Covariant Derivatives of Multivector and Extensor Fields

A. M. Moya\textsuperscript{2}, V. V. Fernández\textsuperscript{1} and W. A. Rodrigues Jr.\textsuperscript{1}
\textsuperscript{1}Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP CP 6065
13083-859 Campinas, SP, Brazil
e-mail: walrod@ime.unicamp.br
\textsuperscript{2}Department of Mathematics, University of Antofagasta, Antofagasta, Chile
e-mail: mmoya@uantof.cl

November 20, 2018

Abstract

We give in this paper which is the fifth in a series of eight a theory of covariant derivatives of multivector and extensor fields based on the geometric calculus of an arbitrary smooth manifold $M$, and the notion of a connection extensor field defining a parallelism structure on $M$. Also we give a novel and intrinsic presentation (i.e., one that does not depend on a chosen orthonormal moving frame) of the torsion and curvature fields of Cartan’s theory. Two kinds of Cartan’s connection fields are identified, and both appear in the intrinsic Cartan’s structure equations satisfied by the Cartan’s torsion and curvature extensor fields.

Contents

1 Introduction 2
2 Parallelism Structure 3
1 Introduction

This is the fifth paper in a series of eight. It is dedicated to a theory of covariant derivatives of multivector and extensor fields. We introduce in Section 2 the notion of a parallelism structure on a smooth manifold $M$ given by a connection extensor field $\gamma$.

In Sections 3 and 4 we introduce the concepts of $a$-directional covariant derivatives of multivector and extensor fields, respectively, and prove the main properties satisfied by these objects. In Section 5 we give a thoughtful study of the so-called symmetric parallelism structures, where among others we present and prove a Bianchi-like identity and give an intrinsic Cartan theory of the torsion and curvature fields. Cartan’s connections of the first and second kind are identified, and both appear on Cartan’s structure equations. We emphasize the novelty of our approach to Cartan theory, namely that it does not depend on a chosen orthonormal moving frame, hence, the name intrinsic used above. In an Appendix several examples are worked in detail in order to show how our theory relates to known theories dealing with the same subject.
2 Parallelism Structure

Let $M$ be a $n$-dimensional smooth manifold. Then, for any point $o \in M$ there exists a local coordinate system $(U_o, \phi_o)$ such that $o \in U_o$ and $\phi_o(o) = (0, \ldots, 0) \in \mathbb{R}^n$. $U_o$ is an open subset of $M$, and $\phi_o$ is an homeomorphism from $U_o$ onto the open subset $\phi_o(U_o) \subseteq \mathbb{R}^n$.

As in [1], let $U_o$ be the canonical space for $(U_o, \phi_o)$, and $U$ be an open subset of $U_o$. We denote the ring (with identity) of smooth scalar fields on $U$, the module of smooth vector fields on $U$ and the module of smooth multi-vector fields on $U$ respectively by $\mathcal{S}(U)$, $\mathcal{V}(U)$ and $\mathcal{M}(U)$. The set of smooth $k$-vector fields on $U$ is denoted by $\mathcal{M}_k(U)$. The module of smooth $k$-extensor fields on $U$ is denoted by $k$-$\text{ext}\!(\mathcal{M}_1^1(U), \ldots, \mathcal{M}_k^2(U); \mathcal{M}_0(U))$, see [1].

Any smooth vector elementary 2-extensor field on $U$ is said to be a connection field on $U$. A general connection field will be denoted by $\gamma$, i.e., $\gamma : U \to 2\text{-ext}^1(U_o)$. The smoothness of such $\gamma$ means that for all $a, b \in \mathcal{V}(U_o)$ the vector field defined by $U \ni p \mapsto \gamma(p)(a(p), b(p)) \in U_o$ is itself smooth.

The open set $U$ equipped with such a connection field $\gamma$, namely $(U, \gamma)$, will be said to be a parallelism structure on $U$.

Let us take $a \in U_o$. The smooth $(1, 1)$-extensor field on $U$, namely $\gamma_a$, defined as $U \ni p \mapsto \gamma_a|_p \in \text{ext}_1^1(U_o)$ such that for all $b \in U_o$

$$\gamma_a|_p(b) = \gamma(p)(a, b), \quad (1)$$

will be called the a-directional connection field on $U$, of course, associated to $(U, \gamma)$.

We emphasize that the $(1, 1)$-extensor character and the smoothness of $\gamma_a$ are immediate consequences of the vector elementary 2-extensor character and the smoothness of $\gamma$.

The smooth $(1, 2)$-extensor field on $U$, namely $\Omega$, defined as $U \ni p \mapsto \Omega(p) \in \text{ext}_1^2(U_o)$ such that for all $a \in U_o$

$$\Omega(p)(a) = \frac{1}{2} \text{biv} \left[ \gamma_a|_p \right], \quad (2)$$

will be called the gauge connection field on $U$.

From the definition of $\text{biv}[t]$ (see [2]), taking any pair of reciprocal frame fields on $U$, say $(\{e_\mu\}, \{e^\mu\})$, and using Eq.$(1)$, we can write Eq.$(2)$ as

$$\Omega(p)(a) = \frac{1}{2} \gamma(p)(a, e^\mu(p)) \wedge e_\mu(p) = \frac{1}{2} \gamma(p)(a, e_\mu(p)) \wedge e^\mu(p). \quad (3)$$
So, we see that the \((1, 2)\)-extensor character and the smoothness of \(\Omega\) are easily deduced from the vector elementary 2-extensor character and the smoothness of \(\gamma\).

Let us take any pair of reciprocal frame fields on \(U\), say \(\left\{ e_\mu \right\}, \left\{ e^\mu \right\}\), i.e., \(e_\mu \cdot e^\nu = \delta^\nu_\mu\). Let \(\Gamma_a\) be the \textit{generalized (extensor field)} of \(\gamma_a\) (see \([2]\)), i.e., \(\Gamma_a\) defined as \(U \ni p \mapsto \Gamma_a|_p \in \text{ext}(\mathcal{U}_a)\), is a smooth extensor field on \(U\) such that for all \(X \in \bigwedge U\):

\[
\Gamma_a|_p(X) = \gamma_a|_p(e^\mu(p)) \wedge (e_\mu(p) \lrcorner X) = \gamma_a|_p(e_\mu(p)) \wedge (e^\mu(p) \lrcorner X). \quad (4)
\]

It is easily seen that (for a given \(X\)) the multivector appearing on the right side of Eq.(4) does not depend on the choice of the reciprocal frame fields. The extensor character and the smoothness of \(\Gamma_a\) follows from the \((1, 1)\)-extensor character and the smoothness of \(\gamma_a\).

We will usually omit the letter \(p\) in writing the definitions given by Eq.(1), Eq.(2) and Eq.(4), and other equations using extensor fields. No confusion should arise with this standard practice. Eq.(4) might be also written in the succinct form \(\Gamma_a(X) = \gamma_a(\partial_b) \wedge (b \lrcorner X)\).

We end this section by presenting some of the basic properties satisfied by \(\Gamma_a\).

i. \(\Gamma_a\) is grade-preserving, i.e.,

\[
\text{if } X \in \mathcal{M}^k(U), \text{ then } \Gamma_a(X) \in \mathcal{M}^k(U). \quad (5)
\]

ii. For any \(X \in \mathcal{M}(U)\),

\[
\Gamma_a(\tilde{X}) = \tilde{\Gamma_a}(X), \quad (6)
\]

\[
\Gamma_a(\hat{X}) = \hat{\Gamma_a}(X), \quad (7)
\]

\[
\Gamma_a(\widehat{X}) = \overline{\Gamma_a}(X). \quad (8)
\]

iii. For any \(f \in \mathcal{S}(U), b \in \mathcal{V}(U)\) and \(X, Y \in \mathcal{M}(U)\),

\[
\Gamma_a(f) = 0, \quad (9)
\]

\[
\Gamma_a(b) = \gamma_a(b), \quad (10)
\]

\[
\Gamma_a(X \wedge Y) = \Gamma_a(X) \wedge Y + X \wedge \Gamma_a(Y). \quad (11)
\]

iv. The adjoint of \(\Gamma_a\), namely \(\Gamma_a^\dagger\), is the generalized of the adjoint of \(\gamma_a\), namely \(\gamma_a^\dagger\), i.e.,

\[
\Gamma_a^\dagger(X) = \gamma_a^\dagger(\partial_b) \wedge (b \lrcorner X). \quad (12)
\]
v. The symmetric (skew-symmetric) part of $\Gamma_a$, namely $\Gamma_a^\pm = \frac{1}{2}(\Gamma_a \pm \Gamma_a^\dagger)$, is the generalized of the symmetric (skew-symmetric) part of $\gamma_a$, namely $\gamma_a^\pm = \frac{1}{2}(\gamma_a \pm \gamma_a^\dagger)$, i.e.,

$$\Gamma_a^\pm(X) = \gamma_a^\pm(\partial_b) \wedge (b \wedge X). \quad (13)$$

vi. $\Gamma_{a-}$ can be factorized by a remarkable formula which only involves $\Omega$. It is

$$\Gamma_{a-}(X) = \Omega(a) \times X. \quad (14)$$

vii. For any $X, Y \in \mathcal{M}(U)$ it holds

$$\Gamma_{a-}(X \star Y) = \Gamma_{a-}(X) \star Y + X \star \Gamma_{a-}(Y), \quad (15)$$

where $\star$ means any suitable product of smooth multivector fields, either $(\wedge)$, $(\cdot)$, $(\lrcorner, \lrcorner)$ or $(b$-Clifford product), see [1, 3].

3 $a$-Directional Covariant Derivatives of Multivector Fields

Given a parallelism structure $(U, \gamma)$, let us take $a \in \mathcal{U}_o$. Then, associated to $(U, \gamma)$ we can introduce two $a$-directional covariant derivative operators ($a$-DCDO’s), namely $\nabla_a^+$ and $\nabla_a^-$, which act on the module of smooth multivector fields on $U$.

They are defined by $\nabla_a^{\pm} : \mathcal{M}(U) \to \mathcal{M}(U)$ such that

$$\nabla_a^+(X)(p) = a \cdot \partial_o X(p) + \Gamma_{a|p}(X(p)) \quad (16)$$

$$\nabla_a^-(X)(p) = a \cdot \partial_o X(p) - \Gamma_{a|p}^\dagger(X(p)) \quad (17)$$

where $a \cdot \partial_o$ is the canonical $a$-DCDO as was defined in [1].

We emphasize that each of $\nabla_a^+$ and $\nabla_a^-$ satisfies indeed the fundamental properties which a well-defined covariant derivative is expected to have. This is trivial to verify whenever we take into account the well-known properties of $a \cdot \partial_o$ (see [1]), and the properties of $\Gamma_a$ given by Eq.(5), Eqs.(9), (10) and (11), and Eq.(12).

As usual we will write Eq.(16) and Eq.(17) by omitting $p$ when no confusion arises.
The smooth multivector fields on \( U \), namely \( \nabla^+_aX \) and \( \nabla^-_aX \), will be respectively called the plus and the minus \( a \)-directional covariant derivatives of \( X \).

We summarize some of the most important properties for the pair of \( a \)-DCDO’s \( \nabla^+_a \) and \( \nabla^-_a \).

i. \( \nabla^+_a \) and \( \nabla^-_a \) are grade-preserving operators on \( \mathcal{M}(U) \), i.e.,

\[
\text{if } X \in \mathcal{M}^k(U), \text{ then } \nabla^\pm_a X \in \mathcal{M}^k(U).
\] (18)

ii. For all \( X \in \mathcal{M}(U) \), and for any \( \alpha, \alpha' \in \mathbb{R} \) and \( a, a' \in \mathcal{U}_o \) we have

\[
\nabla^\pm_{\alpha a + \alpha'a'} X = \alpha \nabla^\pm_a X + \alpha' \nabla^\pm_{a'} X.
\] (19)

iii. For all \( f \in \mathcal{S}(U) \) and \( X, Y \in \mathcal{M}(U) \) we have

\[
\nabla^\pm_a f = a \cdot \partial_o f,
\]
\[
\nabla^\pm_a (X + Y) = \nabla^\pm_a X + \nabla^\pm_a Y,
\]
\[
\nabla^\pm_a (fX) = (a \cdot \partial_o f)X + f(\nabla^\pm_a X).
\] (20, 21, 22)

iv. For all \( X, Y \in \mathcal{M}(U) \) we have

\[
\nabla^\pm_a (X \wedge Y) = (\nabla^\pm_a X) \wedge Y + X \wedge (\nabla^\pm_a Y).
\] (23)

v. For all \( X, Y \in \mathcal{M}(U) \) we have

\[
(\nabla^+_a X) \cdot Y + X \cdot (\nabla^-_a Y) = a \cdot \partial_o (X \cdot Y).
\] (24)

It should be noticed that \( (\nabla^+_a, \nabla^-_a) \) as defined by Eq.(16) and Eq.(17) is the unique pair of \( a \)-DCDO’s associated to \( (U, \gamma) \) which satisfies the remarkable property given by Eq.(24).

We emphasize here that the \( a \)-DO\( DO \) \( a \cdot \partial_o \) acting on \( \mathcal{M}(U) \), is also a well-defined \( a \)-DCDO. In this particular case, the connection field \( \gamma \) is identically zero and the plus and minus \( a \)-DCDO’s are equal to each other, and both of them coincide with \( a \cdot \partial_o \).

We introduce yet another well-defined \( a \)-DCDO which acts also on the module of smooth multivector fields on \( U \), namely \( \nabla^0_a \).

It is defined by

\[
\nabla^0_a X = \frac{1}{2}(\nabla^+_a X + \nabla^-_a X).
\] (25)
But, by using Eqs.\((16)\) and \((17)\), and Eq.\((14)\), we might write
\[
\nabla_0^a X = a \cdot \partial_o X + \Omega(a) \times X.\tag{26}
\]

The \(a\)-DCDO \(\nabla_0^a\) satisfies the same properties which hold for each one of the \(a\)-DCDO’s \(\nabla_0^+\) and \(\nabla_0^-\). But, it has also an additional remarkable property
\[
(\nabla_0^0 X) \cdot Y + X \cdot (\nabla_0^0 Y) = a \cdot \partial_o (X \cdot Y).\tag{27}
\]

Moreover, it satisfies a Leibnitz-like rule for any suitable product of smooth multivector fields, i.e.,
\[
\nabla_0^a (X \ast Y) = (\nabla_0^a X) \ast Y + X \ast (\nabla_0^a Y).\tag{28}
\]

### 3.1 Connection Operators

Associated to any parallelism structure \((U, \gamma)\) we can introduce two remarkable operators which map 2-uples of smooth vector fields to smooth vector fields.

They are defined by \(\Gamma^\pm : \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)\) such that
\[
\Gamma^\pm(a, b) = \nabla_a^\pm b,\tag{29}
\]
and will be called the connection operators of \((U, \gamma)\).

We summarize the basic properties of them.

- **i.** For all \(f \in \mathcal{S}(U)\), and \(a, a', b, b' \in \mathcal{V}(U)\), we have
  \[
  \Gamma^\pm(a + a', b) = \Gamma^\pm(a, b) + \Gamma^\pm(a', b),\tag{30}
  \]
  \[
  \Gamma^\pm(a, b + b') = \Gamma^\pm(a, b) + \Gamma^\pm(a, b'),\tag{31}
  \]
  \[
  \Gamma^\pm(fa, b) = f\Gamma^\pm(a, b),\tag{32}
  \]
  \[
  \Gamma^\pm(a, fb) = (a \cdot \partial_o f)b + f\Gamma^\pm(a, b).\tag{33}
  \]

As we can observe both connection operators satisfy the linearity property only with respect to the first smooth vector field variable. Thus, connection operators are not extensor fields.

- **ii.** For all \(a, b, c \in \mathcal{V}(U)\), we have
  \[
  \Gamma^+(a, b) \cdot c + b \cdot \Gamma^-(a, c) = a \cdot \partial_o (b \cdot c).\tag{34}
  \]

It is an immediate consequence of Eq.\((24)\).
3.2 Deformation of Covariant Derivatives

Let \((\nabla^+_a, \nabla^-_a)\) be any pair of \(a\)-DCDO’s and \(\lambda\) a non-singular smooth \((1, 1)\)-extensor field on \(U\). We define the deformation of these covariant derivatives as the pair \((\lambda \nabla^+_a, \lambda \nabla^-_a)\) by

\[
\lambda \nabla^+_a X = \Delta(\nabla^+_a \Delta^{-1}(X)),
\]
\[
\lambda \nabla^-_a X = \Delta^*(\nabla^-_a \Delta^!(X)),
\]

where \(\Delta\) is the extended\(^1\) of \(\lambda\), is a well-defined pair of \(a\)-DCDO’s, since it satisfies as it is trivial to verify the fundamental properties given by Eqs. (20), (21) and (22), Eq. (23) and Eq. (24). For instance,

\[
\lambda \nabla^+_a f = \Delta(\nabla^+_a \Delta^{-1}(f)) = a \cdot \partial_0 f.
\]

We verify now that the definitions given by Eq. (32) and Eq. (33) also satisfy a property analogous to the one given by Eq. (24). Indeed, we have

\[
(\lambda \nabla^+_a X) \cdot Y + X \cdot (\lambda \nabla^-_a Y) = (\nabla^+_a \Delta^{-1}(X)) \cdot \Delta^!(Y) + \Delta^{-1}(X) \cdot (\nabla^-_a \Delta^!(Y))
\]
\[
= a \cdot \partial_0(\Delta^{-1}(X) \cdot \Delta^!(Y)),
\]
\[
= a \cdot \partial_0 (X \cdot Y).
\]

4 \(a\)-Directional Covariant Derivatives of Extensor Fields

The three \(a\)-DCDO’s \(\nabla^+_a, \nabla^-_a\) and \(\nabla^0_a\) which act on \(\mathcal{M}(U)\) can be extended in order to act on the module of smooth \(k\)-extensor fields on \(U\). For any \(t \in k\text{-ext}(\mathcal{M}^*_1(U), \ldots, \mathcal{M}^*_k(U); \mathcal{M}^\sigma(U))\), we can define exactly \(3^{k+1}\) covariant derivatives, namely \(\nabla^\sigma_1 \ldots \sigma_k \cdot t \in k\text{-ext}(\mathcal{M}^*_1(U), \ldots, \mathcal{M}^*_k(U); \mathcal{M}^\sigma(U))\), where each of \(\sigma_1, \ldots, \sigma_k, \sigma\) is being used to mean either \((+), (-)\) or \((0)\). They are given by the following definition.

For all \(X_1 \in \mathcal{M}^*_1(U), \ldots, X_k \in \mathcal{M}^*_k(U), X \in \mathcal{M}^\sigma(U)\)

\[
(\nabla^\sigma_1 \ldots \sigma_k t)_p(X_1(p), \ldots, X_k(p)) \cdot X(p)
\]
\[
= a \cdot \partial_0(t(p)(\ldots) \cdot X(p)) - t(p)(\nabla_a^\sigma_1 X_1(p), \ldots) \cdot X(p)
\]
\[
- \cdots - t(p)(\ldots, \nabla_a^\sigma_k X_k(p)) \cdot X(p) - t(p)(\ldots) \cdot \nabla_a^\sigma X(p),
\]

\(^1\)Recall that \(\lambda^* = (\lambda^{-1})^\dagger = (\lambda^!)^{-1}\), and \(\Delta^{-1} = (\Delta)^{-1} = (\Delta^1)^{-1}\) and \(\Delta^\dagger = (\Delta)^\dagger = (\Delta^!)\), see [2].
for each \( p \in U \).

As usual when no confusion arises we will write Eq. (39) by omitting \( p \).

We call the reader’s attention that each of \( \nabla_{a}^{\sigma_{1}...\sigma_{k}}t \) as defined by Eq. (39) is in fact a smooth \( k \)-extensor field. Its \( k \)-extensor character and smoothness can be easily deduced from the respective properties of \( t \). We note also that in the first term on the right side of Eq. (39), \( a \cdot \partial \_o \) refers to the canonical \( a-DCDO \) as was defined in \([1]\).

We notice that any smooth \((1,1)\)-extensor field on \( U \), say \( t \), has just \( 3^{1+1} = 9 \) covariant derivatives. For instance, four important covariant derivatives of such \( t \) are given by

\[
\begin{align*}
(\nabla_{a}^{++}t)(X_{1}) \cdot X &= a \cdot \partial \_o(t(X_{1}) \cdot X) \\
&- t(\nabla_{a}^{+}X_{1}) \cdot X - t(X_{1}) \cdot \nabla_{a}^{+}X, \\
(\nabla_{a}^{+-}t)(X_{1}) \cdot X &= a \cdot \partial \_o(t(X_{1}) \cdot X) \\
&- t(\nabla_{a}^{+}X_{1}) \cdot X - t(X_{1}) \cdot \nabla_{a}^{-}X, \\
(\nabla_{a}^{-+}t)(X_{1}) \cdot X &= a \cdot \partial \_o(t(X_{1}) \cdot X) \\
&- t(\nabla_{a}^{-}X_{1}) \cdot X - t(X_{1}) \cdot \nabla_{a}^{+}X, \\
(\nabla_{a}^{-}t)(X_{1}) \cdot X &= a \cdot \partial \_o(t(X_{1}) \cdot X) \\
&- t(\nabla_{a}^{-}X_{1}) \cdot X - t(X_{1}) \cdot \nabla_{a}^{-}X,
\end{align*}
\]

where \( X_{1} \in \mathcal{M}_{1}(U) \) and \( X \in \mathcal{M}(U) \).

We present now some of the basic properties satisfied by these \( a \)-directional covariant derivatives of smooth \( k \)-extensor fields.

**i.** For all \( f \in S(U) \), and \( t, u \in k\text{-}ext(\mathcal{M}_{1}(U), \ldots, \mathcal{M}_{k}(U); \mathcal{M}(U)) \), it holds

\[
\nabla_{a}^{\sigma_{1}...\sigma_{k}}(t + u) = \nabla_{a}^{\sigma_{1}...\sigma_{k}}t + \nabla_{a}^{\sigma_{1}...\sigma_{k}}u,
\]

\[
\nabla_{a}^{\sigma_{1}...\sigma_{k}}(ft) = (a \cdot \partial \_o f)t + f(\nabla_{a}^{\sigma_{1}...\sigma_{k}}t).
\]

**ii.** For all \( t \in 1\text{-}ext(\mathcal{M}_{1}(U); \mathcal{M}(U)) \), it holds a noticeable property

\[
(\nabla_{a}^{\sigma_{1}...\sigma_{k}}t)^\dagger = \nabla_{a}^{\sigma_{1}...\sigma_{k}}t^\dagger.
\]

Note the inversion between \( \sigma_{1} \) and \( \sigma \) into the \( a-DCDO \)’s above. As we can see, the three \( a-DCDO \)’s \( \nabla_{a}^{++}, \nabla_{a}^{--} \) and \( \nabla_{a}^{00} \) commute indeed with the adjoint operator \( \dagger \).

**Proof**
Let us take $X_1 \in \mathcal{M}_1^i(U)$ and $X \in \mathcal{M}^c(U)$. By recalling the fundamental property of the adjoint operator [2], and in accordance with Eq. (39), we can write

$$((\nabla^a \sigma_1 t)^\dagger)(X_1) \cdot X = (\nabla^a \sigma_1 t)(X_1) \cdot X$$

$$= a \cdot \partial_o(t(X_1) \cdot X) - t(\nabla^a_1 X_1) \cdot X - t(X_1) \cdot \nabla^a_1 X$$

$$= a \cdot \partial_o(t^\dagger(X) \cdot X) - t^\dagger(X) \cdot \nabla^a_1 X_1 - t^\dagger(\nabla^a_1 X) \cdot X_1,$$

$$= (\nabla^a \sigma_1 t^\dagger)(X) \cdot X_1.$$  

Hence, by non-degeneracy of scalar product, the expected result immediately follows.

iii. For all $t \in 1$-$ext(\mathcal{M}_1^i(U), \mathcal{M}^c(U))$, it holds

$$(\nabla^{++}_a t)(X_1) = \nabla^{-}_a t(X_1) - t(\nabla^+_a X_1), \quad (47)$$

$$(\nabla^{+-}_a t)(X_1) = \nabla^{++}_a t(X_1) - t(\nabla^-_a X_1), \quad (48)$$

$$(\nabla^{--}_a t)(X_1) = \nabla^{+-}_a t(X_1) - t(\nabla^{--}_a X_1), \quad (49)$$

$$(\nabla^{-+}_a t)(X_1) = \nabla^{-+}_a t(X_1) - t(\nabla^{-+}_a X_1). \quad (50)$$

Proof

We will only try Eq. (47). Let us take $X_1 \in \mathcal{M}_1^i(U)$ and $X \in \mathcal{M}^c(U)$. In accordance with Eq. (39), and by recalling Eq. (24), we have

$$(\nabla^{++}_a t)(X_1) \cdot X = a \cdot \partial_o(t(X_1) \cdot X) - t(\nabla^+_a X_1) \cdot X - t(X_1) \cdot \nabla^+_a X$$

$$= \nabla^-_a t(X_1) \cdot X + t(X_1) \cdot \nabla^+_a X$$

$$- t(\nabla^+_a X_1) \cdot X - t(X_1) \cdot \nabla^+_a X$$

$$= (\nabla^-_a t(X_1) - t(\nabla^+_a X_1)) \cdot X.$$  

Hence, by non-degeneracy of scalar product, it follows what was to be proved.

5 Torsion and Curvature Fields

Let $(U, \gamma)$ be a parallelism structure on $U$. The smooth vector elementary 2-exform field on $U$, namely, $\tau$ such that for all $a, b \in \mathcal{V}(U)$

$$\tau(a, b) = \nabla^+_a b - \nabla^-_b a - [a, b], \quad (51)$$
i.e.,
\[ \tau(a, b) = \gamma_a(b) - \gamma_b(a), \tag{52} \]
will be called the torsion field of \((U, \gamma)\).

The smooth vector elementary 3-extensor field on \(U\), namely \(\rho\), such that for all \(a, b, c \in V(U)\)
\[ \rho(a, b, c) = [\nabla_+ a, \nabla_+ b]c - \nabla_+ [a, b]c, \tag{53} \]
i.e.,
\[ \rho(a, b, c) = (a \cdot \partial_\gamma b)(c) - (b \cdot \partial_\gamma a)(c) + [\gamma_a, \gamma_b](c) - \gamma_{[a,b]}(c), \tag{54} \]
will be called the curvature field of \((U, \gamma)\).

It should be emphasized that the curvature field \(\rho\) is skew-symmetric in first and second variables, i.e.,
\[ \rho(a, b, c) = -\rho(b, a, c). \tag{55} \]

5.1 Symmetric Parallelism Structures

A parallelism structure \((U, \gamma)\) is said to be symmetric if and only if
\[ \gamma_a(b) = \gamma_b(a). \tag{56} \]
As we can easily prove this condition above is completely equivalent to the following condition. For all \(a, b \in V(U)\)
\[ \nabla_+ a - \nabla_+ b = [a, b]. \tag{57} \]

In accordance with Eq.(51) and Eq.(52) we have that a parallelism structure is symmetric if and only if it is torsionless, i.e.,
\[ \tau(a, b) = 0. \tag{58} \]

We now present and prove some basic properties of a symmetric parallelism structure.

i. The curvature field \(\rho\) satisfies the cyclic property
\[ \rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) = 0. \tag{59} \]
**Proof**

By recalling Eq. (53) we can write

\[
\rho(a, b, c) = \nabla^+_a \nabla^+_b c - \nabla^+_b \nabla^+_a c - \nabla^+_c [a, b] c,
\]  

(60)

\[
\rho(b, c, a) = \nabla^+_b \nabla^+_c a - \nabla^+_c \nabla^+_b a - \nabla^+_a [b, c] a,
\]  

(61)

\[
\rho(c, a, b) = \nabla^+_c \nabla^+_a b - \nabla^+_a \nabla^+_c b - \nabla^+_b [c, a] b.
\]  

(62)

By adding Eqs. (60), (61) and (62), wherever by taking into account Eq. (57), we get

\[
\rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) = \nabla^+_a \nabla^+_b c - \nabla^+_c \nabla^+_b a - \nabla^+_b [a, b] c - \nabla^+_c [b, c] a - \nabla^+_a [c, a] b,
\]  

(63)

Hence, by recalling the so-called Jacobi’s identity for the Lie product of smooth vector fields [1], the expected result immediate follows.

ii. The curvature field \(\rho\) satisfies the so-called Bianchi’s identity, i.e.,

\[
(\nabla^{++++}_d \rho)(a, b, c) + (\nabla^{+++-}_a \rho)(b, d, c) + (\nabla^{+++-}_b \rho)(d, a, c) = 0.
\]  

(64)

**Proof**

Let us take \(a, b, c, d, w \in \mathcal{V}(U)\). In accordance with Eq. (39), by using Eq. (24), we have

\[
(\nabla^{++++}_d \rho)(a, b, c) \cdot w
= d \cdot \partial_d (\rho(a, b, c) \cdot w) - \rho(\nabla^+_d a, b, c) \cdot w - \rho(a, \nabla^+_d b, c) \cdot w
- \rho(a, b, \nabla^+_d c) \cdot w
- \rho(a, b, c) \cdot \nabla^+_d w,
\]  

i.e.,

\[
(\nabla^{++++}_d \rho)(a, b, c) = \nabla^+_d \rho(a, b, c) - \rho(\nabla^+_d a, b, c) - \rho(a, \nabla^+_d b, c) - \rho(a, b, \nabla^+_d c).
\]  

(65)

By cycling the letters \(a, b, d\) into Eq. (65), we get

\[
(\nabla^{++++}_a \rho)(b, d, c) = \nabla^+_a \rho(b, d, c) - \rho(\nabla^+_a b, d, c) - \rho(b, \nabla^+_a d, c) - \rho(b, d, \nabla^+_a c),
\]  

(66)

\[
(\nabla^{++++}_b \rho)(d, a, c) = \nabla^+_b \rho(d, a, c) - \rho(\nabla^+_b d, a, c) - \rho(d, \nabla^+_b a, c) - \rho(d, a, \nabla^+_b c).
\]  

(67)
Now, by adding Eqs. (65), (66), and (67), wherever by using Eq. (55) and Eq. (57), we get

\[
(\nabla^+ \rho)(a, b, c) + (\nabla^+ \rho)(b, d, c) + (\nabla^+ \rho)(d, a, c) \\
= \nabla^+_d \rho(a, b, c) + \nabla^+_a \rho(b, d, c) + \nabla^+_b \rho(d, a, c) \\
- \rho([a, b], d, c) - \rho([b, d], a, c) - \rho([d, a], b, c) \\
- \rho(a, b, \nabla^+_d c) - \rho(b, d, \nabla^+_a c) - \rho(d, a, \nabla^+_b c).
\]

(68)

But, in accordance with Eq. (53), we can write

\[
\nabla^+_d \rho(a, b, c) + \nabla^+_a \rho(b, d, c) + \nabla^+_b \rho(d, a, c) \\
= [\nabla^+_d, \nabla^+_a] \nabla^+_d c + [\nabla^+_b, \nabla^+_a] \nabla^+_a c + [\nabla^+_d, \nabla^+_a] \nabla^+_b c \\
- \nabla^+_d \nabla^+_{[a,b]} c - \nabla^+_a \nabla^+_{[d,a]} c - \nabla^+_b \nabla^+_{[d,a]} c.
\]

(69)

and

\[
- \rho([a, b], d, c) - \rho([b, d], a, c) - \rho([d, a], b, c) \\
= -\nabla^+_d \nabla^+_{[a,b]} c - \nabla^+_b \nabla^+_{[d,a]} c - \nabla^+_a \nabla^+_{[d,a]} c \\
+ \nabla^+_d \nabla^+_{[a,b]} c + \nabla^+_a \nabla^+_{[d,a]} c + \nabla^+_b \nabla^+_{[d,a]} c \\
+ \nabla^+_{[a,b],d} c + \nabla^+_{[b,d],a} c + \nabla^+_{[d,a],b} c.
\]

(70)

i.e., by recalling the Jacobi’s identity,

\[
- \rho([a, b], d, c) - \rho([b, d], a, c) - \rho([d, a], b, c) \\
= -\nabla^+_d \nabla^+_{[a,b]} c - \nabla^+_b \nabla^+_{[d,a]} c - \nabla^+_a \nabla^+_{[d,a]} c \\
+ \nabla^+_d \nabla^+_{[a,b]} c + \nabla^+_a \nabla^+_{[d,a]} c + \nabla^+_b \nabla^+_{[d,a]} c.
\]

(70)

Now, by adding Eqs. (69) and (70), wherever by using Eq. (53), we get

\[
\nabla^+_d \rho(a, b, c) + \nabla^+_a \rho(b, d, c) + \nabla^+_b \rho(d, a, c) \\
- \rho([a, b], d, c) - \rho([b, d], a, c) - \rho([d, a], b, c) \\
= \rho(a, b, \nabla^+_d c) + \rho(b, d, \nabla^+_a c) + \rho(d, a, \nabla^+_b c).
\]

(71)

Finally, by putting Eq. (71) into Eq. (68), the expected result immediately follows. ■
5.2 Cartan Fields

The smooth $(1, 2)$-extensor field on $U$, namely $\Theta$, which is defined by

$$\Theta(c) = \frac{1}{2} \partial_a \wedge \partial_b \tau(a, b) \cdot c \quad (72)$$

will be called the Cartan torsion field of $(U, \gamma)$.

We should notice that such $\Theta$ contains all of the geometric information which is just contained in $\tau$. Indeed, Eq.(72) can be inverted in such a way that given any $\Theta$, there is an unique $\tau$ that verifies Eq.(72). We have

$$\tau(a, b) = \partial_c (a \wedge b) \cdot \Theta(c). \quad (73)$$

The smooth bivector elementary 2-extensor field on $U$, namely $\Omega$, which is defined by

$$\Omega(c, d) = \frac{1}{2} \partial_a \wedge \partial_b \rho(a, b, c) \cdot d \quad (74)$$

will be called the Cartan curvature field of $(U, \gamma)$.

Since Eq.(74) can be inverted, by giving $\rho$ in terms of $\Omega$, we see that such $\Omega$ contains the same geometric information as $\rho$. The inversion is realized by

$$\rho(a, b, c) = \partial_d (a \wedge b) \cdot \Omega(c, d). \quad (75)$$

5.3 Cartan’s Structure Equations

Associated to any parallelism structure $(U, \gamma)$ we can introduce two noticeable operators which map 2-uples of smooth vector fields to smooth vector fields. They are:

(a) The mapping $\gamma^+ : \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)$ defined by

$$\gamma^+(b, c) = \partial_a (\nabla^+_a b) \cdot c \quad (76)$$

which will be called the Cartan connection operator of first kind of $(U, \gamma)$.

(b) The mapping $\gamma^- : \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)$ defined by

$$\gamma^-(b, c) = \partial_a b \cdot (\nabla^-_a c) \quad (77)$$

which will be called the Cartan connection operator of second kind of $(U, \gamma)$.

We summarize some of the basic properties which are satisfied by the Cartan operators.
i. For all $f \in S(U)$, and $b, b', c, c' \in V(U)$, we have

\[
\gamma^+(b + b', c) = \gamma^+(b, c) + \gamma^+(b', c), \tag{78}
\]
\[
\gamma^+(b, c + c') = \gamma^+(b, c) + \gamma^+(b, c'). \tag{79}
\]
\[
\gamma^+(fb, c) = \partial_o f b \cdot c + f \gamma^+(b, c), \tag{80}
\]
\[
\gamma^+(b, fc) = f \gamma^+(b, c). \tag{81}
\]

ii. For all $f \in S(U)$, and $b, b', c, c' \in V(U)$, we have

\[
\gamma^-(b + b', c) = \gamma^-(b, c) + \gamma^-(b', c), \tag{82}
\]
\[
\gamma^-(b, c + c') = \gamma^-(b, c) + \gamma^-(b, c'). \tag{83}
\]
\[
\gamma^-(fb, c) = f \gamma^-(b, c), \tag{84}
\]
\[
\gamma^-(b, fc) = \partial_o f b \cdot c + f \gamma^-(b, c). \tag{85}
\]

We have that the Cartan operator of first kind has the linearity property with respect to the second variable, and the Cartan operator of second kind is linear with respect to the first variable.

iii. For any $a, b \in V(U)$,

\[
\gamma^+(b, c) + \gamma^-(b, c) = \partial_o (b \cdot c). \tag{86}
\]

It is an immediate consequence of Eq.(24).

**First Cartan’s structure equation**

For any $c \in V(U)$ it holds

\[
\Theta(c) = \partial_o \wedge c + \partial_s \wedge \gamma^-(s, c). \tag{87}
\]

**Proof**

By using Eq.(51) we can write

\[
\Theta(c) = \frac{1}{2} \partial_a \wedge \partial_b (\nabla^+_a b - \nabla^+_b a - [a, b]) \cdot c,
\]
\[
= \partial_a \wedge \partial_b (\nabla^+_a b - a \cdot \partial_o b) \cdot c. \tag{88}
\]

A straightforward calculation yields

\[
\partial_a \wedge \partial_b (\nabla^+_a b) \cdot c = \partial_a \wedge \partial_b \gamma^+(b, c) \cdot a
\]
\[
= \partial_b \wedge \partial_a (\gamma^-(b, c) \cdot a - a \cdot \partial_o (b \cdot c)),
\]
\[
= \partial_b \wedge \gamma^-(b, c) - \partial_b \wedge \partial_o (b \cdot c). \tag{89}
\]
and
\[
-\partial_a \land \partial_b (a \cdot \partial_a b) \cdot c = \partial_a \land \partial_b (b \cdot (a \cdot \partial_a c) - a \cdot \partial_a (b \cdot c))
\]
\[
= \partial_a \land (a \cdot \partial_a c) + \partial_b \land \partial_a a \cdot \partial_a (b \cdot c),
\]
\[
= \partial_b \land c + \partial_b \land \partial_a (b \cdot c).
\]
(90)

Thus, by putting Eq.(89) and Eq.(90) into Eq.(88), we get the expected result. ■

Second Cartan’s structure equation
For any \( c, d \in \mathcal{V}(U) \) it holds
\[
\Omega(c, d) = \partial_a \land \gamma^+(c, d) + \gamma^+(c, \partial_a) \land \gamma^-(s, d). \tag{91}
\]

Proof
By using Eq.(53) we have
\[
\Omega(c, d) = \frac{1}{2} \partial_a \land \partial_b (\nabla_a^+ \land \nabla_b^+ c - \nabla_{[a,b]}^+ c) \cdot d,
\]
\[
= \partial_a \land \partial_b (\nabla_a^+ (\nabla_b^+ c) - \nabla_{[a,b]}^+ c) \cdot d. \tag{92}
\]
But, by taking a pair of reciprocal frame fields \( \{e_\sigma\}, \{e^\sigma\} \) we can write
\[
\nabla_a^+ (\nabla_b^+ c) \cdot d = \nabla_a^+ (\gamma^+(c, e^\sigma) \cdot b e_\sigma) \cdot d
\]
\[
= a \cdot \partial_a (\gamma^+(c, e^\sigma) \cdot b) e_\sigma \cdot d + \gamma^+(c, e^\sigma) \cdot b \nabla_a^+ e_\sigma \cdot d
\]
\[
= a \cdot \partial_a (\gamma^+(c, e^\sigma) \cdot b) e_\sigma \cdot d + \gamma^+(c, e^\sigma) \cdot b \gamma^+(e_\sigma, d) \cdot a,
\]
\[
= a \cdot \partial_a \gamma^+(c, e^\sigma) \cdot b e_\sigma \cdot d + \gamma^+(c, d) \cdot (a \cdot \partial_a b)
\]
\[
+ \gamma^+(c, e^\sigma) \cdot b \gamma^+(e_\sigma, d) \cdot a. \tag{93}
\]

Now, the first term into Eq.(92), by using Eq.(93) and the well-known identity \( \partial_a \land (fX) = (\partial_a f) \land X + f \partial_a \land X \), where \( f \in \mathcal{S}(U) \) and \( X \in \mathcal{M}(U) \), can be written
\[
\partial_a \land \partial_b \nabla_a^+ (\nabla_b^+ c) \cdot d = \partial_a \land \gamma^+(c, e^\sigma) (e_\sigma \cdot d) + \partial_a \land \partial_b \gamma^+(c, d) \cdot (a \cdot \partial_a b)
\]
\[
- \gamma^+(c, e^\sigma) \land \gamma^+(e_\sigma, d),
\]
\[
= \partial_a \land \gamma^+(c, e^\sigma) (e_\sigma \cdot d)
\]
\[
- \gamma^+(c, e^\sigma) \land (\partial_b (e_\sigma \cdot d) - \gamma^-(e_\sigma, d))
\]
\[
+ \partial_a \land \partial_b \gamma^+(c, d) \cdot (a \cdot \partial_a b),
\]
\[
= \partial_b \land \gamma^+(c, d)
\]
\[
+ \gamma^+(c, e^\sigma) \land \gamma^-(e_\sigma, d)
\]
\[
+ \partial_a \land \partial_b \gamma^+(c, d) \cdot (a \cdot \partial_a b). \tag{94}
\]
It is also
\[-(\nabla^+_a \partial_o c) \cdot d = -\gamma^+(c, d) \cdot (a \cdot \partial_o b).\]  
(95)

Finally, by putting Eq. (94) and Eq. (95) into Eq. (92), we get the expected result.

6 Appendix

Let \(U\) be an open subset of \(U_o\), and let \((U, \phi)\) and \((U, \phi')\) be two local coordinate systems on \(U\) compatibles with \((U_o, \phi_o)\). As we know there must be two pairs of reciprocal frame fields on \(U\). The covariant and contravariant frame fields \(\{b_o \cdot \partial x_o\}\) and \(\{\partial_o x^\alpha\}\) associated to \((U, \phi)\), and those ones \(\{b_\mu \cdot \partial' x_o\}\) and \(\{\partial_o x'^\mu\}\) associated to \((U, \phi')\).

A.1 The \(n^3\) smooth scalar fields on \(U\), namely \(\Gamma^\gamma_{\alpha\beta}\), defined by
\[
\Gamma^\gamma_{\alpha\beta} = \Gamma^+(b_\mu \cdot \partial' x_o, b_\beta \cdot \partial x_o) \cdot \partial_o x^\gamma
\]
(A1)
correspond to the classically so-called coefficients of connection, of course, associated to \((U, \phi)\). The coefficients of connection associated to \((U, \phi')\) are given by
\[
\Gamma^\alpha_{\mu\nu'} = \Gamma^+(b_\mu \cdot \partial' x_o, b_\nu \cdot \partial x_o) \cdot \partial_o x^\nu.'
\]
(A2)

We will check next what is the law of transformation between them. We will employ the simplified notations: \(b_\mu \cdot \partial x_o = \frac{\partial x_o^\sigma}{\partial x^{\mu}} b_\sigma\) and \(\partial_o x^\alpha = b^\gamma \frac{\partial x^\alpha}{\partial x^\gamma}\), etc. Then, by recalling the expansion formulas for smooth vector fields, \(v = (v \cdot \partial_o x^\alpha) b_\alpha \cdot \partial x_o\) and \(v = (v \cdot b_\gamma \cdot \partial x_o) \partial_o x^\gamma\), using Eqs. (30), (31), (32) and (33), and recalling the remarkable identity \((b_\alpha \cdot \partial x_o) \cdot \partial_o X = b_\alpha \cdot \partial X\), where \(X \in \mathcal{M}(U)\) (see [1]), we can write
\[
\Gamma^\alpha_{\mu\nu'} = \Gamma^+(\frac{\partial x^\alpha}{\partial x^{\mu'}} b_\alpha \cdot \partial x_o, \frac{\partial x^\beta}{\partial x^{\nu'}} b_\beta \cdot \partial x_o) \cdot \frac{\partial x^\nu'}{\partial x^\gamma} \partial_o x^\gamma
\]
\[
= \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^\nu'}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial^2}{\partial x^{\nu'} \partial x^\gamma} \Gamma^\gamma_{\alpha\beta}
\]
\[
+ \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial^2}{\partial x^{\nu'} \partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial^2}{\partial x^{\nu'} \partial x^\gamma} \Gamma^\gamma_{\alpha\beta}
\]
\[
= \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^\nu'}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^\nu'}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^\beta},
\]
i.e.,
\[ \Gamma^\nu_{\mu'\nu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^\gamma}{\partial x^{\lambda'}} \Gamma^\gamma_{\alpha\beta} + \frac{\partial^2 x^\beta}{\partial x^{\mu'} \partial x^{\nu'}} \frac{\partial x^\gamma}{\partial x^{\lambda'}}. \]  \hfill (A3)

It is just the well-known law of transformation for the \textit{classical coefficients of connection} associated to each of the \textit{coordinate systems} \( \langle x^\mu \rangle \) and \( \langle x'^\mu \rangle \).

\textbf{A.2} Given a smooth vector field on \( U \), say \( v \), the covariant and contravariant components of \( v \) with respect to \( (U, \phi) \) and \( (U, \phi') \) are respectively given by

\[ v_\alpha = v \cdot (b_\alpha \cdot \partial x_\alpha), \] \hfill (A4)
\[ v^\alpha = v \cdot \partial_\alpha x^\alpha, \] \hfill (A5)
\[ v'_\alpha = v \cdot (b_\alpha \cdot \partial' x_\alpha), \] \hfill (A6)
\[ v'^\alpha = v \cdot \partial'_\alpha x'^\alpha. \] \hfill (A7)

We will check the relationship between the covariant components of \( v \) with respect to each of the \textit{coordinate systems} \( \langle x^\mu \rangle \) and \( \langle x'^\mu \rangle \). By recalling the expansion formula \( w = (w \cdot \partial_\alpha x^\alpha) b_\beta \cdot \partial x_\alpha \), we have

\[ v'_\alpha = v \cdot (b_\alpha \cdot \partial' x_\alpha \cdot \partial_\alpha x^\beta) b_\beta \cdot \partial x_\alpha, \]
\[ v'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} v_\beta. \]  \hfill (A8)

It is the expected law of transformation for the covariant components of \( v \) associated to each of \( \langle x^\mu \rangle \) and \( \langle x'^\mu \rangle \).

Analogously, by using the expansion formula \( w = (w \cdot b_\beta \cdot \partial x_\alpha) \cdot \partial_\alpha x^\beta \), we can get the classical law of transformation for the contravariant components of \( v \), i.e.,

\[ v'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} v_\beta. \]  \hfill (A9)

\textbf{A.3} We will see next which is the meaning of the classical covariant derivatives of the contravariant and covariant components of a smooth vector field. By using the expansion formula \( v = v_\alpha b_\alpha \cdot \partial x_\alpha \), and Eqs. (21) and (22), and recalling once again the remarkable identity \((b_\mu \cdot \partial x_\alpha) \cdot \partial_\alpha X = b_\mu \cdot \partial X\),
where \( X \in \mathcal{M}(U) \), we can write

\[
(\nabla^+_b \cdot \partial x^\lambda) = (b_\mu \cdot \partial x^\mu) (b_\alpha \cdot \partial x^\alpha)
\]

\[
+ v^\alpha (\nabla^+_b \cdot \partial x^\alpha) \cdot \partial x^\lambda \]

\[
= \frac{\partial v^\alpha}{\partial x^\mu} \delta^\lambda_\alpha + v^\alpha \Gamma_\alpha^\beta (b_\mu \cdot \partial x^\alpha) \cdot \partial x^\lambda \]

\[
= \frac{\partial v^\lambda}{\partial x^\mu} + \Gamma^\lambda_\alpha v^\alpha, \tag{A10}
\]

i.e.,

\[
(\nabla^+_b \cdot \partial x^\lambda) = v^\lambda_\mu. \tag{A10}
\]

By using the expansion formula \( v = v_\alpha \partial x^\alpha \), Eqs. (21) and (22), and Eq. (34), we get

\[
(\nabla^-_b \cdot \partial x^\nu) \cdot b_\nu \cdot \partial x^\alpha = \frac{\partial v_\nu}{\partial x^\mu} - \Gamma^\alpha_\mu v_\alpha, \tag{A11}
\]

i.e.,

\[
(\nabla^-_b \cdot \partial x^\nu) \cdot b_\nu \cdot \partial x^\alpha = v_\nu^\alpha. \tag{A11}
\]

**A.4** Now, for instance, let us take a smooth \((1,1)\)-extensor field on \( U \), say \( t \). The covariant and contravariant components of \( t \) with respect to \((U, \phi)\) and \((U, \phi')\) are respectively defined to be

\[
t_{\mu\nu} = t(b_\mu \cdot \partial x^\nu), \tag{A12}
\]

\[
t^\mu_\nu = t(\partial_\nu x^\mu), \tag{A13}
\]

\[
t_{\mu\nu}^\prime = t(b_\mu \cdot \partial' x^\nu), \tag{A14}
\]

\[
t^\mu_\nu^\prime = t(\partial' x^\mu), \tag{A15}
\]

It is also possible to introduce two mixed components of \( t \) with respect to \((U, \phi)\) and \((U, \phi')\). They are defined by

\[
t_{\mu\nu} = t(b_\mu \cdot \partial x^\nu), \tag{A16}
\]

\[
t^\mu_\nu = t(\partial_\nu x^\mu), \tag{A17}
\]

\[
t_{\mu\nu}^\prime = t(b_\mu \cdot \partial' x^\nu), \tag{A18}
\]

\[
t^\mu_\nu^\prime = t(\partial' x^\mu), \tag{A19}
\]
We will try to check the law of transformation for the covariant component of \( t \). We can write
\[
t_{\mu'\nu'} = t(b_{\mu'} \cdot \partial' x_o) \cdot b_\nu \cdot \partial' x_o
\]
\[
= t(\frac{\partial x^\alpha}{\partial x^{\mu'}} b_\alpha \cdot \partial x_o) \cdot \frac{\partial x^\beta}{\partial x^{\nu'}} b_\beta \cdot \partial x_o
\]
\[
= \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} t(b_\alpha \cdot \partial x_o) \cdot b_\beta \cdot \partial x_o,
\]
\[
t_{\mu'\nu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} t_{\alpha\beta}.
\] (A20)

This is the classical law of transformation for the covariant components of a smooth 2-tensor field from \( \langle x^\mu \rangle \) to \( \langle x'^\mu \rangle \).

By following similar steps we can get the laws of transformation for the contravariant and mixed components of \( t \). We have
\[
t_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\beta} t_{\alpha\beta},
\] (A21)
\[
t_{\mu'\nu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\beta} t_{\alpha\beta};
\] (A22)
\[
t_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} t_{\alpha\beta}.
\] (A23)

They perfectly agree with the classical laws of transformation for the respective contravariant and mixed components of a smooth 2-tensor field from \( \langle x^\mu \rangle \) to \( \langle x'^\mu \rangle \).

A.5 We will get next the relationship among the covariant derivatives of \( t \) and the classical concepts of covariant derivatives of the covariant, contravariant and mixed components of a smooth 2-tensor field. For instance, we have
\[
(\nabla^{++}_{b_{\mu'} \partial x_o} t)(b_\alpha \cdot \partial x_o) \cdot b_\beta \cdot \partial x_o
\]
\[
= b_{\mu'} \cdot \partial x_o \cdot \partial_o(t(b_\alpha \cdot \partial x_o) \cdot b_\beta \cdot \partial x_o) - t(\nabla^{++}_{b_{\mu'} \partial x_o} b_\alpha \cdot \partial x_o) \cdot b_\beta \cdot \partial x_o
\]
\[
- t(b_\alpha \cdot \partial x_o) \cdot \nabla^{++}_{b_{\mu'} \partial x_o} b_\beta \cdot \partial x_o
\]
\[
= \frac{\partial t_{\alpha\beta}}{\partial x^{\mu'}} - t(\Gamma^+(b_{\mu'} \cdot \partial x_o, b_\alpha \cdot \partial x_o) \cdot \partial_\sigma x^\sigma b_\beta \cdot \partial x_o)
\]
\[
- t(b_\alpha \cdot \partial x_o) \cdot \Gamma^+(b_{\mu'} \cdot \partial x_o, b_\beta \cdot \partial x_o) \cdot \partial_\sigma x^\sigma b_\tau \cdot \partial x_o
\]
\[
= \frac{\partial t_{\alpha\beta}}{\partial x^{\mu'}} - \Gamma^\alpha_{\mu\sigma} t_{\sigma\beta} - \Gamma^\tau_{\mu\beta} t_{\alpha\tau},
\]
i.e.,
\[(\nabla_{b_\mu \partial x_\alpha}^- t)(b_\alpha \cdot \partial x_\alpha) \cdot b_\beta \cdot \partial x_\alpha = t_\alpha^\beta; \mu.\] (A24)

We can also write
\[(\nabla_{b_\mu \partial x_\alpha}^+ t)(b_\alpha \cdot \partial x_\alpha) \cdot \partial_\alpha x^\beta = t_\alpha^\beta; \mu.\] (A25)

7 Conclusions

We presented in this paper the notion of a parallelism structure on \(M\), i.e.,
a pair \((M, \gamma)\) where \(\gamma\) is a connection extensor field on \(M\). A theory of \(a\)-directional covariant derivatives of multivector and extensor fields is introduced and the main properties satisfied by these objects are proved. We also give a novel and intrinsic presentation (i.e., one that does not depend on a chosen orthonormal moving frame) of Cartan theory of the torsion and curvature fields and of Cartan’s structure equations. Examples as our theory relates to known theories have been worked in detail.

Acknowledgments: V. V. Fernández and A. M. Moya are very grateful to Mrs. Rosa I. Fernández who gave to them material and spiritual support at the starting time of their research work. This paper could not have been written without her inestimable help.

References

[1] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr. Multivector and Extensor Fields on Smooth Manifolds, submitted for publication 1 (2005).
[2] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr., *Extensors in Geometric Algebra*, submitted to publication (2005).

[3] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr., *Geometric Algebras*, submitted for publication (2005).