EXISTENCE RESULTS FOR FRACTIONAL IMPULSIVE DELAY FEEDBACK CONTROL SYSTEMS WITH CAPUTO FRACTIONAL DERIVATIVES

BIAO ZENG

Guangxi University for Nationalities
Faculty of Mathematics and Physics
Nanning 530006, Guangxi Province, P. R. China

(Communicated by Genni Fragnelli)

Abstract. The goal of this paper is to provide systematic approaches to study the feedback control systems governed by fractional impulsive delay evolution equations involving Caputo fractional derivatives in separable reflexive Banach spaces. This work is a continuation of previous work. We firstly give an existence result of mild solutions for the equations by applying the Banach’s fixed point theorem and the Leray-Schauder alternative fixed point theorem. Next, by using the Filippove theorem and the Cesari property, we obtain the existence result of feasible pairs for the feedback control system. Finally, some applications are given to illustrate our main results.

1. Introduction. Impulsive delay differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotics, economics, etc. For the basic theory and recent developments of integral and fractional delay impulsive differential equations, we refer to [1, 8, 11, 12, 15, 20, 21, 22, 30]. In the last years, control theory has been greatly applied in engineering, economics, computers and ecology, especially towards system with controllability, feedback control and optimal control ([4, 6, 10, 14, 23, 24, 25, 27, 28, 29, 32]). The study of impulsive control systems (control systems with impulse effects) has also a long history that can be traced back to the beginning of modern control theory.

Consider an interval \( J = [0, b) (b > 0) \) and a finite set of points

\[ D = \{ t_i \in (0, b), i = 1, 2, \ldots, m \}, \quad 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b. \]

2020 Mathematics Subject Classification. Primary: 34A08, 93B52; Secondary: 49J20, 49J53, 35R12.

Key words and phrases. Impulsive delay evolution equation, Caputo fractional derivatives, feedback control, feasible pair, existence.

The first author is supported by the Natural Science Foundation of Guangxi Province grant No. 2019GXNSFBA185005, the Start-up Project of Scientific Research on Introducing talents at school level in Guangxi University for Nationalities grant No. 2019KJQD04 and the Xiangshui Young Scholars Innovative Research Team of Guangxi University for Nationalities grant No. 2019RSCXSHQN02.

* Corresponding author: Biao Zeng.
In this paper, we firstly consider the following form:

\[
\begin{cases}
C D^q_t x(t) = Ax(t) + f(t, x(t), u(t)), & t \in (0, b] - D, \\
\Delta x(t_k) = G(t_k, x(t_k)), & t_k \in D, \\
x(t) = \psi(t), & t \in [-h, 0],
\end{cases}
\]

(1)

where \( C D^q_t \) denotes the Caputo fractional derivative of order \( q \) \((0 < q \leq 1)\) with the lower limit zero. \( A : D(A) \subseteq X \rightarrow X \), is the infinitesimal generator of a uniformly bounded \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on a reflexive Banach space \( X \). \( \psi : [-h, 0] \rightarrow X \) is a continuous function. Let \( V \) be a separable Banach space. \( f : J \times X \times V \rightarrow X \), \( G : D \times X \rightarrow X \) are given functions to be specified later. Here, \( x(t_k) = x(t^-_k) \) and \( \Delta x(t_k) = x(t^+_k) - x(t^-_k) \), where \( x(t^+_k) \) and \( x(t^-_k) \) denote the right and the left limits of \( x(t) \) at \( t = t_k \in D \), respectively.

Furthermore, we will study the following feedback control system:

\[
\begin{cases}
C D^q_t x(t) = Ax(t) + f(t, x(t), u(t)), & t \in (0, b] - D, \\
u(t) \in U(t, x(t)), & \text{a.e. } t \in J, \\
\Delta x(t_k) = G(t_k, x(t_k)), & t_k \in D, \\
x(t) = \psi(t), & t \in [-h, 0],
\end{cases}
\]

(2)

where \( U : J \times X \rightarrow P(V) \) is a feedback multifunction.

We consider the feedback control problem (2) combining fractional impulsive delay evolution equations and feedback control. In this work, by applying the Banach’s fixed point theorem and the Leray-Schauder alternative fixed point theorem, we obtain some existence results of mild solutions to problem (1). Moreover, the existence result of feasible pairs for the feedback control problem (2) is also considered. Furthermore, we apply our main result to obtain a controllability result for impulsive fractional evolution equations and two existence results for a class of impulsive fractional differential variational inequalities and impulsive fractional Clarke’s subdifferential inclusions.

The rest of this paper is organized as follows. In Section 2, we will present some preliminaries which will be used to prove our main results. In Section 3, at first, some sufficient conditions are established to guarantee some existence of mild solutions of the system (1). Theorems 3.1, 3.3, 3.4 are the main results in this section. Next, we present some sufficient conditions for the existence of feasible pairs of problem (2). Theorems 4.2 and 4.3 are the main results in this section. We present some applications in the last section.

2. Preliminaries. The norm of a Banach space \( X \) will be denoted by \( \| \cdot \|_X \). Let \( P(X) \) be the set of all nonempty subsets of \( X \). For \( h > 0 \), let \( \bar{J} = J \cup [-h, 0] \).

Let \( C(J, X) \) denote the Banach space of continuous functions from \( J \) into \( X \) with the norm \( \| x \|_C = \sup_{t \in J} \| x(t) \|_X \) and \( L^p(J, X) \) denote the Banach space of \( p \)-time integrable functions from \( J \) into \( X \) with the norm \( \| x \|_{L^p} = \left( \int_{t \in J} \| x(t) \|_X^p \right)^{\frac{1}{p}} \). Let \( AC(J, X) \) be the space of functions \( f \) which are absolutely continuous on \( J \) and \( AC^m(J, X) = \{ f : J \rightarrow X \text{ and } f^{(m-1)} \in AC(J, X) \} \). In order to define the mild solutions of problem (1), we also consider the Banach space \( PC(J, X) = \{ x \in C((t_k, t_{k+1}], X) : x(t^-_k), x(t^+_k) \text{ exist, } k = 0, 1, 2, \cdots, m \text{ and } x(t_k) = x(t^-_k) \} \) with the norm \( \| x \|_{PC} = \sup_{t \in J} \| x(t) \|_X \). Let \( C := C([h, 0], X) \cap PC(J, X) \) with the norm \( \| x \|_C = \sup_{t \in [-h, 0]} \| x(t) \|_X \).
Theorem 2.7 ([5], Theorem 6.5.4). Cesari property on $X$ ([9, 18]).

Definition 2.1 ([9, 18]). 

\begin{equation}
I_q^t f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s)ds, \quad q > 0,
\end{equation}

is called Riemann-Liouville fractional integral of order $q$, where $\Gamma$ is the gamma function.

Definition 2.2 ([9, 18]). For a function $f(t)$ given in the interval $[0, \infty)$, the expression

\begin{equation}
C^q D^t f(t) = R^q D^t \left( f(t) - \sum_{k=1}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad 0 \leq n - 1 < q < n.
\end{equation}

where $n = [q] + 1$, $[q]$ denotes the integer part of number $q$, is called the Caputo fractional derivative of order $q$.

Definition 2.3 ([7]). Let $X$ and $Y$ be two Banach spaces. A multifunction $F : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.), if for every open subset $V \subset Y$ the set $F^{-1}(V) = \{ x \in X : F(x) \subset V \}$ is open in $X$;

Definition 2.4 ([10]). Let $X$ and $Y$ be two metric spaces. A multifunction $F : X \to P(Y)$ is said to be pseudo-continuous at $x \in X$ if $\bigcap_{t > 0} F(O_t(x)) = F(x)$, where $O_t(x) = \{ y \in X ||y - x||_X \leq t \}$. We say that $F$ is pseudo-continuous on $X$ if it is pseudo-continuous at each point $x \in X$.

Remark 1 ([10]). (i) Let $F : X \to P(Y)$ be a multifunction taking closed values. Then $F$ is pseudo-continuous if and only if the graph $\{ (x, y) \in X \times Y | y \in F(x) \}$ is closed in $X \times Y$.

(ii) If $F : X \to P(Y)$ is u.s.c. taking closed values, then it is pseudo-continuous.

Definition 2.5 ([10]). Let $X$ be a Banach space, $Y$ be a metric space and $F : X \to P(Y)$ be a multifunction. We say $F$ possesses the Cesari property at $x \in X$, if $\bigcap_{t > 0} \overline{\sigma F(O_t(x))} = F(x)$, where $\overline{\sigma K}$ is the closed convex hull of $K$. If $F$ has the Cesari property at every point $x \in K \subset X$, we simply say that $F$ has the Cesari property on $K$.

Lemma 2.6 ([10], Proposition 4.2). Let $X$ be a Banach space and $Y$ be a metric space. Let $F : X \to P(Y)$ be u.s.c. with convex and closed values. Then $F$ has the Cesari property on $X$.

Theorem 2.7 ([5], Theorem 6.5.4). (Leray-Schauder alternative fixed point theorem) Let $K$ be a closed convex subset of a Banach space $X$ such that $0 \in K$. Let $F : K \to K$ be a compact operator (i.e., a map that restricted to any bounded set in $X$ is relative compact), and let $\Delta = \{ x \in X | \exists \lambda \in (0,1), x = \lambda Fx \}$. Then either the set $\Delta$ is unbounded or there exists $x \in X$ such that $x = Fx$.

Definition 2.8 ([22]). A function $x \in \mathcal{C}$ is called a mild solution to problem (1) if it satisfies

\begin{equation}
x(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
S_q(t)\psi(0) + \int_0^t (t - s)^{q-1} T_q(t - s)f(s, x(s), u(s))ds, & t \in (0, t_1], \\
S_q(t)\psi(0) + \sum_{i=1}^{n-1} S_q(t - t_k)G(t_k, x(t_k)) \\
+ \int_0^t (t - s)^{q-1} T_q(t - s)f(s, x(s), u(s))ds, & t \in (t_k, t_{k+1}], \quad k = 1, \ldots, m,
\end{cases}
\end{equation}
Proof. Define the operator \( \Phi : C \rightarrow C \) by

\[
(\Phi x)(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
S_q(t)\psi(0) + \int_0^t(t-s)^{q-1}T_q(t-s)f(s, x(s), u(s))\,ds, & t \in (0, t_1], \\
S_q(t)\psi(0) + \sum_{i=1}^k S_q(t-t_i)G(t_i, x(t_i)) + \int_0^t(t-s)^{q-1}T_q(t-s)f(s, x(s), u(s))\,ds, & t \in (t_k, t_{k+1}], \quad k = 1, \ldots, m.
\end{cases}
\]

where

\[
S_q(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)d\theta, \quad T_q(t) = q \int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta,
\]

\[
\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{q}{2}} \varpi_q(\theta^{-\frac{q}{2}}),
\]

\[
\varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty),
\]

\( \xi_q \) is a probability density function defined on \((0, \infty)\), that is

\[
\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \text{and} \quad \int_0^\infty \xi_q(\theta)d\theta = 1.
\]

**Lemma 2.9** ([17, 31]). The operators \( S_q(t) \), \( T_q(t) \) satisfy the following properties.

(i) For any fixed \( t \geq 0 \), \( S_q(t) \), \( T_q(t) \) are linear and bounded operators, such that, for any \( x \in X \),

\[
\|S_q(t)x\|_X \leq M \|x\|, \quad \|T_q(t)x\|_X \leq \frac{M}{\Gamma(q)} \|x\|
\]

where \( M = \sup_{t \in [0,T]} \|T(t)\| \).

(ii) \( S_q(t), T_q(t)(t \geq 0) \) are strongly continuous.

(iii) For every \( t > 0 \), \( S_q(t), T_q(t) \) are compact if \( T(t) \) is compact.

3. Existence of mild solutions. In what follows, we will make the following hypotheses on the data of our problems.

\( H(1) \) : \( f : J \times X \times V \rightarrow X \) is Borel measurable on \( J \times X \times V \) and continuous on \( X \times V \).

\( H(2) \) : There exist a function \( \phi \in L^p(J, \mathbb{R}_+) \) \( (p > \frac{1}{q}) \) and constants \( L, N > 0 \) such that

\[
\|f(t, 0, 0)\| \leq \phi(t),
\]

\[
\|f(t, x_1, u_1) - f(t, x_2, u_2)\|_X \leq L \|x_1 - x_2\|_X + N \|u_1 - u_2\|_V
\]

for all \( x_1, x_2 \in X \), \( u_1, u_2 \in V \), a.e. \( t \in J \).

\( H(3) \) : There exist constants \( L_k \geq 0 \) with \( 2M \sum_{i=1}^k L_i < \Gamma(q) \) \( (k = 1, 2, \ldots, m) \) such that

\[
\|G(t_k, x) - G(t_k, y)\|_X \leq L_k \|x - y\|_X
\]

for all \( t_k \in D, x, y \in X \).

**Theorem 3.1.** Assume that the hypotheses \( H(1) - H(3) \) are satisfied. Then for each given control function \( u \in L^p(J, V) \) \( (p > \frac{1}{q}) \), problem (1) has a unique mild solution on \( C \).

**Proof.** Define the operator \( \Phi : C \rightarrow C \) by

\[
(\Phi x)(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
S_q(t)\psi(0) + \int_0^t(t-s)^{q-1}T_q(t-s)f(s, x(s), u(s))\,ds, & t \in (0, t_1], \\
S_q(t)\psi(0) + \sum_{i=1}^k S_q(t-t_i)G(t_i, x(t_i)) + \int_0^t(t-s)^{q-1}T_q(t-s)f(s, x(s), u(s))\,ds, & t \in (t_k, t_{k+1}], \quad k = 1, \ldots, m.
\end{cases}
\]
Then the problem of finding mild solutions for problem (1) is reduced to find fixed points of \( \Phi \). To prove this, we consider the operator \( \Phi \) on the Banach space \( C \) with a weighted norm

\[
\|x\|_r = \sup_{t \in J} e^{-rt} \|x(t)\|_X,
\]

where \( r = \max\left\{ \left( \frac{2ML}{\Gamma(q) - 2M\Gamma(q) \sum_{i=1}^k L_i} \right)^{1/q}, \ k = 1, \ldots, m \right\} \). Now, set

\[
B_r(R) = \{ x \in C : \|x\|_r \leq R \},
\]

where \( R = \max\{ \|\psi\|_{C([-h,0],X)}, 2\omega \} \) and

\[
\omega = M\|\psi(0)\|_X + M \sum_{i=1}^m \|G(t_i,0)\|_X
\]

\[
+ \frac{Mb^\frac{q-1}{p}}{\Gamma(q)} \left( \frac{p-1}{pq-1} \right)^{1-q} \left[ \|\phi\|_{L^p(J,\mathbb{R}^+)} + N\|u\|_{L^p(J,V)} \right].
\]

Next, for the sake of convenience, we subdivide the proof into two steps.

Step 1: We shall prove that the operator \( \Phi \) maps \( B_r(R) \) into itself.

If \( t \in [-h,0] \), then \( x(t) = \psi(t) \), and hence

\[
\|(Fx^\lambda)(t)\|_X = \|\psi(t)\|_X \leq \|\psi\|_{C([-h,0],X)} \leq R.
\]

If \( t \in [0,t_1] \), then by the formula (4), the hypothesis \( H(2) \) and the Hölder inequality, we obtain

\[
\|(\Phi x)(t)\|_X
\]

\[
\leq \|S_q(t)\psi(0)\|_X + \int_0^t (t-s)^{q-1} \|T_q(t-s)f(s,x(s),u(s))\|_X ds
\]

\[
\leq M\|\psi(0)\|_X + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s,0,0)\|_X + L\|x(s)\|_X + N\|u(s)\|_V ds
\]

\[
\leq M\|\psi(0)\|_X + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|\phi(s)\|_X + N\|u(s)\|_V ds
\]

\[
+ \frac{ML\|x\|_r}{\Gamma(q)} \int_0^t (t-s)^{q-1} e^{rs} ds
\]

\[
\leq M\|\psi(0)\|_X + \frac{Mb^\frac{q-1}{p}}{\Gamma(q)} \left( \frac{p-1}{pq-1} \right)^{1-q} \left[ \|\phi\|_{L^p(J,\mathbb{R}^+)} + N\|u\|_{L^p(J,V)} \right]
\]

\[
+ \frac{ML}{\Gamma(q)} e^{rt} r^{-q} \|x\|_r.
\]

In above inequalities, we use the fact that (see (3.1) in [25])

\[
\int_0^t (t-s)^{q-1} e^{rs} ds \leq r^{-q} e^{rt} \Gamma(q).
\]

Thus, we have

\[
\sup_{t \in [0,t_1]} e^{-rt} \|(\Phi x)(t)\|_X \leq \omega + \frac{ML}{\Gamma(q)} r^{-q} \|x\|_r \leq \omega + \frac{1}{2} R \leq R.
\]

If \( t \in (t_k, t_{k+1}) \) (\( k = 1, \ldots, m \)), from the formula (4), the hypotheses \( H(2), H(3) \) and the Hölder inequality, we have

\[
\|(\Phi x)(t)\|_X
\]
Step 2: We show that $\Phi$ is a contraction operator on $B$ which means that $\Phi(B) \subseteq B$. Hence, we get

$$
\sup_{t \in (t_k, t_{k+1})} e^{-rt} \| (\Phi x)(t) \|_X \leq \omega + \left( M \sum_{i=1}^{k} L_i + \frac{ML}{\Gamma(q)} r^{-q} \right) \| x \|_r \leq \omega + \frac{1}{2}R \leq R.
$$

From the above arguments, we know

$$
\| \Phi x \|_r = \sup_{t \in J} e^{-rt} \| (\Phi x)(t) \|_X \leq R,
$$

which means that $\Phi(B_r(R)) \subseteq B_r(R)$.

Step 2: We show that $\Phi$ is a contraction operator on $B_r(R)$.

For any $x, y \in C$, if $t \in (0, t_1]$, then from the formula (4), we get

$$
\| (\Phi x)(t) - (\Phi y)(t) \|_X \leq \int_0^t (t-s)^{q-1} \| T_q(t-s)(f(s,x(s),u(s)) - f(s,y(s),u(s))) \|_X ds
$$

$$
\leq \frac{ML}{\Gamma(q)} \| x - y \|_r \int_0^t (t-s)^{q-1} e^{rs} ds
$$

$$
\leq \frac{ML}{\Gamma(q)} e^{rt} r^{-q} \| x - y \|_r,
$$

which implies

$$
\sup_{t \in [0, t_1]} e^{-rt} \| (\Phi x)(t) - (\Phi y)(t) \|_X \leq \frac{ML}{\Gamma(q)} r^{-q} \| x - y \|_r \leq \frac{1}{2} \| x - y \|_r.
$$

For any $x, y \in C$, if $t \in (t_k, t_{k+1}) (k = 1, \cdots, m)$. From $H(2)$ and $H(3)$, we get

$$
\| (\Phi x)(t) - (\Phi y)(t) \|_X \leq \| \sum_{i=1}^{k} S_q(t-t_i) [G(t_i, x(t_i)) - G(t_i, y(t_i))] \|_X
$$

$$
+ \int_0^t (t-s)^{q-1} \| T_q(t-s)(f(s,x(s),u(s)) - f(s,y(s),u(s))) \|_X ds
$$
\[ \leq \left( M \sum_{i=1}^{k} L_i + \frac{ML}{\Gamma(q)} t^{-q} \right) e^{rt} \|x - y\|_r. \]

Then, one can get
\[
\sup_{t \in (t_k, t_{k+1}]} e^{-rt} \left\| (\Phi x)(t) - (\Phi y)(t) \right\|_X
\leq \left( M \sum_{i=1}^{k} L_i + \frac{ML}{\Gamma(q)} t^{-q} \right) \|x - y\|_r \leq \frac{1}{2} \|x - y\|_r.
\]

Therefore, we obtain
\[
\| \Phi x - \Phi y \|_r = \sup_{t \in J} e^{-rt} \left\| (\Phi x)(t) - (\Phi y)(t) \right\|_X \leq \frac{1}{2} \|x - y\|_r.
\]

Therefore, \( \Phi \) is a contraction operator. According to the Banach’s fixed point theorem, we can deduce that the problem (1) has a unique mild solution on \( J \). The proof is complete.

Next, we consider other hypotheses without Lipschitz conditions in \( H(2) \) and \( H(3) \).

\( H(4) \): \( T(t) \) is compact for every \( t > 0 \).

\( H(5) \): There exist a function \( \phi_1 \in L^p(J, \mathbb{R}^+) \) (\( p > \frac{1}{q} \)) and constants \( L_1, N_1 > 0 \) which for all \( x \in X, u \in V, t \in J \).

\( H(6) \): \( G(t_k, \cdot) : X \to X \) is continuous and there exist constants \( a_k, b_k \geq 0 \) with \( M \sum_{i=1}^{k} a_i < \Gamma(q) \) \( (k = 1, 2, \ldots, m) \) such that
\[
\|G(t_k, x)\|_X \leq a_k \|x\|_X + b_k
\]
for all \( t_k \in D, x, y \in X \).

\textbf{Lemma 3.2} ([24], Lemma 3.5). If \( H(4) \) holds, then the operator \( \pi : L^p(J, X) \to C(J, X) \) for some \( p > \frac{1}{q} \), given by
\[
(\pi h)(\cdot) = \int_0^\cdot (\cdot - s)^{q-1} T_q (\cdot - s) h(s) \, ds, \quad \forall h \in L^p(J, X),
\]
is compact.

Now, we are in the position to prove the following existence result of mild solutions for problem (1) without Lipschitz conditions.

\textbf{Theorem 3.3}. Assume that the hypotheses \( H(1), H(4), H(5), H(6) \) are satisfied. Then for each given control function \( u \in L^p(J, V) \) \( (p > \frac{1}{q}) \), problem (1) has a mild solution on \( PC(J, X) \).

\textit{Proof}. Define the operator \( \Phi : C \to C \) by (3). Then the problem of finding mild solutions for problem (1) is reduced to find fixed points of \( \Phi \). We will apply Theorem 2.7 to obtain the existence of the fixed points of \( \Phi \). Let
\[
\Delta = \{ x \in C | \exists \lambda \in (0, 1), x = \lambda \Phi x \}.
\]

At first, we claim that the set \( \Delta \) is bounded.

Let \( x^\lambda \in \Delta \). For \( t \in [-h, 0] \), it is clear that \( \Delta \) is bounded. If \( t \in (0, t_1] \), then from the hypothesis \( H(5) \) and the Hölder inequality, we obtain
\[
\| x^\lambda(t) \|_X = \| \lambda (\Phi x^\lambda)(t) \|_X
\]
\[
\begin{align*}
&\leq \|S_q(t)\psi(0)\|_X + \int_0^t (t-s)^{q-1}\|T_q(t-s)f(s,x(s),u(s))\|_X ds \\
&\leq M\|\psi(0)\|_X + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1}(\|\phi_1(s)\|_X + L_1\|x^\lambda(s)\|_X + N_1\|u(s)\|_V) ds \\
&\leq M\|\psi(0)\|_X + \frac{Mb^q}{\Gamma(q)} \left(\frac{p-1}{pq-1}\right)^{1-\frac{q}{p}}(\|\phi_1\|_{L^p(J,R^+)} + N_1\|u\|_{L^p(J,V)}) \\
&\quad + \frac{L_1M}{\Gamma(q)} \int_0^t (t-s)^{q-1}\|x^\lambda(s)\|_X ds.
\end{align*}
\]
Thus, by applying the generalized Gronwall inequality in [26], we have
\[
\|x^\lambda(t)\|_X \leq M_1e^{M_2} 
\]
for some \(M_1, M_2 > 0\), which implies that the set \(\Delta\) is bounded.

If \(t \in (t_k, t_{k+1}) (k = 1, \cdots, m)\), from the hypotheses \(H(5), H(6)\) and the Hölder inequality, we have
\[
\|x^\lambda(t)\|_X = \|\lambda(\Phi x^\lambda)(t)\|_X \\
\leq \|S_q(t)\psi(0)\|_X + \|\sum_{i=1}^k S_q(t-t_i)G(t_i, x^\lambda(t_i))\|_X \\
+ \int_0^t (t-s)^{q-1}\|T_q(t-s)f(s,x^\lambda(s),u(s))\|_X ds \\
\leq M\|\psi(0)\|_X + \sum_{i=1}^k (a_i\|x^\lambda(t_i)\|_X + b_i) \\
+ \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1}(\|\phi_1(t)\|_X + L_1\|x^\lambda(s)\|_X + N_1\|u(s)\|_V) ds \\
\leq M\|\psi(0)\|_X + \frac{Mb^q}{\Gamma(q)} \left(\frac{p-1}{pq-1}\right)^{1-\frac{q}{p}}(\|\phi_1\|_{L^p(J,R^+)} + N_1\|u\|_{L^p(J,V)}) + \sum_{i=1}^k b_i \\
+ \sum_{i=1}^k a_i\|x^\lambda(t_i)\|_X + \frac{L_1M}{\Gamma(q)} \int_0^t (t-s)^{q-1}\|x^\lambda(s)\|_X ds.
\]
Thus
\[
\sup_{s \in [0,t]} \|x^\lambda(s)\|_X \\
\leq M\|\psi(0)\|_X + \frac{Mb^q}{\Gamma(q)} \left(\frac{p-1}{pq-1}\right)^{1-\frac{q}{p}}(\|\phi_1\|_{L^p(J,R^+)} + N_1\|u\|_{L^p(J,V)}) + \sum_{i=1}^k b_i \\
+ \sum_{i=1}^k a_i \sup_{s \in [0,t]} \|x^\lambda(s)\|_X + \frac{L_1M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sup_{r \in [0,s]} \|x^\lambda(r)\|_X ds,
\]
and hence
\[
(1 - M \sum_{i=1}^k a_i) \sup_{s \in [0,t]} \|x^\lambda(s)\|_X \\
\leq M\|\psi(0)\|_X + \frac{Mb^q}{\Gamma(q)} \left(\frac{p-1}{pq-1}\right)^{1-\frac{q}{p}}(\|\phi_1\|_{L^p(J,R^+)} + N_1\|u\|_{L^p(J,V)}) + \sum_{i=1}^k b_i
\]
Theorem 3.4. Assume that the hypotheses \( H \) is satisfied. Then for each given control function \( u \) we have
\[
\text{Since } M \sum_{i=1}^{k} a_i < 1, \text{ by applying the generalized Gronwall inequality in [26] again, we have}
\[
\sup_{s \in [0, t]} \|x^\lambda(s)\|_X \leq M_3 e^{M_4} \quad \text{for some } M_3, M_4 > 0,
\]
which implies that the set \( \Delta \) is bounded.

Moreover, by \( H(4), \text{Lemma 2.9(iii)} \) and \( \text{Lemma 3.2} \), it is clear that \( \Phi \) is a compact operator. Therefore, by applying Theorem 2.7, we can deduce that the problem (1) has a mild solution on \( J \). The proof is complete. \( \square \)

Furthermore, we consider following compactness conditions.
\( H(8) \) : For every \( t \in J, u \in V \) and every \( R > 0 \), the set \( Q(t) = \{(t-s)^{q-1} T_q(t-s)f(s, \psi, u) \mid s \in [0, t], \|\psi\|_X \leq R\} \) is relatively compact in \( X \).
\( H(9) \) : \( G(t_k, \cdot) : X \rightarrow X \) is compact for \( t_k \in D \).

**Theorem 3.4.** Assume that the hypotheses \( H(1), H(5), H(6), H(8), H(9) \) are satisfied. Then for each given control function \( u \in L^p(J, V) \) \((p > \frac{1}{3})\), problem (1) has a mild solution on \( PC(J, X) \).

**Proof.** Let \( \Phi \) and \( \Delta \) are defined by (3) and (6). From the proof of Theorem 3.3, we only prove that the operator \( \Phi \) is completely continuous.

Let \( \Phi = \Phi_0 + \Phi_1 + \Phi_2 \), where
\[
(\Phi_0x)(t) = S_q(t)\psi(0), \quad t \in J,
\]
\[
(\Phi_1x)(t) = \int_0^t (t-s)^{q-1} T_q(t-s)f(s, x(s), u(s))ds, \quad t \in J,
\]
\[
(\Phi_2x)(t) = \begin{cases} 0, & t \in [0, t_1], \\ \sum_{i=1}^{k} S_q(t-t_i)G(t_i, x(t_i)), & t \in (t_k, t_{k+1}], \, k = 1, \cdots, m. \end{cases}
\]

It follows from \( H(9) \) that \( \Phi_2 \) is completely continuous. Next, we show that \( \Phi_1 \) is completely continuous. We will prove that \( \Phi_1 \) maps bounded sets into relatively compact ones. Let \( R > 0 \) and
\[
B(R) = \{x \in C : \|x\|_C \leq R\}.
\]
From the mean value theorem, we see that
\[
(\Phi_1x)(t) \in \{x \in C : \|x\|_C \leq R\}, \quad t \in J,
\]
which implies that the set \( \{\Phi_1x(t) : x \in B(R)\} \) is relatively compact for each \( t \in J \). Moreover, from the proof of Theorem 3.2 in [19], we know that the set \( \{\Phi_1x : x \in B(R)\} \) is equicontinuous. Therefore, the relative compactness of the set \( \{\Phi_1x : x \in B(R)\} \) follows from the well known Arzela-Ascoli criterion, which shows that \( \Phi_1 \) is completely continuous. Therefore, the problem (1) has a mild solution on \( J \). The proof is complete. \( \square \)
4. Existence of feasible pairs. In this section, we start to consider the feedback control system (2). To the readers’ convenience, we give the definition as follows.

**Definition 4.1.** A pair \((x, u)\) is said to be feasible if \((x, u)\) satisfies (2).

We denote

\[ V_J = \{ u : J \to V | u(\cdot) \text{ is measurable} \}, \quad \mathcal{H} = \{ (x, u) \in \mathcal{C} \times V_J \mid (x, u) \text{ is feasible} \}. \]

To obtain the existence result of feasible pairs for system (2), we assume the following hypotheses.

\[ H(10) : U : J \times X \to P(V) \text{ is pseudo-continuous and there exist a function } \varphi \in \mathcal{L}^p(J, \mathbb{R}_+) \quad (p > \frac{1}{2}) \quad \text{and a constant } l > 0, \quad \text{such that} \]

\[ \|U(t, x)\| = \sup_{z \in U(t, x)} \|z\|_V \leq \varphi(t) + t\|x\|_X \]

for all \(x \in X\), a.e. \(t \in J\).

\[ H(11) : \text{The set } f(t, x, U(t, x)) \text{ satisfies the following} \]

\[ \bigcap_{\delta > 0} \overline{\omega} f(t, O_{\delta}(x), U(O_{\delta}(t, x))) = f(t, x, U(t, x)) \]

for all \(x \in X\), a.e. \(t \in J\).

We are in the position to present the main results of this section.

**Theorem 4.2.** Assume that all the assumptions of Theorem 3.3 and \(H(10) - H(11)\) are satisfied. Then the set \(\mathcal{H}\) is nonempty.

**Proof.** Case 1. Let \(t \in [0, t_1]\). For any \(n > 0\), let \(\tau_{0,j} = \frac{j}{n}t_1, \ 0 \leq j \leq n - 1\). We set

\[ u_{0,n}(t) = \sum_{j=0}^{n-1} u^{0,j}_{\chi_{[\tau_{0,j}, \tau_{0,j+1}]}(t)}, \quad t \in [0, t_1], \]

where \(\chi_{[\tau_{0,j}, \tau_{0,j+1}]}\) is the character function of interval \([\tau_{0,j}, \tau_{0,j+1}]\). The sequence \(\{u^{0,j}_{\cdot}\}\) is constructed as follows. Firstly, we take \(u^{0,0} \in U(0, \psi(0))\). By Theorem 3.3, there exists \(x_{0,n}(\cdot)\) with \(x_{0,n}(t) = \psi(t)\) for \(t \in [-h, 0]\) and

\[ x_{0,n}(t) = S_q(t)\psi(0) + \int_0^t (t-s)^{q-1} T_q(t-s)f(s, x_{0,n}(s), u^{0,0})ds, \quad t \in (0, \tau_{0,1}). \]

Then take \(u^{0,1} \in U(\tau_{0,1}, x_{0,n}(\tau_{0,1}))\). We can repeat this procedure to obtain \(x_{0,n}\) on \([\tau_{0,1}, \tau_{0,2}]\), etc. By induction, we end up with the following:

\[
\begin{cases}
  x_{0,n}(t) = S_q(t)\psi(0) + \int_0^t (t-s)^{q-1} T_q(t-s)f(s, x_{0,n}(s), u_{0,n}(s)) ds, & t \in (0, t_1], \\
  u_{0,n}(t) \in U(\tau_{0,j}, x_{0,n}(\tau_{0,j})), & t \in [\tau_{0,j}, \tau_{0,j+1}], \ 0 \leq j \leq n-1.
\end{cases}
\]

(10)

By \(H(5), H(6)\) and \(H(10)\) and the proof of Theorem 3.3 there exist \(r_{0,0}, r_{0,1} > 0\) such that

\[ \|x_{0,n}\|_{C([0, t_1], X)} \leq r_{0,0} \]

and

\[ \|f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot))\|_{\mathcal{L}^p([0, t_1], X)} \leq r_{0,1}. \]

Since \(L^p([0, t_1], X)\) is a reflexive Banach space, there is a subsequence of \(\{f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot))\}\), denoted by \(\{f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot))\}\) again, such that

\[ f(\cdot, x_{0,n}(\cdot), u_{0,n}(\cdot)) \to f(\cdot) \text{ in } L^p([0, t_1], X). \]
By $H(4)$ and Lemma 3.2 we obtain that
\[
\int_0^t (t-s)^{q-1} T_q(t-s)f(s , x_{0,n}(s), u_{0,n}(s))ds \to \int_0^t (t-s)^{q-1} T_q(t-s)\tilde{f}_0(s)ds
\]
for $t \in (0,t_1]$ and the set $\{x_{0,n}\}$ is relatively compact in $C((0,t_1], X)$. Then
\[
x_{0,n}(-) \to \tilde{x}_0(-) \text{ in } C((0,t_1], X)
\]
and
\[
\tilde{x}_0(t) = S_q(t)\psi(0) + \int_0^t (t-s)^{q-1} T_q(t-s)\tilde{f}_0(s)ds, \quad t \in (0,t_1].
\]

On the other hand, by the definition of $u_{0,n}(-)$ for $n$ large enough, we have
\[
u_{0,n}(t) \in U(\tau_{0,j}, x_{0,n}(\tau_{0,j})) \subset U(O_\delta(t, \tilde{x}_0(t))),
\]
for all $t \in [\tau_{0,j}, \tau_{0,j+1})$, $0 \leq j \leq n - 1$.

Secondly, by (11) and the Mazur theorem (Chapter 2, Corollary 2.8, [10]), let $q_0^0 \geq 0$ and $\sum_{i \geq 1} q_0^0 = 1$ such that
\[
\psi_{0,l}(-) = \sum_{i \geq 1} q_0^0 f(-, x_{0,i+l}(-), u_{0,i+l}(-)) \to \tilde{f}_0(-) \text{ in } L^p([0,t_1], X).
\]

Then, there is a subsequence of $\{\psi_{0,l}\}$, denoted by $\psi_{0,l}$ again, such that
\[
\psi_{0,l}(t) \to \tilde{f}_0(t) \text{ in } X, \text{ a.e. } t \in [0,t_1].
\]

Hence, from (12) and (13), for $l$ large enough,
\[
\psi_{0,l}(t) \in \text{co} f(t, O_\delta(\tilde{x}_0(t)), U(O_\delta(t, \tilde{x}_0(t)))) \text{, a.e. } t \in [0,t_1].
\]

Thus, for any $\delta > 0$,
\[
\tilde{f}_0(t) \in \text{co} f(t, O_\delta(\tilde{x}_0(t)), U(O_\delta(t, \tilde{x}_0(t)))) \text{, a.e. } t \in [0,t_1].
\]

By $H(11)$, we have
\[
\tilde{f}_0(t) \in f(t, \tilde{x}_0(t), U(t, \tilde{x}_0(t))) \text{, a.e. } t \in [0,t_1].
\]

By $H(10)$ and Corollary 2.18 of [10], we have that $U(-, \tilde{x}_0(-))$ is Souslin measurable. By the Fillippove theorem (Chapter 2, Corollary 2.26, [10]), there exists a measurable function $\tilde{u}_0$ on $[0,t_1]$ such that
\[
\tilde{u}_0(t) \in U(t, \tilde{x}_0(t)), \quad t \in [0,t_1],
\]
and
\[
\tilde{f}_0(t) = f(t, \tilde{x}_0(t), \tilde{u}_0(t)), \quad t \in [0,t_1].
\]

**Case 2.** Let $t \in (t_1,t_2]$. For any $n > 0$, let $\tau_{1,j} = t_1 + \frac{1}{n}(t_2 - t_1)$, $0 \leq j \leq n - 1$. We set
\[
u_{1,n}(t) = \begin{cases}
\tilde{u}_0(t), & t \in [0,t_1], \\
\sum_{j=0}^{n-1} u^{1,j}_\chi_{[\tau_{1,j}, \tau_{1,j+1})}(t), & t \in (t_1,t_2].
\end{cases}
\]

The sequence $\{u^{1,j}\}$ is constructed as follows.
Take $u^{1,0} \in U(t_1, x(t_1))$. By Theorem 3.4, there exists $x_{1,n}(\cdot)$ which is given by

$$
x_{1,n}(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\tau_0(t), & t \in (0, t_1], \\
S_q(t)\psi(0) + S_q(t-t_1)G(t_1, x_{1,n}(t_1)) + \int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_{1,n}(s), u_{1,0}(s))ds, & t \in (t_1, t_2], 
\end{cases}
$$

Then take $u^{1,1} \in U(t_1, x_{1,1}(t_1))$. We can repeat this procedure to obtain $x_{1,n}$ on $(\tau_{1,1}, \tau_{1,2}]$, etc. By induction, we end up with the following:

$$
x_{1,n}(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\tau_0(t), & t \in (0, t_1], \\
S_q(t)\psi(0) + S_q(t-t_1)G(t_1, x_{1,n}(t_1)) + \int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_{1,n}(s), u_{1,n}(s))ds, & t \in (t_1, t_2], 
\end{cases}
$$

Then, there is a subsequence of $u_{1,n}(\cdot)$ such that

$$
\|x_{1,n}\|_{PC([0,t_2], X)} \leq r_{1,0}
$$

and

$$
\|f(\cdot, x_{1,n}(\cdot), u_{1,n}(\cdot))\|_{L^p([0,t_2], X)} \leq r_{1,1}.
$$

Then, there is a subsequence $\{f(\cdot, x_{1,n}(\cdot), u_{1,n}(\cdot))\}$ such that

$$
f(\cdot, x_{1,n}(\cdot), u_{1,n}(\cdot)) \rightharpoonup \tilde{F}_1(\cdot) \quad \text{in } L^p([0, t_2], X).
$$

(14)

It is clear that $\tilde{F}_1|_{[0,t_1]} = \tilde{F}_0$.

By $H(4)$ and Lemma 3.2 again, we have, for $t \in (t_1, t_2]

$$
\int_0^t (t-s)^{q-1}T_q(t-s)f(s, x_{1,n}(s), u_{1,n}(s))ds \to \int_0^t (t-s)^{q-1}T_q(t-s)\tilde{F}_1(s)ds,
$$

$$
x_{1,n}(\cdot) \to \tilde{\tau}_1(\cdot) \quad \text{in } PC([0,t_2], X)
$$

(15)

and

$$
\tilde{\tau}_1(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\tau_0(t), & t \in (0, t_1], \\
S_q(t)\psi(0) + S_q(t-t_1)G(t_1, \tilde{\tau}_1(t_1)) + \int_0^t (t-s)^{q-1}T_q(t-s)\tilde{F}_1(s)ds, & t \in (t_1, t_2]. 
\end{cases}
$$

On the other hand, by the definition of $u_{1,n}(\cdot)$ for $n$ large enough, we have

$$
u_{1,n}(t) \in U(\tau_{1,j}, x_{1,n}(\tau_{1,j})) \subset U(O_\delta(t, \tilde{\tau}_1(t))),
$$

(16)

for all $t \in (\tau_{1,j}, \tau_{1,j+1}]$, $0 \leq j \leq n - 1$.

Secondly, by (14) and the Mazur theorem again, let $q_0^1 \geq 0$ and $\sum_{i \geq 1} q_0^1 = 1$ such that

$$
\psi_{1,l}(\cdot) = \sum_{i \geq 1} q_0^i f(\cdot, x_{1,i+l}(\cdot), u_{1,i+l}(\cdot)) \to \tilde{F}_1(\cdot) \quad \text{in } L^p([0, t_2], X).
$$

Then, there is a subsequence of $\{\psi_{1,l}\}$, denoted by $\{\psi_{1,l}\}$ again, such that

$$
\psi_{1,l}(t) \to \tilde{F}_1(t) \quad \text{in } X, \quad \text{a.e. } t \in [0, t_2].
$$

Hence, from (15) and (16), for $l$ large enough,

$$
\psi_{1,l}(t) \in \cof(t, O_\delta(\tilde{\tau}_1(t)), U(O_\delta(t, \tilde{\tau}_1(\tau_{1,j})))), \quad \text{a.e. } t \in [0, t_2].
$$
Thus, for any $\delta > 0$,
\[
\bar{f}_1(t) \in \mathcal{M}(t, O_\delta(x_1(t)), U(O_\delta(t, \bar{x}_1(\tau_{1,j})))) \quad \text{a.e. } t \in [0, t_2].
\]

By $H(11)$, we have
\[
\bar{f}_1(t) \in f(t, \bar{x}_1(t), U(t, \bar{x}_1(\tau_{1,j}))), \quad \text{a.e. } t \in [0, t_2].
\]

By $H(10)$ and Corollary 2.18 of [10] again, we have that $U(\cdot, \bar{x}_1(\cdot))$ is Souslin measurable. By the Filippov theorem again, there exists a measurable function $\bar{u}_1$ on $[0, t_2]$ such that
\[
\begin{cases}
\bar{u}_1(t) = \bar{u}_0(t), & t \in [0, t_1], \\
\bar{u}_1(t) \in U(t, \bar{x}_1(t)), & t \in (t_1, t_2],
\end{cases}
\]
and
\[
\bar{f}_1(t) = f(t, \bar{x}_1(t), \bar{u}_1(t)), \quad t \in [0, t_2].
\]

**Case 3.** Let $t \in (t_k, t_{k+1})$ ($k = 2, \ldots, m$). For any $n > 0$, let $\tau_{k,j} = t_k + \frac{j}{n}(t_{k+1} - t_k), \ 0 \leq j \leq n - 1$. We set
\[
\begin{cases}
\bar{u}_{k-1}(t), & t \in [0, t_k], \\
\sum_{j=0}^{n-1} u^j \chi(\tau_{k,j}, \tau_{k,j+1})(t), & t \in (t_k, t_{k+1}].
\end{cases}
\]

By induction, we end up with the following:
\[
x_{k,n}(t) = \begin{cases}
\psi(t), & t \in [-h, 0], \\
\bar{x}_{k-1}(t), & t \in (0, t_k], \\
S_q(t)\psi(0) + \sum_{i=1}^{k} S_q(t-t_i)G(t_i, x_{i,n}(t_i)) \\
+ \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{k,n}(s), u_{k,n}(s)) ds, & t \in (t_k, t_{k+1}],
\end{cases}
\]
\[
\begin{cases}
\bar{u}_{k,n}(t), & t \in (\tau_{k,j}, \tau_{k,j+1}), \\
\sum_{j=0}^{n-1} u^j \chi(\tau_{k,j}, \tau_{k,j+1})(t), & t \in (t_k, t_{k+1}].
\end{cases}
\]

By $H(7)$, $H(10)$, and the proof of Theorem 3.3, there exists $r_{k,0} > 0$ such that
\[
\|x_{k,n}\|_{PC([0, t_{k+1}], X)} \leq r_{k,0}.
\]

Moreover, it comes from $H(7)$ and $H(10)$ that there exists $r_{k,1} > 0$ such that
\[
\|f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\|_{L^p([0, t_{k+1}], X)} \leq r_{k,1}.
\]

Then, there is a subsequence $\{f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\}$ such that
\[
f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot)) \rightarrow \bar{f}_k(\cdot) \quad \text{in } L^p([0, t_{k+1}], X).
\]

It is clear that $\bar{f}_k|[0, t_k] = \bar{f}_{k-1}$ and
\[
\int_0^t (t-s)^{q-1} T_q(t-s) f(s, x_{k,n}(s), u_{k,n}(s)) ds \rightarrow \int_0^t (t-s)^{q-1} T_q(t-s) \bar{f}_k(s) ds
\]
for $t \in (t_k, t_{k+1})$.

Similar to the previous proof, we obtain
\[
x_{k,n}(\cdot) \rightarrow \bar{x}_k(\cdot) \quad \text{in } PC([0, t_{k+1}], X).
\]
Therefore, \( \delta > 0 \), for any 

Assume that all the assumptions of Theorem 3.4 and Theorem 4.3.

and

\[
\mathcal{H}(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\mathcal{H}_{k-1}(t), & t \in (0, t_k], \\
S_q(t)\psi(0) + \sum_{i=1}^{k} S_q(t-t_i)G(t_i, \mathcal{H}(t_i)) \\
+ \int_{0}^{t} (t-s)^{q-1} T_q(t-s)\mathcal{H}(s)ds, & t \in (t_k, t_{k+1}]. 
\end{cases}
\]

On the other hand, by the definition of \( u_{k,n}(\cdot) \) for \( n \) large enough, we have

\[
\begin{equation}
\tag{19}
u_{k,n}(t) \in U(\tau_{k,j}, x_{k,n}(\tau_{k,j})) \subset U(O_{\delta}(t, \mathcal{H}(t))),
\end{equation}
\]

for all \( t \in (\tau_{k,j}, \tau_{k,j+1}], \ 0 \leq j \leq n - 1 \).

Secondly, by (17) and the Mazur theorem again, let \( q_{il}^k \geq 0 \) and \( \sum_{i \geq 1} q_{il}^k = 1 \) such that

\[
\psi_{k,i}(\cdot) = \sum_{i \geq 1} q_{il}^k f(\cdot, x_{k,i+l}(\cdot), u_{k,i+l}(\cdot)) \to \bar{\mathcal{H}}_k(\cdot) \quad \text{in } L^p([0, t_{k+1}], X).
\]

Then, there is a subsequence of \( \{\psi_{k,i}\} \), denoted by \( \{\psi_{k,i}\} \) again, such that

\[
\psi_{k,i}(\cdot) \to \bar{\mathcal{H}}_k(t) \quad \text{in } X, \quad \text{a.e. } t \in [0, t_{k+1}].
\]

Hence, from (18) and (19), for \( l \) large enough,

\[
\psi_{k,i}(t) \in \text{co}f(t, O_{\delta}(\mathcal{H}(t)), U(O_{\delta}(t, \mathcal{H}(t)))) \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]

Thus, for any \( \delta > 0 \),

\[
\bar{\mathcal{H}}_k(t) \in \text{co}f(t, O_{\delta}(\mathcal{H}(t)), U(O_{\delta}(t, \mathcal{H}(t)))) \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]

By \( H(11) \), we have

\[
\bar{\mathcal{H}}_k(t) \in f(t, t, \mathcal{H}(t)), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\]

Similarly, there exists a \( \mathcal{H}_k \in V_j \) such that

\[
\begin{cases} 
\mathcal{H}_k(t) = \mathcal{H}_{k-1}(t), & t \in [0, t_k], \\
\mathcal{H}_k(t) \in U(t, \mathcal{H}(t)), & t \in (t_k, t_{k+1}].
\end{cases}
\]

and

\[
\bar{\mathcal{H}}_k(t) = f(t, \mathcal{H}(t), \mathcal{H}_k(t)), \quad t \in [0, t_{k+1}].
\]

Let

\[
\mathcal{H}(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\mathcal{H}_0(t), & t \in (0, t], \\
\mathcal{H}_k(t), & t \in (t_k, t_{k+1}], \ k = 1, \ldots, m,
\end{cases}
\]

\[
\mathcal{H}(t) = \mathcal{H}_{m+1}(t).
\]

Therefore, \( (\mathcal{H}, \mathcal{H}) \in \mathcal{H} \). The proof is complete.

\[ \Box \]

**Theorem 4.3.** Assume that all the assumptions of Theorem 3.4 and \( H(10) - H(11) \) are satisfied. Then the set \( \mathcal{H} \) is nonempty.
Proof. Similar to the proof of Theorem 4.2, we have
\[
x_{k,n}(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\kappa_{k-1}(t), & t \in (0, t_k], \\
S_q(t)\psi(0) + \sum_{i=1}^{k} S_q(t-t_i)G(t_i, x_{i,n}(t_i)) \\
+ \int_{0}^{t} (t-s)^{q-1}T_q(t-s)f(s, x_{k,n}(s), u_{k,n}(s))ds, & t \in (t_k, t_{k+1}], 
\end{cases}
\]
and
\[
\|f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\|_{L^2([0, t_k+1], X)} \leq r_{k,0}
\]
and
\[
\|f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\|_{L^2([0, t_k+1], X)} \leq r_{k,1}
\]
Then, there is a subsequence \( \{f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\} \) such that
\[
f(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot)) \rightharpoonup \overline{f}(\cdot) \text{ in } L^2([0, t_k+1], X).
\]
It is clear that \( \overline{J}_k|_{[0, t_k]} = \overline{J}_{k-1} \). Since the operator \( \pi : L^2(J, X) \to C(J, X) \) defined by (5) is linear and continuous, we obtain
\[
\int_{0}^{t} (t-s)^{q-1}T_q(t-s)f(s, x_{k,n}(s), u_{k,n}(s))ds \rightarrow \int_{0}^{t} (t-s)^{q-1}T_q(t-s)\overline{f}(s)ds
\]
for \( t \in (t_k, t_{k+1}] \). From the proof of Theorem 3.4 it follows that the set \( \{x_{0,n}\} \) is relatively compact in \( PC(J, X) \). Then
\[
x_{k,n}(\cdot) \rightarrow \kappa_k(\cdot) \text{ in } PC((0, t_{k+1}], X),
\]
and hence
\[
\kappa_k(t) = \begin{cases} 
\psi(t), & t \in [-h, 0], \\
\kappa_{k-1}(t), & t \in (0, t_k], \\
S_q(t)\psi(0) + \sum_{i=1}^{k} S_q(t-t_i)G(t_i, \kappa_i(t_i)) \\
+ \int_{0}^{t} (t-s)^{q-1}T_q(t-s)\overline{f}_k(s)ds, & t \in (t_k, t_{k+1}], 
\end{cases}
\]
The rest proof is similar to the proof of Theorem 4.2.

Similarly, we also have the following result.

**Theorem 4.4.** Assume that all the assumptions of Theorem 3.1, and \( H(4), H(10), H(11) \) are satisfied. Then the set \( \mathcal{H} \) is nonempty.

At the end of this section, we consider an optimal control problem stated as follows.

**Problem (\( \varphi \)):** find a pair \((x^0, u^0) \in X_J\) such that
\[
\varphi(x^0, u^0) \leq \varphi(x, u), \quad \forall (x, u) \in X_J,
\]
where \( \varphi(x, u) = \int_{0}^{t} f_0(t, x(t), u(t))dt \).

We make the following assumptions on \( f_0 \):
\( (f_01) \) the functional \( f_0 : J \times X \times V \rightarrow \mathbb{R} \cup \{\pm \infty\} \) is Borel measurable in \((t, x, u)\);
\( (f_02) \) \( f_0(t, \cdot, \cdot) \) is lower semicontinuous on \( X \times V \) for a.e. \( t \in J \) and there exists a constant \( M_1 > 0 \) such that
\[
f_0(t, x, u) \geq -M_1, \quad (t, x, u) \in J \times X \times V.
\]
For any \((t, x) \in J \times X\), we set
\[
\varepsilon(t, x) = \{(z^0, z^1) \in \mathbb{R} \times X | z^0 \geq f_0(t, x, z^1), z^1 = f(t, x, u), u \in U(t, x)\}.
\]

In order to obtain the existence result of optimal state-control pairs for Problem \((\varphi)\), we assume that:

\((H_s)\) : for a.e. \(t \in J\), the map \(\varepsilon(t, \cdot) : X \rightarrow (\mathbb{R} \times X)\) is such that
\[
\bigcap_{\delta > 0} \co \varepsilon(t, O_\delta(x)) = \varepsilon(t, x), \quad \forall x \in X.
\]

Now we have the following existence result of optimal state-control pair for Problem \((\varphi)\), which proof is similar to the proof of Theorem 4.7 in [17], Theorem 4.1 in [24] or Theorem 5.2 in [28].

**Theorem 4.5.** If all the assumptions of one of Theorems 4.2, 4.3, 5.5 and \((f_01), (f_02), (H_s)\) are satisfied, then Problem \((\varphi)\) admits at least one optimal state-control pair.

5. **Applications.** In this section, we apply our previous results to a controllability result for fractional impulsive delay evolution equations and two existence results for a class of fractional delay differential variational inequalities and fractional delay Clarke’s subdifferential inclusions.

### 5.1. Controllability result

Consider the controllability of the following system.

\[
\begin{aligned}
\frac{C D_t^q x(t)}{} &= Ax(t) + f_1(t, x(t)) + Bu(t), \quad t \in (0, b] - D, \\
\Delta x(t_k) &= G(t_k, x(t_k)), \quad t_k \in D, \\
x(t) &= \psi(t), \quad t \in [-h, 0].
\end{aligned}
\]  

(20)

Now we give the definition of controllability as follows.

**Definition 5.1.** System \((22)\) is said to be controllable on the interval \(J\), if for every \(x_b \in X\), there exists a control \(u \in L^2(J, V)\) such that a mild solution \(x\) of system \((22)\) satisfies \(x(b) = x_b\).

We need to make the following assumptions.

\(H(f_1) : f_1 : J \times X \rightarrow X\) is Borel measurable on \(J \times X\) and continuous on \(X\), and there exist a function \(\phi_4 \in L^2(J, \mathbb{R}_+^+)\) and constants \(L_4 > 0\) such that
\[
\|f_1(t, x)\|_X \leq \phi_4(t) + L_4\|x\|_X
\]

for all \(x \in X\), a.e. \(t \in J\).

\(H(W) : W : C^1(J, V) \rightarrow X\) defined by
\[
Wu = \int_0^b (b - s)^{q-1} T_q(b - s)Bu(s)ds
\]

has an invertible operator \(W^{-1}\) which takes values in \(L^q(J, V)/\ker W\), where \(\ker W = \{x \in L^q(J, V) : Wx = 0\}\) and there exists a positive constant \(M_2\) such that \(\|W^{-1}\| \leq M_2\).

The following theorem is the main result of this section.

**Theorem 5.2.** If \(H(4), H(6), H(f_1), H(W)\) hold, then the system \((22)\) is controllable on \(J\).
Proof. For any $x \in C$, $x_b \in X$, we can define the feedback control $U : J \times X \to V$ by

$$U(t, x) = \begin{cases} W^{-1} \left( x_b - S_q(b) \psi(0) - \int_0^b (b-s)^{q-1}T_q(0,b-s)f_1(s,x)ds \right)(t), t \in [0, t_1] \\ W^{-1} \left( x_b - S_q(b) \psi(0) - \sum_{i=1}^k S_q(b-t_i)G(t_i, x(t_i)) \right) - \int_0^b (b-s)^{q-1}T_q(b-s)f_1(s,x, x)ds(t), t \in (t_k, tk_{k+1}], k = 1, \ldots, m. \end{cases}$$

We show that, using this control, the operator $\Phi_1 : C \to C$, defined by

$$\Phi_1(x)(t) = \begin{cases} \psi(t), t \in [-h, 0], \\ S_q(t)\psi(0) + \int_0^t (b-s)^{q-1}T_q(t-s)[f_1(s, x(s)) + BU(t, x(s))]ds, t \in (0, t_1], \\ S_q(t)\psi(0) + \sum_{i=1}^k S_q(t-t_i)G(t_i, x(t_i)) + \int_0^t (b-s)^{q-1}T_q(b-s)[f_1(s, x(s)) + BU(t, x(s))]ds, t \in (t_k, tk_{k+1}], k = 1, \ldots, m. \end{cases}$$

has a fixed point $x$, which is a mild solution of system (22).

It is clear that $U$ is continuous on $J \times X$, and hence $H(10)$ and $H(11)$ are easy to verified. Then, by applying Theorem 4.2, we obtain that the operator $\Phi_1$ has a fixed point. Therefore, system (20) is controllable on $J$. \hfill \Box

5.2. Fractional impulsive differential variational inequalities. Consider the following fractional impulsive differential variational inequality:

$$\left\{ \begin{array}{ll} ^CD^q_t x(t) = Ax(t) + f(t, x(t), u(t)), & t \in (0, b] - D, \\ u(t) \in SOL(K, g(t, x(t), \cdot), \phi), & a.e. t \in J, \\ \Delta x(t_k) = G(t_k, x(t_k)), & t_k \in D, \\ x(t) = \psi(t), & t \in [-h, 0], \end{array} \right.$$  (22)

where $SOL(K, g(t, x(t), \cdot), \phi)$ denotes the solution set of the following mixed variational inequality in $V$: find $u : [0, T] \to K \subset V$ such that

$$\langle g(t, x(t), u(t)), v - u(t) \rangle_E + \phi(v) - \phi(u(t)) \geq 0, \quad \forall v \in K, \quad a.e. \ t \in J.$$ 

Let $F : J \times X \to P(V)$ given by $F(t, x) = f(t, x, U(t, x))$. Then we have the following result.

Lemma 5.3 ([13], Theorem 3.4). Assume that $X, V$ are real separable reflexive Banach spaces and $K$ is a nonempty, compact and convex subset of $V$. Assume that $g : J \times X \times K \to V^*$ is such that $g(\cdot, \cdot, u)$ is continuous from $J \times X$ to $V^*$ endowed with the weak* topology whenever $u \in K$. In addition, we assume that for every $(t, x) \in J \times X$ the mappings $Q := g(t, x, \cdot)$ and $\phi : V \to \mathbb{R} \cup \{+\infty\}$ satisfy the following hypotheses:

(i) $Q : K \to V^*$ is monotone on $K$ and satisfies

$$\lim_{\lambda \to 0^+} \inf \langle Q(\lambda u + (1-\lambda)v), v-u \rangle \leq \langle Qv, v-u \rangle, \quad u, v \in K;$$

(ii) $\phi$ is convex, lower semicontinuous, and $\neq +\infty$.

Then the multifunction $U : [0, T] \times X \to P(K)$ defined by

$$U(t, x) := \{u \in K : \langle g(t, x, u), v - u \rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in K\}. \quad (23)$$
Recall also that the Clarke’s subdifferential or generalized gradient of $j$ there hold:

(i) $\partial j$ is u.s.c.;
(ii) $U$ is superpositionally measurable.

Lemma 5.4 ([13], Lemma 4.2). Assume that $X, V$ are real separable reflexive Banach spaces and $K$ is a nonempty, compact and convex subset of $V$. Assume that the hypotheses of Lemma 5.3 and $H(1)$ and $H(5)$ are satisfied. Suppose that, in addition, $f(t, x, Z)$ is convex for every convex $Z \subset X$, all $x \in X$, a.e. $t \in J$. Then there hold:

(i) $F(t, x)$ is a closed convex subset in $X$ for all $(t, x) \in J \times X$;
(ii) $F(\cdot, x)$ has a strongly measurable selection for a.e. $t \in J$;
(iii) $F(t, \cdot)$ is u.s.c for a.e. $t \in J$.

Let $\mathcal{H}' = \{(x, u) \in C \times V| (x, u)$ is feasible for (22)$. The following result is a consequence of Lemma 5.3, Lemma 5.4 and Proposition 4.5 in [10]

Theorem 5.5. If all the assumptions of Lemma 5.4 are satisfied and $H(4)$ hold, then $\mathcal{H}'$ is nonempty.

5.3. Fractional impulsive subdifferential inclusion. Consider the following Clarke’s subdifferential inclusion:

\[
\begin{aligned}
C \partial^0 x(t) &= Ax(t) + f_1(t, x(t)) + \gamma^* u(t), \quad t \in (0, b] - D, \\
u(t) &\in \partial j(t, x(t)), \quad \text{a.e. } t \in J, \\
\Delta x(t_k) &= G(t_k, x(t_k)), \quad t_k \in D,
\end{aligned}
\]

where $j : J \times Y \to \mathbb{R}$ is a locally Lipschitz function with respect to the second variable with $Y$ being a separable reflexive Banach space, $\partial j(t, \cdot)$ denotes the Clarke’s subdifferential of $j(t, \cdot)$ for $t \in J$ and $\gamma : X \to Y$ is a linear, continuous and compact operator.

Let us recall the definition of the Clarke’s subdifferential for a locally Lipschitz function $j : K \subset X \to \mathbb{R}$, where $K$ is a nonempty subset of a Banach space $X$ (one can see [2, 3, 16]). We denote by $j^0(x; y)$ the Clarke’s generalized directional derivative of $j$ at the point $x \in K$ in the direction $y \in X$, that is

\[
j^0(x; y) := \lim_{\lambda \to 0^+, \zeta \to x} \frac{j(\zeta + \lambda y) - j(\zeta)}{\lambda}.
\]

Recall also that the Clarke’s subdifferential or generalized gradient of $j$ at $x \in K$, denoted by $\partial j(x)$, is a subset of $X^*$ given by

\[
\partial j(x) := \{x^* \in X^* : j^0(x; y) \geq \langle x^*, y \rangle, \forall y \in X\}.
\]

Lemma 5.6 ([16], Proposition 3.23). If $j : K \to \mathbb{R}$ is locally Lipschitz function, then

(i) the function $(x, y) \mapsto j^0(x; y)$ is u.s.c. from $K \times X$ into $\mathbb{R}$;
(ii) for every $x \in K$ the gradient $\partial j(x)$ is a nonempty, convex and weakly* compact subset of $X^*$ which is bounded by the Lipschitz constant $L_x > 0$ of $j$ near $x$;
(iii) the graph of $\partial j$ is closed in $X \times X^*_w$;
(iv) the multi-valued map $\partial j$ is u.s.c. from $K$ into $X^*_w$.

We need to make the following assumption on $j$.

$H(j) : j : J \times Y \to \mathbb{R}$ is continuous on $J$ and locally Lipschitz continuous on $Y$, and there exist a function $\phi_5 \in L^2(J, \mathbb{R}_+)$ and a constant $L_5 > 0$ such that

\[
\|\partial j(t, y)\| \leq \phi_5(t) + L_5\|y\|_Y.
\]
for all \( y \in Y \), a.e. \( t \in J \).

We have the following result by defining the multifunction \( U : J \times X \to P(Y^*) \),
\[ U(t,x) = \partial j(t,\gamma x) \quad \text{for} \quad t \in J, x \in X. \]

**Theorem 5.7.** If \( H(4), H(6), H(f_1), H(j) \) hold, then the system (24) has a solution.

**Acknowledgments.** We would like to thank the editors and the reviewers for their comments and suggestions on the manuscript.

**REFERENCES**

[1] Y.-K. Chang, J. J. Nieto and Z.-H. Zhao, Existence results for a nondensely-defined impulsive neutral differential equation with state-dependent delay, *Nonlinear Anal.: Hybrid Systems*, 4 (2010), 593–599.

[2] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[3] Z. Denkowski, S. Migórski and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[4] G. F. Franklin, J. D. Powell and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, Addison-Wesley, 1986.

[5] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[6] M. I. Kamenskii, P. Nistri and V. V. Obukhovskii and P. Zecca, Optimal feedback control for a semilinear evolution equation, *J. Optim. Theory Appl.*, 82 (1994), 503–517.

[7] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter Series in Nonlinear Analysis and Applications 7, 2001.

[8] N. Kosmatov, Initial value problems of fractional order with fractional impulsive conditions, *Results. Math.*, 63 (2013), 1289–1310.

[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, in *North-Holland Mathematics Studies*, vol. 204, Elsevier Science B.V., Amsterdam, (2006).

[10] X. J. Li and J. M. Yong, *Optimal Control Theory for infinite Dimensional Systems*, Birkhäuser, Boston, 1995.

[11] Z. Liu and X. Li, Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations, *Commun. Nonlinear Sci. Numer. Simulat.*, 18 (2013), 1362–1373.

[12] Z. Liu, X. Li and J. Sun, Controllability of nonlinear fractional impulsive evolution systems, *J. Int. Equ. Appl.*, 25 (2013), 395–405.

[13] Z. Liu, S. Zeng and D. Motreanu, Evolutionary problems driven by variational inequalities, *J. Differential Equations*, 260 (2016), 6787–6799.

[14] A. L. Mees, *Dynamics of Feedback Systems*, John Wiley & Sons, Ltd., New York, 1981.

[15] B. M. Miller and E. Ya. Rubinovich, *Impulsive Control in Continuous and Discrete-Continuous Systems*, Kluwer Academic/Plenum Publishers, New York, 2003.

[16] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics 26, Springer, New York, 2013.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[19] R. Sakthivel, Y. Ren and N. I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Comput. Math. Appl.*, 62 (2011), 1451–1459.

[20] X. J. Wang and C. Z. Bai, Periodic boundary value problems for nonlinear impulsive fractional differential equation, *Electronic Journal of Qualitative Theory of Differential Equations*, (2011), 1–15.

[21] J. R. Wang, M. Fečkan and Y. Zhou, On the new concept of solutions and existence results for impulsive fractional evolution equations, *Dyn. Partial Differ. Equ.*, 8 (2011), 345–362.

[22] J. R. Wang, M. Fečkan and Y. Zhou, A survey on impulsive fractional differential equations, *Fract. Calc. Appl. Anal.*, 19 (2016), 806–831.
[23] W. Wei and X. Xiang, Optimal feedback control for a class of nonlinear impulsive evolution equations, *Chinese J. Engrg. Math.*, **23** (2006), 333–342.

[24] J. R. Wang, Y. Zhou and W. Wei, Optimal feedback control for semilinear fractional evolution equations in Banach spaces, *Syst. Contr. Lett.*, **61** (2012), 472–476.

[25] C. Xiao, B. Zeng and Z. H. Liu, Feedback control for fractional impulsive evolution systems, *Appl. Math. Comput.*, **268** (2015), 924–936.

[26] H. P. Ye, J. M. Gao and Y. S. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, **328** (2007), 1075–1081.

[27] B. Zeng, Feedback control for non-stationary 3D Navier-Stokes-Voigt equations, *Mathematics and Mechanics of Solids*, **25** (2020), 2210–2221.

[28] B. Zeng, Feedback control systems governed by evolution equations, *Optimization*, **68** (2019), 1223–1243.

[29] B. Zeng and Z. H. Liu, Existence results for impulsive feedback control systems, *Nonlinear Analysis: Hybrid Systems*, **33** (2019), 1–16.

[30] W. Zhang and M. Fan, Periodicity in a generalized ecological competition system governed by impulsive differential equations with delays, *Math. Comput. Model.*, **39** (2004), 479–493.

[31] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, **59** (2010), 1063–1077.

[32] Y. Zhou, V. Vijayakumar and R. Murugesu, Controllability for fractional evolution inclusions without compactness, *Evol. Equ. Control Theor.*, **4** (2015), 507–524.

Received May 2020; revised October 2020.

E-mail address: biao_zeng@163.com