Edge irregular reflexive labeling of some tree graphs

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Abstract. Let $G$ be a connected, simple, and undirected graph with a vertex set $V(G)$ and an edge set $E(G)$. Total $k$-labeling is a function $f_e$ from the edge set to the first $k_e$ natural number, and a function $f_v$ from the vertex set to the non negative even number up to $2k_v$, where $k = \max\{k_e, 2k_v\}$).

An edge irregular reflexive $k$-labeling of the graph $G$ is the total $k$-labeling, if for every two different edges $x_1x_2$ and $x_1'x_2'$ of $G$, $wt(x_1x_2) \neq wt(x_1'x_2')$, where $wt(x_1x_2) = f_v(x_1) + f_e(x_1x_2) + f_v(x_2)$. The minimum $k$ for graph $G$ which has an edge irregular reflexive $k$-labelling is called the reflexive edge strength of the graph $G$, denoted by $res(G)$. In this paper, we determined the exact value of the reflexive edge strength of family trees, namely generalized sub-divided star graph, broom graphs, and double star graph.

Keywords: Edge irregular reflexive labeling, reflexive edge strength, generalized sub-divided star graph, broom graph, double star graph.

1. Introduction

A labeling of a graph $G$ is a function from a set of elements graph $G$ to the positive or non-negative numbers, such that these numbers called labels [2], satisfies a special properties of labeling. If the domain of labeling is a set of all vertices, then it is called vertex labeling. If the domain of labeling is a set of all edges, then it is called edge labeling. If the domain of labeling is a set of both vertices and edges, then it is called total labeling. The complete recent survey of graph labeling can be found in [3].

The natural extension of Irregularity strength of a graph $G$, Baca, Jendrol, Miller and Ryan [5] introduced a new notion namely a total vertex and irregularity strength of a graph. In [5], Baca et al also introduced the notion of a total $k$-labeling, an edge irregular total $k$-labeling and a vertex irregular total $k$-labeling. By then, there are a lot of researchers give some results of the irregular labeling. It can be found in [6, 7, 8, 9, 10, 11].

A total $k$-labeling is a total labeling with the first natural number $k$ as a range. Baca et al [12] defined an edge irregular total $k$-labeling is a total $k$-labeling which have the condition every two different edges have different weights. The minimum $k$ for which $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength of a $G$, denoted by tes($G$). Moreover, Tanna et. al [13] develop the research on irregular labeling. They generalized the notion of the edge irregular total labeling $k$-labeling of a graph to be an edge irregular reflexive $k$-labeling.
The Total $k$-labeling defined the function $f_e : E(G) \rightarrow \{1, 2, \ldots, k_e\}$ and $f_v : V(G) \rightarrow \{0, 2, \ldots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. An edge irregular reflexive $k$-labeling of the graph $G$ is the total $k$-labeling, if for every two different edges $x_1x_2$ and $x'_1x'_2$ of $G$, $wt(x_1x_2) \neq wt(x'_1x'_2)$, where $wt(x_1x_2) = f_e(x_1) + f_e(x_1x_2) + f_v(x_2)$. The minimum $k$ for a graph $G$ admitting an edge irregular reflexive $k$-labelling is called the reflexive edge strength of the graph $G$, denoted by $res(G)$ [12]. The following Lemma is the lower bound for the reflexive vertex strength of graph $G$.

**Lemma 1** [12] For every graph $G$,

$$res(G) \geq \begin{cases} \left\lfloor \frac{|E(G)|}{3} \right\rfloor, & \text{if } |E(G)| \neq 2, 3 \text{(mod 6)} \\ \left\lfloor \frac{|E(G)|}{3} \right\rfloor + 1, & \text{if } |E(G)| \equiv 2, 3 \text{(mod 6)} \end{cases}$$

The previous results on edge irregular reflexive are determined the exact value of generalized friendship graph [12], prism and wheel graph [13], cycles, cartesian product of two cycles and join graphs of path graph and cycle graph with $2K_2$ [14]. Therefore, in this paper we determined the exact value of the reflexive edge strength of family trees, namely generalized sub-divided star graph, broom graphs, and double star graph. Double star is a tree obtained from the star $S_n$ by adding a new pendant edge of the existing $n$ pendant vertices. It has $2n + 1$ vertices and $2n$ edges [15]. A broom graph ($B_{n,m}$) is a graph of $n$ vertices, which have a path $P$ with $m$ vertices and $(n - m)$ pendant vertices. All of these vertices are adjacent to either the origin $u$ or the terminus $v$ of the path $P$, see [16]. The subdivision of an edge $e = uv \in E(G)$ with endpoints $u, v$ yields a graph containing one new vertex $x$, and with the edge $uv$ replaced by two new edges, $ux$ and $xv$.

2. Result and Discussion

In this section, we give the results of our research on edge irregular reflexive labeling of some tree graphs. We will use a direct proof to prove each theorem of our results. We start with the following observation.

**Observation 1** Let $T$ be a tree graph with maximum degree $\Delta(T)$. The edge reflexive strength of $T$ satisfies $res(T) \geq \max\left\{ \left\lfloor \frac{\Delta}{2} \right\rfloor + r, \left\lfloor \frac{|E(T)|}{3} \right\rfloor + s \right\}$ where $r = 1$ for $\Delta \equiv 2 \text{(mod 4)}$ and $r = 0$ for otherwise; $s = 1$ for $|E(T)| \equiv 2, 3 \text{(mod 6)}$ and for $|E(T)| = 0$ for otherwise.

By the above observation in hand, we are ready to prove the following theorem.

**Theorem 1** Let $SS_{n,m}$ be a generalize sub-divided star graph. For every natural number $n \geq 3$ and $m \geq 2$, we have the following

$$res(SS_{n,m}) = \begin{cases} \left\lfloor \frac{n(m + 1)}{3} \right\rfloor + 1, & \text{if } n(m + 1) \equiv 2, 3 \text{(mod 6)} \\ \left\lfloor \frac{n(m + 1)}{3} \right\rfloor, & \text{otherwise.} \end{cases}$$
Proof. Let $SS_{n,m}$, for $n \geq 3$ and $m \geq 1$, be a generalized sub divided star graph with the vertex set $V(SS_{n,m}) = \{A, x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq mn\}$ and the edge set $E(SS_{n,m}) = \{x_iy_j; i = j, 1 \leq i \leq n\} \cup \{y_{(j-1)n+i}y_{jn+i}; 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{Ay_{(m-1)n+i}; 1 \leq i \leq n\}$. The order and size of $SS_{n,m}$ are $n(m+1)+1$ and $n(m+1)$, respectively. The maximum degree of $SS_{n,m}$ is $n$. Based on Observation 1, we have $res(SS_{n,m}) \geq \max\left\{\left\lfloor\frac{\Delta(SS_{n,m})}{2}\right\rfloor + r, \left\lceil\frac{|E(T)|}{3}\right\rceil + s\right\}$. If we choose $m = 1$, then we have $\Delta(SS_{n,m}) = n$ and the number of edges in $SS_{n,m}$ is $2n$, thus $E(SS_{n,m}) > \Delta(SS_{n,m})$. It holds for all $m \geq 1$. Thus we have $res(SS_{n,m}) \geq \max\left\{\left\lfloor\frac{\Delta(SS_{n,m})}{2}\right\rfloor + r, \left\lceil\frac{|E(SS_{n,m})|}{3}\right\rceil + s\right\} = \max\left\{\frac{n}{2} + r, \frac{n(m+1)}{3} + s\right\} = \left\lfloor\frac{n(m+1)}{3}\right\rfloor + s$, where $s = 1$ for $n(m+1) \equiv 2, 3 \pmod{6}$ and zero for otherwise.

Furthermore, we will show that $k$ is an upper bound for the edge irregular reflexive total labeling of $SS_{n,m}$, by defining the following:

$$k = \begin{cases} \left\lfloor\frac{n(m+1)}{3}\right\rfloor + 1, & \text{if } n(m+1) \equiv 2, 3 \pmod{6} \\ \left\lfloor\frac{n(m+1)}{3}\right\rfloor, & \text{otherwise.} \end{cases}$$

$$f_1(A) = k$$
$$f_1(x_i) = 0$$
$$f_1(y_j) = \begin{cases} \frac{j-2}{4}, & \text{if } 3 \leq j \leq (m-1)n \\ k, & \text{if } (m-1)n + 1 \leq j \leq mn \end{cases}$$

$$g_1(x_iy_j) = i - f(y_j); i = j, i = 1, 2, 3, \ldots, n$$
$$g_1(y_{(j-1)n+i}y_{jn+i}) = jn + i - f(y_{(j-1)n+i}) - f(y_{jn+i}); i = 1, 2, 3, \ldots, n, j = 1, 2, 3, \ldots, m-1$$
$$g_1(Ay_{(m-1)n+i}) = mn + i - 2k, i = 1, 2, 3, \ldots, n$$

The edge weight from the above function will give a consecutive element of edge weigh set $W = \{1, 2, 3, \ldots, n(m+1)\}$. We can easily see that all elements in $W$ are distinct. It concludes the proof.

We show the illustration of the edge reflexive labeling on $SS_{8,2}$ in Figure 1.
Figure 1. The illustration of labeling on $SS_{8,2}$

Theorem 2 Let $Br_{n,m}$ be a broom graph. For every natural number $n \geq 3$ and $m \geq 2$, the reflexive edge strength of the boom graph is

$$res(Br_{n,m}) = \begin{cases} 
\left\lceil \frac{n+m}{3} \right\rceil + 1, & \text{if } 2m \geq n, \text{ and } n+m \equiv 2,3(\text{mod } 6) \\
\left\lceil \frac{n+m}{3} \right\rceil, & \text{if } 2m \geq n, \text{ otherwise} \\
\left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } 2m < n, \text{ and } m+1 \equiv 2(\text{mod } 4) \\
\left\lceil \frac{n}{2} \right\rceil, & \text{if } 2m < n, \text{ otherwise}
\end{cases}$$

Proof. Let $Br_{n,m}$, for $n \geq 3$ and $m \geq 2$, be a graph with vertex set $V(Br_{n,m}) = \{A, x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(Br_{n,m}) = \{Ax_i; 1 \leq i \leq n\} \cup \{x_ny_j, y_jy_{j+1}; 1 \leq j \leq m-1\}$. Thus the order and size of $n + m + 1$ and $n + m$, respectively. The maximum degree of $Br_{n,m}$ is $n$.

Firstly, we show that $res(Br_{n,m}) \geq max\{\left\lceil \frac{\Delta(Br_{n,m})}{2} \right\rceil + r, \left\lceil \frac{|E(Br_{n,m})|}{3} \right\rceil + s\}$, based on Observation 1. Furthermore, to convince our proof, we split it into two cases: (i) If $2m \geq n$, then $m \geq \frac{n}{2}$. It implies that $m$ is much bigger than $n$, thus it will give $max\{\left\lceil \frac{n+1}{2} + r \right\rceil, \left\lceil \frac{n+m+1}{3} \right\rceil + s\}$.
\[
s = \left\lceil \frac{n + m}{3} \right\rceil + s \quad \text{where} \quad s = 1 \text{ for } n + m \equiv 2, 3 \pmod{6}; \quad \text{(ii) if } 2m < n, \text{ then } m < \frac{n}{2}. \text{ It implies that } m \text{ is much smaller than } n, \text{ thus it will give } \max \{ \left\lceil \frac{n + 1}{2} + r \right\rceil, \left\lceil \frac{n + m + 1}{4} \right\rceil + s \} = \left\lceil \frac{n}{2} \right\rceil + r
\]

where \( r = 1 \) for \( n \equiv 2 \pmod{4} \).

Secondly, we will show the upper bound by defining the following functions of the edge irregular reflexive labeling of \( Br_{n,m} \).

\[
f_2(A) = 0
\]

\[
f_2(x_i) = \begin{cases} 
0, & \text{if } i = 1, 2 \\
2 \left\lceil \frac{i - 2}{4} \right\rceil, & \text{if } 3 \leq i \leq n 
\end{cases}
\]

\[
g_2(Ax_i) = i - f(x_i); \quad i = 1, 2, 3, \ldots n
\]

\[
g_2(x_ny_1) = \begin{cases} 
n + 3 - 4 \left\lceil \frac{n - 2}{4} \right\rceil, & \text{if } n \equiv 3 \pmod{4}, \quad 2m \geq n \\
n + 1 - 4 \left\lceil \frac{n - 2}{4} \right\rceil, & \text{otherwise}
\end{cases}
\]

\[
g_2(y_jy_{j+1}) = n + j + 1 - f(y_j) - f(y_{j+1}); \quad j = 1, 2, 3, \ldots m - 1
\]

Furthermore, for the labeling of \( y_j \), We distinguish two cases.

**Case 1.** For \( 2m \geq n \)

For \( n \equiv 1 \pmod{4} \),

\[
f_2(y_j) = \begin{cases} 
2 \left\lceil \frac{n - 2}{4} \right\rceil, & \text{if } 1 \leq j \leq 2 \left\lceil \frac{n - 2}{4} \right\rceil + 1 \\
2 \left\lceil \frac{n - 2}{4} \right\rceil + 2 \left\lceil \frac{j - 2 \left\lceil \frac{n - 2}{4} \right\rceil - 1}{6} \right\rceil, & \text{if } 2 \left\lceil \frac{n - 2}{4} \right\rceil + 2 \leq j \leq m
\end{cases}
\]

For \( n \equiv 0 \pmod{4} \),

\[
f_2(y_j) = \begin{cases} 
2 \left\lceil \frac{n - 2}{4} \right\rceil, & \text{if } 1 \leq j \leq 2 \left\lceil \frac{n - 2}{4} \right\rceil + 2 \\
2 \left\lceil \frac{n - 2}{4} \right\rceil + 2 \left\lceil \frac{j - 2 \left\lceil \frac{n - 2}{4} \right\rceil - 2}{6} \right\rceil, & \text{if } 2 \left\lceil \frac{n - 2}{4} \right\rceil + 3 \leq j \leq m
\end{cases}
\]

For \( n \equiv 3 \pmod{4} \),
\[
f_2(y_j) = \begin{cases} 
2 \left\lceil \frac{n-2}{4} \right\rceil - 2, & \text{if } j = 1 \\
2 \left\lceil \frac{n-2}{4} \right\rceil, & \text{if } 2 \leq j \leq 2 \left\lceil \frac{n-2}{4} \right\rceil + 3 \\
2 \left\lceil \frac{n-2}{4} \right\rceil + 2 \left\lceil \frac{j-2}{6} \frac{n-2}{4} - 3 \right\rceil, & \text{if } 2 \left\lceil \frac{n-2}{4} \right\rceil + 4 \leq j \leq m 
\end{cases}
\]

For \( n \equiv 2(\text{mod } 4) \),

\[
f_2(y_j) = \begin{cases} 
2 \left\lceil \frac{n-2}{4} \right\rceil, & \text{if } 1 \leq j \leq 2 \left\lceil \frac{n-2}{4} \right\rceil \\
2 \left\lceil \frac{n-2}{4} \right\rceil + 2 \left\lceil \frac{j-2}{6} \frac{n-2}{4} \right\rceil, & \text{if } 2 \left\lceil \frac{n-2}{4} \right\rceil + 1 \leq j \leq m 
\end{cases}
\]

Case 2. For \( 2m < n \)

\[
f_2(y_j) = 2 \left\lceil \frac{n-2}{4} \right\rceil, \quad 1 \leq j \leq m
\]

From the above functions, We get the edge weights as follow:

\[ W = \{1, 2, 3, \ldots, n + m\} \]

Again, it is easy to see that all the elements of \( W \) are distinct number. Thus we get the edge irregular reflexive strength of \( Br_{n,m} \), for \( n \geq 3 \) and \( m \geq 2 \).

**Theorem 3** Let \( DS_{n,m} \) be a double star graph. For every natural number \( n, m \geq 2 \) and \( n \geq m \), the reflexive edge strength of the double star graph is

\[
res(DS_{n,m}) = \begin{cases} 
\left\lceil \frac{n+1}{2} \right\rceil + 1, & \text{if } n \geq 2m - 1, \ (n+1) \equiv 2(\text{mod } 4) \\
\left\lceil \frac{n+1}{2} \right\rceil, & \text{if } n \geq 2m - 1, \text{ otherwise} \\
\left\lceil \frac{n+m+1}{3} \right\rceil + 1, & \text{if } n < 2m - 1, \ (n+m+1) \equiv 2, 3(\text{mod } 6) \\
\left\lceil \frac{n+m+1}{3} \right\rceil, & \text{if } n < 2m - 1, \text{ otherwise.}
\end{cases}
\]

**Proof.** Let \( DS_{n,m} \), \( n, m \geq 3 \), be a graph with the vertex set \( V(DS_{n,m}) = \{A, B, x_i, y_j; 1 \leq i \leq n, 1 \leq j \leq m\} \) and the edge set \( E(DS_{n,m}) = \{AB\} \cup \{Ax_i; 1 \leq i \leq n\} \cup \{By_j; 1 \leq j \leq m\} \).
Thus, the order and size of $DS_{n,m}$ are $m + n + 2$ and $m + n + 1$, respectively. The maximum degree of $DS_{n,m}$ is $\frac{n+1}{2}$. We illustrated the labeling on $DS_{2,2}$ and $DS_{4,3}$ in Figure 2.

Firstly, we show that $res(DS_{n,m}) \geq \max\{\left\lceil \frac{n}{2} \right\rceil + r, \left\lceil \frac{|E(DS_{n,m})|}{3} \right\rceil + s\}$, Based on Observation 1. Furthermore, to convince our lower bound, we split into two cases: (i) If $n \geq 2m - 1$, then $m \leq \frac{n+1}{2}$. It implies that $m$ is much bigger than $n$, thus it will give $\max\{\left\lceil \frac{n+1}{2} \right\rceil + r, \left\lceil \frac{n+1}{2} \right\rceil + r \text{ where } r = 1 \text{ for } n + 1 \equiv 2(\text{mod } 4)\}$; (ii) If $n < 2m - 1$, then $m > \frac{n+1}{2}$. It implies that $m$ is much smaller than $n$, thus it will give $\max\{\left\lceil \frac{n+1}{2} \right\rceil + r, \left\lceil \frac{n+m+1}{3} \right\rceil + s \text{ where } s = 1 \text{ for } n + m + 1 \equiv 2,3(\text{mod } 6)\}$.

Secondly, we will show that $k$ is an upper bound for the edge irregular reflexive total labeling of $DS_{n,m}$, we define by following:

$$k = \begin{cases} 
\left\lceil \frac{n+1}{2} \right\rceil + 1, & \text{if } n \geq 2m - 1, \ (n + 1) \equiv 2(\text{mod } 4) \\
\left\lceil \frac{n+1}{2} \right\rceil, & \text{if } n \geq 2m - 1, \ \text{otherwise} \\
\left\lceil \frac{n+m+1}{3} \right\rceil + 1, & \text{if } n < 2m - 1, \ (n + m + 1) \equiv 2,3(\text{mod } 6) \\
\left\lceil \frac{n+m+1}{3} \right\rceil, & \text{if } n < 2m - 1, \ \text{otherwise}.
\end{cases}$$

$$f_3(A) = 0$$

$$f_3(x_i) = \begin{cases} 
0, & \text{if } i = 1,2 \\
2 \left\lceil \frac{i - 2}{4} \right\rceil, & \text{if } 3 \leq i \leq n
\end{cases}$$

$$g_3(Ax_i) = i - f(x_i), \ 1 \leq i \leq n$$
\[ f_3(B) = \begin{cases} k, & \text{if } k \text{ even} \\ k - 1, & \text{if } k \text{ odd} \end{cases} \]

\[ g_3(AB) = n + 1 - f(B) \]

\[ g_3(By_j) = n + 1 + j - f(B) - f(y_j) \]

Furthermore, for the labeling of \( y_j \), We distinguish two cases

**Case 1.** For \( n \geq 2m - 1 \)

For \( j = 1, 2, 3, \ldots, m \),

\[ f_3(y_j) = \begin{cases} f(B) - 2, & \text{if } m = 2, 3, 4 \\ f(B) + 2 - 2 \left\lceil \frac{m + 1 - j}{2} \right\rceil, & \text{otherwise} \end{cases} \]

**Case 2.** For \( n < 2m - 1 \)

For \( j = 1, 2, 3, \ldots, m \),

\[ f_3(y_j) = \begin{cases} 0, & \text{if } 1 \leq j \leq m - f_3(B) \\ f(B) + 2 - 2 \left\lceil \frac{m + 1 - j}{2} \right\rceil, & \text{if } m - f_3(B) + 1 \leq j \leq m \end{cases} \]

We get the edge weights as follow: \( W = \{1, 2, 3, \ldots, n + m + 1\} \). From this set, we can easily understand that all elements in set \( W \) are distinct. Thus, it gives the desired edge irregular reflexive strength of \( DS_{n,m} \), for \( n \geq 2 \) and \( n \geq m \).

We show the illustration of the edge irregular reflexive labeling on \( Br_{8,3} \) in Figure 3.

![Figure 3](image-url)

**Figure 3.** The illustration of labeling on \( Br_{8,3} \)

3. **Concluding Remarks**

We have determined the reflexive edge strength of three families of tree including their exact values of the reflexive edge strength. To find the an exact values for a bit general graph, for instance any other families of other tree such as lobster graph, caterpillar graph, and banana tree, are still widely open. It is even considered to be NP-problem. Therefore we propose the following open problems.
Open Problem 1 Determine the edge irregular reflexive strength of any other families of tree such as lobster graph, caterpillar graph, banana tree, or spider graph.

Open Problem 2 Determine the edge irregular reflexive strength of any families of regular graphs and try to characterize it.

Open Problem 3 Determine the sharpest upper bound of the edge irregular reflexive strength of any graphs.

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