NON-CYCLOTOMIC PRESENTATIONS
OF MODULES AND PRIME-ORDER
AUTOMORPHISMS OF KIRCHBERG ALGEBRAS

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Dedicated to George Elliott on the occasion of his sixtieth birthday.

Abstract. We prove the following theorem: let $A$ be a UCT Kirchberg algebra, and let $\alpha$ be a prime-order automorphism of $K_*(A)$, with $\alpha([1_A]) = [1_A]$ in case $A$ is unital. Then $\alpha$ is induced from an automorphism of $A$ having the same order as $\alpha$.

This result is extended to certain instances of an equivariant inclusion of Kirchberg algebras. As a crucial ingredient we prove the following result in representation theory: every module over the integral group ring of a cyclic group of prime order has a natural presentation by generalized lattices with no cyclotomic summands.

Introduction

This paper is concerned with Kirchberg algebras satisfying the universal coefficient theorem (UCT). (Following [11] we use the term Kirchberg algebra for a separable nuclear simple purely infinite $C^*$-algebra.) Deep results of Kirchberg, Rørdam, Elliott, and Phillips have made the class of Kirchberg algebras a prominent example of Elliott’s classification program: the algebras are classified by $K$-theory, and homomorphisms at the level of $K$-theory are induced from $*$-homomorphisms of algebras. Because of this classification theorem ([7], [10]), it is possible to prove results about UCT Kirchberg algebras by choosing a convenient model. In this paper we use a construction based on graph $C^*$-algebras (see [15]) to model general UCT Kirchberg algebras.

It is tempting to conjecture that there might be a right inverse to the $K$-theory functor for the class of Kirchberg algebras. As pointed out in [1], this is not possible in general. However if the morphisms between $K$-groups are required to be injective, the conjecture has not yet been contradicted. Nevertheless it seems to be quite a subtle problem. The first step was taken in [1], where it was proved that if the identity element of a (unital UCT) Kirchberg algebra is trivial in $K_0$, then every automorphism of the $K$-theory having order two is induced from an automorphism of the algebra having order two. The proof uses a technical equivariant process for turning a general $C^*$-algebra into a Kirchberg algebra. In order to use this construction, the authors prove a general structure theorem for modules over the group ring of the cyclic group of order two.

1991 Mathematics Subject Classification. Primary 16G30, 20C10, 46L55, 46L80.

Key words and phrases. Kirchberg algebra, $K$-theory, graph algebra, integral representation, lattice, generalized lattice.
The analogous theorem for modules over the group ring of a cyclic group of arbitrary prime order was proved independently in [2]. In this paper we use this theorem to extend the result of [1] to arbitrary prime-order automorphisms of Kirchberg algebras. Our construction is very different from that of [1]. We use the explicit construction of Kirchberg algebras from directed graphs given in [15]. Our strategy is to start with an abelian group with a prime-order automorphism. We then construct a directed graph in which the group appears as a subset of the vertex set, and such that there is an automorphism of the graph extending the automorphism of the group and having the same order. The vertices and edges of the graph are generators of its $C^*$-algebra, and the relations reflect the structure of the graph. Thus there is a homomorphism from the automorphism group of the graph to the automorphism group of its $C^*$-algebra.

An abelian group with an automorphism of prime order defines a module over the integral group ring of the cyclic group of that prime order. Our construction of the directed graph with automorphism requires the solution of a certain problem in integral representation theory that we hope will be of independent interest. It concerns generalized lattices over this group ring. The main result of [2] is that every generalized lattice is a direct sum of (finitely generated) lattices. The (classical) theory of lattices classifies the indecomposable lattices into three types: trivial, cyclotomic and projective. We prove that a certain natural free presentation of the group results in generalized lattices having no cyclotomic summands. Our proof yields a more general result for the simultaneous resolution of an inclusion of modules. We apply this to the problem of lifting prime-order automorphisms to an inclusion of Kirchberg algebras that is equivariant for actions of a cyclic group of prime order.

The first section of the paper is devoted to the precise statement and proof of our results on modules over group rings. Along with the result of [2] already mentioned, we rely heavily on the paper [8], in which all finitely generated indecomposable modules are classified (the case of finite indecomposables was proved in [9]). In the second section, from a given abelian group $G$ with prime-order automorphism we construct the directed graph whose $C^*$-algebra is the (non-unital) UCT Kirchberg algebra having $K$-theory $(G,0)$, and admitting a graph-automorphism of the same prime order. We then use the construction in [15] to treat the general case. In the case of an inclusion of modules, our result in section 1 applies if and only if a certain partial purity condition is satisfied (see Corollary 1.18). In certain cases where this condition fails, however, the result on inclusions of Kirchberg algebras can be established by alternate means.

The figures in this paper were prepared with Xy-pic.

1. Non-cyclotomic presentations of modules

Throughout we let $p$ denote a prime integer, and $C_p = \mathbb{Z}/p\mathbb{Z}$ the cyclic group of order $p$. We let $\alpha$ denote the generator $1 + p\mathbb{Z}$ of $C_p$. Let $R = \mathbb{Z}C_p$ denote the integral group ring of $C_p$. An abelian group $M$ with an automorphism of order $p$ becomes a module over $R$. Throughout this paper the only modules we will consider will be $R$-modules; hence we will usually omit the prefix $R$-. We will make frequent use of two particular elements of $R$. 


Definition 1.1. We let $t$ and $s$ denote the following elements of $R$:

$$
t = \alpha - 1
$$

$$
s = 1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}.
$$

We note that $R$ is isomorphic to $\mathbb{Z}[x]/(x^p - 1)$. We will occasionally let $t$ and $s$ denote the elements $x - 1$ and $1 + x + \cdots + x^{p-1}$ in $\mathbb{Z}[x]$.

For any abelian group $M$, let $\pi_M : \mathbb{Z} M \to M$ denote the canonical surjection

$$
\pi_M(\sum_{x \in M} c_x \widehat{x}) = \sum_{x \in M} c_x x,
$$

where $\{\widehat{x} : x \in M\}$ denotes the canonical basis of the free abelian group $\mathbb{Z} M$. We let $N_M$ denote the kernel of $\pi_M$. If $M$ is a module then $\mathbb{Z} M$ becomes a module via

$$
\alpha : \sum_{x \in M} c_x \widehat{x} = \sum_{x \in M} c_x \alpha \widehat{x},
$$

and $\pi_M$ is a module map. Thus $N_M$ is also a module. Thus we have a presentation of the module $M$ by modules that are free abelian groups:

$$
0 \to N_M \to \mathbb{Z} M \to M \to 0.
$$

Note that this construction respects inclusions of modules. We observe that the decomposition of $M$ into orbits under $\alpha$ determines a decomposition of $\mathbb{Z} M$ as a direct sum of submodules that are isomorphic as modules to $R$ or to the trivial module $\mathbb{Z}$. We conjecture that $N_M$ can be decomposed in a similar manner. We have not been able to prove this. However, for our purposes, the following theorem is sufficient.

Theorem 1.2. $N_M$ can be decomposed as a direct sum of finitely generated projective modules and a trivial module.

We use Theorem 1.2 to prove the following result, which is the main goal of this section.

Theorem 1.3. There is a short exact sequence of modules,

$$
0 \to N_1 \to N_2 \to M \to 0,
$$

where $N_1$ and $N_2$ are free abelian groups, and each is the direct sum of a free module and a trivial module. (Moreover, the modules may be chosen so that the following holds. $N_2$ may be written in the form $N_2 = (\oplus_j R \cdot \xi_j) \oplus (\oplus_k \mathbb{Z} \cdot \eta_k)$ so that every element of $M$ is the image of a basis element of the form $\alpha \xi_j$ or $\eta_k$.)

Proof. We note that if $A$ is a finitely generated projective module there is another module $A'$ such that $A \oplus A'$ is a finitely generated free module. Hence

$$
A \oplus R \oplus R \cdots \cong A \oplus (A \oplus A') \oplus (A \oplus A') \oplus \cdots
$$

$$
\cong (A \oplus A') \oplus (A \oplus A') \oplus \cdots
$$

$$
\cong R \oplus R \oplus \cdots.
$$
Let $R^\infty$ denote the free module with rank equal to $\aleph_0$ times the cardinality of the set of finitely generated projective summands of $N_M$, as provided by Theorem 1.2. Then by Theorem 1.2, $N_1 = N_M \oplus R^\infty$ is isomorphic to the direct sum of a free and a trivial module. Let $N_2 = \mathbb{Z}M \oplus R^\infty$; by the earlier observation this is also the direct sum of a free and a trivial module. Define maps $N_1 \to N_2$ and $N_2 \to M$ by $(x, y) \mapsto (x, y)$, respectively $(x, y) \mapsto \pi_M(x)$. The final claim can be seen by using the elements $\{\hat{x} : x \in M\}$.

A crucial tool for proving Theorem 1.2 is the following recent result of Butler, Campbell and Kovács (the case $p = 2$ of this result was proved independently in [1]).

**Theorem 1.4.** ([2], Theorem 1.1.) Every module whose underlying abelian group is free is a direct sum of finitely generated modules.

It is convenient to use the terminology of [2]. A *generalized lattice* over $R$ is an $R$-module whose underlying abelian group is free. (A lattice over $R$ is then a finitely generated generalized lattice.) According to the Diederichsen-Reiner structure theory of $R$-lattices (see, e.g., [3], section 74), there are finitely many isomorphism classes of indecomposable lattices, classified as projective, trivial or cyclotomic. Thus Theorem 1.2 states that for any $R$-module $M$, the generalized lattice $N_M$ is non-cyclotomic, according to the

**Definition 1.5.** A generalized $R$-lattice is *non-cyclotomic* if it has no cyclotomic summands.

It follows from Theorem 1.4, and the Diederichsen-Reiner structure theory, that a generalized $R$-lattice $N$ is non-cyclotomic if and only if $\ker(s) \cap N = tN$.

**Lemma 1.6.** Suppose that Theorem 1.2 is true for the modules $M_1$ and $M_2$. Then it is true for $M_1 \oplus M_2$.

**Proof.** We first note that for any module $M$, $\mathbb{Z}\hat{0}$ is a direct summand of $N_M$, with complement

$$\tilde{N}_M = \{ \sum_{x \in M} c_x \hat{x} \in N_M : c_0 = 0 \}.$$ 

Now let $M_1$ and $M_2$ be modules for which Theorem 1.2 holds. We will identify $M_1$ and $M_2$ with the corresponding submodules of $M_1 \oplus M_2$. Then in the obvious way we have

$$\tilde{N}_{M_1} \oplus \tilde{N}_{M_2} \subseteq N_{M_1 \oplus M_2}.$$

Let $L = (M_1 \oplus M_2) \setminus (M_1 \cup M_2)$. For $x \in L$, $x = (x_1, x_2)$ with $x_1$ and $x_2$ both nonzero. Define

$$\xi_x = \hat{x} - \hat{x}_1 - \hat{x}_2 \in N_{M_1 \oplus M_2}.$$ 

Let $N_3 = \text{span}\{\xi_x : x \in L\}$. We claim that

(1.1) $$N_{M_1 \oplus M_2} = \mathbb{Z}\hat{0} \oplus \tilde{N}_{M_1} \oplus \tilde{N}_{M_2} \oplus N_3.$$
To see this, let $\xi \in N_{M_1 \oplus M_2}$ be arbitrary. Then

$$\xi = \sum_{x \in M_1 \oplus M_2} c_x \hat{x}$$

$$= \sum_{x \in M_1 \cup M_2} c_x \hat{x} + \sum_{x \in L} (c_{x,1} + c_{x,1} + c_{x,2})$$

$$= c_0 \hat{0} + \sum_{x \in M_1 \setminus \{0\}} \left( c_x + \sum_{y \in M_2 \setminus \{0\}} c_{(x,y)} \right) \hat{x} +$$

$$+ \sum_{y \in M_2 \setminus \{0\}} \left( c_y + \sum_{x \in M_1 \setminus \{0\}} c_{(x,y)} \right) \hat{y} + \sum_{x \in L} \xi_x.$$ 

Writing $\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3$ respecting the above, we have $\xi_0, \xi_3 \in N_M$. We will write $\pi$ for $\pi_{M_1 \oplus M_2}$. We then have

$$\pi(\xi_1) = -\pi(\xi_2) \in M_1 \cap M_2 = \{0\}.$$ 

Therefore $\xi_i \in \tilde{N}_M$ for $i = 1, 2$. It follows that the groups on the right-hand side of 1.1 span the left-hand side. Since these groups are clearly linearly independent, 1.1 is correct as a direct sum of abelian groups. To see that it is a direct sum of modules, note that for $x \in L$,

$$\alpha \xi_x = \hat{\alpha} \hat{0} - \hat{\alpha} x_1 - \hat{\alpha} x_2$$

$$= \xi_{\alpha x}.$$ 

Thus $N_3$ is a module. Finally, since $L$ is a union of $\alpha$-orbits, so is $\{\xi_x : x \in L\}$. Hence $N_3$ is the direct sum of a free and a trivial module. 

Consider an inclusion of modules $M_0 \subseteq M$. The following lemma may be thought of as a partial purity result for $N_{M_0}$ in $N_M$. Lemma 1.13 and Lemma 1.14 below give necessary and sufficient conditions on the inclusion $M_0 \subseteq M$ for $N_{M_0}$ to be a pure submodule of $N_M$.

**Lemma 1.7.** Let $M_0 \subseteq M$ be an inclusion of modules. Then $(tN_M) \cap N_{M_0} = tN_{M_0}$.

**Proof.** We let $\pi$ denote $\pi_M$. Let $\xi \in N_M$ be such that $t\xi \in N_{M_0}$. Write $\xi = \sum_{x \in M} c_x \hat{x}$. Let $\xi_0 = \sum_{x \in M_0} c_x \hat{x} \in \mathbf{Z}M_0$, and $\xi_1 = \sum_{x \notin M_0} c_x \hat{x} \in \mathbf{Z}(M \setminus M_0)$. Let $y = \pi(\xi_1)$. Since $\pi(\xi) = 0$ we have $y = -\pi(\xi_0) \in M_0$. Moreover,

$$ty = t\pi(\xi_1) = \pi(t\xi_1) = \pi(0) = 0,$$

since $t(\xi) \in N_{M_0}$ and $\mathbf{Z}(M \setminus M_0)$ is invariant for $\alpha$. It follows that $\alpha \hat{y} = \hat{\alpha} \hat{y} = \hat{y}$, so that $t\hat{y} = 0$. Let $\eta_0 = \xi_0 + \hat{y} \in \mathbf{Z}M_0$. Then

$$\pi(\eta_0) = \pi(\xi_0) + y = \pi(\xi_0) + \pi(\xi_1) = 0,$$

so $\eta_0 \in N_{M_0}$. Finally,

$$t\eta_0 = t\xi_0 = t\xi.$$

\[\square\]
Lemma 1.8. Let $M_0 \subseteq M$ be an inclusion of modules, and suppose that Theorem 1.2 is true for $M$. Then it is true for $M_0$.

Proof. By the discussion following Definition 1.5, it suffices to show that $\ker(s) \cap N_{M_0} = tN_{M_0}$, assuming that this is true for $N_M$. So let $\xi_0 \in \ker(s) \cap N_M$. Then $\xi_0 \in \ker(s) \cap N_M$, so by hypothesis there is $\xi \in N_M$ such that $\xi_0 = t \xi$. By Lemma 1.7 there is $\eta_0 \in N_{M_0}$ with $\xi_0 = t \eta_0$. ■

Lemma 1.9. Theorem 1.2 is true for the following modules:

1. $R$.
2. $R/(q^k)$, for any prime $q$ and $k > 0$.
3. Any trivial module.

Proof. (1) Let $B = \{e_i : 0 \leq i < p\}$ be the standard basis of $R$ (so $e_i = \alpha^i$). For $x = \sum_{i=0}^{p-1} x_i e_i \in R \setminus B$ let

$$\xi_x = \hat{x} - \sum_{i=0}^{p-1} x_i \hat{e}_i.$$ 

We claim that $\{\xi_x : x \in R \setminus B\}$ is a $\mathbb{Z}$-basis for $N_R$. To see this, note first that the $\hat{x}$ term in $\xi_x$ implies that the collection is linearly independent (over $\mathbb{Z}$). To see that it spans, let $\xi = \sum_{x \in R} c_x \hat{x} \in N_R$. Then

$$\xi = \sum_{x \in B} c_x \xi_x + \sum_{i=0}^{p-1} \left( c_{e_i} + \sum_{x \notin B} x_i c_x \right) \hat{e}_i.$$ 

Applying $\pi$ we find that for each $i$,

$$c_{e_i} + \sum_{x \notin B} x_i c_x = 0.$$ 

Hence $\xi = \sum_{x \notin B} c_x \xi_x$. For $x \notin B$ we have $\alpha \xi_x = \xi_{\alpha x}$. Therefore the partition of $R \setminus B$ into $\alpha$-orbits determines a decomposition of $N_R$ as a direct sum of a free and a trivial module.

(2) Let $M = R/(q^k)$. Let $B$ and $\xi_x$ for $x \in R \setminus B$ be as in the proof of part (1). Then $\{\xi_x : x \notin B\} \cup \{q^k \hat{e}_i : 0 \leq i < p\}$ is a basis for $N_M$. The proof is identical to the proof in part (1), except that the last computation yields, for each $i$,

$$c_{e_i} + \sum_{x \notin B} x_i c_x \equiv 0 \pmod{q^k}.$$ 

Letting this number be denoted $a_i q^k$ we find that $\xi = \sum_{x \notin B} c_x \xi_x + \sum_{i=0}^{p-1} a_i (q^k \hat{e}_i)$. Since $\alpha(q^k \hat{e}_i) = (q^k \hat{e}_{\alpha i} + 1)$, the argument in part (1) shows that $N_M$ is the direct sum of a free and a trivial module.

(3) If $M$ is a trivial module, then so is $N_M$. ■
Lemma 1.10. Theorem 1.2 is true for any finitely generated indecomposable module.

Proof. By Lemma 1.6 and Lemma 1.8 it suffices to prove that any finitely generated indecomposable module is a submodule of a direct sum of modules of the types considered in Lemma 1.9. We rely on the paper [8] of Levy describing all finitely generated indecomposable $R$-modules. (See also [9].) Following [8], we may realize $R$ as a pullback:
\[
R = \{R_1 \overset{\nu_1}{\to} C_p \overset{\nu_2}{\to} R_2\},
\]
where $R_1 = \mathbb{Z}$ and $R_2 = \mathbb{Z}[\zeta]$, $\zeta$ a primitive $p$th root of unity. The maps $\nu_i$ are defined by $\nu_1(1) = 1$ and $\nu_2(\zeta) = 1$, and the generator of $C_p \subseteq R$ is $\alpha = (1, \zeta)$. We let $P_i = \ker \nu_i$, so that $P_1 = p\mathbb{Z}$ and $P_2 = (\zeta - 1)\mathbb{Z}[\zeta]$, and we set $P = P_1 \oplus P_2 \subseteq R$.

Levy calls an $R$-module $M$ $P$-mixed if each torsion element of $M$ has order ideal containing a power of $p$; (equivalently, if the torsion subgroup of the abelian group $M$ is $p$-primary). Proposition 1.3 of [8] states that every finitely generated $R$-module is of the form $M_0 \oplus M_1 \oplus M_2$, where $M_0$ is $P$-mixed, and for $i = 1, 2, M_i$ is an $R_i$-torsion module with no $p$-primary component. It suffices to prove the lemma separately for indecomposable modules of the three types.

We first consider the case of $P$-mixed modules. It is proved in section 1 of [8] that all finitely generated indecomposable $P$-mixed $R$-modules are of two types: deleted cycle and block cycle. We first treat the special case of deleted cycle indecomposables called basic building blocks. A basic building block is a pullback of $R$-modules of the form
\[
M = \{S_1 \overset{f_1}{\to} C_p \overset{f_2}{\to} S_2\},
\]
where for $i = 1, 2$, $S_i = R_i/P_i^{c_i}$ for some $c_i \geq 0$, or $S_2$ is a nonprincipal ideal in $R_2$. We note the following inclusions of $R$-modules.

(i) $M \subseteq S_1 \oplus S_2$.
(ii) $S_2 \subseteq R_2$ when $S_2 \not\subseteq R_2$.
(iii) $R_2 = \mathbb{Z}[x]/(s) \cong t\mathbb{Z}[x]/(ts) = tR \subseteq R$

(\text{thus } f(\zeta) \in R_2 \implies tf(\alpha) \in R).

(iv) For $0 < c_i \leq d_i$, $R_i/P_i^{c_i} \cong P_i^{d_i-c_i}/P_i^{d_i} \subseteq R_i/P_i^{d_i}$.

(v) $R_2/P_2^{k(p^k-1)} = R_2/(p^k) \hookrightarrow R/(p^k)$.

Items (i) – (v) finish the case of a basic building block. We remark that it follows from (iv) that every basic building block which is finite is contained in a module of the form $\mathbb{Z}/(p^k) \oplus R/(p^k)$ for any large enough $k$. To prove (v), we claim that $p$ and $(\zeta - 1)^{p^k-1}$ generate the same ideal in $R_2$. This follows from the following lemma.

Lemma 1.11. There exist $f, g, h \in \mathbb{Z}[x]$ such that $h(1) = -1$, and

1. $t^{p-1} = ph + s$.
2. $p = -t^{p-1} + ptf + sg$.

Proof. Note that $ts = x^p - 1$. Since all but the first and last terms of $t^p$ have coefficients divisible by $p$, there exists $h \in \mathbb{Z}[x]$ such that $t^p - ts = ph$, and hence $t^{p-1} = ph + s$, proving (1). Setting $x = 1$ we find that $h(1) = -1$. Therefore $h = t\beta - 1$, for some $\beta \in \mathbb{Z}[x]$. Substituting for $h$ gives

\[
p = -t^{p-1} + pt\beta + s.
\]
We may replace the coefficient \( p \) on the right by the entire expression on the right. Repeating this procedure \( p - 1 \) times gives equation (2).

We continue with the proof of Lemma 1.10. To describe the remaining two types of indecomposable modules [8] uses the (unique) submodules of \( R_1/P_1^n \) and \( R_2/P_2^n \) isomorphic to \( C_\rho = \mathbb{Z}/(p) \). In the first case, the submodule of \( \mathbb{Z}/(p^n) \) is generated by \( \zeta \) (the coset of) \( p^{n-1} \). In the second case, the submodule of \( \mathbb{Z}[\zeta]/((\zeta - 1)^n) \) is generated by \( \zeta \) (the coset of) \( (\zeta - 1)^{n-1} \). In the case \( n = k(p - 1) \), we compute the image of \( (\zeta - 1)^{n-1} \) in \( R/(p^k) \) under the inclusion (v) above. From inclusion (iii) above we have \( (\zeta - 1)^{n-1} \implies (\alpha - 1)^n = t^n \). Note that for any \( f \in \mathbb{Z}[\alpha] \), \( fs = f(1)s \). From Lemma 1.11 (1), we find that

\[
i^{k(p-1)} = (s + ph)^k = p^k h_k + \sum_{j=1}^{k} \binom{k}{j} s^{j} p^{k-j} h^{k-j}
= p^k h_k + \sum_{j=1}^{k} \binom{k}{j} s^{j} p^{k-j} (-1)^{k-j} = p^k h_k + (-1)^{k-1} p^{k-1} s.
\]

Thus \( (\zeta - 1)^{(k(p-1)-1)} \implies (-1)^{k-1} p^{k-1} s \).

Let \( S = \{S_1 \xrightarrow{f_1} C_p \xleftarrow{f_2} S_2\} \) be a basic building block. If \( S_1 = \mathbb{Z}/(p^n) \) we define a module map \( \lambda: \mathbb{Z}/(p) \to S \) by \( \lambda(j) = (jp^{n-1}, 0) \). If \( S_2 = \mathbb{Z}[\zeta]/((\zeta - 1)^n) \) we define \( \rho: \mathbb{Z}/(p) \to S \) by \( \rho(j) = (0, j((-1)_{n-1}) \).

Now let \( S_j = \{S_{1j} \xrightarrow{f_{1j}} C_p \xleftarrow{f_{2j}} S_{2j}\}, j = 1, \ldots, n \), be basic building blocks, and assume that \( S_{1j} \) is finite for \( j > 1 \) and that \( S_{2j} \) is finite for \( j < n \). Let \( \lambda_j, \rho_j: \mathbb{Z}/(p) \to S_j \) be as above, when defined. The deleted cycle indecomposable \( M \) is constructed by successive push-outs:

\[
M_1 = \{S_1 \xrightarrow{f_1} C_p \xleftarrow{\lambda_1} S_2\}
M_2 = \{M_1 \xrightarrow{f_2} C_p \xleftarrow{\lambda_2} S_3\}
\ldots
M = \{M_{n-1} \xrightarrow{f_n} C_p \xleftarrow{\lambda_n} S_n\},
\]

where for \( 2 \leq j \leq n \) we have used \( \rho_j \) also to denote the composition \( C_p \xrightarrow{\lambda_j} S_j \xleftarrow{f_j} M_{j-1} \). Using the inclusions (i) - (v) we may choose \( k \) such that

\[
S_1 \xrightarrow{} S_{11} \oplus R/(p^k)
S_j \xrightarrow{} \mathbb{Z}/(p^k) \oplus R/(p^k), 1 < j < n
S_n \xrightarrow{} \mathbb{Z}/(p^k) \oplus S_{2n}.
\]

Then in order to embed \( M \) into a direct sum, it suffices to consider the pushout \( \{R/(p^k) \xleftarrow{f^k} C_p \xrightarrow{\lambda^k} \mathbb{Z}/(p^k)\} \), where the map \( \rho \) here is obtained by composing the map \( \rho \) defined above with the inclusion (v). Let us define an epimorphism

\[
R/(p^k) \oplus \mathbb{Z}/(p^k) \to R/(p^k) \oplus \mathbb{Z}/(p^{k-1})
\]
by \((x, y) \mapsto (x + (-1)^{k-1}y, y)\). The kernel of this map is the set of all \((x, y)\)
such that \(y = jp^{k-1}\) and \(x = j(-1)^kp^{k-1}s\), for some \(j\). In other words, \((x, y) = j(-\rho(1), \lambda(1))\). It follows that the image is isomorphic to the push-out. We thus obtain the inclusion

\[
M \hookrightarrow \tilde{M} = S_{11} \oplus \left( \frac{R}{(p^k)} \oplus \frac{Z}{(p^{k-1})} \right)^{n-1} \oplus S_{2n}.
\]

Finally we consider the block cycle indecomposables. Consider the basic building blocks \(S_1, \ldots, S_n\) as before, but assume that \(S_{11}\) and \(S_{2n}\) are also finite modules. Let \(S_{1j} = \mathbb{Z}/(p^{\nu_j})\) and \(S_{2j} = \mathbb{Z}[\zeta]/((\zeta - 1)^{\nu_j})\). To simplify the description of the inclusion, we will consider a larger class of modules, not all of which are indecomposable. Let \(M\) be the deleted cycle indecomposable constructed from \(S_1, \ldots, S_n\). Let \(a_1, \ldots, a_n \in C_p\) with \(a_1 \neq 0\). Let

\[
\omega = ((a_1p^{\nu_1-1}, 0), \ldots, (a_{n-1}p^{\nu_{n-1}-1}, 0), (a_n p^{\nu_n-1}, (\zeta - 1)^{\nu_n-1})) \in M.
\]

The block cycle indecomposable is \(M/\omega\).

Under the inclusion \(M \hookrightarrow \tilde{M}\) we find that

\[
\omega \longmapsto \tilde{\omega} = (a_1 p^{k-1}, (a_2 p^{k-1}s, 0), \ldots, (a_n p^{k-1}s, 0), p^{k-1}s).
\]

Since \(a_1 \neq 0\) there is \(b \in C_p\) such that \(a_1b = 1\). Then \((\tilde{\omega}) = (b\tilde{\omega})\). Write

\[
\tilde{M} = \frac{Z}{(p^k)} \oplus M'
\]

by separating the first summand. Then we may write \(b\tilde{\omega} = (p^{k-1}, p^{k-1}\mu)\). Define an epimorphism

\[
\tilde{M} \twoheadrightarrow \frac{Z}{(p^{k-1})} \oplus M'
\]

by \((y, x) \mapsto (y, x - y\mu)\). As in the case of a deleted cycle indecomposable, we find that the kernel of this map is generated by \(b\tilde{\omega}\), so that

\[
\frac{M}{\omega} \hookrightarrow \frac{\tilde{M}}{(b\tilde{\omega})} \cong \frac{Z}{(p^{k-1})} \oplus M' \cong \left( \frac{Z}{(p^{k-1})} \right)^n \oplus \left( \frac{R}{(p^k)} \right)^n.
\]

This concludes the proof for \(P\)-mixed indecomposables.

A finitely generated \(R_1\)-torsion module with no non-zero \(p\)-torsion elements is a finite abelian group having no \(p\)-primary component, on which the \(C_p\)-action is trivial. Such a module is a direct sum of trivial modules of the form \(\mathbb{Z}/(q^k)\) with \(q \neq p\).

A finitely generated \(R_2\)-torsion module \(M\) with no non-zero \(p\)-torsion elements is a finite module. Write \(M = \oplus_{q \neq p} M_q\) as the direct sum of its primary components. We may view \(M_q\) as a module over \(\mathbb{Z}_q[\zeta] = \mathbb{Z}_q[x]/(s) = \bigoplus_i \mathbb{Z}_q[x]/(\varphi_i)\), where \(s = \prod \varphi_i\); is the factorization of \(s(x)\) into irreducible polynomials over \(\mathbb{Z}_q\), and \(\mathbb{Z}_q\) denotes the \(q\)-adic integers. Then \(M_q = \bigoplus_i M_{q,i}\), where \(M_{q,i}\) is a module over \(\mathbb{Z}_q[x]/(\varphi_i)\). Since \(\mathbb{Z}_q[x]/(\varphi_i)\) is a principal ideal domain (every ideal is generated by a power of \(q\)), \(M_{q,i}\) is a direct sum of cyclic modules \(\mathbb{Z}_q[x]/((\varphi_i) + (q^k))\). Now,

\[
\frac{\mathbb{Z}_q[x]}{(\varphi_i) + (q^k)} \subseteq \frac{\mathbb{Z}_q[x]}{(s) + (q^k)} \subseteq \frac{\mathbb{Z}_q[x]}{(x^p - 1) + (q^k)} \cong \frac{R}{(q^k)}.
\]
Corollary 1.12. Theorem 1.2 is true for any finitely generated module.

Proof of Theorem 1.2. By Theorem 1.4 it is enough to prove that \( \ker s = \im t \) on \( N_M \). Let \( \xi = \sum_{x \in M} c_x \hat{x} \in \ker s \cap N_M \). Let \( M_0 \) be the submodule of \( M \) generated by \( \{ x \in M : c_x \neq 0 \} \). Then \( M_0 \) is finitely generated, so by Corollary 1.12, Theorem 1.2 is true for \( M_0 \). Thus \( \xi \in tN_{M_0} \subseteq tN_M \).

The above results have an unexpected further consequence for inclusions of \( R \)-modules. We first present two lemmas.

Lemma 1.13. Let \( M_0 \subseteq M \) be an inclusion of \( R \)-modules. Suppose that \( (tM) \cap M_0 = tM_0 \). Then \( N_{M_0} \) is a pure submodule of \( N_M \).

Proof. Let \( \lambda \in N_M \) and \( \lambda \in R \) with \( \lambda \xi \in N_{M_0} \). Write \( \lambda = \sum_{i=0}^{p-1} \lambda_i \alpha^i \). Let \( \xi = \xi_0 + \xi_1 \) relative to the decomposition \( ZM = ZM_0 \oplus Z(M \setminus M_0) \) of \( R \)-modules. Then \( \lambda \xi_1 = 0 \). We first consider the case where \( t \) divides \( \lambda \). Write \( \lambda = t^j \mu \) where \( j > 0 \), \( t \) does not divide \( \mu \), and the degree of \( \mu \) is less than \( p - 1 \). Let \( z \in \supp \xi_1 \). If \( z \) is not a fixed point of \( \alpha \) let \( c_i \) be the coefficient of \( \alpha^i z \) in \( \xi_1 \). We have

\[
0 = \lambda \sum_{i=0}^{p-1} c_i \alpha^i z = \sum_{i,j} c_i \lambda_j \alpha^i \alpha^j z = \sum_i \left( \sum_j c_{i-j} \lambda_j \right) \alpha^i z.
\]

It follows that for all \( i \), \( \sum_j c_{i-j} \lambda_j = 0 \). Letting \( W \) be the \( p \times p \) matrix corresponding to the cyclic permutation of the standard basis of \( Z^p \), we have that \( f(W)c = 0 \), where \( c = (c_0, \ldots, c_{p-1})^T \) and \( f(x) = \sum_{i=0}^{p-1} \lambda_i x^i \). Since the degree of \( \mu \) is less than \( p - 1 \), \( \mu(W) \) is an injective linear operator. It follows that \( (W - I)c = 0 \), and hence that \( c_0 = c_1 = \cdots = c_{p-1} \). Then \( \pi(\sum_{i=0}^{p-1} c_i \alpha^i z) = c_0 \sum_{i=0}^{p-1} \alpha^i z \) is a fixed point for \( \alpha \) in \( M \). If \( z \) is a fixed point of \( \alpha \) then also \( \pi(c_0 z) = c_0 z \) is a fixed point. Thus \( \pi(\xi_1) = \pi(\xi_0) \in M_0 \) is a fixed point of \( \alpha \). Then let \( \eta = \xi_0 + \pi(\xi_1) \in N_{M_0} \), and we have that \( t\eta = t\xi_0 = t\xi_1 \).

Now suppose that \( t \) does not divide \( \lambda \). Then there can be no fixed points in \( \supp(\xi_1) \). For, if \( z \) is a fixed point of \( \alpha \), then \( \lambda c_0 \hat{z} = 0 \), and hence \( \sum_{i=0}^{p-1} \lambda_i = 0 \). It follows that

\[
\lambda = \sum_{i=0}^{p-1} \lambda_i \alpha^i = \sum_{i=0}^{p-1} \left( \sum_{j=0}^{i} \lambda_j - \sum_{j=0}^{i-1} \lambda_j \right) \alpha^i \text{, indices taken modulo } p, \\
= \sum_{i=0}^{p-1} \left( \sum_{j=0}^{i} \lambda_j \right) (\alpha^i - \alpha^{i+1}) \\
= (1 - \alpha) \sum_{i=0}^{p-1} \left( \sum_{j=0}^{i} \lambda_j \right) \alpha^i,
\]

and hence \( t \) divides \( \lambda \). Again let \( z \in \supp \xi_1 \), and let \( c_i \) be the coefficient of \( \alpha^i z \) in \( \xi_1 \). We again have \( f(W)c = 0 \). Since \( c \neq 0 \), \( f(x) \) must vanish at some point in the
Let \( x - \text{generated. Let } tM + \text{and also free as an abelian group.} \)

Given \( \xi \in \Sigma \), generated by \( \{ \xi \} \). Since \( (tM) \cap M = tM_0 \), there is \( w_0 \in M_0 \) such that \( tw = tw_0 \). Let \( \eta = \xi_0 + tw_0 \). Then \( \eta \in N_{M_0} \) and \( s\eta = s\xi_0 = s\xi \).

**Lemma 1.14.** Let \( M_0 \subseteq M \) be an inclusion of \( R \)-modules. Suppose that

\[
0 \rightarrow N \rightarrow P \xrightarrow{\pi} M \rightarrow 0
\]

is an exact sequence of \( R \)-modules such that \( N \) and \( P \) are non-cyclotomic generalized lattices. Let \( P_0 \subseteq P \) be a submodule with \( \pi(P_0) = M_0 \), and set \( N_0 = P_0 \cap N \).

Suppose further that \( P_0 \) is a direct summand of \( P \). If \( (tM) \cap M_0 \neq tM_0 \) then \( N_0 \) is not a pure submodule of \( N \).

**Proof.** Let \( P = P_0 \oplus P_1 \). Choose \( z \in M \) such that \( tz \in M_0 \setminus (tM_0) \). Let \( \delta \in M \) with \( \pi(\delta) = z \). Write \( \delta = \delta_0 + \delta_1 \) with \( \delta_i \in P_i \). Let \( x = \pi(\delta_1) \in z + M_0 \). Note that \( tx = t\pi(\delta_1) = tx - t\pi(\delta_0) \in M_0 \). Choose \( \zeta_0 \in P_0 \) with \( \pi(\zeta_0) = tx \). Let \( \xi = \xi_0 - t\delta_1 \). Then \( \pi(\xi) = tx - tx = 0 \), so \( \xi \in N \). Notice that \( \xi = \xi_0 - t\delta_1 \) is also the decomposition of \( \xi \) in \( P_0 \oplus P_1 \). We have \( s\xi = s\zeta_0 \in P_0 \cap N = N_0 \). We claim that \( s\xi \not\in sN_0 \). To see this, suppose to the contrary that there is \( \eta \in N_0 \) with \( s\xi = s\eta \).

Then \( \xi - \eta \in \ker s \cap N = tN \), since \( N \) was assumed to be non-cyclotomic. Choose \( \mu \in N \) such that \( \xi - \eta = t\mu \). Write \( \mu = \mu_0 + \mu_1 \) with \( \mu_i \in P_i \). Then \( t\mu_1 + t\delta_1 \in P_1 \), and also

\[
t\mu_1 + t\delta_1 = t\mu - t\mu_0 + t\delta_1
\]

\[
= \xi - \eta - t\mu_0 + t\delta_1
\]

\[
= \zeta_0 - \eta - t\mu_0 \notin P_0.
\]

Therefore \( t(\mu_1 + \delta_1) = 0 \). Let \( y = \pi(\mu_1 + \delta_1) = \pi(\mu_1) + x \). Then \( ty = 0 \). Since \( x - y = -\pi(\mu_1) = \pi(\mu_0) \in M_0 \), we have \( tx = t(x - y) \in tM_0 \). But then \( tz = tx + t\pi(\delta_0) \in tM_0 \), a contradiction. 

**Example 1.15.** Consider the inclusion \( M_0 \subseteq M \), where \( M = R \) and \( M_0 = tR \cong \mathbb{Z}[\xi] \) (see item (iii) in the proof of Lemma 1.10). We have \( (tM) \cap M_0 = tR \supseteq t^2R = tM_0 \).

**Lemma 1.16.** Let \( M_0 \subseteq M \) be \( R \)-modules with \( M \) countable. Then \( N_{M_0}/N_{M_0} \) is free as an abelian group.

**Proof.** From the proof of Lemma 1.13 we see that \( N_{M_0} \) is always a pure subgroup of \( N_M \). For the rest of this argument, we consider the modules only as abelian groups. Since \( N_M \) is torsion-free, it follows that \( N_{M_0}/N_{M_0} \) is torsion-free. From [6], exercise 52, it suffices to show that every finite rank subgroup of \( N_{M}/N_{M_0} \) is finitely generated. Let \( \xi_1, \ldots, \xi_n \in N_M \), and let \( H \) be the finite-rank subgroup of \( N_M/N_{M_0} \) generated by \( \{ \xi_j + N_{M_0} : 1 \leq j \leq n \} \) over \( \mathbb{Q} \). Write \( \xi_j = \xi_j^0 + \xi_j^1 \) relative to \( \mathbb{Z}M = \).
Let \( \mathbb{Z}M_0 \oplus \mathbb{Z}(M \setminus M_0) \). Let \( E = \bigcup_{j=1}^{\infty} \text{supp} \xi_j \). Then \( E \) is a finite subset of \( M \setminus M_0 \). Let \( G = \{ \xi \in N_M : \text{supp} \xi \subseteq E \cup M_0 \} \). Let \( \eta \in N_M \cap \text{span} \{ \xi_1, \ldots, \xi_n \} + N_{M_0} \). Then there are an integer \( b \neq 0 \) and \( \mu \in N_{M_0} \) such that \( b\eta \in \mu + \text{span} \{ \xi_1, \ldots, \xi_n \} \). Hence there are integers \( a_x \) for \( x \in E \) such that \( b\eta = \sum_{x \in E} a_x \hat{x} \). Hence \( b|a_x \) for all \( x \in E \), so \( \eta \in G \). Therefore \( H \subseteq G / N_{M_0} \subseteq (\mathbb{Z}E + N_{M_0}) / N_{M_0} \), and hence \( H \) is finitely generated.

**Theorem 1.17.** Let \( M_0 \subseteq M \) be \( \mathbb{R} \)-modules with \( M \) countable, and assume that \( (tM) \cap M_0 = tM_0 \). Then \( N_{M_0} \) is a direct summand of \( N_M \).

*Proof.* From Lemma 1.16 and Theorem 1.2 we know that \( N_M / N_{M_0} \) is a direct sum of finitely generate projective modules and copies of the trivial module \( R / (t) \). Since \( N_{M_0} \) is a pure submodule of \( N_M \), by Lemma 1.13, \( N_{M_0} \) is a direct summand of \( N_M \) (as in [6], Notes, section 7).

We now have the following generalization of Theorem 1.3 to inclusions of modules.

**Corollary 1.18.** Let \( M_0 \subseteq M \) be an inclusion of \( \mathbb{R} \)-modules with \( M \) countable. The following are equivalent:

1. There exists a commutative diagram of \( \mathbb{R} \)-modules with exact rows and injective columns
   
   \[
   \begin{array}{c}
   0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0 \\
   0 \rightarrow N_0 \rightarrow P_0 \rightarrow M_0 \rightarrow 0 \\
   \end{array}
   \]

   such that \( N, N_0, P \) and \( P_0 \) are direct sums of free and trivial modules, and such that \( N_0 \), respectively \( P_0 \), is a direct summand of \( N \), respectively \( P \).

2. \( (tM) \cap M_0 = tM_0 \).

*Proof.* If \( (tM) \cap M_0 = tM_0 \), then by Theorem 1.17 we may take \( N_0 = N_{M_0} \) and \( P_0 = \mathbb{Z}M_0 \) and construct such a diagram in which \( N \) and \( P \) (and hence also \( N_0 \) and \( P_0 \)) are non-cyclotomic generalized lattices. The proof of Theorem 1.3 can then be used to add free summands to make \( N, N_0, P \) and \( P_0 \) direct sums of free and trivial modules. If \( (tM) \cap M_0 \neq tM_0 \) then by Lemma 1.14 there can be no such diagram. (We remark that this direction does not require that \( M \) be countable.)

2. **Graphs representing Kirchberg algebras**

We will consider the following situation. Let \( G \) be an abelian group. (In our main application, \( G \) will be countable. However the construction does not require this.) Let \( \Gamma \) be a subgroup of the group of automorphisms of \( G \). Let \( A \) be a \( \Gamma \)-set and \( \pi_0 : A \rightarrow G \) an equivariant map whose range generates \( G \). Define \( \pi : \mathbb{Z}A \rightarrow G \) by \( \pi(\sum_{a \in A} c_a \tilde{a}) = \sum_{a \in A} c_a \pi_0(a) \), where \( \{ \tilde{a} : a \in A \} \) is the canonical basis of \( \mathbb{Z}A \). Then \( \pi \) is a surjective equivariant homomorphism (where the action of \( \Gamma \) on \( \mathbb{Z}A \) is defined by \( \gamma \cdot \tilde{a} = \tilde{\gamma \cdot a} \)). Then \( \ker \pi \) is a subgroup of \( \mathbb{Z}A \) and hence is free abelian. Let \( B \) be a free basis for \( \ker \pi \). We obtain a free presentation of \( G \):

\[
(2.1) \quad 0 \rightarrow \mathbb{Z}B \rightarrow \mathbb{Z}A \xrightarrow{\pi} G \rightarrow 0,
\]

where the map \( \mathbb{Z}B \rightarrow \mathbb{Z}A \) is defined by \( \tilde{b} \mapsto b \). If the basis \( B \) for \( \ker \pi \) can be chosen to be \( \Gamma \)-invariant, then the sequence (2.1) is equivariant. For an element
c = \sum_{a \in A} c_a \hat{a} \in \mathbb{Z}A \text{ we will let } c^{\pm}_a = \pm \max\{\pm c_a, 0\}. \text{ We will also view } c, c^+ \text{ and } c^- \text{ as functions from } A \text{ to } \mathbb{Z}. \text{ In working with graph algebras we will follow [14] in letting vertices of a graph also represent the corresponding projections in the graph algebra.}

\textbf{Theorem 2.1.} Let } G \text{ be an abelian group, let } \Gamma \text{ be a subgroup of the group of automorphisms of } G, \text{ and let } \pi_0 : A \to G \text{ be an equivariant map of a } \Gamma \text{-set } A \text{ to } G \text{ with range generating } G. \text{ Define } \pi \text{ as above, and assume that the basis } B \text{ for } \text{ker } \pi \text{ is } \Gamma \text{-invariant. Then there is a directed graph } E \text{ with the following properties:}

\begin{enumerate}
\item \(E\) is countable if \(G\) is countable.
\item \(E\) is irreducible.
\item There is a unique vertex \(v\) emitting infinitely many edges.
\item \(\Gamma\) acts as automorphisms of \(E\), with \(v\) a fixed point.
\item There is a \(\Gamma\)-equivariant injective map \(A \to E^0\).
\item \(K_0\mathcal{O}(E) \cong G\) and \(K_1\mathcal{O}(E) = (0)\).
\item The isomorphism of \(K_0\mathcal{O}(E)\) with \(G\) is defined by \([a] \mapsto \pi_0(a)\), for \(a \in A\).
\end{enumerate}

\textbf{Proof.} We describe the graph \(E\) in pieces of four types:

\begin{itemize}
\item \(E_A(a)\), for \(a \in A\).
\item \(E_B(b)\), for \(b \in B\).
\item \(E_{AB}(a, b)\), for \(b \in B\) and \(a \in \text{supp } b\).
\item \(E_v\).
\end{itemize}

The four types are depicted in figures 1 – 4. A schematic of how the graph \(E\) is assembled from the pieces is given in figure 5. We remark that in \(E_B(b)\) the vertex \(z_b^\pm\) is present if and only if \(b^\pm \neq 0\) (as a function on \(A\)), and in \(E_{AB}(a, b)\) only half of the pictured graph is present (namely, the half for which the number of edges is nonzero).

We give a brief explanation for the structure of the graph. The purpose of the graphs \(E_{AB}(a, b)\), and the portion of \(E_B(b)\) near the vertex \(z_b\), is to impose the relations \(B\) on the elements \([a] : a \in A\) in \(K_0\mathcal{O}(E)\). (This is a variation on a device used by Szymański, [16].) The loop at the vertex \(a\) in \(E_A(a)\) leaves \([a]\) otherwise unrestricted. The purpose of \(E_A(a)\) is to trivialize the contribution of the vertex \(a\) in \(K_1\mathcal{O}(E)\). The purpose of the right portion of \(E_B(b)\) is to trivialize the class \([z_b]\) in \(K_0\mathcal{O}(E)\). The purpose of \(v\) is to make \(E\) transitive, without affecting the \(K\)-theory. Also, the construction in [15] requires that there be a distinguished vertex emitting infinitely many edges. The vertex \(v\) plays this role. The purpose of \(E_v\) is to trivialize the class \([v]\) in \(K_0\mathcal{O}(E)\). Finally, the loop at the vertex \(a\) imposes conditions in \(K_1\) at the vertices \([z_b] : b \in B\). It is to satisfy these conditions that we are forced to choose the basis \(B\) for \(\text{ker } \pi\) in the first place. It is the difficulty of finding a \(\Gamma\)-invariant basis for \(\text{ker } \pi\) that stands in the way of using the methods of this paper to lift larger groups of automorphisms from the \(K\)-theory of a Kirchberg algebra.
Figure 1. $E_A(a)$

Figure 2. $E_B(b)$

Figure 3. $E_{AB}(a, b)$

Figure 4. $E_c$
The action of $\Gamma$ on $E$ is defined by permuting the pieces of types $E_A$, $E_B$ and $E_{AB}$ according to the actions of $\Gamma$ on $A$ and $B$, and is defined to be trivial on $E_v$. Then properties (1) – (5) are obvious. We compute the $K$-theory of $\mathcal{O}(E)$ by the following formulas. A simple proof may be found in [4]. In the case of a graph without sinks these formulas are equivalent to those given in [5]. (See also [17].)

\[
K_0 \mathcal{O}(E) = \frac{C_c(E^0, \mathbb{Z})}{\langle \delta_x - \sum_{e \in E^1, o(e)=x} \delta_{t(e)} : 0 < \#E^1(x) < \infty \rangle}
\]

(2.2)

\[
K_1 \mathcal{O}(E) = \{ f \in C_c(E^0, \mathbb{Z}) : f(x) = \sum_{e \in E^1, t(e)=x} f(o(e)) \text{ if } 0 < \#E^1(x) < \infty, \]

\[
f(x) = 0 \text{ if } \#E^1(x) = 0 \text{ or } \infty \}
\]

In $K_0$ we let $[x]$ denote the equivalence class of $\delta_x$. We first compute $K_0 \mathcal{O}(E)$.

From $E_B(b)$ we find

\[
[y_{b,i}] = [y_{b,i}] + [y_{b,i-1}],
\]

where we let $z_b = y_{b,0}$. Thus

\[
[z_b] = [y_{b,i}] = 0, \quad i \geq 1.
\]

We have further

\[
0 = [z_b] = [z_b^+] + [z_b^-].
\]

(2.3)

For each $b \in B$ we consider $\{ E_{AB}(a, b) : a \in \text{supp}(b) \}$, together with the leftmost portion of $E_B(b)$, to find that

\[
[z_b^+] = \sum_a b_a^+[a],
\]

\[
[z_b^-] = 2[z_b^-] + \sum_a b_a^-[a].
\]

Combining these with (2.3) gives for each $b \in B$,

\[
0 = \sum_a b_a^+[a] - \sum_a b_a^-[a] = \sum_a b_a[a].
\]

(2.4)

Consideration of $E_v$ gives $[c_i] = [c_i] + [c_{i-1}]$, where we let $v = c_0$, and hence

\[
[v] = [c_i] = 0, \text{ for all } i.
\]

(2.5)
Consideration of \( E_A(a) \) gives \([x'_{a,i}] = 2[x'_{a,i}] + \{v\} \), and hence with (2.5) we get
\[
[x'_{a,i}] = -[v] = 0.
\]
We also have \([a] = [a] + [x_{a,i}] \) and \([x_{a,i}] = [x_{a,i}] + [x'_{a,i}] + [v] \), hence
\[
(x_{a,i}) = 0, \, \text{for all } i.
\]
We thus find that
\[
K_0\mathcal{O}(E) = \left\{ \{a : a \in A\} \left| \{\sum_{a \in A} b_a[a] : b \in B\} \right. \right\} \cong G.
\]
This proves (7) and the first half of (6).

We now compute \( K_1\mathcal{O}(E) \). Let \( f \in K_1\mathcal{O}(E) \) and fix \( a \in A \). From \( E_{AB}(a,b) : b \in B \) \}, and the loop at \( a \) in \( E_A(a) \), we have
\[
f(a) = f(a) + \sum_{b} (b_a^+ f(z_b^+) + b_a^- f(z_b^-)),
\]
and hence
\[
\sum_{b} (b_a^+ f(z_b^+) + b_a^- f(z_b^-)) = 0.
\]
From \( E_B(b) \) we find \( f(z_b^+) = f(z_b) \) and \( f(z_b^-) = 2f(z_b^-) + f(z_b) \), and hence
\[
f(z_b^+) = f(z_b) = -f(z_b^-).
\]
Combining (2.7) and (2.8) gives \( \sum_{b} b_a f(z_b) = 0 \), for \( a \in A \). Viewing \( b \) as an element of \( ZA \) gives
\[
\sum_{b} f(z_b) b = 0.
\]
Since \( B \) is a linearly independent subset of \( ZA \) we conclude that
\[
f(z_b) = 0, \, b \in B.
\]
Since \( E^1(v) \) is infinite we have \( f(v) = 0 \). We also have \( 0 = f(z_b) = f(y_{b,1}). \) For \( i \geq 1 \) we have \( f(y_{b,i}) = f(y_{b,i}) + f(y_{b,i+1}) + f(v) \), and hence \( f(y_{b,i+1}) = 0 \) for \( i \geq 1 \). Thus \( f(y_{b,i}) = 0 \) for all \( i \).

From \( E_A(a) \) we have \( f(x_{a,1}) = f(x_{a,1}) + f(v) \) and \( f(x_{a,i}) = f(x_{a,i}) + f(x_{a,i-1}) \), and hence
\[
f(a) = f(x_{a,i}) = 0.
\]
We have \( f(x'_{a,i}) = 2f(x'_{a,i}) + f(x_{a,i}) \), so
\[
f(x'_{a,i}) = -f(x_{a,i}) = 0.
\]
Consideration of \( E_v \) gives \( f(c_i) = f(c_i) + f(c_{i+1}) + f(v) \), hence \( f(c_{i+1}) = 0 \). Finally, consideration of \( v \) gives
\[
0 = f(v) = f(c_1) + \sum_{i} (f(x_{a,i}) + f(x'_{a,i})),
\]
so that \( f(c_1) = 0 \). Therefore \( f = 0 \), and we have \( K_1\mathcal{O}(E) = (0) \). This concludes the proof of the theorem.

We remark that in the next theorem, if the groups are not countable then the result is a simple purely infinite nuclear \( C^* \)-algebra in the UCT class.
Theorem 2.2. Let $G_0$ and $G_1$ be countable abelian groups, let $\Gamma_i$ be a subgroup of $\text{Aut}(G_i)$, let $\pi_{0,i}: A_i \rightarrow G_i$ be an equivariant map of a $\Gamma_i$-set $A_i$ to $G_i$ with range generating $G_i$, let $\pi_i: Z A_i \rightarrow G_i$ be the associated homomorphism as defined before Theorem 2.1, and assume that a $\Gamma_i$-invariant basis $B_i$ for $\ker \pi_i$ exists. Then there are a non-unital Kirchberg algebra $\Theta$ in the UCT class and a homomorphism $\theta: \Gamma_0 \times \Gamma_1 \rightarrow \text{Aut}(\Theta)$ such that $K_0(\Theta) \cong G_i$ and $\theta(\gamma_0, \gamma_1)_*= (\gamma_0, \gamma_1)_*$ for $\gamma_i \in \Gamma_i$. Moreover, if $x_0 \in A_0$ is fixed by $\Gamma_0$ then there is a full corner $\Theta_0$ of $\Theta$ such that $\Theta_0$ is invariant for $\theta(\Gamma_0 \times \Gamma_1)$ and $[1_{\Theta_0}] = \pi_{0,0}(x_0)$.

Proof. Let $E_i$ be the directed graph constructed in Theorem 2.1 from $G_i$, $\Gamma_i$, $\pi_{0,i}$ and $A_i$. Let $F_0$ be the usual graph describing $O_\infty$: one vertex $w_0$, and denumerably many loops at $w_0$. Let $F_1$ be a graph describing the (non-unital) UCT Kirchberg algebra with $K$-theory $(0, Z)$ (see figure 6. The $K$-theory of the $C^*$-algebra of this graph is easily computed using the formulas (2.2)). The theorem now follows from Theorem 2.1, Proposition 3.20 of [15], and the Künneth formula ([12]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{$K_* = (0, Z)$}
\end{figure}

Corollary 2.3. Let $A$ be a UCT Kirchberg algebra, let $\alpha$ be an automorphism of the $K$-theory of $A$ such that $\alpha^p = id$, where $p$ is prime. Then there is an automorphism $\theta$ of $A$ such that $\theta_* = \alpha$ and $\theta^p = id$.

Proof. This follows from Theorem 2.2 and the Kirchberg-Phillips classification theorem.

There are various alternative corollaries that could be stated. For example, if $A$ is a UCT Kirchberg algebra, and if $\alpha_0$ and $\alpha_1$ are prime order automorphisms of $K_0 (A)$ and $K_1 (A)$ respectively, then there are commuting automorphisms $\theta_0$ and $\theta_1$ of $A$ such that $\theta_* = (\alpha_0, id)$, $\theta_1_* = (id, \alpha_1)$, and $\theta$ has the same order as $\alpha_i$.

The results on $R$-modules from section 1 have implications for inclusions of Kirchberg algebras.

Theorem 2.4. Let $G_0$ and $G_1$ be countable abelian groups and let $\alpha_i$ be an automorphism of $G_i$ having prime order $p_i$. Let $H_i$ be a subgroup of $G_i$ invariant for $\alpha_i$. Let $t_i = \alpha_i - 1$, and suppose that $(t_i G_i) \cap H_i = t_i H_i$ for $i = 0, 1$. Then there is an inclusion of UCT Kirchberg algebras $\iota: B \hookrightarrow A$, and commuting automorphisms $\theta_0$ and $\theta_1$ of $A$, such that $K_* (A) = (G_0, G_1)$, $\iota_*: K_i (B) \rightarrow K_i (A)$ is the inclusion $H_i \subseteq G_i$, and

\[
\begin{align*}
\theta_0 &= \alpha_0, \\
\theta_1 &= \alpha_1,
\end{align*}
\]

and

\[
\begin{align*}
\theta_i^p &= id, \\
\theta_i (B) &= B, \\
\theta_0 &= (\alpha_0, id), \\
\theta_1 &= (id, \alpha_1).
\end{align*}
\]
Moreover, if \( x_0 \in H_0 \) is fixed by \( \alpha_0 \), then the algebras \( A \) and \( B \) and the inclusion \( \iota \) may be taken to be unital, and such that \( [1_A] = x_0 \).

Proof. This follows from Corollary 1.18 and Theorem 2.2.

**Example 2.5.** There is a serendipitous extension of Theorem 2.4 covering, in particular, Example 1.15. In figure 7 we show a graph similar to that of figure 6, but with \( p+1 \) strands attached at the central vertex. The \( C^* \)-algebra of this graph has trivial \( K_0 \), and \( K_1 \) isomorphic to \( \mathbb{Z}^p \), given by \( \{ f \in C_c(\{x_{0,1}, \ldots, x_{p,1}\}) : \sum_{i=0}^p f(x_{i,1}) = 0 \} \). We define an order \( p \) automorphism of the graph by cyclically permuting the strands indexed \( 1, \ldots, p \), and fixing the strand indexed 0. Thus \( K_1 \) becomes the module \( R \). Since the central vertex emits infinitely many edges, the subgraph obtained by deleting the strand indexed 0 has \( C^* \)-algebra contained in the \( C^* \)-algebra of the whole graph. (Its \( C^* \)-algebra is isomorphic to the relative Toeplitz algebra it determines (see [13], Theorem 2.35).) This subgraph is invariant for the automorphism, has \( C^* \)-algebra with trivial \( K_0 \) and \( K_1 \cong \mathbb{Z}^{p-1} \), and together with the automorphism the \( K_1 \) group becomes the module \( \mathbb{Z}[\zeta] \). The modules may be moved to \( K_0 \) by forming the product 2-graph with the graph in figure 6. Moreover, for any group \( G \) with an order \( p \) automorphism, the inclusion \( G \oplus \mathbb{Z}^{p-1} \subseteq G \oplus \mathbb{Z}^p \) may be treated by forming the product 3-graph of the above 2-graph with the graph for \( G \) constructed in Theorem 2.1. We conjecture that Theorem 2.4 holds for any equivariant inclusion of abelian groups.

![Figure 7](image-url)  
**Figure 7.** \( K_* = (0, \mathbb{Z}^p) \)

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