On exhaustive reducible partition of graphs and its application to Hadwiger conjecture

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Abstract

An undirected graph $H$ is called a minor of the graph $G$ if $H$ can be formed from $G$ by deleting edges and vertices and by contracting edges. If $G$ does not have a graph $H$ as a minor, then we say that $G$ is $H$-free. Hadwiger conjecture claim that the chromatic number of $G$ may be closely related to whether it contains $K_{n+1}$ minors. To study the coloring of a $K_{n+1}$-free $G$, we propose a new concept of reducible partition of vertex set $V_G$ of $G$. A reducible partition(RP) of a graph $G$ with $K_n$ minors and without $K_{n+1}$ minors is defined as a two-tuples $\{S_1 \subseteq V_G, S_2 \subseteq V_G\}$ which satisfy the following conditions:

1. $S_1 \cup S_2 = V_G, S_1 \cap S_2 = \emptyset$
2. $S_2$ is dominated by $S_1$,
3. the induced subgraph $G[S_1]$ is a forest,
4. the induced subgraph $G[S_2]$ is $K_n$-free.

We will show that the reducible partition always exist and further we can obtain an exhaustive reducible partition(ERP) of $V_G$: $\{S_1, S_2, \ldots, S_m, m \leq n-1\}$ such that:

1. $\bigcup_{i=1}^{m} S_i = V_G, S_i \cap S_j = \emptyset$ for $i \neq j$,
2. $S_j$ is dominated by $S_i$ if $i \leq j$
3. each induced subgraph $G[S_i]$ is a forest,
4. the induced subgraph $G\left[\bigcup_{j=k}^{m} S_j\right]$ is $K_{n-k+2}$-free.

*Fully documented templates are available in the elsarticle package on CTAN.
With the ERP of a $K_{n+1}$-free graph $G$, one can obtain some useful conclusion of the coloring of $G$.

**Keywords:** Reducible partition, Graph color, four color theorem

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1. **Exhaustive Reducible Partition(ERP) of graphs**

Let $G = (V_G, E_G)$ be a graph with vertex set $V_G$ and edge set $E_G$. A subset $S \subset V_G$ is called a dominating set of $G$ if each vertex in $V_G - S$ is adjacent to at least one vertex of $S$. When the subgraph $G[S]$ induced by $S$ is a forest $F$, then the forest $F$ is called a dominating forest (DF) of the graph $G$. If $G[S]$ is not a subgraph of any other dominating forests of $G$, then $G[S]$ is called a maximal DF.

If $F$ is one of maximal DFs of $G$, then we say $V_G - V_F$ is dominated by $V_F$, where $V_F$ is the vertex set of the graph $F$. The two-tuples $\{V_F, V_G - V_F\}$ is called a dominating forest partition of $G$.

**Theorem 1.** Any graph $G$ possesses a DF.

**Proof.** The theorem can be proved by contradiction. Assume that there are no DFs in $G$ and $F$ is a maximal induced sub-forest of $G$, namely $F$ is not a subgraph of any other induced sub-forest of $G$. As $F$ is not a dominating forest, then there exists some vertex $v$ has no neighbors in $V_F$, then $G[V_F \cup v]$ is a forest and $F$ is a subgraph of $G[V_F \cup v]$, which is a contradiction. $\square$

A minor of a graph $G$ is any graph $H$ that is isomorphic to a graph that can be obtained from a subgraph of $G$ by contracting some edges. If $G$ does not have a graph $H$ as a minor, then we say that $G$ is $H$-free.

**Definition 1.** For a $K_{n+1}$-free graph $G$ with $K_n$ minors and two subset $S_1$ and $S_2$ of $V_G$, $\{S_1, S_2\}$ is called a reducible partition of $V_G$, if the following conditions are satisfied:

1. $S_1 \cup S_2 = V_G, S_1 \cap S_2 = \emptyset$
(2) $S_2$ is dominated by $S_1$,
(3) the induced graph $G[S_1]$ is a forest,
(4) the induced graph $G[S_2]$ is $K_n$-free.

Suppose $S_1$ is a maximal dominating forest of a graph $G$ with $K_n$ and without $K_{n+1}$ as minors, if $G[S_2]$ is $K_n$-free then $\{S_1, S_2\}$ is defined as a reducible partition, where $S_2$ is the complement of $S_1$.

**Lemma 1.** $G$ is a given graph and $S_1 \subseteq V_G$ is a subset of $V_G$. If the induced subgraph $G[S_1]$ is a maximal dominating tree, then for every vertex $v \in V_G - S$ there are at least 2 neighbors in one tree of $S$.

**Proof.** We prove the lemma by contradiction. As $S$ is a dominating tree, then $v$ has at least one neighbor in $S$. Assume that $v \in V_G - S$ only has one neighbor in one tree of $G[S]$, then $v$ can be added into $S$ and $S + v$ is also a dominating tree. Which contradicts that $S$ is maximal.

**Theorem 2.** If $\{S_1, S_2\}$ is a dominating forest partition graph $G$ and $G[S_1]$ is a tree, then $\{S_1, S_2\}$ is a reducible partition of $G$.

**Proof.** The theorem can be proved by contradiction. Assume that a $K_{n+1}$-free $G$ contains $K_n$ minors and $\{S_1, S_2\}$ is not a reducible partition of $G$, then $G[S_2]$ has at least one $K_n$ minor, say $H$. From lemma 1, every vertex of $H$ has two neighbors in $S_1$, hence by contracting all edges of $G[S_1]$, one can obtain a $K_{n+1}$ minor, which contradicts that $G$ is $K_{n+1}$-free.

If the dominating forest $G[S_1]$ contains multiple disjoint trees, then $G[S_2]$ may contains a $K_n$ minor, one example is shown in Fig.1. And by introduce the concept of minimal $K_{n+1}$-free minor, we will show that a reducible partition always exists.

**Definition 2.** If $H$ is a $K_{n+1}$ minor, and $H'$ obtained from $H$ by deleting any edge is $K_{n+1}$-free, then $H$ is called a mimimal $K_{n+1}$ minor.
Figure 1: A $K_5$-free graph $G$. $T_1$ and $T_2$ are two disjoint trees in dominating forest $G[S_1]$, the red vertex set $S_2$ constitute a $K_4$. As both $G$ and $G[S_2]$ contain a $K_4$, then $\{S_1, S_2\}$ is not a reducible partition.

From definition 2, a subgraph obtained by deleting any vertex of a minimal $K_{n+1}$ minor is also $K_{n+1}$-free.

**Theorem 3.** If $G$ contains only one minimal $K_{n+1}$ minor $H$, then $G'$ obtained from $G$ by deleting any edge $e$ of $H$ is $K_{n+1}$-free.

**Proof.** The theorem can be proved by contradiction. Assume that the theorem is not true, then $G'$ has at least one $K_{n+1}$ minor $H'$. Because $H$ is the only one minimal $K_{n+1}$ minor, then $H$ is the intersection of all other subgraphs homeomorphic to $K_{n+1}$. Hence $H$ is a subgraph of $H'$, however $H'$ does not contain the deleted edge $e$, namely $H$ is not a subgraph of $H'$, which a contradiction. \(\Box\)

**Theorem 4.** If $G$ contains $m$ minimal $K_{n+1}$ minors $H_i, i = 1, \cdots, m$ and the intersection of the $m$ $K_{n+1}$ minors $H$ is not null, then $G'$ obtained from $G$ by deleting any edge $e$ of $H$ is $K_{n+1}$-free.

**Proof.** As each $H_i, i = 1, \cdots, m$ is a minimal $K_{n+1}$ minor, then any $K_{n+1}$ minor $H'$ contains at least one of $H_i$ as subgraph. Assume the theorem is not true, then $G'$ has one $K_{n+1}$ minor $H'$ which is different with any $H_i, i = 1, \cdots, m$, which is a contradiction. \(\Box\)
Theorem 5. Let $A_{ij}, i, j = 1, \cdots, m$ are intersections of every pair of minimal $K_{n+1}$ minors $H_i$ and $H_j$ of $G$, if $A_{ij} = \emptyset$ is null then let $A_{ij} = H_i$ and $A_{ji} = H_j$. $G'$ obtained from $G$ by deleting $|A|$ edge $e_k \in E_{Ak}, k = 1, \cdots, |A|$ is $K_{n+1}$-free.

Proof. As $A_{ij}, i = 1, \cdots, |A|$ are intersections of every pair of $K_{n+1}$ minors $H_j$ and $H_k$ of $G$, then when $|A|$ edges $e_i \in E_{A_i}, i = 1, \cdots, |A|$ is deleted, there is no minimal $K_{n+1}$ minors, $G'$ is $K_{n+1}$-free. 

Definition 3. For a $K_{n+1}$-free graph $G$ contains $m$ minimal $K_n$ minors $H_i, i = 1, \cdots, m$. A vertex set $\{v_i \in V_{H_i}, i = 1, \cdots, m\}$ is defined as a critical $K_n$ forest if $G\left[\bigcup_{i=1}^{m} v_i\right]$ is a forest.

Lemma 2. For any $K_{n+1}$-free graph $G$ contains 3 minimal $K_n$ minors $H_i, i = 1, 2, 3$, critical $K_n$ forests always exist.

Proof. We will prove the lemma by contradiction. Assume that there is no critical $K_n$ forests in $G$, then for any vertex set $\{v_i \in V_{H_i}, i = 1, 2, 3\}$, $G\left[\bigcup_{i=1}^{3} v_i\right]$ contains a cycle $C_3$ with 3 vertices, if this is true, then all vertices of $H_k$ are adjacency to $H_j$ for $j \neq k$. That leads to multiple $K_{n+2}$ minors, which is a contradiction as $G$ is $K_{n+1}$-free.

Lemma 3. For any $K_{n+1}$-free graph $G$ contains 4 minimal $K_n$ minors $H_i, i = 1, \cdots, 4$, critical $K_n$ forests always exist.

Proof. We will prove the theorem by contradiction. From lemma 2 one can pick a vertex set $F = \{v_i \in V_{H_i}, i = 1, 2, 3\}$ from any 3 of four minimal $K_n$ minors satisfies that $G[F]$ is a forest. If $F$ is an independent set, then lemma 2 is clear. Suppose $F$ is not an independent set, than (1): $G[F]$ contains a $K_2$ and one isolated vertex, or (2): $G[F]$ is a tree with three vertices. Assume that there is no critical $K_n$ forests in $G$. For case (1), the $K_2$ of $G[F]$ must be the neighbors of each vertex $v_4 \in H_4$. Then the $G$ contains the subgraphs $H_4 + K_2$ which is a $K_{n+2}$ minor and is a contradiction because $G$ is $K_{n+1}$-free. For case (2), there are two vertices of $G[F]$ must be neighbors of each vertex $v_4 \in H_4$. Then the $G$ contains a $K_{n+1}$ minor $G[F \cup V_{H_4}]$ which is a contradiction because $G$ is $K_{n+1}$-free.
Hence, the theorem is true and for \( m = 4 \). And there exists a set \( F = \{ v_i \in V_{H_i}, i = 1, \ldots, 4 \} \) such that \( G[F] \) is either a null subgraph, tree or a forest contains a tree with 3 vertices and an isolated vertex.

**Theorem 6.** For any \( K_{n+1} \)-free graph \( G \) contains \( m \) minimal \( K_n \) minors \( H_i, i = 1, \ldots, m \), there exists a set \( F = \{ v_j \in \bigcup_{i=1}^{m} V_{H_i}, j = 1, \ldots, k, k \leq m \} \) such that \( G[F] \) is a critical \( K_n \) forest, and \( k = m \) when the intersection of any pair \( \{ H_i, H_j \} \) is null.

**Proof.** We will prove the theorem by listing a set \( F = \{ a_i \in V_{H_i}, i = 1, \ldots, k \} \) such that \( G[F] \) is critical \( K_n \) forest from a procedure. In the procedure, we will legally add one vertex of \( H_i \) to \( F \), i.e., after a vertex is added, the result induced subgraph \( G[F] \) does not contain cycles.

Firstly, let \( F = \{ v_1 \in H_1 \} \). Secondly, checking all vertices in \( V_H = \bigcup_{i=2}^{m} V_{H_i} \), if there is no neighbors in \( \bigcup_{i=2}^{m} H_i \), then let \( F = \{ v_1 \in V_{H_1}, v_2 \in V_{H_2} \} \) and \( V_H = \bigcup_{i=3}^{m} V_{H_i} - \bigcup_{v_2 \in V_{H_i}} V_{H_x} \), where \( \bigcup_{v_2 \in V_{H_i}} V_{H_x} \) is the union of all minimal minors which contains \( v_2 \); if there is a neighbor \( v_j \in V_{H_j} \), then let \( F = \{ v_1 \in V_{H_1}, v_j \in V_{H_j} \} \) and replace \( V_H \) by \( V_H - \bigcup_{v_j \in V_{H_j}} V_{H_x} \). Thirdly, check all vertices in \( V_H \), if \( F \) has no neighbors in \( V_H \), then add \( v_x \in V_{H_x} \) to \( F \) and remove all minimal minors which contain \( v_x \) from \( V_H \); if \( F \) has some neighbor \( v_y \in H_y \) such that \( G[F \cup v_y] \) has no cycles, then add \( v_y \) to \( F \) and remove all minimal minors which contain \( v_y \) from \( V_H \). Fourthly, repeat the above procedures until \( V_H \) becomes a null set.

Note that, if there are no neighbors of \( F \) in \( V_H \), then it will produce multiple disjoint trees and cycles can arise only between the latest induced tree and \( V_H \). And in the above procedure, we will firstly check whether \( F \) has neighbors in \( V_H \) or not.

The result induced graph \( G[F] \) from above procedure is obvious a forest. If there are two edges between every vertex of \( H_j \) and the latest tree, then it will produce a \( K_{n+1} \) minor which is a contradiction. That implies that there always exist a vertex \( v_x \in V_H \) has no neighbors in the latest tree of \( F \) or a vertex \( v_y \in V_H \) has only one neighbor in \( F \). \( \square \)
Theorem 7. For any connected graph $G$ has at most $K_{n+1}$ minors, there exists at least one reducible partition $\{S_1, S_2\}$.

Proof. This theorem is clear by using the theorem[6] one can let $S_1 = F$ firstly, where $F$ is defined in the proof of theorem[6]. And then check each vertex $v_j$ in $V_G − S_1$ in sequence, if $G[S_1 ∪ v_j]$ does not contain cycles, then update $S_1$ by adding $v_j$ to $S_1$.

And then let $S_2 = V_G - S_1$, one can obtain a reducible partition $\{S_1, S_2\}$ of $G$.

Now suppose $\{S_1, S_2\}$ is a reducible partition of $G$, if the induced subgraph $G[S_2]$ is a forest then $\{S_1, S_2\}$ is called an exhaustive reducible partition(ERP) of $G$. If $G[S_2]$ is not a forest, one can recursively perform the procedure showed in the proof of theorem[6] to obtain the reducible partition of $G[S_2]$.

Definition 4. For a $K_{n+1}$-free graph $G$ contains $K_n$ minors, a ERP $\{S_1, S_2, \cdots, S_m\}$ is defined as a collection of subset of $V_G$ such that:

1. $\bigcup_{i=1}^{m} S_i = V_G, S_i ∩ S_j = \emptyset, i \neq j$,
2. $S_j$ is dominated by $S_i$ if $i \leq j$
3. the induced graph $G[S_i]$ is a forest,
4. the induced graph $G\left[\bigcup_{j=k}^{m} S_j\right]$ is $K_{n-k+2}$-free.

And $m$ is defined as the depth of ERP.

Proposition 1. For a $K_{n+1}$-free graph $G$ contains $K_n$ minors, the depth of ERP of $G$ is at most $n - 1$.

Proof. As $G\left[\bigcup_{j=k}^{m} S_j\right]$ is $K_{n-k+2}$-free, then take $k = m$ and $n - k + 2 = 3$ to obtain

$$m = n - 1$$

at most.
2. Coloring of planar graphs

In the previous section, we have introduced the conception of ERP of a graph. In this section, we will apply ERP to the problem of coloring of planar graphs. As the depth of ERG of any $K_{n+1}$-free graph is at most $n - 1$, then the depth of ERP is at most 3 for any planar graph. Hence for a planar graph $G$ with cycles, there exists a ERP $\{S_1, S_2, S_3\}$ or $\{S_1, S_2\}$ such that

(1) $\bigcup_{i=1}^{m} S_i = V_G, S_i \cap S_j = \emptyset, i \neq j, m < 4$
(2) each $G[S_i], i = 1, ..., m.$ is a forest;
(3) $S_j$ is dominated by $S_i$, if $i < j$;
(4) $G[S_2 \cup S_3]$ is $K_4$-free, and $G[S_3]$ is $K_3$-free.

When the depth of a planar graph is 2, then $S_3$ is a null set. For this case the chromatic number $\chi(G) \leq 4$ is clear as $G[S_1 \cup S_2]$ is a subgraph of the sum of two complete bipartite graphs. Therefore, only the planar graphs with depth 3 needs to be considered. With the all the above theorem and lemma, we have the four-color theorem which is proved by Appel and Haken with computer[1].

**Theorem 8.** Any planar graph $G$ is 4-colorable.

**Proof.** From proposition[1] we know that the depth of ERP of any planar graph is 3. Assume the ERP of $G$ is $\{S_1, S_2, S_3\}$. It is easy to say that $G[S_2 \cup S_3]$ is 4-colorable as each $G[S_i]$ is a forest. Assume $G[S_2 \cup S_3]$ is labelled by four integers “1”, “2”, “3” and “4”, if we can assign each vertex of $S_1$ by one of integers “1”, “2”, “3” or “4”, then the theorem is proved.

There are two cases in which a planar graph $G$ contains $K_4$ minors.

Case one:
The induced subgraph $G[v \cup N(v; S_2 \cup S_3)]$ has $K_4$ minors, where $N(v; S_1 \cup S_2)$ denotes the neighbors of $v \in S_1$ in $S_2 \cup S_3$.

Case two:
The induced subgraph $G[N(v_1, v_2; S_2 \cup S_3)]$ has $K_4$ minors, where $v_1, v_2 \in S_1$ are two adjacent vertices and $N(v_1, v_2; S_2 \cup S_3)$ is used to denote $N(v_1; S_2 \cup S_3) \cap N(v_2; S_2 \cup S_3)$, namely the common neighbors of $v_1$ and $v_2$. 

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We will show that only four colors should be used for both the two cases.

For case one, \( G[N(S_2 \cup S_3)] \) has \( K_3 \) minors and has no \( K_4 \) minors, this implies that the chromatic number of \( G[N(S_2 \cup S_3)] \) is at most 3. Hence \( G[N(v; S_2 \cup S_3)] \) is 4-colorable.

For case two, \( G[N(v_1, v_2; S_2 \cup S_3)] \) has \( K_2 \) minors and has no \( K_3 \) minors, then the chromatic number of \( G[N(v_1, v_2; S_2 \cup S_3)] \) is 2. Hence, the chromatic number of \( G[v_1 \cup v_2 \cup N(v_1, v_2; S_2 \cup S_3)] \) is at most 4.

As \( G[v_1 \cup v_2 \cup N(v_1, v_2; S_2 \cup S_3)] \) is 4-colorable, then \( G[v_1 \cup v_2 \cup N(v_1) \cup N(v_2)] \) is also 4-colorable, where \( N(v_1) \) and \( N(v_2) \) are the neighbors of \( v_1 \) and \( v_2 \) respectively. This can be showed by contradiction. Assume the chromatic number of \( G[v_1 \cup v_2 \cup N(v_1, v_2; S_2 \cup S_3)] \) is 5, then there exists a vertex \( v_3 \in (N(v_1) \cup N(v_2)) - N(v_1, v_2; S_2 \cup S_3) \) that connecting to both \( v_1 \) and \( v_2 \), which is a contradiction.

Hence, each vertex and the ends of each edge can be labelled by at most 4 colors.

\( \square \)

3. Color of graph without \( K_{n+1} \) minor

Based on ERP of graphs, we have demonstrated that any planar graph \( G \) is 4-colorable in the previous section. Scholars believe that four color theorem is a special case of Hadwiger conjecture which states that if \( G \) is loopless and has no \( K_{n+1} \) minor then its chromatic number satisfies \( \chi(G) \leq n \). It is known to be true for \( 1 \leq t \leq 5 \) \[3, 4\].

**Conjecture 1.** \[3\] Every connected \( n \)-chromatic graph \( G \) contracts to \( K_n \), or to a copy of \( K_n \) with some multiple adjacencies.

Before studying the color of a graph without \( K_{n+1} \), we will firstly discuss the depth of ERP of a graph without \( K_{n+1} \) minor.

Proposition \[3\] tell us that the vertex set of a graph without \( K_{n+1} \) minor can always be divided into at most \( n - 1 \) subsets, and the subgraph induced by vertices in each of the \( n - 1 \) subsets is a collection of disjoint trees, see Fig.2.
For a graph $G$ has no $K_6$ minor, Bela Bollobas and P. Catlin show that $G$ is 5-colorable[5]. Now we show that this conclusion can be obtained by using the ERP of graphs. As the depth of ERP of $G$ without $K_6$ minor is at most 4, hence the vertex set of $G$ can be divided into four subsets, say $S_1, S_2, S_3, S_4$.

**Theorem 9.** A graph $G$ has no $K_6$ minors is 5-colorable.

**Proof.** From Proposition[4] the depth of $G$ is at most 4, that is to say the vertex set $V_G$ can be divided into 4 subsets $S_1, S_2, S_3, S_4$ such that $G[S_j]$ is a forest and $G[S_j]$ is dominated by $G[S_k]$ for $j > k$. Clearly, subgraph $G[S_2 \cup S_3 \cup S_4]$ has no $K_5$ minors, i.e. it is a planar graph. Based on the four color theorem, the chromatic number of $G[S_2 \cup S_3 \cup S_4]$ is at most 4.

A $K_5$ minor can be construct by two ways:

1. the induced subgraph $G[v \cup N(v; S_2 \cup S_3 \cup S_4)]$ has $K_5$ minors, where $v$ belongs to $S_1$ and $N(v; S_2 \cup S_3 \cup S_4)$ is used to denote the neighbors of $v$ in $S_2 \cup S_3 \cup S_4$.

2. the induced subgraph $G[v_1 \cup v_2 \cup N(v_1, v_2; S_2 \cup S_3 \cup S_4)]$ has a $K_5$ minor, where $v_1 \in S_1$ and $v_2 \in S_1$ are two adjacent vertices and $N(v_1, v_2; S_2 \cup S_3 \cup S_4)$ is used to denote the common neighbors of $v_1$ and $v_2$. We claim that less then 6 colors will be used for labelling $G$.

For the first case, $G[N(v; S_2 \cup S_3 \cup S_4)]$ contains no $K_5$ minors and contains $K_4$ minors. By the four color theorem, $G[N(v; S_2 \cup S_3 \cup S_4)]$ is 4-colorable, hence

![Figure 2: Schematic diagram of ERP of a graph G has no $K_{n+1}$ minor. The subgraph $G[S_j]$ induced by $S_k$ only contains disadjoint trees and $G[S_j]$ is dominated by $G[S_k]$ for $j > k$.](image-url)
$G[v \cup N(v; S_2 \cup S_3 \cup S_4)]$ is 5-colorable.

For the second case, $G[N(v_1, v_2; S_2 \cup S_3 \cup S_4)]$ has no $K_4$ minors and has $K_3$ minors. This implies that $G[N(S_2 \cup S_3 \cup S_4)]$ is 3-colorable, then $G[N(v_1, v_2; S_2 \cup S_3 \cup S_4)]$ is 5-colorable.

As $G[v_1 \cup v_2 \cup N(v_1, v_2; S_2 \cup S_3 \cup S_4)]$ is 5-colorable, then $G[v_1 \cup v_2 \cup N(v_1) \cup N(v_2)]$ is also 5-colorable, where $N(v_1)$ and $N(v_2)$ are the neighbors of $v_1$ and $v_2$ respectively. This can be showed by contradiction. Assume the chromatic number of $G[v_1 \cup v_2 \cup N(v_1, v_2; S_2 \cup S_3 \cup S_4)]$ is 6, then there exists a vertex $v_3 \in (N(v_1) \cup N(v_2)) - N(v_1, v_2; S_2 \cup S_3 \cup S_4)$ that connecting to both $v_1$ and $v_2$, which is a contradiction.

With the similar procedure, one can obtain the Hadwiger conjecture.

**Theorem 10.** A graph $G$ has no $K_{n+1}$ minor is $n$-colorable.

**Proof.** We will prove the theorem by induction. Firstly, this theorem is correct for $n = 4$ and $n = 5$ which are four color theorem and theorem respectively. Secondly, we assume the theorem is correct for all $n \leq t$, namely if $G$ has no $K_{t+1}$ then $G$ is $t$-colorable. And we will check whether the theorem is correct.
for \( n = t + 1 \), namely if \( G \) has no \( K_{t+2} \), then whether \( G \) is \((t+1)\)-colorable.

For \( n = t + 1 \), the depth of ERP of \( G \) is at most \( t + 1 \), the vertex set \( V_G \) of \( G \) can be divided into \( t \) subsets \( S_1, S_2, \ldots, S_t \) and \( G[S_j] \) is dominated by \( G[S_k] \) for \( j > k \). Based on the assumption that the theorem is correct for \( n = t \) and \( n = t - 1 \), namely \( G[\bigcup_{j=2}^{t+1} S_j] \) is \( t \)-colorable and \( G[\bigcup_{j=3}^{t+1} S_j] \) is \((t+1)\)-colorable.

A \( K_{t+1} \) minor can be construct by two ways:

1. There exists an induced subgraph \( G[v \cup N(v; \bigcup_{j=2}^{t+1} S_j)] \) has \( K_{t+1} \) minors, where \( v \) belongs to \( S_1 \) and \( N(v; \bigcup_{j=2}^{t+1} S_j) \) is used to denote the neighbors of \( v \) in \( \bigcup_{j=2}^{t+1} S_j \).

2. There exists an induced subgraph \( G[v_1 \cup v_2 \cup N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j)] \) has \( K_{t+1} \) minors, where \( v_1 \in S_1 \) and \( v_2 \in S_1 \) are two adjacent vertices and \( N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j) \) is used to denote the common neighbors of \( v_1 \) and \( v_2 \) in \( \bigcup_{j=2}^{t+1} S_j \).

We claim that less than \( t + 2 \) colors will be used for labelling \( G \).

For the first case, \( G[N(v; \bigcup_{j=2}^{t+1} S_j)] \) contains no \( K_{t+1} \) minors and contains \( K_t \) minors. By the assumption, \( G[N(v; \bigcup_{j=2}^{t+1} S_j)] \) is \( t \)-colorable, hence \( G[v \cup N(v; \bigcup_{j=2}^{t+1} S_j)] \) is \((t+1)\)-colorable.

For the second case, \( G[N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j)] \) has no \( K_t \) minors and has \( K_{t-1} \) minors. By the assumption, \( G[N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j)] \) is \((t-1)\)-colorable, then \( G[v_1 \cup v_2 \cup N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j)] \) is \((t+1)\)-colorable.

As \( G[v_1 \cup v_2 \cup N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j)] \) is \((t+1)\)-colorable, then \( G[v_1 \cup v_2 \cup N(v_1) \cup N(v_2)] \) is also \((t+1)\)-colorable, where \( N(v_1) \) and \( N(v_2) \) are their neighbors of \( v_1 \) and \( v_2 \) respectively. If this is not true, then there exists a vertex \( v_3 \in (N(v_1) \cup N(v_2)) - N(v_1, v_2; \bigcup_{j=2}^{t+1} S_j) \) connecting to both \( v_1 \) and \( v_2 \), which is a contradiction.

Hence, \( G[v_1 \cup v_2 \cup N(v_1) \cup N(v_2)] \) is \((t+1)\)-colorable.

As \( G[S_1] \) is a tree and each vertex and ends of each edge are can be labelled by at most \( t + 1 \) colors, then \( G \) is \((t+1)\)-colorable. \( \square \)
4. Conclusions and Discussions

In this theme, we propose a concept of reducible partition and exhaustive reducible partition of graphs. We showed that any graph $G$ possess exhaustive reducible partitions and the depth of ERP of $G$ is at most $n - 1$ if $G$ doesnot contains $K_{n+1}$ as minor. The theme illustrates that the four color theorem can be proved by using the ERP. If the depth of ERP of a planar graph is 3, then one can obtain that any subgraphs induced by the union of two of these 3 subsets is 4-colorable, to keep the graph is planar, the fifth color is forbidden. If the depth of ERP of a planar graph is 2, the chromatic number is clearly at most 4.

For general graphs, we have shown that the ERP can also work on how to prove Hadwiger conjecture, i.e. it also implies Hadwiger conjecture. The method is similar to the procedure of proving four color theorem of planar graphs. For a $K_{n+1}$-free graph, any two vertices of $S_1$ of the ERP have common neighbors whose induced subgraph is $K_{n-1}$-free, hence it is $n$-colorable.

Some structural information of a graph can be acquired by list the ERP to some degree. With these information, can we design algorithms to improve the proformance for coloring a graph? That will be a work we want to consider in the future.

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