Flux-mediated optomechanics with a transmon qubit in the single-photon ultrastrong-coupling regime

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We propose a scheme for controlling a radio-frequency mechanical resonator at the quantum level using a superconducting qubit. The mechanical part of the circuit consists of a suspended micrometer-long beam that is embedded in the loop of a superconducting quantum interference device (SQUID) and is connected in parallel to a transmon qubit. Using realistic parameters from recent experiments with similar devices, we show that this configuration can enable a tuneable optomechanical interaction in the single-photon ultrastrong-coupling regime, where the radiation-pressure coupling strength is larger than both the transmon decay rate and the mechanical frequency. We investigate the dynamics of the driven system for a range of coupling strengths and find an optimum regime for ground-state cooling, consistent with previous theoretical investigations considering linear cavities. Furthermore, we numerically demonstrate a protocol for generating hybrid discrete- and continuous-variable entanglement as well as mechanical Schrödinger cat states, which can be realised within the current state of the art. Our results demonstrate the possibility of controlling the mechanical motion of massive objects using superconducting qubits at the single-photon level and could enable applications in hybrid quantum technologies as well as fundamental tests of quantum mechanics.

I. INTRODUCTION

The rapid progress in the field of cavity optomechanics and electromechanics over the last decade has enabled the study of massive micro- and nano-mechanical objects in the quantum regime, paving the way for several technological applications as well as fundamental tests of quantum mechanics [1–3]. Important advances include ground-state cooling of mechanical resonators [4, 5], ponderomotive squeezing [6–8], coherent state transfer [9], as well as preparation of quantum states [10, 11]. Such optomechanical setups consist of a mechanical drum or beam resonator that is parametrically coupled to a higher-frequency optical or microwave cavity via radiation-pressure. Typically the coupling $g_0$ is lower than the decay rate of the cavity $\kappa$, limiting the ability to manipulate the mechanical element at single-photon levels. A strong linearised interaction is effectively achieved by driving the cavity with thousands or even millions of photons, which however leads to unresolved heating issues [5, 12, 13] and makes it difficult to couple to artificial atoms working in the single-photon regime. Growing efforts in the field are focusing on reaching the single-photon strong-coupling regime, $g_0 \gg \kappa$, which holds great promise for high-fidelity mechanical state preparation. An even more intriguing prospect is the possibility of reaching the single-photon ultrastrong-coupling regime, where $g_0$ additionally approaches or even exceeds the mechanical frequency $\omega_M$, leading to interesting phenomena such as photon blockade and nonclassical mechanical states [14–18].

A promising playground for enhancing the single-photon coupling is flux-mediated optomechanics, in which a vibrating mechanical element parametrically modulates the inductance of a LC microwave cavity. This can be realised by integrating a mechanical beam into the arms of a superconducting quantum interference device (SQUID), which leads to radiation-pressure-type coupling between the beam resonator and the cavity flux degree of freedom [19–22]. A recent experiment has confirmed the viability of this approach using linear SQUID cavities [23], however, reaching the single-photon strong-coupling still remains a challenge. One limitation of this setup was the suppression of the flux susceptibility, which is proportional to the optomechanical coupling, due to the geometric inductance being a considerable fraction of the total inductance of the linear SQUID. Another limitation of this scheme is related to the fact that the optomechanical coupling is maximised when the applied flux through the SQUID is close to a half-integer flux quantum, which is where the cavity frequency becomes zero. Moreover, the absence of a strong non-linear element, such as a qubit, can limit the range of states that can be created.

Here we show that it is possible to circumvent these issues in a modified circuit that incorporates a superconducting transmon qubit [24] coupled to the mechanical resonator via a flux-mediated radiation-pressure interaction in the single-photon ultrastrong-coupling regime. Using parameters obtained from recent experiments [23], we investigate the possibility of cooling the resonator via sideband driving of the qubit and find that ground-state cooling is possible with less than one drive photons, circumventing the issues associated with strong driving and qubits. Furthermore, we devise a protocol for creating hybrid Bell-cat entanglement and mechanical Schrödinger cat states using flux-pulsing and qubit operations, enabled by the ability to tune the coupling independently of the qubit frequency. Our results pave the way for the successful on-chip integration of mechanical elements with state-of-the-art transmon-based processors and the manipulation of mechanical motion at
The electromagnetic system is described by the Hamiltonian

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \]  
\[ \hat{H}_0 = \hbar \omega_M \hat{b}^{\dagger} \hat{b} + \hbar \omega_T \hat{c}^{\dagger} \hat{c} - \frac{E_c}{2} \hat{c}^{\dagger} \hat{c} \hat{c}^{\dagger} \hat{c}, \]  
\[ \hat{H}_{\text{int}} = \hbar g_0 \hat{c}^{\dagger} \hat{c} (\hat{b} + \hat{b}^{\dagger}) + \hat{H}_{\text{int}}, \]  

where \( \hat{b}^{(1)} \) and \( \hat{c}^{(1)} \) are bosonic operators describing the annihilation (creation) of phonons and qubit excitations, respectively. The effective transmon frequency is given by \( \omega_T = \left(2 \sqrt{E_j E_C - E_C}/\hbar, \right. \) where \( E_j = E_j + E_{J,\max}^M c_3 \cos (\pi \Phi_M/\Phi_0) \) is the modified transmon Josephson energy due to the mechanical SQUID and \( E_C = e^2/2C \) is its charging energy. A detailed derivation of the circuit Hamiltonian is presented in the Appendix.

The qubit is predominantly coupled to the beam via the radiation-pressure interaction, described by the first term in Eq. (4), with single-photon coupling strength

\[ g_0 = \frac{\partial \omega_T}{\partial \Phi_M} X_{\text{ZPF}} = -\frac{\alpha Z}{2 \Phi_0^2} s_3 E_{J,\max}^M \sin (\pi \Phi_M/\Phi_0) X_{\text{ZPF}}, \]  

where \( X_{\text{ZPF}} \) is the zero-point mechanical motion, \( Z = \frac{\hbar}{e} \sqrt{E_C/2E_j} \) is the transmon impedance and \( \phi_0 = \Phi_0/2\pi \) is the reduced flux quantum.

The second term in Eq. (4) describes higher-order contributions to the interaction Hamiltonian (see Appendix for details)

\[ \hat{H}_{\text{int}}' = \hbar g_0' \hat{c}^{\dagger} \hat{c} \hat{c}^{\dagger} \hat{c} (\hat{b} + \hat{b}^{\dagger}) + \hbar g_0'' \hat{c}^{\dagger} \hat{c} (\hat{b} + \hat{b}^{\dagger})^2, \]  

The first part is a non-linear correction to the interaction, stemming from the transmon anharmonicity, with coupling strength \( g_0 = \alpha \hbar Z^2 s_3 E_{J,\max}^M \sin (\pi \Phi_M/\Phi_0) X_{\text{ZPF}}/(16\Phi_0^3) \). Although this term does not impact the dynamics at single-photon levels, it contributes to the radiation-pressure coupling as \( g_0 \rightarrow g_0 + 2g_0' \). The second part stems from a higher-order expansion of \( E_j^M \) to \( O[X^2] \), resulting in a coupling strength \( g_0'' = \alpha^{2} Z s_3 E_{J,\max}^M \sin (\pi \Phi_M/\Phi_0) X_{\text{ZPF}}^2 \) which is three orders of magnitude smaller than \( g_0 \) for the parameters considered here. For the sake of completeness we include all terms of \( \hat{H}_{\text{int}}' \) in the simulations, which however lead to negligible effects on the system dynamics.

The dependence of the radiation-pressure coupling strength \( g_0 \), as well as that of the qubit frequency, on the flux bias \( \Phi_M \) is plotted in Fig. 2(a), for the parameters shown in Table I. The coupling is maximised at the point where the slope of the qubit frequency \( \partial \omega_T/\partial \Phi_M \) is maximum, close to a half-integer flux quantum. Note
that the coupling becomes exactly zero at half-integer flux quanta, as a result of the finite asymmetry of the SQUID ($s_j$ factor in Eq. (5)), which is here chosen to be $s_j = 0.01$, reflecting a 2% fabrication error in junction targeting. Notably, the Josephson inductance of each junction in the SQUID ($L_j = 2\phi_0^2/E_{J,\text{max}}^2$) is chosen to be much smaller than its expected geometric inductance ($L_g \approx 300 \text{ pH}$), such that the screening parameter $\beta_L = L_g/(\pi L_j) \approx 0.06$ [26] is negligible and does not limit the achievable coupling strengths as in Ref. [23].

Another comparative advantage of this proposal is the additional flux-bias degree of freedom provided by the transmon SQUID. More specifically, in implementations using a single SQUID, the frequency of the qubit (or transmon SQUID) becomes zero at the point of maximum coupling $\Phi_M = \Phi_0/2$, as shown in Fig. 2(a). Using a second SQUID, however, entirely circumvents this issue as the minimum qubit frequency is set by $E_j$ and can be non-zero even at $\Phi_M = \Phi_0/2$. Most importantly, it allows to turn the optomechanical coupling on and off while keeping the qubit frequency constant by appropriately adjusting $\Phi_T$, as depicted in Fig. 2(b). This can also ensure that the qubit remains in the transmon regime $E_j \gg E_C$, where it is insensitive to charge noise [24], for the entire coupling range.

We model the dynamical evolution of the system, using the Lindblad master equation

$$
\dot{\rho} = \frac{i}{\hbar}[\rho, \hat{H}] + (n_{\text{th}} + 1)\gamma_m \mathcal{L}[\hat{b}] \rho + n_{\text{th}} \gamma_m \mathcal{L}[\hat{b}^\dagger] \rho + \frac{(n_{\text{th}}^T + 1)}{T_1} \mathcal{L}[\hat{b}] \rho + \frac{n_{\text{th}}^T}{T_1} \mathcal{L}[\hat{b}^\dagger] \rho + \frac{1}{T_2} \mathcal{L}[\hat{c} \hat{c}] \rho,
$$

where $\mathcal{L}[\rho] = (2\rho \hat{b}^\dagger \hat{b} - \hat{b}^\dagger \hat{b} \rho - \rho \hat{b}^\dagger \hat{b})/2$ are superoperators describing each dissipation process, and $n_{\text{th}} = 1/[\exp(\hbar \omega_m/(k_B T)) - 1]$ is the thermal phonon number at temperature $T$. We use the solver package provided by QuTiP [27], including realistic dissipation rates. More specifically, we consider qubit decay and dephasing times $T_1 = T_2 = 10 \mu s$ which are consistent with measured values in a similar tuneable coupling transmon architecture [25] and with transmons operating in 10 mT magnetic fields [28]. We additionally include a thermal transmon occupation $n_{\text{th}} = 5\%$ (effective temperature of 90 mK), corresponding to realistic experimental conditions [29, 30]. The coupling of the mechanical mode to the environment is determined by $\gamma_m = \omega_m/Q$, where the quality factor $Q = 10^6$ is chosen in agreement with experimental observations in recently fabricated SQUID-embedded beams [23].

![FIG. 2. Tuneable radiation-pressure coupling. (a) The orange curve corresponds to the single-photon optomechanical coupling as a function of the flux bias on the mechanical SQUID ($\Phi_M$), while the blue curve depicts the corresponding transmon frequency dependence. The coupling becomes zero at $\Phi_M/\Phi_0 = 0.5$ as a result of a finite SQUID asymmetry. (b) Same plot in the case where an additional flux $\Phi_T$ is applied on the transmon SQUID (tuning $E_j$ from 3 to 10 GHz), such that the qubit frequency remains constant while the coupling is tuned.](image)

| Parameter   | Value |
|-------------|-------|
| $\omega_M/(2\pi)$ | 1 MHz |
| $\omega_f/(2\pi)$ | 5.53 GHz |
| $|g_0|/(2\pi)$ | $\leq 2.4$ MHz |
| $E_{J,\text{max}}/\hbar$ | 200 GHz |
| $E_j/\hbar$ | 3-10 GHz |
| $E_C/\hbar$ | 280 MHz |
| $B$ | 10 mT |
| $\Phi_M/\Phi_0$ | 0.49-0.5 |
| $l$ | 147 $\mu$m |
| $\beta_0$ | $\frac{1}{2}$ |
| $n_{\text{th}}$ | $\sim 200$ (10mK) |
| $T_1$, $T_2$ | 10 $\mu$s |
| $Q_M$ | $10^6$ |

### TABLE I. Parameter set used in the numerical simulations.

III. GROUND-STATE COOLING

Manipulating the mechanical oscillator at the quantum level requires the ability to cool it down to its ground state, where thermal effects are suppressed. In typical optomechanical setups, this is achieved via a red-detuned continuous-wave (CW) tone on the electromagnetic resonator [4, 5]. This leads to an effective linearised interaction that is used to transfer phonons to the resonator, which eventually decay. Typically the single-photon coupling is small and thousands of drive photons are required, therefore the success of these schemes relies heavily on the resonator being linear. CW ground-state cooling via a transmon qubit has the additional disadvantage of the pump power being limited by the critical photon number in Josephson junctions [31]. These issues could be circumvented in a time-domain scheme, by employing an additional qubit and combining tripartite photon-phonon SWAP gates with qubit reset [32], which is however outside the scope of this study.

We investigate the possibility of cooling the beam via sideband driving on the transmon qubit, as depicted...
in Fig. 3(a). More specifically, we add a driving term $\hat{H}_D/\hbar = \mathcal{E}_D(e^{i\Delta t} + \hat{c}e^{-i\omega_D t} + \hat{b}e^{i\omega_p t})$ to the system Hamiltonian, where $\mathcal{E}_D$ and $\omega_p$ denote the amplitude and frequency of the driving tone, respectively. We numerically solve the Lindblad equation (7) for the set of parameters listed in Table I. In Fig. 3(b) we plot the steady-state occupation in the mechanical resonator as a function of the detuning $\Delta = \omega_T - \omega_D$ and single-photon coupling strength for $\mathcal{E}_D = 70$ kHz. The cooling resonances observed at multiples of $\Delta = \omega_M - g_0^2/\omega_M$, are in accordance with predictions for weakly driven optomechanical systems in the single-photon strong-coupling regime [33]. We find an optimal cooling regime around $|g_0| = \omega_M/4$, leading to a phonon occupancy of 3% (assuming a perfectly thermalised transmon $n_{th}^{\text{min}} = 0$). As expected, ground-state cooling with a qubit becomes impossible in the limit of small coupling $|g_0| \ll \omega_M$, or when $|g_0| \sim \omega_M$ where the two modes hybridise. In Fig. 3(c), we plot the steady-state phonon occupancy as we vary the amplitude of the drive, for the optimal cooling condition found in (b) and including a transmon thermal occupancy of $n_{th}^T = 5\%$, leading to $n_{M}^{\text{min}} \approx 10\%$ at $\mathcal{E}_D = 70$ kHz.

**IV. MECHANICAL CAT STATES**

An important feature of the proposed circuit is that it allows for fast tuning of the optomechanical coupling. Practically this can be achieved by applying flux pulses via dedicated on-chip lines, as short as a few nanoseconds, i.e. much shorter than the interaction timescales. One can then use the qubit for creating interesting mechanical states as discussed below.

In Fig. 4(a) we describe an experimentally feasible protocol for generating macroscopic superposition states on the beam resonator. In the first step, starting from the ground-state $|0\rangle_T|0\rangle_M$ and with the coupling off ($\Phi_M/\Phi_0 = 0.5$), a Hadamard gate is applied on the qubit, which creates the superposition state $|+\rangle_T = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_T$. The second step consists of flux-pulsing the mechanical SQUID to $\Phi_M/\Phi_0 = 0.49$ and letting the system evolve for a variable time under the radiation-pressure interaction $U(t) = e^{-i\Delta g_t e^t|\beta|^{+}\langle b+b\rangle}$. The evolution of excitations in the system after one cycle ($t = 1/\omega_M$) is plotted in Fig. 4(b), assuming thermal occupancies of 10% and 5% for the beam and the qubit, respectively. The operation of $U(t)$ results in a coherent displacement on the mechanical resonator depending on the qubit being in the excited state, i.e.

$$U(t)|+\rangle_T|0\rangle_M = \frac{1}{\sqrt{2}}(|0\rangle_T|0\rangle_M + |1\rangle_T|\beta\rangle_M),$$

where $|\beta\rangle_M$ denotes a coherent mechanical state of amplitude $\beta = \sqrt{n_M}$. 

**FIG. 3.** Ground-state cooling. (a) Sideband cooling scheme where a red-detuned drive is applied on the qubit. (b) Steady-state phonon occupancy as a function of the detuning $\Delta$ and flux bias $\Phi_M$ corresponding to different ratios of $|g_0|/\omega_M$ (shown in the top horizontal axis), for a driving amplitude of $\mathcal{E}_D = 70$ kHz. (c) Cooling as a function of $\mathcal{E}_D$ near the optimal condition of $|g_0| \sim \omega_M/4$ found in (b). We additionally include a 5% thermal transmon occupation, which limits the ground-state cooling to a phonon occupancy of $\sim 10\%$.

**FIG. 4.** Generating mechanical cat states. (a) Description of the protocol. In the first step, with the coupling turned off ($\Phi_M/\Phi_0 = 0.5$), the qubit is prepared in the superposition state $|+\rangle_T$ by applying a Hadamard gate. The optomechanical coupling is turned on ($\Phi_M/\Phi_0 = 0.49$) for a variable time such that the system evolves under the radiation-pressure interaction $U(t) = e^{-i\Delta g_t e^t|\beta|^{+}\langle b+b\rangle}$. The coupling is then turned off and a second Hadamard gate is applied on the qubit, followed by a measurement in the computational basis. Measuring the qubit in the ground or excited state results in even or odd Schrödinger cat states in the mechanical oscillator. (b) Evolution of the system excitations after preparing the qubit in a superposition state and turning on the interaction. The blue curve corresponds to the qubit excitation number while the orange curve depicts the phonon number evolution for one cycle ($t = 1/\omega_M$), including 0.1 and 0.05 thermal phonon and qubit occupancy, respectively, as obtained from Fig. 3(c). The dashed curves correspond to the ideal evolution of the system without dissipation. (c) Wigner functions of the mechanical resonator at different times following the protocol in (a) and projecting on $|0\rangle_T$. At $t = 1/2\omega_M$ an even cat state is created with 98% (93%) fidelity, starting from an ideal (attainable) ground state.
The state created above resembles a hybrid Bell-cat state featuring discrete-continuous variable entanglement [34, 35] and can also be written as

\[ |\psi\rangle_{TM} = \frac{1}{\sqrt{2}} |+\rangle_T (|0\rangle + |\beta\rangle)_M + |-\rangle_T (|0\rangle - |\beta\rangle)_M. \] (9)

Turning off the coupling and applying a second Hadamard gate on the qubit, transforms the state into

\[ |\psi\rangle_{TM} = \frac{1}{2\sqrt{2}} (|0\rangle_T (|0\rangle + |\beta\rangle)_M + |1\rangle_T (|0\rangle - |\beta\rangle)_M. \] (10)

By performing a projective measurement on the qubit, the beam collapses in a macroscopic superposition \(|0\pm\beta\rangle\), depending on whether the qubit is measured in its ground/excited state. This state corresponds to an even/odd Schrödinger cat displaced by \(\beta/2\). In Fig. 4(c) we plot the evolution of the even cat state after repeating the above protocol and conditioning on \(|0\rangle_T\).

The size of the cat state is maximum at half a cycle and is determined by \(\beta_{\text{max}} = 2|g_0|/\omega_M\). The fidelity of the prepared state to the ideal Schrödinger cat state is 93% and is mainly limited by the finite thermal occupancy of the initial ground state. Assuming no initial thermal occupancy we find 98% cat state fidelity, while for ideal evolution without dissipation the fidelity is 99.8%. All higher order interaction terms in Eq. (6) are included in the simulations.

V. DISCUSSION

In summary, we have analysed a hybrid system involving a superconducting transmon qubit parametrically coupled to a mechanical beam via radiation-pressure in the ultrastrong-coupling regime, where the coupling strength exceeds the mechanical frequency at the single-photon level. A similar system, considering a carbon nanotube coupled to a transmon qubit, was studied recently in Ref. [36], however it relies on unrealistically optimistic parameters and high magnetic fields \((B = 500 \text{ mT})\) to reach the ultrastrong-coupling regime. We use experimentally feasible parameters, which have been reported in recent experiments combining Aluminium beams with SQUIDs [23], and small magnetic fields that do not compromise the performance of transmon qubits below \(\sim 10 \mu\text{s} [28]\). We have demonstrated numerically the possibility of ground-state cooling, by sideband driving below the single-photon level, for a range of achievable coupling strengths. Additionally, we have investigated the dynamics of the coupled system in the ultrastrong-coupling regime and devised a protocol for preparing mechanical Schrödinger cat states with high fidelity.

Our proposed circuit architecture provides a versatile platform for integrating transmon qubits with long-lived mechanical resonators, and may find interesting applications in hybrid quantum technologies [37]. The prepared Bell-cat states are particularly interesting for several quantum computing schemes and error correcting protocols [38–41]. Additionally, the radiation-pressure interaction can also be employed to prepare mechanical Gottesman–Kitaev–Preskill states, which are useful for fault-tolerant error correction schemes [42, 43]. Finally, the prepared macroscopically distinct massive superposition states are ideally suited for testing fundamental aspects of quantum theory and its relation to gravity [44, 45].

VI. APPENDIX: DETAILED ANALYSIS OF THE ELECTROMECHANICAL SYSTEM

A. Circuit Hamiltonian

The Lagrangian describing the electromechanical circuit in Fig. 1(b) is

\[
\mathcal{L} = \frac{m\dot{X}^2}{2} - \frac{m\omega_x^2 X^2}{2} + \frac{1}{2} C \dot{\phi}^2 \]

\[+ [E_J + E_J^M(X)] \cos \left( \frac{\dot{\phi}}{\phi_0} \right), \] (A11)

where \(X, \phi\) are variables representing the beam displacement and the node flux, respectively, and \(\phi_0 = \hbar/2e\) is the reduced flux quantum. \(C\) denotes the total capacitance of the transmon and Josephson junctions, which are added in parallel. Following a Legrendre transformation we obtain the system Hamiltonian

\[
H = \frac{P^2}{2m} + \frac{m\omega_x^2 X^2}{2} + \frac{Q^2}{2C} - [E_J + E_J^M(X)] \cos \left( \frac{\dot{\phi}}{\phi_0} \right), \] (A12)

where \(\{X, P\}\) and \(\{\phi, Q\}\) are conjugate variable pairs describing the mechanical and the electrical degrees of freedom, respectively.

The optomechanical coupling between the resonator and the qubit can be determined by analysing the term \(E_J^M(X) \cos \left( \frac{\phi}{\phi_0} \right)\) in the above equation. The motion-dependent Josephson energy of the mechanical SQUID is given by

\[
E_J^M(\Phi_b, X) = E_{J,\text{max}}^M [\cos^2(\pi \Phi_b/\Phi_0 + \alpha X) + a_J^2 \sin^2(\pi \Phi_b/\Phi_0 + \alpha X)]^{1/2}, \] (A13)

where \(a_J\) is the SQUID asymmetry. Following the analysis presented in Ref. [32], for \(\alpha X \ll 1, \pi \Phi_b/\Phi_0\), this expression can be approximated by

\[
E_J^M \approx E_{J,\text{max}}^M [c_J \cos(\pi \Phi_b/\Phi_0) - s_J \sin(\pi \Phi_b/\Phi_0) \alpha X], \] (A14)

up to \(O[X]\), where \(c_J = \sqrt{1 + a_J^2 \tan(\pi \Phi_M/\Phi_0)}\) and \(s_J = (1 - a_J^2)/c_J\).
The first term in Eq. (A14) results in an effective transmon Josephson energy given by

$$E_J = E_J + E_{J,max}^M \cos(\pi \Phi_M/\Phi_0),$$

which is responsible for the qubit frequency change as a function of $\Phi_M,$ shown in Fig. 2(a). The second term, combined with an expansion of the cosine term in Eq. (A12) up to $O(\phi^4),$ yields the optomechanical interaction Hamiltonian

$$H_{int} = -\alpha E_{J,max}^M \sin(\pi \Phi_M/\Phi_0) X \left( \frac{\phi^2}{2\phi_0^2} - \frac{\phi^4}{24\phi_0^4} \right),$$

(A16)

We can express the interaction Hamiltonian in second quantisation form by promoting all canonical variables to quantum operators

$$\hat{X} = X_{ZPF} (\hat{b} + \hat{b}^\dagger), \quad \hat{P} = P_{ZPF} i (\hat{b}^\dagger - \hat{b}),$$

$$\hat{\phi} = \sqrt{\hbar Z/2} (\hat{c} + \hat{c}^\dagger), \quad \hat{Q} = \sqrt{\hbar/2Z} i (\hat{c}^\dagger - \hat{c}),$$

(A17)

where $\hat{b}(t), \hat{c}(t)$ are ladder operators describing the annihilation (creation) of phonons and qubit excitations, respectively, satisfying bosonic commutation relations $[\hat{c}, \hat{c}^\dagger] = 1$ and $[\hat{b}, \hat{b}^\dagger] = 1.$ The zero-point fluctuations in the mechanical displacement and momentum are given by $X_{ZPF} = \sqrt{\hbar/(2m\omega_M)}$ and $P_{ZPF} = \sqrt{\hbar m\omega_M/2},$ respectively. $Z = \frac{\hbar}{\phi_0^2} \sqrt{E_C/2E_J}$ denotes the transmon impedance, where $E_C = \frac{\phi_0^2}{2\omega}$ is its charging energy, and the qubit frequency is given by $\omega = \frac{1}{\hbar} \left( \sqrt{8E_JE_C} - E_C \right).$

Replacing the classical variables in Eq. (A16) with the quantum operators introduced in Eq. (A17) we have

$$\hat{H}_{int}/\hbar = g_0 \hat{c}^\dagger \hat{c} (\hat{b} + \hat{b}^\dagger) + g_0' \hat{c}^\dagger \hat{c} (\hat{b} + \hat{b}^\dagger)^2,$$

(A18)

following a rotating wave approximation (RWA) where fast rotating terms $(\hat{c}^\dagger)^n$ are neglected. The first term describes a radiation-pressure interaction between the qubit and the resonator with coupling strength

$$g_0 = \frac{\alpha Z}{2\phi_0^2} E_{J,max}^M \sin(\pi \Phi_M/\Phi_0) X_{ZPF}.$$  

(A19)

The second term is a higher-order correction to the interaction, stemming from the transmon anharmonicity, with coupling strength

$$g_0' = \alpha \hbar Z^2 s_J E_{J,max}^M \sin(\pi \Phi_M/\Phi_0) X_{ZPF} / (16\phi_0^4).$$

(A20)

This term is included in the simulations although it does not lead to substantial contribution in the system dynamics, however, it leads to a correction of the radiation pressure coupling $g_0 \to g_0 + 2g_0'.$

C. Higher-order interaction terms

The next-to-leading order correction in the expansion of $E_J^M(X)$ in Eq. (A14) is given by

$$E_J^M \{O[X^2]\} = -E_{J,max}^M \frac{s_J}{2c_J} \sin^2(\pi \Phi_B/\Phi_0) \alpha^2 X^2.$$  

(A21)

This term, combined with a second-order expansion of the cosine in Eq. (A12), yields the following interaction

$$H_{int}^{(\phi^2 X^2)} = E_{J,max}^M \frac{s_J}{2c_J} \sin^2(\pi \Phi_B/\Phi_0) \alpha^2 X^2 \frac{\phi_0^2}{2\phi_0^4},$$

(A22)

which can be written in second quantisation form (following a RWA) as

$$H_{int}^{(\phi^2 X^2)} = g_0'' \hat{c}^\dagger \hat{c} (\hat{b} + \hat{b}^\dagger)^2.$$  

(A23)

The coupling strength of this interaction is given by

$$g_0'' = \frac{\alpha^2 Z s_J}{4\phi_0^2 c_J} E_{J,max}^M \tan(\pi \Phi_B/\Phi_0) \sin(\pi \Phi_B/\Phi_0) X_{ZPF}^2.$$  

(A24)

The maximum value of this coupling strength is $g_0'' \approx 5$ kHz around $\Phi_M/\Phi_0 = 0.497$ for the parameters considered in this work. For the sake of completeness we include this interaction in the simulations, however, we do not observe any significant impact on the fidelity of the cooling and quantum state preparation protocols.

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