An Inventory Model for Imperfect Quality Products with Rework, Distinct Holding Costs, and Nonlinear Demand Dependent on Price

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Abstract: Traditionally, the inventory models available in the literature assume that all articles in the purchased lot are perfect and the demand is constant. However, there are many causes that provoke the presence of defective goods and the demand is dependent on some factors. In this direction, this paper develops an economic order quantity (EOQ) inventory model for imperfect and perfect quality items, taking into account that the imperfect ones are sent as a single lot to a repair shop for reworking. After reparation, the items return to the inventory system and are inspected again. Depending on the moment at which the reworked lot arrives to the inventory system, two scenarios can occur: Case 1: The reworked lot enters when there still exists inventory; and Case 2: The reworked lot comes into when the inventory level is zero. Furthermore, it is considered that the holding costs of perfect and imperfect items are distinct. The demand of the products is nonlinear and dependent on price, which follows a polynomial function. The main goal is to optimize jointly the lot size and the selling price such that the expected total profit per unit of time is maximized. Some theoretic results are derived and algorithms are developed for determining the optimal solution for each modeled case. It is worth mentioning that the proposed inventory model is a general model due to the fact that this contains some published inventory models as particular cases. With the aim to illustrate the use of the proposed inventory model, some numerical examples are solved.

Keywords: pricing; inventory model; lot size; imperfect quality; non-linear demand

1. Introduction

One of the main goals of any company is to have the right levels of inventory in order to satisfy the demands of the customers, but accomplishing this goal is not an easy task. In this direction, the inventory control helps to managers to control the level of inventories. Basically, the inventory models answer the questions of how much and when to order a lot with the aim of always having good inventory levels that can cover the clients’ demand.

The first inventory model was introduced by Harris [1] more than 100 years ago. This inventory model is well-known as the economic order quantity (EOQ) inventory model. Cárdenas-Barrón et al. [2] celebrated the century anniversary of the EOQ with a special issue.

The EOQ inventory model has several restricted assumptions. For example, one supposition is that the lot does not contain imperfect products; in other words, all products are good. In the real world, not all products will be of high quality; a percentage of the products in the lot. There exist several papers that study the effects of the presence of imperfect quality items.
In recent times, the matter of imperfect quality goods has received great attention from numerous academicians and researchers around the world. For instance, Salameh and Jaber [3] developed an inventory model for imperfect quality items. This inventory model assumes that a proportion of the products in the lot will be of imperfect quality. The buyer conducts an inspection process that screens the whole lot and separates the imperfect quality items from the perfect items. At the end of the inspection process, the imperfect quality goods are sold as a single lot with a discount price. Immediately, Cárdenas-Barrón [4] identified and corrected a flaw in the inventory model by Salameh and Jaber [3]. Goyal and Cárdenas-Barrón [5] proposed another formula to calculate the order quantity that is straightforward and easier to implement than the order quantity formulae given in Salameh and Jaber [3].

Chang [6] built an inventory model, taking into account that the percentage of imperfect goods and the demand are fuzzy variables. Papachristos and Konstantaras [7] revealed that the sufficient conditions provided by Salameh and Jaber [3] and Chan et al. [8] to guarantee that stockouts do not happen may possibly not actually avoid their existence. Jaber et al. [9] pointed out that it is needed to take into consideration that the percentage of imperfect items declines due to the learning curve. They used the logistic learning curve to determine the reduction in the amount of imperfect quality goods.

Maddah and Jaber [10] mentioned that the expected total profit in Salameh and Jaber [3] is not accurate. They used the renewal-reward theorem with the aim of having an exact expected total profit. In this sense, Maddah and Jaber [10] improved the inventory model by Salameh and Jaber [3].

It is important to remark that in the mentioned inventory models, the holding cost for imperfect and perfect products is identical. However, in many situations in the real world, these holding costs are totally distinct. In this direction, Wahab and Jaber [11] reformed and resolved the inventory models of Salameh and Jaber [3], Maddah and Jaber [10], and Jaber et al. [9], taking into account that the holding costs for perfect goods and imperfect goods are not the same.

Hsu and Yu [12] considered the situation in which the vendor offers a special price discount with the aim to influence the purchaser to buy a larger lot than the regular order quantity. They studied three situations for when to give the special price discount. In a subsequent paper, Hsu and Yu [13] investigated the effect of an increase in the price at any moment for the period of the planning horizon. They concluded that an increase in the price motives to the purchaser to acquire a larger lot than the normal order.

On the one hand, Yoo et al. [14] and Khan et al. [15] mentioned that the inspector sometimes commits errors during the inspection process. The two well-known types of screening errors are Type I and Type II. In this context, Yoo et al. [14] included these two types of errors and also considered sales return of defective items due to the errors done in the screening process. Khan et al. [15] introduced an EOQ inventory model with imperfect articles, which includes the two well-known types of errors, Type I and Type II, and their associate costs. On the other hand, Khan et al. [16] pointed out that there always exists a learning curve in the inspection process that makes possible that the proportion of imperfect goods is reduced through time. They took into consideration that there exists a transfer of knowledge in learning when the inspection moves from one period to another. This is modeled using a learning curve for three cases: partial transfer of learning, complete transfer of learning, and no transfer of learning.

Lin [17] developed an EOQ inventory model with imperfect quality with quantity discounts, taking into consideration the lot splitting shipments policy. Later, Chang [18] corrected an error in the inventory model by Lin [17]. The error was in the method of computing holding cost. In the inventory models by Chang [18], there exists a discrete variable and the solution procedure to obtain the integer value for the discrete variable is done with inequalities. Afterward, Cárdenas-Barrón [19] presented a simple formula to compute the integral value of the discrete variable without using inequalities. Khan et al. [20] provided an excellent and complete review of extensions of the EOQ inventory model.
with imperfect quality items. Wahab et al. [21] derived the optimal production–shipment policy for products with imperfect quality when the environmental issue is integrated by considering the fixed and variable carbon emission costs. Rezaei and Salimi [22] derived an EOQ inventory model with imperfect quality items that obtains the maximum purchasing price that a buyer is willing to pay to a supplier with the purpose of evading the presence of imperfect items in the lot.

Jaber et al. [23] presented an entropic EOQ inventory model for goods with imperfect quality. The aforementioned research works suppose that at the end of the screening period, the imperfect articles are removed from the inventory and sold at discount price. Jaber et al. [24] reformed the inventory model by Salameh and Jaber [3] by permitting that the imperfect goods can be reworked in a local repair shop or vended on a secondary market. Paul et al. [25] examined the influence of the proportion of imperfect quality units on the ordering policy. Basically, they studied two distinct scenarios: without price discount and with price discount. Zhou et al. [26] constructed economic production quantity (EPQ) inventory models with imperfect quality products when the supplier offers a one-time only discount. Modak et al. [27] introduced an inventory model which determines the optimal just in time buffer, considering preventive maintenance and the occurrence of imperfect products. Taleizadeh et al. [28] studied the ordering and pricing decisions with imperfect quality goods, screening process, and buyback under a supply chain environment. Taking into consideration that the holding costs for perfect and imperfect articles are different, Shekarian et al. [29] introduced an EOQ inventory model with imperfect goods subject to learning and fuzziness. Rezaei [30] incorporated sampling screening plans for imperfect products into an EOQ inventory model. Alamri et al. [31] stated an efficient inventory control for imperfect goods. Basically, they derived an EOQ inventory model for products with imperfect quality in an environment with fluctuating demand, defective articles, an inspection process, and deterioration. Khan et al. [32] investigated the effect of the vendor managed inventory (VMI) policy with consignment stock contract for a sole supplier single purchaser supply chain in which the supplier sends each production lot in a number of lots to the purchaser’s storeroom.

Lin [33] analyzed distinct carbon tax policies on the performance of an inventory model with imperfect quality products in which the purchaser employs power over its vendor. Recently, Rad et al. [34] derived the optimal production and distribution strategies for imperfect products when the demand is dependent on both selling price and advertising. Tiwari et al. [35] formulated a sustainable inventory model with carbon emissions for deteriorating and imperfect items. Kazemi et al. [36] incorporated the carbon emissions on an EOQ inventory model with imperfect articles. Sebatjane and Adetunji [37] integrated the concept of imperfect quality into an EOQ inventory model for growing items. This inventory model contemplates three different growth functions: linear, split and linear, and logistic. Stopková et al. [38] proposed to change the parameters and the objective function of the EOQ, taking into account other significant characteristics that impact the inventory management. Mashud et al. [39] introduced a sustainable inventory model for imperfect items, considering deterioration and carbon emissions. Recently, Pando et al. [40] developed an inventory model in the case when the demand is stock-dependent under the approach of maximizing the return on investment.

This research work develops an EOQ inventory model for imperfect and perfect quality items, considering that the imperfect ones are sent as a sole batch to a repair shop for reparation. After reworking, the items arrive at the inventory system and are screened again. Depending on the instant at which the reworked batch comes to the inventory system, two cases happen: Case 1: The reworked batch enters when there still exists inventory; Case 2: The reworked batch arrives into when the inventory level is zero. Moreover, it is taken into account that the holding costs of perfect and imperfect items are distinct. The demand of the items is nonlinear and dependent on price according to a polynomial function. The principal objective is to optimize jointly the lot size and the selling price such that the expected total profit per unit of time is maximized.
The rest of this paper is planned as follows. Section 2 introduces the assumptions and notation. Section 3 derives and optimizes the inventory model for imperfect quality products with rework and nonlinear demand dependent on time. Section 4 identifies and presents some special cases. Section 5 solves some numerical examples. Finally, Section 6 presents some conclusions and future research lines.

2. Assumptions and Notation

This section introduces the assumptions and notation for the development of the EOQ inventory model with distinct holding costs for imperfect and perfect quality items.

2.1. Assumptions

The EOQ inventory model with distinct holding costs for imperfect and perfect quality goods is based on the following suppositions:

1. The lot has a proportion of imperfect quality items.
2. The proportion of imperfect quality items follows a known probability density function.
3. A 100% inspection process is done with the intention of identifying the imperfect and perfect quality items and split them.
4. The imperfect quality items are sent as a sole lot to be reworked, after which they are returned.
5. The reworked items return to the inventory system as a single batch when there exists inventory or when the inventory reaches zero.
6. The lot of reworked items contains a proportion of imperfect items, and this proportion has a known probability density function.
7. The proportion of imperfect items in the lot and the reworked lot are independent random variables.
8. The holding costs of the imperfect and perfect quality items are different.
9. The demand is nonlinear and depends on the selling price.
10. The inspection process and demand happen at the same time. The inspection rate is greater than the demand rate.
11. Shortages are not allowed.

2.2. Notation

The inventory model is developed using the following notation.

Parameters:

- \( k \) = Ordering cost ($/order)
- \( c \) = Purchasing cost ($/unit)
- \( h_g \) = Holding cost for perfect items ($/unit/unit of time)
- \( h_d \) = Holding cost for imperfect items ($/unit/unit of time)
- \( d \) = Inspection cost ($/unit)
- \( c_r \) = Rework cost ($/unit)
- \( \nu \) = Selling price for an item of imperfect quality after reworking ($/unit)
- \( x \) = Screening rate (units/unit of time)
- \( p \) = Percentage of imperfect items in the lot \( y \) (%) \( (p) \)
- \( \theta \) = Percentage of imperfect items in the lot reworked \( \theta \) (%) \( (\theta) \)
- \( f(p) \) = Probability density function of \( p \)
- \( f(\theta) \) = Probability density function of \( \theta \)
- \( py \) = Lot size of imperfect items to send to rework (units)
- \( \theta py \) = Imperfect items to sell as a single lot at discount price (units)
- \( D(s) \) = Polynomial function for price-dependent demand; \( D(s) = \alpha - \beta s^n \) (units/unit of time)
- \( \alpha \) = Scale parameter for the price-dependent demand
- \( \beta \) = Sensitivity parameter for the price-dependent demand
- \( n \) = Power index
- \( t_1 \) = Inspection time of lot \( y \) (units of time)
- \( t_2 \) = Inspection time of the reworked lot \( \theta py \) (units of time)
\( t_r = \text{Reworking time (units of time)} \)
\( T = \text{Cycle time (units of time)} \)

Decision variables:
\( y = \text{Lot size (units)} \)
\( s = \text{Selling price for an item of perfect quality ($/unit)} \)

3. Development of the EOQ Inventory Model with Rework, Distinct Holding Costs for Imperfect and Perfect Quality Products, and Nonlinear Demand Dependent on Price

It is supposed that delivery of the lot is done in instantaneous manner. For that reason, at the beginning of the period, a lot size of \( y \) units arrives at the company. Instantly, a 100% screening process of the lot is done at an inspection rate of \( x \) during \( t_1 \). The aim of the inspection process is to detect the imperfect items, which are split from the perfect ones. The imperfect items are placed in a warehouse place with a holding cost of \( h_d \). When the inspection process is completed, the imperfect items \( (py) \) are removed from the warehouse and are sent as a single lot to a repair shop for reworking. After the reworking time \( (t_r) \), the lot of \( py \) units returns to the company, and immediately, the reworked lot is inspected and the imperfect items are separated from the good ones. After the inspection time \( t_2 \), the imperfect units are sold as a single batch at discount price \( v \). There exist two possible cases, depending on the time at which the reworked lot \( (py) \) enters the inventory system: Case 1. The reworked lot \( (py) \) arrives when there exists inventory, and Case 2. The reworked lot \( (py) \) enters when the inventory level is zero. These two cases are illustrated in Figures 1 and 2, respectively. Without loss of generality, and for simplicity, \( D \) is used instead of \( D(s) \).

![Figure 1](image-url). Inventory level through time for Case 1. The reworked lot arrives when there exists inventory.
For the lot $y$ the following is stated:

$N_1(y, p)$ represents the number of perfect items in each lot $y$, which is determined with:

$$N_1(y, p) = y - py = y(1 - p)$$

(1)

To ensure that there are no shortages, the amount of perfect goods must be greater than or equal to the demand during the screening time $t_1$. This is written as follows:

$$N_1(y, p) \geq Dt_1$$

(2)

The screening time of lot $y$ is computed with $t_1 = \frac{y}{x}$. From Equations (1) and (2), and by replacing $t_1$, the following restriction is obtained:

$$p \leq 1 - \frac{D}{x}$$

(3)

For the reworked lot $py$ the following is established:

The $N_2(py, \theta)$ indicates the number of perfect items in each reworked lot $py$ and it is obtained with:

$$N_2(py, \theta) = py - \theta py = py(1 - \theta)$$

(4)

To avoid shortages, the quantity of perfect products must be greater or equal to the demand during the screening time $t_2$. This is written as follows:

$$N_2(py, \theta) \geq Dt_2$$

(5)
The screening time is computed with $t_2 = \frac{py}{\theta}$. From Equations (4) and (5), and by replacing $t_2$, the following constraint is found:

$$\theta \leq 1 - \frac{D}{x}$$

(6)

3.1. Case 1. The Reworked Lot ($py$) Arrives When There Exists Inventory

Here, the inspection time of reworked lot $py$ is $t_2 = \frac{py}{\theta}$. The time $t_3$ is given by $t_3 = \frac{y - Dt_1 - Dh_D - D_2 - \theta py}{\theta}$. TR($y$) and TC($y$) denote the total revenue and the total cost per cycle, respectively. The total revenue TR($y$) is the sum of the revenue from perfect goods in the lot $y$, the revenue from perfect items in the reworked lot $py$, and the revenue of imperfect items, which is computed as:

$$TR(y) = ys(1 - p) + s py(1 - \theta) + \nu(\theta py)$$

(7)

The total cost TC($y$) is the sum of ordering cost $k$, purchasing cost $cy$, reworking cost $c_rpy$, inspection cost $dy(1 + p)$, holding cost of perfect items $H_g(y)$, and holding cost of imperfect items $H_d(y)$.

$H_g(y)$ and $H_d(y)$ are calculated in the following mode. $H_g(y)$ is computed with the multiplication of the holding cost $h_g$ and the total inventory during the cycle, which is determined by the sum of the areas: $\sum_{i=1}^{7} A_i$ in Figure 1. Where: $A_1 = \frac{(D_1 + py)t_1}{2}$, $A_2 = (y - Dt_1 - py)t_1$, $A_3 = \frac{D_2^2}{2}$, $A_4 = (y - Dt_1 - Dt_2 - py)t_r$, $A_5 = \frac{(D_3 + \theta py)t_2}{2}$, $A_6 = (y - Dt_1 - Dt_r - Dt_2 - \theta py)t_2$, and $A_7 = \frac{(T - t_{1} - t_{2} - t_{3})}{2}$.

As a result:

$$H_g(y) = h_g \left\{ \left[ p^2 \left( \frac{\theta}{2x} + \frac{\theta^2}{2D} \right) + p \left( \frac{\theta}{x} - \frac{1}{2x} - \frac{\theta}{D} \right) + \frac{1}{2D} \right]y^2 + [\theta - 1]pyt_r \right\}$$

(8)

$H_d(y)$ is found by the multiplication of $h_d$ and the sum of the areas: $\sum_{i=1}^{9} A_i$ in Figure 1, and it is obtained with:

$$H_d(y) = h_d \left( \frac{py^2}{2x} + \frac{\theta py^2}{2x} \right)$$

(9)

Thus, the total cost per cycle TC($y$) is:

$$TC(y) = k + cy + c_rpy + dy(1 + p) + h_g \left\{ \left[ p^2 \left( \frac{\theta}{2x} + \frac{\theta^2}{2D} \right) + p \left( \frac{\theta}{x} - \frac{1}{2x} - \frac{\theta}{D} \right) + \frac{1}{2D} \right]y^2 + [\theta - 1]pyt_r \right\}$$

(10)

The total profit per cycle, TP($y, B$), is determined as the total revenue per cycle, TR($y$), minus the total cost per cycle, TC($y$):

$$TP(y, B) = s[y(1 - p) + yp(1 - \theta)] + \nu(\theta py) - \left[ k + cy + c_rpy + dy(1 + p) + h_g \left\{ \left[ p^2 \left( \frac{\theta}{2x} + \frac{\theta^2}{2D} \right) + p \left( \frac{\theta}{x} - \frac{1}{2x} - \frac{\theta}{D} \right) + \frac{1}{2D} \right]y^2 + [\theta - 1]pyt_r \right\} \right]$$

(11)

It is important to remark that the proportion of imperfect items ($p$) in the lot $y$ and the proportion of imperfect items ($\theta$) in the reworked lot $py$ are independent random variables. Note that the cycle time $T$ depends on both $p$ and $\theta$. Therefore, the cycle time is also a random variable, which is expressed below:

$$E[T] = \frac{y(1 - E[\theta]E[p])}{D}$$

(12)
By using the renewal theorem, the expected total profit per unit of time is given by:

\[
E[T PU(y)] = \frac{E[TPU(y)]}{E[T]}
\]

(13)

\[
E[T PU(s, y)] = Ds + \frac{D}{1 - E[p|E[y]]} \left[ v_E[p|E[y]] - c - c_E[p] - d(1 + E[p]) - \frac{k}{y} \right] - \frac{h}{2} \left( 1 - E[p|E[y]] \right) y
\]

(14)

The demand is according to a polynomial function which is influenced by the selling price:

\[
D(s) = \alpha - \beta s^n
\]

(15)

Thus, substituting \( D \) by \( D(s) = \alpha - \beta s^n \) into Equation (14) gives:

\[
E[T PU(s, y)] = (\alpha - \beta s^n) s + \frac{(\alpha - \beta s^n)}{1 - E[p|E[y]]} \left[ v_E[p|E[y]] - c - c_E[p] - d(1 + E[p]) - \frac{k}{y} \right] - \frac{h}{2} \left( 1 - E[p|E[y]] \right) y
\]

(16)

The aim is to maximize the expected total profit per unit of time \( E[T PU(s, y)] \). Consequently, the constrained optimization problem is defined below:

\[
Max_{(s,y) \in \Phi} E[T PU(s, y)]
\]

where \( \Phi = \left\{ (s, y) : y > 0, \text{ and } c \leq s \leq \left( \frac{a}{b} \right)^{\frac{1}{n}} \right\} \)

Note that the above optimization problem is highly nonlinear.

Calculating the first and second order derivatives of \( E[T PU(s, y)] \) with respect to \( s \):

\[
\frac{\partial E[T PU(s, y)]}{\partial s} = (\alpha - \beta s^n) - \frac{n \beta s^{n-1}}{1 - E[p|E[y]]} - \frac{n \beta s^{n-1}}{1 - E[p|E[y]]} \left[ v_E[p|E[y]] - c - c_E[p] - d(1 + E[p]) - \frac{k}{y} \right] - \frac{h}{2} \left( 1 - E[p|E[y]] \right) y = 0
\]

(17)

\[
\frac{\partial^2 E[T PU(s, y)]}{\partial s^2} = -n \beta s^{n-1} - n \beta s^{n-1} - (n-1)n \beta s^{n-2} \left[ v_E[p|E[y]] - c - c_E[p] - d(1 + E[p]) - \frac{k}{y} \right] - \frac{h}{2} \left( 1 - E[p|E[y]] \right) y
\]

\[
\frac{\partial^2 E[T PU(s, y)]}{\partial y^2} = \frac{2k(\alpha - \beta s^n)}{(1 - E[p|E[y]])^2} \left\{ \left( \frac{\alpha - \beta s^n}{x} \right) \left[ v_E[p|E[y]] - c - c_E[p] - d(1 + E[p]) - \frac{k}{y} \right] + (1 - E[p|E[y]])^2 \right\} = 0
\]

(19)

Computing the first and second order derivatives of \( E[T PU(s, y)] \) with respect to \( y \):

\[
\frac{\partial E[T PU(s, y)]}{\partial y} = \frac{2k(\alpha - \beta s^n)}{(1 - E[p|E[y]])^2} \left\{ \left( \frac{\alpha - \beta s^n}{x} \right) \left[ v_E[p|E[y]] - c - c_E[p] - d(1 + E[p]) - \frac{k}{y} \right] + (1 - E[p|E[y]])^2 \right\} = 0
\]

(18)

\[
\frac{\partial^2 E[T PU(s, y)]}{y^2} = -\frac{2k(\alpha - \beta s^n)}{(1 - E[p|E[y]])^3} = M_1
\]

(20)
And:

\[
\frac{\partial^2 E[TPU(s,y)]}{\partial y \partial s} = \frac{\partial^2 E[TPU(s,y)]}{\partial s \partial y} = - \frac{n \beta s^{n-1}}{(1 - E[p]E[\theta])} \left[ \frac{k}{p^2} \left( E[p^2E[\theta] + 2E[pE[\theta] - E[p]] \right) \right] = Q_1
\]

In order to ensure the optimal solution, the following conditions must be met: \( L_1 < 0 \), \( M_1 < 0 \) and \( L_1M_1 - Q_1^2 > 0 \). Notice that \( M_1 \) is always negative. The condition \( L_1 < 0 \) is satisfied when:

\[
s \geq - \frac{(n - 1)}{(n + 1)(1 - E[p]E[\theta])} \left[ \nu E[p]E[\theta] - c - \beta c E[p] - d(1 + E[p]) - \frac{c}{y} \right]
\]

The optimal lot size is obtained by solving Equation (19) for \( y \). Thus:

\[
y = \sqrt{\frac{2k(\alpha - \beta s^n)}{h_k \left( \left( \frac{\alpha - \beta s^n}{x} \right) (E[p^2E[\theta] + 2E[pE[\theta] - E[p]] + (1 - E[p]E[\theta] E[p])^2 \right) + h_d \left( \frac{\alpha - \beta s^n}{x} \right) (E[p] + E[p^2E[\theta]) \right) \right)}
\]

The optimal selling price \( s \) and the lot size \( y \) are determined by solving simultaneously Equations (17) and (22) for \( s \) and \( y \). Let \( s_0 \) and \( y_0 \) be the solution of Equations (17) and (22), if \( c \leq s_0 \leq \left( \frac{E[p]}{\beta} \right) \) then \( s^* = s_0 \). If \( s_0 < c \) then set \( s^* = c \). Otherwise, set \( s^* = \left( \frac{E[p]}{\beta} \right)^\frac{1}{2} \). Taking into account the theoretical results obtained above then the following theorem is stated.

**Theorem 1.** If \( L_1 < 0, M_1 < 0 \) and \( L_1M_1 - Q_1^2 > 0 \), then the Hessian matrix associated with the function \( E[TPU(s,y)] \) is negative. Therefore, \( E[TPU(s,y,B)] \), given by Equation (16), is a strictly concave function in \( s \) and \( y \). Thus, there exists a unique optimum.

Note that \( M_1 \) is always negative. Thus, it is enough to demonstrate \( L_1 < 0 \) and \( L_1M_1 - Q_1^2 > 0 \).

With the above theoretic results, the following algorithm is developed for finding the optimal solution. A flow chart of Algorithm 1 is given in Figure 3.
Figure 3. Flow chart of Algorithm 1.
Algorithm 1 Case 1

Step 1. Input the data of the inventory system.
Step 2. Calculate the selling price \( s \) and lot size \( y \) by solving simultaneously Equations (17) and (22).
Step 3. Set \( s_0 \) and \( y_0 \) to be the solutions of Equations (17) and (22); if \( c \leq s_0 \leq \left( \frac{\theta}{\beta} \right)^\frac{1}{2} \), then \( s^* = s_0 \), \( y^* = y_0 \) and go to Step 5. If \( s_0 < c \), then set \( s^* = c \) and go to Step 4. Otherwise, set \( s^* = \left( \frac{\theta}{\beta} \right)^\frac{1}{2} \) and go to Step 4.
Step 4. Compute the lot size \( y^* \) with Equation (22).
Step 5. Determine the expected total profit per unit of time \( E[TPU(s, y)] \) with Equation (16).
Step 6. If \( L_1 < 0 \) and \( L_1 M_1 - Q_1^2 > 0 \), then the solution is optimal. Otherwise, the solution is suboptimal.
Step 7. Report the solution as \( \{s^*, y^*, E[TPU(s^*, y^*)]\} \)
Step 8. Stop.

3.2. Case 2. The Reworked Lot (\( py \)) Enters When the Inventory Level Is Zero

Here, the screening time of lot \( y \) is computed with \( t_1 = \frac{\theta}{2} \). The inspection time of reworked lot \( py \) is \( t_2 = \frac{py}{\theta} \). The time \( t_3 \) is given by \( t_3 = \frac{py - D t_1 - \theta p y}{\theta} \). Note that in this case, the reworked lot (\( py \)) arrives to the inventory system when the inventory level is zero, which implies that the reworking time of the lot \( py \) is \( t_r = \frac{py - D t_1 + py}{\theta} \).

The total cost \( TC(y) \) is calculated in the same manner as in Case 1. It is obtained by the sum of ordering cost \( k \), purchasing cost \( cy \), reworking cost \( c_r py \), inspection cost \( dy(1 + p) \), holding cost of perfect items \( H_g(y) \), and holding cost of imperfect items \( H_d(y) \).

\( H_g(y) \) and \( H_d(y) \) are determined in the following manner. The \( H_g(y) \) is obtained with the multiplication of the holding cost \( h_g \) and the total inventory during the cycle, which is the sum of the areas: \( \sum_{i=1}^{6} A_i \) in Figure 2. Where: \( A_1 = \frac{(D t_1 + p y) t_1}{2} \), \( A_2 = (y - D t_1 - p y) t_1 \), \( A_3 = \frac{(y - D t_1 - p y) (y - D t_1 - p y)}{2} \), \( A_4 = \frac{(D t_2 + \theta p y) t_2}{2} \), \( A_5 = (p y - D t_2 - \theta p y) t_2 \), and \( A_6 = \frac{(p y - D t_2 - \theta p y)^2}{2} \).

Therefore, the holding cost for perfect items is:

\[
H_g(y) = h_g \left[ p^2 \left( \frac{1}{D} + \frac{\theta}{2x} - \frac{\theta}{D} + \frac{\theta^2}{2D} \right) + p \left( \frac{1}{2x} - \frac{1}{D} \right) + \frac{1}{2D} \right] y^2
\]

(23)

\( H_d(y) \) is computed by the multiplication of \( h_d \) and the sum of the areas: \( \sum_{i=7}^{8} A_i \) in Figure 2, which is found with:

\[
H_d(y) = h_d \left( \frac{p y^2}{2x} + \frac{\theta p y^2}{2x} \right)
\]

(24)

The total cost \( TC(y) \) is:

\[
TC(y) = k + cy + c_r p y + dy(1 + p) + h_g \left[ p^2 \left( \frac{1}{D} + \frac{\theta}{2x} - \frac{\theta}{D} + \frac{\theta^2}{2D} \right) + p \left( \frac{1}{2x} - \frac{1}{D} \right) + \frac{1}{2D} \right] y^2 + h_d \left( \frac{p y^2}{2x} + \frac{\theta p y^2}{2x} \right)
\]

(25)

The total profit per cycle, \( TP(y, B) \), is obtained as the total revenue per cycle, \( TR(y) \), minus the total cost per cycle, \( TC(y) \):

\[
TP(y) = s[y(1 - p) + yp(1 - \theta)] + \nu \theta p y - k + cy + c_r p y + dy(1 + p) + h_g \left[ p^2 \left( \frac{1}{D} + \frac{\theta}{2x} - \frac{\theta}{D} + \frac{\theta^2}{2D} \right) + p \left( \frac{1}{2x} - \frac{1}{D} \right) + \frac{1}{2D} \right] y^2 + h_d \left( \frac{p y^2}{2x} + \frac{\theta p y^2}{2x} \right)
\]

(26)
By applying the renewal theorem, the expected total profit per unit of time is given by:

$$E[\text{TPU}(y)] = E[\text{TPU}(y)]$$  \hspace{1cm} (27)

$$E[\text{TPU}(s,y)] = Ds + \frac{D}{1 - E[p|\Theta]} \left[ vE[p|\Theta] - c - c_re[p] - d(1 + E[p]) - \frac{k}{y} \right]$$  \hspace{1cm} (28)

Thus, replacing $D$ by $D(s) = \alpha - \beta s^n$ into Equation (26), gives:

$$E[\text{TPU}(s,y)] = (\alpha - \beta s^n)s + \frac{(\alpha - \beta s^n)}{1 - E[p|\Theta]} \left[ vE[p|\Theta] - c - c_re[p] - d(1 + E[p]) - \frac{k}{y} \right]$$  \hspace{1cm} (29)

Determining the first and second order derivatives of $E[\text{TPU}(s,y)]$ with respect to $s$ and $y$:

$$\frac{\partial E[\text{TPU}(s,y)]}{\partial s} = (\alpha - \beta s^n) - sn\beta s^{n-1} - \frac{n\beta s^{n-1}}{1 - E[p|\Theta]} \left[ vE[p|\Theta] - c - c_re[p] - d(1 + E[p]) - \frac{k}{y} \right] = 0$$  \hspace{1cm} (30)

$$\frac{\partial^2 E[\text{TPU}(s,y)]}{\partial s^2} = -n\beta s^{n-1} - n^2\beta s^{n-2} - \frac{(n-1)n\beta s^{n-2}}{1 - E[p|\Theta]} \left[ vE[p|\Theta] - c - c_re[p] - d(1 + E[p]) - \frac{k}{y} \right] = L_2$$  \hspace{1cm} (31)

$$\frac{\partial E[\text{TPU}(s,y)]}{\partial y} = \frac{k(\alpha - \beta s^n)}{1 - E[p|\Theta]} - \frac{b_s}{1 - E[p|\Theta]} \left[ \frac{\alpha - \beta s^n}{k} + \frac{(h_s + h_d)}{2} \right] \left[ (E[p^2|\Theta] + E[p]) \right] - \frac{b_s}{1 - E[p|\Theta]} \left[ E^2[p|\Theta] + E[p^2|\Theta] \left( 1 - \theta^2 \right) \right] = 0$$  \hspace{1cm} (32)

$$\frac{\partial^2 E[\text{TPU}(s,y)]}{\partial y^2} = -\frac{2k(\alpha - \beta s^n)}{(1 - E[p|\Theta])y^2} = M_2$$  \hspace{1cm} (33)

And:

$$\frac{\partial E[\text{TPU}(s,y)]}{\partial s} = \frac{\partial E[\text{TPU}(s,y)]}{\partial y} = -\frac{n\beta s^{n-1}}{1 - E[p|\Theta]} \left[ k - \frac{(h_s + h_d)}{2x} \left( E[p^2|\Theta] + E[p^2|\Theta] \right) \right] = Q_2$$  \hspace{1cm} (34)

With the aim of guaranteeing the optimal solution, the following conditions must be met: $L_2 < 0$, $M_2 < 0$ and $L_2M_2 - Q_2^2 > 0$. Notice that $M_2$ is always negative. The condition $L_2 < 0$ is satisfied when:

$$s \geq -\frac{(n - 1)}{(n + 1)(1 - E[p|\Theta])} \left[ vE[p|\Theta] - c - c_re[p] - d(1 + E[p]) - \frac{k}{y} \right]$$  \hspace{1cm} (32)
The optimal lot size is obtained by solving Equation (30) for $y$. Therefore:

$$y = \sqrt{\frac{2k(a - \beta s^*)}{h_g(E[(1-p)^2] + E[p^2]E[(1-\theta)^2]) + (h_g + h_d)\left(\frac{a-\beta s^*}{\alpha}\right)(E[\theta]E[p^2] + E[p])}}$$  \hspace{1cm} (35)

The optimal selling price $s$ and the lot size $y$ are determined by simultaneously solving Equations (30) and (35) for $s$ and $y$. Let $s_0$ and $y_0$ be the solution of Equations (30) and (35), if $c \leq s_0 \leq \left(\frac{a}{\beta}\right)^{\frac{1}{2}}$, then $s^* = s_0$. If $s_0 < c$, then set $s^* = c$. Otherwise, set $s^* = \left(\frac{a}{\beta}\right)^{\frac{1}{2}}$. Considering the theoretical results derived above, the following theorem is proposed.

**Theorem 2.** If $L_2 < 0$, $M_2 < 0$, and $L_2M_2 - Q_2^2 > 0$, then the Hessian matrix associated with the function $E[TPU(s, y)]$ is negative. Therefore, the $E[TPU(s, y)]$ given by Equation (29) is strictly concave function in $s$ and $y$. Thus, there exists a sole optimum.

Note that $M_2$ is always negative. Thus, it is enough to demonstrate that $L_2 < 0$ and $L_2M_2 - Q_2^2 > 0$.

To obtain the optimal solution for Case 2, the following algorithm is proposed. A flow chart of Algorithm 2 is given in Figure 4.

**Algorithm 2 Case 2**

**Step 1.** Provide the data of the inventory system.
**Step 2.** Obtain the selling price $s$ and lot size $y$ by simultaneously solving Equation (30) and Equation (35).

**Step 3.** Set $s_0$ and $y_0$ to be the solution of Equation (30) and Equation (35); if $c \leq s_0 \leq \left(\frac{a}{\beta}\right)^{\frac{1}{2}}$, then $s^* = s_0$, $y^* = y_0$, and go to **Step 5**. If $s_0 < c$, then set $s^* = c$ and go to **Step 4**. Otherwise, set $s^* = \left(\frac{a}{\beta}\right)^{\frac{1}{2}}$ and go to **Step 4**.

**Step 4.** Find the lot size $y^*$ with Equation (35).
**Step 5.** Calculate the expected total profit per unit of time $E[TPU(s, y)]$ with Equation (29).
**Step 6.** If $L_2 < 0$ and $L_2M_2 - Q_2^2 > 0$, then the solution is optimal. Otherwise, the solution is suboptimal.
**Step 7.** Report the solution as \{\(s^*, y^*, E[TPU(s^*, y^*)]\)\}
**Step 8.** Stop.
4. Special Cases

It is significant to mention that the proposed inventory model in this research work is a general inventory model due to the fact that some special cases are obtained from it. In this direction, this section identifies and presents some special cases.

Figure 4. Flow chart of Algorithm 2.
(i) \( n = 1 \) signifies that the product has a linear price dependent demand \( (D(s) = \alpha - \beta s) \). Thus, the corresponding equations for the expected total profit, selling price and lot size for this special case are expressed below. For Case 1:

\[
E[TPU(s,y)] = (\alpha - \beta s) + \frac{e^{-\beta s}}{1-E[p]\theta} \left[ vE[p]E[\theta] - c - c_E[p] - d(1 + E[p]) + \frac{k}{2} \left( \frac{h}{2} \{ E[p^2]E[\theta] + 2E[p]E[\theta] - E[p] \} y + 2xE[p]E[\theta]r - 2xE[p]r \} \right) \right] - \frac{h}{2} (1 - E[p]E[\theta]) y
\]

\[
(a - \beta s) - s\beta - \frac{\beta}{(1-E[p]E[\theta])} \left[ vE[p]E[\theta] - c - c_E[p] - d(1 + E[p]) - \frac{k}{2} \left( \frac{h}{2} \{ E[p^2]E[\theta] + 2E[p]E[\theta] - E[p] \} y + 2xE[p]E[\theta]r - 2xE[p]r \} \right) \right] = 0
\]

\[
y = \sqrt{h \left\{ \frac{a - \beta s}{\alpha} (E[p^2]E[\theta] + 2E[p]E[\theta] - E[p]) + (1 - E[p]E[\theta])^2 \right\} + h_d \left( \frac{a - \beta s}{\alpha} (E[p^2]E[\theta]) \right)}
\]

For Case 2:

\[
E[TPU(s,y)] = (\alpha - \beta s) + \frac{e^{-\beta s}}{1-E[p]\theta} \left[ vE[p]E[\theta] - c - c_E[p] - d(1 + E[p]) - \frac{k}{2} \left( \frac{h}{2} \{ E[p^2]E[\theta] + 2E[p]E[\theta] - E[p] \} y + 2xE[p]E[\theta]r - 2xE[p]r \} \right) \right] - \frac{h}{2} (1 - E[p]E[\theta]) y
\]

\[
(a - \beta s) - s\beta - \frac{\beta}{(1-E[p]E[\theta])} \left[ vE[p]E[\theta] - c - c_E[p] - d(1 + E[p]) - \frac{k}{2} \left( \frac{h}{2} \{ E[p^2]E[\theta] + 2E[p]E[\theta] - E[p] \} y + 2xE[p]E[\theta]r - 2xE[p]r \} \right) \right] = 0
\]

\[
y = \sqrt{h \left( E \left[ (1 - p)^2 \right] + E \left[ p^2 \right] E \left( 1 - \theta \right)^2 \right) + h_d \left( \frac{a - \beta s}{\alpha} (E[\theta]E[p^2] + E[p]) \right)}
\]

(ii) When the imperfect items are sold as a single batch, there exists a known and constant demand \( (i.e., \beta = 0) \), and when the selling price is given as input parameter, the inventory model of Wahab and Jaber [11] can be used. The corresponding equations for the expected total profit and lot size are as follows:

\[
E[TPU(y)] = Ds + \frac{D}{1-E[p]} \left[ vE[p] - c - \frac{k}{2} y \right] - \left( \frac{h}{2} \{ E \left[ (1 - p)^2 \right] \} \right) y
\]

\[
y = \sqrt{\frac{2kD}{h \left( E \left[ (1 - p)^2 \right] + (h + h_d) \left( \frac{D}{E[p]} \right) \right)}}
\]

(iii) When the imperfect items are sold as a single batch, there exists a known and constant demand \( D (i.e., \beta = 0) \), and when the selling price is given as an input parameter and the holding cost for perfect and imperfect items are the same (i.e., \( h = h_d = h \)), the inventory model of Maddah and Jaber [10] can be used. Notice that Maddah and Jaber [10] is an improved inventory model of Salameh and Jaber [3]. The corresponding equations for the expected total profit and lot size are as follows:

\[
E[TPU(y)] = Ds + \frac{D}{1-E[p]} \left[ vE[p] - c - \frac{k}{2} y \right] - \left( \frac{h}{2} \{ E \left[ (1 - p)^2 \right] \} \right) y
\]

\[
y = \sqrt{\frac{2kD}{h \left( E \left[ (1 - p)^2 \right] + 2 \left( \frac{D}{E[p]} \right) E[p] \right)}}
\]
(iv) When the imperfect items are not sold as a single batch, there exists a known and constant demand $D$ (i.e., $\beta = 0$); the selling price is not considered as an input parameter and the holding cost for perfect and imperfect items are the same (i.e., $h_g = h_d = h$). In this case, the inventory models of Silver [41] and Shi [42] can be used. The corresponding equations for the total cost and lot size are given below:

$$TC(y) = \frac{D}{(1-E[p])} \left[ c + \frac{k}{y} + \left( \frac{h}{2(1-E[p])} \right) (E[(1-p)^2])y \right]$$

$$y = \sqrt{\frac{2kD}{hE[(1-p)^2]}}$$

(v) When there exists a known and constant demand $D$ (i.e., $\beta = 0$), the selling price is not considered as an input parameter, and all items are of perfect quality (i.e., $p \to 0$), the inventory model of Harris [1] can be used. The corresponding equations for the total cost and lot size are presented below:

$$TC(y) = D \left[ c + \frac{k}{y} \right] + \left( \frac{h}{2} \right) y$$

$$y = \sqrt{\frac{2kD}{h}}$$

5. Numerical Examples

This section solves some numerical examples with the intention of illustrate the applicability of the proposed inventory model.

Example 1. The values for the parameters of this numerical example are as follows: $c = $100 per unit, $k = $3000 per order, $v = $50 per unit, $x = 100,200$ units per unit of time, $d = $0.5 per unit, $h_g = $20 per unit per unit of time, $h_d = $5 per unit per unit of time. These data are obtained from Wahab and Jaber [11]. Additional data are necessary: $c_r = $40 per unit, $\alpha = 12,000$, $\beta = 20$, $t_r = 0.0125$ units of time, and the proportion of imperfect items follows an uniform probability density as $U \sim [a,b]$. Where $f(p)$ is defined as:

$$f(p) = \begin{cases} \frac{1}{b-a} & a \leq p \leq b \\ 0 & \text{otherwise} \end{cases}$$

The $E[p], E[p^2]$ and $E[(1-p)^2]$ are computed as follows:

$$E[p] = \int_a^b pf(p)dp = \frac{a+b}{2}, E[p^2] = \int_a^b p^2f(p)dp = \frac{a^2+ab+b^2}{3}, \text{and} E[(1-p)^2] = \int_a^b (1-p)^2f(p)dp = \frac{a^2+ab+b^2}{3} + 1 - a - b.$$

Considering the following:

$$p \sim f(p) = \begin{cases} 50 & 0 \leq p \leq 0.02 \\ 0 & \text{otherwise} \end{cases}$$

$$\theta \sim f(\theta) = \begin{cases} 10 & 0 \leq \theta \leq 0.1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[p] = \frac{0 + 0.02}{2} = 0.01, E[p^2] = \frac{0^2 + 0(0.02) + 0.02^2}{3} = 0.000133333333,$$

$$E[(1-p)^2] = \frac{0^2 + 0(0.02) + 0.02^2}{3} + 1 - 0 - 0.02 = 0.980133333333$$

$$E[\theta] = \frac{0 + 0.1}{2} = 0.05, E[\theta^2] = \frac{0^2 + 0(0.1) + 0.1^2}{3} = 0.0033333333.$$
\[ E \left[ (1 - \theta)^2 \right] = \frac{0^2 + 0(0.1) + 0.1^2}{3} + 1 - 0 - 0.1 = 0.901333333333 \]

The inventory model is solved for different values of \( n = \{0.5, 0.75, 1\} \). Applying the Algorithm for Case 1, the optimal solution for each value of \( n \) is given in Table 1.

**Table 1.** Optimal solution for Example 1 for \( n = \{0.5, 0.75, 1\} \).

| \( n \) | \( s \)    | \( y \) | \( E[TPU(s, y)] \) |
|------|------|------|------------------|
| 0.5  | 160,069.1 | 1095.898 | 639,574,471,125,697 |
| 0.75 | 2458.558  | 1227.64  | 11,803,761,231,642  |
| 1    | 351.6924  | 1221.399 | 1,220,925,769,487   |

**Example 2.** The data are as follows: \( c = \$15 \) per unit, \( k = \$30 \) per order, \( v = \$6 \) per unit, \( x = 100,200 \) units per unit of time, \( d = \$0.5 \) per unit per unit of time, \( h_k = \$10 \) per unit per unit of time, \( h_d = \$5 \) per unit per unit of time, \( c_r = \$2 \) per unit, \( \alpha = 12,000 \), \( \beta = 20 \), and \( t_r = 0.01 \) units of time. The inventory model is resolved for distinct values of \( n = \{1, 1.5, 2\} \). Using the Algorithm for Case 1, the optimal solution for each value of \( n \) is depicted in Table 2.

**Table 2.** Optimal Solution for Example 2 for \( n = \{1, 1.5, 2\} \).

| \( n \) | \( s \)    | \( y \) | \( E[TPU(s, y)] \) |
|------|------|------|------------------|
| 1    | 307.8445 | 187.3551 | 1,706,160,527,913 |
| 1.5  | 45.13585 | 188.8268 | 173,838,472,462   |
| 2    | 20.32579 | 149.8307 | 16,430,138,762    |

**Example 3.** The values for parameters are the same as in Example 1, but now the time of reworking \( t_r \) is of such a length that the reworked lot returns when the inventory level reaches zero. Employing the Algorithm for Case 2, the optimal solution for each value of \( n \) is shown in Table 3.

**Table 3.** Optimal solution for Example 3 for \( n = \{0.5, 0.75, 1\} \).

| \( n \) | \( s \)    | \( y \) | \( E[TPU(s, y)] \) |
|------|------|------|------------------|
| 0.5  | 160,069.1 | 1105.903 | 639,574,659,8     |
| 0.75 | 2458.549  | 1238.728 | 11,803,968,86     |
| 1    | 351.6837  | 1232.457 | 1,221,132,525     |

**Example 4.** Here, the reworking time \( t_r \) is of such a length that the reworked lot arrives when the inventory level attains zero, and the data of input parameters are as in Example 2. Utilizing the Algorithm for Case 2, the optimal solution for each value of \( n \) is represented in Table 4.

**Table 4.** Optimal Solution for Example 4 for \( n = \{1, 1.5, 2\} \).

| \( n \) | \( s \)    | \( y \) | \( E[TPU(s, y)] \) |
|------|------|------|------------------|
| 1    | 307.8443 | 189.0317 | 1,706,171,579     |
| 1.5  | 45.13573 | 190.5153 | 173,849,5509      |
| 2    | 20.32546 | 151.2076 | 16,440,168,15     |

It is observed that the expected total profit per unit of time of Example 3 is greater than the expected total profit per unit of time of Example 1. Furthermore, the expected total profit per unit of time of Example 4 is greater than the expected total profit per unit of time of Example 2. These results reveal that Case 2 is always a better option than Case 1.
Therefore, it is suggested that managers arrange a contract with the repair shop in order to receive the reworked lot when the inventory level is zero. This is can be done through the implementation of the just in time principle.

The computational runs for solving the numerical examples were done on a computer HP Elite 8300 (Intel CoreTM i5-3470M CPU @3.30 GHz, 8.00 GB RAM, and 64-bit Windows 7 operating system). It is essential to comment that the computational times required to solve the numerical examples are insignificant, as the computer solves them instantaneously.

6. Conclusions

In the literature related to inventory models with imperfect goods, the holding costs for perfect and imperfect articles are regularly assumed to be the same; additionally, the demand is considered as known and constant. By relaxing these impractical assumptions and using the renewal theorem, this paper builds a more realistic inventory model which considers that the holding costs for imperfect and perfect articles are distinct and the demand is influenced by the price. The proposed inventory model is able to obtain the optimal solution for the lot size and selling price which maximize the expected total profit per unit of time. The main aim of this research work is to propose an inventory model that is useful and easy to implement in any firm by practitioners. It is important to remark that the values for the input parameters required by the inventory model are easily determined by any practitioner that wants to implement the inventory model, due to the fact that this type of data are available in all firms.

Obviously, there still remain some topics for future investigation. The following are suggested: (1) permit shortages with full or partial backlogging and lost sales, (2) include the production rate, (3) deteriorating items, (4) discounts, (5) one- or two-level trade credit, (6) carbon emissions, and (7) nonlinear holding cost, among others. These are some interesting research points that can be researched in the near future.

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