FOCK REPRESENTATIONS AND QUANTUM MATRICES

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In this paper we study the Fock representation of a certain $*$-algebra which appears naturally in the framework of quantum group theory. It is also a generalization of the twisted CCR-algebra introduced by W. Pusz and S. Woronowicz. We prove that the Fock representation is a faithful irreducible representation of the algebra by bounded operators in a Hilbert space, and, moreover, it is the only (up to unitary equivalence) representation possessing these properties.

Keywords and phrases: Fock representation, quantum groups, bounded symmetric domain, non-compact Hermitian symmetric spaces

1 Introduction

This work deals with the $*$-algebras $\text{Pol} (\text{Mat}_{m,n})_q$, $0 < q < 1$, $1 \leq m \leq n$, defined by $q$-analogues of the canonical commutation relations (see section 2). Our main result is that the Fock representation of every such algebra is its only faithful irreducible $*$-representation by bounded operators.

Let us explain how the $*$-algebra $\text{Pol} (\text{Mat}_{m,n})_q$ appears in quantum group theory. It was demonstrated by Harish-Chandra that every Hermitian symmetric spaces of non-compact type admits a standard embedding into a vector space as a bounded symmetric domain. A $q$-analogue of the Harish-Chandra embedding was constructed in [25], along with $q$-analogues of the polynomial algebras on vector spaces. The approach, used in [25] in constructing some quantum polynomial algebras, is similar to the suggestion of V. Drinfeld [6] to construct algebras of functions on quantum groups via duality arguments. In the special case of the symmetric space $SU_{n,m}/S(U_n \times U_m)$ one has the $*$-algebra $\text{Pol} (\text{Mat}_{m,n})_q$ described in section 8 of [25]. Our interest in the algebra $\text{Pol} (\text{Mat}_{m,n})_q$ is inspired by the isomorphism

$$\text{Pol} (\text{Mat}_{m,n})_q \simeq \text{Pol} (\text{Mat}_{m,n})_q.$$  \hspace{1cm} (1.1)

In the special cases $m = 1, m = n = 2$, the $*$-algebras we are interested in were considered before by W. Pusz, S. Woronowicz [23] and by D. Proskurin, L. Turowska [22, 28]. Under these restrictions our result is already known. In fact, the principal result of this paper is also valid for all $*$-algebras introduced in [25]. This will be shown in a subsequent work.

Here is the outline of the present paper. In the first part (sections 2 – 6) we sketch the proof of the main result, theorem 2.6. This part presents proofs of only those statements
which do not involve the theory of quantum universal enveloping algebras [8]. The second part (sections 7 – 8), after recalling the principal concepts of that theory, presents a construction of the homomorphism of $*$-algebras

$$\text{Pol}(\text{Mat}_{m,n})_q \xrightarrow{\sim} \text{Pol}(\text{Mat}_{m,n})_q.$$  

The third part (sections 9 – 12) finishes the proof of all the statements formulated before and, in particular, establishes the isomorphism (1.1).

The classical work of W. Arveson [1] initiated the research in non-commutative complex analysis. The study of non-commutative analogues of function algebras on bounded symmetric domains started in [25, 29]. Our goal here is to study representations of $\text{Pol}(\text{Mat}_{m,n})_q$ by bounded operators.

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## 2 Statement of the main results

In what follows $\mathbb{C}$ will be treated as a ground field. We assume that all the algebras under consideration are unital and $q \in (0, 1)$, unless the contrary is stated explicitly. Consider the well known algebra $\mathbb{C}[\text{Mat}_{m,n}]_q$ defined by its generators $z^\alpha_a$, $\alpha = 1, \ldots, m$; $a = 1, \ldots, n$, and the commutation relations

$$z^\alpha_a z^\beta_b - q z^\beta_b z^\alpha_a = 0, \quad a = b \quad \& \quad \alpha < \beta, \quad \text{or} \quad a < b \quad \& \quad \alpha = \beta, \quad (2.1)$$

$$z^\alpha_a z^\beta_b - z^\beta_b z^\alpha_a = 0, \quad \alpha < \beta \quad \& \quad a > b, \quad (2.2)$$

$$z^\alpha_a z^\beta_b - z^\beta_b z^\alpha_a - (q - q^{-1}) z^\alpha_a z^\beta_b = 0, \quad \alpha < \beta \quad \& \quad a < b. \quad (2.3)$$

This algebra is a quantum analogue of the polynomial algebra $\mathbb{C}[\text{Mat}_{m,n}]$ on the space of matrices. It will be convenient for us to introduce an additional assumption $m \leq n$.

The algebra admits a natural gradation given by $\text{deg} z^\alpha_a = 1$. By using the diamond lemma [2], a basis of lexicographically ordered monomials in the vector space $\mathbb{C}[\text{Mat}_{m,n}]_q$ can be constructed [5, p.p. 169 – 171]. This implies that the dimensions of the corresponding graded components of $\mathbb{C}[\text{Mat}_{m,n}]_q$ and $\mathbb{C}[\text{Mat}_{m,n}]$ are the same.

In a similar way, introduce the algebra $\mathbb{C}[\text{\overline{Mat}}_{m,n}]_q$, defined by its generators $(z^\alpha_a)^*$, $\alpha = 1, \ldots, m$, $a = 1, \ldots, n$, and the relations

$$(z^\alpha_a)^* (z^\alpha_b)^* - q (z^\alpha_b)^* (z^\alpha_a)^* = 0, \quad a = b \quad \& \quad \alpha < \beta, \quad \text{or} \quad a < b \quad \& \quad \alpha = \beta, \quad (2.4)$$

$$(z^\alpha_a)^* (z^\alpha_b)^* - (z^\alpha_b)^* (z^\alpha_a)^* = 0, \quad \alpha < \beta \quad \& \quad a > b, \quad (2.5)$$

$$(z^\alpha_a)^* (z^\alpha_b)^* - (z^\alpha_b)^* (z^\alpha_a)^* - (q - q^{-1}) (z^\alpha_b)^* (z^\alpha_a)^* = 0, \quad \alpha < \beta \quad \& \quad a < b. \quad (2.6)$$

A gradation in $\mathbb{C}[\text{\overline{Mat}}_{m,n}]_q$ is given by $\text{deg}(z^\alpha_a)^* = -1$.

Finally, consider the algebra $\text{Pol}(\text{Mat}_{m,n})_q$ whose generators are $z^\alpha_a$, $(z^\alpha_a)^*$, $\alpha = 1, \ldots, m$, $a = 1, \ldots, n$, and the list of relations is formed by (2.1) – (2.6) and

$$(z^\beta_b)^* z^\alpha_a = q^2 \cdot \sum_{a^\prime, b^\prime = 1}^{n} \sum_{\alpha^\prime, \beta^\prime = 1}^{m} R^{\prime \beta^\prime}_{\beta \beta^\prime} R^{\prime \alpha^\prime}_{\alpha \alpha^\prime} \cdot z^\alpha_a (z^\beta_b)^* + (1 - q^2) \delta_{ab} \delta_{\alpha \beta}, \quad (2.7)$$

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with $\delta_{ab}, \delta^{\alpha\beta}$ being the Kronecker symbols, and

$$R_{ij}^{kl} = \begin{cases} 
q^{-1}, & i \neq j \ & i = k \ & j = l \\
1, & i = j = k = l \\
-(q^{-2} - 1), & i = j \ & k = l \ & l > j \\
0, & \text{otherwise.}
\end{cases}$$

The involution in $\text{Pol}(\text{Mat}_{m,n})_q$ is introduced in an obvious way: $*: z^{\alpha}_a \mapsto (z^{\alpha}_a)^*$. 

**Proposition 2.1** The linear map

$$\mathbb{C}[\text{Mat}_{m,n}]_q \otimes \mathbb{C}[\text{Mat}_{m,n}]_q \simeq \text{Pol}(\text{Mat}_{m,n})_q, \quad f \otimes g \mapsto f \cdot g$$

is one-to-one.

**Proof.** See Corollary 10.4.

**Corollary 2.2** The algebra homomorphisms

$$\mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow \text{Pol}(\text{Mat}_{m,n})_q, \quad f \mapsto f \otimes 1,$$

$$\mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow \text{Pol}(\text{Mat}_{m,n})_q, \quad f \mapsto 1 \otimes f$$

are embeddings.

We sketch here an explicit construction for a faithful irreducible $*-$representation of $\text{Pol}(\text{Mat}_{m,n})_q$. Consider a $\text{Pol}(\text{Mat}_{m,n})_q$-module $H$ determined by a single generator $v_0$ and the relations

$$(z^{\alpha}_a)^* v_0 = 0, \quad \alpha = 1, \ldots, m, \ a = 1, \ldots, n.$$  \hspace{1cm} (2.8)

It follows easily from proposition 2.1 that

**Corollary 2.3**

i) $H = \mathbb{C}[\text{Mat}_{m,n}]_q v_0$;

ii) there exists a unique sesquilinear form $(.,.)$ on $H$ with the following properties:

1. $(v_0, v_0) = 1$;

2. $(fu, v) = (u, f^*v), \ f \in \text{Pol}(\text{Mat}_{m,n})_q, \ u, v \in H$.

**Remark.** To write the form $(.,.)$ explicitly, we introduce a bigradation on the vector space $\text{Pol}(\text{Mat}_{m,n})_q$:

$$\text{Pol}(\text{Mat}_{m,n})_q \simeq \bigoplus_{i,j=0}^{\infty} \text{Pol}(\text{Mat}_{m,n})_{q,i,-j}$$  \hspace{1cm} (2.9)

with $\text{Pol}(\text{Mat}_{m,n})_{q,i,-j} = \mathbb{C}[\text{Mat}_{m,n}]_{q,i} \cdot \mathbb{C}[\text{Mat}_{m,n}]_{q,-j}$, and then define a linear functional $\omega : \text{Pol}(\text{Mat}_{m,n})_q \rightarrow \mathbb{C}$ which is just the projection onto the (one-dimensional) homogeneous component $\text{Pol}(\text{Mat}_{m,n})_{q,0,0} \simeq \mathbb{C}$ parallel to direct sum of all other homogeneous components. Evidently, $(fv_0, gv_0) = \omega(g^*f), \ f, g \in \mathbb{C}[\text{Mat}_{m,n}]_q$. 

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Proposition 2.4  The above sesquilinear form $(.,.)$ on $\mathcal{H}$ is positive definite.

Proof. See section 5, in particular, proposition 5.1. □

$\mathcal{H}$ becomes a pre-Hilbert space, so the notion of boundedness for linear maps in $\mathcal{H}$ makes sense.

Proposition 2.5  For every $f \in \text{Pol}(\text{Mat}_{m,n})_q$, the linear map on $\mathcal{H}$, given by $T(f) : v \mapsto fv$, is bounded.

Proof. See corollary 6.3. □

Thus, $T$ extends up to a $*$-representation $\overline{T}$ of $\text{Pol}(\text{Mat}_{m,n})_q$ in the Hilbert space $\overline{\mathcal{H}}$, a completion of $\mathcal{H}$, by bounded operators.

Theorem 2.6  $\overline{T}$ is a faithful irreducible $*$-representation of $\text{Pol}(\text{Mat}_{m,n})_q$ by bounded operators. A representation with these properties is unique up to unitary equivalence.

Proof. See proposition 3.5 (uniqueness), proposition 12.2 (faithfulness), and section 5 (irreducibility). □

3  An auxiliary algebra $\mathbb{C}[\tilde{G}]_q$

To prove theorem 2.6, we need to introduce an auxiliary $*$-algebra $\mathbb{C}[\tilde{G}]_q$. Let $N = m + n$. Consider the Hopf algebra $\mathbb{C}[SL_N]_q$ introduced in the profound works [6, 7] which is defined by its generators $\{t_{ij}\}_{i,j=1,...,N}$ and the relations

\begin{align*}
t_{aa}t_{βb} - qt_{βb}t_{aa} &= 0, & a = b & \& \alpha < β, \text{ or } a < b & \& \alpha = β, \quad (3.1) \\
t_{aa}t_{βb} - t_{βb}t_{aa} &= 0, & a < β & \& a > b, \quad (3.2) \\
t_{aa}t_{βb} - t_{βb}t_{aa} - (q - q^{-1})t_{ab}t_{ab} &= 0, & a < β & \& a < b, \quad (3.3) \\
\det_q t &= 1. \quad (3.4)
\end{align*}

Here $\det_q t$ is a $q$-determinant of the matrix $t = (t_{ij})_{i,j=1,...,N}$:

$$
\det_q t = \sum_{s \in S_N} (-q)^{l(s)} t_{1s(1)} t_{2s(2)} \cdots t_{Ns(N)}, \quad (3.5)
$$

with $l(s) = \text{card}\{(i, j) | i < j \& s(i) > s(j)\}$. The comultiplication $\Delta$, the counit $\varepsilon$, and the antipode $S$ are defined as follows:

$$
\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}, \quad S(t_{ij}) = (-q)^{i-j} \det_q t_{ji},
$$

with $t_{ji}$ being the matrix derived from $t$ by discarding its $j$-th row and $i$-th column.

Let $\mathbb{C}[\tilde{G}]_q \overset{\text{def}}{=} (\mathbb{C}[SL_N]_q, *)$ with the involution $*$ been given by

$$
t_{ij}^* = \text{sign} ((i - m - 1/2)(n - j + 1/2)) (-q)^{j-i} \det_q t_{ij}. \quad (3.6)
$$
The involution is well defined, as one can see by comparing it to the involution on the algebra $\mathbb{C}[SU_N]_q$ of regular functions on the quantum group $SU_N$ (4.3).\footnote{It is worth noting that $\mathbb{C}[	ilde{G}]_q$ is not a Hopf $*$-algebra.}

Recall a standard notation for $q$-minors of $t$:

$$t^\wedge_{i,j} = \sum_{s \in S_k} (-q)^{l(s)} t_{i_1j_1} \cdots t_{i_kj_k},$$

with $I = \{(i_1, i_2, \ldots, i_k) | 1 \leq i_1 < i_2 < \cdots < i_k \leq N\}$, $J = \{(j_1, j_2, \ldots, j_k) | 1 \leq j_1 < j_2 < \cdots < j_k \leq N\}$. Introduce the elements

$$t = t^\wedge_{1,2,\ldots,m} \{n+1,n+2,\ldots,N\}, \quad x = tt^*.$$

It follows from the definitions that $t$, $tt^*$, and $x$ quasi-commute with all the generators of $\mathbb{C}[SL_N]_q$. More precisely,

**Proposition 3.1**

i) $tt^* = t^*t$;

ii) for every polynomial $f$ of a single indeterminate

$$t_{ij} \cdot f(t) = \begin{cases} f(q^\alpha t_{ij}), & i \leq m \quad \& \quad j \leq n, \\ f(q^{-1}t_{ij}), & i > m \quad \& \quad j > n, \\ f(t_{ij}), & \text{otherwise}, \end{cases}$$

$$t_{ij} \cdot f(t^*) = \begin{cases} f(q^\alpha t_{ij}), & i \leq m \quad \& \quad j \leq n, \\ f(q^{-1}t_{ij}), & i > m \quad \& \quad j > n, \\ f(t^*)t_{ij}, & \text{otherwise}, \end{cases}$$

$$t_{ij} \cdot f(x) = \begin{cases} f(q^2t_{ij}), & i \leq m \quad \& \quad j \leq n, \\ f(q^{-2}t_{ij}), & i > m \quad \& \quad j > n, \\ f(x)t_{ij}, & \text{otherwise}. \end{cases}$$

Let $\mathbb{C}[	ilde{G}]_{q,x}$ be the localization of $\mathbb{C}[	ilde{G}]_q$ with respect to the multiplicative set $x, x^2, x^3, \ldots$. An involution in $\mathbb{C}[	ilde{G}]_{q,x}$ is imposed in a natural way: $(x^{-1})^* = x^{-1}$. Of course, $\mathbb{C}[	ilde{G}]_q \hookrightarrow \mathbb{C}[	ilde{G}]_{q,x}$.\footnote{It is well known that $\mathbb{C}[SL_N]_q$ is a domain [9, Lemma 9.1.9].} Note that the element $t$ is invertible in the algebra $\mathbb{C}[	ilde{G}]_{q,x}$:

$$t^{-1} = t^*x^{-1}.$$

**Proposition 3.2** The map

$$i : z^\alpha_a \mapsto t^{-1}t^\wedge_{1,2,\ldots,m} J_{aa},$$

with $J_{aa} = \{n+1,n+2,\ldots,N\} \setminus \{N+1-\alpha\} \cup \{a\}$, admits a unique extension up to an embedding of $*$-algebras $i : \text{Pol(Mat}_{m,n})_q \hookrightarrow \mathbb{C}[	ilde{G}]_{q,x}$.

**Proof.** See proposition 9.8 and the final remarks in section 12. \hfill $\square$

Introduce the element $y \in \text{Pol(Mat}_{m,n})_q$ given by

$$y = 1 + \sum_{k=1}^{m} (-1)^k \sum_{|J'| = k} \sum_{|J''| = k} z^{k_{J'}J''} \left( z^{k_{J'}J''} \right)^*.$$

(The $q$-minors $z^{k_{J'}J''}$ for $z = \{z^a\}$ are defined in the same way as those for $t$.)

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\textit{Note:} The notation $\mathbb{C}[\tilde{G}]_q$ is used for the quantum group $SU_N$ with $q$-analog of the identity matrix $I_q$. The involution $t^*$ is defined using the $q$-minors and quasi-commutes with all the generators. The localization $\mathbb{C}[	ilde{G}]_{q,x}$ is introduced to impose the natural involution on $x, x^2, x^3$, etc. The proposition 3.2 states the existence of an embedding of $*$-algebras $i : \text{Pol(Mat}_{m,n})_q \hookrightarrow \mathbb{C}[	ilde{G}]_{q,x}$.

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Proposition 3.3

\[ i(y) = x^{-1}. \] (3.10)

**Proof.** See section 11.

Proposition 3.1 and formulae (3.9), (3.10) imply

**Corollary 3.4** For all \( \alpha = 1, \ldots, m, \) \( a = 1, \ldots, n, \) one has

\[ z_\alpha^a y = q^{-2} y z_\alpha^a, \quad (z_\alpha^a)^* y = q^2 y (z_\alpha^a)^*. \]

This allows one to prove the uniqueness statement of our main theorem 2.6.

**Proposition 3.5** A faithful irreducible *-representation of \( \text{Pol}(\text{Mat}_{m,n})_q \) by bounded operators in a Hilbert space, if exists, is unique up to unitary equivalence.

**Proof.** Let \( \pi, \pi' \) be two faithful irreducible *-representations of \( \text{Pol}(\text{Mat}_{m,n})_q \) by bounded linear operators in the Hilbert spaces \( H, H' \). In particular, \( \pi(y) \) and \( \pi'(y) \) are non-zero bounded self-adjoint operators. The same standard argument as in [30] can be used to prove that the non-zero spectra of the self-adjoint operators \( \pi(y), \pi'(y) \) are discrete. Consider eigenvectors \( v \) of \( \pi(y) \) and \( v' \) of \( \pi'(y) \) with \( \|v\| = \|v'\| = 1 \), associated to a largest modulus eigenvalue of \( \pi(y) \) and \( \pi'(y) \), respectively. By a virtue of corollary 3.4, \( \pi((z_\alpha^a)^*) v = 0, \pi'((z_\alpha^a)^*) v' = 0, \alpha = 1, 2, \ldots, m, \) \( a = 1, 2, \ldots, n. \) It is easy to show that the kernels of the linear functionals \( \langle \pi(f)v, v \rangle, \langle \pi'(f)v', v' \rangle \) on \( \text{Pol}(\text{Mat}_{m,n})_q \) are just the same subspace \( \bigoplus_{(j,k) \neq (0,0)} \mathbb{C}[\text{Mat}_{m,n}]_{q,j} \mathbb{C}[\text{Mat}_{m,n}]_{q,k}. \) Thus \( \langle \pi(f)v, v \rangle = \langle \pi'(f)v', v' \rangle. \) Hence, by irreducibility, the map \( v \mapsto v' \) admits an extension up to a unitary map which intertwines the representations \( \pi \) and \( \pi'. \)

**4 On a *-representation of \( \mathbb{C}[\widetilde{G}]_q \) in a pre-Hilbert space**

Our purpose is to produce a *-representation \( \widetilde{T} \) of \( \mathbb{C}[\widetilde{G}]_q \) in a pre-Hilbert space such that \( \widetilde{T}(x) \) is invertible. The representation is a tensor product of auxiliary representations \( \widetilde{T}_{(k,k+1)} \) of \( \mathbb{C}[SL_N]_q. \) The latter are indexed by the standard generators \( (k, k + 1) \) of the symmetric group \( S_N. \)

To describe those auxiliary representations, we need the homomorphisms:

\[ \psi_{(k,k+1)} : \mathbb{C}[SL_N]_q \to \mathbb{C}[SL_2]_q, \quad \psi_{(k,k+1)}(t_{ij}) = \begin{cases} t_{i-k+1,j-k+1}, & i, j \in \{k, k + 1\} \\ \delta_{ij}, & \text{otherwise} \end{cases} \]

and the following representation \( \pi_+ \) of \( \mathbb{C}[SL_2]_q \) in a vector space \( \mathcal{L}_+ \) with a basis \( \{e_j\}_{j \in \mathbb{Z}_+}: \)

\[ \pi_+(t_{12}) e_j = q^{-j} e_j, \quad \pi_+(t_{21}) e_j = -q^{-(j+1)} e_j, \]

\[ \pi_+(t_{11}) e_j = e_{j+1}, \quad \pi_+(t_{22}) e_j = \begin{cases} (1 - q^{-2j}) e_{j-1}, & j > 0, \\ 0, & j = 0. \end{cases} \] (4.1)
It is convenient to equip \( \mathcal{L}_+ \) with structure of a pre-Hilbert space as follows

\[
(e_i, e_j) = \begin{cases} 
(q^{-2} - 1)(q^{-4} - 1) \ldots (q^{-2j} - 1)\delta_{ij}, & j > 0, \\
\delta_{i0}, & j = 0.
\end{cases}
\]

The representation \( \tilde{T}_{(k,k+1)} \) of the algebra \( \mathbb{C}[SL_N]_q \) is given by \( \tilde{T}_{(k,k+1)} = \pi_+ \circ \psi_{(k,k+1)} \).

We now turn to a construction of the representation \( \tilde{T} \). Consider the element

\[
u = \left( \begin{array}{cccc}
1 & 2 & \ldots & n \\
m + 1 & m + 2 & \ldots & N \\
1 & 2 & \ldots & m
\end{array} \right),
\]

of the symmetric group \( S_N \). This element is a product of cycles \( s = s_m \cdot s_{m-1} \cdot \ldots \cdot s_1 \), with

\[
s_i = (i, i + 1) \cdot (i + 1, i + 2) \cdot \ldots \cdot (i + n - 1, i + n).
\]

Fix the reduced expression \( u = \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{mn} \), which is just concatenation of the reduced expressions for the \( s_i \). For example, in the case \( m = 2, n = 3 \), one has \( u = (3, 4, 5, 1, 2) \), and the above reduced expression acquires the form \( u = (2, 3)(3, 4)(4, 5)(1, 2)(2, 3)(3, 4) \).

Now we are in a position to introduce the desired representation:

\[ \tilde{T} = \tilde{T}_{\sigma_1} \otimes \tilde{T}_{\sigma_2} \otimes \ldots \otimes \tilde{T}_{\sigma_{mn}}. \]

**Proposition 4.1**

(i) \( \tilde{T} \) is a \(*\)-representation of \( \mathbb{C}[G]_q \) in the pre-Hilbert space \( \mathcal{L} = \mathcal{L}_+^{\otimes mn} \).

(ii)

\[ \tilde{T}(x)e_k = q^{-2\sum k_j} e_k, \]

with \( e_k = e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_{mn}}, \quad k = (k_1, k_2, \ldots, k_{mn}) \in \mathbb{Z}_+^{mn} \).

**Corollary 4.2** \( \tilde{T}(x) \) is invertible.

**Proof** of proposition 4.1. The first problem is to prove that \( \tilde{T} \) is a \(*\)-representation. The method we apply is based on well known results of the theory of compact quantum groups \([14, 3]\). That is why we need an involution \(*\) on \( \mathbb{C}[SL_N]_q \), related to the quantum group \( SU_N \) \([20, 31]\). It is given by

\[
t_{ij}^* = (-q)^{j-i} \det_q t_{ij}.
\]

It is well known that \( \mathbb{C}[SU_N]_q \triangleq (\mathbb{C}[SL_N]_q, \ast) \) is a Hopf \(*\)-algebra. Note that its involution \(*\) is related to the involution \( \ast \) \((3.6)\) as follows:

\[
t_{ij}^* = \lambda_1(i) \lambda_2(j) t_{ij}^*, \quad i, j = 1, \ldots, N,
\]

with

\[
\lambda_1(k) = \text{sign}(k - m - 1/2), \quad \lambda_2(k) = \text{sign}(n - k + 1/2).
\]

Clearly, a representation \( \pi \) of \( \mathbb{C}[SL_N]_q \) in a pre-Hilbert space determines a \(*\)-representation of \( \mathbb{C}[G]_q \) if and only if \( \pi(t_{ij})^* = \lambda_1(i) \lambda_2(j) \pi(t_{ij}^*) \) for all \( i, j = 1, \ldots, N \).
Let $\Lambda = (\lambda'(1),\lambda'(2),\ldots,\lambda'(N))$, $\Lambda'' = (\lambda''(1),\lambda''(2),\ldots,\lambda''(N))$ be two sequences whose entries are $\pm 1$. Suppose we are given a representation $\pi$ of $\mathbb{C}[SL_N]_q$ in a pre-Hilbert space.

**Definition.** $\pi$ is said to be of type $(\Lambda',\Lambda'')$ if

$$\pi(t_{ij})^* = \lambda'(i)\lambda''(j)\pi(t_{ij}^*).$$

**Remark.** This definition is well illustrated by the special case $m = n = 1$. One has $t_{11}^* = -t_{11}^*, t_{12}^* = t_{12}^*, t_{21}^* = t_{21}^*, t_{22}^* = -t_{22}^*$. It is easy to see that $\pi_+$ is of type $(\Lambda_1,\Lambda_2)$ with $\Lambda_1 = (-1,1)$, $\Lambda_2 = (1,-1)$.

**Lemma 4.3** Suppose that representations $\pi'$ and $\pi''$ are of types $(\Lambda',\Lambda'')$ and $(\Lambda'',\Lambda''')$ respectively. Then their tensor product $\pi = \pi' \otimes \pi''$ is of type $(\Lambda',\Lambda''')$.

**Proof.** An application of the relation $(\lambda''(k))^2 = 1$ and the fact that the comultiplication $\Delta : \mathbb{C}[SU_N]_q \rightarrow \mathbb{C}[SU_N]_q^\otimes 2$ is a homomorphism of $*$-algebras yields

$$((\pi(t_{ij}))^*) = \sum_{k=1}^{N} \pi'(t_{ik})^* \otimes \pi''(t_{kj})^* = \lambda'(i)\lambda'''(j) \sum_{k=1}^{N} (\lambda''(k))^2 \pi'(t_{ik}^*) \otimes \pi''(t_{kj}^*) = \lambda'(i)\lambda'''(j) \pi(t_{ij}^*) \quad \square$$

Turn back to the proof of proposition 4.1. Consider the sequence $\Lambda^{(0)}, \Lambda^{(1)}, \ldots, \Lambda^{(mn)}$, given by

$$\Lambda^{(j)} = (\lambda_1(u_j(1)), \lambda_1(u_j(2)), \ldots, \lambda_1(u_j(N))),$$

with $u_0 = e$, $u_1 = \sigma_1$, $u_2 = \sigma_1 \cdot \sigma_2$, $\ldots$, $u_{mn} = u$. Evidently, $\Lambda^{(0)} = \Lambda_1$, $\Lambda^{(mn)} = \Lambda_2$.

Observe that if for some $j \in \{1,\ldots,N-1\}$ the pair $(\Lambda',\Lambda'')$ possesses the properties: $\lambda'(i) = \lambda''(i)$ for $i \notin \{j,j+1\}$, $\lambda'(j) = -1$, $\lambda''(j) = 1$, $\lambda'(j+1) = 1$, $\lambda''(j+1) = -1$ then by the definitions and the remark before lemma 4.3 the representation $\pi_+ \circ \psi_{(j,j+1)}$ of $\mathbb{C}[SL_N]_q$ is of type $(\Lambda',\Lambda'')$. In particular, the representation $\widehat{T}_{(j)+1}$ is of type $(\Lambda^{(j)},\Lambda^{(j+1)})$, and the first statement of proposition 4.1 follows from lemma 4.3.

Turn to the proof of the second statement of the proposition. Recall the notation

$$e_k = e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_{mn}}, \quad k = (k_1,k_2,\ldots,k_{mn}) \in \mathbb{Z}^{mn}_+$$

for the standard orthogonal basis of the pre-Hilbert space $\mathcal{L}$. Let us first demonstrate the idea of the proof in the special case $m = n = 2$. In this case $x = tt^*$ with $t = t_{13}t_{24} - qt_{14}t_{23}$. It follows from the definitions that

$$\psi_{(2,3)} \otimes \psi_{(3,4)} \otimes \psi_{(1,2)} \otimes \psi_{(2,3)}(t_{13}) = 1 \otimes 1 \otimes t_{12} \otimes t_{12},$$

$$\psi_{(2,3)} \otimes \psi_{(3,4)} \otimes \psi_{(1,2)} \otimes \psi_{(3,4)}(t_{24}) = t_{12} \otimes t_{12} \otimes 1 \otimes 1,$$

$$\psi_{(2,3)} \otimes \psi_{(3,4)} \otimes \psi_{(1,2)} \otimes \psi_{(2,3)}(t_{14}) = 0.$$

Thus

$$\psi_{(2,3)} \otimes \psi_{(3,4)} \otimes \psi_{(1,2)} \otimes \psi_{(2,3)}(t) = t_{12} \otimes t_{12} \otimes t_{12} \otimes t_{12}.$$  

and hence for all $k = (k_1,k_2,k_3,k_4)$

$$\widetilde{T}(t)e_k = q^{-(k_1+k_2+k_3+k_4)}e_k. \quad (4.6)$$

Turn to the case of arbitrary $m, n \in \mathbb{N}$. The following statement generalizes (4.6) and implies the second statement of proposition 4.1.
Lemma 4.4 For all \( k \in \mathbb{Z}_{+}^{mn} \)

\[
\tilde{T}(t)e_k = q^{-\sum_j k_j}e_k. \tag{4.7}
\]

Proof. By definition \( \tilde{T} = \pi_{+}^{\otimes mn} \circ \Psi \) with

\[
\Psi = \psi_{\sigma_1} \otimes \psi_{\sigma_2} \otimes \ldots \otimes \psi_{\sigma_{mn}} : \mathbb{C}[SL_N]_q \to \mathbb{C}[SL_2]_q^{\otimes mn}.
\]

To prove (4.7) it suffices to show that

\[
\Psi(t) = \underbrace{t_{12} \otimes \ldots \otimes t_{12}}_{mn}. \tag{4.8}
\]

Let us prove the latter equality. Denote by \( \Psi_{s_i} \) the homomorphism

\[
\psi_{(i,i+1)} \otimes \psi_{(i+1,i+2)} \otimes \ldots \otimes \psi_{(i+n-1,i+n)} : \mathbb{C}[SL_N]_q \to \mathbb{C}[SL_2]_q^{\otimes n}
\]

associated to the cycle \( s_i \) (see (4.2)). Evidently, \( \Psi = \Psi_{s_m} \otimes \Psi_{s_{m-1}} \otimes \ldots \otimes \Psi_{s_1} \). The following equalities may be deduced easily from the definition of comultiplication in \( \mathbb{C}[SL_N]_q \):

i) if \( k < i \) or \( l > i + n \) then

\[
\Psi_{s_i}(t_{kl}) = \delta_{kl} \cdot \underbrace{1 \otimes \ldots \otimes 1}_{n};
\]

ii) if \( k = i \) and \( l = i + n \) then

\[
\Psi_{s_i}(t_{kl}) = \underbrace{t_{12} \otimes \ldots \otimes t_{12}}_{n}.
\]

The latter equalities imply

\[
\Psi(t_{kl}) = \begin{cases} 
0, & l > k + n \\
\underbrace{1 \otimes \ldots \otimes 1 \otimes t_{12} \otimes \ldots \otimes t_{12} \otimes 1 \otimes \ldots \otimes 1}_{(m-k)n} & l = k + n \\
\underbrace{1 \otimes \ldots \otimes 1 \otimes t_{12} \otimes \ldots \otimes t_{12} \otimes 1 \otimes \ldots \otimes 1}_{(k-1)n} & l = k + n 
\end{cases} \tag{4.9}
\]

where \( k \leq m \). Now (4.8) follows from (4.9) and the definition of \( t \). \qed

Since the operator \( \tilde{T}(x) \) is invertible, the representation \( \tilde{T} \) admits a unique extension onto the \( * \)-algebra \( \mathbb{C}[\tilde{G}]_{q,x} \), for which we retain the notation \( \tilde{T} \).

5 Unitary equivalence of \( T \) and \( \tilde{T} \)

Let \( T = \tilde{T} \circ i \) be the \( * \)-representation of \( \text{Pol(Mat}_{m,n})_q \) deduced from the embedding of \( * \)-algebras \( i : \text{Pol(Mat}_{m,n})_q \to \mathbb{C}[\tilde{G}]_{1q,x} \) described in section 3. We are about to produce an isomorphism \( J \) of \( \text{Pol(Mat}_{m,n})_q \)-modules \( \mathcal{H} \) and \( \mathcal{L} \). Equip the spaces \( \mathcal{H}, \mathcal{L} \) and the
algebras $\mathbb{C}[\tilde{G}]_q$, $\text{Pol}(\text{Mat}_{m,n})_q$ with the gradations:

$$\text{deg}(t_{ij}) = \begin{cases} 
1, & i \leq m \& j \leq n \\
-1, & i > m \& j > n \\
0, & \text{otherwise}, 
\end{cases}$$

$$\mathcal{L} = \bigoplus_{j=0}^{\infty} \mathcal{L}_j, \quad \mathcal{L}_j = \left\{ v \in \mathcal{L} | \tilde{T}(x)v = q^{-2j}v \right\},$$

$$\text{deg}(z^a) = 1, \quad \text{deg}(z^a)^* = -1,$$

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} \mathcal{H}_j, \quad \mathcal{H}_j = \mathbb{C}[\text{Mat}_{m,n}]_q v_0.$$

It follows from proposition 3.1 that $\mathcal{L}$ is a graded $\mathbb{C}[\tilde{G}]_q$-module. For any homogeneous vector $v \in \mathcal{L}$ and all $a = 1, \ldots, n$, $\alpha = 1, \ldots, m$, one has

$$\text{deg}(\mathcal{T}(z^a)v) = \text{deg}(v) + 1, \quad \text{deg}(\mathcal{T}(z^a)^*v) = \text{deg}(v) - 1$$

by a virtue of propositions 3.1 and 3.2. Hence,

$$(T(\psi)v_0, v_0) = (T(\psi)e_0, e_0), \quad \psi \in \text{Pol}(\text{Mat}_{m,n})_q. \quad (5.1)$$

Also, since the vector $v_0$ is cyclic, it follows that the map

$$\mathcal{J} : \mathcal{H} \to \mathcal{L}, \quad \mathcal{J} : T(\psi)v_0 \mapsto T(\psi)e_0, \quad \psi \in \mathbb{C}[\text{Mat}_{m,n}]_q,$$

is a well defined morphism of $\text{Pol}(\text{Mat}_{m,n})_q$-modules due to proposition 2.1.

**Proposition 5.1** The above map $\mathcal{J} : \mathcal{H} \to \mathcal{L}$ is an isomorphism of $\text{Pol}(\text{Mat}_{m,n})_q$-modules.

**Remark.** This certainly implies proposition 2.4 and the unitary equivalence of representations $T$ and $\mathcal{T}$ (see (5.1)).

To prove proposition 5.1, we need several lemmas.

**Lemma 5.2** Let $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq m$, $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, $J = \{n+1, n+2, \ldots, N\} \setminus \{n + \alpha_1, n + \alpha_2, \ldots, n + \alpha_k\} \cup \{a_1, a_2, \ldots, a_k\}$. Then

$$i : z^{\wedge k}\{m+1-\alpha_k, m+1-\alpha_k-1, \ldots, m+1-\alpha_1\} \mapsto t^{-1}t^{\wedge m} \{1, 2, \ldots, m\}_J.$$

**Proof.** See section 11.

**Lemma 5.3** If $v$ is a vector in the space of a $*$-representation $\rho$ of $\mathbb{C}[\tilde{G}]_q$ and

$$\rho(t^{\wedge m}_{\{m+1,m+2,\ldots,N\},J})v = 0, \quad J \neq \{1, 2, \ldots, n\} \quad (5.2)$$

then $\rho(x)v = v$. 

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Proof. The following relation is a consequence of a more general formula (6.2) of [20]. For all \( k = 1, \ldots, N - 1 \),
\[
(t^k_{\{1,\ldots,k\}\{N-k+1,\ldots,N\}})^* = (-q)^{k(N-k)} \cdot t^{(N-k)}_{\{k+1,\ldots,N\}\{1,\ldots,N-k\}}.
\]
(5.3)
Recall that \( x = tt^* \), with \( t = t^\otimes_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}} \); it also follows from (4.4), (5.3) that \( t^* = (-q)^{mn}t^\otimes_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}} \). Apply (5.2) and \( \det_q t = 1 \) to obtain
\[
v = \rho(\det_q t)v = ((-q)^{mn}t^\otimes_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}} \cdot t^\otimes_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}) v = \rho(x)v,
\]
which is just our statement. \( \square \)

The next lemma involves

**Definition.** Let \( \text{Pol}(Y)_q \) be a subalgebra of \( \mathbb{C}[\widetilde{G}]_q \) generated by \( t^\otimes_{\{1,2,\ldots,m\}I_j} t^\otimes_{\{m+1,m+2,\ldots,N\}J} \), with \( \text{card } I = m \), \( \text{card } J = n \).

It follows from (4.4), (5.3) that \( \text{Pol}(Y)_q \) is a \( * \)-subalgebra.

**Lemma 5.4** \( \mathcal{L} \) is an irreducible \( \text{Pol}(Y)_q \)-module.

**Proof.** Let
\[
\mathcal{L}^{\text{vac}} \overset{\text{def}}{=} \bigcap_{j \neq \{1,2,\ldots,n\}} \ker \widetilde{T} (t^\otimes_{\{m+1,m+2,\ldots,N\}J}) .
\]
\( \mathcal{L}^{\text{vac}} \) is invariant with respect to \( \widetilde{T} (x) \) due to the commutation relations (3.8). It follows that \( \mathcal{L}^{\text{vac}} = \bigoplus_j (\mathcal{L}^{\text{vac}} \cap \mathcal{L}_j) \). Hence by lemma 5.3, \( \mathcal{L}^{\text{vac}} = \mathcal{L}_0 \). On the other hand, by proposition 4.1, \( \mathcal{L}_0 = \mathbb{C}e_0 \). So, we conclude that \( \mathcal{L}^{\text{vac}} = \mathbb{C}e_0 \).

Turn back to the proof of irreducibility. Assume that there exists a non-trivial \( \text{Pol}(Y)_q \)-invariant subspace \( \mathcal{L}' \). Then \( \mathcal{L}' = \bigoplus_j (\mathcal{L}_j \oplus (\mathcal{L}' \cap \mathcal{L}_j)) \) is also an invariant subspace and \( \mathcal{L} = \mathcal{L}' \oplus \mathcal{L}'' \). Now apply the operators \( \widetilde{T} (t^\otimes_{\{m+1,m+2,\ldots,N\}J}) \), \( J \neq \{1,2,\ldots,n\} \) to find non-zero vectors from \( \mathcal{L}' \cap \mathcal{L}^{\text{vac}} \), \( \mathcal{L}'' \cap \mathcal{L}^{\text{vac}} \), which are linear independent. This contradicts \( \dim \mathcal{L}^{\text{vac}} = 1 \). \( \square \)

Turn back to proving that \( J \) is one-to-one. It follows from lemmas 5.2 and 5.4 that \( J \) is onto. To see that \( J \) is injective, observe that \( J\mathcal{H}_i \subset \mathcal{L}_i \), \( \dim \mathbb{C}[\text{Mat}_{m,n}]_{q,i} = \binom{mn + i - 1}{i} \),
\[
\dim \mathcal{L}_i = \binom{mn + i - 1}{i}, \quad i \in \mathbb{Z}_+ \text{ (the latter equality is due to proposition 4.1)}. \qquad \square
\]

**Remark.** It follows from proposition 5.1 and lemmas 5.2, 5.4 that \( T \) is irreducible.

### 6 Boundedness of the quantum matrix ball

We use here the norm of an \( m \times n \) matrix with entries in \( \text{End} \mathcal{L} \) defined as the norm of the associated linear map \( \bigoplus \mathcal{L} \to \bigoplus \mathcal{L} \). Consider the matrices \( \mathbf{Z} = (z_{aa})_{a=1,\ldots,m, a=1,\ldots,n} \) with \( z_{aa} = (-q)^{a-1}z_a^{m+1-a} \), and \( T(\mathbf{Z}) = (T(z_{aa}))_{a=1,\ldots,m, a=1,\ldots,n} \).

\(^*\text{See }[11, \text{Proposition 1.5}].\)
**Proposition 6.1** $\|T(Z)\| \leq 1$.

We need the following

**Lemma 6.2** In the matrix algebra with entries from $\mathbb{C}[SL_N]_{q,t}$ one has

$$i(Z) = T^{-1}_{12}T_{11},$$

with $i(Z) = (i(z_{aa}))$, $T_{11} = (t_{aa})$, $T_{12} = (t_{a,n+b})$, $\alpha, \beta = 1, \ldots, m$, $a = 1, 2, \ldots, n$.

**Proof.** Let $t = (t_{ij})_{i,j=1,\ldots,m}$ and $\det'_q t = \sum_{s \in S_m} (-q)^{-l(s)} t_{s(m)m} t_{s(m-1)m-1} \cdots t_{s(1)1}$. It is well known [21, section 4] that in $\mathbb{C}[[\text{Mat}_{m,m}]_{q}$ one has

$$\det'_q T \cdot (T^{-1})_{\alpha\beta} = (-q)^{\alpha-\beta} \det_q (T_{\beta\alpha}), \quad \alpha, \beta = 1, \ldots, m$$

(6.3)

(Here, just as in the classical case $q = 1$, $T_{\beta\alpha}$ is a matrix derived from $T$ by discarding the row $\beta$ and column $\alpha$.)

**Proof of proposition 6.1.** Apply (6.3) to invert the matrix $t = (t_{ij})_{i,j=1,\ldots,N}$ in the matrix algebra with entries from $\mathbb{C}[SL_N]_{q,t}$:

$$\sum_{a=1}^N (-q)^{a-\beta} t_{aa} \det_q (t_{\beta a}) = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, N.$$ 

Hence

$$-\sum_{c=1}^n t_{\alpha c} t_{\beta c}^* + \sum_{\gamma=1}^m t_{\alpha,n+\gamma} t_{\beta,n+\gamma}^* = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, m.$$ 

After introducing the notation

$$T_{11} = (t_{aa})_{a=1,\ldots,m, a=1,\ldots,n}, \quad T_{12} = (t_{a,n+b})_{a,b=1,\ldots,m},$$

$$T_{11}^* = (t_{aa}^*)_{a=1,\ldots,m, a=1,\ldots,n}, \quad T_{12}^* = (t_{n+\beta,\alpha}^*)_{a,b=1,\ldots,m},$$

we get

$$-T_{11}T_{11}^* + T_{12}T_{12}^* = I.$$ 

(6.4)

It follows from (6.4) and (6.1) that $i(I - ZZ^*) = T_{12}^{-1}(T_{12}^{-1})^*$. Apply the representation $\tilde{T}$ to both parts of the above relation. By a virtue of $\tilde{T} = \tilde{T} \circ i$ we obtain $\tilde{T}(I - ZZ^*) = \tilde{T}(T_{12}^{-1}) \tilde{T}(T_{12}^{-1})^* \geq 0$. Hence $T(Z)\tilde{T}(Z)^* \leq I$, $\|T(Z)\| = \|\tilde{T}(Z)^*\| \leq 1$. □

Proposition 6.1 and the unitary equivalence of the representations $T$ and $\tilde{T}$ (see section 5) imply

**Corollary 6.3** The operators $T(z_{aa}^\alpha)$, $a = 1, \ldots, n$, $\alpha = 1, \ldots, m$, are bounded.
7 The quantum universal enveloping algebra $U_q\mathfrak{sl}_N$

The Drinfeld-Jimbo quantum universal enveloping algebra is among the basic notions of the quantum group theory. Recall the definition of the Hopf algebra $U_q\mathfrak{sl}_N$ [8]. Let $(a_{ij})_{i,j=1,...,N-1}$ be the Cartan matrix of $\mathfrak{sl}_N$:

$$a_{ij} = \begin{cases} 2, & i-j = 0 \\ -1, & |i-j| = 1 \\ 0, & \text{otherwise}. \end{cases} \quad (7.1)$$

The algebra $U_q\mathfrak{sl}_N$ is determined by the generators $E_i, F_i, K_i, K_i^{-1}, i = 1, \ldots, N-1$, and the relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i$$

$$E_i F_j - F_j E_i = \delta_{ij} (K_i - K_i^{-1})/(q - q^{-1})$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad |i-j| = 1$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \quad |i-j| = 1$$

$$[E_i, E_j] = [F_i, F_j] = 0, \quad |i-j| \neq 1. \quad (7.2)$$

The comultiplication $\Delta$, the antipode $S$, and the counit $\varepsilon$ are determined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad (7.3)$$

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \quad (7.4)$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

We consider in the sequel only $U_q\mathfrak{sl}_N$-modules of the form $V = \bigoplus_{\mu \in \mathbb{Z}^N} V_\mu$, with $\mu = (\mu_1, \ldots, \mu_{N-1})$, $V_\mu = \{ v \in V | K_i v = q^{\mu_i} v, i = 1, \ldots, N-1 \}$, to be referred to as weight modules.\footnote{Note that some authors use the term 'weight' for a larger class of modules [13].} This agreement allows one to introduce the linear operators in $V$

$$H_j v = \mu_j v, \quad v \in V_\mu, \quad j = 1, \ldots, N-1.$$

Note that the defining relations in the classical universal enveloping algebra $U\mathfrak{sl}_N$ can be derived from those in $U_q\mathfrak{sl}_N$ via the substitution $K_i^{\pm 1} = q^{\pm H_i}$ and the formal passage to a limit as $q \to 1$ (e.g. $\lim_{q \to 1} K_i - K_i^{-1} = H_i, i = 1, \ldots, N-1$).

Equip all the weight $U_q\mathfrak{sl}_N$-modules with the gradation $\deg v = j \iff H_0 v = 2j v$, where $H_0$ is the unique element of the standard Cartan subalgebra of $\mathfrak{sl}_N$ with the following properties:

$$[H_0, E_j] = 0, \quad j \neq n; \quad [H_0, E_n] = 2E_n.$$

Let us present an explicit formula for $H_0$:

$$H_0 = \frac{2}{m+n} \left( m \sum_{j=1}^{n-1} j H_j + n \sum_{j=1}^{m-1} j H_{N-j} + mn H_n \right). \quad (7.5)$$

It is easy to prove that
Lemma 7.1 $H_0$ is orthogonal to all the vectors $H_j$, $j \neq n$, with respect to an invariant bilinear form in $\mathfrak{sl}_N$.

The rest of this section is intended to recall some well known results of quantum group theory [8].

Recall that for the standard system of simple roots $\{\alpha_i\}_{i=1,\ldots,n-1}$ of $\mathfrak{sl}_N$ one has $\alpha_i(H_j) = a_{ij}$, $i, j = 1, \ldots, N-1$, with $(a_{ij})$ being the Cartan matrix (7.1). The Weyl group is generated by simple reflections $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. In our case it is canonically isomorphic to $S_N$: $s_i \mapsto (i, i+1)$. Consider the longest element $w_0 = (N, N-1, \ldots, 2, 1) \in S_N$, together with its reduced expression $w_0 = s_{i_1} \ldots s_{i_M}$, $M = N(N-1)/2$, $1 \leq i_k \leq N-1$. One can associate to the reduced expression a total order on the set of positive roots of $\mathfrak{sl}_N$, and then a basis in the vector space $U_q\mathfrak{sl}_N$. The total order is given by

$$\beta_1 = \alpha_1, \quad \beta_2 = s_i(\alpha_{i_2}), \quad \beta_3 = s_i s_{i_2}(\alpha_{i_3}), \ldots \quad \beta_M = s_i \ldots s_{i_{M-1}}(\alpha_{i_M}).$$

Turn to description of the basis in $U_q\mathfrak{sl}_N$ associated to the reduced expression of $w_0$.

G. Lusztig [15] has defined an action of the braid group $B_N$ as a group of automorphisms of the algebra $U_q\mathfrak{sl}_N$ (we follow the definition given in [4]):

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$$

$$T_i(E_j) = \begin{cases} E_j, & |i - j| > 1, \\ q^{-1} E_j E_i - E_i E_j, & |i - j| = 1 \end{cases}, \quad T_i(F_j) = \begin{cases} F_j, & |i - j| > 1, \\ q F_j F_i - F_i F_j, & |i - j| = 1 \end{cases},$$

$$T_i(K_j) = K_j K_i^{-s_{ij}}.$$

Note that the automorphisms $T_i$ permute the weight spaces

$$(U_q\mathfrak{g})_\lambda = \{ \xi \in U_q\mathfrak{g} | K_i \xi K_i^{-1} = q^{\lambda_i} \xi \}, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1}) \in \mathbb{Z}^{N-1},$$

in the following way:

$$T_i : (U_q\mathfrak{g})_\lambda \to (U_q\mathfrak{g})_{s_i(\lambda)}. \quad (7.6)$$

Furthermore,

$$T_i(U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m) = U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m, \quad i \neq n,$$

with $U_q\mathfrak{sl}_n \subset U_q\mathfrak{sl}_N$ being the Hopf subalgebra generated by $E_i, F_i, K_i^{\pm 1}$, $i = 1, 2, \ldots, n-1$, and $U_q\mathfrak{sl}_m \subset U_q\mathfrak{sl}_N$ the Hopf subalgebra generated by $E_{n+i}, F_{n+i}, K_{n+i}^{\pm 1}$, $i = 1, 2, \ldots, m-1$.

We have two maps $\alpha_i \mapsto E_i, \alpha_i \mapsto F_i$, $i = 1, \ldots, N - 1$. These maps, defined on the set of simple roots, are extended onto the set of all positive roots as follows: $E_{\beta_1} = T_1 T_2 \ldots T_{i-1}(E_{i_1}), \quad F_{\beta_1} = T_1 T_2 \ldots T_{i-1}(F_{i_1})$. We use below the notation $U_q\mathfrak{n}_+$ (respectively, $U_q\mathfrak{n}_-$) for the subalgebra in $U_q\mathfrak{sl}_N$ generated by $\{E_i\}$ (respectively, $\{F_i\}$), $i = 1, 2, \ldots, N - 1$.

Proposition 7.2

i) $E_{k_1}^{j_1} \cdot E_{k_2}^{j_2} \ldots \cdot E_{k_M}^{j_M}$, $(k_1, k_2, \ldots, k_M) \in \mathbb{Z}_+^M$, constitute a basis of weight vectors in the vector space $U_q\mathfrak{n}_+$;

ii) $F_{j_1}^{k_1} \cdot F_{j_2}^{k_2} \ldots \cdot F_{j_M}^{k_M}$, $(j_1, j_2, \ldots, j_M) \in \mathbb{Z}_+^M$, constitute a basis of weight vectors in the vector space $U_q\mathfrak{n}_-$;

iii) $F_{j_1}^{k_1} \cdot F_{j_2}^{k_2} \ldots \cdot F_{j_M}^{k_M} \cdot K_1^{j_1} \cdot K_2^{j_2} \ldots \cdot K_{N-1}^{j_{N-1}} \cdot E_{j_1}^{k_1} \cdot E_{j_2}^{k_2} \ldots \cdot E_{j_M}^{k_M}$, $(k_1, k_2, \ldots, k_M), (j_1, j_2, \ldots, j_M) \in \mathbb{Z}_+^M$, $(i_1, i_2, \ldots, i_{N-1}) \in \mathbb{Z}_+^{N-1}$, constitute a basis of weight vectors in the vector space $U_q\mathfrak{sl}_N$. 

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Consider a $U_q\mathfrak{sl}_N$-module $V^h$, determined by its generator $v^h$ and the relations

$$E_j v^h = (K_j^{\pm 1} - 1)v^h = 0, \quad j = 1, \ldots, N - 1,$$

$$F_i v^h = 0, \quad i = 1, \ldots, n - 1, n + 1, \ldots, N - 1.$$  

($V^h$ is canonically isomorphic to the generalized Verma module with zero highest weight.)

We are about to apply proposition 7.2 to produce a basis of the vector space $V^h$ formed by homogeneous vectors. For that, we use the class of reduced expressions for the element $w_0$ described below. Recall the notation $M = N(N - 1)/2$, $M' = M - mn$, $s_j = (j, j + 1)$. Consider the longest element for the subgroup $S_n \times S_m \subset S_N$

$$w'_0 = (n, n - 1, \ldots, 1, N, N - 1, \ldots, n + 1),$$

together with the permutation

$$w''_0 = (n + 1, n + 2, \ldots, N - 1, N, 1, 2, \ldots, n - 1, n).$$

Obviously, $w_0 = w'_0 \cdot w''_0$. Fix the reduced expression $w_0 = s_{i_1}s_{i_2} \cdots s_{i_M}$, given by concatenation of reduced expressions for $w'_0$ and $w''_0$. It follows from the definitions that $\deg (F_{\beta_j}) = -1$ for $j > M'$, and $\deg (F_{\beta_j}) = 0$ for $j \leq M'$. Thus the vectors

$$F^{k_{M'}}_{\beta_{M+1}} F^{k_{M-1}}_{\beta_{M-1}} \ldots F^{k_1}_{\beta_1} v^h, \quad (k_{M'+1}, \ldots, k_M) \in \mathbb{Z}_{+}^{mn},$$

constitute a basis in the vector space $V^h$, and

$$\deg \left( F^{k_{M'}}_{\beta_{M+1}} F^{k_{M-1}}_{\beta_{M-1}} \ldots F^{k_1}_{\beta_1} v^h \right) = - \sum_{j=M'+1}^{M} k_j.$$

It is easy to obtain a similar result for a weight $U_q\mathfrak{sl}_N$-module $V^l$ determined by its generator $v^l$ and the relations

$$F_j v^l = (K_j^{\pm 1} - 1)v^l = 0, \quad j = 1, \ldots, N - 1,$$

$$E_i v^l = 0, \quad i = 1, \ldots, n - 1, n + 1, \ldots, N - 1.$$  

The following vectors form a basis of the graded vector space $V^l$ consisting of homogeneous vectors:

$$S \left( E^{j_{M'+2}}_{\beta_{M'+2}} E^{j_{M'+1}}_{\beta_{M'+1}} \ldots E^{j_M}_{\beta_M} \right) v^l,$$

with $(j_{M'+1}, j_{M'+2}, \ldots, j_M) \in \mathbb{Z}_{+}^{mn}$, and $S$ being the antipode of the Hopf algebra $U_q\mathfrak{sl}_N$.

Recall some notions of the theory of Hopf algebras [3]. Let $A$ be an abstract Hopf algebra and $F$ an algebra equipped also with a structure of $A$-module. $F$ is said to be an $A$-module algebra if the multiplication $F \otimes F \rightarrow F$, $f_1 \otimes f_2 \mapsto f_1 f_2$, is a morphism of $A$-modules. In the case of a unital algebra $F$, the additional assumption is introduced that the embedding $\mathbb{C} \hookrightarrow F$, $1 \mapsto 1$, is a morphism of $A$-modules. A duality argument allows one also to introduce a notion of $A$-module coalgebra.

The results, cited below, are due to S. Levendorskii and Ya. Soibelman [17, 18] and A. Kirillov, N. Reshetikhin [12, 3]. For a good survey of those the reader is referred to [24, 4].
Let $V_1$ and $V_2$ be some $U_q\mathfrak{sl}_N$-modules. It is well known that, in general, the ordinary flip

$$\sigma_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1, \quad \sigma_{V_1,V_2} : v_1 \otimes v_2 \to v_2 \otimes v_1$$

is not a morphism of $U_q\mathfrak{sl}_N$-modules. V. Drinfeld [6] introduced the extremely important notion of the universal $R$-matrix which has lead to appropriate $q$-analogs of the operators $\sigma_{V_1,V_2}$. Let us describe these $q$-analogues.

To start with, recall the standard notation $U_q\mathfrak{b}^+$ (respectively $U_q\mathfrak{b}^-$) for the Hopf subalgebra in $U_q\mathfrak{sl}_N$ generated by $K_i^\pm$, $E_i$, $i = 1, \ldots, N - 1$ (respectively $K_i^\pm$, $F_i$, $i = 1, \ldots, N - 1$). We denote by $\mathcal{C}^+$ (respectively $\mathcal{C}^-$) the category of $U_q\mathfrak{b}^+$-locally finite dimensional (respectively $U_q\mathfrak{b}^-$-locally finite dimensional) weight $U_q\mathfrak{sl}_N$-modules.

Let $V_1$, $V_2$ be weight $U_q\mathfrak{sl}_N$-modules, and either $V_1 \in \mathcal{C}^+$ or $V_2 \in \mathcal{C}^-$. The formula (7.7) involves analogs of root vectors of the Lie algebra $\mathfrak{sl}_N$. Nevertheless, it is well known that the operators $\sigma_{V_1,V_2}$, $\exp$ below determines a linear operator $\check{R}_{V_1, V_2}$ in $V_1 \otimes V_2$:

$$R = \exp_q \left( (q^{-1} - q) E_{\beta_M} \otimes F_{\beta_M} \right) \cdot \exp_q \left( (q^{-1} - q) E_{\beta_{M-1}} \otimes F_{\beta_{M-1}} \right) \cdot \ldots \cdot \exp_q \left( (q^{-1} - q) E_{\beta_1} \otimes F_{\beta_1} \right) q^{-t_0}, \quad (7.7)$$

with $\exp_q(u) = \sum_{k=0}^{\infty} \frac{u^k}{(k)_q^2!}$; $(k)_q^2! = \prod_{j=1}^{k} \frac{1 - q^{2j}}{1 - q^2}$;

$$t_0 = \sum_{i,j=1}^{N-1} c_{ij} H_i \otimes H_j, \quad (7.8)$$

and $(c_{ij})_{i,j=1,\ldots,N-1}$ is inverse to the Cartan matrix $(a_{ij})_{i,j=1,\ldots,N-1}$. It is worthwhile to note that $c_{ij} \in \frac{1}{N}\mathbb{Z}_+$.  

Now we use the relation $\alpha_i(H_j) = a_{ij}$, $i, j = 1, \ldots, N - 1$, to get a different description of $t_0$:

$$\alpha_i \otimes \alpha_j(t_0) = a_{ij}, \quad i, j = 1, \ldots, N - 1.$$  

Recall the definition of the standard inner product in the Cartan subalgebra: $(H_i, H_j) = a_{ij}$, $i, j = 1, \ldots, N - 1$. It allows one to get the third description of $t_0$:

$$(t_0, H_i \otimes H_j) = (H_i, H_j); \quad i, j = 1, \ldots, N - 1.$$  

That is,

$$t_0 = \sum_{k=1}^{N-1} \frac{I_k \otimes I_k}{(I_k, I_k)} \quad (7.9)$$

for any orthogonal basis of the Cartan subalgebra.

The formula (7.7) involves analogs of root vectors of the Lie algebra $\mathfrak{sl}_N$ whose construction, we recall, depends on the choice of a reduced expression for the longest element $w_0 \in S_N$. Nevertheless, it is well known that the operators $\check{R}_{V_1, V_2}$ are independent of that choice.

Let us list some properties of the operators $\check{R}_{V_1, V_2} \overset{\text{def}}{=} \sigma_{V_1,V_2} \cdot R_{V_1,V_2}$. Let again $V_1$, $V_2$ be weight $U_q\mathfrak{sl}_N$-modules, and either $V_1 \in \mathcal{C}^+$ or $V_2 \in \mathcal{C}^-$. Then the operator $\check{R}_{V_1, V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is an invertible operator and a morphism of $U_q\mathfrak{sl}_N$-modules.

Suppose $V$, $V_1$, $V_2$ are weight $U_q\mathfrak{sl}_N$-modules, and either $V_1, V_2 \in \mathcal{C}^+$ or $V \in \mathcal{C}^-$. Then

$$\check{R}_{V_1 \otimes V_2, V} = (\check{R}_{V_1, V} \otimes \text{id}_{V_2}) \cdot (\text{id}_{V_1} \otimes \check{R}_{V_2, V}). \quad (7.10)$$
Finally, suppose $V, V_1, V_2$ are weight $U_q\mathfrak{sl}_N$-modules, and either $V \in \mathcal{C}^+$ or $V_1, V_2 \in \mathcal{C}^-$. Then

$$
\tilde{R}_{V,V_1 \otimes V_2} = (\text{id}_{V_1} \otimes \tilde{R}_{V,V_2}) \cdot (\tilde{R}_{V,V_2} \otimes \text{id}_{V_1}).
$$

(7.11)

The above properties of the operators $\tilde{R}_{V_1,V_2}$ allow one to treat them as $q$-analogues of the ordinary flips $\sigma_{V_1,V_2}$.

Consider the vector representation $\pi$ of $U_q\mathfrak{sl}_N$ in $\mathbb{C}^N$:

$$
\pi(E_i)e_j = \begin{cases} 
q^{-1/2}e_{j-1}, & j = i + 1 \\
0, & \text{otherwise}
\end{cases}
\pi(F_i)e_j = \begin{cases} 
q^{1/2}e_{j+1}, & j = i \\
0, & \text{otherwise}
\end{cases}
\pi(K_i^{\pm 1})e_j = \begin{cases} 
q^{\pm 1}e_j, & j = i \\
q^{\mp 1}e_j, & j = i + 1 \\
e_j, & \text{otherwise}
\end{cases}
$$

with $i = 1, 2, \ldots, N - 1$, $j = 1, 2, \ldots, N$, $\{e_j\}$ being the standard basis in $\mathbb{C}^N$. The linear functionals $l_{jk} \in (U_q\mathfrak{sl}_N)^*$ given by

$$
\pi(\xi)e_k = \sum_{j=1}^N l_{jk}(\xi)e_j, \quad \xi \in U_q\mathfrak{sl}_N,
$$

are called matrix elements of $\pi$ with respect to the basis $\{e_j\}$. There exists a canonical non-degenerate pairing (see e.g. [31])

$$
\mathbb{C}[SL_N]_q \times U_q\mathfrak{sl}_N \rightarrow \mathbb{C}, \quad f \times \xi \mapsto \langle f, \xi \rangle,
$$

which determines an embedding of the Hopf algebras

$$
\mathbb{C}[SL_N]_q \hookrightarrow (U_q\mathfrak{sl}_N)^*, \quad t_{jk} \mapsto l_{jk}, \quad j, k = 1, \ldots, N.
$$

(7.12)

Let $L(\lambda)$ be the simple finite dimensional weight $U_q\mathfrak{sl}_N$-module with highest weight $\lambda$. The embedding $(\text{End}_\mathbb{C}L(\lambda))^* \hookrightarrow (U_q\mathfrak{sl}_N)^*$ allows one to get an isomorphism

$$
\mathbb{C}[SL_N]_q \simeq \bigoplus_{\lambda} (\text{End}_\mathbb{C}L(\lambda))^*.
$$

The embedding (7.12) may be used to equip $\mathbb{C}[SL_N]_q$ with a structure of $U_q\mathfrak{sl}_N$-module algebra:

$$
\langle \xi f, \eta \rangle = \langle f, \eta \xi \rangle, \quad f \in \mathbb{C}[SL_N]_q, \quad \xi, \eta \in U_q\mathfrak{sl}_N.
$$

It is now deducible from the definitions that the generators of $U_q\mathfrak{sl}_N$ act on the generators of $\mathbb{C}[SL_N]_q$ in the following way:

$$
E_i t_{j,k} = \begin{cases} 
q^{-1/2}t_{j,k-1}, & k = i + 1 \\
0, & \text{otherwise}
\end{cases}, \quad F_i t_{j,k} = \begin{cases} 
q^{1/2}t_{j,k+1}, & k = i \\
0, & \text{otherwise}
\end{cases},
$$

(7.13)

$$
K_i^{\pm 1} t_{j,k} = \begin{cases} 
q^{\pm 1}t_{j,k}, & k = i \\
q^{\mp 1}t_{j,k}, & k = i + 1 \\
t_{j,k}, & \text{otherwise}
\end{cases}
$$

(7.14)
8 The algebras $\mathbb{C}[\text{Mat}_{m,n}]_q$, $\text{Pol}(\text{Mat}_{m,n})_q$

In all the above observations we assumed that $q \in (0,1)$ and the ground field is $\mathbb{C}$. Nevertheless, it appears to be much more convenient in this section to replace $\mathbb{C}$ with the field $\mathbb{C}(q^{1/s})$ of rational functions of the indeterminate $q^{1/s}$, $s = 2N$. In the subsequent sections we are going to retrieve our original convention concerning the ground field.

Since our goals are results with $\mathbb{C}$ as a ground field, we need an appropriate procedure for backward passage from $\mathbb{C}(q^{1/s})$ to a ring of Laurent polynomials and finally to $\mathbb{C}$. A passage of that kind could be done via standard techniques well known in quantum group theory [3, §9.2]. In what follows we obtain a number of results for algebras and $*$-algebras over $\mathbb{C}(q^{1/s})$ determined by 'the same' generators and relations as before in the case of the ground field $\mathbb{C}$. It is well known that the results of section 7 are valid also in the case of the ground field $\mathbb{C}(q^{1/s})$ (cf. [4]).

In this section we are going to develop a different approach to the algebra $\mathbb{C}[\text{Mat}_{m,n}]_q$. More precisely, we are about to construct an algebra $\mathbb{C}[\text{Mat}_{m,n}]_q$ which is canonically isomorphic to $\mathbb{C}[\text{Mat}_{m,n}]_q$ and is much more convenient for our further goals.

Consider the Hopf algebra $U_q\mathfrak{sl}_N^{op}$ which differs from $U_q\mathfrak{sl}_N$ by replacing its comultiplication by the opposite one. Equip $V^h$ with a structure of $U_q\mathfrak{sl}_N^{op}$-module coalgebra: $\Delta : v^h \mapsto v^h \otimes v^h$. Consider the graded vector space dual to $V^h$:

$$\mathbb{C}[\text{Mat}_{m,n}]_q = \bigoplus_{j=0}^{\infty} \mathbb{C}[\text{Mat}_{m,n}]_{q,j}, \quad \mathbb{C}[\text{Mat}_{m,n}]_{q,j} = (V^h_{-j})^*, \quad j \in \mathbb{Z}_+.$$

Equip $\mathbb{C}[\text{Mat}_{m,n}]_q$ with a structure of $U_q\mathfrak{sl}_N$-module algebra by the duality:

$$\langle \xi f, v \rangle = \langle f, S(\xi) v \rangle, \quad \langle f_1 f_2, v \rangle = \sum_i \langle f_1, v_i' \rangle \langle f_2, v_i'' \rangle,$$

with $f, f_1, f_2 \in \mathbb{C}[\text{Mat}_{m,n}]_q, v \in V^h, \Delta v = \sum_j v_j' \otimes v_j''$.

Our immediate intention is to describe $\mathbb{C}[\text{Mat}_{m,n}]_q$ in terms of generators and relations. Consider the Hopf subalgebra $U_q\mathfrak{sl}_n \subset U_q\mathfrak{sl}_N$ generated by $E_i, F_i, K_i^{\pm 1}$, $i = 1, 2, \ldots, n - 1$, and the Hopf subalgebra $U_q\mathfrak{sl}_m \subset U_q\mathfrak{sl}_N$ generated by $E_{n+i}, F_{n+i}, K_{n+i}^{\pm 1}, i = 1, 2, \ldots, m - 1$. It follows from the definitions that the homogeneous component $\mathbb{C}[\text{Mat}_{m,n}]_{q,1} = \{ f \in \mathbb{C}[\text{Mat}_{m,n}]_{q} \mid \text{deg } f = 1 \} = (V_{b_1}^h)^*$ is a $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module. We are going to prove that this module splits into the tensor product of a $U_q\mathfrak{sl}_n$-module related to the vector representation and a $U_q\mathfrak{sl}_m$-module related to the covector representation. Consider the $U_q\mathfrak{sl}_m$-module $U$ and the $U_q\mathfrak{sl}_m$-module $V$, determined in the bases

\begin{itemize}
  \item One can observe from the formulation of proposition 8.1 that $s$ should be even, and $s \in \mathbb{N}(0,1)$ due to (8.8), (7.7), (7.8)
  \item non-restricted specialization
\end{itemize}
Proposition 8.1 There exists a unique collection \( \{ z^a \}_{a=1, \ldots, n; \alpha=1, \ldots, m} \), of elements of \( \mathbb{C}[\text{Mat}_{m,n}]_{q,1} \) such that the map \( i : u_a \otimes v^\alpha \mapsto z^a_\alpha, \ a = 1, \ldots, n; \ \alpha = 1, \ldots, m \) admits an extension up to an isomorphism of \( U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m \)-modules \( i : U \otimes V \to \mathbb{C}[\text{Mat}_{m,n}]_{q,1} \), and \( F_n z^m_n = q^{1/2} \).

**Proof.** Let \( V^h_{-k} \) denotes the \((-k)\)-th graded component of the \( U_q \mathfrak{sl}_n \)-module \( V^h \):

\[
V^h_{-k} = \{ v \mid H_0 v = -2kv \}.
\]

It follows from the results of the previous section that the elements \( F_{q^{k_M}} F_{q^{k_{M-1}}} \cdots F_{q^{k_{M+1}}}, k_{M+1} + k_{M+2} + \ldots + k_M = k \), constitute a basis in \( V^h_{-k} \). Hence, the dimension of \( V^h_{-k} \) is just the same as in the classical \((q = 1)\) case:

\[
\dim V^h_{-k} = \binom{mn + k - 1}{k}.
\]

(8.1)

Observe that \( V^h_{-1} \) is non-zero, so \( v' = F_n v^h \neq 0 \) and

\[
E_j v' = \begin{cases} -2v', & j = n \\ v', & |j - n| = 1 \\ 0, & |j - n| > 1 \end{cases}, \quad H_j v' = \begin{cases} -2v', & j = n \\ v', & |j - n| = 1 \\ 0, & |j - n| > 1 \end{cases}
\]

This, together with \( \dim V^h_{-1} = mn \) implies that \( v' \) generates the simple weight \( U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m \)-module \( V^h_{-1} \), and \( \mathbb{C}[\text{Mat}_{m,n}]_{q,1} \simeq U \otimes V \). Of course, the isomorphism \( i : U \otimes V \to \mathbb{C}[\text{Mat}_{m,n}]_{q,1} \) is unique up to a multiple from the ground field, and the elements \( z^a_\alpha = i(u_a \otimes v^\alpha), \ a = 1, \ldots, n; \ \alpha = 1, \ldots, m \), satisfy all the requirements of our proposition, except, possibly, the last property \( F_n z^m_n = q^{1/2} \). One can readily choose the above multiple in the definition of \( i \), which provides this property unless \( F_n z^m_n = 0 \). In the latter case one has \( F_n(E_{i_1}^{k_1} E_{i_2}^{k_2} \cdots E_{i_l}^{k_l} z^m_n) = 0 \) for all \( i_1, \ldots, i_l \) different from \( n \) and all \( k_1, k_2, \ldots, k_l \in \mathbb{Z}_+ \). Hence it follows from the irreducibility of the \( U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m \)-module \( \mathbb{C}[\text{Mat}_{m,n}]_{q,1} \simeq U \otimes V \) that \( F_n z^m_n = 0 \), and thus \( F_n v^h = 0 \). That is, \( \dim V^h = 1 \). On the other hand, it follows from (8.1) that \( \dim V^h = \infty \). This contradiction shows that \( F_n z^m_n \neq 0 \).

**Proposition 8.2** \( z^a_\alpha, \ a = 1, \ldots, n; \ \alpha = 1, \ldots, m \), generate the algebra \( \mathbb{C}[\text{Mat}_{m,n}]_q \).
Proposition 8.3. The elements \( z_\alpha^a \), \( \alpha = 1, \ldots, m \), \( a = 1, \ldots, n \) of the algebra \( \mathbb{C}[\text{Mat}_{m,n}]_q \) satisfy the relations (2.1), (2.2), (2.3).

**Proof.** By a virtue of (8.1), it suffices to prove that for any \( k \in \mathbb{Z}_+ \) the \( \binom{mn + k - 1}{k} \)
monomials
\[
(z_1^1)^{j_1^1}(z_2^1)^{j_2^1} \cdots (z_n^m)^{j_n^m}, \quad j_1^1 + j_2^1 + \ldots + j_n^m = k
\] (8.2)
are linearly independent in \( \mathbb{C}[\text{Mat}_{m,n}]_q, k \). An application of the standard techniques of specialization (see [8, chapter 5]) allows us to reduce this statement to its classical analogue. Namely, consider the basis
\[
F_{\beta_M}^k F_{\beta_{M-1}}^{k-1} \cdots F_{\beta_{M'+1}} F_1^h, \quad k = k_{M-1} + \ldots + k_{M'} - k
\] (8.3)
in \( V_{-k} \). Let us denote by \( \langle , \rangle \) the pairing
\[
\mathbb{C}[\text{Mat}_{m,n}]_q \times V_h \to \mathbb{C}(q^{1/s})
\] (8.4)
which is implicit in the definition of \( \mathbb{C}[\text{Mat}_{m,n}]_q \). Clearly, to prove linear independence of the monomials (8.2), it suffices to show that the determinant of the pairing \( \langle , \rangle : \mathbb{C}[\text{Mat}_{m,n}]_q, k \times V_{-k} \to \mathbb{C}(q^{1/s}) \) in the bases (8.2) and (8.3) is non-zero. An idea, underlying the specialization techniques, may be described roughly as follows. The determinant is a rational function of \( q^{1/s} \). To prove that the function is non-zero, it is enough to demonstrate that the point \( q = 1 \) is neither its pole nor its zero.

Consider the ring \( \mathcal{A} = \mathbb{Q}[q^{1/s}, q^{-1/s}] \) of Laurent polynomials in the indeterminate \( q^{1/s} \) and the \( \mathcal{A} \)-subalgebra \( U_{\mathcal{A}} \) in \( U_q \mathfrak{sl}_N \) generated by the elements \( E_i, F_i, K_i^\pm, L_i = \frac{K_i - K_i^{-1}}{q - q^{-1}} \), \( i = 1, \ldots, N - 1 \). This is a Hopf algebra:
\[
\Delta(L_i) = L_i \otimes K_i + K_i^{-1} \otimes L_i, \quad S(L_i) = -L_i, \quad \varepsilon(L_i) = 0, \quad i = 1, \ldots, N - 1.
\]
Let \( V_{\mathcal{A}} = U_{\mathcal{A}} v^h \). It is easy to show that the basis elements \( F_{\beta_M}^k F_{\beta_{M-1}}^{k-1} \cdots F_{\beta_{M'+1}} F_1^h \) are in \( V_{\mathcal{A}} \). Denote by \( F_{\mathcal{A}} \subset \mathbb{C}[\text{Mat}_{m,n}]_q \) the \( \mathcal{A} \)-module generated by all the monomials \( (z_1^1)^{j_1^1}(z_2^1)^{j_2^1} \cdots (z_n^m)^{j_n^m}, j_1^1, j_2^1, \ldots, j_n^m \in \mathbb{Z}_+ \). It follows from the relations in \( U_q \mathfrak{sl}_N \) and the definitions of modules \( U, V \) that the value of the linear functional \( z_\alpha^a \) on a vector \( v \in V_{\mathcal{A}} \) is in \( \mathcal{A} \). Hence, a similar statement is also valid for all \( f \in F_{\mathcal{A}} \). In particular, the aforementioned determinant belongs to \( \mathcal{A} \). We intend to prove that the determinant is a non-zero element in \( \mathcal{A} \). For that, it suffices to prove that its image under the natural homomorphism \( \mathcal{A} \to \mathbb{Q}, q^{1/s} \mapsto 1 \) is a non-zero number. The latter is a straightforward consequence of non-degeneracy of the natural pairing
\[
\mathbb{C}[z_1^1, \ldots, z_n^m] \times U_{\mathfrak{p}_-} \to \mathbb{C}, \quad f(z_1^1, \ldots, z_n^m) \times \xi \mapsto S(\xi)(f(z_1^1, \ldots, z_n^m))|_{z_1^1=\ldots=z_n^m=0},
\]
where \( U_{\mathfrak{p}_-} \) is the universal enveloping algebra of the Abelian Lie subalgebra
\[
\mathfrak{p}_- = \{ \xi \in \mathfrak{sl}_N \mid [H_0, \xi] = -2\xi \}
\]
(\( H_0 \) is given in (7.5)), and \( S \) is the antipode in \( U_{\mathfrak{sl}_N} \). □

Proposition 8.3. The elements \( z_\alpha^a, \alpha = 1, \ldots, m, a = 1, \ldots, n \) of the algebra \( \mathbb{C}[\text{Mat}_{m,n}]_q \) satisfy the relations (2.1), (2.2), (2.3).
Prove. It is easy to verify that the linear span of the left hand sides in (2.1) – (2.3) corresponds to an $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-submodule of $M^{\otimes 2}$ with $M = C[\text{Mat}_{m,n}]_{q,1}$. Let $M_A \subset M$ be the $A$-module generated by $\{z_a^\alpha\}$, $a = 1, \ldots, n$, $\alpha = 1, \ldots, m$. By a virtue of proposition 8.1, $M^{\otimes 2}$ is decomposed into a direct sum of four simple pairwise non-isomorphic $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-modules.\footnote{The tensor square of the vector (co-vector) representation is isomorphic to the direct sum of its symmetric square and exterior square.} A similar decomposition is also valid for $M_A \otimes M_A$, and a specialization at $q = 1$ leads to four pairwise non-isomorphic $U\mathfrak{sl}_n \otimes U\mathfrak{sl}_m$-modules. By misuse of language, one can say that each submodule of the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module $M^{\otimes 2}$ is unambiguously determined by its specialization at $q = 1$. Consider two such submodules, namely, the kernel of the multiplication operator $C[\text{Mat}_{m,n}]_{q,1}^{\otimes 2} \to C[\text{Mat}_{m,n}]_{q,2}$, $f_1 \otimes f_2 \mapsto f_1 f_2$ and the linear span of the elements given by the left hand sides of (2.1) – (2.3). Their specializations at $q = 1$ coincide, and hence the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-submodules themselves are the same. 

Corollary 8.4 There exists a unique homomorphism of graded algebras

\[
j: C[\text{Mat}_{m,n}]_q \to C[\text{Mat}_{m,n}]_q, \quad j : z^\alpha_a \mapsto z^\alpha_a, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, m. \tag{8.5}\n\]

Proposition 8.5 The homomorphism (8.5) is an isomorphism.

Proof. It is an easy exercise to compute the dimensions of the homogeneous components $C[\text{Mat}_{m,n}]_{q,k} = \{f \in C[\text{Mat}_{m,n}]_q\mid \text{deg } f = k\}$. Specifically,

\[
\dim C[\text{Mat}_{m,n}]_{q,k} = \binom{mn + k - 1}{k}. \tag{8.6}\n\]

It follows from proposition 8.2 that $j$ is onto. What remains is to apply (8.1), (8.6) to observe coincidence of the dimensions of the graded components:

\[
\dim C[\text{Mat}_{m,n}]_{q,k} = \dim C[\text{Mat}_{m,n}]_{q,k}, \quad k \in \mathbb{Z}_+. \tag{8.7}\n\]

So far we considered the $U_q\mathfrak{sl}_N$-module algebra $C[\text{Mat}_{m,n}]_q$ dual to $V^h$. Now turn to producing a $U_q\mathfrak{sl}_N$-module algebra $C[\overline{\text{Mat}}_{m,n}]_q$ dual to $V_l$. Equip $V_l$ with a structure of $U_q\mathfrak{sl}_N^{op}$-module coalgebra: $\Delta : v^l \mapsto v^l \otimes v^l$. Consider the graded vector space dual to $V_l$:

\[
C[\overline{\text{Mat}}_{m,n}]_q = \bigoplus_{j=0}^\infty C[\overline{\text{Mat}}_{m,n}]_{q,-j}, \quad C[\overline{\text{Mat}}_{m,n}]_{q,-j} = (V^l_j)^*, \quad j \in \mathbb{Z}_+. \n\]

Equip $C[\overline{\text{Mat}}_{m,n}]_q$ with a structure of $U_q\mathfrak{sl}_N$-module algebra by the duality:

\[
\langle \xi f, v \rangle = \langle f, S(\xi) v \rangle, \quad \langle f_1 f_2, v \rangle = \sum_i \langle f_1, v_i^l \rangle \langle f_2, v_i'' \rangle, \n\]

with $f, f_1, f_2 \in C[\text{Mat}_{m,n}]_q, \quad v \in V^l, \quad \Delta v = \sum_j v_j^l \otimes v_j''$.

Recall [3] that in the case of $s$-algebras the definition of an $A$-module algebra includes the following compatibility condition for involutions:

\[
(a f)^* = (S(a))^* f^*, \quad a \in A, \quad f \in F. \tag{8.7}\n\]
Let \( U_q\mathfrak{su}_{m,n} \) stands for the Hopf \(*\)-algebra \((U_q\mathfrak{sl}_N, \ast)\) given by

\[
(K_j^{\pm 1})^* = K_j^{\pm 1}, \quad E_j^* = \begin{cases} K_j F_j, & j \neq n \\ -K_j F_j, & j = n \end{cases}, \quad F_j^* = \begin{cases} E_j K_j^{-1}, & j \neq n \\ -E_j K_j^{-1}, & j = n \end{cases},
\]

with \( j = 1, \ldots, N - 1 \). Recall the standard method which was used in [25] to equip the space

\[
\text{Pol}(\text{Mat}_{m,n})_q \overset{\text{def}}{=} \mathbb{C}[\text{Mat}_{m,n}]_q \otimes \mathbb{C}[\text{Mat}_{m,n}]_q.
\]

with a structure of \( U_q\mathfrak{su}_{m,n} \)-module \(*\)-algebra. The involution in question allows one, in particular, to introduce the standard generators \((z_a^\alpha)^*\) of the subalgebra \( \mathbb{C}[\text{Mat}_{m,n}]_q \). Define the product of \( \varphi_+ \otimes \varphi_- \), \( \psi_+ \otimes \psi_- \in \mathbb{C}[\text{Mat}_{m,n}]_q \otimes \mathbb{C}[\text{Mat}_{m,n}]_q \) as follows:

\[
(\varphi_+ \otimes \varphi_-)(\psi_+ \otimes \psi_-) = m_+ \otimes m_- \quad (\varphi_+ \otimes \psi_+ \otimes \psi_-).
\]

Here \( m_+ \), \( m_- \) are the multiplications in \( \mathbb{C}[\text{Mat}_{m,n}]_q \), \( \mathbb{C}[\text{Mat}_{m,n}]_q \) respectively, and

\[
\hat{R} : \mathbb{C}[\text{Mat}_{m,n}]_q \otimes \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,n}]_q \otimes \mathbb{C}[\text{Mat}_{m,n}]_q,
\]

\[
\hat{R} = \sigma \cdot R_{\mathbb{C}[\text{Mat}_{m,n}]_q, \mathbb{C}[\text{Mat}_{m,n}]_q},
\]

(8.8)

with \( \sigma : a \otimes b \mapsto b \otimes a \). The associativity of the multiplication in \( \text{Pol}(\text{Mat}_{m,n})_q \) can be easily derived from (7.10), (7.11) by a standard argument [10]. Note that \( m_+, m_- \), \( \hat{R} \) are morphisms of \( U_q\mathfrak{sl}_N \)-modules. So, \( \text{Pol}(\text{Mat}_{m,n})_q \) is a \( U_q\mathfrak{sl}_N \)-module algebra. We intend to equip \( \text{Pol}(\text{Mat}_{m,n})_q \) with an involution. Consider the antilinear operators \( \ast : V^l \to V^h \), \( \ast : V^h \to V^l \), which are determined by the following properties. Firstly, \((v^h)^* = v^l\), \((v^l)^* = v^h\), and, secondly,

\[
(\xi v)^* = (S^{-1}(\xi))^* v^*,
\]

(8.9)

for all \( v \in V^h \), (resp. \( V^l \)), \( \xi \in U_q\mathfrak{su}_{m,n} \). This is certainly equivalent to

\[
(\xi v^h)^* = (S^{-1}(\xi))^* (v^h)^*; \quad (\xi v^l)^* = (S^{-1}(\xi))^* (v^l)^*, \quad \xi \in U_q\mathfrak{su}_{m,n}.
\]

It follows from the definitions of \( V^h \), \( V^l \) that the operators as above are well defined. In particular, (8.9) can be easily deduced. It also follows from the relation \((S^{-1}(\xi))^*)^* = \xi\) that the operators are mutually inverse. The duality argument allows one to form the mutually inverse antihomomorphisms \( \ast : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,n}]_q \), \( \ast : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,n}]_q :\)

\[
f^*(v) \overset{\text{def}}{=} \overline{f(v^*)}, \quad v \in V^l \quad (\text{resp. } V^h), \quad f \in (V^h)^* \quad (\text{resp. } (V^l)^*).
\]

(8.10)

Now we are in a position to define the antilinear operator \( \ast \) in \( \text{Pol}(\text{Mat}_{m,n})_q \) by

\[
(f_+ \otimes f_-)^* \overset{\text{def}}{=} f_-^* \otimes f_+^*,
\]

for \( f_+ \in \mathbb{C}[\text{Mat}_{m,n}]_q \), \( f_- \in \mathbb{C}[\text{Mat}_{m,n}]_q \). What remains is to verify that \( \ast \) equips \( \text{Pol}(\text{Mat}_{m,n})_q \) with a structure of \( U_q\mathfrak{su}_{m,n} \)-module algebra. For that, the reader is referred to [25, section 8].

We identify \( \mathbb{C}[\text{Mat}_{m,n}]_q \) with its image under the embedding \( \mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow \text{Pol}(\text{Mat}_{m,n})_q \), \( f \mapsto f \otimes 1 \), and \( \mathbb{C}[\text{Mat}_{m,n}]_q \) with its image under the embedding \( \mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow \text{Pol}(\text{Mat}_{m,n})_q \), \( f \mapsto 1 \otimes f \). It follows from proposition 8.2 that \{\( z_a^\alpha \),
\(\alpha = 1, \ldots, m, a = 1, \ldots, n,\) generate the *-algebra \(\text{Pol}(\mathcal{M}_{m,n})_q,\) and the complete list of relations consists of (2.1) – (2.3), together with the following one:

\[
\left(z_b^\beta\right)^* z_a^\alpha = m \tilde{R} \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right) = m \sigma R_{C[\mathcal{M}_{m,n}]_q} C[\mathcal{M}_{m,n}]_q \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right), \tag{8.11}\]

with \(m : \text{Pol}(\mathcal{M}_{m,n})_q \to \text{Pol}(\mathcal{M}_{m,n})_q,\) \(m : f_1 \otimes f_2 \mapsto f_1 f_2\) being the multiplication in \(\text{Pol}(\mathcal{M}_{m,n})_q.\)

Simplify the expression \(R_{C[\mathcal{M}_{m,n}]_q} C[\mathcal{M}_{m,n}]_q \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right)\) in (8.11). Denote by \(U_q\text{su}_n \otimes U_q\text{su}_m\) the subalgebra of the Hopf *-algebra \(U_q\text{su}_n, m\) generated by \(E_j, F_j, K_j, K_j^{-1}\) with \(j \neq n.\) Now an application of proposition 8.1 makes it easy to prove the following

**Lemma 8.6** The sesquilinear form in \(C[\mathcal{M}_{m,n}]_q,\) given by \(\left(z_a^\alpha, z_b^\beta\right) = \delta_{ab} \delta_{\alpha\beta}, a, b = 1, \ldots, n, \alpha, \beta = 1, \ldots, m,\) is \(U_q\text{su}_n \otimes U_q\text{su}_m\)-invariant:

\[
\left(\xi z_a^\alpha, z_b^\beta\right) = \left(z_a^\alpha, \xi z_b^\beta\right), \quad \xi \in U_q\text{su}_n \otimes U_q\text{su}_m, \quad a, b = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, m.\]

**Corollary 8.7** The linear functional \(\mu\) on \(C[\mathcal{M}_{m,n}]_{q,-1} \otimes C[\mathcal{M}_{m,n}]_{q,1}\) given by \(\mu \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right) = \delta_{ab} \delta_{\alpha\beta},\) is invariant:

\[
\mu \left(\xi \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right)\right) = \varepsilon(\xi) \mu \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right), \quad \xi \in U_q\text{su}_n \otimes U_q\text{su}_m, \quad a, b = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, m.\]

**Proof.** Let \(L = C[\mathcal{M}_{m,n}]_{q,1}.\) Consider the anticomodule \(\overline{L}\) which is still \(L\) as an Abelian group, but the actions of the ground field and \(U_q\text{su}_n \otimes U_q\text{su}_m\) are given by \((\lambda, v) \mapsto \overline{\lambda} v, (\xi, v) \mapsto S(\xi)^* v, \xi \in U_q\text{su}_n \otimes U_q\text{su}_m, v \in L.\) It follows from lemma 8.6 that the linear functional \(\overline{\mu} \otimes L \to C(q^{1/3}),\) corresponding to the sesquilinear form in \(L,\) is invariant. \(\Box\)

Let \(L' = C[\mathcal{M}_{m,n}]_{q,-1}, L'' = C[\mathcal{M}_{m,n}]_{q,1},\) and \(R_{L' L''}\) is the linear operator in \(L' \otimes L''\) given by the action of the universal R-matrix of the Hopf algebra \(U_q\text{sl}_n \otimes U_q\text{sl}_m \subset U_q\text{sl}_N,\) determined by a formula similar to (7.7).

**Lemma 8.8** In \(\text{Pol}(\mathcal{M}_{m,n})_q,\) for all \(a, b = 1, \ldots, n, \alpha, \beta = 1, \ldots, m,\)

\[
R_{C[\mathcal{M}_{m,n}]_q C[\mathcal{M}_{m,n}]_q} \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right) = \text{const}_1 \cdot R_{L' L''} \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right) + \text{const}_2 \cdot \delta_{ab} \delta_{\alpha\beta}, \tag{8.12}\]

with \text{const}_1 \text{ and const}_2 \text{ being independent of } a, b, \alpha, \beta.

**Proof.** Reduce the left hand side of (8.12) modulo \(C[\mathcal{M}_{m,n}]_{q,0} \otimes C[\mathcal{M}_{m,n}]_{q,0},\) using (7.7). The ‘redundant’ exponential multiples in the left hand side of the resulting identity can be omitted since

\[
\exp_q (\left(q^{-1} - q\right)E_{\beta_j} \otimes F_{\beta_j}) \left(\left(z_b^\beta\right)^* \otimes z_a^\alpha\right) = \left(z_b^\beta\right)^* \otimes z_a^\alpha
\]
for all $\alpha, \beta = 1, \ldots, m$, $a, b = 1, \ldots, n$, $j > \frac{m(m-1)}{2} + \frac{n(n-1)}{2}$. What remains is to compare the multiple $q^{-t_0}$ related to the Hopf algebra $U_q\mathfrak{sl}_N$ to a similar multiple related to the Hopf subalgebra $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$. It follows from lemma 7.1 and the description of $t_0$ in terms of the orthogonal basis of the Cartan subalgebra (7.9) that their actions on the subspace $\mathbb{C}[\overline{\Mat}_{m,n}]_{q,-1} \otimes \mathbb{C}[\Mat_{m,n}]_{q,1}$ differ only by a constant multiple. This implies the existence of such element const of the ground field that for all $a, b, \alpha, \beta$ one has

$$R_{\mathbb{C}[\overline{\Mat}_{m,n}]_{q} \mathbb{C}[\Mat_{m,n}]_{q}} \left( \left( z^\beta_b \right)^* \otimes z^\alpha_a \right) - \text{const}_1 \cdot R_{L'L''} \left( \left( z^\beta_b \right)^* \otimes z^\alpha_a \right) \in \mathbb{C}[\overline{\Mat}_{m,n}]_{q0} \otimes \mathbb{C}[\Mat_{m,n}]_{q0}.$$ 

Thus we get a linear functional $l$ on $\mathbb{C}[\overline{\Mat}_{m,n}]_{q,-1} \otimes \mathbb{C}[\Mat_{m,n}]_{q,1}$ since

$$\dim(\mathbb{C}[\overline{\Mat}_{m,n}]_{q0}) = \dim(\mathbb{C}[\Mat_{m,n}]_{q0}) = 1.$$

Clearly, the linear maps $\sigma \cdot R_{\mathbb{C}[\overline{\Mat}_{m,n}]_{q} \mathbb{C}[\Mat_{m,n}]_{q}}$ and $\sigma \cdot R_{L'L''}$ are morphisms of $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-modules. So the linear functional $l$ is $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-invariant. What remains is to apply corollary 8.7, together with the fact that the subspace of $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-invariant functionals $\mathbb{C}[\overline{\Mat}_{m,n}]_{q,-1} \otimes \mathbb{C}[\Mat_{m,n}]_{q,1} \rightarrow \mathbb{C}$ is one-dimensional. \hfill \Box

We need an explicit form of the operator $R_{L'L''}$. Let $* : U \rightarrow \overline{U}$, $* : V \rightarrow \overline{V}$, be the identical maps from the above $U_q\mathfrak{sl}_n$-module $U$ and $U_q\mathfrak{sl}_m$-module $V$ onto the associated antimodules. Let $R_{\overline{U}U}$, $R_{\overline{V}V}$ stand for the operators in $\overline{U} \otimes U$, $\overline{V} \otimes V$ respectively, given by the actions of the universal R-matrices of the Hopf algebras $U_q\mathfrak{sl}_n$ and $U_q\mathfrak{sl}_m$. The following result is well known; we reproduce its proof here for the reader’s convenience.

**Lemma 8.9** For all $a, b = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m,$

$$R_{\overline{U}U}(u_b^* \otimes u_a) = \text{const}^' \cdot \begin{cases} q^{-1}u_b^* \otimes u_a, & a \neq b \\ u_b^* \otimes u_a - (q^2 - 1) \sum_{k>a} u_k^* \otimes u_k, & a = b \end{cases},$$

$$R_{\overline{V}V}((v^\beta)^* \otimes v^\alpha) = \text{const}'' \cdot \begin{cases} q^{-1}(v^\beta)^* \otimes v^\alpha, & \alpha \neq \beta \\ (v^\alpha)^* \otimes v^\alpha - (q^2 - 1) \sum_{k>\alpha} (v^k)^* \otimes v^k, & \alpha = \beta \end{cases}$$

with const', const'' being independent of $a, b, \alpha, \beta$.

**Proof.** It suffices to prove the first identity. Consider the linear operator $\sigma \cdot R_{\overline{U}U} : \overline{U} \otimes U \rightarrow U \otimes \overline{U}$, with $\sigma$ being the flip of tensor multiples. This operator is a morphism of $U_q\mathfrak{sl}_n$-modules. Besides, it follows from (7.7) that $\sigma \cdot R_{\overline{U}U}(u_n^* \otimes u_n) = \text{const}' \cdot u_n^* \otimes u_n^*$ since $u_n$ is the lowest weight vector of the $U_q\mathfrak{sl}_n$-module $U$. On the other hand, it is well known (see, for example, [27]) that the composition of $\sigma$ with the operator defined by the right hand side of the first identity in the statement of our lemma possesses the same properties. What remains is to use the fact that each morphism of $U_q\mathfrak{sl}_n$-modules $\overline{U} \otimes U \rightarrow U \otimes \overline{U}$ which annihilates $u_n^* \otimes u_n$, is identically zero (this vector does not belong to any of the two simple components of the $U_q\mathfrak{sl}_n$-module $U \otimes U$, and hence it generates this module). \hfill \Box

\footnote{In fact, the dimensions of isotypic components of the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module $\mathbb{C}[\overline{\Mat}_{m,n}]_{q,-1} \otimes \mathbb{C}[\Mat_{m,n}]_{q,1}$ are the same just as in the case $q = 1$.}
Lemma 8.10 \( R_{\mathbb{C}[\text{Mat}_{m,n}]} \mathbb{C}[\text{Mat}_{m,n}]_q \big((z_n^m)^* \otimes z_n^m\big) = q^2(z_n^m)^* \otimes z_n^m + 1 - q^2 \).

Proof. We are about to apply the explicit formula (7.7) for the universal R-matrix. Prove that

\[
H_j z_n^m = \begin{cases} 
2z_n^m, & j = n \\
-z_n^m, & |j-n| = 1 \\
0, & \text{otherwise}
\end{cases}
\]

The two latter relations follow from the definitions of \( z_n^\alpha \). The first relation follows from \( H_0 z_n^m = 2z_n^m \).

\[
2z_n^m = \frac{2}{m+n}(-m(n-1) - n(m-1))z_n^m + \frac{2mn}{m+n}H_0 z_n^m.
\]

Hence \( z_n^m, (z_n^m)^* \) are weight vectors whose weights are \( \alpha_n, -\alpha_n \) respectively. Thus, we have

\[
t_0((z_n^m)^* \otimes z_n^m) = (\alpha_n, \alpha_n)(z_n^m)^* \otimes z_n^m = -2(z_n^m)^* \otimes z_n^m.
\]

Observe that the only \( q \)-exponent, which survives in (7.7), is \( \exp_q ((q^{-1} - q)E_{\beta_M} \otimes F_{\beta_M}) \).

Of course, \( \beta_M = \alpha_n \), and it is not difficult to prove that \( E_{\beta_M} \otimes F_{\beta_M} = \text{const} \cdot E_n \otimes F_n \).

The constant multiplier in the latter equality is equal to 1 since

\[
(F_{\beta_M}, E_{\beta_M}) = (F_n, E_n) = \frac{1}{q^{-1} - q}
\]

with respect to the well known pairing \( (,): U_q \mathfrak{b}^- \times U_q \mathfrak{b}^+ \to \mathbb{C} \ [8, \text{Chapter 6}] \). Hence

\[
\exp_q ((q^{-1} - q)E_{\beta_M} \otimes F_{\beta_M}) = \exp_q ((q^{-1} - q)E_n \otimes F_n),
\]

and what remains is to use the detailed calculations for the case \( m = n = 1 \) given in [25].

Corollary 8.11 \((z_n^m)^* z_n^m = q^2 z_n^m (z_n^m)^* + 1 - q^2\).
9 The algebras $\mathbb{C}[SL_N]_{q,t}$ and $\mathbb{C}[\widetilde{G}]_{q,x}$

Recall that the $U_q\mathfrak{sl}_N$-module algebra $\mathbb{C}[SL_N]_q$ is a domain, and its element $t = t^{\wedge m}_{1,2,\ldots,m} t^{m+1}_{n+1,a+2,\ldots,N}$ quasi-commutes with all the generators $t_{ij}$. Let $\mathbb{C}[SL_N]_{q,t}$ stand for the localization of the algebra $\mathbb{C}[SL_N]_q$ with respect to the multiplicative set generated by $t$.

**Proposition 9.1** There exists a unique extension of the structure of $U_q\mathfrak{sl}_N$-module algebra from $\mathbb{C}[SL_N]_q$ onto $\mathbb{C}[SL_N]_{q,t}$.

**Proof.** The uniqueness of the extension is obvious. We are going to construct such extension by applying the following statement.

**Lemma 9.2** For every $\xi \in U_q\mathfrak{sl}_N$, $f \in \mathbb{C}[SL_N]_q$, there exists a unique Laurent polynomial $p_{\xi,f}(\lambda)$ with coefficients from $\mathbb{C}[SL_N]_{q,t}$ such that

$$p_{\xi,f}(q^l) = \xi \left( f \cdot t^l \right) \cdot t^{-l}, \quad l \in \mathbb{Z}_+.$$ 

**Proof.** Our statement follows from the definition of a $U_q\mathfrak{sl}_N$-module algebra structure in $\mathbb{C}[SL_N]_q$ (7.13), (7.14), the definition of a comultiplication in $U_q\mathfrak{sl}_N$ (7.3), and (3.7). $\square$

Turn back to the proof of proposition 9.1. We can use the same Laurent polynomials for defining $\xi \left( f \cdot t^l \right)$ for $f \in \mathbb{C}[SL_N]_q$ and all integers $l$:

$$\xi \left( f \cdot t^l \right) \overset{\text{def}}{=} p_{\xi,f}(q^l) t^l.$$ 

Of course, we need firstly to verify that the map $U_q\mathfrak{sl}_N \times \mathbb{C}[SL_N]_{q,t} \to \mathbb{C}[SL_N]_{q,t}$, $\xi \times f \mapsto \xi(f)$ as above is well defined, and secondly that we obtain this way a structure of $U_q\mathfrak{sl}_N$-module algebra. The first item is equivalent to

$$p_{\xi,f}(q^{a+l}) t^a = p_{\xi,f \circ \psi}(q^l), \quad \xi \in U_q\mathfrak{sl}_N, \quad f \in \mathbb{C}[SL_N]_q, \quad a \in \mathbb{Z}_+, \quad l \in \mathbb{Z}.$$ 

This relation is obvious for $l \in \mathbb{Z}_+$, hence it is valid for all integers $l$ due to the well known uniqueness theorem for the Laurent polynomials.

For the second item, we have to prove some identities for $\xi \in U_q\mathfrak{sl}_N$, $f_1 \cdot t^l$, $f_2 \cdot t^l$, $f_1, f_2 \in \mathbb{C}[SL_N]_q$, $l \in \mathbb{Z}$. Observe that the left and right hand sides of those identities (up to multiplying by the same powers of $t$) are just Laurent polynomials of the indeterminate $\lambda = q^l$. So, it suffices to prove them for $l \in \mathbb{Z}_+$ due to the same uniqueness theorem for Laurent polynomials. On the other hand, at all $l \in \mathbb{Z}_+$ one can deduce these identities from the fact that $\mathbb{C}[\mathfrak{sl}_N]_q$ is a $U_q\mathfrak{sl}_N$-module algebra. $\square$

A more general but less elementary approach to proving statements like proposition 9.1 have been obtained in a recent work by V. Lunts and A. Rosenberg [16].

The following result is due to M. Noumi [19] for the case $m = n = 2$. It will be also refined in the sequel. Recall the notation $J_{ao} = \{n+1, n+2, \ldots, N\} \setminus \{N+1-a\} \cup \{a\}$ as in the statement of proposition 3.2.

**Proposition 9.3** The map $i : z_a^\alpha \mapsto t^{-1} \cdot t^{\wedge m}_{1,2,\ldots,m} J_{ao}, \quad \alpha = 1, \ldots, m$, $a = 1, \ldots, n$, admits a unique extension up to a homomorphism of algebras $i : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[SL_N]_{q,t}$. 

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Remark. It will be shown later that in fact $i$ is an embedding (see section 12).

Proof of proposition 9.3. The uniqueness of the extension is obvious.

Let $\mathbb{C}[\text{Mat}_{m,N}]_q$ be the algebra defined by its generators $\{t_{aa}\}$, $\alpha = 1, 2, \ldots, m$, $a = 1, 2, \ldots, N$, and the relations (3.1) – (3.3), together with $t = t_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}}^m$. Consider the localization $\mathbb{C}[\text{Mat}_{m,N}]_{q,t}$ of $\mathbb{C}[\text{Mat}_{m,N}]_q$ with respect to the multiplicatively closed set $t^N$. It suffices to prove that the map

$$i: z_\alpha^a \mapsto t^{-1} \cdot t_{\{1,2,\ldots,m\}a\alpha}^m, \quad \alpha = 1, \ldots, m, \quad a = 1, \ldots, n$$

admits an extension up to a homomorphism of algebras $i: \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,N}]_{q,t}$. Consider an embedding of $U_q\mathfrak{sl}_N$-module algebras $i': \mathbb{C}[\text{Mat}_{m,N}]_q \hookrightarrow (U_q\mathfrak{sl}_N)^*$, which is a composition of the embedding $\mathbb{C}[\text{Mat}_{m,N}]_q \hookrightarrow \mathbb{C}[\mathfrak{SL}_N]_q$, $t_{a,a} \mapsto t_{a+n,a}$, $\alpha = 1, 2, \ldots, m$, $a = 1, 2, \ldots, N$, and the canonical embedding $\mathbb{C}[\mathfrak{SL}_N]_q \hookrightarrow (U_q\mathfrak{sl}_N)^*$. One can use the same argument as in the proof of proposition 9.1 to extend the structure of $U_q\mathfrak{sl}_N$-module algebra and the embedding $i'$ onto the localization $\mathbb{C}[\text{Mat}_{m,N}]_{q,t}$ of $\mathbb{C}[\text{Mat}_{m,N}]_q$.

Consider the embedding $i'': \mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow (U_q\mathfrak{sl}_N)^*$, derived by a duality from the onto morphism of coalgebras

$$j: U_q\mathfrak{sl}_N \to V^h, \quad j: \xi \mapsto S(\xi) v^h; \quad \xi \in U_q\mathfrak{sl}_N,$$

with $v^h$ being the generator of the $U_q\mathfrak{sl}_N$-module $V^h$. Use the embedding $i''$ to get a composition of $i''$ and the homomorphism $\mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,n}]_q$ (see section 8) to obtain a homomorphism of algebras:

$$i''': \mathbb{C}[\text{Mat}_{m,n}]_q \to (U_q\mathfrak{sl}_N)^*.$$ (9.1)

One can easily apply proposition 8.1 to prove that the statement of proposition 9.3 reduces to

Lemma 9.4

$$i' \left( t^{-1} t_{\{1,2,\ldots,m\}J}^m \right) \subset i''(\mathbb{C}[\text{Mat}_{m,n}]_q), \quad \text{card}(J) = m.$$

Proof. It suffices to establish that $i' \left( t^{-1} t_{\{1,2,\ldots,m\}J}^m \right)$ are orthogonal to the kernel of $j$ with respect to the above pairing. Let us agree not to distinguish between the $U_q\mathfrak{sl}_N$-module algebras $\mathbb{C}[\text{Mat}_{m,n}]_q$, $\mathbb{C}[\text{Mat}_{m,N}]_{q,t}$, $\mathbb{C}[\mathfrak{SL}_N]_q$ and their images in $(U_q\mathfrak{sl}_N)^*$. What remains now is to prove that for $\text{card}(J) = m$, $\xi \in U_q\mathfrak{sl}_N$,

$$
\begin{align*}
\langle t^{-1} t_{\{1,2,\ldots,m\}J}^m, (K_i^+ - 1)\xi \rangle &= 0, & i &= 1, 2, \ldots, N - 1, \\
\langle t^{-1} t_{\{1,2,\ldots,m\}J}^m, E_i\xi \rangle &= 0, & i &= 1, 2, \ldots, N - 1, \\
\langle t^{-1} t_{\{1,2,\ldots,m\}J}^m, F_j\xi \rangle &= 0, & j &= 1, 2, \ldots, n - 1, n + 1, \ldots, N - 1.
\end{align*}
$$

These follow from the more general relations

$$
\begin{align*}
\langle t_i^{k+\lambda}, (K_i^{\pm 1} - 1)\xi \rangle &= (q^{\mp(k+1)} - 1) \delta_{im} \langle t_i^{k+\lambda}, (K_i^m)J, \xi \rangle, \\
\langle t_i^{k+\lambda}, E_i\xi \rangle &= 0, & i &= 1, 2, \ldots, N - 1, \\
\langle t_i^{k+\lambda}, F_j\xi \rangle &= 0, & j &= 1, 2, \ldots, n - 1, n + 1, \ldots, N - 1.
\end{align*}
$$
for $\xi \in U_q\mathfrak{sl}_N$. In proving these latter relations, one can restrict matters to the case $k \in \mathbb{Z}_+$ by using the techniques related to Laurent polynomials. Let \( \bar{t} = t_{\{n+1,n+2,\ldots,N\}} \). What remains to prove now is that for all $\xi \in U_q\mathfrak{sl}_N$, $k \in \mathbb{Z}_+$

$$
\begin{align*}
&\langle \bar{t}^k t_{\{n+1,n+2,\ldots,N\}}, (K_i^{\pm 1} - 1)\xi \rangle = (q^{\bar{t}^k (k+1)} - 1) \delta_{im} \langle \bar{t}^k t_{\{1,2,\ldots,m\}}, \xi \rangle, \\
&\langle \bar{t}^k t_{\{n+1,n+2,\ldots,N\}}, F_i \xi \rangle = 0, \quad i = 1, 2, \ldots, N - 1, \\
&\langle \bar{t}^k t_{\{n+1,n+2,\ldots,N\}}, F_{j} \xi \rangle = 0, \quad j = 1, 2, \ldots, n - 1, n + 1, \ldots, N - 1. \quad \square
\end{align*}
$$

**Remark.** The proof of proposition 9.3 involves a construction of the embedding \( i^{-1}i'' \) of $U_q\mathfrak{sl}_N$-module algebras $\mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,N}]_{q,t}$. Hence, the map

$$
z_a^{\alpha} \mapsto t^{-1} \cdot t_{\{1,2,\ldots,m\}} J_{aa}, \quad \alpha = 1, \ldots, m, \quad a = 1, \ldots, n.
$$

admits an extension up to an embedding of $U_q\mathfrak{sl}_N$-module algebras $I : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[N]_{q,t}$.

**Lemma 9.5** For all $1 \leq a < b \leq n$, $1 \leq \alpha < \beta \leq m$,

$$
I : z_a^{\alpha} z_b^{\beta} - q z_a^{\alpha} z_b^{\alpha} \mapsto t^{-1} \cdot t_{\{1,2,\ldots,m\}}^{\{a,b,\ldots,N+1-\beta,\ldots,N+1-\alpha,\ldots,N\}}.
$$

**Proof.** In the same way as in the proof of proposition 9.3, one can establish that

$$
t^{-1} \cdot t_{\{1,2,\ldots,m\}}^{\{a,b,\ldots,N+1-\beta,\ldots,N+1-\alpha,\ldots,N\}} \in I(\mathbb{C}[\text{Mat}_{m,n}]_q).$

This is a weight vector of the $U_q\mathfrak{sl}_N$-module $I(\mathbb{C}[\text{Mat}_{m,n}]_q)$. A computation of the weight yields

$$
I(c_1 z_a^{\beta} z_b^{\alpha} + c_2 z_a^{\alpha} z_b^{\beta}) = t^{-1} \cdot t_{\{1,2,\ldots,m\}}^{\{a,b,\ldots,N+1-\beta,\ldots,N+1-\alpha,\ldots,N\}}
$$

with $c_1, c_2 \in \mathbb{C}$. When computing the constants $c_1, c_2$, one can restrict oneself to the special case $a = n - 1$, $b = n$, $\alpha = m - 1$, $\beta = m$. Even more, one can stick to the case $m = n = 2$ due to the homomorphism:

$$
t_{i,j} \mapsto \begin{cases} t_{i,j-n+2}, & i \leq 2 \quad \& \quad n - 1 \leq j \leq n + 2, \\ 1, & i > 2 \quad \& \quad j = i + n, \\ 0, & \text{otherwise}. \end{cases}
$$

In the special case $m = n = 2$ the result in question is accessible via a direct calculation [19]. \quad \square

A proof of a more general statement is presented in section 11.

The lemma 9.5, together with proposition 8.1, allow one to get a description of the $U_q\mathfrak{sl}_N$-module algebra structure on $\mathbb{C}[\text{Mat}_{m,n}]_q$.

**Corollary 9.6** For $a = 1, \ldots, n; \alpha = 1, \ldots, m$

$$
K_{n}^{\pm 1} z_a^{\alpha} = \begin{cases} q^{\pm 2} z_a^{\alpha}, & a = n \& \alpha = m \\ q^{\pm 1} z_a^{\alpha}, & a = n \& \alpha \neq m \quad \text{or} \quad a \neq n \& \alpha = m \\ z_a^{\alpha}, & \text{otherwise} \end{cases}
$$

$$
F_n z_a^{\alpha} = q^{1/2} \cdot \begin{cases} 1, & a = n \& \alpha = m \\ 0, & \text{otherwise} \end{cases} \quad E_n z_a^{\alpha} = -q^{1/2} \cdot \begin{cases} q^{-1} z_n^{m} z_a^{\alpha}, & a \neq n \& \alpha \neq m \\ (z_n^{m})^2, & a = n \& \alpha = m \\ z_n^{m} z_a^{\alpha}, & \text{otherwise} \end{cases}
$$

The general case is derivable by observing that $I$ is a morphism of $U_q\mathfrak{sl}_N$-modules.
and with $k \neq n$

$$K_k^{±1}z_a^\alpha = \begin{cases} q^{±1}z_a^\alpha, & k < n \& a = k \text{ or } k \geq n \& \alpha = N - k \\ q^{±1}z_a^\alpha, & k < n \& a = k + 1 \text{ or } k \geq n \& \alpha = N - k + 1 \\ z_a^\alpha, & \text{otherwise} \end{cases}$$

$$F_kz_a^\alpha = q^{1/2} \cdot \begin{cases} z_{a+1}^\alpha, & k < n \& a = k \\ z_{a+1}^\alpha, & k > n \& \alpha = N - k \\
0, & \text{otherwise} \end{cases}$$

$$E_kz_a^\alpha = q^{-1/2} \cdot \begin{cases} z_{a-1}^\alpha, & k < n \& a = k + 1 \\ z_{a-1}^\alpha, & k > n \& \alpha = N - k + 1 \\
0, & \text{otherwise} \end{cases}$$

In sections 3, 8 the algebras $\mathbb{C}[SL_N]_q$ and $U_q\mathfrak{sl}_N$ were equipped with involutions. Thus we got *-algebras $\mathbb{C}[^*\tilde{G}]_q$ and $U_q\mathfrak{su}_{n,m}$. Recall that $\mathbb{C}[SL_N]_q$ is a $U_q\mathfrak{sl}_N$-module algebra. It is easy to prove that the involutions in question agree in such a way that $\mathbb{C}[^*\tilde{G}]_q = (\mathbb{C}[SL_N]_q, *)$ is a $U_q\mathfrak{su}_{n,m}$-module algebra.

In section 3 an element $x = tt^*$ and the localization $\mathbb{C}[^*\tilde{G}]_{q,x}$ were introduced. An argument similar to that used in the proof of proposition 9.1 (with the reference to (3.8) instead of (3.7)) allows to get

**Proposition 9.7** There exists a unique extension of the structure of $U_q\mathfrak{su}_{n,m}$-module algebra from $\mathbb{C}[\tilde{G}]_q$ onto $\mathbb{C}[\tilde{G}]_{q,x}$.

**Proposition 9.8** The map

$$i : z_a^\alpha \mapsto t^{-1}t_\wedge^{m\{1,2,\ldots,m\}J_{a\alpha}} \quad (9.2)$$

with $J_{a\alpha} = \{n + 1, n + 2, \ldots, N\} \setminus \{N + 1 - \alpha\} \cup \{a\}$, is uniquely extendable up to a homomorphism of *-algebras $i : \text{Pol}(\text{Mat}_{m,n})_q \to \mathbb{C}[^*\tilde{G}]_{q,x}$.

**Remark.** It will be shown later that in fact $i$ is an embedding (see the conclusion remark in section 12).

**Proof** of proposition 9.8. The uniqueness of the extension is obvious. The existence follows from a construction of a homomorphism of $U_q\mathfrak{su}_{n,m}$-module algebras

$$\text{Pol}(\text{Mat}_{m,n})_q \to \mathbb{C}[\tilde{G}]_{q,x}, \quad z_a^\alpha \mapsto t^{-1}t_\wedge^{m\{1,2,\ldots,m\}J_{a\alpha}}, \quad (9.3)$$

to be described below.

Consider the embedding of $U_q\mathfrak{sl}_N$-module algebras $\mathcal{I} : \mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t}$ (see the previous section) and a similar embedding $\overline{\mathcal{I}} : \mathbb{C}[\text{Mat}_{m,n}]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t^*}$.

$$\overline{\mathcal{I}}f = (\mathcal{I}f^*)^*, \quad f \in \mathbb{C}[\text{Mat}_{m,n}]_q.$$ 

(We use the obvious embeddings of the localizations $\mathbb{C}[SL_N]_{q,t} \subset \mathbb{C}[\tilde{G}]_{q,x}, \mathbb{C}[SL_N]_{q,t^*} \subset \mathbb{C}[\tilde{G}]_{q,x}$.) Consider the linear map $\text{Pol}(\text{Mat}_{m,n})_q \to \mathbb{C}[\tilde{G}]_{q,x}, \quad f_+ \cdot f_- \mapsto \overline{\mathcal{I}}(f_-) \cdot \overline{\mathcal{I}}(f_+),$

*It suffices to use a similar result for the quantum group $SU_N$ and the quantum universal enveloping algebra $U_q\mathfrak{su}_N$, as the compact and the non-compact involutions on the generators differ only by sign change.
If the Bergman diamond lemma. case finite dimensional weighted representations of relations suffices to show that for all $a,b$. In the passage to general q-Cartan domains, the of q-Cartan domains considered in [25]. The only exception here constitute the proofs. It was demonstrated in section 9 that the structure of $U^*$ is a transcendental number then the result of proposition 8.5 is still valid, and we transfer use these formulae to define a $U$ is determined by the relations formulated in corollary 9.6. Our immediate intention is to defined for arbitrary $q$, as well since the coefficients in decompositions of the elements $\tilde{\xi} = 0$. For that, it suffices to establish $t^{-k}(q^{\text{const}})^{-j} n \sigma R_{\mathbb{C}[\text{SL}_N]_{q,t}}(\mathbb{T}(z_\alpha^*) t^{s_j} \otimes t^k \mathcal{I}(z_\alpha^*))$, with $j,k \in \mathbb{Z}$. We may restrict ourselves to the special case $j,k \in \mathbb{N}$ since viewed as a function of the parameter $q^{1/2}$, is a Laurent polynomial. In the above special case $\mathbb{T}(z_\alpha^*) t^{s_j}$, $t^k \mathcal{I}(z_\alpha^*)$, $a,b = 1,2,\ldots, n$, $\alpha, \beta = 1,2,\ldots, m$, are the matrix elements of finite dimensional weighted representations of $U_q \mathfrak{sl}_N$. What remains is to apply the well known [3] R-matrix commutation relations between those matrix elements, (7.7), and the relations $\langle t^k \mathcal{I}(z_\alpha^*), F_i \xi \rangle = \langle \mathbb{T}(z_\alpha^*) t^{s_j}, E_i \xi \rangle = 0,$ for all $\xi \in U_q \mathfrak{sl}_N$, $a,b = 1,2,\ldots, n$, $\alpha, \beta = 1,2,\ldots, m$, $i = 1,2,\ldots, N-1$. □

Remark. A great deal of our techniques can be transferred onto the general case of q-Cartan domains considered in [25]. The only exception here constitute the proofs of proposition 9.3 and lemma 9.4. In the passage to general q-Cartan domains, the displacement of lines in the matrix $t$ used in these proofs is irrelevant. Instead, one should use a q-analogue for the longest element $w_0$ of the Weyl group [3] such that $\Delta(w_0) = w_0 \otimes w_0 \cdot R$, with $R$ being the universal R-matrix (cf. [25, section 16]).

10 $U_q \mathfrak{su}_{n,m}$-module algebra Pol(\text{Mat}_{m,n})_q$

It was demonstrated in section 9 that the structure of $U_q \mathfrak{sl}_N$-module algebra in \text{C}[\text{Mat}_{m,n}]_q is determined by the relations formulated in corollary 9.6. Our immediate intention is to use these formulae to define a $U_q \mathfrak{sl}_N$-module algebra structure in \text{C}[\text{Mat}_{m,n}]_q. If $q \in (0, 1)$ is a transcendental number then the result of proposition 8.5 is still valid, and we transfer the $U_q \mathfrak{sl}_N$-module algebra structure to \text{C}[\text{Mat}_{m,n}]_q via the isomorphism $j : \text{C}[\text{Mat}_{m,n}]_q \to \text{C}[\text{Mat}_{m,n}]_q$
defined in (8.5). Note that this $U_q \mathfrak{sl}_N$-module algebra structure in \text{C}[\text{Mat}_{m,n}]_q is well defined for arbitrary $q$ as well since the coefficients in decompositions of the elements $z_{\alpha_1}^\alpha z_{\alpha_2}^\beta \cdots z_{\alpha_M}^\gamma$ and $\xi(z_{\alpha_1}^\alpha z_{\alpha_2}^\beta \cdots z_{\alpha_M}^\gamma), \xi \in U_q \mathfrak{sl}_N$, in the basis of lexicographically ordered monomials $1$

$$((z_1^a)^j_1 (z_2^b)^j_2 \cdots (z_{n-1}^c)^j_{n-1} (z_n^d)^j_n), \quad j_{1n} \in \mathbb{Z}_+$$

are Laurent polynomials in the parameter $q^{1/2}$.

It was noted in section 2 that the linear independence of these monomials can be proved by an application of the Bergman diamond lemma.
Thus, $\mathbb{C}[\text{Mat}_{m,n}]_q$ becomes a $U_q\mathfrak{sl}_N$-module algebra, and we still have the homomorphism of $U_q\mathfrak{sl}_N$-module algebras

$$j : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,n}]_q, \quad j : z_\alpha^a \mapsto z_\alpha^a, \quad a = 1, \ldots, n, \; \alpha = 1, \ldots, m.$$  \quad (10.1)

The result of proposition 8.5 turns out to hold in the case of arbitrary $q \in (0, 1)$:

**Proposition 10.1** The homomorphism (10.1) is an isomorphism.

**Proof.** Since $j$ respects the gradation and

$$\dim \mathbb{C}[\text{Mat}_{m,n}]_q, i = \dim \mathbb{C}[\text{Mat}_{m,n}]_q = \binom{mn + i - 1}{i}$$

(see section 2 and the equality (8.13)), it suffices to prove injectivity of $j$.

Suppose the kernel of $j$ is a non-trivial ideal $J$. Clearly, $J$ is a $U_q\mathfrak{sl}_N$-submodule of $\mathbb{C}[\text{Mat}_{m,n}]_q$ since $j$ is a morphism of $U_q\mathfrak{sl}_N$-modules. $\mathbb{C}[\text{Mat}_{m,n}]_q$ is a $U_q\mathfrak{b}^\perp$-locally finite dimensional weight $U_q\mathfrak{sl}_N$-module, hence the same is true for $J$. In particular, $J$ contains a non-zero element $f$ satisfying the relations

$$F_i f = 0, \quad i = 1, \ldots, N - 1.$$  

The action of $K_i^{\pm 1}$ respects the above equations, hence we may assume that $f$ is a weight vector in the $U_q\mathfrak{sl}_N$-module $\mathbb{C}[\text{Mat}_{m,n}]_q$, that is a common eigenvector of all $K_i^{\pm 1}$, $i = 1, \ldots, N - 1$.

**Lemma 10.2** Any weight vector in $\mathbb{C}[\text{Mat}_{m,n}]_q$, annihilated by all $F_i$, $i \neq n$, coincides up to a constant with one of the vectors

$$f_{k_1, \ldots, k_m} = (z_n^m)_{k_1} \left( z^2 \right)_{\binom{m-1}{n-1}}^{k_2} \left( z^3 \right)_{\binom{m-2}{n-1}}^{k_3} \cdots \left( z^m \right)_{\binom{m-1}{n-1}}^{k_m}$$

$(k_1, \ldots, k_m \in \mathbb{Z}_+)$.

**Proof** of the lemma. Using the explicit formulae, given in corollary 9.6, one proves that the elements $f_{k_1, \ldots, k_m}$ are weight vectors, annihilated by all $F_i$ with $i \neq n$. Impose the notation $L_{k_1, \ldots, k_m}$ for the finite dimensional simple $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-submodule in $\mathbb{C}[\text{Mat}_{m,n}]_q$ generated by the vector $f_{k_1, \ldots, k_m}$. Clearly,

$$\mathbb{C}[\text{Mat}_{m,n}]_q, i \supset \bigoplus L_{k_1, \ldots, k_m}$$

(10.2)

where the sum is taken over all $m$-tuples $(k_1, \ldots, k_m)$ satisfying $k_1 + 2k_2 + \ldots + mk_m = i$.

In the classical case one has the equality in (10.2). Thus one has the equality in the quantum case as well since the dimensions of the spaces $\mathbb{C}[\text{Mat}_{m,n}]_q, i$, $L_{k_1, \ldots, k_m}$ are just the same as in the classical case. We conclude that any homogeneous (in particular, weight) vector, annihilated by all $F_i$ with $i \neq n$, is a linear combination of some $f_{k_1, \ldots, k_m}$'s as the lowest weight vector in a simple $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module is unique up to a constant. To finish the proof of the lemma it remains to observe that weights of the vectors $f_{k_1, \ldots, k_m}$ are distinct for different $m$-tuples $(k_1, \ldots, k_m)$. \qed
Return to the proof of proposition 10.1. Due to the above lemma we may assume that \( J \) contains \( f_{k_1, \ldots, k_m} \) for some \( k_1, \ldots, k_m \in \mathbb{Z}_+ \), \( k_1 + \ldots + k_m \neq 0 \), which, moreover, satisfies \( F_n f_{k_1, \ldots, k_m} = 0 \). Let
\[
f_{k_1, \ldots, k_m} = \sum_{j=0}^{s} \psi_{j; k_1, \ldots, k_m} \cdot (z_n^m)^j, \quad s \in \mathbb{Z}_+
\]
where \( \psi_{j; k_1, \ldots, k_m} \) are elements of the unital subalgebra in \( \mathbb{C}[\text{Mat}_{m,n}]_q \) generated by \( \{ z_a^\alpha \}_{(\alpha, a) \neq (m,n)} \). Then, by corollary 9.6
\[
F_n f_{k_1, \ldots, k_m} = q^{1/2} \cdot \sum_{j=0}^{s} \frac{1 - q^{2j}}{1 - q^{2}} \cdot \psi_{j; k_1, \ldots, k_m} \cdot (z_n^m)^{j-1}.
\]
So \( F_n f_{k_1, \ldots, k_m} = 0 \) implies \( \psi_{k_1+k_2+\ldots+k_m; k_1, \ldots, k_m} = 0 \) because of \( k_1 + k_2 + \ldots + k_m \neq 0 \). On the other hand, \( \psi_{k_1+k_2+\ldots+k_m; k_1, \ldots, k_m} \neq 0 \) since the image of \( f_{k_1, \ldots, k_m} \) under the homomorphism of algebras
\[
\varphi : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[z], \quad \varphi : z_a^\alpha \mapsto \begin{cases} z, & a = n \& \alpha = m, \\ 0, & a - \alpha \neq n - m, \\ 1, & \text{otherwise}, \end{cases}
\]
is \( z_{k_1+k_2+\ldots+k_m} \neq 0 \). \( \square \)

Let us endow \( \mathbb{C}[\text{Mat}_{m,n}]_q \) with a \( U_q\mathfrak{sl}_N \)-module algebra structure via
\[
(\xi f)^* = (S(\xi))^* f^*, \quad \xi \in U_q\mathfrak{su}_{n,m}, \quad f \in \mathbb{C}[\text{Mat}_{m,n}]_q.
\]
Proposition 10.1 implies that there is a canonical isomorphism of \( U_q\mathfrak{sl}_N \)-module algebras
\[
\mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[\text{Mat}_{m,n}]_q, \quad (z_a^\alpha)^* \mapsto (z_a^\alpha)^*, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, m. \quad (10.3)
\]

**Remark.** Note that for any \( q \in (0, 1) \) the \( * \)-algebra \( \text{Pol}(\text{Mat}_{m,n})_q \) is generated by \( z_a^\alpha, \ a = 1, \ldots, n, \ \alpha = 1, \ldots, m \) due to the isomorphisms (10.1), (10.3). The same fact for transcendental \( q \) is an easy consequence of results of section 8.

**Proposition 10.3**

i) The homomorphism (8.14) is an isomorphism of \( * \)-algebras;

ii) there exists a unique structure of \( U_q\mathfrak{su}_{n,m} \)-module algebra in \( \text{Pol}(\text{Mat}_{m,n})_q \) such that (8.14) is an isomorphism of \( U_q\mathfrak{su}_{n,m} \)-module algebras.

**Proof.** Statement ii) immediately follows from i).

To prove i), consider the filtration
\[
F^k \text{Pol}(\text{Mat}_{m,n})_q = \bigoplus_{i+j \leq k} \text{Pol}(\text{Mat}_{m,n})_{q,i-j}
\]
(see (2.9)) and a similar filtration \( F^k \text{Pol}(\text{Mat}_{m,n})_q, \ k \in \mathbb{Z}_+ \). The homomorphism (8.14) respects these filtrations. Clearly,
\[
\dim F^k \text{Pol}(\text{Mat}_{m,n})_q \leq \sum_{j=0}^{k} \binom{2mn + j - 1}{j}.
\]

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On the other hand, due to the isomorphisms (10.1), (10.3)
\[
\dim F^k \text{Pol}(\mathcal{M}at_{m,n})_q = \sum_{j=0}^{k} \binom{2mn+j-1}{j}.
\]
It remains to observe that the homomorphism (8.14) is surjective as $z^\alpha_a$, $a = 1, \ldots, n$, $\alpha = 1, \ldots, m$, generate the $\ast$-algebra $\text{Pol}(\mathcal{M}at_{m,n})_q$. \qed

**Corollary 10.4** $\mathbb{C}[\text{Mat}_{m,n}]_q \otimes \mathbb{C}\overline{\text{Mat}_{m,n}}_q \cong \text{Pol}(\text{Mat}_{m,n})_q$.

### 11 A proof of proposition 3.3

It follows from the definitions of section 10 and the remark before lemma 9.5 that the homomorphism $i : \mathbb{C}[\text{Mat}_{m,n}]_q \to \mathbb{C}[SL_N]_{q,t}$ is a morphism of $U_q\mathfrak{sl}_N$-module algebras.

**Proposition 11.1** Let $k \in \mathbb{N}$. There exists a constant $c(q, k) \in \mathbb{C}$ such that for all $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq m$, $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, in the algebra $\mathbb{C}[SL_N]_{q,t}$, one has
\[
i \left( z^{\wedge k_{\{1,2,\ldots,k\}}} \right) = c(q, k) t^{-1} t^{\wedge k}_{\{1,2,\ldots,k\},\{1,2,\ldots,k\}} J,
\]
with $J = \{n+1, n+2, \ldots, N\} \setminus \{n + \alpha_1, n + \alpha_2, \ldots, n + a_k\} \cup \{a_1, a_2, \ldots, a_k\}$.

**Proof.** An argument similar to that used while proving lemma 9.5 reduces the general case to the special case $m = n = k$:
\[
i \left( z^{\wedge k_{\{1,2,\ldots,k\}}} \right) = c(q, k) t^{-1} t^{\wedge k}_{\{1,2,\ldots,k\},\{1,2,\ldots,k\}}.
\]
Let $\bar{z} = z^{\wedge k_{\{1,2,\ldots,k\}}}, \bar{t} = t^{\wedge k}_{\{1,2,\ldots,k\},\{1,2,\ldots,k\}}$, and $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k \subset U_q\mathfrak{sl}_2k$ the Hopf subalgebra generated by $E_j, F_j, K_j, j \neq k$.

Consider the subalgebra $\mathcal{F}$ of $\mathbb{C}[SL_{2k}]_q$ generated by the quantum minors $t^{\wedge k}_{\{1,2,\ldots,k\},J}$, $\text{card } J = k$. We need the following

**Lemma 11.2** $\{ t'^{j'} \bar{t}'' \mid j', j'' \in \mathbb{Z}_+ \}$ form a basis in the vector space of $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k$-invariants in $\mathcal{F}$.

**Proof.** It is obvious that $t'^{j'} \bar{t}''$ are $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k$-invariant. They are linear independent since for all $\xi \in U_q\mathfrak{sl}_{2k}$
\[
\left\langle t'^{j'} \bar{t}'', K_k \xi \right\rangle = q^{j'+j''} \left\langle t'^{j'} \bar{t}'', \xi \right\rangle, \quad \left\langle t'^{j'} \bar{t}'', \xi K_k \right\rangle = q^{j''-j'} \left\langle t'^{j'} \bar{t}'', \xi \right\rangle.
\]
Prove that the above vectors generate the space of $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k$-invariants. $\{ \bar{t}^l \mid l \in \mathbb{Z}_+ \}$ form a weight basis of the vector space $\{ f \in \mathcal{F} \mid E_j f = 0, \ j = 1, 2, \ldots, N-1 \}$. Hence the $U_q\mathfrak{sl}_{2k}$-module $\mathcal{F}$ is isomorphic to a sum of the simple $U_q\mathfrak{sl}_{2k}$-modules $U_q\mathfrak{sl}_{2k} \bar{t}^l$, $l \in \mathbb{Z}_+$. The dimensions of their weight subspaces remain intact under the passage from the classical to the quantum case (see [8]). Therefore the dimensions of their $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k$-isotypic components also remain intact. Thus we conclude that $\{ t'^{j'} \bar{t}'' \mid j' + j'' = l \}$ generate the
vector space of $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k$-invariants in $U_q\mathfrak{sl}_{2k}$ and \{ $t^{j'}\overline{t}^{j''} \mid j',j'' \in \mathbb{Z}_+$ \} generate the vector space of $U_q\mathfrak{sl}_k \otimes U_q\mathfrak{sl}_k$-invariants in $\mathcal{F}$.

Turn back to the proof of proposition 11.1 in the special case $m = n = k$. Observe that $i(\overline{z})$ belongs to the localization of $\mathcal{F}$ with respect to the multiplicative set $t^\mathbb{N}$ and is $U_q\mathfrak{sl}_N$-invariant. Hence by lemma 11.2 $i(\overline{z})$ belongs to the linear span of \{ $t^{j'}\overline{t}^{j''} \mid j' \in \mathbb{Z}, j'' \in \mathbb{Z}_+$ \}. What remains is to compare (11.3) and

$$\langle i(\overline{z}), K_k \xi \rangle = 0, \quad \langle i(\overline{z}), \xi K_k \rangle = q^2 \langle i(\overline{z}), \xi \rangle .$$

Proof of lemma 5.2. To compute the constant $c(q,k)$ in (11.1), apply $F_n$ to both sides of this relation. The relations $F_n z^\alpha_a = q^{1/2} \delta_{an} \delta_{am}$ and $H_n z^\beta_b = 0$ for $b \neq n, \beta \neq m$ (see corollary 9.6) imply

$$F_n z^\wedge k \{ m-k+1,...,m \}_{n-k+1,...,n} = q^{1/2} z \{ m-k-1,m-k,...,m-1 \}_{n-k+1,...,n} .$$

It follows from $\Delta(F_n) = F_n \otimes K_n^{-1} + 1 \otimes F_n$, $F_n(t^{-1}) = 0$ that

$$F_n \left( t^{-1} t^{\wedge m} \{ 1,2,...,m \}_{1,n+1,...,n,n+k+1,...,N} \right) = t^{-1} F_n t^{\wedge m} \{ 1,2,...,m \}_{n-k+1,...,n,n+k+1,...,N} = q^{1/2} t^{-1} t^{\wedge m} \{ n-k+1,...,n+1,n+k+1,...,N \} .$$

Hence one has in (11.1) $c(q,k) = c(q,k-1) = \cdots = c(q,1) = 1$.

Lemma 11.3 Let $\text{card}(J) = m$, $J^c = \{1,2,\ldots,N \} \setminus J$, $l(J,J^c) = \text{card}(\{j',j'' \in J \times J^c \mid j' > j''\})$. Then

$$\left( t^{\wedge m} \{ 1,2,...,m \}_J \right)^* = (-1)^{\text{card}(\{1,2,...,n \} \cap J)} (-q)^{l(J,J^c)} t^{\wedge n} \{ m+1,m+2,...,N \}_J .$$

Proof. This lemma is deducible from (4.4) and a general formula of Ya. Soibelman [26], [3, p. 432]:

$$\left( t^{\wedge m} \{ 1,2,...,m \}_J \right)^* = (-q)^{l(J,J^c)} t^{\wedge n} \{ m+1,m+2,...,N \}_J .$$

It now follows from (3.5), (11.4) that

$$\sum_{J \subseteq \{1,2,...,N \} \atop \text{card}(J) = m} (-1)^{\text{card}(\{1,2,...,n \} \cap J)} t^{\wedge m} \{ 1,2,...,m \}_J \left( t^{\wedge m} \{ 1,2,...,m \}_J \right)^* = 1 .$$

Proof of proposition 3.3. Multiply (11.5) by $t^{-1}$ from the left and by $t^{*-1}$ from the right and apply lemma 5.2, together with (11.5). The result is just the statement of proposition 3.3.

12 Faithfulness of the representation $T$

We start with proving the faithfulness of the representation $T$.

Lemma 12.1 For all $k,l \in \mathbb{Z}_+$, the map

$$\mathbb{C}[\text{Mat}_{m,n}]_{q,k} \otimes \mathbb{C}[\text{Mat}_{m,n}]_{q,-l} \to \text{Hom}(\mathcal{H}_l, \mathcal{H}_k); \quad f_+ \otimes f_- \mapsto T(f_+ f_-)|_{\mathcal{H}_l}$$

is one-to-one.
Proof. It follows from corollary 2.3 that

\[ i_1 : \mathbb{C}[\text{Mat}_{m,n}]_{q,k} \to \mathcal{H}_k, \quad i_1 : f_+ \mapsto f_+ v_0, \quad f_+ \in \mathbb{C}[\text{Mat}_{m,n}]_{q,k}, \]

is an isomorphism of vector spaces. Since \( \mathcal{H} \simeq \mathcal{L} \) (see proposition 5.1) and \( \mathcal{H} \) is a pre-Hilbert space, one has an isomorphism

\[ i_2 : \mathbb{C}[\text{Mat}_{m,n}]_{q,-l} \to \mathcal{H}_l^*, \quad \langle i_2(f_-), v \rangle = (v, f_-^* v_0), \quad f_- \in \mathbb{C}[\text{Mat}_{m,n}]_{q,-l}. \]

What remains is to use the canonical isomorphism \( \mathcal{H}_k \otimes \mathcal{H}_l^* \simeq \text{Hom}(\mathcal{H}_l, \mathcal{H}_k) \) and the relation

\[ T(f_+ f_-) v = T(f_+)(T(f_-) v, v_0) v_0 = (v, f_-^* v_0) f_+ v_0 \]

for \( v \in \mathcal{H}_l, f_+ \in \mathbb{C}[\text{Mat}_{m,n}]_{q,k}, f_- \in \mathbb{C}[\text{Mat}_{m,n}]_{q,-l}. \) \( \square \)

**Proposition 12.2** The representation \( T \) is faithful.

**Proof.** Recall that the vector space \( \text{Pol}(\text{Mat}_{m,n})_q \) is equipped with a bigradation (2.9). We need a standard partial order on \( \mathbb{Z}_+^2 \):

\[(k_1, l_1) \leq (k_2, l_2) \iff k_1 \leq k_2 \land l_1 \leq l_2.\]

Assume that our statement is wrong and \( T(f) = 0 \) for some \( f \in \text{Pol}(\text{Mat}_{m,n})_q, f \neq 0 \). Consider a homogeneous component \( f_{kl} \neq 0 \) of \( f \) with minimal bidegree \( (k, l) \). (Such homogeneous component certainly exists, but it may be non-unique for a given \( f \in \text{Pol}(\text{Mat}_{m,n})_q \). Let \( P_k : \mathcal{H} \to \mathcal{H}_k \) be the projection onto \( \mathcal{H}_k \) with kernel \( \bigoplus_{j \neq k} \mathcal{H}_j \). Since \( f_{kl} \) is of a minimal bidegree, one has \( P_k T(f_{kl})|_{\mathcal{H}_l} = P_k T(f)|_{\mathcal{H}_l} = 0, f_{kl} \neq 0 \), which contradicts the statement of lemma 12.1. \( \square \)

**Remark.** Recall that in section 9 we constructed a homomorphism \( i : \text{Pol}(\text{Mat}_{m,n})_q \to \mathbb{C}[\tilde{G}]_{q,x} \) (see proposition 9.8). Now we may establish its injectivity. It follows from proposition 12.2 and the commutative diagram:

\[
\begin{array}{ccc}
\text{Pol}(\text{Mat}_{m,n})_q & \xrightarrow{i} & \mathbb{C}[\tilde{G}]_{q,x} \\
T \downarrow & & \downarrow \tilde{T} \\
\text{End}(\mathcal{H}) & \xrightarrow{~} & \text{End}(\mathcal{L})
\end{array}
\]
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