Exact results and new insights for models defined over small world networks. First and second order phase transitions. II: Applications

M. Ostilli\textsuperscript{1,2} and J. F. F. Mendes\textsuperscript{1}
\textsuperscript{1}Departamento de Física da Universidade de Aveiro, 3810-193 Aveiro, Portugal
\textsuperscript{2}Center for Statistical Mechanics and Complexity, INFMI-CNR SMC, Unità di Roma 1, Roma, 00185, Italy.

We apply a novel method (presented in part I) to solve several small-world models for which the method can be applied analytically: the Viana-Bray model (which can be seen as a 0 or infinite dimensional small-world model), the one-dimensional chain small-world model, and the small-world spherical model in generic dimension. In particular, we analyze in detail the one-dimensional chain small-world model with negative short-range coupling showing that in this case, besides a second-order spin glass phase transition, there are two critical temperatures corresponding to first- or second-order phase transitions.

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I. INTRODUCTION

This paper represents the continuation of another paper \textsuperscript{1} in the following referred as part I - in which a general method to analyze small-world models has been presented. Here (part II) we apply it to some cases of interest which can be fully faced analytically. Even if the exactness of this method is limited to the paramagnetic regions, it gives the exact critical behavior and the exact critical surfaces, and provide a clear and immediate (also in terms of calculation) insight of the physics. As stressed in part I, the underlying structure of the non random part of the model, \textit{i.e.}, the set of spins staying in a given lattice $\mathcal{L}_0$ of dimension $d_0$ and interacting through a fixed coupling $J_0$, is exactly taken into account. When $J_0 \geq 0$, the small-world effect gives rise to the known fact that a second order phase transition takes place, independently of the dimension $d_0$ and of the added random connectivity $c$. However, when $J_0 < 0$, a completely different scenario emerges and - besides a second-order spine glass transition - for a sufficiently large $c$, multiple first and second-order ferromagnetic phase transitions may take place. Here we will emphasize above all this aspect which, to the best of our knowledge, until now was observed only in \textsuperscript{2} and in the context of small-world neural networks in \textsuperscript{3}, and in our opinion represents a new paradigm in view of interesting applications in real small-world networks.

The main advantage of our method lies in its great simplicity. In fact, what is required to apply it, is to solve not the small-world model (a random model) defined over $\mathcal{L}_0$, but a corresponding non random model still defined over $\mathcal{L}_0$ and immersed in an external magnetic field. This implies that we are able to solve analytically small-world models whose underlying lattice $\mathcal{L}_0$ has dimension 0, 1 or infinite, as for an ensemble of non interacting units, the one dimensional chain, and the spherical model, respectively. In fact, in all these cases the non random model can be exactly solved even when immersed in an external magnetic field.

The paper is organized as follows. In Secs. II and III we recall the definition of the small-world models and the method, which mainly consists in finding the solution of the self-consistent equation \textsuperscript{12} minimizing the effective free energy \textsuperscript{24}. In Sec. IV we analyze two simple cases in which $\mathcal{L}_0$ has dimension $d_0 = 0$: an ensemble of non interacting units. The small world-model defined on it gives rise in particular to the well known Viana-Bray model (sometime called random Bethe lattice model). In Sec. V we consider the case in which $\mathcal{L}_0$ is the one dimensional chain. Sec. VI is devoted to the spherical model - an infinite-dimensional model - defined for arbitrary $d_0$ \textsuperscript{4}. Finally, in Sec. VII some conclusions are drawn.

II. SMALL WORLD MODELS

We consider random Ising models constructed by super-imposing random graphs with finite average connectivity onto some given lattice $\mathcal{L}_0$ whose set of bonds $(i,j)$ and dimension will be indicated by $\Gamma_0$ and $d_0$, respectively. Given an Ising model - shortly the unperturbed model - of $N$ spins coupled over $\mathcal{L}_0$ through a coupling $J_0$ and with Hamiltonian

$$H_0 \equiv -J_0 \sum_{(i,j)\in \Gamma_0} \sigma_i \sigma_j - h \sum_i \sigma_i, \quad (1)$$

and given an ensemble $\mathcal{C}$ of unconstrained random graphs $c$, $c \in \mathcal{C}$, whose bonds are determined by the adjacency matrix elements $c_{i,j} = 0, 1$, we define the corresponding small-world model - shortly the random model - as described by the following Hamiltonian

$$H_{c,J} \equiv H_0 - \sum_{i<j} c_{i,j} J_{ij} \sigma_i \sigma_j, \quad (2)$$

the free energy $F$ and the averages $\langle \mathcal{O} \rangle$ being defined in the usual (quenched) way as

$$-\beta F \equiv \sum_{c \in \mathcal{C}} P(c) \int d\mathcal{P} \langle \{ J_{i,j} \} \rangle \log (Z_{c,J}), \quad (3)$$
and

\[ \langle O \rangle_l = \sum_{c \in C} P(c) \int dP(\{J_{i,j}\}) \langle O \rangle_l, \quad l = 1, 2 \quad (4) \]

where \( Z_{c,J} \) is the partition function of the quenched system

\[ Z_{c,J} = \sum_{\{\sigma_i\}} e^{-\beta H_{c,J}(\{\sigma_i\})}, \quad (5) \]

\( \langle O \rangle_{c,J} \) the Boltzmann-average of the quenched system (note that \( \langle O \rangle_{c,J} \) depends on the given realization of the \( J \)'s and of \( c \); \( \langle O \rangle = \langle O \rangle_{c,J} \); for shortness we will often omit to write these dependencies)

\[ \langle O \rangle = \sum_{\{\sigma_i\}} O_{c,J} e^{-\beta H_{c,J}(\{\sigma_i\})} / Z_{c,J}, \quad (6) \]

and \( dP(\{J_{i,j}\}) \) and \( P(c) \) are two product measures given in terms of two normalized measures \( d\mu(J_{i,j}) \geq 0 \) and \( p(c_{i,j}) \geq 0 \), respectively:

\[ dP(\{J_{i,j}\}) = \prod_{(i,j), i < j} d\mu(J_{i,j}), \quad \int d\mu(J_{i,j}) = 1, \quad (7) \]

\[ P(c) = \prod_{(i,j), i < j} p(c_{i,j}), \quad \sum_{c_{i,j}=0,1} p(c_{i,j}) = 1. \quad (8) \]

The variables \( c_{i,j} \in \{0,1\} \) specify whether a “long-range” bond between the sites \( i \) and \( j \) is present \( (c_{i,j} = 1) \) or absent \( (c_{i,j} = 0) \), whereas the \( J_{i,j} \)'s are the random variables of the given bond \( (i,j) \). For the \( c_{i,j} \)'s, we shall consider the following distribution

\[ p(c_{i,j}) = \frac{c}{N} \delta_{c_{i,j},1} + \left( 1 - \frac{c}{N} \right) \delta_{c_{i,j},0}, \quad (9) \]

where \( c > 0 \). This choice leads in the thermodynamic limit \( N \to \infty \) to a number of long range connections per site distributed according to a Poisson law with mean connectivity \( c \).

When we will need to be specific, for the \( J_{i,j} \)'s we will assume either the distribution

\[ \frac{d\mu(J_{i,j})}{dJ_{i,j}} = \delta(J_{i,j} - J), \quad (10) \]

or

\[ \frac{d\mu(J_{i,j})}{dJ_{i,j}} = p\delta(J_{i,j} - J) dJ_{i,j} + (1 - p)\delta(J_{i,j} + J), \quad (11) \]

to consider ferromagnetism or glassy phases, respectively. In Eq. (11) \( p \in [0, 1] \).

### III. AN EFFECTIVE FIELD THEORY

Depending on the temperature \( T \), and on the parameters of the probability distributions, \( d\mu \) and \( p \), the random model may stably stay either in the paramagnetic (P), in the ferromagnetic (F), or in the spin glass (SG) phase. In our approach for the F and SG phases there are two natural order parameters that will be indicated by \( m(F) \) and \( m(SG) \). Similarly, for any correlation function, quadratic or not, there are two natural quantities indicated by \( C(F) \) and \( C(SG) \), and that in turn will be calculated in terms of \( m(F) \) and \( m(SG) \), respectively. To avoid confusion, it should be kept in mind that in our approach, for any observable \( O \) there are - in principle - always two solutions that we label as F and SG, but, for any temperature, only one of the two solutions is stable and useful in the thermodynamic limit.

In the following, we will use the label \( g \) to specify that we are referring to the unperturbed model with Hamiltonian \( H_0 \). Let \( m_0(\beta J_0, \beta h) \) be the stable magnetization of the unperturbed model with coupling \( J_0 \) and in the presence of a uniform external field \( h \) at inverse temperature \( \beta \). Then, the order parameters \( m^{(\Sigma)} \), \( \Sigma = F, SG \), satisfy the following self-consistent decoupled equations

\[ m^{(\Sigma)} = m_0(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h) \quad (12) \]

where the effective couplings \( J^{(F)}_0 \), \( J^{(SG)}_0 \), \( J^{(F)}_0 \) and \( J^{(SG)}_0 \) are given by

\[ \beta J^{(F)}_0 = c \int d\mu(J_{i,j}) \tanh(\beta J_{i,j}), \quad (13) \]

\[ \beta J^{(SG)}_0 = c \int d\mu(J_{i,j}) \tanh^2(\beta J_{i,j}), \quad (14) \]

\[ J^{(F)}_0 = J_0, \quad (15) \]

and

\[ \beta J^{(SG)}_0 = \tanh^{-1}(\tanh^2(\beta J_0)). \quad (16) \]

For the correlation functions \( C^{(\Sigma)} \), \( \Sigma = F, SG \), we have

\[ C^{(\Sigma)} = C_0(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h) + O \left( \frac{1}{N} \right), \quad (17) \]

where \( C_0(\beta J_0, \beta h) \) is the correlation function of the unperturbed (non random) model. For the corrective \( O(1/N) \) term in Eq. (17) we remind the reader to Eq. (31) of part I. Let us indicate by \( C^{(1)} \) and \( C^{(2)} \) the averages and the quadratic averages over the disorder of the correlation function of degree, say, \( k \). \( C^{(1)} \) and \( C^{(2)} \), are related to \( C^{(F)} \) and \( C^{(SG)} \), as follows

\[ C^{(1)} = C^{(F)}, \quad \text{in } F, \quad (18) \]

\[ C^{(1)} = 0, \quad \text{in } F, \quad (19) \]

\[ C^{(1)} = C^{(SG)}, \quad \text{in } SG, \quad (20) \]
and
\[ C^{(2)} = (C^{(F)})^2, \text{ in F}, \]  
\[ C^{(2)} = (C^{(SG)})^2, \text{ in SG}. \]  

Among all the possible stable solutions of Eqs. (12), in the thermodynamic limit, for both \( \Sigma = F \) and \( \Sigma = SG \), the true solution \( \tilde{m}^{(\Sigma)} \), or leading solution, is the one that minimizes \( L^{(\Sigma)} \):
\[ L^{(\Sigma)}(\tilde{m}^{(\Sigma)}) = \min_{m \in [-1,1]} L^{(\Sigma)}(m), \]  
where
\[ L^{(\Sigma)}(m) = \frac{\beta J^{(\Sigma)}(m)^2}{2} + \beta f_0(\beta J^{(\Sigma)}(m) + \beta h) \]  
\( f_0(\beta J_0, \beta h) \) being the free energy density in the thermodynamic limit of the unperturbed model with coupling \( J_0 \) and in the presence of an external field \( h \), at inverse temperature \( \beta \). A necessary condition for a solution \( m^{(\Sigma)} \) to be the leading solution is the stability condition:
\[ \tilde{\chi}_0 \left( \beta^{(\Sigma)} J_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h \right) \beta^{(\Sigma)} J^{(\Sigma)} < 1, \]  
where \( \tilde{\chi}_0(\beta J_0, \beta h) \) stands for the susceptibility \( \chi_0 \) of the unperturbed model divided by \( \beta \).

For the localization and the reciprocal stability between the F and SG phases we remind the reader to Sec. IIID of part I. We recall however that, at least for lattices \( \mathcal{L}_0 \) having only loops of even length, the stable P region is always that corresponding to a P-F phase diagram, so that in the P region the correlation functions must be calculated only through Eqs. (18) and (21).

If \( J_0 \geq 0 \), the inverse critical temperature \( \beta^{(\Sigma)}_c \) is solution of the following exact equation
\[ \tilde{\chi}_0 \left( \beta^{(\Sigma)}_c J_0^{(\Sigma)} 0 \right) \beta^{(\Sigma)}_c J^{(\Sigma)} = 1, \quad \beta^{(\Sigma)}_c < \beta^{(\Sigma)}_{\alpha_0}, \]  
where \( \beta^{(\Sigma)}_{\alpha_0} \) is the inverse critical temperature of the unperturbed model with coupling \( J_0^{(\Sigma)} \). The constrain in Eq. (20) ensures the unicity of the solution.

We end this section by stressing that this method is exact in all the P region and provides the exact critical surface, behavior and percolation threshold, and that, in the absence of frustration, the order parameters \( m^{(\Sigma)} \) become exact also in the limit \( c \to 0^+ \), in the case of second-order phase transitions, and in the limit \( c \to \infty \) (see Sec. IIIC of part I). Note also that the order parameters \( m^{(\Sigma)} \), and then the correlation functions, are by construction always exact in the zero temperature limit.

IV. SMALL WORLD IN \( d_0 = 0 \) DIMENSION

A. The Viana-Bray model

As an immediate example, let us consider the Viana-Bray model. It can be seen as the simplest small-world model in which \( N \) spins with no short-range couplings (here \( J_0 = 0 \)) are randomly connected by long-range connections \( J \) (possibly also random). Note that formally here \( \mathcal{L}_0 \) has dimension \( d_0 = 0 \). Since \( J_0 = 0 \), for the unperturbed model we have
\[ -\beta f_0(0, \beta h) = \log[2 \cosh(\beta h)], \]  
\[ m_0(0, \beta h) = \tanh(\beta h), \]  
\[ \tilde{\chi}_0(\beta J_0, \beta h) = 1 - \tanh^2(\beta h)|_{\beta h = 0} = 1, \]  
It is interesting to check that the first and second derivatives of \( \tilde{\chi}_0 \) in \( h = 0 \) are null and negative, respectively. In fact we have
\[ \frac{\partial}{\partial h} \tilde{\chi}_0(0, \beta h) = -2 \tanh^2(\beta h) \]  
\[ \times \left[ 1 - \tanh^2(\beta h) \right] |_{\beta h = 0} = 0, \]  
and
\[ \frac{\partial^2}{\partial h^2} \tilde{\chi}_0(0, \beta h) = -2 \left[ 1 - \tanh^2(\beta h) \right]^2 + 4 \tanh^2(\beta h) \left[ 1 - \tanh^2(\beta h) \right] |_{\beta h = 0} = -2. \]  

Applying these results to Eqs. (12) we get immediately the self-consistent equations for the F and the SG magnetizations
\[ m^{(F)} = \tanh \left[ m^{(F)} c \int d\mu \tanh(\beta J) \right], \]  
\[ m^{(SG)} = \tanh \left[ m^{(SG)} c \int d\mu \tanh^2(\beta J) \right], \]  
and the Viana-Bray critical surface
\[ c \int d\mu \tanh(\beta^{(F)} J) = 1, \]  
\[ c \int d\mu \tanh^2(\beta^{(SG)} J) = 1. \]  

On choosing for \( d\mu \) a measure having average and variance scaling as \( O(1/c) \), for \( c \propto N \), we recover the equations for the Sherrington-Kirkpatrick model already derived in this form in \( \mathbb{F} \) and \( \mathbb{H} \). In these papers Eqs. (32-35) were derived by mapping the Viana-Bray model and, similarly, the Sherrington-Kirkpatrick model to the non random fully connected Ising model. In this sense it should be also clear that, at least for \( \beta \leq \beta_c \) and zero external field, in the thermodynamic limit, the connected correlation functions (of order \( k \) greater than 1) in the Sherrington-Kirkpatrick and in the Viana-Bray model are exactly zero. In fact, in the thermodynamic limit, the non random fully connected model can be exactly reduced to a model of non interacting spins immersed in an effective medium so that among any two spins there is no correlation. Such a result is due to the fact that, in
these models, all the $N$ spins interact through the same coupling $J/N$, no matter how far apart they are, and the net effect of this is that in the thermodynamic limit the system becomes equivalent to a collection of $N$ non-interacting spins seeing only an effective external field (the medium) like in Eq. (28) with $\beta h \rightarrow \beta J m_0$.

In the limit $\beta \rightarrow \infty$, Eqs. (32) and (33) give the following size (normalized to 1) of the giant connected component

$$m^{(F)} = \tanh(m^{(F)} c),$$

$$m^{(SG)} = \tanh(m^{(SG)} c).$$

These equations are not exact, however, they succeed in giving the exact percolation threshold $c = 1$. In fact, concerning the equation (33) for the $F$ phase, the exact equation for $m^{(F)}$ is (see for example [1] and references therein)

$$1 - m^{(F)} = e^{-c m^{(F)}},$$

which, in terms of the function tanh, becomes

$$\frac{2 m^{(F)} + (m^{(F)})^2}{2 - m^{(F)} + (m^{(F)})^2} = \tanh(m^{(F)} c),$$

so that Eqs. (32) and (33) are equivalent at the order $O(m^{(F)})$. We see also that, as stated in the previous Section, Eqs. (32) and (33) become equal in the limits $c \rightarrow 0$ and $c \rightarrow \infty$.

### B. Gas of Dimers

Let us consider for $L_0$ a set of $2N$ spins coupled through a coupling $J_0$ two by two. The expression “gas of dimers” stresses the fact that the dimers, i.e. the couples of coupled spins, do not interact each other. As a consequence, the free energy, the magnetization, and the susceptibility of the unperturbed model can be immediately calculated. We have

$$-\beta f_0(\beta J_0, \beta h) = \frac{1}{2} \log \left[ 2 e^{\beta J_0} \cosh(2\beta h) + 2 e^{-\beta J_0} \right]$$

$$m_0(\beta J_0, \beta h) = \frac{e^{\beta J_0} \sinh(2\beta h)}{e^{\beta J_0} \cosh(2\beta h) + e^{-\beta J_0}},$$

$$\tilde{\chi}_0(\beta J_0, \beta h) = \frac{2 e^{\beta J_0} + 2 \cosh(2\beta h) e^{-\beta J_0}}{[e^{\beta J_0} \cosh(2\beta h) + e^{-\beta J_0}]^2} \big|_{\beta h = 0} = \frac{e^{\beta J_0}}{\cosh(\beta J_0)}.$$  

Let us calculate also the second derivative of $\tilde{\chi}_0$. From

$$\frac{\partial^2}{\partial \beta h^2} \tilde{\chi}_0(\beta J_0, \beta h) = 4 \sinh(\beta h) \times \frac{e^{-\beta J_0} - 2 e^{\beta J_0} - e^{3\beta J_0} \cosh(2\beta h)}{[e^{\beta J_0} \cosh(2\beta h) + e^{-\beta J_0}]^3},$$

we get

$$\frac{\partial^2}{(\beta h)^2} \tilde{\chi}_0(\beta J_0, \beta h) \bigg|_{\beta h = 0} = -2 \frac{\sinh(\beta J_0) + e^{3\beta J_0}}{[\cosh(\beta J_0)]^3}. \quad (44)$$

We note that, as expected, the second derivative of $\tilde{\chi}_0$ in $h = 0$, for $J_0 \geq 0$ is always negative, whereas, for $J_0 < 0$ it becomes positive as soon as $|\beta J_0| > \log(\sqrt{2})$.

By using the above equations, from Sec. III we get immediately the following self-consistent equation for the magnetizations

$$m^{(\Sigma)} = \frac{\tanh(2\beta J^{(\Sigma)}_0) m^{(\Sigma)} + 2 \beta h}{1 + e^{-2\beta J^{(\Sigma)}_0} \sech(2\beta J^{(\Sigma)}_0) [m^{(\Sigma)} + 2 \beta h]},$$

and - at least for $J_0 \geq 0$ - the equation for the critical temperature

$$\frac{e^{\beta \chi_0^{(\Sigma)}(J_0)\beta h}}{\cosh(\beta \chi_0^{(\Sigma)}(J_0)\beta h)} \beta c \beta J^{(\Sigma)}_0 (\Sigma) = 1. \quad (46)$$

As it will be clear soon, this model lies between the Viana-Bray model and the more complex $d_0 = 1$ dimensional chain small-world model, which will be analyzed in detail in the next section. Our major interest in this simpler gas of dimer small-world model is related to the fact that, in spite of its simplicity and $d_0 = 0$ dimensionality - since the second derivative of $\tilde{\chi}_0$ may be positive when $J_0$ is negative - according to the general result of Sec. III B of part I - it is already able to give rise to also multiple first- and second-order phase transitions.

### V. SMALL WORLD IN $d_0=1$ DIMENSION

In this section we will analyze the case in which $L_0$ is the $d_0=1$-dimensional chain with periodic boundary conditions (p.b.c.). The corresponding small-world model with Hamiltonian (2) in zero field has already been analyzed in [3] by using the replica method. Here we will recover the results found in [3] for $\beta c$ and will provide the self-consistent equations for the magnetizations $m^{(F)}$ and $m^{(SG)}$ whose solution, as expected, turns out to be in good agreement with the corresponding solutions found in [3] for $c$ small. It will be however rather evident how much the two methods differ in terms of simplicity and intuitive meaning. Furthermore, we will derive also an explicit expression for the two-points connected correlation function which, to the best of our knowledge, had not been published yet. Finally, we will analyze in the detail the completely novel scenario for the case $J_0 < 0$ which, as mentioned, produces multiple first- and second-order phase transitions.

In order to apply the method of Sec.III we have to solve the one dimensional Ising model with p.b.c. immersed in an external field. The solution of this non random model is easy and well known (see for example [5]). For the
free energy density, the magnetization and the two-points connected correlation function we have

$$- \beta J_0 (\beta J_0, \beta h) = \log (\lambda_1), \quad (47)$$

$$m_0 (\beta J_0, \beta h) = \frac{e^{\beta J_0} \sinh (\beta h)}{\left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{1}{2}}, \quad (48)$$

$$C_0 (\beta J_0, \beta h; \| i - j \|_0) \equiv \left\langle \sigma_i \sigma_j \right\rangle_0 - \left\langle \sigma \right\rangle_0^2 \sin^2 (2 \varphi) \left( \frac{\lambda_2}{\lambda_1} \right)^{\| i - j \|_0}, \quad (49)$$

where the factor $\varphi$ is defined by

$$\cot (2 \varphi) = e^{2 \beta J_0} \sinh (\beta h), \quad 0 < \varphi < \frac{\pi}{2}, \quad (50)$$

$\| i - j \|_0$ is the (euclidean) distance between $i$ and $j$, and $\lambda_1$ and $\lambda_2$ are the two greatest eigenvalues of the matrix appearing in the transfer matrix method, whose ratio is given by (in the following expression the numerator and the denominator are $\lambda_2$ and $\lambda_1$, respectively)

$$\frac{\lambda_2}{\lambda_1} = \frac{e^{\beta J_0} \cosh (\beta h) - \left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{1}{2}}{e^{\beta J_0} \cosh (\beta h) + \left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{1}{2}}. \quad (51)$$

Let us calculate $\tilde{\chi}_0$ and its first and second derivatives. From Eq. (48) we have

$$\tilde{\chi}_0 (\beta J_0, \beta h) = \frac{e^{-\beta J_0} \cosh (\beta h)}{\left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{1}{2}}, \quad (52)$$

$$\frac{\partial}{\partial \beta h} \tilde{\chi}_0 (\beta J_0, \beta h) = \sinh (\beta h) \times \frac{- e^{-3 \beta J_0} - 2 e^{\beta J_0} \cosh^2 (\beta h) - e^{\beta J_0}}{\left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{3}{2}}, \quad (53)$$

$$\frac{\partial^2}{\partial (\beta h)^2} \tilde{\chi}_0 (\beta J_0, \beta h) = \cosh (\beta h) \times \frac{- e^{-3 \beta J_0} - 2 e^{\beta J_0} \cosh^2 (\beta h) - e^{\beta J_0}}{\left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{3}{2}} + O(\beta h)^2. \quad (54)$$

From Eq. (54) we see that for $J_0 > 0$ and any $\beta$ we have, for sufficiently small $h$,

$$\frac{\partial^2}{\partial (\beta h)^2} \tilde{\chi}_0 (\beta J_0, \beta h) < \cosh (\beta h) \times \frac{1 - 3 e^{\beta J_0}}{\left[ e^{2 \beta J_0} \sinh^2 (\beta h) + e^{-2 \beta J_0} \right]^\frac{3}{2}} + O(\beta h)^2 < 0, \quad (55)$$

whereas for $J_0 < 0$ we have

$$\frac{\partial^2}{\partial (\beta h)^2} \tilde{\chi}_0 (\beta J_0, \beta h) \geq 0 \quad \text{for} \quad e^{-2 \beta J_0} > 3. \quad (56)$$

We see therefore that, according to Sec. IIB of part I, when $J_0 < 0$ for $|\beta J_0| \geq \log (3)/4 = 0.1193...$ we have a first-order phase transition.

From Eqs. (12) and (15), for the magnetizations $m^{(F)}$ and $m^{(SG)}$ at zero external field we have

$$m^{(\Sigma)} = \frac{e^{\beta J_0^{(\Sigma)}} \sinh (\beta J^{(\Sigma)} m^{(\Sigma)})}{\left[ e^{2 \beta J_0^{(\Sigma)}} \sinh^2 (\beta J^{(\Sigma)} m^{(\Sigma)}) + e^{-2 \beta J_0^{(\Sigma)}} \right]^\frac{1}{2}}, \quad (57)$$

From Eqs. (20) and (52) we see that a solution $m^{(\Sigma)}$ becomes unstable at the inverse temperature $\beta_c^{(\Sigma)}$ given by

$$e^{2 \beta_c^{(\Sigma)} J_0^{(\Sigma)}} \beta_c^{(\Sigma)} J_0^{(\Sigma)} = 1. \quad (58)$$

For $J_0 \geq 0$ the above equation gives the exact P-F and P-SG critical temperatures in agreement with [8]. When $J_0 < 0$ - unless the transition be of second-order - Eq. (58) for $\Sigma = F$ does not signal a phase transition. In general, as $J_0 < 0$ the P-F critical temperature must be determined by looking at all the stable solutions $m$ of the self-consistent equation (57) and by choosing the one minimizing the effective free energy $L^{(\Sigma)} (m)$ of Eq. (24).

Finally, for the two-point connected correlation function, from Eqs. (17), (49) and (50), we have

$$C^{(\Sigma)} (\| i - j \|_0) = \sin^2 (2 \varphi^{(\Sigma)}) e^{-\| i - j \|_0 / \xi^{(\Sigma)}}, \quad (59)$$

where

$$2 \varphi^{(\Sigma)} = \cot^{-1} \left[ e^{2 \beta J_0^{(\Sigma)}} \sinh (\beta J^{(\Sigma)} m^{(\Sigma)}) \right], \quad (60)$$

and the correlation length $\xi^{(\Sigma)}$ is given by performing the effective substitutions $\beta J_0 \rightarrow \beta J_0^{(\Sigma)}$ and $\beta h \rightarrow \beta f^{(\Sigma)} m^{(\Sigma)}$ in the logarithm of Eq. (51).

Note that $C_0 (\beta J_0, \beta h)$ is even in $\beta h$ so that $C(-m) = C(m)$.

Near the critical temperature we have

$$\sin (2 \varphi^{(\Sigma)}) = 1 - \frac{(e^{2 \beta J_0^{(\Sigma)} m^{(\Sigma)}})^2}{2} + O(m^{(\Sigma)})^4 \quad (61)$$

and
According to the general result (see Eqs. (51)-(55) of part I), we see that the correlation length remains finite at all temperatures.

In Figs. 1-10 we plot the magnetization \( m^{(F)} \) (solid line), \( \tilde{\chi}_0(\beta^{(F)}J_0^{(F)},0)\beta^{(F)}J^{(F)} \) (dashed line), and \( \tilde{\chi}_0(\beta^{(F)}J_0^{(F)},\beta^{(F)}m^{(F)})\beta^{(F)}J^{(F)} \) (dot-dashed line) for several cases obtained by solving Eq. (57) numerically with \( \Sigma=F \), and by choosing the stable solution minimizing \( L^{(F)}(m) \) (see Eq. (23)). In all these examples we have chosen the measure \( \frac{1}{c} \). Figs. 1 and 2 concern two cases with \( J_0 > 0 \) so that one and only one second-order phase transition is present. The input data of these two cases are the same as those analyzed numerically in [8] (note that in the model considered in [8], the long range coupling \( J \) is divided by \( c \)). As already stated in Sec. III, the self-consistent equations become exact in the limit \( c \rightarrow 0 \), for second order phase transitions, and in the limit \( c \rightarrow \infty \). Therefore, for the magnetization, by comparison with [8], in Fig. 1 and 2, where \( c \) is relatively small and big, respectively, we see good agreement also below the critical temperature.

Figs. 3-10 concern eight cases with \( J_0 < 0 \). As explained above, the critical behavior and the localization of the critical temperatures is more complicated when \( J_0 < 0 \). In particular, given \( |J_0| \), if \( c \) is not sufficiently high the solution \( m^{(F)} = 0 \) remains stable at all temperatures and if it is also a leading solution, no phase transition occurs. Let us consider Eq. (58). For \( J_0 < 0 \) the lhs of this equation has some maximum at a finite temperature.

\[
\left( \xi^{(\Sigma)} \right)^{-1} = \left| \log[\tanh(\beta J_0^{(\Sigma)})] - \frac{(\beta J_0^{(\Sigma)} m^{(\Sigma)})^2}{4 \sinh(\beta J_0^{(\Sigma)})} \left[ (e^{\beta J_0^{(\Sigma)}} + e^{-\beta J_0^{(\Sigma)}}) \tanh(\beta J_0^{(\Sigma)}) + e^{3\beta J_0^{(\Sigma)}} - e^{-\beta J_0^{(\Sigma)}} \right] + O(m^{(\Sigma)})^4 \right|
\]
is the one that is both stable and leading. In fact, when Eq. (58).

\[ \delta(r) \equiv \sqrt{1 + r^2} - r. \]  

(64)

Hence, we see that a sufficient condition for the solution \( m^{(F)} = 0 \) to become unstable is that be

\[ c \left( \frac{1 + \delta(r)}{1 - \delta(r)} \right)^r \delta(r) \geq 1. \]  

(65)

Note that the above represents only a condition for the instability of the solution \( m^{(F)} = 0 \), but the true solution is the one that is both stable and leading. In fact, when \( J_0 < 0 \), a phase transition in general may be present also when Eq. (65) is not satisfied and, correspondingly the possible critical temperatures will be not determined by Eq. (58).

In Fig 3 we report a case with \( J = 1, J_0 = -1/2 \) and \( J = 1 \) and \( J = 1 \), and \( J_0 = -0.9 \) and \( J = 1 \). Here \( T_{c1} = 0.85 \) and \( T_{c2} = 2.81 \).

value \( \tilde{\beta} \) given by

\[ \tilde{\beta} J = \frac{1}{2} \log \left[ \frac{1 + \delta(r)}{1 - \delta(r)} \right], \]  

(63)

where \( r \equiv |J_0|/J \), and we have introduced

\[ \delta(r) \equiv \sqrt{1 + r^2} - r. \]  

(64)

FIG. 5: Magnetization (solid line), and curves of stability (dashed and dot-dashed lines) for the case \( c = 5.5, J_0 = -1 \), and \( J = 1 \). Here \( T_{c1} = 1.02 \) and \( T_{c2} = 2.27 \).

\[ T = 0.728, \text{ first- and second-order phase transitions} \]

\[ T = 0.820, \text{ first- and second-order phase transition} \]

\[ T = 1.820, \text{ first- and second-order phase transition} \]

\[ T = 3.641, \text{ two first-order phase transitions} \]

\[ T = 4.87, \text{ two first-order phase transitions} \]

\[ T = 5.828, \text{ first- and second-order phase transitions} \]

\[ T = 7.55, \text{ two first-order phase transitions} \]

transition is present. Similarly, in Fig. 4 we report again a case in which no phase transition is present due to the fact that here \( r \) is relatively big, \( r = 1.1 \). It is interesting to observe that for \( r = 1 \) Eq. (65) requires a value of \( c \) greater than the limit value \( c = 3 + 2\sqrt{2} = 5.8284... \). In both of Figs. 5 and 6 we report a case in which Eq. (65) is still not satisfied, but nevertheless two first-order phase transitions are present. In both of Figs. 7 and 8 we have one first- and one second-order phase transition. In both of Figs. 9 and 10 we have two second-order phase transitions. The critical temperature of second-order transition can be determined also by Eq. (65). In the top of Figs. 5-10 we write the discriminant temperature \( T_c = |J_0|/\log(3) \) below which a phase transition (if any) is first-order (see Eqs. (64)-(66) (see Sec. IIIIB of part I).

Finally in Figs. 11 and 12 we plot the spin glass order parameter \( m^{(SG)} \) (solid line),

\[ \tilde{\chi}_0 \left( \beta^{(SG)} J_0^{(SG)}, 0 \right) \beta^{(SG)} J^{(SG)} \]  

(dashed line), and

\[ \tilde{\chi}_0 \left( \beta^{(SG)} J_0^{(SG)}, \beta J^{(SG)} m^{(SG)} \right) \beta^{(SG)} J^{(SG)} \]  

(dot-dashed
FIG. 9: Magnetization (solid line), and curves of stability (dashed and dot-dashed lines) for the case \( c = 1.6, J_0 = -0.6 \) and \( J = 7 \). Here \( T_{c1} = 2.58 \) and \( T_{c2} = 7.55 \).

FIG. 10: Magnetization (solid line), and curves of stability (dashed and dot-dashed lines) for the case \( c = 1.4, J_0 = -0.5 \) and \( J = 10 \). Here \( T_{c1} = 3.00 \) and \( T_{c2} = 9.34 \).

line) obtained by solving Eq. (57) numerically with \( \Sigma = SG \). In these two examples we have chosen the measure (11) and, for \( c, J, \) and \( J_0 \), we have considered the same parameters of Figs. 1 and 2 of the ferromagnetic case.

Note that, unlike the P-F critical surface, the P-SG critical surface does not depend on the parameter \( p \) entering in Eq. (11). For the reciprocal stability between the P-F and the P-SG critical surface we remind the reader to the general rules of Sec. IId of part I (see cases (1) and (3)) which, for \( J_0 \geq 0 \), reduce to the results reported in Sec. 6.1 of the Ref. [8]. Here we stress just that, if \( J_0 \geq 0 \), for \( p \leq 0.5 \), only the P-SG transition is possible. However, when \( J_0 < 0 \) and \( c \) is not sufficiently large, the SG phase may be the only stable phase even when \( p = 1 \). In fact, although when \( J_0 < 0 \) the solution \( m^{(F)} \) may have two P-F critical temperatures, in general, if the P-SG temperature is between these, we cannot exclude that the solution \( m^{(SG)} \) starts to be the leading solution at sufficiently low temperatures. In Figs. 13 and 14 on the plane \((T, c)\), we plot the phase diagrams corresponding to the cases of Figs. 9 and 10 respectively. These phase diagrams are obtained by solving Eq. (20) supposing that here, as in the cases of Figs. 9 and 10 where \( c = 1.4 \) and \( c = 0.5 \), respectively, the P-F transition is always second-order. We plan to investigate in more detail the phase diagram in future works.

VI. SMALL-WORLD SPHERICAL MODEL IN ARBITRARY DIMENSION \( d_0 \)

In this section we will analyze the case in which the unperturbed model is the spherical model built up over a \( d_0 \)-dimensional lattice \( \mathcal{L}_0 \) (see [2] and references therein) [4]. In this case the \( \sigma \)'s are continuous “spin” variables ranging in the interval \((-\infty, \infty)\) subjected to the sole constraint \( \sum_{i \in \mathcal{L}_0} \sigma_i^2 = N \), however our theorems and formalism can be applied as well and give results that, within the same limitations prescribed in Sec. III, are exact.

Following [2], for the unperturbed model we have

\[
-\beta f_0(\beta J_0, \beta h) = \frac{1}{2} \log \left( \frac{\pi}{\beta J_0} \right) + \phi(\beta J_0, \beta h, \bar{z}) ,
\]

(66)
from which it follows the equation for \( m \)

where \( g \)

with the measure of Eq. (10).

\[ \frac{\partial}{\partial T} \text{ren} \left( \bar{g} \right) = \frac{1}{2} \left( \frac{\beta J}{g} \right) = 1 \]

\[ \text{log} \left[ \beta J \right] = 0; \]

\[ \text{F and/or SG} \]

\[ \text{F and/or SG} \]

\[ \text{Phase diagram} \]

\[ \text{Phase diagram} \]

\[ \text{FIG. 13: Phase diagram for the case considered in Fig. 9 with the measure of Eq. (10).} \]

\[ \text{FIG. 14: Phase diagram for the case considered in Fig. 10 with the measure of Eq. (10).} \]

\[ m_0(\beta J_0, \beta h) = \frac{\beta h}{2 \beta J_0 z}, \] (67)

where

\[ \phi (\beta J_0, \beta h, z) = \beta J_0 d_0 + \beta J_0 z - \frac{1}{2} g(z) + \frac{(\beta h)^2}{4 \beta J_0 z}, \] (68)

\[ g(z) = \frac{1}{(2\pi)^{d_0}} \int_0^{2\pi} \ldots \int_0^{2\pi} d\omega_1 \ldots d\omega_{d_0} \times \log [d_0 + z - \cos(\omega_1) - \ldots - \cos(\omega_{d_0})], \] (69)

and \( \bar{z} = \bar{z}(\beta J_0, \beta h) \) is the (unique) solution of the equation \( \partial_z \phi (\beta J_0, \beta h, z) = 0 \):

\[ \beta J_0 - \frac{(\beta h)^2}{4 \beta J_0 z^2} = \frac{1}{2} g'(\bar{z}), \] (70)

from which it follows the equation for \( m_0 \)

\[ \beta J_0 \left( 1 - m_0^2 \right) = \frac{1}{2} g' \left( \frac{\beta h}{2 \beta J_0 m_0} \right). \] (71)

The derivative \( g' \) can in turn be expressed as

\[ g'(z) = \int_0^\infty e^{-t(z+d_0)} \left[ J_0 \left( it \right) \right] d_0 \ dt, \] (72)

\( J_0 \) (it) being the usual Bessel function whose behavior for large \( t \) is given by

\[ J_0 (it) = \frac{e^t}{(2\pi)^{1/2}} \left( 1 + O \left( \frac{1}{t} \right) \right). \] (73)

The critical behavior of the unperturbed system depends on the values of \( g'(z) \) and \( g''(z) \) near \( z = 0 \). It turns out that for \( d_0 \leq 2 \) one has \( g'(0) = \infty \) and there is no spontaneous magnetization, whereas for \( d_0 > 2 \) one has \( g'(0) < \infty \) and at \( h = 0 \) the unperturbed system undergoes a second-order phase transition with magnetization given by Eq. (71) which, for \( \beta \) above \( \beta_{c_0} \), becomes

\[ m_0(\beta J_0, 0) = \sqrt{1 - \frac{\beta J_0}{\beta}}, \] (74)

where the inverse critical temperature \( \beta_{c_0} \) is given by

\[ \beta_{c_0} J_0 = \frac{1}{2} g'(0). \] (75)

Furthermore, it turns out that for \( d_0 \leq 4 \) one has \( g''(0) = \infty \), whereas for \( d_0 > 4 \) one has \( g''(0) < \infty \). This reflects on the critical exponents \( c, \gamma \) and \( \delta \), which take the classical mean-field values only for \( d_0 > 4 \).

According to sec. III, to solve the random model - for simplicity - at zero external field, we have to perform the effective substitutions \( \beta J_0 \to \beta J_0^{(\Sigma)} \) and \( \beta h \to \beta J_0^{(\Sigma)} m^{(\Sigma)} \) in the above equations. From Eqs. (67), (70) and (71), we get immediately:

\[ \bar{z}^{(\Sigma)} = \frac{\beta J_0^{(\Sigma)}}{2 \beta J_0^{(\Sigma)}}, \] (76)

the equations for inverse critical temperature \( \beta_{c}^{(\Sigma)} \)

\[ \beta_{c}^{(\Sigma)} J_0^{(\Sigma)} = \frac{1}{2} g' \left( \frac{\beta_{c}^{(\Sigma)} J_0^{(\Sigma)}}{2 \beta_{c}^{(\Sigma)} J_0^{(\Sigma)}} \right), \] (77)

and the magnetizations \( m^{(\Sigma)} \)

\[ m^{(\Sigma)} = \begin{cases} \sqrt{1 - \frac{1}{2 \beta J_0^{(\Sigma)} g' \left( \frac{\beta_{c}^{(\Sigma)} J_0^{(\Sigma)}}{2 \beta_{c}^{(\Sigma)} J_0^{(\Sigma)}} \right)}, & \beta > \beta_{c}^{(\Sigma)}, \\ 0, & \beta < \beta_{c}^{(\Sigma)} \geq 0. \end{cases} \] (78)

Note that, as it must be from the general result of Sec. IIIB of part I, unlike the unperturbed model, as soon as the connectivity \( c \) is not zero, Eq. (77) always has a finite solution \( \beta_{c}^{(\Sigma)} \), independently on the dimension \( d_0 \). In fact, one has a finite temperature second-order phase transition even for \( d_0 \to 0^+ \) where from Eq. (72) we have

\[ g'(z) = \frac{1}{z}, \quad (d_0 = 0) \] (79)

so that the equations for the critical temperature (77) become

\[ \beta_{c}^{(\Sigma)} J_0^{(\Sigma)} = 1, \quad (d_0 = 0) \] (80)
which, as expected, coincide with Eqs. (34) and (35) of the Viana-Bray model.

Similarly, unlike the unperturbed model, in the random model all the critical exponents take the classical mean-field values, independently on the dimension \(d_0\). In the specific case of the spherical model, this behavior is due to the fact that \(g'(z)\) and \(g''(z)\) can be singular only at \(z = 0\) but, as soon as the connectivity \(c\) is not zero, there is an effective external field \(\beta J^{(\Sigma)} m^{(\Sigma)}\) so that \(g^{(\Sigma)}\) is not zero. For the critical behavior, the dependence on the dimension \(d_0\) reflects only in the coefficients, not on the critical exponents. In particular, concerning the argument of the square root of the rhs of Eq. (78), by expanding in the reduced temperature \(t^{(\Sigma)}\), for \(|t^{(\Sigma)}| \ll 1\) we have

\[
1 - \frac{1}{2\beta_c^{(\Sigma)} J_0^{(\Sigma)}} g'' \left( \frac{\beta_c^{(\Sigma)} J^{(\Sigma)}}{2\beta_c^{(\Sigma)} J_0^{(\Sigma)}} \right) = B^{(\Sigma)} t^{(\Sigma)} + O(t^{(\Sigma)})^2,
\]

where

\[
B^{(F)} = -1 + \frac{1}{2\beta_c^{(F)} J_0^{(F)}} g'' \left( \frac{\beta_c^{(F)} J^{(F)}}{2\beta_c^{(F)} J_0^{(F)}} \right)
\]

\[
\times \left( c \int d\mu(J_{i,j}) \left( 1 - \tanh^2(\beta_c^{(F)} J_{i,j}) \right) \beta_c^{(F)} J_{i,j} - c \int d\mu(J_{i,j}) \tanh(\beta_c^{(F)} J_{i,j}) \right),
\]

\[
B^{(SG)} = \frac{1}{2\beta_c^{(SG)} J_0^{(SG)}} \left[ -4 \tanh(\beta_c^{(SG)} J_0) \beta_c^{(SG)} J_0 \frac{g'' \left( \frac{\beta_c^{(SG)} J^{(SG)}}{2\beta_c^{(SG)} J_0} \right)}{2 \beta_c^{(SG)} J_0^{(SG)}} \right.
\]

\[
\times \left( 2c \int d\mu(J_{i,j}) \left( 1 - \tanh^2(\beta_c^{(SG)} J_{i,j}) \right) \tanh(\beta_c^{(SG)} J_{i,j}) \beta_c^{(SG)} J_{i,j}
\]

\[
-4c \int d\mu(J_{i,j}) \tanh^2(\beta_c^{(SG)} J_{i,j}) \beta_c^{(SG)} J_{i,j} \right],
\]

so that from Eqs. (78) and (31) for the critical behavior of the magnetizations we get explicitly the mean-field behavior:

\[
m^{(\Sigma)} = \begin{cases} \sqrt{B^{(\Sigma)} t^{(\Sigma)}} + O(t^{(\Sigma)}), & t^{(\Sigma)} < 0, \\ 0, & t^{(\Sigma)} \geq 0. \end{cases}
\]

VII. CONCLUSIONS

We have applied a novel method to solve analytically several small-world models of interest defined over an underlying lattice \(L_0\) of dimension \(d_0 = 0, 1, \infty\), corresponding to an ensemble of non interacting units (spins, dimers, etc..), the one dimensional chain, and the spherical model, respectively. The long-range couplings may also have an additional own disorder leading, in particular, to spin glass phases. The simplicity of the method allows us to find very easily (just a few line equations) the critical surfaces and the order parameters. In our method the former are exact whereas the latter provide an approximation to the exact order parameters which, in the absence of frustration, become exact in the limit \(c \to 0\), when the transition is of second order, and in the limit \(c \to \infty\). In particular, we have studied in detail the small-world defined over the one dimensional chain with positive and negative short-range couplings, showing explicitly how, in the second case, multicritical points with first- and second-order phase transition arise. Finally, the small-world spherical model - an exact solvable model (in our approach) with continuous spin variables - has provided us an interesting testing bench case to study what happens as \(d_0\) changes continuously from 0 to \(\infty\). As expected from general grounds, unlike the non random version of the model, the small-world model presents always a finite temperature phase transition, even in the limit \(d_0 \to 0^+\). This latter result - besides to be consistent with what we have found in the \(d_0 = 0\)-dimensional discrete models - is easily and physically explained by the method. In fact, it consists on mapping the small-world model (a random model) to a corresponding non random model (no long-range bonds) but immersed in an effective uniform external field which is active as soon as the added random connectivity \(c\) is not zero (see Eqs. (12)–(13)).
able by the method. However, the method can be also applied numerically to study other more complex small-world models for which the corresponding unperturbed model is not analytically solvable. In fact, the numerical complexity in solving such a small-world model, will be still as feasible as a non random model immersed in a uniform external field.

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