1. Introduction

The single particle Anderson model is defined by
\begin{equation}
H_\Lambda = \gamma \Delta + V
\end{equation}
where \(\Delta\) is a kinetic term and \(V\) is an random potential. In the standard formulation, \(\Delta\) is the discrete Laplacian and \(V\) is a diagonal matrix with i.i.d. random entries with bounded density \(\rho\). \(H_\Lambda\) acts on the Hilbert space \(H_\Lambda = \ell^2(\Lambda)\) for \(\Lambda \subset \mathbb{Z}^d\). For sufficiently small \(\gamma\), \(H_\Lambda\) has only pure point spectrum with exponentially localized eigenvectors. The eigenvalues are a random point process, so that eigenvalues corresponding to eigenfunctions localized at sufficient distances are nearly independent.

Our interest is in dynamics of models in the Anderson localized phase possessing highly correlated energies localized at large distances. Our objective here is to consider a simple example of the phenomena. To this end, consider the graph \(\Lambda_2\) created from two copies of \(\mathbb{Z}^d\) and an edge connecting the origins of both lattices. Sites on this lattice may be labeled \(p_{x,i}q\) for \(x \in \mathbb{Z}^d\) and \(i = 1, 2\), a natural lattice distance may be incorporated into \(\Lambda_2\) as
\[d_2((x,i), (y,j)) = \begin{cases} \|x - y\|_1 & \text{if } i = j \\ \|x\|_1 + \|y\|_1 & \text{if } i \neq j \end{cases}\]
A natural basis for \(\ell^2(\Lambda_2)\) is \(|x,i\rangle\) which takes a value of 1 at site \((x,i)\) and 0 at all other sites. Let both copies of \(\mathbb{Z}^d\) be equipped with a copy of the Hamiltonian \(H_\Lambda\) with identical potentials \(V\) and let \(g\) parametrize a hopping between the origins of the respective lattices. That is we define the Hamiltonian,
\begin{equation}
h_g = H_{\mathbb{Z}^d} \oplus H_{\mathbb{Z}^d} + g(|0,1\rangle\langle 0,2| + |0,2\rangle\langle 0,1|)
\end{equation}
acting on \(\ell^2(\Lambda_2)\).

Whereas the rescaled eigenvalues of (1.1) obey Poisson statistics for small \(\gamma\) [8, 5], the rescaled eigenvalues of (2.1) attain a clustering eigenvalue process of eigenvalue pairs. The pairs become degenerate at \(g = 0\) but eigenvalues are almost surely simple at \(g \neq 0\). The lattice distance between sites \((x,1)\) and \((x,2)\) is \(2\|x\|_1\), however, the eigenvalues localized in the region of these sites are highly correlated.

At a first viewing, (1.2) may appear unrelated to the more prominent multiparticle Anderson model [1]. However, we would like to draw attention to the similarities these models possess. Both models should be expected to exhibit resonant tunneling behavior, the dynamics of which have not been explored in the multiparticle Anderson model case.

In the multiparticle model \(N\) copies of (1.1) act on a Hilbert space \(\otimes_{i=1}^N \ell^2(\Lambda)\) for \(N\) particles. The \(\Delta\) term becomes a Laplacian on each copy of \(\mathbb{Z}^d\) and the operator \(V\) is the local potential acting on each particle. An additional term \(U\) is added for interactions between particles. For 2 particles, configurations of the positions of the model are given by \((x,y)\) for \(x, y \in \mathbb{Z}^d\). As discussed in [1], the symmetrized metric is the proper for indistinguishable particles. However, if the particles are distinguishable then it is meaningful to ask about the dynamical properties in the non-symmetrized metric. If the particles are of different species, say having slightly different mass or charge then a
transition of \( (x, y) \sim (y, x) \) will be strongly resonant, and it is interesting to ask how long it takes for the particles to exchange positions.

In a many body setting, similar propagations of excitations may well be infinite dimensional. In a forthcoming paper \cite{7}, we consider the localized phase of a model of a tracer particle interacting with a field of oscillators. A localized particle will affect the excitations at arbitrary distances, and the nature of the statistics of the motion of the excitations is a nontrivial question.

2. Model

The double lattice model above (1.2) may be naturally recast as (1.1) describing state of a spin 1/2 particle perturbed by a transverse magnetic potential near the origin. Thus, we equivalently define the Hilbert space as, \( \mathcal{H}_2 = \ell^2(\bigoplus_{x \in \mathbb{Z}^d} \mathbb{C}^2) \) the Hamiltonian becomes,

\[
(2.1) \quad h_g = (\gamma \Delta + V) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + g \begin{pmatrix} 0 & |\zeta\rangle \langle \xi| \\ |\xi\rangle \langle \zeta| & 0 \end{pmatrix}
\]

where \( \zeta \in \ell^2(\mathbb{Z}^d) \), so that \( \|\zeta\|_2 = 1 \) with bounded support, and \( g \) is coupling to the external magnetic potential. We will alter the above notation of the basis to \( |x, i\rangle \) for \( i = \pm 1 \) and \( x \in \mathbb{Z}^d \). We will slightly abuse notation and write \( \langle \phi, i \rangle \) for a state with entirely spin \( i \) electron for an \( \ell^2 \) function \( \phi \).

It is well known that there exists a \( \gamma_0 > 0 \) so that \( |\gamma| < \gamma_0 \) implies \( H \) as defined in (1.1) obtains pure point spectrum with simple eigenvalues and exponentially decaying eigenvectors. Moreover, \( H \) obeys the fractional moment bound, see for example \cite{2}. That is, there exists a \( C < \infty \) so that, for any \( 0 < s < 1 \) and \( \gamma > 0 \) so that \( \frac{\gamma C}{1-s} < 1 \) and \( z \in \mathbb{C}^+ \)

\[
(2.2) \quad \mathbb{E} \left( |\langle y | (H - z)^{-1} |x\rangle|^s \right) < C \left( \frac{\gamma C}{1-s} \right)^{|x-y|}
\]

Fractional moment bounds for (2.1) may almost immediately be inferred from (2.2).

**Theorem 2.1.** For any \( 0 < s < 1/4 \), there exists a \( C_1 < \infty \) so that, for any \( \gamma > 0 \) so that \( \frac{\gamma C}{1-s} < 1 \) and \( z \in \mathbb{C}^+ \)

\[
\mathbb{E} \left( |\langle y, j | (h_g - z)^{-1} |x, i\rangle|^s \right) < C_1 \left( \frac{\gamma C}{1-s} \right)^{\frac{1}{2}d_2((x,i),(y,j))}
\]

One virtue of fractional moment bounds such as (2.2) is their use in obtaining statements of Anderson Localization. In particular, the property of dynamical localization for (1.1) was first obtained by this method. Dynamical localization for \( H \) states that, there exists \( C < \infty \) and \( \mu > 0 \) so that

\[
(2.3) \quad \mathbb{E} \left( \sup_{t} |\langle y | e^{-itH} |x\rangle| \right) < Ce^{-\mu|x-y|}.
\]

Dynamical localization is a strong form of Anderson Localization, indeed it implies spectral localization and bounds on eigenfunction correlators \cite{2}. No such strong form of Anderson localization follows for (2.1) in the metric \( d_2 \). However, we can extract a bound on rate of flipping spins for bounded times. Let us define the support of \( \zeta \) as \( \text{supp} \zeta = \{ x \in \mathbb{Z}^d : \zeta(x) \neq 0 \} \).

**Corollary 2.2.** There exists \( C < \infty, \mu > 0 \) and \( \gamma_0 > 0 \) so that for any \( 0 < \gamma < \gamma_0 \)

\[
\mathbb{E} \left( |\langle y, -i | e^{-ith_g} |x, i\rangle| \right) < tCe^{-\mu L}
\]

for \( L = \text{dist}(x, \text{supp} \zeta) + \text{dist}(y, \text{supp} \zeta) \).

Bounds of this form may be practically more useful for studying split level resonances in Anderson localized systems as it would be easier to obtain in many cases. Indeed, such a bound can be obtained for the model introduced in \cite{7}. On the other hand, for (2.1) we can study the behavior of states transferring between split energy levels much more precisely.
Our main theorem is the following. Let us define the localization center of a vector \( \phi \in H_2 \) as a point \( x \in \mathbb{Z}^d \) so that \( |\langle e_x, 0 | \phi \rangle| \vee |\langle 0, e_x | \phi \rangle| \) is maximized. Let us define distance
\[
d_1((x, i), (y, j)) = \| x - y \|
\]
Let \( \hat{P}_x \) project to states within distance \( |x|/2 \) of \(|x, +1\rangle\) in the \( d_1 \) distance, that is
\[
(2.4) \quad \hat{P}_x = \sum_{u, j : d_1((u, j), (x, +1)) < |x|/2} |u, j\rangle\langle u, j|.
\]
We will examine the dynamics of a nearly localized eigenvector with spin entirely +1.

**Theorem 2.3.** There is a sequence of vectors \( \phi_k \in H \) with localization centers \( x_k \) with the following properties.

There is an \( \epsilon > 0 \) so that, for \( t < e^{\epsilon|x_k|} \) the packet is stable, almost entirely spin up
\[
(2.5) \quad |\langle \phi_k, +1 | e^{-i\theta_1} | \phi_k, +1 \rangle| > 1 - e^{-\epsilon|x_k|}
\]
There is a time \( t < \infty \) so that the packet reflects almost perfectly to the spin down position
\[
(2.6) \quad |\langle \phi_k, -1 | e^{-i\theta_1} | \phi_k, +1 \rangle| > 1 - e^{-\epsilon|x_k|}.
\]
Finally, the support at all times is almost entirely contained in \( \Lambda_{|x_k|/2}(x_k) \)
\[
(2.7) \quad \inf_{t > 0} \| \hat{P}_{x_k} e^{-i\theta_1} | \phi_k, +1 \rangle \|_2 > 1 - e^{-\epsilon|x_k|}
\]

**Remark 2.4.** The purpose of \((2.7)\) is to emphasize that the particle travels to the second layer only by tunneling, as no significant portion of the wave packet is ever near the edge connecting the two layers.

In fact, \((2.7)\) can be improved to a dynamical localization statement. That is, a statement analogous to \((2.3)\) holds for \( H_g \), which can be seen from Lemma \((3.2)\) which depends on SULE \((3.1)\).

Let us describe the organization of the rest of the paper. In the following subsection \(2.1\) we demonstrate that the model with spin \( h_g \) may be reduced to rank one perturbations of the spinless model and thus obtain the spectral decomposition of \((2.1)\) through that of \((1.1)\). In Section \(3\) we introduce SULE localization and describe the effect of the SULE basis under a rank one perturbation. In Section \(4.1\) we recall the Minami estimate and in Section \(4.2\) we utilize the SULE basis and the Minami estimate to match vectors after a rank one perturbation. Finally in Section \(5\) we study the dynamics of the \((2.1)\) through the inherited SULE basis and matched eigenvalues, this obtains the proof of Theorem \(2.3\). In Appendix \(A\) we conclude with the short proofs of Theorem \(2.1\) and Corollary \(2.2\).

### 2.1. Reduction to \( H \).

\( h_g \) may be analyzed by considering the family of operators, acting on \( H \), defined by,
\[
(2.8) \quad H_g = H + gD
\]
where \( g \in \mathbb{R} \) and \( D = |\zeta\rangle\langle \zeta| \). We will relate the Anderson localized phase of \( h_g \) to \( H_{\pm g} \).

**Proposition 2.5.** (1.) For any \( g > 0 \) and \( \gamma > 0 \) the \( \sigma(h_g) = \sigma(H_g) \cup \sigma(H_{-g}) \).

(2.) Given \( g > 0 \) there is a \( \gamma_0 > 0 \) so that for \( 0 < \gamma < \gamma_0 \), \( h_g \) has simple pure point spectrum with exponentially localized eigenvectors in the \( d_1 \) metric. Moreover, \( \sigma_{pp}(h_g) = \sigma_{pp}(H_g) \cup \sigma_{pp}(H_{-g}) \), and the eigenvectors of \( h_g \) are of the form \( |\phi_\pm, +1\rangle \pm |\phi_\pm, -1\rangle \) where \( \phi_\pm \) is an eigenvector of \( H_{\pm g} \). Conversely, if \( \phi_\pm \) is an eigenvector of \( H_{\pm g} \) then \( |\phi_\pm, +1\rangle \pm |\phi_\pm, -1\rangle \) is an eigenvector of \( h_g \).

This result essentially follows from the fact that \( h_g \) commutes with the ‘rotation’ operator \( U \) on \( H_2 \), defined by \( U (|\phi^{(+1)}, +1\rangle + |\phi^{(-1)}, -1\rangle) = (|\phi^{(+1)}, -1\rangle + |\phi^{(-1)}, +1\rangle) \).
Proof. Recall that generalized eigenvectors with associated generalized eigenvalues in this context are formal solutions, $h\psi = \lambda \psi$ for $h = \mathfrak{h}_g$ or $h = H_{-g}$, not necessarily in $\ell^2$, which are polynomially bounded. The usual arguments imply that the spectrum is exactly the closure of the set of generalized eigenvalues. In fact, the growth of the generalized eigenfunctions may be bounded by a power $d/2$. For details, see [3].

Suppose $\psi = |\psi^{(1)+} \rangle + |\psi^{(1)-} \rangle$ is a generalized eigenvector, with generalized eigenvalue $\lambda$ ie $\mathfrak{h}_g \psi = \lambda \psi$. Then we have $H\psi^{(i)} + gD\psi^{(-i)} = \lambda\psi^{(i)}$, for $i = \pm 1$, from which we have

$$ (H \pm gD)(|\psi^{(1)+} \rangle \pm |\psi^{(1)-} \rangle) = \lambda(|\psi^{(1)+} \rangle \pm |\psi^{(1)-} \rangle). $$

At least one of $(|\psi^{(1)+} \rangle \pm |\psi^{(1)-} \rangle)$ are nonzero, thus, any generalized eigenvalue of $\mathfrak{h}_g$ is a generalized eigenvalue of $H_g$ or $H_{-g}$. Conversely, if $\phi_{\pm}$ is a generalized eigenvector of $H_{\pm g}$ with associated generalized eigenvalue $\lambda$, then $|\phi_{\pm}, +1 \rangle \pm |\phi_{\pm}, -1 \rangle$ is a generalized eigenvalue of $\mathfrak{h}_g$ with generalized eigenvalue $\lambda$. Thus, the set of generalized eigenvalues of $\mathfrak{h}_g$ is exactly given by the union of the generalized eigenvalues of $H_g$ and $H_{-g}$.

We first prove part (1.). The spectrum of $H_g$ (and $\mathfrak{h}_g$) is the closure of its generalized eigenvalues, as the generalized eigenvalues of $\mathfrak{h}_g$ is the union of generalized eigenvalues of $H_g$ and $H_{-g}$, the claim follows.

Let us now prove part (2.). For given $g$ and sufficiently small $\gamma > 0$, $H_g$ has almost surely simple point spectrum and all generalized eigenvectors are exponentially decaying eigenvectors. Moreover, with probability $1$ $\sigma_{pp}(H_g) \cap \sigma_{pp}(H_{-g}) = \emptyset$ [2], thus, in [23.4] at least one of $\psi^{(0)}(0) \pm \psi^{(1)}(0)$ is identically $0$. The symmetry of eigenvectors of $\mathfrak{h}_g$ follows. The converse statement is immediate. \square

3. Rank One perturbations

Let us consider stability of eigen-systems for $H_g$ over $g \in \Gamma \subset \mathbb{R}$. For small enough $\gamma$, the Hamiltonian $H_g = \gamma \Delta + gD + V$ is almost surely Anderson localized. For any eigenvector $\phi$ define the center of localization to be a site $x \in \mathbb{Z}^d$ such that $|x|^{d+1} \phi(x)| = \|X^{d+1} \phi\|_\infty$. Where this definition is ambiguous, select $x$ which is left-most in terms of a lexicographic ordering of elements $(x_1, \ldots, x_d)$. We will always suppose $\Gamma \subset \mathbb{R}$ is finite and $0 \in \Gamma$. For $g \in \Gamma$, let $I^{(g)}$ index the eigenpairs $(\phi_\xi^{(g)}, \lambda_\xi^{(g)})$ with normalized eigenvectors $\|\phi_\xi^{(g)}\|_2 = 1$, so that $(H + gD)\phi_\xi^{(g)} = \lambda_\xi^{(g)}\phi_\xi^{(g)}$. Finally, let $x_\xi^{(g)} \in \mathbb{Z}^d$ denote the center of localization of $\phi_\xi^{(g)}$ selected above.

3.1. SULE Localization. We recall the statement of SULE localization [2](Theorem 7.4). Let $\Gamma \subset \mathbb{R}$ be a fixed finite set.

**Theorem 3.1.** Given $\Gamma$, there is $\gamma_0 > 0$, so that for any $0 < \gamma < \gamma_0$ there exists a set $\Omega_1 \subset \Omega$ so that $P(\Omega_1) = 1$ and for every $\omega \in \Omega_1$ the Hamiltonian $H_g$ has pure point spectrum, dense in $\gamma[0, 4d] + \text{supp}(\rho)$, such that all eigenvectors are exponentially decaying.

**SULE (Semi Uniformly Localized Eigenfunctions)** There exists a constant $A_\omega = A_\omega(\gamma, \Gamma)$ and $\xi > 0$ so that for every $g \in \Gamma$ and $\xi \in I^{(g)}$

$$ |\phi_\xi^{(g)}(x)| \leq A_\omega(1 + |x_\xi^{(g)}|)^{d+1} e^{-\xi|x-x_\xi^{(g)}|}. $$

Let us define the local index set, for $g \in \Gamma$ and $\Lambda \subset \mathbb{Z}^d$, the set is defined as

$$ I^{(g)}_\Lambda := \{i : x_\xi^{(g)} \in \Lambda \} $$

and the local spectrum,

$$ \Sigma^{(g)}_\Lambda := \{\lambda_\xi^{(g)} : i \in I^{(g)}_\Lambda\}. $$
For \( \Lambda \subset \mathbb{Z}^d \) let us write \( P_\Lambda = \sum_{x \in \Lambda} |x\rangle \langle x| \) and for \( g \in \Gamma \) we write the local eigenbasis projection
\[
P_\Lambda^{(g)} = \sum_{i \in I_\Lambda^{(g)}} |\phi_i^{(g)}\rangle \langle \phi_i^{(g)}|.
\]

In the next section we compare projections \( P_\Lambda \) to \( P_\Lambda^{(g)} \).

### 3.2. Concentration of indices of SULE localization

We first show a limit on the fluctuation of concentration of localization centers. Let us define the \( \ell \) neighborhood of a set \( \Lambda \),
\[
\hat{B}_\Lambda^{(\ell)} := \bigcup_{x \in \Lambda} \{ u \in \mathbb{Z}^d : |u - x| \leq \ell \}
\]
the \( \ell \) core of \( \Lambda \) is the set
\[
\hat{\Xi}_\Lambda^{(\ell)} = \mathbb{Z}^d \setminus \hat{B}_\Lambda^{(\ell)}.
\]
Depending on the choice of \( \Lambda \) and \( \ell \), the set \( \hat{\Xi}_\Lambda^{(\ell)} \) may be empty.

Let us define a parameter \( p > 1 \) and a sufficiently large constant \( C_\omega \) depending on \( d, \xi, p \), and the realization of the random field. For \( u \in \mathbb{Z}^d \) let \( \ell_u = \log^p(C_\omega + |u|) \), for \( \Lambda \subset \mathbb{Z}^d \) let \( \ell_\Lambda = \sup_{u \in \Lambda} \ell_u \), we define the ‘standard’ neighborhood and core as
\[
(3.4) \quad B_\Lambda := \hat{B}_\Lambda^{(\ell_\Lambda)}; \quad \Xi_\Lambda := \hat{\Xi}_\Lambda^{(\ell_\Lambda)}.
\]

The following lemma relates the basis of \( I_\Lambda^{(g)} \) to the spatial basis \( \langle |x\rangle \rangle_x \).

**Lemma 3.2.** Let parameters \( p, \xi \) be as above. Moreover, fix \( \alpha < 1 \), then let \( C_\omega \) be sufficiently large depending on \( p, \xi \) and \( C_\omega \). Let \( \Lambda \) be any subset \( \Lambda \subset \mathbb{Z}^d \) and let \( \ell \geq \ell_\Lambda \).

We have,
\[
(3.5) \quad \| (1 - P_{\hat{B}_\Lambda^{(g)}}) P_\Lambda^{(g)} \| < e^{-\xi \ell}.
\]
which implies,
\[
(3.6) \quad \alpha \tr(P_\Lambda^{(g)}) < \tr(P_{B_\Lambda}).
\]
On the other hand,
\[
(3.7) \quad \| (1 - P_{\hat{\Xi}_\Lambda^{(g)}}) P_{\Xi_\Lambda} \| < e^{-\xi \ell}.
\]
which implies,
\[
(3.8) \quad \alpha \tr(P_{\Xi_\Lambda}) < \tr(P_\Lambda^{(g)}).
\]

Observe that \( 3.5 \) states \( P_\Lambda \) projects almost entirely to \( P_{B_\Lambda} \), similarly, \( 3.7 \) states \( P_{\Xi_\Lambda} \) projects almost entirely to \( P_\Lambda^{(g)} \). This observation leads to the implied statements \( 3.6 \) and \( 3.8 \) whose proofs are due to the prefactors in \( 3.1 \).

As a first application of Lemma \( 3.2 \) we bound the number of localization centers contained in a box.

For any \( u \in \mathbb{Z}^d \) and the box \( \Lambda_{L}(u) := \{ x \in \mathbb{Z}^d : |x - u| < L \} \), the nesting
\[
\Lambda_{(L - \log^p(C_\omega + |u| + L))}(u) \subset \Xi_\Lambda \subset \Lambda_{L}(u) \subset \Lambda_{(L + \log^p(C_\omega + |u| + L))}(u)
\]
implies a bound on the number of states in the box
\[
(3.9) \quad \alpha |\Lambda_{(L - \log^p(C_\omega + |u| + L))}(u)| \leq |I_{\Lambda_{L}}^{(g)}(u)| \leq \alpha^{-1} |\Lambda_{(L + \log^p(C_\omega + |u| + L))}(u)|.
\]

Now we prove the lemma.
Proof. The proof of the concentration inequalities (3.8) and (3.6) will follow immediately from estimates (3.5) and (3.7). Indeed, all that is required is a general calculation for a separable Hilbert space $H$. Let $\tilde{P}_a$ and $\tilde{P}_b$ be geometric projectors associated to different orthonormal eigenbases. That is, let $\{\tilde{\phi}_i^{(a)}\}$ and $\{\tilde{\phi}_i^{(b)}\}$ be o.n. bases of $\tilde{H}$. Let $\tilde{P}_a = \sum_{i=1}^N |\tilde{\phi}_i^{(a)}\rangle \langle \tilde{\phi}_i^{(a)}|$ and $\tilde{P}_b = \sum_{j=1}^M |\tilde{\phi}_j^{(b)}\rangle \langle \tilde{\phi}_j^{(b)}|$

$$\text{tr}[\tilde{P}_a] = \text{tr}[\tilde{P}_b \tilde{P}_a] + \text{tr}[(1 - \tilde{P}_b) \tilde{P}_a]$$

We may extract a $\text{tr}[\tilde{P}_a]$ from the second term,

$$\text{tr}[(1 - \tilde{P}_b) \tilde{P}_a] = \text{tr}[(1 - \tilde{P}_b) \tilde{P}_a \tilde{P}_a] \leq \text{tr}[\tilde{P}_a] \| (1 - \tilde{P}_b) \tilde{P}_a \|$$

Thus we have,

$$(3.10) \quad \text{tr}[\tilde{P}_a] \left(1 - \| (1 - \tilde{P}_b) \tilde{P}_a \| \right) \leq \text{tr}[\tilde{P}_b \tilde{P}_a] \leq \text{tr}[\tilde{P}_b]$$

Now let us prove (3.5). Let $i \in I^{(g)}_\Lambda$, a state with localization center $x = x_i$ inside $\Lambda$ then, as $\ell \geq \ell^{\Lambda} \geq \ell_x$,

$$\|(1 - P_{B\Lambda}) P^{(g)}_\Lambda | \phi_i^{(g)} \rangle \|^2 = \sum_{y \notin B\Lambda} |\phi_i^{(g)}(y)|^2 \leq \sum_{y : |x-y| > \ell} |\phi_i^{(g)}(y)|^2$$

Let us use (3.11) to bound the tail of $\phi_i$,

$$\|(1 - P_{B\Lambda}) P^{(g)}_\Lambda | \phi_i^{(g)} \rangle \|^2 \leq \sum_{k=\ell_x}^{\infty} C_d A_\omega^2 (1 + |x|)^{4\nu} k^{d-1} e^{-2\xi k} \leq e^{-\xi \ell}.$$ 

The final inequality follows if $\ell_x = \log^\nu (C_\omega + |x|)$ is taken large enough. We obtain (3.6) via (3.10).

Now let us prove (3.7) for $x \in Z_\Lambda$

$$\|(1 - P^{(g)}_\Lambda) P_{Z\Lambda} | x \rangle \|^2 = \sum_{i \notin I^{(g)}_\Lambda} | \phi_i^{(g)}(x)|^2.$$ 

Let us first expand the sum on the right hand side at each $y \notin \Lambda$, by the eigenfunctions with localization centers at $x_i = y$,

$$(3.11) \quad \sum_{i \notin I^{(g)}_\Lambda} | \phi_i^{(g)}(x)|^2 = \sum_{y : |x-y| > \ell} \sum_{i \notin I^{(g)}_\Lambda} | \phi_i^{(g)}(x)|^2 \leq \sum_{y : |x-y| > \ell} \left( \sum_{i \notin I^{(g)}_\Lambda} | \phi_i^{(g)}(x)|^2 \right).$$ 

From (3.6) the number of states with localization center at site $y$ is bounded by $\alpha^{-1} |B_y| \leq \alpha^{-1} C_d \log^{dp}(C_\omega + |y|)$. Moreover, we use the SULE localization bound for $i \notin I^{(g)}_{(y)}$, thus we have,

$$(3.12) \quad \sum_{i \notin I^{(g)}_\Lambda} | \phi_i^{(g)}(x)|^2 \leq \sum_{y : |x-y| > \ell} \alpha^{-1} C_d \log^{dp}(C_\omega + |y|) A_\omega^2 (1 + |y|)^{2d+2} e^{-2\xi |x-y|}.$$ 

Clearly we have $\alpha^{-1} C_d \log^{dp}(C_\omega + |y|) A_\omega^2 (1 + |y|)^{2d+2} \leq \dot{A}_{\omega}^2 (1 + |y|)^{2d+3}$ for a large constant $\dot{A}_\omega$. For large enough $C_\omega$, which implies $\ell_x$ is large, we have,

$$(3.13) \quad \sum_{y : |x-y| > \ell} \dot{A}_{\omega}^2 (1 + |y|)^{2d+3} e^{-\xi |x-y|} \leq 1.$$ 

Which implies $\sum_{i \notin I^{(g)}_\Lambda} | \phi_i^{(g)}(x)|^2 < e^{-\xi \ell}$ obtaining (3.7) using (3.10). \qed
4. Matching Eigenbases

We are able to match eigenbases under rank for indices in regions with SULE localization and well separated eigenvalues. The first step is to probabilistically control the local separation of eigenvalues. The second step is a relatively standard perturbation argument for a system assuming well separated eigenvalues.

4.1. Minami Estimate. Let us recall the standard Minami estimate for the Anderson model \( H = \gamma \Delta + V \) on a box \( \Lambda \). For an operator \( A \) on \( \mathcal{H} \) and a subset \( \Lambda \subset \mathbb{Z}^d \) let us define the restriction to \( \Lambda \) as \( A_\Lambda = P_\Lambda A P_\Lambda \). We are only interested in sets \( \Lambda \) away from the support of \( \zeta \) so this is equivalent to the local Minami estimates for \( H \). That is, for \( \Lambda = \Lambda_\ell(u) \), so that \( \Lambda \cap \operatorname{supp} \zeta = \emptyset \), we have \((H + gD)_\Lambda = H_\Lambda \). As we will consider only boxes with support away from the support of \( \zeta \), we will simply use \( H_\Lambda \) below without further comment. For \( \phi \in \mathcal{H} \) let us define a restriction and normalization to \( \Lambda \) as,

\[
\phi_\Lambda = \frac{\|\phi\|}{\|P_\Lambda \phi\|} P_\Lambda \phi.
\]

Finally let us define the minimal separation for values in a finite set. For a finite set \( T = \{t_1, \ldots, t_N\} \), we define,

\[
\Delta_{\text{min}}[T] := \min\{|t_i - t_j| : i \neq j\}.
\]

Now let us introduce Minami’s estimate [2] for the local minimum separation of eigenvalues.

Theorem 4.1. For any interval \( J \subset \mathbb{R} \) and subset \( \Lambda \subset \mathbb{Z}^d \),

\[
\mathbb{P}(\text{tr} P_{(H + gD)_\Lambda}(J) \geq 2) \leq \pi^2 \left( \frac{1}{2} \right)(\|\rho\|_\infty |J||\Lambda|)^2.
\]

As an immediate corollary we find a probabilistic bound on \( \Delta_{\text{min}}[\sigma(H_\Lambda)] \).

Corollary 4.2. For \( \rho \) with compact support, there is some finite \( C \) so that for any \( \epsilon > 0 \),

\[
\mathbb{P}(\Delta_{\text{min}}[\sigma(H_\Lambda)] < \epsilon) < C(\|\rho\|_\infty |\Lambda|)^2 \epsilon.
\]

Let \( (u_k)_{k=1}^\infty \) be a sequence of sites \( u_k \in \mathbb{Z}^d \) such that, there is a corresponding sequence \( (L_k)_{k=1}^\infty \) with elements \( L_k \geq 1 \) so that \( L_k \to \infty \) and, for all \( k, k' \), |\( u_k - u_{k'}| > L_k + L_{k'} \). For \( k \in \mathbb{N} \) let \( \Lambda_k = \Lambda_{L_k}(u_k) \) and let \( \epsilon_k \leq L_{k}^{-2d-1} \).

Corollary 4.3. There are infinitely many \( k \in \mathbb{N} \) so that \( \Delta_{\text{min}}[\sigma(H_{\Lambda_k})] > \epsilon_k \).

Proof. The events \( \{\Delta_{\text{min}}[\sigma(H_{\Lambda_k})] > \epsilon_k\} \) are independent, and obey \( \mathbb{P}(\Delta_{\text{min}}[\sigma(H_{\Lambda_k})] > \epsilon_k) > 1 - CL_k^{-1} \).

In particular, \( \mathbb{P}(\Delta_{\text{min}}[\sigma(H_{\Lambda_k})]) \to 1 \) so that the sum of the probabilities over \( k \) is infinite. Thus, the corollary follows from the second Borel-Cantelli theorem. \( \square \)

4.2. Matching Eigenbases. Let us now combine the above Minami estimate with a SULE localized model. We will establish that the labelings are stable under the rank one perturbation.

Let \( C_\omega \) be a sufficiently large constant, and for \( x \in \mathbb{Z}^d \) and \( \Lambda \subset \mathbb{Z}^d \) define \( \ell_x \) and \( \ell_\Lambda \) as discussed in Section 3.2. We will consider \( \ell > \ell_\Lambda \), so that, from the definitions in Section 3.2,

\[
(4.2) \quad \hat{B}_\Lambda^{(\ell)} \supset B_\Lambda \supset \Lambda \supset \Xi_\Lambda \supset \hat{\Xi}_\Lambda^{(\ell)}.
\]

We say an eigenpair of \( H + gD \) (indexed by \( i \in \mathcal{I}(g) \)) \((\epsilon, r)\)-corresponds to an eigenpair of \( H + g'D \) (indexed by \( j \in \mathcal{I}(g') \)) if \(|\lambda_i^{(g)} - \lambda_j^{(g')}| < \epsilon\),

\[
(4.3) \quad |\langle \phi_i^{(g)}, \phi_j^{(g')} \rangle| > 1 - \epsilon,
\]

\( \Box \)
and \(|x_i^{(g)} - x_j^{(g')}| \leq r\). For any \((\epsilon, r)\)-corresponding eigenvectors we will always choose the phase so that 
\[
|\langle \phi_i^{(g)}, \phi_j^{(g')} \rangle - 1| < \epsilon.
\]
Let \(I_{\Lambda}^{(g,g')}(\epsilon, r)\) be the pairs of indices for \((\epsilon, r)\)-corresponding eigenpairs with localization centers contained in \(\Lambda\).

**Theorem 4.4.** Suppose SULE holds for \(H + gD\) for all \(g \in \Gamma\). Given \(\Lambda \subset \mathbb{Z}^d\), let \(\ell > \ell_\Lambda\) and let us write \(B = \hat{B}_\Lambda^{(\ell)}\) and \(\Xi = \hat{\Xi}_\Lambda^{(\ell)}\).

1. For all \(g \in \Gamma\) and every \(i \in I_{\Lambda}^{(g)}\), there is an eigenvalue \(\hat{\lambda}_i\) of \(H_B\) so that
\[
|\lambda_i^{(g)} - \hat{\lambda}_i| < e^{-\xi \ell / 4}.
\]
Moreover, assume that \(H_B\) has simple spectrum satisfying \(\Delta_{\min}[\sigma(H_B)] > \epsilon\) for some \(\epsilon > e^{-\xi \ell / 8}\). Then we have the following.

2. For \(g', g'' \in \Gamma\), suppose there are \(i \in I_{\Lambda}^{(g')}\) and \(j \in I_{\Lambda}^{(g'')}\) associated to the same \(\eta \in \sigma(H_B)\), that is
\[
|\lambda_i^{(g')} - \eta|, |\lambda_j^{(g'')} - \eta| < e^{-\xi \ell / 4},
\]
then
\[
|\langle \phi_i^{(g')}, \phi_j^{(g'')} \rangle| > 1 - e^{-\xi \ell / 16}
\]
(4.4)

3. For any \(g \in \Gamma\), the associated eigenvalues of \(H_B\) are distinct. That is, for \(i, j \in I_{\Lambda}^{(g)}\) so that \(i \neq j\), we have \(\eta_i \neq \eta_j\). Moreover, there is a minimum separation for the local spectrum at \(\Lambda\):
\[
\Delta_{\min}\left[\Sigma_{\Lambda}^{(g)}\right] > \epsilon / 2.
\]

4. For any \(g', g'' \in \Gamma\),
\[
|I_{\Lambda}^{(g',g'')} (e^{-\xi \ell / 16}, \ell_\Lambda)| \geq 2\alpha |\Xi| - |B|
\]
(4.5)
that is there at least \(2\alpha |\Xi| - |B|\) pairs of \((e^{-\xi \ell / 16}, \ell_\Lambda)\)-corresponding eigenstates with centers in \(\Lambda\).

It follows immediately from Corollary 4.3 and 4.4 that infinitely many eigenvectors will be matchable between models. For \(u_0 \in \mathbb{Z}^d \setminus \{0\}\), let \(u_k = 4^{k+1}u_0\) for all \(k \in \mathbb{N}\) and \(L_k = 4^{k-1}|u_0|\), finally let \(\Lambda_k = \Lambda_{L_k}(u_k)\) and \(\ell_k = \ell_{\Lambda_k}\).

**Corollary 4.5.** Suppose SULE holds for \(H + gD\) for \(g = g', g'' \in \Gamma\). There is an \(\epsilon_0 > 0\) so that for any \(\epsilon\) satisfying \(0 < \epsilon < \epsilon_0\), there are infinitely many \(k \in \mathbb{N}\) so that there exist \(i_k \in I_{\Lambda_k}^{(g')}\) and \(j_k \in I_{\Lambda_k}^{(g'')}\) which index \((e^{-\epsilon|x_{i_k}|}, \ell_k)\)-corresponding eigenpairs.

**Proof.** Let \(c > 0\) be sufficiently small and for all \(k\) let \(l_k = cL_k\) let \(B_k = \hat{B}_{\Lambda_k}^{(l_k)} = \Lambda_{L_k + l_k}(u_k)\) and \(\Xi_k = \hat{\Xi}_{\Lambda_k}^{(l_k)} = \Lambda_{L_k - l_k}(u_k)\). Let \(\epsilon_k = L_k^{-2\nu}\), then Corollary 4.3 states there are infinitely many \(k \in \mathbb{N}\) so that \(\Delta_{\min}[\sigma(H_{B_k})] > \epsilon_k\).

For each such \(k\) we apply Theorem 4.4 with \(\Lambda = \Lambda_k\), \(B = B_k\) and \(\Xi = \Xi_k\). From Theorem 4.4 there are \(2\alpha |\Xi_k| - |B_k| \geq \frac{9}{2} |\Xi_k|\) many \((e^{-\xi \ell_k / 16}, \ell_k)\) corresponding pairs of eigenvectors. From each such \(k\) let \(i_k \in I_{\Lambda_k}^{(g')}\) and \(j_k \in I_{\Lambda_k}^{(g'')}\) index one pair of these vectors. Finally, for large enough \(k\), \(\epsilon|x_{i_k}| < \xi \ell_k / 16\) which obtains the result.

We now prove Theorem 4.4.

**Proof.** We begin by truncating the eigensystem on \(\mathbb{Z}^d\) to the subset \(B\) and comparing these to the eigensystem of \(H_B\).

For \(g \in \Gamma\) and \(i \in I_{\Lambda}^{(g)}\) from (3.3) in Lemma 3.2 the normalization factors in (4.1) obey \(\|\phi_i^{(g)}\| < 2\|P_B\phi_i^{(g)}\|\). Moreover, let us write the remainder term
\[
\left( (H)_B - \lambda_i^{(g)} \right) (\phi_i^{(g)})_B = \frac{\|\phi_i^{(g)}\|}{\|P_B\phi_i^{(g)}\|} \sum_{x \in \partial B} |x\rangle \sum_{y \sim x \not\in B} \phi_i^{(g)}(y) =: R_i^{(g)}.
\]
Now we may use (3.5), to bound the norm of the remainder,
\[ \| R_i^{(g)} \|_2^2 \leq \left( \frac{\| \phi_i^{(g)} \|}{\| P_B \phi_i^{(g)} \|} \right)^2 (1 - P_B) \phi_i^{(g)} \| \leq 4e^{-\xi \ell} \]

On the other hand, observe that
\[ \langle (\phi_i^{(g)})_\lambda \rangle (H_A - \lambda_i^{(g)})^2 (\phi_i^{(g)})_\lambda \rangle = \| R_i^{(g)} \|_2^2 \]

Thus, by the min-max theorem, there is an eigenvalue \( \eta \in \sigma(H_B) \), with an associated eigenvector \( \phi \in \mathbb{C}^B \), so that \( |\eta - \lambda_i^{(g)}| < 2e^{-\xi \ell/2} < e^{-\xi \ell/4} \), which proves part 1.

Let us compare the truncated and normalized eigenvectors to the eigenvector of the truncated system. Continue to assume \( g \in \Gamma \) and \( i \in I_{\lambda_i}^{(g)} \) and that \( \eta \) is the \( H_B \) associated to \( \lambda_i^{(g)} \). Now let \( 0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_N \) be the eigenvalues of \( (H_B - \eta)^2 \), and let \( \mathcal{P}_j \) project to the eigenspace of \( H_B \) associated to \( \eta_j \). Now, from the calculation in part 1, we have,
\[ \langle (\phi_i)_B \rangle (H_A - \eta)^2 (\phi_i)_B \rangle = \| (\lambda_i^{(g)} - \eta) (\phi_i^{(g)})_B + R_i^{(g)} \|_2 \leq 4e^{-\xi \ell/2} \]

On the other hand we have,
\[ \langle (\phi_i)_B \rangle (H_A - \eta)^2 (\phi_i)_B \rangle = \sum_{j=1}^{\infty} \eta_j \langle (\phi_i)_B \mathcal{P}_j (\phi_i)_B \rangle \geq \eta_1 \langle (\phi_i^{(g)})_B (1 - P_0)(\phi_i^{(g)})_B \rangle \]

But the \( \phi_i^{(g)} \), and therefore the \( (\phi_i^{(g)})_B \) are normalized, thus we may combine (4.7) and (4.8) to obtain
\[ 1 - \frac{4e^{-\xi \ell/2}}{\eta_1} \leq \langle (\phi)(\phi_i^{(g)})_B \rangle \]

We will obtain both conclusions 2 and 3 of the lemma from equation (4.9). Let us now suppose that \( \sigma(H_B) \) is simple and \( \min^{\Lambda}(\sigma(H_B)) > \epsilon \). Under this assumption \( \eta_1 > \epsilon \).

We now prove part 2. From part 1, we have that every eigenvalue of \( \Sigma^{(g)} \) is within \( e^{-\xi \ell/4} \) of \( \sigma(H_B) \). For an eigenpair \( (\phi, \eta) \) of \( H_A \) and \( g', g'' \in \Gamma \) let us suppose there are two indices \( i \in I_{\lambda_i}^{(g')} \) and \( j \in I_{\lambda_j}^{(g'')} \) so that \( |\lambda_i^{(g')} - \eta|, |\lambda_j^{(g'')} - \eta| \leq e^{-\xi \ell/4} \). From, (4.9) we have for \( (\alpha, g) = (i, g'), (j, g'') \),
\[ 1 - \frac{4}{\epsilon}e^{-\xi \ell/2} \leq \langle \phi | \psi^{(g)}_{\alpha} \rangle \lambda^{(g)} \]

Therefore, as \( \psi^{(g)}_{\alpha} \) is normalized, we may write \( \langle \psi^{(g)}_{\alpha} \rangle_\lambda = \langle \phi | \psi^{(g)}_{\alpha} \rangle \lambda | \phi \rangle + | \phi \rangle \), where \( \| \phi \| \leq \frac{1}{\epsilon}e^{-\xi \ell/2} \) and \( \langle \phi | \phi \rangle = 0 \). It follows that
\[ \left| \langle (\phi_i^{(g')})_\lambda | (\phi_j^{(g'')})_\lambda \rangle \right| \geq \left| \langle \phi | (\phi_i^{(g')})_\lambda \rangle \phi | (\phi_j^{(g'')})_\lambda \rangle \right| - \left| \hat{\phi}_i \hat{\phi}_j \right| \geq \left( 1 - \frac{4}{\epsilon^2}e^{-\xi \ell/2} \right) - \left( \frac{4}{\epsilon^2}e^{-\xi \ell/2} \right) \].

The conclusion, (4.4) follows from the lower bound of \( \epsilon \).

Let us now show part 3, the separation in the local spectrum. From part 1, we have that every eigenvalue of \( \Sigma^{(g)}_A \) is within \( e^{-\xi \ell/4} \) of \( \sigma(H_B) \). Thus, using \( g = g' = g'' \) suppose there is \( i, j \in I_{\lambda_i}^{(g)} \), then
\[ \left| \langle (\phi_i^{(g')})_\lambda | (\phi_j^{(g'')})_\lambda \rangle \right| \leq 4 \left| \langle P_A \phi_i^{(g')} | P_A \phi_j^{(g'')} \rangle \right| = 4 \left| \langle \phi_i^{(g')} | (P_A - 1) \phi_j^{(g'')} \rangle \right| + (P_A - 1) \phi_j^{(g'')} \right| \]

if \( i \neq j \), then \( \langle (\phi_i^{(g')} | \phi_j^{(g'')}} = 0 \) so that, using (3.7)
\[ \left| \langle (\phi_i^{(g')})_\lambda | (\phi_j^{(g'')})_\lambda \rangle \right| \leq 16e^{-\xi \ell} \]

On the other hand, if \( \lambda_i^{(g')} \) and \( \lambda_j^{(g'')} \) are both associated to eigenvalue \( \eta \in \sigma(H_B) \) then \( \phi_i^{(g')}, \phi_j^{(g'')} \) both obey (4.4) which is clearly a contradiction. Thus every eigenvalue of \( \lambda_i^{(g')} \in \Sigma^{(g)}_A \) is associated to a distinct eigenvalue \( \hat{\eta}_i \) of \( \sigma(H_B) \), so that \( |\lambda_i^{(g')} - \hat{\eta}_i| < e^{-\xi \ell/4} \). The desired minimum separation follows from the minimum separation of \( \sigma(H_B) \), that is \( |\lambda_i^{(g')} - \lambda_j^{(g'')}| > \epsilon - 2e^{-\xi \ell/4} \).
Finally, let us prove part 4, again we assume a minimal separation of $\epsilon$ of the spectrum of $H_B$. From part 3 every index of $I_{\Lambda}(g')$ is associated to a distinct eigenvalue of $H_B$. Using the pigeon hole principle we see that there are $|I_{\Lambda}(g')| + |I_{\Lambda}(g'')| - |\sigma(H_B)|$ pairs of indices associated to the same eigenvalue of $H_B$. Of course $|\sigma(H_B)| = |B|$, on the other hand, from (3.8) we have, $|I_{\Lambda}(g')| + |I_{\Lambda}(g'')| \geq 2\alpha|\Xi_{\Lambda}|$. We claim that indices of $I_{\Lambda}(g')$ and $I_{\Lambda}(g'')$ corresponding to the same eigenvalue of $H_B$ are $(2e^{-\xi\ell/4}, \ell)$ corresponding so that (4.5) follows from (4.2).

To obtain the claim, consider $i \in I_{\Lambda}(g')$ and $j \in I_{\Lambda}(g'')$ both associated to the same eigenvalue $\eta \in \sigma(H)$. Closeness of eigenvalues and eigenvectors follow from part 2 of the theorem: indeed, we obtain closeness for

\[|x_i^{(g')} - x_j^{(g'')}| > \log p(C_\omega + |x_i^{(g')}| + |x_j^{(g'')}|)\]

it would then follow from (3.12) that,

\[|\langle \phi_i^{(g')} | \phi_j^{(g'')} \rangle| < e^{-\eta|x_i^{(g')} - x_j^{(g'')}|/8}.
\]

Indeed, writing $y_1 = x_i^{(g')}, y_2 = x_j^{(g'')}$ and $r = |x_i^{(g')} - x_j^{(g'')}|$ note that

\[
\|P^{(g')}_{y_1} P^{(g'')}_{y_2}\| \leq \|P^{(g')}_{y_1} (1 - P_{B_{y_1}^{r/4}}) P^{(g'')}_{y_2}\|^2 + \|P^{(g')}_{y_1} (1 - P_{B_{y_2}^{r/4}}) P^{(g'')}_{y_2}\|^2.
\]

From which we have

\[
\|P^{(g')}_{y_1} P^{(g'')}_{y_2}\|^2 \leq 2e^{-(\epsilon r/4)}
\]

which obtains (4.10).

\[\Box\]

5. Tunneling between corresponding eigenvectors

In this section we utilize the description of $(\epsilon, \ell)$-corresponding pairs of $H_g$ and $H_{-g}$ to analyze the evolution operator $e^{-it\hat{g}_s}$ on selected states initialized in the spin up state. That is, let us assume $H_g$ and $H_{-g}$ both obtain SULE localization.

Let us write the indices of the eigenbases of $H_{\pm g}$ as $i_{\pm} \in I(\pm g)$. The corresponding eigenvectors and eigenvalues are then $\phi^{(\pm g)}_{i_{\pm}}$ and $\lambda^{(\pm g)}_{i_{\pm}}$. By Proposition 2.5 the (normalized) eigenvectors of $h_g$ are exactly the vectors

\[\psi_{i_{\pm}} = \frac{1}{\sqrt{2}} \left( |\phi^{(\pm g)}_{i_{\pm}}, +1\rangle \pm |\phi^{(\pm g)}_{i_{\pm}}, -1\rangle \right)\]

for $i_{\pm} \in I(\pm g)$.

**Proposition 5.1.** Let $i_{\pm} \in I^{(\pm g)}_{\Lambda}$ index a given $(e^{-\epsilon|x_i^{(g')}|}, \ell_{\Lambda})$-corresponding eigenpair. We have the following for all $t > 0$,

\[\left\| e^{-it\hat{g}} |\phi_{i_{+}}^{(g')}, +1\rangle - \frac{1}{\sqrt{2}} \left( Je^{-it\lambda_{i_{+}}^{(g)}} + Ie^{-it\lambda_{i_{-}}^{(g)}} \right) \psi_{i_{+}} \right\| < e^{-\epsilon|x_i^{(g')}|/4}
\]

for $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $I = J^2$.

**Proof.** First let us consider the expression of $|\phi_{i_{+}}^{(g')}, 0\rangle$ in the eigenbasis of $h_g$. For $j_{-} \in I_{-g}$ we may write $a_{j_{-}} = \langle \phi_{j_{-}}^{(-g)} | \phi_{i_{+}}^{(g')} \rangle$ using expression (5.1) we have

\[
\frac{1}{\sqrt{2}} \left( |\phi_{i_{+}}^{(g')}, +1\rangle + |\phi_{i_{+}}^{(g')}, -1\rangle \right) = \sum_{j_{-} \in I_{-g}} a_{j_{-}} \psi_{j_{-}}.
\]
Thus, we may readily represent a spin up state as
\[ |\phi_{t+}^{(g)}, +1 \rangle = \frac{1}{\sqrt{2}} \psi_{t+} + \frac{1}{\sqrt{2}} \sum_{j \in I(\ell_{1})} \frac{1}{\sqrt{2}} a_{j} \psi_{j}. \]

Let us apply the evolution operator to this expansion,
\[ e^{-it(\hbar_{1} - \lambda_{\ell_{1}}^{(g)})} |\phi_{t+}^{(g)}, +1 \rangle = \frac{1}{\sqrt{2}} \psi_{t+} + \frac{1}{\sqrt{2}} \sum_{j \in I(\ell_{1})} e^{-it(\lambda_{\ell_{1}}^{(g)} - \lambda_{j}^{(g)})} a_{j} \psi_{j}. \]

Expanding \( \psi_{t+} \) we find
\[ e^{-it(\hbar_{1} - \lambda_{\ell_{1}}^{(g)})} |\phi_{t+}^{(g)}, +1 \rangle = \frac{1}{\sqrt{2}} \sum_{j \in I(\ell_{1})} \left( J + i e^{-it(\lambda_{\ell_{1}}^{(g)} - \lambda_{j}^{(g)})} \right) a_{j} \psi_{j}. \]
the result follows as
\[ \left\| \frac{1}{\sqrt{2}} \sum_{j \in I(\ell_{1})} \left( J + i e^{-it(\lambda_{\ell_{1}}^{(g)} - \lambda_{j}^{(g)})} \right) a_{j} \psi_{j} \right\|^2 \leq 4 \sum_{j \in I(\ell_{1})} |a_{j}|^2 \leq 8e^{-\epsilon|x_{t+}^{\ell_{1}}|}. \]
and
\[ \left\| \phi_{t+}^{(g)} - \left( \phi_{t+}^{(g)} \right) \phi_{t+}^{(-g)} \phi_{t+}^{(-g)} \right\| < 4e^{\epsilon/2} \]
by the eigenfunction correspondence assumption. \( \square \)

**Proposition 5.2.** Let \( i_{\pm} \in I_{\Lambda}^{(g)} \) index a given \((e^{-\epsilon|x_{t+}^{\ell_{1}}|}, \ell_{1})\)-corresponding eigenpair. For all \( 0 < t < e^{-\epsilon|x_{t+}^{\ell_{1}}|/2} \)
\[ |\langle \phi_{t+}^{(g)}, +1 | e^{it\hbar_{1}} |\phi_{t+}^{(g)}, +1 \rangle - 1 \rangle | \leq e^{-\epsilon|x_{t+}^{\ell_{1}}|/4}. \]

**On the other hand, for** \( x_{t+}^{(g)} \) is sufficiently far from the origin,
\[ \sup_{t>0} \left\| (I - \hat{P}_{x_{t+}^{(g)}}) e^{it\hbar_{1}} |\phi_{t+}^{(g)}, +1 \rangle \right\| < e^{-\epsilon|x_{t+}^{\ell_{1}}|/4}. \]

**Finally, for** \( t = \pi |\lambda_{t_{+}}^{(g)} - \lambda_{\ell_{1}}^{(-g)}|^{-1} \)
\[ e^{-it\hbar_{1}} |\phi_{t+}^{(g)}, +1 \rangle - e^{-it\hbar_{1}} |\phi_{t+}^{(g)}, -1 \rangle \leq e^{-\epsilon|x_{t+}^{\ell_{1}}|/4}. \]

Theorem 2.3 now follows by combining Corollary 4.5 with Theorem 5.2. Indeed, Corollary 4.5 guarantees an infinite sequence of index pairs \( i_{k}, j_{k} \) which have the correspondence property fulfilling the conditions of Theorem 5.2. Thus for each \( i_{k} \) set \( \hat{i} = i_{k} \) then set \( \hat{\phi}_{k} = \phi_{i_{k}}^{(g)} \) which obtains the desired sequence of Theorem 2.3. The result follows as \( e|u| > |u|^r \) for \( u \) far enough from the origin.

**Proof.** For \( t < e^{-|x_{t+}^{\ell_{1}}|/2} \), by the correspondence of eigenvalues we have \( |e^{-it\lambda_{t_{+}}^{(-g)} - e^{-it\lambda_{\ell_{1}}^{(g)}}| < 2t|\lambda_{t_{+}}^{(-g)} - \lambda_{\ell_{1}}^{(g)}| < 2e^{-|x_{t+}^{\ell_{1}}|/2}. \) Therefore,
\[ \left\| \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \left( Je^{-it\lambda_{t_{+}}^{(g)}} + i e^{-it\lambda_{\ell_{1}}^{(-g)}} \right) \right\| < 4e^{-|x_{t+}^{\ell_{1}}|/2} \]
Combining this with, (5.2) we have
\[ \left\| e^{-it\hbar_{1}} |\phi_{t+}^{(g)}, +1 \rangle - |\phi_{t+}^{(g)}, +1 \rangle \right\| \leq 2e^{-|x_{t+}^{(g)}|/4}. \]
Therefore,
\[ |\langle \phi_{t+}^{(g)}, +1 | e^{it\hbar_{1}} |\phi_{t+}^{(g)}, +1 \rangle - 1 \rangle | \leq e^{-|x_{t+}^{(g)}|/4}. \]
which is the statement of (5.4).

The localization statement follows from a combination of (3.5) and (5.2). From (3.5) we have for all \( t > 0 \), and for large enough \( |x_i^{(g)}| \)

\[
\left\| \hat{P}_{x_i^{(g)}} \psi_{i_t} \right\| < 2e^{-\xi|x_i^{(g)}|/2}
\]

applying this bound to (5.2) obtains the result.

Finally, set \( t = \pi|\lambda_i^{(-g)} - \lambda_i^{(g)}|^{-1} \) (5.6)

\[ \square \]

**Appendix A. Fractional moments and dynamical localization**

Here we will recall the results in fractional moment bounds for the standard Anderson model and the related dynamical localization model. In [2](Theorem 6.3) fractional moments are bounded in terms of self avoiding random walks. The following fractional moment bound holds.

**Theorem A.1.** There is a finite constant \( C = C_{p,d} \) so that for any \( 0 < s < 1 \) and any \( z \in \mathbb{C}\setminus\mathbb{R} \)

\[
\mathbb{E}(\langle x \parallel (H - z)^{-1}|y\rangle^s) \leq \frac{C}{1-s} \left( \frac{\gamma^s C}{1-s} \right)^{|x-y|}
\]

Transition probabilities may be bounded in terms of fractional moments. For \( I \supset \sigma(H) \) Theorem 7.7 of [2] states,

**Theorem A.2.** There is \( C < \infty \) depending on \( s \) so that,

\[
\mathbb{E}(\langle x|e^{-itH}|y\rangle) \leq C \lim_{\epsilon \to 0} \int_I \mathbb{E}(\langle x|H - (E + i\epsilon)|y\rangle^s) dE
\]

From these statements we can easily demonstrate Theorem 2.1 and Corollary 2.2.

**Proof.** Let us write \( \mathcal{D} = \begin{bmatrix} 0 & |\zeta\rangle \langle \xi| \\ |\zeta\rangle \langle \xi| & 0 \end{bmatrix} \) then the resolvent equation states, for \( z \in \mathbb{C}\setminus\mathbb{R}, \)

\[
(A.1) \quad (\mathfrak{h}_g - z)^{-1} = (\mathfrak{h} - z)^{-1} - g(\mathfrak{h} - z)^{-1} \mathcal{D}(\mathfrak{h}_g - z)^{-1}.
\]

Thus, if \( i = j \), two applications of of the above equation implies,

\[
\langle x, i | (\mathfrak{h}_g - z)^{-1}|y, i\rangle = \langle x| (H - z)^{-1}|y\rangle + g^2 \langle x| (H - z)^{-1}|\zeta\rangle \langle \zeta, -i | (\mathfrak{h}_g - z)^{-1}|\zeta, i\rangle \langle \zeta| (H - z)^{-1}|y\rangle.
\]

Now we take the \( s^{th} \) moment for \( 0 < s < 1/4 \). For the first term we may apply Theorem A.1 directly. For the second, we apply Holder inequality to find,

\[
\mathbb{E}(\langle x, i | (\mathfrak{h}_g - z)^{-1}|y, i\rangle^s \leq \mathbb{E}(\langle x| (H - z)^{-1}|y\rangle^s + g^{2s}(\mathbb{E}(\langle x| (H - z)^{-1}|\zeta\rangle^4)^{1/4} \mathbb{E}(\langle \zeta, -i | (\mathfrak{h}_g - z)^{-1}|\zeta, i\rangle^2)^{s/2} \mathbb{E}(\langle \zeta| (H - z)^{-1}|y\rangle^4)^{1/4})
\]

Assume \( |x| \gg |y| \), then we may bound the second and third factors of the second term, by a constant, using Theorem 1.1 of [1]. Thus we have, using Theorem A.1,

\[
\mathbb{E}(\langle x, i | (\mathfrak{h}_g - z)^{-1}|y, i\rangle^s \leq \frac{C}{1-s} \left( \frac{\gamma^s C}{1-s} \right)^{|x-y|} + g^{2s}C \sum_{u \in \text{supp} \zeta} \frac{C}{1-s} \left( \frac{\gamma^s C}{1-s} \right)^{|x-u|}
\]

\[
\leq \hat{C} \left( \frac{\gamma^s C}{1-s} \right)^{|x-y|\wedge (|x| - L_\xi)}
\]

where \( L_\xi = \max\{|u| : u \in \text{supp} \zeta\} \). As \( |x - y| \gg 2|x| \),

\[
\mathbb{E}(\langle x, i | (\mathfrak{h}_g - z)^{-1}|y, i\rangle^s \leq \hat{C} \left( \frac{\gamma^s C}{1-s} \right)^{-L_\xi} \left( \frac{\gamma^s C}{1-s} \right)^{|x-y|/2}
\]

concluding the result in this case.
On the other hand, for $i = -j$, (A.1) implies

$$\langle x, i | (\mathfrak{h}_g - z)^{-1} | 0, -i \rangle = -g \langle x, i | (\mathfrak{h} - z)^{-1} | \zeta, i \rangle \langle \zeta, -i | (\mathfrak{h}_g - z)^{-1} | y, -i \rangle$$

Again take the $s$-moment for $0 < s < 1/4$, and apply Holder’s theorem,

$$\mathbb{E} \langle x, 0 | (\mathfrak{h}_g - z)^{-1} | 0, y \rangle^s \leq g^s \left( \mathbb{E} \langle e_x, 0 | (\mathfrak{h} - z)^{-1} | \zeta, 0 \rangle^{2s} \right)^{1/2} \left( \mathbb{E} \langle 0, \zeta | (\mathfrak{h}_g - z)^{-1} | 0, e_y \rangle^{2s} \right)^{1/2}$$

Again, assume $|x| > |y|$ and bound the second term by a constant to obtain,

$$\mathbb{E} \langle x, i | (\mathfrak{h}_g - z)^{-1} | y, -i \rangle^s \leq g^s C \sum_{u \in \text{supp } \zeta} \frac{C'}{1 - s} \left( \frac{\gamma^s C'}{1 - s} \right)^{|x-u|} \leq C_1 \left( \frac{\gamma^s C'}{1 - s} \right)^{|x-L_\xi|},$$

which concludes the proof. □

**Proof.** The perturbation theorem for semigroups obtains,

$$e^{-it\mathfrak{h}_g} = e^{-it\mathfrak{h}} - ig \int_0^t e^{-is\mathfrak{h}_g} \mathfrak{D} e^{-i(t-s)\mathfrak{h}} ds.$$

Now let us take the expectation of the transition probability of (A.2)

$$\mathbb{E} \langle y, -1 | e^{-it\mathfrak{h}_g} | x, +1 \rangle \leq g \int_0^t \mathbb{E} \langle y, -1 | e^{-is\mathfrak{h}_g} | \zeta, -1 \rangle \langle \zeta, +1 | e^{-i(t-s)\mathfrak{h}} | x, +1 \rangle ds$$

$$\leq gt \sup_{s} \mathbb{E} \langle \zeta | e^{-is\mathfrak{h}} | x \rangle ds.$$

The Corollary now follows from Theorem (A.1) and Theorem (A.2). □

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