Exterior algebra methods for the construction of rational surfaces in the projective fourspace

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Abstract

The aim of this paper is to present a construction of smooth rational surfaces in projective fourspace with degree 12 and sectional genus 13. The construction is based on exterior algebra methods, finite field searches and standard deformation theory.

1 Introduction

This paper is dedicated to Gert-Martin Greuel on the occasion of his sixtieth birthday. The use of computer algebra systems is essential for the proof of the main result of this paper. It will become clear that without of computer algebra systems like Singular and Plural developed in Kaiserslautern we could not obtain the main result of this paper at all. We thank the group in Kaiserslautern for their excellent program.

Hartshone conjectured that only finitely many components of the Hilbert scheme of surfaces in \( \mathbb{P}^4 \) correspond to smooth rational surfaces. In 1989, this conjecture was positively solved by Ellingsrud and Peskine [6]. The exact bound for the degree is, however, still open. This motivates our search for smooth rational surfaces in \( \mathbb{P}^4 \). Examples of smooth rational surfaces in \( \mathbb{P}^4 \) prior to this paper were known up to degree 11, see [4]. Our main result is the proof of existence of the following example.

**Theorem 1.1.** There exists a family of smooth rational surfaces in \( \mathbb{P}^4 \) over \( \mathbb{C} \) with \( d = 12 \), \( \pi = 13 \) and hyperplane class

\[
H \equiv 12L - \sum_{i_1=1}^{2} 4E_i - \sum_{i_2=3}^{11} 3E_i - \sum_{i_3=12}^{14} 2E_i - \sum_{i_4=15}^{21} E_i.
\]

in terms of a plane model.

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Abstractly these surfaces arise as the blow up of \( \mathbb{P}^2 \) in 21 points. \( L \) and \( E_i \) in the Theorem denote the class of a general line and the exceptional divisors.

The 21 points lie in special position due to the fact that we need \( h^0(X, \mathcal{O}(H)) = 5 \) and \( h^1(X, \mathcal{O}(H)) = 4 \). Indeed, it will turn out that the component of the Hilbert scheme corresponding to these surfaces has dimension 38, hence up to projectivities this is a 38 − 24 = 14 dimensional family of abstract surfaces. This fits with the fact that the 21 points have to satisfy a condition of codimension \( \leq 20 = 4 \ast 5 \), which leaves us with a family of collections of points in \( \mathbb{P}^2 \) of dimension \( \geq 2 \ast 21 - 20 = 22 \). Up to automorphism of \( \mathbb{P}^2 \) this leads to a family of dimension \( \geq 22 - 8 = 14 \), and hence equality holds. The great difficulty to find points in \( \mathbb{P}^2 \) in very special positions was one of the sources, which led Hartshorne to his conjecture.

We construct these surfaces via their “Beilinson monad”: Let \( V \) be an \( n + 1 \)-dimensional vector space over a field \( K \) and let \( W \) be its dual space. The basic idea behind a Beilinson monad is to represent a given coherent sheaf on \( \mathbb{P}^n = \mathbb{P}(W) \) as a homology of a finite complex of vector bundles, which are direct sums of exterior powers of the tautological rank \( n \) subbundle \( U = \ker(W \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(1)) \) on \( \mathbb{P}(W) \). (Thus \( U \simeq \Omega^1(1) \) is the twisted sheaf of 1-forms. As Beilinson, we will use the notation \( \Omega^p \) for the exterior powers of \( U \).)

The differentials in the monad are given by homogeneous matrices over an exterior algebra \( E = \wedge V \). To construct a Beilinson monad for a given coherent sheaf, we typically take the following steps: Determine the type of the Beilinson monad, that is, determine the vector bundles of the complex, and then find differentials in the monad.

Let \( X \) be a smooth rational surface in \( \mathbb{P}^4 = \mathbb{P}(W) \) with degree 12 and sectional genus 13. The type of a Beilinson monad for the (suitably twisted) ideal sheaf of \( X \) can be derived from the knowledge of its cohomology groups. Such information is partially determined from general results such as the Riemann-Roch formula and the Kodaira vanishing theorem. It is, however, hard to determine the dimensions of all cohomology groups needed to determine the type of the Beilinson monad. For this reason, we assume that the ideal sheaf of \( X \) has the so-called “natural cohomology” in some range of twists. In particular, we assume that in each twist \( -1 \leq n \leq 6 \) at most one of the cohomology groups \( H^i(\mathbb{P}^4, \mathcal{I}_X(n)) \) for \( i = 0 \ldots 4 \) is non-zero. This is an open condition for surfaces in a given component of the Hilbert scheme. Under this assumption the Beilinson monad for the twisted ideal sheaf \( \mathcal{I}_X(4) \) of \( X \) has the following form:

\[
4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O}.
\]

To detect differentials in (1), we use the following techniques developed recently: (1) the first technique is an exterior algebra method due to Eisenbud,
Floystad and Schreyer [5] and (2), the other one is the method using small finite fields and random trials due to Schreyer [9].

(1) Eisenbud, Floystad and Schreyer presented an explicit version of the Bernstein-Gel’fand-Gel’fand correspondence. This correspondence is an isomorphism between the derived category of bounded complexes of finitely generated $S$-graded modules and the derived category of certain “Tate resolutions” of $E$-modules, where $S = \text{Sym}_K(W)$. As an application, they constructed the Beilinson monad from the Tate resolution explicitly. This enables us to describe the conditions, that the differentials in the Beilinson monad must satisfy in an exterior algebra context.

(2) Let $M$ be a parameter space for objects in algebraic geometry such as the Hilbert scheme or a moduli space. Suppose that $M$ is a subvariety of a rational variety $G$ of codimension $c$. Then the probability for a point $p$ in $G(\mathbb{F}_q)$ to lie in $M(\mathbb{F}_q)$ is about $(1 : q^c)$. This approach will be successful if the codimension $c$ is small and the time required to check $p \notin M(\mathbb{F}_q)$ is sufficiently small as compared with $q^c$. This technique was applied first by Schreyer [9] to find four different families of smooth surfaces in $\mathbb{P}^4$ with degree 11 and sectional genus 11 over $\mathbb{F}_3$ by a random search, and he provided a method to establish the existence of lifting these surfaces to characteristic 0. This technique has been successfully applied to solve various problems in constructive algebraic geometry (see [10], [12] and [1]).

Singular or Macaulay2 scripts needed to construct and analyse these surfaces are available at http://www.math.uni-sb.de/~ag-schreyer and http://www.math.colostate.edu/~abo/programs.html.

### 2 The exterior algebra method

Our construction of the rational surfaces uses the “Beilinson monad”. A Beilinson monad represents a given coherent sheaf in terms of direct sums of (suitably twisted) bundles of differentials and homomorphisms between these bundles, which are given by homogeneous matrices over an exterior algebra $E$. Recently, Eisenbud, Floystad and Schreyer [5] showed that for a given sheaf, one can get the Beilinson monad from its “Tate resolution”, that is a free resolution over $E$, by a simple functor. This enables us to discuss the Beilinson monad in an exterior algebra context. In this section, we take a quick look at the exterior algebra method developed by Eisenbud, Floystad and Schreyer.
2.1 Tate resolution of a sheaf

Let $W$ be a $(n+1)$-dimensional vector space over a field $K$, let $V$ be its dual space, and let $\{x_i\}_{0 \leq i \leq n}$ and $\{e_i\}_{0 \leq i \leq n}$ be dual bases of $V$ and $W$ respectively. We denote by $S$ the symmetric algebra of $W$ and by $E$ the exterior algebra $\wedge V$ on $V$. Grading on $S$ and $E$ are introduced by $\deg(x) = 1$ for $x \in W$ and $\deg(e) = -1$ for $e \in V$ respectively. The projective space of 1-quotients of $W$ will be denoted by $P^n = \mathbb{P}(W)$.

Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated $S$-graded module. We set

\[ \omega_E := \text{Hom}_K(E, K) = \wedge W = E \otimes_K \wedge W \cong E(-n-1) \]

and

\[ F^i := \text{Hom}_K(E, M_i) \cong M_i \otimes_K \omega_E. \]

The morphism $\phi_i : F^i \to F^{i+1}$ takes the map $\alpha \in F^i$ to the map

\[ e \mapsto \sum_i x_i \alpha(e_i \wedge e) \in F^{i+1}. \]

Then the sequence

\[ R(M) : \cdots \to F^{i-1} \xrightarrow{\phi_{i-1}} F^i \xrightarrow{\phi_i} F^{i+1} \to \cdots \]

is a complex. This complex is eventually exact. Indeed, $R(M)$ is exact at $\text{Hom}_K(E, M_i)$ for all $i \geq s$ if and only if $s > r$, where $r$ is the Castelnuovo-Mumford regularity of $M$ (see [5] for a detailed proof). So starting from $T(M)^{>r} := T(M_{>r})$, we can construct a doubly infinite exact $E$-free complex $T(M)$ by adjoining a minimal free resolution of the kernel of $\phi_{r+1}$:

\[ T(M) : \cdots \to T^r \to T^{r+1} := \text{Hom}_K(E, M_{r+1}) \xrightarrow{\phi_{r+1}} \text{Hom}_K(E, M_{r+2}) \to \cdots. \]

This $E$-free complex is called the Tate resolution of $M$. Since $T(M)$ can be constructed by starting from $R(M_{>s})$, $s \geq r$, the Tate resolution depends only on the sheaf $\mathcal{F} = \widetilde{M}$ on $\mathbb{P}(W)$ associated to $M$. We call $T(\mathcal{F}) := T(M)$ the Tate resolution of $\mathcal{F}$. The following theorem gives a description of all the terms of a Tate resolution:

**Theorem 2.1** ([5]). Let $M$ be a finitely generated graded $S$-module and let $\mathcal{F} := \widetilde{M}$ be the associated sheaf on $\mathbb{P}(W)$. Then the term of the complex $T(\mathcal{F})$ with cohomological degree $i$ is $\bigoplus_j H^j \mathcal{F}(i-j) \otimes \omega_E$.

Important to us is also the fact the dual complex $\text{Hom}_E(T(\mathcal{F}), E)$ stays exact.
2.2 Beilinson monad

Eisenbud, Floystad and Schreyer [5] showed, that applying a simple functor to the Tate resolution $T(F)$, gives a finite complex of sheaves whose homology is the sheaf $F$ itself: Given $T(F)$, we define $\Omega(F)$ to be the complex of vector bundles on $\mathbb{P}(W)$ obtained by replacing each summand $\omega_E(i)$ by the bundle $\Omega^i(i)$. The differentials of the complex are given by using isomorphisms

$$\text{Hom}_E(\omega_E(i), \omega_E(j)) \cong \bigwedge^{i-j} V \cong \text{Hom}(\Omega^i(i), \Omega^j(j)).$$

Theorem 2.2 ([5]). Let $F$ be a coherent sheaf on $\mathbb{P}(W)$. Then $F$ is the homology of $\Omega(F)$ in cohomological degree 0, and $\Omega(F)$ has no homology otherwise.

We call $\Omega(F)$ the Beilinson monad for $F$.

3 Construction

In this section we will construct our family of rational surfaces $X$ in $\mathbb{P}^4$ with degree $d = 12$, sectional genus $\pi = 13$. The construction takes the following four steps:

1. Analyse the monad and parts of the Tate resolution.

2. Find a smooth surface $X$ with the prescribed invariants over a finite field of a small characteristic.

3. Determine the type of the linear system, which embeds $X$ into $\mathbb{P}^4$ to justify that the surface $X$ found in the previous step is rational.

4. Establish the existence of a lift to characteristic zero.

3.1 Analysis of the monad and Tate resolution

Let $K$ be a field, let $W$ be a five-dimensional vector space over $K$ with basis $\{x_i\}_{0 \leq i \leq 4}$, and let $V$ be its dual space with dual basis $\{e_i\}_{0 \leq i \leq 4}$. Let $X$ be a smooth surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with the invariants given above. The first step is to determine the type of the Beilinson monad for the twisted ideal sheaf of $X$, which is derived from the partial knowledge of its cohomology groups. Such information can be determined from general results such as the Riemann-Roch formula and Kodaira vanishing theorem (see [2] for more detail). We
assume that $X$ has the natural cohomology in the range $-1 \leq j \leq 6$ of twists:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
i & & & & & \scriptstyle{h'^{\mathcal{I}}_X(j)} \\
\hline
13 & 4 & 2 & & & & \scriptstyle{2} \\
\hline
& & & & & \scriptstyle{2} \\
\hline
\end{array}
\]

Here a zero is represented by the empty box. By Theorem 2.1, the Tate resolution $T(\mathcal{I}_X)[4] = T(\mathcal{I}_X(4))$ includes an exact $E$-free complex of the following type:

\[
\to 4\omega_E(3) \to 2\omega_E(2) \oplus 2\omega_E(1) \to 3\omega_E \oplus 5\omega_E(-1) \to 29\omega_E(-2) \to \cdots .
\] (2)

From Theorem 2.2, it follows therefore, that the corresponding Beilinson monad for $\mathcal{I}_X(4)$ is of the following type:

\[
0 \to 4\Omega^3(3) \overset{A}{\to} 2\Omega^2(2) \oplus 2\Omega^1(1) \overset{B}{\to} 3\mathcal{O} \to 0.
\] (3)

The next step is to describe what maps $A$ and $B$ could be the differentials of the monad (3). The identifications

\[
\text{Hom}(\Omega^i(i), \Omega^j(j)) \simeq \text{Hom}_E(\omega_E(i), \omega_E(j)) \simeq \text{Hom}_E(E(i), E(j)),
\]

allow us to think of the maps $A$ and $B$ as homomorphisms between $E$-free modules. Since the Tate resolution and its $E$-dual are exact, the matrix $A$ determines $B$ up to isomorphism.

However, we start with $B$ in our construction. To ease our calculations, we take the map $2\omega_E(1) \overset{B_1}{\to} 3\omega_E$ to be defined by the matrix

\[
B_1 = \begin{pmatrix}
e_0 & e_1 \\
e_1 & e_2 \\
e_3 & e_4
\end{pmatrix},
\]

Since the $\text{GL}(5, K) \times \text{GL}(2, K) \times \text{GL}(3, K)$ orbit of this matrix is dense in $\text{Hom}_E(2\omega_E(1), 3\omega_E)$ this is a reasonable mild additional assumption. The crucial step in the construction is the choice of the map

\[
3\omega_E \overset{C}{\to} 4\omega_E(-2),
\]
Construction of rational surfaces in $\mathbb{P}^4$

where the target $4\omega_E(-2)$ is a free summand of the cokernel $\text{Coker}(5\omega_E(-1) \to 29\omega_E(-2))$. Note that $C \circ B = 0$ must hold in the Tate resolution. The condition $C \circ B_1 = 0$ means that $C$ corresponds to a 4-dimensional quotient space of

$$T = \text{Coker}(2\Lambda^3W \xrightarrow{B_1} 3\Lambda^2W).$$

An exterior algebra computation proves that $\dim T = 10 = 3 \times 10 - 2 \times 10$ as expected. Indeed the map to $T$ is given by the following $10 \times 3$ matrix of two forms in $E$:

$$\varphi = \begin{pmatrix}
0 & 0 & e_3e_4 \\
0 & -e_3e_4 & e_2e_3 - e_1e_4 \\
-e_3e_4 & 0 & e_1e_3 - e_0e_4 \\
0 & e_1e_4 - e_2e_3 & e_1e_2 \\
e_2e_3 - e_1e_4 & e_1e_3 - e_0e_4 & -e_0e_2 \\
e_0e_4 - e_1e_3 & 0 & e_0e_1 \\
0 & e_1e_2 & 0 \\
e_1e_2 & e_0e_2 & 0 \\
e_0e_2 & e_0e_1 & 0 \\
e_0e_1 & 0 & 0
\end{pmatrix}$$

Thus we obtain $C$ from a point $[c] \in \mathbb{G} = \mathbb{G}(10, 4)$ in the Grassmanian as the product $C = \varphi \circ c$, where $c \in K^{4 \times 10}$ denotes a representing $4 \times 10$ matrix. For these $C$ the condition $C \circ B_1 = 0$ will be satisfied.

Consider

$$\mathcal{M} = \{[c] \in \mathbb{G} \mid \exists B_2 \in \text{Hom}(2\omega_E, 3\omega_E(2)) \text{ with } C \circ B_2 = 0\}.$$

More precisely, we consider those $[c] \in \mathbb{G}$ such that

$$0 \to 2\Lambda^4W \xrightarrow{B_1} 3\Lambda^2W \xrightarrow{C} 4W \to 0$$

has two dimensional homology in the middle. The alternating dimensions of the vector spaces in the complex add to zero $2 \times 5 - 3 \times 10 + 4 \times 5 = 0$. The complex is exact for a general choice of $[c] \in \mathbb{G}$ as we see by a computation in an example. Thus $[c] \in \mathbb{G}$, which give the desired two-dimensional homology in the middle, also give two-dimensional homology at the right. We conclude that $\mathcal{M} \subset \mathbb{G}$ has codimension at most $4 = 2 \times 2$ at such points $[c]$.

Once we have chosen a $[c] \in \mathcal{M}$, we can expect, that $B = (B_1, B_2)$ and $C$ determine the monad and hence the desired surface, due to the following Hilbert function argument:

The alternating sum of the dimensions in the complex

$$0 \to 2\Lambda^3W \oplus 2\Lambda^4W \xrightarrow{B} 3\Lambda^2W \xrightarrow{C} 4K \to 0$$
is $2 \times 10 + 2 \times 5 - 3 \times 10 + 4 = 4$. Hence we expect a 4 dimensional homology on the right, which gives the matrix $A$.

In summary we proved the following proposition.

**Proposition 3.1.** There exists a quasi-projective subvariety $\mathcal{M} \subset G(10, 4)$ of codimension at most 4, whose points define a monad of a smooth rational surface in $\mathbb{P}^4$. The $\text{PGL}(5, \overline{K})$ orbit of each family corresponding to a component of $\mathcal{M}$ is an open part of a component of the Hilbert scheme of surfaces.

Here $\overline{K}$ denotes the algebraic closure of our ground field $K$.

**Proof.** Indeed, apart from the condition $[c] \in \overline{\mathcal{M}}$, all other conditions are open conditions. \hfill $\square$

However, this does not prove, that $\mathcal{M}$ is non-empty. Note that $\overline{\mathcal{M}}$ is defined over the integers $\mathbb{Z}$.

### 3.2 Finite Field search

If $\mathcal{M}$ is not empty we can expect to find a point in $\mathcal{M}(\mathbb{F}_q) \subset G(\mathbb{F}_q)$ at a rate of $(1 : q^4)$ by Proposition 3.1. The statistics suggests that there are two different components of $\overline{\mathcal{M}}(\mathbb{F}_5) \subset G(\mathbb{F}_5)$, whose elements have syzygies with Betti table

\[
\begin{array}{cccc}
2 & 4 & . & . \\
1 & . & 3 & 2 \\
0 & . & 2 & 4 \\
-1 & . & . & 5 \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
2 & 4 & . & . \\
1 & . & 3 & 2 \\
0 & . & 2 & 4 \\
-1 & . & . & 10 \\
\end{array}
\]

However, we never obtained a Beilinson monad of a surface from an example with the Betti table of the second type. So these points do not belong to $\overline{\mathcal{M}}(\mathbb{F}_5)$. Examples with the first Betti table appeared 18 times in a test of $5^4 \cdot 10$ examples. It will turn out, that this family has indeed codimension 4.

**Proposition 3.2.** There is a smooth surface in $\mathbb{P}^4$ over $\mathbb{F}_5$ with $d = 12$ and $\pi = 13$.

**Proof.** By random search, we can find $C \in \mathcal{M}(\mathbb{F}_5)$ and hence $B$ and $A$ satisfying the desired conditions. Determine the corresponding maps $A : 4\Omega^3(3) \to 2\Omega^2(2) \oplus 2\Omega^1(1)$ and $B = (B_2, B_1) : 2\Omega^2(2) \oplus 2\Omega^1(1) \to 3\mathcal{O}$. Then compute the homology $\ker(B)/\text{im}(A)$. If the homology is isomorphic to the ideal sheaf of a surface with the desired invariants, then check smoothness of the surface with the Jacobian criterion. If we are lucky, the surface is smooth. If not, we search for a further $C \in \mathcal{M}(\mathbb{F}_5)$. 
For example the point \([c] \in \overline{M}(\mathbb{F}_5)\) represented by the matrix
\[
c = \begin{pmatrix}
2 & 2 & -2 & 0 & -2 & 2 & -1 & 1 & -1 & -2 \\
1 & -1 & 2 & 2 & -1 & 2 & 2 & 0 & 2 & -2 \\
1 & -2 & 1 & -2 & 0 & -1 & -2 & 2 & 1 & -2 \\
-2 & -1 & -2 & -1 & 0 & 2 & 0 & -1 & 2 & 1
\end{pmatrix}
\]
leads to a smooth surface in \(\mathbb{P}^4\) defined over \(\mathbb{F}_5\) of degree \(d = 12\) and sectional genus \(\pi = 13\).

### 3.3 Adjunction process

In this subsection, we spot the surface found in the previous step within the Enriques-Kodaira classification and determine the type of the linear system that embeds \(X\) into \(\mathbb{P}^4\). First of all, we recall a result of Sommese and Van de Ven for a surface over \(\mathbb{C}\):

**Theorem 3.3 ([11]).** Let \(X\) be a smooth surface in \(\mathbb{P}^n\) over \(\mathbb{C}\) with degree \(d\), sectional genus \(\pi\), geometric genus \(p_g\) and irregularity \(q\), let \(H\) be its hyperplane class, let \(K\) be its canonical divisor and let \(N = \pi - 1 + p_g - 1\). Then the adjoint linear system \(|H + K|\) defines a birational morphism
\[
\Phi = \Phi_{|H+K|} : X \to \mathbb{P}^{N-1}
\]
on a smooth surface \(X_1\), which blows down precisely all \((-1)\)-curves on \(X\), unless

(i) \(X\) is a plane, or Veronese surface of degree 4, or \(X\) is ruled by lines;

(ii) \(X\) is a Del Pezzo surface or a conic bundle;

(iii) \(X\) belongs to one of the following four families:

(a) \(X = \mathbb{P}^2(p_1, \ldots, p_7)\) embedded by \(H \equiv 6L - \sum_{i=0}^7 2E_i\);
(b) \(X = \mathbb{P}^2(p_1, \ldots, p_8)\) embedded by \(H \equiv 6L - \sum_{i=0}^7 2E_i - E_8\);
(c) \(X = \mathbb{P}^2(p_1, \ldots, p_8)\) embedded by \(H \equiv 9L - \sum_{i=0}^8 3E_i\);
(d) \(X = \mathbb{P}(\mathcal{E})\), where \(\mathcal{E}\) is an indecomposable rank 2 bundle over an elliptic curve and \(H \equiv B\), where \(B\) is a section \(B^2 = 1\) on \(X\).

**Proof.** See [11] for the proof. \(\square\)
Setting $X = X_1$ and performing the same operation repeatedly, we obtain a sequence

$$X \to X_1 \to X_2 \to \cdots \to X_k.$$  

This process will be terminated if $N - 1 \leq 0$. For a surface with nonnegative Kodaira dimension, one obtains the minimal model at the end of the adjunction process. If the Kodaira dimension equals $-\infty$, we end up with a ruled surface, a conic bundle, a Del Pezzo surface, $\mathbb{P}^2$, or one of the few exceptions of Sommese and Van de Ven.

It is not known, whether the adjunction theory holds over a finite field. However, we have the following proposition:

**Proposition 3.4 ([4], Prop. 8.3).** Let $X$ be a smooth surface over a field of arbitrary characteristic. Suppose that the adjoint linear system $|H + K|$ is base point free. If the image $X_1$ in $\mathbb{P}^N$ under the adjunction map $\Phi_{|H + K|}$ is a surface of the expected degree $(H + K)^2$, the expected sectional genus $\frac{1}{2}(H + K)(H + 2K) + 1$ and with $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_1})$, then $X_1$ is smooth and $\Phi : X \to X_1$ is a simultaneous blow down of the $K_1^2 - K^2$ many exceptional lines on $X$. \hfill \square

**Remark 3.5.** The union of the exceptional divisors contracted in each step is defined over the base field.

In [2] and [4], it is described how to compute the adjunction process for a smooth surface given by explicit equations (see [4] for the computational details). Let $X$ be the smooth surface found in the previous step. The computation for the adjunction process in characteristic 5 gives

$$H \equiv 12L - \sum_{i_1=1}^{2} 4E_{i_1} - \sum_{i_2=3}^{11} 3E_{i_2} - \sum_{i_3=12}^{14} 2E_{i_3} - \sum_{i_4=15}^{21} E_{i_4}, \quad (4)$$

where $L$ is the class of a line in $\mathbb{P}^2$. This process ends with a Del Pezzo surface of degree 7, which is the blowing up of $\mathbb{P}^2$ in two points. Therefore we can conclude that $X$ is rational.

### 3.4 Lift to characteristic 0

In the previous step, we constructed a smooth surface in $\mathbb{P}^4$ defined over $\mathbb{F}_5$. However, our main interest is the field of complex numbers $\mathbb{C}$. In this section, we show the existence of a lift to characteristic 0 as follows: Let $\mathcal{M}$ and $\mathcal{G}$ be given as in the previous subsections.

**Proposition 3.6 ([9]).** Let $[c] \in \mathcal{M}(\mathbb{F}_p)$ be a point, where $\mathcal{M} \subset \mathcal{G}$ has codimension 4. Then there exist a number field $\mathbb{L}$, a prime $\mathfrak{p}$ in $\mathbb{L}$ with residue field
Construction of rational surfaces in \( \mathbb{P}^4 \)

\( \mathcal{O}_{L,p}/p\mathcal{O}_{L,p} \simeq \mathbb{F}_p \) and a family of surfaces \( \mathcal{X} \) defined over \( \mathcal{O}_{L,p} \) with special fiber the surface \( X \) defined over \( \mathbb{F}_p \) corresponding to \([c]\). Furthermore, since the surface \( X/\mathbb{F}_p \) corresponding to \([c]\) is smooth, the surface \( X/L \) corresponding to the generic point of \( \text{Spec} \, \mathbb{L} \subset \text{Spec} \, \mathcal{O}_{L,p} \) is also smooth.

Proof. Let \( p \) be a prime number. If this is not the case, \( Z \) has to be replaced by the ring of integers in a number field which has \( \mathbb{F}_p \) as the residue field.

Since \( M \) has pure codimension 4 in \([c]\), there are four hyperplanes \( H_1, \ldots, H_4 \) in \( G \), such that \([c]\) is an isolated point of \( M(\mathbb{F}_p) \cap H_1 \cap \cdots \cap H_4 \). We may assume that \( H_1, \ldots, H_4 \) are defined over \( \text{Spec} \, Z \) and that they meet transversally in \([c]\). This allows us to think that \( M \cap H_1 \cap \cdots \cap H_4 \) is defined over \( Z \). Let \( Z \) be an irreducible component of \( M_Z \) containing \( C \). Then \( \dim Z = 1 \).

The residue class field of the generic point of \( Z \) is a number field \( \mathbb{L} \) that is finitely generated over \( \mathbb{Q} \), because \( M \) is projective over \( Z \). Let \( \mathcal{O}_L \) be the ring of integers of \( \mathbb{L} \) and let \( p \) be a prime ideal corresponding to \([c] \in Z \). Then \( \text{Spec} \, \mathcal{O}_{L,p} \to Z \subset M \) is an \( \mathcal{O}_{L,p} \)-valued point which lifts \([c]\).

Performing the construction of the surface over \( \mathcal{O}_{L,p} \) gives a flat family \( \mathcal{X} \) of surfaces over \( \mathcal{O}_{L,p} \). Since smoothness is an open property, and since the special fiber \( X = \mathcal{X}_p \) is smooth, the general fiber \( \mathcal{X}_L \) is also smooth.

Next, we argue that the adjunction process of the surface over the number field \( \mathbb{L} \) has the same numerical behavior:

Proposition 3.7 ([4], Cor. 8.4). Let \( \mathcal{X} \to \text{Spec} \, \mathcal{O}_{L,p} \) be a family as in Proposition 3.6. If the Hilbert polynomial of the first adjoint surface of \( X = \mathcal{X} \otimes \mathbb{F}_q \) is as expected, and if \( H^1(X, \mathcal{O}_X(-1)) = 0 \), then the adjunction map of the general fiber \( \mathcal{X}_L \) blows down the same number of exceptional lines as the adjunction map of the special fiber \( X \). 

Last step in the proof of Theorem 1.1. Let \([c]\) be the element of \( M(\mathbb{F}_5) \), which gives the surface in Proposition 3.2. We check, that \([c]\) satisfies the condition of Proposition 3.6 by computing the Zariski tangent space \( T_{M,[c]} \) at \([c]\). Our computation shows that \( \text{codim} \, T_{M,[c]} = 4 \). So \( M \) is smooth of codimension 4 at \([c]\), and \([c]\) and hence the surface lift to a number field.

Finally we count dimension. Our component \( M \subset G(10,4) \) containing \([c]\) has codimension 4, hence dimension \( 4 \times (10 - 4) - 4 = 20 \). The normalization of \( B_1 \) (up to conjugation) gives additional 18 parameters, because the Hilbert scheme of cubic scrolls in \( \mathbb{P}(V) \) has dimension 18. So the component of the Hilbert scheme, that contains our surface, has dimension 38.
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