Quantum State Spectroscopy of Atom-Cavity Systems

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In this work we explore the nature of quantum correlations between two-level atoms (or qubits) and a single mode of a quantum field in phase space. Similar to time-frequency plots, we introduce the notion of a quantum state spectrogram to allow for the visualisation of quantum correlations in coupled systems. By using our method, we are able to gain information on the quantum state in an accessible way that enables an appreciation of subtle quantum effects, without the need for sophisticated mathematics. We apply the method to the Jaynes-Cummings model and some of its extensions to demonstrate that quantum correlations of entanglement, microscopically distinct superposition, and a combination of the two can be readily identified.

I. INTRODUCTION

Macroscopically distinct superpositions of states, also known as Schrödinger cat states, and the existence of entanglement between multiple systems, are the defining features of quantum physics. As technology has moved forward, such quantum correlations have become central to the design and manufacture of quantum technologies, including quantum information systems, quantum computing, and metrology [1–4]. As systems grow in complexity, understanding the quantum correlations becomes highly challenging [5]. Quantitative measures can be used to understand very specific properties from expectation values of observables and entanglement entropies, to deeper system correlations (see, for example, Refs. [6–10], in particular, Ref. [6] demonstrates the vast number of different entanglement measures). Graphical representations of a system’s density matrix can also be used to gain insight into correlations within a system. However, as a system grows in size or complexity, this rapidly becomes too difficult to interpret, let alone display. Phase space methods, providing similar utility to the probability density functions of classical physics, are an alternative approach to gaining an intuitive understanding of a system’s state. Examples include the Husimi Q-functions [11] and Wigner functions [12, 13] of reduced density operators for each subsystem (See, for example, Refs. [14–17]). The drawback of these techniques is that there is always a loss of important information, usually resulting in the inability to represent certain quantum correlations between subsystems. Importantly, however, it has recently been shown how a complete Wigner function may be constructed for any system [18–21].

Wigner functions are frequently used to visualise quantum states because they clearly display the quantum correlations associated with macroscopically distinct superpositions of states (such as those found in Schrödinger cat states). Here, the presence of interference terms in the form of oscillations, that take on both positive and negative values, allow us to see these quantum correlations. In recent work we have explored the visualisation of qubit-cat states, as well as atomic states, allowing us to determine key state characteristics by inspection alone [18–21].

Quantum state tomography seeks to fully reproduce the density matrix of a given system. In this work we explore the possibility of an analogous process using a phase-space representation and how this might be displayed. Specifically, we seek to visualise in an accessible way as much of the full Wigner function of a quantum system as is possible. That is, we propose a form of quantum state spectroscopy. The images this produces are analogous to the usual time-frequency spectrograms, but over the full parametrization of the phase space of the quantum state. These spectrograms allow us to reconstruct important elements of the state, verifying certain quantum properties. We note that experimental realisation would rely on the ability to measure the correct physical quantities, which take the form of displaced parity operators [18–21], for each component of the overall system being studied. Using this approach, we consider how best to graphically represent one of the most important processes in quantum physics – the interaction of light and matter. Here we consider example quantum state spectrograms for the simplest fully quantum mechanical model of a two-level atom (qubit) coupled to a single quantum field mode (the Jaynes-Cummings model), and some of its simple extensions. We show how different quantum states, such as a separable state of a qubit coupled to a Schrödinger cat state and an entangled qubit-field cat, can be clearly identified using this approach.

II. THE WIGNER FUNCTION

The Wigner function $W(q, p)$ for the field mode, in terms of position $q$ and momentum $p$, is traditionally in-
been plotted on the equal-angle slice, $\theta$ for the two-qubit states have four degrees of freedom they have maximally entangled singlet state. As the Wigner functions placed parity is the usual one, explicitly given by

$$\hat{\Pi}_a(\theta, \phi) = \hat{U}(\theta, \phi, \Phi)\hat{\Pi}_a\hat{U}^\dagger(\theta, \phi, \Phi),$$

where the generalised qubit parity is given as $\hat{\Pi}_a = (1 - \sqrt{3}\hat{\sigma}_z)/2$ [18, 20]. The analogue of the displacement operator is given in terms of the standard Euler angles and is explicitly given by $\hat{U}(\theta, \phi, \Phi) = \exp(i\hat{\sigma}_y\phi)\exp(i\hat{\sigma}_z\theta)\exp(i\hat{\sigma}_y\Phi)$, for the standard Pauli matrices $\hat{\sigma}_y$ and $\hat{\sigma}_z$. Note as the parity operator commutes with $\hat{\sigma}_z$, the $\Phi$ terms do not contribute and the qubit Wigner function depends only on $\theta$ and $\phi$, allowing it to be conveniently plotted on the surface of a sphere.

The system’s total generalised parity operator is simply the tensor product of the generalised parity operators for each subsystem. In the case of a qubit coupled to a field mode, this is

$$\hat{\Pi}(q, p, \theta, \phi) = \hat{\Pi}_f(q, p) \otimes \hat{\Pi}_a(\theta, \phi),$$

and the Wigner function, for a density matrix $\rho$, is

$$W(q, p, \theta, \phi) = \text{Tr} \left[ \rho \hat{\Pi}(q, p, \theta, \phi) \right].$$

We can easily extend this to systems with multiple modes or atoms. For instance, the generalised parity, Eq. (5), for $N$ qubits coupled to a field mode is

$$\hat{\Pi}(q, p, \theta, \phi) = \hat{\Pi}_f(q, p) \otimes \hat{\Pi}_a^{\otimes N}(\theta, \phi),$$

where the $N$-qubit displaced parity is generated by

$$\hat{\Pi}_a^{\otimes N}(\theta, \phi) = \bigotimes_{i=1}^{N} \hat{\Pi}_a(\theta_i, \phi_i).$$

A similar procedure can be followed for an arbitrary number of field modes. The Wigner function for the full system is then calculated by taking the expectation value of this generalised parity.

Figure 1 shows a set of qubit Wigner functions that will be useful for interpreting results presented later in this work, following the procedure set out in Ref. [19]. Each of the qubit Wigner functions presented in Fig. 1 is plotted in two different ways. Above the state labels is the more standard plot of the Wigner function, on the surface of a sphere. Below the labels, following Ref. [16], is the corresponding Lambert azimuthal equal-area projection [22]. This projection is area preserving, and maps the surface of a sphere to polar coordinates, where the the north pole is mapped to the centre of the disc and the south pole projects onto the outer boundary. The equator of the sphere is projected to a concentric circle, with a radius $1/\sqrt{2}$ times the radius of the entire circle.
The Lambert azimuthal equal-area projection allows us to view the entire Wigner function in one plot without losing any qubit information. If we choose to use the spherical plots, a hemisphere of the Wigner function is lost from our view together with the associated information. Consequently the Lambert projection is used in the following sections of this paper in order to obtain a more complete visual analysis of the states plotted.

Although we have defined the Wigner function in terms of a density operator \( \rho \), it is also possible to generate a Wigner function for any arbitrary operator. As a demonstration of this, we have shown in Fig. 1(d) the Wigner function for the Pauli \( \sigma_x \) matrix. Any Wigner function has the property that when integrating out all degrees of freedom, the value yielded is the trace of the operator. Since the Pauli operators are all traceless, this integration property is seen in Fig. 1(d) by noting that the negative and positive volumes cancel. We highlight this particular case as it will be key to several of our observations later in this paper.

### III. VISUALISING QUANTUM CORRELATIONS

For one qubit coupled to a single field mode, the full system Wigner function has four degrees of freedom. We could integrate out either qubit or field degrees of freedom to yield a reduced Wigner function for the respective subsystem. Such an approach would be equivalent to constructing Wigner functions from the reduced density matrix of the component parts. As already discussed, the problem with this approach is that it leads to the loss of important correlation information.

Before considering how to best visualise the four-dimensional Wigner function, we show in Fig. 2(a) the usual Wigner function for the iconic optical Schrödinger cat state \( \left( |\alpha\rangle_f + |\alpha\rangle_f \right)/\sqrt{2} \), giving a macroscopically distinct superposition of two coherent states (where each coherent state is generated as a displaced vacuum state such that \( |\alpha\rangle_f = D(\alpha) |0\rangle_f \)). In Fig. 2(b) we show the statistical mixture of the same two coherent states – the lack of quantum coherence is indicated by the absence of interference terms between each of the Gaussian peaks associated with each coherent state. As we are trying to retain the presence of these quantum correlations, the natural choice is to use the Wigner function over other methods, such as the Husimi Q-function. Our method is to plot a sphere at each \((q,p)\), on which we display the \((\theta, \phi)\) dependence and where we set the spheres opacity according to \( max_{\theta, \phi} |W(q, p, \theta, \phi)| \). Unlike in Ref. [21] we have chosen not to integrate out the qubit degrees of freedom, as we wish to retain correlations that would otherwise be lost (because they are traceless - see later discussion). In this way the current work and Ref. [21] present two different forms of quantum state spectroscopy based on the system’s Wigner function. We show in Figs. 2(c) and (e) the full Wigner function of a qubit coupled to a field mode in the separable cat state

\[
|\text{separable cat}\rangle = \frac{1}{2} \left( |\alpha\rangle_f |e\rangle_a + |\alpha\rangle_f |g\rangle_a \right),
\]

where \((c)\) is the iconic optical Schrödinger cat state \(\sqrt{2} (|\alpha\rangle_f |e\rangle_a + |\alpha\rangle_f |g\rangle_a)\). The first thing to notice in Figs. 2(d) and (f) is that the coherent state on the left-hand side is exclusively formed of qubit ground states, identical to the state in Fig. 1(b); likewise the cat on the right is formed of excited qubit states as in Fig. 1(a). It is clear from Eq. [8] that these features are those that we require to see in order for this technique to be of value. Secondly, the quantum correlations are revealed as a set of interference patterns between the two qubit-field coherent states, meeting our expectations about the utility of quantum state spectroscopy. Although not shown, this enables us to distinguish mixed and entangled states, as in the former the interference patterns would not be present. An interesting property of the interference patterns, is that the qubit Wigner functions, at the points of position and momentum within them, are of a similar form to the traceless operator shown in Fig. 1(d). It is worth noting that this traceless behaviour is analogous to integrating out the momentum degree of freedom (or any other marginal) in Fig. 2(a). When this integration is taken, the interference patterns between the two cats are similarly ‘traced’ out. We therefore assert that instances of tracelessness are indicative of quantum correlations.

We note that, for some simple cases, an analytic study of Wigner functions and associated entanglement measures has recently been undertaken in Ref. [24]. Our focus here, on what we have termed quantum state spectroscopy, is likely to be much more amenable to computational analysis for more complex systems.

### IV. SINGLE QUBIT EVOLUTION MODELS

Following the discussion in the preceding section, our attention turns to specific examples of important physical systems. Specifically we study simple models of light-matter interaction. We begin with the Jaynes-Cummings
FIG. 2. Here we show a number of different forms of field mode-qubit cat states, which also include different amounts of quantum correlation. (a) shows the position-momentum Wigner function for a standard Schrödinger cat state consisting of a superposition of two coherent states where quantum correlations appear in the form of positive and negative oscillations between the states. By contrast, (b) is a statistical mixture of the same two coherent states resulting in an absence of quantum correlations. The Wigner function for the field mode in (c) is the same state as in (a), however here it is coupled to the superposition state of the qubit shown in Fig. 1 (c). As expected, the qubit Wigner functions all point in the positive x direction on the Bloch sphere; the sign and magnitude determined by the value of the Wigner function for the field mode in (a). (d) is an entangled cat state; the cat on the left is coupled to a qubit $|g\rangle_a$ state and the cat on the right to a $|e\rangle_a$ qubit. Our technique also allows us to explore the space between the qubit and field mode, where significant quantum correlations can be found. It is particularly interesting to note that the qubit Wigner function at the origin of (d) is identical to that of the qubit Wigner function for the Pauli $\hat{\sigma}_x$ matrix and the implication of this, is that these correlations have the same form as a traceless operator. (e) and (f) are the same states as shown in (c) and (d) respectively using the Lambert azimuthal equal-area projection which allows the entire surface of the sphere to be seen, and not just the top hemisphere. Note the colour bar range is $\pm 2$ for (a) and (b), and $\pm (1 + \sqrt{3})$ for (c) - (f), where blue is positive, red is negative, and white is 0.

model (JCM) [25]. The Hamiltonian for the JCM, taking the rotating wave approximation, is given by

$$\hat{H}_{JC} = \omega_f \hat{a}^\dagger \hat{a} + \frac{\omega_a}{2} \hat{\sigma}_z + \gamma (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+),$$

where $\omega_f$ is the frequency of the field, $\omega_a$ is the qubit transition frequency, and $\gamma$ is the field-qubit coupling constant. These coefficients control the contributions of
the field mode, qubit, and interaction terms of the Hamiltonian respectively. The operators \( \hat{\sigma} \pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2 \) are the qubit raising and lowering operators, which transition the state between eigenstates of \( \hat{\sigma}_z \).

The JCM is well known for its inclusion of collapse and revival phenomena \([7, 26, 27]\). As part of this process a Schrödinger cat forms as quantum information, including entanglement correlations, is transferred from the system into the field.

Snapshots of the JCM are shown in Figs. 3(a) - (d). Figure 3(a) shows the qubit (von Neumann) entropy in cyan and the qubit inversion, \( (\hat{\sigma}_z) \), in red, as the initial state, \( |\alpha = \sqrt{5}\rangle \langle e|_{ao} \), goes through a single collapse and revival. The qubit entropy can be a useful measure for determining the entanglement between the qubit and the field mode. Although this is optimal in the case where there is no decoherence, its addition will lead to mixing during evolution increasing the value of the entropy. This makes it difficult to distinguish whether the state has significantly decohered, or if there is entanglement between the two subsystems from measures such as entanglement entropy.

Figures 3(b) - (d) show several noteworthy points within the evolution. Figure 3(b) corresponds to the vertical blue line in Fig. 3(a) and shows a point of near-maximal entanglement. The field is in the process of becoming a Schrödinger cat state, seen in the formation of kittens due to the separation of the state, and the beginnings of the emergence of interference patterns. We can also see a difference in the direction of the orientation of the qubits in each of the two kittens. Sign inversions due to superposition in the field component, together with quantum correlations from the entanglement of the field and qubit, produce the high intensity oscillations between the kittens.

The point of lowest entropy during the collapse phase of the evolution (corresponding to the pink line in Fig. 3(a)) is given in Fig. 3(c). The state here is directly comparable to Fig. 2(c) showing that there is minimal entanglement between the qubit and the field.

Figure 3(d) corresponds to the green line at the end of Fig. 3(a). Here there is maximal revival in the Rabi oscillation and there remains considerable entanglement between the qubit and the field. The state has two main positive contributions and is comparable to a Fock state. The qubit inversion indicates that, at this point, the state of the qubit is predominantly in the excited state, shown in Fig. 1(a), and is seen through the positive elements of the Wigner functions for the field. The correlation terms display more interesting behaviour similar to the behaviour seen in Fig. 2(d), with respect to the rotation of the qubits. Furthermore, also comparable to Fig. 2(d), the states in the correlation terms possess roughly equal positive and negative amplitudes in the qubit Wigner functions, indicative of the existence of the Wigner functions for traceless operators that we assert are signatures of entanglement.

We now consider the evolution of the Wigner function for the two-photon Rabi model (TPRM) \([28, 29]\). The difference between the models occurs in the interaction term of Hamiltonian and arises from simulating non-linear interaction. The Hamiltonian, in the rotating wave approximation, is given by

\[
\hat{H}_{TPR} = \omega f \hat{a}^\dagger \hat{a} + \frac{\omega_a}{2} \hat{\sigma}_z + \gamma((\hat{a}^\dagger)^2 \hat{\sigma}_- + \hat{a}^2 \hat{\sigma}_+). \quad (10)
\]

This Hamiltonian can be extended to simulate the emission and absorption of any number of photons \([28, 30]\), but here we only consider the two photon exchange.

As in Fig. 3(a), Fig. 3(e) shows the qubit entropy and inversion in cyan and red respectively; again, three points of the evolution that are of interest have been chosen. First where there is little entropy between the qubit and field, followed by where there is maximal entropy, and finally the point where the revival of the Rabi oscillation occurs. Considering the point of low entropy, after the collapse of the oscillations, we see Fig. 3(f) is a state very similar to that in Fig. 2(c). The quantum correlations between the two cats have a similar appearance in both figures, and the negativity in the Wigner function for the field mode has a clear effect on the sign of qubit Wigner functions.

Figure 3(g) shows the point at which there is maximal entropy between the qubit and the field in the evolution. Inspection of Fig. 3(g) reveals the nature of the entanglement. We can see in the cat on the left-hand side, the qubits are close to being the positive eigenstate of \( \hat{\sigma}_z \) shown in Fig. 1(c); whereas on the right-hand side they are the negative eigenstate counterpart. The orthogonal states in each of the cats, the presence of the qubit Wigner function for rotated Pauli matrices in the quantum correlations, and comparison to Fig. 2(d) leads us to the conclusion that this is a manifestation of an entangled cat state.

Finally, in Fig. 3(h) we show the point where the qubit inversion nears 1, signifying the point of the revival in the Rabi oscillations. The entropy at this point is near zero and can be seen from the overwhelming existence of \( |e\rangle_a \) states, but as it does not hit 0, there remains some residual entanglement between the qubit and the field. The signature of the residual entanglement can be seen in the presence of traceless operator Wigner functions, albeit with a low overall amplitude. The traceless operator Wigner functions can be seen between the large area of the revived positive excited states and the smaller area of positive states to the left.

There are many interesting differences between the JCM and TPRM, first of which is the speed of the Rabi oscillations revival, which can easily be seen by comparing the timescales in Figs. 3(a) and (e). Another notable difference between the two models is how the nature of the quantum correlations change during the collapse of the Rabi oscillations, as discussed above. This difference allows us to utilise the states in Figs. 3(c) and (g) as direct comparison to our example states in Figs. 2(c) and (d) respectively, demonstrating the physical model
FIG. 3. Here we show an evolution from the Jaynes-Cummings model, (a) - (d), and from the two-photon Rabi model, (e) - (h), in the rotating wave approximation. (a) and (e) show the entropy of the qubit (cyan) and the qubit inversion, $\langle \hat{\sigma}_z \rangle$ (red) across the evolution. Three noteworthy points from the JCM dynamics are shown in (b) - (d): (b) is a point of maximal entanglement between the qubit and the field, showing quantum correlations between the newly formed kittens; (c) shows the point of lowest entropy between the qubit and field, resulting in a Schrödinger cat state with no quantum correlations between the qubit and the field; (d) is the point of revival for the Rabi oscillations, dominated by qubit excited states in a form closely resembling a Fock state. (f) - (g) show similar evolution: (f) is the point of minimal entropy in the collapse phase of the evolution, note the similarity to Fig. 2(c); (g) is the point of maximal entropy between the qubit and field, features from Fig. 2(d) can be clearly seen in both the nature of the cats and the quantum correlations; (h) is the point of revival for the Rabi oscillations strongly resembling a coherent state, note the similarity to (d).

V. MULTI-QUBIT MODELS

A natural extension of the previous models is to include more qubits in the coupled systems, we now show

required to generate our example states in practice; since, for both, we generate very clear Schrödinger cat states, with differing degrees of qubit-field entanglement.
how our methods can be extended to such systems. Although only one extension is considered here, there are other possibilities that should be chosen depending on the application at hand. Beginning with the JCM, by increasing the number of qubits coupled to the field generates the variant known as the Tavis-Cummings Model (TCM) \cite{tcm} with the Hamiltonian in the interaction picture being

\[
\hat{H}_{TC} = \sum_{i}^{N} (\hat{a}^{\dagger} \hat{\sigma}^{(i)}_{-} + \hat{a} \hat{\sigma}^{(i)}_{+}), \tag{11}
\]

where \( N \) is the total number of qubits. Adding more qubits into the system allows not just for entanglement between the qubits and the field, but also for the two qubits to be entangled with each other. An interesting outcome of entanglement between the qubits, is when setting the initial state to be a maximally entangled qubit state, a process of cat swapping between the qubits and the field occurs \cite{cat-swapping}. Also, similar to adding more photon emission and absorption, increasing the number of qubits coupled to the system speeds up the revival of the Rabi oscillations.

The extension to the TPRM follows in a similar way with the Hamiltonian in the interaction picture being

\[
\hat{H}_{TPTQ} = \sum_{i}^{N} (\hat{a}^{\dagger} \hat{\sigma}^{(i)}_{-} + \hat{a} \hat{\sigma}^{(i)}_{+}). \tag{12}
\]

For this model, we simulate the interaction of the emission and absorption of two photons by multiple qubits. As with Eq. (10), this can be extended to simulate the emission and absorption of any number of photons, while also being able to add as many qubits as desired to the system.

We begin the modelling of the Hamiltonians in Eq. (11) and (12) by considering a field mode coupled to two qubits. When considering two-qubit Wigner functions, we need to choose an appropriate way to represent the information. Since we now have to map four qubit degrees of freedom onto each point in position-momentum phase space, we use exactly the same procedure as before, and following Ref. \cite{wigner}, plot the equal-angle slice of the Wigner function, \( \theta = \theta_1 = \theta_2 \) and \( \phi = \phi_1 = \phi_2 \), and set the opacity according to \( \max_{\phi} |W(q,p,\theta,\phi,\theta,\phi)| \) \cite{wigner}.

We note that since both of the Hamiltonians conserve angular momentum, the \( N \) qubits can be modelled by a single \( j = N/2 \) big spin and its corresponding qudut Wigner function \cite{qdut-wigner}, therefore the equal angle slice is a natural choice as it is closely related to the symmetric subspace. We note that due to this symmetry we could have just used a Wigner function that describes the \( j = N/2 \) subspace \cite{qdut-wigner}, and the spectrogram produced this way would be of a broadly similar form. As mentioned above, the two qubits can now be in an entangled state with each other, producing a different set of qubit Wigner functions. It is worth comparing the two-qubit spin-coherent state, Fig. (e), to that for a single qubit, Fig. (a). With a single qubit, any pure state can be considered a spin-coherent state, shown as a large positive region in the direction of the qubit with a negative region in the orthogonal direction. When we have multiple qubits, a spin-coherent state is given only when the qubits of all the individual qubits are pointing in the same direction. As the number of qubits increases, the Wigner function for a multi-qubit spin-coherent state increases in positive amplitude and decreases in negative. The positive region also becomes more concentrated and begins to approach, in likeness, a coherent state for the field. Since we are now modelling the evolution of two qubits, we can begin the simulations with the two qubits in a maximally entangled state as shown in Fig. (e), giving the initial state as \( |\alpha = \sqrt{5} \rangle \langle |eg\rangle + |ge\rangle \rangle /\sqrt{2} \).

When we begin with this state, the evolution produces a process of disentanglement and re-entanglement, of both the qubits to each other, and the qubits with the field.

Our results of these models are presented in Fig. As before, Figs. (a) and (e) show the entropy and qubit inversion in cyan and red respectively. For these two models, the entropy shown in cyan is just the entropy of the first qubit and the entropy for both qubits is in green. We note that taking the entropy for the second qubit gives the same result. Both variations of the entropy are shown in order to display the degree to which the two qubits are entangled with each other, rather than to the field (as is the case at the beginning of the evolution), and when there is more entanglement between the field mode and qubits. We also note that the qubit inversion is now given by \( \langle \hat{\sigma}^{(1)}_{z} + \hat{\sigma}^{(2)}_{z} \rangle \).

A. Cat Swapping And Sharing

Figures (b) - (d) show the evolution of the TCM from the initial state to the revival. We can see from these figures a process of cat swapping. In the initial state, the cat can be found in the qubits, due to the two-qubit maximally entangled state. During the evolution to Fig. (c), the cat has been swapped to the field mode and is no longer present between the two qubits. In the revival, Fig. (d), the cat has been returned to the qubits and is no longer seen in the field mode.

Likewise, Figs. (f) - (h) are stages of the two-photon, two-qubit Rabi evolution. The collapse and revival in this evolution does not exhibit the cat swapping in the same way as in the TCM. Instead, the cat is briefly swapped from the qubits to the field mode in Fig. (f), but rather than the cat swapping back to the qubits in Fig. (g), the field mode and the qubits share a cat, generating an entangled cat state. Between Figs. (g) and (h) the state returns to one similar to Fig. (f), where the cat can again only be found in the field mode, before swapping back to a qubit cat state in Fig. (h), with no cat in the field mode.
FIG. 4. Here we show the evolutions of the Tavis-Cummings model (TCM), (a) - (d), and two-qubit two-photon Rabi model (TQTP), (e) - (h), in the interaction picture. (a) and (e) show the entropy of the first qubit (cyan), the entropy of both qubits (green), and the qubit inversion, \( \langle \hat{\sigma}^{(1)}_z + \hat{\sigma}^{(2)}_z \rangle \) (red). (b) - (d) show the evolution of the TCM: (b) is a local maximum of entropy, showing quantum correlations between two kittens; (c) shows the point of lowest entropy between the qubits and field with the state in a Schrödinger cat state with little quantum correlation between the qubits and the field; (d) is the point of revival for the Rabi oscillations with the qubits having returned close to their initial state. (f) - (g) show the evolution of the TQTP: (f) is a point of minimal entropy in the collapse phase of the evolution, comparing with Fig. 2(c) highlight features that are seen in both states; (g) shows the point of maximal entropy between the qubit and field, similarities with Fig. 2(d) can be seen in both the nature of the cats and the quantum correlations; (h) is the point of revival for the Rabi oscillations and approaches a coherent state, the qubits are close to a singlet state.

B. Quantum State Spectroscopy

To explore the TCM we begin with a point of high entropy at the beginning of the collapse phase of the evolution, Fig. 4(b). The state here is similar to Fig. 3(b), in that we can see the spin-coherent states in each of the kittens pointing in different directions on the sphere. This
makes it clear that in the two-qubit spin-coherent states, the positive region pointing in the direction of the qubit is more concentrated with a diminishing negative region as compared to the single-qubit analogue; note that the ranges on the colour bar for the two-qubit states have increased in comparison to Fig. 3. In Fig. 4(b), we can also see the emergence of traceless operators in the quantum correlations, seen by the symmetry and equality of positive and negative volumes.

In Figs. 4(c) and (f) we show clear cat states, similar to those in Figs. 3(c) and (f), where there is very little entanglement between the qubit and the field. The qubits’ Wigner functions in both Figs. 4(c) and (f) are mostly pointing in the same direction, indicating low levels of quantum correlation between the qubits and the field mode. Likewise, the revival of the Rabi oscillations is shown in Figs. 4(d) and (h). In both, we see a return to the singlet state in the two-qubit Wigner functions. Further, the entropy of the individual qubits steadily increases towards the revival, with the combined entropy indicating fluctuations of entanglement and disentanglement between the two qubits and the field.

An interesting result is given in Fig. 4(g) where we see a very clear entangled cat state. Here the two-qubit spin-coherent states present in each of the cats are orthogonal and lead to the quantum correlations between the cats. As expected, these quantum correlations manifest as traceless operators, which can be discerned by the preponderance of high amplitudes of the positive and negative terms.

VI. CONCLUSIONS

Extending the ideas of frequency-time spectrograms, we have established the idea of quantum state spectroscopy as an accessible way of identifying entanglement within systems. We have previously shown this approach is useful in understanding the nature of atoms and spin-orbit coupling [21], and here have extended the method to Jaynes-Cummings and Tavis-Cummings type atom-cavity interactions. The visualisation scheme enables us to obtain substantial information and greater insight into the state of quantum systems; including non-classical correlations such as cat swapping/sharing. Extensions to larger systems will undoubtedly shed further insight into the nature of quantum correlations in present and future quantum technologies. In particular we have obtained a useful entanglement signature where the angular dependence of the Wigner function at certain points in the field’s phase space has the same form as the Wigner function for a traceless operator.

As a Wigner function is found by taking expectation values of displaced parity operators, all of which commute and are observable, simultaneous measurement of these quantities might be possible. Should this be realisable, direct measurement of a system’s quantum state spectrogram would become possible.

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