FUNDAMENTAL SOLUTIONS OF PSEUDO-DIFFERENTIAL OPERATORS OVER $p$–ADIC FIELDS

W. A. ZUNIGA-GALINDO

Abstract. We show the existence of fundamental solutions for $p$–adic pseudo-differential operators with polynomial symbols.

1. Introduction

Let $K$ be a $p$–adic field, i.e. a finite extension of $\mathbb{Q}_p$ the field of $p$–adic numbers. Let $R_K$ be the valuation ring of $K$, $P_K$ the maximal ideal of $R_K$, and $\overline{K} = R_K/P_K$ the residue field of $K$. The cardinality of $\overline{K}$ is denoted by $q$. For $z \in K$, $v(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of $z$, $|z|_K = q^{-v(z)}$ and $ac(z) = z\pi^{-v(z)}$ where $\pi$ is a fixed uniformizing parameter for $R_K$. Let $\Psi$ denote an additive character of $K$ trivial on $R_K$ but not on $P_K^{-1}$.

A function $\Phi : K^n \to \mathbb{C}$ is called a Schwartz-Bruhat function if it is locally constant with compact support. We denote by $S(K^n)$ the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over $K^n$. The dual space $S'(K^n)$ is the space of distributions over $K^n$. The space $S'(K^n)$ is the space of distributions over $K^n$. Let $f = f(x) \in K[x]$, $x = (x_1, ..., x_n)$, be a non-zero polynomial, and $\beta$ a complex number satisfying $\Re(\beta) > 0$. If $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in K^n$, we set $[x, y] := \sum_{i=1}^{n} x_i y_i$.

A $p$–adic pseudo-differential operator $f(\partial, \beta)$, with symbol $|f|^\beta_{K^n}$, is an operator of the form

$$f(\partial, \beta) : S(K^n) \to S(K^n)$$

$$\Phi \to \mathcal{F}^{-1} \left( |f|^\beta_{K^n} \mathcal{F}(\Phi) \right),$$

where

$$\mathcal{F} : S(K^n) \to S(K^n)$$

$$\Phi \to \int_{K^n} \Psi(-[x, y]) \Phi(x) dx$$

is the Fourier transform. The operator $f(\partial, \beta)$ has self-adjoint extension with dense domain in $L^2(K^n)$. We associate to $f(\partial, \beta)$ the following $p$–adic pseudo-differential equation:

$$f(\partial, \beta)u = g, \quad g \in S(K^n).$$

A fundamental solution for (1.3) is a distribution $E$ such that $u = E \ast g$ is a solution.

The main result of this paper is the following.

1991 Mathematics Subject Classification. Primary 46S10, 11S40.

Key words and phrases. non-archimedean functional analysis, pseudo-differential operators, Igusa’s local zeta function.
Theorem 1.1. Every $p$-adic pseudo-differential equation $f(\partial, \beta)u = g$, with $f(x) \in K[x_1, \ldots, x_n] \setminus K$, $g \in S(K^n)$, and $\beta \in \mathbb{C}$, $\text{Re}(\beta) > 0$, has a fundamental solution $E \in S'(K^n)$.

The $p$-adic pseudo-differential operators occur naturally in $p$-adic quantum field theory [11], [6]. Vladimirov showed the existence of a fundamental solutions for symbols of the form $|\xi|^\alpha_K$, $\alpha > 0$ [10], [11]. In [7], [6] Kochubei showed explicitly the existence of fundamental solutions for operators with symbols of the form $|f(\xi_1, \ldots, \xi_n)|^\alpha_K$, $\alpha > 0$, where $f(\xi_1, \ldots, \xi_n)$ is a quadratic form satisfying $f(\xi_1, \ldots, \xi_n) \neq 0$ if $|\xi_1|^K + \ldots + |\xi_n|^K \neq 0$. In [8] Khrennikov considered spaces of functions and distributions defined outside the singularities of a symbol, in this situation he showed the existence of a fundamental solution for a $p$-adic pseudo-differential equation with symbol $|f|_K \neq 0$. The main result of this paper shows the existence of fundamental solutions for operators with polynomial symbols. Our proof is based on a solution of the division problem for $p$-adic distributions. This problem is solved by adapting the ideas developed by Atiyah for the archimedean case [1], and Igusa's theorem on the meromorphic continuation of local zeta functions [3], [4]. The connection between local zeta functions (also called Igusa's local zeta functions) and fundamental solutions of $p$-adic pseudo-differential operators has been explicitly showed in particular cases by Jang and Sato [5], [9]. In [9] Sato studies the asymptotics of the Green function $G$ of the following pseudo-differential equation

$$\left(f(\partial, 1) + m^2\right)u = g, \quad m > 0.$$  

The main result in [9, theorem 2.3] describes the asymptotics of $G(x)$ when the polynomial $f$ is a relative invariant of some prehomogeneous vector spaces (see e.g. [3, Chapter 6]). The key step is to establish a connection between the Green function $G(x)$ and the local zeta function attached to $f$.

All the above mentioned results suggest a deep connection between Igusa's work on local zeta functions (see e.g. [3]) and $p$-adic pseudo-differential equations.

2. Local zeta functions and division of distributions

The local zeta function associated to $f$ is the distribution

$$\langle |f|^s_K, \Phi \rangle = \int_{K^n \setminus f^{-1}(0)} \Phi(x) |f(x)|^s_K \, dx,$$

where $\Phi \in S(K^n)$, $s \in \mathbb{C}$, $\text{Re}(s) > 0$, and $dx$ is the Haar of $K^n$ normalized so that $\text{vol}(R^n_K) = 1$. The local zeta functions were introduced by Weil [12] and their basic properties for general $f$ were first studied by Igusa [3], [4]. A central result in the theory of local zeta functions is the following.

Theorem 2.1 (Igusa, [3, Theorem 8.2.1]). The distribution $|f|^s_K$ admits a meromorphic continuation to the complex plane such that $\langle |f|^s_K, \Phi \rangle$ is a rational function of $q^{-s}$ for each $\Phi \in S(K^n)$. In addition the real parts of the poles of $|f|^s_K$ are negative rational numbers.

The archimedean counterpart of the previous theorem was obtained jointly by Bernstein and Gelfand [2], independently by Atiyah [1]. The following lemma is a consequence of the previous theorem.
Lemma 2.1. Let \( f(x) \in K[x_1, ..., x_n] \) be a non-constant polynomial, and \( \beta \) a complex number satisfying \( \text{Re}(\beta) > 0 \). Then there exists a distribution \( T \in \mathcal{S}'(K^n) \) satisfying \( |f|^\beta T = 1 \).

Proof. By theorem 2.1 \( |f|^s_K \) has a meromorphic continuation to \( \mathbb{C} \) such that \( \langle |f|^s_K, \Phi \rangle \) is a rational function of \( q^{-s} \) for each \( \Phi \in \mathcal{S}(K^n) \). Let

\[
|f|^s_K = \sum_{m \in \mathbb{Z}} c_m (s + \beta)^m
\]

be the Laurent expansion at \( -\beta \) with \( c_m \in \mathcal{S}(K^n) \) for all \( m \). Since the real parts of the poles of \( |f|^s_K \) are negative rational numbers by theorem 2.1, it holds that \( |f|^{s+\beta}_K = |f|^\beta |f|^s_K \) is holomorphic at \( s = -\beta \). Therefore \( |f|^\beta c_m = 0 \) for all \( m < 0 \) and

\[
|f|^{s+\beta}_K = c_0 |f|^\beta_K + \sum_{m=1}^{\infty} c_m |f|^\beta_K (s + \beta)^m.
\]

By using the Lebesgue lemma and (2.3) it holds that

\[
\lim_{s \to -\beta} \left\langle |f|^{s+\beta}_K, \Phi \right\rangle = \int_{K^n} \Phi(x) dx = \langle 1, \Phi \rangle
\]

\[
= c_0 |f|^\beta_K.
\]

Therefore we can take \( T = c_0 \).

If \( T \in \mathcal{S}'(K^n) \) we denote by \( \mathcal{F}T = \mathcal{S}'(K^n) \) the Fourier transform of the distribution \( S \), i.e. \( \langle \mathcal{F}T, \Phi \rangle = \langle S, \mathcal{F}(\Phi) \rangle, \Phi \in \mathcal{S}(K^n) \).

3. Proof of the main result

By lemma 2.1 there exists a \( T \in \mathcal{S}'(K^n) \) such that \( |f|^\beta_K T = 1 \). We set \( E = \mathcal{F}^{-1}T \in \mathcal{S}'(K^n) \) and assert that \( E \) is a fundamental solution for (1.3). This last statement is equivalent to assert that \( \mathcal{F}(\Phi) = (\mathcal{F}E) \mathcal{F}(\Phi) \) satisfies \( |f|^\beta_K \mathcal{F}(\Phi) = \mathcal{F}(g) \). Since \( |f|^\beta_K \mathcal{F}(\Phi) = |f|^\beta_K (\mathcal{F}E) \mathcal{F}(\Phi) = |f|^\beta_K T \mathcal{F}(g) = g \), we have that \( E \) is a fundamental solution for (1.3).

4. Operators with twisted symbols

Let \( \chi : R^\times_K \to \mathbb{C} \) be a non-trivial multiplicative character, i.e. a homomorphism with finite image, where \( R^\times_K \) is the group of units of \( K \). We put formally \( \chi(0) = 0 \). If \( f(x) \in K[x_1, ..., x_n] \backslash K \), we say that \( \chi(ac(f))|f|^\beta_K \), with \( \beta \in \mathbb{C}, \text{Re}(\beta) > 0 \), is a twisted symbol, and call the pseudo-differential operator

\[
\Phi \to f(\partial, \beta, \chi)\Phi = \mathcal{F}^{-1} \left( \chi(ac(f))|f|^\beta_K \mathcal{F}(\Phi) \right), \Phi \in \mathcal{S}(K^n),
\]

a twisted operator. Since the distribution \( \chi(ac(f))|f|^\beta_K \) satisfies all the properties stated in theorem 2.1 (cf. [3, Theorem 8.2.1]), theorem 1.1 generalizes literally to the case of twisted operators. In [6, chapter 2] Kochubei showed explicitly the existence of fundamental solutions for twisted operators in some particular cases.
References

[1] M. F. Atiyah, Resolution of singularities and division of distributions, Comm. Pure Appl. Math. 23 (1970), 145-150.
[2] I. N. Bernstein and S. I. Gelfand, Meromorphic property of the functions $P^\lambda$, Functional Anal. Appl., 3 (1969), 68-69.
[3] Igusa Jun-Ichi, An introduction to the theory of local zeta functions, AMS/IP studies in advanced mathematics, v. 14, 2000.
[4] Igusa J., Complex powers and asymptotic expansions, I Crelles J. Math., 268/269 (1974), 110-130; II, ibid., 278/279 (1975), 357-368.
[5] Y. Jang, An asymptotic expansion of the $p$-adic Green function, Tohoku Math. J., 50 (1998), 229-242.
[6] A. N. Kochubei, Pseudodifferential equations and stochastics over non-archimedean fields, Marcel Dekker, 2001.
[7] A. N. Kochubei, Fundamental solutions of pseudo-differential equations associated with $p$-adic quadratic forms, Izvestiya Math., 62 (1998), 1169-1188.
[8] A. Khrennikov, Fundamental solutions over the field of $p$-adic numbers, St. Peterburgh Math. J., 4 (1993), 613-628.
[9] Fumihiro Sato, $p$-adic Green functions and zeta functions, Commentarii Mathematici Universitatis Sancti Pauli, 51 (2002), 79-97.
[10] V. S. Vladimirov, On the spectrum of some pseudo-differential operators over $p$-adic number field, Algebra and Analysis, 2, 107-124 (1990).
[11] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, $p$-adic Analysis and mathematical physics, World Scientific, Singapore, 1994.
[12] A. Weil, Sur la formule de Siegel dans le theorie des groupes classiques, Acta Math. 113 (1965), 1-87.

Department of Mathematics and Computer Science, Barry University, 11300 N.E. Second Avenue, Miami Shores, Florida 33161, USA
E-mail address: wzuniga@mail.barry.edu