On stability of the three-dimensional fixed point in a model with three coupling constants from the $\epsilon$ expansion: Three-loop results.

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Abstract

The structure of the renormalization-group flows in a model with three quartic coupling constants is studied within the $\epsilon$-expansion method up to three-loop order. Twofold degeneracy of the eigenvalue exponents for the three-dimensionally stable fixed point is observed and the possibility for powers in $\sqrt{\epsilon}$ to appear in the series is investigated. Reliability and effectiveness of the $\epsilon$-expansion method for the given model is discussed.

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1 Introduction

In this paper we consider the model given by the effective Landau-Ginzburg-Wilson Hamiltonian with three quartic coupling constants:

\[ H = \int d^D x \left[ \frac{1}{2} (m_0^2 \varphi_\alpha \varphi_\alpha + \nabla \varphi_\alpha \nabla \varphi_\alpha) + \frac{u_0}{4!} (\varphi_\alpha \varphi_\alpha)^2 + \frac{v_0}{4!} \varphi_\alpha^4 + \frac{2z_0}{4!} \varphi_{2\beta}^2 - \varphi_{2\beta}^2 \right]. \quad (1) \]

Here \( \varphi_\alpha \) is a real vector order parameter field in \( D = 4 - 2\epsilon \) dimensions, \( \alpha = 1, 2, \ldots, 2N \), \( \beta = 1, 2, \ldots, N \). The squared "bare mass" \( m_0^2 \) is a linear measure of the temperature, and \( u_0, v_0, \) and \( z_0 \) are the bare coupling constants. Field \( \varphi_\alpha \) can be thought of as possessing two sets of components, even and odd, each may be considered as an \( N \)-component real vector.

The Hamiltonian (1) governs critical thermodynamics in a number of interesting physical systems. For example, when \( N = 2 \) it describes the structural phase transition in \( \text{NbO}_2 \) crystal and antiferromagnetic phase transitions in \( \text{TbAu}_2 \) and \( \text{DyC}_2 \) for \( v = z \). Another physically interesting case \( N = 3 \) is relevant to the antiferromagnetic phase transition in \( \text{K}_2\text{IrCl}_6 \) crystal and, for \( v = z \), to those in \( \text{TbD}_2 \) and \( \text{Nd} \) [1, 2]. The detailed analysis of these systems along with the Landau phenomenological theory can be found in Refs. [1, 3, 4] with references to experimental works therein.

For the first time the renormalization-group (RG) analysis of the model (1) was carried out to the second order of \( \epsilon \) expansion by Mukamel and Krinsky in Refs. [1, 2, 3]. On this ground, it was shown that the \( 2N \)-component real anisotropic model (1) possesses a unique (three-dimensionally) stable fixed point for each \( N \geq 2 \). On the other hand, the critical behavior of this model was studied within the field-theoretical RG approach in three dimensions on the base of two- and three-loop approximations [3, 4]. There were obtained expansions for \( \beta \) functions and critical exponents for arbitrary \( N \). Using the generalized Padé-Borel transformation, the coordinates of the fixed points were found. It was shown that a stable fixed point did exist in the three-dimensional RG flow diagram for \( N \geq 2 \).

Assuming \( v = z \), model (1) formally turns into that with generalized cubic anisotropy and a complex order parameter field. The latter is a specific case \( (m = 2) \) of the well-known \( mn \)-component model. The critical thermodynamics of this model was investigated in detail in Refs. [7, 8, 9]. Direct three-loop calculations for the case \( m = 2, n \geq 2 \) predict stability of the mixed fixed point, the analog of the stable tetragonal fixed point of model (1).

In the meantime, there are general arguments, not relying upon perturbation theory, in favor of that a unique stable fixed point should not be in the physical space although its existence is not forbidden at \( D > 3 \) [10]. The same considerations lead to the conclusion that the only three-dimensionally stable fixed point may be the Bose one.
and it is that point which governs the critical thermodynamics in the phase transitions mentioned. The point is that when \( v = z \), model (1) describes \( N \) interacting Bose systems. As was shown by Sak [11], the interaction term can be represented as the product of energy operators of various two-component subsystems. It was also found that one of the eigenvalue exponents characterizing the evolution of this term under the renormalization group in the neighborhood of the Bose fixed point is proportional to the specific-heat exponent \( \alpha \). Since \( \alpha \) is believed to be negative at this point, as confirmed by highly precise up-to-date experiments with liquid helium [12] including those in outer space [13] and the high-loop RG computations carried out for the simple \( O(n) \)-symmetric model in three dimensions [14, 15], the interaction is irrelevant. Therefore, the Bose fixed point should be stable in three dimensions.

However, the RG approach, when directly applied to model (1) and to the relative \( mn \)-component model, has not yet confirmed this conclusion. On the contrary, all calculations performed up to now indicate the existence of a unique stable fixed point in the physical space, while the Bose point appears to be three-dimensionally unstable [1–3, 5–9]. This may be a consequence of the rather crude approximations used, and the higher order being taken into account the closer the perturbative results could be to the precise results. So, the aim of the paper is to investigate the critical behavior of the three coupling constants model (1) in the next, three-loop, order in \( \epsilon \) and verify compatibility of the predictions given by the \( \epsilon \)-expansion method with the other techniques.

The main result of our study is that the unique fixed point, rather than the Bose one, turns out to be three-dimensionally stable within the given approximation. Calculation of the eigenvalue exponents of this point is a nontrivial task due to their degeneracy in the one-loop approximation. The analysis of the problem fulfilled in this paper shows that such a degeneracy results in substantial reduction of the information obtained from high-loop approximations.

### 2 Three-loop \( \beta \) functions, fixed points, and stability

The character of the critical asymptotics and the flow diagram structure is known to be determined by the RG equations for quartic coupling constants. We calculate the perturbative expansions for the \( \beta \)-functions for arbitrary \( N \) within massless theory using dimensional regularization and the minimal subtraction scheme. The three-loop
results are as follows:

\[
\beta_u = \epsilon u - u^2 - \frac{1}{2(N+4)} \left( 6uv + 2uz \right) + \\
\frac{1}{4(N+4)^2} \left[ 12u^3(3N+7) + 132u^2v + 44u^2z + 30uv^2 + 10uz^2 \right] - \\
\frac{1}{16(N+4)^4} \left[ 4u^4(48\zeta(3)(5N+11) + 33N^2 + 461N + 740) + \\
12u^3v(384\zeta(3) + 79N + 659) + 4u^3z(384\zeta(3) + 79N + 659) + \\
18u^2v^2(96\zeta(3) + N + 321) + 1380u^2vz + 2u^2z^2(288\zeta(3) + 3N + \\
733) + 1512uv^3 + 18uv^2z + 504uvz^2 + 222uz^3 \right],
\]

\[
\beta_v = \epsilon v - \frac{1}{2(N+4)} (12uv + 9v^2 + z^2) + \\
\frac{1}{4(N+4)^2} \left[ 4u^2v(5N+41) + 276uv^2 + 20uvz + 24uz^2 + 102v^2 + 10vz^2 + \\
8z^3 \right] - \frac{1}{16(N+4)^4} \left[ 8u^3v(96\zeta(3)(N + 7) - 13N^2 + 184N + 821) + \\
18u^2v^2(768\zeta(3) + 17N + 975) + 12u^2vz(96\zeta(3) - 13N + 154) + \\
2u^2z^2(576\zeta(3) + 43N + 667) + 108uv^3(96\zeta(3) + 131) + 306uv^2z + \\
12uuz^2(96\zeta(3) + 187) + 2uz^3(384\zeta(3) + 395) + 27v^4(96\zeta(3) + 145) + \\
162v^2z^2 + 8vz^3(48\zeta(3) + 101) + 3z^4(32\zeta(3) + 17) \right],
\]

\[
\beta_z = \epsilon z - \frac{1}{2(N+4)} (12uz + 6vz + 4z^2) + \\
\frac{1}{4(N+4)^2} \left[ 4u^2z(5N+41) + 204uvz + 116uz^2 + 30v^2z + 72vz^2 + 18z^3 \right] - \\
\frac{1}{16(N+4)^4} \left[ 8u^3z(96\zeta(3)(N + 7) - 13N^2 + 184N + 821) + \\
12u^2vz(864\zeta(3) + 4N + 1129) + 4u^2z^2(1440\zeta(3) + 47N + 1796) + \\
18uv^2z(192\zeta(3) + 391) + 72uvz^2(96\zeta(3) + 103) + 2uz^3(960\zeta(3) + 1517) + \\
1512v^3z + 36v^2z^2(48\zeta(3) + 35) + 72vz^3(16\zeta(3) + 25) + 4z^4(48\zeta(3) + 91) \right],
\]

where \(\zeta\) is the Riemann \(\zeta\) function: \(\zeta(3) = 1.20206\). The model under consideration is known to have eight fixed points \[2, 3\]. Below we write out the coordinates of the
most interesting II-tetragonal fixed point only.

\[ u_c = \frac{N+4}{6(N-4)} \epsilon + \frac{N+4}{(4-5N)^3} (70N^2 - 205N + 139)\epsilon^2 + \]
\[ \left( \frac{12(N+4)}{(5N-4)^3} \zeta(3)(64N^3 - 188N^2 + 151N - 23) + \right. \]
\[ \frac{N+4}{4(4-5N)^3} (6370N^4 + 24149N^3 - 144719N^2 + 197208N - 83256) \epsilon^3, \]
\[ v_c = \frac{N+4}{6(N-4)} (N - 2) \epsilon + \frac{N+4}{(5N-4)^3} (30N^3 + 25N^2 - 217N + 166)\epsilon^2 - \]
\[ \left( \frac{24(N+4)}{(5N-4)^3} \zeta(3)(8N^4 + 16N^3 - 88N^2 + 75N - 9) - \frac{N+4}{4(5N-4)^3} (1030N^5 + \right. \]
\[ 2751N^4 + 46033N^3 - 207590N^2 + 267336N - 109808) \epsilon^3, \]
\[ z_c = \frac{N+4}{6(N-4)} (N - 2) \epsilon + \frac{N+4}{(5N-4)^3} (30N^3 + 25N^2 - 217N + 166)\epsilon^2 - \]
\[ \left( \frac{24(N+4)}{(5N-4)^3} \zeta(3)(8N^4 + 16N^3 - 88N^2 + 75N - 9) - \frac{N+4}{4(5N-4)^3} (1030N^5 + \right. \]
\[ 2751N^4 + 46033N^3 - 207590N^2 + 267336N - 109808) \epsilon^3. \]

In order to determine the character of stability of this point we should calculate the stability matrix eigenvalues \( \lambda \)'s. It is convenient, rather, to deal with the quantity \( y = \frac{1}{\epsilon} \) being a root of the reduced characteristic polynomial hereafter denoted \( P(y, \epsilon) \):

\[ -y^3(\epsilon) + a(\epsilon)y^2(\epsilon) - b(\epsilon)y(\epsilon) + c(\epsilon) = 0. \quad (4) \]

The coefficients \( a(\epsilon), b(\epsilon), \) and \( c(\epsilon) \) are the formal series

\[ a(\epsilon) = a_0 + a_1 \epsilon + a_2 \epsilon^2 + \ldots, \]
\[ b(\epsilon) = b_0 + b_1 \epsilon + b_2 \epsilon^2 + \ldots, \]
\[ c(\epsilon) = c_0 + c_1 \epsilon + c_2 \epsilon^2 + \ldots. \quad (5) \]

A solution \( y(\epsilon) \) to Eq. (4) is obtained by consecutive calculating coefficients of the series \( y(\epsilon) = \sum y_k \epsilon^k \) in corresponding orders in \( \epsilon \). Normally, the polynomial \( P(y, 0) \) has three different roots \( y_0 \), and the derivative \( \partial_y P(y_0, 0) \) does not vanish. It implies that the coefficients \( y_k \) are determined in the \( k \)-th order in \( \epsilon \). This customary scheme does not work when the polynomial \( P(y, 0) \) has multiple roots. It takes place for the II-tetragonal fixed point of model (4) since the one-loop approximation, as will be shown below (see formula (4)), yields two equal values of \( y_0 \). To treat this situation properly we have to thoroughly analyze the problem of expanding such twofold degenerate solutions.

If coefficients \( a(\epsilon), b(\epsilon), \) and \( c(\epsilon) \) themselves had been polynomials, the solution \( y(\epsilon) \) would have belonged to the class of so-called algebraic functions. Such a function is
analytical on the complex plane, except for a finite set of isolated points, where it has either poles or branchings of a finite order. The poles are associated with the zeros of the highest coefficient in $P(y, \epsilon)$, while the branching points are associated with those values of $\epsilon$ where $P(y, \epsilon)$ has multiple roots. With twofold degeneracy of $y_0$, branching of order 2 is possible at the point $\epsilon = 0$. But a two-valued function cannot be expanded in integer powers of $\epsilon$. Instead, it should be represented by a Puiseux series in powers of $\sqrt{\epsilon}$. Similar speculations are valid when the dependence of $P(y, \epsilon)$ on $\epsilon$ is of the formal series type since every coefficient of the expansion $y(\epsilon)$ is determined by a finite number of terms in $a(\epsilon), b(\epsilon)$, and $c(\epsilon)$. Therefore, whatever the perturbative order is chosen, there is an algebraic function coinciding with $y(\epsilon)$ modulo higher terms. So, let us formulate the anzatz for $y(\epsilon)$ as

$$y(\epsilon) = y_0 + y_1 \epsilon^{\frac{1}{2}} + y_2 \epsilon + y_3 \epsilon^{\frac{3}{2}} + \ldots.$$  (6)

We shall show that a solution to Eq. (4) in the case of twofold degenerate roots at $\epsilon = 0$ does exist in form (6). We are also interested when noninteger powers in $\epsilon$ do appear in and when they drop from $y(\epsilon)$. The answer will be given by a theorem before which we introduce some notations. Let $\mathbb{Z}^+/2$ be the set of non-negative half-integer numbers. We assume that the infinite point $\infty$ also belongs to $\mathbb{Z}^+/2$. It is convenient to think of it as of an integer number. Denote $[a, b]$ the interval in $\mathbb{Z}^+/2$ with boundaries $a$ and $b$, i.e., the set of points $l \in \mathbb{Z}^+/2$ satisfying inequality $a \leq l \leq b$. To distinguish intervals without one or two boundary points we use parentheses instead of square brackets.

**Theorem.** In the case of twofold degenerate roots in the one-loop approximation, the solutions $y(\epsilon)$ to Eq. (4) are represented by series in powers of $\sqrt{\epsilon}$. There is an alternative: either $P(y, \epsilon)$ has two equal roots in every order of the perturbation theory or the solution $y(\epsilon)$ splits at a finite step $l_s$. Noninteger powers of $\epsilon$ contribute to the expansion $y(\epsilon)$ if and only if $l_s$ is a noninteger number.

Let us define polynomials $A_l(y_0, y_\frac{1}{2}, \ldots, y_l)$, $l \in \mathbb{Z}^+/2$, from the expansion

$$\frac{\partial P(y, \epsilon)}{\partial y} = \sum_{l \in \mathbb{Z}^+/2} A_l(y_0, y_\frac{1}{2}, \ldots, y_l) \epsilon^l.$$  

Another way to introduce them is as follows. Consider the $\epsilon$ expansion of the reduced characteristic equation (ERCE) obtained by substitution of Eqs. (5) and (6) into Eq. (4). At a sufficiently high order the coefficient before $\epsilon^m$ may be shown to be represented as a sum

$$A_0(y_0)y_m + A_\frac{1}{2}(y_0, y_\frac{1}{2})y_{m-\frac{1}{2}} + \ldots + A_l(y_0, y_\frac{1}{2}, \ldots, y_l)y_{m-l} + \ldots = 0, \quad m - l > l.$$
Further, let $I = [0, l_s]$ be the maximum interval containing 0 such that for $l \in I$ the coefficient $y_l$ is found from the order $2l$ of ERCE. It implies, in particular, that the partial solution $[y]_I \equiv (y_0, y_\frac{1}{2}, \ldots, y_l)$ to Eq. (7) exists for all $l \in [0, l_s)$. The upper boundary $l_s$ may be either finite or infinite. First let us prove that the polynomials $A_l(y_0, y_\frac{1}{2}, \ldots, y_l)$ for $l \in [0, l_s)$ turn zero upon substitution of the partial solution $[y]_I$ into them. This is the case, at least, for $l = 0$. Supposing it for all $l$ such that $0 \leq l < m$, where $m$ is some half-integer number strictly below $l_s$, we have

$$[A_0(y_0) y_{2m+\frac{1}{2}} + A_\frac{1}{2}(y_0, y_\frac{1}{2}) y_{2m} + \ldots] + A_{m}(y_0, y_\frac{1}{2}, \ldots, y_m) y_{m+\frac{1}{2}} + \ldots = 0$$

in the order $2m + \frac{1}{2}$ of ERCE. The expression within the square brackets vanishes due to the assumption made, while the rightmost dots stand for the terms depending on $y_l$ with $l$ less than $m + \frac{1}{2}$. Thus, if $A_m(y_0, y_\frac{1}{2}, \ldots, y_m) \neq 0$, the coefficient $y_{m+\frac{1}{2}}$ is determined from the order $2m + \frac{1}{2} \neq 2m + 1$, in contradiction to $m + \frac{1}{2} \leq l_s$ and $m + \frac{1}{2} \in I$. The consequence of the fact just stated is that all $y_l$ with $l \in [0, l_s)$ are found from a quadratic equation

$$Q_l(y_l) \equiv \frac{1}{2} \frac{\partial^2}{\partial y^2} P(y_0, 0) y_l^2 + L_l(y_0, y_\frac{1}{2}, \ldots, y_{l-\frac{1}{2}}) y_l + R_l(y_0, y_\frac{1}{2}, \ldots, y_{l-\frac{1}{2}}) = 0.$$

The highest coefficient $\frac{1}{2} \frac{\partial^2}{\partial y^2} P(y_0, 0)$ is nonzero as only two of the three roots $y_0$ coincide. Now we can give one more characteristic to the polynomial $A_l(y_0, y_\frac{1}{2}, \ldots, y_l)$ for $l \in I$. Namely, it is the derivative of $Q_l(y_l)$ with respect to $y_l$. It follows directly from here that the solution $y(\epsilon)$ does not split at the orders $l \in [0, l_s)$, that is every polynomial $Q_l(y_l)$ has equal roots. Another implication is that all noninteger $l \in [0, l_s)$ give $y_l = 0$. Indeed, every term of the linear coefficient $L_l(y_0, y_\frac{1}{2}, \ldots, y_{l-\frac{1}{2}})$ of $Q_l(y_l)$ depends on some variable $y_k$ with noninteger numbers $k < l$ (because none of the series $a(\epsilon)$, $b(\epsilon)$, and $c(\epsilon)$ involves noninteger powers in $\epsilon$) and vanishes once they turn zero. Assuming recurrently $y_k = 0$ for noninteger $k < l$, we find $y_l$ obeying the equation $Q_l(y_l) = \frac{1}{2} \frac{\partial^2}{\partial y^2} P(y_0, 0) y_l^2 + R_l(y_\frac{1}{2}, \ldots, y_{l-\frac{1}{2}}) = 0$. Since $Q_l(y_l)$ has two equal roots, both are zero. We shall show now that with finite $l_s$, the solution $y(\epsilon)$ splits at the order $l_s$ or, equivalently, that $A_{l_s}(y_0, y_\frac{1}{2}, \ldots, y_{l_s}) \neq 0$. Supposing the opposite, consider the coefficient of ERCE in the noninteger order $2l_s + \frac{1}{2}$ in $\epsilon$:

$$[A_0(y_0) y_{2l_s+\frac{1}{2}} + A_\frac{1}{2}(y_0, y_\frac{1}{2}) y_{2l_s} + \ldots + A_{l_s}(y_0, y_\frac{1}{2}, \ldots, y_l) y_{l_s+\frac{1}{2}}] + \ldots = 0.$$

The expression within the square brackets vanishes as well as the terms depicted by the dots on the right. Those terms contain $y_l$ with noninteger numbers $l$ from the interval $I$. As was stated above, such $y_l$ are equal to zero, hence in the order $2l_s + \frac{1}{2}$, Eq. (7) holds identically. The next order $2l_s + 1$ gives a quadratic equation for $y_{l_s+\frac{1}{2}}$.
with the highest coefficient \( \frac{1}{2} \partial_\epsilon^2 P(y_0,0) \neq 0 \). It means that \( l_s + \frac{1}{2} \) belongs to the interval \( I \) that contradicts the assumption about \( l_s \) being its upper boundary. So, \( A_{l_s}(y_0,y_{\frac{1}{2}},\ldots,y_{l_s}) \neq 0 \) and for arbitrary \( m \in \mathbb{Z}^+ \) the quantity \( y_{l_s+m} \) is determined in the order \( 2l_s + m \) in \( \epsilon \) from a linear equation in which \( A_{l_s}(y_0,y_{\frac{1}{2}},\ldots,y_{l_s}) \) is the highest coefficient. Thus we have proved the existence of solutions to the reduced characteristic equation (7) in the form (8). We have yet to verify disappearance of noninteger powers of \( \epsilon \) from expansion \( y(\epsilon) \) provided \( l_s \) is an integer number. As was shown, the coefficient \( y_{l_s+m} \) is determined in the order \( 2l_s + m \) from a linear equation of the form \( A_{l_s}(y_0,y_{\frac{1}{2}},\ldots,y_{l_s})y_{l_s+m} + B_{l_s+m}(y_0,y_{\frac{1}{2}},\ldots,y_{l_s+m-\frac{1}{2}}) = 0 \). Supposing \( l_s + m \) and therefore \( m \) to be noninteger, we see that each term in \( B_{l_s+m} \) depends on some \( y_l \) with noninteger number \( l < l_s + m \) (because none of \( a(\epsilon), b(\epsilon), \) and \( c(\epsilon) \) contains powers of \( \sqrt{\epsilon} \)). It has already been shown that \( y_l = 0 \) for noninteger \( l \) from the interval \( I \). Assuming recursively \( y_l = 0 \) for \( l < l_s + m \), we have \( B_{l_s+m}(y_0,y_{\frac{1}{2}},\ldots,y_{m-\frac{1}{2}}) = 0 \) and therefore \( y_l = 0 \) for all noninteger \( l \).

Let us apply the theorem proved to the problem of calculating eigenvalue exponents for the II-tetragonal fixed point. The explicit form of the coefficients in Eq. (8) reads

\[
\begin{align*}
    a_0 &= \frac{-1}{(5N-4)^{1/2}}(7N - 8), \\
    a_1 &= \frac{1}{(5N-4)^{1/2}}(270N^3 - 1129N^2 + 1591N - 736), \\
    a_2 &= \frac{1}{2(5N-4)^{1/2}}(48\zeta(3)(5N-4)(144N^4 - 720N^3 + 1289N^2 - 947N + 230) \\
        &+ 10030N^5 - 104229N^4 + 429747N^3 - 804632N^2 + 691620N - 222720), \\
    b_0 &= \frac{1}{(5N-4)^{1/2}}(N - 2)(11N - 10), \\
    b_1 &= \frac{-2}{(5N-4)^{1/2}}(510N^4 - 3157N^3 + 6615N^2 - 5832N + 1868), \\
    b_2 &= \frac{1}{(5N-4)^{1/2}}(48\zeta(3)(5N-4)(272N^5 - 1824N^4 + 4455N^3 - 5095N^2 \\
        &+ 2754N - 558) + 25890N^6 - 338437N^5 + 1547050N^4 - 3437182N^3 \\
        &+ 4044203N^2 - 2430752N + 589412), \\
    c_0 &= \frac{-1}{(5N-4)^{1/2}}(N - 2)^2, \\
    c_1 &= \frac{1}{(5N-4)^{1/2}}(N - 2)(150N^3 - 809N^2 + 1229N - 566), \\
    c_2 &= \frac{-1}{2(5N-4)^{1/2}}(48\zeta(3)(N - 2)(5N-4)(80N^4 - 464N^3 + 865N^2 \\
        &- 641N + 164) + 13950N^6 - 184745N^5 + 887705N^4 - 2072060N^3 \\
        &+ 2541094N^2 - 1575640N + 389512)).
\end{align*}
\]

Substituting this into Eq. (7) and setting \( \epsilon = 0 \) (one-loop approximation) we find \( y_0 \) to be twofold degenerate:

\[
    y_0^{(1)} = -1, \quad y_0^{(2)} = y_0^{(3)} = \frac{2 - N}{5N - 4}.
\]

The solution \( y(\epsilon) \) expanding simple root \( y_0^{(1)} \) is calculated in the conventional way. Our further consideration concerns the multiple root only. The first appearance of the
coefficient $y_{\frac{3}{2}}$ occurs in the order $\frac{1}{2}$ in $\epsilon$ of ERCE, with multiplier $A_0(y_0) = \partial_y P(y_0, 0)$:

$$y_{\frac{3}{2}}A_0(y_0) = 0.$$ 

Due to the degeneracy of $y_0$, $A_0(y_0) = 0$ and $y_{\frac{3}{2}}$ cannot be actually determined from the order $\frac{1}{2}$. To find $y_{\frac{3}{2}}$ we must solve the quadratic equation in the order 1 of ERCE:

$$Q_{\frac{1}{2}}(y_{\frac{3}{2}}) = \frac{1}{2} \partial_y^2 P(y_0, 0) y_{\frac{3}{2}}^2 + (a_1 y_0^2 - b_1 y_0 + c_1) = 0.$$  \hspace{1cm} (9)

The highest coefficient $\frac{1}{2} \partial_y^2 P(y_0, 0) = -(3y_0 - a_0)$ is nonzero because only two of the three roots $y_0$ coincide. Substitution of Eq. (7) into Eq. (9) gives $y_{\frac{1}{2}} = 0$ for all $N$.

The next order $\frac{3}{2}$ of ERCE does not provide $y_1$ because at this step the equation holds identically:

$$A_0(y_0)y_{\frac{3}{2}} + A_{\frac{1}{2}}(y_0, y_{\frac{3}{2}})y_1 + (...)y_{\frac{3}{2}} = 0.$$ 

Here $A_{\frac{1}{2}}(y_0, y_{\frac{3}{2}}) = \partial_y Q_{\frac{1}{2}}(y_{\frac{3}{2}}) = -2(3y_0 - a_0)y_{\frac{3}{2}} = 0$. Considering the factor before $\epsilon^2$ in ERCE, we come to the quadratic equation

$$Q_1(y_1) = \frac{1}{2} \partial_y^2 P(y_0, 0) y_1^2 + (2a_1 y_0 - b_1)y_1 + a_2 y_0^2 - b_2 y_0 + c_2 = 0,$$  \hspace{1cm} (10)

which has the solution

$$y_1 = \frac{3(N - 1)(40N^3 - 208N^2 + 253N - 66)}{(2N - 1)(5N - 4)^3} \pm \frac{4(N - 1)(N - 2)(N + 4)(5N - 4)}{(2N - 1)(5N - 4)^3}.$$  \hspace{1cm} (11)

The order $\frac{5}{2}$ of ERCE gives rise to the relation

$$[A_0(y_0)y_{\frac{3}{2}} + A_{\frac{1}{2}}(y_0, y_{\frac{3}{2}})y_2] + A_1(y_0, y_{\frac{3}{2}}, y_1)y_{\frac{1}{2}} + (...)y_{\frac{3}{2}} = 0,$$

where the expression within the square brackets is zero and

$$A_1(y_0, y_{\frac{3}{2}}, y_1) = -6y_0 y_1 + 2a_0 y_1 + 2a_1 y_0 - b_1.$$  \hspace{1cm} (12)

As was shown when proving the theorem, $A_1(y_0, y_{\frac{3}{2}}, y_1)$ is the derivative of the quadratic polynomial $Q_1(y_1)$ with respect to $y_1$, formula (11). Hence, $A_1(y_0, y_{\frac{3}{2}}, y_1)$ vanishes if and only if $Q_1(y_1)$ has two equal roots. It occurs when $N = 1$ and $N = 2$ in formula (11).
Thus, for $N \neq 1, 2$ the solution $y(\epsilon)$ splits at the integer order in $\epsilon$ and, according to the theorem, would not contain noninteger powers of $\epsilon$. We cannot make such a statement for the physically significant case $N = 2$ because the coefficient $y_2$ is determined in the third order in $\epsilon$ of ERCE involving four-loop contributions. If $y_2$ comes out to be nonzero, the expansion $y(\epsilon)$ will contain noninteger powers. Otherwise, $y_2 = 0$ and only the five-loop approximation, as follows from the theorem, will allow us to calculate the next coefficient $y_2$ in Eq. (11).

To sum up, the three-loop eigenvalue exponents for the II-tetragonal fixed point read

$$
\lambda_1 = -\epsilon + \frac{1}{(5N-4)^2(2N-1)} (60N^3 - 160N^2 + 181N - 85) \epsilon^2 + \frac{1}{2(5N-4)^3(1-2N)^3} (48 \zeta(3)(2N - 1)^2(5N - 4)(32N^4 - 128N^3 + 212N^2 - 153N + 33) + 20560N^7 - 165328N^6 + 644392N^5 - 1406864N^4 + 1756745N^3 - 1224341N^2 + 433704N - 59052) \epsilon^3,
$$

$$
\lambda_2 = \frac{2-N}{5N-4} \epsilon + \frac{1-N}{5N-4} \left( 4 \text{sgn}(N-1) |5N^3 + 6N^2 - 48N + 32| - 3(40N^3 - 208N^2 + 253N - 66) \right) \epsilon^2,
$$

$$
\lambda_3 = \frac{2-N}{5N-4} \epsilon + \frac{N-1}{5N-4} \left( 4 \text{sgn}(N-1) |5N^3 + 6N^2 - 48N + 32| + 3(40N^3 - 208N^2 + 253N - 66) \right) \epsilon^2.
$$

For the physically interesting cases $N = 2$ and $N = 3$ they are presented in Table I. The eigenvalue exponents for the Bose fixed point, the calculation of which is an easy task, are also written out for comparison. It follows from the table that the II-tetragonal fixed point is absolutely stable in three dimensions (3D), in contrast to the Bose one. Obviously, simple resummation procedures, such as the Padé and Padé-Borel methods, being applied to $\lambda$’s, do not change the picture.

Let us now calculate the critical dimensionality $N_c$ of the order parameter. Its $\epsilon$ expansion is found from condition $v_c = z_c = 0$ imposed on the right-hand side of Eq. (11):

$$
N_c = 2 - 2 \epsilon + \frac{5}{6} (6 \zeta(3) - 1) \epsilon^2 + O(\epsilon^3).
$$

The critical dimensionality separates two different regimes of critical behavior of the model. For $N > N_c$ the II-tetragonal rather than the Bose fixed point is three-dimensionally stable in 3D. At $N = N_c$ they interchange their stability so that for $N < N_c$ the stable fixed point is the Bose one.
Table 1: Three-loop eigenvalue exponents for the Bose and the II-tetragonal fixed points ($\epsilon = \frac{1}{2}$ corresponds to the physical space)

| Type of fixed point | $N = 2$                                                                 | $N = 3$                                                                 |
|---------------------|------------------------------------------------------------------------|------------------------------------------------------------------------|
| Bose                | $\lambda_u = \frac{1}{5} \epsilon - \frac{14}{25} \epsilon^2 + \frac{768 \zeta(3)}{625} \epsilon^3$ | $\lambda_1 = -\frac{1}{5} \epsilon + \frac{2}{5} \epsilon^2 - \frac{768 \zeta(3)}{625} + 29 \epsilon^3$ |
|                     | $\lambda_2 = -\epsilon + \frac{6}{5} \epsilon^2 - \frac{384 \zeta(3) + 257}{125} \epsilon^3$             | $\lambda_1 = -\epsilon + \frac{58}{55} \epsilon^2 - \frac{3(123600 \zeta(3) + 71621)}{166375} \epsilon^3$ |
| II-tetragonal       | $\lambda_1 = -\epsilon + \frac{13}{12} \epsilon^2 - \frac{84 \zeta(3) + 65}{36} \epsilon^3$               | $\lambda_1 = -\epsilon + \frac{58}{55} \epsilon^2 - \frac{3(123600 \zeta(3) + 71621)}{166375} \epsilon^3$ |
|                     | $\lambda_2 = -\frac{1}{3} \epsilon^2$                               | $\lambda_2 = -\frac{1}{11} \epsilon - \frac{2}{11} \epsilon^2$        |
|                     | $\lambda_3 = -\frac{1}{3} \epsilon^2$                               | $\lambda_3 = -\frac{1}{11} \epsilon + \frac{2}{605} \epsilon^2$        |

Because the series (14) is alternating, it can be resummed by means of the Padé-Borel method, the result being

$$N_c = a - \frac{2b^2}{c} + \frac{4b^3}{c^2 \epsilon} \exp\left(-\frac{2b}{cc}\right) E_i\left(\frac{2b}{cc}\right).$$ \hspace{1cm} (15)

Here $a$, $b$, and $c$ are the coefficients before $\epsilon^0$, $\epsilon^1$ and $\epsilon^2$ in Eq. (14), respectively, and $E_i(x)$ is the exponential integral. Setting $\epsilon = \frac{1}{2}$, from Eq. (15) we obtain the value of critical dimensionality

$$N_c = 1.50.$$ \hspace{1cm} (16)

Since $N_c$ lies below two, within the given approximation the critical behavior of model (1) for $N = 2$ and $N = 3$ must be governed by the II-tetragonal fixed point. It confirms the deductions given by the analysis of the eigenvalue exponents.

### 3 Discussions

In conclusion, let us briefly discuss the results of the present investigation. The structure of the RG flows of the three quartic coupling constants model was studied within...
the $\epsilon$-expansion method. The three-loop series for $\beta$ functions of the model were obtained. Eigenvalue exponents for the most intriguing II-tetragonal and the Bose fixed points were calculated for arbitrary $N$. For the physically interesting values $N = 2$ and $N = 3$, the II-tetragonal rather than the Bose fixed point was shown to be three-dimensionally stable in 3$D$, and the critical dimensionality $N_c = 1.5$ found has confirmed this conclusion. Consequently, the critical thermodynamics of the antiferromagnetic phase transitions in such substances as $TbAu_2$, $DyC_2$, $K_2IrCl_6$, $TbD_2$, and $Nd$ as well as the structural phase transition in $NbO_2$ crystal should be controlled by the II-tetragonal fixed point. It agrees with the results by Mukamel and Krinsky [1, 2, 3] as well as with calculations performed within the field-theoretical RG approach in 3$D$ (Refs. [5, 6]) but contradicts to the non-perturbative inferences [10]. The distinction of 3$D$ RG predictions from those of the precise theory may be regarded as an effect of low-order approximations. The point is that the three-loop analysis in 3$D$ (Refs. [5, 6]) shows the two rival fixed points (Bose and II-tetragonal) to be close to one another, and it is natural to expect that taking into account next perturbative terms may change their character of stability properly. Such speculations, however, do not suit the $\epsilon$-expansion because we can judge about the closeness of points just indirectly, i.e. by comparing their critical exponents. The $\epsilon$-expansion analysis yields the critical exponents of the II-tetragonal fixed point to be considerably distant from those of the Bose fixed point [1, 2, 3, 17], and hardly one can hope that longer $\epsilon$ series even resummed will bring them close to one another. All this allows us to raise the question whether the $\epsilon$-expansion method is reliable for the given model.

The twofold degeneracy of the solutions to the characteristic equation for the II-tetragonal fixed point in the one-loop approximation worsens the situation. According to the analysis performed, the eigenvalue exponents should be represented as series in powers of $\sqrt{\epsilon}$ instead of $\epsilon$. Even if we adopt the idea that noninteger powers actually drop from the expansions, such a degeneracy decreases the accuracy expected within a given approximation. Namely, in the frame of three-loop approximation we effectively obtain two-loop-like pieces of the series, and to evaluate the next term (of order $\epsilon^3$) we must take into account the five-loop contributions [18]. So, computational difficulties grow faster than the amount of essential information one may extract from the high-loop approximations. This leads to the conclusion that the $\epsilon$-expansion method is not quite effective for the given model.

Of course, the $\epsilon$-expansion method is perfectly reliable in the four-dimensional space time, i.e., in studying field systems. However, as the present investigation shows, one should be careful when applying it to complicated three-dimensional models of statistical physics, especially if insufficiently high approximations are used.

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The critical exponents of the II-tetragonal fixed point for \( n = 4 \) found by Mukamel and Krinsky \[2\] proved to coincide with those of the Heisenberg one within the two-loop approximation. The three-loop calculations performed in Ref. \[6\] showed them to split apart.

Similar phenomenon was observed earlier in the course of studying the impure Ising model (see Ref. \[19\] and references therein). Half-integer powers in \( \epsilon \) arising in that model have different origin but also lead to the loss of accuracy.

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