A THREE LAYER NEURAL NETWORK CAN REPRESENT ANY MULTIVARIATE FUNCTION

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Abstract. In 1987, Hecht-Nielsen showed that any continuous multivariate function can be implemented by a certain type three-layer neural network. This result was very much discussed in neural network literature. In this paper we prove that not only continuous functions but also all discontinuous functions can be implemented by such neural networks.

1. Introduction

Over the past 30 years, the topic of artificial neural networks has been a vibrant area of research. Neural networks are powerful computation devices, which have applications in many fields and problem domains. Application areas range from medicine to petroleum science and geology. In fact neural networks are introduced in any situation where there are problems of prediction, classification or control. Undoubtedly the greatest advantage of neural networks is their ability to be used as an arbitrary function approximation and/or implementation mechanism. In the present article, we are interested in the question of precise representation of multivariate functions by neural networks.

It should be remarked that one of the pioneering papers in neural network theory is the 1987 paper by Hecht-Nielsen [5]. This paper gained tremendous attention of many researchers during the following decades after its publication. The main result of [5] is based on the Kolmogorov Superposition Theorem [15]. It shows that any continuous multivariate function can be implemented by a special three-layer neural network as follows.

Theorem 1 (Hecht-Nielsen [5]). Given a natural number \( d > 1 \) and any continuous function \( f : \mathbb{I}^d \rightarrow \mathbb{R}, y = f(x) \), where \( \mathbb{I} \) is the closed unit interval \([0, 1]\). Then \( f \) can be implemented exactly by three-layer neural network having \( d \) processing elements \( x = (x_1, \ldots, x_d) \) in the first (input) layer, \( 2d + 1 \) processing elements in the middle layer and a processing element \( y \) in the top (output) layer.

The processing elements on the first layer simply distribute the input \( x \)-vector components to the processing elements of the second layer.

The processing elements of the second layer implement the following transfer functions:

\[
z_k = \sum_{j=1}^{d} \lambda^{j-1} \phi(x_j + \epsilon k), \ k = 0, \ldots, 2d,
\]

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where $\phi$ is a universal monotonic Lipschitz function and $\lambda, \epsilon$ are nonzero constants. These $\phi, \lambda$, and $\epsilon$ are independent of $f$. Moreover, the constant $\epsilon$ can be chosen arbitrarily.

The top layer processing element $y$ have the following transfer function:

$$y = \sum_{k=0}^{2d} g_k(z_k),$$

where the functions $g_k$ are real continuous and depend on $f$.

This result is a neural network interpretation of Kolmogorov’s superposition theorem in the form given by Sprecher in [27]. For a comprehensive discussion of this remarkable theorem see the book by Khavinson [14]. A neural network, existence of which is asserted in Theorem 1, is called Kolmogorov’s mapping neural network.

Note that the original theorem of Hecht-Nielsen uses Sprecher’s earlier result from [26]. We have formulated this theorem using Sprecher’s later result [27], which was also extensively discussed in a number of subsequent papers (see, e.g., [2, 17, 28, 29, 30]).

Although the above theorem completely characterizes the power of feedforward neural networks, it was considered by some authors as non-constructive (see, e.g., [3]). Hecht-Nielsen himself wrote in [5] that the theorem tells us that such a three-layer mapping network must exist, but it doesn’t tell us how we can construct it. Nevertheless, Hecht-Nielsen’s theorem stimulated the further research concerning the role of Kolmogorov superposition theorem in neural network theory, which is still active today (see, e.g., [12, 21, 24, 25]). The research on this subject was carried out mainly in two directions. In the first direction, the analysis was concentrated on approximative versions of Kolmogorov’s theorem and similar results on feedforward neural networks (see, e.g., [4, 9, 11, 18, 19, 20, 23]). In the second direction, the precise form of the Kolmogorov superposition theorem and its relationship to neural networks were studied. Hecht-Nielsen’s expectations that “more will be learned about the Kolmogorov mapping network in the years to come” were met by series of works of Sprecher [13, 28, 29, 30]. Due to these works, there is a perspective for a practical usage of the exact representation of continuous functions by Kolmogorov type neural networks. In fact, such a perspective stems from the fact that Sprecher’s function $\phi$, which determines the processing units of the middle layer, could be computed algorithmically (see, e.g., [2, 17, 29]).

Note that the external activation functions $g_k$ in the output layer depend on $f$ and have to be determined by learning procedures. Some practically useful learning algorithms for such networks were discussed in [22]. By using cubic spline technique of approximation, both for external and internal functions in Kolmogorov type networks, more efficient approximation of multivariate functions was achieved (see [6]). In [1], Brattka obtained a computable version of Hecht-Nielsen’s theorem: Every computable multivariate function can be implemented by a three-layer neural network with computable activation functions and computable weights.

It is well known that in nature most functional dependencies are not continuous. Regarding complicated discontinuous functional dependencies the following question arises: can discontinuous functions be implemented by Kolmogorov type
neural networks? In this paper we prove that the answer to this fair and interesting question is positive. More precisely, we prove that Hecht-Nielsen’s theorem can be extended from the class of continuous multivariate functions to the class of all multivariate functions.

2. Main result

The following theorem is valid.

**Theorem 2.** Given a natural number \( d > 1 \) and any function \( F : \mathbb{I}^d \to \mathbb{R}, \ y = F(x), \) where \( \mathbb{I} \) is the closed unit interval \([0,1]\). Then \( F \) can be implemented exactly by three-layer neural network having \( d \) processing elements \( x = (x_1, ..., x_d) \) in the first layer, \( 2d+1 \) processing elements in the middle layer and a single processing element \( y \) in the top layer.

The processing elements on the first layer simply distribute the input \( x \)-vector components to the processing elements of the second layer.

The processing elements of the second layer implement the following transfer functions:

\[
z_k = \sum_{j=1}^{d} \lambda^{j-1} \phi(x_j + \epsilon k), \quad k = 0, ..., 2d,
\]

where the universal monotonic Lipschitz function \( \phi \) and the nonzero constants \( \lambda, \epsilon \) are independent of \( F \). Moreover, the constant \( \epsilon \) can be chosen arbitrarily.

The top layer processing element \( y \) have the following transfer function:

\[
y = \sum_{k=0}^{2d} h_k(z_k),
\]

where the functions \( h_k \) are real and depend on \( F \).

One can observe that the only difference between conclusions of Theorems 1 and 2 are in functions \( g_k \) and \( h_k \). Sprecher’s function \( \phi \) and the constants \( \lambda, \epsilon \) and the number of layers and units in these layers are the same in both theorems. Based on Theorem 2, we can say that not only continuous functions but also all discontinuous functions can be implemented by Kolmogorov’s mapping neural network. Our proof is based on methods and principles of Functional Analysis.

**Proof.** For simplicity of notation put \( r = 2d \). Note that by Theorem 1 every continuous function \( f : \mathbb{I}^d \to \mathbb{R} \) has the form

\[
f(x) = \sum_{k=0}^{r} g_k(z_k(x)). \tag{1}
\]

Using this, we show below that the family of the transfer functions \( z_0, ..., z_r \) satisfies the following condition, which we call Condition (Z):

**Condition (Z):** There is no finite subset \( \{x_1, ..., x_n\} \subset \mathbb{I}^d \) with the property that

\[
\sum_{j=1}^{n} \mu_j \delta_{z_k(x_j)}(t) = 0, \quad k = 0, ..., r, \tag{2}
\]
for some nonzero real numbers $\mu_1, \ldots, \mu_n$ and for any $t \in \mathbb{R}$. Here $\delta_a$ stands for the indicator function of a single point set $\{a\}$. That is,

$$\delta_a(t) = \begin{cases} 
1, & \text{if } t = a \\
0, & \text{if } t \neq a.
\end{cases}$$

Let us first explain Eq. (2) in detail. We see that it stands for a system of certain linear equations. Fix the subscript $k$. Let the set $\{z_k(x_j), j = 1, \ldots, n\}$ have $s_k$ different values, which we denote by $\gamma^k_1, \gamma^k_2, \ldots, \gamma^k_{s_k}$. Take the first number $\gamma^k_1$. Putting $t = \gamma^k_1$, we obtain from (2) that

$$\sum_j \mu_j = 0,$$

where the sum is taken over all $j$ such that $z_k(x_j) = \gamma^k_1$. This is the first linear equation in $\mu_j$ corresponding to $\gamma^k_1$. Take now $\gamma^k_2$. By the same way, putting $t = \gamma^k_2$ in (2), we can form the second equation. Continuing until $\gamma^k_{s_k}$, we obtain $s_k$ linear homogeneous equations in $\mu_1, \ldots, \mu_n$. The coefficients of these equations are the integers 0 and 1. By varying $k$, we finally obtain $s = \sum_{k=0}^{r} s_k$ such equations. Hence (2), in its expanded form, stands for the system of these linear equations.

Thus Condition (Z) means that no system of linear equations of the form (2) has a solution with nonzero components.

It should be remarked that finite sets $\{x_1, \ldots, x_n\}$ satisfying (2) with respect not only the transfer functions $z_k$ but arbitrary multivariate functions were exploited under the name of “closed paths” in several papers of the author (see, e.g., [7, 8, 10]).

Let us now show that if representation (1) is valid for every continuous $f$, then Condition (Z) holds. Assume the contrary. Assume that there is a finite set $p = \{x_1, \ldots, x_n\}$ in $I^d$ with the property (2). Consider the following linear functional

$$G_p(f) = \sum_{j=1}^{n} \mu_j f(x_j).$$

It is not difficult to see that this functional annihilates all sums of the form $\sum_{k=0}^{r} g_k(z_k(x))$, and hence, by the representation (1), every continuous function $f$ on $I^d$. That is, $G_p(f) = 0$ for any $f \in C(I^d)$. On the other hand by Urysohn’s well-known lemma (see, e.g., [10]) there exists a continuous function $f_0$ with the property: $f_0(x_j) = 1$ for indices $j$ such that $\mu_j > 0$; $f_0(x_j) = -1$ for indices $j$ such that $\mu_j < 0$; and $-1 < f_0(x) < 1$ for $x \in I^d \setminus p$. For this function $G_p(f_0) = \sum_{j=1}^{n} |\mu_j| \neq 0$. The obtained contradiction means that Condition (Z) holds for the transfer functions $z_0, \ldots, z_r$.

Now we are going to prove that if Condition (Z) holds for any family of real functions $w_k : I^d \to \mathbb{R}, k = 0, \ldots, r$, with pairwise disjoint ranges, then any multivariate (not necessarily continuous) function $F : I^d \to \mathbb{R}, y = F(x)$, possess the representation

$$F(x) = \sum_{k=0}^{r} s_k(w_k(x)), \quad (3)$$

where the functions $s_k : \mathbb{R} \to \mathbb{R}$ depend on $F$. 
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Introduce the notation

\[ Y_k = w_k(I^d), \quad k = 0, ..., r; \]
\[ \Omega = Y_0 \cup ... \cup Y_r. \]

By our assumption, \( Y_k \), the ranges of the \( w_k \), are pairwise disjoint sets. That is, \( Y_i \cap Y_j = \emptyset \), for all \( i, j \in \{0, ..., r\}, i \neq j \).

Consider the following set

\[ L = \{ Y \in \{ y_0, ..., y_r \} : \text{if there exists } x \in I^d \text{ s.t. } w_k(x) = y_k, \quad k = 0, ..., r \} \]  

Note that \( L \) is not a subset of \( \Omega \). It is a set of some certain subsets of \( \Omega \). Each element of \( L \) is a set \( Y = \{ y_0, ..., y_r \} \subset \Omega \) with the property that there exists at least one point \( x \in I^d \) such that \( w_k(x) = y_k, k = 0, ..., r \). These \( x \) will be called generating points for \( Y \).

It is not difficult to understand that in (4) for each element \( Y \) there exists only one point \( x \in I^d \). This is because if there are two points \( x_1 \) and \( x_2 \) for a single \( Y \) in (4), then \( w_k(x_1) = w_k(x_2), k = 0, ..., r \), and hence for the set \( \{x_1, x_2\} \subset I^d \) we have

\[ 1 \cdot \delta_{w_k(x_1)} + (-1) \cdot \delta_{w_k(x_2)} \equiv 0, \quad k = 0, ..., r. \]

But this contradicts the assumption that Condition (Z) holds for the functions \( w_0, ..., w_r \). Thus we see that the unicity property of generating points holds for each \( Y \) in \( L \).

Since we already know that in (4) for each \( Y \in L \) there exists only one point \( x \in I^d \), we can define the function

\[ t : L \to \mathbb{R}, \quad t(Y) = F(x), \]

where \( x \) is the generating point for \( Y \).

Consider now a class \( S \) of functions of the form \( \sum_{j=1}^{m} r_j \delta_{D_j} \), where \( m \) is a positive integer, \( r_j \) are real numbers and \( D_j \) are elements of \( L \), \( j = 1, ..., m \). We fix neither the numbers \( m, r_j \), nor the sets \( D_j \). Clearly, \( S \) is a linear space. Over \( S \), we define the functional

\[ H : S \to \mathbb{R}, \quad H \left( \sum_{j=1}^{m} r_j \delta_{D_j} \right) = \sum_{j=1}^{m} r_j t(D_j). \]

First of all, we must show that this functional is well defined. That is, once we have the equality

\[ \sum_{j=1}^{m_1} r_j' \delta_{D_j'} = \sum_{j=1}^{m_2} r_j'' \delta_{D_j''}, \]  

we also have the equality

\[ \sum_{j=1}^{m_1} r_j' t(D_j') = \sum_{j=1}^{m_2} r_j'' t(D_j''). \]

But in fact equality (5) can never hold. Suppose the contrary. Suppose that (5) holds. Let us write (5) in the equivalent form

\[ \sum_{j=1}^{m} r_j \delta_{D_j} = 0. \]
Each set $D_j$ consists of $r+1$ real numbers $y^j_0, \ldots, y^j_r, j = 1, \ldots, m$. By our assumption concerning the ranges of the $w_k(x)$, all these numbers are different. Therefore,

$$
\delta_{D_j} = \sum_{i=0}^{r} \delta_{y^j_i}, \quad j = 1, \ldots, m. \tag{7}
$$

Eq. (7) together with Eq. (6) give

$$
\sum_{i=0}^{r} \sum_{j=1}^{m} r_j \delta_{y^j_i} = 0. \tag{8}
$$

Since the sets \{\(y^1_i, y^2_i, \ldots, y^m_i\), \(i = 0, \ldots, r\), are pairwise disjoint, we obtain from (8) that

$$
\sum_{j=1}^{m} r_j \delta_{y^j_i} = 0, \quad i = 0, \ldots, r. \tag{9}
$$

Let now \(x_1, \ldots, x_m\) be generating points for the sets \(D_1, \ldots, D_m\), respectively. Since by (4), \(y^j_k = w_k(x_j)\), for \(k = 0, \ldots, r\), and \(j = 1, \ldots, m\), it follows from (9) that the set \{\(x_1, \ldots, x_m\}\} has property (2), hence Condition (Z) is violated. The obtained contradiction means that Eq. (6), hence Eq. (5) can never hold. Thus, the functional $H$ is well defined. Note that this functional is linear, which can easily be seen from its definition.

Consider now the following space:

$$
S' = \left\{ \sum_{j=1}^{m} r_j \delta_{\omega_j} \right\},
$$

where \(m \in \mathbb{N}, r_j \in \mathbb{R}, \omega_j \subset \Omega\). As above, we do not fix the parameters \(m, r_j\) and \(\omega_j\). Clearly, the space $S'$ is larger than $S$. Let us prove that the functional $H$ can be linearly extended to the space $S'$. So, we must prove that there exists a linear functional $H' : S' \to \mathbb{R}$ such that $H'(x) = H(x)$, for all $x \in S$. Let $A$ denote the set of all linear extensions of $H$ to subspaces of $S'$ containing $S$. The set $A$ is not empty, since it contains the functional $H$. For each functional $v \in A$, let $dom(v)$ denote the domain of $v$. Consider the following partial order in $A$: $v_1 \leq v_2$, if $v_2$ is a linear extension of $v_1$ from the space $dom(v_1)$ to the space $dom(v_2)$. Let now $P$ be any chain (linearly ordered subset) in $A$. Consider the following functional $u$ defined on the union of domains of all functionals $p \in P$:

$$
u : \bigcup_{p \in P} dom(p) \to \mathbb{R}, \quad u(x) = p(x), \quad \text{if} \ x \in dom(p).$$

Obviously, this functional is well defined and linear. Besides, the functional $u$ provides an upper bound for $P$. We see that the arbitrarily chosen chain $P$ has an upper bound. Then by Zorn’s lemma (see, e.g., [16]), there is a maximal element $H' \in A$. We claim that the functional $H'$ must be defined on the whole space $S'$. Indeed, if $H'$ is defined on a proper subspace $D \subset S'$, then it can be linearly extended to a space larger than $D$ by the following way: take any point $x \in S' \setminus D$ and consider the linear space $D' = \{D + \alpha x\}$, where $\alpha$ runs through all real numbers. For an arbitrary point $y + \alpha x \in D'$, set $H'(y + \alpha x) = H'(y) + \alpha b$, where $b$ is any real number considered as the value of $H'$ at $x$. Thus, we constructed a linear functional $H'' \in A$ satisfying $H' \leq H''$. The last contradicts the maximality of $H'$.
This means that the functional $H'$ is defined on the whole $S'$ and $H \leq H'$ ($H'$ is a linear extension of $H$).

Define the following functions by means of the functional $H'$:

$$s_k : Y_k \to \mathbb{R}, \quad s_k(y_k) \overset{\text{def}}{=} H'(\delta_{y_k}), \quad k = 0, ..., r.$$  

Let $x$ be an arbitrary point in $\mathbb{R}^d$. Obviously, $x$ is a generating point for some set $Y = \{y_0, ..., y_r\} \subset \mathcal{L}$. Thus,

$$F(x) = t(Y) = H(\delta_Y) = H\left(\sum_{k=0}^{r} \delta_{y_k}\right) = H'\left(\sum_{k=0}^{r} \delta_{y_k}\right) = \sum_{k=0}^{r} H'(\delta_{y_k}) = \sum_{k=0}^{r} s_k(y_k) = \sum_{k=0}^{r} s_k(w_k(x)).$$

Thus we have proven (3). We stress again that Eq. (3) is valid for any family of real functions $w_k : I^d \to \mathbb{R}, \quad k = 0, ..., r$, which have disjoint ranges and satisfy Condition (Z).

Let us now return to the transfer functions $z_k$. Consider a system of intervals $\{(a_k, b_k) \subset \mathbb{R}\}_{k=0}^{r}$ such that $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for all the indices $i, j \in \{0, ..., r\}$, $i \neq j$. For $k = 0, ..., r$, let $\tau_k$ be one-to-one mappings of $\mathbb{R}$ onto $(a_k, b_k)$. Introduce the following functions on $I^d$:

$$w_k(x) = \tau_k(z_k(x)), \quad k = 0, ..., r.$$  

It is clear that (2) holds if and only if

$$\sum_{j=1}^{n} \mu_j \delta_{w_k(x)} = 0, \quad k = 0, ..., r.$$  

This means that in addition to the transfer functions $z_0, ..., z_r$, Condition (Z) are also valid for the functions $w_0, ..., w_r$.

Note that these new functions $w_k(x)$ have pairwise disjoint ranges. That is, $w_i(I^d) \cap w_j(I^d) = \emptyset$, for all $i, j \in \{0, ..., r\}$, $i \neq j$. Then by Eq. (3) we can write that

$$F(x) = \sum_{k=0}^{r} s_k(w_k(x)) = \sum_{k=0}^{r} s_k(\tau_k(z_k(x))) = \sum_{k=0}^{r} h_k(z_k(x)), \quad (10)$$

where $h_k = s_k \circ \tau_k, \quad k = 0, ..., r$, are real univariate functions depending on $F$. The obtained Eq. (10) proves the theorem.

3. Conclusion

Most multivariate functions that exist in nature and we see in practice are generally not continuous. Although artificial neural networks were proved by many authors to have the capability of representing and approximating all continuous functions, their power to characterize discontinuous functions was not known. This paper shows that Kolmogorov’s mapping three-layer neural network can precisely represent all discontinuous multivariate functions.

It should be remarked that Theorem 2, like Hecht-Nielsen’s theorem, is strictly an existence result. It only states that neural networks implementing discontinuous functions exist and can be obtained by using Kolmogorov’s superposition theorem for continuous functions. The direct application of Theorem 2 to practical problems is doubtful, since our method for determining the functions $h_k$ is substantially based
on Zorn’s lemma and hence highly nonconstructive. However, we hope that efficient learning algorithms for such networks will be developed in the future.

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