VERSAL DEFORMATIONS OF
VECTOR FIELD SINGULARITIES

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ABSTRACT. When a singular point of a vector field passes through resonance, a formal invariant cone appears. In the seventies, Pyartli proved that for \((-1, 1)\)-resonance the cone is in fact analytic and is the degeneration of a family of invariant cylinders. In his thesis, Stolovitch established a new type of normal form and proved that for a simple resonance and under arithmetic conditions the cone is (the germ of) an analytic variety. In this paper, we prove a versal deformation theorem for analytic vector fields with an isolated singularity over Cantor sets. Our result implies that, under arithmetic conditions, the resonant cone is the degeneration of a set of invariant manifolds like in Pyartli’s example. For the multi-Hopf bifurcation, that is for the \((-1, 1)^d\)-resonance, this implies the existence of vanishing tori carrying quasi-periodic motions generalising previous results of Chenciner and Li.

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1. Introduction

In the formal neighbourhood of an analytic planar vector field one may find an invariant algebroid curve, classically called a separatrix. The study of these curves goes back to Briot-Bouquet, Dulac, Picard, Poincaré and was finally solved by Camacho and Sad, who proved their existence in full generality [8, 15, 17, 36, 38].

These early investigations on separatrices were intimately connected to the concept of normal forms and deformations near equilibrium; the idea being that if a vector field can be reduced to a simple form by a change of variables, then it would be easy to find the separatrices. But due to the appearance of small denominators, the issue of convergence of the required transformation turned out to be very non-trivial. In fact, the Poincaré-Siegel divergence theorems suggest that there is no analytic normal forms or versal deformations over a smooth base [37, 41].

In the work of Kolmogorov, Siegel, Bruno and Rüssmann profound results were obtained with regard to convergence questions outside the Poincaré domain [14, 25, 40, 42]. As a rule, the normalising transformations diverge and only in cases of integrability one may expect convergence of the normalising transformations. The dichotomy theorems of Ilyashenko and Perez-Marco show that these integrable cases are exceptional [23, 34, 35]. For excellent overviews on these topics we refer the reader to [24, 29, 46].

So in the light of KAM theory, the best one can hope for is the construction of a convergent versal deformations/normal forms over a Cantor set of positive measure. It is the purpose of this paper to state and proves such a result.

Unlike the case of Hamiltonian dynamics and KAM theory, the Cantor set we construct is not invariant and integral curves may not be all contained inside the set. One may ask: what is the use a vector field over such a set? The answer is quite simple: although the Cantor set is
not invariant, it may very well contain an interesting subset of persistant invariant varieties. In this way we are led back to the very origins of normal forms alluded above.

Hyperbolic vector fields being linearisable in the $C^\infty$-category, divergence and small denominators are invisible in the smooth category [43] (incidentally this is not the case in the Gevrey category see [47]). This underlines the relevance looking at vector fields in the complex domain, as was traditionally done in the early works on the subject.

The simplest example is given by the $(1, -1)$ resonance, leading to the so-called Hopf bifurcation:

\[
\begin{align*}
\dot{x} &= (-\alpha + \mu_1 + \mu_2 xy)x \\
\dot{y} &= (\alpha + \mu_1 + \mu_2 xy)y
\end{align*}
\]

Classically one takes $\alpha$ is purely imaginary and restricts the flow to the real subspace defined by $x = y = z$. For $\mu = 0$, the cone $xy = 0$ is an analytic invariant variety that we call the resonant cone. In the real case, the curves $xy = \varepsilon$, that is the circles $|z|^2 = \varepsilon$, are limit cycles for $\varepsilon > 0$ stable under perturbations [1, 22, 28]. In the complex domain, these limit cycles vanishing at the singular point are real parts of the invariant cylinders $xy = \varepsilon$ that degenerate into the resonant cone [2, 3]. So we find here the classical situation of the vanishing cycles at an $A_1$ singularity: a horizontal family of 1-cycles generating the homology of the cylinder, which vanishes at the singular point (see for instance [5, 6]). Pyartli proved the stability under perturbation of the analytic cylinders degenerating into a cone [39].

It appears that Bruno thought Arnold had conjectured the existence of invariant families in higher dimensions for simple resonances and disproved this conjecture [11]. Incidentally, Bruno’s result was misquoted by Arnold in his famous book on differential equation, where he attributes to Bruno and Pyartli the proof of the existence of such families in dimension $3!$ (see [4, Chapter 6, §36, E]). What Arnold conjectured or at least hoped was the analyticity of the resonant cone for a single resonance and he wrote: However, we can hope that these small perturbations do not destroy the invariant manifold $M_0$.

Apparently unaware of Arnold’s problem, Écalle asserted that the conjecture holds, but only under an arithmetic condition on the eigenvalues of the vector field [18]. Analyticity of the resonant cone was finally proved in 1994 by Stolovitch in his thesis. Stolovitch showed also the necessity of an arithmetic condition [44]. Finally he extended the case of simple resonances to that of so-called positive resonances, but apparently made a (minor) mistake in defining positivity (see 2.3).

More recently Stolovitch proved a KAM theorem for vector fields, namely he showed the existence of a positive measure set of invariant manifolds,
assuming a triviality condition of the Poincaré-Dulac normal form [45].

For the case of the Hopf bifurcation, the triviality condition of Stolovitch means that the Poincaré-Dulac normal form defines a symplectic vector field, so rather than an isolated limit cycle we have a family of circles, the system is integrable and there is no bifurcation at all. Similarly for the multi-dimensional Hopf bifurcation, the condition of Stolovitch means that the system is Hamiltonian. Therefore in this case, the theorem implies the existence of a positive measure set of complex analytic invariant manifolds carrying a complex quasi-periodic motion. In the real elliptic case, these are the complexification of KAM tori vanishing at the singularity and the theorems of Stolovitch imply in particular that these form a set of positive measure around the singularity (see also [19]).

From the standpoint of a singularist, the KAM theorem of Stolovitch means that there are strata in the versal deformation space on which invariant varieties form a positive measure set. So it is natural to ask if such a family could exist over the whole space. Such a result turns out to be a consequence of our versal deformation theorem. If we modify Arnold’s conjecture in the sense that the family of analytic varieties is not analytic but only continuous (and even Whitney $C^\infty$) over a set of positive measure, then it holds true.

The invariant manifolds being toric manifolds, the result might be seen as a complex variant of the theory of quasi-periodic motions in the dissipative context at a singularity. The general theory of such quasi-periodic motions in the dissipative context has been established in the pioneering works of Broer, Bruno, Huitema, Sevryuk and Takens (see [9, 10, 12]). Applying our result to the multiple Hopf bifurcation, we obtain the existence of quasi-periodic motions vanishing at the singularity. This generalises a result obtained by Li in dimension 2 and 3 [26] (see also [16] for the corresponding result for discrete dynamical systems).

In this paper we use, as Stolovitch, positivity assumptions on the linear part of our vector field, but in fact we are confident that the results can be proven under less restrictive conditions. It is for instance plausible that the methods for the classification on planar vector field due Malgrange, Martinet-Ramis, Moussu-Cerveau can be used [27, 31, 32, 33].

From a technical point of view, the proof of our versal deformation theorem is a direct application of the general theory of normal forms developed in [21]. Therefore, with regards to the details of the proof of convergence, we refer to that paper on the theory of Banach space valued functors.
2. DIVERGENT AND CONVERGENT NORMAL FORMS

In the sixties, Bruno started the systematic investigation of differential equations over rings of formal power series, and complemented it with important convergence results. But the formal case in itself is already a rich subject.

2.1. The Poincaré-Dulac normal form. We denote by $\hat{O}$ the local ring of formal power series in $d$-variables $x_1, x_2, \ldots, x_d$:

$$\hat{O} := \mathbb{C}[[x_1, x_2, \ldots, x_d]],$$

and by $\hat{X}$ the corresponding module of derivations

$$\hat{X} := \text{Der}(\hat{O}) = \bigoplus_{i=1}^{d} \hat{O} \partial_{x_i},$$

So a formal vector field $X \in \hat{X}$ is of the form

$$X = \sum_{i=1}^{d} a_i(x) \partial_{x_i}, \quad a_i(x) \in \hat{O},$$

and can be written as an expansion of homogeneous components of increasing degree:

$$X = X_0 + X_1 + X_2 + \ldots$$

Here and in the sequel the dots stand for higher order terms in the Taylor expansion at the origin.

The vector field is called singular if $X_0 = 0$. We say it is an isolated singularity, if the origin is scheme-theoretically the only point for which $X_0 = 0$, or equivalently if the ideal

$$(a_1, a_2, \ldots, a_n) \subset \hat{O}$$

is $\mathcal{M}$-primary, where $\mathcal{M}$ denotes the maximal ideal of $\hat{O}$. Such a vector field $X$ induces derivations $X_k$ of $\mathcal{M}/\mathcal{M}^{k+1}$ and as any linear map, can be decomposed into semi-simple and nilpotent part:

$$X_k = S_k + N_k, \quad [S_k, N_k] = 0,$$

and by taking the limit $k \to +\infty$, we obtain the abstract Poincaré-Dulac normal form

$$X = S + N, \quad [S, N] = 0.$$
2.2. Resonances. From now on we assume that the linear part of the vector field is semi-simple:

\[ X_1 = S_1 = S, \quad N_1 = 0. \]

In appropriate coordinates the semi-simple part takes the form

\[ S = \sum_{i=1}^{d} \lambda_i x_i \partial_{x_i}, \]

and we call

\[ \Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_d) \]

the frequency vector of \( X \). We denote by

\[ \widehat{O}^S := \{ f \in \widehat{O} \mid S(f) = 0 \} \subset \widehat{O} \]

the ring of invariants of \( S \). A monomial

\[ x^R := x_1^{r_1} x_2^{r_2} \ldots x_d^{r_d}, \quad r_i \in \mathbb{N} \]

is invariant if and only if

\[ (\Lambda, R) := \sum_{i=1}^{d} \lambda_i r_i = 0, \]

and the ring \( \widehat{O}^S \) is generated by a finite number of such resonant monomials

\[ x^{R_1}, x^{R_2}, \ldots, x^{R_p}, \quad R_j \in \mathbb{N}^d, \]

so that

\[ \widehat{O}^S = \mathbb{C}[[x^{R_1}, x^{R_2}, \ldots, x^{R_p}}]. \]

The vectors \( R \) for which \( (\Lambda, R) = 0 \) are called resonances and all resonances are non-negative linear combinations of the basic resonances \( R_i \), but this representation in general is not unique.

Similarly, we denote by

\[ \widehat{X}^S := \{ V \in \widehat{X} \mid [S, V] = 0 \} \]

the resonant vector fields, which are invariant under the infinitesimal action by \( S \). A monomial field

\[ x^K \partial_{x_i} = x_1^{k_1} x_2^{k_2} \ldots x_d^{k_d} \partial_{x_i}, \]

belongs to \( \widehat{X}^S \) if and only if

\[ (\Lambda, K) - k_i = 0. \]

Note that the fields \( x_i \partial_{x_i}, i = 1, 2, \ldots, n \), always belong to \( \widehat{X}^S \). The resonant vector fields \( \widehat{X}^S \) form a module over \( \widehat{O}^S \), which in fact is finitely generated. So there are finitely many resonant fields

\[ V_1, V_2, \ldots, V_m, \]
such that any $V \in \hat{X}^S$ can be expressed as

$$V = \sum_{i=1}^{m} \phi_i(x^{R_1}, x^{R_2}, \ldots, x^{R_p})V_i$$

This immediately leads to the following result:

**Theorem** ([13, 17]). Let $X$ be a formal vector field then there exists an automorphism $\varphi \in \text{Aut}(\hat{O})$ and power series $\phi_1, \ldots, \phi_m \in \mathbb{C}[\lbrack [u] \rbrack := \mathbb{C}[u_1, \ldots, u_p]$ such that $\varphi(X)$ is a resonant vector field i.e.:

$$\varphi(X) = S + \sum_{i=1}^{m} \phi_i(x^{R_1}, \ldots, x^{R_p})V_i$$

In the statement of the theorem and in the sequel, the derivation $\varphi(X)$ is defined by the commutative diagram

$$
\begin{array}{ccc}
\hat{O} & \xrightarrow{X} & \hat{O} \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\hat{O} & \xrightarrow{\varphi(X)} & \hat{O}
\end{array}
$$

Note that in differential geometry, a derivation $X$ is interpreted as a vector field and the notation $\varphi_*X$ is used. As $\varphi(X)$ cannot be understood in another way, we adopt this simplified notation which is more adapted to our computations.

### 2.3. Positivity conditions.

We make the following assumptions on the eigenvalue $\Lambda$ of $S$:

**P1)** there are no algebraic relations between a minimal set of generators for the ring of invariants. In other words, we assume that

$$\hat{O}^S = \mathbb{C}[u_1, u_2, \ldots, u_p]$$

is a power series ring.

**P2)** the module of $\hat{X}^S$ is freely generated by the $x_i \partial_{x_i}$'s over the ring $\hat{O}^S$.

The following easy reformulation show why one can think of P1) and P2) as positivity conditions.

**Lemma.** The conditions P1) and P2) are respectively equivalent to the existence of integral vectors $R_1, \ldots, R_p \in \mathbb{N}^d$ with $(L, R_j) = 0$, such that:

1) Each vector $J \in \mathbb{N}^d$ with $(\Lambda, J) = 0$ can be uniquely written as

$$J = \sum_{j=1}^{p} n_j R_j, \quad n_j \in \mathbb{N}.$$
2) Each vector $K \in \mathbb{N}^d$ with $(\Lambda, K) = k_i$ for some $i$, can be uniquely written as

$$K = \sum_{j=1}^{d} m_j R_j + E_j, \quad m_j \in \mathbb{N},$$

where $E_1 = (1, 0, \ldots, 0), \ldots, E_d = (0, \ldots, 0, 1)$ denotes the standard basis vectors.

2.4. Examples and counter-examples. Let us consider a pair of opposite eigenvalues

$$S = \lambda x \partial_x - \lambda y \partial_y$$

The ring of invariants is generated by $xy$, the resonant fields are generated by $x \partial_x$ and $y \partial_y$. Thus it satisfies the positivity conditions. The formal normal form of a vector field with this linear part is

$$(\lambda + g(xy))x \partial_x - (\lambda + h(xy))y \partial_y$$

This vector field is Hamiltonian for the symplectic structure $dx \wedge dy$ precisely when $g = h$. A direct generalisation arises when eigenvalues form opposite pairs

$$S = \sum_{i=1}^{d} \lambda_i x_i \partial_{x_i} - \lambda_i y_i \partial_{y_i},$$

and are generic otherwise, in the sense that

$$\dim \mathbb{Q} \sum_{i=1}^{d} \mathbb{Q} \lambda_i = d.$$

The ring of invariants is generated by $x_i y_i$, $i = 1, 2, \ldots, d$, the resonant fields are the $x_i \partial_{x_i}$, $y_i \partial_{y_i}$, so that it is a case where the the positivity conditions hold. So the formal normal form of a vector field with this linear part is

$$S = \sum_{i=1}^{d} (\lambda_i + g_i(x_1 y_1, \ldots, x_d y_d)) x_i \partial_{x_i} - (\lambda_i + h_i(x_1 y_2, \ldots, x_d y_d)) y_i \partial_{y_i}$$

Again the field is Hamiltonian for the symplectic structure

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

precisely when $g_i = h_i$.

A very different case is the $(1, 1, -2)$-resonance:

$$S = x \partial_x + y \partial_y - 2z \partial_z$$

The ring of invariants is generated by

$$u := x^2 z, \quad v := y^2 z, \quad w := xyz.$$
Among these there is a relation 
\[ uv - w^2 = 0, \]
so the map
\[ \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad (x, y, z) \mapsto (x^2z, y^2z, xyz) \]
is not surjective, but lands on the quadratic cone defined by the above equation. The first positivity condition is not satisfied. Also, the module of resonant vector fields is generated by
\[ x\partial_x, \quad y\partial_y, \quad z\partial_z, \quad x\partial_y, \quad y\partial_x \]
The second positivity condition is not satisfied either and the relations
\[ vx\partial_x = wy\partial_x, \quad uy\partial_y = wx\partial_y \]
show \( X^S \) is not a free module over the invariant ring \( \hat{\mathcal{O}}^S \), showing that P2) does not hold either.

In [44, Section 4], Stolovitch seems to suggests that P1) implies P2). But if we consider the case \( \Lambda = (\alpha + \beta, \alpha, \beta) \), where \( \alpha \) and \( \beta \) are \( \mathbb{Q} \)-independent and positive, there are no non-trivial invariants, \( \hat{\mathcal{O}}^S = \mathbb{C} \), but there is a non-trivial resonant vector field, namely \( yz\partial_z \). So P1) is satisfied, but not P2). By adding extra variables one obtains examples with \( \hat{\mathcal{O}}^S \neq \mathbb{C} \).

2.5. The resonant cone. If \( S \) satisfies the positivity assumptions, then any vector field in Poincaré normal form is of the form
\[ X = S + N = \sum_{i=1}^{d} (\lambda_i + \phi_i(x^{R_1}, x^{R_2}, \ldots, x^{R_p}))x_i\partial_{x_i} \]
In particular, such vector fields are of the form
\[ X = S + T, \quad T = \sum_{i} a_i(x)x_i\partial_{x_i}, \quad a_i(x) \in \hat{\mathcal{O}} \]
that we will call formal logarithmic vector fields. Such a vector field have a special property.

**Definition.** The resonant ideal is the ideal \( I \) generated by the invariants of positive degree.
\[ I = (x^{R_1}, x^{R_2}, \ldots, x^{R_p}) \subset \hat{\mathcal{O}} \]
The resonant cone is the formal scheme defined by the resonant ideal.

**Proposition.** A logarithmic vector field
\[ X = S + T, \quad T = \sum_{i} a_i(x)x_i\partial_{x_i} \]
preserves the the resonant cone.
**Proof.**

\[(S + T)x^{R_j} = \left( (\Lambda_i, R_{ij}) + \sum_{i=1}^{d} a_i(x)r_{i,j} \right) x^{R_j} \in (x^{R_1}, \ldots, x^{R_p}).\]

\[\square\]

**Corollary.** If \( S \) satisfies the positivity assumption, then any vector field \( X = S + \ldots \) admits an invariant ideal isomorphic to the resonant ideal.

**Proof.** By positivity, the normal form of the vector field is logarithmic, so the result from the above proposition. \[\square\]

**2.6. The Stolovitch normal form.** Like for the Poincaré-Siegel theorems, the transformation bringing a vector field to Poincaré Dulac normal form is, as a general rule, divergent [14, 23, 34]. Therefore it is important to have a less restrictive but convergent normal form.

We denote

\[ O := \mathbb{C}\{x_1, x_2, \ldots, x_d\}, \quad X := \text{Der}(O) = \bigoplus_{i=1}^{d} O\partial_{x_i} \]

the local ring of convergent power series and the module of analytic vector fields.

The following fundamental result is due to Stolovitch [44]:

**Theorem.** Let \( S \) be a linear vector field that satisfies the Bruno and positivity conditions. Then for any analytic vector field \( X = S + \cdots \in \text{Der}(O) \)

there exists an analytic automorphism \( \varphi \in \text{Aut}(O) \) such

\[ \varphi(X) = S \mod X_{\text{log}} \]

where

\[ X_{\text{log}} = \sum_{i=1}^{d} O x_i \partial_{x_i} \]

is the \( O \)-module of logarithmic vector fields.

As remarked above, the formal version of this theorem trivially follows from the Poincaré-Dulac normal form. The importance of the theorem lies in the following corollary.

**Corollary.** If \( S \) satisfies the positivity conditions, then any analytic vector field \( X = S + \ldots \) admits an invariant analytic germ, isomorphic to the analytic resonant cone.
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Here by the phrase *analytic resonant cone* we mean of course the germ defined by the ideal in \( \mathcal{O} \) generated by the analytic resonant ideal \((x^{R_1}, x^{R_2}, \ldots, x^{R_p}) \subset \mathcal{O}\).

The above theorem of Stolovitch can be proven using our *Lie iteration*, whose convergence can be shown in great generality (see also 4.3).

3. THE FORMAL VERSAL DEFORMATION

In this section, we consider a derivation with semi-simple linear part \( S \). We assume positivity conditions on \( S \) and choose an integer basis \( R_1, \ldots, R_j \) of resonances.

3.1. Bruno variables. In [11], Bruno extended the ring \( \hat{\mathcal{O}} \) by adding variables \( u_1, \ldots, u_p \) corresponding to the \( p \) monomial invariants \( x^{R_i}, i = 1, 2, \ldots, p \). In the approach of Stolovitch in [45], this can be seen as a generalisation the variables introduced by Moser in KAM-theory to parametrise the invariant tori.

Geometrically, we introduce a space \( \mathbb{C}^p \) with coordinates \( u_1, \ldots, u_p \) and define a map

\[ \mathbb{C}^d \longrightarrow \mathbb{C}^p, x \mapsto (x^{R_1}, x^{R_2}, \ldots, x^{R_p}). \]

The (completion at 0 of the) fibre over 0 is the resonant cone; note that the semi-simple vector field \( S \) preserves the fibres of \( \varphi \). The graph of this map is the space \( \Sigma \subset \mathbb{C}^{n+p} \) defined by the vanishing of the polynomials

\[ J_1 := u_1 - x^{R_1}, \quad J_2 := u_2 - x^{R_2}, \ldots, \quad J_p := u_p - x^{R_p}. \]

Any vector field \( Y \) on \( \mathbb{C}^d \) can be lifted to a vector field \( \tilde{Y} \) on \( \mathbb{C}^{d+p} \), tangent to \( \Sigma \):

\[ Y \mapsto \tilde{Y} := Y + \sum_{j=1}^{p} Y(x^{R_j}) \partial_{u_j}, \]

reflected algebraically by the statement

\[ \tilde{Y}(J_j) = 0, \quad j = 1, 2, \ldots, p. \]

In our formal context we extend the power series ring \( \hat{\mathcal{O}} \) by formal variables and work in

\[ \hat{\mathcal{O}}[[u]] = \mathbb{C}[[x, u]] = \hat{\mathcal{O}}[[u_1, u_2, \ldots, u_p]] = \mathbb{C}[[x_1, \ldots, x_d, u_1, \ldots, u_p]] \]

and rather than the manifold \( \Sigma \), we consider the ideal

\[ J := (J_1, J_2, \ldots, J_p) \subset \hat{\mathcal{O}}[[u]]. \]

Clearly, there is a ring-isomorphism

\[ \hat{\mathcal{O}}[[u]]/J \longrightarrow \hat{\mathcal{O}}, \quad x_i \mapsto x_i, \quad u_j \mapsto x^{R_j}. \]
We will use the lift $Y \mapsto \tilde{Y}$ to identify $\hat{X}, \hat{X}^S$ as subspaces of $\text{Der}(\hat{O}[[u]])$. and suppress the ‘of the notation.

**Definition.** The ideal $J \subset \hat{O}[[u]]$ is called the graph ideal of $S$.

We assign degree 1 to the variables $x_i$ and

$$|R_i| = \sum_{j=1}^{d} R_{i,j}$$

to the variable $u_j$, so that the polynomials $I_j$ are homogeneous of degree $|R_j|$. We filter the ring $\hat{O}[[u]]$, by the order of a series, that is, the smallest degree of a monomial appearing in the series.

This filtration of the ring induces a filtration on $\text{Der}(\hat{O}[[u]])$: a derivation is said to be of order $d$ if it maps terms of order $i$ to terms of order $i + d$, so that the order of $S$ is 0.

3.2. The detuning deformation. We now follow Martinet’s treatment of versal deformations and construct a ring with extra parameters to describe a versal deformation. If we are given an arbitrary deformation, given by a family with still other variables, we also add these to the ring. The inducing maps are then obtained by solving equations by the implicit function theorem [30]. The method is classical and we used this technique also in [20].

Following Bruno, we work in the ring $\hat{O}[[u]]$ and with lifted fields $\hat{X} \subset \text{Der}(\hat{O}[[u]])$.

By a deformation of $S$ we mean a vector field depending on $l$ parameters $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ of the form

$$X = S + T(\mu) \in \hat{X}[[\mu]], \quad T(0) = 0.$$  

Here

$$\hat{X}[[\mu]] \subset \text{Der}(\hat{O}[[u, \mu]])$$

consist of those vector fields of $\hat{X}[[u]]$ that contain no $\partial_{\mu_i}$ (relative vector fields).

A special detuning deformation is obtained by introducing frequency variables $\phi_1, \ldots, \phi_d$ and considering

$$S_v = S + \sum_{i=1}^{d} \phi_i x_i \partial x_i = \sum_{i=1}^{d} (\lambda_i + \phi_i) x_i \partial x_i \in \hat{X}[[\phi]]$$

Under our running positivity assumptions we have

$$\hat{X}^S = \oplus_{i=1}^{d} \hat{O}^S x_i \partial x_i,$$

so the Poincaré-Dulac theorem suggests that the detuning deformation is versal in a formal sense.
If we want to induce an arbitrary deformation \( X = S + T(\mu) \) from the detuning deformation, one first forms the sum of the two deformations, that is we consider

\[
X_v = S_v + T(\mu) = S + \sum_{i=1}^{d} \phi_i x_i \partial_{x_i} + T(\mu),
\]
a deformation of \( S \) with two sets of parameters \( \phi \) and \( \mu \). The versality is then expressed by the existence of a certain automorphism \( \varphi \in \text{Aut}(\mathcal{O}[[u, \phi, \mu]]) \) that maps \( X_v \), considered as a deformation of \( S_v \), back to \( S_v \). So we will have to work with variables \( u, \phi \) and \( \mu \) and vector fields that depend on these variables, but there will be no derivatives in the \( \mu \) or \( \phi \) directions, and thus belong to \( \hat{\mathcal{X}}[[\phi, \mu]] \subset \text{Der}(\mathcal{O}[[u, \phi, \mu]]) \).

We also use the notation

\[
\hat{\mathcal{X}}^S[[\phi, \mu]] \subset \text{Der}(\mathcal{O}[[u, \phi, \mu]])
\]
for the \( \mathcal{O}[[\phi, \mu, u]] \)-module generated by the \( (x^R_t - u_i)x_j \partial_{x_j} \)'s.

We assign degree 2 to the \( \mu \) variables, in this way making a deformation of a linear vector field is obtained by adding terms of higher order.

### 3.3. Formal normal form theorem.

**Theorem.** Let \( S = \sum \lambda_i x_i \partial_{x_i} \) be a linear vector field satisfying the positivity assumptions and let

\[
S_v = \sum_{i=1}^{d} (\lambda_i + \phi_i) x_i \partial_{x_i} \in \hat{\mathcal{X}}[[\phi]]
\]
be its detuning deformation. Then for any perturbation of the form

\[
X_v := S_v + \cdots \in \hat{\mathcal{X}}[[\phi, \mu]],
\]
where the dots stand for higher order terms; there exists an automorphism \( \varphi \) of the algebra \( \hat{\mathcal{O}}[[\phi, \mu, u]] \) which has the following properties:

1) \( \varphi(X_v) = S_v \mod \hat{\mathcal{X}}^S[[\phi, \mu]]. \)
2) \( \varphi(u_j) = u_j \) for \( j = 1, \ldots, k. \)
3) \( \varphi(\mu_j) = \mu_j \) for \( j = 1, \ldots, l. \)
4) \( \varphi(\phi_j) = \phi_j + \cdots \in \mathbb{C}[[\phi, \mu, u]]. \)

**Proof.** The proof is done by applying the Lie iteration. To save ink we set:

\[
R := \hat{\mathcal{O}}[[u, \phi, \mu]], \text{Der} := \hat{\mathcal{X}}[[\phi, \mu]] \subset \text{Der}(R)
\]
The infinitesimal action on the versal deformation is given by map

\[
\text{Der} \longrightarrow \text{Der}, \; X \mapsto [X, S_v].
\]
It is $\mathbb{C}[[u, \phi, \mu]]$-linear and diagonal in the monomial basis:

$$[x^K \partial_{x_i}, S_v] = \lambda_{K,i} x^K \partial_{x_i}, \quad \lambda_{K,i} := (\lambda_i + \phi_i - (\Lambda + \phi, K)).$$

We define a $\mathbb{C}[[u, \phi, \mu]]$-linear map $L$ by putting for non-resonant vector $K$

$$L(x^K \partial_{x_i}) = \lambda_{K,i} x^K \partial_{x_i},$$
$$L(x_i \partial_{x_i}) = \partial_{\phi_i},$$

As the module of resonant vector fields is freely generated by vector fields of the form $x_i \partial_{x_i}$ the map $L$ is well-defined. Note that

$$[u_j \partial_{\phi_i}, S_v] = u_j x_i \partial_{x_i}$$
$$= x^{R_j} x_i \partial_{x_i} + (u_j - x^{R_j}) x_i \partial_{x_i},$$
$$= x^{R_j} x_i \partial_{x_i} \mod J$$

where $J$ is the graph ideal. Using $L$, for $V = S_v + T$, we define the map:

$$j_V : m \mapsto Lm - L(Lm(T)) = L(m - Lm(T)).$$

This map satisfies

$$[j_V(Y), S_v + T] = Y \mod J$$

We define

$$X_0 = X_v, \quad S_0 = 0, \quad A_0 = S_v, \quad v_0 = [j_{S_v}(X_0)]^2,$$

where

$$[-]^b_a$$

denotes the sum of terms of degree $\geq a$ and $< b$ in the Taylor series at 0. The following Lie iteration which brings a vector field $X$ back to its normal form:

$$X_{n+1} = e^{-[-^n v_n]} X_n,$$
$$S_{n+1} = [X_n - [A_n, v_n]]_2^{2n+1},$$
$$A_{n+1} = A_n + S_{n+1},$$
$$v_{n+1} = j_{A_n}([X_n]_2^{2n+2})$$
3.4. Relation to the Poincaré-Dulac normal form. The theorem implies the classical Poincaré-Dulac theorem. In this case, our perturbation does not contain the \( \mu \) parameters.

We start a vector field
\[
X = S + X', \quad X \in \hat{\mathfrak{X}}
\]
where \( X' \) are terms of positive order and \( S \) is semi-simple. We consider the associated perturbation of our corresponding versal deformation:
\[
X_v = S_v + X'
\]
So \( X \) is obtained from \( X_v \) by setting all \( \phi_i = 0 \). The above Theorem 3.3 tells us that there exists an automorphism \( \varphi \in \text{Aut}(\hat{\mathfrak{O}}[[\phi, u]]) \) such that
\[
\varphi(X_v) = S_v \quad \text{mod } \hat{\mathfrak{X}}^S[[\phi]]
\]
This automorphism \( \varphi \in \text{Aut}(R) \) maps \( \phi_j \) to \( \varphi(\phi_j) = \phi_j + \ldots \in \mathbb{C}[[\phi, u]] \).

By the implicit function theorem, we may solve the equations \( \varphi(\phi_j) = 0 \) by \( \phi_j = g_j(u) \) for certain series \( g_j(u) \). So we obtain
\[
\varphi(X) = \sum_{i=1}^{d} (\lambda_i + g_i(u))x_i \partial_{x_i} + \sum_{j=1}^{p} (g, R_j)u_j \partial_{u_j} \mod \hat{\mathfrak{X}}^S
\]
which is the usual Poincaré-Dulac normal form using Bruno variables.

3.5. The versality property. The result implies that the detuning deformation is formally versal. Indeed, let
\[
X = S + T(\mu) \in \text{Der}_\mathbb{C}(\hat{\mathfrak{O}}[[\mu]])
\]
be an arbitrary deformation of \( S \), so
\[
X = \sum_{i=1}^{d} (\lambda_i x_i + T_i(\mu, x)) \partial_{x_i},
\]
such that \( T_i(\mu = 0) = 0 \). We form the sum deformation:
\[
X_v = S_v + T(\mu).
\]

The theorem above states that \( X_v \) is mapped via an automorphism \( \varphi \in \text{Aut}(\hat{\mathfrak{O}}[[u, \phi, \mu]]) \) to \( S_v \). Again the restriction of \( X_v \) to \( \phi = 0 \) is the initial deformation \( X \). Solving the equations \( \varphi(\phi_j) = 0 \) by \( \phi_j = g_j(u, \mu) \), we get that
\[
\varphi(\tilde{X}) = \sum_{i=1}^{d} (\lambda_i + g_i(u, \mu))x_i \partial_{x_i} + (g, R_i)u_i \partial_{u_i} \mod X^S[[\mu]].
\]

If we wish one may eliminate the \( u \)-variables using the ideal \( J \) and get
\[
\varphi(X) = \sum_{i=1}^{d} (\lambda_i + g_i(x^R, \mu))x_i \partial_{x_i}
\]
3.6. The Bruno-Stolovitch ideal. Consider a vector field in normal form

\[ Y = \sum_{i=1}^{d} (\lambda_i + g_i(u, \mu)) x_i \partial x_i + \sum_{j=1}^{p} (g, R_j) u_j \partial u_j \mod X^S[[\mu]] \]

here \( g = (g_1, \ldots, g_p) \), \( R_j = (R_{j,1}, \ldots, R_{j,p}) \) and \( (g, R_j) = \sum_{i=1}^{p} a_i R_{j,i} \). The graph ideal \( J \) is by definition \( Y \)-invariant, but it contains also a bigger invariant ideal \( I_\mu \) obtained by adding the functions \((g, R_j)\)'s.

Definition. The Bruno-Stolovitch ideal of the vector field \( Y \) is the ideal

\[ I_\mu = J + ((g, R_1), \ldots, (g, R_p)) \subset \hat{O}[[u, \mu]] \]

Lemma. The ideal \( I_\mu \) is \( Y \)-invariant:

\[ f \in I_\mu \implies Y(f) \in I_\mu \]

Proof. A direct computations shows that

\[ Y(u_i) = (g, R_i) u_i \mod J \]

and therefore \( Y(u_i) \in I_\mu \). In vector notations

\[ Y(g) = (D_u g) Y(u) \]

which can be interpreted as the image of the vector field \( Y \) under the differential \( D_u g \) of the map \( g \). Therefore

\[ Y((g, R_i)) = ((D_u g) Y(u), R_i) \mod J \]

and the right hand side lies in \( I_\mu \). This proves the lemma. \( \square \)

This means that the ideal \( I_\mu \) generated by the \( x^{R_j} - u_j \)'s and the functions \((g, R_j)\) is an invariant ideal of the vector field.

Assume for a moment that for a fixed value of \( \mu \) the functions \( g_i \)'s are analytic in a small neighbourhood of the origin. Then the vector field \( Y(\mu, -) \) is analytic and tangent to the Bruno-Stolovitch varieties:

\[ V_\mu := \{(x, u) \in \mathbb{C}^{d+p} : (g(u, \mu), R_i) = 0, \ u = x^{R_j}\}. \]

As Bruno showed in [11], as a general rule, one cannot hope to have even a \( \mu \)-dependant analytic family of the varieties. However in the formal case, the varieties \( V_\mu \) are formal schemes and, as one would expect from classical KAM theory, these form a Whitney \( C^\infty \)-family of analytic varieties over a positive measure set.
3.7. Piartly’s example. The versal deformation
\[ Y = (\alpha + \mu + xy + \ldots) x \partial_x + (-\alpha + \mu + xy + \ldots) y \partial_y. \]
is given by:
\[ S_v = (\alpha + \phi_1) x \partial_x + (-\alpha + \phi_2) y \partial_y \]
and by versality of this deformation, there exists an automorphism \( \varphi \) such that
\[ \varphi(\hat{Y}) = (\alpha + g_1(u, \mu)) x \partial_x + (-\alpha + g_2(u, \mu)) y \partial_y + (g_1 + g_2) u \partial_u \mod M \]
As the only resonant exponent is \((1, 1)\), the Bruno-Stolovitch ideal is generated by
\[ ((1, 1), g) = g_1(u, \mu) + g_2(u, \mu) \]
and by the function \( u - xy \) which generates the resonance ideal.

An explicit computations shows that we have the expansion
\[ g_1(u, \mu) + g_2(u, \mu) = 2u + \mu + \ldots \]
The function
\[ g_1(xy, \mu) + g_2(xy, \mu) = 2xy + \mu + \ldots \]
defining the Bruno ideal has an \( A_1 \)-singularity.

If we assume this expansion to be analytic for some small value of \( \mu \) then the manifold \( V_\mu \) is a cylinder for \( \mu \neq 0 \) and a cone for \( \mu = 0 \). Assuming \( \alpha \) imaginary, we may restrict the field to the real plane \( \mathbb{R}^2 \approx \mathbb{C} \subset \mathbb{C}^2 \) defined by the equation
\[ g_1(z \bar{z}, \mu) + g_2(z \bar{z}, \mu) = 2|z|^2 + \mu + \ldots \]
is diffeomorphic to a circle for \( \mu < 0 \). We obtain the standard picture of the family of vanishing cycles of the \( A_1 \)-singularity (see [5, 7]).

3.8. The multi-dimensional Hopf bifurcation. It is easy to generalise the above picture to higher dimensions. We start with a vector field of the form
\[ Y = \sum_{i=1}^{d} (\alpha_i + \mu_i + x_i y_i + \ldots) x_i \partial_{x_i} + (-\alpha_i + \mu_i + x_i y_i + \ldots) y_i \partial_{y_i} \]
where the \( \alpha_i \) are \( \mathbb{Q} \)-independent. We consider the versal deformation:
\[ S_v = \sum_{i=1}^{d} (\alpha_i + \phi_i) x_i \partial_{x_i} + \sum_{i=1}^{d} (-\alpha_i + \phi_{i+d}) y_i \partial_{y_i}. \]
The vector field \( \hat{Y} \) is isomorphic to
\[ Y = \sum_{i=1}^{d} (\alpha_i + g_i(u, \mu)) x_i \partial_{x_i} + \sum_{i=1}^{d} (-\alpha_i + g_{i+d}(u, \mu)) y_i \partial_{y_i} \]
The resonance ideal is generated by the \( u_i - x_i y_i \)’s and the Bruno-Stolovitch ideal is generated by these functions to which we add the
functions \( g_i + g_{i+d} \). In this case, if the function turns out to be analytic form some \( \mu \)-value and if we consider real structure \( x = \mathcal{F} \) as for \( d = 1 \), the real parts of the manifolds \( V_\mu \), for \( \mu_i < 0 \), are tori and the vector field carries a quasi-periodic motion.

4. Versal deformations and invariant varieties

Iteration above was defined on the level of formal power series. But one can define a similar iteration defined for appropriate Banach spaces of analytic functions. The convergence of the iteration procedure then is a direct consequence of the general theory developed in [21] and that we will now briefly recall. We will be very sketchy as full details were already given in [21]. It is likely that one may produce complete and direct proofs along the lines of Stolovitch [44, 45], but this would require a much longer line of arguments and computations.

4.1. Bruno sequences. For convergence, our procedure requires Bruno-type conditions. To fix notations, we briefly recall the standard facts.

Definition ([14]). A strictly monotone positive sequence \( a \) is called a Bruno sequence if the infinite product

\[
\prod_{k=0}^{\infty} a_k^{1/2^k}
\]

converges to a strictly positive number or equivalently if

\[
\sum_{k \geq 0} \left\| \frac{\log a_k}{2^k} \right\| < +\infty.
\]

We denote respectively by \( \mathcal{B}^+ \) and \( \mathcal{B}^- \) the set of increasing and decreasing Bruno sequences.

Attached to a frequency vector \( \Lambda \in \mathbb{C}^d \), we define the sequence \( \sigma(\Lambda) \) with terms

\[
\sigma(\Lambda)_k := \min \{ ||(\Lambda, J - E_i)|| \neq 0 : J \in \mathbb{N}^d \setminus \{0\}, ||J|| \leq 2^k \}
\]

Here \( E_i(= 0, \ldots, 0, 1, 0, \ldots, 0) \) denotes the standard basis of the vector space \( \mathbb{C}^d \). We say that the vector \( \Lambda \) is Bruno if the sequence \( \sigma(\Lambda) \) is a decreasing Bruno sequence.

4.2. Functional analytic setting (part I). We fix the frequency vector \( \Lambda \) of our linear vector field \( S \). We will replace the the ring \( \mathcal{O}[[u]] \) of formal power series by an appropriate system of Banach space of holomorphic functions. To describe these, we first consider the space \( \mathbb{C}^d \times \mathbb{C}^p \), with coordinates \( x, u \). We put

\[
D^d_s := \{ x \in \mathbb{C}^d \mid |x_i| < s, \ i = 1, 2, \ldots, d \}
\]
\[
D^p_s := \{ u \in \mathbb{C}^p \mid |u_j| < s, \ j = 1, 2, \ldots, p \}
\]
We define
\[ U = D^d \times D^k \to \mathbb{N} \times \mathbb{R}_{>0} \]
as a relative set with fibres independent on \( n \in \mathbb{N} \):
\[ U_{n,s} = D^d_s \times D^p_s \]
We then consider the Banach spaces of holomorphic functions and vector fields on these sets (with continuous extension to the boundary),
\[ E_{n,t} := \mathcal{O}^c(U_{n,t}), \quad F_{n,t} := \mathcal{X}^c(U_{n,t}) \]
for which there are corresponding restriction maps of norm \( \leq 1 \)
\[ E_{n,t} \to E_{n,s}, \quad s < t, \quad E_{m,s} \to E_{n,s}, \quad n > m \]
and similarly for \( \mathcal{X} \). In the terminology of [21] such a system of Banach spaces is called an Arnold space and write simply \( E = \mathcal{O}^c(W) \), etc.
The set \( U_{n,s} \) has the Huygens property for the constant sequence \( a_n = 1 \), by which we mean
\[ U_{n,s} + D_{a_n(t-s)} \subset U_{n,t} \]
This guarantees that the locality estimates are satisfied for differential operators and when passing from one Arnold space to another.
In the iteration process, we will also have to choose an appropriate sequence
\[ s_0 > s_1 > s_2 \ldots > s_n > s_{n+1} > \ldots > s_\infty > 0 \]
and form the 'pull-back', i.e. we form
\[ E_n := E_{n,s_n}, n = 0, 1, 2, \ldots \]
for which we still have restriction maps
\[ E_0 \to E_1 \to E_2 \to \ldots \]
As sequence \( (x_n) \) with \( x_n \in E_n \) is called convergent if it is a Cauchy sequence in the sense that
\[ \forall \varepsilon > 0, \exists N, \ n \geq N \implies |\iota x_n - x_{n+p}| \leq \varepsilon \]
where \( \iota = \iota_{n+p,n} \) is the restriction map from \( E_n \) to \( E_{n+p} \).

4.3. The abstract versal deformation theorem.

**Theorem** ([21]). Let \( g \) be an Arnold-Lie algebra, \( X \in g \) and \( h \subset g \) a sub-Arnold-Lie algebra which is a direct summand, that is, the projection on \( h \) is a local operator whose norm is an integrable Bruno sequence. Assume that

1) there exists a a local operator \( L \in \mathcal{L}(g, M) \) and a cutoff operator \( \kappa \in \mathcal{K}(M, M) \) solving approximatively the homological equation modulo \( h \), in the sense that:
\[ [L(Y), X] = Y + \kappa(Y) \mod h \]
2) the norm sequence \( |L|, \ |\kappa| \) are integrable Bruno sequences.
Then there exists \( r \in \mathbb{R}_{>0} \) and \( k \) such that for any \( Y \in r t^k(B_g) \) there exists \( v \in \Gamma(\mathbb{N}, g) \) satisfying

\[
e^v(X + rY) = \iota X \mod \mathfrak{h}
\]

We will not give all the details here but we will indicate which functional space should be considered to deduce the theorem as in the next paragraph we prove a stronger result with parameters for families of vector fields.

Applying the abstract versal deformation theorem with \( g = \mathcal{X}^c(U) \) and \( \mathfrak{h} \) the \( \mathcal{O}^c(U) \)-module generated by \( (\mathcal{X}^S)^c(U) \), we deduce the Stolovitch normal form theorem 2.6.

4.4. **Functional analytic setting (part II).** We will now formulate and sketch the proof of the above theorem with additional parameters \( \phi \) and \( \mu \). We have to make only notational changes in our set-up.

For the domain of \( \phi \)-variables, we have to throw away the appropriate neighbourhoods of the resonant hyperplanes. We fix a decreasing sequence \( a = (a_n) \) and \( s_0 > 0 \) and consider the set

\[
Z_n,s := Z_n,s(\Lambda, a, s_0) := \{ \phi \in D^a_n : \forall k \leq n, \sigma(\Lambda + \phi)_k \geq a_k(s_0 - s) \}
\]

This set has the *Huygens property*, by which we mean

\[
Z_n,s + D_{a^*_n(t-s)} \subset Z_{n,t}, \quad a^*_n := \frac{a_n}{2^n}.
\]

The addition of the \( \mu \)-variables is straightforward. We consider

\[
D^\mu_n := \{ \mu \in \mathbb{C}^l \mid |\mu_j| < s, \ j = 1, 2, \ldots, l \}
\]

and get a relative set

\[
W_n,s := W_n,s(\tau) := D^\mu_n \times Z_n,s \times D^d_s \times D^d_s \subset \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^p \times \mathbb{C}^l,
\]

where \( \tau = t_{n+p,n} \) is the restriction map from \( E_n \) to \( E_{n+p} \).

4.5. **Versal deformations over a Cantor set.** Applying the abstract versal deformation theorem with

\[
g = \mathcal{X}^c(W), \mathfrak{h} = (\mathcal{X}^S)^c(W)
\]

and the map \( L \) defined analogous to the \( L \) in 3.3, we get the following result:
Theorem. Let $\Lambda \in \mathbb{C}^d$ be a Bruno vector satisfying the positivity conditions. Consider a derivation
\[ S_v = \sum_{i=1}^{d} (\lambda_i + \phi_i)x_i \partial_{x_i} \in \mathcal{X}(W)_0 \]
of the algebra $\mathcal{O}(W)$. Then for any other derivation of the form
\[ X_v := S_v + \cdots \in \mathcal{X}(W)_0 \]
there exists a morphism
\[ \varphi : \mathcal{X}(W)_0 \rightarrow \mathcal{X}(W)_{\infty} \]
which has the following properties:
1) $\varphi(X_v) = S_v \mod (\mathcal{X}(W))^S$.
2) $\varphi(u_j) = u_j$ for $j = 1, \ldots, k$.
3) $\varphi(\mu_j) = \mu_j$ for $j = 1, \ldots, l$.
4) $\varphi(\phi_j) = \phi_j + \cdots \in \mathcal{O}(D^{d+p})$

As a corollary we get that the functions defining the Bruno-Stolovitch ideal lie in $\mathcal{O}(W)_\infty$, in particular they define a continuous family of invariant manifolds. Assuming non-degeneracy conditions on the frequency (for instance that the map is a submersion), these are parametrised by a positive measure set. For the particular case of the multi-dimensional Hopf bifurcation, we get a family of tori carrying quasi-periodic motions generalising the theorem of Chenciner-Li [16, 26]. Stolovitch KAM theorem [45] can also be deduced from the convergence of the Lie iteration on this setting.

References
[1] A. Andronov. Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues. Comptes Rendus de l’Académie des Sciences de Paris, 189:559–561, 1929.
[2] V. I. Arnold. Remarks on singularities of finite codimension in complex dynamical systems. Functional Analysis and its Applications, 3(1):1–5, 1969.
[3] V.I. Arnold. Lectures on bifurcations in versal families. In V.I. Arnold Collected Works, pages 271–340. Springer, 1972.
[4] V.I. Arnold. Chapitres supplémentaires de la théorie des équations différentielles ordinaires, MIR, 1980.
[5] V.I. Arnold, A.N. Varchenko, and S. Goussein-Zade. Singularity of differentiable mapping. Vol. II. Nauka:Moscow, 1982. English transl.: Birkhauser, 382p., Basel(1986).
[6] V.I. Arnold, V.A. Vassiliev, V.V. Goryunov, and O.V. Lyashko. Singularity Theory I, Dynamical Systems VI. Springer-Verlag, 1988. English transl., 245p. (1993).
[7] V.I. Arnold, V.A. Vassiliev, V.V. Goryunov, and O.V. Lyashko. Singularity Theory II, Dynamical Systems VIII. VINITI, Moscow, 1989. English transl.:Springer-Verlag, 235p., (1993).
[8] C. Briot and T. Bouquet. Recherches sur les propriétés des fonctions définies par des équations différentielles. J. École Polytech., 21:133–197, 1856.
[9] H.W. Broer, G.B. Huitema, and M.B. Sevryuk. Families of quasi-periodic motions in dynamical systems depending on parameters. In Nonlinear Dynamical Systems and Chaos, pages 171–211. Springer, 1996.
[10] H.W. Broer, G.B. Huitema, and M.B. Sevryuk. Quasi-periodic motions in families of dynamical systems: order amidst chaos. Springer, 2009.
[11] A. D. Bruno. Normal form of differential equations with a small parameter. Matematicheskie Zametki, 16(3):407–414, 1974.
[12] A. D. Bruno. Integral analytic sets. Dokl. Akad. Nauk SSSR, 220(6):1255–1258, 1975. (in Russian).
[13] A.D. Bruno. The normal form of differential equations. Dokl. Akad. Nauk SSSR, 157(6):1276–1279, 1964. (In Russian).
[14] A.D. Bruno. Analytic form of differential equations I. Trans. Moscow Math. Soc., 25:131–288, 1971.
[15] C. Camacho and P. Sad. Invariant varieties through singularities of holomorphic vector fields. Annals of mathematics, 115(3):579–595, 1982.
[16] A. Chenciner. Bifurcations de points fixes elliptiques. Publications Mathématiques de l’Institut des Hautes Études Scientifiques, 61(1):67–127, 1985.
[17] H. Dulac. Solutions d’un système d’équations différentielles dans le voisinage de valeurs singulières. Bulletin de la Société mathématique de France, 40:324–383, 1912.
[18] J. Écalle. Singularités non abordables par la géométrie. Annales de l’institut Fourier, 42(1-2):73–164, 1992.
[19] M. Garay. Degenerations of invariant Lagrangian manifolds. Journal of Singularities, 8:50–67, 2014.
[20] M. Garay and D. van Straten. Hamiltonian normal forms. ArXiv:1909.06053, 2019.
[21] M. Garay and D. van Straten. A category of Banach space functors. ArXiv: 2010.02320, 2020.
[22] E. Hopf. Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystems. Berichten der Mathematisch-Physischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig, 94:1–22, 1942.
[23] Y. Ilyashenko. Divergence of series reducing an analytic differential equation to linear normal form at a singular point. Functional Analysis and its applications, 13(3):227–229, 1979.
[24] Y. Ilyashenko and S Yakovenko. Lectures on Analytic Differential Equations, volume 86 of Graduate Studies in Mathematics. American Mathematical Society, 2008.
[25] A.N. Kolmogorov. On the conservation of quasi-periodic motions for a small perturbation of the Hamiltonian function. Dokl. Akad. Nauk SSSR, 98:527–530, 1954. (In Russian).
[26] X. Li. On the persistence of quasi-periodic invariant tori for double Hopf bifurcation of vector fields. Journal of Differential Equations, 260(10):7320–7357, 2016.
[27] Bernard Malgrange et al. Travaux d’Écalle et de Martinet-Ramis sur les systèmes dynamiques. Astérisque, 92(93):82, 1981.
[28] J.E. Marsden and M. McCracken. The Hopf bifurcation and its applications, volume 19. Springer Science & Business Media, 2012.
[29] J. Martinet. Normalisation des champs de vecteurs holomorphes (d’après A.-D. Brjuno), volume 770 of Sém. Bourbaki, Lecture Notes in Mathematics. Springer, 1981. 272 pp.
[30] J. Martinet. Singularities of smooth functions and maps, volume 58 of Lecture Notes Series. Cambridge University Press, 1982. 272 pp.
[31] J. Martinet and J.-P. Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 55(1):63–164, 1982.

[32] J. Martinet and J.-P. Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Annales scientifiques de l’École Normale Supérieure*, 16(4):571–621, 1983.

[33] R. Moussu and D. Cerveau. Groupes d’automorphismes de $(\mathbb{C}, 0)$ et équations différentielles $y\frac{dy}{d\cdot} = 0$. *Bulletin de la Société Mathématique de France*, 116(4):459–488, 1988.

[34] R. Pérez-Marco. Total Convergence or General Divergence in Small Divisors. *Communications in Mathematical Physics*, 223(3):451–464, 2001.

[35] R. Pérez-Marco. Convergence or generic divergence of the Birkhoff normal form. *Annals of Mathematics*, 2:557–574, 2003.

[36] E. Picard. *Traité d’analyse, Tome 3*. Gauthier-Villars, 3rd edition, 1928. 660 pp.

[37] H. Poincaré. Sur le problème des trois corps et les équations de la dynamique. *Acta mathematica*, 13(1):3–270, 1890.

[38] H. Poincaré. *Sur les propriétés des fonctions définies par les équations aux différences partielles (Première thèse, 1879)*. Oeuvres de Henri Poincaré, Tome I. Gauthiers-Villars, 1951.

[39] A.S. Pyartli. Birth of complex invariant manifolds close to a singular point of a parametrically dependent vector field. *Functional Analysis and Its Applications*, 6(4):339–340, 1972.

[40] H. Rüssmann. Über das Verhalten analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung. *Math. Annalen*, 154:285–306, 1964.

[41] C.L. Siegel. On the integrals of canonical systems. *Annals of Mathematics*, 42(3):806–822, 1941.

[42] C.L. Siegel. Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. *Nach. Akad. Wiss. Göttingen, math.-phys.*, pages 21–30, 1952.

[43] S. Sternberg. On the structure of local homeomorphisms of Euclidean n-space, II. *American Journal of Mathematics*, 80(3):623–631, 1958.

[44] L. Stolovitch. Sur un théorème de Dulac. *Annales de l’institut Fourier*, 44(5):1397–1433, 1994.

[45] L. Stolovitch. A KAM phenomenon for singular holomorphic vector fields. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 102:99–165, 2005.

[46] L. Stolovitch. Progress in normal form theory. *Nonlinearity*, 22(7):R77, 2009.

[47] L. Stolovitch. Smooth Gevrey normal forms of vector fields near a fixed point. *Annales de l’Institut Fourier*, 63(1):241–267, 2013.