Nonconservative Lagrangian mechanics II: purely causal equations of motion

David W. Dreisigmeyer∗
Department of Mathematics
Colorado State University, Fort Collins, CO 80523

Peter M. Young†
Department of Electrical and Computer Engineering
Colorado State University, Fort Collins, CO 80523

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Abstract

This work builds on the Volterra series formalism presented in [D. W. Dreisigmeyer and P. M. Young, J. Phys. A 36, 8297, (2003)] to model nonconservative systems. Here we treat Lagrangians and actions as ‘time dependent’ Volterra series. We present a new family of kernels to be used in these Volterra series that allow us to derive a single retarded equation of motion using a variational principle.

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∗email:dreisigm@math.colostate.edu
†email:pmy@engr.colostate.edu
1 Introduction

The central question addressed in this paper is: ‘How can one have a single retarded equation of motion arise from the use of a variational principle?’ Having a dissipative equation of motion arise from the use of a variational principle has a rather long history in mechanics. One of the central results in this area is Bauer’s 1931 corollary [3]:

**Corollary 1.1** The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle.

Bauer’s proof of this corollary relies on two implicit assumptions. First, there are no extra equations of motion that arise. Also, only integer ordered derivative operators are used in the action and Lagrangian. It is the latter assumption that we will eventually take advantage of to avoid Bauer’s corollary.

There has been some effort devoted to bypassing Bauer’s result. One of the earliest attempts was by Bateman [2]. What he did was allow a dual equation of motion to arise. Let us illustrate this via the dissipative harmonic oscillator. If we start with the Lagrangian

\[ L = m\ddot{x}y + \frac{C}{2}(x\dot{y} - \dot{x}y) - m\omega^2 xy, \]

where \( C \) is a constant, we would have the following equations of motion

\[ m\ddot{x} + C\dot{x} + m\omega^2 x = 0 \quad \text{(2)} \]
\[ m\ddot{y} - C\dot{y} + m\omega^2 y = 0 \quad \text{(3)} \]

Equation (2) is what we want for our harmonic oscillator (i.e., it is retarded or causal). Equation (3) is a time reversed version of (2). Bateman’s method is not particularly general. Also, the appearance of an advanced (time reversed or anti-causal) equation can be considered a drawback to this procedure. Our universe appears to be causal so there does not seem to be any compelling reason why anti-causal effects should arise from any correct action.

A rather novel attempt to get around Bauer’s corollary was explored by Riewe in a series of papers [9, 10]. Riewe tried using fractional derivatives in the action to have nonconservative equations arise. In particular, Riewe did not use Bateman’s
method of having an anti-causal dual equation. However, Riewe’s equations of motion are acausal. In order to circumvent this, it was suggested that all anti-causal operators be replaced with their causal counterparts. (Riewe’s method is rather involved so we do not provide any illustrative examples.) This procedure does allow for much more general equations than Bateman’s method. However, it must be remembered that the actual equations derived are acausal and the procedure to change these into causal equations is rather ad hoc.

In examining Riewe’s approach, Dreisigmeyer and Young showed that his procedure of replacing anti-causal with causal operations may not be a wise idea. While still employing fractional derivatives, Dreisigmeyer and Young instead allowed for a causal equation and an anti-causal dual equation to arise. Thus, their procedure can be considered a generalization of Bateman’s. The appearance of an anti-causal equation is still very troublesome. Since both and deal with fractional derivatives, we will later refer to these techniques as fractional mechanics.

There are some other approaches to our basic problem. Tonti and Arthurs and Jones did develop a procedure based on the convolution product that is useful for the harmonic oscillator. However, this does not seem to generalize to higher ordered potentials. So we need to question if this is the correct procedure to follow. Caldeira and Leggett modelled nonconservative systems by coupling them to an environment. The environment is modelled as a collection of harmonic oscillators which results in the Lagrangian

\[
L = \frac{m}{2} q^2 - V(q) + \sum_{n=1}^{\infty} \left\{ \frac{m_n}{2} \dot{q}_n^2 - \frac{m_n \omega_n^2}{2} (q_n - q)^2 \right\}
\]

where \(q\) is the system’s coordinate and the \(q_n\) are the environment’s coordinates. Here the combined system and environment is conservative while the system alone may be dissipative. This procedure allows for the introduction of very general dissipation terms into the system’s equation of motion. For example, fractional derivatives can be included. However, the microscopic modelling of the environment makes rather complicated. In particular, we would generally have a countably infinite system of equations resulting from (4). Also, the question of causality is ‘put off’ to the end. That is, we enforce causality in \(q\)’s equation of motion by only using the causal solutions of the \(q_n\)’s equations of motion. In other words, we assume the environment acts causally only when we are solving it’s equations of motion. While this is not particularly disturbing, it does mean that causality is not necessarily enforced in the action.
In [6] the idea of treating the actions and Lagrangians as Volterra series was introduced. (Volterra series are the generalization to functionals of the power series concept.) This is a rather powerful framework. It allows us to build up Lagrangians (and hence actions) in a rather systematic way by tailoring the kernels in the Volterra series to our requirements. As originally stated, we desire to derive a single causal equation of motion for a system. We will accomplish this by choosing the correct family of kernels for our Volterra series. Our paper is organized as follows. We review the distributional approach to fractional differentiation in section 2. In section 3 we introduce Volterra series. Our variational principle is developed in section 4. A discussion of our results and possible future research follows in section 5.

2 Fractional derivatives

We will now briefly review fractional derivatives using the distributional approach. (A fuller discussion of this material can be found in [6, 7, 8].) First, define the generalized functions

\[
\Phi^+_{\alpha}(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} t^{\alpha-1} & t > 0 \\
0 & t \leq 0
\end{cases}
\]  
(5)

and

\[
\Phi^-_{\alpha}(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} |t|^{\alpha-1} & t < 0 \\
0 & t \geq 0
\end{cases}
\]  
(6)

where \(\Gamma(\alpha)\) is the gamma function. These distributions will allow us to define two different types of fractional derivatives. One of these will be causal while the other will be anti-causal. The anti-causal derivatives need to be avoided in our equations of motion.

Left fractional derivatives (LFDs) of order \(\alpha\) of a function \(q(t)\) are defined by

\[
a D^\alpha_t [q] := \Phi^+_{-\alpha}(t) \ast q(t)
\]  
(7)

\[
= \frac{1}{\Gamma(-\alpha)} \int_a^t q(\tau)(t - \tau)^{-(\alpha+1)} \, d\tau
\]

where we set \(q(t) \equiv 0\) for \(t < a\). For \(\alpha = n\), \(n\) an integer, (7) becomes

\[
a D^n_t [q] = D^n q
\]  
(8)

4
where \( D \) is the generalized derivative. The LFDs are causal operations. Hence, we will only want LFDs to be in our equations of motion.

Right fractional derivatives (RFDs) of order \( \alpha \) are given by

\[
\mathcal{D}_b^\alpha [q] := \Phi_{-\alpha}(t) * q(t) = \frac{1}{\Gamma(-\alpha)} \int_t^b q(\tau)(\tau - t)^{-(\alpha+1)} d\tau
\]

where now \( q(t) \equiv 0 \) for \( t > b \). Notice that instead of (8), we have

\[
\mathcal{D}_b^n [q] = (-1)^n D^n q .
\]

The RFDs are anti-causal operations. So, we do not want the RFDs to appear in our equations of motion.

The basic problem for fractional mechanics is to avoid the appearance of RFDs in the equations of motion. Neither Riewe [9, 10] nor Dreisigmeyer and Young [6] were able to remove RFDs completely. Riewe suggested replacing RFDs with LFDs in the resulting equations of motion. Dreisigmeyer and Young suggested letting the LFDs appear in one equation while the RFDs appeared in a dual equation. Neither of these methods is entirely satisfactory. We will shortly present a formalism that completely avoids RFDs. Before that, let us look at Volterra series.

### 3 Volterra Series

Now we are going to develop some of the Volterra series concepts that we will need. Our treatment of Volterra series here is somewhat different than that presented in [6]. Here we treat Volterra series as expansions of (generalized) functions that depend on time. This is the viewpoint adopted in the nonlinear systems theory that is somewhat popular in electrical engineering (see, e.g., [11]). In [6] the Volterra series were treated as expansions of functionals only (i.e., no time dependence). This change in viewpoint will make all the difference in achieving our goal of a single causal equation of motion.

For completeness let us first review the Volterra series treatment in [6], see also [12]. (Please note, we will not review the fractional mechanics formalism
developed by Dreisigmeyer and Young in this paper since this will be largely irrelevant for our current work and also take us too far afield.) In [6] only functionals were treated as (time independent) Volterra series. For some functional \( V[q] \) define the kernels

\[
K_n^{(s)}(\tau_1, \ldots, \tau_n) := \frac{\delta^n V[q]}{\delta q(\tau_1) \cdots \delta q(\tau_n)}.
\]  

(11)

Notice that the \( K_n^{(s)}(\cdot) \)'s are symmetric under an interchange of the \( \tau_i \)'s. For example, \( K_2^{(s)}(\tau_1, \tau_2) = K_2^{(s)}(\tau_2, \tau_1) \). Now introduce the convenient notation

\[
K_n^{(s)} q^n := \int_{\tau_1} \cdots \int_{\tau_n} K_n^{(s)}(\tau_1, \ldots, \tau_n) q(\tau_1) \cdots q(\tau_n) \, d\tau_n \cdots d\tau_1.
\]  

(12)

Then we can represent the functional \( V[q] \) as the Volterra series

\[
V[q] = \sum_{n=1}^{\infty} \frac{1}{n!} K_n^{(s)} q^n.
\]  

(13)

The \( V[q] \) were taken as the actions in [6]. (This is why we can take \( K_0^{(s)} = V[0] \equiv 0 \) in (13) without any loss of generality). This use of Volterra series proves to be too restrictive [5]. So let us use a more ‘dynamic’ form of the Volterra series concept.

Instead of expanding the actions in Volterra series like (13), we will now expand the Lagrangians in Volterra series like

\[
L[q; \tau] = \sum_{n=1}^{\infty} \frac{1}{n!} K_n(\tau) q^n
\]  

(14)

where the \( \ast \) notation is as in (12) and the \( K_n(\tau) \) will now be of the form

\[
K_n(\tau) = K_n(\tau, \tau_1, \ldots, \tau_n).
\]  

(15)

(We assume that \( K_0(\tau) \equiv 0 \) in (14).) We call a kernel \( K_n(\tau, \tau_1, \ldots, \tau_n) \) symmetric if it is symmetric under any interchange of the \( \tau_i, i = 1, \ldots, n \). A kernel is called stationary if there exists a kernel \( \kappa_n(\xi_1, \ldots, \xi_n) \) such that

\[
\kappa_n(\tau - \tau_1, \ldots, \tau - \tau_n) = K_n(\tau, \tau_1, \ldots, \tau_n)
\]  

(16)

for all \( \tau, \tau_1, \ldots, \tau_n \).
The $\mathcal{L}[q, \tau]$ will generally be $\tau$-dependent distributions that also depend on the function $q$. So, in a sense, the Lagrangians can also be viewed as $\tau$-dependent functionals. Hence the notation $\mathcal{L}[q; \tau]$ used in (14). This is also why we allow the extra $\tau$ to appear in (15) versus (11).

From (14) we will form the actions

$$V[q; t] = \int_0^t \mathcal{L}[q; \tau] \, d\tau.$$  

The actions given by (17) are $t$-dependent Volterra series. So they can be thought of as distributions or $t$-dependent functionals as the Lagrangians were. It is also useful to view the actions as the anti-derivatives of the Lagrangians.

Our entire problem can now be succinctly stated: ‘Find the correct kernels $K_n(\tau, \tau_1, \ldots, \tau_n)$ to use in (14).’ The Volterra series framework helps clarify the problem of deriving a single causal equation of motion. It reduces the whole question to that of finding a correct family of kernels for our Volterra series. Our attention now turns to this problem.

## 4 Causal fractional Lagrangian mechanics

Systems and control theorists tend to be obsessed with the question of causality. So it seems somewhat natural that the theory and formalism of causal Volterra series reached its highest development due to the work of (mathematically oriented) electrical engineers. This is probably best expressed by the book written by Rugh [11]. We are now going to apply this previous research to analytic mechanics. We will see that enforcing causality in the Lagrangians will result in causal equations of motion. By now we should see that the key to deriving purely causal equations of motion is to find the correct kernels for our Volterra series. What we will do in this section is introduce a new family of kernels to be used in our Volterra series. We then examine these kernels to model the nonconservative harmonic oscillator. Next, a driving force is added on. Finally, higher ordered potentials are dealt with.

First, let us define the symmetric and stationary kernels

$$R_n^\lambda(\tau) := \frac{C_{n-1}}{\Gamma(\lambda + n - 2)} \left(\sqrt{(\tau - \tau_1)^2 + \ldots + (\tau - \tau_n)^2}\right)^{\lambda-1},$$  

(18)
for \( n \geq 2 \), and
\[
 r_n^\lambda(\tau) := \frac{C_n}{\Gamma(\lambda + n - 1)} \left( \sqrt{(\tau - \tau_1)^2 + \ldots + (\tau - \tau_n)^2} \right)^{\lambda - 1},
\]  
(19)
for \( n \geq 1 \), where, in (18) and (19), \( \tau_i \leq \tau \) for \( i = 1, \ldots, n \) and
\[
 C_n := \frac{2^{n-1} \Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}}.
\]  
(20)

(It is interesting to compare the distributions in (18) and (19) with the \( r^\lambda \) examined in (7).) The restrictions on the \( \tau_i \) should be viewed as part of the definitions of the \( R_n^\lambda(\tau) \) and the \( r_n^\lambda(\tau) \). Requiring that \( \tau_i \leq \tau \) is what will give us our causality. At first sight the distributions in (18) and (19) appear to have nothing to do with the fractional derivative operators \( \Phi^\pm_{\alpha}(\tau) \) presented in section (4). This will prove to not be the case presently.

The easiest case to deal with is the nonconservative harmonic oscillator, which we examine now. Consider the Lagrangian
\[
 L_{HO}[q; \tau] = \frac{1}{2} \left[ m R_2^{-2}(\tau) + m C R_2^{-\gamma}(\tau) + m \omega^2 R_0^0(\tau) \right] \star q^2
\]  
(21)
where \( 0 < \gamma < 2 \) and \( C \) is a constant. Remember that the Lagrangian is a \( \tau \)-dependent Volterra series that will be treated as a distribution. From (21) we form the action
\[
 V_{HO}[q; t] := \int_a^t L_{HO}[q; \tau] \, d\tau.
\]  
(22)
In (22) we take \( q(\tau_i) \equiv 0 \) for \( t_i < a \). This implies that \( L[q; \tau] \equiv 0 \) for \( \tau < a \) in (21) since the Lagrangian in (21) is causal. Also, the action in (22) is a \( t \)-dependent Volterra series. Remember that we think of \( V[q; t] \) in (22) as being the anti-derivative of \( L[q; \tau] \) in (21).

In (21) and (22) we should think of \( \tau \) as being ‘now’. When we derive our equations of motion shortly, we will vary the \( q(\tau_i) \) in (22). When we do this, we always consider the perturbations of \( q(\tau_i) \) as happening ‘now’, i.e.,
\[
 q(\tau_i) \longrightarrow q(\tau_i) + h\delta(\tau - \tau_i)
\]  
(23)
where \( h \) is infinitesimal. Using (23) in (22) gives us

\[
\frac{\delta V[q; t]}{\delta q(\tau)} = \int_a^t \left[ mr_1^{-2}(\tau) + m\gamma r_1^{-\gamma}(\tau) + m\omega^2 r_1^0(\tau) \right] * q \, d\tau \\
= \int_a^t \left\{ m_a D^2_\tau [q] + mC_a D^\gamma_\tau [q] + m\omega^2 a D^0_\tau [q] \right\} \, d\tau. \tag{24}
\]

We will require that (24) is the zero distribution for \( t > a \). This means that

\[
m_a D^2_\tau [q] + mC_a D^\gamma_\tau [q] + m\omega^2 a D^0_\tau [q] = 0 \tag{25}
\]

for \( \tau > a \). Equation (25) is the nonconservative harmonic oscillator’s equation of motion. The above can also be extended to \( R^{-\lambda}_2(\tau) \) where \( \lambda > 2 \). So, derivatives of arbitrary order of \( q(\tau) \) can be included in (21).

So how did the restriction \( \tau_i \leq \tau \) result in the causality in (25)? Consider the quantity

\[
S[q; t] = \int R^{-\lambda}_2(\tau)q(\tau_1)\delta(\tau - \tau_2) \, d\tau_2 d\tau_1 d\tau \\
= \int \frac{1}{\Gamma(-\lambda)} \left( \sqrt{(\tau - \tau_1)^2 + (\tau - \tau_2)^2} \right)^{-\lambda-1} q(\tau_1) \delta(\tau - \tau_2) \, d\tau_2 d\tau_1 d\tau \\
= \int \frac{1}{\Gamma(-\lambda)} \left( \sqrt{(\tau - \tau_1)^2} \right)^{-\lambda-1} q(\tau_1) \, d\tau_1 d\tau. \tag{26}
\]

Since \( \tau_1 \leq \tau \), we can rewrite (26) as

\[
S[q; t] = \int_a^t \int_a^\tau \frac{1}{\Gamma(-\lambda)} \left( \frac{(\tau - \tau_1)^{\lambda-1}}{\Gamma(-\lambda)} \right) q(\tau_1) \, d\tau_1 d\tau \\
= \int_a^t \Phi^\pm_\lambda(\tau) \ast q(\tau) \, d\tau \tag{27}
\]

where \( q(\tau) \equiv 0 \) for \( \tau < a \). Hence, the restrictions on the \( \tau_i \) allows us to have the \( a D^\lambda_\tau [q] \) arise from the use of the \( R^{-\lambda}_2(\tau) \) in the Lagrangians.

Driving forces need to be handled somewhat differently than the potential or kinetic energy terms. This is perhaps not surprising since a driving term is, in a sense, outside the ‘universe’ we are modelling. Let us consider the \( q(\tau) \) as our system that moves through some environment described by the kernels in our Volterra series. So the Volterra series includes both our system and environment.
The driving term is neither of these. We could of course expand our view of what the environment or system are to include those mechanisms that give rise to the driving force. This, however, may overly complicate matters. Instead, we will proceed as follows for, e.g., the driven nonconservative harmonic oscillator. Let our new Lagrangians be given by

\[ \mathcal{L}'[q;\tau] = \mathcal{L}_{HO}[q;\tau] - \int_a^\tau f(\tau_1)q(\tau_1)\,d\tau_1. \]  

(28)

Things proceed as before. So our action is given by

\[ \mathcal{V}'[q;t] = \mathcal{V}_{HO}[q;t] - \int_a^t \int_a^\tau f(\tau_1)q(\tau_1)\,d\tau_1\,d\tau. \]  

(29)

Perturbing \( q(\tau_i) \) by \( h\delta(\tau - \tau_i) \) results in

\[ \frac{\delta\mathcal{V}'[q;t]}{\delta q(\tau)} = \frac{\delta\mathcal{V}[q;t]}{\delta q(\tau)} - \int_a^\tau f(\tau)\,d\tau. \]  

(30)

Requiring (30) to be the zero distribution for \( t > a \) gives us

\[ m_a D_\tau^2[q] + mC_a D_\tau^2[q] + m\omega^2_a D_\tau^0[q] = f(\tau) \]  

(31)

for \( \tau > a \). So driving forces are easily included in our equations of motion via the term

\[ -\int_a^\tau f(\tau_1)q(\tau_1)\,d\tau_1 \]  

(32)

in our Lagrangians.

Now we turn our attention to higher ordered potentials. Consider a term like

\[ \mathcal{L}_n[q;\tau] = \frac{1}{n} R_n^\lambda(\tau) \ast q^n \]  

(33)

in our Lagrangians, where \( n \geq 2 \). The term in the action resulting from (33) is given by

\[ \mathcal{V}_n[q;t] = \frac{1}{n} \int_a^t R_n^\lambda(\tau) \ast q^n\,d\tau. \]  

(34)
Perturbing $q(\tau_i)$ as before gives us

$$\frac{\delta \mathcal{V}_n[q; t]}{\delta q(\tau)} = \int_a^t r_{n-1}^\lambda(\tau) \ast q^{n-1} \, d\tau$$

$$= \int_a^t \int_0^{\tau-a} \cdots \int_0^{\tau-a} \frac{C_{n-1}}{\Gamma(\lambda + n - 2)} \left( \sqrt{\tau_1^2 + \cdots + \tau_{n-1}^2} \right)^{\lambda-1} \times q(\tau - \tau_1) \cdots q(\tau - \tau_{n-1}) \, d\tau_{n-1} \, d\tau_1 \, d\tau \tag{35}$$

where (36) follows from (35) by the change of variables $\tau_i \rightarrow \tau - \tau_i$. Now we switch to spherical coordinates so that (36) becomes

$$\frac{\delta \mathcal{V}_n[q; t]}{\delta q(\tau)} = \int_a^t \int_0^\infty C_{n-1} r^{\lambda+n-3} \frac{Q(r, \tau)}{\Gamma(\lambda + n - 2)} \, dr \, d\tau$$

$$= \int_a^t \int_0^\infty r^{\lambda+n-3} \frac{Q(r, \tau)}{\Gamma(\lambda + n - 2)} \, dr \, d\tau \tag{37}$$

where $Q(r, \tau)$ is that part of the integral in (36) that depends on $r$ and $\tau$ after integrating over the angles $\omega_i, i = 1, \ldots, n - 1$. (Note that the restrictions $\tau_i \leq \tau$ become restrictions on the $\omega_i$ when we switch to spherical coordinates. Also, the exact form of $Q(r, \tau)$ is unimportant for our purposes, as we now show.) Rewrite (37) as

$$\frac{\delta \mathcal{V}_n[q; t]}{\delta q(\tau)} = \int_a^t \int_0^\infty \Phi_{\lambda+n-2}(r) Q(r, \tau) \, dr \, d\tau. \tag{38}$$

For the potential energy terms we take $\lambda = 2 - n$. Then (38) becomes

$$\frac{\delta \mathcal{V}_n[q; t]}{\delta q(\tau)} = \int_a^t Q(0, \tau) \, d\tau = \int_a^t q^{n-1}(\tau) \, d\tau \tag{39}$$

From (39) it follows that (33), with $\lambda = 2 - n$, can be rewritten as

$$\mathcal{L}_n[q; \tau] = \frac{1}{n} \delta \left( \sqrt{(\tau - \tau_1)^2 + \cdots + (\tau - \tau_n)^2} \right) \ast q(\tau_1) \cdots q(\tau_n) \tag{40}$$

so that (34) becomes

$$\mathcal{V}_n[q; t] = \frac{1}{n} \int_a^t \delta \left( \sqrt{(\tau - \tau_1)^2 + \cdots + (\tau - \tau_n)^2} \right) \ast q(\tau_1) \cdots q(\tau_n) \, d\tau. \tag{41}$$
All our potential energy terms in the Lagrangian will be as in (40).

So now we can include driving forces, higher ordered potentials and derivatives of arbitrary order in our Lagrangians. We are also able to derive purely causal equations of motion by using the kernels in (13) in our Lagrangians. A general Lagrangian will be given by

\[
\mathcal{L}[q; \tau] = \int_a^\tau f(\tau_1)q(\tau_1) \, d\tau_1 + \frac{1}{2} \sum_{j=1}^k c_j R_2^{-\alpha_j}(\tau) \ast q(\tau_1)q(\tau_2) + \sum_{m=2}^\infty \frac{c_m}{m} \delta \left( \sqrt{(\tau - \tau_1)^2 + \cdots + (\tau - \tau_m)^2} \right) \ast q(\tau_1) \cdots q(\tau_m)
\]

where the \(c_j\) and \(c_m\) are constants and \(0 < \alpha_1 < \ldots < \alpha_n\). Allowing \(\alpha_j < 0\) results in a fractional diffeo-integral equation of motion.

5 Discussion

We have presented a rather general method to model nonconservative systems. There are two key observations that allowed this to be possible. The main point was to find the correct form for the Volterra series expansion of the Lagrangians \(\mathcal{L}[q; \tau]\). This form is given in (42). In particular, allowing the Volterra series kernels to depend on the ‘extra’ parameter \(\tau\) allowed us to get around Bauer’s corollary [3] and the result of Dreisigmeyer and Young in [5]. Because of this expanded definition of the allowed kernels, we were able to find the \(R_1^{\lambda}(\tau)\) that met our requirement for a causal equation of motion. That is, we abandoned using the fractional derivative kernels \(\Phi_\alpha^{\pm}(\tau)\) in our Lagrangians and actions, as in [6]. This is the second observation that allowed us to achieve our goal of a single retarded equation of motion.

There are a few areas of future research that would be interesting to pursue in relation to this paper. Let us initially list the more mathematical of these. First, are the \(R_1^{\lambda}(\tau)\) the only kernels that allow us to derive the correct equations of motion? We have not proved that they are in this paper so, there may be other, more appropriate kernels to use. Also, it would be nice to have a rigorous exploration of the ‘generalized convolution’ operation that appears in the \(\mathcal{L}[q; \tau]\) and \(\mathcal{V}[q; t]\). While we treated both the Lagrangians and the actions as distributions, this still needs to be given a firm mathematical foundation. Finally, we have only let \(\lambda \in \mathbb{R}\)
and \( n = 2, 3, \ldots \) in the \( R^\lambda_n(\tau) \). In fact, for \( n \neq 2 \) only \( \lambda = 0 \) was allowed. It would be interesting to extend this to arbitrary \( \lambda \in \mathbb{R} \) and \( n \geq 2 \). Allowing \( \lambda \in \mathbb{R} \) for any \( n = 2, 3, \ldots \) is relatively straightforward. Allowing for fractional dimensions, e.g., \( n \geq 2 \) with \( n \in \mathbb{R} \), is a more difficult and interesting question.

For physics, the central problem is probably to develop a Hamiltonian formulation that corresponds to our Lagrangian one. Also, it would be interesting to examine what effects special relativity may have in our formalism. Finally, the question of quantization, via, say, the path integral, is also open. All of these areas of research may cast light on the question of whether the \( R^\lambda_n(\tau) \) are the correct, or only, kernels to use in our Lagrangians.

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