ON THE WALDSCHMIDT CONSTANT OF SQUARE-FREE PRINCIPAL BOREL IDEALS

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Abstract. Fix a square-free monomial $m \in S = \mathbb{K}[x_1, \ldots, x_n]$. The square-free principal Borel ideal generated by $m$, denoted $\text{sfBorel}(m)$, is the ideal generated by all the square-free monomials that can be obtained via Borel moves from the monomial $m$. We give upper and lower bounds for the Waldschmidt constant of $\text{sfBorel}(m)$ in terms of the support of $m$, and in some cases, exact values. For any rational $a/b \geq 1$, we show that there exists a square-free principal Borel ideal with Waldschmidt constant equal to $a/b$.

1. Introduction

Introduced in the 1970’s by Waldschmidt [14] to study points in $\mathbb{C}^n$, the Waldschmidt constant pays a pivotal role in the study of the asymptotic properties of homogeneous ideals. More formally, given a homogeneous ideal $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$, the Waldschmidt constant of $I$, denoted $\hat{\alpha}(I)$, is the limit $\lim_{s \to \infty} \frac{\alpha(I^s)}{s}$. Here, $\alpha(J)$ denotes the smallest degree of a generator of the ideal $J$, and $I^s$ denotes the $s$-th symbolic power of $I$. Current interest in this invariant was partially inspired by the work of Bocci and Harbourne [1]. Specifically, the Waldschmidt constant can be used to gain insight into the ideal containment problem, that is, comparing the regular powers of an ideal to those of its symbolic powers. The papers [2, 3, 4, 7] form a small sample of recent work on the Waldschmidt constant; an introduction to this topic can also be found in [6].

The goal of this paper is to investigate the Waldschmidt constant of square-free principal Borel ideals. Given a monomial $m$, if $x_j | m$ and $j < i$, then we call $x_j : \frac{m}{x_i}$ a Borel move of $m$. A monomial ideal is a Borel ideal (or a strongly stable ideal) if for every $m \in I$, all of the Borel moves of $m$ are also in $I$. A monomial ideal $I$ is a principal Borel ideal if there is a single monomial $m$ such that every generator of $I$ is obtained via a Borel move of $m$. A square-free principal Borel ideal generated by $m$, denoted $\text{sfBorel}(m)$, is the square-free monomial ideal generated by all the square-free monomials that can be obtained via Borel moves on $m$. The study of (principal) Borel ideals has a rich history; we point the reader to [10, 11] and the references therein for more on this topic.

Because $\text{sfBorel}(m)$ is a square-free monomial ideal, the results of Bocci, et al. [4] are applicable. For any square-free monomial ideal $I$, it is was shown in [4] how to use the primary decomposition of $I$ to create a linear optimization problem whose optimal
solution is the Waldschmidt constant. Solving this optimization problem, however, may prove to be quite difficult.

As an example, suppose we wish to compute \( \hat{\alpha}(\text{sfBorel}(m)) \) for the monomial

\[
(1.1) \quad m = x_{33215}x_{33216} \cdots x_{104348} \in \mathbb{K}[x_1, \ldots, x_{104348}].
\]

To naively apply [4] to find the Waldschmidt constant for this ideal would involve solving a linear optimization problem in 104348 variables and \( \binom{104348}{33215} \approx 5.1 \times 10^{28347} \) inequalities.

Given the size of this problem, it is natural to ask what information, if any, one can obtain about the Waldschmidt constant for this family of ideals.

The main results of this paper place bounds on the Waldschmidt constant for any square-free principal Borel ideal \( \text{sfBorel}(m) \). Our bounds are expressed in terms of the support of \( m = x_{i_1} \cdots x_{i_s} \), that is, \( \{i_1, \ldots, i_s\} \). Our proofs rely on Francisco, Mermin, and Schweig’s [10] description of the associated primes of \( \text{sfBorel}(m) \). These associated primes then allow us to describe many of the inequalities in the linear optimization problem given in [4], which are then used to bound the Waldschmidt constant. In some cases, we are able to give exact values for \( \hat{\alpha}(\text{sfBorel}(m)) \).

The work in this paper generalizes some of our work in [5]. Previously we showed that the Waldschmidt constant of a principal Borel ideal generated by \( m \) is \( \deg m \). Unlike the case of principal Borel ideals, the Waldschmidt constant of a square-free principal Borel ideal need not be an integer; in fact:

**Theorem 1.1** (Corollary [4.2]). Let \( \frac{a}{b} \geq 1 \) be a rational number. Then there exists a square-free principal Borel ideal \( I \) such that \( \hat{\alpha}(I) = \frac{a}{b} \).

The above result is effective in the sense that we can explicitly construct the required monomial \( m \) so \( \hat{\alpha}(\text{sfBorel}(m)) = \frac{a}{b} \). In fact, the monomial \( m \) of (1.1) was chosen so that

\[
\hat{\alpha}(\text{sfBorel}(m)) = \frac{104348}{33215}.
\]

The astute reader might recognize that this number is a convergent of the continued fraction expansion of \( \pi \). In fact, \( \frac{104348}{33215} = 3.14159265392 \) agrees with \( \pi \) up to 9 digits.

We outline our paper. Section 2 provides the background on square-free principal Borel ideals and the Waldschmidt constant. In Section 3, upper bounds on the Waldschmidt constant are obtained. In Section 4, under certain hypotheses, exact values of \( \hat{\alpha}(\text{sfBorel}(m)) \) are obtained, as well as a recursive approach to finding a lower bound (Theorem 4.4).

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2. Background

Throughout this paper \( S = \mathbb{K}[x_1, \ldots, x_n] \), where \( \mathbb{K} \) is field of characteristic zero. In this section we recall the relevant background on square-free monomial ideals, square-free Borel ideals, and the Waldschmidt constant.

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1As a point of comparison, it is estimated that there are \( 10^{82} \) atoms in the universe!
2.1. **Square-free Borel ideals.** For unexplained terminology about (square-free) monomial ideals, we refer the reader to [12]. We define the main objects of study in this paper.

**Definition 2.1.** Let $X = \{m_1, \ldots, m_t\}$ be a set of square-free monomials in $S$. The **square-free Borel ideal generated by $X$**, denoted $\text{sfBorel}(X)$, is the square-free monomial ideal generated by the square-free monomials that can be obtained via Borel moves from any monomial $m \in X$. If $X = \{m\}$, then we abuse notation and write $\text{sfBorel}(m)$ for $\text{sfBorel}(\{m\})$; furthermore, we call $\text{sfBorel}(m)$ a **square-free principal Borel ideal**.

The *support* of a square-free monomial $m = x_{i_1} \cdots x_{i_s}$ is the set $\text{supp}(m) = \{i_1, \ldots, i_s\}$. For our future arguments, we need two tuples that can be constructed from $\text{supp}(m)$.

**Definition 2.2.** Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial. Let

$$T(m) = (t_0, t_1, \ldots, t_k)$$

where $t_0 = s$ and $t_i = \max\{j < t_{i-1} \mid i_j < i_{j+1} - 1\}$. Furthermore, let

$$IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k}).$$

**Remark 2.3.** The following observations will hopefully help the reader with our notation. The $t_i$'s are recording where the indices of $x_{i_1} x_{i_2} \cdots x_{i_s}$ are “jumping” by more than one. For example, if $m = x_2 x_3 x_5 x_6 x_8 x_{10}$, then

$$T(m) = (t_0, t_1, t_2, t_3) = (6, 5, 4, 2)$$

records the positions where the indices increase by more than one. Note that we are recording this information from right-to-left. Equivalently, we can define the $t_i$’s as follows. Consider the tuple $(i_1 - 1, i_2 - 2, \ldots, i_s - s)$. The $t_i$’s are then the locations where $i_j - j < i_{j+1} - (j + 1)$, again reading right-to-left. In our example, $(2 - 1, 3 - 2, 5 - 3, 6 - 4, 8 - 5, 10 - 6) = (1, 1, 2, 2, 3, 4)$, so $t_0 = 6$, and $t_1 = 5$, $t_2 = 4$ and $t_3 = 2$ since these are the indices where $i_j - j < i_{j+1} - (j + 1)$. Continuing with this example, the tuple

$$IT(m) = (i_{t_0}, i_{t_1}, i_{t_2}, i_{t_3}) = (10, 8, 6, 3)$$

records the indices of the variables where the “jump” occurs.

The following lemma records some facts that follow immediately from the definitions.

**Lemma 2.4.** Let $m$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$. Then

1. $s = t_0 > t_1 > \cdots > t_k \geq 1$,
2. $i_{t_0} > i_{t_1} > \cdots > i_{t_k}$, and
3. $i_{t_j} - t_j > i_{t_{j+1}} - t_{j+1}$ for $j = 0, \ldots, k - 1$.

**Proof.** (1) and (2) are immediate. For (3) we have

$$i_1 - 1 \leq i_2 - 2 \leq \cdots \leq i_j - j \leq \cdots \leq i_s - s$$

for all $j = 1, \ldots, s$. So

$$i_{t_j+1} - t_{j+1} < i_{t_{j+1}+1} - (t_{j+1} + 1) \leq i_{t_j} - t_j$$

since the $t_i$’s are precisely the locations where the inequalities in (2.1) are strict.
Given an ideal $I$, we let $\text{ass}(I)$ denote the set of associated primes of the ideal $I$. When $I$ is a square-free monomial ideal, it is known that all the associated primes are prime monomial ideals. The next result is critical for our arguments, since the associated primes of $\text{sfBorel}(I)$ will be related to an optimization problem to compute the Waldschmidt constant. Note that the original statement of Theorem 2.5 involved the language of Alexander duals. We have given an equivalent expression for this statement.

**Theorem 2.5.** [10] Theorem 3.17 Let $m = x_{i_1}\cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$, and suppose $I = \text{sfBorel}(m)$. Then

$$\langle x_{j_1}, \ldots, x_{j_l} \rangle \in \text{ass}(I)$$

if and only if $x_{j_1}x_{j_2}\cdots x_{j_l}$ is a minimal generator of the square-free Borel ideal

$$\text{sfBorel}(\{x_{t_k}x_{t_k+1}\cdots x_{t_{i_k}}, x_{t_{k-1}}x_{t_{k-1}+1}\cdots x_{t_{i_{k-1}}}, \ldots, x_{t_0}x_{t_0+1}\cdots x_{i_0}\}).$$

The monomials $x_{j_1}\cdots x_{i_j}$ for $j = 0, \ldots, k$ are minimal in the sense that if we remove any of them, we change the generators of the resulting square-free Borel ideal.

**Example 2.6.** In Remark 2.3 it was shown that the monomial $m = x_2x_3x_5x_6x_8x_{10}$ has $T(m) = (6, 5, 4, 2)$ and $IT(m) = (10, 8, 6, 3)$. So the associated primes of $\text{sfBorel}(m)$ are in one-to-one correspondence with the minimal generators of

$$\text{sfBorel}(\{x_2x_3, x_4x_5x_6, x_5x_6x_7x_8, x_6x_7x_8x_9x_{10}\}).$$

2.2. The Waldschmidt constant. We recall the definition of the Waldschmidt constant and a procedure to compute this invariant for square-free monomial ideals.

Given a square-free monomial ideal $I \subseteq S$, let $I = P_1 \cap \cdots \cap P_t$ denote its minimal primary decomposition. The $s$-th symbolic power of the square-free monomial ideal $I$, denoted $I^{(s)}$, is the ideal

$$I^{(s)} = P_1^s \cap P_2^s \cap \cdots \cap P_t^s.$$ 

Note that there is a more general definition of a symbolic power of an ideal (see [6, 8]); our definition is equivalent when restricted to square-free monomial ideals.

For any homogeneous ideal $J \subseteq S$, let $\alpha(J)$ denote the smallest degree of a generator of $J$. The Waldschmidt constant of a square-free monomial ideal $I$, denoted $\hat{\alpha}(I)$, is then

$$\hat{\alpha}(I) = \lim_{s \to \infty} \frac{\alpha(I^{(s)})}{s}.$$ 

Our key tool will be the following result which relates the Waldschmidt constant of a square-free monomial ideal to a linear program.

**Theorem 2.7.** [4] Theorem 3.2 Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ be a square-free monomial ideal with minimal primary decomposition $I = P_1 \cap P_2 \cap \cdots \cap P_t$. Define the $t \times n$ matrix $A$ where

$$A_{i,j} = \begin{cases} 
1 & \text{if } x_j \in P_i \\
0 & \text{if } x_j \notin P_i.
\end{cases}$$

Then $\hat{\alpha}(I)$ is the optimum value of the linear program

$$\min\{1^T y \mid Ay \geq 1, y \geq 0\}.$$
The matrix $A$ in the above theorem will be called the **matrix of associated primes of $I$**.

**Example 2.8.** Let $m$ be the monomial of $\lfloor 1/1 \rfloor$ from the introduction, and so $T(m) = (71134)$ and $IT(m) = (104348)$. The associated primes of $\text{sfBorel}(m)$ are in one-to-one correspondence with the generators of $\text{sfBorel}(x_{71134}x_{71135} \cdots x_{104348})$. But this is the ideal generated by all the square-free monomials of degree 33215, of which there are $\binom{104348}{33215}$. So the matrix of associated primes of $\text{sfBorel}(m)$ will be a $\binom{104348}{33215} \times 104348$ matrix.

### 3. Upper bounds

In this section we give an upper bound on the Waldschmidt constant of a square-free principal Borel ideal. Our strategy is to show that there is enough structure in the optimization problem of Theorem 2.7 that we can bound the Waldschmidt constant.

We begin with a lemma which allows us to reduce to a smaller polynomial ring.

**Lemma 3.1.** Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial in $S = \mathbb{K}[x_1, \ldots, x_n]$ with $I = \text{sfBorel}(m) \subseteq S$. Consider the same monomial $m$, but in the ring $R = \mathbb{K}[x_1, \ldots, x_{i_s}]$, and let $J = \text{sfBorel}(m) \subseteq R$. Then $\hat{\alpha}(I) = \hat{\alpha}(J)$.

**Proof.** By Theorem 2.5, the associated primes of $I$ and $J$ are the same (although viewed in different rings). So the matrix of associated primes of $J$ in Theorem 2.7 is the same as the matrix of the associated primes of $I$, except that the columns in matrix of associated primes of $I$ indexed by the variables $x_{i_s+1}, \ldots, x_n$ all contain zeroes. The result now follows from Theorem 2.7. \qed

Before proceeding, we introduce additional notation. Given a square-free monomial $m$ with $T(m) = (t_0, \ldots, t_k)$ and $IT(m) = (i_{t_0}, \ldots, i_{t_k})$, we have the inequalities

$$t_k < t_{k-1} < \cdots < t_0 = s$$

and

$$i_{t_k} < i_{t_{k-1}} < \cdots < i_{t_0}.$$ 

by Lemma 2.4. Let $\ell$ be smallest integer such that

$$i_{t_{\ell+1}} < t_0 \leq i_{t_\ell}.$$ 

In particular, $\ell$ identifies where in the sequence of $i_{t_j}$’s we would place $t_0 = s$.

Let $A$ be the matrix of associated primes of $I = \text{sfBorel}(m)$. We will let $A_P$ denote the row associated to the associated prime $\langle x_{t_j}, \ldots, x_{i_{t_j}} \rangle$ for $j = 0, \ldots, k$. The row $A_P$ corresponds to a minimal generator $m$ of the ideal in Theorem 2.7.

As the next lemma shows, we can bound the optimal solution of Theorem 2.7 by considering only a submatrix of the matrix of associated primes.

**Lemma 3.2.** Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$, $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$, and $\ell$ as defined above. Let $I = \text{sfBorel}(m)$, and let $A$ denote its matrix of associated primes. Let $B$ be the submatrix of $A$ where the $j$-th row of $B$ corresponds to the associated prime $\langle x_{t_j}, \ldots, x_{i_{t_j}} \rangle$ for $j = 0, \ldots, k$. Suppose $x \in \mathbb{R}^n$ is such that

1. $Bx \geq 1$,
2. $x_j \geq x_{j+1}$ for $1 \leq j \leq i_{t_\ell}$, and
(3) \( x_{i_t} \geq x_j \), for \( i_t \leq j \leq n \).

Then \( Ax \geq 1 \).

Proof. By Lemma 3.1 we can assume \( n = i_s \). Consider any row \( A_P \) of \( A \). By Theorem 2.5, \( P \) corresponds to a monomial \( m \) that is a Borel move of exactly one of \( \{x_{i_t \cdots i_{i_t}}, \ldots, x_{i_0 \cdots i_{i_0}}\} \). Say \( m \) is a Borel move of \( x_{i_t \cdots i_{i_t}} \).

The \( j \)-th row of \( B \) (which corresponds to \( x_{i_t \cdots i_{i_t}} \)) is given by

\[
B_j = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0).
\]

The rows \( A_P \) and \( B_j \) have the same number of 1’s. Additionally, \( A_P \) is formed from \( B_j \) by swapping some of the 1’s with some of the 0’s among the first \( t_j - 1 \) spots.

Since \( B_jx \geq 1 \), we have \( x_{i_t} + \cdots + x_{i_{i_t}} \geq 1 \). Note that \( A_Px \) is formed from \( B_j \) by subtracting some \( x_p \)’s with \( p \in \{t_j, \ldots, i_t\} \) and adding in some \( x_q \)’s with \( q \in \{1, \ldots, t_j - 1\} \).

If \( p \in \{t_j, \ldots, i_t\} \), then the hypotheses imply that \( x_q \geq x_p \) for all \( q \in \{1, \ldots, t_j - 1\} \). If \( p \in \{i_t, \ldots, i_{i_t}\} \), then \( t_j < t_0 \leq i_t \leq p \leq i_{i_t} \). But then \( x_q \geq x_{i_t} \geq x_p \) for all \( q \in \{1, \ldots, t_j - 1\} \). But this means

\[
A_Px \geq B_jx \geq 1
\]

because every time we subtract an \( x_p \) with \( p \in \{t_j, \ldots, i_t\} \) we are replacing it with an \( x_q \) with \( q \in \{1, \ldots, t_j - 1\} \) which is larger.

The result now follows since \( A_Px \geq 1 \) for all rows of \( A \). \( \square \)

We can now bound the Waldschmidt constant of a square-free principal Borel ideal in terms of \( T(m), IT(m) \), and \( \ell \).

**Theorem 3.3.** Let \( m = x_{i_1} \cdots x_{i_s} \) be a square-free monomial with \( T(m) = (t_0, t_1, \ldots, t_k) \) and \( IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k}) \), and suppose \( I = sfBorel(m) \). If \( \ell \) is the smallest integer such that \( i_{t_{\ell+1}} < t_0 \leq i_{t_{t}} \), then

\[
\hat{\alpha}(I) \leq (t_0 - t_{\ell}) \left( \frac{1}{i_{t_{\ell}} - t_{\ell} + 1} \right) + (i_{t_{\ell}} - i_{t_{\ell+1}}) \left( \frac{1}{i_{t_{\ell}} - t_{\ell} + 1} \right) + \cdots + (i_{t_{k-1}} - i_{t_{k}}) \left( \frac{1}{i_{t_{k-1}} - t_{k-1} + 1} \right) + i_{t_{k}} \left( \frac{1}{i_{t_{k}} - t_{k} + 1} \right).
\]
Proof. By Lemma 3.1 we can assume that the number of variables is $i_s = n$. Set $a = i_\ell - t_\ell + 1$, and consider the vector $y \in \mathbb{R}^{i_s = i_0}$ where

$$y^T = \left( \begin{array}{cccccc}
\frac{1}{i_{t_k} - t_k + 1} & \frac{1}{i_{k} - t_k + 1} & \frac{1}{i_{k-1} - t_{k-1} + 1} & \cdots & \frac{1}{i_{k-1} - t_k + 1} & \cdots
\end{array} \right).$$

These inequalities imply that the first $i_\ell$ entries of $y$ form a non-increasing sequence. Thus, condition (2) of Lemma 3.2 holds for $y$. In addition, it follows by Lemma 2.4 that

$$\frac{t_j - 1}{i_t - i_j} < 1$$

for all $j = 1, \ldots, k$. Hence $\frac{1}{a} \geq y_r$ for all $r = i_{t_\ell}, \ldots, i_{t_0}$, and thus condition (3) of Lemma 3.2 also holds for $y$.

Let $B$ be the submatrix of $A$ where the $j$-th row of $B$ corresponds to the associated prime of $I$ that is associated to $x_{t_j} \cdots x_{t_{j+1}}$. That is, written as a row vector:

$$B_j = \left( \begin{array}{c}
0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0
\end{array} \right)_{t_j - 1 \atop i_t - t_j + 1}$$

We now show that $B$ and $y$ satisfy condition (1) of Lemma 3.2, thus completing the proof.

Consider the $j$-th row of $B$, denoted $B_j$. If $j \geq \ell$, we then have

$$B_j y \geq B_j \left( \begin{array}{c}
0
\frac{1}{i_{t_j} - t_j + 1}
\vdots
\frac{1}{i_{t_j} - t_j + 1}
y_{i_{t_j} + 1}
\end{array} \right) \geq 1$$

where $y_{i_{t_j} + 1}$ represents the last $i_{t_0} - i_j$ entries of $y$, the fraction $\frac{1}{i_{t_j} - t_j + 1}$ appears $i_j - t_j + 1$ times, and the 0 is the vector with $t_j - 1$ zeroes. Since every entry of this new vector is less than or equal to the corresponding entry in $y$, the first inequality holds.
Now suppose that \( j < \ell \). Consequently, note that \( t_j < t_0 \leq i_{t_\ell} < i_j \). So we then have

\[
B_j y = \sum_{r=1}^{i_s} (B_j)_r y_r = \sum_{r=t_j}^{i_\ell} y_r + \sum_{r=i_{t_\ell}+1}^{i_j} y_r
\]

\[
\geq \frac{i_{t_\ell} - t_j + 1}{i_{t_\ell} - t_\ell + 1} + \frac{t_{\ell-1} - t_\ell}{i_{t_\ell} - t_\ell + 1} + \cdots + \frac{t_j - t_{j+1}}{i_{t_\ell} - t_\ell + 1}
\]

\[
= \frac{i_{t_\ell} - t_j + 1}{i_{t_\ell} - t_\ell + 1} + \frac{t_j - t_\ell}{i_{t_\ell} - t_\ell + 1} = 1.
\]

The inequality follows from the fact that \( y_r \geq \frac{1}{s} \) for all \( 1 \leq r \leq i_{t_\ell} \). Hence, \( B_j y \geq 1 \) for all rows of \( j \), so condition (1) of Lemma 3.2 also holds. \( \square \)

We derive the following corollary; in the next section we will show that this bound is exact under additional hypotheses.

**Corollary 3.4.** Let \( m = x_{i_1} \cdots x_{i_s} \) be a square-free monomial with \( T(m) = (t_0, t_1, \ldots, t_k) \) and \( IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k}) \), and suppose \( I = \text{sfBorel}(m) \). If \( \ell \) is the smallest integer such that \( i_{t_{\ell+1}} < t_0 \leq i_{t_\ell} \), then

\[
\hat{\alpha}(I) \leq \frac{t_0 - t_\ell + i_{t_\ell}}{i_{t_\ell} - t_k + 1}.
\]

**Proof.** Recall that \( t_0 = s \). Note that

\[
\frac{1}{i_{t_\ell} - t_k + 1} \geq \frac{1}{i_{t_j} - t_j + 1} \quad \text{for} \quad \ell \leq j \leq k.
\]

The result now follows from Theorem 3.3 and this inequality. \( \square \)

**Example 3.5.** Our bound in Theorem 3.3 is sharp. For example, if \( m = x_2x_3x_5x_6x_8x_{10} \), then we have \( T(m) = (6, 5, 4, 2) \) and \( IT(m) = (10, 8, 6, 3) \). For this monomial, \( \ell = 2 \) since \( t_2 = 4 \) and \( t_0 = 6 \leq i_4 = i_{t_2} = 6 \). Then

\[
\hat{\alpha}(\text{sfBorel}(m)) \leq \frac{2}{3} + \frac{3}{3} + \frac{3}{2} = \frac{19}{6},
\]

and this is the actual Waldschmidt constant.

### 4. Some exact values and lower bounds

In this section we compute the exact value of the Waldschmidt constant of principal square-free Borel ideals under some additional hypotheses. We then present a theorem that can be used to find lower bounds recursively.

We begin with the following exact formula.

**Theorem 4.1.** Let \( m = x_{i_1} \cdots x_{i_s} \) be a square-free monomial with \( T(m) = (t_0, t_1, \ldots, t_k) \) and \( IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k}) \), and suppose \( I = \text{sfBorel}(m) \). If \( t_0 \leq i_{t_k} \), then

\[
\hat{\alpha}(I) = 1 + \frac{s - 1}{i_{t_k} - t_k + 1}.
\]
Proof. The hypotheses imply that $\ell = k$, with $\ell$ as in Corollary 3.4. Consequently, Corollary 3.4 then shows that

$$\tilde{\alpha}(I) \leq \frac{t_0 - t_k + i_{t_k}}{i_{t_k} - t_k + 1} = \frac{s - 1 + (i_{t_k} - t_k + 1)}{i_{t_k} - t_k + 1}$$

where we use the fact that $t_0 = s$.

For $j = 0, \ldots, k$, let $P_j$ denote the associated prime of $I$ that is associated with the monomial $x_{t_j} \cdots x_{t_{i_j}}$ using the correspondence of Theorem 2.5. For any other associated prime $P \in \text{ass}(I)$, we will write $P \sim P_j$ if the prime $P$ is associated to a monomial $m$ that can be obtained from $x_{t_j} \cdots x_{t_{i_j}}$ via a Borel move. Note that each associated prime satisfies $P \sim P_j$ for exactly one $j \in \{0, \ldots, k\}$.

Let $A$ be the matrix of associated primes of $I$. There are $|\text{ass}(I)|$ rows, and we write $A_P$ for the row indexed by the associated prime $P$. Let $x = x_t$. We will write $x_P$ to denote the corresponding coordinate in $x$. That is, if $P$ indexes the $i$-th row of $A$, then $x_P$ denotes the $i$-th coordinate of $x$.

With the above notation, we now define the vector $y \in \mathbb{R}^{\text{ass}(I)}$ as follows:

$$y_P = \begin{cases} 
\frac{1}{\binom{i_{t_k}-1}{i_{t_k}}} & P \sim P_k \\
\frac{1}{\binom{i_{t_k}-1}{i_{s-1}}} & P \sim P_0 \text{ and } \langle x_{i_{t_k}+1}, \ldots, x_i \rangle \subseteq P \\
0 & \text{otherwise}.
\end{cases}$$

Note that the second criterion means we are only interested in those prime ideals that arise from Borel moves of $x_{t_0} \cdots x_{t_0}$ that also contain the variables $\{x_{i_{t_k}+1}, \ldots, x_i = i_{t_0}\}$.

The $r$-th row of $A^T$ is such that $(A^T)_r,P = 1$ if $x_r \in P$ and 0 otherwise. Then

$$(A^T y)_r = \sum_{x_r \in P} y_P.$$

So, in order to compute $A^T y$, we have to compute how many times $x_r$ appears in some $P$ such that $P \sim P_k$, or $P \sim P_0$ and $\langle x_{i_{t_k}+1}, \ldots, x_i \rangle \subseteq P$.

Observe that there are $\binom{i_{t_k}}{i_{t_k+1} - s}$ associated primes $P$ such that $P \sim P_k$. This number is the number of Borel moves that can be made from $x_{t_k} \cdots x_{i_{t_k}}$. There are $\binom{i_{t_k}}{i_{s-1}}$ associated primes $P$ such that $P \sim P_0$ and $\langle x_{i_{t_k}+1}, \ldots, x_i \rangle \subseteq P$. To see why this is true, suppose that we consider a Borel move of $x_{t_0} \cdots x_{t_0}$ that is also divisible by $x_{i_{t_k}+1} \cdots x_{i_{t_0}}$, i.e., the Borel move has the form $m'(x_{i_{t_k}+1} \cdots x_{i_{t_0}})$ where $m'$ is a degree $i_{t_k} - t_0 + 1$ monomial in $\{x_1, \ldots, x_i\}$. Since $s = t_0 \leq i_{t_k}$, there are $\binom{i_{t_k}}{i_{s-1}} \geq 1$ possible $m'$.

For $i_{t_k} < r \leq i_{t_0}$ we have that $x_r$ appears in every $P$ such that $y_P \neq 0$ and $P \sim P_0$. Therefore

$$(A^T y)_r = \binom{i_{t_k}}{i_{s-1}} \frac{1}{\binom{i_{t_k}}{i_{s-1}}} = 1.$$
Now, for $1 \leq r \leq \ell$, $x_r$ appears in $\binom{\ell_k - 1}{r}$ elements $P$ such that $P \sim P_k$, and in $\binom{\ell_k - 1}{s-1}$ elements $P$ such that $P \sim P_0$ and $\langle x_{\ell_k+1}, \ldots, x_{i_s} \rangle \subset P$. Therefore

$$
(A^T y)_r = \left( \frac{i_t - 1}{t_k - 1} \right) \left( \frac{1}{\binom{\ell_k - 1}{i_t - 1}} \right) \frac{1}{\binom{\ell_k - 1}{s-1}} + \left( \frac{i_t - 1}{s-1} \right) \frac{1}{\binom{\ell_k - 1}{s-1}} = \frac{s - 1}{i_t} + \frac{i_t - s + 1}{i_t} = 1.
$$

This proves that $A^T y = 1$. Finally

$$
y^T 1 = \left( \frac{i_t}{t_k - 1} \right) \left( \frac{1}{\binom{\ell_k - 1}{i_t}} \right) \frac{1}{\binom{\ell_k - 1}{s-1}} = \frac{i_t}{i_t - t_k + 1} \frac{s - 1}{i_t} + 1 = 1 + \frac{s - 1}{i_t - t_k + 1}.
$$

Due to the duality theorem, we can conclude the result. □

We arrive at the following corollary which was highlighted in the introduction.

**Corollary 4.2.** Let $I = \operatorname{sfBorel}(x_1 x_{i+1} \cdots x_{i+l})$. Then

$$
\tilde{\alpha}(I) = \frac{i + l}{i}.
$$

Consequently, for every rational number $a/b \geq 1$, there exists a square-free principal Borel ideal $I$ such that $\tilde{\alpha}(I) = a/b$.

**Proof.** We have $T(m) = (l + 1)$ and $IT(m) = (i + l)$. Now apply Theorem 4.1.

For the second statement, if $a/b = 1$, we can take $I = \operatorname{sfBorel}(x_1) = \langle x_1 \rangle$, from which it follows that $\tilde{\alpha}(I) = 1$. If $a/b > 1$, i.e., $a > b$, we have $a/b = \frac{b + (a - b)}{b}$. Then the result follows if we take $m = x_{b} x_{b+1} \cdots x_a = x_{b} x_{b+1} \cdots x_{b+(a-b)}$. □

**Remark 4.3.** When $I = \operatorname{sfBorel}(x_1 x_{i+1} \cdots x_n)$, then $I$ is generated by all the square-free monomials of degree $n - i + 1$ in $S$. The Waldschmidt constants for these ideals were first computed in [4, Theorem 7.5].

We now give a lower bound for the Waldschmidt constant of a square-free principal Borel ideal in terms of a smaller square-free principal Borel ideal.

**Theorem 4.4.** Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (\nu, i_t, \ldots, i_{t_k})$, and suppose $I = \operatorname{sfBorel}(m)$. Suppose that $t$ is the smallest integer such that $i_{t+1} < t_0 \leq i_t$. Define $\nu = i_{t+1} + 1$. Then

$$
\tilde{\alpha}(I) \geq \tilde{\alpha}(\operatorname{sfBorel}(x_{i_1} \cdots x_{i_{t+1}})) + 1 + \frac{t_0 - \nu}{i_\nu - \nu + 1}.
$$

**Proof.** By Lemma 3.1 we can assume we are working in the polynomial ring $\mathbb{K}[x_1, \ldots, x_{\ell+1}]$.

Consider the monomials

$$m_1 = x_{i_1} x_{i_2} \cdots x_{i_{t+1}} \in \mathbb{K}[x_1, \ldots, x_{i_{t+1}}]$$
and
\[ m_2 = x_{i_0}x_{i_0+1} \cdots x_{i_0} \in \mathbb{K}[x_{i_0}, x_{i_0+1}]. \]

Observe that while \( m_1 m_2 | m \), \( m \) is not necessarily this product.

Let \( I_1 = \text{sfBorel}(m_1) \) and \( I_2 = \text{sfBorel}(m_2) \), in their respective rings, and furthermore, let \( A(I_1) \) and \( A(I_2) \) be the corresponding matrices of associated primes.

Take \( p \) to be the biggest integer such that \( \nu \leq t_p \). Then
\[ T(m_2) = (t_0 - \nu + 1, t_1 - \nu + 1, \ldots, t_p - \nu + 1) \]

and
\[ IT(m_2) = (i_{t_0} - \nu + 1, i_{t_1} - \nu + 1, \ldots, i_{t_p} - \nu + 1). \]

We have \( p \leq \ell \), then \( i_{t_p} \geq t_0 \). So by Theorem 1.1 we have
\[ \hat{\alpha}(I_2) = 1 + \frac{t_0 - \nu}{i_{t_p} - t_p + 1}. \]

Observe that \( i_{\nu} - \nu + 1 = i_{t_p} - t_p + 1 \), since \( t_p + 1 < \nu \leq t_p \), meaning that \( x_{\nu} \cdots x_{i_\nu} \) is a Borel movement of \( x_{t_p} \cdots x_{i_{t_p}} \).

We claim that
\[ \begin{bmatrix} A(I_1) & 0 \\ 0 & A(I_2) \end{bmatrix} \]

is a submatrix of \( A(I) \), the matrix of associated primes of \( I \).

First notice that any row of \( A(I_1) \) is comes from a Borel movement of a corresponding \( x_{t}x_{t+1} \cdots x_{i_{t}} \) for some \( t \in T(m_1) \). By Theorem 2.5 these are also associated primes of \( I \), implying that \( [A(I_1) \, 0] \) is a submatrix of \( A(I) \).

Now, for any \( \ell + 1 > u > p \), \( x_{i_u} \cdots x_{i_{u'}} \) corresponds to a row of \( A(I) \), and any of its Borel movements contain at least one \( x_{j} \) with \( j < \nu \), otherwise \( \nu \leq u \), contradicting the choice of \( p \). This means for a row \( R \) in \( [0 \, A(I_2)] \), there cannot exist a row \( R' \) of \( A(I) \) with \( \text{supp } R' \subseteq \text{supp } R \). Thus any associated prime of \( I_2 \) can be viewed as an associated prime of \( I \) by Theorem 2.5. Thus \( [0 \, A(I_2)] \) is a submatrix of \( A(I) \) and by our choice of \( \nu \).

Let \( y_1 \) and \( y_2 \) be such that
\[ A(I_1)^T y_1 \leq 1, \quad 1^T y_1 = \hat{\alpha}(I_1) \text{ and } A(I_2)^T y_2 \leq 1, \quad 1^T y_2 = \hat{\alpha}(I_2). \]

After permuting rows, we can assume that
\[ A(I) = \begin{bmatrix} A(I_1) & 0 \\ 0 & A(I_2) \\ B_1 & B_2 \end{bmatrix} \]

where \( B_1, B_2 \) are some appropriately sized matrices. Set \( z = (y_1, y_2, 0) \), where \( 0 \) is a vector of zeroes, where the number of zeroes in this vector are the same as the number of rows as \( B \). Then \( A(I)^T z \leq 1 \). Thus, by the dual version of Theorem 2.7, we have
\[ \hat{\alpha}(I) \geq \hat{\alpha}(I_1) + \hat{\alpha}(I_2) = \hat{\alpha}(I_1) + 1 + \frac{t_0 - \nu}{i_{\nu} - \nu + 1}. \]

\( \square \)
Theorem 4.4 reduces the problem of finding a lower bound on the principal square-free Borel ideal $m = x_{i_1} \cdots x_{i_s}$ to finding a lower bound on the principal square-free Borel ideal of $x_{i_1} \cdots x_{i_{t-1}+1}$. Note one can now reapply Theorem 4.4 to this smaller ideal. At some point, the hypotheses of Theorem 4.1 will hold, which stops our recursive calculation.

This idea can be formally expressed as a formula, provided one is willing to introduce even further notation (involving further subscripts on our subscripts). Instead, we provide the following example in the hope of being more illuminating.

**Example 4.5.** Consider the monomial

$$m = x_3x_4x_5x_8x_{10}x_{48}x_{49}x_{50}x_{98}x_{99}x_{100} \in \mathbb{K}[x_1, \ldots, x_{100}]$$

and let $I = \text{sfBorel}(m)$. For this monomial $T(m) = (12, 9, 6, 3)$ and $IT(m) = (100, 50, 10, 5)$. Since $i_t = 10 < t_0 = 12 < i_{t-1} = 50$, then $\nu = 11$ and Theorem 4.4 gives

$$\hat{\alpha}(I) \geq \hat{\alpha}(I_1) + 1 + \frac{12 - 11}{98 - 10 + 1} = \hat{\alpha}(I_1) + \frac{90}{89}$$

where $I_1 = \text{sfBorel}(x_3x_4x_5x_8x_{10}) = \text{sfBorel}(m_1)$. For this new monomial, we have $T(m_1) = (6, 3)$ and $IT(m_1) = (10, 5)$. Again using Theorem 4.4 we get

$$\hat{\alpha}(I_1) \geq \hat{\alpha}(I_2) + 1 + \frac{6 - 6}{10 - 6 + 1} = \hat{\alpha}(I_2) + 1$$

where $I_2 = \text{sfBorel}(x_3x_4x_5)$. Then by Theorem 4.1 (or in this case, Corollary 4.2), we have $\hat{\alpha}(I_2) = \frac{5}{3}$. Hence

$$\hat{\alpha}(I) \geq \frac{5}{3} + 1 + \frac{90}{89} = \frac{982}{267}.$$ 

Note that if we apply the upper bound of Theorem 3.3 we get

$$3.6904 \approx \frac{155}{42} \geq \frac{982}{267} \approx 3.6779.$$ 

We finish our paper with a result that allows us to make small changes to the generator of the square-free principal Borel without changing the Waldschmidt constant.

**Theorem 4.6.** Let $I = \text{sfBorel}(x_{i_1} \cdots x_{i_{s-1}}x_{i_s})$ and $J = \text{sfBorel}(x_{i_1} \cdots x_{i_{s-1}}x_{i_{s+r}})$ for $r \in \mathbb{N}$. Then $\hat{\alpha}(I) = \hat{\alpha}(J)$.

**Proof.** Let $A$ be the matrix of associated primes of $I$ and let $y = (y_1, \ldots, y_s)$ be an optimal solution to

$$\min\{1^T x \mid Ax \geq 1\}.$$ 

First consider $r = 1$ with $J = \text{sfBorel}(x_{i_1} \cdots x_{i_{s-1}}x_{i_{s+1}})$. Let $A'$ be the matrix of associated primes of $J$. Any element of $\text{ass}(J)$ not in $\text{ass}(I)$ includes both $x_{i_s}$ and $x_{i_{s+1}}$ as generators, so the columns of $A'$ corresponding to $x_{i_s}$ and $x_{i_{s+1}}$ are identical. Let $y' = (y'_1, \ldots, y'_{s+1})$ be an optimal solution to

$$\min\{1^T x \mid A'x \geq 1\}.$$ 

and suppose for contradiction that $y'_1 + \cdots + y'_{s+1} = \hat{\alpha}(J) < \hat{\alpha}(I)$. But this means $(y'_1, \ldots, y'_{s+1})$ is a feasible solution to (4.1), contradicting $y$ being optimal and showing $\hat{\alpha}(J) \geq \hat{\alpha}(I)$. Observing that $(y_1, \ldots, y_s, 0)$ is a feasible solution to (4.2) gives
This shows $\hat{\alpha}(I) = \hat{\alpha}(J)$, and an inductive argument gives the result for $r \in \mathbb{N}$.

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