Entanglement detection: Linear entropy versus Bell-CHSH inequality

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Abstract

The relation between the violation of the Bell-CHSH inequalities and entanglement properties of quantum states is not clear so one may consider the mixedness of the system to understand the entanglement properties better than the Bell-CHSH inequality. In this respect, we prove that if the mixedness of the state measured by the linear entropy is less than $\frac{2}{3}$ but strictly greater than zero then the two qubit states are entangled. But if the linear entropy is greater or equal to $\frac{2}{3}$ then the state may or may not be entangled. Further we show that our entanglement criterion detects larger set of entangled state than Bell-CHSH inequality and Santos’s entropic criterion [Phys. Rev. A 69, 022305 (2004)]. Lastly we illustrate our result by citing few examples.

Quantum entanglement is one of the fascinating feature of quantum mechanics. There is no classical analog of quantum entanglement and that makes it more fascinating than anything else in physics. In the field of quantum information theory entanglement plays a major role. This is also a very useful resource in the sense that using entanglement one can do many things in the quantum world which are usually impossible in ordinary classical world. Some of these tasks are quantum computing [1], quantum teleportation [2], quantum cryptography [3].

The entanglement detection problem is a very genuine and challenging task in quantum information theory. Researchers accept this challenge and proposed many entanglement detection methods by which we could detect the presence of entanglement in a given system. The first successful candidate was J. S. Bell [4] who proposed an entanglement detection scheme (now known as Bell’s inequality), in 1964, when studying the Einstein-Podolsky-Rosen (EPR) paradox [5]. After that many modifications of the original Bell inequality were proposed. Among all these Clauser, Horne, Shimony, Holt (CHSH) inequality [6] is the famous one. In 2002, D. Collins et.al. generalizes the Bell-CHSH inequality for arbitrary d-dimensional systems. This inequality is popularly known as the Collins-Gisin-Linden-Massar-Popescu inequality [7]. Bell inequalities for multipartite arbitrary dimensional system is also studied [8].
Local realism implies constraints on the statistics of two or more physically separated systems. These constraints, called Bell inequalities, can be violated by the statistical predictions of quantum mechanics. A typical Bell-inequality for bipartite two qubits system was derived by Clauser, Horne, Shimony, Holt, allowing more flexibility in local measurement configuration than the original Bell inequality. The Bell-CHSH inequality read as

\[ I_{CHSH} = \langle A_1 B_1 \rangle_{\rho} + \langle A_1 B_2 \rangle_{\rho} + \langle A_2 B_1 \rangle_{\rho} - \langle A_2 B_2 \rangle_{\rho} \leq 2 \]  

where \( \langle A_i B_j \rangle_{\rho} = Tr[\rho(\hat{a}_i \cdot \vec{\sigma}^A)(\hat{b}_j \cdot \vec{\sigma}^B)] \) known as the so-called correlation functions, \( \rho \) is the two-qubit state shared by A and B, \( \vec{\sigma} \) is the Pauli matrix vector, \( \hat{a}_1 \) and \( \hat{a}_2 \) are the unit vectors for the first and the second measurements performed to the subsystem A respectively and so do \( \hat{b}_1 \) and \( \hat{b}_2 \) for the subsystem B. The Bell-CHSH inequality has many merits [9]: (i) It is tight, i.e. it defines one of the facets of the convex polytope of local-realistic (LR) models (ii) It is violated by all the pure two qubit entangled states (iii) It is maximally violated by maximally entangled states. Thus the detection of entanglement in a pure two-qubit system via Bell-CHSH inequality is totally solved. On the contrary if the given system is mixed then Bell-CHSH inequality solves the entanglement problem partially even in two qubit system. The entanglement problem is not completely solved via Bell-CHSH inequality in case of mixed state because of the existence of some entangled states which satisfies the inequality. In this regard Werner [10] proposed a class of mixed spin-\( \frac{1}{2} \) state which satisfies the Bell-CHSH inequality although it is entangled. Thus Bell-CHSH inequality does not detect all entangled state. But if the inequality is violated by mixed state then the state is surely entangled. Though Bell-CHSH inequality fails to detect all two qubits mixed entangled states, it is considered as the most efficient one because until 2004 there was no example of a quantum state not violating the CHSH inequality but violating some other Bell-Inequalities [11]. Later Andás [12] showed that a convex combination of product spin-\( \frac{1}{2} \) state does not violate Bell inequalities for the generalised Bell type observables. Since we are now discussing about the violation of the Bell-CHSH inequality by a given mixed state so in this respect we should mention here that it was hard to say whether a given state violates the CHSH inequality because one had to construct a respective Bell operator for it. So to make this hard task easy, Horodecki family provided an effective criterion (necessary and sufficient condition) for violating the Bell type inequalities by mixed spin-\( \frac{1}{2} \) state [13]. The statement of the criterion is as follows:

**Horodecki criterion [13]**: The two qubit density matrix \( \rho \) violates CHSH inequality for some Bell operator of the form \( B_{CHSH} = \hat{a} \cdot \vec{\sigma} \otimes (\hat{b} + \hat{b}'), \hat{a}' \cdot \vec{\sigma} \otimes (\hat{b} - \hat{b}'), \vec{\sigma} \) iff

\[ M(\rho) = \max_{i>j} (\lambda_i + \lambda_j) > 1 \]  

where \( \hat{a}, \hat{a}', \hat{b}, \hat{b}' \) are unit vectors in \( R^3 \) and \( \lambda_i \)'s are the eigenvalues of the symmetric matrix \( C^T \rho C(\text{T stands for transposition}) \).

Since the relation between the violation of the Bell-CHSH inequalities and entanglement
properties of quantum states is not clear so one may consider the mixedness of the system to understand the entanglement properties better than the Bell-CHSH inequality. It is a known phenomenon that as the mixedness of the system increases, the entanglement decreases. Naturally a question arises: Does there exist any upper bound of the mixedness of the given system up to which the entanglement is stayed in that system and beyond that the entanglement is totally lost? E. Santos [14] studied this problem to some extent and showed that if the linear entropy, which measures the mixedness of the system, is less than $\frac{1}{2}$ then there are states which are entangled. But the bound for the linear entropy given by Santos is weak in the sense that it only detects entangled states which are also detected by Bell-CHSH inequality. Bose and Vedral also studied this type of problem. They gave a lower bound for the von-Neumann entropy and linear entropy and then showed that if the entropy exceeds the given bounds then those states cannot be used as a teleportation channel [15]. In this letter, we provide an upper bound to the mixedness of the state measured by linear entropy and show that the two-qubit states whose mixedness less than the given upper bound are entangled. Our result is interesting in the sense that it detects a larger set of entangled state than any Bell-CHSH inequality and Santos’s entropic criterion.

**Theorem:** The two qubit mixed density matrix $\rho$ is entangled iff

$$S_L(\rho) < \frac{2}{3}$$

where $S_L(\rho)$ is the linear entropy of the density matrix $\rho$.

**Proof:** Any arbitrary state on $H = C^2 \otimes C^2$ can be represented in a Hilbert-Schmidt basis as follows [16]:

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} r_i \sigma_i \otimes I + I \otimes \sum_{i=1}^{3} s_i \sigma_i + \sum_{i,j=1}^{3} c_{ij} \sigma_i \otimes \sigma_j)$$

(4)

where $I$ represents the $2 \times 2$ identity matrix, $\sigma_i \ (i = 1, 2, 3)$ denotes the standard Pauli matrices. The coefficients $r_i$ and $s_i$ are given by

$$r_i = Tr(\rho \sigma_i \otimes I), \quad s_i = Tr[\rho (I \otimes \sigma_i)] \quad (i = 1, 2, 3)$$

(5)

The coefficients $c_{ij}$ form a real matrix which we call as $C_\rho$ and the elements of the matrix can be evaluated by the formula

$$c_{ij} = Tr(\rho \sigma_i \otimes \sigma_j)$$

(6)

The state (1) is pure or mixed according as $Tr(\rho^2) = 1$ or $Tr(\rho^2) < 1$, where

$$Tr(\rho^2) = \frac{1}{4}(1 + \sum_{i=1}^{3} r_i^2 + \sum_{i=1}^{3} s_i^2 + \sum_{i,j=1}^{3} c_{ij}^2)$$

(7)
For a quantum state \( \rho \) in a 4-dimensional Hilbert space \( H \) the linear entropy is defined as follows [17]:

\[
S_L(\rho) = \frac{4}{3}(1 - Tr(\rho^2))
\]  

(8)

Using eq.(7) and eq.(8), we get

\[
S_L(\rho) = 1 - \frac{1}{3}(\sum_{i=1}^{3}(r_i^2 + s_i^2) + \sum_{i,j=1}^{3}c_{ij}^2)
\]

\[\Rightarrow M(\rho) \leq Q(\rho) \]  

(9)

where we introduce the function \( Q(\rho) \) as

\[
Q(\rho) = 3(1 - S_L(\rho)) - \sum_{i=1}^{3}(r_i^2 + s_i^2)
\]

(10)

In eq.(9), we have used the inequality

\[
M(\rho) = \max_{i>j}(\lambda_i + \lambda_j) \leq \sum_{i,j=1}^{3}c_{ij}^2
\]

(11)

Now to prove the theorem, we have to consider two cases. In the first case we considered those states which violates the Bell-CHSH inequality and in the second case we look for those states which satisfies the Bell-CHSH inequality but violate generalised Bell-CHSH inequality.

Case-I: \( M(\rho) > 1 \).
In this case the density matrix \( \rho \) violates the Bell-CHSH inequality and hence entangled. Therefore, using Horodecki criterion and eq. (9), we can say that the state is entangled iff

\[
Q(\rho) > 1
\]

\[\Rightarrow S_L(\rho) < \frac{2}{3} - \frac{1}{3}\sum_{i=1}^{3}(r_i^2 + s_i^2) \leq \frac{2}{3}
\]

(12)

Case-2: \( M(\rho) \leq 1 < Q(\rho) \).
In this case the density matrix \( \rho \) satisfies the Bell-CHSH inequality. Since there exist entangled states which satisfies the Bell-CHSH inequality so it is sufficient to show that the states, for which the relation

\[
M(\rho) \leq 1 < Q(\rho)
\]

(13)
is satisfied, violates the generalised Bell-CHSH inequalities. The generalised Bell-CHSH inequality is violated iff

\[ N(\rho) > 1 \]  \hspace{0.5cm} (14)

where the function \( N(\rho) \) defined as

\[ N(\rho) := \text{Tr} \sqrt{C^T \rho C} = \sum_{i=1}^{3} \sqrt{\lambda_i} \]  

Recalling the definition of the linear entropy for the state \( \rho \), we have

\[ S_L(\rho) = 1 - \frac{1}{3} \left( \sum_{i=1}^{3} (r_i^2 + s_i^2) + \sum_{i,j=1}^{3} c_{ij}^2 \right) \]  \hspace{0.5cm} (15)

The eq. \eqref{eq:15} can be rewritten as

\[ Q(\rho) = \sum_{i,j=1}^{3} c_{ij}^2 = \text{Tr}(C^T \rho C) = \sum_{i=1}^{3} \lambda_i \]  \hspace{0.5cm} (16)

Since \( \lambda_i \leq 1 \) for \( i = 1, 2, 3 \), so

\[ Q(\rho) = \sum_{i=1}^{3} \lambda_i < \sum_{i=1}^{3} \sqrt{\lambda_i} = N(\rho) \]  \hspace{0.5cm} (17)

Combining eq. \eqref{eq:13} and eq. \eqref{eq:17}, we have

\[ M(\rho) \leq 1 < Q(\rho) < N(\rho) \]  \hspace{0.5cm} (18)

Therefore, eq. \eqref{eq:18} tells us that the state \( \rho \) which satisfies the relation \eqref{eq:13} violates the generalised Bell-CHSH inequality and hence entangled. Thus the two qubit state \( \rho \) is entangled iff the linear entropy of the state satisfies

\[ S_L(\rho) < \frac{2}{3} - \frac{1}{3} \sum_{i=1}^{3} (r_i^2 + s_i^2) \leq \frac{2}{3} \]

Hence the theorem is proved.

**Corollary-1:** The density matrix \( \rho \) is useful for teleportation iff

\[ S_L(\rho) < \frac{2}{3} \]  \hspace{0.5cm} (19)

**Proof:** From eq.\eqref{eq:15}, it is clear that the two inequalities \( Q(\rho) > 1 \) and \( N(\rho) > 1 \) simultaneously hold.

(a) \( Q(\rho) > 1 \leftrightarrow S_L(\rho) < \frac{2}{3} \) and

(b) \( N(\rho) > 1 \leftrightarrow \rho \) is useful for teleportation.

Combining the two logics (a) and (b), we can say that the states which have linear entropy less than \( \frac{2}{3} \) are useful for teleportation.
Corollary-2: (a) It may happen that there exist states for which the following inequality holds:

\[ Q(\rho) < 1 < N(\rho) \quad (20) \]

If inequality (20) holds then there exist states with linear entropy exceeds \( \frac{2}{3} \) useful for teleportation.

(b) It may also happen that there exist states for which the inequality given below holds:

\[ Q(\rho) \leq N(\rho) \leq 1 \quad (21) \]

If inequality (21) holds then there always exist some separable states with linear entropy exceeds \( \frac{2}{3} \) and hence useless for teleportation.

Corollary-2 tells us that there exist states with linear entropy greater than \( \frac{2}{3} \) which may or may not be useful for teleportation.

Let us illustrate our results with examples:

Example-1: Let us consider a two-qubit maximally entangled mixed state [20] expressed in the computational basis as follows:

\[
\rho_{MEMS} = \begin{pmatrix}
\frac{p}{2} & 0 & 0 & -\frac{p}{2} \\
0 & 1-p & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{p}{2} & 0 & 0 & \frac{p}{2}
\end{pmatrix}
\]  
(22)

where \( 0 \leq p \leq 1 \).

The real correlation matrix \( C_{\rho_{MEMS}} \) for the state \( \rho_{MEMS} \) is given by

\[
C_{\rho_{MEMS}} = \begin{pmatrix}
-p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & 2p-1
\end{pmatrix}
\]  
(23)

The eigenvalues of the symmetric matrix \( (C_{\rho_{MEMS}}^T C_{\rho_{MEMS}}) \) is given by

\[
\lambda_1 = \lambda_2 = p^2, \lambda_3 = (1-2p)^2
\]  
(24)

The quantity \( M(\rho_{MEMS}) \) is given by

\[
M(\rho_{MEMS}) = \begin{cases}
2p^2 & \text{when } \frac{1}{3} < p \leq 1 \\
5p^2 - 4p + 1 & \text{when } 0 \leq p < \frac{1}{3}
\end{cases}
\]  
(25)

From eq. (25), it can be easily found out that

\[
M(\rho_{MEMS}) \begin{cases}
> 1 & \text{when } \frac{1}{\sqrt{2}} < p \leq 1 \\
\leq 1 & \text{when } 0 \leq p \leq \frac{1}{\sqrt{2}}
\end{cases}
\]  
(26)
Therefore the state $\rho_{MEMS}$ violates the Bell-CHSH inequality when $\frac{1}{\sqrt{2}} < p \leq 1$ and satisfies the inequality when $0 \leq p \leq \frac{1}{\sqrt{2}}$. Now the linear entropy $S_L(\rho_{MEMS})$ is given by

$$S_L(\rho_{MEMS}) = \frac{8p(1-p)}{3}$$  \hspace{1cm} (27)

The function $\frac{8p(1-p)}{3}$ is symmetric with respect to the line $p = \frac{1}{2}$. The function is increasing when $0 \leq p \leq \frac{1}{2}$ and decreasing when $\frac{1}{2} \leq p \leq 1$. The maximum is attained at $p = \frac{1}{2}$. Therefore, we find that

$$S_L(\rho_{MEMS}) < \frac{2}{3}, \quad \text{when} \quad 0 < p \leq 1$$  \hspace{1cm} (28)

Using our theorem we conclude that the state $\rho_{MEMS}$ is entangled for all values of the parameter $p$ except zero. This may also be verified by calculating the concurrence [21] for the state $\rho_{MEMS}$ which in this case is found out to be $p$. Therefore the state is separable only when $p = 0$, otherwise entangled.

Therefore our linear entropic criterion detect entanglement when $0 < p \leq 1$ while Bell-CHSH inequality detect entanglement only when $\frac{1}{\sqrt{2}} < p \leq 1$. Our criterion thus detects larger set of entangled states than Bell-CHSH inequality.

**Example-2:** In example-1, we find that the maximum value of the linear entropy $S_L$ is $\frac{2}{3}$ when $p = \frac{1}{2}$. Now it is very likely to consider the states with $S_L > \frac{2}{3}$ because those states may or may not be entangled. Thus we consider this example to emphasize on the region $S_L > \frac{2}{3}$ by taking a family of Werner state, which can be expressed in the form:

$$\rho_W = \begin{pmatrix}
\frac{1+r}{4} & 0 & 0 & \frac{r}{2} \\
0 & \frac{1-r}{4} & 0 & 0 \\
\frac{r}{2} & 0 & 0 & \frac{1+r}{4} \\
\frac{1+r}{4} & 0 & 0 & \frac{1-r}{4}
\end{pmatrix}$$   \hspace{1cm} (29)

where $0 \leq r \leq 1$.

The linear entropy for $\rho_W$ is given by

$$S_L(\rho_W) = 1 - r^2$$  \hspace{1cm} (30)

In case of Werner state, Bell-CHSH inequality detect the entangled state when $r > \frac{1}{\sqrt{2}}$ but our theorem tells us that the state is entangled iff $r > \frac{1}{\sqrt{3}}$. Therefore, there is a region $\frac{1}{\sqrt{3}} < r < \frac{1}{\sqrt{2}}$ in which our theorem detects the entangled state while Bell-CHSH inequality does not. Thus the condition in terms of linear entropy given in the theorem is more stronger than the condition [2].

The function $N(\rho)$ for the Werner state can be evaluated in terms of the parameter $r$ and is given by

$$N(\rho_W) = 3r$$  \hspace{1cm} (31)
Therefore the state $\rho_W$ is useful for teleportation iff

$$N(\rho_W) > 1 \Rightarrow \frac{1}{3} < r \leq 1 \quad (32)$$

From eq.\((30)\) and eq.\((32)\), we find that the state $\rho_W$ is useful for teleportation iff $0 \leq S_L < \frac{8}{9}$. Thus there exist certain states in the family of Werner states which are useful in teleportation although their linear entropy is greater than $\frac{2}{3}$. This example verifies the first part of the corollary-2. In the region $\frac{8}{9} \leq S_L$, the state $\rho_W$ is useless for teleportation purpose and hence obeying the result obtained by Bose and Vedral.

To summarize, we have investigated the relation between the two properties of the two qubit density operator: mixedness and entanglement. We found that if the linear entropy, which measures the mixedness of the state, is lying between zero and $\frac{2}{3}$ (excluding zero) then the two qubit mixed system is entangled. On one hand, we showed that there exist two qubit entangled system whose linear entropy greater than $\frac{2}{3}$ and useful in teleportation but on the other hand there are mixed system with linear entropy greater than $\frac{2}{3}$ are separable and hence useless in teleportation. Therefore if the given mixed system has linear entropy greater than $\frac{2}{3}$ then we cannot definitely say that the given system is entangled or not. Our criterion not only detects entanglement but also detect larger set of entangled states than Bell-CHSH inequality. Hence our criterion also shed some light on the detection of two qubit entanglement. In future we may opt this criterion to detect the entanglement in d-dimensional and multipartite systems.

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