A practical guide to well roundedness

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Abstract

Let \( G \) be a semisimple algebraic group. We develop a machinery for manipulation and manufacture of well-rounded families \( \{B_T\}_{T>0} \subset G \) as they were defined in a work by A. Gorodnik and A. Nevo. The importance of these types of families is that one can asymptotically count lattice points in them and even obtain an error term. Lattice counting is highly effective for solving asymptotic problems from number theory and the geometry of numbers.

The tools we develop are handy especially when the family is given w.r.t. some decomposition of \( G \) (e.g. Iwasawa or Cartan) and also when it depends upon a sub-quotients of the form \( \mathcal{M}/H \), where \( \mathcal{M} \subset G \) is a submanifold and \( H < G \) is a closed subgroup.

1 Introduction

This text came to life out from the authors’ work [HK19, HK20b, HK20a] on equidistribution problems in geometry of numbers. In the course of our work we relied significantly on counting lattice point results in semisimple algebraic groups. The classical setting for counting lattice points problems is \( \mathbb{R}^d \), where in order to establish an asymptotic formula for the number of lattice points inside an increasing family of compact subsets of \( \mathbb{R}^d \), it is required that the boundary of the sets satisfies certain regularity conditions (e.g. smoothness, finite non vanishing curvature, etc.). With the rise of interest in counting problems in semisimple groups and their affine spaces, e.g. hyperbolic spaces (e.g. [EM93]), where the boundary of a set has fundamentally different properties than in \( \mathbb{R}^d \), a new notion of regularity for a family \( \{B_T\} \) was born: well roundedness. It was introduced in [EM93] (although a similar idea appeared already in [DRS93]), was used e.g. in [GW07], and then defined again in [GN12], for the general setting of lcsc groups. In the latter, the concept of well roundedness was further refined to Lipschitz well roundedness:

Definition 1.1 ([GN12]). Let \( G \) be a locally compact second countable group with a Borel measure \( \mu \), and let \( \{\mathcal{O}_\epsilon\}_{\epsilon>0} \) be a family of identity neighborhoods in \( G \). Assume \( \{B_T\}_{T>0} \subset G \) is a family of measurable domains and denote

\[
B_T^{(+\epsilon)} := \mathcal{O}_\epsilon B_T \mathcal{O}_\epsilon = \bigcup_{u,v \in \mathcal{O}_\epsilon} u B_T v,
\]

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The family \( \{B_T\} \) is Lipschitz well-rounded (LWR) with (positive) parameters \((C, T_0, \epsilon_0)\) if for every \(0 < \epsilon < \epsilon_0\) and \(T > T_0:\)

\[
\mu \left( B_T^{(\epsilon)} \right) \leq (1 + C \epsilon) \mu \left( B_T^{(-\epsilon)} \right). \tag{1.1}
\]

The parameter \(C\) is called the Lipschitz constant of the family \(\{B_T\}\).

The results in [GN12] show that under some conditions on \(G\), which originate from representation theory, one can get an estimate with an error term of the size of the set \(\Gamma \cap B_T\), where \(\Gamma < G\) is a lattice and \(T \to \infty\). To be able to use these results to solve concrete problems, one should have an ample supply of LWR sets as well as a mechanism for checking that a certain family is indeed well rounded. This is the first aim of this work.

Despite their name, simple (and semisimple) groups are quite complicated objects, and they are often studied via their decompositions into “less complicated” groups. A decomposition is when a group (or some “big” subset of it) is written as a product of certain subgroups, e.g. the Cartan, Iwasawa and Bruhat decompositions. In the context of counting lattice points, many natural counting problems translate into counting lattice points in families of increasing sets inside semisimple Lie groups, that are defined via a decomposition of the group. As a baby example, consider the Cartan decomposition \(KAK\) of \( SO_{1,n}(\mathbb{R}) \), with \(K = SO_n(\mathbb{R})\) and \(A = \{a_t : t \in \mathbb{R}\}, a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix} \). The set \(B_T = \{Ka_tK : 0 \leq t \leq T\}\) is the lift of a hyperbolic ball of radius \(T\) in \(H^n \simeq K \setminus SO_{1,n}(\mathbb{R})\), and so counting lattice points in the family \(\{B_T\}_{T>0}\) is essentially equivalent to the hyperbolic sphere problem [LP82, PR94], first stated by Selberg, which concerns counting lattice orbit points in increasing hyperbolic balls.

We refer to [GN12] (see also [GOS10]) for counting in sets that are defined via the Cartan decomposition, to [MMO14] for counting in sets that are defined via the Bruhat decomposition, and to [HN16] for counting in sets that are defined via the Iwasawa decomposition, which is also the setting in our aforementioned ongoing work, for which this text was written.

When verifying the well roundedness of a family of sets inside a semisimple Lie group, it is therefore quite natural to consider a suitable decomposition of the group, and then try and reduce the verification to the that of well roundedness of the projections of the family in each one of the subgroups that appear in the decomposition. The logic being that these subgroups are easier to analyze since they are compact, or abelian, or unipotent, etc. In the example with the lifts of hyperbolic balls, one would like to reduce the well roundedness of the family \(\{B_T\}_{T>0}\) to well roundedness of \(\{a_t : 0 \leq t \leq T\}_{T>0}\) inside the subgroup \(A\) (and of the constant family \(\{K\}\) inside \(K\), which is trivial). However, it is false that well roundedness in the components of the decomposition implies well roundedness of the original family in the group, so this reduction cannot be implemented without further thought. We have considered a systematic approach to this problem, which can be also applied in other situation as well. The idea is a categorical approach of defining morphisms between groups, called roundomorphisms, which pull back a well rounded family in the image into a well rounded family in the domain.

When it comes to counting problems in a semisimple group, many families of interest have the property that their projections to one or more of the components is a fixed set.
A simple example is the projection to the $K$ components of the lifted hyperbolic balls, but in fact such families arise naturally in equidistribution problems (see [Tru13, HN16] and our aforementioned work in progress). Since our method relies on well roundedness in the components, it is helpful to formulate a condition that is easier to verify than the one of well roundedness itself, yet implies well roundedness for a constant family of sets; by that we mean $B_T = B$ for all $T$. We propose the following:

**Definition 1.2.** Let $\mathcal{M}$ be an orbifold. A subset $B$ of $\mathcal{M}$ is called a Boundary Controllable Set, or BCS, if for every $x \in \mathcal{M}$ there is an open neighborhood $U_x$ of $x$ such that $U_x \cap \partial B$ is contained in a finite union of embedded submanifolds of $\mathcal{M}$.

**Remark 1.3.** In most cases (e.g. when $\mathcal{M}$ is a manifold and $B$ is bounded) the global version of the previous definition is sufficient. That means one can take for every point $x \in \mathcal{M}$ the open set $\mathcal{M}$, and so it is sufficient to check that the boundary of $B$ is contained in a finite number of embedded submanifolds with dimension strictly smaller than $\dim(\mathcal{M})$.

Orbifolds arise naturally as a quotient of a submanifold of $G$ by a group with almost free stabilizers. One example would be the space $\text{SO}_n(\mathbb{R}) \backslash \text{SL}_n(\mathbb{R}) / \text{SL}_n(\mathbb{Z})$ which bears great significance in geometry of numbers, being the space of *shapes* of lattices, and when $n = 2$ it is the modular curve. Our aforementioned work in progress concerns equidistribution in spaces of lattices that have the structure of an orbifold, e.g. the space of shapes. Since our approach is lifting the counting problem from the quotient to the group $G$, where we count in well rounded families, we are forced to transfer BCS’s from the quotient space into some convenient fundamental domain inside the group. It turns out that not every fundamental domain is adequate, and in Section 6 we characterize the fundamental domains that are. We call them spread models, since we think of them as fundamental domains that are obtained by cutting the space open and spreading it - here one can think of cutting a two dimensional torus into a parallelogram. We hope that this section could be of further interest in the future, since it is essentially a discussion on fundamental domains for which the quotient map into the space the domain represents, pulls back differential properties from the space to the domain.

This investigation is the second main goal of this work.

Finally in the last section 7.1, we provide some examples for spread models. In particular, we introduce the well known ([Gre93, Sch98]) constructions of fundamental domains coming from geometry of numbers and show that they are indeed spread models for the spaces of lattices that they represent.

## 2 Coordinate balls

Definition 1.1 is w.r.t. a nested family $\{O_{\epsilon}\}_{\epsilon > 0}$ of identity neighborhoods in the group, where by “nested” we mean that $\epsilon_1 < \epsilon_2$ implies $O_{\epsilon_1} \subset O_{\epsilon_2}$. While the definition of well roundedness allows any nested family of identity neighborhoods, we shall work only with neighborhoods that are the images of small balls in the Lie algebra under the exponent map — this is Assumption 2.7 which concludes the current subsection. The advantages of this choice follow from the fact that it is a special case of coordinate balls (Definition 2.2), and this subsection is devoted to investigating the properties of neighborhoods of this sort.
Definition 2.1 (Equivalence of identity neighborhoods). Let $G$ be a Lie group and consider two families $\{O_\epsilon\}_{\epsilon > 0}, \{O'_\epsilon\}_{\epsilon > 0}$ of nested and symmetric identity neighborhoods. We say that these families are \textit{equivalent} if there exist $\epsilon_1, c, C > 0$ such that for every $0 < \epsilon < \epsilon_1$

$$O_{c\epsilon} \subseteq O'_\epsilon \subseteq O_{C\epsilon}. $$

Definition 2.2 (Coordinate balls). A family $\{O_\epsilon\}_{\epsilon > 0}$ of identity neighborhoods inside a Lie group $G$ will be called a family of \textit{coordinate balls} if there exist a ball $B_\epsilon = \{ x \in \mathbb{R}^{\dim(G)} : \|x\| < \epsilon \}$ inside $\mathbb{R}^{\dim(G)}$, and a $C^1$ chart

$$\phi : \bigcup_{g \in G} U \rightarrow \mathbb{R}^m$$

of the identity, such that $\{\phi^{-1}(B_\epsilon)\}_{\epsilon > 0}$ is equivalent to $\{O_\epsilon\}_{\epsilon > 0}$.

Remark 2.3. All coordinate balls of a given Lie group are equivalent. Indeed, if $\phi_1$ and $\phi_2$ are two charts, then $\phi_2\phi_1^{-1}|_{B_1}$ is a bi-Lipschitz map. Hence,

$$\phi_2^{-1}(B_{c\epsilon}) \subseteq \phi_2^{-1}\left(\phi_2\phi_1^{-1}(B_\epsilon)\right) \subseteq \phi_2^{-1}(B_{C\epsilon})$$

for some $c, C > 0$ and $\epsilon < 1$.

The following Lemma specifies two useful features of coordinate balls.

Lemma 2.4. Let $\{O_\epsilon\}_{\epsilon > 0}$ be a family of coordinate balls inside a Lie group $G$. Then for small enough $\epsilon$ and $\delta$, the following two properties hold:

- \textbf{(Connectivity)} $O_\epsilon$ is a connected subset of $G$.
- \textbf{(Additivity)} There exists $c > 0$ such that:

$$O_\epsilon O_\delta \subseteq O_{c(\epsilon + \delta)}. $$

Proof. Connectivity holds since $\phi^{-1}$ ($\phi$ being the associated chart) is continuous. Additivity holds for Riemannian left $G$-invariant balls with $c = 1$ (triangle inequality); these Riemannian balls are indeed coordinate balls, where the implied chart is the Riemannian exponential map. Since all families of coordinate balls are equivalent (Remark 2.3), the statement follows.

One last property of coordinate balls is the following.

Proposition 2.5. Let $\{O_\epsilon\}_{\epsilon > 0}$ be a family of coordinate balls inside a Lie group $G$, and assume

$$\phi : \bigcup_{g \in G} U \rightarrow \mathbb{R}^m$$

is a chart that contains an element $g$. Then, there exist an open set $g \in V \subseteq U$ and positive $\epsilon(g), c(g)$ such that for $\epsilon \leq \epsilon(g)$:

$$O_\epsilon V O_\epsilon \subseteq U$$

and for every $h \in V$

$$\phi(O_\epsilon h O_\epsilon) \subseteq \phi(h) + B_{c(g)}.$$
The proof requires an auxiliary lemma:

**Lemma 2.6** ([HiNT6]). Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. For $\mathcal{O}_\epsilon = \exp(B_\epsilon)$ and every $g \in G$,

$$g^{-1} \mathcal{O}_\epsilon g \subseteq \mathcal{O}_{\epsilon \| \Ad g \|_{\text{op}}} = \exp \left\{ Z \in \mathfrak{g} : \| Z \| \leq \epsilon \cdot \| \Ad g \|_{\text{op}} \right\},$$

where $\| \cdot \|$ is any euclidean norm on $\mathfrak{g}$ and $\| \cdot \|_{\text{op}}$ is the norm on the space of linear operators on $\mathfrak{g}$.

**Proof of Proposition 2.5.** Observe that by the previous lemma and the additivity property in Lemma 2.4, for every $h \in G$ there is a constant $c_1(h)$ such that for $0 < \epsilon \leq c_1(h)$:

$$\phi (\mathcal{O}_\epsilon h \mathcal{O}_\epsilon) = \phi \left( h \cdot h^{-1} \mathcal{O}_\epsilon h \cdot \mathcal{O}_\epsilon \right) \subseteq \phi \left( h \cdot \mathcal{O}_{c_1(h)\epsilon} \right) = \phi \circ L_h \circ \exp (B_{c_1(h)\epsilon}),$$

where $L_h : G \to G$ is the left translation by $h$. By compactness of $D_g := \overline{O_1 g O_1}$ and continuity of $\| \Ad (\cdot) \|_{\text{op}}$, there exist $c_0(g)$ and $\epsilon_0(g)$ for which the above holds uniformly on $D_g$, namely for every $h \in D_g$ and $0 < \epsilon \leq \epsilon_0(g)$:

$$\phi (\mathcal{O}_\epsilon h \mathcal{O}_\epsilon) \subseteq \psi_h \left( B_{c_0(g)\epsilon} \right).$$

We choose $\epsilon (g) > 0$ and $0 < \delta < 1$ small enough so that (using the additivity property again) for $V := \mathcal{O}_{\delta g} \mathcal{O}_{\delta}$ and $0 < \epsilon < \epsilon (g)$ we have

$$\mathcal{O}_\epsilon V \mathcal{O}_\epsilon = \mathcal{O}_\epsilon \mathcal{O}_{\delta g} \mathcal{O}_{\delta} \mathcal{O}_\epsilon \subseteq \mathcal{O}_{\epsilon_0(g)g} \mathcal{O}_{\epsilon_0(g)}.$$

We also assume $\epsilon_0(g)$ is small enough such that $\mathcal{O}_{\epsilon_0(g)g} \mathcal{O}_{\epsilon_0(g)} \subseteq U$.

Since $\psi(h, x) = \psi_h(x)$ is a differentiable map defined on a compact domain $U \times \overline{B_{c_0(g)\epsilon}}$, there exists $c(g) = c(D_g) > 0$ such that for every $h \in U$ and $x \in B_{c_0(g)\epsilon}$:

$$\| \psi_h(x) - \psi_h(0) \| \leq c(g) \| x - 0 \|.$$

Hence,

$$\psi_h \left( B_{c_0(g)\epsilon} \right) \subseteq \psi_h(0) + B_{c(g)\epsilon} = \phi(h) + B_{c(g)\epsilon}.$$

Finally, we fix a choice of coordinate balls that will be used from now on.

**Assumption 2.7.** Unless specified otherwise we will assume that $\mathcal{O}_\epsilon = \exp(B_\epsilon)$, where $\exp$ is the Lie exponent.

## 3 Well rounded sets - criteria and properties

This section is devoted to investigating the concept of well-roundedness for constant families, which are just fixed subsets of $G$: $B_T = B$ for all $T$. It turns out that in the constant case, the LWR property can be reduced to a boundary condition. This enables us to obtain a huge class of Lipschitz well rounded (fixed) sets, which are in fact the BCS's defined in the introduction.
**Lemma 3.1.** Suppose \( \{O_\epsilon\}_{\epsilon>0} \) is a family of coordinate balls, and let \( B \subseteq G \). Then \( B^+ (\epsilon) \setminus B^- (\epsilon) = O_\epsilon \partial B \cdot O_\epsilon \), or equivalently:

\[
B^{(+\epsilon)} = B \cup (O_\epsilon \partial B \cdot O_\epsilon)
\]

and

\[
B^{(-\epsilon)} = B \setminus (O_\epsilon \partial B \cdot O_\epsilon).
\]

**Remark 3.2.** In fact, Lemma 3.1 applies for any family \( \{O_\epsilon\}_{\epsilon>0} \) of connected identity neighborhoods.

**Proof.** We first show that

\[
B^{(+\epsilon)} \setminus B^{(-\epsilon)} = O_\epsilon \partial B \cdot O_\epsilon.
\]

For the inclusion \( \supseteq \), we must show that \( B^{(+\epsilon)} \supseteq O_\epsilon \partial B \cdot O_\epsilon \) and that \( (O_\epsilon \partial B \cdot O_\epsilon) \cap B^{(-\epsilon)} = \emptyset \). For the first, assume \( g \in O_\epsilon \partial B \cdot O_\epsilon \). By symmetry of \( O_\epsilon \), the open set \( O_\epsilon \cdot g \cdot O_\epsilon \) intersects \( \partial B \) non-trivially, and therefore meets \( B \), say in a point \( h \). Then (again by symmetry) \( g \in O_\epsilon \cdot h \cdot O_\epsilon \subset O_\epsilon \partial B \cdot O_\epsilon \). For the latter, note that \( h \in B^{(-\epsilon)} \) if and only if \( h \in uBv \) for all \( u, v \in O_\epsilon \), i.e. if and only if \( u^{-1}h v^{-1} \in B \) for all \( u, v \in O_\epsilon \), which by symmetry of \( O_\epsilon \) is equivalent to \( O_\epsilon \cdot h \cdot O_\epsilon \subset B \). Now if \( g \in O_\epsilon \partial B \cdot O_\epsilon \), then as before the open set \( O_\epsilon \cdot g \cdot O_\epsilon \) intersects \( \partial B \) non-trivially, and in particular meets \( B^c \); then \( O_\epsilon \cdot g \cdot O_\epsilon \not\subseteq B \), namely \( g \not\in B^{(+\epsilon)} \).

For the inclusion \( \subseteq \), let \( g \not\in O_\epsilon \partial B \cdot O_\epsilon \), and we show that \( g \not\in B^{(+\epsilon)} \setminus B^{(-\epsilon)} \). Namely, that either \( g \in B^{(-\epsilon)} \) or that \( g \in (B^{(+\epsilon)})^c \). Indeed, \( g \not\in O_\epsilon \partial B \cdot O_\epsilon \) implies that \( (O_\epsilon g \cdot O_\epsilon) \cap \partial B = \emptyset \), and since \( O_\epsilon g \cdot O_\epsilon \) is connected it follows that either \( O_\epsilon g \cdot O_\epsilon \subseteq B \) or \( O_\epsilon g \cdot O_\epsilon \subseteq B^c \). The first implies (by the equivalence established in the first inclusion) that \( g \in B^{(-\epsilon)} \). The latter implies that \( g \not\in O_\epsilon \partial B \cdot O_\epsilon = B^{(+\epsilon)} \).

The statement of the lemma now follows:

\[
B^{(+\epsilon)} = B^{(-\epsilon)} \cup O_\epsilon \partial B \cdot O_\epsilon \subseteq B \cup O_\epsilon \partial B \cdot O_\epsilon
\]

where the opposite inclusion holds as \( B^{(+\epsilon)} \supseteq O_\epsilon \partial B \cdot O_\epsilon \). Furthermore,

\[
B^{(-\epsilon)} = B^{(+\epsilon)} \setminus O_\epsilon \partial B \cdot O_\epsilon = (B \cup O_\epsilon \partial B \cdot O_\epsilon) \setminus O_\epsilon \partial B \cdot O_\epsilon =
\]

\[
= ((B \setminus O_\epsilon \partial B \cdot O_\epsilon) \cup O_\epsilon \partial B \cdot O_\epsilon) \setminus O_\epsilon \partial B \cdot O_\epsilon = B \setminus O_\epsilon \partial B \cdot O_\epsilon.
\]

\[\square\]

From Lemma 3.1, we deduce the following simple criterion for the Lipschitz well-roundedness of a (fixed) set.

**Lemma 3.3.** Let \( G \) be a Lie group with a Borel measure \( \mu \). If a subset \( B \subset G \) satisfies that \( 0 < \mu (B) < \infty \) and that there exists \( c > 0 \) such that

\[
\mu (O_\epsilon \partial B \cdot O_\epsilon) \leq c \epsilon
\]

for every \( 0 < \epsilon < \frac{\mu (B)}{2c} \), then \( B \) is LWR with

\[
C = \frac{2c}{\mu (B)}.
\]

The converse also holds: suppose \( B \) is LWR with positive measure and parameter \( C \). Then for \( \epsilon < C^{-1} \),

\[
\mu (O_\epsilon \partial B \cdot O_\epsilon) \leq C \mu (B) \epsilon.
\]
Proof. By our assumption, for \( \epsilon < \frac{\mu(B)}{2c} \) and by Lemma 3.1

\[
\mu\left(B^{(\epsilon)}\right) = \mu(\mathcal{O}_\epsilon \mathcal{O}_\epsilon)
\]

\[
= \mu\left(B^{(-\epsilon)}\right) + \mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon)
\]

\[
\leq \mu\left(B^{(-\epsilon)}\right) + c\epsilon
\]

and

\[
\mu(B^{(-\epsilon)}) = \mu\left(B \setminus (\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon)\right)
\]

\[
\geq \mu(B) - \mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon)
\]

\[
\geq \mu(B) - c\epsilon
\]

\[
\left(c < \frac{\mu(B)}{2c}\right) \geq \frac{\mu(B)}{2}
\]

As a result, for \( \epsilon < \frac{\mu(B)}{2c} \),

\[
\frac{\mu(B^{(\epsilon)}) - \mu(B^{(-\epsilon)})}{\mu(B^{(-\epsilon)})} \leq \frac{c\epsilon}{\frac{1}{2}\mu(B)} = \frac{2c}{\mu(B)} \cdot \epsilon.
\]

Regarding the opposite direction, our assumption is that for \( \epsilon < C^{-1} \),

\[
\frac{\mu(B^{(\epsilon)}) - \mu(B^{(-\epsilon)})}{\mu(B^{(-\epsilon)})} \leq C\epsilon.
\]

Hence,

\[
\frac{\mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon)}{\mu(B)} \leq \frac{\mu(B^{(\epsilon)}) - \mu(B^{(-\epsilon)})}{\mu(B^{(-\epsilon)})} \leq C\epsilon.
\]

In other words,

\[
\mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon) \leq \mu(B) C\epsilon.
\]

One consequence of Lemma 3.3 is that finite unions and intersections of LWR sets are in themselves LWR.

Lemma 3.4. Let \( G \) be a Lie group with a Borel measure \( \mu \). If two subsets \( B \) and \( B' \) of \( G \) such that \( 0 < \mu(B \cap B') \) are LWR, then \( B \cap B' \) and \( B \cup B' \) are also LWR with Lipschitz constant

\[
C^B \cap B' = 2 \max \{C, C'\} \cdot \frac{\mu(B) + \mu(B')}{\mu(B \cap B')}; \quad C_{B \cup B'} = 2 \max \{C, C'\} \cdot \frac{\mu(B) + \mu(B')}{\mu(B \cup B')}.
\]

Proof. We prove the lemma only for the intersection \( B \cap B' \); the proof for the union \( B \cup B' \) is similar. By Lemma 3.3 for \( \epsilon < \frac{1}{C_{B \cap B'}} \) (so \( \epsilon < C^{-1}, C'^{-1} \)):

\[
\mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon) \leq C \mu(B) \epsilon,
\]

\[
\mu(\mathcal{O}_\epsilon \cdot \partial B' \cdot \mathcal{O}_\epsilon) \leq C' \mu(B') \epsilon.
\]
Hence, by using the fact that the boundary of an intersection is contained in the union of the boundaries, we obtain that for $\epsilon < \frac{1}{\mu(B \cap B')}$

$$\mu \left( O_\epsilon \cap (B \cap B') \cdot O_\epsilon \right) \leq \mu \left( O_\epsilon \cap B \cdot O_\epsilon \right) + \mu \left( O_\epsilon \cap B' \cdot O_\epsilon \right) \leq \max \left\{ C, C' \right\} \left( \mu(B) + \mu(B') \right) \cdot \epsilon$$

The first direction of Lemma 3.3 yields the desired conclusion.

Using Lemma 3.3, which provides us with an if and only if criterion for Lipschitz well roundedness of a fixed set, we will now obtain that the sets with controlled boundary are indeed LWR.

**Proposition 3.5.** Let $G$ be a Lie group. Assume that $\mu$ is a measure on $G$ that is absolutely continuous w.r.t. Haar measure, and has density that is bounded on compact sets. If $B$ is a compact BCS with $\mu(B) > 0$, then $B$ is Lipschitz well-rounded.

**Proof.** The strategy is to apply Lemma 3.3. This will be done by showing that for a subset $Y$ of $G$ which is compact and consists of a finite union of subsets of embedded submanifolds of strictly smaller dimension (e.g. the boundary of $B$) there exist $c = c(Y), \epsilon(Y) > 0$ such that

$$\mu \left( O_\epsilon Y O_\epsilon \right) \leq c \epsilon$$

for some $0 < \epsilon < \epsilon(Y)$.

It is clearly sufficient to assume that $Y$ is contained in one submanifold. For each point $g \in Y$, there is some chart $\phi_g : U_g \to \mathbb{R}^m$ for which $g \in U_g$ and $\phi \left( U_g \cap Y \right) \subseteq \mathbb{R}^{m-1} \times \{0\}$. Let $V_g$ be the open sets from Proposition 2.5 which satisfy: $g \in V_g \subseteq U_g$. By compactness, there are $g_1, \ldots, g_r \in Y$ for which $V_{g_1}, \ldots, V_{g_r}$ cover $Y$ entirely. In order to establish the inequality in Formula (3.1), it is sufficient to prove it for each $Y \cap U_{g_i}$ separately. Consequentially, we may assume that $r = 1$: $g_1 = g, V_{g_1} = V, Y_0 = Y \cap V$ and $\phi_{g_1} = \phi$.

By Proposition 2.5 there exist $c(g), \epsilon(g) > 0$ such that for $\epsilon < \epsilon(g)$ and $h \in \mathbb{V}$, $\phi \left( O_\epsilon h O_\epsilon \right) \subseteq \phi(h) + B_{c(g)\epsilon}$. In particular

$$\phi \left( O_\epsilon Y_0 O_\epsilon \right) \subseteq \phi \left( Y_0 \right) + B_{c(g)\epsilon}.$$ 

Hence it is sufficient to show that $\phi_\ast \mu \left( \phi \left( Y_0 \right) + B_{c(g)\epsilon} \right) \leq c \epsilon$.

Let $\omega \in L^1 \left( \mathbb{R}^m \right)$ be such that $\phi_\ast \mu = \omega \cdot \mu_{\mathbb{R}^m}$ where $\mu_{\mathbb{R}^m}$ is the Lebesgue measure on $\mathbb{R}^m$. Then, since $\omega$ is bounded on compact sets (and in particular on $\phi \left( O_{c(g)} Y_0 O_{c(g)} \right)$), it is sufficient to show that

$$\mu_{\mathbb{R}^m} \left( \phi \left( Y_0 \right) + B_{c} \right) \leq c \epsilon.$$ 

Indeed, since $Y_0$ is an embedded submanifold, there exists a bounded set $E \subseteq \mathbb{R}^{m-1}$ such that $\phi \left( Y_0 \right) + B_{c} \subseteq E \times [-c_2 \epsilon, c_2 \epsilon]$, which implies the desired result.

**4 Roundomorphisms**

Roughly speaking, the difficulty in checking well roundedness inside a simple non compact Lie group arises from the fact that well roundedness is a multiplicative property, while simple Lie groups are "highly non-abelian". Nevertheless, simple Lie groups have several known
decompositions — Cartan, Iwasawa, etc. — which allow them to be written as the product of more “convenient” subgroups. E.g., in the case of the Iwasawa decomposition, the subgroups $K, A, N$ are compact, abelian and nilpotent respectively, which makes it considerably easier to prove well roundedness inside them. The goal of this section is to reduce the question of whether a family $\mathcal{B}_T \subset G$ is LWR, to verifying LWR of the projections of $\mathcal{B}_T$ to each of the components of $G$ w.r.t. a given decomposition. E.g. when considering the Iwasawa decomposition, the well roundedness of $\mathcal{B}_T$ is reduced to the question of well roundedness of the image of $\mathcal{B}_T$ in the direct product $K \times A \times N$. This can be achieved if the Iwasawa diffeomorphism $G \to K \times A \times N$ preserves well roundedness; maps with this property are the topic of the following definition.

**Definition 4.1** (Roundomorphism). Let $G$ and $H$ be two topological groups with measures $\mu_G$ and $\mu_H$, and let $(O^G_\epsilon)_{\epsilon>0}$ and $(O^H_\epsilon)_{\epsilon>0}$ be two families of identity neighborhoods in $G$ and $H$ respectively. A Borel measurable map $r : G \to H$ will be called an $f$-roundomorphism if it is:

1. **Measure preserving:** $r_* (\mu_G) = \mu_H$.
2. **Locally Lipschitz:** $r (O^G_\epsilon \cdot gO^G_\epsilon) \subseteq O^H_{f(\epsilon)} \cdot r(g) \cdot O^H_{f(\epsilon)}$ for some continuous $f = f (g) : G \to \mathbb{R}_{>0}$ and for every $0 < \epsilon < \frac{1}{T}$.

The following proposition reveals the motivation for defining roundomorphisms, as well as the reason they are called that way: they pull back LWR families to LWR families.

**Proposition 4.2.** Let $r : G \to H$ be an $f$-roundomorphism. Assume that $\{\mathcal{B}_T\}_{T>0}$ is a family of measurable subsets of $H$ such that $f$ is bounded uniformly on $r^{-1} (\mathcal{B}_T)$ by a constant $F$. If $\{\mathcal{B}_T\}$ is LWR with parameters $(T_0, C_0)$, then the pre-image $r^{-1} (\mathcal{B}_T)$ is LWR with parameters $(T_0, F \cdot \max \{C_0, 1\})$.

**Proof.** The strategy of the proof is to show that for $\epsilon < F^{-1}$,

$$\mu_G \left( (r^{-1} (\mathcal{B}_T))^{(+\epsilon)} \right) \leq \mu_H \left( \mathcal{B}_T^{(+F\epsilon)} \right) \quad (4.1)$$

and

$$\mu_H \left( \mathcal{B}_T^{(-F\epsilon)} \right) \leq \mu_G \left( (r^{-1} (\mathcal{B}_T))^{(-\epsilon)} \right). \quad (4.2)$$

It will then follow that for $T > T_0$ and $\epsilon < \frac{1}{F \cdot \max \{C_0, 1\}}$ (so that both $\epsilon < F^{-1}$ and $\epsilon < (FC_0)^{-1}$; the first for inequalities (4.1) and (4.2) to hold, and the second for the LWR of $\{\mathcal{B}_T\}$),

$$\frac{\mu_G \left( (r^{-1} (\mathcal{B}_T))^{(+\epsilon)} \right)}{\mu_G \left( (r^{-1} (\mathcal{B}_T))^{(-\epsilon)} \right)} \leq \frac{\mu_H \left( \mathcal{B}_T^{(+F\epsilon)} \right)}{\mu_H \left( \mathcal{B}_T^{(-F\epsilon)} \right)} \leq 1 + FC_0\epsilon.$$

Inequalities (4.1) and (4.2) follow from measure preservation of $r$, along with the following inclusions:

$$(r^{-1} (\mathcal{B}_T))^{(+\epsilon)} \leq r^{-1} \left( \mathcal{B}_T^{(+F\epsilon)} \right),$$

$$(r^{-1} (\mathcal{B}_T))^{(-\epsilon)} \geq r^{-1} \left( \mathcal{B}_T^{(-F\epsilon)} \right),$$

It will then follow that for $T > T_0$ and $\epsilon < \frac{1}{F \cdot \max \{C_0, 1\}}$ (so that both $\epsilon < F^{-1}$ and $\epsilon < (FC_0)^{-1}$; the first for inequalities (4.1) and (4.2) to hold, and the second for the LWR of $\{\mathcal{B}_T\}$),

$$\frac{\mu_G \left( (r^{-1} (\mathcal{B}_T))^{(+\epsilon)} \right)}{\mu_G \left( (r^{-1} (\mathcal{B}_T))^{(-\epsilon)} \right)} \leq \frac{\mu_H \left( \mathcal{B}_T^{(+F\epsilon)} \right)}{\mu_H \left( \mathcal{B}_T^{(-F\epsilon)} \right)} \leq 1 + FC_0\epsilon.$$
that we now justify. For the first, note that by definition of a roundomorphism, $O_G^G g O_G^G \subseteq r^{-1} \left( O_{F\cdot r}(g) O_{F\cdot r}^H \right)$. Hence, $O_G^G \cdot r^{-1} (B_T) \cdot O_G^G \subseteq r^{-1} (O_{F\cdot r}^H B_T O_{F\cdot r}^H)$. For the second inclusion, suppose $g \in r^{-1} \left( B_T^{(-F\cdot \epsilon)} \right)$. We want to show that if $u, v \in O_G^G$, then $ugv \in r^{-1} (B_T)$. Put differently, $r(ugv) \in B_T$. This is indeed the case, since $r(u'g)v' = r'(g) v'$ for some $u', v' \in O_{F\cdot r}^H$ (local Lipschitzity of $r$), and $u'g v' \in B_T$ since $r(g) \in B_T^{(-F\cdot \epsilon)}$.

The most useful incident of Proposition 4.2 is when $H$ (such that $r : G \to H$ is a roundomorphism) is a direct product of groups. This is what allows us to reduce (under certain conditions) well roundedness in the group $G$ to well roundedness in the components of a decomposition of $G$.

**Corollary 4.3.** Let $r : G \to H = H_1 \times \cdots \times H_q$ be an $f$-roundomorphism and let $B_T = B_T^1 \times \cdots \times B_T^q \subseteq H$. Set

1. $\mu_H = \mu_{H_1} \times \cdots \times \mu_{H_q}$
2. $O_H^H = O_{H_1}^H \times \cdots \times O_{H_q}^H$

and assume that:

1. For $j = 1, \ldots, q$: $B_T^j \subseteq H_j$ is LWR w.r.t. the parameters $(T_j, C_j)$;
2. $f$ is bounded uniformly by $F$ on the sets $r^{-1}(B_T)$.

Then $r^{-1}(B_T)$ is LWR, w.r.t. the parameters

$$T = \max \{T_1, \ldots, T_q\}, \quad C \approx_q F \cdot \max \{C_1, \ldots, C_q, 1\}.$$ 

**Proof.** It is sufficient to prove the claim for $q = 2$, where one then proceeds by induction. According to the previous proposition we only need to show that $B_T$ is Lipchitz well-rounded w.r.t. the parameters $(T, C/F)$. Indeed, since

$$\mu_H \left( B_T^{(\pm \epsilon)} \right) = \mu_{H_1} \left( \left( B_T^1 \right)^{(\pm \epsilon)} \right) \cdot \mu_{H_2} \left( \left( B_T^2 \right)^{(\pm \epsilon)} \right),$$

we obtain

$$\frac{\mu_H \left( B_T^{(\pm \epsilon)} \right)}{\mu_H \left( B_T^{(-\epsilon)} \right)} \leq (1 + C_1 \epsilon) (1 + C_2 \epsilon) \leq (1 + \max \{C_1, C_2\} \epsilon)^2 \leq 1 + 3 \max \{C_1, C_2\} \epsilon$$

for $\epsilon < \frac{1}{\max \{C_1, C_2\}}$. 

**Remark 4.4.** One consequence of Corollary 4.3 is that a direct product of LWR families $B_T^1 \times \cdots \times B_T^q \subseteq H_1 \times \cdots \times H_q$ is LWR. To see this, take $G = H_1 \times \cdots \times H_q$ and $r$ that is the identity map on $G$; it is a roundomorphism with $f \equiv 1$.

The content of the following lemma is that a composition of roundomorphisms is a roundomorphism.
Lemma 4.5. Suppose that \( r_1 : G_1 \to G_2 \) is an \( f_1 \)-roundomorphism and \( r_2 : G_2 \to G_3 \) is an \( f_2 \)-roundomorphism. Then, \( r_2 \circ r_1 \) is an \( f = (f_2 \circ r_1) \cdot f_1 \)-roundomorphism.

Proof. Clearly we only need to check that \( r_2 \circ r_1 \) is locally Lipchitz:

\[
r_2 r_1 \left( \mathcal{O}^{G_1}_\epsilon \cdot g \cdot \mathcal{O}^{G_1}_\epsilon \right) \subseteq r_2 \left( \mathcal{O}^{G_2}_{f_1 \epsilon} \cdot r_1 (g) \cdot \mathcal{O}^{G_2}_{f_1 \epsilon} \right) \subseteq \mathcal{O}^{G_3}_{f \epsilon} \cdot r_2 r_1 (g) \cdot \mathcal{O}^{G_3}_{f \epsilon}.
\]

Finally, any smooth map from \( G_1 \) to \( G_2 \) such that \( r_\#(\mu_{G_1}) = \mu_{H_2} \) is a roundomorphism.

Proposition 4.6. Let \( G_1 \) and \( G_2 \) be Lie groups and \( r : G_1 \to G_2 \) a sooth map. Then \( r \) is locally Lipschitz.

Proof. Let \( g \in G_1 \) with \( \phi_1 : U \to \mathbb{R}^n \) a chart at \( g \). By Proposition 2.5, there is an open neighborhood \( V \subset U \) of \( g \) and \( \epsilon_0, c_1 > 0 \) such that for every \( 0 < \epsilon < \epsilon_0 \),

\[
\phi_1 \left( \mathcal{O}_\epsilon g \mathcal{O}_\epsilon \right) \subseteq \phi (g) + B_{c_1 \epsilon}.
\]

Let \( L_{r(g)}^{-1} : G_2 \to G_2 \) be the left translation by \( r(g)^{-1} \), and let \( W \) be an open neighborhood of \( 1_{G_2} \) such that \( \ln_{G_2} |_W \) is a diffeomorphism onto an open neighborhood of \( G_2 \). We may assume that \( L_{r(g)}^{-1} \circ r (U) \subseteq W \). We get that there is \( c_2 > 0 \) such that for every \( \epsilon < \epsilon_0 \):

\[
\ln_{G_2} \circ L_{r(g)}^{-1} \circ r \left( \mathcal{O}_\epsilon g \mathcal{O}_\epsilon \right) \subseteq \ln_{G_2} \circ L_{r(g)}^{-1} \circ r \circ \phi^{-1} (\phi (g) + B_{c_1 \epsilon}) \subseteq B_{c_2 \epsilon}.
\]

As a result,

\[
r (\mathcal{O}_\epsilon g \mathcal{O}_\epsilon) \subseteq r(g) \cdot \exp_{G_2} (B_{c_2 \epsilon})
\]

\[\square\]

5 Well roundedness of fibered families

In the previous section we have developed a machinery to establish whether a fixed set \( B \subset G \) is LWR (Proposition 3.5), and whether a family of the form \( \{ P_T^1 Q_T^1 \} \) is LWR (Corollary 4.3), where \( P_T^1 \subset P \), \( Q_T^1 \subset Q \) and \( G = PQ \) is a decomposition of \( G \) into subgroups \( P \) and \( Q \). The latter applies for any number of components in the decomposition, but for brevity we wrote it here with two components only. In this section we will extend our machinery to handle families of sets with the more complicated structure of a fiber product, namely sets of the form \( \bigcup_{Z \in P_T^1} Q_T^1 \), where again \( P_T^1 \subset P \), \( Q_T^1 \subset Q \) and \( G = PQ \). The tool of roundomorphisms allows us to reduce to well roundedness in \( P \) and in \( Q \) separately, namely to work in \( P \times Q \), which is what we will do.

We start by formulating a regularity condition on the fibers in \( Q \).

Definition 5.1. Let \( P \) and \( H \) be Lie groups and \( \mathcal{O}_\epsilon^P \) and \( \mathcal{O}_\epsilon^H \) families of coordinate balls. Let \( \mathcal{E} \) be a subset of \( P \), and consider the family \( \mathcal{D}_\mathcal{E} = \{ \mathcal{D}_z \}_{z \in \mathcal{E}} \), where \( \mathcal{D}_z \subseteq H \). We say that the family \( \mathcal{D}_\mathcal{E} \) is bounded Lipschitz continuous (or BLC) w.r.t \( \mathcal{O}_\epsilon^P \) and \( \mathcal{O}_\epsilon^H \) and with parameters \( (C_{\mathcal{D}}, V_{\min}, B) \), where \( C_{\mathcal{D}}, V_{\min} \) are positive real numbers and \( B \) is a bounded subset of \( H \), if for every \( 0 < \epsilon < C_{\mathcal{D}}^{-1} \) the following hold:
1. Every $D_z$ is LWR with parameters $(C, 1/C)$.

2. If $z' \subseteq \mathcal{O}^P_{\epsilon} z \mathcal{O}^P_{\epsilon}$ for $z, z' \in \mathcal{E}$, then $D_z(-C\epsilon) \subseteq D_{z'} \subseteq D_z(1+C\epsilon)$.

3. The volume of $D_z$ (w.r.t. a Haar measure of $H$) is bounded uniformly from below by a positive constant $V_{\text{min}}$.

4. $D_z \subseteq B$ for some bounded set $B$ and every $z \in \mathcal{E}$.

For convenience, we will always assume WLOG that $C \geq 1$.

The following proposition and corollary are concerned with certain manipulations that can be performed on fibered sets, while maintaining the BLC property of the fibers. These manipulations include pulling back the fibers by a locally-Lipschitz map, and enlarging the basis set by taking a product with another set.

**Proposition 5.2.** Let $P, P_0$ be Lie groups and suppose that $r : P_0 \to P$ is an $f$-locally Lipschitz map (Definition 4.1). Let $\mathcal{E} \subseteq P$ and $\mathcal{E}_0 := r^{-1}(\mathcal{E}) \subseteq P_0$. If the family

$$\mathcal{D}_\mathcal{E} = \{D_z\}_{z \in \mathcal{E}}$$

is BLC with parameters $(C, V_{\text{min}}, B)$, then the family

$$\mathcal{D}_\mathcal{E}_0 = \{D_{r(z_0)}\}_{z_0 \in \mathcal{E}_0}$$

is BLC with parameters $(FC, V_{\text{min}}, B)$, where $F = \sup_{g \in r^{-1}(\mathcal{E})} f(g) < \infty$.

**Proof.** Since $\mathcal{D}_\mathcal{E}_0 \subset \mathcal{D}_\mathcal{E}$, properties 1, 3, and 4 of BLC hold automatically in $\mathcal{D}_\mathcal{E}_0$, and it is only left to verify the second property. Indeed, if $z_0' \in \mathcal{O}^P_{\epsilon} z_0 \mathcal{O}^P_{\epsilon}$, then by local Lipschitzity and definition of $F$, $r(z_0') \in \mathcal{O}^P_{\epsilon} r(z_0) \mathcal{O}^P_{\epsilon}$. Since $\mathcal{D}_\mathcal{E}$ is BLC then for $\epsilon \leq 1/FC$ we obtain

$$D_{r(z_0')}(-C\epsilon) \subseteq D_{r(z_0)} \subseteq D_{r(z_0')}(+C\epsilon).$$

\[\square\]

**Corollary 5.3.** Let $P \times Q$ be a product of Lie groups and let $\mathcal{E} \subseteq P$, $\mathcal{E}' = \mathcal{E} \times Q$. If $\mathcal{D}_\mathcal{E} = \{D_z\}_{z \in \mathcal{E}}$ is BLC w.r.t. $\mathcal{O}^P_{\epsilon}$, then

$$\mathcal{D}_{\mathcal{E}'} = \{D_{(z,q)}\}_{(z,q) \in \mathcal{E}'} \text{ such that } D_{(z,q)} = D_z \forall q \in Q$$

is BLC with the same parameters and w.r.t. $\mathcal{O}^P_{\epsilon} \times \mathcal{O}^Q_{\epsilon}$.

**Remark 5.4.** Clearly we can replace the group $Q$ in the definition of $\mathcal{E}'$ with any subset $B \subseteq Q$.

**Proof.** This follows from Proposition 5.2 using the projection map

$$r : P \times Q \to P$$

which is an $f$-local Lipschitz map with $f \equiv 1$.

\[\square\]

We now turn to the concluding result of this section:
Proposition 5.5. Let \( \{ \mathcal{E}_T \}_{T>0} \) be an increasing family inside a Lie group \( P \), and \( \mathcal{E} := \bigcup_{T>0} \mathcal{E}_T \). Let \( \mathcal{D}_z = \{ D_z \}_{z \in \mathcal{E}} \) where \( D_z \subset H \), and consider the family

\[
B_T = \bigcup_{z \in \mathcal{E}_T} z \times D_z \subseteq P \times H.
\]

If \( \{ \mathcal{E}_T \}_{T>0} \) is LWR with parameters \( (T_0, C_\mathcal{E}) \), and \( \mathcal{D}_z \) is BLC w.r.t. a family \( \{ \mathcal{O}_\mathcal{E}^P, \mathcal{O}_\mathcal{E}^H \}_{\epsilon > 0} \) of coordinate balls and with parameters \( (C_{\mathcal{O}_\mathcal{E}}^P, V_{\min}, B) \), then \( B_T \) is LWR w.r.t the coordinate balls \( \mathcal{O}_\mathcal{E}^P \times \mathcal{O}_\mathcal{E}^H \subset P \times H \) and with parameters \( (T_0, C_B) \) where

\[
C_B < C_{\mathcal{O}_\mathcal{E}} c \left( 1 + C_{\mathcal{O}_\mathcal{E}} \right) + \frac{V_{\max}}{V_{\min}} C_\mathcal{E},
\]

\( V_{\max} = \mu_H(B) \) and \( c \geq 1 \) is a constant such that for \( 0 < \epsilon, \delta < \frac{1}{c} \) one has that \( \mathcal{O}_\epsilon^H \mathcal{O}_\delta^H \subseteq \mathcal{O}_{\epsilon(\epsilon + \delta)}^H \) (see Lemma 2.4).

Proof. **Step 1: estimation of** \( B_T^{(+\epsilon)} \). We claim that for \( \epsilon < \frac{1}{cC_{\mathcal{O}_\mathcal{E}}} \) (so \( \epsilon < 1, (cC_{\mathcal{O}_\mathcal{E}})^{-1} \)),

\[
B_T^{(+\epsilon)} \subseteq \left( \bigcup_{z \in \mathcal{E}_T} \left( z \times D_z^{(+c(1+C_{\mathcal{O}_\mathcal{E}})\epsilon)} \right) \right) \cup \left\{ \Delta \mathcal{E}_T \times B^{(1)} \right\} =: Y^+,
\]

where

\[
\Delta \mathcal{E}_T := \mathcal{O}_\epsilon^P \mathcal{E}_T \mathcal{O}_\epsilon^P \setminus \mathcal{E}_T.
\]

We shall first bound the affect of \( \mathcal{O}_\epsilon^P \) perturbations. For that recall that \( D_z \subseteq B \) for all \( z \in \mathcal{E} \). As a result, for \( u, v \in \mathcal{O}_\epsilon^P \) we have

\[
(v, e_H) B_T(u, e_H) = \bigcup_{z \in \mathcal{E}_T} (vzu \times D_z) \subseteq \left( \bigcup_{z \in \mathcal{E}_T} (z \times D_{u^{-1}zu^{-1}}) \right) \cup (\Delta \mathcal{E}_T \times B).
\]

By the second property of BLC, for \( \epsilon < \frac{1}{cC_{\mathcal{O}_\mathcal{E}}} \), this is contained in

\[
\bigcup_{z \in \mathcal{E}_T} \left( z \times D_z^{(+c(1+C_{\mathcal{O}_\mathcal{E}})\epsilon)} \right) \cup (\Delta \mathcal{E}_T \times B).
\]

We will now address the \( \mathcal{O}_\epsilon^H \) perturbations. To this end, note that by the first property of BLC, for \( \epsilon < \frac{1}{cC_{\mathcal{O}_\mathcal{E}}} \)

\[
\left( D_z^{(+c(1+C_{\mathcal{O}_\mathcal{E}})\epsilon)} \right)^{(\epsilon)} \subseteq D_z^{(+c(1+C_{\mathcal{O}_\mathcal{E}})\epsilon)}.
\]

Combining \( \mathcal{O}_\epsilon^P \) and \( \mathcal{O}_\epsilon^H \) perturbations together we obtain,

\[
B_T^{(+\epsilon)} = \mathcal{O}_\epsilon B_T \mathcal{O}_\epsilon \subseteq \bigcup_{z \in \mathcal{E}_T} \left( z \times D_z^{(+c(1+C_{\mathcal{O}_\mathcal{E}})\epsilon)} \right) \cup (\Delta \mathcal{E}_T \times B^{(1)}) = Y^+
\]

(where we have used \( \epsilon < 1 \)).
Step 2: estimation of $B_T^{(-\epsilon)}$. We claim that for $\epsilon < \frac{1}{cG}$,

$$B_T^{(-\epsilon)} \supseteq \bigcup_{z \in E} \left( z \times D_z^{-cG(1+\epsilon)} \right) =: Y^-. $$

First notice that if $0 < a, \frac{b}{c} - a < \frac{1}{c}^2$, then $(D^{-b})^{(a)} \subseteq D^{-\frac{b}{c}}$, since

$$O_H^{-a} O_a^{-b} \subseteq O_H^{-b} \subseteq D^{-b}.$$

Hence, for $\epsilon < \frac{1}{cG}$

$$O_H Y^- \subseteq \bigcup_{z \in E} \left( z \times (D_z^{(-C_{G\epsilon})}) \right)$$

for $u, v \in O_{v,T}$ we have

$$(v, e_H) Y^- \subseteq \bigcup_{z \in E} \left( vzu \times D_z^{(-C_{G\epsilon})} \right) \subseteq \bigcup_{u^{-1}zu^{-1}, z \in E} \left( z \times D_z^{(-C_{G\epsilon})} \right).$$

By the second property of BLC, for $\epsilon < \frac{1}{cG}$ this is contained in

$$\bigcup_{z \in E} \left( z \times D_z \right).$$

All in all, we obtain that $O_H Y^- \subseteq B_T$, proving the claim.

Step 3: estimation of $\mu \left( B_T^{(+-\epsilon)} / \mu \left( B_T^{(-\epsilon)} \right) \right)$. Let $C = C_{G} (c (1 + C_{G}))$. Notice that for $\epsilon < \frac{1}{C}$

$$\mu_G \left( Y^+ \right) = (1 + C) \mu_G (B_T) + \mu_P \left( \Delta E_T \right) \mu_H (B^{(1)}) \leq (1 + C) \mu_G (B_T) + \mu_P \left( \Delta E_T \right) V_{\max}$$

and that

$$\mu_G \left( Y^- \right) = \frac{1}{1 + C} \mu_G \left( \bigcup_{z \in E} \left( z \times D_z \right) \right).$$

Combining what we have shown in the previous steps with estimations for $\mu_G \left( Y^+ \right)$ and $\mu_G \left( Y^- \right)$, we obtain that for $\epsilon < \frac{1}{C}$:

$$\frac{\mu_G \left( B_T^{(+-\epsilon)} \right)}{\mu_G \left( B_T^{(-\epsilon)} \right)} \leq \frac{\mu_G \left( Y^+ \right)}{\mu_G \left( Y^- \right)}$$

$$\leq (1 + C\epsilon)^2 \frac{\mu_G(B_T)}{\mu_G \left( \bigcup_{z \in E} \left( z \times D_z \right) \right)} + V_{\max} (1 + C\epsilon) \cdot \frac{\mu_P \left( \Delta E_T \right)}{\mu_G \left( \bigcup_{z \in E} \left( z \times D_z \right) \right)}$$

where:
• for $\epsilon < 1$, 
  \[(1 + C\epsilon)^2 \leq 1 + 3C\epsilon\]

• for $\epsilon < \frac{1}{C}$, 
  \[V_{\text{max}} (1 + C\epsilon) \leq 2V_{\text{max}}\]

• for $\epsilon < C^{-1}_\epsilon$ and $T > T_0$

\[
\frac{\mu_G(B_T)}{\mu_G\left(\bigcup_{z \in E^{(-)}_T} (z \times D_z)\right)} = 1 + \frac{\mu_G(B_T) - \mu_G\left(\bigcup_{z \in E^{(-)}_T} (z \times D_z)\right)}{\mu_G\left(\bigcup_{z \in E^{(-)}_T} (z \times D_z)\right)} = 1 + \frac{\mu_G\left(\bigcup_{z \in E^{(-)}_T} (z \times D_z)\right)}{\mu_G\left(\bigcup_{z \in E^{(-)}_T} (z \times D_z)\right)} \leq 1 + \frac{\mu_P\left(\mathcal{E}_T \setminus \mathcal{E}_T^{(-)}\right)}{\mu_P\left(\mathcal{E}_T^{(-)}\right)} \frac{V_{\text{max}}}{V_{\text{min}}} \leq 1 + \frac{V_{\text{max}}}{V_{\text{min}}} C\epsilon,
\]

All in all, for $\epsilon < \frac{1}{C+6}\epsilon$ (so that $\epsilon \leq C^{-1}, C^{-1}_\epsilon$) and for $T > T_0$:

\[
\frac{\mu_G(Y^+)}{\mu_G(Y^-)} \leq \left(1 + 3C\epsilon\right) \cdot \left(1 + \frac{V_{\text{max}}}{V_{\text{min}}} C\epsilon\right) + 2V_{\text{max}} \cdot \frac{C\epsilon}{V_{\text{min}}} \epsilon \\
\leq 1 + \left(6V_{\text{max}}V_{\text{min}} C\epsilon + 3C\right) \epsilon.
\]

In order to have that LWR holds for $\epsilon < C^{-1}_B$, we let $C_B = 6\frac{V_{\text{max}}}{V_{\text{min}}} C\epsilon + 3C$. \hfill \bbox

**Proposition 5.6.** Let $H = \mathbb{R}^n$ and $\mathcal{O}_H = B_\epsilon$ be a radius $\epsilon$ euclidean ball. Suppose that $\mathcal{D}_E$ satisfies conditions 3 and 4 of the definition of BLC and instead of condition 1 and 2 it satisfies that for $\epsilon < C^{-1}$ (here $C \geq 1$):

(i) $D_z + B_\epsilon \subseteq (1 + C\epsilon) D_z$

(ii) If $z' \subseteq \mathcal{O}_E z \mathcal{O}_E$ for $z, z' \in \mathcal{E}$, then $D_{z'} \subseteq (1 + C\epsilon) D_z$.

Then $\mathcal{D}_E$ is BLC with parameters $(16^{n+1} R C, V_{\text{min}}, B)$, where $R < \infty$ is the radius of $B$ from property 4 of BLC).

**Proof.** We start with checking the first property of BLC. For this we first show that for all $z \in \mathcal{E}$ and $\epsilon < \frac{1}{4C}$

\[
\frac{1}{1 + 8C\epsilon} D_z \subseteq D_z^{(-\epsilon)}.
\]

(5.1)

Indeed, since $\epsilon < C^{-1}$ then by (i) $D_z + B_\epsilon \subseteq (1 + C\epsilon) D_z$, and so

\[
\frac{1}{1 + 2C\epsilon} D_z + B_\epsilon \subseteq \frac{1}{1 + 2C\epsilon} D_z + B_{\frac{1}{1 + 2C\epsilon}} \subseteq \frac{1}{1 + C\epsilon} (D_z + B_\epsilon) \subseteq D_z.
\]
As a result,
\[ \frac{1}{1 + 8C\epsilon}D_z + B_{2\epsilon} \subseteq D_z \]
which implies \textbf{5.1}. Now, for \( \epsilon < \frac{1}{16C} \) we have that
\[ \frac{\mu_H(D_z^{(+)\epsilon})}{\mu_H(D_z^{(-)\epsilon})} \leq \left( \frac{1 + 2C\epsilon}{1 + 8C\epsilon} \right)^n < (1 + 11C\epsilon)^n < 1 + 16^{n+1}C\epsilon. \]

So \( \{D_z\} \) is LWR with Lipschitz constant \( 16^{n+1}C \).

For the second property of BLC we first show that for all \( \epsilon > 0 \)
\[ D_z^{(-)\epsilon} \subseteq \frac{1}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{\ldots}}}} D_z \quad (1 + C\epsilon) D_z \subseteq D_z^{(+CRe)}. \]
Indeed, If \( x \in D_z^{(-)\epsilon} \), then \( x + \frac{\epsilon}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{\ldots}}} \epsilon \in D_z^{(-)\epsilon} + B_\epsilon \subseteq D_z \). Hence, \( x \in \frac{1}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{\ldots}}} D_z \) and so \( D_z^{(-)\epsilon} \subseteq \frac{1}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{1 + \frac{\epsilon}{\ldots}}} D_z \). Next, if \( y \in (1 + C\epsilon) D_z \), we can write \( y = x + C\epsilon x \) for \( x \in D_z \) and so \( y \in D_z + B_{RC\epsilon} \in D_z^{(+CRe)} \). Now for \( \epsilon < \frac{1}{C} \) let \( z \in E \) and \( z' \subseteq \mathcal{O}_\epsilon^2 \mathcal{O}_\epsilon^P \cap E \), and then by (ii),
\[ D_z^{(-)\epsilon} \subseteq (1 + C\epsilon) D_z \subseteq D_z^{(+CRe)} \]
and
\[ D_z^{(-RCE)} \subseteq \frac{1}{1 + C\epsilon} D_z \subseteq D_z^{(-)\epsilon}. \]
So the second property of BLC holds with the constant \( RC \), and all in all, both first and second properties are satisfied with \( C_g = 16^{n+1}CR \).

**Corollary 5.7.** Assume \( D \subseteq \mathbb{R}^n \) is bounded, convex and has a non-empty interior, then \( D \) is LWR.

**Proof.** It is clearly enough to check the case where the origin is an internal point. We will show that the constant family \( D_E = \{D_z\}_{z \in E} \) with \( D_z = D \) for every \( z \), is BLC as a set in \( \mathbb{R}^n \) using Proposition 5.6. The second property of BLC is trivial since \( D \) is constant, and the third and fourth properties hold since \( D \) is bounded and of positive measure. It remains to show that \( D \) satisfies the first property of BLC. Let \( \alpha > 0 \) be such that \( D \) contains a ball of radius \( > \alpha \) around the origin; we show that \( D + B_\epsilon \subseteq (1 + \alpha^{-1}\epsilon) D \). Indeed, let \( x \in D \) and \( v \in \mathbb{R}^n \) such that \( \|v\| = 1 \). Then
\[ x + \epsilon v = (x + \epsilon v) \left( \frac{1 + \frac{\epsilon}{\alpha}}{1 + \frac{\epsilon}{\alpha}} \right) = \left( 1 + \frac{\epsilon}{\alpha} \right) \cdot \frac{x + \epsilon v}{1 + \frac{\epsilon}{\alpha}} \]
\[ = \left( 1 + \frac{\epsilon}{\alpha} \right) \left[ \frac{1}{1 + \frac{\epsilon}{\alpha}} \cdot x + \frac{\epsilon}{1 + \frac{\epsilon}{\alpha}} \cdot \alpha v \right] (\star) \]
where \( \star \) lies in \( D \), as a convex combination of the two points \( x, \alpha v \) in \( D \).

\[ \square \]
6 Relation between fundamental domains and quotient spaces

As mentioned in the Introduction, this note is meant to support our work on equidistribution in various lattice spaces. All of these spaces are of the form $\mathcal{M}/H$, where $\mathcal{M}$ is a manifold and $H$ is a Lie group acting on it. We are interested in Boundary Controllable Sets (and Bounded Lipschitz Continuous families of sets) in these spaces, which is a differential property; as such, it is easier to check it in concrete manifolds, than in abstract spaces. This brings up the need in finding a set of representatives inside $\mathcal{M}$ for the action of $H$, such that one can move the question of verifying the BCS property from the space $\mathcal{M}/H$ to this subset of $\mathcal{M}$. In the case where $H$ is discrete, one can think of this desired set of representatives as a “nice” fundamental domain in $\mathcal{M}$ (see examples below). The precise definition is the following:

**Definition 6.1.** Let $H$ be a Lie group acting smoothly, freely and properly on a manifold $\mathcal{M}$, and let $\pi : \mathcal{M} \rightarrow \mathcal{M}/H$ denote the associated quotient map. A full set of representatives $F \subset \mathcal{M}$ for $\mathcal{M}/H$ is called a spread model for the quotient space $\mathcal{M}/H$ if the following conditions are met:

1. $F$ is contained in a finite union of embedded submanifolds $\bigcup \alpha V_\alpha$ of $\mathcal{M}$ such that the natural projection $\pi^0_\alpha : V_\alpha \rightarrow \mathcal{M}/H_0$ (here $H_0$ denotes the connected component of $H$) is an open diffeomorphism onto its image;

2. for each $\alpha$, there exists an open set (w.r.t. $V_\alpha$) $F_\alpha \subseteq F \cap V_\alpha$ such that $\overline{F} \cap V_\alpha \subset \overline{F_\alpha}$;

3. $F_\alpha$ is BCS w.r.t. $V_\alpha$. and

4. the quotient map restricted to $\overline{F}$ is proper, namely it pulls back compact sets to compact sets.

We will denote $F \overset{s.m.}{\sim} \mathcal{M}/H$.

**Remark 6.2.** One can extend definition also to the case where $H$ acts almost freely (i.e. the point stabilizer subgroups are finite) on $\mathcal{M}$ by considering the open submanifold $\mathcal{M}_{free}$ on which the action is free. In that case we add the extra condition that $F \subseteq \mathcal{M}$ is a spread model if $F \cap \mathcal{M}_{free}$ is a spread model for the action of $H$ on $\mathcal{M}_{free}$.

**Remark 6.3.** If $H = \Gamma$ is a discrete Lie group, then the conditions in Definition 6.1 are satisfied when: $\Gamma$ acts properly and almost freely on $\mathcal{M}$; $F \subseteq \text{int}(\overline{F})$; the boundary of $F$ is contained in a finite union of lower dimensional submanifolds of $\mathcal{M}$ and the quotient map restricted to $\overline{F}$ is proper. This is indeed the case since the quotient map $\pi : \mathcal{M}_{free} \rightarrow \mathcal{M}_{free}/\Gamma$ restricted to $\text{int}(\overline{F})$ is a diffeomorphism.

**Example 6.4.** Let us mention some examples (an explanation follows).

1. If $\Lambda$ is a lattice in $\mathbb{R}^n$, then any fundamental domain of $\Lambda$ which is a convex polygon with a finite number of edges, is a spread model for $\mathbb{R}^n/\Lambda$. In particular, its fundamental parallelepiped and its Dirichlet domain are spread models.

2. If $\Gamma$ is a discrete group acting properly discontinuously and freely on $\mathbb{H}^n$ by isometries, then any locally finite fundamental polygon $F$ is a spread model. In particular, the (generic) Dirichlet domain of a Fuchsian group is a spread model.
3. Suppose that $G = H \cdot P$, is a product of two closed subgroups having trivial intersection. Then $P$ is a spread model for the action of $H$. In particular, the group of upper triangular matrices in $\text{SL}_n(\mathbb{R})$ is a spread model for the symmetric space $\text{SO}_n(\mathbb{R}) \setminus \text{SL}_n(\mathbb{R})$, and a minimal parabolic group inside a non-compact simple algebraic rank one Lie group is a spread model for the associated hyperbolic space. E.g. the upper triangular matrices in $\text{SL}_2(\mathbb{R})$ (resp. $\text{SL}_2(\mathbb{C})$) form a spread model for the real hyperbolic plane (resp. 3-space).

Indeed, we use Remark 6.3 to justify the first two examples. All the conditions are easily seen to be satisfied except that $\pi|_F$ is proper in the second example. We check it here: suppose that $a_n \to \infty$ in $\overline{F}$, but modulo $\Gamma$, the set $\{\gamma \alpha_n\}$ is bounded. As a result, we may assume that $\gamma_n a_n \to a$ for some $\gamma_n \in \Gamma$. Let $K$ be a compact neighborhood of $a$. We have that $K \cap \gamma_n \overline{F} \neq \emptyset$ for all $n$. Since $F$ is locally finite, we may assume that $\gamma_n = \gamma$ for all $n$. This is however an absurd, since we must have that $\gamma_a_n \to \infty$.

To check example 3 we use the original definition and notice that we may choose $V_a = F_\alpha = P$ (only a single $\alpha$) and since $P = \overline{F}$ is diffeomorphic via $\pi$ to $H\setminus G$ and the boundary of $F_\alpha$ w.r.t. $V_a$ is trivial, we are done.

### 6.1 BCS’s in space and its spread model correspond

We start by claiming that a BCS in the spread model projects modulo $H$ to a BCS in the space $\mathcal{M}/H$.

**Proposition 6.5.** Suppose that $\mathcal{M}$ is a manifold and $\Gamma$ is a discrete group acting on $\mathcal{M}$ freely, properly and smoothly. Let $F \subset \mathcal{M}$ be a spread model for the action of $\Gamma$. If $B \subset F$ is a BCS (w.r.t. $\mathcal{M}$) then so is its projection $\pi(B)$ to $\mathcal{M}/\Gamma$.

The proof requires a lemma.

**Lemma 6.6.** Let $X$ be a topological space together with three subsets $A \subseteq B$ and $C$.

1. $\partial_B A \subseteq \partial A$. If furthermore there is an open subset $W$ of $X$, such that $\overline{A} \subseteq W \subseteq B$, then $\partial_B A = \partial A$.

2. $(\partial C) \cap A \subseteq \partial_B (C \cap A) \cup \partial B$. If furthermore $B$ is open, then $(\partial C) \cap A \subseteq \partial_B (C \cap A)$.

**Proof.** 1) Let $b \in \partial_B A$ and let $U$ be a neighborhood of $b$. The set $U \cap B$ is a neighborhood of $b$ w.r.t. $B$, hence it contains a point of $A$ and a point of $A^c$. As a result, $b \in \partial A$.

We need to show that $\partial A \subseteq \partial_B A$. Let $b \in \partial A$ and let $U'$ be a neighborhood of $b$ w.r.t. $B$. Hence there is a neighborhood $U$ of $b$ such that $U' = U \cap B$. Since $b \in \overline{A} \subseteq W$ and $W$ is open, $U \cap W$ is a neighborhood of $b$ and so $U \cap W = U \cap W \cap B$ is also a neighborhood of $b$ w.r.t. $B$. Hence, $U \cap W$ contains a point of $A$ and a point of $A^c \cap B$. We conclude that indeed $b \in \partial_B A$.

2) Assume $x \in (\partial C) \cap A$. If every neighborhood of $x$ intersects $B^c$, we get that $x \in \partial B$ (as $x \in A \subseteq B$). Otherwise, there is a neighborhood $V$ of $x$ such that $V \subseteq B$. Let $W'$ be a neighborhood of $x$ w.r.t. $B$, so that $W' = W \cap B$ where $W$ is a neighborhood of $x$. Since $W \cap V$ is a neighborhood of $x$ in $X$, we know that it intersects both $C$ and $C^c$. Since $W \cap V = W' \cap V$ we get that it is also a neighborhood of $x$ w.r.t. $B$. Hence, $x \in \partial_B (C \cap A)$.

In the case when $B$ is open, since $x \in A \subset B$, $B$ is a neighborhood of $x$ which does not intersect $B^c$. \[\square\]
Proof of Proposition 6.7. We split $B$ into two parts: $B \cap \partial F$ and $B_{\text{int}} := B \cap F^\circ$. Since, by part (2) of Lemma 6.6, $\partial \pi \left( B \right) \subseteq \pi \left( \partial F \right) \cup \partial_{\pi(F^\circ)} \pi \left( B_{\text{int}} \right)$, it is enough to show that $\pi \left( B_{\text{int}} \right)$ is BCS inside $\pi \left( F^\circ \right)$ and that locally $\pi \left( \partial F \right)$ is contained in a finite union of codimension $\geq 1$ submanifolds.

The first part is clear since $\pi \mid_{F^\circ}$ is an open diffeomorphism onto its image $\pi \left( F^\circ \right)$.

The second part follows from $\partial F$, by assumption, being locally contained in a finite union of codimension $\geq 1$ submanifolds together with $\pi$ being a local diffeomorphism. \hfill \Box

The content of the following result is the converse of Proposition 6.5. In the case where $H$ is discrete, this merely means that the lift of a BCS in $M/H$ is a BCS in $F \subset M$. When $H$ is not discrete, i.e. $\dim \left( M/H \right) < \dim \left( M \right)$, this statement has no actual content since $F$ is its own boundary; a more delicate formulation is therefore in order:

**Proposition 6.7.** Assume that $F$ is a spread model for $M/H$, where $H$ acts almost freely and properly on $M$ and $H_0$ acts freely on $M$. Then if $B \subseteq M/H$ and $B_H \subseteq H$ are BCS (resp. bounded), then so does

$$B_F \cdot B_H \subseteq M,$$

where $B_F = \pi \mid_{F^\circ}^{-1} \left( B \right)$.

The proof requires a lemma. The condition regarding $V_\alpha$ appearing in the first part of Definition 6.1 can be restated as in the following, which is probably already known:

**Lemma 6.8.** Assume that $V_\alpha$ is a submanifold of $M$ and $H$ acts freely and properly on $V_\alpha$. The natural projection $\pi_0^\alpha : V_\alpha \rightarrow M/H_0 \left( \pi_0^\alpha : F_\alpha \rightarrow M/H \right)$ is an open diffeomorphism onto its image iff the map $\theta_\alpha : V_\alpha \times H_0 \rightarrow M \left( F_\alpha \times H \rightarrow M \right)$ given by $\theta_\alpha \left( x, h \right) = x \cdot h$ is an open diffeomorphism onto its image.

**Proof.** It is clearly enough to prove the Lemma for $H_0$. Assume that $\theta_\alpha$ is an open diffeomorphism onto its image. It is sufficient to show that $\pi_0^\alpha$ is an injective (this is clear) submersion and that $\dim V_\alpha = \dim M/H_0$. Consider the diagram:

\[
\begin{array}{ccc}
V_\alpha & \xrightarrow{\iota_h} & V_\alpha \times H_0 \xrightarrow{\theta_\alpha} M \\
\pi_0^\alpha \downarrow & & \downarrow \pi^0 \\
M/H_0 & & \\
\end{array}
\]

where $\iota_h : V_\alpha \rightarrow V_\alpha \times H_0$ is given by $\iota_h \left( v \right) = \left( v, h \right)$. Since $\pi^0 \circ \theta_\alpha \mid_{\left\{ p \right\} \times H_0}$ is a constant map for every $p \in V_\alpha$, then $d \left( \iota_h \circ \pi^0 \circ \theta_\alpha \right)$ and $d \left( \pi^0 \circ \theta_\alpha \right)$ have the same image. The maps $\theta_\alpha$ and $\pi_0^\alpha$ are submersions, hence so is $\pi_0^\alpha$. Finally, since $\dim V_\alpha + \dim H_0 = \dim M = \dim M/H_0 + \dim H_0$, we get the desired dimension equality.

Assume now that $\pi_0^\alpha$ is an open diffeomorphism onto its image. It is sufficient to show that $\theta_\alpha$ is an injective (which is again clear) immersion and that $\dim V_\alpha + \dim H_0 = \dim M$. For every $h \in H_0$ consider the diagram:

\[
\begin{array}{ccc}
V_\alpha & \xrightarrow{\iota_h} & V_\alpha \times H_0 \xrightarrow{\theta_\alpha} M \\
\pi_0^\alpha \downarrow & & \downarrow \pi^0 \\
\frac{\pi_0^\alpha}{M/H_0} & & \frac{\pi^0}{M/H_0}^{-1} \xrightarrow{(\pi_0^\alpha)^{-1}} V_\alpha.
\end{array}
\]

Since the composition of all of the maps in the diagram is the identity map, we get that for any given point $\left( p, h \right) \in V_\alpha \times H_0$ and $\left( X, Y \right) \in T_p V_\alpha \times T_h H_0$, the equality $d_{\left( p, h \right)} \theta_\alpha \left( X, Y \right) = 0$
forces $X = 0$. It is well known that, under the assumption that $H_0$ is a Lie group acting freely and properly on $\mathcal{M}$, $pH_0$ is a closed submanifold of $\mathcal{M}$ and $d\theta_\alpha|_{(p)\times H_0}$ is a diffeomorphism onto $pH_0$. As a result, $Y = 0$ and so $\theta_\alpha$ is an immersion. Furthermore, if $x' = x''$ where $x, x' \in V_\alpha$ and $h, h' \in H_0$, then $\pi_\alpha^0(x') = \pi_\alpha^0(x)$. Since $\pi_\alpha^0$ is injective, we must have that $x = x'$. The action of $H_0$ is free, so $h = h'$ and so $\theta_\alpha$ is injective. Finally, $\dim V_\alpha = \dim \mathcal{M}/H_0$ and $\dim \mathcal{M} = \dim \mathcal{M}/H_0 + \dim H_0$, so $\dim V_\alpha + \dim H_0 = \dim \mathcal{M}$. 

\textbf{Proof of Proposition 6.7.} We first prove that if $B$ and $B_H$ are compact, then so is $\overline{B \times B_H}$. Since $B$ is compact and $\pi_F$ is proper, $\overline{B}$ is compact. As a result, it is enough to show that $\overline{B_F B_H} \subseteq \overline{B_F} \cdot \overline{B_H}$: indeed if $B_F \ni x_n$ and $B_H \ni h_n$ are such that $x_n h_n \to m$, then by compactness we may pass to subsequences and assume also that $x_n \to x$ and $h_n \to h$, so that $x h = m$. This proves the claim.

It remains to show that $\partial (B_F B_H)$ is locally contained in a finite union of lower dimensional submanifolds of $\mathcal{M}$. Let $\{V_\alpha\}$ and $\{F_\alpha\}$ as in the assumptions. Since $F_\alpha$ is open in $V_\alpha$, we have that $\pi_{F_\alpha}$ is a diffeomorphism onto its image (which is an open submanifold of $\mathcal{M}/H$).

To proceed we assume first that the action of $G$ is free. Write $W_\alpha = \pi(F_\alpha)$ and consider the map

$$\tau_\alpha : W_\alpha \times H_0 \to (\pi)^{-1}(W_\alpha) = F_\alpha H_0 \subseteq \mathcal{M}$$

given by

$$\tau_\alpha (u, h) = \left((\pi|_{F_\alpha})^{-1}(u)\right) \cdot h.$$  

By Lemma 6.8 (notice that since $\pi_\alpha^0$ is an open diffeomorphism, then $\tau_\alpha$ must also be such), this is an open diffeomorphism which satisfies $\tau_\alpha (\pi (x), h) = x \cdot h$, where $x \in F_\alpha$. By part 2 of Lemma 6.6

$$\begin{align*}
V_\alpha H \cap \partial (B_F B_H) &\subseteq (\partial (B_F B_H) \cap F_\alpha B_H) \cup \partial ((B_F B_H) \cap (\overline{F_{\alpha}} \setminus F_\alpha) B_H) \\
&\subseteq \partial F_{\alpha,H} (B_F B_H \cap F_\alpha B_H) \cup \partial (F_\alpha B_H) \cup \partial (\partial F_{\alpha} \cdot B_H) \\
&\subseteq \partial F_{\alpha,H} ((B_F \cap F_\alpha) B_H) \cup \partial (F_\alpha B_H).
\end{align*}$$

It is enough to show that the sets $(B_F \cap F_\alpha) B_H$ and $F_\alpha B_H$ are BCS’s. For the first one, we have

$$\partial F_{\alpha,H} ((B_F \cap F_\alpha) B_H) = \tau_\alpha \circ \partial_{W_\alpha \times H} \circ (\pi \cap B \times B_H) \subseteq \tau_\alpha (\partial ((B \times B_H) \cap (W_\alpha \times H))).$$

Since $B \times B_H$ is BCS and $W_\alpha \times H$ is open in $\mathcal{M}/H \times H$, then $(B \times B_H) \cap (W_\alpha \times H)$ is BCS w.r.t. $W_\alpha \times H$. Since $\tau_\alpha$ is a diffeomorphism, $\partial (B_F \cap F_\alpha) B_H$ is BCS w.r.t. $F_\alpha H$. For the second set, write $B_H \subseteq \cup_\beta H_0 h_\beta$ for some $\{h_\beta\}$. By part 2 of Lemma 6.6

$$\partial (F_\alpha B_H) \cap F_\alpha H_0 h_\beta \subseteq V_{\alpha,B_\beta} (F_\alpha B_H \cap V_\alpha H_0 h_\beta) = \left(\theta_\alpha^{-1} \circ (\text{Id}_{F_\alpha} \times R_{h_\beta})\right) (\partial (F_\alpha \times (B_H \cap H_0 h_\beta))).$$

where $R_\beta : \mathcal{M} \to \mathcal{M}$, given by $R_\beta (x) = x h_\beta$, is a diffeomorphism. The claim now follows using the third assumption, $B_H$ being a BCS and $H_0 h_\beta$ being open in $H$.

We now turn to the general case where the $G$ action is almost proper. In that case, by [DK12, Theorem 2.8.5] and [Śni13, Theorem 4.3.5], both $\mathcal{M}_{\text{free}}^c$ and $\pi(\mathcal{M}_{\text{free}}^c)$ are BCS’s, since every point $x$ in them contains an open neighborhood $U_x$, such that $U_x \cap \mathcal{M}_f \cap \pi(\mathcal{M}_{\text{free}}^c)$ is a finite union of codimension $\geq 2$ submanifolds of $\mathcal{M}$. Hence we get that $(B_F \cdot B_H) \cap \mathcal{M}_{\text{free}}^c$ and $\mathcal{M}_f$ are both BCS’s. All in all, $B_F \cdot B_H$ is a BCS. \hfill\square
6.2 Measures on the space and its spread model correspond

We proceed with a short discussion about measures. We will use the following:

Theorem 6.9 ([Jüs18]). Let $H$ be a unimodular Radon lcsc group and let $\mu$ be a $H$-invariant Radon measure on an lcsc space $Y$. Assume that the $H$ action of $H$ on $Y$ is strongly proper (i.e. the action is proper and the quotient space $Y/H$ is lcsc). Then, for a Haar measure $\mu_H$ on $H$ there exists a unique Radon measure $\mu_{Y/H}$ on $Y/H$ such that for all $f \in C_c(Y)$,

$$\int_Y f(y) \, d\mu(y) = \int_{Y/H} \left( \int_H f(yh) \, d\mu_H(h) \right) \, d\mu_{Y/H}(Hy).$$

Proposition 6.10. Let $H$ be a Lie group acting smoothly, almost freely and properly on a manifold $M$ and $F$ a spread model for $H$ in $M$; let $\mu_M$ be an $H$-invariant Radon measure on $M$ and $\mu_H$ a Haar measure on $H$. Finally let $\mu_{M/H}$ be the unique Radon measure on $M/H$ satisfying for every $f \in L^1(M)$

$$\mu_M(f) = \mu_{M/H} \left( \int_H f(xh) \, d\mu_H(h) \right). \tag{6.1}$$

Then, if $B \subseteq M/H$ and $B_H \subseteq H$ are BCS,

$$\mu_M(M_B \cdot B_H) = \mu_{M/H}(B) \cdot \mu_H(B_H).$$

Proof. We clearly may assume that $B$ is small enough so that $M_B$ is contained in a single $F_\alpha$ and $B_H \subseteq H_0$. Let $f = 1_{M_B}1_{B_H}$. Every $x_0 \in M$ such that $\pi(x_0) \in B$ can be written $x_0 = y_0h_0$, where $y_0 \in F$ and $h_0 \in H$. Hence $y_0h_0h \in M_BB_H$ iff $y_0 \in M_B$ and $h_0h \in B_H$. As a result,

$$\int_H 1_{M_B}1_{B_H} (y_0h_0h) \, d\mu_H(h) = 1_{M_B} (y_0) \, \mu_H(B_H) = 1_B(\pi(x_0)) \, \mu_H(B_H).$$

All in all, by equation (6.1)

$$\mu(M_B \cdot B_H) = \mu_{M/H}(B) \cdot \mu_H(B_H).$$

6.3 Spread models for compact quotients

Our main motivation for exploring the compact case is to find a spread model for the sphere $S^{n-1}$, which is diffeomorphic $SO_{n-1}(\mathbb{R}) \setminus SO_n(\mathbb{R})$, inside $SO_n(\mathbb{R})$. Indeed, in Secion 8 we use the results of this part to construct a spread model for the sphere.

Proposition 6.11. Let $K$ be a Lie group. Assume that $K'' < K$ is a closed subgroup such that the quotient space $K/K''$ is compact. Then, there exists a spread model $K'$ for $K/K''$.

Remark 6.12. Since $\partial(A \cup B), \partial(A \cap B) \subseteq \partial A \cup \partial B$, the union, intersection and subtraction of BCSs are in themselves BCS. Also, a direct product of BCS’s is a BCS in the direct product of the manifolds, and a diffeomorphic image of a BCS is a BCS.
Proof of Proposition 6.11. Since \( \pi : K \to K/K'' \) is a fiber bundle with a fiber \( K'' \), there exists an open covering \( \{U_\alpha\} \) of \( K/K'' \) such that for every \( \alpha \) there are local sections \( s_\alpha : U_\alpha \to K \). We may assume that for each \( U_\alpha \) there is an open subset \( W_\alpha \), whose closure lies inside \( U_\alpha \), such that \( \{W_\alpha\} \) is also a covering; the sets \( W_\alpha \) can be chosen to be BCS’s (e.g., by reducing to contained open balls); then, by compactness, this covering can be made finite. Set \( B_\alpha := W_\alpha \setminus \bigcup_{i=1}^{\alpha-1} W_i \). The sets \( B_\alpha \) are disjoint, and they maintain the BCS property (Remark 6.12). It is therefore clear that \( V_\alpha := s_\alpha(U_\alpha) \) and \( F_\alpha := s_\alpha(\text{int}(B_\alpha)) \) satisfy conditions 2 and 3 in Definition 6.1.

Define \( \theta_\alpha : V_\alpha \times K'' \to \pi^{-1}(U_\alpha) = V_\alpha K'' \) by
\[
\theta_\alpha(u,h) = u \cdot h.
\]
This is clearly a diffeomorphism, so by Lemma 6.8, the first condition also holds. The last condition is fulfilled since for a compact \( B \subseteq K/K'' \) we have that
\[
\pi^{-1}|_F(B) = \bigcup_\alpha \pi^{-1}|_F(B \cap B_\alpha) = \bigcup_\alpha \theta_\alpha^{-1}((B \cap B_\alpha) \times H) \cap \bar{F} = \bigcup_\alpha \theta_\alpha^{-1}((B \cap B_\alpha) \times 1_H),
\]
which is clearly compact. \( \Box \)

Remark 6.13. It is clear from the proof that if \( K'' < K''_1 \) are closed Lie subgroups of a Lie group \( K \) such that \( K/K'' \) is compact, then one can find \( F_1 \subseteq F \) such that \( F \) is a spread model of \( K \) w.r.t. \( K'' \) and \( F_1 \) is a spread model of \( K \) w.r.t. \( K''_1 \).

Corollary 6.14. Let \( K \) be a compact Lie group and let \( \Gamma < K \) be a lattice (i.e., a finite subgroup). There exists a fundamental domain \( \mathcal{K} \subseteq K \) for \( \Gamma \) which is BCS.

Proof. This is a direct consequence of Proposition 6.11. \( \Box \)

6.4 Manipulations on spread models

The following result allows us to “compose” spread models in the sense that a spread model for \( \mathcal{M}/G \) “times” a spread model for \( G/H \) is a spread model for \( \mathcal{M}/H \).

Proposition 6.15. Suppose that a Lie group \( G \) acts smoothly, freely and properly on a manifold \( \mathcal{M} \). Let \( H \) be a closed subgroup of \( G \).

1. If \( F^\mathcal{M} \) is a spread model of \( \mathcal{M}/G \) in \( \mathcal{M} \) and \( F^G \) is a spread model of \( G/H \) in \( G \) (considered as a manifold), then \( F = F^\mathcal{M} \cdot F^G \) is a spread model of \( \mathcal{M}/H \) in \( \mathcal{M} \).

2. If \( B^{\mathcal{M}/G} \subseteq \mathcal{M}/G \) and \( B^{G/H} \subseteq G/H \) are BCS’s, then so does
\[
B^{\mathcal{M}/H} := \pi_{\mathcal{M}/H}^\mathcal{M}(B^\mathcal{M} \cdot B^G) \subseteq \mathcal{M}/H,
\]
where \( B^\mathcal{M}, B^G \) are the representatives in \( F^\mathcal{M}, F^G \) of \( B^{\mathcal{M}/G}, B^{G/H} \) respectively.

3. If \( \mu_{G/H}, \mu_{\mathcal{M}/G}, \mu_{\mathcal{M}/H} \) are the measures appearing in Proposition 6.11, then
\[
\mu_{\mathcal{M}/H}(B^{\mathcal{M}/H}) = \mu_{\mathcal{M}/G}(B^{\mathcal{M}/G}) \mu_{G/H}(B^{G/H}).
\]
Proof. We begin with the first part. Let \( F^M_{\alpha} \subseteq V^M_{\alpha} \subseteq \mathcal{M} \) and \( F^G_{\beta} \subseteq V^G_{\beta} \subseteq G \) be as in the definition of a spread model. The sets \( V_{\alpha \beta} = V^M_{\alpha} V^G_{\beta} \) are embedded manifolds as they are the diffeomorphic image of \( V^M_{\alpha} \times V^G_{\beta} \) under the map \( \theta_{\alpha \beta}^{-1} \), which is defined on an open submanifold of \( \mathcal{M} \) (see Lemma 6.8). Moreover, the composition of the diffeomorphisms

\[
\begin{align*}
V_{\alpha \beta} \times H_0 \xrightarrow{\theta_{\alpha \beta}^{-1} \times id_{H_0}} V^M_{\alpha} \times V^G_{\beta} \times H_0 \xrightarrow{id_{V^M_{\alpha}} \times \theta_{\beta}} V^M_{\alpha} \times V^G_{\beta} H_0 \xrightarrow{\theta_{\alpha \beta}} V_{\alpha \beta} H_0
\end{align*}
\]

is a diffeomorphism, and it is \( \theta_{\alpha \beta} \). Since \( V^G_{\beta} H_0 \) is open in \( G \), then \( V^M_{\alpha} \times V^G_{\beta} H_0 \) is open in \( V^M_{\alpha} \times G \); so \( V_{\alpha \beta} \) is open in \( \mathcal{M} \) as the image of the open map \( \theta_{\alpha \beta} \). The first property of a spread model is now established.

Let \( F_{\alpha \beta} = F^M_{\alpha} F^G_{\beta} = \theta_{\alpha \beta}(F^M_{\alpha} \times F^G_{\beta}) \). Since \( \theta_{\alpha \beta} \) is open \( F_{\alpha \beta} \) is also open. \( F^M_{\alpha} \subseteq V^M_{\alpha} \) and \( F^G_{\beta} \subseteq V^G_{\beta} \), hence \( \theta_{\alpha \beta}(F^M_{\alpha} \times F^G_{\beta}) = F^M_{\alpha} \times F^G_{\beta} \subseteq \mathcal{M} \times \mathcal{G} \). As \( \theta_{\alpha \beta}^{-1} \) is continuous we conclude that \( F^M_{\alpha} \times F^G_{\beta} \) is closed and hence \( F^M_{\alpha} \times F^G_{\beta} \). Furthermore,

\[
\cup_{\alpha, \beta}(F^M_{\alpha} \times F^G_{\beta}) = \cup_{\alpha, \beta}(F^M_{\alpha} \times \{ \beta \}) = \cup_{\alpha}(F^M_{\alpha}) \cdot \cup_{\beta}(F^G_{\beta}) = F^M \cdot F^G \supseteq F.
\]

Note for future use that this shows that \( F^M \cdot F^G \) is a closed set, hence \( F^M \cdot F^G \supseteq F \) (in fact we get an equality).

We check the injectivity of \( \pi |_{F_{\alpha \beta}} \). Suppose that \( \pi(p^M_{1} p^G_{1}) = \pi(p^M_{2} p^G_{2}) \), or in other words, \( p^M_{1} p^G_{1} H = p^M_{2} p^G_{2} H \). Hence modulo \( G \) we have that \( p^M_{1} G = p^M_{2} G \). Since the projection to \( \mathcal{M}/G \) is injective when restricted to \( F^M_{\alpha} \), then \( p^M_{1} = p^M_{2} \). Since the \( G \) action on \( \mathcal{M} \) is free, we conclude that \( p^G_{1} H = p^G_{2} H \), but this forces \( p^G_{1} = p^G_{2} \), by the injectivity of the projection to \( G/H \) restricted to \( F^G_{\beta} \).

Next we show that \( \partial \nu_{\alpha \beta} F_{\alpha \beta} \) is contained in a finite union of lower dimensional submanifolds of \( \mathcal{M} \). This follows from the analogous assumptions on \( F^M_{\alpha} \) and \( F^G_{\beta} \), along with the following:

\[
\partial \nu_{\alpha \beta} F_{\alpha \beta} = \left( (id_{V^M_{\alpha}} \times \theta_{\beta}^{-1}) \circ \theta_{\alpha}^{-1} \right) \left( \partial (F^M_{\alpha} \times F^G_{\beta}) \times \{ e \} \right).
\]

Now we show that \( \pi |_{F} \) is a proper map. Suppose that \( p_{n} \to \infty \), where \( p_{n} \in F \). Since \( F = F^M \cdot F^G \), we may decompose \( p_{n} = p^M_{n} p^G_{n} \). Assume by contradiction that \( p_{n} H \) does not tend to infinity. Consequentially, there exist \( h_{n} \in H \) and \( p \in \mathcal{M} \) such that \( p_{n} h_{n} \to p \). For convenience we will write instead that \( p_{n} h_{n} \to p \). Since \( F \subseteq F^M G = F^M \times G \), we can find some \( \alpha \) such that, for almost every \( n \), \( p_{n} h_{n} \in V^M_{\alpha} G \). We clearly have \( p^M_{n} \to p^M \) and \( p^G_{n} h_{n} \to p^G \), where \( p = p^M \cdot p^G \). Since \( G \) is assumed to act properly on \( \mathcal{M} \), we conclude from \( p_{n} \to \infty \) that \( p^G_{n} \to \infty \). However, the restriction to \( F^G \) of the projection \( G \to G/H \) is assumed to be proper, so we get a contradiction with \( p^G_{n} \to G \).

Finally, we need to check that \( F \) is a full set of representatives. Let \( x \in \mathcal{M}/H \) and set \( x' = x G \in \mathcal{M}/G \). There is \( p^M \in F^M \) which projects to \( x' \). Since \( p^M H \) and \( x \) both project to \( x' \), there is some \( g \in G \) such that \( p^M g H = x \). By definition of \( F^G \), there is \( p^G \in F^G \) which projects to \( g H \). As a result, \( p^M p^G H = x \) i.e. \( \pi(p^M p^G) = x \).

We turn to the second part of the Proposition. We know from the first part that \( \pi_{\mathcal{M}/G}(F^M_{\alpha}) \) and \( \pi_{G/H}(F^G_{\beta}) \) are open submanifolds of \( \mathcal{M}/G \) and \( G/H \) respectively, and that the maps

\[
\rho_{\alpha \beta} : \pi_{\mathcal{M}/G}(F^M_{\alpha}) \times \pi_{G/H}(F^G_{\beta}) \to F^M_{\alpha} \times F^G_{\beta} \to F_{\alpha \beta} \to \mathcal{M}/H
\]

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are open diffeomorphisms onto their image. Denote \( L_{\alpha\beta} := \pi_{\mathcal{M}/G}^M(F_{M}^{\alpha}) \times \pi_{G/H}^G(F_{G}^{\beta}) \). Similarly to the proof of Proposition 6.3, we split each

\[
B_{\alpha\beta} := \left( B^{\mathcal{M}/G} \times B^{G/H} \right) \cap L_{\alpha\beta}
\]

into two parts: \( B \cap \partial L_{\alpha\beta} \) and \( B_{\text{int}} := B \cap L_{\alpha\beta}^0 \). Since, by part (2) of Lemma 6.6, it is enough to show that \( \rho_{\alpha\beta}(B_{\text{int}}) \) is BCS inside \( \rho_{\alpha\beta}(L_{\alpha\beta}^0) \) and that locally \( \rho_{\alpha\beta}(\partial L_{\alpha\beta}) \) is contained in a finite union of codimension \( \geq 1 \) submanifolds.

The first statement is clear since \( \rho_{\alpha\beta}|_{L_{\alpha\beta}^0} \) is an open diffeomorphism onto its image \( \rho_{\alpha\beta}(L_{\alpha\beta}^0) \).

For the second statement, first notice that \( \rho_{\alpha\beta}(\partial L_{\alpha\beta}) = \pi_{\mathcal{M}/H_0}^\mathcal{M} \circ \pi_{\mathcal{M}/H_0}^\mathcal{M}(\theta_\alpha^{-1}(\partial_{V_\alpha \times V_\beta}(F_{\alpha} \times F_{\beta}))) \). By assumption, \( \partial_{V_\alpha \times V_\beta}(F_{\alpha} \times F_{\beta}) \) is locally contained in a finite union of codimension \( \geq 1 \) submanifolds (w.r.t. \( V_\alpha \times V_\beta \)). Hence, using the fact that \( \pi_{\mathcal{M}/H_0}^\mathcal{M} \circ \theta_\alpha^{-1} \) is an open diffeomorphism, we conclude that \( \pi_{\mathcal{M}/H_0}^\mathcal{M}(\theta_\alpha^{-1}(\partial_{V_\alpha \times V_\beta}(F_{\alpha} \times F_{\beta}))) \) is also locally contained in a finite union of codimension \( \geq 1 \) submanifolds w.r.t. \( \mathcal{M}/H_0 \). Finally, this property is stable under \( \pi_{\mathcal{M}/H_0}^\mathcal{M} \) since this map is a local diffeomorphism.

It is left to prove the third part of the proposition. Let \( \mu_G, \mu_H, \mu_\mathcal{M} \) be the measures appearing in Proposition 6.10 and assume \( A \subset H \) be a compact neighborhood of \( e_H \) so that \( 0 < \mu_H(A) < \infty \). By Proposition 6.10 we have the following equalities:

\[
\mu_\mathcal{M}(B^M \mathcal{B} G A) = \mu_G(B^G A) \mu_{\mathcal{M}/G}(B^{\mathcal{M}/G}),
\]

\[
\mu_G(B^G A) = \mu_{G/H}(B^{G/H}) \mu_H(A),
\]

\[
\mu_\mathcal{M}(B^M \mathcal{B} G A) = \mu_H(A) \mu_{\mathcal{M}/H}(B^{\mathcal{M}/H}).
\]

It follows that

\[
\mu_{\mathcal{M}/H}(B^{\mathcal{M}/H}) = \mu_{\mathcal{M}/G}(B^{\mathcal{M}/G}) \mu_{G/H}(B^{G/H}).
\]

The content of the following proposition is that if a space can be written as a quotient in two ways, \( \mathcal{M}/G \) and \( \mathcal{M}'/H \), where \( \mathcal{M}' \subset \mathcal{M} \) and \( H < G \), then a spread model for the latter a also a spread model for the first.

**Proposition 6.16.** Suppose that \( \mathcal{M} \) is a manifold and \( \mathcal{M}' \) is an embedded closed submanifold of \( \mathcal{M} \). Suppose that \( G \) is a Lie group and \( H \) is a closed subgroup of \( G \) satisfying \( H_0 = G_0 \cap H \). Assume that \( G \) acts on \( \mathcal{M} \) and \( H \) stabilizes \( \mathcal{M}' \); the action is free, proper and smooth. Finally assume that the map \( \iota : \mathcal{M}'/H \to \mathcal{M}/G \) given by \( m'H \to mG \) is a diffeomorphism. If \( F \) is a spread model of \( \mathcal{M}'/H \) in \( \mathcal{M}' \), then it is also a spread model of \( \mathcal{M}/G \) in \( \mathcal{M} \).
Proof. Let $V_\alpha, F_\alpha \subseteq \mathcal{M}'$ as in the definition of a spread model. The only condition that is not clear is the first one. By Lemma 6.3, it is sufficient to prove that $\theta_\alpha : V_\alpha \times G_0 \to V_\alpha G_0$ is a diffeomorphism whose image is open. For this, one has to show that $\theta_\alpha$ is an injective immersion (since $V_\alpha \times G_0$ and $\mathcal{M}$ both have the same dimension).

If $pg = p_1 \in V_\alpha G_0$, then $p = p_1$ modulo $G_0$. Since $p, p_1 \in V_\alpha$, we conclude that $p = p_1$ modulo $H$. In other words, there is $h \in H$ such that $ph = p_1$. Since the action is free, we get that $g = h \in G_0 \cap H = H_0$. This forces $p = p_1$ and $g = h = e$, since $\theta_\alpha' : V_\alpha \times H_0 \to V_\alpha H_0$ is injective.

The natural map $\pi^0_\alpha : V_\alpha \to \mathcal{M}/G_0$ is an open diffeomorphism onto its image due to the following argument. Consider the following commutative diagram:

$$
\begin{array}{ccc}
V_\alpha & \xrightarrow{\pi^0_\alpha} & \mathcal{M}'/H_0 \\
\downarrow{\pi^0_\alpha} & & \downarrow{\phi} \\
\mathcal{M}/G_0 & \xrightarrow{\iota^0} & \mathcal{M}/G
\end{array}
$$

Since $\pi^0_\alpha, \phi$ and $\iota$ are immersions, we conclude that $\iota^0 \circ \text{proj} \circ \pi^0_\alpha$ is also an immersion. Since $\pi^0_\alpha$ is an open diffeomorphism by assumption, we get that $\text{proj}$ must be an immersion. If $p_1, p_2 \in \mathcal{M}'$ such that $p_2 = p_1 g$ for some $g \in G_0$, it must be that $p_1$ and $p_2$ are equal modulo $H$, since $\iota$ is injective. Freeness of the action implies that $g \in H$, and so $g \in H_0$. As a result, $\text{proj}$ is injective and so $\text{proj} \circ \pi^0_\alpha$ is also. The claim now follows since $V_\alpha$ and $\mathcal{M}/G_0$ have the same dimension. \[ \square \]

7 Construction of fundamental domains for $\text{SL}_m(\mathbb{Z})$

The goal of this section is to recall a construction for fundamental domains of $\text{SL}_m(\mathbb{Z})$ inside $\text{SL}_m(\mathbb{R})$ and inside $\text{SO}_m(\mathbb{R}) \setminus \text{SL}_m(\mathbb{R})$. The motivation for this is that in the next section we will see that these fundamental domains are spread models for two spaces of lattices that we now turn to describe.

Let $\Lambda$ be a (full rank) lattice inside $\mathbb{R}^m$ having the columns of $M \in \text{GL}_m(\mathbb{R})$ as an ordered basis. It is well known that any other basis of $\Lambda$ appears as the columns of a matrix obtained by multiplying $M$ from the right by a matrix from $\text{GL}_m(\mathbb{Z})$. As a result, the space of $m$-lattices can be defined as $\text{GL}_m(\mathbb{R}) / \text{GL}_m(\mathbb{Z})$. One can also consider a more crude space which is the space of shapes of lattices: two lattices $\Lambda_1 = M_1 \cdot \text{GL}_m(\mathbb{Z})$ and $\Lambda_2 = M_2 \cdot \text{GL}_m(\mathbb{Z})$ have the same shape if $\Lambda_1$ differs from $\Lambda_2$ by an orthogonal transformation and rescaling, namely there are $k \in O_m(\mathbb{R})$ and $c > 0$ such that $ckM_1 \cdot \text{GL}_m(\mathbb{Z}) = M_2 \cdot \text{GL}_m(\mathbb{Z})$. As a result, the space of shapes can be defined as

$$
\mathcal{X}_m = \text{PO}_m(\mathbb{R}) \setminus \text{PGL}_m(\mathbb{R}) / \text{PGL}_m(\mathbb{Z}) \cong \text{SO}_m(\mathbb{R}) \setminus \text{SL}_m(\mathbb{R}) / \text{SL}_m(\mathbb{Z}).
$$

Notice that in the right hand side we consider unimodular lattices (i.e. lattices with covolume 1), since clearly every lattice can be rescaled to a unimodular lattice. We let

$$
\mathcal{L}_m := \text{SL}_m(\mathbb{R}) / \text{SL}_m(\mathbb{Z})
$$

denote the space of unimodular lattices in $\mathbb{R}^m$. 

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In Subsection 7.1 we introduce a variant of Siegel sets inside $\text{SL}_m(\mathbb{R})$, which contain a finite number of representatives from every (right) orbit of $\text{SL}_m(\mathbb{Z})$, and therefore a fundamental domain; in Subsection 7.2 we define the fundamental domain $\bar{F}_m \subset \text{SL}_m(\mathbb{R})$ inside the Siegel set, as well as the resulting fundamental domain $F_m$ inside $\text{SO}_m(\mathbb{R}) \setminus \text{SL}_m(\mathbb{R})$, or more precisely in $P_m$, the group of upper triangular matrices of determinant 1 with positive diagonal entries; indeed the Iwasawa decomposition of $\text{SL}_m(\mathbb{R})$ implies that $\text{SO}_m(\mathbb{R}) \setminus \text{SL}_m(\mathbb{R})$ is diffeomorphic to $P_m$.

Let us stress on the fact that the construction for $F_m$ is not new (Sch98, Gre93), but we bring it here for completeness, as well as for adding the way to obtain $\bar{F}_m$ from $F_m$.

### 7.1 Reduced bases

Write $P_m = A_m N_m$ where $A_m$ is the subgroup of diagonal matrices, and $N_m$ the subgroup of unipotent matrices.

Let $\Lambda$ be a lattice in $\mathbb{R}^m$ (not necessarily unimodular). We describe an inductive method to construct an ordered basis $\{v_1, \ldots, v_m\}$ for $\Lambda$ as follows. Let $v_1$ be a shortest nonzero element of $\Lambda$. For future reference, we denote its length by $a_1$ and its direction $v_1/a_1$ by $\phi_1$. Next, write $V_1$ for $\text{span}_\mathbb{R}(v_1)_{\mathbb{R}}$ and consider the projection of $\Lambda$ to $V_1^\perp$, which is a lattice of dimension $m - 1$. One can find a vector $v_2 \in \Lambda$ whose projection to $V_1^\perp$ is of nonzero minimal length $a_2$. Since actually all the elements in $\{v_2 + n v_1 : n \in \mathbb{Z}\}$ share this property of having their projection to $V_1^\perp$ be of length $a_2$, we may assume that $v_2$ also satisfies that its projection to $V_1$ is $n_{1,2} a_1 \cdot \phi_1$ with $|n_{1,2}| \leq \frac{1}{2}$. We proceed by induction:

**Definition 7.1 (and notations).** A *Reduced* basis for a lattice $\Lambda$ is a basis $\{v_1, \ldots, v_m\}$ in which for all $j \in \{1, \ldots, m\}$, the basis element $v_j$ is chosen such that:

1. The projection of $v_j$ to $V_{j-1}^\perp = (\text{span}_\mathbb{R}(v_1, \ldots, v_{j-1}))^\perp$ has minimal non-zero length $a_j$ (here $V_0 = \{0\}$); denote this projection by $a_j \phi_j$, where $\phi_j$ is a unit vector.

2. The projection of $v_j$ to $V_{j-1} = \text{span}_\mathbb{R}(v_1, \ldots, v_{j-1}) = \text{span}_\mathbb{R}(\phi_1, \ldots, \phi_{j-1})$ is

$$
\sum_{i=1}^{j-1} (n_{i,j} a_i) \phi_i \text{ with } |n_{i,j}| \leq \frac{1}{2} \text{ for all } i = 1, \ldots, j - 1.
$$

The matrix $M = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}$ is called a *reduced matrix* of $\Lambda$.

We note that in the case of unimodular bases (bases of co-volume 1), one may need to replace $v_1$ by $-v_1$ in order for the reduced matrix $M$ to have determinant 1 (and not $-1$).

The parameters $\{a_j\}, \{n_{i,j}\}$ and $\{\phi_j\}$ involved in the process of constructing a reduced basis $\{v_1, \ldots, v_m\}$ are interpreted via the KAN coordinates of the associated reduced matrix as follows. Let

$$
a = \text{diag}(a_1, \ldots, a_m), \ k = [\phi_1 \cdots \phi_m]
$$

and

$$
n = \begin{pmatrix} 1 & n_{1,1} & \cdots & n_{1,m} \\ & 1 & \vdots \\ & & \ddots & n_{m-1,m} \\ & & & 1 \end{pmatrix}.
$$
Then, since the $i$-th column of $ka$ is $a\phi_i$ = the projection of $v_i$ to $V_{i-1}$, and the $i$-th column of $n$ is exactly the coordinates of $v_i$ w.r.t. the orthogonal set $\{a_1\phi_1, \ldots, a_m\phi_m\}$, we obtain that the reduced matrix is

$$M = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} = kan.$$

**Lemma 7.2.** Suppose $M = k_m a_m n_m$ is -reduced w.r.t. some lattice $\Lambda$, where $k_m$, $a_m$ and $n_m$ are as above.

1. $n_M$ is a unipotent upper triangular matrix whose entries are bounded in $[-\frac{1}{2}, \frac{1}{2}]$ (in particular, $\|n_{M}^{\pm 1}\|, \|n_{M}^{\pm 2}\| < 1$).

2. $a_m = \text{diag}(a_1, \ldots, a_m)$ is a diagonal matrix which satisfies that $a_1 \prec \cdots \prec a_m$. Specifically, $\sqrt{3} a_j \leq a_{j+1}$.

3. If $\lambda \in \Lambda$ (i.e. $\lambda = Mv$ for some $v \in \mathbb{Z}^m$) satisfies $\lambda \notin V_{j-1}$, then

$$\|\lambda\| \geq \text{dist}(\lambda, V_{j-1}) \geq \text{dist}(v_j, V_{j-1}) = a_j.$$

4. If $x \in V_j$, then $\|a_M x\| < a_j \|x\|$.

**Proof.** Parts 1 and 3 are immediate from the construction of $M$. For part 2 recall that $\{a_1\phi_1, \ldots, a_m\phi_m\}$ is an orthogonal set in $\mathbb{R}^m$ and that $v_{j+1} = a_{j+1}\phi_{j+1} + \sum_{i=1}^{j} n_{i,j+1} a_i\phi_i$. Then,

$$a_j^2 \leq \text{dist}(v_{j+1}, V_{j-1})^2 = \|n_{j,j+1} (a_j\phi_j) + (a_{j+1}\phi_{j+1})\|^2 = a_j^2 |n_{j,j+1}|^2 + a_{j+1}^2.$$

Now, since $|n_{j,j+1}| \leq \frac{1}{2}$ (by part 1), we obtain:

$$\frac{1}{4} a_j^2 + a_{j+1}^2 \geq a_j^2$$

and therefore

$$a_{j+1} \geq \sqrt{3} a_j.$$

As for part 4

$$\|a x\| = \left\| \begin{pmatrix} a_1 & \cdots & a_m \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \end{pmatrix} \right\| = \left\| \begin{pmatrix} a_1 x_1 \\ \vdots \\ a_j x_j \end{pmatrix} \right\| \leq \max_{1 \leq i \leq j} |a_i| \|x\| \text{ part 2} \times a_j \|x\|.$$

**Definition 7.3.** We refer to the sets

$$\left\{ M \in \text{GL}_m(\mathbb{R}) : M \text{ is reduced for the lattice } \Lambda \right\}$$

as reduced Siegel sets.

**Remark 7.4.** The reduced Siegel sets are contained in the well-known Siegel sets (e.g., [BM00, Rag72, Chapter X]).
We note that parts 1 and 3 of Lemma 7.2 are the defining conditions of the reduced Siegel sets (the inequalities in parts 2 and 4 are redundant, since they follow from part 3). Observe that these defining inequalities depend only on the entries of the $N$ and $A$ components of the matrix. Indeed, let $M = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} = kan$ and $z = an$; the inequalities in 4 are on the entries of $n$, and the inequalities in 3 translate into

\[ a_j \leq \| \text{projection of } zv \|_{\text{to span}_{\mathbb{R}}(e_j, \ldots, e_m)}, \]

for every $v = (\alpha_1, \ldots, \alpha_m)^t \in \mathbb{Z}^n$ and $j = 1, \ldots, m$. This is because:

\[ a_j = \text{dist}(v_j, V_{j-1}) \leq \text{dist}\left(\sum_{i=j}^{m} \alpha_i v_i, V_{j-1}\right) = \text{dist}\left(\sum_{i=j}^{m} \alpha_i z_i, E_{j-1}\right) \]

where $E_{j-1} := \text{span}_{\mathbb{R}}(e_1, \ldots, e_{j-1})$ and $z = [z_1, \ldots, z_m]$,

\[ = \text{dist}(zv, E_{j-1}) = \| \text{projection of } zv \|_{\text{to span}_{\mathbb{R}}(e_j, \ldots, e_m)}. \]

We also note that the number of the defining inequalities for these reduced Siegel sets is infinite: indeed, every $v \in \mathbb{Z}^n$ yields an inequality in formula 7.1 (resp. part 3 of Lemma 7.2). However, it is shown in [Sch98, p.49] that it is actually sufficient to consider the inequalities 7.1 for only finitely many $v \in \mathbb{Z}^n$. We state it for future reference:

**Proposition 7.5.** The reduced Siegel sets are defined by a finite number of inequalities in (the entries of) the $N$ and $A$ components of a matrix.

**Remark 7.6.** The finitely many integral vectors $u$ that imply the sufficient inequalities from 7.1 are as follows. For any $j = 1, \ldots, m$ one considers the $v \in \mathbb{Z}^n$ which satisfy

\[ \max (|\alpha_j|, \ldots, |\alpha_m|) \leq C a_j \| \text{projection of } zv \|_{\text{to span}_{\mathbb{R}}(e_j, \ldots, e_m)} \]

where $C$ is some constant that depends only on $m$ and can be computed explicitly from [Sch98]; clearly, there is only a finite number of integral vectors $u$ which satisfy this condition. In particular, the reduced Siegel sets can be computed explicitly.

### 7.2 Fundamental domains of $\text{SL}_m(\mathbb{Z})$ inside $\text{SL}_m(\mathbb{R})$ and $P_m$

From now on we shall only consider unimodular lattices (resp. bases) in $\mathbb{R}^m$. Since these unimodular lattices are identified with cosets in $\text{SL}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z})$, where a representative for a coset is a choice of a basis for the corresponding lattice, it follows that a fundamental domain for (the right action of) $\text{SL}_m(\mathbb{Z})$ inside $\text{SL}_m(\mathbb{R})$ consists of a unique choice of a basis for every unimodular lattice $\Lambda < \mathbb{R}^m$.

We know (by the construction presented in the previous subsection) that every lattice has a reduced basis, and therefore the reduced Siegel set contains a fundamental domain for $\text{SL}_m(\mathbb{Z})$. We turn to describe a closure of such a fundamental domain, namely a choice of a unique reduced basis for almost every unimodular lattice $\Lambda < \mathbb{R}^m$. 

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Remark 7.7. It is shown in [Sch98] that the number of reduced bases for a lattice $\mathbb{R}^m$ is finite, where a bound on this number depends only on $m$, and not on the lattice. Intuitively, this is due to the fact that a given lattice has only a finite number of shortest vectors (where the bound on this number depends only on the dimension), and a reduced basis is constructed such that in every step, one chooses a shortest vector from some lattice.

Given a reduced basis $\{v_1, \ldots, v_m\}$, one can clearly obtain further reduced bases for the same lattice by alternating the signs of the elements $v_j$. Note that the corresponding reduced matrices $M$ will have the same $A$ components, and in fact they will vary from each other only by the signs of the entries of $n$ and $k$ as follows:

- replacing $\{v_1, \ldots, v_m\}$ by $\{-v_1, \ldots, -v_m\}$ is done by multiplying from $M$ from the left by $-I$ (replacing $k$ by $-k$);
- replacing $v_j$ by $-v_j$ for $j = 2, \ldots, m$ is done by changing the sign of the $j$-th row and column of $n$ (above the diagonal) and changing the sign of the $j$-th column of $k, \phi_j$.

In order to preserve the property $\det(M) = 1$, one is only allowed to alternate the sign of an even number among the $v_j$’s. In particular, one is allowed to change all signs simultaneously if and only if $-I \in K = \text{SO}_m(\mathbb{R})$.

Definition 7.8. We let $\tilde{F}_m \subset \text{SL}_m(\mathbb{R})$ denote the closed subset of the reduced Siegel set (Definition 7.3) which satisfies the following conditions on the $N$ and $K$ components:

1. Condition on the sign of the first row of $n$:
   \[
   2 \mid m \implies n_{1,j} \geq 0 \quad \text{for} \quad j > 2 \implies n_{1,j} \in \left[0, \frac{1}{2}\right] \quad \text{for} \quad j > 2
   \]
   \[
   2 \nmid m \implies n_{1,j} \geq 0 \quad \text{for} \quad j > 1 \implies n_{1,j} \in \left[0, \frac{1}{2}\right] \quad \text{for} \quad j > 1
   \]

2. Condition on the $K$-components: they lie in a closure of a fundamental domain of the lattice $\mathbb{Z}(K)$ in $K$.

We also denote by $F_m \subseteq P_m$ the projection of $\tilde{F}_m$ mod $K$, namely the set of upper triangular matrices whose columns are reduced bases for the lattice spanned by their columns, and whose $N$ components satisfy condition 1.

We state the following for future reference:

Corollary 7.9. The boundary of $\tilde{F}_m$ (resp. $F_m$) is contained in a finite union of lower-dimensional manifolds in $\text{SL}_m(\mathbb{R})$ (resp. $P_m$).

Proof. According to Proposition 7.5 and Definition 7.8, $\tilde{F}_m$ and $F_m$ are defined by a finite number of polynomial inequalities. \hfill \square

The significance of $\tilde{F}_m$ and $F_m$ stems from the following:

Theorem 7.10. $\tilde{F}_m \subset \text{SL}_m(\mathbb{R})$ and $F_m \subset P_m$ are the closures of fundamental domains for the right actions of $\text{SL}_m(\mathbb{Z})$ on $\text{SL}_m(\mathbb{R})$ and on $P_m$ respectively.
Proof. Since \( \tilde{F}_m \) is a subset of the reduced Siegel set, the latter containing a basis for every lattice, defined by the additional conditions from Definition 7.8 which merely impose a choice of signs for the basis elements — we conclude that \( \tilde{F}_m \) contains a representative (basis) for every unimodular lattice in \( \mathbb{R}^m \).

It is now sufficient to show that every \( M \in \text{int} \left( \tilde{F}_m \right) \) is the unique representative for the lattice spanned by its columns. In other words, if a given lattice has more than one representative in \( \tilde{F}_m \), then these representatives (that are reduced matrices for the lattice) lie in the boundary of \( \tilde{F}_m \).

Let \( M \in \text{int} \left( \tilde{F}_m \right) \). Then, \( M \) satisfies a strict version of the inequalities in Formula 7.1 and in Definition 7.8. Write \( M = kan \). Using induction and (the strict version of) inequality 7.1, it is clear that \( v_1 \) is uniquely determined up to a sign; \( v_2 \) is uniquely determined up to a sign and modulo \( V_1 \); and so forth, every \( v_j \) is uniquely determined up to a sign and modulo \( V_{j-1} \). As a result, \( a \) is uniquely determined, and the columns of \( k, \phi_1, \ldots, \phi_m \), are determined up to a sign. Since \( v_j = \sum_{i=1}^{j} n_{i,j} (a_i \phi_i) \), one can show using reverse induction (from \( i = j - 1 \) to \( i = 1 \)) that there are unique \( n_{i,j} \) satisfying the strict version of condition 4 from Definition 7.8 so that \( v_1, \ldots, v_m \) are determined up to a sign. According to the explanation in the beginning of this section, the inequalities in Definition 7.8 impose a unique choice of signs, and therefore a unique representative in the interior of \( \tilde{F}_m \).

Remark 7.11 (Declaring abuse of notation). As mentioned in Theorem 7.10, \( \tilde{F}_m \) and \( F_m \) are closures of fundamental domains; in order to obtain actual fundamental domains, one should remove parts of their boundaries, leaving a unique representative for every lattice. However, we will abuse notation and denote \( \tilde{F}_m \) and \( F_m \) for the actual fundamental domains, and not their closures.

Notation 7.12. For a matrix \( M \), denote by \( K_M \) a fundamental domain in \( \text{SO}_m (\mathbb{R}) \) for the finite group \( \text{Sym}^+(M) \), which is the stabilizer in \( \text{SO}_m (\mathbb{R}) \) of the lattice spanned by the columns of \( M \).

Proposition 7.13. The relation between the fundamental domains \( \tilde{F}_m \) and \( F_m \) is given by

\[
\tilde{F}_m = \bigcup_{z \in F_m} K_z \cdot z,
\]

and when \( z \in \text{int} (F_m) \) it holds that \( \text{Sym}^+(M) = Z(K) \), the center of \( K = \text{SO}_m (\mathbb{R}) \).

Remark 7.14. In fact, it is shown in [Sch98, p.50] that the interior of any fundamental domain of \( \text{SL}_m (\mathbb{Z}) \) consists of matrices \( M \) for which the \( \text{Sym}^+(M) \) is \( Z(K) \).

Corollary 7.15. From Proposition 7.13 it follows that the measure of \( \tilde{F}_m \) is the measure of \( F_m \) times the measure of a generic fiber in \( \text{SO}_m (\mathbb{R}) \) (namely the fiber of the interior points), which is the measure of \( \text{SO}_m (\mathbb{R}) \) divided by the index of its center: 2 when \( m \) is even, and 1 when \( m \) is odd.
8 Special examples of spread models in $SL_m(\mathbb{R})$

8.1 Space of $d$-dimensional lattices in $\mathbb{R}^n$

In a similar fashion to the construction of the space of full dimensional lattices in $\mathbb{R}^n$, we can construct the space of unimodular rank $d$ lattices with orientation in $\mathbb{R}^n$:

$$\mathcal{L}_{d,n} := SL_n(\mathbb{R})/T,$$

where

$$T = \left[ SL_d(\mathbb{Z}) \mathbb{R}_{d,n-d} \atop 0_{n-d,d} SL_{n-d}(\mathbb{R}) \right] \times \left\{ \begin{bmatrix} \alpha^{-\frac{1}{d}} I_d & 0_{d,n-d} \\ 0_{n-d,d} & \alpha^{\frac{1}{d-n}} I_{n-d} \end{bmatrix} : \alpha > 0 \right\}.$$  

One can also get a fundamental domain for $\mathcal{L}_{d,n}$ in $SL_n(\mathbb{R})$ using a variant of Iwasawa decomposition. For this denote by $P''$ the subgroup $\left[ P_d 0_{d,n-d} \atop 0_{n-d,d} P_{n-d} \right]$ in $SL_n(\mathbb{R})$, $K = SO_n(\mathbb{R})$, $K'' = \left[ SO_d(\mathbb{R}) 0_{d,n-d} \atop 0_{n-d,d} SO_{n-d}(\mathbb{R}) \right]$ and let $K'$ be a spread model for $K/K'' \simeq Gr(d,n)$, the Grassmannian of oriented $d$-dimensional subspaces of $\mathbb{R}^n$ (see Proposition 6.11). It is easy to conclude that

$$\mathcal{L}_{d,n} = SL_n(\mathbb{R})/T \cong K P''/\left[ SL_d(\mathbb{Z}) 0_{d,n-d} \atop 0_{n-d,d} SO_{n-d}(\mathbb{R}) \right]$$

and that $K' \left[ \tilde{F}_d 0_{d,n-d} \atop 0_{n-d,d} I_{n-d} \right]$ is a set of representatives for $T$ inside $SL_n(\mathbb{R})$, and for $\left[ SL_d(\mathbb{Z}) 0_{d,n-d} \atop 0_{n-d,d} SO_{n-d}(\mathbb{R}) \right]$ inside $K P''$.

Next, we have the space of oriented normalized quotient lattices of $\Lambda/\Lambda_d$, where $\Lambda$ is a full lattice in $\mathbb{R}^n$ and $\Lambda_d$ is a $d$-dimensional sub-lattice of $\Lambda$. It is more convenient to identify $\Lambda/\Lambda_d$ with the orthogonal projection of $\Lambda$ onto the subspace orthogonal to span$_{\mathbb{R}}(\Lambda)$. Now it is easy to see that this space, say $\mathcal{L}^\pi_{d,n}$, can be presented as the quotient

$$\mathcal{L}^\pi_{d,n} = SL_n(\mathbb{R})/T^\# = K P''/\left[ SL_d(\mathbb{Z}) 0_{d,n-d} \atop 0_{n-d,d} SL_{n-d}(\mathbb{Z}) \right],$$

where

$$T^\pi = \left[ SL_d(\mathbb{R}) \mathbb{R}_{d,n-d} \atop 0_{n-d,d} SL_{n-d}(\mathbb{Z}) \right] \times \left\{ \begin{bmatrix} \alpha^{-\frac{1}{d}} I_d & 0_{d,n-d} \\ 0_{n-d,d} & \alpha^{\frac{1}{d-n}} I_{n-d} \end{bmatrix} : \alpha > 0 \right\}.$$  

It is easy to see that $K' \left[ I_d 0_{d,n-d} \atop 0_{n-d,d} \tilde{F}_{n-d} \right]$ is a set of representatives for $T^\pi$ inside $SL_n(\mathbb{R})$ and for $\left[ SL_d(\mathbb{Z}) 0_{d,n-d} \atop 0_{n-d,d} SL_{n-d}(\mathbb{Z}) \right]$ inside $K P''$.

In fact this space is diffeomorphic to $\mathcal{L}_{n-d,n}$. To see this write $g \in SL_n(\mathbb{R})$ as $g = (A|B)$, where $A \in \mathbb{R}^{n,d}$ and $B \in \mathbb{R}^{n-d,n}$ and consider the map from $\mathcal{L}^\pi_{d,n}$ to $\mathcal{L}_{n-d,n}$ given by

$$(A|B) \cdot T^\pi_d \mapsto \left( (I - A^t A)^{-1} A^t \right) B | A \right) \cdot T_{n-d}$$

and the inverse map is given by

$$(\hat{A}\hat{B}) \cdot T_{n-d} \mapsto \left( (I - \hat{A}^\dagger \hat{A})^{-1} \hat{A}^\dagger \right) \hat{B} | \hat{A} \right) \cdot T^\pi_d,$$

where $\hat{B} \in \mathbb{R}^{n,d}$ and $\hat{A} \in \mathbb{R}^{n-d,n}$. We leave it to the reader to check that these maps are well-defined and are inverses of each other (see also the appendix of our forthcoming work [HK2005]).
A third and final space is the space of normalized pairs \((\Lambda, \Lambda^z)\), where \(\Lambda\) is a rank \(d\) oriented unimodular lattice and \(\Lambda^z\) is an oriented unimodular quotient lattice \(\Lambda_0/\Lambda\), with \(\Lambda_0\) being a full lattice containing \(\Lambda\). The resulting space is modeled by

\[
\mathcal{P}_{d,n} := \text{SL}_n (\mathbb{R}) / T_d^\kappa = KP" / \left[ \begin{smallmatrix} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{smallmatrix} \right],
\]

where

\[
T_d^\kappa = \left[ \begin{smallmatrix} \text{SL}_d (\mathbb{Z}) & \mathbb{R}^{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{smallmatrix} \right] \times \left\{ \begin{bmatrix} \alpha^{-1} I_d & 0_{d,n-d} \\ 0_{n-d,d} & \alpha^{-1} I_{n-d} \end{bmatrix} : \alpha > 0 \right\}.
\]

It is easy to see that \(K' \left[ \begin{smallmatrix} \tilde{F}_d & 0_{d,n-d} \\ 0_{n-d,d} & \tilde{F}_{n-d} \end{smallmatrix} \right]\) is a set of representatives for \(T_d^\kappa\) inside \(\text{SL}_n (\mathbb{R})\) and for \(\left[ \begin{smallmatrix} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{smallmatrix} \right]\) inside \(KP"\).

### 8.2 Spread models for lattice spaces

The goal of this final part is to provide concrete examples for spread models. These examples are all for spaces of lattices, which is where the authors’ interest in spread models stems from.

**Proposition 8.1.** The following pairs consist of quotient spaces and their spread models in the corresponding manifolds:

1. \(\mathcal{L}_m = \text{SL}_m (\mathbb{R}) / \text{SL}_m (\mathbb{Z})\), and \(\tilde{F}_m\) inside \(\mathcal{M} = \text{SL}_m (\mathbb{R})\);
2. \(\mathcal{X}_m = \text{SO}_m (\mathbb{R}) \setminus \text{SL}_m (\mathbb{R}) / \text{SL}_m (\mathbb{Z}) \simeq P_m / \text{SL}_m (\mathbb{Z})\), and \(F_m\) inside \(\mathcal{M} = P_m\).
3. \(\mathcal{P}_{d,n} = KP" / \left[ \begin{smallmatrix} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{smallmatrix} \right]\) and \(K' \left[ \begin{smallmatrix} \tilde{F}_d & 0_{d,n-d} \\ 0_{n-d,d} & \tilde{F}_{n-d} \end{smallmatrix} \right]\) inside \(KP"\)
4. \(\mathcal{L}_d,n = KP" / \left[ \begin{smallmatrix} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SO}_{n-d} (\mathbb{R}) \end{smallmatrix} \right]\) and \(K' \left[ \begin{smallmatrix} \tilde{F}_d & 0_{d,n-d} \\ 0_{n-d,d} & \tilde{F}_{n-d} \end{smallmatrix} \right]\) inside \(\mathcal{M} = KP"\).
5. \(\mathcal{L}^z_{d,n} = KP" / \left[ \begin{smallmatrix} \text{SO}_d (\mathbb{R}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{smallmatrix} \right]\) and \(K' \left[ \begin{smallmatrix} I_d & 0_{d,n-d} \\ 0_{n-d,d} & \tilde{F}_{n-d} \end{smallmatrix} \right]\) inside \(\mathcal{M} = KP"\).

For the proof, we will use the following.

**Assumption 8.2.** According to Corollary 6.14, we can choose all \(K_z\)'s to be BCS’s. We choose the same BCS fundamental domain \(K_z\) as \(K_z\) for all \(z \in P\) such that \(\text{Sym}^+ (z)\) is \(Z(K)\), and then apply Remark 6.13 to choose a BCS fundamental domain \(K_z\) for the remaining \(z\)'s, such that \(K_z \subseteq K\).

**Proof of Proposition 8.1.** **Parts 1 and 2.** The spaces \(\mathcal{L}_m\) and \(\mathcal{X}_m\) are dealt similarly; since the quotients are by discrete subgroups, we will verify that the fundamental domains are indeed spread models using Remark 6.13. For brevity, let \(\mathcal{M} := \text{SO}_m (\mathbb{R}) \setminus \text{SL}_m (\mathbb{R}) = P_m\). The boundary \(F_m \cap \mathcal{M}_{\text{free}}\) is contained in a finite union of lower dimensional submanifolds according to Corollary 7.9 for the boundary of \(\tilde{F}_m\), we should use also the fact that all the \(K\)-fibers in the expression for \(\tilde{F}_m\) given in Proposition 7.13 are BCS’s, by Assumption 8.2. In more details,

\[
\tilde{F}_m = (K \cdot (P_m \cap \text{int} (F_m))) \cup \bigcup_{i=1}^{q(m)} K_{z_i} \cdot \{ z \in P_m \cap \partial F_m : \text{Sym}^+ (A_z) = \text{Sym}^+ (A_{z_i}) \}
\]
where \( \{ z : \text{Sym}^+ (A_z) = \text{Sym}^+ (A_{z_0}) \} \) is contained in \( \partial F_m \), and is therefore a BCS in \( P_m \). Since the fibers in \( \text{SO}_m (\mathbb{R}) \) are BCS’s due to Assumption \( \text{S.2} \), then by Fact \( \text{S.12} \) and the fact that \( \text{SO}_m (\mathbb{R}) \times P_m \) is homeomorphic to \( \text{SL}_m (\mathbb{R}) \) we get that the boundary of \( \tilde{F}_m \) is also contained in a finite union of lower dimensional manifolds.

The fact that the quotient map \( \pi \) restricted to the closure is proper is a consequence of the Mahler compactness criterion, as we now explain. Assume that \( B \subset L_m \) is compact, which by Mahler’s criterion means that there exists a positive constant \( \beta \) such that for every \( \Lambda \in B \), the length of the shortest vector in \( B \) is at least \( \beta \). Let \( g = \pi (\Lambda) \in \tilde{F}_m \) and write \( g = kan \) where \( a = \text{diag} (a_1, \ldots, a_m) \). By construction of \( \tilde{F}_m \), \( a_1 \) is the length of a shortest vector in \( \Lambda \), and therefore \( \beta \leq a_1 \). Also by construction of \( F_m \), the columns of \( g \) are a reduced basis (Definition \( \text{7.1} \)) to \( \Lambda \); hence by part \( \text{2} \) Lemma \( \text{7.2} \)

\[
0 < \beta \leq a_1 \leq (\sqrt{3}/2) a_2 \leq \cdots \leq (\sqrt{3}/2)^{m-1} a_m
\]

\[
(\sqrt{3}/2)^{m-1} a_m = (\sqrt{3}/2)^{m-1} \cdot \frac{1}{a_1 \cdot a_m} \leq (\sqrt{3}/2)^{m-1} \cdot \frac{1}{\beta^{m-1}}.
\]

We conclude that \( (a_1, \ldots, a_m) \) lies in a bounded subset of \( \mathbb{R}^{m-1} \), namely the \( A \) components of the elements in \( \pi^{-1} (B) \) lie in a compact set. The \( N \) and \( K \) components of the elements of \( \tilde{F}_m \) are bounded uniformly, so in particular it holds for the elements of \( \pi^{-1} (B) \); we conclude therefore that it is a compact set in \( \tilde{F}_m \), so the map \( \pi \rceil_{\tilde{F}_m} \) is proper. The proof of properness in the case of \( F_m \) is identical.

Finally, it is clear that \( F_m \cap M_{\text{free}} \subseteq \text{int} (F_m \cap M_{\text{free}}) \); for \( \tilde{F}_m \), we have by Assumption \( \text{S.2} \) that every \( K_z \) lies in \( K \) and therefore,

\[
\text{int} \left( \tilde{F}_m \right) = \tilde{K} \cdot \text{int} (F_m) = \tilde{K} \cdot F_m \supseteq \bigcup_{z \in F_m} K_z \cdot z = \tilde{F}_m,
\]

where \( \tilde{K} \) is as in Assumption \( \text{S.2} \). Here the second equality follows from \( \tilde{K} \) being compact and the following inclusion is because \( K_z \subseteq K \).

**Part 3.** Let us introduce the notation \( G'' \) for the group \( \left[ \begin{array}{cc} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{array} \right] \). As for the space \( \mathcal{P}_{d,n} \) we are going to use Proposition \( \text{6.15} \) with the space \( KP'' \), the group \( G'' \) and its closed subgroup \( \left[ \begin{array}{cc} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{array} \right] \). Notice that \( KP'' / G'' = K / K'' \) so that, by Proposition \( \text{6.16} \) it follows that \( K' \) is a spread model in \( KP'' \) w.r.t. \( G'' \). By part 2 it is clear that \( \left[ \begin{array}{cc} \tilde{F}_d & 0_{d,n-d} \\ 0_{n-d,d} & \tilde{F}_{n-d} \end{array} \right] \) is a spread model in \( G'' \) w.r.t. \( \left[ \begin{array}{cc} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{array} \right] \). As a result, Proposition \( \text{6.15} \) implies that \( K' \left[ \begin{array}{cc} \tilde{F}_d & 0_{d,n-d} \\ 0_{n-d,d} & \tilde{F}_{n-d} \end{array} \right] \) is indeed a spread model in \( KP'' \) for the group \( \left[ \begin{array}{cc} \text{SL}_d (\mathbb{Z}) & 0_{d,n-d} \\ 0_{n-d,d} & \text{SL}_{n-d} (\mathbb{Z}) \end{array} \right] \).

**Part 4 and 5.** These parts are proved in a similar fashion as part 3.

\[\square\]

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