TENSOR PRODUCTS OF IDEMPOTENT SEMIMODULES.
AN ALGEBRAIC APPROACH

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Abstract. We study idempotent analogs of topological tensor products in the sense of A. Grothendieck. The basic concepts and results are simulated on the algebraic level. This is one of a series of papers on idempotent functional analysis.

Key words: idempotent functional analysis, idempotent semiring, idempotent semimodule, tensor product, polylinear mapping, nuclear operator.

Dedicated to S.G. Krein on the occasion of his 80th birthday

Introduction

We construct and study tensor products in some natural categories of idempotent semimodules. In idempotent analysis, these tensor products play a role similar to that of the topological tensor products constructed by A. Grothendieck [1] in functional analysis. However, we point out that the idempotent versions of the basic notions and results are fairly different from their conventional analogs.

The basic concepts and results (including those of “topological” nature) are simulated on the algebraic level; the point is that the operation of idempotent addition can be defined for infinitely many summands. The present paper is one of a series of papers on idempotent functional analysis, which is an abstract version of idempotent analysis in the sense of [2–9]. In the subsequent publications, we intend to study the links between idempotent tensor products, idempotent linear integral and nuclear operators, and traces of operators.

§1. Basic Concepts

1.1. We recall that an idempotent semigroup is an additive semigroup $S$ with commutative addition $\oplus$ such that $x \oplus x = x$ for all elements $x \in S$. If $S$ has a neutral element, then this element is denoted by $0$. Any idempotent semigroup is a partially ordered set with respect to the standard order defined as follows: $x \preceq y$ if and only if $x \oplus y = y$. This is obviously well defined, and $x \oplus y = \sup \{x, y\}$. Thus each idempotent semigroup is an upper semilattice, and moreover, the notions of an idempotent semigroup and an upper semilattice essentially coincide [10].

The definitions given below are partly borrowed from [11, 12]. An idempotent semigroup $S$ is said to be $\alpha$-complete (or algebraically complete) if it is complete

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as an ordered set, i.e., each subset $X \subset S$ has the least upper bound $\text{sup} X$, denoted also by $\oplus X$, and the greatest lower bound $\inf X$, denoted also by $\wedge X$. This semigroup is said to be $b$-complete (or boundedly complete) if each subset $X \subset S$ (possibly, empty) that is bounded above has the least upper bound $\oplus X$ (in this case each nonempty subset $Y \subset S$ has the greatest lower bound $\wedge Y$, and $S$ is a lattice). Note that each $a$- or $b$-complete idempotent semigroup has a zero $0$, which coincides with $\oplus \emptyset$, where $\emptyset$ is the empty set. Needless to say, $a$-completeness implies $b$-completeness. The cut completion procedure [10] provides an embedding $S \rightarrow \hat{S}$ of an arbitrary idempotent semigroup $S$ in an $a$-complete idempotent semigroup $\hat{S}$ (called the normal completion of $S$); furthermore, $\hat{S} = S$. In a similar way, one defines the $b$-completion $S \rightarrow \hat{S}_b$: if $S \ni \infty = \text{sup} S$, then $\hat{S}_b = S$; otherwise, $\hat{S} = \hat{S}_b \cup \{\infty\}$. An arbitrary $b$-complete idempotent semigroup $S$ also can differ from $\hat{S}$ only by the element $\infty = \text{sup} S$.

Let $S$ and $T$ be $b$-complete idempotent semigroups. A homomorphism $f : S \rightarrow T$ will be called a $b$-homomorphism if $f(\oplus X) = \oplus f(X)$ for each bounded subset $X$ of $S$. If a $b$-homomorphism $f$ extends to be a homomorphism $\hat{S} \rightarrow \hat{T}$ of the normal completions, and moreover, if $f(\oplus X) = \oplus f(X)$ for all $X \subset S$, then $f$ will be called an $a$-homomorphism. Each $a$-homomorphism is a $b$-homomorphism. If the semigroup $S$ is $a$-complete, then each $b$-homomorphism is an $a$-homomorphism. If $S$ and $T$ are topological idempotent semigroups, then a homomorphism $f : T \rightarrow S$ that takes zero to zero is an $a$-homomorphism if and only if $f$ is lower semicontinuous [11, 12].

1.2. An idempotent semiring (for brevity, we sometimes simply say “semiring”) is an idempotent semigroup $(K, \oplus)$ equipped with an associative multiplication $\circ$ such that both distributivity conditions hold. If the multiplication is commutative, then the idempotent semiring is said to be commutative. An element $1 \in K$ is called the unit of the semirings $K$ if $x \circ 1 = 1 \circ x = x$ for all $x \in K$. An element $0 \in K$, $0 \neq 1$, is called the zero of the semirings $K$ if $x \oplus 0 = x$ and $x \circ 0 = 0 \circ x = 0$ for all $x \in K$. In this paper we consider only idempotent semirings with unit. The presence of zero is usually also assumed (unless specified otherwise). An idempotent semifield (or simply semifield) is a commutative idempotent semiring in which every nonzero element is invertible with respect to multiplication. An idempotent semiring $K$ is said to be $a$-complete (respectively, $b$-complete) if $K$ is an $a$-complete (respectively, $b$-complete) idempotent semigroup and if the generalized distributivity laws $k \circ (\oplus X) = \oplus (k \circ X)$ and $(\oplus X) \circ k = \oplus (X \circ k)$ hold for each subset (respectively, each bounded subset) $X \subset K$ and each $k \in K$. It follows from the generalized distributivity that each $a$-complete or $b$-complete idempotent semiring has a zero element, which coincides with $\oplus \emptyset$. The notion of an $a$-complete idempotent semiring coincides with that of a complete dioid in the sense of [13, 14].

The set $\mathbb{R}(\text{max}, +)$ of real numbers equipped with the idempotent addition $\oplus = \text{max}$ and the multiplication $\circ = +$ is an example of an idempotent semiring; in this case $1 = 0$. By supplementing this idempotent semiring with the element $0 = -\infty$, we obtain the $b$-complete semiring $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$ with the same operations and with zero. By supplementing $\mathbb{R}_{\text{max}}$ with the element $+\infty$ and by setting $0 \circ (+\infty) = 0$, $x \circ (+\infty) = +\infty$ for $x \neq 0$, $x \oplus (+\infty) = +\infty$ for all $x$, we obtain the $a$-complete idempotent semiring $\mathbb{R}_{\text{max}} = \mathbb{R}_{\text{max}} \cup \{+\infty\}$. The standard order on $\mathbb{R}(\text{max}, +)$, $\mathbb{R}_{\text{max}}$, and $\mathbb{R}_{\text{max}}$ coincides with the usual order. The
idempotent semirings \(\mathbb{R}(\max, +)\), \(\mathbb{R}_{\text{max}}\) are semifields. On the other hand, an \(a\)-complete idempotent semiring other than \(\{0, 1\}\) cannot be a semifield. There is an important class of examples related to (topological) vector lattices (e.g., see [15, Chap. V] and [10]). By defining the sum \(x \oplus y\) as \(\sup\{x, y\}\) and the multiplication \(\odot\) as addition of vectors, one can interpret vector lattices as idempotent semifields. By supplementing a complete vector lattice (in the sense of [10, 15]) with the element \(0\), we obtain a \(b\)-complete semifield. Next, we can add the “infinite” element, thus obtaining an \(a\)-complete idempotent semiring (coinciding as the ordered set with the normal completion of the original lattice).

1.3. Let \(V\) be an idempotent semigroup and \(K\) an idempotent semiring. Suppose that a multiplication \(k, x \mapsto k \odot x\) of elements of \(V\) by elements of \(K\) is given, and moreover, this multiplication is associative, distributes over the addition in \(V\), and satisfies \(1 \odot x = x\) for all \(x \in V\). In this case the semigroup \(V\) is called an idempotent semimodule (or simply semimodule) over \(K\). An element \(0_V \in V\) is called the zero of the semimodule \(V\) if \(k \odot 0_V = 0_V\), \(0_k \odot x = 0_V\), and \(0_V \oplus x = x\) for all \(k \in K\) and \(x \in V\). Let \(V\) be a semimodule over a \(b\)-complete idempotent semiring \(K\). This semimodule is said to be \(b\)-complete if it is \(b\)-complete as an idempotent semigroup and if the generalized distributivity laws \((\oplus Q) \odot x = \oplus (Q \odot x)\) and \(k \odot (\oplus X) = \oplus (k \odot X)\), \(k \in K\), \(x \in X\), hold for any bounded subsets \(Q \subset K\) and \(X \subset V\). This semimodule is said to be \(a\)-complete if it is \(b\)-complete and contains the element \(\infty = \sup V\).

Let \(V\) and \(W\) be idempotent semimodules over an idempotent semiring \(K\). A mapping \(p: V \to W\) is said to be linear (over \(K\)) if \(p(x \oplus y) = p(x) \oplus p(y)\) and \(p(k \odot x) = k \odot p(x)\) for all \(x, y \in V\) and \(k \in K\). Let the semimodules \(V\) and \(W\) be \(b\)-complete. A linear mapping \(p: V \to W\) is said to be \(b\)-linear if it is a \(b\)-homomorphism of idempotent semigroups and \(a\)-linear if it extends to be an \(a\)-homomorphism of the normal completions \(\hat{V}\) and \(\hat{W}\). Every \(a\)-linear mapping is \(b\)-linear. If a mapping \(V \to W\) is \(b\)-linear and the semimodule \(V\) is \(a\)-complete, then this mapping is \(a\)-linear. The \(a\)-linearity is similar to continuity (semicontinuity) for linear mappings. A semiring \(K\) or the normal completion \(\hat{K}\) of a semifield \(K\) is a semimodule over \(K\). If \(W\) coincides with \(K\) (or \(\hat{K}\)), then the linear mapping \(p\) is called a linear functional. Linear (respectively, \(b\)-linear) mappings will also be referred to as linear (respectively, \(b\)-linear) operators.

In analysis, the most important examples of idempotent semimodules and spaces are either subsemimodules of topological vector lattices [15] (possibly coinciding with these) or their duals (that is, semimodules of linear functionals with some regularity condition, say, of \(a\)-linear functionals).

**Remark 1.** Clearly, idempotent semimodules over a given idempotent semiring form a category with linear mappings as morphisms. In the present paper we mainly deal with another category, which is formed by \(b\)-complete idempotent semimodules over a given \(b\)-complete idempotent semiring with \(b\)-linear mappings as morphisms, and also with the full subcategory of \(a\)-complete semimodules with \(a\)-linear mappings as morphisms. Note that these categories are not additive (it is natural to say that they are semiadditive); for the basic notions of the category theory, e.g., see [16].

In what follows, unless specified otherwise, we consider only \(b\)-complete (including \(a\)-complete) idempotent semirings and idempotent semimodules.
§2. Direct Products and Sums of $b$-Complete Idempotent Semimodules

Let $K$ be a $b$-complete idempotent semiring, and let $\{V_\alpha\}_{\alpha \in A}$ be a family of $b$-complete idempotent semimodules over $K$. We represent an element of the direct product of all sets of this family as a (generally speaking, infinite) sequence (“vector”) $x = \{x_\alpha\} = (\ldots, x_\alpha, \ldots)$, where $x_\alpha \in V_\alpha$. The componentwise operations $x \oplus y = \{x_\alpha + y_\alpha\} = (\ldots, x_\alpha + y_\alpha, \ldots)$ and $k \odot x = \{k \odot x_\alpha\} = (\ldots, k \odot x_\alpha, \ldots)$, $k \in K$, make the direct product of the sets $V_\alpha$, $\alpha \in A$, an idempotent semimodule, which is called the direct product of the semimodules $V_\alpha$, $\alpha \in A$, and is denoted by $\prod_{\alpha} V_\alpha$. The direct product of semimodules $V_1, \ldots, V_n$ will also be denoted by $V_1 \times \cdots \times V_n$. For each index $\alpha \in A$, we define a projection $p_\alpha : \prod_{\alpha} V_\alpha \to V_\alpha$ by setting $p_\alpha(x) = x_\alpha$ and an embedding $i_\alpha : V_\alpha \to \prod_{\alpha} V_\alpha$ by setting $(i(x))_\alpha = x$ and $(i(x))_\beta = 0$ for $\alpha \neq \beta$.

**Proposition 1.** The semimodule $\prod_{\alpha} V_\alpha$ is $a$-complete (respectively, $b$-complete) if so are all semimodules $V_\alpha$.

This assertion readily follows from the definitions.

**Proposition 2.** Let $\{V_\alpha\}_{\alpha \in A}$ be a family of $b$-complete semimodules over $K$, and let $V$ be a $b$-complete semimodule over $K$. Then the following assertions hold:

1) for each family of $b$-linear mappings $f_\alpha : V \to V_\alpha$ there exists a unique $b$-linear mapping $f : V \to \prod_{\alpha} V_\alpha$ such that $p_\alpha f = f_\alpha$ for each index $\alpha \in A$;

2) if the indexing set $A$ is finite, then for each family of $b$-linear mappings $f_\alpha : V_\alpha \to V$ there exists a unique $b$-linear mapping $f : \prod_{\alpha} V_\alpha \to V$ such that $f i_\alpha = f_\alpha$ for each index $\alpha \in A$;

3) if all semimodules $V$ and $V_\alpha$, $\alpha \in A$, are $a$-complete, then for any indexing set $A$ and any family of $a$-linear mappings $f_\alpha : V_\alpha \to V$ there exists a unique $a$-linear mapping $f : \prod_{\alpha} V_\alpha \to V$ such that $f i_\alpha = f_\alpha$ for each $\alpha \in A$.

**Proof.** We start from assertion 1). We set $f(x) = \{f_\alpha(x)\}$. One can readily see that a subset of $\prod_{\alpha} V_\alpha$ is bounded if and only if the projections of this subset on all components are bounded. This, together with the definitions, readily implies that $f$ is $b$-linear; moreover, by construction, $p_\alpha f = f_\alpha$ for all $\alpha \in A$. The uniqueness of the desired mapping is also obvious, and the proof of assertion 1) is complete. The mapping $f$ is called the direct product of the mappings $f_\alpha$.

To prove assertion 2), we set $f(\{x_\alpha\}) = \bigoplus_{\alpha} f_\alpha(x_\alpha)$. One can readily see that if $X \subset \bigprod_{\alpha} V_\alpha$ is a bounded subset, then so is the set $\{f_\alpha(x_\alpha) \mid \alpha \in A, x_\alpha \in p_\alpha(X)\}$. It follows that the mapping $f$ is well defined; the relations $f i_\alpha = f_\alpha$ for all $\alpha \in A$ and the fact that $f$ is $b$-linear and unique can be verified directly. This mapping is called the direct sum of the mappings $f_\alpha$.

The proof of assertion 3) is similar. The finiteness of the indexing set was used only in the proof of the boundedness of the set $\{f_\alpha(x_\alpha) \mid \alpha \in A, x_\alpha \in p_\alpha(X)\}$. However, for an $a$-complete semimodule $V$ this holds automatically. It remains to notice that under the assumptions of this assertion $a$-linearity coincides with $b$-linearity.

Thus the proof of Proposition 2 is complete. □

**Remark 2.** In the language of the category theory, Proposition 2 has the following meaning. In the category of $b$-complete idempotent semimodules there exists a
(categorical) direct product of an arbitrary family of semimodules $V_\alpha$; this product coincides with $\prod_\alpha V_\alpha$. If the family is finite, then there also exists a (categorical) direct sum $\sum_\alpha V_\alpha$ semimodules $V_\alpha$, which coincides with the direct sum $\prod_\alpha V_\alpha$. In the category of $a$-complete semimodules, for an arbitrary family of semimodules $V_\alpha$ there exist a (categorical) direct sum $\sum_\alpha V_\alpha$ and a direct product $\prod_\alpha V_\alpha$, which coincide with each other and with the direct product in the category of $b$-complete semimodules.

§ 3. Tensor Products of $b$-Complete Semimodules

Let $\{V_\alpha\}_{\alpha \in A}$ be a family of $b$-complete idempotent semimodules over a given $b$-complete commutative idempotent semiring $K$, and let $V = \prod_\alpha V_\alpha$.

By $T$ we denote the set of formal sums of the form

$$t = \bigoplus_{x = \{x_\alpha\} \in X} \lambda(x) \odot \bigotimes_\alpha x_\alpha,$$

where $X \subset V$, $x = \{x_\alpha\} = (\ldots, x_\alpha, \ldots)$, and $\lambda$ is an arbitrary mapping of $X$ into $K$. An element of the form (1) will be called a representation of a tensor (in what follows we define a tensor as a class of equivalent representations). The natural (formal) idempotent addition (which should not be confused with the addition in $V = \prod_\alpha V_\alpha$) and the multiplication by elements $k \in K$ (which takes each function $\lambda(x)$ to $x \mapsto k \odot \lambda(x)$) make $T$ a semimodule over $K$.

We say that a representation of the form (1) is bounded if $X$ is bounded in $V$ and $\lambda(x)$ is a bounded function on $X$ (ranging in $K$). The set of all bounded representations will be denoted by $T_b$. Clearly, $T_b$ is a subsemimodule of $T$. One can readily see that the semimodule $T_b$ is $b$-complete.

As usual, we equip $T$ and $T_b$ with the equivalence relation generated by the identities

$$k \odot (\cdots \otimes x_\alpha \otimes \cdots) = (\cdots \otimes k \odot x_\alpha \otimes \cdots),$$

$$\bigotimes (\cdots \otimes (\oplus X_\alpha) \otimes \cdots) = \bigoplus \bigotimes (\cdots \otimes x \otimes \cdots),$$

where $k \in K$, $X_\alpha \subset V_\alpha$, some value of the index $\alpha$ is chosen, and all respective components (formal factors) on the right- and left-hand sides in (2) and (3) coincide except for those written out explicitly. Relation (2) permits one to replace the sum (1) by an equivalent representation of the form

$$t = \bigoplus_{x = \{x_\alpha\} \in X} \otimes_\alpha x_\alpha.$$

In what follows we deal with representations of the form (4); in this case, by some abuse of speech, the set $X \subset \prod_\alpha V_\alpha$ occurring in (4) will also be called a representation of a given tensor. The addition of tensors of the form (4) corresponds to the union of their representations (summands). This addition must be idempotent; hence, by summing a given representation with all equivalent representations, we obtain the same tensor (up to equivalence). The corresponding representation is naturally called complete; we identify a tensor, i.e., a class of equivalent representations, with the corresponding complete representation. We shall assume that the complete representation always contains the element $0_V = \otimes_\alpha 0_{V_\alpha} \in V$. 

Let us proceed to precise definitions. For each element $k \in K$, we denote by $k_\alpha$ the self-mapping of $V = \prod_\alpha V_\alpha$ that acts as the multiplication by $k$ of the $\alpha$th component of each element $x \in V$ and does not alter any other components. A subset $S \subset V$ will be called an $\alpha$-fiber or simply a fiber if the projection $p_\beta(S)$ is a singleton for $\alpha \neq \beta$ and if $p_\alpha(S) = V_\alpha$. Thus, all but one components of elements of a fiber $S$ are fixed, whereas the $\alpha$th component can take arbitrary values in $V_\alpha$.

A tensor, or a complete representation of a tensor, is an arbitrary subset $X \subset V = \prod_\alpha V_\alpha$ with the following properties:

1) if $k_\alpha(x) \in X$ for some index $\alpha \in A$ and $k \in K$, then $k_\beta(x) \in X$ for all indices $\beta \in A$;
2) $\oplus (X \cap S) \in X$ for each fiber $S \subset V$;
3) if $y \in S$, $x \in X \cap S$, and $y \preceq x$, then $y \in X$ for each fiber $S \subset V$.

Needless to say, property 1) corresponds to Eq. (2), and properties 2) and 3) correspond to Eq. (3) with regard to the fact that $x \oplus y = x$ for $y \preceq x$. We shall assume that a complete representation $X$ always contains $0_V = \{0_\alpha\}$. Formally, this corresponds to the identity $\sup \emptyset = 0_V$.

We define the $\tau$-hull $X^\tau$ of an arbitrary subset $X \subset V = \prod_\alpha V_\alpha$ as the intersection of all tensors (that is, their complete representations) containing $X$. Two subsets $X, Y \subset V$ are said to be equivalent (we denote this equivalence relation by $\sim$) if their $\tau$-hulls $X^\tau$ and $Y^\tau$ coincide.

An elementary analysis shows that the following assertion holds.

**Proposition 3.** The equivalence generated by (2) and (3) with regard to the fact that the addition is idempotent on the set of representations of the form (4) coincides with the equivalence $\tau$. Every representation of the form (1) is equivalent to a unique complete representation of the form (4). This equivalence is consistent with the structure of a semimodule over $K$ on $T$ and $T_b$.

It follows that the quotient semimodules $\widetilde{T}$ and $\widetilde{T}_b$ are well defined. One can readily see that the semimodule $\widetilde{T}_b$ is $b$-complete; in any case, this follows from Proposition 4 below. The quotient semimodule $\widetilde{T}_b$ will be denoted by $\bigotimes_\alpha V_\alpha$ and called the $b$-tensor (or simply tensor) product of the $b$-complete semimodules $V_\alpha$.

Recall that we identify a tensor with its complete representation, i.e., a subset $X X^\tau \subset V$.

We say that a tensor $X$ is a bounded or $b$-tensor if it is contained (as a set) in a tensor $\{x\}^\tau$, where $x \in V$. In this case we say that $X$ is bounded by the tensor $\{x\}^\tau$, i.e., the tensor with representation $\bigotimes_\alpha x_\alpha$, where $x = \{x_\alpha\}$.

Let $T_b(V)$ be the set of all $b$-tensors, ordered by inclusion. This order gives rise to an idempotent addition in $T_b(V)$. The sum of an arbitrary bounded family $\{X_\alpha\}$ of tensors coincides with the $\tau$-hull $\bigcup_\alpha X_\alpha^\tau$ of their unions. If a $b$-tensor $X$ is bounded by a tensor $\{x\}^\tau$ and a $b$-tensor $Y$ is bounded by a tensor $\{y\}^\tau$, then the tensor $X \oplus Y = \{X \cup Y\}^\tau$ is bounded by the tensor $\{x \oplus y\}^\tau$, where $x \oplus y \in V$. Clearly, the intersection of an arbitrary family of $b$-tensors is again a $b$-tensor, which coincides with the greatest lower bound of this family of tensors (with respect to the order defined by the set-theoretic inclusion of subsets of $V$). It follows that every subset $M \subset T_b(V)$ that is bounded above has the least upper bound

$$\sup M = \inf \{X \mid Y \preceq X \text{ for all } Y \in M\} = \bigcap_{X \supset Y \in M} X$$
class in $T$ that this completes the verification of the second distributivity law. Thus we have proved that these mappings are mutually inverse and the structures of semimodules over $K$ etc. (e.g., see [10]). Thus $T_b(V)$ is a lattice and a $b$-complete idempotent semigroup.

One can multiply elements $X \in T_b(V)$ by elements $k \in K$ according to the formula $k \odot X = (k_\alpha(X))^\tau$ for a given index $\alpha \in A$; by property 1 of the complete representation of a tensor, this multiplication is independent of the choice of $\alpha$.

**Proposition 4.** The idempotent semigroup $T_b(V)$ with the above-defined multiplication by elements of $K$ is a $b$-complete semimodule over $K$. This semimodule is isomorphic to the semimodule $\bigotimes_\alpha V_\alpha$.

**Proof.** Let us verify that $T_b(V)$ is a $b$-complete semimodule. We have already seen that $T_b(V)$ is a $b$-complete idempotent semigroup. Hence it suffices to verify the distributivity laws and the associativity of the multiplication by elements of $K$. One can readily verify that for each $b$-homomorphism $g: V \to V$ and each $b$-tensor $X \subset V$, the set $g^{-1}(X)$ is also a $b$-tensor. It follows that $g(X^\tau) \subset (g(X))^\tau$ and $(g(X))^\tau = (g(X))^\tau$ for each subset $X \subset V$. By applying these relations to $b$-homomorphisms of the form $k_\alpha$, we find that $(k_\alpha(X))^\tau = (k_\alpha(X)^\tau))^\tau$ for all $X \subset V$, $k \in K$, $\alpha \in A$, whence it follows that the multiplication by elements of $K$ is associative.

Let us verify the distributivity. For each subset $\mathcal{T} \subset T_b(V)$ that is bounded above, one has $\bigoplus \mathcal{T} = \left(\bigcup_{t \in \mathcal{T}} t\right)^\tau$. Consequently,

$$k \odot (\bigoplus \mathcal{T}) = k \odot \left(\bigcup_{t \in \mathcal{T}} t\right)^\tau = k_\alpha \left(\left(\bigcup_{t \in \mathcal{T}} t\right)^\tau\right) = k_\alpha \left(\bigcup_{t \in \mathcal{T}} k_\alpha(t)^\tau\right) = \bigoplus_{t \in \mathcal{T}} k \odot t = \bigoplus (k \odot \mathcal{T}),$$

Since

$$\left(\bigcup_{t \in \mathcal{T}} k_\alpha(t)^\tau\right)^\tau = \left(\bigcup_{t \in \mathcal{T}} (k_\alpha(t))^\tau\right)^\tau = \bigoplus_{t \in \mathcal{T}} k \odot t = \bigoplus (k \odot \mathcal{T}),$$

we find that $k \odot (\bigoplus \mathcal{T}) = \bigoplus (k \odot \mathcal{T})$, so that one of the distributivity laws follows. Here the small letter $t$ stands for a $b$-tensor, whereas above (and below) tensors are denoted by capital letters.

Let us now verify the second distributivity law $(\bigoplus Q) \odot X = \bigoplus (Q \odot X)$, where $Q$ is an arbitrary bounded subset in $K$ and $X$ is an arbitrary $b$-tensor. It suffices to verify that the intersections of these sets with the $\alpha$-fiber $S$ coincide for an arbitrary index $\alpha \in A$. However, by construction, $(\bigoplus Q)_\alpha(x) = \bigoplus (Q \odot x) \in \{k \odot x \mid k \in Q\}^\tau$ for any $x \in S$. Consequently, $(\bigoplus Q)_\alpha(x) \in \left(\bigcup_{k \in Q} k_\alpha(X)^\tau\right)$ for $x \in S \cap X$, whence it follows that $$\left(\bigcup_{k \in Q} k_\alpha(X)^\tau\right)^\tau \subset \left(\bigcup_{k \in Q} k_\alpha(X)\right)^\tau$$ and hence $(\bigoplus Q) \odot X = ((\bigoplus Q)_\alpha(X))^\tau \subset \left(\bigcup_{k \in Q} k_\alpha(X)\right)^\tau$. However, $$\left(\bigcup_{k \in Q} k_\alpha(X)\right)^\tau = \left(\bigcup_{k \in Q} k_\alpha(X)\right)^\tau = \bigoplus_{k \in Q} k \odot X = \bigoplus (Q \odot X)$$. Thus $(\bigoplus Q) \odot X \subset \bigoplus (Q \odot X)$. Since the opposite inequality is obvious, this completes the verification of the second distributivity law. Thus we have proved that $T_b(V)$ is a $b$-complete semimodule over $K$.

It remains to indicate (and verify) an isomorphism between $T_b(X)$ and $\bigoplus_\alpha V_\alpha$. Basically, now we can readily derive this from Proposition 3. We obtain a mapping $T_b(V) \to \bigoplus_\alpha V_\alpha$ by taking each $b$-tensor $X$ to its (complete) formal representation $\bigoplus_{x \in \{x_\alpha\}_{\alpha \in X}} \bigotimes_\alpha x_\alpha$ of the form (4) and by passing to the corresponding equivalence class in $T_b$. The inverse mapping takes an arbitrary representation of the form (1) or (4) to the complete representation of the form (4). The verification of the fact that these mappings are mutually inverse and the structures of semimodules over $K$ are consistent (coincide) is elementary. The proof of the proposition is complete. \(\square\)

**Remark 3.** Relation (2) and property 1 of the complete representation of a tensor show why we require the basic semiring $K$ to be commutative.
§ 4. Basic Results on $b$-Polylinear Mappings

Let $\{V_\alpha\}_{\alpha \in A}$ be an arbitrary family of $b$-complete idempotent semimodules over a $b$-complete commutative semiring $K$, and let $W$ be an arbitrary $b$-complete semimodule over $K$. We set $V = \prod_\alpha V_\alpha$; thus, $V$ is the direct product of the semimodules $V_\alpha$; by $\bigotimes_\alpha V_\alpha$ we denote the $b$-tensor product of these semimodules.

A mapping $f: \prod_\alpha V_\alpha \to W$ is said to be $b$-polylinear if it is separately $b$-linear in every component $V_\alpha$; this means that for every given index $\beta \in A$ the mapping $x_\beta \mapsto f(\ldots, x_\beta, \ldots)$ for arbitrary fixed values of all other coordinates is a $b$-linear mapping $V_\beta \to W$.

We define a canonical mapping $\pi$ of the direct product $V = \prod_\alpha V_\alpha$ into the tensor product $T_b(V) = \bigotimes_\alpha V_\alpha$ by setting $\pi(x) = \{x\}^\tau$, i.e., $\pi(\{x_\alpha\}) = \bigotimes_\alpha x_\alpha$.

Remark 4. The range of the canonical mapping $\pi$ generates the semimodule $\bigotimes_\alpha V_\alpha$, i.e., each element of this semimodule is a linear combination (not necessarily finite) of elements of the form $\bigotimes_\alpha x_\alpha$; moreover, each element is a sum (not necessarily finite) of elements of the form $\bigotimes_\alpha x_\alpha$. In other words, the set $\pi(\prod_\alpha V_\alpha)$ is a system of generators of the semimodule $\bigotimes_\alpha V_\alpha$ and even of the corresponding idempotent semigroup.

**Theorem 1.** The canonical mapping $\{x_\alpha\} \mapsto \bigotimes_\alpha x_\alpha$ of the direct product $V = \prod_\alpha V_\alpha$ of the semimodules $V_\alpha$ into the tensor product $T_b(V) = \bigotimes_\alpha V_\alpha$ of these semimodules is $b$-polylinear. For each $b$-polylinear mapping $f: \prod_\alpha V_\alpha \to W$ there exists a unique $b$-linear mapping $f_\otimes: \bigotimes_\alpha V_\alpha \to W$ such that $f = f_\otimes \pi$.

To prove this theorem, we need some auxiliary assertions. Just as above, we identify a tensor with its complete representation of the form (4), i.e., with the corresponding subset in $\prod_\alpha V_\alpha$. For each element $w$ of an arbitrary idempotent semigroup $W$, we denote by $\text{Low}(w)$ the set $\{u \in W \mid u \preceq w\}$.

**Lemma 1.** Let $W$ be an arbitrary $b$-complete semimodule over $K$. For each $b$-polylinear mapping $f: \prod_\alpha V_\alpha \to W$, the complete preimage of each set of the form $\text{Low}(w)$, where $w \in W$, is a $b$-tensor.

**Proof.** Let $X = f^{-1}(\text{Low}(w))$. If $x$ and $y$ are elements of some fiber, $x \in X$, and $y \preceq x$, then $f(y) \preceq f(x) \preceq w$, whence $y \in X$. If $S \subset X$ and $S$ is contained in a single fiber, then $f(\oplus S) = \oplus f(S) \preceq w$, whence $\oplus S \subset X$. If $y = k_\alpha(x)$ and $k_\beta(x) \in X$ for some index $\beta$, then $f(y) = k \circ f(x) = f((k_\beta(x))) \preceq w$, whence $y \in X$. Thus conditions 1)–3) from the definition of a tensor are satisfied, which completes the proof of Lemma 1. □

**Lemma 2.** For any $b$-polylinear mapping $f: \prod_\alpha V_\alpha \to W$ and any subset $X \subset V = \prod_\alpha V_\alpha$, one has $\oplus f(X) = \oplus f(X^\tau)$.

**Proof.** It follows from Lemma 1 that the set $\{x \in V \mid f(x) \preceq w\}$ is a $b$-tensor for each $w \in W$. In particular, the set $\{x \in V \mid f(x) \preceq \oplus f(X)\}$ is a $b$-tensor containing $X$. It follows that it contains the set $X^\tau$, whence $\oplus f(X) \supseteq \oplus f(X^\tau)$. Since the opposite inequality is obvious, the proof of Lemma 2 is complete. □

**Proof of Theorem 1.** Let us verify that the canonical mapping is $b$-polylinear. To this end, let us verify that $\pi(\oplus X) = \oplus \pi(X)$ for each subset $X \subset V = \prod_\alpha V_\alpha$ lying in an $\alpha$-fiber $S \subset V$. Since $\oplus X \subset X^\tau$, we have the inclusion $\pi(\oplus X) \subset X^\tau$. On the other hand, if $x \in X$, then $x \in S$, $\oplus X \in S$, and $x \preceq \oplus X$, whence
Thus the homogeneity of the mapping $f$ follows from Theorem 1. □

Remarks. Theorems 1 and 2 show that in the category of $b$-complete semimodules (with $b$-linear mappings as morphisms) we have constructed a natural tensor product. This tensor product is also a natural tensor product in the full subcategory of $a$-complete semimodules, since one can readily see that the $b$-tensor product of an arbitrary family of $a$-complete semimodules (over a given idempotent semiring) is an $a$-complete semimodule.

By using similar constructions, one can readily construct a tensor product (with finitely many factors) in the category of arbitrary idempotent semimodules over a given commutative idempotent semiring (with linear mappings as morphisms).
§5. Tensor Products of \(b\)-Linear Mappings

Let \(\{V_\alpha\}_{\alpha \in A}\) and \(\{W_\alpha\}_{\alpha \in A}\) be families of \(b\)-complete semimodules over a given \(b\)-complete commutative semiring \(K\). Suppose that for each index \(\alpha \in A\) there is a \(b\)-linear mapping \(f_\alpha : V_\alpha \to W_\alpha\). By \(f\) we denote the direct product \(\prod_\alpha f_\alpha\) of these mappings, i.e., the mapping \(f : \prod_\alpha V_\alpha \to \prod_\alpha W_\alpha\) such that \(f(\{x_\alpha\}) = \{f(x_\alpha)\}\).

One can readily see that this mapping extends to be a \(b\)-polylinear mapping

\[
\prod_\alpha V_\alpha \to \bigotimes_\alpha W_\alpha.
\]

It follows from Theorem 1 that the mapping (5) extends to be a \(b\)-linear mapping

\[
\bigotimes V_\alpha \to \bigotimes W_\alpha.
\]

The mapping (6) is called the \(b\)-tensor (or simply tensor) product of the \(b\)-linear mappings \(f_\alpha\) and is denoted by \(\bigotimes_\alpha f_\alpha\).

§6. The Tensor Algebra

Let \(U, V\) and \(W\) be \(b\)-complete semimodules over a \(b\)-complete commutative idempotent semiring \(K\). We use the symbol \(\oplus\) to denote the direct sum of two semimodules and the symbol \(\sum_\alpha V_\alpha\) to denote the direct sum of an arbitrary family of \(a\)-complete semimodules. In these cases the direct sum coincides with the direct product (see Remark 2).

**Theorem 3.** The following \(b\)-complete semimodules are isomorphic:

1) \(U \otimes V\) and \(V \otimes U\);
2) \((U \oplus V) \otimes W\) and \(U \otimes W \oplus V \otimes W\);
3) \((U \otimes V) \otimes W\) and \(U \otimes (V \otimes W)\);
4) \((\sum_\alpha V_\alpha) \otimes W\) and \(\sum_\alpha V_\alpha \otimes W\),

where \(\{V_\alpha\}_{\alpha \in A}\) is an arbitrary family of \(a\)-complete semimodules over \(K\).

**Corollary.** The set of isomorphism classes of \(b\)-complete semimodules (over a given \(b\)-complete commutative idempotent semiring) is a commutative associative semiring with respect to the operations of direct sum and \(b\)-tensor product.

**Remark 6.** This semiring of semimodules is not idempotent.

**Proof of Theorem 3.** Assertion 1) is trivial and readily follows from the definition of the \(b\)-tensor product. Assertion 3) follows from the fact that the direct products \(U \times (V \times W)\) and \((U \times V) \times W\) are isomorphic, and moreover, the set of \(b\)-tensors (i.e., of their complete representations of the form (4)) is the same in both cases and coincides with the set of \(b\)-tensors in \(U \times V \times W\), i.e., with \(U \otimes V \otimes W\), as desired.

Let us prove assertions 2) and 4). Suppose that the direct sum \(\sum_\alpha V_\alpha = V\) of the family of semimodules \(V_\alpha\) exists and coincides with the direct product \(\prod_\alpha V_\alpha\) (see Remark 2); for assertion 2) this family is finite (or even consists of two semimodules). Let \(i_\alpha : V_\alpha \to V\) and \(p_\alpha : V \to V_\alpha\) be the corresponding canonical embeddings and projections, and let \(i : \sum_\alpha (V_\alpha \otimes W) \to V \otimes W\) be the direct sum of the mappings \(i_\alpha \otimes I_W\), where \(I_W\) is the identity operator in \(W\) (the tensor product of \(b\)-linear
mappings was defined in §5). Thus the restriction of the mappings \( i \) to \( V_\alpha \otimes W \) coincides with \( i_\alpha \otimes I_W \). Likewise, let \( p: V \otimes W \to \sum_\alpha (V_\alpha \otimes W) = \prod_\alpha (V_\alpha \otimes W) \) be the direct product of the mappings \( p_\alpha \otimes I_W \). A routine verification shows that \( i \) and \( p \) are mutually inverse isomorphisms. \( \square \)

§7. Tensor Products of Semimodules of Bounded Functions

Let \( K \) be a \( b \)-complete commutative idempotent semiring and \( X \) an arbitrary set. By \( \mathcal{B}(X,K) \) we denote the set of bounded functions on \( X \) ranging in \( K \), i.e., mappings \( X \to K \) with bounded range. If \( f,g \in \mathcal{B}(X,K) \), then the pointwise addition and multiplication

\[(f \oplus g)(x) = f(x) \oplus g(x), \quad (f \odot g)(x) = f(x) \odot g(x),\]

where \( x \) is an arbitrary element of \( X \), define a structure of a \( b \)-complete semimodule over \( K \) on \( \mathcal{B}(X,K) \) (e.g., see [17, 18, 5, 12]).

**Proposition 5.** For any sets \( X \) and \( Y \), the \( b \)-complete semimodules \( \mathcal{B}(X \times Y,K) \) and \( \mathcal{B}(X,K) \otimes \mathcal{B}(Y,K) \) are isomorphic.

**Proof.** Let \( \delta_x \in \mathcal{B}(X,K) \) be the function equal to \( 1 \) at the point \( x \in X \) and \( 0 \) at all other points. One can readily verify that the mapping

\[(f,g) \mapsto \oplus \{ f(x) \odot f(y) \odot \delta_{(x,y)} \}\]

is a \( b \)-linear mapping \( \mathcal{B}(X,K) \times \mathcal{B}(Y,K) \to \mathcal{B}(X \times Y,K) \); hence it extends to be a \( b \)-linear mapping

\[(7) \quad i: \mathcal{B}(X,K) \otimes \mathcal{B}(Y,K) \to \mathcal{B}(X \times Y,K).\]

We define a \( b \)-linear mapping

\[(8) \quad j: \mathcal{B}(X \times Y,K) \to \mathcal{B}(X,K) \otimes \mathcal{B}(Y,K)\]

by setting \( j(f) = \oplus \{ f(x,y) \odot \delta_x \odot \delta_y \} \); this is obviously well defined. One can readily see that the mappings (7) and (8) are the inverses of each other and specify the desired homomorphism. It suffices to verify this on a system of generators. For an element \( (x,y) \in X \times Y \), one readily has \( i(j(\delta_{(x,y)})) = i(\delta_x \odot \delta_y) = \delta_{(x,y)} \), so that \( ij \) is the identity mapping on \( \mathcal{B}(X \times Y,K) \). On the other hand, elements of the form \( \delta_x \odot \delta_y \) obviously generate \( \mathcal{B}(X,K) \otimes \mathcal{B}(Y,K) \), and

\[j(i(\delta_x \odot \delta_y)) = j(\delta_{(x,y)}) = \delta_x \odot \delta_y;\]

it follows that \( ji \) is the identity mapping. The proof of the proposition is complete. \( \square \)

§8. Nuclear Operators on \( b \)-Complete Semimodules

Let \( V \) and \( W \) be \( b \)-complete semimodules over a \( b \)-complete commutative idempotent semiring \( K \). By \( W^* \) we denote the semimodule of all \( b \)-linear mappings \( W \to K \) (\( b \)-linear functionals). The set of all \( b \)-linear mappings \( W \to V \) is denoted by \( \operatorname{Hom}_b(W,V) \). This set bears a natural structure of a \( b \)-complete semimodule over \( K \). Thus \( W^* = \operatorname{Hom}_b(W,K) \).

For \( v \in V \) and \( w^* \in W^* \) we define an operator (i.e., a \( b \)-linear mapping) \( p_{w^*,v}: W \to V \) by the formula

\[(9) \quad p_{w^*,v}: x \mapsto w^*(x) \odot v.\]

Operators of the form (9) will be called operators of rank 1 (or one-dimensional operators).
Proposition 6. There exists a unique b-linear mapping \( p: W^* \otimes V \to \text{Hom}_b(W, V) \) such that \( p(w^* \otimes v) = p_{w^*, v} \). The range of the operator \( p \) is the subsemimodule of \( \text{Hom}_b(W, V) \) generated by the set of all operators of the form (9), i.e., by operators of rank 1.

To prove this, it suffices to notice that the mapping \( W^* \times V \to \text{Hom}_b(W, V) \) taking each pair \((w^*, v)\) to the operator \( p_{w^*, v} \) is b-bilinear and its range consists of all operators of rank 1; then we can apply Theorem 1. The mapping \( p \) described in Proposition 6 will be called canonical.

By analogy with [1], we refer to a b-linear mapping \( n: W \to V \) as a b-nuclear operator (or a nuclear mapping) if \( n \) lies in the range of the canonical mapping \( p \). It follows from Proposition 6 that a nuclear mapping \( W \to V \) can be represented as a sum of operators of rank 1. It readily follows from this proposition and from our definitions that the following assertion holds.

Corollary. If the semimodules \( V \) and \( W \) are a-complete, then the range of the canonical mapping \( p \) is an a-complete subsemimodule of \( \text{Hom}_b(W, V) \); a b-linear mapping \( W \to V \) is nuclear if and only if it can be represented as a sum (possibly, infinite) of operators of rank 1.

The authors intend to carry out a detailed study of b-nuclear operators in connection with idempotent analogs of kernel theorems (in the spirit of Schwartz and Grothendieck) in one of the subsequent papers. Here we only consider the following example.

Example (see [17, 18]). Let \( W = \mathcal{B}(X, K) \) and \( V = \mathcal{B}(Y, K) \) (as described in §7). By the kernel theorem [18], every b-linear operator \( f: W \to V \) has the form

\[
(10) \quad f: \varphi(x) \mapsto \tilde{\varphi}(y) = \int_{X} K_f(x, y) \circ \varphi(x) \, dx = \sup_{x \in X} (K_f(x, y) \circ \varphi(x)),
\]

where \( \varphi \in W, K_f(x, y) \in \mathcal{B}(X \times Y, K) \) and the “idempotent integral” \( \int_{X} \psi(x) \, dx \) (e.g., see [2, 4–8]) is defined as \( \sup_{x \in X} \psi(x) \) for each function \( \psi \in \mathcal{B}(X, K) \). We define a b-linear functional \( K_{f, y} \) on \( W \) by setting

\[
K_{f, y}(\varphi) = \int_{X} K_f(x, y) \varphi(x) \, dx.
\]

The integral representation (10) can be rewritten in the form \( f(\varphi) = \bigoplus_{y \in Y} K_{f, y}(\varphi) \delta_y \). It follows from the kernel theorem that each b-linear operator \( f: \mathcal{B}(X, K) \to \mathcal{B}(Y, K) \) is b-nuclear. It follows from the theorem on the structure of a b-linear functional [18] that each b-nuclear operator \( f: W \to V \) can be specified by an “integral” kernel \( K_f(x, y) \). Thus, the existence of an integral representation of an operator is provided by the fact that the operator is nuclear and by the existence of integral representations of b-linear functionals. This scheme can be generalized to a wide class of b-complete semimodules.

Remark 7. Note that \( W^* = \hat{(W_b)^*} \), where \( \hat{W_b} \) is the b-completion of the semimodule \( W \) (the procedure of b-completion for semimodules is discussed, for example, in [12]). Hence the requirement that the semimodule \( W \) be b-complete is in fact not very important. In general, if semimodules \( V \) and \( W \) over \( K \) admit b-completions
\( \hat{V}_b \) and \( \hat{W}_b \) over \( \hat{K}_b \), then the “completed” \( b \)-tensor product \( V \otimes W \) can be defined as \( \hat{V}_b \otimes \hat{W}_b \).

Remark 8. For the case of \( a \)-complete idempotent semimodules (and tensors products of finite families of semimodules of this type), some results of the present paper are also contained (in other terms and under a different approach) in [19], where a rather general categorical approach to tensor products, suggested in [20] (see also [21]), was studied. Note that the approach of [20] is not always convenient for applications to analysis in the spirit of A. Grothendieck [1]. For example, for the category of locally convex (or Banach) spaces this approach provides only one of all possible (and important in analysis) topological tensor products in the sense of [1]. In forthcoming papers, we shall consider various versions of idempotent analogs of topological tensor products.

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