A categorification of the polynomial ring

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Categorification

Integers $\Rightarrow$ Abelian groups $\Rightarrow$ Abelian categories

**Decat**  Compute the Grothendieck group of abelian category.

**Cat**    Given an abelian group with additional data, such as a collection of its endomorphisms, realize it as a Grothendieck group of some interesting category equipped with exact endofunctors that descend to the endomorphisms.

**Goal:** Diagrammatic categorification of $\mathbb{Z}[x]$
Algebras with planar interpretation

Group algebra \( \mathbb{C}[S_n] \)

\[
T_i^2 = 1 \\
T_i T_j = T_j T_i, \quad |i - j| > 1 \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}
\]

Hecke algebra \( H_n \)

\[
T_i^2 = (q - 1)T_i + q \\
T_i T_j = T_j T_i, \quad |i - j| > 1 \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}
\]

Categorification

Category of Soergel bimodules categorifies \( \mathbb{Z}[S_n] \) and, considered as a graded category, it gives \( H_n \).
From algebras to categories

Temperley-Lieb algebra $TL_n$

TL category

- **Objects**
  Non-negative integers

- **Morphisms** $n \rightarrow m$
  Given by plane diagrams with $n$ bottom and $m$ top endpoints i.e. linear combination of planar diagrams over $\mathbb{Z}[q, q^{-1}]$ or a field $\mathbb{Q}(q)$ up to isotopies.

Subject to isotopy relations & $\bigcirc = q + q^{-1}$
Category as an algebra

Temperley Lieb algebra on \( n \) strands \( TL(n) = Hom_{TL}(n, n) \)

\( TL \) category can be viewed as algebra without a unit 1 but with system of mutually orthogonal idempotents
\( 1_n \in Hom_{TL}(n, n), \ \forall n:\)

\[
TL = \bigoplus_{n,m \geq 0} Hom_{TL}(n, m)
\]
Goal: Diagrammatic categorification of $\mathbb{Z}[x]$

- $\mathbb{Z}[x]$ is a ring: we need a monoidal category
- Monomial $x^n \leftrightarrow$ Indecomposable projective module $P_n$
- Integral inner product $(x^n, x^m) = \dim \text{Hom}(P_n, P_m)$

Rotate diagrams $90^\circ$ clockwise so that diagrams match left/right action of algebra on itself.
Categorification \( \mathbb{Z}[x] \)

Functorification

Chebyshev polynomials

Hermite polynomials

**SLarc diagrams**

\[ nB_m \overset{\text{def}}{=} \text{set of isotopy classes of planar diagrams} \]

\[ |nB_m^-| = \sum_{k=0}^{\min(n,m)} \binom{n}{k} \binom{m}{k} = \binom{n+m}{n}. \]
**SLarc diagrams**

\[ B_m^{-} \overset{\text{def}}{=} \bigsqcup_{n \geq 0} nB_m^{-} \]

\[ B^{-} \overset{\text{def}}{=} \bigsqcup_{n,m \geq 0} nB_m^{-} \]

\[ nB_m^{-}(k) \text{ diagrams in } nB_m^{-} \text{ of width } k \]

\[ nB_m^{-}(\leq k) \text{ diagrams in } nB_m^{-} \text{ of width less than or equal to } k. \]
If assume $d \in \mathbb{C}$, up to rescaling, the value of the floating arc $d$ can be set to 0 or 1.

- If $d = 1$ we get two orthogonal idempotents, so $\text{Hom}(1, 1) \cong \mathbb{C} \oplus \mathbb{C} \Rightarrow$ semisimple! to be continued....

- Set the value of the floating arc to zero $d = 0$, get only one idempotent $\text{Hom}(1, 1) \cong \mathbb{C}[\alpha]/(\alpha^2)$. 
**SLarC algebra $A^-$**

$k$ a field and $A^-_k$-vector space with the basis $B^-_k$.

**Multiplication:**

- generated by the concatenation of elements of $B^-_k$

- if $y \in nB^-_m$, $z \in kB^-_l$ and $m \neq k$, then the concatenation is not defined and we set $yz = 0$.

- product is zero if the resulting diagram has an arc which is not attached to the lines $x = 0$ or $x = 1$, called *floating arc.*
SLarc algebra $A^-$

$$A^- = \bigoplus_{n,m \geq 0} nA^-_m \text{ where } nA^-_m \text{ is spanned by diagrams in } nB^-_m.$$  

- associative
- non-unital with a system of orthogonal idempotents $\{1_n\}_{n \geq 0}$.
Examples

Diagrams $i b_n$ and $b^i_n$ composed with diagram $a \in B^-$. Left multiplication cannot increase width.
Modules over $A^-$

Consider
left modules $M$ over $A^-$ with the property $M = \bigoplus_{n \geq 0} 1_n M$.

Definition
A left $A^-$-module $M$ is called finitely-generated if and only if it’s isomorphic to a quotient of a direct sum of finitely many indecomposable projective modules with finite multiplicities.

Notation
$A^- \text{mod}$ the category of finitely-generated left $A^-$-modules

$A^- \text{pmod}$ the category of finitely-generated projective left $A^-$-modules.
Projective, standard and simple modules over $A^-$

$P_n = A^-1_n$ indecomposable projective left $A^-$-modules.
Basis: all diagrams in $B^-$ with $n$ right endpoints.

$M_n$ standard module is the quotient of $P_n$ by the submodule spanned by diagrams which have right sarcs.
Basis: diagrams in $B_n^-$ with no right sarcs.

$L_n$ simple 1-dim module
Categorification $\mathbb{Z}[x]$  

Functorification  

Chebyshev polynomials  

Hermite polynomials

$P_0: \quad \ldots $  

$P_1: \quad \ldots $  

$M_0$  

$M_1$  

$P_0$
Module homomorphisms

Diagrams in $nB_m^-$ constitute a basis for $\text{Hom}(P_n, P_m)$.

Remark
All diagrams in $B^-$ except $1_n$ act trivially on simple module $L_n$. 
Properties

Proposition

$\text{Hom}_{A^{-}}(M, N)$ is a finite-dimensional $k$-vector space for any $M, N \in A^{-}-\text{mod}$.

Corollary

The category $A^{-}-\text{mod}$ is Krull-Schmidt.

Proposition

Any $P \in A^{-}-\text{pmod}$ is isomorphic to a direct sum $P \cong \bigoplus_{i=0}^{N} P_{i}^{n_{i}}$ with the multiplicities $n_{i}$’s being invariants of $P$.

Proposition

A submodule of a finitely-generated left $A^{-}$-module is finitely-generated.

Corollary

The category $A^{-}-\text{mod}$ is abelian.
Grothendieck group/ring

Definition
Grothendieck group $K_0(A)$ of finitely generated projective $A$-modules is a group generated by symbols of projective modules $[P]$, such that

$$[P] = [P'] + [P''] \text{ if } P \cong P' \oplus P''$$

Theorem
$K_0(A^-)$ is a free group with basis $\{[P_n]\}_{n \geq 0}$.

$$K_0(A^-) \cong \mathbb{Z}[x] \text{ via } [P_n] \leftrightarrow x^n.$$  

If a category is monoidal, Grothendieck group becomes a ring.
Monoidal structure on $A^{-}\text{pmod}$

Tensor product bifunctor

$A^{-}\text{pmod} \times A^{-}\text{pmod} \to A^{-}\text{pmod}$

- $P_n \otimes P_m = P_{n+m}$ and extend to all projective modules
- on basic morphisms of projective modules $\alpha : P_n \to P_{n'}$ and $\beta : P_m \to P_{m'}$ by placing $\alpha$ on top of $\beta$ and then extending it to all morphisms and objects using bilinearity.
Relations between $P_n$ and $M_n$

Left multiplication by a basis vector cannot increase the width
⇒ $P_n(\leq m)$ is a submodule of $P_n$.

$$P_n = P_n(\leq n) \supset P_n(\leq n - 1) \supset \cdots \supset P_n(\leq 0)$$

$P_n(\leq m)/P_n(\leq m - 1)$ is spanned by diagrams in $P_n(m)$.

These diagrams can be partitioned into $\binom{n}{m}$ classes enumerated by positions of the $n - m$ right sarcs.
Relations between $P_n$ and $M_n$

$$P_n(\leq m)/P_n(\leq m-1) \cong M_m$$

In the Grothendieck group of finitely-generated $A^-$-modules

$$[P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m] \quad (1)$$
Projective resolution of $M_m$

\[ x^n = [P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m] \leftrightarrow [M_n] = \sum_{m \leq n} (-1)^{n+m} \binom{n}{m} [P_m] \]

Expect a finite projective resolution of $M_m$

\[ \begin{array}{ccccccccc}
\oplus & \left( \begin{array}{c} m \\ n \end{array} \right) & \rightarrow & P_n & \rightarrow & \ldots & \rightarrow & P_{m-2}^{\oplus \frac{m(m-1)}{2}} & \rightarrow & P_{m-1}^{\oplus m} & \rightarrow & P_m & \rightarrow & M_m & \rightarrow & 0
\end{array} \]

Proposition

*The complex with the differential defined above is exact.*

Corollary

*Homological dimension of standard module $M_m$ is $m.$*
Projective resolutions of $M_0$ and $M_1$

$$0 \rightarrow P_0 \xrightarrow{\iota} M_0 \rightarrow 0$$

$$0 \rightarrow P_0 \rightarrow P_1 \xrightarrow{\text{pr}} M_1 \rightarrow 0$$
Resolution of simple modules $L_k$ by $M_m$

Resolution of simple $L_k$ by standard modules $M_m$ for $m \geq k$:

$$d \rightarrow M_{k+m} \oplus \binom{k+m}{m} \rightarrow \cdots \rightarrow d \rightarrow M_{k+2} \oplus \binom{k+2}{2} \rightarrow d \rightarrow M_{k+1} \oplus M_k \rightarrow \cdots \rightarrow L_k \rightarrow 0.$$
Projective resolution of simple modules $L_k$

\[ \ldots \xrightarrow{d_H} P_{k+m-1} \xrightarrow{d_H} P_{k+m} \oplus \binom{k+m}{m} \xrightarrow{d_H} P_{k+m+1} \xrightarrow{d_H} P_{k} \xrightarrow{d_H} 0 \]

Lemma

Simple modules $L_k$ have infinite homological dimension.
$C(A^-)$ category of bounded complexes of projective modules modulo chain homotopies

- $C(A^-)$ is monoidal
- $C(A^-)$ contains $M_n$ but not $L_n$.
- $C(A^- \pmod) \times C(A^- \pmod) \to C(A^- \pmod)$
- $P(M_n) \otimes P(M_m) \cong P(M_{m+n})$
- $M_n \otimes M_m \cong M_{m+n}$, when viewed as objects of $C(A^- \pmod)$

$K_0(C(A^-)) \cong K_0(A^-)$

$X = (\ldots \to X^i \to X^{i+1} \to \ldots) \Rightarrow [X] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [X^i]$. 
Categorification of polynomial ring \( \mathbb{Z}[x] \)

\[
[P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m] \quad \leftrightarrow \quad x^n = \sum_{m=0}^{n} \binom{n}{m} (x - 1)^m
\]

\[
[M_n] = \sum_{m \leq n} (-1)^{n+m} \binom{n}{m} [P_m] \quad \leftrightarrow \quad (x - 1)^n = \sum_{m \leq n} (-1)^{n+m} \binom{n}{m} x^m
\]

\[
[L_n] = \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} [M_{n+k}] \quad \leftrightarrow \quad \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} (x - 1)^{n+k}
\]

\[
\leftrightarrow \quad \frac{(x - 1)^n}{x^{n+1}}.
\]
Categorifying multiplication in the ring $\mathbb{Z}[x]$

In $K_0(\mathcal{C}(A^-))$

$P(M_n) \otimes P(M_m) \cong P(M_{m+n})$ categorifies multiplication

$[M_n] \cdot [M_m] = (x - 1)^{n+m} = [M_{n+m}]$

**Generalization**

$\otimes$ for $A^-$ modules admitting a finite filtration by $M_n$

- Need to construct and tensor their projective resolutions
- derived tensor product $\hat{M} \hat{\otimes} N$ has cohomology only in degree zero and $H^0(\hat{M} \hat{\otimes} N) \cong_{D_b} \hat{M} \hat{\otimes} N$ has a filtration by standard modules.
Approximations of identity

$A^{-}(\leq k)$ spanned by diagrams in $B^{-}$ of width $\leq k$

$kP = 1_k A^{-}$ right projective module

$kM$ is spanned by diagrams $k B^{-}$ without left sarcs

Lemma

$A^{-}(\leq k)/A^{-}(\leq k - 1) \cong M_k \otimes_k M$ as an $A^{-}$-bimodule.
Approximations of identity

Definition
For a given $k \geq 0$ define a functor $F_k : A^- \mod \rightarrow A^- \mod$ by

$$F_k(M) = A(\leq k) \otimes_{A^-} M$$

for any $A^-$-module $M$.

Lemma

$$F_k(M_m) = \begin{cases} M_m, & \text{if } m \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

$$F_k(P_n) = \begin{cases} P_n, & \text{if } n \leq k; \\ P_n(\leq k), & \text{if } n > k. \end{cases}$$

Proof.

$$A^- (\leq k) \otimes_{A^-} P_m = A^- (\leq k) \otimes_{A^-} A^- 1_m = A^- (\leq k) 1_m$$
Approximations of identity

On the level of Grothendieck group $F_k$ corresponds to operator $[F_k]$:

$$[F_k][P_n] = \begin{cases} [P_n] = x^n, & \text{if } n \leq k; \\ \sum_{m=0}^{k} \binom{n}{m} [M_m] = \sum_{m=0}^{k} \binom{n}{m} (x - 1)^m, & \text{if } n > k. \end{cases}$$

Lemma

$$L^i F_k(M_m) = \begin{cases} M_m, & \text{if } i = 0, k \geq m; \\ 0, & \text{otherwise.} \end{cases}$$

$[F_k]$ approximates identity

- for $n \leq k$ it is $\text{Id}$ on $P_n$
- for $n > k$, it is like taking $k + 1$ terms in the expansion of $[P_n]$ in the basis $\{[M_m]\}_{m \geq k}$

$$f(x) = \sum_{m \geq 0} a_m (x - 1)^m \rightarrow \sum_{m=0}^{k} a_m (x - 1)^m$$
Restriction and induction functors

Let \( \iota : B \hookrightarrow A \) be a unital inclusion of arbitrary rings \( A, B \).

\[
\text{Ind} : B - \text{mod} \hookrightarrow A - \text{mod} \text{ given by } \text{Ind}(M) = A \otimes_B M
\]
is left adjoint to the restriction functor

\[
\text{Hom}_A(\text{Ind}(M), N) \cong \text{Hom}_B(M, \text{Res}(N)).
\]

Non-unital inclusion \( \iota(1_B) = e \neq 1_A, \ e^2 = e \in A \)

For \( A \)-module \( N \) define \( \text{Res}(N) = eN \) with \( B \subset eAe \) acting via \( \iota \).

\[
\text{Ind}(M) = A \otimes_B M \cong Ae \otimes_B M \oplus A(1 - e) \otimes_B M = Ae \otimes_B M.
\]

A similar construction works for non-unital \( B \) and \( A \) equipped with systems of idempotents.
Restriction and induction functors on $A^-$

$\iota : A^- \leftrightarrow A^-$ induced by adding a straight through line at the top of every diagram

- $d \in mB_n \Rightarrow \iota(d) \in m+1B^-_{n+1}$.
- $\{1_n\}_{n \geq 0} \leftrightarrow \{1_{n+1}\}_{n \geq 0}$ missing $1_0$.
- $\iota$ gives rise to both induction and restriction functors, with

$$Res(N) \cong N/1_0 N \cong \bigoplus_{k>0} 1_k N$$

and $A^-$ acting on the left via $\iota$. 
Restriction functor on $A^{-}$

Figure: (a) is $P_{12}^{\empty}$ and (b) is $P_{12}^{(i)}$

Decomposition of $P_n$ as a sum of vector spaces spanned by diagrams of type

(a) where left sarc is attached to the top left point $P_n^{\empty}$

(b) where the top left point is connected by larc to the i-th point on the right $P_n^{i}$. 
Restriction functor on $A^-$

$P_n^\emptyset \simeq P_n$

$P_n^{(i)} \simeq P_{n-i}$
Restriction functor on $A^-$

- $\text{Res}(L_n) = L_{n-1}$ if $n > 0$ and $\text{Res}(L_0) = 0$
- $\text{Res}(M_n) \cong M_n \oplus M_{n-1}$ for $n > 0$, and $\text{Res}(M_0) \cong M_0$.
- $\text{Res}(P_n) \cong \bigoplus_{k=0}^{n} P_k$ for $n > 0$, and $\text{Res}(P_0) \cong P_0$.

On the Grothendieck group, restriction takes:

$$[P_n] = x^n \mapsto \sum_{i=0}^{n} [P_i] = \sum_{i=0}^{n} x^i$$

$$[M_n] = (x - 1)^n \mapsto [M_i] + [M_{i-1}] = x(x - 1)^{n-1}.$$
Induction functor on $A^-$

- $\text{Ind}(P_n) \cong P_{n+1}$ for $n \geq 0$.
- $\text{Ind}(M_n) \cong M_n \oplus M_{n+1}$ for $n \geq 0$.

**Lemma**

*Higher derived functors of the induction functor applied to a standard module are zero:* $L^i \text{Ind}(M_n) = 0$, for every $i > 0$.

Induction corresponds to the multiplication by $x$ as:

\[
\begin{align*}
[P_n] &= x^n & \iff & & [P_{n+1}] &= x^{n+1} \\
[M_n] &= (x - 1)^n & \iff & & [M_n] + [M_{n+1}] &= x(x - 1)^n
\end{align*}
\]
Bernstein–Gelfand–Gelfand (BGG) reciprocity

- A finite–dimensional $A^–$–module $M$: $[M : L_n] = \dim 1_n M$
- A finitely-generated $A^–$–module $M$: locally finite–dimensional property:
  \[ \dim(1_n M) < \infty, \text{ for } n \geq 0 \]
- Multiplicity of $L_n$ in $M$ def. by $[M : L_n] := \dim(1_n M)$

\[
[M_m : L_n] = \dim(1_n M_m) = \begin{cases} 
\binom{n}{m}, & \text{for } n \geq m; \\
0, & \text{if } n < m.
\end{cases}
\]

Recall $[P_n : M_m] = \binom{n}{m}$, hence $[P_n : M_m] = [M_m : L_n]$
Chebyshev polynomials of the second kind $U_n$

Recursive definition

$U_{n+1}(x) = xU_n(x) - U_{n-1}(x)$

Initial conditions: $U_0(x) = 1, U_1(x) = x$

Inner product \{ $U_n$ \} form an orthogonal set on $[-1, 1]$

$$(f, g) = \frac{2}{\pi} \int_{-1}^{1} f(x)g(x)\sqrt{1 - x^2}dx$$

hence

$$(x^n, x^m) = C_{\frac{n+m}{2}}$$

$U_0(x) = 1$

$U_1(x) = x$

$U_2(x) = x^2 - 1$

$U_3(x) = x^3 - 2x$

$U_4(x) = x^4 - 3x^2 + 1$

$U_5(x) = x^5 - 4x^3 + 3x$

$U_6(x) = x^6 - 5x^4 + 4x^2 - 1$

$U_7(x) = x^7 - 6x^5 + 5x^3 - 4x$
Representations of $sl(2)$

- All finite dimensional representations of $sl(2)$ are completely reducible

**Def.** $Rep(sl(2))$ the Grothendieck ring of $sl(2)$, generated by symbols $[V]$ corresponding to representations $V$ satisfying:

$$[V \oplus W] = [V] + [W] \quad (2)$$
$$[V \otimes W] = [V] \cdot [W] \quad (3)$$

- Basis: $[V_0],[V_1],\ldots,[V_n],\ldots$
- Multiplication: $1 = [V_0]$

$$[V_n][V_m] = [V_n \otimes V_m] = \sum_{k=|n-m|,\text{parity}}^{n+m} [V_k] \quad (4)$$
Choose a different basis: 1, \([V_1], [V_1 \otimes 2], \ldots\)

\[x^n = [V_1 \otimes^n] = [V_1]^n\]

\[\text{Rep}(sl(2)) \cong \mathbb{Z}[x]\]

**Correspondence**

Monomials \(x^n \leftrightarrow V_1 \otimes^n\)

Chebyshev polynomials \(U_n(x) \leftrightarrow V_n\)

**Examples:** \(V_1 \otimes^2 \cong V_2 \otimes V_0\)

\[
[V_2] = [V_1]^2 - [V_0] \\
U_2(x) = x^2 - 1
\]
**Goal:** another categorification of $\mathbb{Z}[x]$

- non-semisimple
- such that $\{x^n\}_{n \geq 0}$, $\{U_n(x)\}_{n \geq 0}$ correspond to natural objects.

$\text{Hom}(V_1^\otimes n, V_1^\otimes m)$ has a pictorial interpretation via Temperley-Lieb algebra and its relatives.

Basis in $\text{Hom}(V_1^\otimes n, V_1^\otimes m)$ given by crossingless $(n, m)$-matchings.
\[ = 2 \text{ isotopy invariance} \]

\[ V_1^\otimes 2 \to V_0 \]

\[ V_0 \to V_1^\otimes 2 \]

**Quantum deformation**

\[ = q + q^{-1} \text{ Jones polynomial} \]

**Another deformation: maximally degenerate non-semisimple.**

If \( = \alpha \) then \( e = \frac{1}{\alpha} \)

\[ e^2 = \frac{1}{\alpha^2} \]

\[ = \frac{1}{\alpha} e. \]

Remove idempotents: the analogue of \( V_1^\otimes n \) becomes indecomposable.
Algebra $A^c$

$nA_m^c$ a $k$-vector space with basis $nB_m$. 
Algebra $A^c$

Multiplication: $nA^c_m \times mA^c_l \rightarrow nA^c_l$

Analogous to the SLarc case, on the level of pictures, multiplication is just a horizontal composition of diagrams, when number of endpoints match, satisfying relations:

\[
\begin{align*}
\begin{array}{c}
\text{circle} \\
= 0
\end{array}
\quad & \quad \begin{array}{c}
\text{horizontal composition} \\
= 0
\end{array}
\quad & \quad \begin{array}{c}
\text{diagram composition} \\
= 0
\end{array}
\end{align*}
\]

Get an associative ring $A^c = \bigoplus_{m,n \geq 0} nA^c_m$

$A^c$ is a non-unital distributed ring: $\{1_n\}_{n \geq 1}$ are mutually orthogonal idempotents.
Standard modules

$$M_n = \bigoplus_{m \geq 0} 1_m M_n$$
where $1_m M_n$ has basis of diagrams in $m B_n^c$ without returns on the right.

Action of $A^c$: Composition with the additional condition: if a diagram contains right return it equals zero.
On the level of Grothendieck group we have:

\[
[M_n] = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} [P_{n-2k}]
\]

\[
U_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}
\]

\[
K_0(A^c) \cong Z[x]
\]

\[
[P_n] = x^n
\]

\[
[M_n] = U_n(x)
\]

Unlike \( sl(2) \) case, where \( P_n \) corresponds to \([V_1^\otimes n] \), \( P_n \) are indecomposable so the category is non-semisimple.
Hermite Polynomials

There are a few equivalent ways of defining Hermite polynomials:

- Rodrigues’s representation

\[ H_n(x) = (-1)^n e^{x^2/2} \frac{\partial^n}{\partial x^n} e^{-x^2/2}. \]

- \( H_n(x) \) is the unique degree \( n \) polynomial with the top coefficient one and orthogonal to \( x^m \) for all \( 0 \leq m < n \) with respect to the inner product

\[
(f, g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(x)e^{-\frac{x^2}{2}} \, dx
\]

- \((x^m, x^n) = (n + m - 1)!!\)
$H_n(x)$ contains only powers of $x$ of the same parity as $n$. For small values of $n$ the Hermite polynomials are:

\[
\begin{align*}
H_0(x) &= 1, \\
H_1(x) &= x, \\
H_2(x) &= x^2 - 1, \\
H_3(x) &= x^3 - 3x, \\
H_4(x) &= x^4 - 6x^2 + 3, \\
H_5(x) &= x^5 - 10x^3 + 15x, \\
H_6(x) &= x^6 - 15x^4 + 45x^2 - 15.
\end{align*}
\]

\[
H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k u_{n,k} x^{n-2k}
\]

\[
x^n = \sum_{k=0}^{n/2} u_{n,k} H_{n-2k}(x).
\]

where $u_{n,k} = \binom{n}{n-2k} (2k-1)!! = \frac{n!}{2^k k!(n-2k)!}$
Diagrammatics for the categorification of $H_n(x)$

- Each arc is simple, i.e. without self-intersections.
- Each pair of arcs has at most one intersection.
- Allow only isotopies that preserve these conditions and triple intersections of three distinct arcs are allowed during isotopies.
Categorification of Hermite polynomials

Projective module \( P_n \leftrightarrow x^n \)
Big standard module \( \tilde{M}_n \leftrightarrow H_n(x) \)
Standard module \( M_n \leftrightarrow \frac{H_n(x)}{n!} \)
References and future directions

• Generalize to the categorification of other classes of orthogonal polynomials.

• Topological interpretation of the Bernstein–Gelfand–Gelfand reciprocity property

• Find a categorical lifting of more complicated parts of the orthogonal polynomials theory.

• Categorification of Knot and Graph Polynomials and the Polynomial Ring, GWU Electronic dissertation published by ProQuest, 2010 http://surveyor.gelman.gwu.edu/

• arXiv:1101.0293

THANK YOU