Dynamics of primordial fields in quantum cosmological spacetimes

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Quantum cosmological models are commonly described by means of semiclassical approximations in which a smooth evolution of the expectation values of geometry operators replaces the classical and singular dynamics. The advantage of such descriptions is that they are relatively simple and display the classical behavior for large universes. However, they may “smooth out” an important inner structure to include which a more nuanced treatment is needed. The purpose of the present work is to investigate this inner structure and its influence on primordial gravitational waves. To this end we quantize a model of the Friedmann-Lemaître-Robertson-Walker universe filled with a “linear” barotropic cosmological fluid and with gravitational waves. The quantization yields an equation of motion for the Fourier modes of gravitational radiation, which is a quantum extension to the usual parametric oscillator equation for gravitational waves propagating in an expanding universe. The two quantum effects from the cosmological background that enter the enhanced equation of motion are (i) a repulsive potential resolving the big bang singularity and replacing it with a Big Bounce; and (ii) uncertainties in the numerical values for the background spacetime dynamical variables. First we study the former effect and its consequences for the primordial amplitude spectrum and carefully discuss the physical scales and parameters of the model. Next we investigate the latter effect, in particular the extent to which it may affect the primordial amplitude of gravitational waves.

I. INTRODUCTION

Theories of the origin of primordial structure which are based on models of a quantum bounce replacing the big bang singularity (see e.g. [1,2]) are often formulated in terms of the effective or trajectory dynamics of early universe. The goal of the present work is to construct and study an enhanced framework which incorporates a full quantum description of the homogenous cosmological spacetime and its full action on the perturbations to homogeneity propagating thereon (see e.g. [3–6] for alternative proposals).

There are two distinct consequences of the description of the background spacetime by means of a wave function. First, the singular dynamics of classical variables is replaced by nonsingular dynamics of quantum expectation values yielding semi-classical bouncing trajectories. This aspect of quantum cosmological spacetimes and its effect on the propagation of quantum fields has been widely studied for cosmological applications. Second, the background spacetime wave function implies some spreading in the background dynamical variables and, in particular, in the coupling between the perturbations and the background mode. The consequences of the latter are rarely studied [7,8]. We illustrate the origin of this effect with a simple example. Note that there are many ways in which the classical cosmological evolution in terms of the scale factor $a$ may be replaced by a semi-classical evolution of $a$. For instance, the classical scale factor may be replaced with the expectation values of various powers of the quantum scale factor as follows: $a(\eta) = \langle \hat{a}^n(\eta) \rangle^{1/n}$, where $n$ is any non-zero value. In Fig. 1 we plot the evolution of the scale factor in conformal time for a unique wave function and a few values of $n$. The plot shows in particular that for negative values of $n$, the universe generically undergoes a phase of accelerated contraction before being decelerated, halted and pushed into expansion, and that the dynamics may exhibit a degree of asymmetry between contracting and expanding phases. This is a purely quantum spread effect which demonstrates that “quantum forces” are not necessarily purely repulsive even when they ultimately revert the dynamics of the universe. The ambiguity illustrated by this example is neglected by “semi-classical” trajectories in which all the above scale factors evolve the same. It is therefore necessary to find if this neglected structure could produce some observable cosmological effects.

In this work we consider cosmological implications of the presence of quantum uncertainties in a universe undergoing a bounce. We omit the non-essential, though possible, phase of inflation and instead focus on fluid-dominated universes. Moreover, we restrict our attention to the universe from which the density perturbations are absent. In other words, we investigate the effect of quantum uncertainties of the background spacetime on the dynamical law of primordial gravitational waves in fluid-driven bouncing universes. Quantum bounces in such universes have been previously studied within the Bohm-de Broglie approach in [9]. The results obtained therein are in fact reproduced in a semi-classical limit of our model. We go beyond the semi-classical and add spreading to the background which produces an extra structure in the dynamical coupling between the gravitational waves and the background. As we shall see,
it influences the evolution of the amplitude of primordial gravitational waves. It is clear that the existence of this influence must be universal to all quantum cosmological models irrespectively of the employed quantization procedure or the assumed background symmetries.

The outline of the paper is as follows. In Section II we briefly describe the Hamiltonian formalism for the investigated cosmological model and its quantization. Our discussion includes the issue of back-reaction and entanglement between the background spacetime and the perturbations. We also discuss the existence of the classical limit which is necessary for cosmological applications. The main result of this section is the quantum evolution equation for the modes of gravitational radiation. In Section III we first employ a semi-classical method based on infinitesimally narrow wave packets to study the quantum bounce and the resultant quantum evolution equation. We numerically solve that equation and discuss the cosmological implications of the obtained result. Then we employ the full quantum approach and discuss the new qualitative features that it brings in at the level of the aforementioned equation. We resort to the WKB approximation in order to analytically investigate the evolution of the gravity-wave amplitude in the spreading background. The main findings are summarized and discussed in Section IV.

II. QUANTUM COSMOLOGICAL MODEL

A. Classical and quantum Hamiltonian

Let us assume a flat universe with toroidal topology $\Sigma = T^3$ and the line element,

$$ds^2 = -N^2dt^2 + a^2(\delta_{ab} + h_{ab}(x))dx^a dx^b,$$

where the coordinate volume equals $\int_\Sigma d^3x = V_0$ and the physical volume equals $V = a^3 V_0$. The metric perturbations $h_{ab}$ and their conjugate momenta $\pi^{ab}$ are resolved into the Fourier coefficients,

$$\tilde{h}_{ab}(\vec{k}) = V_0^{-1} \int_\Sigma h_{ab}(x)e^{-i\vec{k} \cdot \vec{x}} d^3x,$$

$$\tilde{\pi}^{ab}(\vec{k}) = \int_\Sigma \pi^{ab}(x)e^{-i\vec{k} \cdot \vec{x}} d^3x,$$

which are next expressed in a new tensorial basis with two distinct polarization modes of the gravitational wave,

$$\tilde{h}_\pm = \tilde{h}_{ab} A^a_{\pm} A^b_{\pm}, \quad \tilde{\pi}_\pm = \tilde{\pi}^{ab} A^a_{\pm} A^b_{\pm},$$

where $A^a_{\pm} = \frac{1}{\sqrt{2}}(\nu^a \omega^b - \nu^b \omega^a)$, $\nu^a$ and $\omega^a$ are such that $k^{-1}\vec{k}$, $\nu^a$ and $\omega^a$ form an orthonormal frame with respect to the fiducial metric, $\delta_{ab}$. The new variables satisfy the usual commutation relation, $\{\tilde{h}_+,\tilde{h}_\pm\} = \delta_{-,l}\delta_{+,l}$, and the reality condition for the field $h_{ab}(x)$ implies $\tilde{h}_\mp(\vec{k}) = \tilde{h}_\pm(\vec{k})$ and $\tilde{\pi}_\mp(\vec{k}) = \tilde{\pi}_\pm(\vec{k})$.

The physical Hamiltonian for the fluid-driven homogeneous and isotropic universe with linear tensor perturbations thereon reads [10],

$$H = H^{(0)} + \sum_\vec{k} H^{(2)}_\vec{k},$$

where

$$H^{(0)} = gp^2,$$

$$H^{(2)}_\vec{k} = -g \left( \frac{q}{\gamma} \right)^{-2} |\tilde{\pi}_\pm(\vec{k})|^2 - \frac{k^2}{4g} \left( \frac{q}{\gamma} \right)^{\frac{6+2\gamma}{2\gamma}} |\tilde{h}_\pm(\vec{k})|^2,$$

where $g = \frac{16\pi G}{V_0}$, $w$ is the ratio of the fluid’s pressure to its energy density, $\gamma = \frac{4\sqrt{3}}{3(1-w)}$, whereas $q = \gamma a^{-3+3w}$ and $p = \frac{3(1-w)}{8\pi G} a^{3+3w} H$ (where $H = \frac{\dot{a}}{a}$ is the Hubble rate) are canonical background variables. It follows from the Friedmann equation that the background Hamiltonian, $H^{(0)} = a^{3+3w} H^2 = \frac{4^{3+3w}}{16g} \rho V_0$, equals $1/96$ of the energy of matter in the entire universe when its physical and coordinate volumes are equal, $V = V_0$. The Hamiltonian [4] generates the dynamics with respect to a fluid variable which has been removed from the phase space. This choice of internal clock variable yields the lapse $N = a^{3w}$.

We fix the coordinates by setting $V_0 = l_p^3$, i.e. the coordinate volume equals the Planck volume. Furthermore, we assume that the present volume of the universe equals $V_0 = r \cdot 1.25 \cdot 10^{185} l_p$, where $r > 1$ is the ratio of the volume of the universe to the volume of its observable patch. This implies the present value of the scale factor to be $a_0 = 5 \cdot 10^{61} r^{1/3}$. We set the pivot scale to correspond
to a tenth of the diameter of the observable universe, i.e. \( \lambda_{\text{phys}} = 5 \cdot 10^6 \text{P} \), which yields the coordinate pivot wavenumber \( k_c = 20 \text{P}^{-1/3} \). Given the present value of the Hubble rate, \( H = 11.5 \cdot 10^{-62} \text{P}^{-1} \), and the redshift of the matter-radiation equality era, \( z_{eq} = 3400 \), we are able to estimate the value of the Hamiltonian for the radiation-dominated universe (i.e. \( w = \frac{4}{3} \)),

\[
H^{(0)} = 2.3 \cdot 10^{120} r^4 m_P,
\]

where \( r \) needs still to be determined. It follows that if the radiation-dominated era in the expanding universe begins at the volume \( V_T \) with a transition from another fluid-dominated era with \( w \) then the primordial value of the Hamiltonian must read\(^1\)

\[
H_{w_p}^{(0)} = 2.3 \cdot 10^{120} r^4 m_P V_T^{w-\frac{1}{2}}
\]

(7)

(where \( V_T \) is given in Planck volumes). Although the value of \( r \) is irrelevant for the classical dynamics of the model, the quantum corrections that we study below must depend on it as does the value of the canonical variable \( q \propto r^{1/3} \). Therefore, quantum cosmological dynamics depends on the size of the entire universe.

The Hamilton equations generated by the classical Hamiltonian \([9]\) yield the following gravitational wave propagation equation in conformal time, \( \eta = \int \left( \frac{q}{T} \right)^{2/3} dt \),

\[
\gamma^\mu_{\pm,k} + \left( k^2 - \frac{(q \mp \pi)}{q \mp \pi} \right) \mu_{\pm,k} = 0,
\]

where \( \mu_{\pm,k} = \left( \frac{q}{T} \right)^{2/3} h_{\pm,k} \). As we show below, introducing quantum effects to the background dynamics changes this equation in a significant way.

Quantization of the Hamiltonian \([9]\) may be carried out as follows. The phase space is the Cartesian product of the homogeneous and inhomogeneous sector, \( (q,p) \times \prod (\tilde{h}_{\pm,k}, \tilde{\pi}_{\pm,k}) \). Note that the background canonical variables have a non-trivial range, \( (q,p) \in \mathbb{R}_+ \times \mathbb{R} \). In this case, the canonical prescription which tells us to replace \( q \) and \( p \) with the usual position and momentum operators, \( \hat{Q} \) and \( \hat{P} \), does not work properly for the following reasons: (i) the momentum operator on the half-line is not self-adjoint and thus it cannot be considered as an elementary observable, (ii) the Hamiltonian operator as the square of the momentum operator is not self-adjoint either and requires imposing a suitable boundary condition on the wave functions. It seems more appropriate to use the dilation instead of the momentum operator, \( \hat{D} = \frac{1}{2} (\hat{Q} \hat{P} + \hat{P} \hat{Q}) \). The dilation operator is self-adjoint and the Hamiltonian operator which is the square of the ratio of dilation to position, \( (\frac{\hat{D}}{Q})^2 \) is self-adjoint for a wide class of symmetric orderings. The quantum zero-order Hamiltonian can be shown to generically contain a purely quantum term,

\[
p^2 \mapsto \hat{D}^2 + \hbar^2 \frac{\hat{K}}{Q^2}, \quad K > 0,
\]

(9)

which is a repulsive potential \( \propto \hat{Q}^{-2} \). The new term prevents the universe from reaching the singularity and generically replaces it with a bounce. More details on the above quantization and the unitary dynamics generated by the quantum Hamiltonian \([9]\) may be found in \([11]\).

Quantization of the perturbation variables is straightforward as they have the usual ranges which means that the canonical prescription works well in their case. Thus, \( \hat{h}_+ (\tilde{k}) \) and \( \hat{\pi}_+ (\tilde{k}) \) are replaced with the usual position and momentum operators on the real line. Finally, the total quantum Hamiltonian reads \([10]\),

\[
H \mapsto \hat{H} = \hat{H}^{(0)} + \sum_k \hat{H}_k^{(2)},
\]

\[
\hat{H}_k^{(0)} = g \left( \hat{D}^2 + \hbar^2 \frac{\hat{K}}{Q^2} \right),
\]

\[
\hat{H}_k^{(2)} = -g \left( \frac{\hat{Q}}{\gamma} \right)^{-2} |\hat{\pi}_+ (\tilde{k})|^2 - \frac{k^2}{4g} \left( \frac{\hat{Q}}{\gamma} \right)^{6w+2} \frac{m_p^2}{3-2w} |\hat{h}_+ (\tilde{k})|^2.
\]

(10)

B. Quantum dynamics

The Hamiltonian \([10]\) is valid if the perturbation variables and their spatial derivatives are much smaller than the unity, and their energy satisfies the following relation,

\[
|H^{(2)}| \ll H^{(0)}.
\]

(11)

At the quantum level there are two assumptions made about the total system: the lack of back-reaction and the lack of entanglement between the perturbations and the background. The lack of entanglement can be made consistent with the dynamical law if the latter is suitably redefined by means of the variational method. Then the back-reaction term can be consistently removed from the new dynamical law.

The lack of entanglement between the perturbations and the background is removed if the total space of states is given by the products of elements of the two respective spaces of states,

\[
|\psi \rangle = |\psi_0 \rangle \cdot |\psi_1 \rangle \in \mathcal{H} \subset \mathcal{H}_{\text{hom}} \otimes \mathcal{H}_{\text{inhom}},
\]

(12)

where \( \mathcal{H}_{\text{hom}} \) and \( \mathcal{H}_{\text{inhom}} \) stand for the homogeneous and inhomogeneous Hilbert spaces, respectively. This as-

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\(^1\) It follows from the Israel junction conditions at the transition between different fluid-dominated cosmological spacetimes.
Assumption breaks the Schrödinger equation and we determine the new dynamical law by applying the variational method. We introduce the quantum action,
\[
S_Q(\psi_0, \psi_1) := \int \langle \psi_0, \psi_1 | i \hbar \frac{\partial}{\partial t} - \hat{H} \rangle \psi_0, \psi_1 \rangle dt,
\]
whose variation leads to the following dynamical law,
\[
\frac{i \hbar}{\partial t} \langle \psi_1 | \psi_1 \rangle = \langle \psi_1 | \hat{H} \rangle \psi_1 \rangle + \langle \psi_0 | \hat{H}^{(2)} \rangle \psi_0 \rangle \psi_1 \rangle,
\]
which may be further simplified,
\[
\frac{i \hbar}{\partial t} \langle \psi_1 | \psi_1 \rangle = \langle \psi_1 \rangle \hat{H}^{(0)} \rangle \psi_0 \rangle + \langle \psi_0 | \hat{H}^{(2)} \rangle \psi_0 \rangle \psi_1 \rangle,
\]
if the term \( \langle \psi_1 | \hat{H}^{(2)} \rangle \psi_1 \rangle \psi_1 \rangle \) is included.

In analogy with the classical case, we introduce a new time parameter and a new dynamical variable,
\[
\eta = \int \langle \frac{\hat{Q}}{\gamma} \rangle^{-2} \rho_{\gamma}^{-1} dt,
\]
\[
\mu_{\pm, k} = \langle \frac{\hat{Q}}{\gamma} \rangle^{-2} \rho_{\gamma}^{-1} \hat{h}_{\pm, k},
\]
The above quantum-level definitions lead, as shown below, to the form of the dynamical law which closely resembles the classical counterpart \([8]\). Interestingly, these definitions emphasize the role played by the quantum uncertainty in the background Hamiltonian, which replaces the classical singularity with a bouncing behaviour of the expectation values of dynamical variables such as \( \langle \hat{Q}^{-2} \rangle \) or \( \langle \hat{Q}^{6w+2} \rangle \). The second effect is due to the quantum uncertainty in the background spacetime. The latter influences both the speed of gravitational waves, \( c_g^2 \), and the interaction potential, \( V \), which is defined as
\[
V = \frac{\langle \hat{Q}^{-2} \rangle^{6w+2}}{\langle \hat{Q}^{-2} \rangle^{3w+1}}. \tag{19}
\]
Notice that the uncertainty effect vanishes at the semi-classical level where all the expectation values \( \langle \hat{Q}^n \rangle \) are replaced by the respective semi-classical expressions, \( \langle \hat{Q}^n \rangle \). Moreover, both effects vanish away from the bounce when \( \langle \hat{Q}^{-2} \rangle \) becomes large as we show below.

Let us switch to the Heisenberg form for the equation of motion and solve the dynamics for the operator \( \hat{Q}^2 \).

\[\hat{Q}^2, \hat{H}^{(0)} = 4i \hat{D}, \quad [\hat{D}, \hat{H}^{(0)}] = 2i \hat{H}^{(0)}, \quad [\hat{Q}^2, \hat{D}] = 2i \hat{Q}^2, \]
which allows to immediately integrate the dynamics,
\[
\hat{D}(t) = 2 \hat{H}^{(0)} t + \hat{D}(0), \quad \hat{Q}^2(t) = 4 \hat{H}^{(0)} t^2 + 4 \hat{D}(0) t + \hat{Q}^2(0). \tag{21}
\]
Thus, for large $|t|$ we find,

$$\lim_{t \to \pm \infty} \delta^2_t = \left( \frac{1}{\mathbf{H}(0)^{\frac{3n+1}{2n}}} \right)_{\pm \infty} \approx \text{const}. \tag{22}$$

The above limits show that the oscillation frequency of every mode is asymptotically fixed and well-defined vacuum states for both remote past and remote future exist.

### C. Gravitational wave amplitude

Our convention for the physical dimensions is as follows: the spacetime coordinates are given in units of length, whereas the scale factor, and thus $q$, are dimensionless. The momentum coordinate $p$ has the dimension of mass times length. Analogously, the perturbation variables $h_{\pm}(\vec{k})$ and $\pi_{\pm}(\vec{k})$ have no dimension and the dimension of mass times length, respectively.

Let us introduce the annihilation and creation operators,

$$\hat{a}_{\pm}(\vec{k}) = \frac{1}{\sqrt{2}} \left( \hat{a}_{\pm} h_{\pm}(t) + \hat{a}^\dagger_{\pm} \pi_{\pm}(t) \right),$$

$$\hat{a}_{\pm}(\vec{k}) = \frac{1}{\sqrt{2}} \left( \hat{a}_{\pm} h_{\pm}(t) + \hat{a}^\dagger_{\pm} \frac{1}{\sqrt{2}} \hat{a}_{\pm} h_{\pm}(t) \right), \tag{23}$$

where $\hat{a}_{\pm}$’s are constant, whereas $h_{\pm}(t)$ are the isotropic mode functions which solve the isotropic Eq. [17]. Upon setting $h_{\pm}(t) = h_{\pm}(t)$, the Hamiltonian $\mathbf{H}_{\pm}$ becomes minimal at $t_0$ on the vacuum state $|0\rangle$ such that $\hat{a}_{\pm,k}|0\rangle = 0$ if

$$h_{\pm}(t_0) = \sqrt{\frac{4\Omega_{\pm}(t_0)}{\Omega_{\pm}(t_0)}}, \quad \hat{h}_{\pm}(t_0) = i \sqrt{\hbar \Omega_{\pm}(t_0)}. \tag{24}$$

The predictions for primordial gravitational radiation are often given in terms of the amplitude spectrum which is deduced from the equal-time correlation function, where we assume that the interesting coordinate distances $|\vec{x} - \vec{y}| < 1$ (or, $k > 1$) are small in comparison to the size of the universe. Furthermore, the isotropy $\mu_{\pm} = \mu_k$ is assumed. Following the convention of [12], we define the spectrum of amplitude of quantum fluctuations of the gravitational waves (per each polarization mode) as:

$$\delta^2_{\pm}(k) = \left( \frac{\sqrt{\gamma}}{\langle Q^2 \rangle_{\pm}^{\gamma}} \right)^{\frac{1}{n}} \frac{\mu_k}{k^{3/2}}. \tag{25}$$

The amplitude $\delta^2_{\pm}(k)$ is in general time-dependent. However, for long wavelength modes once their amplitude is set after the bounce, it remains to a large degree constant during a substantial part of the subsequent cosmological evolution.

### III. INTERNAL STRUCTURE OF THE BOUNCE

In this section we argue in favour of the dynamical significance of the inner structure of quantum bounce. We demonstrate the effect of the background wave function on the form of the gravitational wave propagation equation [19]. First, however, we discuss a semi-classical description of the quantum bounce which neglects its inner structure and the interaction potential it leads to.

#### A. Semi-classical description

In what follows we derive the gravitational wave propagation equation [19] by means of the Ehrenfest equations. Given the dynamics of the expectation values of elementary variables, $\langle \hat{Q} \rangle(t)$ and $\langle \hat{P} \rangle(t)$, we form the approximate dynamics of the expectation values of the relevant compound observables. According to the Ehrenfest theorem, the dynamics of the elementary expectation values generated by the zero order Hamiltonian $\mathbf{H}^{(0)}$ of Eq. (10) reads,

$$\frac{d}{dt} \langle \hat{Q} \rangle = 2g \langle \hat{P} \rangle, \quad \frac{d}{dt} \langle \hat{P} \rangle = 2gh^2K\langle \hat{Q}^2 \rangle. \tag{26}$$

We assume the probability density in the scale-factor representation,

$$\rho(x, t) = \delta(x - q(t)), \tag{27}$$

which is a mathematical idealization of a probability density peaked around a semi-classical solution for which $\langle \hat{Q} \rangle(t) = q(t)$. It is not an exact solution to the Schrödinger equation (15a), but an approximate one which allows immediately to obtain the dynamics of the expectation values of all functions of $Q$ once the dynamics of $q$ is known. Discarding the higher moments of $Q$ in the wave function may only be temporarily a valid approximation due to the natural spreading of the probability distribution with time.

We find the solution to (26) to read,

$$\langle \hat{Q} \rangle(t) = q_0 \sqrt{(k_{\text{max}}t)^2 + 1}, \tag{28a}$$

$$\langle \hat{P} \rangle(t) = \frac{q_0 k_{\text{max}}^2 t}{2g \sqrt{(k_{\text{max}}t)^2 + 1}}. \tag{28b}$$

where $q_0 = \sqrt{\frac{2gh^2K}{\sqrt{(Q^2)^2}}}$, $k_{\text{max}} = \frac{\mathbf{H}^{(0)}}{h\sqrt{K}}$ and $\mathbf{H}_{\text{sem}}^{(0)} = g \left( (\hat{P})^2 + \frac{K_\gamma}{Q^2} \right)$ is assumed to be equal to the classical value, $\mathbf{H}_{\text{sem}}^{(0)} = \mathbf{H}^{(0)}$. We plot a typical solution in Fig. 2. The classical and semi-classical trajectories are the same away from the singularity, which proves the correct behaviour of the semi-classical model. Close to the singularity, the classical and semi-classical trajectories diverge as the former terminates (or, originates) in the singularity, whereas the latter
There is no universe had not undergone the fluid transition, then we avoid the singularity through a bounce.

Making use of the relations below Eq. 5, we obtain the scale factor of the universe at the bounce, from which we infer the redshift at the bounce,

$$z_b = \left( \frac{10^{120} \rho}{(1 - w) \sqrt{K}} \right)^{\frac{1}{3}w} z_T^{\frac{1}{3}w},$$

where $z_T$ is the fluid-transition redshift. If the early universe had not undergone the fluid transition, then we have $z_b \approx 10^{120} \frac{\rho}{\sqrt{K}}$ and the physical wavelength of the pivot mode at the bounce reads $\lambda_{\text{phys}} \approx 5 \times 10^{-60} \sqrt{K} l_P$. This implies a huge value of $\lambda_{\text{phys}}$ for a cosmological scenario in which the observable cosmological scales are around the order of $l_P$ at the bounce. In general, we note that the bigger the universe is and the more energy it contains, the smaller the volume at which it bounces. The inverse is true for the value of $\sqrt{K}$. Because the amount of energy in the observable universe is so huge, the quantum correction preventing the singularity comes to dominate the dynamics at the Planck volume only if the value of $\sqrt{K}$ is very large. Nevertheless, we may fine-tune the model to yield a bounce exactly at Planck scale.

We shall now turn to the evolution of the coefficients $c_g^2(t)$ and $V(t)$ in the gravitational wave propagation equation 19. It is straightforward to obtain them in the current approximation,

$$c_g^2(t) \bigg|_{\text{sem}} = \frac{q(0)}{q(t)} = 1,$$
$$V(t) \bigg|_{\text{sem}} = \frac{q''(t)}{q(t)} = k_{\text{max}}^2 \left[ \frac{\gamma}{\gamma - 2} (1 + (k_{\text{max}} t)^2) \right]^{\frac{6(3w - 2)}{4w - 2}},$$

where prime $'$ denotes differentiation with respect to conformal time $\eta$ defined in Eq. 18. We note that the typical length scale influenced by the bounce is given by the factor $\left( \frac{2}{\gamma - 2} \right)^{\frac{6w - 2}{4w - 2}} k_{\text{max}}$ and thus, we introduce a dimensionless quantity, $\bar{k} = \left( \frac{2}{\gamma - 2} \right)^{\frac{6w - 2}{4w - 2}} \frac{k}{k_{\text{max}}}$, to express the scale-dependence of the gravity-wave amplitude. Similarly, the typical timescale at which the bounce operates is given by $k_{\text{max}}$, hence we introduce a dimensionless quantity $\tilde{t} = k_{\text{max}} t$.

In the semi-classical treatment of Eq. 19, the gravitational waves propagate at the speed of light and are influenced by the evolution of the universe in the vicinity of the bounce by the interaction potential 30b. A similar potential was obtained within the Bohm-de Broglie approach in 9 where the long wavelength amplitude spectrum was found to have the spectral index $n_\gamma = \frac{6w - 2}{4w - 2}$.

In Fig. 3 we provide an independent verification of their result by numerical integration of the primordial amplitude for a few values of $w$ and a range of modes $\bar{k}$. The analytical computation of the primordial spectrum for this and other interaction potentials is discussed in the subsection III C. The time evolution of a few modes of the primordial gravitational wave is plotted in Fig. 4.

Let us describe the relation between the primordial amplitude and the value of $K$ and $w$. One might think that since the larger the value of $K$ the less redshifted and milder the bounce is, the amplitude should decrease as $K$ increases. However, it can be shown that the amplitude scales with $K$ as $A_K \propto K^{\frac{3w - 2}{4w - 2}}$ (see the subsection III C for the explicit formulae). It follows that for $w > \frac{1}{3}$, the larger the value of $K$ (and the stronger the quantum effect) the smaller the primordial amplitude as one would expect. On the other hand, when $w > \frac{1}{3}$, this relation becomes inverted, that is, the larger the value of $K$ the larger the primordial amplitude. Hence, respecting the upper bound on the amplitude one may decrease the value of $K$ as much as one wishes for $w > \frac{1}{3}$. Note that the case $w = \frac{1}{3}$ is a borderline for which the primordial amplitude does not actually depend on $K$. Its value

\[\text{FIG. 2: The evolution of } q \text{ in the classical (dashed line, } K = 0) \text{ and semi-classical (solid line, } K = \frac{1}{3}) \text{ background model.}\]
for $w = \frac{1}{5}$, $z_T = 10^{28}$ and $r = 2$ reads $A_t \approx 10^{14.6}$.

Let us assume that the gravitational wave amplitude at the pivot scale should not exceed $10^{-5}$, which is consistent with the Planck data for $k_* = 0.002 Mpc^{-1}$ [14]. This in turn puts constraints on the free parameter $K$. In Fig. 5 we plot the required value of $K$ as a function of the fluid, $w$. We find huge values irrespectively of the assumed fluid. In Fig. 6 we plot the redshift (and the energy density in Fig. 7) at the bounce if the pivot scale amplitude reads $10^{-5}$. These results clearly indicate that for the modes of interests we avoid the so-called trans-Planckian problem as the observable modes when propagating through the bounce, where they are the shortest, exceed the Planck length by a number of orders of magnitude. On the other hand, they may indicate the problem of unnaturally large value of $K$. On the grounds that it is a quantum correction one might expect that it should be of order of unity in Planck units while it has to be of many, many orders of magnitude larger for most $w$'s in order to produce $A_t = 10^{-5}$.

There are two ways to argue for the possibility of a large $K$ in our quantum model. Both arguments refer to the ways in which we think about quantization of gravitational systems. First, note that we do not know which choice of basic variables is correct for quantization of gravitational systems. In the preceding section we chose dilation $\hat{D}$ and position $\hat{Q}$ but we did not specify the ordering of these operators in the Hamiltonian. In [15] it was actually shown that $-\frac{1}{2} < K < \infty$ depending on the chosen ordering. Thus, large values of $K$ can be easily accommodated by theory. The second argument is more subtle and is based on the nature of dynamics in quantum gravity. It is known (see e.g. [16] and references therein) that quantities like the scale of the bounce are not physically meaningful (or, unambiguous) in quantum gravity unless one indicates the internal clock used for computing those quantities. This property is referred to as the time problem. It follows that the scale of the bounce obtained in the present model is tied to the specific choice of clock $t$ that we have made for the derivation of the model. One might have chosen another clock and found much more Planckian, or even sub-Planckian, scale of the bounce issued from a weaker repulsive potential, i.e. a smaller value of $K$. The contradiction between those two conclusions would be, however, only illusory as it was shown in [17] that the physical predictions for the classical phase of the cosmological evolution derived from both models must agree with each other, as for instance, in regard to the predicted value of the amplitude of primordial gravitational waves in a large expanding universe. Finally, let us note that the value of $K$ allowed by the cosmological observations can be extremely large [18].

Above we have described the semi-classical model. It remains to verify whether adding a substantial amount of quantum spread to the cosmological background can alter the model in some important ways. In particular, whether the final gravity-wave amplitude is modified in this case due to some modifications of the gravity-wave propagation speed (30a) or modifications of the interaction potential (30b). We shall investigate this issue now.

### B. Quantum description

The mathematical idealization of the probability density made above yields immediately the quantum dynamics of the universe with the classical behaviour for large volumes. The interaction potential issued from such an
In what follows we solve the complete dynamics of the background model without any approximation and plot the resulting interaction potential. The analytically integrable solutions are very few and they require numerical integration of the expectation value of $\hat{Q}^{-2}$. We use an analytical three-parameter solution to the background Schrödinger equation (15a).

$$\langle q | \psi_0 \rangle \propto \sqrt{q} e^{-\frac{1}{2\sigma^2}(ip_0q_0 - q^2/4 - q_0^2/4)}/\sqrt{2\pi(\sigma^2 + it)}$$

$$\times I \sqrt{N^+} \left( \frac{q(2ip_0\sigma^2 + q_0)}{2(\sigma^2 + it)} \right),$$

where $h = 1 = g$ and $q_0, p_0, \sigma$ are free parameters. The evolution of the associated density distribution is plotted in Fig. 9. We see a wave packet moving towards the boundary $q = 0$, and strongly self-interfering as it bounces against the repulsive potential. The spread of the wave packet is growing as it moves away from the boundary.

The evolutions of the coefficients $c_q^2(t)$ and $V(t)$ obtained from the solution (31) are plotted in Fig. 8 and Fig. 10 respectively. The speed of waves squared, $c_q^2(t)$, consists of two maxima separated by a minimum exactly at the bounce and it rapidly decreases to the value $c_q^2 = 1$ as $t \to \pm \infty$. The fact that $c_q^2(t) \geq 1$ follows from the Schwarz inequality. The growth of $c_q^2$ is natural when the wave packet with a finite spread moves towards smaller $q$’s. The brief decline in $c_g^2$ exactly at the moment of the bounce is due to the momentary reduction of the spread as the wave packet bounces off the potential. The fact that $c_g^2$ becomes larger than unity is interpreted as the breakdown of the “semi-classical spacetime” interpretation of the model rather than as a superluminal propagation of the gravitational waves. We do not expect, however, a significant influence of the dynamical $c_g^2$ on the amplification of long-wavelength gravitational waves precisely because they are assumed to satisfy $k^2 \ll V$ at the bounce and the term $\propto k^2$ in Eq. (19) is simply negligible.

Similarly, the interaction potential $V(t)$ in Fig. 10 displays an extra structure which does not occur for the

approximation seems rather universal as it was also found in another trajectory approach [9]. Trajectories are the usual way in which quantum cosmological bounces are described. However, this description completely neglects the quantum uncertainty in the numerical values for the size and the expansion rate of the universe close to the bounce. It is legitimate to ask whether the amount of uncertainty that is completely negligible for the presently large universe might have played a significant role when the universe was small. The non-vanishing uncertainty should be reflected in the dynamics of the coefficients $c_q^2(t)$ and $V(t)$ of Eq. (19) as they depend on higher order moments. As a result, the primordial structure and gravitational waves could be influenced by this purely quantum effect.

In what follows we solve the complete dynamics of the background model without any approximation and plot the resulting interaction potential. The analytically integrable solutions are very few and they require numerical integration of the expectation value of $\hat{Q}^{-2}$. We use an analytical three-parameter solution to the background Schrödinger equation (15a).

$$\langle q | \psi_0 \rangle \propto \sqrt{q} e^{-\frac{1}{2\sigma^2}(ip_0q_0 - q^2/4 - q_0^2/4)}/\sqrt{2\pi(\sigma^2 + it)}$$

$$\times I \sqrt{N^+} \left( \frac{q(2ip_0\sigma^2 + q_0)}{2(\sigma^2 + it)} \right),$$

where $h = 1 = g$ and $q_0, p_0, \sigma$ are free parameters. The evolution of the associated density distribution is plotted in Fig. 9. We see a wave packet moving towards the boundary $q = 0$, and strongly self-interfering as it bounces against the repulsive potential. The spread of the wave packet is growing as it moves away from the boundary.

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Similarly, the interaction potential $V(t)$ in Fig. 10 displays an extra structure which does not occur for the
FIG. 8: The speed of gravitational waves squared $c_g^2$ in the propagation equation \([19]\) generated by the background dynamics in the analytical state \([31]\). We set $p_0 = -4$, $q_0 = 30$, $\sigma = 2$, $K = \frac{3}{4}$.

FIG. 9: The evolution of the density distribution in $q$ yielded by the exact wave packet \([31]\) with $p_0 = -4$, $x_0 = 30$ and $\sigma = 2$ ($K = \frac{3}{4}$).

which when expanded in $\hbar$ yields at lowest order,

$$
\partial_t S = -g \left( S_x^2 + \frac{\hbar^2 K}{x^2} \right), \quad \partial_t A^2 = -2g \partial_x (A^2 S_x),
$$

where $S$ is the Hamilton’s principal function,

$$
S(t,x) = g \int_{x',t'}^{x,t} \left( \frac{1}{4} x'^2 - \frac{K}{x'^2} \right) dt',
$$

where the integral is taken over the semi-classical trajectories with fixed initial condition and $A^2$ behaves like the density of particles following the semi-classical trajectories.

Let us assume the probability distribution at the moment of the bounce to read,

$$
\rho(x,0) = \rho(x).
$$

The solution \([32]\) reads now

$$
\langle x|\psi_0(t) = \sqrt{\rho(x_0(x,t))} \frac{\partial x_0}{\partial x} \exp \left[ iS(x,t)/\hbar \right],
$$

where

$$
x_k^2(x,t) = \frac{1}{2} \left( x^2 + \sqrt{x^4 - 16g^2\hbar^2Kt^2} \right),
$$

$$
S(x,t) = \frac{g^2\hbar^2 K}{x_k^2(x,t)} - \frac{\sqrt{K(1 + g^2\hbar^2)}}{2\hbar} \arctan \left( \frac{2g\sqrt{K}}{x_k^2(x,t)} \right).
$$

Within the WKB approximation the sought expectation value $\langle \hat{Q}^{-2} \rangle$ reads,

$$
\langle \hat{Q}^{-2} \rangle(t) = \int_0^x x^{-2} \rho(x_0(x,t)) \frac{\partial x_0}{\partial x} dx,
$$

$$
= \int_0^{\infty} \frac{\rho(x_0) dx_0}{x_k^2 + \frac{4g^2\hbar^2 K}{x_k^2} t^2},
$$

FIG. 10: The interaction potential $V$ of Eq. \([19]\) issued from the fully quantum background dynamics described by \([31]\). We set $p_0 = -4$, $q_0 = 30$, $\sigma = 2$, $K = \frac{3}{4}$, $\omega = \frac{1}{3}$.

semi-classical solution given by Eq. \([27]\).

C. WKB approximation

The available analytical solutions do not allow for obtaining an analytical formula for the interaction potential $V$ and the numerical integration of $V$ is cumbersome for large $K$. Therefore we resort to the WKB approximation \([19]\). We assume the solution to the Schrödinger equation \([15a]\),

$$
\langle x|\psi_0(t) = A(x,t) \exp \left[ iS(x,t)/\hbar \right], \quad A, S \in \mathbb{R},
$$
where in the last line we switched to the Heisenberg picture. The formula (38) yields an analytical expression for some choices of \( \rho(x) \) and thereby it yields an analytical expression for the interaction potential \( V \).

Let us assume the density distribution at the bounce to read,

\[
\rho(x) = \frac{x}{2q_b^2}\chi_{[q_b(1-\sigma),q_b(1+\sigma)]}(x), \tag{39}
\]

where \( \chi_{[q_b(1-\sigma),q_b(1+\sigma)]}(x) \) is the characteristic function, \( q_b \) is a fixed bouncing point and \( 0 < \sigma < 1 \) is a free dimensionless parameter. We then find,

\[
\langle \hat{Q}^2 \rangle(t) = q_b^2 \left( 1 + \sigma^2 + \frac{\ln \left| \frac{1+\sigma}{1-\sigma} \right|}{2\sigma} (k_{\text{max}}t)^2 \right),
\]

\[
\langle \hat{Q}^{-2} \rangle(t) = 1 + \sigma^2 + \frac{\ln \left| \frac{1+\sigma}{1-\sigma} \right|}{2\sigma} (k_{\text{max}}t)^2, \tag{40}
\]

\[
\langle \hat{Q} \rangle |_{t=0} = q_b,
\]

\[
(\Delta \hat{Q})^2 |_{t=0} = \langle \hat{Q}^2 \rangle - q_b^2 = q_b^2 \sigma^2.
\]

It follows from the last equality that \( \sigma \) has the interpretation of the relative volume dipersion. Notice that for \( \sigma \to 0 \) one naturally retrieves the semi-classical description of the subsection IIIA as

\[
\rho(x) \to \delta(x-q_b),
\]

\[
\langle \hat{Q}^2 \rangle(t) \to q_b^2 \left( 1 + (k_{\text{max}}t)^2 \right),
\]

\[
\langle \hat{Q}^{-2} \rangle(t) \to \frac{1}{q_b^2(1+(k_{\text{max}}t)^2)}, \tag{41}
\]

\[
(\Delta \hat{Q})^2 |_{t=0} \to 0.
\]

Hence, Eqs (40) provide a one-parameter extension to the semi-classical model with the free parameter being the spread of the wave function. The resultant formulae for the interaction potential and the speed of gravitational waves are given in Appendix A.

The interaction potential \( V \) in time \( \tilde{t} = k_{\text{max}}t \) produced by the density distribution (39) is plotted for a few values of \( \sigma \) in Fig. 11. It is apparent that the potential is very sensitive to the value of \( \sigma \). In fact, the height and the width of its peak can be altered by many orders of magnitude by the spread of the wave packet. Thus, it is natural to expect that the amplitude spectrum generated by these potentials can also be substantially altered.

In Fig. 12 we plot the evolution of the speed of gravitational waves. For the studied WKB states \( (\sigma > 0) \) the speed decreases just before the bounce and then at the bounce it increases back to its asymptotic value. It behaves symmetrically in time after the bounce. As before, we interpret this behavior as a breakdown of the semi-classical interpretation. This behavior suppresses the value of the \( k^2 \) term in Eq. (8) precisely at the moment when it is already subdominant and therefore, the dynamical effect of the varying speed of gravitational waves is negligible. The speed has been normalized so that it asymptotically converges to unity. The fact that it converges to a different value than unity is not physically relevant as this discrepancy is removed by simply redefining the length scale so that the measured wavenumber is \( k_{\text{eff}} = c_g^\infty k \), where \( c_g^\infty \) is the asymptotic value of \( c_g \). In this way the measured speed of gravitational waves equal to unity is retrieved.

In Fig. 13 we plot the numerically integrated evolution of the amplitude of a selected mode both in the semi-classical and the WKB approximation. We obtain a suppression of the amplitude in the WKB approximation. Now we turn to the analytical computation of the primordial amplitude spectrum for the interaction potentials issued from the WKB approximation. We solve the wave propagation equation (19) by employing the
The dynamics is solved separately in two distinct evolution regimes. The first regime spans from the remote past up to the moment when a particular mode crosses the interaction potential, $-\tilde{t}_c$. In this regime we solve the wave equation (42) in its asymptotic form,

$$\tilde{\mu}''_{\pm,k} + \left( (e_\infty^N k)^2 + \frac{2(3w-1)}{(1+3w)^2\eta^2} \right) \tilde{\mu}_{\pm,k} = 0,$$

(42)

to which analytical solution in terms of the Hankel functions is known (see Appendix B for details).

The other regime spans the time interval during which a particular mode is inside the potential, between $-\tilde{t}_c$ and $t_c$. Inside this evolution regime we use the integral form of Eq. (19) expanded in powers of $k_{\text{eff}}$ and compute only the lowest order term,

$$\mu(\hat{Q}^{-2}) \frac{1}{\sqrt{4\pi w}} = A_1(k_{\text{eff}}) + A_2(k_{\text{eff}}) \int d\eta_1 (\hat{Q}^{-2})^{-\frac{1}{2w}}.$$

(43)

One may show that this solution exhibits the following late time behavior,

$$\lim_{\tilde{t} \to +\infty} \mu(\hat{Q}^{-2}) \frac{1}{\sqrt{4\pi w}} = \hat{A}_1 - \pi \hat{A}_2 + O(\tilde{t}^{n<0})$$

(44)

where $\hat{A}_1 = A_1 - \frac{2}{3} A_2$ and $\hat{A}_2 = A_2$. Hence the amplitude spectrum in this approximation must be proportional to this particular linear combination of constants $A_1$ and $A_2$.

The values of $\hat{A}_1$ and $\hat{A}_2$ are obtained from matching solutions from the two regimes at the time of the potential crossing, $-t_c$. On the other hand, the first regime solution is chosen by the demand that it corresponds to the Bunch-Davies vacuum in the remote past. This procedure yields $A_1$ subdominant and $A_2$ dominant. The final result is the primordial amplitude spectrum, which is proportional to $|\hat{A}_2|$, and reads,

$$\delta_h(k_{\text{eff}}) = \left( \frac{\sqrt{2}|1-3w|}{3(1-w)} \right)^{\frac{3w}{2w+1}} \left[ \frac{2C}{\sqrt{2|1-3w|}} + D \right] \left( \frac{\gamma}{g_0} \right)^{\frac{3w}{2w+1}} k_{\text{max}} \sqrt{V_0(1+\sigma^2)}^{-\frac{1}{2w+1}} k_{\text{eff}}^{\frac{3w}{2w+1}},$$

(45)

where $C$ and $D$ are constants in the general solution to Eq. (42) for the first evolution regime,

$$C = c_2 \sqrt{e_\infty^N k_\eta H^{(2)}_\nu (e_\infty^N k_\eta)},$$

$$D = \frac{c_2}{2} H^{(2)}_\nu (e_\infty^N k_\eta)$$

(46)

where $c_2 = \sqrt{\frac{\pi gh}{3}} e^{-\frac{i}{2} \nu (\nu + \frac{1}{2})}$ and $\nu = \frac{3(1-w)}{2(3w+1)}$.

$^5$ See in particular Section IV B. Piecewise approximation and matching in the flat spatial section case.
quantum spread. We found that the spread induces scales. We found a large space of admissible parameters gravitational waves amplitude at the physically relevant the bounce making use of the bound on the primordial K\[9\]. Next we studied the free parameter of our model, amplitude spectra in agreement with the results of the Bohm-de Broglie trajectory approach. We obtained tional waves which had been previously obtained within reproduced a class of interaction potentials for gravita-
tional waves propagating across the primordial universe. We first analyzed the semi-classical description in which the spreading cosmological background. As \( k_{\text{eff}} \eta_{c} \) is independent of \( \sigma \), the relation between the “semiclassical” and the “quantum” amplitude spectrum reads:

\[
\delta_{h}(\tilde{k}_{\text{eff}}) = (1 + \sigma^{2})^{-\frac{3w}{3w+1}} \cdot \delta_{h}(\tilde{k}_{\text{eff}})_{\sigma=0}.
\]

The “quantum” factor in the above equation takes values from the interval \( 0 < (1 + \sigma^{2})^{-\frac{3w}{3w+1}} \leq 1 \) for fluids with \( -\frac{1}{3} \leq w \leq 1 \). In Fig. 14 we plot the dependence of the gravity-wave amplitude on the dispersion \( \sigma \) for selected values of \( w \). It universally leads to the quantum damp-
ing of the amplitude. The suppression may be very mild and rather irrelevant or, in case when \( w \approx -\frac{1}{3} \), very large and very significant leading to observable effects.

IV. CONCLUSIONS

We studied the effect of the bounce and quantum un-
certainties in the background geometry on the gravita-
tional waves propagating across the primordial universe. We first analyzed the semi-classical description in which the wave packets are assumed to be infinitely narrow. We reproduced a class of interaction potentials for gravitational waves which had been previously obtained within the Bohm-de Broglie trajectory approach. We obtained the amplitude spectra in agreement with the results of [9]. Next we studied the free parameter of our model, \( K \), the redshift of the bounce and the matter density at the bounce making use of the bound on the primordial gravitational waves amplitude at the physically relevant scales. We found a large space of admissible parameters that lead to plausible cosmological scenarios.

We then enhanced the treatment by the inclusion of quantum spread. We found that the spread induces qualitative changes to the dynamical law of gravitational waves. One way the uncertainties enter the dynamical law is by varying the speed of gravitational waves as the universe bounces. The other way they manifest themselves is by altering the interaction potential. Surprisingly, we found that these important changes of the dynamical law ultimately have no effect on the spectral index of the primordial amplitude for long wavelength modes. The amplitude on the other hand becomes multiplied by an overall factor independent of the wave-
length. The factor, however, is rather irrelevant for most cosmological fluids, nevertheless it can cause a signifi-
cant suppression of the amplitude in cases when \( w \approx -\frac{1}{3} \).

The finding that the quantum spread does not influ-
ence the cosmological predictions for most cosmological fluids is very important theoretically. It implies that the semi-classical analysis is completely sufficient in those cases at linear order. It might also imply that it will never be possible to discern any difference between classi-
cal and quantum bouncing scenarios. Cosmology in those cases becomes insensitive to quantum gravity effects.

The finding that the quantum spread may significantly suppress the primordial amplitude for some cosmological fluids, even if they themselves are not physically appealing, indicates that one should verify the possible effect of quantum spread every time that one introduces a form of matter not included in our work. We note that it is not clear from our work why the amplitude is suppressed rather than amplified. It might be the case that there exist quantum states of the background spacetime which amplify the amplitude of gravitational waves.

The effect of uncertainties illustrates the basic fact about quantum mechanics and semi-classical descriptions thereof. Namely, there are infinitely many ways in which one can replace a given classical observable with a function of expectation values of operators that behaves like the classical observable for large universes. All such functions provide semi-classical expressions for a given clas-
cical observable but with a different behaviour exactly in the regime where classical mechanics breaks down. This is illustrated by the examples of the scale factor described in the introduction and shows a serious limitation on the physical interpretation of semi-classical descriptions. It is also illustrated by the non-trivial evolution of the cou-
pling of the gravitational waves and indicates that any semi-classical description must be verified whether it in-
deed reproduces the correct quantum behavior.

Finally, we wish to emphasize that the presented model could be used for modelling the primordial universe after it is extended to include density perturbations. At the moment we are not able to predict whether the tensor-to-
scalar ratio is going to be sufficiently small and whether the scalar spectral index is going to be consistent with observations. It is also presently unclear to which de-
gree density perturbations can be affected by quantum uncertainties of the cosmological background. We shall investigate these issues in future.
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\[ V(t) = \frac{-256(k_{\text{max}}t)^2(\sigma + \sigma^3)^2(-5 + 6w)(8\sigma)^{2-w}}{((k_{\text{max}}t)^2 + (1 - \sigma)^4)(k_{\text{max}}t)^2 + (1 + \sigma)^4)(3w - 3)^2} \]

\[ + 16\sigma(1 + \sigma^2)((1 - \sigma^2)^4 - 2(k_{\text{max}}t)^2(1 + 6\sigma^2 + \sigma^4) - 3(k_{\text{max}}t)^4)(8\sigma)^{2-w} \Lambda \frac{9w-5}{10w-3}k_{\text{max}}^2 \frac{q^2}{q'_2} \frac{6w-2}{3w-3} \]

where \( \Lambda = \ln \left| \frac{(k_{\text{max}}t)^2 + (\sigma + 1)^4}{(k_{\text{max}}t)^2 + (\sigma - 1)^4} \right| \),

\[ c_g^2(t) = \frac{4^{1+w}}{1 - \sigma^2} \frac{3-3w}{1-w} \left( F_+ - F_- \right) \left( \frac{\ln (1 + \eta) + (k_{\text{max}}t)^2}{(\sigma - 1)^4 + (k_{\text{max}}t)^4} \right) \frac{3w-1}{3w-3} \]

where \( F_\pm = (1 \pm \sigma) \frac{8}{1 - \sigma^2} F_1 \left( \frac{3w-1}{3w-3}, \frac{2}{3w-3}, \frac{2}{3w-3} \right) \).  

Appendix A: Gravity-wave propagation equation in WKB approximation

In the WKB approximation the gravity-wave propagation equation \([19]\) reads,

\[ \hat{\mu}''_{\pm,k} + \left( k^2 c_g^2 (\hat{t}(\eta)) - V(t(\eta)) \right) \hat{\mu}_{\pm,k} = 0, \]  

(A1)

where \( \eta(t) = \int^t \left( \frac{\gamma^2}{8w_0^2 \sigma} \ln \left| \frac{(1 + \sigma)^4 + (k_{\text{max}}t)^2}{(1 - \sigma)^4 + (k_{\text{max}}t)^2} \right| \right) \frac{3w-1}{3w-3} \) dt and

\[ \langle \hat{Q}^{-2} \rangle = \frac{1 + \sigma^2}{q_0^2 \hat{t}^2}, \]  

(B1)

which, through \([18]\), yields the asymptotic relation between the time parameters \( \hat{t} \) and \( \eta \),

\[ \lim_{\hat{t} \to \pm \infty} k_{\text{max}} \eta(\hat{t}) = \left\{ \begin{array}{ll}
\pm \frac{3(1 - w)}{3w + 1} \left[ \hat{t} \left( \frac{1 + 3w}{1 - \sigma^2} \right) \frac{3w-1}{3w-1} \right]^{\frac{3w-1}{2}} \left( \frac{\gamma^2}{q_0^2} \right)^{\frac{3w-1}{3w-1}} & \text{if } \hat{t} > \hat{t}_c \\ \\
\left| \hat{t} \right| \frac{1 + 3w}{1 - \sigma^2} & \text{if } \hat{t} < \hat{t}_c \end{array} \right. \]

Hence, the asymptotic form of the equation \([19]\) reads

\[ \hat{\mu}''_{\pm,k} + \left( c_g^2 k^2 + \frac{2(3w - 1)}{(1 + 3w)^2 \gamma^2} \right) \hat{\mu}_{\pm,k} = 0, \]  

(B3)

solution of which is

\[ \mu = \sqrt{\eta} \left[ c_1(k) H^{(1)}(c_g^2 \eta \eta) + c_2(k) H^{(2)}(c_g^2 \eta \eta) \right], \]  

(B4)

where \( \nu = \frac{3(1 - w)}{2(3w + 1)} \). The solution \([19] 4\) is a sufficiently accurate solution of the equation \([19]\) for \( |\hat{t}| \gg 1 \), and is used at all times for which the potential is subdominant, that is, before (after) the potential crossing time \(-\hat{t}_c (\hat{t}_c)\), which asymptotically reads

\[ \hat{t}_c = \left[ \frac{9}{2} \left( \frac{1 - w}{1 - 3w} \right) \frac{3w-1}{3w-1} \left( k_{\text{eff}}^2 \right)^{\frac{3w-1}{3w-1}} \right] \]  

(B5)

where \( k_{\text{eff}} = \tilde{k} c_g^2 \).

On the other hand, at times \(-\hat{t}_c \leq \hat{t} \leq \hat{t}_c \) when the interaction potential is dominant, the solution is approximated by the lowest order in \( k_{\text{eff}} \) terms of the formal solution of \([19]\),

\[ \mu(\hat{Q}^{-2}) \gamma^{\frac{1}{w-1}} = A_1(k_{\text{eff}}) + A_2(k_{\text{eff}}) \int d\eta_1 (\hat{Q}^{-2}) \gamma^{-\frac{1}{w-1}} - k_{\text{eff}}^2 \int d\eta_2 (\hat{Q}^{-2})^{-\frac{1}{w-1}} \int d\eta_3 \langle \hat{Q}^{-2} \rangle \gamma^{\frac{1}{w-1}} \mu, \]  

(B6)

where the quantity \( \mu(\hat{Q}^{-2}) \gamma^{\frac{1}{w-1}} \) corresponds to the classical variable \( h = \frac{k}{a} \). Because the change in \( c_g^2 \) does not break the dominance of the interaction potential, the present approximation neglects the evolution of \( c_g^2 \) and picks its value at infinity where the effective
wavenumber \( \tilde{k}_{\text{eff}} = \tilde{k} c_{g}^{\infty} \) is defined. The asymptotic value of speed of gravitational waves (see the definition below Eq. (19)),

\[
c_{g}^{\infty} = (\langle \tilde{Q} \rangle_{\infty})_{\infty} (\tilde{Q})^{-\frac{3w+1}{3w-1}},
\]

(B7)
is easily found in the WKB approximation. Indeed, as one may show,

\[
(\tilde{Q}^{n})_{\infty} = \frac{\tilde{t}^{n} q_{b}^{n}}{2(2-n)} [(1+\sigma)^{2-n} - (1-\sigma)^{2-n}] .
\]

(B8)

After neglecting higher order terms in the equation [(B6)], the only integral left reads

\[
\begin{align*}
8\sigma q_{b}^{2} & \int_{0}^{1} d\eta (\tilde{Q}^{-2})^{-\frac{\sigma}{1+w}} = 2(1+\sigma)^{2} \arctan \left[ \frac{\tilde{t}}{(1+\sigma)^{2}} \right] \\
-2(1-\sigma)^{2} & \arctan \left[ \frac{\tilde{t}}{(1-\sigma)^{2}} \right] + \tilde{t} \ln \left[ \frac{(1+\sigma)^{4} + \tilde{t}^{2}}{(1-\sigma)^{4} + \tilde{t}^{2}} \right].
\end{align*}
\]

(B9)

Therefore, the solution far away from the bounce, in the leading terms, is

\[
\begin{align*}
\lim_{\tilde{t} \rightarrow +\infty} \mu(\tilde{Q}^{-2})^{-\frac{\sigma}{1+w}} & = \tilde{A}_{1} + \tilde{A}_{2} \frac{1+\sigma^{2}}{\tilde{t}}, \\
\lim_{\tilde{t} \rightarrow -\infty} \mu(\tilde{Q}^{-2})^{-\frac{\sigma}{1+w}} & = \tilde{A}_{1} - \pi \tilde{A}_{2} + O(\tilde{t}^{n<0}).
\end{align*}
\]

(B10)

We match the solutions at the point \(-\tilde{t}_{c}\), where \(\mu\) can be approximated by

\[
\mu(-\tilde{t}_{c}) = \tilde{A}_{1}(-\tilde{t}_{c})^{\frac{3w-1}{3w-w}} \left( \frac{1+\sigma^{2}}{q_{b}^{3}} \right)^{\frac{1}{3w-w}} \\
+ \tilde{A}_{2}(-\tilde{t}_{c})^{\frac{3w-1}{3w-w}} \left( \frac{1+\sigma^{2}}{q_{b}^{3}} \right)^{1-\frac{3w-1}{3w-w}} \\
\text{and, from [B4], we also have } (\eta_{c} = \eta(-\tilde{t}_{c}))
\]

\[
\mu(-\eta_{c}) = \frac{C}{\sqrt{c_{g}^{\infty}}}, \quad \mu'(\eta_{c}) = D \sqrt{c_{g}^{\infty}}
\]

(B12)

Assuming the Bunch-Davies vacuum normalization \(c_{1} = 0\) and \(c_{2} = \sqrt{\pi g_{b} e^{-\frac{\gamma}{2} (\nu+\frac{1}{2})}}\), the constants are

\[
C = c_{2} \sqrt{c_{g}^{\infty} k_{\eta_{c}}} H_{u-1}^{(2)}(c_{g}^{\infty} k_{\eta_{c}}), \\
D = \frac{c_{2} H_{v+1}^{(2)}(c_{g}^{\infty} k_{\eta_{c}})}{2} \sqrt{c_{g}^{\infty} k_{\eta_{c}}}
\]

(B13)

Combining the equations (B11) and (B12) we obtain

\[
\begin{align*}
\tilde{A}_{1} & = - \left( -\sqrt{\frac{2}{3(1+w)}} \right) \frac{\tilde{A}_{1}}{3(w-1)} \\
& \times \frac{1}{q_{b} \sqrt{k_{\text{max}}}} \left( (1-3w)C \right) \left( \sqrt{\frac{2}{3(1-w)}} - D \right) \tilde{k}_{\text{eff}}^{3(w-1)}; \\
\tilde{A}_{2} & = \left( -\sqrt{\frac{2}{3(1+w)}} \right) \frac{3(w-1)}{3(1-w)} \\
& \times \frac{1}{q_{b} \sqrt{k_{\text{max}}}} \left( -2C \right) \left( \frac{1+\sigma^{2}}{3(w-1)} \right) \tilde{k}_{\text{eff}}^{3(w-1)}.
\end{align*}
\]

(B14)

Note that for cosmological fluids with \(-\frac{1}{3} \leq w \leq 1\) the coefficient \(\tilde{A}_{1}\) scales with a negative power of \(\tilde{k}\), therefore it is subdominant for \(\tilde{k} \ll 1\). The dominant part of the amplitude spectrum is determined by \(\tilde{A}_{2}\). Making use of Eq. (B11), we find the spectrum of amplitude (25)

\[
\delta k (\tilde{k}_{\text{eff}}) = \left( \frac{2 \sqrt{2(1-3w)}}{3(1-w)} \right) \frac{2C}{\sqrt{21-3w}} + D \left( \frac{3(w-1)}{q_{b} \sqrt{k_{\text{max}}}} \right) \frac{1}{k_{\text{max}}^{2}} \sqrt{\nu_{0}(1+\sigma^{2})^{-\frac{3w-1}{3w-1}}} \tilde{k}_{\text{eff}}^{3w-1}.
\]

(B15)

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