Simple protocol for generating W states in resonator-based quantum computing architectures

Andrei Galiautdinov
Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602, USA

(Dated: May 5, 2014)

We describe a simple, practical scheme for generating multi-qubit W states in resonator-based architectures, in which N Josephson phase qubits are capacitively coupled to a common resonator bus. The entire control sequence consists of three pulses: a local Rabi pulse that excites a single qubit in the circuit; a coupling pulse that transfers the qubit excitation to the resonator bus; and the main, entangling operation that simultaneously couples the bus to all N qubits. If the qubit-resonator coupling strength g is much smaller than the qubit energy splitting \( E_{10} \), the system initially excited into the near-degenerate single-excitation subspace stays within that subspace, while smoothly evolving toward the fully uniform W state superposition. The duration of the final entangling operation is found to decrease with the total number of the qubits according to \( t = \pi/(2g\sqrt{N}) \), in agreement with some of the previously proposed cavity QED W state generation schemes.

I. DESCRIPTION OF THE W PROTOCOL

Our control sequence consists of the following three steps:

1. First, the initial local Rabi pulse is applied to one of the qubits in the circuit, bringing the qubit from its ground state \( |0\rangle \) to the excited state \( |1\rangle \),

\[
|00\ldots00_{r}\rangle \rightarrow |00\ldots01_{r}\rangle.
\]  

(1)

2. The corresponding qubit-resonator coupling is then turned on, which transfers the excitation state from the qubit to the bus,

\[
|00\ldots01_{r}\rangle \rightarrow |00\ldots001_{r}\rangle.
\]  

(2)

3. The second entangling pulse is applied, which couples the bus to all the qubits in the system. If the coupling g is much smaller than the qubit energy splitting \( E_{10} \), the system initially prepared in the near-degenerate single-excitation subspace stays within that subspace, while smoothly evolving toward the fully uniform W-state superposition. The duration of the final entangling operation is found to decrease with the total number of the qubits according to \( t = \pi/(2g\sqrt{N}) \), in agreement with some of the previously proposed cavity QED W state generation schemes.

This last step is similar to the W state generation scheme proposed for cavity QED in Ref. [3]. In the terminology of reference [2], our resonator bus plays the role of the entanglement mediator.

II. MATHEMATICAL PRELIMINARY

It is well-known how to perform the first two operations of the W sequence described above [3]. We can therefore assume that the circuit was initially prepared in the state \( |00\ldots00_{r}\rangle \), with only the bus excited. By simultaneously turning on the N couplings, the W state of the N-qubit network can then be generated using a single entangling operation (cf. [2]).

In order to see how this works, we consider a formal problem of an “effective” Hamiltonian,

\[
H^{(N+1)} = g \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},
\]  

(4)

which operates within a certain \( (N+1) \)-dimensional Hilbert space \( H^{(N+1)} \), whose physical significance will be clarified below. The spectrum of \( H^{(N+1)} \) is found to be

\[
E^{(N+1)} = \mp \sqrt{N}, 0, \ldots, 0.
\]  

(5)

The corresponding eigenvector matrix \( S^{(N+1)} \), which diagonalizes \( H^{(N+1)} \) via \( H^{(N+1)} = S^{(N+1)}H^{(N+1)}S^{(N+1)} \), is given by

\[
S^{(N+1)} \approx \begin{pmatrix} -\sqrt{N} & \sqrt{N} & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & -1 & -1 & -1 & \ldots & -1 \\ 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix},
\]  

(6)

where we left the columns of \( S^{(N+1)} \) unnormalized for notational simplicity. Direct exponentiation then shows

*Electronic address: ag@physast.uga.edu
that the $N$-dimensional uniform superposition state in this "effective" $(N+1)$-dimensional system can be generated via

$$
\frac{1}{\sqrt{N}} \begin{pmatrix}
0 \\
1 \\
1 \\
\vdots \\
1
\end{pmatrix} = i e^{-i H^{(N+1)} t^{(N)}} \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
$$

where

$$
t^{(N)} \equiv \frac{\pi}{2g \sqrt{N}}.
$$

III. $W_N$ STATE GENERATION

Our $W$ state generation scheme is based on the idea that the effective Hamiltonian $H^{(N+1)}$ considered above should be viewed as operating within the single-excitation subspace of a network consisting of $N$ qubits coupled to a common resonator bus. The corresponding mapping (extended by linearity) between the "effective" Hilbert space $H^{(N+1)}$ and the $(N+1)$-dimensional single-excitation subspace of the system may be chosen to be

$$
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix} \to |00\ldots01_r\rangle,
\begin{pmatrix}
0 \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \to |00\ldots010_r\rangle, \ldots
$$

The uniform $N$-dimensional superposition state generated in accordance with Eq. (22) then corresponds to the $W_N$ state of the $N$ qubits attached to the common bus.

In order for this approach to work, the single-excitation subspace has to be well isolated from the rest of the system’s Hilbert space. This near-degeneracy condition is typically well satisfied in various superconducting qubit architectures whose couplings, $g \simeq 100$ MHz, are much smaller than the qubit and resonator level splittings of $E_{10} \simeq 10$ GHz.

Let us check that the Hamiltonian $H^{(N+1)}$ arises naturally within the single-excitation subspace of a capacitively coupled network consisting of superconducting phase qubits and a resonator bus. When projected into the computational subspace spanned by the eigenfunctions $|0\rangle$, $|1\rangle$, $|2\rangle$ of the individual Josephson phase qubits (as well as the resonator), the Hamiltonian of such a network is given by

$$
H = \sum_{i=1}^{N} H_i + H_r + \sum_{i=1}^{N} g_{ir} p_i p_r,
$$

where the index $i$ numbers the qubits and $r$ labels the bus, with

$$
H_1 = \begin{pmatrix}
-E_{10} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & E_{10} - \Delta_1
\end{pmatrix},
$$

$$
H_j = \begin{pmatrix}
-E_{10} & 0 & 0 \\
0 & \epsilon_j & 0 \\
0 & 0 & E_{10} + 2\epsilon_j - \Delta_j
\end{pmatrix}, \quad j = 2, \ldots, N,
$$

$$
H_r = \begin{pmatrix}
-E_r & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & E_r + 2\epsilon_r
\end{pmatrix}.
$$

The generalized momenta $p_i$ and $p_r$ are given by

$$
p_i = \lambda_2 + b\lambda_5 + c\lambda_7 = i \begin{pmatrix}
0 & -1 & b_i \\
1 & 0 & -c_i \\
b_i & c_i & 0
\end{pmatrix},
p_r = \lambda_2 + \sqrt{2}\lambda_7 = i \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -\sqrt{2} \\
0 & \sqrt{2} & 0
\end{pmatrix},
$$

where $\lambda_k$, $k = 1, 2, \ldots, 8$, are the standard Gell-Mann generators of the Lie algebra $su(3)$. In the above, $g_{ir}$ are the qubit-bus coupling constants, $E_{10}$ is the energy splitting of the first (reference) qubit, $\epsilon_j$, $\epsilon_r$, $j = 2, 3, \ldots, N$, are the energy shifts relative to the single-excitation energy of the first qubit, $\Delta_i$, $i = 1, 2, \ldots, N$, are the qubit anharmonicities, $b_i$ and $c_i$ are the off-diagonal matrix elements of the $i$th qubit momentum, and $E_r$ is the resonator energy splitting. The $(N+1) \times (N+1)$ block of the Hamiltonian $H$ acting within the $(N+1)$-dimensional single-excitation subspace spanned by $|00\ldots01_r\rangle$, $|00\ldots010_r\rangle, \ldots, |10\ldots000_r\rangle$, is then given by the real symmetric matrix,

$$
H^{(N+1)} = \begin{pmatrix}
\epsilon_r & g_{Nr} & g_{N-1r} & g_{N-2r} & \cdots & g_{3r} & g_{2r} & g_{1r} \\
g_{Nr} & \epsilon_N & 0 & 0 & \cdots & 0 & 0 & 0 \\
g_{N-1r} & 0 & \epsilon_{N-1} & 0 & \cdots & 0 & 0 & 0 \\
g_{N-2r} & 0 & 0 & \epsilon_{N-2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g_{2r} & 0 & 0 & 0 & \cdots & 0 & \epsilon_2 & 0 \\
g_{1r} & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}.
$$

This immediately shows that the $W_N$ state can be generated if we place all system elements on resonance with each other by choosing

$$
ge_2 = e_3 = e_4 = \cdots = e_N = e_r = 0,
ge_{1r} = g_{2r} = g_{3r} = \cdots = g_{Nr} = g.
$$
IV. NUMERICAL SIMULATION RESULTS

We tested this scheme on an $N = 4$ qubit network with $g = 100$ MHz, $E_{10} = E_r = 10$ GHz, $\Delta_j = 250$ MHz, and

$$p_i = i \begin{pmatrix} 0 & -1 & -0.08 \\ 1 & 0 & -1.43 \\ 0.08 & 1.43 & 0 \end{pmatrix}. \quad (15)$$

Assuming the system starts in the excited state $|00\ldots001_{r}\rangle$, the simulated final state of the system is found to be

$$|W_N\rangle_{\text{sim}} = \begin{pmatrix} -0.0003i \\ 0.4999 \\ 0.4999 \\ 0.4999 \end{pmatrix}, \quad (16)$$

with the corresponding entangling time being $t = 1.2500$ ns. Ignoring the decoherence effects, the intrinsic fidelity $\mathcal{F}$ of the found state $|W_N\rangle_{\text{sim}}$ relative to the ideal $W_N$ state, is

$$\mathcal{F}_N = |\langle W_N | W_N\rangle_{\text{sim}}|^2 = 0.9994. \quad (17)$$

V. $W_{N+1}$ STATE GENERATION

In a similar manner, the $W_{N+1}$ state can also be generated, in which the resonator is maximally entangled with the qubits. This corresponds to the sequence of operations

$$|00\ldots000_{r}\rangle \to |00\ldots010_{r}\rangle \to$$

$$|00\ldots001_{r}\rangle + |00\ldots010_{r}\rangle + \cdots + |10\ldots000_{r}\rangle$$

$$\sqrt{N + 1}. \quad (18)$$

In this scenario, we take full advantage of qubit tunability to construct the “effective” single-excitation Hamiltonian of the form

$$H^{(N+1)} = g \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \quad (19)$$

Its spectrum and the diagonalizing transformation matrix $S^{(N+1)}$ (here shown unnormalized) are

$$E^{(N+1)} = 1 \mp \sqrt{N + 1}, 0, \ldots, 0, \quad (20)$$

and

$$\begin{pmatrix} 1 - \sqrt{N + 1} & 1 + \sqrt{N + 1} & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & -1 & -1 & -1 & \ldots & -1 \\ 1 & 1 & 0 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad (21)$$

respectively. The corresponding $W$ state is generated via

$$\frac{1}{\sqrt{N + 1}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} = i e^{i \alpha^{(N+1)} e^{-i H^{(N+1)} t^{(N+1)}}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \ldots \end{pmatrix}, \quad (22)$$

where

$$\alpha^{(N+1)} = \pi \frac{2}{2\sqrt{N + 1}}, \quad t^{(N+1)} = \frac{\pi}{2g\sqrt{N + 1}}, \quad (23)$$

provided we detune the qubits from the resonator by $2g$,

$$\epsilon_2 = \epsilon_3 = \epsilon_4 = \cdots = \epsilon_N = 0, \quad \epsilon_r = 2g, \quad g_{1r} = g_{2r} = g_{3r} = \cdots = g_{Nr} = g. \quad (24)$$

The duration of the entangling pulse is now $t = 1.1180$ ns, with the simulated single-excitation final state of the network being

$$|W_{N+1}\rangle_{\text{sim}} = \begin{pmatrix} 0.4472 - 0.0003i \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{pmatrix}, \quad (25)$$

with fidelity

$$\mathcal{F}_{N+1} = 0.9997. \quad (26)$$

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