A SLOW BLOW UP SOLUTION FOR THE FOUR DIMENSIONAL ENERGY
CRITICAL SEMI LINEAR HEAT EQUATION

TONGTONG LI, LIMING SUN, AND SHUMAO WANG

ABSTRACT. We consider the energy critical four dimensional semi-linear heat equation
\[ \partial_t v - \Delta v - v^3 = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^4. \]

Formal computation of Filippas et al. (R. Soc. Lond. Proc. 2000) conjectures the existence of a sequence of type II blow-up solutions with various blow-up rates
\[ \| v(t) \|_{L^\infty(\mathbb{R}^4)} \approx \frac{\log(T - t)}{(T - t)^L}, \quad L = 1, 2, \ldots. \]

Schweyer (J. Funct. Anal. 2012) rigorously constructs a type II blow-up solution for the case \( L = 1 \). In this paper, we show the existence of type II blow-up solution for \( L = 2 \). The method here could be generalized to deal with all the cases \( L \geq 2 \).

1. INTRODUCTION

1.1. Setting of the problem. Consider the following semi-linear heat equation
\[
\begin{align*}
\partial_t v - \Delta v &= |v|^{p-1}v, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
v \mid_{t=t_0} &= v_0.
\end{align*}
\] (1.1)

Because of the simplicity of the nonlinearity, problem (1.1) has been widely considered as a popular model for testing the methods designed to analysis the behavior of solutions near singularity formation. It has been extensively studied in the literature, for example [9, 10, 15, 24, 3]. It is well-known that for a large class of initial data (for instance, bounded and continuous) there is a unique maximal classical solution \( v(t, x) \) for \( t \in (0, T) \). If \( T \) is finite, then \( u \) will blow up at time \( t = T \). There are two types of blow-ups depending on the rate
\[
\begin{align*}
\limsup_{t \to T} (T - t)^{\frac{1}{p-1}} \|v(t)\|_{L^\infty(\mathbb{R}^d)} &< +\infty \quad \text{type I}, \\
\limsup_{t \to T} (T - t)^{\frac{1}{p-1}} \|v(t)\|_{L^\infty(\mathbb{R}^d)} &= +\infty \quad \text{type II}.
\end{align*}
\]

In this article, we will focus on the radial type II blow-up solution in energy critical case, that is \( p = p_S := \frac{d+2}{d-2} \). In this situation, the total dissipated energy
\[
E(v) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^d} |v|^{p+1} \, dx
\] (1.2)
is left invariant by the scaling symmetry of the problem
\[ v(t, x) \mapsto \lambda^{\frac{2}{p-1}} v(\lambda^2 t, \lambda x). \]

The blow-up of (1.1) is almost completely understood in the sub-critical range \(1 < p < p_S\), for instance, by [7, 9, 10, 11, 25, 29]. The solution always blows up in type I in this range. The existence of type II blow-up has been established in various settings, for instance by [15, 14, 24] when \(p > p_{JL}\), where
\[
p_{JL} = \begin{cases} 
\infty & \text{if } d \leq 10, \\
1 + \frac{4}{d-4-2\sqrt{d-1}} & \text{if } d \geq 11.
\end{cases} 
\] (1.3)

On the other hand, when \(p_S < p < p_{JL}\), Matano and Merle [16] excludes the occurrence of a type II blow-up for radial solutions. Therefore, in dimension \(3 \leq d \leq 10\), the choice \(p = p_S\) is the only one for which a type II blow-up occurs for radial data.

Recently, there are active researches in the energy critical case \(p = p_S\). In the pioneering work by Filippas, Herrero, and Velázquez [8], they find that \(u\) can exhibit type II blow-up in finite time in lower dimensions in the energy critical case \(p = p_S\). They formally obtain sign changing type II blow-up solutions using the matched asymptotic expansion technique. Also, they give a sequence of blow-up speeds (corrected by Harada [13])
\[
\|v(t)\|_{L^\infty(\mathbb{R}^d)} \approx \begin{cases} 
(T-t)^{L} & d = 3, \\
\left|\log(T-t)^{\frac{d-1}{d+1}}\right| & d = 4, \\
(T-t)^{-L} & d = 5, \\
(T-t)^{-\frac{1}{d-4-2\sqrt{d-1}}} & d = 6,
\end{cases} 
\] (1.4)

where \(L \geq 1\) is an integer. Recently, there is a surge of interest in constructing such type II blow-up solutions as predicted by [8].

Schweyer [28] first rigorously constructs a radial blow-up solution in the case \(d = 4\) and \(L = 1\). He uses a strategy developed in the study of geometrical dispersive problems by Merle and Raphaël [17, 19], Merle et al. [20] and Raphaël and Rodnianski [26]. The nature of his approach is energy estimates and making no use of the maximum principle. Meanwhile, del Pino, Wei and their collaborators develop an inner-outer gluing method and lead to a series of works on construction. They apply their methods to construct type II blow-up solutions in several cases. To be more precise, del Pino et al. [6] constructs solutions of \(d = 3\) and all \(L \geq 1\). del Pino et al. [5] establishes the existence of solutions of \(d = 5\) and \(L = 1\). Later on, Harada [12] completes the construction for \(d = 5\) and all \(L \geq 2\). Using this inter-outer gluing method, Harada [13] also shows the existence of type II blow-up solution with the specific rate for \(d = 6\) in (1.4).

One may wonder what will happen for \(d \geq 7\). Collot, Merle, and Raphaël [3] proves no existence of type II blow-up solution in \(d \geq 7\) cases near the ground state solitary wave and gives a complete classification of its asymptotic behavior. Recently, Wang and Wei [30] precludes the type II blow-up for all positive solutions in \(d \geq 7\).

After all the works mentioned above, it seems that the cases \(d = 4\) and \(L \geq 2\) are still unsettled. The goal of this paper is to fill the gap of these remaining cases.
1.2. **Statement of the result.** We consider the energy critical semi-linear heat equation in dimension $d = 4$

\[
\begin{align*}
\frac{\partial_t v}{(t,x)} &\in \mathbb{R} \times \mathbb{R}^4, \\
v \mid_{t=t_0} &= v_0.
\end{align*}
\] (1.5)

In this article, we construct a radial type II blow-up solution based on the Talenti-Aubin soliton

\[
Q(r) = \frac{1}{1 + \frac{r^2}{8}}, \quad \Delta Q + Q^3 = 0.
\] (1.6)

Our result is the following

**Theorem 1.1.** For any $\alpha^* > 0$, there exists $C^\infty$ radial initial data $v_0$ with

\[
E(Q) < E(v_0) < E(Q) + \alpha^*
\] (1.7)

such that the solution to (1.5) blows up in finite time $T = T(v_0) < \infty$ in a type II regime: there exists $v^* \in \dot{H}^1$ such that

\[
v(t,x) - \frac{1}{\lambda(t)} Q \left( \frac{x}{\lambda(t)} \right) \rightarrow v^* \quad \text{in } \dot{H}^1 \text{ as } t \to T,
\] (1.8)

and for some $c(v_0) > 0$,

\[
\lambda(t) = c(v_0)(1 + \alpha_{t\to T}(1)) \frac{(T-t)^2}{|\log(T-t)|^\frac{3}{2}}.
\] (1.9)

**Remark 1.2.** (i) Our method relies on the approach to construct slow blow-up dynamics for the corotational energy-critical harmonic heat flow of Raphaël and Schweyer [27]. People have observed a deep connection between dimension four energy critical semi-linear heat equation and two-dimensional harmonic map flows, and interested readers can check [27, 4] and references therein. Our result verifies the existence of the blow-up speed corresponding to $L = 2$ as conjectured in [8] because $v$ will blow up in the speed of the reciprocal of $\lambda$.

(ii) The construction to the case of $L = 2$ is much more complicate than the one of $L = 1$. First, the approximate solution in the case of $L = 1$ needs to be sharpened here in order to get (1.9). This requires a better approximation to the blow-up solution to further reduce the errors which are produced in the case of $L = 1$. See Step 1 in subsection 1.4 for more details. Second, we need to deal with two different “unstable directions” in our setting. One is from the Schrödinger operator $\mathcal{H}$ in (2.4), and the other one is from modulation parameter $b = (b_1, b_2)$ in (1.22). In this problem, we can deal with them at the same time. See Step 3 in subsection 1.4. For the case of $L > 2$, these two types of difficulty persist and are all the essential ones. Actually, one can introduce an appropriate class of functions to continue the approximate process (see a similar argument in [27]). This will also produce more unstable directions but can be handled similarly as here. Since the proof of $L > 2$ is just a tautology of the idea here, we will not present it in this paper.

(iii) The method we rely on is a powerful tool. It has been applied to construct finite time blow-up solutions in Schrödinger map [21], focusing energy supercritical Schrödinger equation [22], defocusing energy supercritical Schrödinger equation in [23], energy supercritical wave equation in [2], nonradial energy supercritical heat equation in [1].
1.3. **Notations.** We introduce the differential operator
\[ \Lambda f := f + y \cdot \nabla f \] (energy critical scaling).

Given a positive number \( b_1 > 0 \), we let
\[ B_0 := \frac{1}{\sqrt{b_1}}, \quad B_1 := \frac{\log b_1}{\sqrt{b_1}}. \]

Given a parameter \( \lambda > 0 \), we let
\[ u_\lambda(r) := \frac{1}{\lambda} u(y) \quad \text{with} \quad y = \frac{r}{\lambda}. \]

We let \( \chi \) be a smooth non-increasing cutoff function with
\[ \chi(y) = \begin{cases} 
1 & \text{for} \quad y \leq 1, \\
0 & \text{for} \quad y \geq 2,
\end{cases} \]
and use the notation
\[ \int f := \int_{0}^{+\infty} f(r)r^3 dr. \]

1.4. **Strategy of the proof.** Now we sketch the main points in the proof of Theorem 1.1.

**Step 1: Approximate solution** \( Q + b_1 T_1 + b_2 T_2 \). We first reparametrize (1.5) by
\[ v(t, r) = \frac{1}{\lambda} u(s, y), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)} \] (1.10)
which leads to
\[ \partial_s u + b_1 \Lambda u - \Delta u - u^3 = 0, \quad b_1 = -\frac{\lambda_s}{\lambda}. \] (1.11)

We look for blow-up solution \( u \) close to \( Q \) in \( \dot{H}^1 \) topology. In this case, \( b_1 \) remains small and the flow is controlled by the linearized Hamiltonian
\[ H = -\Delta - V = -\Delta - 3Q^2 \] (1.12)
which has a resonance by the scaling symmetry
\[ H \Lambda Q = 0, \quad \Lambda Q \sim -\frac{8}{y^2} \quad \text{as} \quad y \to +\infty. \] (1.13)

Besides, we need \( T_1, T_2 \) given by
\[ HT_2 = -T_1, \quad HT_1 = -\Lambda Q \] (1.14)
with asymptotics
\[ T_1 \sim -4 \log y + 2, \quad T_2 \sim y^2 \left( \frac{1}{2} \log y - \frac{5}{8} \right) \] (1.15)
as \( y \) tends to infinity.

The linearization of the flow implies a possible approximate solution
\[ Q + b_1 T_1 + b_2 T_2 \] (1.16)
with a priori bound \( |b_2| \lesssim b_1^2 \). Indeed, the \( O(b_1) \) error vanishes
\[ b_1 (\Lambda Q + HT_1) = 0 \] (1.17)
by the definition of $T_1$. At $O(b_1^2)$ level, the leading term is
\[(b_1)_s T_1 + b_1^2 \Lambda T_1 - b_2 T_1,\] (1.18)
which can, since $\Lambda T_1 \sim T_1$, be canceled by setting
\[(b_1)_s + b_1^2 - b_2 = 0.\] (1.19)
Similar calculation at $O(b_1^3)$ level suggests that we take
\[(b_2)_s + 3b_1 b_2 = 0.\] (1.20)
Certainly some other terms $S_j(b, y)$ are needed to further reduce the error, and we construct the approximate solution as
\[Q_b(y) = Q(y) + b_1 T_1 + b_2 T_2 + S_2 + S_3 + S_4 = Q(y) + \alpha(b, y).\] (1.21)
which generates a small error $\Psi_b$. More importantly, from flux computation, some $\log b_1$ term should be introduced to the dynamical system for $b = (b_1, b_2)$. That is, we should take
\[
\begin{aligned}
\lambda + b_1 &= 0, \\
(b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 &= 0, \\
(b_2)_s + b_1 b_2 (3 + c_{b_1}) &= 0.
\end{aligned}
\] (1.22)
Indeed, this ODE system has a solution with $\lambda \to 0^+$ at finite time $T = T(v_0) < +\infty$ and
\[
\lambda \sim c_1(v_0) \frac{(\log s)^{\frac{1}{2}}}{s^\frac{3}{2}}, \quad b_1 \sim b_1^c = \frac{2}{3s} - \frac{4}{9s \log s}, \quad b_2 \sim b_2^c = -\frac{2}{9s^2} + \frac{20}{27s^2 \log s}. \] (1.23)
The problem is that this solution might be unstable. In fact, we define $U$ by
\[
b_k = b_k^c + \frac{U_k}{s^k (\log s)^{\frac{3}{2}}}, \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.
\] (1.24)
Then the dynamical system for $b$ above implies
\[
s \frac{dU}{ds} = AU + \text{error terms}, \] (1.25)
which is, after diagonalization, equivalent to
\[
s \frac{dV}{ds} = D_A V + \text{error terms}, \quad D_A = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}, \quad V = PU
\] (1.26)
where the first unstable direction $V_2$ occurs corresponding to the positive eigenvalue $\frac{2}{3}$ of $A$. And it must be controlled a priori to avoid disrupting the dynamic for $b$. For technical reasons we modify it to $\bar{V}_2$, see Proposition 3.5 and Lemma 4.3.

**Step 2: Decomposition of the flow.** To get (1.22), we decompose $u$ as
\[u = Q_b + \varepsilon\] (1.27)
for some small $\varepsilon(s, y)$ subject to the orthogonal conditions
\[(\varepsilon, H^k \Phi_M) = 0 \quad \text{for } 0 \leq k \leq 2.\] (1.28)
where $(\cdot, \cdot)$ denotes $L^2$ inner product and $\Phi_M$ is a compactly supported substitute for $\Lambda Q$ (since $\Lambda Q \notin L^2$). Then Implicit Function Theorem ensures the existence and uniqueness of the decomposition as long as $\|\varepsilon\|_{\dot{H}^1}$ remains small. Moreover, it turns out that (1.28) is enough to derive the expected modulation equations

$$\left| \frac{\lambda_s}{\lambda} + b_1 \right| + |(b_1)_s + b_1^2(1 + c_{b_1}) - b_2| + |(b_2)_s + b_1 b_2(3 + c_{b_1})| \lesssim \|\varepsilon\|_{\text{loc}} + \frac{b_1^3}{|\log b_1|} \quad (1.29)$$

for some local-in-space norm $\|\varepsilon\|_{\text{loc}}$.

So we need to bound $\|\varepsilon\|_{\text{loc}}$. Thanks to the Hardy type bounds below ensured by (1.28)

$$\Xi_{2k} := \int |H^k\varepsilon|^2 \gtrsim \int \frac{|\varepsilon|^2}{(1 + y^{4k})(1 + |\log y|^2)}, \quad 1 \leq k \leq 3, \quad (1.30)$$

we turn to bound $\Xi_{2k}$. Here comes the second unstable direction, i.e. the projection

$$\tau(t) = (\varepsilon, \psi). \quad (1.31)$$

Here $\psi$ corresponds to the only non-positive spectrum of $H$

$$H\psi = -\varsigma\psi, \quad \varsigma > 0. \quad (1.32)$$

If uncontrolled, its growth would destroy any bound of $\Xi_{2k}$. Assuming suitable a priori bound on $\tau$, we can derive some Lyapounov monotonicity

$$\frac{d}{dt} \Xi_6 \lesssim \frac{1}{\lambda^{10}} \frac{b_1^6}{|\log b_1|^2} \quad (1.33)$$

which leads to the estimate

$$\Xi_6 \lesssim \frac{b_1^6}{|\log b_1|^2}. \quad (1.34)$$

This together with similar bounds on $\Xi_4$ and $\Xi_2$ serves to control $\|\varepsilon\|_{\text{loc}}$. The required smallness of $\|\varepsilon\|_{\dot{H}^1}$ is ensured by the dissipation of energy and the sub-coercivity of $H$.

**Step 3: Control of the two unstable directions.** With the above analysis, we see the core of the proof is to control the unstable models $\tilde{V}_2$ and $\tau$ at the same time. This is possible since $\tilde{V}_2$ comes from the development of modulation parameter $b$ while $\tau$ comes from the projection of $\varepsilon$ to the non-positive eigenvalue direction of $H$. Hence they are (almost) independent of each other as long as both are reasonably small so as not to destroy the bounds for $b$ and $\varepsilon$.

The method is to apply a Brouwer type argument. We set initial data $v_0$ as

$$v_0 = Q_{b(0)} + \varepsilon(0) = Q_{b(0)} + \tau(0) \tilde{\psi} \quad (1.35)$$

where $\tilde{\psi}$ satisfies

$$(\tilde{\psi}, \psi) = 1, \quad (\tilde{\psi}, H^k\Phi_M) = 0, \quad 0 \leq k \leq 2. \quad (1.36)$$

so that (1.28) and (1.31) are satisfied at $t = 0$. It can be seen that $v_0$ is uniquely determined by $U_1(0), \tilde{V}_2(0)$ and $\tau(0)$. We simply take $U_1(0) = 0$ and assume a priori

$$\tilde{V}_2(t) \in [-1, 1], \quad \tilde{\tau}(t) := \tau(t) \frac{\log b_1(t)}{b_1(t)^{3+\frac{1}{2}}} \in [-1, 1]. \quad (1.37)$$
If (1.37) fails at \( t = \tilde{T}_{\text{exit}}(v_0) < T(v_0) \) for any reasonable initial data \( v_0 \), we will have a map
\[
\mathbb{D} = [-1,1] \times [-1,1] \to \partial \mathbb{D},
\]
\[
(\tilde{V}_2(0), \tilde{\tau}(0)) \mapsto (\tilde{V}_2(\tilde{T}_{\text{exit}}), \tilde{\tau}(\tilde{T}_{\text{exit}})).
\]
Moreover, the dynamics for \( b \) and \( \varepsilon \) yield, respectively,
\[
|s(\tilde{V}_2)s - \frac{2}{3} \tilde{V}_2| \lesssim (\log s)^{-\frac{1}{4}}, \quad |\tau_s - \varsigma \tau| \lesssim \frac{b_1^4}{|\log b_1|}.
\]
which ensure the strictly outgoing behavior
\[
\frac{d}{ds} \tilde{V}_2^2(\tilde{T}_{\text{exit}}) > 0, \quad \frac{d}{ds} \tilde{\tau}^2(\tilde{T}_{\text{exit}}) > 0.
\]
Hence classical PDE theory ensures that the map above is continuous and leaves the boundary points fixed, which contradicts Brouwer fixed-point theorem. So we conclude that some \( v_0 \) exists such that (1.37) holds for all \( t < T(v_0) \), hence deduce (1.22) and Theorem 1.1.

The article is organized as follows. In Section 2, we first construct the approximate self-similar solution \( Q_b \), give the sharp estimates about the error term \( \Psi_b \) and supply a local version of \( Q_b \). Then we derive the dynamical system of \( b = (b_1, b_2) \) in Section 2.3, which corresponds to one of the unstable directions. In Section 3, we first give a suitable decomposition of the solution \( v(t, x) \) and design the bootstrap regime. Then we derive the modulation equations in Section 3.2. Finally, in the end of this section, we derive the fundamental monotonicity of the Sobolev-type norms \( \Xi_2, \Xi_4 \) and \( \Xi_6 \). In Section 4, we first get improved control of \( \Xi_2, \Xi_4 \) and \( \Xi_6 \). Then by a standard Brouwer argument, we control the two unstable directions at the same time by suitably choosing the initial data, which is the heart of our analysis. The above analysis finishes the bootstrap regime, which easily implies the blow up statement of Theorem 1.1.

2. APPROXIMATE PROFILE

Introduce a parameter \( \lambda = \lambda(t) > 0 \). Let \( v(t, \cdot) = u_\lambda(s, \cdot) \), or
\[
v(t, r) = \frac{1}{\lambda} u(s, y), \quad y = \frac{r}{\lambda}
\]
where
\[
s = s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \quad s_0 > 0.
\]
then (1.5) becomes
\[
\partial_s u - \frac{\lambda_s}{\lambda} Au - \Delta u - u^3 = 0.
\]
In this section we construct an explicit approximate solution to (2.3) close to \( Q \). The construction relies on the spectral properties of linearized Hamiltonian
\[
H = -\Delta - V = -\Delta - 3Q^2
\]
which are well known and are summarized below:

(i) \( H \) has a unique negative eigenvalue
\[
H \psi = -\varsigma \psi, \quad \varsigma > 0
\]
and $\psi$ decays exponentially.

(ii) $H$ has a resonance at the origin induced by the scaling symmetry

$$H \Lambda Q = 0$$

with $\Lambda Q \notin L^2$, or more precisely

$$\Lambda^i Q = \begin{cases} 1 + O(y^2) & \text{as } y \to 0, \\ (-1)^i \frac{8}{y^2} + O\left(\frac{1}{y}\right) & \text{as } y \to +\infty. \end{cases}$$

(2.6)

ODE theory provides another solution to $H \Gamma = 0$ for $y > 0$:

$$\Gamma(y) = -\Lambda Q(y) \int_1^y \frac{dx}{x^2[\Lambda Q(x)]^2}$$

$$= \frac{y^2 - 8}{(y^2 + 8)^2} \left( \frac{y^2}{16} + 6 \log y - \frac{583}{112} - \frac{4}{y^2} \right) - \frac{64}{(y^2 + 8)^2}$$

(2.7)

and the asymptotic behavior

$$\Lambda^i \Gamma(y) = \begin{cases} (-1)^i \frac{8}{y^2} + O(|\log y|) & \text{as } y \to 0, \\ \frac{1}{16} + O\left(\frac{\log y}{y^2}\right) & \text{as } y \to +\infty. \end{cases}$$

(2.8)

(iii) By (ii), the solution to $Hu = f$ is given by

$$u = H^{-1} f = \Gamma(y) \int_0^y f(x) \Lambda Q(x) x^3 dx - \Lambda Q(y) \int_0^y f(x) \Gamma(x) x^3 dx$$

(2.9)

up to the addition of $\alpha Q(y) + \beta \Gamma(y)$. We restrict ourselves to the case where $f$ is smooth and take $\alpha = \beta = 0$, which is equivalent to require $u$ is smooth and $u(0) = 0$.

Now we turn to the construction. In the following subsection, we assume

$$\begin{cases} \frac{2}{8} + b_1 = 0, \\ (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 = 0, \\ (b_2)_s + b_1 b_2 (3 + c_{b_1}) = 0. \end{cases}$$

(2.10)

where $c_{b_1}$ is defined by (2.26).

### 2.1. Construction of the approximate blow-up profile.

**Proposition 2.1. (Construction of the approximate profile).** Let $M > 0$ be large enough. Then there exists a small enough universal constant $b^*(M) > 0$, such that the following holds true. Let there be a $C^1$ map

$$b = (b_k)_{1 \leq k \leq 2} : [s_0, s_1] \mapsto (-b^*(M), b^*(M)) \times (-b^*(M), b^*(M))$$

with a priori bounds on $[s_0, s_1]$

$$0 < b_1 < b^*(M), \quad |b_2| \lesssim b_1^2.$$

(2.11)

Then there exist profiles $T_1, T_2, S_2, S_3$ and $S_4$, such that

$$Q_6(y) = Q(y) + b_1 T_1 + b_2 T_2 + S_2 + S_3 + S_4 = Q(y) + \alpha(b, y)$$

(2.12)
generates an error
\[
\Psi_b := \partial_s Q_b - \frac{\lambda_s}{\lambda} \Lambda Q_b - \Delta Q_b - Q_b^3
\] (2.14)
which satisfies:
\[
\int_{y \leq 2B_1} |H^k \Psi_b|^2 \leq \frac{b_{1,k+2}}{\log b_1} \text{ for } 1 \leq k \leq 2,
\] (2.15)
\[
\int_{y \leq 2B_1} |H^3 \Psi_b|^2 \leq \frac{b_8^8}{|\log b_1|^2},
\] (2.16)
\[
\int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^{12-2i}} |\partial^i \Psi_b|^2 \leq b_8^i |\log b_1|^C \text{ for } 0 \leq i \leq 4,
\] (2.17)
\[
\int_{y \leq 2M} |H^k \Psi_b|^2 \leq b_{1,k+1}^M M^C \text{ for } 0 \leq k \leq 3.
\] (2.18)

**Proof. Step 1: Computation of the error.** Take \(T_1, T_2\) to be the solution to
\[
HT_1 + \Lambda Q = 0, \quad HT_2 + T_1 = 0,
\] (2.19)
given by (2.10). Then as \(y \to +\infty\), we obtain
\[
\text{for } 0 \leq i \leq 10, \quad \Lambda^i T_1 = -4 \log y + 2 - 4i + O\left(\frac{|\log y|^2}{y^2}\right),
\] (2.20)
\[
\Lambda^i T_2 = O(y^2 \log y).
\] (2.21)
There holds the behavior at \(y \to 0\)
\[
\text{for } 0 \leq i \leq 10, \quad \Lambda^i T_1 = O(y^2), \quad \Lambda^i T_2 = O(y^4).
\] (2.22)

Note that \(T_1, T_2\) are independent of \(b_1\) and \(b_2\). In the following we shall find \(S_i\) of order \(b_1^i\) for \(i = 2, 3, 4\). We expand \(Q^3_b\) and rearrange them to the polynomial of \(b_1\):
\[
Q^3_b = Q^3 + 3Q^2\alpha + R_2 + R_3 + R_4 + R
\] (2.23)
where \(R\) is a polynomial of \(Q, T, S_j\) with \(O(b_1^i)\) coefficients and
\[
\begin{align*}
R_2 &= 3b_2^3 Q T_1^2 \\
R_3 &= 6b_1 b_2 T_1 T_2 + 6b_1 Q T_1^2 S_2 + b_1^3 T_1^3 \\
R_4 &= 6b_1 Q T_1^2 S_3 + 3b_2^2 Q T_2^2 + 6b_2 Q T_2 S_2 + 3Q S^2_2 + 3b_1^2 b_2 T_1^2 T_2 + 3b_1^2 T_1^2 S_2.
\end{align*}
\] (2.24)

Now a direct computation leads to
\[
\Psi_b = b_1 \Lambda Q + [(b_1)_1 T_1 + b_1^2 \Lambda T_1 + b_1 HT_1] + [(b_2)_1 T_2 + b_1 b_2 \Lambda T_2 + b_2 HT_2] + \sum_{j=2}^4 \left[\partial_s S_j + b_1 \Lambda S_j + HS_j\right] - (Q^3_b - Q^3 - 3Q^2\alpha)
\] (2.25)
\[
= b_1^2 [\Lambda T_1 - (1 + c_1) T_1] + H S_2 - R_2 \\
+ b_1 b_2 [\Lambda T_2 - (3 + c_1) T_2] + H S_3 + (\partial_s S_2 + b_1 \Lambda S_2) - R_3 \\
+ H S_4 + (\partial_s S_3 + b_1 \Lambda S_3) - R_4 \\
+ (\partial_s S_4 + b_1 \Lambda S_4) - R.
\]
\textbf{Step 2: Construction of the radiation} $\Sigma_{b_1}$. We introduce a radiation term to cancel the 1-growth in $\Lambda T_1 - T_1$ (the $\log y$ growth vanishes by (2.20)). Let

$$c_{b_1} = \frac{64}{\int \chi_{\frac{B_0}{4}}(\Lambda Q)^2} = \frac{2}{\log b_1} \left( 1 + O \left( \frac{1}{\log b_1} \right) \right), \quad (2.26)$$

$$d_{b_1} = c_{b_1} \int \chi_{\frac{B_0}{4}} \Gamma \Lambda Q = O \left( \frac{1}{b_1 \log b_1} \right). \quad (2.27)$$

Let $\Sigma_{b_1}$ solves

$$H \Sigma_{b_1} = c_{b_1} \chi_{\frac{B_0}{4}} \Lambda Q + d_{b_1} H[(1 - \chi_{3B_0})\Lambda Q], \quad (2.28)$$

that is,

$$\Sigma_{b_1} = c_{b_1} \Gamma \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 - c_{b_1} \Lambda Q \int_0^y \chi_{\frac{B_0}{4}} \Gamma \Lambda Q + d_{b_1} (1 - \chi_{3B_0})\Lambda Q. \quad (2.29)$$

The choice of $c_{b_1}$ and $d_{b_1}$ yields

for $0 \leq i \leq 10$, \quad $\Lambda^i \Sigma_{b_1} = \begin{cases} 
- c_{b_1} \Lambda^i T_1 & \text{for } y \leq \frac{B_0}{4}, \\
64\Lambda^i \Gamma & \text{for } y \geq 6B_0,
\end{cases} \quad (2.30)$

and for $\frac{B_0}{4} \leq y \leq 6B_0$,

$$\Sigma_{b_1} = c_{b_1} \left( \frac{1}{16} + O \left( \log y \frac{y^2}{y^2} \right) \right) \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 + c_{b_1} O \left( \frac{1}{y^2} \right) O(y^2)$$

$$= 4 \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 + O \left( \frac{1}{\log b_1} \right), \quad (2.31)$$

$$\Lambda \Sigma_{b_1} = c_{b_1} \Gamma \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 - c_{b_1} \Lambda Q \int_0^y \chi_{\frac{B_0}{4}} \Gamma \Lambda Q + d_{b_1} \Lambda [(1 - \chi_{3B_0})\Lambda Q],$$

$$= c_{b_1} \left( \frac{1}{16} + O \left( \log y \frac{y^2}{y^2} \right) \right) \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 + c_{b_1} O \left( \frac{1}{y^2} \right) O(y^2)$$

$$= 4 \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 + O \left( \frac{1}{\log b_1} \right). \quad (2.32)$$

Similarly one can establish,

for $0 \leq i \leq 10$, \quad $\Lambda^i \Sigma_{b_1} = 4 \int_0^y \chi_{\frac{B_0}{4}}(\Lambda Q)^2 + O \left( \frac{1}{\log b_1} \right). \quad (2.33)$

We need also to bound $b_1^j \partial_{b_1}^j \Lambda^i \Sigma_{b_1}$. Simple calculation reveals that

for $1 \leq j \leq 10$, \quad $|b_1^j \partial_{b_1}^j \chi_{\frac{B_0}{4}}| \leq \frac{1}{4} \frac{B_0}{4} y \leq \frac{B_0}{2}$, \quad $|b_1^j \partial_{b_1}^j \chi_{3B_0}| \leq \frac{1}{3} \frac{B_0}{3} y \leq \frac{B_0}{3}$. 

Since $\Lambda Q$ is independent of $b_1$, then
\[
\left| b_i^j \partial_{b_1}^j \int_0^y \chi_{B_0/4}(\Lambda Q)^2 \right| \lesssim \int_{B_0/4}^y (\Lambda Q)^2 \cdot 1_{y \geq b_0} \lesssim 1_{y \geq b_0},
\]
\[
\left| b_i^j \partial_{b_1}^j \int_0^y \chi_{B_0/4} \Gamma \Lambda Q \right| \lesssim \int_{B_0/4}^y |\Gamma \Lambda Q| \cdot 1_{y \geq b_0} \lesssim \frac{1}{b_1} 1_{y \geq b_0}
\]
which in particular yield
\[
\text{for } 1 \leq j \leq 10, \quad |b_i^j \partial_{b_1}^j c_{b_1}| \lesssim \frac{1}{|\log b_1|^2}, \quad |b_i^j \partial_{b_1}^j d_{b_1}| \lesssim \frac{1}{b_1 |\log b_1|}.
\]
Taking these into the definition of $\Sigma_{b_1}$, we conclude
\[
\text{for } 0 \leq i \leq 10, \quad 1 \leq j \leq 10, \quad |b_i^j \partial_{b_1}^j \Lambda^i \Sigma_{b_1}| \lesssim \frac{1}{|\log b_1|} 1_{y \leq 6B_0}. \quad (2.34)
\]

**Step 3: Construction of $S_2$.** Let
\[
\Theta_2 := \Lambda T_1 - T_1 + \Sigma_{b_1},
\]
then by (2.20), (2.9), (2.30) and (2.31), we obtain
\[
\Lambda^i \Theta_2 = O \left( \frac{|\log y|^2}{y^2} \right) \quad \text{for } y \geq 6B_0,
\]
\[
\Lambda^i \Theta_2 = O(1) \quad \text{for } y \leq \sqrt{B_0},
\]
\[
\Lambda^i \Theta_2 = 4 \int_0^y \chi_{B_0/4}(\Lambda Q)^2 - 4 + O \left( \frac{1}{|\log b_1|} \right) + O \left( \frac{|\log y|^2}{y^2} \right) \quad \text{for } \sqrt{B_0} \leq y \leq 6B_0.
\]
Combining the above estimates, we have
\[
\text{for } 0 \leq i \leq 10, \quad |\Lambda^i \Theta_2| \lesssim 1_{y \leq 1} + \left( 1 + \frac{|\log \sqrt{b_1} y|}{|\log b_1|} \right) 1_{1 \leq y \leq 6B_0} + \frac{|\log y|^2}{y^2} 1_{y \geq 6B_0}. \quad (2.36)
\]
where we used
\[
\frac{|\log y|^2}{y^2} \lesssim 1 \lesssim \frac{|\log b_1 y|}{|\log b_1|} \quad \text{for } 1 \leq y \leq \sqrt{B_0},
\]
\[
\frac{|\log y|^2}{y^2} \lesssim \sqrt{b_1} |\log b_1|^2 \lesssim \frac{1}{|\log b_1|} \quad \text{for } \sqrt{B_0} \leq y \leq 6B_0.
\]
Since $T_1$ is independent of $b_1$, then $b_i^j \partial_{b_1}^j \Lambda^i \Theta_2 = b_i^j \partial_{b_1}^j \Lambda^i \Sigma_{b_1}$. Together with (2.34) we get
\[
|b_i^j \partial_{b_1}^j \Lambda^i \Theta_2| \lesssim 1_{y \leq 1} + \left( 1 + \frac{|\log \sqrt{b_1} y|}{|\log b_1|} \right) 1_{1 \leq y \leq 6B_0} + \frac{|\log y|^2}{y^2} 1_{y \geq 6B_0} \quad (2.37)
\]
for $0 \leq i, j \leq 10$.

Now let $S_2$ be the solution to
\[
b_i^2 \Theta_2 + HS_2 - R_2 = 0. \quad (2.38)
\]
Recall that $R_2 = 3b_1^2QT_1^2$. Since $H$ commutes with multiplication of $b_1$, then $S_2 = b_1^2\tilde{S}_2$ where $\tilde{S}_2$ solves $H\tilde{S}_2 = 3QT_1^2 - \Theta_2$. The following claim follows from simple calculus.

**Claim 1.** Suppose $H^{-1}f$ defined in (2.10). One has

$$H^{-1}\Theta_2 \lesssim 1_{y \leq 1} + y^2 \left( \frac{1 + |\log \sqrt{b_1}y|}{|\log b_1|} \right) 1_{1 \leq y \leq 6B_0} + \frac{1}{b_1|\log b_1|} 1_{y \geq 6B_0},$$

$$H^{-1}[QT_1^2] \lesssim H^{-1}[1_{y \leq 1} + y^{-2}|\log y|^21_{y \geq 1}] \lesssim 1_{y \geq 1} + |\log y|^31_{y \geq 1}. \quad (2.39)$$

Above estimates for $\Theta_2$ and $R_2$ imply

$$|\tilde{S}_2| \lesssim 1_{y \leq 1} + \frac{1}{b_1|\log b_1|} 1_{y \geq 1} \quad \text{for } y \leq 2B_1. \quad (2.40)$$

and the rough bound

$$|\tilde{S}_2| \lesssim 1 + y^2, \quad \text{for } y \leq 2B_1. \quad (2.41)$$

In general, taking $\Lambda^i$ operation on (2.38) and applying the commutator

$$[H, \Lambda^i] = H\Lambda^i - \Lambda^iH = 2H + (V + \Lambda V), \quad (2.42)$$

we obtain inductively similar estimates of $\Lambda^i S_2$ for $1 \leq i \leq 10$. Also, taking $\partial_{b_1}$ operation and (since it commutes with $H$) using (2.37), then one has

$$\text{for } 0 \leq i, j \leq 10, y \leq 2B_1, \quad |b_1^j\partial_{b_1}^i\Lambda S_2| \lesssim b_1^j(1_{y \leq 1} + \frac{1}{b_1|\log b_1|} 1_{y \geq 1});$$

$$\text{for } 0 \leq i, j \leq 10, y \leq 2B_1, \quad |b_1^i\partial_{b_1}^j\Lambda S_2| \lesssim b_1^i(1 + y^2). \quad (2.44)$$

Note that $\partial_{b_2} S_2 = 0$.

**Step 4: Construction of $S_3$.** Here we use $H^{-1}\Sigma_{b_i}$ to cancel the leading order growth in $\Lambda T_2 - 3T_2$. Define

$$\Theta_3 := \Lambda T_2 - 3T_2 - H^{-1}\Sigma_{b_i}. \quad (2.45)$$

then using (2.43) and $HT_2 = -T_1$ we get

$$H\Theta_3 = -\Lambda T_1 - 2T_1 + (V + \Lambda V)T_2 + 3T_1 - \Sigma_{b_i} = -\Theta_2 + (V + \Lambda V)T_2, \quad (2.46)$$

hence with bounds (2.37) and (2.21) we derive

$$\text{for } 0 \leq i, j, k \leq 10, y \leq 2B_1, \quad |b_1^{i+2k}\partial_{b_1}^j\partial_{b_2}^k\Lambda^i\Theta_3| \lesssim 1_{y \leq 1} + \frac{1}{b_1|\log b_1|} 1_{y \geq 1}. \quad (2.47)$$

Next turn to $R_3$. From (2.20), (2.21), (2.45) and a priori bound $|b_2| \lesssim b_1^2$, we get

$$\text{for } 0 \leq i, j, k \leq 10, y \leq 2B_1, \quad |b_1^{i+2k}\partial_{b_1}^j\partial_{b_2}^k\Lambda^i R_3| \lesssim b_1^3(1_{y \leq 1} + |\log y|^3 1_{y \geq 1}) \quad (2.48)$$

Let $S_3$ be the solution to

$$b_1b_2\Theta_3 + HS_3 + [-b_1^2(1 + c_{b_1}) + b_2]\partial_{b_1} S_2 + b_1\Lambda S_2 - R_3 = 0, \quad (2.49)$$

then estimates (2.48), (2.49) and (2.44) yield

$$\text{for } 0 \leq i, j, k \leq 10, y \leq 2B_1, \quad |b_1^{i+2k}\partial_{b_1}^j\partial_{b_2}^k\Lambda^i S_3| \lesssim b_1^3 \left( 1_{y \leq 1} + \frac{y^2}{b_1|\log b_1|} 1_{y \geq 1} \right);$$

$$\text{for } 0 \leq i, j, k \leq 10, y \leq 2B_1, \quad |b_1^{i+2k}\partial_{b_1}^j\partial_{b_2}^k\Lambda^i S_3| \lesssim b_1^3(1 + y^4). \quad (2.50)$$
**Step 5: Construction of $S_4$.** From (2.45), (2.52), we get

for $0 \leq i, j, k \leq 10$, $y \leq 2B_1$, \[|b_1^{i+j+2k}\partial_{\beta_j}^i \partial_{\beta_k}^k \Lambda^i R_4| \lesssim b_4^i (1_{y \leq 1} + y^2 \log y)^C 1_{y \geq 1}. \hspace{1cm} (2.53)\]

Let $S_4$ be the solution to

$$HS_4 + [-b_1^2(1 + c_b) + b_2] \partial_{\beta_1}^2 S_3 + [-b_1 b_2(3 + c_b)] \partial_{\beta_2} S_3 + b_1 \Lambda S_3 - R_4 = 0, \hspace{1cm} (2.54)$$

then similar to Step 4, we have

for $0 \leq i, j, k \leq 10$, $y \leq 2B_1$, \[|b_1^{i+j+2k}\partial_{\beta_j}^i \partial_{\beta_k}^k \Lambda^i S_4| \lesssim b_4^i \left(1_{y \leq 1} + \frac{y^4}{b_1 |\log b_1|} 1_{y \geq 1}\right), \hspace{1cm} (2.55)\]

\[|b_1^{i+j+2k}\partial_{\beta_j}^i \partial_{\beta_k}^k \Lambda^i S_4| \lesssim b_4^i (1 + y^6). \hspace{1cm} (2.56)\]

**Step 6: Estimation of the error.** According to our constructions above and assumption (2.11), we get

$$\Psi_b = -b_4^2(\Sigma_{b_1} + c_b T_1) + b_1 b_2 H^{-1}(\Sigma_{b_1} + c_b T_1) + [-b_1^2(1 + c_b) + b_2] \partial_{\beta_1}^2 S_4 + [-b_1 b_2(3 + c_b)] \partial_{\beta_2} S_4 + b_1 \Lambda S_4$$

We split it into three parts and estimate as follows:

(i) The first line: Write $\tilde{\Sigma}_{b_1} := \Sigma_{b_1} + c_b T_1$. Note that $\tilde{\Sigma}_{b_1} = 0$ for $y \leq \frac{B_0}{4}$.

By (2.28), we estimate for $\frac{B_0}{4} \leq y \leq 2B_1$

$$|H^k \tilde{\Sigma}_{b_1}| \lesssim \frac{1}{|\log b_1|^2 y^2 k} \text{ for } 1 \leq k \leq 3,$$

$$|\partial_{\beta}^i \tilde{\Sigma}_{b_1}| \lesssim y^{-i}, \hspace{1cm} |\partial_{\beta}^i H^{-1} \tilde{\Sigma}_{b_1}| \lesssim y^{2-i} \text{ for } 0 \leq i \leq 10,$$

hence we conclude

\[\int_{y \leq 2B_1} |H^k \tilde{\Sigma}_{b_1}|^2 \lesssim \int_{\frac{B_0}{4} \leq y \leq 2B_1} \frac{1}{y^{4k}} \lesssim b_1^{2k-2} |\log b_1|^C \text{ for } k = -1, 0, 1, \hspace{1cm} (2.58)\]

\[\int_{y \leq 2B_1} |H^k \tilde{\Sigma}_{b_1}|^2 \lesssim \frac{1}{|\log b_1|^2} \int_{\frac{B_0}{4} \leq y \leq 2B_1} \frac{1}{y^{4k}} \lesssim \frac{b_1^{2k-2}}{|\log b_1|^2} \text{ for } k = 2, 3. \hspace{1cm} (2.59)\]

(ii) The second line: By the rough bound (2.56), we have

\[\int_{y \leq 2B_1} |H^k \Lambda S_4|^2 + |H^k (b_1 \partial_{\beta_1} S_4)|^2 + |H^k (b_1^2 \partial_{\beta_2} S_4)|^2 \lesssim b_1^8 \int_{y \leq 2B_1} (1 + y^{6-2k})^2 \lesssim b_1^{2k} |\log b_1|^C \text{ for } k = 1, 2. \hspace{1cm} (2.60)\]

The crucial $H^3$ level bound requires more effort. Apply operator $H$ twice to (2.54) and use (2.43), then we find

$$H^3 S_4 = O(b_1)(H^2 S_3 + \Lambda^2 S_3 + b_1 \partial_{\beta_1} H^2 S_3 + b_1^2 \partial_{\beta_2} H^2 S_3) + O(b_1^4)(1_{y \leq 1} + \frac{|\log y|^C}{y^2} 1_{y \geq 1}).$$
Again, apply $H$ to (2.50) and use (2.43)

$$H^2 S_3 = O(b_1^3)(\Theta_2 + \Lambda \Theta_2 + b_1 \partial_{b_1} \Theta_2) + O(b_1^4) \left(1_{y \leq 1} + \frac{\log y}{y^2} 1_{y \geq 1}\right).$$

Now the bound (2.37) for $\Theta_2$ implies

$$|H^3 S_4| \lesssim b_1^4 \left(1_{y \leq 1} + \left(1 + \frac{\log \sqrt{b_1} y}{\log b_1}\right) 1_{1 \leq y \leq 6B_0} + \frac{\log y}{y^2} 1_{y \geq 6B_0}\right)$$

thus

$$\int_{y \leq 2B_1} |H^3 S_4|^2 \lesssim b_1^8 \left(\int_{y \leq 1} 1 + \int_{1 \leq y \leq 6B_0} \frac{(1 + |\log \sqrt{b_1} y|)^2}{\log b_1^2} + \int_{6B_0 \leq y \leq 2B_1} \frac{|\log y|^C}{y^4}\right) \lesssim \frac{b_1^6}{|\log b_1|^2}.\quad (2.61)$$

Similarly we have

$$\int_{y \leq 2B_1} |H^3 \Lambda S_4|^2 + |H^3 (b_1 \partial_{b_1} S_4)|^2 + |H^3 (b_1^2 \partial_{b_2} S_4)|^2 \lesssim \frac{b_1^6}{|\log b_1|^2}.\quad (2.62)$$

(iii) The third line: From rough bounds (2.45), (2.52), (2.56) for $S_j$ we derive

$$|\partial_y R| \lesssim b_1^5 1_{y \leq 1} + \sum_{j=5}^{12} b_1^j y^{2j-6-i} (1 + |\log y|^C) 1_{y \geq 1} \lesssim b_1^5 (1_{y \leq 1} + y^{4-i} |\log b_1|^C 1_{y \geq 1}) \text{ for } y \leq 2B_1.$$

Hence

$$\int_{y \leq 2B_1} |H^b R|^2 \lesssim b_1^{10} \int_{y \leq 1} 1 + b_1^{10} |\log b_1|^C \int_{1 \leq y \leq 2B_1} y^{8-4k} \lesssim b_1^{2k+4} |\log b_1|^C \text{ for } 1 \leq k \leq 3.\quad (2.62)$$

The bounds (2.58)-(2.62) together yield (2.15) and (2.16). (2.17) can be proved in similar way. (2.18) follows from $\tilde{S}_{b_1} = 0$ for $y \leq 2M \leq B_0/4$. □

2.2. localization. The approximate solution $Q_b$ we constructed above stays close to $Q$ only in the parabolic zone $y \leq 2B_1$, and certain localization is needed to avoid the growth at infinity.

**Proposition 2.2. (Localization)** Under the assumptions of Proposition 2.1, assume further that

$$|(b_1)|_1 \lesssim b_1^2.\quad (2.63)$$

Let $\tilde{Q}_b := Q + \bar{a}, \bar{a} := \chi_{B_1}$. Then

$$\partial_t \tilde{Q}_b - \frac{\lambda}{\lambda} \Delta \tilde{Q}_b - \Delta \tilde{Q}_b - \tilde{Q}_b^2 = \tilde{\Psi}_b + \text{Mod},\quad (2.64)$$
We estimate the

\[ \text{Mod} := - \left( \frac{\chi_{s}}{\chi} + b_{1} \right) \Lambda \hat{Q}_{b} + [(b_{1})_{s} + b_{1}^{2}(1 + c_{b_{1}}) - b_{2}] \left[ \hat{T}_{1} + \chi_{B_{1}} \sum_{j=2}^{4} \frac{\partial S_{j}}{\partial y_{1}} \right] \]\n
and \( \hat{\Psi}_{b} \) satisfies

\[ \int |H^{k} \hat{\Psi}_{b}|^{2} \lesssim b_{1}^{2k+2} \log b_{1}^{C} \quad \text{for } k = 1, 2, \] (2.65)

\[ \int |H^{3} \hat{\Psi}_{b}|^{2} \lesssim \frac{b_{1}^{8}}{|\log b_{1}|^{2}}, \] (2.66)

\[ \int \frac{1 + |\log y|^{2}}{1 + y^{12-2r}} |\partial_{y}^{i} \hat{\Psi}_{b}|^{2} \lesssim b_{1}^{8} \log b_{1}^{C} \quad \text{for } 0 \leq i \leq 4, \] (2.67)

\[ \int_{y \leq 2M} |H^{k} \hat{\Psi}_{b}|^{2} \lesssim b_{1}^{10} M^{C} \quad \text{for } 0 \leq k \leq 3. \] (2.68)

**Proof.** From localization we compute

\[ \hat{\Psi}_{b} = \chi_{B_{1}} \Psi_{b} + b_{1}(1 - \chi_{B_{1}}) \Lambda Q + \alpha \partial_{s} \chi_{B_{1}} - \alpha \Delta \chi_{B_{1}} - 2 \partial_{y} \alpha \partial_{y} \chi_{B_{1}} \]

\[ - [(Q + \chi_{B_{1}} \alpha)^{3} - Q^{3}] + \chi_{B_{1}}[(Q + \alpha)^{3} - Q^{3}]. \]

We estimate the \( H^{3} \) level bound (2.67) term by term, as follows.

(i) Start with \( \chi_{B_{1}} \hat{\Psi}_{b} \). From (2.57) we estimate for \( 0 \leq i \leq 6, B_{1} \leq y \leq 2B_{1} \)

\[ |\partial_{y}^{i} \hat{\Psi}_{b}| \lesssim b_{1}^{2i} y^{-i} + b_{1}^{3} y^{2-i} + \frac{b_{1}^{4} y^{4-i}}{|\log b_{1}|} + b_{1}^{5} y^{4-i} \log b_{1} \supseteq b_{1}^{4} y^{4-i} \log b_{1}^{1} \]

Together with (2.16) we get

\[ \int |H^{3}(\chi_{B_{1}} \hat{\Psi}_{b})|^{2} \lesssim \int_{y \leq B_{1}} |H^{3} \hat{\Psi}_{b}|^{2} + \frac{b_{1}^{8}}{|\log b_{1}|^{2}} \int_{B_{1} \leq y \leq 2B_{1}} \frac{1}{y^{4}} \lesssim \frac{b_{1}^{8}}{|\log b_{1}|^{2}}. \]

(ii) The second term is easily controlled by

\[ \int |H^{3}[(1 - \chi_{B_{1}}) \Lambda Q]|^{2} \lesssim \int_{y \geq B_{1}} \frac{1}{y^{16}} \lesssim \frac{b_{1}^{8}}{|\log b_{1}|^{2}}. \]

(iii) For the third term \( \alpha \partial_{s} \chi_{B_{1}} \), with \textit{a priori} bound \( |(b_{1})_{s}| \lesssim b_{1}^{5} \), we have

\[ 0 \leq i \leq 10, \quad |\partial_{y}^{i} \partial_{s} \chi_{B_{1}}| = |(b_{1})_{s} \partial_{b_{1}} \partial_{y}^{i} \chi_{B_{1}}| \lesssim \frac{b_{1}}{y^{3} 1_{B_{1} \leq y \leq 2B_{1}}}, \]

and our calculations in the previous subsection imply

\[ |\partial_{y}^{i} \alpha| \lesssim b_{1}^{2} y^{2-i} \log y \quad \text{for } B_{1} \leq y \leq 2B_{1}. \]
So we have
\[ \int |H^3(\alpha \partial_s \chi_{B_1})|^2 \lesssim b_1^4 \int_{B_1 \leq y \leq 2B_1} \frac{\log y}{y^8} \lesssim \frac{b_1^8}{|\log b_1|^2}. \]

(iv) The next two terms are bounded by
\[ \int |H^3(\alpha \Delta \chi_{B_1} + 2\partial_y \alpha \partial_y \chi_{B_1})|^2 \lesssim b_1^4 \int_{B_1 \leq y \leq 2B_1} \frac{\log y}{y^8} \lesssim \frac{b_1^8}{|\log b_1|^2}. \]

(iv) Finally, note that
\[-[(Q + \chi_{B_1})^3 - Q^3] + \chi_{B_1}[(Q + \alpha)^3 - Q^3] = 3(\chi_{B_1} - \chi_{B_1}^2)Q\alpha^2 + (\chi_{B_1} - \chi_{B_1}^3)\alpha^3, \]
and
\[ \int |H^3[3(\chi_{B_1} - \chi_{B_1}^2)Q\alpha^2 + (\chi_{B_1} - \chi_{B_1}^3)\alpha^3]|^2 \]
\[ \lesssim \int_{B_1 \leq y \leq 2B_1} \left| \frac{1}{y^6} \left[ b_1^4 y^2 (\log y)^2 + b_1^6 (\log y)^3 \right] \right|^2 \]
\[ \lesssim b_1^{10} |\log b_1|^C \lesssim \frac{b_1^8}{|\log b_1|^2}. \]
This concludes the proof of (2.67). Proofs for the other three are similar (and simpler).

2.3. **Dynamical system for** \( b = (b_1, b_2) \). In the first subsection, we have seen the importance of the (modulation) assumption (2.11). Thus \( b = (b_1, b_2) \) should approximately satisfy
\[ (b_k)_s + (2k - 1 + \frac{2}{\log s})b_1 b_k - b_{k+1} = 0, \quad k = 1, 2, \quad b_3 = 0. \]
Indeed, this equation has an approximate solution
\[ b_1^e = \frac{2}{3s} - \frac{4}{9s \log s}, \quad b_2^e = -\frac{2}{9s^2} + \frac{20}{27s^2 \log s}, \quad b_3^e = 0 \] (2.70)
in the sense that
\[ (b_k^e)_s + \left(2k - 1 + \frac{2}{\log s}\right)b_k^e b_k - b_{k+1}^e = O \left(\frac{1}{s^{k+1}(\log s)^2}\right), \quad k = 1, 2. \] (2.71)
The proof is by direct calculation. Now we look for \( b \) near this approximate solution.

**Proposition 2.3.** Let
\[ b_k = b_k^e + \frac{U_k}{s^k(\log s)^2}, \quad k = 1, 2, \quad b_3 = 0 \] (2.72)
and \( U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \), \( b_k^e \) in (2.70). Then we have
\[ (b_k)_s + \left(2k - 1 + \frac{2}{\log s}\right)b_1 b_k - b_{k+1} \]
\[ = \frac{1}{s^{k+1}(\log s)^2} \left[ s(U_k)_s - (AU)_k + O \left(\frac{1}{\sqrt{\log s}} + \frac{|U| + |U|^2}{\log s}\right) \right] \] (2.73)
where \( A = \begin{bmatrix} -\frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix} = P^{-1} D_A P \) with \( P = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \), \( D_A = \begin{bmatrix} -1 & 2 \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} \).

**Proof.** In fact, direct computation yields

\[
(b_1)_s + \left(1 + \frac{2}{\log s}\right)b_1^2 - b_2
= \frac{1}{s^2(\log s)^{\frac{3}{2}}} \left[s(U_1)_s - U_1 + O\left(\frac{|U|}{\log s}\right)\right] \\
+ \frac{4}{3} U_1 - U_2 + O\left(\frac{|U| + |U|^2}{\log s}\right) + O\left(\frac{1}{s^2|\log s|^2}\right)
\] (2.74)

and

\[
(b_2)_s + (3 + \frac{2}{\log s})b_1b_2
= \frac{1}{s^3(\log s)^{\frac{3}{2}}} \left[s(U_2)_s - 2U_2 + O\left(\frac{|U|}{\log s}\right)\right] \\
+ 3 \left(-\frac{2}{9} U_1 + \frac{2}{3} U_2\right) + O\left(\frac{|U| + |U|^2}{\log s}\right) + O\left(\frac{1}{s^3|\log s|^2}\right).
\] (2.75)

which can be arranged to (2.73). The diagonalization of \( A \) is simple linear algebra. \(\square\)

### 3. THE TRAPPED REGIME

From now on, we assume that the initial data \( v_0 \in \dot{H}^1 \cap \dot{H}^6 \). From standard local well posedness theory, (1.5) has a solution \( v \in C([0, T), \dot{H}^1 \cap \dot{H}^6) \) with lifetime \( T = T(v_0) \leq +\infty \).

In this section we describe our choice of initial data and design a bootstrap regime to control the behavior of the corresponding solution. Our main analysis is on the Lyapounov monotonicity in subsection 3.3.

#### 3.1. Modulation theory

We first try to decompose the solution as

\[
v(t, \cdot) = (\tilde{Q}_{b(t)} + \varepsilon)_{\lambda(t)}, \quad \text{or} \quad u = \tilde{Q}_b + \varepsilon. \tag{3.1}
\]

where \( \varepsilon(s, y) \) satisfies the orthogonality conditions

\[
(\varepsilon, H^k \Phi_M) = 0 \quad \text{for} \quad 0 \leq k \leq 2.
\] (3.2)

Here \( \Phi_M \) is a substitute for \( \Lambda Q \) supported on \( y \leq 2M \) (this is necessary since \( \Lambda Q \notin L^2 \)), as described below.

Given \( M > 0 \) large enough, define

\[
\Phi_M := \chi_M \Lambda Q + c_{M,1} H(\chi_M \Lambda Q) + c_{M,2} H^2(\chi_M \Lambda Q) \tag{3.3}
\]

where

\[
c_{M,1} = \frac{(\chi_M \Lambda Q, T_1)}{(\chi_M \Lambda Q, \Lambda Q)} = O(M^2), \quad c_{M,2} = \frac{-(\chi_M \Lambda Q, T_2) + c_{M,1}(\chi_M \Lambda Q, T_1)}{(\chi_M \Lambda Q, \Lambda Q)} = O(M^4)
\]
are chosen to ensure the cancellation
\[(\Phi_M, T_i) = 0, \quad i = 1, 2\]  \hfill (3.4)
and non-degeneracy
\[(247,3.4)\]
\[(\Phi_M, \Lambda Q) = (\chi M \Lambda Q, \Lambda Q) = 64 \log M (1 + o_{M \to \infty} (1)), \]  \hfill (3.5)
\[||\Phi_M||^2_{L^2} \lesssim \int |\chi M \Lambda Q|^2 + c_{M,1}^2 \int |H(\chi M \Lambda Q)|^2 + c_{M,2}^2 \int |H^2(\chi M \Lambda Q)|^2 \lesssim \log M. \]  \hfill (3.6)
Now at the point \(\lambda = 1, b = (b_1, b_2) = (0, 0)\), we have
\[(\partial_{\lambda} (\tilde{\Omega} b), \partial_{b_1} (\tilde{\Omega} b), \partial_{b_2} (\tilde{\Omega} b),) = (\Lambda Q, T_1, T_2), \]
hence the non-degeneracy of Jacobian
\[
\left| \left( \frac{\partial}{\partial (\lambda, b_j)}(\tilde{\Omega} b), H^2 \Phi_M \right)_{1 \leq j \leq 2, 0 \leq k \leq 2} \right|_{\lambda=1, b=0} = \left| \begin{array}{ccc}
(\Lambda Q, \Phi_M) & 0 & 0 \\
0 & - (\Lambda Q, \Phi_M) & 0 \\
0 & 0 & (\Lambda Q, \Phi_M) 
\end{array} \right| \neq 0.
\]
Now we introduce some notations:
(i) The energy norm
\[\Xi_1(t) := \int |\partial_y \epsilon|^2 \]  \hfill (3.7)
and higher Sobolev norms
\[\Xi_{2k}(t) := \int |\epsilon_{2k}|^2, \quad 1 \leq k \leq 3 \text{ with } \epsilon_{2k} := H^k \epsilon. \]  \hfill (3.8)
(ii) The unstable models
\[V(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = PU, \quad \text{where } P, U \text{ from Proposition 2.3.} \]  \hfill (3.9)
\[\tau(t) = (\epsilon(t), \psi), \quad \text{where } \psi \text{ is from (2.5).} \]  \hfill (3.10)
With these preparations, we turn to the construction of initial data. Set \(v_0\) in the decomposition form (3.1) as
\[v_0 = \tilde{\epsilon}_{b(0)} + \tau(0)\tilde{\psi}\]  \hfill (3.11)
where \(\tilde{\psi}\) satisfies
\[(\tilde{\psi}, \psi) = 1, \quad (\tilde{\psi}, H^k \Phi_M) = 0, \quad 0 \leq k \leq 2. \]  \hfill (3.12)
This way, the orthogonal conditions (3.2) are automatically satisfied at \(t = 0\).
Besides, \(b(0)\) is determined by \(V(0)\) (or \(U(0)\)) through (2.72)
\[b_k = b_k^e + \frac{U_k}{s^k (\log s)^4}, \quad k = 1, 2. \]  \hfill (3.13)
We fix \(U_1(0) = 0\), hence
\[b_1(0) = \frac{2}{3s_0} - \frac{4}{9s_0}. \]  \hfill (3.14)
Choose \(s_0\) large enough and \(V(0), \tau(0)\) properly so that the following bounds hold:
(i) Initial smallness:
\[ 0 < b_1(0) < b^*(M) \ll 1, \quad \Xi_1(0) \leq b_1(0), \]  
\[ \Xi_2(0) + \Xi_4(0) + \Xi_6(0) \leq b_1(0)^7. \]  
(3.15)  
(3.16)

(ii) Control of the unstable models:
\[ |V_1(0)| \leq 1, \quad |V_2(0)| \leq 1, \quad |\tau(0)| \leq \frac{b_1(0)^{3+\frac{3}{2}}}{|\log b_1(0)|} \]  
(3.17)

(iii) Without loss of generality we assume
\[ \lambda(0) = 1. \]  
(3.18)

By Implicit Function Theorem, (3.15) ensures that the decomposition (3.1) exists and is unique near \( t = 0 \). Moreover, \( \lambda, b \in C^1 \).

Given another large enough universal constant \( K > 0 \), independent of \( M \), the continuity of the flow implies the following proposition.

**Proposition 3.1.** *(Bootstrap)* There exists some maximal time \( T_{exit} \in [0, T(v_0)] \), called exit time, such that for all \( t \in [0, T_{exit}) \) the following bounds hold:

(i) Control of \( \varepsilon \):
\[ \Xi_1(t) \leq 10 \sqrt{b_1(0)}, \]  
\[ \Xi_2(t) \leq b_1^4(t)|\log b_1(t)|^K, \]  
\[ \Xi_4(t) \leq b_1^4(t)|\log b_1(t)|^K, \]  
\[ \Xi_6(t) \leq K \frac{b_1^6(t)}{|\log b_1(t)|^2}. \]  
(3.19)  
(3.20)  
(3.21)  
(3.22)

(ii) Control of unstable models:
\[ |V_1(t)| \leq 2, \quad |V_2(t)| \leq 2, \quad |\tau(t)| \leq \frac{b_1(t)^{3+\frac{3}{2}}}{|\log b_1(t)|}. \]  
(3.23)

**Remark 3.2.** Bound (3.23) on \( V \) ensures \( |b_2| \lesssim b_1^2 \), so (2.12) holds.

We now describe bootstrap regime. First, use the control of unstable models to improve the bounds (3.19)-(3.22). Next, for \( V \) and \( \tau \), our primary observation is that the two unstable directions are in some sense independent of each other, hence a Brouwer argument works to provide some initial data with \( T_{exit} = T(v_0) \). The proof of Theorem 1.1 follows from Proposition 3.1 easily.

In the next two subsections we assume \( t \in [0, T_{exit}) \) and deduce some key tools to close the bootstrap.
3.2. Modulation equations. Bring (3.1) and (2.64) into (2.3), we find
\[
\partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon + \varepsilon \mathcal{F} = -\mathcal{Q}_b - \text{Mod} + L(\varepsilon) + N(\varepsilon)
\]  
where
\[
L(\varepsilon) = 3(\tilde{Q}_b - Q)^2 \varepsilon, \quad N(\varepsilon) = 3\tilde{Q}_b \varepsilon^2 + \varepsilon^3.
\]
We now derive the modulation equations for \( b, \lambda \) as a consequence of the orthogonality conditions (3.2).

**Proposition 3.3.** (Modulation equations) We have the bounds on the modulation parameters
\[
\left| \frac{\lambda_s}{\lambda} + b_1 \right| + \left| (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 \right| \lesssim b_1^{3 + \frac{1}{2}},
\]
\[
\left| (b_2)_s + b_1 b_2 (3 + c_{b_1}) \right| \lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{\Xi_0} + \frac{b_1^3}{|\log b_1|} \right),
\]
with constants independent of \( M \) and \( K \), as long as the \( b^*(M) \) in (2.12) is small enough.

**Remark 3.4.** (3.26) shows \(|(b_1)_s| \leq b_1^2 \) and hence (2.63) holds.

**Proof.** Let
\[
D(t) := \left| \frac{\lambda_s}{\lambda} + b_1 \right| + \left| (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 \right| + \left| (b_2)_s + b_1 b_2 (3 + c_{b_1}) \right|.
\]

**Step 1: Law for \( b_2 \).** Take the inner product of (3.24) with \( H^2 \Phi_M \). Using (3.2) we get
\[
\text{Mod, } H^2 \Phi_M = -(H^3 \varepsilon, \Phi_M) - (H^2 \tilde{Q}_b, \Phi_M) + \left( \frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H^2 \Phi_M \right).
\]
On the other hand, by the definition (2.65) of Mod,
\[
\text{Mod, } H^2 \Phi_M = - \left( \frac{\lambda_s}{\lambda} + b_1 \right) (\Lambda \tilde{Q}_b, H^2 \Phi_M) + \left[ (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 \right] \left( \tilde{T}_1 + \chi_{B_1} \sum_{j=2}^4 \frac{\partial S_j}{\partial b_1}, H^2 \Phi_M \right)
\]
\[
+ \left[ (b_2)_s + b_1 b_2 (3 + c_{b_1}) \right] \left( \tilde{T}_2 + \chi_{B_1} \sum_{j=3}^4 \frac{\partial S_j}{\partial b_2}, H^2 \Phi_M \right)
\]
\[
= \left[ (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 \right] \left( \sum_{j=2}^4 \frac{\partial S_j}{\partial b_1}, H^2 \Phi_M \right)
\]
\[
+ \left[ (b_2)_s + b_1 b_2 (3 + c_{b_1}) \right] \left( \sum_{j=3}^4 \frac{\partial S_j}{\partial b_2}, H^2 \Phi_M \right) + \left[ (b_2)_s + b_1 b_2 (3 + c_{b_1}) \right] (\Lambda Q, \Phi_M)
\]
where we used \( H^2 \lambda \tilde{Q}_b = H^2 \tilde{T}_1 = 0 \) and \( H^2 \tilde{T}_2 = \lambda Q \) for \( y \leq 2M \).

We now estimate these terms respectively. By Cauchy–Schwarz inequality and (2.69)
\[
||H^3 \varepsilon, \Phi_M|| \lesssim ||H^3 \varepsilon||_{L^2} ||\Phi_M||_{L^2} \lesssim \sqrt{\log M} \sqrt{\Xi_0},
\]
\[(|H^2 \tilde{\Psi}_b, \Phi_M| \lesssim ||H^2 \tilde{\Psi}_b||_{L^2(y \leq 2M)}||\Phi_M||_{L^2} \lesssim b_1^5 M^C).\]

By interpolation bounds in the Appendix B and (3.22), we obtain
\[
\left| \left( \frac{\lambda}{\chi} \Lambda \varepsilon, H^2 \Phi_M \right) \right| \lesssim (|D(t)| + b_1) C(M) \sqrt{\Xi_0} \lesssim b_1 M^C |D(t)| + \sqrt{\log M} \sqrt{\Xi_0},
\]
\[
\left| (L(\varepsilon) + N(\varepsilon), H^2 \Phi_M) \right| \lesssim b_1 C(M) \sqrt{\Xi_0} \lesssim \sqrt{\log M} \sqrt{\Xi_0}.
\]

Using rough bound of \(S_j\) in (2.45), (2.52), (2.56), one has
\[
\left| (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 \right| \leq \left( \sum_{j=2}^{4} \frac{\partial S_j}{\partial b_1}, H^2 \Phi_M \right) \left( \sum_{j=3}^{4} \frac{\partial S_j}{\partial b_2}, H^2 \Phi_M \right)
\]
\[
\lesssim b_1 M^C |D(t)|.
\]

So we conclude that
\[
\left| (b_2)_s + b_1 b_2 (3 + c_{b_1}) \right| = \frac{1}{|(\Lambda Q, \Phi_M)|} \left( \sqrt{\log M} \sqrt{\Xi_0} + b_1^3 M^C + b_1 M^C |D(t)| \right)
\]
\[
\lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{\Xi_0} + \frac{b_1^3}{|\log b_1|} \right) + b_1 M^C |D(t)|.
\]

**Step 2: Law for \(b_1\) and \(\lambda\).** Similarly, take the inner product of (3.24) with \(H \Phi_M\) (note that \((H \varepsilon, H \Phi_M) = (\varepsilon, H^2 \Phi_M) = 0)\):
\[
(\text{Mod}, H \Phi_M) = -(H \tilde{\Psi}_b, \Phi_M) + \left( \frac{\lambda}{\chi} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H \Phi_M \right),
\]
and again by the definition of \(\text{Mod}\)
\[
(\text{Mod}, H \Phi_M)
\]
\[
= - \left( \frac{\lambda}{\chi} + b_1 \right) (\Lambda \tilde{Q}_b, H \Phi_M) + [(b_1)_s + b_1^2 (1 + c_{b_1})] - b_2 \left( \tilde{T}_1 + \chi b_1 \sum_{j=2}^{4} \frac{\partial S_j}{\partial b_1}, H \Phi_M \right)
\]
\[
+ [(b_2)_s + b_1 b_2 (3 + c_{b_1})] \left( \tilde{T}_2 + \chi b_1 \sum_{j=3}^{4} \frac{\partial S_j}{\partial b_2}, H \Phi_M \right)
\]
\[
= [(b_1)_s + b_1^2 (1 + c_{b_1})] - b_2 \left( -\Lambda Q, \Phi_M \right) + [(b_1)_s + b_1^2 (1 + c_{b_1})] - b_2 \left( \sum_{j=2}^{4} \frac{\partial S_j}{\partial b_1}, H \Phi_M \right)
\]
\[
+ [(b_2)_s + b_1 b_2 (3 + c_{b_1})] \left( \sum_{j=3}^{4} \frac{\partial S_j}{\partial b_2}, H \Phi_M \right)
\]

The above computation yields
\[
\left| (b_1)_s + b_1^2 (1 + c_{b_1}) - b_2 \right| \lesssim b_1^{3 + \frac{1}{2}} + b_1 M^C |D(t)|.
\]
Finally, taking the inner product of (3.24) with $\Phi_M$, we obtain
\[
\left| \frac{\lambda_s}{\lambda} + b_1 \right| \lesssim b_1^{3+\frac{1}{2}} + b_1 M^C |D(t)|. \tag{3.31}
\]

**Step 3: Conclude estimates.** From (3.29)-(3.31), it is not hard to get
\[
|D(t)| \lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{\Xi_6} + \frac{b_1^3}{|\log b_1|} \right) + b_1^{3+\frac{1}{2}}. \tag{3.32}
\]
Inject this into (3.29)-(3.31) and we get the desired results. \(\square\)

Unfortunately, (3.27) is not enough to derive the sharp blow-up rate of $\lambda$, because $b_1 b_2 c_{b_1}$ (up to $\sqrt{\log M}$) is about the same size with $b_1^3 / |\log b_1|$ on the right hand side. To get the sharp blow-up rate and close the bootstrap, we need the following improved bound for $b_2$.

**Proposition 3.5.** (Improved modulation) Let $\delta > 0$ be small enough, $B_\delta := \frac{1}{b_1}$ and
\[
\bar{b}_2 := b_2 + \frac{(H^2 \varepsilon, \lambda b_1 \lambda Q)}{64 \delta |\log b_1|}, \tag{3.33}
\]
then
\[
|\bar{b}_2 - b_2| \lesssim b_1^{2+\frac{1}{2}}, \tag{3.34}
\]
\[
|(\bar{b}_2)_s + b_1 \bar{b}_2 (3 + c_{b_1})| \lesssim \frac{C(M)}{\sqrt{|\log b_1|}} \left[ \sqrt{\Xi_6} + \frac{b_1^3}{|\log b_1|} \right]. \tag{3.35}
\]

**Proof.** As above, we replace $H^2 \Phi_M$ by $H^2 \chi_{B_\delta} \lambda Q$ and take the inner product with (3.24)
\[
(\text{Mod}, H^2 \chi_{B_\delta} \lambda Q) = - \frac{d}{ds} (H^2 \varepsilon, \chi_{B_\delta} \lambda Q) + (H^2 \varepsilon, (\partial_s \chi_{B_\delta}) \lambda Q) - (H^2 \varepsilon, \chi_{B_\delta} \lambda Q)
- (H^2 \tilde{\psi}_{b_1} \chi_{B_\delta} \lambda Q) + \left( \frac{\lambda_s}{\lambda} \lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H^2 \chi_{B_\delta} \lambda Q \right).
\]
Since $B_\delta \leq B_0 \leq B_1$ for $\delta$ small enough, we get
\[
(\text{Mod}, H^2 \chi_{B_\delta} \lambda Q)
= [(b_1)_s + b_1^2 (1 + c_{b_1}) - b_2] \left( \sum_{j=2}^{4} \frac{\partial S_j}{\partial b_1} H^2 \chi_{B_\delta} \lambda Q \right)
+ [(b_2)_s + b_1 b_2 (3 + c_{b_1})] \left( \sum_{j=3}^{4} \frac{\partial S_j}{\partial b_2} H^2 \chi_{B_\delta} \lambda Q \right)
+ [(b_2)_s + b_1 b_2 (3 + c_{b_1})] (\lambda Q, \chi_{B_\delta} \lambda Q).
\]
\[
= \frac{b_1}{b_1^2} O \left( \sqrt{\Xi_6} + \frac{b_1^3}{|\log b_1|} \right) + [(b_2)_s + b_1 b_2 (3 + c_{b_1})] (64 \delta |\log b_1| + O(1))
= [(b_2)_s + b_1 b_2 (3 + c_{b_1})] \cdot 64 \delta |\log b_1| + O \left( \sqrt{\Xi_6} + \frac{b_1^3}{|\log b_1|} \right)
\]
where we used (3.26) and (3.27). We estimate
\[ |(H^3\varepsilon, \chi_{B_i}\Lambda Q)| \leq |(H^3\varepsilon, (b_2)_{s + b_1b_2(3 + c_b)}| \lesssim \frac{b_1}{b_1^{l_8}} C(M) \sqrt{\Xi_6} \lesssim \sqrt{\Xi_6}, \]
\[ |(H^3\varepsilon, \chi_{B_i}\Lambda Q)| \lesssim ||H^3\varepsilon||_{L^2}||\chi_{B_i}\Lambda Q||_{L^2} \lesssim \sqrt{\log b_1} \sqrt{\Xi_6}, \]
\[ |(H^2\tilde{\psi}_b, \chi_{B_1}\Lambda Q)| \lesssim ||H^2\tilde{\psi}_b||_{L^2(\gamma \preceq B_2)}||\chi_{B_1}\Lambda Q||_{L^2} \lesssim \frac{b_1^5}{b_1^{l_8}} \sqrt{\log b_1} \lesssim \frac{b_1^2}{|\log b_1|}, \]
\[ \left| \left( \lambda_s \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H^2\chi_{B_1}\Lambda Q \right) \right| \lesssim \frac{b_1}{b_1^{l_8}} C(M) \sqrt{\Xi_6} \lesssim \sqrt{\Xi_6}. \]

The collection of these bounds yields the preliminary estimate
\[ \frac{d}{ds} (H^2\varepsilon, \chi_{B_1}\Lambda Q) + [(b_2)_{s + b_1b_2(3 + c_b)}| \cdot 64\delta] |\log b_1| \lesssim C(M) \sqrt{\log b_1} \left( \sqrt{\Xi_6} + \frac{b_1^3}{|\log b_1|} \right). \]  
(3.36)

We also have
\[ (H^2\varepsilon, \chi_{B_1}\Lambda Q) \lesssim \frac{1}{b_1^{l_8}} \sqrt{\Xi_6} \lesssim b_1^{2 + \frac{1}{2}}. \]  
(3.37)

which bounds the deviation
\[ |\tilde{b}_2 - b_2| = \frac{|(H^2\varepsilon, \chi_{B_1}\Lambda Q)|}{64\delta |\log b_1|} \lesssim b_1^{2 + \frac{1}{2}}. \]

The proposition follows from
\[ |(b_2)_{s + b_1b_2(3 + c_b)}| = \left| \frac{d}{ds} (H^2\varepsilon, \chi_{B_1}\Lambda Q) + [(b_2)_{s + b_1b_2(3 + c_b)}] + O(b_1^{\frac{3}{2} + \frac{1}{2}}) \right| \]
\[ = \left| \frac{1}{64\delta |\log b_1|} \frac{d}{ds} (H^2\varepsilon, \chi_{B_1}\Lambda Q) + [(b_2)_{s + b_1b_2(3 + c_b)}] + O(b_1^{\frac{3}{2} + \frac{1}{2}}) \right| \]
\[ \lesssim \frac{C(M)}{|\log b_1|} \left( \sqrt{\Xi_6} + \frac{b_1^3}{|\log b_1|} \right). \]

This completes the proof. □

3.3. Lyapounov monotonicity. We now turn to derive a suitable Lyapounov functional for \( \Xi_6 \) energy. This is crucial to close the bootstrap in Proposition 3.1.

**Proposition 3.6.** We have
\[ \frac{d}{dt} \left[ \frac{1}{\lambda^{10}} \left( \Xi_6 + O(b_1^{\frac{1}{2}} \frac{b_1^6}{|\log b_1|^2}) \right) \right] \leq C \cdot \frac{b_1}{\lambda^{1/2}} \left[ \frac{b_1^6}{|\log b_1|^2} + \frac{\Xi_6}{\sqrt{\log M}} + \frac{b_1^3}{|\log b_1| \sqrt{\Xi_6}} \right] \]  
(3.39)

for some constant \( C > 0 \) independent of \( M \) and \( K \) if \( b_1 \) is small enough.

**Proof.** Step 1: Suitable derivatives. Define \( w(t, r) = \frac{1}{\lambda} \varepsilon(s, y) \), in abbreviation \( w(t, \cdot) = (\varepsilon(s, \cdot))_{\lambda} \). We also denote
\[ w_{2\lambda} = H_{\lambda}^k w, \quad \text{where } H_{\lambda} := -\Delta - \frac{1}{\lambda} V_{\lambda} := -\Delta - \bar{V}. \]

(3.40)
Then (3.24) becomes
\[ \partial_t w + H_\lambda w = \lambda^{-2} \mathcal{F}_\lambda. \] (3.41)

Using commutator identity
\[ [\partial_t, H_\lambda] = \partial_t H_\lambda - H_\lambda \partial_t = -\partial_t \hat{V}, \] (3.42)
we further derive
\[ \partial_t w_2 + H_\lambda w_2 = -\partial_t \hat{V} w + H_\lambda \left( \lambda^{-2} \mathcal{F}_\lambda \right), \] (3.43)
\[ \partial_t w_4 + H_\lambda w_4 = -\partial_t \hat{V} w_2 - H_\lambda (\partial_t \hat{V} \cdot w) + H_\lambda^2 (\lambda^{-2} \mathcal{F}_\lambda), \] (3.44)
\[ \partial_t w_6 + H_\lambda w_6 = -\partial_t \hat{V} w_4 - H_\lambda (\partial_t \hat{V} \cdot w_2) - H_\lambda^2 (\partial_t \hat{V} \cdot w) + H_\lambda^3 (\lambda^{-2} \mathcal{F}_\lambda). \] (3.45)

**Step 2: Energy identity.** We first note
\[ \partial_t \hat{V} = \frac{\lambda s}{\lambda} \hat{V}, \quad \text{with} \quad \hat{V} = \frac{-1}{\lambda^3} (V + \Lambda V)_\lambda, \]
then using (3.45) we compute
\[ \frac{1}{2} \frac{d}{dt} \int w_6^2 \]
\[ = \int w_6 \partial_t w_6 \]
\[ = - \int w_6 \left[ H_\lambda w_6 + \partial_t \hat{V} w_4 + H_\lambda (\partial_t \hat{V} \cdot w_2) + H_\lambda^2 (\partial_t \hat{V} \cdot w) - H_\lambda^3 (\lambda^{-2} \mathcal{F}_\lambda) \right] \]
\[ = - \int w_6 H_\lambda w_6 + \left( \frac{\lambda s}{\lambda} + b_1 \right) \int w_6 \left[ \hat{V} w_4 + H_\lambda (\hat{V} w_2) + H_\lambda^2 (\hat{V} w) \right] \]
\[ - \int w_6 \left[ b_1 \hat{V} w_4 + H_\lambda (b_1 \hat{V} w_2) + H_\lambda^2 (b_1 \hat{V} w) - H_\lambda^3 (\lambda^{-2} \mathcal{F}_\lambda) \right]. \] (3.46)

We further process those terms in the last line of (3.46). By (3.44), the first term becomes
\[ - \int b_1 \hat{V} w_4 w_6 = \int b_1 \hat{V} w_4 \left[ \partial_t w_4 + \partial_t \hat{V} \cdot w_2 + H_\lambda (\partial_t \hat{V} \cdot w) - H_\lambda^2 (\lambda^{-2} \mathcal{F}_\lambda) \right] \]
which produces a boundary term from integration-by-parts
\[ \int b_1 \hat{V} w_4 \partial_t w_4 = \frac{1}{2} \frac{d}{dt} \int b_1 \hat{V} w_4^2 - \frac{1}{2} \int \partial_t (b_1 \hat{V}) w_4^2. \]
Similarly the third term becomes
\[ - \int w_6 H_\lambda^2 (b_1 \hat{V} w) = \int H_\lambda^2 (b_1 \hat{V} w) \left[ \partial_t w_4 + \partial_t \hat{V} \cdot w_2 + H_\lambda (\partial_t \hat{V} \cdot w) - H_\lambda^2 (\lambda^{-2} \mathcal{F}_\lambda) \right], \]
and applying (3.42) after integration-by-parts, we have
\[ \int H_\lambda^2 (b_1 \hat{V} w) \partial_t w_4 = \frac{d}{dt} \int H_\lambda^2 (b_1 \hat{V} w) w_4 - \int w_4 \partial_t H_\lambda^2 (b_1 \hat{V} w) \]
\[ = \frac{d}{dt} \int H_\lambda^2 (b_1 \hat{V} w) w_4 + \int w_4 (\partial_t \hat{V}) H_\lambda (b_1 \hat{V} w) + \int w_4 H_\lambda [ (\partial_t \hat{V}) b_1 \hat{V} w] \]
\[ - \int w_4 H_\lambda^2 (\partial_t (b_1 \hat{V}) w) - \int w_4 H_\lambda (b_1 \hat{V} \partial_t w). \]
Now use (3.41) to replace $\partial_t w$ and we find
\[
- \int w_4 H_2^2(b_1 \dot{V} \partial_t w) = \int w_4 H_2^2(b_1 \dot{V} (w_2 - \lambda^{-2} \mathcal{F}_\lambda))
\]
\[
= \int w_6 H_\lambda (b_1 \dot{V} w_2) - \int w_4 H_2^2(b_1 \dot{V} \cdot \lambda^{-2} \mathcal{F}_\lambda).
\]
which cancels the second term in the last line of (3.46).

To sum up, there holds the energy identity
\[
\frac{1}{2} \frac{d}{dt} \left[ w_6^2 - b_1 \dot{V} w_4^2 - 2 w_4 H_2^2(b_1 \dot{V} w) \right]
= - \int w_6 H_\lambda w_6 + \left( \frac{\lambda s}{\lambda} + b_1 \right) \int w_6 \left[ \dot{V} w_4 + H_\lambda(\dot{V} w_2) + H_2^2(\dot{V} w) \right]
+ \int (b_1 \dot{V}) w_4 (\partial_t V) w_2 + \int (b_1 \dot{V}) w_4 H_\lambda (\partial_t V \cdot w) - \frac{1}{2} \int \partial_t (b_1 \dot{V}) \cdot w_4^2
+ \int (b_1 \dot{V}) w_4 (\partial_t V) w_2 + \int (b_1 \dot{V}) w_4 H_\lambda (\partial_t V \cdot w) - \frac{1}{2} \int \partial_t (b_1 \dot{V}) \cdot w_4^2
+ \int H_2^2(b_1 \dot{V} w) (\partial_t V) w_2 + \int H_2^2(b_1 \dot{V} w) H_\lambda (\partial_t V \cdot w)
+ \int w_4 (\partial_t V) H_\lambda (b_1 \dot{V} w) + \int w_4 H_\lambda \left( \partial_t V \cdot b_1 \dot{V} w \right) - \int w_4 H_2^2 \left[ \partial_t (b_1 \dot{V}) \cdot w \right]
- \int b_1 \dot{V} w_4 H_2^2(\lambda^{-2} \mathcal{F}_\lambda) - \int H_2^2(b_1 \dot{V} w) H_2^2(\lambda^{-2} \mathcal{F}_\lambda) - \int w_4 H_2^2(b_1 \dot{V} \lambda^{-2} \mathcal{F}_\lambda)
+ \int w_6 H_2^3(\lambda^{-2} \mathcal{F}_\lambda).
\]
(3.47)

Below we estimate (3.47) term by term to derive (3.39). The estimates use heavily the interpolation bounds in Appendix B.

**Step 3: Lower order quadratic terms.**

(i) The first term on the RHS of (3.47) is controlled by (A.1) and our assumption (3.23) of $\tau$
\[
- \int w_6 H_\lambda w_6 = - \frac{1}{\lambda^{12}} \int \varepsilon_6 H \varepsilon_6 \lesssim \frac{1}{\lambda^{12}} (\varepsilon_6, \psi)^2 \lesssim \frac{1}{\lambda^{12}} \varepsilon^6(\varepsilon, \psi)^2 \lesssim b_1 \frac{b_1^6}{\lambda^{12} \log b_1}. \]

(ii) Next, since
\[
|\partial_y^i \dot{V}| \lesssim \frac{1}{\lambda^{12}} \frac{1}{1 + y^{4+i}},
\]
we estimate the second term using Cauchy-Schwarz and interpolation bounds in Appendix B
\[
\left| \left( \frac{\lambda s}{\lambda} + b_1 \right) \int w_6 \left[ \dot{V} w_4 + H_\lambda(\dot{V} w_2) + H_2^2(\dot{V} w) \right] \right|
\lesssim b_1^{3+\frac{4}{s}} \frac{1}{\lambda^{12}} \int \sum_{i=0}^4 \frac{\varepsilon_6 ||\partial_y^i \varepsilon||}{1 + y^{8-i}} \lesssim b_1 \frac{b_1^6}{\lambda^{12} \log b_1}. \]
(3.48)
(iii) From the definition of $\bar{V}$, $\tilde{V}$ and modulation equation \eqref{eq:3.26}, we get

$$|\partial_y \partial_t \tilde{V}| \lesssim \frac{b_1}{\lambda^4} \frac{1}{1 + y^{4+i}}, \quad |\partial_y \partial_t (b_1 \tilde{V})| \lesssim \frac{b_1}{\lambda^6} \frac{1}{1 + y^{4+i}},$$

so the next three lines in (3.47) is bounded by

$$\left| \int (b_1 \tilde{V}) w_4 (\partial_t \tilde{V}) w_2 + \int (b_1 \tilde{V}) w_4 H_\lambda (\partial_t \tilde{V}) \cdot w \right| - \frac{1}{2} \int \partial_t (b_1 \tilde{V}) \cdot w^2_4$$

$$+ \int (b_1 \tilde{V}) w_4 (\partial_t \tilde{V}) w_2 + \int (b_1 \tilde{V}) w_4 H_\lambda (\partial_t \tilde{V}) \cdot w \quad - \frac{1}{2} \int \partial_t (b_1 \tilde{V}) \cdot w^2_4$$

$$+ \int H_\lambda^2 (b_1 \tilde{V}) w (\partial_t \tilde{V}) w_2 + \int H_\lambda^2 (b_1 \tilde{V}) w H_\lambda (\partial_t \tilde{V}) \cdot w$$

$$\lesssim \frac{b_1^2}{\lambda^{12}} \cdot \sum_{0 \leq i,j \leq 4} |\partial_y^{i+1} \partial_t^{j+1}| \frac{b_1^6}{1 + y^{12+i-j}} \lesssim \frac{b_1}{\lambda^{12}} \cdot \frac{b_1^6}{|\log \lambda|^2}.$$

(iv) The boundary terms are estimated similarly

$$\left| \int -\frac{1}{2} b_1 \tilde{V} w^2_4 - w_4 H^2 (b_1 \tilde{V}) w \right| \lesssim \frac{b_1}{\lambda^{10}} \int \frac{|\varepsilon|}{1 + y^4} + \sum_{i=0}^4 |\varepsilon| |\partial_y \varepsilon| \lesssim \frac{b_1^7}{\lambda^{10}} \cdot \frac{b_1^6}{|\log \lambda|^2}.$$

**Step 4: Further use of dissipation.** Finally we deal with the $\mathcal{F}$ terms. We need to treat the term \( \int w_6 H^3_\lambda (\lambda^{-2} \mathcal{F}_\lambda) \) carefully. Let

$$\mathcal{F} = \mathcal{F}^0 + \mathcal{F}^1, \quad \mathcal{F}^0 = -\tilde{V}_b - \text{Mod}, \quad \mathcal{F}^1 = L(\varepsilon) + N(\varepsilon),$$

then

$$\int w_6 H^3_\lambda (\lambda^{-2} \mathcal{F}_\lambda)$$

$$= \int w_6 H^3_\lambda \left[ \lambda^{-2} (\mathcal{F}^0_\lambda + \mathcal{F}^1_\lambda) \right]$$

$$= \int w_6 H^3_\lambda (\lambda^{-2} \mathcal{F}^0_\lambda)$$

$$- \int H^3_\lambda (\lambda^{-2} \mathcal{F}^1_\lambda) \left[ \partial_t w_4 + \partial_t \tilde{V} \cdot w_2 + H_\lambda (\partial_t \tilde{V} \cdot w) - H^2_\lambda \lambda^{-2} \mathcal{F} \right].$$

As before, we integrate by parts and use \eqref{eq:3.42} to compute

$$- \int \partial_t w_4 H^3_\lambda (\lambda^{-2} \mathcal{F}^1_\lambda)$$

$$= - \frac{d}{dt} \int w_4 H^3_\lambda (\lambda^{-2} \mathcal{F}^1_\lambda) + \int w_4 \partial_t H^3_\lambda (\lambda^{-2} \mathcal{F}^1_\lambda)$$

$$= - \frac{d}{dt} \int w_6 H^2_\lambda (\partial_t \tilde{V}) (\mathcal{F}^1_\lambda) - \int w_4 (\partial_t \tilde{V}) H^2_\lambda (\mathcal{F}^1_\lambda) - \int w_4 H_\lambda [\partial_t \tilde{V} H_\lambda (\lambda^{-2} \mathcal{F}^1_\lambda)]$$

$$- \int w_4 H^2_\lambda (\partial_t \tilde{V} \cdot \lambda^{-2} \mathcal{F}^1_\lambda) + \int w_4 H^3_\lambda [\partial_t (\lambda^{-2} \mathcal{F}^1_\lambda)],$$

(3.50)
where by direct calculation
\[ \int w_4 H_\lambda^3 (\partial_\ell (\lambda^{-2} \mathcal{F}_\lambda)) = - \int w_6 H_\lambda^2 \left( \frac{\lambda_s}{\lambda^5} (2 \mathcal{F}_\lambda + \Lambda \mathcal{F}_\lambda) \right). \]

We now claim the following bounds:
\[ \sum_{i=0}^4 \int \left( 1_{y \leq 1} + \frac{|\log y|^2}{y^{12-2t}} 1_{y \geq 1} \right) (|\partial_y \mathcal{F}|^2 + |\partial_y \mathcal{F}|^2) + \int |H^2 \Lambda \mathcal{F}|^2 \lesssim \frac{b_1^6}{|\log b_1|^2} + \frac{\Xi_6}{|\log M|}, \quad (3.51) \]
\[ \int |H^3 \mathcal{F}_0|^2 \lesssim b_1^2 \left[ \frac{b_1^6}{|\log b_1|^2} + \frac{\Xi_6}{\log M} \right], \quad (3.52) \]
\[ \int |H^2 \mathcal{F}|^2 \lesssim b_1^{1/4} \frac{b_1^6}{|\log b_1|^2}, \quad (3.53) \]
\[ \int |H^3 \mathcal{F}|^2 \leq C \gamma b_1^{2-\gamma}, \quad \frac{b_1^6}{|\log b_1|^2}, \quad \gamma > 0 \text{ can be taken arbitrarily small.} \quad (3.54) \]

With these bounds we estimate terms concerning \( \mathcal{F} \). First, for those in the next-to-last line of (3.47) we have
\[ - \int b_1 \hat{V} w_4 H_\lambda^3 (\lambda^{-2} \mathcal{F}_\lambda) - \int H_\lambda^3 (b_1 \hat{V} w) H_\lambda^2 (\lambda^{-2} \mathcal{F}_\lambda) - \int w_4 H_\lambda^2 (b_1 \hat{V} \lambda^{-2} \mathcal{F}_\lambda) \]
\[ \lesssim \frac{b_1}{\lambda^{12}} \int \sum_{0 \leq i,j \leq 4} \frac{|\partial \mathcal{F}|^2 |\partial \mathcal{F}|^2}{1 + y^{12-i-j}} \lesssim \frac{b_1}{\lambda^{12}} \left[ \frac{\Xi_6}{\sqrt{\log M}} + \frac{b_1^3}{|\log b_1|^2} \sqrt{\Xi_6} \right]. \]

To estimate \( \int w_6 H_\lambda^3 (\mathcal{F}_\lambda) \), we use (3.49) and (3.50). From (3.52) we derive
\[ \int w_6 H_\lambda^3 (\lambda^{-2} \mathcal{F}_\lambda) \lesssim \frac{1}{\lambda^{12}} \int |\mathcal{F}| H^3 (\mathcal{F}_\lambda) \lesssim \frac{b_1}{\lambda^{12}} \left( \frac{\Xi_6}{\sqrt{\log M}} + \frac{b_1^3}{|\log b_1|^2} \sqrt{\Xi_6} \right), \]
and from (3.51) and (3.53)
\[ - \int w_4 (\partial_\ell \hat{V}) H_\lambda^3 (\mathcal{F}_\lambda) - \int w_4 H_\lambda (\partial_\ell \hat{V} H_\lambda (\lambda^{-2} \mathcal{F}_\lambda)) - \int w_4 H_\lambda^2 (\partial_\ell \hat{V} \cdot \lambda^{-2} \mathcal{F}_\lambda) \]
\[ \lesssim \frac{b_1}{\lambda^{12}} \int \sum_{i=0}^4 \frac{|\mathcal{F}| |\partial \mathcal{F}|}{1 + y^{8-i}} + |\mathcal{F}| H^2 (\mathcal{F} + H^2 \mathcal{F}_\lambda) \]
\[ \lesssim \frac{b_1}{\lambda^{12}} \left( \frac{\Xi_6}{\sqrt{\log M}} + \frac{b_1^3}{|\log b_1|^2} \sqrt{\Xi_6} \right). \]
Picking $\gamma = 1/2$ in (3.54), other non-boundary terms are controlled by
\[
- \int H_\lambda^3(\lambda^{-2} \mathcal{F}_{\lambda}^1)(\partial_t V \cdot w_2 + H_\lambda(\partial_t V \cdot w) - H_\lambda^2(\lambda^{-2} \mathcal{F}_\lambda)) \leq \int H_\lambda^2(\lambda^{-2} \mathcal{F}_{\lambda}^1)(H_\lambda(\partial_t V \cdot w_2) + H_\lambda^2(\partial_t V \cdot w) - H_\lambda^3(\lambda^{-2} \mathcal{F}_\lambda)) \leq \frac{1}{12} \int |H^2 \mathcal{F}_\lambda^1| \left( b_1 \sum_{i=0}^4 \frac{|\partial_y \varepsilon|}{1 + y^{8-i}} + |H^3 \mathcal{F}_\lambda| \right) \leq \frac{b_1}{12} \cdot \frac{b_1^6}{|\log b_1|^2}.
\]
We can also estimate the new boundary term as
\[
\int w_6 H_\lambda^2(\lambda^{-2} \mathcal{F}_{\lambda}^1) \leq \frac{1}{10} \int |\varepsilon_6| |H^2 \mathcal{F}_\lambda^1| \leq \frac{b_1^4}{10} \cdot \frac{b_1^6}{|\log b_1|^2}.
\]
The proposition follows from these estimates. Now we turn to the proof of the claim.

**Step 5: $\tilde{\Psi}_b$ terms.** The contribution of $\tilde{\Psi}_b$ in (3.51), (3.52) is estimated in (2.16) and (2.17).

**Step 6: Mod terms.** Recall (2.65) that
\[
\text{Mod} = - \left( \frac{\lambda}{\lambda} + b_1 \right) \Lambda \tilde{Q}_b + [(b_1)_s + b_2^2(1 + c_{b_1}) - b_2] \left[ T_1 + \chi_{B_1} \sum_{j=2}^4 \frac{\partial S_j}{\partial b_1} \right] + [(b_2)_s + b_1 b_2(3 + c_{b_1})] \left[ T_2 + \chi_{B_2} \sum_{j=3}^4 \frac{\partial S_j}{\partial b_2} \right].
\]

**Proof of (3.52) for Mod.** Thanks to modulation equations (3.26), (3.27), it suffices to show
\[
\int |H^3 \Lambda \tilde{Q}_b|^2 + \int |H^3(T_1 + \chi_{B_1} \sum_{j=2}^4 \frac{\partial S_j}{\partial b_1})|^2 \leq b_1^3 |\log b_1|^C, \quad (3.55)
\]
\[
\int |H^3(T_2 + \chi_{B_2} \sum_{j=3}^4 \frac{\partial S_j}{\partial b_2})|^2 \leq b_1^2, \quad (3.56)
\]
For (3.55), since $H^3 \Lambda Q = H^3 T_1 = 0$, we have
\[
\int |H^3 \Lambda \tilde{Q}_b|^2 + \int |H^3(T_1 + \chi_{B_1} \sum_{j=2}^4 \frac{\partial S_j}{\partial b_1})|^2 \leq \int |H^3 T_1|^2 + b_1^3 |\log b_1|^C \int_{y \leq 2B_1} \frac{1}{1 + y^{12}} 
\leq \int_{B_1 \leq y \leq 2B_1} |H^3 T_1|^2 + b_1^3 |\log b_1|^C \leq b_1^3 |\log b_1|^C.
\]
To prove (3.56), we use $H^3 T_2 = 0$ to estimate
\[
\int |H^3 T_2|^2 \leq \int_{B_1 \leq y \leq 2B_1} \frac{|\log y|^2}{y^{12}} \leq b_1^2, \quad (3.57)
\]
Note that (2.55) implies that
\[
\int |H^3(\chi B_1 \partial S_3/b_2)|^2 \lesssim \frac{b_1^2}{|\log b_1|^2} \int_{y \leq 2B_1} \frac{1}{1 + y^4} \lesssim b_1^2. \tag{3.58}
\]

Finally we deal with the term $H^3 \partial b_2 S_3$. From (2.50), (2.38), (2.47)
\[
H^3 \partial b_2 S_3 = O(b_1)(H \Theta_2 + b_1 \partial b_1 H \Theta_2) + O(b_1) \left(1_{y \leq 1} + \frac{|\log y|^C}{y^4} \right) \text{ for } y \geq 1,
\]
where by the definition of $\Theta_2$
\[
H \Theta_2 = H(\Lambda T_1 - T_1 + \Sigma b_1) = -\Lambda^2 Q - \Lambda Q + H_b + O\left(\frac{|\log y|^C}{y^4} \right) \text{ for } y \geq 1.
\]
Now we estimate from (2.7) and (2.28)
\[
|\Lambda^2 Q - \Lambda Q| \lesssim \frac{1}{1 + y^4}, \quad |H \Sigma b_1| + |b_1 \partial b_1 H \Sigma b_1| \lesssim \frac{1}{|\log b_1| (1 + y^2)} \text{ for } y \leq 6B_0
\]
and hence obtain
\[
|H^3 \partial b_2 S_3| \lesssim b_1 \left(1_{y \leq 1} + \frac{1}{|\log b_1| y^2} 1_{1 \leq y \leq 6B_0} + \frac{|\log b_1|^C}{y^4} 1_{y \geq 6B_0}\right).
\]
Together with (2.51) we conclude
\[
\int \left|H^3(\chi B_1 \partial S_3/b_2)\right|^2 \lesssim \int_{y \leq B_1} \left|H^3 \partial S_3/b_2\right|^2 + \frac{1}{|\log b_1|^2} \int_{B_1 \leq y \leq 2B_1} \frac{1}{y^8} \lesssim b_1^2. \tag{3.59}
\]
Combining (3.57), (3.58), and (3.59), we derive (3.56) and finish the proof of (3.52).

**Proof of (3.51) for Mod.** Using rough bounds for $S_j$ we simply estimate for $0 \leq i \leq 4$
\[
\int \frac{1 + |\log y|^2}{1 + y^{12 - 2i}} \left[|\partial_y^i \Lambda \tilde{Q}_b|^2 + |\partial_y^i (\tilde{T}_1 + \chi B_1 \sum_{j=2}^4 \partial S_j/b_1)|^2 + |\partial_y^i (\tilde{T}_2 + \chi B_1 \sum_{j=3}^4 \partial S_j/b_2)|^2\right] \lesssim \int \frac{1 + |\log y|^C}{1 + y^8} \lesssim 1
\]
which combined with modulation equations implies the desired result:
\[
\sum_{i=0}^4 \int \frac{1 + |\log y|^2}{1 + y^{12 - 2i}} |\partial_y^i \text{Mod}|^2 \lesssim \frac{b_1^6}{|\log b_1|^2} + \frac{\Xi_6}{\log M}. \tag{3.61}
\]

**Step 7: $L(\epsilon)$ terms.** Recall that $L(\epsilon) = 3(\tilde{Q}_b - Q^2)\epsilon$. We have
\[
|\partial_y^i (\tilde{Q}_b^2 - Q^2)| \lesssim b_1 \left(1_{y \leq 1} + \frac{|\log y|^C}{y^{2+i}} 1_{1 \leq y \leq 2B_1}\right). \tag{3.62}
\]
Using Leibniz rule we estimate
\[
\sum_{i=0}^{4} \int \left( 1_{y \leq 1} + \frac{\log y}{y^{12-2i}} 1_{y \geq 1} \right) |\partial_y^i L(\varepsilon)|^2 + \int |H^2 \Lambda L(\varepsilon)|^2 
\]
\[\lesssim b_1^2 \sum_{i=0}^{5} \left( \int_{y \leq 1} |\partial_y^i \varepsilon|^2 + \int_{1 \leq y \leq 2B_1} \frac{|\log y|^C}{y^{12-2i}} |\partial_y^i \varepsilon|^2 \right) \]
\[\lesssim b_1^2 (1 + |\log b_1|^C) \Xi_6 \lesssim \frac{b_1^6}{|\log b_1|^2}, \tag{3.63} \]
\[
\int |H^2 L(\varepsilon)|^2 \lesssim b_1^2 \sum_{i=0}^{4} \left( \int_{y \leq 1} |\partial_y^i \varepsilon|^2 + \int_{1 \leq y \leq 2B_1} \frac{|\log y|^C}{y^{12-2i}} |\partial_y^i \varepsilon|^2 \right) 
\]
\[\lesssim b_1^2 (1 + |\log b_1|^C) \Xi_6 \lesssim \frac{b_1^6}{|\log b_1|^2}, \tag{3.64} \]
\[
\int |H^3 L(\varepsilon)|^2 \lesssim b_1^2 \sum_{i=0}^{6} \left( \int_{y \leq 1} |\partial_y^i \varepsilon|^2 + \int_{1 \leq y \leq 2B_1} \frac{1 + |\log y|^C}{y^{16-2i}} |\partial_y^i \varepsilon|^2 \right) 
\]
\[\lesssim b_1^2 \Xi_6 \lesssim C_6 b_1^{2-\gamma} \frac{b_1^6}{|\log b_1|^2}, \tag{3.65} \]
which bound the $L(\varepsilon)$ terms in (3.51), (3.53) and (3.54) respectively.

**Step 8: $N(\varepsilon)$ terms.** Recall that $N(\varepsilon) = 3 \hat{Q}_b \cdot \varepsilon^2 + \varepsilon^3$. We split the integral into two parts:

**Control for $y \leq 1$.** We know $|\partial_y^i \hat{Q}_b| \lesssim 1$ and thus
\[
\int_{y \leq 1} |H^2 (\hat{Q}_b \cdot \varepsilon^2)|^2 \lesssim \sum_{i+j \leq 4} \int_{y \leq 1} |\partial_y^i \varepsilon|^2 |\partial_y^j \varepsilon|^2 
\]
\[\lesssim \sum_{i+j \leq 4} \int_{y \leq 1} |\partial_y^i \varepsilon|^2 \|\partial_y^j \varepsilon\|_{L^\infty(y \leq 1)}^2 
\]
\[\lesssim \Xi_6^2 \lesssim \frac{b_1^6}{|\log b_1|^2}, \tag{3.66} \]
where we assume $j \leq i$ (hence $j \leq 2$) and use (B.10). In the following we assume $i \geq j \geq k$ and derive
\[
\int_{y \leq 1} |H^2 \varepsilon|^2 \lesssim \sum_{i+j+k \leq 4} \int_{y \leq 1} |\partial_y^i \varepsilon|^2 |\partial_y^j \varepsilon|^2 |\partial_y^k \varepsilon|^2 
\]
\[\lesssim \sum_{i+j+k \leq 4} \int_{y \leq 1} |\partial_y^i \varepsilon|^2 \|\partial_y^j \varepsilon\|_{L^\infty(y \leq 1)}^2 \|\partial_y^k \varepsilon\|_{L^\infty(y \leq 1)}^2 \]
\[\lesssim \Xi_6^3 \lesssim \frac{b_1^6}{|\log b_1|^2}. \tag{3.67} \]
Similar calculations imply
\[
\sum_{i=0}^{4} \int_{y \leq 1} |\partial_{y}^{i} N(\varepsilon)|^2 + \int_{y \leq 1} |H^{2} \Lambda N(\varepsilon)|^2 \lesssim \frac{b_{1}^{6}}{|\log b_{1}|^2}, \tag{3.68}
\]
\[
\int_{y \leq 1} |H^{3} N(\varepsilon)|^2 \lesssim b_{1}^{2} \frac{b_{1}^{6}}{|\log b_{1}|^2}. \tag{3.69}
\]
These bounds verify the \( y \leq 1 \) part of (3.51), (3.53) and (3.54).

**Control for \( y \geq 1 \).** Now we have
\[
|\partial_{y}^{j} \tilde{Q}_{b}| \lesssim \frac{1}{y^{2+i}} + b_{1} \frac{1 + |\log y|^C}{y^{i}}, \quad 1_{y \leq 2B_{1}} \lesssim \frac{1 + |\log y|^C}{y^{2+i}}. \tag{3.70}
\]

**Proof of (3.53) for \( N(\varepsilon), y \geq 1 \).** First for \( H^{2}(\tilde{Q}_{b}, \varepsilon^{2}) \) the bound above yields
\[
\int_{y \geq 1} |H^{2}(\tilde{Q}_{b}, \varepsilon^{2})|^{2} \lesssim \sum_{i+j \leq 4} \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{12-2i-2j}} |\partial_{y}^{i} \varepsilon^{2}| |\partial_{y}^{j} \varepsilon^{2}| \tag{3.71}
\]
\[
\lesssim \sum_{i+j \leq 4} \int_{y \geq 1} \frac{|\partial_{y}^{i} \varepsilon^{2}|^{2}}{y^{8-2i}(1 + |\log y|^{2})} \cdot |y^{j-2} \partial_{y}^{j} \varepsilon^{2}(1 + |\log y|^{C})|_{L^{\infty}(y \geq 1)}^{2} \lesssim |\log b_{1}|^{C} \Xi^{2} \lesssim b_{1} \frac{b_{1}^{6}}{|\log b_{1}|^2}.
\]

Next we treat the term \( H^{2}(\varepsilon^{3}) \). Since \( |\partial_{y}^{i} V| \lesssim \frac{1}{1+y^{4+i}} \), we get
\[
|H^{2}(\varepsilon^{3})|^{2} \lesssim \sum_{i+j+k \leq 4} \frac{|\partial_{y}^{i} \varepsilon^{2}|^{2} |\partial_{y}^{j} \varepsilon^{2}| |\partial_{y}^{k} \varepsilon^{2}|}{y^{8-2i-2j-2k}}. \tag{3.72}
\]

Again assume \( i \geq j \geq k \). If \((i, j, k) = (2, 2, 0)\) we estimate
\[
\int_{y \geq 1} |\partial_{y}^{2} \varepsilon^{2}|^{2} |\partial_{y}^{2} \varepsilon^{2}| |\partial_{y}^{2} \varepsilon^{2}| = \int_{y \geq 1} |\partial_{y}^{2} \varepsilon^{2}|^{2} \cdot |\partial_{y}^{2} \varepsilon^{2}|^{2} \lesssim |\partial_{y}^{2} \varepsilon^{2}|^{2} \cdot |\partial_{y}^{2} \varepsilon^{2}|^{2} \|\log y\|_{L^{\infty}(y \geq 1)}^{2} \|\varepsilon\|_{L^{\infty}(y \geq 1)}^{2}, \tag{3.73}
\]
only otherwise \( k \leq j \leq 1 \) and
\[
\int_{y \geq 1} \frac{|\partial_{y}^{i} \varepsilon^{2}|^{2} |\partial_{y}^{j} \varepsilon^{2}|^{2} |\partial_{y}^{k} \varepsilon^{2}|^{2}}{y^{8-2i-2j-2k}} \lesssim \int_{y \geq 1} \frac{|\partial_{y}^{i} \varepsilon^{2}|^{2}}{y^{8-2i}(1 + |\log y|^{2})} \cdot |y^{j} \partial_{y}^{j} \varepsilon^{2}|^{2} \|\log y\|_{L^{\infty}(y \geq 1)}^{2} \|\varepsilon\|_{L^{\infty}(y \geq 1)}^{2} \|\varepsilon\|_{L^{\infty}(y \geq 1)}^{2}. \tag{3.74}
\]

Hence we conclude with the help of Appendix B that
\[
\int_{y \geq 1} |H^{2}(\varepsilon^{3})|^{2} \lesssim |\log b_{1}|^{C} \Xi^{2} \Xi^{4} \lesssim b_{1}^{\frac{i}{2}} \frac{b_{1}^{6}}{|\log b_{1}|^2} \tag{3.75}
\]
which is where the exponent \( 1/2 \) in (3.53) comes from. This completes the proof of (3.53).
Proof of (3.51) for \( N(\varepsilon), y \geq 1 \). Note that the term involving \( \partial_y^5 \varepsilon \) in \( \int_{y \geq 1} |H^2 \Lambda(\tilde{Q}_b \cdot \varepsilon^2)|^2 \) is bounded by

\[
\int_{y \geq 1} \frac{1 + |\log y|^C}{y^2} |\partial_y^5 \varepsilon|^2 \lesssim \int_{y \geq 1} \frac{|\partial_y^5 \varepsilon|^2}{y^2 (1 + |\log y|^2)} \cdot ||\varepsilon(1 + |\log y|^C)||^2_{L^\infty(y \geq 1)} \lesssim \frac{b_1^6}{|\log b_1|^2}
\]

and in \( \int_{y \geq 1} |H^2 \Lambda(\varepsilon^3)|^2 \) by

\[
\int_{y \geq 1} |\partial_y^5 \varepsilon|^2 |\varepsilon|^4 \lesssim \int_{y \geq 1} \frac{|\partial_y^5 \varepsilon|^2}{y^2 (1 + |\log y|^2)} \cdot ||\varepsilon(1 + |\log y|^C)||^2_{L^\infty(y \geq 1)} \cdot ||\varepsilon||_{L^\infty(y \geq 1)}^2 \lesssim \frac{b_1^6}{|\log b_1|^2},
\]

where we use (A.2) and smallness of energy (3.19). Other lower order terms are estimated as before in (3.71), (3.73) and (3.74).

Proof of (3.54) for \( N(\varepsilon), y \geq 1 \). For \( H^3(\tilde{Q}_b \cdot \varepsilon^2) \) we expand

\[
|H^3(\tilde{Q}_b \cdot \varepsilon^2)|^2 \lesssim \sum_{i+j \leq 6} \frac{1 + |\log y|^C}{y^{16-2i-2j}} |\partial_y^i \varepsilon|^2 |\partial_y^j \varepsilon|^2,
\]

assume \( i \geq j \), and then estimate

\[
\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{16-2i-2j}} |\partial_y^i \varepsilon|^2 |\partial_y^j \varepsilon|^2
\]

\[
\lesssim \begin{cases} 
\int_{y \geq 1} \frac{1 + |\log y|^C}{y^6} |\partial_y^3 \varepsilon|^2 \cdot \frac{||y \partial_y^3 \varepsilon||^2_{L^\infty(y \geq 1)}}{1 + |\log y|}, & i = j = 3, \\
\int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2}{y^{12-2i(1 + |\log y|^2)}} \cdot ||y^{j-2} \partial_y^j \varepsilon(1 + |\log y|^C)||^2_{L^\infty(y \geq 1)}, & \text{otherwise.}
\end{cases}
\]

In both cases, we have

\[
\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{16-2i-2j}} |\partial_y^i \varepsilon|^2 |\partial_y^j \varepsilon|^2 \lesssim |\log b_1|^C \Xi_4 \Xi_6 \lesssim b_1^2 \frac{b_1^6}{|\log b_1|^2}.
\]

Besides, we need to control the term \( H^3(\varepsilon^3) \). First as in (3.72)

\[
H^3(\varepsilon^3) \lesssim \sum_{i+j+k \leq 6} \frac{|\partial_y^i \varepsilon|^2 |\partial_y^j \varepsilon|^2 |\partial_y^k \varepsilon|^2}{y^{12-2i-2j-2k}}.
\]
Assuming $i \geq j \geq k$, we divide the summand into several cases and summarize the estimates as follows:

$$
\int_{y \geq 1} \frac{|\partial_y^j \varepsilon|^2 |\partial_y^k \varepsilon|^2 |\partial_y^k \varepsilon|^2}{y^{12-2i-2j-2k}} \lesssim \begin{cases}
\int_{y \geq 1} \frac{|\partial_y^j \varepsilon|^2}{y^{12-2i-2|1+\log y|^2}} ||y^j \partial_y^j \varepsilon||^2_{L^\infty(y \geq 1)}, & j \leq 1, \\
\int_{y \geq 1} \frac{|\partial_y^j \varepsilon|^2}{y^{12-2i-2|1+\log y|^2}} ||\partial_y^j \varepsilon||^2_{L^\infty(y \geq 1)}, & j = 2, k \leq 1, \\
\int_{y \geq 1} \frac{|\partial_y^j \varepsilon|^2}{y^{12-2i-2|1+\log y|^2}} ||\partial_y^j \varepsilon||^2_{L^\infty(y \geq 1)}, & (i, j, k) = (2, 2, 2), \\
\int_{y \geq 1} \frac{|\partial_y^j \varepsilon|^2}{y^{12-2i-2|1+\log y|^2}} ||\partial_y^j \varepsilon||^2_{L^\infty(y \geq 1)}, & (i, j, k) = (3, 3, 0).
\end{cases}
$$

Using interpolation bounds in Appendix B, we conclude

$$
\int_{y \geq 1} |H^2 \Omega(\varepsilon)|^2 \lesssim b_1^2 \frac{b_1^2}{|\log b_1|^2}
$$

and hence prove (3.54). As a result, we finish the whole proof of Proposition 3.6. □

Similarly we have Lyapunov monotonicity for $\Xi_4$ and $\Xi_2$:

**Proposition 3.7.** For some constant $C$ independent of $M$ and $K$, there holds

$$
\frac{d}{dt} \left( \frac{1}{\lambda^6} \Xi_4 \right) \leq C \frac{b_1^2}{\lambda^8},
$$

$$
\frac{d}{dt} \left( \frac{1}{\lambda^2} \Xi_2 \right) \leq C \frac{b_1^2}{\lambda^3}.
$$

**Proof.** To prove (3.80), we compute

$$
\frac{1}{2} \frac{d}{dt} \int w_4^2 = \int w_4 (-H_{\lambda} w_4 - \partial_t \bar{V} w_2 - \dot{H}_{\lambda}(\partial_t \bar{V} w) + H_{\lambda}^2(\lambda^{-2} F_{\lambda})).
$$

Unlike in the proof of (3.39), here we could estimate these terms directly. The first term is estimated from (3.23)

$$
- \int w_4 H_{\lambda} w_4 = - \frac{1}{\lambda^8} \int \varepsilon_4 H \varepsilon_4 \lesssim \frac{1}{\lambda^8} (\varepsilon, \psi)^2 \lesssim \frac{b_1^2}{\lambda^8},
$$

and the next two follow from $|\partial_t \bar{V}| \lesssim \frac{b_1}{\lambda^2} \frac{1}{1+y}$

$$
\left| - \int w_4 \partial_t \bar{V} w - \int w_4 H_{\lambda}(\partial_t \bar{V}) w \right| \lesssim \frac{b_1}{\lambda^7} \int \frac{\varepsilon_2 ||\varepsilon_4| + |\varepsilon||\varepsilon_6||}{1 + y^4} \lesssim \frac{b_1^2}{\lambda^8}.
$$

The last term is controlled thanks to (3.53)

$$
\left| \int w_4 H_{\lambda}^2(\lambda^{-2} F_{\lambda}) \right| \lesssim \frac{1}{\lambda^6} \int |\varepsilon_4||H^2 F_{\lambda}| \lesssim \frac{b_1^2}{\lambda^8}.
$$

hence (3.80) is proved.
Now we turn to the proof of (3.81). Similarly
\[ \frac{1}{2} \frac{d}{dt} \int w_2^2 = \int w_2 [-H_\lambda w_2 - \partial_t \bar{V} w + H_\lambda (\lambda^{-2} F_\lambda)]. \] (3.86)
and it is not hard to show
\[ - \int w_2 H_\lambda w_2 \leq \frac{b^2}{\lambda^4}, \quad \text{and} \quad - \int w_2 \partial_t \bar{V} w \leq \frac{b^2}{\lambda^4}. \] (3.87)
To estimate the last term we need to bound \( \int |H_\mathcal{F}|^2 \). Recall that \( \mathcal{F} = -\bar{\Psi} - \text{Mod} - L(\varepsilon) + N(\varepsilon) \). By (2.66) we have
\[ \int |H \bar{\Psi}|^2 \leq b_1^4 |\log b_1|^C \leq b_1^\frac{11}{4}. \]
For the Mod term, since
\[ |H A \tilde{Q}_b|^2 + |H \left( \tilde{T}_1 + \chi_{B_1} \sum_{j=2}^4 \frac{\partial S_j}{\partial b_1} \right)|^2 + |H \left( \tilde{T}_2 + \chi_{B_1} \sum_{j=3}^4 \frac{\partial S_j}{\partial b_2} \right)|^2 \leq |\log b_1|^C 1_{y \leq 2B_1}, \]
we could derive
\[ \int |H (\text{Mod})|^2 \leq b_1^4 |\log b_1|^C \leq b_1^\frac{11}{4}. \]
For the error term, we directly estimate them, i.e. for \( L(\varepsilon) \) we have
\[ \int |HL(\varepsilon)|^2 \leq b_1^4 \left( \sum_{i=0}^2 \int_{y \leq 1} |\partial_j^i \varepsilon|^2 + \sum_{i=0}^2 \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{8-2i}} |\partial_j^i \varepsilon|^2 \right) \leq b_1^\frac{11}{4}. \]
For \( N(\varepsilon) = 3 \tilde{Q}_b, \varepsilon^2 + \varepsilon^3 \) we have
\[ \int |H (\tilde{Q}_b \varepsilon^2)|^2 \leq \sum_{i+j \leq 2} \left( ||\partial_j^i \varepsilon||_{L^\infty(0 \leq 1)} \cdot ||\partial_j^i \varepsilon||_{L^\infty(0 \leq 1)}^2 + \int_{y \geq 1} \frac{|\partial_j^i \varepsilon|^2}{y^{8-2i}} \cdot ||y^j \partial_y^i \varepsilon||_{L^\infty(0 \geq 1)} \cdot |\log b_1|^C \right) \]
\[ \leq b_1^\frac{11}{4} + \Xi_4 \cdot \Xi_2 \cdot |\log b_1|^C \leq b_1^\frac{11}{4}, \]
\[ \int |H (\varepsilon^3)|^2 \leq ||\varepsilon||_{L^\infty}^2 \sum_{i+j \leq 2} \left( ||\partial_j^i \varepsilon||_{L^\infty(0 \leq 1)} \cdot ||\partial_j^i \varepsilon||_{L^\infty(0 \leq 1)}^2 + \int_{y \geq 1} \frac{|\partial_j^i \varepsilon|^2}{y^{4-2i}} \cdot ||y^j \partial_y^i \varepsilon||_{L^\infty(0 \geq 1)} \cdot ||\log b_1|^C \right) \]
\[ \leq b_1^6 + |\log b_1|^C \Xi_3^3 \leq b_1^\frac{11}{4}. \]
Summing up the above estimates, we get
\[ \int |H \mathcal{F}|^2 \leq b_1^\frac{11}{4}. \] (3.88)
Thus
\[ \left| \int w_2 H_\lambda (\lambda^{-2} F_\lambda) \right| = \frac{1}{\lambda^3} \left| \int \varepsilon_2 H \mathcal{F} \right| \leq \frac{b^7}{\lambda^3}, \] (3.89)
which completes the proof of (3.81). \( \square \)
4. CLOSING THE BOOTSTRAP AND PROOF OF THEOREM 1.1

4.1. Closing the bootstrap. Now we are able to close the bootstrap. As stated in section 3, this consists of two steps, the first of which is the following:

**Proposition 4.1.** *(Improved control)* Assume $K > 0$ has been chosen large enough in Proposition 3.1, then for all $t \in [0, T_{exit})$ there hold

\[
\Xi_1(t) \leq \sqrt{b_1(0)},
\]
\[
\Xi_2(t) \leq b_1^4 |\log b_1|^{\frac{K}{2}}, \quad \Xi_4(t) \leq b_1^4 |\log b_1|^{\frac{K}{2}}, \quad \Xi_6(t) \leq \frac{K}{2} |\log b_1|^2.
\]

**Proof.** Step 1: Improved energy bound. (4.1) results from the decrease of $E(u)$. Indeed, let $\tilde{\varepsilon} = \tilde{\alpha} + \varepsilon$, then

\[
E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{4} \int |u|^4
\]
\[
= \frac{1}{2} \int |\nabla (Q + \tilde{\varepsilon})|^2 - \frac{1}{4} \int (Q + \tilde{\varepsilon})^4
\]
\[
= E(Q) + (H\tilde{\varepsilon}, \tilde{\varepsilon}) - \frac{1}{4} \int (4Q\tilde{\varepsilon}^3 + \tilde{\varepsilon}^4).
\]

We estimate these terms respectively. Using Cauchy-Schwarz inequality and the bound

\[
|\partial_y \tilde{\alpha}| \lesssim \frac{b_1 |\log b_1|^C}{1 + y^4} 1_{y \leq 2B_1},
\]

the linear part is

\[
(H\tilde{\varepsilon}, \tilde{\varepsilon}) = (H\varepsilon, \varepsilon) + (H\tilde{\alpha}, \tilde{\alpha}) + 2(H\varepsilon, \tilde{\alpha}) = (H\varepsilon, \varepsilon) + O(b_1^{\frac{3}{2} + \gamma}).
\]

for some $\gamma > 0$ small enough. Moreover, the sub-coercivity (A.1) of $H$ and (3.23) yield

\[
(H\varepsilon, \varepsilon) \geq c \int |\partial_y \varepsilon|^2 - \frac{1}{c} (\varepsilon, \psi)^2 \gtrsim \Xi_1 + O(\frac{b_1^2}{|\log b_1|^2}).
\]

For nonlinear terms, from (A.2), Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ and assumption (3.19) on $\Xi_1$

\[
\int |Q\tilde{\varepsilon}^3| \lesssim \int |\varepsilon|^3 + |\tilde{\alpha}|^3 \lesssim \int \frac{|\varepsilon|^2}{y^2} |\varepsilon||L^\infty + b_1^4 |\log b_1|^C \lesssim b_1^{\frac{3}{2} + \gamma},
\]
\[
\int |\tilde{\varepsilon}|^4 \lesssim \int |\varepsilon|^4 + \int |\tilde{\alpha}|^4 \lesssim \Xi_1^2 + b_1^2 |\log b_1|^C \lesssim b_1(0)
\]

With the above analysis and the smallness of $b_1$ ensured by (3.23), we conclude

\[
E(u) - E(Q) \gtrsim \Xi_1 + O(b_1(0)^{\frac{3}{2} + \gamma}).
\]

(4.4)

On the other hand, by our construction of initial data and dissipation of energy, we have

\[
E(u) - E(Q) \lesssim b_1(0) |\log b_1(0)|^C,
\]

(4.5)

and (4.1) follows by taking $b_1(0)$ small enough.
Step 2: Control of $\Xi_6$. Now we use proposition 3.6 to obtain the improved control for $\Xi_6$. To this end, we first derive an explicit formula for $\lambda$, which is also useful for the later proof.

By modulation equation (3.26) and explicit formula (2.72), we obtain

$$\frac{\lambda_s}{\lambda} = -b_1 + O(b_1^{3+\frac{1}{2}}) = -\frac{2}{3s} + \frac{4}{9s \log s} + O\left(\frac{1}{s(\log s)^{\frac{3}{2}}}\right)$$

or equivalently

$$\frac{d}{ds} \log \left[\frac{\lambda_s}{(\log s)^{\frac{3}{2}}}\right] \leq \frac{1}{s(\log s)^{\frac{3}{2}}}.$$  \hspace{1cm} (4.7)

Recall we assume $\lambda(0) = 1$. Simple integration shows

$$\lambda = \frac{(\log s)^{\frac{3}{2}}}{s^{\frac{3}{2}}} \cdot \frac{s_0^{\frac{3}{2}}}{(\log s_0)^{\frac{3}{2}}} \left[1 + O(\log s_0)^{-\frac{3}{2}}\right]. \hspace{1cm} (4.8)$$

Now integrate (3.39) on $[0, t]$, we derive

$$\Xi_6(t) \leq \lambda^{10}(t) \left[\Xi_6(0) + \frac{b_1^6(0)}{|\log b_1(0)|^2}\right] + \frac{b_1^6(t)}{|\log b_1(t)|^2}$$

$$+ C \left[1 + \frac{K}{\sqrt{\log M}} + \sqrt{K}\right] \lambda^{10}(t) \int_0^t \frac{b_1}{\lambda^{12} |\log b_1|^2} (\tau) d\tau$$

where $C > 0$ is some constant independent of $M$ and $K$.

Together with (4.8), (2.72) and initial bound (3.16), we estimate the first term of (4.9) as

$$\lambda^{10}(t) \left[\Xi_6(0) + \frac{b_1^6(0)}{|\log b_1(0)|^2}\right] \leq \lambda^{10}(t) \frac{b_1^6(0)}{|\log b_1(0)|^2} \lesssim \frac{b_1^6(t)}{|\log b_1(t)|^2}. \hspace{1cm} (4.9)$$

For the integral term, we use (3.26), (2.72) to get

$$\frac{(b_1)_s}{b_1} = -\frac{3}{2} b_1 + O\left(\frac{b_1}{|\log b_1|}\right).$$

Combining with (4.6), this implies that

$$\frac{d}{dt} \left(\frac{1}{\lambda^{10} |\log b_1|^2}\right) = \frac{1}{\lambda^{12} |\log b_1|^2} \left[-10 \frac{\lambda_s}{\lambda} + (6 + \frac{2}{|\log b_1|}) \frac{(b_1)_s}{b_1}\right] \gtrsim \frac{b_1}{\lambda^{12} |\log b_1|^2}. \hspace{1cm} (4.10)$$

Therefore

$$\int_0^t b_1 \frac{b_1^6}{\lambda^{12} |\log b_1|^2} (\tau) d\tau \lesssim \frac{1}{\lambda^{10}(t) |\log b_1(t)|^2}. \hspace{1cm} (4.11)$$

Injecting (4.9), (4.11) into (4.9), we conclude

$$\Xi_6 \leq C \left[1 + \frac{K}{\sqrt{\log M}} + \sqrt{K}\right] \frac{b_1^6}{|\log b_1|^2}, \hspace{1cm} (4.12)$$

with $C > 0$ independent of $M$ and $K$, and derive

$$\Xi_6 \leq \frac{K}{2} \frac{b_1^6}{|\log b_1|^2} \hspace{1cm} (4.13)$$

by choosing $K$ large enough.
Step 3. Control of $\Xi_4$ and $\Xi_2$. Now we integrate (3.80) and (3.81) on $[0, t]$, and obtain

$$
\Xi_4(t) \leq \lambda^6(t)\Xi_4(0) + C\lambda^6(t) \int_0^t \frac{b_1^5}{\lambda^8}(\tau)d\tau,
$$

$$
\Xi_2(t) \leq \lambda^2(t)\Xi_2(0) + C\lambda^2(t) \int_0^t \frac{b_1^5}{\lambda^4}(\tau)d\tau.
$$

We estimate the first term of each by (4.8) and (3.16)

$$
\lambda^6(t)\Xi_4(0) \lesssim \lambda^6(t)b_1^7(0) \lesssim b_1^4(t)|\log b_1(t)|^C,
$$

$$
\lambda^2(t)\Xi_2(0) \lesssim \lambda^2(t)b_1^7(0) \lesssim b_1^4|\log b_1|^C.
$$

Consequently, for some large constant $C > 0$ (independent of $M$ and $K$), one has

$$
\frac{d}{dt}\left(\frac{1}{\lambda^6}b_1^4|\log b_1|^C\right) = \frac{1}{\lambda^8}b_1^4|\log b_1|^C \left[-6\frac{\lambda_s}{\lambda} + \frac{4C}{|\log b_1|}\frac{(b_1)_s}{b_1}\right] \gtrsim \frac{b_1^5}{\lambda^8}
$$

$$
\frac{d}{dt}\left(\frac{1}{\lambda^2}b_1^4|\log b_1|^C\right) = \frac{1}{\lambda^4}b_1^4|\log b_1|^C \left[-2\frac{\lambda_s}{\lambda} + \frac{4}{3} - \frac{C}{|\log b_1|}\frac{(b_1)_s}{b_1}\right] \gtrsim \frac{b_1^5}{\lambda^4}
$$

which bound the integral terms. To sum up, we obtain

$$
\Xi_4 \leq b_1^4|\log b_1|^C, \quad \Xi_2 \leq b_1^4|\log b_1|^C
$$

where $C$ is independent of $M$ and $K$. By taking $K$ large enough, we conclude

$$
\Xi_4 \leq b_1^4|\log b_1|^\frac{4}{5}, \quad \Xi_2 \leq b_1^4|\log b_1|^\frac{4}{5}
$$

which finishes the whole proof of Proposition 4.1.

Finally, we come to the heart of this paper, i.e. the control of unstable models. The main point of our analysis is to separate the two unstable directions $V_2$ and $\tau$ and to control both of them at the same time based on a Brouwer argument. First, we shall analyse the dynamics of these two unstable directions as a preparation.

Lemma 4.2. There holds, for all $t \in [0, T_{exit})$,

$$
\left|\frac{d\tau}{ds} - \varsigma\tau\right| \lesssim \frac{b_1^4}{|\log b_1|}.
$$

Proof. By the dynamic equation (3.24) for $\varepsilon$, and the fact $H\psi = -\varsigma\psi$, we get

$$
\tau_s - \varsigma\tau = (\partial_s\varepsilon, \psi) + (\varepsilon, H\psi) = (\partial_s\varepsilon + H\varepsilon, \psi) = (\mathcal{F}, \psi) + \frac{\lambda_s}{\lambda}(\Lambda\varepsilon, \psi).
$$

In the proof of Proposition 3.6, in particular see (3.65), (3.69), (3.79) and (3.52), we have actually shown

$$
\int |H^3\mathcal{F}|^2 \leq C(M, K)\frac{b_1^8}{|\log b_1|^2},
$$

which gives the desired upper bound

$$
|\mathcal{F}, \psi) = \frac{1}{\varsigma^3} |(\mathcal{F}, H^3\psi)| = \frac{1}{\varsigma^3} |(H^3\mathcal{F}, \psi)| \lesssim |H^3\mathcal{F}|_{L^2}\|\psi\|_{L^2} \lesssim \frac{b_1^4}{|\log b_1|}.
$$
Thus (4.15) follows.

For the dynamic of \( V \), we use the improved \( \tilde{b}_2 \) in (3.33). Let

\[
\tilde{b}_1 := b_1, \quad \tilde{b}_2 \text{ as in (3.33),} \quad \tilde{b}_3 := 0,
\]

with associated unstable models

\[
\tilde{b}_k = b_k^e + \frac{\tilde{U}_k}{s^k(\log s)^{\frac{4}{3}}}, \quad k = 1, 2, \quad \tilde{U}_3 = 0, \quad \tilde{V} = P\tilde{U} = P \left[ \tilde{U}_1 \tilde{U}_2 \right]
\]

and

\[
\tilde{\tau}(t) := \tau(t) \cdot \frac{|\log b_1(t)|}{b_1(t)^{3+\frac{1}{2}}}
\]

Moreover, we modify (3.23) to

\[
|\tilde{V}_1(t)| \leq 1, \quad |\tilde{V}_2(t)| \leq 1, \quad |\tilde{\tau}(t)| \leq 1.
\]

We define

\[
\tilde{T}_{\text{exit}} := \sup \{ 0 \leq t_1 \leq T(v_0) : \forall t \in [0, t_1], \ (3.19)-(3.23) \text{ and (4.20) hold} \}.
\]

Recall the definition of \( T_{\text{exit}} \) in Proposition 3.1. We have the following lemma.

**Lemma 4.3.** For all \( t \in [0, T_{\text{exit}}) \), there holds that

\[
|\tilde{V} - V| \lesssim s^{-\frac{1}{4}}
\]

and

\[
|s(\tilde{V}_k)_s - (DA\tilde{V})_k| \lesssim (\log s)^{-\frac{1}{4}}, \quad k = 1, 2.
\]

**Proof.** (4.21) is immediately from (3.34) and \( b_1 \lesssim \frac{1}{s} \). To prove (4.22), we compute using the above definitions

\[
(\tilde{b}_k)_s - (2k - 1 + c_{b_1})\tilde{b}_1 \tilde{b}_k + \tilde{b}_{k+1} = \frac{1}{s^{k+1}(\log s)^{\frac{4}{3}}} \left[ s(\tilde{U}_k)_s - (A\tilde{U})_k + O(\frac{1}{\sqrt{\log s}} + \frac{|\tilde{U}| + |\tilde{U}|^2}{\log s}) \right].
\]

Using (3.23) and (4.21), we know \( |\tilde{U}| \lesssim 1 \). Then we use the modulation equations and \( b_1 \lesssim \frac{1}{s} \) to conclude

\[
|s(\tilde{U}_k)_s - (A\tilde{U})_k| \lesssim |(\tilde{b}_k)_s - (2k - 1 + c_{b_1})\tilde{b}_1 \tilde{b}_k + \tilde{b}_{k+1}| \cdot s^{k+1}(\log s)^{\frac{5}{4}} + \frac{1}{\sqrt{\log s}} \lesssim (\log s)^{-\frac{1}{4}}.
\]

This is equivalent to (4.22).

Now we are ready to complete the second step towards closing the bootstrap.

**Proposition 4.4.** There exists some initial data \( v_0 \) such that \( T_{\text{exit}} = T(v_0) \).
Proof. We first construct proper \( v_0 \) so that (3.19)-(3.23) and (4.20) hold at \( t = 0 \). Recall \( v_0 = \tilde{Q}_{b(0)} + \tau(0) \tilde{\psi} \) where \( b_k(0) \) is determined by \( U_k(0) \) and we have set \( U_1(0) = 0 \).

Given a pair \((\tilde{V}_2(0), \tilde{\tau}(0)) \in \mathbb{D} := [-1, 1] \times [-1, 1]\), we get the following from (4.19)

\[
\tau(0) = \tilde{\tau}(0) \left( \frac{b_1(0)^{3+\frac{1}{2}}}{\log b_1(0)} \right). \tag{4.23}
\]

and from (3.33) and above definitions, we can determine \( \tilde{V}_1(0) \) and \( U_2(0) \) by solving

\[
\tilde{V}(0) = P\tilde{U}(0), \quad \text{with} \quad \begin{cases}
\tilde{U}_1(0) = U_1(0) = 0,
\tilde{U}_2(0) = U_2(0) - \tau(0) \frac{(H^2 \tilde{\psi}, \chi_{B\delta} \Lambda Q)}{64b_1 \log b_1(0)} \cdot s_0^2(\log s_0)^{\frac{3}{4}}.
\end{cases} \tag{4.24}
\]

It is not hard to derive

\[
\begin{cases}
\tilde{V}_1(0) = -\frac{1}{3} \tilde{V}_2(0), \\
U_2(0) = \frac{1}{3} \tilde{V}_2(0) + \tau(0) \frac{(H^2 \tilde{\psi}, \chi_{B\delta} \Lambda Q)}{64b_1 \log b_1(0)} \cdot s_0^2(\log s_0)^{\frac{3}{4}}.
\end{cases} \tag{4.25}
\]

With \( U_1(0), U_2(0) \) and \( \tau(0) \) at hand, \( v_0 \) is constructed satisfying (3.19)-(3.22) and (4.20).

By contradiction, we assume \( T_{exit} < T(v_0) \) for all such \( v_0 \). Proposition 4.1 and (4.21) imply \( \tilde{T}_{exit} \leq T_{exit} < T(v_0) \leq +\infty \). Hence we obtain a map:

\[
\mathbb{D} \rightarrow \partial \mathbb{D} \\
(\tilde{V}_2(0), \tilde{\tau}(0)) \mapsto (\tilde{V}_2(\tilde{T}_{exit}), \tilde{\tau}(\tilde{T}_{exit})). \tag{4.26}
\]

Moreover, the following strictly outgoing behavior is satisfied: For \((\tilde{V}_2(\tilde{T}_{exit}), \tilde{\tau}(\tilde{T}_{exit})) \in \partial \mathbb{D}\), there are two cases:

(i) If \(|\tilde{V}_2(\tilde{T}_{exit})| = 1\), using (4.22), we get

\[
|s(\tilde{V}_2)_s - \tilde{\frac{2}{3} \tilde{V}_2}| \lesssim (\log s)^{-\frac{1}{4}},
\]

thus

\[
\frac{d}{ds} \tilde{V}_2(\tilde{T}_{exit}) > 0. \tag{4.27}
\]

(ii) If \(|\tilde{\tau}(\tilde{T}_{exit})| = 1\) we resort to (4.15) to compute

\[
\tilde{\tau}_s = \tilde{\tau} \left[ \frac{\tau_s}{\tau} + \frac{(b_1)_s}{b_1} \left(-3 - \frac{1}{2 + \frac{1}{|\log b_1|}} \right) \right] = \zeta \tilde{\tau} \left( 1 + O(b_1^{\frac{3}{4}}) \right), \tag{4.28}
\]

which implies

\[
\frac{d}{ds} \tilde{\tau}_2(\tilde{T}_{exit}) > 0. \tag{4.29}
\]

Summing up the above analysis, we conclude that, the map defined by (4.26) is continuous. Combining with the Brouwer theorem, we get a contradiction and thus finish the proof. \( \square\)
4.2. Proof of Theorem 1.1. We choose initial data $v_0$ such that $T_{\text{exit}} = T(v_0) \leq +\infty$. As a result, our previous calculations are valid. Recall

$$
\frac{d}{ds} \log \left( \frac{\lambda s^{\frac{4}{3}}}{(\log s)^{\frac{4}{9}}} \right) = O \left( \frac{1}{s(\log s)^{\frac{4}{9}}} \right),
$$

and by integration on $[s, +\infty)$, we get

$$
\lambda = c_1(v_0) \frac{(\log s)^{\frac{4}{9}}}{s^{\frac{4}{3}}} \left[ 1 + O((\log s)^{-\frac{4}{3}}) \right]
$$

for some constant $c_1(v_0) > 0$. Using $dt = \lambda^2 ds$ and (3.26), one has

$$
-\lambda \lambda_t = -\frac{\lambda}{\lambda} = \frac{2}{3s} \left[ 1 + O \left( \frac{1}{\log s} \right) \right] = \frac{c_2(v_0) \lambda^{\frac{2}{3}}}{|\log \lambda|^{\frac{2}{3}}} \left[ 1 + O((\log s)^{-\frac{4}{3}}) \right],
$$

that is

$$
-\lambda^{-\frac{2}{3}} |\log \lambda|^{\frac{2}{3}} \lambda_t = c_2(v_0)(1 + O(1)).
$$

From this it is not hard to show $\lambda$ touches zero at some finite time $T = T(v_0) < \infty$. Moreover

$$
T(v_0) - t = \int_s^{+\infty} \lambda^2 d\sigma = c_3(v_0) \frac{(\log s)^{\frac{4}{9}}}{s^{\frac{4}{3}}} \left[ 1 + O((\log s)^{-\frac{4}{3}}) \right].
$$

(4.31)

Finally, we can conclude from (4.30) and (4.31) that

$$
\lambda(t) = c(v_0) \frac{(T(v_0) - t)^2}{|\log(T(v_0) - t)|^{\frac{4}{3}}} (1 + o_{t \rightarrow T(v_0)}(1))
$$

(4.32)

This verifies (1.9).

To prove (1.8), We adapt the strategy from [18]. First, direct computation using (4.14) implies

$$
\forall t \in [0, T), \quad \| \Delta \tilde{v}(t, x) \|_{L^2(\mathbb{R}^4)} < C(v_0) < +\infty
$$

(4.33)

where

$$
\tilde{v}(t, x) = v(t, x) - \frac{1}{\lambda(t)} Q \left( \frac{x}{\lambda(t)} \right) = (\tilde{\alpha} + \varepsilon) \lambda(t).
$$

(4.34)

Now standard parabolic theory ensures the regularity of $v(t, x)$ away from the origin. Hence

$$
\forall R > 0, \quad \tilde{v}(t, x) \rightarrow v^* \text{ in } \dot{H}^1(|x| > R)
$$

(4.35)

for some $v^* \in \dot{H}^1(|x| > R)$. Moreover, (4.33) ensures the energy of $\tilde{v}(t, x)$ does not concentrate at the origin, that is

$$
\forall t \in [0, T), \quad \| \tilde{v}(t, x) \|_{\dot{H}^1(|x| < R)} \rightarrow 0 \text{ uniformly as } R \rightarrow 0
$$

(4.36)

Combined with the boundedness of energy, we have $v^* \in \dot{H}^1$ and

$$
v(t, x) - \frac{1}{\lambda(t)} Q \left( \frac{x}{\lambda(t)} \right) \rightarrow v^* \text{ in } \dot{H}^1
$$

(4.37)

as stated in (1.8). This finishes the proof of Theorem 1.1.
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APPENDIX A. SOME SOBOLEV LEMMAS

Recall the definition of $\psi$ and $\Phi_M$ in (2.5) and (3.3). The following lemma A.1-A.4 can be found in [28]. Readers can refer to the proof therein.

**Lemma A.1.** (Sub-coercivity of $H$) There exists $M_0 > 0$ and $c > 0$ such that for $M > M_0$ and any $u \in \dot{H}^1_{rad}(\mathbb{R}^4)$ with $(u, \Phi_M) = 0$, we have

$$ (Hu, u) \geq c \int |\partial_y u|^2 - \frac{1}{c} (u, \psi)^2. \quad \text{(A.1)} $$

**Lemma A.2.** (Hardy inequality) For all $u \in \dot{H}^1_{rad}(\mathbb{R}^4)$, we have

$$ \int \frac{|u|^2}{y^2} + \sup_{y \in \mathbb{R}^4} |yu|^2 \lesssim \int |\partial_y u|^2. \quad \text{(A.2)} $$

**Lemma A.3.** For all $u \in \dot{H}^1_{rad}(\mathbb{R}^4) \cap \dot{H}^2_{rad}(\mathbb{R}^4)$ and $\gamma > 0$, we have

$$ \int \frac{|\partial^2_y u|^2}{y^4(1 + |\log y|^2)} \lesssim \int \frac{|\partial_y u|^2}{y^2} + \int_{1 \leq y \leq 2} |u|^2, \quad \text{(A.3)} $$

$$ \int_{y \geq 1} \frac{|u|^2}{y^{4+\gamma}(1 + |\log y|^2)} \lesssim \int_{y \geq 1} \frac{|\partial_y u|^2}{y^{2+\gamma}(1 + |\log y|^2)} + \int_{1 \leq y \leq 2} |u|^2. \quad \text{(A.4)} $$

**Lemma A.4.** (Weighted coercivity of $H$) Let $M \geq 1$ be large enough, then there exists $C(M) > 0$ such that for any $u \in \dot{H}^1_{rad}(\mathbb{R}^4) \cap \dot{H}^2_{rad}(\mathbb{R}^4)$ with $(u, \Phi_M) = 0$, we have

$$ \int |Hu|^2 \geq C(M) \left[ \int y^4(1 + |\log y|^2) \frac{|u|^2}{y^4} + \int \frac{|\partial_y u|^2}{y^2} + \int |\partial^2_y u|^2 \right]. \quad \text{(A.6)} $$

$$ \int_{y \leq 1} |Hu|^2 + \int_{y \geq 1} \frac{|Hu|^2}{y^{4+4k}(1 + |\log y|^2)} \geq C(M) \left[ \int_{y \leq 1} (|u|^2 + |\partial_y u|^2) + \int_{y \geq 1} \frac{|u|^2}{y^{8+4k}(1 + |\log y|^2)} + \frac{|\partial_y u|^2}{y^{6+4k}(1 + |\log y|^2)} \right]. \quad \text{(A.7)} $$

In the interior domain, we have the following estimates:
Lemma A.5. (i) For any \( u \in H^{2k}_{rad}(y \leq 1) \), we have
\[
\| u \|_{H^{2k}(y \leq 1)} \lesssim \| u \|_{L^2(y \leq 1)} + \| \Delta^k u \|_{L^2(y \leq 1)}. \tag{A.8}
\]

(ii) For any \( u \in H^{2k+1}_{rad}(y \leq 1) \), we have
\[
\| u \|_{H^{2k+1}(y \leq 1)} \lesssim \| u \|_{L^2(y \leq 1)} + \| \nabla \Delta^k u \|_{L^2(y \leq 1)}. \tag{A.9}
\]

Proof. (i) Using the standard PDE theory, we know that for \( v \in H^{2k}(y \leq 1) \) with \( v|_{y=1} = 0 \) there holds
\[
\| v \|_{H^{2k}(y \leq 1)} \lesssim \| v \|_{L^2(y \leq 1)} + \| \Delta^k v \|_{H^0(y \leq 1)}. \tag{A.10}
\]
By induction, it is not hard to show that for any \( v \in H^{2k}(y \leq 1) \) with
\[
v = \Delta v = \Delta^2 v = \ldots = \Delta^{k-1} v = 0 \quad \text{for } y = 1,
\]
we have
\[
\| v \|_{H^{2k}(y \leq 1)} \lesssim \| v \|_{L^2(y \leq 1)} + \| \Delta^k v \|_{L^2(y \leq 1)}. \tag{A.11}
\]
Now we define \( \varphi_m(y) = y^{2m} \) so that
\[
\Delta^m \varphi_m \neq 0 \quad \text{and} \quad \Delta^n \varphi_m = 0, \quad \forall \ n > m.
\]
Let \( v := u - \sum_{i=0}^{k-1} c_i \varphi_i \), where \( c_i \) is chosen such that
\[
v = \Delta v = \ldots = \Delta^{k-1} v = 0 \quad \text{for } y = 1.
\]
which is equivalent to
\[
\begin{bmatrix}
\varphi_0 & \varphi_1 & \ldots & \varphi_{k-1} \\
\Delta \varphi_0 & \Delta \varphi_1 & \ldots & \Delta \varphi_{k-1} \\
\Delta^{k-1} \varphi_0 & \Delta^{k-1} \varphi_1 & \ldots & \Delta^{k-1} \varphi_{k-1}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\ldots \\
c_{k-1}
\end{bmatrix}
= \begin{bmatrix}
u(1) \\
\Delta u(1) \\
\Delta^2 u(1) \\
\ldots \\
\Delta^{k-1} u(1)
\end{bmatrix}. \tag{A.12}
\]
Note that the matrix above is upper triangular with nonzero diagonal elements, so \( c_i \) can be solved. Besides \( \sum_{i=0}^{k-1} |c_i| \lesssim \sum_{i=0}^{k-1} |\Delta^i u(1)| \).

We apply (A.11) to \( v \) and conclude
\[
\| u \|_{H^{2k}(y \leq 1)} \lesssim \| v \|_{H^{2k}(y \leq 1)} + \sum_{i=0}^{k-1} |c_i| \tag{A.13}
\]
\[
\lesssim \| v \|_{L^2(y \leq 1)} + \| \Delta^k v \|_{L^2(y \leq 1)} + \sum_{i=0}^{k-1} |c_i| \]
\[
\lesssim \| u \|_{L^2(y \leq 1)} + \| \Delta^k u \|_{L^2(y \leq 1)} + \sum_{i=0}^{k-1} |\Delta^i u(1)|.
\]
Elementary Sobolev interpolation shows
\[
\sum_{i=0}^{k-1} |\Delta^i u(1)| \lesssim \| u \|_{W^{2k-2, \infty}(y \leq 1)} \lesssim \gamma \| u \|_{H^{2k}(y \leq 1)} + C(\gamma) \| u \|_{L^2(y \leq 1)}, \tag{A.14}
\]
for any \( \gamma > 0 \). Picking \( \gamma \) small enough we obtain (A.8).
(ii) The proof is completely similar based on the fact
\[ \|v\|_{H^{2k+1}(y \leq 1)} \lesssim \|v\|_{L^2(y \leq 1)} + \|\nabla \Delta^k v\|_{L^2(y \leq 1)}; \quad \forall v \in H^{2k+1}(y \leq 1), \; v\big|_{y=1} = 0, \]
so we omit the details here. \qed

\section*{Appendix B. Interpolation Bounds}

We recall that $\varepsilon$ satisfies the following orthogonality conditions
\[ (H^i \varepsilon, \Phi_M) = 0, \quad i = 0, 1, 2 \]
and upper bounds
\[ \Xi_1 := \int |\partial_y \varepsilon|^2 \leq 10 \sqrt{b_1(0)}, \]
\[ \Xi_2 \leq b_4^4 |\log b_1|^K, \quad \Xi_4 \leq b_4^4 |\log b_1|^K \quad \text{and} \quad \Xi_6 \leq K \frac{b_6^6}{|\log b_1|^2}, \]
where $\Xi_{2k} = \int |H^k \varepsilon|^2$. We shall collect relevant bounds for $\varepsilon$ needed in this paper.

\begin{lemma}
We have the following estimates with constants dependent of $M$,

(i) Weighted bounds for $H^i \varepsilon$: For $1 \leq k \leq 3, \; 0 \leq i \leq k-1$,
\[ \int_{y \leq 1} (|H^1 \varepsilon|^2 + |\partial_y H^1 \varepsilon|^2) + \int_{y \geq 1} \left( \frac{|H^1 \varepsilon|^2}{y^{4k-4i}(1 + |\log y|^2)} + \frac{|\partial_y H^1 \varepsilon|}{y^{4k-4i}(1 + |\log y|^2)} \right) \lesssim \Xi_{2k}. \]

(ii) Weighted bounds for $\partial_y^i \varepsilon$:
\[ \int \left( \frac{|\varepsilon|^2}{y^4(1 + |\log y|^2)} + \frac{|\partial_y \varepsilon|^2}{y^2} + \frac{|\partial_y^2 \varepsilon|^2}{y^2} \right) \lesssim \Xi_2, \tag{B.1} \]
\[ \int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2}{y^{4k-2i}(1 + |\log y|^2)} \lesssim \Xi_{2k}, \quad \text{for } k = 2, 3, \; 0 \leq i \leq 2k, \tag{B.2} \]
\[ ||\varepsilon||^2_{H^i(y \leq 1)} \lesssim \Xi_6. \tag{B.3} \]

(iii) Lossy bounds for $y \geq 1$:
\[ \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{12-2i}} |\partial_y^i \varepsilon|^2 \lesssim |\log b_1|^{C_1(C)} \Xi_6, \quad \text{for } 0 \leq i \leq 4, \tag{B.4} \]
\[ \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{8-2i}} |\partial_y^i \varepsilon|^2 \lesssim |\log b_1|^{C_1(C)} \Xi_4, \quad \text{for } 0 \leq i \leq 2, \tag{B.5} \]
\[ \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{4-2i}} |\partial_y^i \varepsilon|^2 \lesssim |\log b_1|^{C_1(C)} \Xi_2, \quad \text{for } i = 0, 1. \tag{B.6} \]

(iv) Point-wise bounds for $y \geq 1$:
\[ ||\varepsilon(1 + |\log y|^C)||^2_{L^\infty(y \geq 1)} + ||y \partial_y \varepsilon(1 + |\log y|^C)||^2_{L^\infty(y \geq 1)} \lesssim |\log b_1|^{C_1(C)} \Xi_2, \tag{B.7} \]
\[ \left\| \frac{1 + |\log y|^C}{y^2} \right\|^2_{L^\infty(y \geq 1)} + \left\| \frac{1 + |\log y|^C}{y} \partial_y \varepsilon \right\|^2_{L^\infty(y \geq 1)} \]
\[ + \left\| (1 + |\log y|^C) \partial_y^2 \varepsilon \right\|^2_{L^\infty(y \geq 1)} \lesssim \left| \log b_1 \right|^{C_1(C)} \Xi_4, \quad (B.8) \]
\[ \left\| \frac{y \partial_y^3 \varepsilon}{1 + |\log y|} \right\|^2_{L^\infty(y \geq 1)} \lesssim \Xi_4. \quad (B.9) \]

(v) **Point-wise bounds for \( y \leq 1 \):**
\[ \left\| \varepsilon \right\|^2_{L^\infty(y \leq 1)} + \left\| \partial_y \varepsilon \right\|^2_{L^\infty(y \leq 1)} + \left\| \partial_y^2 \varepsilon \right\|^2_{L^\infty(y \leq 1)} + \left\| \partial_y^3 \varepsilon \right\|^2_{L^\infty(y \leq 1)} \lesssim \Xi_6. \quad (B.10) \]

**Proof.** The proof is parallel to the Appendix B in [28] with the help of Appendix A in our article, so we omit the details here. □

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