Reentrant phase transitions and van der Waals behaviour for hairy black holes

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Black hole chemistry

- There is a close relationship between the laws of **thermodynamics** and the laws of **black hole mechanics** [Hawking et al; 1973]

\[ dE = TdS - PdV + \text{work terms} \quad \leftrightarrow \quad dM = \left( \frac{\kappa}{8\pi} \right) dA + \text{work terms} \]
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- Geometric and scaling arguments suggest that the cosmological constant, \( \Lambda \), should be considered as a thermodynamic parameter in the first law [Kastor et al; 0904.2765],

\[ dM = TdS + VdP + \sum_{i} \Omega_i dJ_i + \Phi dQ \]

where

\[ P = -\frac{\Lambda}{8\pi} = \frac{(d - 1)(d - 2)}{16\pi\ell^2} \quad (\text{here } \Lambda < 0) \]

⇒ mass is enthalpy.

- Why? The first law and Smarr formula are related via Eulerian scaling

\[ (d - 3)M = (d - 2)TS - 2VP + (d - 2)\sum_{i} \Omega_i J_i + (d - 3)\Phi Q \]
Black hole chemistry: results

- Can write black hole equations of state which lead to precise physical analogies with thermal systems: van der Waals behaviour, triple points, (multiple)-reentrant phase transitions [Mann, Kubiznak; 1401.2586].

- Thermodynamically inspired black hole solutions [Mann, Kubiznak; 1408.1105].

- Here we study the “chemistry” of AdS black holes with conformal scalar hair for the first time [Hennigar, Mann; 1509.06798].

Figure: Gibbs free energy for the charged AdS black hole [1205.0559].
\[ I = \int d^d x \sqrt{-g} \left[ \frac{1}{2\kappa} [R - 2\Lambda] - \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \frac{d - 2}{4(d - 1)} R\phi^2 \right] \]

- For Einstein gravity conformally coupled to a scalar field in asymptotically flat spacetime there exist well-known no-hair theorems. “Black holes have no hair.”
- In the presence of a cosmological constant, the no hair theorems can be evaded. Hairy black holes with conformal scalar hair are known to exist in \( d = 3 \) and \( d = 4 \) [Martinez et al; hep-th/0205319, hep-th/0406111].
- In \( d > 4 \) no go results had been reported \( \Rightarrow \) a new approach required. Idea: couple the scalar field to dimensionally extended Euler densities [Oliva, Ray; 1112.4112].
An aside: Lovelock gravity

- The focus here will be on a special class of hairy black holes where the scalar field is conformally coupled to higher curvature terms.
- Higher curvature gravity modifies the standard Einstein-Hilbert action through the addition of higher curvature terms,

\[ \mathcal{I} = \int d^d x \sqrt{g} \left( c_0 + c_1 R + c_2 \mathcal{L}(R^2) + c_3 \mathcal{L}(R^3) + \cdots \right) \]

- Generically, bad things will happen e.g. the field equations will no longer be second order differential equations.
- Lovelock gravity is the most general higher curvature theory of gravity which maintains second order field equations,

\[ \mathcal{L}^{(k)} = \frac{2k!}{2^k} \delta^{a_1}_{c_1} \delta^{b_1}_{d_1} \cdots \delta^{a_k}_{c_k} \delta^{b_k}_{d_k} R_{a_1 b_1} \cdots R_{a_k b_k} c_1 d_1 \cdots c_k d_k \]

- Appear in perturbative approaches to quantizing gravity.
Hairy black holes

- Design a tensor,
  \[ S_{\mu\nu} \gamma^\delta = \phi^2 R_{\mu\nu} \gamma^\delta + \text{necessary terms} \]
  which transforms as \( S_{\mu\nu} \gamma^\delta \rightarrow \Omega^{-8/3} S_{\mu\nu} \gamma^\delta \) when \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \) and \( \phi \rightarrow \Omega^{-1/3} \phi \).

- Can build terms where the scalar field is conformally coupled to the Euler densities [Oliva et al; 1112.4112, 1508.03780],
  \[ \mathcal{L}^{(k)}(\phi, \nabla \phi) = b_k \frac{2k!}{2^k} \phi^{3d-8k} \delta^{a_1}_{[c_1} \delta^{b_1}_{d_1} \cdots \delta^{a_k}_{c_k} \delta^{b_k}_{d_k}] S_{a_1 b_1 c_1 d_1} \cdots S_{a_k b_k c_k d_k} \]

- Here [Hennigar, Mann; 1509.06798] we consider solutions of the theory,
  \[ \mathcal{I} = \frac{1}{\kappa} \int d^5x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F^2 + \kappa \left( b_0 \phi^{15} + b_1 \phi^7 S_{\mu\nu}^{\mu\nu} 
  + b_2 \phi^{-1} (S_{\mu\gamma} \mu\gamma S_{\nu\delta}^{\nu\delta} - 4S_{\mu\gamma}^{\nu\gamma} S_{\nu\delta}^{\mu\delta} + S_{\mu\nu} \gamma^\delta S^{\nu\mu} \gamma^\delta) \right) \right] \]
Metric & properties

- The metric is given by

\[ ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Sigma_{(k)}^2, \quad f = k - \frac{m}{r^2} - \frac{q}{r^3} + \frac{e^2}{r^4} + \frac{r^2}{\ell^2} \]

with \( k = -1, 0, 1 \) while

\[ q = \frac{64\pi}{5} k b_1 n^9, \quad n = \epsilon \left( -\frac{18kb_1}{5b_0} \right)^{1/6}, \quad \phi = \frac{n}{r^{1/3}}, \quad A = \frac{\sqrt{3}e}{r^2} dt, \quad F = dA, \]

here \( \epsilon = -1, 0, 1 \). The couplings have to obey the constraint \( 10b_0 b_2 = 9b_1^2 \).

- Thermodynamic quantities which satisfy the first law & Smarr formula,

\[
M = \frac{3\omega_3(k)}{16\pi} m, \quad Q = -\frac{\sqrt{3}\omega_3(k)}{16\pi} e, \quad S = \omega_3(k) \left( \frac{r_+^3}{4} - \frac{5}{8} q \right), \quad V = \frac{\omega_3(k)}{4} r_+^4
\]

\[
T = \frac{1}{\pi \ell^2 r_+^4} \left[ r_+^5 + \frac{k \ell^2 r_+^3}{2} + \frac{q \ell^2}{4} - \frac{e^2 \ell^2}{2r_+} \right], \quad \Phi = -\frac{2\sqrt{3}}{r_+^2} e
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Metric & properties

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\[ T = \frac{1}{\pi\ell^2 r_+^4} \left[ r_+^5 + \frac{k\ell^2 r_+^3}{2} + \frac{q\ell^2}{4} - \frac{e^2\ell^2}{2r_+} \right], \quad \Phi = -\frac{2\sqrt{3}}{r_+^2} e \]
The equation of state is,

\[ P = \frac{T}{v} - \frac{2k}{3\pi v^2} + \frac{512}{243 \pi v^6} - \frac{64}{81 \pi v^5} \]

where the specific volume is \( \nu = 4r_+/3 \).

The Gibbs free energy is,

\[ G = M - TS = \omega_{3(k)} \left[ \frac{9k
\nu^2}{256\pi} - \frac{27\nu^4 P}{1024} + \frac{40q^2}{81\pi \nu^4} + \left( \frac{5P\nu}{8} + \frac{5k - 4}{12\pi \nu} \right) q \right. \\
+ \left. \left( \frac{5}{9\pi \nu^2} - \frac{320q}{243\pi \nu^5} \right) e^2 \right] . \]
Results I: van der Waals behaviour \((k = 1, e = 0, q < 0)\)

- For \(e = 0, q < 0\) and \(k = 1\) there is a single critical point,

\[
T_c = -\frac{3}{20\pi} \frac{(-5q)^{2/3}}{q}, \quad \nu_c = \frac{4}{3}(-5q)^{1/3}, \quad P_c = \frac{9}{200\pi} \left( -\frac{\sqrt{5}}{q} \right)^{2/3}
\]

with mean field theory critical exponents, \(\alpha = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3\), \(P_c\nu_c/T_c = 2/5\).

Figure: \(q = -1\): \(G\) vs. \(T\), \(P\) vs. \(\nu\) and \(P\) vs. \(T\) (left to right). red ⇒ less than critical value, black ⇒ critical value, blue ⇒ greater than critical value.
Results II: reentrant phase transition \((k = 1, \ e = q = 1)\)

**Figure:** Gibbs free energy: \(k = 1, \ e = q = 1\). Pressures \(P = 0.031, 0.0313,\) and \(0.032\) (left to right). The dashed black line corresponds to parameters yielding negative entropy. We see a large/small/large BH reentrant phase transition.

- **Reentrant phase transition:** A monotonic variation of a thermodynamic parameter yields two (or more) phase transitions, with the final state macroscopically similar to the initial state.
Results II: reentrant phase transition \( (k = 1, e = q = 1) \)

Figure: Coexistence plots: \( k = 1, e = q = 1 \). **Left:** \( P - T \) coexistence plot showing zeroth and first order phase transitions (red and black curves, respectively). To the left of the blue line the black holes have negative entropy. **Right:** Zoomed version of the left plot.
Results III: zero entropy limit

\[ S = \omega_3(k) \left( \frac{r^3}{4} - \frac{5}{8} q \right) \]

- Expanding the equation of state near the critical point in terms of 
  \[ \omega = \frac{\nu}{\nu_c} - 1, \rho = \frac{P}{P_c} - 1 \text{ and } \tau = \frac{T}{T_c} - 1 \] 
  gives

\[ \rho \approx A\tau - B\omega\tau - C\omega^3 \]

with \( A, B, C > 0 \). Solving for \( \omega \) there are three real solutions:

\[ \omega_1 = \frac{2}{3} \sqrt{-\frac{3B\tau}{C}} \quad \omega_2 = \omega_3 = -\frac{1}{3} \sqrt{-\frac{3B\tau}{C}} \]

- Then the entropy is,

\[ q = \frac{27}{160} \nu_c^3 \Rightarrow S_i = \frac{27\pi^2 \nu_c^3}{128} ((\omega_i + 1)^3 - 1) \]

- Only one branch has positive entropy, so there are no phase transitions.
Results IV: summary

- No interesting results in the $k = 0, -1$ cases.

| Case summary for $k = +1$ | $q > 0$ | $q < 0$ |
|---------------------------|--------|--------|
| $e = 0$                   | no criticality | van der Waals |
| $e \neq 0$                | reentrant phase transition | van der Waals |

**Table:** $q$ is the hair parameter; $e$ is electric charge.

- Criticality ceases in the zero entropy limit.
Conclusions

- We considered, for the first time, the extended phase space thermodynamics of black holes with conformal scalar hair.
- In the hyperbolic and flat cases, there were no interesting results.
- In the spherical case, we see van der Waals behaviour and reentrant phase transitions
- In the case of zero entropy, the criticality ceases.

Thank you!
Example of “no criticality” behaviour

**Figure:** Uncharged case: Gibbs free energy: \( k = 1, q = 1 \). A representative \( G \) vs. \( T \) plot for \( P = 0.05 \) and \( q = 1 \). The red lines represent physical branches of the Gibbs energy, while the dashed black line corresponds to negative entropy black holes. The Gibbs free energy displays a cusp, the upper branch of which terminates at finite temperature due to enforcing positivity of entropy. At temperatures below the cusp, no black hole solutions exist.
Hairy black holes

- The tensor,

\[ S_{\mu\nu}^{\gamma\delta} = \phi^2 R_{\mu\nu}^{\gamma\delta} - 12\delta^{[\gamma}_{[\mu} \delta^{\delta]}_{\nu]} \nabla_{\rho} \phi \nabla^{\rho} \phi - 48\phi \delta^{[\gamma}_{[\mu} \nabla_{\nu]} \nabla^{\delta]} \phi + 18\delta^{[\gamma}_{[\mu} \nabla_{\nu]} \phi \nabla^{\delta]} \phi \]

transforms as \( S_{\mu\nu}^{\gamma\delta} \rightarrow \Omega^{-8/3} S_{\mu\nu}^{\gamma\delta} \) when \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \) and \( \phi \rightarrow \Omega^{-1/3} \phi \).

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Results III: zero entropy limit

Figure: $k = 1$: $q$-$e$ parameter space: blue dots indicate physical critical points, green dots indicate unphysical critical points (negative entropy). Black line: $q \approx 1.3375e^{3/2}$. 