Five types of blow-up in a semilinear fourth-order reaction–diffusion equation: an analytic–numerical approach

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Abstract

Five types of blow-up patterns that can occur for the 4th-order semilinear parabolic equation of the reaction–diffusion type

$$u_t = -\Delta^2 u + |u|^{p-1} u \quad \text{in } \mathbb{R}^N \times (0, T), \quad p > 1,$$

is discussed. For the semilinear heat equation $u_t = \Delta u + u^p$, various blow-up patterns were under scrutiny since the 1980s, while the case of higher order diffusion was studied much less, regardless of the wide range of its application. The types of blow-up include the following:

(i) Type I(ss): patterns of self-similar single point blow-up, including those for which the final time profile $|u(\cdot, T^-)|^{N(p-1)/4}$ is a measure;

(ii) Type I(log): self-similar non-radial blow-up with angular logarithmic TW swirl;

(iii) Type I(Her): non-self-similar blow-up close to stable/centre subspaces of Hermitian operators obtained via linearization about constant uniform blow-up pattern;

(iv) Type II(sing): non-self-similar blow-up on stable/centre manifolds of a singular steady state in the supercritical Sobolev range $p > p_S = (N + 4)/(N - 4)$ for $N > 4$ and

(v) Type II(LN): non-self-similar blow-up along the manifold of stationary generalized Loewner–Nirenberg type explicit solutions in the critical Sobolev case $p = p_S$, when $|u(\cdot, T^-)|^{N(p-1)/4}$ contains a measure as a singular component.

There is some evidence that Type I(ss) are structurally stable (generic) patterns. However, justifying this and the existence of other proposed types
of blow-up remain difficult open problems, so formal analytic and numerical methods are key in supporting some theoretical judgements.

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(Some figures in this article are in colour only in the electronic version)

1. From second-order to higher order blow-up R–D models: towards a PDE route from the 20th to 21st century

1.1. The RDE-4 and applications

This paper is devoted to a description of blow-up patterns for the fourth-order reaction–diffusion equation (the RDE-4 in short)

\[ u_t = -\Delta^2 u + |u|^{p-1}u \quad \text{in } \mathbb{R}^N \times (0, T), \quad \text{where } p > 1, \tag{1.1} \]

where \( \Delta \) stands for the Laplacian in \( \mathbb{R}^N \). This has the biharmonic diffusion \(-\Delta^2\) and is a higher order counterpart of the classic second-order PDEs, which we begin our discussion with. For applications of such higher-diffusion models, see short surveys and references in [7, 46]. In general, higher order semilinear parabolic equations arise in many physical applications such as thin film, convection–explosion, and lubrication theory, flame and wave propagation (the Kuramoto–Sivashinsky equation and the extended Fisher–Kolmogorov equation), phase transition at critical Lifschitz points, bi-stable systems and applications to structural mechanics. The effect of fourth-order terms on self-focusing problems in nonlinear optics is also well known in the applied and mathematical literature. For a systematic treatment of extended KPPF-equations, see Peletier–Troy [88].

Note that another related fourth-order one-dimensional semilinear parabolic equation

\[ u_t = -u_{xxxx} - [(2 - (u_x)^2)u_x]_x - \alpha u + q e^u, \tag{1.2} \]

where \( \alpha, q \) and \( s \) are positive constants obtained from physical parameters, occurs in the Semenov–Rayleigh–Benard problem [60], where the equation is derived for studying the interaction between natural convection and the explosion of an exothermically reacting fluid confined between two isothermal horizontal plates. This is an evolution equation for the temperature fluctuations in the presence of natural convection, wall losses and chemistry. It can be considered as a formal combination of the equation derived in [50] (see also [9]) for the Rayleigh–Benard problem and of the Semenov-like energy balance [22, 91] showing that the natural convection and the explosion mechanism may reinforce each other; see more details on the physics and mathematics of blow-up for (1.2) in [46]. In a special limit, (1.2) reduces to the generalized Frank-Kamenetskii equation (see [7] for basic blow-up results)

\[ u_t = -u_{xxxx} + e^u, \tag{1.3} \]

which is a natural extension of the classic Frank-Kamenetskii equation presented below.

Equation (1.1) can be considered as a non-mass-conservative counterpart of the well-known limit unstable Cahn–Hilliard equation from phase transition,

\[ u_t = -u_{xxxx} - (|u|^{p-1}u)_x, \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{1.4} \]

which is known to admit various families of blow-up solutions; see [14] for a long list of references. Somehow, (1.1) is related to the famous Kuramoto–Sivashinsky equation from flame propagation theory

\[ u_t = -u_{xxxx} - u_{xx} + uu_x, \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{1.5} \]

which always admits global solutions, so no blow-up for (1.5) exists.
1.2. On second-order reaction–diffusion (R–D) equations: a training ground of blow-up PDE research in the 20th century

Blow-up phenomena, as examples of extremely non-stationary behaviour of nonlinear mechanical and physical systems, have become more natural in PDE theory since the systematic development of combustion theory in the 1930s. This essential combustion influence began with the derivation of the semilinear parabolic reaction–diffusion PDE such as the classic Frank-Kamenetskii (F-K) equation (1938) \[ \text{F-K equation} \]

\[ u_t = \Delta u + e^u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad (1.6) \]

which occurs in combustion theory of solid fuels and is often also called the solid fuel model. First blow-up results in related ODE models are due to Todes in 1933; see the famous monograph \[101\] for details of the history and applications. The related model with a power superlinear source term takes the form (also available among various nonlinear combustion models \[101\])

\[ u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad (1.7) \]

For such typical models, blow-up means that in the Cauchy problem\(^1\) the classic bounded solution \(u(x, t)\) exists in \(\mathbb{R}^N \times (0, T)\), while

\[ \sup_{x \in \mathbb{R}^N} |u(x, t)| \to +\infty \quad \text{as} \quad t \to T^-, \quad (1.8) \]

where \(T \in \mathbb{R}_+ = (0, +\infty)\) is then called the blow-up time of the solution \(u(x, t)\).

During the last 50 years of very intensive research starting from seminal Fujita results in 1966 (inducing Fujita exponents), we have currently got a rather complete understanding of the types of blow-up for the semilinear (1.6), (1.7) and other models. This is very well explained in a number of monographs; see \[3, 27, 43, 45, 78, 86, 89, 90\]. However, one should remember that even for simple R–D equations such as (1.6) and (1.7), there are blow-up scenarios in the multi-dimensional geometries, which still did not get a proper rigorous mathematical justification. For instance, there are a number of surprises even in the radial geometry for (1.7), which reads for \(r = |x| > 0\) as

\[ u_t = \frac{1}{r^{N-1}}(r^{N-1}u_r)_r + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+ \quad (u_r|_{r=0} = 0, \text{symmetry}), \quad (1.9) \]

in the supercritical Sobolev range

\[ p > p_S = \frac{N+2}{N-2}, \quad \text{where} \quad N > 2. \quad (1.10) \]

Several critical exponents, which may essentially change blow-up evolution, appear for (1.9) in the range (1.10). Among those let us mention the most exotic and amazing Joseph–Lundgren’s \[59\] and Lepin’s \[64, 65\] ones:

\[ p_{JL} = 1 + \frac{4}{N-4-2\sqrt{N-1}} \quad \text{and} \quad p_{L} = 1 + \frac{6}{N-10} \quad (N \geq 11). \quad (1.11) \]

In particular, this shows that, in the parameter range

\[ N \geq 11 \quad \text{and} \quad p \geq p_S, \quad (1.12) \]

new principal issues of blow-up evolution for (1.9) essentially take place. Note that, in \[44\], some critical blow-up exponents were shown to exist for the quasilinear combustion equation with a porous medium diffusion:

\[ u_t = \Delta u^m + u^p, \quad \text{where} \quad p > m > 1 \quad (u(x, t) \geq 0). \quad (1.13) \]

\(^1\) For simplicity, we avoid using the initial-boundary value problems, where boundary conditions can affect the principles of singularity formation.
This shows certain universality of formation of blow-up singularities for a wider class of R–D equations, which we are now going to extend to the RDE-4 (1.1).

We do not plan to give a detailed enough review of such a variety of these delicate and becoming diverse (rather surprisingly) in the 21st century mathematical results for second-order R–D models (1.6), (1.7) and (1.13), which quite recently attracted the attention of several remarkable mathematicians from various areas of PDE theory. We refer to [4–6, 44, 55, 77, 97] for earlier results since the 1980s and the 1990s, and to more recent papers [19, 72] and [79–81] as a guide to the research, which was essentially intensified in the last few years. Further results can be traced out by the MathSciNet, using the most recent papers of the authors mentioned above.

It is worth mentioning that most of these results have been obtained for non-negative blow-up solutions of (1.6), (1.7) and (1.10), since the positivity property is naturally supported by the Maximum Principle (the MP) for such second-order parabolic equations. For instance, a full classification of such non-negative blow-up patterns for (1.7) (all of them belong to the family Type I(Her)) in the subcritical range $1 < p < p_S$ was obtained in [97]. For (1.6), this happens in dimension $N = 1$ and $2$. In other words, the family of blow-up patterns for (1.6) and subcritical (1.7) first formally introduced in [98] is evolutionary complete (a notion from [28], where further references can be found). In the range $p \geq p_S$ for (1.7) and from $N = 3$ for (1.6), there occur self-similar patterns of Type I(ss) and many others being non-self-similar, which makes the global blow-up flow much more complicated.

In contrast, for the quasilinear R–D equation (1.13), the generic blow-up is self-similar of Type I(ss) (as for (1.1)); see [90, chapter 4] for details and references.

For $p \geq p_S$, such a complete classification for (1.7) is far from being complete. For example, for $p \geq p_S$ in (1.7), non-symmetric blow-up patterns are practically unknown. (1.14)

Moreover, for solutions of changing sign, the results are much more rare and are essentially incomplete. It is worth mentioning the surprising blow-up patterns of changing sign constructed in [20], with the structure to be used later on for (1.1), where we comment on this Type II(LN) blow-up patterns for (1.1) in greater detail.

1.3. Back to the RDE-4: five types of blow-up patterns and layout of the paper

We shall discuss possible types of blow-up behaviour for the RDE-4 (1.1). In what follows, we will use the auxiliary classification from Hamilton [52], where Type I blow-up means the solutions satisfying, for some constant $C > 0$ (depending on $u$),

\begin{align*}
\text{Type I:} \quad (T - t)^{-\frac{1}{p-1}} |u(x, t)| &\leq C \quad \text{as} \quad t \to T^-, \quad \text{and, otherwise} \\
\text{Type II:} \quad \limsup_{t \to T^-} (T - t)^{-\frac{1}{p-1}} \sup_x |u(x, t)| &= +\infty
\end{align*}

(Type II also called slow blow-up in [52]). In R–D theory, blow-up with the dimensional estimate (1.15) was usually called of self-similar rate, while Type II was referred to as fast and non-self-similar; see [45, 90].

Thus, we plan to describe the following five types of blow-up with an extra classification issue in each of them (this list also shows the overall layout of the paper):

1. Type I(ss): various patterns of self-similar single point blow-up mainly in radial geometry, including those, for which $|u(\cdot, T^-)|^{N(p-1)/4}$ can be a measure (section 2). This kind of blow-up admits self-similar extension for $t > T$ [35]. Almost nothing is known for non-radial similarity blow-up patterns for $N \geq 2$, which are expected to exist.
Five types of blow-up in a semilinear fourth-order reaction–diffusion equation

(ii) Type I(log): non-radial self-similar blow-up with angular logarithmic travelling wave \((\log TW)\) swirl, which in the similarity rescaled variables corresponds to periodic orbits as \(o\)-limit sets (section 3);

(iii) Type I(Her): non-self-similar blow-up close to stable/centre subspaces of \(\text{Hermitian}\) operators (and generalized \(\text{Hermite}\) polynomials) obtained via linearization about constant uniform blow-up and matching with a Hamilton–Jacobi region (section 4). This kind of blow-up is complete and does not admit a regular extension for \(t > T\) [35];

(iv) Type II(sing): non-self-similar blow-up on stable/centre manifolds of \(\text{singular steady state}\) (SSS) in the supercritical Sobolev (and Hardy) range \(p > p_S = \frac{N+4}{N-4}\) for \(N > 4\) with matching to a central quasi-stationary region (section 5) and

(v) Type II(LN): non-self-similar blow-up along the manifold of stationary generalized \(\text{Loewner–Nirenberg type explicit solutions}\) in the critical Sobolev case \(p = p_S\), when final time profiles \(|u(\cdot, T^-)|^{N(p-1)/4}\) contain measures in the singular component (section 6, a natural bounded extension for \(t > T\) is then assumed).

There is already some analytic and numerical evidence (PDE numerics) [7] supporting the fact that Type I(ss) similarity patterns represent a structurally stable (generic) type of blow-up, though there is no proof. Therefore, other types of possible blow-up (if any) are expected to be unstable.

We must admit that the analysis of all the blow-up types indicated above is very difficult mathematically, so we present practically no rigorous results. Recall that, even for the second-order equation (1.6), all these types excluding Type I(Her) not only did not have any complete classification, but some of them were not even detected at all. For (1.1), the best known critical exponent is obviously Sobolev’s one

\[
p_S = \frac{N + 4}{N - 4}, \quad \text{where } N \geq 5, \text{ and also } p_* = \frac{N}{N - 4}, \tag{1.16}
\]

while the others, as counterparts of those in (1.11), need further study and understanding. However, many critical exponents for (1.1) cannot be explicitly calculated. Overall, we aim that our approaches to blow-up patterns can be extended to \(2m\)th-order parabolic equations such as

\[
u_t = -(-\Delta)^m u + |u|^{p-1} u, \tag{1.17}
\]

though the case \(m = 2\) (the first even \(m\)s) already contains some surprises.

Nevertheless, it seems that, at some stage of struggling to develop new concepts, it is inevitable to attempt to perform a formal classification under the clear danger of a lack of any rigorous justification\(^2\). In this rather paradoxical connection, it is also worth mentioning that the most well-known nowadays and the fundamental open problem of fluid mechanics\(^3\) and PDE theory on global existence or non-existence (blow-up) of bounded smooth \(L^2\)-solutions of the Navier–Stokes equations (the NSEs)

\[
u_t + (u \cdot \nabla)u = -\nabla p + \Delta u, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \quad u|_{t=0} = u_0 \in L^2 \cap L^\infty, \tag{1.18}
\]

from one side belongs to a ‘blow-up configurational’ type: \(to \text{ predict possible swirling ‘twistor-tornado’ type of blow-up patterns.}\) Moreover, it seems that the NSEs (1.18) was

\(^2\) Actually following Kolmogorov’s legacy from the 1980s sounding not completely literally as: ‘The main goal of a mathematician is not proving a theorem, but an effective investigation of the problem...’. This interplays well with Tao’s position of 2007 [96, p 624]: ‘...mathematical rigour, while highly important, is only one component of what determines a quality piece of mathematics’.

\(^3\) The Millennium Prize Problem for the Clay Institute; see Fefferman [16].
the first model, for which Leray in 1933–1934 (see [66, 67, p 245]) formulated the so-called Leray’s scenario of self-similar blow-up as \( t \to T^- \) and a similarity continuation beyond for \( t > T \).

Non-existence of such similarity blow-up for the NSEs (1.18) was proved in 1996 in Nečas–Ružička–Šverák [83]. However, for the semilinear heat equations (1.7) and (1.13), the validity of Leray’s scenario of blow-up was rigorously established; see [44, 80] and references therein.

In general, we observe certain similarities between these two blow-up problems; see [33], where Type I(log) patterns were introduced for (1.18) and [34] for more details and references on other related exact blow-up solutions. Overall, we claim that equations (1.6), (1.7), (1.1) and (1.18) admit some similar principles of constructing various families of blow-up patterns, although, of course, for the last two, the construction gets essentially harder and many steps are made formally, without any proper justification. This is especially so for the NSEs (1.18), which are composed of a non-local solenoidal parabolic equation:

\[
\mathbf{u}_t + \mathbb{P}(\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u}, \tag{1.19}
\]

where the integral operator \( \mathbb{P} = I - \nabla \Delta^{-1}(\nabla \cdot) \) is Leray–Hopf’s projector onto the solenoidal vector field. More precisely, the RDE-2 (1.7) obeying the MP is indeed too simple to mimic any NSEs blow-up patterns, while (1.1), which is similar to (1.19) traces no MPs, can be about right (possibly, still illusionary). Then (1.1) stands for an auxiliary ‘training ground’ to approach an understanding of mysterious and hypothetical blow-up for (1.18).

As another important application, we note the recent result in Masmoudi–Zaag [71] describing a single point blow-up for the complex Ginzburg–Landau equation

\[
u_t = (1 + \beta i)\Delta u + (1 + \delta i)|u|^{p-1}u - \gamma u \quad \text{in} \; \mathbb{R}^N \times \mathbb{R}, \quad u_0 \in L^\infty(\mathbb{R}^N), \; p > 1, \tag{1.20}
\]

with real constants \( \beta, \delta, \gamma \), where \( p - \delta^2 - \beta \gamma(p + 1) > 0 \). Blow-up for (1.20) was open for some time that generated some new scenarios such as convergence as \( t \to T^- \) after proper blow-up scaling of the solution to a periodic orbit. This idea was also treated as a blow-up possibility in the 3D NSEs (1.18); see [34, section 2.7] for the original references and a discussion.

However, the reality of blow-up for (1.20) turned out to be easier and more natural: at least for \( N = 1 \), single point blow-up for (1.20) was proved in [71] to correspond well to the linearized patterns of Type I(Her) in our classification, i.e. constructed via a standard Hermite polynomial \( H_2 \) structure with extension to a nonlinear Hamilton–Jacobi region. We recall that for the 1D R–D equation (1.7), this idea was formally proposed in Hocking–Stuartson–Stuart in 1972 [56], with a full description of a whole countable set on blow-up patterns already in the 1980s [98] and full proofs achieved in the 1990s (see multiple references above). Typical details of a similar construction are pointed out in section 4 for (1.1), which remain analogous for various PDEs. Paper [71] also contains a very useful survey concerning blow-up in various types of nonlinear PDEs and touches the problem of classification of Type I blow-up.

In appendix A, we present other families of PDEs, which expose a similar open problem on the existence/non-existence of \( L^\infty \)-blow-up of solutions from bounded smooth initial data. Overall, it is worth saying that the problem of a description of blow-up patterns and their evolution completeness takes and shapes certain universality features and becomes an inevitable part of the general existence-uniqueness-regularity-non-existence— PDE theory of the 21st century.
2. Type I(ss): self-similar blow-up

This is the simplest and most natural type of blow-up for scaling invariant equations such as (1.1), where the behaviour as \( t \to T^- \) is given by a self-similar solution:

\[
\begin{align*}
  u_S(x,t) &= (T-t)^{-\frac{1}{p+1}} f(y), \\
  y &= \frac{x}{(T-t)^{1/4}}
\end{align*}
\]

where a non-constant function \( f \neq 0 \) is a proper solution of the elliptic problem:

\[
A(f) \equiv -\Delta^2 f - \frac{1}{4} y \cdot \nabla f - \frac{1}{p-1} f + |f|^{p-1} f = 0 \quad \text{in } \mathbb{R}^N, \ f(\pm \infty) = 0.
\]

(2.2)

We recall that, for (1.7), such non-trivial self-similar Type I blow-up is non-existent in the subcritical range \( p \leq (N+2)/(N-2) \). But this is not the case for the RDE-4 (1.1). Note that (2.2) is a difficult elliptic equation with non-coercive and non-monotone operators, which are not variational in any weighted \( L^2 \)-spaces. There are still no sufficiently general results of solvability of (2.2) in higher dimensions, so our research is a first attempt.

In what follows, for any dimension \( N \geq 1 \), by \( f_0(y) \) we will denote the first monotone radially symmetric blow-up profile, which, being on the lower \( N \)-branch (so \( f_0 \) is not unique, see explanations below), is expected to be generic, i.e. structurally stable in the rescaled sense. We also deal with the second symmetric profile \( f_1(y) \), which seems to be unstable, or, at least, less stable than \( f_0 \). There are also other similarity solutions concentrated about the singular SSS \( U(y) \) (see section 5), but those, being adjacent to the unstable equilibrium \( U \), are expected to be unstable also.

It is worth mentioning that self-similar blow-up for (1.1) is incomplete, i.e. blow-up solutions, in general, admit global similarity extension for \( t > T \) in the form (2.3) with the change \( T-t \mapsto t-T \). The related principal questions of global existence and uniqueness of such extended solutions are studied in [35] and will not be treated here.

2.1. One dimension: first examples of non-uniqueness

Thus, for \( N = 1 \), (2.2) becomes the ODE

\[
A(f) \equiv -f^{(4)} - \frac{1}{4} y f' - \frac{1}{p-1} f + |f|^{p-1} f = 0 \quad \text{in } \mathbb{R}, \ f(\pm \infty) = 0,
\]

(2.3)

which was studied in [7] by a number of analytic-branching and numerical methods. It was shown that (2.3) admits at least two different blow-up profiles with algebraic decay at infinity. See [46, section 3] for further centre manifold-type arguments supporting this multiplicity result in a similar 4th-order blow-up problem. Without going into the details of such a study, we present a few illustrations only and will address the essential dependence of similarity profiles \( f(y) \) on \( p \). In figure 1, we present those pairs of solutions of (2.3) for \( p = \frac{7}{2} \) and \( p = 2 \). All the profiles are symmetric (even), and so satisfy two symmetry conditions:

\[
f'(0) = f''(0) = 0.
\]

(2.4)

No non-symmetric blow-up was detected in numerical experiments (though there is no proof that such ones are non-existent: recall that the ‘moving plane’ and Aleksandrov’s Reflection Principle methods do not apply to the parabolic PDE (1.1) without the MP). Figure 2 shows two similar blow-up profiles for \( p = 5 \).
2.2. On the existence of similarity profiles for $N = 1$: classification of blow-up and oscillatory bundles

We now provide extra details concerning the existence of at least a single blow-up profile $f(y)$ satisfying (2.3), (2.4). We perform shooting from $y = +\infty$, by using the 2D bundle (see further comments below expansion (2.19) for any $N \geq 1$)

$$f(y) = [C_1 y^{-1/p} + \cdots] + [C_2 y^{-1/(p-1)} e^{-a_0 y^{p/(p-1)}} + \cdots], \quad a_0 = 3 \cdot 2^{-1}, \quad C_{1,2} \in \mathbb{R}, \quad (2.5)$$

to $y = 0$, where the symmetry conditions (2.4) are posed (or to $y = -\infty$, where the same bundle (2.5) with $y \mapsto -y$ takes place). By $f = f(y; C_1, C_2)$, we denote the corresponding solution defined on some maximal interval

$$y \in (y_0, +\infty), \quad \text{where} \quad y_0 = y_0(C_1, C_2) \geq -\infty. \quad (2.6)$$
If $y_0(C_1, C_2) = -\infty$, then the corresponding solution $f(y; C_1, C_2)$ is global and can represent a proper blow-up profile (but not often, see below). Otherwise
\[ y_0(C_1, C_2) > -\infty \implies f(y; C_1, C_2) \to \infty \quad \text{as} \quad y \to y_0^+. \] (2.7)

Note that ‘oscillatory blow-up’ for the ODE close to $y = y_0^+$,
\[ f^{(4)} = |f|^{p-1} f \left( 1 + o(1) \right), \] (2.8)
where $\lim sup f(y) = +\infty$ and $\lim inf f(y) = -\infty$ as $y \to y_0^+$, is non-existent. The proof is easy and follows by multiplying (2.8) by $f'$ and integrating between the two extremum points $(y_1, y_2)$, where the former one $y_1$ is chosen to be sufficiently close to the blow-up value $y_0^+$, whence the contradiction
\[ 0 < \frac{1}{2} (f')^2(y_1) \sim -\frac{1}{p+1} |f|^{p+1}(y_1) < 0. \]

We first study this set of blow-up solutions. These results are well understood for such fourth-order ODEs; see [48] and also [17, 18], so we omit some details.

**Proposition 2.1.** The set of blow-up solutions (2.8) is four-dimensional.

**Proof.** The first parameter is $y_0 \in \mathbb{R}$. Others are obtained from the principal part of equation (2.8) describing blow-up via (2.3) as $y \to y_0^+$. We apply a standard perturbation argument to (2.8). Omitting the $o(1)$-term and assuming that $f > 0$, we find its explicit solution
\[ f_0(y) = A_0(y - y_0)^{-\frac{1}{p+1}}, \quad A_0^{p-1} = \Phi \left( -\frac{4}{p-1} \right), \quad \Phi(m) = m(m-1)(m-2)(m-3). \] (2.9)
For convenience, the graph of $\Phi(m)$ is shown in figure 3. Note that it is symmetric relative to $m_0 = \frac{1}{2}$, at which $\Phi(m)$ has a local maximum
\[ \Phi \left( \frac{1}{2} \right) = \frac{9}{16}. \] (2.10)
By linearization, \( f = f_0 + Y \), we get Euler’s ODE:

\[
(y - y_0)^4 Y^{(4)} = pA_0^{p-1} \equiv p \Phi \left( -\frac{4}{p - 1} \right).
\]

It follows that the general solution is composed of the polynomial ones with the following characteristic equation:

\[
Y(y) = (y - y_0)^m \implies \Phi(m) = p \Phi \left( -\frac{4}{p - 1} \right).
\]

Since the multiplier \( p > 1 \) in the last term in (2.12) and \( m = -4/(p - 1) \) is a solution if this ‘\( p \times \)’ is omitted, this algebraic equation for \( m \) admits a unique positive solution \( m^+ > 3 \), a negative one \( m^- < -4/(p - 1) \), which is not acceptable by (2.9), and two complex roots \( m_{1,2} \) with Re \( m_{1,2} = \frac{3}{2} > 0 \). Therefore, the general solution of (2.8) about the blow-up one (2.9), for any fixed \( y_0 \), has a 3D stable manifold.

Thus, according to proposition 2.1, the blow-up behaviour with a fixed sign (2.7) (i.e. non-oscillatory) is generic for the ODE (2.3). However, this 4D blow-up bundle together with the 2D bundle of good solutions (2.5) as \( y \to \pm \infty \) are not enough to justify the shooting procedure. Indeed, by a straightforward dimensional estimate, an extra bundle at infinity is missing.

To introduce this new oscillatory bundle, we begin with the simpler ODE (2.8), without the \( o(1) \)-term, and present in figure 4 the results of shooting of a ‘separatrix’ that lies between orbits, which blow up to \( \pm \infty \). Obviously, this separatrix is a periodic solution of this equation with a potential operator. Such variational problems are known to admit periodic solutions of arbitrary period.

Thus, figure 4 fixes a bounded oscillatory (periodic) solution as \( y \to +\infty \). When we return to the original equation (2.3), which is not variational, we are still able to detect more complicated oscillatory structures at \( y = \infty \). Namely, these are generated by the principal terms in

\[
f^{(4)} = -\frac{1}{2} f'y + |f|^{p-1} f + \cdots \quad \text{as } y \to \infty.
\]

(2.13)

Similar to figure 4, in figure 5, we present the result of shooting (from \( y = -\infty \), which is the same by symmetry) of such oscillatory solutions of (2.3) for \( p = 5 \). It is easy to see that
such oscillatory solutions have increasing amplitude of their oscillations as $y \to \infty$, which, as above, is proved by multiplying (2.13) by $f'$ and integrating over any interval $(y_1, y_2)$ between the two extrema. Figure 6 shows shooting of similar oscillatory structures at infinity for $p = 7$ (a) and $p = 2$ (b). It is not difficult to prove that the set of such oscillatory orbits at infinity is 1D and this well corresponds to the periodic one in figure 4 depending on the single parameter being its arbitrary period.

By $C_2^\pm (C_1)$ in figure 5, we denote the values of the second parameters $C_2$ such that, for a fixed $C_1 \in \mathbb{R}$, the solutions $f(y; C_1, C_2^\pm)$ blow up to $\pm \infty$ respectively. These values are necessary for shooting the symmetry conditions (2.4).
Figure 6. Shooting an oscillatory solution at infinity of (2.3): $p = 7$ (a) and $p = 2$ (b).

Thus, overall, using two parameters $C_{1,2}$ in the bundle (2.5) for $y \gg 1$ leads to a well-posed problem of a 2D–2D shooting:

\[
\begin{align*}
\text{find } C_{1,2} \text{ such that: } & \\
& \begin{cases}
    y_0(C_1, C_2) = -\infty, & \text{and} \\
    \text{no oscillatory behaviour as } y \to -\infty.
\end{cases}
\end{align*}
\] (2.14)

Concerning the actual proof of existence via shooting of at least a single blow-up pattern $f_0(y)$, by construction and oscillatory properties of equation (2.3), we first claim that, in view of continuity relative to the parameters, the following holds:

for any $C_1 > 0$, there exists $C_2^*(C_1) \in (C_2^-(C_1), C_2^+(C_1))$ such that $f''''(0) = 0$.  (2.15)
We next change $C_1$ to prove that, at this $C^*_1(C_1)$, the derivative $f'(0)$ also changes sign. Indeed, one can see that
\[
f'(0; C_1, C^*_2(C_1)) > 0 \quad \text{for} \quad C_1 \ll 1 \quad \text{and} \quad f'(0; C_1, C^*_2(C_1)) < 0 \quad \text{for} \quad C_1 \gg 1.
\]

Actually, this means that, for such essentially different values of $C_1$, the solution $f(y; C_1, C^*_2(C_1))$ has first oscillatory 'humps' for $y > 0$ and $y < 0$, respectively. By continuity in $C_1$, (2.16) implies the existence of a $C^*_1$ such that
\[
f'(0; C^*_1, C^*_2(C^*_1)) = 0,
\]
which together with (2.15) induced the desired solution. Overall, the above geometric shooting corresponds well to that applied in the standard framework of classic ODE theory, so we do not treat this in greater detail. However, we must admit that proving analogously the existence of the second solution $f_1(y)$ (detected earlier by not fully justified arguments of homotopy and branching theory and confirmed numerically) is an open problem. A more difficult open problem is to show why problem (2.14) does not admit non-symmetric (non-even) solutions $f(y)$ (or does it?).

2.3. Dimensions $N \geq 2$: on 2D shooting and analogous non-uniqueness

In higher dimensions, it is easier to describe Type I(ss) blow-up in radial geometry, where (2.2) also becomes an ODE of the form (now $y$ stands for $|y| > 0$)
\[
- f^{(4)} - \frac{2(N - 1)}{y} f''' - \frac{(N - 1)(N - 3)}{y^2} f'' + \frac{(N - 1)(N - 3)}{y^3} f' - \frac{1}{4} y f'' + \frac{1}{p - 1} f + |f|^{p-1} f = 0
\]

in $\mathbb{R}$, with the same two symmetry conditions (2.4). To explain the nature of difficulties in proving the existence of solutions of (2.18), let us describe the admissible behaviour for $y \gg 1$. Similar to (2.5), there exists a 2D bundle of such asymptotics (see details in [15, section 3.3]): as $y \to +\infty$,
\[
f(y) = \left[ C_1 y^{-\frac{1}{p-1}} + \cdots \right] + \left[ C_2 y^{-\frac{1}{2(N-1)}} \right] e^{-a_0 y^{\frac{N}{3}}} + \cdots, \quad \text{where} \quad a_0 = 3 \cdot 2^{-\frac{4}{3}}
\]

and $C_1$ and $C_2$ are arbitrary parameters. This somehow recalls a typical centre manifold structure of the origin $\{f = 0\}$ at $y = \infty$: the first term in (2.19) is a node bundle with algebraic decay, while the second one corresponds to a 'non-analytic' exponential bundle around any of algebraic curves. Thus, a dimensionally well-posed shooting is

\textit{Shooting}: using 2 parameters $C_{1,2}$ in (2.19) to satisfy 2 conditions(2.4).

In the case of analytic dependence of solutions of (2.18) on parameters $C_{1,2}$ in the bundle (2.19) (this is rather plausible via standard trends of ODE theory, but difficult to prove), the problem cannot have more than a countable set of solutions. Actually, our numerics confirm that, in wide parameter ranges of $p > 1$ and $N \geq 1$, there exist not more than two solutions (up to other more unstable ones about the SSS; see section 5):
\[
f_0(y) \text{ with } [C_{10}(p, N), C_{20}(p, N)] \quad \text{and} \quad f_1(y) \text{ with } [C_{11}(p, N), C_{21}(p, N)].
\]

The rest of this section is devoted to justify this.
The eventual similarity blow-up patterns can be characterized by their final time profiles: passing to the limit $t \to T^-$ in (2.1) and using the expansion (2.19) yields

\[
\text{if } C_1(p, N) \neq 0, \text{ then } u(x, t) \to C_1|x|^{-\frac{1}{N}} \text{ as } t \to T^- \tag{2.22}
\]

uniformly on any compact subset of $\mathbb{R}^N \setminus \{0\}$. If $C_1 = 0$ in (2.19), i.e. $f(y)$ has an exponential decay at infinity, then the limit is different: in the sense of distributions,

\[
C_1(p, N) = 0 \implies |u(x, t)|^{\frac{N(p-1)}{N-1}} \to C_3 \delta(x), \quad t \to T^-; \quad C_3 = \int |f|^{\frac{N(p-1)}{N-1}} < \infty. \tag{2.23}
\]

It is very difficult to prove that (2.23) actually takes place at some $p = p_a(N) > 1$ (even for $N = 1$), and we will justify this numerically for some not very large dimensions $N \leq 11$.

We now start to describe various similarity blow-up profiles for $N \geq 2$. As a first and analogous to $N = 1$ example, in figure 7, we construct numerically first two profiles, $f_0(y)$ and $f_1(y)$, for the three-dimensional case $N = 3$ and $N = 10$ for $p = 2$, which look rather similar to those in figure 1 for $N = 1$.

2.4. $N \geq 2$: $p$-branches of the profile $f_0(y)$ and $f_1(y)$

Such $p$-branches of solutions are a convenient way to describe families of profiles $f_0(y)$ depending on the exponent $p$; cf [36, 47]. In figure 8, we present such a branch of $f_0$ for $N = 4$, where (a) shows the actual smooth deformation of $f_0(y)$ with changing $p$, while (b) is the corresponding $p$-branch. In figure 9, the same is done for $N = 8$. Note that both figures 7(b) and 9(b) show that $\|f\|_\infty = f(0)$ approaches 1 for large $p$, which is a general phenomenon for such ODEs described first in [36, section 5]. Similarly, figure 10 shows $p$-branches of the second blow-up profile $f_1(y)$ for $p = 2$ in the cases $N = 1 (a)$ and $N = 12 (b)$ (the critical dimension, where $p_b = 2$).

It is well understood that, for equations such as (2.18), the solutions $f(y)$ blow-up as $p \to 1^+$ with a super-exponential rate $\sim(p - 1)^{-1/(p-1)}$; see [36, 47]. As an example, in figure 11, we present such a blowing up behaviour of the $p$-branch of $f_1(y)$ for $N = 6$.

2.5. $N \geq 2$: $N$-branches of blow-up profiles

Firstly, in some $N$-intervals, there is a continuous dependence of $f_0(y)$ on the dimension, as figure 12 clearly shows for $p = 2$ and figure 13 for $p = 5$ (those values of $p$ will be constantly used later on for the sake of comparison).

However, we found that there are other solutions of the monotone type $f_0$, which are shown in figure 14 for $p = 5 (a)$ and $p = 2 (b)$, where in the latter one the profile from the lower $N$-branch is not shown as being too relatively small.

Thus, secondly, this non-uniqueness demands another approach to branching, namely, the $N$-branching that we perform next. In figure 15, we show the lower $N$-branch of solutions $f_0(y)$ for $p = 5$, where (a) describes the deformation of $f_0$ and (b) gives the actual $N$-branch. Blow-up of the upper $N$-branch as

\[
N \to N^*_p, \quad \text{where } N_p : \frac{N + 4}{N - 4} = p, \tag{2.24}
\]

so that $N_3 = 6$ for $p = 5$, is shown in figure 16, with the same meaning of (a) and (b). A general view of the whole $N$-branch of $f_0$ for $p = 5$ is schematically explained in figure 17, where by means of a dotted line we draw a possible expected, but still hypothetical, connection of the lower (stable) and the upper (more unstable, plausibly) $f_0$-branches, which we were not able to reconstruct numerically. Numerical continuation in the parameter $N$ is quite a challenging problem in some $N$-ranges.
Thus, we expect that there exists a saddle–node bifurcation at some
\[ p = 5 : \, N_{sn} \in (14.979, \, 15) \] (2.25).

In figure 18 for \( p = 2 \), we show blow-up of the upper \( N \)-branch as \( N \to N_2 = 12^* \). We then expect that the lower and upper branches have a turning (saddle–node) bifurcation point at some
\[ p = 2 : \, N_{sn} \in (20.3, \, 23) \].

We hope that such an interesting saddle–node branching phenomenon will attract true experts in numerical methods, bearing in mind that numerical experiments might be for a long time the only tool for the study of such blow-up phenomena.

Finally, in figure 19, we present the numerical results confirming \( N \)-branching for \( p = 2 \) of the second blow-up profile \( f_1 \), where, as usual, (a) describes smooth deformation of \( f_1(y) \),

Figure 7. Two self-similar blow-up solutions of (2.18) for \( p = 2 \): \( N = 3 \) (a) and \( N = 10 \) (b).
while (b) shows the $N$-branch. It seems that $N$-branches of $f_1$ are global and do not suffer from a saddle–node bifurcation.

2.6. On sign changes of $f_0(y)$ and $f_1(y)$

We now study some particularly important properties of blow-up similarity profiles. We begin with the easier property of sign changes. We have already seen several strictly positive profiles $f_0(y)$ for some $p$s, which is rather surprising since the equations do not obey the MP. However, we will show that, for smaller $p$, the similarity profiles can gain extra zeros as sign changes. Such $p$s, when a new zero is gained, we denote by $p_0(N)$.

Consider the one dimension $N = 1$. Firstly, the attentive reader can see that in figure 1(a), already for $p = \frac{3}{2}$ the profile $f_0(y)$ changes sign, while for $p = 2$ it is positive.
Hence, \( p_0(1) \in \left( \frac{3}{2}, 2 \right) \). Secondly, more thorough numerics are presented in figure 20, where (a) shows \( f_0 \) in a vicinity of
\[
p_0(1) = 1.7358 \ldots,
\]
while (b) shows a sharp shooting of the critical value (2.26). Figure 21 shows shooting
\[
p_0^{(1)}(1) = 1.23 \ldots,
\]
at which the second profile \( f_1(y) \) gets a new zero. By the boldface line we denote a new ‘\( p \)-undetected’ solution with extra zeros gained at another \( p_{01} \), showing that such roots are not unique. Since \( f_1(y) \) is expected to be less stable, we will concentrate on the roots \( p_0 \) for the generic blow-up profile \( f_0(y) \).

Thus, similarly, figure 22(a) yields
\[
p_0(3) = 1.446 \ldots \quad \text{and} \quad p_0(9) = 1.204 \ldots
\]
Figure 10. The $p$-branches of blow-up self-similar profiles $f_1(y)$ for $p = 2$: $N = 1$ (a) and $N = 12$ (b).

In figure 23(a), a similar phenomenon is checked for $N = 10$, with $p_0(10) = 1.188 \ldots$. In (b), we see no sign changes of $f_0(y)$ for $N = 12$, but this can happen for smaller $p \approx 1^+$, when $f(0)$ gets $10^7 \sim 10^8$, while their negative counterparts take values $\sim -10^5$, and numerics become rather unreliable. The overall numerical results for shooting $p_0(N)$ are shown in table 1.

A proper asymptotic theory for $p \approx 1^+$ involving expansions such as (2.19) and $(p - 1)^{-1}$-scaling of the ODE (2.18) (cf [36, section 5]) would be rather fruitful. Note that we have observed some numerical evidence for existence of the second root $p_0^1$ for $N = 8$ and 9 (see the dotted line in figure 22), which is surprising in view of non-oscillating of the exponential term in (2.19), but numerics were too difficult and rather poor to identify the new root if any.
2.7. On the final time measure-like Type I(ss) blow-up

This is a much more difficult problem, which we resolve numerically for $N = 1$ only. For $N \geq 2$, we got no sufficiently reliable results (rather plausibly, such a self-similar phenomenon may be unavailable in some higher dimensions).

We will refine figure 20. We claim that equation (2.23) has the following root:

$$p_1(1) = 1.40 \ldots < p_0(1) = 1.7358 \ldots ,$$

at which the coefficient $C_1(p, N)$ vanishes, so that $f_0(y)$ has exponential decay at infinity. To see this, we show in figure 24 with the scale of $10^{-20}$ how the coefficient $C_1(p, 1)$ changes sign around (2.28):

$$C_1(1.39, 1) > 0, \quad \text{while } C_1(1.40, 1) < 0.$$
Non-vanishing of these two profiles in smaller scales up to $10^{-40}$ was checked in the logarithmic scale (we do not present here a number of such numerics).

2.8. On non-radial self-similar blow-up patterns in dimensions $N \geq 2$

This question was not studied in the literature at all and indeed is very difficult. We make a slight observation only: the linearization (4.2) performed below about the constant equilibrium in the elliptic equation (2.2) leads to the perturbed linear elliptic equation

$$\left(B^* + I\right)Y + D(Y) = 0$$

(2.30)

(on spectral properties of $B^*$, see lemma 4.1). Then, $B^* + I$ has a large unstable subspace

$$E^u(0) = \text{Span}\{\psi_{\beta} : \lambda_{\beta} + 1 > 0\},$$

(2.31)
so that the corresponding eigenfunctions may characterize possible shapes of various similarity solutions (actually, this is true for \( N = 1 \) \[7\]). Roughly speaking, we claim that

\[
M(p, N) = \dim E^u(0) - (N + 1)
\]

can characterize the total number of blow-up similarity patterns as solutions of (2.2).

We subtract \((N + 1)\)-dimensions corresponding to the natural instabilities relative to shifting the blow-up point \(0 \in \mathbb{R}^N\) (\(N\) dimensions) and blow-up time \(T\) (1 dimension). These unstable modes are not available if the blow-up point \((0, T)\) is fixed. The dimension \(M = M(p, N)\) can characterize the total number of solutions \(f(y)\) of the elliptic problem (2.2) including many non-radial ones. In other words, we expect that those unstable \(M\) modes initiate heteroclinic connections through the corresponding unstable manifold \(W^u(0)\) to the set of steady solutions \(\{f_k(y)\}, k = 1, 2, \ldots, M\). In section 4, we show that stable modes from \(E^\pm(0)\) with \(\lambda_{0} + 1 < 0\) and the centre ones \(E^c(0)\) with \(\lambda_{0} + 1 = 0\) will lead to other ‘linearized’ blow-up patterns, so that \(\{f_k\}\) are ‘nonlinear eigenfunctions’. 

**Figure 14.** Two monotone blow-up profiles of \(f_0\)-type: for \(p = 5\) (a) and for \(p = 2\) (b).
Proving any part of claim (2.32) is a difficult open problem for any $N \geq 2$. Note also that, for the second-order quasilinear counterpart (1.13) ($m > 1$ is essential!), non-radially symmetric self-similar blow-up patterns have been known for more than 30 years; see [63] and a survey [62] for extra details.

3. Type I(log): self-similar patterns with angular logTW swirl

This is a simple idea for producing non-radial blow-up patterns, but its consistency is quite questionable.

3.1. Non-stationary rescaling

Dealing with non-self-similar blow-up, instead of (2.1), we use the full similarity scaling

$$u(x, t) = (T - t)^{-\frac{1}{m+1}} v(y, \tau), \quad y = \frac{x}{(T - t)^{1/m}},$$

$$\tau = -\ln(T - t) \rightarrow +\infty, \quad t \rightarrow T^{-}. \quad (3.1)$$
Then \( v(y, \tau) \) solves the following parabolic equation:
\[
v_{\tau} = A(v) \equiv -\Delta^2 v - \frac{1}{4} y \cdot \nabla v - \frac{1}{p - 1} v + |v|^{p-1} v \quad \text{in } \mathbb{R}^N \times (\tau_0, \infty), \quad \tau_0 = -\ln T,
\]
(3.2)
where \( A \) is the stationary elliptic operator in (2.2), so that similarity profiles (if any) are just stationary solutions of (3.2).

3.2. Blow-up angular swirling mechanism

We begin with \( N = 2 \), where \( y = (y_1, y_2) \), and, in the corresponding polar coordinates \( \{\rho, \phi\} \), with \( \rho^2 = y_1^2 + y_2^2 \),
\[
\Delta = \Delta_\rho + \frac{1}{\rho^2} D_\phi^2, \quad \text{where } \Delta_\rho = D_\rho^2 + \frac{1}{\rho} D_\rho \text{ and } y \cdot \nabla = \rho D_\rho.
\]
(3.3)
We next consider a TW in the angular direction by fixing the angular dependence
\[ \varphi = \sigma \tau + \mu \equiv -\sigma \ln(T - t) + \mu, \quad \mu \in (0, 2\pi), \tag{3.4} \]
where \( \sigma \in \mathbb{R} \) is a constant (a nonlinear eigenvalue). In the original independent variables \( \{x, t\} \), (3.4) represents a blowing up logarithmic TW in the angular direction with unknown wave speeds \( \sigma \). In other words, (3.4) assumes that blowing up as \( t \to T^- \) is accompanied by a focusing TW-angular behaviour also in a logarithmic blow-up manner.

Thus, assuming the logTW angular dependence (3.4) of the solution \( v = v(y, \mu, \tau) \), \( \varphi = \sigma \tau + \mu \), yields the equation
\[ v_{\tau} = A(v) - \sigma v_{\mu} \equiv -\Delta^2 v - \frac{1}{4} y \cdot \nabla v - \frac{1}{p - 1} v + |v|^{p-1}v - \sigma v_{\mu} \quad \text{in} \quad \mathbb{R}^N \times (\tau_0, \infty), \tag{3.5} \]
where \( \tau_0 = -\ln T \). In particular, this non-radial self-similar blow-up may be generated by bounded steady profiles satisfying
\[ A'(f_0)f_{\mu} - \sigma f_{\mu} \equiv -\Delta^2 f - \frac{1}{4} y \cdot \nabla f - \frac{1}{p - 1} f + |f|^{p-1}f - \sigma f_{\mu} = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{3.6} \]
For \( \sigma \neq 0 \), which, as we have mentioned, plays the role of a nonlinear eigenvalue, the blow-up behaviour with swirl corresponds to periodic orbits as \( \omega \)-limit sets; see a discussion in [33] to the NSEs (1.18). As a first approach to the solvability of (3.6), one can assume branching of a solution \( f(\rho, \mu) \) from the radial one \( f_0 \) at \( \sigma = 0 \). Then setting \( f = f_0 + \sigma \psi^* + \cdots \) yields that \( \psi^*(\rho, \mu) \) must be a non-trivial non-radial eigenfunction for \( \lambda = 0 \):
\[ A'(f_0)\psi^* = 0. \tag{3.7} \]
On the other hand, branches of solutions \( f \) of (3.6) may occur at a saddle–node bifurcation \( \sigma = \sigma_\ast \neq 0 \), where \( \sigma_\ast \) belongs to the spectrum of the linear pencil \( A'(f) - \sigma D_\mu \). Both eigenvalue problems are very difficult, and we do not exclude the possibility that, overall,
Five types of blow-up in a semilinear fourth-order reaction–diffusion equation

problem (3.6) for $\sigma \neq 0$ may admit only such solutions that are singular at the origin $y = 0$. Anyway, even in this unfortunate case, we believe that introducing such rather unknown types of non-radial blow-up with swirl deserves mentioning among other more practical patterns. Let us also mention that, in $\mathbb{R}^N$, one can distribute the variables as

$$y = (y_1, y_2, y') \in \mathbb{R}^N, \quad \text{where} \quad y' \in \mathbb{R}^{N-2},$$

and arrange a $\sigma_1$-log$\text{TW}$ in variables $(y_1, y_2)$ only to get a periodic blow-up behaviour. Choosing other disjoint pairs $(y_k, y_{k+1})$ and constructing the corresponding periodic swirl in these variables, in particular, it is formally possible to produce a \textit{quasi-periodic blow-up swirl} with arbitrary number $\sigma_1, \ldots, \sigma_n$, $n \leq \lfloor N/2 \rfloor$, of fundamental frequencies. Of course, this leads to complicated nonlinear eigenvalue problems, which are open even for $n = 1$, i.e. for the periodic motion introduced above first.

Figure 18. Lower and upper $N$-branches of $f_0$ for $p = 2$: the lower branch for $N \in [1, 20.3]$ (a) and blow-up of the upper one as $N \to 12^+$ (b).
3.3. Remark 1: logarithmic twistors for the NSEs

Note that such an idea of a blow-up angular logarithmic swirl applies to the 3D NSEs (1.18) in cylindrical or spherical coordinates, generating a number of open problems, [34, section 5, 8].

3.4. Remark 2: on the origin of logTWs and invariant solutions

The scaling group-invariant nature of such logTWs was first obtained by Ovsianikov in 1959 [85], who performed a full group classification of the nonlinear heat equation

\[ u_t = (k(u)u_x)_x, \]
for arbitrary functions $k(u)$. In particular, such invariant solutions appear for the porous medium and fast diffusion equations for $k(u) = u^n$, $n \neq 0$:

$$u_t = (u^n u_x)_x \implies \exists u(x, t) = t^{-\frac{1}{n}} f(x + \sigma \ln t), \quad \text{where} \quad -\frac{1}{n} f + \sigma f' = (f^n f')'.$$

Blow-up angular dependence as $t \rightarrow T^-$ such as in (3.4) was studied later on in [1], where the corresponding similarity solutions for the reaction–diffusion equation with source (1.13) in $\mathbb{R}^2 \times (0, T)$ were indicated by reducing the PDE to a quasilinear elliptic problem (it seems that there is no rigorous proof yet of the existence of such patterns). For parabolic models such as (1.13), that are order-preserving via the MP and do not have a natural ‘vorticity’ mechanism, such ‘spiral waves’ as $t \rightarrow T^-$ must be generated by large enough initial data specially ‘rotationally’ distributed in $\mathbb{R}^2$. For the biharmonic operator as in (1.1) with no MP, such a swirl blow-up dependence may be more relevant; see below.
Figure 21. Shooting the root $p_0^{(1)}(1)$ for $N = 1$.

4. Type I(Her): non-self-similar ‘linearized’ patterns with a local generalized Hermite polynomial structure

For the classic R–D equation (1.7), a countable set of non-self-similar blow-up patterns of a similar structure was first formally introduced in [98], although the history of such non-self-similar blow-up asymptotics goes back to Hocking–Stuartson–Stuart in 1972 [56]. These authors invented an interesting novel formal technique of analytic expansions (in fact, an analogy of a centre manifold analysis) to confirm that blow-up occurs on subsets governed by the ‘hot spot’ variables, as $t \to T^-$:

$$u(x, t) = (T - t)^{-\frac{1}{p-1}} [f_*(\xi) + o(1)], \quad \text{where } \xi = \frac{x}{\sqrt{(T - t) \ln(T - t)}}$$ (4.1)
and $f_\ast$ is a unique solution of a Hamilton–Jacobi equation of the form (4.15) (with $\frac{1}{2} \mapsto \frac{1}{2}$).

A justified construction of such patterns and other applications were performed a year later by Bressan [4, 5] for the first time (the first generic pattern) and in dozens of papers by Herrero and Velázquez and other authors in the 1990s; see [6, 53, 77, 97] as a guide together with other papers traced by the MathSciNet. It is curious that earlier, in 1987, a sharp upper bound of such a non-similarity blow-up evolution (4.1) ('first half of blow-up') was proved in [42] by a modification of the Friedman–McLeod gradient estimate [23], though the ‘second half of blow-up’ took an extra 10 years to complete along similar lines [24, section 7].

For the RDE-4, it seems that there is no hope to get an easy and fast rigorous justification of such a non-self-similar blow-up scenario, although the main idea remains the same. We follow [25] and also [26], where such a construction is applied to non-singular absorption phenomena (regular flows with no blow-up), so a full mathematical justification is available.
therein. On the other hand, Bressan’s original topological construction of the first stable blow-up pattern (repeating locally the structure of the Hermite polynomial $H_2$) via an ‘isolating blocks’ technique [4, 5] for the Frank-Kamenetskii equation (1.6) looks rather promising, since it does not use somehow essentially typical MP tools. Moreover, in [2], this kind of ‘centre manifold’ (a ‘conditional invariant set’) topological analysis was successfully applied to the quasilinear heat equations

$$u_t = \nabla \cdot (k(u) \nabla u) + Q(u),$$

being a ‘nonlinear perturbation’ of (1.7), where $k(u)$ and $Q(u)/u^p$ are not asymptotically positive constants as $u \to +\infty$. The finite-dimensional $H_2$-like blow-up evolution close to a ‘centre subspace’ was rigorously justified without using classic barrier and MP techniques. This analysis also applied to the perturbed F-K equation (1.6).

Therefore, we expect that such geometric-like techniques can be used for (1.1) on the basis of the known generalized Hermitian spectral theory for the rescaled operator [13] (see the next subsection), but will demand much more complicated mathematics. In addition, the recent matching construction [71] of the first stable blow-up pattern corresponding to the
Table 1. Roots $p_0(N)$ for profiles $f_0(y)$ to get a new zero.

| $N$ | $p_0(N)$ |
|-----|----------|
| 1   | 1.7358... |
| 2   | 1.53...   |
| 3   | 1.446...  |
| 4   | 1.377...  |
| 5   | 1.320...  |
| 6   | 1.28...   |
| 7   | 1.25...   |
| 8   | 1.226...  |
| 9   | 1.204...  |
| 10  | 1.188...  |
| 11  | 1.16...   |

Hermite polynomial $H_2$ for the Ginzburg–Landau equation (1.20) also inspires some optimism concerning technique extensions to higher order PDEs.

However, we do not intend here to prove the existence of such blow-up patterns and concentrate on the formal general principles of their formation, which is also not that straightforward.

4.1. Linearization and spectral properties

The construction of such blow-up patterns is as follows. Performing the standard linearization about the constant equilibrium in equation (3.2) yields the following perturbed equation:

$$v = f_s + Y, \quad f_s = (p - 1)^{-1} \implies Y_1 = (B^* + I)Y + D(Y),$$

where $D(Y) = c_0 Y^2 + \cdots, c_0 = \frac{2}{p - 1} \frac{1}{1} \text{ is a quadratic perturbation as } Y \to 0$ and

$$B^* = -\Delta^2 - \frac{1}{4} y \cdot \nabla \text{ in } L^2(\mathbb{R}^N), \quad \rho^*(y) = e^{-a|y|^{1/\nu}}, \quad a \in (0, 3 \cdot 2^{-\frac{1}{3}}),$$

is the adjoint Hermite operator with some good spectral properties [13].
Lemma 4.1. \( B^* : H^4_\rho(X) \to L^2_\rho(Y) \) is a bounded linear operator with the spectrum
\[
\sigma(B^*) = \left\{ \lambda_\beta = -\frac{|\beta|}{4}, |\beta| = 0, 1, 2, \ldots \right\} \quad (\equiv \sigma(B), \ B = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I).
\]

Eigenfunctions \( \psi^\ast_\beta(y) \) are \(|\beta|\)th-order generalized Hermite polynomials
\[
\psi^\ast_\beta(y) = \frac{1}{\sqrt{\beta!}} \left[ y^{\beta} + \sum_{j=1}^{[\beta/4]} \frac{1}{j!} (\Delta)^j y^{\beta} \right], \quad |\beta| = 0, 1, 2, \ldots.
\]
and the subset \( \{ \psi^\ast_\beta \} \) is complete in \( L^2_\rho(Y) \).

As usual, if \( \{ \psi_\beta \} \) is the adjoint basis of eigenfunctions of the adjoint operator
\[
B = -\Delta^2 + \frac{1}{4} y \cdot \nabla + \frac{N}{4} I \quad \text{in} \ L^2_\rho(X), \quad \text{with} \ \rho = \frac{1}{\rho^*},
\]
with the same spectrum (4.4), the bi-orthonormality condition holds in \( L^2(Y) \):
\[
\langle \psi_\mu, \psi^\ast_\nu \rangle = \delta_{\mu\nu} \quad \text{for any} \ \mu, \nu.
\]

4.2. Inner expansion

Thus, in the Inner Region characterized by compact subsets in the similarity variable \( y \), we assume a centre or a stable subspace behaviour as \( \tau \to +\infty \) for the linearized operator \( B^* + I \):
\[
\text{centre : } Y(y, \tau) = a(\tau) \psi^\ast_\beta(y) + w^\perp, \quad \lambda_\beta = -1, |\beta| = 4,
\]
\[
\text{stable : } Y(y, \tau) = -C e^{(\lambda_\beta+1)\tau} \psi^\ast_\beta(y) + w^\perp, \quad \lambda_\beta < -1, |\beta| > 4.
\]

For the centre subspace behaviour in (4.8), substituting the eigenfunctions expansion into equation (4.2) yields the following coefficient:
\[
\dot{a} = a^2 \gamma_0 + \cdots, \quad \text{where} \ \gamma_0 = c_0 \langle (\psi^\ast_\beta)^2, \psi_\beta \rangle \implies a(\tau) = -\frac{1}{\gamma_0} \tau + \cdots.
\]
Note that, for matching purposes, we have to assume that (see details in [25])
\[
\text{if} \ \gamma^\ast_0(0) > 0, \quad \text{then} \ \gamma_0 > 0 \ \text{and} \ C > 0.
\]
Actually, for \( k = 4 \) and \( N = 1 \), it is calculated explicitly that
\[
\gamma_0 = -c_0 136 \sqrt{6} < 0,
\]
so that the centre manifold patterns with the positive eigenfunction
\[
\psi^\ast_4(y) = \frac{1}{\sqrt{24}} (y^4 + 24)
\]
correspond to the solutions that blow up on finite interfaces; see [46, section 3]. A full justification of such a behaviour can be done along the lines of classic invariant manifold theory (see e.g. [70]), but can be very difficult.

Actually, we can construct more general asymptotics by taking an arbitrary linear combination of eigenfunctions from the centre subspace. Overall, the whole variety of such asymptotics is characterized as follows:
\[
Y(y, \tau) = a(\tau) |y|^\beta \chi(\varphi) + \cdots, \quad \text{where} \ a(\tau) = -\begin{cases} \frac{1}{\gamma_0} \tau + \cdots \quad & \text{for} \ |\beta| = 4, \\ C e^{(\lambda_\beta+1)\tau} + \cdots \quad & \text{for} \ |\beta| > 4. \end{cases}
\]
In general, here, \( \chi(\varphi) > 0 \) is an arbitrary smooth function on the sphere \( S^{N-1} \), where its positivity is induced by matching issues to be revealed below.
We follow [25], where it is shown that the asymptotics (4.8) admit matching with the Outer Region, being a Hamilton–Jacobi (H–J) one. More precisely, in the centre case with $|\beta| = 4$, according to (4.8), (4.9), we introduce the outer variable and obtain from (3.2) the following perturbed H–J equation:

$$
\xi = \frac{y}{\tau^{1/4}} \implies v_\tau = -\frac{1}{4} \xi \cdot \nabla v - \frac{1}{p-1} v + |v|^{p-1} v + \frac{1}{\tau} \left( \frac{1}{4} \xi \cdot \nabla v - \Delta^2 v \right). \quad (4.14)
$$

Passing to the limit as $\tau \to +\infty$ in such singularly perturbed PDEs is not easy at all even in the second-order case (see a number of various applications in [45]). Though currently not rigorously (this looks still illusive), we assume stabilization as $\tau \to +\infty$ to the stationary solutions $f(\xi)$ satisfying the unperturbed H–J equation

$$
-\frac{1}{4} \xi \cdot \nabla f - \frac{1}{p-1} f + |f|^{p-1} f = 0 \quad \text{in } \mathbb{R}^N. \quad (4.15)
$$

This is solved via characteristics, where we have to choose the solution satisfying (4.13):

$$
f(\xi) = f_\ast - \frac{1}{\gamma_0} |\xi|^4 \chi(\varphi) + \cdots \quad \text{as } \xi \to 0 \implies f(\xi) = f_\ast (1 + c_\ast |\xi|^4 \chi(\varphi))^{-\frac{1}{p-1}}, \quad (4.16)
$$

where $c_\ast = (1/\gamma_0)(p - 1)^{p/(p-1)}$. Since $\gamma_0 < 0$ according to (4.11), the resulting profile satisfies $f(\xi) \geq f_\ast$ and blows up on the surface $\{c_\ast |\xi|^4 \chi(\varphi) = -1\}$. Note that this actual non-existence of a bounded centre subspace pattern plus the known unstable eigenspace of $B^\ast + I$ in the linearized equation (4.2) somehow reflect the existence of two self-similar solutions $f_0(y)$ and $f_1(y)$ as ‘nonlinear eigenfunctions’; see [46] for relevant arguments.

Thus, these centre subspace patterns are not bounded and should be excluded from consideration. On the other hand, for the 2mth-order PDE (1.17) with odd $m = 3, 5, \ldots$, we have $\gamma_0 > 0$ and then (4.16) can represent standard blow-up patterns [25].

Similarly, for the stable behaviour for $|\beta| > 4$ in (4.8), we use the following change:

$$
\xi = e^{\frac{1}{|\beta|} \tau} y \implies v_\tau = -\frac{1}{|\beta|} \xi \cdot \nabla v - \frac{1}{p-1} v + |v|^{p-1} v - e^{\frac{1}{|\beta|} \tau} \Delta^2 v. \quad (4.17)
$$

Passage to the limit $\tau \to +\infty$ and matching with the Inner Region are analogous and lead to truly existent blow-up patterns for any $m \geq 2$ in (1.17), [25].

Overall, according to matching conditions (4.13) and (4.17), the whole set of possible blow-up patterns of Type II(Her) is composed of a countable set for $|\beta| = 4 (m \text{ odd}), 5, 6, \ldots$ of continuous (uncountable) families induced by smooth functions $\chi$ on $\mathbb{S}^{N-1}$.

5. Type II(sing): linearization about the SSS and matching

The idea of such Type II blow-up patterns for the RDE-2 (1.7) is due to Herrero–Velázquez [55], where a justification of existence was achieved (see [81] for extra details). We apply this method to the RDE-4 (1.1) and, for the same reasons, we are not obliged to concentrate on a proof. Thus, instead of the linearization (4.2) about the constant equilibrium, we perform it about a singular one.

5.1. Singular stationary solution (SSS)

Consider the stationary equation

$$
-\Delta^2 U + |U|^{p-1} U = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (5.1)
$$
The explicit radial SSS has the standard scaling invariant form

\[ U(y) = C_* |y|^{-\mu}, \quad \text{where } \mu = \frac{4}{p-1}, \quad C_* = D^{\frac{1}{p-1}}, \]

and \( D = \mu(\mu + 2)\[(\mu + 1)(\mu + 3) + (N - 1)(N - 5 - 2\mu)\]. \]

It follows that such SSS exists, i.e. \( D > 0 \), in the following parameter ranges:

\[ p > \frac{N}{N-4}, \quad N > 4 \quad \text{or} \quad p < \frac{N+2}{N-2}, \quad N > 2. \]

5.2. Linearization in Inner Region: discrete spectrum by Hardy–Rellich inequality

Thus, we perform linearization in (3.2) about the SSS:

\[ v = U + Y \Rightarrow v_\tau = \hat{B}^* Y + D(Y), \]

where, as usual, \( D(Y) \) is a quadratic perturbation as \( Y \to 0 \) and

\[ \hat{B}^* = H^* - \frac{1}{4} y \cdot \nabla - \frac{1}{p-1} I, \quad H^* = -\Delta + \frac{pD}{|y|^4} I. \]

Similar to lemma 4.1, the operator \( \hat{B}^* \) at infinity admits a proper functional setting in the same metric of \( L^2_{\rho^*} \). However, it is also singular at the origin \( y = 0 \), where its setting depends on the principal part \( H^* \).

**Proposition 5.1.** The symmetric operator \( H^* \) admits a Friedrich’s self-adjoint extension with the domain \( H_0^2(B_1) \), discrete spectrum and compact resolvent in \( L^2_{\rho^*}(B_1) \), where \( B_1 \subset \mathbb{R}^N \) is the unit ball, iff

\[ pD \leq c_H = \frac{[N(N-4)]^2}{16}. \]

**Proof.** Indeed, (5.6) is just a corollary of the classic Hardy–Rellich-type inequality\(^4\)

\[ \frac{[N(N-4)]^2}{16} \int_{B_1} \frac{u^2}{|y|^2} \leq \int_{B_1} |\Delta u|^2 \quad \text{for } u \in H_0^2(B_1), \]

where the constant is sharp. For compact embedding of the corresponding spaces, see Maz’ja [73, p 65, etc.]. \( \square \)

The necessary inequality (5.6) takes the form

\[ G_p(N) \equiv \frac{[N(N-4)]^2}{16} - \frac{4p}{p-1} \left( 2 + \frac{4}{p-1} \right) \times \left[ 1 + \frac{4}{p-1} \right] \left( 3 + \frac{4}{p-1} \right) + (N-1)(N-5 - \frac{8}{p-1}) \] \[ \geq 0 \] \]

and does not admit an easy analytic analysis. In figure 25, numerics show that

(5.8) holds for \( N \geq 24 \) if \( p = 2 \), and \( N \geq 19 \) if \( p = 3 \).

In particular, checking (5.8) at \( p = +\infty \) yields the inequality

\[ G_\infty(N) \equiv \frac{[N(N-4)]^2}{16} - 8[3 + (N-1)(N-5)] > 0. \]

If this is true, then (5.8) holds for all \( p \gg 1 \), so

**Proposition 5.2.** For any \( N \geq 13 \), there exists a \( p_H(N) > 1 \) such that

(5.8) holds for all \( p \geq p_H(N) \),

and hence the operator \( \hat{B}^* \) in (5.5) has a discrete spectrum in \( L^2_{\rho^*}(\mathbb{R}^N) \).

\(^4\) This was derived by Rellich already in 1954; see [49, 102] for further references and full history.
Thus, we assume that, under certain conditions, (5.11) holds and $\sigma(\hat{B}^*) = \{\hat{\lambda}_k\}$ is discrete, with the eigenfunctions $\{\hat{\psi}_k^\beta, |\beta| = k\}$. Furthermore, it is also convenient to assume that the spectrum is (at least partially) real. To justify such an assumption for this non-self-adjoint operator, we rewrite (5.5) in the form

$$\hat{B}^* = B^* + \frac{c}{|y|^4} I - \frac{1}{p - 1} I,$$

where $c = p D$ (5.12)

and $B^*$ is the previous operator (4.3) with the real spectrum shown in lemma 4.1 (actually, this means that $B^*$ admits a natural self-adjoint representation in the space $l^2_{\rho}$ of sequences, where it is also sectorial, [31]). Therefore, the real spectrum of (5.12) can be obtained by branching-perturbation theory (see Kato [61]) from that $\{\lambda_{k\beta} = -(k/4) - 1/(p - 1), k = |\beta| \geq 0\}$ of $B^* - 1/(p - 1)I$ at $c = 0$. Next, the branch must be extended to $c = p D$, which is also
a difficult mathematical problem; see [37, section 6] for some extra details, which are not necessary here in such a formal blow-up analysis.

Thus, we fix a certain exponentially decaying pattern in Inner Region I:

\[ Y(y, \tau) = C e^{\lambda(y) \tau} \hat{\psi}^*_{\beta}(y) + \cdots \quad \text{as } \tau \to +\infty \left( \hat{\lambda}_\beta < 0 \right). \] (5.13)

If there exists \( \hat{\lambda} = 0 \in \sigma(B^*) \), the expansion will mimic that in (4.8) for the centre subspace case. Note that (5.13) includes all the non-radial linearized blow-up patterns.

### 5.4. Matching with Inner Region II close to the origin

In order to match (5.13) with a smooth bounded flow close to \( y = 0 \), which we call Inner Region II, one needs the behaviour of the eigenfunction \( \hat{\psi}^*_{\beta}(y) \) as \( y \to 0 \). To get this, without loss of generality, we assume the radial geometry. Then, the principal operator in the eigenvalue problem

\[ \mathbf{H}^* \hat{\psi}^* + \cdots = \lambda \hat{\psi}^* \quad \text{as } y \to 0 \] (5.14)

yields the following characteristic polynomial (see [29]):

\[ \hat{\psi}^* = |y|^n + \cdots \implies H_r(y) = -\gamma(y - 2)(y + N - 2)(y + N - 4) + c = 0. \] (5.15)

Consider the most interesting critical and extremal case

\[ c \equiv pD = c_{11} = \frac{[N(N - 4)]^2}{16} \implies H_r(y) \equiv -\left[ y + \frac{(N - 4)}{2} \right]^2 \left[ y^2 + (N - 4)y - \frac{N^2}{4} \right]. \] (5.16)

There exists the double root \( \gamma_{1, 2} = -(N - 4)/2 < 0 \), which generates two \( L^2 \)-behaviours:

\[ \hat{\psi}^*_{1}(y) = |y|^{\frac{N + 4}{2}} \ln |y|(1 + o(1)) \quad \text{and} \quad \hat{\psi}^*_{2}(y) = |y|^{-\frac{N + 4}{2}}(1 + o(1)) \quad \text{as } y \to 0. \] (5.17)

Note that \( H_{10}^2 \)-approximations of \( \hat{\psi}^*_{2} \) establish that \( c_{11} \) is the best constant in (5.6). The other two roots of the characteristic equation in (5.16) are

\[ \gamma_{3,4} = \frac{1}{2}[4 - N \mp \sqrt{(N - 4)^2 + N^2}], \] (5.18)

where \( \gamma_3 < \gamma_{1,2} < 0 \) and \( \gamma_4 > 0 \) correspond to \( L^2 \)-solutions. We have

\[ \hat{\psi}^*_{2}(y) = |y|^{\gamma_4}(1 + o(1)) \notin L^2, \] (5.19)

so that in \( L^2 \) the deficiency indices of \( \mathbf{H}^* \) are \( (3, 3) \) and cannot be equal to \( (4, 4) \). Unlike the second-order case, the straightforward conclusion on the discreteness of the spectrum in the case \( (4, 4) \) [82, p 90] does not apply, so Friedrich’s extension of \( \mathbf{H}^* \) is constructed by other arguments [29] and include settings where two most singular behaviour in (5.17) and (5.19) are excluded; this leads to the so-called principal solution with the minimally possible growth at the singular point.

Overall, this gives the following behaviour of the proper eigenfunctions at the origin:

\[ \hat{\psi}^*_{\beta}(y) = -\nu^{\beta} |y|^{-\frac{N + 4}{2}} + \cdots \quad \text{as } y \to 0 \left( \nu^{\beta} > 0 \text{ are normalization constants} \right). \] (5.20)

This allows to detect the rate of blow-up of such patterns by estimating the maximal value of the expansion near the origin:

\[ \nu^{\beta}(y, \tau) = C_\nu|y|^{\frac{N - 4}{2}} - \nu^{\beta} C e^{\lambda^{\beta}(\tau)|y|^{-\frac{N + 4}{2}}} + \cdots \quad \text{as } y \to 0 \text{ and } \tau \to +\infty, \] (5.21)

where we observe the natural condition of matching

\[ \nu^{\beta} C > 0. \] (5.22)
Calculating the absolute maximum in $y$ of the function on the right-hand side of (5.22) (this is a standard and justified trick in some R–D problems; see e.g. [12]) yields an exponential divergence:

$$\|v_{\beta}(\cdot, \tau)\|_\infty = d_{\beta} e^{\rho_{\beta} \tau} + \cdots,$$

where $\rho_{\beta} = \frac{8|\hat{\lambda}_{\beta}|}{(N-4)(p-p_S)} > 0$ ($p > p_S$), (5.23)
where $d_{\beta} > 0$ are some constants. Note that, depending on the spectrum $\{\hat{\lambda}_{\beta} < 0\}$, (5.23) can determine a countable set of various Type II blow-up asymptotics.

Let us define more clearly the necessary matching procedure. In a standard manner, we return to the original rescaled equation (3.2) and perform the rescaling in Region II according to (5.23):

$$v(y, \tau) = e^{\rho_{\beta} \tau} w(\xi, s), \quad \xi = e^{\mu_{\beta} \tau} y, \quad \mu_{\beta} = \frac{(p-1)\rho_{\beta}}{4},$$

(5.24)
Then $w$ solves the following exponentially perturbed uniformly parabolic equation:

$$w_s = -\Delta^2 w + |w|^{p-1} w - \frac{1}{(p-1)\rho_{\beta}} \left[ \left( \frac{1}{4} + \mu_{\beta} \right) \xi \cdot \nabla w + \left( \frac{1}{p-1} + \rho_{\beta} \right) w \right].$$

(5.25)
As above, we arrive at a stabilization problem as $s \to +\infty$ to a bounded stationary solution, which is widely used in blow-up applications (see examples in [45]). In general, once the uniform boundedness of the orbit $\{w(s), s > 0\}$ is established (an open problem), the passage to the limit in (5.25) as $s \to +\infty$ is a standard issue of asymptotic parabolic theory.

Our blow-up patterns correspond to the stabilization uniformly on compact subsets

$$w(\xi, s) \to W(\xi), \quad s \to +\infty,$$

where $-\Delta^2 W + |W|^{p-1} W = 0$,

$$\xi \in \mathbb{R}^N, \quad W(0) = d_{\beta},$$

(5.26)
for all admissible $|\beta| = 0, 1, 2, \ldots$. We next discuss a crucial issue on such a matching.

5.5. Matching: on necessary structure of global bounded stationary solutions

There are two issues associated with the stationary problem (5.26).

1. Firstly and fundamentally, one can see that, bearing in mind the matching of Regions I and II, the bounded stationary solutions $W(\xi)$ defined by (5.26) must be positive and non-oscillatory as $\xi \to \infty$. Otherwise, such a matching with positive SSS $U(\xi)$ is impossible. There exists a definite negative result in the subcritical Sobolev range (there is a diverse literature on this subject that is currently popular nowadays, so we refer to a recent paper [51] as a guide):

a solution $W > 0$ of the problem in (5.26) is non-existent for $p \in (1, p_S)$. (5.27)

Actually, this means that all the entire (i.e. without singularities) solutions of (5.26) are oscillatory, as figure 26 shows for $p = 2, N = 3$. Note that we are restricted by (5.6).

2. Secondly and fortunately, existence of such positive solutions $W(\xi)$ is already well established [48, p 908]:

for $p > p_S$, for any $d_{\beta} > 0$, there exists a unique positive solution $W(\xi)$. (5.28)

Here we exclude the critical case $p = p_S$, where exact positive solutions exist to be used in section 6. As a numerical illustration, figure 27 shows two such results for $N = 13$ (a),
Figure 26. Shooting an oscillatory solution of (5.26) for $p = 2$ and $N = 3$.

where the dotted line denotes the explicit solution for $N = 12$. It is clearly seen that $W(\xi)$ for $N = 13$ lies above this, and so remains positive. In (b), we show the positivity of the solution $W$ for $N = 24$, where by (5.9), the spectrum is guaranteed to be discrete.

3. Of course, the above is not sufficient for matching of Inner Regions I and II to get a blow-up pattern. More importantly, we have the following:

**Proposition 5.3.** The entire solutions $W(\xi)$ of the radial ODE (5.26) are not oscillatory as $y \to +\infty$ about the SSS (5.2) iff (5.11) holds, and then

$$p \geq p_H : \quad W(\xi) \text{ has at most finite intersections with } U(\xi) \text{ on } \xi \in (0, +\infty). \quad (5.29)$$

**Proof.** It suffices to observe that, as customary, the oscillatory behaviour as $y \to +\infty$ is governed by the linearized operator therein, which is (5.5) (the limit (5.31) below justifies the linearization). Hence, in the critical Hardy case, the characteristic polynomial (5.16) has real roots only (actually, all of them, and this is quite a general property [29, 30]). Obviously, the same holds in the subcritical range $pD < c_H$, meaning that $W(\xi)$ is not oscillatory about $U(\xi)$ as $\xi \to +\infty$. Clearly, if $pD > c_H$, (5.15) and (5.16) imply the existence of a proper root $\gamma \in \mathbb{C}$ with a not that large negative real part. □

The condition in (5.29) is optimal: in the complement range $p < p_H$, the entire radial solutions are oscillatory, [18]. Thus, (5.29) implies that, for the present problem

discrete spectrum and non-oscillation occur in the same range $p \geq p_H(N)$. \quad (5.30)

Indeed, this has some natural roots in general spectral theory of ordinary differential operators. For instance, for second-order singular operators, the non-oscillating behaviour at singular endpoints always implies the existence of a self-adjoint extension in $L^2$ with a discrete spectrum; see lemma 3.1.1 in [68, p 74]. For higher-order symmetric operators [82], such a universal conclusion is not that clear, but is easily observed in particular problems related to simpler homogeneous operators for Hardy’s inequalities as in [29, 30].
Checking the positivity of the solutions of (5.26) for $p = 2$: $N = 13$ (a) and $N = 24$ (b).

Proposition 5.3 for $p > p_H(N)$ was proved in [48, p 909], where other important properties of entire solutions $W(\xi)$ of (5.26) have been established. So we do not need to mention them here in detail and will use the following only (see also [11]):

$$\left( \frac{p + 1}{2} \right)^{\frac{1}{p - 1}} W(\xi) \to U(\xi) \quad \text{as } \xi \to +\infty. \quad (5.31)$$

However, a number of problems concerning (5.2) remain open. For instance, proving that (cf Open Problem 3 in [48, p 915] on ordering of the family $\{W(\xi), d_B > 0\}$)

$$\text{for } p \geq p_H(N), \quad W(\xi) \text{ does not intersect } U(\xi). \quad (5.32)$$

Note that in view of inevitable using shooting techniques, the property (5.32) is very difficult to check numerically.
Fortunately, as the standard topology suggests, solving the open problem (5.32) is not necessary for the validity of the matching of Inner Regions I and II, since the non-oscillating of $W(\xi)$ as $\xi \to +\infty$ is in principal demand (one can see that existence of a finite number of intersections cannot spoil the matching). Thus, we conclude that

for $p \geq p_H(N)$, matching of two flows (5.13) and (5.26) is plausible

though an essential extra mathematical work is necessary to prove this (the author still believes that this can be done in a reasonably finite period of time, but its scale can be beyond any expectation).

5.6. On new blow-up similarity solutions in the oscillatory range $p < p_H$

Thus, (5.16) clearly shows that, for $p < p_H$, the solutions $W(\xi)$ are oscillatory about the SSS $U(\xi)$. Such bundle behaviour (as in the second-order case, see [44] and later publications) suggests that in this subcritical Hardy range, there may be a sequence of similarity profiles satisfying (2.18) and exhibiting arbitrary finite oscillations about $U(\xi)$ for sufficiently small radial $\xi > 0$. Such self-similar blow-up profiles concentrated in a neighbourhood of the unstable singular equilibrium $U$ (above $U$, a.a. solutions must blow-up) are expected to be also highly unstable, at least in comparison with the previous profiles $f_0$ and $f_1$ studied in section 2. Therefore, we ignore such new families (possibly countable depending on parameter ranges) of the s-s blow-up.

5.7. On related non-radial blow-up patterns

These can be predicted in various ways. Firstly, one can start with a non-radial SSS solving the elliptic equation (5.1), but surely such ones are unknown. Secondly, under condition (5.6), a non-radial eigenfunction $\psi^*_\beta(y)$ (e.g. corresponding to an 'angular' swirl obtained by angular separation of variables) of $\hat{B}$ can be taken into account. Then matching will assume using non-radial entire solutions of (5.13), which then deserve further difficult study.

6. Type I(LN): non-self-similar blow-up evolution on a manifold of generalized Loewner–Nirenberg stationary solutions

6.1. Classic Loewner–Nirenberg (L–N) conformally invariant exact solutions

These are classic solutions obtained in Loewner–Nirenberg [69], 1974, for the second-order elliptic equation with the critical Sobolev exponent

$$\Delta W + W^p = 0 \quad \text{in } \mathbb{R}^N, \quad W(0) = d > 0, \quad \text{for } p = p_S = \frac{N + 2}{N - 2} (N > 2),$$

which are invariant under conformal and projective transformations. The corresponding symmetries of (6.1) were earlier detected by Ibragimov in 1968 [57]. These solutions are given by

$$W_0(\xi) = d \left[ \frac{N(N - 2)}{N(N - 2) + d^{4/(N-2)}|\xi|^2} \right]^{\frac{2}{N-2}} > 0 \quad \text{in } \mathbb{R}^N$$

and exhibit a number of uniqueness and other exceptional properties of equation (6.1).
6.2. Generalized L–N solutions for the biharmonic equation

For the critical biharmonic counterpart of (6.1)

\[-\Delta^2 W + |W|^{p-1} W = 0 \quad \text{in } \mathbb{R}^N, \quad W(0) = d > 0; \quad p = p_S = \frac{N + 4}{N - 4} (N > 4), \]  

the corresponding exact solutions are known from the 1980s at least, which we call the generalized L–N ones:

\[ W_0(\xi) = d \left[ \frac{\sqrt{(N + 2)(N^2 - 4)(N - 4)}}{\sqrt{(N + 2)(N^2 - 4)(N - 4) + d^4/(N - 4)}} \right]^{\frac{N-4}{2}} > 0 \quad \text{in } \mathbb{R}^N. \]  

The earliest references to the exact expressions (6.4) we have found are [39, p 1057] in 1985 and [84, 94] in 1992, where in the latter important properties of \( W_0 \) have been proved (see also [51] for further references). For the 2\( m \)-th order polyharmonic extension, the corresponding positive entire solutions look similar:

\[ (\Delta^2)^m W + |W|^{p-1} W = 0, \quad p = \frac{N + 2m}{N - 2m} \Rightarrow W_0(\xi) = d \left[ \frac{B}{B + d^4/(N - 2m)} \right]^{\frac{N-2m}{2}} \]  

where \( N > 2m \) and \( B^m = (N + 2(m - 1))/((N - 2(m + 1))!! \). See Svirshchevskii, 1993, [93] (in a preprint, the solutions were published as early as in 1989 [92]), and more related exact solutions of other critical elliptic PDEs (e.g. with the \( p \)-Laplacian) and extra references in [38, section 5].

6.3. Formal construction of Type II(LN) blow-up patterns for \( p = p_S \)

Let \( v(y, \tau) \) be the rescaled solution of (3.2) in, say, radial geometry at the moment. Let us assume that \( v(y, \tau) \) behaves for \( \tau \gg 1 \) being close to the stationary manifold composed of the explicit equilibria (6.4), i.e. for some unknown function \( \varphi(\tau) \to +\infty \) as \( \tau \to +\infty \)

\[ v(y, \tau) = \varphi(\tau) W_0 \left( \frac{\varphi^{-1}(\tau)}{N} y \right) + \cdots \]  

on the corresponding shrinking compact subsets in the new variable \( \xi = \varphi^{-1}(\tau)y \). It then follows that, on the solutions (6.4) in terms of the original rescaled variable \( y \) (cf computations in [20, p 2963]; our notations have been slightly changed)

\[ |v(y, \tau)|^p v(y, \tau) \to \frac{e_N}{\varphi(\tau)} \delta(y) \quad \text{as } \tau \to +\infty \]  

in the sense of distributions, where \( e_N > 0 \) is some constant. Therefore, on this manifold of solutions, the rescaled equation (3.2) takes asymptotically the form

\[ v_\tau = A(v) = -\Delta^2 v - \frac{1}{4} y \cdot \nabla v - \frac{N - 4}{8} v + \frac{e_N}{\varphi(\tau)} \delta(y) + \cdots \]  

for \( \tau \gg 1 \). (6.7)

According to lemma 4.1, we are looking for Type II patterns of the form

\[ v_\beta(y, \tau) = c_\beta(\tau) \varphi_\beta^\delta(y) + \cdots \Rightarrow \dot{c}_\beta = -\alpha_\beta c_\beta + h_\beta \frac{1}{\varphi(\tau)} + \cdots, \]  

where \( \alpha_\beta = (2|\beta| + N - 4)/8 > 0 \) and \( h_\beta = e_N \varphi(\tau) \). Simple particular ‘resonance’ solutions correspond to the exponential functions

\[ \varphi(\tau) = e^{|\beta| \tau} + \cdots \quad \text{and} \quad c_\beta(\tau) = h_\beta \tau e^{-|\beta| \tau} + \cdots \]  

for \( \tau \gg 1, \quad |\beta| > 0 \). (6.9)

Bearing in mind the scaling in (6.5), this yields a countable family of distinct complicated blow-up structures, where most of them are not radially symmetric. To reveal the actual space–time and changing sign structures of such Type II patterns, special matching procedures apply.
In [20], this analysis has been performed in the radial geometry for (1.7), although still no rigorous justification of the existence of such blow-up scenarios is available. Thus, the first Fourier coefficient in (6.8) implies a complicated structure of the pattern around the formed Dirac’s $\delta(y)$ according to (6.6). However, since these expansions are given by the generalized Hermite polynomials $\{\psi_\beta\}$, this matching is expected not to impose more difficulties as those similar in section 4. In any case, more matching details for the much harder PDE (1.1) then seem excessive here.

7. A final conclusion on blow-up PDE issues

Thus, we have listed several types of blow-up singularities that may (or may not) occur for the RDE-4 (1.1). Explaining the general principles of their formation, we have omitted any attempt to prove their existence or non-existence, just solely mentioning that this is expected to be very difficult and can take a really long time.

Overall, we would again like to mention that the RED-4 (1.1) can be considered as a proper training ground in attempting to guess generic or non-generic blow-up patterns for other, more complicated nonlinear PDEs (see appendix A as a guide to these existence-non-existence and uniqueness–non-uniqueness issues of general PDE theory), including, in particular, the 3D NSEs (1.18) in order to eventually settle this amazing open Millennium Problem. It seems that these first and better developed types of blow-up, which are well known in R–D theory for decades (though remaining very difficult still mathematically), should be the first ones to be taken into account before proposing new and more involved blow-up scenarios for specific solenoidal vector fields; see a survey in [34] for main references and more detailed discussions.

In this ‘blow-up’ connection, it is worth mentioning that novel singularity and other PDE techniques (including refined gradient system monotonicity properties and blow-up surgery) were crucial in Perel’man’s proof, in 2003, of Poincaré conjecture (another Millennium Prize Problem and the only one solved among other seven) by studying evolution and singularities of the Ricci flow as a parabolic system of three equations. It is remarkable that Tao’s comments on Perel’man’s proof [95] also contain the NSEs regularity problem (the fourth footnote in [95, p 2]), though not in a direct blow-up context, but indeed in a strong PDE one. Overall, these again underline how deeply blow-up singularity theory enters the core of fundamental open problems of mathematics of the twenty-first century, where, most plausibly, only first steps have been taken and on which more attention should be focused, and, it seems, the sooner the better.

Appendix A: On the universality of the open $L^p \Rightarrow L^\infty$ problem in the PDE theory

The Millennium Prize Problem, posed specially for the NSEs (1.18), states the open question on the existence or non-existence of single point $L^\infty$ blow-up. With respect to this context, it is worth mentioning that somehow similar open regularity problems (sometimes not of much lighter significance) occur for many evolution PDEs of various types. We list a few of them, where the difficult open mathematical aspects of global existence and/or blow-up are associated with the following factors:

(i) supercritical Sobolev parameter range of the principal operator (hence, standard or very enhanced embedding-interpolation techniques fail), and, in fact, as a corollary,
(ii) multi-dimensional space $x \in \mathbb{R}^N$, with $N \geq 3$, at least (this leaves a lot of room for constructing various $L^\infty$ blow-up patterns via self-similarity, angular swirl, axis precessions, linearization, matching, etc).
The semilinear supercritical wave equations

A.  The author would like to thank I V Kamotski, who first attracted his attention to this problem.

(II) 2mth-order supercritical semilinear heat equation with absorption (m = 1 is covered by the MP; see [10, 47], where the result in section 4 for \( p > p_{S}(2m) \) applies to small solutions only):

\[ u_{t} = -(-\Delta)^{m}u - |u|^{p-1}u, \quad \text{with } p > p_{S}(2m) = \frac{N + 2m}{N - 2m} \quad (N > 2m, \ m \geq 2); \]

(A.2)

(III) The semilinear supercritical wave equations (see [58, 100], as most recent guides)

\[ u_{tt} = \Delta u - |u|^{p-1}u, \quad \text{with } p > p_{S}(2) = \frac{N + 2}{N - 2} \quad (N \geq 3). \]  

(A.3)

Possibly, here the Maximum Principle kind of arguments associated with the single Laplacian \( \Delta \) can still play a role; then it is to be replaced by \( -\Delta^{2} \); see below.

One can add to those ‘supercritical’ PDEs some others of a different structure such as the Kuramoto–Sivashinsky equations for \( l = 1, 2, \ldots \) [40]

\[ u_{t} = -(-\Delta)^{2}u + (-\Delta)^{l}u + \frac{1}{p} \sum_{(k)} dx \cdot D_{x}(|u|^{p}), \quad |d| = 1, \]

\[ p > p_{0} = 1 + \frac{2(4l - 1)}{N}. \]  

(A.4)

Here, \( p_{0} \) is not the Sobolev critical exponent, though precisely for \( p > p_{0} \), \( L^{2} \not\supset L^{\infty} \) by blow-up scaling, [40, section 5]. On the other hand, more exotic applied models exhibit similar fundamental difficulties such as the following nonlinear dispersion equation (see [32, 41] for references and some details)

\[ u_{t} = -D_{x}((-\Delta)^{m}u) - D_{x}(|u|^{p-1}u), \quad \text{with } p > p_{S}(2m) = \frac{N + 2m}{N - 2m}. \]  

(A.5)

In view of the conservation properties for models (A.4) and (A.5), these, though being local, can be more adequate to the non-local NSEs (1.18), than the others above.

In most of the cases, the operator on the right-hand sides satisfying for \( u \in C_{0}^{\infty}(\mathbb{R}^{N}) \)

\[ A(u) = -(-\Delta)^{m}u - |u|^{p-1}u \implies \langle A(u), u \rangle = -\int |D^{m}u|^{2} - \int |u|^{p+1} \leq 0, \]  

(A.6)

is indeed coercive and monotone in the metric of \( L^{2}(\mathbb{R}^{N}) \), which always helps for global existence–uniqueness of sufficiently smooth solutions of these evolution PDEs. For the NSE (A.1), this gives a stronger conservation law than for the focusing equation with the ‘source-like’ term \( = |u|^{p-1}u \). Evidently, replacing \( \Delta \) in (A.1) and (A.3) by \( -(-\Delta)^{m}, \ m \geq 2 \) moves the supercritical range to that in (A.2). On the other hand, introducing quasilinear differential operators \( -(-\Delta)^{m}|u|^{\sigma}u \) with \( \sigma > 0 \) moves the critical exponent to \( p_{S}(2m, \ \sigma) = (\sigma + 1)(N + 2m)/(N - 2m) \). Similar supercritical PDEs can contain 2mth-order \( p \)-Laplacian operators, such as the one for \( m = 2, \ \sigma > 0 \):

\[ A(u) = -\Delta(|u|^{\sigma}u) - |u|^{p-1}u, \quad \langle A(u), u \rangle = -\int |\Delta u|^{\sigma+2} - \int |u|^{p+1} \leq 0. \]  

(A.7)

\(^{5}\) The author would like to thank I V Kamotski, who first attracted his attention to this problem.
However, the lack of embedding-interpolation techniques to get $L^\infty$-bounds, which can be expressed as the lack of compact Sobolev embedding of the corresponding spaces for bounded domains $\Omega \subset \mathbb{R}^N$ (this analogy is not straightforward and is used as a certain illustration only)

$$H^m(\Omega) \not\subset L^{p+1}(\Omega) \quad \text{for} \quad p > ps(2m),$$

actually presents the core of the problem: it is not clear how and when bounded solutions can attain in a finite blow-up time a ‘singular blow-up component’ in $L^\infty$. For the operator in (A.7), a similar supercritical demand reads as

$$W^{2,\sigma}_\sigma(\Omega) \not\subset L^{p+1}(\Omega) \quad \text{for} \quad p > p_S(4, \sigma) = \frac{(\sigma + 1)N + 2(\sigma + 2)}{N - 2(\sigma + 2)}, \quad N > 2(\sigma + 2).$$

In the given supercritical Sobolev ranges, finite mass/energy blow-up patterns for (A.1)–(A.5) are unknown, as well as global existence of arbitrary (non-small) solutions. It is curious that for the NSEs with the same absorption mechanism as above,

$$u_t + (u \cdot \nabla)u = -\nabla p + \Delta u - |u|^{p-1}u, \quad \text{div} \ u = 0 \quad \text{in} \ \mathbb{R}^3 \times \mathbb{R}^+, \quad \text{(A.10)}$$

by the same reasons and similar to (A.2), the global existence of smooth solutions is guaranteed [8] in the subcritical Sobolev range only: for

$$p \leq 5 = \frac{N + 2}{N - 2} \quad \text{for} \quad p \geq \frac{7}{2} \quad \text{by another natural reason}$$

We thus claim that, even for the PDEs with local nonlinearities (A.1)–(A.3) and similar higher order others, the study of the admissible types of possible blow-up patterns can represent an important and constructive problem, with the results that can be key also for the non-local parabolic flows such as (1.18), (A.10), etc. Moreover, it seems reasonable first to clarify the blow-up origins in some of the local supercritical PDEs, looking similar and simpler (hopefully, since (1.18) is both non-local and vector-valued unlike the others) and next to extend the approaches to the non-local NSEs (1.18). Some of these problems, though, are not that attractive and, unfortunately, are not related to the ‘millennium’ issues. However, the PDE experts clearly and very well recognize how important these are for general PDE theory.

References

[1] Bakirova M I, Dimova S N, Dorodnitsyn V A, Kurdyumov S P, Samarskii A A and Svirshchevski S R 1988 Invariant solutions of the heat equation that describe the directed propagation of combustion and spiral waves in a nonlinear medium Sov. Phys.—Dokl. 33 187–9
[2] Bebernes J, Bressan A and Galaktionov V A 1996 On symmetric and nonsymmetric blow-up for a weakly quasilinear heat equation NoDEA 3 269–86
[3] Bebernes J and Eberly D 1989 Mathematical Problems in Combustion Theory (Applied Mathematical Sciences vol 83) (Berlin: Springer)
[4] Bressan A 1990 On the asymptotic shape of blow-up Indiana Univ. Math. J. 39 947–60
[5] Bressan A 1992 Stable blow-up patterns J. Diff. Equat. 98 57–75
[6] Bricmont J and Kupiainen A 1994 Universality in blow-up for nonlinear heat equations Nonlinearity 7 539–75
[7] Budd C J, Galaktionov V A and Williams J F 2004 Self-similar blow-up in higher-order semilinear parabolic equations SIAM J. Appl. Math. 64 1775–809
[8] Cai X and Jiu Q 2008 Weak and strong solutions for the incompressible Navier–Stokes equations with damping J. Math. Anal. Appl. 343 799–809
[9] Chapman C J and Proctor M R E 1980 Nonlinear Rayleigh–Benard convection between poorly conducting boundaries J. Fluid Mech. 101 759–82
[10] Chaves M and Galaktionov V A 2008 $L^\infty$ and decay estimates for higher-order semilinear diffusion-absorption equations J. Math. Anal. Appl. 341 575–87
[11] Dalmasso R 1991 Positive entine solutions of superlinear biharmonic equations Funkcial. Ekvac. 34 403–22
Five types of blow-up in a semilinear fourth-order reaction–diffusion equation

[12] Dold J W, Galaktionov V A, Lacey A A and Vazquez J L 1998 Rate of approach to a singular steady state in quasilinear reaction–diffusion equations *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 24 663–87

[13] Egorov Yu V, Galaktionov V A, Kondratiev V A and Pohozaev S I 2004 Global solutions of higher-order semilinear parabolic equations in the supercritical range *Adv. Diff. Eqns* 9 1009–38

[15] Evans J D, Galaktionov V A and Williams J F 2006 Blow-up and global asymptotics of the limit unstable Cahn–Hilliard equation *SIAM J. Math. Anal.* 38 64–102

[16] Fefferman C 2004 Existence & Smoothness of the Navier–Stokes Equation (The Clay Mathematics Institute) http://www.esi2.us.es/~mbilbao/claymath.htm

[17] Ferrero A and Grunau H-C 2007 The Dirichlet problem for supercritical biharmonic equations with power-type nonlinearity *J. Diff. Eqns* 234 582–606

[18] Ferrero A, Grunau H-C and Karageorgis P 2009 Supercritical biharmonic equations with power-type nonlinearity *Ann. Mat.* 188 171–85 (arXiv:0711.2202v1)

[19] Fila M, Matano H and Poláčik P 2005 Immediate regularization after blow-up *SIAM J. Math. Anal.* 37 752–76

[20] Filippas S, Herrero M A and Velázquez J L 2000 Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity *Proc. R. Soc. Lond. A* 456 2957–82

[24] Galaktionov V A 1999 Dynamical systems of inequalities and nonlinear parabolic equations *Commun. Partial Diff. Eqns* 24 2191–36

[25] Galaktionov V A 2001 On a spectrum of blow-up patterns for a higher-order semilinear parabolic equations *Proc. R. Soc. Lond. A* 457 1–21

[26] Galaktionov V A 2003 Critical global asymptotics in higher-order semilinear parabolic equations *Int. J. Math. Math. Sci.* 60 3809–25

[27] Galaktionov V A 2004 *Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications* (Boca Raton, FL: Chapman&Hall/CRC)

[28] Galaktionov V A 2004 Evolution completeness of separable solutions of non-linear diffusion equations in bounded domains *Math. Methods Appl. Sci.* 27 1755–70

[29] Galaktionov V A 2005 On extensions of Hardy’s inequalities *Commun. Contemp. Math.* 7 97–120

[30] Galaktionov V A 2006 On extensions of higher-order Hardy’s inequalities *Diff. Integr. Eqns* 19 327–44

[31] Galaktionov V A 2007 Sturmian nodal set analysis for higher-order parabolic equations and applications *Adv. Diff. Eqns* 12 669–720

[32] Galaktionov V A 2009 Shock waves and compactons for fifth-order nonlinear dispersion equations *Eur. J. Appl. Math.* submitted (arXiv:0902.1632)

[33] Galaktionov V A 2008 On blow-up space jets for the Navier–Stokes equations in $\mathbb{R}^3$ with convergence to Euler equations *J. Math. Phys.* 49 113101

[34] Galaktionov V A 2009 On blow-up ‘twistors’ for the Navier–Stokes equations in $\mathbb{R}^3$: a view from reaction–diffusion theory arXiv:0901.4286

[35] Galaktionov V A 2009 *Incomplete self-similar blow-up in a semilinear fourth-order reaction–diffusion equation* arXiv:0902.1090

[36] Galaktionov V A and Harwin P J 2005 Non-uniquness and global similarity solutions for a higher-order semilinear parabolic equation *Nonlinearity* 18 717–46

[37] Galaktionov V A and Kamotski I V 2009 On nonexistence of Baras–Goldstein type for higher-order parabolic equations with singular potentials *Trans. Am. Math. Soc.* at press (arXiv:0901.4171)

[38] Galaktionov V A and King J R 2003 Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents *J. Diff. Eqns* 189 199–233

[39] Galaktionov V A, Kurdyumov S P and Samarskii A A 1985 A parabolic system of quasilinear equations: II. *Diff. Eqns* 21 1049–62

[40] Galaktionov V A, Mitidieri E and Pohozaev S I 2009 On global solutions and blow-up for Kuramoto–Sivashinsky-type models and well-posed Burnett equations *Nonlinear Anal.* 70 2930–52

[41] Galaktionov V A and Pohozaev S I 2008 Third-order nonlinear dispersive equations: shocks, rarefaction, and blow-up waves *Comput. Math. Phys.* 48 1784–810 (arXiv:0902.0253)
[42] Galaktionov V A and Posashkov S A 1987 Estimates of localized unbounded solutions of quasilinear parabolic equations Diff. Eqns 23 1133–43
[43] Galaktionov V A and Svirshchevski i S R 2007 Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics (Boca Raton, FL: Chapman&Hall/CRC)
[44] Galaktionov V A and Vazquez J L 1997 Continuation of blow-up solutions of nonlinear heat equations in several space dimensions Commun. Pure Appl. Math. 50 1–68
[45] Galaktionov V A and Vazquez J L 2004 A Stability Technique for Evolution Partial Differential Equations a Dynamical Systems Approach (Progress Nonlinear Differential Equations and their Applications vol 56) (Boston/Berlin: Birkhäuser)
[46] Galaktionov V A and Williams J F 2003 Blow-up in a fourth-order semilinear parabolic equation from explosion-convection theory Eur. J. Appl. Math. 14 745–64
[47] Galaktionov V A and Williams J F 2004 On very singular similarity solutions of a higher-order semilinear parabolic equation Nonlinearity 17 1075–99
[48] Gazzola F and Grunau H-C 2006 Radial entire solutions for supercritical biharmonic equations Math. Ann. 334 905–36
[49] Gazzola F, Grunau H-C and Mitidieri E 2004 Hardy inequalities with optimal constants and remainder terms Trans. Am. Math. Soc. 356 2149–68
[50] Gertsberg V L and Sivashinsky G I 1981 Large cells in nonlinear Rayleigh–Benard convection Prog. Theor. Phys. 66 1219–29
[51] Guo Y and Liu J 2008 Liouville-type theorems for polyharmonic equations in $\mathbb{R}^N$ and in $\mathbb{R}^N_+$ Proc. R. Soc. Edinb. A 138 339–59
[52] Hamilton R 1995 The Formation of Singularities in the Ricci Flow (Surveys in Differential Geometry vol II) (Cambridge, MA: International Press) pp 7–136
[53] Herrero M A and Velázquez J J L 1993 Blow-up behaviour of one-dimensional semilinear parabolic equations Ann. Inst. Henry Poincaré 10 131–89
[54] Herrero M A, Ughi M and Velázquez J J L 2004 Approaching a vertex in a shrinking domain under a nonlinear flow NoDEA 11 1–28
[55] Herrero M A and Velázquez J J L 1994 Blow-up of solutions of supercritical semilinear parabolic equations C.R. Acad. Sci. Paris Sér. I Math. 319 141–5
[56] Hocking L M, Stewartson K and Stuart J T 1972 A nonlinear instability burst in plane parallel flow J. Fluid Mech. 51 705–35
[57] Ibragimov N H 1968 On the group classification of differential equations of second order Sov. Math.—Dokl. 9 1365–9
[58] Ikeda R and Inoue Y 2008 Total energy decay for semilinear wave equations with a critical potential type of damping Nonlinear Anal. 69 1396–401
[59] Joseph D D and Lundgren T S 1973 Quasilinear Dirichlet problems driven by positive sources Arch. Ration. Mech. Anal. 49 241–69
[60] Joulin G, Mikishev A B and Sivashinsky G I A Semenov–Rayleigh–Benard problem Preprint
[61] Kato T 1976 Perturbation Theory for Linear Operators (Berlin/New York: Springer)
[62] Kurdyumov S P 1990 Evolution and self-organization laws in complex systems Int. J. Mod. Phys. C1 299–327
[63] Kurdyumov S P, Kurkina E S, Potapov A B and Samarskii A A 1984 The architecture of multidimensional thermal structures Sov. Phys.—Dokl. 29 106–8
[64] Lepin L A 1988 A countable spectrum of eigenfunctions of a nonlinear heat-conduction equation with distributed parameters Diff. Eqns 24 799–805
[65] Lepin L A 1990 Self-similar solutions of a semilinear heat equation Mat. Modelirov. 2 63–74
[66] Leray J 1933 Sur le mouvement d’un liquide visqueux emplissant l’espace C. R. Acad. Sci. Paris 196 527
[67] Leray J 1934 Sur le mouvement d’un liquide visqueux emplissant l’espace Acta Math. 63 193–248
[68] Lewitt B M and Sargsjan I S 1975 Introduction to Spectral Theory: Self-Adjoint Ordinary Differential Operators (Transl. Math. Mon. vol 39) (Providence, RI: American Mathematical Sociey)
[69] Loewner C and Nirenberg L 1974 Partial differential equations invariant under conformal or projective transformations Contributions to Analysis (New York: Academic) pp 245–72
[70] Lunardi A 1995 Analytic Semigroups and Optimal Regularity in Parabolic Problems (Basel/Berlin: Birkhäuser)
[71] Masmoudi N and Zaag H 2008 Blow-up profile for the complex Ginzburg–Landau equation J. Funct. Anal. 255 1613–66
[72] Matano H and Merle F 2004 On nonexistence of type II blow-up for a supercritical nonlinear heat equation Commun. Pure Appl. Math. LVII 1494–541
[73] Maz’ja V 1985 Sobolev Spaces (Berlin/Tokyo: Springer)
Five types of blow-up in a semilinear fourth-order reaction–diffusion equation

[74] Merle F and Raphael P 2005 On a sharp lower bound on the blow-up rate for the $L^2$ critical nonlinear Schrödinger equation. J. Am. Math. Soc. 19 37–90

[75] Merle F and Raphael P 2005 The blow-up dynamics and upper bound on the blow-up rate for critical nonlinear Schrödinger equation Ann. Math. 161 157–222

[76] Merle F and Raphael P 2005 Profiles and quantization of the blow-up mass for critical nonlinear Schrödinger equation Commun. Math. Phys. 253 675–704

[77] Merle F and Zaag H 1997 Stability of blow-up profiles for equations of the type $u_t = \Delta u + |u|^{p-1}u$ Duke Math. J. 86 143–95

[78] Mitidieri E and Pohozaev S I 2001 Apriori Estimates and Blow-up of Solutions to Nonlinear Partial Differential Equations and Inequalities (Proceedings of the Steklov Institute of Mathematics vol 234) (Moscow: International Academic Publishing Company Nauka/Interperiodica)

[79] Mizoguchi N 2004 Blow-up behaviour of solutions for a semilinear heat equation with supercritical nonlinearity J. Diff. Eqns 205 298–328

[80] Mizoguchi N 2006 Multiple blow-up of solutions for a semilinear heat equation: II J. Diff. Eqns 231 182–94

[81] Mizoguchi N 2007 Rate of Type II blowup for a semilinear heat equation Math. Ann. 339 839–77

[82] Nair M A 1967 Linear Differential Operators Part 1 (New York: Frederick Ungar Publishing Company)

[83] Nečas J, Ružiˇcka M and ˇSver´ak V 1996 On Leray’s self-similar solutions of the Navier–Stokes equations Acta Math. 176 283–94

[84] Nussair E S, Swanson C A and Yang J F 1992 Critical semilinear biharmonic equations in $\mathbb{R}^N$ Proc. R. Soc. Edinb. A 121 139–48

[85] Ovsiannikov L V 1959 Group properties of a nonlinear heat equation Dokl. Akad. Nauk SSSR 125 492–5

[86] Pa o C V 1992 Nonlinear Parabolic and Elliptic Equations (New York: Plenum)

[87] Planchon F and Raphaël P 2007 Existence and stability of the log–log blow-up dynamics for the $L^2$-critical nonlinear Schrödinger equation in a domain Ann. Henri Poincaré 8 1177–219

[88] Peletier L A and Troy W C 2001 Spatial Patterns. Higher Order Models in Physics and Mechanics (Boston/Berlin: Birkhäuser)

[89] Quittner P and Souplet P 2007 Superlinear Parabolic Problems Blow-up, Global Existence and Steady States (Birkhäuser Advanced Texts: Basler Lehrbücher) (Basel: Birkhäuser)

[90] Samarskii A A, Galaktionov V A, Kurdyumov S P and Mikhailov A P 1995 Blow-up in Quasilinear Parabolic Equations (Berlin/New York: Walter de Gruyter)

[91] Semenov N 1935 Chemical Kinetics and Chain Reaction (Oxford: Clarendon)

[92] Svirshchevskii S R 1989 Symmetry of nonlinear elliptic equations for critical values of the parameter Akad. Nauk SSSP Inst. Prikl. Mat. (Preprint) 118 16pp (see also MathSciNet)

[93] Svirshchevskii S R 1993 Group classification and invariant solutions of nonlinear polyharmonic equations Diff. Eqns 29 1538–47

[94] Swanson C A 1992 The best Sobolev constant Appl. Anal. 47 227–39

[95] Tao T 2006 Perelman’s proof of the Poincaré conjecture: a nonlinear PDE perspective arXiv:math/0610903v1[math.DG]

[96] Tao T 2007 What is good mathematics? Bull. Am. Math. Soc. (N.S.) 44 623–34

[97] Velázquez J J L 1993 Estimates on $(N – 1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation Indiana Univ. Math. J. 42 445–76

[98] Velázquez J J L, Galaktionov V A and Herrero M A 1991 The structure of the space near a blow-up point for semilinear heat equations: a formal approach Comput. Math. Math. Phys. 31 46–55

[99] Visan M 2007 The defocusing energy-critical nonlinear Schrödinger equation in higher-dimension Duke Math. J. 138 281–374

[100] Yang L and Zhong C-K 2008 Global attractor for plate equation with nonlinear damping Nonlinear Anal. 69 3802–10

[101] Zel’doˇvich Ya B, Barenblatt G I, Librovich J V and Makhviladze G M 1985 The Mathematical Theory of Combustion and Explosions (New York: Consultants Bureau [Plenum])

[102] Yafaev D 1999 Sharp constants in the Hardy–Rellich inequalities J. Funct. Anal. 168 121–44