NORMALITY OF VERY EVEN NILPOTENT VARIETIES IN $D_{2l}$

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ABSTRACT. For the classical groups, Kraft and Procesi [4], [5] have resolved the question of which nilpotent orbits have closures which are normal and which are not, with the exception of the very even orbits in $D_{2l}$ which have partition of the form $(a^2k, b^2)$ for $a, b$ distinct even natural numbers with $ak + b = 2l$.

In this article, we show that these orbits do have normal closure. We use the technique of [8].

1. SOME LEMMAS IN $A_l$

We retain the notation of [8]. Throughout, $G$ is a connected simple algebraic group over $C$, $B$ a Borel subgroup, $T$ a maximal torus in $B$. The simple roots are denoted by $\Pi$, and they correspond to the Borel subgroup opposite to $B$. Let $\{\omega_i\}$ be the fundamental weights of $G$ corresponding to $\Pi$. If $\alpha \in \Pi$, then $P_\alpha$ denotes the parabolic subgroup of semisimple rank one containing $B$ and corresponding to $\alpha$. If $P$ is a parabolic subgroup of $G$, we denote by $u_P$ the Lie algebra of its unipotent radical.

We recall Proposition 1.1. [3] Let $V$ be a rational representation of $B$ and assume that $V$ extends to a representation of the parabolic subgroup $P_\alpha$ where $\alpha$ is a simple root. Let $\lambda \in X^*(T)$ be such that $m = \langle \lambda, \alpha^\vee \rangle \geq -1$. Then there is a $G$-module isomorphism

$$H^i(G/B, V \otimes \lambda) = H^{i+1}(G/B, V \otimes \lambda - (m + 1)\alpha) \text{ for all } i \in \mathbb{Z}. $$

In particular, if $m = -1$, then all cohomology groups vanish.

For the rest of this section and the next, let $G = SL_{l+1}(C)$. We index the simple roots $\Pi = \{\alpha_j\}$ so that $\alpha_1$ is an extremal root and $\alpha_j$ is next to $\alpha_{j+1}$ in the Dynkin diagram of type $A_l$.

The following lemma follows easily from several applications of the previous proposition.

Lemma 1.2. [2] Let $V$ be a rational representation of $B$ which extends to a representation of $P_{\alpha_j}$ for $a \leq j \leq b$. Let $\lambda \in X^*(T)$ be such that $\langle \lambda, \alpha_j^\vee \rangle = 0$ for $a < j < b$. Set $r = \langle \lambda, \alpha_a^\vee \rangle$ and assume that $a - b - 1 \leq r \leq -1$. Then $H^*(V \otimes \lambda) = 0$.

A similar statement holds by applying the non-trivial automorphism to the Dynkin diagram of type $A_l$. We use this lemma to prove

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Lemma 1.3. Let $V$ be a representation of $B$ which is stable under the parabolic subgroups $P_{\alpha_j}$ for $1 \leq j \leq b$. Let $\lambda \in X^+(T)$ be such that $\langle \lambda, \alpha_i^\vee \rangle = 1$ for some $a$ satisfying $1 \leq a < b$. Assume that $\langle \lambda, \alpha_j^\vee \rangle = 0$ for $1 \leq j \neq a < b$. Set $k = \langle \lambda, \alpha_b^\vee \rangle$. If $-b-1 \leq k \leq -1$ and $k + b - a \neq -1$, then $H^s(V \otimes \lambda) = 0$.

Proof. If $k + b - a \geq 0$, the result follows directly from Lemma 1.2. On the other hand, if $k + b - a \leq -2$, then as in the proof of Lemma 1.2 in [7],

$$H^i(V \otimes \lambda) = H^{i+b-a}(V \otimes \mu)$$

where

$$\mu = \lambda + (-k-1)\alpha_b + (-k-2)\alpha_{b-1} + \cdots + (-k-b+a)\alpha_{a+1}.$$ 

Now $\langle \mu, \alpha_j^\vee \rangle = 0$ for $1 \leq j < a$ and $\langle \mu, \alpha_a^\vee \rangle = k + b - a + 1$. By the hypothesis on $\lambda$ and the present assumption about $k + b - a$, we have

$$-a \leq k + b - a + 1 \leq -1.$$ 

Then Lemma 1.2 yields the desired vanishing. $\square$

2. A THEOREM FOR $A_1$ (REVIEW)

Let $P_m$ denote the maximal proper parabolic subgroup of $G = SL_{l+1}(\mathbb{C})$ containing $B$ corresponding to all the simple roots except $\alpha_m$. Denote the Lie algebra of the unipotent radical of $P_m$ by $u_m$. The action of $P_m$ on $u_m$ gives a representation of $P_m$ (and also $B$). Denote the dual representation by $u_m^*$. Set $m' = \min\{m, l+1-m\}$. In [7], Lemma 1.2 and Proposition 1.1 were used to prove

Theorem 2.1. [7] Let $r$ be an integer in the range $2m' - 2 - l \leq r \leq 0$. Then there is a $G$-module isomorphism

$$H^i(G/B, S^n u_m^* \otimes r\omega_m) = H^i(G/B, S^{n+m'} u_{l+1-m}^* \otimes -r\omega_{l+1-m}) \text{ for all } i, n \geq 0.$$

3. A THEOREM FOR $D_{2l+1}$

Theorem 2.1 has an analog in type $D_{2l+1}$. We label the simple roots of $G$ of type $D_{2l+1}$ as in [6], so $\alpha_{2l-1}$ lies at the branched vertex of the Dynkin diagram. Let $P$ be the maximal proper parabolic subgroup containing $B$ corresponding to all the simple roots except $\alpha_{2l}$. And let $P'$ be the maximal proper parabolic subgroup containing $B$ corresponding to all the simple roots except $\alpha_{2l+1}$ (so $P$ and $P'$ are interchanged by an outer automorphism of $G$).

Theorem 3.1. Let $r$ be an integer in the range $-3 \leq r \leq 0$. Then there is a $G$-module isomorphism

$$H^i(G/B, S^n u_P^* \otimes r\omega_{2l}) = H^i(G/B, S^{n+r} u_{P'}^* \otimes -r\omega_{2l+1}) \text{ for all } i, n \geq 0.$$ 

Proof. Step 1.

In this step, $r$ may be an arbitrary integer. Consider the intersection $V = u_P \cap u_{P'}$. We will show in Step 1 that for all $i, n$

$$H^i(S^n u_P^* \otimes r\omega_{2l}) = H^i(S^n V^* \otimes r\omega_{2l}).$$
We begin by taking the Koszul resolution of the short exact sequence
\[ 0 \to U \to u_p^* \to V^* \to 0 \]
(this defines \( U \)) and tensoring it with \( r \omega_{2l} \). This gives
\[ 0 \to \cdots \to S^{n-j}u_p^* \otimes \wedge^j U \otimes r \omega_{2l} \to \cdots \to S^n u_p^* \otimes r \omega_{2l} \to S^n V^* \otimes r \omega_{2l} \to 0. \]

We claim that \( H^*(S^{n-j}u_p^* \otimes \wedge^j U \otimes r \omega_{2l}) = 0 \) for \( 1 \leq j \leq \dim U \) from which Equation \( \text{[II]} \) will follow. The \( T \)-weights of \( U \) are those of the form \( \alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l} \), where \( 1 \leq k \leq 2l \). Therefore, if \( \lambda \) is a \( T \)-weight of \( \wedge^j U \), then \( \lambda \) is of the form
\[ (0, 0, 1, \ldots, 1, 2, 2, \ldots, j - 1, \ldots, j - 1, j, \ldots, j, 0) \]
in the basis of simple roots. If this expression contains a subsequence of the form \( m, m, m, 1, \ldots, 1 \), then \( \lambda \) will have inner product \(-1\) with the simple coroot corresponding to the middle \( m \). Hence \( H^*(Q \otimes \lambda) = 0 \) where \( Q \) is any \( P \)-representation by Proposition \( \text{[I.1]} \) The same result holds if there are any 0’s in the initial part of the expression. Therefore, we are reduced to considering those \( \lambda \) of the form
\[ (1, 2, 3, \ldots, j - 1, j, j, \ldots, j, 0). \]
Such a \( \lambda \) satisfies \( \langle \lambda, \alpha_{2l+1}^\vee \rangle = -j \) with the exception of the case \( j = 2l \), where instead \( \langle \lambda, \alpha_{2l+1}^\vee \rangle = -j + 1 = -2l + 1 \). In the latter case \( H^*(Q \otimes \lambda) = 0 \) by Lemma \( \text{[I.2]} \) applied to the parabolic subgroup with Levi factor of type \( A_{2l} \) consisting of all simple roots except \( \alpha_{2l} \). For the cases where \( j < 2l \), we can apply Lemma \( \text{[I.2]} \) also for the \( A_{2l} \) consisting of all simple roots except \( \alpha_{2l} \). In that case, \( a = j, b = 2l, k = -j \) and so \( k + b - a = 2l - 2j \), which, being an even number, is never -1. Also, clearly \( -b - 1 \leq k \leq -1 \). Thus we conclude that for all weights \( \lambda \) appearing in \( \wedge^j U \), we have \( H^*(Q \otimes \lambda) = 0 \) for any \( P \)-representation \( Q \). Hence for \( Q := S^{n-j} u_p^* \otimes r \omega_{2l} \), it follows that \( H^*(Q \otimes \wedge^j U) = 0 \) by the usual filtration argument.

**Step 2.**
Let \( V_1 \) be the \( B \)-stable subspace of \( u \) consisting of the direct sum of all root spaces \( g_\alpha \) where \(-\alpha\) is bigger than or equal to the root
\[ (0, \ldots, 0, 1, 2, 1, 1) \]
in the usual partial ordering on roots. Let \( V_2 \) be the \( B \)-stable subspace of \( u \) consisting of the direct sum of all root spaces \( g_\alpha \) where \(-\alpha\) is bigger than or equal to the root
\[ (0, 0, \ldots, 0, 1, 2, 2, 1, 1). \]
Let \( \mu \) be a weight of the form \( r \omega_{2l} + s \omega_{2l+1} \) where \( r, s \) are integers. Assume that \(-3 \leq r \leq -1 \) and that \( s = 0 \) if \( r = -3 \). In this step we show for all \( n \geq 0 \) that
\[ H^*(S^n V_1^* \otimes \mu) = 0. \]

Take the Koszul resolution
\[ 0 \to U_2 \to V_1^* \to V_2^* \to 0 \]
(this defines \( U_2 \)) and tensor it with \( \mu \). We will show that
\[ H^*(S^n V_2^* \otimes \mu) = 0 \]
and
\[ H^s(S^{n-j}V^*_1 \otimes \wedge^j U_2 \otimes \mu) = 0 \]
for \( 1 \leq j \leq 2l - 2 \) and then Equation (2) will follow (the dimension of \( U_2 \) is \( 2l - 2 \) as shown below).

The subspace \( V^*_2 \) is stable under the minimal parabolic subgroups \( P_{\alpha_m} \) for \( m = 2l - 1, 2l, \) and \( 2l + 1 \). It follows from the assumption on \( \mu \) that \( H^s(S^n V^*_2 \otimes \mu) = 0 \) by Lemma 1.2 applied to the \( A_3 \) determined by the simple roots \( \alpha_m \) for \( m = 2l - 1, 2l, \) and \( 2l + 1 \).

Now the \( T \)-weights of \( U_2 \) are \( \alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1} \) where \( 1 \leq k \leq 2l - 2 \). If \( \lambda \) is a weight of \( \wedge^j U_2 \), then \( \lambda \) is of the form
\[(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, j-1, j, \ldots, j, 2j, j, j)\]
in the basis of simple roots. As in the previous step, if there are any 0’s present or if any of the integers between 1 and \( j - 1 \) inclusive are repeated, then
\[ H^s(Q \otimes \lambda) = 0 \]
where \( Q := S^{n-j}V^*_1 \otimes \mu \) since \( Q \) is stable under the action of the parabolic subgroups \( P_{\alpha_k} \) for \( 1 \leq k \leq 2l - 2 \). Hence we are reduced to considering those \( \lambda \) of the form
\[(1, 2, 3, \ldots, j-2, j-1, j, \ldots, j, 2j, j, j)\]
for \( 1 \leq j \leq 2l - 2 \). Such a \( \lambda \) satisfies \( \langle \lambda, \alpha_{2l-2} \rangle = -j \) with the exception of \( j = 2l - 2 \) where \( \langle \lambda, \alpha_{2l-2} \rangle = -2l + 3 \). In the latter case \( H^s(Q \otimes \lambda) = 0 \) by Lemma 1.2 applied to the \( A_{2l-2} \) consisting of the first \( 2l - 2 \) simple roots. For the cases where \( j < 2l - 2 \), we can apply Lemma 1.3 also for the \( A_{2l-2} \) consisting of the first \( 2l - 2 \) simple roots. In that case, \( a = j, b = 2l - 2, \) \( k = -j \) and so \( k + b - a = 2l - 2j - 2 \), which is never \(-1 \). Also, clearly \(-b - 1 \leq k \leq -1 \). We therefore also have \( H^s(Q \otimes \lambda) = 0 \).

Consequently, if we filter \( \wedge^j U_2 \) by \( B \)-submodules such that the quotients are one-dimensional, we deduce that
\[ H^s(S^{n-j}V^*_1 \otimes \wedge^j U_2 \otimes \mu) = 0 \]
for \( 1 \leq j \leq 2l - 2 \). Hence Equation (2) follows.

**Step 3.**
In this step, we show that for all \( i, n \)
\[ H^i(S^n V^*_1 \otimes \mu) = H^i(S^{n-i}V^*_1 \otimes \mu + \omega_{2l} + \omega_{2l+1}) \]
for \( \mu \) as in Step 2.

We take the Koszul resolution of the short exact sequence
\[ 0 \to U_1 \to V^* \to V^*_1 \to 0 \]
(this defines \( U_1 \)) and tensor it with \( \mu \) arriving at
\[ 0 \to S^{n-2l+1} V^* \otimes \wedge^{2l-1} U_1 \otimes \mu \to \cdots \to S^{n-j} V^* \otimes \wedge^j U_1 \otimes \mu \to \cdots \to S^n V^* \otimes \mu \to S^n V^*_1 \otimes \mu \to 0 \]

We first show that \( H^s(S^{n-j}V^*_1 \otimes \mu \otimes \lambda) = 0 \) for any \( \lambda \) appearing in \( \wedge^j U_1 \) for \( j \neq 0, l \). The weights of \( U_1 \) are
\[ \alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1} \]
where $1 \leq k \leq 2l - 1$. If $\lambda$ is a weight of $\wedge^j U_1$, then as in the previous steps we are quickly reduced to those $\lambda$ of the form

$$(1, 2, 3, \ldots, j - 2, j - 1, j, \ldots, j, j, j)$$

for $1 \leq j \leq 2l - 1$. Such a $\lambda$ satisfies $\langle \lambda, \alpha^{(2l - 1)}_j \rangle = -j$ with the exception of $j = 2l - 1$ where $\langle \lambda, \alpha^{(2l - 2)}_j \rangle = -2l + 2$. The latter vanishing follows from Lemma \[2.2\] applied to the $A_{2l - 1}$ consisting of the first $2l - 1$ simple roots. For the cases where $j < 2l - 1$, we can apply Lemma \[1.3\] also for the $A_{2l - 1}$ consisting of the first $2l - 1$ simple roots. In that case, $a = j$, $b = 2l - 1$, $k = -j$ and so $k + b - a = 2l - 2j - 1$, which is $-1$ only when $j = l$. Therefore, we deduce that

$$H^*(S^{n-j} V^* \otimes \wedge^j U_1 \otimes \mu) = 0$$

when $j \neq 0, l$. And furthermore,

$$H^i(S^{n-l} V^* \otimes \wedge^l U_1 \otimes \mu) = H^i(S^{n-l} V^* \otimes \lambda \otimes \mu),$$

where $\lambda = (1, 2, 3, \ldots, l - 1, l, \ldots, l, l, l)$. Now $S^{n-l} V^* \otimes \mu$ is stable under $P_{\alpha_m}$ for $1 \leq m \leq 2l - 1$. Hence $l - 1$ applications of Proposition \[1.1\] yields

$$H^i(S^{n-l} V^* \otimes \lambda \otimes \mu) = H^{i+l-1}(S^{n-l} V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}).$$

By breaking Equation \[4\] into short exact sequences and taking cohomology on $G/B$, we conclude that

$$H^i(S^n V^* \otimes \mu) = H^i(S^n V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}),$$

where we are using

$$H^*(S^n V^*_1 \otimes \mu) = 0$$

from Step 2.

**Step 4.** We obtain the theorem by using Step 3 repeatedly, starting with $\mu = r\omega_m$ with $r$ in the prescribed range of the statement of the theorem. After $-r$ steps we arrive at

$$H^i(S^n V^* \otimes r\omega_{2l}) = H^i(S^{n+rl} V^* \otimes -r\omega_{2l+1}),$$

for all $i, n$. The proof is completed by using Step 1 and the symmetric version of Equation \[1\] (obtained by applying an outer automorphism of $G$) which gives

$$H^i(S^{n+rl} V^* \otimes -r\omega_{2l+1}) = H^i(S^{n+rl} \mu_{P_r} \otimes -r\omega_{2l+1})$$

for all $i, n$. □

In what follows, we will use Theorem \[2.1\] in the more general situation of Section 4 in \[3\]. Similarly we can apply Theorem \[3.1\] in an analogous general situation. Namely, suppose $G$ is of general type and $P$ is a parabolic subgroup of $G$ containing $B$ with Levi factor $L$ containing a simple factor of type $A_{2l}$. Furthermore, suppose this simple factor belongs to a Levi subgroup $L'$ of $G$ of type $D_{2l+1}$ and $[L, L'] \subset L'$. Then the analog in $G$ of Theorem \[3.1\] holds just as the analog of Theorem \[2.1\] does in Proposition 6 in \[3\].
4. Main Theorem

For the rest of the paper $G$ is connected of type $D_{2l}$. We want to show that both nilpotent orbits in $\mathfrak{g}$ with partition $(a^{2k}, b^2)$ for $a, b$ distinct even natural numbers with $ak + b = 2l$ (see [2]) have normal closure. Let $O$ denote one of these two orbits.

Following the idea of [3], we find a nilpotent orbit $O'$ which we already know has normal closure and which contains $O$ in its closure. If we can show that the regular functions on $O$ are naturally a quotient of the regular functions on $O'$, then it follows that $O$ also has normal closure. To that end we consider the nilpotent orbit $O'$ in $\mathfrak{g}$ with partition $\lambda = (a^{2k}, b+1, b-1)$.

Lemma 4.1. The closure of $O'$ is normal.

Proof. The only minimal degenerations of $O'$ in $\mathfrak{g}$ are the two orbits with partition $\mu = (a^{2k}, b^2)$ (which together are one orbit for the full orthogonal group of rank $2l$). Hence by [3] the singularity of the closure of $O'$ along the union of these two orbits is smoothly equivalent to the singularity of the closure of the orbit with partition $(2)$ along the orbit with partition $(1, 1)$ in type $A_1$ (we remove the first $2k$ rows from $\lambda$ and $\mu$, and then remove the first $b - 1$ columns from the resulting partitions). Hence this is a singularity of type $A_1$ and so by [3], $O'$ has normal closure.

Lemma 4.2. The orbit $O'$ is a Richardson orbit for any parabolic with Levi factor of type

$$\frac{a-b}{2} - 1 \times A_{2k-1} \times \cdots \times A_{2k-1} \times A_{2k} \times A_{2k+1} \times \cdots \times A_{2k+1} \times \frac{b}{2} - 1.$$

Any parabolic with Levi factor of type

$$\frac{a-b}{2} \times A_{2k-1} \times \cdots \times A_{2k-1} \times A_{2k+1} \times \cdots \times A_{2k+1} \times \frac{b}{2}$$

has Richardson orbit one or the other of the two nilpotent orbits with partition $(a^{2k}, b^2)$.

Proof. Both statements follow from Section 7 in [2].

It will be convenient to represent parabolic subgroups containing $B$ by the simple roots of $G$ which are not simple roots of their Levi factors. Thus we can speak of such a parabolic subgroup as a subset of the numbers $1$ to $2l + 1$, with each number $i$ corresponding to the simple root $\alpha_i$.

Set $d = a - b$ and let $P'$ be the parabolic represented by

$$\{2k + 1, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l\}$$

and let $P''$ be represented by

$$\{2k + 1, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l - 1\},$$

so $P'$ are $P''$ are interchanged by an outer automorphism of $D_{2l+1}$. By the previous lemma $O'$ is Richardson for both $P'$ and $P''$. Let $P$ be the parabolic represented by

$$\{2k, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l\}.$$
Theorem 4.3. There is a short exact sequence
\begin{equation}
0 \to H^0(S^{n-2l-k(a-4)-1}u_{P_0}^* \otimes \nu) \to H^0(S^n u_{P_0}^*) \to H^0(S^n u_P^*) \to 0,
\end{equation}
where \( \nu = \omega_{4k+2} \) if \( a > 4 \) and \( \nu = 2\omega_{2l-1} \) if \( a = 4 \) (and hence \( b = 2 \)).

Proof. We use two elements from the proof of Theorem 2.1 in [7]. Let \( P_1 \) be the parabolic represented by
\[
\{2k + 2, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l\}
\]
and set \( V = u_P \cap u_{P_1} \). Then Step 1 of the proof of Theorem 2.1 (for a group of type \( A_{4k+1} \) applied to the first \( 4k + 1 \) simple roots of \( G \)) yields the isomorphism \( H^i(S^n u_{P_1}^*) = H^i(S^n V^*) \) for all \( i, n \). And Step 3 of the proof Theorem 2.1 yields the long exact sequence
\[
\ldots \to H^i(S^{n-2k-1} u_{P_0}^* \otimes \mu) \to H^i(S^n u_{P_0}^*) \to H^i(S^n V^*) \to H^{i+1}(S^{n-2k-1} u_{P_0}^* \otimes \mu) \to \ldots
\]
where \( \mu \) equals
\[
(1, 2, 3, \ldots, 2k, 2k + 1, 2k, \ldots, 2, 1, 0, 0, \ldots, 0).
\]
This is obtained by taking the Koszul resolution of
\[
0 \to U \to u_{P_0}^* \to V^* \to 0
\]
(this defines \( U \)) and simplifying the terms.

The remainder of the proof involves showing that
\[
H^i(S^{n-2k-1} u_{P_0}^* \otimes \mu) = H^i(S^{n-2l-k(a-4)-1} u_{P_0}^* \otimes \nu)
\]
for all \( i, n \).

This is carried out by using Theorem 2.1 numerous times (for \( r = -1 \) and the \( l \) in that theorem equal to either \( 4k \) or \( 4k + 1 \) and \( m = 2k \) or \( 2k + 1 \), respectively) and Theorem 3.1 once (for \( r = -2 \) and the \( l \) in that theorem equal to \( k \)).

After \( \frac{a-b-2}{2} \) applications of Theorem 2.1 with \( r = -1, l \) there equal to \( 4k \), and \( m' = 2k \), we have
\[
H^i(S^{n-2k-1} u_{P_0}^* \otimes \mu) = H^i(S^{n-k(a-b)-1} Q_1^* \otimes \mu_1)
\]
where \( \mu_1 \) equals
\[
(1, 2, 3, \ldots, 2k, 2k + 1, 2k, \ldots, 2, 1, 0, 0, \ldots, 0),
\]
and \( Q_1 \) is the Lie algebra of the unipotent radical of
\[
\{2k+1, 4k+1, 6k+1, \ldots, k(d-2)+1, kd+1, k(d+2)+2, k(d+4)+4, k(d+6)+6, \ldots, 2l-2k-2, 2l\}.
\]

Next, we apply Theorem 2.1 \( \frac{b-2}{2} \) more times with \( r = -1, l \) there equal to \( 4k + 1 \), and \( m' = 2k + 1 \), to obtain
\[
H^i(S^{n-k(a-b)-1} Q_1^* \otimes \mu_1) = H^i(S^{n-ka+2k-b/2} Q_2^* \otimes \mu_2)
\]
where \( \mu_2 \) equals
\[
(1, 2, 3, \ldots, 2k, 2k + 1, 2k, \ldots, 2k + 1, 2k, \ldots, 2, 1, 0),
\]
and $Q_2$ is the Lie algebra of the unipotent radical of
\[ \{2k+1, 4k+1, 6k+1, \ldots, k(d-2)+1, kd+1, k(d+2)+3, k(d+4)+5, k(d+6)+7, \ldots, 2l-2k-1, 2l\} \]

Next, we use Theorem 3.1 with $r = -2$ for the case $D_{2k+1}$ applied to the simple roots $\alpha_i$ of $G$ with $2l - 2k \leq i \leq 2l$. This yields
\[ H^i(S^{n-ka+2k-b/2}Q_2^{'\prime} \otimes \mu_2) = H^i(S^{n-ka-b/2}Q_3^\prime \otimes \mu_3) \]
where $\mu_3$ equals
\[
\begin{align*}
(1, 2, 3, & \ldots, 2k, 2k + 1, \ldots, 2k + 1, 2k + 2, 2k + 3, 2k + 4, \ldots, 4k, 2k + 1, 2k),
\end{align*}
\]
and $Q_3$ is the Lie algebra of the unipotent radical of
\[ \{2k+1, 4k+1, 6k+1, \ldots, k(d-2)+1, kd+1, k(d+2)+3, k(d+4)+5, k(d+6)+7, \ldots, 2l-2k-1, 2l-1\} \]

If $2l - 4k - 1 = 1$, which is the case if and only if $a = 4$ and $b = 2$, we have $\mu_3 = 2\omega_{2l-1}$ and the latter parabolic subgroup is $P''$.

On the other hand, if $a > 4$, we continue by using Theorem 2.1 another $\frac{a-b}{2}$ times followed by another $\frac{b-2}{2}$ times (in reverse of how we have just used it). The result is that
\[ H^i(S^{n-ka-b/2-2}Q_3^\prime \otimes \mu_3) = H^i(S^{n-2ka+4k-b-1}Q_4^\prime \otimes \mu_4) \]
where $\mu_4$ equals
\[
\begin{align*}
(1, 2, 3, & \ldots, 4k + 1, 4k + 2, \ldots, 4k + 2, 2k + 1, 2k + 1),
\end{align*}
\]
and $Q_4$ is the Lie algebra of the unipotent radical of
\[ \{2k+1, 4k+2, 6k+2, \ldots, k(d-2)+2, kd+2, k(d+2)+2, k(d+4)+4, k(d+6)+6, \ldots, 2l-2k-2, 2l-1\} \]
The latter parabolic is exactly $P''$ and $\mu_4 = \omega_{4k+2}$. Furthermore, $n - 2ka + 4k - b - 1 = n - 2l - ka + 4k - 1$ since $ak + b = 2l$.

Hence when $a = 4$ or $a > 4$, we have shown that
\[ H^i(S^{n-2k-1}u_{P''}^* \otimes \mu_i) = H^i(S^{n-2l-k(a-4)-1}u_{P''}^* \otimes \nu) \]
for all $i, n$. We finish the proof by observing that $\nu$ extends to a character of $P''$ and it is dominant. Hence $H^i(S^{n-2l-k(a-4)-1}u_{P''}^* \otimes \nu) = 0$ for $i > 0$ as in [11]. Similarly, $H^i(S^n u_{P''}^*) = 0$ and $H^i(S^n u_{P'}^*) = 0$ for $i > 0$ and the proof is complete. \hfill \Box

**Corollary 4.4.** The closure of $\mathcal{O}$ is normal.

**Proof.** We only need to note that the functions of degree $n$ on $\mathcal{O}'$ (and also its closure since the closure is normal) as a $G$-module are isomorphic to $H^0(S^n u_{P''}^*)$. This follows since $\mathcal{O}'$ has trivial $G$-equivariant fundamental group when $G$ is adjoint (see [2]). Hence the moment map determined by $P'$ must be birational. Thus the short exact sequence of the theorem together with the discussion in Section 3 of [8] yields the result. \hfill \Box
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