NONEMPTINESS OF BRILL-NOETHER LOCI

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Abstract. Let $X$ be a non-singular algebraic curve of genus $g$. We prove that the Brill-Noether locus $W_{n,d}^{k-1}$ is non-empty if $d = nd' + d''$ with $0 < d'' < 2n$, $1 \leq s \leq g$, $d' \geq (s - 1)(s + g)/s$, $n \leq d'' + (n - k)g$, $(d'', k) \neq (n, n)$. These results hold for an arbitrary curve of genus $\geq 2$, and allow us to construct a region in the associated “Brill-Noether $(\mu, \lambda)$-map” of points for which the Brill-Noether loci are non-empty. Even for the generic case, the region so constructed extends beyond that defined by the so-called “Teixidor parallelograms.” For hyperelliptic curves, the same methods give more extensive and precise results.

1. Introduction

Brill-Noether theory is concerned with the study of the subvarieties of the moduli space of stable bundles, determined by bundles having at least a specified number of sections. More precisely, if $\mathcal{M}(n, d)$ is the moduli space of stable vector bundles of rank $n$ and degree $d$ over a non-singular algebraic curve $X$ of genus $g \geq 2$ over $\mathbb{C}$, and $k \geq 1$, the corresponding Brill-Noether locus is

$W_{n,d}^{k-1} := \{ E \in \mathcal{M}(n, d) | h^0(E) \geq k \}$.

The main questions in Brill-Noether theory regard the nonemptiness, dimension, connectedness, irreducibility, cohomology classes, etc., of these varieties. (Similar statements can be made for semistable bundles.)

For line bundles, Brill-Noether theory has been studied since the last century, and for a generic curve the basic questions have been answered (see [1]). However, the corresponding theory for vector bundles of higher rank is far from being complete even for the generic case. (In section 2 we will recall the known results for this case.)

In this paper we will be concerned with the nonemptiness question. We will prove (see Corollary 3.2 and Theorem 3.9) that

if $d = nd' + d''$, $0 < d'' < 2n$, and $d' \geq 0$ then, for any $X$,

$W_{n,d}^{k-1} \neq \emptyset$ if $n \leq d'' + (n - k)g$ and $(d'', k) \neq (n, n)$. (A)

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More generally, if \(1 \leq s \leq g\) and there exists a line bundle \(L\) on \(X\) of degree \(\text{deg}(L) \geq s\), then
\[
\mathcal{W}_{n,d}^{s+1} \neq \emptyset \quad \text{if} \quad 0 < d'' < 2n, \quad n \leq d'' + (n-k)g \quad \text{and} \quad (d'', k) \neq (n, n). \quad (B)
\]

Observe that such a line bundle \(L\) always exists if \(d'' \geq (s-1)(s+g)/s\). If \(X\) is hyperelliptic, we see by considering powers of the hyperelliptic line bundle that \(L\) exists if \(d'' \geq 2s - 2\).

To get a clear picture of the triples \((n, d, k)\) for which \(\mathcal{W}_{n,d}^{k-1}\) is not empty, as Alastair King has pointed out, it is easier to represent this kind of result if we write \(\mu = d/n\), \(\lambda = k/n\) and plot points in the \((\mu, \lambda)\)-plane; we will refer to this representation as the Brill-Noether map (or BN map). It will be convenient to call a point \((\mu, \lambda)\) \(\in \mathbb{Q}^2\) a \(n\)-Brill-Noether point (or \(n\)-BN point), \(n \in \mathbb{N}\), if \(d = n\mu\) and \(k = n\lambda\) are both integers and \(\mathcal{W}_{n,d}^{k-1} \neq \emptyset\). If \((\mu, \lambda)\) is \(n\)-BN for all \(n\) such that \(d = n\mu\) and \(k = n\lambda\) are both integers, we just say it is a BN point.

The results (A) and (B) define a region in the \((\mu, \lambda)\)-plane, which we denote by \(BMNO\), where the points are \(n\)-BN for “many values” of \(n\), and thus the corresponding Brill-Noether loci are non-empty (for an explanation of what we mean by “many values” see Remarks 4.3 and 4.4). Actually, in the hyperelliptic case we give a precise description of which points are \(n\)-BN. The results, in this case, come close to a complete solution of the nonemptiness problem. In particular, the boundary of the region in which stable bundles of rank \(> 1\) can exist is completely determined, and, as might be expected, it is close to the Clifford line.

On the other hand, the results in [16], later refined in [8], show that one can define a polygonal region \(T\), the so-called “Teixidor’s parallelograms,” such that all the points \((\mu, \lambda)\) inside \(T\) are BN, except perhaps at certain vertices. The region \(BMNO\) covers a large part of \(T\), but more importantly it extends beyond \(T\). Our methods, and especially Theorem 3.9, give stronger results for special curves. Furthermore, since we do not use the results of [16] and [8], our results give another proof of nonemptiness for those parts of \(T\) which are included in \(BMNO\).

The paper is organized as follows: In section 2 we give a brief survey of what is known about \(\mathcal{W}_{n,d}^{k-1}\). In section 3, we prove assertions (A) and (B). In section 4 we describe the region \(BMNO\) in the \((\mu, \lambda)\)-plane for the general case. In section 5 we compare the regions \(BMNO\) and \(T\). In section 6 we study the case where \(X\) is hyperelliptic.

2. A survey of the known theory

In this section we recall the known results of Brill-Noether theory for vector bundles of higher rank (see also [3] and [4]), and thereby also fix notations. We will use the
Brill-Noether map to indicate the regions where not only the nonemptiness is known but also some of the topology of \( W_{n,d}^{k-1} \).

Let \( X \) be a non-singular algebraic curve of genus \( g \geq 2 \) over \( \mathbb{C} \) and \( \mathcal{M}(n,d) \) the moduli space of stable vector bundles over \( X \) of rank \( n \) and degree \( d \). We define the Brill-Noether loci for \((n, d, k)\),

\[
\mathcal{W}_{n,d}^{k-1} := \{ E \in \mathcal{M}(n,d) | h^0(E) \geq k \}
\]

as in the introduction. Denoting by \( \tilde{\mathcal{M}}(n,d) \) the moduli space of equivalence classes of semistable bundles over \( X \) of rank \( n \) and degree \( d \), we can define similarly Brill-Noether loci \( \tilde{\mathcal{W}}_{n,d}^{k-1} \) in \( \tilde{\mathcal{M}}(n,d) \). In what follows we will concentrate on stable bundles, but we will also explicitly indicate where semistable bundles are allowed. Since for \( k > 0, d < 0 \), \( \mathcal{W}_{n,d}^{k-1} = \emptyset \) and for \( k \leq 0 \), \( \mathcal{W}_{n,d}^{k-1} \) is the whole moduli space \( \mathcal{M}(n,d) \), we will assume that \( d \geq 0 \) and \( k \geq 1 \).

It follows from the theory of determinantal varieties that every non-empty component of \( \mathcal{W}_{n,d}^{k-1} \) has dimension greater than or equal to the Brill-Noether number

\[
\rho_{n,d}^{k-1} := n^2(g-1) + 1 - k(k-d+n(g-1)),
\]

and for generic \( X \) this number is the expected dimension of \( \mathcal{W}_{n,d}^{k-1} \). However, there is no similar formula for the expected dimension of \( \tilde{\mathcal{W}}_{n,d}^{k-1} \).

For \( n = 1 \), it is a classical result (see [1]) that \( \mathcal{W}_{1,d}^{k-1} \neq \emptyset \) if \( \rho_{1,d}^{k-1} \geq 0 \); moreover, for a generic curve the converse is also true. However, for \( n \geq 2 \), it is known (see [3]) that \( \rho_{n,d}^{k-1} \geq 0 \) does not imply that \( \mathcal{W}_{n,d}^{k-1} \neq \emptyset \).

A point \((\mu, \lambda)\) \( \in \mathbb{Q}^2 \) will be called a \( n \)-Brill-Noether point (or \( n \)-BN point), \( n \in \mathbb{N} \), if \( d = n\mu \) and \( k = n\lambda \) are both integers and \( \mathcal{W}_{n,d}^{k-1} \neq \emptyset \). If it is \( n \)-BN for all \( n \) such that \( d = n\mu \) and \( k = n\lambda \) are both integers we just say it is a BN point.

In the Brill-Noether map, using the Riemann-Roch Theorem and Clifford’s Theorem, one can define a region such that outside this region the problem becomes trivial, in the sense that \( \mathcal{W}_{n,d}^{k-1} \) is either empty or the whole moduli space \( \mathcal{M}(n,d) \). More precisely, consider the following lines:

\begin{enumerate}
  \item \( \mu = \lambda + g - 1 \) \quad (Riemann-Roch line)
  \item \( \mu = 2\lambda - 2 \) \quad (Clifford line)
\end{enumerate}

These lines, together with the positive axes and the line \( \mu = 2g-2 \), define a bounded pentagonal region, which we denote by \( P \). That is, we define \( P \) to be the region defined by the inequalities

\[
\mu < \lambda + g - 1, \mu \geq 2\lambda - 2, 0 \leq \mu \leq 2g-2, \lambda > 0
\]

(see Figure 1).
Below and to the right of $P$, Riemann-Roch implies that $W_{n,d}^{k-1}$ is the whole space. Above and to the left of $P$, Riemann-Roch and Clifford’s Theorem, together with the definition of stability, imply that $W_{n,d}^{k-1}$ is empty. Thus, we are interested in studying only the points inside $P$.

**Remark 2.1.**

1) For $\mu = 0$, the only stable bundle in $P$ is the trivial line bundle $\mathcal{O}$ at the point $(0, 1)$, while for $\mu = 2g - 2$, the only such bundle is the canonical line bundle $K$ at the point $(2g - 2, g)$. However semistable bundles exist at all points of these two edges of $P$ [3]. Note that, according to our definition, the points $(0, 1)$ and $(2g - 2, g)$ are only 1-BN.

2) The inequalities defining $P$ are all sharp except for the Clifford bound. The exact rôle of this bound is not clear but it has, for example, been improved by Re (see [13]) for non-hyperelliptic curves. In this case, if we restrict to the range $1 \leq \mu \leq 2g - 3$, we can replace the Clifford line by the line $\mu = 2\lambda - 1$. For further improvements, see for example [3] and [10].

By Serre duality we know that, if $(\mu, \lambda)$ is BN, then so is

$$\sigma(\mu, \lambda) := (2g - 2 - \mu, \lambda + g - 1 - \mu).$$

Though it is not readily apparent from Figure 1, this gives a symmetry in $P$ through the line $\mu = g - 1$. For later purposes, it will be convenient to write

$$R = \{(\mu, \lambda) \in P : \mu \leq g - 1\},$$

so that in particular $P = R \cup \sigma(R)$ (see Figure 1).

An important feature of the BN map is the curve defined by the equation

$$\tilde{\rho} = \frac{1}{n^2}(\rho_{n,d}^{k-1} - 1) = 0,$$

called the Brill-Noether curve (or BN curve). From what was said earlier, this represents the boundary of the region where one would expect the Brill-Noether loci to have positive dimension, though it is known that this analogy to the case of line bundles is not valid in general (see [3], [7], etc.). The BN curve is a portion of a hyperbola, with equation

$$\tilde{\rho}(\mu, \lambda) = (g - 1) - \lambda(\lambda - \mu + g - 1) = 0.$$

The results of [16] and [3] allow us to define a polygonal region which we denote by $T$, contained in the interesting region $P$ in the $(\mu, \lambda)$-plane, such that any point $(\mu, \lambda)$ in $T$ is BN except possibly for certain vertices. This region was described in detail in [3] and [7] in its original form, and we will recall its construction in section 4 incorporating the results of [8]. For the time being, we just point out that $T$ has sides parallel to the lines $\lambda = 0, \mu = \lambda$, and vertices at points with integer coordinates, on or below the BN curve (see Figure 2).
The most significant results for our purposes are the following, which hold for slopes restricted to $0 \leq \mu < 2$:

1. For $0 < \mu \leq 1$, Brambila-Paz, Grzegorczyk and Newstead proved in [3] that $(\mu, \lambda)$ is BN if and only if $1 \leq \mu + (1 - \lambda)g$ and $(\mu, \lambda) \neq (1, 1)$.
2. For $1 < \mu < 2$, Mercat in [7] proved that $(\mu, \lambda)$ is BN if and only if $1 \leq \mu + (1 - \lambda)g$.

In the BN map these results define two trapezoidal regions inside $R$, which we will denote by $BGN$ and $M$ respectively (see Figure 3).

Ballico, Mercat and Newstead have recently proved the existence of stable bundles at some points outside the regions defined above; in particular, these bundles can have negative Brill-Noether number [4].

For $X$ generic, Teixidor also proved that $W_{k-1}^{n,d}$ has an irreducible component of dimension $\rho_{n,d}^{k-1}$. For any curve, Brambila-Paz, Grzegorczyk and Newstead in [3] also proved that for $0 < \mu \leq 1$, if $W_{n,d}^{k-1}$ is nonempty, then it is irreducible of dimension $\rho_{n,d}^{k-1}$, and $\text{Sing} W_{n,d}^{k-1} = W_{n,d}^k$ ([3], Theorem A). For $1 < \mu < 2$, Mercat in [7] also proved that, if $n = d + (n - k)g$ or $n < d < n + g$, $W_{n,d}^{k-1}$ is irreducible ([7], 2-B-1 and 3-A-1); in any case all components have the expected dimension and $\text{Sing} W_{n,d}^{k-1} = W_{n,d}^k$ ([7], 2-C-1). So, for slopes $0 \leq \mu < 2$, the results are very complete.

For $k = 1$, Sundaram [14] proved that $W_{0,d}^0$ is irreducible of dimension $\rho_{n,d}^0$, and Laumon [6] showed $\text{Sing} W_{n,d}^0 = W_{n,d}^1$. For rank 2 there are also results of nonemptiness and irreducibility in [13], [7] and [14] and for rank 3 in [12].

**Remark 2.2.** If $X$ is not hyperelliptic, the results of [6] can be extended to cover the case $\mu = 2$ [3]. For further details, see Remark 4.8.

From the results of [3] and the symmetry of the region $P$, for $g = 2$ one has a complete description of Brill-Noether loci: nonemptiness, irreducibility and singularities.

### 3. NEMHENTINESS OF BRILL-NOETHER LOCI

In this section we will prove assertions (A) and (B).

Note first that, for any line bundle $L$ of degree $d'$, the formula $E \mapsto E \otimes L$ defines an isomorphism

$$\Phi_L : \mathcal{M}(n, d) \to \mathcal{M}(n, nd' + d).$$  \hspace{1cm} (1)

We shall make repeated use of this idea of tensoring stable bundles by line bundles. If $V$ is a subvariety of $\mathcal{M}(n, d)$, then $\Phi_L(V)$ is a subvariety of $\mathcal{M}(n, nd'+d)$; in particular, if $V$ is a non-empty Brill-Noether locus and $h^0(L) > 0$, then $\Phi_L(V)$ will meet certain Brill-Noether loci in $\mathcal{M}(n, nd'+d)$, which will therefore be non-empty.

Actually, if $d' \geq 0$ then there always exists a line bundle $L$ of degree $d'$ that has at least one section, so we have the following theorem.
**Theorem 3.1.** If $W^{k-1}_{n,d} \neq \emptyset$ (respectively $\tilde{W}^{k-1}_{n,d} \neq \emptyset$), then $W^{k-1}_{n,d+nd'} \neq \emptyset$ (respectively $\tilde{W}^{k-1}_{n,d+nd'} \neq \emptyset$) for any $d' \geq 0$.

**Proof:** Choose a line bundle $L$ of degree $d'$ with $h^0(L) > 0$. Then, for any $E$ with $h^0(E) \geq k$ we have $h^0(E \otimes L) \geq k$.

**Corollary 3.2.** (Assertion (A)) Suppose $d = nd' + d''$ with $0 < d'' < 2n$, $d' \geq 0$, $n \leq d'' + (n-k)g$ and $(d'', k) \neq (n, n)$. Then $W^{k-1}_{n,d} \neq \emptyset$.

**Proof:** This follows from the results of [3] and [7] and Theorem 3.1.

**Corollary 3.3.** If $k < n$, then $W^{k-1}_{n,rn} \neq \emptyset$ for $r \geq 1$.

**Proof:** Take $d'' = n$ and $d' = r - 1$ in Corollary 3.2.

In general, the multiplication map

$$\mu_{E,F} : H^0(E) \otimes H^0(F) \to H^0(E \otimes F)$$

is not injective and $h^0(E) \cdot h^0(F)$ does not give a lower bound for $h^0(E \otimes F)$. However, if $E$ is a point in $W^{k-1}_{n,d}$ with $d/n < 2$ and $L$ a line bundle of degree $d' \geq 0$ with at least $s$ independent sections, then we have the following lemmas.

**Lemma 3.4.** If $d < n + g$, then $h^0(E \otimes L) \geq ks$.

**Proof:** From [3] (for $d \leq n$) and [4], 3-A-1 (for $n < d < n + g$) we know that any such bundle has $\mathcal{O}^k$ as a subsheaf. Hence, $\oplus^k L$ is a subsheaf of $E \otimes L$. Therefore $h^0(E \otimes L) \geq h^0(\oplus^k L) \geq ks$.

**Remark 3.5.** Note that the existence of $E$ implies that $n \leq d + (n-k)g$, so the hypothesis $d < n + g$ implies that $k \leq n$.

**Lemma 3.6.** If $k > n$, $d = n + (k-n)g$ and $d' \leq 2g$, then $h^0(E \otimes L) \geq ks$.

**Proof:** From [7], 2-B-1 we know that any such bundle fits in an exact sequence

$$0 \to F^* \to \mathcal{O} \otimes H^0(X, F)^* \to E \to 0 \tag{2}$$

where $F$ is a stable bundle of slope $> 2g$ and $h^0(F) = h^0(E) = k$. Tensor (2) by $L$ and take the cohomology sequence. Since $\deg(F^* \otimes L) < 0$, we have $H^0(X, F^* \otimes L) = 0$ and hence

$$h^0(E \otimes L) \geq h^0(L) \cdot h^0(F) \geq ks.$$
Lemma 3.7. If \( n + g \leq d < 2n \) and \( d' \leq 2g \), then there exists \( E \in \mathcal{W}_{n,d}^{k-1} \) with \( h^0(E \otimes L) \geq ks \).

Proof: In this case we have two exact sequences

\[
0 \to \mathcal{O}' \to E' \to E \to 0 \quad (3)
\]

\[
0 \to D(E')^* \to \mathcal{O}^{n+l+l'} \to E' \to 0 \quad (4)
\]

where \( k \leq n + l \), \( E \in \mathcal{W}_{n,d}^{m+l-1} \), \( E' \) and \( D(E') \) are stable and \( \mu(D(E')) > 2g \) (see [7], 3-B-1 and its proof). Tensor both sequences by \( L \) and take the cohomology sequences.

Since \( H^0(X, L \otimes E' \otimes L) = 0 \),

\[
h^0(E' \otimes L) \geq h^0(\oplus^{n+l+l'} L) = (n + l + l')h^0(L)
\]

Thus

\[
h^0(E \otimes L) \geq (n + l)h^0(L) \geq ks.
\]

Remark 3.8. i) From the Brill-Noether theory for line bundles we know that, if \( d' \geq \eta(s) := (s - 1)(s + g)/s \), then there exists a line bundle \( L \) of degree \( d' \) with at least \( s \) independent sections. Actually, \( \mathcal{W}_{k,d'}^{s-1} \) is the variety of such bundles and, for \( X \) generic, it has dimension \( g - s(s - d' + g - 1) = s(d' - \eta(s)) \).

ii) If \( X \) is hyperelliptic, we see by considering powers of the hyperelliptic line bundle that \( L \) exists if \( d' \geq 2s - 2 \).

We deduce the following theorem that proves assertion (B):

Theorem 3.9. Suppose \( d = nd' + d'' \) with \( d' \geq 0 \), \( 0 < d'' < 2n \) and that \( 1 \leq s \leq g \). If \( n \leq d'' + (n - k)g \) and \( (d'', k) \neq (n, n) \) and there exists a line bundle \( L \) on \( X \) of degree \( d' \) with \( h^0(L) \geq s \), then \( \mathcal{W}_{n,d''}^{sk-1} \neq \emptyset \).

Proof: Note first that, for fixed \( n, k, d'' \), \( s \), if the theorem is true for one value of \( d' \), it is true for all larger values. Since \( \eta \) is an increasing function of \( s \) and \( \eta(g) = 2g - 2 \), it is therefore sufficient by Remark 3.8 to prove the theorem with the additional hypothesis that \( d' \leq 2g - 2 \). Now, from [3] and [7] we know that under the given hypotheses \( \mathcal{W}_{n,d''}^{sk-1} \) is non-empty. It follows from the lemmas that \( \mathcal{W}_{n,d''}^{sk-1} \) is non-empty.

Remark 3.10. In the semistable case, the theorem can be extended to the cases \( d'' = 0 \), \( k \leq n \) and \( (d'', k) = (n, n) \).

Corollary 3.11. If \( k < n \) and \( r \geq s + g - g/s \), then \( \mathcal{W}_{r,n}^{sk-1} \neq \emptyset \).
Proof: Take \( d'' = n \) and \( d' = r - 1 \) in Theorem 3.9.

We finish this section by describing the above results for \( g = 3 \).

Example 3.12. Suppose \( X \) has genus 3. It follows from [3], [7] and Corollary 3.3 that a point \( (\mu, \lambda) \in R \) is BN if \( \mu > 0, 1 \leq \mu + 3(1 - \lambda), (\mu, \lambda) \neq (1, 1) \), except possibly when \( \mu = 2 \) and \( 1 \leq \lambda \leq 4/3 \). In fact \((2,1)\) is also BN by [10] and [8]. Moreover, any \( W_{n,d}^{k-1} \) corresponding to such a point has pure dimension \( 2n^2 + 1 - k(k - d + 2n) \) and \( \text{Sing} W_{n,d}^{k-1} = W_{n,d}^k \); it is irreducible if \( 0 < d < n + 3 \) or if \( d = n + 3(k - n) \).

If \( X \) is not hyperelliptic, then by [9] we can remove the exceptional case and say that \( (\mu, \lambda) \in R \) is BN if and only if \( \mu > 0, 1 \leq \mu + 3(1 - \lambda), (\mu, \lambda) \neq (1, 1) \). The statements about dimension and singularities still apply. Apart from the trivial bundle \( \mathcal{O} \), there is also precisely one further stable bundle on \( X \) in \( R \), namely the bundle \( E_K \) with \( n = 2, d = 4, k = 3 \) (see [7], 2-A-4).

4. The region BMNO in the \((\mu, \lambda)\)-plane

In this section, using the results of the previous section, combined with Serre duality, we describe the region \( \text{BMNO} \). Throughout the section \( X \) is an arbitrary non-singular algebraic curve of genus \( g \geq 3 \).

In order to translate the results of section 2 into geometric form, we introduce for each \( d' \) and \( s \) such that \( d' \geq \eta(s) = (s - 1)(s + g)/s \), the affine map \( T_{d',s}(\mu, \lambda) \) given by

\[
T_{d',s}(\mu, \lambda) = (\mu + d', s\lambda).
\]

Notice that it shifts points to the right, and, if \( s > 1 \), also expands in the \( \lambda \) direction. (We shall refer to these maps as translations although strictly speaking only the \( T_{d',1} \) are translations.) The idea is to use the regions \( BGN \) and \( M \) as “tiles” to cover a larger region, the tiling being obtained by translating \( BGN \) and \( M \) by the maps \( T_{d',s} \). Then we will apply Serre duality to obtain the \( \text{BMNO} \) region.

More precisely, recall that \( \eta(s) = (s - 1)(s + g)/s \), and set \( \hat{\eta}(s) = \lceil \eta(s) \rceil \) (we will use the notation \( \lceil \cdot \rceil \), and \( \lfloor \cdot \rfloor \), respectively, for the least integer not smaller, and the largest integer not greater than a given number, the so-called “ceiling” and “floor” functions). We will consider \( d' \) and \( s \) such that \( d' \geq \hat{\eta}(s) \) for the affine maps \( T_{d',s} \). For the description of the regions \( BGN \) and \( M \), we consider the trapezia

\[
BGN' = \{(\mu, \lambda) : 0 < \mu \leq 1, \quad 0 < \lambda \leq \frac{1}{g}(\mu + g - 1)\}
\]

\[
M' = \{(\mu, \lambda) : 1 < \mu \leq 2, \quad 0 < \lambda \leq \frac{1}{g}(\mu + g - 1)\}.
\]
Note however that the point \((1, 1) \in BGN'\) is only 1-BN, so we define
\[
BGN = BGN' - \{(1, 1)\}.
\]
Moreover, a geometrical interpretation of Corollary 3.3 shows that, if we translate \(BGN\) by \(T_{1,1}\), we obtain the points in the boundary of \(M'\) with \(\mu = 2, 0 < \lambda < 1\), and so these also give BN points. We therefore define
\[
M = M' \cup \{(2, \lambda) : 0 < \lambda < 1\}.
\]

**Remark 4.1.** Notice that there are still some points in the boundary line \(\mu = 2\) of \(M\) that are not covered this way, namely those for which \(1 \leq \lambda \leq 1 + 1/g\), and therefore we do not know whether they are BN points or not. For the non-hyperelliptic case see Remark 4.3.

From Theorem 3.9 and Remark 3.8 we have

**Theorem 4.2.** If \((\mu, \lambda) \in BGN \cup M, 1 \leq s \leq g\) and \(d' \geq \hat{\eta}(s)\), then \(T_{d',s}(\mu, \lambda)\) is \(n\)-BN for all \(n\) such that \((\mu, \lambda)\) is \(n\)-BN.

**Remark 4.3.** One can in fact obtain many points in \(T_{d',s}(BGN \cup M)\) which are BN points. Let \((\mu, \lambda)\) \(\in T_{d',s}(BGN \cup M)\) and write \(\mu = \frac{a}{b}, \lambda = \frac{c}{e}\) in their lowest terms. To get bundles of rank \(n\), the conditions we need are that \(b \mid n\) and \(es \mid cn\). If \(s \mid c\), these conditions reduce to \(b \mid n\) and \(e \mid n\), so \((\mu, \lambda)\) is BN. Points of this form are dense in \(T_{d',s}(BGN \cup M)\).

**Remark 4.4.** i) If we take a point \((\mu, \lambda)\) in \(T_{d',s}(BGN \cup M)\) lying strictly below the top boundary, we can obtain an improvement to Theorem 4.2. Suppose \(n\) is a positive integer such that \(n\mu\) and \(n\lambda\) are both integers and define \(\lambda' = \frac{s}{n} \left\lceil \frac{n\lambda}{s} \right\rceil\). If \((\mu, \lambda') \in T_{d',s}(BGN \cup M)\), then \(W^{n\lambda'-1}_{n,n\mu} \neq \emptyset\); hence \(W^{n\lambda'-1}_{n,n\mu} \neq \emptyset\). Now
\[
\lambda' \leq \frac{s}{n} \left( \frac{n\lambda + s - 1}{s} \right) = \lambda + \frac{s - 1}{n};
\]
so this holds for all sufficiently large \(n\) (even if \(n\lambda/s\) is not an integer).

ii) By a similar method, taking \(\lambda' = s\), we can show also that the region
\[
\mu > \hat{\eta}(s) + 1, \; \lambda \leq s, \; \mu \notin N
\]
consists entirely of BN points.

The translates of the regions \(BGN\) and \(M\) can be described explicitly as follows:
\[
T_{d',s}(BGN) = \{(\mu, \lambda) : d' < \mu \leq d' + 1, \; 0 < \lambda \leq \frac{s}{g}(\mu - d' - 1) + s,
\]
\[
(\mu, \lambda) \neq (d' + 1, s)\}.
\]
\[ T_{d',s}(M) = \{(\mu, \lambda) : d' + 1 < \mu < d' + 2, \ 0 < \lambda \leq \frac{s}{g}(\mu - d' - 1) + s \} \]

Furthermore, it follows from these formulae that \( T_{d'+1,s}(BGN) \) is strictly included in \( T_{d',s}(M) \), while \( T_{d'+1,s+1}(BGN) \) strictly includes \( T_{d',s}(M) \), i.e.

\[ T_{d'+1,s}(BGN) \subset T_{d',s}(M) \subset T_{d'+1,s+1}(BGN). \] (5)

Therefore, if \( d' \geq \hat{\eta}(s) \), the translate \( T_{d',s}(BGN) \) covers a larger region than does \( T_{d'-1,s-1}(M) \).

We will use these relations to translate, in a convenient way, the known regions. From Example 3.12, we can assume \( g > 3 \).

We can obtain a new region of BN points, either by translating \( BGN \) by \( T_{2,1} \), or \( M \) by \( T_{1,1} \); by (5), the latter covers a larger area than the former, so we use \( T_{1,1}(M) \) to enlarge the region.

We now continue the process, translating \( M \) by \( T_{d',1} \) with increasing \( d' \) (but always keeping \( d' < g - 2 \) to remain in \( R \)) (as illustrated in Figure 4; there, \( T_{2,1}(BGN) \) is represented by the lighter part in the first diagram, so we can compare it to \( T_{1,1}(M) \)).

Now, for exactly the same reasons as before, this is the best we can do as long as \( d' < \hat{\eta}(2) - 1 \). However, when \( d' = \hat{\eta}(2) - 1 \), \( T_{\hat{\eta}(2)-1,1}(M) \) covers a smaller region than \( T_{\hat{\eta}(2),2}(BGN) \), so we now use the latter (see Figure 5). Of course we can now only guarantee to get \( n \)-BN points, for some values of \( n \); however, see Remarks 4.3 and 4.4.

If we have not yet arrived at \( d' = g - 2 \), at the next step we cover a larger region using now \( T_{\hat{\eta}(2),2}(M) \). We then continue the process until we arrive at \( \hat{\eta}(3) - 1 \) or \( g - 2 \). In the latter case we stop, in the former we use now \( T_{\hat{\eta}(3),3}(BGN) \), and repeat the process.

The union of trapezia obtained in this way is therefore a polygonal region which consists entirely of \( n \)-BN points for some values of \( n \). This is the best we can do purely by translating, but there is a possibility of obtaining further \( n \)-BN points by first translating beyond \( \mu = g - 1 \) and then applying the Serre duality map \( \sigma \).

Thus we consider the affine maps \( U_{d',s} = \sigma \circ T_{d',s} \), given explicitly by

\[ U_{d',s}(\mu, \lambda) = (2g - 2 - \mu - d',\ s\lambda + g - 1 - \mu - d'). \]

We now have \( d' \geq g - 2 \) and \( T_{d',s} \) maps part of \( BGN \cup M \) below the Riemann-Roch line and hence outside \( P \). So \( U_{d',s}(\mu, \lambda) \) will not lie entirely in \( R \); in fact the second coordinate in the above formula can be \( \leq 0 \). However it is easy to see that

\[ U_{d',s}(BGN) \cap R \neq \emptyset \iff d' \geq \hat{\eta}(s) \text{ and } g - 1 \leq d' \leq \min\{s + g - 2, 2g - 3\}, \]

\[ U_{d',s}(M) \cap R \neq \emptyset \iff d' \geq \hat{\eta}(s) \text{ and } g - 2 \leq d' \leq s + g - 3. \]
If \( s \geq g \), then \( \eta(s) \geq s + g - 2 \), so there are no \( d' \) satisfying the above conditions; we shall therefore assume that \( s < g \).

We write for convenience \( U_{d',s}(1, 1) = (d_1, s_1) \), so that
\[
d_1 = 2g - 3 - d', \quad s_1 = s + g - 2 - d'.
\]
Then \( U_{d',s}(BGN) \cap R \) is given by
\[
d_1 \leq \mu < d_1 + 1, \quad 0 < \lambda \leq (1 - \frac{s}{g})(\mu - d_1) + s_1
\]
with the point \((d_1, s_1)\) omitted, while \( U_{d',s}(M) \cap R \) is given by
\[
d_1 - 1 < \mu < d_1, \quad 0 < \lambda \leq (1 - \frac{s}{g})(\mu - d_1) + s_1
\]
together with the line segment
\[
\ell = \{(d_1 - 1, \lambda) : 0 < \lambda < s_1 - 1\}.
\]

The following lemmas will show that we can gain an extra triangle by replacing \( T_{\hat{\eta}(s_1 + 1) - 2, s_1}(M) \) by the appropriate \( (U_{d',s}(BGN) \cap R) \cup \ell'' \) where \( \ell'' \) is a line segment.

**Lemma 4.5.** Suppose \( d' \geq \hat{\eta}(s) \) and \( g - 2 \leq d' \leq s + g - 3 \). Then \( s_1 \geq 1 \), \( d_1 - 1 \geq \hat{\eta}(s_1) \) and
\[
U_{d',s}(M) \cap R \subset T_{d_1 - 1, s_1}(BGN) \cup \ell.
\]

**Proof:** Note first that \( s_1 = s + g - 2 - d' \geq 1 \).

The inequality \( d_1 - 1 \geq \hat{\eta}(s_1) \) is equivalent to
\[
s_1(d_1 - 1) \geq (s_1 - 1)(s_1 + g),
\]
or, substituting for \( s_1, d, \)
\[
(s + g - 2 - d')(2g - 4 - d') \geq (s + g - 3 - d')(s + 2g - 2 - d').
\]
This simplifies to
\[
(s + 1)d' \geq s^2 + (g - 1)s - 2 = (s - 1)(s + g) + g - 2.
\]
But by hypothesis \( sd' \geq (s - 1)(s + g) \) and \( d' \geq g - 2 \). This proves the inequality.

Comparing the formulae for \( U_{d',s}(M) \) and \( T_{d_1 - 1, s_1}(BGN) \), we see that it is now sufficient to prove that \( \frac{s}{g} \leq 1 - \frac{s}{g} \), i.e. \( s_1 + s \leq g \). Now \( s_1 + s = 2s + g - 2 - d' \), so we need to show that \( d' \geq 2s - 2 \). Since \( s < g \), this follows from the hypothesis \( d' \geq \hat{\eta}(s) \).

Note that, if \( s_1 = 1 \), then \( \ell = \emptyset \), while, if \( s_1 > 1 \),
\[
\ell \subset T_{d_1 - 2, s_1 - 1}(BGN).
\]
Combined with the lemma, this tells us that \( U_{d',s}(M) \) gives nothing new.
Lemma 4.6. Suppose \( d' \geq \hat{\eta}(s) \) and \( g - 1 \leq d' \leq s + g - 2 \). If \( d_1 \geq \hat{\eta}(s_1 + 1) \), then
\[
U_{d',s}(BGN) \cap R \subset T_{d_1,s_1+1}(BGN) \cup \ell'
\]
where
\[
\ell' = \{(d_1, \lambda) : 0 < \lambda < s_1\}.
\]

Proof: This follows easily from the formulae for the two sets.

If \( s_1 = 0 \), then \( \ell' = \emptyset \), while, if \( s_1 \geq 1 \),
\[
\ell' \subset T_{d_1-1,s_1}(BGN).
\]
So again we get nothing new.

It remains therefore to consider the case where \( d_1 < \eta(s_1 + 1) \). Now \((d_1 + 1, s_1 + 1) = \sigma(d', s)\). Since the Brill-Noether number \( \rho \) is invariant under \( \sigma \) and \( d' \geq \eta(s) \), it follows that \( d_1 + 1 \geq \eta(s_1 + 1) \). So the only case we need to consider is
\[
d_1 + 1 = \hat{\eta}(s_1 + 1) \leq g - 1.
\]

Lemma 4.7. In the above circumstances,
\[
T_{d_1-1,s_1}(M) \subset (U_{d',s}(BGN) \cap R) \cup \ell''
\]
where
\[
\ell'' = \{(d_1 + 1, \lambda) : 0 < \lambda < s_1\}.
\]

Proof: From the formulae for the two sets, we see that it is sufficient to prove that \( \frac{s_1}{g} \leq 1 - \frac{s_1}{g} \). For this, see the proof of Lemma 4.5.

Thus in each chain
\[
T_{\hat{\eta}(s_1),s_1}(M), \ldots, T_{\hat{\eta}(s_1+1)-2,s_1}(M)
\]
in the construction described earlier, we can gain an extra triangle by replacing \( T_{\hat{\eta}(s_1+1)-2,s_1}(M) \) by the appropriate \((U_{d',s}(BGN) \cap R) \cup \ell''\).

Finally, then, we define \( BMNO \) to be the union of the trapezia constructed above together with their Serre duals. The region \( BMNO \cap R \) is bounded from below by \( \lambda = 0 \), on the sides by \( \mu = 0 \) and \( \mu = g - 1 \), and from above by the graph of a seesaw-like function \( f_g \) defined on the interval \((0, g - 1]\) by
\[
f_g(\mu) = \begin{cases} 
\frac{s_1}{g}(\mu - \lceil \mu \rceil) + s & \mu \in (\hat{\eta}(s), \hat{\eta}(s) + 1] \\
\frac{s_1}{g}(\mu - \lceil \mu \rceil + 1) + s & \mu \in (\hat{\eta}(s) + 1, \hat{\eta}(s + 1) - 1] \\
\frac{\hat{\eta}(s+1)-s}{g}(\mu - \lceil \mu \rceil + 1) + s & \mu \in (\hat{\eta}(s + 1) - 1, \hat{\eta}(s + 1)].
\end{cases}
\]

We extend \( f_g \) to the whole interval \((0, 2g - 2]\) by insisting that its graph is invariant under \( \sigma \); the graph of \( f_g \) is then the top boundary of \( BMNO \).
Figure 6 shows a typical \textit{BMNO} region.

We stress the fact that we have to exclude from \textit{BMNO} those points corresponding to translates of those parts of the boundaries of \textit{BGN} or \textit{M} which are not included in the original regions, and we can summarize as follows:

If \((\mu, \lambda)\) lies in or on the polygon defined above, \((\mu, \lambda)\) is \(n\)-BN, for many values of \(n\), except for \(\mu = 0\) and \((\mu, \lambda) = (1, 1)\), and possibly for \(\mu \in \mathbb{N} \setminus \{0, 1\}, \mu \in (\hat{\eta}(s), \hat{\eta}(s+1)]\), \(\lambda \geq s\).

**Remark 4.8.** When \(X\) is not hyperelliptic, Mercat has proved recently [9] that the results of [7] extend to the case \(\mu = 2\). The constructions of [7] are the same as those of [9], so the proofs of section 2 still work, except that in Lemmas 3.6 and 3.7, we should replace the condition \(d' \leq 2g\) by \(d' \leq 2g - 1\). The effect of this is that those of the points excluded from \textit{BMNO} as above which arise as translates of the right-hand boundary of \(M\) can be restored. However those points arising from the left-hand boundary of \(BGN\) cannot be restored. Thus the only points of the boundary which must be excluded are those of the form \((\hat{\eta}(s), \lambda)\) with \(\lambda > (s - 1)(1 + \frac{1}{g})\) and the points \((\hat{\eta}(s) + 1, s)\) which arise as translates of \((1, 1)\).

**Remark 4.9.** For semistable bundles, the results of [3] and [9] allow us to include both left-hand and right-hand boundaries of \(BGN \cup M\) (and indeed the point \((1, 1)\)). So in this case the whole boundary of \textit{BMNO} can be included. Moreover one can include the whole of the line segments \(\{((\hat{\eta}(s), \lambda) : 0 < \lambda \leq s\}\).

**Remark 4.10.** The analysis in Remarks 4.3 and 4.4 works also for \(U_{d',s}\) and hence whenever \(\lambda < f_g(\mu)\) (with the usual exceptions for integral values of \(\mu\)). So there is certainly a dense subset of \textit{BMNO} consisting of BN points.

Finally, we have the following proposition, showing that the region \textit{BMNO} always “stays close” to the BN curve:

**Proposition 4.11.** Let

\[
\rho_g(\mu) = \sqrt{\frac{(\mu - g + 1)^2 + 4(g - 1) + \mu - g + 1}{2}}
\]

denote the function whose graph is the BN curve. Then, for \(\mu \in (0, 2g - 2)\),

\[
0 \leq \rho_g(\mu) - f_g(\mu) < 1.
\]

**Proof:** Since the graphs of \(\rho_g\) and \(f_g\) are both invariant under \(\sigma\), it is sufficient to prove this for \(\mu \leq g - 1\).
For the first inequality we need to prove that every point of $BMNO$ lies on or below the BN curve. Since this is certainly true for points of $BGN \cup M$ and $\tilde{\rho}$ is invariant under $\sigma$, it is sufficient to prove that, whenever $(\mu, \lambda) \in BGN \cup M$ and $d' \geq \eta(s)$,

$$I = \frac{1}{\lambda} (\tilde{\rho}(T_{d,s}(\mu, \lambda)) - \tilde{\rho}(\mu, \lambda)) \geq 0.$$ 

A simple calculation shows that

$$I = sd' - (s - 1)((s + 1)\lambda - \mu + g - 1) \geq (s - 1)(s - (s + 1)\lambda + \mu + 1) \geq (s - 1) \left( \mu + \frac{s + 1}{g}(1 - \mu) \right) \geq 0.$$

For the second inequality, note first that both $\rho_g$ and its derivative $\rho'_g$ are strictly increasing (this is easy to see either geometrically or by calculus). It follows from the formulae for $f_g(\mu)$ and the fact that, by definition of $\hat{\eta}(s)$,

$$\rho_g(\mu) < s + 1 \text{ for } \mu \leq \hat{\eta}(s + 1) - 1,$$

that it is sufficient to prove the inequalities

$$\rho_g(\hat{\eta}(s + 1)) - \left( \frac{\hat{\eta}(s + 1) - s}{g} + s \right) < 1$$

and

$$\rho_g(\hat{\eta}(s + 1)) - \left( s + 1 - \frac{s + 1}{g} \right) < 1$$

for $s \geq 1$ and $\hat{\eta}(s + 1) \leq g - 1$.

Since $\rho'_g(g - 1) = \frac{1}{2}$ and $\rho'_g$ is strictly increasing,

$$\rho_g(\hat{\eta}(s + 1)) < \rho_g(\hat{\eta}(s + 1) - 1) + \frac{1}{2} < s + \frac{3}{2},$$

On the other hand

$$\frac{\hat{\eta}(s + 1) - s}{g} + s \geq \frac{\eta(s + 1) - s}{g} + s = \frac{s}{s + 1} + s \geq s + \frac{1}{2},$$

proving the first of the required inequalities. Also $\eta(s + 1) \leq g - 1$ implies that $(s + 1)^2 \leq g$; hence $s + 1 \leq \frac{1}{s + 1} \leq \frac{1}{2}$. Thus

$$s + 1 - \frac{s + 1}{g} \geq s + \frac{1}{2},$$

and we are done.
Remark 4.12. A careful analysis of this proof shows that the worst cases for $\rho_g(\mu) - f_g(\mu)$ are as $\mu \to \tilde{\eta}(s+1) - 1$ from above. Thus in fact
\[
\rho_g(\mu) - f_g(\mu) < \max[\rho_g(\tilde{\eta}(s+1) - 1) - s]
\]
taken over values of $s \geq 1$ for which $\tilde{\eta}(s+1) \leq g - 1$, and this inequality is best possible. The best possible inequality which is independent of $g$ is the one stated in the proposition.

Examples of stable bundles which are outside the range to which the constructions of this section apply are given in [2] and [11], and some different examples in [4].

5. Comparison with Teixidor’s region

We now compare BMNO with the corresponding region $T$ constructed by the results of Teixidor [16] and Mercat [8], mainly by means of some examples.

In the stable case, Teixidor’s original result excluded from $\tilde{\W}_{k-1}^{n,d}$ the vertical segments of length 1, with upper end at a point on the BN curve $\tilde{\rho} = 0$ with integer coordinates. However Mercat in [8] removed this restriction except for the topmost point of each segment, although he needs also to exclude all the points described in the last sentence of the following theorem, while Teixidor excluded only those segments whose topmost point lies on the BN curve. We will quote the results of both as follows:

Theorem 5.1. (Teixidor/Mercat) A point $(\mu, \lambda)$ determines a non-empty locus $\tilde{\W}_{k-1}^{n,d}$ if any of the following three conditions holds:
\[
\tilde{\rho}([\mu], [\lambda]) \geq 0 \text{ and } 0 \neq \lambda - [\lambda] \leq \mu - [\mu] \\
\tilde{\rho}([\mu], [\lambda]) \geq 0 \text{ and } \lambda - [\lambda] > \mu - [\mu] \\
\tilde{\rho}([\mu], [\lambda]) \geq 0 \text{ and } \lambda = [\lambda].
\]
Moreover, under the same conditions, $\W_{k-1}^{n,d}$ is non-empty except possibly for points $(\mu, \lambda)$ with $\mu, \lambda$ integers and $\tilde{\rho}(\mu - 1, \lambda) < 0$.

Remark 5.2. In the semistable case, this theorem is a mere translation of a result of Teixidor ([11], Theorem 1, p. 386) to the $(\mu, \lambda)$ language; note that Teixidor’s result is stated for $X$ generic, but for semistable bundles this automatically implies the result for any $X$. Observe that conditions (1) and (2) in fact define triangles in the $(\mu, \lambda)$-plane, with all their vertices at points with integer coordinates, as illustrated in Figure 7, where the lighter area corresponds to the first condition and the darker to the second. Condition 3 describes a horizontal segment of length 1, starting at the point $([\mu], [\lambda])$.

As shown in Figure 7, for any point with integer coordinates on or below the BN curve, the first two conditions together determine a parallelogram; hence, the region defined by Theorem 5.1 is sometimes referred to as “Teixidor’s parallelograms”. We denote this region by $T$. 
Then $d' \geq \hat{\eta}(s)'$ if and only if \( \tilde{\rho}(d' + 1, s) \geq 0 \). (Recall that \( d' \geq \hat{\eta}(s) \) if and only if \( \tilde{\rho}(d', s) \geq -1 \).) The region \( T \) is then bounded below by \( \lambda = 0 \), on the sides by \( \mu = 0 \) and \( \mu = 2g - 2 \) and from above by the graph of a function \( t_g \) defined by

\[
t_g(\mu) = \begin{cases} 
\mu - [\mu] + s & \mu \in (\hat{\eta}(s)', \hat{\eta}(s)' + 1] \\
s & \mu \in (\hat{\eta}(s)' + 1, \hat{\eta}(s) + 1)'.
\end{cases}
\]

Unlike \( f_g \), the function \( t_g \) is in fact continuous and non-decreasing, so the shape of \( T \) is simpler than that of \( BMNO \). Note also that the region covered by Teixidor’s parallelograms is invariant under \( \sigma \), so we do not obtain anything new by using Serre duality. Finally it is easy to check that \( 0 \leq \rho_g(\mu) - t_g(\mu) < 1 \) (compare Proposition 4.11).

Figure 8 shows a typical Teixidor polygon (here, \( g = 10 \) and the only vertex on the BN curve is \( (3, 9) \), since 3 is the only divisor of \( g - 1 = 9 \).)

To compare the upper boundaries of \( T \) and \( BMNO \), we first note that

\[
\hat{\eta}(s)' = \begin{cases} 
\hat{\eta}(s) & \text{if } \hat{\eta}(s) = \eta(s) \\
\hat{\eta}(s) - 1 & \text{otherwise}.
\end{cases}
\]

For \( \mu \leq g - 1 \), it follows that \( f_g(\mu) \geq t_g(\mu) \) except possibly in the intervals \((\hat{\eta}(s) - 1, \hat{\eta}(s) + 1)\). If \( \hat{\eta}(s) = \eta(s) \) (or equivalently \( \tilde{\rho}(\hat{\eta}(s), s) = -1 \)), then \( f_g(\mu) \geq t_g(\mu) \) in this interval as well. On the other hand, if \( \hat{\eta}(s) \neq \eta(s) \), then \( t_g(\mu) > f_g(\mu) \) on \((\hat{\eta}(s) - 1, \hat{\eta}(s) + 1)\). Thus \( BMNO \) always extends outside \( T \) and, for almost all values of \( g \), \( T \) also extends outside \( BMNO \).

At any rate, for a given (small) genus, it is easy to compute both \( \hat{\eta}(s) \) and \( \hat{\eta}(s)' \) explicitly. The figures 9, 10 and 11, illustrate the cases \( g = 10, g = 12, \) and \( g = 13 \), respectively, where different situations can be appreciated. There the shaded area is \( BMNO \), and Teixidor’s polygons are only outlined.

6. The hyperelliptic case

Suppose now that \( X \) is a non-singular hyperelliptic curve of genus \( g \geq 3 \). If we denote by \( L \) the hyperelliptic line bundle on \( X \) then \( h^0(L^\otimes(s-1)) = s \) for \( 1 \leq s \leq g \), so we can take \( d' = 2s - 2 \) in Theorem 3.9. The analogue of Theorem 4.2 is

**Theorem 6.1.** Let \( X \) be a non-singular hyperelliptic curve of genus \( g \geq 3 \). If \( (\mu, \lambda) \in BGN \cup M \) and \( 1 \leq s \leq g \), then \( T_{2s-2,s}(\mu, \lambda) \) is \( n \)-BN for all \( n \) such that \( (\mu, \lambda) \) is \( n \)-BN.
We now define
\[ BMNO_h = \bigcup_{1 \leq s \leq g-1} (T_{2s-2,s} (BGN \cup M) \cap P). \]

It will be convenient to include the point \((2, 1)\) in \(M\) (see [8]).

This region is already invariant under Serre duality, so we do not need to invoke the transformations \(U_{d,s}'\) in this case. The top boundary of \(BMNO_h\) is given by the graph of the function \(h_g\) defined on \((0, 2g - 2)\) by
\[ h_g(\mu) = \frac{s}{g}(\mu - 2s + 1) + s \text{ for } \mu \in (2s - 2, 2s]. \]

The analogues of Remarks 4.3 and 4.4 hold and indeed we can improve Remark 4.4 (ii). For \(1 \leq s \leq g - 1\), the region
\[ 2s - 1 < \mu \leq 2s, \quad \lambda \leq s \]
consists entirely of BN points. By Serre duality, so also does
\[ 2g - 2 - 2s < \mu < 2g - 1 - 2s, \quad \lambda \leq s + \mu - g + 1, \]
i.e. (replacing \(s\) by \(g - s\))
\[ 2s - 2 < \mu < 2s - 1, \quad \lambda \leq \mu - s + 1. \]
Of course, all points of \(BGN \cup M\) are BN, hence also all points of its Serre dual. These results are illustrated in Figure 12.

In the semistable case, we can include the points \((2s - 1, s)\) and also the line segments \(\{(2s, \lambda) : s < \lambda \leq s + 1\}\).

The next step is to show that all special stable bundles, except for certain line bundles, lie in \(BMNO_h\).

**Theorem 6.2.** Let \(X\) be a hyperelliptic curve, \(E\) a stable bundle on \(X\) of rank \(n\), degree \(d\) and slope \(\mu = \frac{d}{n}\), and \(s\) an integer.

1) If \(0 \leq s \leq g\) and \(2s - 2 < \mu < 2s\), then
\[ h^0(E) \leq sn + \frac{s}{g}(d - (2s - 1)n). \]

2) If \(0 \leq s \leq g - 1\), \(\mu = 2s\) and \(E \not\sim L^s\), then \(h^0(E) \leq sn\).

**Proof:** (1) We begin by writing
\[ F_s(n, d) = sn + \frac{s}{g}(d - (2s - 1)n). \]
We check easily that
\[ 2F_s(n, d) = F_{s-1}(n, d - 2n) + F_{s+1}(n, d + 2n). \]
To prove the theorem, we argue by induction on $s$. For $s = 0$, the result is obvious, since $E$ stable with $\mu < 0$ implies $h^0(E) = 0$. The result for $s = g$ follows from this by Serre duality and Riemann-Roch.

Now suppose $0 < s < g$. Suppose that there exists a stable bundle $E$ of slope $\mu$ with $2s - 2 < \mu < 2s$ and such that $H^0(E) = F_s(n, d) + b_0$ with $b_0 > 0$. Tensoring the exact sequence $0 \to L^* \to H^0(L) \otimes \mathcal{O} \to L \to 0$ by $E$, we get

$$0 \to L^* \otimes E \to H^0(L) \otimes E \to L \otimes E \to 0.$$ 

Since $h^0(L) = 2$, this gives

$$2h^0(E) \leq h^0(E \otimes L^*) + h^0(E \otimes L).$$

By inductive hypothesis, we have

$$h^0(E \otimes L^*) \leq F_{s-1}(n, d - 2n);$$

hence

$$h^0(E \otimes L) \geq 2F_s(n, d) + 2b_0 - F_{s-1}(n, d - 2n) = F_{s+1}(n, d + 2n) + 2b_0.$$ 

Thus $h^0(E \otimes L) = F_{s+1}(n, d + 2n) + b_1$, with $b_1 \geq 2b_0$. Continuing in this way, we construct a sequence $(b_i)$, defined by

$$h^0(E \otimes L^{\otimes i}) = F_{s+i}(n, d + 2in) + b_i;$$

with

$$b_{i+1} \geq 2b_i - b_{i-1}.$$ 

We deduce that this sequence is strictly increasing.

On the other hand, by the result for $s = g$, we have

$$h^0(E \otimes L^{\otimes (g-s)}) \leq F_g(n, d + 2(g - s)n).$$

So $b_{g-s} = 0$, which is a contradiction. The result follows.

\(\text{(2)}\) Again we proceed by induction. For $s = 0$, the only stable bundle of slope $0$ with $h^0(E) > 0$ is $\mathcal{O}$. Similarly, the only stable bundle of slope $2g - 2$ with $h^0(E) > (g-1)n$ is $K$.

For $0 < s < g - 1$, we proceed as in (1). If there exists a stable bundle $E$ of slope $2s$ such that $h^0(E) = sn + b_0$ with $b_0 > 0$, we define the sequence $(b_i)$ for $1 \leq i \leq g - s - 1$ by $h^0(E \otimes L^{\otimes i}) = (s + i)n + b_i$ and prove that $(b_i)$ is strictly increasing. On the other hand, since by hypothesis $E \otimes L^{\otimes (g-s-1)} \not\cong K$, it follows that $b_{g-s-1} = 0$. Again we have a contradiction.

**Remark 6.3.** It follows from the proof of Theorem 6.2 that, if $1 \leq s \leq g - 1$ and $h^0(E)$ takes its maximum value $F_s(n, d)$ (or $sn$), then also $h^0(E \otimes L^*) = F_{s-1}(n, d - 2n)$ (or $(s - 1)n$) and $h^0(E \otimes L) = F_{s+1}(n, d + 2n)$ (or $(s + 1)n$).
Corollary 6.4. If \((\mu, \lambda) \in BMNO_h\), then \((\mu, \lambda)\) is \(n\)-BN for infinitely many values of \(n\). The only special stable bundles which lie outside \(BMNO_h\) are the line bundles \(L^{(s-1)}\) for \(1 \leq s \leq g\) and \(L^{(s-1)}(p)\) for \(1 \leq s \leq g-1\) and \(p \in X\).

Proof: By Theorems 6.1 and 6.2, it is sufficient to prove that the points \((2s-1, s)\) are only \(1\)-BN. By [3] Theorem B, \((1, 1)\) is only \(1\)-BN; hence, by Remark 6.3, \((2s-1, s)\) is also only \(1\)-BN.

According to this Corollary, there do not exist stable bundles of rank \(n > 1\) and slope \(2s-1\) with \(1 \leq s \leq g-1\) and \(h^0(E) = sn\). However

Proposition 6.5. Let \(X\) be a hyperelliptic curve. For any integers \(n, s\) with \(n > 0, 1 \leq s \leq g-1\), there exist stable bundles \(E\) of rank \(n\) and slope \(2s-1\) with \(h^0(E) = sn-1\).

Proof: For \(s = 1\), this is a special case of [3], Theorem B. If \(1 < s \leq g-1\), a result of [4] says that, if \(\Delta\) is a torsion sheaf of length \(n\) with support \(n\) distinct points of \(X\), and if \(M\) is a line bundle of degree 2 on \(X\) such that \(h^0(M) = 1\) then a sufficiently general extension

\[
0 \to L^{(s-1)} \oplus \cdots \oplus L^{(s-1)} \oplus L^{(s-2)} \otimes M \to E \to \Delta \to 0
\]

is stable, and clearly \(h^0(E) = sn-1\).

We have now completely settled the nonemptiness problem for bundles of integral slope. For bundles of non-integral slope, however, we still have an indeterminate region of points which we know to be \(n\)-BN but which may fail to be BN. The next example shows that this can indeed happen.

Example 6.6. Suppose that \(X\) has genus \(g \geq 4\). Suppose that \(1 \leq s \leq g-1\) and that \(E\) is a stable bundle of rank \(n\) and degree \(d\) with \(2s-1 < \mu = \frac{d}{n} < 2s\). Write

\[
d - (2s-1)n = gl + l' \text{ with } 0 \leq l' < g.
\]

By Theorem 6.2, we have

\[
h^0(E) \leq sn + \frac{s}{g}(d - (2s-1)n) = sn + sl + \frac{sl'}{g},
\]

in other words

\[
h^0(E) \leq sn + sl + \left\lfloor \frac{sl'}{g} \right\rfloor.
\]

If \(\left\lfloor \frac{sl'}{g} \right\rfloor < 1\), then Theorem 6.1 gives the existence of a bundle \(E\) with the maximum possible number of sections.

Suppose now that \(s = 2\) and \(\frac{g}{2} \leq l' < g - 1\). We claim that, in this case,

\[
2h^0(E) < 2n + 2l + \left\lfloor \frac{2l'}{g} \right\rfloor = 2n + 2l + 1.
\]
Proof of the claim: Suppose that there exists a stable bundle $E$ as above with $h^0(E) = 2n + 2l + 1$. We know that
\[ 2h^0(E) \leq h^0(E \otimes L) + h^0(E \otimes L^*). \]
Hence
\[ 4n + 4l + 2 \leq 3n + 3l + \left\lfloor \frac{3l'}{g} \right\rfloor + n + l. \]
So $\left\lfloor \frac{3l'}{g} \right\rfloor = 2$ and $h^0(E \otimes L) = 3n + 3l + 2$. Beginning again with $E \otimes L$ and continuing in this way for a total of $g - 3$ steps, we obtain
\[ \left\lfloor \frac{(g - 1)l'}{g} \right\rfloor = g - 2, \]
hence $l' = g - 1$. This contradicts our assumption and proves that there are points which fail to be BN.

Remark 6.7. In the exceptional case $l' = g - 1$ of Example 6.6, we can prove that $E$ does exist. In fact, since $3 < \mu < 4$, by [3] we can find a stable bundle $F$ of rank $n$ and slope $4 - \mu$ with $h^0(F) = n - l - 1$. Then $K \otimes F^*$ has slope $2g - 6 + \mu$ and $h^0(K \otimes F^*) = (2g - 3)n + gl + l' - (g - 1)n + n - l - 1 = (g - 1)n + (g - 1)l + g - 2$. Now take $E = K \otimes F^* \otimes L^{\otimes (g-3)} = F^* \otimes L^{\otimes 2}$ and use the argument of Example 6.6 in reverse. We obtain $h^0(E) = 2n + 2l + 1$ as required.

Re’s improvement of the Clifford bound for $X$ non-hyperelliptic [13] is intriguingly close to the boundary of $BMNO_h$. The results of this section show the extent to which Re’s bound fails for a hyperelliptic curve.

We finally remark that, in the hyperelliptic case, the upper boundary of the region where $n$-BN points exist is not the graph of a continuous function; possibly this extends to other cases.

Fig. 1 The BN map

Fig. 2 Teixidor’s parallelograms

Fig. 3 The regions $BGN$ and $M$

Fig. 4 First steps in the construction of the region $BMNO$
Fig. 5 Gain by translating $BGN$ with bundle with 2 sections

Fig. 6 Construction of a typical $BMNO$ region (genus 10)

Fig. 7 Teixidor’s triangles

Fig. 8 Teixidor’s region for genus 10

Fig. 9 $BMNO$ and $T$ regions for genus 10

Fig. 10 $BMNO$ and $T$ regions for genus 12 (restricted to $R$)

Fig. 11 $BMNO$ and $T$ regions for genus 13 (restricted to $R$)

Fig. 12 The hyperelliptic case.

**REFERENCES**

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris *Geometry of Algebraic Curves*, Vol. 1, Springer-Verlag, New York, 1985.

[2] A. Bertram and B. Feinberg *On stable rank two bundles with canonical determinant and many sections* in Algebraic Geometry (Lecture Notes in Pure and Applied Mathematics, Vol. 200), Marcel Dekker, 1998, pp. 259-269.

[3] L. Brambila Paz, I. Grzegorczyk and P.E. Newstead *Geography of Brill-Noether loci for small slope*, J. Algebraic Geometry 6 (1997), 645-669.

[4] E. Ballico, V. Mercat and P.E. Newstead *Vector bundles on curves with too many sections*, in preparation.

[5] E. Ballico and P. E. Newstead *On Clifford’s theorem for vector bundles on algebraic curves*, preprint (1998).

[6] G. Laumon *Fibrés vectoriels spéciaux*, Bull. Soc. Math. France 119 (1991), 97-119.

[7] V. Mercat *Le problème de Brill-Noether pour les fibrés stables de petite pente*, J. Reine Angew. Math., 506 (1999), 1-14.
[8] V. Mercat *Le problème de Brill-Noether et le théorème de Teixidor*, Manuscripta Math., 98 (1999), 75-85.
[9] V. Mercat *Fibrés vectoriels de pente 2*, preprint (1998).
[10] V. Mercat *Le théorème de Clifford révisité*, in preparation.
[11] S. Mukai *Non-abelian Brill-Noether theory and Fano threefolds*, preprint.
[12] P. E. Newstead and L. Brambila-Paz *Subvariedades del espacio moduli*, Aportaciones Matemáticas, Serie Comunicaciones 16 (1995), 43-53.
[13] R. Re *Multiplication of sections and Clifford bounds for stable vector bundles on curves*, Comm. in Algebra 26 (1998), 1931-1944.
[14] N. Sundaram *Special divisors and vector bundles*, Tohoku Math. J. 39 (1987), 175-213.
[15] Xiao Jiang Tan *Some results on the existence of rank two special stable vector bundles*, Manuscripta Math. 75 (1992), 365-373.
[16] M. Teixidor i Bigas *Brill-Noether theory for stable vector bundles*, Duke Math. J. 62 (1991), 385-400.
[17] M. Teixidor i Bigas *Brill-Noether theory for vector bundles of rank 2*, Tohoku Math. J. 43 (1991), 123-126.
Figure 1

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Figure 5

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Figure 6

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Condition 1, theorem 5.1

Condition 2, theorem 5.1

Figure 7
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Figure 8

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Figure 9

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Figure 10

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Figure 11

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Figure 12

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Figure 1. The BN map

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