THRESHOLD DYNAMICS OF A DELAYED MULTI-GROUP HEROIN EPIDEMIC MODEL IN HETEROGENEOUS POPULATIONS

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(Communicated by Yuan Lou)

Abstract. The aim of this paper is to investigate the threshold dynamics of a heroin epidemic in heterogeneous populations. The model is described by a delayed multi-group model, which allows us to model interactions both within-group and inter-group separately. Here we are able to prove the existence of heroin-spread equilibrium and the uniform persistence of the model. The proofs of main results come from suitable applications of graph-theoretic approach to the method of Lyapunov functionals and Krichhoff’s matrix tree theorem. Numerical simulations are performed to support the results of the model for the case where \( n = 2 \).

1. Introduction. Over the past two decades, illicit drug abuse have received much attentions, which brings tremendous pressures and damages to social and public health system due to its prevalence all over the world. Epidemics modeling, using theory of systems of differential equations, has been widely used to describe heroin addiction and spread in epidemic way, which is termed as an important tool to understand the mechanism of heroin spread. We refer to [13, 28, 15, 18, 12, 8] and the references cited therein. Some qualitative properties of models, such as the

2010 Mathematics Subject Classification. Primary: 34D23; Secondary: 92B30.

Key words and phrases. Multi-group heroin epidemic model, delay, heterogeneous populations, graph-theoretic approach, global stability.

X. Liu is supported by National Natural Science Foundation of China (No. 11271303). J. Wang is supported by National Natural Science Foundation of China (Nos. 11401182 and 11471089), Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province (No. 2014TD005), project funded by China Post-doctoral Science Foundation (No. 2014M552295) and project funded by Chongqing Postdoctoral Foundation (No. Xm2014024).
existence, uniqueness, and the stability of equilibria, provide the theoretical basis for the specialist teams to design suitable and reasonable strategies for the control of heroin.

Most results on heroin epidemic model focus on the existence and long time behavior of solutions of single group. There are few results on the heroin epidemic in heterogeneous populations. The purpose of this paper is to establish the qualitative properties of a delayed multi-group heroin epidemic model. To this end, we would like to mention a recent work of Fang et al. in [4]. Denote by \( S(t), U_1(t) \) and \( U_2(t) \) the numbers of susceptible individuals, heroin users not in treatment, and heroin users undergoing treatment, at time \( t \), respectively. The delayed heroin epidemic model formulated in [4] takes the following form

\[
\begin{aligned}
S'(t) &= \Lambda - \mu S(t) - \beta S(t) \int_0^{\tau_1} \hat{f}(\theta) U_1(t - \theta) e^{-(\mu + \delta_1 + p)\theta} d\theta, \\
U'_1(t) &= \beta S(t) \int_0^{\tau_1} \hat{f}(\theta) U_1(t - \theta) e^{-(\mu + \delta_1 + p)\theta} d\theta - (\mu + \delta_1 + p) U_1(t) \\
&\quad + p \int_0^{\tau_2} \hat{g}(\theta) U_1(t - \theta) e^{-(\mu + \delta_2)\theta} d\theta, \\
U'_2(t) &= p U_1(t) - p \int_0^{\tau_2} \hat{g}(\theta) U_1(t - \theta) e^{-(\mu + \delta_2)\theta} d\theta - (\mu + \delta_2) U_2(t),
\end{aligned}
\]

where the parameters \( \Lambda, \beta, \mu, p, \delta_1, \delta_2 \) represent the entering flux of susceptible individuals, the rate of becoming heroin users through contacts between susceptible individuals and heroin users, the natural death rate of populations, the rate of heroin users becoming treatment individuals, the removal rate including recovery and heroin-related death of heroin users not in treatment, the removal rate including immunity to heroin addiction during treatment period and heroin-related death of heroin users in treatment. The time that a susceptible individual becomes a heroin user is described by

\[ \int_0^{\tau_1} \hat{f}(\theta) U_1(t - \theta) e^{-(\mu + \delta_1 + p)\theta} d\theta, \]

where \( \tau_1 \) is the maximum of the delay, the kernel function \( \hat{f}(\theta) \) denotes the distribution of the infectivity of heroin users at time \( \theta \), and \( e^{-(\mu + \delta_1 + p)\theta} \) accounts the probability surviving the progression stage to be a heroin user. The time that a heroin user who is under treatment relapsing to untreated compartment is described by

\[ p \int_0^{\tau_2} \hat{g}(\theta) U_1(t - \theta) e^{-(\mu + \delta_2)\theta} d\theta, \]

where \( \tau_2 \) is the maximum of this delay, the kernel function \( \hat{g}(\theta) \) denotes the distribution function over the interval \([0, \tau_2]\), and \( e^{-(\mu + \delta_2)\theta} \) denotes the probability surviving untreated until relapsing to untreated compartment at time \( \theta \).

Furthermore, the non-negative and continuous functions \( \hat{f}(\theta) \) and \( \hat{g}(\theta) \) satisfy \( \int_0^{\tau_1} \hat{f}(\theta) d\theta = 1 \) and \( \int_0^{\tau_2} \hat{g}(\theta) d\theta = 1 \). Using the direct Lyapunov method, the threshold dynamics of system (1) are obtained, which are completely determined by the basic reproduction number.

Multi-group epidemic models have been developed to explore transmission dynamics of infectious diseases in heterogeneous populations. This multi-group feature or phenomenon can be particularly shown for some diseases, such as heroin users, measles, HIV/AIDS or malaria (vector borne disease). The heterogeneity
of population allows us to divide population into several homogeneous groups according to epidemiological or geographical such as communities, cities, and countries. Thus, we can model interactions both within-group and inter-group separately. Li et al. [11] investigated a class of multi-group epidemic models with distributed delays to describe the disease spread in a heterogeneous host population with general age-structure and varying infectivity. Feng et al. [5] established a general class of multi-group epidemic models with latency and relapse in heterogeneous populations. The global dynamics of equilibria of multi-group models involving basic reproduction number and a graph-theoretic approach to the method of global Lyapunov functionals can be found in the literatures. We refer to [6, 10, 19, 20, 3, 16, 17, 25, 9, 24, 26, 23] and references therein.

Motivated by the above works, it comes natural questions that: How does finite time delay affect the heroin spread in heterogeneous populations? and whether sustained oscillations can occur by introducing heterogeneity? To this end, we will decompose the population into \( n \) homogeneous groups. Within \( k \)-th group, \( S_k, U_{1k} \) and \( U_{2k} \) denote the numbers of susceptible, heroin users not in treatment, and heroin users undergoing treatment at time \( t \), respectively.

Thus, for \( k = 1, \ldots, n \), we formulate the following delayed multi-group epidemic model (based on (1)) to describe the spread of heroin,

\[
\begin{cases}
S_k' = \Lambda_k - \mu_k S_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\tau_1} \hat{f}_j(\theta) U_{1j}(t - \theta) e^{-(\mu_j + \delta_j + p_j)\theta} d\theta, \\
U_{1k}' = \sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\tau_1} \hat{f}_j(\theta) U_{1j}(t - \theta) e^{-(\mu_j + \delta_j + p_j)\theta} d\theta - (\mu_k + \delta_{1k} + p_k) U_{1k} \\
+ p_k \int_0^{\tau_2} \hat{g}_k(\theta) U_{1k}(t - \theta) e^{-(\mu_k + \delta_{2k})\theta} d\theta, \\
U_{2k}' = p_k U_{1k} - p_k \int_0^{\tau_2} \hat{g}_k(\theta) U_{1k}(t - \theta) e^{-(\mu_k + \delta_{2k})\theta} d\theta - (\mu_k + \delta_{2k}) U_{2k},
\end{cases}
\]

where all parameters are assumed to be nonnegative, and their epidemic meanings are described as follows:

- \( \Lambda_k \): the influx of individuals in \( k \)-th group;
- \( \beta_{kj} \): the rate of becoming heroin users into \( k \)-th group through contacts between susceptible individuals in \( k \)-th group and heroin users in \( j \)-th group;
- \( \mu_k \): the natural death rate in \( k \)-th group;
- \( p_k \): the rate of heroin users entering into the treatment compartment in \( k \)-th group;
- \( \delta_{1k} \): the removal rate including recovery and heroin-related death of heroin users not in treatment in \( k \)-th group;
- \( \delta_{2k} \): the removal rate including immunity to heroin addiction during treatment period and heroin-related death of heroin users in treatment in \( k \)-th group.

We assume that \( \beta_{kj} \) is nonnegative and \( n \)-square matrix \((\beta_{kj})_{n \times n}\) is irreducible and provides the patterns of contact and transmission among groups, which means that for any two distinct groups \( k \) and \( j \), individuals in \( U_{1k} \) can infect those in \( S_j \) directly or indirectly. The term

\[
\sum_{j=1}^{n} \beta_{kj} S_k \int_0^{\tau_1} \hat{f}_j(\theta) U_{1j}(t - \theta) e^{-(\mu_j + \delta_j + p_j)\theta} d\theta
\]
denotes the infection incidence in \( k \)-th group and
\[
\int_0^{\tau_2} \hat{g}_k(\theta)U_{1k}(t - \theta)e^{-(\mu_k + \delta_{2k})\theta}d\theta
\]
denotes the rate of heroin users in treatment relapsing to heroin users in \( k \)-th group. Furthermore, for all \( j, k = 1, \cdots, n \), we always assume that \( \int_0^{\tau_1} f_j(\theta)d\theta = 1 \) and \( \int_0^{\tau_2} \hat{g}_k(\theta)d\theta = 1 \) hold true.

The initial conditions for model (2) take the form
\[
S_k(0) = \varphi_{0k}(\theta), \ U_{1k}(\theta) = \varphi_{1k}(\theta), \ U_{2k}(\theta) = \varphi_{2k}(\theta), \ \theta \in [-\tau, 0], \ 1 \leq k \leq n, \quad (3)
\]
where \( \varphi_{ik}(\theta) \in C_k([-\tau, 0], \mathbb{R}^+) \), \( i = 0, 1, 2 \), the space of continuous functions mapping \([-\tau, 0]\) into \( \mathbb{R}^+ \) with the norm \( ||\varphi_{ik}||_{C_k} = \sup_{\theta \in [-\tau, 0]}|\varphi_{ik}(\theta)| \), where \( \tau = \max\{\tau_1, \tau_2\} \). Furthermore, define \( Y_\Delta = \{\varphi_k \in C_k : \varphi_k(\theta) \geq 0 \text{ for } \theta \in [-\tau, 0]\} \.

According to the standard theory of functional differential equations [7], model (2) has a unique solution \((S_1, U_{11}, U_{21}, \cdots, S_n, U_{1n}, U_{2n})\) with the initial conditions (3). Note that the case \( n = 1 \) is exactly the model investigated in Fang et al. [4].

The organization of this paper is as follows. In Section 2, some preliminary results are presented for model (2). The main results concerning on the global attractivity are shown in Section 3. Finally, a brief conclusion is made in Section 4.

2. Preliminaries. This section is devoted to the positivity and boundedness of solutions, the existence of equilibria and the explicit expression of the basic reproduction number.

It follows from the first equation of system (2), we have
\[
S_k' \leq \Lambda_k - \mu_k S_k,
\]
which implies that \( \limsup_{t \to \infty} S_k \leq \Lambda_k/\mu_k \). Furthermore, adding all equations of system (2), we have
\[
S_k' + U_{1k}' + U_{2k}' \leq \Lambda_k - \mu_k(S_k + U_{1k} + U_{2k}),
\]
which implies that
\[
\limsup_{t \to \infty}(S_k + U_{1k} + U_{2k}) \leq \frac{\Lambda_k}{\mu_k}. \quad (4)
\]

For easy of presentations, we denote that
\[
f_k(\theta) = \hat{f}_k(\theta)e^{-(\mu_k + \delta_{1k} + p_k)\theta}, \quad g_k(\theta) = \hat{g}_k(\theta)e^{-(\mu_k + \delta_{2k})\theta}, \quad \lambda_k = \mu_k + \delta_{1k} + p_k \quad (5)
\]
and
\[
\sigma_k = \int_0^{\tau_1} f_k(\theta)d\theta, \quad \delta_k = \int_0^{\tau_2} g_k(\theta)d\theta. \quad (6)
\]
Clearly, \( \sigma_k, \delta_k \in (0, 1] \) for all \( 1 \leq k \leq n \). Note that the variable \( U_{2k} \) do not appear in the first and the second equations of system (2). Therefore, in the rest of the paper, we can consider the following subsystem of system (2)
\[
\begin{aligned}
S_k' &= \Lambda_k - \mu_k S_k - \sum_{j=1}^{n} \beta_{kj}S_k \int_0^{\tau_1} f_j(\theta)U_{1j}(t - \theta)d\theta, \\
U_{1k}' &= \sum_{j=1}^{n} \beta_{kj}S_k \int_0^{\tau_1} f_j(\theta)U_{1j}(t - \theta)d\theta - \lambda_k U_{1k} + p_k \int_0^{\tau_2} g_k(\theta)U_{1k}(t - \theta)d\theta,
\end{aligned}
\quad (7)
\]
with the initial conditions
\[ S_k(0) = \varphi_{0k}(\theta), \quad U_{1k}(\theta) = \varphi_{1k}(\theta), \quad \theta \in [-\tau, 0], \quad 1 \leq k \leq n. \] (8)

We consider system (7) in the phase space
\[ X = \Pi_{k=1}^{n} (\mathbb{R}^+ \times C_k). \]

It is easy to check that all solutions of system (7) in \( X \) with initial conditions (8) remain nonnegative for all \( t \geq 0 \). Similarly, it follows from the first equation of system (7) that one has that
\[ \lim \sup_{t \to \infty} S_k \leq \frac{\Lambda_k}{\mu_k}. \]

Furthermore, (4) implies that
\[ \lim \sup_{t \to \infty} (S_k + U_{1k}) \leq \frac{\Lambda_k}{\mu_k}. \]

Therefore, the following set
\[ \Gamma = \left\{ (S_1, U_{11}, \ldots, S_n, U_{1n}) : 0 \leq S_k \leq \frac{\Lambda_k}{\mu_k}, 0 \leq S_k + U_{1k} \leq \frac{\Lambda_k}{\mu_k}, 1 \leq k \leq n \right\} \]
is positively invariant set of system (7). Here, we define \( \overset{\circ}{\Gamma} \) as the interior of \( \Gamma \).

Thus, summarizing the above discussions yield the following result.

**Theorem 2.1.** All solutions of system (7) with the initial conditions (8) remain nonnegative and bounded for all \( t \geq 0 \).

System (7) always has the heroin-free equilibrium (HFE) \( E_0 = (S_1^0, 0, \ldots, S_n^0, 0) \in \Gamma \), where \( S_k^0 = \frac{\Lambda_k}{\mu_k} \). The heroin-spread equilibrium (HSE),
\[ E^* = (S_1^*, U_{11}^*, \ldots, S_n^*, U_{1n}^*) \in \overset{\circ}{\Gamma} \], where \( S_k^*, U_{1k}^* > 0 \), if it exists, should satisfy
\[ \begin{align*}
\Lambda_k &= \mu_k S_k^* + \sum_{j=1}^{n} \beta_{kj} S_k^* U_{1j}^* \sigma_j, \\
\lambda_k U_{1k}^* &= \sum_{j=1}^{n} \beta_{kj} S_k^* \sigma_j U_{1j}^* + p_k \delta_k U_{1k}^*.
\end{align*} \] (9)

According to the concept of [2, 22, 27], the next generation matrix for system (7) is given by
\[ M^0 = \left( \beta_{kj} \sigma_k S_k^0 D_k \right)_{n \times n}, \quad 1 \leq k, j \leq n, \] (10)

where \( D_k = 1/(\lambda_k - p_k \delta_k) \). The basic reproduction number can be defined as the spectral radius of the matrix \( M^0 \), i.e.,
\[ R_0 = \rho(M^0). \] (11)

Let \( S = (S_1, \ldots, S_n) \) and \( S^0 = (S_1^0, \ldots, S_n^0) \), then \( M^0 = M(S^0) \). Since \( B = (\beta_{kj})_{n \times n} \) is irreducible, we have \( M(S) \) and \( M^0 \) are irreducible. It follows from \( 0 \leq S_k \leq S_k^0 \) with \( 1 \leq k \leq n \) that one has
\[ 0 \leq M(S) \leq M(S^0) = M^0. \] (12)
If $S \neq S^0$, then $M(S) < M^0$. Furthermore, $M(S) + M^0$ is also irreducible. Therefore, according to the properties of nonnegative matrices, one has $\rho(M(S)) < \rho(M^0)$ if $S \neq S^0$. In literatures of epidemic modeling, $R_0$ often serves as a threshold condition for global attractivity of the system.

**Remark 1.** Here, $R_0$ is the spectral radius of the matrix $(\beta_{kj}\sigma_k S^0_k D_k)_{n \times n}$ and has a clearly biological meaning. Note that $1/\lambda_k$ is the average time in the untreated compartment on the first pass, and $p_k/\lambda_k$ denotes the probability of entering the treatment compartment, and $\delta_k$ is the probability of relapsing into the untreated compartment. Thus, the total average time $D_k$ in the untreated compartment at the $k$-th group on multiple passes

$$\frac{1}{\lambda_k} \left[ 1 + \frac{p_k \delta_k}{\lambda_k} + \left( \frac{p_k \delta_k}{\lambda_k} \right)^2 + \cdots \right] = \frac{1}{\lambda_k - p_k \delta_k}. \quad (13)$$

Multiplying (13) by the adequate contact rate $\beta_{kj}\sigma_k$ and the susceptible $S^0_k$ at the $k$-th group produces each entry of $M^0$, and thus gives $R_0$. It denotes the average number of new heroin users produced by one drug user not in treatment compartment at the $j$-th group introduced into the susceptible compartment at the $k$-th group.

**Remark 2.** Consider the case where $n = 1$, according to (11), $R_0$ reduces to

$$R_{01} = \frac{\beta \sigma S^0}{\lambda - p \delta},$$

where

$$S^0 = \frac{\Lambda}{\mu}, \quad \sigma = \int_0^{\tau_1} f(\theta)d\theta, \quad \delta = \int_0^{\tau_2} g(\theta)d\theta.$$  

Actually, $R_{01}$ is exactly the same as that in [4].

**Remark 3.** Consider the case where $n = 2$. The next generation matrix $(M^0_2)$ has the following form

$$M^0_2 = \frac{\sigma_1 S^1_1}{\lambda_1 - p_1 \delta_1} \frac{\sigma_2 S^2_2}{\lambda_2 - p_2 \delta_2} \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix},$$

where

$$S^0_k = \frac{\Lambda_k}{\mu_k}, \quad \sigma_k = \int_0^{\tau_1} f_k(\theta)d\theta, \quad \delta_k = \int_0^{\tau_2} g_k(\theta)d\theta, \quad k = 1, 2.$$ 

Direct calculations show that

$$R_{02} = \frac{\sigma_1 S^1_1}{\lambda_1 - p_1 \delta_1} \frac{\sigma_2 S^2_2}{\lambda_2 - p_2 \delta_2} \left( \beta_{11} + \frac{\sigma_1 S^1_1}{\lambda_1 - p_1 \delta_1} \frac{\sigma_2 S^2_2}{\lambda_2 - p_2 \delta_2} \beta_{22} \right) + \sqrt{\left( \beta_{11} - \beta_{22} \right)^2 + 4 \beta_{12} \beta_{21} \frac{\sigma_1 S^1_1}{\lambda_1 - p_1 \delta_1} \frac{\sigma_2 S^2_2}{\lambda_2 - p_2 \delta_2}}.$$  

3. **Main results.** In this section, we devote to the global attractiveness of equilibria by constructing proper Lyapunov functionals and using Krichhoff’s matrix tree theorem.

Now, we are in the position to state our main results.

**Theorem 3.1.** Assume that $B = (\beta_{kj})_{n \times n}$ is irreducible. If $R_0 \leq 1$, then the HFE is unique and globally attracting in $\Gamma$ and if $R_0 > 1$, it is unstable.

**Proof.** We will solve this problem by using the Volterra type function which takes the following form

$$H(x) = x - 1 - \ln x, \quad x > 0.$$
Obviously, \( H(x) \geq 0 \) for \( x > 0 \) and \( H'(x) = 1 - 1/x \). Thus, \( H(x) \) has its unique global minimum at \( x = 1 \) with \( H(1) = 0 \).

Constructing the following Lyapunov functional
\[
L_{HFE} = S_k^0 H \left( \frac{S_k}{S_k^0} \right) + U_{1k} + \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_{0}^{\tau_1} f_j(\theta) \int_{0}^{\theta} U_{1j}(t-r) dr d\theta \\
+ p_k \int_{0}^{\tau_2} g_k(\theta) \int_{0}^{\theta} U_{1k}(t-r) dr d\theta.
\]

Calculating the time derivative of \( L_{HFE} \) along with the solutions of system \( (7) \), we have
\[
L'_{HFE|}(7) = \left( 1 - \frac{S_k^0}{S_k} \right) S_k' + U_1' + \frac{d}{dt} \left[ \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_{0}^{\tau_1} f_j(\theta) \int_{0}^{\theta} U_{1j}(t-r) dr d\theta \right] \\
+ \frac{d}{dt} \left[ p_k \int_{0}^{\tau_2} g_k(\theta) \int_{0}^{\theta} U_{1k}(t-r) dr d\theta \right].
\]

Note that
\[
\frac{d}{dt} \left[ \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_{0}^{\tau_1} f_j(\theta) \int_{0}^{\theta} U_{1j}(t-r) dr d\theta \right] \\
= - \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_{0}^{\tau_1} f_j(\theta) \int_{0}^{\theta} \frac{d}{dr} U_{1j}(t-r) dr d\theta \\
= - \sum_{j=1}^{n} \beta_{kj} S_k^0 U_{1j} - \sum_{j=1}^{n} \beta_{kj} S_k^0 \int_{0}^{\tau_1} f_j(\theta) U_{1j}(t-\theta) d\theta,
\]
and similarly,
\[
\left[ p_k \int_{0}^{\tau_2} g_k(\theta) \int_{0}^{\theta} U_{1k}(t-r) dr d\theta \right]' = p_k \delta_k U_{1k} - p_k \int_{0}^{\tau_2} g_k(\theta) U_{1k}(t-\theta) d\theta. \tag{15}
\]

Then, combining \( \Lambda_k = \mu_k S_k^0 \) and \( (14)-(15) \) into \( L'_{HFE|}(7) \) yield
\[
L'_{HFE|}(7) = - \mu_k S_k^0 \left( H \left( \frac{S_k^0}{S_k} \right) + H \left( \frac{S_k}{S_k^0} \right) \right) \\
+ (\lambda_k - p_k \delta_k) \left( \sum_{j=1}^{n} \frac{\beta_{kj} S_k^0 \sigma_j}{\lambda_k - p_k \delta_k} - 1 \right) U_{1k}. \tag{16}
\]

Since \( B \) is irreducible, we know that matrix \( M^0 \) is also irreducible, thus \( M^0 \) has a positive left eigenvector \( (\omega_1, \omega_2, ..., \omega_n) \) corresponding to the spectral radius \( \rho(M^0) \leq 1 \). Let
\[
c_k = \frac{\omega_k}{\lambda_k - p_k \delta_k} > 0.
\]

Define
\[
V_{HFE} = \sum_{k=1}^{n} c_k L_{HFE}.
\]
Using (16), the time derivative of $\mathcal{V}_{HFE}$ along with the solutions of system (7) is

$$
\mathcal{V}_{HFE}'(t) = - \sum_{k=1}^{n} c_k \mu_k S_k^0 \left( H \left( \frac{S_k}{S_k^0} \right) + H \left( \frac{S_k}{S_k^0} \right) \right) + \sum_{k=1}^{n} \omega_k \left( \sum_{j=1}^{n} \beta_{kj} S_k^0 \sigma_j - 1 \right) U_{1k} 
$$

$$
= - \sum_{k=1}^{n} c_k \mu_k S_k^0 \left( H \left( \frac{S_k}{S_k^0} \right) + H \left( \frac{S_k}{S_k^0} \right) \right) + (\omega_1, \ldots, \omega_n)(M^0 - I)U_1 \quad (17)
$$

$$
= - \sum_{k=1}^{n} c_k \mu_k S_k^0 \left( H \left( \frac{S_k}{S_k^0} \right) + H \left( \frac{S_k}{S_k^0} \right) \right) + (\mathcal{R}_0 - 1)(\omega_1, \ldots, \omega_n)U_1 \leq 0
$$

holds if $\mathcal{R}_0 \leq 1$. Here $U_1 = (U_{11}, \ldots, U_{1n})^T$.

If $\mathcal{R}_0 < 1$, then $\mathcal{V}_{HFE}'(7)$ implies $S_k = S_k^0$ and $U_1 = 0$.

If $\mathcal{R}_0 = 1$, then $\mathcal{V}_{HFE}'(7)$ implies

$$(\omega_1, \ldots, \omega_n)M(S)U_1 = (\omega_1, \ldots, \omega_n)U_1. \quad (18)$$

If $S \neq S^0$, then it follows from (12) that we have

$$(\omega_1, \ldots, \omega_n)M(S) < (\omega_1, \ldots, \omega_n)M^0 = (\omega_1, \ldots, \omega_n),$$

which contradicts to (18). Then, system (7) only has the solution $S = S^0$ and $U_1 = 0$. Thus, it can be verified that $M_0 = \{E_0\}$ is the largest compact invariant subset of $\{S_1, U_{11}, \ldots, S_n, U_{1n}\} \in \Gamma : \mathcal{V}_{HFE}'(7) = 0$. By the Lyapunov-LaSalle invariance principle, the HFE is globally attracting when $\mathcal{R}_0 \leq 1$.

If $\mathcal{R}_0 > 1$ and $U_1 \neq 0$, we have

$$(\omega_1, \ldots, \omega_n)(M^0 - I) = (\rho(M^0) - 1)(\omega_1, \ldots, \omega_n) > 0,$$

which implies that $\mathcal{V}_{HFE}'(7) > 0$ in a sufficiently small neighborhood of the HFE in $\Gamma$, by continuity. Thus, the HFE is unstable when $\mathcal{R}_0 > 1$. The proof is standard and we omit it here. This finishes the proof. \hfill \square

**Remark 4.** Biologically, Theorem 3.1 implies that the heroin epidemic will be extinct in all groups if the basic reproduction number is not greater than unity, i.e., $\mathcal{R}_0 \leq 1$.

Furthermore, we obtain some informations about the heroin epidemic in terms of persistence, i.e., the heroin will persist in all groups if $\mathcal{R}_0 > 1$. Here, we denote the positive solution $(S_1(t, S(0), \varphi), U_{11}(t, S(0), \varphi), \ldots, S_n(t, S(0), \varphi), U_{1n}(t, S(0), \varphi))$ of system (7) by $(S(t, S(0), \varphi), U_1(t, S(0), \varphi))$.

**Theorem 3.2.** Assume that $\mathcal{R}_0 > 1$. Then there exists $\eta > 0$ such that for every $(S(0), \varphi) \in \mathbb{R}^+ \times Y_\Delta$ with $\varphi(0) > 0$, the solution $(S(t, U_1(t))$ of system (7) satisfies

$$
\liminf_{t \to \infty} U_{1i}(t, S(0), \varphi) \geq \eta.
$$

**Proof.** Define

$$
Y = \{(S(0), \varphi) \in \mathbb{R}^+ \times Y_\Delta\},
$$

$$
Y_0 = \{(S(0), \varphi) \in \mathbb{R}^+ \times Y_\Delta : \varphi_i(0) > 0, \ i = 1, 2, \ldots, n\},
$$

$$
\partial Y_0 = Y \setminus Y_0.
$$

It then suffices to prove that system (7) is uniformly persistent with respect to $(Y_0, \partial Y_0)$.
Define a continuous semiflow $\pi(t) : Y \times \mathbb{R}^+ \to Y$ obtained by system (7) such that
\[
\pi(t)(S(0), \varphi) = (S(t, S(0), \varphi), U_1(t, S(0), \varphi)).
\]
Firstly, both $Y$ and $Y_0$ are positively invariant with respect to $\pi(t)$. Clearly, $\partial Y_0 = \{(S(0), \varphi) \in Y : \varphi_k(0) = 0\}$ for at least one $k \in \{1, 2, ..., n\}$ and it is relatively closed in $X$. Furthermore, system (7) is point dissipative in $\Gamma$. Define
\[
\Sigma_\partial = \{(S(0), \varphi) \in Y : (S(t, S(0), \varphi), U_1(t, S(0), \varphi)) \in \partial Y_0\}.
\]
We now show that
\[
\Sigma_\partial = \{(S(0), \varphi) \in \partial Y_0 : U_{1k}(t, S(0), \varphi) = 0 \text{ for } t \geq 0, k = 1, 2, ..., n\}. \tag{19}
\]
Assume that $(S(0), \varphi) \in \Sigma_\partial$. It suffices to show $U_{1k}(t, S(0), \varphi) = 0$ for $t \geq 0$. If it is not true, then there exist $t_0, 0 \leq k_0 \leq n$ and $t_0 \geq 0$ such that $U_{1k_0}(t_0, S(0), \varphi) > 0$. Thus the set $\{1, 2, ..., n\}$ can be departed into $M_1$ and $M_2$ such that
\[
U_{1k}(t_0, S(0), \varphi) = 0, \quad \forall k \in M_1; \quad U_{1k}(t_0, S(0), \varphi) > 0, \quad \forall k \in M_2.
\]
Clearly, $M_1$ is non-empty due to the definition of $\Sigma_\partial$ and $M_2$ is also non-empty since $U_{1k_0}(t_0, S(0), \varphi) > 0$. For any $j \in M_1$, by the irreducibility of the matrix $(\beta_{kj})_{n \times n}$, there is an $k_1 \in M_2$ such that
\[
U_{1j}^*(t) \bigg|_{t = t_0} = \sum_{j=1}^n \beta_{kj}S_k U_{1k_1} \int_{0}^{\tau_2} f_j(\theta)d\theta - \lambda_k U_{1j} + p_k U_{1k_1} \int_{0}^{\tau_2} g_k(\theta)d\theta > 0.
\]
It follows that there is an $\eta_0 > 0$ such that $U_{1j} > 0$ for $j \in M_1$ and $t \in (t_0, t_0 + \eta_0)$. Clearly, we can restrict $\eta_0 > 0$ small enough such that $U_{1k} > 0$ for $k \in M_2$ and $t \in (t_0, t_0 + \eta_0)$. This means that $(S(t, S(0), \varphi), U_1(t, S(0), \varphi)) \notin \partial Y_0$ for $t \in (t_0, t_0 + \eta_0)$, which contradicts the assumption that $(S(0), \varphi) \in \Sigma_\partial$. This proves (19) holds.

Furthermore, it remains to show that
\[
\limsup_{t \to \infty} \max_{\eta > 0} \{U_{1k}(t, S(0), \varphi)\} > \eta, \quad \text{for } (S(0), \varphi) \in Y_0, \quad k = 1, 2, ..., n, \tag{20}
\]
where every $\eta > 0$ is small enough such that the following system
\[
\dot{S}_k^* = \Lambda_k - \mu_k S_k - \sum_{j=1}^n \beta_{kj}S_k \sigma_j \eta, \quad k = 1, 2, ..., n \tag{21}
\]
admits a unique positive equilibrium $(S_1^0(\eta), ..., S_n^0(\eta))$ which is globally attracting. The implicit function theorem implies that $(S_1^0(\eta), ..., S_n^0(\eta))$ is continuous in $\eta$. Thus, we can further restrict $\eta$ small enough such that $(S_1^0(\eta), ..., S_n^0(\eta)) > (S_1^0 - \eta_1, ..., S_n^0 - \eta_n)$ for $\eta_1 > 0$.

Suppose that (20) does not hold, then there exists a large enough $t_1 > 0$ such that $0 < U_{1k}(t, S(0), \varphi) \leq \eta$ for $t \geq t_1$ and $k = 1, 2, ..., n$. Then for $t \geq t_1$ and $k = 1, 2, ..., n$, one has
\[
S_k^t \geq \Lambda_k - \mu_k S_k - \sum_{j=1}^n \beta_{kj}S_k \sigma_j \eta, \quad k = 1, 2, ..., n. \tag{22}
\]
It follows from the global attractiveness of $(S_1^0(\eta), ..., S_n^0(\eta))$ of system (21) and $S_0(\eta) > S_0 - \eta_1$ that there exists a $t_2$ such that $S(t) > S_0 - \eta_1$ for $t \geq t_3 = t_1 + t_2$. Thus, one gets
\[
\rho(M_0(\eta_1)) = \rho \left( \beta_{kj} \sigma_k (S_k^0 - \eta_1)D_k \right)_{n \times n} > 1
\]
for sufficiently small $\eta_1$. Consequently, for $t \geq t_3$ and $k = 1, 2, \ldots, n$, we obtain

$$U_{1k}^t \geq \sum_{j=1}^n \beta_{kj}(S_k^0 - \eta_1) \int_0^{t_3} f_j(\theta)U_{1j}(t - \theta) d\theta - \lambda_k U_{1k} + p_k \int_0^{t_3} g_k(\theta)U_{1k}(t - \theta) d\theta$$

$$= \sum_{j=1}^n \beta_{kj}(S_k^0 - \eta_1)\sigma_k U_{1k}(\bar{t}_1) - \lambda_k U_{1k}(\bar{t}_1) + p_k \delta_k U_{1k}(\bar{t}_1).$$

Here, we use the mean value theorem for integrals and then obtain for any $t$, there exists a $\bar{t}_1 \in [t - t_1, t]$ such that

$$\int_0^{t_3} f_j(\theta)U_{1j}(t - \theta) d\theta = \sigma_k U_{1k}(\bar{t}_1), \quad \int_0^{t_3} g_k(\theta)U_{1k}(t - \theta) d\theta = \delta_k U_{1k}(\bar{t}_1).$$

Then it follows from a standard comparison argument and the nonnegativity that $\rho(M^0(\eta_1)) > 1$ implies that there is at least one $k \in \{1, 2, \ldots, n\}$ such that $U_{1k}(t) \to \infty$ as $t \to \infty$, which contradicts to the boundedness of solutions. Therefore (20) holds.

Note that $(S_1^0, \ldots, S_n^0)$ is globally attracting in $\mathbb{R}^+ / \{0\}$ for system (22). By the afore-mentioned claim, $(S^0, 0)$ is an isolated invariant set in $\mathcal{Y}$, and $\mathcal{W}^s(S^0, 0) \cup \mathcal{Y}_0 = \emptyset$. Clearly, every orbit in $\Sigma$ converge to $(S^0, 0)$ which is the only invariant set in $\Sigma$. By [21, Theorem 4.6], for a stronger repelling property of $\partial \mathcal{Y}_0$, we conclude that system (7) is indeed uniformly persistent with respect to $(\mathcal{Y}_0, \partial \mathcal{Y}_0)$. This finishes the proof. 

By Theorem 3.2 and using similar arguments that in [14, Lemma 4.1], the following corollary holds.

**Corollary 1.** Assume that $\mathcal{R}_0 > 1$. Let $(S(t), \varphi_{id})$ be a solution of system (7), then there exists a positive constant $\eta > 0$ such that $S(t), U_1(t) > \eta$ for all $t > 0$.

Uniform persistence of system (7) from Theorem 3.1 and uniform boundedness of solutions in $\bar{\Gamma}$ imply that there exists at least a heroin-spread equilibrium (positive equilibrium) of system (7) in $\bar{\Gamma}$ (see Theorem 2.8.6 in [1]). Thus, we have the following corollary.

**Corollary 2.** Assume that $B = (\beta_{kj})_{n \times n}$ is irreducible. If $\mathcal{R}_0 > 1$, then system (7) has at least one positive equilibrium.

Finally, we have the following result for system (7).

**Theorem 3.3.** Assume that $B = (\beta_{kj})_{n \times n}$ is irreducible. If $\mathcal{R}_0 > 1$, then there exists a unique HSE, which is globally attracting in $\bar{\Gamma}$.

**Proof.** Let us mention that the global attractiveness of the HSE implies that the HSE is unique in the $\bar{\Gamma}$ whenever it exists.

Constructing a Volterra-type Lyapunov functional as follows

$$\mathcal{L}_{HSE} = S_k^* H\left(\frac{S_k}{S_k^*}\right) + U_{1k}^* H\left(\frac{U_{1k}}{U_{1k}^*}\right) + \int_0^{\tau_2} g_k(\theta)\int_0^\theta H\left(\frac{U_{1k}(t - r)}{U_{1k}^*}\right) dr d\theta$$

$$+ \sum_{j=1}^n \beta_{kj} S_k^* U_{1j}^* \int_0^{\tau_1} f_j(\theta)\int_0^\theta H\left(\frac{U_{1j}(t - r)}{U_{1j}^*}\right) dr d\theta.$$
Calculating the time derivative of $\mathcal{L}_{HSE}$ along with the solutions of system (7) yields

$$
\mathcal{L}'_{HSE}(7) = (1 - \frac{S_k^*}{S_k})S_k + (1 - \frac{U_{1k}^*}{U_{1k}})U_{1k} + \frac{d}{dt} \left[ p_k U_{1k}^* \int_0^{\tau_2} g_k(\theta) \int_0^{\theta} H \left( \frac{U_{1k}(t-r)}{U_{1k}^*} \right) dr d\theta \right] \\
+ \frac{d}{dt} \left[ \sum_{j=1}^n \beta_{kj} S_k^* U_{1j}^* \int_0^{\tau_1} f_j(\theta) \int_0^{\theta} H \left( \frac{U_{1j}(t-r)}{U_{1j}^*} \right) dr d\theta \right].
$$

Note that

$$
- \frac{\beta_{kj} S_k^* U_{1j}^*}{\beta_{kj} S_k^* U_{1j}^*} = \frac{p_k}{\beta_{kj} S_k^* U_{1j}^*} + \frac{d}{dt} \left[ \sum_{j=1}^n \beta_{kj} S_k^* U_{1j}^* \int_0^{\tau_1} f_j(\theta) \int_0^{\theta} H \left( \frac{U_{1j}(t-r)}{U_{1j}^*} \right) dr d\theta \right]
$$

and similarly, we have

$$
\left[ p_k U_{1k}^* \int_0^{\tau_2} g_k(\theta) \int_0^{\theta} H \left( \frac{U_{1k}(t-r)}{U_{1k}^*} \right) dr d\theta \right]' = p_k U_{1k} \delta_k - p_k \int_0^{\tau_2} g_k(\theta) U_{1k}(t-\theta)d\theta + p_k U_{1k}^* \int_0^{\tau_2} g_k(\theta) \ln \frac{U_{1k}(t-\theta)}{U_{1k}^*} d\theta.
$$

Thus, together with (9) and (23)-(23), we have

$$
\mathcal{L}'_{HSE}(7) = - \mu_k S_k \left( H \left( \frac{S_k^*}{S_k} \right) + H \left( \frac{S_k}{S_k^*} \right) \right) + \sum_{j=1}^n \beta_{kj} S_k^* U_{1j}^* \sigma_j \left( 2 - \frac{S_k^*}{S_k} - \frac{U_{1k}^*}{U_{1k}} + \frac{U_{1j}}{U_{1j}^*} \right) \\
- \frac{U_{1k}^*}{U_{1k}} \sum_{j=1}^n \beta_{kj} S_k \int_0^{\tau_1} f_j(\theta) U_{1j}(t-\theta)d\theta \\
+ p_k U_{1k} \int_0^{\tau_2} g_k(\theta) \left( 1 - \frac{U_{1k}(t-\theta)}{U_{1k}} + \ln \frac{U_{1k}(t-\theta)}{U_{1k}} \right) d\theta.
$$

Let $\bar{\beta}_{kj} = \beta_{kj} S_k^* U_{1j}^* \sigma_j$ for all $k,j$, we denote

$$
\bar{B} = \begin{bmatrix}
\sum_{i \neq 1} \bar{\beta}_{1i} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{21} \\
-\bar{\beta}_{12} & \sum_{i \neq 2} \bar{\beta}_{2i} & \cdots & -\bar{\beta}_{n2} \\
-\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{i \neq n} \bar{\beta}_{ni}
\end{bmatrix}.
$$

Note that $\bar{B}$ is the Laplacian matrix of the matrix $B$ (see [10] for more details). Since $B$ is irreducible, matrix $\bar{B}$ also is irreducible. Let $C_{kj}$ denote the cofactor
of the \((k,j)\) entry of \(B\). We know that system \(Bv = 0\) has a positive solution 
v = \((v_1, \cdots, v_n)\), where \(v_k = C_{kk} > 0\) for \(k = 1, 2, \cdots, n\). Define
\[ V_{HSE} = \sum_{k=1}^{n} v_k \mathcal{L}_{HSE}. \]

Using (24), the time derivative of \(V_{HSE}\) along with the solutions of system (7) is
\[
V'_{HSE}(t) = -\sum_{k=1}^{n} v_k \mu_k S_k^* \left( H \left( \frac{S_k^*}{S_k} \right) + H \left( \frac{S_k}{S_k^*} \right) \right) \\
- \sum_{k=1}^{n} v_k \beta_{k} S_k^* U_{1k}^* \int_{0}^{T} \int_{0}^{T} f_j(\theta) \left( 2 - \frac{S_k^*}{S_k} - \frac{U_{1k}^*}{U_{1k}} + \frac{U_{1j}^*}{U_{1j}} - \frac{S_k U_{1k}^* U_{1j}^*}{S_k^* U_{1k} U_{1j}} \right) \, d\theta \\
+ \sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* S_k^* U_{1j}^* \sum_{k=1}^{n} \int_{0}^{T} \int_{0}^{T} f_j(\theta) \left[ H \left( \frac{S_k^*}{S_k} \right) + H \left( \frac{S_k}{S_k^*} \right) \right] \, d\theta \\
- \sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* S_k^* U_{1j}^* \sigma_j \left( \frac{U_{1k}^*}{U_{1k}} - \frac{U_{1j}^*}{U_{1j}} - \ln \frac{U_{1k}^* U_{1j}^*}{U_{1k} U_{1j}} \right)
\]

Here, we denote that
\[
\Phi = \sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* S_k^* U_{1j}^* \sigma_j \left( \frac{U_{1k}^*}{U_{1k}} - \frac{U_{1j}^*}{U_{1j}} - \ln \frac{U_{1k}^* U_{1j}^*}{U_{1k} U_{1j}} \right)
\]
\[
= \sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* S_k^* U_{1j}^* \sigma_j \left( \frac{U_{1k}^*}{U_{1k}} - \frac{U_{1j}^*}{U_{1j}} \right) - \sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* S_k^* U_{1j}^* \sigma_j \ln \frac{U_{1k}^* U_{1j}^*}{U_{1k} U_{1j}}
\]
\[
= \Phi_1 - \Phi_2.
\]

In the following, we will show that \(\Phi_1 \equiv 0\) and \(\Phi_2 \equiv 0\).

We firstly claim that \(\Phi_1 \equiv 0\). It follows from \(Bv = 0\) that \(\sum_{j=1}^{n} \beta_{jk} v_j = \sum_{i=1}^{n} \beta_{ki} v_k\).

Since \(\bar{\beta}_{jk} = \beta_{jk} S_j^* U_{1k}^* \sigma_k\), then we have
\[
\sum_{j=1}^{n} \beta_{jk} S_j^* U_{1k} \sigma_k v_j = \sum_{i=1}^{n} \beta_{ki} S_k^* U_{1i} \sigma_i v_k, \quad k = 1, 2, \cdots, n.
\] (25)

Therefore, it follows from (25) that
\[
\sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* S_k^* U_{1j}^* \sigma_j \left( \frac{U_{1k}^*}{U_{1k}} - \frac{U_{1j}^*}{U_{1j}} \right) = \sum_{k=1}^{n} U_{1k} \sum_{j=1}^{n} v_j \beta_{jk} S_j^* U_{1k}^* \sigma_k = \sum_{k=1}^{n} U_{1k} \sum_{j=1}^{n} v_j \beta_{jk} S_j^* U_{1k}^* \sigma_k
\]
\[
\Phi_i = \sum_{k=1}^{n} \frac{U_{1k}}{U_{1k}^*} \sum_{k=1}^{n} v_k \beta_{ki} S_k^* U_{11}^* \sigma_i = \sum_{k,j=1}^{n} v_k \beta_{kj} S_k^* U_{1j}^* \sigma_j \frac{U_{1k}}{U_{1k}^*},
\]
which implies that \( \Phi_i \equiv 0 \) for all \( U_{1k}, k = 1, 2, \ldots, n \).

Next we will use Krichhoff’s matrix tree theorem (see [10, 11]) to show that \( \Phi_2 \equiv 0 \). The matrix \((\beta_{kj})_{n\times n}\) can be viewed as a directed graph denoted by \( G \) with vertices 1, 2, \ldots, \( n \) and a directed arc \((k, j)\) from \( k \) to \( j \) iff \( \beta_{kj} = 0 \). Furthermore, the set of all directed arcs of \( G \) is denoted by \( E(G) \). The directed spanning subtree of \( G \) that is rooted at vertex \( k \) is denoted by \( T \), and Krichhoff’s matrix tree theorem implies that the sum of weights of all \( T \) can be expressed as \( v_k \beta_{kj} \). According to adding a directed arc \((k, j)\) from the root \( k \) to vertex \( j \) in a subtree \( T \), one can obtain such a unicyclic subgraph \( Q \) of \( G \) with the weight \( w(Q) = v_k \beta_{kj} \). There exists a unique cycle \( CQ \) of \( Q \) and the arc \((k, j)\) is part of the unique cycle \( CQ \), thus, the fact that each arc of \( CQ \) is added to a corresponding rooted subtree \( T \) can form the same unicyclic subgraph \( Q \). Therefore, the double sum in \( \Phi_2 \) can be reorganized as a sum over all unicyclic subgraphs \( Q \) containing vertices 1, 2, \ldots, \( n \), that is, \( \Phi_2 = \sum_Q \Phi_Q \), where

\[
\Phi_Q = w(Q) \cdot \sum_{(k,j) \in E(CQ)} \ln \frac{U_{1k} U_{1j}^*}{U_{1k}^* U_{1j}} = w(Q) \cdot \ln \left( \prod_{(k,j) \in E(CQ)} \frac{U_{1k} U_{1j}^*}{U_{1k}^* U_{1j}} \right).
\]

Since \( E(CQ) \) denotes the set of all arcs of \( CQ \), we have

\[
\prod_{(k,j) \in E(CQ)} \frac{U_{1k} U_{1j}^*}{U_{1k}^* U_{1j}} = 1,
\]
which implies that \( \Phi_Q = 0 \) for each \( Q \), and thus \( \Phi_2 \equiv 0 \) for each \( U_{1k}, k = 1, \ldots, n \).

Therefore, we have \( \forall HSE(7) \leq 0 \) for all solutions \((S_1, U_{11}, \ldots, S_n, U_{1n}) \in \hat{\Gamma} \).

Here, we use the fact that for all \( \theta \in [0, \tau] \)

\[
H \left( \frac{U_{1k}(t - \theta)}{U_{1k}^*} \right) \geq 0, \quad H \left( \frac{S_k U_{1k} U_{1j}(t - \theta)}{U_{1k} U_{1j}^*} \right) \geq 0.
\]
Furthermore, \( \forall HSE(7) = 0 \) if and only if \( S_k = S_k^*, S_k U_{1k} U_{1j}(t - \theta) = S_k^* U_{1k} U_{1j}^* \)
and \( U_{1j}(t - \theta) = U_{1j}^* \), which implies \( U_{1k}(t - \theta) = U_{1k}^* \). Hence, the largest compact invariant subset of \( \{(S_1, U_{11}, \ldots, S_n, U_{1n}) \in \hat{\Gamma} : \forall HSE(7) = 0\} \) is the singleton \( \{E^*\} \).

By the Lyapunov-LaSalle invariance principle, the HSE is globally attracting in \( \hat{\Gamma} \). That is, \( \lim_{t \to \infty} (S(t), \varphi(t)) = E^* = (S_1^*, U_{11}^*, \ldots, S_n^*, U_{1n}^*) \).

**Remark 5.** Biologically, Theorem 3.3 implies that the heroin epidemic will remain spread and persist at the level of unique HSE in all groups, independent of the initial values if \( \Re_0 > 1 \).

4. **Numerical simulation.** This section is devote to numerical simulation to support our analysis. We focus on the effects of two delays on the level of heroin spread. We consider the case where \( n = 2 \). Note that the expression of \( \Re_{02} \) (described in Remark 3) takes the following form

\[
\Re_{02} = \frac{\sigma_1 S_1^0}{\lambda_1 - p_1 \delta_1} \frac{\sigma_2 S_2^0}{\lambda_2 - p_2 \delta_2} (\beta_{11} + \beta_{22}) + \sqrt{(\beta_{11} - \beta_{22})^2 + 4 \beta_{12} \beta_{21}}.
\]
It is easy to check that
\[ \frac{\partial \mathcal{R}_{02}}{\partial \sigma_k} > 0, \quad \frac{\partial \mathcal{R}_{02}}{\partial \delta_k} > 0, \quad k = 1, 2, \]
and
\[ \frac{d\sigma_k}{d\tau_1} > 0, \quad \frac{d\delta_k}{d\tau_2} > 0, \quad k = 1, 2, \]
which imply that both \( \tau_1 \) and \( \tau_2 \) have positive influence on the value of \( \mathcal{R}_{02} \). In order to describe the biological implications, we give some numerical examples by choosing parameters without considering the justification.

Fix the parameters as
\[ \Lambda_1 = \Lambda_2 = 3, \quad \mu_1 = \mu_2 = 0.71, \quad \delta_{11} = \delta_{12} = 0.85, \quad \delta_{21} = \delta_{22} = 0.2 \]
\[ p_1 = p_2 = 0.8, \quad \beta_{11} = 0.8, \beta_{12} = 0.9, \beta_{21} = 1.3, \beta_{22} = 0.45 \]
and assume \( \delta_1 = \delta_2 = \delta \) and \( \sigma_1 = \sigma_2 = \sigma \).

Taking \( \tau_2 = 1, \hat{g}(s) = 1 \). According to the effects of \( \tau_1 \) and \( \tau_2 \) on \( \sigma \) and \( \delta \), we simulate that \( \mathcal{R}_{02} \) increases as the values of \( \sigma \) and \( \delta \) increase (see Fig. 1). Furthermore, Fig. 2 shows that \( \tau_1 \) has the positive effects on the values of \( \sigma \) and \( \mathcal{R}_{02} \).

Taking \( \tau_1 = 1 \) and \( \hat{f}(s) = 1 \). The other parameter values are the same as in Fig. 1. Simulation result shows that \( \tau_2 \) has the positive effects on the values of \( \delta \) and \( \mathcal{R}_{02} \) (see Fig. 3). Thus, the increasing delays will intense the spread heroin and raise the ultimately spread level.

![Figure 1. Influence of \( \sigma \) and \( \delta \) on \( \mathcal{R}_{02} \).](image)

5. Conclusions and discussions. In this paper, we have analyzed a delayed multi-group heroin epidemic model to describe the spread of heroin in heterogeneous populations. We provide a rigorous analysis of the model and establish its global dynamics. The threshold parameter, which corresponds to the well-known basic reproduction number \( \mathcal{R}_0 \), completely determines the global dynamics of the model. That is, if \( \mathcal{R}_0 \leq 1 \), then the HFE is globally attracting; while if \( \mathcal{R}_0 > 1 \), then the HSE is globally attracting by combining a proper Lyapunov functional and Krichhoff’s matrix tree theorem.
Theorem 3.1 and Theorem 3.3 show that the model has the sharp threshold property determined by the basic reproduction number $R_0$. Thus, the heroin either is extinct or remains spread completely depending on the value of the basic reproduction number. Biologically, Theorem 3.2 implies that the heroin will persist in all groups of the population and will eventually settle at a constant level in each group. Since the heterogeneity of populations plays an important role on $R_0$, it affects the global dynamics of the model. However, it does not cause any sustained oscillations.

**Acknowledgments.** We are grateful to the two anonymous referees for their valuable comments and suggestions that led to an improvement of the presentation.

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