THE FUNDAMENTAL GROUPS OF CONTACT TORIC MANIFOLDS

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Abstract. Let $M$ be a connected compact contact toric manifold. Most of such manifolds are of Reeb type. We show that if $M$ is of Reeb type, then $\pi_1(M)$ is finite cyclic, and we describe how to obtain the order of $\pi_1(M)$ from the moment map image.

Let $M$ be a contact manifold, and $\alpha$ be a contact 1-form on $M$. Let $T^k$ be a connected compact $k$-dimensional torus. If $T^k$ acts on $M$ preserving the contact form $\alpha$, then it preserves the contact structure $\xi = \ker(\alpha)$.

A contact manifold $M$ of dimension $2n+1$ with an effective $T^{n+1}$-action preserving the contact structure is called a contact toric manifold. If the Reeb vector field of a contact form on $M$ is generated by a one parameter subgroup action of $T^{n+1}$, then the contact $T^{n+1}$-manifold $M$ is called a contact toric manifold of Reeb type.

Recall that a $2n$-dimensional symplectic manifold equipped with an effective Hamiltonian $T^n$-action is called a symplectic toric manifold. Contact toric manifolds are the odd dimensional analog of symplectic toric manifolds. Compact symplectic toric manifolds and compact contact toric manifolds are both classified (\cite{3} and \cite{4}). Most of the compact contact toric manifolds are of Reeb type.

Compact symplectic toric manifolds are simply connected (\cite{1} p235, \cite{6}, \cite{7}). In contrast, the fundamental groups of connected compact contact toric manifolds are finite abelian if they are of Reeb type (\cite{5}), and are infinite abelian if they are not of Reeb type. (The latter fact can be derived by listing the non-Reeb type manifolds using the classification in [4].)

In this paper, we prove a result on the fundamental groups of compact contact toric manifolds of Reeb type. To describe the result, we define some terms and state a known result as follows. Let $(M, \alpha)$ be a connected compact contact toric manifold of dimension $2n+1$. Let $t$ be the Lie algebra of the torus $T^{n+1}$, and $t^*$ be the dual Lie algebra. The contact moment map $\Phi: M \to t^*$ is defined to be

$$(\Phi(x), X) = \alpha_x(X_M(x)), \forall x \in M, \text{ and } \forall X \in t,$$

where $X_M$ is the vector field on $M$ generated by the $X$-action. The moment cone of $\Phi$ is defined as

$$C(\Phi) = \{ t\Phi(x) | t \geq 0, x \in M \}.$$ 

It is known (\cite{2}, \cite{5}, \cite{4} etc.) that, if $M$ is of Reeb type, then $C(\Phi)$ is a strictly convex rational good polyhedral cone (of dimension $n+1$). Strictly convex means that $C(\Phi)$ contains no linear subspaces of $t^*$ of positive dimension, polyhedral means that $C(\Phi)$

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is a cone over a polytope, *rational* means that the normal vectors of the facets of the cone lie in the integral lattice of $t$, and *good* means that for any codimension $l$ face $F_l$ of $C(\Phi)$, the normal vectors of the facets which intersect at $F_l$ form a $\mathbb{Z}$-basis of the lattice of an $l$-dimensional linear subspace of $t$.

**Theorem.** Let $(M, \alpha)$ be a connected compact contact toric manifold of Reeb type with dimension $2n + 1$. Let

$$I = \{v_1, v_2, \cdots, v_d\}$$

be the set of primitive inward normal vectors of the facets of the moment cone, ordered in the way that the first $n$ vectors are the normal vectors of the facets which intersect at a (any) 1-dimensional face of the moment cone. Then

$$\pi_1(M) = \mathbb{Z}_k,$$

where

$$k = \gcd(\det[v_1, v_2, \cdots, v_n, v_{n+1}], \det[v_1, v_2, \cdots, v_n, v_{n+2}], \cdots, \det[v_1, v_2, \cdots, v_n, v_d]).$$

The 3-dimensional lens spaces are compact contact toric manifolds of Reeb type. Hence any finite cyclic group can be the fundamental group of a contact toric manifold of Reeb type.

**Proof of Theorem.** Let $\mathbb{Z}_T \subset t$ be the integral lattice of the torus $T^{n+1}$, and $\mathcal{L}$ the sublattice of $\mathbb{Z}_T$ generated by the elements in $I$. By Lerman’s Theorem [5],

$$\pi_1(M) = \mathbb{Z}_T / \mathcal{L}.$$

We identify $\mathbb{Z}_T = \mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$. Since the moment cone $C(\Phi)$ is a good cone, $v_1, \cdots, v_n$ is a $\mathbb{Z}$-basis of an $n$-dimensional subspace of $\mathbb{Z}^{n+1}$. So there exists another vector $u \in \mathbb{Z}^{n+1}$ such that $\{v_1, \cdots, v_n, u\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$. Let $\mathcal{L}'$ be the sublattice generated by the elements in $\{v_1, \cdots, v_n\}$. Then

$$\mathbb{Z}_T / \mathcal{L}' = \mathbb{Z}^{n+1} / \mathbb{Z}^n = \mathbb{Z} = \mathbb{Z}\langle u \rangle.$$

Since $\{v_1, \cdots, v_n, u\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$, for $\forall$ $n + 1 \leq j \leq d$, we have

$$v_j = l_j u \mod \mathcal{L},$$

where $l_j \in \mathbb{Z}$.

Let $k = \gcd(l_j)_{j=n+1}^d$. Since the elements in $I$ span an $n + 1$-dimensional vector space, at least one $l_j \neq 0$. So $k \neq 0$. Then

$$\mathbb{Z}_T / \mathcal{L} = \mathbb{Z}\langle u \rangle / k\mathbb{Z}\langle u \rangle = \mathbb{Z}_k$$

is finite cyclic. Moreover, notice that

$$l_j = \pm \det[v_1, \cdots, v_n, v_j], \forall \ n + 1 \leq j \leq d,$$

where $[v_1, \cdots, v_n, v_j]$ denotes the matrix with column vectors $v_1, \cdots, v_n$ and $v_j$.

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