In this brief comment, we consider the exact, deterministic, and nonasymptotic transformation of multiple copies of pure states under LOCC. It was conjectured in quant-ph/0103131 that, if $k$ copies of $|\psi\rangle$ can be transformed to $k$ copies of $|\phi\rangle$, the same holds for all $r \geq k$. We present counterexamples to the above conjecture.

1 Background and notations

We consider pure bipartite states shared by Alice and Bob. Let $A$ and $B$ be the labels of their respective quantum systems, each with $d$ dimensions. Suppose Alice and Bob want to transform a state $|\psi\rangle$ to another state $|\phi\rangle$. They each have complete classical descriptions of the states, but they can only perform Local quantum Operations and Classical Communication (LOCC) \[1\]. The transformation can be performed under many scenarios. The final state can be exactly or approximately the desired output, and the transformation can work deterministically or with certain probability. It is also possible to consider converting between an asymptotically large numbers of two different states.

We consider here only the scenario of exact, deterministic, and nonasymptotic transformation using LOCC. If such transformation is possible, we write $|\psi\rangle \rightarrow |\phi\rangle$, otherwise, we write $|\psi\rangle \not\rightarrow |\phi\rangle$. A necessary and sufficient condition for $|\psi\rangle \rightarrow |\phi\rangle$ was found by Nielsen \[2\], with beautiful connections to the theory of majorization. For a pure state $|\eta\rangle$, let $\lambda_{\eta}$ denote the list of eigenvalues of the reduced density matrix $\text{Tr}_A(|\eta\rangle\langle\eta|)$ in descending order, and let $\lambda_{\eta}^{(i)}$ be the $i$-th element in $\lambda_{\eta}$. Nielsen’s criterion can be stated as

$$\lambda_{\psi} \leq \lambda_{\phi} \quad \text{iff} \quad \forall m \leq d \sum_{i=1}^{m} \lambda_{\psi}^{(i)} \leq \sum_{i=1}^{m} \lambda_{\phi}^{(i)}$$

(1)

An intriguing consequence of the criterion is that, even when $|\psi\rangle \not\rightarrow |\phi\rangle$, it may still be possible to transform the state $|\psi\rangle$ to the state $|\phi\rangle$ as part of a larger transformation. For example, $|\psi\rangle|\alpha\rangle \rightarrow |\phi\rangle|\alpha\rangle$ may be true for some state $|\alpha\rangle$, which functions as a catalyst \[3\]. It is also possible for $|\psi\rangle^\otimes 2 \rightarrow |\phi\rangle^\otimes 2$ \[4\] or even $|\psi\rangle^\otimes 4 \rightarrow |\phi\rangle^\otimes 5$ \[5\].

In Ref. \[6\], interesting properties about multiple-copy transformation between $|\psi\rangle^\otimes k$ and $|\phi\rangle^\otimes k$ under various scenarios are investigated. In the scenario of exact, deterministic, asymptotic transformation considered in this paper, they have derived a necessary condition for $|\psi\rangle^\otimes k \rightarrow |\phi\rangle^\otimes k$ for some $k$, $\lambda^{(1)}_{\psi} \leq \lambda^{(1)}_{\phi}$ and $\lambda^{(d)}_{\psi} \geq \lambda^{(d)}_{\phi}$. They have also made a very plausible conjecture,

$$|\psi\rangle^\otimes k \rightarrow |\phi\rangle^\otimes k \quad \Rightarrow \quad \forall r \geq k \quad |\psi\rangle^\otimes r \rightarrow |\phi\rangle^\otimes r.$$  

(2)

The conjecture is obviously true when $k$ divides $r$. However, we find counterexamples for $k = 2$, $r = 3$, and
\( d \geq 5 \). One example for \( d = 5 \) is given by

\[
\lambda_{\phi} = \{0.493, 0.284, 0.158, 0.035, 0.030\} \\
\lambda_{\psi} = \{0.598, 0.145, 0.129, 0.125, 0.003\}
\]

and a simple example for \( d = 6 \) is given by

\[
\lambda_{\psi} = \{0.24, 0.22, 0.22, 0.10, 0.10, 0.03\} \\
\lambda_{\phi} = \{0.27, 0.25, 0.16, 0.16, 0.15, 0.01\}
\]

The conjecture fails because of the exact and nonasymptotic nature of the transformation.

We remark that, for fixed \( k \), a counterexample in \( d_0 \) dimensions can always be turned into one in \( d \geq d_0 \) dimensions. The reason is that the criteria for a counterexample involve only continuous functions of \( \lambda_{\psi} \) and \( \lambda_{\phi} \). Therefore, the above counterexamples suffice to disprove the conjecture for \( k = 2 \) and \( \forall d \geq 5 \).

In the following, we describe numerical results that provide not only counterexamples but also statistics of violations of the conjecture.

## 2 Counterexamples

We performed a numerical search for \( d = 4, 5, \ldots, 9 \) and \( k = 2 \). The \( d = 3 \) case is uninteresting, as \( |\psi\rangle \not\rightarrow |\phi\rangle \) implies \( \forall k \ |\psi\rangle^\otimes k \not\rightarrow |\phi\rangle^\otimes k \). For each \( d \), a large number of random pairs \( \{\lambda_{\psi}, \lambda_{\phi}\} \) are sampled, and without loss of generality, \( \lambda_{\phi}^{(1)} \leq \lambda_{\psi}^{(1)} \). We search for \( \{\lambda_{\psi}, \lambda_{\phi}\} \) corresponding to \( |\psi\rangle, |\phi\rangle \) satisfying

\[
\begin{align*}
\text{I} & \quad |\psi\rangle \not\rightarrow |\phi\rangle \, \text{and} \, |\psi\rangle^\otimes 2 \not\rightarrow |\phi\rangle^\otimes 2 \\
\text{II} & \quad |\psi\rangle^\otimes 3 \not\rightarrow |\phi\rangle^\otimes 3
\end{align*}
\]

The search parameters and the results can be summarized:

| \( d \) | 4 \times 10^8 | 2 \times 10^8 | 10^7 | 10^7 | 10^7 | 10^7 |
|---|---|---|---|---|---|---|
| Sample size | \( \times 10^{-3} \) | 1.56 \times 10^{-2} | 2.16 \times 10^{-2} | 2.63 \times 10^{-2} | 3.01 \times 10^{-2} | 3.31 \times 10^{-2} |
| Fraction of sample satisfying I | 8.31 \times 10^{-7} | 3.21 \times 10^{-7} | 3.94 \times 10^{-4} | 3.12 \times 10^{-4} | 1.60 \times 10^{-4} | 1.63 \times 10^{-4} |
| Fraction of sample satisfying II given I | 0 |

The fractions of sample satisfying I have negligible statistical uncertainties, while the fractions satisfying II conditioned on I have about 11-14\% uncertainties. For each \( d \geq 6 \), tens of counterexamples for the conjecture are found. For \( d = 5 \), exactly one counterexample is found in \( 2 \times 10^8 \) random samples. For \( d = 4 \) no counterexample is found in \( 4 \times 10^8 \) random samples, so that the conjecture may still be true for this value of \( d \).

1Except for \( d = 4 \) and 5 for which few counterexamples are found.
3 Discussion

The conjecture investigated in this paper fails because of the exact, deterministic, and nonasymptotic nature of the transformation. The conclusion drawn from our results, along with Refs. [4, 5, 6], is that even if one knows for all \( k < r \), whether \(|\psi\rangle^\otimes k \rightarrow |\phi\rangle^\otimes k\), one still cannot infer whether \(|\psi\rangle^\otimes r \rightarrow |\phi\rangle^\otimes r\) without additional information on the states. This is in sharp contrast with the much simpler conditions for asymptotic manipulations. In particular, asymptotically \(|\psi\rangle^\otimes m\) and \(|\phi\rangle^\otimes n\) are interconvertible with high fidelity and probability if \( mH(\lambda_\psi) = nH(\lambda_\phi) \) where \( H \) is the entropy function.

Nielsen’s criterion on transformation of pure bipartite states has led to many important results in the various possible scenarios, some of which are counterintuitive. The results in the exact, deterministic, nonasymptotic scenario are mathematically interesting, yet their very oddity suggests that the asymptotic or probabilistic scenarios are perhaps more physically relevant.

4 Acknowledgments

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References

[1] See, for example, C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A, 54 3824 (1996).

[2] M.A. Nielsen, Phys. Rev. Lett. 83 436 (1999).

[3] D. Jonathan and M.B. Plenio, Phys. Rev. Lett. 83 3566 (1999).

[4] Many researchers have found such examples. A partial list includes D. Beckmann, D. Jonathan, M.A. Nielsen, M.B. Plenio, and J.A. Smolin.

[5] D. Beckman and M.A. Nielsen, personal communications. In our notations, their example is represented by

\[
\begin{align*}
\lambda_\psi &= \{0.6465, 0.3406, 0.0075, 0.0053\} \\
\lambda_\phi &= \{0.9557, 0.0269, 0.0174, 0\}
\end{align*}
\]

[6] S. Bandyopadhyay, V. Royhowdhury, and U. Sen, arXive e-print quant-ph/0103131.