $(\text{hMon}\mathcal{B})[\Sigma^{-1}]$ and $(\text{hMon}\mathcal{F})[\Sigma^{-1}]$. The proposition below illustrates that our results of Section 2 apply also to categories where the latter cannot be applied a priori. Let $\mathcal{L}\mathcal{B}$ denote the category of LB-spaces, i.e., of locally convex spaces which appear as a countable inductive limit of Banach spaces, with continuous linear maps as morphisms.

**Proposition 14.** The category of LB-spaces is a Waelbroeck category, but it is not quasiabelian.

**Proof.** If $f: X \to Y$ is a morphism in $\mathcal{L}\mathcal{B}$, then its cokernel is given by $\text{cok } f = Y/f(X)$ endowed with the quotient topology. Its kernel is given by $\ker f = f^{-1}([0])^\circ$, where we use the notation $U^\circ := \text{ind}_{n \in \mathbb{N}} U \cap X_n$, if $U \subseteq X = \text{ind}_{n \in \mathbb{N}} X_n$ is a closed subspace of an LB-space, cf. [9, Remark 3.1.1]. We thus get that the induced map $\overline{f}: \text{coim } f \to \text{im } f$, given by

$$
\overline{f}: X/f^{-1}([0]) \to Y/\overline{f}(X)^\circ, \overline{f}([x]_{f^{-1}([0])}) := f(x),
$$

is always injective, i.e., a monomorphism. It is an epimorphism if and only if $f(X) \subseteq \overline{f}(X)^\circ$ is dense. An example due to Grothendieck allows to construct a map $f: X \to Y$ such that the latter is not the case. We refer to [27, Example 4.2], where the same example was used but in the framework of a different category. We thus get that $\mathcal{L}\mathcal{B}$ is left-semiabelian but not semiabelian and thus in particular not quasiabelian, cf. [16].

Using the closed graph and open mapping theorems, it is straightforward to check that a map $f: X \to Y$ is a kernel in $\mathcal{L}\mathcal{B}$ if and only if $f$ is injective and $f(X) \subseteq Y$ is closed, and that it is a cokernel if and only if it is surjective, cf. again [9, Remark 3.1.1]. We see that the forgetful functor $\mathcal{L}\mathcal{B} \to \text{Ab}$ preserves kernels. Given $f: X \to Y$, its range $r_f: R_f \to Y$ is given by

$$
r_f: X/f^{-1}([0]) \to Y, r_f([x]_{f^{-1}([0])}) := x,
$$

according to Lemma 11. This shows that the forgetful functor also preserves ranges. As in our remarks after Lemma 11 we observe, that $r_f: f(X) \to Y$, $r_f(x) = x$, where $f(X)$ carries the topology of $X/f^{-1}([0])$, is another and more intuitive realization of the range.
Finally, let a sequence of morphisms

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \]  

(21)
in \( \mathcal{LB} \) be given. We claim that \((f,g)\) is a kernel-cokernel pair in \( \mathcal{LB} \) if and only if this is true in \( \text{Ab} \).

\[ \Rightarrow \] Let \( f = \ker g \) and \( g = \text{cok} f \). In view of the above we can assume w.l.o.g. that \( X = g^{-1}(\{0\}) \subseteq Y \) is a linear subspace and that \( f \) is the inclusion map of this subspace. Moreover, we may assume that \( Z = Y/X \) is the quotient and that \( g \) is the quotient map. This, however, means that we have a short exact sequence of abelian groups.

\[ \Leftarrow \] Let (21) in \( \mathcal{LB} \) be given and assume that it is a short exact sequence when we only consider the group structure. That is, \( f \) is injective, \( g \) is surjective and \( f(X) = g^{-1}(\{0\}) \) holds algebraically. Since \( g \) is continuous, the latter equality provides that \( f \) is a kernel in \( \mathcal{LB} \) if we use our observations from above. We claim that \( g \) is a cokernel of \( f \) in \( \mathcal{LB} \). We have \( g \circ f = 0 \) and take a linear and continuous map \( j : Y \to J \) with \( j \circ f = 0 \). As \( g = \text{cok} f \) holds in \( \text{Ab} \), there exists a unique linear map \( h : Z \to J \) with \( h \circ g = j \), i.e., the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \text{id} \\
0 & \xrightarrow{g} & Z \\
\downarrow & & \downarrow h \\
0 & \xrightarrow{j} & J \\
\end{array}
\]

commutes. By the open mapping theorem, \( h \) is continuous. Indeed, take a 0-neighborhood \( U \subseteq J \). We claim that \( h^{-1}(U) \subseteq Z \) is a 0-neighborhood. Therefore, we consider \( g(j^{-1}(U)) \subseteq Z \), which is open and contains zero. If \( z \in g(j^{-1}(U)) \) is given, we find \( y \in Y \) such that \( g(y) = z \) and \( h(z) = h(g(y)) = j(y) = j(y) \in U \), i.e., \( z \in h^{-1}(U) \). Thus \( g(j^{-1}(U)) \) is contained in \( h^{-1}(U) \).

Now we want to explore the relation between Waelbroeck categories, quasiabelian categories and categories that satisfy the assumptions of [2, Exemple 1.3.22]. The difference of the latter two is precisely the existence of arbitrary cokernels.

**Lemma 15.** A category \( \mathcal{A} \) is quasiabelian if and only if the following two conditions hold.

(i) The assumptions of Bernstein, Bellenso, Deligne [2, Exemple 1.3.22] are satisfied. That is, \( \mathcal{A} \) is an exact category and every morphism \( f : X \to Y \) in \( \mathcal{A} \) has a kernel \( \ker f \to X \) that fits into a conflation \( \ker f \to X \to \text{coim} f \).

(ii) Every morphism in \( \mathcal{A} \) has a cokernel.

**Proof.** The category \( \mathcal{A} \) is quasiabelian if and only if all kernels and all cokernels exist and the set of all kernel-cokernel pairs forms an exact structure. The latter follows from [27, Theorem 3.2] and [26, Remark 1.1.2]. It is now enough to use that, in a category which has all kernels and cokernels, a morphism is a cokernel if and only if it is the cokernel of its kernel.

It follows from the above that the category of \( \text{LB} \)-spaces, studied in Proposition 14, also does not satisfy the assumptions of Bernstein, Bellenso, Deligne [2, Exemple 1.3.22]. We finally want to show that categories that satisfy these assumptions, in particular, all quasiabelian categories, can be treated with the theory of Section 2.

Due to set-theoretic problems, we are unfortunately not able to prove that each of these categories has the Waelbroeck property. One can, however, see that for those parts of the proofs in Section 2 where we argue in the category of abelian groups, it would be enough to have a functor \( F : \mathcal{A} \to \text{Ab} \), where \( \mathcal{A} \) is a small category that contains the diagram which is studied in the corresponding part of the proof. Then one could use the classical trick, see, e.g., [32, Remark on p. 12], of applying an embedding theorem only to a suitable small subcategory in order to chase elements. Indeed, we have the following.
Proposition 16. Every small category that satisfies the assumptions in [2, Exemple 1.3.22] is a Waelbroeck category. In particular, every small quasiabelian category is a Waelbroeck category.

Proof. By definition, the set $\mathcal{E}$ of all kernel-cokernel pairs in $\mathcal{A}$ forms an exact structure. By the Gabriel-Quillen embedding theorem, see, e.g., [5, Theorem A.1], there exists a ring $R$ and a fully faithful functor $G: (\mathcal{A}, \mathcal{E}) \to \text{Mod} R$ which is exact and reflects exactness. For $F$ we can take the composition of $G$ with the forgetful functor $\text{Mod} R \to \text{Ab}$. In particular, $F$ preserves kernels and cokernels in the sense of Lemma 11.

In [30] Waelbroeck presents his construction of the category of quotients in the language of bornological vector spaces. The other categories he studies, e.g., the category of Banach spaces, are treated as special cases. In the remainder of this section we show that Waelbroeck’s construction in the case of bornological vector spaces coincides precisely with our approach in the preceding sections. Therefore, our main results all hold true also for bornological vector spaces although the latter might not form a Waelbroeck category in the sense of Definition 7.

We start by fixing our notation. There are several different definitions of “bornological vector spaces” in the literature [12, 13, 14, 24, 30]. Some authors use attributes like convex, separable or complete to consider subclasses of general bornological vector spaces, while others assume these properties from the beginning and include them in the very first definition. We concentrate on the two classes studied in Waelbroeck’s final monograph [30], the “$b_0$-spaces” and the “$b$-spaces”.

Definition 17. Let $X$ be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We call $\mathcal{B} \subseteq \mathcal{P}(X)$ a bornology on $X$ if the conditions below are satisfied.

(B1) $\forall x \in X: \{x\} \in \mathcal{B}$.

(B2) $\forall B \in \mathcal{B}: B' \subseteq B \Rightarrow B' \in \mathcal{B}$.

(B3) $\forall B_1, B_2 \in \mathcal{B}: B_1 \cup B_2 \in \mathcal{B}$.

(B4) $\forall B \in \mathcal{B}, \lambda \in \mathbb{K}: \lambda B \in \mathcal{B}$.

(B5) If $B$ belongs to $\mathcal{B}$ then the same is true for its absolutely convex hull $\Gamma B$.

(B6) If $B \in \mathcal{B}$ is a linear subspace of $X$ then $B = \{0\}$ holds.

The pair $(X, \mathcal{B})$ is a bornological vector space and the elements of $\mathcal{B}$ are said to be the bounded subsets of $X$.

Definition 18. A bornological vector space $(X, \mathcal{B})$ is complete if the following holds.

(B7) For every $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}$ such that $B' \subseteq B$, $B'$ is absolutely convex and

$$X_{B'} = (\text{span } B', \| \cdot \|_{B'}) \quad \text{with} \quad \| x \|_{B'} = \inf \{ \lambda > 0 : x \in \lambda B' \}$$

is a Banach space.

The conditions (B1)–(B6) are equivalent to Waelbroeck’s definition of a $b_0$-space, see [30, Definitions 1.1.1, 1.1.22, 1.1.40 and 1.1.53]. The conditions (B1)–(B7) are equivalent to Waelbroeck’s definition of a $b$-space, see [30, Proposition 1.1.68 and Definition 1.1.67 and Notation 0.4.10].

Hogbe-Nlend [13, Chapters 1.1 and 3] considers the same types of spaces under the names of convex separated bornological vector spaces and complete convex bornological vector spaces, respectively. In the French version of his book [12, Chapter I.5, III.3 and IV.2], an “ebc régulièrement séparé” is a space satisfying (B1)–(B6). An “ebc complet” is a space satisfying (B1)–(B7). We point out that Meyer [19, 20] includes also the completeness in the term “bornological vector space”.

Definition 19. Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be bornological spaces. A linear map $f: X \to Y$ is bounded if $f(B) \in \mathcal{B}_Y$ holds for every $B \in \mathcal{B}_X$.

Below we want to employ results of Prosmans, Schneiders [24]. They consider first the category $\mathcal{B}_C$ which has as objects the spaces satisfying (B1)–(B5) and as morphisms the bounded linear maps. Then they use the two full subcategories
(i) \( \mathcal{B}_C \) of bornological vector spaces,
(ii) \( \mathcal{B}_C \) of complete bornological spaces.

By [24, Propositions 1.8, 4.10 and 5.6(b)] all three categories \( \mathcal{B}_C, \mathcal{B}_C \) and \( \mathcal{B}_C \) are quasiabelian. Therefore, [26, Corollary 1.2.20] implies that the pulations in the homotopy categories of the corresponding monomorphism categories form multiplicative systems and that the localizations are abelian.

**Proposition 20.** (Schneider [26], Prosmans, Schneider [24]) Let \( A \) be either the category of bornological vector spaces or the category of complete bornological vector spaces with bounded linear maps as morphisms. Let \( \Sigma = \{(\alpha', \alpha) : f \to g \text{ is a pulation in } A\} \). Then \( \Sigma \subseteq \text{hMon}_A \) is a multiplicative system and \( (\text{hMon}_A)[\Sigma^{-1}] \) is abelian.

In contrast to Banach spaces, Fréchet spaces or LB-spaces we cannot consider the forgetful functor \( F : A \to \text{Ab} \) in order to see that the categories above are Waelbroeck categories. Indeed, the class of kernel-cokernel pairs in the category of bornological vector spaces as well as in the category of complete bornological vector spaces is strictly smaller than the class of composable morphisms \((f, g)\) which are exact as abelian groups.

**Lemma 21.** Let \( A \) be either the category of bornological vector spaces or the category of complete bornological vector spaces with bounded linear maps as morphisms. Two morphisms \( f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y) \) and \( g : (Y, \mathcal{B}_Y) \to (Z, \mathcal{B}_Z) \) form a kernel-cokernel pair in \( A \) if and only if the following two conditions hold.

(i) \((f, g)\) is a kernel-cokernel pair in \( \text{Ab} \).

(ii) \( \mathcal{B}_X = \{f^{-1}(B) ; B \in \mathcal{B}_Y\} \) and \( \mathcal{B}_Z = \{g(B) ; B \in \mathcal{B}_Y\} \).

**Proof.** We consider the sequence

\[
0 \to u \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0
\]

and get from [26, Remark 1.1.10] and [24, Lemma 4.8 and Corollary 5.7] that \((f, g)\) is a kernel-cokernel pair in \( A \) if and only if it is a kernel-cokernel pair in \( \mathcal{B}_C \). The latter is equivalent to \( u, f, g \) being strict and the canonical maps \( \text{im} u \to \text{ker} f, \text{im} f \to \text{ker} g \) and \( \text{im} g \to \text{ker} v \) being isomorphisms, see [26, Definition 1.1.9 and Remark 1.1.10]. In \( \mathcal{B}_C \) we can employ [24, Proposition 1.5] to see that the latter is equivalent to \( f \) being injective and strict, \( g \) being surjective and strict and \( f(X) = g^{-1}(0) \) algebraically.

We recall that a morphism \( f : X \to Y \) in a quasiabelian category is strict if \( \overline{f} : \text{cok} \ker f \to \ker \text{cok} f \) is an isomorphism [26, Definition 1.1.1]. By [24, Proposition 1.5] the map \( f : X \to Y \) is strict in \( \mathcal{B}_C \) if and only if

\[
\forall B \in \mathcal{B}_Y \exists B' \in \mathcal{B}_X : B \cap f(X) = f(B')
\]

holds. Now it is straightforward to show that an injective map \( f : X \to Y \) is strict if and only if \( \mathcal{B}_X = \{f^{-1}(B) ; B \in \mathcal{B}_Y\} \) holds and that a surjective map \( g : Y \to Z \) is strict if and only if \( \mathcal{B}_Z = \{g(B) ; B \subseteq \mathcal{B}_Y\} \) holds.

We thus showed that \((f, g)\) is a kernel-cokernel pair in \( A \) if and only if \( f \) is injective, \( g \) is surjective, \( f(X) = g^{-1}(0) \) holds algebraically, and \( \mathcal{B}_X = \{f^{-1}(B) ; B \in \mathcal{B}_Y\} \) and \( \mathcal{B}_Z = \{g(B) ; B \subseteq \mathcal{B}_Y\} \) hold.

Using Lemma 21 we show that the morphisms that Waelbroeck [30, Chapter 2] makes invertible in his construction of the categories of quotient \( b_0 \)-spaces and quotient \( b \)-spaces are precisely the pulations.
**Proposition 22.** Let $\mathcal{A}$ be either the category of bornological vector spaces or the category of complete bornological vector spaces with bounded linear maps as morphisms. The diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\alpha'} & Y' \\
\downarrow{f} & & \downarrow{g} \\
X & \xrightarrow{\alpha} & Y 
\end{array}
$$

(22)

is a pulation in $\mathcal{A}$ if and only if $\alpha^{-1}(g(Y')) = f(X')$ and $Y = \alpha(X) + g(Y')$ hold in the sense of bornological spaces, see [30, p. 80].

**Proof.** In [30, Definitions 1.1.28, 1.1.30 and 1.1.34] Waelbroeck defines (i) bornological subspaces, (ii) a bornology on the range of a bounded map, (iii) a bornology on the preimage of a bornological subspace under a bounded map, and (iv) a bornology on the sum of two bornological subspaces. Using these definitions successively we get that

$$
\alpha^{-1}(g(Y')) = f(X') \text{ and } Y = \alpha(X) + g(Y')
$$

(23)

hold as bornological spaces if and only if the latter holds algebraically and we have

$$
\{ B \subseteq \alpha^{-1}(g(Y')) ; B \in \mathcal{B}_X \text{ and } \exists B'' \in \mathcal{B}_{Y'} : g(B'') = \alpha(B) \} = \{ f(B') ; B' \in \mathcal{B}_{X'} \}
$$

(24)

as well as

$$
\mathcal{B}_Y = \{ B \subseteq g(Y') + \alpha(X) ; \exists B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_{Y'} : B \subseteq \alpha(B_1) + g(B_2) \}
$$

(25)

for the bornologies. We mention that (24) and (25) can be simplified as $B \subseteq g(Y') + \alpha(X)$ and $B \subseteq \alpha^{-1}(g(Y'))$ follow automatically from the conditions on $B$ on the left respectively right hand side.

We know that (22) is a pulation in $\mathcal{A}$ if and only if $[[f], \alpha g]$ is a kernel-cokernel pair in $\mathcal{A}$. By Lemma 21 this is true if and only if the latter is a kernel-cokernel pair in $\mathcal{A}$. From (23) hold algebraically, and

$$
\mathcal{B}_{X'} = \{ [f]^{-1}(B) ; \exists B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_{Y'} : B \subseteq B_1 + B_2 \}
$$

(26)

as well as

$$
\mathcal{B}_Y = \{ \alpha g(B') ; \exists B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_{Y'} : B' \subseteq B_1 + B_2 \}
$$

(27)

hold. Here we used the direct sum bornology $\mathcal{B}_{X \oplus Y'}$, see [24, Remark 1.3 and Propositions 4.5 and 5.5] to write down the right hand sides of Lemma 21(ii) explicitly.

From now on, we assume that (23) holds algebraically. We first establish that (25) is equivalent to (27). For this it is enough to show that the right hand sides coincide. The inclusion “$\subseteq$” is easy to see. For “$\supseteq$” let $B \subseteq \alpha(B_1) + g(B_2)$ be given with $B_1 \in \mathcal{B}_{Y'}$ and $B_2 \in \mathcal{B}_X$. Put $B' := B_1 + B_2$ and observe that then $B \subseteq [\alpha g](B')$ follows. Since the right hand side of (27) is a bornology the desired conclusion follows from (B2).

It remains to establish that (24) is equivalent to (26). We first claim that

$$
\mathcal{B}_{X'} = \{ B' \subseteq X' ; f(B') \in \{ f(B) ; B \in \mathcal{B}_{X'} \} \}
$$

(28)

holds. Indeed, “$\subseteq$” is trivial. For “$\supseteq$” let $B' \subseteq X'$ be given with $f(B') \in \{ f(B) ; B \in \mathcal{B}_{X'} \}$. That is, we find $B \in \mathcal{B}_{X'}$ with $f(B') = f(B)$ from whence it follows that $B' = B$ holds as $f$ is injective. Therefore, $B' \in \mathcal{B}_{X'}$ holds. Next, we claim that (24) is equivalent to

$$
\mathcal{B}_{X'} = \{ B' \subseteq X' ; f(B') \in \{ B \in \mathcal{B}_X ; \exists B'' \in \mathcal{B}_{Y'} ; g(B'') = \alpha(B) \} \}
$$

(29)

In view of (28) it is clear that (24) implies (29). For the other direction, assume that (29) holds.
“⊂” Let $B \subseteq \alpha^{-1}(g(Y'))$ be given. Assume that $B \in \mathcal{B}_X$ and let $B'' \in \mathcal{B}_Y$, be such that $g(B'') = \alpha(B)$. We put $B' := f^{-1}(B)$ and obtain $f(B') = f(f^{-1}(B)) = B$. The last equality is true as (23) holds algebraically from whence it follows that $B \subseteq f(X')$. Using (29) it follows that $B' \in \mathcal{B}_X$ holds. So we have $B = f(B')$ with $B' \in \mathcal{B}_X$ and are done.

“⊃” Let $B' \in \mathcal{B}_X$. Then $f(B') \subseteq f(X')$ holds and it follows $f(B') \subseteq \alpha^{-1}(g(Y'))$ since (23) holds algebraically. As $f$ is bounded we have that $f(B') \in \mathcal{B}_X$ holds. By (29) we get that there exists $B'' \in \mathcal{B}_Y$ such that $g(B'') = \alpha(f(B'))$ is valid. Consequently, $f(B')$ belongs to the set on the left hand side of (24).

Reformulating (29) we obtain that (24) is equivalent to

$$\mathcal{B}_X = \{ B' \subseteq X' ; f(B') \in \mathcal{B}_X \text{ and } \exists B'' \in \mathcal{B}_Y : g(B'') = \alpha(f(B')) \} \tag{30}$$

and deduce that for the equivalence of (24) and (26) it is enough to show the right hand sides of (30) and (26) coincide.

“⊂” Let $B_1 \in \mathcal{B}_X$, $B_2 \in \mathcal{B}_Y$ and $B \subseteq B_1 + B_2$. We put $B' := (\alpha f)^{-1}(B_1 + B_2) = \{ f(x) : \exists x \in X' : -f(x) \in B_1 \text{ and } \alpha'(x) \in B_2 \} \subseteq B_1$, and by (B2) we obtain that $f(B') \in \mathcal{B}_X$. Using $\alpha \circ f = g \circ \alpha'$, we also obtain that

$$(\alpha \circ f)(B') \subseteq g(\{ \alpha'(x) : \exists x \in X' : -f(x) \in B_1 \text{ and } \alpha'(x) \in B_2 \}) \subseteq g(B_2)$$

holds. Now we put $B'' := \{ x \in B_2 : g(x) \in (\alpha \circ f)(B') \}$, which belongs to $\mathcal{B}_Y$, and satisfies $g(B'') = \alpha(f(B'))$. That is, $B'$ belongs to the set on the right hand side of (30).

“⊃” Let $B' \subseteq X'$ be given and assume that $f(B') \in \mathcal{B}_X$ holds and that there exists $B'' \in \mathcal{B}_Y$, such that $g(B'') = \alpha(f(B'))$ is valid. We put $B_1 := -f(B')$, which is in $\mathcal{B}_X$ by (B4). Moreover, we put $B_2 := \alpha'(B')$ and observe that then $g(B'') = \alpha(f(B')) = g(\alpha'(B'))$ holds, which implies $B'' = \alpha'(B')$ and thus yields $B_2 \in \mathcal{B}_Y$. We compute

$$[\alpha f](B') = \{ -f(x) + \alpha'(x) : x \in B' \} \subseteq -f(B') + \alpha'(B') = B_1 + B_2,$$

which implies $B' \subseteq [\alpha f]^{-1}(B)$. Therefore, $B'$ belongs to the set on the right hand side of (26) because of (B2) and we are done. □

We conclude by pointing out that Proposition 20, Lemma 21 and Proposition 22 hold verbatim for the category $\mathcal{B}_C$. Therefore, we have also for the category of possibly non-separable bornological vector spaces that the pulsations can be described by the two equations in Proposition 22, that they form a multiplicative system in the homotopy category of the monomorphism category, and that the localization is abelian.

4. THE CLASSICAL WAEELBROECK CONSTRUCTION

Waelbroeck began his research on categories of formal quotients in 1962 and published until 2005 a large number of articles on this subject. We refer to his last publication, the book [30], for a complete list of references and historical background information. In [30], Waelbroeck develops his theory at the same time for different categories. His account involves the concept of bornologies, see Section 3, and he uses a substantial amount of “...non-standard terminology concerning topological vector spaces”, cf. the review [3] by Bonet. In order to prevent confusion, we restrict ourselves in this section again to the category $\mathcal{B}$ of Banach spaces with linear and continuous maps as morphisms.

Firstly, we have to comment on Waelbroeck’s terminology of “quotient Banach spaces”. In [30, Definition 2.1.1] the symbol $X[X']$ refers to what we denote by $f : X' \rightarrow X$. Waelbroeck here drops $f$ from the notation by assuming w.l.o.g. that $X' \subseteq X$ is a linear subspace. His morphisms [30, Definition 2.1.2] correspond to those of $h\text{Mon}\mathcal{B}$. His equivalent of the latter category
he denotes by $\tilde{Q}$. In order to define the category $Q$, which corresponds to $(\text{hMon}\mathcal{B})[\Sigma^{-1}]$, Waelbroeck makes the pulations—"pseudoisomorphisms" in his notation—invertible but does not use the abstract theory of localization. His result [30, Theorem 2.1.25], however, corresponds precisely to the universal property of $(\text{hMon}\mathcal{B})[\Sigma^{-1}]$, see [10, p. 6] and [21, Theorem 1.1.1].

Secondly, we have to emphasize that the summary [28], which might appear more attractive to read than the book [30], as it also restricts to Banach spaces, follows in several aspects different lines than Waelbroeck’s final monograph. In [28], Waelbroeck does not introduce an analogue of our category $\text{hMon}\mathcal{B}$ but uses tacitly that equalities might only hold up to homotopy. Moreover, the term “pseudoisomorphism” here refers only to those pulations $(\alpha', \alpha)$ where $\alpha$ is surjective. In [28, p. 554], Waelbroeck first states that $\tilde{q}\mathcal{B}$, which here corresponds to $\text{Mon}\mathcal{B}$, is a subcategory of $\text{Mod} k$. Then he defines $q\mathcal{B}$ to be the subcategory generated by $q\mathcal{B}$ and the inverses of pseudoisomorphisms. In his book [30, p. 77], he later corrects the first statement and works around the second. The proofs in [28] show that Waelbroeck probably wanted $q\mathcal{B}$ to be the free category generated by the graph $\Gamma(V, E)$ with

$$V := \tilde{q}\mathcal{B}, \ E := \{ s^{-1} \circ u; \ s \text{ is a pseudoismorphism and } \text{dom} \ s = \text{dom} \ u \}$$

$$\partial_0(s^{-1} \circ u) := \text{cod} \ s \text{ and } \partial_1(s^{-1} \circ u) := \text{cod} \ u$$

in the notation of Mac Lane [17, p. 48ff]. This would coincide with the way localizations were defined by Gabriel, Zisman, see Milić [21, Chapter 1.1].

In the remainder we address the question in which sense the categories $\text{Mon}\mathcal{B}$, $\text{hMon}\mathcal{B}$ and $(\text{hMon}\mathcal{B})[\Sigma^{-1}]$ can or cannot be considered as subcategories of $\text{Mod} k$. This relates to our discussion above and to the functor $R$: $\text{Mon}\mathcal{B} \to \text{Mod} k$ defined via

$$R(f) = X/f(X') \quad \text{and} \quad R(\alpha', \alpha)([x]_{f(X')}) = [f(x)]_{g(Y')}$$

for $f: X' \to X$, $g: Y' \to Y$ and $(\alpha', \alpha): f \to g$. Note that Waelbroeck uses this functor in [30, Definition 2.1.6 and proof of Theorem 2.1.25] to define his category $Q$.

We first observe that $R$: $\text{Mon}\mathcal{B} \to \text{Mod} k$ is not injective on objects. This can be seen by means of linear algebra. For instance, we have $R(0: 0 \to k) = k/0 \cong k^2/k \oplus 0 = R(\left[\begin{smallmatrix}1 \\ 0 \end{smallmatrix}\right]: k \to k^2)$ but $0: 0 \to k$ and $\left[\begin{smallmatrix}1 \\ 0 \end{smallmatrix}\right]: k \to k^2$ cannot be isomorphic in $\text{Mon}\mathcal{B}$. This would mean that there exists a morphism $(\alpha', \alpha)$ with both entries being isomorphisms in $\mathcal{B}$. We see further that $R$: $\text{Mon}\mathcal{B} \to \text{Mod} k$ is also not faithful. Consider the two objects from above and the diagram

$$\begin{array}{ccc}
0 & \xrightarrow{0} & k \\
0 & \downarrow & \downarrow \\
k & \xrightarrow{[\frac{1}{0}]} & k^2
\end{array}$$

which represents the morphism $(0, [\frac{1}{0}]): 0 \to [\frac{1}{0}]$. This morphism is non-zero in $\text{Mon}\mathcal{B}$, but $R(0, [\frac{0}{1}]) = 0$ holds. Next, we show that we can consider $R$ also as a functor $\text{hMon}\mathcal{B} \to \text{Mod} k$.

**Lemma 23.** Let $f: X' \to X$ and $g: Y' \to Y$ be monomorphisms in $\mathcal{B}$ and let $(\alpha', \alpha): f \to g$ be a morphism in $\text{Mon}\mathcal{B}$. The map $R(\alpha', \alpha)$ is zero in $\text{Mod} k$ if and only if the morphism $(\alpha', \alpha)$ is zero in $\text{hMon}\mathcal{B}$.

**Proof.** \(\Rightarrow\) Assume $R(\alpha', \alpha) = 0$. We fix $x \in X$. By assumption, we have $\alpha(x) \in g(Y')$. Since $g$ is injective, we find precisely one $y' \in Y'$ with $g(y') = \alpha(x)$. We define $\rho: X \to Y'$ via $\rho(x) = y'$ and obtain a linear map which is continuous by the closed graph theorem. Let $x_n \to x$ and $\rho(x_n) \to y'$. We have $\alpha(x_n) = g(\rho(x_n))$, where the first sequence converges to $\alpha(x)$ and the second to $g(y')$. Thus $\alpha(x) = g(y')$ is valid, which means $\rho(x) = y'$. By construction, we have $\alpha = g \circ \rho$, from whence it follows that $(\alpha, \alpha')$ is zero in $\text{hMon}\mathcal{B}$.

\(\Leftarrow\) Select $\rho: X \to Y'$ with $\alpha = g \circ \rho$. For $x \in X$ we have $\alpha(x) = g(\rho(x)) \in g(Y')$ and thus $R(\alpha', \alpha)([x]_{f(X')}) = [\alpha(x)]_{g(Y')} = 0$. \qed

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If we now take two morphisms \((\beta', \beta), (\alpha', \alpha): f \to g\) with \(R(\alpha', \alpha) = R(\beta', \beta)\), then \(R((\alpha', \alpha)(\beta', \beta))\) is zero in \(\text{Mod}_k\) by definition. From Lemma 23 we get \((\alpha', \alpha) = (\beta', \beta)\) in \(\text{hMon}\). This shows that \(R: \text{hMon} \to \text{Mod}_k\) is faithful. However, the latter is again not injective on objects. We need now an analytic counterexample. Consider the morphism

\[
\begin{array}{ccc}
\ell^\infty & \xrightarrow{\alpha'} & \ell^\infty \\
\downarrow i & & \downarrow i \\
\ell^\infty/c_0 & \xrightarrow{\beta} & \ell^\infty
\end{array}
\]

where \(i\) is the inclusion and \(q\) is the quotient map. Then \(R(i, 0) = \text{id}_{\ell^\infty/c_0}\) but \(i: c_0 \to \ell^\infty\) and \(0: 0 \to \ell^\infty/c_0\) are not isomorphic. Assume, to the contrary, that there exist \((\alpha', \alpha)\) and \((\beta', \beta)\) such that their composition

\[
\begin{array}{ccc}
c_0 & \xrightarrow{\alpha'} & 0 \\
\downarrow & & \downarrow \\
\ell^\infty & \xrightarrow{\beta} & \ell^\infty/c_0
\end{array}
\]

is equal to \(\text{id}\) in \(\text{hMon}\). That is, there exists \(\rho: \ell^\infty \to c_0\) such that \(i \circ \rho = \text{id}_{\ell^\infty} - \beta \circ \alpha\). In view of \(i \circ \rho \circ i = i \circ (\text{id}_{c_0} - \beta \circ \alpha') = i \circ (\text{id}_{c_0} - i) = i\), we get that \(p := i \circ \rho\) is a projection from \(\ell^\infty\) onto \(c_0\). Contradiction. Our last aim is to extend \(R\) to \((\text{hMon})[\Sigma^{-1}]\).

**Lemma 24.** Let \(f: X' \to X\) and \(g: Y' \to Y\) be monomorphisms in \(\mathcal{B}\) and let \((\alpha', \alpha): f \to g\) be a morphism in \(\text{Mon}\). The map \(R(\alpha', \alpha)\) is an isomorphism if and only if the morphism \((\alpha', \alpha)\) is a pullation.

**Proof.** Below we use the forgetful functor \(G: \mathcal{B} \to \text{Mod}_k\) and Lemma 8. For the sake of readability we drop the letter \(G\) from the notation.

“\(\Rightarrow\)” Let \(x \in \alpha^{-1}(g(Y'))\) be given, i.e., \(\alpha(x) \in g(Y')\) holds. We select \(y' \in Y'\) such that \(\alpha(x) = g(y')\) holds. This means that \([x]_{f(X')} = 0\) and thus \(x \in f(X')\). Let now \(y \in Y\) be given. Then we find \(x \in X\) with \([\alpha(x)]_{g(Y')} = R(\alpha', \alpha)(x) = [y]_{g(Y')}\). Consequently, we find \(y' \in Y'\) such that \(\alpha(x) - y = g(y')\) holds. As we just proved the inclusions \(\alpha^{-1}(g(Y')) \subseteq f(X')\) and \(Y \subseteq g(Y')\), we are done in view of Lemma 8.(iii).

“\(\Leftarrow\)” In view of Lemma 8.(iii), we may assume that \(\alpha^{-1}(g(Y')) = f(X')\) and \(Y = g(Y') + \alpha(X)\) hold. If now \(R(\alpha', \alpha)((x)_{f(X')}) = 0\) is valid, then \(\alpha(x) \in g(Y')\) holds and by the above it follows that \(x \in f(X')\). This means that \([x]_{f(X')} = 0\). Given \([y]_{g(Y')}\), we may select \(y' \in Y'\) and \(x \in X\) with \(y = g(y') + \alpha(x)\), i.e., \(R(\alpha', \alpha)((x)_{f(X')}) = [y]_{g(Y')}\).

By Lemma 24 and the universal property of the localization, see [21, Theorem 1.1.1], we get a functor \(S: (\text{hMon})[\Sigma^{-1}] \to \text{Mod}_k\) which makes the diagram

\[
\begin{array}{ccc}
\text{hMon} & \xrightarrow{R} & \text{Mod}_k \\
\downarrow Q & & \downarrow \rotatebox{90}{S} \\
(\text{hMon})[\Sigma^{-1}] & & \end{array}
\]

commutative. Here \(Q\) denotes the canonical functor. As (31) is a pullation, we see that the two monomorphisms \(i: c_0 \to \ell^\infty\) and \(0: 0 \to \ell^\infty/c_0\) are now isomorphic in \((\text{hMon})[\Sigma^{-1}]\), cf. [28, Remark after Definition 2]. Considering the functor \(S\), we thus improved in some sense on the non-injectivity of \(R\). However, in the category \(\text{Mod}_k\) there are just too many isomorphisms to get injectivity on objects. Let \(Z\) be an infinite-dimensional vector space and let \(\|\cdot\|_X\) and \(\|\cdot\|_{Y'}\) be two non-equivalent norms on \(Z\) such that \(X := (Z, \|\cdot\|_X)\) and \(Y := (Z, \|\cdot\|_{Y'})\) are both Banach spaces. Then \(0: 0 \to X\) and \(0: 0 \to Y\) are not isomorphic in \((\text{hMon})[\Sigma^{-1}]\) but \(S(0: 0 \to X) \cong Z \cong S(0: 0 \to Y)\) holds in \(\text{Mod}_k\). We have however the following result.
Proposition 25. The functor $\Sigma : \mathrm{(hMonB)}_{\Sigma^{-1}} \to \mathbf{Mod}_k$ is faithful and conservative.

Proof. The functor $S$ is the identity on objects and sends a morphism $\varphi$ which is represented by the left roof

$$
\begin{array}{ccc}
(x', x) & \xrightarrow{f} & (x', x) \\
\sim & & \sim \\
R(x', x)^{-1} & \xrightarrow{g} & R(x', x)^{-1}
\end{array}
$$

to $S(\varphi) = R(x', x) \circ R(x', x)^{-1}$. The linear map $R(x', x)^{-1}$ is an isomorphism and thus $S(\varphi) = 0$ holds if and only if $R(x', x) = 0$ is valid. It follows with Lemma 23 that $S$ is faithful. Furthermore, if $S(\varphi)$ is an isomorphism, then $R(x', x)$ is an isomorphism. Lemma 24 implies that $(x', x)$ is a pullback. Thus $\varphi$ is an isomorphism in $(\mathrm{hMonB})_{\Sigma^{-1}}$. \hfill \Box

We conclude this article by observing that the results and remarks of this section can be transferred verbatim to the category of Fréchet spaces, where they correspond to Waelbroeck’s consideration in [29], or to other categories of locally convex spaces, e.g., to the category of LB-spaces mentioned in Section 3, where they are completely new.

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