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On Ampleness of vector bundles

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Abstract. In this article, we give a necessary and sufficient condition for ampleness of semistable vector bundles with vanishing discriminant on a smooth projective variety \(X\). As an application, we show ampleness of some special vector bundles on certain ruled surfaces. We prove similar results for parabolic ampleness.

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1. Introduction

Let \(X\) be a complex manifold of dimension \(n\), and \(E\) be a holomorphic vector bundle of rank \(r\) on \(X\) endowed with a hermitian metric \(h\). The hermitian bundle \((E, h)\) determines a unique hermitian connection compatible with the complex structure on \(X\) and \(E\), called as Chern connection, and it is denoted by \(D_E\). This connection \(D_E\) in turn gives rise to a curvature tensor, called as Chern curvature tensor and denoted by \(\Theta(E, h)\) \(\in C^\infty(X, \wedge^{1,1} T_X^* \otimes \text{End}(E))\) a \(\text{End}(E)\)-valued \((1, 1)\) form on \(X\). If \(z_1, z_2, \ldots, z_n\) are local coordinates on \(X\), and if \((e_\lambda)_{1 \leq \lambda \leq r}\) is a local orthonormal frame on \(E\), then one can write

\[
i \Theta(E, h) = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j k \lambda \mu} dz_j \wedge d\bar{z}_k \otimes e^*_\lambda \otimes e_{\mu},
\]

where \(c_{j k \lambda \mu} = c_{k j \lambda \mu}\). One looks at the associated quadratic form on \(S = T_X \otimes E\) as follows:

\[
\tilde{\Theta}_{E,h}(\xi \otimes v) = \langle \Theta_{E,h}(\xi, \xi) \cdot v, v \rangle_h = \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j k \lambda \mu} \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu.
\]

The hermitian bundle \((E, h)\) is said to be Griffiths positive if at any point \(z \in X\), we have \(\tilde{\Theta}_{E,h}(\xi \otimes v) > 0\) for all \(0 \neq \xi \in T_{X,z}\) and for all \(0 \neq v \in E_z\).

A holomorphic vector bundle \(E\) on a complex projective manifold is called ample in the sense of Hartshorne if the tautological line bundle \(\mathcal{O}_{\mathbb{P}(E)}(1)\) is ample, i.e. there exists a smooth hermitian metric on \(\mathcal{O}_{\mathbb{P}(E)}(1)\) with positive curvature.

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It is always true that if a hermitian holomorphic vector bundle \( (E, h) \) on a complex projective manifold \( X \) is Griffiths positive, then \( E \) is ample in the sense of Hartshorne. A famous conjecture of Griffiths asks whether ample bundles in the sense of Hartshorne admit Griffiths positively curved metrics. Also it is well known that if \( E \) is ample, then \( \text{det}(E) \) is ample. However, ampleness of \( \text{det}(E) \) does not ensure ampleness of \( E \) in general.

For a vector bundle of rank \( r \) on a complex manifold \( X \), the characteristic class

\[
c_2(\text{End}(E)) = 2r c_2(E) - (r - 1)c_1^2(E) \in H^4(X, \mathbb{Q})
\]

is called the discriminant of \( E \), denoted by \( \Delta(E) \).

In Section 3, we prove the following.

**Theorem 1.** Let \( X \) be a projective variety of dimension \( n \) and \( (E, h) \) be a hermitian holomorphic bundle of rank \( r \) on \( X \). Further assume that \( E \) is a semistable vector bundle with \( \Delta(E) = 0 \). Then the following are equivalent:

(i) \( (E, h) \) is Griffiths positive.
(ii) \( E \) is ample in the sense of Hartshorne.
(iii) \( \text{det}(E) \) is ample.

The Nakai–Moishezon criterion for ampleness says that a line bundle \( L \) on a projective variety \( X \) is ample if and only if \( L^{\dim Y} \cdot Y > 0 \) for every positive dimensional subvarieties \( Y \) of \( X \). Mumford gave an example of a non-ample line bundle on a ruled surface whose intersection with every curve is positive (see [14, Chapter 1]). Therefore, in general, it is not sufficient to check the condition only for curves in Nakai–Moishezon criterion. However, in some special cases, to check ampleness of a line bundle \( L \) on \( X \), it is enough to check that \( L \cdot C > 0 \) for every irreducible curve \( C \subset X \) (e.g., on abelian varieties [21], on flag bundles [7]). One must also note that for a globally generated vector bundle \( E \) on \( X \), \( E \) is ample if and only if it’s restriction to every curve \( C \subset X \) is ample. This follows easily from Gieseker’s Lemma (see [15, Proposition 6.1.7]). In general, there is no straightforward way to check ampleness of a given vector bundle on a projective variety \( X \). In [12], it is proved that an equivariant vector bundle on a toric variety \( X \) is ample if and only if its restriction to finitely many invariant curves in \( X \) are ample. Similar result holds for torus equivariant vector bundles on certain homogenous variety (see [6]). In [1], a sufficient condition is given to check ampleness of a vector bundle of rank 2 on some specific smooth surfaces with Picard rank 1.

We recall from [11, Chapter 5] that a vector bundle \( W \) of rank 2 on a smooth projective curve \( C \) is said to be normalized if \( H^0(W) \neq 0 \), but \( H^0(W \otimes L) = 0 \) for all line bundle \( L \) on \( C \) with \( \text{deg}(L) < 0 \). We notice that a normalized bundle \( W \) is semistable if and only if \( \text{deg}(W) \geq 0 \). An important consequence of Theorem 1 is the following.

**Corollary 2.** Let \( \rho : X = \mathbb{P}(W) \to C \) be a ruled surface defined by a normalized rank 2 bundle on a smooth curve \( C \) such that \( \mu_{\text{min}}(W) = \text{deg}(W) \). Let \( E \) be a semistable vector bundle of rank \( r \) on \( X \) with discriminant \( \Delta(E) = 0 \). Then, \( E \) is ample if and only if \( E|_{\sigma} \) and \( E|_{f} \) are ample, where \( \sigma \) is the smooth section of \( \rho \) such that \( \Theta_{X}(\sigma) \equiv \Theta_{\mathbb{P}(W)}(1) \) and \( f \) is a fibre of \( \rho \).

The above Corollary 2 implies the following:

**Corollary 3.** Let \( \rho : X = \mathbb{P}(W) \to C \) be a ruled surface on a smooth curve \( C \) defined by a normalized rank 2 bundle \( W \) on \( C \) with \( \mu_{\text{min}}(W) = \text{deg}(W) \), and \( E \) be a vector bundle on \( C \). Then the vector bundle \( E = \rho^*(V) \otimes \Theta_{\mathbb{P}(W)}(m) \) is ample on \( X \) if and only if \( m > 0 \) and \( \mu_{\text{min}}(E) > -m\text{deg}(W) \).

We also prove similar result for parabolic ampleness in Section 4.
2. Preliminaries

2.1. Harder–Narasimhan Filtration

A non-zero torsion-free coherent sheaf $\mathcal{G}$ on $X$ is said to be $H$-semistable if

$$
\mu_H(\mathcal{F}) = \frac{\ell_1(\mathcal{F}) \cdot H^{n-1}}{\text{rank}(\mathcal{F})} \leq \mu_H(\mathcal{G}) = \frac{\ell_1(\mathcal{G}) \cdot H^{n-1}}{\text{rank}(\mathcal{G})}
$$

for all subsheaves $\mathcal{F}$ of $\mathcal{G}$. For every vector bundle $E$ on $X$, there is a unique filtration

$$
0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{k-1} \subseteq E_k = E
$$

of subbundles of $E$, called the Harder–Narasimhan filtration of $E$, such that $E_i/E_{i-1}$ is $H$-semistable torsion free sheaf for each $i \in \{1, 2, \ldots, k\}$ and $\mu_H(E_i/E_{i-1}) > \mu_H(E_{i+1}/E_i)$ for each $i \in \{1, 2, \ldots, k-1\}$. We define $Q_k := E_k/E_{k-1}$ and $\mu_{\min}(E) := \mu_H(Q_k) = \mu_H(E_k/E_{k-1})$.

Let $N_1(X)_{\mathbb{R}}$ be the set of all numerical equivalence classes of real one cycles on $X$. Inside $N_1(X)_{\mathbb{R}}$, the closure of the convex cone generated by effective one cycles is called the closed cone of curves and it is denoted by $\overline{NE}(X)$. By Theorem 1.4.29 of [14], a divisor $D$ is ample if and only if $D \cdot \gamma > 0$ for all $\gamma \in \overline{NE}(X) - \{0\}$.

3. Main result and applications

We begin this section by proving our main result.

**Proof of Theorem 1.** (i) $\implies$ (ii). See Theorem 6.1.25 in [15] for a proof.

(ii) $\implies$ (iii). See Corollary 5.3 in [10] for a proof.

(iii) $\implies$ (i). There exists a filtration

$$
0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = E
$$

such that on each $G_j = E_j/E_{j-1}$, there exists a hermitian metric $h_j$ on $G_j$ for which the curvature tensor is equal to $\frac{1}{2} \gamma \otimes \text{Id}_{G_j}$ where $\gamma$ is a $(1,1)$-form representing the first Chern class $c_1(E)$ (see [19]). Since $\det(E)$ is ample, each $(G_j, h_j)$ is Griffiths positive. As extension of two Griffiths positive bundles is again Griffiths positive, we have inductively each $E_i$ is Griffiths positive and thus $E$ is also Griffiths positive.

**Remark 4.** Theorem 1 can be thought of as a generalization of Gieseker’s ampleness criterion for semistable vector bundles on smooth curves (see [13, Theorem 3.2.7]). However, the condition about vanishing discriminant is not essential for both $V$ and $\det(V)$ to be ample. For example, consider the tangent bundle $T_{p_2}$. Then $T_{p_2}$ sits in the following exact sequence:

$$
0 \longrightarrow \mathcal{O}_{p_2} \longrightarrow \mathcal{O}_{p_2}(1)^{\oplus 3} \longrightarrow T_{p_2} \longrightarrow 0.
$$

Hence, $T_{p_2}$ being quotient of an ample bundle is ample and $\det(T_{p_2}) \equiv \mathcal{O}_{p_2}(3)$ is also ample. But $T_{p_2}$ is semistable with $\Delta(T_{p_2}) \neq 0$.

**Remark 5.** Note that for a vector bundle $E$ on a smooth projective curve $C$, we have $\Delta(E) = 0$. Hence our result Theorem 1 is analogous to the result in [22]. Also one can compare our result with the results in [16] and [20].

A vector bundle $V$ on an abelian variety $X$ is called weakly-translation invariant (semi-homogeneous in the sense of Mukai) if for every closed point $x \in X$, there is a line bundle $L_x$ on $X$ depending on $x$ such that $T_x^* (V) \cong V \otimes L_x$ for all $x \in X$, where $T_x$ is the translation morphism given by $x \in X$. 

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Corollary 6. A semi-homogeneous vector bundle $E$ of rank $r$ on an abelian variety $X$ is ample if and only if $\det(E)$ is ample if and only if $\det(E) \cdot C > 0$ for all irreducible curve $C$ in $X$.

Proof. Mukai proved that $E$ is Gieseker semistable (see [13, Chapter 1] for definition) with respect to some polarization and it has projective Chern classes zero, i.e., if $c(E)$ is the total Chern class, then $c(E) = (1 + c_1(E)/r)^r$ (see [18, Theorem 5.8, p. 260], [18, Proposition 6.13, p. 266]; also see [17, p. 2]). Gieseker semistability implies slope semistability (see [13]). So, in particular, we have $E$ is slope semistable with $\Delta(E) = 2r(c_1(E) - (r-1)c_1^2(E)) = 0$. Hence, the result follows from Theorem 19 and Proposition 1.4 in [21].

Corollary 7. Let $W$ be a vector bundle of rank $m$ over a smooth complex projective curve $C$ and $\rho : \mathbb{P}(W) \to C$ be the projectivisation map. Let $E$ be a semistable vector bundle on $\mathbb{P}(W)$ of rank $r$ with discriminant $\Delta(W) = 0$, and $c_1(E) = x\xi + yf$, where $\xi$ and $f$ are the numerical classes of $\mathcal{O}_{\mathbb{P}(W)}(1)$ and a fibre of $\rho$ respectively. Then, $E$ is ample if and only if $x > 0$ and $(x\mu_{\text{min}}(W) + y) > 0$.

Proof. We note that by Lemma 2.1 of [9], the nef cone of divisors in $\mathbb{P}(W)$ is given by

$$\text{Nef}(\mathbb{P}(W)) = \{a(\xi - \mu_{\text{min}}(W)f) + b f \mid a, b \in \mathbb{R}_{\geq 0}\}.$$

Applying duality (see [14, Proposition 1.4.28]), we get

$$\overline{\text{Nef}}(\mathbb{P}(W)) = \{a(\xi^{m-1} - (\deg(W) - \mu_{\text{min}}(W))\xi^{m-2}f) + b\xi^{m-2}f \mid a, b \in \mathbb{R}_{\geq 0}\}.$$

Hence, $\det(E)$ is ample if and only if

- $c_1(E) \cdot (\xi^{m-1} - (\deg(W) - \mu_{\text{min}}(W))\xi^{m-2}f) = (x\mu_{\text{min}}(W) + y) > 0$ and
- $c_1(E) \cdot \xi^{m-2}f = x > 0$.

Therefore, the result follows from the previous theorem.

Corollary 8. Let $\rho : X = \mathbb{P}(W) \to C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve $C$ such that $\mu_{\text{min}}(W) = \deg(W)$. Let $E$ be a semistable vector bundle of rank $r$ on $X$ with discriminant $\Delta(E) = 0$. Then, $E$ is ample if and only if $E|_\sigma$ and $E|_f$ are ample, where $\sigma$ is the smooth section of $\rho$ such that $\mathcal{O}_X(\sigma) = \mathcal{O}_{\mathbb{P}(W)}(1)$ and $f$ is a fibre of $\rho$.

Proof. Let $c_1(E) = x\xi + yf$, where $\xi = [\sigma] \in N^1(X)$. Note that, by the given hypothesis, both $E|_\sigma$ and $E|_f$ are semistable, and hence both are ample if and only if

- $\deg(E|_\sigma) = (x\xi + yf) \cdot \xi = (x\deg(W) + y) > 0$, and
- $\deg(E|_f) = (x\xi + yf) \cdot f = x > 0$.

But, in that case, $(x\mu_{\text{min}}(W) + y) = (x\deg(W) + y) > 0$. Therefore, the result follows from the previous corollary.

Remark 9. Let $\rho : X = \mathbb{P}(W) \to C$ be a ruled surface on a smooth curve $C$ as in Corollary 8. Then, for any semistable vector bundle $R$ on $C$ and any integer $m$, $E := \rho^*(R) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is a semistable vector bundle with vanishing discriminant. Hence by Corollary 8, any semistable vector bundle $V$ on $X$ of this form $\rho^*(R) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample if and only if $E|_\sigma$ and $E|_f$ are ample if and only if $m > 0$ and $\deg(R) > -m\deg(W)$.

For example, we consider the ruled surface $\rho : X = \mathbb{P}(W) \to C$ over the elliptic curve $C$ defined by the nonsplit extension $0 \to \mathcal{O}_C \to W \overset{\sigma}{\to} \mathcal{O}_C \to 0$. Then for any semistable bundle $R$ on $C$ of positive degree, $E := \rho^*(R) \otimes \mathcal{O}_X(m)$ is ample for every positive integer $m$.

Corollary 10. Let $\rho : X = \mathbb{P}(W) \to C$ be a ruled surface on a smooth curve $C$ defined by a normalized rank 2 bundle $W$ on $C$ with $\mu_{\text{min}}(W) = \deg(W)$, and $V$ be a vector bundle on $C$. Then the vector bundle $E = \rho^*(V) \otimes \mathcal{O}_{\mathbb{P}(W)}(m)$ is ample on $X$ if and only if $m > 0$ and $\mu_{\text{min}}(V) > -m\deg(W)$. 

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Proof. Let
\[ 0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{k-1} \subsetneq V_k = V \]
be the Harder–Narasimhan filtration of \( V \), and \( Q_i = V_i/V_{i-1} \) for each \( i \). Since \( \rho \) is a smooth map, in particular it is flat and hence \( \rho^* \) is an exact functor. We also observe that for any ample line bundle \( H \) on \( X \), we have \( \mu_H(\rho^* Q_i) = \mu(Q_i) (f \cdot H) \) and \( f \cdot H > 0 \), where \( f \) denotes a fiber of \( \rho \). Fix \( E_i := \rho^*(V_i) \otimes \mathcal{O}_{\mathbb{P}(E)}(m) \). Then above observation and the uniqueness of Harder Narasimhan filtration imply that
\[ 0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = E \]
is the Harder Narasimhan filtration of \( E \) with respect to any polarization \( H \).

Now, suppose \( m \) satisfies \( m > 0 \) and \( \mu_{\min}(V) > -m \deg(W) \). Then by the previous remark, we conclude that each \( R_i := \rho^*(Q_i) \otimes \mathcal{O}_{\mathbb{P}(W)}(m) \) is ample. Inductively, each \( E_i \) is ample. In particular \( E \) is also ample.

Conversely, if \( V \) is ample for some \( m \), then \( R_k = \rho^*(Q_k) \otimes \mathcal{O}_{\mathbb{P}(W)}(m) \) is ample for each \( k \). Thus \( m \) must satisfy \( m > 0 \) and \( \mu_{\min}(E) > -m \deg(W) \). \( \square \)

Example 11. Let us consider the ruled surface \( \rho : X = \mathbb{P}(W) \to C \) over a curve \( C \) where \( W = \mathcal{O}_C \oplus \mathcal{L} \) for some line bundle \( \mathcal{L} \) on \( C \) with \( \deg(\mathcal{L}) < 0 \). Then for any vector bundle \( E \) on \( C \) with \( \mu_{\min}(E) > -m \deg(\mathcal{L}) \) for some positive integer \( m \), the bundle \( V = \rho^*(E) \otimes \mathcal{O}_X(m) \) is ample.

Let \( \rho : \mathbb{F}_e = \mathbb{P}([\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-e)]) \to \mathbb{P}^1 \) be a Hirzebruch surface for some \( e > 0 \). Let \( C_0 \) be its normalized section such that \( \mathcal{O}_{\mathbb{F}_e}(C_0) \) is the tautological bundle on \( \mathbb{F}_e \), and \( f \) be a fibre of \( \rho \). We recall the following results from [11, Chapter 5].

Theorem 12. Let \( D = aC_0 + bf \) be a divisor on \( \mathbb{F}_e \). Then
(a) If \( D \) is an irreducible curve \( \neq C_0, f \), then \( a > 0 \) and \( b \geq ae \).
(b) The linear system \( |D| \) contains a section of \( \rho \) if and only if \( a = 1 \) and either \( b = 0 \) or \( b \geq e \).
(c) The linear system \( |D| \) contains an irreducible non-singular curve if and only if it contains an irreducible curve if and only if \( a = 0 \), \( b = 1 \) (namely \( f \) ); or \( a = 1, b = 0 \) (namely \( C_0 \) ); or \( a > 0, b > ae \); or \( e > 0, a > 0, b = ae \).
(d) \( D \) is very ample if and only if \( D \) is ample if and only if \( a > 0 \) and \( b > ae \).

Lemma 13. Any irreducible curve of \( \mathbb{F}_e \) other than the fibers of \( \rho \) is linearly equivalent to an effective curve which is a union of sections of the map \( \rho \).

Proof. Let \( C \) be an irreducible curve in \( \mathbb{F}_e \) other than a fibre and the section \( C_0 \). Then \( C \sim xC_0 + yf \) for some \( x > 0 \) and \( y \geq xe \). Let \( y = mxr + e \) for some \( m > 0 \) and \( 0 \leq r < xe \). Now, \( C \sim xC_0 + ye \sim (x-1)(C_0 + ye) + (y-1)(C_0 + ye) \). This proves the result. \( \square \)

Proposition 14. Let \( C \) be an irreducible curve in \( \mathbb{F}_e \) and \( C = C_1 + \cdots + C_r \) where \( C_i \)'s are sections of the map \( \rho \). Let \( E \) be a vector bundle on \( \mathbb{F}_e \) such that for any two curves \( B \) and \( B' \) in \( \mathbb{F}_e \) with \( B \sim B' \), we have \( E|_B \cong E|_{B'} \). Then \( \rho^*(E|_C) \cong \bigoplus_i \rho^*(E_{C_i}) \) as vector bundles on \( \mathbb{P}^1 \).

Proof. We first observe that for any two curves \( B \) and \( B' \) in \( \mathbb{F}_e \) which are linearly equivalent to each other,
\[ \rho^*(E \otimes \mathcal{O}_B) \cong \rho^*(E \otimes \mathcal{O}_{B'}) \] on \( \mathbb{P}^1 \).
In other words, \( \rho^*(E|_B) \cong \rho^*(E|_{B'}) \) on \( \mathbb{P}^1 \). So, without loss of generality we assume that \( C = C_1 + \cdots + C_r \) and \( C_i \to C \) be an irreducible component of it. Then, \( \mathcal{O}_C \to \mathcal{O}_{C_i} \), which induces a sheaf map \( \rho_*(E \otimes \mathcal{O}_C) \to \rho_*(E \otimes \mathcal{O}_{C_i}) \) on \( \mathbb{P}^1 \) for all \( i \), and hence induces a map \( \rho_*(E \otimes \mathcal{O}_C) \to \bigoplus_i \rho_*(E \otimes \mathcal{O}_{C_i}) \) as well.

We claim that
\[ \rho^*(E \otimes \mathcal{O}_C) \to \bigoplus_i \rho^*(E \otimes \mathcal{O}_{C_i}) \]
is an isomorphism on $\mathbb{P}^1$. Indeed, for any $y \in \mathbb{P}^1$,
\[
\left(\rho_*(E \otimes \mathcal{O}_C)\right)_y \cong \bigoplus_{x \in \mathcal{C}(\rho^{-1}(y))} E_x.
\]
On the other hand,
\[
\left(\bigoplus_i \rho_i(E \otimes \mathcal{O}_{C_i})\right)_y \cong \bigoplus_{x \in \mathcal{C}_i, \rho(x) = y} E_x.
\]
Hence, the map is isomorphic at the stalk level. This proves our claim and the result. $\square$

Any rank two vector bundle $E$ on $\mathbb{F}_e$ has two numerical invariants describing it as an extension in a canonical manner. The first invariant $d_E$ is defined by the splitting type of $E$ on a general fiber $f$, i.e., if $E|_f = \mathcal{O}_{p_1}(d) \oplus \mathcal{O}_{p_1}(d')$ and $d \geq d'$, then $d_E = d$. The second invariant $r_E = r = \deg(\rho(E(-dC_0)))$. See [8] for more information about these numerical invariants $d$ and $r$. Note that, if $E$ is globally generated, then for a generic fibre $f$, $E|_f = \mathcal{O}_{p_1}(d) \oplus \mathcal{O}_{p_1}(d')$, where $d \geq d' \geq 0$.

**Theorem 15.** Let $E$ be a globally generated rank two bundle on $\mathbb{F}_e$ with numerical invariants $d$ and $r$, and $E$ sits in the exact sequence
\[
0 \to \mathcal{O}(dC_0 + rf) \to E \to \mathcal{O}((d' + r)f) \to 0.
\]
Further assume that for any two curves $B$ and $B'$ in $\mathbb{F}_e$ with $B \sim B'$, we have $E|_B \cong E|_{B'}$. Then, $E$ is ample if and only if $E|_f$, $E|_{C_0}$ and $E|_{C_0 + nf}$ are ample on a generic fibre $f$, on $C_0$ and sections of $\rho$ of the forms $C_0 + nf$ with $d(n-e) + r \leq 0$ respectively.

**Proof.** Restriction of ample bundle being ample, $E|_C$ is ample for any curve $C$ in $\mathbb{F}_e$ whenever $E$ is ample.

Conversely, let $E|_f$, $E|_{C_0}$ and $E|_{C_0 + nf}$ are ample on a generic fibre $f$, on $C_0$ and sections of $\rho$ of the forms $C_0 + nf$ with $d(n-e) + r \leq 0$ respectively. Now, if
\[
E|_f = \mathcal{O}_{p_1}(d) \oplus \mathcal{O}_{p_1}(d'),
\]
for a generic fibre $f$ of $\rho$ with $d \geq d' \geq 0$, then the ampleness of $E|_f$ implies that $d, d' > 0$.

Let $f'$ be a fibre among those finitely many fibre which has different splitting type of $E$ than that of a generic fibre. Restricting the exact sequence (1) to $f'$, we get
\[
0 \to \mathcal{O}(d) \to E|_{f'} \to \mathcal{O}_{p_1}(d') \to 0.
\]
Hence, $E|_{f'}$ being an extension of two ample line bundle, is also ample.

Let $C \sim C_0 + nf$ be any section of $\rho$, where either $n = 0$ or $n \geq e$. Now, restricting the exact sequence (1) to $C$, we get
\[
0 \to \mathcal{O}(d(n-e) + r) \to E|_C \to \mathcal{O}_{p_1}(d'(n-e) + r') \to 0.
\]
As $E|_{C_0}$ is ample on $C_0$, and $\mathcal{O}_{p_1}(-d'e + r')$ being the quotient is also ample. Hence, $(-d'e + r') > 0$, which implies $d'(n-e) + r' > 0$ for any $n \geq 1$. Note that, if $d(n-e) + r > 0$ then $E|_{C_0 + nf}$ is ample, as it is then an extension of two ample bundles. If $d(n-e) + r \leq 0$ then $E|_{C_0 + nf}$ is also ample by the given hypothesis. Therefore, we conclude that restriction of $E$ onto each fibre and each section is ample.

Let $C$ be any curve of $\mathbb{F}_e$ other than a fibre of $\rho$, and $C \sim C_1 + \cdots + C_r$ where $C_i$’s are sections. Now, using Proposition 14, we get that $\rho_*(E|_C)$ is an ample vector bundle on $\mathbb{P}^1$.

If $E$ is not ample, then by Gieseker’s lemma [15, Proposition 6.1.7], there exists an irreducible curve $C$ in $\mathbb{F}_e$ other than the fibres and a surjective homomorphism $u: E|_C \to \mathcal{O}_C$. This induces the surjection $\rho_*(E|_C) \to \rho_*(\mathcal{O}_C) \cong \mathcal{O}_{p_1}$, as well as the injection $\mathcal{O}_{p_1} \hookrightarrow (\rho_*(E|_C))^*$ which contradicts the fact that $\rho_*(E|_C)$ is an ample bundle on $\mathbb{P}^1$. Therefore, $E$ is ample. This completes the proof. $\square$
4. Remark about Parabolic Ampleness

Let $X$ be a connected smooth complex projective variety of dimension $d$ and $D \subset X$ be an effective divisor on $X$.

**Definition 16.** A quasi parabolic structure on a coherent sheaf $E$ with respect to $D$ is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = \mathcal{F}_1(E) \supset \mathcal{F}_2(E) \supset \cdots \supset \mathcal{F}_l(E) \supset \mathcal{F}_{l+1}(E) = E(-D)$$

where $E(-D) = \mathcal{O}_X \otimes \mathcal{O}_X(-D)$. The integer $l$ is called the length of the filtration.

A parabolic structure is a quasi-parabolic structure, as above, together with a system of parabolic weights $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1$, where each $\alpha_i$ is attached to $\mathcal{F}_i(E)$.

We shall denote the parabolic sheaf by $(E, \mathcal{F}, \alpha_*)$ or simply by $E_*$ when there is no confusion. For any parabolic sheaf $E_*$ defined as above, for any $t \in \mathbb{R}$, we define the following filtration $\{E_t\}_{t \in \mathbb{R}}$ of coherent sheaves parametrized by $\mathbb{R}$:

$$E_t = \mathcal{F}_i(E)(-\lfloor t \rfloor D)$$

where $\lfloor t \rfloor$ is the integral part of $t$ and $\alpha_{i-1} < t - \lfloor t \rfloor \leq \alpha_i$ with $\alpha_0 = \alpha_{l-1} = 1$ and $\alpha_{l+1} = 1$. Note that, any coherent subsheaf $M$ of $E$ has an induced parabolic structure such that if $\{M_t\}_{t \in \mathbb{R}}$ is the corresponding filtration then $M_t = E_t \cap M$ for any $t \geq 0$.

The parabolic degree of $E_*$ with respect to a fixed ample bundle $L$ on $X$, denoted by $\text{par}_\text{deg}(E_*)$ is defined as follows:

$$\text{par}_\text{deg}(E_*) := \int_{-1}^0 \deg(E_t) \, dt$$

The parabolic slope of $E_*$, denoted by $\text{par}_\mu(E_*)$ is the quotient $\text{par}_\text{deg}(E_*)/\text{rank}(E)$.

**Definition 17.** The parabolic sheaf $E_*$ is called parabolic semistable (resp. parabolic stable) if for every subsheaf $M$ of $E$ with $0 < \text{rank}(M) < \text{rank}(E)$, and $E/M$ being torsion-free sheaf, the inequality $\text{par}_\mu(M_*) \leq \text{par}_\mu(E_*)$ (resp. $\text{par}_\mu(M_*) < \mu(E_*)$) is satisfied.

Consider the decomposition

$$D = \sum_{i=1}^n n_i D_i$$

where any $D_i$ is a reduced irreducible divisor and $n_i \geq 1$. Let

$$f_i : n_i D_i \rightarrow X$$

denote the inclusion of the subscheme $n_i D_i$. For $1 \leq i \leq n$, let

$$0 = F^i_{l_i+1} \subset F^i_{l_i} \subset F^i_{l_i-1} \subset \cdots \subset F^i_1 = f^*_i E$$

(3)

Let $\alpha^i_j$, $1 \leq j \leq l_i + 1$ be real numbers satisfying

$$1 = \alpha^i_{l_i+1} > \alpha^i_l > \alpha^i_{l_i-1} > \cdots \alpha^i_2 > \alpha^i_1 \geq 0.$$

From now on we will always impose the following three conditions on the parabolic bundles $E_*$ that we will consider:

(a) the parabolic divisor $D = \sum_{i=1}^n n_i D_i$ is a normal crossing divisor, i.e., all $n_i = 1$ and $D_i$ are smooth divisors and they intersect transversally.

(b) all $F^i_j$ on $D_i$ in sequence (3) are subbundles of $f^*_i E$ for every $i$.

(c) all the weights $\alpha^i_j$ are rational numbers; so $\alpha^i_j = m^i_j/N$, where $N$ is a fixed integer and $m^i_j \in \{0, 1, \ldots, N - 1\}$.
In [2], parabolic tensor product has been defined. The parabolic $m$-fold symmetric product $S^m(E_*)$, is the invariant subsheaf of the $m$-fold parabolic tensor product of $E_*$ for the natural action of the permutation group for the factors of the tensor product. The underlying sheaf of the parabolic sheaf $S^m(E_*)$ will be denoted by $S^m(E_*)_0$. We recall the definition of parabolic ampleness from [3].

**Definition 18.** The parabolic sheaf $E_*$ is called parabolic ample if for any coherent sheaf $F$ on $X$ there is an integer $m_0$ such that for any $m \geq m_0$, the tensor product $F \otimes S^m(E_*)_0$ is globally generated.

Parabolic Chern classes $c_i(E_*) \in H^{2i}(X, \mathbb{Q})$ has been introduced in [3]. For a parabolic vector bundle $E_*$ of rank $r$ we define the parabolic discriminant, denoted by $\Delta_{par}(E_*)$ as follows:

$$\Delta_{par}(E_*) := 2r c_2(E_*) - (r - 1)c_1^2(E_*)$$

**Theorem 19.** Let $E_*$ be a semistable parabolic vector bundle of rank $r$ on a smooth complex projective variety $X$ such that $\Delta_{par}(E_*) = 0$. Then, $E_*$ is parabolic ample if and only if its parabolic first Chern class $c_1(E_*)$ is in the ample cone of $X$.

**Proof.** Let $p : Y \to X$ be the Kawamata cover, $V$ be the corresponding orbifold bundle on $Y$ with $c_1(V) = p^* c_1(E_*)$ (see [2] and [4]). So if $E_*$ is ample, then $V$ is also ample (see [3]) and thus $c_1(V)$ is also ample. Using the finiteness of the surjective map $p$, we conclude that $c_1(E_*)$ is in the ample cone of $X$.

Conversely, if $c_1(E_*)$ is in the ample cone of $X$, then $\text{det}(V)$ is also ample. Also, by the given hypothesis, $V$ is orbifold semistable and hence semistable (in the usual sense) with $\Delta(V) = p^* \Delta_{par}(E_*) = 0$. Hence $V$ is ample and thus $E_*$ is parabolic ample. □

**Proposition 20.** Let $\pi : X \to Y$ be a smooth surjective morphism between two smooth connected complex projective varieties $X$ and $Y$. Let $E_*$ be a parabolic semistable bundle on $Y$ with parabolic divisor $D \subset Y$ and $\Delta_{par}(E_*) = 0$. Then, the pullback bundle $\pi^*(E_*)$ under the map $\pi$ is also parabolic semistable on $X$ with parabolic divisor $\pi^*(D) \subset X$ and $\Delta_{par}(\pi^*(E_*)) = 0$.

Conversely, if $E_*$ be a parabolic semistable bundle on projective bundle $\pi : X = \mathbb{P}(E) \to Y$ with parabolic divisor $D' = \pi^{-1}(D)$, with $\Delta_{par}(E_*) = 0$ and the parabolic first Chern class $c_1(E_*) = \pi^*(c_1(L))$ for some line bundle $L$ on $Y$, then there exists a semistable parabolic bundle $E'_*$ on $Y$ with parabolic divisor $D$ and $\Delta_{par}(E'_*) = 0$ such that $E_* = \pi^*(E'_*)$.

**Proof.** Let $D = \sum_{i=1}^n D_i$ be the normal crossing divisor on $Y$ and $D' = \pi^*(D)$. Since $\pi$ is smooth, the pullback divisor $D'$ on $X$ is also a normal crossing divisor satisfying condition (a).

Let $p : Y' \to Y$ be a Kawamata cover with Galois group $G$ such that $p^* D_i = k_i N(p^* D_i)_{\text{red}}$ for some positive integers $k_i$ and $N$. Consider the following fibre product diagram

$$
\begin{array}{ccc}
X' & \to & X \\
\downarrow \pi & & \downarrow \pi \\
Y' & \to & Y
\end{array}
$$

Then $\tilde{p} : X' \to X$ is a Galois cover with the same Galois group $G$. Let $V$ be the orbifold bundle on $Y'$ associated to the parabolic bundle $E_*$ on $Y$ (see [4]). The pullback orbifold bundle $V' := \pi^*(V)$ then corresponds to the parabolic pullback bundle $\pi^* (E_*)$ on $X$ with parabolic divisor $D'$.

We note that $\Delta(V) = \Delta_{par}(E_*) = 0$. Since $E_*$ is parabolic semistable, by using the correspondence in [4], $V$ is also orbifold semistable, and hence semistable (in the usual sense). Therefore
the pullback bundle $V'$ is also orbifold semistable with $\Delta(V') = 0$, proving that $\pi^*(E_\ast)$ is parabolic semistable with $\Delta_{\text{par}}(\pi^*(E_\ast)) = 0$.

Conversely, let $V$ be the orbifold bundle on $X'$ associated to the parabolic bundle $E_\ast$ on $X$. Then,

$$c_1(V) = \overline{\pi}^*c_1(E_\ast) = \overline{\pi}^*\pi^*(\mathcal{L}) = \overline{\pi}^*(\mathcal{L})$$

Now by the given hypothesis, $V$ is orbifold semistable and hence semistable (in the usual sense). Since $\Delta(V) = 0$, by Theorem 1.2 in [5] $V|_f$ is semistable for every fibre $f$ of the map $\overline{\pi}$ (Here $\text{rank}(\mathcal{E}) = m + 1$) and $\deg(V|_f) = 0$. This implies $V \simeq \overline{\pi}^*(W)$ for some orbifold bundle $W$ on $Y'$ which must be semistable. Let $E'_\ast$ be the associated semistable parabolic bundle on $Y$. Note that $\Delta(V) = \overline{\pi}^*(\Delta(W)) = 0$ and $\overline{\pi}^*$ is injective. Hence $\Delta(W) = 0$. By a similar argument we have $\Delta_{\text{par}}(E'_\ast) = 0$. Then by the construction of $E'_\ast$, the result follows.

**Corollary 21.** Let $W$ be a vector bundle of rank $m$ over a smooth complex projective curve $C$ and $\rho : X = \mathbb{P}(W) \to C$ be the projection map. Let $E_\ast$ be a semistable vector bundle on $X$ of rank $r$ with parabolic discriminant $\Delta_{\text{par}}(E_\ast) = 0$, and parabolic $1$st Chern class $c_1(E_\ast) = x\xi + yf$, where $\xi$ and $f$ are the numerical classes of $\Theta_{\mathbb{P}(W)}(1)$ and a fibre of $\rho$ respectively. Then, $E_\ast$ is ample if and only if $x > 0$ and $(x\mu_{\text{min}}(W) + y) > 0$.

**Proof.** We note that

$$\overline{\mathcal{NE}}(\mathbb{P}(W)) = \left\{a(\xi^{m-1} - (\deg(W) - \mu_{\text{min}}(W))\xi^{m-2}f) + b\xi^{m-2}f \mid a, b \in \mathbb{R}_{\geq 0}\right\}$$

Hence, $c_1(E_\ast)$ is in the ample cone if and only if

- $c_1(E_\ast) \cdot \{\xi^{m-1} - (\deg(W) - \mu_{\text{min}}(W))\xi^{m-2}f\} = (x\mu_{\text{min}}(W) + y) > 0$
- $c_1(E_\ast) \cdot \xi^{m-2}f = x > 0$.

Therefore, the result follows from the previous theorem.

**Example 22.** Let $\rho : X = \mathbb{P}(W) \to Y$ be a projective bundle on a smooth projective variety $Y$. Let $D \subset Y$ be a normal cross divisor in $Y$ and $F_\ast$ be a semistable parabolic bundle of rank $r$ on $Y$ with parabolic divisor $D$. Then $\rho^*(F_\ast)$ is a parabolic semistable bundle on $X$ with parabolic divisor $D' = \rho^*(D)$. Let $D' = \sum_{i=1}^n D'_i$ be the decomposition into irreducible components of $D'$. A parabolic line bundle with parabolic divisor $D'$ is a data of the form $L_\ast = \{L, \{\alpha_1, \ldots, \alpha_n\}\}$, where $L$ is a line bundle on $X$ and each $0 \leq \alpha_i < 1$ corresponds to the divisor $D'_i$. Assume $\alpha_i \in \mathbb{Q}$ for all $i$. Then $E_\ast = \rho^*(F_\ast) \otimes L_\ast$ is parabolic semistable with $\Delta_{\text{par}}(E_\ast) = 0$. Note that $c_1(L_\ast) := c_1(L) + \sum_{i=1}^n \alpha_i[D_i]$. One can choose $L_\ast$ in such a way that $c_1(E_\ast) = c_1(\rho^*F_\ast) + r c_1(L_\ast)$ is in the ample cone of $X$. This way one can produce parabolic ample bundles on $X$.

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