A Note on Decoding Order in Optimizing Multi-Cell NOMA

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Abstract—In this technical note, we present a new theoretical result for resource optimization with non-orthogonal multiple access (NOMA). For multi-cell scenarios, a so-called load-coupling model has been proposed to characterize the presence of mutual interference for NOMA, and resource optimization relies on the use of fixed-point iterations [1], [2] across cells. One difficulty here is that the order of decoding for successive interference cancellation (SIC) in NOMA is generally not known a priori. This is because the decoding order in one cell depends on interference, which, in turn, is governed by resource allocation in other cells, and vice versa. To achieve convergence, previous works have used workarounds that pose restrictions to NOMA, such that the SIC decoding order remains in optimization. As a comment to [1], [2], we derive and prove the following result: The convergence is guaranteed, even if the order changes over the iterations. The result not only waives the need of previous workarounds, but also implies that a wide class of resource optimization problems for multi-cell NOMA is tractable, as long as that for single cell is.

Index Terms—SIC, NOMA, interference, multi-cell

I. INTRODUCTION

Non-orthogonal Multiple Access (NOMA) with successive interference cancellation (SIC) allows more than one user to share resource in the time-frequency domain. With superposition coding, each user performs signal decoding of interference signals. The decoding needs to follow signal strength so as to make SIC succeed. In the simplest case, the decoding order is determined by channel gains of users. NOMA in single-cell scenarios has been widely addressed [3]–[9]. In multi-cell NOMA, inter-cell interference has an influence on the decoding order, which has to be accounted for [10], [11]. Besides, NOMA requires user grouping for resource sharing. The candidate options for user grouping is exponential in the number of users. Third, power allocation (a.k.a power split) affects resource efficiency. Both user grouping and power split are intertwined with the decoding order.

The decoding order in one cell depends on the interference from other cells; the interference, in term, depends on the allocated power and time-frequency resources. A cell that allocates more resource for serving users generates more interference to others. One approach for multi-cell optimization for orthogonal multiple access (OMA) consists of fixed point iterations for the so-called load coupling equation system [12]–[34]. In every iteration, the resource allocation of one cell is computed, with the allocation in other cells temporarily being fixed. At convergence, an equilibrium with respect to resource allocation and the resulting interference is obtained. However, in NOMA, applying the type of iterative method is challenging, since the decoding order is not known beforehand, but subject to change during the iterative process. Some references [35]–[39] do not explicitly address the interaction between decoding order and interference. To the best of our knowledge, only [1], [2], [40] consider the type of dependency resource optimization of multi-cell NOMA. In [40], the authors investigate multi-cell NOMA power control without user grouping. References [1], [2] have used workarounds such that some pre-computed decoding order remains NOMA-compliant in optimization, which on the other hand, poses limitations on the applicable scenarios. To be specific, [40] requires that there is only one candidate group consisting of all users, and [1], [2] require: 1) there are up to two users in each group and 2) the candidate groups are selected such that the NOMA-compliant decoding order can be pre-determined no matter the interference. Without these conditions, neither the convergence nor the optimality of their proposed algorithm is guaranteed.

This technical note serves as a comment to [1], [2], though our main results are not necessarily bound to the specific system setups in [1] and [2]. The contributions are:

- We show a general conclusion with respect to the formulation of multi-cell NOMA resource optimization problems, such that the decoding order needs not to be explicitly ensured by constraints.
- We use a so-called load-coupling model as an example, to showcase our conclusion. The load coupling model has been widely adopted in OMA scenarios [12]–[34] and has been extended to NOMA in [1], [2].
- We formally proved that, the convergence and the optimality are guaranteed without imposing the limitations on the candidate user groups, even if the decoding order, due to variable inter-cell interference, changes from iteration to the next in a fixed-point method. Furthermore, the result implies that a wide class of resource optimization problem for multi-cell NOMA is tractable as long as that for single cell is.

We clarify that in this technical note, the term “correct decoding order”, unless otherwise stated, always refers to the NOMA-compliant decoding order. That is a receiver decodes the signals in descending order of signal strengths.

II. SYSTEM MODEL

A. Preliminaries

Denote by $\mathcal{I} = \{1, 2, \ldots, n\}$ the set of cells, and $\mathcal{J}$ the set of user equipments (UEs). For each cell $i (i \in \mathcal{I})$, denote by $\mathcal{J}_i$ the set of UEs served by cell $i$. Denote by $d_j$ the bits demand
of UE \( j \) \((j \in J) \). Denote by \( p_i \) the transmission power of cell \( i \) on each resource unit (RU). By using SIC, multiple UEs can access one RU simultaneously, with \( p_i \) split among these UEs. We refer to the UEs sharing the same RUs as a group, and the process of selecting UEs to form groups as user grouping. We use \( u \) to refer to a generic group. For any UE \( j \in u \), denote by \( q_{ju} \) the portion of power \( p_i \) used for UE \( j \) on each RU allocated to group \( u \). For cell \( i \) \((i \in I) \), denote by \( \mathcal{U}_i \) the set of all groups of UEs in \( J_i \). In analogy with this, we use \( \mathcal{U}_i \) to refer to the set of all groups that UE \( j \) belongs to. In order to keep generality, we allow also singleton group \( u \). In this case, the UE in the group does not share RU with others (and hence no SIC), i.e., the UE is allocated with resource with \( \text{SIC} \), i.e., the UE is allocated with resource with.

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**B. NOMA with SIC**

We consider downlink and use \( g_{ij} \) to denote the gain from cell \( i \) to UE \( j \). For any cell \( i \), the signal-to-interference-and-noise ratio (SINR) of UE \( j \) in group \( u \), denoted by \( \gamma_{ju} \), is given below.

\[
\gamma_{ju} = \frac{q_{ju}g_{ij}}{\sum_{h \in u, \text{intra-cell}} q_{hu}g_{hj} + \sum_{k \in \mathcal{T}_j \setminus \{i\}, \text{inter-cell}} p_kg_{kj}\rho_k + \sigma^2}, \quad j \in u, \quad u \in \mathcal{U}_i
\]  

(1)

The denominator of \( \gamma_{ju} \) consists of three parts: intra-cell interference, inter-cell interference, and noise power \( \sigma^2 \). The variables in resource allocation are the power split \( q_{ju} \), the decoding order indicator \( \theta_{hj} \) (discussed below), and the cell-level resource allocation \( \rho_k \) that is served as a scaling parameter for the inter-cell interference (discussed in Section II-C). For each group \( u \), note that a UE \( j \) of this group decodes the data of UEs with stronger signal in \( u \), and receives interference from the other UEs of \( u \). We use \( \theta_{hj} \) as a binary indicator: \( \theta_{hj} = 1 \) if and only if UE \( j \) receives intra-cell interference from UE \( h \), and \( \theta_{hj} = 0 \) if and only if UE \( j \) can decode the signal of UE \( h \). As a convention, \( \theta_{hj} = 0 \) if \( h = j \). We remark that \( \theta_{hj} \) is subject to the correct decoding order, which is determined by the channel condition, inter-cell interference, and noise [41].

For any UE \( j \), define

\[
w_j = \left( \sum_{k \in \mathcal{T}_j \setminus \{i\}} p_kg_{kj}\rho_k + \sigma^2 \right) / g_{ij}.
\]  

(2)

Then \( \theta_{hj} = 1 \) if and only if \( w_h \geq w_j \), i.e., UE \( j \) has better signal strength than UE \( h \), and hence the latter can decode the former with SIC.

**C. Cell Load Coupling**

To ease our presentation, define \( c_{ju} \) as the achievable capacity for UE \( j \) in group \( u \) on any RU, namely,

\[
c_{ju} = \log(1 + \gamma_{ju}) = \log \left( 1 + \frac{q_{ju}}{\sum_{h \in u} q_{hu}\theta_{hj} + w_j} \right).
\]  

(3)

1Strictly speaking, if \( w_h = w_j \), then either \( \theta_{hj} = 1 \) or \( \theta_{hj} = 0 \) holds.

The load coupling model defines \( \rho_i \) to be the load of cell \( i \), which represents the proportion of cell \( i \)’s allocated time-frequency resource. Denote by \( x_u \) the proportion of RUs allocated to group \( u \). We have \( \rho_i = \sum_{u \in \mathcal{U}_i} x_u \). The cell load \( \rho_i \) indicates the likelihood that a UE receives interference from \( i \), and is hence used as an interference scaling parameter \([1], [2], [12]–[14]\), see [1]. We remark that, by [2], for any UE \( j \) served by cell \( i \), \( w_j \) changes with other cells’ resource allocation \( \rho_k \) \((k \in \mathcal{T} \setminus \{i\})\). Hence the decoding order depends on the network-wide resource allocation.

We use \( M \) and \( B \) to represent the total number of RUs in one cell and the bandwidth of each RU, respectively. To satisfy UE \( j \)’s demand \( d_j \), we have

\[
\sum_{u \in \mathcal{U}_i} MBc_{ju}x_u \geq d_j, \quad j \in J
\]  

(4)

imposing that \( d_j \) is satisfied by the sum of the demand delivered to UE \( j \) over all groups in \( \mathcal{U}_j \). In the discussion below, we use normalized \( d_j \) such that the two notations \( M \) and \( B \) are not necessary in our presentations.

**III. PROBLEM FORMULATION**

**A. Mathematical Formulation**

Consider optimizing resource allocation in NOMA networks for resource efficiency. As mentioned in Section II-C, \( \theta \) is the decoding order indicator. We optimize power split \( q_{u} \), group-level resource allocation \( x \), user group selection \( y \), and cell-level resource allocation \( \rho \). The objective function \( F \) is a generic cost function of the cell loads (i.e. time-frequency resource usage of cells) and \( F \) is monotonically increasing in \( \rho_1, \rho_2, \ldots, \rho_n \) element-wisely. The formulation is given in (5) below.

\[
\min_{\theta \in \{0,1\}} F(\rho_1, \rho_2, \ldots, \rho_n) \quad \text{s.t.} \quad \sum_{u \in \mathcal{U}_i} \log \left( 1 + \sum_{h \in u} q_{hu}\theta_{hj} + w_j \right) x_u \geq d_j, \quad \forall j
\]  

(5a)

\[
w_j = \sum_{k \in \mathcal{T}_j \setminus \{i\}} p_kg_{kj}\rho_k + \sigma^2 / g_{ij}, \quad j \in J_i, \quad \forall i
\]  

(5b)

\[
\sum_{j \in u} q_{ju} \leq p_i, \quad u \in \mathcal{U}_i, \quad \forall i
\]  

(5c)

\[
\rho_i = \sum_{u \in \mathcal{U}_i} x_u, \quad \forall i
\]  

(5d)

\[
\theta_{hj} \geq \min\{1, w_h - w_j\}, \quad h \neq j, \quad h, j \in u, \quad \forall u
\]  

(5e)

\[
\theta_{hj} + \theta_{j} = 1, \quad h \neq j, \quad h, j \in u, \quad \forall u
\]  

(5f)

\[
\theta_{hj} \in \{0,1\}, \quad h \neq j, \quad h, j \in u, \quad \forall u
\]  

(5g)

2With \( d_j \) being normalized by \( M \times B \) in this formulation. One can refer to [1] and [2] to verify that constraints [5d] along with [5c] are equivalent to [1] in Section II-C.

The user demands constraints are [5b] and [5c]. Constraints [5d] and [5e] impose the cell power limit and cell load.
limit, respectively. Constraints (51)–(55) are for the decoding order. Specifically, by (51) and (55) we have \( \theta_{hj} = 1 \) if \( w_h > w_j \), which is the rule of the correct decoding order in Section II-B. Constraints (52) are imposed for the case of \( w_h = w_j \) for any \( j \) and \( h \): the equation \( \theta_{hj} + \theta_{jh} = 1 \) makes sure that one must decode the other. Note that the decoding order depends on the cell level resource allocation, i.e., \( \rho_1, \rho_2, \ldots, \rho_n \).

We remark that if necessary, user group selection constraints can be added to (5), and our conclusion in this paper still holds.

**B. Obstacles of Solving (5)**

We remark that (5) is highly non-linear. In addition, a major obstacle for some iterative algorithms for (5) is that, the variation of resource allocation in each iteration leads to the change of decoding order for each group. There is an algorithmic framework derived in [1, 2], which uses a top-down paradigm (detailed in Section III-C). The basic idea is to break down (5) to single cell level and then solves the single-cell problems iteratively. For each iteration, there is an inner loop over the cells. This inner loop can be performed in parallel or sequentially. In the former case, the optimized resource allocation of all cells serves as the input of the next iteration. In the latter case, the optimized resource allocation of one cell is part of the input, when the subproblem of another cell is solved. Then, by fixed-point theory, the authors proved the convergence of the algorithm, as well as the optimality of the solution at the convergence.

However, the algorithmic framework relies on two restrictions of candidate groups, ([1] Lemma 1) and ([2] Lemma 1), respectively: Only those groups of which the decoding orders can be pre-determined, are considered for optimization. The other groups are eliminated from \( \mathcal{U} \). The limitation states that the decoding orders of all the candidate groups must be independent to the inter-cell interference such that they remain all the time, resulting in sub-optimality. To have a high probability of forming such groups, ([1] Lemma 1) and ([2] Lemma 1) require each group consists of up to two UEs. (Open Problem) We remark that, if the restrictions are dropped, then in each iteration, the variation of cell loads may lead to the change of decoding order. In this case, neither convergence nor optimality is known.

**C. Solution of ([1], [2]) for (5)**

By considering only groups for which decoding order is independent of interference, the variable \( \mathbf{\theta} \) along with (51)–(55) can be dropped from (5), since \( \mathbf{\theta} \) can be pre-determined in this special case. The algorithmic framework in ([1], [2]) is detailed as follows. Consider any cell \( i \), one can define the single-cell load minimization problem as a function of the other cells’ loads \( \mathbf{\rho} - i = [\rho_1, \rho_2, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_n] \), denoted by \( f_i \):

\[
\hat{f}_i(\mathbf{\rho} - i) = \min_{q_i, x_i, w_j} \rho_i \text{ s.t. (55) of cell } i \quad (6)
\]

The authors proved that, if the problem (6) is solvable, then (5) amounts to obtaining the fixed point of \( f_i \) (\( i \in \mathcal{I} \)). To be specific, the authors first proved that \( f_i \) (\( i \in \mathcal{I} \)) is standard interference function (SIF) ([42]), of which the definition is given below.

**Definition 1.** Any function \( f(\mathbf{\rho}) \) that has the following two properties, is an SIF, where \( \mathbf{\rho} \) is an arbitrary non-negative vector.

1) (Scalability) \( \alpha f(\mathbf{\rho}) > f(\alpha \mathbf{\rho}), \ \mathbf{\rho} \geq 0, \ \alpha > 1. \)
2) (Monotonicity) \( f(\mathbf{\rho}) \geq f(\mathbf{\rho'}), \ \mathbf{\rho} \geq \mathbf{\rho'}, \ \mathbf{\rho}, \mathbf{\rho'} \geq 0. \)

Based on the fact that \( f_i(\mathbf{\rho} - i) \) (\( i \in \mathcal{I} \)) is SIF, one can obtain the unique fixed point \( \mathbf{\rho}^* \) with \( \mathbf{\rho}^* = f(\mathbf{\rho}^*) \), by fixed-point iterations on \( f_i \), where \( f(\mathbf{\rho}) = [f_1(\mathbf{\rho} - 1), f_2(\mathbf{\rho} - 2), \ldots, f_n(\mathbf{\rho} - n)] \quad ([42]). \)

Namely, for the iterative process \( \mathbf{\rho}^{(k+1)} = f(\mathbf{\rho}^{(k)}) \) (\( k \geq 0 \)), we have \( \lim_{k \to \infty} \mathbf{\rho}^{(k)} = \mathbf{\rho}^* \), for arbitrary non-negative starting point \( \mathbf{\rho}^{(0)} \). The convergence rate is geometric ([43]). Based on the convergence of the fixed-point iterations, the authors proved that \( \mathbf{\rho}^* \) along with the other variables \( q, x, w \) that are obtained by solving (6) at \( \mathbf{\rho}^* \), for all \( i \in \mathcal{I} \), is optimal to (5).

**IV. RESULTS**

This section derives our theoretical results, which give the answer to the open problem in Section III. Our main conclusion is that ([1] Lemma 1) and ([2] Lemma 1) can be dropped, without loss of optimality or convergence of the proposed solution methods. To show this, we first prove a general conclusion in Section IV-A that is not tied to the load coupling system. The conclusion states that, even if algebraically one allows the capacity formula \( c_{j, u} = \log(1 + \gamma_{j, u}) \) with “decoding orders” in \( u \) to be all possible permutations of \( U \), the correct decoding order leads to the largest \( c_{j, u} \). Based on this, we prove in Section IV-B the convergence of the solution methods. We then show the optimality after the convergence proof.

**A. Pseudo Rate Region**

Consider rate region at the RU level. In this subsection, the interference is given, and hence the correct decoding order as well. We use \( \mathbf{\theta}^* \) to refer to this order:

\[
\theta_{hj}^* = 1 \iff w_h \geq w_j, \ \forall h, j.
\]

Consider one RU. Suppose there are \( K \) (\( K \geq 2 \)) UEs multiplexed on this RU. The UEs are indexed by following their correct decoding order. That is, UE 1 decodes UEs 2, . . . , \( K \). UE 2 receives interference from UE 1, and decodes UEs 3, . . . , \( K \), and so on. In this case, we have \( w_1 \leq w_2 \leq \ldots \leq w_K \). The capacity of UE \( j \) (\( j = 1, 2, \ldots, K \)), denoted by \( c_j \), is

\[
c_j = \log \left( 1 + \frac{q_j}{\sum_{h=1}^{j-1} q_h + w_j} \right).
\]

Considering RU power limit \( p \), the power split constraint reads

\[
\sum_{j=1}^{K} q_j \leq p. \quad (7)
\]

\[\text{If } w_h = w_j, \text{ then one of } \theta_{hj}^* \text{ and } \theta_{jh}^* \text{ equals 1 and the other equals 0.} \]
The rate of UEs $1, 2, \ldots, K$ are as follows.

\[
c_1 = \log \left( 1 + \frac{q_1}{w_1} \right), \\
c_2 = \log \left( 1 + \frac{q_2}{q_1 + w_2} \right), \\
\vdots \\
c_K = \log \left( 1 + \frac{q_K}{\sum_{h=1}^{K-1} q_h + w_K} \right)
\]

For user 1, we have

\[
c_1 = \log \left( 1 + \frac{q_1}{w_1} \right) \Rightarrow q_1 = w_1 e^{c_1} - w_1
\]

For user 2, we have

\[
c_2 = \log \left( 1 + \frac{q_2}{q_1 + w_2} \right) \\
\Rightarrow q_1 + q_2 = w_1 e^{c_1+c_2} + (w_2 - w_1)e^{c_2} - w_2
\]

By successively applying the same formula until the last user $K$, we obtain the equation below, where $w_0 = 0$.

\[
R_{\theta'}(c) = \sum_{j \in \mathcal{B}} q_j = \sum_{t=1}^{K} (w_t - w_{t-1}) e^{\sum_{\ell=t}^{K} c_{\ell}} - w_K
\]

where $c = [c_1, c_2, \ldots, c_K]$, and $\theta'$ indicates the correct decoding order. Consequently, the power split constraint (9) is equivalent to (8) below

\[
R_{\theta'}(c) \leq p,
\]

where the power split variables $q_1, q_2, \ldots, q_j$ are replaced by variables $c_1, c_2, \ldots, c_k$ that represent the rates, respectively for UEs $1, 2, \ldots, K$. The inequality (9) forms a bounded area and is the rate region of all the $K$ UEs.

We remark that though the discussion above is based on applying the successive rule on UEs by following their correct decoding order, the rule is also applicable for the case that UEs are ordered arbitrarily. Namely, for a group of UEs that are ranked indexed in any given permutation of the UEs, this successive rule also gives a formula with the same form of (8). We introduce notations to represent this formula in general. Define $\mathcal{B}$ as a domain of $\theta$, which is formed by constraints (5g) and (5h), i.e.,

\[
\mathcal{B} = \{ \theta : 5g \text{ and } 5h, \quad \theta_{hj} = 0 \quad \forall h = j \}\.
\]

It is easy to verify that there is a one-to-one mapping between all $\theta$ in $\mathcal{B}$ and all permutations of UEs $1, 2, \ldots, K$: If $\theta_{hj} = 1$ (indicating $\theta_{jk} = 0$), then $h$ is before $j$ in indexing, and vice versa. We use $R_{\theta}$ as a generic notation to represent (8) defined for the order indicated by $\theta$ ($\theta \in \mathcal{B}$), so as to distinguish from the formula $R_{\theta'}$ that is specified for the correct decoding order.

We name the region defined by $R_{\theta}(c) \leq p^{\max}$ with any $\theta \in \mathcal{B}$ as pseudo rate region.

\[
R_{\theta}(c) \leq p, \quad \theta \in \mathcal{B}
\]

The reason for the name “pseudo” is because, under $\theta$ ($\theta \neq \theta'$), the SIC cannot be successfully performed for all UEs.

**Theorem 1.** Any pseudo rate region is a subset of the rate region of the correct decoding order. Namely,

\[
\{ c : R_{\theta}(c) \leq p \} \subseteq \{ c : R_{\theta'}(c) \leq p \}
\]

or equivalently,

\[
R_{\theta'}(c) \leq R_{\theta}(c), \quad \forall c \geq 0
\]

holds for any $\theta \in \mathcal{B}$.

**Proof.** Consider the pseudo rate region for $\theta$ ($\theta \in \mathcal{B}$), i.e. $R_{\theta}(c) \leq p$. We index the UEs from 1 to $K$ by following the order indicated by $\theta$. We remark that if $\theta$ is not the correct decoding order (i.e. $\theta \neq \theta'$), then there must exist two UEs that are adjacent in the list, denoted by $\ell$ and $\ell + 1$, such that $w_\ell > w_{\ell+1}$. We swap the order of the two, and denote by $\theta'$ the new decoding order. Below, we prove $R_{\theta'}(c) \leq R_{\theta}(c)$ for any non-negative $c$.

To ease our representation, we define $w_0 = 0$ and $w_{K+1} = w_{K+2} = w_{K+3} = 0$. We also explicitly impose that for any summation notation $\sum_{t=a}^{b}$ in our expression, if $b < a$, then this term in the sum equals zero.

For $\ell$ and $\ell + 1$ ($\ell = 1, 2, \ldots, K - 1$), we have

\[
R_{\theta}(c) = \sum_{t=1}^{\ell-1} (w_t - w_{t-1}) e^{\sum_{\ell=t+1}^{K} c_k} + (w_{\ell+1} - w_\ell) e^{c_{\ell+1} + \sum_{k=\ell+2}^{K} c_k} + (w_{\ell+2} - w_{\ell+1}) e^{\sum_{k=\ell+2}^{K} c_k} + \sum_{t=\ell+2}^{K} (w_{t+1} - w_t) e^{\sum_{k=t+1}^{K} c_k}
\]

and

\[
R_{\theta'}(c) = \sum_{t=1}^{\ell-1} (w_t - w_{t-1}) e^{\sum_{\ell=t+1}^{K} c_k} + (w_{\ell+1} - w_\ell) e^{c_{\ell} + \sum_{k=\ell+2}^{K} c_k} + (w_{\ell+2} - w_{\ell+1}) e^{\sum_{k=\ell+2}^{K} c_k} + \sum_{t=\ell+2}^{K} (w_{t+1} - w_t) e^{\sum_{k=t+1}^{K} c_k}
\]

We remark that, the difference $R'(c) - R(c)$ makes the two summation terms in both the head and tail (if either exists) disappear. See (11) below.

In the result of (11), because $w_\ell > w_{\ell+1}$ and $c_\ell \geq 0$ ($\ell = 1, 2, \ldots, K$), we conclude

\[
R_{\theta'}(c) \leq R_{\theta}(c), \quad \forall c \geq 0.
\]

As a result,

\[
\{ c : R_{\theta}(c) \leq p \} \subseteq \{ c : R_{\theta'}(c) \leq p \}
\]

(12)

The result in (12) shows that, for two adjacent UEs $\ell$ and $\ell + 1$ with $w_\ell > w_{\ell+1}$, swapping the order of the two UEs
for any Lemma 1. variable in the fixed-point iterations, for which we will prove the convergence and optimality.

\[ R_\theta(c) - R_\theta(c') = (w_{\ell+1} - w_{\ell-1})e^{\sum_{k=1}^{K} c_k} + (w_{\ell} - w_{\ell+1})e^{\sum_{k=1}^{K} c_k} + (w_{\ell+2} - w_{\ell})e^{\sum_{k=1}^{K} c_k} - (w_{\ell} - w_{\ell+1})e^{\sum_{k=1}^{K} c_k} - (w_{\ell+1} - w_{\ell})e^{\sum_{k=1}^{K} c_k} 
+ (w_{\ell+2} - w_{\ell})e^{\sum_{k=1}^{K} c_k} - (w_{\ell} - w_{\ell+1})e^{\sum_{k=1}^{K} c_k} - (w_{\ell+1} - w_{\ell})e^{\sum_{k=1}^{K} c_k} 
+ e^{\sum_{k=1}^{K} c_k} ((w_{\ell+1} - w_{\ell-1}) - (w_{\ell} - w_{\ell-1}) + (w_{\ell+2} - w_{\ell}) - (w_{\ell+2} - w_{\ell+1})) 
+ e^{\sum_{k=1}^{K} c_k} (w_{\ell+1} - w_{\ell}) + e^{\sum_{k=1}^{K} c_k} (w_{\ell+1} - w_{\ell}) - e^{\sum_{k=1}^{K} c_k} 
= (w_{\ell+1} - w_{\ell})e^{\sum_{k=1}^{K} c_k} + e^{\sum_{k=1}^{K} c_k} (e^{c_{\ell}} - e^{c_{\ell+1}} + 1) 
= (w_{\ell+1} - w_{\ell})e^{\sum_{k=1}^{K} c_k} (e^{c_{\ell}} - 1)(e^{c_{\ell+1}} - 1) \] (11)

enlarges the pseudo rate region. We therefore conclude that the correct decoding order yields the largest rate region, namely, both

\[ \{ c : R_\theta(c) \leq \rho_i \} \subseteq \{ c : R_\theta(c) \leq \rho_i \} \]

and

\[ R_\theta(c) \leq R_\theta(c), \forall c \geq 0 \]

hold for any \( \theta \in B \). \( \square \)

B. Convergence and Optimality of Fixed-Point Algorithm for Load Coupling with NOMA

In this section, we investigate the convergence of the approach for solving (5) as outlined in Section III without any restriction/limitation. We first re-define the problem \( f_i \) in (6) in Section III by taking into consideration the variable \( \theta \) that characterizes the dependency between decoding orders and interference.

\[ f_i(\rho_{-i}) = \min_{q, x, w, \theta} \rho_i \text{ s.t. } (5b)-(5g) \text{ of cell } i \] (13)

We remark that, though \( \theta \) is variable in (13), it will induce the correct decoding order \( \theta^* \) by constraints (5i)-(5n), as \( \rho_{-i} \) and hence interference are known.

Therefore, \( \theta \) is determined for any given \( \rho_{-i} \) in (13). However, we do not eliminate the \( \theta \) variables from (13), because even though \( \theta^* \) is directly induced by \( \rho_{-i} \), the latter is variable in the fixed-point iterations, for which we will prove the convergence and optimality.

We first prove Lemma 1 below, which will be used later to prove the convergence of fixed-point iterations on \( f_i \).

**Lemma 1.** Given non-negative \( w_j \), the inequalities below hold for any \( \alpha > 1 \).

\[ \frac{1}{\alpha} c_{j\alpha}(w_j) < c_{j\alpha}(\alpha w_j) \]

\[ 1/c_{j\alpha}(w_j) < \frac{\alpha}{c_{j\alpha}(\alpha w_j)} \]

\[ 1/c_{j\alpha}(w_j) < \frac{\alpha}{c_{j\alpha}(\alpha w_j)} \]

\[ \frac{1}{\alpha} c_{j\alpha}(w_j) < c_{j\alpha}(\alpha w_j) \]

\[ \square \]

**Proof.** Since \( 1/c_{j\alpha}(w_j) \) is strictly concave in \( \rho_{-i} \), we have

\[ \frac{1}{\alpha} c_{j\alpha}(w_j) < \frac{\alpha}{c_{j\alpha}(\alpha w_j)} \]

We use \( f_i(\rho_{-i}, \theta) \) to represent the optimization problem defined in (13) under any given \( \theta \) \((\theta \in B)\). Mathematically:

\[ f_i(\rho_{-i}, \theta) = \min_{q, x, w, \theta} \rho_i \text{ s.t. } (5b)-(5e) \text{ of cell } i \] (14)

We remark that by variable substitution as in Section IV-A one has a reformulation of \( f_i(\rho_{-i}, \theta) \), with \( q \) replaced by \( c \):

\[ f_i(\rho_{-i}, \theta) = \min_{c, x, w, \theta} \rho_i \text{ s.t. } R_\theta(c, w) \leq p_i \] (15a)

\[ \text{s.t. } R_\theta(c, w) \leq p_i \text{ s.t. } R_\theta(c, w) \leq p_i \] (15b)

\[ \text{and } R_\theta(c, w) \leq p_i \] (15c)

**Lemma 2.** For any given \( \theta \) \((\theta \in B)\), \( f_i(\rho_{-i}, \theta) \) is an SIF of \( \rho_{-i} \) \((\rho_{-i} \geq 0)\).

**Proof.** (Monotonicity) Consider any \( f_i(\rho_{-i}, \theta) \) \((\theta \in B)\). Consider (14). For any \( \rho_{-i} \) and \( \rho_{-i}^\prime \) with \( \rho_{-i}^\prime \leq \rho_{-i} \), we have \( w_j(\rho_{-i}) \geq w_j(\rho_{-i}^\prime) \) \((j \in J_i)\). Therefore \( c_{j\alpha}(\rho_{-i}) \leq c_{j\alpha}(\rho_{-i}^\prime) \). Replacing \( c_{j\alpha}(\rho_{-i}) \) by \( c_{j\alpha}(\rho_{-i}^\prime) \) leads to a relaxation on the constraints (5b), resulting in lower objective value. We then conclude \( f_i(\rho_{-i}^\prime, \theta) \leq f_i(\rho_{-i}, \theta) \) for any \( \theta \in B \).

(Scalability) Denote the value of \( f_i(\rho_{-i}, \theta) \) by \( \rho_i^\prime \). i.e. \( f_i(\rho_{-i}, \theta) = \rho_i^\prime \). Denote the optimal solution of \( f_i(\rho_{-i}, \theta) \) by \( (\theta', \alpha x') \). Under \( \rho_{-i} \), consider the following minimization problem. Denote its optimal objective value by \( z \).

\[ z = \min_{q, x, w, \theta} \rho_i \text{ s.t. } \frac{1}{\alpha} \sum_{u \in U_i} c_{j\alpha}(q, w_j) x_u \geq d_j, j \in J_i \] (16a)

\[ \text{s.t. } \frac{1}{\alpha} \sum_{u \in U_i} c_{j\alpha}(q, w_j) x_u \geq d_j, j \in J_i \] (16b)

\[ \text{in } (5c)-(5e) \text{ of cell } i \] (16c)

It is straightforward to verify that \( (\theta'', \alpha x'') \) is feasible to (16), with the objective value equaling \( \alpha f_i(\rho_{-i}, \theta) \). We conclude that the optimum of (16) is no higher than \( \alpha f_i(\rho_{-i}, \theta) \). Namely, we have

\[ z \leq 0 f_i(\rho_{-i}, \theta) \].

(17)

For \( f_i(\alpha \rho_{-i}, \theta) \), the corresponding formulation is as follows, where we remark that multiplying \( \alpha \) on \( \rho_{-i} \) is equivalent to performing the multiplication on \( w_j \) for all \( j \in J_i \).

\[ f_i(\alpha \rho_{-i}, \theta) = \min_{q, x, w, \theta} \rho_i \text{ s.t. } \sum_{u \in U_i} c_{j\alpha}(q, \alpha w_j) x_u \geq d_j, j \in J_i \] (18a)

\[ \text{s.t. } \sum_{u \in U_i} c_{j\alpha}(q, \alpha w_j) x_u \geq d_j, j \in J_i \] (18b)

\[ \text{in } (5c)-(5e) \text{ of cell } i \] (18c)

Note that (18) differs from (16) only in (18b). Note that (16b) is equality at the optimum. By Lemma 1 for any solution
We then conclude both the monotonicity and the scalability fixed-point iterations for SIF is proved in [43].

Based on Theorem 2. One can refer to [1, Theorem 3].

Lemma 3. $\theta^*$ is optimal to $\min_{\theta \in B} f_i(\rho_{-i}, \theta)$ (i ∈ I), i.e.,

$$f_i(\rho_{-i}, \theta^*) = \min_{\theta \in B} f_i(\rho_{-i}, \theta), \quad i \in I.$$ 

Proof. Consider any cell $i$ (i ∈ I) and any decoding order $\theta$ other than the correct one. By Theorem 1 under fixed $c$ and $w$, replacing $R\theta(c, w)$ by $R\theta\prime(c, w)$ makes the constraint (15b) remain satisfied (or relaxed if $R\theta(c, w) < R\theta\prime(c, w)$), such that one will not get a worse objective value under the correct decoding order. This conflicts with that $\theta$ is the optimal solution to $\min_{\theta \in B} f_i(\rho_{-i}, \theta)$. Hence, we conclude that, at the optimum of $\min_{\theta \in B} f_i(\rho_{-i}, \theta)$, $\theta$ is (or can be replaced by) $\theta^*$. Hence $\theta = \theta^*$ at the optimality of $\min_{\theta \in B} f_i(\rho_{-i}, \theta)$. □

We then prove Theorem 2 below.

Theorem 2. The function $f_i(\rho_{-i})$ (i ∈ I) in (13) is SIF.

Proof. First, note that $f_i(\rho_{-i}) = f_i(\rho_{-i}, \theta^*)$, by the definitions of the two functions $f_i(\rho_{-i})$ and $f_i(\rho_{-i}, \theta)$, and $\theta^*$. Second, by Lemma 3 we have $f_i(\rho_{-i}, \theta^*) = \min_{\theta \in B} f_i(\rho_{-i}, \theta)$.

Therefore, we conclude $f_i(\rho_{-i}) = \min_{\theta \in B} f_i(\rho_{-i}, \theta)$.

Consequently, consider any non-negative $\rho_{-i}^\prime \leq \rho_{-i}$ with $\rho_{-i}^\prime \leq \rho_{-i}$ and $\alpha > 1$, by applying Lemma 2 we have

$$\min_{\theta \in B} f_i(\rho_{-i}^\prime, \theta) \leq \min_{\theta \in B} f_i(\rho_{-i}, \theta)$$

and

$$\min_{\theta \in B} \alpha f_i(\rho_{-i}, \theta) < \min_{\theta \in B} \alpha f_i(\rho_{-i}, \theta).$$

We then conclude both the monotonicity and the scalability hold for $f_i(\rho_{-i})$.

The following corollary shows an algorithmic framework for optimally solving problem (5). Briefly, one only needs to apply fixed-point iterations on all $f_i$ (i ∈ I) to reach the optimality. Given cell loads $\rho_{-i}$, evaluating $f_i(\rho_{-i})$ (i ∈ I) submits to solving a single-cell load minimization problem.

Corollary 1. The iterations $\rho^{(k+1)} = f(\rho^{(k)})$, with arbitrary starting point $\rho^{(0)} = 0$, converge to a unique fixed-point $\rho^*$, such that $\rho^* = f(\rho^*)$, with linear convergence rate. Let $q^*, x^*, w^*, y^*, \theta^*$ be the solution obtained by solving the problems $f_i(\rho_{-i})$ for all $i \in I$. Then for problem (5), we have

1) The optimal solution is $q^*, x^*, w^*, y^*, \theta^*$.
2) The optimal objective value is $F(\rho^*_1, \rho^*_2, \ldots, \rho^*_n)$.

The proof of Corollary 1 can be straightforwardly derived based on Theorem 2. One can refer to [1] Theorem 3] or [2] Theorem 6] for more details. The convergence rate of the fixed-point iterations for SIF is proved in [43].

V. Discussion

This section discusses the potential application of our derived results in Section IV with respect to NOMA resource optimization, along three dimensions: problem formulation, tractability, and optimality.

A. Decoding Order Constraints

It is worth noting that constraints (5b) are redundant for (5). Consider the formulation below.

$$\min_{q, x, \rho, w \geq 0} F(\rho_1, \rho_2, \ldots, \rho_n) \text{ s.t. } (5b)-(5e), \quad (5g), \quad \text{and } (5h)$$

Theorem 1 along with the analysis in Section IV indeed reveals that at its optimum, $\theta = \theta^*$. Namely, for any $\theta_{hj}$ at the optimum, if $\theta_{hj} = 1$, we must have $w_h \geq w_j$, meaning that $\theta_{hj}$ satisfies $\theta_{hj} \geq \min\{1, w_h - w_j\}$. Hence the non-linear constraints (5b) are redundant and can be removed from the formulation (5).

B. Tractability of (5)

We remark that even though (5) is a resource optimization problem in multi-cell scenarios, the difficulty indeed lies in its corresponding single-cell load minimization problems, i.e., (13). By Lemma 3 we reach the optimum at $\theta^*$. Once $f_i(\rho_{-i}, \theta^*)$ can be solved to optimality, then as pointed out by Corollary 1 the optimum of (5) can be straightforwardly obtained in linear convergence. There are some special cases of (14) that submit to a polynomial-time solution, briefly discussed below.

If the power allocation $q$ is fixed, then the single-cell load minimization problem (5) is a linear programming problem in $x$ and $w$ [1]. We remark that (14) can be reformulated to (15) by using the successive rule in Section IV-A. As the second case, if the demand on each user group is given, then the variable $x$ can be eliminated, such that (15) is a convex programming formulation of $c$ and $w$ [44].

Consider another case where the number of UEs in each group is no more than two (i.e., $|u| \leq 2$ for all $u \in U$), and there is no overlapping UE for any two selected groups. Then (5) can be solved optimally within polynomial time by combinatorial optimization [2]. The basic idea is to bring the single-cell load optimization down to the group level, and then prove that the optimal group selection amounts to solving a maximum independent edge set problem.

For the general case of (14), whether it is tractable or not, remains open, suggesting future research to be done along this direction.

C. A Comment on [1], [2]

In [1], [2], the proposed solution has guaranteed convergence if some restrictions, [1] Lemma 1 and [2] Lemma 1, are imposed. The results derived in this technical note provide a complementary theoretical insight, namely, the restrictions can be dropped without any loss of optimality or convergence.

We remark that the convexity of constraints (15b) holds only if $\theta$ is set to be the correct decoding order (i.e., $\theta_{hj} = 1$ if and only if $w_h > w_j$ (h, j ∈ J)). By Lemma 3 we know that this is the only case that needs to be taken into account.
This technical note has addressed the convergence and optimality of an algorithmic framework for solving a class of resource optimization problems in multi-cell NOMA networks. The note proved that the correct decoding order corresponds to the largest region. Then, results for convergence and optimality, with variable decoding order in the iterative process, are formally established. The note has also discussed the tractability of multi-cell resource optimization with load coupling, and reveals that solving the single-cell problem is the key in terms of tractability.

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