MONOKINETIC CHARGED PARTICLE BEAMS:
QUALITATIVE BEHAVIOR OF THE SOLUTIONS
OF THE CAUCHY PROBLEM
AND 2d TIME-PERIODIC SOLUTIONS
OF THE VLASOV-POISSON SYSTEM

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Let \((f, U)\) be a solution of the 2d Vlasov-Poisson system for charged particles:

\[
\begin{aligned}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left( \partial_x U(t, x) + \partial_z U_0(x) \right) \cdot \nabla_v f &= 0 \\
-\Delta U &= \rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) \, dv
\end{aligned}
\]  

\(f = f(t, x, v)\) is the distribution function, defined on the phase space \(\mathbb{R}^2 \times \mathbb{R}^3\)
\((x\) represents the position and \(v\) the velocity of the particles) and \(U = U(t, x)\) is
the self-consistent potential given by the Poisson equation. \(U_0\) is an external
potential which depends only on \(|x|\).

Such a simplified problem appears when one considers the free propagation of a particle beam in \(\mathbb{R}^3\) and when the particles are monokinetic in the
\(z\)-direction of the axis of the beam and when one considers a distribution function averaged other the velocities in the \(z\)-direction. The beam is assumed
to be infinite in that direction, with a spatial density not depending on \(z\) (see
Appendix A for a formal derivation of the model). This model may be used in high energy plasma physics (see [18]) to describe 3-dimensional electron beams with high density. It takes the dispersion of the velocities in the orthogonal to the z-direction plane as well as the self-consistent nonlinear interaction into account, which seems to be crucial for the understanding of the experiments and of the numerical simulations.

The two main motivations are indeed the following: first to understand how the support of a beam grows when it is initially compactly supported and more generally, what is its behaviour, how the tail (which corresponds to large velocities and is not easily computed by particle simulation methods) of the distribution function evolves, what kind of dispersion is obtained... and second, to give a systematic method for the study of the time-periodic solutions that clearly appear for large times in numerical simulations.

A natural asymptotic boundary condition for such a model would be to consider only a logarithmic growth of the potential $U(t,x)$ as $|x| \to +\infty$, but since we are interested in the local behavior of such a solution, we will relax this assumption and consider more general cases in part II. In part I, we will assume that

$$U(t,x) = -\frac{1}{2\pi} \ln |x| \ast \rho(t,x)$$

and study the problem in a framework for which this formula makes sense.

For the same reason, we will also frequently assume throughout the paper (but in most cases it is not essential) that the external potential – when there is one – is harmonic: there exists some $\rho_0 \in \mathbb{R}$ such that

$$U_0(x) = \frac{\rho_0}{2} |x|^2 \quad \forall x \in \mathbb{R}^2.$$  \hspace{1cm} (H)

The paper is divided into two parts. The first one is concerned with the evolution problem (existence, regularity or uniqueness as well as more qualitative aspects, like dispersion results, growth of the support for compactly supported distribution functions or equipartition of the energy). The main estimate is a Lyapunov functional which is used to control the energy (and provides an existence result for non compactly supported distribution functions) and to give an estimate for the dispersion ($N = 2$ appears to be the critical dimension, and we have to use logarithmic estimates). In the second
part, we focus on time-periodic (and stationary) solutions and present results in two directions: first, we give an (uncomplete) classification of the solutions that are time-periodic (but radially symmetric if they are averaged over one time period); then we study a special class of solutions which is the counterpart of the class of solutions of the Vlasov-Poisson system in dimension three that satisfy the so-called Ehlers & Rienstra ansatz.

Three sections, which are of general interest but rather technical, are rejected at the end of the paper: Appendix A deals with a formal derivation of the two dimensional model, which main interest is to show that the model is local, with two consequences: it is not restrictive to take the confining potential harmonic, and there is no a priori natural boundary condition for $U$. In Appendix B are stated two interpolation lemmas, with an explicit computation of the constants. Appendix C provides explicit and detailed statements for Jeans' theorem.

Following B. Perthame’s definitions (see [36]), a weak solution is a solution in the sense of the distributions such that the energy is bounded but not necessarily constant. A strong solution (in dimension $N = 2$) is a solution that has moments of order $2 + \varepsilon$ in $v$ and $x$ (at least when the external potential is harmonic) and such that the energy does not depend on $t$. In this paper, we will make use of an intermediate notion of solutions corresponding to the case $f(U_0 |v|^2)$ in $L^1$ with $U_0$ growing at least logarithmically at infinity (confined case) or such that $f$ has a moment in $x$ of order $m \in [1,2]$. For such solutions, the self-consistent potential energy is continuous w.r.t. the time (in dimension $N = 2$), but the kinetic energy and the external potential energy are only bounded w.r.t. the time.

Since the number of references in this work is quite huge and concerns very different subjects, the references will be mentioned throughout the paper.

For general results on the evolution problem, one has to mention at least the papers by S. Ukai & T. Okabe [40] and S. Wollman [41,42] in dimension two, and the papers by C. Bardos & P. Degond [7], P-L. Lions & B. Perthame [33] and K. Pfaffelmoser [37] in dimension three. The dispersion relations are strongly related to the work of R. Illner & G. Rein [31], P-L. Lions & B. Perthame [34] and B. Perthame [36].
What concerns the time-periodic solutions and the Jeans' theorem is directly inspired respectively by J. Batt, H. Berestycki, P. Degond & B. Perthame [6] and by J. Batt, W. Faltenbacher, & E. Horst [8].

Part I:
Qualitative behavior of the solutions of the Cauchy problem

Introduction

The first part of this paper is devoted to various results on the initial value problem for the two dimensional Vlasov-Poisson system in the presence (or not) of a confining potential.

In section 1, we introduce two notions of Lyapunov functionals. In section 2 (see the complete list of references therein), an existence result is given, in a more general framework than what had been established by S. Ukai & T. Okabe in [40] and S. Wollman in [41,42]. This result essentially benefits of recent papers on the (more difficult) theory for the three dimensional problem. The questions of the regularity and of the uniqueness are treated with the approach developed by P.-L. Lions and B. Perthame in [33].

A dispersion result is given in section 3: if there is no confining potential, the solution is vanishing for large time. This result is obtained using the same methods as R. Illner & G. Rein in [31] or B. Perthame in [36] for the case of the dimension three. Dimension two corresponds to a limit case for this method (use of logarithmic estimates).

The question of the growth of the support of an initially compactly supported distribution function is studied in section 4. One has to mention that the method, which is strongly dependant of the potential when it is applied to the computation of the size of the support in the phase space, gives the growth in the velocity space even if there is no confining potential, as in the paper [38] by G. Rein, but with a different method.

An equipartition of the energy result is given in section 5. The estimate obtained there is in fact the keypoint of Part I since it allows the computations
on the Lyapunov functional. Section 5 also contains the moment estimates that are needed to compute the Lyapunov functional as well as to define the notion of solutions.

1. A Lyapunov functional and a priori estimates

Following the same idea as in B. Perthame [36] and R. Illner and G. Rein in [31], we first derive a Lyapunov functional for the Vlasov-Poisson system in dimension 2 with an external potential. In the following, we shall assume that \( f \) and \( U \) are smooth, and that \( f \) is compactly supported, in order to perform any integration by part that is needed. We shall see later how to handle the non smooth case.

Multiplying the Vlasov equation by \(|v|^2\), \((x \cdot v)\) and \(|x|\), we can obtain respectively

\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^2 \times \mathbb{R}^3} f(t,x,v) \cdot |x|^2 \, dz \, dv \right] = 2 \int_{\mathbb{R}^2 \times \mathbb{R}^3} f(t,x,v) \cdot (x \cdot v) \, dx \, dv , \tag{1.1}
\]

\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^2 \times \mathbb{R}^3} f(t,x,v) \cdot (x \cdot v) \, dz \, dv \right]
+ \int_{\mathbb{R}^2 \times \mathbb{R}^3} f(t,x,v) \cdot |v|^2 \, dz \, dv - \int_{\mathbb{R}^3} (x \cdot \nabla U_0) \rho(t,x) \, dx + \frac{M^2}{4\pi} , \tag{1.2}
\]

\[
\int_{\mathbb{R}^3} f(t,x,v) \cdot ([|v|^2 + U(t,x) + 2U_0(x))] \, dz \, dv = \int_{\mathbb{R}^3} f(t=0,x,v) \cdot ([|v|^2 + U(t=0,x) + 2U_0(x))] \, dz \, dv = E_0 , \tag{1.3}
\]

using the fact that

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^3} |v|^2 \cdot \nabla U(t,x) \cdot \partial_x f(t,x,v) \, dz \, dv
= 2 \int_{\mathbb{R}^2} \, dx \, U(t,x) \, \nabla f(t,x,v)
= -2 \int_{\mathbb{R}^2} \, dx \, U(t,x) \, \frac{\partial \rho}{\partial t}(t,x)
= - \int_{\mathbb{R}^2} \rho(t,x) U(t,x) \, dx
= - \int_{\mathbb{R}^2 \times \mathbb{R}^3} f(t,x,v)U(t,x) \, dz \, dv ,
\]

and that

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^3} (x \cdot v) \cdot \nabla U(t,x) \cdot \partial_x f(t,x,v) \, dz \, dv
= - \int_{\mathbb{R}^2} \, dx \, z \cdot \nabla U(t,x) \rho(t,x)
= \frac{M^2}{4\pi} ,
\]
because of Poisson's equation (see section 5 for the proof of this identity).

In dimension $N = 3$ (see [36] and [31]), it is enough to compute (for some $\alpha > 0$)

$$\frac{d}{dt} \left( \frac{1}{t + \alpha} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \cdot |x - v(t + \alpha)|^2 \, dx dv + (t + \alpha) \int_{\mathbb{R}^N} |\nabla U(t, x)|^2 \, dx \right)$$

$$= - \frac{1}{(t + \alpha)^{\frac{3}{2}}} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v)|x - v(t + \alpha)|^2 \, dx dv$$

to get a decay estimate since the Lyapunov functional is a positive quantity. In dimension $N = 2$ and in the presence of an external potential $U_0$, the analogous computation would give

$$\frac{d}{dt} \left( \frac{1}{t + \alpha} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \cdot (|x - v(t + \alpha)|^2 + U(t, x) + 2U_0(x)) \, dx dv \right.$$ 

$$+ \frac{M^2}{2\pi} \ln(t + \alpha) \right)$$

$$= - \frac{2}{(t + \alpha)^{\frac{3}{2}}} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v)|x - v(t + \alpha)|^2 \, dx dv$$

$$+ \frac{2}{t + \alpha} \int_{\mathbb{R}^N} (x \cdot \nabla U_0(x)) \, \rho(t, x) \, dx ,$$

which is not enough to conclude (even if $x \cdot \nabla U_0(x) \leq 0$) since $\int f(t, x, v)U(t, x) \, dx dv$ is not necessarily a positive quantity. We will distinguish two cases:

- $U_0(x)$ is growing when $|x| \to +\infty$ like at least $\ln|x|$ (confinement case)
- $x \mapsto \frac{x \cdot \nabla U_0}{|x|^2}$ is bounded (and, for instance, $x \cdot \nabla U_0 \leq 0$ - dispersive case. In this case, $L_\alpha$ is decreasing)

Case 1: (confinement case)

Proposition 1.1 : Assume that $f_0$ is a nonnegative function in $L^1 \cap L^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$ such that

$$E_0 = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v) \cdot (|v|^2 + U(t = 0, x) + U_0(x)) \, dx dv < +\infty$$

with $U_0 \geq 0$, $\nabla U_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$ and $U(t = 0, x)$ given by the Poisson equation. Assume also that

$$L = \liminf_{|x| \to +\infty} \frac{U_0(x)}{\ln|x|} > \frac{M}{2\pi}$$

with $M = ||f_0||_{L^1(\mathbb{R}^2 \times \mathbb{R}^2)}$. If $f$ is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to the initial data $f_0$, then:
There exists a constant $C \in [0, 2 - \frac{M^2}{\pi}]$ such that
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) \cdot (U(t, x) + 2U_0(x)) \, dx dv \geq C \cdot \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) U_0(x) \, dx dv \geq 0.
\]

(ii) If
\[
\exists m \in [1, 2] \quad \forall t > 0 \quad \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) (|v|^2 + |x|^m) \, dx dv < +\infty,
\]
then
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) \cdot (U(t, x) + (2 - C)V_0(x)) \, dx dv \leq E_0,
\]
and
\[
L_0(t) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) \cdot \left( \frac{|v - v(t + \alpha)|^2}{(t + \alpha)^2} + U(t, x) + 2U_0(x) \right) \, dx dv + M^2 \ln(t + \alpha)
\]
is bounded from below for any $\alpha > 0$, $t > 0$. Moreover, if $f$ is a strong solution, i.e. a solution such that
\[
\exists \varepsilon > 0 \quad \forall t > 0 \quad \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) (|v|^{2+\varepsilon} + |x|^{2+\varepsilon}) \, dx dv < +\infty,
\]
then
\[
\frac{dL_0}{dt}(t) = -2 \frac{2}{(t + \alpha)^3} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) |v - v(t + \alpha)|^2 \, dx dv \\
+ \frac{2}{t + \alpha} \int_{\mathbb{R}^2} (x \cdot \nabla U_0(x)) \rho(t, x) \, dx. \tag{1.4}
\]

Proof of Proposition 1.1: We have to prove that the self-consistent potential energy is from below
\[
-\int_{\mathbb{R}^2} \rho(t, x) U(t, x) \, dx \leq \frac{M^2}{\pi} \ln 2 + \frac{M}{\pi} \int_{\mathbb{R}^2} \rho(t, x) \ln |x| \, dx
\]
since
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(t, x) \rho(t, y) \ln |x - y| \, dx dy \leq 2 \int_{|y| \leq |x|} \rho(t, x) \rho(t, y) \ln(2|x|) \, dx dy.
\]

Then
\[
\frac{M}{\pi} \int_{\mathbb{R}^2} \rho(t, x) \ln |x| \, dx \leq \frac{M}{\pi} \int_{|x| < \lambda} \rho(t, x) \ln |x| \, dx + \frac{M}{\pi} \int_{|x| \geq \lambda} \rho(t, x) \ln |x| \, dx
\]
\[
\leq \frac{M^2}{\pi} \ln \lambda + \frac{M}{\pi} \int_{\mathbb{R}^2} \rho(t, x) U_0(x) \, dx \cdot \sup_{|x| \geq \lambda} |U_0(x)|
\]
for $k$ large enough.

$$
\int_{\mathbb{R}^{2}} \rho(t,x)U(t,x) \, dx \geq \frac{-M^2}{\pi} (\ln 2 + \ln k) - \frac{M}{\pi \inf_{|x|>k} \frac{U(t,x)}{|x|}} \int_{\mathbb{R}^{2}} \rho(t,x)U_0(x) \, dx.
$$

The proof of (ii) is contained in Remark 1.2 below (see also Lemma 5.4 in Section 5).

**Remark 1.2**:

(i) We will prove in section 5 that the condition on $f$ holds as soon as the initial data has moments of order $m$ and 2 respectively in $x$ and $v$. $f$ is a strong solution if the initial data has moments of order $2+r$.

(ii) For a solution $f \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ with moments of order 2 in $v$ and such that

- either (case a)

  $$
  \lim_{|x| \to +\infty} \frac{U_0(x)}{|x|} = +\infty,
  $$

- or (case b) $f$ has a moment of order $m$ in $x$,

  no concentration or vanishing of the self-consistent potential energy may occur: if we split

  $$
  \int_{\mathbb{R}^{2}} \rho(t,x)U(t,x) \, dx = \frac{1}{2\pi} \int_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy
  $$

  into three parts corresponding to $|x-y| < \epsilon$, $\epsilon < |x-y| < \epsilon^{-1}$ and $|x-y| < \epsilon$, we just have to control the first one and the third one:

  $$
  \int_{|x-y| < \epsilon} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy
  \leq ||\rho(t,\cdot)||_{L^\infty(\mathbb{R}^2)} \cdot ||\rho(t,\cdot)||_{L^1(\mathbb{R}^2)} \cdot \left(\int_0^{\epsilon} 2\pi (\ln r)^3 \, dr \right)^{1/2},
  $$

  and (case a), using the symmetry $(x,y) \mapsto (y,x)$,

  $$
  \int_{|x-y| > \epsilon^{-1}} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy
  = 2 \int_{|x-y| > \epsilon^{-1}} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy
  + 2 \int_{|x-y| > \epsilon^{-1}} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy
  \leq 2M \int_{|x| > \epsilon^{-1}} \rho(t,x) \ln (|x|/(1+\theta)) \, dx
  \to 0 \quad \text{as } \epsilon \to 0^+.
  $$
for any $\theta \in [0, 1]$.

In (case b), if $p(t, \cdot)$ has a moment of order $m > 0$, using again the symmetry $(x, y) \mapsto (y, x)$,

$$\int_{|x-y|>\varepsilon^{-1}} \rho(t, x) p(t, y) \ln |x-y| \, dx \, dy \leq \varepsilon^m \ln(1/\varepsilon) \int_{|x-y|>\varepsilon^{-1}} \rho(t, z) p(t, y) |x-y|^m \, dx \, dy$$

for $\varepsilon > 0$ small enough, and the conclusion holds using the identity $|x-y|^m \leq 2^m (|x|^m + |y|^m)$. (Case a) as well as (Case b) correspond to cases where the self-consistent potential energy is continuous w.r.t. the time.

(iii) If $U_0(x) = K \ln(1 + |x|)$ for some $K < \frac{M}{M_0}$, for any $\alpha > 0$, we can get the following dispersion-type estimate:

$$\liminf_{t \to +\infty} \int_{\mathbb{R}^2} f(t, x, v) \frac{|x-(t+\alpha)\nu|^2}{(t+\alpha)^2} \, dx \, dv \leq 2M \left( K - \frac{M_0^2}{4\pi} \right).$$

Let us prove it: according to Proposition 1.2 (i), with $L = K$, $C = 2 - \frac{M}{M_0}$, as $t \to +\infty$,

$$\frac{M^2}{2\pi} \ln(t+\alpha) \left( 1 + o(1) \right) \leq L(0) - \int_0^t \left( \int_{\mathbb{R}^2} f(s, x, v) \frac{|x-(s+\alpha)\nu|^2}{(s+\alpha)^2} \, dx \, dv \right) \, ds + 2KM,$$

$$\frac{M^2}{2\pi} \ln(t+\alpha) \leq 2KM - \liminf_{t \to +\infty} \int_{\mathbb{R}^2} f(t, x, v) \frac{|x-(t+\alpha)\nu|^2}{(t+\alpha)^2} \, dx \, dv \ln(t+\alpha) + o(\ln(t+\alpha)).$$

Case 2: Assume that $x \mapsto \frac{\nabla U_0}{|\nabla U_0|}$ belongs to $L^\infty(\mathbb{R}^2)$. Let us compute (with $\alpha > 1$ and $t > 0$)

$$H_\alpha(t) = \int_{\mathbb{R}^2} f(t, x, v) \cdot \left( \frac{1}{(t+\alpha)^2 \ln(t+\alpha)} |x|^2 + \frac{1}{(t+\alpha)^2} |x-v(t+\alpha)|^2 ight)$$

$$+ U(t, x) + U_0(x) + \frac{M}{2\pi} \ln(t+\alpha) \, dx \, dv,$$

$$\frac{d}{dt} H_\alpha(t) = -\frac{2}{(t+\alpha)^3} \int_{\mathbb{R}^2} f(t, x, v) \cdot |(1 + \frac{1}{2\ln(t+\alpha)}) x - u(t+\alpha)|^2 \, dx \, dv$$

$$- \frac{1}{(t+\alpha)^2 \ln(t+\alpha)^2} \int_{\mathbb{R}^2} f(t, x, v)|x|^2$$

$$+ \frac{2}{t+\alpha} \int_{\mathbb{R}^2} (x \cdot \nabla U_0) \rho(t, x) \, dx.$$
The reason why such a quantity is decreasing if \( x \cdot \nabla U_0(x) \leq 0 \) for almost all \( x \in \mathbb{R}^d \) and why one has to introduce a term \( \int \int f(t,z,v)|x|^2 \, dx \, dv \) is related to the notion of asymptotic dispersion profile. This is the subject of a paper in preparation with G. Rein [20].

The Lyapunov functional is finite for any \( t > 0 \): we have indeed

\[
\int_{\mathbb{R}^d} \rho(t,x)U(t,x) \, dx = -\frac{1}{2\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy
\]

and

\[
\int \int_{|x-y| < h} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy \leq M^2 \ln k,
\]

\[
\int \int_{|x-y| \geq h} \rho(t,x)\rho(t,y) \ln |x-y| \, dx \, dy \leq \int \int_{|x-y| \geq h} \rho(t,x)\rho(t,y)|x-y|^2 \, dx \, dy \cdot \frac{\ln k}{k^2}
\]

provided \( k \geq \sqrt{e} \), since \( k \to \ln k \) is decreasing on \([\sqrt{e}, +\infty)\). Then

\[
\int \int_{|x-y| \geq h} \rho(t,x)\rho(t,y)|x-y|^2 \, dx \, dy \leq 4M \int \rho(t,x)|x|^2 \, dx
\]

because \( |x-y|^2 \leq 2(|x|^2 + |y|^2) \). Thus

\[
\int_{\mathbb{R}^d} \rho(t,x)U(t,x) \, dx \geq -\frac{M^3}{2\pi} \ln k - \frac{2M}{\pi} \int_{\mathbb{R}^d} \rho(t,x)|x|^2 \, dx \cdot \frac{\ln k}{k^2},
\]

and

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,z,v) \cdot \left( \frac{1}{(t+\alpha)^2 \ln(t+\alpha)}|z|^2 + U(t,z) + \frac{M}{2\pi} \ln(t+\alpha) \right) \, dx \, dv \\
\geq \int_{\mathbb{R}^d} \rho(t,x)|x|^2 \, dx \cdot \left( \frac{1}{(t+\alpha)^2 \ln(t+\alpha)} - \frac{2M}{\pi} \frac{\ln k}{k^2} \right) \\
+ \frac{M^2}{2\pi} \left( \ln(t+\alpha) - \ln k \right).
\]

Let \( k = k(t) \) be such that

\[
\frac{1}{(t+\alpha)^2 \ln(t+\alpha)} = \frac{2M}{\pi} \frac{\ln k}{k^2}.
\]

As \( t \to +\infty \),

\[
k(t) = \sqrt{\frac{M}{2\pi} (t+\alpha) \ln(t+\alpha)} \cdot (1 + o(1))
\]

and then

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,z,v) \cdot \left( \frac{1}{(t+\alpha)^2 \ln(t+\alpha)}|z|^2 + U(t,z) + \frac{M}{2\pi} \ln(t+\alpha) \right) \, dx \, dv \\
\geq -C \left( 1 + \ln \ln(t+\alpha) \right) \tag{1.7}
\]
for some constant $C > 0$. We can summarize these properties in the

**Proposition 1.3**: Assume that $f_0$ is a nonnegative function in $L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^2)$ such that

$$
\int \int_{\mathbb{R}^3 \times \mathbb{R}^2} f_0(x, v) \cdot |x|^2 + |v|^2 \, dx \, dv < +\infty
$$

with $U_0 \geq 0$, $\nabla U_0 \in W^{1,\infty}(\mathbb{R}^2)$. Assume also that $z \mapsto \frac{\nabla U_0}{1 + |z|^2}$ is bounded. Then, if $f$ is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to the initial data $f_0$,

(i) there exists some $\alpha_0 > 0$ and a constant $C > 0$ such that for any $\alpha > \alpha_0$,

$$
\int \int_{\mathbb{R}^3 \times \mathbb{R}^2} f(t, x, v) \cdot \left( \frac{1}{(t + \alpha)^2 \ln(t + \alpha)} |x|^2 + \frac{1}{(t + \alpha)^2} |x - v(t + \alpha)|^2 
+ U(t, x) + U_0(x) + \frac{M}{2\pi \ln(t + \alpha)} \right) \, dx \, dv \geq - C \left( 1 + \ln(\ln(t + \alpha)) \right) (1.9)
$$

(ii) For any $\alpha > 1$, if $x \cdot \nabla U_0(x) \leq 0$ for almost all $x \in \mathbb{R}^2$, then

$$
H_\alpha(t) = \int \int_{\mathbb{R}^3 \times \mathbb{R}^2} f(t, x, v) \cdot \left( \frac{1}{(t + \alpha)^2 \ln(t + \alpha)} |x|^2 + \frac{1}{(t + \alpha)^2} |x - v(t + \alpha)|^2 
+ U(t, x) + U_0(x) + \frac{M}{2\pi \ln(t + \alpha)} \right) \, dx \, dv \leq H_\alpha(0) \quad \forall t > 0 ,
$$

and, if $f$ is a strong solution (i.e. has a moment of order $2 + \epsilon$ in $x$ and $v$), then

$$
\frac{d}{dt} H_\alpha(t) = - \frac{2}{(t + \alpha)^3} \int \int_{\mathbb{R}^3 \times \mathbb{R}^2} f(t, x, v) \cdot \left[ (1 + \frac{1}{2\ln(t + \alpha)}) |x - v(t + \alpha)|^2 
- \frac{1}{(t + \alpha)^3 (\ln(t + \alpha))^2} \int \int_{\mathbb{R}^3 \times \mathbb{R}^2} f(t, x, v) |x|^2 
+ \frac{2}{t + \alpha} \int_{\mathbb{R}^3} (x \cdot \nabla U_0) \rho(t, x) \, dx \right].
$$

**Proof of Proposition 1.3**: To prove (1.8), one may use a regularized problem (i.e. a smooth approximation of $f_0$ and a regularized kernel instead of Poisson's...
kernel) and pass to the limit:

\[
H_n(t) \leq H_0(0) - \int_0^t \frac{2}{(s + \alpha)^3} \left\{ \int \int_{R^2 \times R^2} f(s, z, v) \cdot \left[(1 + \frac{1}{2\ln(s + \alpha)})^2 \right] |z - v(s + \alpha)|^2 \, dz \, dv \\
+ \frac{1}{(s + \alpha)^3 (\ln(s + \alpha))^3} \int \int_{R^2 \times R^2} f(s, z, v)|z|^3 \\
+ \frac{2}{s + \alpha} \int_{R^2} (z \cdot \nabla U_0) \rho(s, z) \, dz \right\} \, ds. \quad \forall \ t > 0
\]

(1.10)

(1.9) then holds without any further computations. We refer to section 5 for the justifications of (1.6) and (1.8) depending on the moments one knows to exist.

\[\square\]

Remark 1.4 : For a strong solution corresponding to an external potential $U_0$ such that $x \cdot \nabla U_0$ is bounded but for which $x \cdot \nabla U_0$ changes its sign, Equation (1.6) still holds.

2. Existence, regularity and uniqueness results

Existence results of the Cauchy problem for the Vlasov-Poisson system are now well known (see [37], [38], [39], [36], [29]) in dimension three without external potential. The situation in dimension two is easier than in dimension three (see [3], [40], [41, 42], [7]). However, since we consider here the case with an external potential and since we assume weaker assumptions on $f_0$ than in the previous papers, let us give an existence result.

Theorem 2.1 : Assume that $U_0$ is nonnegative and that $f_0$ is a nonnegative function in $L^1 \cap L^\infty(R^2 \times R^2)$ such that $(x, v) \mapsto f_0(x, v) \cdot (|v|^2 + U_0(x) + U(t = 0, x))$ belongs to $L^1(R^2 \times R^2)$. If $\nabla U_0$ belongs to $W^{1,\infty}(R^2)$, then there exists a nonnegative solution $f \in C^0(R^+; L^1(R^2 \times R^2))$ in the sense of the distributions of the Vlasov-Poisson system with initial data $f_0$ such that

\[
\int \int_{R^2 \times R^2} f(t, x, v) \cdot (|v|^2 + |U(t, x)| + U_0(x) + U(t = 0, x)) \, dz \, dv < +\infty \quad (2.1)
\]

provided

- either (confinement)

\[
\lim_{|x| \to \infty} \frac{U_0(x)}{|x|} = +\infty
\]
or
\[ x \mapsto \frac{\nabla U_0(x)}{1 + |x|} \in L^\infty(\mathbb{R}^2), \]
and
\[ \exists m \in [1, 2] \int_{\mathbb{R}^2} f_0(x, v) \left( |x|^m + |v|^2 \right) \, dx \, dv < +\infty, \]

or
\[ x \mapsto \frac{x \cdot \nabla U_0(x)}{1 + |x|^2} \in L^\infty(\mathbb{R}^2), \]
and
\[ \int_{\mathbb{R}^2} f_0(x, v) \left( |x|^2 + |v|^2 \right) \, dx \, dv < +\infty. \]

Note that the energy estimate holds as an inequality
\[ \int_{\mathbb{R}^2} f(t, x, v) \cdot (|v|^2 + U(t, x) + U_0(x)) \, dx \, dv \]
\[ \leq \int_{\mathbb{R}^2} f_0(x, v) \cdot (|v|^2 + U(t = 0, x) + U_0(x)) \, dx \, dv < +\infty, \quad (2.2) \]
\[ \left| \int_{\mathbb{R}^2} f(t, x, v) U(t, x) \, dx \, dv \right| < +\infty \]
\forall t > 0 \text{ in both cases.}

These solutions are weak solutions but it is easy to prove that they are in fact strong as soon as \( f_0 \) satisfies the condition
\[ \exists \varepsilon > 0 \int_{\mathbb{R}^2} f_0(x, v) \left( |x|^{2+\varepsilon} + |v|^{2+\varepsilon} \right) \, dx \, dv < +\infty \]
since the momenta of order \( 2 + \varepsilon \) are finite for any \( t > 0 \) (see lemma 5.2). In that case (2.2) becomes an equality, while estimates like bounds on the energy are enough for weak solutions.

The conditions on \( f \) could be weakened using the notion of renormalized solutions (see [11, 12]) while the notion of solutions in the sense of the characteristics still holds (in the renormalized sense) as soon as \( \nabla U + \nabla U_0 \) at least belongs to \( W^{1,1}_{bc}(\mathbb{R}^+ \times \mathbb{R}^2) \) (see Remark 2.2 below). Since the proof of such results rely on now classical methods, let us only mention the main ingredients.

Sketch of the proof:
1) \textit{a priori estimates}: Assume first that \( f \) is a classical solution, smooth enough to justify the integrations by parts.

a) for any \( C^1 \) convex function \( s \) defined on \( [0, +\infty[ \)
\[ \frac{d}{dt} \int_{\mathbb{R}^2} s(f(t, x, v)) \, dx \, dv = 0 \]
since
\[
0 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\partial f}{\partial t}(t, z, v) \cdot s'(f(t, z, v)) \, dz \, dv \\
+ + \int_{\mathbb{R}^3 \times \mathbb{R}^3} v \cdot \partial_x f(t, z, v) \cdot s'(f(t, z, v)) \, dz \, dv \\
+ + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla(U(t, x) + U_0(x)) \cdot \partial_x f(t, z, v) \cdot s'(f(t, z, v)) \, dz \, dv \\
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\partial f}{\partial t}(t, z, v) \, dz \, dv + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_x \cdot \left( v \, s(f(t, z, v)) \right) \, dz \, dv \\
+ + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_v \cdot \left( \nabla(U(t, x) + U_0(x)) \, s(f)(t, z, v) \right) \, dz \, dv .
\]

(Of course the result also holds for a non convex function $s$, but the result does not pass to the limit: see below).

This proves for example that
\[
\frac{d}{dt} \|f(t, \cdot, \cdot)\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} = 0
\]
for any $p \in [1, +\infty[$ and

\[
\|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leq \|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \quad \forall \ t \in [0, +\infty[ .
\]  

(2.3)

For $p = 1$, this proves the conservation of the mass:

\[
\|\rho(t, \cdot)\|_{L^1(\mathbb{R}^3)} = \|f(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} = \|\rho_0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \quad \forall \ t \in [0, +\infty[ .
\]

(2.4)

b) conservation of the energy:

\[
\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( |v|^2 + U(t, x) + 2U_0(x) \right) f(t, z, v) \, dz \, dv = 0 .
\]

(multiply the Vlasov equation by $|v|^2$, integrate by parts and use the Poisson equation.)

c) Lyapunov's functional: equations (1.3) and (1.5) hold as in section 1 (see also sections 3 and 5). They are needed only in case 2.

d) an interpolation lemma (see Appendix B for more details): with the notation $\rho(t, z) = \int_{\mathbb{R}^3} f(t, z, v) \, dv$,

\[
\|\rho(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq 2\sqrt{\pi} \cdot \|f(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^{1/2} \cdot \|\rho(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^{1/2} .
\]

We can indeed split the integral defining $\rho$ into two integrals and evaluate these integrals in different ways:

\[
\rho(t, z) = \int_{|v| < R} f(z, v) \, dv + \int_{|v| \geq R} f(z, v) \, dv ,
\]
If we optimize on $R = R(t, z)$, then we get
\[
\int_{|v| < R} f(x, v) \, dv \leq \pi R^2 \cdot ||f(t, \cdot, \cdot)||_{L^1(R^2 \times R^2)}^2,
\]
\[
\int_{|v| \geq R} f(x, v) \, dv \leq \frac{1}{R^2} \int_{R^2} f(x, v) \, |v|^2 \, dv.
\]
If we optimize on $R = R(t, z)$, then we get
\[
\rho(t, z) \leq 2^{\sqrt{\pi}} \cdot ||f(t, \cdot, \cdot)||_{L^1(R^2 \times R^2)}^{1/2} \cdot \left( \int_{R^2} f(x, v) \, |v|^2 \, dv \right)^{1/2},
\]
which easily proves the estimate. Using now (2.3) and (2.5), we get
\[
||\rho(t, \cdot)||_{L^1(R^2)} \leq 2^{\sqrt{\pi}} \cdot ||f_0||_{L^1(R^2 \times R^2)}^{1/2} \cdot ||f(t, z, \cdot)|v|^2||_{L^1(R^2 \times R^2)}^{1/2} \quad \forall \, t \in [0, +\infty[.
\]
\[
(2.6)
\]
2) **Compactness:** A simple method to pass to the limit in the equation is to notice that as soon as $f \psi(t, \cdot, \cdot) \Delta \psi \, dv$ strongly converges in $L^3$ for any $L^\infty$ with compact support function $\psi$, then the limit $f$ is a solution of the Vlasov-Poisson system in the distribution sense. This is easily obtained using the averaging lemmas (see [28], [24], [25], [13]) since
\[
\frac{\partial f}{\partial t} + v \cdot \Delta f = \nabla_v \cdot [\nabla(U + U_0) \cdot f],
\]
and since $\nabla(U + U_0) \cdot f \psi(t, \cdot, \cdot) \, dv$ belongs to $L^1$ for any $L^\infty$ function $\psi$ (take for example $p = \frac{3}{2}$ and $q = 3$ in (2.7)).
the existence result for the regularized problem is easily obtained using the
characteristics and a fixed point method.

**Remark 2.2:** The assumption $f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ can also be removed and
replaced by the condition $f_0 \ln f_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ giving an existence result for
renormalized solutions as in [12].

Regularity and uniqueness results are obtained in a classical way (see [4], [40]
and [41,42] or [7] and [33] for results respectively in dimension two and three).
The results we present here follow the strategy of proof used by P.-L. Lions &
B. Perthame [33], except that in dimension $N = 2$, there is no need of high order
moments. These results extend the ones obtained by S. Ukai & T. Okabe [40]
and S. Wollman [41,42] (without external potential) to a more general class of
solutions (with eventually an external potential).

**Proposition 2.3:** Let $f_0$ and $U_0$ satisfy the same assumptions as in Theorem 2.1 and
assume that $t \mapsto (X_0(s, t, x, v), V_0(s, t, x, v))$ are the characteristics in the external
potential $U_0$ defined by

\[
\begin{align*}
\frac{d}{dt} X_0(s, t, x, v) &= V_0(s, t, x, v), \\
X_0(t, t, x, v) &= x,
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} V_0(s, t, x, v) &= -\nabla_x U_0(X_0(s, t, x, v)), \\
V(t, t, x, v) &= v.
\end{align*}
\]

(i) Regularity:

If for any $T > 0$, there exists an $R_0 > 0$ such that

\[
(t, x, v) \mapsto \sup_{(s, t) \in (0, T)} \{|y - X_0(0, t, x, v)| < \frac{R^2}{2}, |w - V_0(0, t, x, v)| < Rt\}
\]

belongs to $L^\infty((0, T) \times \mathbb{R}^2; L^1(\mathbb{R}^2)) \forall R > R_0$

\[
\lim_{R \to +\infty} \left( \frac{e^{-|t|}}{1 + e^{-|t|}} \right) \sup_{(s, t) \in (0, T)} \{|y - X_0(0, t, x, v)| < \frac{R^2}{2}, |w - V_0(0, t, x, v)| < R t\}
\]

for some $\varepsilon > 0$ which may depend on $R$ (and $T$), then for any $T > 0$, there exists
a solution $f$ satisfying the same properties as in Theorem 2.1 and such that $(t, x) \mapsto
\rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) \, dv$ belongs to $L^\infty((0, T) \times \mathbb{R}^2)$. If moreover $\nabla U_0$ belongs to $C^2$
and $\forall R, T > 0$,

\[
\sup_{(s, t) \in (0, T)} \{|y - X_0(0, t, x, v)| \leq R, |w - V_0(0, t, x, v)| \leq R\}
\]

belongs to $L^\infty((0, T) \times \mathbb{R}^2; L^1 \cap L^2(\mathbb{R}^2))$, (2.9)
then for any $T > 0$, $\rho$ belongs to $L^\infty((0,T);C^{\beta,1}(\mathbb{R}^2_+))$ and $(t,x) \mapsto \nabla U(t,x) = -\frac{2}{\pi \sqrt{\pi}} * (t,.)$ belongs to $L^\infty((0,T);C^{\beta,1}(\mathbb{R}^2_+))$ for any $\beta \in ]0,1].$

(ii) Uniqueness:
If $f_0$ satisfies conditions (2.8) and (2.9), then the solution of the Vlasov-Poisson system such that $\rho$ belongs to $L^\infty((0,T) \times \mathbb{R}^2)$ is unique (of course, the potential $U$ is always defined up to an additive constant).

Examples: If $f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ has a compact support, then (2.8) is automatically satisfied. If it is a Lipschitz function, then (2.9) also holds true. Conditions (2.8) and (2.9) are related to the asymptotic behavior of $f_0$ as $|x,v| \to +\infty$. Assume for example that $f_0$ is dominated by a gaussian in $x$ and $v$:

$$f_0(x,v) \leq C \cdot e^{-\frac{1}{\sigma^2}(v^2 + |x|^2)} \quad \forall (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2.$$ 

1) Assume first that $U_0 \equiv 0$. A straightforward computation gives

$$\int_{\mathbb{R}^2} \left( \sup_{y \in \mathbb{R}^2} \{ f_0(y + vt, y) : |y - x| < \frac{1}{2} R t^2, |v - w| < R t \} \right)$$

$$\leq C \cdot \varepsilon \frac{e^{\frac{1}{4\varepsilon} R^2 t^2}}{\varepsilon^2 + 4} \int_{\mathbb{R}^2} e^{-\frac{1}{\sigma^2}(v^2 + |x|^2)}$$

$$\leq \frac{8\sigma C}{R^2 + 4} e^{-\frac{1}{4\varepsilon}(v^2 + |x|^2)} ,$$

which proves (2.8) by taking for instance

$$\varepsilon = \frac{8 \ln R}{R^2(T^2 + 4)} .$$

2) If $U_0$ is an harmonic potential (assume $U_0(x) = |x|^2$ to avoid technicalities), then

$$|V_0(0, t, x, v)^2 + X_0(0, t, x, v)|^2 = |v|^2 + |x|^2 .$$

$$\int_{\mathbb{R}^2} \left( \sup_{y \in \mathbb{R}^2} \{ f_0(y, y) : |y - x_0(0, t, x, v)| < \frac{1}{2} R t^2, |v - v_0(0, t, x, v)| < R t \} \right)$$

$$\leq C \cdot \varepsilon \frac{e^{\frac{1}{4\varepsilon} R^2 t^2}}{\varepsilon^2 + 4} \int_{\mathbb{R}^2} e^{-\frac{1}{\sigma^2}(v^2 + |x|^2)}$$

$$\leq 2\pi C \cdot \varepsilon \frac{e^{\frac{1}{4\varepsilon}(v^2 + |x|^2)}}{\varepsilon^2 + 4} ,$$

and the result holds again with

$$\varepsilon = \frac{8 \ln R}{R^2 \max(T, 1)^2(\max(T, 1)^2 + 4)} .$$
As in [33], we could relax the assumption (2.8) and replace it by the condition: \( \forall T > 0, \forall R > 0, \)

\[
(t, x, v) \mapsto \sup \{ f_0(y, w) : |y - X_0(0, t, x, v)| < \frac{1}{2} R t^2, |w - V_0(0, t, x, v)| < R t \}
\]

belongs to \( L^\infty((0, T) \times \mathbb{R}^2_+; L^2(\mathbb{R}^2_+)) \),

but in this case, one would have to prove first an estimate on a momentum of order higher than 2. This can be avoided in dimension 2 (see part 1 of the proof below) if (2.8) is satisfied.

**Proof of Proposition 2.3**: For the details of the proof, one may refer to [33] and check that the proof can be adapted to the dimension \( N = 2 \) in the presence of an external potential. We give here a sketch of that proof and refer to [33] for more details.

1) Since \( \rho \) belongs to \( L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2_+)) \), \((t, x) \mapsto E(t, x) + \nabla_x U_0(x) = -\nabla_x U(t, x)\) belongs to \( L^\infty((0, T); W^{1,2}_0(\mathbb{R}^2_+)) \) by Agmon, Douglis & Nirenberg's theorem. Using characteristics (for the following truncated transport problem)

\[
\begin{align*}
\dot{X}(s) &= V(s), \\
\dot{V}(s) &= E^R(s, X(s)), \\
X(t) &= x, \\
V(t) &= v,
\end{align*}
\]

in the sense of R.-J. DiPerna and P.-L. Lions [11] with

\[
E^R(s, x) = -\nabla U_0(x) - \min(R, |\nabla U(s, x)|) \frac{\nabla U(s, x)}{|\nabla U(s, x)|},
\]

one proves that

\[
|V(0) - V_0(0, t, x, v)| \leq R t,
\]

\[
|X(0) - X_0(0, t, x, v) + tV_0(0, t, x, v)| \leq \frac{R t^2}{2},
\]

where \( t \mapsto (X_0(s, t, x, v), V_0(s, t, x, v)) \) are the characteristics in the external potential \( U_0 \) defined by

\[
\begin{align*}
\frac{d}{ds} X_0(s, t, x, v) &= V_0(s, t, x, v), \\
X_0(0, t, x, v) &= x, \\
\frac{d}{ds} V_0(s, t, x, v) &= -\nabla_x U_0(X_0(s, t, x, v)), \\
V(0, t, x, v) &= v.
\end{align*}
\]

The solution \( f^R \) of

\[
\partial_t f^R + v \cdot \partial_x f^R + E^R(t, x) \cdot \partial_v f^R = 0
\]
satisfies
\[ f_R(t, x, v) = f_0(X(0), V(0)) \]
\[ \leq \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \rho(t, y) \, dy \cdot \frac{R^2}{2}, \quad |x - X_0(0, t, x, v)| < \frac{R^2}{2}, \quad |w - V_0(0, t, x, v)| < R t \).

As in the proof of Theorem 2.1, a direct computation shows that, for any \( \epsilon > 0 \),
\[ 2\pi \| \mathcal{E}_t \|_{L^p(\mathbb{R}^2, dx)} + \frac{1}{2\pi} \left( \frac{1}{2\pi} \right) \frac{1}{\epsilon} \| \rho(t, \cdot) \|_{L^p(\mathbb{R}^2, dx)} \left\| \rho(t, \cdot) \right\|_{L^{p'}(\mathbb{R}^2, dx)} \]
\[ \leq \| \rho(t, \cdot) \|_{L^p(\mathbb{R}^2, dx)} + \frac{1}{2\pi} \left( \frac{1}{2\pi} \right) \frac{1}{\epsilon} \| \rho(t, \cdot) \|_{L^{p'}(\mathbb{R}^2, dx)} \left\| \rho(t, \cdot) \right\|_{L^p(\mathbb{R}^2, dx)} \]
(split the integral \( \int_{\mathbb{R}^2} \rho(t, y) \, dy \) into two parts corresponding respectively to \(|x - y| > 1\) and \(|x - y| \leq 1\) and use Hölder’s inequality with \( p = \frac{p + 1}{p} \) and \( p' = 2 + \epsilon \) for the second one). Assumption (2.8) just asserts that \( \| \mathcal{E}_t \|_{L^\infty(\mathbb{R}^2)} \) does not depend on \( R \) for \( R \) large enough. Taking then \( \rho = f_R \) proves that \( \rho \) belongs to \( L^\infty((0, T) \times \mathbb{R}^2) \).

2) There exist an \( \alpha > 0 \) (\( \alpha = e^{-CT} \) for some \( C > 0 \)) such that the characteristic curves
\[ \dot{X}(s, t, x_0, v_0) = V(s, t, x_0, v_0), \quad V(t, x_0, v_0) = v_0, \quad i = 1, 2, \]
are uniquely defined and satisfy for any \( t, s \in [0, T] \)
\[ |X(s, t, x_0, v_0) - X(s, t, x_3, v_3)| + |V(s, t, x_0, v_0) - V(s, t, x_3, v_3)| \leq C \left[ |x_1 - x_2|^\alpha + |v_1 - v_2|^\alpha \right], \]
whenever \( E + \nabla U_0 = -\nabla U \) is given by \( \rho \) through the Poisson equation.

The proof relies on the estimate
\[ |(E(t,x) + \nabla U_0(x)) - (E(t,y) + \nabla U_0(y))| \leq C|x - y| \ln \frac{1}{|x - y|} \]
for any \( x, y \in \mathbb{R}^2 \) such that \(|x - y| < 1/2\), which is proved exactly in the same way as in [33]: since
\[ \left| \frac{x - z}{|x - z|^2} - \frac{y - z}{|y - z|^2} \right| = \left| \frac{x - y}{|x - y|} \right| \cdot \frac{1}{|x - z||y - z|}, \]
\[ 2\pi \left| \nabla U(t, x) - \nabla U(t, y) \right| \]
\[ = 2\pi \left| \rho(t) \right|_{L^\infty(\mathbb{R}^2)} \left( \int_{|x - z| \leq 1} \frac{dz}{|x - z|^2} + \int_{|x - z| > 1} \frac{dz}{|x - z|^2} - \frac{y - z}{|y - z|^2} |\rho(t, z)| \, dz \right) \]
\[ \leq 4\pi \left| \rho(t) \right|_{L^\infty(\mathbb{R}^2)} + |x - y| \cdot \left( \int_{|x - z| \leq 1} \frac{dz}{|x - z|^2} \left| \left| \rho(t) \right|_{L^\infty(\mathbb{R}^2)} + \left| \rho(t) \right|_{L^{p'}(\mathbb{R}^2)} \right) \right) \]
\[ = 4\pi \left| \rho(t) \right|_{L^\infty(\mathbb{R}^2)} + |x - y| \cdot \left( 2\pi |x - y| \left| \rho(t) \right|_{L^\infty(\mathbb{R}^2)} + \left| \rho(t) \right|_{L^{p'}(\mathbb{R}^2)} \right), \]
and the result holds with $\epsilon = |x - y| \cdot |\ln(\epsilon + 1)|$. Then

$$\ln \left( |X_1(s) - X_2(s)|^2 + |V_1(s) - V_2(s)|^2 \right) \leq \left( |X_1(s) - X_2(s)|^2 + |V_1(s) - V_2(s)|^2 \right) e^{-\alpha t},$$

which gives the result with $\alpha = e^{-\alpha t}$.

3) Since

$$|\rho(t, x_1) - \rho(t, x_2)| \leq \int \sup \left| \nabla_x f_0(y, \omega) \right| \cdot |\nu - \lambda(0)| \leq R \cdot \left( |y - \lambda(0)| \leq R \right) \cdot \sup \left| \left(V(0, t, x_1, \omega) - V(0, t, x_2, \omega)\right)|X(0, t, x_1, \omega) - X(0, t, x_2, \omega)| + |V(0, t, x_1, \omega) - V(0, t, x_2, \omega)| \right),$$

(2.10)

(for some $R$ eventually depending on $t$) again like in [33], this proves that $\rho$ belongs to $L^\infty((0, T); C^\infty(\mathbb{R}^2))$, which ensures that $\varphi$ belongs to $L^\infty((0, T); C^1(\mathbb{R}^2))$ using Schauder's theorem (see [1], [27]): the characteristic curves are Lipschitz continuous and (2.10) written with $\alpha = 1$ proves that $\rho$ belongs to $L^\infty((0, T); C^0(\mathbb{R}^2))$ and that $E$ belongs to $L^\infty((0, T); C^{1,\beta}(\mathbb{R}^2))$ for any $\beta \in [0, 1]$.

4) the uniqueness result can be shown in the same way as in [33] (with the simplification that we don't have to care about the field term coming from the initial data since we only have to consider the difference between two solutions - the argument is not used for getting estimates on "higher moments"). Assume that there exist two solutions $f_1$ and $f_2$ corresponding to the same initial data $f_0$ and define

$$D(t) = \sup_{0 \leq s \leq t} \|f_1(s, \ldots) - f_2(s, \ldots)\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2; \text{d}s \text{d}x)}.$$

Then

$$\frac{d}{dt} D^2(t) \leq 2C(T) \cdot \sup_{0 \leq s \leq t} \|E_1(s, \ldots) - E_2(s, \ldots)\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2; \text{d}s \text{d}x)} \cdot D(t)$$

since condition (2.9) provides

$$\nabla_x f \in L^\infty((0, T) \times \mathbb{R}^2; L^1(\mathbb{R}^2)).$$

Then

$$\frac{d}{dt} D(t) \leq C(T) \cdot \Delta(t) \quad (2.11)$$
Performing an integration by parts, this gives the following expressions for $\rho_i$ and $E_1 - E_2$

\[
\rho_i(t, z) = \int_{\mathbb{R}^3} f_0(x - vt, v) \, dv + \int_0^t \left( \int_{\mathbb{R}^3} \left( E_i(t - s, z - vs, v) \cdot f_i(t - s, z - vs, v) \right) \, dv \right) \, ds , \\
E_1(t, z) - E_2(t, z) = \frac{x}{2 \pi |\mathbb{R}^3|^2} \cdot \int_0^t \left( \int_{\mathbb{R}^3} \left( (E_1 - E_2) f_1 + E_2 f_1 - f_2 \right)(t - s, z - vs, v) \, dv \right) \, ds .
\]

Using the fact that $\text{div}_v (\mathcal{E})$ is the Dirac distribution,

\[
\|E_1(t, \cdot) - E_2(t, \cdot)\|_{L^2(\mathbb{R}^3, dt)} \leq \|\int_0^t \left( \int_{\mathbb{R}^3} \left( (E_1 - E_2) f_1 + E_2 f_1 - f_2 \right)(t - s, z - vs, v) \, dv \right) \, ds\|_{L^2(\mathbb{R}^3, dt)}.
\]

Applying the Cauchy-Schwarz inequality, we get

\[
\int_0^t \int_{\mathbb{R}^3} \left( |E f|^2(t - s, z - vs, v) \right) \, dv \, ds \leq \int_0^t \int_{\mathbb{R}^3} \left( |E|^2(t - s, z - vs, v) \right) \, dv \, ds ,
\]

\[
\|E_i(t, \cdot) - E_j(t, \cdot)\|_{L^2(\mathbb{R}^3, dt)} \leq \sup_{0 \leq s \leq t} \|E_i(t, \cdot) - E_j(t, \cdot)\|_{L^2(\mathbb{R}^3, dt)} .
\]

Applying successively this computation to $(E = E_1 - E_2, f = f_1)$ and $(E = E_2, f = f_1 - f_2)$, we get

\[
\|E_1(t, \cdot) - E_2(t, \cdot)\|_{L^2(\mathbb{R}^3, dt)} \leq C \cdot D(t) ,
\]

where $C$ is a constant which only depends on the energy: this computation is summarized in the identity

\[
\Delta(t) \leq t \cdot \Delta(t) \cdot \|f_0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} + C \cdot D(t) ,
\]

\[
\Delta(t) \leq \frac{C}{1 - t \cdot \|f_0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}} \cdot D(t) .
\]
Combining this with (2.11), we get

\[ \frac{d}{dt} D(t) \leq t \cdot \left( \frac{C}{1 - t \cdot \|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}} \right) D(t) \]

which proves that \( D(t) \equiv 0 \) for any \( t \in ]0, t_0[ \) with

\[ t_0 = \left( \frac{\|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}}{C} \right)^{-1} . \]

**Remark 2.4:** As in [33], condition (2.9) is satisfied as soon as for example the \( L^1 \) bound is satisfied and \( f_0 \) is Lipschitz continuous in \( x \) and \( v \).

### 3. A dispersion result

When there is no confining potential, the solution of the Vlasov-Poisson system is vanishing for large time. We present here a dispersion result which is the analogous in dimension 2 of the result obtained by R. Illner & G. Rein and B. Perthame in dimension 3 (see [31], [36]). It is essentially based on the computation of the Lyapunov functional of section 1.

**Proposition 3.1:** Assume that \( f \) is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to the initial data \( f_0 \), where \( f_0 \) is a nonnegative function in \( L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^d) \) such that

\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) \cdot \left( |x|^2 + |v|^2 + U_0(x) \right) \, dx \, dv < +\infty \]

with \( U_0 \geq 0, \nabla U_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^d), \nabla \cdot \nabla U_0 \leq 0 \) a.e. Then there exists some \( \alpha_0 > 0 \) and a constant \( C_\alpha > 0 \) (for any \( \alpha > \alpha_0 \)) such that

(i) for any \( t > 0 \),

\[ -C_\alpha \left( 1 + \ln(\ln(t + \alpha)) \right) \leq H_\alpha(t) \]

\[ = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot \left( \frac{1}{(t + \alpha)^2 \ln(t + \alpha)} |x|^2 + \frac{1}{(t + \alpha)^2} |x - v(t + \alpha)|^2 
\]

\[ + U(t, x) + U_0(x) + \frac{M}{2\pi} \ln(t + \alpha) \right) \, dx \, dv \leq H_\alpha(0) , \]
(ii) for any \( T > 0 \), for a strong solution,

\[
\int_0^T \frac{1}{(t + \alpha)^2 (\ln(t + \alpha))^\gamma} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|x|^2 \, dx \, dv \right) \, dt \leq C_\alpha \left( 1 + \ln(\ln(T + \alpha)) \right),
\]

\[
\int_0^T \frac{1}{t + \alpha} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|u - \frac{x}{t + \alpha}|^2 \, dx \, dv \right) \, dt \leq C_\alpha \left( 1 + \ln(\ln(T + \alpha)) \right). \tag{3.2}
\]

As a straightforward consequence, we have the

**Corollary 3.2**: With the same assumptions as in Proposition 3.1, there exists some \( \alpha_0 > 0 \) and a constant \( C_\alpha > 0 \) (for any \( \alpha > \alpha_0 \)) such that:

(i) For any \( t > 0 \),

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|v|^2 \, dx \, dv \leq C_\alpha \left( 1 + \ln(t + \alpha) \ln(\ln(t + \alpha)) \right)
\]

and

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|x|^2 \, dx \, dv \leq C_\alpha \left( 1 + (t + \alpha)^2 \ln(t + \alpha) \ln(\ln(t + \alpha)) \right),
\]

\[
\frac{1}{(t + \alpha)^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|x - (t + \alpha)|^2 \, dx \, dv \leq C_\alpha \left( 1 + \ln(\ln(t + \alpha)) \right). \tag{3.4}
\]

If \( f \) is a strong solution, then

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|U(t, x)| \, dx \, dv \sim -\frac{M^2}{2\pi} \ln(t + \alpha).
\]

If moreover \( U_0 \equiv 0 \), then

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|v|^2 \, dx \, dv \sim \ln(t + \alpha),
\]

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)|x|^2 \, dx \, dv \sim (t + \alpha)^2 \ln(t + \alpha).
\]

(ii) for any \( T > 1 \),

\[
\int_0^T \frac{1}{t + \alpha} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v) \cdot \left[ (1 + \frac{1}{2\ln(t + \alpha)}) \frac{x}{t + \alpha} - v \right]^2 \, dx \, dv \leq C_\alpha \left( 1 + \ln(\ln(T + \alpha)) \right), \tag{3.5}
\]

\[
\frac{1}{\ln(\ln(T + \alpha))} \int_0^T \frac{1}{t + \alpha} \left\| \rho(t, \cdot) \right\|_{L^1(\mathbb{R}^3)} \, dt < C_\alpha \tag{3.6}
\]

and

\[
\frac{1}{\ln(\ln(T + \alpha))} \int_0^T \frac{1}{t + \alpha} \left\| \nabla U(t, \cdot) \right\|_{L^1(\mathbb{R}^3)} \, dt < C_\alpha \tag{3.7}
\]
for any \( q \in [2, +\infty] \).

(iii) There exists a sequence \((t_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to +\infty} t_n = +\infty \) such that, for any \( \varepsilon > 0 \),

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t_n, x, v) v - \frac{v}{t_n + \alpha}^2 \, dx \, dv \cdot (\ln(t_n))^{1-\varepsilon} = 0 ,
\]

\[
\lim_{n \to +\infty} \|\rho(t_n, \cdot)\|_{L^1(\mathbb{R}^d)} \cdot (\ln(t_n))^{1-\varepsilon} = 0 ,
\]

\[
\lim_{n \to +\infty} \|\nabla \Upsilon(t_n, \cdot)\|_{L^1(\mathbb{R}^d)} \cdot (\ln(t_n))^{\frac{1}{2} - (1-\varepsilon)} = 0 ,
\]

for any \( q \in [2, +\infty] \).

(iv) if \( f_0 \) is compactly supported, then for any \( t > 0 \) \( f(t, \cdot, \cdot) \) is compactly supported too. If \( R(t) \) is the minimal radius of the balls centered at the origin and containing the support of \( f(t, \cdot, \cdot) \), then

\[
R(t) \geq C \alpha(t + \alpha) .
\]

**Proof of Corollary 3.2**: In this proof we will denote all the positive constants by the same symbol \( C \).

(i) is obtained through a refinement of the proof of (1.3) in section 1: let \( k = k(t) \) be such that

\[
\frac{1}{2} \cdot \frac{1}{(t + \alpha)^2 \ln(t + \alpha)} = \frac{2M}{\pi} \cdot \frac{\ln k}{k^2} .
\]

As \( t \to +\infty \),

\[
k(t) = \sqrt{\frac{M}{4\pi} (t + \alpha) \ln(t + \alpha) \cdot (1 + O(1))}
\]

and then

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot \left( \frac{1}{(t + \alpha)^2 \ln(t + \alpha)} |x|^2 + U(t, x) + \frac{M}{2\pi} \ln(t + \alpha) \right) \, dx \, dv \geq \frac{1}{2} \cdot \frac{1}{(t + \alpha)^2 \ln(t + \alpha)} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) |x|^2 \, dx \, dv - C \left( 1 + \ln(\ln(t + \alpha)) \right) ,
\]

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) |x|^2 \, dx \, dv \leq C \alpha \left( 1 + (t + \alpha)^2 \ln(t + \alpha) \ln(\ln(t + \alpha)) \right) .
\]

On the other side, because of (i) in Proposition 3.1,

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot \frac{1}{(t + \alpha)^2} |x - v(t + \alpha)|^2 \, dx \, dv \leq C \alpha \left( 1 + \ln(\ln(t + \alpha)) \right) .
\]

Thus

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) |v|^2 \, dx \, dv \leq C \alpha \left( 2 + \ln(t + \alpha) \ln(t + \alpha) \right) ,
\]
which proves (3.4).

Proof of (ii): because of (1.5), for any $T > 1$,
\[
\int_0^T \frac{1}{t + \alpha} \int_{U \times \mathbb{R}^d} f(t, x, \eta) \left[ \ln \left( 1 + \frac{1}{2 \ln(t + \alpha)} \right) \right] \frac{x}{t + \alpha} - v^2 \, dv \, dx \leq C \left( 1 + \ln(\ln(T + \alpha)) \right).
\]

(3.5)

Since
\[
\rho(t, x) = \int_{|x - (t + \frac{1}{2 \ln(t + \alpha)})\mathbb{R}|} f(t, x, \eta) \, dv + \int_{|x - (t + \frac{1}{2 \ln(t + \alpha)})\mathbb{R}|^c} f(t, x, \eta) \, dv,
\]
the classical method of interpolation provides
\[
\rho(t, x) \leq \pi R^2 ||f(t, ..., \eta)||_{L^\infty(U \times \mathbb{R}^d)} + \frac{1}{R} \int_{\mathbb{R}^d} f(t, x, \eta) \cdot |v - (1 + \frac{1}{2 \ln(t + \alpha)})^\frac{x}{t + \alpha}|^2 \, dv.
\]

Optimizing on $R$,
\[
||\rho(t, ...)||_{L^\infty(U \times \mathbb{R}^d)} \leq 2 \sqrt{\pi} ||f(t, ..., \eta)||_{L^\infty(U \times \mathbb{R}^d)}
\]
\[
\cdot \int_{\mathbb{R}^d} f(t, x, \eta) \cdot |v - (1 + \frac{1}{2 \ln(t + \alpha)})^\frac{x}{t + \alpha}|^2 \, dv,
\]
we get easily estimate (3.6).

(3.7) is then deduced from the Hardy-Littlewood-Sobolev inequality:
\[
||\nabla U(t, ...)||_{L^p(U \times \mathbb{R}^d)} \leq c(q) ||\rho(t, ...)||_{L^q(U \times \mathbb{R}^d)} \quad \text{with} \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{2} \quad \text{for any} \quad q \in [2, +\infty[,
\]
and the Hölder inequality for $\rho$:
\[
||\rho(t, ...)||_{L^p(U \times \mathbb{R}^d)} \leq ||\rho(t, ...)||_{L^q(U \times \mathbb{R}^d)}^{1-2/p} \cdot ||\rho(t, ...)||_{L^{(1-2/p)q}(U \times \mathbb{R}^d)}^{2(1-1/p)}
\]
\[
= ||\rho(t, ...)||_{L^{q}(U \times \mathbb{R}^d)}^{1/q} \cdot ||\rho(t, ...)||_{L^{2q}(U \times \mathbb{R}^d)}^{2/q}.
\]
\[
||\nabla U(t, ...)||_{L^2(U \times \mathbb{R}^d)}^{1/2} \leq c(q) M^{1/2} \cdot ||\rho(t, ...)||_{L^2(U \times \mathbb{R}^d)}
\]
which proves (3.7).

Proof of (iii): the method is the same for (3.8), (3.9) and (3.10). Let us prove for example (3.9). Assume that
\[
\lim_{t \to +\infty} ||\rho(t, ...)||_{L^2(U)}^{1/2} \cdot (\ln(t))^k = k > 0.
\]

For $T \to +\infty$,
\[
\frac{1}{\ln(\ln(T + \alpha))} \int_0^T \frac{1}{t + \alpha} ||\rho(t, ...)||_{L^2(U)} \, dt \geq \frac{k}{\ln(t_0)} \frac{1}{\ln(\ln(T + \alpha))} \int_0^T \frac{1}{t + \alpha} \, dt = \frac{k}{c} \left[ \ln(\ln(T + \alpha)) \right]^T_0.
\]
giving a contradiction with (3.6).

Proof of (iv): the fact that \( f(t,.,.) \) is compactly supported if \( f_0 \) is compactly supported will be proved in section 4. \( f \) is then a strong solution:

\[
- \frac{M^2}{2\pi} \ln(2R(t)) \leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) U(t, x) \, dxdv = - \frac{M^2}{2\pi} \ln(t + \alpha) \left( 1 + o(t) \right) .
\]

4. Growth of the support

This part is devoted to the special case corresponding to an initial data with compact support. The following theorem gives an upper bound for the growth of the support of the distribution function with respect to the time. Since it remains of finite size for any positive time, the moments in \( v \) are finite (which also means that the distribution function is a strong solution: see [36]). We assume here that \( U_0 \) is harmonic (see Appendix A for the justifications of the model):

\[
U_0(x) = \frac{\rho_0}{2} |x|^2 \quad \forall x \in \mathbb{R}^3 .
\]

Theorem 4.1: Assume that \( f \) is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to an initial data \( f_0 \) in \( L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) with compact support. Then \( f \) is a strong solution (in Perthame's sense) and for any \( t > 0 \), \( f(t,.,.) \) has a compact support, and for any \( \varepsilon > 0 \), there exists a constant \( C(\varepsilon) > 0 \) such that

\[
R(t) = \text{diam}(\text{supp}(f(t,.,.))) \leq \text{diam}(\text{supp}(f_0)) + C(\varepsilon)(1 + t)^{1+\varepsilon} . \tag{4.1}
\]

Proof: As before the proof is established in the context of classical smooth solutions. The result is then obtained by passing to the limit for a well chosen approximating sequence. It is not very difficult to check that the estimates on the support may be evaluated uniformly (for a well chosen approximating sequence).

Consider \( E(t, x, v) = \frac{|v|^2}{2} + U(t, x) + U_0(x) \):

\[
\frac{d}{dt} E(t, x(t), v(t)) = \frac{\partial U}{\partial t} (t, x(t)) \tag{4.2}
\]
for any characteristics $t \mapsto (x(t), v(t))$. The main idea is to evaluate the $L^\infty$-norm of $\partial U(\cdot, \cdot)$ using the Poisson equation (derived with respect to the time):

$$-\Delta \left( \frac{\partial U}{\partial t} \right) = \frac{\partial \rho}{\partial t}.$$  \hfill (4.3)

An integration of the Vlasov equation with respect to $v$ shows that (local conservation of the mass)

$$\int_{\mathbb{R}^3} \rho \, dv \left[ \partial_t f + v \cdot \partial_v f - (\partial_x U(t, x) + \partial_y U_0(z)) \cdot \partial_v f \right] = 0 ,$$

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot \int_{\mathbb{R}^3} f(t, x, v) \, dv = 0 .$$  \hfill (4.4)

Combining (4.3) and (4.4), we get

$$-\Delta \left( \frac{\partial U}{\partial t} \right) = -\nabla_x \cdot \int_{\mathbb{R}^3} f(t, x, v) \, dv ,$$

$$\left\| \frac{\partial U}{\partial t} \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{d}{2\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \int_{\mathbb{R}^3} f(t, y, v) \, dv \, dv ,$$

$$\left\| f(t, \cdot, v) \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{2\pi} \left( \frac{2\pi}{\epsilon (R(t))^\beta} \right)^{1/\beta} \left( \int_{\mathbb{R}^3} f(t, x, v) \, dv \right)^{1-\beta} .$$  \hfill (4.5)

Using now the interpolation lemma given in Appendix B (Lemma B.2), we get

$$\left\| f(t, \cdot, v) \right\|_{L^\infty(\mathbb{R}^3)} \leq K(2, \infty, 1 - \beta) \cdot \left\| f(t, \cdot, v) \right\|_{L^\infty(\mathbb{R}^3)} \cdot \left( \int_{\mathbb{R}^3} f(t, x, v) \, dv \right)^{1-\beta} .$$

with

$$\beta = \frac{1}{2 - \epsilon} \quad \text{and} \quad 1 - \beta = \frac{1 - \epsilon}{2 - \epsilon} .$$

This finally proves that

$$\left\| \frac{\partial U}{\partial t} (t, \cdot) \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{K(2, \infty, 1 - \beta)}{2\pi} \left( \frac{2\pi}{\epsilon (R(t))^\beta} \right)^{1/\beta} \left( \int_{\mathbb{R}^3} f(t, x, v) \, dv \right)^{1-\beta} ,$$

$$\left\| \frac{\partial U}{\partial t} (t, \cdot) \right\|_{L^\infty(\mathbb{R}^3)} \leq C(\epsilon) \cdot \left( \frac{R(t)}{\epsilon} \right)^{1/\beta} \cdot \left( \int_{\mathbb{R}^3} f(t, x, v) \, dv \right)^{1-\beta} .$$
for some constant
\[ C(\varepsilon) = \varepsilon^{\frac{1}{2}} \cdot \left( \frac{M}{2\pi} \right)^{\frac{1}{2}} \left( \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} f_0(x, v) \left( |v|^2 + U(t = 0, x) + 2U_0(x) \right) dv \right) \]

which only depends on the initial data.

Using the energy estimate,
\[ E(t, x(t), v(t)) = E(t, x(t), v(t))_{t=0} = C(\varepsilon) \cdot \left( R(t) \right)^{\frac{1}{2} + \frac{1}{2}} \cdot t, \]

and the fact that \( U \) is bounded from below on \( \text{supp}(\rho) \) by \( -\frac{M}{2\pi} \ln(2R(t)) \) and that \( U_0 \) is harmonic,
\[ \min(\rho_0, 1) \cdot \left( |x(t)|^2 + |v(t)|^2 \right) - \frac{M}{2\pi} \ln(2R(t)) \leq E(t, x(t), v(t)) \leq E(0, 0, 0) + C(\varepsilon) \cdot \left( R(t) \right)^{\frac{1}{2} + \frac{1}{2}} \cdot t, \]

\[ \min(\rho_0, 1) \cdot \left( R(t) \right)^2 - \frac{M}{2\pi} \ln(2R(t)) \leq E(t, x(t), v(t)) \leq \sup_{x_0, v_0 \in \text{supp}(f_0)} E(0, x_0, v_0) + C(\varepsilon) \cdot \left( R(t) \right)^{\frac{1}{2} + \frac{1}{2}} \cdot t, \]

which essentially proves the result for any \( \epsilon > \frac{M}{2\pi} \).

**Remark 4.2:**
1) The precise form of the confining potential \( U_0 \) is used in the proof of Theorem 4.1 only to prove (4.6). The proof is easily extended to the case when \( U_0 \) satisfies an other explicit behavior like \( U_0(x) \sim |x|^\alpha \) as \( |x| \to +\infty \) with \( \alpha > 0 \), \( \alpha \neq 2 \). Whatever \( U_0 \) behaves like, the estimate holds for the velocity:
\[ \hat{R}(t) = \sup \{|v| : \exists x \in \mathbb{R}^d \text{ s.t. } \langle x, v \rangle \in \text{supp}(f(t, \cdot, \cdot)) \} \]

also satisfies
\[ \hat{R}(t) \leq C(\varepsilon) \cdot (1 + t)^{\frac{1}{2} + \varepsilon} \]

for some constant \( C(\varepsilon) > 0 \) (and for any \( \varepsilon > 0 \)) as soon as \( U_0 \) is bounded from below and satisfies the other conditions of Theorem 2.1 (one has to add a term \( \ln(\ln(x + a)) \) in the estimate of \( \int_{\mathbb{R}^d} f(t, x, v) |v| dv \) which essentially does not change the result if for instance \( x \cdot \nabla U \leq 0 \) a.e. (dispersive case). Even
if $U_0$ is not bounded from below, the method may still apply but the estimate may be much worse. For example, if there exists some direction $\nu \in S^1$ such that $\nabla U_0(t\nu) \cdot \nu < 0$ for any $t > 0$ large enough, and $\lim_{t \to +\infty} U_0(t\nu) = -\infty$, the growth of the size of the support will surely be given by the linear motion in the potential $U_0$.

2) In a recent work, G. Rein [38] showed that in three dimensions, the growth (in the velocities) of the support is of order $(1+t)^{2/3}$ when $U_0 \equiv 0$, but the method is rather different. The bound obtained here is probably too large (since it corresponds — roughly spoken — to the growth given by the free motion, i.e. what one can get when $U_0 \equiv 0$, for the velocities), but as well as in dimension three, the question of the optimal growth is clearly open. One may conjecture that the optimal growth is at least logarithmic.

In dimension $N = 1$, the method also applies:

$$\| \frac{\partial U}{\partial t} \|_{L^2(\mathbb{R})} \leq \| j \|_{L^1(\mathbb{R})} \leq \| f \|_{L^1(\mathbb{R})} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t,x,v)|v|^2 \, dx \, dv \leq C(1 + R(t)),$$

$$(R(t))^3 - R(t) \leq C(1 + t\sqrt{1 + R(t)})$$

which gives for $R(t)$ an estimate of order $(1 + t)^{2/3}$ as $t \to +\infty$.

One can notice that the method we apply here fails in dimension $N = 3$.

3) One could ask if there were other interpolation inequalities that would improve the result. This does not seem very clear. Let us try to optimize the estimate of the $L^2$-norm of $j(t,x) = \int_{\mathbb{R}^3} f(t,x,v) \, dv$. In the following computations, $C$ is a bound for different constants depending only on the initial data.

$$\| j(t,.) \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq \| j(t,.) \|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{1 - w/2} \cdot \| j(t,.) \|_{L^w(\mathbb{R}^3;\mathbb{R}^3)}^{w/2}.$$

A direct computation of $\| j(t,.) \|_{L^w(\mathbb{R}^3;\mathbb{R}^3)}$ gives

$$\| j(t,.) \|_{L^w(\mathbb{R}^3;\mathbb{R}^3)} \leq \int_{|v| < R(t)} f(t,x,v) \, dv \leq C \cdot (R(t))^3.$$

Using lemma B.2, we can interpolate $\| j(t,.) \|_{L^w(\mathbb{R}^3;\mathbb{R}^3)}$ between $\| f(t,.) \|_{L^p(\mathbb{R}^3;\mathbb{R}^3)}$ and
with \( u = \frac{2(\eta - \frac{1}{2})^{+}}{\frac{1}{2}(\eta - 1)} \). Putting these two estimates together, we get

\[
\|j(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq C \|f(t, \cdot)|v|^4\|_{L^2(\mathbb{R}^d)} \leq C \frac{R(t)^{\frac{1}{2} + \frac{1}{\eta - 1} + 1}}{\ln(t + \alpha)},
\]

Optimizing with respect to \( p \) and \( k \), we again get

\[
\|j(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq C \cdot R(t).
\]

4) Assume that \( U_0 \equiv 0 \). Like in [35], it is possible to give an higher moment estimate for \( f \) (which depends on \( R(t) \) only through a term of order \( \frac{3}{2} \eta \) as \( \eta \to 0 \)) provided this moment is bounded for the initial data: for any \( \eta > 0 \), there exists a constant \( C(q) \) such that

\[
\int \int f(t, x, v)|v|^q \, dx \, dv \leq \int \int f_0(x, v)|v|^q \, dx \, dv + C(q) \cdot R(t)^{\frac{3}{2} + \frac{1}{\eta}} \cdot t \left( 1 + \ln(t + \alpha) \ln(\ln(t + \alpha)) \right)^{\frac{1}{2} + \frac{1}{\eta}},
\]

but such estimates are not useful for estimating the support because of their dependence in \( t \): reinjecting it in equation (4.5) does not improve the estimate on \( R(t) \).

Because of its general interest, let us indicate the idea of the proof of (4.7) and (4.8). In the following, \( C(q) \) denotes various constant which may depend on \( \eta \) and on the initial data \( f_0 \). We also assume that the solution \( f \) is smooth enough to allow the integrations by parts performed below, but the computation can easily be justified in the general setting.

Since

\[
|\nabla U(t, z)| \leq \frac{1}{2\pi} \int \int \frac{1}{|x - y|} f(t, y, v) \, dv,
\]
like in the proof of Theorem 4.1,

$$||\nabla U(t,.)||_{L^\infty(R^2)} \leq \frac{1}{2\pi} \left( 2\pi \langle \hat{R}(t) \rangle \right)^{\frac{n}{2}} \cdot ||f(t,.,v)||_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)}$$

$$= C(\eta) \cdot \left( \hat{R}(t) \right)^{\frac{n}{2}} \cdot ||f(t,.)||_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)}.$$  \hspace{1cm} (4.9)

The $L^{\frac{2n}{n-2}}$ norm of $f(t,.)$ is then evaluated by the interpolation inequality (see Appendix B)

$$||f(t,.)||_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)} \leq C(\eta) \cdot ||f(t,.,v)||^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)} \cdot ||f(t,.,v)||^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)}.$$  \hspace{1cm} (4.10)

with $\alpha = \frac{1}{2n}$ and $k = \frac{2n}{1-\eta}$. Putting (4.9) and (4.10) together, we get

$$||\nabla U(t,.)||_{L^\infty(R^2)} \leq C(\eta) \cdot ||f(t,.,v)||^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)} \cdot ||f(t,.,v)||^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}(R^2 \times R^2, dv)} \cdot \left( \hat{R}(t) \right)^{\frac{n}{2}}.$$  \hspace{1cm} (4.10)

Multiplying (formally) the Vlasov equation by $|v|^a$ and integrating by parts, we get

$$\int \int f(t,z,v) |v|^a \, dz \, dv \leq 3||\nabla U(t,.)||_{L^\infty(R^2)} \cdot \int \int f(t,z,v) |v|^a \, dz \, dv$$

which proves (4.7). (4.8) is obtained in the same way using an integration w.r.t. $t$. This method can also easily be generalized and gives for the moment of order $n$ the estimate

$$\int \int f(t,z,v) |v|^a \, dz \, dv \leq C(\eta) \cdot (1 + t)^{n-2+a}.$$  \hspace{1cm} (4.10)

It is also possible to give for any $\zeta > 0$ an estimate of the momentum

$$\int \int f(t,z,v) |v|^a \, dz \, dv$$

depending only on the initial data and on $t^n$ (no more dependance in $\hat{R}(t)$) using the Hardy-Littlewood-Sobolev inequality in the limit case $L^1(R^2)$.

5) The size of the support grows only because of the tail of the distribution (when $U_0$ is confining or say, at least, growing to $+\infty$ as $|x| \to +\infty$: let us consider here the case when $U_0$ is harmonic - for more details on confining potentials see [19]). For any large $A > 0$ we have indeed the following inequalities

$$\int_{|x| > A} \rho(t,x) \, dx = 0.$$
if $A > R(t)$ (and $t > 0$ is small enough), and for any $A > 0$

$$\int_{|x| > A} \rho(t,x) \, dx \leq \frac{1}{A^3} \int_{R^d} |x|^2 \rho(t,x) \, dx \leq \frac{C}{A^3 \min(R_0, 1)}$$

if $A < R(t)$, for some constant $C$ (the proof that $\int_{R^d} |x|^2 \rho(t,x) \, dx$ is bounded relies on the energy estimate).

5. Equipartition of the energy and moments

5.1 Stationary solutions

Assume first that $f$ is a smooth classical stationary solution of the Vlasov-Poisson system

$$\begin{cases}
    v \cdot \partial_x f - (\partial_x U(x) + \partial_x U_0(x)) \cdot \partial_v f = 0 \\
    - \Delta U = \rho(x) = \int_{R^d} f(x,v) \, dv
\end{cases} \quad (V)$$

Multiplying by $(x \cdot v)$, integrating with respect to $x$ and $v$ and performing integrations by parts, we easily get

$$\int \int_{R^d \times R^d} f(x,v) |v|^2 \, dx \, dv = \int_{R^d} \left( x \cdot (\nabla U + \nabla U_0) \right) \rho(x) \, dx .$$

Using now the Poisson equation to compute $\int_{R^d} \left( x \cdot \nabla U(x) \right) \rho(x) \, dx$, we successively get, with the convention of summation over repeated indices,

$$\int_{R^d} \left( x \cdot \nabla U(x) \right) \rho(x) \, dx = \int_{R^d} \left( x \cdot \nabla U(x) \right) \left( -\Delta U(x) \right) \, dx$$

$$= - \int_{R^d} \left( x_i \frac{\partial U}{\partial x^i} \frac{\partial U}{\partial x^j} \right) \, dx$$

$$= \int_{R^d} |\nabla U(x)|^2 \, dx + \int_{R^d} \sum_{i,j} x_i \frac{\partial U}{\partial x^i} \frac{\partial U}{\partial x^j} \, dx$$

$$= \int_{R^d} |\nabla U(x)|^2 \, dx + \int_{R^d} (x \cdot \nabla) \left( \frac{|\nabla U|^2}{2} \right) \, dx$$

$$= 0$$

provided $U$ is smooth enough and such that $|x| \nabla U(x)$ tends to $0$ as $|x| \to +\infty$ (this identity is known as Rellich's identity and is frequently used in Pohozaev's method for elliptic problems).
In dimension two, the assumption on the decay of $\nabla U$ is not satisfied. One has therefore to take asymptotic boundary terms into account. Assume that $x \mapsto (1 + |x|)\rho(x)$ is bounded in $L^1(\mathbb{R}^2)$.

$$\left| \nabla U (x) + \frac{x}{2\pi |x|^2} \int_{\mathbb{R}^2} \rho(y) \, dy \right| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \left( \frac{x}{|x|^2} - \frac{x-y}{|x-y|^2} \right) \rho(y) \, dy \right|$$

$$\leq \frac{1}{2\pi |x|} \int_{\mathbb{R}^2} \frac{|y|\rho(y)}{|x-y|} \, dy$$

since

$$\left| \frac{x}{|x|^2} - \frac{x-y}{|x-y|^2} \right| = \frac{|y|^2}{|x|^2 \cdot |x-y|^2}.$$ 

Then

$$\nabla U(x) = -\frac{x}{2\pi |x|^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y,v) \, dy \, dv \cdot (1 + o(1))$$

as $|z| \to +\infty$ if $f$ has a compact support. Integrating over a ball of radius $R$ and taking the large $R$ limit gives

$$\lim_{R \to +\infty} \int_{B(0,R)} \frac{df(x)}{2\pi |x|} \int_{\mathbb{R}^2} \frac{|y|\rho(y)}{|x-y|} \, dy = 0,$$

(5.1)

$$\int_{\mathbb{R}^2} \left( x \cdot \nabla U (x) \right) \rho(x) \, dx$$

$$= \lim_{R \to +\infty} \left( -\int_{B(0,R)} \frac{(x \cdot \nabla U(x))^2}{|x|} \, dx + \int_{B(0,R)} |\nabla U(x)|^2 \, dx \right)$$

$$= -\frac{1}{4\pi} \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, v) \, dx \, dv \right)^2$$

(5.2)

This proves that

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, v) |v|^2 \, dx \, dv = \int_{\mathbb{R}^2} (x \cdot \nabla U_0) \rho(x) \, dx - \frac{1}{4\pi} \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, v) \, dx \, dv \right)^2.$$ 

When $U_0$ is harmonic,

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, v) |v|^2 \, dx \, dv = 2 \int_{\mathbb{R}^2} U_0 \rho(x) \, dx - \frac{M^2}{4\pi},$$

with $M = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, v) \, dx \, dv$. These results are still true even if $\rho$ has a non compact support.

Proposition 5.1 : Let $U_0 \geq 0$ be such that $z \mapsto \frac{\nabla U_0}{|z|^2}$ belongs to $L^\infty(\mathbb{R}^2)$. Assume that $f$ is a $L^1 \cap L^\infty(\mathbb{R}^2)$ nonnegative stationary solution of the Vlasov-Poisson system such that

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, v) (|x|^2 + |v|^2) \, dx \, dv < +\infty$$
and \( \rho(x) = \int_{\mathbb{R}^n} f(x, v) \, dv \) satisfies 
\[ \nabla U(x) = \frac{1}{M} \frac{\nabla f(x, v)}{dv} \]
for some \( \epsilon > 0 \). Then

\[
\int_{\mathbb{R}^n} \left| \nabla U(x) \right|^2 \, dx = \int_{\mathbb{R}^n} \left( -\frac{\nabla f(x, v)}{dv} \right)^2 \, dv = -\frac{M^2}{4\pi}
\]

with \( \rho(x) = \int_{\mathbb{R}^n} f(x, v) \, dv \) and \( M = \int_{\mathbb{R}^n} f(x, v) \, dv \).

**Proof of Proposition 5.1:** It relies on Lebesgue’s theorem of dominated convergence and on Equation (5.1) which is still true even if \( f \) has a non compact support: assume that

\[
\left| \int_{\partial B(0, R)} \frac{\nabla f(x, v)}{|x|^2} \, dv \right| \leq \frac{1}{M} \left| \int_{\partial B(0, R)} \frac{\nabla U(x)}{|x|^2} \, dx \right|
\]

as \( R \to +\infty \). On one hand,

\[
\left| \int_{\partial B(0, R)} \frac{\nabla f(x, v)}{|x|^2} \, dv \right| \leq \frac{1}{M} \left| \int_{\partial B(0, R)} \frac{\nabla f(x, v)}{|x|^2} \, dv \right|
\]

This can be evaluated by

\[
\left| \int_{\mathbb{R}^n} f(x, v) \, dv \right| = \frac{M}{2\pi}
\]

and

\[
2\pi \left| \frac{x \cdot \nabla U}{|x|^2} \right| = \sup_{|x|=R} \left| \int_{\mathbb{R}^n} \frac{x \cdot (x-y)}{|x|^2} \rho(y) \, dy \right|
\]

using

\[
\int_{\mathbb{R}^n} \frac{|y|}{|x-y|} \rho(y) \, dy = \int_{|y| \leq 1} \frac{|y|}{|x-y|} \rho(y) \, dy + \int_{|y| > 1} \frac{|y|}{|x-y|} \rho(y) \, dy
\]

On the other hand,

\[
\left| \int_{\partial B(0, R)} \frac{\nabla f(x, v)}{|x|^2} \, dv \right| \leq \frac{1}{2} \left| \nabla f(x, v) \right| \leq \frac{M}{2\pi} \left| \frac{\nabla f(x, v)}{|x|^2} \right|
\]

\[
\left| \int_{\partial B(0, R)} \frac{\nabla f(x, v)}{|x|^2} \, dv \right| \leq \frac{M}{2\pi} \left| \frac{\nabla f(x, v)}{|x|^2} \right|
\]

or

\[
\left| \int_{\partial B(0, R)} \frac{\nabla f(x, v)}{|x|^2} \, dv \right| \leq \frac{M}{2\pi} \left| \frac{\nabla f(x, v)}{|x|^2} \right|
\]
Equation (5.3) has still to be proved. It relies on the

**Lemma 5.2**: Let \( g \in L^1(\mathbb{R}^2) \) be a nonnegative function such that

\[
\int_{\mathbb{R}^2} \frac{g(y)}{|y|} \, dy < +\infty.
\]

Then

\[
\lim_{R \to \infty} \int_{S^1} d\nu \int_{\mathbb{R}^2} \frac{g(y)}{|xR - y|} \, dy = 0.
\]

**Proof of Lemma 5.2**: Since

\[
\int_{S^1} 2\pi \int_{\mathbb{R}^2} \frac{g(y)}{|Rv - y|} \, dy = \int_{\mathbb{R}^2} \frac{g(y)}{|y|} \, dy
\]

for any \( v_0 \in S^1 \), provided

\[
g(y) = \int_{S^1} \frac{d\nu}{2\pi} g(v|y|),
\]

it is enough to prove the lemma when \( g \) is radially symmetric.

\[
V(r) = \int_{\mathbb{R}^2} \frac{g(y)}{2\pi|y - v_0|} \, dy
\]

is a solution of

\[
\begin{cases}
- \frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = \bar{g}, \\
\frac{dV}{dr}(0) = 0, \\
V(0) = \int_{\mathbb{R}^2} \frac{g(y)}{2\pi|y|} \, dy.
\end{cases}
\]

Since

\[
\frac{dV}{dr}(r) = -\frac{1}{r^2} \int_0^r s \bar{g}(s) \, ds,
\]

there exists a limit

\[
V(\infty) = \lim_{r \to \infty} V(r).
\]

Thus

\[
V(r) = V(0) - \int_0^1 \frac{1}{s^2} \int_0^s t^2 \bar{g}(t) \, dt \, ds
\]

\[
= V(\infty) + \frac{1}{r} \int_0^r \bar{g}(s) \, ds + \int_r^{+\infty} \bar{g}(s) \, ds,
\]

and

\[
\left| \frac{1}{r} \int_0^r \bar{g}(s) \, ds \right| \leq \int_0^r \bar{g}(s) \, ds \int_{B(0, r)} \frac{g(y)}{2\pi|y|} \, dy \to 0 \quad \text{as } r \to 0.\]

Taking \( r = 0 \) gives therefore \( V(\infty) = 0. \) \( \blacksquare \)
Remark 5.3:

(i) Lemma 5.2 is the analogous in dimension $N = 2$ of Newton's identity. Equation (5.4) can also be written as

$$
\int_{\mathbb{R}^2} \frac{\partial \eta(y)}{|x - y|} \, dy = \int_{\mathbb{R}^2} \frac{\partial \eta(y)}{\max(|x|, |y|)} \, dy \quad \forall x \in \mathbb{R}^2
$$

(provided $\eta$ is radially symmetric).

(ii) If the stationary solution is a strong solution that can be approximated by strong (eventually time dependant) solutions of the Vlasov-Poisson system with compact support, then the result also holds without the assumption that $x \mapsto |x| \cdot \rho(x) \in L^2_{\text{reg}}(\mathbb{R}^2)$.

5.2 The evolution problem

This result may be extended to the evolution problem. On one hand, for a smooth and sufficiently decaying at infinity solution,

$$
\frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{R}^N} f(t,x,v) (x \cdot v) \, dz \, dv - \int_{\mathbb{R}^2 \times \mathbb{R}^N} f(t,x,v) |v|^2 \, dz \, dv
$$

$$
+ \int_{\mathbb{R}^2} \left( x \cdot (\nabla U + \nabla U_0) \right) \rho(t,x) \, dx = 0,
$$

and on the other hand (apply Cauchy-Schwarz inequality to $x \sqrt{f}$ and $z \sqrt{f}$),

$$
\left( \int_{\mathbb{R}^2 \times \mathbb{R}^N} f(t,x,v) (x \cdot v) \, dz \, dv \right)^2 \leq \int_{\mathbb{R}^2 \times \mathbb{R}^N} f(t,x,v) |v|^2 \, dz \, dv
$$

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^N} f(t,x,v) |x|^2 \, dz \, dv,
$$

which is bounded since the kinetic energy and the external potential energy are bounded for any $t > 0$ (we assume here that $U_0$ is harmonic). This can be proved with the same method as in Section 1, Equation (1.3). The energy estimate gives (see Section 3 and Equations (2.3) and (2.5))

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^N} f(t,x,v) |v|^2 \, dz \, dv + \rho_0 \int_{\mathbb{R}^2} \rho(t,x) |x|^2 \, dx + \int_{\mathbb{R}^2} \rho(t,x) U(t,x) \, dx
$$

$$
\leq \int_{\mathbb{R}^2 \times \mathbb{R}^N} f_0(x,v) (|v|^2 + \rho_0 |x|^2 + U(t = 0,x)) \, dz \, dv.
$$

Then, as for Equation (1.3),

$$
\int_{\mathbb{R}^2} \rho(t,x) U(t,x) \, dx \geq - \frac{M^2}{2\pi} \ln k - \frac{2M}{\pi} \int_{\mathbb{R}^2} \rho(t,x) |x|^2 \, dx
$$
for some $k \geq \sqrt{2}$. Taking now $k = k(\rho_0) \geq \sqrt{2}$ such that
\[
\frac{2M \ln k + \rho_0}{\pi k^2} < \frac{\rho_0}{2},
\]
we get
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|v|^2 \, dx \, dv + \frac{\rho_0}{2} \int_{\mathbb{R}^2} \rho(t, x)|x|^2 \, dx \leq 2E_0 + \frac{M^2}{2\pi} \ln k(\rho_0)
\]
which gives a uniform (in $t$) bound on $\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)(|v|^2 + |x|^2) \, dx \, dv$.

This proves that
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v) (x \cdot v) \, dx \, dv - \int_0^t \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(s, x, v) |v|^2 \, dx \, dv \right) \, ds
\]
\[+ \int_{\mathbb{R}^2} \left( 2U_0 - \frac{M}{4\pi} \right) \rho(s, x) \, dx \right) \, ds
\]
\[= \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v) (x \cdot v) \, dx \, dv
\]
and then
\[
\lim_{t \to +\infty} \frac{1}{t} \int_0^t \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(s, x, v) |v|^2 \, dx \, dv - \int_{\mathbb{R}^2} 2U_0(x) \rho(s, x) \, dx \right) \, ds
\]
\[= -\frac{1}{4\pi} \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v) \, dx \, dv \right)^2.
\]

For a strong solution, i.e., a solution such that (see [36])
\[
3\varepsilon > 0 \quad \forall \, t > 0 \quad \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)(|v|^{2+\varepsilon} + |x|^{2+\varepsilon}) \, dx \, dv < +\infty,
\]
Equations (5.5) and (5.6) are still true. For example, solutions corresponding to initially compactly supported distribution functions are strong solutions (see Section 4). Property (5.7) is in fact true if it holds for the initial data, and it is also possible to prove the following lower order moment estimate (see [10] for a detailed study in dimension 3):

**Lemma 5.4**: Let $U_0 \in W_{loc}^{3,\infty}(\mathbb{R}^2)$, $U_0 \geq 0$. Assume that $f$ is a weak solution of the Vlasov-Poisson system corresponding to a nonnegative initial data $f_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

(i) If $U_0$ is such that $x \mapsto \frac{\nabla U_0}{1 + U_0}$ belongs to $L^\infty(\mathbb{R}^2)$ and if for some $\varepsilon > 0$,
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v)(|v|^{2+\varepsilon} + |x|^{2+\varepsilon}) \, dx \, dv < +\infty,
\]
then for any $t > 0$,
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)(|v|^{2+\varepsilon} + |x|^{2+\varepsilon}) \, dx \, dv < +\infty.
\]
(ii) If $U_0$ is such that $z \mapsto \frac{\nabla U_0}{1 + |z|}$ belongs to $L^m(\mathbb{R}^2)$ and if for some $m \in [1, 2]$,
\[ \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v)(|v|^2 + |x|^m) \, dx \, dv < +\infty, \]
then for any $t > 0$,
\[ \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|x|^m \, dx \, dv < +\infty \]
provided $t \mapsto \int \int f(t, x, v)|v|^2 \, dx \, dv$ is bounded.

(iii) If $z \mapsto \frac{\nabla U_0}{1 + |z|}$ belongs to $L^m(\mathbb{R}^2)$ and if
\[ \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x, v)(|v|^2 + |x|^3) \, dx \, dv < +\infty, \]
then for any $t > 0$,
\[ \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)(|v|^2 + |x|^3) \, dx \, dv < +\infty. \]

Proof of Lemma 5.4: Let
\begin{align*}
I(t) &= \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|x|^{2+\epsilon} \, dx \, dv, \\
J(t) &= \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|v|^{2+\epsilon} \, dx \, dv, \\
K(t) &= \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|v|^{3} \, dx \, dv, \\
\rho_0 &= \| \frac{\nabla U_0}{1 + |z|} \|_{L^m(\mathbb{R}^2)}. \end{align*}

We will prove the estimates only for smooth solutions. Using truncations (like $|x|^{2+\epsilon}$ for $|x| < R$, $R^{2+\epsilon}$ for $|x| \geq R$ instead of $|x|^{2+\epsilon}$), it will then be easy to check that they still hold for weak solutions. As usual, if we multiply the Vlasov equation respectively by $|x|^{2+\epsilon}$ and $|v|^{2+\epsilon}$, we get
\begin{align}
\frac{dI}{dt} &= (2 + c) \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|x|^{1}(x \cdot v) \, dx \, dv \\
&\leq (2 + c) \left( \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|x|^{2+\epsilon} \, dx \, dv \right)^{\frac{1+\epsilon}{1+\epsilon}} \left( \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(t, x, v)|v|^{2+\epsilon} \, dx \, dv \right)^{\frac{\epsilon}{2+\epsilon}} \\
&= (2 + c) \left( I(t) \right)^{\frac{1+\epsilon}{1+\epsilon}} \left( J(t) \right)^{\frac{\epsilon}{2+\epsilon}}, \tag{5.8}
\end{align}
\[
\frac{dJ}{dt} = -(2+\epsilon) \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) [\nabla U_0(x) + \nabla U(t, x)] \cdot v \, dv \, dx \right] \\
\leq (2+\epsilon) \left[ \rho_0 \left( \| f \|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} + \| f(t) \|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \right) \right]^{\frac{\theta}{1+\theta}} \times \\
\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \, dv \, dx \\
+ \int_{\mathbb{R}^d} \| \nabla U(t, x) \| \cdot \left( \int_{\mathbb{R}^d} f(t, x, v) |v|^{1+\theta} \, dv \right) \, dx.
\] (5.9)

\textbf{Case (i)}

\[
\frac{dJ}{dt} = -(2+\epsilon) \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) |v|^\gamma [\nabla U_0(x) + \nabla U(t, x)] \cdot v \, dv \, dx \right] \\
\leq (2+\epsilon) \left[ \rho_0 \left( \| f \|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} + \| f(t) \|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \right) \right]^{\frac{\theta}{1+\theta}} \times \\
\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \, dv \, dx \\
+ \| f(t, \cdot) \|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \| \nabla U(t, \cdot) \|_{L^\infty(\mathbb{R}^d)} \left( K(t) \right)^{\frac{\theta}{1+\theta}}.
\]

Then

\[\| \nabla U(t, \cdot) \|_{L^\infty(\mathbb{R}^d)} \leq \frac{M}{2\pi} + C(\eta) \| \rho(t, \cdot) \|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^d)},\]

with

\[C(\eta) = \left( 2\pi \cdot \frac{2+\epsilon}{\eta} \right)^{\frac{\theta}{1+\theta}},\]

and (interpolation inequality – see Appendix B)

\[\| \rho(t, \cdot) \|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^d)} \leq C \cdot \| f(t, \cdot) \|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}^{\theta} \cdot \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot |v|^\gamma \, dv \, dx \right)^{\frac{\theta}{1+\theta}},\]

for some \( C > 0 \) (not depending on \( \eta > 0 \), small). Hölder’s inequality gives

\[\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot |v|^{2+\epsilon} \, dv \, dx \leq \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot |v|^\gamma \, dv \, dx \right)^\theta \cdot \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, v) \cdot |v|^{2+\epsilon} \, dv \, dx \right)^{1-\theta},\]

with \( \theta = 1 - \frac{\gamma}{\gamma} \cdot \epsilon \in [0, \epsilon]. \) Plugging these estimates into (5.8) and (5.9) and using the bounds on \( K(t) \) obtained in Section 1, one can prove that \( I(t) \) and \( J(t) \) have an at most exponential growth in \( t \).

\textbf{Case (ii)} : The same computation holds with \( m = 2 + \epsilon \in [1, 2] \), except that we have to estimate (5.9) using directly an interpolation inequality:

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla U(t, x)| \cdot \left( \int_{\mathbb{R}^d} f(t, x, v)^{1/k} \cdot (f(t, x, v))^{1-1/k} |v|^{m-1} \, dv \right) \, dx \\
\leq \| \nabla U(t, \cdot) \|_{L^1(\mathbb{R}^d)} \cdot \| \rho(t, \cdot) \|_{L^1(\mathbb{R}^d)} \cdot \| f(t, \cdot) \|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^d \times \mathbb{R}^d)} \cdot \| f(t, \cdot) \|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^d \times \mathbb{R}^d)}.
\]
provided \( \frac{1}{\rho} + \frac{1}{\theta} + \frac{1}{1} = 1 \). Thus (see Appendix B),
\[
\|\rho(t, \cdot)\|_{L^1(R^d)} + \|f(t, \cdot, \cdot)\|_{(m-1)^2/(m+1)^2 L^2(R^d \times R^d)} \\
\leq C \cdot \|f_0\|_{L^1(R^d)} + \|f(t, \cdot, \cdot)\|_{L^1(R^d \times R^d)}^{1/2} \\
\]
for some constant \( C > 0 \), provided
\[
\frac{q}{k} = 2 \quad \text{and} \quad \frac{1}{r} = \frac{m}{4} - \frac{1}{2k} \in [0, 1] 
\]
(take for instance \( k = \frac{2}{3 - m} + 1 \)). Since \( \rho(t, \cdot) \in L^1 \cap L^2(R^d) \), \( \nabla \rho \in L^p(R^d) \) for any \( p \in [2, +\infty] \) with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{1} = 1 \) if and only if \( m < 3 \) (which is of course true if \( m \in [1, 2] \)). The conclusion then holds as before:
\[
\frac{dI}{dt}(t) \leq \int f_0(x, v)(|v|^2 + U(t = 0, z) + 2U_0(z)) \, dx 
\]
for some constant \( C > 0 \) which depends only on \( f_0 \).

Case (iii) : \( m = 2, \, \varepsilon = 0 \). Estimate (5.9) is now replaced by the usual energy estimate:
\[
J(t) \leq \int \int f_0(x, v)(|v|^2 + U(t = 0, z) + 2U_0(z)) \, dx 
\]
and like in Section 1 (Remark 1.2, (ii)), we get
\[
J(t) \leq C(1 + I(t)) 
\]
for some constant \( C > 0 \) which depends only on \( f_0 \).

As a consequence, we have for strong solutions the following proposition, which generalizes one of the properties obtained by J. Batt in [5].

**Proposition 5.5** : Let \( U_0 \in W_0^{2,\infty}(R^d) \), \( U_0 \geq 0 \). Assume that \( f \) is a nonnegative strong solution of the Vlasov-Poisson system. Then
\[
\frac{d}{dt} \int f(t, x, v) \, (x \cdot v) \, dx = -\int \int f(t, x, v) \, (x \cdot v) \, dx 
\]
and like in Section 1 (Remark 1.2, (ii)), we get
\[
J(t) \leq C(1 + I(t)) 
\]
Remark 5.6: Other moment estimates are easily obtained (see also [34], [36], [10]): for example, if $U_0$ is harmonic, then

$$\int_0^{10^6} dt \int_{\mathbb{R}^2} \left( \frac{|x|^2|v|^2}{|x|^2} - \frac{x}{|x|} \cdot (\nabla U(t, x) + \nabla U_0(x)) \right) f(t, x, v) \, dx \, dv < +\infty,$$

which is easily proved by multiplying the Vlasov equation by $\frac{x}{|x|}$ and integrating with respect to $t$, $x$ and $v$.

Part II:

2d time-periodic solutions

1. Introduction: some classifications results and a model for time-periodic solutions

Since the problem for stationary solutions is easier than for time-periodic solutions, we present first some classification results for the solutions of the stationary Vlasov-Poisson system. We extend then the ideas developed for these solutions to the time-periodic case. A detailed version of these result is given in Appendix C.

Section 2 of this part is devoted to the study of a subclass of these time-periodic solutions.

1.1. A classification result for stationary solutions

We first explicit the special class of solutions satisfying the weak Ehlers & Rienstra ansatz (see [23], [6]), and then give a factorization result which proves that these solutions are in fact the generic ones that have a radial spatial density.

It is easy to realize that any distribution function depending on $x$ and $v$ only through the quantities

$$E(x, v) = \frac{1}{2}|v|^2 + U(x) + U_0(x) \quad \text{and} \quad F(x, v) = x \wedge v$$
(we shall say that \( f \) satisfies the weak Ehlers & Rienstra ansatz) i.e. such that

\[
f(x, v) = h(E(x, v), F(x, v)) \quad \text{(weak ER)}
\]

for some function \( h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ \), is a solution of the stationary Vlasov equation,

\[
v \cdot \partial_x f - (\partial_x U(x) + \partial_x U_0(x)) \cdot \partial_v f = 0 \quad \text{(sV)}
\]

provided \( U \) is radially symmetric. The problem is then reduced to the Poisson equation

\[
-\Delta U = H(U, x), \quad \text{(P)}
\]

where \( H(U, x) = \int_{0}^{+\infty} d\alpha_1 \int_{-\infty}^{+\infty} d\alpha_2 \ h(\sqrt{\alpha_1^2 + \alpha_2^2} + U, |x|) \). It is then not difficult to give existence results for such a class of solutions. An interesting point is the fact that these solutions are the most general one can construct (up to some regularity assumptions and a technical non-resonance criterion) that have a radial spatial density.

To be more precise, assume that \((f, U)\) does not depend on \( t \) and that \( U \) is radially symmetric, i.e. that there exists a function \( u : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
U(x) = u(|x|) \quad \forall x \in \mathbb{R}^2. \quad \text{(S1)}
\]

The factorization result, known as Jeans' theorem (see [8] and [14, 15] for various applications to kinetic equations in plasma physics), gives the form of the generic smooth solution.

The weak form of the result says that the averaged distribution function \( \overline{f} \) defined by \( \overline{f}(x, v) = \int_{E \in S} f(|x| \cdot v, v) \ dv \) for all \((x, v) \in \mathbb{R}^2 \times \mathbb{R}^2 \) satisfies (locally w.r.t \( x \) and \( v \)) the weak Ehlers & Rienstra ansatz: for any \((x_0, v_0) \in \text{supp}(f)\), there exist a neighbourhood \( V \) and a function \( h : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) such that

\[
\overline{f}(x, v) = h(E(x, v), F(x, v)) \quad \forall (x, v) \in V. \quad \text{(weak ER)}
\]

If moreover a non resonance condition is satisfied (see Appendix C for a precise statement), and provided \( f \) is continuous, then the factorization result holds for \( f \):

\[
f(x, v) = \overline{f}(x, v) \quad \forall (x, v) \in V.
\]

A sufficient condition for these two results to be global is \( r \mapsto \overline{r} \left( \frac{\partial}{\partial r} + \frac{\partial}{\partial \overline{r}} \right) \) is monotone increasing : in this case, one may take \( V = \mathbb{R}^2 \times \mathbb{R}^2 \).
1.2. Time-periodic solutions

For time-periodic solutions, we may proceed exactly in the same way. Consider now

\[ E(t, x, v) = \frac{1}{2} |v|^2 + U(t, x) + U_0(x) \quad \text{and} \quad F(x, v) = x \wedge v. \]

The same kind of factorization result as for stationary solutions (under regularity and non-resonance assumptions) shows that a time-periodic solution of period \( T \) such that the average over one period of the self-consistent potential is radially symmetric satisfies a factorization property, which is global w.r.t. \( t \) and local in the phase space \( \mathbb{R}_+^2 \times \mathbb{R}_+^2 \). If it is global, then the factorization result may be written as: there exists a function \( g : \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}_+ \) such that for almost all \((t, x, v) \in [0, T] \times \text{supp}(f),\)

\[ f(t, x, v) = g(t, E(t, x, v), F(x, v)). \]

A detailed version of this result is given in Appendix C. In the following, we will make one more assumption and assume that \( g \) does not depend on \( t \). The class of solutions we shall consider is therefore defined by: there exist constant \( T > 0 \) and a function \( g : \mathbb{R}_+^2 \to \mathbb{R}_+ \) such that for almost all \((t, x, v) \in \mathbb{R} \times \mathbb{R}_+^2 \times \mathbb{R}_+^2,\)

\[ f(t, x, v) = g(E(t, x, v), F(x, v)) \quad \text{and} \quad f(t + T, x, v) = f(t, x, v). \] (weak ER)

In the next section we will prove that under a technical but generic assumption, the solution satisfies the (strong) Ehlers & Rienstra ansatz

\[ f(t, x, v) = g(E(t, x, v) - \omega F(x, v)) \] (ER)

for some function \( g \) and for \( \omega = \frac{v}{v} \). In that case, exactly as in the paper by J. Batt, H. Berestycki, P. Degond & B. Perthame [6] (devoted to the study of the 3d-solutions of the gravitational Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz) the Vlasov equation is reduced to

\[ \partial_t U(t, x) - \omega(Ax \cdot \nabla U(t, x)) = 0, \]

where the linear operator \( A \) is such that \( v \cdot Ax = x \wedge v \) for any \((x, v) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2;\)

\[ A(x_1, x_2) = -(x_2, x_1) \] in a cartesian system of coordinates.
$f$ and $U$ are in a solid motion of rotation around the $z$-axis with a constant angular velocity $\omega$ and take the following form

$$f(t,x,v) = g\left(\frac{|v - \omega A x|^2}{2} + U_0(x) + \omega\left(\frac{x^t A x}{2}\right)\right)$$

and

$$U(t,x) + U_0(x) - \frac{\omega^2}{2} |x|^2 = \omega\left(\frac{x^t A x}{2}\right),$$

and the problem is reduced to the nonlinear Poisson equation for $w$

$$-\Delta w + 2(\rho_0 - \omega^2) = G(w)$$

with $G(w) = \pi \int_{-\infty}^{\infty} g(s) \, ds$ (we assume here that $U_0$ is an harmonic potential).

We may first mention that such a formulation provides a very simple way for constructing time-periodic solutions to the Vlasov-Poisson system: since the equation for $w$ does not depend on $x$, for any solution $w$, $w$, defined by $x \rightarrow w(x) = w(x + \tau)$ is also a solution for any $\tau \in \mathbb{R}^2$, which clearly does not have the same symmetry properties as $w$ (see Remark 3.2). Thus the potential $U$ and the distribution function $f$ are time-periodic solutions which are not radially symmetric and depend therefore explicitly of $t$. We will also consider the case where the confining potential $U_0$ is not radially symmetric, and exhibit a branch of solutions that have a logarithmic growth, starting from the solutions that are radially symmetric up to a translation. These solutions are time-periodic (and generically explicitly time-dependent (i.e. non stationary) solutions. Adequate conditions on $G$ ensure that they have a finite mass.

But this part of the paper will be mainly devoted to the class of non-isotropic solutions with quadratic growth, i.e. the solutions such that

$$w \sim \delta(\theta x_1^2 + (1 - \theta)x_2^2) \text{ as } |x| \to +\infty$$

for some $\theta \in [0,1]$, $(x_1, x_2)$ being a system of cartesian coordinates of $x \in \mathbb{R}^2$. In [6], the solutions that were considered were $3d$-solutions of the gravitational Vlasov-Poisson system corresponding in the $2d$-case to solutions such that $\theta = 0$ or $1$ ($4d$-solutions), or such that $\theta = \frac{1}{2}$ (radially symmetric solutions). We will adapt their results to the $2d$-electrostatic Vlasov-Poisson system with a confining potential (Section 3, Proposition 3.4: existence of $1d$-solutions, and Proposition 3.5: existence of radially symmetric solutions), but also study the general case: $\theta \in (0,1)$, $\theta \neq \frac{1}{2}$. 
The spatial density \((t,x) \mapsto \int_{-\infty}^{\infty} f(t,x,v) \, dv\) is, up to a rotation of angle \(\omega t\) given by
\[
G \left( u(x) + \frac{s}{2} |x|^2 \right) = G \left( u(x) + \delta(\theta x_1^2 + (1-\theta)x_2^2) \right) \sim G \left( \delta(\theta x_1^2 + (1-\theta)x_2^2) \right) \quad \text{as} \quad |x| \to +\infty .
\]
It belongs to \(L^1(\mathbb{R}^2)\) if \(s \mapsto G(s)\) is sufficiently decreasing for \(s \to +\infty\). Note that the asymptotic behaviour of \(u\) for such solutions is
\[
u(x) = \delta(\theta - 1/2)(x_1^2 - x_2^2) + o(|x|^2) \quad \text{as} \quad |x| \to +\infty ,
\]
for any \(\theta\) belonging to \([0,1]\), which corresponds to a non standard asymptotic behaviour (see Theorem 3.3: asymptotic behaviour, necessary conditions for the existence of time-periodic solutions and consequences).

For this study, our main mathematical reference is a paper by J. Batt, H. Berestycki, P. Degond & B. Perthame [6] (three-dimensional Vlasov-Poisson system in the gravitational case). Compared to it the main results of this paper are the following:
- the class of solutions is larger \((\theta \in [0,1] \text{ instead of } \theta = 0, \frac{1}{2} \text{ or } 1)\) and the symmetry assumptions are weaker ("weak" Ehlers and Rienstra ansatz instead of "strong" Ehlers and Rienstra ansatz) - roughly spoken, we prove that it corresponds to the class of the solutions that are in a solid motion of rotation with a constant angular velocity and such that the self-consistent potential has an at most quadratic growth,
- since the confinement of the particles is due to an external potential and not to the self-consistent potential (the force between the particles is repulsive), it is possible to perturb it and build a branch of solutions starting from the radially symmetric solutions (up to a translation),
- the asymptotic boundary conditions are systematically explored; choosing a quadratic growth for the self-consistent potential (like in BBDP) is not absurd (one has to keep in mind that the model corresponds to the study of a beam locally near its axis, as shown in Appendix A).

The study of the stationary solutions has been neglected. Most of the results for time-periodic solutions are easily extended to the stationary case by simply taking \(\omega = 0\). One has also to refer to J. Batt, W. Faltenbacher, & E. Horst [8] for this point. An attempt of a general classification of the time periodic solutions by the mean of Jeans' theorem is presented in Appendix C.
Ideas, which are very popular among astrophysicists (see [14,15] for a review) have been introduced from a mathematical point of view in [8], from which some notations are taken.

2. The Ehler-Rienstra ansatz for time-dependant solutions

In the rest of the paper, we will consider the special class of solutions of the Vlasov-Poisson system that are such that

\[ f(t,x,v) = g(E(t,x,v) - \omega F(x,v)) \]  

(this ansatz will be referred as the Ehlers & Rienstra ansatz) for some \( \omega \in \mathbb{R} \), and where \( E \) and \( F \) are defined by

\[ E(t,x,v) = \frac{1}{2} |v|^2 + U(t,x) + U_0(x), \]
\[ F(x,v) = x \wedge v. \]

Such solutions are also called in the physical literature "locally isotropic solutions" (see [23]). Before studying these solutions in details, we will notice that this class of solutions corresponds to a priori larger class of solutions, the class of solutions satisfying only the weak Ehlers & Rienstra ansatz i.e. such that

\[ f(t,x,v) = h(E(t,x,v), F(x,v)) \]  

(weak ER)

provided they are explicitly time-dependant. This result will be important in view of an attempt of classification of all the time-periodic solutions of the Vlasov-Poisson system given in Appendix C.

**Theorem 2.1 :** Assume that \((f,U)\) is a solution of the Vlasov-Poisson system satisfying the weak Ehlers & Rienstra ansatz with \( h \) a nonnegative function of class \( C^3 \) defined on \( \mathbb{R}^2 \) such that \( \{(E,F) \in \mathbb{R}^2 \ : \ \frac{\partial h}{\partial F}(E,F) = 0 \} \) is a finite union of \( 1 \)-d \( C^3 \) manifolds, and such that \( z \rightarrow \rho(t,z) = \int_{\mathbb{R}^2} f(t,z,v) \ dv \) is a nonnegative continuous function with mean \( \left( \int_{\mathbb{R}^2} f(t,z,v) \ dv \right) = 0 \) \((U_0 \) is an harmonic potential such that \( U_0(x) = \frac{\omega^2}{2} |x|^2 \), where \( \rho_0 \) is positive real constant). Assume that \( \partial_t U \neq 0 \). Then there exists an \( \omega \in \mathbb{R} \) such that \( f \) is time-periodic of period \( \frac{2\pi}{\omega} \), \( f \) satisfies the Ehlers & Rienstra ansatz, and there exists a \( C^3 \) function \( g \) such that

\[ h(E,F) = g(E - \omega F). \]
Moreover \( f \) and \( U \) may be written in the following form
\[
f(t,x,v) = g \left( \frac{|v - \omega A x|^2}{2} + U_0(x) + w \left( e^{i\tau A x} - \frac{\omega^2}{2} |x|^2 \right) \right),
\]
(2.3)
and
\[
U(t,x) = w \left( e^{i\tau A x} \right),
\]
(2.4)
where \( w \) is a solution of the nonlinear Poisson equation
\[
-\Delta w + 2(\rho_0 - \omega^2) = \rho,
\]
(2.5)
and \( A \) is the linear operator such that \( v \cdot A x = x \wedge v \) for any \((x,v) \in \mathbb{R}^2 \times \mathbb{R}^2\).

**Proof**:

1) A simple computation shows that
\[
\left( \partial_t + v \cdot \partial_x - (\partial_x U(t,x) + \partial_v U_0(x)) \cdot \partial_v \right) E(t,x,v) = \frac{\partial U}{\partial t}(t,x).
\]
2) Let \((x_1,x_2)\) be cartesian coordinates of \( x \in \mathbb{R}^2 \), so that \( x \wedge v = x_1 v_2 - x_2 v_1 \), and denote by \( A \) the linear operator such that \( A x \) is represented by \((-x_2,x_1)\). Let us define \( F \) by:
\[
F(x,v) = x \wedge v = (v \cdot A x).
\]
Then, with these notations
\[
\left( \partial_t + v \cdot \partial_x - (\partial_x U(t,x) + \partial_v U_0(x)) \cdot \partial_v \right) F(x,v) = -(A x \cdot \nabla_x U(t,x)).
\]
3) If \( f \) satisfies the weak Ehlers & Rienstra ansatz, it is therefore a solution of the Vlasov equation if and only if
\[
\frac{\partial h}{\partial E} \left( E(t,x,v), F(x,v) \right) \cdot \partial_t U(t,x) + \frac{\partial h}{\partial F} \left( E(t,x,v), F(x,v) \right) \cdot (A x \cdot \nabla_x U(t,x)) = 0.
\]
(2.6)
If \( \omega(E,F) = \left( \frac{\partial h}{\partial E}, \frac{\partial h}{\partial F} \right)(E,F) \), then
\[
\partial_t U(t,x) - \omega(E,F)(A x \cdot \nabla_x U(t,x)) = 0.
\]
Since \( U \) is a solution of the Poisson equation and does not depend on \( v \),
\[
(z,v) \mapsto \omega(E(t,x,v), F(x,v))
\]
is locally a constant on \( \mathbb{R}^2 \times \mathbb{R}^2 \), with \( \Sigma = \{(z,v) : \omega(E(t,x,v), F(x,v)) = 0\} \), and \( \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Sigma \) is connected. Let us prove it.
4) According to the assumptions on \( h \), one may assume either that there locally exists a \( C^1 \) function \( E \mapsto F(E) \) such that

\[
\frac{\partial h}{\partial E}(E, F(E)) = 0, \tag{Case 1}
\]

or that there locally exists a \( C^1 \) function \( F \mapsto E(F) \) such that

\[
\frac{\partial h}{\partial E}(E(F), F) = 0, \tag{Case 2}
\]

on \( \{(E, F) \in \mathbb{R}^2 : \frac{\partial h}{\partial E}(E, F) = 0\} \). Let us look at the first case. According to the expressions of \( E \) and \( F \) given by equations (2.1) and (2.2), either

\[ x \cdot v = 0, \]

or one can (locally) find a function \( E \mapsto v(E) = (v_1(E), v_2(E)) \) (with values in \( \mathbb{R}^2 \)) such that

\[
\begin{align*}
v_1 \frac{dv_1}{dE} + v_2 \frac{dv_2}{dE} &= 1, \\
v_2 \frac{dv_1}{dE} - v_1 \frac{dv_2}{dE} &= \frac{dF}{dE},
\end{align*}
\]

and \( \Sigma \) is therefore tangent to

\[ \text{Vect}(\frac{dv_1}{dE} \times (\nabla U + \nabla U_0)^\perp), \]

provided \( \nabla U + \nabla U_0 \neq 0 \), or to

\[ \text{Vect}(\frac{dv}{dE}) \times \text{Vect}(\tau) \]

if \( \nabla U + \nabla U_0 = 0 \). Here \( \tau \) is the unit tangent vector to the set \( \{x \in \mathbb{R}^2 : \nabla U(x) + \nabla U_0(x) = 0\} \). It is well defined since it is a unit vector belonging to \( \text{Ker}(D^2(U + U_0)) \) if \( D^2(U + U_0) \neq 0 \), or is defined by continuity if \( D^2(U + U_0) = 0 \).

If \( D^2(U + U_0) \equiv 0 \) on a neighbourhood \( \mathcal{V} \), then

\[
\frac{\partial^2 U}{\partial x_1^2} + \rho_0 = 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial x_2^2} + \rho_0 = 0 \quad \text{on} \; \mathcal{V},
\]

which would imply that \( \rho \equiv \rho_0 \) on \( \mathcal{V} \), in contradiction with the assumptions on \( \rho \).

In case 2, the proof is exactly the same except that one has to exchange the roles of \( E(t, x, v) \) and \( F(x, v) \), \( E(F) \) and \( F(E) \).

\( \Sigma \) is therefore contained in a finite union of manifolds of dimension 2: \( (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Sigma \) is connected.
5) For all \( t \in \mathbb{R} (x,v) \mapsto \omega(E(t,x,v), F(x,v)) \) is a constant. Obviously, \( \omega \) does not depend either on \( t \): \( \omega = \omega(E,F) \) is therefore almost everywhere w.r.t. \((E,F)\) equal to a constant, which we still denote by \( \omega \), when \( E,F \) belong to the set

\[
\mathcal{X} = \{(E,F) : E = E(t,x,v), \quad F = F(x,v), \quad (t,x,v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3\}.
\]

On \( \mathcal{X} \),

\[
\frac{\partial h}{\partial F} + \omega \frac{\partial h}{\partial E} = 0,
\]

so that there exists a function \( g : \mathbb{R} \mapsto \mathbb{R} \) such that

\[
h(E,F) = g(E - \omega F).
\]

6) Equation (2.6) now reads

\[
\partial_t U - (\omega A \cdot \nabla) U = 0,
\]

which is solved by

\[
U(t,x) = w(e^{\omega t} x),
\]

where \( w \) is a solution of the Poisson equation

\[
-\Delta w + 2(\rho_0 - \omega^2) = \rho ,
\]

and replacing \( E \) and \( F \) by their values expressed in terms of \( \omega \) and \( w \):

\[
E(t,x,v) = \frac{|v|^2}{2} + U_0(x) + w(e^{\omega t} x),
\]

\[
F(x,v) = (Ax \cdot v).
\]

gives for \( f \) the expression

\[
f(t,x,v) = g\left( \frac{|v - \omega A t|^2}{2} + U_0(x) + w\left(e^{\omega t} x\right) - \frac{\omega^2}{2} |x|^2 \right).
\]

\[
\square
\]

3. The nonlinear Poisson equation and time-periodic solutions

This section contains the main results of the paper. Theorem 3.3 (Asymptotic behaviour, necessary conditions for the existence of time-periodic solutions and consequences) and Theorem 3.7 (Existence of time-periodic anisotropic solutions with finite mass) are completely new results. They present a
priori considerations on the asymptotic behaviour of the solutions, and existence results for a new class of solutions, which includes the solutions given by J. Batt, H. Berestycki, P. Degond & B. Perthame in [6]. These solutions have the property that they are explicitly time-dependant and may have a finite $U^1$-norm (mass), which was not the case for the solutions given in [6].

The other results of this section (Proposition 3.1: equivalence with a nonlinear Poisson equation, Proposition 3.4: existence of radially symmetric solutions and Proposition 3.5: existence of 1d-solutions) are more or less an adaptation of some of the results given in [6] (for 3d-solutions of the gravitational Vlasov-Poisson system) to the 2d-solutions of the Vlasov-Poisson system with a confining potential.

First of all, if $f$ satisfies the Ehlers & Rienstra ansatz, we can give a complete characterization of the time-dependant solutions of the Vlasov-Poisson system in terms of an equivalent nonlinear Poisson equation.

**Proposition 3.1:** (Equivalence with a nonlinear Poisson equation) Assume that $g$ belongs to $L^1 \cap W^{1,\infty}(\mathbb{R})$, that $g$ is nonnegative and not identically equal to 0. $(f, U)$ is a solution in $C^0(\mathbb{R}; L^1(\mathbb{R}^2_{\text{loc}} \times \mathbb{R}^2_{\text{loc}})) \times C^1(\mathbb{R}; W^{2,1}_{\text{loc}}(\mathbb{R}^2_{\text{loc}}))$ of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz $(f, U$ are weak solutions respectively in the sense of the characteristics as defined by R. DiPerna & P.-L. Lions in [11], and in the sense of the distributions) if and only if there exists a solution $w \in W^{2,1}_{\text{loc}}(\mathbb{R}^2)$ of

$$-\Delta w + 2(\rho_0 - \rho^w) = G(w), \quad (NLP)$$

with $G(w) = \pi \int_{0}^{+\infty} g(s) \, ds$. The relation between $(f, U)$ and $w$ is given by Equations (2.3) and (2.4). $x \mapsto \rho(t, x)$ is locally Lipshitz, and $U$ belongs to $C^1(\mathbb{R} \times \mathbb{R}^2)$.

**Proof:**

1) Assume first that $(f, U)$ is a solution of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz. According to this ansatz

$$g'(E - \omega F) \cdot \left( \partial_t U - \omega A x \cdot \nabla U \right) = 0,$$

with

$$(E - \omega F)(t, x, v) = \frac{|v - \omega A x|^2}{2} + U_0(x) + U(t, x) - \frac{|x|^2}{2}.$$

Assume that $t$ and $x$ are fixed, such that $x$ belongs to the support of $\rho(t, \cdot)$, i.e. the support of

$$x \mapsto \rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v) \, dv.$$
Because of the assumptions on \( g \),

\[
\{ v \in \mathbb{R}^2 : g'(E - \omega F)(t, x, v) \neq 0 \}
\]

has a strictly positive measure:

\[
\frac{d}{dt} U(t, e^{-ut} \mathbb{A} x) = 0,
\]

so that \( w(x) = U(t, e^{-ut} \mathbb{A} x) + U_0(x) - \frac{e^2}{2} |x|^2 \) does not depend on \( t \) as long as \( x \) belongs to the support of \( \rho(t, \cdot) \). Since

\[
\rho(t, z) = \int_{\mathbb{R}^2} \rho \left( (E - \omega F)(t, x, v) \right) dv = \pi \int_{-\infty}^{\infty} \rho \left( s + U(t, x) + U_0(x) - \frac{|\omega|^2}{2} \right) ds ,
\]

we get

\[
\rho(t, e^{-ut} \mathbb{A} x) = \pi \int_{0}^{\infty} \rho \left( s + w(x) \right) ds = G(w(x)) .
\]

\( w \) is therefore a solution of

\[-\Delta w + 2(\rho_0 - \omega^2) = G(w) .\]

Let us define \( w \) on \( \mathbb{R}^2 \setminus \text{supp}(\rho(t, \cdot)) \) by

\[-\Delta w + 2(\rho_0 - \omega^2) = -\Delta U|_{(t, e^{-ut} \mathbb{A} x)} = 0 .\]

\( w \) does not depend on \( t \) for any \( x \in \mathbb{R}^2 \), and the first part of the theorem holds.

The other side of the proof is obvious. \( \square \)

Remark 3.2: Since the (NLP) equation does not depend on \( x \), it immediately follows that for any solution \( w \), \( w_\tau \), defined by

\[
w_\tau(x) = w(x + \tau) \quad \forall x \in \mathbb{R}^2
\]
is still a solution for any \( \tau \in \mathbb{R}^2 \). This very simple fact allows us to exhibit explicitly time-dependent periodic solutions as soon as \( w \) is not constant, even in the case when \( w \) is spherically symmetric, which means that the corresponding solution \((f, U)\) of the Vlasov-Poisson system is rotationally symmetric and stationary (see Proposition 3.4). It is possible to consider such solutions because the domain \( \mathbb{R}^2 \) is of course translation invariant and because the boundary conditions are specified only at infinity. In the following, we will not consider
such a cause of time-dependance (except for the study of 1-d solutions, see Proposition 3.5) since it is a consequence of the assumption that the confining potential takes the very special form

$$U_0(x) = \frac{\rho_0}{2} \cdot |x|^2 \quad \forall x \in \mathbb{R}^2,$$

but we will concentrate our attention on another cause of time-dependance, which is much more fundamental: the anisotropy of the solutions (see Proposition 3.5 and Theorem 3.7) which is clearly related to the asymptotic boundary conditions (see Theorem 3.3) and to the asymptotic behaviour of $U_0$.

The asymptotic condition on the behaviour of the density

$$\limsup_{|x|\to+\infty} \rho(t, x) = 0$$

gives a lower bound on

$$\liminf_{|x|\to+\infty} \frac{U(t, x)}{|x|^2}.$$  

We will see in Proposition 3.5 (Existence of 1-d solutions) that this bound is optimal. Such an asymptotic boundary condition is far from the usual one

$$\lim_{|x|\to+\infty} U(t, x) = \text{Constant},$$

but we will see that there are no solutions satisfying such a condition.

**Theorem 3.3**: (Asymptotic behaviour, necessary conditions for the existence of time-periodic solutions and consequences)

(i) Under the same assumptions as in Proposition 3.1, there exists a nontrivial (i.e. $f \not\equiv 0$, $f \not\equiv 0$) solution $(f, U)$ of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz such that

$$\limsup_{|x|\to+\infty} \rho(t, x) = 0$$

if one of the two following conditions is satisfied:

either

$$\liminf_{|x|\to+\infty} \frac{U(t, x)}{|x|^2} > -\frac{\delta}{2} \quad \forall t \in \mathbb{R},$$

or

$$\omega^2 < \rho_0 \quad \text{and} \quad \liminf_{|x|\to+\infty} \frac{U(t, x)}{|x|^2} = -\frac{\delta}{2} \quad \forall t \in \mathbb{R},$$

where $\delta = \rho_0 - \omega^2$.

(ii) If $\liminf_{|x|\to+\infty} \frac{U(t, x)}{|x|^2} > -\frac{\delta}{2}$, then $\omega^2 < \rho_0$, and $(t, x) \mapsto \rho(t, x) = \int_{\mathbb{R}} f(t, x, v) dv$ belongs to $C^0(\mathbb{R}, L^1(\mathbb{R}^2))$ if and only if $\int_{t_0}^{t_0+\infty} G(s) ds < +\infty$.

(iii) A necessary condition for the existence of a nontrivial solution $(f, U)$ of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz and such that

$$\limsup_{|x|\to+\infty} \rho(t, x) = 0$$

is $\omega^2 < \rho_0$. 

(iv) Assume moreover that for some \( \epsilon > 0 \), \((t, x) \mapsto \rho(t, x) = (1 + |x|^\epsilon) \int_{\mathbb{R}^3} f(t,x,v) \, dv \) belongs to \( C^0(\mathbb{R}, L^1(\mathbb{R}^2)) \), i.e. \( \int_0^{+\infty} G(s)(1 + s^\epsilon) \, ds < +\infty \). Then there is no solution such that

\[
\lim_{|x| \to +\infty} U(t, x) = U_\infty \text{ for all } t \in \mathbb{R}, \text{ for some constant } U_\infty \in \mathbb{R}.
\]

(v) If \( \limsup_{|x| \to +\infty} \rho(t, x) = 0 \) and \( \limsup_{|x| \to +\infty} \frac{U(t,x)}{|x|^2} < +\infty \), or \( \limsup_{|x| \to +\infty} \frac{w(x)}{|x|^2} < +\infty \), where \( w \) and \( U \) are related by \( U(t, x) = w \left( e^{\epsilon tA} x \right) \), then there exist \( \theta \in [0, 1] \) and a system of cartesian coordinates such that

\[
w(x) = U \left( t, e^{-\epsilon tA} x \right) = \theta \pi^2 + (1 - \theta) x_2^2 + o(|x|^2) \quad \text{as} \quad |x| \to +\infty.
\]

Proof:

Since \( G \) is positive decreasing and not identically equal to zero, and since

\[
\rho(t, x) = G(U(t, x) + \frac{\delta}{2}) \quad \text{with} \quad \delta = \rho_0 - \omega^2,
\]

the condition

\[
\limsup_{|x| \to +\infty} \rho(t, x) = 0
\]

is equivalent to

\[
w \leq \liminf_{|x| \to +\infty} w(x) = \liminf_{|x| \to +\infty} \left( U(t, e^{-\epsilon tA} x) + \frac{\delta}{2} |x|^2 \right),
\]

with

\[
w = \inf \{ w \in \mathbb{R} : G(w) = 0 \} \in ]-\infty, +\infty],
\]

a condition that is obviously violated if

\[
\liminf_{|x| \to +\infty} \frac{U(t, x)}{|x|^2} < -\frac{\delta}{2} \quad \forall t \in \mathbb{R}.
\]

If \( \liminf_{|x| \to +\infty} \frac{U(t, x)}{|x|^2} = -\frac{\delta}{2} > 0 \), then \( w(x) = U(t, e^{-\epsilon tA} x) + \frac{\delta}{2} |x|^2 \) is such that

\[
-\Delta \left( \frac{\delta}{2} |x|^2 - w \right) = -G(w) \leq 0,
\]

and

\[
\limsup_{|x| \to +\infty} \frac{1}{|x|^2} \left( \frac{\delta}{2} |x|^2 - w \right) = \frac{\delta}{2} < 0
\]

implies that

\[
h(R) = \sup_{|x| = R} \left( \frac{\delta}{2} |x|^2 - w \right) = -R^2 \inf_{|x| = R} \left( \frac{\delta}{2} |x|^2 - w \right)
\]
is such that there exists a sequence \((R_n)_{n \in \mathbb{N}}\) satisfying
\[
\lim_{n \to +\infty} R_n = +\infty \quad \text{and} \quad h(R_n) = \frac{\delta}{2} R_n^2 \to -\infty.
\]
Applying the Maximum Principle, we obtain for all \(n \in \mathbb{N},\)
\[
\frac{\delta}{2} |x|^2 - w(x) \leq h(R_n) \quad \forall x \in B(0,R),
\]
which is impossible:
\[
\liminf_{|x| \to +\infty} \frac{U(t,x)}{|x|^2} = -\frac{\delta}{2} \implies \omega^2 \leq \rho_0,
\]
which proves (i).

Assume now that \(\liminf_{|x| \to +\infty} \frac{U(t,x)}{|x|^2} > -\frac{\delta}{2} > 0.\)
\[
\limsup_{|x| \to +\infty} \frac{1}{|x|^2} \left( \frac{\delta}{2} |x|^2 - w \right) = -\liminf_{|x| \to +\infty} \frac{U(t,x)}{|x|^2} < 0
\]
and
\[
-\Delta \left( \frac{\delta}{2} |x|^2 - w \right) = -G(w) \leq 0
\]
again imply that
\[
w(x) \geq \frac{\delta}{2} |x|^2 + R^2 \inf_{|x| = R} \frac{U(t,x)}{|x|^2} \quad \forall x \in B(0,R), \quad \forall R > 0,
\]
which also gives a contradiction, and proves that
\[
\liminf_{|x| \to +\infty} \frac{U(t,x)}{|x|^2} > -\frac{\delta}{2} \implies \omega^2 \leq \rho_0,
\]
which proves (ii).

(iii) is an immediate consequence of (i) and (ii).

(iv) is easily deduced from the asymptotic equivalence
\[
U(t,e^{-w\Lambda z}) = w(x) - \frac{\delta}{2} |x|^2 \sim -\frac{M}{2\pi} \ln |x| \quad \text{with} \quad M = \int_{\mathbb{R}^2} \rho(t,x) \, dx \quad \forall t \in \mathcal{R}.
\]

Assume that a system of polar coordinates \((r,\varphi)\) is given, and define \(\tilde{w}\) by
\[
\tilde{w}(r,\varphi) = r^2 w(r^{-1} \cos \varphi, r^{-1} \sin \varphi)
\]
for all \((r,\varphi) \in [0, +\infty) \times [0, 2\pi].\) \(\tilde{w}\) is solution of
\[
4\tilde{w} - 3\tilde{w} r^2 - r^3 \Phi^2 \Phi^2 \tilde{w} - \Phi^2 \Phi^2 \tilde{w} = 2\Phi (\tilde{w}) - \Phi^2 \tilde{w} \sim 2\Phi \quad \text{as} \quad r \to 0 + .
\]
Because of the assumption

$$\limsup_{|x| \to +\infty} \frac{w(z)}{|z|^2} = \text{Const},$$

the function $a = \tilde{w}(0,.) : [0,2\pi] \to \mathbb{R}$ such that

$$\tilde{w}(r,\varphi) \sim a(\varphi) \quad \text{as} \quad r \to 0^+$$

satisfies the equation

$$4a + \frac{\partial^{2}a}{\partial \varphi^{2}} = 2\delta.$$
solutions (even when they are not translated), but are always of infinite mass (Proposition 3.5).

The last result of this section will be devoted to solutions intermediate between the ones of Proposition 3.4 (radially symmetric solutions) and the ones of Proposition 3.5 (1-d solutions), which appear to form (Theorem 3.7) a new class of solutions (as far as the author knows). These solutions have the interesting property that they are time-periodic non stationary solutions, and that they may have a finite total mass. They moreover have a non standard asymptotic behaviour:

\[ U(t, e^{-t/2} x) \sim \delta(t - 1/2)(x_1^2 - x_2^2) + o(|x|^2) \quad \text{as} \quad |x| \to +\infty. \]

in a well chosen system of cartesian coordinates.

**Proposition 3.4**: *(Existence of radially symmetric solutions)*

Assume that \( G \) is a Lipschitz decreasing function, such that

\[
\lim_{w \to +\infty} G(w) = 0 \quad \text{and} \quad \lim_{w \to -\infty} G(w) = G^{\infty} < \infty.
\]

For any \( \delta > 0 \),

\[
\begin{align*}
- \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) + 2\delta &= G(w) \\
w(0) &= w_0 \\
dw/dr(0) &= 0
\end{align*}
\]

has a unique solution \( r \mapsto w(r) \) for \( r \in [0, +\infty] \). Let us define \( \bar{w} \) by

\[
\bar{w} = \sup \{ w \in \mathcal{R} : G(w) > 2\delta \} \quad \text{if} \quad G^{\infty} \geq 2\delta,
\]

\[
\bar{w} = -\infty \quad \text{if} \quad G^{\infty} < 2\delta.
\]

Three cases may occur:

(i) \( w_0 = \bar{w} \): then \( w(r) = w_0 \) for all \( r > 0 \).

(ii) \( w_0 > \bar{w} \): then \( w \) is strictly increasing on \( [0, +\infty[ \) and

\[
\frac{dw}{dr}(r) \sim \delta r \quad \text{and} \quad w(r) \sim \frac{\delta}{2} r^2 \quad \text{as} \quad r \to +\infty.
\]

(iii) \( w_0 < \bar{w} \): then \( w \) is strictly decreasing on \( [0, +\infty[ \) and

\[
\frac{dw}{dr}(r) \sim \frac{1}{2}(2\delta - G^{\infty}) r \quad \text{and} \quad w(r) \sim \frac{1}{4}(2\delta - G^{\infty}) r^2 \quad \text{as} \quad r \to +\infty.
\]
If \( g = -\frac{1}{2}G' \), then \( f \) belongs to \( C^0(\mathbb{R}, L^1(\mathbb{R}_0^2 \times \mathbb{R}^2)) \), \( U \) belongs to \( C^1(\mathbb{R}, C^1(\mathbb{R}^2)) \) and
\[
f(t, x, v) = g \left( \frac{1}{2} |v - \omega Ax|^2 + w(\rho + \delta z) \right),
\]
\[
U(t, x) = w(\rho + \delta z),
\]
with \( \omega = \pm \sqrt{\rho^2 - \delta} \), are solutions of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (in the sense that \( f \) is solution in the sense of the characteristics as defined by R.J. DiPerna and P.-L. Lions). \( f \) and \( U \) are in fact stationary solutions:
\[
U(t, x) = w(|x|) - U_0(x) + \frac{1}{2} \alpha^2 |x|^2 = U(x)
\]
and
\[
f(t, x, v) = g \left( \frac{1}{2} |v|^2 + w(|x|) - \omega(v \cdot Ax) + \frac{1}{2} \omega^2 |x|^2 \right) = f(x, v)
\]
do not depend on \( t \): \( f \) (resp. \( U \)) depend only on \( |x|, |v| \) and \( (v \cdot Ax) = z \wedge v \) (resp. \(|z|\)). They are rotationally invariant: for any \( r \in \mathbb{R}^\ast \),
\[
U(x) = U(e^{\alpha} x) \quad \forall x \in \mathbb{R}^2,
\]
(\( U \) is radially symmetric)
and
\[
f(x, v) = f(e^{\alpha} x, e^{\alpha} v) \quad \forall x \in \mathbb{R}^2.
\]
\( f \) belongs to \( L^1(\mathbb{R}^2 \times \mathbb{R}^2) \) if and only if \( G \) belongs to \( L^1(\mathbb{R}^+ \times \mathbb{R}) \) and \( w_0 > \bar{w} \).

**Proof:**
The existence of a solution for any \( r \in [0, +\infty[ \) is obvious since \( G \) is bounded, and the other properties follow from the formula
\[
\frac{dw}{dr} = \frac{1}{r} \int_0^r r \left( 2\delta - G(w(s)) \right) ds,
\]
which implies that \( \frac{dw}{dr} \) has the same sign as \( w - w_0 \), and that
\[
0 \leq (2\delta - G(w_0))^\frac{r}{2} \leq \frac{dw}{dr} \leq \delta r \quad \text{if} \quad w_0 \geq \bar{w},
\]
and
\[
0 \leq (2\delta - G(w_0))^\frac{r}{2} \leq \frac{dw}{dr} \leq (2\delta - G(w_0))^\frac{r}{2} \quad \text{if} \quad w_0 \leq \bar{w}.
\]
The behaviour when \( r \to +\infty \) then follows:

if \( w_0 > \bar{w} \), then \( \lim_{r \to +\infty} w(r) = +\infty \) and \( G(w(r)) - 2\delta \sim -2\delta \).
if $w_0 < \bar{w}$, then $\lim_{r \to +\infty} w(r) = -\infty$ and $G(w(r)) - 2\delta \sim G^\infty - 2\delta$.

if $w_0 = \bar{w}$, then $w(r) \equiv w_0$.

The rest of the proof is a simple computation. □

Remarks:

1) for radially symmetric solutions, $U$ has a logarithmic growth:

$$U \left( t, e^{-\omega t^2} x \right) = w(|x|) - \frac{\delta}{2} |x|^2 \equiv -\frac{M}{2\pi} t_i |x| \quad \text{as} \quad |x| \to +\infty$$

since $-\nabla U \left( t, e^{-\omega t^2} x \right) = \frac{1}{|\omega|} \int_0^{|x|} r G(w(r)) \, dr$.

This property is characteristic of the radially symmetric solutions among the class of the solutions considered in this paper (see Theorem 3.7 and Remark 3.8).

2) Using the fact that the equation $-\Delta w + 2\delta = G(w)$ is obviously translation invariant, i.e. that for any solution $w$ and for any $x_0 \in \mathbb{R}^d$, $x \mapsto (w(x) + x_0)$ is also a solution, it is therefore easy to find solutions which are not radially symmetric and therefore explicitly time-dependent. But the translation invariance is of course strongly related to the fact $U_0$ has been chosen to be an harmonic potential. When this is not the case, see Remark 3.8.

We will now consider 1-d solutions in the following sense: look for a function $s \mapsto w(s)$, $s \in \mathbb{R}$, solution of

$$U(t, e^{-\omega t^2} x) + \frac{\delta}{2}(x_1^2 - x_2^2) = w(x_1) \quad \forall \ x = (x_1, x_2) \in \mathbb{R}^2,$$

with $\delta = \rho_0 - \omega^2$. $U$ is a solution of the Poisson equation if and only if

$$\frac{d^2 w}{ds^2} + 2\delta = G(w(s)). \quad (3.1)$$

Note again that this solution is invariant under translations: if $w$ is a solution, then $s \mapsto w(s + \tau)$ is also a solution for any fixed $\tau \in \mathbb{R}$.

Proposition 3.5: (Existence of 1-d solutions)

Assume that $G$ is a Lipshitz decreasing function, such that

$$\lim_{w \to +\infty} G(w) = 0 \quad \text{and} \quad \lim_{w \to -\infty} G(w) = G^\infty < \infty.$$ 

Let us define $\bar{w}$ by

$$\bar{w} = \sup \{ w \in \mathbb{R} : G(w) > 2\delta \} \quad \text{if} \quad G^\infty \geq 2\delta.$$
For any \( \delta > 0 \), Equation
\[
\frac{d^2 w}{ds^2} + 2\delta = G(w(s))
\] (3.1)
has a unique solution for \((w(0), \frac{dw}{ds}(0)) \in \mathbb{R}^2 \) given. For any \( \delta > 0 \), any solution of Equation (3.1) satisfies one of the two following properties:

(i) There exists \( s_0 \in \mathbb{R} \) such that \( \frac{dw}{ds}(s_0) = 0 \). If \( w_0 = w(s_0) \), then 3 cases may occur

- \( w_0 = \min_{s \in \mathbb{R}} w(s) \) if \( w_0 > \bar{w} \) (Case 1.a)
- \( w(s) = w_0 \) \( \forall s \in \mathbb{R} \) if \( w_0 = \bar{w} \) (Case 1.b)
- \( w_0 = \max_{s \in \mathbb{R}} w(s) \) if \( w_0 < \bar{w} \) (Case 1.c)

\( w \) is convex in Case 1.a and concave in case 1.c.

(ii) There exists \( s_0 \in \mathbb{R} \) such that \( w(s_0) = \bar{w} \) and either

- \( \frac{dw}{ds} \equiv 0 \) and \( w(s) \equiv \bar{w} \) \( \forall s \in \mathbb{R} \) (Case 2.a)

or

- \( \frac{dw}{ds}(s) \not= 0 \) \( \forall s \in \mathbb{R} \) and \( \left( \frac{dw}{ds}(s) - \frac{dw}{ds}(s_0) \right) \cdot \left( w(s) - \bar{w} \right) > 0 \) \( \forall s \not= s_0 \) (Case 2.b)

In Case 2.b, \( w \) is convex on \( \{ s \in \mathbb{R} : w(s) > \bar{w} \} \) and concave on \( \{ s \in \mathbb{R} : w(s) < \bar{w} \} \).

Case (ii) occurs only if \( G^\infty > 2\delta \) or if

\( G^\infty = 2\delta \) and \( G(w) = 2\delta \) (Case 3)

for some \( w \in \mathbb{R} \). In both cases (and when \( w \not= \bar{w} \) for case 3), \( \lim_{s \to \pm \infty} w(s) \in \{ \pm \infty \} \), and as \( s \to \pm \infty \),

- if \( w \to +\infty \), then \( w(s) \sim \delta s^2 \),
- and if \( w \to -\infty \), then \( w(s) \sim \frac{1}{2}(2\delta - G^\infty)s^2 \).

In case 3, the same result occurs as \( s \to +\infty \), but as \( s \to -\infty \), \( s \to \frac{dw}{ds}(s) \) is constant.

If \( g = -\frac{1}{2}G \), then \( f \) and \( U \) respectively defined by

\[
f(t, x, v) = g \left( \frac{1}{2} |v - \omega Ax|^2 + w(|v - \omega Ax|) \right),
\]

and

\[
U(t, x) = w(|v - \omega Ax|),
\]
belong to $C^0(\mathbb{R}, L^1_\text{loc}(\mathbb{R}^2 \times \mathbb{R}^2))$ and $C^1(\mathbb{R}, C^2(\mathbb{R}^2))$, with $\omega = \pm \sqrt{\rho_0 - \delta}$, are solutions of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (in the sense that $f$ is solution in the sense of the characteristics as defined by R.J. DiPerna and P.-L. Lions).

This solution is explicitly time-dependent, even if $w$ is a constant function (Cases 1.b / Case 2.a), and time-periodic of period $T = \frac{\pi}{\delta}$ in Case (i), when $w$ is even (i.e. when $\frac{\partial w}{\partial s}(0) = 0$) and of period $T = \frac{2\pi}{\delta}$ in the other cases. $f \not\equiv 0$ never belongs to $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

Remark 3.6: The same results hold in Case (i) even if the condition $G_m < +\infty$ is not satisfied. When $G_m = +\infty$, it is not very difficult to extend some of the results of Proposition 3.4.

Proof of Proposition 3.5: The formula
\[
\frac{dw}{ds}(s) = \frac{dw}{ds}(s_0) + 2\delta(s - s_0) - \int_0^s G(w(\sigma)) \, d\sigma \quad \forall (s, s_0) \in \mathbb{R}^2
\]
and concavity or convexity properties are enough to prove directly the results on $w$. The rest of the proof is a simple computation again.

Let us notice that the solutions corresponding to Case (i) of Proposition 3.5 are "radially symmetric" 1-d solutions, i.e. even, up to a translation such that $s_0 = 0$. We will now exhibit a third class of solutions, that will be intermediate between 1-d solutions and 2-d radially symmetric solutions, so that they will be time-periodic (with period $\frac{\pi}{\delta}$) and have a finite total mass. Before giving the result, let us introduce some notations (that have already been used in the introduction): assume that $(x_1, x_2)$ are cartesian coordinates of $x \in \mathbb{R}^2$. Let us choose $\theta \in [0, 1]$ and consider a solution $v$ of
\[
-\Delta v = G\left(v(x) + Q_\theta\right) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,
\] (3.1)

with
\[
Q_\theta(x) = \delta(\theta \delta x_1^2 + (1 - \theta)x_2^2) = \delta'(x_1^2 + x_2^2).
\]

Then $u$ defined by
\[
u(\delta x) + \frac{\delta}{2}|x|^2 = u(x) + \frac{\delta}{2}(x_1^2 + x_2^2) = v(x) + Q_\theta(x)
\]
is also a solution of
\[
-\Delta u = G(u(x)) + \frac{\delta}{2}|x|^2
\]
since
\[ -\Delta Q_\delta(x) = -\delta(2\theta + 2(1 - \theta)) = -2\delta. \]

It immediately follows that
\[ u(x) = v(x) + \frac{\delta}{2}(2\theta - 1)(x_1^2 - x_2^2) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2. \]

We can notice that if \( \theta = 0 \) or \( \theta = 1 \) these solutions correspond to the 1d-solutions, and that if \( \theta = 1/2 = 1 - \theta \), then they correspond to radially symmetric 2d-solutions.

**Theorem 3.7**: (Existence of time-periodic anisotropic solutions with finite mass)
Assume that \( G \) is a continuous decreasing function, such that \( \lim_{w \to +\infty} G(w) = 0 \). Let us define \( \bar{w} \) by
\[ \bar{w} = \sup\{w \in \mathbb{R} : G(w) > 2\delta\} \quad \text{if} \quad G \geq 2\delta \quad \text{and} \quad \bar{w} = -\infty \quad \text{if} \quad G < 2\delta, \]
and assume that \( G \) belongs to \( L^q(\mathbb{R}^+ \cap [1,2]) \) for some \( q \in [1,2] \) and is such that \( \int_{1}^{\bar{w}} G(s) \ln s \, ds < +\infty \). For any \( \delta > 0 \), any \( C^2 \) solution of
\[ -\Delta v = G(v + Q_\delta) \quad (3.1) \]
such that \( \inf_{x \in \mathbb{R}^2} \left( v(x) + Q_\delta(x) \right) > \bar{w} \) and \( \inf_{|x| \to +\infty} \frac{v(|x|)}{|x|^2} = 0 \) is also a solution of
\[ T(v + Q_\delta) = v + Q_\delta, \]
where \( Q_\delta(x) = Q_\delta((x_1, x_2)) = \delta(\theta x_1^2 + (1 - \theta)x_2^2) \) for all \( x = (x_1, x_2) \in \mathbb{R}^2 \), the operator \( T : L^\infty_{\text{loc}}(\mathbb{R}^2) \to L^\infty_{\text{loc}}(\mathbb{R}^2) \) being defined by
\[ T(u)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x - y|)G(w(y)) \, dy + Q_\delta(x) + w_0 - I(u) \quad \forall x \in \mathbb{R}^2, \]
with \( I(u) = \inf_{x \in \mathbb{R}^2} \left( -\frac{1}{2\delta} \int_{\mathbb{R}^2} \ln(|x - y|)G(w(y)) \, dy + Q_\delta(x) - q_\delta(x) \right), \min_{x \in \mathbb{R}^2} (w(x) - q_\delta(x)) = w_0 \), and
\[ q_\delta(x) = \frac{1}{2} \left( 2 - \frac{G(w_0)}{\delta} \right) Q_\delta(x) = \frac{1}{2} \left( 2\delta - G(w_0) \right) (\theta x_1^2 + (1 - \theta)x_2^2) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2. \]

Then for any \( w_0 > \bar{w} \), for any \( \theta \in [0,1] \), there exists a solution \( v \) such that
\[ \inf_{x \in \mathbb{R}^2} \left( v(x) + Q_\delta(x) - q_\delta(x) \right) = w_0. \]
Let us define \( g = -\frac{1}{2}G' \) and assume that it belongs to \( L^1_\infty (\mathbb{R}) \). \( f \) defined by

\[
 f(t, x, v) = g \left( \frac{1}{2} |v - \omega Ax|^2 + w(e^{itA} x) + U_0(x) - \frac{1}{2} \omega^2 |x|^2 \right),
\]
and

\[
 U(t, x) = w(|e^{itA} x|),
\]
respectively belong to \( C^0(\mathbb{R}, L^1_{\text{loc}}(\mathbb{R}^2)) \) and \( C^1(\mathbb{R}, C^2(\mathbb{R}^2)) \), with \( \omega = \pm \sqrt{\rho_0 - \delta} \), and are solutions of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (in the sense that \( f \) is solution in the sense of the characteristics as defined by R.J. DiPerna and P.-L. Lions).

This solution is explicitly time-dependent provided \( \theta \neq \frac{1}{2} \) and time-periodic of period \( T = \frac{2\pi}{\omega} \) (or \( T = \frac{\pi}{\omega} \)). Under the above conditions, \( f \) belongs to \( C^0(\mathbb{R}; L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \) if and only if \( G[w_0, +\infty) \) belongs to \( L^1([w_0, +\infty]) \).

**Proof:**

The first part of the proposition is obvious. The last part does not present any difficulty: the time-dependance property of \( f \) is easily derived from the fact that the asymptotic form of \( w = v + Q_\theta \) (as \( |x| \to +\infty \)) is clearly not invariant under rotations. The solution is by construction \( \frac{2\pi}{\omega} \) periodic. In fact the proof below also works on the subset of the functions which are symmetric with respect to the origin, and proves that \( (f, U) \) can be chosen \( \frac{\pi}{\omega} \) time-periodic.

We therefore only have to prove the existence of a solution. We will do it in four steps.

1st step: more definitions and immediate consequences:

Let us consider the space

\[
 X = \{ w \in C^0(\mathbb{R}^2; \mathbb{R}) \mid \limsup_{|x| \to +\infty} \frac{w(x)}{|x|^2} < +\infty \},
\]
with the norm

\[
 ||w||_X = \frac{||w(x)||_{L^\infty(\mathbb{R}^2)}}{1 + |x|^2},
\]
and the subset

\[
 K_{w_0, C} = \{ w \in X \mid w(x) \sim Q_\theta(x) \text{ as } |x| \to +\infty, \quad \inf_{x \in \mathbb{R}^2} \left( w(x) - Q_\theta(x) \right) = w_0, \quad w(x) \leq C + Q_\theta(x) \forall x \in \mathbb{R}^2 \}.
\]

The operators \( T_0, T : X \to X \) defined by

\[
 (T_0 w)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x - y|)G(w(y)) \ dy + Q_\theta(x),
\]

and

\[
 (T w)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x - y|)G(w(y)) \ dy + \Delta \theta(x).
\]
and
\[(Tw)(x) = (T_{w_0}) (x) + w_0 - I(w) \quad \forall x \in \mathbb{R}^2,\]
with \(I(w) = \inf_{x \in \mathbb{R}^d} \left( (Tw)(x) - q_\theta (x) \right)\), are continuous on \(K_{w_0,C}\), which is a convex closed subset of \(X\).

\((Tw)(x) \sim Q_\theta (x)\) as \(|x| \to +\infty\), because \(\int_{\mathbb{R}^d} \ln(\lvert x - y \rvert) G(w(y)) \, dy = o(\lvert x \rvert^2)\) as \(|x| \to +\infty\).

\(I(w)\) is therefore well defined and
\[\inf_{x \in \mathbb{R}^d} \left( (Tw)(x) - q_\theta (x) \right) = w_0.\]

**2nd step : estimates :**

First of all, for any \(w \in K_{w_0,C}\), since \(G\) is decreasing and \(w_0 \leq w_0 + q_\theta (y) \leq w(y)\),
\[
k(w_0) = -\frac{1}{2\pi} \int_{|x-y|>1} \ln(|x-y|)G(w_0 + q_\theta (y)) \, dy
\leq -\frac{1}{2\pi} \int_{|x-y|>1} \ln(|x-y|)G(w(y)) \, dy
\leq -\frac{1}{2\pi} \int_{|x-y|<1} \ln(|x-y|)G(w(y)) \, dy
\leq -\frac{1}{2\pi} \int_{|x-y|<1} \ln(|x-y|)G(w(y)) \, dy \leq \frac{G(w_0)}{4},
\]
which gives for \(\frac{1}{2\pi} \ln|x| * G(w(x))\) the estimate
\[k(w_0) \leq -\frac{1}{2\pi} \int_{\mathbb{R}^d} \ln(|x-y|)G(w(y)) \, dy \leq \frac{G(w_0)}{4}.\]

Moreover
\[I(w) = \inf_{x \in \mathbb{R}^d} \left( -\frac{1}{2\pi} \int_{\mathbb{R}^d} \ln(|x-y|)G(w(y)) \, dy + Q_\theta (x) - q_\theta (x) \right)
\geq \inf_{x \in \mathbb{R}^d} \left( -\frac{1}{2\pi} \int_{\mathbb{R}^d} \ln(|x-y|)G(w(y)) \, dy \right) \geq k(w_0),\]
and
\[Tw - Q_\theta \leq w_0 - I(w) - \frac{1}{2\pi} \int_{\mathbb{R}^d} \ln(|x-y|)G(w(y)) \, dy \leq w_0 - k(w_0) + \frac{G(w_0)}{4} = C(w_0).\]

For any \(C \geq C(w_0)\), \(K_{w_0,C}\) is non-empty and stable under the action of \(T\). In the following, we shall consider \(K = K_{w_0,C(w_0)}\) and get a fixed point result on \(T(K) \subset K\) using Schauder's theorem.

**3rd step : a compactness result :**
According to its definition, \( K \) is bounded for the norm \( \| \cdot \|_X \). For the moment, let us forget the problem of the non-compactness of \( \mathbb{R}^2 \) and try to apply Ascoli's lemma. We have to prove that \( T(K) \) is uniformly equicontinuous. As in [21] or in [9], it is a consequence of the inequality

\[
\| \nabla T w \| _{L^q(\mathbb{R}^2)} \leq \frac{1}{2\pi} \int_{|x-y| \leq 1} \frac{G(u_0 + q_y(y))}{|x-y|} \, dy + \frac{1}{2\pi} \int_{|x-y| > 1} \frac{G(u_0 + q_y(y))}{|x-y|} \, dy
\]

\[
\leq G(u_0) + \frac{1}{2\pi} \int_{|x-y| > 1} \frac{G(u_0 + q_y(y))}{|x-y|} \, dy
\]

With \( c = 2(\delta - G(u_0)) \cdot \min(\theta, (1 - \theta)) \) if \( q = 1 \), and

\[
\| \nabla T w \| _{L^q(\mathbb{R}^2)} \leq G(u_0) + \frac{c^{1/q}}{2\pi(p-2)^{1/q}} \left( \int_0^{+\infty} |G(u_0 + s)|^p \, ds \right)^{1/p}
\]

with \( \frac{1}{p} + \frac{1}{q} = 1 \) if \( q \in [1, 2] \).

Applying Ascoli's lemma, \( T(K) \) would be relatively compact if the functions of \( K \) had been defined on a compact set.

4th step: conclusion:

Since the set \( K \) is a set of functions defined on \( \mathbb{R}^2 \), one has to be careful. Let \( R \) be a strictly positive real number.

1) The operator \( T^R : C^\infty(B(0, R)) \rightarrow C^\infty(\mathbb{R}^2; \mathbb{R}) \) defined by

\[
(T^R w)(x) = (T_0^R w)(x) + u_0 - I^R(w) \quad \forall x \in B(0, R),
\]

\[
(T_0^R w)(x) = -\frac{1}{2\pi} \int_{B(0, R)} \ln(|x-y|)G(w(y)) \, dy + Q_0(x) \quad \forall x \in B(0, R),
\]

with \( I^R(w) = \inf_{x \in B(0, R)} \left( (T_0 w)(x) - q_y(x) \right) \), is continuous on the set \( K^R \) defined by

\[
K^R = \{ w|_{B(0, R)} : w \in K \},
\]

and applying Ascoli's lemma, one proves that its restriction \( \bar{T}^R \) defined by

\[
\bar{T}^R w = (T w)|_{B(0, R)}
\]

is such that \( T^R(K^R) \) is precompact; Schauder's theorem then implies that \( T^R \) has a fixed-point on \( K^R \) in the following sense: there exists a function \( w^R \) of \( K \) such that

\[
(T^R w^R)(x) = w^R(x) \quad \forall x \in B(0, R).
\]
2) We just have to compute the difference between $w^R$ and $T w^R$:

$$\|w^R - T w^R\|_x \leq \|w^R - T w^R\|_{L^\infty(B(0,R))} + \|(w^R)_{B^c(0,R)} - (T w^R)_{B^c(0,R)}\|_x.$$

But on one side

$$\|(w^R)_{B^c(0,R)} - (T w^R)_{B^c(0,R)}\|_x = O\left(\frac{\ln R}{1 + R^2}\right)$$

uniformly in $w^R \in K^R$ as $R \to +\infty$, because

$$(w^R)_{B^c(0,R)}(x) - (T w^R)_{B^c(0,R)}(x) = O\left(\frac{\ln |x|}{1 + |x|^2}\right)$$

uniformly in $w^R \in K^R$ as $|x| \to +\infty$, and on the other side

$$\|w^R - T w^R\|_{L^\infty(B(0,R))} = \|T^R w^R - T w^R\|_{L^\infty(B(0,R))} \leq \frac{1}{2} \sup_{x \in B(0,R)} \int_{|y| > R} |\ln(|x - y|)| G(w_0 + q \phi(y)) \, dy$$

$$\leq \frac{1}{2} \int_{|y| > R} |\ln(|y| + R)| G(w_0 + q \phi(y)) \, dy \to 0$$

as $R \to +\infty$. Because of the uniform estimate on $\|\nabla T w\|_{L^\infty(B^R)}$ for all $w \in K$, up to the extraction of a subsequence $(R_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to +\infty} R_n = +\infty,$$

there exists a function $w \in K$ such that

$$w^{R_n} \to w \quad \text{in} \quad X \quad \text{as} \quad n \to +\infty,$$

and using Lebesgue's theorem of dominated convergence,

$$T w^{R_n} \to T w \quad \text{in} \quad X \quad \text{as} \quad n \to +\infty.$$

Passing then to the limit, $w$ is such that

$$\|w - T w\|_x = 0,$$

which gives the result.

**Remark 3.8**:

1) The solutions of Theorem 3.7 may have a finite total energy

$$\frac{1}{2} \int_{B^R \times R^D} (|w|^2 + U(t,x) + 2u_0(x))f(t,x,v) \, dx \, dv$$
provided $s \mapsto g(s)$ is sufficiently decreasing as $s \to +\infty$. 2)
Last part of the proof shows that $T(K)$ is precompact. An alternative method
would be to apply directly Schauder's fixed-point theorem to $T$ and $K$.
3) The question of the uniqueness seems to be more technical than difficult.
For example, assuming that $g$ is decreasing and such that
$$
sup_{x \in \mathbb{R}^d} \frac{1}{1+|x|^2} \int_{\mathbb{R}^d} \left(1 + |y|^2\right) \left|\ln|x-y|\right| g(w_0 + q\delta(y)) \, dy < 2\pi,$$
the operator $T$ is then contracting on the set $K$. Of course, this only proves
the uniqueness of the solution of the fixed point equation since the asymptotic
boundary condition is not sufficient to determine the location of the minimum.
It is for example clear that the solution of the following fixed point equation
$$T(v + \hat{Q}_d) = v + \hat{Q}_d, \quad \lim_{|x| \to +\infty} \frac{v(x)}{|x|^2} = 0,$$
where $\hat{Q}_d(x) = Q_d(x + x_0)$ for some $x_0 \in \mathbb{R}^d$ provides a solution to
$$-\Delta \tilde{v} = G(\tilde{v} + \delta^\frac{d}{2} |x|^2)$$
with $\tilde{v}(x) = v(x + x_0)$ and $v$ solution to $-\Delta v = G(u + \hat{Q}_d)$ which gives a solution
to $-\Delta u = G(u + \frac{1}{r}|x|^2)$ satisfying the asymptotic boundary condition provided
$u(x) + \frac{1}{r}|x|^2 = \tilde{v}(x) + \hat{Q}_d(x)$.
4) Using a development of $w(x)$ for $|x| \to +\infty$, and the techniques developed
in [26] and adapted in [9], it is probably not very difficult to prove that up
to a translation, the solution $w$ with the anisotropic asymptotic boundary condition is, up to a translation, symmetric with respect to the origin and
therefore periodic of period $\frac{\pi}{2}$.
5) Instead of $q_\epsilon(x) = \frac{1}{2} (2\delta - G(w_0))(\theta x_1^2 + (1 - \theta)x_2^2)$, we may prove the theorem with
$$q_\epsilon(x) = q_\epsilon^*(x) = \epsilon(\theta x_1^2 + (1 - \theta)x_2^2),$$
for any $\epsilon \in (0, G(w_0)], w_0 > \bar{w}$. This allows to give an existence result of a solution
$W$ such that $W_0 > \inf_{x \in \mathbb{R}^d} w(x) > \bar{w}$. There exists indeed a solution $w^\epsilon$ for which
$w_0 - \epsilon = \inf_{x \in \mathbb{R}^d} \left(w(x) + q_\epsilon^*(u)\right)$, for any $\epsilon > 0$ small enough. But the passage to
the limit $\epsilon \to 0+$ is not clear since
$$K^*(w_0) = -\frac{1}{2\pi} \int_{|x-y|>1} \ln|x-y|G(w_0 + q_\epsilon^*(y)) \, dy.$$
is obviously not bounded from below. For the same reason, we cannot apply directly Lebesgue’s theorem of dominated convergence.

Note that the meaning of \( w_0 \) in Theorem 3.7 is the same as in Propositions 3.4 and 3.5 if \( \epsilon \in (\frac{1}{2} G(w_0) + 2\delta), G(w_0) \). The proof is easy. If we denote by \( w' \)

\[
w' = w - q_0^2,
\]

then \( -\Delta w' = G(w) - 2\delta + 2\epsilon - G(w) \geq 0 \) if \( G(w_0) - 2\delta + 2\epsilon \geq 0 \) (this implies indeed \( G(w(x)) - 2\delta + 2\epsilon \geq 0 \) for any \( x \in \mathbb{R}^2 \)). Thus, if \( \epsilon \in [\delta - G(w_0), \delta] \), then \( w_0 = \inf_{x \in \mathbb{R}^2} w'(x) = \inf_{x \in \mathbb{R}^2} w(x) \) for any solution \( w \) in the sense of Proposition 3.4 or 3.5.

6) The method also applies when the confining potential is not radially symmetric (but has an asymptotic quadratic growth). For instance, we may consider the case where

\[
U_0(x) = \frac{\rho_0}{2} |x|^2 + \lambda_0(x)
\]

where \( V_0 \) is a continuous compactly supported radially symmetric perturbation. When the solution (obtained by the fixed point method) is unique, this allows to prove the existence of a branch of solutions (parametrized by \( \lambda \) - this can be seen as an implicit function theorem) and gives an existence result when \( U_0 \) is not an harmonic potential.

It is worth to notice that this provides a branch of solutions starting from the radial solutions (plus a translation) which may have finite mass, a logarithmic growth, and are generically time-periodic non stationary solutions.

5. Conclusion and some open questions

First, let us summarize the results obtained in part II.

The Jeans’ Theorem given in Proposition C.2 proves that generically any time-periodic solution such that its average is radially symmetric in fact satisfies the weak Ehlers & Rienstra ansatz, provided it does not explicitly depends on time.

But according to Theorem 2.1, any solution satisfying the weak Ehlers & Rienstra ansatz is time-periodic and obeys to the usual (strong) Ehlers & Rienstra ansatz. The dependance in time is obtained through a rotation with a constant angular velocity. Solving the Vlasov-Poisson system is completely
equivalent to solving a nonlinear Poisson equation (Proposition 3.1), whose nonlinearity can be chosen in an arbitrary way.

Whatever the nonlinearity is, if the spatial density belongs to $L^1$, the angular velocity is bounded by a quantity depending on the confining potential. The asymptotic behaviour of the potential at infinity is such that the level curves are (asymptotically) ellipses, provided the potential is at most quadratic at infinity, which is a reasonable assumption (Theorem 3.3). It is also proved that the potential cannot be constant at infinity.

Special solutions have then been constructed (Proposition 3.4: radially symmetric solutions, and Proposition 3.5: ld-solutions) using the ideas of [6]. The most general class of solutions compatible with the natural asymptotic boundary conditions is studied in Theorem 3.7: these solutions are non-isotropic solutions and therefore naturally time-periodic; they may have a finite mass and are explicitly time-dependant, which was not the case for the radially symmetric solutions.

In view of simplifying the computations, the confining potential was supposed to be harmonic. This very special form allows to introduce another method for giving time-dependant solutions: one can indeed use the invariance by translation of the equation for $w$ (from which the potential is deduced) to get nonsymmetric solutions that will therefore be explicitly time-dependant, but this is a very special property of the harmonic potential. However this allows to build a branch of non trivial solutions by perturbing the harmonic potential. On the other side, the nonisotropic solutions given in Theorem 3.7 are independant of the local form of the confining potential provided its asymptotic behaviour is quadratic.

A complete study of general confining potentials has still to be done. The question of the uniqueness of the solutions (up to a translation, and up to the addition of a constant to the potential) and of their stability would be of the greatest interest. This last question may not be out of reach in view of the characterization that has been given for the time-periodic solutions, but probably implies first the removal of the technical assumptions used in this paper, and a new approach for the time-dependant Vlasov-Poisson system. It is also probably possible to prove symmetry results, as mentioned in Remark 3.8 (this is related to the questions on the uniqueness).
As a concluding remark, let us note that the Ehlers & Rienstra ansatz and the nonlinear Poisson equation have been (from a mathematical point of view) used in many mathematical papers for the study of the 3d Vlasov-Poisson gravitational system and that the results given in the paper may easily be adapted to this case.

Appendix A: Formal derivation of a simple 2d model for monokinetic charged particle beams

Consider a solution $F^\lambda$ of the Vlasov-Poisson equation in $\mathbb{R}^3$

$$\partial_t F^\lambda + V \cdot \partial_x F^\lambda - \partial_x U_0^\lambda(x) \cdot \partial_v F^\lambda + \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{X - X'}{|X - X'|^3} F^\lambda(t, X', V') dX' dV' \cdot \partial_v F^\lambda = 0$$

with the notations $X = (x, z) \in \mathbb{R}^2 \times \mathbb{R}$ and $V = (v, w) \in \mathbb{R}^2 \times \mathbb{R}$, and assume that $F^\lambda$ obeys to the scaling

$$F^\lambda(t, X, V) = \lambda^{\alpha-2} \cdot G^\lambda\left(t; \left(\frac{1}{\lambda} x, \lambda z\right); \left\{\frac{1}{\lambda} v; \lambda^{\alpha-2}(w - w_0)\right\}\right)$$

with $\alpha \in [0, 2]$ and $w_0 \in \mathbb{R}$. We look for the limit $\lambda \to +\infty$, which corresponds to a beam with a symmetry under translations in the $z$-direction, monokinetic with a velocity $w_0$ in that direction. The limit, if it exists, has then an infinite mass since $\|F^\lambda\|_{L^1(\mathbb{R}^2 \times \mathbb{R})} = \lambda \|G^\lambda\|_{L^1(\mathbb{R}^2 \times \mathbb{R})}$. $G^\lambda$ is solution of

$$\partial_t G^\lambda + v \cdot \partial_x G^\lambda + \lambda^{\alpha-1} w \cdot \partial_x G^\lambda - \partial_x U_0^\lambda(2x) \cdot \partial_v G^\lambda$$

$$+ \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(\frac{\lambda^2}{|x - x'|^2} + \frac{|x - x'|^2}{\lambda^2}\right)^{3/2} \cdot \partial_v G^\lambda$$

$$+ \frac{(x - x')^2}{(\lambda^2)|x - x'|^2 + |x - x'|^2} \cdot \lambda^{2-\alpha} \partial_w G^\lambda \right) G^\lambda(t, X', V') \, dx' \, dz' \, dv' \, dw' = 0 .$$

Assume now that each term in this equation remains bounded, that there exists a potential $U_0$ such that

$$\partial_x U_0(\lambda x) = \lim_{\lambda \to +\infty} \partial_x U_0^\lambda(x) \quad \forall x \in \mathbb{R}^2 ,$$

and consider

$$g(t, \pi, v) = \lim_{\lambda \to +\infty} \frac{1}{2\lambda} \int_\lambda^{+1} G^\lambda(t; (x, v); (\pi, w)) \, dx .$$
It is not very difficult to check that -- provided it is well defined -- \( g \) is then solution of
\[
\frac{\partial g}{\partial t} + v \cdot \nabla g + \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{z - z'}{|z - z'|^2} \left( \int_{\mathbb{R}} g(t, z', v', w') \, dw' \right) \cdot \nabla g = 0,
\]
since
\[
\int_{-\infty}^{+\infty} \frac{\lambda(x - x')}{\left( \lambda^2|x - x'|^2 + \frac{\delta}{\gamma} \right)^{3/2}} \, dZ = 2 \frac{x - x'}{|x - x'|^2}.
\]
Here \( w \) represents the asymptotic dispersion (of order \( \frac{1}{\sqrt{\lambda^2 + \frac{\delta}{\gamma}} \}) \) of the velocities around \( w_0 \) of the original distribution function, which justifies the word "monokinetic". \( w \) is a conserved quantity of the microscopic dynamics and the averaged density function
\[
f(t, x, v) = \int_{\mathbb{R}} g(t, x, v, w) \, dw
\]
is a solution of the 2d Vlasov-Poisson system
\[
\begin{cases}
\frac{\partial f}{\partial t} + v \cdot \nabla f = (\partial_x U(t, x) + \partial_x U_0(x)) \cdot \nabla f = 0 \\
-\Delta U = \rho(t, x) = \int_{\mathbb{R}} f(t, x, v) \, dv
\end{cases}
\]
It makes sense to ask that \( f \) belongs to \( L^\infty(\mathbb{R}^2; L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \), which means that one looks for beams with a finite mass per unit length along the axis, but the behavior in \( x \) modelizes the local distribution function and there is therefore no a priori natural boundary condition for \( U \). For the same reason, it is also assumed that the exact form of \( U_0 \) is not important and that one can take an harmonic potential for modelizing it near its bottom.

**Appendix B : two interpolation lemmas**

In the two following lemmas, the relations between the norms and the exponents are easily recovered using scalings in \( x \) and \( v \). The first lemma can be found for instance in [32,33]. The second one is a generalization of the first lemma to moments higher than one. These lemmas are related to the estimates used by B. Perthame [36] or R. Illner & G. Rein [31] for the question of the dispersion in dimension three (see concluding remark). Detailed proofs are
given here, including the explicit form of the constants which appear in the inequalities.

Lemma B.1: Let \( f \) be a nonnegative function belonging to \( L^p(\mathbb{R}^N \times \mathbb{R}^N) \) for some \( p \in [1, +\infty] \) such that

\[
x \mapsto \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x, v) \, |v|^k \, dv
\]

belongs to \( L^r(\mathbb{R}^N) \) for some \((r, k) \in [1, +\infty[ \times [0, +\infty[\). Then the function

\[
x \mapsto \rho(x) = \int_{\mathbb{R}^N} f(x, v) \, dv
\]

belongs to \( L^q(\mathbb{R}^N) \) with

\[
q = r \cdot \frac{N(p-1) + kp}{N(p-1) + kr}
\]

and satisfies

\[
\|\rho\|_{L^q(\mathbb{R}^N)} \leq C(N, p, k) \cdot \|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \cdot \||f(x, v) \, |v|^k\, dv\|_{L^{\frac{r}{r-1}}(\mathbb{R}^N)}^{\frac{r}{r-1}},
\]

with

\[
\alpha = \frac{kp}{N(p-1) + kp} \quad \text{and} \quad 1 - \alpha = \frac{N(p-1)}{N(p-1) + kp},
\]

and

\[
C(n, p, k) = \left( |\mathbb{S}^{N-1}| \right)^{\frac{k(p-1)}{N(p-1) + kp}} \cdot \left( \frac{kp}{p-1} \right)^{\frac{N(p-1)-1}{N(p-1) + kp}} \cdot \frac{N(p-1)-1}{N(p-1) + kp}.
\]

When \( r \) varies between 1 and \( +\infty \), \( q \) varies between \( 1 + \frac{kp}{N(p-1) + kp} \) and \( p + \frac{N(p-1)}{kp} \).

Proof: Assume to simplify that \( p < +\infty \). Let \( \rho \) be defined by

\[
\rho(x) = \int_{\mathbb{R}^N} f(x, v) \, dv \quad \forall x \in \mathbb{R}^N.
\]

We can split the integral defining \( \rho \) into two integrals and evaluate these integrals in different ways

\[
\rho(x) = \int_{|v| < R} f(x, v) \, dv + \int_{|v| \geq R} f(x, v) \, dv,
\]

\[
\int_{|v| < R} f(x, v) \, dv \leq \left( \frac{1}{N} |\mathbb{S}^{N-1}| \right)^{1/p} \left( \int_{\mathbb{R}^N} |f(x, v)|^p \, dv \right)^{1/p},
\]

\[
\int_{|v| \geq R} f(x, v) \, dv \leq \frac{1}{R^k} \int_{\mathbb{R}^N} f(x, v) \, |v|^k \, dv.
\]
If we optimize on $R$, then we get

$$\rho(x) \leq C(N, p, k) \cdot \left( \int_{R^n} |f(x, v)|^p \, dv \right)^{\frac{k}{p(N-1)}} \cdot \left( \int_{R^n} f(x, v) \, dv \right)^{\frac{k}{k(N-1)}} ,$$

with

$$C(n, p, k) = \left( \frac{2^{N-1}}{2^{N-1}} \right)^{\frac{n}{p(N-1)}} \cdot \left( \frac{N}{2^{N-1}} \right)^{\frac{n}{p(N-1)}} + \frac{N}{2^{N-1}} \cdot \frac{k}{k(N-1)} \cdot \frac{N}{2^{N-1}} .$$

The $L^q$-norm of $\rho$ is now bounded by

$$\|\rho\|_{L^q(R^n)} \leq C(N, p, k) \cdot \left[ \int_{R^n} \left( \int_{R^n} |f(x, v)|^p \, dv \right)^{\frac{k}{p(N-1)} q} \, dx \right]^{\frac{1}{q}} ,$$

and using Hölder's inequality, we obtain

$$\|\rho\|_{L^q(R^n)} \leq C(N, p, k) \cdot \left[ \int_{R^n} \left( \int_{R^n} |f(x, v)|^p \, dv \right)^{\frac{k}{p(N-1)} q} \, dx \right]^{\frac{1}{q}} \cdot \left[ \left( \int_{R^n} f(x, v) \, dv \right)^{\frac{k}{k(N-1) q}} \, dx \right]^{\frac{1}{q}} ,$$

with

$$\frac{1}{q} + \frac{1}{t} = 1 .$$

If we assume that

$$\frac{kq}{N(p-1) + kp} = 1$$

$$\frac{N(p-1)q}{N(p-1) + kp} = r ,$$

then

$$\frac{1}{q} = \frac{k}{N(p-1) + kp}$$

and

$$\frac{1}{qt} = \frac{N(p-1)}{N(p-1) + kp} \cdot \frac{1}{r}$$

with

$$q = r \cdot \frac{N(p-1) + kp}{N(p-1) + kr} ,$$

which proves the lemma:

$$\|\rho\|_{L^q(R^n)} \leq C(N, p, k) \left[ \int_{R^n} \int_{R^n} |f(x, v)|^p \, dv \, dx \right]^{\frac{k}{p(N-1)}} \cdot \left[ \left( \int_{R^n} f(x, v) \, dv \right)^{\frac{k}{k(N-1)}} \, dx \right]^{\frac{1}{q}} .$$
Lemma B.2: Let $f$ be a nonnegative function belonging to $L^p(R^N \times R^N)$ for some $p \in [1, +\infty]$ such that

$$z \mapsto \int_{R^N \times R^N} f(z, v) |v|^m \, dv$$

belongs to $L^r(R^N)$ for some $(r, k) \in [1, +\infty] \times [0, +\infty]$. Let $m \in [0, k]$ and assume that

$$m < \frac{p-1}{p-r} \left( N(r-1) + kr \right)$$

if $r < p$. Then the function

$$z \mapsto \int_{R^N} f(z, v) |v|^m \, dv$$

belongs to $L^s(R^N)$ with

$$u = \frac{r}{k} \cdot \frac{N(p-1) + kp}{N(p-1) + m(p-r) + kr}$$

and satisfies

$$\| f(z, v) |v|^m \|_{L^r(R^N)} \leq K(N, p, k, m) \cdot \| f \|^\beta_{L^s(R^N)} \cdot \| f(z, v) |v|^k \|_{L^r(R^N)}^{1-\beta}$$

with

$$\beta = \frac{(k-m)p}{N(p-1) + kp} \quad \text{and} \quad 1 - \beta = \frac{N(p-1) + kp}{N(p-1) + kp},$$

and

$$K(n, p, k, m) = \left( \frac{N}{N-1} \right)^{\frac{k-m-1}{N-1}} \cdot \left( \frac{k}{N-1} \right)^{\frac{N(p-1)}{N-1}} \cdot \frac{N^{p-1} + kp}{(kp)^{p-1}} \left( \frac{k-m}{kp} \right)^{(k-m)/k}.$$

Proof: As in lemma B.1 we assume to simplify that $p < +\infty$.

1st step: We prove that

$$\| f(z, v) |v|^m \|_{L^s(R^N)} \leq \| f(z, v) |v|^k \|_{L^r(R^N)} \cdot \| f(z, v) |v|^m \|_{L^r(R^N)}^{(k-m)/k},$$

$$\int_{R^N} f(z, v) |v|^m \, dv = \int_{R^N} (f(z, v))^{(k-m)/k} \cdot (f(z, v))^{m/k} \, dv.$$
Applying Hölder’s inequality, we first get
\[
\int_{\mathbb{R}^N} f(x, v) |v|^m \, dv \leq \left( \int_{\mathbb{R}^N} f(x, v) \, dv \right)^{(k-m)/k} \cdot \left( \int_{\mathbb{R}^N} f(x, v) |v|^k \, dv \right)^{m/k}.
\]
With the notation \( \rho(x) = \int_{\mathbb{R}^N} f(x, v) \, dv \),
\[
\left\| \int_{\mathbb{R}^N} f(x, v) |v|^m \, dv \right\|_{L^q(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \rho(x)^{(k-m)/k} \cdot \left( \int_{\mathbb{R}^N} f(x, v) |v|^k \, dv \right)^{m/k} \, dx.
\]
Applying again Hölder’s inequality, we get
\[
\left\| \int_{\mathbb{R}^N} f(x, v) |v|^m \, dv \right\|_{L^q(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \cdot \left\| \int_{\mathbb{R}^N} f(x, v) |v|^k \, dv \right\|_{L^{q/k}(\mathbb{R}^N)},
\]
with
\[
q = \frac{(k-m)ur}{kr - um},
\]
\[
\frac{1}{u} = m \cdot \frac{1}{k} + \frac{k - m}{k} \cdot \frac{N(p - 1) + kp}{N(p - 1) + m(p - r) + kr} \cdot \frac{1}{r}.
\]
2nd step: Applying now Lemma B.1, we get
\[
\|\rho\|_{L^1(\mathbb{R}^N)} \leq C(N, p, k) \frac{(k-m)\|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}^r}{\|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}^{p-1} \|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}} \cdot \left\| \int_{\mathbb{R}^N} f(x, v) |v|^k \, dv \right\|_{L^{q/k}(\mathbb{R}^N)},
\]
which gives
\[
\left\| \int_{\mathbb{R}^N} f(x, v) |v|^m \, dv \right\|_{L^q(\mathbb{R}^N)} \leq C(N, p, k) \frac{(k-m)\|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}^r}{\|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}^{p-1} \|f\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}} \cdot \left\| \int_{\mathbb{R}^N} f(x, v) |v|^k \, dv \right\|_{L^{q/k}(\mathbb{R}^N)}.
\]
\[\Box\]

Remark: Of course, it is possible to use the method of Lemma B.1 to establish directly the result of Lemma B.2 (but it is more complicated and it does not improve the constants).

The interpolation method can also be used for any moment in \(|x - vt|\) (as in [36], [31] for \(|x - vt|^p\)). The integration with respect to \(v\) of \(f\) on \(|x - vt| < R\) instead of the integration of \(f\) on \(|v| < R\) provides an explicit dependance in \(t\) for the optimal \(R\), which gives a decay for the interpolated quantity. The result is (formally) easily recovered by the change of variables \(v \mapsto (x - vt)\).
Appendix C: Jeans’ theorem for stationary and time-periodic solutions

Consider first the case of the stationary Vlasov-Poisson system,

\[ v \cdot \partial_t f - (\partial_x U(x) + \partial_y U_0(x)) \cdot \partial_y f = 0 \]  
\[ -\Delta U = \rho(x) = \int f(x, v) \, dv, \]

and assume that \((f, U)\) does not depend on \(t\) and that \(U\) (and \(U_0\)) is radially symmetric, i.e. that there exists a function \(u : \mathbb{R}^+ \to \mathbb{R}\) such that

\[ U(x) = u(|x|) \quad \forall x \in \mathbb{R}^2. \] (S1)

Proposition C.1: Jeans’ theorem for stationary solutions

(Weak Formulation) Any \(C^1 \cap L^1\) stationary solution \((f, U)\) of (VP) satisfying (S1) is such that \((\tilde{f}, U)\) defined by \(\tilde{f}(x, v) = \int_{\mathbb{R}^2} f(|x| - v, v) \, dv\) for all \((x, v) \in \mathbb{R}^2 \times \mathbb{R}^2\) is also a solution of (VP), \(\tilde{f}\) satisfies locally the weak Ehlers & Riemann ansatz: for any \((x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}^2\), there exists a neighbourhood \(V\) of \((x_0, v_0)\) and a function \(h : \mathbb{R}^2 \to \mathbb{R}^+\) such that

\[ f(x, v) = h(E(x, v), F(x, v)) \quad \forall (x, v) \in V, \quad \text{(weak ER)} \]

and, if \(V = \mathbb{R}^3 \times \mathbb{R}^2\), with \(H\) defined as in section II.1.1 by

\[ H(U, x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\frac{\xi^2 + E^2}{2} + U, s_0|x|) \, ds_0 \, dx, \]

then

\[ -\frac{1}{r} \frac{d}{dr} (r \frac{du}{dr}) = H(u(r), r). \]

(Strong Formulation) Assume moreover that the following non-resonance condition is satisfied

\[ \text{mes} \{ (x, v) \in \text{supp}(f) : \frac{1}{2\pi} \Theta(E(x, v), F(x, v)) \in \mathbb{Q} \} = 0 \] (NR)

with

\[ \Theta(E, F) = 2 \int_{r_1(E, F)}^{r_2(E, F)} \int_{-r_1(E, F)}^{r_2(E, F)} \frac{dr}{\sqrt{2E - 2u(r) - 2u_0(r) - \frac{E^2}{2}}} \]

where \([r_1(E, F), r_2(E, F)]\) is the maximal interval such that

\[ \forall r \in [r_1(E, F), r_2(E, F)], \quad 2E - 2u(r) - 2u_0(r) - \frac{E^2}{2} \leq 0. \]

Then for \((x_0, v_0) \in \text{supp}(f)\) a.e., the support of the trajectory of \((x(t))_{t \in \mathbb{R}^+}\) defined by

\[ \frac{dx}{dt} = -\nabla_x (U(x) + U_0(x)) , \quad x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = v_0 \]
is dense (ergodicity property) in the annulus \(C(r_1, r_2)\) with \(r_i = r_i(E(x_0, v_0), F(x_0, v_0))\), \(i = 1, 2\). As a consequence, \(f(x, v) = \mathcal{F}(x, v)\) for any \((x, v)\) in a neighbourhood \(V\) in \(\mathbb{R}^2 \times \mathbb{R}^3\) of the orbit \(t \mapsto \{(x(t), \frac{dx}{dt}(t))\}\).

(Global result) If \(r \mapsto r^3(\frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta})\) is monotone increasing, then the above factorization result is global:

\[ f(x, v) = h(E(x, v), F(x, v)) \quad \forall (x, v) \in V = \mathbb{R}^2 \times \mathbb{R}^3, \quad \text{(weak ER)} \]

and \(f\) satisfies the global \((V = \mathbb{R}^2 \times \mathbb{R}^3)\) weak Ehlers & Rienstra ansatz.

We have to notice that \((\mathcal{F}, U)\) is a solution of the stationary Vlasov-Poisson system (with the same spatial density) as soon as \((f, U)\) is a solution such that \(U\) is radially symmetric.

For the proof of this proposition and for more realistic conditions on \(U\) than the \((NR)\) condition, one has to refer to [14,15]. The weak formulation of Jeans’ theorem was given in [8] (for the special class of distribution functions such that \(f = \mathcal{F}\) in the three-dimensional Vlasov-Poisson case – in that case, the result is automatically global). The difficult part of this theory is to check whether there are resonances or not. A classical result (see for example [2]) says that if bounded orbits exist and are all closed, then \(U + U_0\) is either keplerian or harmonic, which gives an explicit condition for resonances to appear. We can for example state the following result (see [14,15] for a proof):

**Proposition : Non Resonance Condition**

Assume that \(U\) is radially symmetric (i.e. satisfies condition \((S1)\)), that \(u\) is at least of class \(C^4\) on \([0, +\infty[\) and that \((E, F) \mapsto \Theta(E, F)\) is analytic. If \(U_0(x) = \Theta^\ast|x|^2\) and

\[ \text{mes}\{x \in \supp(f) : \nabla_x(\int_{\mathbb{R}^3} f(x, v) \, dv) = 0\} = 0, \]

then the non resonance condition \((NR)\) is satisfied.

We now state an equivalent (in the strong formulation) result for time-periodic solutions.

**Proposition C.2 : Jeans’ theorem for time-periodic solutions**

Assume now that \((f, U)\) is a \(C^1(\mathbb{R}; L^1(\mathbb{R}^2 \times \mathbb{R}^3)) \times C^1(\mathbb{R} \times \mathbb{R}^2)\) time-periodic explicitly time-dependant solution of \((VP)\) of period \(T\) satisfying the following symmetry assumption

\[ \forall x \in \bigcup_{t \in [0, T]} \text{supp}(f(t, \ldots)), \quad x \mapsto \int_0^T U(t, x) \, dt \text{ is radially symmetric.} \quad \text{(S2)} \]
With the same notations as in Proposition C.1, if \( U \) satisfies the following ergodicity property,

\[
\left( x(t + nT), \text{sgn} \left( \frac{dx}{d\tau} (t + nT) \right) \right)_{n \in \mathbb{N}} \text{ is dense in } C(\tilde{\tau}(t), \tilde{\tau}(t)) \times \{ \pm 1 \},
\]

with \( \tilde{\tau}_i = \tilde{\tau}_i(t, E(x_0, v_0), F(x_0, v_0)) \), \( i = 1, 2 \), then for almost all \((t, x, v) \in [0, T] \times \text{supp}(f)\) \( f \) satisfies the factorization identity:

\[
f(t, x, v) = g(t, E(t, x, v), F(x, v)).
\]

The factorization identity is not exactly the same as in the weak Ehlers & Rienstra ansatz. In part II, the study has been restricted to the subclass of \( g \) which do not depend on \( t \).

**Remark**: The method applies in the same way when one assumes that

\[
U_0(x) \sim \rho_0 \cdot (\theta_0 x_1^2 + (1 - \theta_0)x_2^2) \text{ as } |x| \to +\infty,
\]

for some \( \theta_0 \in [0, 1] \) (but one has then to make some essentially technical modifications on the assumptions on \( \rho \); see [8], [14] for similar problems in dimension 3). It also applies to the more realistic but also more technical case corresponding to the situation when \( \theta_0 \) depends on \( t \) (and is time-periodic of period \( T = \frac{\pi}{\omega} \)). The set of time-periodic solutions we have found can be extended to the case when the angular velocity is equal to zero (infinite period) and provides a new class of stationary solutions which seems not to be known up to now. These solutions are such that the asymptotic behaviour (for large \( |x| \)) of the potential is not isotropic. Among the distribution functions satisfying the Ehlers & Rienstra ansatz, and if we forget about the explicit dependence in \( t \) for the factorized distribution function \( g \), the set of stationary solutions (i.e. the set of time-dependent solutions of zero angular velocity) is much wider than the set of time-periodic solutions with strictly positive period, even in the limit when \( \omega \) goes to zero. It is not very difficult indeed to prove the existence of stationary solutions for large classes of \( \lambda \) that do not satisfy the (strong) Ehlers & Rienstra ansatz, in the same way as it has been done in [8] or [6] (see also references inside).
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NEW EXISTENCE RESULTS FOR
SINGULAR SOLUTIONS OF SOME
SEMILINEAR ELLIPTIC EQUATIONS

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Abstract

Let $\Omega$ be a smooth open bounded subset of $\mathbb{R}^n$ with $n \geq 4$. Given $\Sigma \subset \Omega$ any closed, smooth $k$-dimensional submanifold, $1 \leq k < n - 2$, with smooth boundary, we prove the existence of infinitely many positive weak solutions of

$$
\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

whose singular set is $\Sigma$, provided the exponent $p$ is larger than and close to $\frac{n-k}{n-2}$.

1 Introduction and the statement of results

The singular Yamabe problem can be stated as follows [SY]: Can one find a complete metric $\tilde{g}$ on the complement of a closed subset $\Sigma$ of a compact
Riemannian manifold \((M, g)\), which is pointwise conformally equivalent to the metric \(g\) and whose scalar curvature is constant?

If one looks for the metric \(\tilde{g} = u^{\frac{2}{n-2}} g\), where \(u\) is some positive function on \(M \setminus \Sigma\), then the problem reduces to the existence of positive solutions of the following semilinear elliptic equation

\[
\Delta_p u - \frac{n - 2}{4(n - 1)} R_g u + \frac{n - 2}{4(n - 1)} R_\tilde{g} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad M \setminus \Sigma,
\]

which blow up sufficiently fast at any point of \(\Sigma\) in order to ensure the completeness of the corresponding metric. In the above equation, \(R_g\) and \(R_\tilde{g}\) are the scalar curvature of the metric \(g\) and \(\tilde{g}\).

In the special case where \(M\) is the unit \(n\)-sphere with the canonical metric, one can use the conformal invariance of (2) to show that, when \(R_\tilde{g} \neq 0\), the study of this equation is equivalent to the study of one of the following semilinear elliptic equations

\[
\Delta u \pm u^p = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Sigma,
\]

with \(p = \frac{n+2}{n-2}\). Depending on the sign of the scalar curvature there are two different equations which we will refer to as the "positive" or "negative" case.

Motivated by this geometric considerations, the existence and properties of singular solutions of (3) have been extensively studied over the past years.

In the negative case, many existence results are known since the pioneer work of Loewner and Nirenberg [LN] where they have established the existence of solutions of \(\Delta u - u^p = 0\) provided \(\Sigma\) is a smooth submanifold of dimension \(k > k_0\) where \(k_0 = n - 2 - \frac{2}{p-1}\). McOwen has shown in [Mc] the existence of a solution to

\[
\begin{align*}
\Delta_p u &= u^p + Su & \text{on} \quad M \setminus \Sigma \\
\end{align*}
\]

when \(\Sigma\) is a smooth submanifold without boundary of dimension \(k > k_0\) and \(S\) is a continuous function on \(M\), which satisfies

\[
c_1 \text{dist}_g(x, \Sigma)^{-\frac{2}{p-1}} \leq u(x) \leq c_2 \text{dist}_g(x, \Sigma)^{-\frac{2}{p-1}}.
\]

Finn has established [Fn1] the existence of a solution of (4) which satisfies (5) when \(\Sigma\) is a smooth submanifold of dimension \(k > k_0\) with boundary. For recent progresses we refer to the work of Finn [Fn2] in which the author has recently narrowed the gap between the known necessary and sufficient conditions for the existence of solutions of (4). In this work, Finn shows the
existence of a solution of (4) when $\Sigma$ possesses at every point a regular tangent cone of dimension greater than $k_0$.

In the positive case, extending previous partial results of Mazzeo and Smale [MS], Pacard [P1], [P2], Rebai [R], Mazzeo and Pacard [MP] have proved the existence of infinitely many positive weak solutions of (1) which are singular on the union of finitely many disjoint smooth submanifolds without boundaries, provided the dimensions of the submanifolds belong to $(n - \frac{2p+1}{p-1}, n - \frac{2p}{p-1})$.

In the remaining of the paper, we will focus our attention on the positive case. In this framework and in the light of both the work of Mazzeo and Pacard and the work of Finn, one can ask the following question:

Do there exist solutions of (3), if the singular set $\Sigma$ has a boundary or is the union of non disjoint submanifolds?

In this paper we will partially answer this question. Our main result reads

**Theorem 1.** Let $\Omega$ be a regular open bounded subset in $\mathbb{R}^n$, with $n \geq 4$, and let $\Sigma$ a smooth $k$-dimensional submanifold of $\Omega$ with smooth boundary, $1 \leq k < n - 2$. For all $p$ greater than but close to $\frac{n-k}{n-k-2}$, there exists infinitely many positive weak solutions of

$$\Delta u + u^p = 0,$$

in $\Omega \setminus \Sigma$ with $u = 0$ on $\partial \Omega$, whose singular set is $\Sigma$. In addition, for any such solution $u$, there exists a constant $c > 0$ such that

$$\frac{1}{c} \text{dist}(x, \Sigma)^{-\frac{1}{n-1}} \leq u(x) \leq c \text{dist}(x, \Sigma)^{-\frac{1}{n-1}},$$

as $x$ tends to $\Sigma$.

Let us emphasize that, in the Theorem, the constant $c$ does depend on the solution $u$. We now further comment the restriction "$p$ close to $\frac{n-k}{n-k-2}$". This assumption is probably not necessary and might be removed by adapting the proof of Mazzeo and Pacard [MP], though this seems far from obvious since major technical difficulties have to be overcome. Moreover, with this assumption, one can avoid almost all of the technical difficulties and therefore, make the construction more transparent.

**Organization of the paper:** In Section 2 we recall some properties of radial singular solutions of (6). In Section 3, using the results from [MP] we define singular solutions of (6) whose singular set is a half line. In Section 4, we give the definition of appropriate spaces and describe the mapping properties of the Laplacian operator in these spaces, furthermore we construct approximate solutions and we obtain the existence of the desired solutions by using a fixed point argument.
2 Solution with one isolated singularity

In this section, we assume that \( N \geq 3 \) and we recall some well known results concerning the existence and properties of radial, positive, singular solutions of

\[
(7) \quad \Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

We will restrict our attention to the case where \( p > \frac{N}{N-2} \), since otherwise singular solutions do not extend to weak solutions in all \( \mathbb{R}^N \). In the remaining \( \tilde{r} \) will denote \(|\tilde{x}|\) where \( \tilde{x} = (x_1, x_2, \ldots, x_N) \). And through all the paper the constant \( c_p \) is defined to be

\[
c_p := \left( \frac{2}{p-1} \left( N - \frac{2p}{p-1} \right) \right)^{-\frac{1}{2}}.
\]

**Proposition 1.** For all \( p \in (\frac{N}{N-2}, \frac{N+2}{N-2}) \), there exists a unique positive singular radial solution \( u_0 \) of (7) such that

\[
u_0(\tilde{x}) = (c_p + o(1)) |\tilde{x}|^{-\frac{2}{p-1}},
\]

in the neighborhood of the origin and

\[
u_0(\tilde{x}) = (1 + o(1)) |\tilde{x}|^{2-N}
\]
as \(|\tilde{x}|\) tends to \(+\infty\).

**Proof:** The proof of this result is in [MP]. \(\square\)

We will also need the following estimates:

**Proposition 2.** There exists a constant \( c > 0 \) such that the function \( u_0 \) is bounded by \( c \tilde{r}^{-\frac{2}{p-1}} \) for all \( \tilde{r} \leq 1 \) and by \( c \tilde{r}^{2-N} \) for all \( \tilde{r} \geq 1 \). And there exists \( p^* \in (\frac{N}{N-2}, \frac{N+2}{N-2}) \) such that for all \( p \in (\frac{N}{N-2}, p^*) \)

\[
u_0(\tilde{r}) \leq c_p |\tilde{r}|^{-\frac{2}{p-1}},
\]

for all \( \tilde{r} \in (0, \infty) \). There exist \( \gamma_0 > 0 \) and a constant \( c > 0 \) such that the function

\[
\phi(\tilde{r}) = \frac{2}{p-1} u_0(\tilde{r}) - \tilde{r} \partial_\tilde{r} u_0(\tilde{r})
\]
is bounded (in absolute value) by \( c \tilde{r}^{-\frac{2}{p-1}+\gamma_0} \) for all \( \tilde{r} \leq 1 \) and by \( c \tilde{r}^{2-N} \) for all \( \tilde{r} \geq 1 \).

**Remark :** Let us notice that

\[
\phi(\tilde{r}) = -\partial_\xi \left( \xi^{-\frac{2}{p-1}} u_0(\tilde{r}/\xi) \right) \bigg|_{\xi=1}.
\]

**Proof :** The fact that \( u_0(\tilde{x}) \leq c_p |\tilde{x}|^{-\frac{2}{p-1}} \) when \( p \) is close to \( \frac{N}{N-2} \) is proven in the work of Chen and Lin [CL]. The bound for \( \phi \) is proven in [MP]. \(\square\)
3 Singular solutions with a half line as singular set

We still assume that \( p > \frac{N}{N-2} \). We now look for positive weak solutions of \( \Delta u + u^p = 0 \) in \( \mathbb{R}^{N+1} \) which are singular on a half line. To this aim, we look for our solution \( u \) in the form

\[
u(x) = |x|^{-\frac{2}{p-1}} w(\theta),\]

where \( \theta = x/|x| \in S^N \) and where \( w \) has an isolated singularity at \( \theta_0 \in S^N \). This special form of singular solutions already appears in the negative case, in the recent work of Finn [Fn2]. If one can find a solution of this form, then the singular set of \( u \) will be given by \( D_0 = \{ r\theta_0 : r \geq 0 \} \subset \mathbb{R}^{N+1} \). A simple computation shows that the equation

\[
\Delta u + u^p = 0, \quad \text{in} \quad \mathbb{R}^{N+1} \setminus D_0,
\]

reduces to

\[
\Delta_{S^N} w - \frac{2}{p-1} \left( N - \frac{p+1}{p-1} \right) w + w^p = 0 \quad \text{in} \quad S^N \setminus \{ \theta_0 \}.
\]

Up to a rigid motion, we can assume that \( \theta_0 \) is the north pole of \( S^N \) and that, the upper hemisphere of \( S^N \) is parameterized near \( \theta_0 \) by

\[
\bar{x} \in \mathbb{R}^N \longrightarrow (\bar{x}, (1 - |\bar{x}|^2)^{1/2}) \in S^N.
\]

In the remaining, \( \pi \) will denote the orthogonal projection from the upper hemisphere of \( S^N \) over \( \mathbb{R}^N \setminus \{0\} \), namely \( \pi(\theta) = \bar{x} \) if \( \theta = (\bar{x}, (1 - |\bar{x}|^2)^{1/2}) \in S^N \).

We now define the weighted spaces we will work with in this section

**Definition 1.** Given \( \nu \in \mathbb{R} \), we define \( C^0_\nu(S^N \setminus \{\theta_0\}) \) to be the set of all \( w \in C^0(S^N \setminus \{\theta_0\}) \) for which the norm

\[
\|w\|_\nu \equiv \sup_{\theta \in S^N \setminus \{\theta_0\}} \text{dist}(\theta, \theta_0)^{-\nu} |w(\theta)|,
\]

is finite.

We should keep in mind that, for all \( w \in C^0_\nu(S^N \setminus \{\theta_0\}) \) there exists a constant \( c > 0 \) such that \( |w(\theta)| \leq c|\pi(\theta)|^\nu \) near \( \theta_0 \). Now we set \( \bar{c}_p = \frac{2}{p-1} \left( N - \frac{p+1}{p-1} \right) \) and we investigate the mapping properties of the operator \( \Delta_{S^N} - \bar{c}_p \), when acting on the above defined weighted spaces.

**Proposition 3.** Assume that \( 2 - N < \nu < 0 \) and that \( p > \frac{N+1}{N-2} \). Then, for all
$f \in C^0_{\nu-2}(S^N \setminus \{\theta_0\})$, there exists a unique $w \in C^0_\nu(S^N \setminus \{\theta_0\})$ solution of

$$\Delta_{S^N} w - \bar{c}_p w = f \quad \text{in} \quad S^N \setminus \{\theta_0\}.$$  

In addition, the linear operator

$$G_p : f \in C^0_{\nu-2}(S^N \setminus \{\theta_0\}) \rightarrow w \in C^0_\nu(S^N \setminus \{\theta_0\}),$$

is bounded independently of $p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$.

**Proof:** We will denote for short $\rho(\theta) = \text{dist}(\theta, \theta_0)$. To begin with, let us remark that in the coordinates defined above, the operator $\Delta_{S^N}$ has the following form

$$\Delta_{S^N} w = \Delta_{R^N} w + O(|x|^4 \nabla^2 w) + O(|x| \nabla w),$$

in the neighborhood of $\theta_0$. For a proof of this expansion, we refer to [FM]. Hence, we obtain

$$\Delta_{S^N} |x|^\nu = \nu (\nu + N - 2) |x|^{\nu-2} + O(|x|^\nu).$$

Since $\nu \in (2-N, 0)$, there exists $\delta > 0$ such that, $\nu(\nu+N-2)|x|^{\nu-2}+O(|x|^\nu) \leq 0$ if $\rho(\theta) \leq \delta$. Thus we obtain $\Delta_{S^N} |x|^\nu - \bar{c}_p |x|^\nu \leq -\bar{c}_p |x|^{\nu-2}$, as long as $\rho(\theta) \leq \delta$. Let $f$ be a function in $C^0_{\nu-2}(S^N)$. Since $\nu > 2 - N$, we see that $f \in L^{1+\alpha}(S^N)$ for a small $\alpha > 0$, in addition, the constant $\bar{c}_p > 0$ since we have assumed that $p > \frac{N+1}{N-1}$. In particular, we know that there exists $w \in W^{1,1+\alpha}(S^N)$, solution of (10)?]. And using Green’s formula we get the pointwise estimate

$$|w(\theta)| \leq c \|f\|_{\nu-2} \text{ provided } \rho(\theta) \geq \delta.$$

Using the preliminary computation and the fact that the operator $\Delta_{S^N} - \bar{c}_p$ satisfies the maximum principle, we see that, $|x|^\nu$ multiplied by some suitable constant will be a supersolution for (10) when $\rho(\theta) \leq \delta$. Similarly, $-|x|^\nu$ multiplied by some suitable constant will be a subsolution for (10) in the same set. Therefore we conclude that $|w(\theta)| \leq c \|f\|_{\nu-2} \rho(\theta)^\nu$, for some constant independent of $f$. Which is the desired estimate. Finally, the fact that the operator $G_p$ is uniformly bounded as the parameter $p$ is chosen in $(\frac{N}{N-2}, \frac{N+2}{N-2})$ follows at once from the observation that, under such an assumption, the constant $\bar{c}_p$ remains bounded away from 0. Thus, all constants are bounded independently of $p$ chosen in this interval.

We are now in a position to prove the main result of this section:

**Theorem 2.** Assume that $\nu \in (2-N, \inf(0, 4-N))$. Then, there exists $\bar{p} \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ such that for all $p \in (\frac{N}{N-2}, \bar{p})$, there exist $\varepsilon_0 > 0$, $\delta_1 > 0$, $(w_k)_{k \in \mathbb{N}}$ a one parameter family of solutions of (9) and for all $k \geq 0$, there exists some constant $\alpha_k > 0$ independent of $\varepsilon$ such that
\[
|\nabla^k (u_\varepsilon(\theta) - e^{-\frac{2}{p-1} \varepsilon u_0(\pi(\theta)/\varepsilon))} | \leq c_k \varepsilon^n \rho(\theta)^{\nu-k} \quad \text{if } \rho(\theta) \leq \delta_1
\]
\[
|\nabla^k u_\varepsilon(\theta) | \leq c_k \varepsilon^n \quad \text{if } \rho(\theta) > \delta_1,
\]
where \( \gamma_1 = \inf(2 - \nu - \frac{2}{p-1}, N - \frac{2p}{p-1}) > 0. \)

**Proof:** The existence part of the Theorem follows for example from [MP]. But since the estimates (12) which are not given by the result of [MP], are a key point of our proof of Theorem 1, we felt the necessity to give a short proof of the Theorem 2. The idea is first to construct an approximate solution of (9) and then to perturb it.

To construct an approximate solution of (9), we proceed as follows. Let \( u_0 \) be the function defined in Proposition 1. Since the (7) is invariant under dilation, for all \( \varepsilon > 0 \) the function \( u_\varepsilon \) defined by

\begin{equation}
(13) \quad u_\varepsilon(\varepsilon) \equiv e^{-\frac{2}{p-1} \varepsilon u_0(\varepsilon)/\varepsilon),
\end{equation}

is also a solution of (7). Now, fix a parameter \( \delta > 0 \) small enough and choose a cut-off function \( \eta \), defined on \( S^N \), such that \( \eta(\theta) = 0 \) if \( \rho(\theta) \geq 2\delta \) and \( \eta(\theta) = 1 \) if \( \rho(\theta) \leq \delta \). Then, we define a one parameter family of approximate solutions by \( \tilde{w}_\varepsilon(\theta) \equiv \eta(\theta) u_\varepsilon(\pi(\theta)) \) and we set \( \tilde{f}_\varepsilon \equiv \Delta_S \tilde{w}_\varepsilon - \tilde{c}_p \tilde{w}_\varepsilon + \tilde{w}_\varepsilon^p \). We make use of the expansions given in Proposition 1 as well as of the one in (11) in order to conclude that

\[
\left\{ \begin{array}{l}
|\tilde{f}_\varepsilon| \leq c \rho(\theta)^{-\frac{2}{p-1}} \varepsilon^{N-\frac{2N}{p-1}} \varepsilon^{2-N} \quad \text{if } \rho(\theta) \leq \varepsilon \\
|\tilde{f}_\varepsilon| \leq c \varepsilon^{N-\frac{2N}{p-1}} \varepsilon^{2-N} \quad \text{if } \varepsilon \leq \rho(\theta) \leq 2\delta.
\end{array} \right.
\]

Now fix \( \nu \in (2 - N, \inf(0, 4 - N)) \), since \( \nu \leq 2 - \frac{2}{p-1} \), we can conclude that there exists a constant \( c > 0 \) such that, for all \( \varepsilon \in (0, 1) \), we have

\[
\|\tilde{f}_\varepsilon\|_{L^2} \leq c \varepsilon^n, \quad \text{where } \gamma_1 \equiv \inf(2 - \nu - \frac{2}{p-1}, N - \frac{2p}{p-1}) > 0.
\]

Now, using a perturbation argument, we will construct an exact solution of (9) which is close to \( \tilde{w}_\varepsilon \).

We have to find a function \( v \) such that \( w = \tilde{w}_\varepsilon + v \) satisfies

\begin{equation}
(14) \quad \Delta_S (\tilde{w}_\varepsilon + v) - \tilde{c}_p (\tilde{w}_\varepsilon + v) + |\tilde{w}_\varepsilon + v|^p = 0,
\end{equation}

in \( S^N \setminus \{\theta_0\} \). Let us rewrite this equation as

\[
\Delta_S v = - (q_\varepsilon(v) + \tilde{f}_\varepsilon),
\]

where \( \tilde{f}_\varepsilon \) is the error term, which has already been defined, and where \( q_\varepsilon(v) \equiv \)
In order to solve \((14)\), it is enough to find a fixed point for the nonlinear mapping

\[
N_\epsilon(v) \equiv -G_p \left( q_\epsilon(v) + f_\epsilon \right),
\]

in the space \(C_0^0(S^N \setminus \{\theta_0\})\).

Let us now fix \(\varepsilon > 0\) and, for all \(\kappa > 0\), we define

\[
B_\kappa = \{ v \in C_0^0(S^N \setminus \{\theta_0\}) : \|v\|_\nu \leq \kappa \varepsilon\gamma_1 \},
\]

where the constant \(\gamma_1\) is the one which has been defined before.

Now, Taylor's formula yields

\[
q_\epsilon(v_2) - q_\epsilon(v_1) = p(v_2 - v_1) \int_0^1 |\tilde{u}_\epsilon + sv_2 + (1 - s)v_1|^{p-2} (\tilde{u}_\epsilon + sv_2 + (1 - s)v_1) \, ds.
\]

In particular, there exists some constant \(c > 0\) such that

\[
|q_\epsilon(v_2) - q_\epsilon(v_1)| \leq c|v_2 - v_1|( |\tilde{u}_\epsilon|^{p-1} + |v_2|^{p-1} + |v_1|^{p-1}) \, ds.
\]

This is at this stage that the assumption \(p\) close to \(\frac{N}{p-2}\), which we have not used so far, simplifies the proof of \([MP]\). Indeed, \(\nu\) being fixed greater than 2 \(- N\), we have \(\nu > \frac{N}{p-1}\) for \(p\) close enough to \(\frac{N}{p-2}\). Say \(p \leq \bar{p}\). The fact that \(\nu > \frac{N}{p-1}\) will be used in many estimates. For example we get readily the following estimate

\[
\sup_{\rho \leq \varepsilon \rho^2 - \nu} |v_2 - v_1| v_1^{\nu} \leq \varepsilon(\nu - 1) \sup_{\rho \leq \varepsilon \rho^2 \nu - 1} |v_2 - v_1| = \varepsilon\nu - 1
\]

since \(\nu(p - 1) > -2\). Therefore, using \((8)\) in proposition \(2\) we get

\[(15) \quad \|q_\epsilon(v_2) - q_\epsilon(v_1)\|_{p-2} \leq c\|v_2 - v_1\|_\nu (c_{\bar{p}}^{p-1} + \varepsilon\gamma_1(\nu - 1)).\]

Let us emphasize that at this stage that we need the operator \(G_p\) to be bounded uniformly on \(p\) but may be depending on \(\nu\). This is why, instead of making the assumption \(\nu > \frac{N}{p-1}\), we have fixed \(\nu\) in the interval \((2 - N, \inf(0, 4 - N))\).

Now let's \(C_1\) be an upper bound of \(G_p\) for \(p \in (\frac{N}{p-2}, \frac{N}{p-1})\), then we get from \((15)\)

\[
\|N_\epsilon(v_2) - N_\epsilon(v_1)\|_{\nu} \leq C_1\|v_2 - v_1\|_\nu (c_{\bar{p}}^{p-1} + \varepsilon\gamma_1(\nu - 1)).
\]

Notice that \(c_{\bar{p}}\) tends to 0 as \(p\) tends to \(\frac{N}{p-2}\). Thus for \(p\) close enough to \(\frac{N}{p-2}\), say \(p \in (\frac{N}{p-2}, \bar{p})\) there exists \(\alpha_0 > 0\) and \(\varepsilon_0 > 0\) such that for all \(\varepsilon \in (0, \varepsilon_0]\), \(N_\epsilon\) is a contraction mapping from the ball \(B_{\alpha_0}\) into itself and therefore, \(N_\epsilon\) has a unique fixed point \(v_\epsilon\) in this set.
The fact that \( w = \tilde{w} + v \) is then a weak solution of (9) is standard (see for example the end of section 4.4).

This ends the proof of the existence of \( w \) as well as (12) for \( k = 0 \). The estimate of (12) for \( k \neq 0 \) then follows from Schauder’s estimates.

As an immediate corollary, we get

**Corollary 1.** There exist \( \varepsilon_0 > 0 \) and a one parameter family \((\tilde{u}_\varepsilon)_{\varepsilon \leq \varepsilon_0}\) of solutions of

\[
\Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^{N+1} \setminus D_0,
\]

which is of the form

\[
\tilde{u}_\varepsilon(x) = |x|^{-2} w_\varepsilon(\theta),
\]

where \( w_\varepsilon \) is the one parameter family of solutions given in Theorem 2.

4 **Singular solutions with a submanifold with boundary as singular set**

Given our preliminary results, we are now in position to give the proof of Theorem 1. We will pay special attention to the novel features and to the most important points of the proof: the derivation of the estimates which ensure that the approximate solution is good enough to be perturbed into a solution of our problem and the mapping properties of the Laplacian when defined between suitably chosen weighted spaces. These are the main contributions of our paper.

4.1 **Notations and different coordinate systems**

We set \( n = N + k \).

From now on, \( \Sigma \) is assumed to be a closed smooth submanifold of \( \Omega \subset \mathbb{R}^n \) with smooth boundary \( \partial \Sigma \). It is always possible to find \( \tilde{\Sigma} \) an open (at least \( C^4 \)) smooth submanifold of \( \Omega \subset \mathbb{R}^n \) which is an extension of \( \Sigma \) in the sense that the closure of \( \Sigma \) is included in \( \tilde{\Sigma} \).

For all \( \delta > 0 \) small enough, \( T_\delta \) will denote a tubular neighborhood of radius \( \delta \) around \( \Sigma \) and \( \tilde{T}_\delta \) a tubular neighborhood of radius \( \delta \) around \( \tilde{\Sigma} \). It is easy to see that, provided \( \delta > 0 \) is small enough, the set \( T_\delta \) can be decomposed into two subsets, one of which is a disk bundle over \( \Sigma \) and will be denoted by \( T_\delta^0 \), the other being a half of a tubular neighborhood of radius \( \delta \) around \( \partial \Sigma \) and will be denoted \( B_\delta \). A similar decomposition holds for \( \tilde{T}_\delta \). We will finally assume that \( \delta \) is chosen small enough so that \( T_\delta \subset \tilde{T}_\delta^0 \).
In $\mathcal{T}_\delta^0 \setminus \Sigma$ we define Fermi coordinates $(\rho, \theta, \tilde{y}) \in (0, +\infty) \times S^{N-1} \times \tilde{\Sigma}$ where $\rho$ is the distance to $\tilde{\Sigma}$, $\tilde{y}$ is the coordinate on $\tilde{\Sigma}$ and $\theta$ is the angular coordinate in $N_\delta \tilde{\Sigma}$ the normal space to $\tilde{\Sigma}$ at $\tilde{y}$. Reducing $\delta > 0$ further if this is necessary, we may always assume that $\rho$, the distance to $\tilde{\Sigma}$ is a $C^3$ function in $\mathcal{T}_\delta^0$.

Finally, in $B_\delta \setminus \tilde{\Sigma}$, we choose Fermi coordinates $(r, \theta, z) \in (0, +\infty) \times S^N \times \partial \Sigma$.

To summarize, any point $x \in \mathcal{T}_\delta^0 \setminus \tilde{\Sigma}$ can be represented by its Fermi coordinates $(\rho, \theta, \tilde{y}) \in (0, +\infty) \times S^{N-1} \times \tilde{\Sigma}$. And any point in $B_\delta$ can be represented either by the coordinates $(r, \theta, z) \in (0, +\infty) \times S^N \times \partial \Sigma$ or by the coordinates $(\rho, \theta, \tilde{y}) \in (0, +\infty) \times S^{N-1} \times \tilde{\Sigma}$, if it does not belong to $\tilde{\Sigma}$.

We define
\[
dc(x) = \text{dist}(x, \Sigma),
\]

### 4.2 Mapping properties of the Laplacian

Granted these definitions, we now define the weighted spaces we will work with, in this section.

**Definition 2.** Assume that $\mu \in \mathbb{R}$. Let $\mathcal{D}_\mu^0(\Omega \setminus \Sigma)$ be the set of all $w \in C^0(\Omega \setminus \Sigma)$ for which the norm
\[
|w|_\mu \equiv \sup_{x \in \Omega \setminus \Sigma} d_\Sigma(x)^{-\mu} |w(x)|,
\]
is finite.

In the following Proposition we study the mapping properties of $\Delta$ when defined in the weighted space $\mathcal{D}_\mu^0(\Omega \setminus \Sigma)$.

**Proposition 4.** Assume that $\mu \in (2 - N, 0)$. Then for all $f \in \mathcal{D}_\mu^{-2}(\Omega \setminus \Sigma)$ there exists a unique $w \in \mathcal{D}_\mu^0(\Omega \setminus \Sigma)$ solution of
\[
\begin{cases}
\Delta w = f & \text{in } \Omega \setminus \Sigma \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
In addition, the linear operator
\[
G : f \in \mathcal{D}_\mu^{-2}(\Omega \setminus \Sigma) \longrightarrow w \in \mathcal{D}_\mu^0(\Omega \setminus \Sigma)
\]
is bounded.

**Proof:** To begin with, we use a result of Finn and McOwen [FM] which states that, in term of the Fermi coordinates $(\rho, \theta, \tilde{y})$ which have been introduced above, the Laplacian in $\mathcal{T}_\delta^0$ can be locally written in terms of the Laplacians on $\tilde{\Sigma}$ and on the normal space $N_\delta \tilde{\Sigma}$ (which will be identified with $\mathbb{R}^N$). Namely
\[
\Delta u = \Delta_{\mathbb{R}^N} u + \Delta_{\tilde{\Sigma}} u + e_1 \nabla^2 u + e_2 \cdot \nabla u,
\]
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with

\[ |e_1| + |e_2| \leq c. \]

Using this, it is now a simple exercise to compute

\[ \Delta p^\mu = \mu(N - 2 + \mu)p^\mu - 2 + e_1 \cdot \nabla^2 p^\mu + e_2 \cdot \nabla p^\mu. \]

Therefore, reducing \( \delta \) if this is necessary, we can state that

\[ \Delta p^\mu \leq \frac{\mu(N - 2 + \mu)}{2} p^\mu - 2, \]

in \( T_\delta \).

Similarly, in terms of the Fermi coordinates \((r, \theta, z)\) which have been introduced above, the Laplacian in \( B_\delta \) can be locally written in terms of the Laplacians on \( \partial \Sigma \) and on the normal space \( N, \partial \Sigma \) (which will be identified with \( \mathbb{R}^{N+1} \)). Namely

(20) \[ \Delta u = \Delta_{\mathbb{R}^{N+1}} u + \Delta_{\partial \Sigma} u + \tilde{e}_1 \cdot \nabla^2 u + \tilde{e}_2 \cdot \nabla u, \]

with

\[ \sup_{B_\delta \cap \partial \Sigma} |r^{-1} \tilde{e}_1| + \sup_{B_\delta \cap \partial \Sigma} |\tilde{e}_2| \leq c. \]

Thus we find

\[ \Delta \tau^\mu = \mu(N - 1 + \mu) \tau^\mu - 2 + O(\tau^{\mu - 1}), \]

in \( B_\delta \). Again, reducing \( \delta \) if this is necessary, we can state that

\[ \Delta \tau^\mu \leq \frac{\mu(N - 1 + \mu)}{2} \tau^\mu - 2, \]

in \( B_\delta \).

Now, let \( f \in D^-_{\mu - 2}(\Omega \setminus \Sigma) \) be given. Since we have assumed that \( \mu > 2 - N \), it is easy to see that \( f \) belongs to \( L^1(\Omega) \). Moreover, there exists \( c > 0 \) such that

\[ \|f\|_{L^1(\Omega)} \leq c |f|_{\mu - 2}. \]

In particular, this already guarantees the existence of \( w \in \cap_{\mu > 1, \nu > (n - 1)} W^{1, \nu}(\Omega) \) solution of (18). Moreover, Green's representation formula (see for example [GT]) yields the existence of a constant \( c > 0 \) independent of \( f \) such that

(21) \[ |w(x)| \leq c |f|_{\mu - 2}, \]

for all \( x \in \Omega \) such that \( d_\Sigma(x) \geq \delta \). In fact, we have that
where $w_n$ is the volume of the unit n-sphere. If $x_0 \in \Omega \setminus \tilde{T}_s^0$, the above estimate allows us to conclude that

$$|w(x)| \leq \frac{1}{n(n - 2)w_n} \int_{\Omega} |x - x_0|^{2-n} |f| \, dx,$$

In particular $|w(x)|$ is bounded by a constant times $|f|_{\mu-2}$ on $\partial \tilde{T}_s^0$.

Using the preliminary computation and the fact that the operator $\Delta$ satisfies the maximum principle, we see that $v = c_\mu |f|_{\mu-2}^\rho$ with $c_\mu = \frac{2}{\mu(n - 2 + \rho)} > 0$ is a supersolution for (18) in $\tilde{T}_s^0$. Similarly, $-v$ is a subsolution for (18) in the same set. Therefore we conclude that $|w(x)| \leq c_\mu |f|_{\mu-2}^\rho$ in $\tilde{T}_s^0$. Combining this together with the first estimate, we obtain

$$|w(x)| \leq c |f|_{\mu-2}^\rho,$$

for some constant $c$ independent of $f$ and for all $x \in \Omega \setminus B_s$. Which, in particular implies that

$$|w(x)| \leq c |f|_{\mu-2}^\rho,$$

on $\partial B_s$.

Finally, in $B_s$ we use $r^\mu$ multiplied by some suitable constant, more exactly $	ilde{v} = \tilde{c}_\mu |f|_{\mu-2}r^\mu$ as a supersolution for (18) and $-\tilde{v}$ as a subsolution to conclude that

$$|w(x)| \leq \tilde{c}_\mu |f|_{\mu-2} r^\mu$$

in $B_s$. Here $\tilde{c}_\mu = \max(\frac{2}{\mu(n - 1 + \rho)}, c) > 0$. The proof of the result is therefore complete.

4.3 Construction of the approximate solutions

In this section, we construct a sequence of approximate solutions $S_t$ and derive some important estimates. To this aim, we will use both $u_t$ the radial singular solution of $\Delta \mathbf{R} u + u^p = 0$ in $\mathbf{R}^N \setminus \{0\}$ which has been defined in (13) and $u_x$ the sequence of solutions of $\Delta \mathbf{R} u + u^p = 0$ in $\mathbf{R}^{N+1} \setminus D_0$ which has been defined in Corollary 1.

Throughout this section, $\eta$ will denote a smooth cutoff function defined in $\mathbf{R}$, such that
\[ \eta(t) = 1 \quad \text{if} \quad t \leq 1/2, \]

and

\[ \eta(t) = 0 \quad \text{if} \quad t \geq 1. \]

We will also need a smooth positive function \( \alpha : \Sigma \rightarrow \mathbb{R} \) which is equal to the geodesic distance from \( \tilde{y} \in \Sigma \) to \( \partial \Sigma \) in a tubular neighborhood of radius \( 2\delta \) around \( \partial \Sigma \) and which is greater than \( \delta \) away from this neighborhood. This function \( \alpha \) can be extended to \( \tilde{\Sigma} \) by setting for all \( \tilde{y} \in \tilde{\Sigma} \setminus \Sigma \), \( \alpha \) to be equal to minus the geodesic distance from \( \tilde{y} \) to \( \partial \Sigma \).

If \((\tilde{r}, \tilde{\theta}, \tilde{y}) \in (0, \infty) \times S^{N-1} \times \Sigma\) denote the Fermi coordinates of a point \( x \) in \( T^0_d \), we define

\[ S^0_{\tilde{r}}(x) = \eta(\tilde{r}/\delta) \, u_{\alpha(\tilde{y})}(\tilde{r}). \]

If \((\tilde{r}, \tilde{\theta}, \tilde{y}) \in (0, \infty) \times S^{N-1} \times \tilde{\Sigma}\) and \((r, \theta, x) \in (0, \infty) \times S^N \times \partial \Sigma\) are the coordinates of a point \( x \in B_d \), we define

\[ S^1_{\tilde{r}}(x) = \eta(r/\delta) \, u_\alpha(\tilde{x}, \alpha(\tilde{y})), \]

where by definition \( \tilde{x} = \tilde{r} \tilde{\theta} \in \mathbb{R}^N \).

Then, the one parameter family of approximate solutions is defined by

\[ S_y(x) = \eta(\alpha(\tilde{y})/\delta) \, S^1_{\tilde{x}}(x) + (1 - \eta(\alpha(\tilde{y})/\delta)) \, S^0_{\tilde{r}}(x), \]

if \( x \in T^0_d \) and \( S_y(x) = 0 \) elsewhere.

The remaining of the section is devoted to the careful derivation of estimates for \( \Delta S_y + S^0_{\tilde{r}} \mu \) with which we will use the inequality \( \nu > \frac{2}{p-1} \) without explicit mention. For example in II-2-3 and in II-3 specially in the estimate of (III) and (IV).

**Proposition 5.** Assume that \( \mu < \inf(\frac{2-\nu}{p-1}, \nu) \). Then there exists a constant \( c > 0 \) such that

\[ |\Delta S_y + S^0_{\tilde{r}}|_{\mu=2} \leq c \, \varepsilon^{\gamma_2} \]

with \( \gamma_2 = \inf(\gamma_1, \frac{2-\nu}{p-1} - \mu) \).

Let us before giving the proof, comment our choice of the parameter \( \mu \). The main reason for choosing an other parameter than \( \nu \), say \( \mu \) is that on one hand we need that \( \frac{2-\nu}{p-1} - \mu > 0 \) in the estimate of the error term \( |\Delta S_y + S^0_{\tilde{r}}|_{\mu=2} \) but on an other hand the parameter \( \nu \) was be chosen, as we have seen, independent of \( p \).
Proof: Let \( f_x = \Delta S_x + S_x^T \). Since \( S_x \equiv 0 \) outside \( T_x \), we may assume from now on that \( x \in T_x \).

Case I - First of all, let us assume that \( \delta/2 \leq d_{\Sigma}(x) \leq \delta \). In this case, it follows directly from the second estimate of (12) as well as from Proposition 1, which provides the asymptotic behavior of \( u_0 \) near \( \infty \), that

\[
|\Delta S_x + S_x^T| \leq c(\varepsilon^{7/4} + \varepsilon^{N-2\varepsilon_1}).
\]

Therefore, we always have

\[
(d_{\Sigma})^{2\mu} |\Delta S_x + S_x^T| \leq c \varepsilon^{7/4},
\]

when \( \delta/2 \leq d_{\Sigma}(x) \leq \delta \) and where \( c \) is some constant depending only on \( \delta \).

Case II - Next, we assume that \( d_{\Sigma}(x) \leq \delta/2 \). In this case let us further distinguish three subcases, according to the value of \( \alpha(\hat{y}) \).

Case II.1 - Assume, in addition that \( \alpha(\hat{y}) > \delta \). In this case, we simply have \( S_x(x) = S_x^T(x) = (\varepsilon \alpha(\hat{y}))^{-1/2} u_0(\hat{f}/(\varepsilon \alpha(\hat{y}))) \) and given (19) we find that

\[
\Delta S_x + S_x^T = e_1 \cdot \nabla^2 S_x + e_2 \cdot \nabla S_x + \Delta S_x.
\]

Using Proposition 2, we find provided \( d_{\Sigma}(x) \leq \varepsilon \),

\[
|e_2 \cdot \nabla S_x| + |e_1 \cdot \nabla^2 S_x| \leq c \varepsilon^{7/4},
\]

and

\[
|\Delta S_x| \leq c \varepsilon^{-2\varepsilon_1}.
\]

Still using Proposition 2, we find, provided \( \varepsilon \leq d_{\Sigma}(x) \leq \delta/2 \)

\[
|e_2 \cdot \nabla S_x| + |e_1 \cdot \nabla^2 S_x| \leq c \varepsilon^{N-2\varepsilon_1} r^{1-N},
\]

and

\[
|\Delta S_x| \leq c \varepsilon^{N-2\varepsilon_1} r^{2-N}.
\]

In fact, \( |S_x^T| \leq c \varepsilon^{N-2\varepsilon_1} r^{2-N} \alpha(\hat{y})^{N-2\varepsilon_1} \) with \( \alpha(\hat{y}) > \delta \). Therefore, we may conclude that

\[
(d_{\Sigma})^{2\mu} |\Delta S_x + S_x^T| \leq c \sup_{r(|\alpha(\hat{y})| \geq \delta)} \varepsilon^{N-2\varepsilon_1} \varepsilon^{2-N} \leq c \varepsilon^{7/4},
\]

provided \( d_{\Sigma}(x) \leq \delta/2 \) and \( \alpha(\hat{y}) \geq \delta \).

Case II.2 - Assume, in addition that \( \alpha(\hat{y}) \leq \delta/2 \). In this case \( S_x(x) = S_x^T(x) = u_0(x, \alpha(\hat{y})) \). Since, \( \alpha(\hat{y}) \) is a geodesic distance from \( \hat{y} \) to \( \partial \Sigma \), we see that
\[ \Delta \zeta(\zeta, \alpha(\zeta)) = (\partial_{x_N^*}, \zeta)(\zeta, \alpha(\zeta)). \]

Next we use the fact that the function \( \zeta \) is an exact solution of \( \Delta_{\mathbb{R}^N} u + u^p = 0 \), in \( \mathbb{R}^N \), which, together with (19) implies that
\[ \Delta \zeta + S_{\zeta} = e_1 \cdot \nabla^2 \zeta + e_2 \cdot \nabla \zeta. \]

In order to estimate properly this expression, three subcases have to be distinguished. For convenience, we define
\[ \tilde{\zeta} = (\zeta^2 + \alpha^2(\zeta))^{1/2}. \]

With this definition, we see that
\[ \tilde{\zeta}(\zeta, \alpha(\zeta)) = \tilde{\zeta}(\zeta, \alpha(\zeta))/\tilde{\zeta}(\zeta, \alpha(\zeta)). \]

**Case II.2.1** - If \( \alpha(\zeta) < 0 \) or if \( \tilde{\zeta} \geq \delta_1 \), where \( \delta_1 \) is the constant given in Theorem 2, then we obtain from the second estimate in (12)
\[ |e_1 \cdot \nabla^2 \zeta| + |e_2 \cdot \nabla \zeta| \leq c \varepsilon \tilde{\zeta}^{\frac{N+1}{2}} - 1, \]
and therefore,
\[ (d_2)^{2-\mu} \Delta \zeta + S_{\zeta} \leq c \varepsilon^{\frac{1}{2}}. \]

**Case II.2.2** - If \( \varepsilon \leq \tilde{\zeta} \leq \delta_1 \). Then we obtain from the first estimate in (12)
\[ |e_1 \cdot \nabla^2 \zeta| + |e_2 \cdot \nabla \zeta| \leq c \varepsilon^{\frac{N+1}{2}} \tilde{\zeta}^{\frac{1}{2}} - 1 \left( e^{-\tilde{\zeta}^2} (\tilde{\zeta}/\varepsilon) + \varepsilon (\tilde{\zeta}/\varepsilon) \right), \]
and hence
\[ (d_2)^{2-\mu} \Delta \zeta + S_{\zeta} \leq c \varepsilon^{\frac{N+1}{2}} + \varepsilon^{\frac{N-2\delta_1}{2}}. \]

**Case II.2.3** - If \( \tilde{\zeta} \leq \varepsilon \). Then we obtain from the first estimate in (12)
\[ |e_1 \cdot \nabla^2 \zeta| + |e_2 \cdot \nabla \zeta| \leq c \varepsilon^{\frac{N+1}{2}} \left( \tilde{\zeta}^{-\frac{1}{2}} + \varepsilon (\tilde{\zeta}/\varepsilon) \right)^{\frac{1}{2}}. \]

Therefore, we obtain
\[ (d_2)^{2-\mu} \Delta \zeta + S_{\zeta} \leq c \varepsilon^{\frac{N+1}{2}} - \mu, \]
in this case.

Collecting (25), (26) and (27), we conclude that
\[ (d_2)^{2-\mu} \Delta \zeta + S_{\zeta} \leq c \varepsilon^{\frac{N+1}{2}} + \mu, \]
provided \( d_E(x) \leq \delta/2 \) and \( \alpha(y) \leq \delta/2 \).

**Case 11.3** - Let us finally assume that \( \alpha(y) \in (\delta/2, \delta) \). In this case \( S_\varepsilon = \eta S_\varepsilon^1 + (1 - \eta) S_\varepsilon^0 \). We may now compute

\[
\Delta S_\varepsilon + S_\varepsilon^T = e_1 \cdot \nabla^2 S_\varepsilon + e_2 \cdot \nabla S_\varepsilon 
+ (1 - \eta) \Delta E S_\varepsilon^0
+ 2 \nabla_\varepsilon \eta \nabla_\varepsilon (S_\varepsilon^1 - S_\varepsilon^0) + \Delta E \eta (S_\varepsilon^1 - S_\varepsilon^0)
- \eta (S_\varepsilon^1)^p - (1 - \eta) (S_\varepsilon^0)^p + (\eta S_\varepsilon^1 + (1 - \eta) S_\varepsilon^0)^p. 
\]

(1)

(II)

(III)

(IV)

In this expression, the quantities (I) and (II) can be estimated as before and one obtains

\[(d_E)^{2-\mu}(I) + (II) \leq c \varepsilon^\mu.\]

It remains to estimate (III) and (IV). To this aim, we need some information concerning \( S_\varepsilon^1 - S_\varepsilon^0 \). Notice already that

\[S_\varepsilon^1(x) - S_\varepsilon^0(x) = (\varepsilon \bar{r})^{-\frac{2-d}{p+1}} u_0(\varepsilon \bar{r}) - (\varepsilon \alpha(y))^{-\frac{2-d}{p+1}} u_0(\varepsilon \alpha(y)) + O\left(\varepsilon^{-\frac{2-d}{p+1}} \varepsilon^\nu \varepsilon^\mu \right),\]

(i) thanks to (12).

The estimates for (ii) follow simply from (12) and, without any difficulty, one finds that

\[|ii| \leq c \varepsilon^\mu \bar{r}^{-\mu},\]

since \( \bar{r} \geq \delta/2 \).

Concerning (i), we write

\[i = -\int_{\alpha(y)}^{\bar{r}} \phi(\bar{r}/\xi) \xi^{-\frac{2-d}{p+1}} d\xi,\]

where the function \( \phi \) has been defined in Proposition 2. Using the estimates of Proposition 2 as well as the fact that

\[|\alpha(y) - \bar{r}| \leq c \bar{r}^2,\]

we conclude that

\[|(i)| \leq c (\bar{r}/\varepsilon)^{\mu} \bar{r}^{-\frac{2-d}{p+1}},\]

if \( \bar{r}/\varepsilon \leq 1 \), while
if $r/e \geq 1$. We also obtain similar estimates for $\nabla \bar{q}(\mu)$. Using these estimates we get

$$|(d_2)^{2-\mu}(III)| \leq c(e^{\eta_1} + e^{4-\frac{2}{r-1} - \mu} + e^{N-\frac{2r}{r-1}}),$$

and

$$|(d_2)^{2-\mu}(IV)| \leq c(e^{\eta_1} + e^{2-\frac{2}{r-1} - \mu} + e^{N-\frac{2r}{r-1}}).$$

Thus, we conclude that

$$(d_2)^{2-\mu}|\Delta S_e + S_p^e| \leq c(e^{\eta_1} + e^{\frac{4}{r-1} - \mu}),$$

provided $d_2(x) \leq \delta/2$ and $a(\bar{q}) \in (\delta/2, \bar{\delta})$.

The result then follows from (24), (28) and (30).

4.4 The fixed point argument and the proof of the main Theorem

For the time being, let us prove the existence of $v \in D_1(\Omega \setminus \Sigma)$ solution of

$$(31) \begin{cases} \Delta v + Q_\epsilon(v) + f_\epsilon = 0 & \text{in } \Omega \setminus \Sigma \\ v = 0 & \text{on } \partial \Omega, \end{cases}$$

where $f_\epsilon = \Delta S_e + S_p^e$, is the quantity which has been estimated in the previous section and where $Q_\epsilon \equiv |S_e + v|^p - S_p^e$. As in Section 3, the existence of $v$ will be a consequence of the contraction mapping argument.

Let now define for all $\kappa > 0$

$$B_\kappa = \{v \in D_1(\Omega \setminus \Sigma) : |v|_\mu \leq \kappa e^{\eta_1}\}.$$

In order to obtain a solution of (31) it is enough to find a fixed point for the nonlinear mapping

$$M_\epsilon(v) = -G(f_\epsilon + Q_\epsilon(v)).$$

We need the following

**Lemma 1.** Assume that $\mu \in (2 - N, \inf(\frac{N-2}{p-1}, \nu))$. Then, there exists $\bar{p} \in (\frac{N-2}{N-1}, \frac{N-4}{N-2})$ and for all $p \in (\frac{N-2}{N-1}, \bar{p})$, there exists $\kappa > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and for all $v_2, v_1 \in B_\kappa$

$$|M_\epsilon(0)|_\mu \leq \frac{\kappa}{2} \epsilon^{\eta_1},$$
In particular, $M_\varepsilon$ is a contraction from the ball $B_\varepsilon$ into itself and therefore, $M_\varepsilon$ has a unique fixed point $v_\varepsilon \in B_\varepsilon$.

Let us emphasize that the restriction $p < \beta$ is necessary in our method since we use the estimation of the error $\Delta S_\varepsilon + S_\varepsilon^p$ given in which itself uses the assumption $p < \beta$. Furthermore, as we will see in a while that the condition $\mu > \frac{\beta - 1}{p - 1}$ is compatible with $\mu \in (2 - N, \inf(\frac{p - 1}{p - 1}, \nu))$ only for $p$ lying in the interval $(\frac{N - 1}{N - 2}, \beta)$.

**Proof:** First of all, it follows from Proposition 5 and Proposition 4 that

$$\|M_\varepsilon(0)\|_\mu \leq \frac{\varepsilon}{2}^\tau,$$

for some constant $\kappa > 0$, which does not depend on $\varepsilon$. Notice that, if $\mu \in (2 - N, \inf(\frac{p - 1}{p - 1}, \nu))$, then for all $p$ close enough to $\frac{N}{N - 2}$, say $\frac{N}{N - 2} < p \leq \beta$, we have $\mu > \frac{\beta - 1}{p - 1}$. Then, we use Taylor expansion to get

$$Q_\varepsilon(v_2) - Q_\varepsilon(v_1) = p \{v_2 - v_1\} \int_0^1 |S_\varepsilon + sv_2 + (1 - s)v_1|^{p - 2}(S_\varepsilon + sv_2 + (1 - s)v_1) \, ds.$$

In particular, there exists some constant $c > 0$ such that

$$|Q_\varepsilon(v_2) - Q_\varepsilon(v_1)| \leq c |v_2 - v_1| \{[S_\varepsilon]^{p - 1} + |v_2|^{p - 1} + |v_1|^{p - 1}\}.$$

It is then a simple exercise to check, using the fact that $[S_\varepsilon]^{p - 1}(x) \leq c_\varepsilon^p - 1 (d_\varepsilon(x))^{-2}$ for $\varepsilon$ small enough, and $\mu > \frac{\beta - 1}{p - 1}$ that

$$|Q_\varepsilon(v_2) - Q_\varepsilon(v_1)| \leq c \max(c_\varepsilon^p - 1, \varepsilon^{\tau(p - 1)}) |v_1 - v_2|_\mu.$$

The conclusion follows at once from the fact that the constant $c_\mu$ tends to 0 as $p$ tends to $\frac{N}{N - 2}$.

In order to complete the proof of Theorem 1, it remains to prove that $U_\varepsilon \equiv S_\varepsilon + v_\varepsilon > 0$, where $v_\varepsilon$ is the fixed point we have obtained in Lemma 1. On one hand, there exists some constant $c$ such that

$$|S_\varepsilon(x)| \leq c(d_\varepsilon(x))^{\frac{2\mu}{\mu - 1}},$$

in a neighborhood of $\Sigma$. On the other hand, since $v_\varepsilon \in B_\varepsilon$, we have $|v_\varepsilon(x)| \leq \kappa_\varepsilon^{\tau\varepsilon} (d_\varepsilon(x))^\mu$. But since we have chosen $\mu$ in a such a way that $\mu > \frac{\beta - 1}{p - 1}$, it follows that $U_\varepsilon$ is positive in neighborhood of $\Sigma$. Thus, the maximum principle together with the fact that
\[ \Delta U_\epsilon = -|U_\epsilon|^p < 0 \quad \text{in} \quad \Omega \setminus \Sigma, \]

implies that \( U_\epsilon > 0 \) everywhere. Thus \( U_\epsilon \) is a positive weak solution of

\[
\begin{align*}
\Delta U_\epsilon + U_\epsilon^p &= 0 \quad \text{in} \quad \Omega \setminus \Sigma \\
U_\epsilon &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

This will therefore end the proof of Theorem 1.

\[ \square \]

5 Open problem

It follows from the result of Mazzeo and Pacard [MP] and from our construction in Section 3, that given any \( k_1 \)-dimensional submanifold without boundary \( M^{k_1} \subset \mathbb{S}^N \) there exists a solution of \( \Delta_{R^{k_1+1}} u + u^p = 0 \) whose singular set is a cone over \( M^{k_1} \):

\[ C^{k_1+1} = \{ r \theta : \theta \in M^{k_1}, \, r \geq 0 \}, \]

provided \( p \in \left( \frac{N-k_1}{N-k_1-2}, \frac{N-k_1+2}{N-k_1-2} \right) \).

Now, assume that \( \Sigma^{k_1+k_2} \) is a closed \( k_1+k_2 \)-dimensional submanifold of \( \mathbb{R}^n \) (with \( n = N+k_2 \)), and assume that the boundary of \( \Sigma^{k_1+k_2} \) is locally given by \( C^{k_1+1} \times \mathbb{R}^{k_2-1} \) (up to a diffeomorphism), can one find a solution of \( \Delta_{\mathbb{R}^n} u + u^p = 0 \) whose singular set is \( \Sigma^{k_1+k_2} \)?

Even if we restrict our attention to \( p \) close to \( \frac{N-k_1}{N-k_1-2} \), the linear analysis of Section 4.2 is missing.

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GLOBAL EXISTENCE FOR WAVE MAPS WITH TORSION

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Abstract
Wave maps (i.e. nonlinear sigma models) with torsion are considered in 2+1 dimensions. Global existence of smooth solutions to the Cauchy problem is proven for certain reductions under a translation group action: invariant wave maps into general targets, and equivariant wave maps into Lie group targets. In the case of Lie group targets (i.e. chiral models), a geometrical characterization of invariant and equivariant wave maps is given in terms of a formulation using frames.

1 INTRODUCTION
There has been considerable progress during the last ten years in the mathematical study of long-time existence properties of solutions of geometrically-based classical field theories. A significant portion of this work has focussed
on the study of what are called wave maps in the mathematics literature and nonlinear sigma models (or, in certain special cases, chiral models) in the physics literature. These are defined as maps $\psi$ from a Lorentzian geometry $(M^{m+1}, \eta)$, e.g. Minkowski space, to a Riemannian geometry $(N^n, g)$, e.g. a symmetric space or a compact Lie group, with $\psi$ being a critical point for the functional

$$ S[\psi] = \int_{M^{m+1}} \eta^{\mu\nu} g_{AB}(\psi) \partial_\mu \psi^A \partial_\nu \psi^B, \tag{1} $$

and hence satisfying the wave map equation

$$ \eta^{\mu\nu} \nabla_\mu \psi^A + \Gamma^A_{BC}(\psi) \partial_\mu \psi^B \partial_\nu \psi^C \eta^{\mu\nu} = 0; \tag{2} $$

here $\nabla$ is the (torsion-free) derivative operator determined by the metric $\eta$, and $\Gamma$ are the (torsion-free) connection coefficients compatible with $g$.

Wave maps have a well-posed Cauchy problem, and it is known that for $1+1$ dimensional base geometries $(M^{1+1}, \eta)$, every choice of smooth initial data evolves into a global smooth solution [1, 2, 3], while for $3+1$ (or higher) dimensional base geometries, certain smooth initial data leads to solutions with singularities [3, 4]. Not yet understood is what happens in general for $2+1$ dimensional base geometries. This is the "critical dimension" (see [5, 6]), where global smooth solutions are expected, at least, for all smooth initial data of sufficiently small energy. While global existence results are known to hold for certain classes of rotationally-symmetric wave maps in $2+1$ dimensions (without restrictions on the energy) [7, 4], not much is known otherwise for critical wave maps [6].

An interesting modification to the wave map equations can be obtained by adding torsion. This can be done in $2+1$ dimensions, without adding extra dynamical fields, as follows. One fixes a pair of background fields: a closed one-form field $\omega$ on the base manifold $M^{2+1}$ and a non-closed two-form field $p$ on the target $N^n$. The field $p$ serves as a "torsion potential" in the sense that the torsion tensor on $N^n$ is defined as

$$ Q^A_{BC} = 3/2 g^{AP} \partial_D p_{BC} \tag{3} $$

A map $\psi : M^{2+1} \rightarrow N^n$ is defined to be a torsion wave map if it is a critical point for the functional

\[ \int_{M^{2+1}} \eta^{\mu\nu} \nabla_\mu \psi^A + \Gamma^A_{BC}(\psi) \partial_\mu \psi^B \partial_\nu \psi^C \eta^{\mu\nu} + 3/2 g^{AP} \partial_D p_{BC} \partial_\mu \psi^A \partial_\nu \psi^B = 0. \]

\[ ^1 \text{Integrals over } M^{m+1} \text{ are understood to use the natural volume form compatible with the metric } \eta. \]
where $\lambda$ is a coupling constant and $\epsilon$ is the 2+1 volume tensor normalized with respect to $\eta$. The torsion wave map equation obtained from (4) is given by
\[
\tau^{\mu \nu} \nabla_{\mu} \psi^{A} + \bar{\Gamma}^{A}_{BC}(\psi) \partial_{\mu} \psi^{B} \partial_{\nu} \psi^{C} \left( \eta^{\mu \nu} + \lambda \epsilon^{\mu \nu \sigma} v_{\sigma} \right) = 0
\]
where
\[
\bar{\Gamma}^{A}_{BC} = \Gamma^{A}_{BC} + Q^{A}_{BC}
\]
are the connection coefficients compatible with $g$, with torsion $Q$. Note that the effect of the torsion is to add the nonlinear term
\[
\lambda \epsilon^{\mu \nu \sigma} v_{\sigma} Q^{A}_{BC}(\psi) \partial_{\mu} \psi^{B} \partial_{\nu} \psi^{C}
\]
to the wave map equation (2).

Wave maps without torsion have a conserved, symmetric stress-energy tensor [5] arising from the functional $S[\psi]$. With the addition of torsion, the corresponding symmetric stress-energy tensor obtained from the functional $S_{\text{tor}}[\psi]$ is no longer conserved. This has presented an obstacle for the analysis of torsion wave maps. However, we point out that a non-symmetric stress-energy tensor can be derived by considering the variation of $S_{\text{tor}}[\psi]$ under infinitesimal diffeomorphisms of $M^{2+1}$ acting on $\eta, \epsilon, v,$ and $\psi$. This leads to
\[
T^{\mu}_{\alpha} = \tau^{\mu \nu} g_{AB} \partial_{\nu} \psi^{A} \partial_{\alpha} \psi^{B} - 1/2 \delta^{\mu}_{\alpha} \eta^{\nu \rho} g_{AB} \partial_{\nu} \psi^{A} \partial_{\rho} \psi^{B} \\
+ 1/2 \lambda \epsilon^{\mu \nu \rho} v_{\rho} P_{AB} \partial_{\nu} \psi^{A} \partial_{\mu} \psi^{B}
\]
which satisfies
\[
\nabla_{\mu} T^{\mu}_{\alpha} = 1/2 \lambda \epsilon^{\mu \nu \rho} \partial_{\mu} \psi^{A} \partial_{\nu} \psi^{B} \partial_{\rho} \psi^{C} - \nabla_{\mu} \nabla_{\nu} v^{\alpha}.
\]
Hence $T^{\mu}_{\alpha}$ is conserved if $v$ is covariantly constant on $(M^{2+1}, \eta)$. Furthermore, $T^{\mu}_{\alpha}$ reduces to the standard symmetric stress-energy tensor for wave maps without torsion when $v$ is set to zero. The stress-energy tensor (8) is central to investigating global existence for torsion wave maps.

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2 The contorsion coefficients $\bar{\Gamma}^{A}_{BC}$ compatible with $g$ as determined by the torsion are identically equal to $Q$. 

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The critical dimension for torsion wave maps, just as for standard wave maps, is $2+1$. While we do not attempt here to investigate the general class of critical torsion wave maps, we are able to prove global existence for various reductions of critical wave maps, with and without torsion, where the base geometry is Minkowski space. These reductions are defined by the invariance or equivariance of the wave map $\psi$ under a one-dimensional group of translations acting on $M^{2+1}$. More specifically, choose Cartesian coordinates $(x, y, t)$ for $(M^{2+1}, \eta)$ and denote the translation group action by $(x, y, t) \rightarrow (x, y + \lambda, t)$. Then, for any target $N^n$, a wave map $\psi$ is translation invariant if

$$\psi^A(x, y + \lambda, t) = \psi^A(x, y, t).$$  \hspace{1cm} (10)

Translation equivariant wave maps require that the target $N^n$ admit a translation group action. Let $\rho^A_B(\lambda)$ denote a representation of the translation group action on the base $M^{2+1}$ acting on the target $N^n$. Then a wave map $\psi$ is translation equivariant if

$$\psi^A(x, y + \lambda, t) = \rho^A_B(\lambda)\psi^B(x, y, t).$$  \hspace{1cm} (11)

Note that translation equivariance (11) reduces to translation invariance (10) when (and only when) the representation $\rho(\lambda)$ is chosen to be trivial, $\rho^A_B(\lambda) = \delta^A_B$.

One class of targets for which there is a natural translation group action available are Lie groups, $G$. For a Lie group target $N^n = G$, left and right multiplication on $G$ by a one-parameter exponential subgroup $\exp(\lambda A)$ define translation group actions, where $A$ is any element in the Lie algebra of $G$. This leads to three types of equivariance as follows. Let $\Psi$ denote a matrix representation of the wave map $\psi : M^{2+1} \rightarrow G$ and let $L$ and $R$ be matrix representations of elements of the Lie algebra of $G$. Then $\psi$ is said to be, respectively, left-translation equivariant if

$$\Psi(x, y + \lambda, t) = \exp(\lambda L)\Psi(x, y, t),$$  \hspace{1cm} (12)

or right-translation equivariant if

$$\Psi(x, y + \lambda, t) = \Psi(x, y, t)\exp(\lambda R),$$  \hspace{1cm} (13)

or conjugate-translation equivariant if

$$\Psi(x, y + \lambda, t) = \exp(\lambda L)\Psi(x, y, t)\exp(\lambda R).$$  \hspace{1cm} (14)
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Corresponding to invariant wave maps (10) and equivariant wave maps (12), (13), (14), we have the following four classes of reductions:

Invariant Wave maps (Any target)

\[ \psi = \phi(x, t) \]  

Left-Equivariant Wave maps (Lie group target)

\[ \Psi = \exp(yL)\Phi_L(x, t) \]

Right-Equivariant Wave maps (Lie group target)

\[ \Psi = \Phi_R(x, t)\exp(yR) \]

Conjugate-Equivariant Wave maps (Lie group target)

\[ \Psi = \exp(yL)\Phi_C(x, t)\exp(yR) \]

In each case the 2+1 wave map equation for \( \psi \) yields a 1+1 reduced equation for \( \phi, \Phi_L, \Phi_R, \Phi_C \), respectively.

We establish global existence of solutions to the Cauchy problem for the class of translation-invariant wave maps with torsion in Section 2. While the proof for these wave maps is very similar to that for 1+1 wave maps with no torsion, the torsion terms do introduce some subtleties into the analysis, which we highlight.

In order to prove global existence of solutions to the Cauchy problem for the three classes of translation equivariant wave maps with torsion, we find it useful to work with a frame formulation for 2+1 wave maps. In Section 3 we introduce the frame formulation for general targets and then proceed to relate wave map equivariance for Lie group targets to frame invariance and equivariance. In particular, our global existence theorems for equivariant wave maps have a natural formulation and proof using frames.

The proof for the left equivariant, right equivariant, and conjugate equivariant wave maps with torsion is fairly similar in each case. We focus on the left equivariant case (which corresponds to invariant frames) and carry out the global existence proof in detail, in Section 4. We then briefly note in Section 5 the differences entailed in proving global existence for the other two cases. We make a few concluding remarks in Section 6.

2 INVARIANT WAVE MAPS WITH TORSION

The translation invariance condition (10) is characterized by the wave map
functions (15) being independent of $y$. Under this reduction the torsion wave map equation (5) becomes
\[
\gamma^{\alpha\beta} \partial_\alpha \partial_\beta \phi^A + \Gamma^A_{BC}(\phi) \partial_\alpha \phi^B \partial_\beta \phi^C (\gamma^{\alpha\beta} + \lambda v_y e^{\alpha\beta}) = 0
\]
where $\gamma^{\alpha\beta}$ is the 1 + 1 Minkowski metric ($\alpha, \beta$ run over $x$ and $t$) and $e^{\alpha\beta}$ is the 1 + 1 Levi-Civita tensor. We hereafter take $v_y$ to be constant, but we make no further restrictions: The target $(N^n, g)$ can be any Riemannian geometry, and the torsion potential $p$ can be any non-closed two form on $N^n$.

Interestingly, while the torsion term
\[
\lambda v_y Q^A_{BC}(\phi) \partial_\alpha \phi^B \partial_\beta \phi^C e^{\alpha\beta}
\]
appears in a nontrivial way in the reduced wave map equation (19), and while the stress-energy tensor (8) generally contains a torsion term, for translation invariant wave maps the torsion drops out of many of the stress-energy tensor components. We have
\[
T_{tt} = T_{xx} = \frac{1}{2}(\partial_t \phi^2 + |\partial_x \phi|^2),
\]
\[
T_{xt} = T_{tx} = \partial_t \phi^A \partial_x \phi^B g_{AB},
\]
all of which contain no torsion, along with
\[
T_{yy} = \frac{1}{2}(\partial_y \phi^2 - |\partial_x \phi|^2) + \frac{1}{2} \lambda v_y e^{\alpha\beta} \partial_\alpha \phi^A \partial_\beta \phi^B p_{AB},
\]
\[
T_{yy} = 0,
\]
\[
T_{yx} = \frac{1}{2} \lambda v_y e^{\alpha\beta} \partial_\alpha \phi^A \partial_\beta \phi^B p_{AB},
\]
\[
T_{yy} = 0,
\]
\[
T_{yt} = \frac{1}{2} \lambda v_y e^{\alpha\beta} \partial_\alpha \phi^A \partial_\beta \phi^B p_{AB}.
\]
Note that $v_x$ and $v_t$ do not appear in the reduced wave map equation (19); setting them to zero does not affect (19), but it does result in $T_{yx}$ and $T_{yt}$ vanishing.
We now consider the Cauchy problem for translation invariant wave maps (15) with torsion. Initial data at \( t = t_0 \) consists of a pair of maps
\[
\dot{\phi} : \Sigma \to N, \quad \dot{\theta} : \Sigma \to TN
\] (here \( \Sigma = R^1 \) or \( S^1 \) allowing for periodic boundary conditions). A solution to the Cauchy problem is then a map \( \phi : \Sigma \times R^1 \simeq M^{2+1} \to N \) which satisfies (19) along with the initial conditions
\[
\phi(x, t_0) = \dot{\phi}(x), \quad \partial_t \phi(x, t_0) = \dot{\theta}(x).
\] Note that there are no constraints on the choice of initial data \( \{ \dot{\phi}, \dot{\theta} \} \). Global existence of initial value solutions is established by the following theorem.

**Theorem 1.** For any smooth compact support initial data, the Cauchy problem (19) and (29) has a unique smooth global solution \( \phi(x, t) \) for all \( t \in R^1 \).

**Proof:** The PDE system (19) is manifestly hyperbolic; hence, local existence and uniqueness are immediate [9]. To prove global existence, it is sufficient (by the usual open-closed arguments [9]) to show that if \( \phi(x, t) \) satisfies (19) on \( \Sigma \times I \), with \( I \) a bounded open interval in \( R^1 \), then \( \phi(x, t) \) and all its derivatives are bounded on \( \Sigma \times I \).

To show that \( \phi \) and its first derivatives \( \partial_t \phi \) are bounded, we use an argument based on stress-energy conservation (see [3]). From the form of the stress-energy components (21) to (27), together with the conservation equations
\[
\partial_t T^t_t + \partial_x T^x_t = 0, \quad \partial_t T^x_x + \partial_x T^t_x = 0,
\] we find that
\[
\gamma^{\alpha \beta} \partial_\alpha \partial_\beta T_{tt} = 0.
\] It then follows from standard results (see [6]) for the wave equation on 1 + 1 Minkowski space that \( T_{tt} \) is bounded on \( I \). Thus the first derivatives of \( \phi \) are bounded. As a consequence of the mean value theorem and the assumed compact support of the initial data, \( \phi \) is then bounded as well.

There are a number of ways of proceeding to argue that second and higher order derivatives of \( \phi \) are bounded. Here we use an argument which

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\( \dot{\phi} \) is compactly supported if it is constant everywhere outside a compact region in \( \Sigma \); \( \dot{\theta} \) is compactly supported if it zero outside such a region.
is adapted from Shatah [6] based on bounding successive kth order energies

\[ E_k(t) = \frac{1}{2} \int_{\Sigma} (\partial_\tau \partial_x^k \phi|^2 + |\partial_x^{k+1} \phi|^2) dx. \]  

(32)

Note that the ordinary energy

\[ E_0(t) = \int_{\Sigma} T_\mu dx = \frac{1}{2} \int_{\Sigma} (|\partial_\tau \phi|^2 + |\partial_\tau \phi|^2) dx \]

(33)

is bounded and independent of t, \( E_0(t) = E_0(t_0) \), for smooth compact support initial data.

We start by rewriting the torsion wave map equation (19) in the form

\[ D^a V^A_a = 0 \]

(34)

where \( V^A_a = \partial_a \phi^A \), \( D^a = \gamma^{ab} D_b \), and \( D^a = \partial_a + \Gamma^a_{bc} V^C_b + \lambda Q^A_a \phi^A V^C_c \)

defines a covariant derivative operator which includes the connection with torsion. If we now apply \( D^a \) to equation (34) and commute \( D^a \) past the derivative operators, keeping track of the various curvature and torsion terms which arise, then we obtain a nonlinear wave equation for \( V^A_a \):

\[ D^a D_a V^A_a + P^A_a(V, V, V) = 0 \]

(35)

where \( P(V, V, V) \) denotes an expression which is trilinear in \( V^A_a \) and involves no higher derivatives of \( \phi^A \).

By multiplying (35) by \( \gamma^{ab} g_{ca} D_a V^D_c \), we straightforwardly derive the conservation equation

\[ D(t) \frac{1}{2} |D_a V|^2 + \frac{1}{2} |D_a V|^2 - \frac{1}{2} D_a (D_a V \cdot D_a V) = \tilde{P}(V, V, V) \cdot D V \]

(36)

where \( \tilde{P}(V, V, V) \) is, like \( P(V, V, V) \), trilinear in \( V \) with no higher derivatives of \( \phi^A \). Now, integrating (36) over \( \Sigma \), we obtain

\[ \partial_t E_1(t) = \int_{\Sigma} \tilde{P}(V, V, V) \cdot D V dx \]

(37)

for the 1st order energy defined in (32). Estimating the right hand side of (37), we find

\[ \partial_t E_1(t) \leq C \|V\|^2_{L^2} \|D_a V\|_{L^2} \]

\[ \leq C \|V\|^2_{L^2} / E_1(t) \]

(38)

and hence
It follows from Sobolev inequalities that
\[ \|V\|_{L^6} \leq C\|V\|_{L^3}^{1/3}\|DV\|_{L^2}^{1/3}, \]  
so we have
\[ \partial_t \sqrt{E_1} \leq C\|V\|_{L^2}^{1/2}\|DV\|_{L^2} \leq C\mathcal{E}_0 \sqrt{E_1}. \]  
Since \( \mathcal{E}_0 \) is bounded, it follows from (41) that
\[ \sqrt{E_1(t)} \leq Ce^{kt} \]  
which bounds \( E_1(t) \), and therefore bounds the \( L^2 \) norm of \( DV \). Hence \( \|\partial^2\phi\|_{L^2} \) is bounded.

To bound \( E_2(t) \), we start from the wave equation (35) for \( V \) and repeat the previous argument. Setting \( W_\gamma := D\psi A \), we derive
\[ D^\alpha D_\gamma W\gamma A + R_\gamma A(V, V, W) = 0 \]  
where \( R(V, V, W) \) is bilinear in \( V \), linear in \( W \), and involves no other derivatives of \( \phi \). From (43) we obtain the conservation equation
\[ D_t \left( \frac{1}{2} |D_t W|^2 + \frac{1}{2} |D_t W|^2 \right) - D_x (D_t W \cdot D_t W) = R(V, V, W) \cdot DV \]  
where \( R(V, V, W) \) has the same properties as \( R(V, V, W) \). Integrating over \( \Sigma \) and estimating, we obtain
\[ \partial_t \sqrt{E_2(t)} \leq C\|V\|_{L^2}^{1/2}\|W\|_{L^4}\sqrt{E_2(t)}. \]  
Since \( V \) and \( W = DV \) are \( L^2 \) bounded, by applying Sobolev inequalities to (45) we obtain
\[ \partial_t \sqrt{E_2} \leq C\sqrt{E_2} \]  
Hence we have that \( \mathcal{E}_2(t) \) is bounded and therefore so is the \( L^2 \) norm of \( DV \). Thus, since \( DV = D\phi \), it follows that \( \|\partial^2\phi\|_{L^2} \) is bounded.

The argument proceeds to all successively higher orders and we thereby determine that all derivatives of \( \phi \) are \( L^2 \) bounded. It follows from Sobolev embedding that all derivatives of \( \phi \) are pointwise bounded, which completes the proof of Theorem 1.
3 EQUIVARIANT WAVE MAPS WITH TORSION
AND THE FRAME FORMULATION

We begin by setting up a frame formulation for wave maps with and without torsion. (See also [7]). We first choose a frame basis \( \{ e^a \} (a = 1, \ldots, n) \) for the target geometry \( (\mathbb{N}^n, g) \) and let \( e^a(\psi) \) denote the frame associated to \( \psi \).

We now define the “frame fields”

\[
K^a_\mu := e^a(\psi) \partial_\mu \psi^A
\]

(47)

where \( \{ e^a \} \) are the components of the dual basis to \( \{ e_a \} \). These frame fields \( K^a_\mu \) may be viewed either as the pull-back of the dual frame \( \{ e^a \} \) from \( \mathbb{N}^n \) to \( M^{2+1} \) along the map \( \psi : M^{2+1} \to \mathbb{N}^n \), or as the frame components of the wave map gradient on the tangent space of the target geometry \( (\mathbb{N}^n, g) \). In any case, it follows from (47) that \( K \) satisfies the identity

\[
\nabla^\mu K^a_\mu = -1/2 C^a_{bc}e^b(\psi)K^b_\mu K^c_\mu
\]

(48)

where \( C^a_{bc} \) are the frame commutator coefficients defined by

\[
[e_a, e_b] = C^c_{ab}e_c.
\]

Moreover, one verifies that if \( \psi \) satisfies the wave map equation (2), then \( K \) satisfies

\[
\nabla^\mu K^a_\mu = -C^a_{bc}(\psi)K^b_\mu K^c_\mu \eta^{\mu\nu}
\]

(50)

where \( C^a_{bc} := g^{ad}C_{db}g_{ac} \), with \( g^{ad} := e^a_Bg^{AB} \) and \( g_{ab} := e_a^Bg_{AB} \); or if \( \psi \) satisfies the torsion wave map equation (5), then \( K \) satisfies

\[
\nabla^\mu K^a_\mu = -C^a_{bc}(\psi)K^b_\mu K^c_\mu \eta^{\mu\nu} - \lambda e^{av}_{\mu}[\omega_\nu Q^a_{bc}(\psi)K^b_\mu K^c_\mu]
\]

(51)

where \( Q^a_{bc} := e^b_{A}Q_A^{BC}e^c_{C} \).

Up to this point in setting up the frame formulation, we have made no restrictions on the choice of the target or on the nature of the wave maps. We now focus on equivariant wave maps (12) to (14) and their corresponding frame formulations, so we assume the target geometry to be a Lie group \( G \).

While \( K \) can be defined for any frame basis on \( G \), the frame field equations are simplest if we require that \( \{ e_a \} \) be a left-invariant basis for \( G \). It then follows that the commutator coefficients \( C^a_{bc} \) are independent of \( \psi \) and are constant. If we make the further restrictions that the metric \( g \) be a left-invariant tensor on \( G \),
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\[ g_{AB} = \epsilon^a_{\ A} \epsilon^b_{\ B} g_{ab}, \]  
(52)

and the torsion potential \( p \) be a left invariant two-form on \( G \),

\[ p_{AB} = \epsilon^a_{\ A} \epsilon^b_{\ B} p_{ab}, \]  
(53)

so that the components \( g_{ab} \) and \( p_{ab} \) are constant, then the coefficients \( C^a_{\ bc}(\psi) \) are independent of \( \psi \) and constant while so are the frame components \( Q^a_{\ bc}(\psi) \) as well; in particular, we have

\[ Q^a_{\ bc} = -3/2 g^{ad} p_{cd} C^d_{bc}. \]  
(54)

with

\[ C^a_{bc} = 2 \epsilon^a_{\ [a} \epsilon^b_{\ b}] \epsilon^c_{\ c]. \]  
(55)

**Remark 1:** Every nonabelian Lie group admits both a left-invariant metric \( g \) and a left-invariant two-form \( p \). However, for semi-simple Lie groups \( G \), if \( G \) has dimension three then all left-invariant two-forms \( p \) are necessarily closed, and consequently \( Q = 0 \) so there is no torsion. This is not the case if \( G \) has larger dimension. In particular, a non-closed left-invariant two-form \( p \) and hence non-zero torsion \( Q \) is admitted by all nonabelian semi-simple Lie groups \( G \) other than the three-dimensional ones (namely \( SU(2) \) and its real forms \( SO(3), SO(1,2), SO(2,1) \)). See Proposition A in the appendix.

We now find that, assuming the restrictions just noted, we can write equations (48), (50) and (51) strictly in terms of the frame fields \( K \), with no explicit \( \psi \) dependence:

\[ \nabla_\nu K^a_{\ \mu} = -1/2 C^a_{bc} K^b_{\ \nu} K^c_{\ \mu}, \]  
(56)

\[ \nabla^\nu K^a_{\ \mu} = -C^a_{bc} K^b_{\ \nu} K^c_{\ \mu}, \]  
(57)

\[ \nabla^\nu K^a_{\ \mu} = -C^a_{bc} K^b_{\ \nu} K^c_{\ \mu} - \lambda \epsilon^{\nu\mu\nu\eta} Q^a_{\ bc} K^b_{\ \nu} K^c_{\ \mu}. \]  
(58)

The field equations (56) and (57) together are a self-contained PDE system for \( K \) which is equivalent to the wave map equation (2); the field equations (56) and (58) likewise are a self-contained PDE system for \( K \) which is equivalent to the wave map equation with torsion (5). Note that the system with torsion reduces to the system without torsion when \( \lambda = 0 \).

**Proposition 1.** Let \((M^{2+1}, \eta)\) be a Lorentzian geometry, and let \( N^a = G \) be a Lie group target.

1. Suppose that \( \psi^A \) is a solution of the torsion wave map equation (5). Then \( K^a_{\ \mu} \) defined by (47) satisfies the field equations (56) and (58).
2. Suppose that \( K^\mu \) is a solution of the field equations (56) and (58). If \( M^{2+1} \) is simply connected, then there exists a torsion wave map \( \psi^A \), satisfying equation (5), which is related to \( K^\mu \) by (47).

Proof: To prove part (1), we first note that for \( K^\mu \) given by (47), the field equation (56) is an identity. We then verify that, through the torsion wave map equation (5), the substitution of (47) for \( K^\mu \) satisfies the field equation (58).

For the converse, to prove part (2), we note the field equation (56) shows that \( K^\mu \) can be viewed as a Lie-algebra valued connection one-form on the trivial bundle \( M^{2+1} \times G \), with zero curvature. Since the bundle is trivial and the manifold \( M^{2+1} \) is assumed to be simply connected, there exists a global parallel section. Correspondingly, there exists a smooth map \( U : M^{2+1} \rightarrow G \) (called a "gauge transformation") in terms of which we have

\[
K^\mu = (U^{-1} \partial_\mu U)^A e_A^\mu (I) \tag{59}
\]

where \( I \) is the identity element of \( G \).

Now let \( \psi^A \) denote \( U \) written in terms of local coordinates on \( G \). It follows that \( U^{-1} \) pulls back \( e_A^\mu (I) \) to \( e_A^\mu (\psi) \), at the Lie group element specified by \( U \). Hence we have

\[
(U^{-1} \partial_\mu U)^A e_A^\mu (I) = e_A^\mu (\psi) \partial_\mu \psi^A. \tag{60}
\]

Combining (60) with (59), we obtain equation (47). Then by substituting (47) into the field equation (58), we verify that \( \psi \) satisfies (5).

We note that, independent of their usefulness for the study of wave maps, these field theories in terms of \( K \) viewed as a Lie-algebra valued one-form field on \( M^{2+1} \) have some interest as a nonlinear generalization of Maxwell’s equations. Indeed, for the abelian case \( C^{ab}_a = 0 \), the field equations (56) and (57) are exactly Maxwell’s equations in 2+1 dimensions, while the field equations (56) and (58) are a modification of Maxwell’s equations by adding torsion. This relationship is explored elsewhere [10, 11, 12].

Since we will use frame fields to study translation equivariant wave maps, we now characterize frame fields which correspond to the three classes of equivariant wave maps (12), (13), (14). We begin with the following definitions of invariant and equivariant frame fields under a translation group action.

**Invariant Frame Field:**

\[
K(x, y + \lambda, t) = K(x, y, t) \tag{61}
\]
Equivariant Frame Field:

\[ K(x, y + \lambda, t) = \exp(-\lambda A)K(x, y, t)\exp(\lambda A) \]  \hspace{1cm} (62)

Here \( A \) is an element of the Lie algebra of the target Lie group \( G \), and \((x, y, t)\) are standard coordinates for the Minkowski space base geometry \((M^{2+1}, \eta)\).

Geometrically, the translation equivariant group action (62) on \( K \) arises via the pull-back of the dual frame components \( \{e^*_a\} \) under right multiplication in \( G \) by the one-parameter exponential subgroup generated from the Lie algebra element \( R \). When \( R = 0 \) this group action reduces to the translation invariant group action (61) on \( K \). (Alternatively, note that the translation invariant group action arises directly by left multiplication in \( G \) since the dual frame is left-invariant.)

Based on Proposition 1, the correspondence between invariant/equivariant frame fields and wave maps is summarized by the following two results.

**Proposition 2.**

1. If \( \psi \) is left equivariant (12), then the corresponding frame field \( K \) is invariant (61).
2. If \( \psi \) is right equivariant (13), then the corresponding frame field \( K \) is equivariant (62), with the components \( K^*_y \) constant.
3. If \( \psi \) is conjugate equivariant (14), then the corresponding frame field \( K \) is equivariant (62).

The proof of these correspondences amounts to a direct calculation using a matrix representation for \( \psi \) and \( K \). There are straightforward converse correspondences as well.

**Proposition 3.**

1. If \( K \) is invariant (61), then the corresponding wave map \( \psi \) is left equivariant (12).
2. If \( K \) is equivariant (62), then the corresponding wave map \( \psi \) is conjugate equivariant (14).
3. If \( K \) is equivariant (62) with the components \( K^*_y \) constant, then the corresponding wave map \( \psi \) is right equivariant (13).

**Proof:** Let \( U \) denote a the matrix representation of the wave map \( \psi \) cor-
responding to $K$. We first prove part (1). It follows from the definition of frame field invariance, together with relation (59), that $U$ satisfies

$$\partial_y(U^{-1}\partial_y U) = 0. \quad (63)$$

Integrating the $y$ component of this equation, and then multiplying both sides by $U$, we obtain the linear matrix ordinary differential equation

$$\partial_y U(x, y, t) = U(x, y, t)f(x, t) \quad (64)$$

where $f$ is an arbitrary Lie-algebra matrix valued function (independent of $y$). The general solution to (64) is

$$U(x, y, t) = \exp(yA(x, t))V(x, t) \quad (65)$$

where $V$ is an arbitrary Lie-algebra nonsingular matrix valued function (independent of $y$), and $A := VF^{-1}$.

We now impose the $x, t$ components of equation (63). Calculating $U^{-1}\partial_t U$ with $U$ from (65), we find

$$U^{-1}\partial_t U = V^{-1}\partial_t V + V^{-1}\exp(-yA)(\partial_t \exp(yA))V, \quad (66)$$

so

$$0 = \partial_y(U^{-1}\partial_t U) = V^{-1}\exp(-yA)\partial_t A \exp(yA)V, \quad (67)$$

which implies that

$$\partial_t A = 0. \quad (68)$$

Similarly, working with $U^{-1}\partial_x U$ and imposing $0 = \partial_x(U^{-1}\partial_t U)$ we determine that

$$\partial_x A = 0. \quad (69)$$

Thus $A$ must be a constant Lie-algebra valued matrix, which we denote $L$; then (65) becomes

$$U(x, y, t) = \exp(yL)V(x, t). \quad (70)$$

Condition (12) immediately follows, so the wave map corresponding to $K$ is left equivariant.

We now prove part (2). From the definition of frame equivariance, there is a $y$-independent Lie-algebra matrix valued field $f_\mu(x, t)$ and a constant Lie-algebra matrix $A$ such that
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\[ K_\mu(x, y, t) = \exp(-yA) f_\mu(x, t) \exp(yA). \]  

(71)

Hence, from relation (59), \( U(x, y, t) \) must satisfy

\[ U^{-1} \partial_y U = \exp(-yA) f_\mu \exp(yA). \]  

(72)

The \( y \)-component of this equation yields

\[ (\partial_y U) \exp(-yA) = U \exp(-yA) f_y, \]  

(73)

and after some manipulation we obtain the linear matrix ODE

\[ \partial_y W = W(f_y - A) \]  

(74)

where \( W(x, y, t) := U(x, y, t) \exp(-yA) \). The general solution to (74) is

\[ W(x, y, t) = \exp(yB(x, t)) V(x, t) \]  

(75)

where \( V \) is an arbitrary Lie-algebra matrix valued function, and \( B \) is defined as

\[ B := V(f_y - A)V^{-1}. \]  

(76)

Working with the other components of equation (72) we derive

\[ \partial_t W = W f_t, \]  

(77)

and

\[ \partial_x W = W f_x. \]  

(78)

Then rearranging (77) and using (75), we obtain

\[ f_t(x, t) = W^{-1} \partial_t W = V^{-1} \partial_t V + V^{-1} \exp(-yB) \partial_t (\exp(-yB)) V. \]  

(79)

Since both \( f_t \) and \( V \) are independent of \( y \), if we take \( \partial_y \) of both sides of equation (79) we have

\[ 0 = \partial_y (V^{-1} \exp(-yB) \partial_t (\exp(yB)) V) = V^{-1} \exp(-yB) \partial_y B \exp(yB) V \]  

(80)

which implies that \( \partial_y B = 0 \). Similarly, using (78), we find that \( \partial_x B = 0 \). Hence \( B \) is a constant Lie-algebra matrix, which we denote \( L \). Thus, after combining (75) with the definition \( W = U \exp(-yA) \), we see that
\[ U(x, y, t) = \exp(y L) V(x, t) \exp(y A) \tag{81} \]

so \( U(x, y, t) \) is conjugate equivariant (14) with \( R = A \).

Finally, we prove part (3). From (81) we have
\[ K_y = U^{-1} \partial_y U = \exp(-y A) V^{-1} L V \exp(y A) + A \tag{82} \]

which is assumed to be constant. By differentiating with respect to \( y \), we obtain \( [V^{-1} L V, A] = 0 \), and hence (81) becomes \( K_y = V^{-1} L V + A \). Thus, it follows that \( B := V^{-1} L V \) defines a constant Lie-algebra matrix which commutes with \( A \). We then have
\[
\begin{align*}
U(x, y, t) &= \exp(y L) V(x, t) \exp(y A) \\
&= V(x, t) \exp(y B) \exp(y A) = V(x, t) \exp(y R) \tag{83}
\end{align*}
\]

where \( R := A + B \). Hence \( U(x, y, t) \) is right equivariant (13).

As a consequence of Propositions 2 and 3, we can prove global existence of solutions to the Cauchy problem for the three classes of translation equivariant wave maps (with or without torsion) by using invariant or equivariant frame fields. We do this first for the invariant frame fields in the next section.

Our analysis makes essential use of the wave map stress-energy tensor (8). Through the relation (47) for \( K \) in terms of \( \psi \), we obtain
\[
T^\mu{}_{\alpha} = \eta^{\mu\nu} K^a_0 K^b_0 g_{ab} - 1/2 \delta^\mu{}_{\alpha} \eta^{\nu\rho} K^a_0 K^b_0 g_{ab} + 1/2 \lambda \epsilon^{\mu\nu\rho\sigma} \nu a p_{ab} K^a_0 K^b_0. \tag{84}
\]

One verifies that, for solutions \( K \) of (56) and (58) in which \((M^{2+1}, \eta)\) is Minkowski space, this non-symmetric stress-energy tensor satisfies the conservation equation
\[
\partial_\mu T^\mu{}_{\alpha} = 1/2 \lambda \epsilon^{\mu\nu\rho\sigma} \nu a p_{ab} K^a_0 K^b_0 \partial_\alpha \nu a. \tag{85}
\]

Hereafter we specialize to the situation where \( v \) is constant on \( M^{2+1} \). This makes the analysis of the field equations considerably simpler. In particular, the stress-energy is strictly conserved, \( \partial_\mu T^\mu{}_{\alpha} = 0 \).

4 \quad GLOBAL EXISTENCE FOR INVARIANT FRAME FIELD EQUATIONS WITH TORSION

By definition (61) of translation invariance for frame fields, the component functions \( K^a_\mu \) are independent of \( y \). Then, adopting the convenient notation
\[
E^a := K^a_0, H^a := K^a_y, B^a := K^a_t, \tag{86}
\]

we find that the translation-invariant frame field equations take the form
\[
\partial^a H^a = -C_{\nu \rho}^a E^\rho H^\nu \tag{87}
\]
for the functions \( (E^a(x, t), H^a(x, t), B^a(x, t)) \). Note that, in this system of field equations, (87) is a constraint equation while (88) to (90) are evolution equations.

Initial data at \( t = t_0 \) for the Cauchy problem is specified by choosing (on \( \Sigma = R^1 \) or \( S^1 \) allowing for periodic boundary conditions) Lie-algebra valued functions \( \{E^a(x), H^a(x), B^a(x)\} \) which satisfy the constraint

\[ \partial_t \hat{E}^a = -C_{bc}^a \hat{E}^b \hat{E}^c. \]  

A solution to the Cauchy problem is then a set of fields \( \{E^a(x, t), H^a(x, t), B^a(x, t)\} \) satisfying (88) to (90) and the initial conditions

\[ E^a(x, t_0) = \hat{E}^a(x), H^a(x, t_0) = \hat{H}^a(x), B^a(x, t_0) = \hat{B}^a(x). \]  

To show that the Cauchy problem is well-posed, we note that well-posedness is known for the wave map equation without torsion \( [6] \), which is equivalent to the system (87) to (90) up to the addition of the torsion terms involving \( \lambda \). These terms do not involve any derivatives of the fields and hence do not effect the well-posedness. Alternatively, we note that, up to such terms, the system is equivalent to the Maxwell equations in 2+1 dimensions, which constitute a well-posed system. It follows that the system (87) to (90) is well-posed and, moreover, is first-order hyperbolic.

In this section we prove global existence of smooth solutions to the Cauchy problem for the 1+1 field equations (87) to (90). The proof relies on the use of the stress-energy tensor (84) along with light cone estimates.

To proceed we write out the components of the stress-energy tensor (8) in terms of \( E^a, H^a, B^a \). Using the coordinates \( (x, y, t) \) for \( M^{2+1} \) we have

\[ T_{xx} = \frac{1}{2}(E_x^2 - E_y^2 + B^2) + \lambda \nu_x H^a B^a p_{ab} \]  
\[ T_{yy} = \frac{1}{2}(-E_x^2 + E_y^2 + B^2) + \lambda \nu_y H^a B^a p_{ab} \]  
\[ T_{xz} = E \cdot B + \lambda \nu_x H^a E^a p_{ab} \]  
\[ T_{yt} = E \cdot B + \lambda \nu_t H^a B^a p_{ab} \]  
\[ T_{xy} = H \cdot B + \lambda \nu_y H^a E^a p_{ab} \]  
\[ T_{xt} = H \cdot B + \lambda \nu_t H^a B^a p_{ab} \]
\[ T_{xy} = E \cdot H + \lambda v_y H^a B^b p_{ab} \]  
\[ T_{yz} = E \cdot H + \lambda v_z B^a E^b p_{ab} \]  
where \( E^2 := E^a E^b g_{ab} \) and \( E \cdot B = E^a B^b g_{ab}, \) etc.

For derivation of light cone estimates, it is useful to work with null components of the stress-energy tensor. We introduce null coordinates which mix \( t \) and \( x \) (but not \( y \)):
\[ \ell = t + x \quad \quad x = \frac{1}{2}(\ell + n) \]
\[ n = -t + x \quad \quad t = \frac{1}{2}(\ell - n) \]

Then we find (for the components we will need):
\[ T_{\ell \ell} = K^2 + \lambda v_x H^a K^b p_{ab} \]
\[ = \frac{1}{4}(B + E)^2 + \frac{\lambda}{4} (v_x + v_y) H^a (B^b + E^b) p_{ab} \]  
\[ T_{nn} = K^2 - \lambda v_x H^a K^b p_{ab} \]
\[ = \frac{1}{4} (-B + E)^2 - \frac{\lambda}{4} (v_x + v_y) H^a (-B^b + E^b) p_{ab} \]
\[ T_{\ell n} = -\frac{1}{2} K^2 - \lambda v_x K^a H^b p_{ab} \]
\[ = -\frac{1}{2} E^2 + \frac{\lambda}{4} (-v_x + v_y) H^a (B^b + E^b) p_{ab} \]
\[ T_{nt} = -\frac{1}{2} K^2 + \lambda v_x K^a H^b p_{ab} \]
\[ = -\frac{1}{2} E^2 - \frac{\lambda}{4} (v_x + v_y) H^a (-B^b + E^b) p_{ab}. \]

For these components the stress-energy conservation equation (85) has the null component form
\[ \partial_n T_{\ell \ell} + \partial_\ell T_{nn} = 0, \]  
\[ \partial_\ell T_{\ell n} + \partial_n T_{nt} = 0. \]

These equations are essential for the derivation of the light cone estimates we will need.

Also important for our analysis is the energy function
\[ E(t) = \int_{\Sigma} T_{tt} dx \]
\[ = \int_{\Sigma} \left( \frac{1}{2}(E^2 + H^2 + B^2) + \lambda v_x H^a E^b p_{ab} \right) dx \]  
(108)
We note that for certain values of the coupling constant \( \lambda \), the energy \( E(t) \) can be negative, and it therefore does not in general control the \( L^2 \) norm of \( E^a, H^a, \) or \( B^a \). However, for sufficiently small \( \lambda \), there is a constant \( k > 0 \) such that
\[
\frac{1}{k}(E_x^2 + E_y^2) \leq E_x^2 + E_y^2 + 2\lambda v_t H^a E^a p_{ab} \leq k(E_x^2 + E_y^2) \tag{109}
\]
and hence the energy is positive, so that \( E(t) \) does consequently control \( \|E\|_{L^2}, \|H\|_{L^2} \), and \( \|B\|_{L^2} \). We assume henceforth that \( \lambda \) is sufficiently small for this to be the case. \(^4\)

We now state our main results. Let \( \Sigma \) denote \( R^1 \) or \( S^1 \), and introduce coordinates \((x, t)\) for \( \Sigma \times R^1 \simeq M^{2+1} \). Fix constants \( v_t, v_x, v_y \). Let \( G \) be a Lie group with \( C_{bc} \) denoting the Lie-algebra commutator structure tensor. Fix on the Lie algebra of \( G \) a positive definite metric tensor \( g_{ab} \) (it need not necessarily be compatible with the commutator) and a skew-tensor \( p_{ab} \). Let \( Q^a_{bc} \) be the tensor defined by (54).

**Theorem 2.** Let \( \lambda \) be a small constant. \(^4\) For any smooth compact support initial data \((92)\) satisfying \((91)\), the Cauchy problem \((87)\) to \((90)\) has a unique smooth global solution \( \{E^a(x, t), H^a(x, t), B^a(x, t)\} \) for all \( t \in R^1 \).

Combining this result with Propositions 2 and 3 from Section 3, we have a corresponding result for wave maps.

**Theorem 3.** The Cauchy problem for left-translation equivariant Lie group wave maps \((12)\), with or without torsion, has a unique smooth global solution for all smooth compact support initial data.

**Proof of Theorem 2:**
Local existence and uniqueness of smooth solutions of the PDE system \((87)\) to \((90)\) follows from standard results (see, for example,[9]) for first-order hyperbolic systems in \( 1+1 \) dimensions. In order to prove global existence, it is sufficient by the usual "open-closed" arguments[9] to establish the following: For \( \{E^a(x, t), H^a(x, t), B^a(x, t)\} \) satisfying equations \((87)\) to \((90)\) for \( t \in I \), with \( I \) a bounded open interval in \( R^1 \), each component of these fields is

\[^4\text{It is sufficient that } \lambda \text{ satisfy} \]
\[
|\lambda| \leq 1/\sqrt{|v_t|}\] where \( |p|^2 = |p_{ab}p_{cd}g^{ac}g^{bd}| \).
bounded for $t \in I$, as are all orders of their derivatives. We prove this boundedness result as follows:

**Step 1: Conserved Energy**

It follows from the stress-energy conservation equation $\partial_t T_{tt} - \partial_z T_{zt} = 0$ that the energy $\mathcal{E}(t)$ satisfies $\frac{d}{dt} \mathcal{E}(t) = \mathcal{F}(t)$ where

$$\mathcal{F}(t) := \int_\Sigma \partial_z T_{zt} dx = T_{zt}\big|_{\partial \Sigma}$$

(110)

is the flux. If we are working on $\Sigma = S^1$, then $\partial \Sigma$ is empty, so $\mathcal{F}(t) = 0$. If instead $\Sigma = R^1$, then we note that as a consequence of hyperbolicity of the system (87) to (90), the fields $\{E^a, H^a, B^a\}$ have compact support on $\Sigma$ for all $t \in I$, and hence $\mathcal{F}(t) = 0$. Thus, the energy is conserved, $\mathcal{E}(t) = \mathcal{E}(t_0)$, for all $t \in I$.

As we noted earlier, the energy controls the $L^2$ norm of the fields $\{E^a, H^a, B^a\}$, so long as $\lambda$ is sufficiently small (as assumed in the theorem). Hence we have

$$\|E^a\|_{L^2(\Sigma)} < k, \quad \|H^a\|_{L^2(\Sigma)} < k, \quad \|B^a\|_{L^2(\Sigma)} < k$$

(111)

for some constant $k$ (depending on $\mathcal{E}(t_0)$), for all $t \in I$.

**Step 2: Bounded $H^a$**

In the system (87) to (90), the field $H^a$ enters in a different way from $E^a$ and $B^a$, since the evolution equation (89) for $H^a$ involves no spatial derivative terms, and the constraint equation (87) has $\partial_a H^a$ appearing, but no spatial derivatives of $E^a$ or $B^a$. Consequently, we treat $H^a$ differently from the other two fields: we first show that $H^a$ is bounded, and then use this in showing that $E^a$ and $B^a$ are bounded.

To start, we integrate the absolute values of both sides of the constraint equation (87) over $\Sigma$, obtaining

$$\int_\Sigma |\partial_z H^a| dx = \int_\Sigma |C_{bc} E^b H^c| dx.$$ 

(112)

Since $C_{bc} E^b H^c$ is constant, there exists a constant $k_1$ such that

$$|C_{bc} E^b H^c| \leq |C_{bc}| \|E^b\| \|H^c\| \leq k_1 (E^2 + H^2)$$

(113)

by standard algebraic inequalities. It follows from (112) and (113) together with the bounds (111) that
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(114)

\[ \int_{\Sigma} |\partial_x H^a| dx \leq k_2 \]

for a constant \( k_2 \). Combining (114) with the mean value theorem, we obtain controls on the spatial variation of \( E^a(x, t) \) for any fixed time \( t \). In particular, for any \( x_1, x_2 \in \Sigma \) with fixed \( t \), we have

\[
|H^a(x_2, t) - H^a(x_1, t)| = \left| \int_{x_1}^{x_2} \partial_x H^a(x, t) dx \right| \\
\leq \int_{x_1}^{x_2} |\partial_x H^a(x, t)| dx \leq k_2.
\]

(115)

If we are working on \( \Sigma = R^1 \), we can choose \( x_1 \) outside the support of \( H^a(x, t) \) for all \( t \in I \), and therefore it follows from (115) that \( |H^a(x, t)| \leq k_2 \) for all \( (x, t) \in \Sigma \times I \). Hence, \( H^a(x, t) \) is bounded on \( \Sigma \times I \).

If instead we are working on \( \Sigma = S^1 \), we need to do more to bound \( H^a(x, t) \). Consider \( \int_{S^1} H^a(x, t) dx \), which is the spatial average of \( H^a \) on \( S^1 \). From the fundamental theorem of calculus, and from the evolution equation (89), we obtain (for \( t \in I \))

\[
\int_{S^1} H^a(x, t) dx = \int_{t_0}^{t} \frac{d}{ds} \int_{S^1} H^a(x, s) dx ds + \int_{S^1} H^a(x, t_0) dx \\
= -\int_{t_0}^{t} \int_{S^1} C_k B^k(x, s) H^a(x, s) dx ds + \int_{S^1} H^a(x, t_0) dx.
\]

(116)

Next, using standard quadratic algebraic inequalities, we note that \( \int_{S^1} C_k B^k H^a dx \) is bounded in terms of the energy,

\[
\left| \int_{S^1} C_k B^k H^a dx \right| \leq k_3 E(t) = k_3 E(t_0)
\]

(117)

for some constant \( k_3 \). Hence, \( \int_{t_0}^{t} \int_{S^1} C_k B^k(x, s) H^a(x, s) dx ds \) is bounded above and below,

\[
\left| \int_{t_0}^{t} \int_{S^1} C_k B^k(x, s) H^a(x, s) dx ds \right| \leq (t - t_0) k_3 E(t_0) \leq k_4
\]

(118)

for some constant \( k_4 \) for all \( t \in I \). Then since \( \int_{S^1} H^a(x, t_0) dx \) involves initial data only, it also is bounded above and below. Therefore, from (116) we have that

\[
\left| \int_{S^1} H^a(x, t) dx \right| \leq k_5
\]

(119)

and so the average of \( H^a \) over \( S^1 \) is bounded above and below, for all \( t \in I \).
Combining this result with the spatial variance control (115), we conclude that \( H^a(x, t) \) is bounded (above and below) on \( \Sigma \times I \).

**Step 3: Bounded \( E^a \) and \( B^a \)**

While standard 1+1 light cone arguments do not directly apply to the system (87) to (90), a modified argument can be used with the pointwise bounds on \( H^a \) achieved in Step 2.

Using the null form of the stress-energy conservation laws (106)-(107), along with the expressions (102)-(105) for the stress-energy components, we have

\[
\partial_t (B + E)^2 = 2\partial_t H^2 - \lambda (v_t + v_x)p_{ab} \partial_n \left( H^a (B^b + E^b) \right) + \lambda (v_t + v_x)p_{ab} \partial_t \left( H^a (-B^b + E^b) \right), \tag{120}
\]

\[
\partial_t (-B + E)^2 = 2\partial_n H^2 - \lambda (-v_t + v_x)p_{ab} \partial_n \left( H^a (B^b + E^b) \right) + \lambda (-v_t + v_x)p_{ab} \partial_t \left( H^a (-B^b + E^b) \right). \tag{121}
\]

We use the field equations (87) to (90) to remove all of the derivatives which appear on the right-side of these equations. Thus

\[
\partial_n (B + E)^2 = -2H^b H^c (B^a + E^a) C_{abc} - 2\lambda (v_t + v_x) H^a (-B^b + E^b)(B^c + E^c)Q_{bca} \tag{122}
\]

and

\[
\partial_t (-B + E)^2 = -2H^b H^c (-B^a + E^a) C_{abc} - 2\lambda (-v_t + v_x) H^a (-B^b + E^b)(B^c + E^c)Q_{bca}. \tag{123}
\]

It is convenient here to let \( \alpha^a := B^a + E^a \) and \( \beta^a := -B^a + E^a \), and so we have

\[
\partial_n \alpha^2 = -2C_{abc} H^b H^c \alpha^a - 2\lambda (v_t + v_x)Q_{bca} H^a \beta^c \tag{124}
\]

and

\[
\partial_t \beta^2 = -2C_{abc} H^b H^c \beta^a - 2\lambda (-v_t + v_x)Q_{bca} H^a \alpha^c. \tag{125}
\]

Since \( C_{abc}, \ Q_{bca}, \lambda, \ v_t \) and \( v_x \) are constant, and since \( H^a \) is bounded on \( \Sigma \times I \), we immediately have the following estimates for the right-sides of (124) and (125):

\[
\partial_n \alpha^2 \leq k_6 \sqrt{\alpha^2} + k_7 \sqrt{\alpha^2} \sqrt{\beta^2} \tag{126}
\]
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with some constants \( k_6, k_7, k_8, \) and \( k_9. \)

We now apply a light cone argument to the differential inequalities (126) and (127). First, choose an arbitrary point \((\hat{x}, \hat{t})\) in \( \Sigma \times I \) to the future of the initial surface \( \Sigma, \) so \( \hat{t} > t_0, \) and integrate \( \alpha^2 \) back along the light ray parallel to \( \partial_x \) via (126) and also integrate \( \beta^2 \) back along the light ray parallel to \( \partial_t \) via (127). This yields

\[
\alpha^2(\hat{x}, \hat{t}) \leq \alpha^2(\hat{x} + \hat{t} - t_0, t_0) + k_6 \int_{t_0}^{\hat{t}} \sqrt{\alpha^2(\hat{x} + \hat{t} - s, s)} ds + k_7 \int_{t_0}^{\hat{t}} \sqrt{\alpha^2(\hat{x} + \hat{t} - s, s)} \sqrt{\beta^2(\hat{x} + \hat{t} - s, s)} ds
\]

and

\[
\beta^2(\hat{x}, \hat{t}) \leq \beta^2(\hat{x} + t_0 - \hat{t}, t_0) + k_9 \int_{t_0}^{\hat{t}} \sqrt{\beta^2(\hat{x} - \hat{t} + s, s)} ds + k_9 \int_{t_0}^{\hat{t}} \sqrt{\alpha^2(\hat{x} - \hat{t} + s, s)} \sqrt{\beta^2(\hat{x} - \hat{t} + s, s)} ds.
\]

Next, take the supremum of these expressions over \( \Sigma. \) Letting \( \hat{\alpha}^2(t) := \sup_{x \in \Sigma} \alpha^2(x, t) \) and \( \hat{\beta}^2(t) := \sup_{x \in \Sigma} \beta^2(x, t), \) we obtain from (128)

\[
\hat{\alpha}^2(t) \leq \hat{\alpha}^2(t_0) + k_6 \sup_{x \in \Sigma} \int_{t_0}^{t} \sqrt{\alpha^2(x, s)} ds + k_7 \sup_{x \in \Sigma} \int_{t_0}^{t} \sqrt{\alpha^2(x, s)} \sqrt{\beta^2(x, s)} ds
\]

\[
\leq \hat{\alpha}^2(t_0) + k_6 \int_{t_0}^{t} \sqrt{\alpha^2(s)} ds + k_7 \int_{t_0}^{t} \sqrt{\alpha^2(s)} \sqrt{\beta^2(s)} ds
\]

\[
\leq \hat{\alpha}^2(t_0) + k_9 \int_{t_0}^{t} \sqrt{\alpha^2(s)} ds + k_1 \int_{t_0}^{t} \sqrt{\beta^2(s)} ds
\]

where the last step is a consequence of the Holder inequality. If we define

\[
a(t) := \int_{t_0}^{t} \alpha^2(s) ds
\]
and
\[ b(t) := \int_{t_0}^t \beta^2(s) ds \] (132)

then (130) can be written as (with \( t \) replaced by \( t \))
\[ \frac{d}{dt} a(t) \leq a(t_0) + k_{10}(t - t_0)^{1/2} a^{1/2}(t) + k_{11} a^{1/2}(t) b^{1/2}(t). \] (133)

Similarly, from (129), we derive
\[ \frac{d}{dt} b(t) \leq b(t_0) + k_{12}(t - t_0)^{1/2} b^{1/2}(t) + k_{13} a^{1/2}(t) b^{1/2}(t). \] (134)

We want to show \( a(t) \) and \( b(t) \) are bounded functions of \( t \) by applying a Gronwall type argument to the coupled inequalities (133),(134). It is useful first to divide by \( a^{1/2}(t) \) in (133) and by \( b^{1/2}(t) \) in (134), yielding
\[ \frac{d}{dt} a^{1/2}(t) \leq a(t_0) a^{-1/2}(t) + k_{10}(t - t_0) a^{1/2} + k_{11} b^{1/2}(t), \] (135)
\[ \frac{d}{dt} b^{1/2}(t) \leq b(t_0) b^{-1/2}(t) + k_{12}(t - t_0) b^{1/2} + k_{13} a^{1/2}(t). \] (136)

We estimate the term \( a(t_0) a^{-1/2}(t) \) by using the fact that \( a(t) \) is a monotonic increasing function of \( t \), due to positivity of \( \frac{d^2}{dt^2} \) in (131). Thus, \( a(t_0) a^{-1/2}(t) \) is bounded by \( a^{-1/2}(t_0) \). In addition, we note the term \( k_{10}(t - t_0)^{1/2} \) is bounded since \( t \in I \) is bounded. We thereby obtain
\[ \frac{d}{dt} a^{1/2}(t) \leq k_{14} + k_{11} b^{1/2}(t). \] (137)

Similarly, we obtain
\[ \frac{d}{dt} b^{1/2}(t) \leq k_{15} + k_{13} a^{1/2}(t). \] (138)

Adding (137) and (138), and defining \( c(t) := a^{1/2}(t) + b^{1/2}(t) \), we derive
\[ \frac{d}{dt} c(t) \leq k_{16} + k_{17} c(t). \] (139)

Gronwall's inequality immediately applies to (139), and so we determine that \( c(t) \) is bounded for all \( t \in I \). Then \( a^{1/2}(t) \) and \( b^{1/2}(t) \), which are positive, are bounded.
Returning to the inequalities (133)-(134), it follows that \( \frac{d}{dt}a(t) \) and \( \frac{d}{dt}b(t) \) are each bounded. Hence, from the definitions of \( a \) and \( b \), we obtain that \( \sup_{t \in I} \alpha^2 \) and \( \sup_{t \in I} \beta^2 \) are bounded for all \( t \in I \). Since \( \alpha^2 = (B^a + E^a)^2 \) and \( \beta^2 = (-B^a + E^a)^2 \), we conclude that \( B^a(x,t) \) and \( E^a(x,t) \) are bounded on \( \Sigma \times I \).

Step 4: Bounded Derivatives

Now that we have determined that \( E^a, H^a, \) and \( B^a \) are bounded on \( \Sigma \times I \), we proceed to show that the first derivatives of these functions, and subsequently all higher order derivatives, are bounded on \( \Sigma \times I \).

We start with \( H^a \). From (87) and (89), it follows that since \( E^a, H^a, \) and \( B^a \) are bounded, then \( \partial_x H^a \) and \( \partial_t H^a \) are bounded. Similarly, if the order \( n \) derivatives of \( E^a, H^a, \) and \( B^a \) are bounded, then it follows from (derivatives of) (87) and (89) that the order \( n+1 \) derivatives of \( H^a \) are bounded. Hence, (by induction on \( n \)), the derivatives of \( H^a \) to all orders are bounded.

For \( E^a \) and \( B^a \), we use light cone arguments much like step 3, but involving a "derivative stress-energy" tensor. Specifically, let

\begin{equation}
T_{(1)\mu} = \frac{1}{4} (\partial_x B + \partial_x E)^2 + \frac{\lambda}{4} (v_1 + v_n) \partial_x H^a (\partial_x B^b + \partial_x E^b)_{ab} \tag{140}
\end{equation}

\begin{equation}
T_{(1)\nu} = \frac{1}{4} (-\partial_x B + \partial_x E)^2 - \frac{\lambda}{4} (-v_1 + v_n) \partial_x H^a (-\partial_x B^b + \partial_x E^b)_{ab} \tag{141}
\end{equation}

\begin{equation}
T_{(1)\tau} = -\frac{1}{2} (\partial_x H)^2 + \frac{\lambda}{4} (v_1 + v_n) \partial_x H^a (\partial_x B^b + \partial_x E^b)_{ab} \tag{142}
\end{equation}

\begin{equation}
T_{(1)\tau} = -\frac{1}{2} (\partial_x H)^2 - \frac{\lambda}{4} (v_1 + v_n) \partial_x H^a (-\partial_x B^b + \partial_x E^b)_{ab} \tag{143}
\end{equation}

as defined analogously to the stress-energy components (102) to (105). Then, as a consequence of the field equations (87) to (90), we find

\begin{equation}
\partial_n T_{(1)\mu} + \partial_\mu T_{(1)\tau} = Y_1(E, H, B; \partial_x E, \partial_x B) \tag{144}
\end{equation}

and

\begin{equation}
\partial_n T_{(1)\nu} + \partial_\nu T_{(1)\tau} = Y_2(E, H, B; \partial_x E, \partial_x B) \tag{145}
\end{equation}

where \( Y_1 \) and \( Y_2 \) are homogeneous quadratic in \( \partial_x E \) and \( \partial_x B \), with bounded coefficients. Although (144) and (145) are not strict conservation equations, we can nevertheless proceed similarly to step 3.
Through use of the field equations, we can express (144) and (145) as

\[ \partial_t (\partial_x B + \partial_x E)^2 = \tilde{Y}_1 (E, H, B; \partial_x E, \partial_x B) \] (146)

and

\[ \partial_t (-\partial_x B + \partial_x E)^2 = \tilde{Y}_2 (E, H, B; \partial_x E, \partial_x B) \] (147)

with \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) of the same nature as \( Y_1 \) and \( Y_2 \). It then follows from standard algebraic inequalities that

\[ \partial_t (\partial_x B + \partial_x E)^2 \leq k_{10} |\partial_x E|^2 + k_{20} |\partial_x E|^2, \] (148)

\[ \partial_t (-\partial_x B + \partial_x E)^2 \leq k_{21} |\partial_x B|^2 + k_{22} |\partial_x E|^2. \] (149)

We now apply the light cone arguments of step 3 to the differential inequalities (148) and (149): Starting at an arbitrary point in \( \Sigma \times I \), we integrate (148) and (149) back to the initial surface along light rays generated by \( \partial_x \) and \( \partial_t \). Taking suprema over \( \Sigma \) and adding the resulting inequalities, we obtain

\[ \sup_{x \in \Sigma} [(\partial_x B(x, t))^2 + (\partial_x E(x, t))^2] \leq k_{23} + k_{24} \int_0^t \sup_{x \in \Sigma} [(\partial_x B(x, s))^2 + (\partial_x E(x, s))^2] ds. \] (150)

Applying the Gronwall inequality to (150) shows that \( |\partial_x B| \) and \( |\partial_x E| \) are bounded for all \( t \in I \). With \( E^a, H^a, B^a, \partial_x E^a \) and \( \partial_x B^a \) bounded, it follows from the evolution equations (88) and (90) that \( \partial_t E^a \) and \( \partial_t B^a \) are bounded as well.

The previous argument can be applied to all orders of derivatives of the fields. This establishes that \( E^a, H^a, B^a \), and all of their derivatives are bounded on \( \Sigma \times I \). Global existence now follows from the usual "open-closed" arguments.

5 GLOBAL EXISTENCE FOR EQUIVARIANT FRAME FIELD EQUATIONS WITH TORSION

As discussed in Section 3, while left-equivariant wave maps (12) correspond to invariant frame fields (61), conjugate-equivariant wave maps (14) and right-equivariant wave maps (13) correspond to equivariant frame fields (62). In this section we show that global existence holds for smooth solutions to the Cauchy problem for translation equivariant frame fields.
We first note that by definition of translation equivariance, \( K^a_{\nu}(x, y, t) \) can be expressed as

\[
K^a_{\nu}(x, y, t) = \exp(-yR)E^a(x, t) \exp(yR) \tag{151}
\]

\[
K^a_{\nu}(x, y, t) = \exp(-yR)H^a(x, t) \exp(yR) \tag{152}
\]

\[
K^a_{\nu}(x, y, t) = \exp(-yR)B^a(x, t) \exp(yR) \tag{153}
\]

in terms of some Lie-algebra valued fields \( \{ E^a(x, t), H^a(x, t), B^a(x, t) \} \) which do not depend on \( y \). Here \( R \) is a fixed (constant) element in the Lie algebra; the left multiplication by \( \exp(-yR) \) combined with right multiplication by \( \exp(yR) \) denotes the adjoint action of a one-parameter Lie subgroup on the Lie algebra.

Substituting expressions (151) to (153) into the frame field equations with torsion (56) and (58) on Minkowski space, we obtain the following 1+1 reduced PDE system provided that \( C_{\alpha \beta} \) and \( Q^{\alpha}_{\beta \nu} \) are invariant under the adjoint action of \( \exp(yR) \).

We note that the only difference between these equations for translation equivariant frame fields and equations (87) to (90) for translation invariant frame fields is the presence of the commutator terms involving \( R \).

While the expressions for the field equations are changed somewhat in passing from invariant to equivariant frame fields, the expressions for the stress-energy components (93) to (100) and (102) to (105) do not change at all. (In particular, while \( K^a_{\nu} \) is not independent of \( y \), the quadratic expressions \( K^a_{\nu}K^\beta_{\nu \beta} \) and \( Q^{\alpha}_{\beta \nu}K^\beta_{\nu \beta} \) are invariant, and consequently so is \( T_{\mu \nu} \).)

Initial data for the Cauchy problem for translation equivariant frame fields consists of Lie-algebra valued functions \( \{ E^a(x), H^a(x), B^a(x) \} \) on \( \Sigma \) which satisfy the constraint

\[
\partial_\nu H^a = -C_{\beta \epsilon} E^\beta (H^\epsilon - R^\epsilon) \tag{158}
\]

A solution to the Cauchy problem is a set of fields \( \{ E^a(x, t), H^a(x, t), B^a(x, t) \} \) satisfying equations (154) to (157) and the initial conditions (92).

The global existence result, and its corollary, are stated as follows. Let \( \Sigma \) denote \( R^1 \) or \( S^1 \), and introduce coordinates \( (x, t) \) for \( \Sigma \times R^1 \). Fix constants
Let $G$ be a Lie group and let $R^a$ be a fixed (constant) vector in the Lie algebra of $G$. Assume $G$ admits on its Lie algebra a positive definite metric tensor $g_{ab}$ and a skew tensor $p_{ab}$ which are each invariant under the adjoint action of the Lie subgroup generated by $R^a$:

\[
g_{ac}C_{bc}^a R^c = -g_{bc}C_{ac}^b R^c \quad (159)
\]
\[
p_{ac}C_{bc}^a R^c = p_{bc}C_{ac}^b R^c \quad (160)
\]

where $C_{bc}^a$ denotes the Lie-algebra commutator structure tensor. Let $Q_{bc}^a$ be the tensor defined by (54).

**Theorem 4.** Let $\lambda$ be a small constant. For any smooth compact support initial data (92) satisfying (158), the Cauchy problem (154) to (157) has a unique smooth global solution $\{E^a(x,t), H^a(x,t), B^a(x,t)\}$ for all $t \in \mathbb{R}^1$.

From Propositions 2 and 3 we obtain a corresponding result for wave maps.

**Theorem 5.** The Cauchy problems for conjugate-translation equivariant wave maps (14) and right-translation equivariant wave maps (13), with or without torsion, have unique smooth global solutions for all smooth compact support initial data.

**Remark 2:** Under the translation invariant form (61) for frame fields, which corresponds to left-translation equivariant (12) or translation invariant (10) wave maps, the reduction of the frame field equations and corresponding wave map equation is consistent for any Lie group target. However, this is not the case under the translation equivariant form (62) for frame fields, which corresponds to conjugate-translation equivariant (14) or right-translation equivariant (12) wave maps. The translation equivariance ansatz gives a consistent reduction of the frame field equations and corresponding wave map equation only if the target geometry $(G, g, p)$ is invariant under right multiplication by the translation group generated by the Lie algebra element $R$ appearing in (12) to (14) for wave maps and (62) for frame fields. We refer to this condition, given by (159) and (160), as translation invariance of the target. As shown in Proposition A in the appendix, every compact semi-simple Lie group $G$ admits a translation invariant geometry $(G, g, p)$, except that the dimension of $G$ must be greater than three to support a non-zero torsion $Q$ (see Remark 1).
Proof of Theorem 4:

The proof of Theorem 4 is very similar to that of Theorem 2. We summarize the differences (if any) in each step.

Step 1: Conserved Energy

Since the expression for the energy is unchanged and since it is conserved, there are no changes in obtaining $L^2$ bounds for $E^a(x,t), H^a(x,t), B^a(x,t)$.

Step 2: Bounded $\mathcal{H}^a$

Instead of (112), we have

$$\int_\Sigma |\partial_x H^a| dx \leq \int_\Sigma |C_{bc}^a E^b H^c| dx + \int_\Sigma |C_{bc}^a E^b R^c| dx.$$  \hspace{1cm} (161)

The first of the two terms on the right hand side of (161) may be handled as in (113). As for the second term, we have

$$\int_\Sigma |C_{bc}^a E^b R^c| dx \leq k_{25} \int_\Sigma |E^b| dx$$
$$< k_{25} (\int_\Sigma |E^b| dx)^{1/2}$$
$$\leq k_{26}$$ \hspace{1cm} (162)

where the second inequality uses the compact support of $E^b$ together with the Holder inequality, and the last inequality follows from the $L^2$ bound on $E^a$. Hence we obtain

$$\int_\Sigma |\partial_x H^a| dx \leq k_{27}$$ \hspace{1cm} (163)

analogous to (114).

If $\Sigma = R^2$, the argument leading to a pointwise bound on $H^a(x,t)$ for $t \in I$ proceeds exactly as in the proof of Theorem 2. If $\Sigma = S^1$, then we need to modify the argument which begins with (116). We have

$$\int_{S^1} H^a(x,t) dx = - \int_0^t \int_{S^1} C^a_{bc} B^b(x,s) (H^c(x,s) + R^c) dx ds$$
$$+ \int_{S^1} H^a(x,t_0) dx.$$  \hspace{1cm} (164)

The term $\int_0^t \int_{S^1} C^a_{bc} B^b(x,s) R^c dx ds$ can be bounded from above using the same quadratic inequality that is used in (162), and so we obtain
\[ \left| \int_{S^1} H^a(x, t) \, dx \right| \leq k_{28} \]  

(165)

analogous to (119). The argument for pointwise bounds on \( H^a(x, t) \) for \( \Sigma = S^1 \) can then be completed as in the proof of Theorem 2.

Step 3: Bounded \( E^a \) and \( B^a \)

From inequalities (126) and (127) onward, the light-cone arguments used to bound \( E^a(x, t) \) and \( B^a(x, t) \) in the proof of Theorem 2 work identically to bound \( E^a(x, t) \) and \( B^a(x, t) \) here. To arrive at (126) and (127) we use the following equations, analogous to (122) and (123),

\[
\partial_a (B - E)^2 = -2H^b(H^c + R^c) (B^a + E^a) C_{abc} \\
-2\lambda(v_t + v_z) H^b (-B^b + E^b)(B^c + E^c) Q_{bca} \\
+2\lambda(v_t - v_z) C_{bc} a R^b(B^a + E^a)(B^b - E^b)p_{ab}, \quad (166)
\]

\[
\partial_a (-B + E)^2 = -2H^b(H^c + R^c) (-B^a + E^a) C_{abc} \\
-2\lambda(-v_t + v_z) H^b (-B^b + E^b)(B^c + E^c) Q_{bca} \\
+2\lambda(v_t + v_z) C_{bc} a R^b(-B^a + E^a)(B^b + E^b)p_{ab}, \quad (167)
\]

Adopting the notation \( \alpha^a := B^a + E^a \), \( \beta^a := -B^a + E^a \), these equations become

\[
\partial_a \alpha^2 = -2C_{abc} H^b(H^c + R^c) \alpha^a \\
-2\lambda(v_t + v_z) Q_{bca} a^2 \alpha^c + 2\lambda(v_t + v_z) R^b \alpha^c \beta^b C_{bca} a^2 p_{ab}, \quad (168)
\]

\[
\partial_t \beta^2 = -2C_{abc} H^b(H^c + R^c) \beta^a \\
-2\lambda(-v_t + v_z) Q_{bca} a^2 \beta^c + 2\lambda(v_t + v_z) R^b \beta^c \alpha^b C_{bca} a^2 p_{ab}. \quad (169)
\]

Then, noting that \( C_{abc} \), \( Q_{abc} \), \( \lambda \), \( v_t \), \( v_z \) and \( R \) are constant, and recalling that \( H^a \) is bounded on \( \Sigma \times I \), we obtain

\[
\partial_a \alpha^2 \leq k_{29} \sqrt{\alpha^2} + k_{30} \sqrt{\alpha^2} \sqrt{\beta^2} \quad (170)
\]

and

\[
\partial_t \beta^2 \leq k_{31} \sqrt{\beta^2} + k_{32} \sqrt{\alpha^2} \sqrt{\beta^2} \quad (171)
\]

which are identical to (126) and (127).
Step 4: Bounded Derivatives

One can see in Step 2 and Step 3 that the presence of the commutator terms involving $R$ in the field equations (154) to (157) changes little in the arguments for boundedness, since these extra terms are easily controlled by the analogous quadratic terms appearing in the equations. The same holds true for Step 4. We can define the derivative stress-energy components just as in (140) to (143) and then obtain conservation equations similar to (144) to (145), with small modifications in the expressions $Y_1$ and $Y_2$ which appear there. These modifications are readily handled in deriving the estimate (148) and (149). The rest of the argument proceeds unchanged.

Hence we obtain global existence.

6 CONCLUDING REMARKS

The wave map global existence results we have obtained here extend previous work in two significant ways. First, our study of translation equivariant critical wave maps for Lie group targets (Theorems 3 and 5) provides a counterpart to work on rotationally equivariant critical wave maps for symmetric-space targets (see [4,6]). Second, our inclusion of torsion gives an interesting generalization of critical wave maps for arbitrary targets, which ties into current work on integrable chiral models in 2+1 dimensions in the case of Lie group targets [8].

Furthermore, our results demonstrate the utility of the frame formulation of wave maps for Lie group targets (Proposition 1). The translation-equivariant reduction of critical wave maps studied here is motivated by this formulation and the analysis is especially straightforward in terms of frames. An important question to investigate for future work is how the frame formulation might help in understanding the unreduced critical wave map equation for general Lie group targets and symmetric-space targets.

APPENDIX

Proposition A. Let $G$ be a semi-simple Lie group with commutator structure tensor $C_{bc}^a$.

1. The Lie algebra of $G$ admits a translation invariant (159) positive-definite metric $g_{ab}$ if $G$ is compact.

2. The Lie algebra of $G$ admits a translation invariant (160) skew-tensor
$p_{ab}$ with non-zero torsion (54) if $G$ is compact and has dimension greater than three.

3. If $G$ has dimension three then the torsion (54) is zero for every skew-tensor $p_{ab}$ on the Lie algebra of $G$.

Proof of 1: If $G$ is compact then its Lie algebra admits an invariant positive-definite metric $g_{ab}$ (see, e.g. [13]), which satisfies

$$g_{ac} C_{bc}^e = -g_{bc} C_{ac}^e.$$  \hspace{1cm} (172)

(In particular, the Cartan-Killing metric given by $g_{ab} := -C_{ac}^e C_{bc}^e$ is both invariant and positive-definite.) Hence condition (159) holds.

Proof of 2 and 3: Hereafter $g_{ab}$ denotes the Cartan-Killing metric. We first remark that, for any $G$, the natural construction

$$p_{ab} := C_{ab}^d g_{de} R^e$$  \hspace{1cm} (173)

is easily seen to yield a translation invariant skew-tensor. But the resulting torsion tensor (54) is always zero, since

$$p_{e[a} C_{bc]}^e = C_{e[a}^d C_{bc]}^e g_{df} R^f = 0$$  \hspace{1cm} (174)

by the Jacobi identity.

In three dimensions it is easy to show that $C_{ab}^e g_{ae}$ must be proportional to the totally-skew Levi-Civita tensor $\epsilon_{abc}$, while any skew-tensor $p_{ab}$ can be expressed in the form

$$p_{ab} = \epsilon_{abc} p^c$$  \hspace{1cm} (175)

for some vector $p^c$ in the Lie algebra of $G$. Thus, it follows that $p_{ab}$ must have the form (173) where $R^e$ is proportional to $p^e$, and hence from (173) and (174) we have that the torsion tensor (54) is zero. This shows that there is no torsion for any three-dimensional $G$ (and hence none in particular with $p_{ab}$ being translation invariant).

Now suppose $G$ has dimension greater than three. In this case, $G$ must have rank greater than one and hence the Lie algebra of $G$ possesses an abelian subalgebra of dimension at least two (see, e.g. [13]). This allows the explicit construction of a translation invariant skew-tensor $p_{ab}$ as follows. Let $p^a$, $q^a$ be any two (linearly independent) commuting vectors in the Lie algebra of $G$, so $p^a q^b C_{ab}^e = 0$, and let $p_e := g_{ea} p^a$, $q_e := g_{ea} q^a$. Set $R^a := \alpha p^a + \beta q^a \neq 0$.
with constants $\alpha, \beta$. Then it is straightforward to show that the skew-tensor defined by

$$p_{ab} := 2[p_a q_b]$$  \hspace{1cm} (176)$$

is translation invariant as a consequence of $p$ and $q$ commuting with $R$. Now it remains to show that the torsion tensor given by (54) and (176) is non-zero.

We have

$$g_{ad} Q^d_{bc} = 3/2(p_e q_a C_{be} - q_e p_a C_{be})$$  \hspace{1cm} (177)$$

To show that the tensor (177) is non-zero when $G$ is compact, we contract (177) with the vector $s^a = p^a q^e - q^a p^e$ satisfying $s^a q_a = 0$. This yields

$$s^a g_{ad} Q^d_{bc} = -1/2 s^a p_a C_{be} q_e$$  \hspace{1cm} (178)$$

with $s^a p_a = p^a p_a q^d q_d - (p^a q_a)^2 \neq 0$ due to positive-definiteness of $g_{ab}$. Moreover, since $G$ is semi-simple, its Lie algebra has empty center and so $C_{be} q_e = C_{ab} q^a q^b$ is non-zero (that is, there exists a vector $v^b$ so that $C_{ab} q^a v^b \neq 0$). Therefore, $s^a g_{ad} Q^d_{bc}$ is non-zero and thus so is the torsion tensor (177).

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NON-EXISTENCE RESULTS FOR SEMILINEAR KOHN-LAPLACE EQUATIONS IN UNBOUNDED DOMAINS

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1 Introduction and main results

In this paper we prove several non-existence theorems for the boundary value problem

\[
\begin{cases}
  -\Delta_{\mathbb{H}^n} u = f(u) & \text{in } \Omega, \\
  u \in S^1_0(\Omega), & u \neq 0,
\end{cases}
\]

where \( \Omega \) is a connected unbounded open subset of the Heisenberg group \( \mathbb{H}^n \), \( \Delta_{\mathbb{H}^n} \) denotes the Kohn Laplacian on \( \mathbb{H}^n \) and \( S^1_0(\Omega) \) is the Sobolev-Stein space, completion of \( C^\infty_0(\Omega) \) with respect to the norm

\[
\|u\|_{S^1(\Omega)} := \left( \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 \right)^{\frac{1}{2}}.
\]

\( \nabla_{\mathbb{H}^n} \) stands for the intrinsic gradient on \( \mathbb{H}^n \). We shall denote by \( Q := 2n + 2 \) the homogeneous dimension of \( \mathbb{H}^n \) and by \( Q^* := \frac{2Q}{Q-2} \) the exponent of the Sobolev-type embedding.
We directly refer to section 2 for more details about the notation and definitions we need.

Throughout the paper we shall assume the function $f$ locally lipschitz-continuous on $\mathbb{R}$ and satisfying the following conditions:

\[(H) \quad f(s) = O(|s|^m) \text{ as } s \to 0 \quad \text{and} \quad f(s) = O(|s|^\mu) \text{ as } |s| \to \infty,\]

for suitable positive constants $m, \mu > 1$.

Our aim is to prove non-existence results for problem (1.1), when $\Omega$ belongs to certain classes of open subsets of $\mathbb{H}^n$, by means of some techniques first introduced in [26] and [28]. For example, when $f(u) = u^m$ and $\Omega$ is a halfspace of $\mathbb{H}^n$ we obtain the following result.

**Theorem 1.1** Let $u$ be a nonnegative weak solution of

\[
\begin{cases}
-\Delta_{\mathbb{H}^n} u = u^m & \text{in } \Omega, \\
u \in S^1_0(\Omega).
\end{cases}
\]

(1) If $m > \frac{Q}{2}$ and $u^{m-1} \in L^\frac{Q}{m-1}(\Omega)$, then $u \equiv 0$.

(2) If $1 < m \leq \frac{Q}{2}$ and $u^{m-1} \in L^\frac{Q}{m-1}(\Omega)$, then $u \equiv 0$.

In Theorem 1.1 and in what follows, we agree to let $v \in L^p(\Omega)$ if there exists $p < q$ such that $v \in L^p(\Omega)$. We would like to remark that in the critical case $m = \frac{Q+2}{Q-2}$ the condition $u^{m-1} \in L^\frac{Q}{m-1}(\Omega)$ follows from the embedding $S^1_0 \hookrightarrow L^Q$.

This special case has been studied in the previous papers [26] and [28].

In the classical context of the Laplace operator, results like that of Theorem 1.1 are well-known and have been established in [13] and [18, 19]. However, we stress that novel and significant difficulties arise in the setting of the operator $\Delta_{\mathbb{H}^n}$. These difficulties are mainly due to the lack of good a priori estimates for the derivative $\partial_\nu u$ of a weak solution $u$ to (1.1). Indeed such a derivative, in the Heisenberg group setting, should be considered a second order derivative, since $\partial_\nu = -\frac{i}{4}[X_j,Y_j]$ being $X_j, Y_j, j = 1, \ldots, n$, a basis for the Lie algebra of $\mathbb{H}^n$. The key point of our approach is to deduce the needed estimate of $\partial_\nu u$ from the hypothesis $u^{m-1} \in L^\frac{Q}{m-1}$. 

\[S^1_0(\mathbb{H}^n) \hookrightarrow L^Q(\mathbb{H}^n).\]
Theorem 1.1 holds for a more general class of domains $\Omega$ and nonlinearities $f$.

Definition 1.2 We shall say that a smooth open set $\Omega$ is of class $A$ if
\[(A)\quad \Omega = \hat{\Omega} \times \mathbb{R}, \text{ where } \hat{\Omega} \subset \mathbb{R}^2.\]

We also suppose the boundary $\partial \hat{\Omega}$ regular enough to assure that every bounded weak solution $u$ to (1.1) is Hölder-continuous up to the boundary with an exponent $\alpha \in [0,1]$. This property holds if there exist two positive constants $\delta$ and $r_0$ such that
\[
|B_d(\xi, r) \setminus \Omega| \geq \delta |B_d(\xi, r)| \quad \forall \xi \in \partial \Omega \quad \forall r \in ]0, r_0[. \tag{1.2}
\]
Here $B_d(\xi, r)$ denotes the intrinsic ball with center at $\xi$ and radius $r$, while $|\cdot|$ stands for the Lebesgue measure in $\mathbb{H}^n$.

Definition 1.3 We will say that a domain $\Omega$ of class $A$ is $\tau$-starshaped if there exists $z_0 \in \mathbb{R}^{2n}$ such that, denoting by $\nu \in \mathbb{R}^{2n}$ the outer unit normal to $\partial \hat{\Omega}$, $\langle z_0, \nu \rangle \geq 0$ everywhere on $\partial \hat{\Omega}$ and $\langle z_0, \nu \rangle > 0$ in at least one point of $\partial \hat{\Omega}$. Moreover, we will say that $\Omega$ is $\delta$-starshaped if $\hat{\Omega}$ is starshaped in the usual sense.

Definition 1.4 Finally, we shall say that an open set $\Omega \subseteq \mathbb{H}^n$ is of class $B$ if there exists $\lambda \geq 0$ such that
\[(B)\quad \Omega = \{(z,t) \in \mathbb{H}^n | t > \lambda |z|^2\}.\]

We explicitly remark that every open set of class $B$ satisfies the regularity assumption (1.2). Actually, we shall prove our results for a class $B$ of domains slightly more general than this one. We refer to section 4 for more details.

In order to state our main results, next Theorems 1.5 and 1.6, it is convenient to define, for any $s \in \mathbb{R}$, $F(s) := \int_0^s f(r)dr$ and $G(s) := 2QF(s) - (Q - 2)sf(s)$.

Theorem 1.5 Suppose one of the following conditions holds:
(a) $\Omega$ is a domain of class $B$;
(b) $\Omega$ is a $\tau$-starshaped domain of class $A$ and $f \geq 0$;
(c) $\Omega$ is a $\delta$-starshaped domain of class $A$, $f \geq 0$, $G(s) \leq 0$ for any $s \geq 0$ and $n \geq 2$;
(d) $\Omega$ is of class $A$, $\mathbb{H}^n \setminus \overline{\Omega}$ is $\delta$-starshaped, $f \geq 0$, $G(s) \geq 0$ for any $s \geq 0$ and $n \geq 2$.

Then, problem (1.1) has no weak nonnegative solutions $u$ such that

$$\frac{f(u)}{u} \in L^\frac{q}{q-1}(\Omega) \quad \text{if} \quad m > \frac{Q^*}{2}, \quad \frac{f(u)}{u} \in L^\frac{q}{q-1} \cap L^\frac{q}{q-1}(\Omega) \quad \text{if} \quad 1 < m \leq \frac{Q^*}{2}. \quad (1.3)$$

In (1.3) $m$ denotes the exponent in (H); moreover we agree to let $\frac{Q^*}{2}$ when $u = 0$. We explicitly remark that (1.3) is always true for a function $u \in \mathcal{S}_0^1(\Omega)$ if $m$ and $\mu$ in (H) satisfy $m \geq \frac{2+\mu}{Q^*} \geq \mu$.

A similar result holds when $\Omega = \mathbb{H}^n$.

**Theorem 1.6** Suppose one of the following conditions holds:
(a) $G \geq 0$, $G(s) > 0$ a.e. in a neighborhood of $s = 0$ and $n + m > 1 + \frac{3}{2}$;
(b) $G \leq 0$, $G(s) < 0$ a.e. in a neighborhood of $s = 0$ and $n + m > 1 + \frac{3}{2}$.

Then, if $\Omega = \mathbb{H}^n$, problem (1.1) has no weak solutions $u$ such that

$$\frac{f(u)}{u} \in L^\frac{q}{q-1}(\mathbb{H}^n) \quad \text{if} \quad m > \frac{Q^*}{2}, \quad \frac{f(u)}{u} \in L^\frac{q}{q-1} \cap L^\frac{q}{q-1}(\mathbb{H}^n) \quad \text{if} \quad 1 < m \leq \frac{Q^*}{2}. \quad (1.4)$$

When $f(s) = |s|^{m-1}s$ we have

$$G(s) = 2Q\left(\frac{1}{m+1} - \frac{1}{Q^*}\right)|s|^{m+1}.$$

From the previous theorem we then obtain the following corollary.

**Corollary 1.7** Let $u \in \mathcal{S}_0^1(\mathbb{H}^n)$ be a weak solution to

$$-\Delta_{\mathbb{H}^n} u = |u|^{m-1}u, \quad m \neq \frac{Q^*}{Q-2}.$$

Let us also suppose $n + m > 1 + \frac{3}{2}$ and that (1.4) is satisfied. Then $u \equiv 0$.

From Theorem 1.6 we also obtain nonexistence results for open subsets of $\mathbb{H}^n$ of class $B$. Indeed, if $u \in \mathcal{S}_0^1(\Omega)$ is a weak solution to (1.1) and $\Omega$ is of class $B$, in section 4 we will show that, setting $u \equiv 0$ outside $\Omega$, $u$ is a solution to the equation in (1.1) on the whole $\mathbb{H}^n$. As a consequence, from Theorem 1.6 we straightforwardly obtain the following result.
Corollary 1.8 Let \( \Omega \subseteq \mathbb{R}^n \) be an open set of class \( B \). Let us also suppose \( G(s) \) of constant sign and a.e. different from zero in a neighborhood of \( s = 0 \). Finally, let \( n + m > 1 + \frac{3}{2} \). Then problem (1.1) has no weak solutions satisfying (1.3).

We explicitly remark that in Theorem 1.6, and in its Corollaries 1.7 and 1.8, we deal with solutions which may change sign, while in Theorem 1.5 we only consider nonnegative solutions. We need this sign restriction in order to apply the unique continuation results in [17] and the monotone techniques in [26].

Any halfspace of \( \mathbb{R}^n \) is either a \( \tau \)-starshaped domain of class \( A \) or equivalent to a domain of class \( B \) (namely to \( \Omega_{0} = \{ t > 0 \} \) up to a left translation and possibly to the "reflection" \( (x, y, t) \mapsto (y, x, -t) \)). Hence Theorem 1.1 is a corollary of Theorem 1.5, the latter holding for any halfspace of \( \mathbb{R}^n \).

When \( f(u) = u^{Q'-1} \) and \( \Omega \) is a cone of \( \mathbb{R}^n \), non-existence theorems for weak nonnegative solutions to (1.1) play a crucial role in studying Palais-Smale condition for the functional

\[
J(u) = \frac{1}{2} \int_{D} |\nabla u|^2 - \int_{D} F(u),
\]

where \( D \) is a bounded open set of \( \mathbb{R}^n \) and \( F(u) = \frac{1}{Q'} u^{Q'} \). This problem arises when solving the Dirichlet problem for critical semilinear equations on \( \mathbb{R}^n \), by means of variational critical point methods (see [11, 12, 31]). Equations like that in (1.1), with \( f(u) = u^{Q'-1} \), naturally appear in studying the Yamabe problem for CR manifolds (see [22, 23, 24]). Non-existence and existence results for Yamabe-type equations on the whole \( \mathbb{R}^n \) have been recently proved in [8, 27, 32]. We would also like to quote two papers by Birindelli-Capuzzo Dolcetta-Cutri [4, 5] containing non-existence results for semilinear Kohn-Laplace equations in some subcritical cases. Related results are also contained in [16, 17, 1, 6, 2, 3, 29, 30, 10, 7].

The starting point of our method are some identities of Rellich-Pohozaev type, due to Garofalo and Lanconelli [17]. By using these identities and with arguments similar to those employed in [17], we prove some general non-existence results that require strong asymptotic behavior at infinity of the derivative \( \partial_{\nu} u \).
of the solution $u$ (see Theorems 3.3, 3.5 and 4.2). We are able to find the needed estimate of $\partial_t u$ when $\Omega$ is invariant with respect to the translations in the $t$-direction or $\Omega$ is of type $B$. These two cases are studied with completely different techniques in section 3 and 4, respectively. Both techniques rely on the following general theorem for Schrödinger-type inequalities proved in [29] which is a crucial tool in our proofs.

**Theorem 1.9** Let $\Omega$ be an unbounded open subset of $\mathbb{R}^n$ and let $V \in L^p(\Omega) \cap L^2(\Omega)$ for some $q_1 < \frac{n}{2} < q_2$. If $u$ is a classical solution of

$$
\begin{cases}
|\Delta u| \leq |Vu| & \text{in } \Omega, \\
u = 0 & \text{in } \partial \Omega, \\
u(\xi) \to 0 & \text{as } d(\xi) \to \infty,
\end{cases}
$$

then $u(\xi) = O\left(\frac{1}{d^s}\right)$ as $d(\xi) \to \infty$, for every $s < Q - 2$.

The paper is organized as follows. In section 2 we complete the list of the notation and establish some preliminary results.

In section 3 we study the case when $\Omega$ is a domain of class $A$. We use an argument introduced in [26] based on the representation of the harmonic part of $\partial_t u$ as limit of a sequence of iterated mean value operators modeled on the geometry of $\Omega$.

In section 4 the domains $\Omega$ of class $B$ are considered. Exploiting the particular symmetries of $\Omega$ we estimate the behavior of $\partial_t u$ by means of a systematic use of ad hoc barrier functions. This method, introduced in [28], also requires a subtle study of the regularity of $u$ at the origin, the only characteristic point of $\partial \Omega$.

The short section 5 is devoted to the proof of Theorem 1.6 and some its variants.

Finally, in the Appendix, we prove $L^p$-estimates of weak solutions $u$ to (1.1) for any unbounded open set $\Omega \subseteq \mathbb{R}^n$. The main result, Theorem A.3, establishes that $u \in L^p(\Omega)$ for $\frac{n}{2} < p \leq \infty$ and $u(\xi) \to 0$ at infinity, if $\frac{\partial u}{\partial \nu} \in L^\frac{\infty}{p}$. If we read the equation (1.1) as a Schrödinger-type equation with
potential \( V := \frac{f(u)}{u} \). Theorem 1.9 together with Theorem A.3 provide optimal asymptotic behavior at infinity for the weak solution \( u \). Another consequence of Theorem A.3 is the following: if the constant \( m \) in the hypothesis (H) is greater than \( \frac{Q}{2} \), from the condition \( \frac{f(u)}{u} \in L^{q} \) it follows \( \frac{f(u)}{u} \in L^{q-} \). For this reason, when \( m > \frac{Q}{2} \) in our uniqueness conditions (1.3) and (1.4) we may only require \( \frac{f(u)}{u} \in L^{q} \).

We close this introduction by stressing that the assumption \( m > 1 \) in the hypothesis (H) is almost necessary in order to obtain non-existence results for (1.1) when \( \Omega \) is a domain of class \( A \). Indeed, if \( f(u) = -u + u^{m}, \ 1 < m < \frac{Q+2}{Q-2} \) and \( \Omega = \{ |z| < 1 \} \times \mathbb{R} \), problem (1.1) has a strictly positive solution. This assertion can be proved by using the same techniques as in the work by Biagini [1].

2 Notation and preliminary results

The Heisenberg group \( \mathbb{H}^{n} \), whose points will be denoted by \( \xi = (z, t) = (x, y, t) \), is the Lie group \((\mathbb{R}^{2n+1}, 0)\) with composition law defined by

\[
\xi \circ \xi' = (z + z', t + t' + 2\langle z', y \rangle - \langle z, y' \rangle),
\]

where \( \langle , \rangle \) denotes the inner product in \( \mathbb{R}^{n} \). The Kohn Laplacian on \( \mathbb{H}^{n} \) is the operator

\[
\Delta_{\mathbb{H}^{n}} = \sum_{j=1}^{n} (X_{j}^{2} + Y_{j}^{2}),
\]

where \( X_{j} = \partial_{x_{j}} + 2y_{j}\partial_{t} \), \( Y_{j} = \partial_{y_{j}} - 2x_{j}\partial_{t} \), \( j \in \{1, \ldots, n\} \), generate the real Lie algebra of left-invariant vector fields on \( \mathbb{H}^{n} \). The operator \( \Delta_{\mathbb{H}^{n}} \) is the prototype of the so-called sublaplacians on stratified nilpotent Lie groups. We set

\[
\nabla_{\mathbb{H}^{n}} = (X_{1}, \ldots, X_{n}, Y_{1}, \ldots, Y_{n}).
\]

A natural group of dilations on \( \mathbb{H}^{n} \) is given by

\[
\delta_{\lambda}(\xi) = (\lambda z, \lambda^{2}t), \ \lambda > 0.
\]

The Jacobian determinant of \( \delta_{\lambda} \) is \( \lambda^{Q} \) where
is the homogeneous dimension of $\mathbb{H}^n$. The operators $\nabla_{\mathbb{H}^n}$ and $\Delta_{\mathbb{H}^n}$ are invariant w.r.t. the left translations $\tau_\xi$ of $\mathbb{H}^n$ and homogeneous w.r.t. the dilations $\delta_\alpha$ of degree one and of degree two, respectively.

A remarkable analogy between the Kohn Laplacian and the classical Laplace operator is that a fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by [14]

$$ F(\xi) = \frac{c_Q}{d(\xi)^{n/2}} , \quad (2.3) $$

where $c_Q$ is a suitable positive constant and

$$ d(\xi) = (|\xi|^4 + t^2)^{1/4} . \quad (2.4) $$

Moreover, if we define $d(\xi, \xi') = d_{\mathbb{H}^n}(\xi) = d(\xi'^{-1} \circ \xi)$, then $d$ is a distance on $\mathbb{H}^n$. We shall denote by $B_d(\xi, r)$ the $d$-ball of center $\xi$ and radius $r$. By the left translation invariance of the distance $d$, we have $\tau_\xi(B_d(0, r)) = B_d(\xi, r)$. Moreover, since $d$ is homogeneous of degree 1 with respect to the dilations $\delta_\alpha$, we also have $\delta_\alpha(B_d(0, r)) = B_d(0, \alpha r)$ and $|B_d(\xi, r)| = r^{Q} |B_d(0, 1)|$. Here $| \cdot |$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. We also recall that the Lebesgue measure is a Haar measure on $\mathbb{H}^n$.

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality:

$$ \| \varphi \|_{Q^*}^2 \leq B_Q \| \nabla_{\mathbb{H}^n} \varphi \|_2^2 \quad \forall \varphi \in C_0^\infty(\mathbb{H}^n) , \quad Q^* := \frac{2Q}{Q-1} . \quad (2.5) $$

where $B_Q$ is a positive constant whose best value has been determined by Jerison and Lee in [24]. If $\Omega$ is an open subset of $\mathbb{H}^n$, we shall denote by $S^1(\Omega)$ the Sobolev space of the functions $u \in L^Q(\Omega)$ such that $\nabla_{\mathbb{H}^n} u \in L^2(\Omega)$. The norm in $S^1(\Omega)$ is given by

$$ \| u \|_{S^1(\Omega)} = \| u \|_{Q^*} + \| \nabla_{\mathbb{H}^n} u \|_2 . \quad (2.6) $$

We denote by $S_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to (2.6). By means of (2.5), this norm is equivalent in $S_0^1(\Omega)$ to that generated by the inner product
\( \langle u, v \rangle_{S_0^1} = \int_{\Omega} \langle \nabla_{\mathbb{R}^n} u, \nabla_{\mathbb{R}^n} v \rangle. \) 

(2.7)

Thus \( S_0^1(\Omega) \) is a Hilbert space. We emphasize that, for general unbounded domains, the space \( S_0^1(\Omega) \) is not embedded in \( L^2(\Omega) \). Moreover we denote by \( \Gamma^a(\Omega) \) the intrinsic Hölder spaces introduced in [15] (see also [17]).

A weak solution to the Dirichlet problem (1.1) is a function \( u \in S_0^1(\Omega) \) (\( u \neq 0 \)) such that

\[
\int_{\Omega} \langle \nabla_{\mathbb{R}^n} u, \nabla_{\mathbb{R}^n} \varphi \rangle = \int_{\Omega} f(u) \varphi \quad \forall \varphi \in S_0^1(\Omega).
\]

(2.8)

We remark that \( f(u) \in \mathcal{L}^0 \) ensures \( f(u) \varphi \in L^1 \), since \( u, \varphi \in S_0^1 \subseteq L^2(\mathbb{R}^n) \).

For every \( \xi_0 \in \mathbb{H}^n \) we define the vector field

\[
P^{\xi_0} = (x_0, t_0 + 2i(y_0, x) - (x_0, y)).
\]

(2.9)

We say that a smooth unbounded open subset \( \Omega \) of \( \mathbb{H}^n \) is \( \tau \)-starshaped with respect to a point \( \xi_0 \in \mathbb{H}^n \) if denoting by \( N \) the outer unit normal to \( \partial \Omega \) we have

\[
\langle P^{\xi_0}, N \rangle \geq 0
\]

(2.10)
everywhere on \( \partial \Omega \), and \( \langle P^{\xi_0}, N \rangle > 0 \) in at least one point of \( \partial \Omega \). Moreover we set \( X = (z, \tau t) \) and we say that a smooth open subset \( \Omega \) of \( \mathbb{H}^n \) is \( \delta \)-starshaped with respect to a point \( \xi_0 \in \Omega \) if, denoting by \( N \) the outer unit normal to the boundary of \( \tau_{\xi_0}^{-1}(\Omega) \), we have

\[
\langle X, N \rangle \geq 0
\]

(2.11)
everywhere on \( \partial(\tau_{\xi_0}^{-1}(\Omega)) \) (see [17]). We remark that these definitions agree with the ones given in section 1 when \( \Omega \) is of class \( A \).

We recall the following representation formula which can be found in [16]: if \( v \) is a \( C^2 \) function in an open subset \( A \) of \( \mathbb{H}^n \) and \( B_0(\xi, r) \subseteq A \) then

\[
v(\xi) = (M, v)(\xi) = \frac{Q}{r^{Q-1}} \int_0^{Q-1} \int_{B_0(\xi, \theta)} \left( \Gamma(\xi, \xi') - \frac{c_{Q-1}}{(Q-1)^2} \right) \Delta_{\mathbb{H}^n} v(\xi') d\xi' d\theta
\]

(2.12)
where $M$, is the mean value operator defined by

$$ (Mv)(ξ) = \frac{1}{α Timer} \int_{B_d(1,r)} ψ(ξ, ξ')v(ξ') dξ'. \quad (2.13) $$

Here we have set $Γ(ξ, ξ') = Γ(ξ) = Γ(ξ^{*-1} o ξ)$ and $ψ(ξ, ξ') = ψ(ξ^{*-1} o ξ)$,

where $ψ(ξ) = \frac{|ξ|^2}{d(ξ)^2}$. Moreover $α_q = \int_{B_d(0,1)} ψ(ξ)dξ$.

In section 3 and section 4 we shall need the following result.

Lemma 2.1 Let $g ∈ Γ^n(Δ^n) \cap L^p(Δ^n)$ for some $α ∈ [0, 1]$ and $p ∈ [1, 2]$. Let $w = Γ * g := \int_{Δ^n} Γ(ξ, ξ') g(ξ') dξ'$. Then $w ∈ Γ^n(Δ^n)$ and

$$ δ(ξ, ξ') = \int_{B_d(ξ, r)} δ(ξ, ξ') |g(ξ') - g(ξ)| dξ' + \int_{B_d(ξ, r)} δ(ξ, ξ') g(ξ') dξ', \quad \text{for every } ξ ∈ Δ^n \text{ and } r > 0. \quad (2.14) $$

for every $ξ ∈ Δ^n$ and $r > 0$. Moreover, if $g(ξ) = O(d(ξ)^{-s})$ as $d(ξ) \to \infty$ for some $s ∈ R$, then

$$ δ(ξ, ξ') = O(d(ξ)^{-r}), \text{ as } d(ξ) \to \infty \quad (2.15) $$

for every $r ∈ (0, 1 - α) \cup [1, s + 1] = \bigcup_{p = 1}^{s + 2}$.\n
Proof The proof of the first part is standard (see e.g. [26]). Let us prove (2.15). Let $ζ ∈ [0, 1]$. From (2.14), for every $ξ, ξ' ∈ Δ^n$ and $r > 0$, we get

$$ |δ(ξ, ξ')| ≤ c \int_{B_d(ξ, r)} d(ξ, ξ')^{-q} |g(ξ') - g(ξ)|^{ζ} \left( \sup_{B_d(ξ, r)} |g| \right)^{1-ζ} dξ' $$

$$ + c \int_{Δ^n \setminus B_d(ξ, r)} d(ξ, ξ')^{-q} |g(ξ')| dξ' $$

$$ \leq c \left( \sup_{B_d(ξ, r)} |g| \right)^{ζ} \int_{B_d(ξ, r)} d(ξ, ξ')^{-q} dξ' + c |||d(ξ')^{-q}||_{L^p(Δ^n \setminus B_d(ξ, r))} ||g||_p $$

$$ = c r^{α(1 - ζ)} \left( \sup_{B_d(ξ, r)} |g| \right)^{1-ζ} + cr^{-q}. $$

From this estimate, since $g(ξ) = O(d(ξ)^{-s})$ as $d(ξ) \to \infty$, choosing $r = \frac{d(ξ)}{2}$, we get $|δ(ξ, ξ')| ≤ cd(ξ)^{αs}d(ξ)^{(s-1)ζ} + cd(ξ)^{-q}$ at infinity. This proves (2.15).
3 Domains of class $A$

Let $\Omega$ be a domain of class $A$, $\Omega = \tilde{\Omega} \times \mathbb{R} \subseteq \mathbb{H}^n$, where $\tilde{\Omega} \subseteq \mathbb{R}^{2n}$. For every $\omega \in L^1_{\text{loc}}(\Omega)$ we define

$$T\omega : \Omega \to \mathbb{R}, \quad (T\omega)(\xi) = (M_{\tilde{\Omega}}(\omega))(\xi),$$

(3.1)

where $M_{\tilde{\Omega}}$ is the mean value operator introduced in (2.13) and

$$r(z,t) := \frac{1}{2}\text{dist}(z,\mathbb{R}^{2n} \setminus \tilde{\Omega}).$$

(3.2)

We explicitly remark that $\overline{B_d(\xi,r(\xi))} \subseteq \Omega$ for every $\xi \in \Omega$, $r \in C(\Omega, \mathbb{R}^+)$ and $r(z,t) = r(z,0)$ for any $(z,t) \in \Omega$, since $\partial \Omega$ is parallel to the $t$-axis. The following proposition states all the properties of the operator $T$ that we will use in this section.

**Proposition 3.1**

1. $T$ is a linear operator from $L^1_{\text{loc}}(\Omega)$ into $C(\Omega)$.

2. The operators $T$ and $\partial_t$ commute. More precisely if $\partial_t \omega \in C(\Omega)$ then also $\partial_t(T\omega) \in C(\Omega)$ and $\partial_t(T\omega) = T(\partial_t \omega)$.

3. $T$ is an increasing operator, i.e. $T\omega_1 \leq T\omega_2$ if $\omega_1 \leq \omega_2$.

4. If $\omega \in C^2(\Omega)$ and $\Delta_{\mathbb{H}^n} \omega \leq 0$ then $T\omega \leq \omega$.

5. If $\omega \in C^2(\Omega)$, $\Delta_{\mathbb{H}^n} \omega \leq 0$ and $\omega(\xi) \to 0$ as $\xi \to \partial \Omega \cup \{\infty\}$, then $(T^k \omega)_{k \in \mathbb{N}}$ is a monotone decreasing sequence pointwise convergent to zero in $\Omega$ ($T^k$ denotes the $k$-th iterated of the operator $T$).

**Proof**

1. We prove that, for every fixed $\omega \in L^1_{\text{loc}}(\Omega)$ and $\xi_0 \in \Omega$,

$$|T\omega(\xi) - T\omega(\xi_0)| \leq |M_{\tilde{\Omega}}(\omega(\xi)) - M_{\tilde{\Omega}}(\omega(\xi_0))| + |M_{\tilde{\Omega}}(\omega(\xi_0)) - M_{\tilde{\Omega}}(\omega(\xi))|,$$

On the one hand (2.13) gives

$$|M_{\tilde{\Omega}}(\omega(\xi)) - M_{\tilde{\Omega}}(\omega(\xi_0))| = \frac{1}{a \sqrt{t}} \left| \int_{B_d(\xi_0, r(\xi_0))} \psi(\xi, \eta) \omega(\eta) d\eta - \int_{B_d(\xi_0, r(\xi_0))} \psi(\xi_0, \zeta) \omega(\zeta) d\zeta \right|$$

$$= \frac{1}{a \sqrt{t}} \left| \int_{B_d(\xi_0, r(\xi_0))} \psi(\xi_0, \zeta)(\omega(\xi_0) \circ \xi_0^{-1} \circ \zeta - \omega(\zeta)) d\zeta \right|$$

$$\leq \frac{1}{a \sqrt{t}} \int_{B_d(\xi_0, r(\xi_0))} |\omega(\xi_0) \circ \xi_0^{-1} \circ \zeta - \omega(\zeta)| d\zeta.$$
The continuity of \( r \) and the \( L^1 \)-continuity theorem ensure that the far right hand term in the previous inequalities goes to zero as \( \xi \to \xi_0 \). On the other hand, the continuity of \( r \) yields \( \lim_{\xi \to \xi_0} M_{r(\xi)}(\omega(\xi_0)) = M_{r(\xi_0)}(\omega(\xi_0)) \). It then follows that \( T \omega \in C(\Omega) \).

(3)-(4) immediately follow from (3.1), (2.12) and (2.13).

(5) We first remark that, being \( \Delta_{\partial \Omega} \omega \leq 0 \) and \( \omega \to 0 \) at any point of \( \partial \Omega \cup \{\infty\} \), by the maximum principle \( \omega \geq 0 \). Moreover, since \( T \omega \leq \omega \) and \( T \) is increasing (see the previous assertions (3) and (4)), \( \omega \geq T^k \omega \geq T^{k+1} \omega \geq 0 \) for any \( k \in \mathbb{N} \). Then, there exists a function \( h : \Omega \to \mathbb{R} \) such that

\[
T^k \omega \searrow h \quad \text{and} \quad 0 \leq h \leq \omega. \tag{3.3}
\]

Hence \( h \in L^1_{loc}(\Omega) \). Recalling (3.1) and (2.13), we see that (3.3) implies

\[
h(\xi) = \lim_{k \to \infty} \left( T^{k+1} \omega(\xi) \right) = \lim_{k \to \infty} \left( \frac{1}{\alpha \Omega r(\xi)^{\frac{\alpha}{\gamma}}} \int_{B(\xi,r(\xi))} \psi(\xi,x) T^k \omega(\xi') dx' \right)
\]

\[
= \frac{1}{\alpha \Omega r(\xi)^{\frac{\alpha}{\gamma}}} \int_{B(\xi,r(\xi))} \psi(\xi,x') h(\xi') dx' = Th(\xi)
\]

for every \( \xi \in \Omega \). Therefore, (1) provides \( h = Th \in C(\Omega) \). Moreover (3.3) yields

\[
h \in C(\Omega), \quad h \equiv 0 \text{ in } \partial \Omega, \quad h(\xi) \to 0 \text{ as } d(\xi) \to \infty. \tag{3.4}
\]

Let us now assume by contradiction that \( h \) is not identically 0. From (3.3) and (3.4) there exists \( \xi_0 \in \Omega \) such that \( h(\xi_0) = \max_{\Omega} h > 0 \). Hence \( A = h^{-1}(\{h(\xi_0)\}) \) is closed and non empty. Moreover for every \( \xi \in A \), \( h = Th \) yields

\[
0 = h(\xi) - Th(\xi) = \frac{1}{\alpha \Omega r(\xi)^{\frac{\alpha}{\gamma}}} \int_{B(\xi,r(\xi))} \psi(\xi,x') (\max_{\Omega} h - h(\xi')) dx'.
\]

Then \( h(\xi') = \max_{\Omega} h = h(\xi_0) \) for every \( \xi' \in B_d(\xi, r(\xi)) \), i.e. \( B_d(\xi, r(\xi)) \subseteq A \). Therefore \( A \) is also open. This yields \( A = \Omega \) since \( \Omega \) is connected. In other words \( h \equiv h(\xi_0) > 0 \) in \( \Omega \), in contradiction with (3.4). Hence it has to be \( h \equiv 0 \) in \( \Omega \).

(2) We set \( F : \Omega \times B_d(0,1) \to \mathbb{R} \),

\[
F(\xi,\xi') = (\omega \circ r \circ \delta_{\tau(\xi)})(\xi') = \omega(z + r(\xi)z', t + r(\xi)^2 t' + 2r(\xi)(z' - x, y - \langle x, y \rangle)).
\]
From (3.2) we have $\partial_t r \equiv 0$ so that

$$\partial_t F(\xi', \xi') = ((\partial_\omega \circ \tau_\xi) \circ \delta_{\tau(\xi)})(\xi').$$

(3.5)

By means of a change of variable we obtain

$$T\omega(\xi) = \frac{1}{\alpha Q} \int_{B_d(\xi')} (\psi \circ \tau_\xi^{-1})\omega = \frac{1}{\alpha Q} \int_{B_d(\xi')} (\psi \circ \delta_{\tau(\xi)}) (\omega \circ \tau_\xi \circ \delta_{\tau(\xi)})$$

$$= \frac{1}{\alpha Q} \int_{B_d(\xi')} \psi(\xi') F(\xi', \xi') d\xi'.$$

Hence, from (3.5) and the continuity of $\partial_t \omega$, we get

$$\partial_t (T\omega)(\xi) = \frac{1}{\alpha Q} \int_{B_d(\xi')} \psi((\partial_\omega \circ \tau_\xi \circ \delta_{\tau(\xi)})) = \frac{1}{\alpha Q} \int_{B_d(\xi')} (\psi \circ \tau_\xi^{-1})\partial_\omega = T(\partial_t \omega)(\xi)$$

and the assertion follows, also keeping in mind (1).

$\square$

**Proposition 3.2** Let $\Omega$ be a domain of class $A$. Let $g \in \Gamma^{\alpha}_{\infty}(\mathbb{H}^n) \cap L^p(\mathbb{H}^n)$ for some $\alpha \in ]0,1[$ and $p \in [1, \frac{Q}{2}]$. Moreover assume that $g \geq 0$ and $g(\xi) = O(\frac{1}{d(\xi)^s})$ as $d(\xi) \to \infty$, for some $s > 0$. If $u$ is a solution of

$$\begin{align*}
-\Delta_{\mathbb{H}^n} u &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \partial \Omega, \\
 u(\xi) &\to 0 \quad \text{as } d(\xi) \to \infty,
\end{align*}$$

then

$$\partial_\tau u(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as } d(\xi) \to \infty,$$

for every $0 < \tau < s, \tau \leq \min\{Q - 2, \frac{Q}{p}\}$.

**Proof** We set $w = \Gamma^* g$. By Lemma 2.1, there exist $M > 0$ and $\xi_0 \not\in \Omega$ such that

$$-M d_{\xi_0}^{\tau} \leq \partial_\tau w \leq M d_{\xi_0}^{\tau}.$$

(3.6)

Since $0 < \tau \leq Q - 2$, we have $\Delta_{\mathbb{H}^n}(d_{\xi_0}^{-\tau}) \leq 0$ in $\Omega$. Thus, by means of Proposition 3.1-(3)-(4), for every $k \in \mathbb{N}$
\[ -Md_{R}^{-} \leq T^{k}(-Md_{R}^{-}) \leq T^{k}(\partial_{t}w) \leq T^{k}(Md_{R}^{+}) \leq Md_{R}^{+} \quad \text{in } \Omega. \quad (3.7) \]

We denote by \( v \) the \( \Delta_{R} \)-harmonic part of \( u \), i.e. \( v = w - u \). By Proposition 3.1-(4), \( Tv = v \). Moreover \( g \geq 0 \) and Proposition 3.1-(5) give \( T^{k}u \searrow 0 \). Since \( T \) is a linear operator we obtain

\[ T^{k}w = T^{k}v + T^{k}u = v + T^{k}u \searrow v. \quad (3.8) \]

Proposition 3.1-(2) and (3.8) yield

\[ \int_{\Omega} \Phi T^{k}(\partial_{t}w) = \int_{\Omega} \Phi \partial_{t}(T^{k}w) = -\int_{\Omega} (\partial_{t}\Phi) T^{k}w \xrightarrow{k \to \infty} -\int_{\Omega} v \partial_{t}\Phi = \int_{\Omega} \Phi \partial_{t}v \]

for every \( \Phi \in C_{0}^{\infty}(\Omega) \). From (3.9) and (3.7) we finally obtain \( |\int_{\Omega} \Phi \partial_{t}v| \leq M \int_{\Omega} \Phi d_{R}^{+} \) for every \( \Phi \in C_{0}^{\infty}(\Omega) \), \( \Phi \geq 0 \), which implies \( |\partial_{t}v| \leq Md_{R}^{+} \) in \( \Omega \).

Since also (3.6) holds, we conclude that \( |\partial_{t}u| \leq 2Md_{R}^{+} \) in \( \Omega \).

By means of the Rellich-Pohozaev type identities in [17] we can prove the following result.

**Theorem 3.3** Let \( \Omega \) be an unbounded \( \tau \)-starshaped domain of \( \mathbb{R}^{n} \). Then the problem

\[ \begin{cases} -\Delta_{\mathbb{R}^{n}} u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \quad (3.10) \]

has no nontrivial nonnegative solutions \( u \in \Gamma^{2}(\Omega) \) such that \( |\nabla_{\mathbb{R}^{n}} u| \in L^{2}(\Omega) \), \( F(u) \in L^{1}(\Omega) \) and \( \partial_{u}(\xi) = O\left( \frac{1}{d(\xi)^{2} + 1} \right) \), as \( d(\xi) \to \infty \).

**Proof** The proof follows the lines of the proof of [17, Theorem 2.4]. It is sufficient to use the condition \( \partial_{u}(\xi) = O\left( \frac{1}{d(\xi)^{2} + 1} \right) \) instead of the hypothesis \( (P_{\\mathbb{R}^{n}})/1 + d) \in L^{2} \) of [17]. We omit the details.

\[ \square \]

From Proposition 3.2, Theorem 1.9 and Theorem 3.3 we get the following theorem.
Theorem 3.4 Let $\Omega$ be a $\tau$-starshaped domain of class $A$ and assume $f \geq 0$. Then (1.1) has no nonnegative weak solutions $u \in L^\infty(\Omega)$ such that $u \to 0$ at $\infty$ and $\frac{u}{|u|} \in L^{\frac{Q}{Q-2}}(\Omega)$.

Proof Let $u$ be a solution to (1.1) satisfying the hypothesis of the theorem. We set $u = 0$ outside $\Omega$ and $g = f(u)$. Since $u \in S^1(\Omega) \cap L^\infty(\Omega)$, $g \in L^\infty(\Omega)$ and (1.2) holds, there exists $\alpha \in [0,1]$ such that $u \in \Gamma^\alpha(\mathbb{H}^n)$ (see [26, Th. A.1]). Hence $g \in \Gamma^\alpha(\mathbb{H}^n)$ ($f$ is locally lipschitz-continuous, $u \in L^\infty$) and $u \in \Gamma^{2+\alpha}_{\text{loc}}(\Omega)$ (see [26]). Moreover, since $\Omega$ is of class $A$, $\partial \Omega$ has no characteristic points and then $u \in \Gamma^2(\bar{\Omega})$ (see [25], [21]). By Theorem 3.3, we now only need to prove that $F(u) \in L^1(\Omega)$ and $\partial u(u) = O\left(\frac{1}{d(\xi)^{\frac{Q-2}{2}}(1+|\xi|)}\right)$, as $d(\xi) \to \infty$. We set $V = \frac{u}{|u|}$. Since $u$ is bounded and (H) holds, there exists $M > 0$ such that

$$0 \leq \frac{f(u)}{u} \leq Mu^{m-1} \in L^\infty.$$  

Hence $-\Delta g + u = 1'$, with $V \in L^0(\Omega) \cap L^p(\Omega)$, $q_1 < \frac{Q}{2} < q_2$, and by Theorem 1.9 we obtain

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as} \quad d(\xi) \to \infty, \quad \forall s < Q - 2.$$

Then $u \in L^p$ for every $p \in ]\frac{Q}{Q-2}, +\infty]$. In particular $u \in L^{m+1}$ and from (H) it follows $F(u) \in L^1(\Omega)$. Moreover $g \geq 0$ and (3.11)-(3.12) imply

$$g(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as} \quad d(\xi) \to \infty, \quad \forall s < m(Q - 2)$$

and $g \in L^p(\mathbb{H}^n)$ for some $p < \frac{Q}{Q-2} \leq \frac{Q}{2}$ (since $m > 1$). We can now apply Proposition 3.2 and obtain

$$\partial u(u) = O\left(\frac{1}{d(\xi)^{Q-2}}\right) \quad \text{as} \quad d(\xi) \to \infty.$$ 

Since $Q - 2 \geq \frac{Q-2}{2} + 1$, this completes the proof.

We now consider $\delta$-starshaped domains. Arguing as in [17] we can prove the following theorem.
Letting \( g = \frac{\sum u^2 + |P(u)|}{d^2} + |\nabla u|^2 \), we deduce that there exist a divergent sequence \((R_k)_{k \in \mathbb{N}}\) and a vanishing sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) such that
\[
\left\{ \begin{array}{l}
\int_{\Omega \cap B_{R_k}} \frac{\sum u^2 + |P(u)|}{d^2} + |\nabla u|^2 \frac{d\sigma}{d_0} = o\left(\frac{1}{R_k}\right), \quad \text{as } k \to +\infty, \\
\int_{\Omega \cap B_{R_k}} \left(\sum u^2 + |P(u)|\right) \frac{d\sigma}{d_0} = o\left(\frac{1}{\varepsilon_k}\right), \quad \text{as } k \to +\infty.
\end{array} \right. 
\]
(4.5)

Moreover, letting \( g \) be the characteristic function of the set \( B_r \), we get
\[
\int_0^r \left( \int_{\partial B_r} \frac{d\sigma}{d_0} \right) d\theta = \int_{B_r} d\xi = cr^Q,
\]
and, by differentiation, \( \int_{\partial B_r} \frac{d\sigma}{d_0} = cQr^{Q-1} \). Recalling (4.4) and (4.5) we finally obtain
\[
\int_{B_{R_k} \cap \Omega} |\nabla u|^2 (P, N) d\sigma \leq o(1) + \frac{c}{R_k^{Q+2}} \left( \int_{\Omega \cap B_{R_k}} \frac{|\nabla u|^2}{d_0} d\sigma \right)^{\frac{1}{2}} \left( \int_{\Omega \cap B_{R_k}} \frac{1}{d_0} d\sigma \right)^{\frac{1}{2}}
+ \frac{c}{\varepsilon_k^{Q+2}} \left( \int_{\partial \Omega \cap B_{R_k}} \frac{|\nabla u|^2}{d_0} d\sigma \right)^{\frac{1}{2}} \left( \int_{\partial \Omega \cap B_{R_k}} \frac{1}{d_0} d\sigma \right)^{\frac{1}{2}}
\leq o(1) + \frac{1}{R_k^{Q+2}} \left( o\left(\frac{1}{R_k}\right) \right)^{\frac{1}{2}} R_k^{\frac{Q}{2}+1} + \frac{1}{\varepsilon_k^{Q+2}} \left( o\left(\frac{1}{\varepsilon_k}\right) \right)^{\frac{1}{2}} \varepsilon_k^{\frac{Q}{2}+1}
= o(1), \quad \text{as } k \to +\infty.
\]

Now, as in [17] we can conclude that \( u \equiv 0 \) in \( \Omega \).

\( \square \)

**Remark 4.3** Since \((P, N) > 0\) everywhere on \( \partial \Omega \), the above proof shows that a solution \( u \) with no sign restriction, satisfying the other hypothesis of Theorem 4.2, is actually a solution to the equation on the whole \( \mathbb{H}^n \) (setting \( u = 0 \) outside \( \Omega \)).

**Proposition 4.4** Let \( u \in L^\infty(\Omega) \) be a weak solution to (1.1) such that \( u \to 0 \) at \( \infty \) and \( \frac{\partial u}{\partial n} \in L^2(\Omega) \). Then

(1) \( u \in \Gamma^+(\Omega \setminus \{0\}) \cap C(\Omega) \) is a classical solution to (3.10) and for every \( s < Q - 2 \),

(2) \( u \) is a weak solution to the equation on the whole \( \mathbb{H}^n \).
Moreover, if we continue \( u \) on \( \mathbb{H}^n \) by setting \( u = 0 \) outside \( \Omega \), there exist \( \beta_0 \in [0, 1] \) and \( M > 0 \) such that

\[
|u(\xi) - u(\xi')| \leq M d(\xi, \xi')^{\beta_0} \quad \forall \xi, \xi' \in \mathbb{H}^n \quad (4.7)
\]

(i.e. \( u \in \Gamma^{\beta_0}(\mathbb{H}^n) \)).

2. Setting \( w = \Gamma * (f(u)) \) we have \( w \in \Gamma^2(\mathbb{H}^n) \) and

\[
\partial_t w(\xi) = O(d(\xi)^{-2}), \quad \text{as } d(\xi) \to \infty. \quad (4.8)
\]

3. \( v := w - u \) is a classical solution of

\[
\begin{aligned}
-\Delta_{\mathbb{H}^n} v &= 0 & \text{in } \Omega \\
v &= w & \text{in } \partial \Omega
\end{aligned}
\quad (4.9)
\]

and for every \( s < Q - 2 \)

\[
v(\xi) = O(d(\xi)^{-s}), \quad \text{as } d(\xi) \to \infty, \xi \in \Omega. \quad (4.10)
\]

**Proof** It follows from Theorem 1.9 and Lemma 2.1, arguing as in the proof of Theorem 3.4 and recalling that \( \partial \Omega \) is characteristic only at the origin. To prove (4.10) we only need to notice that, setting \( V = \frac{f_{\xi} \Omega}{u} \), we have \( V \in L^{\frac{Q}{2}} \cap L^\infty \), so that

\[
|v(\xi)| \leq \int_{\xi}^{\Gamma_\xi} V \, \text{d}w + \int_{\xi}^{\Gamma_\xi} V \, \text{d}u \leq \frac{c}{d(\xi)}, \quad \text{for large } d(\xi).
\]

We now want to prove that a solution \( u \) as in Proposition 4.4 verifies the hypothesis of Theorem 4.2.

From now on (in this section) we shall then assume (1)-(2)-(3) of Proposition 4.4 and prove the following propositions.

**Proposition 4.5** If (4.1) holds, then \( \partial_t u(\xi) = O(d(\xi)^{\frac{2+Q}{2}}) \), as \( d(\xi) \to \infty \), \( \xi \in \Omega \).

**Proposition 4.6** If (4.2) holds, then \( |\nabla u|, |\partial_t u| \in L^\infty(\Omega \cap B_d(0,1)) \).
In the sequel we shall always assume (4.1) when looking for estimates at infinity (respectively (4.2) when looking for estimates at the origin). The first step of our approach consists in finding an estimate of \( \partial_t u \) at the boundary of \( \Omega \).

**Proposition 4.7** We have \( \partial_t u(\xi) = O(d(\xi)^{\frac{Q-a}{2}}) \), as \( d(\xi) \to \infty \), \( \xi \in \partial \Omega \).

We will use cylindrically symmetric barrier functions and split the proof in a remark and two lemmas. For every \( R \geq 1 \) we set

\[ A_R = \{ \xi = (z,t) \in \mathbb{R}^n \mid 0 < t - H(|z|) < 1, |z| > R \} \]

and for every \( \alpha > 0 \) and \( \beta \in [0,1] \) we define

\[ F_{\alpha,\beta}: A_1 \to \mathbb{R}^+, \quad F_{\alpha,\beta}(z,t) = \frac{(\sin(t - H(|z|)))^\beta}{|z|^\alpha}. \]

We also set \( F_\alpha = F_{\alpha,1} \).

**Remark 4.8** If \( \Phi(z,t) = \varphi(|z|,t) = \varphi(r,t) \) is a cylindrically symmetric regular function, then, as it has been noticed by Garofalo-Lanconelli in [17], we have

\[ \Delta_{\mathbb{S}^1} \Phi = \partial_r^2 \varphi + \frac{Q-3}{r} \partial_r \varphi + 4\partial^2 \varphi. \quad (4.11) \]

**Lemma 4.9** Let \( \alpha = \frac{Q-a}{2} \). For every \( \beta \in [0,1] \) there exist \( \delta_3 > 0 \) and \( R_3 \geq 1 \) such that

\[ -\Delta_{\mathbb{S}^1} F_{\alpha,\beta} \geq \delta_3 |z|^{2-\alpha} \quad \text{in} \quad A_{R_3}. \quad (4.12) \]

Moreover, there exists \( R_\alpha \geq 1 \) such that

\[ -\Delta_{\mathbb{S}^1} F_{\alpha} \geq |z|^2 F_\alpha \quad \text{in} \quad A_{R_\alpha}. \quad (4.13) \]

**Proof** Using (4.11), a computation yields

\[ -\Delta_{\mathbb{S}^1} F_{\alpha,\beta}(z,t) = F_{\alpha,\beta}(z,t) \left( \frac{Q(Q-4-\alpha)}{|z|^2} + \beta(H'(|z|)^2 + 4|z|^2) \right. \]

\[ + \beta \cot(t - H(|z|)) \left( \frac{H''(|z|)}{|z|} (Q - 3 - 2\alpha) + H''(|z|) \right) \]
Lemma 4.10 There exist $R \geq 1$ and $M > 0$ such that

$$|u| \leq MF_{\alpha,\beta} \quad \text{in } A_R. \quad (4.15)$$

Proof Let $\alpha = \frac{Q-2}{2}$. We first prove that for every large $R > 0$ there exists $M > 0$ such that

$$|u| \leq MF_{\alpha,\beta} \quad \text{in } \partial A_R \cup \{\infty\} \quad \forall \beta \in [0,1]. \quad (4.16)$$

Since $u = 0$ in $\partial \Omega$ and $u \to 0$ at infinity, we only need to prove (4.16) in $\partial_1 = \{t - H(|z|) = 1, |z| \geq R\}$ and $\partial_2 = \{0 \leq t - H(|z|) \leq 1, |z| = R\}$. From (4.6) it follows that for every $\xi = (z,1 + H(|z|)) \in \partial_1$ we have

$$|u(\xi)| \leq \frac{c}{d(\xi)^{\alpha}} \leq \frac{c}{|z|^\alpha} \leq M \frac{\sin 1}{|z|^\alpha} = MF_{\alpha}(\xi) \leq MF_{\alpha,\beta}(\xi).$$

On the other hand, $\partial_t u \in C(\partial_2)$. Hence for every $\xi \in \partial_2$

$$|u(\xi)| = |u(z,t) - u(z,H(|z|))| \leq (\max_{\partial_2} |\partial_t u|)(t - H(|z|))$$

$$\leq c_\alpha (\sin (t - H(|z|)))^\beta = M_{R,\alpha} \frac{(\sin (t - H(|z|)))^\beta}{R^\alpha} = MF_{\alpha,\beta}(\xi).$$

Therefore (4.16) holds.

We now set $\beta = \frac{1}{m}$. From (4.6), (4.12), (4.16) and (H) we deduce the existence of $R' \geq 1$ and $M' > 0$ such that

$$\begin{cases}
|u| \leq MF_{\alpha,\beta} \quad \text{in } \partial A_R \cup \{\infty\}, \\
|\Delta_{E^*} u| \leq c|u|^m \leq c_\alpha^{d-\alpha} \leq -\Delta_{E^*} (M'F_{\alpha,\beta}) \quad \text{in } A_{R'}.
\end{cases} \quad (4.17)$$

Hence, the maximum principle for $\Delta_{E^*}$ yields

$$|u| \leq M'F_{\alpha,\beta} \quad \text{in } A_{R'}. \quad (4.17)$$
From (4.17) and (4.13) we get
\[ |\Delta_{\Omega} u| \leq c|u|^2 \leq cM^{1/2}F_{\alpha}\frac{1}{|z|^\frac{\alpha}{2^*}} \leq c|z|^{\frac{2}{2^*}}F_{\alpha} \leq -\Delta_{\Omega}(cF_{\alpha}) \]
in $A_R$, for $R$ large enough. Because of (4.16) we conclude $|u| \leq MF_{\alpha}$ in $A_R$.

**Proof of Proposition 4.7** Since $u = 0$ in $\partial \Omega$, from (4.15) we get
\[ \left| \frac{u(z, s + H(|z|)) - u(z, H(|z|))}{s} \right| \leq \frac{M}{|z|^\frac{2}{2^*}} \leq \frac{M}{|z|^\frac{2}{2^*}} \]
for $|z| > R$ and $0 < s < 1$. Letting $s \to 0^+$ we finally obtain (for large $|z|$)
\[ |d_t u(z, H(|z|))| \leq \frac{M}{|z|^\frac{2}{2^*}} \leq \frac{c}{d(z, H(|z|))^{\frac{2}{2^*}}}. \]
We have used here that $H(r) = O(r^2)$ as $r \to +\infty$ (which follows from (4.1)).

The next step is to extend inside $\Omega$ the estimate obtained in Proposition 4.7 and prove Proposition 4.5. We need some lemmas. From (4.1) it follows that there exist $r_0 > 0$ and $\delta \in [0, \frac{1}{2}]$ such that
\[ \max_{|z| = r_0} |H'| \leq \frac{r}{\delta} \quad \forall r \geq r_0. \]
We define $\Omega_0 = \{ \xi = (z, t) \in \Omega \mid |z| > r_0 \}$ and
\[ \begin{align*}
\tau : \Omega &\to \mathbb{R}^+, \quad \tau(z, t) = \delta(t - H(|z|)), \\
\varphi : \Omega_0 &\to \mathbb{R}^+, \quad \varphi(z, t) = \frac{r}{\sqrt{(t-H(|z|))^2 + |z|^2}}, \quad \varphi = \min\{\varphi, 1\}. \end{align*} \tag{4.18} \]

**Lemma 4.11** For every $\xi \in \Omega_0$ we have
\[ B_d(\xi, \varphi(\xi)) \subseteq \Omega. \tag{4.19} \]

**Proof** We fix $\xi \in \Omega_0$ and for sake of brevity we set $\tau = \tau(\xi), \varphi = \varphi(\xi)$. Let $\xi' \in B_d(\xi, \varphi)$ and let us denote $\zeta = (h, k) = (x - x', y - y') = z - z', s = t - t'$. We have (see (2.1) and (2.4))
\[ g^2 > d(\xi, \xi')^2 = |\zeta|^4 + (s + 2(|x - h, y - y|) \leq |\zeta|^4 + (s + 2(z, (k, -h)))^2. \]

Hence $|\xi| < \varphi$ and $|s| \leq |s + 2(z, (k, -h))| + 2|z||\xi| < \varphi^2 + 2|z|\varphi$. On the other hand (4.18) yields $\varphi^2 + 2|z|\varphi = \tau$. Therefore $|t - t'| = |s| < \tau$ and

$$t' - H(|z'|) \geq t' - H(|z|) - \|z| - |z'|| \max_{|z| < \varphi, |z'|} |H'| \geq t' - t + \frac{\tau}{\delta} - \frac{|z|}{\delta} \varphi$$

$$\geq \frac{\tau}{\delta} - \frac{\tau}{\delta} \frac{|z|}{|z| + \sqrt{\tau + |z|^2}} \geq \frac{\tau}{\delta} \left(1 - \frac{1}{2\delta}\right) > 0.$$

☐

We now consider the function $v$ defined in Proposition 4.4. Since $\Delta_{\Omega \varphi} v = 0$ in $\Omega$ and (4.19) holds, we can give an estimate of $\partial v$ in terms of $v$ and $\varphi$.

**Lemma 4.12** There exists $c > 0$ such that for every $\xi \in \Omega_0$ we have

$$|\partial_v(\xi)| \leq \frac{c}{\theta(\xi)^2} \sup_{B_v(\xi, \delta \varphi)} |v|$$

(4.20)

**Proof** It follows from the $\Delta_{\Omega \varphi}$-harmonicity of $v$. We refer to [33, Proposition 2.1], for a complete proof.

☐

We define

$$D = \{\xi = (z, t) \in \Omega_0 \mid 0 < t - H(|z|) < |z|\},$$

$$E = \{\xi = (z, t) \in \Omega \mid t - H(|z|) \geq \max\{|z|, 1\}\}.$$  

From (4.10), (4.18) and (4.20) the next lemma easily follows.

**Lemma 4.13** We have

$$\partial_t v = O(d^1 t^{-2}), \quad \text{as } d(\xi) \to \infty, \xi \in D,$$  

(4.21)

$$\partial_t v \equiv O(d^{2+\beta} t^\alpha), \quad \text{as } d(\xi) \to \infty, \xi \in E.$$  

(4.22)

We now set, for any large $R > 0$,

$$D_R = \{\xi = (z, t) \in D \mid \tau(z, t) < 2R - |z|\}.$$  

We also set $\beta = \frac{1}{2}$, $\alpha = 2 + \beta$ and define

$$G_R : D_R \to \mathbb{R}^+, \quad G_R(z, t) = \frac{\tau(z, t)^\beta}{(2R - |z|)^{2\alpha}}.$$
Lemma 4.14 There exist $M > 0$ and $R_0 > r_0$ such that for every $R > R_0$ we have
\[
|\partial_t v| \leq M\left(\Gamma^\frac{1}{2} + R^\frac{1}{2}G_R\right) \quad \text{in } D_R := D_R \cap \{|z| > R_0\}. \tag{4.23}
\]

Proof Using (4.11) and (4.1) it is not difficult to verify that, for a large $R_0$,
\[-\Delta_{\mathbb{H}^n} G_R \geq 0 \text{ in } D_R.\]
Since $\Delta_{\mathbb{H}^n}(\partial_{\mathbb{H}^n} v) = \partial_{\mathbb{H}^n}(\Delta_{\mathbb{H}^n} v)$, by (4.9) $\partial_{\mathbb{H}^n} v$ is $\Delta_{\mathbb{H}^n}$-harmonic in $\Omega$. Moreover $\Delta_{\mathbb{H}^n}(\Gamma^\frac{1}{2}) \leq 0 \text{ in } D_R$. Then, if we prove that
\[
|\partial_{\mathbb{H}^n} v| \leq M\left(\Gamma^\frac{1}{2} + R^\frac{1}{2}G_R\right) \quad \text{in } \partial D_R, \tag{4.24}
\]
(4.23) will follow from the maximum principle. From (4.8) and Proposition 4.7 we get
\[
|\partial_{\mathbb{H}^n} v| \leq |\partial_{\mathbb{H}^n} w| + |\partial_t v| \leq M\Gamma^\frac{1}{2} \quad \text{in } \partial D_R \cap \partial \Omega.
\]
On the other hand (4.21) yields
\[
|\partial_{\mathbb{H}^n} v| \leq \frac{MR^\frac{1}{2}}{t(z,t)} = M R^\frac{1}{2}G_R \quad \text{in } \partial D_R \cap \{|t - H(z)| = 2R - |z|\}
\]
and (4.22) gives
\[
|\partial_{\mathbb{H}^n} v| \leq M\Gamma^\frac{1}{2} \quad \text{in } \partial D_R \cap \{t - H(|z|) = |z|}\]
where $M$ is a constant not depending on $R$. Moreover $|\partial_{\mathbb{H}^n} v|$ is a continuous function on the set $\{|z| = R_0, \quad 0 \leq t - H(R_0) \leq R_0\}$. Therefore (4.24) holds.

\[\square\]

Corollary 4.15 We have $|\partial_{\mathbb{H}^n} v(\xi)| \to 0$, as $d(\xi) \to \infty$, $\xi \in \Omega$.

Proof From (4.23) it follows that, for $R$ large enough,
\[
|\partial_{\mathbb{H}^n} v| \leq M\left(\Gamma^\frac{1}{2} + R^\frac{1}{2}G_R\right) \quad \text{in } D \cap \{|z| = R\}.
\]
Then, if $\xi = (z, t) \in D$ and $|z|$ is sufficiently large, we have
\[
|\partial_{\mathbb{H}^n} v(\xi)| \leq M\left(\Gamma(\xi)^\frac{1}{2} + \frac{|\xi|^\frac{1}{2}t(\xi)^2}{(2|z| + |z|^2)^\frac{1}{2}}\right) \leq M\left(\Gamma(\xi)^\frac{1}{2} + \frac{1}{|z|^\frac{1}{2}}\right).
\]
Hence $|\partial_{\mathbb{H}^n} v(\xi)| \to 0$, as $d(\xi) \to \infty$, $\xi \in D$. Since also (4.22) holds, the corollary is proved.

\[\square\]
Proof of Proposition 4.5 We set $\tilde{\Omega} = \Omega \setminus \overline{B_d(0,1)}$. Due to (4.8), (4.9) and Proposition 4.7 we have $|\partial_t v| \leq M\Gamma^{\frac{1}{2}}$ in $\partial \tilde{\Omega}$ and $\Delta_{\EuScript{B}^*} (\partial_t v) = \partial_t (\Delta_{\EuScript{B}^*} v) = 0 \geq \Delta_{\EuScript{B}^*} (M\Gamma^{\frac{1}{2}})$ in $\tilde{\Omega}$. Then, by using Corollary 4.15 and the maximum principle, we get $|\partial_t v| \leq M\Gamma^{\frac{1}{2}}$ in $\tilde{\Omega}$. This estimate and (4.8) finally give $|\partial_t u| \leq M\Gamma^{\frac{1}{2}}$ in $\tilde{\Omega}$.

We now want to estimate $\partial_t u$ in a neighborhood of the origin, the only characteristic point of $\partial \tilde{\Omega}$. We recall (4.2) and choose $r_0 > 0$ small enough. Then we define

$$A = \{ \xi = (z, t) \in \mathbb{H}^n | 0 < t - H(|z|) < 1, |z| < r_0 \}.$$ 

Lemma 4.16 There exists $M > 0$ such that $|u(z, t)| \leq M(t - H(|z|))$ for every $(z, t) \in A$.

Proof For every $\beta \in [0, 1]$ we define

$$(P)_{\beta} = (\exists M_{\beta} > 0 \text{ such that } |u| \leq M_{\beta} (t - H(|z|))^\beta \text{ in } A).$$

From the Hölder continuity of $u$ (see (4.7)) we obtain

$$|u(z, t)| = |u(z, t) - u(z, H(|z|))| \leq Md((z, t), (z, H(|z|)))^\beta = M(t - H(|z|))^\beta,$$

i.e. $(P)_{\beta'}$ holds. Therefore it is sufficient to show that, setting $\beta' = \frac{\beta}{m}$,

$$(P)_{\beta'} \Rightarrow (P)_{\beta} \quad \forall \beta \in [0, 1]$$

and we will get $(P)_1$ and prove the lemma. Let us then fix $\beta \in [0, 1]$ and assume $(P)_{\beta'}$. We set $F : A \to \mathbb{R}^+$, $F(z, t) = (t - H(|z|))^\beta \exp \{-|z|^2\}$. Using formula (4.11) a computation yields

$$-\Delta_{\EuScript{B}^*} F = F \left( 2Q - 4 - 4|z|^2 + \beta \frac{t - H(|z|)}{t - H(|z|)} \left( H''(|z|) \right. \right.$$

$$+ \left. \frac{Q - 3 - 4|z|^2}{|z|} H'(|z|) + \frac{1 - \beta}{t - H(|z|)} (H'(|z|)^2 + 4|z|^2) \right).$$

Hence, from (4.2) we get $-\Delta_{\EuScript{B}^*} F \geq F$ in $A$, so that, from $(P)_{\beta'}$ we obtain
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\[ |\Delta_{\mathbb{R}^n} u| \leq c |u|^m \leq c (M_\beta (t - H(|z|))^p)^m = c (t - H(|z|))^p \leq c F \leq -\Delta_{\mathbb{R}^n} (M_\beta F) \]
in \mathcal{A}. On the other hand, since \( u \in \Gamma^2(\overline{\Omega} \setminus \{0\}) \), it is easily seen that \( M_\beta \) can be chosen such that \( |u| \leq M_\beta F \) in \( \partial \mathcal{A} \). Therefore, from the maximum principle \((P)_\beta\) follows.

**Proposition 4.17** There exists \( M > 0 \) such that \( |\partial_\nu u(z, H(|z|))| \leq M \) for every \( z \in \mathbb{R}^n \) such that \( 0 < |z| < r_0 \).

**Proof** It is an immediate consequence of Lemma 4.16.

Since \( H \) is smooth, arguing as in the proof of Lemma 4.11, it is not difficult to see that there exists \( \delta > 0 \) such that, if we define \( \tau, \varrho \) as in (4.18), we have \( B_\delta (\xi, \varrho (\xi)) \subseteq \Omega \) for every \( \xi \in \Omega_0 := \Omega \cap \) a small neighborhood of the origin. The only thing which is to notice is that, for small \( |z| > 0 \),

\[ \varrho \max_{|z| - \varrho |z| + \varrho} |H| \leq c (|z| + \varrho) \varrho \leq c (\frac{\tau}{2} + \tau) = c \tau. \]

Hence we get

**Lemma 4.18** There exists \( c > 0 \) such that for every \( \xi \in \Omega_0 \) we have

\[ |\nabla_{\mathbb{R}^n} \nu (\xi)| \leq \frac{c}{\varrho (\xi)} \sup_{B_\delta (\xi, \varrho (\xi)^2)} |{\nu}|, \quad |\partial_\nu \nu (\xi)| \leq \frac{c}{\varrho (\xi)^2} \sup_{B_\delta (\xi, \varrho (\xi)^2)} |{\nu}|. \quad (4.25) \]

**Proof** See the proof of Lemma 4.12.

**Lemma 4.19** There exists \( M > 0 \) such that

\[ |\nabla_{\mathbb{R}^n} u| \leq M, \quad |\partial_\nu u| \leq \frac{M}{\varrho} \quad \text{in } \Omega_0. \quad (4.26) \]

**Proof** We fix \( \xi_0 \in \Omega_0 \) and define \( \nu_0 = v - v(z_0, H(|z_0|)) \). For sake of brevity we also set \( \varrho = \varrho (\xi_0), \tau = \tau (\xi_0), \) and \( B = B_\delta (\xi_0, \varrho) \). For every \( \xi \in B \) we have

\[ |\nu_0 (\xi)| = |u (\xi) - u (z_0, H(|z_0|))| \leq \left( \max_{B_\delta (\xi_0, \varrho)} |{\nabla u}| \right) |\xi - (z_0, H(|z_0|))| + |u (\xi)| \]

\[ \leq c (|z_0 - z_0| + |t - H(|z_0|)) + c |t - H(|z|)| \quad \text{(by Lemma 4.16)} \]

\[ \leq c (\varrho + |t - H(|z_0|)) \leq c (\varrho + |t - z_0| + |t_0 - H(|z_0|)) \leq c (\varrho + \tau) \leq c \varrho. \]
Hence \( \sup_B |v_0| \leq c_0\). Since \( v_0 \) is \( \Delta_{\mathcal{R}} \)-harmonic as well as \( v \), (4.25) holds also replacing \( v \) with \( v_0 \). Therefore we obtain

\[
|\nabla_{\mathcal{R}} v(\xi_0)| = |\nabla_{\mathcal{R}} v_0(\xi_0)| \leq \frac{c}{\varrho} \sup_B |v_0| \leq c,
\]

\[
|\partial_t v(\xi_0)| = |\partial_t v_0(\xi_0)| \leq \frac{c}{\varrho^2} \sup_B |v_0| \leq \frac{c}{\varrho},
\]

where \( c \) is a positive constant not depending on \( \xi_0 \). Since \( w \in \Gamma^2(\mathcal{H}^n) \) we finally get (4.26).

We now fix \( \beta \in [0,1] \) and set \( \alpha = 2 + \beta \).

**Lemma 4.20** There exist \( \varepsilon_0 > 0, \gamma > 0 \) and \( M > 0 \) such that, setting for every \( \varepsilon \in (0, \varepsilon_0) \), \( K_\varepsilon = \{ \xi = (z, t) \in \mathcal{H}^n \mid |z| < \varepsilon_0, \ 0 < t - H(|z|) < \gamma |z|(|z| - \varepsilon) \} \) and \( \Psi_\varepsilon : K_\varepsilon \to \mathbb{R}^+, \Psi_\varepsilon(z, t) = \frac{1}{\varepsilon} + \frac{\varepsilon - t + H(|z|)}{(|z| - \varepsilon)^\alpha} \), we have

\[
|\partial_t v| \leq M\Psi_\varepsilon \quad \text{in } K_\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0), \tag{4.27}
\]

**Proof** Using (4.11) one can check that \( -\Delta_{\mathcal{R}} \Psi_\varepsilon \geq 0 \) in \( K_\varepsilon \). Then, since \( \Delta_{\mathcal{R}}(\partial_t v) = \partial_t (\Delta_{\mathcal{R}} v) = 0 \), if we prove that

\[
|\partial_t v| \leq M\Psi_\varepsilon \quad \text{in } \partial K_\varepsilon, \tag{4.28}
\]

the maximum principle will give (4.27). From Proposition 4.17 we get

\[
|\partial_t v| \leq |\partial_t w| + |\partial_t u| \leq M \leq M\Psi_\varepsilon \quad \text{in } \partial K_\varepsilon \cap \partial \Omega.
\]

Moreover \( \partial_t v \in C(\overline{\Omega} \setminus \{0\}) \) implies

\[
|\partial_t v| \leq c \leq M\Psi_\varepsilon \quad \text{in } \partial K_\varepsilon \cap \{|z| = \varepsilon_0\}.
\]

On the other hand from (4.26) it follows that

\[
|\partial_t v| \leq M\Psi_\varepsilon \quad \text{in } \partial K_\varepsilon \cap \{t - H(|z|) = \gamma |z|(|z| - \varepsilon)\},
\]

with \( M \) not depending on \( \varepsilon \). Indeed, keeping in mind the very definition (4.18) of the function \( \varphi \), if \( t - H(|z|) = \gamma |z|(|z| - \varepsilon) \) and \( \varepsilon < |z| \leq 2\varepsilon \), we have

\[
\frac{1}{\varphi(\xi)} \leq \frac{|z|}{t - H(|z|)} \leq c^{-\beta} \frac{\varepsilon^{1-\beta}(t - H(|z|))^{\beta}}{(|z| - \varepsilon)^{\beta+\alpha}} \leq c\Psi(\xi).
\]
Moreover, if $t - H(|z|) = \gamma |z||z| - \varepsilon$ and $2\varepsilon \leq |z| \leq \varepsilon_0$, then $\frac{1}{d(t)} \leq c^{-1}t |z||z| - \varepsilon \leq \frac{\varepsilon_0}{\beta}$, where $\varepsilon \leq c \Phi_*(\xi)$. Therefore (4.28) holds.

\[ \square \]

**Corollary 4.21** We have $\partial_t \nu(\xi) = O(\frac{1}{d(t)})$, as $d(\xi) \to 0$, $\xi \in \Omega$.

**Proof** From (4.27) it follows that for every $\varepsilon \in [0, \frac{\varepsilon_0}{2}]$,

$$|\partial_t \nu| \leq M \Psi \quad \text{in} \quad \{|z| = 2\varepsilon, 0 < t - H(|z|) < \frac{\gamma}{2}|z|^2\}.$$

This means that for every $\xi \in \{0 < |z| < \varepsilon_0, 0 < t - H(|z|) < \frac{\gamma}{2}|z|^2\}$ we have

$$|\partial_t \nu(\xi)| \leq M \Psi^{\frac{1}{\beta}}(\xi) = \frac{2M}{|z|} + c \frac{|z|^{1-\beta}(t - H(|z|))|z|}{|z|^3} \leq \frac{c}{|z|} \leq \frac{c}{d(\xi)}.$$

We have used here that $H(r) = O(r^2)$ as $r \to 0$, which follows from $H$ smooth and even. On the other hand (4.18) and (4.26) give

$$|\partial_t \nu(\xi)| \leq \frac{M}{\Theta(\xi)} \leq \frac{c}{\sqrt{t - H(|z|)}} \leq \frac{c}{d(\xi)} \quad \text{in} \quad \Omega_0 \cap \{t - H(|z|) \geq \frac{\gamma}{2}|z|^2\}.$$

\[ \square \]

Finally, we are in a position to complete the proof of Proposition 4.6.

**Proof of Proposition 4.6** Thanks to (4.26) we only need to prove that $|\partial_t \nu| \in L^\infty(B)$, $B = \Omega \cap B_d(0, 1)$. From Proposition 4.17 it follows that there exists $M > 0$ such that $|\partial_t \nu| \leq M$ in $\partial B \setminus \{0\}$. Moreover, by Corollary 4.21, for every $\varepsilon > 0$ we have $\lim_{\varepsilon \to 0^+} (M + \varepsilon \Gamma - |\partial_t \nu|) = +\infty$. Since $\Delta \nu(\partial_t \nu) = 0 = \Delta \nu(\nu + \varepsilon \Gamma)$ in $B$, the maximum principle gives $|\partial_t \nu| \leq M + \varepsilon \Gamma$ in $B$. Letting $\varepsilon \to 0$ we finally obtain $|\partial_t \nu| \leq M$ in $B$.

\[ \square \]

**Proof of Theorem 4.1** It directly follows from Theorem 4.2 and Propositions 4.4, 4.5 and 4.6.

\[ \square \]

**Proof of Theorem 1.5-(a)** The proof directly follows from Theorem 4.1 and Theorem A.3 in the Appendix.

\[ \square \]
5 Proof of Theorem 1.6

The proof follows from Theorem A.3 in the Appendix, Theorem 1.9 and the next lemma.

Lemma 5.1 Assume \( n + m > 1 + \frac{3}{2} \). If \( u \in \Gamma^2(\mathbb{H}^n) \) is a solution of \(-\Delta_{\mathbb{H}^n} u = f(u) \) in \( \mathbb{H}^n \), such that

\[
 u(\xi) = O(d(\xi)^{-r}), \quad \text{as} \; d(\xi) \to \infty, \; \forall s < Q - 2, \tag{5.1}
\]

then \( \int_{\mathbb{H}^n} G(u) = 0 \). Moreover, if \( G \) is as in (a) (or as in (b)) of Theorem 1.6, then \( u \equiv 0 \).

Remark 5.2 In particular, if \( n + m > 1 + \frac{3}{2} \) (or 1) and \( m \neq \frac{Q+2}{Q-2} \), then \(-\Delta_{\mathbb{H}^n} u = |u|^{m-1} u \) has no entire solutions \( u \) satisfying (5.1).

Proof of Lemma 5.1 Since \( n+m > 1 + \frac{3}{2} \) implies \( m > \frac{Q+2}{Q-2} \), from (5.1) and (H) we get \( F(u) \in L^1 \) and \( f(u) = O\left(\frac{1}{d^{Q+1}}\right) \) at infinity, for some \( \varepsilon > 0 \). It is now standard to prove that \( u = \Gamma \ast (f(u)) \) and \( |D_{\mathbb{H}^n} u|, |D_{\mathbb{H}^n} u||\partial u|d, u f(u) \in L^1(\mathbb{H}^n) \). Arguing as in the proof of Theorem 2.3 of [17] we finally get \( \int_{\mathbb{H}^n} G(u) = 0 \).

A Appendix

Throughout this section we denote by \( \Omega \) an arbitrary open subset of \( \mathbb{H}^n \).

Lemma A.1 Let \( V \in L^q(\Omega) \), \( g \in L^{\frac{Q+2}{2}}(\Omega) \) and let \( u \) be a weak solution of

\[
\begin{cases}
-\Delta_{\mathbb{H}^n} u = Vu + g & \text{in } \Omega, \\
u \in S_0^2(\Omega). 
\end{cases}
\tag{A.1}
\]

If \( g \in L^r(\Omega) \) for some \( r \in [Q^*, +\infty) \) and \( \|V^+\|_q < \frac{1}{B^*} \) (see (2.5)), then \( u \in L^p(\Omega) \) for every \( p \in [Q^*, \frac{Q+2}{2}] \). Moreover, if \( g \in L^\infty(\Omega) \) and \( V \in L^1(\Omega) \) for some \( q > \frac{Q^*}{2} \), then \( u \in L^p(\Omega) \) for every \( p \in [Q^*, +\infty) \).

Proof The above lemma is analogous to Lemma 4.1 in [17]. Here we work in
the Sobolev space $S^1_0$ instead of $S^2_0$ (being $S^1_0$ the completion of $C_0^\infty$ w.r.t. the norm $\|u\|_2 + \|\nabla u\|_2$). We adapt the proof of [17] which relies on some boot-strap and iteration techniques inspired to those of Brezis-Kato [9] and Moser ([20, Chapter 8]).

There exists $a \in ]0,1]$ such that

$$\int_\Omega V \varphi^2 \leq \|V^+\|_q \|\varphi\|_{Q'}^2 \leq B_Q \|V^+\|_q \|\nabla \varphi\|_{L^2}^2 < a \|\nabla \varphi\|_{L^2}^2 \quad (A.2)$$

for every $\varphi \in S^1_0(\Omega)$. For every $k \in \mathbb{N}$ we define $V_k = \text{sgn}(V) \min \{V, k\}$. Since (A.2) holds, Lax Milgram's Theorem implies the existence of exactly one weak solution $u_k$ of

$$\begin{cases}
-\Delta \varphi_k u_k = V_k u_k + g & \text{in } \Omega, \\
u_k \in S^1_0(\Omega). 
\end{cases} \quad (A.3)$$

Moreover

$$\|u_k\|_{Q^*} \leq \sqrt{B_Q} \|u_k\|_{Q'} \leq \frac{1}{1-a} \sqrt{B_Q} \|g\|_{S^1} \leq \frac{1}{1-a} B_Q \|g\|_{Q^*}. \quad (A.4)$$

The same argument also yields uniqueness for the problem (A.1). Hence, by the boundedness of $(u_k)$ in $S^1_0(\Omega)$, taking a subsequence if necessary, we have

$$u_k \to u \quad \text{weakly in } S^1_0(\Omega). \quad (A.5)$$

We now want to prove that, for every $p \in [Q^*, \frac{Q^*}{Q^* - 2}]$ there exists $c_p > 0$ such that

$$\sup_{k \in \mathbb{N}} \|u_k\|_p \leq c_p. \quad (A.6)$$

We set $\beta = \frac{Q^*}{Q^* - 2}$. Since $\beta > 1$ and (A.6) holds for $p = Q^*$ (see (A.4)), we only need to prove (A.6) for $\beta p$, under the hypothesis that (A.6) holds for a $p \in [Q^*, r]$. We fix $k \in \mathbb{N}$ and, for sake of brevity, we set $v = u_k$. We then define, for every $m \in \mathbb{N}$, $v_m = \min \{v^+, m\}$, $\varphi_m = v_m^{p-1}$, $\psi_m = v_m$. We remark that $v, v_m, \varphi_m, \psi_m \in S^1_0(\Omega)$, since $p \geq Q^* > 2$. Moreover $v_m, \varphi_m, \psi_m$ are all nonnegative. Choosing $\varphi_m$ as a test function in the weak formulation of (A.3) and setting $\alpha_p = \frac{4\beta - 1}{\beta^p}$, we obtain
\[ \alpha_p \int \frac{|\nabla B^+ \psi_m|^2}{n} = \int \frac{\langle \nabla B^+ v_m, \nabla B^+ \varphi_m \rangle}{n} = \int \frac{\langle \nabla B^+ v, \nabla B^+ \varphi_m \rangle}{n} = \int \frac{(V_k v \varphi_m + g \varphi_m)}{n} \leq \int \frac{|V_k| \varphi_m^2}{n} + k \int \frac{v^p + k}{} + \frac{g \varphi_m}{n}. \]

(A.7)

Since \( V \in L^\frac{q}{2}(\Omega) \), we can choose \( M_p > 0 \) such that \( \int \frac{|V|^q}{n} \leq \frac{d_p}{2B^q} \).

Then

\[ \int \frac{|V|^2}{n} \leq M_p \int \frac{\psi_m^2}{n} + \int \frac{|V|^2}{n} \]

\[ \leq M_p \left\| \psi_m \right\|_{L^2}^2 + \frac{\alpha_p}{2B_q} \left\| \psi_m \right\|_{L^q}^q \leq M_p \left\| \psi_m \right\|_{L^2}^2 + \frac{\alpha_p}{2B_q} \left\| \nabla B^+ \psi_m \right\|_{L^2}^2. \]

From (A.7) and (A.8) it follows

\[ \frac{\alpha_p}{2B_q} \left\| \nabla \psi_m \right\|_{L^p}^p \leq \frac{\alpha_p}{2B_q} \left\| \nabla B^+ \psi_m \right\|_{L^2}^2 \leq M_p \left\| \psi_m \right\|_{L^2}^2 + k \int \frac{v^p + |g|}{n} \left\| \psi_m \right\|_{L^2}^2. \]

From these inequalities, as \( m \) goes to infinity, we obtain

\[ \frac{\alpha_p}{2B_q} \left\| v^+ \right\|_{L^p}^p \leq \frac{\alpha_p}{2B_q} \left\| \nabla B^+ \psi_m \right\|_{L^2}^2 \leq M_p \left\| \psi_m \right\|_{L^2}^2 + k \int \frac{v^p + |g|}{n} \left\| \psi_m \right\|_{L^2}^2. \]

since (A.6) holds for \( p \) and \( p \in [\beta_p, \gamma] \). Since a similar estimate can be proved for \( \nu^- \), (A.6) holds when replacing \( p \) with \( \beta_p \). Therefore (A.6) holds for every \( p \in [\beta_p, \gamma] \). We now fix \( p \in [\beta_p, \gamma] \) and we define \( C = \{ h \in \mathcal{S}^\prime(\Omega) | \| h \|_p \leq c_p \}. \) Futo's Lemma ensures that \( C \) is a closed convex subset of \( \mathcal{S}^\prime(\Omega) \). By (A.6) \( (u_k) \) is contained in \( C \). Hence, by (A.5), \( u \in C \). In particular \( u \in L^p(\Omega) \). This proves the first part of the lemma. We omit, for brevity, the proof of the second part (see e.g. [26]).

Next lemma allows to obtain \( u \in L^p \) for some \( p < Q^* \). The proof is essentially contained in [26]. However we include it below, since this is a crucial step of our technique.

**Lemma A.2** Let \( V \in L^\frac{q}{2}(\Omega) \), \( g \in L^{\frac{q}{2}}(\Omega) \) and let \( u \) be a weak solution to (A.1). Let \( \varepsilon \in [0, 1] \) and set \( M_\varepsilon = (1 - 2\varepsilon)^{-2} \). If \( g \in L^{\frac{q}{2}}(\Omega) \) and \( \| V^+ \|_{L^q}^q < \frac{1}{B^q M_\varepsilon} \) (see (2.5)), then \( u \in L^p(\Omega) \) for every \( p \in [Q^*(1 - \varepsilon), Q^*] \).
Proof. We have

$$\int_\Omega |V^* \phi|^2 \leq \|V^*\|_Q \|\phi\|_Q^2 \leq \frac{1}{M_\epsilon} \|\nabla_{y^*} \phi\|_Q^2 \quad \forall \phi \in S_0^2(\Omega),$$

(A.9)

with $\frac{1}{M_\epsilon} < 1$. For every $k \in \mathbb{N}$ let $\eta_k \in C^\infty([0, +\infty))$ be such that $0 \leq \eta_k \leq 1$, $\eta_k \equiv 0$ in $[0, \frac{1}{k+1}]$, $\eta_k \equiv 1$ in $[\frac{1}{k}, +\infty]$. We define $V_k = \eta_k(V^*)$ so that

$$V_k \in L^1(\Omega) \cap L^\infty(\Omega)$$

(A.10)

and $V_k \to V^*$. Exactly in the same way as in the proof of Lemma A.1, we can see that for every $k \in \mathbb{N}$ problem (A.3) admits a weak solution $u_k$ such that

$$\sup_{k \in \mathbb{N}} \|u_k\|_{S_0^2(\Omega)} \leq c = c(Q, g, \epsilon)$$

(A.11)

and $u_k \rightharpoonup u$ weakly in $S_0^2(\Omega)$. We now want to prove that

$$\sup_{k \in \mathbb{N}} \|u_k\|_{Q^* (1-\epsilon)} \leq c_0 = c_0(Q, g, \epsilon).$$

(A.12)

We fix $k \in \mathbb{N}$ and, for sake of brevity, we set $v = u_k$. Then we define, for every $m \in \mathbb{N}$, $u_m = m^{-\frac{1}{2\epsilon}} (v^*)^{\frac{1}{1-2\epsilon}}$ where $v \leq \frac{1}{m}$, $u_m = v$ where $v > \frac{1}{m}$, $\psi_m = v^{1-2\epsilon}$, $\phi_m = v^{1-2\epsilon}$. We remark that $\psi, \psi_m, \phi, \psi_m \in S_0^2(\Omega)$, since $0 < 1-2\epsilon < 1-\epsilon < 1$. Moreover $\psi_m, \phi_m, \psi_m \geq 0$ and $\psi_m \to v^+$ pointwise, as $m \to +\infty$. Choosing $\psi_m$ as a test function in the weak formulation of (A.3) and setting $c_\epsilon = \left(\frac{1-\epsilon}{1-2\epsilon}\right)^2$, we obtain

\[
\begin{aligned}
\int_\Omega |\nabla_{x^*} u_m|^2 &= (1 - 2\epsilon)c_\epsilon \int_\Omega \langle \nabla_{x^*} u_m, \nabla_{x^*} \psi_m \rangle \\
&\quad + c_\epsilon \int_\Omega |m^{\frac{1}{2\epsilon}} v^{1-2\epsilon} - 1 \langle \nabla_{x^*} u, \nabla_{x^*} \psi_m \rangle \\
&\leq c_\epsilon \int_\Omega \langle \nabla_{x^*} u, \nabla_{x^*} \psi_m \rangle + c_\epsilon \int_\Omega \langle \nabla_{x^*} u, \nabla_{x^*} \phi_m \rangle.
\end{aligned}
\]

(since $\langle \nabla_{x^*} u, \nabla_{x^*} \phi_m \rangle \geq 0$ in $\Omega$)

\[
\begin{aligned}
&= c_\epsilon \int_\Omega \langle \nabla_{x^*} u, \nabla_{x^*} \psi_m \rangle = c_\epsilon \int_\Omega (V_k u \phi + g \psi_m) \\
&\leq c_\epsilon \left( \int_\Omega |V_k u|^2 + \int_\Omega |g|^2 \right) \\
&\leq c_\epsilon \left( \frac{1}{M_\epsilon} \|\nabla_{y^*} \phi\|_Q^2 + \frac{1}{m^{1-2\epsilon}} \|V_k u\|_Q^2 + \|g\|_{Q_0^2}^2 \|\psi_m\|_{Q_0^2}^{1-2\epsilon}. \right)
\end{aligned}
\]
Here we have used (A.9). Since $\frac{d}{M^*} < 1$, from (A.10), (A.11) and (2.5), we obtain $\|\nabla_{\text{Kohn}} \psi_m\|_{L^2}^2 < mB^2 + c_2$, with $c_1 = c_1(V, \epsilon, k)$ and $c_2 = c_2(Q, g, \epsilon)$. Hence

$$\|\psi_m\|_{L^2(Q'(1-\epsilon))}^2 = \|\psi_m\|_{L^2}^2 \leq B_02\|\nabla_{\text{Kohn}} \psi_m\|_{L^2}^2 \leq \frac{c_1B_0}{m^2-2M} + c_2B_0$$

and letting $m \to +\infty$, Fatou's Lemma yields $\|\psi^+\|_{L^2(Q'(1-\epsilon))} \leq (c_2B_0)^{\frac{1}{m^2-2M}} = c_0 = c_0(Q, g, \epsilon)$. A similar estimate can be proved for $\psi^-$. Then, inequality (A.12) holds. Arguing as in the proof of Lemma A.1, we finally obtain $u \in L^2(Q'(1-\epsilon)(\Omega))$.

\[\square\]

**Theorem A.3** Let $u$ be a weak solution to (1.1). If $\frac{U(u)}{u} \in L^{\frac{d}{2}}(\Omega)$, then $u \in L^p(\Omega)$ for every $p \in \left[\frac{Q^*}{Q^* - 1}, +\infty\right]$ and $u(\xi) \to 0$, as $d(\xi) \to \infty$.

**Proof** From $\frac{U(u)}{u} \in L^{\frac{d}{2}}$ it follows that $\int_{|u|>M} |\frac{U(u)}{u}|^\frac{d}{2} \to 0$, as $M \to +\infty$. Hence there exists $M > 0$ such that, setting $V = \frac{U(u)}{u} \chi_{|u|<M}$ and $g = f(u)\chi_{|u|<M}$, we have $\|V\|^\frac{d}{2} < \frac{1}{B_0}$ and $g \in L^2(\Omega) \cap L^\infty(\Omega)$. Moreover $f(u) = V + g$. Then, by Lemma A.1, $u \in L^p$ for every $p \in [Q^*, +\infty]$. Since (H) holds, we get $V \in L^p$ for every $p \in [\frac{d}{2}, +\infty]$. Again by Lemma A.1 we then obtain $u \in L^\infty$.

We now fix $\varepsilon \in (0, \frac{1}{2}]$ and we set $M_\varepsilon = (1-2\varepsilon)^{-1}$. From $\frac{U(u)}{u} \in L^{\frac{d}{2}}$ it follows that $\int_{|u|<\varepsilon} |\frac{U(u)}{u}|^\frac{d}{2} \to 0$, as $\varepsilon \to 0$. Hence there exists $\delta = \delta_\varepsilon > 0$ such that, setting $V = \frac{U(u)}{u} \chi_{|u|<\varepsilon}$ and $g = f(u)\chi_{|u|>\varepsilon}$, we have $\|V\|^\frac{d}{2} < \frac{1}{B_0}$ and $g \in L^2(\Omega)$. We also have $g \in L^1(\Omega)$ since

$$\int_{\Omega} |g| \leq \int_{|f| \leq 1} \chi_{|u| \geq \varepsilon} + \int_{|f| > 1} |f(u)|^\frac{d}{2} \leq \|f(u)\|^\frac{d}{2} \leq |\{|u| \geq \delta\}| + c < +\infty,$$

recalling that $u \in L^{Q^*}(\Omega)$. Moreover $f(u) = V + g$. Hence $u \in L^Q(1-\epsilon)$ by Lemma A.2. Since $\varepsilon \in (0, \frac{1}{2}]$ is arbitrary, we get $u \in L^p$ for every $p > \frac{Q^*}{d}$.

We now set $V = \frac{U(u)}{u}$. Then $V \in L^\frac{d}{2} \cap L^\infty$ and $u \in S^d \cap L^\infty$ is a weak solution to $-\Delta_{\text{Kohn}} u = V u$. It is now standard to prove that there exists a positive constant $c$ such that $\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{L^{Q^*}(1+\epsilon)}$ for every $\xi \in \Omega$ (here we have set $u = 0$ outside $\Omega$). Hence $u(\xi) \to 0$, as $d(\xi) \to \infty$.

\[\square\]
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GLOBAL WELL-POSEDNESS FOR SEMI-LINEAR WAVE EQUATIONS

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§1. INTRODUCTION

In this paper we shall study global well-posedness of initial value problem (IVP) associated to the semi-linear wave equation

\[
\begin{align*}
\partial_t^2 u - \Delta u &= -|u|^{\rho-1}u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3, \ \rho \in [2,5), \\
u(x,0) &= u_0(x), \\
\partial_t u(x,0) &= u_1(x),
\end{align*}
\]

(1.1)

under minimal regularity assumptions on the real valued data \((u_0, u_1)\).

It is known that for

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(1.2) \((u_0, u_1) \in \dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3), \ s \in (\gamma(\rho), 1),\)

with \(\dot{H}^s = D^{-s}L^2 = (-\Delta)^{-s/2}L^2,\) and

(1.3) \[\gamma(\rho) = \begin{cases} 
3/2 - 2/(\rho - 1), & 3 \leq \rho < 5, \\
1 - 1/(\rho - 1), & 2 \leq \rho \leq 3,
\end{cases}\]

the IVP (1.1) is locally well posed in a time interval \([0, T_0],\) with \(T_0 = T_0(\|(D^s u_0, D^{s-1} u_1)\|_2),\) and that the lower bound \(\gamma(\rho)\) in (1.3) is optimal and can be reached when \(\rho \neq 2,\) see [7] and references therein.

Solutions of (1.1), at least formally, satisfy the conservation law

(1.4) \[\int_{\mathbb{R}^3} ((\partial_t u)^2 + (\nabla u)^2 + \frac{2}{\rho + 1}|u|^{\rho+1})(x, t)dx = \text{constant}.\]

Thus for data

(1.5) \((u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \cap L^{\rho+1}(\mathbb{R}^3) \times L^2(\mathbb{R}^3),\)

the conservation law (1.4) allows to extend the local solution of (1.1) to any time interval.

Our aim is to establish global well-posedness results for data with less regularity that the one described in (1.5). To illustrate our results first we consider the particular case of (1.1)

(1.6) \[\begin{cases} 
\partial_t^2 u - \Delta u = -u^3, & t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
u(x, 0) = u_0(x), \\
\partial_t u(x, 0) = u_1(x),
\end{cases}\]

which is locally well posed for data

(1.7) \((u_0, u_1) \in \dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3), \ s \in [1/2, 1],\)

and globally well posed for data

(1.8) \((u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \times L^3(\mathbb{R}^3).\)
Theorem 1.1.

Let $s \in (3/4, 1)$. Then for any $T > 0$ and $(u_0, u_1) \in H^s(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ the IVP (1.6) has a unique solution $u(t)$ defined in time interval $[0, T]$. Moreover,

$$u(t) = W'(t)u_0 + W(t)u_1 + z(t),$$

with

$$\sup_{[0,T]} \|z(t)\|_{H^s} \leq cT^{(1-s)/(4s-3)}.$$  

Above we have introduced the notation

$$W(t)u_1 = \frac{\sin(Dt)}{D} u_1,$$

In the general case our results can be stated as follows.

Theorem 1.2.

Let $s \in (\alpha(\rho), 1)$, with

$$\alpha(\rho) = \begin{cases} (2\rho - 3\rho^2 - 39)/(16\rho - 2\rho^2 - 14), & 3 \leq \rho < 5, \\ (5\rho - 9)/(4\rho - 4), & 2 \leq \rho \leq 3, \end{cases}$$

Then for any $T > 0$ and $(u_0, u_1) \in H^s(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ the IVP (1.1) has a unique solution $u(t)$ defined in time interval $[0, T]$. Moreover,

$$u(t) = W'(t)u_0 + W(t)u_1 + z(t),$$

with

$$\sup_{[0,T]} \|z(t)\|_{H^s} \leq cT^{(1-s)/(1-s-\beta)},$$

and

$$\beta = \begin{cases} \frac{\rho - 3}{2} + \rho(1-s) - 1 + \frac{2(1-s)(\rho - 1)}{5 - \rho}, & 3 \leq \rho < 5, \\ \rho(1-s) - 1 + \frac{(1-s)(\rho - 1)^2}{5 - \rho}, & 2 \leq \rho \leq 3. \end{cases}$$
For the cases $\rho = 2$ and $\rho = 3$ the index provided by Theorems 1.1 and 1.2, resp. is half way between what the scaling argument suggested for the local well posedness and that given by the energy estimate. Nevertheless, as it was shown in [6]-[7], when the index suggested by the scaling argument is negative (as in the case $\rho = 2$) this argument does not provide the correct value of the best Sobolev index for the local well posedness.

The above results apply to the semilinear Klein-Gordon, i.e. $\partial_t^2 - \Delta + 1$ instead of the wave operator $\partial_t^2 - \Delta$ as the associated linear operator in (1.1), by substituting the homogeneous Sobolev spaces $H^s(\mathbb{R}^3)$ by the inhomogeneous ones $H^s(\mathbb{R}^3) = (1 - \Delta)^{-s/2}L^2(\mathbb{R}^3)$.

Also the techniques below extend to dimensions $n = 2, 4, ...$. However, for the sake of simplicity in the exposition we do not consider these cases here.

The assumptions $u_0 \in L^4(\mathbb{R}^3)$, $u_0 \in L^{\rho+1}(\mathbb{R}^3)$ in Theorem 1.1 and Theorem 1.2, resp. can be removed by working locally in the space variable, cutting off the data and using the finite speed of propagation of the solution. In this case the results in (1.10) and (1.13) will be in local Sobolev spaces in the space variables.

Our proof follows Bourgain's ideas. In [1] he introduced a general scheme to establish global well-posedness of nonlinear evolution equations in $H^s$ for the values of $s$ between those determined by conservation laws. It was first used for the 2-D Schrödinger equation, see [1]. It was also applied in [2] to the PBVP for the semi-linear Klein-Gordon equation in one and two space dimensions, and later in other models, see [3], [4], [9].

Roughly speaking, one splits the data into two pieces: high and low frequencies (smooth part). For the latter one solves the original problem in a time interval $[0, \Delta T]$ with $\Delta T$ depending on the regularity of the original data. Its solution $v(t)$ has enough regularity so it satisfies the conservation
law. One uses $v(t)$ to find $y(t) = (u - v)(t)$ in the time interval $[0, \Delta T]$. We observe that the inhomogeneous part $z(t)$ of $y(t)$, see (2.12), is in $\dot{H}^1$. Thus, we add $z(\Delta T)$ to $v(\Delta T)$, and repeat the argument in the time interval $[\Delta T, 2\Delta T]$. In each step of this process the estimates for the involved norms grow. This has to be taken into account to make the process uniform. It is here where the restriction on $s$ appears.

In the next section we shall prove Theorem 1.1. It will be clear from our exposition how the argument extends to the general case in Theorem 1.2.

§2. **Proof of Theorem 1.1**

We start by recalling the homogeneous version of Strichartz’ estimate, [10], for the associated linear problem

$$\begin{cases}
\partial_t^2 w - \Delta w = 0, & t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
w(x, 0) = w_0(x), \\
\partial_t w(x, 0) = w_1(x).
\end{cases}$$

(2.1)

**Theorem 2.1.**

The solution $w = w(x, t)$ of the IVP (2.1) satisfies

$$\|D^{1-\theta}w\|_{L^2_t\dot{H}^{2/(1-\theta)}_x} \equiv \left( \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left| D^{1-\theta}w(x, t) \right|^2 \right)^{1/(1-\theta)}$$

$$\leq c_\theta \| (Dw_0, w_1) \|_2,$$

for any $\theta \in [0, 1)$.

For the proof of (2.2) we refer to [8]. It was proven in [5] that the estimate for the extremal case $\theta = 1$ in (2.2) fails, and that it still holds for radial data, i.e. $(w_0(|x|), w_1(|x|))$.

Next, we split the initial data $(u_0, u_1) \in \dot{H}^s \cap L^4 \times \dot{H}^{s-1}$.
Let $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi(\xi) = 1$, $|\xi| \leq 1$, $\phi(\xi) = 0$, $|\xi| \geq 2$, $\phi_N(\xi) = \phi(\xi/N)$, and define

$$u_0 = u_{0,1} + u_{0,2}, \quad u_1 = u_{1,1} + u_{1,2},$$

(2.1) where

$$u_{0,1}(x) = (\phi_N(\xi)u_0)'(x), \quad u_{1,1}(x) = (\phi_N(\xi)u_1)'(x),$$

(2.2) with $N$ to be chosen. Thus,

$$\|(D^l u_{0,1}, D^{l-1} u_{1,1})\|_2 \leq cN^{1-s}, \quad \text{for} \quad l \geq s,$$

(2.3) and

$$\|(D^l u_{0,2}, D^{l-1} u_{1,2})\|_2 \leq cN^{1-s}, \quad \text{for} \quad l \in [0, s].$$

(2.4)

First we consider the IVP with regular data

$$\begin{cases}
\partial_t^2 v - \Delta v = -v^3, & t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
v(x, 0) = u_{0,1}(x), \\
\partial_t v(x, 0) = u_{1,1}(x),
\end{cases}$$

(2.5) and its integral version

$$v(t) = W'(t)u_{0,1} + W(t)u_{1,1} - \int_0^t W(t - t')v^3(t')dt',$$

(2.6) whose solution $v(x, t)$ satisfies the conservation law (1.4). Thus, from (2.3)

$$\|(\partial_t v, \nabla v)(t)\|_2 + \|v(t)\|_2^2 \sim N^{1-s}, \quad t \geq 0.$$  

(2.7) From (2.2) it follows that for any $\theta \in [0, 1)$

$$\|D^{1-\theta}v\|_{L^{2\theta/(1-\theta)}_x L^2_t} \leq c_{\theta} N^{1-s} + c_{\theta}\|v^3\|_{L^3_{x,t} L^2_t} \leq c_{\theta} N^{1-s} + c_{\theta}\|v\|_{L^3_{x,t} L^2_t}^3 \leq c_{\theta} N^{1-s} + c_{\theta}\Delta T\|v\|_{L^3_{x,t} L^2_t}^3.$$  

(2.8)
Hence, for any $\theta \in [0, 1)$

$$\|D^{1-\theta} v\|_{L_\Delta^\theta L_2^{(1-\theta)}} \leq c_\theta N^{1-\theta},$$

with

$$\Delta T \sim N^{-2(1-\theta)}.$$  

Next we consider the IVP for $y(t) = u(t) - v(t)$

$$\left\{ \begin{array}{l} \partial_t^2 y - \Delta y = -(y + v)^3 + v^3, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\ y(x, 0) = u_{0,2}(x), \\ \partial_t y(x, 0) = u_{1,2}(x), \end{array} \right.$$  

and its integral version

$$y(t) = W'(t)u_{0,2} + W(t)u_{1,2} - \int_0^t W(t - t') F(t')dt'$$

$$= W'(t)u_{0,2} + W(t)u_{1,2} + z(t),$$

where

$$F = (y + v)^3 - v^3 = y^3 + 3y^2v + 3vy^2 = F_1 + F_2 + F_3.$$  

From (2.2) and (2.4) it follows that for any $\ell \in [0, 1)$ and any $\theta \in [0, 1)$

$$\|D^{1-\theta} y\|_{L_\Delta^\theta L_2^{(1-\theta)}} \leq c_\theta N^{1-\theta} + c_\theta \int_0^{\Delta T} \|D^{1-\theta} F(t)\|_2 dt$$

$$\leq c_\theta N^{1-\theta} + c_\theta \|F\|_{L_\Delta^\theta L_2^{(1-\theta)}}.$$  

To estimate $F$, for any $\ell \in [4/7, 1)$, it suffices to consider $F_1$ and $F_3$ in

(2.13). Thus,

$$\|F_1\|_{L_\Delta^\theta L_2^{(1-\theta)}} = \|y^2\|_{L_\Delta^\theta L_2^{(1-\theta)}} = \|y\|_{L_\Delta^\theta L_2^{(1-\theta)}}^2$$

$$\leq (\Delta T)^{2\ell-1}\|y\|_{L_\Delta^\theta L_2^{(1-\theta)}}^2 \|D^j y\|_{L_\Delta^\theta L_2^{(1-\theta)}}^{(2j-4)/j},$$
and
\begin{equation}
\|F_5\|_{L^1_{xT}L_{x}^{3/\left(8-2l\right)}} = 3\|u^2y\|_{L^1_{xT}L_{x}^{3/\left(8-2l\right)}} \\
\leq c\Delta T\|v\|_{L^1_{xT}L_{x}^2}^{\frac{2}{3}}\|y\|_{L^2_{xT}L_{x}^{3/\left(8-2l\right)}} \leq c\Delta T\|v\|_{L^2_{xT}L_{x}^4}\|D^l y\|_{L^2_{xT}L_{x}^2},
\end{equation}
for \( l \geq 4/7 \). Therefore, defining
\begin{equation}
\|\|y\|\|_l \equiv \|D^l y\|_{L^2_{xT}L_{x}^2} + \|y\|_{L^2_{xT}L_{x}^{2/\left(1-l\right)}}, \quad l \in \left[1/2, 1\right),
\end{equation}
taking \( l = l_0 \) in \eqref{2.14}, and collecting \eqref{2.14}-\eqref{2.16} and \eqref{eq:2.7} one finds that for \( l \in \left[4/7, s\right] \)
\begin{equation}
\|\|y\|\|_l \leq c_\theta (N^{1-s} + (\Delta T)^{2l-1}\|\|y\|\|_l^3 + \Delta T^N N^{2(1-s)}\|\|y\|\|_l).}
\end{equation}

Hence, using \eqref{2.10} one has that for \( l \geq 4/7 \)
\begin{equation}
\|\|y\|\|_l \leq c_\theta N^{1-s}.\]
\end{equation}

For \( l \in \left[1/2, 4/7\right) \) the above argument leads to
\begin{equation}
\|\|y\|\|_l \leq c_\theta (N^{1-s} + (\Delta T)^{1/3}\|\|y\|\|_l^2 + \Delta T^N N^{2(1-s)}\|\|y\|\|_l),
\end{equation}
which combined with \eqref{2.19} allows us to conclude that if \( s \geq 2/3 \) then for \( l \in \left[1/2, s\right] \)
\begin{equation}
\|\|y\|\|_l \leq c_\theta N^{1-s}.\]
\end{equation}

We need an estimate for the lower derivatives of \( y \).
Using \eqref{eq:2.4}, \eqref{eq:2.7}, \eqref{2.14}, \eqref{2.21} and assuming \( s \geq 2/3 \) one finds that
\begin{equation}
\|y\|_{L^2_{xT}L_{x}^2} \leq cN^{-s} + c\|F\|_{L^1_{xT}L_{x}^2},
\end{equation}
\begin{equation}
\leq cN^{-s} + c(\Delta T)^{1/3}\|y\|_{L^2_{xT}L_{x}^2}\|y\|_{L^2_{xT}L_{x}^2} + c\Delta T\|y\|_{L^2_{xT}L_{x}^2}\|y\|_{L^2_{xT}L_{x}^2} \\
\leq cN^{-s} + c\Delta T N^{2(1-s)}\|y\|_{L^2_{xT}L_{x}^2},
\end{equation}
which implies that for $s \geq 2/3$

\begin{equation}
(2.23) \quad \|y\|_{L^s_{\Delta T}L^2} \leq cN^{-s}, \quad \text{with} \quad \Delta T \sim N^{-2(1-s)}.
\end{equation}

A combination of (2.21) and (2.23) yields

\begin{equation}
(2.24) \quad \|D^l y\|_{L^s_{\Delta T}L^2} \leq cN^{l-s}, \quad \text{for} \quad l \in [0, 1/2].
\end{equation}

Next, we shall estimate $\|(\partial_t z, \nabla z)(\Delta T)\|_{L^1}$, with $z(t)$ defined in (2.12).

Thus,

\begin{equation}
(2.25) \quad \|(\partial_t z, \nabla z)(\Delta T)\|_{L^2} \leq c\|F\|_{L^2_{\Delta T}L^2} + c\|y\|^3_{L^2_{\Delta T}L^2} + c\|y\|^2_{L^2_{\Delta T}L^2}.
\end{equation}

From (2.17) and (2.21) one has that

\begin{equation}
(2.26) \quad \|y\|_{L^2_{\Delta T}L^2} \leq \|y\|_{L^{2/3}_{\Delta T}L^3} \leq cN^{2-3s}.
\end{equation}

To estimate the last term in (2.25) we combine (2.9)-(2.10), (2.24), Sobolev and Hölder inequalities to obtain that for any $\theta \in (7/9, 1)$, (we recall that $s \geq 2/3$)

\begin{equation}
(2.27) \quad \|y\|_{L^2_{\Delta T}L^2} \leq c_\theta \|\nabla y\|_{L^{10/3}_{\Delta T}L^{10/3-1}_{\Delta T}} \|y\|_{L^{20/3}_{\Delta T}L^{20/3-1}_{\Delta T}} \leq c_\theta \|\Delta y\|_{L^{10/3}_{\Delta T}L^{10/3-1}_{\Delta T}} \|\nabla y\|_{L^{10/3}_{\Delta T}L^{10/3-1}_{\Delta T}} \|y\|_{L^{20/3}_{\Delta T}L^{20/3-1}_{\Delta T}} \leq c_\theta N^{3\theta-1-s(6\theta-3)}.
\end{equation}

Hence, (2.25)-(2.27) imply that

\begin{equation}
(2.28) \quad \|(\partial_t z, \nabla z)(\Delta T)\|_{L^1} \sim c_\theta N^{3\theta-1-s(6\theta-3)}.
\end{equation}

Finally we estimate $\|z_t(\Delta T)\|_{L^1}$.
Combining (2.12), Minkowski’s integral inequality, Sobolev’s embedding and fractional integration we get that

\[
\|z(\Delta T)\|_{L^1} \leq \int_0^{\Delta T} \|D^{3/4} \sin(t - t') F(t')\|_{L^2} dt'
\]

\[
\leq c \int_0^{\Delta T} \|D^{3/4} \sin(t - t') F(t')\|_{L^2} dt'
\]

\[
\leq c \|D^{-1/4} F(t)\|_{L^3} \leq c \|F(t)\|_{L^3}^{12/7}.
\]

To bound the last expression we use (2.9)-(2.10), (2.17), and (2.21) to write

\[
\|y\|_{L^3}^{12/7} = \|y\|_2^{3/11} \leq c(\Delta T)^{1/12} \|y\|_{L^2}^{3/11} \|y\|_2^{3/11} \leq c(\Delta T)^{1/12} \|y\|_{L^2}^{3/11} \|y\|_2^{3/11} \leq c N^{(10 - 17s)/6},
\]

and

\[
\|y^2\|_{L^3}^{12/7} \leq c(\Delta T)^{3/4} \|y\|_{L^2} \|y\|_{L^3} \leq c N^{(2 - 3s)/2}.
\]

Therefore, collecting the information in (2.29)-(2.31) we find that

\[
\|z(\Delta T)\|_{L^1} \leq c N^{(2 - 3s)}.
\]

From (2.28) and (2.32) we conclude that

\[
\|(\partial_t z, \nabla z)(\Delta T)\|_{L^2} + \|z(\Delta T)\|_2^2 \sim c_N N^{3\theta - 1 - s(6\theta - 3)}.
\]

Next we solve the IVP (2.5) in the time interval \([\Delta T, 2\Delta T]\) with data

\[
(v(\Delta T) + z(\Delta T), \partial_t v(\Delta T) + \partial_t z(\Delta T))
\]

and then we insert its solution \(v(t)\) in (2.11) to solve this IVP in the time interval \([\Delta T, 2\Delta T]\) with data

\[
(W'(\Delta T) u_{0,2} + W(\Delta T) u_{1,2}, -\Delta W(\Delta T) u_{0,2} + W'(\Delta T) u_{1,2}).
\]
and repeat the above estimates.

To reach the time $T$ we have to apply the above argument

\begin{equation}
\frac{T}{\Delta T} = TN^{2(1-\epsilon)} - \text{times.}
\end{equation}

Then, the total added to the expression in the right hand side of (2.7) will be (see (2.33))

\begin{equation}
c_\theta TN^{2(1-\epsilon)} N^{3\theta - 1 - s(\theta - 3)}.
\end{equation}

To make the above computations uniform we just need (see (2.7)) that

\begin{equation}
c_\theta TN^{2(1-\epsilon)} N^{3\theta - 1 - s(\theta - 3)} \leq N^{1-s}.
\end{equation}

If $s > 3/4$ the inequality (2.38) holds by fixing the value of $\theta$ sufficiently close to 1. Taking now $N$ sufficiently large we complete the proof.

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ANALYTICITY PROPERTIES AND ESTIMATES OF RESOLVENT KERNELS NEAR THRESHOLDS

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Abstract- Resolvent estimates are derived for the family of ordinary differential operators
\[
\left\{-c^2(y) \left[ \rho(y) \frac{d}{dy} \left( \frac{1}{\rho(y)} \frac{d}{dy} \right) - p^2 \right] \right\}, \quad p \in [0, \infty), \quad y \in \mathbb{R}.
\]

It is assumed that \( c(y) = c_k > 0, \rho(y) = \rho_k \) for \( \pm y > y_\varepsilon \), and the kernels are studied in neighborhoods of the points \( \{ c_k^2 p^2 \} \), uniformly in compact intervals of \( p \). This family arises in the direct integral decomposition of the acoustic propagator in layered media, \(-c^2(y) \rho(y) \nabla x,y \left( \frac{1}{\rho(y)} \nabla x,y \right), \) \( x \in \mathbb{R}^n \), and the
1 Introduction

Consider the operator $H = -\mathcal{A}^2(y) \rho(y) \nabla \cdot \left( \frac{1}{\rho(y)} \nabla \right)$, where $\nabla = \nabla_{x,y}$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}$, $n \geq 1$. This operator is known as the "acoustic propagator" in (unperturbed) stratified media ([4, 7, 9, 19]). In what follows we impose the following assumptions on the real functions $\rho(y), c(y)$ ($y \in \mathbb{R}$):

\begin{enumerate}
  \item $\rho(y) \in C^1(\mathbb{R})$, $c(y)$ is piecewise continuous with finitely many jump discontinuities, and there exist positive constants $\rho_m, \rho_M, c_m, c_M$ such that
    \[ 0 < \rho_m \leq \rho(y) + |\rho'(y)| \leq \rho_M, \quad 0 < c_m \leq c(y) \leq c_M. \]
  \item There exist positive constants $y_c, \rho_\pm, c_\pm$ such that $c_+ < c_-$ and
    \[ \rho(y) = \rho_\pm, \quad c(y) = c_\pm \quad \text{for} \quad \pm y > y_c. \]
\end{enumerate}

It is well-known that the operator $H$ can be defined as a self-adjoint operator in the space $L^2(\mathbb{R}^{n+1}, c^{-2}(y) \rho^{-1}(y) dx dy)$. Taking the (partial) Fourier transform with respect to the $x$-coordinates, we have the unitary equivalence.

\begin{equation}
H \cong \int_{\mathbb{R}^n} \oplus H_{\lambda} \, d\zeta,
\end{equation}

where

\begin{equation}
H_p = -c^2(y) \left[ \rho(y) \frac{d}{dy} \left( \frac{1}{\rho(y)} \frac{d}{dy} \right) - p^2 \right], \quad 0 \leq p < \infty.
\end{equation}

For every $p$, the operator $H_p$ is self-adjoint in $L^2(\mathbb{R}, c^{-2}(y) \rho^{-1}(y) dy)$ and its essential spectrum $[c_+^2 p^2, \infty)$ is absolutely continuous (in view of (1.1)-(1.2)). The spectral multiplicity of $H_p$ jumps at $c_+^2 p^2$ from 1 to 2.

It is the purpose of this paper to study the behavior of the resolvent kernels associated with the family $\{H_p\}$, specifically near the points $\{c_+^2 p^2\}$. We obtain estimates on the kernels and their derivatives (in terms of the spectral parameter), uniformly in $p$ in compact intervals of $[0, \infty)$. In particular, these estimates hold at "thresholds" (see Remark 2.6).
Such estimates constitute the main technical tools needed in the study of the
resolvent of $H$ near the continuous spectrum, and in particular the “limiting
absorption principle” ([4]) and “low energy estimates”.

To illustrate the situation at hand we take the Laplacian $-\Delta$ in $\mathbb{R}^{n+1}$ and let
$T$ be its self-adjoint realization in $L^2(\mathbb{R}^{n+1})$. Then, parallel to (1.3) -(1.4), we have

$$ T \cong \int_{\mathcal{R}} \Theta_{T\xi} d\zeta, $$

(1.5)

$$ T_p = -\frac{d^2}{dy^2} + p^2, \quad 0 \leq p < \infty. $$

(1.6)

For $z \in \mathbb{C}, \text{Im} \ z \neq 0$, we denote the resolvents by $M(z) = (T - z)^{-1}, M_p(z) = (T_p - z)^{-1}$, so that

$$ M(z) \cong \int_{\mathcal{R}} \Theta_{M\xi} (z) d\zeta. $$

(1.7)

The kernel (Green's function) $L_p(\xi, \eta; z)$ associated with $M_p(z)$ is given by

$$ L_p(\xi, \eta; z) = (2i)^{-1}(z - p^2)^{-1/2} \exp[i(\xi - \eta)(z - p^2)^{1/2}], \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}, $$

(1.8)

where $\text{Im}(z - p^2)^{1/2} > 0$. Various estimates and properties of $-\Delta$ can be derived via (1.7)-(1.8). In particular, this applies to estimates near $z = 0$, the “low energy estimates”. We shall derive such estimates in the Appendix, as a motivation for our treatment of $H$ in terms of (1.3).

As we show in this work, the kernels associated with $(H_p - z)^{-1}$ share some of the main properties of the kernels $L_p$, thus permitting analogous results for $H$.

We note that in particular we give here new proofs to Propositions A.2, A.4 in [4] and a correction to an error (pointed out to us by Professor J. Ralston) in the proof of Proposition A.1 (b) there (see Remark 2.3 for more details).

Decompositions of the type (1.3)-(1.4) are also encountered in a variety of examples in mathematical physics, such as acoustic or electromagnetic propagators [2, 7, 9, 15, 17] and linear elasticity theory [6, 11, 16]. It is expected that the techniques developed here will serve in the spectral study of those cases as well.

Our method of proof is based on a study of the analyticity properties of the kernels in terms of the variable $k = (z\xi - \eta^2)^{1/2}$ (see next section for details). This is closely related to the work of Cohen and Kappeler [8]. In particular,
their treatment of the Wronskian of Jost functions (Proposition 2.4 in [8]) corresponds to our conclusion (2.12). However, since we are dealing with families of operators, we need to study the way in which the zeros of the Wronskian "move around" (see (2.10)) and get uniform estimates on a family of kernels. Furthermore, in order to get estimates on the \( z \)-derivatives of the kernels, we need to study analytic extensions of the kernels to sectors in the lower \((k-)\)half-plane. We refer the reader to [1, 18] for works concerning such extensions.

To formulate our results, fix \( L > 0, a > 0 \) and \( 0 < \beta < \pi \). For every \( p \in [0, L] \)
\begin{equation}
I_p = (c^2_p, p^2 + a), \quad \Omega^+_p = \{ z \in \mathbb{C}; z - c^2_p = re^{i\theta}, \ 0 < r < a, \ 0 < \theta < \beta \}.
\end{equation}

Let \( \Phi_p(\xi, \eta; z) \) be the resolvent kernel associated with \((H - z)^{-1}\).

**Theorem A** There exists a constant \( C > 0 \), depending only on \( L, a, \beta \), such that
\begin{equation}
|\Phi_p(\xi, \eta; z)| \leq C|z - c^2_p|^{-1/2} + |z - c^2_p|^{-1/2},
\end{equation}
for \( z \in \Omega^+_p, \ \ p \in [0, L] \), \( (\xi, \eta) \in \mathbb{R} \times \mathbb{R} \). Furthermore, \( \Phi_p(\xi, \eta; z) \) can be extended continuously to \( \mathbb{R} \times \mathbb{R} \times I_p \), and the estimate (1.10) is valid for \( z \in I_p \setminus \{c^2_p\} \).

**Remark 1.1** Remark that \( \Phi_p(\xi, \eta; z) \) is continuous at \( z = c^2_p (p > 0) \), but (1.10) expresses the fact that

\[ \sup_{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}} |\Phi_p(\xi, \eta; z)| \]

may blow-up as \( z \) approaches \( c^2_p \).

It follows from (1.10), (1.8), that the "blow-up" rate for the kernel \( \Phi_p \) at the bottom of the essential spectrum is no worse than that of \( L_p \), associated with \(-\Delta\). In particular, repeating verbatim the proofs of Lemmas A.1, A.2 in the Appendix, we obtain a "low energy estimate" for the resolvent \((H - z)^{-1}\). In its formulation, we use the weighted-\( L^2 \) norm introduced in the Appendix, and also the "limiting absorption principle", which allows the extension of \((H - z)^{-1}\) to positive \( z \) (continuously in the same operator space).

**Corollary** Fix \( 0 < \tau < \pi \) and let \( \Lambda_\tau = \{ z; 0 \leq \arg z \leq \tau \} \). Then,
(a) For any \( s > 1, \ n \geq 1, \)
We shall now state a theorem concerning the differentiability of $a,((, q;z)$ with respect to real $z$ embedded in the continuous spectrum. Recall (4.1) that such a result implies the "limiting absorption principle" as well as the Hölder continuity of the limiting values of the resolvent.

**Theorem B** Fix $a > 0$, $0 < \ell < L$, and let $I_p$ be as in (1.9) with $p \in [l, L]$. Then there exists a constant $C = C(\ell, L, a) > 0$ such that, for $\lambda \in I_p \setminus \{c_+^2 p^2\}$,

$$\left| \frac{\partial}{\partial \lambda} \Phi_p(\xi; \eta; \lambda) \right| \leq C \left[ |\lambda - c_+^2 p^2|^{-3/2} + |\lambda - c_+^2 p^2|^{-3/2} \right] (1 + |\xi| + |\eta|).$$

The proofs of Theorems A, B are given in the next section.

**Remark 1.2** Results similar to those of Theorems A, B can be obtained for an operator of the form $-\frac{d^2}{dp^2} + V(y)$, where $V(y)$ is constant for large $|y|$. See also Remark 2.9.

## 2 Resolvent Kernels and proofs of Theorems A, B

Given $p \in [0, \infty)$, $z \in C^+ \equiv \{w; \text{Im } w > 0\}$, we are interested in estimating the kernels $\Phi_p$ associated with $(H_p - z)^{-1}$. However, it will be more convenient to replace $z$ by another parameter $k$, defined by

$$k = (zc_+^2 - p^2)^{1/2}.$$

We shall always take $\text{Im } \zeta^{1/2} \geq 0$ for $\zeta \in C$.

The Jost functions $\varphi, \psi$, which are solutions of

$$\begin{aligned}
(H_p - z) u &= 0, \\
\varphi(y; k, p) &= \exp(iky), \quad y > y_c, \\
\psi(y; k, p) &= \exp[-i((k^2 + p^2) c_+^2 - p^2)^{1/2} y], \quad y < -y_c.
\end{aligned}$$

\[(2.3)
\]
Note that the exponent of $\psi$ vanishes at $k = \pm \bar{k}$, where
\begin{equation}
\bar{k} = \bar{k}(p) = (c^2 \bar{c}_+^2 - 1)^{1/2}p, \quad \bar{k} \geq 0,
\end{equation}
so that
\begin{equation}
\psi(y; k, p) = \exp[-ic_D c_+^{-1}(k^2 - \bar{k}^2)^{1/2}y], \quad y < -y_c.
\end{equation}
Fix $L > 0, a > 0$. For any $p \in [0, L]$ take $I_p, \Omega_p^+$ as in (1.9). Note that $z \in \Omega_p^+$ implies $k \in \mathbb{C}^+ \backslash \{\Re k > 0\}$. However, using the asymptotic behavior (2.3) for $\varphi, \psi$, it is obvious that both functions can be extended to $k \in \mathbb{C}^+$, so that they solve (2.2) with $z = c_+^2(k^2 + p^2)$. In view of standard theorems about solutions of ordinary differential equations we can summarize the properties of $\varphi, \psi$ and their $y$-derivatives $\varphi', \psi'$ as follows.

**Claim 2.1.** The functions $\varphi(y; k, p), \psi(y; k, p), \varphi'(y; k, p), \psi'(y; k, p)$ are continuous in $\mathbb{R} \times \mathbb{C}^+ \times [0, L]$ and analytic in $k \in \mathbb{C}^+$ for any fixed $(y, p) \in \mathbb{R} \times [0, L]$.

(Remark that for the continuity of $\psi$ for real $k < -\bar{k}$ one takes in (2.3), (2.5) the root satisfying $\arg((k^2 + p^2) c_+^2 c_-^2 - p^2)^{1/2} = \arg((k^2 - \bar{k}^2)^{1/2} = \pi$).

For any two functions $\tau(y), \theta(y)$ we define the Wronskian $[\tau, \theta]$ by
\begin{equation}
[\tau, \theta] = \rho^{-1}(y)(\tau' \theta - \tau \theta').
\end{equation}
Recall that $[\tau, \theta]$ is independent of $y$ if $\tau, \theta$ are solutions of (2.2). In particular, by evaluating the following Wronskians for $\pm y > y_c$, taking (2.3) into account, we get, for $k \in \mathbb{R}$,
\begin{equation}
[\varphi, \tilde{\varphi}] = 2i \rho_+^{-1}k, \quad [\psi, \tilde{\psi}] = \begin{cases} 0, & |k| \leq \bar{k}, \\ -2i \rho_-^{-1}c_+ c_-^{-1}(k^2 - \bar{k}^2)^{1/2}, & \text{sgn} k, \quad |k| > \bar{k}. \end{cases}
\end{equation}
(Note that $\psi = \tilde{\psi}$ for $|k| \leq \bar{k}$).

The resolvent kernel (Green's function), $\Phi_p(\xi, \eta; k), k \in \mathbb{C}^+$, is given by [12],
\begin{equation}
\Phi_p(\xi, \eta; k) = \begin{cases} \frac{\varphi(\xi; k, p) \psi(\eta; k, p)}{[\varphi, \psi]} & \xi \geq \eta, \\ \frac{\varphi(\eta; k, p) \psi(\xi; k, p)}{[\varphi, \psi]} & \xi \leq \eta. \end{cases}
\end{equation}
In view of Claim 2.1, the kernel $\Phi_p$ is analytic in $k \in \mathbb{C}^+$ and can be extended continuously to $k \in \mathbb{C}^+$ except for zeros of $[\varphi, \psi]$. It follows from our next claim that in fact $[\varphi, \psi]$ does not vanish for $k \in \mathbb{R} \setminus \{0\}$.
Claim 2.2  (a) The Jost functions \( \varphi, \psi \) are linearly independent (hence \( [\varphi, \psi] \neq 0 \)) for every \((k, p) \in \mathbb{C}^+ \times [0, L], \Re k \neq 0\).

(b) Let \( \Gamma \subseteq \mathbb{C}^+ \setminus \{0\} \) be compact and such that \( \pm k(p) \not\in \Gamma \) for \( p \in [0, L] \). Then,

\[
\sup_{x \in \Gamma, y \in [0, L]} |\varphi(x; k, p) \psi(y; k, p)| < \infty.
\]

**Proof.** (a) Observe that if \( \Im k > 0, \Re k \neq 0 \), then \( z = (k^2 + p^2) \xi \) is non-real. Thus, if \( \varphi = \alpha \psi, \alpha \in \mathbb{C} \), then in view of (2.3) the function \( \varphi \) is an \( L^2 \)-eigenfunction of \( H_p \), associated with a non-real eigenvalue \( z \). Thus, it remains to consider \( k \in \Re \setminus \{0\} \). But, if \( \varphi = \alpha \psi, \alpha \in \mathbb{C} \), then \( [\varphi, \psi] = |\alpha|^2 [\psi, \psi] \), which, in view of (2.7), is a contradiction for all \( k \in \Re \setminus \{0\} \).

(b) In view of (2.3) the boundedness claim in (2.9) is clear if \( \Gamma \) is compact and \( \sup_{x \in \Gamma, y \in [0, L]} |\varphi(x; k, p) \psi(y; k, p)| < \infty \). It thus suffices to look at the cases \( y_c < \eta \leq \xi \) or \( \eta \leq \xi \). In the first case, let \( \varphi_1(y; k, p) \) be the solution of (2.2) such that \( \varphi_1(y; k, p) = \exp(-i k y) \) for \( y > y_c \). Then \( \psi(y; k, p) = \alpha(k, p) \varphi(y; k, p) + \beta(k, p) \varphi_1(y; k, p) \) where \( \alpha, \beta \) are continuous for \((k, p) \in R \times [0, L]\) (since \( \varphi \) and \( \varphi_1 \) are linearly independent for such \((k, p)\)) and the boundedness of \( \psi(y; k, p) \varphi_1(y; k, p), \xi \geq \eta \), follows from (2.3). The second case follows in exactly the same way, since \( k^2 - k^2 \) does not vanish for \((k, p) \in R \times [0, L]\).

**Remark 2.3** In Claim 2.2 (a) we correct the statement of Proposition A.1 (b) in [4], where the linear independence of \( \varphi, \psi \) was stated also for \( k = 0 \) (i.e., \( z = c_i^2 p^2 \)). However, for this value the Jost functions may be linearly dependent, corresponding to a solution of \((H_p - c_i^2 p^2) u = 0 \) which is constant for \( y > y_c \) and decays exponentially as \( y \to -\infty \). This is the case labeled as "exceptional" in [8], in contrast to the "generic" case, where \( [\varphi, \psi] \neq 0 \) at \( c_i^2 p^2 \). Since the proof of Proposition A.2 in [4] relies on Proposition A.1 (b), we need to modify it, which we do below in the proof of Theorem A. We are grateful to J. Raistin for bringing this error to our attention.

It follows from Claim 2.1 that the Wronskian \([\varphi, \psi](k, p)\) is continuous in \( \mathbb{C}^+ \times [0, L] \) and analytic in \( k \in \mathbb{C}^+ \), for every fixed \( p \in [0, L] \). By Claim 2.2 (a) it does not vanish if \( \Re k \neq 0 \). Its zeros on the positive imaginary \( k \)-axis correspond to the \( L^2 \)-eigenvalues of \( H_p \) below the threshold \( c_i^2 p^2 \). In fact, we can obtain more information by noting the following claim, which extends the analyticity (in \( k \)) domain of \([\varphi, \psi]\). Its proof is immediate, in view of (2.3) and standard theorems concerning ordinary differential equations.

**Claim 2.4** Fix \( 0 < p < L \). Then the Jost functions \( \varphi(\eta; k, p), \psi(y; k, p) \) and
their $y$-derivatives $\varphi', \psi'$, can be extended analytically to the disk $|k| < \frac{1}{2} \hat{k}(p)$. If $p = 0$ these functions can be extended analytically to all of $\mathbb{C}$.

In particular, the Wronskian $[\varphi, \psi](k, p)$ is analytic in $k$ in $\mathbb{C}^+ \cup \{ |k| < \frac{1}{2} \hat{k}(p) \}$, for any fixed $p \in [0, L]$ (in $\mathbb{C}$ for $p = 0$).

We infer immediately from this claim that there are only finitely many zeros (including multiplicity) of $[\varphi, \psi](k, p)$ for $\text{Im } k > 0$, for any fixed $p \in [0, L]$.

Note that $[\varphi, \psi](i\zeta, p) = 0$, $\zeta > 0$, implies that $\lambda = c^2_1(p^2 - \zeta^2)$ is an eigenvalue of $H_p$, and since $H_p \geq 0$ we have $\zeta \leq p$. Let, for $p \in [0, L]$, $\{\zeta_{j}(p)\}_{j=1}^{J(p)}$ be the finitely many zeros of $[\varphi, \psi](\cdot, p)$ in $\mathbb{C}^+$ (including multiplicity. Actually, following the argument of Lemma 1.3 in [8], we see that they are all simple, but we shall not need this fact). We arrange these zeros such that

$$p \geq \zeta_1(p) \geq \ldots \geq \zeta_J(p) > 0, \quad J = J(p).$$

The continuity of $[\varphi, \psi](k, p)$ in $p$ and standard analyticity arguments imply that the integer function $J(p)$ is constant in intervals where $[\varphi, \psi](0, p) \neq 0$. However, the following claim shows that this is the case except possibly for a finite number of values.

Claim 2.5 There are at most finitely many points $0 \leq p_1 < p_2 < \ldots < p_m \leq L$ such that $[\varphi, \psi](0, p_j) = 0$, $j = 1, \ldots, m$.

Proof. As is seen from (1.4) and (2.3) the function $[\varphi, \psi](0, p)$ can be extended analytically to $p \in \mathbb{C}$.

Remark 2.6 The values $\{c^2_1, p_j^2\}$ are called the “thresholds” of $H$. Generically, as in Remark 2.3, they correspond to a “semi-bound” state and mark the “birth” of a new branch of proper eigenvalues (i.e., $\zeta_{J(p)}(p) \to 0$ as $p \to p_j + 0$). We refer to [4] for details.

From Claim 2.5 and the discussion preceding it we conclude that there exists $M = M(L) > 0$ such that

$$J(p) \leq M, \quad p \in [0, L].$$

We now turn to the study of $[\varphi, \psi](k, p)$ for real $k$ near $k = 0$. Our next claim will show, in particular, that $k = 0$ is at most a simple zero for $p = p_j$, $j = 1, \ldots, m$. For a single operator, it follows from Proposition 2.4 in [8].

Claim 2.7 Fix $\delta > 0$. Then there exists a positive constant $\gamma > 0$ such that

$$|[\varphi, \psi](k, p)| \geq \gamma |k|, \quad k \in [-\delta, \delta], p \in [0, L].$$
Proof. As in the proof of Claim 2.2 (b), we write for $k \neq 0$,

\begin{equation}
\psi(y; k, p) = \alpha(k, p) \varphi(y; k, p) + \beta(k, p) \varphi_1(y; k, p).
\end{equation}

so that by (2.7),

\begin{equation}
[\varphi, \psi](k, p) = 2i\rho_{+}^{-1}k\beta(k, p).
\end{equation}

On the other hand, using (2.7) once again to evaluate $[\psi, \check{\psi}]$ by (2.13),

\begin{equation}
-2i\rho_{+}^{-1}c_{+}c_{-}^{-1}(k^{2} - \bar{k}^{2}) \cdot (1 - \chi(k))\text{sgn} k = 2ik\rho_{+}^{-1}|\alpha|^{2} - |\beta|^{2}
\end{equation}

$$\chi(k) = 1 \text{ for } k \in [-\bar{k}, \bar{k}], = 0 \text{ otherwise}.$$ We conclude that

\begin{equation}
|\alpha(k, p)| \leq |\beta(k, p)| \quad (\alpha(k, p) = \beta(k, p) \text{ for } k \in (-\bar{k}, \bar{k})).
\end{equation}

Suppose that there exist sequences $\{k_{n}\} \subseteq [-\delta, \delta] \setminus \{0\}$, $\{p_{n}\} \subseteq [0, L]$ such that $\beta(k_{n}, p_{n}) \to 0$. Necessarily $\alpha(k_{n}, p_{n}) \to 0$. Without loss of generality assume $k_{n} \to K \in [-\delta, \delta]$, $p_{n} \to q \in [0, L]$. By continuity we infer from (2.13) that $\psi(y; k, q) \equiv 0$, a contradiction to (2.3). Thus,

\begin{equation}
\inf_{K}(|\beta(k, p)| > 0 \text{ in } \left([-\delta, \delta] \setminus \{0\}\right) \times [0, L].
\end{equation}

which proves (2.12) in view of (2.14).

We may now conclude the proof of Theorem A.

Proof of Theorem A. Take $\delta \geq 2 \left(L + \bar{k}(L)\right)$, and let $\Gamma_{\delta} = \overline{C^{T}} \cap \{|k| \leq \delta\}$. In what follows we use $C > 0$ to denote positive constants not depending on $k, p$. In view of (2.9) we have, for $k \in \overline{C^{T}}, \{k\} = \delta$,

\begin{equation}
|\varphi(\xi; k, p) \psi(\eta; k, p)| \leq C, \quad p \in [0, L], \quad -\infty < \eta \leq \xi < \infty,
\end{equation}

and, by (2.10) and Claim 2.2 (a), $[\varphi, \psi](k, p) \neq 0$ for $k \in \overline{C^{T}}, \{k\} = \delta, p \in [0, L]$. The expression (2.8) now yields, on the circular part of $\partial \Gamma_{\delta}$,

\begin{equation}
|\Phi(p(\xi; \eta, k))| \leq C, \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}, \quad p \in [0, L], \quad k \in \partial \Gamma_{\delta} \cap \overline{C^{T}}.
\end{equation}

Next, we estimate $\Phi(p(\xi; \eta, k))$ for $k$ in the real part of $\partial \Gamma_{\delta}$. This means that we need again to estimate products of the type $\varphi(\xi; k, p) \psi(\eta; k, p). \eta \leq \xi$, this
time for real \( k \in [-\delta, \delta] \). Note first that, by (2.3) and Claim 2.1, we have the individual uniform boundedness,

\[
|\varphi(\xi; k, p)| \leq C, \quad |\psi(\eta; k, p)| \leq C, \quad k \in [-\delta, \delta], \quad \xi \geq -\eta, \quad \eta \leq \eta_c.
\]

Thus, it suffices to treat the cases (a) \( \eta_c \leq \eta \leq \xi \) and (b) \( \eta \leq \xi \leq -\eta_c \). In case (a) we use (2.13) to write

\[
\varphi(\xi; k, p) \psi(\eta; k, p) = [\alpha(k, p) \varphi(\eta; k, p) + \beta(k, p) \varphi_1(\eta; k, p)] \varphi(\xi; k, p),
\]

and in conjunction with (2.14), for \( k \neq 0 \),

\[
\frac{\varphi(\xi; k, p) \psi(\eta; k, p)}{[\varphi, \psi](k, p)} = -\frac{i}{2} r_+ k^{-1} \left[ \frac{\alpha(k, p)}{\beta(k, p)} \varphi(\eta; k, p) + \varphi_1(\eta; k, p) \right] \varphi(\xi; k, p)
\]

so that (2.16) and (2.3) yield

\[
|k \Phi_p(\xi, \eta; k)| \leq C, \quad k \in [-\delta, \delta] \setminus \{0\}, \quad \eta_c \leq \eta \leq \xi, \quad p \in [0, L].
\]

In case (b) we first define (analogously to \( \varphi_1 \)) a second solution \( \psi_1 \) to (2.2), such that, for \( y < -\eta_c \),

\[
\psi_1(y; k, p) = \exp[i((k^2 + p^2)c_0^2 c_2^2 - p^2)^{1/2} y] = \exp[ic_0 c_2^{1/2}(k^2 - \tilde{k}^2)^{1/2} y],
\]

(Recall that \( \arg(k^2 - \tilde{k}^2)^{1/2} = \pi \) if \( k < -\tilde{k} \)).

The solutions \( \psi, \psi_1 \) are linearly independent if \( k \neq \pm \tilde{k} \), but coincide if \( k = \pm \tilde{k} \) (and are identically 1 for \( y < -\eta_c \)). However, the pair \( \{ \psi, (k^2 - \tilde{k}^2)^{-1/2}(\psi - \psi_1) \} \) is linearly independent if \( k \neq \pm \tilde{k} \) and remains linearly independent when extended continuously to \( k = \pm \tilde{k} \). We may therefore write,

\[
\varphi(y; k, p) = \theta(k, p) \psi(y; k, p) + \tau(k, p) \frac{(\psi - \psi_1)(y; k, p)}{(k^2 - \tilde{k}^2)^{1/2}}
\]

(with suitable modification at \( k = \pm \tilde{k} \)), with \( \theta(k, p), \tau(k, p) \) continuous in \([\eta_c, \xi] \times [\eta_c, \xi] \). We obtain readily,

\[
[\varphi, \psi](k, p) = -2ic_0 c_2^{-1} \rho_+^{-1} \tau(k, p).
\]

We write by (2.8), for \( \eta \leq \xi \leq -\eta_c, k \neq 0 \),

\[
\Phi_p(\xi, \eta; k) = \frac{\psi(\eta; k, p) [\theta(k, p) \psi(\xi; k, p) + \tau(k, p) \frac{(\psi - \psi_1)(\xi; k, p)}{(k^2 - \tilde{k}^2)^{1/2}}]}{[\varphi, \psi](k, p)}
\]
so that in view of (2.12), (2.24) and the uniform boundedness of $\psi(\eta; k, p)$ $\psi(\xi; k, p)$, $\eta \leq \xi \leq -\gamma_\varepsilon$, $(k, p) \in [-\delta, \delta] \times [0, L]$, we get

\[
\begin{align*}
(2.25) \quad & \left| [\varphi, \psi](k, p) \left[ \Phi_p(\xi, \eta; k) - \frac{i}{2} c_+^{-1} c_- \rho \frac{\psi(n; k, p) - \psi(n; k, \pm k^2)}{(k^2 - k^2)^{1/2}} \right] \right| \leq C, \\
& \eta \leq \xi \leq -\gamma_\varepsilon, \quad (k, p) \in \{[\delta, \delta] \backslash \{0, \pm \hat{k}\} \} \times [0, L].
\end{align*}
\]

Note that by Claims 2.4 and 2.7 the function $k \Phi_p(\xi, \eta; k)$, for fixed $\xi, \eta, p$, is analytic near $k = 0$, so that the estimates (2.21), (2.25) are valid also at $k = 0$. Furthermore, the function

\[
(2.26) \quad F_p(\xi, \eta; k) := \prod_{j=1}^{J(p)} \frac{k - i\zeta_j(p)}{k + i\zeta_j(p)} k \Phi_p(\xi, \eta; k)
\]

(see (2.10) for $\zeta_j(p)$) is analytic in $\Gamma_\delta$ and continuous in $\Gamma_\delta$. It therefore attains its maximum (absolute) value on $\partial \Gamma_\delta$. Take $0 < \varepsilon < \frac{\delta}{2}$. Due to (2.11) the product $\Lambda = \prod_{j=1}^{J(p)} \frac{k - i\zeta_j(p)}{k + i\zeta_j(p)}$ is uniformly bounded on the line segments $\{\arg k = \varepsilon\} \cap \Gamma_\delta$ and $\{\arg k = 0\} \cap \Gamma_\delta$, for all $p \in [0, L]$. The same is clearly true for $\Lambda^-1$. Noting the boundary estimates (2.19), (2.20) and (2.21), we get, in the sector $\{0 \leq \arg k \leq \varepsilon\} \cap \Gamma_\delta$,

\[
(2.27) \quad |k \Phi_p(\xi, \eta; k)| \leq C.
\]

$(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$, $\max(\xi, \eta) > -\gamma_\varepsilon$, $p \in [0, L]$, $0 \leq \arg k \leq \varepsilon$, $k \neq \hat{k}(p)$ and $|k| \leq \delta$, where $C = C(L, \varepsilon)$. In the case $\eta \leq \xi \leq -\gamma_\varepsilon$, we repeat the argument above, replacing in (2.26) the function $k \Phi_p$ by (see (2.25)),

\[
[\varphi, \psi](k, p) \left[ \Phi_p(\xi, \eta; k) - \frac{i}{2} c_+^{-1} c_- \rho \frac{\psi(n; k, p) - \psi(n; \xi, \eta) (\xi, k, p)}{(k^2 - k^2)^{1/2}} \right].
\]

We conclude that this function is uniformly bounded in $\Gamma_\delta \cap \{0 \leq \arg k \leq \varepsilon\}$, so that the uniform boundedness of $\psi(\eta; k, p)$ $\psi(\xi; k, p)$, $\eta \leq \xi \leq -\gamma_\varepsilon$, yields now

\[
(2.28) \quad |\Phi_p(\xi, \eta; k)| \leq C \left( \frac{1}{|k|} + \frac{1}{|k^2 - k^2|^{1/2}} \right),
\]

$\eta \leq \xi \leq -\gamma_\varepsilon$, $0 \leq \arg k \leq \varepsilon$, $0 < |k| \leq \delta$, $k \neq \hat{k}$, $p \in [0, L]$. We can now conclude easily the proof of Theorem A. Indeed, if $z \in \Omega_p^+$, $p \in [0, L]$, then the associated $k$ (by (2.1)) satisfies $0 < \arg k < \varepsilon$, for $\varepsilon = \frac{\delta}{2}$. 

\[
\begin{align*}
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& \text{1763}
\end{align*}
\]
Since $|k| = |zc^{-2} - p^2|^{1/2}$ and $|k^2 - \tilde{k}^2|^{1/2} = c_+^{-1}c_-|zc^{-2} - p^2|^{1/2}$, the estimate (1.10) follows from (2.27), (2.28), which also imply the same estimates for the extension to $I_p$.

\[\square\]

**Remark 2.8** Observe that the fact that we are dealing with a family of operators $H_p$, $p \in [0, L]$, complicates the proof. Indeed, in our estimates we had to account for the various positions of $\tilde{k} = \tilde{k}(p)$ (approaching 0 as $p \to 0$) and $\tilde{i}c_j(p)$, the zeros of the Wronskian $[\varphi, \psi](k, p)$. The latter could also vanish as $p$ approaches a threshold.

As noted in Remark 1.1, the kernel $\Phi_p(\xi, \eta; k)$ is continuous at $k = \tilde{k}$ if $p > 0$ (Claim 2.2 (a)), but $\sup_{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}} |\Phi_p(\xi, \eta; k)|$ may blow-up as $k$ approaches $k(p)$.

**Remark 2.9** The method used here can clearly be used in the study of resolvent kernels for an operator of the form $-\frac{d^2}{dp^2} + V(y)$, where $V(y) = V_\pm$ for $y > y_0$. The case $V_+ = V_-$ corresponds to $p = 0$ while $V_+ > V_-$ corresponds to $p > 0$.

Similar estimates are obtained for the kernel in the neighborhood of $V_+$.

**Proof of Theorem B.** By Claim 2.4 we have, for $0 < \delta < \frac{1}{2} \tilde{k}(\ell)$, that $\varphi(y; k, p), \psi(y; k, p)$ and their $y$-derivatives are analytic (in $k$) in the disk $B_\delta = \{ |k| < \delta \}$. In particular, the derivative $\frac{d}{dk} [\varphi, \psi](k, p)$ is bounded for $(k, p) \in B_\delta \times [\ell, L]$. This implies, in view of (2.12), that for some $0 < \varepsilon < \delta$,

$$
(2.29) \quad |[\varphi, \psi](k, p)| \geq \gamma |k|, \quad k \in B^\varepsilon := B_\varepsilon \cap \{ |\arg k| < \varepsilon \}, \quad p \in [\ell, L],
$$

where $\gamma = \gamma(\varepsilon, \delta, \ell, L) > 0$. It follows ((2.8)) that $\Phi_p(\xi, \eta; k)$ can be extended analytically to $B^\varepsilon \setminus \{ 0 \}$. Remark that for $k < 0$ it is no longer the resolvent kernel associated with $H_p - z = (k^2 + p^2) c_+^2$, but just its analytic continuation (from $C^+ \cap B^\varepsilon_+$). However, the equality (2.8) remains valid throughout $B^\varepsilon_+$, and $\varphi, \psi$ are solutions of (2.2), satisfying the asymptotic behavior (2.3) in $k \in B^\varepsilon_+$. We also note that, as before, the exponent of $\psi$,

$$
(2.30) \quad \text{Im}((k^2 + p^2) c_+^2 c_-^2 - p^2)^{1/2} > 0 \quad \text{for} \quad p \in [\ell, L], k \in B^\varepsilon_+,
$$

and $\psi$ decays exponentially as $y \to -\infty$, uniformly in $k \in B^\varepsilon_+$. We have the representation (2.13) for $k \in B^\varepsilon_+ \setminus \{ 0 \}, p \in [\ell, L]$, hence also (2.14). By (2.29), (2.14),

$$
(2.31) \quad |\beta(k, p)| \geq \frac{\gamma}{2} \rho_* = \gamma_1.
$$
Evaluating (2.13) at \( y = y_c \), we get for \( k \neq 0 \),
\[
\beta(k;p)^{-1}\psi(y_c; k, p) = \frac{\alpha(k;p)}{\beta(k;p)} \phi(y_c; k, p) + \varphi_1(y_c; k, p),
\]
and in view of (2.31) and \( |\varphi_1(y_c; k, p)| = 1 \), we conclude
\[
(\text{2.32}) \quad \sup \left\{ \left| \frac{\alpha(k;p)}{\beta(k;p)} \right| : \; k \in B^\epsilon_c \setminus \{0\}, \; p \in [\ell, L] \right\} < \infty.
\]
We can now establish the key estimate
\[
(\text{2.33}) \quad |k\Phi_p(\xi, \eta; k)| \leq C \exp\left[-\operatorname{Im} k \right]^{\xi\eta^+}((\xi, \eta) \in \mathbb{R} \times \mathbb{R}, (k, p) \in B^\epsilon_c \times [\ell, L]),
\]
where \( b^+ = \max(b, 0) \). Indeed, for \( \operatorname{Im} k \geq 0 \) we have the stronger estimate
\[
(2.27)-(2.28) \quad \text{(note that \( |k^2 - \overline{k}|^2 > \frac{1}{2} \overline{k}(\ell)^2 \))}. \quad \text{In general, using (2.13), (2.14), (2.8), we get, for \( k \neq 0 \),}
\]
\[
(\text{2.34}) \quad k\Phi_p(\xi, \eta; k) = \frac{\rho}{2i} \varphi(\xi, k, p) \left[ \frac{\alpha(k;p)}{\beta(k;p)} \phi(\eta; k, p) + \varphi_1(\eta; k, p) \right], \; \xi \geq \eta,
\]
so that (2.32) and the asymptotic behavior (2.3) yield (2.33) in the case \(-y_c \leq \eta \leq \xi \). Obviously, (2.33) is valid if \( \eta < -y_c < \xi \). Finally, if \( \eta \leq \xi \leq -y_c \), the estimate (2.33) follows from (2.29) and the uniform exponential decay of \( \psi \) (as \( y \to -\infty \)).

Turning back to the proof of the theorem, we take \( k_0 \in (0, \frac{1}{2}) \) and let \( \Lambda_0 \subseteq B^\epsilon_c \) be the circle centered at \( k_0 \) with radius equal to \((2 + \xi^+ + \eta^+)k_0 \). By (2.33) the analytic function \( k\Phi_p(\xi, \eta; k) \) is bounded on \( \Lambda_0 \) with a bound which is independent of \( (\xi, \eta) \), so that the Cauchy formula implies
\[
(\text{2.35}) \quad \left| \frac{\partial}{\partial k} \left( k\Phi_p(\xi, \eta; k) \right) \right|_{k=k_0} \leq C \left| k_0^{-1}(1 + \xi^+ + \eta^+) \right| \in \mathbb{R} \times \mathbb{R}, \; p \in [\ell, L], \; k_0 \in (0, \frac{1}{2}).
\]
Changing back to \( \lambda = (k^2 + p^2)^{c_2^+} \in I_\rho \), \( \lambda < c^2 p^2 \gamma \), and noting (2.33), we obtain (1.13) away from \( c^2 \rho^2 \).

To prove (1.13) for \( \lambda \) near \( c^2 \rho^2 \), we need to study \( \frac{\partial}{\partial k} \Phi_p(\xi, \eta; k) \) near \( k = \hat{k}(p) \). We observe first that while \( \phi, \phi' \) can be extended analytically to a full neighborhood of \( \hat{k}(p) \), the functions \( \psi, \psi' \) (see (2.22)) can be extended analytically to sectors
\[
(\text{2.36}) \quad B^\epsilon_{\delta_0} = \{ k; \left| k - \hat{k}(p) \right| < \delta \}, \quad -\varepsilon < \arg(\pm(k - \hat{k})) < \varepsilon,
\]
for \( \delta, \varepsilon > 0 \) sufficiently small (uniformly in \( p \in [\ell, L] \)). In order to maintain continuity we need to take now \( \operatorname{Im}(k \pm \hat{k})^{1/2} < 0 \) for \( k \in B^\epsilon_{\delta_0}, \; \operatorname{Im} k < 0 \).
Using the analytic extensions, we can now proceed as in the study above near \( k = 0 \). First, by \( (2.12) \), \([\varphi, \psi](k, \rho)\) is bounded away from zero in \( B_{\delta k} \times [\epsilon, L] \). Then, using \( (2.34) \) for \( \xi \geq \eta \geq -\gamma \) and the expression following \( (2.24) \) for \( \eta \leq \xi \leq \gamma \) we obtain

\[
|\Phi_p(\xi, \eta; k)| \leq C\{\exp[-(1 - \text{Im} k^+)(1 + \xi^+ + \eta^+)] + |k^2 - \tilde{k}^2|^{-1/2} \exp[-(\text{Im}(k^2 - \tilde{k}^2)^{1/2})(1 + \xi^- + \eta^-)]\}
\]

\((k, \rho) \in (B_{\delta k} \setminus \{\tilde{k}\}) \times [\epsilon, L], (\xi, \eta) \in \mathbb{R} \times \mathbb{R} \) and we denoted \( b^- = -\min(b, 0) \).

We now proceed as in the argument leading up to \( (2.35) \) with suitable modifications. So, taking \( k_0 \in (k, k + \frac{\delta}{2}) \) (the case \( k_0 \in (k - \frac{\delta}{2}, \tilde{k}) \) is treated similarly), let \( \Lambda_0 \) be the circle centered at \( k_0 \), having radius \( (2 + |\xi| + |\eta|)^{-1}(k_0 - \tilde{k}) \) \sin \epsilon, so that \( \Lambda_0 \subset B_{\frac{\delta}{4}k} \). In order to estimate \( \text{Im}(k^2 - \tilde{k}^2)^{1/2} \) we note that

\[
2\text{Im}(k^2 - \tilde{k}^2)^{1/2} = \text{Im}(k^2 - \tilde{k}^2)(\text{Re}(k^2 - \tilde{k}^2)^{1/2})^{-1},
\]

the numerator of which is less than \( C|k - k_0| \).

The denominator is greater than \( |\text{Re}(k^2 - \tilde{k}^2)|^{1/2} \) and this last one greater than \( C|\text{Re}(k - \tilde{k})\text{Re}(k + \tilde{k}) - \text{Im}(k - \tilde{k})\text{Im}(k + \tilde{k})|^{1/2} \). For \( k \in \Lambda_0 \) (\( \epsilon \) is small enough and depends on \( k(l) \)), we have

\[
|\text{Im}(k^2 - \tilde{k}^2)|^{1/2} \leq C|k - \tilde{k}|^{-1/2}|k - k_0| \leq C(k_0 - \tilde{k})^{1/2}(2 + |\xi| + |\eta|)^{-1}\sin \epsilon,
\]

where \( C > 0 \) depends only on \( l, L, \delta \). We can now use \( (2.38) \) in \( (2.37) \) to conclude that the analytic function \( (k^2 - \tilde{k}^2)^{1/2}\Phi_p(\xi, \eta; k) \) is bounded on \( \Lambda_0 \), with a bound independent of \( (\xi, \eta) \in \mathbb{R} \times \mathbb{R} \), so that the Cauchy formula yields,

\[
\left| \frac{\partial}{\partial k}\left((k^2 - \tilde{k}^2)^{1/2}\Phi_p(\xi, \eta; k)\right)|_{k=k_0}\right| \leq C(k_0 - \tilde{k})^{-1}(1 + |\xi| + |\eta|)^{1/2}\sin \epsilon.
\]

Observe that \( \lambda \cdot c_2^2 \rho^2 = c_2^2(k^2 - \tilde{k}^2) \), so that \( \frac{\partial}{\partial k} \) and \( \frac{\partial}{\partial \lambda} \) are uniformly bounded for \( k \in [\tilde{k}, L] \) and \( k \in (\tilde{k} - \frac{\delta}{2}, \tilde{k} + \frac{\delta}{2}) \). Replacing the \( k \)-derivative in \( (2.39) \) by \( \lambda \)-derivative we obtain \( (1.13) \). Observe that outside neighborhoods of \( c_2^2 \rho^2 \) the theorem is obvious.

\[\square\]

**Appendix**

In this appendix we show how the decomposition \((1.3)-(1.7)\), combined with estimates on the kernel \( L_p \), leads to estimates on the behavior of the resolvent.
of the Laplacian $-\Delta$ near the lower endpoint of its spectrum, $\lambda = 0$. For this purpose, we shall use only the global bound of $L_p$.

For $\sigma \in \mathbb{R}$ we denote by $L^{2,\sigma}(\mathbb{R}^n)$ the weighted-$L^2$ space normed by

$$||f||^2_{L^{2,\sigma}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |x|^2)^{\sigma} |f(x)|^2 dx.$$  

Since $|L_p(\xi, \eta; z)| \leq C |z - p|^{-1/2}$, we get

$$||M_p(z)||_{L^{2,\sigma}(\mathbb{R}^n)} \leq C_s |z - p|^{-1/2} ||f||_{L^{2,\sigma}(\mathbb{R}^n)}, \quad s > \frac{1}{2}.$$  

We shall now use this estimate to establish an estimate for $M(z)$ near $z = 0$. So, using the notation in the Introduction, let $f(x, y), g(x, y) \in C^\infty_0(\mathbb{R}^{n+1})$ and let $\hat{f}(\theta, y), \hat{g}(\theta, y)$ be their partial Fourier transforms with respect to $x \in \mathbb{R}^n$.

Then, by (1.7),

$$\int_{\mathbb{R}^n} (M_p(z) f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (M_p(z) \hat{f}(\theta, \cdot), \hat{g}(\theta, \cdot)) dS_\theta dp,$$

where $\Sigma_\theta = \{ \theta \in \mathbb{R}^n; \theta = p \}$ and $dS_\theta$ is the Lebesgue surface measure.

Now fix $\beta > 1$ and let $|z| < \beta^2 - 1$. The family $\{M_p(z), p > \beta\}$ is uniformly bounded in $L^2(\mathbb{R})$, so that

$$\int_0^\beta \int_{\Sigma_\theta} (M_p(z) \hat{f}(\theta, \cdot), \hat{g}(\theta, \cdot)) dS_\theta dp \leq C ||f||_{L^2(R^{n+1})} ||g||_{L^2(R^{n+1})}.$$  

Thus, in (A.2), it suffices to estimate the right-hand side with $\int_0^\beta$ replaced by $\int_0^\beta$.

Let $H^s(\mathbb{R}^n)$ be the Sobolev space of order $s \in \mathbb{R}$, so that $||\tilde{u}||_{H^s(\mathbb{R}^n)} = ||u||_{L^{2,s}(\mathbb{R}^n)}$.

We recall the classical trace lemma

$$||\tilde{u}||_{L^2(\Sigma_\theta)} \leq C_{s,n} ||u||_{H^s(\mathbb{R}^n)}, \quad s > \frac{1}{2},$$  

where $C_{s,n} > 0$ is independent of $p \in (0, \infty)$.

Combining (A.1)-(A.4) we obtain, for $|z| < \beta^2 - 1$,

$$|M(z) f, g| \leq C_{s,n,0} \left( ||f||_{L^2(R^{n+1})} ||g||_{L^2(R^{n+1})} + \int_0^\beta |z - p|^{|1/2|} ||f||_{L^2(\Sigma_\theta) \otimes L^{2,s}(\mathbb{R})} ||g||_{L^2(\Sigma_\theta) \otimes L^{2,s}(\mathbb{R})} dp \right).$$  

\[A.5\]
We can simplify (A.5) by noting that the norm of $L^2_s(\mathbb{R}^{n+1})$ is stronger than that of $L^2_s(\mathbb{R}^n) \otimes L^2_s(\mathbb{R})$, so that, for $|z| < \beta^2 - 1$, $s > \frac{1}{2}$,

$$\leq C_{s,n,p} \left[ \|f\|_{L^2_s(\mathbb{R}^{n+1})} \|g\|_{L^2_s(\mathbb{R}^{n+1})} + \int_0^\beta |z - p^2|^{-1/2} dp \|f\|_{L^2_s(\mathbb{R}^n) \otimes L^2_s(\mathbb{R})} \|g\|_{L^2_s(\mathbb{R})} \right].$$

We can simplify (A.5) by noting that the norm of $L^2_s(\mathbb{R}^{n+1})$ is stronger than that of $L^2_s(\mathbb{R}^n) \otimes L^2_s(\mathbb{R})$, so that, for $|z| < \beta^2 - 1$, $s > \frac{1}{2}$,

$$\left\| M(z) \right\|_{B_s(L^2_s(\mathbb{R}^{n+1}), L^2_s(\mathbb{R}^{n+1}))} \leq C_{s,n,A} \left[ 1 + \int_0^\beta |z - p^2|^{-1/2} dp \right].$$

($B(X,Y)$ is the space of bounded linear operators from $X$ to $Y$, equipped with the operator norm).

The estimate (A.6) holds for $z \in \mathbb{C}^+$. However, we know that the operator-valued function $M(z)$ can be extended continuously, in the same operator space, to positive $z$ (the “limiting absorption principle”). In fact, this can be proved using the above arguments ([4]). Retaining the notation $M(z)$ for the extended function, we have the validity of (A.6) for $|z| < \beta^2 - 1$, $0 \leq \arg z \leq \tau$, where $\beta > 1$ and $0 < \tau < \pi$ are fixed. (In the absence of negative eigenvalues, we could actually take here $\tau = \pi$. However, this is not true in the case of Hypothesis (1.3), where $H_\rho$ possesses, in general, negative eigenvalues).

Inspecting the behavior of the integral in (A.6) as $z$ approaches $0$, $0 \leq \arg z \leq \pi$, we easily obtain the following “low energy estimate” for the resolvent $M(z) = (-\Delta - z)^{-1}$.

**Lemma A.1** For $0 \leq \arg z \leq \pi$, $s > 1$, $n \geq 1$,

$$\left\| M(z) \right\|_{B_s(L^2(\mathbb{R}^n))} = O(\log |z|), \quad z \to 0.$$  

Certainly (A.7) is not “optimal”, at least for $n \geq 2$ (see [3, 5, 13, 14]). However, the simplicity of its derivation makes it easy to generalize to cases such as those mentioned in the Introduction. Also, if $n \geq 3$ we can improve the trace estimate (A.4) to read ([5])

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_{s,n} p^{-1}\|u\|_{H^s(\mathbb{R}^n)}, \quad \frac{1}{2} < s \leq 1, \quad n \geq 3.$$  

Using this in (A.5) we obtain, instead of (A.6),

$$\left\| M(z) \right\|_{B_s(L^2(\mathbb{R}^n))} \leq C_{s,n} \left[ 1 + \int_0^\beta |z - p^2|^{-1/2} dp \right].$$

where $s > \frac{1}{2}$, $n \geq 3$. Replacing $L^2_s(\mathbb{R}^n) \otimes L^2_s(\mathbb{R})$ by the stronger norm of $L^{2,1+s}(\mathbb{R}^{n+1})$, we can modify Lemma A.1 as follows.
Lemma A.2  For $0 \leq \arg z \leq \tau < \pi$, $s > \frac{3}{2}$, $n \geq 3$.

\begin{equation}
\|M(z)\|_{B(L^2_{[x]}(\mathbb{R}^{*+1}),L^2_{[x]}(\mathbb{R}^{*+1}))} = O(1) \quad \text{as} \quad z \to 0.
\end{equation}

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THE INHOMOGENEOUS NEUMANN PROBLEM
IN LIPSCHITZ DOMAINS

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Abstract. Existence and uniqueness is established for the solution to the inhomogeneous Neumann problem for Laplace's equation in Lipschitz domains with data in $L^p$ Sobolev spaces.

In [7], Jerison and Kenig studied the inhomogeneous Dirichlet problem

$$\Delta w = F \text{ on } \Omega$$  \hspace{1cm} (1)
$$w = 0 \text{ on } \partial\Omega$$  \hspace{1cm} (2)

with data in Sobolev spaces of domains $\Omega$ with Lipschitz boundary and obtained the best possible range of spaces for which estimates hold. In the present article, we consider the corresponding problem for Neumann boundary conditions

$$\Delta w = F \text{ on } \Omega$$  \hspace{1cm} (3)
$$\frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega$$  \hspace{1cm} (4)
wherein $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$. In our main theorem, Theorem 1.6, we prove estimates for the solution in Sobolev spaces and establish uniqueness as well. We find in fact that estimates are true within exactly the same range of spaces as for the problem (1)-(2). Though we do not address the corresponding optimality assertion here, this range has been conjectured to be essentially the best possible, as it is for Dirichlet boundary conditions.

That estimates for the inhomogeneous Neumann problem hold within the same range as those for the Dirichlet problem is not too surprising, inasmuch as we actually invoke estimates for the homogeneous Dirichlet problem used by Jerison and Kenig in their analysis of the problem (1)-(2) in the proof for the case of Neumann conditions we present. The general outline of our approach is quite classical. Given sufficiently smooth data $F$ on $\Omega$ with $\int_{\Omega} F \, dV = 0$, it involves extending $F$ by 0 to all of $\mathbb{R}^n$ and then writing the proposed solution $w$ as

$$w = (F \ast N)|_{\Omega} - u,$$

(5)

wherein $N$ denotes the Newtonian potential on $\mathbb{R}^n$ so that $\Delta (F \ast N) = F$, and $u$ satisfies the homogeneous Neumann problem

$$\Delta u = 0 \quad \text{on } \Omega$$

$$\frac{\partial u}{\partial n} = \frac{\partial (F \ast N)}{\partial n} \quad \text{on } \partial \Omega.$$

(6) (7)

From (6)-(7) it is clear that in order to obtain estimates for the inhomogeneous Neumann problem it is necessary to combine estimates for the homogeneous Dirichlet problem with those for the operator sending Neumann boundary values to the Dirichlet boundary values of the harmonic function exhibiting those Neumann boundary values, loosely speaking the inverse of the operator known as the Calderón operator and which we denote by $T$. The estimates for this operator, of interest in their own right, are given in
Theorem 3.15, and constitute the key step in the proof of the main theorem regarding the problem (3)-(4). Their proof depends on estimates due to Dahlberg and Kenig [2] and Verchota [15] which imply that, letting $L^p(\partial\Omega)_{1+}$ denote the subspace of $L^p(\partial\Omega)$ of functions with mean-value-0,

$$\Upsilon : L^p(\partial\Omega)_{1+} \to L^q_1(\partial\Omega), 1 < p \leq 2,$$

and similar estimates in boundary Hardy spaces due to Brown [1] from which it follows that

$$\Upsilon : H^p(\partial\Omega) \to H^q_1(\partial\Omega), 1 - \epsilon < p < 1.$$

We obtain estimates for $\Upsilon$ for the entire range of spaces required to prove the main theorem on the inhomogeneous Neumann problem via interpolation of operators. Of paramount importance here is that for the positive Sobolev and Besov boundary spaces defined in §2 surface area density is never present as a weight factor when the defining norm for these spaces is written in terms of the norm for the corresponding space on $R^{n-1}$, whereas for all of the negative spaces it is. Another way of saying this is that if $\partial\Omega$ is the boundary of a domain above a Lipschitz graph $\psi$ on $R^{n-1}$ then the boundary Sobolev space $L^s_2(\partial\Omega)$, $s \geq 0$, for example, is identical to the space $L^s_2(R^{n-1})$, but, for the negative range, if $g \in L^s_2(\partial\Omega)$, $s \geq 0$, then we have instead $g\omega \in L^s_2(R^{n-1})$, where $\omega = \sqrt{1 + |\nabla\psi|^2}$ denotes surface area density. (The two boundary spaces are identical, however, when $s = 0$.) This compatibility guarantees that interpolation between any two of the nonnegative spaces or between any two of the nonpositive ones will always be legitimate. An analogous observation holds for Brown’s boundary Hardy spaces as well.

Finally, within the last section, we establish uniqueness of the solution $w$ to the problem (3)-(4). When $w$ lies in $L^2(\Omega)$ uniqueness is quite straightforward and is a consequence of the $L^2$ expansion of $w$ as an infinite series of
eigenfunctions. When \( w \) is not a member of \( L^2 \), we call upon \( L^\infty \) estimates for the Neumann eigenfunctions of the Laplacian on \( \Omega \) which follow from our own existence results to recast the problem as one involving solutions to the heat equation. Invoking a renowned theorem of Beurling and Deny, we are able to prove the convergence of the heat semigroup as \( t \to 0 \) and deduce uniqueness in \( L^p, 1 < p < 2 \).

The results contained within this article appeared in the author’s Ph.D. thesis [16]. We note that the regularity results described here overlap considerably with those in recent work of Fabes, Mendez, and Mitrea [3].

1 Statement of the Estimates

We begin with some basic definitions. We say that a bounded, open, connected subset \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain if to each \( P \in \partial \Omega \) may be associated a Lipschitz function \( \psi : \mathbb{R}^{n-1} \to \mathbb{R} \) with Lipschitz constant less than some fixed \( m = m(\Omega) > 0 \) (called the Lipschitz constant of \( \Omega \)), an isometry \( \phi : \mathbb{R}^n \to \mathbb{R}^n \), and a radius \( r > 0 \) for which

\[
\phi(B(P,r) \cap \Omega) = \{(x', x_n) \in \mathbb{R}^n | x_n > \psi(x'), |(x', x_n) - \phi(P)| \leq r\}
\]

\[
\phi(B(P,r) \cap \partial \Omega) = \{(x', x_n) \in \mathbb{R}^n | x_n = \psi(x'), |(x', x_n) - \phi(P)| \leq r\},
\]

where \( B(P,r) \) is the closed ball in \( \mathbb{R}^n \) of radius \( r \) about \( P \).

Surface measure on \( \partial \Omega \) will be denoted by \( d\sigma \) and the usual \( n \)-dimensional Lebesgue (volume) measure in \( \mathbb{R}^n \) by \( dV \). The boundary spaces \( L^p(\partial \Omega), 1 \leq p \leq \infty \), will have their traditional meanings.

Let \( 1 < p < \infty \) and \(-\infty < \alpha < \infty \). We recall that the Sobolev (potential) space \( L^p_\alpha = L^p_\alpha(\mathbb{R}^n) \) is the collection of all tempered distributions \( f \) on \( \mathbb{R}^n \) for which
for some $g \in L^p = L^p(\mathbb{R}^n)$, where, as usual, the circumflex operator above denotes Fourier transformation of tempered distributions. We impose a norm on this space by setting

$$\|f\|_{L^p_{\alpha}} = \|\hat{g}\|_{L^p}.$$  

Furthermore we recall that the dual space of $L^p_{\alpha}$ is of course the space $L^{p',\alpha}$, where $1/p + 1/p' = 1$. Now let $R_0f$ denote the restriction of a function $f$ on $\mathbb{R}^n$ to $\Omega$. For $1 < p < \infty$ and $\alpha \geq 0$ we define the Sobolev space on $\Omega$ with indices $p$ and $\alpha$ as $L^p_\alpha(\Omega) = R_0L^p_{\alpha}$ with the standard quotient norm

$$\|f\|_{L^p_\alpha(\Omega)} = \inf\{\|g\| : R_0g = f\}. \quad (10)$$

We also require the familiar extension operator of Stein [12].

**Theorem 1.1.** For any bounded Lipschitz domain $\Omega$, there is a bounded linear extension operator $E_0$ mapping $L^p_{\alpha}(\Omega)$ into $L^p_{\alpha}(\mathbb{R}^n)$ simultaneously for all nonnegative integers $k$ and every $p, 1 < p < \infty$.

By complex interpolation we have (See Jerison and Kenig [7])

**Proposition 1.2.** The map $E_0$ extends to a bounded linear operator mapping $L^p_{\alpha}(\Omega)$ into $L^p_{\alpha}(\mathbb{R}^n)$ simultaneously for all $\alpha \geq 0$ and all $p$ with $1 < p < \infty$.

**Remark 1.3.** Notice that $R_0 : L^p_{\alpha} \to L^p_{\alpha}(\Omega)$ induces a bounded linear map $R_0^* : (L^p_{\alpha}(\Omega))^* \to L^{p'}_{-\alpha}$ between corresponding dual spaces via

$$(R_0^*f)(g) = F(R_0(g))$$

for $g \in L^p_{\alpha}$. Similarly, $E_0 : L^{p'}_{\alpha}(\Omega) \to L^{p'}_{\alpha}$ induces a bounded linear map $E_0^* : L^{p'}_{-\alpha} \to (L^p_{\alpha}(\Omega))^*$. Since $R_0 \circ E_0 = \text{Id}$, we have $E_0^* \circ R_0^* = \text{Id}$ as well.

Using a well-known characterization of $L^p_{\alpha}$ due to Strichartz (see Theorem 3.4 of Jerison and Kenig [7]), it is possible to establish the veracity of a
certain boundedness property of the truncation operator \( \chi_n \) which operates
on functions \( f \) defined on \( \mathbb{R}^n \) by setting \( (\chi_n f)(x) = f(x) \) for \( x \in \Omega \) and
\( (\chi_n f)(x) = 0 \) for \( x \in \mathbb{R}^n \setminus \Omega \).

**Proposition 1.4.** Let \( \Omega \) be a bounded Lipschitz domain. Suppose that \( 1 < p < \infty \) and \( 0 \leq \alpha < 1/p \). Then
\[
\|\chi_n f\|_{L^p} \leq C\|f\|_{L^p}.
\] (11)

For the proof, see [7], Proposition 3.5.

Of course the dualized statement also holds and will be very useful.

**Corollary 1.5.** Assume that \( 1 < p < \infty \) and that \(-1/p' < \alpha \leq 0\). Then, for
\( f \in L^p_\alpha \),
\[
\|\chi_n f\|_{L^p} \leq C\|f\|_{L^p}.
\] (12)

It is proven first for elements of \( \mathcal{C}_c^\infty(\mathbb{R}^n) \) and then extended to all of \( L^p_\alpha \)
via density.

We are now ready to describe precisely the sense in which the Neumann
problem will be solved. Suppose that \( 1 < p < \infty \), \( \frac{1}{p} < \alpha < 1 + \frac{1}{p} \), and
\( F \in (L^p_{2-\alpha}(\Omega))_{1,1}^* \), wherein by definition
\[
(L^p_{2-\alpha}(\Omega))_{1,1}^* = \{ F \in (L^p_{2-\alpha}(\Omega))^* \mid F(1) = 0 \}.
\]
Then a function \( w \in L^p_\alpha(\Omega) \) satisfies the **generalized inhomogeneous**
Neumann problem \( \text{NP}(p, \alpha, F) \) with data \( F \) if
\[
F(v) = -< \chi_n \nabla(E_\alpha(w)), \nabla(E_\alpha(v)) >
\] (13)
for all \( v \in L^p_{2-\alpha}(\Omega) \).

Notice that when \( w \in L^p_\alpha(\Omega), v \in L^p_\alpha(\Omega), \) and \( F \in L^p(\Omega) \), the standard
definitions of elementary distribution theory show that (13) may be written
in a simpler, more transparent form:
By density and continuity, this implies, among other things, that the bilinear form on the right-hand side in (13) is independent of the particular extension operators used as long as they share with Stein's the boundedness properties expressed in Prop. 1.2.

We present, at last, our principal result.

**Theorem 1.6.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, and let $1 < p < \infty$ and $1/p + 1/p' = 1$. There is $\epsilon, 0 < \epsilon \leq 1$, depending only on the Lipschitz constant of $\Omega$, such that, for every $F \in (L^p_{2-\alpha}(\Omega))^*_\perp$, there exists a solution $w \in L^p_0(\Omega)$ to the generalized inhomogeneous Neumann problem $NP(p, \alpha, F)$, provided one of the following holds:

1. $p_0 < p < p_0'$ and $1/p < \alpha < 1 + 1/p$
2. $1 < p \leq p_0$ and $3/p - 1 - \epsilon < \alpha < 1 + 1/p$
3. $p_0' \leq p < \infty$ and $1/p < \alpha < 3/p + \epsilon$

wherein $1/p_0 = 1/2 + \epsilon/2$ and $1/p_0' = 1/2 - \epsilon/2$. Moreover, for all $F \in (L^p_{2-\alpha}(\Omega))^*_\perp$, we have the estimate

$$\|w\|_{L^p_0(\Omega)} \leq C\|F\|_{(L^p_{2-\alpha}(\Omega))^*_\perp}. \quad (14)$$

Finally, modulo constants, this solution $w$ is unique.

The range for $\alpha$ and $p$ for which this theorem holds is best understood as the open hexagon in $(\alpha, 1/p)$-space with vertices $A_1 = (0, 0), A_2 = (\epsilon, 0), A_3 = (2 - 1/p_0, 1/p_0'), A_4 = (2, 1), A_5 = (2 - \epsilon, 1), \text{ and } A_6 = (1/p_0, 1/p_0)$:
2 Some Boundary Function Spaces and Estimates for the Dirichlet Problem

Denote by $F_p^{s,q} = F_p^{s,q}(\mathbb{R}^{n-1})$ the inhomogeneous Triebel-Lizorkin spaces on euclidean space. When $1 < p < \infty$ and $q = 2$ these are just the standard Sobolev spaces $L_p^s$ that we have already encountered, when $q = p$ they become the familiar Besov spaces $B_p^s = B_p^{s,p}$, and when $0 < p < 1$, $q = 2$, and $s = 0$ they are quasi-normed spaces identical to the local Hardy spaces $h_p$. We will not explicitly define them here but refer the reader to Triebel [13,14] or Frazier-Jawerth [4].

First consider a region $\Omega$ above the graph of a Lipschitz function $\psi$. For $1 < p < \infty$, $q = p$, $0 < s < 1$ or else for $1 < p < \infty$, $q = 2$, $0 \leq s \leq 1$ define the boundary Triebel-Lizorkin space $F^{s,q}_p(\partial \Omega)$ as the space of functions $f(x, \psi(x)) = g(x)$, where $g \in F^{s,q}_p(\mathbb{R}^{n-1})$. To extend this to boundaries $\partial \Omega$ of all bounded Lipschitz domains, let $\{B_j = B(P_j, r) ; j = 1, 2, \ldots, N\}$ be a
covering of $\partial \Omega$ by balls as in the definition of a Lipschitz domain and let $\eta_j \in C_0^\infty(\mathbb{R}^n)$ be such that $\text{supp}\, \eta_j \subset B_j$, $0 \leq \eta_j \leq 1$, and $\sum_j \eta_j = 1$ on $\partial \Omega$. Let $\phi_j$ be an isometry of $\mathbb{R}^n$ such that

$$\phi_j(B_j \cap \Omega) = \{(x', x_n) \mid x_n > \psi_j(x'), |(x', x_n) - \phi_j(P_j)| \leq r\}. \quad (15)$$

When $1 < p < \infty, q = p, 0 < s < 1$ or else when $1 < p < \infty, q = 2, 0 \leq s \leq 1$, define the **boundary Triebel-Lizorkin space** $F^s_{p,q}(\partial \Omega)$ as the space of all functions $g$ on $\partial \Omega$ for which the norm

$$\|g\|_{F^s_{p,q}(\partial \Omega)} = \sum_{j=1}^N \|\eta_j g\|_{F^s_{p,q}(\partial \Omega)} \|$$

is finite and endow the space with this norm. The definition (16) is independent of the choice of coordinate system. One way to see this is to consider that it is at $q = 2$, $s = 0$ since $F^0_{p,2} = L^p$, and direct differentiation as well as the standard properties of Lipschitz functions show that it is at $q = 2$, $s = 1$ since $F^1_{p,2} = L^p_1$. (Alternately, in this latter case, there is a definition involving suitably defined tangential derivatives along the boundary. See Verchota [15], §1.) Invariance under coordinate change then follows from complex interpolation on $\mathbb{R}^{n-1}$ for $q = 2, 0 < s < 1$ and from real interpolation on $\mathbb{R}^{n-1}$ for $q = p, 0 < s < 1$.

We may also define dual spaces to our positive boundary Triebel-Lizorkin spaces. When $1 < p < \infty, q = p, -1 < s < 0$ or else when $1 < p < \infty, q = 2, -1 \leq s \leq 0$, we set

$$F^s_{p,q}(\partial \Omega) = (F^{-s}_{p,q}(\partial \Omega))^*$$

where $1/p + 1/p' = 1$. $L^p(\partial \Omega)$ lies in $F^s_{p,q}(\partial \Omega)$ under the pairing

$$\langle g, f \rangle = \int_{\Omega} gf \, d\sigma, \quad f \in F^s_{p,q}(\partial \Omega), \quad g \in L^p(\partial \Omega).$$

In other words the corresponding norm is
\[ \|g\|_{F^{s,q}_{\Psi}(\partial \Omega)} = \sup \{ |\int_{\partial \Omega} g f \, d\sigma| \mid f \in F_{\Psi}^{s,q}(\partial \Omega), \|f\|_{F_{\Psi}^{s,q}(\partial \Omega)} \leq 1 \}, g \in L^p(\partial \Omega). \]

Since, on euclidean space, \( F^{s,q}_p = (F^{-s,q}_p)^* \) for the index ranges specified, it follows that, employing coordinate mappings, we may redefine the negative Triebel-Lizorkin spaces in terms of corresponding euclidean space norms. When \( 1 < p < \infty, q = p, -1 < s < 0 \) or else when \( 1 < p < \infty, q = 2, -1 \leq s \leq 0, F^{s,q}_p(\partial \Omega) \) is the completion of the space of all \( g \in L^1(\partial \Omega) \) such that

\[
(\eta_j g)(\phi_j^{-1}(\cdot), \psi_j(\cdot))\sqrt{1 + |\nabla \psi_j(\cdot)|^2} \in F^{s,q}_p
\]

for all \( j = 1, \ldots, N \) with respect to the norm

\[
\|g\|_{F^{s,q}_{\Psi}(\partial \Omega)} = \sum_{j=1}^{N} \| (\eta_j g)(\phi_j^{-1}(\cdot), \psi_j(\cdot))\sqrt{1 + |\nabla \psi_j(\cdot)|^2} \|_{F^{s,q}_p}.
\]  

(17)

As noted within the introductory remarks, the observation that surface area density is included as a weight factor in the equivalent definition (17) of the norm for the negative Triebel-Lizorkin spaces but is absent from the definition of the norm (16) for the positive spaces will be of critical importance to the veracity of Corollaries 3.10 and 3.13 and of Theorem 3.15, since, in all of these cases, presence or absence of the weight factor gives a bounded, linear mapping between, respectively, any of the nonpositive boundary spaces we have defined or any of the nonnegative ones and the corresponding space on \( \mathbb{R}^{n-1} \).

As we have already noted, the euclidean Triebel-Lizorkin spaces coincide with euclidean Sobolev and Besov spaces for appropriate values of the indices. Acknowledging this fact, we will also call the space \( F^{s,2}_p(\partial \Omega), 1 < p < \infty, -1 \leq s \leq 1 \), a boundary Sobolev space and interchangeably employ the notation \( L^p(\partial \Omega) \) for it. Similarly, when \( 1 < p < \infty, -1 < s < 1 \), we will often write \( B^s_p(\partial \Omega) \) for \( F^{s,p}_p(\partial \Omega) \), and call \( F^{s,p}_p(\partial \Omega) \) a boundary Besov space.
It will in addition be important that Hölder’s inequality implies that $L^p(\partial \Omega)$ acts on $B^s_p(\partial \Omega)$, $-1 < s < 0$, via integration:

$$g \mapsto \int_{\partial \Omega} fg d\sigma, g \in B^s_p(\partial \Omega),$$

and thus $L^p(\partial \Omega)$ is naturally embedded in $B^s_p(\partial \Omega)$ for this range of $s$.

Since we are interested in the solution to the Neumann problem we will consider mean-value-0 versions of these boundary function spaces. Recall that any $f \in B^s_p(\partial \Omega)$, $1 < p \leq \infty$, $-1 < s < 1$, $s \neq 0$, acts on the real line either via integration over $\partial \Omega$ (i.e., $< f, r > = \int_{\partial \Omega} r f d\sigma$ for all real numbers $r$) if $s \geq 0$ or by virtue of the identification of $B^s_p(\partial \Omega)$ as a space of linear functionals on an appropriate function space if $s < 0$. Thus we set

$$B^s_p(\partial \Omega)_1 = \{ f \in B^s_p(\partial \Omega) | f(1) = 0 \},$$

and similarly for our boundary Sobolev spaces (as well as for the spaces $L^1(\partial \Omega), L^\infty(\partial \Omega)$) for the ranges for which they have been defined.

We remark that if $\mathcal{F} \subseteq L^1(\partial \Omega)$ is a dense subset of some space $B^s_p(\partial \Omega)$ (resp. $L^s_p(\partial \Omega)$) then $\mathcal{F}_1 = \{ f - \frac{1}{|\partial \Omega|} \int_{\partial \Omega} f d\sigma | f \in \mathcal{F} \}$ is a dense subset of $B^s_p(\partial \Omega)_1$ (resp. $L^s_p(\partial \Omega)_1$). One easily shows this by using the fact that $L^p(\partial \Omega)$-convergence implies $L^1(\partial \Omega)$-convergence (in the case for which $s \geq 0$) or that norm convergence implies weak-* convergence (in the case for which $s < 0$).

**Theorem 2.1 (Trace Theorem for Lipschitz Domains).** Let $1 < p < \infty$ and suppose that $1/p < \alpha < 1 + 1/p$ and $s = \alpha - 1/p$, then the mapping $\text{Tr}$, initially defined on $C^\infty(\Omega)$ as the restriction to $\partial \Omega$, extends to a bounded linear operator from $L^p(\Omega)$ to $B^s_p(\partial \Omega)$.

Theorem 2.1 follows easily from a special case of a theorem due to Jonsson and Wallin [8], Theorem 1, p.182 (also see [7]). Now let’s review the estimates for the homogeneous Dirichlet problem.
Theorem 2.2. Consider \( \epsilon \) such that \( 0 < \epsilon \leq 1 \). Define \( p_0 \) and \( p'_0 \) by \( 1/p_0 = (1+\epsilon)/2 \) and \( 1/p'_0 = (1-\epsilon)/2 \). Let \( s \) and \( p \) be numbers satisfying one of the following:

(a) \( p_0 < p < p'_0 \) and \( 0 < s < 1 \)
(b) \( 1 < p \leq p_0 \) and \( 2/p - 1 - \epsilon < s < 1 \)
(c) \( p'_0 \leq p < \infty \) and \( 0 < s < 2/p + \epsilon \).

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \) for some \( n \geq 3 \). There exists \( \epsilon \) depending only on the Lipschitz constant of \( \Omega \) such that for every \( g \in B^p_\sigma(\partial \Omega) \) there exists a unique harmonic function \( u \) such that \( \text{Tr } u = g \) and \( u \in L^p_{s+1/p}(\Omega) \). Moreover,

\[
\|u\|_{L^{p}_{s+1/p}(\Omega)} \leq C\|g\|_{B^p_\sigma(\partial \Omega)}.
\]

For the proof we refer the reader to Jerison-Kenig [7].

3 Estimates for the Inverse Calderón Operator

Let \( \Omega \) be a bounded Lipschitz domain and suppose that \( Q \in \partial \Omega \). We define the (interior) nontangential cone \( \Gamma_\alpha(Q) \) for \( \alpha > 0 \) via

\[
\Gamma_\alpha(Q) = \{X \in \Omega | |X - Q| < (1 + \alpha)\text{dist}(X, \partial \Omega)\}.
\]

If \( u \) is a function on \( \partial \Omega \) we may define its nontangential maximal function \( M(u) \) by setting

\[
M(u)(Q) = \sup\{|u(P)| | P \in \Gamma_1(Q)\}.
\]

We say that \( u \) has a nontangential limit at \( Q \in \partial \Omega \) if there is a finite,
well-defined limit (which we will call) $u(Q)$ as $P \to Q$ from within $\Gamma_\alpha(Q)$ for all $\alpha > 0$.

Next recall the classical method of layer potentials to solve Laplace’s equation with Neumann boundary conditions. Given $g \in L^1(\partial \Omega)$, its **single layer potential** (SLP) is the function defined via

$$Sg(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial \Omega} \frac{g(Q)}{|X - Q|^{n-2}} d\sigma(Q), \quad (20)$$

where $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. We will occasionally write $N(X, Q)$ for the kernel $|X - Q|^{2-n}$ which is the Newtonian potential. The integral in (20) is in general pointwise well-defined on all of $\mathbb{R}^n \setminus \partial \Omega$ and well-defined a.e. $d\sigma$ on $\partial \Omega$ in the sense that it converges absolutely a.e. $d\sigma$ (since the singularity present is locally integrable). Moreover, differentiating twice under the integral ensures that $Sg$ is harmonic on $\mathbb{R}^n \setminus \partial \Omega$.

Of course, since we are interested in the inverse of the Calderon operator, we will be interested in both the Neumann and Dirichlet boundary values of these single layer potentials. To bring the Neumann boundary values into consideration, define operators

$$K^*_\varepsilon f(P) = \frac{1}{\omega_n} \int_{\{|Q| < |P-Q| \geq \varepsilon\}} \frac{\langle P - Q, n(P) \rangle}{|P - Q|^n} f(Q) d\sigma(Q) \quad (21)$$

for $f \in L^1(\partial \Omega), \varepsilon > 0$, and set

$$K^* f(P) = \text{p.v.} \frac{1}{\omega_n} \int_{\partial \Omega} \frac{\langle P - Q, n(P) \rangle}{|P - Q|^n} f(Q) d\sigma(Q) \quad (22)$$

$$\equiv \lim_{\varepsilon \to 0} K^*_\varepsilon f(P), \quad (23)$$

(wherein $n(P)$ denotes the outward unit normal vector to $\partial \Omega$ defined a.e. $d\sigma$ on any Lipschitz domain) whenever $f \in L^1(\partial \Omega)$ is such that the limit in (23) converges for a.e. $d\sigma$ $P \in \partial \Omega$. Also write

$$T = \frac{1}{2} I - K^*. \quad (24)$$
See [2] or [9] for more information on the operators (21), (22)-(23), and (24).

In 1987 Dahlberg and Kenig (See [2], Theorems 4.17 and 4.18 as well as equations (1.1) and (1.2)) extended an $L^2$ result of Verchota’s [15] by establishing the following theorem.

**Theorem 3.1.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain whose complement is connected. Then there is $\varepsilon = \varepsilon(\Omega) > 0$ such that, whenever $1 < p < 2 + \delta$, $T$ is an invertible mapping from $L^p(\partial\Omega)_+ \subseteq L^p(\partial\Omega)_+$, and $S$ is an invertible mapping from $L^p(\partial\Omega)$ onto $L^p(\partial\Omega)$.

Moreover given $f \in L^p(\partial\Omega)_+$ with $1 < p < 2 + \delta$ and writing $u = ST^{-1}f$, i.e.,

$$u(X) = \frac{1}{\omega_n(n-2)} \int_{\partial\Omega} |X - Q|^{3-n} (\frac{1}{2} I - K^*)^{-1}(f)(Q)d\sigma(Q),$$

(25)

it follows that $u$ is the unique (modulo constants) harmonic function on $\Omega$ such that the nontangential maximal function $M(\nabla u)$ is bounded in $L^p(\partial\Omega)$ and $\frac{\partial u}{\partial n} = f$ nontangentially a.e. on $\partial\Omega$ in the sense that $<\nabla u(X), n(Q)> \rightarrow f$ as $X \rightarrow Q$ nontangentially for a.e. $Q \in \partial\Omega$. Finally, we have

$$||M(\nabla u)||_{L^p(\partial\Omega)} \leq C||f||_{L^p(\partial\Omega)}$$

(26)

for all $f \in L^p(\partial\Omega)_+$.

**Remark 3.2.** From (26) and the Fundamental Theorem of Calculus we have that the maximal function $M(u)$ on $\partial\Omega$ is also bounded in $L^p$.

In order to determine the Dirichlet boundary values of our single layer potentials we have

**Proposition 3.3.** If $h \in L^1(\partial\Omega)$ then $Sh(Q) \rightarrow Sh(Q)$ as $X \rightarrow Q$ nontangentially for a.e. $Q \in \partial\Omega$. In particular, if $1 < p < 2 + \delta$, then for all $f \in L^p(\partial\Omega)_+$, $u(X) = ST^{-1}f(X) \rightarrow ST^{-1}f(Q)$ for a.e. $Q$.

For the proof, consult Zanger [15].
Now for $1 < p < 2 + \delta$ define the inverse Calderón or Neumann to Dirichlet operator $\Upsilon : L^p(\partial \Omega) \to L^2(\partial \Omega)$ by setting

$$\Upsilon(f) = (ST^{-1}(f))|_{\partial \Omega}$$

(27)

for $f \in L^p(\partial \Omega)$. Then Theorem 3.1 and Proposition 3.3 imply a corollary

**Corollary 3.4.** Given $f \in L^p(\partial \Omega)$, $1 < p < 2 + \delta$, we have the bound

$$\|\Upsilon f\|_{L^2(\partial \Omega)} \leq C\|f\|_{L^p(\partial \Omega)}.$$  

In addition, the harmonic function $u = ST^{-1}f$ has nontangential Neumann boundary data $f$ and nontangential Dirichlet boundary data $\Upsilon(f) = (ST^{-1}f)|_{\partial \Omega}$.

Theorem 3.1 tells us that the function $u$ has certain nontangential Neumann boundary values but we are, of course, also interested in weaker solution properties as well. For these we have

**Theorem 3.5.** Suppose that $1 < p < 2 + \delta$. Assume that $f \in L^p(\partial \Omega)$ and let $u = ST^{-1}f$ be the single layer potential of $T^{-1}f$ considered as a function on $\Omega$. Then $u \in L^p(\Omega)$ and

$$\int_{\partial \Omega} f Tr \, d\sigma = \int_{\Omega} < \nabla u, \nabla v > \, dV,$$  

(29)

for all $v \in L^p(\Omega)$.

Given Theorem 3.1, the proof of this is quite standard, so we do not tarry here to address details.

**Remark 3.6.** Suppose that $1 < p < \infty$, and assume that the function $u \in L^p(\Omega)$ whose maximal function $M(u)$ is bounded in $L^p(\partial \Omega)$ has nontangential boundary values $g$ in $L^p(\partial \Omega)$, i.e., that $g$ is the nontangential limit of $u$ on $\partial \Omega$. Then $g = Tr u$ a.e. $d\sigma$.

The proof of this fact is rather similar in spirit to that of Theorem 3.5 and is left to the reader.
As its title suggests, the principal goal of this section is to state and prove a theorem yielding estimates for the inverse Calderón operator for a range sufficiently large to prove Thm. 1.6. Using interpolation, the estimates (28) of Cor. 3.4 are nearly all we need. However, to state the complete theorem, we will require an additional estimate, which is in the nature of an "ε -improvement" over what would have been possible with the estimates (28) alone. This estimate involves the Neumann problem for data in Hardy spaces on Lipschitz hypersurfaces and will require some preliminary definitions.

Let \( \Delta(Q_0, r) = \{ P \in \partial \Omega \mid |P - Q_0| < r \} \), and assume that \( r \) is less than \( \text{diam}(\partial \Omega) \). Also let \( d = n - 1 \) denote the dimension of \( \partial \Omega \). Following Brown [1], we say that the function \( \alpha \) is an atom for \( H^p(\partial \Omega) \), with \( (n-1)/n < p < 1 \) if for some \( Q_0 \) and \( r \) we have

\[
\begin{align*}
(i) & \quad \text{supp } \alpha \subset \Delta(Q_0, r) \\
(ii) & \quad \int_{\Delta(Q_0, r)} \alpha(Q) d\sigma(Q) = 0 \\
(iii) & \quad \|\alpha\|_{L^2(\Delta(Q_0, r))} \leq C r^{-d(1/p-1/2)}.
\end{align*}
\]

We denote by \( H^p(\partial \Omega) \) the Hardy space on \( \partial \Omega \) (with index \( p \)) and define it as the collection

\[
\{ g \mid g = \sum \lambda_j a_j \text{ with } \Sigma |\lambda_j|^p < \infty \}
\]

for some sequence of atoms \( a_j \) and complex numbers \( \lambda_j \). The quasi-norm for \( H^p(\partial \Omega) \) is given by

\[
\|g\|_{H^p(\partial \Omega)}^p = \inf \{ \Sigma |\lambda_j|^p \mid g = \sum \lambda_j a_j \}.
\]

Recall that for \( 0 < p < 1 \) the local Hardy space \( h^p = h^p(\mathbb{R}^{n-1}) \) (see Goldberg [5] or Stein [11]) is identical to the Triebel-Lizorkin space \( F_p^{0,2} = F_p^{0,2}(\mathbb{R}^{n-1}) \) (see Triebel [13][14]). In light of this, for \( (n-1)/n < p < 1 \) we may identify \( H^p(\partial \Omega) \) with the space \( F_p^{0,2}(\partial \Omega)_{1+} \), itself defined as the closure
in $F_{p,2}^0(\partial \Omega)$ of the set $F_{p,2}^0(\partial \Omega) \cap L^1(\partial \Omega)_{1,1}$, where $F_{p,2}^0(\partial \Omega)$ is in turn defined, invoking notations used in our earlier definitions of boundary Triebel-Lizorkin spaces in §2, as the completion of the space of all $g \in L^1(\partial \Omega)$ such that

$$\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot))) \sqrt{1 + |\nabla \psi_j(\cdot)|^2} \in F_{p}^{0,2}$$

for all $j = 1, \ldots, N$ with respect to the (quasi-)norm

$$||g||_{F_{p}^{0,2}(\partial \Omega)} = \sum_{j=1}^{N} ||(\eta_j g)(\phi_j^{-1}(\cdot, \psi_j(\cdot))) \sqrt{1 + |\nabla \psi_j(\cdot)|^2}||_{F_{p}^{0,2}}.$$  (30)

Note the appearance of the weight factor on the right in (30). This identification is a straightforward consequence of standard atomic decompositions for the space $h^p = F_{p}^{0,2}(\mathbb{R}^{n-1})$ (see Stein [11]) analogous to the one used by Brown in his definition of $H^p(\partial \Omega)$.

The space of distributions on $\partial \Omega$ with one Hardy space derivative, is denoted $H^1_p(\partial \Omega)$, $(n - 1)/n < p < 1$. Once again following Brown [1], we give a precise definition by first defining atoms. Thus we say that $A$ is an atom for $H^1_p(\partial \Omega)$ if for some $Q_0 \in \partial \Omega$ and $r > 0$, we have

(i) $\text{supp} A \subset \Delta(Q_0, r) \cap \partial \Omega$

(ii) $||\nabla \text{tan} A||_{L^1(\partial \Omega)} \leq r^{-d(1/p - 1/2)},$

wherein the tangential derivative $\nabla \text{tan} g$ is defined for smooth $g$ in a neighborhood $B(P_j, r) \cap \partial \Omega$ of $P_j = P \in \partial \Omega$ as the vector

$$\nabla \text{tan} g(P) = (\frac{\partial g}{\partial t_1}(P), \ldots, \frac{\partial g}{\partial t_{n-1}}(P))$$

where

$$\frac{\partial g}{\partial t_i}(P) = \frac{\partial}{\partial x_i}(g \circ \phi_j^{-1}(\cdot, \psi_j(\cdot)))|_{x = x'} \; \text{for} \; i = 1, \ldots, n - 1, \; P = \phi_j^{-1}(x', \psi_j(x')).$$

Note the independence from the choice of coordinate system of the size of the quantity $||\nabla \text{tan} A||_{L^1(\partial \Omega)}$ on the left-hand side of the inequality in (ii). Just
as in the case above for $H^p(\partial \Omega)$ we now define $H^p_1(\partial \Omega)$ to be the $p$-span of these atoms and endow it with the analogous (quasi-)norm.

In analogy with the situation for $H^p(\partial \Omega)$, we may define $H^p_1(\partial \Omega)$ in terms of the Triebel-Lizorkin space $F^1_p = F^1_p(\mathbb{R}^{n-1})$ on euclidean space (once again see Triebel [13],[14] or Frazier-Jawerth [4] for detailed definitions). In fact for $(n-1)/n < p < 1$ we have $H^p_1(\partial \Omega) = F^{1,2}_p(\mathbb{R}^{n-1})$, where this latter space is the completion of the space of all $g \in L^1(\partial \Omega)$ such that

$$
\eta_j g(\phi_j^{-1}(-, \psi_j(\cdot))) \in F^{1,2}_p,
$$

$j = 1, ..., N$, with respect to the norm

$$
\|g\|_{F^{1,2}_p(\partial \Omega)} = \sum_{j=1}^N \|\eta_j g(\phi_j^{-1}(-, \psi_j(\cdot)))\|_{F^{1,2}_p}.
$$

Once again this identification is a consequence of standard atomic decompositions for the space $F^{1,2}_p(\mathbb{R}^{n-1})$ (see Frazier-Jawerth [4], sections 7 and 12) analogous to the one used by Brown in his definition of $H^p_1(\partial \Omega)$.

Once again the key point of the definition for the space $H^p_1(\partial \Omega)$ is that the way in which its atoms have been defined makes them essentially equivalent to atoms on $\mathbb{R}^{n-1}$. This observation is very apparent when $\partial \Omega$ is the domain above a Lipschitz graph $\psi$ defined on $\mathbb{R}^{n-1}$, with surface density $\omega = \sqrt{1 + |\nabla \psi|^2}$. Then, defining the spaces $H^p_1(\partial \Omega)$ and $H^p(\partial \Omega)$ just as we have defined them above for boundaries of bounded Lipschitz domains, the derivatives of the resulting atoms for the space $H^p_1(\partial \Omega)$ will in fact be members of the standard euclidean Hardy space $H^p(\mathbb{R}^{n-1}, dx)$, where $dx$ is Lebesgue measure. In contrast $H^p(\partial \Omega)$ is identical to the space $H^p(\mathbb{R}^{n-1}, \omega dx)$.

Having set forth these definitions, we may now state an existence theorem for the Neumann problem in Hardy spaces.
Theorem 3.7[1] Let $1 > p > 1 - \delta_m > (n - 1)/n$ for some constant $\delta_m > 0$ depending on the Lipschitz constant $m$ of $\partial \Omega$, and suppose that $g \in H^p(\partial \Omega)$. Then the interior Neumann problem with data $g$ has a solution $u$ which satisfies
\[ \|u|\partial \Omega\|_{H^p(\partial \Omega)} + \|M(\nabla u)\|_{L^p(\partial \Omega)} \leq C\|g\|_{H^p(\partial \Omega)}. \] (32)
when $u$ is normalized by $u(0) = 0$.

We shall also need R. Brown's corresponding uniqueness result.

Theorem 3.8[1] If $1 - \delta_m < p < 1$, and $u$ satisfies
\[ \begin{cases} 
\Delta u = 0 \\
M(\nabla u) \in L^p(\partial \Omega)
\end{cases} \] (33)
with $\frac{\partial u}{\partial n}$ vanishing in the $H^p$-sense, then $u$ is a constant.

We have not defined here what it means to "vanish in the $H^p$-sense" (though of course the interested reader may find a definition for this in Brown's paper). Suffice it to say that the condition that $\frac{\partial u}{\partial n} = 0$ nontangentially a.e. on $\partial \Omega$, which we will have, will be strong enough to imply it.

We will also shortly find occasion to exploit the space $(H^p(\partial \Omega))^*$, $(n - 1)/n < p < 1$, which we define as the vector space dual of the space $H^p(\partial \Omega)$ under the pairing
\[ \langle g, f \rangle = \int_{\partial \Omega} g f \, d\sigma, \] (34)
with $f \in L^\infty(\partial \Omega)$, $\int_{\partial \Omega} f \, d\sigma = 0$, $g \in L^1(\partial \Omega)$ The norm is
\[ \|g\|_{(H^p(\partial \Omega))^*} = \sup\{|\int_{\partial \Omega} g f \, d\sigma| : f \in L^\infty(\partial \Omega), \int_{\partial \Omega} f \, d\sigma = 0, \|f\|_{H^p(\partial \Omega)} \leq 1\} \]
for $g$ for which this norm is finite, and of course the pairing (34) extends to all of $H^p(\partial \Omega)$ and $(H^p(\partial \Omega))^*$ via continuity.

The space $(H^p(\partial \Omega))^*$, $(n - 1)/n < p < 1$, is defined as the vector space dual of the space $H^p(\partial \Omega)$ under the pairing
\[<g, f> = \int_{\partial \Omega} g f \, d\sigma\]  
with \(f \in L^\infty(\partial \Omega), g \in L^1(\partial \Omega)\). Once again the corresponding norm is

\[\|g\|_{(H^s(\partial \Omega))^*} = \sup\{|\int_{\partial \Omega} g f \, d\sigma| \mid f \in L^\infty(\partial \Omega), \|f\|_{H^s(\partial \Omega)} \leq 1\}\]

for \(g\) for which this norm is finite, and the pairing (35) extends to all of \(H^s(\partial \Omega)\) and \((H^s(\partial \Omega))^*\) via continuity.

Once again these dual spaces may be redefined in terms of certain Triebel-Lizorkin spaces \(F^s_p = F^s_p(\mathbb{R}^{n-1})\) on euclidean space, in fact in terms of spaces of the form \(B_{s}^\infty = F_{s}^\infty = F_{s}^{s,\infty}(\mathbb{R}^{n-1}), s\) real. As usual consult Triebel [13],[14] or Frazier-Jawerth [4] for precise definitions. We identify \((H^0(\partial \Omega))^*\) with \(B_{s(p)}^\infty(\partial \Omega) = F_{s(p)}^{s(p),\infty}(\partial \Omega), s(p) = (n-1)(1-p)/p,\)

defined as the space of all functions \(g\) on \(\partial \Omega\) such that

\[\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot))) \in F_{s(p)}^{s(p),\infty},\]  
\[(36)\]

\(j = 1, \ldots, N,\) with norm

\[\|g\|_{F_{s(p)}^{s(p),\infty}(\partial \Omega)} = \sum_{j=1}^{N} \|\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot)))\|_{F_{s(p)}^{s(p),\infty}},\]

and identify \((H^0(\partial \Omega))^*\) with \(F_{s(p)-1,\infty}^{s(p)-1,\infty}(\partial \Omega), s(p) = (n-1)(1-p)/p,\)

defined as the completion of the space of all \(g \in L^1(\partial \Omega)\) such that

\[\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot))) \sqrt{1 + |\nabla \psi_j(\cdot)|^2} \in F_{s(p)-1,\infty}^{s(p)-1,\infty},\]  
\[(37)\]

\(j = 1, \ldots, N,\) with respect to the norm

\[\|g\|_{F_{s(p)-1,\infty}^{s(p)-1,\infty}(\partial \Omega)} = \sum_{j=1}^{N} \|\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot))) \sqrt{1 + |\nabla \psi_j(\cdot)|^2}\|_{F_{s(p)-1,\infty}^{s(p)-1,\infty}}.\]

These identifications follow from (30) and (31) and the facts that, on euclidean space, \((F^0_p)^* = F_{s(p)}^{s(p),\infty}\) and \((F^0_p)^* = F_{s(p)-1,\infty}^{s(p)-1,\infty}\) with \(s(p) = (n -\)
1)$(1-p)/p$ for the range of $p$ relevant to us (see Triebel [13], Theorem 2.11.3, p. 180).

We can also consider, if we wish, the mean-value-0 counterparts of each of the spaces $F_p^{1,2}(\partial \Omega)$, $F_p^{\infty,\infty}(\partial \Omega)$, and $F_p^{\infty,1,\infty}(\partial \Omega)$, denoted $F_p^{1,2}(\partial \Omega)_0$, $F_p^{\infty,\infty}(\partial \Omega)_0$, and $F_p^{\infty,1,\infty}(\partial \Omega)_0$ respectively, which are defined in the same fashion except that one takes only $g \in L^1(\partial \Omega)$, with mean-value-0 in (31), (36), and (37).

The final ingredients required prior to stating our estimates for the inverse Calderón operator are the necessary interpolation results. These we collect within Corollaries 3.10 and 3.13, after introducing the requisite interpolation spaces. The first interpolation method for the euclidean Triebel-Lizorkin spaces that we will invoke to obtain interpolation results for our boundary spaces is simply the familiar real interpolation method for Sobolev spaces.

We can define boundary interpolation spaces corresponding to this method in a familiar fashion. For $1 < p < \infty$, $0 \leq s_0 \neq s_1 \leq 1$, $s = (1-\theta)s_0 + \theta s_1$, and $0 < \theta < 1$, we define $(F_p^{s_0,2}(\partial \Omega), F_p^{s_1,2}(\partial \Omega))_{\theta,p}$ as the space of all functions $g$ on $\partial \Omega$ such that

$$\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot))) \in (F_p^{s_0,2}, F_p^{s_1,2})_{\theta,p},$$

$j = 1, \ldots, N$, with norm

$$\|g\|_{(F_p^{s_0,2}(\partial \Omega), F_p^{s_1,2}(\partial \Omega))_{\theta,p}} = \sum_{j=1}^{N} \|\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot)))\|_{(F_p^{s_0,2}, F_p^{s_1,2})_{\theta,p}}.$$

For the negative range of spaces, the surface area density weight factor must as usual be included. For $1 < p < \infty$, $-1 \leq s_0 \neq s_1 \leq 0$, $s = (1-\theta)s_0 + \theta s_1$, and $0 < \theta < 1$, we define $(F_p^{s_0,2}(\partial \Omega), F_p^{s_1,2}(\partial \Omega))_{\theta,p}$ as the completion of the space of all $g \in L^1(\partial \Omega)$ such that

$$\eta_j g(\phi_j^{-1}(\cdot, \psi_j(\cdot)))\sqrt{1 + |\nabla \psi_j(\cdot)|^2} \in (F_p^{s_0,2}, F_p^{s_1,2})_{\theta,p},$$
\[ j = 1, \ldots, N, \text{ with respect to the norm} \]
\[ \|q\|_{(p^{1}, p^{2} ; (\Omega), p^{1}, p^{2} ; (\Omega))_{\theta, p}} = \sum_{j=1}^{N} \|\eta_{j} q(\phi_{j}^{-1}(\cdot), \psi_{j}(\cdot))\|_{(p^{1}, p^{2} ; (\Omega), p^{1}, p^{2} ; (\Omega))_{\theta, p}}. \]

The interpolation property (which says of course that a linear operator bounded on two indexed function spaces should be bounded on the spaces in between) for the spaces \( (F^{a, 2}_{p} (\Omega), F^{s, 2}_{p} (\Omega))_{\theta, p} \) is directly inherited from that for the spaces \( (F^{a, 2}_{p}, F^{s, 2}_{p})_{\theta, p} \). The relevant interpolation theorem is

**Theorem 3.9.** Let \( 1 < p < \infty, \infty < s_{0} \neq s_{1} < \infty, s = (1 - \theta)s_{0} + \theta s_{1}, \text{ and } 0 < \theta < 1. \) Then

\[ (F^{a, 2}_{p}, F^{s, 2}_{p})_{\theta, p} = F^{s, p}_{p}. \]  (38)

For this result, see Theorem 2.4.2 in Triebel [13]. Note that for this range of indices, the spaces on the left in (38) are Sobolev spaces and that on the right is a Besov space.

**Corollary 3.10.** Let \( 1 < p < \infty, \text{ either } -1 \leq s_{0} \neq s_{1} \leq 0 \text{ or } 0 \leq s_{0} \neq s_{1} \leq 1, s = (1 - \theta)s_{0} + \theta s_{1}, \text{ and } 0 < \theta < 1. \) Then

\[ (F^{a, 2}_{p} (\Omega), F^{s, 2}_{p} (\Omega))_{\theta, p} = F^{s, p}_{p} (\partial \Omega). \]  (39)

**Remark 3.11.** This corollary remains valid when each of the three spaces \( F^{a, 2}_{p} (\Omega), F^{s, 2}_{p} (\Omega), \text{ and } F^{p, 2}_{p} (\partial \Omega) \) is replaced by its mean-value-0 counterpart. The interpolation space \( (F^{a, 2}_{p} (\Omega))_{1}, (F^{s, 2}_{p} (\Omega))_{1} )_{\theta, p} \) is of course defined in the same way as \( (F^{a, 2}_{p} (\Omega), F^{s, 2}_{p} (\Omega))_{\theta, p} \) except that one considers only functions \( g \) on \( \partial \Omega \) with mean-value-0.

The second and final method is called the \( \pm \)-method of interpolation and is denoted \( \langle F^{a, 0}_{p_{0}}, F^{s, 0}_{p_{1}}, \theta \rangle \). This method features the interpolation property for the entire spectrum of Triebel-Lizorkin spaces, including the quasi-normed end. For more information about it – as well as direction to precise definitions of the interpolation spaces involved – see Frazier-Jawerth [4].
We can now define boundary interpolation spaces in the natural way. For the positive range of spaces, we define \( \eta_j g(\phi_j^{-1}(\cdot), \psi_j(\cdot)) \in < F_{p_0}^{p_0, q_0}(\partial \Omega), F_{p_1}^{q_1, q_1}(\partial \Omega), \theta > \) as the completion of the space of all \( g \in L^1(\partial \Omega) \) such that
\[
\eta_j g(\phi_j^{-1}(\cdot), \psi_j(\cdot)) \in < F_{p_0}^{p_0, q_0}(\partial \Omega), F_{p_1}^{q_1, q_1}(\partial \Omega), \theta > ,
\]
j = 1, ..., N, with respect to the norm
\[
\|g\|_{< F_{p_0}^{p_0, q_0}(\partial \Omega), F_{p_1}^{q_1, q_1}(\partial \Omega), \theta >} = \sum_{j=1}^{N} \|\eta_j g(\phi_j^{-1}(\cdot), \psi_j(\cdot))\|_{< F_{p_0}^{p_0, q_0}, F_{p_1}^{q_1, q_1}, \theta >},
\]
whenever the indices involved remain within one of the following ranges:

- \((a)\) \( 1 < p_0 \leq p_1 < \infty, q_0 = q_1 = 2, 0 \leq s_0, s_1 \leq 1, 0 < \theta < 1 \)
- \((b)\) \( 1 < p_0 \leq p_1 \leq \infty, q_0 = p_0, q_1 = p_1, 0 < s_0, s_1 \leq 1, 0 < \theta < 1, \)
- \((c)\) \( \frac{n-1}{n} < p_0 < 1, 1 < p_1 \leq 2, q_0 = q_1 = 2, s_0 = 1, s_1 = 0, \)
\[ \frac{1}{p_1} < (1 - \theta)/p_0 + \theta/p_1 < 1. \]

As discussed in the introductory paragraphs to this article, interpolation is possible between spaces whose indices lie within the stated ranges \((a), (b), \) and \((c)\) since all such spaces may be defined without including the surface area density factor.

For the negative spaces, we define \( \eta_j g(\phi_j^{-1}(\cdot), \psi_j(\cdot)) \sqrt{1 + |\nabla \psi_j(\cdot)|^2} \in < F_{p_0}^{p_0, q_0}(\partial \Omega), F_{p_1}^{q_1, q_1}(\partial \Omega), \theta > \) as the completion of the space of all \( g \in L^1(\partial \Omega) \) such that
\[
\eta_j g(\phi_j^{-1}(\cdot), \psi_j(\cdot)) \sqrt{1 + |\nabla \psi_j(\cdot)|^2} \in < F_{p_0}^{p_0, q_0}, F_{p_1}^{q_1, q_1}, \theta > ,
\]
j = 1, ..., N, with respect to the norm
\[
\|g\|_{< F_{p_0}^{p_0, q_0}(\partial \Omega), F_{p_1}^{q_1, q_1}(\partial \Omega), \theta >} = \sum_{j=1}^{N} \|\eta_j g(\phi_j^{-1}(\cdot), \psi_j(\cdot)) \omega_j(\cdot)\|_{< F_{p_0}^{p_0, q_0}, F_{p_1}^{q_1, q_1}, \theta >} ,
\]
where \( \omega_j(x) = \sqrt{1 + |\nabla \psi_j(x)|^2} \) for \( x \in \mathbb{R}^{n-1} \), whenever the indices involved remain within one of the following ranges:
Once again, the interpolation property for the function spaces
\( < F_{s_0, \infty}^{\alpha, \theta}(\partial \Omega), F_{p_1, q_1}^{\alpha, \theta}(\partial \Omega), \theta > \) follows directly from that for the spaces
\( < F_{s_0, \infty}^{\alpha, \theta}(\partial \Omega), F_{p_1, q_1}^{\alpha, \theta}, \theta > . \)

**Theorem 3.12.** Suppose \(-\infty < s_0, s_1 < \infty, 0 < p_0, q_0, p_1, q_1 \leq \infty, s = (1 - \theta)s_0 + \theta s_1, 1/p = (1 - \theta)/p_0 + \theta/p_1, \) and \( 1/q = (1 - \theta)/q_0 + \theta/q_1. \) Then

\[
< F_{p_0, q_0}^{s_0, \theta}, F_{p_1, q_1}^{s_1, \theta}, \theta > = F_{p, q}^{s, \theta}.
\]  

This is the inhomogeneous version of Theorem 8.5 in the aforementioned reference [4] (see also §12 there).

**Corollary 3.13.** Suppose \( s_0, s_1, p_0, p_1, q_0, q_1, \) and \( \theta \) fall within one of the ranges specified in (a), (b), (c), (a'), (b') or (c'). Then

\[
< F_{p_0, q_0}^{s_0, \theta}(\partial \Omega), F_{p_1, q_1}^{s_1, \theta}(\partial \Omega), \theta > = F_{p}^{s, \theta}(\partial \Omega).
\]  

**Remark 3.14.** Once again, this corollary remains valid when each of the three spaces \( F_{p_0, q_0}^{s_0, \theta}(\partial \Omega), F_{p_1, q_1}^{s_1, \theta}(\partial \Omega), \) and \( F_{p_0, q_0}^{s_0, \theta}(\partial \Omega), \) is replaced by its mean-value-0 counterpart.

Our Besov space estimates for the inverse Calderón operator are contained in the following theorem.

**Theorem 3.15.** There exists \( \epsilon \) with \( 0 < \epsilon \leq 1 \) so that the inverse Calderón operator \( T \) introduced in (27) satisfies

\[
\| Tg \|_{B_{p}^{s}(\partial \Omega)} \leq C \| g \|_{B_{p, \infty}^{s}(\partial \Omega)}.
\]  

provided \( p \) and \( s \) lie within the ranges
(a) $p_0 < p < p'_0$ and $0 < s < 1$
(b) $1 < p \leq p_0$ and $2/p - 1 - \epsilon < s < 1$
(c) $p'_0 < p < \infty$ and $0 < s < 2/p + \epsilon$,

wherein $1/p_0 = \frac{1 + \varepsilon}{2}$, $1/p'_0 = \frac{1 - \varepsilon}{2}$.

The range for $p$ and $s$ for which estimates hold may be described as the open hexagon in $(s, 1/p)$-space with vertices $A_1 = (0, 0), A_2 = (\varepsilon, 0), A_3 = (1, 1/p'_0), A_4 = (1, 1), A_5 = (1 - \varepsilon, 1)$, and $A_6 = (0, 1/p_0)$:

---

**The Proof of Theorem 3.15.** Recall from Corollary 3.4 that, for $1 < p < 2 + \delta$, we have the operator $T : L^p(\partial\Omega)_{1+} \to L^p_1(\partial\Omega)$. Considering the Banach space adjoints of these operators we get mappings $T'_p : L^p_{1+}(\partial\Omega) \to L^{p'}(\partial\Omega)_{1+}$ where $1/p + 1/p' = 1$ and hence mappings $T''_{p'} : L^{p'}_{1+}(\partial\Omega)_{1+} \to L^{p'}(\partial\Omega)$ for $2 - \delta < p' < \infty$. We will obtain our theorem by interpolating between a rather wide selection of pairs of these operators. Thus we wish to prove that, wherever their domains of definition intersect, any two of these operators coincide.
Our strategy will be to show that all of these maps agree with \( \Upsilon : L^2(\partial \Omega)_{1,1} \rightarrow L^2_1(\partial \Omega) \), whenever their domains intersect with \( L^2(\partial \Omega)_{1,1} \), and this will suffice because \( C^\infty(\bar{\Omega})|\partial \Omega \) is a common dense subset of all of the spaces \( L^p(\partial \Omega), L^p_{-1}(\partial \Omega), 1 < p < \infty \). Of course all of the maps \( \Upsilon : L^p(\partial \Omega)_{1,1} \rightarrow L^2_1(\partial \Omega), 1 < p < 2 + \delta \), coincide on their intersections, since the definitions of the single layer potential \( S \) and the operator \( T = \frac{1}{2}I - K^* \) do not depend on the particular value of \( p \).

Now let's show that \( \Upsilon_2^* : L^2_1(\partial \Omega)_{1,1} \rightarrow L^2(\partial \Omega) \) restricted to \( L^2(\partial \Omega)_{1,1} \) is compatible with \( \Upsilon : L^2(\Omega)_{1,1} \rightarrow L^2_1(\partial \Omega) \). Suppose that \( f, g \in L^2(\partial \Omega)_{1,1} \), and let \( u_f \) and \( u_g \) be the corresponding unique harmonic functions of Cor. 3.4 with Neumann data \( f \) and \( g \) respectively. Then from Theorem 3.5 and Remark 3.6 it follows that

\[
\Upsilon_2^* f(g) = \int_{\partial \Omega} f \Upsilon(g) d\sigma = \int_{\partial \Omega} <\nabla u_f, \nabla u_g> dV = \int_{\partial \Omega} \Upsilon(f)gd\sigma = (\Upsilon f)(g),
\]

which is as we wished.

To show that \( \Upsilon : L^2(\partial \Omega)_{1,1} \rightarrow L^2(\partial \Omega) \) is compatible with \( \Upsilon_2^* : L^2_{-1}(\partial \Omega)_{1,1} \rightarrow L^p(\partial \Omega) \) for all values of \( p' \in (2 - \epsilon, \infty) \), we can use what we just proved in the previous paragraph, since, for \( f, g \in C^\infty(\bar{\Omega})|\partial \Omega \),

\[
(\Upsilon_2^* f)(g) = \int_{\partial \Omega} f \Upsilon(g) d\sigma = (\Upsilon_2^* f)(g) = (\Upsilon f)(g).
\]

Furthermore, we note that, from the uniqueness result (33) of Thm. 3.8 it follows that, whenever \( 1 < p < 2 + \delta \) and \( 1 - \delta_m < p < 1 \), we have that \( \Upsilon|[(L^p(\partial \Omega))_{1,1} \cap H^p(\partial \Omega)] = \Upsilon_p|[(L^p(\partial \Omega))_{1,1} \cap H^p(\partial \Omega)] \), provided \( \Upsilon_p \) refers to the mapping

\[
\Upsilon_p : H^p(\partial \Omega) \rightarrow H^1_p(\partial \Omega)
\]

defined by setting \( \Upsilon_p f = u|\partial \Omega \), wherein \( u|\partial \Omega \) is as in Theorem 3.7.
Finally, we dualize (45) to obtain an operator

\[ \mathcal{T}_\phi^* : (H^1_0(\partial \Omega))^* \to (H^\alpha(\partial \Omega))^*. \]  

(46)

Our identifications \((H^1_0(\partial \Omega))^* = B_{-1+s(p)}(\partial \Omega)\) and \((H^\alpha(\partial \Omega))^* = B_{s(p)}^\infty(\partial \Omega)\) allow us to rewrite (46) as

\[ \mathcal{T}_\phi^* : B_{-1+s(p)}(\partial \Omega) \to B_{s(p)}^\infty(\partial \Omega), \]  

(47)

with \(s(p) = (n-1)(1-\bar{p})/\bar{p}\) and thus restriction yields a mapping

\[ \mathcal{T}_\phi : B_{-1+s(p)}^\infty(\partial \Omega)_{1+} \to B_{s(p)}^\infty(\partial \Omega). \]  

(48)

That this mapping is also consistent with \(\mathcal{T} : L^2(\partial \Omega)_{1+} \to L^2(\partial \Omega)\) for the purposes of interpolation one readily sees by noting that \(L^2(\partial \Omega)_{1+} \cap B_{-1+s(p)}^\infty(\partial \Omega)\) is dense in \(B_{-1+s(p)}^\infty(\partial \Omega)_{1+}\) and considering that

\[ \mathcal{T}_\phi^*(f)(a) = f(\mathcal{T}_\phi(a)) = f(\mathcal{T}(a)) = \mathcal{T}(f)(a), \]  

(49)

for all \(f \in L^2(\partial \Omega)_{1+}\) and all \(a \in L^2(\partial \Omega)_{1+}\) an atom for \(H^\alpha(\partial \Omega)\), using our conclusions from the preceding paragraph.

The fact that any two of these myriad operators coincide on the intersection of their domains now justifies our eliminating all subscripts as well as the superscript \(*\) wherever they occur and simply referring to all of these mappings as \(\mathcal{T}\).

Having settled all relevant compatibility questions, it is at last time to interpolate. Recalling the identifications and interpolation results discussed prior to the statement of Thm. 3.15, let \(1/p_0 = 1/2 + \epsilon/2\) and \(1/p = 1 + \epsilon/2\) where \(\epsilon > 0\) is chosen small enough that, setting \(p_0 = \bar{p}, p_1 \equiv p_0, q_0 \equiv q_1 \equiv 2, s_0 \equiv 1, s_1 \equiv 0\) and \(p_0 = \bar{p}, p_1 \equiv p_0, q_0 \equiv q_1 \equiv 2, s_0 \equiv 0, s_1 \equiv -1\), these two sets of indices lie within the ranges described in (c) and \((c')\) respectively.
Thus Cor. 3.13 may be applied, and so use (41) to interpolate between 
\( \mathcal{Y} : L^p_{\alpha}(\partial \Omega)_{1+} \to L^p(\partial \Omega) \) and \( \mathcal{Y} : H^p(\partial \Omega) \to H^p(\partial \Omega) \) to obtain an operator

\[
\mathcal{Y} : L^p_{\alpha,p-2,-}(\partial \Omega)_{1+} \to L^p_{\alpha,p-1,-}(\partial \Omega)
\]

(50)

for \( 1 < p \leq p_0 \).

Now to obtain (42) in case (a) fix \( p \in (p_0,p_0') \) and use (39) to interpolate between \( \mathcal{Y} : L^p(\partial \Omega)_{1+} \to L^p(\partial \Omega) \) and \( \mathcal{Y} : L^p_{\alpha}(\partial \Omega)_{1+} \to L^p(\partial \Omega) \). For case (b), we fix \( p \in (1, p_0) \) and use (39) to interpolate between (50) and the operator \( \mathcal{Y} : L^p(\partial \Omega)_{1+} \to L^p(\partial \Omega) \), recalling Cor. 3.4.

To obtain estimates for the range \( p'_0 \leq p < \infty \) of case (c), we first observe that \( \mathcal{Y} \) satisfies

\[
\mathcal{Y} : B^p_{s-1}(\partial \Omega)_{1+} \to B^p_s(\partial \Omega)
\]

(51)

for \( 0 < s < \frac{2 + \epsilon s}{p} \) and \( p'_0 \leq p < \infty \). This is because, fixing \( p \) and \( s \) within this range, we can choose \( p_1 \) so large that the straight line \( l \) joining \( (0,1/p_1) \) and \( (1,1/p'_0) \) (in \( (s,1/p) \)-space) intersects the horizontal line passing through \( (0,1/p) \) at a point \( (s_0,1/p) \) to the right of \( (s,1/p) \). Then of course \( s < s_0 \) and we can use (41) for ranges \( (a') \) and \( (a) \) to interpolate between \( \mathcal{Y} : L^p_{\alpha}(\partial \Omega)_{1+} \to L^p_{\alpha}(\partial \Omega) \) and \( \mathcal{Y} : L^p_{\alpha}(\partial \Omega)_{1+} \to L^p(\partial \Omega) \) to obtain an operator

\[
\mathcal{Y} : L^p_{s-1}(\partial \Omega)_{1+} \to L^p_{s}(\partial \Omega).
\]

(52)

To realize (51) we can then interpolate between (52) and the map \( \mathcal{Y} : L^p_{s-1}(\partial \Omega)_{1+} \to L^p(\partial \Omega) \) using (39).

Using (41) with ranges \( (b') \) and \( (b) \) and our estimate \( \mathcal{Y} : B^p_{\infty}(\partial \Omega)_{1+} \to B^p_{\infty}(\partial \Omega) \) for \( 1 - \delta_m < \rho < 1 \) while noting that \( \epsilon \equiv 2(1/\rho - 1) \leq s(\rho) \equiv (n - 1)(1/\rho - 1) \), it is easy to see how to extend the bound (51) to the full range in case (c), thereby proving Theorem 3.15.
4 Proof of Existence

In this section, we combine the tools we have developed to establish the estimates contained in Theorem 1.6.

Let \( F \in (L^p_{-\alpha}(\Omega))^* \), with \( 1/p + 1/p' = 1 \) and \( p \) and \( \alpha \) falling within the appropriate range for the theorem, and note the symmetry under the transformation \((p, \alpha) \to (p', 2 - \alpha)\). As in Remark 1.3, \( R_{\Omega}^p F \in L^p_{-2} \) with \( \text{supp} \, R_{\Omega}^p F \subseteq \overline{\Omega} \). Setting \( h = N^* (R_{\Omega}^p F) \), where \( N \) is the Newtonian potential on \( \mathbb{R}^n \), it follows that \( \Delta h = R_{\Omega}^p F \) on \( \mathbb{R}^n \) and, via the Elliptic Regularity Theorem, that \( h = h|_{\Omega} \in L^p_{\alpha} \); in fact we have \( \|h\|_{L^p_{\alpha}(\Omega)} \leq C \|R_{\Omega}^p F\|_{L^p_{-2}(-\alpha)} \).

Moreover,
\[
\|h\|_{L^p_{\alpha}(\Omega)} \leq C \|F\|_{(L^p_{-\alpha}(\Omega))^*}.
\]

(53)

Now define the normal derivative \( L_h \in B_{\alpha-1/p}^{p'}(\partial\Omega) \) of the function \( h \) by setting
\[
<L_h, g> = <F, v_g> + <\chi_\Omega \nabla(E_\Omega(h)), \nabla(E_\Omega(v_g))>
\]
for all \( g \in B_{\alpha-1/p}^{p'}(\partial\Omega) \), where \( v_g \) is the unique harmonic function in \( L^p_{-\alpha}(\Omega) \) with \( \text{Tr} \, v_g = g \) given by the estimate for the homogeneous Dirichlet problem, Thm. 2.2. The normal derivative \( L_h \) is a bounded linear functional on \( B_{\alpha-1/p}^{p'}(\partial\Omega) \). In fact we even have

Proposition 4.1. With \( f, h \) and \( L_h \) as above,
\[
\|L_h\|_{B_{\alpha-1/p}^{p'}(\partial\Omega)} \leq C \|F\|_{(L^p_{-\alpha}(\Omega))^*}.
\]

(55)

Proof. Consider first the leading term in (54). The estimate for the homogeneous Dirichlet problem gives
\[
|<F, v_g>| \leq \|F\|_{(L^p_{-\alpha}(\Omega))^*} \|v_g\|_{L^p_{\alpha}(\Omega)} \\
\leq C \|F\|_{(L^p_{-\alpha}(\Omega))^*} \|g\|_{B_{\alpha-1/p}^{p'}(\partial\Omega)}.
\]

(56)
Meanwhile, taking stock of the second term, we find that

\[
| \langle \chi_\Omega \nabla (E_\Omega (h)), \nabla (E_\Omega (v)) \rangle | \leq C \| \chi_\Omega \nabla (E_\Omega (h)) \|_{L^{2}_i} \| \nabla (E_\Omega (v)) \|_{L^{2}_r}
\]

upon employing, respectively, the boundedness of the truncation operator (Prop. 1.4 and Cor. 1.5), Stein's Extension Thm. (See Prop. 1.2), and the estimate for the homogeneous Dirichlet problem (Theorem 2.2), proving Prop. 4.1.

Recall that we are attempting to find a solution to the generalized inhomogeneous Neumann problem \( NP(p, \alpha, F) \). Considering (13), it seems clear that we must show that we can replace \( v_g \) in (54) with general \( v \in L^2_{-\alpha} (\Omega) \), i.e., not just those functions in \( L^2_{-\alpha} (\Omega) \) which happen to be harmonic.

To accomplish this, we shall also require another integration-by-parts formula on Lipschitz domains. For direction to its proof consult Grisvard [6], Thm. 1.5.3.1.

**Lemma 4.2.** For every \( u \in L^2 (\Omega) \) and \( v \in L^1 (\Omega) \) we have

\[
\int_\Omega v \Delta u \, dV = - \int_\Omega < \nabla u, \nabla v > \, dV + \int_{\partial \Omega} \text{Tr} v < \text{Tr}(\nabla u), n > \, d\sigma. \tag{58}
\]

Now we show, as promised, that

\[
< L_h, \text{Tr} v > = < F, v > + \chi_\Omega \nabla (E_\Omega (h)), \nabla (E_\Omega (v)) > \tag{59}
\]

for all \( v \in L^2_{-\alpha} (\Omega) \). Let's first prove this for \( F \in L^p (\Omega) = (L^p (\Omega))^* \) and \( v \in C^\infty (\overline{\Omega}) \). The same arguments used in the first paragraph of this section
show that \( \|h\|_{L_p^2(\Omega)} \leq C\|F\|_{L^p(\Omega)} \). Thus we can apply Lemma 4.2, which implies that

\[
\int_{\partial\Omega} \langle \text{Tr}(\nabla h), n \rangle (v \partial \Omega) d\sigma = \int_{\Omega} F v dV + \int_{\Omega} \langle \nabla h, \nabla v \rangle dV
\]

\[
= \langle F, v \rangle + \langle \nabla (E_\Omega(v)), \chi_\Omega \nabla (E_\Omega(h)) \rangle.
\]

On the other hand, setting \( g = v \partial \Omega, C \in C^\infty(\Omega) \partial \Omega C \in L^{p'}_{\Delta-\alpha}(\partial \Omega) \), we recall that there exists a unique harmonic function \( v_g \) in \( L^p(\Omega) \) with \( v_g \partial \Omega = g \). Consequently, employment of Lemma 4.2 once again gives

\[
\int_{\partial\Omega} \langle \text{Tr}(\nabla h), n \rangle (v \partial \Omega) d\sigma = \int_{\Omega} \langle \text{Tr}(\nabla h), n \rangle (\text{Tr} v_g) d\sigma
\]

\[
= \int_{\Omega} F v_g dV + \int_{\Omega} \langle \nabla v_g, \nabla h \rangle dV
\]

\[
= \langle F, v_g \rangle + \langle \chi_\Omega \nabla (E_\Omega(h)), \nabla (E_\Omega(v_g)) \rangle
\]

\[
= \langle L_h, g \rangle.
\]

so that comparison of (60) and (61) yields the result, as long as \( F \in L^p(\Omega) \) and \( v \in C^\infty(\Omega) \). Since \( L^p(\Omega) \) is dense in \( (L^{p'}(\Omega))^* \) for \( s \geq 0 \) and it is well-known that \( C^\infty(\Omega) \) is dense in \( L^{p'}(\Omega) \) for \( s \geq 0 \), a simple density-continuity argument now establishes (59).

To complete our proof of the existence assertion of Theorem 1.6 we seek to apply our estimates for the inverse Calderon operator (Theorem 3.15) and the estimates for the homogeneous Dirichlet problem (Theorem 2.2). First consider the case \( 1 < p < p_0 \) and select \( \alpha \) from within the range designated by Theorem 1.6. Since \( L^p(\partial \Omega) \) is dense in \( B^p_{\alpha-1/p}(\partial \Omega) \), it follows that \( L^p(\partial \Omega)_+ \) is dense in \( B^p_{\alpha-1/p}(\partial \Omega)_+ \).

Of course, given \( f \in L^p(\partial \Omega)_+ \), Cor. 3.4 gives us a unique harmonic function \( u \in L^p_1(\Omega) \) with nontangential Neumann boundary data \( f \) and nontangential Dirichlet data \( \Upsilon(f) \in L^p_1(\partial \Omega) \). By Theorem 3.5 we have
\[
\int_{\partial \Omega} f(\text{Tr} v) d\sigma = \int_{\Omega} \nabla u, \nabla v > dV
\] (62)
for all \( v \in C^\infty(\overline{\Omega}) \), which may be rewritten
\[
\int_{\partial \Omega} f(\text{Tr} v) = \langle \chi_\Omega \nabla (E_\Omega(u)), \nabla (E_\Omega(v)) \rangle.
\] (63)

By Theorems 2.2 and 3.15 we have
\[
||v||_{L^p(\partial \Omega)} \leq C||\Upsilon(f)||_{B^{p}_{n-1,n}(\partial \Omega)} \leq ||f||_{B^{p}_{n-1,n}(\partial \Omega)}
\] (64)

Thus by density and continuity we can extend our mapping \( f \mapsto u \) to a linear operator mapping all of \( B^{p}_{n-1,n}(\partial \Omega) \) into \( L^p(\Omega) \) for which
\[
||u_L||_{L^p(\partial \Omega)} \leq C||L||_{B^{p}_{n-1,n}(\partial \Omega)}
\] (65)
for all \( L \in B^{p}_{n-1,n}(\partial \Omega) \) and
\[
L(\text{Tr} v) = \langle \chi_\Omega \nabla (E_\Omega(u_L)), \nabla (E_\Omega(v)) \rangle.
\] (66)

for all \( v \in L^2_{2,\alpha}(\Omega) \).

To extend the results just obtained to the range \( p'_0 < p < \infty \), fix such a \( p \) and once more select \( \alpha \) from within the range of Theorem 1.6. Note that \( L^\infty(\partial \Omega)_{1\perp} \) is a dense subset of \( B^{p}_{n-1,n}(\partial \Omega)_{1\perp} \) that is also a subset of all of the spaces \( L^p(\partial \Omega)_{1\perp} \), \( 2 < p < p'_0 \). But, as before, for each \( f \in L^\infty(\partial \Omega)_{1\perp} \), there exists a unique harmonic function \( u_p \in L^p(\partial \Omega) \) with nontangential Neumann boundary data \( f \) and Dirichlet data \( \Upsilon(f) \in L^p_{1\perp}(\partial \Omega) \) in the sense of the Trace Theorem such that (66) holds with \( u \equiv u_p \) and \( v \in C^\infty(\overline{\Omega}) \).

On the other hand, by Theorems 2.2 and 3.15, we have
\[
||u_p||_{L^p(\partial \Omega)} \leq C||\Upsilon(f)||_{B^{p}_{n-1,n}(\partial \Omega)} \leq C||f||_{B^{p}_{n-1,n}(\partial \Omega)}.
\] (67)

Here \( u_p \) is the unique harmonic function in \( L^p(\Omega) \) for which \( \text{Tr} u_p = \Upsilon(f) \),
considered as a member of $B_{n-1/p}(\partial \Omega)$. But upon examination of the proof of Theorem 2.2 in Jerison-Kenig [7] (Theorem 5.1 in their paper) we find that the estimate for the homogeneous Dirichlet problem in the range \([p'_0, \infty)\) is actually obtained via interpolation from that for the range \([2, p'_0)\). What this means is that we must have \(u_p = u_p\). In this way we obtain (65) and (66) for the entire range permitted for \(p\) and \(\alpha\) described in Theorem 1.6.

The final step in our proof of the existence assertion of Theorem 1.6 is merely to combine all of the preceding work in the appropriate fashion. Given \(F \in (L^p_0(\Omega))^*_\ast\), with \(1/p + 1/p' = 1\) and \(p\) and \(\alpha\) falling within the stated ranges, let \(h = (N \ast (R_\alpha^\ast F))|\Omega\) be as in the opening paragraph of the present section and define \(L_h\) by (54). Using the Sobolev Embedding Theorem to show that \(\nabla(E_\Omega(1))|\Omega = 0\), it follows from the definition that \(<L_h, 1> = 0\). Therefore, combining (55) and (65) (with \(L \equiv L_h\)) and equating (59) with (66) (once more with \(L \equiv L_h\)), the entire existence assertion of Theorem 1.6 is proven upon defining \(w \equiv h - u_L\).

5 Proof of Uniqueness

Recall that the well-known methods of Lax-Milgram ensure that there exists a sequence of eigenfunctions \(u_k \in C^\infty(\Omega) \cap L^2(\Omega), k = 0, 1, 2, \ldots\), with corresponding eigenvalues \(\lambda_k\) satisfying the (weak) Neumann eigenvalue problem for the Laplacian

\[
\lambda_k \int_\Omega u_k v dV = \int_\Omega \nabla u_k, \nabla v > dV,
\]

for all \(v \in L^2(\Omega)\).

To establish uniqueness for the inhomogeneous Neumann problem, we first observe that for all \(k = 0, 1, 2, \ldots\) we have \(u_k \in L^p_0(\Omega) \cap L^p_{n-\alpha}(\Omega)\) with
1 < p < \infty, \alpha chosen from within the admissible range for Theorem 1.6, and 
1/p + 1/p' = 1, and that, moreover, we have the \(L^\infty(\Omega)\)-bound

\[\|u_k\|_{L^\infty(\Omega)} \leq C(1 + \lambda_k)^{[n/2]+1}, \tag{69}\]

where \([\cdot]\) denotes the greatest integer function. These assertions are verified by first noting that (68) implies that

\[\|u_k\|_{L^2(\Omega)} \leq (1 + \lambda_k). \tag{70}\]

Note as well that \(u_k\) may be viewed as the unique Lax-Milgram solution in \(L^2(\Omega)\) to the weak inhomogeneous Neumann problem (68) with data \(\lambda_k u_k\) and that this is precisely the solution \(w \in L^2(\Omega)\) to the generalized inhomogeneous Neumann problem with data \(F \equiv \lambda_k u_k\) that we have constructed in our own existence proof.

Now one uses the Sobolev inclusions

(i) \(L^p \subset C \subset L^r\) for \(1 < p < r < \infty\) and \(1/p - 1/r = (1/n)(s - t)\)

(ii) \(L^p \subset L^\infty\) for \(p > n/s\)

in conjunction with the estimates in our own existence theorem to bootstrap up to the estimate (69). To illustrate, we use the estimate (70) as our starting point and we obtain an \(L^{p_1}(\Omega)\)-estimate on \(u_k\), where \(p_1\) satisfies \(1/p_1 = 1/2 - 1/n\), in the following way (writing \(p'_1\) for the conjugate exponent to \(p_1\) and \(\bar{u}_k\) for the extension of \(u_k\) by 0 to a function on \(\mathbb{R}^n\)):

\[\|u_k\|_{L^{p_1}(\Omega)} \leq C(1 + \lambda_k)||u_k||_{(L^p_1(\Omega))^{p'_1}},\]

(\text{using (3.9) in Thm. 3.6})

\[= C(1 + \lambda_k)||E_0(\bar{u}_k)||_{(L^{p'_1}(\Omega))^{p'_1}},\]
We iterate this procedure at most \( \lfloor \frac{n}{2} \rfloor + 1 \) times with \( p_0 = 2 \) and \( \frac{1}{p_i + 1} = \frac{1}{p_i} - \frac{1}{n} \) to obtain

\[
\|u_k\|_{L^\infty(\Omega)} \leq C(1 + \lambda_k)^{\lfloor \frac{n}{2} \rfloor + 1}, \tag{71}
\]

with \( \tilde{p} \geq n \). One final application of the Sobolev inclusion \((i)\) above then gives (71) for all \( p \geq \tilde{p} \), and hence yet another application of the estimates in the existence theorem gives

\[
\|u_k\|_{L^p(\Omega)} \leq C(1 + \lambda_k)^{\lfloor \frac{n}{2} \rfloor + 1}
\]

with \( \epsilon \) as in the statement of the existence theorem and \( p \) as large as one may desire. In particular, if \( p > n/\epsilon \) then the Sobolev inclusion \((ii)\) above implies (69). Also, since \( u_k \in L^p(\Omega) \subseteq (L^q_{\infty-\alpha}(\Omega))^* \), applying (14) once more shows that \( u_k \in L^p_0(\Omega) \) for all values for \( p \) and \( \alpha \) in the ranges \((a),(b),(c)\) of Theorem 1.6. Finally, notice that \( 2 - \alpha \) falls within the corresponding range for the conjugate exponent \( q \). Thus \( u_k \) is a member of all of the spaces \( L^q_{2-\alpha}(\Omega) \) as well.

Now let \( w \in L^p_0(\Omega) \), and assume that

\[
< \chi_n \nabla(E_n(w)), \nabla(E_n(v)) > = 0 \tag{72}
\]

for all \( v \in L^q_{2-\alpha}(\Omega) \). Since, as just noted, \( u_k \in L^q_{2-\alpha}(\Omega) \), it follows from (68) that
for all $k$. In case $w \in L^2(\Omega)$, this means that for $k > 0$ the coefficient of $u_k$ in the $L^2(\Omega)$-eigenfunction expansion of $w$ is 0. Consequently, in this case $w$ is a multiple of the constant eigenfunction and so must be constant.

For uniqueness within the range $1 < p < 2$, let $f \in L^p(\Omega), 1 < p < 2$. Write

$$c_k = \int_\Omega f u_k \, dV,$$

and set

$$T_t f(x) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} u_k(x)$$

for $t > 0$. I claim that $T_t f \to f$ as $t \to 0$ in $L^p(\Omega)$-norm. First note that the series in (74) converges absolutely for all $x \in \Omega$, and this follows very easily from the $L^\infty$ estimates (69) for the eigenfunctions and Weyl's asymptotic formula (on Lipschitz domains) for the distribution of eigenvalues, which implies that

$$0 < C(\Omega) \leq \lambda_k / k^{2/n} \leq C(\Omega) < \infty$$

as $k \to \infty$. When $f \in L^2(\Omega)$ the function $T_t f$ is in fact the solution to the heat equation on $\Omega$ with initial temperature distribution $T_0 f(x) = f(x)$.

Due to a well-known theorem of Beurling-Deny (see the second Beurling-Deny criterion which appears as Theorem XIII.51 in Reed and Simon [10], more specifically the implication within the statement of that theorem that (d)$\Rightarrow$(a)),

$$\|T_t f\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$  \hspace{1cm} (75)

If in fact $g \in L^2(\Omega)$ then, since $\sum_k c_k^2 < \infty$ in this case, we know that

$$\lim_{t \to 0^+} T_t g = g$$  \hspace{1cm} (76)

in $L^2(\Omega)$-norm. Because $L^2(\Omega)$ is dense in $L^p(\Omega)$, using (75), we have $T_t f \to f$. 

\[ \lambda_k \int_\Omega w u_k \, dV = 0 \]  \hspace{1cm} (73)
as \( t \to 0^+ \) in \( L^p(\Omega) \)-norm for all \( f \in L^p(\Omega), 1 < p < 2 \).

Finally let \( f \equiv w \in L^p_\alpha \) with \( 1 < p < 2 \) and \( \alpha \) lying within the admissible range for the existence theorem, and assume that (72) is satisfied. Equation (73) then implies that \( c_k = 0 \) for all \( k > 0 \). Hence \( T_tw \equiv c_0 w_0 \) for all \( t > 0 \), where \( w_0 \) is the constant (normalized) eigenfunction. Therefore \( w \) itself is constant, and uniqueness in Theorem 1.6 is established.

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ON ABSENCE OF EMBEDDED EIGENVALUES FOR SCHRÖDINGER OPERATORS WITH PERTURBED PERIODIC POTENTIALS

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Abstract

The problem of absence of eigenvalues imbedded into the continuous spectrum is considered for a Schrödinger operator with a periodic potential perturbed by a sufficiently fast decaying "impurity" potential. Results of this type have previously been known for the one-dimensional case only. Absence of embedded eigenvalues is shown in dimensions two and three if the corresponding Fermi surface is irreducible modulo natural symmetries. It is conjectured that all periodic potentials satisfy this condition. Separable periodic potentials satisfy it, and hence in dimensions two and three Schrödinger operator with a separable periodic potential perturbed by a sufficiently fast decaying "impurity" potential has no embedded eigenvalues.

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1809
1 Introduction

Consider the stationary Schrödinger operator

$$H_0 = -\Delta + q(x)$$  \hspace{1cm} (1)

in $L^2(\mathbb{R}^n)$ ($n = 2$ or $3$), where the real potential $q(x) \in L^\infty(\mathbb{R}^n)$ is periodic with respect to the integer lattice $\mathbb{Z}^n$:

$$q(x + l) = q(x), l \in \mathbb{Z}^n, x \in \mathbb{R}^n.$$  

The spectrum of this operator is absolutely continuous and has the well-known band-gap structure (see [10], [13], [22], [31], [34], [38], [40]):

$$\sigma(H_0) = \bigcup_{i \in \mathbb{Z}} [a_i, b_i],$$

where $a_i < b_i$, and $\lim_{i \to \infty} a_i = \infty$. Let us introduce also a perturbed operator with an impurity potential $v(x)$:

$$H = H_0 + v(x) = -\Delta + q(x) + v(x),$$  \hspace{1cm} (2)

where $v(x)$ is compactly supported or sufficiently fast decaying at infinity (the exact conditions will be introduced later). It is known (see [5], Section 18 in [16], and references therein) that the continuous spectrum of $H$ coincides with the spectrum of $H_0$, and only some additional "impurity" point spectrum $\{\lambda_j\}$ can arise. A general understanding is that eigenvalues $\lambda_j$ should normally arise only in the gaps of the continuous spectrum (i.e. in the gaps of $\sigma(H_0)$). In other words, the case of embedded eigenvalues when one of the eigenvalues $\lambda_j$ belongs to one of the open segments $(a_i, b_i)$ is prohibited (at least for a sufficiently fast decaying impurity potential $v(x)$). Physically speaking, the existence of an embedded eigenvalue means a strange situation when an electron is confined, in spite of having enough energy for propagation. There are known examples of embedded eigenvalues, first of which was suggested by J. von Neumann and E. Wigner (see [11] and section XIII.13 in [34] for discussion of this topic and related references). The problem of the absence of embedded eigenvalues has been intensively studied. There are many known results on non-existence of embedded eigenvalues for the case of zero underlying potential $q(x)$ (see, for instance, books [11], [19] Section 14.7, and [34] Section XIII.13). The case of a periodic potential $q(x)$ is much less studied. In [11] a one-dimensional example is provided where a not very fast decaying perturbation of a periodic potential creates embedded eigenvalues. There are many papers devoted to studying the behavior of the point spec-
trum in the gaps of the continuous spectrum (see, for instance, [1], [2], [4] - [6], [9], [15], [18], [25] - [27], [33], and [35] - [37]). Apparently the only known result on the absence of embedded eigenvalues relates to the one-dimensional case of the Hill's operator

\[ H_0 = -\frac{d^2}{dx^2} + q(x) + \nu(x). \]

(see [35], [36]). However, it was shown in [14] and [20] that in dimension higher than 1 the set of embedded eigenvalues must be discrete. The purpose of this paper is to address the problem of absence of embedded eigenvalues in the multi-dimensional situation. Section 2 contains some necessary notions, auxiliary information, and our main condition on the periodic potential. It is conjectured that the condition is always satisfied. According to [3], this condition is satisfied at least when the potential is separable in 2D or of the form \( q_1(x_1) + q_2(x_2, x_3) \) in 3D. In Section 3 we prove a conditional statement (in dimension less than four) on the absence of embedded eigenvalues for a periodic potential satisfying this condition. Finally, the Section 4 contains the main unconditional result. In our considerations we follow an approach that was suggested long ago by the second author for the case of zero underlying potential \( q(x) \) (see [39]). Some recent developments made it possible to adjust this method to the periodic case. In particular, results on allowed rate of decay of solutions obtained in [12] and [29] play a crucial role.

A more limited version of the main result was announced by the authors without proof in [23].

2 Bloch and Fermi varieties

Let us introduce some notions and notations. We assume that \( q(x) \in L^\infty(\mathbb{R}^n) \) is a periodic potential.

**Definition 1** The (complex) Bloch variety \( B(q) \) of the potential \( q \) consists of all pairs \( (k, \lambda) \in \mathbb{C}^{n+1} \) (where \( k \in \mathbb{C}^n \) is a quasimomentum and \( \lambda \in \mathbb{C} \) is an eigenvalue) for which there exists a non-zero solution of the equation

\[ H_0u = \lambda u \] (3)

satisfying the so called Floquet-Bloch condition:
Here \( k \cdot l = \sum k_i l_i \) is the standard dot-product.

**Definition 2** The projection \( F_c(q) \) onto \( \mathbb{C}^n \) of the intersection of the Bloch variety \( B(q) \) with the subspace \( \lambda = c \) is called the (complex) Fermi variety of the potential \( q \) at the level of energy \( c \):

\[
F_c(q) = \{ k \in \mathbb{C}^n \mid (k, c) \in B(q) \}.
\]

In other words, \( F_c(q) \) consists of all quasimomenta \( k \in \mathbb{C}^n \) for which there exists a non-zero solution of the equation

\[
H_0 u = cu
\]
satisfying (4).

When \( c = 0 \) we will use the notation \( F(q) \) for \( F_0(q) \). Let us notice the following obvious relation between the Bloch and Fermi varieties:

\[
\{(k, c) \in B(q)\} \iff \{k \in F_c(q)\} \iff \{k \in F(q - c)\}.
\]

We will use the following notations for the real parts of the above varieties:

\[
B_{R}(q) = B(q) \cap \mathbb{R}^{n+1}, F_{R,\lambda}(q) = F_{\lambda}(q) \cap \mathbb{R}^n.
\]

The varieties \( B_R(q) \) and \( F_{R,\lambda}(q) \) are called the real Bloch variety and the real Fermi variety respectively. The following statement is contained in Theorems 3.17 and 4.4.2 of [22] (with remarks of Section 3.4.D in [22] about conditions on potentials taken into account).

**Lemma 3** The Bloch variety \( B(q) \subset \mathbb{C}^{n+1} \) is the set of all zeros of a non-zero entire function \( f(k, \lambda) \) on \( \mathbb{C}^{n+1} \) of order \( n \) (see [24]), i.e.

\[
|f(k, \lambda)| \leq C \cdot \exp \left( \epsilon |k| + |\lambda|^{n+\epsilon} \right)
\]

for any \( \epsilon > 0 \). The Fermi variety \( F_{\lambda}(q) \) is the set of all zeros of a non-zero entire function of order \( n \) on \( \mathbb{C}^n \).

Lemma 3 implies in particular that both Bloch and Fermi varieties are
examples of what is called in complex analysis analytic sets (see for instance [8], [17], and [30]). Moreover, these are principal analytic sets in the sense that they are sets of all zeros of single analytic functions, while general analytic sets might require several analytic equations for their (local) description. Analyticity of these varieties (without estimates on the grows of the defining function) was obtained in [40].

**Definition 4** An analytic set \( A \subseteq \mathbb{C}^m \) is said to be irreducible, if it cannot be represented as the union of two non-trivial analytic subsets.

Irreducibility of the zero set of an analytic function can be understood as absence of non-trivial factorizations of this function (i.e., of a factorization into analytic factors that have smaller zero sets).

**Definition 5** A point of an analytic set \( A \subseteq \mathbb{C}^m \) is said to be regular, if in a neighborhood of this point the set \( A \) can be represented as an analytic submanifold of \( \mathbb{C}^m \). The set of all regular points of \( A \) is denoted by \( \text{reg}A \).

We collect in the following lemma several basic facts about analytic sets that we will need later. The reader can find them in many books on several complex variables. In particular, all these statements are proven in sections 2.3, 5.3, 5.4, and 5.5 of Chapter 1 of [8].

**Lemma 6** Let \( A \) be an analytic set.

a) The set \( \text{reg}A \) is dense in \( A \). Its complement in \( A \) is closed and nowhere dense in \( A \).

b) The set \( A \) can be represented as a (maybe infinite) locally finite union of irreducible subsets

\[
A = \bigcup_i A_i
\]
called its irreducible components.

c) Irreducible components are closures of connected components of \( \text{reg}A \). In particular, set \( A \) is irreducible if and only if \( \text{reg}A \) is connected.

d) Let \( A \) be irreducible and \( A_1 \) be another analytic set such that \( A \cap A_1 \) contains a non-empty open portion of \( A \). Then \( A \subseteq A_1 \). In particular, if \( f \) is
an analytic function that vanishes on an open portion of $A$, then $f$ vanishes on $A$.

e) Any analytic set $A$ has a stratification $A = \cup A_j$ into disjoint complex analytic manifolds (strata) $A_j$ such that the union $\cup A_j$ is locally finite, the closure $\overline{A_j}$ of each $A_j$ and its boundary $\partial A_j \setminus A_j$ are analytic subsets, and such that if the intersection $A_j \cap \overline{A_k}$ of two different strata is not empty, then $A_j \subset \overline{A_k}$ and $\dim A_j < \dim A_k$.

We will need the following simple corollary from this lemma.

**Corollary 7** Let $A$ be an irreducible proper analytic subset of $\mathbb{C}^n$ such that the intersection $A \cap \mathbb{R}^n$ contains an open part of a smooth $(n-1)$-dimensional submanifold $M \subset \mathbb{R}^n$. If $f$ is an analytic function in a complex neighborhood of $M$ such that $f = 0$ on $M$, then $f = 0$ on an open subset of $A$. In particular, if $B$ is another analytic subset of $\mathbb{C}^n$ such that $M \subset B$, then $A \subset B$.

**Proof.** Considering stratification $A = \cup A_j$ (see the last statement in the preceding lemma), one can conclude that only strata of dimension $n - 1$ can contain open pieces of $M$. In fact, if there is an $A_j$ of complex dimension at most $n - 2$ containing an open piece of $M$, we get the contradiction as follows. Consider a point of $M$ and the tangent space to $A_j$ at this point. This is a complex linear subspace of complex dimension at most $n - 2$, which contains a real subspace (tangent space to $M$) of real dimension $n - 1$. This is obviously impossible. So, now we can assume that instead of $A$ we are dealing with one of its strata of dimension $n - 1$, i.e. we may assume that $A$ is smooth. In appropriate analytic coordinates in a complex neighborhood of a point of $M$ one can represent $A$ locally as a complex hyperplane with the real part $M$. Then standard uniqueness theorem for analytic functions implies that any function $f$ analytic on $A$ and vanishing on $M$ is identically equal to zero on $A$. If now $B$ is an analytic subset of $\mathbb{C}^n$ such that $M \subset B$, then each of the analytic functions locally defining $B$ has the properties of the function $f$ above. Hence, we conclude that $B$ contains an open part of $A$. Then statement d) of the lemma implies that $A \subset B$. This finishes the proof of the corollary.

After this brief excursion into complex analysis we return now to Bloch and Fermi varieties. The next statement follows by inspection of (4):
Lemma 8 The sets \( B(q) \) and \( F_\lambda(q) \) are periodic with respect to the quasi-momentum \( k \) with the lattice of periods \( 2\pi Z^n \subseteq C^n \) (i.e., the dual lattice to \( Z^n \)).

We choose as a fundamental domain of the group \( 2\pi Z^n \) acting on \( R^n \) the following set called the (first) Brillouin zone:

\[
B = \{ k = (k_1, \ldots, k_n) \in R^n | 0 \leq k_j \leq 2\pi, j = 1, \ldots, n \}.
\]

It has been known for a long time (though not always formulated in these terms) that one can tell where the spectrum \( \sigma(H_0) \) lies by looking at the Fermi variety. We will now briefly remind the reader this classical result and, at the same time, the definition of spectral bands.

Consider the problem (3) - (4). It has a discrete real spectrum \( \{ \lambda_j(k) \} \), where \( \lambda_j \to \infty \) as \( j \to \infty \). We number the eigenvalues in the increasing order. This way we obtain a sequence of continuous functions \( \lambda_j(k) \) on the Brillouin zone \( B \). They are usually called band functions, or branches of the dispersion relation. The values of the function \( \lambda_j(k) \) for a fixed \( j \) span the \( j \)th band \([a_j, b_j]\) of the spectrum \( \sigma(H_0) \). A reformulation of this statement is the following theorem, which can be found in equivalent forms in [10] (Section 6.6), [13], [22] (Theorem 4.1.1), [31], and [34] (Theorem XII.98).

Theorem 9 A point \( \lambda \) belongs to the spectrum \( \sigma(H_0) \) of the operator \( H_0 = -\Delta + q(x) \) if and only if the real Fermi variety \( F_{R,\lambda}(q) \) is non-empty.

In other words, by changing \( \lambda \) one observes the Fermi variety \( F_\lambda(q) \) and notices the moments when it touches the real subspace. This set of values of \( \lambda \) is the spectrum. Now the question arises how can one distinguish the interiors of the spectral bands. The natural idea is that when \( \lambda \) is in the interior of a spectral band, then the real Fermi variety will be massive. "Massive" means here "of dimension \( n - 1 \)" i.e. of maximal possible dimension for a proper analytic subset in \( R^n \). This is confirmed by the following statement.

Lemma 10 If \( \lambda \) belongs to the interior of a spectral band, then the real Fermi variety \( F_{R,\lambda}(q) \) contains an open piece of a submanifold \( M \) of dimension \( n - 1 \) in \( R^n \).

Proof of the lemma follows from the stratification of the real Fermi variety \( F_{R,\lambda}(q) \) into smooth manifolds (see for instance propositions 17 and 18 of the
Chapter V in [30]) and from the obvious remark that when $\lambda$ belongs to the interior of a spectral band, then $F_{R,\lambda}(q)$ must separate $\mathbb{R}^n$. These two observations imply existence of a smooth piece in $F_{R,\lambda}(q)$ of dimension at least $(n - 1)$.

Now we introduce our basic condition:

**Condition 11** Assume that for any $\lambda$ that belongs to the interior of a spectral band of the Schrödinger operator

$$H_0 = -\Delta + q(x)$$

any irreducible component of the Fermi variety $F_\lambda(q)$ intersects the real space $\mathbb{R}^n$ by a subset of dimension $n - 1$ (i.e. by a subset that contains a piece of a smooth hypersurface).

In fact, the Lemma 10 says that for $\lambda$ in the interior of a spectral band the Fermi variety $F_\lambda(q)$ does intersect the real space $\mathbb{R}^n$ over a subset of dimension $n - 1$. We, however, need more, that every irreducible component of $F_\lambda(q)$ does the same. In other words, there are no "hidden" components that do not show up on the real subspace in any significant way. Thus, Lemma 10 and irreducibility of $F_\lambda(q)/2\pi\mathbb{Z}^n$ would imply validity of the Condition 11. In fact, we believe that (modulo the action of the dual lattice) the Fermi surface is irreducible. The following conjecture formulated in [3] is probably correct:

**Conjecture 12** For any periodic potential (from an appropriate functional class, for instance continuous, or locally square integrable) the surface $F_\lambda(q)/2\pi\mathbb{Z}^n$ is irreducible.

It looks like this conjecture is very hard to prove (see related discussion in [3] and [21]). As the rest of the paper shows, proving it would lead to a result on the absence of embedded eigenvalues. The following weaker conjecture must be easier to prove:

**Conjecture 13** A generic periodic potential (from an appropriate functional class) satisfies the Condition 11.
The word "generic" could mean "from a residual set", or something of this sort.

One can easily prove using separation of variables and simple facts about the Hill’s equation irreducibility of $F_x(q)/2\pi Z^n$ for separable periodic potentials $q(x) = \sum_i q_i(x_i)$ in any dimension [3]. The paper [3], however, also contains a stronger non-trivial result:

**Lemma 14 ([3])** If $d = 3$ and $q(x) = q_1(x_1) + q_2(x_2, x_3)$, then $F_x(q)/2\pi Z^n$ is irreducible.

The proof of this lemma relies on the deep study of the Bloch variety done in [21].

## 3 The conditional result

We are ready to prove our main conditional statement.

**Theorem 15** If a real periodic potential $q(x) \in L^\infty(\mathbb{R}^n)$ ($n \leq 3$) satisfies the Condition 11 and an impurity potential $v(x)$ is measurable and satisfies the estimate

$$|v(x)| \leq C e^{-|x|^r}, r > 4/3$$

almost everywhere in $\mathbb{R}^n$, then the spectrum of $H$ contains no embedded eigenvalues. In other words,

$$\{\lambda_j\} \cap \bigcup_{i \geq 1} (a_i, b_i) = \emptyset,$$

where $\{\lambda_j\}$ is the impurity point spectrum of $H$, and

$$\bigcup_{i \geq 1} [a_i, b_i] = \sigma(H_0)$$

is the band structure of the essential spectrum of $H$.

**Proof.** Let us assume that there exists a $\lambda$ that belongs to some $(a_i, b_i)$ and to the point spectrum of $H$ simultaneously. Then there exists a non-zero function $u(x) \in L^2(\mathbb{R}^n)$ (an eigenfunction) such that
\[ -\Delta u + qu + vu = \lambda u, \]

or

\[ (H_0 - \lambda)u = -vu. \]

Let us denote the function in the right hand side by \( \psi(x) : \psi(x) = -v(x)u(x). \)

Consider the fundamental domain \( K \) of the group \( \mathbb{Z}^n \) of periods:

\[ \mathcal{K} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | 0 \leq x_i \leq 1, i = 1, \ldots, n\}. \]

Then (6) implies that the function \( \psi(x) \) satisfies the estimate

\[ ||\psi||_{L^1(K+x)} \leq C e^{-l r} \tag{7} \]

for all \( l \in \mathbb{Z}^n \), where \( r > 4/3 \). Consider the following Floquet transform of functions defined on \( \mathbb{R}^n \) (see section 2.2 in [22] about properties of this transform):

\[ \mathcal{F} : f(x) \rightarrow \hat{f}(k, x) = \sum_{l \in \mathbb{Z}^n} f(x - l) e^{-ik \cdot (x - l)}. \]

The result is a function, which is \( \mathbb{Z}^n \)-periodic with respect to \( x \). We can consider the resulting function as a function of \( k \) with values in a space of functions on the \( n \)-dimensional torus \( T^n = \mathbb{R}^n / \mathbb{Z}^n \). If we apply this transform to the function \( \psi(x) \), then we get a function of \( k \in \mathbb{C}^n \)

\[ \phi(k) = \hat{\psi}(k, x) \tag{8} \]

with values in the space \( L^2(T^n) \).

**Lemma 16** Function \( \phi(k) \) is an entire function on \( \mathbb{C}^n \) of the order \( s = r/(r - 1) < 4 \).

**Proof of the lemma.** From (7) we get

\[ ||\psi(x - l)e^{-ik \cdot (x - l)}||_{L^1(K)} \leq C e^{C|l|+Im(k \cdot l) - l r}. \]

Then

\[ ||\phi(k)||_{L^2(K)} \leq C e^{C|k|} \sum_{l \in \mathbb{Z}^n} e^{-0.5l r} e^{Im(k \cdot l) - 0.5l r}. \]
A simple Legendre transform type estimate (finding extremal values of the exponent) shows that

$$e^{\text{Im}(\lambda(k)) - 0.5|\lambda|^2} \leq C e^{C|\lambda|/|r - 1|},$$

which proves that the function $\phi(k)$ is an entire function of the order $s = r/(r - 1)$ in $\mathbb{C}^n$. The lemma is proven.

As it is well known (see e.g. Chapter 2 of [22] or XIII in [34]), the transform $F$ leads to the following operator equation with a parameter $k$:

$$(H_0(k) - \lambda) \tilde{u}(k) = \phi(k),$$

where

$$\tilde{u}(k) = (F u)(k, x)$$

is defined for $k \in \mathbb{R}^n$ and

$$H_0(k) = (i\nabla - k)^2 + q(x)$$

is considered as an operator on the torus $T^n$. According to Theorem 2.2.5 in [22],

$$\tilde{u}(k) \in L^2_{\text{loc}}(\mathbb{R}^n, L^2(T^n)).$$

In fact, standard interior elliptic estimates show that

$$\tilde{u}(k) \in L^2_{\text{loc}}(\mathbb{R}^n, H^2(T^n)),$$

where $H^2(T^n)$ is the Sobolev space of order two on $T^n$. Theorem 3.1.5 of [22] implies that the operator

$$(H_0(k) - \lambda) : H^2(T^n) \to L^2(T^n)$$

is invertible if and only if $k \notin F_{R, \lambda}(q)$. As it is shown in the proof of Theorem 3.3.1 and in Lemma 1.2.21 of [22] (see also comments in Section 3.4.D on reducing requirements on the potential, in particular Theorem 3.4.2), the inverse operator can be represented as follows:

$$(H_0(k) - \lambda)^{-1} = B(k)/\zeta(k),$$

where $B(k)$ is a bounded operator from $L^2(T^n)$ into $H^2(T^n)$ and $B(k)$ and $\zeta(k)$ are correspondingly an operator and a scalar entire functions of order $n$ in $\mathbb{C}^n$. Besides, the zeros of $\zeta(k)$ constitute exactly the Fermi variety $F_\lambda(q)$. We conclude now that the following representation holds on the set $\mathbb{R}^n \setminus F_{R, \lambda}(q)$:
where $g(k)$ is a $H^2(T^n)$-valued entire function of order $n$.

Now we need the following auxiliary result:

**Lemma 17** Let $Z$ be the set of all zeros of an entire function $\zeta(k)$ in $\mathbb{C}^n$ and $Z_j$ be its irreducible components. Assume that the real part $Z_{j,R} = Z_j \cap \mathbb{R}^n$ of each $Z_j$ contains a submanifold of real dimension $n-1$. Let also $g(k)$ be an entire function in $\mathbb{C}^n$ with values in a Hilbert space $H$ such that on the real subspace $\mathbb{R}^n$ the ratio

$$\tilde{u}(k) = \frac{g(k)}{\zeta(k)}$$

belongs to $L_{2,loc}(\mathbb{R}^n, H)$. Then $\tilde{u}(k)$ extends to an entire function with values in $H$.

**Proof of the lemma.** Applying linear functionals, one can reduce the problem to the case of scalar functions $g$, so we will assume that $g(k) \in \mathbb{C}$.

According to Lemma 6, the sets of regular points of components $Z_j$ are disjoint. Hence, the traces of these sets on $\mathbb{R}^n$ are also disjoint. Consider one component $Z_j$. The intersection of $\text{reg}Z_j$ with the real space $\mathbb{R}^n$ contains a smooth manifold of dimension $(n-1)$. Namely, we know that $Z_{j,R}$ contains such a manifold, which we will denote $M_j$. The only alternative to our conclusion would be that the whole $M_j$ sits inside the singular set of $Z_j$. The proof of Corollary 7 shows that this is impossible, since lower dimensional strata cannot contain any open pieces of $M_j$.

Let us denote by $m_j$ the minimal order of zero of function $\zeta(k)$ on $Z_j$. (We remind the reader that the order of zero of an analytic function at a point is determined by the order of the first non-zero term of the function's expansion at this point into homogeneous polynomials.) Since the condition that an analytic function has a zero of order higher than a given number can be written down as a finite number of analytic equations, one can conclude that the order of zero of $\zeta(k)$ equals $m_j$ on a dense open subset of $Z_j$, whose complement is an analytic subset of lower dimension. As it was explained in the proof of Corollary 7, lower dimensional strata cannot contain $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$. This means that one can find a point $k^j \in Z_{j,R}$ such that $Z_{j,R}$ is a smooth hypersurface in a neighborhood $U$ of $k^j$ in $\mathbb{R}^n$ and such that the order of zero of $\zeta(k)$ on $Z_{j,R}$ equals $m_j$ in...
this neighborhood. Let us now prove that \( g(k) \) has zeros on \( Z_{j,R} \cap U \) of at least the same order as \( \zeta(k) \). If this were not so, then in appropriate local coordinates \( k = (k_1, \ldots, k_n) \) the ratio \( g/\zeta \) would have a singularity of at least the order \( k_i^{-1} \), which implies local square non-integrability of the function \( \tilde{u}(k) \). This contradicts our assumption, so we conclude that \( g(k) \) has zeros on \( Z_{j,R} \cap U \) of at least the same order as \( \zeta(k) \). Consider the (analytic) set \( A \) of all points in \( \mathbb{C}^n \) where \( g(k) \) has zeros of order at least \( m_j \). We have just proven that the intersection of \( A \) with \( Z_j \) contains an \((n-1)\)-dimensional smooth submanifold of \( \mathbb{R}^n \). Then Corollary 7 implies that \( Z_j \subset A \), or \( g(k) \) has zeros on \( Z_j \) of at least the order \( m_j \). Hence, according to Proposition 3 in section 1.5 of Chapter 1 in [8] the ratio \( g/\zeta \) is analytic everywhere in \( \mathbb{C}^n \) except maybe at the union of subsets of \( Z_j \), where the function \( \zeta \) has zeros of order higher than \( m_j \). This subset, however, is of dimension not higher than \( n-2 \). Now a standard analytic continuation theorem (see for instance Proposition 3 in Section 1.3 of the Appendix in [8]) guarantees that \( g/\zeta \) is an entire function. This concludes the proof of the lemma.

Returning to the proof of the theorem and using the result of the Lemma 10 and the assumption that the Condition 11 is satisfied for the potential \( q(x) \), we conclude that \( \tilde{u}(k) \) is an entire function and that it is the ratio of two entire functions of order at most \( w = \max(n, s) < 4 \), where \( s \) is defined in Lemma 16. Using (in the radial directions) the estimate of entire functions from below contained in section 8 of Chapter 1 in [28] (see also Theorem 1.5.6 and Corollary 1.5.7 in [22] or similar results in [7]), we conclude that \( \tilde{u}(k) \) is itself an entire function of order \( w \) with values in \( H^2(T^n) \). Now Theorem 2.2.2 of [22] claims that the solution \( u(x) \) satisfies the decay estimate:

\[
||u||_{H^2(K+a)} \leq C_p \exp(-c_p |a|^p)
\]

for any \( p < w/(w-1) \), where \( K \) is an arbitrary compact in \( \mathbb{R}^n \), and \( a \in \mathbb{R}^n \). Using standard embedding theorems, we conclude that

\[
|u(x)| \leq C_p \exp(-c_p |x|^p)
\]

for any \( p < w/(w-1) \). Since \( w < 4 \), one can choose a value \( p \) such that

\[
4/3 < p < w/(w-1).
\]

However, Remark 2.6 in [12] and Theorem 1 in [29] state that there is no non-trivial solution of the equation

\[-\Delta u + qu + vu - \lambda u = 0\]

with the rate of decay
This contradiction concludes the proof of the theorem.

Remark 1 The condition $p > 4/3$ (and hence the dimension restriction $n < 4$) is essential for the validity of the result of [12] and [29], so this is the place where our argument breaks down for dimensions four and higher even if we require compactness of support of the perturbation potential $v(x)$. The rest of the arguments stays intact (the embedding theorem argument, which also depends on dimension, is not really necessary).

4 Separable potentials

The result of the previous section leads to the problem of finding classes of periodic potentials that satisfy the Condition 11. As we stated in conjectures 12 and 13, we believe that all (or almost all) of periodic potentials satisfy this condition. Although we were not able to prove these conjectures, as an immediate corollary of the Lemma 14 and Theorem 15 we get the following result:

**Theorem 18** If for $n < 4$ the background periodic potential $q(x) \in L^\infty(\mathbb{R}^n)$ is separable for $n = 2$ or $q(x) = q_1(x_1) + q_2(x_2, x_3)$ for $n = 3$, and the perturbation potential $v(x)$ satisfies the estimate 6, then there are no eigenvalues of the operator $H$ in the interior of the bands of the continuous spectrum.

5 Comments

1. We have only proven the absence of eigenvalues embedded into the interior of a spectral band. It is likely that eigenvalues cannot occur at the ends of the bands either (maybe except the bottom of the spectrum), if the perturbation potential decays fast enough. This was shown in the one-dimensional case in [35] under the condition

$$\int (1 + |x|)|v(x)|dx < \infty$$
on the perturbation potential. On the other hand, if \( v(x) \) only belongs to \( L^1 \), then the eigenvalues at the endpoints of spectral bands can occur [36].

2. Most of the proof of the conditional Theorem 15 does not require the unperturbed operator to be a Schrödinger operator. One can treat general selfadjoint periodic elliptic operators as well. The only obstacle occurs at the last step, when one needs to conclude the absence of fast decaying solutions to the equation. Here we applied the results of [12] and [29], which are applicable only to the operators of the Schrödinger type. Carrying over these results to more general operators would automatically generalize Theorem 15. The restriction that the dimension \( n \) is less than four also comes from the allowed rate of decay stated in the result of [12] and [29].

3. It might seem that the irreducibility condition 11 arises only due to the way the proof is done. We believe that this is not true, and that the validity of the condition is essentially equivalent to the absence of embedded eigenvalues. To be more precise, we conjecture that existence for some \( \lambda \) in the interior of a spectral band of an irreducible component \( A \) of the Fermi surface such that \( A \cap \mathbb{R}^n = \emptyset \) implies existence of a localized perturbation of the operator that creates an eigenvalue at \( \lambda \). As a supporting evidence of this one can consider fourth order periodic differential operators, where the Fermi surface contains four points. In this case one can have \( \lambda \) in the continuous spectrum, while some points of the Fermi surface being complex. Then one can use these components of the Fermi surface "hidden" in the complex domain to cook up a localized perturbation that does create an eigenvalue at \( \lambda \) [32].

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ON NONLINEAR SCHröDINGER EQUATIONS

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Dedicated to Walter Strauss on his sixtieth birthday.

Introduction: Statement of the problem.
In the present work I would like to address the question of regularity of a semilinear Schrödinger equation, with a critical nonlinear term. The equation to be considered is

\[ \begin{align*}
    i\partial_t z - \Delta z + |z|^4 z &= 0 \quad ; \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 , \\
    z(0, x) &= z_0(x) \in C^\infty \cap L^3 , \quad x \in \mathbb{R}^3 .
\end{align*} \quad (0.1) \]

Presently I would like to show that, assuming radial symmetry, the solution remains regular for all time. This is a different proof of a recent result by J. Bourgain [B]. The interest in the proof rests in the use of a new a priori estimate derived in Chapter 2.

The important observation for equation (0.1) is that the total energy, namely the quantity,

\[ E(t) := \int_{\mathbb{R}^3 \setminus \{0\}} \left( |\nabla z|^2 + \frac{1}{3} |z|^6 \right) dx , \quad (0.2) \]

is conserved in time i.e. \( E(t) = E(0) \). The energy density is positive definite, which justifies the term repulsive for the nonlinear potential. The energy estimate however, does not preclude the possibility that the solution explodes in finite time. It is my aim to show that this cannot happen. The crucial step in the argument is to show that the solution of (0.1) is

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apriori bounded in $L^\infty$. Once this is known further regularity can be obtained by standard arguments. The proof proceeds in two steps. First, I will obtain an apriori energy type estimate which can be stated as follows. The time average of the energy density over a parabolic cylinder of height $r^2$ and radius $r$,

$$C([t_0,x_0];r) := \{(t,x) : |x-x_0| \leq r ; \ t_0 \leq t \leq t_0 - r^2 \} \quad (0.3,a)$$

is controlled by the quantity

$$\frac{1}{R^2(r)} \int_{B(x_0,R(r))} |z(t_0,x)|^3 \, dx \quad \text{where} \ \lim_{r \to 0} R(r) = 0 \quad (0.3,b)$$

This means that the energy cannot concentrate on the average. Since I consider only radially symmetric functions it is enough to take $x_0 = 0$. Using the integral representation for the solution one can obtain an inequality for $|z(t,x)|$. If the energy is small in a time interval, call it $I$, then one can show that $\sup_{t \in I} |z| \leq C|I|^{1/4}$. On the other hand if the next time interval, call it $J$, where the energy might be large, is relatively small in size i.e. $|J| < \delta |I|$, for some $\delta$ sufficiently small but apriori fixed, then one can show that $\sup_{t \in J} |z| \leq C|J|^{1/4}$. This completes the proof since the measure of the union of the time intervals where the energy is large divided by the measure of the union of the time intervals where the energy is small can be made arbitrarily small.

Equations of type (0.1) have been proposed as multiparticle approximations in Quantum Mechanics, when one considers a large number of quantum particles acting in unison and takes into account only a finite number of particle-particle interactions. These approximations are called mean-field theories and the condensate wavefunction $z(t,x)$ obeys a nonlinear Schrödinger equation of the general form,

$$iz_t - \Delta z + f(|z|^2)z = 0 \quad (0.4)$$

One well known example is the Bose-Einstein condensation, where the nonlinear potential is usually assumed to be cubic. It is natural to think of (0.4) as describing the evolution of a fluid with density $\rho := |z|^2/2$ and momenta $p_{\alpha} := (iz_{x_{\alpha}} - z\bar{z}_{,\alpha})/2$, $\alpha = 1,2,\ldots,n$.

In order to set the problem let me mention that, by standard arguments, one can show that a smooth solution exists locally in time, see [CZ]. Thus, I can assume without loss of generality that $z(t,x)$ is a smooth solution of (0.1) in the domain $R^3 \times [0,T)$ for some $T > 0$. The time $t = T$ is the final time of existence. The crucial step is an apriori bound of the form

$$\sup_{R^3 \times [0,T]} |z(t,x)| \leq K \quad (0.5)$$

for some finite number $K$. Below I would like to explain why this bound implies that the solution is regular for all times and to introduce certain auxiliary tools regarding equation (0.1).
Consider the linear equation with a forcing term,

$$ix_t - \Delta x = f ; \quad x(0,x) := x_0(x) .$$

(0.6, a)

For equation (0.3) above, the Strichartz estimate reads, see [STR],

$$\|z\|_{L^{10/3}(\mathbb{R}^2 \times \mathbb{R})} \leq C_1 \|f\|_{L^{12/7}(\mathbb{R}^2 \times \mathbb{R})} + C_2 \|x_0\|_{L^6(\mathbb{R})} .$$

(0.6, b)

The standard energy-type estimate reads,

$$\sup_t \|z(t,.)\|_{L^2} \leq \int dt \|f(t,.)\|_{L^2} + \|x_0\|_{L^2} .$$

(0.6, c)

Let me differentiate equation (0.1) and call

$$\psi := \partial_x .$$

(0.7, a)

The equation that \(\psi\) satisfies is,

$$i\psi_t - \Delta \psi + |z|^4 \psi + 2z |z|^2 (\bar{\psi} + z\psi) = 0 .$$

(0.7, b)

Applying Strichartz's estimate, see (0.6,b) above, and Hölder's inequality I have,

$$\|\psi\|_{L^{10/3}} \leq C \|z\|_{L^{10/3}} \|\psi\|_{L^{10/3}} + E^\psi \quad \text{where,} \quad E^\psi := \int_{\mathbb{R}^2} \|\nabla z\|^2 dx .$$

(0.7, c)

On the other hand let me call,

$$w = z^3 .$$

(0.8, a)

The equation that \(w\) satisfies is

$$iw_t - \Delta w = -3|w|^4 w + 6z (\nabla z)^2 .$$

(0.8, b)

Applying again Strichartz's inequality and Hölder, I obtain

$$\|w\|_{L^{10/3}} \leq C \|w\|_{L^{10/3}} \|\psi\|_{L^{10/3}}^2 + E^w \quad \text{where,} \quad E^w := \int_{\mathbb{R}^2} |z|^4 dx .$$

(0.8, c)

Let me call, for simplicity,

$$N := \|\psi\|_{L^{10/3}} ; \quad M := \|z\|_{L^{10}} .$$

(0.9)

Equations (0.7,c) and (0.8,c) can be written as a system of inequalities

$$N \leq C_1 N M^4 + E^\psi$$

(0.10, a)

$$M^3 \leq C_2 M^7 + C_3 N M^2 + E^w .$$

(0.10, b)
From (0.10,a,b) one can show that small energy implies that the solution is regular, moreover one can see that the problem is well posed for sufficiently small time. The solution can be constructed by an iterative process which will converge if a certain quantity is small. The relevant quantity is the following. Consider the linear equations

\[ i\partial_t \Delta z = 0 \quad z(0, x) := z_0(x) \quad (0.11, a) \]
\[ i\partial_t \Delta w = 0 \quad w(0, x) := w_0(x) \quad (0.11, b) \]

and call

\[ D[0, T] := R^3 \times [0, T] \quad (0.11, c) \]

The interval of existence can be estimated by the requirement that

\[ \|\nabla z\|_{L^{16/5}(D[0, T])} \leq E_0 \quad \text{and} \quad \|w\|_{L^{16/5}(D[0, T])} \leq E_0 \quad (0.11, d) \]

where \( E_0 \) is some universal constant. The argument above implies that equation (0.1) is locally well posed, where the interval of existence is determined by (0.11,d) for \( E_0 \) a universal constant. In general, it is desirable to estimate \( T \) in terms of \( z_0(x) \), assuming only that \( z_0 \in H^1(R^3) \).

Finally differentiating the equation (0.1) and applying either the Strichartz or the energy estimate one can improve the regularity and show that the solution of (0.1) is \( C^\infty \). Existence and regularity for subcritical equations of the type (0.1) was studied in [GV1,2], see also [CZ]. Blow up for repulsive potentials was demonstrated in [GL]. Various aspects of regularity and scattering were considered in [K], [S], [LS], [W] and more recently in [B1,2]. [B] has different proof for the same equation.

Chapter 1 : Main argument.

Equation (0.1) has, local in time, smooth solutions, thus I can assume that \( z(t, x) \) satisfies

\[ i\partial_t \Delta z + |z|^4 z = 0 \quad ; \quad (t, x) \in R^3 \times [0, T) \quad (1.1, a) \]
\[ z(0, x) := z_0(x) \quad . \quad (1.1, b) \]

The time \( t = T \), where \( T > 0 \), is the maximal time of existence. I want to demonstrate the following result.

**Theorem 1.1 :** There exists a finite constant \( K \) such that

\[ \sup_{R^3 \times [0, T]} |z(t, x)| \leq K \quad . \quad (1.1, c) \]
Once the apriori bound in Theorem 1.1 is known, further regularity can be obtained in the manner explained in the introduction. The first step is to write the solution of (1.1, a, b) in integral form. The Green's function for the Schrödinger equation is

$$S(t, x) := \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp \left[-\frac{|x|^2}{4t}\right].$$  \hfill (1.2, a)

For convenience let me call the nonlinear term by $N[z]$ and the phase factor in $S$ by $\Phi$ i.e.

$$N[z](t, x) := |z(t, x)|^4 z(t, x) ; \quad \Phi(t, x) := \exp \left[-\frac{|x|^2}{4t}\right].$$ \hfill (1.2, b)

The solution of (1.1, a, b) can be written implicitly in integral form as follows,

$$z(t_0, x_0) = \int_{R^d} S(t_0, x_0 - x)z_0(x) \, dx + \int_{R^d \times [0, t]} S(t_0 - t, x_0 - x)N[z](t, x) \, dxdt.$$ \hfill (1.2, c)

Let me give a name to the two terms in the expression above,

$$U(t_0, x_0) := \int_{R^d} S(t_0, x_0 - x)z_0(x) \, dx \quad (1.2, d)$$

$$F(t_0, x_0) := \int_{R^d \times [0, t]} S(t_0 - t, x_0 - x)N[z](t, x) \, dxdt.$$ \hfill (1.2, e)

The key ingredients in the argument are, first the conservation of energy i.e.

$$E(t) := \int_{R^d \times t} \left\{ |\nabla z(t, x)|^2 + \frac{1}{3} |z(t, x)|^4 \right\} \, dx := E.$$ \hfill (1.3, a)

The second ingredient is a Hardy-type inequality, namely

$$\sup_{x_0 \in R^d} \int_{R^d \times t} \frac{|z(t, x)|^2}{|x - x_0|^2} \, dx \leq CE.$$ \hfill (1.3, b)

The third ingredient is the identity,

$$\int_{R^d \times t} \frac{|z(t, x)|^2}{|x|^2} \nabla_x \left\{ \exp \left[-\frac{|x|^2}{4t}\right] \right\} = \exp \left[-\frac{|x|^2}{4t}\right].$$ \hfill (1.3, c)

For a fixed point $(t_0, x_0)$ and some $t_1$ such that $t_1 < t_0$, let me define the functions

$$r := |x - x_0|, \quad u_0 := \nabla_r \frac{x^2 - x_0^2}{r} \quad (1.4, a)$$

$$R(t_0 - t) := (t_0 - t)^{\frac{1}{1+2\delta}} (t_0 - t)^{\delta}, \quad s := \frac{r}{R(t_0 - t)} \quad (1.4, b)$$

where $0 < |a| < 1/4$. I would like to separate the inner and outer regions of integration, for this reason I want to introduce a smooth cut-off function,

$$\psi(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq 1/2 \\ 0 & \text{if } 1 \leq s < +\infty \end{cases} \text{ where } s = \frac{r}{R(t_0 - t)}.$$ \hfill (1.4, c)
Now I can use the cut off function $\psi(\epsilon s)$, where $\epsilon$ is a small parameter, to write
\[
F(t, x) = F_o(t, x) + F_i(t, x) ,
\]
where
\[
F_o(t, x) := \int_{R^3 \times [0, t_1]} (1 - \psi(\epsilon s)) N[z](t, x) S(t_0 - t, x_0 - x) \, dx \, dt \quad (1.4, e)
\]
\[
F_i(t, x) := \int_{R^3 \times [0, t_1]} \psi(\epsilon s) N[z](t, x) S(t_0 - t, x_0 - x) \, dx \, dt \quad (1.4, f)
\]
The term $F_o$ can be integrated by parts using (1.3, ~) to obtain,
\[
F_o(t_0, x_0) = - i \int_{R^3 \times [0, t_1]} \left[ a(\epsilon s) \frac{N[z](t, x)}{|x - x_0|} + b(\epsilon s) u^a \nabla \cdot N[z](t, x) \right] \Phi(t_0 - t, x_0 - x) \, dx \, dt \quad (1.4, g)
\]
where, recall that $u^a := (x^a - x_0^a)/|x - x_0|$ and
\[
a(s) := 2(1 - \psi(s)) - 2\psi'(s)s ; \quad b(s) := 1 - \psi(s) . \quad (1.4, h)
\]
Next I want to separate the domain of integration in the region near $t_0$ and the region far from $t_0$. Pick $t_1$ such that $0 < t_1 < t_0$ and a parameter $a$ such that $0 < a < 1/4$, for example take $a = 1/6$.

For $t_1 \leq t \leq t_0$ let $R_1(t_0 - t) := (t_0 - t)^{1/2} a (t_0 - t_1)^{a} \quad (1.5, a)$

For $0 \leq t \leq t_1$ let $R_2(t_0 - t) := (t_0 - t)^{1/2} a (t_0 - t_1)^{-a} \quad (1.5, b)$

Define the sets
\[
D_1 := \{ (t, x) : \epsilon r \leq R_1(t_0 - t) ; \quad t_1 \leq t \leq t_0 \} ; \quad C_1 := R^3 \times [t_1, t_0] \setminus D_1 \quad (1.5, c)
\]
\[
D_2 := \{ (t, x) : \epsilon r \leq R_2(t_0 - t) ; \quad 0 \leq t \leq t_1 \} ; \quad C_2 := R^3 \times [0, t_1] \setminus D_2 \quad (1.5, d)
\]
I will also make use of the sets
\[
R_1(x_0, R) := \{ (t, x) : |x - x_0| \leq R \} \quad (1.5, e)
\]
\[
R_1^2 := \{ (t, x) : x \in R^3 \} . \quad (1.5, f)
\]
Let me define the density function
\[
\rho(z(t, x)) := \frac{1}{2} |z(t, x)|^2 . \quad (1.5, a)
\]
Observe that for any $R > 0$ and $x_0 \in R^3$, I have
NONLINEAR SCHR"ODINGER EQUATIONS

\[ \frac{1}{R^2} \int_{B(t_0, R)} \rho \, dx \leq \left( \int_{B(t_0, R)} \rho^3 \, dx \right)^{\frac{1}{3}} := P(t) \quad \text{(1.6, b)} \]

I want to define an auxiliary quantity

\[ \mu_\epsilon(t_0, x_0; [a, b]) := \frac{\sqrt{t_0 - t_1}}{\epsilon^2} \int_a^b \frac{R_2^2(t_0 - t)}{(t_0 - t)^{\frac{3}{2}}} \left( \int_{B(t_0, R)} \rho \, dx \right) \, dt \quad \text{(1.6, c)} \]

which can be used to obtain the bound for the integral over \( D_1 \), see (1.4,f),

\[ \left| \int_{D_1} \text{integrand} \right| \leq \mu_\epsilon(t_0, x_0; t_1, t_0) \sqrt{t_0 - t_1} \left( \sup_{D_1} |\rho|^3 \right) \quad \text{(1.6, d)} \]

The quantity \( \mu_\epsilon \) can be bounded by the energy via (1.6,b). The integrand over \( D_2 \), see (1.4,f), can be split into two parts \( D_2 \cap [t_1, t_2] \times \mathbb{R}^3 \) and \( D_2 \cap [t_2, 0] \times \mathbb{R}^3 \) for some \( t_2 \) such that \( 0 \leq t_2 < t_1 \). Using the inequality

\[ \int_{B(t_0, R)} |x|^3 \, dx \leq \left( \int_{B(t_0, R)} |x|^6 \, dx \right)^{\frac{1}{3}} R^4 \quad \text{(1.6, e)} \]

we have the bound, see (1.6,b),

\[ \left| \int_{D_2} \text{integrand} \right| \leq \sup_{[t_1, t_2]} P(t) \frac{E}{\epsilon} \left( \frac{t_0 - t_1}{t_0 - t_2} \right)^{\frac{3}{2}} \quad \text{(1.6, f)} \]

The rest of the integrals, see (1.4,g), can be bounded using the fact that \( |x - x_0| \geq R(t_0 - t)/\epsilon \) as follows

\[ \left| \int_{C_1} \text{integrand} \right| \leq \epsilon E \left( \sup_{C_1} |z| \right) \quad \text{and} \quad \left| \int_{C_3} \text{integrand} \right| \leq \epsilon E \left( \sup_{C_3} |z| \right) \quad \text{(1.6, g)} \]

Combining all the above estimates I obtain an inequality of the form

\[ |z(t_0, x_0)| \leq \mu_\epsilon(t_0, x_0; [t_1, t_0]) \sqrt{t_0 - t_1} \left[ \sup_{D_1} |\rho|^3 \right] + \epsilon E \left( \sup_{C_1 \cup C_3} |z| \right) + \\
+ \frac{\sup_{[t_1, t_2]} P(t)}{\epsilon(t_0 - t_2)^{\frac{3}{2}}} + \frac{E}{\epsilon(t_0 - t_2)^{\frac{3}{2}}} \left( t_0 - t_2 \right)^{\frac{3}{2}} \quad \text{(1.7)} \]

At this point I will have to use the radial symmetry, the fact that \( x(t, x) \) is radially symmetric implies that

\[ |z(t_0, x_0)| \leq \frac{E}{|x_0|^\frac{3}{2}} \quad \text{(1.8, a)} \]
hence WLOG the domain $D_z$ can be assumed to contain the origin i.e. $\sqrt{t_1 - t_0} > |x_0|$, otherwise inequality (1.7) is trivially true. In conclusion, I can assume that

$$\sup_{\mathcal{O}_z} |x| \leq \frac{E}{(t_0 - t_1)^{\frac{1}{4}}}.$$  

(1.8, b)

Let me define

$$D[t_1, t_0] := \mathbb{R}^3 \times [t_1, t_0] ; \quad L[t_1, t_0] := \sup_{D[t_1, t_0]} |x| ; \quad M(t) := \sup_{\mathbb{R}^3} |x|.$$  

(1.9)

Theorem 2.1 and Corrolary 2.1, from the next chapter, provide the following local energy estimate. Call

$$e(t) := \int_{B_1(0, \sqrt{T-t_1})} \left( |\nabla x|^2 + |x|^2 \right) dx,$$  

(1.10, a)

then

$$\frac{1}{T - t_1} \int_{t_1}^{T} e(t) \, dt = o(T - t_1) \quad \text{where} \quad \lim_{T \to T^{-}} o(T - t) = 0.$$  

(1.10, b)

Let me define the sets, for some $\eta$ small fixed,

$$J := \{ t ; \quad e(t) \geq \eta^2 \} ; \quad I := [0, T] \setminus J.$$  

(1.11, a)

then from (1.10,b) we have

$$|J \cap [t, T]| \sim T - t \quad \text{and} \quad |J \cap [t, T]| = o(T - t)(T - t).$$  

(1.11, b)

The sets $I, J$ can be written as unions of intervals i.e.

$$I = \bigcup_{k=1}^{\infty} [a_k, b_k] \quad \text{call} \quad I_k := [a_k, b_k]$$  

(1.11, c)

$$J = \bigcup_{k=1}^{\infty} [b_k, a_{k+1}] \quad \text{call} \quad J_k := [b_k, a_{k+1}].$$  

(1.11, d)

The idea in (1.11,a) is to consider $I$ to be the times that the energy inside a ball of radius $\sqrt{T - t}$ is small and $J$ the complement where the energy might be large. The estimate in (1.11,b) implies that $|J| = o(T - t)|I|$. The first step is to show that $M(t)$ can be estimated in the intervals $I_k$.

**Lemma 1.1:** With the notation introduced in (1.9), the following estimate holds

$$L \left[ \frac{a_k + b_k}{2}, b_k \right] \leq \frac{C(E)}{|I_k|^\frac{1}{4}} ; \quad C(E) \sim C E \frac{1}{4}. \quad (1.12)$$
Proof: The proof is by contradiction. Let \( t_0 \) be such that \( M(t_0) = L[(a_k + b_k)/2, b_k] \), then \((a_k + b_k)/2 \leq t_0 \leq b_k\). Call \( I_k := [a_k, b_k] \) and assume that
\[ M(t_0) |I_k| \gg E^{\beta}. \tag{1.13,a} \]

Choose \( t_1 \) such that \( \Delta t := t_0 - t_1 = M^{-1/4}(t_0) \) and \( t_2 = a_k \). Because of (1.13,a) we have that \( t_1 < a_k := t_2 \). Let us write \( M := M(t_0) \) and \( L := L[t_0, t_1] \) for simplicity. Using (1.11,a), I can estimate
\[ \mu_i(t_0, x_0; t_1, t_0) \leq \frac{n^3}{\epsilon^2} \tag{1.13,b} \]
and (1.7) can be written as
\[ M \leq \left( \frac{n^3 L^3}{\epsilon^2 M^{3\beta}} + \epsilon E \frac{L}{M} + \frac{n^3}{\epsilon^2} \right) M + \frac{E^{1/3} M^{2\gamma}}{\epsilon^{1/3} |I_k|^{(1-2\alpha)/4}}. \tag{1.13,c} \]

Choose \( \epsilon << 1/E \), for example set,
\[ \epsilon(E) := \frac{\epsilon_0}{E}; \quad \epsilon_0 = \text{small}. \tag{1.13,d} \]

If we assume that
\[ \frac{L}{M} \leq 10 \tag{1.13,e} \]
we obtain from (1.13,c) that, for \( \eta(E) \sim \eta_0 \epsilon_0 / E \) and \( \eta_0 \) sufficiently small, \( M \leq C(E) / |I_k|^{\beta/4} \)
contradicting (1.13,a). Hence, there exists some \( \tilde{t} \) such that \( t_1 \leq \tilde{t} < t_0 \) and
\[ M(\tilde{t}) > 10 M(t_0) \quad \text{with} \quad |t_0 - \tilde{t}| \leq \frac{C(E)}{M^{\beta/4}(t_0)}. \tag{1.13,f} \]

Now let me call \( \tilde{t} := t_1 \) and repeat the argument in order to find \( t_2 \) etc. such that eventually we construct a sequence \( \{t_n\} \) with
\[ M(t_{n+1}) > 10 M(t_n) \quad \text{with} \quad |t_{n+1} - t_n| \leq \frac{1}{10} |t_n - t_{n-1}|. \tag{1.13,g} \]

This leads to a contradiction with (1.13,a) since \( \lim_{n \to \infty} t_n \leq a_k \) and \( \lim_{n \to \infty} M(t_n) = \infty \), and completes the proof of Lemma 1.1.

The next step is to show that if \( |I_k| \) is much smaller than \( |I_k| \), then \( \sup_{I_k} M(t) \sim \sup_{I_k} M(t) \).

Lemma 1.2 Assume that
\[ |J_k| \leq \delta |I_k|; \quad \delta := \frac{\delta_0}{E^{\beta/4}} \tag{1.14,a} \]
and let me choose
\[ \epsilon := \frac{\epsilon_0}{E} \quad \text{and} \quad \eta^2 := \frac{n^3 \epsilon_0^3}{E^{1/4}} \tag{1.14,b} \]
where $\epsilon_0$, $\eta_0$ and $\delta_0$ are small and independent of $E$. Under the assumption (1.14,a) and the choice (1.14,b) the following holds true

$$L[b_k, a_{k+1}] \leq \frac{C(E)}{|I_k|^4} ; \quad C(E) \sim CE^\frac{1}{4} . \quad (1.14,c)$$

**Proof of Lemma 1.2:** Because of Lemma 1.1 I have that $M(b_k) \leq C(E)/|I_k|^{1/4}$. Assume that there exists some $t_0$ such that $b_k < t_0 \leq a_{k+1}$ and

$$L \left[ \frac{b_k + b_k}{2}, t_0 \right] = M(t_0) \leq \frac{C(E)}{t_0 - b_k} . \quad (1.15,a)$$

otherwise Lemma 1.2 holds. Choose $t_1$ variable such that $t_1 < b_k \leq t_0 \leq a_{k+1}$ From (1.6,c) and (1.11,a) I can bound $\mu_\epsilon$ as follows

$$\mu_\epsilon \{ t_0, x_0; t_1, t_0 \} \leq d \frac{E}{\epsilon_2^2} \quad \text{where} \quad d := \left[ \frac{t_0 - b_k}{t_0 - t_1} \right]^{(1-a_0)/2} \quad (1.15,b)$$

while

$$\mu \{ t_0, x_0; t_1, b_k \} \leq \frac{\eta_0^2}{\epsilon_2^2} . \quad (1.15,c)$$

Then for any $t_1$ such that $(a_k + b_k)/2 \leq t_1 \leq t_0$ with the notation $\Delta t = t_0 - t_1$ and the choice $t_2 = 0$, inequality (1.7) becomes

$$M(t_0) \leq d \frac{E + \eta_0^2}{\epsilon_2^2} \sqrt{\Delta t} M^{1/3}(t_0) + \epsilon E M(t_0) + \frac{E^{5/6}}{\sqrt{\Delta t}^{1/2}} . \quad (1.15,d)$$

With the choice of $\epsilon$ as in (1.14,b), inequality (1.15,d) implies that

$$M(t_0) \leq C \frac{E^{5/3}}{(\Delta t)^{1/4}} \quad \text{provided that} \quad (dE + \eta_0^2) < \frac{\epsilon_2^2}{E^{17/3}} . \quad (1.15,e)$$

In view of (1.14,b) for the choice of $\eta$, the first $t_1$ such that (1.15,e) holds is when

$$d \sim \frac{\epsilon_2^2}{E^{17/3}} . \quad (1.15,f)$$

Choose $a := 1/6$ in (1.15,b) so that (1.15,f) reads

$$\frac{t_0 - b_k}{t_0 - t_1} \sim \frac{\epsilon_2^6}{E^{18}} \quad \text{hence} \quad M(t_0) \leq \left( \frac{\epsilon_2^6}{E^{18}} \right)^{1/2} \frac{C(E)}{t_0 - b_k} . \quad (1.15,g)$$

The inequality for $M(t_0)$ is satisfied, because of (1.15,a), if we choose $\delta_0 \sim \epsilon_2^6$. Next let $t_1$ decrease until $t_1 = (a_k + b_k)/2$. The restriction on $\delta$, $\eta$ and $\epsilon$ imposed by (1.15,e) is always satisfied and eventually we obtain from (1.15,e) that $M(t_0) \leq C(E)/|I_k|^{1/4}$. This implies
Lemma 1.2 since in this way we can exaust the interval $J_k$.

**Proof of Theorem 1.1**: From Lemma 1.2, one can conclude that as long as

$$|J_k| \leq \delta \sum_{k=k_0}^l |I_k| \quad (1.16, a)$$

we have that

$$L[\alpha_k, \alpha_{k+1}] \leq C \left( \sum_{k=k_0}^l |I_k| \right)^{-1/4} \quad (1.16, b)$$

The only way to violate this is, if for every $k$ integer there exists $l(k) > k$ such that

$$|J_{l(k)}| \geq \delta \sum_{m=k}^{m=l(k)} |I_m| \quad (1.16, c)$$

but (1.16,c) contradicts (1.11,b). Hence $M(t)$ has to stay bounded and this concludes the proof of Theorem 1.1.

**Chapter 2: Apriori Energy estimates.**

I would like to consider a nonlinear Schrödinger equation in an arbitrary space dimension greater or equal than three. The unknown is a smooth complex function $z(t, x)$ satisfying a Schrödinger equation with nonlinear potential $F(|z|^2)$.

$$z(t, x) : [0, T) \times \mathbb{R}^n \mapsto \mathbb{C} ; \quad n \geq 3$$

$$iz_t - \Delta z + F'(|z|^2)z = 0 \quad (2.1, a)$$

I will write $x = (x^\alpha)$ with $\alpha = 1, 2, \ldots, n$ to denote the space variables. The solution exists for $(t, x) \in [0, T) \times \mathbb{R}^n$ and $t = T$ is the final time of existence.

The idea is to view equation (2.1) as the evolution equation of a fluid with density $\rho$ and momenta $p_\alpha$. Let me start by defining the following relevant quantities

$$\rho[z(t, x)] := |z|^2/2 \quad (2.2, a)$$

$$p_\alpha[z(t, x)] := \frac{i}{2} \{ \bar{z}_\alpha \bar{z} - z \bar{z}_\alpha \} , \quad \alpha = 1, 2, \ldots, n \quad (2.2, b)$$

$$p_\alpha[z(t, x)] := \frac{i}{2} \left( z_\alpha \bar{z} - \bar{z}_\alpha \bar{z} \right) \quad (2.2, c)$$

$$\sigma_\alpha[z(t, x)] := \{ z_\alpha \bar{z} + \bar{z}_\alpha \bar{z} \} , \quad \alpha = 1, 2, \ldots, n \quad (2.2, d)$$

$$\sigma[z(t, x)] := \sigma_\alpha \alpha^\alpha \quad (2.2, e)$$

$$\sigma_\alpha[z(t, x)] := \{ z_\alpha \bar{z} + \bar{z}_\alpha \bar{z} \} , \quad \alpha = 1, 2, \ldots, n \quad (2.2, f)$$

$$\lambda[z(t, x)] := p_0 + \frac{\sigma}{2} + F(2\rho) \quad (2.2, g)$$

Equation (2.1,b) is the evolution equation derived from the Lagrangian.
\[ \mathcal{L}(\tau, \tilde{x}) := \int_{\mathbb{R}^n \times \mathbb{R}} \lambda(\tau) \, dx \, dt \quad . \tag{2.3} \]

The conservation laws for the mass, momenta and energy can be expressed as follows:

\[
\begin{align*}
\partial_t \{ \rho \} + \nabla \cdot \{ \rho \, \mathbf{v} \} &= 0 \quad \text{(2.4a)} \\
\partial_t \{ \rho \mathbf{v} \} + \nabla \cdot \{ \rho \mathbf{v} \otimes \mathbf{v} - \rho \mathbf{g} \} &= 0 \quad \mathbf{a} = \{1, 2, \ldots, n\} \quad \text{(2.4b)} \\
\partial_t \left\{ \frac{1}{2} \rho \mathbf{v}^2 + F(2\rho) \right\} - \nabla \cdot \{ \rho \mathbf{v} \mathbf{v} \} &= 0 \quad . \quad \text{(2.4c)}
\end{align*}
\]

Notice that \( \sigma_{\mathbf{a}} \) can be written as:

\[
\sigma_{\mathbf{a}} = -2 \{ \nu_{\mathbf{a}} + F'(2\rho) \rho_{\mathbf{a}} \} \quad \text{where} \quad \nu_{\mathbf{a}}[\tau] := \frac{1}{2} \{ \mathbf{v}_{\tau} \mathbf{v}_{\tau} - \mathbf{v}_{\tau} \mathbf{v}_{\tau} \} . \quad \text{(2.4d)}
\]

Integrating equation (2.4c) gives the energy bound:

\[
\int_{\mathbb{R}^n} \left\{ \frac{1}{2} \rho \mathbf{v}^2 + F(2\rho) \right\} \, dx := E = \text{constant} \quad . \quad \text{(2.5a)}
\]

Where \( \mathbb{R}^n = \{ (\tau, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^n \} \) is the time-slice at time \( \tau \). I will use the well-known Hardy-type inequality:

\[
\sup_{\mathbf{x}_0 \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho}{|\mathbf{x} - \mathbf{x}_0|^2} \, dx \leq E \quad . \quad \text{(2.5b)}
\]

The following constitutive equation relates the Lagrangian density \( \lambda \) with the density \( \rho \), defined in (2.2,a) respectively:

\[
\lambda = \Delta \rho - f(\rho) ; \quad f(\rho) := F'(2\rho)\rho - F(2\rho) \quad . \quad \text{(2.6a)}
\]

Equations (2.4a,b) together with the constitutive equation (2.6a) are equivalent to (2.1,b). The problem is that one cannot localize the energy, see equation (2.4c), since we do not know how to estimate \( \sigma_{\mathbf{a}} \). It is my goal to estimate the time average of the energy density over a parabolic cylinder, employing only equations (2.4a,b) and (2.6a).

**Assumption**: I will assume in what follows that:

\[
f(\rho) := F'(2\rho)\rho - F(2\rho) \geq 0 \quad . \quad \text{(2.6b)}
\]

I would like to introduce first the technical tools needed for the intended estimates. Consider a cut-off function \( \psi \) such that:

\[
\psi(\tau) := \begin{cases} 1 & \text{if } 0 \leq \tau \leq 1 \\ 0 & \text{if } 2 \leq \tau < +\infty \end{cases} \quad . \quad \text{(2.7a)}
\]

and let me use the notation,
Also, I will use the sets defined below,

\[ R^n_t := \{ t \} \times R^n \]  

\[ B_t(x_0, R) := \{ (t, x) : |x - x_0| \leq R \} \]  

\[ D(x_0) := \{ T - x_0, T \} \times R^n \]  

Multiplying (2.4,a) with \( \psi^2_r(|x - x_0|) \) I have the identity

\[
\frac{\partial_t}{2} \left\{ \psi^2_r(|x - x_0|) \rho \right\} + \nabla_x \left\{ \psi^2_r(|x - x_0|) p^\alpha \right\} = 2 \left\{ \frac{\partial}{\partial x} \psi^2_r(|x - x_0|) \right\} \psi^\rho_r(|x - x_0|)u^\alpha(x - x_0)p_\alpha .
\]  

Observe that, since \( z(t, x) \) is a complex function, I can write,

\[ z = \sqrt{2p} e^{i\theta} \]  

\[ p_\alpha = -2\rho_{\alpha} \]  

Integrating over \( D(x_0) \) and using (2.5,a,b) I obtain the bound

\[
\lim_{T \to t} \int_{R^n_t} \psi^2_r(|x - x_0|) \rho \, dx - \int_{R^n_T-x_0} \psi^2_r(|x - x_0|) \rho \, dx \leq C \rho_0 E .
\]  

Because of (2.11) the limit in (2.12,c) is unique, hence there exists \( \rho(T, x) \) and (2.11) can be written

\[
\int_{R^n_T-x_0} \psi^2_r(|x - x_0|) \rho \, dx \leq C \rho_0 E .
\]  

In the present chapter, I would like to demonstrate the following theorem.

**Theorem 2.1:** Fix a point in \( x_0 \in R^n \), and let \( \psi(r) \) be a smooth cut-off function defined in (2.7,a). Define the quantities

\[
\psi^\rho_r(|x - x_0|) \rho \, dx
\]

\[
\Phi^\rho(|x|) := \delta^\rho - u^\rho(x) u^\rho(x) \]  

\[
u^\rho(|x|) := \frac{2^\rho}{|x|}
\]
\[ \Lambda(x_0; t, x) := \frac{1}{|x - x_0|^2} \left[ \frac{(\Pi^{22}(x - x_0)\sigma_{a,b} + f(\rho))}{|x - x_0|^2} + \delta(n - 3)\frac{\rho}{|x - x_0|^2} \right] + \delta(x - x_0)\rho, \] 

where \( \delta(\cdot) \) denotes the delta function. With the notation in (2.14, a,b,c), I have the estimates

\[ \frac{1}{\sqrt{s_0}} \int_{\mathcal{C}(s_0)} \psi(x_0) \Lambda(x_0; t, x) \, dx \, dt \leq C(E)Q((T, x_0); r_0), \] 

where \( s_0 \) and \( r_0 \) are related by

\[ \frac{s_0}{r_0} := Q^{1/2}(T, x_0; r_0). \] 

If there exists an \( R > 0 \) such that \( Q((T, x_0); R) = 0 \) then in the right hand side of (2.15, a,b) the quantity \( Q((T, x_0); r_0) \) should be replaced by \( C(E)(s_0/R^4) \) with \( r_0 \sim R/2 \).

A corollary of the above theorem is the following.

**Corollary 2.1:** For a fixed point \( (T, x_0) \) consider the parabolic cylinder of height \( s_0 \) and radius \( \sqrt{s_0} \),

\[ C((T, x_0); s_0) := \{(t, x) : |x - x_0| \leq \sqrt{s_0}; \quad T - s_0 \leq t \leq T\}. \] 

The following estimates hold,

\[ \frac{1}{s_0} \int_{C((T, x_0); s_0)} [\rho + f(\rho)] \, dx \, dt \leq C(E)Q^{1/2}(T, x_0; r_0), \] 

where again

\[ \frac{s_0}{r_0} := Q^{1/2}(T, x_0; r_0). \] 

Theorem 2.1 estimates only the angular derivatives with respect to \( x_0 \), however by considering three distinct points \( x_0, x_1, x_2 \) not on the same line and distance \( \sqrt{s_0} \) apart one can estimate the full derivative in the manner stated in the corollary. Notice that (2.16, b) follows directly from (2.15, a) and (2.15, c).

I would like to multiply (2.4, a) with the following function

\[ \Phi := \frac{1}{s_0} \phi \left( \frac{|x - x_0|}{s_0}, \frac{T - t}{s_0} \right) \] 

where \( \phi(r, t) := \frac{\psi^2(r)}{t + 1} \). 

The resulting identity is
\[ \partial_t \{ \Phi \rho \} + \nabla_x \{ \Phi \rho^2 \} = G \rho + 2H^\alpha \rho_0 , \]  
(2.18)

where \( G \) and \( H^\alpha \) are given by

\[ G := \frac{1}{\mu_0} \begin{pmatrix} |x-x_0| & T-t \end{pmatrix} \begin{pmatrix} 0 \ 0 \end{pmatrix} \] where \( g(r,t) := \frac{\psi^2(r)}{(t+1)^2} \) \( (2.19, a) \)

\[ H^\alpha := \frac{1}{\mu_0 \alpha} \begin{pmatrix} |x-x_0| & T-t \end{pmatrix} \begin{pmatrix} 0 \ 0 \end{pmatrix} \] where \( u^\alpha(x-x_0) \) \( (2.19, b) \)

\[ h(r,t) := \frac{\psi(r) \psi'(r)}{t+1} \] ; \( u^\alpha(x) := \frac{x^\alpha}{|x|} \) \( (2.19, c) \)

Contracting \((2.4, b)\) with a vector field \( X^\alpha(x) := \Phi^\alpha(x) \)

I have the identity

\[ \partial_t \{ P \} + \nabla \{ M^\rho \} = R \] , \( (2.20) \)

where

\[ P := \Phi^\alpha \rho_0 \] \( (2.21, a) \)

\[ M^\rho := \Phi^\alpha \rho_0^\beta - \Phi^\beta \lambda + (\Delta \Phi) \nabla^\rho \rho - (\Delta \Phi)^\beta \rho \] \( (2.21, b) \)

\[ R := \Phi_{\alpha \beta \rho} \rho_0^\alpha + (\Delta \Phi) \rho - (\Phi^\beta) f(\rho) \] \( (2.21, c) \)

and I used the fact that \( \lambda = \Delta \rho - f(\rho) \), see \((2.6)\). A simple choice for \( \Phi(x) \) is

\[ \Psi(x) := \psi^2_\alpha (|x-x_0|) |x-x_0| \] \( (2.22, a) \)

In order to simplify the presentation, let me call

\[ r := |x-x_0| \] ; \( u^\alpha(x-x_0) := \nabla^\alpha r \) \( (2.22, b) \)

Now it is straightforward to compute

\[ \Psi_{\alpha \beta} = \frac{\psi^2_\alpha (r)}{r} (\delta_{\alpha \beta} - u_\alpha u_\beta) + 2u_\alpha \nabla \psi_\alpha^2 (r) + r \nabla_\alpha \psi_\beta^2 (r) \] \( (2.23, a) \)

\[ \Delta \Psi = (n-1) \psi^2_\alpha (r) + r \Delta \psi^2 (r) + 2u_\alpha \nabla_\alpha \psi^2 (r) \] \( (2.23, b) \)

\[ \Delta^2 \Psi = -\frac{(n-1)(n-3)}{3} \psi^2_\alpha (r) + r \Delta^2 \psi_\alpha^2 (r) + 4u_\alpha \left( \nabla_\alpha \Delta \psi_\alpha^2 (r) - \frac{n-1}{r} \nabla_\alpha \psi^2_\alpha (r) \right) \]
\[ \quad + 2 \frac{n+2}{2} \Delta \psi^2 (r) - \frac{4}{r} u_\alpha u_\beta \nabla_\alpha \psi_\beta^2 (r) \] \( (2.23, c) \)

\[ \nabla_\rho \Delta \psi = -\psi^2_\alpha (r) (n-1) \frac{u_\alpha}{r^2} + \frac{n-1}{r} \nabla_\rho \psi_\alpha^2 (r) + \nabla_\rho (r \Delta \psi_\alpha^2 (r) + 2u_\alpha \nabla_\alpha \psi_\beta^2 (r)) \] \( (2.23, d) \)

Equations \((2.23)\) can be written

\[ \Psi_{\alpha \beta} = \frac{\psi^2_\alpha (r)}{r} (\delta_{\alpha \beta} - u_\alpha u_\beta) + \frac{1}{r_0} \times \text{bounded terms} \] \( (2.24, a) \)
(i) periodic boundary conditions on a rectangle $R = [0, \omega_1] \times [0, \omega_2]$

or

(ii) anti-periodic boundary conditions on one pair of sides of $R$ and we may impose $\partial u/\partial n = 0 = \partial \Delta u/\partial n$ on the remaining sides of $R$.

With these choices the set of equilibria is certainly at least one-dimensional.

For, if $u$ is a steady state, then a shift $u(\cdot + \tau, \cdot)$ yields another steady state.

Two reasons motivated us to exploit boundary data of the above type:

(a) Micro-graphs of real microstructure look periodic at least away from grain boundary or boundaries of incompatible phases (see e.g. [27]).

(b) The minimizers of the one-dimensional energy functional

$$\int_0^1 [W(u_x) + \frac{1}{2} \delta^2 u_{xx}^2] \, dx$$

are periodic, moreover the period is of order $\delta^{1/3}$ (see [21]) for small values of $\delta$.

We stress that $W$ is a double well potential here. Our prototype $W$ is smoothed out version of

$$\tilde{W}(\xi) = \min\{(|\xi - a|^2, |\xi + a|^2) \quad a \neq 0.$$

We have in mind the following examples

$$W_1(\xi) = \frac{1}{4} (\xi_1^2 - 1)^2 + d \xi_2^2 + \frac{1}{4} \xi_2^4, \quad d \geq 0 \quad (1.2)$$

$$W_2(\xi) = \frac{1}{4} (\xi_1^2 + \xi_2^2 - 1)^2. \quad (1.3)$$

The feature of $W_1$ is that it has just two minima, while $W_2$ has a circle of minima. Our analysis will be applicable to both of these examples but not for the same set of boundary conditions. We collect our assumption on $W$ as well as on the boundary conditions in Section 2 below.

One of our results presented here is existence of solutions to (1.1) for $\rho > 0$ (the case $\rho = 0$ was treated in [26]). We do this in Section 3. We present details because of our special needs. Namely, we have to show high regularity as well as a priori estimates guaranteeing existence of the $\omega$-limit set. We use the standard tool of theory of analytic semigroups as exposed in [17] (with the changes made in the Russian translation). We assume little smoothness of initial data, namely $u(0, x) = u_0 \in W^{2,2}(\Omega), \ u_t(0, x) = u_1 \in L^2(\Omega)$. We can do so because it turns out that the equation smoothes out the data
to some extent permitted by the nonlinear term. Our basic method is a modified version of a transformation due to Andrews-Ball and Pego (see [1] and [23]) which was successfully applied not only to one-dimensional equation of viscoelasticity and its modifications (see [4], [11], [12], [18], [19], [23]) but also to multidimensional system of viscoelasticity, (see [25], [29]). In our problem global in time existence is obtained due to the energy identity

\[ \frac{1}{2} \int_{\Omega} u_t^2 \, dx + E(u(t)) + \int_0^t \int_{\Omega} |\nabla u_t|^2 \, dx \, dt = E(u_0) + \frac{1}{2} \int_{\Omega} u_t^2 \, dx, \quad (1.4) \]

where

\[ E(u) = \int_{\Omega} [W(\nabla u) + \frac{1}{2} \delta^2 |\Delta u|^2] \, dx \quad (1.5) \]

The identity (1.4) is also a source of further a priori estimates, which lead to existence of the \( \omega \)-limit set in the \( C^4(\Omega) \) topology. They are proved by a patient boot-strapping argument. They are collected in Theorem 3.5 of Section 3.2.

Our main result, however, is Theorem 4.1 stating the convergence of each solution to an equilibrium point as \( t \) goes to infinity. The main hypothesis is that \( W \) is analytic. We also impose on \( W \) some non-degeneracy conditions. We require that \( u_0 \) belong to some special neighborhood of a connected component of the set of local minimizers of \( E \). The neighborhood in question is defined in terms of \( E \) alone. We also need that the kinetic energy \( \frac{1}{2} \int_{\Omega} u_t^2 \, dx \) is small. The analyticity assumptions is used to guarantee that critical values of \( E \) do not have an accumulation point. In this way we circumvent one of the difficulties here which is the lack of knowledge of the structure of the equilibria of (1.1). This problem seems to be much less investigated than its counterpart for the Cahn-Hilliard equation, see [7], [8], [33], [34], [35].

It seems rather certain that the set of equilibria is large. This is true for periodic boundary conditions. Let us note that boundary condition (i) (respectively (ii)) imply that the set \( \mathcal{M} \) of equilibria of (1.1) is at least two-dimensional (respectively, one-dimensional), because shifts of \( u \) in \( x_1 \) or \( x_2 \) (respectively \( x_1 \)) yield new equilibria. Thus, if we fix \( u \in \mathcal{M} \) and set \( Lh = \text{div}(D\sigma(\nabla u)\nabla h) + \delta^2 \Delta^2 h \), then we note that

\[ \dim \ker L \geq \dim \mathcal{M}, \]

But we do not know if the equality holds. If we knew that \( \dim \ker L = \dim \mathcal{M} \), then we could apply the theory of Hale-Raugel [16] to deduce convergence. However, we do not know this. What we do here is we circumvent this
problem. We analyze the behavior of the flow on the local center manifold $W^c(u)$, where $u$ is a local minimizer of $E$. Section 4.2 is devoted to establishing existence of $W^c(u)$. The assumption that $u$ is a minimizer permits us to deduce that it is stable with respect to the flow restricted to the local center manifold $W^c(u)$. The idea that it is sufficient to study behavior of the flow on $W^c(u)$ has been present in the theory of dynamical systems for some time, e.g. we refer to [5]. Our proof of stability of the restricted flow requires some non-degeneracy condition on $W$, meaning $D^4W$ must be positive. The analysis is performed in Section 4.3. It is facilitated by a special scaling of the flow which is explained in Section 4.1. The proof of our convergence result is carried out in Section 4.4.

We present our analysis for $n = 2$. This restriction comes from the embedding theorem which guarantees that for two space dimensions we have $W^{1,2}(\Omega) \subset L^p(\Omega)$, for all $p < \infty$. This makes the energy finite for any polynomial growth of $W$ provided that $u \in W^{2,2}(\Omega)$. One may wonder, if for higher space dimensions, working with $L^p(\Omega)$, where $p = n$, is an option. However, this is not the case since the available energy estimates yield boundedness of solutions only in $W^{2,2}(\Omega)$. This is not sufficient to prove global-in-time existence of solutions for higher space dimensions and $W$ of arbitrary polynomial growth.

Let us finally comment on convergence results for closely related problems. Most of them are one-dimensional.

(a) The case $\rho, \nu > 0, \delta = 0$ i.e. equation of viscoelasticity, has been extensively studied. Convergence of strain was established in one dimension by Pego [23], later his results were refined by Friesecke and McLeod [11], [12]. In the multi-dimensional case the numerical simulations of [29], [20] and [14] suggest that convergence may hold, however we are not aware of any analytical result. The difficulty comes from the fact that the equation

$$\text{div}(\sigma(\nabla u)) = 0$$

tends to have a big set of solutions for interesting boundary conditions (see e.g. [10]).

(b) The case $\rho = 0, \delta = 0, \nu > 0$, i.e. the equation of quasi-steady approximation to (1.1) is better understood. If the number of the space dimensions is one, then this problem is a diffused in space reaction-diffusion equation for $u_\nu$, for which we have convergence results due to Chen-Matano [6] for a variety of boundary conditions and little smoothness of $W$. On the other hand the convergence result of [26] in multidimensional case
requires that $W$ is analytic. It is so, because the main tool of [6], i.e. the maximum principle is not applicable to the quasi-steady approximation of (1.1). It is not clear to us if the analyticity assumption on $W$ is essential.

2. Preliminaries and existence

We present here our assumptions on the domain $\Omega$ and on the boundary conditions. We provide here a list of boundary conditions for which the subsequent analysis applies. They all have one common ingredient, namely the bi-harmonic operator is the square of a sectorial one (actually, a self-adjoint one). We use this assumption in the process of transforming our problem to a simpler one.

We also state here our assumption on the nonlinear term, $\sigma = D\xi W$. In particular they allow $W$ to have quite arbitrary set of minimizers. We will indicate that the energy functional, when considered on functions on a torus has infinite number of families of critical points. This is implied by symmetries of $T^2$, however we do not pursue the task of studying the structure of the set of critical points of $E$.

We assume that $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. Our specific hypotheses on $W : \mathbb{R}^2 \rightarrow \mathbb{R}$, are:

(H1) $W(\xi) \geq 0$ for all $\xi \in \mathbb{R}^2$.

(H2) There exist $r \in \mathbb{N}$, $r \geq 2$, $K > 0$, such that $|D^i W(\xi)| \leq K(|\xi|^{r-i} + 1)$ for all $\xi \in \mathbb{R}^2$, $i = 0, 1, 2, 3$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$.

In particular we make no assumptions on the set of minima of $W$, it might be a curve (as in case (1.3)) as well as a discrete set of points (case (1.2)). These two conditions along with appropriate smoothness of $W$ (say, $W$ is $C^5$) are sufficient to prove existence of solutions. However, our analysis requires more, namely:

(H3) for all $R > 0$ there exist $C_R > 0$ and $B_R > 0$ such that for all $\xi \in B(0, R)$ we have

$$|D^3 W(\xi)| \leq C_R (|\xi| + |\xi|^2)$$

$$D^4 W(\xi) \geq B_R, \quad \text{i.e.} \quad D^4 W(\xi)(v, v, v, v) \geq B_R |v|^4, \quad \forall v \in \mathbb{R}^2.$$

It is crucial for our studies to assume that $W$ is analytic, meaning that for any $u \in C^{4,\alpha}(\Omega)$, $W$ satisfies:

(H4) $W(\nabla u(x) + \xi) = \sum_{|k|=0}^{\infty} a_k(x)\xi^k$
uniformly in $x \in \Omega$ for sufficiently small $\xi \in \mathbb{R}^2$, i.e., $|\xi| < \beta(u)$, $\beta(u) > 0$.

Let us now turn our attention to the boundary conditions to which our subsequent analysis applies. We stress that some of them induce additional restrictions on $W$. Here is a partial list of some admissible boundary data:

(B1) $\Omega = [0, \omega]^2$, and $u$ is periodic (i.e., $u$ is well defined on torus $T^2$) and $\int_\Omega u\,dx = 0$.

(B2) $\partial \Omega$ is smooth and $u$ satisfies

\[
\begin{align*}
    u &= 0 \quad \text{on } \partial \Omega & \text{(B2a)} \\
    \Delta u &= 0 \quad \text{on } \partial \Omega & \text{(B2b)}
\end{align*}
\]

(B3) $\partial \Omega$ is smooth, $\nu$ is the outer normal to $\partial \Omega$ and $u$ satisfies

\[
\begin{align*}
    \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \quad \text{and} \quad \int_\Omega u\,dx = 0 & \text{(B3a)} \\
    \frac{\partial \Delta u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega. & \text{(B3b)}
\end{align*}
\]

This condition, however, requires an additional constraint on $W$, namely we need (H5) below to be able to perform integration by parts in the proof of the energy identity (see Theorem 3.7 (b) further in the text) and to prove that operator $L$ in §4.2 is self-adjoint:

\[(H5) \quad W = W(|\xi|^2).\]

We may also consider yet another boundary condition:

(B4) $u : \Omega \to \mathbb{R}$, where $\Omega = (0, \omega_1) \times (0, \omega_2)$ and $u$ is anti-periodic in $x_1$, namely it satisfies

\[
    u(x_1 + \omega_1, x_2) = -u(x_1, x_2) \quad \text{(B4a)}
\]

and $u$ also satisfies

\[
\begin{align*}
    u_{x_2} &= 0 \quad \text{on } x_2 = 0 \text{ and } x_2 = \omega_2 & \text{(B4b)} \\
    \Delta u_{x_2} &= 0 \quad \text{on } x_2 = 0 \text{ and } x_2 = \omega_2. & \text{(B4c)}
\end{align*}
\]

The same reason as in the case of boundary condition (B3) forces us to adopt the following restriction on $W$:

\[(H6) \quad \frac{\partial W}{\partial \xi_2}(\xi_1, 0) = 0.\]
Let us note here that anti-periodicity of $u$ where $u \in W^{1,p}(\Omega)$, $p \geq 1$, determines its average over $\Omega = (0,\omega_1) \times (0,\omega_2)$. Namely, we have
\[
\int_\Omega u(x_1, x_2) \, dx_1 dx_2 = \int_\Omega (u_1/2 - x_1)u_{x_1}(x_1, x_2) \, dx_1 dx_2.
\] (2.1)
It is also easy to see that if $u \in C^4$ satisfies (B1) and it is an equilibrium point, i.e.
\[
-\text{div} \sigma(u) + \delta^2 \Delta^2 u = 0,
\] (2.2)
then for all $\tau \in \mathbb{R}$, $u(\cdot + \tau, \cdot)$ and $u(\cdot, \cdot + \tau)$ are also equilibria. However, if $u$ satisfies (2.2) and (B4), then $u(\cdot + \tau, \cdot)$ will also be an equilibrium if we assume that $W$ satisfies in addition:

(H7) $W(-\xi) = W(\xi)$ for all $\xi \in \mathbb{R}^2$.

We remark that other combinations of boundary conditions in the spirit of (B4) are possible, but we leave the details to the interested reader. We will concentrate our attention on (B1) (or (B4)) since this assumption implies that the set of equilibria is at least two-dimensional (respectively, one-dimensional). We stress, that (B4) compared to (B1) requires additional conditions, namely (H6) and (H7).

Let us now state the fundamental property of all these boundary conditions

**Lemma 2.1.** Let us set $Z = \{ u \in L^2(\Omega) : \int_\Omega u = 0 \}$, $X_{p,1} = X_{p,3} = L^p(\Omega) \cap Z$, $X_{p,2} = X_{p,4} = L^p(\Omega)$. We define $D(A_{p,i}) \subset X_{p,i} \rightarrow X_{p,i}$ by
\[
A_{p,i}u = -\Delta u, \quad i = 1, \ldots, 4,
\]
where
\[
D(A_{p,i}) = \begin{cases}
\{ u \in W^{2,p}(\Omega) : u \text{ satisfies (B1)} \} & \text{if } i = 1; \\
\{ u \in W^{2,p}(\Omega) : u \text{ satisfies (B2 a)} \} & \text{if } i = 2; \\
\{ u \in W^{2,p}(\Omega) : u \text{ satisfies (B3 a)} \} & \text{if } i = 3; \\
\{ u \in W^{2,p}(\Omega) : u \text{ satisfies (B4 a), and (B4 b)} \} & \text{if } i = 4.
\end{cases}
\]
Then for all $p \in [2, \infty)$, $A_{p,i}$ are sectorial on $X_{p,i}$, $i = 1, 2, 3, 4$. Moreover, if $p = 2$, then $A_{2,i}$ is self-adjoint and positive definite, i.e.
\[
\min \text{ spectrum}(A_{2,i}) \geq \lambda > 0, \quad i = 1, \ldots, 4.
\]

The proof is not difficult (it follows basically the lines of [17, §1.6]) and it is left to the reader.

We also note here that the above Lemma implies that the fractional powers of $A_{p,i}$ are well defined, so are $X_{p,i}^\alpha$ the fractional powers of $X_{p,i}$. We apply the convention: $X_{p,i}^0 = X_{p,i}$, $X_{p,i}^1 = D(A_{p,i})$. We also have
Lemma 2.2.

(a) \[X_{p,1}^{1/2} = W^{1,p}(\Omega) \cap Z, \quad X_{p,2}^{1/2} = W^{1,p}(\Omega), \quad X_{p,3}^{1/2} = W^{1,p}(\Omega) \cap Z,\]

\[X_{p,4}^{1/2} = W^{1,p}_{sp}(\Omega) := \{ u \in W^{1,p}(\Omega) : u \text{ satisfies (B4a)} \};\]

(b) \[X_{p,i}^{3/2} = W^{3,p}(\Omega) \cap D(A_{p,i}), \quad i = 1, 2, 3, 4.\]

Proof. (a) If \( p = 2 \), then these statements are easy and left to the reader. The general case is shown in the course of proof of Lemma 3.6 in Section 3.3. Part (b) follows immediately from the first one and from the observation that

\[u \in D(A_{2,A}^{3/2}) \iff A_{2,A}u \in D(A_{2,A}^{1/2}).\]

Subsequently for the sake of the simplicity of notation we drop the subscript \( i \) since the particular form of the boundary conditions is unimportant.

We close this section with remarks on the structure of the set of critical points of \( E \). We have presented in Section 1 two specific examples of \( W \) (formulae (1.2) and (1.3)). It is easy to check that (1.2) satisfies (H1)-(H4) and (H6)-(H7) but not (H5). Thus \( W \) defined in (1.2) may not be considered in conjunction with boundary conditions (B3). On the other hand \( W \) defined by (1.3) satisfies all of (H1)-(H7).

Nonetheless, in general, the structure of the set of solutions to (2.2) does not seem to be known. But let us make some simple observations.

**Proposition 2.3.** If \( W \) is given by (1.2) and \( E \) is defined by (1.5) on \( D(A_{2,A}) \), then the global minimizers of \( E \) are independent of \( x_2 \).

We leave an easy proof to the reader.

It is also not difficult to notice that the periodic boundary conditions imply that the set of families of equilibria is large. What we have in mind in particular is that each finite subgroup of a torus generates a family of solutions to (2.2). Namely, if \( G \) is a subgroup of \( T^2 \), then it defines in a natural way an action on \( T^2 \) which leaves \( E \) invariant. By the Symmetric Criticality Principle of Palais (see [22]) it follows that the critical points of \( E \) are invariant under \( G \). Since the number of finite subgroups of \( T^2 \) is infinite, then we expect existence of an infinite number families of solutions to (2.2).

3. Existence and smoothness of solutions

We first establish existence and uniqueness of strong solutions to
CONVERGENCE OF SOLUTIONS

\[ u_{tt} = \text{div}(\nabla u) + \Delta u - \delta^2 \Delta^2 u, \]
\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad (3.1) \]
boundary conditions: any of (B1), (B2), (B3) or (B4).

Comparing (3.1) with (1.1) we see that \( \rho \) and \( \nu \) are set to 1. This is possible after an appropriate scaling. We will describe this scaling in Section 4.1, and we will see that \( \delta = \delta \nu^{-1} \rho^{1/2} \), but for the development of the present section the particular transformation of coordinates and the value of \( \delta \) is unimportant (except for \( \delta > 0 \)). From now on in this Section we will drop the bar over \( \delta \) and we will write \( \delta \) instead of \( \bar{\delta} \).

In order to achieve the main goal of the Section we will use a version of the Andrews-Ball-Pego transformation. It had been applied to one-dimensional equation of viscoelasticity but later it was adapted to a multidimensional setting (see [25], [29]). This transformation was also used to study one-dimensional viscoelasticity with capillarity, see [18], [19]. We will describe this transformation in Section 3.1. We will see that the problem is reduced then to two coupled parabolic equations, to which the theory of analytic semigroups may be applied yielding existence and uniqueness of solutions to (3.1). We use the semigroup theory as exposed in the book by Henry, [17], with the updates made in the Russian translations. Since we intend to make rather straightforward applications of this theory the arguments we present are sometimes sketchy.

The second subsection is devoted to establishing a priori estimates on solutions. In particular, we need to establish higher regularity of solutions. These facts are required not only for a proof of the existence of the \( \omega \)-limit set but also we need them in our analysis of the flow on the center manifold. However, studying higher regularity leads to questions on the behavior of elements of \( X^\alpha \) for small \( \alpha > 0 \). This is so because we avoid using the standard argument with flattening out portions of the boundary of \( \Omega \). We relegate our remarks on \( X^\alpha \) to the Appendix in §3.3.

3.1. Existence and uniqueness of solutions

By a strong solution to (3.1) we mean \( u \) belonging to
\[ u \in C([0, \infty); D(A_2)) \cap C((0, \infty), W^{4,2}(\Omega) \cap D(A_3^2)), \]
\[ u_t \in C([0, \infty); W^{1,2}(\Omega)) \cap C((0, \infty), W^{2,2}(\Omega) \cap D(A_2)), \]
\[ u_{tt} \in C((0, \infty), L^2(\Omega)) \]
and satisfying (3.1) in the \( L^2(\Omega) \) sense.
We will prove global-in-time existence utilizing a version of the transformation due to Andrews-Ball-Pego. We will describe it momentarily.

If \( u \) is a strong solution to (3.1) we define \( P \in D(A_2) \) as a solution to

\[
\Delta P = u_t, \tag{3.2a}
\]

by Lemma 2.1 \( P \) is well-defined and unique. We introduce the second variable \( Q \in D(A_2) \) by

\[
Q = u - P. \tag{3.2b}
\]

If we insert the new variables into (3.1), then we discover that it turns into

\[
\Delta P_t = \text{div}(\nabla(P + Q)) + (1 - \delta^2)\Delta^2 P - \delta^2 \Delta^2 Q.
\]

By Lemma 2.1 and the definition of strong solutions we may apply \( A^{-1} \) to both sides of this equation. This yields

\[
P_t = A^{-1}\text{div}(\nabla(P + Q)) + (1 - \delta^2)\Delta P - \delta^2 \Delta Q,
\]

where \( P(x, 0) = P_0(x) = A^{-1}u_1(x) \). For the sake of convenience let us set

\[
H(z_1, z_2) = -A^{-1}\text{div}(\nabla z_1 + \nabla z_2). \tag{3.3}
\]

The equation for \( Q \) is obtained by differentiation in time of (3.2b)

\[
Q_t = u_t - P_t = H(P, Q) + \delta^2 \Delta P + \delta^2 \Delta Q,
\]

where \( Q(x, 0) = Q_0(x) = u_0(x) - A^{-1}u_1(x) \). We thus have to deal with the system

\[
\begin{align*}
P_t &= A^{-1}\text{div}(\nabla(P + Q)) + (1 - \delta^2)\Delta P - \delta^2 \Delta Q, \\
Q_t &= -A^{-1}\text{div}(\nabla(P + Q)) + \delta^2 \Delta P + \delta^2 \Delta Q,
\end{align*}
\]

and \( P(x, 0) = P_0(x), Q(x, 0) = Q_0(x), P_0, Q_0 \in D(A_2) \).

We can write (3.4) in an abstract form as an equation in the Banach space \( Y_p = X_p \times X_p, p \geq 2 \)

\[
z_t + B_p z = F(z), \quad z(0) = z_0, \tag{3.5}
\]

where \( z = (P, Q), F(z) = (-H(z), H(z))^T \), \( H \) is defined by (3.3), \( z_0 = (P_0, Q_0) \) and \( B_p : D(B_p) \subset Y_p \to Y_p \) is given by

\[
B_p = \begin{bmatrix}
(1 - \delta^2)A_p & -\delta^2 A_p \\
\delta^2 A_p & \delta^2 A_p
\end{bmatrix}
\]
and \( D(B_p) = D(A_p) \times D(A_p) \). We will need

**Lemma 3.1.** (cf. [18, Lemma 2.2.1]) If \( \delta > 0, \ p \geq 2 \), then the operator \( B_p \) defined above is sectorial on \( Y_p \). Moreover,

\[
\min \text{Re spectrum } (B_p) \geq \lambda_0 := \frac{1}{2} \Re (1 - \sqrt{1 - 4\delta^2}) \lambda,
\]

where \( \lambda = \min \text{ spectrum } (A_2) \).

**Proof.** In effect we repeat the argument of [18]. We may write \( B_p = MA \),

where

\[
A = \begin{bmatrix} A_p & 0 \\ 0 & A_p \end{bmatrix}, \quad M = \begin{bmatrix} (1 - \delta^2) & -\delta^2 \\ \delta^2 & \delta^2 \end{bmatrix}.
\]

By elementary calculations we check that there exist \( 2 \times 2 \) matrices \( D \) and \( \Lambda = \text{diag}(\lambda_+, \lambda_-) \), \( \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 - 4\delta^2}) \) such that

\[
B_p = D \Lambda AD^{-1}.
\]

Thus, it follows immediately that \( B_p = \mu \) is invertible if and only if \( \mu \notin \lambda_- \text{ spectrum } (A_p) \cup \lambda_+ \text{ spectrum } (A_p) \). Moreover since \( A_p \) is sectorial \( B_p \) is sectorial too.

The proof of this Lemma implies that the fractional powers of \( Y_p \) coincide with Cartesian square of fractional powers of \( X_p \). In particular we have

**Lemma 3.2.** Let us assume boundary conditions \( (B4) \), then

\[
D(B_p^{1/2}) = (W_{ap}^{1/2}(\Omega))^2, \quad D(B_p^{3/2}) = (W_{ap}^{3/2}(\Omega) \cap D(A_p))^2
\]

**Proof.** Let us take any \( \alpha > 0 \). By definition of the corresponding quantities we infer that (3.7) implies (cf. [17, Theorem 1.3.1] and [17, Def. 1.4.1])

\[
e^{-B_p t} = D e^{-\lambda t} D^{-1}, \quad B_p^{-\alpha} = D \Lambda^{-\alpha} A^{-\alpha} D^{-1}.
\]

We may set in particular \( \alpha = \frac{1}{2} \). By definition \( D(B_p^{1/2}) = \text{Rg}(B_p^{-1/2}) \), thus the second identity above implies that

\[
D(B_p^{1/2}) = \text{Rg}(D \Lambda^{-1/2} A^{-1/2} D^{-1}) = \text{Rg}(D \Lambda^{-1/2} A^{-1/2}).
\]

By Lemma 2.2, \( D(A_p^{1/2}) = W_{ap}^{1/2}(\Omega) \). We also notice that \( D(W_{ap}^{1/2}(\Omega))^2 = (W_{ap}^{1/2}(\Omega))^2 \), so

\[
D(B_p^{1/2}) = (W_{ap}^{1/2}(\Omega))^2.
\]

The second part of the Lemma follows from the first one if we notice that

\[
D(B_p^{3/2}) = (W_{ap}^{3/2}(\Omega))^2.
\]
Remark. Only minor changes are required for other boundary conditions. We leave them to the reader.

We need two sorts of existence theorems: (i) for $W$ satisfying \((H1)-(H2)\), implying polynomial growth of $D_\xi W$ at infinity; (ii) for cut-off systems. We use the latter in the construction of the local center manifold.

**Proposition 3.3.** We consider boundary conditions \((B4)\). Let us assume that $W$ satisfies \((HI-H2), (H6), (H7)\) and it is smooth (say $C^5$). (If we impose another boundary condition, then we must adjust the requirements on $W$ accordingly). We set

\[(i) \quad \sigma(\nabla u) = \psi(\|\nabla u\|_{X_2^2}) (D_\xi W(\nabla U + \nabla u) - D_\xi W(\nabla U) - D_\xi^2 W(\nabla U) \nabla u),\]

where $\psi \in C^\infty_0(\mathbb{R})$, $U$ is a critical point of $E$, $p = 2$, $z_0 \in Y_2^\alpha$, $\alpha \in (\frac{1}{2}, 1)$;

or

\[(ii) \quad \sigma(\nabla u) = D_\xi W(\nabla u), \quad z_0 \in Y_2^\alpha, \quad p \geq 2 \quad \text{and} \quad \alpha \in (1/p + (r - 2)/2(r - 1), 1),\]

where $r$ is provided in \((H2)\).

Then there exist $T > 0$ ($T = \infty$ if (i) holds) and $z$ a unique strong solution to \((3.4)\) on $[0, T)$ such that

\[z \in C^1((0, T); Y_p) \cap C([0, T); Y_2^\alpha) \cap C((0, T); Y_1^1).\]

**Remarks.** (1) The proof hinges on $\sigma(\nabla u) \in L^2(\Omega)$. This leads to restrictions on $\alpha, r, p$ and the number of space dimensions $n$. In higher dimension $\sigma(\nabla u) \in L^2(\Omega)$ is not true without $\alpha > 1$, if $r \geq 4$ and $p = 2$. Moreover, if $n > 2$ then the energy estimates alone will not be sufficient to establish global-in-time existence.

(2) The Proposition requires only minor changes for the other boundary conditions. We leave them to the reader.

**Proof.** We will apply the theory of analytic semigroups of [17]. We have already noted in Lemma 2.1 that $B_p$ is sectorial. We now claim that

\[z \mapsto H(z) = -\Delta^{-1} \text{div}(\nabla (P + Q))\]

is locally Lipschitz continuous from $Y_2^\alpha$ to $L^2(\Omega)$. The Embedding Theorem [17, Theorem 1.6.1] and \((H2)\) imply that $\sigma(\nabla (P+Q)) \in L^2(\Omega)$, hence $H(z) \in X_2^{1/2}$. It is not difficult to check that

\[
\|H(z_1) - H(z_2)\|_{L^2(\Omega)} \leq K_0 \|H(z_1) - H(z_2)\|_{Y_2^{1/2}} \leq K_1 \|\nabla z_1 - \nabla z_2\|_{L^2(\Omega)} \\
\leq K_2 \|z_1 - z_2\|_{Y_2^{1/2}},
\]

\((3.8)\)
where in general $K_2$ depends on $\|z_1\|_{Y_2^1/2}, \|z_2\|_{Y_2^1/2}$ except the case (i), when $K_2$ is independent of $z_1, z_2$. An application of [17, Theorem 3.3.3] completes the proof. If (i) holds, then $H$ is globally Lipschitz continuous and [17, Corollary 3.3.5] yields $T = \infty$.

**Remark.** Let us note that if $z_0 \in Y_2^\alpha$, $\alpha < 1$, then the initial data do not have enough regularity to guarantee that $E(u_0)$ is finite. However, we shall see that the data get smoothed out to some extent. Hence subsequently, we consider much smoother solutions.

**Theorem 3.4.** We assume that $u_0 \in D(A_2)$, $u_1 \in L^2(\Omega)$ (thus $z_0 \in D(B_2)$), $W$ satisfies (H1–H2) (and also (H5) or (H6–H7) depending on the boundary conditions) and $\sigma(\xi) = D_\xi W(\xi), \delta > 0$. If $z_0 = (P_0, Q_0) = (\Delta^{-1} u_1, u_0 - \Delta^{-1} u_1)$, and

$$u(t) = P(t) + Q(t),$$

then

(a) $u$ is a unique strong solution of (3.1)

(b) the energy identity holds, i.e.,

$$E(u(t)) + \int_0^1 \int_\Omega \frac{1}{2} u_t^2(t) \, dx + \int_0^t \int_\Omega |\nabla u_t|^2 \, dx \, dt = E(u_0) + \int_0^1 \int_\Omega \frac{1}{2} u_0^2 \, dx \quad \forall t \geq 0;$$

(c) $u$ is global in time.

**Proof.** We carry it out in a few steps. Since the argument we use is standard, we present only a sketch while asking the reader to provide missing calculations. In the sequel, for the sake of simplicity of notation we write $\sigma(t)$ for $\sigma(\nabla P(t) + \nabla Q(t))$.

(1) Let us fix $\alpha \in (1/2 + (r - 2)/(r - 1), 1)$. We note that $z_0 = (P_0, Q_0) \in X_2^\alpha$, thus Proposition 3.3 yields that $P + Q \in C([0, T); X_2^\alpha)$. By [17, Theorem 1.6.1] and (H2) we conclude that $H \in C([0, T); X_2^1)$.

We now proceed to show smoothness of $P$ and $Q$.

(2) We recall the constant variation formula, here $\epsilon \geq 0$

$$z(t) = e^{-B_2(t - \epsilon)} z(\epsilon) + \int_\epsilon^t e^{-B_2(t - s)} F(z(s)) \, ds.$$  \hspace{1cm} (3.9)

We apply $B_2^{1 + \eta}, 0 \leq \eta < 1/2$ to both sides. Step (1) above and $z(\epsilon) \in Y_2^\eta$ combined yield

$$B_2^{1 + \eta} z(t) = B_2^\eta e^{-B_2(t - \epsilon)} B_2 z(\epsilon) + \int_\epsilon^t B_2^{1/2 + \eta} e^{-B_2(t - s)} B_2^{1/2} F(z(s)) \, ds.$$  

We thus conclude using [17, Theorem 1.4.3] that
\[ z \in C((0, T); Y \_2^{1+\eta}). \]

But in particular, \( z_0 \in Y \_2^1 \) so for \( \eta = 0 \), we notice that \( z \in C((0, T); Y \_2^1). \)

(3) We now claim that for all \( \varepsilon > 0 \)
\[ H(z) \in C((\varepsilon, T); X \_2^1). \]

We note that [17, Theorem 1.6.1] yields for any \( \eta > 0 \) that \( X \_2^{1+\eta} \subset W \_1^\infty(\Omega) \). Thus, by step (2) the term \( \text{div} \sigma(t) \) is a well defined element of \( L^2(\Omega) \). Thus, \( H(z) \in X \_2^1 \).

We may now argue as in (2) and apply \( B \_2^{1+\eta}, \gamma < 1 \) to both sides of (3.9). We conclude that
\[ z \in C((\varepsilon, T); Y \_2^{1+\gamma}), \quad 0 \leq \gamma < 1. \quad (3.10) \]

(4) It follows from (3.10) after simple calculations that
\[ \text{div} \sigma(\nabla(P(t) + Q(t))) \in W \_1^2(\Omega), \]
for \( t \geq \varepsilon > 0 \), but we do not know if \( \sigma(\nabla(P(t) + Q(t))) \in X \_2^{1/2} \), i.e. the problem is we do not know if the element \( \sigma(P(t) + Q(t)) \) satisfies the proper boundary conditions for membership in \( X \_2^{1/2} \). But, by Lemma 3.6 in \$3.3 \text{div} \sigma(t) \in X \_2^{\eta}, \) for some \( 0 < \eta < 1/2 \), so if we apply \( B \_2^\eta \) to both sides of (3.9) we see that
\[ B \_2^3 z(t) = B \_2^{1-\gamma} e^{-B \_2(t-s)} B \_2^{1+\gamma} z(t) \]
\[ + \int_\varepsilon^t B \_2^{1-\eta} e^{-B \_2(s-t)} B \_2^\eta M(\sigma(s), -\text{div} \sigma(s))^T ds, \quad (3.11) \]

where \( M \) is defined by (3.6). Hence,
\[ z \in C((0, T); Y \_2^3). \]

(5) We now differentiate (3.8) with respect to time. This yields
\[ z_t(t) = e^{-B \_2(t-s)} B \_2 z(t) + \int_\varepsilon^t e^{-B \_2(t-s)} B \_2 F(z(s)) ds + F(z(t)). \quad (3.12) \]

The argument as in (4) implies that
\[ z_t \in C((0, T); Y \_2^1). \]
(6) We now define \( u = P + Q \). It is easy to see from steps (3), (4) and (5) that \( u \) enjoy the regularity required a strong solution. One can also check if \( u \) satisfies our equation
\[
\begin{align*}
    ut = \Delta P_t &= \Delta(\Delta^{-1}\text{div}(\nabla u) + u_t - \delta^2(\Delta P + \Delta Q)) \\
    &= \text{div}(\nabla u) + \Delta u_t - \delta^2 \Delta^2 u.
\end{align*}
\]

(7) We are in a position to prove uniqueness. Suppose now we have two strong solutions \( u^i, \ i = 1, 2 \). If we set \( v = u^1 - u^2 \), then the difference \( v \) satisfies
\[
\begin{align*}
    vtt = \text{div}(\sigma(\nabla u^1) - \sigma(\nabla u^2)) + \Delta v_t - \delta^2 \Delta v
\end{align*}
\]
\( \forall (x,0) = 0, \ v_t(x,0) = 0. \)

We now proceed in a usual way: we multiply the above equation by \( v_t \), we integrate the result over \( \Omega \times [t_0,t] \), \( (t_0 > 0) \) and we integrate by parts. Smoothness of \( \sigma \) and Young’s inequality imply
\[
\begin{align*}
    \frac{1}{2} \int_\Omega v_t^2(x,t) dx + \delta^2 \int_\Omega |\Delta v(x,t)|^2 dx &\leq \frac{b(t)^2}{4} \int_{t_0}^t \int_\Omega |\nabla v|^2 dxdt \\
&\quad + \delta^2 \int_\Omega |\Delta v(x,t_0)|^2 dx + \frac{1}{2} \int_\Omega v_t^2(x,t_0) dx,
\end{align*}
\]
where
\[
b(t) = \max_{0 \leq \tau \leq t} \| \int_0^1 \sigma(s \nabla u^1(\tau) + (1-s) \nabla u^2(\tau)) ds \|_{L^2(\Omega)}.
\]

We note that for all considered boundary conditions Poincaré’s inequality is applicable (for (B4) this is guaranteed by (2.1)), which yields \( \int_\Omega |\nabla u|^2 \leq C_P \int_\Omega |\Delta u|^2 \) for some \( C_P > 0 \). After letting \( t_0 \to 0 \) we arrive at
\[
\delta^2 \int_\Omega |\nabla v|^2 dx ds \leq C_P \frac{b(t)^2}{4} \int_0^t \int_\Omega |\nabla v|^2 dx ds.
\]
So, Gronwall inequality implies that \( v \equiv 0 \), or \( u^1 = u^2 \), as desired.

(8) We shall show the energy identity. In step (6) we established enough smoothness of solutions for differentiation of the left-hand-side of (b) to be permitted on \( (0,T) \). We leave to the reader checking that the derivative is zero. Continuity of \( (u_0, u_1) \to E(u_0) + \frac{1}{2} \int u_1^2 \) implies (b).

(9) In order to show (c) it is enough to prove that \( P \) and \( Q \) are defined globally. By part (b) we know \( E(u(t)) < E(u_0) < \infty \). This implies in
particular that
\[ \|\Delta(P + Q)\|_{L^2(\Omega)}^2 \leq 2\delta^{-2}E(u_0) < \infty, \quad t \geq 0. \]
By the Zygmund–Calderon inequality we have
\[ \|P + Q\|_{W^{2,2}(\Omega)} \leq K_1 < \infty, \quad t \geq 0 \]
with a bound independent of time. The Sobolev Embedding Theorem (see, e.g. Theorem 7.10 in [13]) provides us with
\[ W^{2,2}(\Omega) \subset W^{1,p}(\Omega), \]
for all \( p \in [1, \infty) \). It follows that
\[ \|\nabla(P + Q)(t)\|_{L^p(\Omega)} \leq K_2 < \infty, \quad t \geq 0 \quad (3.13) \]
independently of time. Hence, by (H2), [17, Theorem 1.4.3], Lemma 3.2 and (3.13)
\[
\begin{align*}
\|B^\alpha_\gamma \int_0^t e^{-B_2(t-s)}F(z(s))\|_{L^2(\Omega)} \\
\leq \int_0^t C(t-s)^{1/2-\alpha}e^{-\lambda(t-s)}\|
abla(P + Q)(s)\|_{L^2(\Omega)} \, ds \\
\leq C(\max_{s \leq t} \|
abla(P + Q)(s)\|_{L^2(\Omega)}^{\gamma-1} + 1) \leq K_3(W, \Omega, \lambda, r, \delta^2) < \infty.
\end{align*}
\]
This combined with (3.9) implies in particular that \( \|z\|_{X_2^\gamma} \) is bounded on \([0,T]\) only in terms of initial data. Moreover, if \( G \subset X_2^\gamma \) is an arbitrary bounded set, then inequality (3.8) applied to \( z_1 \in G, z_2 = 0 \) implies that \( F(G) \) is bounded in \( X_2 \). Thus, we are in a position to invoke [17, Theorem 3.3.4] in order to conclude that \( T = +\infty \). \( \Box \)

### 3.2 A priori estimates

We have not yet established higher smoothness of solutions. We will do this now. We essentially use a bootstrapping type argument here. While the idea is simple its implementation requires some work and rather lengthy calculations. For the sake of readability we break our presentation in a number of steps. We also define
\[
E_D(u, u_t, u_{tt}) = \frac{1}{2} \int_\Omega \left[ v_{tt}^2 + \frac{1}{2} W(\nabla u)(\nabla u_t, \nabla u_{tt}) + \delta^2|\Delta u|^2 \right] ds.
\]

**Theorem 3.5.** We assume that the assumptions of Theorem 3.4 hold. We assume that \( t_0 > 0 \), then for all \( \mu \in (0,1) \) there exist \( K = K(t_0) > 0 \) such that
In particular, for \( t > 0 \), \( u \) is a classical solution of (3.1). Moreover, we have an energy inequality

\[
E_D(u, u_t, u_{tt})(t) + \int_0^t \int_\Omega |\nabla u_t|^2 \leq E_D(u, u_t, u_{tt})(t_0)
\]

(b) \[ + K(\sigma, u_0, u_1, t_0) \int_0^t \int_\Omega |\nabla u_t|^2, \]

where \( K(\sigma, u_0, u_1, t_0) = \frac{1}{2} \max_{\tau \geq t_0} \| D_2^2 \sigma(\nabla u(\tau)) \|_{L^\infty(\Omega)} \| \nabla u_t(\tau) \|_{L^\infty(\Omega)} \).

**Proof.** Step (a). We have established in step (4) of proof of Theorem 3.4 that \( z \in C((0, \infty); Y_p^2) \). Hence by Sobolev Embedding Theorem we conclude that \( z \in C([\epsilon, T); W^{3,p}(\Omega)) \) for any \( p \in (2, \infty), 0 < \epsilon < T \). We may now repeat the steps (2-6) from the proof of Theorem 3.4 replacing \( X_p^2 \) with \( X_p^p \).

We end up with

\[
z \in C((0, \infty); Y_p^2),
\]

\[
z_t \in C((0, \infty); Y_p^1).
\]

In order to establish

\[
\| \nabla u \|_{L^\infty(\Omega)} \leq K < \infty \quad \text{for } t \geq t_0
\]

for some \( K > 0 \) it is enough to show that

\[
\| z \|_{Y_p^1} \leq K, \quad \text{for } t \geq t_0.
\]

Subsequently, for the sake of brevity we shall write \( \sigma(t) \) for \( \sigma(\nabla P(t) + \nabla Q(t)) \) and \( H(t) \) for \( H(z(t)) \). We use (3.9) for our purposes while replacing \( B_2 \) with \( B_p \). Due to [17, Theorem 1.4.3] we obtain:

\[
\| B_p z(t) \|_{L^p(\Omega)} \leq C e^{-\lambda t} \| z(t_0) \|_{X_p^2} + C \int_{t_0}^t (t-s)^{-1/2} e^{-\lambda(t-s)} \| \sigma(s) \|_{L^2(\Omega)} ds.
\]

We thus infer using (H2), [17, Theorem 1.4.3] and (3.13) that for \( t_0 > 0 \),

\[
\| \nabla u \|_{L^\infty(\Omega)} \leq C \| B_p z(t) \|_{L^p(\Omega)} \leq C \| z(t_0) \|_{X_p^2} + K_4(t_0, E(u_0), \delta).
\]

Step (b). Proving that \( \| \nabla u_t \|_{L^\infty(\Omega)} \) stays bounded for \( t \geq t_0 > 0 \) requires a uniform bound on \( \| P \|_{X_p^p} \) for \( t \geq t_0 > 0, p > 2 \), because of \( \Delta P = u_t \). We
do this in two steps. We show first that \( \|B_p^{3/2} z\|_{L^p(\Omega)} \leq K \), for \( t \geq t_0 > 0 \).

We apply \( B_p^{3/2} \) to (3.9) (with \( B_2 \) replaced by \( B_p \)), we see that

\[
\|B_p^{3/2} z(t)\|_{L^p(\Omega)} \leq C\|B_p^{3/2} z(t_0)\|_{L^p(\Omega)}
+ C \max_{s \in [t_0, t]} \|D\sigma(s)\|_{L^{\infty}(\Omega)} \|D^2 u(s)\|_{L^p(\Omega)} \int_{t_0}^t e^{-\lambda(t-s)}(t-s)^{-1/2} ds.
\]

Inequality (3.14) guarantees that the factor in front of the integral is finite, hence

\[
\|B_p^{3/2} z(t)\|_{L^p(\Omega)} \leq K_5(\|B_p^{3/2} z(t_0)\|_{L^p(\Omega)}, t_0, E(u_0), \delta) < \infty \quad \text{for } t \geq t_0 > 0
\]  

(3.15)

The estimate for \( \|B^2_p z\|_{L^p(\Omega)} \) requires a bit more work since a priori we do not know if \( F(z) \) belongs to \( Y_p^{3/2} \), i.e. if it satisfies the appropriate boundary condition. We note here that \( F(z) \in D(B_p^{3/2}) \) automatically for periodic boundary conditions (B1).

Formula (3.11), where \( B_2 \) is replaced by \( B_p \) yields for \( t > 0 \)

\[
\|B_p^2 z(t)\|_{L^p(\Omega)} \leq C\|B_p^2 z(t_0)\|_{L^p(\Omega)} + C \int_{t_0}^t \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\eta}} \|A_p^{\eta} \sigma(s)\|_{L^p(\Omega)} ds,
\]

where \( \eta > 0 \) is sufficiently small given by Lemma 3.6. So, we have

\[
\|A_p^{\eta} \sigma(s)\|_{L^p(\Omega)} \leq C\|\sigma(s)\|_{W^{1,2}(\Omega)} \leq K_6(t_0, \|u\|_{W^{1,2}(\Omega)}) \|u\|_{W^{1,2}(\Omega)}.
\]

Thus, estimate (3.15) combined with (3.16) yields the required bound:

\[
\|B_p^2 z(t)\|_{L^p(\Omega)} \leq K(t_0, z(t_0)) < \infty \quad \text{for } t \geq t_0 > 0,
\]  

(3.17)

and finally by the Sobolev Embedding Theorem we obtain

\[
\|\nabla u_t\|_{L^\infty(\Omega)} \leq K < \infty, \quad \text{for } t \geq 1.
\]  

(3.18)

Step (c). We claim that

\[
\|B_p F(s) - B_p F(t)\|_{L^p(\Omega)} \leq C|t-s| \quad \forall p < \infty, \quad t, s \geq 1,
\]

(3.19)

where \( C \) is independent of \( s, t \). By inspection,

\[
\|B_p F(s) - B_p F(t)\|_{L^p(\Omega)} \leq C\|\text{div} (\sigma(s) - \sigma(t))\|_{L^p(\Omega)}
\leq C \max_{s \in \Omega} \|D^2 \sigma(s)\|_{L^{\infty}(\Omega)} \|D^2 u(s)\|_{L^p(\Omega)} \int_s^t \|\text{div} u(\tau)\|_{L^p(\Omega)} d\tau
\]

\[
+ C \max_{s \in [t_0, s]} \|D^2 \sigma(s)\|_{L^{\infty}(\Omega)} \int_s^t D^2 u(\tau) d\tau \|L^p(\Omega).
\]
We now recall that $D^2 u_t = D^2 \Delta P$, so by (3.17) and (3.18) our claim follows.

Step (d) We are now in a position to show that $z \in C^{4, \mu}(\Omega)$, $\mu \in (0,1)$ and that
\[ \| z \|_{C^{4, \mu}(\Omega)} \leq K, \quad \text{for } t \geq 1. \]

We note that (3.9) and (3.17) imply
\[
B^2_p z(t) = e^{-B_p(t-t_0)}B^2_p z(t_0) + \int_{t_0}^{t} B_p e^{-B_p(t-s)} (B_p F(z(s)) - B_p F(z(t))) \, ds \\
+ (I - e^{-B_p(t-t_0)}) B_p F(z(t)) = J_1(t) + J_2(t) + J_3(t). \tag{3.20}
\]

We show that for $t \geq t_0 > 0$ each $J_i$, $i = 1, 2, 3$ belongs to $C^0, \mu(\Omega)$. By [17, Theorem 1.4.3] we see
\[
\|B_p J_1(t)\|_{L^p(\Omega)} \leq C(t-t_0)^{-1} \|B^2_p z(t_0)\|_{L^p(\Omega)},
\]
so that the embedding $X^1_p \subset C^{0, \mu}(\Omega)$ ensures that $J_1(t) \in C^{4, \mu}(\Omega)$ with a uniform bound for $t \geq 1$ and $t_0 = 1/2$.

Let us pick $y \in (0,1)$ and we calculate using (3.19)
\[
\|B^2_p J_2(t)\|_{L^p(\Omega)} \leq C \int_{t_0}^{t} (t-s)^{-1-\gamma} e^{-\lambda(t-s)} |t-s| \, ds \\
\leq K(\gamma, t_0, p) < \infty \quad \text{for } t_0 > 0.
\]

but by [17, Theorem 1.6.1] we know that $X^1_p \subset C^{0, \mu}(\Omega)$ for $\mu > 0, \gamma < 1$ such that $\mu < 2\gamma - 2/p$.

Finally we look at $J_3$. It is sufficient to estimate $\|J_3(t)\|_{W^{2,p}(\Omega)}$:
\[
\|J_3(t)\|_{W^{2,p}(\Omega)} \leq \|B_p F(z(t)) - e^{-B_p(t-t_0)}B_p z(t)\|_{W^{2,p}(\Omega)} \\
\leq \|\text{div}(\nabla u(t))\|_{W^{2,p}(\Omega)} + C \|B^2_p z(t)\|_{L^p(\Omega)}.
\]

We note that because of (3.17) and (3.18) we have a bound
\[
\|\text{div}(\nabla u(t))\|_{W^{2,p}(\Omega)} \leq K(\sigma, \|z\|_{L^2}, t_0) \quad \text{for } t \geq t_0 > 0. \tag{3.21}
\]

Hence,
\[
\|J_3(t)\|_{W^{2,p}(\Omega)} \leq K(\sigma, \|z\|_{L^2}, t_0) \quad \text{for } 0 < t_0 \leq t < \infty.
\]

We are thus in a position to conclude that if we choose $t_0 = 1/2$, then
\[
\|z\|_{C^{4, \mu}(\Omega)} \leq K < \infty,
\]
and the bound is independent of $t \geq 1$.

Step (e). The argument in step (d) shows also that for $\epsilon > 0$ given by Lemma 3.6 in the Appendix we have that

$$z(t) \in D(B_p^{2+k}) \quad t > 0.$$  

for small $\epsilon > 0$. Moreover, the reasoning in step (d) shows that we have a bound

$$\|z\|_{L_p^{2+k}} \leq K_8 < \infty,$$

and the constant $K_8$ is independent of $t \geq 1$.

Our goal now is to show that $z \in W^{6,p}(\Omega)$, $z_t \in W^{4,p}(\Omega)$, hence we could perform differentiation with respect to time of equation (3.1). In step (f) we show first that $z \in C^\ast([1, \infty); D(B_p^2))$, where $\epsilon > 0$ is as in step (e) above. We set up the difference

$$z(t + h) - z(t) = (e^{-hB_p} - I)e^{-(t-t_0)B_p}z(t_0)$$

$$+ \int_{t_0}^t e^{-B_p(t-s)}F(z(s))\,ds$$

$$+ \int_t^{t+h} e^{-B_p(t+h-s)}F(z(s))\,ds.$$

We re-write it in similar fashion as we did with (3.20) and we come to

$$z(t + h) - z(t) = (e^{-hB_p} - I)e^{-(t-t_0)B_p}z(t_0)$$

$$+ \int_{t_0}^t e^{-B_p(t-s)}(F(z(s)) - F(z(t)))\,ds$$

$$+ \int_t^{t+h} e^{-B_p(t+h-s)}(F(z(s)) - F(t + h))\,ds$$

$$+ (e^{-hB_p} - I)B_p^{-1}F(z(t + h)) = \sum_{i=1}^5 I_i.$$

Now, we have to estimate $\|B_p^2I_i\|_{L_p^r(\Omega)}$, $i = 1, \ldots , 5$. We notice that $B_p^2I_1$, $B_p^2I_3$ and $B_p^2I_5$ are easy to estimate, by [17, Theorem 1.4.3]

$$\|B_p^2I_1\|_{L_p^r(\Omega)} \leq C\|B_p^{2+k}(z(t_0))\|_{L_p^r(\Omega)},$$

$$\|B_p^2I_3\|_{L_p^r(\Omega)} \leq C\|B_p^{1+k}F(z(t))\|_{L_p^r(\Omega)},$$
\[
\|B_p^2 I_0\|_{L^r(\Omega)} \leq C h^\alpha \|B_p^{1+\varepsilon} F(z(t+h))\|_{L^r(\Omega)}.
\]

Due to (3.22) \(\|B_p^2 I_1\|_{L^r(\Omega)}\) is bounded for \(t \geq 1\). The desired estimates for \(\|B_p^2 I_0\|_{L^r(\Omega)}, \|B_p^2 I_3\|_{L^r(\Omega)}\) are furnished by (3.16).

We now examine \(I_2\), due to [17, Theorem 1.4.31], for \(\alpha < 1\) and (3.19) above we have for \(t > t_0 > 0\)

\[
\|B_p^2 I_2\|_{L^r(\Omega)} \leq C h^\alpha \int_{t_0}^t e^{-\lambda(t-s)} \frac{e^{\lambda(t-s)}}{(t-s)^{1+\alpha}} (t-s) \, ds \leq C h^\alpha.
\]

Finally we look at \(I_4\), by (3.19) we obtain:

\[
\|B_p^2 I_4\|_{L^r(\Omega)} \leq C \int_t^{t+h} \frac{e^{-\lambda(t+h-s)}}{(t+h-s)^{1+\alpha}} (t+h-s) \, ds \leq C h.
\]

Hence, we may conclude that for some small positive \(\varepsilon:\)

\[
z \in C^\varepsilon([1, \infty); X_p^2)
\]  

(3.23)

Step (g). We improve here the estimates in step (d) by showing that \(z \in C([0, \infty); W^{6,p}(\Omega))\). It is clear from the reasoning in step (d) that it is sufficient to show only that \(J_2 \in W^{6,p}(\Omega)\), because we have already shown this for \(J_1\) and \(J_3\). We look again at \(B_p(F(z(s) - F(z(t))))\), we note that

\[
B_p(F(z(s) - F(z(t)))) = \int_0^1 D^2 \sigma(w) \left(Dw, \int_s^t \nabla u(t) \, dt\right) \, dr
\]

\[
+ \int_0^1 D^2 \sigma(w) : \int_s^t D^2 u(t) \, dp \, dr,
\]

where if \(A, B \in \mathbb{R}^{n \times n}\), then \(A : B\) means \(\text{tr} A^T B\) and we set

\[
w := \tau \nabla u(s) + (1 - \tau) \nabla u(t).
\]

We need in particular more information on \(\int_s^t D^2 u(t) \, dp\):

\[
\int_s^t D^2 u(t) \, dp = \int_s^t (D^2 u(t) - D^2 u(s)) \, dp + (t-s) D^2 u(s),
\]

and similarly
\[
\int_s^t \nabla u_t(\rho) \, d\rho = \int_s^t (\nabla u_t(\rho) - \nabla u_t(s)) \, d\rho + (t-s) \nabla u_t(s)
\]

So, after recalling that \( u_t = \Delta P \) we obtain

\[
\begin{align*}
\|B_p J_2\|_{L^p(\Omega)} & \leq \\
& \|B_p \int_{t_0}^t B_p e^{-B_p(t-s)} \int_0^1 D_\xi^2 \sigma(w) \left( D w, \int_s^t -t_s \nabla \Delta (P(\rho) - P(s)) \, d\rho \right) \, ds \|_{L^p(\Omega)} \\
& \quad + \|B_p \int_{t_0}^t B_p e^{-B_p(t-s)} \int_0^1 D_\xi \sigma(w) \left( D w, \nabla \Delta P(s) \right) \, ds \|_{L^p(\Omega)} \\
& \quad + \|B_p \int_{t_0}^t B_p e^{-B_p(t-s)} \int_0^1 D_\xi^2 \sigma(w)(D w, \nabla \Delta P(s)) \, ds \|_{L^p(\Omega)} \\
& \quad + \|B_p \int_{t_0}^t B_p e^{-B_p(t-s)} \int_0^1 D_\xi \sigma(w) : D^2 \Delta P(s) \, ds \|_{L^p(\Omega)} \\
& = J_{21} + J_{22} + J_{23} + J_{24}.
\end{align*}
\]

By (3.23) we have

\[
J_{21}, J_{22} \leq C \int_{t_0}^t e^{-\lambda(t-s)} \int_{t_0}^t (t-s)^t ds \leq C < \infty.
\]

In order to estimate \( J_{23}, J_{24} \) we use inequality (3.22) established in step (e).

\[
\begin{align*}
I_{23} & \leq C \int_{t_0}^t e^{-\lambda(t-s)} \int_0^1 B_p D_\xi \sigma(w)(D w, \nabla \Delta P(s)) \, ds \\
I_{24} & \leq C \int_{t_0}^t e^{-\lambda(t-s)} \int_0^1 B_p D_\xi \sigma(w) : D^2 \Delta P(s) \, ds.
\end{align*}
\]

We notice that if we set \( \phi = D_\xi^2 \sigma D w \) (resp. \( \phi = D_\xi \sigma \)) and \( v = \nabla \Delta P(s) \) (resp. \( v = D^2 \Delta P(s) \)) then they satisfy the assumptions of Lemma 3.7 of §3.3. Hence, we can conclude that

\[
J_{23}, J_{24} \leq C \int_{t_0}^t e^{-\lambda(t-s)} \int_0^1 B_p \|D^2 \Delta P(s)\|_{L^p(\Omega)} \, ds \leq K < \infty \quad \text{for} \ t \geq 1
\]

and finally

\[
z \in C([1, \infty); W^{6,p}(\Omega)).
\]

Step (h). We may now show that \( z_t \in C([1, \infty); W^{4,p}(\Omega)) \). By formula (3.12) we have
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\[ z_t(t) = e^{-B_p(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{-B_p(t-s)}B_p(F(z(s)) - F(z(t))) \, ds \\
+ (I - e^{-B_p(t-t_0)})F(z(t)) + F(z(t)). \]

The argument presented in step (g) applied to the above formula shows that indeed \( z_t \in Y_{p, q}^2 \).

Step (i). We conclude from step (g) and step (h), that for \( p \geq 2 \)

\[ \Delta^2 u_t = \Delta^2 P_t \in L^p(\Omega) \]

\[ \Delta u_{tt} = \Delta^2 P_t \in L^p(\Omega) \]

\[ \frac{\partial}{\partial t} \text{div}_{\sigma}(\nabla u) = D^2 \sigma(\nabla u) : \nabla \Delta P + D^2 \sigma(\nabla u) : D^2 \Delta P \in L^p(\Omega). \]

Hence we may differentiate in time the RHS of (3.1) for \( t > 0 \). After a series of integration by parts where we used the specific form of our boundary conditions (and (H5) or (H6) if necessary depending on the boundary conditions) we can easily see

\[ \frac{d}{dt} \Delta P(t, u, u_t, u_{tt}) = \int_{\Omega} [-|\nabla u_{tt}|^2 + \frac{1}{2} D^2 \sigma(\nabla u)(\nabla u_t, \nabla u_t, \nabla u_t)] \, dx. \]

The last term is easy to estimate

\[ |-\int_{\Omega} D^2 \sigma(\nabla u)(\nabla u_t, \nabla u_t, \nabla u_t) \, dx| \]

\[ \leq \max_{t_0 \leq \tau \leq t} \| D^2 \sigma(\nabla u) \|_{L^\infty(\Omega)} \| \nabla u_t \|_{L^\infty(\Omega)} \int_{\Omega} |\nabla u_t|^2 \, dx. \]

By (3.14) and (3.18) the RHS is bounded. So, after integrating \( d\Delta P/dt \) on \([t_0 + \epsilon, t], \epsilon > 0 \) and passing with \( \epsilon \) to 0, we come to the desired result. □

As an immediate corollary to this Theorem we obtain

\[ \int_{t_0}^\infty \int_{\Omega} |\nabla \Delta P|^2 \, dx \, dt < \infty. \] (3.24)

Remark. It is apparent from the proof of Theorem 3.5 that if \( z_0 \in W^{6, p}(\Omega) \) and \( z_0 \in D(B^2_p) \) for \( p > 2 \), then \( z \in C([0, \infty); W^{6, p}(\Omega) \cap D(B^2_p)) \). We will use this fact in §4.4.

3.3 Appendix

Our objective is to make clear that the spaces \( X_\alpha^p \) are independent of the boundary conditions for small \( \alpha > 0 \). This fact is well-know for domains with
smooth boundary, but in (B1) and (B4) the boundary is not smooth. We also state a proposition on multiplication of element of $X^\alpha_p$ by H"older continuous functions. This fact may be found in [24] but in a slightly different context.

We will use the notation introduced in Lemma 2.1. Let us recall that $F_{p,q}^\alpha$ stands for Triebel-Lizorkin spaces, (see [24], [30], [31]).

**Lemma 3.6.** If $p \in [2, \infty)$, then for all $\eta \in (0, \frac{1}{2p})$ we have

(a) $X^\eta_{p,1} = F_{p,2}^\eta(T^2) \cap Z$, $X^\eta_{p,2} = F_{p,2}^\eta(\Omega)$, $X^\eta_{p,3} = F_{p,2}^\eta(\Omega) \cap Z$, $X^\eta_{p,4} = F_{p,2}^\eta(T^2)$;

(b) $W^{1,p}(\Omega) \subset X^\eta_p$.

**Proof.** We know that $X^\eta_p = [L^p(\Omega), D(A_{p,1})\eta]$, (see [30, Theorem 1.15.3]) where the square bracket denotes complex interpolation. We consider all boundary conditions one by one. We consider first $i = 2, 3$. In this case the boundary conditions do not survive the process of interpolation if $2\eta < 1/p$, see ([30, Theorem 4.3.3/1]), so

$$X^\eta_{p,2} = [L^p(\Omega), D(A_{p,2})\eta] = [L^p(\Omega), W^{2,p}(\Omega)]_\eta.$$

Moreover, $P_0 : L^p(\Omega) \rightarrow L^p(\Omega)$ given by

$$P_0 v = v - \frac{1}{|\Omega|} \int_\Omega v \, dx$$

is a projection, hence by [30, Theorem 1.17.1/1] we conclude that

$$X^\eta_{p,3} = [L^p(\Omega), D(A_{p,3})\eta] = [L^p(\Omega), W^{2,p}(\Omega)]_\eta \cap Z.$$

We recall that by [30, Theorem 4.3.2/2] we have

$$X^{1/2}_{p,3} = W^{1,p}(\Omega) \cap Z.$$

Thus, (a) and (b) follow for $i = 2, 3$.

In the case of $i = 1, 4$ we cannot apply the above results directly because of lack of smoothness of boundary of $\Omega$. We use instead the fact that the boundary conditions allow us to interpret $D(A_{p,1})$ as a function space on a torus, and $D(A_{p,4})$ as a function space on a cylinder – both are smooth manifolds. Let us look at the case $i = 1$, we have

$$D(A_{p,1}) = W^{2,p}(T^2) \cap Z.$$

By [30, Theorem 1.17.1/1] we see that

$$X^\eta_{p,1} = [L^p(T^2), W^{2,p}(T^2)]_\eta \cap Z.$$

Hence, $X^{1/2}_{p,1} \subset X^\eta_{p,1}$, if $\eta < 1/2$, and (b) will follow after we establish that
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\( X_{p,1}^{1/2} = W^{1,p}(T^2) \cap Z \). By the definition of \( F_{p,q}^s(T^2) \), (see [31, 7.2.2]) and the identifications \( W^{k,p}(\mathbb{R}^n) = F_{p,2}^k(\mathbb{R}^n) \) we deduce using [31, Theorem 7.4.4] that

\[ X_{p,1}^{1/2} = \left( F_{p,2}^0, F_{p,2}^2 \right)_{1/2} \cap Z = W^{1,p}(T^2) \cap Z, \]

(see [24, Proposition 2.12 (iv)]).

At last we consider \( i = 4 \). We define \( E_1 : L^p(\Omega) \to L^p((0, 2\omega_1) \times (0, \omega_2)) \) by formula

\[
E_1 u(x_1, x_2) = \begin{cases} 
  u(x_1, x_2) & \text{if } x_1 < \omega_1; \\
  -u(x_1 - \omega_1, x_2) & \text{if } x_1 \geq \omega_1,
\end{cases}
\]

and we also set \( E_2 : L^p((0, 2\omega_1) \times (0, \omega_2)) \to L^p(T^2) \), where \( T^2 = (0, 2\omega_1) \times (-\omega_2, \omega_2) \).

\[
E_2 u(x_1, x_2) = \begin{cases} 
  u(x_1, x_2) & \text{if } x_2 > 0; \\
  u(x_1, -x_2) & \text{if } x_2 \leq 0.
\end{cases}
\]

We set \( E := E_2 E_1 \). We note that \( Eu \in Z \). We also define two projections \( A, S : L^p(T^2) \cap Z \to L^p(T^2) \cap Z \):

\[
Au(x_1, x_2) = \frac{1}{2}(u(x_1, x_2) - u(x_1 + \omega, x_2)); \\
Su(x_1, x_2) = \frac{1}{2}(u(x_1, x_2) + u(x_1, -x_2)).
\]

It is easy to establish that \( AS = SA \), hence \( P := AS \) is a projection, moreover

\[
\text{Rg } E = \text{Rg } P.
\]

It is also easy to show that

\[
\Delta E u = E \Delta u, \quad P \Delta u = \Delta P u,
\]

so

\[
E(A_{p,4} + \lambda)^{-1} u = (A_{p,4} + \lambda)^{-1} E u, \quad P(A_{p,4} + \lambda)^{-1} u = (A_{p,4} + \lambda)^{-1} P u.
\]

Hence we reach

\[
E A_{p,4}^\alpha = A_{p,4}^\alpha E, \quad P A_{p,4}^\alpha = A_{p,4}^\alpha P.
\]

Our conclusion is that the spaces \( X_{p,4}^\alpha \) may be identified with complemented subspaces of \( X_{p,4}^\alpha \). Thus, (a) and (b) follow for \( i = 4 \) from the results for \( i = 1 \). \( \square \)

Here is our result on multiplication.
Lemma 3.7. If $\phi \in C^{0,\mu}(\Omega)$ and $v \in X^\eta_{p,i}$, where $\mu > 2/p$ and $1/(2p) > \eta$, and
\[
\phi \cdot v \in Z \quad \text{(required only if } i = 1, 3),
\]
then
\[
\phi \cdot v \in X^\eta_{p,i} \quad \text{and} \quad \|\phi \cdot v\|_{X^\eta_{p,i}} \leq C\|\phi\|_{C^{0,\mu}(\Omega)}\|v\|_{X^\eta_{p,i}}.
\]

**Proof.** In the previous Lemma we characterized $X^\eta_{p,i}$ in terms of Triebel-Lizorkin space on domains or manifolds. Because of the definition of spaces $F^\mu_{p,q}$ on manifolds it is sufficient to establish
\[
C^{0,\mu}(\Omega) \cdot F^{2\mu}_{p,2}(\Omega) \subset F^{2\mu}_{p,2}(\Omega),
\]
where $\Omega$ is a domain with smooth boundary. But spaces $F^{2\mu}_{p,2}(\Omega)$ are defined as restriction of $F^{2\mu}_{p,2}$ to $\Omega$. So, it is enough to prove (3.26) for $\Omega = \mathbb{R}^2$.

Moreover, if $\mu > 2/p$, then by the difference characterization of $F^{2\mu}_{p,2}$ (see [24, §2.3.1]) one can easily see that $C^\mu \subset F^{\mu'}_{p,2}$ and $2/p < \mu' < \mu$. Finally, by [24, §4.6.1, Theorem 1] for $\mu' > 2/p$ we obtain that
\[
X^\eta_{p} \cdot C^\mu \subset F^{2\mu}_{p,2} \cdot F^{\mu'}_{p,2} \subset F^{2\mu}_{p,2} = X^\eta_{p},
\]
as required. \(\square\)

4. Convergence

We state and prove here our main result. First, for the sake of convenience we introduce some more notation. Namely, we denote by $M$ a connected component of the set of local minimizers of $E$. It is thus clear that $M$ is contained in some level set of $E$. We notice that if $U$ is any element of $M$, then $U$ is quite smooth meaning e.g. $U \in W^{6,p}(\Omega)$, $p > 2$. It is so because by the proof of Theorem 3.5 the solutions to (3.1) belong to this space for $t > 0$. Here is our main result.

**Theorem 4.1.** Let us assume (H1–H4) and either of the boundary conditions (B1–B4) (condition (B3) requires also (H5), while (B4) must be supplemented by (H6) as well as (H7)). Let us suppose that $M$ is defined as above and $c_0 := E(u)$ for $u \in M$, and $p > 2$ is arbitrary. Then there exist $m > c_0$ such that $\mathcal{U} := \{u \in W^{6,p}(\Omega) : E(u) < m\}$ is a neighborhood of $M$ in $W^{6,p}(\Omega)$ with the following property. Namely, if $u_0 \in \mathcal{U}$ and $u_1 \in W^{4,p}(\Omega)$ satisfy
\[
E(u_0) + \int_{\Omega} \frac{1}{2} u_1^2 \, dx < m,
\]
then the unique solution $u(t)$ to (3.1) with initial data $u_0$ and $u_1$ belongs to $\mathcal{U}$ and $u(t)$ converges as $t \to \infty$ to a unique point of $\mathcal{M}$. In addition, the convergence is in the $C^{4,\mu}(\Omega)$ norm, $u_4 \to 0$ in $C^{2,\mu}(\Omega)$, and $u_4 t \to 0$ in $C^{0,\mu}(\Omega)$.

Let us make some comments before proving this Theorem. This result is particularly interesting when the set $\mathcal{M}$ is large namely $\dim \mathcal{M} > 0$. The boundary conditions (B1) (respectively, (B4)) imply that $\dim \mathcal{M} \geq 2$ (respectively, $\dim \mathcal{M} \geq 1$). The Theorem above implies that if the initial conditions are close to a global minimum of $E$, then solutions must converge to a possibly different global minimum (this is in stark contrast to viscoelasticity in one-dimension, see [23], [11]). We note that in some circumstances the theory of [16] seems applicable, e.g. for $W$ given by (1.2). We have seen in Proposition 2.3 that the global minimizers are independent of $x_2$, so it is relatively easy to analyze the resulting problem and to show that actually $\dim \ker L = 1 = \dim \mathcal{M}$, where $Lh = -\text{div}(D_2^2W(\nabla U)\nabla h) + \delta^2 \Delta^2 h)$ and $U$ is a global minimizer of $E$. But it is not clear if this kind of reasoning may be carried out for any local minimizer of $E$. What we do here is we circumvent this problem. Namely, we look at the behavior of the flow of (3.1) on a local center manifold $W^c(U)$ for a fixed $U \in \mathcal{M}$. Our main goal is to show that $U$ is stable with respect to the flow restricted to $W^c(U)$. The general theory (e.g. [17, Chapter 9]) implies then that $U$ is stable, hence the $\omega$-limit set of $(u_0, u_1)$ must be a singleton. Of course, it is not true in general that a trajectory starting close to a manifold of equilibria will converge to a unique equilibrium point.

The difficulty in carrying out our program is the presence of the third order terms in the expansion of $E$ around a critical point $U \in \mathcal{M}$. We show that we can make these terms negligible by properly scaling the equations. We introduce those scalings in Section 4.1. Next, we prove existence of the local center manifold in Section 4.2 and in Section 4.3 we show that if $U$ is a local minimum and $U + w$ belongs to the local center manifold and the appropriate norm of $w$ is sufficiently small. We will show that inequality (4.1) implies stability of the restricted flow. Let us note that the role of the analyticity assumption (H4) is to make sure that there is no other critical point of $E$ in $\mathcal{U}$ a neighborhood of $\mathcal{M}$, precisely we use for this purpose a version of an inequality of Lojasiewicz, see [26, Theorem 3.2]. On the other hand the calculations we perform on $W^c(U)$ require only that $W \in C^5$. 

$$DE(U + w)w \geq 0$$ (4.1)
4.1 Scalings

We introduce two scalings of equation (1.1) and we collect their properties. The first of the scalings is

\[ t \to \rho^{-1/2} t =: \tau, \quad x \to \rho^{1/4} \nu^{-1/2} x =: y. \]

and we also set

\[ \bar{u}(\tau, y) = u(\rho^{1/2} \tau, \rho^{-1/4} \nu^{1/2} y). \]

It is now easy to see that with these definitions equation (1.1) becomes

\[ \bar{u}_{\tau\tau} = \text{div}_y \bar{\sigma}(\nabla_y \bar{u}) + \Delta_y \bar{u} - \bar{\delta}^2 \Delta^2 \bar{u}, \tag{4.2} \]

where

\[ \bar{\Omega} = \rho^{1/4} \nu^{-1/2} \Omega, \quad \bar{\delta} = \rho^{1/2} \nu^{-1} \delta, \quad \bar{W}(\xi) = W(\rho^{1/4} \nu^{-1/2} \xi). \]

Thus, we obtained precisely (3.1) for which we have already shown existence and uniqueness in Proposition 3.3. For the notational convenience let us rename the variables of the scaled equation back to \( x \) and \( t \).

However, our convergence result of Theorem 4.1 requires another scaling of equation (3.2). Let us fix \( a > 0 \) and set

\[ y := xa, \quad \tau := ta^2, \quad \bar{u}(\tau, y) = u(\tau/a^2, y/a). \tag{4.3} \]

Then we obtain equation of the same form of equation but with \( \bar{\sigma} \) replaced by \( \bar{\sigma} \) etc., namely

\[ \bar{\sigma} = D_\xi \bar{W}, \quad \bar{W}(\xi) = \frac{1}{a^4} W(a \xi), \quad \bar{\Omega} = a \Omega, \]

\[ D_\xi^i \bar{W}(\xi) = \frac{1}{a^{2+i}} D_\xi^i W(\xi) \quad i = 1, \ldots \tag{4.4} \]

If in particular \( i = 4 \) and the assumption (H3) is in force, then

\[ D_\xi^4 \bar{W}(\xi) \geq B_R, \tag{4.5} \]

with the bound independent of \( a \). We also notice that the boundary conditions do not change under these transformations.

Let us note that the following generalized eigenvalue problem will be important in our considerations

\[ -\text{div}_x(D_\xi \sigma(\nabla_x u) \nabla h) + \bar{\delta}^2 \Delta^2 h = \lambda \Delta \Delta h, \quad \text{in } \Omega \tag{4.6} \]

and its scaled version
We need to establish their relationship. It turns out that we have

**Proposition 4.2.** If \( u \) and \( \bar{u} \) are related by (4.3), then \( \lambda \in \mathbb{C} \) is an eigenvalue of (4.6) if and only if \( \lambda/a^2 \) is an eigenvalue of (4.7).

**Proof.** Let us suppose that \( (\lambda, h) \) is an eigenpair of (4.6). Let us consider
\[
\tilde{h}(y) = h(y/a),
\]
then
\[
\text{div}_y(\partial h/\partial y) = -\text{div}_y(\partial h/\partial y),\]
and by (4.5) and (4.4) for \( i = 2 \)
\[
D_\xi \tilde{h}(y) = \frac{1}{a^2} D_\xi h(y/a),
\]
hence
\[
\text{div}_y(D_\xi \partial(\nabla_y u)\nabla_y h) = \frac{1}{a^4} \text{div}_y(D_\xi^2 W(\nabla_x u)\nabla_x h).
\]
It follows that \( (\lambda, h) \) satisfies (4.6) if and only if \( (\lambda/a^2, \tilde{h}) \) satisfies (4.7). \( \Box \)

We close this subsection by noting that for \( p \geq 1 \) we have
\[
\|\nabla_y h\|_{L^p(\Omega)} = a^{1/2-1}\|\nabla_x u\|_{L^p(\Omega)}
\]
\[
\|\tilde{h}\|_{L^p(\Omega)} = a^{1/2}\|h\|_{L^p(\Omega)}.
\]

### 4.2 Existence of a local center manifold

Let us fix an equilibrium point \((0, U)\) of the \( P, Q\)-system (3.4). We want to analyze the behavior of solutions (3.4) near \((0, U)\). A part of this task is to establish the existence of a local center manifold \( W^c(U) \). For this reason in equation (3.4) we replace the nonlinear term by the one multiplied by a cut-off function. We introduce \( \psi_\eta : X^p_\eta \to \mathbb{R} \) by \( \psi_\eta(x) = \varphi(\eta^{-1}\|x\|_{X^p_\eta}) \), where \( \eta > 0, 0 < \alpha < 1, \varphi \in C_0^\infty(\mathbb{R}), \varphi \geq 0, \)
\[
\varphi(t) = \begin{cases} 0 & \text{for } t \geq 2, \\ 1 & \text{for } t \leq 1. \end{cases}
\]

We note that \( \psi_\eta \) is differentiable and its derivative is given by
\[
D\psi_\eta(x) h = \frac{1}{\eta}\varphi'(\eta^{-1}\|x\|_{X^p_\eta})(A^\alpha x, A^\alpha h)/\|A^\alpha x\|_{L^2(\Omega)}.
\]

We want to look at the dynamics of the cut-off system introduced below,
\[
\hat{P}_t = \hat{H}(\bar{P} + \bar{Q}) + \Delta \hat{P} - \delta \Delta (\bar{P} + \bar{Q}) + \Delta^{-1}\text{div}D_\xi \sigma(\nabla U)(\nabla \bar{P} + \nabla \bar{Q})
\]
\[
\hat{Q}_t = -\hat{H}(\bar{P} + \bar{Q}) + \delta \Delta (\bar{P} + \bar{Q}) - \Delta^{-1}\text{div}D_\xi \sigma(\nabla U)(\nabla \bar{P} + \nabla \bar{Q}),
\]
where by definition \( \bar{Q} := Q - U, \bar{P} := P, \) and
\[
\hat{H}(h) := \Delta^{-1}\text{div}(\psi_\eta(h)S(h)), \quad S(h) = \sigma(\nabla h + \nabla U) - \sigma(U) - D_\xi \sigma(\nabla U)\nabla h.
\]
Thus, equations (4.9) and (3.4) coincide when \( \|P + Q\|_{X_p^2} \leq \eta \). We shall need

**Lemma 4.3.** Let us assume that \((H1)\) and \((H2)\) hold and \(\alpha \in \left(\frac{1}{2} + \frac{r - 2}{2(r - 1)}, 1\right)\) (hence \(X_p^2 \subset W^{1,2(r-1)}(\Omega)\)), then

(a) \( \hat{H} : X_p^2 \rightarrow X_p^{1/2}, \|\hat{H}(h)\|_{X_p^{1/2}} \leq C\|h\|_{X_p^2}, \) and \( \hat{H} \) is differentiable;

(b) \( \text{Lip}(\hat{H}) \leq a_{\text{Lip}}\eta_n \), for some positive \( a_{\text{Lip}} \) which is independent of \( \eta \).

**Proof.** The definition of \( \hat{H} \) implies that it is enough to check that

(a') \( \psi_n S \) maps \( X_p^2 \) into \( L^2(\Omega) \), \( \|\psi_n(h)S(h)\|_{L^2(\Omega)} \leq C\|h\|_{X_p^2} \), and \( \psi_n S \) is differentiable;

(b') \( \text{Lip}(\psi_n S) \leq a'_{\text{Lip}}\eta_n \), where positive \( a'_{\text{Lip}} \) is independent of \( \eta \).

We first establish the inequality in (a')

\[
\|\psi_n(h)S(h)\|_{L^2(\Omega)} \leq \left( \int_{\Omega} \left| \psi_n^2(h) \int_0^1 \int_0^1 D^2_1 \sigma(\nabla U + st\nabla h) \nabla h \, dt \right|^2 \, dx \right)^{1/2} \leq \left( \int_{\Omega} \int_0^1 |D^2_1 \sigma(\nabla U + st\nabla h)|^{r'} \, ds \, dx \right)^{1/r'} \|\nabla h\|_{L^{2(r-1)}(\Omega)}^2,
\]

where we used \((H2)\) for \( i = 3 \) and \( r' = 2(r - 1)/(r - 3) \). Hence,

\[
\|\psi_n(h)S(h)\|_{L^2(\Omega)} \leq C(\|\nabla U\|_{L^{2(r-1)}} + \|\nabla h\|_{L^{2(r-1)}})^{r-3}\|\nabla h\|_{L^{2(r-1)}(\Omega)}^2 \leq C\|\nabla h\|_{L^{2(r-1)}(\Omega)}^2.
\]

As far as differentiability of \( \psi_n S \) is concerned we note that it is easy to check that

\[
\|\psi_n(x + h)S(x + h) - \psi_n(x)S(x) - D\psi_n(x)hS(x) + \psi_n(x)DS(x)h\|_{L^2(\Omega)} \leq C(x, h)\|\nabla h\|_{L^{2(r-1)}(\Omega)}^2 \leq C'\|x, h\|_{X_p^2}^2,
\]

where \( DS(x)h = D_1 \sigma(\nabla U + \nabla x)\nabla h - D_2 \sigma(\nabla U)\nabla h \).

We now show (b'), first we set \( F_i = \nabla U + st\nabla h_i, \ i = 1, 2 \)

\[
\|\psi_n(h_1)S(h_1) - \psi_n(h_2)S(h_2)\|_{L^2(\Omega)} \leq \frac{\|S(h_1)\|_{L^2(\Omega)}}{\eta} \int_0^1 \left| \psi_n \left( \frac{1}{\eta} ||sh_1 + (1-s)h_2||_{X_p^2} \right) \right| \, ds \|h_1 - h_2\|_{X_p^2} + \psi_n(h_2) \int_0^1 \int_0^1 \|D^2_1 \sigma(F_1)\nabla h_1^2 - D^2_1 \sigma(F_2)\nabla h_2^2\|_{L^2(\Omega)}^2 \, ds \, dt =: I.
\]

If \( \|h_i\|_{X_p^2} \geq 2\eta, i = 1, 2 \) or if \( W \) is a polynomial of order 4, then (b') follows.
Otherwise we may assume that
\[ \|h_1\|_{X^2_2} \leq 2\eta \leq \|h_2\|_{X^2_2} \quad \text{or} \quad \|h_i\|_{X^2_2} \leq 2\eta, \quad i = 1, 2 \]
and \( r > 4 \).

It is not difficult to see that
\[ I \leq \frac{C}{\eta} \|h_1 - h_2\|_{X^2_2} \|h_2\|_{X^2_2}^2 \]
\[ + C\psi_\eta(h_2) \|h_1 - h_2\|_{X^2_2} (\|U\|_{X^2_2} + \|h_1\|_{X^2_2})^{r-3} \|h_1 + h_2\|_{X^2_2}^3 \]
\[ + C\psi_\eta(h_2) \|h_1 - h_2\|_{X^2_2} (\|U\|_{X^2_2} + \|h_1\|_{X^2_2} + \|h_2\|_{X^2_2})^{r+2} \|h_1\|_{X^2_2}^2. \]
Thus, \((b')\) follows immediately.

By similar calculations we establish our next Lemma, however, the detailed considerations are left to the reader.

**Lemma 4.4.**
\[ \text{Lip}(D\bar{H}) \leq C. \]

We are now in a position to state existence of the center manifold.

**Theorem 4.5.** Let us suppose that \( U \) is a local minimum of \( E \) (for either of the boundary conditions). We set \( V := \ker DE(U), \) (clearly, \( \dim V \geq \dim M \)) and we equip \( V \) with the \( X^{1/2}_2 \) norm. We fix \( \alpha \) as in Lemma 4.3.

Then, there exists \( \eta > 0, \) and \( h : V \to Y^2_2, \) \( h \in C^{1,1}(V, Y^2_2), \) such that the set
\[ W^c(U) = \{ z \in Y : z = y + h(y) \} \]
is invariant for the cut-off system \((4.9)\). In addition
\[ h(0) = 0 = Dh(0), \quad (4.11) \]
i.e. \( W^c(U) \) is a local center manifold for \((3.1)\). Moreover, \( W^c(U) \) is asymptotically stable.

**Proof.** We first note that by Proposition 3.3 there exists a unique global in time solution to the cut-off system \((4.9)\). But for the purpose of proving existence of a local center manifold \( W^c(U) \) we have to rewrite \((4.9)\) in a form to which [17, Theorem 9.1.1] is applicable. We consider here \( X^{1/2}_2 \) equipped with the inner product \( (u, v) = \int_\Omega \nabla u \nabla v \) which yields a norm equivalent to \( \| \cdot \|_{X^{1/2}_2}. \) We define \( L : D(L) \subset X^{1/2}_2 \to X^{1/2}_2 \) by formula
\[ Lu = \Delta^{-1} DE(U)v = -\Delta^{-1} \text{div}(D\zeta\sigma(\nabla U)\nabla v) + \delta^2 \Delta v, \]
here $D(L) = X_2^{3/2}$. Furthermore we set

$$V := \text{span} \{\psi_j\} \subset X_2^{1/2},$$

where $\psi_j, j = 1, \ldots, N$ form an orthonormal in $X_2^{1/2}$ basis of ker $L$. We also introduce $\pi_i : X_2^{1/2} \to X_2^{1/2}$ by

$$\pi_1 \phi = -\sum_{j=1}^{N} (\psi_j, \phi) \psi_j, \quad \pi_2 = I - \pi_1,$$

Hence, they are orthogonal projections in $X_2^{1/2}$. In particular we have $V = \pi_1 X_2^{1/2}$. We choose a new underlying Banach space

$$X := X_2^{1/2} \times \pi_2 X_2^{1/2}$$

and we define $x = (x_1, x_2) \in X, \ y \in V$ by

$$x_1 = \bar{P}, \quad x_2 = \pi_2 \bar{Q}, \quad y = \pi_1 \bar{Q}.$$

We also set

$$f(x, y) = (\bar{H}(x + y), -\pi_2 \bar{H}(x + y))^T,$$

$$g(x, y) = -\pi_1 \bar{H}(x + y), \quad g_0(y) = -\pi_1 \bar{H}(y).$$

These mappings are differentiable by Lemma 4.3. And we finally define

$$\mathcal{G} : D(\mathcal{G}) \subset X \to X, \quad \mathcal{G} x = \begin{bmatrix} -L(x_1 + x_2) + \Delta x_1 \ 
L(x_1 + x_2) \end{bmatrix},$$

where $D(\mathcal{G}) = X_2^{3/2} \times \pi_2 X_2^{3/2}$. We remark that $\mathcal{G}$ is well-defined. Indeed, we notice that $L$ is a self-adjoint operator. It is so due to the boundary conditions and to the restrictions we have imposed on $W$. The detailed calculations we leave to the reader. Thus, if $x \in D(\mathcal{G})$, then $\text{Rg} L = (\ker L)^{\perp} = (\text{Rg} \pi_1)^{\perp}$ and

$$L(x_1 + x_2) = (\pi_1 + \pi_3) L(x_1 + x_2) = \pi_2 L(x_1 + x_2),$$

meaning that in fact $\mathcal{G} x \in X$.

If we keep in mind all these definitions, then the system, (4.9) may be rewritten as

$$x' + \mathcal{G} x = f(x, y),$$

$$y' = g(x, y).$$

Our present task is to check that the assumptions of [17, Theorem 9.1.1] are fulfilled. We first show that $\mathcal{G}$ is sectorial on $X$. 
It is a well-known fact that since $B_2$ is sectorial on $Y_2$, then it is also sectorial on $Y_2^{1/2}$. On the other hand
\[ Gx = B_2x + \left[ -\Delta^{-1}\text{div}(D_\mathcal{C}\sigma(\nabla U)(\nabla(x_1 + x_2))) \right], \]
where in fact the mapping
\[ X_2^{1/2} \ni v \mapsto \Delta^{-1}\text{div}(D_\mathcal{C}\sigma(\nabla U)\nabla v) \in X_2^{1/2} \]
is continuous. Thus, by [17, Theorem 1.3.2] $G$ is sectorial on $X_2^{1/2} \times X_2^{1/2}$ too. But we noticed that $V = \ker L$ is an invariant space for $G$, hence $G$ is sectorial on $X$.

We now check that the assumption (i) of [17, Theorem 9.1.1] holds. We have to guarantee existence of $\mu \geq 0$ and $N \geq 1$ such that if $y_1$, $y_2$ are arbitrary solutions of
\[ y' = g_0(y) \quad (4.12) \]
in $-\infty < t < +\infty$, then
\[ \|y_1(t) - y_2(t)\|_V \leq Ne^{\mu(t-\tau)}\|y_1(\tau) - y_2(\tau)\|_V \]
for all $t$, $\tau$. We shall see that the proof of this fact requires
\[ \text{Re spectrum}(G) \geq \lambda_0 > 0. \quad (4.13) \]
We postpone the proof of (4.13) for a while and we pick any $0 < \mu < \lambda_0$, $N = 1$, then we choose $\eta$ such that for all $v \in V$, $\mu = \|\pi_1\|_{\text{Lip}}\kappa\eta$, where $\kappa$ is such that $\kappa^{-1}\|v\|_X \leq \|v\| \leq \kappa\|v\|_X$. Equation (4.12) yields
\[ y_1(t) - y_2(t) = y_1(\tau) - y_2(\tau) + \int_{\tau}^{t} \pi_1(\hat{H}(y_2(s)) - \hat{H}(y_1(s))) \, ds. \]
By Lemma 4.3 (b) we have that
\[ \|\pi_1(\hat{H}(y_2(s)) - \hat{H}(y_1(s)))\| \leq \|\pi_1\|_{\text{Lip}}\eta\|y_1(s) - y_2(s)\|_X \leq \mu\|y_1(s) - y_2(s)\|_V. \]
Furthermore by Gronwall inequality we obtain for $t > \tau$, as well as for $\tau > t$:
\[ \|y_1(t) - y_2(t)\|_V \leq \|y_1(\tau) - y_2(\tau)\|_V e^{\mu(t-\tau)}. \]
Now we shall show (4.13), it will follow that (ii) of [17, Theorem 9.1.1] holds. In the proof of $G$ being sectorial we have actually shown that $G$ is a
compact perturbation of $B_2$ restricted to a smaller space, thus their essential spectra agree. Hence the essential spectrum of $G$ is empty. It remains to check that the real parts of all eigenvalues are positive. Let us suppose to the contrary, i.e. $(x_1, x_2)$ is an eigenvector such that the corresponding eigenvalue $\lambda$ has $\text{Re} \lambda \leq 0$. Then

$$
\begin{cases}
-L(x_1 + x_2) + \Delta x_1 = \lambda x_1, \\
L(x_1 + x_2) = \lambda x_2.
\end{cases}
$$

After adding them up we obtain

$$
\Delta x_1 = \lambda (x_1 + x_2),
$$

and it also follows that $x_1$ is smooth. We can solve this equation for $x_2$ and feed the result into the first equation of the above system:

$$
-\frac{1}{\lambda} L \Delta x_1 + \Delta x_1 = \lambda x_1.
$$

We now take the inner product in $L^2(\Omega)$ with $x_1$, after rearranging the terms we see

$$
-\frac{1}{\lambda} (L \Delta x_1, x_1) = \lambda \|x_1\|^2_{L^2(\Omega)} - \|\nabla x_1\|^2_{L^2(\Omega)}.
$$

Because of the definition of $L$ and its self-adjointness we obtain

$$
(L \Delta x_1, x_1) = (\Delta x_1, L x_1) = D^2 E(U)(x_1, x_1) \geq 0.
$$

So, if $\text{Re} \lambda \leq 0$, then

$$
0 \leq -\text{Re} \frac{1}{\lambda} (\Delta x_1, L x_1) = \text{Re} \lambda \|x_1\|^2_{L^2(\Omega)} - \|\nabla x_1\|^2_{L^2(\Omega)} < 0,
$$

which is a contradiction; (4.13) holds.

By the very definition, the functions $f, g, D_x g, D_y g, D_y g_0, D_x g_0, D_x f, D_y f$ are uniformly bounded. Moreover, the mappings $(x, y) \mapsto D_x f, D_y f, D_x g, D_y g, D_y g_0, D_y g_0$ are Lipschitz continuous by Lemma 4.4. Thus the assumption (iii) of [17, Theorem 9.1.1] holds. Since the above transformations are independent of $t$ then (iv) of [17, Theorem 9.1.1] is fulfilled too.

Summarizing, [17, Theorem 9.1.1] guarantees existence of an invariant manifold

$$
W^c(U) = \{(x, y): x = h(y), \ y \in V\},
$$

where $h \in C^{1,1}(V, X^c_2)$. Next, we notice that $f(0, 0) = 0 = g(0, 0)$, and
Lemma 4.3 (a) implies that $Df(0,0) = 0$. It is not then difficult to establish using the facts that

$$h(0) = 0, \quad Dh(0) = 0.$$  \hfill (4.14)

Hence, $W^c(U)$ is indeed a center manifold, i.e. it is tangent to the flow at $(0,0)$ in the $x, y$ coordinates.

Moreover, by the same Theorem, because of (4.13), the set $W^c(U)$ is asymptotically stable, meaning that if $\beta_0 < \lambda_1, \|x(\tau)\|_x^2$ is sufficiently small, then there exists a solution $(\tilde{x}(t), \tilde{y}(t))$ in $W^c(U)$ such that

$$\|x(t) - \tilde{x}(t)\|_x^2 + \|y(t) - \tilde{y}(t)\|_y \leq K e^{\beta_0(t-\tau)} \|x(\tau) - h(y(\tau))\|_x^2.$$  

\[ \square \]

Remark. The membership of $h$ in $C^{1,1}$ and (4.14) imply that

$$\|h(y)\|_x^2 = \|h(y) - h(0) - Dh(0)y\|_x^2 = \| \int_0^1 (Dh(sy) - Dh(0)y) \, ds \|_x^2$$

$$\leq \frac{1}{2} \text{Lip}(Dh) \|y\|_x^2.$$  \hfill (4.15)

4.3 Behavior of $\delta E$ on the center manifold

We prove here an inequality which is central to our proof of stability of the flow restricted to the local center manifold. We are able to show this only for certain scalings of (3.1). On the other hand it is sufficient to show stability of the flow for some scaling parameter $a$, because if the flow is stable for some $a$ it is stable for all positive $a$. Let us recall that $V, h: V \to Y^0$ have the same meaning as in the previous subsection and tilde over a letter denotes the transformation defined in (4.3). To be precise we should distinguish between $\tilde{V}$ and $V$, however (4.3) defines an isomorphism between them.

For our convenience we also introduce more notation:

$$\tilde{E}(\tilde{v}) = \int_0^1 [\tilde{W}(\nabla \tilde{v}) + \frac{1}{2} \delta^2 |\Delta \tilde{v}|^2] \, dy.$$  

Lemma 4.6 Let us assume that $W$ satisfies (H1-H2) (and (H5) or (H6-7) depending on assumed boundary conditions), $U$ is fixed local minimum. Then, there exists $a > 0$ such that there exists $U$ a neighborhood of 0 in $V (= \ker D\tilde{E}(U))$ such that if $\tilde{w} = h(v), \tilde{v} \in U$ and $\tilde{v} + h(v) \in W^c(U)$, then

$$D\tilde{E}(\tilde{U} + \tilde{v} + h(v))(\tilde{U} + \tilde{v} + h(v)) \geq 0 \quad \text{for all } \tilde{v} \in U. \hfill (4.16)$$

Remark. If $\dim V = \dim \mathcal{M}$, then (4.16) is trivial because $\mathcal{M} = W^c(U)$.
and the LHS is identically 0. But if \( \dim V > \dim W^{\infty}(U) \), then the above inequality is not obvious.

**Proof.** In our calculations we use that \( u(t) \in W^{4,p}(\Omega) \). This degree of smoothness was established in the proof of Theorem 3.5. We write a Taylor expansion of \( \tilde{E} \), where we use that \( \tilde{U} \) is an equilibrium.

\[
D \tilde{E}(\tilde{U} + \tilde{v} + \tilde{h}(v)) = I_1 + I_2 + I_3 + R,
\]

where

\[
I_1 = D^2 \tilde{E}(\tilde{U})(\tilde{v} + \tilde{h}(v))^2,
\]

\[
I_2 = \frac{1}{2} \int_{\Omega} D^3 \tilde{W}(\nabla_y \tilde{U})(\nabla_y (\tilde{v} + \tilde{h}(v)))^2 dy
\]

\[
I_3 = \frac{1}{3!} \int_{\Omega} D^4 \tilde{W}(\nabla_y \tilde{U})(\nabla_y (\tilde{v} + \tilde{h}(v)))^3 dy
\]

\[
R = \int_{\Omega} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} D^2 \tilde{W}(F) s_1^2 s_2^2 s_3 ds_1 \ldots ds_4 (\nabla_y (\tilde{v} + \tilde{h}(v)))^5 dy,
\]

where \( F = \nabla_y \tilde{U} + s_1 s_2 s_3 s_4 \nabla_y (\tilde{v} + \tilde{h}(v)) \). We will estimate these terms from below.

If we recall that \( D^2 \tilde{E}(\tilde{U})\tilde{v} = 0 \), then it is not difficult to check that

\[
I_1 = \int_{\Omega} [D^2 \tilde{W}(\nabla_y \tilde{U})(\nabla_y \tilde{h}(v))]^2 \, dx \geq \lambda_0 \| \nabla_y \tilde{h}(v) \|_{L^2(\Omega)}^2
\]

where \( \lambda_0 = \lambda_0/a^2 > 0 \), see Proposition 4.2.

Let us look now at \( I_2 \). It is easy to notice that \( D^3 E(U)v^3 = 0 \), hence it follows that

\[
2I_2 = \int_{\Omega} D^3 \tilde{W}(\nabla_y \tilde{U})(3 \nabla_y \tilde{v} \nabla_y \tilde{h}(v) + 3 \nabla_y \tilde{v} \nabla_y \tilde{h}(v)^2 + \nabla_y \tilde{v} \tilde{h}(v)^3) \, dy
\]

\[= I_{21} + I_{22} + I_{23}.
\]

Assumption (H3) for \( R = \| \nabla_x U \|_{L^\infty(\Omega)} \) and \( \tilde{C}(R) := C_R R(1 + R) \) implies that

\[
|I_{21}| \leq \frac{3}{a} \tilde{C}(R) \int_{\Omega} |\nabla_y \tilde{v}|^2 |\nabla_y \tilde{h}(v)| \, dy.
\]

It turns out that \( I_{21} \) is the most difficult term to estimate. One may think that this term scales in \( a \) as a fourth order term because \( \tilde{h}(v) \) is quadratic in \( \tilde{v} \) (see (4.15)). However it is not quite true, i.e. we will show that its scaling
behavior is rather different from the behavior of $I_3$ which is of degree four. Our calculations are not straightforward. We have to exploit in an essential way the fact that $\dim V < \infty$. We note that by Young inequality we obtain

$$|I_{21}| \leq \frac{3}{a} \tilde{C}(R) \left( \frac{1}{\varepsilon} \int_{\Omega} |\nabla_{y} \tilde{v}|^4 \, dy + \varepsilon \int_{\Omega} |\nabla_{y} h(v)|^2 \, dy \right),$$

where $\varepsilon > 0$ is to be specified later. Now, due to (4.8) we see

$$\|\nabla_{x} h(v)\|_{L^2(\Omega)} = \|\nabla_{x} h(v)\|_{L^2(\Omega)}.$$

Subsequently, by (4.15) and the fact that all norm on $V$ are equivalent we come to

$$\|\nabla_{x} h(v)\|_{L^2(\Omega)} \leq C\|h\|_{X^2} \leq C\|
abla_{x} v\|_{L^{4}(\Omega)},$$

so, again by (4.8)

$$\|\nabla_{x} h(v)\|_{L^2(\Omega)} \leq Ca^{1/2}\|\nabla_{x} \tilde{v}\|_{L^4(\Omega)}.$$

Therefore, for some $C > 0$, $|I_{21}|$ may be estimated by

$$|I_{21}| \leq 3\tilde{C}(R) \left( \frac{1}{a\varepsilon} + cC \right)\|\nabla_{x} \tilde{v}\|_{L^4(\Omega)}^{4}.$$

Now, we estimate $I_{22}$ using the equivalence of all norms on $V$ and (4.17)

$$|I_{22}| \leq \frac{3}{a} \tilde{C}(R) \frac{\|\nabla_{x} v\|_{L^\infty(\Omega)}}{a} \int_{\Omega} |\nabla_{y} h(v)|^2 \, dy$$

$$\leq 3Ca^{-3/2}\tilde{C}(R) \|\nabla_{x} \tilde{v}\|_{L^4(\Omega)}\|\nabla_{x} h(v)\|_{L^2(\Omega)}.$$

In a similar way we obtain the estimate for $I_{23}$:

$$|I_{23}| \leq \frac{\tilde{C}(R)}{a} \int_{\Omega} |\nabla_{y} h(v)|^3 \, dy = \frac{\tilde{C}(R)}{a^2}\|\nabla_{x} h(v)\|_{L^2(\Omega)}^3$$

$$\leq \frac{C}{a^2}\tilde{C}(R)\|h(v)\|_{L^2(\Omega)}^3 \leq \frac{C}{a^2}\tilde{C}(R)\|\nabla_{x} v\|_{L^4(\Omega)}^3 \leq aC\tilde{C}(R)\|\nabla_{x} \tilde{v}\|_{L^4(\Omega)}^3.$$

We have to bound $I_3$ carefully. By (4.5) we have

$$D^4\hat{W}(\nabla_{x} U)\xi^4 \geq B_{R}|\xi|^4 > 0, \quad \text{for all } \xi \in \mathbb{R}^2, \quad \xi \neq 0,$$

where $R = \|\nabla_{x} U\|_{L^\infty(\Omega)}$. Thus,
provided that \( \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})} - \| \nabla_y \tilde{h}(\tilde{v}) \|_{L^4(\bar{\Omega})} > 0 \). By (4.8), (4.15) and the equivalence of norms in \( V \) we come to

\[
\| \nabla_y \tilde{h}(\tilde{v}) \|_{L^4(\bar{\Omega})} \leq C a^{1/2} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^2,
\]

that is we need

\[
\frac{1}{2} > C a^{1/2} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}.
\]

Hence, continuing the estimate of \( I_3 \) we end up with

\[
I_3 \geq \frac{B_R}{24} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^4,
\]

provided that (4.18) holds.

We finally estimate the remainder in the Taylor expansion. Using (H2) and the same methods as above, for \( \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})} \leq 1 \) we come to

\[
|R| \leq K a (R^{-5} + 1) \| \nabla_y (\tilde{v} + \tilde{h}(\tilde{v})) \|_{L^5(\bar{\Omega})}^5
\]

\[
\leq a K (R^{-5} + 1) (a^{3/2-1} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})} + a^{3/2-1} \| \nabla_y h(\tilde{v}) \|_{L^4(\bar{\Omega})})^5
\]

\[
\leq K (R^{-5} + 1) a^{1/2} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^6.
\]

We may now combine all the estimate, which yields for some constant \( C > 0 \) independent of \( a \)

\[
I_1 + I_2 + I_3 + R \geq \tilde{\lambda}_0 \| \nabla_y \tilde{h}(\tilde{v}) \|_{L^4(\bar{\Omega})}^2 - 2C(R) \left( \frac{1}{\epsilon a} + \epsilon C \right) \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^4
\]

\[
- 2C \tilde{C}(R) a^{-3/2} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})} \| \nabla_y \tilde{h}(\tilde{v}) \|_{L^4(\bar{\Omega})}^2
\]

\[
- C \tilde{C}(R) a \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^6 + \frac{B_R}{24} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^4
\]

\[
- K (R^{-5} + 1) a^{1/2} \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^6
\]

\[
= \| \nabla_y \tilde{h}(\tilde{v}) \|_{L^4(\bar{\Omega})}^2 \left( \tilde{\lambda}_0 - 2C a^{-3/2} \tilde{C}(R) \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})} \right)
\]

\[
+ \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^4 \left( \frac{B_R}{24} - 2C(R) \left( \frac{1}{\epsilon a} + \epsilon C \right) \right)
\]

\[
- C \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})}^2 \left( \tilde{C}(R) a \| \nabla_y \tilde{v} \|_{L^4(\bar{\Omega})} \right).
\]

We first choose \( \epsilon > 0 \) so that

\[
2\epsilon \tilde{C}(R) < \frac{B_R}{24} \cdot \frac{1}{4},
\]
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next we pick $a > 0$ such that
\[
\frac{2\overline{C}(R)}{ea} < \frac{B_R}{2^4} \cdot \frac{1}{4}.
\]
We next choose $1 > \rho > 0$, such that for $\|\nabla \bar{v}\|_{L^4(\tilde{\Omega})} < \rho$ inequalities (4.18) and the following ones
\[
\lambda_0 - C\overline{a}^{-3/2} \overline{C}(R)\|\nabla \bar{v}\|_{L^4(\tilde{\Omega})}^2 \geq \frac{1}{2} \lambda_0,
\]
\[
C\overline{C}(R)\|\nabla \bar{v}\|_{L^4(\tilde{\Omega})}^2 + ((R^{-5} + 1)\overline{a}^{1/2}\|\nabla \bar{v}\|_{L^4(\tilde{\Omega})}) < \frac{B_R}{2^6}
\]
are fulfilled. Thus, the desired inequality holds
\[
D\bar{E}(\bar{U} + \bar{\theta} + \bar{h}(v))(\bar{U} + \bar{\theta} + \bar{h}(v)) \geq \frac{1}{2} \lambda_0\|\nabla \bar{v}\|_{L^2(\tilde{\Omega})}^2 + \frac{B_R}{2^6}\|\nabla \bar{v}\|_{L^4(\tilde{\Omega})}^4 > 0.
\]

4.4 Proof of the convergence result

We are now ready for the proof of our main Theorem. Let us explain the idea. We first show that the $\omega$-limit set exists, this is guaranteed by the a priori bounds of §3.2. We pick a point $(0, U) \in \omega(u_0, u_1)$. Analyticity of $W$ guarantees that in a neighborhood of $M$ there are no other critical points of $E$. We analyze the flow restricted to the local center manifold $W^c(U)$ – for this we have to consider the cut-off system (4.9). We want to show that the flow restricted to $W^c(U)$ is such that all points in the $\omega(u_0, u_1)$-limit set are stable, hence they are all stable. This implies that $\omega(u_0, u_1)$-set must be a singleton. The proof of stability of the restricted flow depends very much on the inequality from Lemma 4.6. We stress that our analysis does not go through if $U$ is just a saddle point of $E$ - we would have to deal with "bad" third order terms in the Taylor expansion of $E$ near $U$.

Proof of Theorem 4.1. We divide our reasoning in a number of steps. We stress that we work with the $(P, Q)$ variables which are easier to deal with than the original ones.

(1) We first establish existence of the $\omega$-limit set. By Theorem 3.5 (a) the set $\{(P, Q) : t \geq 1\}$ is bounded in $C^{4,\mu'}(\Omega)$, $\mu' > 0$. But the embedding $C^{4,\mu'}(\Omega) \subset C^{4,\mu}(\Omega)$, for $\mu' > \mu$ is compact. Thus existence of a connected compact $\omega$-limit set follows from the standard theory of dynamical systems (c.g. see [17, Theorem 4.3.3] or [15, Theorem 3.8.2]). We also know that
\[
\text{dist}_{C^{4,\mu}(\Omega)}((P, Q), \omega(u_0, u_1)) \rightarrow 0. \tag{4.19}
\]
We note that $E$ is a Liapunov function of the system, hence $\omega(u_0, u_1)$ must be a subset of the set of equilibria, say $\{0\} \times \mathcal{M}$. We also notice that
(2) We claim that there exists $V$ a neighborhood of $M \subset \{E(u) = e_0\}$ in $W^{0,p}(\Omega)$ such that the only critical point of $E$ in $V$ belong to $M$. We work with this function space because of the embedding $W^{0,p}(\Omega) \subset C^4(\Omega)$ and the Remark at the end of §3.2.

In order to check our claim we use the Lojasiewicz inequality
\begin{equation}
\|\nabla E(\xi + u)\|_{W^{-1,2}(\Omega)} \geq |E(\xi + u) - e_0|^{1-\theta(u)},
\end{equation}
where $\|\xi\|_{C^4(\Omega)} < \beta_0(u)$ and $\beta_0(u) > 0$ is sufficiently small, $\theta(u) \in (0, \frac{1}{2})$, and $\|u\|_{W^{-1,2}(\Omega)}$ is defined by
\[ \|u\|^2_{W^{-1,2}(\Omega)} = -\int_{\Omega} u\Delta u \, dx. \]

Inequality (4.20) has been proven in [26, Theorem 3.2].

It is not difficult to check that $M$ is bounded and closed set in $C^4,\mu(\Omega)$, for all $\mu \in (0,1)$. Hence $M$ is compact in $C^4,\mu(\Omega)$, $\mu < 1$, and in fact in $W^{0,p}(\Omega)$. By the very fact that each $\xi \in M$ is a local minimizer of $E$ there exist a $\bar{\xi}$ such that for all $\bar{v} \in B_{W^{0,p}(\Omega)}(\xi, \bar{\xi})$ we have $E(\bar{v}) \geq e_0$. Let us now take $\gamma = \min\{\bar{\xi}(x), \beta_0(x)\}$. By compactness of $M$ we can find a finite covering of $M$ with balls
\[ \mathcal{W} = \bigcup_{i=1}^n B_{W^{0,p}(\Omega)}(x_i, r(x_i)), \]
and (4.20) holds in $\mathcal{W}$. We set $V = \mathcal{U} \cap \mathcal{W}$. Since $\partial \mathcal{W} \cap M = \emptyset$ there exists $d > 0$ such that the set $M_d = \{x \in W^{0,p}(\Omega) : \text{dist}_{W^{0,p}(\Omega)}(x, M) < d\}$ is disjoint from $\partial \mathcal{W}$. Let us set $\mathcal{V} = M_d$. Let us suppose that we have a critical point in $\mathcal{V}$. It is of the form $u + \xi$ where $u \in M$ and $\|\xi\|_{W^{0,p}(\Omega)} < \beta(u)$, then by (4.20) $E(u + \xi) = e_0$ implying $u + \xi \in M$. Our claim follows.

(3) (a) Let us notice that $E^{-1}\{e_0\} \cap \mathcal{W} = M$. For, if for all $d$ such that $0 < d < \bar{d}$ and $x_d$ existed such that $\text{dist}_{W^{0,p}(\Omega)}(x_d, M) < d$ and $x_d \notin M$ and $E(x_d) = e_0$, then by step (2) $DE(x_d) \neq 0$. This would mean that in any neighborhood of $M$ one could find another $y_d$ such that $E(y_d) < E(x_d) = e_0$, which is impossible.

(b) Let us set
\[ m = \min_{u \in \partial \mathcal{W}} E(u). \]
By compactness of $M$ there is $u_m \in \partial \mathcal{W}$ such that $m = E(u_m)$. Sub-step (a) implies that $m > e_0$.
(c) For \( e_0 < e < m \) let us set \( \mathcal{V}_e \) to be a connected component of \( \{ x \in W^{6,p}(\Omega) : E(x) < e \} \) which contains \( \mathcal{M} \). We claim that \( \mathcal{V}_e \subset \mathcal{W} \). If \( x \in W^{6,p}(\Omega) \) is such that \( \text{dist}_{W^{6,p}(\Omega)}(x, \mathcal{M}) = \tilde{d} \), then \( E(x) > e \), hence if \( E(x) \leq e \) then, \( \text{dist}_{W^{6,p}(\Omega)}(x, \mathcal{M}) < \tilde{d} \) or \( \text{dist}_{W^{6,p}(\Omega)}(x, \mathcal{M}) > \tilde{d} \), in the latter case \( x \notin \mathcal{V}_e \). Our claim follows.

We are now in a position to define the set \( \mathcal{U} \). Namely, we choose \( \mathcal{U} := \mathcal{V}_m \).

If \( u_0 \in W^{6,p}(\Omega) \) is such that \( m - E(u_0) > 0 \), so if we take any \( u_1 \in W^{4,p}(\Omega) \) such that \( \int_0^1 u_1^2 < 2(m - E(u_0)) \), then by Theorem 3.5 (see also the remark at the end of §3.2) \( P(t) \), \( Q(t) \) belong to \( W^{6,p}(\Omega) \) for all \( t \geq 0 \) and

\[
m > E(u_0) + \int_0^1 \frac{1}{2} u_1^2 = E(u(t)) + \int_0^1 \frac{1}{2} u_1^2 + \int_0^t \int_\Omega |\nabla u|^2 \, dx \, dt.
\]

Thus \( E(u(t)) < m \) and \( u(t) \in \mathcal{U} \) for all \( t \geq 0 \).

(4) Let us fix a point \((0, U)\) in \( \omega(u_0, u_1) \)-limit set. We subsequently choose \( \eta > 0 \) provided by Theorem 4.5. We conclude existence of the center manifold \( W^c(U) \) for the cut-off system (4.9).

(5) We prove stability of the flow scaled first then restricted to the \( W^c(U) \).

The scaling parameter is given by Lemma 4.6. For the sake of simplicity of notation we suppress the dependence upon \( \eta \) of variables \( \tilde{P} \), \( \tilde{Q} \). We set \( \tilde{Q} = \tilde{Q} - \tilde{U} \), and we look at the equation for \( \tilde{Q} \) if \( \| \tilde{P} + \tilde{Q} \|_{X^2} \leq \eta \).

\[
\frac{d}{dt} \tilde{Q}_t = -\Delta u^{-1} \text{div}_x \sigma(\nabla_y (\tilde{Q} + \tilde{P} + \tilde{U})) + \delta \Delta u^{-1} \sigma(\tilde{Q} + \tilde{P})
\]
or

\[
\Delta u \tilde{Q}_t = -\text{div}_x \sigma(\nabla_y (\tilde{Q} + \tilde{P} + \tilde{U})) + \delta \Delta u^{-1} \sigma(\tilde{Q} + \tilde{P}). \tag{4.21}
\]

We now write \( \tilde{P} + \tilde{Q} = \tilde{U} + \tilde{v} + \tilde{h}(v) \), where \( \tilde{v} \in \tilde{V} \). Thus, \( \Delta u \tilde{Q}_t = \Delta u (\tilde{U} + \tilde{v} + \tilde{h}(v) - \tilde{P}) \), and we take inner product of (4.21) with \( \tilde{v} + \tilde{h}(v) \). We end up with

\[
(\Delta u (\tilde{U} + \tilde{v} + \tilde{h}(v) - \tilde{P}), \tilde{v} + \tilde{h}(v)) = \nabla \tilde{E}(\tilde{U} + \tilde{v} + \tilde{h}(v) + \tilde{h}(v)),
\]

by Lemma 4.6 the RHS in non-negative. So, we have

\[
-\frac{1}{2} \frac{d}{dt} \left\| \nabla_y (\tilde{v} + \tilde{h}(v)) \right\|_{L^2(\tilde{U})}^2 \geq \int_\Omega \Delta u \tilde{P}_t (\tilde{v} + \tilde{h}(v)) \, dy.
\]

We integrate this in time over \([t_0, t]\) and subsequently we integrate the right-hand-side by parts. This yields,
It does not matter whether or not \( P \) is scaled, by (4.19) and \( \omega(u_0, u_1) \subset \{0\} \times \mathcal{M} \) for any \( \varepsilon \) one can find \( t_0 > 0 \) such that
\[
\|P\|_{W^{1,2}(\Omega)} \leq \varepsilon \quad \text{for } t \geq t_0.
\]
We claim that we have (possibly after increasing \( t_0 \))
\[
I(t_0) := \int_{t_0}^{\infty} (\|\tilde{Q}_t\|_{L^2(\Omega)}^2 + \|\Delta_y \tilde{P}\|_{L^2(\Omega)}^2) < \varepsilon.
\]
We note that \( \tilde{Q}_t = \tilde{P}_t - \Delta_y \tilde{P} \), so
\[
I(t_0) \leq \int_{t_0}^{\infty} (2\|\tilde{P}_t\|_{L^2(\Omega)}^2 + 3\|\Delta_y \tilde{P}\|_{L^2(\Omega)}^2) \, dt.
\]
The second integral is finite due to Theorem 3.4 (b). The first one is finite because of Poincaré inequality and (3.24)
\[
\int_{t_0}^{\infty} \int_\Omega |P_t|^2 < C \int_{t_0}^{\infty} \int_\Omega |
abla \Delta P_t|^2 < \infty,
\]
so
\[
\lim_{t_0 \to \infty} \int_{t_0}^{\infty} \int_\Omega |P_t|^2 = 0.
\]
Hence, for sufficiently large \( t_0 \)
\[
2 \left| \int_{t_0}^{\infty} \int_\Omega \Delta_y \tilde{P}_t \, dy \right| \leq I(t_0) \leq \varepsilon.
\]
(6) We now combine these estimates to get
\[
\|\nabla_y (\tilde{u} + \tilde{h}(v))\|_{L^2(\Omega)}(t) \leq 3\|\nabla_y (\tilde{u} + \tilde{h}(v))\|_{L^2(\Omega)}(t_0) + 4\varepsilon + 5\varepsilon^2.
\]
The above inequality shows that for some $\alpha > 0$, the local minimizer $U$ is stable with respect to the flow restricted to the center manifold. But stability is independent of the scaling, so in particular $U$ is stable with respect to the unscaled flow restricted to the local center manifold.

(7) We show that $(0, U)$ is stable. By Theorem 4.5 $W^c(U)$ is asymptotically stable. So, if $(P_0, Q_0)$ is sufficiently close to $(0, U)$ then for some $\alpha < 1$ (see Theorem 4.5)

$$
\|(P(t), Q(t)) - (0, U)\|_{X^2} \\
\leq \|(P(t), Q(t)) - (v + h(v))(t)\|_{X^2} + \|(v + h(v))(t) - (0, U)\|_{X^2} \\
\leq e^{-\beta_0 t}\|(P(0), Q(0)) - (v + h(v))(0)\|_{X^2} \\
+ \|(v + h(v))(t) - (0, U)\|_{X^2},
$$

where $v + h(v)$ is some trajectory in $W^c(U)$. By step (6) stability of $(0, U)$ follows.

(8) We conclude the proof. Let us suppose that there are two different $U_i$, $i = 1, 2$, in $\omega(u_0, u_1)$. Hence there exist sequences $\{t_i\}_{i=1}^\infty$ converging to infinity such that $(P(t_i), Q(t_i)) \to (0, U_i)$. Because they are stable for given $\epsilon_1$ there exists $\epsilon_2 > 0$ such that, the flow originating in $B_{C^4,\nu}(0, U_i, \epsilon_2)$ will stay in $B_{C^4,\nu}(0, U_i, \epsilon_1)$. But this is impossible if we choose, $\epsilon_1 < \frac{1}{2}\|U_1 - U_2\|_{C^4,\nu}(\Omega)$. Hence

$$
\omega(u_0, u_1) = \{(0, U)\}
$$

for some $U \in C^4,\nu(\Omega)$. \hfill \Box

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EFFET RÉGULARISANT MICROLOCAL ANALYTIQUE
POUR L'ÉQUATION DE SCHRÖDINGER :
LE CAS DES DONNÉES OSCILLANTES

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0. Introduction et résultats

L'effet régularisant microlocal pour l'équation de Schrödinger à coefficients variables a fait l'objet de travaux récents dus principalement à Kapitanski-Safarov [KS], Craig-Kappele-Strauss [CKS], Doi [D], Shanarin [Sh], Wunsch [W] et Robbiano-Zuily [RZ]. Dans tous ces travaux il s'agit de comprendre l'influence du comportement à l'infini de la donnée à l'instant initial sur la régularité microlocale de la solution pour des temps ultérieurs. Dans [RZ] nous examinions la régularité analytique microlocale de la solution pour des données à décroissance exponentielle à l'infini. Dans le présent travail, qui constitue une suite à [RZ], nous considérons le cas d'une donnée $u_0$ oscillante de la forme $u_0(x) = e^{iv(x)}a(x)$. Dans le cas du front d'onde $C^\infty$ le type de résultats que nous obtenons ont été prouvés par Shanarin [Sh] lorsque les coefficients sont constants et par Wunsch [W] lorsqu'ils sont $C^\infty$. Décrivons maintenant plus précisément ces résultats.

Soient $g_0$ la métrique plate sur $\mathbb{R}^n$ et $g$ une métrique Riemannienne analytique telle que $g \geq \nu g_0$ pour un certain $\nu > 0$. Désignons par $\Delta_g$...
le Laplacien associé et par $p$ son symbole principal. On considérera dans ce qui suit l'opérateur à coefficients analytiques

\begin{equation}
P = P(y, D_y) = -\Delta_y + \sum_{|\beta| \leq 1} a_\beta(y) D_y^\beta.
\end{equation}

Si $\rho_0 = (y_0, \eta_0)$ est un point de $T^*\mathbb{R}^n \setminus 0$ on note $\gamma^+_{\rho_0}$ la bicaractéristique future de $p$ issue de $\rho_0$. Plus précisément on considère le problème

\begin{equation}
\begin{cases}
\dot{Y}(s) = -\frac{\partial p}{\partial \eta}(Y(s), \Theta(s)) \\
\dot{\Theta}(s) = -\frac{\partial p}{\partial y}(Y(s), \Theta(s))
\end{cases}
(0.2)
(Y(0), \Theta(0)) = \rho_0
\end{equation}

et alors

\begin{equation}
\gamma^+_{\rho_0} = \{(Y(s), \Theta(s)), \ s \in [0, s^*], \ ou (Y, \Theta) \text{ est solution de } (0.2)\}.
(0.3)
\end{equation}

Notons que la projection sur la base de $\gamma^+_{\rho_0}$ une géodésique future issue de $y_0$. Nous faisons l'hypothèse suivante

\begin{equation}
\begin{cases}
\text{la géodésique future issue de } y_0 \text{ est définie pour tout } s \in [0, +\infty[ \text{ et } \lim_{s \to +\infty} |Y(s)| = +\infty.
\end{cases}
(0.4)
\end{equation}

La bicaractéristique $\gamma^+_{\rho_0}$ est alors elle aussi définie pour tout $s \geq 0$.

A cette géodésique et à $\varepsilon > 0$ on associe l'ensemble

\begin{equation}
\Gamma_{\varepsilon} = \bigcup_{s \geq 0} \{x \in \mathbb{R}^n : |x - Y(s)| < \varepsilon (1 + s)\}.
(0.5)
\end{equation}

La deuxième hypothèse que nous ferons exprime que dans un certain $\Gamma_{\varepsilon}$ l'opérateur $P$ défini en (0.1) est asymptotiquement égal à $-\Delta_{y_0} = -\sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$.

Plus précisément on supposera que

\begin{equation}
\begin{cases}
\text{il existe des constantes positives } \varepsilon_0, \ C_0, \ K_0, \ R_0, \ \sigma_0 \text{ telles que pour tout } \alpha \in \mathbb{N}^n \text{ et tout } y \in \Gamma_{\varepsilon_0} \text{ tel que } |y| \geq R_0
\end{cases}
(0.6)
\end{equation}

\begin{equation}
\sum_{j,k=1}^n |D^j_y (g^{jk}(y) - \delta_{jk})| + \sum_{|\beta| \leq 1} |D^\alpha_y a_\beta(y)| 
\leq C_0 K_0^{(\alpha \|\alpha\| + \sigma_0)}|y|^{-((1 + |\alpha| + \sigma_0))}
\end{equation}
où δ_{jk} est le symbole de Kronecker et \((g^{jk}) = (g_{jk})^{-1}\) où \(g = g_{jk} dx^j \otimes dx^k\).

On montre alors sous les hypothèses (0.4) et (0.6) que

\[
\lim_{s \to +\infty} \Theta(s) = \eta_\infty \in \mathbb{R}^n \setminus 0,
\]

où \(\Theta\) a été défini dans (0.2).

Précisons maintenant la forme de notre donnée de Cauchy. Soit \(\psi\) une fonction réelle et analytique sur \(\mathbb{R}^n\) et \(a \in L^2(\mathbb{R}^n)\) telles que \(\psi\) et \(a\) se prolongent en des fonctions holomorphes dans un ensemble de la forme

\[
\Gamma_{\varepsilon_0} = \{ z \in \mathbb{C}^n : \text{Re} z < \delta_0 |\text{Re} z|, \ |z| > R \}
\]

et vérifient

\[
\text{il existe } m \geq 1, \ M \geq 1, \ R > 0 \text{ telles que pour } z \in \Gamma_{\varepsilon_0}, \ |\psi(z)| \leq C|z|^m, \ |a(z)| \leq C|z|^M.
\]

Remarquons qu'il existe alors \(\varepsilon'_0 < \varepsilon_0\) et \(R' \geq R\) tel que

\[
|\partial^\alpha \psi(z)| \leq C_\alpha |z|^{m-|\alpha|}, \ z \in \Gamma_{\varepsilon'_0}, \ |z| \geq R'.
\]

La donnée de Cauchy sera prise de la forme

\[
u_0(y) = e^{i\psi(y)} a(y), \ y \in \mathbb{R}^n.
\]

Soit \(u \in C^0(]-\infty, 0[, L^2(\mathbb{R}^n))\) une solution du problème

\[
\begin{cases}
\frac{\partial u}{\partial t} + i P(y, D_y) u = 0, & t < 0 \\
u_{|t=0} = u_0.
\end{cases}
\]

Le résultat principal de ce travail est alors le

**Théorème 0.1.** Soient \(P\) défini en (0.1), \(\rho_0 \in T^* \mathbb{R}^n \setminus 0\) et \(t_0 < 0\). On suppose (0.4) et (0.6) satisfaites. Soit \(u\) une solution de (0.11), (0.10) où \(a\) et \(\psi\) vérifient (0.9). On suppose, avec \(\eta_\infty\) défini en (0.7), qu'il existe \(C > 0\), \(\lambda_0 > 0\) telles que soit \(m = 1\), soit \(m \geq 2\) et

\[
\left| \frac{\partial \psi}{\partial y} (2\lambda |t_0| \eta_\infty) - \lambda \eta_\infty \right| \geq C \lambda^{m-1}, \ \lambda \geq \lambda_0.
\]

Alors \(\rho_0 \notin WF_a(u(t_0, \cdot))\).
On déduit facilement de ce résultat le

**Corollaire 0.2.** Soient $P$, $\rho_0$, $t_0$ comme au théorème 0.1. Soit $a$ vérifiant (0.9) et supposons que $\psi(y) = P_m(y) + R_m(y)$ où $P_m$ est un polynôme homogène de degré $m \geq 1$ et $R_m(z) = a(|z|^m)$ dans $\Gamma_{\epsilon_0}$ pour $|z| \to +\infty$. Alors $\rho_0 \notin WF_\alpha(u(t_0, \cdot))$ dans chacun des cas suivants

a) $m = 1$.

b) $m = 2$, $P_2(y) = (Ay, y)$ et $\eta_\infty$ n'est pas vecteur propre de la matrice $A$ pour la valeur propre $(4|t_0|)^{-1}$.

c) $m \geq 3$ et $\frac{\partial P_m}{\partial y}(\eta_\infty) \neq 0$.

En particulier si $m = 2$ et si $A$ a toutes ses valeurs propres négatives ou si $m \geq 3$ et $\frac{\partial P_m}{\partial y}$ n'a pas de zéro réel alors, pour tout $t_0 < 0$, $\rho_0 \notin WF_\alpha(u(t_0, \cdot))$.

I. Rappels

Nous rappelons dans ce paragraphe un certain nombre de résultats obtenus dans [RZ] qui nous seront utiles dans la suite.

1) Le front d'onde analytique uniforme et sa propagation

Ce front d'onde sera défini, en suivant Sjöstrand [Sj], par la transformation de FBI (Fourier-Bros-Iagolnitzer).

Soit $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$. Soient $x_0 \in \mathbb{C}^n$ et $\varphi$ une fonction holomorphe dans $U_{x_0} \times V_{y_0}$ (des voisinages de $x_0$ et $y_0$ dans $\mathbb{C}^n$) telle que

$$\frac{\partial \varphi}{\partial y}(x_0, y_0) = -\eta_0,$$

$$\operatorname{Im} \frac{\partial^2 \varphi}{\partial y^2}(x_0, y_0) \text{ est définie positive,}$$

$$\det \left( \frac{\partial^2 \varphi}{\partial x \partial y} \right)(x_0, y_0) \neq 0.$$

Alors $\Phi(x) = \max_{y \in V_{y_0}} \operatorname{Re}(i\varphi(x, y))$ est bien définie pour $x \in U_{x_0}$.

Soit $a(x, y, \lambda) = \sum_{k \geq 0} \lambda^{-k} a_k(x, y)$ un symbole analytique d'ordre zéro
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elliptique en \((x_0, y_0)\); ceci veut dire que les \(a_k\) sont holomorphes dans
\(U_{x_0} \times V_{y_0}\), vérifiant \(|a_k(x, y)| \leq C^{k+1} k^k\) et \(a_0(x_0, y_0) \neq 0\).

Soit \(\chi \in C_0^\infty(\mathbb{R}^n)\) à support près de \(y_0\), \(\chi = 1\) au voisinage de \(y_0\).

Soit \(u(t, \cdot)\) une famille de distributions sur \(\mathbb{R}^n\) dépendant d'un paramètre réel \(t\). Soit \(t_0 \in \mathbb{R}\).

On dira que \(\rho_0 \notin \overline{WF_a}(u(t_0, \cdot))\) si il existe \(\varphi, a, \chi\) comme ci-dessus, un
voisinage \(W_{x_0}\), des constantes positives \(C, \mu, \varepsilon\) telles que

\[
|Tu(t, x, \lambda)| \leq C e^{\lambda \Phi(x) - \mu \lambda}
\]

pour tout \(x \in W_{x_0}\) et tout \(t \in \mathbb{R}\) tel que \(|t - t_0| \leq \varepsilon\). Ceci définit le front
d'onde analytique uniforme \(WF_a(u(t_0, \cdot))\) et c'est un résultat fondamental de
la théorie de SJÖSTRAND [S] que cette définition est indépendante du choix
de \(\varphi, a, \chi\) vérifiant les propriétés ci-dessus. Evidemment \(WF_a(u(t_0, \cdot)) \subset \overline{WF_a}(u(t_0, \cdot))\).

On a alors le

**Théorème 1.1** ([RZ] Théorème 6.1). Soit \(\rho_0 \in T^*\mathbb{R}^n \setminus 0\) et \(\gamma_{\rho_0}\) la bica-
ratéristique de \(P\) passant par \(\rho_0\). Soit \(t_0 \in \mathbb{R}\). Soit \(u \in C^0(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))\) une
solution de l'équation \(\frac{\partial u}{\partial t} + i P(y, D_y) u = 0\). Alors si \(\rho_0 \notin \overline{WF_a}(u(t_0, \cdot))\) on a
\(\gamma_{\rho_0} \cap \overline{WF_a}(u(t_0, \cdot)) = \emptyset\).

2) L'opérateur \(S\)

Rappelons tout d'abord qu'un point \(\rho_1 = (y_1, \eta_1) \in T^*\mathbb{R}^n \setminus 0\) est dit
"sortant" si \(y_1 > R\) (où \(R\) est assez grand en fonction des données \(C_0, K_0, R_0\)
(0.6)) et \(y_1 \cdot \eta_1 \geq 0\).

La bicaractéristique \((y(s), \eta(s))\) de \(P\) issue d'un point sortant est définie
pour tout \(s \geq 0\) et vérifie ([RZ] Théorème 2.1)

\[
y(s) = y_1 + 2s \eta_1 + Z(s), \quad \eta(s) = \eta_1 + \zeta(s)
\]

\[
|Z(s)| \leq \frac{C}{R^{1+\sigma_0}} s, \quad |\zeta(s)| \leq \frac{C}{R^{1+\sigma_0}}
\]

où \(C\) est une constante qui ne dépend que de \(C_0, K_0\) (définis en (0.6)) et \(\eta_1\).
Si \(\rho_1 \in \gamma_{\rho_0}\) avec \(\rho_1 = (Y(s_1), \Theta(s_1))\), on a

\[
y(s) = Y(s + s_1), \quad \eta(s) = \Theta(s + s_1).
\]
Soit $\Gamma_{e_0}$ l'ensemble défini en (0.5), (0.6) et prenons $v \in L^2(\Gamma_{e_0})$. Posons $x_0 = (x_0',0)$ où $x_0' = y_1' - i\eta_1'$. On a alors la

**Proposition 1.2** ([RZ] Proposition 5.1). Il existe un opérateur $S$ tel que si l'on pose pour $v \in L^2(\Gamma_{e_0})$, $I(x) = \lambda^{-1}D_{x_1}Sv(x) - \lambda^{-2}SPv(x)$ il existe $\varepsilon^*$. $\mu > 0$, $N \in \mathbb{N}$, $C > 0$ telles que pour $|x' - x_0'| < \varepsilon^*$, $|\text{Im} x_n| < \varepsilon^*$, $\text{Re} x_n \geq 0$.

\[ |I(x)| \leq C e^{\lambda \Phi(x') - \lambda \mu (1 + |x_n|)^N} \|v\|_{L^2(\Gamma_{e_0})}. \]

L'opérateur $S$ est de la forme suivante

\[ S v(x) = \int_{\mathbb{R}^n} e^{i\lambda \Phi(x,y)} f(x,y,\lambda) \chi \left( \frac{y - y(\text{Re} x_n)}{1 + |x_n|} \right) v(y) dy. \]

Ici $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 1$ si $|x| \leq \frac{1}{2} \varepsilon$, $\chi(x) = 0$, $|x| \geq \varepsilon$ où $\varepsilon \leq \varepsilon_0$, $y(s)$ est la projection de la bicaractéristique future issue de $\rho_1$, $\varphi$ est une fonction holomorphe dans un voisinage de l'ensemble

\[ E = \left\{ (x,y) \in C^{2n} : |x' - x_0'| < \varepsilon_1, |\text{Im} x_n| < \varepsilon_1, \text{Re} x_n > 0 \right\}, \]

où $\varepsilon_1 < \varepsilon_0$ et $\varphi$ vérifie (1.1), (1.2), (1.3) ; enfin $f$ est un symbole analytique d'ordre zéro elliptique en $(x_0,y_1)$.

Rappelons les propriétés de $\varphi$. Tout d'abord ([RZ], (3.31), (2.23)) pour $(x,y) \in E$

\[ \frac{\partial \varphi}{\partial y}(x,y) = \frac{i}{1 - 2ix_n} \left\{ y - \left( y_{1n} - i\eta_{1n} \right) + F(x,y) \right\}, \]

\[ |F(x,y)| \leq \frac{C}{\text{Re} x_n}. \]

D'autre part si $\varepsilon_1$ (dans (1.10)) est assez petit il existe une unique fonction $x \mapsto tz(x)$ de l'ensemble $\Omega = \{ x \in \mathbb{C}^n : |x' - x_0'| < \varepsilon_1, |\text{Im} x_n| < \varepsilon_1, \text{Re} x_n > 0 \}$ à valeurs dans $\mathbb{R}_0^n$ telle que ([RZ] formule (3.36)).

\[ \text{Im} \frac{\partial \varphi}{\partial y}(x,z(x)) = 0, \quad x \in \Omega. \]

De plus ([RZ], (3.38))

\[ z(x) = \left( \frac{\text{Re} x' - B}{y_{1n} + \frac{B}{A} \eta_{1n}} \right)^2 + \frac{A^2 + B^2}{A} G(x) \]
où

\[
\begin{align*}
A = 1 + 2 \operatorname{Im} x_n, & \quad B = 2 \operatorname{Re} x_n \\
|G(x)| \leq \frac{C}{R^{\epsilon_0}(1 + |x_n|)}, & \quad x \in \Omega.
\end{align*}
\]

La fonction \( \varphi \) vérifie en outre les estimations suivantes dans \( E \) (cf. (2.5))

\[
\begin{align*}
\left| \frac{\partial^2 \varphi}{\partial y^2}(x, y) \right| & \leq \frac{C_1}{1 + |x_n|} \\
\left| \frac{\partial^2 \varphi}{\partial y^2}(x, y) \right| & \leq \frac{C_2}{(1 + |x_n|)^2} \\
\left| \frac{\partial^2 \varphi}{\partial y^2}(x, y) \right| & \geq \frac{C_3}{(1 + |x_n|)^2}
\end{align*}
\]

(voir [RZ] Lemme 3.7 et (3.32)), où les \( C_i \) dépendent des constantes données dans (0.6).

Enfin toujours pour \( \epsilon \) assez petit, la fonction

\[
\Phi(x) = -\operatorname{Im} \varphi(x, z(x)) = \max \{ \operatorname{Re} \varphi(x, y), \quad y \in \mathbb{R}^n, \quad (x, y) \in E \}
\]
est bien définie et vérifie ([RZ] Lemme 3.8)

\[
\frac{\partial \Phi}{\partial \operatorname{Re} x_n}(x) = 0, \quad x \in \Omega.
\]

Comme, par définition, \( x'_0 = y'_1 - i\eta'_1 \), il résulte de (1.14) que

\[
z(x) = y_1 + \left( \begin{array}{c}
\operatorname{Re}(x' - x'_0) \\
0
\end{array} \right) + AG \\
+ B \left\{ \eta_1 - \frac{2 \operatorname{Im} x_n}{A} = y_1 - \frac{1}{A} \left( \operatorname{Im}(x' - x'_0) \right) + BG \right\}
\]
de sorte que l'on peut écrire

\[
\begin{align*}
z(x) = z + B \eta, & \quad B = 2 \operatorname{Re} x_n \quad \text{où} \\
|z - y_1| & \leq C \varepsilon_1, \quad |\eta - \eta_1| \leq C \varepsilon_1 \text{ si } |x' - x'_0| + |\operatorname{Im} x_n| + \frac{1}{R^{\epsilon_0}} \leq \varepsilon_1.
\end{align*}
\]

D'autre part, en utilisant (1.11), on peut écrire
\[
\frac{\partial \varphi}{\partial y}(x, z(x)) = -\frac{1}{2x_n + i} \left\{ z(x) - \left( \frac{x'}{y_{1n} - i \eta_{1n}} \right) + F(x) \right\}.
\]

Compte tenu de (1.14) on obtient

\[
\frac{\partial \varphi}{\partial y}(x, z(x)) = \frac{-1}{2x_n + i} \left\{ \left\{ i + \frac{B}{A} \right\} \left( -\text{Im} \ \frac{x'}{\eta_{1n}} \right) + \frac{A^2 + B^2}{A} G + F \right\}
\]

e t i + \frac{B}{A} = \frac{2ix_n + i}{A}. Par conséquent

\[
\frac{\partial \varphi}{\partial y}(x, z(x)) = -\frac{1}{A} \left( -\text{Im} \ \frac{x'}{\eta_{1n}} \right) - \frac{A^2 + B^2}{A(2x_n + i)} G - \frac{F}{2x_n + i}.
\]

Or

\[
-\frac{1}{A} \left( -\text{Im} \ \frac{x'}{\eta_{1n}} \right) = -\eta_1 + \frac{2 \text{Im} x_n}{1 + 2 \text{Im} x_n} \eta_1 + \frac{1}{A} \left( \text{Im} \left( x' - x'_0 \right) \right).
\]

On déduit de (1.12), (1.15) que

\[
\begin{cases}
\frac{\partial \varphi}{\partial y}(x, z(x)) = -\eta_1 + H(x) \quad \text{où} \\
|H(x)| \leq C \varepsilon_1 \quad \text{si} \quad |x' - x'_0| + |\text{Im} x_n| + \frac{1}{R^{\sigma_0}} \leq \varepsilon_1. \quad \text{Re} x_n \geq 0.
\end{cases}
\]

II. Preuve du théorème 0.1

Soient \( P, t_0, \rho_0 \) comme au théorème 0.1. Nous allons démontrer que sous la condition (0.12) on a \( \rho_0 \notin \overline{W \Phi}(u(t_0, \cdot)) \), ce qui est un résultat plus fort que celui annoncé. On raisonne par l’absurde. Supposons que \( \rho_0 \in \overline{W \Phi}(u(t_0, \cdot)) \). D’après le théorème 1.1, \( \gamma_{\rho_0}^+ \subset \overline{W \Phi}(u(t_0, \cdot)) \). On utilise alors le

Lemme II.1 ([RZ] Lemme 7.1). Soient \( \rho_0 \in T^* \mathbb{R}^n \setminus 0 \) et \( \gamma_{\rho_0}^+ \) la bica-ractéristique future issue de \( \rho_0 \). Si \( \lim_{s \to +\infty} |Y(s)| = +\infty \), pour tout \( R \) assez grand il existe \( \rho_1 = (y_1, \eta_1) \in \gamma_{\rho_0}^+ \) tel que \( |y_1| > R \) et \( y_1 \cdot \eta_1 \geq 0 \) (i.e. \( \rho_1 \) est un point sortant).

Pour \( R \) assez grand (choisi ultérieurement en fonction des données) on prend un point \( \rho_1 \in \gamma_{\rho_0}^+ \) sortant. D’après ci-dessus on a \( \rho_1 \in \overline{W \Phi}(u(t_0, \cdot)) \). Nous allons montrer que ceci est impossible.
Soit $S$ l'opérateur défini au paragraphe 1.2. Alors, avec les notations de la Proposition 1.2, $Su(t, x)$ est solution du problème

\[
\begin{cases}
\left( \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x_n} \right) Su(t, x) = i\lambda^2 I(t, x), & t < 0 \\
Su|_{t=0} = Su_0(x)
\end{cases}
\]

où $I$ vérifie (1.8).

On en déduit que pour $|x' - x_0| + |x_n| < \varepsilon_2$, $|t - t_0| < \delta_0$, ou $\varepsilon_2 \leq \varepsilon_1$. $\varepsilon_2 \leq \varepsilon_1$ et $\delta_0$ est assez petit, on a

\[
Su(t, x) = Su_0(x', x_n - \lambda t) + i\lambda^2 \int_0^t I \left( \sigma; (x', x_n + \lambda(\sigma - t)) \right) d\sigma.
\]

Comptant tenu de (1.8) et (1.18) on a pour $|x - x_0| < \varepsilon_2$, $|t - t_0| < \delta_0$

\[
\lambda^2 \int_0^t I \left( \sigma; (x', x_n + \lambda(\sigma - t)) \right) d\sigma \leq C e^{\lambda \Phi(x') - \mu_0 \lambda} \sup_{|t - t_0| \leq \delta} \|u(t)\|_{L^2}
\]

où $\mu_0 > 0$.

Nous allons montrer le

**Lemme 11.2.** Sous les hypothèses du théorème 0.1, il existe $\varepsilon_3 > 0$ tel que pour tout $t_0 < 0$ il existe des constantes positives $C$, $\delta_1$, $\lambda_0$, $\mu$ telles que pour $|x - x_0| + \frac{1}{|X_n|} < \varepsilon_3$, $|t - t_0| < \delta_1$ et $\lambda > \lambda_0$

\[
|Su_0(x', x_n - \lambda t)| \leq C e^{\lambda \Phi(x') - \mu \lambda}.
\]

**PRÉUVE.** Nous poserons dans ce qui suit,

\[
X = (x', x_n - \lambda t) \in \mathbb{C}^n.
\]

D'après (0.10) et (1.9) on peut écrire

\[
Su_0(X) = \int_{\mathbb{R}^n} e^{i\lambda (\varphi(X, y) + \lambda \psi(y))} f(X, y, \lambda) \left( \frac{y - y(\text{Re}X_n)}{1 + |X_n|} \right) a(y) dy.
\]

où $\text{supp} \, \chi \subset \{|x| \leq \varepsilon_0\}$, $\chi = 1$ si $|x| \leq \frac{1}{2} \varepsilon_0$ où $\varepsilon_0 < \varepsilon_0$ est tel que (0.9)' soit vérifiée.

Soit $\chi_1 \in C_0^\infty(\mathbb{R}^n)$, $\text{supp} \, \chi_1 \subset \{|x| \leq \frac{1}{2} \varepsilon_0\}$, $\chi_1 = 1$ si $|x| \leq \frac{1}{3} \varepsilon_0$. Fixons $x, t, \lambda$ et posons

\[
\theta(X, y) = \varphi(X, y) + \frac{1}{\lambda} \psi(y).
\]
Soit $m \geq 1$ le réel intervenant dans les conditions (0.9), (0.12). Soit $\delta > 0$ un (petit) réel à déterminer. Considérons le contour dans $\mathbb{C}^n$

$$\Sigma = \left\{ z \in \mathbb{C}^n : z = y + i \chi \left( \frac{y - y(\text{Re}X_n)}{1 + |X_n|} \right) \frac{\delta}{\lambda^{2(m-1)}} \right\}.$$  

D’après (1.19) on a $z(X) = z + 2(\text{Re}X_n - \lambda t)\eta$ où $|z - y_1| \leq C_3 \varepsilon_3$, $|\eta - \eta_1| \leq C_3 \varepsilon_3$ si $|x' - x'_0| + |\text{Im} y_1| + \frac{1}{R_\varepsilon} \leq \varepsilon_3$. On en déduit

$$\left\{ \begin{array}{l} z(X) = z - 2\lambda t \eta \text{ où } |\tilde{z} - y_1| \leq C_3 \varepsilon_3, \ |\eta - \eta_1| \leq C_3 \varepsilon_3 \\ \text{si } |x' - x'_0| + |y_1| + \frac{1}{R_\varepsilon} \leq \varepsilon_3 \text{ et } |t - t_0| \leq \delta_1. \end{array} \right.$$  

D’après (1.20), (2.8) et (2.6) on peut écrire

$$\left\{ \begin{array}{l} \frac{\partial \theta}{\partial y}(X, z(X)) = -\eta_1 + \frac{1}{\lambda} \frac{\partial \psi}{\partial y}(\tilde{z} + 2\lambda t \eta) + H(X) \\ |H(X)| \leq C_3 \varepsilon_3 + 2C_3 \delta_1 \chi^{-2} \text{ si } |x' - x_0| + |x_0| \\ \text{si } |x' - x'_0| + |x_0| + \frac{1}{R_\varepsilon} \leq \varepsilon_3, \ |t - t_0| \leq \delta_1. \end{array} \right.$$  

D’après (1.5), $\eta(s) \rightarrow \eta_{\infty}$ si $s \rightarrow +\infty$ et $|\eta_{\infty} - \eta_1| \leq \frac{C_3 \varepsilon_3}{R_\varepsilon}$. Compte tenu de l’hypothèse (0.12) si $\varepsilon_3$ est assez petit et si $\varepsilon_3 \leq \varepsilon_1$, (2.9) implique que

$$\left\{ \begin{array}{l} \left| \frac{\text{Re} \ \partial \theta}{\partial y}(X, z(X)) \right| \geq \frac{1}{2} \lambda^{m-2}, \ \lambda \geq \lambda_0, \ \text{si} \\ |x - x_0| + \frac{1}{R_\varepsilon} \leq \varepsilon_3. \end{array} \right.$$  

D’autre part, d’après (0.9), (0.9)', (1.11), (1.14) et (1.15) on a

$$\left| \frac{\partial \theta}{\partial y}(X, z(X)) \right| \leq C_4 \lambda^{m-2}.$$  

Il résulte de (2.7) que sur $\Sigma$ on a

$$\left| \text{Im } z \right| \leq \frac{C_4 \delta}{\lambda^m} \leq C_4 \delta.$$
On choisit $\delta$ assez petit pour que $C_1 \delta \leq \delta_0$ où $\delta_0$ a été introduit dans (0.8).

Posons alors pour $z \in \mathbb{C}^n$, $|\text{Im } z| \leq C_1 \delta$, $\text{Re } z \in \Gamma_{\epsilon_0}$

\begin{equation}
\omega = e^{i\lambda \theta(X,z)} f(X,z,\lambda) \chi \left( \frac{z - y(\text{Re } X_n)}{1 + |X_n|} \right) a(z) dz.
\end{equation}

Nous voulons montrer que

\begin{equation}
\int_{\Gamma} \omega = \int_{\mathbb{R}^n} \omega.
\end{equation}

Posons pour cela $I_1 = \int_{\Gamma_1} \omega$, $J = \int_{\Gamma_2} \omega$ et considérons l'ouvert borné $\Omega = \left\{ y \in \mathbb{R}^n : |y - y(\text{Re } X_n)| < \frac{1}{2} \epsilon_0'(1 + |X_n|) \right\}$.

On écrit $I = \int_{\Gamma_1} \omega + \int_{\Gamma_2} \omega = I_1 + I_2$ et $J = \int_{\Gamma_1 \setminus \Omega} \omega + \int_{\Gamma_2 \setminus \Omega} \omega = J_1 + J_2$

Si $y \in \Omega$ alors $\chi_1 = 0$ et $\Gamma \cap \Omega = \Omega^c$ donc $I_2 = J_2$. Si $y \in \Omega$ alors $\chi = 1$ et la forme $\omega$ est holomorphe. D'après la formule de Stokes on a $I_1 = J_1$. Ceci prouve (2.14). Par conséquent, d'après (2.7), (2.13), (2.14) on a avec $X = (x', x_n - \lambda t)$

\begin{equation}
S_u_0(X) = \int_{\Gamma} \omega.
\end{equation}

Posons

\begin{equation}
\begin{cases}
\Sigma_1 = \left\{ z \in \Sigma : |\text{Re } z - y(\text{Re } X_n)| \leq \frac{1}{4} \epsilon_0'(1 + |X_n|) \right\} \\
\Sigma_2 = \left\{ z \in \Sigma : \frac{1}{4} \epsilon_0'(1 + |X_n|) < |\text{Re } z - y(\text{Re } X_n)| \leq \epsilon_0'(1 + |X_n|) \right\}.
\end{cases}
\end{equation}

Sur $\Sigma_1$ on a $\chi = \chi_1 = 1$ de sorte que

\begin{equation}
\Sigma_1 = \left\{ z = y + i \frac{\delta}{\lambda^2(\lambda - 1)} \text{Re } \frac{\partial}{\partial y} (X_n z(X)), \quad y \in \mathbb{R}^n \right\}.
\end{equation}

En outre on a d'après (2.15), (2.16)

\begin{equation}
S_u_0(X) = I_1 + I_2, \quad I_j = \int_{\Sigma_j} \omega, \quad j = 1, 2.
\end{equation}
On écrit alors

\[
\begin{aligned}
- \text{Im} \theta(X, z) &= - \text{Im} \theta(X, z(X)) - \text{Im} \frac{\partial \theta}{\partial y}(X, z(X))(z - z(X)) - \\
- \frac{1}{2} \text{Im} \left[ \frac{\partial^2 \theta}{\partial y^2}(X, z^*)(z - z(X))^2 \right] &= (1) + (2) + (3)
\end{aligned}
\]

où \( z^* = \alpha z + (1 - \alpha) z(X), \) \( 0 < \alpha < 1. \)

Le terme (1) : on a \( \theta(X, z(X)) = \varphi(X, z(X)) + \frac{1}{\lambda} \psi(z(X)). \) Comme \( z(X) \) appartient à \( \mathbb{R}^n \) et que \( \psi \) est réelle sur \( \mathbb{R}^n \) on a \( \text{Im} \psi(z(X)) = 0. \) D'autre part - \( \text{Im} \varphi(X, z(X)) = \Phi(X) \) où \( X = (x', x_n - \lambda t). \) Il résulte de (2.13) que

\[
(1) = \Phi(z).
\]

Le terme (2) : on a, puisque \( z(X) \in \mathbb{R}^n, \)

\[
\text{Im} \frac{\partial \theta}{\partial y}(X, z(X))(z - z(X)) = \text{Im} \frac{\partial \theta}{\partial y}(X, z(X))(\text{Re} z - z(X)) \\
+ \text{Re} \frac{\partial \theta}{\partial y}(X, z(X)) \cdot \text{Im} z.
\]

Ensuite

\[
\text{Im} \frac{\partial \theta}{\partial y}(X, z(X)) = \text{Im} \frac{\partial \varphi}{\partial y}(X, z(X)) + \frac{1}{\lambda} \text{Im} \frac{\partial \psi}{\partial y}(z(X)).
\]

Par définition de \( z(X) \) (cf. (1.13)) on a \( \text{Im} \frac{\partial \varphi}{\partial y}(X, z(X)) = 0. \) D'autre part comme \( \psi \) est holomorphe \( \frac{\partial \psi}{\partial y} = \nabla_y \psi \) où \( \nabla_y \) est le gradient réel. Alors \( \text{Im} \frac{\partial \psi}{\partial y}(z(X)) = \text{Im} \nabla_y \psi(z(X)) = 0 \) car \( z(X) \in \mathbb{R}^n \) et \( \psi \) est réelle sur le réel. On en déduit

\[
(2) = - \text{Re} \frac{\partial \theta}{\partial y}(X, z(X)) \cdot \text{Im} z.
\]

Le terme (3) : on a

\[
(3) = - \frac{1}{2} \text{Im} \left[ \frac{\partial^2 \theta}{\partial y^2}(X, z^*)(\text{Re} z - z(X))^2 \right. \\
- \left. \text{Im} \left[ i \frac{\partial^2 \theta}{\partial y^2}(X, z^*)(\text{Re} z - z(X)) \text{Im} z \right] \right. \\
- \frac{1}{2} \left. \text{Im} \left[ \frac{\partial^2 \theta}{\partial y^2}(X, z^*)(\text{Im} z)^2 \right] \right] = A_1 + A_2 + A_3.
\]
Considérons le terme $A_3$. On a

$$(2.23) \quad \text{Im} \frac{\partial^2 \theta}{\partial y^2} (X, z^*) = \text{Im} \frac{\partial^2 \varphi}{\partial y^2} (X, z^*) + \frac{1}{\lambda} \text{Im} \frac{\partial^2 \psi}{\partial y^2} (z^*) = a + b.$$ 

Le terme $a$) est majoré, d’après (1.16), par $C \lambda^{-2}$ et le terme $b$), d’après (0.9)' et (2.8), par $C \lambda^{-m-3}$. Ensuite, d’après (2.12) on a

$$(2.24) \quad |\text{Im} z| \leq \frac{C \delta}{\lambda^m}.$$ 

On déduit

$$(2.25) \quad |A_3| \leq \frac{C' \delta^2}{\lambda^{m+1}}$$

où $C'$ peut dépendre de $t_0$ mais est indépendante de $\delta$, $\lambda$.

Considérons le terme $A_2 = \text{Re} \frac{\partial^2 \theta}{\partial y^2} (X, z^*) (\text{Re} z - z(X)) \cdot \text{Im} z$. D’après (2.8), (1.16) et (0.9)' on a $|\text{Re} \frac{\partial^2 \theta}{\partial y^2} (X, z^*)| \leq C (\lambda^{-1} + \lambda^{-m-3})$; en utilisant (2.24) on en déduit que pour tout $\varepsilon > 0$ il existe $C_\varepsilon > 0$ telle que

$$(2.26) \quad |A_2| \leq \varepsilon \frac{1}{\lambda^2} |\text{Re} z - z(X)|^2 + C \varepsilon \delta^2.$$ 

Quant au terme $A_1$, en utilisant (2.8), (1.16) on peut écrire

$$A_1 \leq -\frac{C}{\lambda^2} (\text{Re} z - z(X))^2 - \frac{1}{2\lambda} \text{Im} \frac{\partial^2 \psi}{\partial y^2} (z^*) (\text{Re} z - z(X))^2$$

où $z^*$ a été défini en (2.19). Alors $\text{Im} z^* = \alpha \text{Im} z$. Comme $\text{Im} \frac{\partial^2 \psi}{\partial y^2} (z^*) = 0$ on a $|\text{Im} \frac{\partial^2 \psi}{\partial y^2} (z^*)| \leq C \lambda^{-m-3} |\text{Im} z|$, d’après (0.9)' . Donc $\frac{1}{\lambda^3} |\text{Im} \frac{\partial^2 \psi}{\partial y^2} (z^*) (\text{Re} z - z(X))^2| \leq \frac{C \delta^2}{\lambda^2} |\text{Re} z - z(X)|^2$, d’après (2.24). Si $\delta$ est assez petit on obtient

$$(2.27) \quad A_1 \leq -\frac{C}{\lambda^2} |\text{Re} z - z(X)|^2.$$ 

En utilisant les informations fournies par (2.22), (2.25), (2.26) et (2.27) on obtient

$$A \geq -\frac{C}{\lambda^2} |\text{Re} z - z(X)|^2 + C' \delta^2.$$ 

A. Estimation de $I_1$.

On déduit de (2.19), (2.20), (2.21) et (2.28) que sur $\Sigma_1$
\[-\text{Im} \theta(X, z) \leq \Phi(x) - \frac{\delta}{\lambda^{(m-1)}} \left| \text{Re} \frac{\partial \theta}{\partial y} (X, z(X)) \right|^2 - \frac{C}{\lambda^3} |\text{Re} z - z(X)|^2 + C' \delta^2.\]

On utilise alors (2.10). Il vient
\[-\text{Im} \theta(X, z) \leq \Phi(x) - C_0 \delta + C' \delta^2 - \frac{C}{\lambda^3} |\text{Re} z - z(X)|^2\]
d'où, en prenant \(\delta\) assez petit
\[(2.29)\quad -\text{Im} \theta(X, z) \leq \Phi(x) - C \delta.\]

On déduit de (2.18) que
\[(2.30)\quad |I_1| \leq C e^{\lambda \Phi(x) - \mu \lambda}.\]

\textbf{B. Estimation de } I_2. \]

Rappelons que
\[I_2 = \int_{\Sigma_2} e^{i \lambda \Phi(x, z)} f(X, z, \lambda) \chi \left( \frac{z - y(\text{Re} X_n)}{1 + |X_n|} \right) a(z) \, dz\]
où \(\Sigma_2\) a été défini en (2.16) et \(\theta\) en (2.6).

Sur \(\Sigma_2\) on a \(|\text{Re} z - y(\text{Re} X_n)| \geq \frac{\varepsilon_0'}{4} (1 + |X_n|)\) et d'après (2.24) on a encore \(|\text{Im} z| \leq \frac{C_0}{\lambda^2}\). Comme \(y(\text{Re} X_n) = y_1 + 2 \text{Re} X_n \eta_1 + Z(\text{Re} X_n)\) (voir (1.5)) il résulte de (1.19) que \(|y(\text{Re} X_n) - z(X)| \leq C \varepsilon_3 (1 + |x_n|)\) si \(|x - x_0| + \frac{1}{R_0} \leq \varepsilon_3 \ll \varepsilon_0'.\) On en déduit que sur \(\Sigma_2\) on a
\[(2.31)\quad |\text{Re} z - z(X)| \geq \frac{1}{6} \varepsilon_0' (1 + |X_n|) \geq C \varepsilon_0' (1 + \lambda).\]

On reprend (2.19) où \(-\text{Im} \theta(X, z) = (1) + (2) + (3). On a toujours
\[(2.32)\quad (1) = \Phi(x).\]

Ensuite
\[(2.33)\quad (2) = -\frac{\delta}{\lambda^{(m-1)}} \chi_1 \left( \frac{y - y(\text{Re} X_n)}{1 + |X_n|} \right) \left| \text{Re} \frac{\partial \theta}{\partial y} (X, z(X)) \right|^2 \leq 0.\]

Enfin
On déduit de (2.31) et (2.34) que

\[ (3) \leq -\frac{C}{\delta^2} |\text{Re}z - z(X)|^2 + C'\delta^2. \]

Par conséquent

\[ (3) \leq -C\varepsilon_0^2 + C'\delta^2 \leq -\frac{1}{2} C\varepsilon_0^2 \text{ si } \delta \ll \varepsilon_0'. \]

On déduit de (2.2), (2.3) et du lemme 11.2 que

\[ \text{Posons maintenant pour } |x - x_0| + \frac{1}{\rho_0} \leq \varepsilon_0, |t - t_0| \leq \delta_1 \]

\[ Tu(t, x) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(x,y)} f(x, y, \lambda) e^{i\psi(y)} a(y) \chi(y - y_0) dy. \]

On a alors le

**Lemme II.3.** Il existe des constantes positives \( \varepsilon_4, C \) telles que pour tout \( t_0 < 0 \) il existe \( \mu_0 > 0, \delta_2 > 0 \) tels que pour \( |x' - x'_0| + |x_0| + \frac{1}{\rho_0} \leq \varepsilon_4 \) et \( |t - t_0| \leq \delta_2 \)

\[ (2.37) \quad |Tu(t, x)| \leq C e^{i\lambda\phi(x,y)} \left( \sup_{|t - t_0| \leq \delta_2} ||u(t, \cdot)||_{L^2} + 1 \right). \]

**PREUVE.** Il suffit d’estimer la différence \( Tu(t, x) - Su(t, x) \). On procède comme dans [RZ] Proposition 5.3 et on utilise le fait que sur le support de \( \chi(y - y_0) - \chi \left( \frac{y - \text{Re}z}{1 + |x_0|} \right) \) on a \( \frac{1}{\rho_0} \varepsilon_0 \leq |y - y_0| \leq C\varepsilon_0. \)

**Fin de la preuve du théorème 0.1.**

Comptant tenu de (2.38), (1.4) montre que \( \rho_1 \notin \widehat{W}_{2,\delta}(u(t_0, \cdot)) \) ce qui contredit notre hypothèse et termine la preuve du théorème 0.1.

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EXPLICITLY RECOVERING ASYMPTOTICS OF SHORT RANGE POTENTIALS

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Abstract. Inverse scattering for real-valued short range potentials on $\mathbb{R}^n$ is studied. It is shown that the scattering matrix at fixed energy is the pull-back of a pseudo-differential operator and that the symbol of the operator determines the asymptotics of the potential. This is done by an explicit construction of the Poisson operator for the scattering problem as an oscillatory integral.

1. Introduction

Our objectives in this note are to present a short clear proof that the asymptotics of a short range potential are determined by fixed energy scattering data and to exposit on some notions of Melrose and Zworski relating to the ideas of propagation of growth at infinity and the concept of a Legendrian distribution. This result has already been proven in [4] by the author and Sa Barreto, using Hörmander’s calculus of Fourier integral operators and the calculus of Legendrian distributions developed by Melrose and Zworski in [7]. The proof we present here is in some sense the same as the one in [4], however instead of using the calculus to avoid explicit representation of the distributions involved, we solve the eikonal equation and work...
concretely at all stages thus avoiding any machinery except some basic facts about pseudo-differential operators. As well as enabling those who are not well-versed in micro-local analysis to understand our result, we also hope that this paper may serve as a useful introduction and motivation to the ideas of [7]. The fact that we work explicitly also brings the work closer to the goal of concretely recovering asymptotics from real-world measurements.

We define a short range potential to be a real-valued function which is a classical symbol of order $-2$. Recall that the class, $S^\alpha_n(\mathbb{R}^n)$, of classical symbols of order $m$ is the set of smooth functions $f(z)$ such that there exist a sequence of smooth, homogeneous functions, $f_{m-j}$, of order $m-j$ such that

$$
\left| D^\alpha \left( f(z) - \sum_{j<N} f_{m-j}(z) \right) \right| \leq C(z)^{m-|\alpha|-N},
$$

for $|z|$ large where $(z) = (1 + |z|^2)^{1/2}$.

Let $\Delta$ be the positive Laplacian on Euclidean space, $\mathbb{R}^n$. We are interested in studying solutions of

$$(\Delta + V - \lambda^2)u = 0,$$  \hfill (1.1)

with $\lambda$ a fixed positive number. It is well-known (see for example [6]) that given a function $f_0 \in C^\infty(S^{n-1})$, then there exists a unique solution $u$ of (1.1) of the form,

$$e^{i\lambda|z|}f(z) + e^{-i\lambda|z|}g(z),$$  \hfill (1.2)

with $f, g$ classical symbols of order $-\frac{n-1}{2}$, with the lead term of $f$ equal to the function $|z|^{-\frac{n+1}{2}}f_0(z/|z|)$. The scattering matrix, $S(\lambda)$, is then defined to be the map taking $f_0$ to the lead term of $g$ multiplied by $|z|^{n+1}$. The scattering matrix is then a map on $C^\infty(S^{n-1})$ and extends to a unitary map on $L^2(S^{n-1})$. We define the Poisson operator, $P(\lambda)$, to be the map taking $f_0$ to $u$. Throughout, we shall think of $S^{n-1}$ as being the sphere at infinity and we can think of $f_0$ as being incoming data on this sphere and $S(\lambda)f_0$ as being corresponding outgoing data. We will reserve $x$ for $|z|^{-1}$ which should be thought of as the boundary defining function of infinity. Note that if we put $\omega = z/|z|$, then

$$dz^2 = \frac{dx^2}{x^4} + \frac{d\omega^2}{x^2}.$$
and so the compactified $\mathbb{R}^n$ is a special case of the scattering metrics discussed in [7].

We prove,

**Theorem 1.1.** Let $V_1, V_2$ be short range potentials and let $S_j(\lambda)$ be the associated scattering matrices. Then $S_j = P_j a^*$, with $P_j$ a zeroth order classical pseudodifferential operator and $a^*$ pull-back by the antipodal map and if in addition $P_1 - P_3 \in \Psi DO^{-\infty}(S^{n-1})$ then $V_1 - V_2 \in S^{-\infty}(\mathbb{R}^n)$.

Observe,

$$(\Delta - \lambda^2)e^{ik \cdot w} = 0$$

and so if $f$ is an integrable function on $S^{n-1}$, we can define

$$u = \int f(\omega)e^{ik \cdot w}d\omega = 0,$$

and we have $(\Delta - \lambda^2)u = 0$. If $f$ is smooth we can apply stationary phase and conclude that $u$ is an eigenfunction of the form

$$Ce^{ik|z|}f(|z|)|z|^{-\frac{n-1}{2}} + O(|z|^{-\frac{n+1}{2}}),$$

and in fact we have a complete asymptotic expansion. This show that $e^{ik \cdot w}$ is the Schwartz kernel of the Poisson operator. It also shows that the scattering matrix for the free problem is just pull-back by the antipodal map.

As in [7], our approach here is to construct the Poisson operator for the scattering problem for $\Delta + V - \lambda^2$ and then the properties of the scattering matrix can be read off directly. As in [4], we shall see that in showing the recovery result it is easier to examine the difference of the Poisson operators associated to two potentials than to compute lower order terms of a single one - however we shall indicate how the lower order terms can be computed if one so desired.

We construct the kernel of the Poisson operator, $P(\lambda)$, as a perturbation of the one for the free problem; in particular we look for it in the form $e^{ik \cdot w}(1 + a(z, \omega))$ with $a \in S^{n-1}_{\gamma} (\mathbb{R}^n)$ and depending smoothly on $\omega$. We successively obtain transport equations on the asymptotic expansion of $a$ which are along the great circles running from $\omega$ to $-\omega$. The transport equations degenerate on approach to
and following [7], we need a different ansatz there which is really a micro-local blow-up. In particular, we look for $P(\lambda)$ in the form,

$$
\int_0^\infty e^{i\lambda(S|z| - \sqrt{\lambda^2 - |z|^2})} |z|^\lambda S^\lambda S_{S^\lambda} a \left( \frac{-1}{S|z|}, S, \mu \right) dS d\mu, \tag{1.3}
$$

with $a \in \mathcal{C}^\infty([0, \epsilon) \times [0, \epsilon) \times S^{n-2})$, where we have rotated $z$ so that $\omega$ is the north pole and $z = (z', z_n)$.

Once we have constructed $P(\lambda)$ we see easily that the difference of the scattering matrices associated to two potentials which agree so some order is a lower order pseudo-differential operator (pulled-back by the antipodal map) and that the principal symbol of this operator contains precisely the data obtained by integrating the lead term of the difference of the potentials with a sine function weighting, over geodesics of length $\pi$. We then give a simple argument showing the injectivity of this transformation. An induction on the order of the difference then completes the argument.

We refer the reader to [4] for a brief history of this problem. The recovery result of [4] was extended to cover potentials of the form $C/|z| + V$ with $V$ short range in [3] by using complex orders of homogeneity and that could be done by the techniques here also. Vasy, [8], has shown that the techniques of [7] can be applied to real-valued potentials in $S^{-1}_{1/2}(\mathbb{R}^n)$ by considering complex orders of homogeneity which vary from point to point. It is therefore likely that a recovery result can be proved in that case also by a combination of those techniques with those of this paper and [4]. We also remark that the techniques developed here could also be applied to reprove the result of [5] with little change but we leave this to the enthusiastic reader.

The work here bears some similarities to that of Agmon and Hörmander, [1]. They constructed solutions of higher order constant coefficient partial differential equations by using oscillatory integral techniques.

I would like to thank Maciej Zworski for pointing out the connections between [7] and the problem of recovering asymptotics. I also thank the referee for his alacrity and care.
2. The Poisson operator away from the antipodal point.

In this section, we construct the Poisson operator away from the antipodal point and look at how a perturbation affects it.

Suppose $V \in S^{-2}_{\text{cl}}(\mathbb{R}^2)$ then we have,

$$(\Delta + V - \lambda^2)e^{i\lambda x} = Ve^{i\lambda x}.$$  

So we look for a Poisson operator of the form $e^{i\lambda x}(1 + a)$ with $a$ a classical symbol of order $-1$ in $x$ depending parametrically in $\omega$. Applying the operator we obtain

$$e^{i\lambda x}(-2i\lambda \omega \frac{\partial a}{\partial z} + \Delta a + V + Va).$$

We wish to pick the asymptotic expansion so that this is Schwartz. We solve iteratively. Let $a \sim \sum_{j=1}^{\infty} a_{-j}$ and $V \sim \sum_{j=2}^{\infty} V_{-j}$. The lead term is

$$-2i\lambda \omega \frac{\partial a_{-1}}{\partial z} + V_{-2}.$$  

Note as these are homogeneous functions this is really an equation on the sphere.

We therefore take coordinates $(r, s, \theta)$ where $r = |z|, s$ is the geodesic distance of $z/|z|$ from $\omega$ and $\theta$ is the angular coordinate about $\omega$. Note the coordinate system depends on $\omega$ but we shall suppress $\omega$ in our notation most of the time.

Now with out loss of generality, we can take $\omega$ to be the north pole. Then

$$\omega \partial_z = \partial_s.$$  

Now $\cos(s) = \frac{r}{|r|}, r = |z|$. The $\theta$ coordinate will be purely parametric. We have

$$\frac{\partial s}{\partial z_n} = \frac{-1}{\sqrt{1 - \frac{1}{|r|^2}}} \left( \frac{1}{|r|^2} - \frac{z_n^2}{|r|^2} \right) = -r^{-1} \sin(s),$$

and

$$\frac{\partial r}{\partial z_n} = \cos(s).$$

So applying $\omega \partial_z$ to $a_{-j}(s, \theta; \omega)r^{-j}$, we obtain

$$r^{-j-1} \left[ -\sin(s) \frac{\partial a_{-j}}{\partial s} - j \cos(s) \right] a_{-j}.$$  

So for $j = 1$, we have taking $r = 1$
which is equivalent to

$$2i\lambda \partial_s (\sin(s)a_{-1}) = V_{-2}.$$  

We want \(a\) to be smooth at \(s = 0\), as this is the place where we want a delta function so

$$\sin(s)a_{-1} = -\frac{1}{2i\lambda} \int_0^s V_{-2}(s', \theta; \omega) ds',$$

which implies that

$$a_{-1}(s, \theta; \omega) = \frac{i}{2\lambda \sin(s)} \int_0^s V_{-2}(s', \theta; \omega) ds'.$$  \hspace{1cm} (2.1)

This will be singular as \(s \to \pi\), but let's ignore that for now. Now suppose we have chosen the first \(j\) terms so we have an error \(d \in S^{-j-1}\) with lead term \(d_{-j-1}(s, \theta; \omega)r^{-j-1}\). We then want to solve the transport equation,

$$-2i\lambda \partial_s (a_j r^{-j}) + d_{-j-1} r^{-j-1} = 0,$$

as above we get

$$2i\lambda(\sin(s)) \partial_s [j \cos(s)] a_j + d_{-j-1} = 0.$$  

We solve this to obtain,

$$a_j(s, \theta, \omega) = \frac{i}{2\lambda(\sin(s))^j} \int_0^s (\sin(s'))^{j-1} d_{-j-1}(s', \theta; \omega) ds'.$$  \hspace{1cm} (2.2)

So away from \(s = \pi\), we can achieve an error in \(S^{-\infty}\) by applying Borel’s lemma. Recall \(s = \pi\) corresponds to the antipodal point, so in conic neighbourhoods disjoint from \(-\omega\) we have an error in \(S^{-\infty}\). Note that we have a focusing of the geodesics at \(-\omega\) and our solutions fail to be smooth there in two different ways. The first is that they blow up as \(s \to \pi\) and the second is that the lead term on approach to \(-\omega\) will depend on the angle as the forcing term in the transport equation will depend on the angle. In particular provided \(d_{-j-1}\) does not grow faster than \((\pi - s)^{-j}\) we have that \(a_j\) grows as \((\pi - s)^{-j}\).

Before dealing with this problem, note that if we have two potentials \(V_1, V_2\), such that \(V_1 - V_2 = W \in S^{-k}\) then the first \(k - 2\) terms in the construction above are the same and the difference in forcing of the \(k - 1\) term will be \(W_{-1}\), the leading term of \(W\). So they will differ by
This fact is what makes the inverse problem argument work as this essentially is the principal symbol of the difference of the scattering matrices.

3. The Ansatz at the Antipodal Point

We need a different ansatz to deal with the singularity at $s = \pi - \theta$. Consider

$$e^{i s' - u - \sqrt{u^2 + |z|^2}} (|z|, u) du,$$

with a smooth, symbolic in $|z|$ and compactly supported in $u$.

Let $\phi$ denote the phase function. We then have that

$$\frac{\partial \phi}{\partial u} = \frac{u}{\sqrt{1 + |u|^2}},$$

so the phase is stationary if and only if

$$\frac{z'}{|z|} = \frac{u}{\sqrt{1 + |u|^2}}.$$

We deduce that at stationary points,

$$1 + |u|^2 = \frac{1}{1 - |\theta|^2}$$

and thus that $\frac{\partial \phi}{\partial u} = 0$ if and only if

$$u = \frac{1}{\sqrt{1 - |\theta|^2}} \frac{z'}{|z|}.$$

We therefore compute that the phase at critical points is $-|z_n|$ and so is $z_n$ in $z_n < 0$.

This is $\omega \cdot z$ if we take $\omega$ to be the north pole. The Hessian of $\phi$ is

$$-\frac{\delta_{jk}}{\sqrt{1 + |u|^2}} \frac{n_j n_k}{(1 + |u|^2)^{3/2}}$$

which equals

$$-\sqrt{1 - |\theta|^2} (\delta_{jk} - \theta_j \theta_k)$$

where $\theta = z' / |z|$. Observing that $\theta$ is an eigenvector and also that any vector orthogonal to $\theta$ is an eigenvector we have that the determinant is
So provided we are in \( z_n < 0 \) we can use stationary phase to transform from form (3.1) to the original ansatz. It's important to realize that we can also do the opposite. To see this observe that if we have a function \( e^{i\lambda z}w a(z, \omega) \) with lead term \( a_m(z, \omega) \) then letting \( b(|z|, u) = C|z|^{\frac{n}{2}}(1 - \frac{|z|^2}{|y|^2})a_m(z, \omega) \) where \( z \) satisfies (3.2) and \( C \) is the constant arising from the stationary phase, we have that

\[
\int e^{i\lambda z - \sqrt{1 + |u|}} b(|z|, u) du = e^{i\lambda z}a(z, \omega) = e^{i\lambda z}c(z, \omega)
\]

with \( c \) of order \( m - 1 \) so repeating and asymptotically summing we can achieve equality modulo an error of order \(-\infty\). Note also that the amplitudes in each form are unique modulo terms of order \(-\infty\) and that there is a simple relation between their lead terms or principal symbols. Whilst this argument was carried out for \( \omega \) the north pole, we can always, at least locally, in \( \omega \) redefine our \( z \) coordinate system to make this true and patch together.

So close to the south pole, we look for the Poisson operator in the form,

\[
\int_0^\infty \int \left( \frac{1}{|z|} \right)^\gamma S^n e^{i|z| - \sqrt{1 + |u|}} a \left( \frac{1}{|z|}, S, \mu \right) dS d\mu,
\]

with \( a(t, S, \mu) \) a smooth function compactly supported on \([0, \varepsilon] \times [0, \varepsilon] \times S^{n-2} \). We assume that \( \omega \) has been rotated to the north pole. Note that for \(|z|\) in a compact set the integral is supported on a compact set and so we have no problems with convergence - in particular the integral yields a smooth function. (Though not of course smooth up to the sphere at infinity.) We denote the class of functions which can be written in this form plus a Schwartz error by \( \Gamma^\gamma \).

**Proposition 3.1.** We have

\[
\Delta - \lambda^2 : \Gamma^\gamma \rightarrow \Gamma^{\gamma+1, \alpha+1}.
\]

**Proof.** We obtain various pieces according to how the derivatives fall.

The first (and of highest growth) piece is that obtained when we differentiate the phase twice. ie we gain a term \( |\partial_\gamma \phi|^2 - \lambda^2 \). Of course we have chosen \( \phi \) to satisfy an eikonal equation that is to vanish on \( \partial_\gamma \phi = 0 \). In particular, we have
\[ |\partial_x \phi|^2 - \lambda^2 = \frac{-2i \lambda S \sqrt{1 + S^2} \partial \phi}{S |x| \lambda} \]

So integrating by parts we obtain the same form but now with the amplitude,

\[ 2i \lambda \partial_S \left( \sqrt{1 + S^2} b \right), \]

with \( b \) now a symbol of order \((\gamma + 1, \alpha + 2)\). Differentiating this yields a symbol of order \((\gamma + 1, \alpha + 1)\).

The second piece is obtained by applying the Laplacian to the phase. This multiplies the symbol by \(-\sqrt{1 + S^2} (n-1) |z|^{-1}\) and so again we get a term in \(I^{\gamma+1, \alpha+1}\).

The third piece is obtained by applying the Laplacian to the amplitude, it is straightforward to see that this is in \(I^{\gamma+2, \alpha+2}\).

The final piece and the most troublesome is obtained by differentiating the phase and the amplitude once each. This yields an amplitude:

\[ \sum_{j=1}^{n-1} -\mu_j \frac{z_j}{|z|^2} \partial_b - S \sqrt{1 + S^2} |z|^{-2} \partial_a. \]

The second term is a symbol of order \((\gamma+2, \alpha+3)\). The first term is more troublesome as we have a \(z_j\) which is not allowed for in our definition. Before multiplication by \(z_j\) the first term is in \(I^{\gamma+3, \alpha+3}\). So the proof is complete subject to the proof of

**Lemma 3.1.** Multiplication by \(z_j\) maps \(I^{\gamma, \alpha}\) into \(I^{\gamma-1, \alpha} + I^{\gamma, \alpha-1}\).

\[ \square \]

To prove Lemma 3.1, we change coordinates out of polars and put \(u = \mu s\), to obtain,

\[ \int z_j e^{i \psi u - \sqrt{1 + |u|^2} s} \left( \frac{1}{|u||s|} \right)^{\frac{\gamma}{2} + a} \left( \frac{1}{|u||s|} \right)^{\frac{\alpha}{2} + a} \right) \, du. \]

If \(\psi\) denotes the phase function, we have

\[ \frac{\partial \psi}{\partial u_j} = z_j - \frac{u_j}{\sqrt{1 + |u|^2}} |z|. \]

So we can replace \(z_j\) by \(\frac{\partial \psi}{\partial u_j} + \frac{u_j}{\sqrt{1 + |u|^2}} |z|\). Returning to polar coordinates, the second term will decrease the \(\gamma\) by one. Integrating by parts in \(u_j\) and returning to polar
coordinates we see that the second term decreases the $\alpha$ order by one. The lemma follows.

Of course we also need the mapping properties under multiplication by short range potentials. Away from the antipodal point, these are immediate as we can just convert to the form $e^{i\alpha^z \omega} a(x,\omega)$, with a symbolic in $z$.

At the antipodal point, the result is trickier. If $V \in S^{-k}_d(\mathbb{R}^n)$ then we obtain an integral of the form

$$\int_0^\infty e^{i\alpha(S^t t - \sqrt{1 + |\mu|^2})} \left( \frac{1}{S|x|} \right)^{\gamma + k} \left( \frac{1}{\mu} \right)^{\gamma + k} \mu d\mu,$$

with $b(t, S, r, \theta, \mu)$ smooth and compactly supported in all variables. If we let $a(t, S, \theta, \mu) = b(t, S, S t, \theta, \mu)$ we obtain

$$\int_0^\infty e^{i\alpha(S^t t - \sqrt{1 + |\mu|^2})} \left( \frac{1}{S|x|} \right)^{\gamma + k} \left( \frac{1}{\mu} \right)^{\gamma + k} \mu d\mu,$$

which is in the correct form except for dependence of $a$ on $z/|z|$. As we are working close to the antipodal point, we can assume $a$ is a function of $z'/|z|$, that is eliminate the dependence on $z_n$. However, this can be eliminated by converting to non-polar coordinates and expanding the symbol in a Taylor series about

$$z' = \frac{u}{\sqrt{1 + |z|^2}},$$

and repeatedly integrating by parts as in the proof of Lemma 3.1. So we conclude that,

**Proposition 3.2.** If $V \in S^{-k}_d(\mathbb{R}^n)$ then multiplication by $V$ maps $\Gamma^{\gamma,\alpha}$ into $\Gamma^{\gamma+k,\alpha+k}$.

We will also need a standard asymptotic summation result.

**Proposition 3.3.** If $u_j \in \Gamma^{\gamma+k,\alpha}$, $j \in \mathbb{N}$, then there exists $u \in \Gamma^{\gamma,\alpha}$ such that

$$u - \sum_{j \in \mathbb{N}} u_j \in \Gamma^{\gamma+k,\alpha}$$

for all $N$.

This is immediate from applying Borel's lemma to the amplitudes.

We want to examine the distributional asymptotics of our ansatz. For this it will be easier to work with $z = |z|^{-1}$ and think in terms of smoothness up to $x = 0$.
We expect a lead term with coefficient \( e^{-iA/Z} \) so multiplying by \( e^{iXlzl} \) we have,

\[
x^n \int_{S^{n-1}} \int_0^\infty e^{i\lambda k - (\lambda^2 + \lambda^2 - 1)}(x/\hat{S})^{\gamma - \alpha} a(x, s, \mu) dSd\mu.
\]

Now doing a change of variables \( \eta = S/x \), we have,

\[
x^{n+1} \int_{S^{n-1}} \int_0^\infty e^{i\lambda k - (\lambda^2 + \lambda^2 - 1))\eta^{\gamma - \alpha} a(\eta^{-1}, x, \eta, \mu)d\eta d\mu.
\]

Now we want the distributional asymptotics so consider integrating against a function of \( \theta' \). Provided the original \( a \) was supported sufficiently close to \( s = 0 \) we have that the derivative of the phase in \((\theta', \mu)\) is non-zero so interchanging the order of integration, which we can do as everything is compactly supported for \( x > 0 \), we see that the integral can be made rapidly decreasing in \( \eta \). So we will have \( x^{n+1} \) times a smooth function of \( x \) and the lead coefficient can be obtained by taking \( x = 0 \). So we have that the lead distributional coefficient is the distribution,

\[
e^{-iA|z|^{\gamma - \alpha}} \int_0^\infty \int_{S^{n-1}} e^{i\lambda k - (\lambda^2 + \lambda^2 - 1))\eta^{\gamma - \alpha} a(\eta^{-1}, 0, \mu) d\eta d\mu.
\]

Now if we go back to non-polar coordinates we obtain,

\[
e^{-iA|z|^{\gamma - \alpha}} \int_{\mathbb{R}^{n-1}} e^{i\lambda k - (\lambda^2 + \lambda^2 - 1))\eta^{\gamma - \alpha} a(\eta^{-1}, 0, \mu) d\eta d\mu.
\]

where \( b(\xi) = a(\xi^{-1}, 0, 1) \) and is therefore a symbol of order 0.

So bringing \( \omega \) back in, we conclude that the lead term is a pseudo-differential operator (after pull-back by antipodal map) of order \( \alpha - \gamma - (n - 2) \). Note that the principal symbol of this operator is the lead term of \( a \) as \( t \) goes to 0.

So to summarize, we have

**Proposition 3.4.** If \( u(z, \omega) \in I^{\gamma, \alpha} \) then \( e^{iA|\omega|} \int (|z|^{\gamma - \alpha} f(\omega) d\omega \) is a smooth symbolic function in \( |z| \) of order \( -1 - \alpha \) and its lead coefficient is \( |z|^{\gamma - \alpha - 1} (K, f) \) where \( K \) is the pull-back of the Schwartz kernel of a pseudo-differential operator of order \( \alpha - \gamma - (n - 2) \) by the map \( \theta \mapsto \theta \). The principal symbol of \( K \) determines and is determined by the lead term of the symbol, \( a(t, x, \mu) \), of \( u \) as \( S \to 0 \).

This result on asymptotics also allows us to prove a slightly surprising but very
useful fact about the mapping properties under the Laplacian which we will need later. This is what makes this ansatz work, for example one could have blown-up the symbol at $z/|z| = -\omega, |z|^{-1} = 0$ instead but then one would not have this property and the argument would not go through.

Lemma 3.2. If $u \in I^{s, \frac{d}{2} + \delta}$ and $(\Delta + V - \lambda^2)u \in I^{s+\delta, \frac{d}{2} + \delta}$ then $(\Delta + V - \lambda^2)u \in I^{s+\delta, \frac{d}{2} + \delta}$.

Proof. Note that the Laplacian in polar coordinates is

$$\Delta = -\frac{\partial^2}{\partial r^2} + \frac{1-n}{r} \frac{\partial}{\partial r} + r^{-2} \Delta_{S^\infty}.$$

Now if we apply $\Delta - \lambda^2$, to a function $e^{-i\lambda|z|^\alpha}$, using polar coordinates we obtain,

$$-2i\lambda(2\alpha + n-1)e^{-i\lambda|z|^\alpha} + C_\alpha e^{-i\lambda|z|^\alpha}.$$

So in particular if $\alpha = -\frac{n-1}{2}$, then the lead term vanishes. More generally, if we consider a symbol of order $-\frac{n-1}{2}$, then the lead term is multiplied by $-2i\lambda(2\alpha + n-1)$ and multiplied by $|z|^{-1}$. So the lead term will be killed if $\alpha = -\frac{n-1}{2}$, and there will be an improvement of two orders. Note that this is why the generalized eigenfunctions take the form (1.2).

Now if we consider, $\int u(|z|, \omega) f(\theta, \omega) d\theta d\omega$ then by Proposition 3.4 we have the lead term is $e^{-i\lambda|z|^\alpha} (K, f)$, and $K$ is a pulled-back pseudo-differential operator of order $-\frac{n-1}{2} - (\gamma + j)$, and that the principal symbol is equivalent to the lead term of the amplitude of $u$ as $S \to 0$.

Now we have with $r = |z|$ that

$$\int (\Delta + V - \lambda^2) u(r\theta, \omega) f(\theta, \omega) d\theta d\omega = \left( -\frac{\partial^2}{\partial r^2} + \frac{1-n}{r} \frac{\partial}{\partial r} - \lambda^2 \right) \int u(r\theta, \omega) f(\theta, \omega) d\theta d\omega$$

$$+ \int u(r\theta, \omega) (r^{-2} \Delta_{S^\infty} + V) f(\theta, \omega) d\theta d\omega, \quad (3.4)$$
so we conclude that \( f(\Delta + V - \lambda^2)u(|z|, \omega) \) decays to order \(-\frac{n-1}{2} - 2\) rather than \(-\frac{n-1}{2} - 1\) as predicted by Prop 3.4. This means that the lead term of the distributional asymptotics is zero and that the lead term of the amplitude of \((\Delta + V - \lambda^2)u\) vanishes as \(S \to 0\). This means that \((\Delta + V - \lambda^2)u \in I^{\gamma+j+1,\alpha+1} + I^{\gamma+j,\alpha+2}\). If repeat our argument on the first piece, our result follows. 

As we are attempting to solve by getting improvement in the \(\gamma\) filtration we note that

**Proposition 3.5.** If \(u \in I^{\gamma,\alpha} = \bigcap_{\gamma} I^{\gamma,\alpha}\) then \(u = e^{-i\lambda|z|} f(z)\), with \(f\) a classical symbol of order \(-\alpha - 1\).

After this one gets an error of the form \(e^{-i\lambda|z|} c\) with \(c\) a symbol of order \(-\frac{n-1}{2}\), this can then be removed by iteration. (Indicial equation argument.) Note that this term makes a smooth contribution to the scattering matrix in any case. The Schwartz error is then removed by applying the resolvent and also gives a smooth contribution.

So first we need to solve the transport equations right up to \(S = 0\). As we have behaviour of the form \(\frac{1}{\sqrt{|z|}}\) in the \(t\) variable the order of singularity allowed in \(s\) will increase by one as the order of \(|z|\) decreases by one. In particular, the lead term will have a singularity of order \(\alpha - \gamma\) as \(s \to 0^+\). Now as we have to convert from our previous ansatz, using the stationary phase computation above, we must take \(\gamma = -\frac{n-1}{2}\), and \(\alpha = \frac{n-3}{2}\), as the principal term should look like \(S^{n-2}\) as \(S\) goes to 0. We then have from Prop 3.4 that the distributional asymptotics have lead term \(e^{-i\lambda|z|} |z|^{-\frac{n-3}{2}} K\), with \(K\) the kernel of a zeroth order, pseudo-differential operator composed with the pull-back of the antipodal map. The term, \(K\), will be the Schwartz kernel of the scattering matrix.

Now we carry out our construction. We look for our solution to be in \(I^{-\frac{n-1}{2}, \frac{n-3}{2}}\), close to the south pole and we know from above that \(u_0 = e^{i\lambda t} \omega\) will solve for the leading term so applying \(\Delta + V - \lambda^2\), we obtain an error in \(e_1 \in I^{-\frac{n-1}{2}+2, \frac{n-3}{2}+2}\), using Prop 3.4. Now the symbol of this will blow-up to order \(n-2\) as \(S \to 0^+\). Converting back to the form of the first ansatz the symbol blows up to order 0 as
we see that the quantity for $k - 2$ is determined also. So if $k$ is even we can reduce to $k - 2 = 0$ and if $k$ is odd to $k - 2 = 1$. Now if we have no sin weighting and we differentiate, we obtain the difference of the function at the endpoints of the geodesic that is the odd part of the function. If we have a sin$^1$ weighting differentiating a couple of times we see that we can similarly recover the odd part of the function.

Now if $k$ is even, we can join together two half geodesics to obtain the integral of $W_{-k}$ around the full closed geodesic. Now it is a theorem of Funk that the integral of a smooth function around every closed geodesic on $S^{n-1}$, is zero if and only if the function is odd for $n \geq 3$, see for example [2], so we conclude that if the principal symbol is identically zero for $k$ even then $W_{-k}$ is identically zero.

If $k$ is odd, we observe that if a geodesic starts on $z_n = 0$ then the $z_n$ coordinate will be a constant multiple of $\sin s$, with $s$ the geodesic distance. So can then apply Helgasson's result to $z_n W_{-k}$ and conclude that if all the integrals vanish then $W_{-k}$ is even and as we already knew it had to be odd, we conclude that it is zero.

Hence inducting on $k$ we deduce that if the difference of the scattering matrices for one energy is smoothing then the difference of the potentials is Schwartz.

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GLOBAL SOLVABILITY OF MONGE-AMPÈRE TYPE EQUATIONS

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Dedicated to Professor Norio Shimakura on his sixtieth birthday

Abstract

We study the global solvability of Monge-Ampère equations of mixed type by "blowing up" the problem onto the torus embedded at the singular point of the equations.

1 Introduction

In this paper we are interested in the global solvability of fully nonlinear equations including Monge-Ampère equations of mixed type degenerating on nor-
mally crossing lines. As far as the author knows these types of equations are not well solved even in the analytic class due to the lack of estimates of the linearized equations. We shall present a new method for such equations. Our main idea is to reduce the equations, by use of the Cauchy-Riemann equation, onto the (product of) torus embedded at points on which the equations change type. The resultant equations on the torus contain no singularities. After solving them we extend the solutions inside the torus by harmonic extensions, and the maximal principle yields the unique solution for the given problems. This method enables us to work outside "singular" set instead of working directly in a neighborhood of such a point. Because we can obtain a good estimate of the linearized operators on the torus we can use an elementary iteration scheme instead of a Nash-Moser one although the original equations degenerate.

Our main results in this paper are the global solvability of a Monge-Ampère type equations $M(u) = f$ in a bounded complete Reinhardt domain in $\mathbb{C}^n$ not necessarily convex. (cf. §2 and §5). More precisely, let the order of a formal power series $f$, ord $f$ be defined as the smallest degree of its constituent monomials, that is, the smallest integer $k$ such that $\partial^{\alpha} f(0) \neq 0$ for some $|\alpha| = k$ and $\partial^\beta f(0) = 0$ for all $|\beta| \leq k-1$. For $u_0$ and $f_0$ satisfying $M(u_0) = f_0$, we want to solve $M(u_0 + v) = f_0 + g$ for analytic $g$ satisfying ord $g \geq$ ord $f_0$.

Here the localizing function $u_0$ corresponds to initial values in the case of initial value problems, while in some cases the equations are Tricomi mixed type at $u = u_0$ in the real domain. In §2 we show the global solvability under the Riemann-Hilbert factorization condition of the corresponding symbol on tori (Toeplitz symbol) in case $n = 2$. In §5 we study the global solvability in the case $n > 2$ under certain ellipticity conditions like (5.4) and (5.20) of the reduced symbol on tori.

Another interesting application of the method in this paper is the convergence of all formal power series solutions of fully nonlinear equations not satisfying a Poincaré condition. (cf. [3], [5], [6]). This is an extension of
a classical theorem of Kashiwara-Kawai-Sjöstrand's (cf. [4]) for linear equations to fully nonlinear equations. In fact, we have different phenomena in the nonlinear case. (See Corollary 2.5 and §6).

In §6 we study the global solvability of Monge-Ampère equations in an unbounded outer domain. We will show new phenomena as to the structure of regular and singular solutions in the case \( n = 2 \). Namely, we will show the (essentially) linear structure of solutions of \( M(u) = 0 \) at infinity (cf. Theorem 6.2.), the parametrization theorem of all solutions of \( M(u) = f_0 + g \) by a linear subspace (Theorems 6.5). As to the solvability, we will show unique global solvability of mixed type equations under a Riemann-Hilbert factorization condition (Theorem 6.8) and the one under a so-called spectral condition (Corollary 6.9. See also Remark 6.10).

2 Global Solvability and Riemann-Hilbert factorization

1.1. Statement of results. For \( x = (x_1, x_2) \in C^2 \) and \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \) we define \( \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \) and \( |\alpha| = \alpha_1 + \alpha_2 \), where \( \mathbb{Z}_+ \) denotes the set of nonnegative integers. We set \( \mathbb{R}^2_+ = \{(\eta_1, \eta_2) \in \mathbb{R}^2; \eta_1 \geq 0, \eta_2 \geq 0\} \). For \( a = (a_1, a_2) \in C^2 \), let \( D_a := \{|z_1| \leq |a_1| \times \{|z_2| \leq |a_2|\} \} \) and \( T^2_a = \{|z_1| = |a_1| \times \{|z_2| = |a_2|\} \} \) be the closed disk and the torus, respectively. We say that \( \mathbb{R} \subseteq C^2 \) is a Reinhardt domain if \( T^2_a \subseteq \Omega \) for each \( a = (a_1, a_2) \in \Omega \). If, in addition, \( D_a \subseteq \Omega \), we say that \( \Omega \) is complete. Let \( \Omega \) be a bounded complete Reinhardt domain containing the origin not necessarily convex. We denote by \( \mathcal{O}(\Omega) \), the set of holomorphic functions in \( \Omega \).

For \( m \in \mathbb{Z}, m \geq 1 \), let \( M(u) \) be a fully nonlinear operator

\[
M(u) := \sum_{|\alpha| = |\beta| = m} a_{\alpha \beta} (\partial^\alpha u)(\partial^\beta u) + \sum_{|\alpha| = |\beta| = m} b_{\alpha \beta} x^\alpha \partial^\beta u, \quad a_{\alpha \beta}, b_{\alpha \beta} \in C. \quad (2.1)
\]

Let \( u_0(x) \) be a homogeneous polynomial of degree \( 2m \), and set \( f_0(x) = M(u_0) \).
For \( g \in \mathcal{O}(\Omega) \) such that \( \text{ord} \, g > 2m \) we consider
\[
M(u_0 + w) = f_0(x) + g(x), \quad \text{in } \Omega.
\]
Let \( P := M_{u_0} = \sum_\alpha (\partial M/\partial z_\alpha)(u_0)\partial^\alpha \) be the linearized operator of \( M \) at \( u = u_0 \). We denote by \( p_\alpha(x, \xi) \) the principal symbol of \( P \), where \( \xi = (\xi_1, \xi_2) \) is the covariable of \( x \). The Toeplitz symbol \( \sigma(z, \eta) \), \((z \in \Omega, \eta \in \mathbb{R}^2)\) at \( u_0 \) is defined by
\[
\sigma(z, \eta) = p_\alpha(z_1, z_2; z_1^{-1} \eta_1, z_2^{-1} \eta_2),
\]
namely, we set \( x = (z_1, z_2) \) and \( \xi = (z_1^{-1} \eta_1, z_2^{-1} \eta_2) \) in \( p_\alpha(x, \xi) \).

We denote by \( \breve{\Omega} \) the real representation of \( \Omega \), and by \( \partial \breve{\Omega} \) its boundary.

We assume the following conditions

\[
\text{(A.1)} \quad \sigma(z, \eta) \neq 0 \quad \forall z = (z_1, z_2) \in \Omega, \quad (|z_1|, |z_2|) \in \partial \breve{\Omega}, |z_j| > 0, j = 1, 2,
\]
\[
\forall \eta \in \mathbb{R}^2, |\eta| = 1.
\]

\[
\text{(A.2)} \quad \text{ind}_1 \sigma = \text{ind}_2 \sigma = 0.
\]

Here \( \text{ind}_1 \sigma \) (resp. \( \text{ind}_2 \sigma \)) is defined by
\[
\text{ind}_1 \sigma = \frac{1}{2\pi i} \oint_{| \zeta_1 | = 1} d\zeta_1 \log \sigma(R_1 \zeta_1, R_2 z_2, \xi), \quad (R_1, R_2) \in \partial \breve{\Omega}.
\]

**Remark.** The right-hand side of (2.4) is an integer-valued continuous function of \( z_2 \ (|z_2| = R_2), \xi \ (\xi \in \mathbb{R}^2) \) and \( R = (R_1, R_2) \in \partial \breve{\Omega} \ (R_j > 0) \). By the connectedness, the integral is independent of \( z_2, \xi \) and \( R = (R_1, R_2) \in \partial \breve{\Omega} \).

The condition (A.1)-(A.2) is called a Riemann-Hilbert factorization condition.

Then we have

**Theorem 2.1** Suppose (A.1) and (A.2). Let \( \Omega' \subset \subset \Omega \) be arbitrarily given. Then there exist \( \varepsilon > 0 \) and an integer \( N \geq 2m \) such that, for any \( g \in \mathcal{O}(\Omega) \) satisfying \( \sup_{\Omega} |g(x)| < \varepsilon \) and \( \text{ord} \, g \geq N \), (2.2) has a unique analytic solution \( w \in \mathcal{O}(\Omega') \) satisfying \( \text{ord} \, w \geq N \).

We define the set of holomorphic functions \( W_R \) by
In case $\partial \Omega$ consists of one point, $R = (R_1, R_2)$, $R_j > 0$ we have

\[ R := \{ u = \sum_{n \geq 0} u_n x^n : ||u||_R := \sum_{n} |u_n|R^n < \infty \}. \tag{2.5} \]

**Theorem 2.2** Suppose that $\partial \Omega = \{ (R_1, R_2) \}$ $(R_j > 0)$, and assume (A.1) and (A.2). Then there exist $\epsilon > 0$ and an integer $N \geq 2m$ depending only on $u_0$ such that, for any $g \in W_R$ satisfying $||g||_R < \epsilon$ and $\text{ord} g \geq N$, (2.2) has a unique analytic solution $w \in W_R$ satisfying $\text{ord} w \geq N$.

**Remark 2.3** In case the local solvability of (2.2) is concerned we need not the smallness of $g$, and $u_0$ may be any polynomial of degree $2m$. Indeed, if we replace $u_0$ in (2.3) with $2m$ homogeneous part of $u_0$, a similar argument as in the proof of Theorem 2.2 yields the following

**Proposition 2.4** Suppose (A.1) and (A.2). Then there exist an integer $N \geq 2m$ depending only on $u_0$ such that, for any $g$ analytic in some neighborhood of the origin satisfying $\text{ord} g \geq N$, (2.2) has a unique analytic solution $w$ in some neighborhood of the origin such that $\text{ord} w \geq N$.

We shall prove a nonlinear version of a Kashiwara-Kawai-Sjöstrand theorem.

**Corollary 2.5** Suppose (A.1) and (A.2). Then, for every $g$ analytic at the origin such that $\text{ord} g \geq 2m$ all formal power series solutions of (2.2) converge in some neighborhood of the origin.

**Remark 2.6** a) The condition (A.1) in Corollary 2.5 cannot be dropped in general. (See Corollary 6.6 and Remark 6.7.)

b) In 1974, Kashiwara-Kawai-Sjöstrand showed the convergence of all formal power series solutions of the following linear partial differential equations of regular singular type, $\sum_{|\alpha| = |\beta| \leq m} a_{\alpha\beta}(x)x^\alpha(\partial/\partial x)^\beta u = f(x)$, where $m$ is an integer and $a_{\alpha\beta}(x)$ and $f(x)$ are analytic in some neighborhood of the origin.
of $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. They gave a sufficient condition for the convergence of all formal power series solutions, a certain ellipticity of the equation. (cf. (0.2) in [4]). It contains, as a special case, a so-called Poincaré condition. Corollary 2.5 gives a generalization of [4] to fully nonlinear equations.

Let $M \equiv M(u)$ be a Monge-Ampère operator in $(x, y) \in \mathbb{R}^2$

$$M(u) := u_{xx} u_{yy} - u_{xy}^2 + c(x, y)u_{xy},$$

where $c(x, y)$ is a homogeneous polynomial of degree 2, and we abbreviate $u_{xx} = \partial_x^2 u$, $u_{yy} = \partial_y^2 u$, and so on. Let $u_0(x, y)$ be a homogeneous polynomial of degree 4 and set $f_0(x, y) = M(u_0)$. For an analytic function $g(x, y)$ such that $\text{ord} g > 4$ we consider

$$M(u_0 + w) = f_0(x, y) + g(x, y).$$

In the following we denote by $P$ the linearized operator of $M(u)$ at $u = u_0$. We also set $R_1 = R_2 = 1$ for the sake of simplicity.

**Example 2.1.** Let $u_0(x, y) = x^2y^2$ and $c(x, y) = kxy \ (k \in \mathbb{R})$. It follows that $f_0(x, y) = 4(k - 3)x^2y^2$. The operator $P$ and the Toeplitz symbol are given, respectively, by

$$P = 2x^2 \partial_x^2 + 2y^2 \partial_y^2 + (k - 8)xy \partial_x \partial_y,$$

$$\sigma(z, \eta) = 2(\eta_1^2 + \eta_2^2) + (k - 8)\eta_1 \eta_2.$$  

The condition (A.1) reads $2 + (k - 8)\eta_1 \eta_2 \neq 0$ for all $\eta \in \mathbb{R}_+^2$, $|\eta| = 1$. Because $0 \leq \eta_1 \eta_2 \leq 1/2$, it follows that (A.1) is equivalent to $k > 4$. We can easily verify (A.2) if $k > 4$. Note that if $k > 4$ (A.1) is a so-called Poincaré condition for $P$. We note that because the Toeplitz symbol is independent of $z$, we can apply Theorem 2.1 as well as Theorem 2.2 if $k > 4$.

By simple computations of the characteristic polynomials, we see that $M(u)$ at $u = u_0$ is elliptic outside the set $xy = 0$ if and only if $4 < k < 12$. Note that if $k < 4$ or $k > 12$ the equation is weakly hyperbolic and degenerates on the lines $x = 0$ and $y = 0$. 


Next we will estimate the integer \( N \) in Theorem 2.2 when \( k > 4 \). Because \( P \) preserves homogeneous polynomials we study the injectivity of \( P \) on the set of homogeneous polynomials of degree greater than 5. By definition, a monomial \( x^\mu y^\nu (\nu + \mu > 5) \) is in the kernel of \( P \) if and only if

\[
2\nu(\nu - 1) + 2\mu(\mu - 1) + (k - 8)\nu\mu \neq 0. \tag{2.10}
\]

Since \( k > 4 \), we easily see that (2.10) implies that \( k \neq 16/3, 9/2 \) if \( \nu = 2 \) or \( \mu = 2 \). If \( \nu = \mu = 3 \) it follows from (2.10) that \( k \neq 16/3 \). Similarly, if \( \nu = 3, \mu = 4 \) or \( \nu = 4, \mu = 3 \) we get \( k \neq 5 \). More generally, if \( \nu + \mu = n \) (\( n \geq 5 \)) the condition (2.10) is equivalent to \( k \neq 8 - 2(n^2 - n)/(\nu\mu) \leq 4 + 8/n \). The equality holds when \( \nu = \mu = n/2 \). Consequently, the injectivity of \( P \) on the set of homogeneous polynomials of degree \( n \) holds in each of the following cases: a) \( k > 16/3, n \geq 5 \), b) \( k > 5, n \geq 7 \), c) \( k > 4 + 8/n, n \geq 8 \).

**Example 2.2.** Let \( u_0(x, y) = x^4 + kx^2y^2 + y^4 \), \( c(x, y) \equiv 0 \) and \( f_0(x, y) = M(u_0) = 12(2kx^4 + 2ky^4 + (12 - k^2)x^2y^2) \). The operator \( P \) and its Toeplitz symbol \( \sigma(z, \eta) \) are given, respectively by

\[
P = 12y^2\partial^2_y + 12x^2\partial^2_x + 2k(y^2\partial^2_y + x^2\partial^2_x) - 8xy\partial_x\partial_y \tag{2.11}
\]
\[
\sigma(z, \eta) = 2k(\eta_1^2 + \eta_2^2) - 8\eta_1\eta_2 + 12(z_1^2z_2\eta_1^2 + z_1^2z_2^2\eta_2^2). \tag{2.12}
\]

Clearly, Poincaré condition does not hold for \( P \). In order to apply Theorem 2.2, the condition (A.1) with \( \bar{\Omega} = \{(1, 1)\} \) reads

\[
k - 4\eta_1\eta_2 + 6(\eta_1^2t^2 + \eta_2^2t^{-2}) \neq 0 \quad \forall t \in C, \quad |t| = 1 \forall \eta \in \mathbb{R}^2_+, \quad |\eta| = 1. \tag{2.13}
\]

In case \( \eta_1 = \eta_2 \), it follows from \( |\eta| = 1 \) that \( \eta_1 = \eta_2 = 1/\sqrt{2} \). Hence (2.13) implies that \( k \not\in [-4, 8] \). If \( \eta_1 \neq \eta_2 \) we have that \( 2i \Re (\eta_1^2t^2 + \eta_2^2t^{-2}) = (\eta_1^2 - \eta_2^2)(t^2 - t^{-2}) \), which vanishes only if \( t^2 = \pm 1 \). Because \( k \) is real (2.13) is verified if \( t^2 \neq \pm 1 \). In the case \( t^2 = \pm 1 \), (2.13) implies that \( k \neq 4\eta_1\eta_2 \pm 6 \). It follows that \( k \not\in [-6, -4] \) and \( k \not\in [6, 8] \) since \( 0 \leq \eta_1\eta_2 \leq 1/2 \). Therefore (A.1) is equivalent to \( k < -6 \) or \( k > 8 \). We can easily verify (A.2) for \( \eta_1 = 0, \eta_2 = 1 \) under these conditions.
On the other hand, if we want to apply Theorem 2.1, it is necessary to verify (A.1). By simple computations this is equivalent to verify (2.13) for $|t| = \rho$, where $\rho$ varies in $0 < \Re \rho_1 \leq \rho \leq \Re \rho_2 < \infty$. Hence the condition is valid if $k$ is sufficiently large.

We will show that the type of $M$ changes near the origin. Indeed, if $k > 8$ $f_0$ changes sign on the four lines $f_0(x, y) = 0$ in $\mathbb{R}^2$ intersecting at the origin. Hence the equation is hyperbolic-elliptic near the origin. If $k < -6$ the origin $x = y = 0$ is the only zero of $f_0$ in $\mathbb{R}^2$, i.e., $P$ is degenerate hyperbolic.

**Example 2.3.** Let $u_0(x, y) = x^4 + bx^2y^2$, $c(x, y) = cxy$, where $b > 6$ and $c$ is a constant chosen later. The operator $P$ is given by $P = 2b x^2 \partial_\xi^2 + 2(6x^2 + by^2) \partial_\eta^2 + (c - 8b)xy \partial_\xi \partial_\eta$, which is degenerate elliptic with degeneracy on the line $x = 0$. Because $\sigma(x, \eta) = 2b(\eta_1^2 + \eta_2^2) + 12\eta_2^2 x_1^2 x_2^2 + (c - 8b)\eta_1 \eta_2$, (A.1) reads

$$2b + 12\eta_2^2 t^2 + (c - 8b)\eta_1 \eta_2 \neq 0, \quad \forall t, |t| = 1, \forall \eta \in \mathbb{R}_1^+, |\eta| = 1. \quad (2.14)$$

This is easily verified if $\eta_2 = 0$ or $t^2$ is not real. Now, suppose that $t^2 = \pm 1$. Because $b > 6$ and $0 \leq \eta_1 \eta_2 \leq 1/2$, (2.14) holds if $c - 8b > 0$. The condition (A.2) for $\eta_1 = 1, \eta_2 = 0$ is easily checked by definition. It follows from Theorem 2.2 that (2.7) has a solution if the order of $g$ is sufficiently large. If we want to apply Theorem 2.1 we have to verify (2.14) for $|t| = \rho$ with $0 < \Re \rho_1 \leq \rho \leq \Re \rho_2 < \infty$. Clearly, this condition is valid for sufficiently large $b$ depending on $\rho_1$ and $\rho_2$.

## 3 Reduction to the boundary

In this section we obtain a crucial estimate for the linearized operator $P$ in (2.1) under (A.1) and (A.2).

### 2.1. Restriction to the boundary and Toeplitz operators.

Let $W_R(T_R^2)$ be the restriction of $W_R$ in (2.5) to the torus $\{|x_1| = R_1\} \times \{|x_2| = R_2\}$
The space $W_R(T^2_R) := \left\{ u = \sum_{\eta \geq 0} u_\eta R^\eta e^{i\eta \theta}; \|u\|_R := \sum_{\eta} |u_\eta| R^\eta < \infty \right\}$. (3.1)

The space $W_R(T^2_R)$ is a Banach space with the norm $\| \cdot \|_R$ and each $u \in W_R(T^2_R)$ can be extended to an analytic function on the polydisk $D_R$ by a harmonic extension. We denote by $L^1_R(T^2_R)$ the space of integrable functions on $\{ |x_1| = R_1 \} \times \{ |x_2| = R_2 \}$. We denote by $\pi$ the projection from $L^1_R(T^2_R)$ to $W_R(T^2_R)$.

For $\alpha \in \mathbb{N}^2, \beta \in \mathbb{N}^2$ and a smooth function $u$ we have $x^{\alpha-\beta} \partial_x^\alpha u = x^{\alpha-\beta} \partial_x^\alpha u$. Moreover, by the commutator relation $[t, \partial_j] = -1$ we have $t^{k} \partial_k^j = t \partial_j (t \partial_k - 1) \cdots (t \partial_k - k + 1)$. It follows that, by setting $\delta_j = x_j \partial_j$ and $j = 1, 2$,

$$x^{\alpha-\beta} \partial_x^\alpha = c_{\alpha, \beta} \delta \cdot p_\delta(e),$$ (3.2)

where $\delta = (\delta_1, \delta_2)$. Hence we can write $P$ in the form $P = \sum_{\alpha, \beta} c_{\alpha, \beta} x^{\alpha-\beta} \partial \cdot p_\delta(e)$, where $c_{\alpha, \beta}$ are appropriately chosen constants.

Suppose now that $P$ acts on holomorphic functions $u$, $\partial_j u = 0$, where $\partial_j$ is the Cauchy-Riemann operator with respect to $x_j$. If we introduce the polar coordinates $x_j = r_j \exp(i\theta_j)$ ($j = 1, 2$) we have that $x_j \partial_j = (r_j \partial_{r_j} - i \partial_{\theta_j})/2$, ($j = 1, 2$). It follows from Cauchy-Riemann equation that $(r_j \partial_{r_j} + i \partial_{\theta_j})u = 0$. Because the normal derivative $r_j \partial_{r_j}$ can be expressed by a tangential derivative $\partial_r$, we can restrict $P$ to the torus $\{ |x_1| = R_1 \} \times \{ |x_2| = R_2 \}$. Namely, we can regard $P$ as the operator on $W_R(T^2_R)$. By definition we have $\pi P = P$ on $W_R(T^2_R)$.

Let $< D_\theta >$ be a pseudodifferential operator on $T^2_R$ with symbol $< \eta > := (1 + |\eta|^2)^{1/2}$, where $\eta$ is the covariable of $\theta$. Then the operator $\pi P < D_\theta >^{-m}$ on $W_R(T^2_R)$ is called a Toeplitz operator on $W_R(T^2_R)$. Note that the Fredholmness of $P < x \partial_x >^{-m}$ on $W_R$ is equivalent to that of the Toeplitz operator $\pi P < D_\theta >^{-m}$ on $W_R(T^2_R)$.

2.1. Fredholm property of Toeplitz operators. Let $H$ be a Banach space with norm $\| \cdot \|$. We denote by $L(H)$ the space of linear continuous operators
on $H$. An operator $L \in \mathcal{L}(H)$ is said to be a Fredholm operator if the range $LH$ of $L$ is closed in $H$, the kernel and cokernel of $L$ is of finite dimension, i.e., $\dim \ker L < \infty$ and $\dim \text{Coker } L < \infty$, where $\text{Coker } L = H/LH$. We denote the space of Fredholm operators by $\Psi(H)$. For $L \in \Psi(H)$ we define the index of $L$ by $\text{ind } L := \dim \ker L - \dim \text{Coker } L$. A linear operator $C$ is said to be a compact operator if it maps every bounded set into a precompact set. For $L \in \Psi(H)$ and a compact operator $C$, the operator $L + C$ is a Fredholm operator such that $\text{ind } (L + C) = \text{ind } L$.

The following elementary lemma is useful in the following arguments. (See also Lemma 3.4 in [6])

**Lemma 3.1** Let $q(\eta)$ be a function on $\mathbb{N}^2$ such that $\sup_{|\eta| \geq N} |q(\eta)| \to 0$ when $N \to \infty$. Then the pseudodifferential operator $q(D_\theta) : W_\pi(T_0^2) \to W_\pi(T_0^2)$ is a compact operator.

Let $\lambda_\theta(D_\theta)$ be a pseudodifferential operator with symbol $\lambda_\theta(\eta) := \eta^{|\eta| - |\theta|}$ ($\eta \neq 0$) and $\lambda_\theta(0) = 0$. By Lemma 3.1, the Fredholmness of $\pi P$ is equivalent to that of the operator $T$,

$$T = \pi \sum_{|\alpha|,|\theta| = m} c_{\alpha\theta} e^{i(\alpha - \theta)\theta} \lambda_\theta(D_\theta) : W_\pi(T_0^2) \to W_\pi(T_0^2).$$

(3.3)

We note that, by (2.3), $T = \pi \sigma(e^{i\theta}D_\theta/D_\theta)$. For every $u \in W_\pi(T_0^2)$ the order of $u$, $\text{ord } u$ is defined as the order of $\hat{u}$, where $\hat{u}$ is an analytic extension of $u$ to $W_\pi$. The following theorem is crucial in our argument.

**Theorem 3.2** Suppose (A.1) and (A.2). Then there exists an integer $k_0$ such that $\pi P < D_\theta >^{-m}$ is invertible as a map on $W_\pi(T_0^2) \cap \{u; \text{ord } u \geq k_0\}$ into itself.

For the proof we prepare

**Proposition 3.3** Suppose (A.1) and (A.2). Then $T : W_\pi(T_0^2) \to W_\pi(T_0^2)$ in (3.3) is a Fredholm operator of index zero.
The proof is done by exactly the same method as that of Theorem 3.1 in [6].

Proof of Theorem 3.2. By (A.1), (A.2), Proposition 3.3 and Lemma 3.1 we see that $T$ is a Fredholm operator of index zero on $W_R(T^2_\mathbb{R})$. Because $T$ preserves the homogeneity there exists an integer $k_1$ such that $T$ is a bijection on $W_R(T^2_\mathbb{R}) \cap \{u; \text{ord } u = n\}$ into itself if $n \geq k_1$. Because $\pi P < D_\theta >^{-m}$ - $T$ is a pseudodifferential operator with negative order times a projection we can choose $k_0$ such that $\pi P < D_\theta >^{-m}$ is invertible as a map on $W_R(T^2_\mathbb{R}) \cap \{u; \text{ord } u \geq k_0\}$ into itself. □

4 Proof of Theorems

Proof of Theorem 2.2. We note that $W_R(T^2_\mathbb{R})$ is a Banach algebra by the usual multiplication of functions. We restrict the equation (2.2) on $W_R$ to $W_R(T^2_\mathbb{R})$. Let the polynomial $p_\eta(\theta)$ be given by (3.2). Then, by restriction we replace $\partial^\alpha u$ in (2.1) by

$$e^{-i\theta p_\eta(D_\theta)}u = \pi e^{-i\theta p_\eta(D_\theta)}u.$$  \hspace{1cm} (4.1)

Because $M(u_0 + u) = M(u_0) + Qw + M(w)$ for the linearization $Q$ of $M$ at $u = u_0$, the restriction of (2.2) to the torus can be written in the following form

$$\pi Qw + \sum_{\alpha,\beta} a_{\alpha,\beta}(\pi e^{-i\theta p_\alpha(D_\theta)}w)(\pi e^{-i\theta p_\beta(D_\theta)}w) = g.$$  \hspace{1cm} (4.2)

Let $k_0$ be given in Theorem 3.2 and consider (4.2) on $X := W_R(T^2_\mathbb{R}) \cap \{w; \text{ord } w \geq k_0\}$. It follows from Theorem 3.2 that there exists $(\pi Q)^{-1} := S$ on $X$. We denote the nonlinear part in (4.2) by $K(w)$ and we introduce a new unknown function $v$ by $w = Sv$. Then by recalling $g = \pi g$ we have

$$v + K(Sv) = \pi g.$$  \hspace{1cm} (4.3)

We define the sequence $\{v_k\}$ by $v_0 = 0$, $v_1 = \pi g - K(Sv_0) = \pi g$, $v_{k+1} = $
\[ \pi g - K(Sv_k) \text{ for } k = 0, 1, \ldots \] If \( \|g\|_R \) is sufficiently small, it follows from the definition of \( K \) that the apriori estimate \( \|v_k\|_R < \rho \) \( (k = 1, 2, \ldots) \) holds for small \( \rho > 0 \). Hence, the limit \( v := \sum_{k=0}^{\infty} (v_{k+1} - v_k) = \lim_k v_k \) exists in \( X \). We have
\[
v = \lim v_k = \pi g - \lim K(Sv_{k-1}) = \pi g - K(Sv).
\]
Hence there exists a solution \( v \in X \) to \( (4.2) \).

As to the uniqueness, let \( \|w_1\|_R < \rho, \|w_2\|_R < \rho \) be the solutions of \( (4.3) \). Then we see that \( v = w_1 - w_2 \) satisfies \( v + \tilde{K}(Sw_1, Sw_2)v = 0 \), where \( \tilde{K}(Sw_1, Sw_2) := K(Sw_1) - K(Sw_2) \) is a polynomial of \( Sw_1 \) and \( Sw_2 \). Because \( \|\tilde{K}(Sw_1, Sw_2)\|_R < 1 \) for sufficiently small \( \rho \), we have \( \|v\|_R \leq \|\tilde{K}v\|_R \leq \|\tilde{K}\|_R\|v\|_R < \|v\|_R \). Hence \( v = 0 \).

Let \( \dot{w} \) \( (w = Sv) \) be the analytic extension of \( w \) into the polydisk \( D_R \). The analytic function \( M(\dot{w} + \dot{w}) - f_0(x) - g(x) \) in \( D_R \) vanishes on the Silov boundary of \( D_R \) by the construction of \( w \). Hence the maximal principle implies that it vanishes in \( D_R \). Hence \( \dot{w} \) is the solution of \( (2.2) \).

Suppose that there exist two solutions \( w_1 \) and \( w_2 \) of \( (2.2) \). By the unique solvability of the restricted equation we see that \( w_1 = w_2 \) on the boundary. The maximal principle implies that \( \dot{w}_1 = \dot{w}_2 \) in \( D_R \). \( \square \)

**Proof of Theorem 2.1.** By definition of \( u_0 \) the Toeplitz symbol is in variant if we replace \( R \) by \( Rp \) for \( \rho \leq 1 \). It follows that the conditions \( (A.1) \) and \( (A.2) \) hold true for \( R = Rp \). We cover \( \Omega' \) by a finite number of polydisks \( D_R \)'s such that for each \( D_R \) \( (A.1) \) and \( (A.2) \) holds. In each \( D_R \) there exists a unique solution of our equation if \( g \) is sufficiently small. In order to prolong analytic solution, suppose that there exist solutions in \( D_R \) and \( D_{R'} \). By assumption the solution is unique on \( D_R \cap D_{R'} \). Hence we can prolong the solution to \( D_R \cup D_{R'} \). It follows that we have a unique solution in \( \Omega' \). \( \square \)

**Proof of Corollary 2.5.** Let \( w = \sum_{j=0}^{\infty} w_j \) be a formal solution of \( (2.2) \), where \( \text{ord} g \geq 2m + 1 \), and \( w_j \) is homogeneous degree \( j \). For \( k \) chosen later, we
set \( w = w_0 + U \), where \( U = \sum_{j=1}^{\infty} w_j \). In order to show the convergence of \( U \), define a polynomial \( h \) by \( M(w_0 + w) = f_0 + h \) and rewrite (2.2) in the form \( M(w_0 + w + U) = f_0 + h + g - h \). By taking \( k \) sufficiently large, one can make the order of \( g - h \) arbitrarily large. Because (A.1) and (A.2) are invariant if we replace \( w_0 \) by \( w_0 + w \), it follows from by Proposition 2.4 that the equation has a unique analytic solution \( U \) near the origin. This proves the convergence of \( U \). \( \Box \)

5 Extension to higher dimensions

In this section we will generalize the results in §2 to a general dimension. Let \( x = (x_1, \ldots, x_n) \in \mathbb{C}^n \) and define \( \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} \), \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+ \). For \( R = (R_1, \ldots, R_n) \) \( (R_j > 0) \), let \( D_R \) be the disk \( D_R = \{|x_1| \leq R_1\} \times \cdots \times \{|x_n| \leq R_n\} \). We define the torus \( T_R^\alpha \) by \( T_R^\alpha = \{|x_1| = R_1\} \times \cdots \times \{|x_n| = R_n\} \).

Let \( \Omega \subset \mathbb{C}^n \) be a bounded complete Reinhardt domain containing the origin not necessarily convex. We say that \( \Omega \) is a Reinhardt domain if for each \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega \) the torus \( \{(a_1 e^{\theta_1}, \ldots, a_n e^{\theta_n}); 0 \leq \theta_j \leq 2\pi, j = 1, \ldots, n\} \) is contained in \( \Omega \). A Reinhardt domain \( \Omega \) is said to be complete if for each \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega \) the disk \( D_R, R = (|a_1|, \ldots, |a_n|) \) is contained in \( \Omega \). We denote by \( \tilde{\Omega} \) the real representation of a Reinhardt domain, namely, \( \tilde{\Omega} := \{r \in \mathbb{R}^n_+; r = (|x_1|, \ldots, |x_n|), x \in \Omega\} \). We denote by \( \mathcal{O}(\Omega) \) the set of holomorphic functions in \( \Omega \).

Let \( M(x, z) \) be an analytic function of \( (x, z) \in \Omega \times W \) holomorphic in \( x \in \Omega \) and polynomial in \( z \). We consider the operator \( M(u) := M(x, \partial^\alpha u) \). Let \( u_0(x) \in \mathcal{O}(\Omega) \) and set \( f_0(x) = M(u_0) \). For \( g \in \mathcal{O}(\Omega) \) we consider

\[
M(u_0 + v) = f_0(x) + g(x) \tag{5.1}
\]

Let \( P = M_{w_0} = \sum_{|\alpha| \leq m} (\partial M/\partial x_\alpha)(x, w_0) \partial^\alpha \) be the linearized operator of \( M(u) \).
at $u = u_0$. We write it in the form $P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$, where $m \in \mathbb{N}$, and $a_\alpha(x)$ is holomorphic in the closure of $\Omega$. We define the Toeplitz symbol $\sigma(z, \xi)$ by
\[
\sigma(z, \xi) := \sum_{|\alpha| \leq m} a_\alpha(z) z^{-\alpha} \rho_\alpha(\xi) <\xi>^{-m},
\]
where $<\xi> = (1 + |\xi|^2)^{1/2}$ and $\rho_\alpha(\xi) = \prod_{j=1}^n (\xi_j - 1) \cdots (\xi_j - \alpha_j + 1)$. We decompose
\[
\sigma(z, \xi) = \sigma'(z, \xi) + \sigma''(z, \xi), \quad \sigma'(z, \xi) = \int_{\mathbb{T}^n} \sigma(Re^\theta, \xi) d\theta.
\]
We assume that there exist constants $c \in \mathbb{C}$, $|c| = 1$ and $d > 0$ such that
\[
\text{Re} \sigma'(\xi) \geq d > 0 \quad \text{for all} \quad \forall \xi \in \mathbb{Z}^n_+.
\]
Then we have

**Theorem 5.1** Let $\Omega' \subset \subset \Omega$ be arbitrarily given. Suppose that (5.4) is satisfied. Then there exist $d_0$ and $\varepsilon > 0$ such that if $d \geq d_0$ in (5.4) the equation (5.1) in $\Omega'$ has a unique solution $v$ in $\Omega'$ for any $g$ holomorphic in $\Omega$ such that $\sup_{\Omega'} |g(x)| \leq \varepsilon$.

We define the set of holomorphic functions $W_R \equiv W_R(D_R)$ by
\[
W_R(D_R) := \{ u = \sum_{n \geq 0} u_n x^n; \|u\|_R := \sum_{n} |u_n| R^n < \infty \}. \tag{5.5}
\]
We denote by $W_R(T^n_R)$ the restriction of $W_R$ to the torus $T^n_R$, $|x_j| = R_j$ by Fourier series expansion. Every element in $W_R(T^n_R)$ can be extended as a holomorphic function in $D_R$, uniquely. Let $\ell^1_R$ be the space of all absolutely convergent sequences, namely $(u_n)_{n \in \mathbb{Z}^n}$ such that $\sum_n |u_n| R^n < \infty$. Let $\ell^1_{k, \eta}$ be the subspace of $\ell^1_R$ such that $u_\eta = 0$ for $\eta \not\in \mathbb{Z}^n_+$. We note that $W_R(T^n_R)$ is isomorphic to the space $\ell^1_{k,\eta}$ in terms of Fourier series expansion. We denote by $\pi$ the projection $\pi : \ell^1_R \mapsto \ell^1_{k,\eta}$. Clearly, the map $\pi$ also defines the natural projection on the space of all weighted absolutely convergent Fourier series on...
T \mathbb{T}_R^m$ onto $W_R(\mathbb{T}_R^m)$. Because every element of $W_R$ has a Taylor expansion we can define a pseudodifferential operator $<D_x>$ by the same way as before.

Lemma 5.2 Suppose that $(5.4)$ is satisfied for some $\Omega = (R_1, \ldots, R_n)$ with $R_j > 0$ $(j = 1, \ldots, n)$. Then there exist $d_0$ and $\varepsilon > 0$ such that if $d \geq d_0$ in $(5.4)$ the equation $Pu = f$ in $W_R$ has a unique solution for any $f \in W_R$ such that $\|f\|_R \leq \varepsilon$.

Proof. We restrict the operator $P < D_x >^{-m}$ to the torus by the help of $\overline{\partial}$ operator. Then the Toeplitz symbol of the restricted operator is given by (5.2).

It follows from (5.4) that, if $d$ is sufficiently large,

$$Re \sigma(z, \xi) = Re \sigma'(\xi) + Re \sigma''(z, \xi) > d - K_0, \quad \forall z \in \mathbb{T}_R^n, \forall \xi \in \mathbb{Z}_+^n,$$

for some $K_0 > 0$ such that $d > K_0$. Hence, if $\varepsilon > 0$ is sufficiently small we have

$$\|1 - \varepsilon \sigma(\cdot, \xi)\|_{L^\infty} < 1 - \varepsilon d + \varepsilon K_0, \quad \forall \xi \in \mathbb{Z}_+^n. \quad (5.6)$$

For $u \in W_R(\mathbb{T}_R^n)$ we have

$$\|(1 - \varepsilon \sigma)u\|_R = \|\pi(1 - \varepsilon \sigma)u\|_R \leq \|(1 - \varepsilon \sigma)u\|_{L^\infty}, \quad (5.7)$$

where we used the boundedness of $\pi : \ell^1_R \to \ell^1_R$. If we can show that $\|(1 - \varepsilon \sigma)u\|_{L^\infty} < \|u\|_R = \|u\|_{L^\infty}$ we have that $\|(1 - \varepsilon \sigma)u\|_R < \|u\|_R$. This proves that $\varepsilon \sigma$ is invertible on $W_R(\mathbb{T}_R^n)$. Hence $\pi \sigma$ is invertible.

We set $Q(\theta, \xi) = 1 - \varepsilon \sigma(Re^{i\theta}, \xi)$ and we expand $Q(\cdot, \xi)$ into Fourier series, $Q(\theta, \xi) = \sum_\nu \hat{b}_\nu(\xi) R^\nu e^{i\theta \nu}$. The sum is convergent by the smoothness of $Q$.

In the following we denote by $Q$ the pseudodifferential operator with symbol given by $Q(\theta, \xi)$. For $u = \sum_\xi \hat{u}(\xi) R^\xi e^{i\theta \xi} \in W_R(\mathbb{T}_R^n)$, we have

$$(Qu)(\theta) = \sum_\xi Q(\theta, \xi) \hat{u}(\xi) R^\xi e^{i\theta \xi} = \sum_\xi \sum_\nu e^{i\theta \nu} \hat{b}_\nu(\xi) \hat{u}(\xi) R^{\nu+\xi} e^{i\theta \xi}. \quad (5.8)$$

By changing the summation, the right-hand side is equal to

$$\sum_\nu \sum_\xi \hat{b}_\nu(\xi) \hat{u}(\xi) R^{\nu+\xi} e^{i\theta (\xi + \nu)} = \sum_\nu T^\nu u, \quad (5.9)$$
where
\[
(T^v u)(\theta) := \sum_{\xi} \hat{b}_v(\xi) \hat{u}(\xi) R^v e^{im(\xi + v)} = \sum_{\xi} \hat{b}_v(\xi - \nu) \hat{u}(\xi - \nu) R^v e^{im\xi}. \tag{5.10}
\]

Hence we have
\[
\|T^v u\|_R = \sum_{\xi} |\hat{b}_v(\xi - \nu)||\hat{u}(\xi - \nu)||R^{v'} \leq \sup_{\xi} |\hat{b}_v(\xi)R^{v'}| \sum_{\xi} |\hat{u}(\xi - \nu)||R^{v'-\nu} = \|u\|_R \sup_{\xi} |\hat{b}_v(\xi)|R^{v}. \tag{5.11}
\]

Let \(\sigma(Re^\theta, \xi) = \sum_{v} \delta_v(\xi) R^v e^{iu\theta}\) be the Fourier expansion of \(\sigma(Re^\theta, \xi)\). Then we have
\[
Q(\theta, \xi) = 1 - \varepsilon \cos(Re^\theta, \xi) = 1 - \varepsilon \cos(\xi) - \varepsilon \sum_{v \neq 0} \delta_v(\xi) R^v e^{iu\theta}. \tag{5.12}
\]

Suppose that we have the estimate
\[
\sum_{v} \sup_{\xi} |\hat{b}_v(\xi)||R^v = \sup_{\xi} |1 - \varepsilon \cos(\xi)| + \varepsilon \sum_{v \neq 0} \sup_{\xi} |\delta_v(\xi)||R^v < 1. \tag{5.13}
\]

Then we get, from (5.11) that
\[
\|1 - \varepsilon \cos\|_{L^1} \leq \sum_{v} \|T^v u\|_R \leq \|u\|_R \sum_{v} \sup_{\xi} |\hat{b}_v(\xi)||R^v < \|u\|_R. \tag{5.14}
\]

In order to prove (5.13) we note that (5.13) is clearly satisfied for sufficiently small \(\varepsilon\) if the summation in \(\nu\) is finite. (See also (5.6). In the general case we note
\[
R^v \delta_v(\xi) = (2\pi)^{-n} \int_{T^n} \sigma''(Re^\theta, \xi) e^{-iu\theta} d\theta. \tag{5.15}
\]

Hence by the partial integration we have
\[
R^v \delta_v(\xi) = (2\pi)^{-n} \int_{T^n} D^\nu \sigma''(Re^\theta, \xi) e^{-iu\theta} d\theta. \tag{5.16}
\]

It follows that \(\sup_{\xi} |\nu^m \delta_v(\xi)||R^v \leq \sup_{\xi} |D^\nu \sigma''(Re^\theta, \xi)| =: K_n\). Therefore the quantity \((1 + |\nu|)^{m+1} R^v |\delta_v(\xi)|\) \((|\nu| = |\nu_1| + \cdots + |\nu_n|)\) is bounded by some constant \(K\) depending only on \(R\), \(\sigma''\) and \(n\). By (5.4) the left-hand side of (5.13) is bounded by
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\[ \sup \left| -\varepsilon \sigma'(\xi) + \varepsilon \sum_{\xi \neq 0} \partial \iota_{\sigma}(\xi) \right| R' \leq 1 - \varepsilon d + \varepsilon \sum_{\xi \neq 0} (1 + |\nu|)^{n-1} \leq 1 - \varepsilon d + \varepsilon K' \]

\[ \text{(5.17)} \]

for some \( K' > 0 \) depending on \( K \). If \( d \) is sufficiently large we have that \( 1 - \varepsilon d + \varepsilon K' < 1 \). \( \square \)

**Proof of Theorem 5.1.** Let \( \hat{\Omega} \) be a real representation of \( \Omega \). By assumption \( \hat{\Omega} \) is covered by a finite number of \( D_R \)'s with \( R = (R_1, \ldots, R_n) \in \hat{\Omega} \) such that \( R_j \geq \rho_j > 0 \). In each \( D_R \) we will solve (5.1). Indeed, by Lemma 5.2 and the argument of the proof of Theorem 2.2 (5.1) has a unique holomorphic solution if \( d \) in (5.4) is sufficiently large and \( \sup_{\Omega} |f| \) is sufficiently small. The intersection of two disks \( D_R \) and \( D_{R'} \) is equal to \( D_{R'} \) with \( R' = (R''_1, \ldots, R''_n) \), \( R''_j = (R_j, R_j') \geq \rho_j \). Because \( \Omega \) is a complete Reinhardt domain it follows that \( R' \in \hat{\Omega} \). Hence, by the unique solvability of the holomorphic solution in \( D_{R'} \), the solutions in \( D_R \) and \( D_{R'} \) coincide in \( D_{R'} \) if \( d \) in (5.4) is sufficiently large and \( \sup_{\Omega} |f| \) is sufficiently small. It follows that we can continue a unique holomorphic solution over a finite union of \( D_R \)'s. Therefore we have a unique solution in \( \hat{\Omega} \). \( \square \)

We will apply the above method to Monge-Ampère type equations under a weaker condition than (5.4). With the same notations as above we consider

\[ M(u) := \det(u_{jz,k}) + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} z^\alpha \partial^\beta_z, \]

\[ \text{(5.18)} \]

where \( a_{\alpha\beta} \in \mathbb{C} \) and \( u_{jz,k} = \partial^\alpha u / \partial x_j \partial x_k \). Let \( u^0 \) be holomorphic in \( \Omega \) and define \( M(u^0) = f_0 \). We want to solve the equation (5.1) for \( g \in \mathcal{O}(\Omega) \) such that \( \text{ord} g \) is sufficiently large. Here the order of \( g \) is defined in the introduction.

The Toeplitz symbol \( \sigma(z, \xi) \) of the linearized operator \( P \) is given by

\[ \sigma(z, \xi) = (z_1 \cdots z_n)^{-2} \det(\xi_j z_k + z_j z_k u_{jz,k}(z)) - f_0(z) + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} z^\alpha \partial^\beta_z. \]

\[ \text{(5.19)} \]

We make the decomposition (5.3) and assume that there exist constants \( c \in \mathbb{C}, |c| = 1, N \geq 1 \) and \( d > 0 \) such that
Re \omega'(\xi) \geq d > 0 \quad \text{for all} \quad \forall \xi \in \mathbb{Z}_+^n, |\xi| \geq N. \quad (5.20)

The condition (5.20) is slightly weaker than (5.4). Then we have

**Theorem 5.3** Let \( \Omega' \subset \subset \Omega \) be arbitrarily given. Suppose that (5.20) is satisfied. Then there exist \( d_0, N_0 \geq 1 \) and \( \varepsilon > 0 \) such that if \( d \geq d_0 \) in (5.20) the equation (5.1) in \( \Omega' \) has a unique holomorphic solution \( v \) in \( \Omega' \) for any \( g \) holomorphic in \( \Omega \) such that \( \text{ord} \, g \geq N_0 \), \( \sup \{ |g(x)| \leq \varepsilon \}. \)

The proof of Theorem 5.3 is almost the same as that of Theorem 5.1 except for that we work in the class of functions \( v \) such that \( \text{ord} \, v \geq N_0 \). (See also the proof of Theorem 2.2.)

**Example.** We consider the case \( n = 3 \), \( u^0 = x_1x_2x_3 \) and the linear part is equal to \(-2((x_1\partial_1)^2 + (x_2\partial_2)^2 + (x_3\partial_3)^2)\). We can easily verify the condition (5.20).

## 6 Monge-Ampère equations in an outer domain

Let \( \Omega \) be a simply connected complete Reinhardt domain containing the infinity, \( x_1 = \infty, \ldots, x_n = \infty \). We say that \( \Omega \) is complete if for each \((a_1, \ldots, a_n) \in \Omega\) the disk at \( \infty \), \( D_a = \{|x_1| \geq |a_1|^{-1}\} \times \cdots \times \{|x_n| \geq |a_n|^{-1}\} \) is contained in \( \Omega \). Let \( \tilde{\Omega} \) be the real representation of \( \Omega \). We say that a Reinhardt domain \( \Omega' \) is compactly contained in \( \Omega \) and denote it by \( \Omega' \subset \subset \Omega \) if there exist \( \rho_j > 0 \) \((j = 1, \ldots, n)\) such that the real representation \( \tilde{\Omega}' \) is contained in the set \( \{ R_+^n; x_j \geq \rho_j, j = 1, \ldots, n \} \) and \( \tilde{\Omega}' \subset \subset \tilde{\Omega} \).

We consider the operator given by the left-hand side of (5.18). By the transformation \( x_j = y_j^{-1} \) \((j = 1, \ldots, n)\) we have that \( u_{x_j x_k} = y_j y_k (\delta_j \delta_k u + \varepsilon_{jk} \delta_j u), \) where \( \delta_j = y_j \partial_j \) and \( \varepsilon_{jk} \) is a Kronecker delta. Hence it follows that \( \det(u_{x_j x_k}) = (y_1 \cdots y_n)^2 \det(\delta_j \delta_k u + \varepsilon_{jk} \delta_j u) \). Because \( \det(\delta_j \delta_k u + \varepsilon_{jk} \delta_j u) \) is divisible by \( y_1 \cdots y_n \), we set \( M := (y_1 \cdots y_n)^{-1} \det(\delta_j \delta_k u + \varepsilon_{jk} \delta_j u) \) and we have
det(u_{x_1x_n}) = (y_1 \cdots y_n)^3 \hat{M}. Hence, if \( a_{\alpha \beta} = 0 \) in (5.18), (5.18) is equivalent to \( \hat{M} = g \) for appropriately chosen \( g \). In view of this we consider the following operator

\[
M(u) := (x_1 \cdots x_n)^3 \det(u_{x_1x_n}) + \sum_{|\alpha| = |\beta| = 2} a_{\alpha \beta} x^\alpha \xi^\beta = f(x). \tag{6.1}
\]

Let \( u^0 \) be a holomorphic function in \( \Omega \) such that \( \text{ord} u^0 \geq 2 \) and set \( f_0 = M(u^0) \). Let \( P \) be the linearized operator of \( M(u) \) at \( u = u_0 \). The Toeplitz symbol \( \sigma(z, \xi) \) is then given by

\[
\sigma(z, \xi) = \frac{\det(\xi_j \xi_k + z_j z_k u_{x_jx_k}(z))}{z_1 \cdots z_n} h(z) + \sum_{|\alpha| = |\beta| = 2} a_{\alpha \beta} (-1)^{|\beta|} z^\beta \xi^\alpha, \tag{6.2}
\]

where

\[
h(z) = (z_1 \cdots z_n)^{-1} \det(\delta_j \delta_k u^0). \tag{6.3}
\]

We have the following theorem.

**Theorem 6.1** Let \( \Omega' \subset \subset \Omega \) be an arbitrarily Reinhardt domain. We make the decomposition (5.3) for \( \sigma(z, \xi) \) in (6.2), and we assume (5.20). Then there exist \( d_0, N_0 \geq 1 \) and \( \epsilon > 0 \) such that if \( d \geq d_0 \) in (5.20) the equation (6.1) with \( u = u^0 + v \) and \( f = f_0 + g \) in \( \Omega' \) has a unique holomorphic solution \( v \) for any \( g \) holomorphic in \( \Omega' \) such that \( \text{ord} g \geq N_0 \), \( \sup_{\Omega'} |g| < \epsilon \).

**Proof.** We make the change of variables \( x_j = y_j^{-1} \) in (6.1) to obtain

\[
\hat{M} + Q = g(y), \quad g(y) = f(y^{-1}), \tag{6.4}
\]

for some \( Q \). By simple calculations the Toeplitz symbol of the linearized operator of the left-hand side of (6.4) at \( u = u^0 \) is given by (6.2) and (6.3). Moreover, we may solve (6.4) in a bounded complete Reinhardt domain because \( \Omega' \subset \subset \Omega_\rho \). By the similar argument as in Theorem 5.3, we can prove the existence of a solution for (6.4). \( \square \)

In the above theorem the existence of a linear part is crucial in actual applications. In the following, we suppose that the linear part in (6.1) is zero,
namely \( a_{\alpha} = 0 \). Furthermore, we assume \( n = 2 \), and we shall study regular
and singular solutions to the equation

\[
u_{x_1x_1} u_{x_2x_2} - u_{x_1x_2}^2 = f(x), \tag{6.5}\]

where \( x = (x_1, x_2) \in \mathbb{R}^2 \). Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \). Let \( X_j \) be the linear
space of all formal power series in \( x_j \) given by \( X_j = \{ u; \, v = x_j^{\alpha_j} \sum_{k=0}^{\infty} v_k x_j^{-k} \} \)
\((j = 1, 2)\). Then we have

**Theorem 6.2** Suppose that \( f = 0 \) and \( |\alpha| = \alpha_1 + \alpha_2 \neq -1 \). Then every
solution of (6.5) of the form \( u = x^{-\alpha} \sum_{n \in \mathbb{Z}_+} u_n x^{-n} \) is contained either in \( X_1 \)
or in \( X_2 \). Hence they depend only on one variable.

**Remark 6.3** Theorem 6.2 can be stated in a more general setting. Indeed,
consider general Monge-Ampère equation

\[
A r + B s + C t + D(r - s^2) - E = 0, \quad r = z_1 x_1, s = z_2 x_2, t = z_2 x_2^2, \\
\text{where } A, B, C, D \text{ and } E \text{ are smooth functions of } x_1, x_2, z, p = \partial z/\partial x_1, q = \partial z/\partial x_2. \text{ Let } \lambda_1 \text{ and } \lambda_2 \text{ be the roots of the equation } \lambda^2 + B \lambda + (AC + DE) = 0.
\] and consider the Monge system

\[
M : dz - p dx_1 - q dx_2 = D dp + C dx_1 + \lambda_1 dx_2 = D dq + \lambda_2 dx_1 + Adx_2 = 0.
\]

It is well-known that if \( M \) has three independent first integrals, then the above
Monge-Ampère equation is mapped to (6.5) with \( f = 0 \) by contact transforma-
tions.

We first note that the change of variables \( x_j = z_j^{-1}, (j = 1, 2) \) in (6.5) yields

\[
M(u) \equiv (\delta_1^2 u + \delta_1 u)(\delta_2^2 u + \delta_2 u) - (\delta_1 \delta_2 u)^2 = g(z), \tag{6.6}
\]

where \( \delta_j = z_j(\partial/\partial z_j) \) \((j = 1, 2)\) and \( g(z) = z_1^{-2} z_2^{-2} f(z_1^{-1}, z_2^{-1}) \).

**Proof of Theorem 6.2.** We set \( |\alpha| = m \) and \( g = 0 \). By substituting the expansion
\( u = z^m \sum_{n \in \mathbb{Z}_+} u_n z^n \) \((u_0 \neq 0)\) in (6.6) and by comparing the homogeneous
part of degree $2m$ we obtain $(\alpha_2^2 + \alpha_1)(\alpha_2^2 + \alpha_2) - \alpha_1^2 \alpha_2^2 = 0$. It follows from the condition $|\alpha| \neq -1$ that $\alpha_1 \alpha_2 = 0$, which implies $\alpha = (m, 0)$ or $\alpha = (0, m)$.

Suppose that $m \leq -2$ and $\alpha = (m, 0)$. Let $u = z_1^m + u_1 + u_2 + \cdots$ be the homogeneous expansion of $u$ with $u_j$ being homogeneous degree $m + j$. By comparing $2m+1$ homogeneous part of (6.6) we have $(\delta_1^2 + \delta_2)u_1 = 0$. Because $u_1$ contains only nonnegative powers of $z_2$ by assumption $\alpha = (m, 0)$ it follows that $u_1 = c_1 z_1^{-m+1}$ for some constant $c_1$. Suppose that $u_j = c_j z_1^{m+j}$ for $j \leq k$. By comparing the homogeneous part of degree $2m + k + 1$ in (6.6) we obtain $(\delta_1^2 + \delta_2)u_{k+1} = 0$, to yield $u_{k+1} = c_{k+1} z_1^{m+k+1}$. This proves that $u \in X_1$. We can similarly prove that $u \in X_2$ in case $\alpha = (0, m)$. In view of $|\alpha| \neq -1$, the other case $m \geq 0$ is contained in the following proposition.

**Proposition 6.4** Let $u = \sum_{\nu=1}^{\infty} u_\nu$ be the homogeneous expansion of $u$ in (6.6). Suppose that $u_j = 0$ for any $j < k$ and that $u_k \neq 0$. Then we have the followings

a) Either $u_k = cz_1^k$ or $u_k = cz_2^k$ holds for some $c \neq 0$.

b) Either $u_\nu = cz_1^\nu (\nu > k)$ or $u_\nu = cz_2^\nu (\nu > k)$ holds for some $c_\nu$.

**Proof.** If we prove a), b) follows by the same argument as in Theorem 6.2. In order to prove a) we note that the comparison of terms of homogeneous order $2k$ in (6.6) implies that $u_k$ satisfies (6.6) with $g = 0$. We substitute the expansion of $u_k$, $u_k = \sum_{j=0}^k a_j z_2^{k-j}$ into (6.6). Since the term $z_2^j$ does not appear we compare the coefficients of $z_2^{k-1} z_2$, to obtain $a_k a_{k-1} = 0$.

If $a_k \neq 0$ it follows that $a_{k-1} = 0$. Next, by comparing the terms of $z_1^{2k-2} z_2^2$, we obtain $a_k a_{k-2} = 0$, that is $a_{k-2} = 0$. In the same way we can show that $u_k$ depends only on $z_1$.

We consider the case $a_k = 0$. Because the assertion is trivial for $k = 1$ we assume $k \geq 2$. By comparing the coefficients of $z_1^{2k-2} z_2^2$ in $M(u_k) = 0$ we have that $a_{k-1} = 0$. Suppose that $a_{k-\nu} = 0$ for $\nu \leq j \leq k - 2$. It follows that $u_k$ is of degree greater than $j + 1$ in $z_2$. Hence, by comparing the coefficients of
degree $2(j + 1)$ in $z_2$, we obtain $a_{k-1}^2 = 0$. It follows that $a_k = \cdots = a_1 = 0$. Hence $u_k$ depends only on $z_2$. □

We will study the solutions of the form $u = \sum_{n=1}^{\infty} u_n z^{-n}$ of (6.5) or (6.6). By writing $g(z) = \sum_{j=1}^{\infty} g_j$, with $g_j$ being homogeneous degree $j$, we seek $u = \sum_{j=1}^{\infty} u_j$ where $u_1 = a z_1 + b z_2$ and $u_j$ is homogeneous degree $j$. We assume $ab \neq 0$. If there exists a formal solution of (6.6), we can easily see that $g(z)$ satisfies the condition $g(z) = O(z_1 z_2)$. By comparing terms of homogeneous degree 2 in (6.6) we have $g_2(z) = 4ab z_1 z_2$. We assume these compatibility conditions. By the scale change of variables we may assume $a = b = 1/2$.

**Theorem 6.5** Let $g$ be an arbitrarily power series satisfying $g(z) = z_1 z_2$, $g(z) = O(z_1 z_2)$. For any formal power series $v$ such that $M(v) = 0$, ord $v \geq 2$ there exists a unique formal power series $\phi$ such that ord $\phi \geq 2$ and $u := u_1 + v + \phi$ satisfies (6.6). We write $\phi = Sv$. Conversely, let $u = u_1 + w$, ord $w \geq 2$ be a solution of (6.6) and let $j = 1$ or $j = 2$. Then there exists a unique power series $v$ of $z_j$ such that, $M(v) = 0$, ord $v \geq 2$ and $u = u_1 + v + Sv$. Moreover, $Sv$ does not contain the powers of $z_j$ only.

**Proof.** We use the same notations as above. In order to determine $u_2$ we compare the terms of homogeneous degree 3 in (6.6). We then have $P_1 u_2 = g_3(z)$, where $P_1 = z_1 (z_2^2 + k_2) + z_2 (z_1^2 + k_1)$. If we substitute the expansions $u_2(z) = c_0 z_1^2 + c_1 z_1 z_2 + c_2 z_2^2$ and $g_3(z) = d_1 z_1^2 + d_2 z_2^2$ into the equation $P_1 u_2 = g_3(z)$ we obtain that $6c_0 + 2c_1 = d_1$ and $2c_1 + 6c_2 = d_2$. These equations have a unique solution once we give $c_0$ or $c_2$, which is a kernel element of $M(v) = 0$ in view of Theorem 6.2.

Suppose that we have determined $u_j$ for $j < k$. By comparing $(k + 1)$-th homogeneous part of (6.6) we see that $u_k$ satisfies $P_1 u_k + (\cdots) = g_{k+1}(z)$, where the dots denotes the terms determined by $u_j$ with $j < k$. Because we can easily see, from the definition of $M$ that these term are divisable by $z_1 z_2$ the recurrence relations has the same structures as for $u_2$ and we can determine $u_k$ if we assign the kernel element in $u_k$. The rest of Theorem 6.5 is clear. □
Corollary 6.6 Let \( u = u_1 + v + Sv \) be a formal power series solution of (6.6). If \( u \) converges, it follows that the kernel element \( v \) converges.

Proof. If otherwise, \( u \) does not converge because \( Sv \) does not contain the powers of \( x \) only. \( \Box \)

Remark 6.7 We note that the operator \( P_1 \) in the proof of Theorem 6.5 is degenerate elliptic-hyperbolic near the origin. Indeed, the principal symbol of \( P_1 \) is given by \(-z_1 z_2 (z_2 \xi_1^2 + z_1 \xi_2^2)\). We remark that Corollary 2.5 cannot be applied since \( P_1 \) does not satisfy (A.1).

Concerning the convergence of formal solutions \( u \) in Theorem 6.5 we note that the existence of a nontrivial kernel element \( v \), \( M(v) = 0 \) leads to the divergence in general.

We define the space \( W_R \) as in Theorem 2.2 with respect to the variable \( z \). Now, we drop the assumption \( ab \neq 0 \) in Theorem 6.5. For simplicity we assume \( u_1 = 0 \) and we look for the solution of (6.6) in the form \( u = \sum_{j=2}^\infty u_j \), where \( u_2(z) = a z_2^2 + b z_1 z_2 + c z_1^2 \), \( u_j \) being homogeneous degree \( j \). We define \( g_4 = M(u_2) \).

Theorem 6.8 We define the Toeplitz symbol by \( \sigma(t, \xi) := 3 a z_2^2 t + b(z_2^2 + \xi_1 \xi_2 + 3 c z_1^2) \). Suppose that

\[
\sigma(t, \xi) \neq 0 \quad \text{for all } t \in \mathbb{C}, \quad |t| = 1, \quad \forall \xi \in \mathbb{R}^2, \quad |\xi| = 1, \quad (6.7)
\]

\[
\text{ind}_{|t|=1} \sigma(t, \xi) \equiv \frac{1}{2 \pi i} \oint_{|t|=1} d\xi \log \sigma(t, \xi) = 0 \quad \exists \xi \in \mathbb{R}^2, \quad |\xi| = 1. \quad (6.8)
\]

Then there exists \( N \geq 3 \) and \( r > 0 \) such that for every \( g = g_4 + \hat{g} \in W_R \) such that \( \text{ord} \hat{g} \geq N \) and \( \| \hat{g} \|_R < r \) the equation (6.6) has a unique solution \( u \in W_R \) such that \( u = u_2 + v \), \( \text{ord} v \geq N \).

Proof. Because both sides of (6.6) is divisable by \( z_1 z_2 \) by definition we consider \( z_1^{-1} z_2^{-1} M(u_2 + v) / 2 = z_1^{-1} z_2^{-1} g / 2 \). By simple computation we have that \( M(u_2 + v) = M(u_2) + 2 z_1 z_2 M(v) \), where
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