On the Approximation of Quantum Gates using Lattices

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Abstract

A central question in Quantum Computing is how matrices in $SU(2)$ can be approximated by products over a small set of ”generators”. A topology will be defined on $SU(2)$ so as to introduce the notion of a covering exponent $[11]$, which compares the length of products required to covering $SU(2)$ with $\varepsilon$ balls against the Haar measure of $\varepsilon$ balls. An efficient universal set over $PSU(2)$ will be constructed using the Pauli matrices, using the metric of the covering exponent. Then, the relationship between $SU(2)$ and $S^3$ will be manipulated to correlate angles between points on $S^3$ and give a conjecture on the maximum of angles between points on a lattice. It will be shown how this conjecture can be used to compute the covering exponent, and how it can be generalized to universal sets in $SU(2)$.

1 Introduction

A classical bit is the basic unit of information used in classical computing, which has the states 1 or 0. Quantum computing extends this concept using the notion of Quantum bits. Dirac notation is used to denote the basic states $|0\rangle$ and $|1\rangle$. Then a quantum bit, or qubit, is a pair of complex numbers $\alpha, \beta$ which correspond to the probability of the qubit being in the states $|0\rangle$ or $|1\rangle$. Thus, the quantum bit is represented as $\alpha|0\rangle + \beta|1\rangle$. Since $\alpha$ and $\beta$ represent the probability of the qubit being in a particular state, it must hold that $|\alpha|^2 + |\beta|^2 = 1$. Therefore, qubits can be represented by unit vectors in $\mathbb{C}^2$.

This construction can be compounded to construct $n$-qubits, which are ordered collections of $n$-qubits. An $n$-qubit relates to the probability of $n$ different qubits are in a particular configuration. Therefore, any $n$-qubit is thus taken to be the tensor product of these qubits,

$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes \cdots \otimes (\alpha_n|0\rangle + \beta_n|1\rangle)$$

As mentioned above 1-qubits form the unit circle in $\mathbb{C}^2$, and it follows that $n$-qubits form vectors in $(\mathbb{C}^2)^\otimes n$. In classical computers, gates are linear operators over classical bits. Examples of classical gates include the AND, OR, and NOT gates. Thus, a quantum gate follows naturally as a linear function over the vector space $\mathbb{C}^{2^n}$. However, a prime function of quantum gates is that they are reversible. That is they are invertible, and more specifically they are unitary.

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Since each 1-qubit is a unit vector, then 1-qubit quantum gates should take unit vectors to unit vectors. Thus, quantum gates are taken to have determinant 1. Let $SU(2)$ denote the collection of $2 \times 2$ Hermitian matrices with determinant 1. Then, $n$-qubit gates can be formed by tensoring 1-qubit matrices with the Controlled NOT gate:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, questions that are harder to answer for $n$-qubit gates can be extrapolated from the answer over 1-qubit gates \[12\].

The main goal of quantum computing is obviously to build a quantum computer. In order for a quantum computer to be constructed, a finite base set of quantum gates must be chosen so that they generate $SU(2)$. However, it turns out that it is not practical to consider this problem. See the papers \[1\] \[6\] \[7\] \[13\] \[14\] and the references cited therein for examples of coverings of compact sets in Euclidean space, as well as some of their applications. Thus, gate sets are constructed so that the elements they generate can approximate any quantum gate. Defining how a gate approximates another gate is half the battle, which usually entails constructing a metric-induced topology on $SU(2)$, but in general gate sets which generate dense subsets of $SU(2)$ are chosen. In a given context, a set which generates a dense subset of $SU(2)$ is referred to as a universal set in $SU(2)$, and by definition gives that it can approximate any gate in $SU(2)$ with arbitrary precision according to the chosen measure of approximation. However, as in all computing, the question becomes how to choose efficient universal sets to approximate elements in $SU(2)$. Efficient meaning that it requires the least amount of matrices (or some generalized notion of cost) to approximate all elements of $SU(2)$.

In this paper, the goal is to construct a universal gate set $T$ in $SU(2)$ that efficiently approximates all of $SU(2)$ using a natural and simple notion of distance. A quantity called the covering exponent given by \[11\] will be used to measure the efficiency of a gate set in approximating every element of $SU(2)$. In general, $T$ will be constructed to minimize the maximal cost of approximating any gate. Since quantum gates are very similar up to scalars, it is also useful to consider how $T$ approximates the subset $PSU(2) \subset SU(2)$ (the equivalence classes of $SU(2)$ under multiplication by $-1$). It will be shown that $T$ can efficiently approximate $PSU(2)$, but does not quite efficiently approximate $SU(2)$.

## 2 Background

As shown in \[12\], gates such as the controlled NOT gate can be tensored with 1-qubit gates without much cost to approximate 2-qubit gates very well. Thus, a universal gate set on $SU(2)$ can easily be extended to a universal gate set on $SU(2^n)$. Furthermore, gates do not vary greatly up to constants. Thus, it is often convenient to study approximation over $PSU(2)$. The projective special linear group, $PSU(n)$, is defined as $SU(n)/Z(SU(n))$. For $n = 2$, $Z(SU(2)) \cong \mathbb{Z}/2\mathbb{Z}$. Another reason to use the case of 1-qubit quantum gates is this property, which gives that $PSU(2) \cong SU(2)/\{I, -I\}$. Additionally, it is useful that $PSU(2) \cong SO(3)$. Thus, the choice may not be largely important in a given application, but the choice must be consistent in order for the math to work. For constructions not dependent on the choice, $G$ will be used to represent either $SU(2)$ and $PSU(2)$. It will be apparent from the context, and reiterated when necessary, which choice is being used.
2.1 Structure of $SU(2)$

It is an elementary fact that any element $M \in SU(2)$ can be written in terms of $\alpha, \beta \in \mathbb{C}$ as

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

Thus, $M$ can be associated with some vector $(x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. In turns out, that the map $M \mapsto (x_1, x_2, x_3, x_4)$ is a diffeomorphism. Note that

$$\det M = \alpha \bar{\alpha} + \beta \bar{\beta} = |\alpha|^2 + |\beta|^2 = 1$$

This relation allows sets in $SU(2)$ to be related to sets on $S^3$. It is a powerful tool in computing the efficiency of universal sets of $SU(2)$. However, unlike $S^3$, $SU(2)$ does not have a standard topology by convention. Before notions of universality and closeness can be used, $SU(2)$ must be set up as a metric space with an induced topology. Define the distance between two matrices $M, N$ as

$$d_G(M, N) = \sqrt{1 - \frac{|Tr(M^\dagger N)|}{2}}$$

(1)

where $M^\dagger$ represents the conjugate transpose of $M$. Let $M, N, P \in SU(2)$. Most of the conditions for a metric are straightforwardly derivative of basic properties from the trace function and $SU(2)$. More interestingly, it is invariant under left and right multiplication as shown below

$$d_G(PM, PN) = \sqrt{1 - \frac{|Tr((PM)^\dagger(PN))|}{2}}$$

$$= \sqrt{1 - \frac{|Tr(M^\dagger P^\dagger PN)|}{2}}$$

$$= \sqrt{1 - \frac{|Tr(M^\dagger N)|}{2}}$$

$$= d_G(M, N)$$

$$d_G(MP, NP) = \sqrt{1 - \frac{|Tr((MP)^\dagger(NP))|}{2}}$$

$$= \sqrt{1 - \frac{|Tr((NP)(MP)^\dagger)|}{2}}$$

$$= \sqrt{1 - \frac{|Tr(NPP^\dagger M^\dagger)|}{2}}$$

$$= \sqrt{1 - \frac{|Tr(NM^\dagger)|}{2}}$$

$$= \sqrt{1 - \frac{|Tr(M^\dagger N)|}{2}}$$

$$= d_G(M, N)$$

Thus, $d_G(MN, N) = d_G(M, I)$. That implies that a matrix $M$ acting on $N$ can only move it as far as $d_G(M, I)$. This is convenient since

$$d_G(M, M) = d_G(I, I) = \sqrt{1 - \frac{|Tr(I^\dagger I)|}{2}} = \sqrt{1 - \frac{2}{2}} = 0$$
With this metric, then there is an induced topology from the metric space \((G, d_G)\) using balls as open sets. A Haar measure on \(G\) is a measure \(\mu : G \to \mathbb{R}_{>0}\) so that \(\mu(G) = 1\) and \(\mu(MS) = \mu(SM) = \mu(S)\) where \(M \in G\) and \(S \subset G\) is a Borel subset of \(G\). Then for \(M \in G\) and \(\varepsilon > 0\), the size of a ball \(B_G(M, \varepsilon)\) will be evaluated as \(\mu(B_G(M, \varepsilon))\). Thus, every time \(G\) is referenced, the measure space \((G, d_G, \mu)\) will be the object being used.

2.2 Universal Sets

Let \(\Gamma\) be a finite subset of \(G\). The set \(\Gamma\) is said to be universal in \(G\), with respect to a chosen topology, if the subgroup of \(G\) generated by \(\Gamma\) is dense. If \(\Gamma\) is not universal, then there will be open balls that contain no elements generated by \(\Gamma\). A well known theorem cited in [4] expands on the importance of universal sets.

**Theorem 2.1.** (Solovay-Kitaev) Let \(\Gamma\) be a finite universal set in \(SU(n)\) and \(\varepsilon > 0\). Then there exists a constant \(c\) such that for any \(M \in SU(n)\), there is a finite product \(S\) of gates in \(\Gamma\) of length \(O(\log^c \left( \frac{1}{\varepsilon} \right))\) such that \(d_G(S, M) < \varepsilon\).

Universality of \(\Gamma\) gives that any one matrix can be approximated with arbitrary precision. Theorem 2.1 gives that \(\Gamma\) can approximate \(SU(n)\) with arbitrary efficiency and provides a estimation for the maximum length required to achieve this approximation. This theorem provides justification for studying the efficiency of universal gate sets in approximating all of \(SU(2)\), instead of specific matrices. As computers are not typically constructed to perform single calculations, this is much more useful.

To consider the efficiency of a universal set, first the idea of cost must be developed. In this paper, the notion of height from [11] will be used. Let \(w\) be a weight function on \(\Gamma\). Then \(\forall \gamma \in \langle \Gamma \rangle\) define the height of \(\gamma\) in \(\Gamma\) as

\[
h(\gamma) = \min \left\{ \sum_i w(c_i) : c_i \in \Gamma, \gamma = \prod c_i \right\}
\]

(2)

Note that this notion of height is heavily dependent on the choice of \(w\). Thus all results should be taken into the context of the choice of weight, and that all weights have good motivation behind them. Given a choice of weight, then define the following sets for \(t > 0\)

\[
U_\Gamma(t) = \{ \gamma \in \langle \Gamma \rangle : h(\gamma) = t \}
\]

\[
V_\Gamma(t) = \{ \gamma \in \langle \Gamma \rangle : h(\gamma) \leq t \}
\]

Thus, if one is continuously taking products in \(\Gamma\), then \(U_\Gamma(t)\) are the gates added after the \(t\)th product and \(V_\Gamma(t)\) are the gates that have been generated after \(t\) products. Thus, \(U_\Gamma(t - 1)\) and \(U_\Gamma(t)\) are disjoint, which gives a useful identity relating the two:

\[
V_\Gamma(t) = \bigsqcup_{0 \leq k \leq t} U_\Gamma(k)
\]

(3)

Let \(\varepsilon > 0\). Define the covering length of \(\Gamma\) within \(\varepsilon\), denoted \(t_\varepsilon\) as in [11], as follows

\[
t_\varepsilon = \min \left\{ t \in \mathbb{N} : G \subset \bigcup_{\gamma \in V_\Gamma(t)} B_G(\varepsilon) \right\}
\]

(4)
The calculation of \( t_\varepsilon \) is the ultimate prize. Especially, if it can be computed or even bounded as a function of \( \varepsilon \), then \( t_\varepsilon \) can provide an explicit measure of how much cost it takes to approximate \( SU(2) \). However, it doesn’t quite give the whole picture. For one, comparing the covering lengths of universal sets is complicated. It is within the realm of reason that perhaps \( t_\varepsilon \) does not grow uniformly or otherwise behaves pathologically (although at a minimum non-decreasing), which could complicate comparisons.

### 2.3 Covering Exponent

Let \( \Gamma \) be a universal set in \( G \), and \( \varepsilon > 0 \). Per the definition of a Haar measure, for any \( t > 0 \) such that

\[
G \subseteq \bigcup_{\gamma \in \mathcal{V}_G(t)} B_G(\varepsilon)
\]

it follows

\[
\mu \left( \bigcup_{\gamma \in \mathcal{V}_G(t)} B_G(\gamma) \right) \geq \mu(G) = 1
\]

Then by construction \( t_\varepsilon \) minimizes this gap. Let \( B_G(\varepsilon) \) denote \( B_G(I, \varepsilon) \). Now, breaking the left hand side down,

\[
\mu \left( \bigcup_{\gamma \in \mathcal{V}_G(t_\varepsilon)} B_G(\varepsilon) \right) = \sum_{\gamma \in \mathcal{V}_G(t_\varepsilon)} \mu(B_G(\gamma, \varepsilon)) = \sum_{\gamma \in \mathcal{V}_G(t_\varepsilon)} \mu(B_G(I, \varepsilon)) = |\mathcal{V}_G(t_\varepsilon)| \mu(B_G(\varepsilon))
\]

Thus, substituting this last form into the inequality,

\[
|\mathcal{V}_G(t_\varepsilon)| \mu(B_G(\varepsilon)) \geq 1
\]

If \( \Gamma \) approximates \( G \) optimally, then inequality (5) becomes an equality. In general, as \( |\mathcal{V}_G(t_\varepsilon)| \) becomes close to \( \frac{1}{\mu(B_G(\varepsilon))} \), the overlap between the balls centered at points in \( \mathcal{V}_G(t_\varepsilon) \) is minimized. Thus, \( \Gamma \) becomes more efficient at approximating \( G \). For a universal set \( \Gamma \) in \( G \) and a Haar measure \( \mu \) on \( G \), the covering exponent as given as in [11] is defined as

\[
K(\Gamma) = \limsup_{\varepsilon \to 0} \frac{\log |\mathcal{V}_G(t_\varepsilon)|}{\log \left( \frac{1}{\mu(B_G(\varepsilon))} \right)}
\]

Note that \( K \) is heavily dependent on \( t_\varepsilon \), and does vary with a choice of \( G \). The second part is to be expected, \( PSU(2) \) is almost half the elements of \( SU(2) \) and should typically be easier to generate. The dependence of \( K \) on \( t_\varepsilon \) is more convenient than impeding, as it sticks close to the original idea of directly comparing lengths of products to measure efficiency. The covering exponent will be used in this paper to measure and construct efficient universal sets in \( PSU(2) \) and \( SU(2) \).
3 An Efficient Universal Set in PSU(2)

What makes a universal set optimal, or even efficient in approximating SU(2)? There are many theories and methods behind this question, however the angle taken here will be that of well distributed points on the sphere. There are many suitable choices for these points, as explored in [9][7]. However, the one that will be explored is a solution set to the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5k$ for different integers $k > 0$. These points are fairly evenly distributed, but conveniently have a very simple structure. This allows the calculations for $K(T)$ to be simplified using a handful of results. The preliminary result will mirror the analysis of a similar set in [11], which takes the form as the following theorem

**Theorem 3.1.** $K(T) \leq 2$

Then a conjecture is proposed, which would improve this upper bound. However, the resulting theorem is a little less set in stone, taking the form as the following theorem.

**Theorem 3.2.** For any $\delta > 0$ such that Conjecture 3.4 holds, $K(T) \leq 2 - \delta$

3.1 Construction of $T$

To construct an efficient universal set, lattices in $\mathbb{R}^4$ will be projected onto $S^3$ and then related to quantum gates. To do this, some additional framework specific to this construction is needed. First, for any set $S \subset \mathbb{R}$ let

$$H(S) = \{a + bi + cj + dk : a, b, c, d \in S\}$$

be the set of quaternions with coefficients in $S$. Define the map

$$\Phi : SU(2) \rightarrow H(\mathbb{R})$$

$$\begin{bmatrix} \alpha \\ -\beta \\ \alpha \\ \beta \end{bmatrix} \mapsto \alpha + \beta j$$

Note that $\Phi$ forms an injective homomorphism, as

$$\Phi(MN) = \Phi \left( \begin{bmatrix} \alpha_M \alpha_N - \beta_M \beta_N & \alpha_M \beta_N + \beta_M \alpha_N \\ -\beta_M \alpha_N - \alpha_M \beta_N & -\beta_M \beta_N + \alpha_M \alpha_N \end{bmatrix} \right)$$

$$= \alpha_M \alpha_N - \beta_M \beta_N + (\alpha_M \beta_N + \beta_M \alpha_N) j$$

$$= (\alpha_M + \beta_M j)(\alpha_N + \beta_N j)$$

$$= \Phi(M)\Phi(N)$$

To construct the universal set, consider integer quaternion factors of the integer 5. Listed out, they are

$$1 \pm 2i, 1 \pm 2j, 1 \pm 2k, 2 \pm i, 2 \pm j, 2 \pm k, 5$$

Note that,

$$2 + i = (1 - 2i)i$$

Thus, the factors of 5 can be generated by

$$1 + 2i, 1 + 2j, 1 + 2k, 1 - 2i, 1 - 2j, 1 - 2k, i, j, k$$
Let $T = \Phi^{-1}(\{1 + 2i, 1 + 2j, 1 + 2k, 1 - 2i, 1 - 2j, 1 - 2k, i, j, k\})$. Then the space spanned by $T$ consists of quaternion factorizations of $5^k$ for all $k \in \mathbb{Z}$. Products of length $k$ will have a Euclidean norm of $5^k$. Thus, the factorizations of $5^k$ correspond exactly to representations of $5^k$ as a sum of 4 squares. Moreover, any factorization of $5^k$ is represented as a factorization of $5^{k+1}$ by adding a factor of 5 to the beginning. Thus, up to factors of 5, the collection of factorizations for $5^k$ contains all factorizations of $5^j$ for any $j < k$. Define the weight $w$ on $T$ so that

$$w(A) = \begin{cases} 
1 & A = i, j, k \\
0 & A = 1 \pm 2i, 1 \pm 2j, 1 \pm 2k
\end{cases}$$

Using this weight, the previous argument shows that $U_T(k)$ corresponds to all factorizations of $5^k$ over the quaternions, and $V_T(k)$ is in bijection with $U_T(k)$. As a result of Lagrange’s Four Squares theorem, the function

$$r(n) = \sum_{m | n} m$$

is the number of ways to write $n$ as a sum of four integers. Thus, $r(n)$ also counts all quaternions of norm $n$. So,

$$|V_T(k)| = r(5^k) = \sum_{m | 5^k} m = \sum_{j=0}^{k} 5^j = \frac{5^{k+1} - 1}{5-1} = \frac{1}{4}(5^{k+1} - 1) \quad (7)$$

### 3.2 Upper Bound of $K(T)$

Recall from \[6\],

$$K(T) = \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left(\frac{1}{\mu(B_G(\varepsilon))}\right)}$$

Some well known calculations give that $\mu(B_G(\varepsilon))$ is approximately $\varepsilon^2$ when $\varepsilon$ is small only when $G = PSU(2)$. It will be shown that $|V_T(t_\varepsilon)|$ can be bounded in a manner such that it makes allows the terms in $K(T)$ to be sufficiently simplified. The following proposition accomplishes this feat.

**Proposition 3.3.** There exists a constant $c > 0$ such that for any $\varepsilon > 0$,

$$|V_T(t_\varepsilon)| \geq \frac{ct^2}{\varepsilon^4}$$

**Proof.** In \[11\], the same lower bound was shown for $T \setminus \{X, Y, Z\}$. It will be shown that the bound above persists when these elements are included. Consider $\mathbb{R}^3$ as the subspace of $H$ generated by

\[\text{For an example, see } [2]\]

\[\text{Note that the exclusion of these elements make it such that } \Phi \text{ is no longer bijective over } T\]
Then for any $v \in \mathbb{R}^3$, $a \in H$ can act on $v$ by conjugation in $H$. Note that $a$ and $-a$ correspond to the same transformation. Thus, the choice of $G = PSU(2)$ allows for $G$ to be put in a 1-to-1 correspondence with elements of $SO(3)$. Thus, the action of $\gamma \in V_T(t)$ on a vector $v \in \mathbb{R}^3$ in this manner will be represented by juxtaposition. Let $k_\varepsilon$ be a point pair invariant kernel on $S^2$ so that the following hold:

- $k_\varepsilon(x, y) \geq 0$ for any $x, y \in S^2$
- $\int_{S^2} k_\varepsilon(x, y) dy = 1$
- $k_\varepsilon(x, y) = 0$ when $d_{S^2}(x, y) \geq \varepsilon$
- There is a non-zero constant $c' \in \mathbb{C}$ so that $k_\varepsilon(x, x) \leq \frac{c'}{\varepsilon^2}$ for any $x \in S^2$

Additionally, let $h_{k_\varepsilon}$ be the spherical transform of $k_\varepsilon$ such that $h_{k_\varepsilon}(j) \geq 0$ for any $j \geq 0$. Then Hecke Operators are constructed as follows,

$$(T_t f)(x) = \sum_{\gamma \in V_T(t)} f(\gamma x)$$

Then from [11] and the spectral theorem, there is a sequence of real eigenvalues for the $T_t$

$$\lambda_0(t), \lambda_1(t), \ldots$$

and an orthonormal basis of $L^2(S^2)$ of corresponding eigenfunctions

$$\phi_0, \phi_1, \ldots$$

Then, [11] gives that

$$\phi_j(x)h_{k_\varepsilon}(j) = \int_{S^2} k_\varepsilon(x, y)\phi_j(y) dy = 1$$

In particular, $\phi_0$ is the basis vector of the 1-dimensional space of spherical harmonics of degree 0. Then

$$\phi_0(x) = \frac{1}{\sqrt{4\pi}}$$

Thus,

$$h_{k_\varepsilon}(0) = \int_{S^2} k_\varepsilon(x, y) dy = 1$$

Then, as the $\phi_j$ form an orthonormal basis, $k_\varepsilon$ can be written as

$$k_\varepsilon(x, y) = \sum_{j=0}^{\infty} h_{k_\varepsilon}(j) \phi_j(x)\phi_j(y)$$

Fix a point $x_0$ in $S^2$. Then for any $\gamma \in V_T(t)$,

$$k_\varepsilon(\gamma x_0, y) = \sum_{j=0}^{\infty} h_{k_\varepsilon}(j) \phi_j(\gamma x_0, y)\phi_j(y)$$
Hence,

\[ \sum_{\gamma \in V_T(t)} k_\varepsilon(\gamma x_0, y) = \sum_{\gamma \in V_T(t)} \sum_{j=0}^{\infty} h_{k_\varepsilon}(j) \phi_j(\gamma x_0) \phi_j(y) \]
\[ = \sum_{\gamma \in V_T(t)} h_{k_\varepsilon}(0) \phi_0(s x_0) \phi_0(y) + \sum_{j=1}^{\infty} \sum_{\gamma \in V_T(t)} h_{k_\varepsilon}(j) \phi_j(s x_0) \phi_j(y) \]
\[ = \sum_{\gamma \in V_T(t)} 1 \cdot \frac{1}{\sqrt{4\pi}} \cdot \frac{1}{\sqrt{4\pi}} + \sum_{j=1}^{\infty} h_{k_\varepsilon}(j) \phi_j(y) \sum_{\gamma \in V_T(t)} \phi_j(\gamma x_0) \]
\[ = \frac{|V_T(t)|}{4\pi} + \sum_{j=1}^{\infty} h_{k_\varepsilon}(j) \phi_j(y)(T_t \phi_j)(x_0) \]

By construction, \( \phi_j \) is the eigenfunction of \( T_t \) with eigenvalue \( \lambda_j(t) \). Thus,

\[ \sum_{\gamma \in V_T(t)} k_\varepsilon(\gamma x_0, y) = \frac{|V_T(t)|}{4\pi} + \sum_{j=1}^{\infty} h_{k_\varepsilon}(0) \phi_j(y) \lambda_j(t) \phi_j(x_0) \]

If \( d_{S^2}(\gamma x_0, y) > \varepsilon \) for all \( \gamma \in V_T(t) \), then by construction \( k_\varepsilon(\gamma x_0, y) = 0 \) for all \( \gamma \in V_T(t) \). If that were true,

\[ \sum_{\gamma \in V_T(t)} k_\varepsilon(\gamma x_0, y) = \frac{|V_T(t)|}{4\pi} + \sum_{j=1}^{\infty} \lambda_j(t) h_{k_\varepsilon}(j) \phi_j(x_0) \phi_j(y) = 0 \]

which gives

\[ \frac{|V_T(t)|}{4\pi} = - \sum_{j=1}^{\infty} \lambda_j(t) h_{k_\varepsilon}(j) \phi_j(x_0) \phi_j(y) \leq \sum_{j=1}^{\infty} h_{k_\varepsilon}(j) |\lambda_j(t)| |\phi_j(x_0)| |\phi_j(y)| \]

It is an elementary identity that

\[ |\phi_j(x_0)| |\phi_j(y)| \leq \frac{1}{2} (|\phi_j(x_0)|^2 + |\phi_j(y)|^2) \]

Since \( T \) only has 6 elements of non-zero weight, a theorem from [11] can be applied with the value of \( q=5 \).

\[ |\lambda_j(t)| \leq 2tq^\frac{1}{2} \]

Combining these two,

\[ \frac{|V_T(t)|}{4\pi} \leq \sum_{j=1}^{\infty} t5^\frac{1}{2} h_{k_\varepsilon}(j) (|\phi_j(x)|^2 + |\phi_j(y)|^2) \]

Then, from the expression of \( k_\varepsilon \) in terms of the basis \( \phi_j \), it follows

\[ \sum_{j=1}^{\infty} t5^\frac{1}{2} h_{k_\varepsilon}(j) (|\phi(x_0)|^2 + |\phi_j(y)|^2) = t5^\frac{1}{2} \sum_{j=1}^{\infty} h_{k_\varepsilon}(j) |\phi_j(x_0)|^2 + h_{k_\varepsilon}(j) |\phi_j(y)|^2 \]
\[ = t5^\frac{1}{2} (k_\varepsilon(x_0, x_0) + k_\varepsilon(y, y)) \]
Choose $z \in \{x_0, y\}$ so that $k_\varepsilon(z, z) = \min \{k_\varepsilon(x_0, x_0), k_\varepsilon(y, y)\}$. From \ref{eq:V}, $|V_T(t)| = \frac{1}{4}(5^{t+1} - 1)$. Therefore,

$$5^t = 4|V_T(t)| + 1$$

Which gives

$$t5^{t/2}(k_\varepsilon(x_0, x_0) + k_\varepsilon(y, y)) \leq 2tk_\varepsilon(z, z) \sqrt{4|V_T(t)| + 1}$$

However, from the construction of $k_\varepsilon$,

$$2tk_\varepsilon(z, z) \sqrt{4|V_T(t)| + 1} \leq 2t\frac{c'}{\varepsilon^2} \sqrt{4|V_T(t)| + 1}$$

Note that the choice of $c$ is not vital, as long as the inequality in the definition of $k_\varepsilon$ holds. Thus, take $c$ to be a constant such that

$$t5^{t/2}(k_\varepsilon(x_0, x_0) + k_\varepsilon(y, y)) \leq \frac{2tc'}{\varepsilon^2} \sqrt{4|V_T(t)|}$$

Stringing it all together,

$$\frac{|V_T(t)|}{4\pi} \leq t5^{t/2}(k_\varepsilon(x_0, x_0) + k_\varepsilon(y, y)) \leq \frac{2tc'}{\varepsilon^2} \sqrt{4|V_T(t)|}$$

Isolating the $|V_T(t)|$ term from the equation above then gives

$$\sqrt{|V_T(t)|} \leq \frac{16\pi c't}{\varepsilon^2}$$

Which implies

$$|V_T(t)| \leq \frac{256\pi^2(c')^2t^2}{\varepsilon^4}$$

Take $c = 256\pi^2(c')^2$. Then if $d_{S^2}(\gamma x_0, y) > \varepsilon$ for all $\gamma \in V_T(t)$,

$$|V_T(t)| \leq \frac{ct^2}{\varepsilon^4}$$

Thus, if there is some $\gamma \in V_T(t)$ so that $d_{S^2}(\gamma x_0, y) < \varepsilon$, then the contrapositive of the above statement holds as

$$|V_T(t)| \geq \frac{ct^2}{\varepsilon^4}$$

By construction, if $t = t_\varepsilon$ then for any $y \in S^2$ there is a $\gamma \in V_T(t)$ so that $d_{S^2}(\gamma x_0, y) < \varepsilon$. Thus, for all $\varepsilon > 0$,

$$|V_T(t_\varepsilon)| \geq \frac{ct_\varepsilon^2}{\varepsilon^4}$$

Thus rearranging the inequality in theorem \ref{thm:main} yields

$$\frac{1}{\varepsilon^2} \geq \sqrt{\frac{|V_T(t_\varepsilon)|}{ct_\varepsilon^2}}$$
Then \( K(T) \) can be calculated as

\[
K(T) = \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{1}{\mu(B_G(\varepsilon))} \right)} \\
= \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{1}{\varepsilon^2} \right)} \\
\leq \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \sqrt{\frac{|V_T(t_\varepsilon)|}{ct^2}} \right)} \\
= \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\frac{1}{2} \log |V_T(t_\varepsilon)| - \frac{1}{2} \log (c) - \log (t_\varepsilon)} \\
= \limsup_{\varepsilon \to 0} \left( \frac{\log (6 \cdot 5^{t_\varepsilon} - 2)}{\log (6 - \frac{2}{5^{t_\varepsilon}})} \right) \\
= \limsup_{\varepsilon \to 0} \frac{\log_5 (5^{t_\varepsilon}) + \log_5 (6 - \frac{2}{5^{t_\varepsilon}})}{t_\varepsilon + \log_5 (6 - \frac{2}{5^{t_\varepsilon}})} \\
= \limsup_{\varepsilon \to 0} \frac{1}{2} + \frac{1}{2} \log_5 (6 - \frac{2}{5^{t_\varepsilon}}) - \frac{1}{2} \log_5 (c) - \frac{1}{2} \log_5 (t_\varepsilon) \\
= \limsup_{\varepsilon \to 0} \frac{1 + 0}{\frac{1}{2} + 0 - 0 - 0} \\
= 2
\]

Note that \( \mu(B_G(\varepsilon)) = O(\varepsilon^2) \) when \( G = PSU(2) \) is used with the constant in the \( O \) term equal to 1. This is because any variations by a constant factor \( k \) would split into a term \( \log \left( \frac{1}{k} \right) \) which, when divided by \( t_\varepsilon \) as in the calculations above, would tend to zero.

### 3.3 Refining the Upper Bound

The general framework of the upper bound would hopefully provide for an improved upper bound for \( T \). Since \( V_T(t_\varepsilon) \) provides a connection to \( S^3 \), an obvious place to look is the distribution of points on \( S^3 \). As with many point distributions on \( S^3 \), it is valuable to consider a way of calculating a mesh norm as done in [3][5][8][10].

Recall that in [11], the metric was defined as

\[
d_G(X, Y) = \sqrt{1 - \frac{|Tr(X^t Y)|}{2}}
\]
Thus, consider \( Tr(X^\dagger Y) \). Some simple algebra yields

\[
Tr(X^\dagger Y) = Tr \left( \frac{1}{\sqrt{|x_1|^2 + |x_2|^2}} \frac{1}{\sqrt{|y_1|^2 + |y_2|^2}} \begin{bmatrix} x_1 & -x_2 \\ x_2 & y_1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix} \right)
\]

\[
= Tr \left( \frac{1}{|\Phi(X)||\Phi(Y)|} \begin{bmatrix} x_1 y_1 + x_2 y_2 & x_1 y_2 - x_2 y_1 \\ x_2 y_1 + x_1 y_2 & x_2 y_2 + x_1 y_1 \end{bmatrix} \right)
\]

Thus if \( \langle \Phi(X), \Phi(Y) \rangle \) is defined on \( H \) just as it is on \( \mathbb{R}^4 \), then the formula for the dot product on \( \mathbb{C} \) as \( \mathbb{R}^2 \) can be extended as

\[
\langle \Phi(X), \Phi(Y) \rangle = Re((x_1 + x_2 j)(y_1 + y_2 j))
\]

\[
= Re((x_1 + x_2 j)(\overline{y_1} - \overline{y_2} j))
\]

\[
= Re(x_1 \overline{y_1} + x_2 \overline{y_2} - x_1 \overline{y_2} j + x_2 j \overline{y_1})
\]

\[
= Re(x_1 \overline{y_1} + x_2 \overline{y_2})
\]

where the conjugate in the first line is the quaternion conjugate (they match on \( \mathbb{C} \)). Thus, combined

\[
Tr(X^\dagger Y) = \frac{2\langle \Phi(X), \Phi(Y) \rangle}{|\Phi(X)||\Phi(Y)|}
\]

which gives

\[
d_G(X, Y) = \sqrt{1 - \frac{|Tr(X^\dagger Y)|}{2}}
\]

\[
d_G(X, Y)^2 = 1 - \frac{\langle \Phi(X), \Phi(Y) \rangle}{|\Phi(X)||\Phi(Y)|}
\]

With this relation \( d_G(X, Y) \) is equal to \( 1 - \cos \theta \) where \( \theta \) is the angle between \( \Phi(X), \Phi(Y) \). Thus, the angular distribution of \( V_T(t) \) on \( S^3 \) can give a bound on \( d_G(X, Y) \). As in section \ref{sec:3.2} a bound on \( \varepsilon \) gives a bound on \( K(T) \). Hence, the goal is to provide an upper bound on \( d_G(\Phi(X), \Phi(Y)) \) for \( \varepsilon > 0 \), and then use that to provide an upper bound for \( \varepsilon \). Since \( d_G(\Phi(X), \Phi(Y))^2 = 1 - \cos \theta \), then any lower bound of \( \cos \theta \) will bound \( 1 - \varepsilon^2 \) from above.

Now refer back to Section \ref{sec:3.1} where it was noted that \( V_T(t) \) is in a bijection with solutions to the family of quadratic forms \( x_1^2 + x_2^2 + x_3^3 + x_4^4 = 5^t \) where \( l \leq t \) is an integer. Then note that for any
distinct \(X, Y \in SU(2)\), \(d_G(X, Y)^2\) can be calculated in terms of \(|\Phi(X)|, |\Phi(Y)|\), and \(\langle \Phi(X), \Phi(Y) \rangle\). Suppose \(X \in \langle T \rangle\) and \(Y \in PSU(2)\) so that \(X\) approximates \(Y\) within \(\varepsilon\). Then

\[
d_G(X, Y) \leq \varepsilon
\]

which implies

\[
d_G(X, Y)^2 = 1 - \frac{\langle \Phi(X)\Phi(Y) \rangle}{|\Phi(X)||\Phi(Y)|} \leq \varepsilon^2
\]

Thus,

\[
\frac{\langle \Phi(X), \Phi(Y) \rangle}{|\Phi(X)||\Phi(Y)|} \geq 1 - \varepsilon^2
\]

Note that this is the formula for the cosine of the angle between \(\Phi(X)\) and \(\Phi(Y)\). Thus, we formulate a conjecture on the necessary angles to calculate the covering exponent.

**Conjecture 3.4.** There is some \(0 < \delta < 1\) so that for any \(\varepsilon > 0\) and \(a \in S^3\) there is a point \(b \in \mathbb{Z}^4\) with some \(k \in \mathbb{Z}\) so that \(|b| = 5^k\) and \(\langle a, \frac{b}{5^k} \rangle \geq 1 - 5^{\frac{k}{2\delta}}\).

Suppose the conjecture holds. For any matrix \(M \in SU(2)\), \(\Phi(M) \in S^3\). Thus, by the conjecture, there is a \(b \in H(\mathbb{Z})\) and \(k \in \mathbb{Z}\) so that \(|b| = 5^k\) and \(\langle \Phi(M), \frac{b}{5^k} \rangle > 1 - 5^{\frac{k}{2\delta}}\). Let \(N = \frac{1}{\sqrt{5^k}} \Phi^{-1}(b)\). Then

\[
d_G(M, N) = \sqrt{1 - \frac{|Tr(M^\dagger N)|}{2}}
\]

\[
= \sqrt{1 - \frac{\langle \Phi(M), \Phi(N) \rangle}{|\Phi(M)||\Phi(N)|}}
\]

\[
= \sqrt{1 - \frac{\langle \Phi(M), \frac{b}{5^k} \rangle}{5^k \cdot \frac{1}{5^k}}}
\]

\[
= \sqrt{1 - \langle \Phi(M), \frac{b}{5^k} \rangle}
\]

\[
\leq \sqrt{1 - (1 - 5^{\frac{k}{2\delta}})}
\]

\[
= 5^{\frac{k}{2\delta}}
\]

However, as noted earlier, \(k\) can always be chosen to be \(t_\varepsilon\) since \(\Phi(V_T(t_\varepsilon))\) contains copies of \(\Phi(U_T(k))\) for all \(k\). Thus, for all \(M \in SU(2)\), \(b\) can be chosen such that \(k = t_\varepsilon\) giving

\[
d_G(M, N) \leq 5^{\frac{t_\varepsilon}{2\delta}}
\]

By construction, \(t_\varepsilon\) is the smallest height such that \(V_T(t_\varepsilon)\) can cover \(SU(2)\) within a tolerance of \(\varepsilon\). Then if \(d_G(M, N) < \varepsilon\) it necessarily follows that \(d_G(M, N) \leq 5^{\frac{t_\varepsilon}{2\delta}}\). Since this holds for all \(\varepsilon > 0\), then

\[
\varepsilon \leq 5^{\frac{t_\varepsilon}{2\delta}}
\]

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Given this result, the upper bound from Section 3.2 can be rewritten as

\[ K(T) = \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{1}{\mu(B_G(\varepsilon))} \right)} \]

\[ = \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{1}{\varepsilon^2} \right)} \]

\[ \leq \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{5}{2\varepsilon^2} \right)} \]

\[ = \limsup_{\varepsilon \to 0} \frac{\log_5 \left( \frac{1}{\varepsilon^2} \left( 5^{t_\varepsilon+1} - 1 \right) \right)}{\log_5 \left( \frac{5}{2\varepsilon^2} \right)} \]

\[ = \limsup_{\varepsilon \to 0} t_\varepsilon + 1 + \log_5 \left( 1 - 5^{-t_\varepsilon - 1} \right) + \log_5 \left( \frac{1}{4} \right) \]

\[ = \limsup_{\varepsilon \to 0} \frac{t_\varepsilon}{2 - \delta} \]

\[ = \limsup_{\varepsilon \to 0} \frac{1 + \frac{1}{t_\varepsilon} \left( 1 + \log_5 \left( 1 - 5^{-t_\varepsilon - 1} \right) + \log_5 \left( \frac{1}{4} \right) \right)}{2 - \delta} \]

\[ = \limsup_{\varepsilon \to 0} \frac{1 + 0}{2 - \delta} \]

\[ = 2 - \delta \]

The conjecture suggests that such a δ exists per this construction of T, and that it directly gives the covering exponent. Thus, when the case of G = PSU(2) is acceptable, this set T provides a tangible and effective universal gate set for G. However, this setup described in this paper can be extrapolated to SU(2) and other more general universal sets. It is important to note the construction of T can be replicated by using different primes p ≡ 1(mod4). Furthermore, |V_T(t_\varepsilon)| = \frac{1}{p-1}(p^{t_\varepsilon+1}-1).
Thus, assuming Conjecture 3.4 holds

\[
K(T) = \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{1}{\mu(B_G(\varepsilon))} \right)} \\
= \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log \left( \frac{1}{\varepsilon^2} \right)} \\
\leq \limsup_{\varepsilon \to 0} \frac{\log |V_T(t_\varepsilon)|}{\log(p^{\frac{t_\varepsilon}{2\pi}})} \\
= \limsup_{\varepsilon \to 0} \frac{\log_p \left( \frac{1}{4} (p^{2t_\varepsilon+1} - 1) \right)}{\log_p \left( p^{\frac{t_\varepsilon}{2\pi}} \right)} \\
= \limsup_{\varepsilon \to 0} \frac{\log_p \left( p^{t_\varepsilon+1} \right) + \log_p \left( 1 - p^{-t_\varepsilon-1} \right) + \log_p \left( \frac{1}{4} \right)}{\frac{t_\varepsilon}{2\pi}} \\
= \limsup_{\varepsilon \to 0} \frac{t_\varepsilon + 1 + \log_p \left( 1 - p^{-t_\varepsilon-1} \right) + \log_p \left( \frac{1}{4} \right)}{\frac{t_\varepsilon}{2\pi}} \\
= \limsup_{\varepsilon \to 0} \frac{1 + \frac{1}{t_\varepsilon} \left( 1 + \log_p \left( 1 - p^{-t_\varepsilon-1} \right) + \log_p \left( \frac{1}{4} \right) \right)}{\frac{1}{2\pi}} \\
= \limsup_{\varepsilon \to 0} \left( 1 + \frac{0}{2\pi} \right) \\
= 2 - \delta
\]

Thus, the choice of \( p \) has no impact on the covering exponent for \( T \) except for which \( \delta \) a choice of \( p \) allows the conjecture to hold. How \( \delta \) changes with a change in \( p \) is still an open question. Nevertheless, this framework allows for those computations on the 3-sphere to directly correlate to the efficiency of this construction.

### 3.4 The Next Steps

The calculation of \( \delta \) is not as straightforward as it seems, especially in the case of \( T \). For very small values of \( t_\varepsilon \), the largest holes form around the axes. However, the holes then shift towards the center of each sedecant (1/16th) of \( S^3 \). This shifting nature of the holes implies that any bounds on their size must be checked across the entirety of one sedecant. Since the sign of coordinates is irrelevant in whether they are a solution to a quadratic form, each sedecant should be identical. However, this calculation is still not simple by any means. Thus, other forms of well distributed points with more inherent bounds on their holes may yield better results.

For these reasons, Conjecture 3.4 is quite open ended but opens the door for some concrete improvements on the upper bound of 2. This framework can be generalized to many of the other point sets mentioned in papers such as [1][6][7][8][9]. Each point set is still bounded by the same notion of ”holes” in the distributions, which when bounded provide similar estimations on the covering exponent for the universal sets they generate. However, many of these point distributions use things like energy minimalization which, by construction minimize these holes. Work on this conjecture will provide a deeper understanding between competing quantum algorithms.
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