Predictor-corrector scheme for simulating wave propagation on shallow water region

Rifky Fauzi\textsuperscript{a}, L. Hari Wiryanto\textsuperscript{b}
Institut Teknologi Bandung, Jalan Ganesha 10, Bandung, Indonesia
E-mail: \textsuperscript{a}rifkyfauzi9@gmail.com, \textsuperscript{b}leo@math.itb.ac.id

Abstract. In this paper, a well-known Saint-Venant’s Equation is solved by using Predictor-Corrector method to simulate the propagation of wave in shallow water region with several type of bottom friction effects. The predictor step is chosen to be Adam-Bashforth, meanwhile Adam-Moulton for the corrector step. Numerical stability is also discussed by applying von Neumann stability into the linear equations.

1. Introduction
Shallow water equation or Saint Venant’s equation (SVE) is a system of equations that describe fluid flow in relatively long channel. This system though is able to perform simulation for geophysical flow such as ocean lake and atmosphere and even to simulate tsunami waves. The followings are shallow water system of equations

\begin{align}
\frac{\partial \eta}{\partial t} + \frac{\partial ((d_0 + \eta)u)}{\partial x} &= 0 \quad (1) \\
\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x} \left( hu^2 + \frac{1}{2} g \cos(\theta) h^2 \right) &= g \sin(\theta) h - C_f \frac{|u|}{h^{\alpha}} \quad (2)
\end{align}

where \( \eta \) surface elevation, \( h = d_0 + \eta \) denoting height of water, \( u \) is horizontal water velocity, \( \theta \) is bed inclination, \( d_0 \) is water depth and \( g \) is acceleration due to gravity. Meanwhile \( C_f \) is bottom friction parameter that depends on the choice of constant \( \alpha \). SVE is a limiting case from Navier-Stokes equation by assuming that the horizontal length scale is much larger than vertical length scale. This shallowness assumption leads the SVE to be a hydrostatic model.

Bottom friction constant \( C_f \) is non-physical parameter that can be obtained by empirical approach based on mathematical model \cite{1}. However this constant is prescribed without any definite and complete physical concept. Thus in nature the value of this constant cannot be uniform. In \cite{2} different value of \( C_f \) define different areas such as smooth area, coral area and scattered tree or building in performing tsunami flooding simulation. One of effort in obtaining constant \( C_f \) can be seen in \cite{1}. The determination of \( C_f \) is based on the optimization from SVE solution comparing to observational data. This approach shows that there is spatial variation in obtained value.

The bottom friction effect plays a significant role in the tidal phenomenon \cite{3}. It gives rise to shallow area let the waves interact with bottom \cite{4}. In other words, as the waves propagate from deeper area to shallower area those waves start to interact with bottom. Such frictional
effect may appear as the existence of pores of the sandy bottom, muddy bottom or bottom irregularities [4]. This effect affects momentum transfer of the propagating waves since one has already familiar from basic mechanics that friction effect acts opposite to the motion.

Wave-bottom interaction rises in the momentum equation modeled as the resistive force fluid over the bottom. The resistance rises from the characteristics of the bottom which can sometimes also be called roughness coefficient that resist the momentum transport. The form of the bottom friction effect or simply the resistance force varies due to the existence of the internal resistance. In this paper, it is sufficient to assume that there is no internal resistance the bottom friction effect term in Equation 4 is valid. Otherwise, this form has to be changed to include the internal resistance effect.

To simulate wave propagation in shallow water region the inclination effect in the SVE need to be vanished. Thus by taking $\theta = 0$ the SVE reads

$$\frac{\partial \eta}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$

(3)

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x} \left( hu^2 + \frac{1}{2} gh^2 \right) = -C_f \frac{|u|}{h^\alpha}$$

(4)

The objectives of this paper are for simulating wave propagation in shallow water region with several value of friction coefficient. The inclination effect is neglected due to physical reason that in shallow water region (coastal region to be specific) this effect plays insignificant role than bottom topography. This effect in fact has an important role in the instability of uniform fluid flow which can generate rollwaves or progressing periodic waves see [5] and [6].

The purpose of this study is to simulate wave propagating in shallow water region by taking several value of friction coefficient which obtained from several literature. A good approximation solution of Equation 3 and Equation 4 is extremely needed to obtain best result. The solution of the equations obtained by solving it using predictor-corrector scheme. Additionally, to prevent instability in the numerical iteration, numerical stability is also discussed. Once the numerical scheme has arranged the effect of bottom friction is investigated by varying the value of the parameter.

2. Bottom Friction Effect

There are two choices of bottom friction i.e. analytical approach and empirical approach. The bottom friction effect from empirical approach contains non-physical parameter that can be obtained by trial and error comparing to the experiment [8]. In contrast with the empirical approach, the analytical approach of bottom friction can be obtained from Poiseuille flow which hold the analytical meaning that this effect depends on physical parameter such as viscosity and gravitational constant. It is, however, the analytical approach is not the discussion in this study.

The development of bottom friction approach frequently found in hydraulics problem. Nevertheless, the approach can be used in coastal problem [1], [4] and [9]. Since it is derived from hydraulics problem the bottom friction effect varies as the choice of $\alpha$ in the following term

$$(h, u) = C_f \frac{|u|}{h^\alpha}$$

(5)

The friction constant $C_f$ depends on the choice of $\alpha$. There are three main approach of the choice $\alpha$ that derived by Manning, Chezy and Darcy-Weisbach with their own approach. The differences arise from the experiment that they did in the beginning of the development of bottom friction effect.

For Manning’s approach the $\alpha = \frac{4}{3}$ with the corresponding $C_f = n^2$ that can be obtained experimentally depending on the type of the bottom whether the bottom is muddy bottom or
there is strong irregularity on the bottom profile \[8\]. This value also appears in Chezy approach that is for \(\alpha = 0\) where \(C_f = n^2\) called Chezy’s coefficient \[6\]. Meanwhile for the Darcy-Weisbach that is \(\alpha = 1\) the value of bottom friction \(C_f\) only depends on the Reynold number (Re) of the flow that is

\[
C_f = \begin{cases} \frac{\Delta f}{f} & \text{for } Re < 48 \\ \frac{f}{16g} & \text{for } Re \geq 48 \end{cases}
\] (6)

3. Numerical Procedure

Predictor-Corrector scheme has been succeeded simulating coupled dispersive shallow water equation in \[10\] by using Adam-Bashforth for predictor and Adam-Moulton for corrector. In this paper, similar schemes are also used to simulate wave propagation in shallow water region. Firstly, one need to rewrite the SWE into following form

\[
\frac{\partial h}{\partial t} = H\left(h, u, \frac{\partial h}{\partial x}, \frac{\partial u}{\partial x}\right)
\]

\[
\frac{\partial v}{\partial t} = U\left(h, u, \frac{\partial h}{\partial x}, \frac{\partial u}{\partial x}\right)
\] (7)

Then discretize right hand side Equation 7 by using central difference

\[
H^n_j = -h^n_j \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x} - \frac{h^n_{j+1} - h^n_{j-1}}{2\Delta x}
\] (8)

\[
U^n_j = -u^n_j \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x} - \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x} + C_f \frac{|u^n_j| |u^n_j|}{(h^n_j)^\alpha}
\] (9)

Subsequently, the mass equation i.e. Equation 3 and Equation 4 discretized by using Adam-Bashforth time integration gives

\[
\frac{\bar{\eta}^{n+1}_j - \eta^n_j}{\Delta t} = -\frac{1}{2}(3H^n_j - H^{n-1}_j)
\] (10)

\[
\frac{\bar{u}^{n+1}_j - u^n_j}{\Delta t} = -\frac{1}{2}(3U^n_j - U^{n-1}_j)
\] (11)

meanwhile for Adam-Moulton time integration gives

\[
\frac{\bar{h}^{n+1}_j - h^n_j}{\Delta t} = -\frac{1}{2}(\bar{H}^{n+1}_j + H^n_j)
\] (12)

\[
\frac{\bar{u}^{n+1}_j - u^n_j}{\Delta t} = -\frac{1}{2}(\bar{U}^{n+1}_j + U^n_j)
\] (13)

\(\bar{H}^{n+1}_j\) and \(\bar{U}^{n+1}_j\) are obtained from predictor scheme.

In these schemes, one need two initial condition. In \[10\], it is suggested that for such multistep method need special treatment. The initial condition is given by \(n = 0\), meanwhile for \(n = -1\) is defined as zero.

4. Numerical Stability

Since it has been a problematic matter to investigate the stability of nonlinear equation. A strategic step should be taken to manage the stability analysis of the numerical scheme. The strategy is to perturb the stationary solution of Equation 3 and Equation 4 by giving a small parameter namely \(\epsilon \ll 1\) then analyze the numerical solution of \(O(\epsilon)\) equations.
The effect of bottom friction only appears in nonlinear form by taking the perturbed uniform solution from Equation 3 and 4. The perturbed solutions read
\begin{align*}
u(x,t) &= 0 + \epsilon u_1(x,t) \\
\eta(x,t) &= \eta_0 + \epsilon \eta_1(x,t)
\end{align*}
(14)

Substituting (14) into Equation 3 and Equation 4 and neglecting $O(\epsilon^2)$ gives
\begin{align*}
\frac{\partial \eta_1}{\partial t} + d_0 \frac{\partial u_1}{\partial x} &= 0 \\
\frac{\partial u_1}{\partial t} + g \frac{\partial \eta_1}{\partial x} &= 0
\end{align*}
(15)

which are set of linear coupled equation. From (15) one can clearly see that the bottom friction does not appear. In contrast with the case for $\theta > 0$, the bottom friction appears within the linear form [5].

Equation 15 in predictor scheme reads
\begin{align*}
\bar{\eta}_{j}^{n+1} &= \eta_j^n - d_0 \left( \frac{3 u_{j+1}^{n+1} - 2 u_j^n - u_{j-1}^n}{2 \Delta x} \right) \\
\bar{u}_{j}^{n+1} &= u_j^n - g \left( \frac{3 \eta_{j+1}^{n+1} - 2 \eta_j^n - \eta_{j-1}^n}{2 \Delta x} \right)
\end{align*}
(16) \hspace{1cm} (17)

and the corrector scheme
\begin{align*}
\eta_{j}^{n+1} &= \eta_j^n - d_0 \left( \frac{\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1}}{2 \Delta x} \right) \\
u_{j}^{n+1} &= u_j^n - g \left( \frac{\bar{\eta}_{j+1}^{n+1} - \bar{\eta}_j^{n+1}}{2 \Delta x} \right)
\end{align*}
(18) \hspace{1cm} (19)

The stability analysis can be done by using Von Nuemann stability
\begin{align*}
\nu_j^n &= \tilde{\nu} e^{i \beta \Delta x j + i \beta n \Delta t} \\
\nu_j^n &= \tilde{\nu} e^{i \beta \Delta x j + i \beta n \Delta t}
\end{align*}
(20) \hspace{1cm} (21)

By substituting into predictor scheme one gets
\begin{align*}
\begin{pmatrix}
\lambda (\lambda - 1) & -ir(3\lambda - 1)
\end{pmatrix}
\begin{pmatrix}
\tilde{\nu}
\end{pmatrix} &= 0
\end{align*}
(22)

Meanwhile for the corrector scheme
\begin{align*}
\begin{pmatrix}
\lambda - 1 & -ir(\lambda + 1)
\end{pmatrix}
\begin{pmatrix}
\tilde{\nu}
\end{pmatrix} &= 0
\end{align*}
(23)

where
\begin{align*}
\lambda &= e^{i \beta \Delta t}
\end{align*}
(24)

is amplification factor and
\begin{align*}
r &= gd_0 \frac{\Delta t}{2 \Delta x} \sin \theta
\end{align*}
(25)
The main idea to analyze the numerical scheme is by designing the amplification so that it is not larger than 1, otherwise the numerical solution would blow up. We know that Equation 22 and 23 give nontrivial solution only if we have singular matrices. Thus, for the amplification for 22 is given by fourth order polynomial as follows

$$\lambda^2(\lambda - 1)^2 + r^2(3\lambda - 1)^2 = 0 \quad (26)$$

Equation 26's root can be found numerically and is a function of phase angle $\theta$. Here we employ Courant number as the numerical stability criterion

$$Cr = \sqrt{gd_0 \Delta t \Delta x} \quad (27)$$

and by setting $\Delta x = 0.1$ and choose $Cr = \{0.01, 0.05, 0.1, 0.5, 1.0\}$. Figure 4 shows the maximum of norm of the roots of Equation 26 from figure 4, it can be seen that the numerical scheme is stable for predictor if we choose $Cr = 0.1$.

Meanwhile for the corrector scheme, the amplification is given by the roots following polynomial

$$\lambda^2 + 2(r^2 - 1)r^2 + 1 = 0 \quad (28)$$

The maximum modulus of the root is given by

$$|\lambda| = 1 \leq 1 \quad (29)$$

Thus the numerical scheme is stable if

$$gd_0 \frac{\Delta t}{2\Delta x} \leq 1 \quad (30)$$

5. Results

In the simulation, waves are propagating from left to right. Both spatial and temporal steps are chosen so that satisfy the numerical stability. The effect of bottom friction is investigated by analyzing the propagating waves for a given bottom friction constant. For convenient comparison, waves propagate without bottom friction i.e. $C_f = 0$ is considered. In Figure 2, monochromatic waves propagating from left to right. The waves profile deform due to nonlinear effect in the model. As we can see that the most severe effect from bottom effect is sought in result for $\alpha = 0$. The biggest the value of $\alpha$ gives less damping effect.
6. Conclusions

Numerical code for waves propagation in shallow water region has been developed by using Adam-Bashforth and Adam-Moulton as Predictor and Corrector respectively. The numerical stability based on von Neumann stability analysis has also been studied to prevent infinite numerical result from finite initial condition. The numerical results show that for any type of bottom friction has decay effect to wave amplitude. This fact shows that rollwaves in shallow water region is different comparing to rollwaves down inclined channel i.e. this phenomenon does not arise from instability of a particular solution. Alternatively, rollwaves can be simulated by taking small but not too small amplitude thus the nonlinear effect still can affect at the same time it can maintain its shape so that it does not break.

References

[1] Ding Y Jia Y and Wang S S Y 2004 Identification of mannings roughness coefficients in shallow water flows Journal of Hydraulic Engineering 130 501-510

[2] Mader C L 2004 Numerical Modeling of Water Waves

[3] Wang D, Liu Q and Lv X 2014 Mathematical Problems in Engineering 2014 1-7

[4] Luo W and Monbaliu J 1994 Journal of Geophysical Research 99 18501-18511
[5] Fauzi R and Wiryanto L H 2017 AIP Conf. Proceedings 1867 1-8
[6] Dressler R F 1949 Communications on Pure and Applied Mathematics 2 149-194
[7] Needham D J and Merkin J H 1984 Proc. R. Soc. Lond. A 394 259-278
[8] Kirstetter G, Hu J, Delestre F, Darboux F, Lagree P Y, Popinet S, Fullana J M and Josserand C 2016 Journal of Hydrology 536 1-9
[9] Liu Y, Shi Y, Yuen D A, Sevre E O D, Yuan X and Xing H L 2009 Acta Geotechnica 4 129-137
[10] Wiryanto L H and Mungkasi S 2014 Applied Mathematical Sciences 8 5293-5302