3D Quantum Hyperbolic Field Theory

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Abstract

We construct a new family of exact quantum field theories modeled on hyperbolic geometry, called quantum hyperbolic field theories (QHFTs). All the QHFTs are defined for a same (2 + 1)-bordism category, based on the set of compact oriented 3-manifolds $Y$, equipped with properly embedded framed links $L_F$ and with flat connections $\rho$ of principal $PSL(2, \mathbb{C})$-bundles over $Y \setminus L_F$, with arbitrary holonomy at the link meridians. The QHFTs generalize our previous works [3, 4, 5] on volumes, Chern-Simons invariants and quantum hyperbolic invariants of $PSL(2, \mathbb{C})$-characters (i.e. conjugacy classes of $PSL(2, \mathbb{C})$-valued representations of the fundamental group) of closed 3-manifolds. A main part of the paper consists in specifying the marked surfaces that make the objects of the bordism category. This marking includes the introduction of adequate parameters for the space of all $PSL(2, \mathbb{C})$-characters of a punctured surface, which is the fundamental phase space of the QHFTs. Each QHFT associates to a triple $(Y, L_F, \rho)$ as above with marked boundary components a tensor, which is generically holomorphic w.r.t. the parameters for the restriction of $\rho$ to the punctured boundary $\partial Y \setminus L_F$. As a first application, we get new numerical invariants of 3-manifolds, such as Chern-Simons invariants of $PSL(2, \mathbb{C})$-characters of arbitrary link complements, or quantum invariants of compact hyperbolic cone manifolds. Another application is the construction, for any $PSL(2, \mathbb{C})$-character of a surface of finite topological type, of new conjugacy classes of linear representations of the mapping class group. Finally, we discuss some evidences showing that the QHFTs are pertinent to 3D gravity.

Keywords: quantum field theory, geometric invariants of 3-manifolds, matrix dilogarithms, $PSL(2, \mathbb{C})$-character variety, classical and quantum 3D gravity.

1 Introduction

This paper is the continuation of our previous works [3, 4, 5]. It completes the construction of a new family of 3D-quantum field theories (QFTs), whose fundamental ingredients are, on the combinatorial side, suitably structured families of hyperbolic ideal tetrahedra, and, on the functional side, matrix versions of the dilogarithm functions. As both play a central role in 3-dimensional hyperbolic geometry and volume computations, we call these QFTs quantum hyperbolic field theories (QHFTs).
More precisely, having as model Atiyah's formalization of the *topological quantum field theories* (TQFTs) \[1, 26\], we use the terms “3D-quantum field theory” as synonymous of:

**Monoidal functor from some \((2 + 1)\)-bordism category to the tensorial category of complex linear spaces.**

Recall that it means a correspondence (a “representation”) sending marked surfaces to complex linear spaces, and 3-dimensional manifolds to linear morphisms between the linear spaces associated to the marked boundary components. This correspondence maps the gluing of bordisms to the morphism composition, and respects as well certain tensor products in both categories. How the marking of surfaces is rich does reflect in how the part of the QFT supported by the product bordisms is non trivial; at least it should include projective representations of the appropriate mapping class groups.

In the case of TQFTs one uses essentially the bare topological bordism category, but this set up can be extended to bordism categories supported by suitably “equipped” 3-manifolds. The family of QHFTs that we construct in this paper is defined for a \((2 + 1)\)-bordism category based on oriented compact 3-manifolds \(Y\), which are equipped with properly embedded framed links \(L_F\) and with flat connections (up to gauge equivalence) on principal \(PSL(2, \mathbb{C})\)-bundles over \(Y \setminus L_F\) (i.e. with \(PSL(2, \mathbb{C})\)-characters of \(Y \setminus L_F\)), having arbitrary holonomy at the meridians of the link components. Recall that \(PSL(2, \mathbb{C})\) can be identified with the group of direct isometries of the hyperbolic 3-space.

In fact, the main themes of Thurston’s geometrization program, such as the Teichmüller spaces with the action of the modular groups, the hyperbolic volume, or more generally numerical invariants of \(PSL(2, \mathbb{C})\)-characters of 3-manifolds like the volume or the Chern-Simons invariants, have quantum analogs which are contained in the QHFTs, in the sense that they are completely described by these field theories. So the differential geometry of these central classical objects should hopefully reflect deeply in the QHFTs.

For instance, we have defined in \[4, 5\] new families of complex valued (up to a determined phase ambiguity) quantum invariants for cusped hyperbolic 3-manifolds, and for arbitrary triples \((W, L, \rho)\), where \(W\) is a compact closed oriented 3-manifold, \(L\) is an unframed link in \(W\), and \(\rho\) is a \(PSL(2, \mathbb{C})\)-character defined on the whole of \(W\) (hence it is trivial at the link meridians). We also obtained new simplicial formulas for the volume and the Chern-Simons invariants of such characters. As explained in Section 4 of the present paper, all these invariants are specific QHFT *partition functions* (using the classical terminology of the physics litterature), or variations of them. The celebrated Volume Conjectures state precise relationships between the ’semi-classical limit’ of the quantum invariants and the volume and the Chern-Simons invariants, when the manifold admits a hyperbolic structure and the \(PSL(2, \mathbb{C})\)-character is the hyperbolic holonomy. We refer to Section 5 of \[4\], Sections 6-7 of \[5\], and to \[2\] for details on these conjectures.

In this paper, we are mainly concerned with the extension of the heavy apparatus of combinatorial structures underlying the simplicial formulas of these invariants to manifolds *with marked boundary components*. In particular, we introduce several parameter spaces for the basic *phase space* of the theory, which is the space \(R(g, r)\) of \(PSL(2, \mathbb{C})\)-characters of punctured surfaces \(\Sigma_{g,r}\). Each parameter space is an algebraic variety that fibers over \(R(g, r)\) and admits a filtration, with, for instance, different strata being bundles over the Teichmüller spaces of hyperbolic closed surfaces, punctured surfaces, or surfaces with totally geodesic boundary.

The parameter’s construction is self-contained, has a purely 3-dimensional interpretation, and is naturally adapted to the QHFTs. However, it is remarkable that
the $I\partial$-parameter spaces (on which the QHFT morphisms are eventually defined) are highly reminiscent of the well-known shear-bend coordinates for pleated hyperbolic surfaces with punctures. On another hand, the full QHFT marking of surfaces, including these spaces of parameters, encode the irreducible representations of a “quantum moduli space” of $PSL(2,\mathbb{C})$-characters of punctured surfaces, very similar to the Kashaev or (exponential version of the) Chekhov-Fock quantum Teichmüller spaces (see [20], [13]). We plan to investigate both facts in a separate paper.

The QHFTs are exact (in principle, every QHFT morphism can be explicitly computed), finite dimensional (i.e. the linear space associated to any marked surface is finite dimensional), and hermitian. They form a family indexed by the odd positive integers $N \geq 1$. Each QHFT associates to a triple $(Y, L_F, \rho)$ as above with marked boundary a tensor, which is holomorphic (up to a determined ambiguity) on a dense subset of the $I\partial$-parameter space for the $PSL(2,\mathbb{C})$-characters of the (punctured) boundary of $Y \setminus L_F$. In particular, we get tensor valued holomorphic functions on the bundles of $I\partial$-parameters over the Teichmüller spaces for the boundary components.

For instance, in the case of product bordisms, by varying the marking simultaneously on both boundary components, letting fixed the character, these tensors define conjugacy classes of projective representations of the mapping class groups. Also, in the “classical” case when $N = 1$, these tensors are just scalars. By extending the results of [5], Section 6 (which hold for closed manifolds or cusped hyperbolic manifolds), they can be interpreted as the evaluation of a second Cheeger-Chern-Simons class for manifolds $Y$ with parametrized boundary, i.e. $CS(\rho) + \sqrt{-1}Vol(\rho)$, where $\rho$ is a $PSL(2,\mathbb{C})$-character of $Y$, and Vol and CS are respectively a volume and Chern-Simons invariant of $\rho$ on the marked bordism $Y$. Finally, the QHFT morphisms for triples $(W, L_F, \rho)$, where $W$ is a closed manifold, are always scalars. It is an open problem to understand their dependence w.r.t. the framing of $L_F$, as well as the relationship between them and the invariants of cusped hyperbolic 3-manifolds defined in [5] (compare with Section 4.6).

The ultimate building blocks of the QHFTs are so called matrix dilogarithms, which are determined automorphisms $R_N, N \geq 1$, of $\mathbb{C}^N \otimes \mathbb{C}^N$ associated to hyperbolic ideal tetrahedra equipped with an elaborated extra-decorations, and that satisfy certain fundamental five term identities. The matrix dilogarithms have been introduced, formalized and widely studied in [5]. They are “quantum” versions of the classical dilogarithm functions (see Section 8 of [5] and the references therein, and [2]).

The above deep interaction between classical objects coming from differential geometry and analysis, and quantum algebraic objects, is not the only motivation for studying the QHFTs. Another one is the fact that the whole family of QHFTs forms a unified theory that could be understood as a finite regularization of quantum 3D gravity. This is discussed in Subsection 1.2 below. Before that, we describe the content of the paper in the next Subsection 1.1.

Let us conclude this introduction by noting that Turaev has formalized in [27] a notion of Homotopic QFT (HQFT), which provides a general framework for QFTs based on 3-dimensional cobordisms equipped with a representation of their fundamental group in a fixed group $G$. It is tempting to look at the QHFTs as examples of HQFTs for $G=PSL(2,\mathbb{C})$, but some key points in our construction show that this cannot be exactly the case. For instance, there is the needed “link fixing”, which

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1The resulting invariants of surface diffeomorphisms should be closely related to those obtained recently by Bonahon-Liu [14].
implies that we have to use punctured surfaces as objects. However the relationship between the QHFTs and Turaev’s HQFTs certainly deserves further investigations.

1.1 Description of the paper

The adequate sets of parameters for the spaces of $PSL(2, \mathbb{C})$-characters of punctured surfaces are developed in Section 2. This section is self-contained; only some notions, which we recall when needed, are taken from [4, 5].

After some preliminaries on the variety of $PSL(2, \mathbb{C})$-characters, in Subsection 2.2, we introduce for any closed compact surface $S$ with a finite set $V = \{ p_i \}_{i=1}^r$ of framed (i.e. with a fixed segment $l_i$ in a disk neighborhood) marked points a notion of efficient triangulation for the surface with boundary $F = S \setminus \coprod D_i$, where $D_i$ is a small open disk with center on $l_i$. These triangulations have two main features. First, they are naturally adapted to the 3-dimensional machinery of $I$-cusps developped in Section 3. Second, they allow to define charts for the whole space of $PSL(2, \mathbb{C})$-characters of $F$, for any kind of boundary holonomies. Namely, given any efficient triangulation $T$ of $F$ with a suitable system $b$ of orientation of the edges called branching, we produce bundles of cocycle $D$-parameters (Subsection 2.3)

\[ Z(T, b) \longrightarrow R(g, r) \]

and $I\partial$-parameters (Subsection 2.4)

\[ W(T, b) \longrightarrow R_I(g, r) \]

where $R(g, r)$ denotes the space of $PSL(2, \mathbb{C})$-characters of $F$ and $R(g, r)_I$ is a Zariski open subset of $R(g, r)$. The total spaces of these bundles are algebraic varieties, which admit partitions into subbundles, according to the type of the holonomies at the boundary components of $F$ (trivial, parabolic, or else). The total spaces of these subbundles form a filtration by the dimension. Examples are the parameters for the Teichmuller space of the closed surface with marked points $(S, V)$ (in the deepest part of the filtration), and the parameters for the Teichmuller space of finite area complete hyperbolic metrics on the punctured surface $S \setminus V$ (in the quasi-regular part of the filtration).

The QHFT morphisms eventually depend on the $I\partial$-parameters, but the cocycle $D$-parameters come at first naturally from the combinatorial presentation of manifolds we need for all the construction. Namely, they are the matrix entries of $PSL(2, \mathbb{C})$-valued 1-cocycles on efficient triangulations that represent the elements of $R(g, r)$, with a preferred “normalized” form into which any cocycle can be put by conjugation. The $I\partial$-parameters are products of cross-ratio moduli of certain families of decorated hyperbolic ideal tetrahedra. These hyperbolic tetrahedra are associated to the 3-simplices of standard triangulated cylinders over $F$, by suitably extending the above normalized cocycles to the cylinders, and then using an idealization procedure reminiscent of the construction of piecewise-straight developing maps for geometric structures on 3-manifolds. The $I\partial$-parameters can also be viewed as describing representations of the groupoid of paths transverse to the given efficient triangulation (see the end of Subsection 2.4).

For the convenience of the reader, we present in an Appendix the relationship between the cocycle $D$-parameters and the Kashaev-Penner coordinates for the moduli space of irreducible $PSL(2, \mathbb{C})$-characters on punctured surfaces, with parabolic holonomies at the punctures (this includes the Teichmuller space of $S \setminus V$).

In Section 3 we define the QHFT $(2 + 1)$-bordism category, based on triples $(Y, L_F, \rho)$ (Subsection 3.1). In particular we carefully describe the marked surfaces that make the (elementary) objects of the category. In fact, it is more convenient
to deal with an equivalent category based on 3-manifolds with corners, at the intersection of a closed tubular neighborhood of $L_F$ with $\partial Y$. This makes the inclusion of the results of Section 2 immediate, as the marking of surfaces shall incorporate the phase space parameters. Finally, we introduce in Subsection 3.2 the notion of $I$-cusps: these are standard forms to represent pairs $(U(L_F), \rho)$, where $U(L_F)$ is a closed tubular neighborhood of $L_F$ and $\rho$ is the restriction of $\rho$ onto it, as the gluing of suitably decorated hyperbolic ideal tetrahedra. This notion is essential to the QHFTs, for instance to obtain numerical invariants for closed manifolds $Y$, in the case when the character $\rho$ is non trivial at the meridians of $L_F$.

In Section 4 we quickly review the matrix dilogarithms (the explicit formulas are given in an Appendix) and we define the tensors that represent the bordisms. This completes the construction of the QHFTs. We heavily refer to the notions and results developed in [3, 4, 5], avoiding too many unnecessary repetitions, and pointing out the substantial new achievements.

In particular, we discuss the corresponding QHFT partition functions for closed manifolds $Y$, and a variation of the QHFT construction (Subsection 4.4). Namely, we consider a more restricted bordism category such that the associated partition functions for triples $(W, L, \rho)$, where $W$ is closed, $L$ is an unframed link, and $\rho$ is trivial at the link meridians, coincide with the dilogarithmic invariants already defined in [4, 5]. We set the relationship between the two kinds of partition functions that are available for triples $(W, L_F, \rho)$, for such a special $\rho$.

We consider also the part of the QHFTs supported by the trivial (product) bordisms (Subsection 4.5). As mentioned in the Introduction, it contains interesting conjugacy classes of linear representations of the mapping class groups of punctured surface (defined up to a determined phase ambiguity).

Finally we indicate in Subsection 4.6 a so called universal QHFT environment, that is the most general set up where our constructions formally makes sense. The specialized QHFTs previously constructed naturally map into this universal environment, and have a clear intrinsic topological/geometric meaning. This suggests the possibility of other meaningful specializations of the universal QHFT environment.

1.2 QHFT and 3D gravity

There are some evidences that the QHFTs are pertinent to 3D gravity (see [6] for further comments on this points). Thanks to the 3-dimensional peculiar fact that the Ricci curvature tensor completely determines the Riemann curvature tensor, classical 3D (pure) gravity concerns the study of Riemannian or Lorentzian 3-manifolds of constant curvature. The sign of the curvature coincides with the sign of the cosmological constant for the theory. We stipulate that all manifolds are oriented and that the Lorentzian space-times are also time-oriented. We also include in the picture the presence of world lines of “particles”; the singularities of the metric are concentrated along these lines. Typical examples are the cone manifolds of constant curvature with a properly embedded link as cone locus, where the cone angles reflect the “mass” of the particles. In the Lorentzian case we also require that the world lines are of causal type (see e.g. [9]).

The hyperbolic 3-manifolds are the classical solutions of Riemannian (sometimes called Euclidean) 3D gravity with (normalized) negative cosmological constant. Geometrically finite hyperbolic 3-manifold, or more generally topologically tame ones, possibly with links of concentrated singularities (hence with non necessarily trivial holonomy at the link meridians), give fundamental examples of supports for the QHFT bordism category. Here it is understood that these manifolds are equipped with the holonomies of the hyperbolic structures (remind that $PSL(2, \mathbb{C})$ is iden-
tified with \( Isom^+(\mathbb{H}^3) \), the group of direct isometries of the hyperbolic 3-space). Moreover, for compact hyperbolic 3-manifolds \( W \), a deep volume rigidity result (see e.g. \([13, 16]\)) tells us that a volume function is well defined on the space \( \mathcal{R}(W) \) of conjugacy classes of \( \text{PSL}(2, \mathbb{C}) \)-valued representations of \( \pi_1(W) \), and that, if \( \rho \) is the holonomy of a hyperbolic structure \( h \) on \( W \), then:

1. \( \text{Vol}(\rho) = \text{Vol}(W, h) \);
2. \( \rho \) is the unique maximum of the volume function.

(With some technical complication this result holds also for cusped manifolds, i.e. for non compact finite volume complete hyperbolic 3-manifolds). This geometric result is strictly related to Euclidean 3D gravity with negative cosmological constant, when formulated in terms of a Chern-Simons type action for the so called “new variables”, which are the connections on principal \( \text{PSL}(2, \mathbb{C}) \)-bundles, instead of the metrics and the framings (see \([28]\)). The “constraint” equations for this action imply that the phase space of the theory becomes the space of flat \( \text{sl}(2, \mathbb{C}) \)-connections (up to gauge equivalence). In fact, the Chern-Simons action for flat \( \text{sl}(2, \mathbb{C}) \)-connections equals a constant times \( \text{CS}(\rho) + i \text{Vol}(\rho) \), where \( \text{CS}(\rho) \) denotes the Chern-Simons invariant of the flat connection \( \rho \). This is a natural complexification of the above volume function on \( \mathcal{R}(W) \), and hyperbolic manifolds, the classical solutions of Riemannian 3D gravity, maximize the norm of \( \exp((1/2i\pi)(\text{CS}(\rho) + i \text{Vol}(\rho))) \). It is a fact (see \([5]\) and Section 5 of this paper) that the “classical member” \( \text{QHFT}_1 \) of the family actually computes this exponentiated classical complex action for pairs \( (W, \rho) \). Moreover, different so called “Volume Conjectures” should identify \( \text{QHFT}_1 \) with \( \text{QHFT}_\infty \), i.e. the “classical limit” of the “quantum” theories \( \text{QHFT}_N \), \( N > 1 \), when \( N \to \infty \) (see \([4, 5]\) or \([2]\) for a discussion on this point). Remind also that, in the particular case of a link \( L \) in \( S^3 \) equipped with the trivial flat bundle, \( \text{QHFT}_N \), \( N > 1 \), computes the Kashaev’s \([18]\) invariant \( < L >_N \), later identified by Murakami-Murakami \([23]\) with \( J_N(L)(\exp(2\pi i/N)) \), where \( J_N \) denotes a suitably normalized colored Jones invariant.

Another intriguing fact is that bordisms supported by hyperbolic 3-manifolds are not only pertinent to Euclidean 3D gravity with negative cosmological constant. This claim comes from the following few facts; we refer to \([22, 8, 12, 7]\) for the details and more articulated statements. We recall, for example, that geometrically finite hyperbolic 3-manifolds with incompressible ends of infinite volume can be concretely interpreted as interactions between Lorentzian space-times of arbitrarily fixed constant curvature. More precisely, we can canonically associate to every end of such a hyperbolic 3-manifold a domain of dependence of a compact Cauchy surface, of arbitrarily fixed constant curvature \( \kappa \). A key point is that these Lorentzian space-times, independently on their constant curvature, share the same “parameter space” \( T_g \times \mathcal{MLE}_g \), where \( T_g \) denotes the Teichmüller space of hyperbolic structure on a fixed surface \( S \) of genus \( g \geq 2 \), and \( \mathcal{MLE}_g \) is the space of measured geodesic laminations (see e.g. \([15]\)) on these hyperbolic surfaces. Moreover this is also the parameter space of projective structures on \( S \) \([25]\). The holonomy of the projective structure related to a hyperbolic end as above is just the restriction of the holonomy of the hyperbolic 3-manifold. Moreover these space-times have a very explicit geometric description. In particular, they admit a canonical cosmological time: the proper time that every event has been in existence and that coincides with its finite Lorentz distance from the initial singularity. This initial singularity has a rich geometry (“dual” to the geodesic lamination) which is the “past limit” in an appropriate sense of the geometry of the level surfaces of the cosmological time. Every such a level surface is a Cauchy surface. Moreover, when \( \kappa \leq 0 \), there is a canonical Wick rotation directed by the gradient of the cosmological time (which is in general a \( C^0 \) vector field) that converts the future of a determined level surface...
into the whole associated hyperbolic end. Remind that the very basic example of
Wick rotation directed by the field $\partial/\partial x_3$ converts the Minkowski metric on $\mathbb{R}^3$
with signature $(+,+,−)$ into the Euclidean metric; sometimes one refers to it as
“passing to the imaginary time”. Wick rotation is a basic procedure for interplaying
Riemannian and Lorentzian geometry, including the global causality of Lorentzian
space-times.

In a sense, this behaviour confirms the intuition at page 72 of \cite{39}: these
Lorentzian space-times should be considered not really as “space-times”, but rather
as mere “world sheets”; hence it does not really make sense to ask about their cur-
vature. The latter is matter of a “universe” where they should be embedded. The
above considerations show in particular that hyperbolic universes can concretely
realize the changes of topology of these world sheets in a purely classical 3D gravity
set up, providing that we avoid any (somewhat misleading) separation in differ-
ent sectors, accordingly to the metric signature and the sign of the cosmological
constant.

2 Phase space parameters

To orient the boundary $\partial Y$ of any oriented $n$-manifold $Y$, we adopt the convention:
last is the ingoing normal.

For every $(g,r) \in \mathbb{N} \times \mathbb{N}$, such that $g \geq 0$, $r > 0$, and $r > 2$ if $g = 0$, we fix
a compact closed oriented base surface $S = S_g$ of genus $g$, with a set $V = V_{g,r} =$
$\{v_1,\ldots,v_r\}$ of $r$ marked points. Our basic “phase space” is

$$\mathcal{R}(g,r) = \text{Hom}(\pi, PSL(2,\mathbb{C}))/PSL(2,\mathbb{C})$$

that is the “space” of all $PSL(2,\mathbb{C})$-valued representations of the fundamental group
$\pi = \pi_1(S \setminus V)$, up to conjugation.

In this section we produce the bundles of cocycle $D$-parameters and $I\partial$-parameters
over $\mathcal{R}(g,r)$. It is convenient to replace $S \setminus V$ with an oriented compact surface
with isomorphic fundamental group. So we fix a compact oriented surface $F$ with $r$
boundary components, obtained by removing from $S$ the interior of a small 2-disk
$D_i$ such that, for every $i$, $v_i \in \partial D_i$. Clearly the inclusions of $\text{Int}(F)$ into $F$ and
$S \setminus V$ respectively induce the identification $\pi = \pi_1(F)$. We stress that there are no
restrictions on the values of the representations at the boundary loops of $F$.

2.1 Preliminaries on the character variety

As $\pi$ is a free group with $\kappa = 2g + r − 1$ elements, the variety of representations
$\text{Hom}(\pi, PSL(2,\mathbb{C}))$ is naturally identified with $PSL(2,\mathbb{C})^\kappa$. Any choice of free
generators of $\pi$ determines such an identification, and the identifications associated
to different choices are related by algebraic automorphisms of $PSL(2,\mathbb{C})^\kappa$. Moreover,
the isomorphism $PSL(2,\mathbb{C}) \cong SO(3,\mathbb{C})$ implies that $\text{Hom}(\pi, PSL(2,\mathbb{C}))$ is an
affine complex algebraic variety, with the complex algebraic action of $PSL(2,\mathbb{C})$.
But the quotient space $\mathcal{R}(g,r)$ is a much more delicate object. This rough topo-
logical quotient space is not even Hausdorff and it is more convenient to consider
the algebraic quotient $X(\pi) = \text{Hom}(\pi, PSL(2,\mathbb{C}))/PSL(2,\mathbb{C})$ of invariant theory,
called the variety of $PSL(2,\mathbb{C})$-characters. We refer to \cite{28} for a careful treat-
ment of this matter. We recall that $X(\pi)$ is an affine complex algebraic set to-
gether with a surjective regular map $t : \text{Hom}(\pi, PSL(2,\mathbb{C})) \to X(\pi)$, which induces
an isomorphism $t^*$ between the regular functions on $X(\pi)$ and the regular
functions on $\text{Hom}(\pi, PSL(2,\mathbb{C}))$ invariant by conjugation. In general $t(\gamma) = t(\sigma)$
does not imply that $\gamma$ and $\sigma$ are conjugate, i.e. that the quotient set $\mathcal{R}(g,r)$ is
of representations, up to suitable gauge transformations, that are counterparts of the conjugacy action. It shall be useful to consider these objects as geometric bundles over \( R(\pi,PSL(2,\mathbb{C})) \), to treat the “complex dimension” and so on. We will do it somewhat formally, being aware that everything can be substantiated in terms of the variety of characters, or by restriction to the irreducible representations. We prefer to treat the whole \( R(\pi,PSL(2,\mathbb{C})) \) anyway, because the construction of the QHFTs does not really require any restriction on the flat \( PSL(2,\mathbb{C}) \)-connections. So, as the group \( PSL(2,\mathbb{C}) \) has trivial centre and complex dimension equal to 3, we can say that the complex dimension of \( R(\pi,PSL(2,\mathbb{C})) \) is equal to \( 3\kappa - 3 = -3\chi(F) \).

2.2 Efficient triangulations

The first step is to select a class of efficient triangulations of \( F \).

We use possibly singular triangulations of compact oriented \( n \)-manifolds \( Y \). Any such a triangulation \( T \) can be described as a finite family of oriented abstract \( n \)-simplices, together with the identification of some pairs of abstract \((n-1)\)-faces, in such a way that \( Y \) is the quotient space. The face identifications are orientation reversing so that the orientations of the \( n \)-simplices match to produce the given orientation of \( Y \). Multiply adjacent as well as self-adjacent \( n \)-simplices are allowed.

Let us start with any branched triangulation \((T',b')\) of \( S \) having \( V \) as set of vertices. A branching \( b' \) is a system of orientations of the edges of \( T' \) such that the induced orientations on the edges of each abstract triangle is compatible with a total ordering of its vertices, via the rule: each edge is directed towards the biggest end-point. Hence no abstract triangle of \( T' \) inherits from \( b' \) an orientation of its boundary: only two edges have a compatible prevailing orientation. If \( x_0,x_1,x_2 \) are the \( b' \)-ordered vertices of a triangle, we name and order its \( b' \)-oriented edges as: \( e_0 = [x_0,x_1], e_1 = [x_1,x_2], e_2 = [x_0,x_2], \) so that \( e_0, e_1 \) have the prevailing orientation. This also induces a \( b' \)-orientation on every triangle, that is the orientation which induces the prevailing edge orientation. The \( b' \)-orientation may or may not agree with the given orientation of \( S \). We encode it via a sign function \( \sigma = \sigma(T',b') \), defined on the set of triangles of \( T' \), by stipulating that the sign of a triangle is \( \pm 1 \) if the two orientations do or do not agree respectively. Note that such a \((T',b')\) exists due to the assumption we have made on the pair \((g,r)\).

Given any \((T',b')\) as above, we consider corner maps \( v \mapsto c_v \) which associate to each vertex of \( T' \) one corner in its star \( \text{Star}(v) \), and we denote by \( v \mapsto t_v \) the induced map that associates to \( v \) the (abstract) triangle that contains the corner \( c_v \). We say that \( v \mapsto c_v \) is \( t \)-injective if \( v \mapsto t_v \) is injective.

Lemma 2.1 For every \((g,r)\) as above, let us assume furthermore that \( r > 3 \) if \( g = 0 \). Then every triangulation \( T' \) of \( S \) with \( r \) vertices admits \( t \)-injective corner maps.

Proof. First we show that for every \((g,r)\) as in the statement of the lemma there exist triangulations of \( S \) with \( r \) vertices admitting \( t \)-injective corner maps. We do it by induction on \( r \). For \((g = 0, r = 4)\) and \((g > 0, r = 1)\), it is evident that \( T' \)
with $t$-injective $c_v$ do exist. Clearly, a $t$-injective $c_v$ exists on any $T''$ obtained from $T'$ via a $1 \to 3$ move, i.e. a move that subdivides one triangle of $T'$ by 3 triangles, introducing one new vertex. So we conclude by induction on $r$.

![Figure 1: The flips with marked corners.](image)

In figure 1 the corner selection $v \mapsto c_v$ is specified by a $\ast$, and the rows show essentially all possible flips, up to some evident variations, that preserve the property that $v \mapsto t_v$ is injective. Consider any triangulation $T'$ of $S$ with $r$ vertices, and let $T''$ be a triangulation with the same number of vertices and which admits a $t$-injective corner map. It is well known that $T''$ is connected to $T'$ via a finite sequence of (naked) flips. The $t$-injective corner map $v \mapsto c''_v$ for $T''$ transits to a $t$-injective $v \mapsto c_v$ for $T'$, by decorating these flips as in Fig. 1. The only case excluded by the above lemma is $(g = 0, r = 3)$; in this case we have 2 triangles, hence $t$-injective maps cannot exist.

In the generic cases when the lemma applies, let us fix a $t$-injective corner map $v \mapsto c_v$ for $(T', b')$. In the interior of every triangle $t = t_v$ of $T'$ that contains a selected corner $c_v$ corresponding to a vertex $v$, consider two nested bigons $D_v \subset D'_v$ with one common vertex at $v$. Call $v' \in D_v$ and $v'' \in D'_v$ the other two vertices of the bigons. Remove from $t_v$ the interior of $D_v$, obtaining $s_v$. Triangulate $s_v$ by making the cone with base $v''$. We find a triangulation of $s_v$ with 5 triangles, 5 vertices and 10 edges. Repeating this procedure independently on every $s_v$, we get a triangulation $T$ of $F$, with $3r$ vertices and $p + 4r$ triangles, where $p$ denotes the number of triangles of $T'$. The set of edges of $T$, $E(T)$, contains $E(T')$, and $|E(T)| = |E(T')| + 7r$.

Now we fix a way of extending the branching $b'$ to a branching $b$ on $T$. This is shown in in Fig. 2. With this choice there is a clear transition from $(T, b)$ to $(T', b')$: first zip the two boundary components of the inner bigon, and get a branched triangulation $(T'', b'')$ of $S$ with $\bar{V} = V \cup V' \cup V''$ as set of vertices. Then collapse each bigon pattern of $T''$ to the corresponding $v$, and get back the initial $(T', b')$ of $(S, V)$.
In the generic case, the triangulations \((T, b)\) of \(F\) thus obtained are by definition our efficient \(c\)-triangulations. The triangle sign function \(\sigma\) naturally extends to \((T, b)\). For each \(s_v\), we select a base triangle among the 2 containing a boundary edge. For example, in Fig. 2, we take the triangle \(\tau_v\) which contains the boundary edge such that the \(b\)-orientation and the boundary orientation do agree. In general we stipulate that \(\sigma(\tau_v) = 1\).

In the special case \((g = 0, r = 3)\) we have to consider the further situation of a triangle containing two selected corners. This is shown on the left of the first row of Fig. 3. In fact this figure shows essentially all the possible configurations that we obtain by using arbitrary corner maps. There are no conceptual obstructions to use arbitrary corners maps in what follows. We prefer to specialize the corner maps just to limit the configurations and simplify the exposition. Moreover, we will limit
ourselves to give the details in the generic case (and referring to Fig. 2), as the extension to the special case or to the other positions of the selected corner shall be straightforward.

2.3 $\mathcal{D}$-parameters

Fix an $e$-triangulation $(T, b)$ of $F$. Let us denote by $Z(T, b)$ the space of $PSL(2, \mathbb{C})$-valued 1-cocycles on $(T, b)$. We use the $b$-orientation of the edges, so that on each triangle with ordered $b$-oriented edges $e_0, e_1, e_2$ the cocycle condition reads: 
$$z(e_0)z(e_1)z(e_2)^{-1} = 1.$$ 

We write $C(T, b)$ for the space of $PSL(2, \mathbb{C})$-valued 0-cochains, that is the $PSL(2, \mathbb{C})$-valued functions defined on the set of vertices of $T$. Two 1-cocycles $z$ and $z'$ are said equivalent up to gauge transformation if there is a 0-cochain $\lambda$ such that, for every (abstract) oriented edge $e = [x_0, x_1]$, we have $z'(e) = \lambda(x_0)^{-1}z(e)\lambda(x_1)$. Possibly $\lambda(x_0) = \lambda(x_1)$, when the two abstract vertices are identified to one vertex of $T$.

We denote by $H(T, b) = Z(T, b)/C(T, b)$ the quotient set. It is well known that $H(T, b)$ is in one-to-one correspondence with $\mathcal{R}(g, r)$. More precisely, fix a vertex $x_0$ of $T$ as base point and set $\pi = \pi_1(F, x_0)$. Then there is a natural surjective map $f : Z(T, b) \to \text{Hom}(\pi, PSL(2, \mathbb{C}));$ $f(z)$ and $f(z')$ represent the same point in $\mathcal{R}(g, r)$ if they are related by gauge transformations. The complex dimension of $C(T, b)$ is equal to $3im(PSL(2, \mathbb{C})) = 9r$. The algebraic set $Z(T, b)$ is defined by $(p + 4r)\dim(PSL(2, \mathbb{C})) = 3(p + 4r)$ relations on $|E(T)|\dim(PSL(2, \mathbb{C})) = 3|E(T)|$ variables ($p$ is the number of triangles of the initial triangulation $T'$ of $S$ with $r$ vertices). Remind that $PSL(2, \mathbb{C})$ has trivial centre. Hence we find that the (formal) complex dimension of $H(T, b)$ is just

$$3(|E(T)| - (p + 4r) - 3r) = -3\chi(F)$$

that is the dimension of $\mathcal{R}(g, r)$. This essentially means that there are no negligible relations defining $Z(T, b)$.

Let us denote by $p : Z(T, b) \to H(T, b)$ the natural projection. A way to get $\mathcal{D}$-parameters for $\mathcal{R}(g, r) \cong H(T, b)$, that is parameters based on 1-cocycle coefficients, should be to construct nicely parametrized global sections of $p$. Although this is too optimistic, we will specialize anyhow the cocycles to reduce as much as possible the set of residual gauge transformations.

The conjugacy class of every element $g \in PSL(2, \mathbb{C})$ can be specified by a symbol $c(g)$, as follows. Set $c(id) := I$, otherwise set either $c(g) = (a, \text{diag})$, or $c(g) = 1$, where: $a \in \mathbb{C} \setminus \{0, 1\}$, “diag” means that $g$ is represented (up to conjugation) by a diagonal matrix with $a$ as first eigenvalue; 1 means that $g$ is represented by the unipotent upper triangular matrix with 1 as upper triangular coefficient. In other words, $c(g)$ determines one distinguished representative in the conjugacy class of $g$. Sometimes we will say that $g$ is of trivial, parabolic or generic type, respectively.

In what follows, we will confuse any element of $PSL(2, \mathbb{C})$ with its $SL(2, \mathbb{C})$-representatives. If $B = B(2, \mathbb{C})$ denote the Borel subgroup of upper-triangular matrices of $SL(2, \mathbb{C})$, every $g \in B$ is written in the form $g = [a, b]$, where $a \in C^*$ is the first eigenvalue of $g$, and $b$ is the upper-diagonal entry of $g$.

Define 
$$\beta' : \mathcal{R}(g, r) \to [\{I, 1\} \cup (C^* \times \{\text{diag}\})]'$$

as the map which associates to every holonomy $\rho$ the $r$-uple $(c(\rho(\gamma_1)), \ldots, c(\rho(\gamma_r)))$, where $\gamma_i$ is the oriented boundary loop of $F$ at the vertex $v_i$. Consider $H(T, b)$ as a set realization of $\mathcal{R}(g, r)$, and lift $\beta'$ to $Z(T, b)$ via the composition $\beta = \beta' \circ p$. 

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For every $x \in \beta(Z(T,b))$, put $Z(T,b,x) := \beta^{-1}(x)$. This is mapped by $p$ onto $H(T,b,x) = \mathcal{R}(g,r,x) = \beta^{-1}(x)$. So, by varying $x$, we get a partition of the projection $p$, and we consider each piece $p_x : Z(T,b,x) \to \mathcal{R}(g,r,x)$.

Denote by $Z(T,b,x)_{\beta}$ the subset of $Z(T,b,x)$ made by the cocycles $z$ such that, along every oriented boundary loop $\gamma_i$, the product of the cocycle values, starting from the vertex $v_1$, exactly equals the distinguished representative $c(p(z)(\gamma_i))$. Any cocycle $z \in Z(T,b,x)$ can be modified to a cocycle in $Z(T,b,x)_{\beta}$ via a gauge transformation associated to a suitable 0-cochain with support at $V$. So the algebraic set $Z(T,b,x)_{\beta}$ is non empty and the restriction of $p_x$ maps it onto $\mathcal{R}(g,r,x)$.

Hence we further restrict ourselves to $p_x : Z(T,b,x)_{\beta} \to \mathcal{R}(g,r,x)$. The set of residual gauge transformations is already smaller. In fact, to stay in $Z(T,b,x)_{\beta}$ we must act with 0-cochains $\lambda$ such that, for every $v_i$, $\lambda(v_i)$ belongs to the stabilizer $\text{Stab}(x_i)$ (while $\lambda$ is arbitrary at the other vertices $v'_i$ and $v''_i$ of $T$).

By using the notations introduced in Subsection 2.2, let us consider $\lambda \in Z(T,b,x)_{\beta}$ of the base triangle $\tau_v$ of $s_v$ with its $b$-ordered edges $e_0(v), e_1(v)$. Define

$$\delta_x : Z(T,b,x)_{\beta} \to (PSL(2,\mathbb{C})^2)^r$$

$$\delta_x(z) = ([z(e_0(v_1)), z(e_1(v_1))], \ldots, [z(e_0(v_r)), z(e_1(v_r))]).$$

For every $y \in (PSL(2,\mathbb{C})^2)^r$, set $Z(T,b,x,y)_{\beta} := \delta_x^{-1}(y)$. As above, any cocycle $z \in Z(T,b,x)_{\beta}$ can be modified to one in $Z(T,b,x,y)_{\beta}$ just by acting with a suitable 0-cochain with support at the vertices $v'_i$ and $v''_i$. Hence we have:

**Lemma 2.2** The map $\delta_x$ and the restriction of $p_x$ to $Z(T,b,x,y)_{\beta}$ are surjective.

We restrict once more to $p_{x,y} : Z(T,b,x,y)_{\beta} \to \mathcal{R}(g,r,x)$. Let us determine the residual gauge transformations. Fix a cell $s_v$ as defined in Subsection 2.2. Consider a 0-cochain $\alpha$ with support at $v$, $v'$ and $v''$. Set $g_h = z(e_h(v)), h = 0, 1$. We stay in $Z(T,b,x,y)_{\beta}$ iff:

$$a(v) \in \text{Stab}(x_v), \quad a(v')^{-1}g_0a(v') = g_0, \quad a(v')^{-1}g_1a(v) = g_1.$$

Then it is clear that, once $a(v)$ is fixed, then the rest of the cochain is uniquely determined. As the $s_v$'s contribute independently each to the other, the set $\mathcal{G}(T,b,x,y)$ of residual gauge transformations is parametrized by

$$\mathcal{G}(T,b,x,y) \cong \text{Stab}(x_1) \times \cdots \times \text{Stab}(x_r).$$

Note that we have $\text{Stab}(I) = PSL(2,\mathbb{C})$, $\text{Stab}(1) = \text{Par}(2,\mathbb{C})$ and $\text{Stab}(a,\text{diag}) = \text{Diag}(2,\mathbb{C})$, the image in $PSL(2,\mathbb{C})$ of the upper triangular parabolic (resp. diagonal) subgroup of $SL(2,\mathbb{C})$. Hence $\mathcal{G}(T,b,x,y)$, in particular its dimension, can be easily determined, and depend only on the types, say $t(x)$, of the boundary loops.

From this we can derive a rather neat qualitative description of $\mathcal{R}(g,r)$. Denote by $Z(T,b)_{\beta}$ the union of all $Z(T,b,x,y)_{\beta}$'s, with the natural projection $p_{\beta} : Z(T,b)_{\beta} \to \mathcal{R}(g,r)$. Let

$$t : ([I,1] \cup (\mathbb{C}^* \times \{\text{diag}\})^r) \to ([I,1], \text{diag})^r$$

be the natural forgetting map which associates to each boundary conjugacy class its type. Define $\phi' = t \circ \beta'$ and denote by $\phi_{\beta}$ the restriction to $Z(T,b)_{\beta}$ of $\phi = t \circ \beta$. For every $w \in ([I,1], \text{diag})^r$ set $\mathcal{R}(g,r,w-\text{type}) = \phi'^{-1}(w)$ and $Z(T,b,w-\text{type})_{\beta} = \phi_{\beta}^{-1}(w)$. The above constructions eventually give:

**Proposition 2.3** (1) By varying $w$ we get a partition of the projection $p_{\beta}$ by the maps $p_{w,\beta} : Z(T,b,w-\text{type})_{\beta} \to \mathcal{R}(g,r,w-\text{type})$. Moreover, each space $\mathcal{R}(g,r,w-$
by \( \sqcup \) \( R \)

The equivalence classes of cocycle \( D \)

Definition 2.4

Set \( \overline{I} \)

T, b

\((\) precisely on each open set in \( R \) dim(\( Z \)) get one “bundle” \( R \) of its actual closure in stratification triangles, so that \( 2 \mid p \) and \( p + 4r = 4r - 2\chi(F) \) triangles, and \( |E(T)| = 7r - 3\chi(F) \).

Before proceeding, we recall few facts about 3-dimensional branchings (see \[1\] for full details).

2.4 \( \mathcal{I} \)-parameters

Let \( \langle T, b \rangle \) be an e-triangulation of \( F \) as above. Recall that: \( \langle T, b \rangle \) is obtained from a branched triangulation \( \langle T', b' \rangle \) of \( S \); \( \chi(F) = \chi(S) - r \); \( T' \) has \( r \) vertices and \( p \) triangles, so that \( 2|E(T')| = 3p = -6\chi(F) \). The triangulation \( T \) has \( 3r \) vertices and \( p + 4r = 4r - 2\chi(F) \) triangles, and \( |E(T)| = 7r - 3\chi(F) \).

Before proceeding, we recall few facts about 3-dimensional branchings (see \[1\] for full details).
Branchings. Given a triangulation $T$ of an oriented compact 3-manifold $Y$, a branching $b$ on $T$ is a system of orientations of the edges of $T$ which induces on each abstract tetrahedron $\Delta$ of $T$ a total ordering $x_0, x_1, x_2, x_3$ of its vertices. Note that each 2-face of $\Delta$ inherits a 2-dimensional branching in the sense just defined. The ambient orientation of $Y$ induces an orientation on each $\Delta$. Also the branching induces a $b$-orientation on $\Delta$: the $b$-orientation coincides with the ambient orientation iff the $b$-orientation of the 2-face $f(3)$ opposite to the vertex $x_3$ of $\Delta$ coincides with the boundary orientation (i.e. $\sigma(f(3)) = 1$). The sign function $\ast_b$ for the tetrahedra of $(T, b)$ is defined by $\ast_b(\Delta) := \sigma(f(3))$. For every $(\Delta, b)$ we denote by $e_0, e_1, e_2$ the $b$-ordered and oriented edges of $f(3)$.

It is convenient to give also an encoding of these 3-dimensional branched triangulations $(T, b)$ in terms of their dual cell decompositions. In Fig. 4 we see an enriched version of the 1-skeleton of such a dual cell decomposition, localized at branched tetrahedra of sign $\ast_b = \pm 1$ (ignore the symbols $x, \alpha, \ldots, \delta$ for the moment, as they refer to later considerations).

In the picture we see in fact a planar realization of this 1-skeleton; its four branches at the vertex dual to $\text{int}(\Delta)$ are arranged to form a normal crossing with an under/over arc specification (like for ordinary link diagrams). The $b$-sign $\ast_b = \pm 1$ is encoded by the usual normal crossing index. We have omitted to draw any arrows on two branches, as it is understood that they are incoming at the crossing. Note that these decorated graphs contain all the information in order to reconstruct the corresponding dual branched tetrahedron $(\Delta, b)$. The oriented branches are outgoing exactly when the corresponding dual 2-face has $b$-sign $\sigma = 1$.

The cylinder $\mathcal{C}(T, b)$. Consider the cylinder $\mathcal{C} = \mathcal{C}_F = F \times [-1, 1]$, oriented in such a way that the oriented surface $F$ is identified with the oriented “horizontal” boundary component $F_- = F \times \{-1\}$ of $\mathcal{C}$. Sometimes we write $(T_-, b_-)$ for the branched triangulation on this boundary component, via this identification. Similarly we write $(T_+, b_+)$ for the copy of $(T, b)$ on the other horizontal boundary component $F_+$, which has the opposite orientation. Finally, we denote by $\mathcal{C}(T, b) := (\mathcal{C}(T), \mathcal{C}(b))$ the branched triangulation of $\mathcal{C}$ obtained as follows.

Consider first the natural product cell decomposition $P(T, b)$ of $\mathcal{C}$, made by $p+4r$ prisms with triangular base. We stipulate that all the $3r$ vertical edges of $P(T, b)$ are oriented to point towards $F_+$. For every abstract prism $P$, every “vertical” quadrilateral, $R$ say, on its boundary has both the two horizontal and the two vertical edges endowed with parallel orientations. So exactly one vertex of $R$ is a source (that belongs to $F_-$), and exactly one is a pit (that belongs to $F_+$). Then
we triangulate each $R$ via the oriented diagonal going from the source to the pit. Finally we extend the so obtained triangulation of $\partial P$, to a triangulation of $P$ by 3 tetrahedra, just by making the cone from the $b$-first vertex of the bottom base triangle of $P$ (note that no further vertices nor further edges have been introduced). Repeating this for every prism, we finally get our branched triangulation $\mathcal{C}(T, b)$ of $\mathcal{C}$.

Let us list few properties of $\mathcal{C}(T, b)$:

1. $\mathcal{C}(T, b)$ contains $6r$ vertices (that are all on $F_\pm$) and $3(4r - 2\chi(F))$ tetrahedra; moreover, it has $2(7r - 3\chi(F))$ horizontal edges on $F_\pm$, $r$ vertical edges over the vertices of $T$, and $7r - 3\chi(F)$ diagonal edges on the vertical rectangles. Note that each prism contains 10 triangles.

2. For every prism $P$, denote:
   - $t_\pm = t_\pm(P)$ the base triangle contained in $F_\pm$;
   - $\Delta_\pm = \Delta_\pm(P)$ the tetrahedron based at $t_\pm$;
   - $\Delta_0 = \Delta_0(P)$ the interior tetrahedron.
Assume that $\sigma_b(t_-) = 1$ (so that $\sigma_b(t_+) = -1$). Then both $*(\cdot)_b(\Delta_\pm) = 1$. The tetrahedron $\Delta_0$ shares one edge with each base triangle respectively; these are opposite edges of $\Delta_0$. We have $*(\cdot)_b(\Delta_0) = -1$. The $b$-oriented dual graph of $\mathcal{C}(T, b)|P$ points outside at $t_-$, and points inside at $t_+$. If $*_{b}(t_-) = -1$ the same facts hold, providing that all the signs are inverted.

The $\mathcal{I}$-parameters for $\mathcal{R}(g, r)$ shall result from a suitable idealization procedure of the $\mathcal{D}$-parameters discussed in Subsection 2.3. As explained there, we will work separately on each $\mathcal{R}(g, r, x)$ by using the surjective projections $p_{x,y} : Z(T, b, x, y)_\beta \to \mathcal{R}(g, r, x)$. The rough idea is to extend each $z \in Z(T, b, x, y)_\beta$ to some cocycle $z(\cdot) \in Z(\mathcal{C}(T, b))$ and take (if possible) its idealization, thus obtaining the corresponding cross-ratio $\mathcal{I}$-parameters.

**Idealization.** Let us briefly recall few general facts about the idealization procedure. We refer to [4, 5] for the full details. Let $(T, b, z)$ be a branched triangulation of an oriented compact 3-manifold $Y$, equipped with a $PSL(2, \mathbb{C})$-valued 1-cocycle $z \in Z(T, b)$. We fix once for ever $0 \in \mathbb{C}$ as base point of our idealization procedure. We say that an abstract tetrahedron $(\Delta, b, z)$ of $(T, b, z)$ (with the induced branching and cocycle), is idealizable if

$$u_0 = 0, \quad u_1 = z_0(0), \quad u_2 = z_0z_1(0), \quad u_3 = z_0z_1z_2(0)$$

are 4 distinct points in $\mathbb{C} \subseteq \mathbb{HP}^1 = \partial \mathbb{H}^3$. Here $z_i = z(e_i)$. These 4 points span a (possibly flat) hyperbolic ideal tetrahedron with ordered vertices. We call $(T, b, z)$ a $D$-triangulation if all its tetrahedra are idealizable. The idealization $(T, b, w)$ of a $D$-triangulation consists of the family $\{(\Delta, b, w)\}$, where $\Delta$ spans the 3-simplices of $T$, and each edge $e$ of $\Delta$ is now decorated by the the appropriate cross-ratio modulus $w(e) \in \mathbb{C} \setminus \{0, 1\}$ of the above hyperbolic ideal tetrahedron. In fact the $w(e)$’s are specified by the modular triple $w = (w_0, w_1, w_2)$, $w_i = w(e_i)$, as opposite edges share the same cross-ratio modulus. The idealization $(T, b, w)$ gives a so called $\mathcal{I}$-triangulation of $Y$. This means that at each internal (i.e. not contained in $\partial Y$) edge $e$ of $T$ it is satisfied the edge compatibility condition

$$\prod_{h \in e^+(e)} w^j(h)^*e^j = 1$$

where $e_T$ is the map that associates to every abstract edge the corresponding edge in $T$ (via the face identifications), and $*_{b^j} = \pm 1$ according to the $b^j$-orientation of the tetrahedron $\Delta^j$ that contains the abstract edge $h$. 

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\textit{I}-triangulations actually encode $PSL(2, \mathbb{C})$-valued representations of the fundamental group of $Y$ up to conjugation. More precisely, by lifting a given \textit{I}-triangulation of $Y$ to its universal covering $Y'$, we can construct, by “developing” the hyperbolic ideal tetrahedra of the triangulation in the naturally compactified hyperbolic space $\mathbb{H}^3$, a pseudo developing map $d : Y' \rightarrow \mathbb{H}^3$ and a representation $h : \pi_1(Y) \rightarrow PSL(2, \mathbb{C})$, such that $d(\gamma(y)) = h(\gamma)(d(y))$ for every $\gamma \in \pi_1(Y)$ and $y \in Y'$. The pseudo-developing map $d$ is unique up to post-composition with the action of $PSL(2, \mathbb{C})$ on $\mathbb{H}^3$, and $h$ is unique up to conjugation.

**Definition 2.5** We say that a cocycle $z \in Z(T, b, x, y)_\beta$ is idealizable if for any triangle of $(T, b)$ the points $u_0 = 0$, $u_1 = z_0(0)$ and $u_2 = z_0z_1(0)$ are distinct in $\partial \mathbb{H}^3$. We denote by $Z_1(T, b, x, y)_\beta$ the set of these idealizable cocycles.

Note that if there exists an idealizable extension $C(z)$ of $z$ to $C(T)$, then $z \in Z_1(T, b, x, y)_\beta$. We construct such extensions as follows.

- Take first the following trivial extension $C^0(z)$ of $z$ to $C(T, b)$.
- Copy $z$ on the triangulations $(T_\pm, b_\pm)$ of the horizontal boundary components $F_\pm$ of $C$.
- Every vertical quadrilateral $R$ of $C(T, b)$ has the bottom and top horizontal edges endowed with parallel orientations and with the same cocycle value, say $g$. Give each vertical edge the value $1 = [1, 0]$; there is a unique way to complete the cocycle, just by giving each diagonal edge the corresponding value $g$.

Evidently $C^0(z)$ is not idealizable. We have to perturb it. For every $a \in \mathbb{C}^*$, consider the $0$-cochain $c_a^-$ that gives each vertex of $T_-$ the value $[1, 0]$, and each vertex of $T_+$ the value $[1, a]$. Finally let $C^-(z, a)$ be the cocycle on $C(T, b)$ obtained by perturbing $C^0(z)$ via the gauge transformation corresponding to $c_a^-$.

The proof of the following lemma is easy. Recall that a triangulation is said quasi-regular if every edge has distinct end-points.

**Lemma 2.6** For every $(T, b)$, $x$, $y$ and $a$ as above we have:

1. $Z_1(T, b, x, y)_\beta$ is a non empty dense open subset of $Z(T, b, x, y)_\beta$.
2. If $(T, b)$ is quasi-regular, then the projection of $Z_1(T, b, x, y)_\beta$ covers the whole of $R(g, r, x)$.
3. We can remove from $Z_1(T, b, x, y)_\beta$ a finite number of complex algebraic hypersurfaces, to obtain an algebraic set $Z_1,a(T, b, x, y)_\beta$ such that for every $z$ in $Z_1,a(T, b, x, y)_\beta$ the cylinder cocycle $C^-(z, a)$ is idealizable.
4. There is a finite number of non zero complex numbers $s$ such that the corresponding $Z_1,a(T, b, x, y)_\beta$’s cover the whole of $Z_1(T, b, x, y)_\beta$.

We are interested to the portion $R_I(g, r, x)$ of $R(g, r, x)$ covered by the projections of these $Z_1(T, b, x, y)_\beta$’s. In order to make everything more definite, we are going now to specify a normalized choice $y_x$ of $y$.

**Normalizing** $y = y_x$. Let $S$, $F$ (identified with $F_-$), $(T', b')$, with associated $e$-triangulation $(T, b)$ of $F$, be as usual. For every vertex $v$ of $(T', b')$, consider the corresponding cell $s_v$ containing the other two vertices $v'$ and $v''$ of $T$. Recall the base triangle $\tau = \tau_v$ defined in Subsection 2.2. Denote by $\tau' = \tau_{v'}$ the triangle in $s_v$ that contains the other boundary edge. Call $e_0 = e_0(v), e_1 = e_1(v), e_2 = e_2(v)$ the $b$-ordered edges of $\tau$. Set similarly $e'_{j}$ for $\tau'$. Note that $e_0 = e'_{0}$.

All $z \in Z(T, b, x, y)_\beta$ share, by definition, the same contribution of $y$ at $v$, that is $y_x = (g_0, g_1) := (z(e_0), z(e_1))$. Set $g_2 = g_0g_1$. Similarly denote $h_0, h_1, h_2$ on $\tau'$. Recall that $v_x = (h_1)^{-1}g_1$, and that

- $x_v = [s, 0]$, $s \in \mathbb{C} \setminus \{0, 1\}$, in the case of generic boundary holonomy;
- \(x_v = [1, 1]\), in the case of parabolic boundary holonomy;
- \(x_v = [1, 0]\), in the trivial case.

For any 0-cochain \(c\) with support in \(s_v\), write \(c = c(v), c' = c(v')\) and \(c'' = c(v'')\). If \(c\) leads to residual gauge transformations for \(Z(T, b, x, y)\beta\), then \(c \in \text{Stab}(x_v)\), that is

- \(c = [d, 0], d \in \mathbb{C}^*\), in the generic case;
- \(c = [1, b], b \in \mathbb{C}\), in the parabolic case;
- \(c\) is an arbitrary element of \(\text{PSL}(2, \mathbb{C})\) in the trivial case.

Normalization in the generic case. Set \(g_0 = [1, 1]\) and \(g_1 = [f, 0]\), with \(f \neq 0\), \(f \neq -1\), \(f \neq +1\), \(f \neq -1\) and \(g_2 = [1, f + 1]\). As \(c = [1, b]\), then \(c' = [1, f][1, b][1, -f] = [1, b] = c\). Also, we have \(h_0 = [1, 1]\), \(h_1 = g_1[1, -1] = [1, f - 1]\), and \(h_2 = [1, f]\). The vertices of the idealization of \(\tau\) are \((0, 1, 1 + f)\), the ones of \(\tau'\) are \((0, 1, f)\). In the case of a trivial boundary loop we have \(g_i = h_i, i = 0, 1, 2\). Then we can impose that by working both in the generic and parabolic styles, we eventually get the same idealization for \(\tau\). So we impose \(f' = 1 + f\), that is \(f = (1 + \sqrt{5})/2\). We take the same normalization also in the parabolic case.

For every \(x\), we denote by \(Z(T, b, x, y_x)\beta\) the subset of \(Z(T, b, x)\beta\) obtained by performing on each \(s_v\) the above normalization. The surjective projections \(\bar{p}_x = \pi_{x, y_x} : Z(T, b, x, y_x)\beta \to R(g, \tau, x)\) form the bundle of cocycle \(D\)-parameters of Definition 2.4. From now on we will apply the previously discussed idealization procedure to this normalized situation.

Construction of the \(I\)-parameters. The image of the sets \(Z_{I,a}(T, b, x, y_x)\beta\), defined in Lemma 2.6 by the projections \(\bar{p}_x\) give our favourite patches for \(R_I(g, \tau, x)\). On each \(Z_{I,a}(T, b, x, y_x)\beta\) we have a “change of coordinates”

\[i : Z_{I,a}(T, b, x, y_x)\beta \to I_{\alpha}(T, b, x, y_x)\beta\]

that passes from the cocycle \(D\)-parameters to \(I\)-parameters, namely the cross-ratio moduli of the tetrahedra in the cylinder \(C(T, b)\), obtained via the idealization of the \(C^{-}(z, a)\)’s. Every space of \(I\)-parameters \(I_{\alpha}(T, b, x, y_x)\beta\) is a Zariski open set of an algebraic subvariety of \((\mathbb{C} \setminus \{0, 1\})^{3(2r - 2\chi(F))}\).

Indeed, there are in total \(3(4r - 2\chi(F))\) tetrahedra in \(C(T, b)\), but the moduli of the tetrahedra over each pair of triangles at the boundary loops of \(F\) are fixed by the normalization.

This subvariety is defined by the following set of algebraic equations:

- Diagonal relations. These are \(|E(T)| - 3r = 4r - 3\chi(F)\) edge compatibility conditions at the internal diagonal edges of the vertical quadrilaterals. Again because of the normalization, we can ignore, for every vertex \(v \in V\), the 3 quadrilaterals that lie over the two loop boundary edges and over the edge connecting \(v'\) and \(v''\) respectively.
Vertical relations. These are $2r$ relations at the vertical edges of $C(T, b)$ over the $v$- and $v''$-vertices respectively. The second ones are again $I$-triangulation edge compatibility conditions \( \Pi \) at these interior edges. The first ones are also edge compatibility conditions, once we have filled each vertical boundary tube of $\partial F \times [-1, 1]$ by a suitable $I$-cusp, so that also the vertical edges over the $v$-vertices become interior edges (the needed cusp machinery is developed in the next Section 3.2). These relations actually depend on the values $x_v$ at the corresponding boundary loops.

So, we have in total $6r - 3\chi(F)$ relations defining $I_a(T, b, x, y_x)$. This gives

$$\dim(I_a(T, b, x, y_x)) \geq 3(2r - 2\chi(F)) - (6r - 3\chi(F)) = -3\chi(F).$$

On the other hand, $I_a(T, b, x, y_x)$ is an open set of a space of complex dimension $-3\chi(F)$. It projects onto $R_I(g, r, x)$, since $R(g, r)$ is encoded by the $I$-triangulations of the cylinder $C_F$. Hence we eventually get the following remarkable facts:

**Proposition 2.7** (1) We have $\dim(I_a(T, b, x, y_x)) = -3\chi(F)$, i.e. the system of equations defining $I_a(T, b, x, y_x)$ is not overdetermined.

(2) The residual $D$-gauge transformations $G(T, b, t(x))$ transit via the idealization map $i : Z_{I,a}(T, b, x, y_x) \to I_a(T, b, x, y_x)$ onto a space of residual $I$-gauge transformations $G_I(T, b, t(x))$ of the same dimension. So we have a bundle of cross-ratio $I$-parameters

$$p_{I,a} : I_a(T, b, x, y_x) \to R_I(g, r, x)$$

with structural group $G_I(T, b, t(x))$.

**Remark 2.8** We can replace $PSL(2, \mathbb{C})$ by $PSL(2, \mathbb{R})$, and almost everything can be repeated verbatim. A main difference is that in the case of parabolic ends we have $c(g) = \pm 1$, that is to every parabolic end there is an associated sign. Moreover, via the idealization, we get only degenerate tetrahedra, that is only real $I$-moduli.

The $I$-boundary map. We are interested now to the two-dimensional “trace” on $(T, b)$ of the cross-ratio $I$-parameters. For every $a \in \mathbb{C}^*$, we can use the same formula which enters the edge compatibility condition \( \Pi \) for the interior edges of any $I$-triangulation, to define a map

$$W_a : Z_{I,a}(T, b, x, y_x) \to (\mathbb{C} \setminus \{0, 1\})^{E(T)} \times \mathbb{R}^{-3r}$$

which factorizes via the idealization as

$$W_a = W^I_a \circ i : Z_{I,a}(T, b, x, y_x) \to I_a(T, b, x, y_x) \to (\mathbb{C} \setminus \{0, 1\})^{E(T)} \times \mathbb{R}^{-3r}.$$

More precisely, as usual let us identify $(T, b)$ with $(T_-, b_-)$, so that the edges of $T$ are contained in the horizontal boundary component $F_-$ of $C$. The map $W^I_a$ associates to each edge $e$ of $T = T_-$ the signed product

$$\prod_{h \in e \cap \tilde{I}^{-1}(e)} w^I(h)^{\gamma(h)}$$

of the cross-ratio moduli of $i(C^+(z, a))$ at the abstract edges $h$ of $C(T)$ descending onto $e$. (We can impose in Definition 2.5 the genericity condition that $W^I_a(z)(e) \neq 1$ for all $e$). Note that for every $v \in V$, the products $2$ for the two boundary edges of $F$ at $v$ and the edge connecting $v'$ and $v''$ are fixed by the normalization.

Here is an interpretation of the $W_a(z)(e)$’s. Consider the two (branched) triangles adjacent to an edge $e$ of $T$. Let us order, if needed, the two vertices opposite to
by orienting the edge $e'$ connecting them (i.e. dual to $e$), so that $(e', e)$ defines the positive orientation of $F$. Then the vertices in the star of $e$ are totally ordered.

As in the above discussion about the idealization procedure, consider the $z$-orbit of $0$ along the oriented edges of both triangles. The meaning of $W_a(z)(e)$ is that of a cross-ratio for the quadrilateral in $\mathbb{C}P^1 = \partial \mathbb{C}P^3$, with ordered vertices the so obtained four orbit points. We have:

**Lemma 2.9** The maps $W_a$’s match on the overlaps of the spaces $Z_{I,a}(T, b, x, y_x)_\beta$, so that we have a well defined $\mathcal{I}$-boundary map

$$W : Z_I(T, b, x, y_x)_\beta \to (\mathbb{C} \setminus \{0, 1\})^{\left|E(T)\right| - 3r}.$$

**Proof.** We have to show that two cocycles $\mathcal{C}^-(z, a)$ and $\mathcal{C}^-(z, a')$ that differ only for the coefficients $a$ and $a'$ lead to $W_a^I = W_a'^I$. Consider the gluing $\mathcal{C}(T, b) \cup -\mathcal{C}(T, b)$, where we have inverted the orientation of the second copy, and the gluing is made by identifying the two copies of $F_-$. Also $\mathcal{C}^-(z, a)$ and $\mathcal{C}^-(z, a')$ glue together and give us an idealizable cocycle on $\mathcal{C}(T, b) \cup -\mathcal{C}(T, b)$. Every edge $e$ on $F_-$ is now an interior edge, and the usual edge compatibility condition exactly means that $W_a^I(e)W_a'^I(e)^{-1} = 1$.

Recall that $\mathcal{I}_a(T, b, x, y_x)_\beta$ is an open affine algebraic set of complex dimension $-3\chi(F)$. As the maps $W_a^I$ are given by monomials, $\text{Im}(W)$ is also open with $\text{dim}(\text{Im}(W)) \leq -3\chi(F)$, Moreover, $\mathcal{R}_I(g, r, x)$ is encoded just by the products of $\mathcal{I}$, rather than the whole set of cross-ratio $\mathcal{I}$-parameters (see the next paragraph). A direct computation shows that the group $\mathcal{G}(T, b, t(x))$ of residual $\mathcal{D}$-gauge transformations transit via the map $W_a$ to a group of the same dimension. So we eventually get:

**Proposition 2.10** The set $\text{Im}(W)$ is an affine complex algebraic set of dimension $-3\chi(F)$, that is the total space of a bundle over $\mathcal{R}_I(g, r, x)$.

**Definition 2.11** We call $W(T, b, x) = \text{Im}(W)$ the space of $\mathcal{I}\partial$-parameters for $\mathcal{R}(g, r, x)$, and denote $\pi : W(T, b, x) \to \mathcal{R}_I(g, r, x)$ the associated bundle.

Recall from Lemma (2), that if $T$ is quasi-regular, then $\mathcal{R}_I(g, r, x) = \mathcal{R}(g, r, x)$. We note that the relations between the $\mathcal{I}\partial$-parameters are very implicit compared to the very transparent edge compatibility relations in $\mathcal{I}_a(T, b, x, y_x)_\beta$.

**Holonomies from $\mathcal{I}\partial$-parameters.** We can describe explicitly the bundle map $\pi : W(T, b, x) \to \mathcal{R}_I(g, r, x)$. For a conjugacy class of representations $\rho \in \mathcal{R}(g, r, x)$, take an $e$-triangulation $T = T_-$ for the surface $F = F_-$ (viewed as the lower horizontal boundary component of $\mathcal{C}$), such that there exists an idealizable cocycle $z \in Z_I(T, b, x, y_x)_\beta$ representing $\rho$. For instance, such a $z$ exists if $T$ is quasi-regular; if $\rho$ is quasi-Fuchsian, or more generally if there exists a non-empty domain of $\mathbb{C}P^1$ on which $\rho(\pi_1(F))$ acts freely, then any $T$ works. We construct representatives $\tilde{\rho}$ of $\rho$ from any point $W = W(z)$ in the fiber $\pi^{-1}(\rho)$ as follows.

Choose a base point $q$ in $F$ not in the 1-skeleton of $T$. Given an element of $\pi_1(F, q)$, represent it by a closed curve $\gamma \subset F$ transverse to $T$, and which do not backtrack (i.e. it never departs from an edge it just entered). Assume that $\gamma$ intersects an edge $e$ of $T$ positively w.r.t. the orientation of $F$. Fig. 4 shows the three possible branching configurations for the two triangles glued along $e$.

Fix arbitrarily a square root $W'(e)$ of $W(e)$. Consider the elements of $\text{PSL}(2, \mathbb{C})$ given by

$$\gamma(e) = \left( \begin{array}{cc} W'(e) & 0 \\ 0 & W'(e)^{-1} \end{array} \right), \quad p = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad l = \left( \begin{array}{cc} -1 & 1 \\ -1 & 0 \end{array} \right).$$
and $r = l^{-1}$. Recall that $PSL(2, \mathbb{C})$ is isomorphic to $\text{Isom}^+(\mathbb{H}^3)$, with the natural conformal action on $\mathbb{CP}^1 = \partial \mathbb{H}^3$ via linear fractional transformations. The matrix $\gamma(e)$ represents the isometry with fixed points $0, \infty \in \mathbb{CP}^1$ and mapping $1$ to $W(e)$.

The elliptic elements $p$ and $l$ send $(0, 1, \infty)$ to $(\infty, 1, 0)$ and $(\infty, 0, 1)$ respectively.

The flat principal $PSL(2, \mathbb{C})$-bundles associated to $\rho$ carry parallel transport operators, that we may compute along $\gamma$ by using the cocycle $z$. For the portion of $\gamma$ represented in the left configuration of Fig. 5, if $\gamma$ turns to the left after crossing $e$ the parallel transport operator is $\gamma(e) \cdot p \cdot l$, while it is $\gamma(e) \cdot p \cdot r$ if $\gamma$ turns to the right. (The composition is on the right, as is the action of $PSL(2, \mathbb{C})$ on the total spaces of the bundles given by $\rho$). Similarly, in the middle and right pictures the parallel transport operators are given by $\gamma(e) \cdot l$ or $\gamma(e) \cdot p \cdot l$, and $\gamma(e) \cdot p \cdot r$ or $\gamma(e) \cdot r$ respectively. The action of $p$, $l$ and $r$ depends on the reordering of the vertices after the mapping $\gamma(e)$. Note that the whole branching configuration enters the computation of $W(e)$. If $\gamma$ intersects $e$ negatively, essentially we have to replace $\gamma(e)$ with $\gamma(e)^{-1}$ in the above expressions. Continuing this way each time $\gamma$ crosses an edge of $T$ until it comes back to $q$, we get an element of $PSL(2, \mathbb{C})$.

This element does depend only on the homotopy class of $\gamma$ (based at $q$), for any $W \in \pi^{-1}(\rho)$, and we eventually obtain a well-defined representation $\tilde{\rho} : \pi_1(F, q) \to PSL(2, \mathbb{C})$ in the conjugacy class of $\rho$. Indeed, we can push a little $\gamma$ in the interior of the cylinder $C$, and then use the encoding of $R(g, r)$ via $I$-triangulations of $C$ to check the claim. The point is that the very definition of the maps $W^I_a$ implies that the parallel transport operator along $\gamma$ is not altered when we perturb it this way.

3 The QHFT bordism category, and $I$-cusps

In this section we define the (2+1)-bordism category at the basis of the QHFTs. We start with a naked, “coordinate free” category, then we progressively reach the final elaborated marking. This shall incorporate the phase space parameters discussed in Sections 2.3-2.4. Finally, we introduce the notion of $I$-cusps.

3.1 The (2+1)-bordism category

Naked bordism category. Like in Section 2 for every $(g, r) \in \mathbb{N} \times \mathbb{N}$, such that $g \geq 0$, $r > 0$, and $r > 2$ if $g = 0$, we fix a compact closed oriented base surface $S = S_g$, of genus $g$ with a set of $r$ marked points $V = V_{g,r}$. We denote by $-S$ the same surface with the opposite orientation. An elementary object of our naked category either is the empty set, or is represented by a diffeomorphism $\phi : \pm S \to \Sigma$. In other words it is a parametrized surface $\Sigma$. Both the orientation and the marked points $V$ transit from $\pm S$ to $\Sigma$ via $\phi$. Later the surface $\pm S$ shall be equipped
with further extra-structures such as triangulations; we always stipulate that these extra-structures transit from from $\pm S$ to $\Sigma$ via $\phi$.

The pairs $(\pm S, \phi)$ are considered up to the following equivalence relation: $(\pm S, \phi_1)$ is identified with $(\pm S, \phi_2)$ (i.e. they represent the same elementary object) if there is an oriented diffeomorphism $h : \Sigma_1 \to \Sigma_2$, such that $(\phi_2)^{-1} \circ h \circ \phi_1$ pointwise fixes $V$ and is isotopic to the identity automorphism of $S$ relatively to $V$. An object is a finite union of elementary ones, where $(g, r)$ varies.

We define now the bordisms between objects, i.e. the morphisms of the naked category. Let $Y$ be an oriented compact 3-manifold with (possibly empty) boundary $\partial Y$. It is given as an input vs output bipartition of the boundary components so that $\partial Y = \partial_- Y \cup \partial_+ Y$. We can imagine that $\partial_- Y$ is “at the bottom” of $Y$, while $\partial_+ Y$ is “on the top”. Each boundary component inherits the boundary orientation, via the usual convention “last is the ingoing normal”. We assume also that it is given a properly embedded link $L \subset Y$, considered up to proper ambient isotopy. Sometimes it is convenient to look at $L$ as $L = L_i \cup L_b$, where $L_i$ is the internal part of $L$ made by its closed connected components, while $L_b$ is the union of the components homeomorphic to the interval $[0,1]$, with end-points at some boundary components of $Y$ (possibly the same), and transverse to $\partial Y$. For every boundary component $\Sigma$ of $Y$, we assume that $|L_b \cap \Sigma| > 0$, and that $|L_b \cap \Sigma| > 2$ if $g(\Sigma) = 0$. Note that we do not require that every component of $L_b$ connects $\partial_-$ with $\partial_+$. As the base surfaces $S_g$ and the boundary components $\Sigma$ of $Y$ have given orientations, we need to specify the “sign” of an object $[\phi : \pm S \to \Sigma]$. Hence we can associate objects $\alpha_{\pm}$ to both $\partial_{\pm}$. We get in this way the bordism from $\alpha_{-}$ to $\alpha_{+}$ with support $(Y, L)$. We also allow that $\partial Y = \emptyset$, i.e. $Y = W$ is a closed manifold, and $(W, L)$ is a morphism from the empty set to itself. We stress that $L$ is non empty in any case.

**Introducing framings.** For every $(S, V)$ as above, we introduce a framing at each marked point $v \in V$. This means that we fix a system of disjoint embedded segments $a_v$ in $S$ having the $v$’s as “first” end-point. We denote by $v''$ the other end-point. We adapt the above definition of the objects, by requiring that $(\phi_2)^{-1} \circ h \circ \phi_1$ is the identity on the $a_v$’s, and the isotopies are relative to them.

We assume now that the above link $L \subset Y$ is framed, and we denote it by $L_F$. This means that $L_F$ is a disjoint union of properly embedded orientable ribbons. Each interior component of $L_F$ is homeomorphic to the annulus $S^1 \times [0,1]$, the other components are homeomorphic to $I \times [0,1]$. On the boundary of each ribbon we keep track of a core line of the form $X \times \{0\}$ (here $X = S^1$ or $I$ resp.), for the corresponding component of the unframed link $L$, and there is a longitudinal line $X \times \{1\}$ that specifies the framing of the normal bundle of the parallel core line. This induces on each boundary component $\Sigma$ of $Y$ a system of framed marked points (i.e. $L \cap \Sigma$ is framed by $L_F \cap \Sigma$). Bordisms supported by $(Y, L_F)$ are defined similarly as above.

It is convenient to reformulate the bordism category with framed links in an equivalent but slightly different way, which is closer to the phase space parameters up.

**Zipping-unzipping.** Fix a mid-point $v'$ on each arc $a_v$. Then, let us unzip (cut open) each $a_v$ at the open sub-interval $(v, v')$. In this way we get from $\pm S = \pm S_g$ an oriented surface $\pm F = \pm F_{g,r}$ with $r$ boundary components. Each boundary component of $F$ is a bigon with vertices $v$ and $v'$; $v'$ is connected to $v''$ by the sub-interval $[v', v'']$ of $a_v$. We can use these $\pm F$’s as sources of elementary objects $[\phi : \pm F \to \Sigma]$. Naturally, also $\Sigma$ has now $r$ boundary components. To define equivalent
parametrized surfaces, we use (homotopy classes of) the oriented diffeomorphisms of $\pm F$ that are the identity on the boundary.

Consider now $(Y, L_F)$ as above. For every ribbon component of $L_F$ take a mid-line corresponding to $X \times \{1/2\}$. This eventually gives us a triple $\Lambda = (\lambda, \lambda', \lambda'')$ of parallel unframed links in $Y$: the core line $\lambda = X \times \{0\}$, this just introduced mid-line $\lambda' = X \times \{1/2\}$, and the longitudinal boundary line $\lambda'' = X \times \{1\}$. The trace of $(L_F, \Lambda) \cap \partial Y$ at each component $\Sigma$ of $\partial Y$ makes a system $\{a_u\}$ of disjoint segments; each one has a marked end-point $u$, a mid-point $u'$ and the other end-point $u''$. Let us unzip each ribbon band at the open sub-band $X \times (0,1/2)$. We get in this way a 3-manifold “with corners” $\tilde{Y}$. Its boundary $\partial \tilde{Y}$ has two horizontal parts $\partial_\pm \tilde{Y}$ contained in $\partial_\pm Y$, and a tunnel part $\tilde{L}_F$. The horizontal parts intersect the tunnel part at the corner locus; this is a union of bigons contained in $\partial Y$. Each boundary component of $Y$ corresponds to a horizontal boundary component of $\tilde{Y}$, still denoted by $\Sigma \in \partial_\pm$. Each internal tunnel component is homeomorphic to the torus $T = S^1 \times S^1$; the other tunnel components are homeomorphic to $A = S^1 \times I$. Every tunnel component is made by the union of two copies of $X \times (0,1/2)$, glued each to the other at $\lambda \cup \lambda'$.

The horizontal boundary components can be considered as targets of elementary objects $[\phi : \pm F \to \Sigma]$, each $(\tilde{Y}, \tilde{L}_F)$ can be considered as the support of a morphism between such objects. All this is straightforward. Clearly, we can zip back $(\tilde{Y}, \tilde{L}_F)$ to reobtain the initial $(Y, L_F)$, so we have two equivalent settings to describe the same bordism category.

The QHFT bordism category. To define the (elementary) objects of our final bordism category, for every $F$ as above, we fix as part of a marking an $e$-triangulation $(T, b)$ of $F$, and a cocycle $z \in Z_1(T, b, x, y_x)\rho$ (for some $x$). So an elementary object is of the form $[(\pm F, (T, b), x, z), \phi]$; we keep $(T, b), x$ and $z$ fixed when we define equivalent marked surfaces. Let $(\tilde{Y}, \tilde{L}_F)$ be a naked bordism as above, and assume furthermore that the boundary objects $\alpha_\pm$ are equipped to be objects of the present QHFT bordism category. Assume also that it is given a conjugacy class $\rho$ of $PSL(2, \mathbb{C})$-valued representations of $\pi_1(\tilde{Y} \setminus \tilde{L}_F)$. Then $(\tilde{Y}, \tilde{L}_F, \rho)$ is the support of a QHFT bordism from $\alpha_- \to \alpha_+$ iff, for every elementary boundary object $[(\pm F, (T, b), x, z), \phi]$ we have $\phi^* (\rho) = [z]$.

Bordism Composition. Consider a QHFT bordism $B$ from $\alpha_- \to \alpha_+$, with support $(\tilde{Y}, \tilde{L}_F, \rho)$, and another bordism $B'$ from $\alpha'_- \to \alpha'_+$, with support $(\tilde{Y}', \tilde{L}_F', \rho')$. Assume that $\beta_+$ and $\beta'_-$ are sub-objects of $\alpha_+$ and $\alpha'_-$ respectively, such that $\beta_+ = - \beta'_-$. Then we can glue the two bordisms at the common sub-objects. We get a new bordism (morphism) $B'' := B * B'$ with support $(\tilde{Y}'', \tilde{L}_F'', \rho'')$, from $\alpha''_- \to \alpha''_+$, where $\alpha''_\pm = (\alpha_- \cup (\alpha'_- \setminus \beta'_-)) \cup (\alpha'_+ \setminus \beta_+)$. We say that $B''$ is the composition of the bordism $B$ followed by the bordism $B'$.

3.2 Cusps

The cusps that we are going to introduce will allow us, in Section 4, to turn the tunnel part of QHFT bordisms supported by a triple $(\tilde{Y}, \tilde{L}_F, \rho)$ into “toric ends” or “annular ends” at infinity, thus replacing the corresponding markings with the weaker, purely geometric, dependence w.r.t. the $\rho$-holonomy at the meridians of $L_F$.

Triangulated cusps. We consider topological oriented cusps $C = C_A = A \times [0, +\infty[, \ C_T = T \times [0, +\infty[ \text{ with annular base } A = S^1 \times [-1,1] \text{ and toric base} \ $
\( T = S^1 \times S^1 \) respectively.

We fix first a specific class of branched triangulations of the bases \( A, T \). The idea is that of taking a branched triangulation \((T', b')\) of the band \( B = [0, 1] \times X \), with \( X = [-1, 1] \) and \( X = S^1 \) respectively, having the vertices on the line \( L = \{0, 1\} \times X \); then we unzip, i.e. cut open, \( B \) along \((0, 1) \times X \), thus getting a triangulation \((T, b)\) of \( A \) or \( T \) made by two copies of \((T', b')\) that coincide on the line \( L \). More precisely \( B \) is subdivided by a certain number of quadrilaterals \( R \) having parallel “vertical” sides on \( L \) and two interior parallel “horizontal” sides. Parallel sides of each \( R \) have parallel orientation. Finally a branched triangulation \((T', b')\) of \( B \) is obtained by introducing (in some way) an oriented diagonal on each \( R \). In Fig. 6 we see a fundamental domain of the base with a triangulation \((T, b)\) obtained by starting with 2 quadrilaterals.

![Figure 6: A special triangulation of the torus.](image)

We consider now the one point-compactification \( C^* \) of a given cusp \( C \). From \((T, b)\) we get the a branched triangulation \((T^*, b^*)\) of \( C^* \), just by taking the one-point compactification of \((T, b) \times [0, \infty]\); more precisely we stipulate that the point \( \infty \) is a common vertex of all tetrahedra of \((T^*, b^*)\), and is the opposite vertex to each triangle of \((T, b)\). The branching \( b^* \) extends \( b \), by imposing that \( \infty \) is a pit for every branched tetrahedron. In Fig. 7 we see an example of cusp triangulation (starting with one quadrilateral).

![Figure 7: An example of triangulated cusp.](image)

**I-cusps.** Let \( z \) be any idealizable \( PB(2, \mathbb{C}) \)-valued cocycle (i.e. with values in the Borel subgroup of \( PSL(2, \mathbb{C}) \)) on some \((T, b)\) as above. The idealization of each triangle is an ideal triangle with 3 distinct ordered vertices belonging to \( \mathbb{C} \subset \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 \). We look at such an ideal triangle as the base of an ideal hyperbolic tetrahedron which as the further vertex at \( \infty \), and is oriented like the corresponding tetrahedron of \((T^*, b^*)\). Doing it for every triangle, we obtain, by
definition, the idealization of \([T^*, b^*, z]\), which is called an \(\mathcal{I}\)-cusp. We have in fact an \(\mathcal{I}\)-triangulation, that is:

**Lemma 3.1** The cross-ratio moduli of every \(\mathcal{I}\)-cusp verify the edge compatibility condition \((1)\) at each interior “vertical edge” (i.e. not contained in a base triangle).

**Proof.** The cross-ratio moduli define a similarity structure for each triangle of \(T\). These extend to a pseudo-similarity structure of the whole base (see above the discussion of the idealization procedure), with holonomy given by the values of \(z\) on simplicial representatives of generators of the fundamental group, or as at the end of Section \(1\). In particular, the holonomy is trivial on small closed loops on \(T\) winding once about the endpoint of an interior vertical edge of the \(\mathcal{I}\)-cusp. Hence \((1)\) is satisfied. 

Finally we can precise the vertical relation over any \(v\)-vertex used in Subsection \(2.4\), before Proposition \(2.7\). Given \(z \in \mathcal{I}_a(T, b, x, y) \beta\), the vertical tube over the boundary loop associated to \(v\) can be considered as the base of an annular \(\mathcal{I}\)-cusp. Glue this \(\mathcal{I}\)-cusp to the tube, so that the vertical edge over \(v\) becomes interior. Then the corresponding vertical relation is just an usual edge compatibility condition.

### 4 QHFT morphisms

Consider a morphism \(B\) of the QHFT bordism category from the object \(\alpha_-\) to the object \(\alpha_+\), with support \((\tilde{Y}, \tilde{L}, \rho)\), as in the previous Section. For every odd integer \(N \geq 1\), we will show how to associate explicitly to \(\alpha_{\pm}\) a finite dimensional complex linear space \(E_N(\alpha_{\pm})\), and to the bordism a linear morphism

\[
\mathcal{H}_N(B) : E_N(\alpha_-) \to E_N(\alpha_+)
\]

in a functorial way w.r.t. the gluing of bordisms and morphism composition. This is the technical core of the construction of the QHFTs. As usual we associate to the empty object the ground field \(\mathbb{C}\).

#### 4.1 Matrix dilogarithms and trace tensors

**Review of the matrix dilogarithms.** The building blocks of the QHFT morphisms are the matrix dilogarithms that we have introduced and studied in [5]. Here we limit ourselves to recall some of their qualitative properties, what is just enough to follow the logic of the construction.

An \(\mathcal{I}\)-tetrahedron \((\Delta, b, w)\) consists of an oriented tetrahedron \(\Delta\) equipped with a branching \(b\), and a modular triple \(w = (w_0, w_1, w_2) = (w(e_0), w(e_1), w(e_2)) \in (\mathbb{C} \setminus \{0, 1\})^3\) such that (indices mod(\(\mathbb{Z}/3\mathbb{Z}\))):

\[
w_{j+1} = 1/(1 - w_j).
\]

Hence \(w_0w_1w_2 = -1\), and this gives a cross-ratio modulus \(w(e)\) to each edge \(e\) of \(\Delta\) by imposing that \(w(e) = w(e')\). We have already used such notions in Subsection \(2.3\). We will use the notations and conventions established there.

Given any \(\mathcal{I}\)-tetrahedron \((\Delta, b, w)\), we consider an extra-decoration made by two \(\mathbb{Z}\)-valued functions defined on the edges of \(\Delta\), called flattening and integral charge respectively. These functions share the property that opposite edges take the same value, hence it is enough to specify their values on the edges \(e_0, e_1, e_2\). We denote by \(\log\) the standard branch of the logarithm which has the arguments in \([-\pi, \pi]\).
For every \( f = (f_0, f_1, f_2) \) with \( f_i = f(e_i) \in \mathbb{Z} \), set
\[
l_j = l_j(b, w, f) = \log(w_j) + i\pi f_j
\]
for \( j = 1, 2, 3 \). We call \( l_j \) a log-branch of \((\Delta, b, w)\) for the edge \( e_j \). We say that \((f_0, f_1, f_2)\) is a flattening of \((\Delta, b, w)\) if
\[
l_0 + l_1 + l_2 = 0.
\]

An integral charge of \((\Delta, b)\) is a function \( c = (c_0, c_1, c_2) \) with \( c_i = c(e_i) \in \mathbb{Z} \), such that \( c_0 + c_1 + c_2 = 1 \). An \( I \)-tetrahedron endowed with a flattening and an integral charge is said flat/charged.

For every \( N > 0 \), any map \( A : \mathbb{C} \setminus \{0, 1\} \to \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N) \) can be interpreted as a function of \( I \)-tetrahedra via the formula:
\[
A(\Delta, b, w) := A(w_0)^* b.
\]
Namely, put the standard tensor product basis on \( \mathbb{C}^N \otimes \mathbb{C}^N \), so that \( A = A(x) \in \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N) \) is given by its matrix elements \( A^\beta_\gamma^\alpha_\delta \), where \( \alpha, \ldots, \delta \in \{0, \ldots, N-1\} \).

We denote by \( \bar{A} = \bar{A}(x) \) the inverse of \( A(x) \), with entries \( \bar{A}^\beta_\gamma^\alpha_\delta \). The branching \( b \) selects \( w_0 \) among the triple of cross-ratio moduli. We use it also to associate to each 2-face of \( \Delta \) one index among \( \gamma, \delta, \alpha, \beta \). The rule is shown in Fig. 4.

The matrix dilogarithm of rank \( N \), \( N \geq 1 \) being any odd integer, is an explicitly given \( \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N) \)-valued function
\[
\mathcal{R}_N(\Delta, b, w, f, c) = \mathcal{R}_N(w_0, f, c)^* b
\]
defined on flat/charged \( I \)-tetrahedra. The explicit formula is given in \( \S \). Note that the matrix elements are holomorphic w.r.t. the log-branches (up to sign when \( N = 1 \)); in particular, they depend on the whole decoration, not only on \( w_0 \), so flattenings and charges are incorporated in the above identification between tensors and decorated tetrahedra.

Each matrix dilogarithm \( \mathcal{R}_N \) satisfies a finite set of fundamental five term identities. These identities are supported by suitable \( I \)-flat/charged versions, called transit configurations, of the basic \( 2 \to 3 \) bistellar move on 3-dimensional triangulations (sometimes called Pachner or Matveev-Piergallini move). This bare move is shown on the top row of Fig. 4. We postulate that all the 5 tetrahedra involved in the move are oriented and that they induce opposite orientations on every common 2-face. Hence, we have two triangulations \( T \) and \( T' \) (by 2 and 3 tetrahedra resp.) of a same oriented polyhedron, and each tetrahedron inherits the induced orientation. The \( 2 \to 3 \) transit configurations involve an appropriate procedure to transfer branchings, moduli, flattenings and integral charges from the tetrahedra of \( T \) to those of \( T' \).

The matrix dilogarithms satisfy also other relations corresponding to transit configurations associated to few other local 3-dimensional triangulation moves, such as the so called bubble move which increases by one the number of interior vertices (see the bottom row of Fig. 4).

The matrix dilogarithms, as well as the elaborated extra decoration on \( I \)-tetrahedra, arise from the solution of a symmetrization problem for a family of basic matrix dilogarithms, which satisfy only one peculiar five term identity supported by the top row move of Fig. 4 called the matrix Schaeffer’s identity. This identity
Figure 8: The moves between singular triangulations.

is characterized by determined geometric constraints on the cross ratio moduli. In the “classical” case $N = 1$, the basic dilogarithm coincides with (the exponential of) the classical Rogers dilogarithm. The quantum ($N > 1$) basic dilogarithms are derived from the 6$j$-symbols for the cyclic representation theory of a Borel quantum subalgebra $B_\zeta$ of $U_\zeta(sl(2, \mathbb{C}))$, where $\zeta = \exp(2i\pi/N)$ (see [19], Section 8 of [5] or the Appendix of [4]).

**Trace tensors.** Assume that we dispose of an $I$-triangulation $(T, b, w)$ for a compact oriented 3-manifold $M$ (see Section 2.4). Remind that this means that the edge compatibility condition (1) is satisfied at each interior edge. Assume also that every (abstract) $I$-tetrahedron $(\Delta, b, w)$ of the triangulation has a flat/charge $(f, c)$. Then, for every fixed odd integer $N$, we can associate to each such $(\Delta, b, w, f, c)$ the corresponding matrix dilogarithm $R_N(\Delta, b, w, f, c)$. A $N$-state of $(T, b, w, f, c)$ is a function which associate to every triangle of the 2-skeleton of $T$ a value in $\{0, \ldots, N - 1\}$. So, every $N$-state determines indeed a matrix element of each matrix dilogarithm. As two tetrahedra induce opposite orientations on any common face, our identification rules (4) together with Fig. 4 imply that an index at a common face is “down” for the $R_N$ of one tetrahedron while it is “up” for the other. By applying Einstein’s rule of “summing on repeated indices” to the matrix elements selected by each possible $N$-state of $T$, we get the contraction (i.e. the trace) of these pattern of tensors $R_N(\Delta, b, w, f, c)$. We denote this trace by

$$\prod_{\Delta \subset T} R_N(\Delta, b, w, f, c).$$

The type of the trace tensor depends on the $b$-sign $\sigma$ of the boundary triangles of $(T, b)$. The construction of $\mathcal{H}_N(B)$ shall result from a specific implementation of this procedure of taking the trace, that includes suitable global constraints on flattenings and integral charges.
4.2 Bordism globally flat/charged $\mathcal{I}$-triangulations.

Let $\mathcal{B}, \alpha_\pm$, and $(\bar{\mathcal{Y}}, \bar{L}_\mathcal{I}, \rho)$ be as at the beginning of this Section. Recall from Section 3 that we have associated to $\mathcal{B}$ the pairs $(Y, L_\mathcal{I})$, $(Y, \bar{\Lambda})$ and $(Y, \Lambda)$. Here $L \subset Y$ is the initial unframed link, the ribbon link $L_\mathcal{I}$ encodes a framed version of $L$, $\bar{\Lambda} = (\lambda, \lambda, \lambda')$ is made by 3 unframed parallel copies of $L$, which lie on $L_\mathcal{I}$. The links $\Lambda$ and $\Lambda'$ are on the tunnel boundary of $\bar{\mathcal{Y}}$, while $\Lambda''$ is transversal to the horizontal boundary of $\bar{\mathcal{Y}}$ at the image of the $v''$-vertices.

The first step consists in taking a so-called distinguished $\mathcal{D}$-triangulation for $\mathcal{B}$, that we are going to define. For every elementary object $[(F, (T, b), x, z), \phi]$ of $\alpha_-$ we take some idealizable extension $C^-(z, a)$ of $z$ to $C(T, b)$ (see Section 2.4). Then we extend $\phi$ to a parametrization $\Phi$ of a collar in $\bar{\mathcal{Y}}$ of the corresponding horizontal boundary component $\Sigma = \phi(F) \subset \partial_- \bar{\mathcal{Y}}$. For the elementary objects in $\alpha_+$ of the form $[(-F, (T, b), x, z), \phi]$, we do similarly, but we use $C^+(z, a)$, that is the idealizable extensions of $z$ obtained by exchanging the role of $F_-$ and $F_+$. Hence we work with 0-cochains supported at $\partial_- C(T, b)$ instead of $\partial_+ C(T, b)$, as we did to define $C^-(z, a)$. By doing this for all elementary objects, we get a collar $\mathcal{D}$-triangulation, that is a branched triangulation of a collar of the whole horizontal boundary of $\bar{\mathcal{Y}}$, equipped with a 1-cocycle $z_{coll}$ that represents the restriction of $\rho$ to this collar.

**Definition 4.1** A distinguished $\mathcal{D}$-triangulation for $\mathcal{B}$ consists of a 4-uplet $\mathcal{K} = (K, \bar{H}, b, z)$ where:

(a) $(K, b)$ is a branched triangulation of $\bar{\mathcal{Y}}$ that extends a collar $\mathcal{D}$-triangulation.

(b) $(K, b)$ induces on each tunnel boundary component of $\bar{\mathcal{Y}}$ a triangulation of the type specified in Section 3.2 for the cusp bases.

(c) The link $\bar{\Lambda} = \lambda \cup \lambda' \cup \lambda''$ is realized by the sub-complex $\bar{H} = H \cup H' \cup H''$ of $K$, and $\bar{H}$ is Hamiltonian, that is it contains all the vertices of $K$. Because of (a) and (b), this is equivalent to require that $H''$, which lies in the interior of $K$, contains all the internal vertices of $K$.

(d) $z$ is a $PSL(2, \mathbb{C})$-valued 1-cocycle on $(K, b)$ such that: $z$ extends the collar cocycle $z_{coll}$; $\rho = [z]$ on the whole $\bar{\mathcal{Y}}$; $z$ is idealizable; the restriction of $z$ to each tunnel component of $\partial \bar{\mathcal{Y}}$ is of the kind specified in Section 3.2 for the bases of $\mathcal{I}$-cusps.

The next step consists in passing from the above $\mathcal{D}$-triangulation $\mathcal{K} = (K, \bar{H}, b, z)$ to an $\mathcal{I}$-triangulation $\mathcal{K}_\mathcal{I} = (K_\mathcal{I}, \bar{H}, b, w)$ for $\mathcal{B}$. To do it, Consider every tunnel boundary component as the base of a topological cusp, oriented in such a way that, by glueing these cusps, we get the oriented 3-manifold $Y \setminus L$. Now take the idealization of $z$, and glue the corresponding $\mathcal{I}$-cusps (see Section 3.2). The resulting family of $\mathcal{I}$-tetrahedra defines an $\mathcal{I}$-triangulation $\mathcal{K}_\mathcal{I}$ for $\mathcal{B}$; at every internal edge it satisfies the edge compatibility condition 11.

We specify now which kind of global flattening and integral charge are carried by $\mathcal{K}_\mathcal{I} = (K_\mathcal{I}, \bar{H}, b, w)$. For every elementary object $[(\pm S, (T, b), x, z), \phi]$ in $\alpha_\pm$, and for every edge $e$ of $(T_\pm, b_\pm)$, we have the values $W_\pm(z)(e)$ of the $\mathcal{I}$-boundary function defined in Subsection 2.4. We can extend the definition of $W_\pm(z)$ to the edges in the tunnel boundary components of $K_\mathcal{I}$. By taking the union over the elementary objects we get the $\mathcal{I}$-boundary data $W_\pm$ of the objects $\alpha_\pm$.

Recall that we denote by $\log$ the standard branch of the logarithm with arguments in $]-\pi, \pi]$.

**Definition 4.2** A global flattening on $(K_\mathcal{I}, \bar{H}, b, w)$ is a collection $f$ of flattenings for the $\mathcal{I}$-tetrahedra of $(K_\mathcal{I}, \bar{H}, b, w)$ such that:
(1) At each internal edge $e$ of $K_I$, the associated log-branches formally satisfy the log of the edge compatibility condition \[ \sum_{h \in e^{-1}(e)} * l(h) = 0 = \log(1) \] where $\epsilon$ is the identification map that associates to every abstract edge its image in $K_I$ (via the face pairings), and $* = \pm 1$ according to the $b$-orientation of the tetrahedron that contains the abstract edge $h$.

(2) The condition extends at every boundary edge $e$ of $K_I$ by requiring that \[ \sum_{h \in \epsilon^{-1}(e)} * l(h) = \log(W_{\pm}(e)). \]

A global integral charge on $(K_I, \hat{H}, b, w)$ is a collection of integral charges over the tetrahedra of $K_I$ such that:

(a) The sum of charges about every internal edge of $K_I \setminus \hat{H}$ is equal to 2.

(b) The sum of charges at every boundary edge of $K_I$ is equal to 1.

(c) The sum of charges about every edge of $H''$ equals 0.

An $\mathcal{I}$-triangulation $(\mathcal{K}_I, f, c)$ equipped with a global flattening $f$ and a global integral charge $c$ is said (globally) flat/charged.

**The cohomological charge.** Let $(\mathcal{K}_I, f, c)$ be a flat/charged $\mathcal{I}$-triangulation for $B$ as above. The reductions mod(2) of both $f$ and $c$ induce cohomology classes $[f], [c] \in H^1(\bar{Y}; \mathbb{Z}/2\mathbb{Z})$. Moreover, we have integral classes $<f>, <c> \in H^1(\partial Y; \mathbb{Z})$ on $\partial Y$. These classes are defined as follows. For any mod(2) (resp. integral) 1-homology class $a$ in $Y \setminus L$ (resp. $b$ in $\partial Y$), realize it by a disjoint union of (resp. oriented and essential) closed paths transverse to the triangulation $\mathcal{K}_I$ and ‘without back-tracking’, i.e. such that it never departs from a 2-face of a tetrahedron (resp. 1-face of a triangle) by which it just entered. Then the mod(2) sum of the flattenings or charges associated to the angular sectors that we cross when following such paths in the interior of $\bar{Y}$ define the value of the corresponding class on $a$.

Similarly, the signed sum of the log-branches or charges of the corners that we cross when following such paths on $\partial Y$ define the value of the corresponding class on $b$; for each vertex $v$ the sign is $\epsilon_b$ (resp. $-\epsilon_b$) if, with respect to $v$, the path goes in the direction (resp. opposite to the one) given by the orientation of $\partial Y$.

The set $\{[f], [c], <f>, <c>\}$ is called the cohomological charge of $(\mathcal{K}_I, f, c)$. It is a fact (see [31]) that this cohomological charge is preserved by the transit configurations mentioned in Subsection [4.4]. Let us denote by $\partial_{tunnel}Y$ the tunnel boundary components of $\bar{Y}$. It is convenient to normalize the cohomological charge by requiring that:

(i) $<f> = <c> = k$, and $k$ is identically 0 except possibly on $\partial_{tunnel}Y$;

(ii) $[f] = [c] = h$, and $i^*(h) = k \mod(2)$, where $i : \partial_{tunnel}Y \rightarrow Y$ is the inclusion map.

With this normalization the whole boundary configuration of a flat/charged triple $(\mathcal{K}_I, f, c)$ is completely determined by the objects $a_{\pm}$, and the cohomological charge reduces to $(h, k)$. From now on we consider only triples $(\mathcal{K}_I, f, c)$ with such normalized cohomological charge. The results below can be extended to the case when the pairs $([f], <f>)$ and $([c], <c>)$ are independent, or not equal to 0 on $\partial Y \setminus \partial_{tunnel}Y$; the only difference lies in the constraints on the cohomological charge when gluing QHFT bordisms (see Remark [17]).
4.3 The QHFT bordism tensors $\mathcal{H}_N(K_I, f, c)$

Fix an odd integer $N \geq 1$. Let $\mathcal{B}, \alpha_\pm$ and $(Y, \tilde{L}_F, \rho)$ be as usual, and let $(K_I, f, c)$ be a flat/charged $\mathcal{I}$-triangulation with normalized cohomological charge for $\mathcal{B}$.

First we specify the linear spaces $E_N(\alpha_\pm)$. Recall that it is defined a sign function $\sigma_\pm$ on the set of triangles of $\alpha_\pm$. Let us fix an ordering of the elementary objects, together with an ordering of the triangles of each elementary object. Thus we get a lexicographical order on the whole set of triangles of $\alpha_\pm$. Fix a complex linear space $E = E_N$ of dimension $N$, endowed with a given basis, so that it is identified with $\mathbb{C}^N$. Write $E = E^1$ and $E^{-1} = E'$ for its dual space. Also $E'$ is canonically identified with $\mathbb{C}^N$. Then set $E_N(\alpha_\pm)$ to be the tensor product of the $|J_\pm|$ ordered spaces $E^{\sigma_\pm(t)}$, where $t$ spans the set of triangles of $\alpha_\pm$ and $|J_\pm|$ is the number of these triangles. The space $E_N(\alpha_\pm)$ is identified with the tensor product of $|J_\pm|$ copies of $\mathbb{C}^N$.

We consider every matrix dilogarithm $R_N \in \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$ as an element of $(E_N^r)^{\otimes 2} \otimes (E_N^r)^{\otimes 2}$. The trace tensor (5) for the pattern of matrix dilogarithms associated to $(K_I, f, c)$ gives us a morphism

$$\mathcal{H}_N(K_I, f, c) \in \text{Hom}(E_N(\alpha_-), E_N(\alpha_+))$$

We can state now the main technical result in the construction of QHFT$_N$.

**Theorem 4.3** Let $\mathcal{B}$ be a QHFT bordism between objects $\alpha_\pm$. Then:

1. Flat/charged $\mathcal{I}$-triangulations $(K_I, f, c)$ for $\mathcal{B}$, with any prescribed normalized cohomological charge $(h, k)$, do exist.

2. Let $(K_I, f, c)$ be such a triangulation with cohomological charge $(h, k)$. Let $v_0$ and $v_1$ be respectively the number of boundary and internal vertices of the triangulation. Then, for every odd integer $N \geq 1$, up to a sign and a $N$th-root of unity multiplicative factor, the normalized linear map

$$N^{-(v_0/2+v_1)}\mathcal{H}_N(K_I, f, c) \in \text{Hom}(E_N(\alpha_-), E_N(\alpha_+))$$

only depends on the triple $(\alpha_\pm, h, k)$, so that the bordism tensor

$$\mathcal{H}_N(B, h, k) := N^{-(v_0/2+v_1)}\mathcal{H}_N(K_I, f, c)$$

is well defined (up to the above phase ambiguity).

**Proof.** The proof of this theorem is technically demanding, but it follows strictly the arguments of [4, §4]. As there are only slight differences, we limit ourselves to few comments.

Point (1). The existence of $\mathcal{D}$-triangulations $\mathcal{K}$ is an essentially straightforward adaptation. We know that integral charges and flattenings, with arbitrary cohomological charge, exist on any closed 3-dim. $(\mathcal{D})$-triangulation. Moreover, they make an affine space over a same integral lattice $\mathcal{L}$, generated by determined vectors attached to the abstract stars of all the edges (see [4, §4] and [5, §6]). The lattice is fixed by the cohomological charge. Consider the double $\partial K_I := \partial K_I \cup \partial (-K_I)$. By symmetry of the triangulation about $\partial K_I \subset \partial K_I$, the above facts imply that we can find a flat/charge on $\partial K_I$ that induces one on $K_I$; in particular (2) and (c) in Definition 4.2 are satisfied. An easy Mayer-Vietoris argument show that we recover the cohomological charge $(h, k)$ by properly choosing it on $\partial K_I$ (this agrees with the normalization $< f > = < k >$). Finally, Definition 4.2 (2)-(c) just kill the generating vectors of $\mathcal{L}$ at the edges of $\partial K_I$. So the sets of flattenings and integral charges on $K_I$ are affine spaces over a lattice $\mathcal{L}'$ generated by the interior edges of $K_I$ only.

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For (2), the key point consists in showing that any two arbitrary flat/charged $I$-triangulations for $B$ are connected via a finite sequence of transit configurations (see Subsection 4.1) in the interior of $K_I$, so that after each transit we still have a flat/charged $I$-triangulation of $B$. Such sequences preserve both the whole boundary configuration and the normalized cohomological charge, and they allow also to vary the flattenings and integral charges by any multiple of the generators of the corresponding lattice $L'$.

Finally, the fundamental identities satisfied by the matrix dilogarithms (the five term ones as well as the ones supported by other triangulation local moves) imply that the trace tensor is transit invariant, up to the specified phase ambiguity. In particular, the normalization factor $N^{-v_I}$ is needed for the bubble transit invariance (the factor $N^{-v_{I\partial}/2}$ is introduced to have a good behaviour with respect to bordism composition, see Prop. 4.6 below).

A new ingredient w.r.t. to [4, 5] is the presence of $I$-cusps. The transits preserve the germ at infinity of each cusp, that is the topological singularity corresponding to the collapsed tunnel boundary components of $\tilde{Y}$, and allow also to modify the base triangulation. So the bordism tensors do not depend on any fixed representative of $\rho$ at the link meridians.

Recall the $I$-boundary data $W_\pm$ before Definition 4.2. We have:

**Corollary 4.4** The QHFT morphisms supported by a bordism between objects $\alpha_\pm$ define complex analytic functions (up to the phase ambiguity mentioned in Th. 4.3) on open dense subsets of the space of $I\partial$-parameters $W_- \times W_+$ for $\alpha_+ \cup \alpha_-$. 

**Proof.** Given a QHFT bordism $B$, fix an arbitrary branched triangulation $(K, b)$ as in Definition 4.1. The space of log-branches supported by $(K, b)$ is linear, and the matrix dilogarithms (4) depend holomorphically on the log-branches (up to sign when $N = 1$). By definition, the QHFT tensors $H_N(B)$ are specific (normalized) contractions of matrix dilogarithms, and Theorem 4.3 (2) implies that they are eventually functions of the right-hand side members of the identities (7). As these are logarithms of the $I\partial$-parameters, we deduce that the QHFT tensors are indeed analytic functions on a dense subset of $W_- \times W_+$. □

To treat the morphism composition, we incorporate in the gluing set up the fixed ordering on the set of triangles; moreover we include in the definition of $Z_\alpha(T, b, x, y_2)$ (see Section 2.3-2.4) the **genericity assumption** that for no boundary edge $e$ of $(K, b)$ the $I\partial$-parameter $W_\pm(e)$ belongs to the negative real ray. This is a mild assumption, and it is completely uninfluential if the surface triangulations are quasi-regular. The advantage of this assumption is the following fact, which follows from the proof of Lemma 2.9:

If $B'' = B' \ast B$ is a bordism obtained as in Section 3 and $(K, f, c)$, $(K', f', c')$ are flat/charged $I$-triangulations with normalized cohomological charges $(h, k)$ and $(h', k')$, they automatically glue to give a flat/charged $I$-triangulation $(K'', f'', c'')$ for $B''$, for some $(h'', k'') := (h' \ast h, k' \ast k)$.

**Remark 4.5** Even if $h = h' = 0$ it is possible that $h'' \neq 0$. In general $h''$ depends on the whole pair $(B, B')$. This follows from an easy application of a Mayer-Vietoris argument. But if the glued part of the boundary is connected, or is a boundary in $B \ast B'$, then $h = h' = 0$ implies $h'' = 0$.

The following functoriality result is a consequence of Theorem 4.3.
Proposition 4.6 With the above hypothesis we have:
\[
\mathcal{H}_N(\mathcal{B}^0, h'', k'') = \mathcal{H}_N(\mathcal{B}'', h', k') \circ \mathcal{H}_N(\mathcal{B}, h, k)
\]
where “≡_N” denotes the phase ambiguity mentioned in Theorem 4.3 (2), and \( \circ \) is the morphism composition.

There is also a Hermitian property of QHFT morphisms (proved as for closed manifolds, see Prop. 4.29 of [5]):

Proposition 4.7 Write \( \mathcal{B} \) for the QHFT bordism with opposite total space orientation for \( \rho \) (i.e. opposite orientation for \( Y \) and complex conjugate holonomy \( \bar{\rho} \)). Then:

\[
\mathcal{H}_N(\mathcal{B}, h, -k) = \mathcal{H}_N(\mathcal{B}, h, k)^*\text{T}
\]

where \( ^*\text{T} \) denotes the matrix hermitian conjugation.

4.4 QHFT partition functions

Here we discuss numerical invariants, called QHFT partition functions, for closed manifolds, i.e. manifolds with empty boundary.

Assume that \( Y = W \) is closed, and that it contains a non empty link \( L_F \). Then \((W, L_F)\) supports a QHFT bordism from the empty object to itself. The associated tensor \( \mathcal{H}_N(W, L_F, \rho, h, k) \) is a scalar (well defined up to “≡_N”). We can also set the cohomological charge to be \((0, 0)\). So we get an invariant \( \mathcal{H}_N(W, L_F, \rho) \) for the triple \((W, L_F, \rho)\). Typical examples of such triples are given by cone hyperbolic manifolds \( W \) with cone locus an unframed link \( L \). The character \( \rho \) is just the hyperbolic holonomy of \( W \setminus L \), and we take an arbitrary framing for \( L \).

Here is another way to express these partition functions, in terms of manifolds with toric boundary. In that situation, fix an ordered basis \((m_i, l_i)\) for the integral homology of each boundary torus. Then we can construct the pair \((W, L_F)\), where \( W \) is obtained by Dehn filling of \( Y \), along the \( m_i \)'s, and \( L_F \) is the disjoint union of the cores of the surgered solid tori, framed by the \( l_i \)'s. In this way, the invariants \( \mathcal{H}_N(W, L_F, \rho) \) are viewed as invariants of \((Y, \{(m_i, l_i)\})\).

Let us introduce a QHFT bordism category that covers a more restricted range of topological/geometric situations. We propose this variation because the corresponding partition functions exactly equal the invariants for triples \((W, L, \rho)\) already constructed in [3, 5].

Consider the naked bordism category with unframed links introduced at the beginning of Section 2. We assume also that \((Y, L)\) is equipped with a \( \text{PSL}(2, \mathbb{C}) \)-character \( \rho \) which is defined on \( \pi_1(Y) \); hence \( \rho \) is trivial at the link meridians. To define the elementary objects, we use triangulations \((T', b')\) of \((S, V)\) of the type discussed in Section 2. Moreover we consider cocycles \( z \) which have idealizable extensions \( \mathcal{C}^- (z, a) \) to the (triangulated) cylinder \( S \times [-1, 1] \). This give us a notion of “QHFT" bordism category. Note that the \( \mathcal{L} \)-cusps are no longer present. We consider \( \mathcal{D} \)-triangulations for which the link \( L \) is realized as a Hamiltonian subcomplex, and we have only the mod \((2)\) cohomological charge \( h \). The analog of Theorem 4.4 associates tensors \( H_N(\mathcal{B}, h) \) (still defined mod \("\equiv_N\")) to every QHFT bordism \( \mathcal{B} \).

For triples \((W, L, \rho)\) with \( W \) closed, we obtain partition functions \( H_N(W, L, \rho) \).

So, we dispose, for every \( N \), of the two partitions functions \( H_N(W, L_F, \rho) \) and \( H_N(W, L, \rho) \). The next result shows that they are very similar indeed.

Proposition 4.8 We have \( H_N(W, L_F, \rho) =_N H_N(W, \Lambda, \rho) \), where, as in Section 3, the link \( \Lambda \) is made by 3 specific parallel copies of \( L \) given by the framing \( L_F \).
Proof. We consider supports that are very close each to the other. Recall the zipping procedure from the subsections \[22\] and \[51\]. Since \(\rho\) has trivial holonomy at the meridians of \(L_F\), we can zip back any \(\mathcal{D}\)-triangulation of \((W, L_F, \rho)\), in particular the cocycle, to get a \(\mathcal{D}\)-triangulation used to compute \(H_{N}(W, \Lambda, \rho)\) (passing to its idealization). After the zipping there are no longer cusps, and the cross-ratio moduli of the survived tetrahedra are unchanged.

So, to conclude we must check that the cusp contributions to \(\mathcal{H}_{N}(W, L_F, \rho)\) are trivial. By looking at Figure \[6\] we see that each cusp tensor is supported by two patterns of flat/charged \(I\)-tetrahedra glued in a mirror-like fashion w.r.t. the central vertical line. Mirror \(I\)-tetrahedra have the same cross-ratio moduli. So we may choose exactly the same flattenings (resp. integral charges) for them; one example is where each right-angle in the figure is given the flattening \(\pm 1\) (according to the sign of minus the imaginary part of the corresponding cross-ratio modulus), whereas the acute angles are given the value 0. For the charges we put correspondingly the values 1 or 0. It is easily seen that such a choice makes a flat/charged \(I\)-tetrahedra, where the cohomological charge is identically 0. The branching induces opposite orientations of the mirror patterns of flat/charged \(I\)-tetrahedra. So the respective automorphisms are inverse one to the other, and each cusp contribution to \(\mathcal{H}_{N}(W, L_F, \rho)\) is an identity tensor. \(\square\)

Remark 4.9 It is known \[11, 5\] that in the “classical” case \(N = 1\), the link is immaterial in the construction of \(H_{N}(W, L, \rho)\). But in the “quantum cases” \(N > 1\), the presence of a non empty link is essential for the theory, and its choice alters the value of \(H_{N}(W, L, \rho)\). Also, when \(N = 1\), it is known that \(H_{1}(W, \rho)\) equals \(\exp(1/8\pi \text{CS}(\rho) + i\text{Vol}(\rho))\) (see \[5\], Section 6).

4.5 QHFT supported by product bordisms

In this section we consider the product bordisms \(\tilde{Y} = F \times [-1, 1]\), containing the tunnel boundary \(\partial F \times [-1, 1]\), and equipped with a conjugacy class \(\rho\) of \(\text{PSL}(2, \mathbb{C})\)-valued representations of \(\pi_{1}(F) = \pi_{1}(\tilde{Y})\). This corresponds to triples \((Y, L_F, \rho)\), where \(Y = S \times [-1, 1]\) and \(L_F\) is associated to \(L = V \times [-1, 1]\).

We are interested in the families of QHFT morphisms supported by \(\tilde{Y}\), when the marking of the boundary components \(F \times \{-1\}\) and \(F \times \{1\}\) vary. In fact, we can consider separately the variations that concern the parametrizing homeomorphisms, the \(e\)-triangulations, and the cocycles, respectively. For each one, we can also restrict to elementary generating variations such as Dehn twists, flips, and gauge transformations induced by 0-cochains supported at one vertex, respectively.

Consider in particular objects of the form \([([\pm F, (T, b), x, z, \psi])\), where we stipulate that \((\pm F, (T, b), x, z)\) is fixed and only the homeomorphism \(\psi\) varies. Given \(\psi_{-}\) and \(\psi_{+}\), set \(\psi = \psi_{+}^{-1}\psi_{-}\) and \([\psi]\) for the corresponding element in the mapping class group \(\text{Mod}(g, r)\).

Denote by \((T_{[\psi]}, L_{[\psi]}\) the mapping torus of \(\psi\), which only depends on \([\psi]\). Here \(L_{[\psi]}\) is the tunnel boundary of \(T_{[\psi]}\); its components are tori. Denote again \(\rho\) the pull-back on \(T_{[\psi]}\) of the given conjugacy class \(\rho\), via the natural projection onto \(F\). For the cylinders and the mapping tori, the cohomological charge shall be trivial so we omit to mention it. Fix an odd integer \(N \geq 1\). We denote by \(q_{N}(\rho, [\psi])\) the QHFT partition function for the triple \((T_{[\psi]}, L_{[\psi]}, \rho)\). The cylinder \(\tilde{Y}\) only depends on the pair \((g, r)\), and the objects only depend on \([\phi_{\pm}]\) (as we are assuming that \((\pm F, (T, b), x, z)\) is fixed). So we write \(\mathcal{H}_{N}(r, g, [\psi_{-}], [\psi_{+}])\) for \(\mathcal{H}_{N}(Y, L_F, \rho)\). Set \(d_{g, r}(N) = \dim (E_{N}(\alpha_{\pm})) = N^{2/2g-2+3r}\).
Lemma 4.10  (1) The QHFT tensor $\mathcal{H}_N(g, r, [\psi_-], [\psi_+])$ only depends on $[\psi]$. So we denote it by $\mathcal{H}_N(g, r, [\psi])$.

(2) $\mathcal{H}_N(g, r, [id]) = N \text{Id}$, and $\text{Trace } \mathcal{H}_N(g, r, [id]) = N d_{g, r}(N) = N q_N(\rho, [id])$.

(3) $\mathcal{H}_N(g, r, [h_1]) \circ \mathcal{H}_N(g, r, [h_2]) = \mathcal{H}_N(g, r, [h_2 h_1])$. Hence, in particular: $\mathcal{H}_N(g, r, [\psi^{-1}]) \circ \mathcal{H}_N(g, r, [\psi]) = N \text{Id}$.

(4) $\text{Trace } \mathcal{H}_N(g, r, [\psi]) = N q_N(\rho, [\psi])$.

Proof. By Theorem 4.3 (2), the tensor $\mathcal{H}_N(g, r, [\psi_-], [\psi_+])$ is equal (up to the “$\equiv_N$” ambiguity) to the one supported by the mapping cylinder of $[\psi]$, where $F_-$ is parametrized by the identity and $F_+$ is parametrized by $\psi$. Indeed, the homeomorphism $\psi^{-1} \times \text{Id}$ relates the two supporting cylinders. Point (1) follows. Also, this gives

$$\mathcal{H}_N^2(g, r, [id]) = N \mathcal{H}_N(g, r, [id])$$

i.e. $\mathcal{H}_N(g, r, [id])$ is an idempotent. By using a triangulation of the trivial mapping cylinder of the form $C^+(z, a) \ast C^-(z, a)$ and the fact that the $R_N$’s are automorphisms, we see from (b) that $\mathcal{H}_N(g, r, [id])$ read as a tensor product of invertible matrices. Hence it is invertible. The claims (3)-(4) are consequences of Prop. 4.6. \hfill $\Box$

If, with the usual notations, we vary now only the cocycles $z_\pm$ in specifying the different objects, we still get invertible tensors $\mathcal{H}_N(g, r, z_\pm)$ by the same arguments. If we change the triangulation $(T_-, b_-)$, for instance, by a branched elementary flip, getting $(T_+, b_+)$, there is a unique way to define $z_+$ in such a way that it coincides with a given $z_-$ on the common edges. Generically also $z_+$ belong to $Z_f(T_+, b_+, x, y_\beta)$, and again we get an invertible representing tensor $\mathcal{H}_N(T_+, b_+, z_\pm)$. The effect of such changes of the marking (for a given $\rho$) on a bordism tensor $\mathcal{H}_N(g, r, [\psi])$ is a conjugation by $\mathcal{H}_N(g, r, z_\pm)$ or $\mathcal{H}_N(T_+, b_+, z_\pm)$. So we have:

Corollary 4.11 For every $N$, $\rho$, and $(T, b, z)$ as above, the tensors $\mathcal{H}_N(g, r, [\psi])$ realize a $d_{g, r}(N)$-dimensional linear representation mod ($\equiv_N$) of the mapping class group $\text{Mod}(g, r)$. Hence, by varying $(T, b, z)$, there is a well-defined conjugacy class of representations mod ($\equiv_N$) of $\text{Mod}(g, r)$ associated to QHFT$_N$.

4.6 Universal QHFT environment

Here we indicate the most general environment where the matrix dilogarithm technology applies, and QHFT, formally at least, makes sense.

We can associate trace tensors to any roughly flat/charged $\mathcal{I}$-triangulated compact oriented pseudo-manifold $Z$. Such a triangulation is given, as usual, by a family of oriented abstract flat/charged $\mathcal{I}$-tetrahedra equipped with certain orientation reversing identifications of some 2-faces, in such a way that $Z$ is the quotient space. “Pseudo-manifold” means that the non-manifold points of $Z$ are contained in the set of vertices of the triangulation. “Roughly” means that we require neither that the edge compatibility condition is satisfied at the internal edges, nor that the collection of local flattenings and integral charges satisfy any global constraint. So we are in a situation much more general than that of Section 4.3.

We consider these triangulated and decorated pseudo-manifolds up to the equivalence relation generated by the transit configurations mentioned before Fig. 5 (i.e. the flat/charged $\mathcal{I}$-transits described in detail in [4, 5]).

The notion of boundary is well defined for this class of triangulated pseudo-manifolds, so we can construct a consistent bordism category. In fact, we can also relax the requirement that the input/output bipartition of the boundary of the
bordisms is made by the disjoint union of closed boundary components. We can consider adequate triangulated portions of the boundaries, such that the category is stable under bordism composition (see for instance [26] for such a general notion of bordism category).

In this way, a single flat/charged \( I \)-tetrahedron can be considered as a bordism between the two boundary quadrilaterals, that are triangulated by the couple of 2-faces having \( b \)-sign equal to \( \pm \), respectively. The associated matrix dilogarithms are, by definition, the \( \text{QHFT} \) tensors representing this bordism. They represent the elementary flip on the quadrilateral triangulations with two triangles. Any roughly flat/charged \( I \)-triangulated oriented pseudo-manifold \( Z \) can be considered as the result of a composition of several such tetrahedral elementary bordisms. The fundamental transit invariance of the matrix dilogarithms ensures tautologically that we have a well defined so called \( \text{universal} \) \( \text{QHFT} \) for this very general bordism category.

The \( \text{QHFT} \)s constructed in the previous sections, which lead, in particular, to the partition functions \( H_N(W, L_F, \rho) \) or \( H_N(W, L, \rho) \) of Subsection 4.4 naturally map into the universal \( \text{QHFT} \). Indeed, they are obtained via a specialization of its setup (recall the further constraints of Def. 4.2 for flattenings and integral charges). The corresponding refined flat/charged \( I \)-equivalence classes have a clear, intrinsic topological/geometric meaning, described in Section 7 of [5].

Another remarkable specialization of the universal \( \text{QHFT} \) environment is given by the dilogarithmic invariants \( H_N(M) \) of oriented non compact complete hyperbolic 3-manifolds \( M \) of finite volume (for short: \( \text{cusped manifolds} \)), defined in [5]. These invariants are constructed by using geodesic ideal triangulations with non negative volume tetrahedra, which can be considered as triangulations of the pseudo-manifold \( \tilde{M} \) obtained by taking the one point compactification at each cuspidal end of \( M \), where the vertices are just the non-manifold points. Also in that situation there is a natural notion of flattening and integral charge; the essential difference is that for the integral charges we require only the global constraint (a) in Definition 4.2 (there is no link and no boundary). By using a volume rigidity result for cusped manifolds, we proved in [5], Th. 6.8, that the (normalized) flat/charged \( I \)-triangulations of a given cusped manifold \( M \) all belong to the same flat/charged \( I \)-equivalence class in the universal \( \text{QHFT} \) environment. Again, these classes have a clear, intrinsic topological/geometric meaning.

This is not evident for the equivalence class of an arbitrary roughly flat/charged \( I \)-triangulated pseudo-manifolds. However, the above examples suggest the possibility that the universal \( \text{QHFT} \) supports other \( \text{geometrically meaningful} \) specializations.

For instance, by using a suitable restricted (but non trivial) subset of flat/charged \( I \)-transits, we can formally construct a \( \text{QHFT free from the "=N" phase ambiguity} \). Clearly, the problem of understanding the geometric meaning (if any) of this specialization of the universal \( \text{QHFT} \) remains open. It is clearly related to the ultimate understanding of the nature of the phase ambiguity itself.

Appendix

A.1 relationship between the QHFT marking of surfaces and the Penner-Kashaev’s coordinates

As before, let \( S \) be a closed compact surface of genus \( g \) with a set \( V \) of \( r \) marked points, and write \( \hat{S} = S \setminus V \). Put a finite area complete hyperbolic metric on \( \hat{S} \). Denote by \( T^r_g \) the marked Teichmüller space of \( \hat{S} \). Recall that \( T^r_g \) is homeomorphic
to $\mathbb{R}^{6g-6+2r}$, and to the subspace of $\text{Hom}(\pi_1(\bar{S}), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$ made of \textit{admissible} isomorphisms (i.e. mapping parabolic elements to parabolics and induced by orientation preserving homotopy equivalences), up to conjugation; this subspace lies in the manifold part of $\text{Hom}(\pi_1(\bar{S}), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$.

In \cite{21}, R.C. Penner constructed for any ideal triangulation $T'$ of $\bar{S}$ an $\mathbb{R}_+^r$-principal bundle $\bar{T}_g^r \to \bar{T}_g^r$ called the “decorated” Teichmüller space. The total space $\bar{T}_g^r$ is parametrized by $-3\chi(\bar{S})$ coordinates associated to the edges of $T'$, called the $\lambda$-\textit{lengths}. By fixing horocycles about the punctures of $\bar{S}$ (more precisely, at the lifts to the universal cover $\mathbb{D}^2$), the $\lambda$-lengths read as $\sqrt{2}\exp(\delta)$, where $\delta$ is the algebraic distance between the horocycles at the endpoints of the edges, counted positively if they do not intersect. The radii of the horocycles form the semi-group $\mathbb{R}_+$ and such that the values belong to the parabolic subgroup $\text{PSL}(2, \mathbb{R})$ such that the values belong to the parabolic subgroup $P$ for all short edges, and $3$ short edges contained in the boundary link, $\text{link}(v)$ of $\text{star}(v)$.

Fix a parabolic subgroup $P$ of $\text{PSL}(2, \mathbb{R})$, say upper-triangular, with normalizer $B = \text{N}(P)$, the Borel subgroup of upper-triangular matrices. We have the Bruhat decomposition $\text{PSL}(2, \mathbb{R}) = B \bigsqcup B\theta B$, where $\theta$ is the $(2 \times 2)$-matrix with $1$ on the bottom left, $-1$ on the top right, and $0$ elsewhere. Then, suitable gauge transformations at the vertices of $C'$ allow us to construct $1$-cocycles on $C'$ representing $\rho$ and such that the values belong to the parabolic subgroup $P$ for all short edges, and to the subset $\theta H$ for all long edges, where $H = B/P \cong \mathbb{R}_+$. This space of cocycles defines the fiber over $\rho$ in Kashaev’s bundle $\tilde{\mathcal{M}}$, and the residual gauge transformations turn out to be isomorphic to a semi-group of $\tilde{P}$ isomorphic to $H^r \cong \mathbb{R}_+^r$.

In fact, for each hexagon in $C'$ there exists a sign $\varepsilon = \pm 1$ such that the cocycle values on the short edges are uniquely defined in terms of $\varepsilon$ and the cocycle values on the long edges. So Kashaev’s coordinates are eventually given by:

- a sign $\varepsilon(t) \in \{-1, 1\}$ for each triangle $t$ of $T'$ (i.e. each hexagon of $C'$);
- an $\mathbb{R}_+$-valued function defined on the edges of $T'$ (i.e. the long edges of $C'$);

Kashaev described a partition of the fibration $\tilde{\mathcal{M}} \to \mathcal{M}$ in terms of the above sign function $\varepsilon$. The component where all the $\varepsilon$-values are positive or negative is isomorphic to Penner’s decorated Teichmüller space $\bar{T}_g^r \to \bar{T}_g^r$.

**Embedding $\tilde{\mathcal{M}}$ into $Z(T, b, 1, y_1)$.** By the two-dimensional version of \cite[Th. 3.4.9]{24}, we can turn $T'$ into an ideal triangulation of $F$ supporting a branching, via a finite sequence of elementary flips. Fix a total ordering on the set $V$ of marked points. By using the orientation of the short edges of $C'$, an injective map $v \mapsto t_v$ as in Lemma \ref{24} selects one vertex $x_v$ of $\text{link}(v)$. We stipulate that this is the second endpoint of the short edge of $C'$ contained in $t_v$, and we use $x_v$ as base point on $\text{link}(v)$.

Starting from $x_v$, order the vertices of $\text{link}(v)$ in accordance with its orientation. Then, in the reverse order, ‘slide’ the vertices of $\text{link}(v)$ one by one to $x_v$, except the
biggest vertex $x''_v$, next before $x_v$. This removes all the vertices on link($v$) but two. After each sliding, the value of the cocycle on the new edge with one endpoint at $x_v$ is forced. (The ordering on $V$ is needed because each long edge shall eventually slide along two link($v$)'s). Do this procedure for all the vertices of $T'$. We end up with a cellulation of $F'$ where each initial triangle $t_v \subset T'$ appears as the union of one triangle, one quadrilateral (with one short edge exactly), and one bigon. We get also a well-determined, automatically defined, $PSL(2,\mathbb{C})$-valued 1-cocycle on this cellulation.

Remove a small monogon inside each bigon and put a vertex $x'_v$ on its boundary, at the ‘midpoint’ of the segment $[x_v, x''_v]$. Triangulate the resulting cellulation without adding new vertices. Clearly, we can extend the branching of $T$ during the above procedure. So we eventually find an $e$-triangulation $(T, b)$ of $F$. At each step the cocycle values are forced, except for the loop boundary edges, but also there the holonomy about the loop is forced.

The constructions of Section 2.3 imply that this cocycle can be turned into one of the space $Z(T, b, 1, y_1)_\beta$ via a suitable minimal sequence of gauge transformations (recall that 1 denotes the type of the representations with only parabolic holonomies about the punctures). Here we use the $PSL(2,\mathbb{R})$-version of the theory, mentioned in Remark 2.8. Also, the residual gauge transformations in Kashaev’s bundle $\mathcal{M}$ transit to the semi-group of $\mathcal{G}(T, b, 1) \cong \text{Par}(2,\mathbb{C})^r$ specified by the signs $c(g) = \pm 1$, for all the boundary loop holonomies $g$. So we eventually get an embedding $\mathcal{M} \to Z(T, b, 1, y_1)_\beta$.

A.2 Formulas for the matrix dilogarithms

Here we give the explicit formulas for the automorphisms $R_N(\Delta, b, w, f, c)$ associated in Section 4.1 to flat/charged $I$-tetrahedra, $N \geq 1$ being any odd positive integer.

For $N = 1$, we forget the integral charge $c$, so that $R_1$ is defined simply on flattened $I$-tetrahedra. Namely, we have

$$
R_1(\Delta, b, w, f) = \exp \left( \frac{\log(b, t, f)}{i \pi} \right)
$$

where $\log(b, t, f)$ is a log-branch as defined in (8).

For $N = 2m + 1 > 1$ and every complex number $x$ set $x^{1/N} = \exp(\log(x)/N)$, where log is the standard branch of the logarithm with arguments in $]-\pi, \pi]$ (by convention we put $0^{1/N} = 0$). Denote by $g$ the complex valued function, analytic over the complex plane with cuts from the points $x = \zeta^k$ to infinity ($k = 1, \ldots, N - 1$), defined by

$$
g(x) := \prod_{j=1}^{N-1} (1 - x\zeta^{-j})^{1/N}
$$

and set $h(x) := g(x)/g(1)$ (we have $|g(1)| = N^{1/2}$). For any $u \in \mathbb{C} \setminus \{0, 1\}$ and $n \in \mathbb{N}$, put

$$
\omega(u', v'|n) = \prod_{j=1}^{n} \frac{v'}{1 - u'\zeta^j}
$$

where $v = 1 - u$, and $u'$, $v'$ are arbitrary $N$th roots of $u$ and $v$. These functions $\omega$ are periodic in their integer argument, with period $N$. Finally, write $[x] = N^{-1}(1 - x^N)/(1 - x)$. Given a flat/charged $I$-tetrahedron $(\Delta, b, w, f, c)$, set

$$
w'_j = \exp((1/N)(\log(w_j) + (f_j - s_b c_j)(N + 1)\pi i)).
$$
We define

\[ R_N(\Delta, b, w, f, c) = \frac{(w'_0)^{-c_1} (w'_1)^{c_0}}{(w'_1)^{1}} (L_N)^* b (w'_0, (w'_1)^{-1}) \]  

(9)

where

\[
L_N(u', v')_{k,l}^{i,j} = h(u') \zeta^{k+j} \omega(u', v' | i-k) \delta(i+j-l) \]

\[
(L_N(u', v')^{-1})_{k,l}^{i,j} = \frac{h(u')}{h(w')} \zeta^{-i-l} \omega(w'/\zeta, v' | k-i) \]

and \( \delta \) the \( N \)-periodic Kronecker symbol, i.e. \( \delta(n) = 1 \) if \( n \equiv 0 \mod(N) \), and \( \delta(n) = 0 \) otherwise.

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