Proceedings of the II Amazonian Symposium on Physics

August 10, 2012
Chapter 1

Analog cosmology with spinor BECs

Esteban Calzetta
Departamento de Física, FCEyN UBA and IFIBA, CONICET, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina

Abstract: We show that the properties of spinor Bose-Einstein condensates allow us to build an analog Taub (axisymmetric Bianchi IX) Universe. We shall develop this proposal on the example of a rubidium condensate, where the relevant experiments are well within present day capabilities. A better Taub analog however would be built out of a collective Rydberg excitation.

1.1 Introduction

Analog models \cite{1} are increasingly used to test fundamental issues in quantum and gravitational physics. While the quest for an experimental demonstration of the Unruh and Hawking effects dominates the field, analogs to cosmological space times are also of great interest. A large variety of physical systems have been proposed as the physical support for the analog models, from plain water \cite{2} to truly astrophysical systems \cite{3}. Bose-Einstein condensates (BECs) stand out for the excellent control one may achieve at the experimental level and the relatively thorough understanding of their physics at the theoretical one. There are several proposals to build black and white holes \cite{4-13} and expanding Universes \cite{14-16} (including anisotropic Bianchi I Universes \cite{17}) out of BECs, as well as proposals to test specific mechanisms without reproducing a complete cosmological scenario \cite{18-22}. Following up on an earlier communication, in this contribution we shall describe a proposal to build an analog Taub Universe from a spinor BEC \cite{23}.

\footnote{calzetta@df.uba.ar}
The proposal relies on experiments which are clearly within present day capabilities, since similar ones have been already carried out.

The Bianchi IX metric describes the homogeneous vacuum solution to the classical Einstein equations with the largest number of free parameters. It has been thoroughly investigated as a likely description of the Universe close to the initial singularity. Moreover, the model displays dynamical chaos and as such it has been the principal source of our understanding of the interaction between gravity and chaos, both at the classical and quantum levels [24]. The Taub Universe is a particular case of the Bianchi IX one, where the evolution is axisymmetric besides homogeneous.

It is well known that both a particle with spin [25, 26] and a quantum field in a Bianchi type IX Universe [27–33] are related to the quantum mechanical top. In an earlier communication we combined these insights to show that the dynamics of a cold gas of atoms with total momentum $F \neq 0$ may be used to explore the behavior of quantum matter in the Taub space time [34]. Indeed, a spinor Bose-Einstein condensate may be described as a field defined both in ordinary space time and in an internal space with the geometry of a sphere. A magnetic field deforms the internal sphere into an homogeneous space of the Taub class [35–42].

However, in that work we did not go beyond the test field approximation. Since then, it has been realized that analog gravitational models need not be confined to the kinematics of gravitation. They may be used to explore certain aspects of space-time dynamics as well as fundamental issues such as the nature of semiclassical approximations [43–49]. With this in mind we shall reexamine our earlier proposal to show that the same effect occurs due to the magnetic field generated by the condensate itself. This will open the possibility of writing a self-consistent evolution for the model.

1.2 Field theory of BEC

For a field-theoretic description of BEC we begin with a second - quantized field operator $\Psi (x, t)$ which removes an atom at the location $x$ at times $t$ [50, 51]. It obeys the canonical commutation relations

$$[\Psi (x, t), \Psi (y, t)] = 0$$  \hspace{1cm} (1.1)

$$[\Psi (x, t), \Psi^\dagger (y, t)] = \delta (x - y)$$  \hspace{1cm} (1.2)

The dynamics of this field is given by the Heisenberg equations of motion

$$- i\hbar \frac{\partial}{\partial t} \Psi = [H, \Psi]$$  \hspace{1cm} (1.3)

$$H = \int d^4x \left\{ \Psi^\dagger H \Psi + V_{int} \left[ \Psi^\dagger, \Psi \right] \right\}$$  \hspace{1cm} (1.4)
1.2. FIELD THEORY OF BEC

\[ H\Psi = -\frac{\hbar^2}{2M} \nabla^2 \Psi + V_{\text{trap}}(x) \Psi \]  
\hspace{1cm} (1.5)

The Heisenberg equation of motion

\[ i\hbar \frac{\partial}{\partial t} \Psi = H\Psi + \frac{\partial V_{\text{int}}}{\partial \Psi^\dagger} \]  
\hspace{1cm} (1.6)

is also the classical equation of motion derived from the action

\[ S = \int d^{d+1}x \, i\hbar \Psi^\dagger \frac{\partial}{\partial t} \Psi - \int dt \, H \]  
\hspace{1cm} (1.7)

The theory is invariant under a global phase change of the field operator

\[ \Psi \rightarrow e^{i\theta} \Psi, \quad \Psi^\dagger \rightarrow e^{-i\theta} \Psi^\dagger \]  
\hspace{1cm} (1.8)

The spontaneous breaking of this symmetry signals Bose-Einstein condensation, and the order parameter is identified with the wave function of the condensate \[52\].

1.2.1 Spinor BECs

In a spinor BEC, the interaction of the nuclear spin \( I \) and the electron spin \( S \) for an electron in the ground state introduces in the Hamiltonian a hyperfine term \((A/2)F^2\), where \( F = I + S \) and \( A \) is a constant (for rubidium, \( A \approx 2 \times 10^{-6}\) eV), leading to a ground state with \( F^2 = F(F + 1) \neq 0 \) \[50, 53–55\]. Spinor BEC have been realized experimentally with several different species: rubidium \(^{87}\)Rb with \( F = 1 \) \[56–59\] and \( F = 2 \) \[60, 61\]; sodium \(^{23}\)Na with \( F = 1 \) \[62\] and \( F = 2 \) \[63\] and chromium \(^{52}\)Cr with \( F = 3 \) \[64, 65\]. Spinor BEC can also be achieved with Rydberg atoms and molecular condensates \[66\]. See also \[67–80\]. For concreteness, we shall consider the case of rubidium \[81\]. For this element, \( I = 3/2 \) and the valence electron carries no orbital angular momentum so the atom can have either \( F = 1 \) or \( F = 2 \), with the former being the lowest modes, and so those that condensate.

Spinor condensates show local point interactions, spin-exchange interactions and dipole-dipole interactions. The strength of each of these can be controlled independently, and we shall neglect the local point interactions.

In principle we could classify atomic states by the \( z \) projection of the total angular momentum \( F \) and describe their associated destruction operators as so many independent fields \[82, 83\]. However, to emphasize the symmetries linking these states, we shall seek an alternative description, describing the spinning atom as a particle in an enlarged configuration space.

The space of states of any spinning particle is a subspace of the Hilbert space of a quantum top \[84\]. The configuration space of the top is parameterized by Euler angles \( \theta, \varphi \) and \( \psi \), and the spin operators are identified with differential operators.
\[
F_x = \frac{\hbar}{i} \left\{ \cos \varphi \frac{\partial}{\partial \theta} - \frac{\sin \varphi \cos \theta}{\sin \theta} \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \psi} \right\}
\]
(1.9)

\[
F_y = \frac{\hbar}{i} \left\{ \sin \varphi \frac{\partial}{\partial \theta} + \frac{\cos \varphi \cos \theta}{\sin \theta} \frac{\partial}{\partial \varphi} - \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \psi} \right\}
\]
(1.10)

\[
F_z = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}
\]
(1.11)

L. S. Schulman used this to develop a path integral for spin \[25\].

Because of the hyperfine splitting, the Hamiltonian \(H\) for a spinor condensate contains a term \((A/2)F^2\) (for rubidium, \(A = 210^{-6}\) eV). After second quantization, this term becomes (\(\chi^a = (\theta, \varphi, \psi)\))

\[
H_0 = \frac{A}{2} \int d^3\chi \sqrt{g_0} g_0^{ab} \frac{\partial \Psi^\dagger}{\partial \chi^a} \frac{\partial \Psi}{\partial \chi^b}
\]
(1.12)

where \(g_{0ab}\) is the metric of the sphere, given by

\[
ds_0^2 = d\theta^2 + d\varphi^2 + d\psi^2 + 2 \cos \theta d\varphi d\psi
\]
(1.13)

### 1.3 Self-consistent magnetic field

Since all retardation effects are negligible, to couple a magnetic field to the condensate we add new terms to the Hamiltonian

\[
\int d^4x \left\{ \frac{1}{2\mu_0} \tilde{B}^2 - \frac{1}{2\alpha} \left( \tilde{\nabla} \cdot \tilde{A} \right)^2 - \tilde{B} \tilde{M} \right\}
\]
(1.14)

where \(\tilde{B} = \tilde{\nabla} \times \tilde{A}\) and \(\tilde{M}\) is the magnetization; \(\mu_0\) is the vacuum permeability and \(\alpha\) the gauge fixing parameter.

The point is that the nuclear and electron spin contribute in a different way to the magnetization

\[
\tilde{M} = \frac{1}{\hbar} \left( \mu_N \tilde{I} + \mu_e \tilde{S} \right) \approx \frac{1}{\hbar} \mu_e \tilde{S}
\]
(1.15)

and since \([F^2, \tilde{S}] \neq 0\), it mixes \(F = 1\) and \(F = 2\) states. Observe that \(\mu_e = -2\mu_B\), where \(\mu_B\) is Bohr’s magneton \(e/2m_e\), the factor of 2 comes from the electron’s g-factor, and the sign because the electron is negatively charged.

In general, the matrix elements of the magnetization may be computed from the Wigner-Eckart theorem [85]. However, in this simple case a direct evaluation is easiest. Let us adopt a local frame such that \(\tilde{B}\) points in the
1.3. SELF‐CONSISTENT MAGNETIC FIELD

$z$ direction. Then we only need the matrix elements of $S_z$, which moreover commutes with $F_z$. Obviously, these matrix elements are easiest to compute in states $|I_z = m_1, S_z = m_z\rangle$ where the nuclear and electron spin are well defined; we need them for states $|F, F_z\rangle$ where total angular momentum and its $z$ projection are well defined.

Since

$$|2, 2\rangle = |I_z = 3/2, S_z = 1/2\rangle \quad (1.16)$$

we have

$$\frac{1}{\hbar} S_z |2, 2\rangle = 1/2 |2, 2\rangle \quad (1.17)$$

Applying $F_‐$ to both sides of eq. (1.16) we get

$$|2, 1\rangle = \frac{\sqrt{3}}{2} |I_z = 1/2, S_z = 1/2\rangle + \frac{1}{2} |I_z = 3/2, S_z = -1/2\rangle \quad (1.18)$$

The other linearly independent combination is

$$|1, 1\rangle = -\frac{1}{2} |I_z = 1/2, S_z = 1/2\rangle + \frac{\sqrt{3}}{2} |I_z = 3/2, S_z = -1/2\rangle \quad (1.19)$$

Therefore

$$\frac{1}{\hbar} S_z |2, 1\rangle = \frac{\sqrt{3}}{4} |I_z = 1/2, S_z = 1/2\rangle - \frac{1}{4} |I_z = 3/2, S_z = -1/2\rangle$$

$$= \frac{1}{4} |2, 1\rangle - \frac{\sqrt{3}}{4} |1, 1\rangle \quad (1.20)$$

$$\frac{1}{\hbar} S_z |1, 1\rangle = -\frac{\sqrt{3}}{4} |2, 1\rangle - \frac{1}{4} |1, 1\rangle \quad (1.21)$$

Applying $F_‐$ to eqs. (1.18) and (1.19) we get

$$|20\rangle = \frac{1}{\sqrt{2}} \left(|I_z = -1/2, S_z = 1/2\rangle + |I_z = 1/2, S_z = -1/2\rangle\right)$$

$$|10\rangle = \frac{1}{\sqrt{2}} \left(-|I_z = -1/2, S_z = 1/2\rangle + |I_z = 1/2, S_z = -1/2\rangle\right) \quad (1.22)$$

Therefore

$$\frac{1}{\hbar} S_z |2, 0\rangle = -\frac{1}{2} |10\rangle$$

$$\frac{1}{\hbar} S_z |1, 0\rangle = -\frac{1}{2} |20\rangle \quad (1.23)$$
Finally, by symmetry
\[
\frac{1}{\hbar} S_z |2, -1\rangle = -\frac{1}{4} |2, -1\rangle + \frac{\sqrt{3}}{4} |1, -1\rangle
\]
\[
\frac{1}{\hbar} S_z |1, -1\rangle = \frac{\sqrt{3}}{4} |2, -1\rangle + \frac{1}{4} |1, -1\rangle
\]
(1.24)
\[
\frac{1}{\hbar} S_z |2, -2\rangle = -\frac{1}{2} |2, -2\rangle
\]
(1.25)

1.4 Low energy effective Hamiltonian

We are now ready to obtain the dynamics of the low energy \( F = 1 \) modes by explicitly integrating out the energetic \( F = 2 \) modes. If moreover we neglect the kinetic energy of the \( F = 2 \) atoms, then this integration can be carried out independently at each point in the trap. We shall consider only one such point, where moreover we assume \( \vec{B} = B \hat{z} \). Let us expand
\[
\Psi = \sum_{F=1,2} \sum_{M=-F}^{F} \sum_{K=-F}^{F} f_{FMK}(t) Y_{FMK}(\theta, \varphi, \psi)
\]
(1.26)
where the modes \( Y_{FMK} \) are common eigenfunctions of \( F^2, F_z \) and \( F_\xi = (\hbar/i) (\partial/\partial \psi) \) with eigenvalues \( \hbar^2 F (F + 1), \hbar M \) and \( \hbar K \) respectively.

We shall consider only the one-loop approximation whereby the \( F = 2 \) modes are considered as free except for their coupling to the \( F = 1 \) modes [51]. Within this approximation, the \( F = 2, M = \pm 2 \) and the \( F = 2, K = \pm 2 \) modes may be ignored. For the other modes we obtain the Hamiltonian
\[
H_2 = \sum_{M=-1}^{1} \sum_{K=-1}^{1} \left\{ \hbar \omega_2 - M \chi_2 \right\} f_{2MK}^\dagger f_{2MK} + f_{1MK}^\dagger J_{1MK} + J_{1MK}^\dagger f_{2MK}
\]
(1.27)
where
\[
\omega_2 = \frac{3A}{\hbar} \\
\chi_2 = \frac{\mu_e B}{4} \\
J_{1MK} = \frac{\mu_e B}{2} \left[ 1 + \frac{\sqrt{3}}{2} M - M^2 \right] f_{1MK}
\]
(1.28)

Integrating out the \( F = 2 \) modes within the IN-OUT formalism [51] yields a non-local effective Hamiltonian.
1.5. THE EFFECTIVE DYNAMICS AS A SELF-CONSISTENT METRIC

\[ H_{\text{eff}} = -\frac{i}{\hbar} \sum_{M=-1}^{1} \sum_{K=-1}^{1} \int_{t' > t} dt' \left\langle T \left[ f_{2MK} (t') f_{2MK}^\dagger (t) \right] \right\rangle J_{1MK}^\dagger (t') J_{1MK} \]  

(1.29)

Where \( T \) stands for time ordering. Keeping only the lowest order term in \( B \) and assuming the \( F = 2 \) modes are in their vacuum state we get

\[ \left\langle T \left[ f_{2MK} (t') f_{2MK}^\dagger (t) \right] \right\rangle = e^{-i\omega_2(t' - t)} (t' - t) \]  

(1.30)

To obtain a local effective Hamiltonian we further approximate

\[ J_{1MK}^\dagger (t') = e^{i\omega_1(t' - t)} J_{1MK}^\dagger (t) \]  

(1.31)

where \( \omega_1 = A/\hbar \). Then the \( t' \) integral may be evaluated and we get

\[ H_{\text{eff}} = \frac{\mu_B^2 B^2}{8A} \sum_{M=-1}^{1} \sum_{K=-1}^{1} \left[ M^2 - 4 \right] f_{1MK}^\dagger f_{1MK} \]  

(1.32)

1.5 The effective dynamics as a self-consistent metric

To be able to interpret the effective dynamics of the \( F = 1 \) modes as the evolution of a field in a nontrivial space time, we write again the Hamiltonian as an integral over Euler angles

\[ H_{\text{eff}} = \frac{\mu_B^2 B^2}{8A} \int d^3 \chi \sqrt{g_0} \left[ \frac{\partial \Psi^\dagger}{\partial \varphi} \frac{\partial \Psi}{\partial \varphi} - 4 \Psi^\dagger \Psi \right] \]  

(1.33)

where only the \( F = 1 \) components of \( \Psi \) are retained. Comparing to the case where only the hyperfine term is present, we see that the metric of the sphere has been deformed into

\[ ds_B^2 = \left[ 1 + B^2 \right] d\theta^2 + d\varphi^2 + \left[ 1 + B^2 \sin^2 \theta \right] d\psi^2 + 2 \cos \theta \, d\varphi \, d\psi \]  

(1.34)

where

\[ B = \frac{\mu_B B}{2A} \]  

(1.35)

To see that this is indeed a Taub metric, identify
CHAPTER 1. ANALOG COSMOLOGY WITH SPINOR BECS

\[ \omega_1 = d\varphi + \cos \theta \, d\psi \]  
(1.36)

\[ \omega_2 = \cos \varphi \, d\theta + \sin \varphi \sin \theta \, d\psi \]  
(1.37)

\[ \omega_3 = -\sin \varphi \, d\theta + \cos \varphi \, \sin \theta \, d\psi \]  
(1.38)

The metric can then be written in terms of Misner parameters as

\[ ds_B^2 = e^{2\Omega} \left\{ e^{-2\beta_-} \omega_1^2 + e^{2\beta_+} + \sqrt{3} \beta_- \omega_2^2 + e^{2\beta_-} - \sqrt{3} \beta_- \omega_3^2 \right\} \]  
(1.39)

where \( \beta_- = 0, \beta_+ = \Omega \) and

\[ \Omega = \frac{1}{3} \ln \left[ 1 + B^2 \right] \]  
(1.40)

Physically, the physical degree of freedom in the metric is the same as in the magnetic field, and so there is no “Einstein” equation independent of Maxwell’s equations. Moreover, this equation is not local in the internal space, rather the magnetic field is coupled to two global quantities. The dynamic equation is the Ampere Law \( \nabla \times \vec{H} = 0 \), where \( \vec{H} \) is defined as the variational derivative of the total effective Hamiltonian with respect to \( \vec{B} \).

Therefore we get

\[ H^i = \frac{1}{\mu_0} B^i - M^i - c_0 B^i + C^i_j B^j \]  
(1.41)

\[ c_0 = \frac{\mu_0^2}{\Lambda} \int d^3 \chi \sqrt{g_0} \Psi^\dagger \Psi \]  
(1.42)

\[ C^i_j = \frac{\mu_0^2}{8 \Lambda h^2} \int d^3 \chi \sqrt{g_0} \Psi^\dagger \left\{ \hat{F}^i, \hat{F}^j \right\} \Psi \]  
(1.43)

Although very different in form to Einstein dynamics, this equation could be used, for example, to explore the validity of the semiclassical approximation in this problem.

1.6 Final remarks

We have shown that a spinor BEC with dipolar interactions has a natural representation as a field on an internal space whose scale factor and shape depend upon a self-consistent magnetic field. In this sense, there is a well defined sense in which the “field” backreacts on the magnetic field and we have a self consistent dynamics coupling the matter field and the Maxwell
field. This can be used to explore both the quantum dynamics of internal spaces with nontrivial geometry and fundamental issues in semiclassical theories.

One important difference between this problem and the full field theoretic ones is that only a handful of modes really represent physical configurations. The problem becomes more “field like” the larger the number of modes. This can be accomplished, for example, by considering the dipole interaction of Rydberg collective excitations $[86-89, 91]$. Such excitations have already been built in the laboratory in $D$ states with very large values of $n (n > 50)$ $[92, 93]$, and therefore couple to a much larger number of modes.

The work presented in this communication can be extended in several ways. In a dynamical setting, we expect the $F = 2$ modes we have integrated out will behave as an environment for the relevant $F = 1$ modes. Therefore, besides the effective field we have studied, we expect there will arise dissipation, noise and decoherence effects $[94, 95]$. These are major considerations for the assessment of current proposals for building analog models in the laboratory $[95, 96]$.

Acknowledgments

We warmly thank the organizers for the invitation to be part of the “II Amazonian Symposium on Physics - Analogue Models of Gravity 30 Years Celebration” where this work was presented.

This work has been supported by CONICET, UBA and ANPCyT (Argentina)
Bibliography

[1] W. G. Unruh, *Experimental Black-Hole Evaporation?*, Phys. Rev. Lett. 46, 1351 (1981).

[2] S. Weinfurtner, E. W. Tedford, M. C. J. Penrice, W. G. Unruh and G. A. Lawrence, *Measurement of stimulated Hawking emission in an analogue system*, Phys. Rev. Lett. 106, 021302 (2011).

[3] T. K. Das, *Analogue Hawking radiation from astrophysical black-hole accretion*, Class. Quantum Grav. 21 5253 (2004).

[4] R. Balbinot, A. Fabbri, S. Fagnocchi, A. Recati and I. Carusotto, *Non-local density correlations as signal of Hawking radiation in BEC acoustic black holes*, Phys. Rev. A78, 021603 (2008).

[5] I. Carusotto, S. Fagnocchi, A. Recati, R. Balbinot and A. Fabbri, *Numerical observation of Hawking radiation from acoustic black holes in atomic Bose-Einstein condensates*, New J. Phys. 10, 103001 (2008).

[6] J. Macher and R. Parentani, *Black-hole radiation in Bose-Einstein condensates*, Phys.Rev. A80, 043601 (2009).

[7] A. Recati, N. Pavloff, I. Carusotto, *Bogoliubov Theory of acoustic Hawking radiation in Bose-Einstein Condensates*, Phys.Rev. A80, 043603 (2009).

[8] Oren Lahav, Amir Itah, Alex Blumkin, Carmit Gordon, Shahar Rinott, Alona Zayats, Jeff Steinhauer, *Realization of a sonic black hole analogue in a Bose-Einstein condensate*, Phys.Rev.Lett. 105, 240401 (2010).

[9] A. Fabbri, C. Mayoral, *Step-like discontinuities in Bose-Einstein condensates and Hawking radiation: the hydrodynamic limit*, Phys.Rev. D83, 124016 (2011).

[10] C. Mayoral, A. Fabbri and M. Rinaldi, *Step-like discontinuities in Bose-Einstein condensates and Hawking radiation: dispersion effects*, Phys.Rev. D83, 124047 (2011).
[11] C. Mayoral, A. Recati, A. Fabbri, R. Parentani, R. Balbinot and I. Carusotto, *Acoustic white holes in flowing atomic Bose-Einstein condensates*, New J. Phys. **13**, 025007 (2011).

[12] I. Zapata, M. Albert, R. Parentani and F. Sols, *Resonant Hawking radiation in Bose-Einstein condensates*, New J. Phys. **13**, 063048 (2011).

[13] P.-É. Larré, A. Recati, I. Carusotto, N. Pavloff, *Quantum fluctuations around black hole horizons in Bose-Einstein condensates*, Phys. Rev. A **85**, 013621 (2012).

[14] J. Lidsey, *Cosmic dynamics of Bose-Einstein condensates*, Class. Q. Grav. **21**, 777 (2004).

[15] Y. Kurita, M. Kobayashi, T. Morinari, M. Tsubota and H. Ishihara, *Spacetime analogue of Bose-Einstein condensates: Bogoliubov-de Gennes formulation*, Phys.Rev.A **79**, 043616 (2009).

[16] Chi-Yong Lin, Da-Shin Lee, Ray J. Rivers, *Mimicking Friedmann-Robinson-Walker universes with tunable cold Fermi atoms: Galilean invariance fights back*, arXiv: 1205.0133

[17] J. D’Ambroise and F. L. Williams, *A dynamic correspondence between Bose-Einstein condensates and Friedmann-Lemaître-Robertson-Walker and Bianchi I cosmology with a cosmological constant*, J. Math. Phys. **51**, 062501 (2010).

[18] E. Calzetta and B-L. Hu, *Bose-Einstein Condensate Collapse and Dynamical Squeezing of Vacuum Fluctuations*, Phys. Rev. A **68**, 043625 (2003).

[19] E. Calzetta and B-L. Hu, *Early Universe Quantum Processes in BEC Collapse Experiments*, Int. J. Theor. Phys. **44**, 1691 (2005)

[20] P. Jain, S. Weinfurtner, M. Visser and C. W. Gardiner, *Analogue model of a FRW universe in Bose?Einstein condensates: Application of the classical field method*, Phys. Rev. A **76**, 033616 (2007).

[21] U. R. Fischer, R. Schützhold and M. Ulhmann, *Bogoliubov theory of quantum correlations in the time-dependent Bose-Hubbard model*, Phys. Rev. A **77**, 043615 (2008).

[22] A. Retzker, J. I. Cirac, M. B. Plenio, B. Reznik, *Detection of acceleration radiation in a Bose-Einstein condensate*, Phys. Rev. Lett. **101**, 110402 (2008).

[23] Dan M. Stamper-Kurn and M. Ueda, *Spinor Bose gases: Explorations of symmetries, magnetism and quantum dynamics*, arXiv:1205.1888
[24] E. Calzetta, *Chaos, decoherence and quantum cosmology*, Class. Q. Grav. **29**, 143001 (2012).

[25] L. Schulman, *A Path Integral for Spin*, Phys. Rev. **176**, 1558 (1968).

[26] L. S. Schulman, *Techniques and Applications of Path Integration* (John Wiley & Sons, New York, 1981).

[27] B.L. Hu, *Scalar Waves and Tensor Perturbations in the Mixmaster Universe*, Ph. D. Thesis, Princeton University, 1972.

[28] B. L. Hu, S. A. Fulling, and Leonard Parker, *Quantized Scalar Fields in a Closed Anisotropic Universe*, Phys. Rev. **D 8**, 2377 (1973).

[29] B.L. Hu, *Scalar waves in the mixmaster universe I: the Helmholtz equation in a fixed background*, Phys. Rev. D **8**, 1048 (1973).

[30] B.L. Hu, *Scalar waves in the mixmaster universe II: Particle creation*, Phys. Rev. D **9**, 3263 (1974).

[31] T. C. Shen, B. L. Hu, and D. J. O’Connor, *Symmetry behavior of the static Taub universe: Effect of curvature anisotropy*, Phys. Rev. D **31**, 2401 (1985).

[32] B. L. Hu and D. J. O’Connor, *Mixmaster inflation*, Phys. Rev. D **34**, 2535 (1986).

[33] E. Calzetta, *Particle Creation, Inflation and Cosmic Isotropy*, Physical Review D **44**, 3043 (1991).

[34] E. Calzetta, *Analog Cosmology with Spinor Bose-Einstein Condensates*, talk given at the Nonequilibrium problems in particle physics and cosmology workshop, KITP, USB (2008) (arXiv:0712.0376).

[35] L. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975).

[36] C. Misner, K. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1972).

[37] M. Ryan and L. Shepley, *Homogeneous Relativistic Cosmology* (Princeton University Press, Princeton, 1975).

[38] V.A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, *Oscillatory approach to a singular point in the Relativistic Cosmology*, Adv. in Phys. **19**, 525 (1970).

[39] C. Misner, *Classical and Quantum Dynamics of a Closed Universe*, in M. Carmeli (ed.), *Relativity* (Plenum Press, New York, 1970).
[40] C. Misner, *Minisuperspace*, in J. Klauder (ed.), *Magic Without Magic* (Freeman, San Francisco, 1972).

[41] J. D. Barrow, *Chaotic Behavior in General Relativity*, Phys. Rep. **85**, 1 (1982)

[42] D. Hobill, A. Burd and A. Coley, *Deterministic Chaos in General Relativity* (Plenum Press, New York, 1994)

[43] U. R. Fischer, *Dynamical Aspects of Analogue Gravity: The Backreaction of Quantum Fluctuations in Dilute Bose-Einstein Condensates*, Lect. Notes Phys. **718**, 93 (2007).

[44] R. Balbinot, S. Fagnocchi and A. Fabbri, *The depletion in Bose Einstein condensates using Quantum Field Theory in curved space*, Phys. Rev. A **75**, 043622 (2007).

[45] Florian Girelli, Stefano Liberati, Lorenzo Sindoni, *Gravitational dynamics in Bose Einstein condensates*, Phys.Rev.D **78**, 084013 (2008)

[46] Stefano Liberati, Florian Girelli, Lorenzo Sindoni, *Analogue Models for Emergent Gravity*, in Proceedings of the XVIII SIGRAV Conference, Cosenza, September 22-25, 2008 [arXiv:0909.3834].

[47] Lorenzo Sindoni, Florian Girelli, Stefano Liberati, *Emergent gravitational dynamics in Bose-Einstein condensates*, AIP Conf. Proc. **1196**, 258 (2009).

[48] Stefano Finazzi, Stefano Liberati, Lorenzo Sindoni, *The cosmological constant: a lesson from Bose-Einstein condensates*, Phys. Rev. Lett. **108**, 071101 (2012).

[49] Stefano Finazzi, Stefano Liberati, Lorenzo Sindoni, *The analogue cosmological constant in Bose-Einstein condensates: a lesson for quantum gravity*, this volume [arXiv:1204.3039]

[50] C. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases* (Cambridge University Press, Cambridge, England, 2002)

[51] E. Calzetta and B-L. Hu, *Nonequilibrium Quantum Field Theory*, Cambridge University Press, Cambridge, Inglaterra (2008)

[52] V.I. Yukalov, *Basics of Bose-Einstein Condensation*, Phys. Part. Nucl. **42** (2011) 460-513

[53] T. L. Ho, *Spinor Bose Condensates in Optical Traps*, Phys. Rev. Lett. **81**, 742 (1998).
[54] T. Ohmi and K. Machida, *Bose-Einstein condensation with internal degrees of freedom*, J. Phys. Soc. Jpn. 67, 1822 (1998).

[55] T Lahaye, C Menotti, L Santos, M Lewenstein, T Pfau, *The physics of dipolar bosonic quantum gases*, Rep. Prog. Phys. 72, 126401 (2009)

[56] L. E. Sadler, J. M. Highbie, S. R. Leslie, M. Vengalattore and D. M. Stamper-Kurn, *Spontaneous symmetry breaking in a quenched ferromagnetic spinor Bose-Einstein condensate*, Nature 443, 312 (2006).

[57] M. Anderlini, P. Lee, B. Brown, J. Sebby-Strabley, W. Phillips and J. Porto, *Controlled exchange interaction between pairs of neutral atoms in an optical lattice*, Nature 448, 452 (2007).

[58] M. Vengalattore, S. R. Leslie, J. Guzman and D. M. Stamper-Kurn, *Spontaneously modulated spin textures in a dipolar spinor Bose-Einstein condensate*, Phys. Rev. Lett. 100, 170403 (2008)

[59] M.-S. Chang, C. D. Hamley, M. D. Barrett, J. A. Sauer, K.M. Fortier, W. Zhang, L. You, and M. S. Chapman, *Observation of Spinor Dynamics in Optically Trapped 87Rb Bose-Einstein Condensates*, Phys. Rev. Let. 92, 140403 (2004)

[60] T. Kuwamoto, K. Araki, T. Eno, and T. Hirano, *Magnetic field dependence of the dynamics of 87Rb spin-$2$ Bose-Einstein condensates* Phys. Rev. A 69, 063604 (2004)

[61] H. Schmaljohann, M. Erhard, J. Kronjäger, M. Kottke, S. van Staa, L. Cacciapuoti, J. J. Arlt, K. Bongs, and K. Sengstock, *Dynamics of $F = 2$ Spinor Bose-Einstein Condensates*, Phys. Rev. Let. 92, 040402 (2004)

[62] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur and W. Ketterle, *Spin domains in ground-state Bose Einstein condensates* Nature 396, 345 (1998)

[63] A. Görlitz, T. L. Gustavson, A. E. Leanhardt, R. Löw, A. P. Chikkatur, S. Gupta, S. Inouye, D. E. Pritchard, and W. Ketterle, *Sodium Bose-Einstein Condensates in the $F = 2$ State in a Large-Volume Optical Trap*, Phys. Rev. Let. 90, 090401 (2003)

[64] Thierry Lahaye, Tobias Koch, Bernd Fröhlich, Marco Fattori, Jonas Metz, Axel Griesmaier, Stefano Giovanazzi and Tilman Pfau, *Strong dipolar effects in a quantum ferrofluid*, Nature 448, 672 (2007).

[65] Q. Beaufils, R. Chicireanu, T. Zanon, B. Laburthe-Tolra, E. Maréchal, L. Vernac, J.-C. Keller, and O. Gorceix *All-Optical Production of Chromium Bose-Einstein Condensates*, Phys. Rev. A 77, 061601(R) (2008).
[66] C. Menotti, M. Lewenstein, T. Lahaye and T. Pfau, *Dipolar interaction in ultra-cold atomic gases*, Proceedings of the Workshop ”Dynamics and Thermodynamics of Systems with Long Range Interactions” (Assisi, July 2007), AIP Conference Proceedings (arXiv:0711.3422).

[67] A. Lamacraft, *Quantum quenches in a spinor condensate*, Phys. Rev. Lett 98, 160404 (2007).

[68] M. Uhlmann, R. Schützhold and U. Fischer, *Vortex quantum creation and winding number scaling in a quenched spinor Bose gas*, Phys. Rev. Lett 99, 120407 (2007).

[69] B. Damski and W. Zurek, *Dynamics of a quantum phase transition in a ferromagnetic Bose-Einstein condensate*, Phys. Rev. Lett. 99, 130402 (2007).

[70] B. Damski and W. Zurek, *Quantum phase transition in a ferromagnetic Bose-Einstein condensate: dynamics from the broken-symmetry to the polar phase*, New J. Phys. 10, 045023 (2008).

[71] H. Saito, Y. Kawaguchi and M. Ueda, *Topological defect formation in a quenched ferromagnetic Bose-Einstein condensate*, Phys. Rev. A75, 013621 (2007).

[72] H. Saito, Y. Kawaguchi and M. Ueda, *Kibble-Zurek mechanism in a quenched ferromagnetic Bose-Einstein condensate*, Phys. Rev. A 76, 043613 (2007).

[73] G. I. Mias and S. M. Girvin, *Quantum noise, scaling and domain formation in a spinor BEC*, Phys. Rev. A77, 023616 (2008).

[74] M. Vengalattore, J. M. Higbie, S. R. Leslie, J. Guzman, L. E. Sadler, D. M. Stamper-Kurn, *High-Resolution Magnetometry with a Spinor Bose-Einstein Condensate*, Phys. Rev. Lett. 98, 200801 (2007)

[75] Thierry Lahaye, Jonas Metz, Bernd Froehlich, Tobias Koch, Maximilian Meister, Axel Griesmaier, Tilman Pfau, Hiroki Saito, Yuki Kawaguchi, Masahito Ueda, *d-wave collapse and explosion of a dipolar Bose-Einstein condensate*, Phys. Rev. Lett. 101, 080401 (2008)

[76] S. R. Leslie, J. Guzman, M. Vengalattore, J. D. Sau, M. L. Cohen, D. M. Stamper-Kurn, *Amplification of Fluctuations in a Spinor Bose Einstein Condensate*, arXiv:0806.1553

[77] M. Vengalattore, J. Guzman, S. Leslie, F. Serwane, D. M. Stamper-Kurn, *Periodic spin textures in a degenerate F=1 87Rb spinor Bose gas*, arXiv:0901.3800
[78] J Metz, T Lahaye, B Fröhlich, A Griesmaier, T Pfau, H Saito, Y Kawaguchi, M Ueda, *Coherent collapse of a dipolar Bose-Einstein condensate for different trap geometries*, New J. Phys. 11, 055032 (2009)

[79] T. Lahaye, J. Metz, T. Koch, B. Fröhlich, A. Griesmaier, T. Pfau, *A purely dipolar quantum gas*, PUSHING THE FRONTIERS OF ATOMIC PHYSICS, Proceedings of the XXI International Conference on Atomic Physics, editors R Cote, P L Gould, M G Rozman, W. W. Smith, p. 160, World Scientific, Singapore (2009) [arXiv:0808.3876]

[80] Ryan Barnett, Anatoli Polkovnikov, Mukund Vengalattore, *Do Quenched Spinor Condensates Thermalize?*, Phys. Rev. A 84, 023606 (2011)

[81] R. Löw, *A versatile setup for experiments with Rubidium Bose Einstein condensates: From optical lattices to Rydberg matter*, Ph. D. Thesis (Universitat Stuttgart, 2006)

[82] L. Sindoni, *Emergent gravitational dynamics from multi-Bose-Einstein-condensate hydrodynamics?*, Phys. Rev. D83, 024022 (2011).

[83] M. Fizia and K. Sacha, *Inert-states of spin-5 and spin-6 Bose-Einstein condensates*, J. Phys. A: Math. Theor. 45, 045103 (2012)

[84] N. Rosen, *Particle spin and rotation*, Phys. Rev. 82, 621 (1951).

[85] A. R. Edmonds, *Angular momentum in quantum mechanics* (Princeton UP, Princeton, 1957)

[86] T. Gallagher, *Rydberg Atoms* (Cambridge University Press, London, 1994).

[87] R. Heidemann, *Rydberg Excitation of Bose-Einstein Condensates: Coherent Collective Dynamics*, Ph. D. Thesis (Universitat Stuttgart, 2008).

[88] R. Heidemann, U. Raitzsch, V. Bendkowsky, B. Butscher, R. Low, and T. Pfau, *Rydberg Excitation of Bose-Einstein Condensates*, Phys. Rev. Lett. 100, 033601 (2008).

[89] N. Henkel, R. Nath and T. Pohl, *Three-Dimensional Roton Excitations and Supersolid Formation in Rydberg-Excited Bose-Einstein Condensates*, Phys. Rev. Lett. 104, 195302 (2010)

[90] T. E. Lee, H. Häffner and M. C. Cross, *Antiferromagnetic phase transition in a nonequilibrium lattice of Rydberg atoms*, Phys. Rev. A 84, 031402 (2011).
[91] W. Li, L. Hamadeh, and I. Lesanovsky, *Probing the interaction between Rydberg-dressed atoms through interference*, Phys. Rev A 85, 053615 (2012).

[92] M. Viteau, M. G. Bason, J. Radogostowicz, N. Malossi, D. Ciampini, O. Morsch and E. Arimondo, *Rydberg Excitations in Bose-Einstein Condensates in Quasi-One-Dimensional Potentials and Optical Lattices*, Phys. Rev. Lett. 107, 060402 (2011).

[93] M. Viteau et al., *Rydberg excitation of a Bose-Einstein condensate*, Laser Physics in press [arXiv:1203.1261].

[94] E. Calzetta and B-L. Hu, *Stochastic Behavior of Effective Field Theories Across Threshold*, Phys. Rev. D55, 3536 (1997).

[95] S. Wuester, *Phonon background versus analogue Hawking radiation in Bose-Einstein condensates*, Phys. Rev. A 78, 021601(R) (2008).

[96] F. Lombardo and G. Turiaci, *Decoherence and loss of entanglement in acoustic black holes*, Phys. Rev. Lett. (in press) [arXiv:1206.1351].