On separable Pauli equations

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Abstract

We classify (1+3)-dimensional Pauli equations for a spin-$\frac{1}{2}$ particle interacting with the electro-magnetic field, that are solvable by the method of separation of variables. As a result, we obtain the eleven classes of vector-potentials of the electro-magnetic field $A(t, \vec{x}) = (A_0(t, \vec{x}), \vec{A}(t, \vec{x}))$ providing separability of the corresponding Pauli equations. It is established, in particular, that the necessary condition for the Pauli equation to be separable into second-order matrix ordinary differential equations is its equivalence to the system of two uncoupled Schrödinger equations. In addition, the magnetic field has to be independent of spatial variables. We prove that coordinate systems and the vector-potentials of the electro-magnetic field providing the separability of the corresponding Pauli equations coincide with those for the Schrödinger equations. Furthermore, an efficient algorithm for constructing all coordinate systems providing the separability of Pauli equation with a fixed vector-potential of the electro-magnetic field is developed. Finally, we describe all vector-potentials $A(t, \vec{x})$ that (a) provide the separability of Pauli equation, (b) satisfy vacuum Maxwell equations without currents, and (c) describe non-zero magnetic field.

1 Introduction

A quantum mechanical system consisting of a spin-$\frac{1}{2}$ charged particle, moving with momentum $\vec{p}$ in a time-dependent electro-magnetic field with the four-component vector-potential $(A_0, \vec{A})$, is described in a non-relativistic approximation by the Pauli equation (see, e.g., [1])

$$
\left( p_0 - eA_0(t, \vec{x}) - \left( \vec{p} - e\vec{A}(t, \vec{x}) \right)^2 + e\vec{\sigma}\vec{H} \right) \psi(t, \vec{x}) = 0.
$$

Here $\psi(t, \vec{x})$ is the two-component wave function in three space dimensions $\vec{x} = (x_1, x_2, x_3)$, $\vec{H} = \text{rot} \, \vec{A}$ is the magnetic field, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector consisting of three Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

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Hereafter we use the notations
\[ p_0 = i \frac{\partial}{\partial t}, \quad \vec{p} = -i \vec{\nabla}, \quad a = 1, 2, 3, \]
and summation over the repeated Latin indices from 1 to 3 is implied.

As the Pauli equation has variable coefficients, we cannot apply the standard Fourier transformation. The only regular way for solving (1) is the classical method of separation of variables in curvilinear coordinate systems. In this respect, a natural question arises, which equations of the form (1) are separable, namely, which potentials \( A_0, \vec{A} \) allow for separability of the Pauli equation in some curvilinear coordinate system? One of the principal objectives of the present article is to provide an efficient algorithm for answering these kinds of questions for systems of partial differential equations. It is essentially based on the results of the paper [2], where the problem of separation of variables in the Schrödinger equation has been analyzed in detail. As the Pauli equation (1) differs from the Schrödinger equation by the term \( e \vec{\sigma} \vec{H} \) only, it is natural to attempt modifying the technique employed in [2] in order to make it applicable to system of partial differential equations (PDEs) (1).

Integrable Hamiltonian systems with velocity–dependent potentials have been studied for the case \( n = 2 \), i.e., in a Euclidean plane by Winternitz with co-authors [3, 4]. Recently Benenti with co-authors [5] studied the problem of separation of variables in the stationary Hamilton-Jacobi equation with vector-potential from a geometrical point of view.

The problem of separation of variables for linear systems of first-order partial differential equations such as the Dirac equation has been repeatedly addressed by Shapovalov and Bagrov with co-authors [6, 7] and by Kalnins and Miller with co-authors [8, 9, 10]. They developed a symmetry approach to the separation of variables in the Dirac equation where separability is characterized by the existence of a complete set of first-order matrix symmetry operators.

Symmetry and supersymmetry properties of the Pauli equations are studied in [11, 12, 13]. Let us also mention the paper [14], where physical aspects of the problem of separation of variables in some (1+3)-dimensional Pauli equations with time dependent potentials are studied, and the monograph [7], where some classes of exact solutions of the Pauli equation are presented.

With all the variety of approaches to separation of variables in PDEs one can notice the three generic principles, namely,

a) Representation of a solution to be found in a separated (factorized) form via several functions of one variable.

b) Requirement that the above mentioned functions of one variable should satisfy some ordinary differential equations.

c) Dependence of so found solution on several arbitrary (continuous or discrete) parameters, called spectral parameters or separation constants.

By a proper formalizing of the above features we have formulated in [2] an algorithm for variable separation in the Schrödinger equation with vector-potential. Below we generalize this algorithm for the case of system of PDEs (1).
To have a right to talk about description of all potentials and all coordinate systems enabling us to separate the Pauli equation, one needs to provide a rigorous definition of separation of variables. The definition we intend to use is based on ideas contained in the paper by Koornwinder [15].

Let us introduce a new coordinate system \( t, \omega_a = \omega_a(t, \vec{x}) \), \( a = 1, 2, 3 \), where \( \omega_a \) are real-valued functions, functionally independent with respect to the spatial variables \( x_1, x_2, x_3 \), i.e.:

\[
\det \left\| \frac{\partial \omega_a}{\partial x_b} \right\|_{a,b=1}^3 \neq 0. \tag{4}
\]

For a solution to be found we adopt the following separation Ansatz:

\[
\psi(t, \vec{x}) = Q(t, \vec{x}) \varphi_0(t) \prod_{a=1}^3 \varphi_a \left( \omega_a(t, \vec{x}), \lambda \right) \chi, \tag{5}
\]

where \( Q, \varphi_\mu, (\mu = 0, 1, 2, 3) \) are non-singular \( 2 \times 2 \)–matrix functions of the indicated variables and \( \chi \) is an arbitrary two-component constant column. What is more, the condition of commutativity of the matrices \( \varphi_\mu \) is imposed, namely,

\[
[\varphi_\mu, \varphi_\nu] = \varphi_\mu \varphi_\nu - \varphi_\nu \varphi_\mu = 0, \quad \mu, \nu = 0, 1, 2, 3. \tag{6}
\]

Note that the restriction (6) is an extra requirement, which narrows the class of separable Pauli equations. However, without this condition an efficient handling of the Ansätze of the form (6) seems to be impossible. At least, in all papers devoted to variable separation in a systems of PDEs the condition of commutativity is imposed (explicitly or implicitly).

**Definition 1** We say that the Pauli equation (1) admits separation of variables in a coordinate system \( t, \omega_a = \omega_a(t, \vec{x}) \), \( a = 1, 2, 3 \), if there are non–singular \( 2 \times 2 \)–matrix function \( Q(t, \vec{x}) \) and four matrix ordinary differential equations

\[
i \dot{\varphi}_0 = -(P_{00}(t) + P_{0b}(t)\lambda_b) \varphi_0,
\]

\[
\ddot{\varphi}_a = (P_{a0}(\omega_a) + P_{ab}(\omega_a)\lambda_b) \varphi_a, \quad a = 1, 2, 3, \tag{7}
\]

jointly depending in an analytic way on three independent complex parameters \( \lambda_1, \lambda_2, \lambda_3 \) (separation constants), such that, for each triplet \( (\lambda_1, \lambda_2, \lambda_3) \) and for each set of solutions \( \varphi_0(t), \varphi_1(\omega_1), \varphi_2(\omega_2), \varphi_3(\omega_3) \) of (7), function (6) under condition (6) is a solution of (1).

In the above formulas \( P_{\mu\nu}, \mu, \nu = 0, 1, 2, 3 \) are some complex \( 2 \times 2 \)–matrix functions of the indicated variables.

**Definition 2** Three complex parameters \( \lambda_1, \lambda_2, \lambda_3 \) in (6) are called independent, if the equality

\[
\text{rank } \left\| P_{\mu a} \right\|_{\mu=0}^3 = 6. \tag{8}
\]

holds, whenever \( \varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)\varphi_3(\omega_3) \neq 0 \).
Condition (8) secures essential dependence of a solution with separated variables on the separation constants $\vec{\lambda}$.

Note, that putting $Q = I, \omega_a = x_a, a = 1, 2, 3$ in (8) yields the standard separation of variables in the Cartesian coordinate system. Next, choosing the spherical coordinates as $\omega_1, \omega_2, \omega_3$ we arrive at the variable separation in the spherical coordinate system and so on. The principal task is describing all possible forms of the functions $Q, \omega_a, a = 1, 2, 3$, that provide separability of the Pauli equation in the sense of the definition given above. Solution of this problem, in its turn, requires describing the functions $A_0, \ldots, A_3$ that enable variable separation in the Pauli equation in the corresponding coordinate system. More precisely, we will need to solve the two mutually connected principal problems:

– to describe all cases of coefficients, for which the corresponding Pauli equation (1) is separable (in the sense of definition 1) in at least one coordinate system;

– to construct all coordinate systems that allow for separation of variables (in the sense of definition 1) in the Pauli equation (1) with some fixed vector-potential $(A_0, \vec{A})$.

Note, that formulas (5)–(8) form the input data of the method. We can change these conditions and thereby modify the definition of separation of variables. For instance, we can change the order of the reduced equations (7) or the number of essential parameters $\lambda_a$ (a more detailed analysis of this problem for the Schrödinger equation can be found in [14]). So, our claim of obtaining the complete description of vector-potentials and coordinate systems providing separation of variables in (1) makes sense only within the framework of definition 1. If one uses a more general definition, it might be possible to construct new coordinate systems and vector–potentials providing separability of equation (1). But all solutions of the Pauli equation with separated variables known to us fit into the above suggested scheme.

Transformations

$$\lambda_a \rightarrow \lambda'_a = \Lambda_a(\lambda_1, \lambda_2, \lambda_3), \quad a = 1, 2, 3$$

under condition

$$\det \left| \frac{\partial \Lambda_a}{\partial \lambda_b} \right|_{a,b=1}^3 \neq 0.$$  

(10)

preserve the form of relations (5)–(8). So we can regard the corresponding spectral parameters $\vec{\lambda}$ and $\vec{\lambda}'$ as equivalent ones. Within the framework of this equivalence relation we can choose $\vec{\lambda}$ in such a way that all matrices $P_{\mu\nu}, \mu, \nu = 0, 1, 2, 3$ in reduced equations (11) are Hermitian ones and parameters $\lambda_1, \lambda_2, \lambda_3$ are real numbers.

Next, we introduce an equivalence relation on the set of all vector–potentials $A_0(t, \vec{x}), \vec{A}(t, \vec{x})$ providing separability of equation (11), on the sets of solutions with separated variables and corresponding coordinate systems.

Definition 3 We say that two vector–potentials $A(t, \vec{x})$ and $A'(t, \vec{x})$ are equivalent if they are transformed one into another by the gauge transformation

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} f, \quad A_0 \rightarrow A'_0 = A_0 - \frac{\partial f}{\partial t},$$

(11)

where $f = f(t, \vec{x})$ is an arbitrary smooth function.
For the Pauli equations to be invariant with respect to the above transformation, the wave function $\psi(t, \vec{x})$ is to be transformed according to the rule

$$\psi \rightarrow \psi' = \psi \exp(ief)$$  \hspace{1cm} (12)

Indeed, if the transformations (11)–(12) in the Pauli equation (1) are performed one after another, we obtain the initial equation, provided we replace the functions $\vec{A}, A_0, \psi$ with $\vec{A}', A_0', \psi'$.

Note that the system of PDEs (1) admits a wider equivalence group from the point of view of the standard theory of partial differential equations (Shapovalov and Sukhomlin [17]). However, this group cannot be regarded as an equivalence group within the context of quantum mechanics, since allowed transformations of the wave function must preserve the probability density $\psi^*\psi$. And it is straightforward to check that the wider Shapovalov and Sukhomlin equivalence group breaks this rule, because it, generally speaking, does not preserve $\psi^*\psi$. By this very reason, we restrict our considerations to the gauge transformations only.

**Definition 4** Two solutions of the Pauli equation with separated variables are called equivalent if they can be transformed one into another by group transformations from the Lie transformation group admitted by Pauli equation (1). Moreover, solutions of the Pauli equation with separated variables having equivalent (in the sense of equivalence relation (9)–(10)) spectral parameters $\vec{\lambda}$ are equivalent.

**Definition 5** Two coordinate systems $t, \omega_1, \omega_2, \omega_3$ and $t', \omega_1', \omega_2', \omega_3'$ are called equivalent if they give equivalent solutions with separated variables. In particular, two coordinate systems are equivalent if the corresponding Ansätze (5) are transformed one into another by reversible transformations of the form

$$t \rightarrow t' = f_0(t), \hspace{0.5cm} \omega_a \rightarrow \omega'_a = f_a(\omega_a), \hspace{0.5cm} a = 1, 2, 3,$$

$$Q \rightarrow Q' = Q l_0(t) l_1(\omega_1) l_2(\omega_2) l_3(\omega_3),$$  \hspace{1cm} (13) \hspace{1cm} (14)

where $f_0, \ldots, f_3$ are some smooth functions and $l_0, \ldots, l_3$ are some smooth $2 \times 2$–matrix functions of the indicated variables.

Indeed, transformations (13) and (14) preserve the form of Ansätze (3). So after completing the procedure of separation of variables in these coordinate systems we obtain the same solutions with separated variables.

These equivalence relations reflect the freedom in choice of the functions $Q, \omega_1, \omega_2, \omega_3$ and separation constants $\lambda_1, \lambda_2, \lambda_3$ preserving the form of the conditions (3)–(8). They split the set of all possible vector–potentials, providing separability of equation (1), and sets of solutions with separated variables and corresponding coordinate systems into equivalence classes. In a sequel, when presenting the corresponding lists we will give only one representative for each equivalence class.
2 Classification of separable Pauli equations (1)

In this section we obtain an exhaustive classification of the Pauli equations solvable within the framework of the approach described in the Introduction. Furthermore, we describe curvilinear coordinate systems enabling separation of variables in (1).

Using the equalities (7) and (6) we get

\[ [P_{\mu 0} + P_{\mu a} \lambda_a, P_{\nu 0} + P_{\nu a} \lambda_a] = 0. \]

Splitting the expression with respect to \( \lambda_a \) yields

\[ [P_{\mu \alpha}, P_{\nu \beta}] + [P_{\mu \beta}, P_{\nu \alpha}] = 0, \quad (15) \]

where \( \mu, \nu, \alpha, \beta = 0, 1, 2, 3 \) and henceforth summation over repeated Greek indices is not used. Choosing \( \alpha = \beta \) we have

\[ [P_{\mu \alpha}, P_{\nu \alpha}] = 0. \]

Taking into account this equality and the fact that any Hermitian \((2 \times 2)\)–matrix can be represented as a linear combination of the unit and Pauli matrices (2), we get the following form of \( P_{\mu \alpha} \):

\[ P_{\mu \alpha} = F_{\mu \alpha}(\omega_\mu I + G_{\mu \alpha}(\omega_\mu) \vec{s}_\alpha \vec{\sigma}), \quad (16) \]

where \( F_{\mu \alpha}, G_{\mu \alpha} \) are some smooth scalar functions of the indicated variables, \( \omega_0 = t \) and \( \vec{s}_\alpha \) is a constant three-component vector. Substitution of expression (16) into (15) yields

\[ (G_{\mu \alpha} G_{\nu \beta} - G_{\mu \beta} G_{\nu \alpha})[\vec{s}_\alpha \vec{\sigma}, \vec{s}_\beta \vec{\sigma}] = 0. \]

From this equality we conclude that there are two distinct cases: either \( \vec{s}_\alpha \sim \vec{s}_\beta \) or \( G_{\mu \alpha} \sim G_{\mu \beta} \). In view of this fact we get the two possible forms for the equations (1):

\[ i \dot{\varphi}_0 = -\left( F_{00}(t) + F_{0b}(t) \lambda_b + (G_{00}(t) + G_{0b}(t) \lambda_b) \vec{s}_b \vec{\sigma} \right) \varphi_0, \quad (17) \]

\[ \ddot{\varphi}_a = (F_{ab}(\omega_a) + F_{b0}(\omega_a) \lambda_b + (G_{a0}(\omega_a) + G_{ab}(\omega_a) \lambda_b) \vec{s}_a \vec{\sigma}) \varphi_a, \]

and

\[ i \dot{\varphi}_0 = -\left( F_{00}(t) + F_{0b}(t) \lambda_b + G_0(t)(\vec{s}_b \lambda_b) \vec{\sigma} \right) \varphi_0, \quad (18) \]

\[ \ddot{\varphi}_a = (F_{ab}(\omega_a) + F_{b0}(\omega_a) \lambda_b + G_a(\omega_a)(\vec{s}_b \lambda_b) \vec{\sigma}) \varphi_a, \]

with \( a = 1, 2, 3. \)

Definition 1 is quite algorithmic in the sense that it contains a regular algorithm of variable separation in Pauli equation (1). Formulas (5), (17)–(18) form the input data of the method. The principal steps of the procedure of variable separation in Pauli equation (1) are as follows.

1. We insert the Ansatz (5) into the Pauli equation and express the derivatives \( \dot{\varphi}_0, \dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3 \) in terms of functions \( \varphi_0, \varphi_1, \varphi_2, \varphi_3 \), using equations (17)–(18).
2. We regard \( \varphi_0, \varphi_1, \varphi_2, \varphi_3, \lambda_1, \lambda_2, \lambda_3 \) as new independent variables \( y_1, \ldots, y_7 \). As the functions \( Q, \omega_1, \omega_2, \omega_3, A_0, A_1, A_2, A_3 \) are independent on the variables \( y_1, \ldots, y_7 \), we can demand that the obtained equality is transformed into identity under arbitrary \( y_1, \ldots, y_7 \). In other words, we should split the equality with respect to these variables under condition of commutativity \( (\|) \). After splitting we get an overdetermined system of nonlinear partial differential equations for unknown functions \( Q, \omega_1, \omega_2, \omega_3, A_0, A_1, A_2, A_3 \).

3. After solving the above system we get an exhaustive description of vector–potentials \( A(t, \vec{x}) \) providing separability of the Pauli equation and corresponding coordinate systems.

Having performed the first two steps of the above algorithm we obtain the system of nonlinear matrix PDEs:

\[
\begin{align*}
(i) & \quad \frac{\partial \omega_b}{\partial x_a} \frac{\partial \omega_c}{\partial x_a} = 0, \quad b \neq c, \quad b, c = 1, 2, 3; \\
(ii) & \quad \sum_{a=1}^{3} F_{ab}(\omega_a) \frac{\partial \omega_a}{\partial x_c} \frac{\partial \omega_a}{\partial x_c} = F_{0b}(t), \quad b = 1, 2, 3; \\
(iii) & \quad \sum_{a=1}^{3} G_{a\mu}(\omega_a) \frac{\partial \omega_a}{\partial x_c} \frac{\partial \omega_a}{\partial x_c} = G_{0\mu}(t), \quad \mu = 0, 1, 2, 3; \\
(iv) & \quad 2 \left( \frac{\partial Q}{\partial x_b} - i e Q A_b \right) \frac{\partial \omega_a}{\partial x_b} + Q \left( i \frac{\partial \omega_a}{\partial t} + \Delta \omega_a \right) = 0, \quad a = 1, 2, 3; \\
(v) & \quad Q \sum_{a=1}^{3} F_{a0}(\omega_a) \frac{\partial \omega_a}{\partial x_b} \frac{\partial \omega_a}{\partial x_b} + i \frac{\partial Q}{\partial t} + \Delta Q - 2 i e A_b \frac{\partial Q}{\partial x_b} + \\
& \quad \quad \quad + \left( -F_{00}(t) - i e \frac{\partial A_b}{\partial x_b} - e A_0 - e^2 A_b A_b + e \vec{\sigma} \vec{H} \right) Q = 0.
\end{align*}
\]

Thus the problem of variable separation in the Pauli equation reduces to integrating of a system of nonlinear PDEs for eight unknown functions \( A_0, A_1, A_2, A_3, Q, \omega_1, \omega_2, \omega_3 \) of four variables \( t, \vec{x} \). What is more, some coefficients are arbitrary matrix functions which should be determined in the process of integrating of the system of PDEs \((i) - (v))\). We succeeded in constructing the general solution of the latter which yields, in particular, all possible vector-potentials \( A(t, \vec{x}) = (A_0(t, \vec{x}), \ldots, A_3(t, \vec{x})) \) such that Pauli equation \( (1) \) is solvable by the method of separation of variables.

In view of \( (\|) \) we can always choose from each set of the equations \((ii) - (iii)\) and \((ii) - (iiib)\) three such equations that the matrix of coefficients of \( \omega_{ax}, \omega_{ax} \) (\( a=1,2,3 \)) is non-singular. It is called the Stäckel matrix \( [18] \). The system consisting of these three
equations and of the equations (i) was integrated in [2]. Its general solution \( \vec{x} = \vec{x}(t, \vec{x}) \) is given implicitly within the equivalence relation (13) by the following formulas:

\[
\vec{x} = \mathcal{O}(t) \mathcal{L}(t) (\vec{x}(\vec{\omega}) + \vec{v}(t)),
\]

(19)

Here \( \mathcal{O}(t) \) is a time-dependent \( 3 \times 3 \) orthogonal matrix with Euler angles \( \alpha(t), \beta(t), \gamma(t) \):

\[
\mathcal{O}(t) = \begin{pmatrix}
\cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \gamma & \sin \alpha \sin \gamma \\
\sin \alpha \cos \beta + \cos \alpha \sin \beta \cos \gamma & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \cos \gamma & -\cos \alpha \sin \gamma \\
\sin \beta \sin \gamma & \cos \beta \sin \gamma & \cos \gamma
\end{pmatrix};
\]

(20)

\( \vec{v}(t) \) stands for the vector-column whose entries \( v_1(t), v_2(t), v_3(t) \) are arbitrary smooth functions of \( t \); \( \vec{x} = \vec{x}(\vec{\omega}) \) is given by one of the eleven formulas:

1. Cartesian coordinate system,
   \( z_1 = \omega_1, \quad z_2 = \omega_2, \quad z_3 = \omega_3, \quad \omega_1, \omega_2, \omega_3 \in \mathbb{R}. \)
2. Cylindrical coordinate system,
   \( z_1 = \cos \omega_1 \cos \omega_2, \quad z_2 = \cos \omega_1 \sin \omega_2, \quad z_3 = \omega_3, \quad 0 \leq \omega_2 < 2\pi, \quad \omega_1, \omega_3 \in \mathbb{R}. \)
3. Parabolic cylindrical coordinate system,
   \( z_1 = (\omega_1^2 - \omega_2^2)/2, \quad z_2 = \omega_1 \omega_2, \quad z_3 = \omega_3, \quad \omega_1 > 0, \quad \omega_2, \omega_3 \in \mathbb{R}. \)
4. Elliptic cylindrical coordinate system,
   \( z_1 = \cosh \omega_1 \cos \omega_2, \quad z_2 = \sinh \omega_1 \sin \omega_2, \quad z_3 = \omega_3, \quad \omega_1 > 0, \quad -\pi < \omega_2 \leq \pi, \quad \omega_3 \in \mathbb{R}, \quad a > 0. \)
5. Spherical coordinate system,
   \( z_1 = \omega_1^{-1} \text{sech} \omega_2 \cos \omega_3, \quad z_2 = \omega_1^{-1} \text{sech} \omega_2 \sin \omega_3, \quad z_3 = \omega_1^{-1} \text{tanh} \omega_2, \quad \omega_1 > 0, \quad \omega_2 \in \mathbb{R}, \quad 0 \leq \omega_3 < 2\pi. \)
6. Prolate spheroidal coordinate system,
   \( z_1 = a \csch \omega_1 \text{sech} \omega_2 \cos \omega_3, \quad a > 0, \quad z_2 = a \csch \omega_1 \text{sech} \omega_2 \sin \omega_3, \quad z_3 = a \coth \omega_1 \tanh \omega_2, \quad \omega_1 > 0, \quad \omega_2 \in \mathbb{R}, \quad 0 \leq \omega_3 < 2\pi. \)
7. Oblate spheroidal coordinate system,
   \( z_1 = a \csc \omega_1 \text{sech} \omega_2 \cos \omega_3, \quad a > 0, \quad z_2 = a \csc \omega_1 \text{sech} \omega_2 \sin \omega_3, \quad z_3 = a \cot \omega_1 \tanh \omega_2, \quad 0 < \omega_1 < \pi/2, \quad \omega_2 \in \mathbb{R}, \quad 0 \leq \omega_3 < 2\pi. \)
8. Parabolic coordinate system,
\[ z_1 = e^{\omega_1 + \omega_2} \cos \omega_3, \quad z_2 = e^{\omega_1 + \omega_2} \sin \omega_3, \]
\[ z_3 = (e^{2\omega_1} - e^{2\omega_2})/2, \]
\[ \omega_1, \omega_2 \in \mathbb{R}, \quad 0 \leq \omega_3 \leq 2\pi. \]

9. Paraboloidal coordinate system,
\[ z_1 = 2a \cosh \omega_1 \cos \omega_2 \sinh \omega_3, \quad a > 0, \]
\[ z_2 = 2a \sinh \omega_1 \sin \omega_2 \cosh \omega_3, \]
\[ z_3 = a(\cosh 2\omega_1 + \cos 2\omega_2 - \cosh 2\omega_3)/2, \]
\[ \omega_1, \omega_2 \in \mathbb{R}, \quad 0 \leq \omega_2 < \pi. \]

10. Ellipsoidal coordinate system,
\[ z_1 = a \frac{1}{\text{sn}(\omega_1, k)} \text{dn}(\omega_2, k') \text{sn}(\omega_3, k), \quad a > 0, \]
\[ z_2 = a \frac{\text{dn}(\omega_1, k)}{\text{sn}(\omega_1, k)} \text{cn}(\omega_2, k') \text{cn}(\omega_3, k), \]
\[ z_3 = a \frac{\text{cn}(\omega_1, k)}{\text{sn}(\omega_1, k)} \text{sn}(\omega_2, k') \text{dn}(\omega_3, k), \]
\[ 0 < \omega_1 < K, \quad -K' \leq \omega_2 \leq K', \quad 0 \leq \omega_3 \leq 4K. \]

11. Conical coordinate system,
\[ z_1 = \omega_1^{-1} \text{dn}(\omega_2, k') \text{sn}(\omega_3, k), \]
\[ z_2 = \omega_1^{-1} \text{cn}(\omega_2, k') \text{cn}(\omega_3, k), \]
\[ z_3 = \omega_1^{-1} \text{sn}(\omega_2, k') \text{dn}(\omega_3, k), \]
\[ \omega_1 > 0, \quad -K' \leq \omega_2 \leq K', \quad 0 \leq \omega_3 \leq 4K; \]

and \( \mathcal{L}(t) \) is a 3 \times 3 diagonal matrix
\[
\mathcal{L}(t) = \begin{pmatrix}
  l_1(t) & 0 & 0 \\
  0 & l_2(t) & 0 \\
  0 & 0 & l_3(t)
\end{pmatrix},
\]
where \( l_1(t), l_2(t), l_3(t) \) are arbitrary non-zero smooth functions that satisfy the following conditions

- \( l_1(t) = l_2(t) \) for the partially split coordinate systems (cases 2–4 from (21)),
- \( l_1(t) = l_2(t) = l_3(t) \) for non-split coordinate systems (cases 5–11 from (21)).

Here we use the usual notations for the trigonometric, hyperbolic and Jacobi elliptic functions, number \( k \) \((0 < k < 1)\) being the modulus of the latter and \( k' = (1 - k^2)^{1/2}. \)

From a geometric point of view the right-hand side of formula \((19)\) is a result of application to vector \( \vec{z}(\vec{\omega}) \) of the following time-dependent transformations performed one after another:

1. translations \( \vec{z} \rightarrow \vec{z}' = \vec{z} + \vec{v}(t), \)
2. dilatations $\vec{z} \rightarrow \vec{z}' = \mathcal{L}(t)\vec{z}$,

3. three-dimensional rotations $\vec{z} \rightarrow \vec{z}' = \mathcal{O}(t)\vec{z}$ with Euler angles $\alpha(t), \beta(t), \gamma(t)$.

Together with the rotations the following vector $\vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3)$ is considered \([19, \S 35]\)

\[
\begin{align*}
\Omega_1(t) &= \dot{\gamma}(t) \cos \alpha(t) + \dot{\beta}(t) \sin \alpha(t) \sin \gamma(t), \\
\Omega_2(t) &= \dot{\gamma}(t) \sin \alpha(t) - \dot{\beta}(t) \cos \alpha(t) \sin \gamma(t), \\
\Omega_3(t) &= \dot{\alpha}(t) + \dot{\beta}(t) \cos \gamma(t),
\end{align*}
\]

that is directed along momentary axis of rotation and called *angular velocity vector*.

Note that we have chosen the coordinate systems $\omega_1, \omega_2, \omega_3$ by means of the equivalence relation \([13]\) in such a way that the relations

\[\Delta \omega_a = 0, \quad a = 1, 2, 3\]  \(24\)

hold for all the cases 1–11 in \([21]\).

After integration of system (i)–(iii) it is not difficult to integrate the remaining equations (iv) and (v) from the system under study, since they can be regarded as algebraic equations for the functions $A_a(t, \vec{x}), (a = 1, 2, 3)$ and $A_0(t, \vec{x})$, correspondingly.

Multiplying equation (iv) from the right by $Q^{-1}$ we obtain for each component of matrices $\frac{\partial Q}{\partial x_b}Q^{-1}, b = 1, 2, 3$ the systems of three linear algebraic equations. The determinants of the systems do not vanish according to \([1]\). So, they have the unique solution

\[\frac{\partial Q}{\partial x_b}Q^{-1} = f_b(t, \vec{x})I, \quad b = 1, 2, 3, \]  \(25\)

where $f_b(t, \vec{x})$ are scalar smooth functions and $I$ is unit $2 \times 2$–matrix. From the compatibility conditions

\[\frac{\partial f_a}{\partial x_b} = \frac{\partial f_b}{\partial x_a}, \quad a, b = 1, 2, 3\]

of the above system of PDEs we obtain that there exists such function $g(t, \vec{x})$ that the equalities $f_a = \partial g/\partial x_a, a = 1, 2, 3$ hold. So \((25)\) takes the form

\[\frac{\partial Q}{\partial x_b} = \frac{\partial g}{\partial x_b}Q, \quad b = 1, 2, 3.\]

The general solution of this system of matrix PDEs is

\[Q = \mathcal{U}(t) \exp g(t, \vec{x}), \]  \(26\)

where $\mathcal{U}(t)$ is arbitrary $2 \times 2$–matrix function of $t$.

Let us represent the complex-valued function $g(t, \vec{x})$ in \((26)\) as $g = S_1 + iS$, where $S_1, S$ are real-valued functions. Now, if we take into account that the components of the vector potential $A(t, \vec{x})$ and functions $\omega_1, \omega_2, \omega_3$ are real-valued functions, then after
inserting (26) into (iv) with the use of (24) we can split the obtained equations into real and imaginary parts:

\[
\frac{\partial S_1}{\partial x_b} \frac{\partial \omega_a}{\partial x_b} = 0, \quad a = 1, 2, 3; \quad (27)
\]

\[
2 \left( \frac{\partial S}{\partial x_b} - eA_b \right) \frac{\partial \omega_a}{\partial x_b} + \frac{\partial \omega_a}{\partial t} = 0, \quad a = 1, 2, 3. \quad (28)
\]

Taking into account the equality (4), we obtain from (27) the equalities

\[
\frac{\partial S_1}{\partial x_b} = 0, \quad b = 1, 2, 3.
\]

It gives that

\[
S_1 = S_1(t).
\]

Let us denote \( \vec{A} = e\vec{A} - \vec{\nabla}S \). Then the system (28) takes the form of three linear algebraic equations for functions \( A_1, A_2, A_3 \):

\[
\frac{\partial \omega_a}{\partial t} = 2e \frac{\partial \omega_a}{\partial x_b} A_b, \quad a = 1, 2, 3.
\]

The determinant of this system does not vanish due to (4). Consequently, it has a unique solution. Making in this solution the hodographic transformation

\[
t = t, \quad x_a = u_a(t, \omega_1, \omega_2, \omega_3), \quad a = 1, 2, 3, \quad (29)
\]

we get the following expressions for \( A_1, A_2, A_3 \):

\[
\vec{A} = -\frac{1}{2e} \frac{\partial \vec{u}(t, \vec{\omega})}{\partial t}.
\]

After substitution into this formula expression for \( \vec{u}(t, \vec{\omega}) \) (19), we return to variables \( t, x_1, x_2, x_3 \) and thus obtain the following system:

\[
2(-e\vec{A}(t, \vec{x}) + \vec{\nabla}S) = \mathcal{M}(t)\vec{x} + \mathcal{O}(t)\mathcal{L}(t)\vec{v}. \quad (30)
\]

Here we use the designation

\[
\mathcal{M}(t) = \hat{\mathcal{O}}(t)\mathcal{O}^{-1}(t) + \mathcal{O}(t)\hat{\mathcal{L}}(t)\mathcal{L}^{-1}(t)\mathcal{O}^{-1}(t), \quad (31)
\]

where \( \mathcal{O}(t), \mathcal{L}(t) \) are variable \( 3 \times 3 \) matrices defined by formulas (21) and (22), correspondingly, \( \vec{v} = (v_1(t), v_2(t), v_3(t))^T \). Note that \( \hat{\mathcal{O}}\mathcal{O}^{-1} \) is antisymmetric and \( \mathcal{O}\hat{\mathcal{L}}\mathcal{L}^{-1}\mathcal{O}^{-1} \) is symmetric part of matrix \( \mathcal{M} \).

The direct calculation shows that equations (v) and (30) are invariant under gauge transformations (11). Thus the function \( S \) is transformed by the rule

\[
S \rightarrow S' = S + ef, \quad (32)
\]

which follows from (12). In other words, if the transformations (11), (32) in equations (v) and (30) are performed one after another, we obtain the initial equations where functions \( \vec{A}, A_0, S \) should be replaced with functions \( \vec{A}', A_0', S' \). So, if the Pauli equation (1) with potential \( \vec{A}, A_0 \) admits separation of variables in some coordinate system, then the Pauli equation with potential \( \vec{A}', A_0' \) admits separation of variables in the same coordinate system (the multiplier \( Q(26) \) is changed only). Therefore, it is worthwhile to fix some gauge
and to work only with representatives of the equivalence classes of potentials $A(t, \vec{x})$ (in the sense of equivalence relation (11)).

We choose the gauge in a way that the equality

$$2\nabla S = \mathcal{O}(t) \dot{\mathcal{L}}(t) \mathcal{L}^{-1}(t) \mathcal{O}^{-1}(t) \vec{x} + \mathcal{O}(t) \mathcal{L}(t) \dot{v}.$$  

(33)

holds. After integration of this system of PDEs we obtain the expression for $S$:

$$S = \frac{1}{4} \sum_{a=1}^{3} \left( \frac{i_a}{x_a^2} + 2 \hat{v}_a x_a' \right),$$  

(34)

where we use the notations

$$\vec{x}' = \mathcal{O}^{-1} \vec{x}.$$  

(35)

Next, we obtain from equation (30) the explicit form for space-like components of vector-potential of electromagnetic field

$$\vec{A}(t, \vec{x}) = -\frac{1}{2e} \dot{\mathcal{O}} \mathcal{O}^{-1} \vec{x},$$  

(36)

where the explicit form of matrix $\dot{\mathcal{O}} \mathcal{O}^{-1}$ is given by the formula

$$\dot{\mathcal{O}} \mathcal{O}^{-1} = \begin{pmatrix} 0 & -(\dot{\alpha} + \dot{\beta} \cos \gamma) & \dot{\gamma} \sin \alpha - \dot{\beta} \cos \alpha \sin \gamma \\ \dot{\alpha} + \dot{\beta} \cos \gamma & 0 & -(\dot{\gamma} \cos \alpha + \dot{\beta} \sin \alpha \sin \gamma) \\ -(\dot{\gamma} \sin \alpha - \dot{\beta} \cos \alpha \sin \gamma) & \dot{\gamma} \cos \alpha + \dot{\beta} \sin \alpha \sin \gamma & 0 \end{pmatrix}.$$  

(37)

where $\alpha, \beta, \gamma$ are arbitrary functions of $t$.

Thus formula (36) means that the space-like components of electromagnetic field $A(t, \vec{x})$ are linear with respect of spatial variables. So the magnetic field $\vec{H} = \text{rot} \ \vec{A}$ should be homogeneous, i.e., independent of spatial variables $\vec{x}$. From formulas (33), (37) we can obtain its explicit form

$$eH_1 = -\dot{\gamma}(t) \cos \alpha(t) - \dot{\beta}(t) \sin \alpha(t) \sin \gamma(t),$$

$$eH_2 = -\dot{\gamma}(t) \sin \alpha(t) + \dot{\beta}(t) \cos \alpha(t) \sin \gamma(t),$$

$$eH_3 = -\dot{\alpha}(t) - \dot{\beta}(t) \cos \gamma(t).$$  

(38)

Now the space-like components of the electromagnetic field take the final form

$$\vec{A}(t, \vec{x}) = \frac{1}{2} \begin{pmatrix} 0 & -H_3(t) & H_2(t) \\ H_3(t) & 0 & -H_1(t) \\ -H_2(t) & H_1(t) & 0 \end{pmatrix} \vec{x} = \frac{1}{2} \vec{H}(t) \times \vec{x},$$  

(39)

where symbol $\times$ denotes cross product.

Within the equivalence relation (14) we can always choose the function $U(t)$ to be a solution of matrix ODE

$$iU = (-e\vec{\sigma} \vec{H}(t))U$$  

(40)

with the initial conditions $U(0) = I$. Due to the theorem of existence and the uniqueness of the solution of the Cauchy problem for the system of ODEs there is unique solution...
\( \mathcal{U}(t) \) of system (40) for each fixed configuration of magnetic field \( \vec{H}(t) \). Moreover, matrix \( \mathcal{U}(t) \) is a unitary one. Indeed, taking into account (40), we have the equality

\[
\frac{d}{dt}(\mathcal{U}^* \mathcal{U}) = \mathcal{U}^*(ie\vec{\sigma}\vec{H})\mathcal{U} + \mathcal{U}^*(-i\vec{\sigma}\vec{H})\mathcal{U} = 0,
\]

i.e., \( \mathcal{U}^* \mathcal{U} = \text{const} \). The initial conditions give \( \mathcal{U}^* \mathcal{U} = I \).

Thus we can consider the following change of variables in the Pauli equation (1)

\[
\psi = \mathcal{U}(t)\tilde{\psi}.
\] (41)

Due to the unitarity of the matrix \( \mathcal{U} \) the quantity \( \psi^*\psi \), which is regarded in quantum mechanics as the probability density, is not changed. So the change of variables (41) is the correct one. As a result, the term \( e\vec{\sigma}\vec{H} \) in Pauli equation (1) vanishes, and we obtained a system of two Schrödinger equations for the function \( \tilde{\psi} \).

Thus we proved the following assertion.

**Lemma 1** A necessary condition for the Pauli equation (1) to be separable (in the sense of definition (1)) is that it has to be equivalent (in the sense of equivalence relation (41)) to a system of two uncoupled Schrödinger equations.

Let us substitute the equality (26) into equation (v), taking into account equations (40) and \( S_1 = S_1(t) \). Splitting the equation obtained into real and imaginary parts (note that all functions \( F_{00}, F_{a0}, a = 1, 2, 3 \) are real-valued ones), we obtain the equalities

\[
\sum_{a=1}^{3} F_{a0}(\omega_a) \frac{\partial \omega_a}{\partial x_b} \frac{\partial \omega_a}{\partial x_b} - \frac{\partial S}{\partial t} - \frac{\partial S}{\partial x_b} \frac{\partial S}{\partial x_b} + 2eA_b \frac{\partial S}{\partial x_b} - F_{00}(t) - eA_0 - e^2A_bA_b = 0
\] (42)

\[
\dot{S}_1 + \Delta S - eA_b \frac{\partial A_b}{\partial x_b} = 0.
\] (43)

Inserting into equation (42) expressions for \( S \) (34) and \( A_1, A_2, A_3 \) (39), we obtain the explicit form of \( A_0 \):

\[
eA_0(t, \vec{x}) = \sum_{a=1}^{3} F_{a0}(\omega_a) \frac{\partial \omega_a}{\partial x_b} \frac{\partial \omega_a}{\partial x_b} - F_{00}(t) - e^2A_bA_b - \frac{1}{4} P.
\] (44)

Here \( A_bA_b \) follows from (39), (37):

\[
4A_bA_b = (H_2x_3 - H_3x_2)^2 + (H_3x_1 - H_1x_3)^2 + (H_2x_1 - H_1x_2)^2,
\] (45)

where \( H_1, H_2, H_3 \) are components of magnetic field (38); function \( P \) has the form

\[
P = \sum_{a=1}^{3} \left( \frac{i}{l_a} x'^2_a + 2(l_a \ddot{v}_a + 2l_a \dot{v}_a)x'_a + l_a^2 \dot{v}_a^2 \right),
\] (46)

where \( x'_1, x'_2, x'_3 \) are given by formula (35) and \( l_a = l_a(t), v_a = v_a(t), a = 1, 2, 3 \), are arbitrary smooth functions, which define new coordinate system (19).
Let us emphasize that the expression for \( A_0 \) includes arbitrary functions \( F_{10}(\omega_1), F_{20}(\omega_2), F_{30}(\omega_3), F_{00}(t) \), where functions \( \omega_a = \omega_a(t, \vec{x}) \), \( a = 1, 2, 3 \), belong to one of 11 classes, whose representatives are given implicitly by the formulas (19)–(22).

Below we give explicit forms of the eikonalas \( R^{-2}_a = \frac{\partial \omega_a}{\partial x_b} \frac{\partial \omega_a}{\partial x_b} \) for each class of \( \omega_a \) (see, also [2]):

1. \( R^{-2}_i = h^{-2}_i, \quad i = 1, 2, 3; \)
2. \( R^{-2}_1 = R^{-2}_2 = h^{-2}_1 e^{-2\omega_1}, \quad R^{-2}_3 = h^{-2}_3; \)
3. \( R^{-2}_1 = R^{-2}_2 = h^{-2}_1 (\omega^2_1 + \omega^2_2)^{-1}, \quad R^{-2}_3 = h^{-2}_3; \)
4. \( R^{-2}_1 = R^{-2}_2 = h^{-2}_1 a^{-2} (\cosh^2 \omega_1 - \cos^2 \omega_2)^{-1}, \quad R^{-2}_3 = h^{-2}_3; \)
5. \( R^{-2}_1 = h^{-2}_1 \omega^4_1, \quad R^{-2}_2 = R^{-2}_3 = h^{-2}_1 \omega^2_1 \cosh^2 \omega_2; \)
6. \( R^{-2}_1 = h^{-2}_1 a^{-2} \sinh^2 \omega_1 (\sinh^{-2} \omega_1 + \cosh^{-2} \omega_2)^{-1}, \quad R^{-2}_2 = h^{-2}_1 a^{-2} \cosh^2 \omega_2 (\sinh^{-2} \omega_1 + \cosh^{-2} \omega_2)^{-1}, \quad R^{-2}_3 = h^{-2}_1 a^{-2} \sinh^2 \omega_1 \cosh^2 \omega_2; \)
7. \( R^{-2}_1 = h^{-2}_1 a^{-2} \sin^2 \omega_1 (\sin^2 \omega_1 - \cosh^{-2} \omega_2)^{-1}, \quad R^{-2}_2 = h^{-2}_1 a^{-2} \cosh^2 \omega_2 (\sin^2 \omega_1 - \cosh^{-2} \omega_2)^{-1}, \quad R^{-2}_3 = h^{-2}_1 a^{-2} \sin^2 \omega_1 \cosh^2 \omega_2; \)
8. \( R^{-2}_1 = h^{-2}_1 e^{-2\omega_1} (e^{2\omega_1} + e^{2\omega_2})^{-1}, \quad R^{-2}_2 = h^{-2}_1 e^{-2\omega_2} (e^{2\omega_1} + e^{2\omega_2})^{-1}, \quad R^{-2}_3 = h^{-2}_1 e^{-2(\omega_1 + \omega_2)}; \)
9. \( R^{-2}_1 = h^{-2}_1 a^{-2} (\cosh 2\omega_1 - \cos 2\omega_2)^{-1} (\cosh 2\omega_1 + \cosh 2\omega_3)^{-1}, \quad R^{-2}_2 = h^{-2}_1 a^{-2} (\cosh 2\omega_1 - \cos 2\omega_2)^{-1} (\cosh 2\omega_2 + \cosh 2\omega_3)^{-1}, \quad R^{-2}_3 = h^{-2}_1 a^{-2} (\cosh 2\omega_1 + \cosh 2\omega_3)^{-1} (\cosh 2\omega_2 + \cosh 2\omega_3)^{-1}; \)
10. \( R^{-2}_1 = h^{-2}_1 a^{-2} \left( \frac{\dn^2 (\omega_1, k)}{\sn^2 (\omega_1, k)} - k^2 \cn^2 (\omega_2, k') \right)^{-1} \left( \frac{\dn^2 (\omega_1, k)}{\sn^2 (\omega_1, k)} + k^2 \cn^2 (\omega_3, k) \right)^{-1}, \quad R^{-2}_2 = h^{-2}_1 a^{-2} \left( \frac{\dn^2 (\omega_1, k)}{\sn^2 (\omega_1, k)} - k^2 \cn^2 (\omega_2, k') \right)^{-1} \left( k^2 \cn^2 (\omega_2, k') + k^2 \cn^2 (\omega_3, k) \right)^{-1}, \quad R^{-2}_3 = h^{-2}_1 a^{-2} \left( \frac{\dn^2 (\omega_1, k)}{\sn^2 (\omega_1, k)} + k^2 \cn^2 (\omega_3, k) \right)^{-1} \left( k^2 \cn^2 (\omega_2, k') + k^2 \cn^2 (\omega_3, k) \right)^{-1}; \)
11. \( R^{-2}_1 = h^{-2}_1 \omega^4_1, \quad R^{-2}_2 = R^{-2}_3 = h^{-2}_1 \omega^2_1 (k^2 \cn^2 (\omega_2, k') + k^2 \cn^2 (\omega_3, k))^{-1}. \)

At last, let us find the multiplier \( Q \). Substituting the formulas (14) and (39) into equation (43) gives

\[
S_1 = -\frac{1}{2} \sum_{a=1}^{3} \frac{\dot{I}_a}{I_a},
\]

whence it follows that

\[
S_1 = -\frac{1}{2} \sum_{a=1}^{3} \ln l_a.
\]

Taking into account expression for \( S \) (14), we obtain from formula (20) the explicit form
of $Q$

$$Q = U(t) \frac{1}{\sqrt{l_1^2 l_2 l_3}} \exp \sum_{a=1}^{3} \frac{i}{4} \left( \frac{j_a}{l_a} x'^2_a + 2l_a v_a x'_a \right) ,$$

(49)

where $U(t)$ is given by the equation (40), and $x'_1, x'_2, x'_3$ are given by formula (33).

Thus we have proved the main result of the article:

**Theorem 1** Pauli equation (1) admits separation of variables (in the sense of definition 1) if and only if it is gauge equivalent to Pauli equation where

- the magnetic field $\vec{H} = \text{rot} \vec{A}$ is independent of the spatial variables,

- the space–like components $A_1, A_2, A_3$ of the vector–potential of the electromagnetic field are given by (39),

- the time–like component $A_0$ is given by formulas (44)–(47).

Comparing the components of magnetic field (38) with components of angular velocity vector (23) of rotation of coordinate system (19), we obtain the equality $e\vec{H} = -\vec{Ω}$. So, we prove the following assertion:

**Corollary 1** Let Pauli equation (1) admit separation of variables in some non-stationary coordinate system $t, \omega_a = \omega_a(t, \vec{x})$, $a = 1, 2, 3$, where functions $\omega_1(t, \vec{x}), \omega_2(t, \vec{x}), \omega_3(t, \vec{x})$ are given implicitly by formulas (19)–(22). Then angular velocity vector (23) of rotation of this coordinate system equals $-e\vec{H}$, where $\vec{H} = \text{rot} \vec{A}$ is magnetic field.

It follows from the corollary that a necessary condition for the Pauli equation (1) with non-zero magnetic field $\vec{H}$ to be separable (in the sense of our definition 1) is that the angular velocity vector (23) of rotation of the separation coordinate system (19)–(22) has to be non-zero.

Summing up we conclude that coordinate systems and vector-potentials of the electromagnetic field $A(t, \vec{x}) = (A_0(t, \vec{x}), \vec{A}(t, \vec{x}))$ providing separability of the corresponding Pauli equations coincide with those for the Schrödinger equations. Namely, we prove that the magnetic field $\vec{H} = \text{rot} \vec{A}$ has to be independent of the spatial variables. Next, we have eleven classes of potentials $A_0(t, \vec{x})$, corresponding to eleven classes of coordinate systems $t, \omega_a = \omega_a(t, \vec{x})$, $a = 1, 2, 3$, where the functions $\omega_1(t, \vec{x}), \omega_2(t, \vec{x}), \omega_3(t, \vec{x})$ are given implicitly by formulas (19)–(22). Pauli equation (1) for each class of the functions $A_0(t, \vec{x}), \vec{A}(t, \vec{x})$ defined by (39), (44) and (17) under arbitrary $F_{00}(t), F_{a0}(\omega_a)$ and fixed arbitrary functions $\alpha(t), \beta(t), \gamma(t), v_a(t), l_a(t)$, $a = 1, 2, 3$, separates in exactly one coordinate system.

The solutions with separated variables are of the form (3), where $Q$ is given by (10). The separation equations read as (17) or (18), where the functions $F_{\mu0}, \mu = 0, 1, 2, 3$, are arbitrary smooth functions defining the form of the time-like component of the vector-potential $A(t, \vec{x})$ (see, (14)). The explicit forms of other coefficients $F_{\mu a}, G_{\mu a}, G_{\mu}$ of reduced equations can be obtained by splitting relations (ii) and (iii) with respect to independent variables $\omega_1, \omega_2, \omega_3, t$ for each class of the functions $\vec{z} = \vec{z}(\vec{ω})$ given in (21). Let
us denote

\[ S = \begin{pmatrix} T_1 & T_2 & T_3 \\ S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}, \tag{50} \]

where the functions \( S_{ab}(\omega_a) \) \( (a, b = 1, 2, 3) \) are given below as entries of \( 3 \times 3 \) Stäckel matrices, whose structure is determined by the choice of the functions \( \vec{z} = \vec{z}(\vec{\omega}) \):

\[
\begin{align*}
F_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & F_2 &= \begin{pmatrix} e^{2\omega_1} & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & F_3 &= \begin{pmatrix} \omega_1^{-4} & -\omega_1^{-2} & 0 \\ 0 & \cosh^{-2}\omega_2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\
F_4 &= \begin{pmatrix} a^2 \cosh^2\omega_1 & 1 & 0 \\ -a^2 \cos^2\omega_2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & F_5 &= \begin{pmatrix} -\omega_1^{-2} & 0 \\ 0 & \cosh^{-2}\omega_2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\
F_6 &= \begin{pmatrix} a^2 \sinh^{-4}\omega_1 & -\sinh^{-2}\omega_1 & -1 \\ a^2 \cosh^{-4}\omega_2 & \cosh^{-2}\omega_2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, & F_7 &= \begin{pmatrix} a^2 \sinh^{-4}\omega_1 & -\sinh^{-2}\omega_2 & -1 \\ -a^2 \cosh^{-4}\omega_2 & \cosh^{-2}\omega_2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\
F_8 &= \begin{pmatrix} e^{4\omega_1} & -e^{2\omega_1} & -1 \\ e^{4\omega_2} & e^{2\omega_2} & -1 \\ 0 & 0 & 1 \end{pmatrix}, & F_9 &= \begin{pmatrix} a^2 \cosh^2 2\omega_1 & -a \cosh 2\omega_1 & -1 \\ -a^2 \cosh^2 2\omega_2 & a \cosh 2\omega_2 & 1 \\ a^2 \cosh^2 2\omega_3 & a \cosh 2\omega_3 & -1 \end{pmatrix}, \\
F_{10} &= \begin{pmatrix} a^2 \frac{dn^4(\omega_1, k)}{sn^4(\omega_1, k)} & -\frac{dn^2(\omega_1, k)}{sn^2(\omega_1, k)} & 1 \\ -a^2 k^4 cn^4(\omega_1, k') & k^2 \cosh^2(\omega_2, k') & -1 \\ a^2 k^4 cn^4(\omega_3, k) & k^2 \cosh^2(\omega_3, k) & 1 \end{pmatrix}, & F_{11} &= \begin{pmatrix} \omega_1^{-4} & -\omega_1^{-2} & 0 \\ 0 & k^2 \cosh^2(\omega_3, k') & -1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{51} \]

The functions \( T_1(t), T_2(t), T_3(t) \) are expressed in terms of the functions \( h_1(t), h_2(t), h_3(t) \):

\[
\begin{align*}
1 &. \quad T_i = h_i^{-2}, & i &= 1, 2, 3; \\
2 - 4. \quad T_1 = h_1^{-2}, & T_2 = 0, & T_3 = h_3^{-2}; \\
5 - 11. \quad T_1 = h_1^{-2}, & T_2 = T_3 = 0. \tag{52} \end{align*}
\]

Let \( K \) and \( M \) be \( 3 \times 3 \) constant matrices. Now, if the reduced equations are given by (57), then

\[ F = \|F_{\mu a}\|_{\mu=0}^3 a=1^3, \quad G = \|G_{\mu a}\|_{\mu=0}^3 a=1^3 \]

are block \((6 \times 8)\)-matrices, where \( F_{\mu a} \) and \( G_{\mu a} \) are \( 2 \times 2 \)-matrices that are equal to products of the corresponding entries of the matrices \( SK \) and \( SM \) by the unit (in the case of the matrix \( F \)) or \( \vec{s} \vec{\sigma} \) (in the case of the matrix \( G \)) matrices. Accordingly, equation (8) takes the form

\[ \text{rank}(F + G) = 6. \tag{53} \]

If \( \text{rank } K = 3 \), then we can always rearrange \( \lambda_1, \lambda_2, \lambda_3 \) with the use of the equivalence relation (8) in order to get \( K = I \). Analogously, without loss of generality we may put \( M = I \), provided \( \text{rank } M = 3 \) and \( \vec{s}^2 \neq 0 \).
If rank $M = 0$, then the column $\|G_\mu 0\|_{\mu=0}^3$ has necessarily the form $S\vec{g}$, where $\vec{g}$ is a constant three-component column. If rank $M \neq 0$, then we can always kill this column by a proper rearranging of $\lambda_1, \lambda_2, \lambda_3$ with the use of the equivalence relation (9).

Next, if the reduced equations are given by ([18]), then the matrix $F$ is defined in the same way as in the previous case. Furthermore,

$$G = \|G_\mu \vec{s}_a \vec{g}\|_{\mu=0 a=1}^3$$

is a block $(6 \times 8)$-matrix, where $G_\mu$ are the three-component columns $S\vec{g}_\mu$ ($\vec{g}_\mu$ is a constant three-component column). In addition, in this case identity (53) holds, so that we can put $K = I$, when rank $K = 3$. If $\vec{s}_a, (1 = 1, 2, 3)$ are three linear independent vectors, then we can always put $\vec{s}_0 = 0$.

We will finish this section with the following remark. It follows from theorem 1 that a choice of magnetic fields $\vec{H}$ allowing for variable separation in the corresponding Pauli equation is very restricted. Namely, the magnetic field should be independent of spatial variables $x_1, x_2, x_3$ in order to provide the separability of Pauli equation ([1]) into three second-order matrix ordinary differential equations of the form (7). However, if we allow for separation equations to be of a lower order, then additional possibilities for variable separation in the Pauli equation arise. As an example, we give the vector potential

$$A(t, \vec{x}) = \left( A_0 \left( \sqrt{x_1^2 + x_2^2} \right), 0, 0, A_3 \left( \sqrt{x_1^2 + x_2^2} \right) \right),$$

where $A_0, A_3$ are arbitrary smooth functions. The Pauli equation ([1]) with this vector-potential separates in the cylindrical coordinate system

$$t, \quad \omega_1 = \ln \left( \sqrt{x_1^2 + x_2^2} \right), \quad \omega_2 = \arctan(x_1/x_2), \quad \omega_3 = x_3$$

into two first-order and one second-order matrix ordinary differential equations. The corresponding magnetic field $\vec{H} = \text{rot} \vec{A}$ is evidently $x$-dependent. In this respect, let us also mention the recent paper by Benenti with co-authors ([3]), where the problem of separation of variables in the stationary Hamilton-Jacobi equation with vector-potential has been studied. They have presented a number of vector-potentials, for which the Hamilton-Jacobi equation is separable, and the corresponding magnetic fields are inhomogeneous ones. These potentials allow for separation of variables in the stationary Schrödinger and Pauli equations with vector-potentials as well (see, e.g., ([13]) concerning the relationship between the separation of variables in the Schrödinger and Hamilton-Jacobi equations). These facts imply an importance of application of our approach to classify the non-stationary Pauli equations of the form ([1]), which admit separation of variables into first- and second-order matrix ordinary differential equations. We remind that here we give the classification results for the case, when all the reduced equations are second-order ones. We intend to address this problem in one of our future publications.
3 Algorithm of separation of variables in the Pauli equation with fixed potential

Theorem 1 gives the solution of the problem of classification of the Pauli equations (1) with variable coefficients that are separable (in the sense of definition 1) at least in one coordinate system.

Let us consider the problem of classification of coordinate systems that allow for separation of variables (in the sense of definition 1) in the Pauli equation (1) with fixed vector-potential \( A_0, \vec{A} \).

Let some fixed vector-potential \( \vec{A}(t, \vec{x}), A_0(t, \vec{x}) \) be given. The scheme of finding all coordinate systems providing separation of variables is as follows:

1. With help of gauge transformations (11) we reduce the space-like components of vector-potential \( \vec{A}(t, \vec{x}) \) to the form (39). If it is impossible, then Pauli equation (1) with this vector-potential is not solvable by the method of separation of variables in the framework of our approach.

2. We solve the system of ODE (38) for given magnetic field \( \vec{H}(t) \) and obtain the explicit form of functions \( \alpha(t), \beta(t), \gamma(t) \).

3. For each of 11 classes of coordinate systems \( t, \omega_a = \omega_a(t, \vec{x}), a = 1, 2, 3 \), which are given by formulas (19)–(22), taking into account restrictions obtained on the first step of the algorithm, we find the explicit form of

   (a) the time-like component \( A_0 \) of the vector-potential in terms of \( \vec{\omega} \);
   (b) function \( P \), substituting in (46) the expression for \( \vec{x}' \) in terms of \( \vec{\omega} \) (see formulas (35) and (13)):

   \[
   \vec{x}' = \mathcal{L}(t) (\vec{z}(\vec{\omega}) + \vec{v}(t));
   \]  
   (54)
   (c) quantity \( e^2 A_b A_b \) by the formula

   \[
   4e^2 A_b A_b = (n_2 x'_3 - n_3 x'_2)^2 + (n_3 x'_1 - n_1 x'_3)^2 + (n_2 x'_1 - n_1 x'_2)^2,
   \]

   where \( x'_1, x'_2, x'_3 \) are given by the formula (54), and functions \( n_1, n_2, n_3 \) are as follows

   \[
   \begin{align*}
   n_1 &= \dot{\gamma}(t) \cos \beta(t) + \dot{\alpha}(t) \sin \beta(t) \sin \gamma(t), \\
   n_2 &= -\dot{\gamma}(t) \sin \beta(t) + \dot{\alpha}(t) \cos \beta(t) \sin \gamma(t), \\
   n_3 &= \dot{\beta}(t) + \dot{\alpha}(t) \cos \gamma(t);
   \end{align*}
   \]

   (d) eikonals \( \frac{\partial \omega_a}{\partial x_b} \frac{\partial \omega_a}{\partial x_b} = R_a^{-2}, a = 1, 2, 3 \), which are determined from the list (17) for given class of coordinates.

4. We substitute the equalities obtained into equation (44) and obtain 11 equations for each of 11 classes of coordinate systems \( t, \omega_1, \omega_2, \omega_3 \). For each of these equalities we find all possible functions \( F_{a0}(\omega_a), a = 1, 2, 3, F_{00}(t) \) that reduce it to the identity.
by the independent variables $t$, $\omega_1$, $\omega_2$, $\omega_3$ (i.e. we split this equality with respect to these variables). It gives, in its turn, the explicit form of the functions $v_a(t)$, $l_a(t)$, $a = 1, 2, 3$ and additional restriction on $\alpha(t)$, $\beta(t)$, $\gamma(t)$, giving the form of the coordinate system in question. All obtained coordinates for which the functions $F_{a0}(\omega_a)$, $a = 1, 2, 3$, $F_{00}(t)$ exist are only coordinate systems providing separability of Pauli equations in the sense of definition [1].

**Example.** As illustration of this algorithm consider the problem of separation of variables in Pauli equation (1) for a particle interacting with a constant magnetic field. Without loss of generality we can always choose it as directed along axes $OZ$: $\mathbf{eH} = (0, 0, c)^T$, where $c$ is a non-zero real constant. The vector-potential of electro-magnetic field has the form

$$
2eA = \begin{pmatrix}
0 & -c & 0 \\
c & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \mathbf{x}, \quad eA_0 = \frac{q}{|\mathbf{x}|} - \frac{c^2}{12} \left(x_1^2 + x_2^2 - 2x_3^2\right),
$$

where $q$ is a non-zero real constant and $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

A direct check shows this vector-potential satisfies the vacuum Maxwell equations without currents

$$
\square A_0 - \frac{\partial}{\partial t} \left(\frac{\partial A_0}{\partial t} + \text{div} \ A\right) = 0,
$$
$$
\square A + \mathbf{\nabla} \left(\frac{\partial A_0}{\partial t} + \text{div} \ A\right) = \mathbf{0},
$$

where $\square = \partial^2/\partial t^2 - \Delta$ is d’Alembert operator. Therefore, it is a natural generalization of the standard Coulomb potential, which is obtained from (56) under $c \to 0$.

**Proposition 1** The set of inequivalent coordinate systems providing separability of the Pauli equation (1) with vector potential of electromagnetic field (56) is exhausted by the following ones:

$$
\mathbf{x} = \mathcal{O}(t) \vec{z},
$$

where $\mathcal{O}$ is a time-dependent $3 \times 3$ orthogonal matrix (20), with Euler angles

$$
\alpha(t) = -ct, \quad \beta = \text{const}, \quad \gamma = \text{const},
$$

and $\vec{z}$ is one of the following coordinate systems:

1. spherical (formula 5 from list (21)),
2. prolate spheroidal II (formula 6 from (21), where one should replace $z_3$ with $z_3 = a(\coth \omega_1 \tanh \omega_2 \pm 1)$),
3. conical (formula 11 from (21)).

**Proof.** The space-like component $A(t, \vec{x})$ of the given vector-potential (56) is already reduced to form (58).
The system of ODE (38) for given magnetic field takes the form:

\[ \dot{\gamma} \cos \alpha + \dot{\beta} \sin \alpha \sin \gamma = 0, \quad \dot{\gamma} \sin \alpha - \dot{\beta} \cos \alpha \sin \gamma = 0, \quad \dot{\alpha} + \dot{\beta} \cos \gamma = -c. \]

This implies the equivalent system

\[ \dot{\gamma} = 0, \quad \dot{\beta} \sin \gamma = 0, \quad \dot{\alpha} + \dot{\beta} \cos \gamma = -c. \]

Its general solution up to translation by \( t \) is given by formulas (59) (solution \( \alpha \pm \beta = -ct \) for case \( \sin \gamma = 0 \) is included into (59) as a particular case after denoting \( \alpha \pm \beta \rightarrow \alpha \)).

The steps 3 and 4 of the above algorithm will be illustrated by the case of spherical coordinate system 5 from (21) (for other coordinate systems this procedure is an analogous one). For this case the equality (44) in terms \( \omega_1, \omega_2, \omega_3 \) takes the form

\[ l^{-2} \left( F_{10}(\omega_1)\omega_1^4 + (F_{20}(\omega_2) + F_{30}(\omega_3))\omega_1^2 \cosh^2 \omega_2 \right) - F_{00}(t) = \]

\[ = \frac{q}{|\vec{x}|} + \left( \frac{c^2}{6} + \frac{1}{4 l} \right) |\vec{x}|^2 + \frac{3}{6} \left( 2(l\ddot{v}_a + 2l\dot{v}_a)x'_a + l^2 \dot{v}_a^2 \right), \]

where \( l = l_1 = l_2 = l_3, l \neq 0 \) (because of the spherical coordinate system is non-split one), and

\[ x'_1 = l(\omega_1^{-1} \text{sech} \omega_2 \cos \omega_3 + v_1(t)), \]
\[ x'_2 = l(\omega_1^{-1} \text{sech} \omega_2 \sin \omega_3 + v_2(t)), \]
\[ x'_3 = l(\omega_1^{-1} \tanh \omega_2 + v_3(t)). \]

Next we perform on both parts of equality (60) the following step by step operations:

1. multiplying by \( \omega_1 \),
2. differentiation with respect to \( \omega_1 \),
3. division by \( \omega_1^2 \),
4. differentiation with respect to \( \omega_1 \),
5. differentiation with respect to \( \omega_2 \),
6. multiplying by \( l\omega_1^7 |\vec{x}|^7 \),
7. twice multiplying by \( \omega_1 \).

As result we get

\[ -24q \text{sech}^3 \omega_2 (v_1^2 + v_2^2 + v_3^2)(v_1 \cos \omega_3 + v_2 \sin \omega_3 + v_3 \sinh \omega_2) \times \]
\[ (v_3(1 + \cos \omega_3) + v_2 \sin \omega_3 \sinh \omega_2) = 0. \]

The equality obtained is transformed into an identity with respect to independent variables \( \omega_1, \omega_2, \omega_3 \) if and only if the condition \( v_1 = v_2 = v_3 = 0 \) holds. Now the equality (60) takes the form

\[ l^{-2} \left( F_{10}(\omega_1)\omega_1^4 + (F_{20}(\omega_2) + F_{30}(\omega_3))\omega_1^2 \cosh^2 \omega_2 \right) - F_{00}(t) = \]

\[ = \frac{q}{l} \omega_1 + \left( \frac{c^2}{6} + \frac{1}{4 l} \right) \frac{l^2}{\omega_1^2}. \]

Performing on both parts of equality (61) the following step by step operations:
1. differentiation with respect to $\omega_1$,
2. multiplying by $l^2$,
3. differentiation with respect to $t$,
4. multiplying by $\omega_3^2$,
5. differentiation with respect to $\omega_1$,

we get the equality $3q\dot{\omega}_1^2 = 0$. This implies $l = \text{const}$ and with the help of dilatations we can put without loss of generality $l = 1$. Thus the coordinate system takes the form (58).

The equation (61) yields

$$F_{10}(\omega_1)\omega_1^4 + (F_{20}(\omega_2) + F_{30}(\omega_3))\omega_1^2 \cosh^2 \omega_2 - F_{00}(t) = q\omega_1 + \frac{c^2}{6} \omega_1^{-2}.$$ 

We can split the equation obtained by the independent variables $\omega_1, \omega_2, \omega_3$. As a result we get

$$F_{10} = q\omega_1^{-3} + \frac{c^2}{6} \omega_1^{-6} + k_1\omega_1^{-4} - k_2\omega_1^{-2},$$

$$F_{20} = k_2 \text{sech}^2 \omega_2 - k_3, \quad F_{30} = k_3, \quad F_{00} = k_1.$$ 

The theorem is proved. ▷

4 Separation of variables in the Pauli-Maxwell system

The expressions (39), (44)–(47) give the most general form of the vector-potential of the electromagnetic field, providing separability of the corresponding Pauli equations. But, because of generality of the results, these expressions are too cumbersome, and their physical interpretation is somewhat difficult. Therefore it would be interesting to know the form of these potentials under certain physical restrictions. The most natural restriction is that the vector-potential satisfies the vacuum Maxwell equations without currents (57).

In this section we describe all explicit forms of the vector-potentials $A(t, \vec{x})$ that

a) provide separability of Pauli equation,

b) satisfy vacuum Maxwell equations without currents (57) and

c) describe the non-zero magnetic field.

Furthermore, we construct inequivalent coordinate systems enabling us to separate variables in the corresponding Pauli equation.

The similar problem with more strong restrictions was analyzed in [20] for a two-dimensional Schrödinger equation with vector-potential. Note that an analogous problem for the Dirac equation for an electron was analyzed in [21].
Taking into account the form of $\vec{A}$ (33), the Maxwell equations (57) take the form
\[ \Delta A_0 = 0, \] (62)
and
\[ \frac{\partial^2 A_0}{\partial t \partial x_1} = -\ddot{l}_3 x_2 + \ddot{l}_2 x_3, \quad \frac{\partial^2 A_0}{\partial t \partial x_2} = \ddot{l}_3 x_1 - \ddot{l}_1 x_3, \quad \frac{\partial^2 A_0}{\partial t \partial x_3} = -\ddot{l}_2 x_1 + \ddot{l}_1 x_2. \]

From the compatibility conditions of the above system of PDEs we get
\[ \ddot{l}_1 = \ddot{l}_2 = \ddot{l}_3 = 0, \quad \frac{\partial^2 A_0}{\partial t \partial x_a} = 0, \quad a = 1, 2, 3. \] (63)

Inserting expression for potential $A_0(t, \vec{x})$ (44) into (32) with subsequent change of independent variables (19) yields (we use the relations $\Delta \omega_i = 0$, $\omega_{ix_a} \omega_{jx_a} = 0$, $i \neq j$, $i, j = 1, 2, 3$)
\[ \sum_{j=1}^{3} \frac{\partial^2}{\partial \omega_j^2} \left( \sum_{i=1}^{3} F_{i0}(\omega_i) R_i^{-2} \right) R_j^{-2} = \frac{1}{2} \sum_{i=1}^{3} \frac{\ddot{l}_i}{l_i} + e^2(H_1^2 + H_2^2 + H_3^2), \] (64)
where the eikonals
\[ R_i^{-2} = \frac{\partial \omega_i}{\partial x_a} \frac{\partial \omega_i}{\partial x_a}, \quad i = 1, 2, 3 \] (65)
are given in the list (47).

Thus we get eleven functional relations $\mathcal{P}_1, \ldots, \mathcal{P}_{11}$ for each class of coordinate system (19), whose form is determined by the form of one of the eleven expressions $z_1(\omega_1, \omega_2, \omega_3)$, $z_2(\omega_1, \omega_2, \omega_3)$, $z_3(\omega_1, \omega_2, \omega_3)$ from the list (21). As $t, \omega_1, \omega_2, \omega_3$ are functionally independent, we can split the above relations with respect to the variables $t, \omega_1, \omega_2, \omega_3$, thus getting ordinary differential equations for the functions $F_{i0}(\omega_i), l_i(t), i = 1, 2, 3$. After solving them the formula (44) yields the expressions for $A_0$ in terms of variables $t, \omega_1, \omega_2, \omega_3$. Returning to variables $t, x_1, x_2, x_3$ (with the aid of (19)), we should split the expression obtained for $A_0(t, \vec{x})$ with respect to $t$. Indeed, the general solution of the equation (33) is
\[ A_0(t, \vec{x}) = f_1(\vec{x}) + f_2(t). \]

At the expense of the gauge invariance of the Pauli equation we may choose $f_2(t) = 0$. Thus the potential $A_0$ should be a function of $\vec{x}$ only. This condition restricts the choice of $A_0$, thus giving ordinary differential equations for the functions $l_i(t), v_i(t), i = 1, 2, 3$. Solving them we obtain the explicit forms of the function $F_{i0}(t)$ and coordinate systems (19). After simplifying these coordinate systems with the aid of equivalence transformations we get a full description of the vector-potentials $A(t, \vec{x})$ and coordinate systems, giving the solution of the problem under study.

Omitting the details of the calculations (they are very cumbersome) we present below the results. Note, when presenting lists of the vector-potentials $A(t, \vec{x})$ and coordinate systems we use invariance of the system of the Pauli and Maxwell equations with respect to the groups of rotations by spatial variables $x_1, x_2, x_3$ and translations by all variables $t, x_1, x_2, x_3$ (see, e.g., [22]).
1. Case of non-stationary magnetic field:

\[ e\vec{H} = (0, 0, At + B), \]
\[ eA_0 = -\frac{k}{2}(x_1^2 + x_2^2 - 2x_3^2) + a_1x_1 + a_2x_2 + a_3x_3, \]

where \( A, B, k, a_1, a_2, a_3 \) are arbitrary real constants.

The coordinate system is

\[ \vec{x} = L\mathcal{O}(\vec{z} + \vec{v}). \]

Here \( \mathcal{O} \) is a time-dependent \( 3 \times 3 \) orthogonal matrix \( \mathcal{O}(\alpha, \beta, \gamma) \), where

\[ \alpha = -\frac{1}{2}At^2 - Bt, \quad \beta = 0, \quad \gamma = 0; \]

\( \vec{z} \) is Cartesian, cylindrical or elliptic cylindrical coordinate system (formulas 1, 2, 4 from [21]); \( L \) is the \( 3 \times 3 \) diagonal matrix

\[ L = \begin{pmatrix} l(t) & 0 & 0 \\ 0 & l(t) & 0 \\ 0 & 0 & l_3(t) \end{pmatrix}, \]

and \( \vec{v}(t) \) is vector-column \( \vec{v}(t) = (v_1, v_2, v_3)^T \) where functions \( l(t), l_3(t), v_1(t), v_2(t), v_3(t) \) are solutions of the following system of ordinary differential equations:

\[ \frac{2c}{l^4} - \frac{1}{2}\frac{\dot{l}}{l} + k = \frac{1}{2}(At + B)^2, \quad \frac{c_3}{l_3^3} - \frac{1}{4}\frac{\dot{l}_3}{l_3} = \frac{1}{2}k, \]
\[ l\ddot{v}_1 + 2\dot{l}\dot{v}_1 + 4c\frac{v_1}{l_3} - 2c_{11}\frac{1}{l} = -2(a_1 \cos \alpha + a_2 \sin \alpha), \]
\[ l\ddot{v}_2 + 2\dot{l}\dot{v}_2 + 4c\frac{v_2}{l_3} - 2c_{12}\frac{1}{l} = -2(-a_1 \sin \alpha + a_2 \cos \alpha), \]
\[ l_3\ddot{v}_3 + 2\dot{l}_3\dot{v}_3 + 4c_3\frac{v_3}{l_3^3} - 2c_{13}\frac{1}{l_3} = -2a_3. \]

Here \( c, c_3, c_{11}, c_{12}, c_{13} \) are arbitrary real constants.

2. Cases of stationary magnetic field.

Case 1:

\[ e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0; \]
\[ eA_0 = -\frac{k^2}{12}(x_1^2 + x_2^2 - 2x_3^2) + a_1x_1 + a_2x_2 + a_3x_3, \]

where \( \vec{a} = (a_1, a_2, a_3) \) is constant vector.

The coordinate system is

\[ \vec{x} = l\mathcal{O}(\vec{z} + \vec{v}). \]
Here $\mathcal{O}$ is a time-dependent $3 \times 3$ orthogonal matrix $\mathcal{O}(\alpha, \beta, \gamma)$, where $\alpha = -kt$, $\beta = \text{const}$, $\gamma = \text{const}$; $\vec{z}$ is one of coordinate system, given by formulas 1-11 from (21); function $l(t)$ is solution of the equation

$$k^2 + \frac{3}{2} \frac{\dot{l}}{l} = \frac{c}{l^3}$$

given by one of the formulas:

$$c = \pm 1, \quad l^2 = \sqrt{C_1^2 + \frac{1}{k^2} \sin \left(2 \sqrt{\frac{3}{2} k t} \right) + C_1},$$

for coordinate system $\vec{z}$ given by the formulas 1, 2, 4, 5, 6, 7, 10, 11 from the list (21) and

$$c = 0, \quad l = C_1 \sin \left(\sqrt{\frac{2}{3} k t} \right)$$

for coordinate system $\vec{z}$ given by the formulas 1-11 from the list (21). Here $C_1$ is an arbitrary real constant. Vector $\vec{v}$ is a solution of the following system of ordinary differential equations:

$$3l\ddot{\vec{v}} + 6\dot{l}\dot{\vec{v}} + \frac{2c}{l^3} \vec{v} = -6\mathcal{O}^{-1} \vec{a}.$$

**Case 2:**

$$e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0;$$

$$eA_0 = \frac{a}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{k^2}{12} (x_1^2 + x_2^2 - 2x_3^2), \quad a = \text{const} \neq 0.$$

The coordinate system is

$$\vec{x} = \mathcal{O}\vec{z}.$$

Here $\mathcal{O}$ is a time-dependent $3 \times 3$ orthogonal matrix $\mathcal{O}(\alpha, \beta, \gamma)$, where $\alpha = -kt$, $\beta = \text{const}$, $\gamma = \text{const}$ and $\vec{z}$ is one of the following coordinate systems:

1. spherical (formula 5 from (21)),
2. prolate spheroidal II (formula 6 from (21), where one should replace $z_3$ with $z_3 = a(\coth \omega_1 \tanh \omega_2 \pm 1)$),
3. conical (formula 11 from (21)).

**Case 3:**

$$e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0;$$

$$eA_0 = -\frac{k^2}{12} (x_1^2 + x_2^2 - 2x_3^2) + \frac{a_1}{r} + \frac{a_2 x_3}{r^3} + \frac{a_3}{r^2} \left(\frac{x_3}{2r} \ln \frac{r + x_3}{r - x_3} - 1 \right),$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $a_1, a_2, a_3$ are real constant numbers.
The coordinate system is \( \vec{x} = l \mathcal{O} \vec{z} \). Here \( \mathcal{O} \) is a time-dependent 3×3 orthogonal matrix \( \mathcal{O}(\alpha, \beta, \gamma) \), where \( \alpha = -kt, \beta = \gamma = 0 \); \( \vec{z} \) is the spherical coordinate system, given by formula 5 from (21) and function \( l(t) \) is given by

\[
l^2 = \sqrt{C_1^2 + \frac{1}{k^2}} \sin \left( \frac{2 \sqrt{\frac{2}{3}} kt}{2} \right) + C_1, \quad \text{or} \quad l = C_1 \sin \left( \sqrt{\frac{2}{3}} kt \right)
\]

under condition \( a_1 = 0 \) and \( l = 1 \) under condition \( a_1 \neq 0 \). Here \( C_1 \) is an arbitrary real constant.

Case 4:

\[
e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0; \quad eA_0 = -\frac{k^2}{12} (x_1^2 + x_2^2 - 2x_3^2) + \frac{a_1}{r^+} + \frac{a_2}{r^-} + a_3 \left( \frac{1}{r^+} \arctanh \frac{x_3^+}{r^+} - \frac{1}{r^-} \arctanh \frac{x_3^-}{r^-} \right),
\]

where \( x_3^\pm = x_3 \pm a \) and \( r^\pm = \sqrt{x_1^2 + x_2^2 + (x_3 \pm a)^2} \), and \( a, a_1, a_2, a_3 \) are arbitrary real constants. The coordinate system is \( \vec{x} = \mathcal{O} \vec{z} \).

Here \( \mathcal{O} \) is a time-dependent 3×3 orthogonal matrix \( \mathcal{O}(\alpha, \beta, \gamma) \), where \( \alpha = -kt, \beta = \gamma = 0 \) and \( \vec{z} \) is a prolate spheroidal coordinate system, given by formula 6 from (21).

Case 5:

\[
e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0; \quad eA_0 = -\frac{k^2}{12} (x_1^2 + x_2^2 - 2x_3^2) + \frac{a_1}{r^+} + \frac{a_2}{r^-} + a_3 \left( \frac{1}{r^+} \arccot f_1 - \frac{x_3}{f_1} \arctanh \frac{x_3}{a f_1} \right),
\]

where

\[
f = \sqrt{(a^2 - r^2)^2 + 4a^2 x_3^2}, \quad f_1 = \sqrt{-\frac{a^2 + r^2 + f}{2a^2}}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},
\]

and \( a, a_1, a_2, a_3 \) are arbitrary real constants. The coordinate system is \( \vec{x} = \mathcal{O} \vec{z} \).

Here \( \mathcal{O} \) is a time-dependent 3×3 orthogonal matrix \( \mathcal{O}(\alpha, \beta, \gamma) \), where \( \alpha = -kt, \beta = \gamma = 0 \) and \( \vec{z} \) is an oblate spheroidal coordinate system, given by formula 7 from (21).

Note that expression for \( A_0 \) can be rewritten in the form

\[
eA_0 = -\frac{k^2}{12} (x_1^2 + x_2^2 - 2x_3^2) + \frac{a_1 + ia_2}{\tilde{r}^+} + \frac{a_1 - ia_2}{\tilde{r}^-} + ia_3 \left( \frac{1}{\tilde{r}^+} \arctanh \frac{\tilde{x}_3^+}{\tilde{r}^+} - \frac{1}{\tilde{r}^-} \arctanh \frac{\tilde{x}_3^-}{\tilde{r}^-} \right),
\]

where \( \tilde{x}_3^\pm = x_3 \pm ia \) and \( \tilde{r}^\pm = \sqrt{x_1^2 + x_2^2 + (x_3 \pm ia)^2} \).
Case 6:

\[ e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0; \]

\[ eA_0 = -\frac{k^2}{6} (x_1^2 + x_2^2 - 2x_3^2) + \frac{a_1}{r} + a_2x_3 + \frac{a_3}{r} \ln \frac{r + x_3}{r - x_3}, \]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \) and \( a_1, a_2, a_3 \) are arbitrary real constants.

The coordinate system is

\[ \vec{x} = O\vec{z}. \]

Here \( O \) is a time-dependent \( 3 \times 3 \) orthogonal matrix \( O(\alpha, \beta, \gamma) \), where \( \alpha = -kt, \beta = \gamma = 0 \) and \( \vec{z} \) is a parabolic coordinate system, given by formula 8 from (21).

Case 7:

\[ e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0; \]

\[ eA_0 = -\frac{q}{2} (x_1^2 + x_2^2 - 2x_3^2) + a \ln(x_1 + x_2) + a_3x_3, \]

where \( k, a, a_3 \) are arbitrary real constants.

The coordinate system is

\[ x_1 = e^{\omega_1} \cos(\omega_1 - kt), \quad x_2 = e^{\omega_1} \sin(\omega_1 - kt), \quad x_3 = l_3\omega_3 + v_3; \]

where \( l_3, v_3 \) are solutions of the system of ordinary differential equations

\[ \frac{c_3}{l_3^4} - \frac{1}{4} \frac{l_3}{l_3} = q, \quad l_3\ddot{v}_3 + 2\dot{l}_3\dot{v}_3 + 4c_3\frac{v_3}{l_3} - 2c_{13} \frac{1}{l_3} = -2a_3. \]

Note that some of the potentials obtained have the clear physical meaning. For instance, cases 2 and 3 under condition \( k = a_2 = a_3 = 0 \) give the standard Coulomb potential. Case 4 under condition \( k = a_3 = 0 \) gives the potential for a well-known two-center Kepler problem, i.e., the problem of finding wave functions of electron moving in the field of two fixed Coulomb centres with charges \( a_1, a_2 \) and intercenter distance \( 2a \) (the model of ionized hydrogen molecule). Coulson and Joseph [23] showed that the corresponding Schrödinger equation admits separation of variables in the prolate coordinate system only. We obtained this potential as a particular case of the more general potential.

5 Concluding Remarks

Theorem 1 provides the complete solution of the problem of classification of the Pauli equations (1), which are solvable within the framework of the method of separation of variables in the sense of our definition 1. According to these theorems the coordinate systems and the vector-potentials of the electromagnetic field \( A(t, \vec{x}) = (A_0(t, \vec{x}), \vec{A}(t, \vec{x})) \) providing separability of the corresponding Pauli equations coincide with those for the Schrödinger equations with vector-potential. So the results obtained in the article are valid for the Schrödinger equation as well.
It is well-known that the possibility of variable separation in a system of PDEs is closely connected to its symmetry properties [9, 10]. Namely, solutions with separated variables are common eigenfunctions of three matrix mutually commuting symmetry operators of the equation under study. For all the cases of variable separation in Pauli equation (1) such matrix second-order operators can be constructed in the explicit form, by analogy to what has been done in [24] for the (1+2)-dimensional Schrödinger equation. They are expressed in terms of the matrix coefficients of the separation equations (17)–(18).

A promising development of the research is classification and study of superintegrable (admitting sufficiently many higher symmetries) cases of Pauli equation. Notice that the notions of separability and superintegrability are closely related. By now, superintegrable physical systems can be regarded as one of the most intensively developed and significant fields of mathematical physics. The problem of classifying superintegrable stationary Schrödinger equations with scalar potential has been solved by Winternitz with co-workers [25] and Evans [26] for space dimensions $n = 2$ and $n = 3$ (see also [4]). They have found all potentials that allow for separability of the corresponding Schrödinger equation in more than one coordinate system. We intend to modify and generalize this approach to (1 + 3)-dimensional Pauli equation (1). A study of the problem is in progress now and will be reported in our future publications.

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