Almost complete analytical integration in Galerkin BEM

Daniel Seibel *

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In this work, semi-analytical formulae for the numerical evaluation of surface integrals occurring in Galerkin boundary element methods (BEM) in 3D are derived. The integrals appear as the entries of BEM matrices and are formed over pairs of surface triangles. Since the integrands become singular if the triangles have non-empty intersection, the transformation presented in [1] is used to remove the singularities. It is shown that the resulting integrals admit analytical formulae if the triangles are identical or share a common edge. Moreover, the four-dimensional integrals are reduced to one- or two-dimensional integrals for triangle pairs with common vertices or disjoint triangles respectively. The efficiency and accuracy of the formulae is demonstrated in numerical experiments.

1. Introduction

Whenever unbounded domains appear in the modelling of physical problems, boundary element methods (BEM) present a particularly effective tool for the numerical simulation. Instead of discretising the underlying boundary value problem directly, BEM operate on the corresponding integral equations posed on the boundary. Hence, shape functions are defined on the surface and not in the volume, which results in less degrees of freedoms overall. BEM are therefore applied in numerous fields of science, ranging from computational elasticity to electromagnetic and acoustic scattering.

However, the price one pays is the occurrence of dense matrices that are expensive to calculate. A Galerkin approximation of the Laplace equation with trial and test functions \( \varphi \) and \( \psi \) requires the calculation of integrals over surface triangles \( \sigma \) and \( \tau \),

\[
I = \int_{\tau} \int_{\sigma} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{y}) dS(\mathbf{y}) \psi(\mathbf{x}) dS(\mathbf{x}),
\]

*Faculty of Mathematics and Computer Science, Campus E1.1, Saarland University, 66123 Saarbrücken, Germany, E-Mail: seibel@num.uni-sb.de
which do not vanish. In addition, the kernel function is singular at \( x = y \), so standard quadrature rules for the approximation of \( I \) may perform poorly. Whereas the problem of fully populated matrices can be solved by hierarchical low-rank approximation \([2]\), several algorithms for the efficient calculation of \( I \) are available in the literature. The general strategy is to use coordinate transformations that render the integral suitable for numerical integration. Approaches based on polar or Duffy coordinates \([3, 4]\) yield regular integrals for a wide selection of kernel functions \([5]\), which can be approximated by product quadrature rules \([6]\). However, since the integrals are essentially four-dimensional, numerical quadrature is expensive. In order to reduce the computational effort, analytical integration can be carried out for specific kernels to obtain lower dimensional integrals \([7, 8]\). If the integral is only defined in weak sense as a finite-part or Hadamard integral, then integration by parts is often the appropriate solution \([9]\).

The main contribution of this article is the derivation of analytical formulae for the complete integration of \( I \) for singular cases. Our approach is based on the regularisation method by S. Erichsen and S. A. Sauter \([6]\) which removes the singularities by applying a variant of the Duffy transformation. We show that the resulting representation admits closed formulae of \( I \) for identical triangles as well as triangles with a common edge and reduce it to a one-dimensional integral for triangles with a common vertex.

2. Preliminaries

We consider the numerical solution of the Laplace problem

\[
-\Delta u = 0 \quad \text{in } \Omega,
\]
\[
u = g \quad \text{on } \Gamma = \partial \Omega, \tag{1}
\]
in a domain \( \Omega \) with bounded Lipschitz boundary \( \Gamma \). If \( \Omega \) is unbounded, we assume the radiation condition

\[ |u(x)| \in \mathcal{O}(|x|^{-1}) \quad \text{for } |x| \to \infty. \]

The representation formula expresses the solution \( u \) in terms of its boundary values only,

\[
u(x) = \int_{\Gamma} u^*(x, y) \partial_n u(y) dS(y) - \int_{\Gamma} \partial_n(y) u^*(x, y) g(y) dS(y), \quad x \in \Omega.
\]

Here, \( n \) is the unit normal to \( \Gamma \) pointing outwards \( \Omega \) and \( u^* \) is the fundamental solution of the Laplace operator,

\[ u^*(x, y) = \frac{1}{4\pi |y - x|}. \]

The boundary value problem is hence reduced to the problem of finding the unknown Neumann trace \( \nu = \partial_n u \). To this end, we take the traces in the representation formula and insert the Dirichlet condition to obtain the boundary integral equation

\[ \forall \nu = \left( \frac{1}{2} I + K \right) g \quad \text{on } \Gamma, \tag{2} \]
where the layer potentials are defined by
\[(Vw)(x) = \int_\Gamma u^*(x, y) w(y) dS(y), \quad (Kw)(x) = \int_\Gamma \partial_n(y) u^*(x, y) w(y) dS(y).\]

Neumann or mixed boundary conditions can be treated similarly. We refer to [10] for more details.

For the numerical solution of (2) with BEM, we discretise the boundary with finite elements.

**Definition 1 (Mesh).** A mesh \((\Gamma_h, T_h)\) (or simply \(\Gamma_h\)) is a finite collection of non-empty and open elements \(\tau \subset \Gamma_h\) which satisfies:

1. \(T_h = \{\tau_n\}_{n=1}^N\) is a triangulation of \(\Gamma_h\), i.e.
   \[\Gamma_h = \bigcup_{n=1}^N \bar{\tau}_n.\]
   The intersection \(\bar{\tau}_n \cap \bar{\tau}_m\) of two distinct elements is either empty or consists of a common vertex or edge.

2. Each \(\tau\) in \(T_h\) is a flat triangle with vertices \(p_1, p_2, p_3\). The reference mapping \(\chi_\tau : \pi \to \tau, \quad \chi_\tau(x_1, x_2) = p_1 + x_1(p_2 - p_1) + x_2(p_3 - p_2),\) parametrises \(\tau\) by the reference triangle
   \[\pi = \{(x_1, x_2) \mid 0 < x_1 < 1, 0 < x_2 < x_1\} \subset \mathbb{R}^2.\]
   We denote by
   \[J_\tau = (p_2 - p_1 \mid p_3 - p_2) \in \mathbb{R}^{3 \times 2}, \quad g_\tau = \sqrt{\text{det}(J_\tau^T J_\tau)}\]
   the Jacobian and the Gram determinant of \(\chi_\tau\) respectively and assume \(g_\tau \neq 0\).

**Remark 1.** Certainly, the particular choice of the reference element \(\pi\) is not important for our approach. The reason why we use a non-standard \(\pi\) nonetheless lies in the fact that the presentation in Section 3 becomes simpler and it is in accordance with the literature referenced there.

On the triangular mesh, we define piece-wise constant and piece-wise linear ansatz functions.

**Definition 2 (Boundary element spaces).** For \(p = 0, 1\), we denote by
\[S^0_h(\pi) = \{1\}, \quad S^1_h(\pi) = \{1 - x_1, x_1 - x_2, x_2\}\]
the set of reference functions and by
\[S^p_h(\tau) = \text{span}\{\varphi \circ \chi_\tau^{-1} : \varphi \in S^p_h(\pi)\}\]
the local boundary element space on $\tau$. We define the global space by gluing the local spaces together, i.e.

$$S^p_h = \{ \varphi : \Gamma_h \to \mathbb{R} : \varphi|_\tau \in S^p_h(\tau) \ \forall \tau \in T_h \}.$$ 

For $p = 1$, we moreover require that the functions $\varphi$ are continuous.

We choose the Lagrangian basis

$$\varphi^0_n(x) = \begin{cases} 1 & \text{if } x \in \tau_n, \\ 0 & \text{else} \end{cases}, \quad \varphi^1_m(x_j) = \begin{cases} 1 & \text{if } j = m, \\ 0 & \text{else} \end{cases},$$

where $\{x_j\}_{j=1}^M$ denotes the set of vertices in $\Gamma_h$. Then, the ansatz

$$t_h = \sum_{n=1}^N t_n \varphi^0_n \in S^0_h, \quad t \in \mathbb{R}^N, \quad g_h = \sum_{m=1}^M g_m \varphi^1_m \in S^1_h, \quad g \in \mathbb{R}^M,$$

for the approximate boundary data leads to the Galerkin approximation of (2): Find $t \in \mathbb{R}^N$ such that

$$Vt = \left( \frac{1}{2} M + K \right) g,$$

where the matrices $M \in \mathbb{R}^{N \times M}, V \in \mathbb{R}^{N \times N}$ and $K \in \mathbb{R}^{N \times M}$ are given by

$$M[n, m] = \langle \varphi^0_n, \varphi^1_m \rangle, \quad V[n, i] = \langle \varphi^0_n, \psi^0_i \rangle, \quad K[n, m] = \langle \varphi^0_n, \varphi^1_m \rangle,$$

with $i, n = 1, \ldots, N$ and $m = 1, \ldots, M$. The brackets symbolise the usual $L_2$-inner product

$$\langle u, v \rangle = \int_{\Gamma_h} u(x) v(x) \, dS(x).$$

The boundary integral equation is now reduced to a system of linear equations, which can be solved efficiently with direct or iterative methods.

### 3. Integral Regularisation

The entries of $V$ an $K$ are of the form

$$\int_{\Gamma_h \times \Gamma_h} k(x, y) \varphi(y) dS(y) \psi(x) dS(x) = \sum_{\sigma, \tau \in T_h} \int_\tau k(x, y) \varphi(y) dS(y) \psi(x) dS(x),$$

where $k = u^*, \partial_n(u^*)u^*$ is the kernel function and $\varphi, \psi$ are trial and test functions respectively. Let $I$ be one of the summands for the non-trivial case $\tau \subset \text{supp } \varphi$ and $\sigma \subset \text{supp } \psi$. We transform back to the reference element $\pi$,

$$I = \int_{\sigma \tau} \int k(x, y) \varphi(y) dS(y) \psi(x) dS(x)$$

$$= \int_{\pi \times \pi} g_\sigma g_\tau k(\chi_\sigma(x), \chi_\tau(y)) \varphi(\chi_\sigma(x)) \psi(\chi_\tau(y)) d(y) d(x),$$

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and abbreviate the integrand by $q$. Since the kernel function $k(x, y)$ is singular at $x = y$, the integral needs to be regularised. We distinguish between four different cases: the intersection $\bar{\sigma} \cap \bar{\tau}$ may consist either of

1. the whole element,
2. exactly one edge,
3. exactly one point,
4. be empty.

In the following, we summarise the regularisation introduced in [6]. For the most part, we adhere to the version of [1, Chapter 5].

### 3.1. Identical elements

For identical elements $\sigma = \tau$, we substitute

$$ z = x - y, \quad Z = (x + y)/2 $$

such that

$$ I = \int_\Pi \int_{\pi_z} q(Z + z/2, Z - z/2) \, dZ \, dz $$

with

$$ \Pi = \{ z = x - y \mid x, y \in \pi \}, \quad \pi_z = (\pi - z/2) \cap (\pi + z/2). $$

The singularity of the integrand is now located at $z = 0$.

As shown in Figure 1, we decompose $\Pi$ into six triangles $\pi_i = A_i \pi$ with

$$ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} $$
and \( A_{i+3} = -A_i \) for \( i = 1, 2, 3 \). Thus, we obtain

\[
I = \sum_{i=1}^{6} \int \int q(Z + A_i z/2, Z - A_i z/2) dZ \, dz.
\]

In the last step, we parametrise \( \pi \) by \((0, 1)^2\) using the Duffy transformation

\[
z(\eta) = \begin{pmatrix} \eta_1 \\ \eta_1 \eta_2 \end{pmatrix}, \quad A_i(\eta) = A_i z(\eta)
\]

with Jacobian \( \eta_1 \) and conclude that \( I \) has the representation

\[
I = \int_{(0,1)^4} \eta_1 \sum_{i=1}^{6} \int q(Z + A_i(\eta)/2, Z - A_i(\eta)/2) dZ \, d\eta.
\]

In Section 4, we will see that the integrand is smooth since the Jacobian \( \eta_1 \) cancels out the singularity of \( q \).

### 3.2. Common edge

If the two triangles intersect at exactly one edge, we can proceed similarly to the first case. Let \( \chi_\sigma \) and \( \chi_\tau \) be chosen in such a way that the common edge is parametrised by

\[
\chi_\tau(x_1, 0) = \chi_\sigma(x_1, 0), \quad x_1 \in (0, 1).
\]

The kernel function is singular at this edge, i.e. at \((x, y)\) with \( x_2 = y_2 = 0 \) and \( x_1 = y_1 \). The regularisation is carried out via the mappings

\[
A_1(\eta) = \eta_1 \begin{pmatrix} 1 \\ \eta_2 \eta_4 \\ 1 - \eta_2 \eta_3 \end{pmatrix}, \quad A_2(\eta) = \eta_1 \begin{pmatrix} 1 \\ \eta_2 \\ 1 - \eta_2 \eta_3 \end{pmatrix}, \quad A_3(\eta) = \eta_1 \begin{pmatrix} 1 - \eta_2 \eta_3 \\ \eta_2 (1 - \eta_3) \\ 1 - \eta_2 \eta_3 \end{pmatrix},
\]

\[
A_4(\eta) = \eta_1 \begin{pmatrix} 1 - \eta_2 \eta_3 \eta_4 \\ \eta_2 \eta_3 (1 - \eta_4) \\ \eta_2 (1 - \eta_3 \eta_4) \end{pmatrix}, \quad A_5(\eta) = \eta_1 \begin{pmatrix} 1 - \eta_2 \eta_3 \eta_4 \\ \eta_2 (1 - \eta_3 \eta_4) \\ \eta_2 \eta_3 \end{pmatrix},
\]

and reads

\[
I = \int_{(0,1)^4} \eta_1^3 \eta_2^2 \left( q(A_1(\eta)) + \eta_3 \sum_{i=2}^{5} q(A_i(\eta)) \right) d\eta.
\]
3.3. Common vertex

Let the origin in the reference domain be mapped to the common vertex, i.e. 
\[ \chi_\tau(0,0) = \chi_\sigma(0,0). \]

By virtue of the mappings 
\[ A_1(\eta) = \eta_1 \begin{pmatrix} 1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \quad A_2(\eta) = \eta_1 \begin{pmatrix} \eta_3 \\ \eta_4 \\ 1 \end{pmatrix}, \]

we obtain 
\[ I = \int_{(0,1)^4} \eta_1^3 \eta_3 (q(A_1(\eta)) + q(A_2(\eta))) \, d\eta. \]

In summary, the regularisation yields integral representations with smooth integrands on the unit cube. In this form, the integral can be approximated efficiently by quadrature rules and the quadrature error decays exponentially with the quadrature order.

4. Calculation of integrals

Instead of applying quadrature rules directly, we calculate parts of the regularised integrals analytically.

4.1. Single layer potential

With the discretisation provided in Section 2, the entries of the single layer potential \( V \) are of the form 
\[ I = \int_\sigma \int_\tau \frac{1}{4\pi |y-x|} dS(y) \, dS(x) = \frac{g_\tau g_\sigma}{4\pi} \int_{\pi \times \pi} \frac{1}{|\chi_\tau(y) - \chi_\sigma(x)|} \, dy \, dx. \quad (5) \]

We proceed like in Section 3 and begin with the case of identical elements.

4.1.1. Identical Elements

Let \( v \) and \( w \) be the edges of \( \tau = \sigma \) with starting point \( p \). Then, the triangle is parameterised by 
\[ \chi_\sigma(y) = \chi_\tau(y) = p + y_1 v + y_2 w. \]
and the regularisation of (5) reads

\[
I = \int_{(0,1)^2} \eta_1 \sum_{i=1}^{6} \int_{\pi A_i(\eta)} q(Z + A_i(\eta)/2, Z - A_i(\eta)/2) \, dZ \, d\eta
\]

\[
= \frac{g_2^2}{2\pi} \int_{(0,1)^2} \eta_1 \left( \left| \frac{\pi A_1(\eta)}{\eta_1 \eta_2 v + \eta_1 w} \right| + \left| \frac{\pi A_2(\eta)}{\eta_1 \eta_2 w + \eta_1 v} \right| + \left| \frac{\pi A_3(\eta)}{\eta_1 \eta_2 (w + v) - \eta_1 w} \right| \right) \, d\eta,
\]

where the area is \( |\pi A_i(\eta)| = (1 - \eta_1)^2/2 \) for \( i = 1, 2, 3 \). Hence, we obtain

\[
I = \frac{g_2^2}{12\pi} \int_{(0,1)} \left( \frac{1}{|\eta_2 v + w|} + \frac{1}{|\eta_2 w + v|} + \frac{1}{|\eta_2 (w + v) - w|} \right) \, d\eta. \quad (6)
\]

Figure 2 depicts the complex continuation of the integrand for concrete values of \( v \) and \( w \). It is smooth on the real axis, since the edges are linearly independent, but has poles and branch cuts in the complex domain.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Visualisation of the integrand \( f(z) \) of (6) in the complex plane.}
\end{figure}

The three terms in the integrand are of the form

\[
\frac{1}{\sqrt{\gamma + \beta \eta_2 + \alpha \eta_2^2}}, \quad \text{with } \alpha > 0, \ 4\alpha \gamma - \beta^2 > 0.
\]

The anti-derivative is given by

\[
F(\eta_2) = \frac{1}{\sqrt{\alpha}} \ln \left( 2\sqrt{\alpha} \sqrt{\gamma + \beta \eta_2 + \alpha \eta_2^2 + 2\alpha \eta_2 + \beta} \right), \quad (7)
\]

see [11, Section 1.2.52.8] and [12, Section 2.261]. Thus, the integral reduces to

\[
I = \frac{g_2^2}{12\pi} \left[ F_1(\eta_2) + F_2(\eta_2) + F_3(\eta_2) \right]_0^1.
\]
where \( F_i \) is \( F \) with the parameters
\[
\begin{align*}
\alpha_1 &= |v|^2, & \beta_1 &= 2v \cdot w, & \gamma_1 &= |w|^2, \\
\alpha_2 &= |w|^2, & \beta_2 &= 2w \cdot v, & \gamma_2 &= |v|^2, \\
\alpha_3 &= |w+v|^2, & \beta_3 &= -2(w+v) \cdot w, & \gamma_3 &= |w|^2.
\end{align*}
\]

4.1.2. Common edge
Let the reference mappings be given by
\[
\chi_{\tau}(y) = p + y_1 v + y_2 u, \quad \chi_{\sigma}(x) = p + x_1 v + x_2 w,
\]
such that \( v \) is the common edge of \( \sigma \) and \( \tau \) starting from \( p \). Then, the integral (5) reduces to
\[
I = \int_{(0,1)^4} \eta_3 \left( q(A_1(\eta)) + \eta_3 \sum_{i=2}^{5} q(A_i(\eta)) \right) d\eta
\]
\[
= \frac{g_2 g_3}{24\pi} \int_{(0,1)^2} \left( \frac{1}{|\eta_3 u + v + \eta_4 w - u|} + \frac{\eta_3}{|\eta_4 \eta_3 u + \eta_3 (w + v) - \eta_3 u + w|} \right)
\]
\[
+ \frac{\eta_3}{|\eta_4 \eta_3 (w + v) - \eta_3 w + u|} + \frac{\eta_3}{|\eta_4 \eta_3 w + \eta_3 (w + v) - w|} \right) d\eta_3 d\eta_4
\]
\[
= \frac{g_2 g_3}{24\pi} \sum_{i=1}^{5} I_i.
\]

In comparison to the previous case, the integrand is not necessarily smooth in the real domain. When the two triangles lie in the same plane, it has poles as seen in Figure 3. However, they only occur outside of \((0,1)^2\) since the triangles do not overlap.

**First integral \( I_1 \)**

Let us introduce the variables
\[
a = w, \quad b = v, \quad c = u + v.
\]

We integrate with respect to \( \eta_3 \) by using (7) and obtain for the first integral
\[
I_1 = \frac{1}{c} \int_0^1 \ln \left( \frac{|\eta_4 a + b|}{|\eta_4 a + b - c|} + \frac{(\eta_4 a + b) \cdot c}{|\eta_4 a + b - c|} \right) d\eta_4.
\]
Integration by parts leads to

\[ I_1 = \frac{1}{|c|} \ln \left( \frac{|a + b| |c| + (a + b) \cdot c}{|a + b - c| |c| + (a + b - c) \cdot c} \right) - \frac{1}{|c|} \int_0^1 (h_1(\eta_4) - h_0(\eta_4)) d\eta_4, \quad (9) \]

where

\[ h_0(\eta_4) = \frac{(\eta_4 a + b) \cdot b + |\eta_4 a + b| b \cdot \hat{c}}{|\eta_4 a + b|^2 + |\eta_4 a + b| (\eta_4 a + b) \cdot \hat{c}}, \]

\[ h_1(\eta_4) = \frac{(\eta_4 a + b - c) \cdot (b - c) + |\eta_4 a + b - c| (b - c) \cdot \hat{c}}{|\eta_4 a + b - c|^2 + |\eta_4 a + b - c| (\eta_4 a + b - c) \cdot \hat{c}} \]

with \( \hat{c} = c / |c| \). We note that \( h_1 \) coincides with \( h_0 \) when \( b \) is replaced by \( b - c \) and proceed with integrating \( h = h_0 \). We follow the approach of [7, Appendix C.2] and define

\[ p = \frac{a \cdot b}{|a|^2}, \quad q^2 = \frac{|b|^2}{|a|^2} - p^2 \geq 0 \]

such that

\[ |\eta_4 a + b| = |a| \sqrt{(\eta_4 + p)^2 + q^2}. \]

If \( q = 0 \) then \( a = p b \) and the integral simplifies to

\[ \int h(\eta_4) d\eta_4 = p \ln(1 + 1/p). \]

Otherwise, we have \( q > 0 \) and the substitution

\[ \eta_4 = -p + q \sinh(s), \quad d\eta_4 = q \cosh(s) ds, \]
yields for the indefinite integral
\[ \int h(\eta_4) \, d\eta_4 = \int f(s) \, ds = q \int \frac{p |a| \sinh(s) + q |a| + \cosh(s) b \cdot c}{q |a| \cosh(s) + q |a| \cosh(s) [(p-a) + (p-a) \cdot c]} \, ds. \]

We use a variant of the Weierstraß substitution,
\[ \tanh(s/2) = t, \quad \sinh(s) = \frac{2t}{1-t^2}, \quad \cosh(s) = \frac{1+t^2}{1-t^2}, \quad ds = \frac{2}{1-t^2} \, dt, \]
and obtain
\[ \int h(\eta_4) \, d\eta_4 = \int \frac{2q}{1-t^2} \frac{2p |a| t + q |a| (1-t^2) + (b \cdot c) (1+t^2)}{q |a| (1+t^2) + 2q(a \cdot c) t + ((p-a) + (p-a) \cdot c) (1-t^2)} \, dt \]
\[ = 2q \int \frac{1}{1-t^2} \frac{b \cdot c + q |a| + 2p |a| t + ((b \cdot c) - q |a|) t^2}{q |a| + (p-a) \cdot c + 2q(a \cdot c) t + (q |a| + (p-a) \cdot c) t^2} \, dt. \]

The integrand is now a rational function and we abbreviate it by
\[ \frac{1}{1-t^2} \frac{\beta_0 + \beta_1 t + \beta_2 t^2}{2 \alpha_0 + \alpha_1 t + \alpha_2 t^2}. \]

We decompose it into partial fractions,
\[ \frac{\gamma_1}{1-t} + \frac{\gamma_2}{1+t} + \frac{\gamma_3 + \gamma_4 t}{\alpha_0 + \alpha_1 t + \alpha_2 t^2}, \]
where
\[ \gamma_1 = \frac{1}{2} \beta_0 + \beta_1 + \beta_2, \quad \gamma_2 = \frac{1}{2} \beta_0 - \beta_1 + \beta_2, \quad \gamma_3 = \beta_0 - (\gamma_1 + \gamma_2) \alpha_0, \quad \gamma_4 = \alpha_2 (\gamma_1 - \gamma_2). \]

The first two terms yield
\[ F(t) = \int \left( \frac{\gamma_1}{1-t} + \frac{\gamma_2}{1+t} \right) \, dt = \gamma_2 \ln |1+t| - \gamma_1 \ln |1-t|. \]

The third term depends on the discriminant \( D = 4\alpha_0 \alpha_2 - \alpha_1^2 \) of the denominator, which is non-negative due to
\[ D = |\det(a | b| c)|^2 / |a|^2. \]

For \( D > 0 \) we have
\[ G(t) = \int \frac{\gamma_3 + \gamma_4 t}{\alpha_0 + \alpha_1 t + \alpha_2 t^2} \, dt = \frac{\gamma_4}{2\alpha_2} \ln |\alpha_0 + \alpha_1 t + \alpha_2 t^2| \]
\[ + \frac{2\gamma_3 \alpha_2 - \gamma_4 \alpha_1}{\alpha_2 \sqrt{D}} \arctan \left( \frac{\alpha_1 + 2\alpha_2 t}{\sqrt{D}} \right) \]

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and for $D = 0$

$$G(t) = \int \frac{\gamma_3 + \gamma_4 t}{\alpha_0 + \alpha_1 t + \alpha_2 t^2} \, dt = \frac{\gamma_4}{\alpha_2} \ln \left| t + \frac{\alpha_1}{2\alpha_2} \right| - \frac{2\gamma_3 \alpha_2 - \gamma_4 \alpha_1}{\alpha_2 (2\alpha_2 t + \alpha_1)}.$$ 

We resubstitute

$$t = \tanh \left[ \frac{1}{2} \text{arcsinh} \left( \frac{p + \eta_4}{q} \right) \right] = \frac{\sinh \left[ \text{arcsinh} \left( \frac{p + \eta_4}{q} \right) \right]}{1 + \cosh \left[ \text{arcsinh} \left( \frac{p + \eta_4}{q} \right) \right]} = \frac{p + \eta_4}{q + \sqrt{(p + \eta_4)^2 + q^2}},$$

and set

$$t_0 = \frac{p}{q + \sqrt{p^2 + q^2}}, \quad t_1 = \frac{p + 1}{q + \sqrt{(p + 1)^2 + q^2}}.$$

Finally, we obtain

$$\int_0^1 h(\eta_4) \, d\eta_4 = 2q (F(t_1) - F(t_0) + G(t_1) - G(t_0)).$$

(10)

Note that the value of the integral only depends on the vectors $a, b, c$. Since it is of importance for the other cases as well, we abbreviate it by

$$H(a, b, c) = \int_0^1 h(\eta_4) \, d\eta_4.$$ 

We conclude that $I_1$ can be expressed in closed form as

$$I_1 = \frac{1}{|c|} \ln \left( \frac{|a + b| |c| + (a + b) \cdot c}{|a + b - c| |c| + (a + b - c) \cdot c} \right) - \frac{1}{|c|} (H(a, b - c, c) - H(a, b, c))$$

(11)

with $a = w, b = v, c = u + v$.

**Remaining integrals**

For the remaining integrals, we integrate with respect to the fourth variable firstly. With

$$a = v, \quad b = w, \quad c = u + v,$$

we have

$$I_2 = \frac{1}{|c|} \int_0^1 \ln \left( \frac{\eta_3 a + b}{|\eta_3(a - c) + b| |c| + (\eta_3(a - c) + b) \cdot c} \right) \, d\eta_3.$$ 

Integration by parts yields an expression almost identical to (9),

$$I_2 = \frac{1}{|c|} \ln \left( \frac{|a + b| |c| + (a + b) \cdot c}{|a - c + b| |c| + (a - c + b) \cdot c} \right) - \frac{1}{|c|} \int_0^1 (h_1(\eta_3) - h_0(\eta_3)) \, d\eta_3,$$

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where $h_0$ and $h_1$ are given by

$$h_0(\eta_3) = \frac{(\eta_3 a + b) \cdot b + |\eta_3 a + b| b \cdot \hat{c}}{\eta_3 a + b |^2 + \eta_3 a + b | (\eta_3 a + b) \cdot \hat{c}},$$

$$h_1(\eta_3) = \frac{(\eta_3 (a - c) + b) \cdot b + |\eta_3 (a - c) + b| b \cdot \hat{c}}{\eta_3 (a - c) + b |^2 + \eta_3 (a - c) + b | (\eta_3 (a - c) + b) \cdot \hat{c}}.$$

Thus, $I_2$ can be computed analogously to (11) by

$$1 \left| \frac{a + b}{c} \right| \ln \left( \frac{|\eta_3 a + b|}{|\eta_3 (a - c) + b|} \right) - 1 \left| \frac{a + b}{c} \right| (H(a - c, b, c) - H(a, b, c)). \quad (12)$$

Because this applies to the other integrals as well, we only list the parameters $a, b$ and $c$ in Table 1.

| $j$ | $a$ | $b$ | $c$ |
|-----|-----|-----|-----|
| $2$ | $v$ | $w$ | $u + v$ |
| $3$ | $v$ | $u$ | $w + v$ |
| $4$ | $u + v + w$ | $-w$ | $u$ |
| $5$ | $u + v + w$ | $-w$ | $v + w$ |

Table 1: Values for $a, b, c$ in (12) to compute $I_j$.

### 4.1.3. Common vertex

We consider the configuration

$$\chi_\tau(y) = p + y_1u_1 + y_2u_2, \quad \chi_\sigma(x) = p + x_1v_1 + x_2v_2$$

with common vertex $p$. Then, integration with respect to $\eta_1$ results in

$$I = \frac{g_\tau g_\sigma}{12\pi} \int_{(0,4)^3} \left( \frac{\eta_3}{|\eta_3 u_1 + \eta_3 u_2 - v_1 - \eta_2 v_2|} + \frac{\eta_3}{|u_1 + \eta_2 u_2 - \eta_3 v_1 - \eta_3 v_2|} \right) d\eta_2 d\eta_3 d\eta_4$$

$$= \frac{g_\tau g_\sigma}{12\pi} (I_1 + I_2).$$

We only consider $I_1$, since $I_2$ is obtained by swapping $u_i$ and $v_i$. Similar to the previous section, we introduce the variables

$$a = u_1 + u_2, \quad b(\eta_2) = -v_1 - \eta_2 v_2, \quad c = u_2,$$

and integrate with respect to $\eta_4$ to obtain

$$I_1 = \frac{1}{|c|} \int_{(0,1)^2} \ln \left( \frac{|\eta_3 a + b(\eta_2)| |c|}{|\eta_3 (a - c) + b(\eta_2)| |c| + (\eta_3 (a - c) + b(\eta_2)) \cdot c} \right) d\eta_3 d\eta_2.$$
We insert Formula (12) for the inner integral, which yields

\[
I_1 = \frac{1}{|c|} \int_0^1 \ln \left( \frac{|a + b(\eta_2)| |c| + (a + b(\eta_2)) \cdot c}{|a - c + b(\eta_2)| |c| + (a - c + b(\eta_2)) \cdot c} \right) \, d\eta_2
\]

\[
- \frac{1}{|c|} \int_0^1 (H(a - c, b(\eta_2), c) - H(a, b(\eta_2), c)) \, d\eta_2.
\]

The first integral can be written in the form of \(I_1\) from Section 4.1.2, i.e.

\[
\frac{1}{|\tilde{c}|} \int_0^1 \ln \left( \frac{|\eta_2 \tilde{a} + \tilde{b}| |\tilde{c}| + (\eta_2 \tilde{a} + \tilde{b}) \cdot \tilde{c}}{|\eta_2 \tilde{a} + \tilde{b} - \tilde{c}| |\tilde{c}| + (\eta_2 \tilde{a} + \tilde{b} - \tilde{c}) \cdot \tilde{c}} \right) \, d\eta_2
\]

with \(\tilde{a} = -v_2, \tilde{b} = u_1 + u_2 - v_1, \tilde{c} = u_2\), and its value is hence given by (11). Because it is not possible to integrate the remaining integral analytically, we approximate it numerically with a quadrature rule

\[
\int_0^1 (H(a - c, b(\eta_2), c) - H(na, b(\eta_2), c)) \, d\eta_2 \approx \sum_{i=1}^n \omega_i \left( H(a - c, b(\eta^{(i)}), c) - H(a, b(\eta^{(i)}), c) \right)
\]

with weights \(\omega_i > 0\) and nodes \(\eta^{(i)} \in [0, 1]\).

### 4.1.4. Far-field

Although the far-field does not constitute a singular case, the analytical formulae are still applicable. Let the elements be given by

\[
\chi_\tau(y) = p_1 + y_1 u_1 + y_2 u_2, \quad \chi_\sigma(x) = p_2 + x_1 v_1 + x_2 v_2,
\]

and set \(p = p_1 - p_2\). Analogously to the previous cases, we pull the region of integration back to \((0, 1)^4\) by

\[
A : (0, 1)^4 \rightarrow \pi \times \pi, \quad A(\eta) = \begin{pmatrix} \eta_1 \\ \eta_1 \eta_2 \\ \eta_3 \\ \eta_3 \eta_4 \end{pmatrix},
\]

leading to

\[
I = \frac{g_\tau g_\sigma}{4\pi} \int_{(0,1)^4} \frac{\eta_1 \eta_3}{|p + \eta_3 u_1 + \eta_3 \eta_4 u_2 - \eta_1 v_1 - \eta_1 \eta_2 v_2|} \, d\eta.
\]
Of the four iterated integrals, we compute two analytically and two by numerical quadrature, e.g.

\[
I \approx \sum_{k,\ell=1}^{n} \omega_k \omega_\ell \int_{(0,1)^2} \left| p + \eta^{(k)} u_1 + \eta_{4\ell} \eta^{(\ell)} u_2 - \eta^{(k)} v_1 - \eta_{2\ell} \eta^{(\ell)} v_2 \right| d\eta_2 d\eta_1,
\]

where the two-dimensional integral is calculated analytically using (11) with

\[
a = \eta^{(k)} v_2, \quad b = p - \eta^{(k)} v_1 + \eta^{(\ell)} (u_1 + u_2), \quad c = \eta^{(\ell)} u_2.
\]

4.2. Double layer potential

For the double layer potential \( K \), we need to compute integrals of the form

\[
J = \int_{\sigma} \int_{\tau} \frac{(x - y) \cdot n}{4\pi |x - y|^3} \varphi(y) dS(y) dS(x) = \frac{g_\sigma g_\tau}{4\pi} \int_{\pi \times \pi} \frac{(\chi_\sigma(x) - \chi_\tau(y)) \cdot n}{|\chi_\sigma(x) - \chi_\tau(y)|^3} \varphi(\chi_\tau(y)) \, dx \, dy,
\]

where \( n \) is the outer unit normal vector at \( \tau \) and \( \varphi \in S^1_h(\tau) \), i.e.

\[
\varphi(\chi_\tau(y)) = a_0 + a_1 y_1 + a_2 y_2
\]

with coefficients \( a_0, a_1, a_2 \in \mathbb{R} \).

4.2.1. Identical elements

For identical elements \( \sigma = \tau \), we simply have \( J = 0 \) due to

\[
(y - x) \cdot n = 0, \quad \text{for} \ x, y \in \tau.
\]

4.2.2. Common edge

We assume that the triangles are parametrised by

\[
\chi_\tau(y) = p + y_1 v + y_2 u, \quad \chi_\sigma(x) = p + x_1 v + x_2 w.
\]

Applying the regularisation to \( J \) and integrating with respect to \( \eta_1 \) and \( \eta_2 \) leads to

\[
J = \frac{g_\sigma g_\tau}{4\pi} w \cdot n \sum_{i=1}^{5} J_i,
\]

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where the integrals $J_i$ are given by

\[
J_1 = \int_{(0,1)^2} \frac{\eta_4}{|\eta_3(u + v) + \eta_4 w - u|} \left(c_0 + c_1 \eta_3 + c_2(1 - \eta_3)\right) d\eta_3 d\eta_4,
\]

\[
J_2 = \int_{(0,1)^2} \frac{\eta_3}{|\eta_3\eta_4(u + v) - \eta_3 u + w|} \left(c_0 + c_1 \eta_3 \eta_4 + c_2 \eta_3(1 - \eta_4)\right) d\eta_3 d\eta_4,
\]

\[
J_3 = \int_{(0,1)^2} \frac{\eta_4^2(1 - \eta_4)}{|\eta_3\eta_4(u + v) - \eta_3 w + u|} \left(c_0 + c_2 \right) d\eta_3 d\eta_4,
\]

\[
J_4 = \int_{(0,1)^2} \frac{\eta_3(1 - \eta_3)}{|\eta_3\eta_4(u + \eta_3 w + w) - \eta_3 w|} \left(c_0 + c_2 \eta_3 \eta_4\right) d\eta_3 d\eta_4,
\]

\[
J_5 = \int_{(0,1)^2} \frac{\eta_3(1 - \eta_3\eta_4)}{|\eta_3\eta_4(u + \eta_3 w + w) - \eta_3 w|} \left(c_0 + c_2 \eta_3\right) d\eta_3 d\eta_4.
\]

with $c_0 = a_0/2 + a_1/3$, $c_1 = -a_1/6$ and $c_2 = a_2/6$. In contrast to the respective case of the single layer potential, the integrand of $J$ is always smooth in the real domain as shown in Figure 4. Indeed, if the two triangles lie in the same plane, then $J = 0$ due to $w \cdot n = 0$.

![Figure 4: Visualisation of the integrand $f(\eta_3, \eta_4)$ of (14).](image)

In the following, we derive an analytic expression for the integrals by the example of $J_1$. We define

\[
R(\eta_3, \eta_4) = \sqrt{\gamma(\eta_3) + \beta(\eta_3) \eta_4 + \alpha \eta_4^2}
\]

with

\[
\gamma(\eta_3) = |\eta_3(u + v) - u|^2, \quad \beta(\eta_3) = 2(\eta_3(u + v) - u) \cdot w, \quad \alpha = |w|^2,
\]

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such that $R^3$ equals the denominator. Denoting the discriminant by $D = 4\alpha\gamma - \beta^2$, we integrate with respect to $\eta_4$ using [12, Section 2.264],

$$J_1 = -2 \int_0^1 (c_0 + c_2 + (c_1 - c_2)\eta_3) \left[ \frac{2\gamma(\eta_3) + \beta(\eta_3)\eta_4}{D(\eta_3)R(\eta_3, \eta_4)} \right] d\eta_3$$

$$= -2 \int_0^1 (c_0 + c_2 + (c_1 - c_2)\eta_3) \left[ \frac{2\gamma(\eta_3) + \beta(\eta_3)}{D(\eta_3)\sqrt{\gamma(\eta_3)} + \alpha} - \frac{2\gamma(\eta_3)}{D(\eta_3)\sqrt{\gamma(\eta_3)}} \right] d\eta_3.$$ 

Hence, the integral reduces to

$$J_1 = \int_0^1 \left( h_1^{(1)}(\eta_3) d\eta_3 - h_0^{(1)}(\eta_3) \right) d\eta_3$$

with $h_0^{(1)}$ and $h_1^{(1)}$ of the form

$$h(\eta_3) = \frac{4P(\eta_3)}{D(\eta_3)\sqrt{Q(\eta_3)}},$$

where $P(\eta_3)$ is a cubic and $Q(\eta_3)$ a positive quadratic polynomial respectively,

$$P(\eta_3) = p_0 + p_1\eta_3 + p_2\eta_3^2 + p_3\eta_3^3, \quad Q(\eta_3) = q(q_0 + q_1\eta_3 + \eta_3^2) > 0.$$ 

In order to find the anti-derivative of $h$, we write $D(\eta_3)$ as

$$D(\eta_3) = 4d(d_0 + d_1\eta_3 + \eta_3^2)$$

and decompose into partial fractions,

$$h(\eta_3) = \frac{1}{d\sqrt{Q(\eta_3)}} \left( p_2 - d_1p_3 + p_3\eta_3 + \right.$$

$$\left. \frac{p_0 - d_0p_2 + d_0d_1p_3 + (p_1 - d_1p_2 - d_0p_3 + d_1^2p_3)\eta_3}{d_0 + d_1\eta_3 + \eta_3^2} \right).$$

We are familiar with the first term and recall that

$$\int \frac{1}{\sqrt{q_0 + q_1\eta_3 + \eta_3^2}} d\eta_3 = F(\eta_3) = \ln \left( 2\sqrt{q_0 + q_1\eta_3 + \eta_3^2} + 2\eta_3 + q_1 \right)$$

and [12, Section 2.264] also gives

$$\int \frac{\eta_3}{\sqrt{q_0 + q_1\eta_3 + \eta_3^2}} d\eta_3 = \sqrt{q_0 + q_1\eta_3 + \eta_3^2} - \frac{q_1}{2} F(\eta_3).$$

For the remaining term, we abbreviate the constants in the numerator by

$$n = p_0 - d_0p_2 + d_0d_1p_3, \quad m = p_1 - d_1p_2 - d_0p_3 + d_1^2p_3.$$
Following [13, Chapter 3], we substitute
\[ \eta_3 = \frac{\nu + \mu t}{1 + t}, \quad d\eta_3 = \frac{\mu - \nu}{(1 + t)^2} dt, \]
where \( \mu \) and \( \nu \) are the real and distinct solutions of the quadratic equation
\[ (d_1 - q_1)z^2 + 2(d_0 - q_0)z + (q_1d_0 - d_1q_0) = 0. \]
In this way, the linear terms of the denominator vanish and we obtain
\[
\omega \int \text{sgn}(1 + t) \frac{n + m\nu + (n + m\mu)t}{(\lambda + t^2)\sqrt{\zeta + t^2}} \, dt
\]
with
\[
\omega = \frac{\mu - \nu}{(\mu^2 + d_1\mu + d_0)\sqrt{\mu^2 + q_1\mu + q_0}},
\]
\[
\lambda = \frac{\nu^2 + d_1\nu + d_0}{\mu^2 + d_1\mu + d_0}, \quad \zeta = \frac{\nu^2 + q_1\nu + q_0}{\mu^2 + q_1\mu + q_0}.
\]
We have
\[
\int \frac{(n + m\mu)t}{(\lambda + t^2)\sqrt{\zeta + t^2}} \, dt = (n + m\mu) \int \frac{1}{\lambda - \zeta + s^2} \, ds
\]
by means of \( s_1 = \sqrt{\zeta + t^2} \). For the remaining term, we use the substitution
\[
s_0 = \frac{t}{\sqrt{\zeta + t^2}}, \quad \frac{dt}{\sqrt{\zeta + t^2}} = \frac{ds_0}{1 - s_0^2},
\]
such that
\[
\lambda + t^2 = \frac{\lambda + (\zeta - \lambda)s_0^2}{1 - s_0^2}
\]
and
\[
\int \frac{n + m\nu}{(\lambda + t^2)\sqrt{\zeta + t^2}} \, dt = (n + m\nu) \int \frac{1}{\lambda + (\zeta - \lambda)s_0^2} \, ds_0.
\]
The integrals are of the form
\[
\int \frac{1}{\rho + s^2} \, ds, \quad \text{where } \rho = \lambda - \zeta \text{ or } \rho = \frac{\lambda}{\zeta - \lambda},
\]
and the anti-derivative depends on the sign of \( \rho \),
\[
\int \frac{1}{\rho + s^2} \, ds = G(\rho, s) = \begin{cases} 
\frac{\arctan \left( \frac{s}{\sqrt{\rho}} \right)}{\sqrt{\rho}}, & \rho > 0, \\
-\frac{1}{s}, & \rho = 0, \\
\frac{1}{2\sqrt{-\rho}} \ln \left| \frac{s - \sqrt{-\rho}}{s + \sqrt{-\rho}} \right|, & \rho < 0,
\end{cases}
\]

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see [12, Section 2.103]. The value of the indefinite integral is therefore
\[
\int \frac{n + m \eta_{3}}{(d_{0} + d_{1}\eta_{3} + \eta_{3}^{2})\sqrt{q_{0} + q_{1}\eta_{3} + \eta_{3}^{2}}} \, d\eta_{3} = \\
\omega \text{sgn}(1 + t) \left[ \frac{n + m\nu}{\kappa - \lambda} G\left( \frac{\lambda}{\kappa - \lambda}, s_{0}(t) \right) + (n + m\mu) G(\lambda - \kappa, s_{1}(t)) \right]
\]
(16)
and we write \( H(t) \) for short. It remains to determine the domain of the integral in terms of \( t \). We resubstitute

\[
t^{-1}(0, 1) = \begin{cases} 
(t_{0}, t_{1}), & t_{0} > -1 \text{ or } t_{1} < -1, \\
(-\infty, t_{0}) \cup (t_{1}, \infty), & t_{0} < -1 < t_{1},
\end{cases}
\]
with \( t_{0} = \min\{t(0), t(1)\} \), \( t_{1} = \max\{t(0), t(1)\} \) and

\[
t(0) = -\frac{\nu}{\mu}, \quad t(1) = -\frac{\nu - 1}{\mu - 1}.
\]
The second case requires the calculation of improper integrals. We have

\[
\lim_{t \to \pm\infty} G(\rho, s_{0}(t)) = \lim_{s \to \pm 1} G(\rho, s) = G(\rho, \pm 1)
\]
and

\[
\lim_{t \to \pm\infty} G(\rho, s_{1}(t)) = \lim_{s \to \infty} G(\rho, s) = \begin{cases} 
\frac{\pi}{2\sqrt{\rho}}, & \rho > 0, \\
0, & \rho \leq 0.
\end{cases}
\]
We combine the results and obtain
\[
\int_{0}^{1} h(\eta_{3}) \, d\eta_{3} = \frac{1}{d\sqrt{q}} \left[ p_{3}\sqrt{q_{0} + q_{1}\eta_{3} + \eta_{3}^{2}} + \left( p_{2} - d_{1}p_{3} - p_{3}\frac{q_{1}}{2} \right) F(\eta_{3}) \right]_{0}^{1} \\
+ \begin{cases} 
H(t_{1}) - H(t_{0}), & t_{0} > -1 \text{ or } t_{1} < -1, \\
H(t_{0}) - \lim_{t \to \infty} H(t) + \lim_{t \to -\infty} H(t) - H(t_{1}), & t_{0} < -1 < t_{1}.
\end{cases}
\]
(17)
Finally, the integral \( J_{1} \) can now be computed by applying the formula to the integrals of \( h_{1}^{(1)} \) and \( h_{0}^{(1)} \). The other \( J_{i} \) can be calculated in the same way. After integrating with respect to \( \eta_{4} \), we have

\[
J_{i} = \int_{0}^{1} \left( h_{1}^{(i)}(\eta_{3}) \, d\eta_{3} - h_{0}^{(i)}(\eta_{3}) \right) \, d\eta_{3},
\]
where \( h_{0}^{(i)} \) and \( h_{1}^{(i)} \) are again of the form (15). Thus, we conclude that all integrals are expressible analytically via (17) in terms of certain parameters, which are listed in Tables 2, 3 and 4.
4.2.3. Common vertex

Like in Section 4.1.3, let $u_1, u_2$ be the edges of $\tau$ and $v_1, v_2$ the edges of $\sigma$. We integrate with respect to $\eta_1$ and obtain

$$J = \frac{g_\tau g_\sigma}{4\pi} (J_1 + J_2)$$

$$= \frac{g_\tau g_\sigma}{4\pi} \left( \int_{(0,4)^3} \frac{\eta_3(v_1 + \eta_2v_2) \cdot n}{|\eta_3 u_1 + \eta_3\eta_4 u_2 - v_1 - \eta_2 v_2|^3} (c_0 + c_1\eta_3 + c_2\eta_3\eta_4) \, d\eta_2 \, d\eta_3 \, d\eta_4 \right)$$

$$+ \int_{(0,4)^3} \frac{\eta_3^2(v_1 + \eta_2v_2) \cdot n}{|u_1 + \eta_2 u_2 - \eta_3 v_1 - \eta_3\eta_4 v_2|^3} (c_0 + c_1 + c_2\eta_2) \, d\eta_2 \, d\eta_3 \, d\eta_4,$$

where $c_0 = a_0/2$, $c_1 = a_1/3$ and $c_2 = a_2/3$. With $v(\eta_2) = v_1 + \eta_2 v_2$ the first integral reads

$$J_1 = \int_0^1 v(\eta_2) \cdot n \int_{(0,1)^2} \frac{\eta_3(c_0 + c_1\eta_3 + c_2\eta_3\eta_4)}{|\eta_3 u_1 + \eta_3\eta_4 u_2 - v(\eta_2)|^3} \, d\eta_3 \, d\eta_2,$$

where the inner integral has the same form as the integrals of the previous section. Therefore, we use (17) to compute the inner integral and approximate the outer integral with numerical quadrature. We summarise the parameters in Tables 5, 7 and 8. Note that for $J_2$ only $u_i, v_i$ need to be exchanged.

4.2.4. Far-field

With the notation of Section 4.1.4, the integral on $(0,1)^4$ reads

$$J = \frac{g_\tau g_\sigma}{4\pi} \int_{(0,1)^4} \eta_1\eta_3(a_0 + a_1\eta_3 + a_2\eta_3\eta_4) \frac{(-p + \eta_1 v_1 - \eta_1\eta_2 v_2) \cdot n}{|p + \eta_3 u_1 + \eta_3\eta_4 u_2 - \eta_1 v_1 - \eta_1\eta_2 v_2|^3} \, d\eta.$$

We integrate analytically with respect to $\eta_1$ and $\eta_2$ and numerically with respect to $\eta_3$ and $\eta_4$, i.e. we approximate $J$ by

$$\sum_{k,\ell=1}^n \omega_k \omega_\ell \eta^{(k)}(a_0 + a_1\eta^{(k)} + a_2\eta^{(k)}\eta^{(\ell)}) \int_{(0,1)^2} \eta_1(-p + \eta_1 v_1 - \eta_1\eta_2 v_2) \cdot n \left| u(\eta^{(k)}, \eta^{(\ell)}) - \eta_1 v_1 - \eta_1\eta_2 v_2 \right|^3 \, d\eta_1 \, d\eta_2,$$

with $u(\eta^{(k)}, \eta^{(\ell)}) = p + \eta^{(k)} u_1 + \eta^{(k)}\eta^{(\ell)} u_2$. The parameters are listed in Tables 6, 9 and 10.

5. Numerical experiments

In this final section, we verify the correctness of the analytical formulae in numerical examples. To this end, we consider two different geometries for $\Gamma_h$, namely a triangulated
unit sphere $\Gamma_h^{(1)}$ with $N = 4608$ triangles and the surface $\Gamma_h^{(2)}$ of a transformer visualised in Figure 5.

By $A_{r,s}$ we denote the approximation of the exact boundary element matrix $A = V, K$ computed by the semi-analytical formulae. We use tensorised Gauss-Legendre quadrature rules with $r \times r$ points for the far-field and $s$ points in the singular vertex case. Hence, the computation of $V_{r,s}$ requires $O(Ns + N^2r^2)$ evaluations of anti-derivatives. Note that the conventional approach based on four-dimensional quadrature involves $O(Ns^4 + N^2r^4)$ kernel evaluations.

We measure the relative error

$$e = \|A_{r,s} - A\|_F / \|A\|_F$$

in the Frobenius norm defined by

$$\|A\|_F^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} (A[m, n])^2, \quad A \in \mathbb{R}^{M \times N}.$$ 

Since the exact boundary element matrix $A$ is not available, we compute a reference approximation with four-dimensional quadrature of order $r = 22$ and $s = 24$. Moreover, we set $s = r + 2$ in all experiments.

Figures 6 and 7 show that the error $e$ decreases exponentially in the quadrature order $r$ as expected. It reaches $10^{-12}$ and $10^{-8}$ respectively and we see that $V_{r,s}$ is slightly more accurate than $K_{r,s}$ for identical $r$ and $s$. Overall, we conclude that the semi-analytical formulae produce numerically correct results.
6. Conclusion

In comparison to black-box numerical quadrature, analytical integration exploits the specific structure of the discretisation to reduce the computational costs while preserving the level of accuracy. Since the regularisation method is not limited to the Laplace equation, it is promising to extend the strategy to other relevant kernel functions. Whereas the application to linear elasticity should follow directly from the results presented here, the situation of time-harmonic wave problems is less straightforward due to oscillatory integrands. Whether analytical integration is possible there needs to be investigated in future work.

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Figure 7: Relative error $e$ for increasing quadrature order $r$ for the transformer $\Gamma_h^{(2)}$.

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A. Appendix

In the following, the parameter tables for the formulae of the double layer potential are listed.

| $i$ | $j$ | $q$          | $q \cdot q_0$ | $2q \cdot q_1$ |
|-----|-----|--------------|----------------|----------------|
| 1   | 0   | $u + v$      | $u^2$          | $-(u + v) \cdot u$ |
| 1   | 0   | $u + v$      | $u - w$        | $-(u + v) \cdot (u - w)$ |
| 2   | 0   | $v^2$        | $w^2$          | $-u \cdot w$     |
| 3   | 0   | $w^2$        | $u^2$          | $-u \cdot w$     |
| 4   | 0   | $v + w$      | $w^2$          | $-(v + w) \cdot w$ |
| 5   | 0   | $u + v + w$  | $w^2$          | $-(u + v + w) \cdot w$ |

Table 2.: Parameters $q$ and $q_k$ in (17) for the integrals of $h_j^{(i)}$ for the edge case.

| $i$ | $d$          | $d \cdot d_0$ | $2d \cdot d_1$ |
|-----|--------------|----------------|----------------|
| 1   | $|u + v|^2 |w|^2 - ((u + v) \cdot w)^2$ | $|u|^2 |w|^2 - (u \cdot w)^2$ | $(u \cdot w) (u + v) \cdot w - |w|^2 (u + v) \cdot u$ |
| 2   | $|u|^2 |v|^2 - (u \cdot v)^2$ | $|u + v|^2 |w|^2 - ((u + v) \cdot w)^2$ | $v \cdot w (u + v) \cdot u - u \cdot w (u + v) \cdot v$ |
| 3   | $|v|^2 |w|^2 - (v \cdot w)^2$ | $|v + w|^2 |u|^2 - ((v + w) \cdot u)^2$ | $u \cdot v (v + w) \cdot w - u \cdot w (v + w) \cdot v$ |
| 4   | $|u|^2 |v + w|^2 - (v \cdot w)^2$ | $|u|^2 |w|^2 - (u \cdot w)^2$ | $u \cdot w (v + w) \cdot u - |u|^2 (v + w) \cdot w$ |
| 5   | $|u|^2 |v + w|^2 - (v \cdot w)^2$ | $|v|^2 |w|^2 - (v \cdot w)^2$ | $u \cdot v (v + w) \cdot w - u \cdot w (v + w) \cdot v$ |

Table 3.: Parameters $d$ and $d_k$ in (17) for the integrals of $h_j^{(i)}$ for the edge case.
| i | j | \( p_0 \) | \( p_1 \) | \( p_2 \) | \( p_3 \) |
|---|---|---|---|---|---|
| 1 | 0 | \(- (c_0 + c_2) |u|^2\) | \((2c_0 - c_1 + 3c_2)|u|^2 \) | \((2c_1 - c_0 - 3c_2) |u|^2 - (c_0 + c_2)|v|^2\) | \((c_2 - c_1) |u + v|^2\) |
| 1 | 1 | \((c_0 + c_2)(w - u) \cdot u\) | \(2(c_0 + c_2) u \cdot v - (c_0 + c_2)v \cdot w\) | \((2c_1 - c_0 - 3c_2) |u|^2 - (c_0 + c_2)|v|^2\) | \((c_2 - c_1) |u + v|^2\) |
| 2 | 0 | \((c_0 + c_2)(u + v) \cdot w\) | \((2c_1 - c_2)u \cdot w + c_2v \cdot w\) | \(-c_0(u + v) \cdot u\) | \(-c_1 |u|^2 + c_2u \cdot v\) |
| 3 | 0 | \((c_0 + c_2) |u|^2\) | \((c_0 + c_2)(v - w) \cdot u\) | \(-c_0 + c_2)v \cdot w\) | \(0\) |
| 4 | 0 | \(-c_0u \cdot w - c_2 |w|^2\) | \(c_0(v + 2w) \cdot u\) | \(-c_0(v + w) \cdot u\) | \(c_2 |v + w|^2\) |
| 5 | 0 | \(-c_0v \cdot w\) | \(c_0(v - w) \cdot u - c_2v \cdot w\) | \(c_0(u + v + w) \cdot u\) | \(c_2(u + v + w) \cdot u\) |

Table 4: Parameters \( p_k \) in (17) for the integrals of \( h_j^{(i)} \) for the edge case.
\[ i \quad j \quad p_0 \quad p_1 \quad p_2 \quad p_3 \\
1 \quad 0 \quad -c_0 u_2 \cdot v - c_2 |v|^2 \quad c_0 u_1 \cdot u_2 + 2c_2 u_1 \cdot v - c_1 u_2 \cdot v \quad -c_2 |u_1|^2 + c_1 u_1 \cdot u_2 \quad 0 \\
1 \quad 1 \quad -c_0 u_2 \cdot v - c_2 |v|^2 \quad 2c_2 u_1 \cdot v + c_0 u_2 \cdot (u_1 + u_2) + (c_2 - c_1) u_2 \cdot v \quad -c_2 u_1 \cdot (u_1 + u_2) + c_1 u_2 \cdot (u_1 + u_2) \quad 0 \\

Table 5: Parameters \( p_k \) in (17) for the integrals of \( h^{(i)}_j \) for the vertex case.

\[ i \quad j \quad p_0 \quad p_1 \quad p_2 \quad p_3 \\
1 \quad 0 \quad -n \cdot v_2 |u|^2 + (n \cdot p)(u \cdot v_2) \quad 2(n \cdot v_2)(u \cdot v_1) - (n \cdot v_1)(u \cdot v_2) - (n \cdot p)(v_1 \cdot v_1) \quad -(n \cdot v_2)|v_1|^2 + (n \cdot v_1)(v_1 \cdot v_2) \quad 0 \\
1 \quad 1 \quad -n \cdot v_2 |u|^2 + (n \cdot p)(u \cdot v_2) \quad (n \cdot v_2)(2u \cdot v_1 + u \cdot v_2) - (n \cdot v_1)(u \cdot v_2) - (n \cdot p)(v_1 \cdot v_2 + |v_2|^2) \quad -(n \cdot v_2)(|v_1|^2 + v_1 \cdot v_2) + (n \cdot v_1)(|v_2|^2 + v_1 \cdot v_2) \quad 0 \\

Table 6: Parameters \( p_k \) in (17) for the integrals of \( h^{(i)}_j \) for the far-field case.
Table 7: Parameters $q$ and $q_k$ in (17) for the integrals of $h_j^{(i)}$ for the vertex case.

| $i$ | $j$ | $q$       | $q \cdot q_0$ | $2q \cdot q_1$ |
|-----|-----|-----------|----------------|----------------|
| 1   | 0   | $|u_1|^2$  | $|v|^2$         | $u_1 \cdot v$  |
| 1   | 1   | $|u_1 + u_2|^2$ | $|v|^2$         | $-(u_1 + u_2) \cdot v$ |

Table 8: Parameters $d$ and $d_k$ in (17) for the integrals of $h_j^{(i)}$ for the vertex case.

| $i$ | $d$       | $d \cdot d_0$ | $2d \cdot d_1$ |
|-----|-----------|----------------|----------------|
| 1   | $|u_1|^2 |u_2|^2 - (u_1 \cdot u_2)^2$ | $|u_2|^2 |v|^2 - (u_2 \cdot v)^2$ | $(u_1 \cdot u_2)(u_2 \cdot v) - |u_2|^2 u_1 \cdot v$ |

Table 9: Parameters $q$ and $q_k$ in (17) for the integrals of $h_j^{(i)}$ for the far-field case.

| $i$ | $j$ | $q$       | $q \cdot q_0$ | $2q \cdot q_1$ |
|-----|-----|-----------|----------------|----------------|
| 1   | 0   | $|u + v|^2$ | $|u|^2$         | $-(u + v) \cdot u$ |
| 1   | 1   | $|u + v|^2$ | $|u - w|^2$     | $-(u + v) \cdot (u - w)$ |

Table 10: Parameters $d$ and $d_k$ in (17) for the integrals of $h_j^{(i)}$ for the far-field case.

| $i$ | $d$       | $d \cdot d_0$ | $2d \cdot d_1$ |
|-----|-----------|----------------|----------------|
| 1   | $|u + v|^2 |w|^2$ | $|u|^2 |w|^2 - (u \cdot w)^2$ | $(u \cdot w)(u + v) \cdot w$ |
|     | $-(|u + v) \cdot w|^2$ | $|u|^2 |w|^2 - (u \cdot w)^2$ | $-|w|^2 (u + v) \cdot u$ |