ISOMORPHISM BETWEEN ONE-DIMENSIONAL AND MULTIDIMENSIONAL FINITE DIFFERENCE OPERATORS

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(Communicated by Yuri Latushkin)

Abstract. Finite difference operators are widely used for the approximation of continuous ones. It is well known that the analysis of continuous differential operators may strongly depend on their dimensions. We will show that the finite difference operators generate the same algebra, regardless of their dimension.

1. Introduction. There is an obvious difference between linear continuous ordinary differential equations and partial differential systems, both with non-constant periodic coefficients. In general, while ODEs is a part of PDEs formally, often books are written either about ODEs or PDEs, see, e.g., [10, 3]. In the continuous case, there are many reasons for this separation. For example, there is no full analogue of the Picard-Lindelöf theorem even for linear PDEs. We will consider linear differential operators acting on one- and multidimensional tori. We will show that if the continuous derivatives are replaced by their discrete analogues, then the corresponding one- and multidimensional differential operators will generate the same algebra, namely, the universal uniformly hyperfine (UHF) $C^*$-algebra. In this sense, topological and algebraic properties of algebras of one- and multidimensional finite difference (FD) operators are identical. Moreover, we will show that for any $N, M \in \mathbb{N}$ there is a unitary isomorphism $\mathcal{U}_{N,M}$ between the spaces of one-dimensional scalar functions and $N$-dimensional $M$-vector-valued functions, which makes the algebra $\mathcal{H}_{1,1}$ of one-dimensional scalar FD operators and the algebra $\mathcal{H}_{N,M}$ of $N$-dimensional $M \times M$-matrix-valued FD operators similar $\mathcal{H}_{1,1} = \mathcal{U}_{N,M}^{-1} \mathcal{H}_{N,M} \mathcal{U}_{N,M}$. The similarity may allow the development of unified methods and tools for the global analysis of both: one- and multidimensional FD operators. Algebras of discrete and continuous differential operators have numerous applications including design of symbolic and numerical solvers of various ODEs and PDEs, see, e.g., [9, 5, 1, 6]. Most of the articles and monographs devoted to FD operators study their numerical aspects. Abstract algebraic properties of some FD operators are often studied in the context of so-called rotation algebras, see, e.g.,

2020 Mathematics Subject Classification. Primary: 35P99, 16G99; Secondary: 39A05, 39A14.

Key words and phrases. Representation of finite difference operators, UHF algebras, ODE and PDE.

This paper is a contribution to the project M3 of the Collaborative Research Centre TRR 181 “Energy Transfer in Atmosphere and Ocean” funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 274762653. This work is also supported by the RFBR (RFFI) grant No. 19-01-00094.
functions of the form (1).

where $P \in \mathbb{N}$, $S_n \in \mathbb{C}^{M \times M}$, and $\chi_{J_n}$ is the characteristic function of the parallelepiped $J_n = \prod_{i=1}^{N} [p_{in}, q_{in})$ with rational end points $p_{in}, q_{in} \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$. In particular, continuous matrix-valued functions belong to $R_{N,M}^\infty$ because they can be uniformly approximated by the functions of the form (1).

For $S \in R_{N,M}^\infty$, the operator of multiplication by the function $M_S : L_2^N \to L_2^{N,M}$ is defined by

$$M_S u(x) = S(x)u(x), \quad u(x) \in L_2^{N,M}, \quad x \in \mathbb{T}^N. \quad (2)$$

For $i \in \mathbb{N} = \{1, ..., N\}$, $h \in \mathbb{Q}$, the elementary finite difference operator $D_{i,h} : L_2^{N,M} \to L_2^{N,M}$ is defined by

$$D_{i,h}u(x) = \frac{u(x + he_i) - u(x)}{h}, \quad u(x) \in L_2^{N,M}, \quad x \in \mathbb{T}^N, \quad (3)$$

where the standard basis vector $e_i = (\delta_{ij})_{j=1}^N$, and $\delta_{ij}$ is the Kronecker delta. Recall that the addition on the torus is assumed by modulo 1. The finite difference operators with bounded coefficients have the usual form

$$A u = \sum_{n=1}^{P_b} \left( \prod_{j=1}^{P_b} M_{jn}D_{jn} \right) u + M_{00} u, \quad u \in L_2^{N,M}, \quad (4)$$

where $P, P_b \in \mathbb{N}$, and $M_{jn}, D_{jn}$ are some operators of the form (2), (3) respectively. The $C^*$-algebra $\mathcal{H}_{N,M}$ of finite difference operators with bounded coefficients is generated by all the possible operators $A$ given by (4), i.e.

$$\mathcal{H}_{N,M} = \prod_{i \in \mathbb{N}}^{\mathbb{B}_{N,M}} \{ D_{i,h}, M_S : i \in \mathbb{N}, \quad h \in \mathbb{Q}, \quad S \in R_{N,M}^\infty \}, \quad (5)$$

where $\mathbb{B}_{N,M} = \mathbb{B}(L_2^{N,M})$ is the $C^*$-algebra of bounded operators acting on $L_2^{N,M}$. It is seen that $\mathcal{H}_{N,M}$ is a unital $C^*$-subalgebra in $\mathbb{B}_{N,M}$. Also, $\mathcal{H}_{N,M}$ is a minimal $C^*$-algebra that contains all the operators listed in (2) and (3).

Let us recall a definition of the universal UHF $C^*$-algebra $\mathcal{U}$. One of this definitions is based on the inductive limit

$$\mathbb{C}^{1 \times 1} \xrightarrow{\varphi_1} \mathbb{C}^{2 \times 2} \xrightarrow{\varphi_2} \mathbb{C}^{6 \times 6} ... \xrightarrow{\varphi_{n-1}} \mathbb{C}^{n! \times n!} ... \to \mathcal{U},$$

where $\varphi_n$ are unital *-embeddings, or, formally,

$$\mathcal{U} = \mathbb{C}^{1 \times 1} \otimes \mathbb{C}^{2 \times 2} \otimes \mathbb{C}^{3 \times 3} \otimes ... \mathbb{C}^{n \times n} \otimes ...$$

The corresponding supernatural number $\Omega_{\mathcal{U}} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot ... = p_1^{\infty} p_2^{\infty} p_3^{\infty} \cdot ...$ contains all the prime numbers $p_1 = 2, \ p_2 = 3, \ p_3 = 5, \ ...$ infinitely many times. Any UHF algebra is a subalgebra of $\mathcal{U}$ since any supernatural number divides $\Omega_{\mathcal{U}}$. Recall
that there is one-to-one correspondence between UHF algebras and supernatural numbers, see, e.g., [4, 2, 8]. Let us formulate our main result.

**Theorem 1.1.** For any \( N, M \in \mathbb{N} \), the \( C^* \)-algebra \( \mathcal{H}_{N,M} \) is \(*\)-isomorphic to \( \mathcal{U} \). Moreover, there is a unitary operator \( \mathcal{U}_{N,M} : L^2_{1,1} \to L^2_{N,M} \) such that \( \mathcal{H}_{1,1} = \mathcal{U}^{-1}_{N,M} \mathcal{H}_{N,M} \mathcal{U}_{N,M} \).

Note that by Theorem 1.1 there is a unitary spatial conjugation between any \( \mathcal{H}_{N,M} \) and \( \mathcal{H}_{N_1,M_1} \):

\[
\mathcal{H}_{N,M} = (\mathcal{U}_{N,M} \mathcal{U}_{N_1,M_1}^{-1}) \mathcal{H}_{N_1,M_1} (\mathcal{U}_{N,M} \mathcal{U}_{N_1,M_1}^{-1})^{-1}.
\]

This means that there is no difference between \( \mathcal{H}_{N,M} \) for different \( N, M \). For example, if \( \mathcal{A} \in \mathcal{H}_{N,M} \) then there is \( \mathcal{B} \in \mathcal{H}_{N_1,M_1} \) with the same spectrum \( \text{sp}_{\mathcal{H}_{N,M}}(\mathcal{A}) = \text{sp}_{\mathcal{H}_{N_1,M_1}}(\mathcal{B}) \), and the \( C^* \)-algebras generated by \( \mathcal{A} \) and \( \mathcal{B} \) are \(*\)-isomorphic

\[
\mathcal{A}_{N,M} \{1, \mathcal{A}, \mathcal{A}^*\} \cong \mathcal{A}_{N_1,M_1} \{1, \mathcal{B}, \mathcal{B}^*\}.
\]

In particular, \( \mathcal{A} \) is invertible if and only if \( \mathcal{B} \) is invertible. Thus, there are no special difficulties in the analysis of partial finite difference operators in comparison with ordinary finite difference ones. Moreover, due to Theorem 1.1 there is a unitary transform \( \mathcal{U} \) between solutions of partial finite difference equations and ordinary finite difference ones: if \( u \) is a solution of finite PDE \( \mathcal{A}u = f \) then \( v = \mathcal{U}^{-1}u \) is a solution of finite ODE \( \mathcal{B}v = g \), where \( \mathcal{B} = \mathcal{U}^{-1} \mathcal{A} \mathcal{U} \) and \( g = \mathcal{U}^{-1}f \), and vice-versa.

It is useful to take into account the following remark. Using

\[
L^2_{N,M} = \bigoplus_{m=1}^{M} (L^2_{1,1})^\otimes N = \mathbb{C}^N \otimes (L^2_{1,1})^\otimes N,
\]

we conclude that

\[
\mathcal{H}_{N,M} \cong \mathbb{C}^{N \times M} \otimes \mathcal{H}_{1,1}^\otimes N \cong \mathcal{H}_{1,1}^\otimes M \otimes \mathcal{U}^\otimes N \cong \mathcal{U} \cong \mathcal{H}_{1,1}.
\]

since \( \mathcal{U} \otimes \mathcal{V} \cong \mathcal{U} \otimes \mathcal{V} \cong \mathcal{V} \otimes \mathcal{U} \) for any UHF algebra \( \mathcal{V} \).

It is already mentioned above that any UHF algebra is a subalgebra of the universal UHF algebra. Hence \( \mathcal{H}_{1,1} \) contains all the UHF algebras as subalgebras. Let us construct instances of these subalgebras explicitly. Let \( \mathfrak{M}_\mathcal{V} = \prod_{n=1}^\infty \mathcal{H}_{p_n^{N_n}} \) be some supernatural number corresponding to the UHF algebra \( \mathcal{V} \). Some of \( N_n \) can be infinite or zero. In addition to the notation \( \mathbb{N}_N = \{1, \ldots, N\} \), we use also \( \mathbb{N}_\infty = \mathbb{N} \) and \( \mathbb{N}_0 = \{0\} \). Then

\[
\mathcal{V} \cong \mathcal{H}_{1,1}(\mathfrak{M}_\mathcal{V}) = \mathcal{A}_{1,1} \{D_{1,h}, \mathcal{M}_S : h = p_n^{-j}, j \in \mathbb{N}_{N_n}, n \in \mathbb{N}, S \in R_{1,1}^\infty(\mathfrak{M}_\mathcal{V})\},
\]

where \( R_{1,1}^\infty(\mathfrak{M}_\mathcal{V}) \) is the \( C^* \)-algebra of scalar one-dimensional regulated functions which can be uniformly approximated by the step functions of the form (1) but with \( p_{1n}, q_{1n} \) equal to \( p_r^{-s} \) for \( r \in \mathbb{N} \) and \( s \in \mathbb{N}_{N_r} \). In particular, the CAR-algebra (canonical anticommutation relations in quantum mechanics), which has the supernatural number \( 2^{\infty} \), can be represented as the differential algebra generated by the dyadic derivatives \( D_{1,2^{-n}}, n \in \mathbb{N} \) and the operators of multiplication by dyadic regulated functions. The proof of (7) is similar to the first part of the proof of Theorem 1.1.

Let \( \mathfrak{N} \) be some supernatural number. For the multidimensional case, it is natural to define

\[
\mathcal{H}_{N,M}(\mathfrak{N}) = \mathcal{C}^{N \times M} \otimes \mathcal{H}_{1,1}(\mathfrak{N})^\otimes N.
\]
see the first identity in (6). Then the corresponding supernatural number is
\[ \mathcal{N}_{\mathcal{H}_{N,M}(\mathfrak{F})} = M \mathcal{N} \]
since \( \mathfrak{F} \otimes \mathfrak{F} = \mathfrak{F} \mathfrak{F} \) for any UHF algebras \( \mathfrak{F} \) and \( \mathfrak{F} \). For example, \( \mathcal{H}_{N,M}(\mathfrak{F}) \) is the CAR-algebra if and only if \( \mathfrak{F} = 2^\infty \) and \( M = 2^m \) for some \( m \in \mathbb{N} \cup \{0\} \).

2. Proof of Theorem 1.1. We fix \( N, M \in \mathbb{N} \) and, for convenience, we will often omit these indices below. Let \( h = 1/p \) for some \( p \in \mathbb{N} \). Denote \( R^h \) the \( C^* \)-subalgebra of \( R^\infty \) consisting of step functions constant on each \( J = \prod_{i=1}^N [hp_i, hp_i + h) \subset T^N \), where \( p_i \in \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, \ldots, p - 1\} \). Note that the abelian group \( (h\mathbb{Z}_p, +) \) is a subgroup of \( (T, +) \). Consider the finite-dimensional \( C^* \)-subalgebra \( \mathcal{H}^h \subset \mathcal{H} \) defined by
\[ \mathcal{H}^h = \text{Alg}\{ \mathcal{D}_{i,h}, \mathcal{M}_S : i \in \mathbb{N}_N, \, S \in R^h \}. \]
It is seen that any operator \( A \in \mathcal{H}^h \) has the form
\[ Au(x) = \sum_{j \in \mathbb{Z}_p^N} A_j(x) u(x + h j), \quad u \in L^2_{N,M}, \quad (8) \]
with some \( A_j(x) \in R^h \). This is because all such \( A \) belongs to \( \mathcal{H}^h \) since \( \mathcal{D}_{i,h} \) is generated by shift operators, and all such \( A \) form an algebra containing \( \mathcal{D}_{i,h} \) for \( i \in \mathbb{N}_N \) and \( \mathcal{M}_S \) for \( S \in R^h \). The next step is to find the convenient representation of \( \mathcal{H}^h \). Using (8), we have
\[ (Au)(y + hr) = \sum_{j \in \mathbb{Z}_p^N} A_j(y + hr) u(y + hj + hr) \]
\[ = \sum_{j \in \mathbb{Z}_p^N} A_{j-r}(y + hr) u(y + hj), \quad y \in I_h = [0, h)^N \subset T^N, \quad r \in \mathbb{Z}_p^N. \quad (9) \]
The Hilbert space \( L^2 \) is naturally isomorphic to the direct sum of Hilbert spaces of functions defined on (shifted) cubes \( I_h \):
\[ L^2 \cong L^2_{h} = \bigoplus_{j \in \mathbb{Z}_p^N} L^2(I_h \to \mathbb{C}^M) \]
with the isomorphism \( \mathcal{F}_h : L^2 \to L^2_{h} \) defined by
\[ u(x) \leftrightarrow (u(y + hj))_{j \in \mathbb{Z}_p^N}, \quad u \in L^2, \quad x \in T^N, \quad y \in I_h. \]
Then, by (9) the operator \( \mathcal{F}_h A \mathcal{F}_h^{-1} : L^2_h \to L^2_h \) is the operator of multiplication by \( M_{p^N} \times M_{p^N} \)-matrix-valued function \( B_A(y), \, y \in I_h \) of the form
\[ B_A(y) = (A_{j-r}(y + hr))_{r,j \in \mathbb{Z}_p^N}. \quad (10) \]
It is seen that
\[ B_{\alpha A + \beta B} = \alpha B_A + \beta B_B, \quad B_{AB} = B_A B_B, \quad B_A^* = B_A^*, \]
Moreover, \( B_A(y) \) is constant for \( y \in I_h \) since all \( A_j \in R^h \). Finally, note that for any \( B \in \mathbb{C}^{M_{p^N} \times M_{p^N}} \) there is a unique \( A \in \mathcal{H}^h \), such that \( B = B_A \). We can explicitly and uniquely recover \( A \) from \( B_A \), using (10) and (9), (8). Thus, \( A \mapsto B_A \) is \( * \)-isomorphism between \( \mathcal{H}^h \) and \( \mathbb{C}^{M_{p^N} \times M_{p^N}} \).

Taking \( h_n = 1/n! \), \( n \in \mathbb{N} \), we can write
\[ \mathcal{H}^h_1 \overset{\psi_1}{\longrightarrow} \mathcal{H}^h_2 \overset{\psi_2}{\longrightarrow} \mathcal{H}^h_3 \overset{\psi_3}{\longrightarrow} \cdots \to \mathcal{H}, \quad (11) \]
where \( \psi_n \) is the natural embedding \( \mathcal{H}^{h_n} \) into \( \mathcal{H}^{h_{n+1}} \). Such embedding exists since the partition of \( T^N \) onto \( ((n+1)!)^N \) identical cubes contains the partition of \( T^N \) onto \( (n!)^N \) identical cubes. The embeddings \( \psi_n \) are unital. The inductive limit in (11) is \( \mathcal{H} \) since any \( D_{i,h} \) with \( h = p/q \in \mathbb{Q} \) \( (p, q \in \mathbb{N}) \) belongs to \( \mathcal{H}^{h_n} \) and \( R^\infty \) can be uniformly approximated by \( R^{h_n} \) following the definition of regulated functions.

Remembering \( \mathcal{H}^{h_n} \cong \mathbb{C}^{M(n!)^N \times M(n!)^N} \), we can conclude that the supernatural number \( N_{\mathcal{H}} = M^2 p_3^3 p_4^5 ... \) of the UHF algebra \( \mathcal{H} \) contains all the prime numbers \( p_1 = 2, p_2 = 3, p_3 = 5, ... \) infinitely many times. Thus, \( \mathcal{H} \) is \( \ast \)-isomorphic to the universal UHF algebra \( \mathcal{Y} \).

**Figure 1.** Two first partitions for the unitary transform \( U_{2,1}^{-1} \) between \( L^2 \) and \( L^2 \) are shown. The characteristic functions of squares and intervals with the same “blue” and “red” numbers are transformed into each other under the action of \( U_{2,1} \).

**Figure 2.** The unitary transform \( U_{2,1}^{-1} \), see Fig. 1, applied to the function \( z(x, y) = 1 + \sin(\pi(x^2 + y^2)) \).

Let us construct the unitary operator \( U : L^2_{1,1} \to L^2_{N,M} \) such that \( \mathcal{H}_{1,1} = U^{-1} \mathcal{N}_{N,M} U \). The matrix units in the representation (10) correspond to the “elementary translators” – operators, that move (shift) values \( u_m \) in one cube \( I_h + ha \) to \( u_m' \) in a cube \( I_h + hb \), setting other values to zero. In the previous sentence, \( a, b \in \mathbb{Z}^N \), and \( 1 \leq m, m' \leq M \) since \( u = (u_m)_{m=1}^M \). It is well known that matrix units generate a full matrix algebra. Hence the elementary translators generate \( C^* \)-algebra \( \mathcal{H}^h \) since \( \mathcal{H}^h \) is isomorphic to the corresponding full matrix algebra. The
unitary $\mathcal{U}$ may be an operator that transforms characteristic functions of intervals to characteristic functions of cubic cells preserving an order, see Fig. 1. This is not a unique class of unitary operators realizing the similarity between $\mathcal{H}_{N,M}$ and $\mathcal{H}_{1,1}$, but, perhaps, one of the simplest ones. The preservation of an order means that $\mathcal{U}^{-1}\mathcal{T}\mathcal{U}$ is an elementary translator between intervals if $\mathcal{T}$ is an elementary translator between cubes. For example, if a red square will move to the position of another red square then the corresponding blue sub-squares will move to the new positions with the same numbers as before moving, see Fig. 1. Since the elementary translators generate $\mathcal{H}^h$, we can state that

$$\mathcal{H}_{1,1}^{M^{-1}h^N} = \mathcal{U}^{-1}\mathcal{H}_{N,M}^h\mathcal{U}, \ \forall n \geq 1,$$

that leads to $\mathcal{H}_{1,1} = \mathcal{U}^{-1}\mathcal{H}_{N,M}\mathcal{U}$, because the inductive limit of $\mathcal{H}_{1,1}^{M^{-1}h^N}$ is $\mathcal{H}_{1,1}$ for the same reasons as in (11). Below, we compute explicitly the $\mathcal{U}$-conjugations of elementary translators or matrix units, and discuss (12) in details.

The unitary operator $\mathcal{U}$ is, in fact, the operator of changing the variable since it transforms characteristic functions of arbitrary small cubes. Let us provide an explicit computation of some class of operators $\mathcal{U}$ following the idea illustrated on Fig. 1. Any number $x \in [0,1)$ can be expanded as follows

$$x = \frac{x_1}{M} + \frac{x_2}{M(2)^N} + \frac{x_3}{M(3)^N} + \ldots, \quad (13)$$

where

$$x_1 \in \{0,\ldots,M-1\}, \quad x_j \in \{0,\ldots,j^N-1\}, \quad j > 1.$$

The coefficients $x_j$ can be found recurrently

$$x_1 = \left\lfloor Mx \right\rfloor, \quad x_2 = \left\lfloor (Mx - x_1)2^N \right\rfloor, \quad x_3 = \left\lfloor (M2^Nx - 2^Nx_1 - x_2)3^N \right\rfloor, \quad \ldots,$$

where $\lfloor \cdot \rfloor$ is the floor function. Now, let

$$\vartheta_j : \{0,\ldots,j^N-1\} \to \{0,\ldots,j-1\}^N \quad (14)$$

be some 1-1 mappings. Mappings $\vartheta_j$ correspond to the numeration of sub-cubes of the “range” $j$, e.g. red (range 2) and blue (range 3) numbers on Fig. 1. They can be arbitrary 1-1 mappings. The key point is that fixing such one numeration $\vartheta_j$ for sub-cubes of one cube of the range $j-1$, it should be the same for sub-cubes of other cubes of the range $j-1$. It is exactly the preservation of order described above. This remark determines completely the structure of the resulting mapping $\vartheta$. Define the mapping $\vartheta : [0,1/M) \to [0,1]^N$ by

$$\vartheta(x) = \vartheta_2(x_2) + \vartheta_3(x_3) + \vartheta_4(x_4) + \ldots \quad (15)$$

The mapping $\vartheta$ is 1-1 mapping up to a set of zero measure. This is a full analogue of the fact that a digital expansion is unique to all the real numbers except some set of zero measure. Some of rational numbers admit simultaneously two digital expansions: finite and infinite. Up to a set of zero measure, the inverse mapping $\mu : [0,1)^N \to [0,M^{-1}]$ can be computed in the same way as a direct mapping since expansions (13) and (15) are similar. Let $x \in [0,1)^N$ and define $x_j = x_j(x)$ by

$$x_2 = \left\lfloor 2x \right\rfloor, \quad x_3 = \left\lfloor (2x - x_2)3 \right\rfloor, \quad x_4 = \left\lfloor (2 \cdot 3 \cdot x - 3x_2 - x_3)4 \right\rfloor, \quad \ldots \quad (16)$$

$$\vartheta_2(x_2) = \left\lfloor 2x \right\rfloor, \quad \vartheta_3(x_3) = \left\lfloor (2x - x_2)3 \right\rfloor, \quad \vartheta_4(x_4) = \left\lfloor (2 \cdot 3 \cdot x - 3x_2 - x_3)4 \right\rfloor, \quad \ldots \quad (16)$$
where the floor function is applied to each of the vector components. Then
\[ \mu(x) = \frac{\vartheta_2^{-1}(x_2)}{M(2!)^N} + \frac{\vartheta_3^{-1}(x_3)}{M(3!)^N} + \frac{\vartheta_4^{-1}(x_4)}{M(4!)^N} + ... \] (17)

Identities
\[ \vartheta(\mu(x)) = x, \quad \mu(\vartheta(x)) = x \] (18)
are true at least for \( x \in \mathbb{T}, x \in \mathbb{T}^N \) such that \( x, \vartheta^{-1}(x), \mu(x) \) are irrational and \( x, \vartheta(x), \mu^{-1}(x) \) have no rational components. The sets of \( x, x \) that may violate (18) have the corresponding Lebesgue measure equal to zero. Indeed, \( \mathbb{Q} \) and \( \vartheta(\mathbb{Q}) \) are countable and hence have a zero Lebesgue measure. Similarly, measure of \( \mu^{-1}(\mathbb{Q}) \) is 0. The set \( S_0 \) of \( x \) that have some rational components has also zero \( \mathbb{N} \)-dimensional Lebesgue measure as a countable union of \( \mathbb{N} - 1 \)-dimensional affine hyperplanes.

Let \( S_1 = \{(x_i)_{i=1}^N : x_{i_0} = a_1 \} \), where \( i_0 \in \{1, ..., N\} \) and \( a \in \mathbb{Q} \) are fixed, be one of such affine hyperplanes. Then the \( i_0 \)-component of \( x \) is the same for all \( x \in S_1 \) and for any fixed \( j \geq 2 \). This means that \( x \) can take at most \( j^{-N-1} \) different values among of \( j^N \) possible ones. Identity (17) yields
\[ \mu(S_1) \subset S_2 = \left\{ x : x = \sum_{j=2}^{\infty} \sum_{j=2}^{\infty} \frac{t_j}{M(j!)^N}, t_j \in A_j \right\} \]
for some \( A_j \subset \{0, ..., j^N - 1\} \) with the number of elements \# \( A_j \leq j^{-N-1} \). Thus, the Lebesgue measure
\[ \text{Leb}_1(S_2) \leq \left( \prod_{j=2}^{\infty} \# A_j \right) \text{Leb}_1(S_n) = (n!)^{N-1}\text{Leb}_1(S_n), \] (19)

where
\[ S_n = \left\{ x : x = \sum_{j=n+1}^{\infty} \frac{t_j}{M(j!)^N}, t_j \in \{0, ..., j^N - 1\} \right\}, \]
for any \( n > 2 \). The minimal element in \( S_n \) is \( x_{\text{min}} = 0 \), the maximal is
\[ x_{\text{max}} = \sum_{j=n+1}^{\infty} \frac{j^N - 1}{M(j!)^N} = \sum_{j=n+1}^{\infty} \frac{1}{M(j!)^N} = \frac{1}{M(n!)^N}. \]

Hence, \( S_n \subset [x_{\text{min}}, x_{\text{max}}] = [0, M^{-1}(n!)^{-N}] \) and \( \text{Leb}_1(S_n) \leq M^{-1}(n!)^{-N} \), which with (19) leads to \( \text{Leb}_1(S_2) \leq (n!)^{-N}M^{-1}(n!)^{-N} = (n!)^{-N} \rightarrow 0 \) for \( n \rightarrow \infty \). Thus, \( \text{Leb}_1(\mu(S_1)) = 0 \) and hence \( \text{Leb}_1(\mu(S_0)) = 0 \). Similarly \( \text{Leb}_1(\vartheta^{-1}(S_0)) = 0 \). Therefore, identity (18) is valid a.e.. Let us agree to formulate the subsequent statements implicitly up to a set of zero measure, if necessary.

The linear operator \( \mathcal{U} : L^2_{1,1} \rightarrow L^2_{N,M} \) is defined by
\[ \mathcal{U}u = \psi = (v_i)_{i=0}^{M-1}, \quad \text{where} \quad v_i(x) = M^{-\frac{i}{2}}u(\mu(x) + \frac{i}{M}), \quad x \in [0,1)^N, \] (20)
or, by (18),
\[ u(x)|_{\left[iM, \frac{i+1}{M}\right)} = M^\frac{i}{2}v_{i+1}\left(\vartheta(x - \frac{i}{M})\right), \quad \text{if} \ x \in \mathbb{Z}_M, \] (21)
for \( u \in L^2_{1,1}, \psi \in L^2_{N,M} \). While (20) defines \( \mathcal{U} \), identity (21) defines \( \mathcal{U}^{-1} \). Note that we defined \( \vartheta \) on \( [0,1/M) \) since there are \( M \) components of the vector \( \psi = (v_i)_{i=0}^{M-1} \) each of which is “mapped” on the corresponding interval \([i/M, (i+1)/M)\). Also, we need the factor \( M^{-\frac{i}{2}} \) since \( \mathcal{U} \) should be a unitary operator. Up to the factor \( M^{-1} \), the mapping \( \mu \) is measure-preserving. The unitarity of \( \mathcal{U} \) and the preservation of
the measure will be discussed in details at the end of this Chapter. The mapping \( \vartheta \) is discontinuous. In some sense, ‘Cantor-type’ mapping \( \vartheta \) has a fractal nature, see Fig. 2.

Let us compute the image of the matrix units under the action of \( \mathcal{U} \). Recall that the matrix unit is a matrix whose entries are all 0 except in one cell, where it is 1. We fix \( h_n = 1/n! \) for some \( n > 2 \). Let \( \mathcal{M} \in \mathcal{H}_{N,M}^{h_n} \) be a matrix unit. Due to (10) and (8), for any \( v = (v_{i+1})_{i=0}^{M-1} \in L_{N,M}^2 \) the operator \( \mathcal{M} \) has the form

\[
(\mathcal{M}v)(x + h_n p) = \begin{cases} 
v_{m+1}(x + h_n q)e_{k+1}, & x \in I_{h_n}, \\ 0, & \text{otherwise} \end{cases}
\]

for some fixed \( p, q \in \mathbb{Z}_{h_n}^N \) and \( k, m \in \mathbb{Z}_M \). Here, \( e_{k+1} = (\delta_{ik})_{i=0}^{M-1} \in \mathbb{C}_M \) is the standard basis vector. Any operator of the form (22) is a matrix unit in \( \mathcal{H}_{N,M}^{h_n} \). Let \( u \in L_{L,1}^2 \) be some function. We denote \( w = \mathcal{U}^{-1} \mathcal{M} \mathcal{U} u \). Using (21), we can state that

\[
w(x)|_{\left[\left(L_{L,1}^2, u\right)\right]} = 0, \quad i \in \mathbb{Z}_M \setminus \{k\}
\]

since all of the components of \( \mathcal{M} \mathcal{U} u \) are 0 except maybe for the \( k + 1 \)-th, see (22). Let us denote \( w = (w_{i+1})_{i=0}^{M-1} = \mathcal{M} \mathcal{U} u \). Then using (22) along with (20), we obtain

\[
w_{k+1}(x + h_n p) = M^{-\frac{1}{2}} u\left(\mu(x + h_n q) + \frac{m}{M}\right), \quad x \in I_{h_n}.
\]

On the other hand, using \( \mathcal{U} w = w \) along with (20), we obtain

\[
w_{k+1}(x + h_n p) = M^{-\frac{1}{2}} w\left(\mu(x + h_n p) + \frac{k}{M}\right), \quad x \in I_{h_n}.
\]

Equations (24) and (25) lead to

\[
w(\mu(x + h_n p) + \frac{k}{M}) = w\left(\mu(x + h_n p) + \frac{m}{M}\right), \quad x \in I_{h_n}.
\]

Let \( x \in I_{h_n} = [0, h_n)^N \subset \mathbb{T}^N \). Due to (16), we have

\[
k_j(x + h_n p) = k_j(x + h_n q) = 0, \quad j = 2, \ldots, n - 1,
\]

\[
k_n(x + h_n p) = p, \quad k_n(x + h_n q) = q.
\]

Thus, substituting (27) into (17), we deduce that

\[
\mu(x + h_n p) = \frac{a}{M(n!)^N} + x, \quad \mu(x + h_n q) = \frac{b}{M(n!)^N} + x,
\]

where \( a, b \in \mathbb{Z}_{h_n}^N \) are constants independent on \( x \) and

\[
x = \frac{\vartheta^{-1}_{n+1}(\mu_{n+1}(x))}{M((n+1)!)^N} + \frac{\vartheta^{-1}_{n+2}(\mu_{n+2}(x))}{M((n+2)!)^N} + \ldots.
\]

Definition (14) allows us to estimate bounds of \( x \) in (29):

\[
x_{\text{min}} = 0, \quad x_{\text{max}} = \frac{1}{M(n!)^N}.
\]

Moreover, it is seen that for \( x \in I_{h_n} \), the element \( x \) runs through the almost whole set \([x_{\text{min}}, x_{\text{max}}] = [0, 1/M(n!)^N]\), since for any \( j > 0 \) the value \( \vartheta_j(x) \) runs through the whole set \([0, \ldots, j - 1]^N\) except maybe some elements that correspond to \( x \).
belonging to a set of zero measure. Using this fact in (28) and then in (26), we deduce that
\[ w(x + M^{-1}h_n^N a_1) = u(x + M^{-1}h_n^N b_1), \quad x \in [0, M^{-1}h_n^N), \quad (30) \]
where constants \(a_1, b_1 \in \mathbb{Z}_{Mh_n^N}\) do not depend on \(x\) and \(u\). For other \(x \in \mathbb{T} \setminus [0, M^{-1}h_n^N)\) we have \(w(x) = 0\) a.e., see (23) and similar arguments related to (23) and (22). Thus, (30) shows us that \(\mathcal{U}^{-1}\mathcal{M}\mathcal{U}\) is a matrix unit in \(\mathcal{H}_{1,1}^{M^{-1}h_n^N}\), by the same reason as discussed in (22). Noting that \(\theta\) and its inverse \(\mu\) have similar structure, see (15) and (16), we may apply the same arguments as above to prove that for a matrix unit \(\mathcal{M}_1 \in \mathcal{H}_{1,1}^{M^{-1}h_n^N}\) the operator \(\mathcal{U}\mathcal{M}_1\mathcal{U}^{-1}\) is a matrix unit in \(\mathcal{H}_{N,M}^{h_n^N}\). Hence \(\mathcal{H}_{1,1}^{M^{-1}h_n^N} = \mathcal{U}^{-1}\mathcal{H}_{N,M}^{h_n^N}\mathcal{U}\) because the matrix units generate the corresponding algebras. As already mentioned above, this fact leads to \(\mathcal{H}_{1,1} = \mathcal{U}^{-1}\mathcal{H}_{N,M}\mathcal{U}\), see after (12).

We already have shown that \(\mu(I_{h_n}) = [0, M^{-1}h_n^N] + M^{-1}h_n^N a\) for some \(a \in \mathbb{Z}_{h_n^{-N}}\), see above (30). Using the same arguments, it can be shown that \(\mu\) performs \(1 - 1\) mapping between shifted cubes \(I_{h_n} + h_n a \subset \mathbb{T}^N\), \(a \in \mathbb{Z}_{h_n}^N\) and shifted intervals \([0, M^{-1}h_n^N] + M^{-1}h_n^N b \subset [0, M^{-1})\), \(b \in \mathbb{Z}_{h_n^{-N}}\). This property is valid for any \(n \in \mathbb{N}\). Thus, we can state that for any Borel set \(S \subset \mathbb{T}^N\) the Lebesque measure \(\text{Leb}_1(\mu(S)) = M^{-1}\text{Leb}_N(S)\), since this identity is true for all the elementary cubes discussed above and for their unions. Thus, using definition (20), we get the identity for the inner products
\[
(\mathcal{U}u, \mathcal{U}v)_{L_{N,M}^2} = M^{-1}\sum_{i=0}^{M-1} \int_{\mathbb{T}^N} u(\mu(x) + \frac{i}{M})\bar{v}(\mu(x) + \frac{i}{M})dx
\]
\[
= \sum_{i=0}^{M-1} \int_{0}^{1} u(x + \frac{i}{M})\bar{v}(x + \frac{i}{M})dx = \int_{0}^{1} u(x)\bar{v}(x)dx = (u, v)_{L_{1,1}^2},
\]
where \(u, v \in L_{1,1}^2\) and the bar denotes complex conjugation. This means that \(\mathcal{U}\) is a unitary operator.

**Data Availability Statement.** The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Received July 2020; revised September 2020.

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