Existence to nonlinear parabolic problems with unbounded weights

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Abstract

We consider the weighted parabolic problem of the type

\[
\begin{cases}
    u_t - \text{div}(\omega_2(x)|\nabla u|^{p-2}\nabla u) = \lambda \omega_1(x)|u|^{p-2}u, & x \in \Omega, \\
    u(x, 0) = f(x), & x \in \Omega, \\
    u(x, t) = 0, & x \in \partial \Omega, \ t > 0,
\end{cases}
\]

for quite a general class of possibly unbounded weights \(\omega_1, \omega_2\) satisfying the Hardy-type inequality. We prove existence of a global weak solution in the weighted Sobolev spaces provided that \(\lambda\) is smaller than the optimal constant in the inequality.

Key words and phrases: existence of solutions, Hardy inequalities, parabolic problems, weighted \(p\)–Laplacian, weighted Sobolev spaces

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1 Introduction

Our aim is to provide an existence result for a broad class of nonlinear parabolic equations

$$u_t - \text{div}(\omega_2(x)|\nabla u|^{p-2}\nabla u) = \lambda \omega_1(x)|u|^{p-2}u, \quad \text{in} \quad \Omega$$  \hspace{1cm} (1)

where $p > 2$, weight functions $\omega_1, \omega_2 \geq 0$ are possibly unbounded, $\Omega \subseteq \mathbb{R}^N$ is a bounded open set. We develop the previous results [24] by allowing $\omega_1$ to be unbounded, which entail challenges in functional analysis of the two-weighted Sobolev spaces $W^{1,p}_{(\omega_1, \omega_2)}(\Omega)$.

We impose the restrictions on the weights in order to control the structure of the two-weighted Sobolev spaces, as well as to ensure monotonicity of the leading part of the operator. Namely, we assume

- (W1) $\omega_1, \omega_2 : \overline{\Omega} \rightarrow \mathbb{R}_+ \cup \{0\}$ and $\omega_1, \omega_2 \in L^1_{\text{loc}}(\Omega)$;
- (W2) $\omega_1^{\frac{2}{p-2}} \in L^1(\Omega)$;
- (W3) for any $U \subset \subset \Omega$ there exists a constant $\omega_2(x) \geq c_U > 0$ in $U$;
- (W4) $(\omega_1, \omega_2)$ is a pair of weights in Hardy inequality

$$K \int_{\Omega} |\xi|^p \omega_1(x) dx \leq \int_{\Omega} |\nabla \xi|^p \omega_2(x) dx; \hspace{1cm} (2)$$

Furthermore, assume that there exists $s > p$ such that

- (W5) for any $U \subset \subset \Omega$ we have a compact embedding

$$W^{1,p}_{(\omega_1, \omega_2), 0}(U) \subset \subset L^s_{\omega_1}(U);$$

- (W6) there exists $q \in (p, s)$, such that

$$\omega_1^{\frac{q}{s-q}} \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \omega_2^{\frac{q}{q-p}} \in L^1_{\text{loc}}(\Omega).$$

The assumptions are discussed in Subsection 2.3.
The existence of solutions to problems

\[ u_t - \text{div}(a(x,t,u,\nabla u)) = f, \]

where the involved operator is monotone and has \( p \)-growth, is very well understood, e.g. [6, 7, 8]. Nonetheless, this research concerns the autonomic case, i.e. when the right–hand side does not depend on the solution itself.

Various physical models (combustion models) involve semilinear parabolic problems of the form

\[ u_t - \Delta u = f(u). \]

Fujita’s Theory, developed since 1960s, analyses the possible singularities of solutions. There are known examples of problems, where solutions explode (blow-up) to infinity in finite time. More recent research in that directions was carried out by Giga and Kohn.

In [25] Vazquez and Zuazua, generalizing the seminal paper by Baras and Goldstein [3], describe the asymptotic behaviour of the heat equation that reads

\[ u_t = \Delta u + V(x) u \quad \text{and} \quad \Delta u + V(x) u + \mu u = 0, \]

where \( V(x) \) is an inverse–square potential. The key tool is an improved form of the Hardy–Poincaré inequality. The optimal constant in Hardy-type inequality indicates the critical \( \lambda \) for blow-up or global existence, as well as the sharp decay rate of the solution.

This phenomenon is observed in wide range of parabolic problems, including semilinear equations, see e.g. [11, 12, 13, 14, 15, 16, 18, 20, 25]. In several papers, e.g. [4, 5, 10], dealing with the rate of convergence of solutions to fast diffusion equations \( u_t = \Delta u^m \), the authors study the estimates for the constants in Hardy-Poincaré-type inequalities and their application. The weighted fast diffusion equation is getting attention [11, 12].

In general, application of the general Hardy inequalities is expected to infer certain properties of solution to wide class of parabolic problems. The inspiration of our research was the paper of García Azorero and Peral Alonso [18], who apply the Hardy inequality [18, Lemma 2.1] of the form

\[ \lambda_{N,p} \int_{\mathbb{R}^N} |\xi|^p |x|^{-p} \, dx \leq \int_{\mathbb{R}^N} |\nabla \xi|^p \, dx, \]

where \( \lambda_{N,p} \) is optimal, to obtain the existence of weak solutions to the corresponding parabolic problem

\[ u_t - \Delta_p u = \frac{\lambda}{|x|^p} |u|^{p-2} u, \quad 1 < p < N. \]
We adapt some ideas of Anh, Ke [2], who consider the initial boundary value problem for a class of quasilinear parabolic equations involving weighted $p$-Laplace operator

$$u_t - \text{div}(\sigma(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u - f(x,u), \quad 2 \leq p < N.$$ 

Our major difficulties are of technical nature and require more advanced setting than classical one in [2, 18]. We employ the two-weighted Sobolev spaces $W^{1/p}(\omega_1, \omega_2)(\Omega)$, due to presence of general class of weights both in the leading part of the operator and on the right-hand side of (1). The key tool is truncation method of Boccardo, Murat [8].

Our main result is the following theorem.

**Theorem 1.1.** Let $p > 2$, $\Omega \subseteq \mathbb{R}^N$ be an open subset, $f \in L^2(\Omega)$. Assume that $\omega_1, \omega_2$ satisfy conditions (W1)–(W6).

There exist $\lambda_0 = \lambda_0(p, N, \omega_1, \omega_2)$ such that for all $\lambda \in (0, \lambda_0)$, the parabolic problem

$$\begin{cases}
  u_t - \Delta_p^{\omega_2}u = \lambda \omega_1(x)|u|^{p-2}u \quad &x \in \Omega, \\
  u(x, 0) = f(x) \quad &x \in \Omega, \\
  u(x, t) = 0 \quad &x \in \partial \Omega, \ t > 0,
\end{cases}
$$

has a global weak solution $u \in L^p(0, T; W^{1,p}(\omega_1, \omega_2, 0)(\Omega))$, such that $u_t \in L^p(0, T; W^{-1,p'}(\omega_1, \omega_2, 0')(\Omega))$, i.e.

$$\int_{\Omega_T} (u_t \xi + \omega_2|\nabla u|^{p-2}\nabla u\nabla \xi - \lambda \omega_1(x)|u|^{p-2}u \xi) \, dx \, dt = 0,$$

holds for each $\xi \in L^p(0, T; W^{1,p}(\omega_1, \omega_2, 0)(\Omega))$. Moreover, $u \in L^\infty(0, T; L^2(\Omega_T))$.

**Remark 1.1.** In fact, the proof of the above theorem implies the existence to

$$\begin{cases}
  u_t - \Delta_p^{\omega_2}u = \lambda W(x)|u|^{p-2}u \quad &x \in \Omega, \\
  u(x, 0) = f(x) \quad &x \in \Omega, \\
  u(x, t) = 0 \quad &x \in \partial \Omega, \ t > 0,
\end{cases}
$$

with any $W(x) \leq \omega_1(x)$ without assumption $||W||_{L^\infty(\Omega_T)} < \infty$. 

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Remark 1.2 (Examples of admissible weights). The conditions (W1)-(W6) are satisfied by the following pairs of weights:

- \( \omega_1(x) = |x|^{-p}, \omega_2(x) \equiv 1. \) The compact embedding (W5) is given by [17, Theorem 3.4] by Franchi, Serapioni, and Serra Cassano. This example retrieves the result of [18];

- \( \omega_1(x) = (\text{dist}(x, \partial \Omega))^{\gamma-p}, \omega_2(x) = (\text{dist}(x, \partial \Omega))^{\gamma}, \) with some \( \gamma < 0. \) The compact embedding (W5) is given by [20, Example 18.15] by Kufner and Opic and holds for \( \Omega \) with sufficiently regular boundary.

The paper is organised as follows. Section 2 provides discussion on properties of the two-weighted Sobolev spaces and assumptions on the admissible weights. In Section 3 we stand the relation between the first eigenvalue of the elliptic operator and the optimal constant in the Hardy inequality. After the compactness results in Section 4 the proof of Theorem 1.1 is given.

2 Preliminaries

2.1 Notation

In the sequel we assume that \( p > 2, \frac{1}{p} + \frac{1}{p'} = 1, \Omega \subset \mathbb{R}^N \) is an open subset not necessarily bounded. For \( T > 0 \) we denote \( \Omega_T = \Omega \times (0, T). \)

We denote \( p \)-Laplace operator by

\[ \Delta_p u = \text{div}(\nabla |\nabla u|^{p-2} \nabla u) \]

and \( \omega \)-\( p \)-Laplacian by

\[ \Delta_{p \omega} u = \text{div}(\omega |\nabla u|^{p-2} \nabla u), \] (4)

with a certain weight function \( \omega : \Omega \to \mathbb{R}. \)

We use truncations \( T_k(f)(x) \) defined as follows

\[ T_k(f)(x) = \begin{cases} f & |f| \leq k; \\ \frac{f}{k} & |f| \geq k. \end{cases} \] (5)

By \( \langle f, g \rangle \) we denote the standard scalar product in \( L^2(\Omega). \)

Let \( B(r) \subset \mathbb{R}^N \) denote the ball with the radius \( r, \) whose center shall be clear from the context. Then \( |B(r)| \) is its Lebesgue’s measure, \( \omega(B(r)) \) its \( \omega \)-measure, i.e. \( \omega(B(r)) = \int_{B(r)} \omega(x) \, dx. \)
2.2 Weighted Lebesgue and Sobolev spaces

Suppose $\omega$ is a positive, Borel measurable, real function defined on an open set $\Omega \subset \mathbb{R}^N$. Let

$$\omega' = \omega^{-1/(p-1)}.$$  \hfill (6)

**Definition 2.1** ($B_p$-condition, [20]). We say that $\omega$ satisfies the $B_p$-condition on $\Omega$ ($\omega \in B_p(\Omega)$), if

$$\omega' \in L^1_{\text{loc}}(\Omega).$$  \hfill (7)

Note that any $\omega \in L^1_{\text{loc}}(\Omega)$, which is strictly positive inside $\Omega$ satisfies $B_p$-condition on $\Omega$.

**Remark 2.1.** When $1 < p < \infty$ and $\omega \in B_p$, we have $L^p_{\omega,\text{loc}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$, see [20]. Moreover, for any $\omega \in L^1_{\text{loc}}(\Omega)$ and $s > p$ we have

$$L^s_{\omega,\text{loc}}(\Omega) \subset L^p_{\omega,\text{loc}}(\Omega).$$  \hfill (8)

If $\nabla$ denotes distributional gradient, we denote

$$W^{1,p}_{(\omega_1,\omega_2)}(\Omega) := \{ f \in L^p_{\omega_1}(\Omega) : \nabla f \in (L^p_{\omega_2}(\Omega))^N \}$$  \hfill (9)

with the norm

$$\|f\|_{W^{1,p}_{(\omega_1,\omega_2)}(\Omega)} := \|f\|_{L^p_{\omega_1}(\Omega)} + \|\nabla f\|_{(L^p_{\omega_2}(\Omega))^N}$$

$$= \left( \int_\Omega |f|^p \omega_1(x) \, dx \right)^{\frac{1}{p}} + \left( \int_\Omega \sum_{i=1}^N \left| \frac{\partial f}{\partial x_i} \right|^p \omega_2(x) \, dx \right)^{\frac{1}{p}}.$$  \hfill (10)

**Fact 2.1** (e.g. [20]). If $p > 1$, $\Omega \subset \mathbb{R}^N$ is an open set, $\omega_1, \omega_2$ satisfy $B_p$-condition (7), then

- $W^{1,p}_{(\omega_1,\omega_2)}(\Omega)$ defined by (9) equipped with the norm $\| \cdot \|_{W^{1,p}_{(\omega_1,\omega_2)}(\Omega)}$ is a Banach space;

- $\bar{L^p_{0}}(\Omega) = C^{\infty}_0(\Omega) = W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$, where the closure is in the norm $\| \cdot \|_{W^{1,p}_{(\omega_1,\omega_2)}(\Omega)}$.
• if $\omega_1, \omega_2$ are a pair in the Hardy-Poincaré inequality of the form (2), we may consider the Sobolev space $W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$ equipped with the norm

$$\|f\|_{W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)} = \|
abla f\|_{L^p_{\omega_2}(\Omega)}.$$

**Fact 2.2.** Operator $\Delta_{\omega_2}^p$, given by (4), is hemicontinuous, i.e. for all $u, v, w \in W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$ the mapping $\lambda \mapsto \langle \Delta_{\omega_2}^p (u + \lambda v), w \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.

We look for solutions in the space $L^p(0,T; W^{1,p}_{(\omega_1,\omega_2)}(\Omega))$, i.e.

$$L^p(0,T; W^{1,p}_{(\omega_1,\omega_2)}(\Omega)) = \{ f \in L^p(0,T; L^p_{\omega_1}(\Omega)) : \nabla f \in (L^p(0,T; L^p_{\omega_2}(\Omega)))^N \},$$

where $\nabla$ denotes distributional gradient with respect to the spatial variables, equipped with the norm

$$\|f\|_{L^p(0,T; W^{1,p}_{(\omega_1,\omega_2)}(\Omega))} := \left( \int_0^T \|f\|_{L^p_{\omega_1}(\Omega)}^p dt \right)^{\frac{1}{p}} + \left( \int_0^T \|
abla f\|_{(L^p_{\omega_2}(\Omega))^N}^p dt \right)^{\frac{1}{p}}.$$

**Dual spaces**

Let us stress that

$$(L^p_{\omega}(\Omega))^* \neq L^{p'}_{\omega'}(\Omega), \quad \text{but} \quad (L^p_{\omega}(\Omega))^* = L^{p'}_{\omega'}(\Omega)$$

with $\omega'$ given by (6).

By $W^{-1,p'}_{(\omega_1',\omega_2')}(\Omega)$ we denote the dual space to $W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$ and the duality pairing is given by the standard scalar product. We note that $L^{p'}(0,T; W^{-1,p'}_{(\omega_1',\omega_2')}(\Omega))$ is the dual space to $L^p(0,T; W^{1,p}_{(\omega_1,\omega_2),0}(\Omega))$.

**2.3 Comments on admissible weights**

We give here the reasons for which we assume the conditions (W1)-(W6).

Ad. (W1) It is general assumption on the spaces: $L^p_{\omega_1}(\Omega)$ and $W^{1,p}_{(\omega_1,\omega_2)}(\Omega)$.

Ad. (W2) To ensure that the weighted Sobolev space $W^{1,p}_{(\omega_1,\omega_2)}(\Omega)$ is a Banach space, we need to assume $\omega_1 \in B_p(\Omega)$, cf. (7). It is necessary to assume a stronger condition $\omega_1^{-\frac{2}{p-2}} \in L^1_{loc}(\Omega)$, to obtain the embedding

$$L^p_{\omega_1,loc}(\Omega) \subset L^{p'}_{\omega_1',loc}(\Omega)$$
\[ \omega_{1}^{-\frac{2}{s-q}} \in L^{1}(\Omega), \text{ to obtain} \]
\[ W_{(\omega_{1},\omega_{2}),0}^{1,p}(\Omega) \subset L^{2}(\Omega). \]

Ad. (W3) It guarantees strict monotonicity of the operator.
Moreover, it implies that \( \omega_{2} \in B_{p}(\Omega), \text{ cf. [1]}, \) which is necessary to ensure that \( W_{(\omega_{1},\omega_{2}),0}^{1,p}(\Omega) \) is a Banach space.

Ad. (W4) It is counterpart of the Poincaré inequality in \( W_{(\omega_{1},\omega_{2}),0}^{1,p}(\Omega) \).
We shall stress that there are multiple methods of deriving weights admissible in the Hardy inequalities having the form [2]. In particular, the results of the first author [23, Theorem 4.1] show that the weights may be generated by nonnegative solutions to the elliptic problem and the regularity conditions imposed on the weights are in fact expected regularity properties of the solutions.

Ad. (W5) It is necessary for the compactness method of Boccardo and Murat [8].
To obtain (W5) the result by Franchi, Serapioni and Serra Cassano [17], Theorem 3.4 can be applied. If one is equipped with another continuous embedding of the weighted Sobolev space into the weighted Lebesgue space, they may apply the results by Opic and Kufner [21], Sections 17 and 18 to obtain compact embedding on domains similar to John domains. For other ideas on compact embeddings in weighted Sobolev spaces we refer to [2, Proposition 2.1] by Anh and Ke.

Ad. (W6) Those are technical assumptions. Note that \( \omega_{1}^{-\frac{2}{s-q}} \in L_{loc}^{1}(\Omega) \) may follow from (W2). It depends on the possible values of exponents \( s \) and \( q \).
As mentioned before the integrability assumption on \( \omega_{1}^{-\frac{2}{s-q}} \in L_{loc}^{1}(\Omega) \) implies [12].

**Remark 2.2.** If \( \Omega \) is bounded, \( p \geq 2 \) and \( \omega_{1},\omega_{2} \) satisfy (W1)-(W4), then
\[ W_{(\omega_{1},\omega_{2})}^{1,p}(\Omega) \subset L_{\omega_{1}}^{p'}(\Omega) = (L_{\omega_{1}}^{p}(\Omega))^{*} \subset (W_{(\omega_{1},\omega_{2}),0}^{1,p}(\Omega))^{*} = W_{(\omega_{1},\omega_{2})}^{-1,p'}(\Omega). \]
and
\[ L^{p}(0,T;W_{(\omega_{1},\omega_{2})}^{1,p}(\Omega)) \subset L^{p}(0,T;L_{\omega_{1}}^{p'}(\Omega)) \subset L^{p'}(0,T;W_{(\omega_{1},\omega_{2})}^{-1,p'}(\Omega)). \]
Remark 2.3. If $\Omega$ is bounded, $2 < p < s$ and $\omega_1, \omega_2$ satisfy (W1)-(W5), then
\[ W_{(\omega_1, \omega_2)}^{1,p}(\Omega) \subset \subset L_{\omega_1}^s(\Omega) \subset L_{\omega_1}^{p'}(\Omega) \subset W_{(\omega_1', \omega_2')}^{-1,p'}(\Omega). \] (10)
Furthermore,
\[ W_{(\omega_1, \omega_2),0}^{1,p}(\Omega) \subset \subset L^2(\Omega) \]
and
\[ L^p(0,T; W_{(\omega_1, \omega_2),0}^{1,p}(\Omega)) \subset L^2(0,T; L^2(\Omega)) = L^2(\Omega_T). \] (11)
Moreover, if additionally we have (W6), then
\[ L_{\omega_1,loc}^s(\Omega) \subset \subset L_{loc}^q(\Omega) \text{ for } q \in (p,s). \] (12)

2.4 Auxiliary tools

For the sake of completeness we recall the general analytic tools necessary in our approach.

Theorem 2.1 (The Vitali Convergence Theorem). Let $(X, \mu)$ be a positive measure space. If $\mu(X) < \infty$, \(\{f_n\}\) is uniformly integrable, $f_n(x) \to f(x)$ a.e. and $|f(x)| < \infty$ a.e. in $X$, then $f \in L^1_\mu(X)$ and $f_n(x) \to f(x)$ in $L^1_\mu(X)$.

For the Aubin–Lions Lemmas we refer e.g. to [22].

Theorem 2.2 (The Aubin Lions Lemma 1). Suppose $1 < p < \infty$, $X, B, Y$ are the Banach spaces, $X \subset \subset B \subset Y$, $F$ is bounded in $L^p(0,T;X)$ and relatively compact in $L^p(0,T;Y)$ then $F$ is relatively compact in $L^p(0,T;B)$.

Theorem 2.3 (The Aubin Lions Lemma 2). Suppose $1 \leq p < \infty$, $X, B, Y$ are the Banach spaces, $X \subset \subset B \subset Y$. If $F$ is bounded in $L^p(0,T;X)$ and $\frac{dF}{dt}$ is bounded in $L^r(0,T;Y)$, where $r > 1$, then $F$ is relatively compact in $L^p(0,T;B)$.

For the Brezis Lieb Lemma we refer to [13].

Theorem 2.4 (The Brezis Lieb Lemma). Suppose $\Omega \subset \mathbb{R}^N$, $1 \leq p < \infty$, and $\mu \geq 0$ is a Radon measure. If $f_n \to f$ a.e. in $\Omega$ and $(f_n)_n$ is bounded in $L^p_\mu(\Omega)$, then the following limit exists
\[ \lim_{n \to \infty} \left( \|f_n\|_{L^p_\mu(\Omega)}^p - \|f - f_n\|_{L^p_\mu(\Omega)}^p \right) = \|f\|_{L^p_\mu(\Omega)}^p \]
and the equality holds.
We have the following corollary of the above theorem.

**Corollary 2.1.** Suppose \( \Omega \subset \mathbb{R}^N \), \( 1 \leq p < \infty \), and \( \omega_1 : \Omega \to \mathbb{R} \cup \{0\} \) is measurable. If \( u_m \to u \) strongly in \( L^p(0,T;L^p_{\omega_1}(\Omega)) \), then

\[
\omega_1 |u_m|^{p-2} u_m \to \omega_1 |u|^{p-2} u \quad \text{strongly in } L^{p'}(0,T;L^{p'}_{\omega'_1}(\Omega)).
\]

**Proof.** If \( u_m \to u \) strongly in \( L^p(0,T;L^p_{\omega_1}(\Omega)) \) and a.e. in \( \Omega \), then Theorem 2.4 yields that

\[
\int_{\Omega_T} \omega_1 |u_m|^p \, dx \, dt \to \int_{\Omega_T} \omega_1 |u|^p \, dx \, dt.
\]

Equivalently,

\[
\int_{\Omega_T} \omega_1 |u_m|^{p-2} u_m \left| |u|^{p-1} - |u_m|^{p-1} \right|^{\frac{p}{p-1}} \, dx \, dt \to \int_{\Omega_T} \omega_1 |u_m|^{p-2} u \left| |u|^{p-1} - |u_m|^{p-1} \right|^{\frac{p}{p-1}} \, dx \, dt,
\]

which, once again by Theorem 2.4 implies

\[
|u_m|^{p-2} u_m \to |u|^{p-2} u \quad \text{strongly in } L^{p'}(0,T;L^{p'}_{\omega_1}(\Omega)).
\]

When we observe that

\[
\int_{\Omega_T} \omega_1 \left( |u_m|^{p-1} - |u|^{p-1} \right)^{\frac{p}{p-1}} \, dx \, dt
= \int_{\Omega_T} \omega_1' \left( \omega_1 |u_m|^{p-1} - \omega_1 |u|^{p-1} \right)^{\frac{p}{p-1}} \, dx \, dt,
\]

we conclude that

\[
\omega_1 |u_m|^{p-2} u_m \to \omega_1 |u|^{p-2} u \quad \text{strongly in } L^{p'}(0,T;L^{p'}_{\omega_1}(\Omega)).
\]

\( \square \)

3  **The nonlinear eigenvalue problem**

The optimal constant in the Hardy–type inequality provides a spectral information for weighted problems. For the nonlinear eigenvalue problem

\[
- \text{div}(|\nabla u|^{p-2} u \omega_2) = \lambda |u|^{p-2} u \omega_1,
\]

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we have the following variational characterisation of the first eigenvalue by the Rayleigh quotient

$$
\lambda_1 = \inf \left\{ \frac{\int_\Omega \omega_2 |\nabla \phi|^p \, dx}{\int_\Omega \omega_1 |\phi|^p \, dx} : \phi \in W^{1,p}_{(\omega_1,\omega_2),0}(\Omega) \right\},
$$

considered e.g. in [11, 2, 16, 18, 25].

Via the method of [18], we obtain the following results.

**Theorem 3.1 (The first eigenvalue).** Suppose $1 < p < \infty$, $\Omega \subseteq \mathbb{R}^N$. Assume further that $\omega_1, \omega_2 : \Omega \to \mathbb{R}_+$ satisfy conditions (W1)-(W4) and $\lambda_{N,p}$ is the optimal left-hand side constant in the Hardy inequality (2).

Consider $\lambda_1(m)$ — the first eigenvalue to the problem

$$
\begin{cases}
-\text{div}(\omega_2 |\nabla \psi|^{p-2} \nabla \psi) = \lambda W_m |\psi|^{p-2} \psi & x \in \Omega \subset \mathbb{R}^N, \\
\psi(x) = 0 & x \in \partial \Omega,
\end{cases}
$$

where $W_m(x) = T_m(\omega_1(x))$ and $T_m$ is given by (5). Then $\lambda_1(m) \geq \lambda_{N,p}$ and moreover $\lim_{m \to \infty} \lambda_1(m) = \lambda_{N,p}$.

**Proof.** We define the first eigenvalues by the following Rayleigh quotients

$$
\lambda_1(m) = \inf \left\{ \frac{\int_\Omega \omega_2 |\nabla \phi|^p \, dx}{\int_\Omega \omega_1 |\phi|^p \, dx} : \phi \in W^{1,p}_{(\omega_1,\omega_2),0}(\Omega) \right\},
$$

$$
\lambda_{N,p} = \inf \left\{ \frac{\int_\Omega \omega_2 |\nabla \phi|^p \, dx}{\int_\Omega \omega_1 |\phi|^p \, dx} : \phi \in W^{1,p}_{(\omega_1,\omega_2),0}(\Omega) \right\}.
$$

In particular, according to [2] for each $\phi \in W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$ we have

$$
\lambda_{N,p} = \inf \frac{\int_\Omega \omega_2 |\nabla \phi|^p \, dx}{\int_\Omega \omega_1 |\phi|^p \, dx} \leq \frac{\int_\Omega \omega_2 |\nabla \phi|^p \, dx}{\int_\Omega \omega_1 |\phi|^p \, dx}.
$$

Then $\lambda_{N,p} \leq \lambda_1(m)$, $(\lambda_1(m))_{m \in \mathbb{N}}$ is a nonincreasing sequence, and $\lim_{m \to \infty} \lambda_1(m)$ exists.

We prove that $\lim_{m \to \infty} \lambda_1(m) = \lambda_{N,p}$ by contradiction. Suppose $\lim_{m \to \infty} \lambda_1(m) = \lambda_{N,p} + 2\varepsilon$ with a certain $\varepsilon > 0$. Let us take $\phi_0 \in W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$ such that

$$
\frac{\int_\Omega \omega_2 |\nabla \phi_0|^p \, dx}{\int_\Omega \omega_1 |\phi_0|^p \, dx} < \lambda_{N,p} + \varepsilon.
$$
On the other hand, due to the Lebesgue Monotone Convergence Theorem we notice that
\[
\frac{\int_{\Omega} \omega_2 |\nabla \phi_0|^p \, dx}{\int_{\Omega} W_m |\phi_0|^p \, dx} \quad \xrightarrow{m \to \infty} \quad \frac{\int_{\Omega} \omega_2 |\nabla \phi_0|^p \, dx}{\int_{\Omega} \omega_1 |\phi_0|^p \, dx},
\]
thus there exists \(m_0\), such that
\[
\lambda_{N,p} \leq \frac{\int_{\Omega} \omega_2 |\nabla \phi_0|^p \, dx}{\int_{\Omega} W_{m_0} |\phi_0|^p \, dx} \leq \lambda_{N,p} + \frac{3}{2} \varepsilon.
\]
and
\[
\lambda_{N,p} + 2 \varepsilon \leq \lambda_1(m_0) \leq \lambda_{N,p} + \frac{3}{2} \varepsilon.
\]

Theorem 3.2 (Positivity). Suppose \(1 < p < \infty\), \(\Omega \subseteq \mathbb{R}^N\). Assume further that \(\omega_1, \omega_2 : \Omega \to \mathbb{R}_+\) satisfy conditions (W1)-(W4), \(\lambda_{N,p}\) is the optimal left-hand side constant in the Hardy inequality (2) and the nonlinear operator \(\mathcal{L}_\lambda\) in \(W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)\) is given by
\[
\mathcal{L}_\lambda u = -\text{div}(\omega_2 |\nabla u|^{p-2} \nabla u) - \lambda \omega_1(x)|u|^{p-2}u.
\]
For \(\lambda \leq \lambda_{N,p}\), we have positivity of the operator. Moreover, for sufficiently big \(\lambda\), the operator is unbounded from below.

Proof. The result for small \(\lambda\)s results from the Hardy inequality [2]. If \(\lambda\) is bigger than the optimal constant in the already mentioned inequality, we reach the goal as an easy consequence of a density argument and the existence of \(\phi \in C_0^\infty(\Omega)\), such that \(\langle \mathcal{L}_\lambda \phi, \phi \rangle < 0\). We can assume that \(||\phi||_p = 1\). We define \(u_\mu(x) = \mu^\frac{N}{p} \phi(\mu x)\) and we have \(||u_\mu||_p = 1\). Due to the homogeneity of the operator we conclude that \(\langle \mathcal{L}_\lambda u_\mu, u_\mu \rangle = \mu^{p} \langle \mathcal{L}_\lambda \phi, \phi \rangle < 0\).

4 Existence

This section is divided into two subsections. The first one concerns necessary compactness properties, while the second one provides the proof of the main result.
4.1 Compactness results

Before we start the proof of the main theorem we need to adjust [8, Lemma 4.2] in the following way.

**Theorem 4.1.** Suppose \( p > 2 \) and \( \omega_1, \omega_2 \) satisfy (W1)-(W5). Assume further that

\[
(u_m)_t = h_m \quad \text{in} \quad D'(\Omega),
\]

where \( h_m \) — bounded in \( L^p(0, T; W^{-1,p'}(\omega_1, \omega_2)) \) and \( u_m \xrightarrow{m \to \infty} u \)
in \( L^p(0, T; W_{(\omega_1, \omega_2),0}^1(U)) \).

Then

(a) \( u_m \xrightarrow{m \to \infty} u \) strongly in \( L^p(0, T; L_{\omega_1}^s(U)) \);

(b) \( u_m \xrightarrow{m \to \infty} u \) a.e. in \( \Omega_T \) (up to a subsequence).

**Proof.** Let us consider a function \( \phi(x, t) = \psi(x)\eta(t) \), where \( \psi \in D(\Omega) \) and \( \eta \in D(0, T) \), and set \( v_m = \phi u_m \). For any bounded open subset \( U \), such that \( \text{supp} \phi \subset U \subset \Omega \), we have

\[
(v_m)_t = (\phi u_m)_t = \phi(u_m)_t + \phi_t u_m = \phi h_m + \phi_t u_m.
\]

Then \( v_m \) is bounded in \( L^p(0, T; W_{(\omega_1, \omega_2),0}^1(U)) \) and, due to (13), \( (v_m)_t \) is bounded in \( L^p(0, T; W_{(\omega_1, \omega_2),0}^{-1,p'}(\Omega)) \). We are going to apply the Aubin–Lions Lemma (Theorem 2.3). Let us note that if \( p > 2 \), then (W5) and (10) gives \( W_{(\omega_1, \omega_2),0}^1(U) \subset W_{(\omega_1, \omega_2),0}^{-1,p'}(\Omega) \).

Therefore \( v_m \) is relatively compact in \( L^p(0, T; L_{\omega_1}^s(U)) \).

Moreover, since we know (11), strong convergence in Lebesgue’s space implies convergence almost everywhere. \( \square \)

For the convenience of the reader, we provide the following extension of [9, Lemma 5] with the proof.

**Theorem 4.2.** Let \( U \) be a bounded open subset in \( \mathbb{R}^N \), \( U_T := U \times (0, T) \), \( 2 < p < \infty \) and \( \omega_1, \omega_2 \) satisfy (W1)-(W4). Assume that \( \nu_m \rightharpoonup \nu \) weakly in \( L^p(0, T; W_{(\omega_1, \omega_2),0}^1(U)) \) and a.e. in \( U_T \), and

\[
\int_{U_T} \omega_2 \left[ |\nabla \nu_m|^{p-2} \nabla \nu_m - |\nabla \nu|^{p-2} \nabla \nu \right] \nabla (\nu_m - \nu) \, dx \, dt \to 0. \quad (14)
\]

Then \( \nabla \nu_m \to \nabla \nu \) strongly in \( L^p(0, T; (L_{\omega_2}^p(U))^N) \), when \( m \to \infty \).
Proof. We adapt the proof of [9, Lemma 5] to the weighted setting. Let $D_m$ be defined by

$$D_m(x) = [\|\nabla \nu_m\|^{p-2}\nabla \nu_m - \|\nabla \nu\|^{p-2}\nabla \nu] \nabla (\nu_m - \nu).$$

By the monotonicity of $\Delta^\omega$ we note that $\omega_2(x)D_m \geq 0$. Since (14), observe that $D_m \to 0$ in $L^1(0, T; L^1_\omega(U))$ strongly. Thus, up to a subsequence $D_m \to 0$ a.e. in $U_T$. Recall $U_T$ is bounded. Suppose $X \subset U$ is a maximal set of full Lebesgue’s measure (and therefore of full $\omega_2$-measure), where for each $x \in X$ we have

$$|\nu(x)| < \infty, \ |
abla \nu(x)| < \infty, \ \nu_m(x) \to \nu(x), \ D_m(x) \to 0.$$

Clearly $\omega_2|\nabla \nu_m|^p \geq 0$ and $0 \leq D_m(x)$. Moreover,

$$D_m(x) = |\nabla \nu_m|^p + |\nabla \nu|^p - |\nabla \nu_m|^{p-2}\nabla \nu_m \nabla \nu - |\nabla \nu|^p \nabla \nu \nabla \nu_m \geq |\nabla \nu_m|^p - c(x)(|\nabla \nu_m|^{p-1} + |\nabla \nu_m|),$$

with $c(x)$ dependent on $X$, but not on $m$. As $D_m(x) \to 0$, we infer that $|\nabla \nu_m|$ is uniformly bounded on $X$.

Let us take arbitrary $x_0 \in X$ and denote

$$\zeta_m = \nabla \nu_m(x_0), \ \zeta = \nabla \nu(x_0).$$

Observe that $\omega_2(x_0) > 0$ and $(\zeta_m)$ is a bounded sequence. Set $\zeta_*$ as one of its cluster points. Recall $D_m(x_0) \to 0$ and note that

$$D_m(x_0) \to (|\zeta_*|^{p-2}\zeta_* - |\zeta|^{p-2}\zeta)(\zeta_* - \zeta).$$

Thus, $\zeta = \zeta_*$ is a unique cluster point of whole the sequence and $\nabla \nu_m(x_0) \to \nabla \nu(x_0)$ for arbitrary $x_0 \in X$. Then

$$\omega_2|\nabla \nu_m|^p \to \omega_2|\nabla \nu|^p \text{ in } X.$$

It implies uniform integrability of the sequence $|\nabla u_m|^p$ in $L^1_\omega(X)$, which implies uniform integrability in $L^1_\omega(U)$.

Therefore, Vitali’s Convergence Theorem (Theorem 2.1) yields that

$$\int_U \omega_2 (|\nabla \nu_m|^p - |\nabla \nu|^p) \, dx \to 0 \text{ for } m \to \infty$$

and the claim follows. \qed
Next we use the following modification of [8, Theorem 4.1].

**Theorem 4.3.** Assume \( p > 2, \omega_1, \omega_2 \) satisfy (W1)-(W6). Suppose

\[
(u_m)_t - \Delta_p^{\omega_2}(u_m) = g_m \quad \text{in} \quad D'(\Omega),
\]

moreover \( g_m \xrightarrow{m \to \infty} g \) in \( L^p(0,T;W^{-1,p'}(\omega_1,\omega_2)') (\Omega) \) and \( u_m \xrightarrow{m \to \infty} u \) in \( L^p(0,T;W^{1,p}_{(\omega_1,\omega_2)}(\Omega)) \).

Then, for any fixed \( k > 0 \), we have a strong convergence the gradients

\[
\nabla T_k(u_m) \xrightarrow{m \to \infty} \nabla T_k(u) \quad \text{in} \quad L^p(0,T;(L^p(\omega_2(U))')^N).
\]

**Proof.** We define \( S_k(s) = \int_0^s T_k(r) \, dr \), where \( T_k \) is given by [5]. Then for any \( \phi \in D(\Omega_T) \) and any \( \zeta \in L^p(0,T;W^{1,p}_{(\omega_1,\omega_2)}(\Omega)) \) such that \( \zeta \in L^p(0,T;W^{-1,p'}_{(\omega_1,\omega_2)}(\Omega)) \) we have

\[
\int_{\Omega_T} \zeta_t \phi T_k(\zeta) dxdt = -\int_{\Omega_T} \phi_t S_k(\zeta) dxdt.
\]

We fix compact sets \( K \subset \Omega_T \) and \( U \subset \Omega \), such that \( K \subset (0,T) \times U \subset \Omega_T \). We take an arbitrary function \( \phi_K \in D(\Omega_T) \) with \( \text{supp} \phi_K \subset K \subset \subset \Omega_T \), such that \( 0 \leq \phi_K \leq 1 \) with \( \phi_K = 1 \) on \( K \). Then we test [15] by

\[
w_m = (T_k(u_m) - T_k(u)) \phi_K
\]

getting

\[
0 = \int_{\Omega_T} (u_m)_t \phi_K [T_k(u_m) - T_k(u)] dxdt \\
+ \int_{\Omega_T} \phi_K \omega_2 |\nabla u_m|^{p-2} \nabla u_m [\nabla T_k(u_m) - \nabla T_k(u)] dxdt \\
+ \int_{\Omega_T} \omega_2 |\nabla u_m|^{p-2} \nabla u_m [T_k(u_m) - T_k(u)] \nabla \phi_K dxdt \\
- \int_{\Omega_T} g_m [T_k(u_m) - T_k(u)] \phi_K dxdt \\
= J^1_m + J^2_m + J^3_m + J^4_m.
\]
We deal with $J_m^1$ and $J_m^4$ in the similar way. We note that either sequence $$((u_m)_m)$$ or $$(g_m)_m$$ are bounded sequences in $L^p(0,T;W_{(\omega_1,\omega_2)}^{-1,p'}(\Omega))$. Therefore, Theorem 4.1 implies that up to a subsequence $T_k(u_m) \to T_k(u)$ strongly in $L^p(0,T;L^p_{\omega_1,\text{loc}}(\Omega))$, as we have (W5) and (8). Then $J_m^1, J_m^4 \to 0$ as $m \to \infty$. As for $J_m^3$, we apply the Hölder inequality, to get

$$J_m^3 = \int_{\Omega_T} \omega_2 |\nabla u_m|^{p-2} \nabla u_m [T_k(u_m) - T_k(u)] \nabla \phi_K dxdt$$

$$= \int_{U_T} \omega_2 |\nabla u_m|^{p-2} \nabla u_m [T_k(u_m) - T_k(u)] \nabla \phi_K dxdt$$

$$\leq \text{const} \left[ \int_0^T \left( \int_U [T_k(u_m) - T_k(u)]^q dx \right)^{\frac{p}{q}} dt \right]^{\frac{1}{p}} \cdot \left[ \int_0^T \int_U \omega_2 |\nabla u_m|^p dx \ dt \right]^{\frac{p-1}{p}} \left( \int_U \omega_2^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{qp}},$$

where $c_H > 0, U \subseteq \Omega$ such that $\text{supp} \phi_K \subset (0,T) \times U$, and $q$ comes from (W6). Then $J_m^3$ tends to zero. Indeed,

- by Theorem 4.1 we obtain $T_k(u_m) - T_k(u) \to 0$ strongly in $L^p(0,T; L^\omega_{\omega_1}(U))$. Notice that (W6) ensures that there exists $q$ such that

$$L^p(0,T; L^\omega_{\omega_1}(U)) \subset L^p(0,T; L^q(U)).$$

- weak convergence of $(u_m)$ in $L^p(0,T; W^{1,p}_{(\omega_1,\omega_2),0}(\Omega))$ implies its uniform boundedness in this space (up to a subsequence), thus $\int_{U_T} \omega_2 |\nabla u_m|^p dx dt < C$, with a constant $C$ independent of $m$;

- $\int_U \omega_2^{\frac{p}{p-1}} dx < \infty$ due to (W6).

As $J_m^1 + J_m^2 + J_m^3 + J_m^4 = 0$ and $\lim_{m \to \infty}(J_m^1 + J_m^2 + J_m^4) = 0$, then also $\lim_{m \to \infty} J_m^2 = 0$, i.e.

$$\int_{\Omega_T} \phi_K \omega_2 |\nabla u_m|^{p-2} \nabla u_m [\nabla T_k(u_m) - \nabla T_k(u)] dxdt \xrightarrow{m \to \infty} 0. \quad (16)$$

Let us observe that if $m \to \infty$, then $E_m$, given by

$$E_m = \int_{\Omega_T} \phi_K \omega_2 \left[ |\nabla T_k(u_m)|^{p-2} \nabla T_k(u_m) - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right] [\nabla T_k(u_m) - \nabla T_k(u)] dxdt,$$
tends to 0. Indeed,

\[ E_m = \int_K \phi_K \omega_2 |\nabla u_m|^{p-2} \nabla u_m [\nabla T_k(u_m) - \nabla T_k(u)] dx dt + \]
\[ - \int_{\Omega} \phi_K \omega_2 |\nabla u_m|^{p-2} \nabla u_m [\nabla T_k(u_m) - \nabla T_k(u)] \chi_{\{u_m > k\}} dx dt + \]
\[ - \int_{\Omega} \phi_K \omega_2 |\nabla T_k(u)|^{p-2} \nabla T_k(u) [\nabla T_k(u_m) - \nabla T_k(u)] dx dt, \]

(17)

where the first term converges to zero because of (16). Since \( \nabla T_k(u_m) \chi_{\{u_m > k\}} = 0 \), the second term reads

\[ \int_{\Omega} \phi_K \omega_2 |\nabla u_m|^{p-2} \nabla u_m [\nabla T_k(u_m) - \nabla T_k(u)] \chi_{\{u_m > k\}} dx dt, \]

where \( |\nabla u_m| \) is bounded in \( L^p(0, T; L^p_{\omega_2}(\Omega)^N) \) and for \( m \to \infty \) we have \( \nabla T_k(u_m) \chi_{\{u_m > k\}} \to \nabla T_k(u) \chi_{\{u > k\}} \) strongly in \( L^p(0, T; L^p_{\omega_2}(U)^N) \). Then the Monotone Convergence Theorem and fact that \( u_m \) is nondecreasing give the point. The third term in (17) converges to zero, because

\[ T_k(u_m) - T_k(u) \xrightarrow{m \to \infty} 0 \quad \text{weakly in } L^p(0, T; W^{1,p}_{(\omega_1, \omega_2)}(\Omega)). \]

We have proven that for \( m \to \infty \) we have \( E_m \to 0 \). Recall weak convergence \( u_m \xrightarrow{m \to \infty} u \) in \( L^p(0, T; W^{1,p}_{(\omega_1, \omega_2, 0)}(\Omega)) \) and a.e. in \( \Omega_T \). Therefore, Theorem 4.2 for \( \nu = T_k(u) \) and \( \nu_m = T_k(u_m) \) yields

\[ \nabla u_m \xrightarrow{m \to \infty} \nabla u \quad \text{strongly in } L^p(0, T; (L^p_{\omega_2}(U))^N). \]

\[ \Box \]

4.2 Proof of the main result

In order to construct a weak solution to (3) we first consider a truncated problem

\[ (u_m)_t - \Delta^\omega_2 u_m = \lambda T_m(\omega_1) |u_m|^{p-2} u_m, \]

where \( T_m \) is given by (5). Existence of the solution to the truncated problem is a consequence of the following result.
Theorem 4.4 ([24, Theorem 3.1]). Let $p > 2$, $\Omega \subseteq \mathbb{R}^N$ be an open subset, $f \in L^2(\Omega)$ and $\omega_1, \omega_2$ satisfy (W1)-(W5).

There exists $\lambda_0 = \lambda_0(p, N, \omega_1, \omega_2)$, such that for all $\lambda \in (0, \lambda_0)$ the parabolic problem

$$
\begin{cases}
  u_t - \Delta_p^\omega_2 u = \lambda W(x)|u|^{p-2}u & x \in \Omega, \\
  u(x, 0) = f(x) & x \in \Omega, \\
  u(x, t) = 0 & x \in \partial\Omega, \ t > 0,
\end{cases}
$$

where $W : \Omega \to \mathbb{R}_+$ is such that

$$
W(x) \leq \min\{m, \omega_1(x)\}
$$

with a certain $m \in \mathbb{R}_+$, has a global weak solution $u \in L^p(0, T; W^{1,p}_0(\partial\Omega))$, such that $u_t \in L^p(0, T; W^{-1,p'}_0(\partial\Omega))$, i.e.

$$
\int_{\Omega_T} (u_t \xi + \omega_2|\nabla u|^{p-2}\nabla u \nabla \xi + \lambda W(x)|u|^{p-2}u \xi) \, dx \, dt = 0,
$$

holds for each $\xi \in L^p(0, T; W^{1,p}_0(\partial\Omega))$. Moreover, $u \in L^\infty(0, T; L^2(\Omega))$.

Remark 4.1. In our previous paper [24] another embedding result was used, namely [2, Proposition 2.1]. The theorem holds true as well, when we assume (W5) instead of that one.

Let us present the proof of the main result.

Proof of Theorem 1.1. We consider $u_m$ — the solution to the truncated problem

$$
\begin{cases}
  w_t - \Delta_p^\omega_2 w = \lambda T_m(\omega_1)|w|^{p-2}w & x \in \Omega \\
  w(x, 0) = f(x) & x \in \Omega \\
  w(x, t) = 0 & x \in \partial\Omega, \ t > 0,
\end{cases}
$$

where $T_m$ is given by (5). Due to Theorem 4.4 there exists a solution $u_m$ to the problem (18) such that

$$
u_m \in L^p(0, T; W^{1,p}_{(\omega_1, \omega_2)}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \quad (u_m)_t \in L^p(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega)).$$

We are going to let $m \to \infty$. To obtain a priori estimate we test the problem (18) by $u_m$ getting

$$
\frac{1}{2} \frac{d}{dt} \|u_m\|^2_{L^2(\Omega)} + \int_{\Omega} \omega_2|\nabla u_m|^p \, dx = \lambda \int_{\Omega} T_m(\omega_1)|u_m|^p \, dx \leq \lambda \int_{\Omega} \omega_1|u_m|^p \, dx \leq \frac{\lambda}{K} \int_{\Omega} \omega_2|\nabla u_m|^p \, dx,
$$

18
where the last inequality is allowed due to the Hardy inequality $[2]$. Note that the density of Lipschitz and compactly supported functions in $W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)$ is given by Fact $2.1$. Therefore,

$$
\frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 + \left(1 - \frac{\lambda}{K}\right) \int_\Omega \omega_2 |\nabla u_m|^p dx \leq 0.
$$

Note that

$$
\int_0^T \frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 dt = \|u_m(\cdot, T)\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)}^2.
$$

Summing up, we obtain

$$
\frac{1}{2} \|u_m(\cdot, T)\|_{L^2(\Omega)}^2 + \left(1 - \frac{\lambda}{K}\right) \int_0^T \|\nabla u_m(\cdot, t)\|_{L^p(\Omega)}^p dt \leq \frac{1}{2} \|f\|_{L^2(\Omega)}^2.
$$

In particular, this implies

- $(u_m)_{m \in \mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$;
- $(u_m)_{m \in \mathbb{N}}$ is bounded in $L^p(0,T;W^{1,p}_{(\omega_1,\omega_2),0}(\Omega))$.

Thus, there exists a function $u \in L^p(0,T;W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $u_t \in L^p(0,T;W^{-1,p'}_{(\omega_1,\omega_2)}(\Omega))$, such that and up to a subsequence, we have

$$
(19) \quad u_m \xrightarrow{\ast} u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)),
$$

$$
(19) \quad u_m \xrightarrow{m \to \infty} u \quad \text{in} \quad L^p(0,T;W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)).
$$

We know that for each $\xi \in L^p(0,T;W^{1,p}_{(\omega_1,\omega_2),0}(\Omega))$ the following equality holds

$$
\int_{\Omega_T} \left( (u_m)_t \xi + \omega_2 |\nabla u_m|^{p-2} \nabla u_m \nabla \xi + \lambda T_m(\omega_1) |u_m|^{p-2} u_m \xi \right) dx \, dt = 0. \quad (20)
$$

We have to show that the limit function $u$ from $\mathbf{(19)}$ is the weak solution to $\mathbf{(3)}$, i.e.

$$
\int_{\Omega_T} \left( u_t \xi + \omega_2 |\nabla u|^{p-2} \nabla u \nabla \xi + \lambda \omega_1 |u|^{p-2} u \xi \right) dx \, dt = 0
$$
holds for each \( \xi \in L^p(0, T; W^{1,p}_{(\omega_1, \omega_2)}(\Omega)) \). Let us note that the integral above is well-defined on this class, in particular \( L^p(0, T; W^{1,p}_{(\omega_1, \omega_2)}(\Omega)) \subset L^2(\Omega_T) \). The weak convergence of gradients is not enough to pass to the limit with \( \int_{Q_T} \omega_2 |\nabla u_m|^{p-2} \nabla u_m \nabla \xi \). Thus, the first step is to get strong convergence of gradients. We follow the spirit of Boccardo and Murat to obtain a strong convergence of the gradients of trucations and apply it in (20) splitted into

\[ 0 = \int_{\Omega_T} (u_m) t \xi dx dt + \int_{\Omega_T \cap \{|u_m| \leq k\}} \omega_2 |\nabla u_m|^{p-2} \nabla u_m \nabla \xi dx dt + \]

\[ + \int_{\Omega_T \cap \{|u_m| > k\}} \omega_2 |\nabla u_m|^{p-2} \nabla u_m \nabla \xi dx dt + \int_{\Omega_T} \lambda T_m(\omega_1)|u_m|^{p-2} u_m \xi dx dt \]

\[ = A^m_1 + A^m_2 + A^m_3 + A^m_4. \]

The convergence of \( A^m_1 \) to \( \int_{\Omega_T} u_t \xi dx dt \) can be obtained by integrating by parts and by the Lebesgue’s Monotone Convergence Theorem since \((u_m)_m \) is a nondecreasing sequence.

In the case of \( A^m_2 \) we first remark that over this set we can write

\[ A^m_2 = \int_{\Omega_T} \omega_2 |\nabla T_k(u_m)|^{p-2} \nabla T_k(u_m) \nabla \xi dx dt. \]

Now we engage the method of Boccardo and Murat via Theorem 4.3 to get

\[ \nabla T_k(u_m) \xrightarrow{m \to \infty} \nabla T_k(u) \text{ in } L^p(0, T; (L^p_{\omega_2}(U))^N). \]

The assumptions of Theorem 4.3 are satisfied, because besides the weak convergence of functions, we have

\[ g_m = \lambda \omega_1 |u_m|^{p-2} u_m \xrightarrow{m \to \infty} \lambda \omega_1 |u|^{p-2} u = g \]  \hspace{1cm} (21)

in \( L^{p'}(0, T; W^{-1,p'}(\omega_{1}'(\omega_2)'(\Omega))) \). To justify this we apply the Aubin–Lions Lemma (Theorem 2.2). Since we assume (W5) and we know (11), we have

\[ W^{1,p}_{(\omega_1, \omega_2),0}(U) \subset L^p_{\omega_1}(U) \subset W^{-1,p'}_{(\omega_1', \omega_2')}(\Omega). \]

Then we infer that \( u_m \to u \) strongly in \( L^p(0, T; L^p_{\omega_1}(U)) \). Strongly convergent sequence has a subsequence convergent almost everywhere. If it is necessary,
we pass to such subsequence, but we do not change the notation. Note that

$$
\|g_m\|_{L^p(0,T;L^p_1(\Omega))} = \lambda \int_{\Omega_T} \omega_1^m |u_m|^{p-1} \frac{\partial u_m}{\partial t} \, dx \, dt =
$$

$$
= \lambda \int_{\Omega_T} \omega_1^m \frac{1}{p-1} \omega_1^{\frac{p}{p-1}} |u_m|^p \, dx \, dt = \lambda \int_{\Omega_T} \omega_1 |u_m|^p \, dx \, dt < \infty
$$

and thus

$$
g_m \in L^p(0,T;L^p_1(U)) \subset L^p(0,T;W^{-1,p}_1(\omega_1^{1/2}))(\Omega).
$$

According to the Brezis–Lieb Lemma (Corollary 2.1) the strong convergence of $u_m \to u$ in $L^p(0,T;L^p_1(U))$ implies the strong convergence

$$
\omega_1 |u_m|^{p-2} u_m \to \omega_1 |u|^{p-2} u \quad \text{in} \quad L^p(0,T;L^p_1(U)),
$$

which entails strong convergence $g_m \to g$ in $L^p(0,T;L^p(0,T;L^p_1(U)))$ and in turn also (21). This finishes the case of $A_2^m$.

We easily show that the Hölder inequality implies that $A_3^m \leq s(k)$, with a certain constant $s$ depending on $k$. Indeed,

$$
\int_{\Omega_T \cap \{|u_m| > k\}} \omega_2 |\nabla u_m|^{p-2} \nabla u_m \nabla \xi \, dx \, dt \leq
$$

$$
\leq \left( \int_{\Omega_T \cap \{|u_m| > k\}} \omega_2 |\nabla u_m|^p \, dx \, dt \right)^{\frac{p-1}{p}} \left( \int_{\Omega_T \cap \{|u_m| > k\}} \omega_2 |\nabla \xi|^p \, dx \, dt \right)^{\frac{1}{p}} \leq \text{const} \left( \int_{\Omega_T \cap \{|u| > k\}} \omega_2 |\nabla \xi|^p \, dx \, dt \right)^{\frac{1}{p}} = s(k).
$$

Note that the integral on the right–hand side above is finite even for $k = 0$ and that the sequence $(u_m)$ is nondecreasing (and thus $\{|u_m| > k\} \subset \{|u| > k\}$).

It suffices to show that $A_4^m - \int_{\Omega_T} \lambda \omega_1 |u|^{p-2} u \xi \, dx \, dt \to 0$, when $m \to \infty$.

We show that both expressions below tend to zero

$$
\int_{\Omega_T} T_m(\omega_1) |u_m|^{p-2} u_m \xi \, dx \, dt - \int_{\Omega_T} \omega_1 |u|^{p-2} u \xi \, dx \, dt =
$$

$$
= \int_{\Omega_T} (T_m(\omega_1) - \omega_1) |u_m|^{p-2} u_m \xi \, dx \, dt + \int_{\Omega_T} (|u_m|^{p-2} u_m - |u|^{p-2} u) \omega_1 \xi \, dx \, dt =
$$

$$
= B_1^m + B_2^m.
$$
To deal with $B_1^m$ we recall that $(|u_m|^{p-2}u_m)_m$ is bounded in $L^p(0,T;W^{-1,p'}(\Omega))$ (cf. the case of $A_2^m$), while $T_m(\omega_1) \nrightarrow \omega_1$, so the Lebesgue Monotone Convergence Theorem implies $B_1^m \to 0$ as $m \to \infty$.

Let us concentrate on $B_2^m$. We have

$$|B_2^m| \leq \left( \int_{\Omega_T} |u_m|^{p-2}u_m - |u|^{p-2}u \right)^{\frac{p-1}{p}} \left( \int_{\Omega_T} |\omega_1|_p dx dt \right)^{\frac{1}{p}}.$$  

We employ the Brezis–Lieb Lemma (Corollary 2.1) to get

$$\omega_1|u_m|^{p-2}u_m \rightharpoonup \omega_1|u|^{p-2}u_m \quad \text{in} \quad L^p(0,T;L^p(U)),$$

leading to

$$|u_m|^{p-2}u_m \rightharpoonup |u|^{p-2}u_m \quad \text{in} \quad L^p(0,T;L^p_{\omega_1}(U)).$$

which implies that $B_2^m \to 0$ as $m \to \infty$.

We have shown that for every $k \in \mathbb{N}$

$$\int_{\Omega_T} (u_t \xi + \omega_2|\nabla u|^{p-2}\nabla u \nabla \xi + \lambda \omega_1|u|^{p-2}u \xi) \ dx \ dt = s(k).$$

Since $s(k) \to 0$ when $k \to \infty$, we conclude that $u$ is the desired solution. □

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