3-dimensional affine hypersurfaces admitting a pointwise $SO(2)$- or $\mathbb{Z}_3$-symmetry

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November 5, 2018

Abstract

In (equi-)affine differential geometry, the most important algebraic invariants are the affine (Blaschke) metric $h$, the affine shape operator $S$ and the difference tensor $K$. A hypersurface is said to admit a pointwise symmetry if at every point there exists a linear transformation preserving the affine metric, the affine shape operator and the difference tensor $K$. In this paper, we consider the 3-dimensional positive definite hypersurfaces for which at each point the group of symmetries is isomorphic to either $\mathbb{Z}_3$ or $SO(2)$. We classify such hypersurfaces and show how they can be constructed starting from 2-dimensional positive definite affine spheres.

Subject class: 53A15

Keywords: affine differential geometry, affine spheres, reduction theorems, pointwise symmetry, 3-dimensional affine hypersurfaces, Calabi product of hyperbolic affine spheres

1 Introduction

In this paper we study nondegenerate (equi-)affine hypersurfaces $F: M^n \to \mathbb{R}^{n+1}$. In that case, it is well known that there exists a canonical choice of

\hnote{*}Partially supported by the DFG-project SI 163-7

\hnote{†}Partially supported by a research fellowship of the Alexander von Humboldt Stiftung (Germany)
transversal vector field $\xi$ called the affine (Blaschke) normal, which induces a connection $\nabla$, a symmetric bilinear form $h$ and a 1-1 tensor field $S$ by

$$D_X Y = \nabla_X Y + h(X,Y)\xi,$$

$$D_X \xi = -SX,$$  

for all $X, Y \in \mathcal{X}(M)$. The connection $\nabla$ is called the induced affine connection, $h$ is called the affine metric (or Blaschke metric) and $S$ is called the affine shape operator. In general $\nabla$ is not the Levi Civita connection $\hat{\nabla}$ of $h$. The difference tensor $K$ is defined as

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y,$$  

for all $X, Y \in \mathcal{X}(M)$. Moreover the form $h(K(X,Y), Z)$ is a symmetric cubic form with the property that for any fixed $X \in \mathcal{X}(M)$, trace $K_X$ vanishes. This last property is called the apolarity condition. The difference tensor $K$, together with the affine metric $h$ and the affine shape operator are the most fundamental algebraic invariants for a nondegenerate affine hypersurface (more details in Sec. 2.1). We say that $M$ is positive definite if the affine metric $h$ is positive definite. For the basic theory of nondegenerate affine hypersurfaces we refer to [6] and [8].

A hypersurface is said to admit a pointwise symmetry if at every point there exists a linear transformation preserving the affine metric, the affine shape operator and the difference tensor $K$. The study of submanifolds which admit pointwise isometries was initiated by Bryant in [1] where he studied 3-dimensional Lagrangian submanifolds of $\mathbb{C}^3$. Following essentially the same approach, a classification of 3-dimensional affine hyperspheres admitting pointwise isometries was obtained in [11].

In [9], for 3-dimensional positive definite hypersurfaces, the possible groups which can act on the algebraic invariants as well as the canonical forms for $S$, $K$ and $h$ were computed. In this paper, we consider the 3-dimensional positive definite hypersurfaces for which at each point the group of symmetries is isomorphic to either $\mathbb{Z}_3$ or the group of rotations $SO(2)$. The paper is organized as follows. First in Section 2, we shortly recall the basic equations of Gauss, Codazzi and Ricci for an affine hypersurface and use those equations, together with the canonical form of $h$, $S$ and $K$, to obtain information about the coefficients of the connection. In particular, it follows that such a hypersurface $M$ admits a warped product structure. In Section 3, we classify such hypersurfaces by showing how they can be constructed starting from 2-dimensional positive definite affine spheres. This classification can be seen as a generalisation of the well known Calabi product of hyperbolic affine spheres and of the constructions for affine spheres considered in [1]. Note that affine
hypotheses, i.e. affine hypersurfaces for which all affine normals are parallel or pass through a fixed point, are without any doubt the most studied class of affine hypersurfaces. They are closely related to solutions of Monge Ampère equations. The following natural question for a (de)composition theorem, related to the Calabi product and its generalisations in \[4\], gives another motivation for studying 3-dimensional hypersurfaces admitting a \(Z_3\)-symmetry or an \(SO(2)\)-symmetry:

\textbf{Question.} Let \(M^n\) be a nondegenerate affine hypersurface in \(\mathbb{R}^{n+1}\). Under what conditions do there exist affine hyperspheres \(M^r_1\) in \(\mathbb{R}^{r+1}\) and \(M^s_2\) in \(\mathbb{R}^{s+1}\), with \(r + s = n - 1\), such that \(M = I \times f_1 M_1 \times f_2 M_2\), where \(I \subset \mathbb{R}\) and \(f_1\) and \(f_2\) depend only on \(I\) (i.e. \(M\) admits a warped product structure)? How can the original immersion be recovered starting from the immersion of the affine spheres?

Of course the first dimension in which the above problem can be considered is three and our study of 3-dimensional affine hypersurfaces with \(Z_3\)-symmetry or \(SO(2)\)-symmetry provides an answer in that case.

\section{Structure equations and integrability conditions}

\subsection{Preliminaries}

We consider 3-dimensional affine hypersurfaces \(F : M^3 \to \mathbb{R}^4\). Assume that \(M^3\) has at every point a \(Z_3\)-symmetry or an \(SO(2)\)-symmetry. Then we recall from [9] the following: At every point \(p\) of \(M^3\) there exists a basis \(\{e_1, e_2, e_3\}\) which is orthonormal with respect to the affine metric \(h\) such that the difference tensor \(K\) and the shape operator \(S\) are respectively given by:

\[
K_{e_1} = \begin{pmatrix}
2\mu_1 & 0 & 0 \\
0 & -\mu_1 & 0 \\
0 & 0 & -\mu_1
\end{pmatrix}, \quad K_{e_2} = \begin{pmatrix}
0 & -\mu_1 & 0 \\
-\mu_1 & \mu_2 & 0 \\
0 & 0 & -\mu_2
\end{pmatrix},
\]

\[
K_{e_3} = \begin{pmatrix}
0 & 0 & -\mu_1 \\
0 & 0 & -\mu_2 \\
-\mu_1 & -\mu_2 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a
\end{pmatrix}.
\]  

(4)

We have that \(\mu_1\) is nonzero. Moreover, \(\mu_2\) vanishes if and only if \(M^3\) admits a 1-parameter group \(SO(2)\) of isometries. In that case the form of \(K\) and \(S\) remains invariant under rotations in the \(e_2e_3\)-plane. In case that \(\mu_2\) is
different from zero, the group \( \mathbb{Z}_3 \) of rotations leaving \( K \) and \( S \) invariant is generated by the rotation with angle \( \frac{2\pi}{3} \) in the \( e_2e_3 \)-plane.

We recall some of the fundamental equations, which a nondegenerate hypersurface has to satisfy; see also [8] or [6]. These equations relate \( S \) and \( K \) with amongst others the curvature tensor \( R \) of the induced connection \( \nabla \) and the curvature tensor \( \hat{R} \) of the Levi Civita connection \( \hat{\nabla} \) of the affine metric \( h \). We respectively have the Gauss equation for \( \nabla \), which states that:

\[
R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,
\]

and the Codazzi equation

\[
(\nabla_X S)Y = (\nabla_Y S)X.
\]

The fundamental existence and uniqueness theorem, see [2] or [3], states that given \( h, \nabla \) and \( S \) such that the difference tensor is symmetric and traceless with respect to \( h \), on a simply connected manifold \( M \) an affine immersion of \( M \) exists if and only if the above Gauss equation and Codazzi equation are satisfied. From these the Codazzi equation for \( K \) and the Gauss equation for \( \hat{\nabla} \) follow.

\[
(\hat{\nabla}_X K)(Y,Z) = \frac{1}{2}(h(Y,Z)SX - h(X,Z)SY)
\]

\[
- h(SY,Z)X + h(SX,Z)Y),
\]

and

\[
\hat{R}(X,Y)Z = \frac{1}{2}(h(Y,Z)SX - h(X,Z)SY)
\]

\[
+ h(SY,Z)X - h(SX,Z)Y) - [K_X,K_Y]Z
\]

2.2 An adapted frame

From now on, we assume that \( M^3 \) admits a \( \mathbb{Z}_3 \)-symmetry or an \( SO(2) \)-symmetry. The first meaning that at every point of \( M^3 \) the group of isometries preserving \( S \) and \( K \) is isomorphic to \( \mathbb{Z}_3 \), whereas in the second case, we assume that that group of isometries is at every point isomorphic to \( SO(2) \).

We define the Ricci tensor of the connection \( \hat{\nabla} \) by:

\[
\hat{\text{Ric}}(X,Y) = \text{trace}\{Z \mapsto \hat{R}(Z,X)Y\}.
\]

It is well known that \( \hat{\text{Ric}} \) is a symmetric operator. Then, we have

**Lemma 1.** Let \( p \in M \) and \( \{e_1,e_2,e_3\} \) the basis constructed earlier. Then

\[
\hat{\text{Ric}}(e_1,e_1) = (a + \lambda) + 6\mu_1^2,
\]

\[
\hat{\text{Ric}}(e_1,e_2) = 0,
\]

\[
\hat{\text{Ric}}(e_1,e_3) = 0,
\]

\[
\hat{\text{Ric}}(e_2,e_2) = \frac{3}{2}a + \frac{1}{2}\lambda + 2(\mu_1^2 + \mu_2^2),
\]

\[
\hat{\text{Ric}}(e_2,e_3) = 0,
\]

\[
\hat{\text{Ric}}(e_3,e_3) = \frac{3}{2}a + \frac{1}{2}\lambda + 2(\mu_1^2 + \mu_2^2).
\]
Proof. We use the Gauss equation (8) for $\hat{R}$. It follows that

\[
\hat{R}(e_2, e_1)e_1 = \frac{1}{2}(a + \lambda)e_2 - K_{e_3}(2\mu_1 e_1) + K_{e_1}(-\mu_1 e_2) \\
= \left(\frac{1}{2}(a + \lambda) + 3\mu_1^2\right) e_2,
\]

\[
\hat{R}(e_3, e_1)e_1 = \frac{1}{2}(a + \lambda)e_3 - K_{e_3}(2\mu_1 e_1) + K_{e_1}(-\mu_1 e_3) \\
= \left(\frac{1}{2}(a + \lambda) + 3\mu_1^2\right) e_3,
\]

\[
\hat{R}(e_3, e_1)e_2 = -K_{e_3}(-\mu_1 e_2) + K_{e_1}(-\mu_2 e_3) = 0.
\]

From this it immediately follows that

\[
\hat{\text{Ric}}(e_1, e_1) = (a + \lambda) + 6\mu_1^2
\]

and

\[
\hat{\text{Ric}}(e_1, e_2) = 0.
\]

The other equations follow by similar computations. \qedhere

Now, we want to show that the basis we have constructed at each point $p$ can be extended differentiably to a neighborhood of the point $p$ such that at every point the components of $S$ and $K$ with respect to the frame $\{e_1, e_2, e_3\}$ have the previously described form.

**Lemma 2.** Let $M^3$ be an affine hypersurface of $\mathbb{R}^4$ which admits a pointwise $\mathbb{Z}_3$-symmetry or a pointwise $SO(2)$-symmetry. Let $p \in M$. Then there exists a frame $\{e_1, e_2, e_3\}$ defined in a neighborhood of the point $p$ such that the components of $K$ and $S$ are respectively given by:

\[
K_{e_1} = \begin{pmatrix} 2\mu_1 & 0 & 0 \\ 0 & -\mu_1 & 0 \\ 0 & 0 & -\mu_1 \end{pmatrix}, \quad K_{e_2} = \begin{pmatrix} 0 & -\mu_1 & 0 \\ -\mu_1 & \mu_2 & 0 \\ 0 & 0 & -\mu_2 \end{pmatrix},
\]

\[
K_{e_3} = \begin{pmatrix} 0 & 0 & -\mu_1 \\ 0 & 0 & -\mu_2 \\ -\mu_1 & -\mu_2 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}.
\]

**Proof.** First we want to show that at every point the vector $e_1$ is uniquely defined and differentiable. We introduce a symmetric operator $\hat{A}$ by:

\[
\hat{\text{Ric}}(Y, Z) = h(\hat{A}Y, Z).
\]

Clearly $\hat{A}$ is a differentiable operator on $M$. On the set of points where $-\frac{1}{2}(a - \lambda) + 4\mu_1^2 - 2\mu_2^2 \neq 0$, the operator has two distinct eigenvalues. The eigendirection which corresponds with the 1-dimensional eigenvalue corresponds with the vector field $e_1$. 

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On the set of points where \(-\frac{1}{2}(a - \lambda) + 4\mu_1^2 - 2\mu_2^2 = 0\), we have that \(a \neq \lambda\) (cp. [9]), as otherwise we would have an \(A_4\)-symmetry. In this case, the differentiable operator \(S\) has two distinct eigenvalues (cp. (10)) and \(e_1\) is uniquely determined as the eigendirection corresponding to the 1-dimensional eigenvalue. This shows that taking at every point \(p\) the vector \(e_1\) yields a differentiable vector field.

To show that \(e_2\) and \(e_3\) can be extended differentiably, we consider two cases. First we assume that \(M\) admits a pointwise \(SO(2)\)-symmetry. In that case we have that \(\mu_2 = 0\) and we take for \(e_2\) and \(e_3\) arbitrary orthonormal differentiable local vector fields which are orthogonal to the vector field \(e_1\). In case that \(M\) admits a pointwise \(\mathbb{Z}_3\)-symmetry we proceed as follows. We start by taking arbitrary orthonormal differentiable local vector fields \(u_2\) and \(u_3\) which are orthogonal to the vector field \(e_1\). It is then straightforward to check that we can write

\[
K_{u_2}u_2 = -\mu_1 e_1 + \nu_1 u_2 + \nu_2 u_3
\]

\[
K_{u_2}u_3 = \nu_2 u_2 - \nu_1 u_3
\]

\[
K_{u_3}u_3 = -\mu_1 e_1 - \nu_1 u_2 - \nu_2 u_3
\]

for some differentiable functions \(\nu_1\) and \(\nu_2\) with \(\nu_1^2 + \nu_2^2 \neq 0\). Therefore, if necessary by interchanging the role of \(u_2\) and \(u_3\), we may assume that in a neighborhood of the point \(p\), \(\nu_1 \neq 0\). Rotating now over an angle \(\theta\), thus defining

\[
e_2 = \cos \theta u_2 + \sin \theta u_3,
\]

\[
e_3 = -\sin \theta u_2 + \cos \theta u_3,
\]

we get that

\[
h(K(e_2, e_2), e_2) = (\cos^3 \theta - 3 \cos \theta \sin^2 \theta)\nu_1 + (- \sin^3 \theta + 3 \cos^2 \theta \sin \theta)\nu_2
\]

\[
= \cos 3\theta \nu_1 + \sin 3\theta \nu_2
\]

\[
h(K(e_3, e_3), e_3) = (- \sin^3 \theta + 3 \cos^2 \theta \sin \theta)\nu_1 + (- \cos^3 \theta + 3 \cos \theta \sin^2 \theta)\nu_2
\]

\[
= \sin 3\theta \nu_1 - \cos 3\theta \nu_2.
\]

Therefore, taking into account the symmetries of \(K\), in order to obtain the desired frame, it is sufficient to choose \(\theta\) in such a way that

\[
\sin 3\theta \nu_1 - \cos 3\theta \nu_2 = 0,
\]

and \(\cos 3\theta \nu_1 + \sin 3\theta \nu_2 > 0\). As this is always possible, the proof is completed. 

\[ \square \]
Remark. It actually follows from the proof of the previous lemma that the vector field $e_1$ is globally defined on $M$, and therefore the function $\mu_1$, too. This in turn implies that the functions $\mu_2$ (as it can be expressed in terms of $\mu_1$ and the Pick invariant $J$ and $J$ is either identically zero or nowhere zero), $\lambda$ and $a$ (as it can be expressed in terms of the mean curvature and $\lambda$) are globally defined functions on the affine hypersurface $M$.

From now on we will always work with the local frame constructed in the previous lemma. We introduce the connection coefficients with respect to this frame by $\hat{\nabla}_{e_i} e_j = \sum_{k=1}^{3} \varphi_{ijk}^k e_k$. As the connection $\hat{\nabla}$ is metrical, we have the usual symmetries.

2.3 Codazzi equations for $K$

An evaluation of the Codazzi equations (7) for $K$ using the computer program mathematica (see also: www-sfb288.math.tu-berlin.de/~cs/symm2.nb resp. symm6.nb) results in the following equations:

$$e_2(\mu_1) = 2\mu_1 \varphi_{11}^2, \quad (eq.1)$$
$$\mu_2 \varphi_{11}^3 = 4\mu_1 \varphi_{21}^3, \quad (eq.1)$$
$$e_1(\mu_1) = \frac{1}{2}(a - \lambda) - \mu_2 \varphi_{11}^2 - 4\mu_1 \varphi_{21}^2, \quad (eq.1)$$
$$e_1(\mu_1) = \frac{1}{2}(a - \lambda) + \mu_2 \varphi_{11}^2 - 4\mu_1 \varphi_{31}^3, \quad (eq.2)$$
$$e_3(\mu_1) = 2\mu_1 \varphi_{11}^3, \quad (eq.2)$$
$$\mu_2 \varphi_{11}^3 = 4\mu_1 \varphi_{21}^3, \quad (eq.2)$$
$$e_1(\mu_2) + e_2(\mu_1) = 3\mu_1 \varphi_{11}^2 - \mu_2 \varphi_{21}^2, \quad (eq.3)$$
$$0 = -\mu_1 \varphi_{11}^3 + 3\mu_2 \varphi_{12}^3 - \mu_2 \varphi_{31}^3, \quad (eq.3)$$
$$e_3(\mu_1) = -\mu_2 (\varphi_{21}^3 + \varphi_{31}^2), \quad (eq.4)$$
$$e_3(\mu_2) = 3\mu_2 \varphi_{22}^3 - \mu_1 (\varphi_{21}^3 - 3\varphi_{31}^2), \quad (eq.4)$$
$$e_2(\mu_2) = -\mu_1 (\varphi_{21}^3 - \varphi_{31}^2) - 3\mu_2 \varphi_{32}^3, \quad (eq.4)$$
$$e_1(\mu_2) = -\mu_1 \varphi_{11}^2 - \mu_2 \varphi_{31}^3, \quad (eq.5)$$
$$e_3(\mu_1) = 3\mu_1 \varphi_{11}^3 + \mu_2 (3\varphi_{12}^3 + \varphi_{31}^2), \quad (eq.5)$$
$$e_2(\mu_1) = \mu_2 (\varphi_{21}^3 - \varphi_{31}^2), \quad (eq.6)$$
$$e_3(\mu_2) = 3\mu_2 \varphi_{22}^3 + \mu_1 (3\varphi_{21}^3 - \varphi_{31}^2), \quad (eq.6)$$
$$e_2(\mu_1) - e_1(\mu_2) = \mu_1 \varphi_{11}^2 + \mu_2 \varphi_{21}^2, \quad (eq.7)$$
$$4\mu_1 (\varphi_{21}^3 - \varphi_{31}^2) = 0, \quad (eq.8)$$
$$e_3(\mu_1) = \mu_1 \varphi_{11}^3 - \mu_2 (3\varphi_{12}^3 + \varphi_{31}^2). \quad (eq.9)$$

In the above expressions, the equation numbers refer to corresponding
equations in the mathematica program. In order to simplify the above equations, we now distinct two cases.

**Lemma 3.** An evaluation of the Codazzi equations for $K$ gives:

\[
\begin{align*}
\varphi^2_{11} &= 0, & \varphi^3_{11} &= 0, & \varphi^3_{21} &= 0, & \varphi^2_{31} &= 0, & \varphi^2_{21} = \varphi^3_{31} &=: \eta, \\
e_1(\mu_1) &= \frac{1}{2}(a - \lambda) - 4\mu_1\eta, & e_2(\mu_1) &= 0 = e_3(\mu_1).
\end{align*}
\]

If $\mu_2 \neq 0$, we get in addition that $\varphi^3_{12} = 0$ and

\[
e_1(\mu_2) = -\mu_2\eta, & e_2(\mu_2) = -3\mu_2\varphi^3_{32}, & e_3(\mu_2) = 3\mu_2\varphi^3_{22}. \tag{27}
\]

**Proof.** First, we assume that $\mu_2 = 0$ (thus $\mu_1 \neq 0$). In that case, it follows from (10) (resp. (14)) that $\varphi^3_{21} = 0$ (resp. $\varphi^2_{31} = 0$), whereas (19) implies that $\varphi^2_{21} = \varphi^3_{31}$. As it now follows from (17) and (22) that $e_2(\mu_1) = e_3(\mu_1) = 0$, (9) and (13) imply that $\varphi^2_{11} = \varphi^3_{11} = 0$. Finally (12) now reduces to

\[
e_1(\mu_1) = \frac{1}{2}(a - \lambda) - 4\mu_1\varphi^2_{21}.
\]

Next, we want to deal with the case that $\mu_2 \neq 0$. First it follows from (2), (22) and (13), taking also into account (20), that

\[
e_2(\mu_1) = 2\mu_1\varphi^2_{11} = \mu_2(\varphi^2_{21} - \varphi^3_{31}) = 4\mu_1\varphi^2_{11} - \mu_2(\varphi^2_{21} - \varphi^3_{31}).
\]

Therefore we get by (11) and (12) that $\varphi^2_{21} = \varphi^3_{31}$ and thus $e_2(\mu_1) = 0 = \varphi^2_{11}$. From (17) and (13) it follows that

\[
e_3(\mu_1) = -\mu_2(\varphi^3_{21} + \varphi^2_{31}) = 2\mu_1\varphi^3_{11}.
\]

From (25), (10) and the previous equation it follows that $\varphi^3_{21} = \varphi^2_{31} = \varphi^3_{11} = 0$ and $e_3(\mu_1) = 0$. From (16) it then follows that $\varphi^3_{12} = 0$. From (15), (19) and (18) we obtain the equations for $e_i(\mu_2)$, $i = 1, 2, 3$ and from (11) it follows that

\[
e_1(\mu_1) = \frac{1}{2}(a - \lambda) - 4\mu_1\varphi^2_{21}.
\]

As a direct consequence we write down the Levi-Civita connection:
Lemma 4.

\[ \hat{\nabla}_e e_1 = 0, \]
\[ \hat{\nabla}_e e_2 = \varphi_{12}^3 e_3, \]
\[ \hat{\nabla}_e e_3 = -\varphi_{12}^3 e_2, \]
\[ \hat{\nabla}_e e_1 = \eta e_2, \]
\[ \hat{\nabla}_e e_2 = -\eta e_1 + \varphi_{22}^3 e_3, \]
\[ \hat{\nabla}_e e_3 = -\varphi_{22}^3 e_2, \]
\[ \hat{\nabla}_e e_1 = \eta e_3, \]
\[ \hat{\nabla}_e e_2 = \varphi_{32}^3 e_3, \]
\[ \hat{\nabla}_e e_3 = -\eta e_1 - \varphi_{32}^3 e_2, \]

where in case that \( \mu_2 \neq 0 \), we have in addition that \( \varphi_{12}^3 = 0 \).

2.4 Gauss for \( \nabla \)

Taking into account the previous results, we then proceed with an evaluation of the Gauss equations (5) for \( \nabla \):

\[ \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = h(Y, Z)SX - h(X, Z)SY, \]

again using the computer program mathematica (see also: www-sfb288.math.tu-berlin.de/~cs/symm2.nb resp. symm6.nb). This results amongst others in the following equations (cp. equations 11, 13, 14 and 16 in the mathematica program):

\[ e_1(\eta) = -\eta^2 - 3\mu_1^2 - \frac{1}{2}(a + \lambda), \]
\[ e_2(\eta) = 0, \]
\[ e_3(\eta) = 0. \]

2.5 Codazzi for \( S \)

An evaluation of the Codazzi equations (6) for \( S \):

\[ (\nabla_X S)(Y) = (\nabla_Y S)(X) \]
by mathematica (see also: www-sfb288.math.tu-berlin.de/~cs/sym2.nb resp. symm6.nb, equations 20 - 22)) then yields:

\[ e_1(a) = (\mu_1 - \eta)(a - \lambda), \]
\[ e_2(a) = 0, \]
\[ e_3(a) = 0, \]
\[ e_2(\lambda) = 0, \]
\[ e_3(\lambda) = 0. \]

### 2.6 Structure equations

Summarized we have obtained the structure equations (cp. (1), (2) and (3)):

\[ D e_1 e_1 = 2\mu_1 e_1 + \xi, \]  
(28)
\[ D e_1 e_2 = -\mu_1 e_2 + \varphi_{12}^3 e_3, \]  
(29)
\[ D e_1 e_3 = -\varphi_{12}^3 e_2 - \mu_1 e_3, \]  
(30)
\[ D e_2 e_1 = (\eta - \mu_1)e_2, \]  
(31)
\[ D e_3 e_1 = (\eta - \mu_1)e_3, \]  
(32)
\[ D e_2 e_2 = -(\eta + \mu_1)e_1 + \mu_2 e_2 + \varphi_{22}^3 e_3 + \xi, \]  
(33)
\[ D e_2 e_3 = -\varphi_{22}^3 e_2 - \mu_2 e_3, \]  
(34)
\[ D e_3 e_2 = (\varphi_{32}^3 - \mu_2)e_3, \]  
(35)
\[ D e_3 e_3 = -(\eta + \mu_1)e_1 - (\varphi_{32}^3 - \mu_2)e_2 + \xi, \]  
(36)

\[ D e_1 \xi = -\lambda e_1, \]  
(37)
\[ D e_2 \xi = -ae_2, \]  
(38)
\[ D e_3 \xi = -ae_3, \]  
(39)

Moreover, the functions \( a, \lambda, \mu_1 \) and \( \eta \) are all constant in the \( e_2 \) and \( e_3 \)-directions and the \( e_1 \)-derivatives are determined by (cp. Section 2.5 and 2.4 and Lemma 3):

\[ e_1(a) = (\mu_1 - \eta)(a - \lambda), \]  
(40)
\[ e_1(\eta) = -\eta^2 - 3\mu_1^2 - \frac{1}{2}(a + \lambda), \]  
(41)
\[ e_1(\mu_1) = -4\mu_1 \eta - \frac{1}{2}(\lambda - a). \]  
(42)

### 3 Main results

As the vector field \( e_1 \) is globally defined, we can define the distributions \( H_1 = \text{span}\{e_1\} \) and \( H_2 = \text{span}\{e_2, e_3\} \). In the next lemmas we will investigate
some properties of these distributions following from Lemma 4. For the terminology we refer to [7].

**Lemma 5.** The distribution $H_1$ is autoparallel with respect to $\hat{\nabla}$.

**Proof.** From $\hat{\nabla}_{e_1}e_1 = 0$ the claim follows immediately. 

**Lemma 6.** The distribution $H_2$ is spherical with mean curvature normal $H = -\eta e_1$.

**Proof.** For $H = -\eta e_1 \in H_1 = H_2^\perp$ we have $h(\hat{\nabla}_{e_a}e_b, e_1) = h(e_a, e_b)h(H, e_1)$ for all $a, b \in \{2, 3\}$, and $h(\hat{\nabla}_{e_a}H, e_1) = h(-e_a(\eta)e_1 - \eta \hat{\nabla}_{e_a}e_1, e_1) = 0$.

**Remark.** $\eta \ (= \varphi^2_{21} = \varphi^3_{31})$ is independent of the choice of $e_2$ and $e_3$. It therefore is a globally defined function on $M$.

We introduce a coordinate function $t$ by $\frac{\partial}{\partial t} := e_1$. Using the previous lemma, according to [4], we get:

**Lemma 7.** $(M, h)$ admits a warped product structure $M^3 = \mathbb{R} \times f^f N^2$ with $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{\partial f}{\partial t} = \eta. \tag{43}$$

**Remark.** $a, \eta$ and $\mu_1$ are functions of $t$, they satisfy by (40), (41) and (42):

$$\frac{\partial a}{\partial t} = (\mu_1 - \eta)(a - \lambda),$$
$$\frac{\partial \eta}{\partial t} = -\eta^2 - 3\mu_1^2 - \frac{1}{2}(a + \lambda),$$
$$\frac{\partial \mu_1}{\partial t} = -4\eta \mu_1 + \frac{1}{2}(a - \lambda),$$

To compute the curvature of $N^2$ we use the gauss equation (8) and obtain:

$$K(N^2) = e^{2f}(a - \mu_1^2 + 2\mu_2^2 + \eta^2), \tag{44}$$

which we verify by a straightforward computation is indeed independent of $t$.

Our first goal is to find out how $N^2$ is immersed in $\mathbb{R}^4$, i.e. to find an immersion independent of $t$. A look at the structure equations (28) - (39) suggests to start with a linear combination of $e_1$ and $\xi$.

We will solve the problem in two steps. First we define $v := Ae_1 + \xi$ for some function $A$ on $M^3$. Then $\frac{\partial}{\partial t}v = \alpha v$ iff $\alpha = A$ and $\frac{\partial}{\partial t}A = A^2 - 2\mu_1 A + \lambda$, 

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and \( A := -(\eta + \mu_1) \) solves the latter differential equation. Next we define a positive function \( \beta \) on \( \mathbb{R} \) as solution of the differential equation:
\[
\frac{\partial}{\partial t} \beta = \beta(\eta + \mu_1)
\]  
(45)
with initial condition \( \beta(t_0) > 0 \). Then \( \frac{\partial}{\partial t} (\beta e) = 0 \) and by (31), (38), (32) and (39) we get (since \( \beta, \eta \) and \( \mu_1 \) only depend on \( t \)):
\[
\begin{align*}
D_{e_1}(\beta(-\eta + \mu_1) e_1 + \xi)) &= 0, \\
D_{e_2}(\beta(-\eta + \mu_1) e_1 + \xi)) &= -\beta(a + \eta^2 - \mu_1^2) e_2, \\
D_{e_3}(\beta(-\eta + \mu_1) e_1 + \xi)) &= -\beta(a + \eta^2 - \mu_1^2) e_3.
\end{align*}
\]  
(46) (47) (48)

Lemma 8. Define \( \nu := a + \eta^2 - \mu_1^2 \) on \( \mathbb{R} \). \( \nu \) is globally defined, \( \frac{\partial}{\partial t} (e^2 \nu) = 0 \) and \( \nu \) vanishes identically or nowhere on \( \mathbb{R} \).

**Proof.** Since \( 0 = \frac{\partial}{\partial t} K(N^2) = \frac{\partial}{\partial t} (e^2 (\nu + 2\mu_1^2)) \) (cp. (27) and (43)) and \( \frac{\partial}{\partial t} (e^2 f 2 \mu_1^2) = 0 \), we get that \( \frac{\partial}{\partial t} (e^2 \nu) = 0 \). Thus \( \frac{\partial}{\partial t} \nu = -2(e^2 f) \nu = -2\eta \nu. \)

Now we consider different cases depending on the behaviour of \( \nu \).

### 3.1 The first case: \( \nu \neq 0 \) on \( M^3 \)

We may, by translating \( f \), i.e. by replacing \( N^2 \) with a homothetic copy of itself, assume that \( e^2 \nu = \epsilon \), where \( \epsilon = \pm 1 \).

Lemma 9. \( \phi := \beta(-\eta + \mu_1) e_1 + \xi) : N^2 \to \mathbb{R}^4 \) defines a proper affine sphere in a 3-dimensional linear subspace of \( \mathbb{R}^4 \). \( \phi \) is part of a quadric iff \( \mu_2 = 0 \).

**Proof.** By construction we ensured that \( \frac{\partial}{\partial t} \phi = 0 \) (cp. (46)), thus \( \phi \) is defined on \( N^2 \). Furthermore it is an immersion, since \( \phi_a(e_a) = -\nu e_a \) for \( a = 2, 3 \) by (17) and (48). A further differentiation, using (33) (\( \beta \) and \( \nu \) only depend on \( t \)), gives:
\[
\begin{align*}
D_{e_2} \phi_a(e_2) &= -\beta(a + \eta^2 - \mu_1^2) D_{e_2} e_2 \\
&= -\beta(a + \eta^2 - \mu_1^2)(-\eta + \mu_1) e_1 + \mu_2 e_2 + \varphi^3_{22} e_3 + \xi) \\
&= \mu_2 \phi_a(e_2) + \varphi^3_{22} \phi_a(e_3) - (a + \eta^2 - \mu_1^2) \phi \\
&= \mu_2 \phi_a(e_2) + \varphi^3_{22} \phi_a(e_3) - \epsilon e^{-2f} \phi.
\end{align*}
\]

Similarly, we obtain the other derivatives, using (33) - (36), thus:
\[
\begin{align*}
D_{e_2} \phi_a(e_2) &= \mu_2 \phi_a(e_2) + \varphi^3_{22} \phi_a(e_3) - e^{-2f} \epsilon \phi, \\
D_{e_2} \phi_a(e_3) &= -\varphi^3_{22} \phi_a(e_2) - \mu_2 \phi_a(e_3), \\
D_{e_3} \phi_a(e_2) &= (\varphi^3_{32} - \mu_2) \phi_a(e_3), \\
D_{e_3} \phi_a(e_3) &= -\varphi^3_{32} + \mu_2) \phi_a(e_3) - e^{-2f} \epsilon \phi, \\
D_{e_a} \phi &= -\beta e^{-2f} \epsilon e_a, \quad a = 2, 3.
\end{align*}
\]
We can read off the coefficients of the difference tensor $K^\phi$ of $\phi$ (cp. (1) and (3)): $(K^\phi)^{22}_2 = \mu_2$, $(K^\phi)^{23}_3 = -\mu_2$, $(K^\phi)^{33}_3 = 0 = (K^\phi)^{33}_3$, and see that $\text{trace}(K^\phi)_X$ vanishes. The affine metric introduced by this immersion corresponds with the metric on $N^2$. Thus $-\epsilon\phi$ is the affine normal of $\phi$ and $\phi$ is a proper affine sphere. Finally the vanishing of the difference tensor characterizes quadrics.

Our next goal is to find another linear combination of $e_1$ and $\xi$, this time only depending on $t$. (Then we can express $e_1$ in terms of $\phi$ and some function of $t$.)

**Lemma 10.** Define $\delta := ae_1 + (\eta - \mu_1)\xi$. Then there exist a constant vector $C \in \mathbb{R}^4$ and a function $g(t)$ such that

$$\delta(t) = g(t)C.$$

**Proof.** Using (31) and (38) resp. (32) and (39) we obtain that $D_{e_2}\delta = 0 = D_{e_3}\delta$. Hence $\delta$ depends only on the variable $t$. Moreover, we get by (40), (28), (41) and (37) that

$$\frac{\partial}{\partial t}\delta = D_{e_1}(ae_1 + (\eta - \mu_1)\xi)
= (\mu_1 - \eta)(a - \lambda)e_1 + 2a\mu_1 e_1 + a\xi - (\eta - \mu_1)\lambda e_1
+ (-\eta^2 - 3\mu_1^2 - a + 4\mu_1\eta)\xi
= (3\mu_1 - \eta)(ae_1 + (\eta - \mu_1)\xi)
= (3\mu_1 - \eta)\delta.$$

This implies that there exists a constant vector $C \in \mathbb{R}^4$ and a function $g(t)$ such that $\delta(t) = g(t)C$. \qed

Combining $\phi$ and $\delta$ we obtain for $e_1$ (cp. Lem. 9 and 10) that

$$e_1(t,u,v) = -\frac{1}{\beta \nu}(\beta g C + (\eta - \mu_1)\phi(u,v)).$$

In the following we will use for the partial derivatives the abbreviation $F_x := \frac{\partial}{\partial x}F$, $x = t, u, v$.

**Lemma 11.**

$$F_t = -\frac{g}{\nu}C - \frac{\partial}{\partial t}(\frac{1}{\beta \nu})\phi,$$

$$F_u = -\frac{1}{\beta \nu}\phi_u,$$

$$F_v = -\frac{1}{\beta \nu}\phi_v.$$
Proof. As by (45) and Lem. \(\partial_t \frac{1}{\beta \nu} = \frac{1}{\beta \nu} (\eta - \mu_1)\), we obtain the equation for \(F_t = e_1\) by (49). The other equations follow from (47) and (48).

It follows by the uniqueness theorem of first order differential equations and applying a translation that we can write

\[ F(t, u, v) = \tilde{g}(t) C - \frac{1}{\beta \nu} (t) \phi(u, v) \]

for a suitable function \(\tilde{g}\) depending only on the variable \(t\). Since \(C\) is transversal to the image of \(\phi\) (cp. Lem. 9 and 10), we obtain that after applying an equiaffine transformation we can write: \(F(t, u, v) = (\gamma_1(t), \gamma_2(t) \phi(u, v))\). Thus we have proven the following:

**Theorem 1.** Let \(M^3\) be an affine hypersurface of \(\mathbb{R}^4\) which admits a pointwise \(SO(2)\)- or \(\mathbb{Z}_3\)-symmetry and with the globally defined function \((a + \eta^2 - \mu_2^2)\) not identically zero on \(M^3\). Then \(M^3\) is affin equivalent to

\[ F : I \times N^2 \to \mathbb{R}^4 : (t, u, v) \mapsto (\gamma_1(t), \gamma_2(t) \phi(u, v)), \]

where \(\phi : N^2 \to \mathbb{R}^3\) is an elliptic or hyperbolic affine sphere and \(\gamma : I \to \mathbb{R}^2\) is an affine curve. Moreover, if \(M^3\) admits a pointwise \(SO(2)\)-symmetry then \(N^2\) is either an ellipsoid or a hyperboloid.

In the next theorem we deal with the converse.

**Theorem 2.** Let \(\phi : N^2 \to \mathbb{R}^3\) be an elliptic or hyperbolic affine sphere (scaled such that the absolute value of the mean curvature equals 1) and let \(\gamma : I \to \mathbb{R}^2\) be an affine curve. Then if

\[ F(t, u, v) = (\gamma_1(t), \gamma_2(t) \phi(u, v)) , \]

defines a nondegenerate positive definite affine hypersurface, it admits a pointwise \(\mathbb{Z}_3\)- or \(S_1\)-symmetry.
Proof. We have

\[
\begin{align*}
F_t &= (\gamma_1', \gamma_2') \phi, \\
F_u &= (0, \gamma_2 \phi_u), \\
F_v &= (0, \gamma_2 \phi_v), \\
F_{tt} &= (\gamma''_1, \gamma''_2 \phi) = \left(\frac{\gamma''_1 \gamma_1' - \gamma''_2 \gamma_2'}{\gamma_1}ight)(0, \phi) + \frac{\gamma''_2}{\gamma_1} F_t, \\
F_{ut} &= \frac{\gamma''_2}{\gamma_2} F_u, \\
F_{vt} &= \frac{\gamma''_2}{\gamma_2} F_v, \\
F_{uu} &= (0, \gamma_1 \phi_{uu}), \\
F_{uv} &= (0, \gamma_1 \phi_{uv}), \\
F_{vv} &= (0, \gamma_1 \phi_{vv}).
\end{align*}
\]

This implies that \( F \) defines a nondegenerate affine immersion provided that \( \gamma_2 \gamma_1 (\gamma''_2 \gamma_1' - \gamma''_1 \gamma_2') \neq 0 \). We moreover see that this immersion is definite provided that the affine sphere is hyperbolic and \( \gamma_1 \gamma_1' (\gamma''_2 \gamma_1' - \gamma''_1 \gamma_2') > 0 \) or when the proper affine sphere is elliptic and \( \gamma_1 \gamma_1' (\gamma''_2 \gamma_1' - \gamma''_1 \gamma_2') < 0 \). As the proof in both cases is similar, we will only treat the first case here. An evaluation of the conditions for the affine normal \( \xi \ (\xi_t, \xi_u, \xi_v) \) are tangential and \( \det(F_t, F_u, F_v, \xi) = \sqrt{\det h} \) leads to:

\[
\xi = \alpha(t)(0, \phi(u, v)) + \beta(t) F_t,
\]

where \( (\gamma''_2 \gamma_1' - \gamma''_1 \gamma_2') \gamma_1^2 = \gamma_2^4 \gamma_1^3 \alpha^5 \) and \( \alpha' + \frac{3(\gamma''_2 \gamma_1' - \gamma''_1 \gamma_2')}{\gamma_1^2} = 0 \). Taking \( e_1 \) in the direction of \( F_t \), we see that \( F_u \) and \( F_v \) are orthogonal to \( e_1 \). It is also clear that \( S \) restricted to the space spanned by \( F_u \) and \( F_v \) is a multiple of the identity, and \( S(F_t) = \lambda F_t \), since \( S \) is symmetric. Moreover, we have that

\[
\begin{align*}
(\nabla h)(F_t, F_u, F_u) &= \left(\frac{\gamma_2}{\gamma_1} - \frac{\gamma'}{\alpha} - \frac{2 \gamma'}{\gamma_2}\right) h(F_u, F_u), \\
(\nabla h)(F_t, F_u, F_v) &= \left(\frac{\gamma_2}{\gamma_1} - \frac{\gamma'}{\alpha} - \frac{2 \gamma'}{\gamma_2}\right) h(F_u, F_v), \\
(\nabla h)(F_t, F_v, F_v) &= \left(\frac{\gamma_2}{\gamma_1} - \frac{\gamma'}{\alpha} - \frac{2 \gamma'}{\gamma_2}\right) h(F_v, F_v), \\
(\nabla h)(F_u, F_t, F_t) &= 0 = (\nabla h)(F_v, F_t, F_t),
\end{align*}
\]

implying that \( K_{F_t} \) restricted to the space spanned by \( F_u \) and \( F_v \) is a multiple of the identity. Using the symmetries of \( K \) it now follows immediately that \( F \) admits an \( \mathbb{Z}_3 \)-symmetry or an \( SO(2) \)-symmetry.

3.2 The second case: \( \nu \equiv 0 \) and \( \mu_1 \neq \eta \) on \( M^3 \)

Next, we consider the case that \( a = \mu_2^2 - \eta^2 \) and \( \eta \neq \mu_1 \) on \( M^3 \). Since by (12) and (11) \( e_1(\eta - \mu_1) = -\eta^2 - 3\mu_2^2 - a + 4\mu_1 \eta = 4\mu_1 (\eta - \mu_1) \) we see that
\( \eta \neq \mu_1 \) everywhere on \( M^3 \) or nowhere.

We already have seen that \( M^3 \) admits a warped product structure. The map \( \phi \) we have constructed in Lemma 9 will not define an immersion (cp. (47) and (48)). Anyhow, for a fixed point \( t_0 \), we get from (33) - (36), (47), and (48), using the notation \( \tilde{\xi} = - (\eta + \mu_1) e_1 + \xi \):

\[
\begin{align*}
D_{e_2} e_2 &= \varphi_{22}^3 e_3 + \mu_2 e_2 + \tilde{\xi}, \\
D_{e_2} e_3 &= - \varphi_{22}^3 e_2 - \mu_2 e_3, \\
D_{e_3} e_2 &= \varphi_{33}^2 e_3 - \mu_2 e_3, \\
D_{e_3} e_3 &= \varphi_{33}^2 e_2 - \mu_2 e_2 + \tilde{\xi}, \\
D_{e_a} \tilde{\xi} &= 0, \quad a = 2, 3.
\end{align*}
\]

Thus, if \( u \) and \( v \) are local coordinates which span the second distribution, then we can interprete \( F(t_0, u, v) \) as an improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that \( \mu_2 \) at \( t_0 \) vanishes identically (as a function of \( u \) and \( v \)). From the differential equations (27) determining \( \mu_2 \), we see that this is the case exactly when \( \mu_2 \) vanishes identically, i.e. when \( M \) admits a pointwise \( SO(2) \)-symmetry.

After applying a translation and a change of coordinates, we may assume that

\[
F(t_0, u, v) = (u, v, f(u, v), 0),
\]

with affine normal \( \tilde{\xi}(t_0, u, v) = (0, 0, 1, 0) \). To obtain \( e_1 \) at \( t_0 \), we consider (31) and (32) and get that

\[
D_{e_a} (e_1 - (\eta - \mu_1)F) = 0, \quad a = 2, 3.
\]

Evaluating at \( t = t_0 \), this means that there exists a constant vector \( C \) such that \( e_1(t_0, u, v) = (\eta - \mu_1)(t_0) F(t_0, u, v) + C \). Since \( \eta \neq J \) everywhere, we can write:

\[
e_1(t_0, u, v) = \alpha_1(u, v, f(u, v), \alpha_2), \tag{50}
\]

where \( \alpha_1 \neq 0 \) and we applied an equiaffine transformation so that \( C = (0, 0, 0, \alpha_1 \alpha_2) \). To obtain information about \( \frac{\partial}{\partial t} e_1 \) we have that \( D_{e_1} e_1 = 2 \mu_1 e_1 + \xi \) (cp. (28)) and \( \xi = \tilde{\xi} + (\eta + \mu_1) e_1 \) by the definition of \( \tilde{\xi} \). Also we know that \( \tilde{\xi}(t_0, u, v) = (0, 0, 1, 0) \) and by (46) - (48) that \( D_{e_i} (\beta \xi) = 0, \quad i = 1, 2, 3 \). Taking suitable initial conditions for the function \( \beta (\beta(t_0) = 1) \), we get that \( \beta \xi = (0, 0, 1, 0) \) and finally the following vector valued differential equation:

\[
\frac{\partial}{\partial t} e_1 = (\eta + 3 \mu_1) e_1 + \beta^{-1}(0, 0, 1, 0). \tag{51}
\]
Solving this differential equation, taking into account the initial conditions (50) at \( t = t_0 \), we get that there exist functions \( \delta_1 \) and \( \delta_2 \) depending only on \( t \) such that

\[
e_1(t, u, v) = (\delta_1(t)u, \delta_1(t)v, \delta_1(t)(f(u, v) + \delta_2(t)), \alpha_2\delta_1(t)),
\]

where \( \delta_1(t_0) = \alpha_1, \delta_2(t_0) = 0, \delta'_1(t) = (\eta + 3\mu_1)\delta_1(t) \) and \( \delta'_2(t) = \delta^{-1}_1(t)\beta^{-1}(t) \).

As \( e_1(t, u, v) = \frac{\partial F}{\partial t}(t, u, v) \) and \( F(t_0, u, v) = (u, v, f(u, v), 0) \) it follows by integration that

\[
F(t, u, v) = (\gamma_1(t)u, \gamma_1(t)v, \gamma_1(t)f(u, v) + \gamma_2(t), \gamma_1(t) - 1)),
\]

where \( \gamma'_1(t) = \delta_1(t), \gamma_1(t_0) = 1, \gamma_2(t_0) = 0 \) and \( \gamma'_2(t) = \delta_1(t)\delta_2(t) \). After applying an affine transformation we have shown:

**Theorem 3.** Let \( M^3 \) be an affine hypersurface of \( \mathbb{R}^4 \) which admits a pointwise \( SO(2) \)- or \( \mathbb{Z}_3 \)-symmetry and with the globally defined functions satisfying \( a = -\eta^2 + \mu^2 \) but \( a \neq 0 \) on \( M^3 \). Then \( M^3 \) is affine equivalent with

\[
F : I \times N^2 \to \mathbb{R}^4 : (t, u, v) \mapsto (\gamma_1(t)u, \gamma_1(t)v, \gamma_1(t)f(u, v) + \gamma_2(t), \gamma_1(t)),
\]

where \( \psi : N^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v)) \) is an improper affine sphere with affine normal \((0, 0, 1)\) and \( \gamma : I \to \mathbb{R}^2 \) is an affine curve. Moreover, if \( M^3 \) admits a pointwise \( SO(2) \)-symmetry then \( N^2 \) is an elliptic paraboloid.

In the next theorem we again deal with the converse.

**Theorem 4.** Let \( \psi : N^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v)) \) be an improper affine sphere with affine normal \((0, 0, 1)\) and let \( \gamma : I \to \mathbb{R}^2 \) be an affine curve. Then if

\[
F(t, u, v) = (\gamma_1(t)u, \gamma_1(t)v, \gamma_1(t)f(u, v) + \gamma_2(t), \gamma_1(t)),
\]

defines a nondegenerate positive definite affine hypersurface, it admits a pointwise \( \mathbb{Z}_3 \)- or \( S_1 \)-symmetry.
Proof. We have
\[ F_t = (\gamma'_1 u, \gamma'_1 v, \gamma'_1 f(u, v) + \gamma'_2, \gamma'_1), \]
\[ F_u = (\gamma_1, 0, \gamma_1 f_u, 0), \]
\[ F_v = (0, \gamma_1, \gamma_1 f_v, 0), \]
\[ F_{tt} = (\gamma''_1 u, \gamma''_1 v, \gamma''_1 f(u, v) + \gamma''_2, \gamma''_1) = \frac{\gamma''}{\gamma'_1} F_t + \frac{\gamma''_2}{\gamma'_1} (0, 0, 1, 0), \]
\[ F_{uv} = (0, 0, f_{uu} \gamma_1, 0), \]
\[ F_{vu} = (0, 0, f_{uv} \gamma_1, 0), \]
\[ F_{vv} = (0, 0, f_{vv} \gamma_1, 0). \]

This implies that \( F \) defines a nondegenerate affine immersion provided that \( \gamma_1 \gamma'_1 (\gamma''_2 - \gamma''_1) \neq 0 \). We moreover see that this immersion is definite provided that the improper affine sphere is positive definite and \( \gamma_1 \gamma'_1 (\gamma''_2 - \gamma''_1) > 0 \) or when the improper affine sphere is negative definite and \( \gamma_1 \gamma'_1 (\gamma''_2 - \gamma''_1) < 0 \). As the proof in both cases is similar, we will only treat the first case here. It easily follows that we can write the affine normal \( \xi \) as:
\[ \xi = \alpha(t)(0, 0, 1, 0) + \beta(t) F_t, \]
where \((\gamma''_2 - \gamma''_1) = \gamma'_1 (\gamma'_1)^3 \alpha^5 \) and \( \alpha' + \frac{\beta(\gamma''_2 - \gamma''_1)}{\gamma'_1} = 0 \). Taking \( e_1 \) in the direction of \( F_t \), we see that \( F_u \) and \( F_v \) are orthogonal to \( e_1 \). It is also clear that \( S \) restricted to the space spanned by \( F_u \) and \( F_v \) is a multiple of the identity, and \( S(F_t) = \lambda F_t \), since \( S \) is symmetric. Moreover, we have that
\[ (\nabla h)(F_t, F_u, F_u) = (-\frac{\gamma'_1}{\gamma'_1} - \frac{\alpha'}{\alpha}) h(F_u, F_u), \]
\[ (\nabla h)(F_t, F_u, F_v) = (-\frac{\gamma'_1}{\gamma'_1} - \frac{\alpha'}{\alpha}) h(F_u, F_v), \]
\[ (\nabla h)(F_t, F_v, F_v) = (-\frac{\gamma'_1}{\gamma'_1} - \frac{\alpha'}{\alpha}) h(F_v, F_v), \]
\[ (\nabla h)(F_u, F_t, F_t) = 0 = (\nabla h)(F_v, F_t, F_t), \]
implying that \( K_{F_t} \) restricted to the space spanned by \( F_u \) and \( F_v \) is a multiple of the identity. Using the symmetries of \( K \) it now follows immediately that \( F \) admits an \( \mathbb{Z}_3 \)-symmetry or an \( SO(2) \)-symmetry. \( \square \)

3.3 The third case: \( \nu \equiv 0 \) and \( \mu_1 = \eta \) on \( M^3 \)

The final case now is that \( a = \mu^2_1 - \eta^2 \) and \( \eta = \mu_1 \) on the whole of \( M^3 \). This is dealt with in the following theorem:
Theorem 5. Let $M^3$ be an affine hypersurface of $\mathbb{R}^4$ which admits a pointwise $SO(2)$- or $\mathbb{Z}_3$-symmetry and with the globally defined functions satisfying $a = -\eta^2 + \mu_1$ and $\eta = \mu_1$ on $M^3$. Then $M^3$ is affine equivalent to

\[ F : I \times N^2 \to \mathbb{R}^4 : (t, u, v) \mapsto (u, v, f(u, v) + \gamma_2(t), \gamma_1(t)), \]

where $\psi : N^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v))$ is an improper affine sphere with affine normal $(0, 0, 1)$ and $\gamma : I \to \mathbb{R}^2$ is an affine curve. Moreover, if $M^3$ admits a pointwise $SO(2)$-symmetry then $N^2$ is an elliptic paraboloid.

Proof. We proceed in the same way as in Theorem 3. We again use that $M^3$ admits a warped product structure and we fix a parameter $t_0$. At the point $t_0$, we have for $\tilde{\xi} = - (\eta + \mu_1)e_1 + \xi = -2\mu_1 e_1 + \xi$:

\[
\begin{align*}
D_{e_2}e_2 &= \varphi_{22}^{3}e_3 + \mu_2 e_2 + \tilde{\xi}, \\
D_{e_2}e_3 &= -\varphi_{22}^{3}e_2 - \mu_2 e_3, \\
D_{e_3}e_2 &= \varphi_{33}^{2}e_3 - \mu_2 e_3, \\
D_{e_3}e_3 &= \varphi_{33}^{2}e_2 - \mu_2 e_2 + \tilde{\xi}, \\
D_{e_a}\tilde{\xi} &= 0, \quad a = 2,3.
\end{align*}
\]

Thus, if $u$ and $v$ are local coordinates which span the second distribution, then we can interprete $F(t_0, u, v)$ as an improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that $\mu_2$ at $t_0$ vanishes identically (as a function of $u$ and $v$). From the differential equations (27) determining $\mu_2$, we see that this is the case exactly when $\mu_2$ vanishes identically, i.e. when $M$ admits a pointwise $SO(2)$-symmetry.

After applying a translation and a change of coordinates, we may assume that

\[ F(t_0, u, v) = (u, v, f(u, v), 0), \]

with affine normal $\tilde{\xi}(t_0, u, v) = (0, 0, 1, 0)$. To obtain $e_1$ at $t_0$, we consider (31) and (32) and get that

\[ D_{e_a} e_1 = (\eta - \mu_1)e_a = 0, \quad a = 2, 3. \]

It follows that $e_1(t_0, u, v)$ is a constant vector field. As it is transversal, we may assume that there exists an $\alpha \neq 0$ such that

\[ e_1(t_0, u, v) = (0, 0, 0, \alpha). \]
As $e_1$ is determined by the differential equation (cp. [51]):

$$\frac{\partial e_1}{\partial t} = 4\mu_1 e_1 + \beta^{-1}(0, 0, 1, 0),$$

it follows that

$$e_1(t, u, v) = (0, 0, \delta_2(t), \delta_1(t)),$$

where $\delta_2(t_0) = 0$, $\delta'_2(t) = 4\mu_1(t)\delta_2(t) + \beta^{-1}(t)$, $\delta_1(t_0) = \alpha$ and $\delta'_1(t) = 4\mu_1(t)\delta_1(t)$. Integrating once more with respect to $t$ we obtain that

$$F(t, u, v) = (u, v, f(u, v) + \gamma_2(t), \gamma_1(t)),$$

for some functions $\gamma_1$ and $\gamma_2$ with $\gamma'_i = \delta_i$ and $\gamma_i(t_0) = 0$ for $i = 1, 2$.

In the next theorem we deal with the converse.

**Theorem 6.** Let $\psi : N^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v))$ be an improper affine sphere with affine normal $(0, 0, 1)$ and let $\gamma : I \to \mathbb{R}^2$ be an affine curve. Then if

$$F(t, u, v) = (u, v, f(u, v) + \gamma_2(t), \gamma_1(t))$$

defines a nondegenerate positive definite affine hypersurface, it admits a pointwise $\mathbb{Z}_3$- or $S_1$-symmetry.

**Proof.** We have

$$F_t = (0, 0, \gamma'_2, \gamma'_1),$$

$$F_u = (1, 0, f_u, 0),$$

$$F_v = (0, 1, f_v, 0),$$

$$F_{tt} = (0, 0, \gamma''_2, \gamma''_1) = \frac{\gamma''}{\gamma_1} F_t + \frac{(\gamma''_2 \gamma'_1 - \gamma''_1 \gamma'_2)}{\gamma_1^3}(0, 0, 1, 0),$$

$$F_{ut} = F_{vt} = 0,$$

$$F_{uu} = (0, 0, f_{uu}, 0),$$

$$F_{uv} = (0, 0, f_{uv}, 0),$$

$$F_{vv} = (0, 0, f_{vv}, 0).$$

This implies that $F$ defines a nondegenerate affine immersion provided that $\gamma'_1(\gamma''_2 \gamma'_1 - \gamma''_1 \gamma'_2) \neq 0$. We moreover see that this immersion is definite provided that the improper affine sphere is positive definite and $(\gamma''_2 \gamma'_1 - \gamma''_1 \gamma'_2) \gamma'_1 > 0$ or when the improper affine sphere is negative definite and $(\gamma''_2 \gamma'_1 - \gamma''_1 \gamma'_2) \gamma'_1 < 0$. As the proof in both cases is similar, we will only treat the first case here. It easily follows that we can write the affine normal $\xi$ as:

$$\xi = \alpha(t)(0, 0, 1, 0) + \beta(t)F_t.$$
where \((\gamma''_2 \gamma'_1 - \gamma''_1 \gamma'_2) = (\gamma'_1)^3 \alpha^5\) and \(\alpha' + \frac{\beta (\gamma''_2 \gamma'_1 - \gamma''_1 \gamma'_2)}{\gamma'_1} = 0\). Taking now \(e_1\) in the direction of \(F_t\), we see that \(F_u\) and \(F_v\) are orthogonal to \(e_1\). It is also clear that \(SF_u = SF_v = 0\), and \(S(F_t) = \lambda F_t\), since \(S\) is symmetric. Moreover, we have that

\[
(\nabla h)(F_t, F_u, F_u) = -\frac{\omega}{\alpha} h(F_u, F_u) \\
(\nabla h)(F_t, F_u, F_v) = -\frac{\omega}{\alpha} h(F_u, F_v) \\
(\nabla h)(F_t, F_v, F_v) = -\frac{\omega}{\alpha} h(F_v, F_v), \\
(\nabla h)(F_u, F_t, F_t) = 0 = (\nabla h)(F_v, F_t, F_t),
\]

implying that \(K_{F_t}\) restricted to the space spanned by \(F_u\) and \(F_v\) is a multiple of the identity. Using the symmetries of \(K\) it now follows immediately that \(F\) admits an \(\mathbb{Z}_3\)-symmetry or an \(SO(2)\)-symmetry. \(\square\)

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