UNIFIED GEOMETRIC MULTIGRID ALGORITHM FOR HYBRIDIZED HIGH-ORDER FINITE ELEMENT METHODS

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Abstract. We consider a standard elliptic partial differential equation and propose a geometric multigrid algorithm based on Dirichlet-to-Neumann (DtN) maps for hybridized high-order finite element methods. The proposed unified approach is applicable to any locally conservative hybridized finite element method including multinumerics with different hybridized methods in different parts of the domain. For these methods, the linear system involves only the unknowns residing on the mesh skeleton, and constructing intergrid transfer operators is therefore not trivial. The key to our geometric multigrid algorithm is the physics-based energy-preserving intergrid transfer operators which depend only on the fine scale DtN maps. Thanks to these operators, we completely avoid upscaling of parameters and no information regarding subgrid physics is explicitly required on coarse meshes. Moreover, our algorithm is agglomeration-based and can straightforwardly handle unstructured meshes. We perform extensive numerical studies with hybridized mixed methods, hybridized discontinuous Galerkin method, weak Galerkin method, and a hybridized version of interior penalty discontinuous Galerkin methods on a range of elliptic problems including subsurface flow through highly heterogeneous porous media. We compare the performance of different smoothers and analyze the effect of stabilization parameters on the scalability of the multigrid algorithm.

Key words. Iterative solvers, Geometric multigrid, Hybridized methods, HDG, High-Order, Dirichlet-To-Neumann maps, multinumerics

AMS subject classifications. 65N30, 65N55, 65N22, 65N12, 65F10

1. Introduction. Hybridization of finite element methods was first introduced in 1965 [25] for solving linear elasticity problems. Hybridized finite element methods have come a long way since then and a vast amount of research has been done over the past few decades (see, e.g. [11, 20, 21, 27, 37]). Hybridized methods offer a number of significant advantages over the original ones, and some of which are: 1) the resulting linear system can be significantly smaller and sparser [14, 21]; 2) hp-adaptivity is natural for hybridized methods due to the already available skeletal space; 3) it offers a natural way to couple different numerical methods in different parts of the domain (multinumerics) and hence exploiting their individual strengths; and 4) for certain problems the trace unknowns can be used to post-process the solution to obtain super-convergence [2,21].

The main challenge facing hybridized methods is however the construction of scalable solvers for the linear system involving only trace unknowns on the mesh skeleton. Over the past 30 years, a tremendous amount of research has been devoted to the convergence of multigrid methods for such linear systems, both as iterative methods and as preconditioners for Krylov subspace methods. Optimal convergence with respect to the number of unknowns is usually obtained under mild elliptic regularity assumptions [6–8]. Multigrid algorithms have been developed for mortar domain
decomposition methods [3, 43]. Several multigrid algorithms have been proposed for hybridized mixed finite element methods [9, 16]. Most of them are based on an equivalence between the interface operator and a nonconforming finite element method, and for these methods optimal convergence has already been established [4, 10]. Multigrid algorithms based on restricting the trace (skeletal) space to linear continuous finite element space has been proposed for hybridized mixed methods [28], hybridized discontinuous Galerkin methods [19] and weak Galerkin methods [15]. These algorithms fall under the non-inherited category, i.e., the coarse scale operators do not inherit all the properties of the fine scale ones. To the best of our knowledge, no multigrid algorithm has been developed for the case of multinumerics.

The objective of this work is to develop a multigrid algorithm that applies to both hybridized formulations and multinumerics. The algorithm applies to both structured and unstructured grids. At the heart of our approach is the energy-preserving intergrid transfer operators which are a function of only the fine scale DtN maps (to be discussed in details in section 3). As such they avoid any explicit upscaling of parameters, and at the same time allow for the use of multinumerics throughout the domain. The multigrid algorithm presented in this paper thus differs from the existing approaches in that the Galerkin coarse grid operator is a discretized DtN map on every level.

This paper is organized as follows. Section 2 introduces the model problem, notation, and hybridized methods considered in this paper. In Section 3, we define the necessary ingredients for our geometric multigrid algorithm, that is, the coarsening strategy, the intergrid transfer operators, the local correction operator, and the smoothing operator. These operators are then used to define the multigrid algorithm. Section 4 presents several numerical examples to study the robustness of the proposed algorithm for different hybridized methods, smoothers and test cases. Finally, Section 5 summarizes our findings and discusses future research directions.

2. Model Problem, Notation and Hybridized methods. In this section we will first introduce the notations and discuss some of the locally conservative hybridized methods proposed in the recent years for the second order elliptic equation.

Consider the following second order elliptic equation

\[-\nabla \cdot (K \nabla q) = f, \quad \text{in } \Omega, \quad (2.1a)\]
\[q = g_D, \quad \text{on } \partial \Omega. \quad (2.1b)\]

where \( \Omega \) is an open, bounded, and connected subset of \( \mathbb{R}^d \), with \( d \in \{2, 3\} \). Here, \( K \) is a symmetric, bounded, and uniformly positive definite tensor, \( f \in L^2(\Omega) \), and \( g_D \in H^{3/2}(\partial \Omega) \). Let \( T_h \) be a conforming partition of \( \Omega \) into \( N_T \) non-overlapping elements \( T_j \), \( j = 1, \ldots, N_T \), with Lipschitz boundaries such that \( T_h := \bigcup_{j=1}^{N_T} T_j \) and \( \overline{\Omega} = T_h \). The mesh size \( h \) is defined as \( h := \max_{j \in \{1, \ldots, N_T\}} \text{diam}(T_j) \). We denote the skeleton of the mesh by \( E_h := \bigcup_{j=1}^{N_T} \partial T_j \): the set of all (uniquely defined) interfaces \( e \) between elements. We conventionally identify \( n^- \) as the outward normal vector on the boundary \( \partial T \) of element \( T \) (also denoted as \( T^- \)) and \( n^+ = -n^- \) as the outward normal vector of the boundary of a neighboring element (also denoted as \( T^+ \)). Furthermore, we use \( n \) to denote either \( n^- \) or \( n^+ \) in an expression that is valid for both cases, and this convention is also used for other quantities (restricted) on a face \( e \in E_h \).

For simplicity, we define \( (\cdot, \cdot)_T \) as the \( L^2 \)-inner product on a domain \( T \subset \mathbb{R}^d \) and \( \langle \cdot, \cdot \rangle_T \) as the \( L^2 \)-inner product on a domain \( T \) if \( T \subset \mathbb{R}^{d-1} \). We shall use \( \| \cdot \|_T := \| \cdot \|_{L^2(T)} \) as the induced norm for both cases. Boldface lowercase letters are conventionally used for vector-valued functions and in that case the inner prod-
uct is defined as \((u, v)_T := \sum_{i=1}^{m} (u_i, v_i)_T\), and similarly \((u, v)_T := \sum_{i=1}^{m} (u_i, v_i)_T\), where \(m\) is the number of components \((u_i, i = 1, \ldots, m)\) of \(u\). Moreover, we define \((u, v)_{T_h} := \sum_{T \in T_h} (u, v)_T\) and \((u, v)_{\mathcal{E}_h} := \sum_{e \in \mathcal{E}_h} (u, v)_e\) whose induced norms are clear, and hence their definitions are omitted. We employ boldface uppercase letters, e.g. \(K\), to denote matrices and tensors. We denote by \(\mathcal{P}^p(T)\) the space of polynomials of degree at most \(p\) on a domain \(T\). We use the terms “skeletal unknowns”, “trace unknowns” and “Lagrange multipliers” interchangeably and they all refer to the unknowns on the mesh skeleton.

First, we cast equation (2.1) into the following first-order or mixed form:

\[
\begin{align*}
\mathbf{u} &= -K \nabla q & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= f & \text{in } \Omega, \\
q &= g_D & \text{on } \partial \Omega.
\end{align*}
\]  

(2.2)

The hybrid mixed DG method or hybridized DG (HDG) method for the discretization of equation (2.2) is defined as

\[
\begin{align*}
\left( K^{-1} \mathbf{u}, \mathbf{v} \right)_T - (q, \nabla \cdot \mathbf{v})_T + \langle \lambda, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial T} &= 0, \\
- (\mathbf{u}, \nabla w)_T + \langle \mathbf{u} \cdot \mathbf{n}, w \rangle_{\partial T} &= (f, w)_T, \\
\left\langle [\mathbf{u} \cdot \mathbf{n}], \mu \right\rangle_e &= 0,
\end{align*}
\]

(2.3)

where the numerical flux \(\mathbf{u} \cdot \mathbf{n}\) is given by

\[
\mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} + \tau(q - \lambda).
\]  

(2.4)

For simplicity, we have ignored the fact that the equations (2.3a), (2.3b) and (2.3c) must hold for all test functions \(\mathbf{v} \in V_h(T), w \in W_h(T),\) and \(\mu \in M_h(e),\) respectively (this is implicitly understood throughout the paper), where \(V_h, W_h,\) and \(M_h\) are defined as

\[
\begin{align*}
V_h(T_h) &= \left\{ \mathbf{v} \in [L^2(T_h)]^d : \mathbf{v}|_T \in [\mathcal{P}^p(T)]^d, \forall T \in T_h \right\}, \\
W_h(T_h) &= \left\{ w \in L^2(T_h) : w|_T \in \mathcal{P}^p(T), \forall T \in T_h \right\}, \\
M_h(e) &= \left\{ \lambda \in L^2(E_h) : \lambda|_e \in \mathcal{P}^p(e), \forall e \in E_h \right\},
\end{align*}
\]

and similar spaces \(V_h(T), W_h(T)\) and \(M_h(e)\) on \(T\) and \(e\) can be defined by replacing \(T_h\) with \(T\) and \(E_h\) with \(e,\) respectively.

Next, we consider the hybridized interior penalty DG (IPDG) schemes posed in the primal form. To that end, we test equation (2.2a) with \(\mathbf{v} = \nabla w\) and then integrate by parts twice the terms on the right hand side to obtain

\[
\left( \mathbf{u}, \nabla w \right)_T = -(K \nabla q, \nabla w)_T + \langle (q - \lambda), K \nabla w \cdot \mathbf{n} \rangle_{\partial T}.
\]

Substituting this in equation (2.3b) gives

\[
(K \nabla q, \nabla w)_T - \langle (q - \lambda), K \nabla w \cdot \mathbf{n} \rangle_{\partial T} + \langle \mathbf{u} \cdot \mathbf{n}, w \rangle_{\partial T} = (f, w)_T.
\]  

(2.5)

For IPDG schemes, the numerical flux takes the form

\[
\mathbf{u} \cdot \mathbf{n} = -K \nabla q \cdot \mathbf{n} + \tau(q - \lambda),
\]  

(2.6)
and it is required to satisfy the conservation condition (2.3c). Substituting the numerical flux (2.6) in (2.5) we get the following primal form of the hybridized IPDG scheme

\[
(K \nabla q, \nabla w)_T - \langle (q - \lambda), K \nabla w \cdot n \rangle_{\partial T} - \langle K \nabla q \cdot n, w \rangle_{\partial T} + \langle \tau (q - \lambda), w \rangle_{\partial T} = (f, w)_T.
\tag{2.7}
\]

This scheme is called the hybridized symmetric IPDG scheme (SIPG-H) and has been studied in \[21,27,42\]. In order to include the other hybridized IPDG schemes, we can generalize (2.7) to

\[
(K \nabla q, \nabla w)_T - s_f \langle (q - \lambda), K \nabla w \cdot n \rangle_{\partial T} - \langle K \nabla q \cdot n, w \rangle_{\partial T} + \langle \tau (q - \lambda), w \rangle_{\partial T} = (f, w)_T.
\tag{2.8}
\]

where \(s_f \in \{-1, 0, 1\}\). The scheme corresponding to \(s_f = -1\) is called the hybridized non-symmetric IPDG scheme (NIPG-H) and the one with \(s_f = 0\) is called the hybridized incomplete IPDG scheme (IIPG-H) \[27\]. It is interesting to note that in the HDG scheme, if we do not integrate by parts equation (2.3a) and substitute \(u = -K \nabla q\) in equations (2.3b) and (2.4) we obtain the IIPG-H scheme. On the other hand, if we do not integrate by parts equation (2.3b) and take \(\hat{u} \cdot n = u \cdot n\) in equation (2.3c) we obtain the hybridized mixed methods as follows

\[
(K^{-1} u, v)_T - (q, \nabla \cdot v)_T + \langle \lambda, v \cdot n \rangle_{\partial T} = 0,
\tag{2.9a}
\]

\[
(\nabla \cdot u, w)_T = (f, w)_T,
\tag{2.9b}
\]

and we need to enforce the conservation condition \(\langle [u \cdot n], \mu \rangle_e = 0\). However, in this case we can no longer choose the same solution order for both \(u\) and \(q\) due to the lack of stabilization. Indeed, inf-sup stable mixed finite element spaces have to be chosen for \(V_h\) and \(W_h\). For the details, we refer the readers to \[20\] for the hybridized Raviart-Thomas (RT-H) and the hybridized Brezzi-Douglas-Marini (BDM-H) mixed finite element spaces.

We note in passing that the hybridized weak Galerkin mixed finite element methods introduced in \[32\] follows similar hybridization as done in the hybridized mixed methods but with a different choice for local spaces accompanied with a stabilization term. In many cases one can show that HDG and weak Galerkin methods coincide, and we do not attempt to distinguish them in this paper.

The common solution procedure for all of these hybridized methods can now be described. First, we express the local volume unknowns \(u\) and/or \(q\), element-by-element, as a function of the skeletal unknowns \(\lambda\). Then, we use the conservation condition to construct a global linear system involving only the skeletal unknowns. Once they are solved for, the local volume unknowns can be recovered in an element-by-element fashion completely independent of each other. The main advantage of this Schur complement approach is that, for high-order methods, the global trace system is much smaller and sparser compared to the linear system for the volume unknowns \[14,21\]. The question that needs to be addressed is how to solve the trace system efficiently. In the next section we develop a geometric multigrid algorithm to provide an answer.

3. Geometric multigrid algorithm based on DtN maps. For all of the hybridized methods described in the previous section, the resulting linear systems for
the skeletal unknowns $\lambda$, in operator form, can be written as

$$A\lambda = g. \quad (3.1)$$

The well-posedness of the trace system (3.1) for the hybridized methods discussed in the previous section has been shown in [21, 27, 32].

The concept of a Dirichlet-to-Neumann (DtN) map is essential to our approach, so we briefly describe it here. A map $A$ is called a DtN map/operator if it maps Dirichlet data on a domain boundary to Neumann data. It is a particular type of Poincaré-Steklov operator, which contains a large class of operators which map one type of boundary condition to another for elliptic PDEs. For hybridized methods in equation (2.3), the trace variable $\lambda$ plays the role as the flux for the auxiliary equation (2.3a) and hence the operator $A$ in equation (3.1) is a DtN operator which maps Dirichlet boundary data of the original PDE (2.1) to Neumann data $\lambda$. In finite-dimension, the Schur complement of the linear system with volume unknowns condensed out is also known as discrete DtN map. We refer the readers to [33] for more information.

To define our multigrid algorithm we start with a sequence of partitions of the mesh $T_h$:

$$T_1, T_2, \ldots, T_N = T_h,$$

where each $T_k$ contains $N_{T_k}$ closed (not necessarily convex, except on the finest level where each $N_{T_k}$ is in fact some element $T_j$ in $T_h$) macro-elements consisting of unions of macro-elements from $T_{k+1}$. Associated with each partition, we define interface grids $E_1, E_2, \ldots, E_N = E_h$, where each $e \in E_k$ is the intersection of two macro-elements in $T_k$. Each partition $E_k$ is in turn associated with a skeletal (trace) space $M_k$ (to be defined in Section 3.1).

Remark 3.1. We note that if each $T_k$ consists of simplices or parallelopipeds, then each $M_k$ is straightforward to define. In general, however, the macro-elements need not be one of these standard shapes, and we need to define a parameterization of each macro-edge to define $M_k$. While this is certainly nontrivial, this calculation only needs to be performed one time for each mesh and can be reused for many simulation or time steps as long as the macro-elements (agglomeration) do not change. For all of the results presented in Section 4, the time required to calculate these parameterizations is negligible in comparison with the total simulation time.

We are now in the position to introduce necessary ingredients to define our geometric multigrid algorithm.

3.1. Coarsening strategy. In the multigrid algorithm we first coarsen in $p$, followed by the coarsening in $h$. We explain these two coarsening procedures in detail as follows.

3.1.1. $p$-coarsening. In our $p$-coarsening strategy, we restrict the solution order $p$ on the finest level $N$ to $p = 1$ on level $N - 1$ with the same number of elements. In this case the Lagrange multiplier spaces become

$$M_k = \begin{cases} \{ \eta \in \mathcal{P}^p(e), \forall e \in E_k \} & \text{for } k = N, \\ \{ \eta \in \mathcal{P}^1(e), \forall e \in E_k \} & \text{for } k = 1, 2, \ldots, N - 1. \end{cases}$$

Let us comment on one subtle point. It would be ideal if we could use $\mathcal{P}^0$ in coarser levels because it is trivial to define piecewise constant functions along arbitrary macro-edges (2D) or macro-faces (3D), i.e., no parameterization is needed. However, multilevel algorithms using restriction and prolongation operators based on piecewise constant interpolation typically have been shown not to perform well [5, 31].
For high-order methods, the numerical examples in Section 4 indicate that our strategy gives scalable results in solution order $p$ when strong smoothers such as block-Jacobi or Gauss-Seidel is used. This is also observed in [29], for a $p$-multigrid approach applied to DG methods for Poisson equation using block-Jacobi smoother. Other $p$-multigrid strategies, such as $p_{k-1} = p_k/2$ ($p_k$ is the solution order on the $k$th level) or $p_{k-1} = p_k - 1$, can be straightforwardly incorporated within our approach. Our future work will include theoretical and numerical comparisons among these strategies. Once we interpolate to $p = 1$ we carry out the $h$-coarsening as described in the next section.

Fig. 3.1: A demonstration of $h$-coarsening strategy. (a) Level $k$ mesh; (b) An identification of interior $\mathcal{E}_{k,I}$ (blue) and boundary $\mathcal{E}_{k,B}$ (red) edges; (c) Coarsening of boundary edges; and (d) Level $k-1$ mesh after interior edges are statically condensed out. The numbers in (a) and (d) represent the number of (macro) elements in levels $k$ and $k-1$, respectively.

3.1.2. $h$-coarsening. We adopt an agglomeration-based $h$-coarsening, allowing intergrid transfer and coarse grid operators to be defined using the fine scale DtN maps. By taking this approach, no upscaling of parameters is required and our coarse operators are discretized DtN maps at any mesh level (see Proposition 3.3). In Figure 3.1, we show an $h$-coarsening strategy between two consecutive levels $k$ and $k-1$. First, we identify the interior (blue) and boundary (red) edges in Figure 3.1(b). We decompose $\mathcal{E}_k = \mathcal{E}_{k,I} \oplus \mathcal{E}_{k,B}$, where $\mathcal{E}_{k,I}$ consists of edges interior to macro elements in the coarser partition $T_{k-1}$ and $\mathcal{E}_{k,B}$ contains edges common to their boundaries. We also decompose the trace space $M_k$ on $\mathcal{E}_k$ into two parts $M_{k,I}$ and $M_{k,B}$ corresponding to $\mathcal{E}_{k,I}$ and $\mathcal{E}_{k,B}$, respectively. Specifically, we require $M_k = M_{k,I} \oplus M_{k,B}$ such that each $\lambda_k \in M_k$ can be uniquely expressed as $\lambda_k = \lambda_{k,I} + \lambda_{k,B}$, where

$$
\lambda_{k,I} = \begin{cases} 
\lambda_k, & \text{on } M_{k,I}, \\
0, & \text{on } M_{k,B},
\end{cases} \quad \text{and} \quad
\lambda_{k,B} = \begin{cases} 
0, & \text{on } M_{k,I}, \\
\lambda_k, & \text{on } M_{k,B}.
\end{cases}
$$

Given the decomposition $M_k = M_{k,I} \oplus M_{k,B}$, the trace system (3.1) at the $k$th level can be written as

$$
A_k \lambda_k = g_k \iff \begin{bmatrix} A_{k,II} & A_{k,IB} \\
A_{k,BI} & A_{k,BB} \end{bmatrix} \begin{bmatrix} \lambda_{k,I} \\
\lambda_{k,B} \end{bmatrix} = \begin{bmatrix} g_{k,I} \\
g_{k,B} \end{bmatrix}.
$$

The coarser space $M_{k-1}$ is defined such that $M_{k-1} \subset M_{k,B}$. This is done by first agglomerating the boundary edges of level $k$ as in Figure 3.1(c), and then statically condensing out the interior edges to obtain the mesh on level $k-1$ in Figure 3.1(d). The elements in level $k$ are numbered from 1 to 42 in Figure 3.1(a) and the macro elements in level $k-1$ are numbered from 1 to 6 in Figure 3.1(d).
Clearly, for a unstructured mesh the identification of interior and boundary edges is non-trivial and non-unique. In this paper we use an ad-hoc approach which is sufficient for the purpose of demonstrating our proposed algorithm. Specifically, we first select a certain number of levels $N$ and seed points $N_{T_1}$. The locations of these seed points are chosen based on the geometry of the domain and the original fine mesh, such that we approximately have equal number of fine mesh elements in each macro-element at level 1. Based on these selected seed points, we agglomerate the elements in the finest mesh ($k = N$) to form $N_{T_1}$ macro-elements at level 1. We then divide each macro-element in level 1 into four approximately equal macro-elements to form level 2. This process is recursively repeated to create macro-elements in finer levels $k = 3, \ldots, N-1$. As an illustration we consider the mesh in Figure 3.2 (this mesh and the corresponding coarsening strategies will be used in numerical example II in Section 4.2) and in Figures 3.3 and 3.4 we show two different coarsening strategies with $N = 7$ obtained from seven and four seed points.

![Figure 3.2](image-url)

**Fig. 3.2:** Example II: Unstructured mesh for a rectangular box with eight holes.

![Figure 3.3](image-url)

**Fig. 3.3:** Example II. Coarsening strategy 1: the seed points are marked with green squares in level 1.

Part of our future work is to explore the coarsening strategies similar to the ones in [40, 41]. This may give better coarsened levels with the number of macro-elements reduced/increased by a constant factor between successive levels. The intergrid transfer operators necessary for moving residuals/errors between levels is described next.

### 3.2. Intergrid transfer operators

In standard nested multigrid algorithms, the intergrid transfer operators are straightforward. A popular approach is to take injection for the prolongation and its adjoint for the restriction. For skeletal system care must be taken in the construction of these operators to ensure the convergence of the multigrid algorithm under consideration. In the following we propose “physics-based energy-preserving” operators using the fine scale DtN maps.
3.2.1. Prolongation. The prolongation operator, $I_k : M_{k-1} \rightarrow M_k$, transfers the error from the grid level $k-1$ to the finer grid level $k$. Note that standard prolongation using injection would set the values on the interior edges to zero, and thus does not work. For our multigrid algorithm it is defined as a function of the solution order $p$ using DtN maps. In particular:

* for $p = 1$,
\[
I_k := \left[ -A_{k,11}^{-1} A_{k,1B} J_k \right] \quad \text{for} \quad k = 2, \cdots, N, \tag{3.3}
\]

* and for $p > 1$,
\[
I_k := \begin{cases} 
J_N \\ -A_{k,11}^{-1} A_{k,1B} J_k & \text{for } k = N, \\
J_k & \text{for } k = 2, \cdots, N - 1. \end{cases} \tag{3.4}
\]

Here, $J_k : M_{k-1} \rightarrow M_k$ is the injection (interpolation). To understand the idea behind the prolongation operator, let us consider $p = 1$ in (3.3). First, through $J_k$, we interpolate the error from the grid level $k - 1$ to obtain error on the boundary edges of the finer grid level $k$. Then we solve (via the first block in the definition of $I_k$), for the error on the interior edges as a function of the interpolated error on the boundary. For $p > 1$ the prolongation is defined in (3.4), namely, we use the same prolongation as in the case of $p = 1$ for all levels $k \leq N - 1$ and interpolate error from piecewise linear polynomials at level $N - 1$ to piecewise $p$th-order polynomials at level $N$.

3.2.2. Restriction. The restriction operator, $Q_{k-1} : M_k \rightarrow M_{k-1}$, restricts the residual from level $k$ to coarser level $k-1$. The popular idea is to define the restriction operator as the adjoint (with respect to the $L^2$-inner products $\langle \cdot, \cdot \rangle_{E_k}$ and $\langle \cdot, \cdot \rangle_{E_{k-1}}$ in $M_k$ and $M_{k-1}$) of the prolongation operator, i.e. $Q_{k-1} = I_k^\ast$, and from our numerical studies (not shown here) it still works well. However, this purely algebraic procedure, though convenient, is not our desire. Here, we construct the restriction operator $Q_{k-1}$ such that the coarse grid problem, via the Galerkin approximation, is exactly a discretized DtN problem on level $k - 1$. To that end, let us define $Q_{k-1}$ as:
\[ Q_{k-1} := \begin{bmatrix} -J_k^* A_{k,BI} A_{k,II}^{-1} & J_k^* \\ \end{bmatrix} \quad \text{for} \quad k = 2, \ldots, N, \quad (3.5) \]

\[ Q_{k-1} := \begin{cases} J_N^* \\ \begin{bmatrix} -J_k^* A_{k,BI} A_{k,II}^{-1} & J_k^* \\ \end{bmatrix} \end{cases} \quad \text{for} \quad k = N, \quad k = 2, \ldots, N - 1. \quad (3.6) \]

Note that if \( A \) is symmetric, then our definition of the restriction operator \( Q_{k-1} \) is indeed the adjoint of the prolongation operator \( I_k \).

Given the prolongation and restriction operators, we obtain our coarse grid equation using the discrete Galerkin approximation \[ Q_{k-1} A_k I_k \lambda_{k-1} = Q_{k-1} g_k, \quad (3.7) \]
where the coarse grid operator
\[ A_{k-1} := Q_{k-1} A_k I_k, \quad (3.8) \]
for either \( p = 1 \) and \( k \leq N \) or \( p > 1 \) and \( k \leq N - 1 \), reads
\[ A_{k-1} = \begin{bmatrix} -J_k^* A_{k,BI} A_{k,II}^{-1} & J_k^* \\ A_{k,II} & A_{k,BI} \\ \end{bmatrix} \begin{bmatrix} A_{k,II} & A_{k,BI} \\ A_{k,BI} & A_{k,BB} \\ \end{bmatrix} \begin{bmatrix} -A_{k,II}^{-1} A_{k,IB}J_k \\ J_k \\ \end{bmatrix} = J_k^* \left( A_{k,BB} - A_{k,BI} A_{k,II}^{-1} A_{k,IB} \right) J_k, \quad (3.9) \]
and
\[ A_{N-1} = J_N^* A_N J_N, \quad (3.10) \]
for \( p > 1 \) and \( k = N \).

**Energy preservation:** In order for the multigrid algorithms to converge the intergrid operators should be constructed in such a way that the “energy” does not increase when transferring information between a level to a finer one \[6,28\]. Note that “energy” here need not necessarily be associated with some physical energy. Indeed, if we associate \( A_k \) with a bilinear form \( a_k(\ldots) \) such that \( a_k(\kappa,\mu) = \langle A_k \kappa, \mu \rangle_{E_k} \), \( \forall \kappa, \mu \in M_k \), where again \( \langle \ldots \rangle_{E_k} \) represents the \( L^2 \) inner product on \( E_k \), then we call \( a_k(\kappa,\kappa) \) the energy on level \( k \) associated with \( \kappa \). Non-increasing energy means
\[ a_k(I_k \lambda, I_k \lambda) \leq a_{k-1}(\lambda, \lambda) \quad \forall \lambda \in M_{k-1}, \quad \forall k = 2, 3, \ldots, N. \]

**Proposition 3.2 (Energy preservation).** The proposed multigrid approach preserves the energy in the following sense: \( \forall k = 2, 3, \ldots, N \),
\[ a_k(I_k \lambda, I_k \lambda) = a_{k-1}(\lambda, \lambda) \quad \forall \lambda \in M_{k-1}. \quad (3.11) \]
Proof. We proceed first with $p = 1$. From the definition of $A_k$ in (3.2) and the definition of the prolongation operator $I_k$ in (3.3) we have

$$a_k (I_k \lambda, I_k \lambda) = \left( \left[ -A^{-1}_{k,II} A_{k,IB} J_k \lambda, J_k \lambda \right], \left[ A_{k,II}, A_{k,IB} \right] \left[ -A^{-1}_{k,II} A_{k,IB} J_k \lambda \right] \right)_{E_k} = \left( \left( A_{k,IB} - A_k B A^{-1}_{k,II} A_{k,IB} \right) J_k \lambda, J_k \lambda \right)_{E_k} = a_{k-1} (\lambda, \lambda),$$

where the last equality comes from the definition of the coarse grid operator $A_{k-1}$ in (3.9). For $p > 1$, we need to prove (3.11) only for $k = N$, but this is straightforward since from (3.4), (3.6), and (3.10) we have

$$a_N (I_N \lambda, I_N \lambda) = \langle A_N J_N \lambda, J_N \lambda \rangle_{E_k} = \langle J^*_N A_N J_N \lambda, \lambda \rangle_{E_k} = a_{k-1} (\lambda, \lambda).$$

We now prove that the coarse grid operator (3.8) is also a discretized DtN map on every level.

**Proposition 3.3.** At every level $k = 1, \ldots, N$, the Galerkin coarse grid operator (3.8) is a discretized DtN map on that level.

**Proof.** The proof is a straightforward induction using the proposed coarsening strategy and the definition of the intergrid transfer operators. First, consider the case of $p = 1$. The fact that $A_N = A$ in (3.1) is a discretized DtN map on the finest level is clear by the definition of the hybridized methods. Assume at level $k$ the operator $A_k$ in (3.2) is a discretized DtN map. Taking $\lambda_{k,B} = J_k \lambda_{k-1}$ and condensing $\lambda_{k,I}$ out yield

$$\left( A_{k,IB} - A_k B A^{-1}_{k,II} A_{k,IB} \right) J_k \lambda_{k-1} = g_k \beta - A_k B A^{-1}_{k,II} g_k \beta,$$

which, after restricting on the coarse space $M_{k-1}$ using $J^*_k$, becomes

$$J^*_k \left( A_{k,IB} - A_k B A^{-1}_{k,II} A_{k,IB} \right) J_k \lambda_{k-1} = J^*_k \left( g_k \beta - A_k B A^{-1}_{k,II} g_k \beta \right),$$

which is exactly the coarse grid equation (3.7). That is, $A_{k-1}$ is the Schur complement obtained by eliminating all the trace unknowns inside all the macro-elements on level $k - 1$. By definition, the coarse grid operator $A_{k-1}$ on level $(k - 1)$ is also a discrete DtN map.

For the case of $p > 1$, it is sufficient to show that $A_{N-1}$ is a discrete DtN map, but this is clear by: 1) taking $\lambda = J_N \lambda_{N-1}$ in (3.1), 2) restricting both sides to $M_{N-1}$ using $J^*_N$, 3) recalling that $A_N$ is a discretized DtN map, and 4) observing that the resulting equation coincides with the coarse grid equation (3.10).

### 3.3. Smoothing

Let us denote the smoothing operator at level $k$ by $G_{k,m_k}$, where $m_k$ stands for the number of smoothing steps performed at level $k$. We take $m_k$ to satisfy

$$\beta_0 m_k \leq m_{k-1} \leq \beta_1 m_k,$$

with $\beta_1 \geq \beta_0 > 1$. That is, the number of smoothing steps are increased as the mesh is coarsened. The reason for this choice is based on the theoretical analysis for non-nested multigrid methods in [7]. However, as numerically shown later in Section 4.5, even constant number of smoothing steps at all levels, e.g. 2, works well.
3.4. Local correction operator. To motivate the need of local correction operator, let us consider the following decomposition of the skeletal space $M_k = \overline{M}_k \oplus \hat{M}_k$ such that

$$\lambda_k = \overline{\lambda}_k + \hat{\lambda}_k, \quad \overline{\lambda}_k \in \overline{M}_k \text{ and } \hat{\lambda}_k \in \hat{M}_k,$$

where $\overline{\lambda}_k = \overline{\lambda}_{k,I}$ is given by the local correction $T_k : M_k \rightarrow M_k$:

$$\overline{\lambda}_k := T_k \left[ \begin{array}{c} g_{k,I} \\ 0 \end{array} \right] := \left[ \begin{array}{cc} A_{k,II}^{-1} & 0 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{c} g_{k,I} \\ 0 \end{array} \right], \quad (3.12)$$

and the “global component” $\hat{\lambda}_k = \left[ \begin{array}{c} \hat{\lambda}_{k,I} \\ \hat{\lambda}_{k,B} \end{array} \right]$ by

$$\left[ \begin{array}{cc} A_{k,II} & A_{k,IB} \\ A_{k,BI} & A_{k,BB} \end{array} \right] \left[ \begin{array}{c} \hat{\lambda}_{k,I} \\ \hat{\lambda}_{k,B} \end{array} \right] = \left[ \begin{array}{c} 0 \\ g_{k,B} - A_{k,BI}A_{k,II}^{-1}g_{k,I} \end{array} \right] .$$

Let us now consider $p = 1$. Standard two-grid cycles, for example, include: 1) smoothing iterations on the system $(3.1)$ for the fine grid, 2) restricting the residual to the coarse grid, 3) performing the coarse grid correction, and 4) prolongating the error back to the fine grid. However, with the prolongation operator defined in equation $(3.3)$, only $\hat{\lambda}_k$ can be recovered directly. In other words, the burden of capturing $\overline{\lambda}_k$ is left for the smoother and/or multigrid iterations. Indeed, in our numerical studies we found that this standard algorithm takes more multigrid iterations and still converges with Gauss-Seidel and Chebyshev accelerated Jacobi smoothers. However, it diverges with block-Jacobi smoother at low orders as shown in Table 4.11.

This issue can be fixed using the concept of subspace correction [39, 44]. We can perform a local correction in the subspace $\overline{M}_k$, as in $(3.12)$, either before or after the coarse-grid correction (or both if symmetry is desirable). With this local correction, we have observed that even with Jacobi-type smoothers the multigrid algorithm converges. It is important to point out that the application of the local correction operator $T_k$ is completely local to each macro-element at the $(k - 1)$th level, and thus can be carried out in parallel.

3.5. Relationship to AMG operators. The intergrid transfer operators in section 3.2 and the local correction operator in 3.4 are closely related to the ideal interpolation and smoothing operators in algebraic multigrid (AMG) literature [39]. However, the ideal operators in AMG lead to a direct solver strategy, namely nested dissection [26] and suffer from large memory requirements. This renders the algorithm impractical for large scale simulations especially in 3D. In that respect, the coarsening in our operators is the key as it leads to an $O(N)$ iterative algorithm and can be applied to large scale problems. In fact, this approach is pursued in the context of finite differences in [23, 24, 34, 35] under the name of Schur complement multigrid methods. However, our approach is more general in the sense that it can be applied to any unstructured mesh, whereas all the previous works deal with structured meshes. As shown in Figure 3.2, it does not require the meshes to be nested. In fact, our approach can be considered as a combination of AMG and geometric multigrid methods. As such it benefits from the robustness of AMG and the fixed coarse grid construction costs of geometric multigrid.

3.6. Multigrid V-cycle algorithm. We are now in the position to define our multigrid algorithm. We begin with a fixed point scheme on the finest level:

$$\lambda_{i+1}^i = \lambda_i + B_N(r_N), \quad i = 0, \ldots$$
where \( r_N = g_N - A_N \lambda^i \) and the action of \( B_N \) on a function/vector is defined recursively using the multigrid algorithm 1.

**Algorithm 1** Multigrid v-cycle algorithm

1: Initialization:
\[ e^{(0)} = 0, \]
2: Presmoothing:
\[ e^{(1)} = e^{(0)} + G_{k,m_k} (r_k - A_k e^{(0)}), \]
3: Local Correction:
\[ e^{(2)} = e^{(1)} + T_k (r_k - A_k e^{(1)}), \]
4: Coarse Grid Correction:
\[ e^{(3)} = e^{(2)} + I_k B_{k-1} (Q_{k-1} (r_k - A_k e^{(2)})), \]
5: Local Correction:
\[ e^{(4)} = e^{(3)} + T_k (r_k - A_k e^{(3)}), \]
6: Postsmoothing:
\[ B_k (r_k) = e^{(5)} = e^{(4)} + G_{k,m_k} (r_k - A_k e^{(4)}). \]

At the coarsest level \( M_1 \), we set \( B_1 = A_1^{-1} \) and the inversion is computed using direct solver. Note that both local correction steps 3 and 5 are presented for the sake of symmetry of the algorithm. In practice, we use either step 3 or step 5. For example, numerical results in Section 4 use only step 3.

4. **Numerical results.** In this section, we study the performance of multigrid both as a solver and as a left preconditioner. Since the global trace system of NIPG-H and IIPG-H methods are not symmetric, we use preconditioned GMRES for all the methods to enable a direct comparison. For all the numerical examples: 1) the stopping tolerance is taken as \( 10^{-9} \) for the normalized residual (normalized by the norm of the right hand side of (3.1)), and 2) following [7] we choose the smoothing parameters \( \beta_0 \) and \( \beta_1 \) (see Section 3.3) as 2. In all the tables, “*” stands for either the divergence of the GMRES/multigrid solver or 200 iterations was already taken but the normalized residual was still larger than the tolerance.

In examples 1-5 in Sections 4.1-4.5, we consider HDG and hybridized IPDG methods. In example 6 in Section 4.6, we consider the case of multnumerics with mixed methods (RT-H) in some parts of the domain and NIPG-H in other parts. Though we have also tested the multigrid algorithm for hybridized RT1 and RT0 methods for example I and obtained scalable results, they are not shown here due to the page limitation.

4.1. **Example I: Poisson equation in a unit square.** In this first example, we consider the Poisson equation in a unit square. We take the exact solution to be of the form \( q = xy e^{x^2 y^3} \). Dirichlet boundary condition using the exact solution is applied directly (strongly) via the trace unknowns. We consider hybridized DG methods and study the performance of multigrid both as a solver and as a preconditioner. The number of levels in the multigrid hierarchy and the corresponding number of quadrilateral elements are shown in Table 4.1.

We consider the following four different smoothers and compare their performance for each of the hybridized DG methods.

- Damped point-Jacobi with the relaxation parameter \( \omega = 2/3 [38] \).
- Chebyshev accelerated point-Jacobi method. This smoother requires the estimates of extreme eigenvalues and we compute an approximate maximum
Table 4.1: Example I: The multigrid hierarchy.

The eigenvalue $\lambda_{max}$ with $O(10^{-2})$-accuracy. Following [1], we estimate the smallest eigenvalue using $\lambda_{max}/30$.

- Lower-upper symmetric point Gauss-Seidel method with one forward and one backward sweep during each iteration, and we name this smoother as LU-SGS for simplicity.
- Block-Jacobi smoother where one block consists of all the degrees of freedom corresponding to an edge $e \in \mathcal{E}_k$ at the $k$th level. Here, we do not use any damping as it does not make any difference in the number of iterations in our numerical studies.

For the stabilization parameters, we consider a mesh-independent value $\tau = 1$ and mesh-dependent form $\tau = 1/h_{min}$ for HDG. The former corresponds to the upwind HDG proposed in [12, 13]. Following [36], we take the stabilization to be $\tau = (p+1)(p+2)/h_{min}$ for the hybridized IPDG schemes. In later examples, when the permeability $K$ is spatially varying we also consider $\tau$ as a function of $K$. We refer to [12,18,22] for the $h$-convergence order of HDG schemes and [36] for the IPDG schemes.

Table 4.2: Example I. HDG with stabilization $\tau = 1/h_{min}$: the number of iterations for multigrid with point-Jacobi smoother as solver and preconditioner.

Table 4.3: Example I. HDG with stabilization $\tau = 1/h_{min}$: the number of iterations for multigrid with GMRES as solver and preconditioner.

In Tables 4.2-4.5, we study the performance of multigrid as a solver and as a preconditioner for HDG with stabilization $\tau = 1/h_{min}$. We now present a few important observations from these tables. The stabilization $\tau = 1/h_{min}$ gives $h$-scalable results with all the smoothers, i.e. the number of required multigrid/GMRES iterations is almost unchanged when the mesh is refined. However, the actual number of iterations depends on the effectiveness of the smoother under consideration. As can be seen in Table 4.2 with point-Jacobi smoother, the number of iterations increases with high-order solutions when multigrid is used as a solver. In contrast, the multigrid pre-
conditioner is effective in keeping the number of GMRES iterations almost unchanged (the increase is negligible). The results in Table 4.3 shows that Chebyshev acceleration for point-Jacobi smoother behaves the same as the point-Jacobi smoother in terms of scalability. However, in terms of the number of iterations it takes either more (for multigrid as solver) or about the same (for multigrid preconditioned GMRES). Table 4.4 shows that LU-SGS is the most effective smoother, requiring the least number of multigrid/GMRES iterations. Moreover, the number of both multigrid and GMRES iterations are almost constant as either the mesh is refined or the solution order increases. However, it should be pointed out that the LU-SGS smoother requires twice the amount of work compared to point-Jacobi for each iteration due to one forward and one backward sweep. Table 4.5 shows that block-Jacobi is the second best smoother in terms of the number of iterations; otherwise its scalability behavior is similar to the point-Jacobi smoother as the mesh or the solution order is refined.
In Tables 4.6 and 4.7, we present results for the multigrid solver and GMRES with multigrid preconditioner when $\tau = 1$ is used for the HDG discretization. With the point-smoothers\(^1\) (Table 4.6), the number of iterations for the multigrid solver scales like $O(1/h)$ (since the number of iterations is nearly doubled each time $h$ reduces by half). Even for GMRES with multigrid as preconditioner, we do not obtain $h$-scalability with point smoothers. On the other hand, the block-Jacobi smoother in Table 4.7 for $p \geq 4$ provides $h$-scalable results and with solution orders greater than 5 the results are exactly the same as those for $\tau = 1/h_{\text{min}}$. With respect to solution orders, similar to the case of $\tau = 1/h_{\text{min}}$, we do not obtain perfect $p$-scalability with any smoother though the growth of the number of iterations is very slow with block-Jacobi smoother.

We present in Tables 4.8-4.10 the results for the hybridized NIPG-H scheme with

\(^1\)We do not show results for point-Jacobi and Chebyshev accelerated point-Jacobi smoothers as the behavior of the number of iterations is very similar to that of the LU-SGS smoother.
\[ \tau = (p + 1)(p + 2)/h_{\text{min}}. \] As can be seen, the number of iterations for both multigrid as a solver and as a preconditioner is similar to those with HDG using \( \tau = 1/h_{\text{min}}. \)

The results using IIPG-H and SIPG-H are similar and hence are omitted for brevity. For all of the IPDG schemes, using the Chebyshev acceleration increases the number of iterations and we do not include these results here.

Finally, we test the performance of multigrid without the local correction operator in step 3 of Algorithm 1. The results are shown for HDG with \( \tau = 1/h_{\text{min}} \) and block-Jacobi smoother in Table 4.11. As can be seen, for \( p \leq 3 \) multigrid as solver and also as a preconditioner fails to converge for fine mesh sizes. For \( p \geq 4 \) we see scalable results although the number of iterations are slightly more than that in Table 4.5 with local correction. We also see an odd-even behavior, with even orders giving better iteration counts than odd ones. For \( \tau = 1 \), we observed similar results and hence not shown. For the hybridized IPDG schemes, with NIPG-H and IIPG-H from \( p \geq 2 \) onwards we observed scalable results whereas for SIPG-H for all orders we obtained
Unified Geometric Multigrid

| $p$ | Levels | Levels |
|-----|--------|--------|
| 2   | 3 4 5 6 7 | 2 3 4 5 6 7 |
| 1   | 4 4 4 4 4 4 | 4 4 4 4 4 5 |
| 2   | 4 4 4 5 5 5 | 4 4 4 4 4 4 |
| 3   | 4 5 5 5 5 5 | 4 4 4 5 5 4 |
| 4   | 5 5 5 5 5 6 | 4 4 5 5 5 5 |
| 5   | 5 6 6 6 6 6 | 4 5 5 5 5 5 |
| 6   | 6 6 6 6 6 6 | 5 5 5 5 5 5 |
| 7   | 6 7 7 7 7 7 | 5 5 5 5 5 5 |
| 8   | 7 7 7 7 7 7 | 5 6 6 6 6 6 |
| 9   | 7 7 8 8 8 8 | 5 6 6 6 6 6 |
| 10  | 7 8 8 8 8 8 | 6 6 6 6 6 6 |

Table 4.9: Example I. NIPG-H: the number of iterations for multigrid with LU-SGS smoother as solver and preconditioner.

| $p$ | Levels | Levels |
|-----|--------|--------|
| 2   | 3 4 5 6 7 | 2 3 4 5 6 7 |
| 1   | 5 7 8 8 8 8 | 4 6 6 6 6 6 |
| 2   | 6 6 6 6 7 7 | 5 5 5 5 5 5 |
| 3   | 7 8 8 8 8 8 | 5 6 6 6 6 6 |
| 4   | 9 9 10 10 10 10 | 6 7 7 7 7 7 |
| 5   | 10 11 11 11 11 11 | 6 7 7 7 7 7 |
| 6   | 11 12 12 12 12 12 | 7 8 8 8 8 7 |
| 7   | 12 13 13 13 13 13 | 7 8 8 8 8 8 |
| 8   | 13 14 14 14 14 14 | 7 8 8 8 8 8 |
| 9   | 14 15 15 15 15 15 | 8 9 9 9 8 9 |
| 10  | 15 16 16 16 16 16 | 8 9 9 9 9 9 |

Table 4.10: Example I. NIPG-H: the number of iterations for multigrid with block-Jacobi smoother as solver and preconditioner.

scalability without local correction. The iteration counts are again slightly more than the ones obtained with local correction. With increase in order, since block-Jacobi smoother gets stronger, even without the local correction operator it is sufficient to give scalable results. However, in order to provide more robustness and since the local correction operator is inexpensive, in the following test cases we always include it in the step-3 of Algorithm 1.

In summary, for HDG with $\tau = 1/h_{\min}$ and multigrid preconditioned GMRES, almost perfect $hp$-scalability has been observed for all the smoothers, while for $\tau = 1$ this scalability holds with only block smoothers together with high solution orders. All the hybridized IPDG schemes give scalable results with $\tau = (p + 1)(p + 2)/h_{\min}$ and the number of iterations is very similar to that of HDG with $\tau = 1/h_{\min}$. Of all the smoothers considered in this paper, LU-SGS seems to be the best smoother in terms of the number of iterations. However, block-Jacobi smoother, the second best, is more convenient from the implementation point of view (especially on parallel
computing systems), and for that reason we show numerical results only for block-Jacobi smoother from now on. Since the Chebyshev acceleration does not seem to improve the performance of multigrid by a significant margin, we will not use it in the subsequent sections. We would like to mention that the performance of our multigrid approach in terms of iteration counts is better or at least comparable to other existing multigrids proposed in the literature [15,19,28,30]. Since the time to solution depends on many other factors such as the choice of smoother, the number of smoothing steps, and the architecture of the machine we are not able to comment on that aspect at this moment. However, in future work we plan to carry out a detailed study between different multigrid algorithms for hybridized methods with respect to simulation of challenging problems.

4.2. Example II: Unstructured mesh. The goal of this section is to test the robustness of the multigrid algorithm for a highly unstructured mesh. To that end consider the mesh in Figure 3.2, which consists of 6699 simplices with an order of magnitude variation in the diameter of elements i.e. \( h_{\text{max}} \approx 10 h_{\text{min}} \). Unlike structured meshes, for unstructured meshes with local clustering of elements (local refinements) it is not straightforward to select the number of levels and the “best” coarsening strategy that well balances the number of elements (in the finer level) for each macro-element. Ideally, given a number of levels we wish to minimize the number of multigrid/GMRES iterations. Here, we compare two coarsening strategies with the same number of levels and study the performance of multigrid as a solver and as a preconditioner. In our future work, we will explore the coarsening strategies similar to the ones in [40, 41]. Figures 3.3 and 3.4 show different levels corresponding to the two coarsening strategies. The total number of levels in both the strategies is seven and the last level corresponds to the original fine mesh as shown in Figure 3.2. For solution orders \( p > 1 \), again, we first restrict the residual to \( p = 1 \) and then carry out the geometric coarsening. There are seven macro-elements in the first level for coarsening strategy 1, and four macro-elements for strategy 2.

We take zero Dirichlet boundary condition, \( K = I \) (I is the identity), and \( f = 1 \) for this example. We now study the performance of HDG and hybridized IPDG schemes. In Table 4.12, we compare the number of iterations taken for HDG with \( \tau_1 = 1 \) and

| \( p \) | Levels | MG as solver | Levels | MG with GMRES |
|---|---|---|---|---|
| 1 | * | * | * | 8 | 18 | * | * | * |
| 2 | 8 | 13 | 21 | 26 | 27 | 28 | 6 | 7 | 9 | 11 | 11 | 11 |
| 3 | 16 | 60 | * | * | * | 7 | 11 | 20 | 39 | 97 | * |
| 4 | 10 | 11 | 12 | 13 | 13 | 13 | 7 | 7 | 8 | 8 | 8 |
| 5 | 11 | 12 | 12 | 12 | 12 | 12 | 6 | 8 | 8 | 8 | 7 |
| 6 | 12 | 12 | 13 | 13 | 13 | 13 | 7 | 8 | 8 | 8 | 8 |
| 7 | 15 | 20 | 23 | 25 | 26 | 26 | 9 | 10 | 11 | 11 | 11 |
| 8 | 14 | 15 | 15 | 15 | 15 | 15 | 8 | 9 | 9 | 9 | 8 |
| 9 | 15 | 16 | 16 | 16 | 16 | 17 | 8 | 9 | 9 | 9 | 9 |
| 10 | 18 | 19 | 19 | 20 | 20 | 20 | 9 | 10 | 10 | 10 | 10 |

Table 4.11: Example I. HDG with stabilization \( \tau = 1/h_{\text{min}} \): the number of iterations for multigrid with block-Jacobi smoother as solver and preconditioner without local correction.
\( \tau_2 = 1/h_{\min} \) and solution orders \( p = 1, \ldots, 8 \). As can be seen, for \( p \in \{1, 2, 3\} \), using \( \tau_1 \) takes less number of iterations, and for \( p \geq 4 \) both stabilization parameters require almost the same iteration counts. For coarsening strategy 2, since the coarsening happens slightly more aggressive (i.e. less macro elements at any level) the iteration counts in general are higher. However, using multigrid as a preconditioner for GMRES the difference in the iteration counts for the two strategies is negligible.

In Table 4.13, we consider the NIPG-H scheme with \( \tau_1 = 1/h_{\min} \) and \( \tau_2 = (p + 1)(p + 2)/h_{\min} \). As can be seen, the multigrid solver diverges for many values of solution order \( p \). The iterations for \( \tau_1 \) are less than that for \( \tau_2 \). We observe that the iteration counts of multigrid preconditioned GMRES with \( \tau_1 \), except for solution orders \( p = 2 \) and \( p = 3 \) (which require more iterations), are similar to those for HDG. When multigrid is used as a preconditioner, except for solution orders equal to two and three, both the coarsening strategies give similar iteration counts with \( \tau_1 \). It turns out that the iteration counts for IIPG-H and SIPG-H are higher than that of HDG and NIPG-H and hence are omitted for brevity.

Within the settings of this example we conclude that multigrid, both as a solver and as a preconditioner, is more effective for HDG than for hybridized IPDG schemes. It is also relatively more robust with respect to the coarsening strategies and values of stabilization parameter when used as a preconditioner.

| \( p \) | \( \tau_1 \) | \( \tau_2 \) | \( \tau_1 \) | \( \tau_2 \) | \( \tau_1 \) | \( \tau_2 \) |
|------|------|------|------|------|------|------|
| 1    | 25   | *    | 41   | 108  | 10   | 21   | 13   | 18   |
| 2    | 20   | 63   | 33   | 37   | 10   | 12   | 12   | 13   |
| 3    | 23   | 24   | 39   | 41   | 10   | 11   | 13   | 14   |
| 4    | 27   | 27   | 43   | 45   | 11   | 11   | 14   | 14   |
| 5    | 30   | 30   | 48   | 49   | 11   | 12   | 15   | 15   |
| 6    | 33   | 33   | 51   | 52   | 12   | 12   | 15   | 15   |
| 7    | 36   | 36   | 55   | 55   | 13   | 13   | 16   | 16   |
| 8    | 38   | 38   | 58   | 58   | 13   | 13   | 16   | 16   |

Table 4.12: Example II. HDG with \( \tau_1 = 1, \tau_2 = 1/h_{\min} \): the number of iterations for multigrid as solver and preconditioner for both coarsening strategies.

4.3. Example III: Smoothly varying permeability. In this example we consider a smoothly varying permeability of the form

\[
K = \kappa I,
\]

where \( \kappa = 1 + 0.5 \sin(2\pi x) \cos(3\pi y) \). The domain considered is a circle and we take the exact solution again to be of the form \( q = xye^{x^2+y^2} \). The forcing and the Dirichlet boundary condition are chosen corresponding to the exact solution. The number of levels and the corresponding number of triangular elements in the multigrid hierarchy are shown in Table 4.14.

Table 4.15 shows the number of multigrid and GMRES iterations for HDG with \( \tau = 1 \). Note that perfect \( h \)-scalability with multigrid as a solver is not observed, and this is mainly due to the fact that the number of elements between successive levels
is not exactly increased/decreased by a constant\(^2\). Again, we see a little growth in the number of iterations as \(p\) increases for the multigrid solver, while for GMRES with multigrid preconditioner we obtain almost perfect \(hp\)-scalability. We have also conducted numerical examples with \(\tau_1 = 1/h_{\text{min}}, \tau_2 = \kappa/h_{\text{min}}\) and observed that the iteration counts (not shown here) are similar to those with \(\tau = 1\).

Next we consider the IIPG-H scheme with \(\tau = \kappa(p + 1)(p + 2)/h_{\text{min}}\). Table 4.16 shows that the iteration counts are comparable with HDG results in Table 4.15 for GMRES with multigrid preconditioner. The multigrid solver for IIPG-H, on the other hand, has slightly less iteration counts. An extra penalization factor of 1.5 is necessary to obtain well-posed linear systems for SIPG-H, but otherwise the corresponding

\(^2\)This, as argued before, is not trivial for unstructured meshes.
results for NIPG-H and SIPG-H are similar and hence are omitted.

| $p$ | MG as solver | MG with GMRES |
|-----|--------------|---------------|
|     | Levels       | Levels        |
| 2   | 3 4 5 6 7    | 2 3 4 5 6 7   |
| 1   | 14 15 17 16 18 20 | 8 9 10 10 10 10 |
| 2   | 12 13 16 15 17 18 | 8 9 9 9 10 10  |
| 3   | 14 16 18 18 20 21 | 9 10 10 10 11 11 |
| 4   | 17 19 21 21 23 24 | 10 10 11 11 12 12 |
| 5   | 19 21 23 23 25 26 | 11 11 12 12 12 12 |
| 6   | 21 24 26 25 28 29 | 11 12 12 12 13 13 |
| 7   | 22 26 27 27 30 31 | 11 13 13 13 13 13 |
| 8   | 24 29 29 29 32 33 | 12 13 13 13 13 14 |

Table 4.16: Example III. IIPG-H with stabilization $\tau = \kappa(p + 1)(p + 2)/h_{\min}$: the number of iterations for multigrid as solver and preconditioner.

4.4. Example IV: Highly discontinuous permeability. In this example, we test the robustness of the multigrid solver on a highly discontinuous and spatially varying (six orders of magnitude) permeability field given by $K = \kappa I$ with

$$\kappa = \begin{cases} 
10^6, & (x, y) \in (0, 0.56) \times (0, 0.56) \text{ or } (x, y) \in (0.56, 1) \times (0.56, 1), \\
1, & \text{otherwise},
\end{cases}$$

in a unit square.

We choose zero Dirichlet boundary condition and $f = 1$. In Table 4.17, we study the performance of multigrid as a solver and as a preconditioner for GMRES using HDG with $\tau = \kappa/h_{\min}$. The number of elements and the number of levels are same as in example I. Similar to the previous examples, almost perfect $h$- and $p$-scalabilities are observed with multigrid preconditioned GMRES, whereas the number of iterations increases with the solution order $p$ for the multigrid solver. For $\tau = 1/h_{\min}$ and $\tau = 1$, the multigrid solver takes too many iterations to converge for finer meshes, and for that reason we report the iteration counts only for multigrid preconditioned GMRES.

| $p$ | MG as solver | MG with GMRES |
|-----|--------------|---------------|
|     | Levels       | Levels        |
| 2   | 3 4 5 6 7    | 2 3 4 5 6 7   |
| 1   | 6 8 8 9 9 9  | 2 5 5 5 5 5  |
| 2   | 6 7 9 10 10 11 | 3 4 5 5 5 5  |
| 3   | 8 9 10 10 10 10 | 3 5 6 6 5 5  |
| 4   | 10 11 11 11 11 | 4 6 6 6 6 5  |
| 5   | 11 13 13 13 13 | 4 6 7 7 6 6  |
| 6   | 13 14 14 14 14 | 4 6 7 7 7 6  |
| 7   | 14 15 16 16 15 15 | 5 7 7 7 7 7  |
| 8   | 15 16 17 17 17 17 | 5 7 8 8 7 7  |

Table 4.17: Example IV. HDG with stabilization $\tau = \kappa/h_{\min}$: the number of iterations for multigrid as solver and preconditioner.
Table 4.18: Example IV. HDG with stabilizations $\tau = 1/h_{\text{min}}$ and $\tau = 1$: the number of iterations for multigrid preconditioned GMRES.

Table 4.19: Example IV. SIPG-H with $\tau = \kappa/(p+1)(p+2)/h_{\text{min}}$: the number of iterations for multigrid as solver and preconditioner.

for NIPG-H and IIPG-H are almost identical and are omitted. As can be observed, as a preconditioner for GMRES the multigrid algorithm is both $h$- and $p$-scalable for all the hybridized DG methods.

4.5. Example V: SPE10 test case. In this example, we consider the benchmark problem Model 2 from the Tenth Society of Petroleum Comparative Solution Project (SPE10) [17]. We consider the permeability field $K = \kappa I$, where $\kappa$ corresponding to the 75th layer is shown in left of Figure 4.1. The permeability field varies by six orders of magnitude and is highly heterogeneous which gives rise to extremely complex velocity fields. The domain is $1200 \times 2200 \text{ [ft]}^2$. The mesh has
60 × 220 quadrilateral elements and the element edges align with the discontinuities in the permeability. We choose \( f = 0 \), which corresponds to no source or sink. For the boundary conditions we take the pressure \( (q) \) on the left and right faces to be 1 and 0, respectively. On the top and bottom faces no flux boundary condition \( u \cdot n = 0 \) is applied. From extensive numerical examples we have observed that the multigrid hierarchy with seven levels results in the least number of GMRES iterations. From numerical examples I-IV, we see that HDG method is relatively the most robust and scalable. In addition, it provides simultaneous approximations for both velocity and pressure. Thus, we consider only HDG for this example.

Since multigrid as a solver either converges very slowly or diverges for a number of cases in this example, we report results exclusively for the multigrid preconditioned GMRES. The pressure field is shown in the right of Figure 4.1 for \( p = 1 \). In Table 4.20 are the number of iterations for solution orders \( p \in \{1, 2, 3, 4\} \) with \( \tau = 1/h_{\text{min}} \) and \( \tau = 1 \). The second row shows that, for \( p = 2 \), the iterations are much larger compared to other solution orders. By the time of writing, we have not yet found the reason for this behaviour. For other solution orders, using \( \tau = 1 \) results in more iterations for \( p = 1 \) and less for \( p = 3, 4 \) compared to \( \tau = 1/h_{\text{min}} \). Again, we like to emphasize that in this example we completely avoid upscaling of the permeability field as our multigrid algorithm is based only on the fine scale DtN maps.

The results in columns two and three of Table 4.20 correspond to geometrically increasing smoothing steps with respect to levels as explained in Section 3.3. In columns four and five, we simply take two pre- and post-smoothing steps in all levels and, as can be seen, the iteration counts are marginally different compared to those resulted from the increasing smoothing steps. This implies that we can achieve similar accuracy with less computational cost using constant (here two) pre- and post-smoothing steps in all levels.

Fig. 4.1: Example V: Permeability (\( \kappa \)) in log scale (left) and pressure field (right) using solution order \( p = 1 \).

4.6. Example VI: Multinumerics. The goal of this section is to demonstrate that the proposed multigrid algorithm can handle different fine scale DtN maps (multinumerics) on unstructured grids without explicit upscaling for macro-elements. In the following, we choose a combination of RT1 and NIPG as a representative for other
Table 4.20: Example V. HDG with stabilizations $\tau = 1/h_{\min}$, $\tau = 1$: the number of iterations for multigrid preconditioned GMRES with variable smoothing and constant smoothing.

We consider the following permeability field

$$K = (1 + 1/2 \sin(4\pi x) \cos(3\pi y)) I,$$

and $f = 1$. We consider unit square and use either RT1 mixed finite elements or first order NIPG discontinuous Galerkin as shown in Figure 4.2 (left). We plot the finest mesh in the left of Figure 4.2 and a coarsening strategy with 4 levels in Figure 4.3. We point out that (many or all) macro-elements on the coarser grids contain both RT1 and NIPG fine grid elements.

![Image](image_url)

Fig. 4.2: Example VI: On the left, NIPG-H elements are purple and RT1 elements are yellow. On the right are the number of iterations for the multigrid solver using multinumerics.

On the right of Figure 4.2 are the iteration counts using the multigrid solver. It can be observed that the performance, i.e. the number of iterations, using multinumerics is nearly identical to those for NIPG-H in Section 4.1 despite the fact that different numerical methods are used throughout the domain and the macro-elements are neither triangles nor quadrilaterals. This flexibility is one of the highlighted characteristics of our multigrid algorithm.

5. Conclusion. We have proposed a unified DtN-based geometric multigrid algorithm for hybridized high-order finite element methods. Our approach differs significantly from the previous attempts in the sense that the intergrid transfer operators are physics-based energy-preserving and the coarse grid operators are discretized DtN combinations of hybridized methods. We expect similar results using other combinations, as our multigrid algorithm operates on the trace system. Also we would like to point out that the other multigrid algorithms for hybridized methods [15, 19, 28, 30] may also work for the case of multinumerics. However, to the best of our knowledge we are not able to find such a study yet and it is beyond the current scope of this paper. In our future work we intend to compare these different algorithms along with our approach in the context of multinumerics.

We consider the following permeability field

$$K = (1 + 1/2 \sin(4\pi x) \cos(3\pi y)) I,$$

and $f = 1$. We consider unit square and use either RT1 mixed finite elements or first order NIPG discontinuous Galerkin as shown in Figure 4.2 (left). We plot the finest mesh in the left of Figure 4.2 and a coarsening strategy with 4 levels in Figure 4.3. We point out that (many or all) macro-elements on the coarser grids contain both RT1 and NIPG fine grid elements.

![Image](image_url)

Fig. 4.2: Example VI: On the left, NIPG-H elements are purple and RT1 elements are yellow. On the right are the number of iterations for the multigrid solver using multinumerics.

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![Image](image_url)

Fig. 4.2: Example VI: On the left, NIPG-H elements are purple and RT1 elements are yellow. On the right are the number of iterations for the multigrid solver using multinumerics.
maps on every level. These operators completely avoid upscaling of parameters, such as permeability and source, to coarse scales. As our numerical results have indicated, they also allow for multimodules and variable coefficients without deteriorating the performance of the multigrid algorithm. We have presented several numerical examples with hybridized discontinuous Galerkin (HDG) methods, hybridized interior penalty discontinuous Galerkin (IPDG) methods and reported several observations. For HDG methods, with stabilization \( \tau = 1 \) or \( 1/h_{\min} \) (depending on the example), we obtain almost perfect scalability in mesh size and solution order using multigrid preconditioned GMRES. For all the other hybridized IPDG methods, with a stabilization of the form \( \tau = O\left(p^2/h_{\min}\right) \) similar scalability is observed. However, the multigrid algorithm applied to IPDG seems to be less robust compared to HDG with respect to stabilization parameters and coarsening strategies especially for highly unstructured meshes. We have also reported scalable results for the proposed multigrid method when applying to multimodular discretizations with mixed methods in certain parts of the domain and DG methods in other parts. We are currently extending our multigrid algorithm to convection-diffusion, Stokes, Oseen and incompressible resistive magnetohydrodynamic (MHD) equations. Ongoing research is to provide a rigorous analysis of the proposed multigrid algorithm and we will report our findings elsewhere in the near future.

Acknowledgements. T. Wildey’s work is supported by the Office of Science Early Career Research Program. S. Muralikrishnan and T. Bui-Thanh are partially supported by the DOE grant DE-SC0018147 and NSF grant NSF-DMS1620352. We are grateful for the support. The authors would like to thank the anonymous referees for their critical and useful comments that significantly improved the paper.

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