UNIQUENESS AND DECAY RESULTS FOR A BOUSSINESQUIAN NANOFUID

A. Borrelli¹, G. Giantesio², M.C. Patria³

¹Department of Mathematics and Computer Science
University of Ferrara
via Machiavelli 35, 44121 Ferrara, ITALY

²Department of Mathematics and Physics
Catholic University of the Sacred Heart
via Musei 41, 25121 Brescia, ITALY

³Department of Mathematics and Computer Science
University of Ferrara
via Machiavelli 35, 44121 Ferrara, ITALY

Abstract: In this paper a uniqueness theorem for classical solutions is proved in the case of the evolution of a nanofluid filling a bounded domain under the Boussinesq approximation. The mass density of the nanofluid depends on the temperature and on the nanoparticle volume fraction. A decay in time of a suitable energy is achieved assuming that the material parameters satisfy some conditions. These results are then generalized in the presence of a magnetic field.

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1. Introduction

Nanofluids are characterized by a high thermal conductivity obtained by suspending nanometer-sized (1-100 nm) solid metal (Cu, Fe, Au) or metal oxide (CuO, Al₂O₃) particles into a base liquid with low thermal conductivity, such
as water, oils, ethylene glycol, etc. The presence of a few percents of nanoparticles produces a relevant enhancement of the effective thermal conductivity of nanofluids.

The concept of nanofluid was first introduced by Choi [5] in 1995 and it has been extensively studied recently (see for example [6, 7, 8, 9, 11, 12, 14, 13, 15, 16, 17, 22, 19, 23, 18, 25]) because of its application to a wide class of engineering problems, such as chemical processes and cooling of electronic equipments.

There are two main theories of modeling nanofluids: single-phase and two-phase. For an exhaustive review of the theory and the applications we refer to [20]. In this paper we adopt the two-phase model formulated by Buongiorno [4, 2] in which the nanofluid is considered as a two-component mixture and thermophoresis and Brownian motion are the principal mechanisms that produce a relative velocity between the nanoparticles and the base fluid.

While there is a vast literature concerning the study of specific problems involving nanofluids, to the best of our knowledge there are no results of uniqueness and decay in time of the energy.

In this paper we consider the evolution of a nanofluid filling a bounded domain under the Boussinesq approximation taking account that the mass density of the nanofluid depends on the temperature and on the nanoparticle volume fraction.

We first formulate the initial-boundary problem in dimensionless form in Section 2, we then state in Section 3 the uniqueness theorem for classical solutions without any restriction on the material parameters by introducing a suitable energy.

Section 4 is devoted to obtain a decay in time result of the energy supposing that the material parameters satisfy some inequalities.

Finally, in Section 5 we extend the previous results to the case in which a magnetic field is impressed to the nanofluid without modifying the hypotheses on the material parameters. To this regard, we notice that significant enhancement in the thermal conductivity of nanofluids has been experimentally shown when magnetic fields are applied (see for example [1, 21, 3, 10]).

2. Statement of the problem

In this section we introduce the initial-boundary value problem which we would like to study in the details.

Let us consider the evolution of a Boussinesquian homogeneous nanofluid
filling a bounded domain \( \Omega \). We denote by \( \mathbf{v}, p, T, \Phi \) the velocity, the modified pressure (given by the difference between the pressure and the hydrostatic pressure), the temperature and the nanoparticle volume fraction, respectively. We assume that the gravity \( \mathbf{g} \) is aligned with the unit vector \( \mathbf{e}_3 \).

The governing field equations are

\[
\rho_R \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \Delta \mathbf{v} - \rho_R [\alpha_T(T - T_R) - \alpha_\Phi(\Phi - \Phi_R)] \mathbf{g},
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

\[
\rho_R c \left( \frac{\partial T}{\partial t} + \nabla T \cdot \mathbf{v} \right) = k \Delta T + \rho_p c_p \left( D_B \nabla T \cdot \nabla \Phi + D_T \frac{|\nabla T|^2}{T} \right),
\]

\[
\frac{\partial \Phi}{\partial t} + \nabla \Phi \cdot \mathbf{v} = \nabla \cdot \left( D_B \nabla \Phi + D_T \frac{\nabla T}{T} \right),
\]

for all \((x, t) \in \Omega \times (0, +\infty)\).

We have followed the Buongiorno two-phase model ([4]) for the description of the nanofluid, while, unlike Buongiorno, we suppose slightly compressibility in order to apply the Boussinesq approximation. More precisely, we take into account that the mass density \( \rho \) of the nanofluid depends on \( T \) and \( \Phi \) and that it is the average of the nanoparticle density \( \rho_p \) and base fluid density \( \rho_{bf} \):

\[
\rho = \Phi \rho_p + (1 - \Phi) \rho_{bf}.
\]

In (1) we have assumed that there exists a configuration in which the temperature \( T \) and the nanoparticle volume fraction \( \Phi \) take the constant values \( T_R \) and \( \Phi_R; \rho_R = \rho(T_R, \Phi_R) \) and

\[
\alpha_T = -\frac{1}{\rho_R} \frac{\partial \rho}{\partial T}(T_R, \Phi_R), \quad \alpha_\Phi = \frac{1}{\rho_R} \frac{\partial \rho}{\partial \Phi}(T_R, \Phi_R).
\]

We remark that from physical considerations the thermal coefficient of volume expansion \( \alpha_T \) and the composition coefficient of volume expansion \( \alpha_\Phi \) are positive. Moreover, we have \( \rho \simeq \rho_{bf} \) because \( \Phi \ll 1 \) ([24]).

In \((1)_3,4\) the parameter \( \mu \) is the fluid dynamical viscosity coefficient, \( c, c_p \) are the nanofluid and nanoparticle specific heat and \( k \) is the nanofluid thermal conductivity. We take into account Brownian diffusion and thermophoresis by means of the coefficients \( D_B \) and \( D_T \) and, as it is usual, we neglect dissipative terms in \((1)_3\).
To (1) we associate the initial conditions
\[ v(x, 0) = v_0(x), \quad T(x, 0) = T_0(x), \quad \Phi(x, 0) = \Phi_0(x), \quad \forall x \in \Omega \] (2)
and the boundary conditions
\[ \begin{align*}
  v |_{\partial \Omega \times (0, +\infty)} &= v^*, \\
  T |_{\partial \Omega \times (0, +\infty)} &= T^*, \\
  \Phi |_{\partial \Omega \times (0, +\infty)} &= \Phi^*.
\end{align*} \] (3)

The fields \( v_0, T_0, v^*, T^*, \Phi^* \) are prescribed on the appropriate domains and satisfy suitable regularity and compatibility conditions. Moreover we suppose that the domain \( \Omega \) is bounded and that it is possible to apply the divergence theorem.

It is convenient to rewrite problem (1) in dimensionless form by using the following transformations:

\[ \begin{align*}
  \mathbf{V} &= \frac{v}{V}, \quad \vartheta = \frac{T - T_R}{T}, \quad \varphi = \frac{\Phi - \Phi_R}{\Phi}, \\
  P &= \frac{p}{\rho_R V^2}, \quad \nu = \frac{\mu}{\rho R}, \\
  x' &= \frac{x}{L}, \quad t' = \frac{V t}{L}, \quad \mathbf{V} = \frac{\nu}{L}, \quad T = \sqrt{\frac{\nu^3 \Delta T}{\alpha T \gamma L^3 g}}, \\
  \Phi &= \sqrt{\frac{\nu^3 \Phi_R}{\alpha \Phi D_B L^3 g}},
\end{align*} \] (4)

where \( V, L, \Delta T > 0 \) are the reference velocity, length, characteristic temperature difference, respectively, while the thermal diffusivity \( \gamma \) is given by
\[ \gamma = \frac{k}{\rho_R c}. \] (5)

We assume that the change of temperature in the nanofluid is small comparing to the temperature \( T_R \) so that we can replace \( T \) by \( T_R \) in the denominator of \((1)_{3.4}\).

Problem (1) in dimensionless form becomes
\[ \begin{align*}
  \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} &= -\nabla P + \Delta \mathbf{V} + (R_T \vartheta - R_\Phi \varphi)e_3, \\
  \nabla \cdot \mathbf{V} &= 0, \\
  P_r \left( \frac{\partial \vartheta}{\partial t} + \mathbf{V} \cdot \nabla \vartheta \right) &= \Delta \vartheta + \frac{1}{Le} \left( \nabla \vartheta \cdot \nabla \varphi + \frac{1}{N_{BT}} |\nabla \vartheta|^2 \right),
\end{align*} \]
UNIQUENESS AND DECAY RESULTS...  567

\[ Sc \left( \frac{\partial \varphi}{\partial t} + \mathbf{V} \cdot \nabla \varphi \right) = \Delta \varphi + \frac{1}{N_{BT}} \Delta \vartheta, \] (6)

where we used the following dimensionless numbers:

\[
\begin{align*}
P_r & = \frac{\nu}{\gamma} \quad \text{(Prandtl number),} \\
R_T^2 & = \frac{\alpha_T L^3 \Delta T g}{\nu \gamma} \quad \text{(temperature Rayleigh number),} \\
R^2 & = \frac{\alpha \Phi L^3 \Phi R g}{\nu \gamma} \quad \text{(nanoparticle volume fraction Rayleigh number),} \\
Le & = \frac{k}{\rho_p c_p D_B \Phi} \quad \text{(Lewis number),} \\
N_{BT} & = \frac{D_B T_R \Phi}{D_T T}, \quad Sc = \frac{\nu}{D_B} \quad \text{(Schmidt number).} 
\end{align*}
\] (7)

For the sake of brevity, we continue to denote by \( \Omega \) the transformed of the domain occupied by the nanofluid, we omit to write the initial \((2)\) and boundary conditions \((3)\) in dimensionless form and in the sequel we will imply that they are dimensionless.

We confine ourselves to consider classical solutions to the initial-boundary value problem \((\mathbf{V}, P, \vartheta, \varphi)\), i.e.

\[
\begin{align*}
\mathbf{V},\, \vartheta,\, \varphi & \in C^{2,1}(\Omega \times (0, +\infty)) \cap C(\overline{\Omega} \times [0, +\infty)), \\
P & \in C^{1,0}(\Omega \times (0, +\infty)) \cap C(\overline{\Omega} \times [0, +\infty)), \\
\nabla \mathbf{V},\, \nabla \vartheta,\, \nabla \varphi & \in C(\overline{\Omega} \times [0, +\infty)).
\end{align*}
\]

3. Uniqueness Theorem

In this section we prove the uniqueness of the solution of the problem \((6), (2), (3)\) without any restriction on the material parameters.

**Theorem 1.** Let \((\mathbf{V}_i, P_i, \vartheta_i, \varphi_i)\) \(i = 1, 2\) be two solutions of the problem \((6), (2), (3)\). Then

\[
\mathbf{V}_1 = \mathbf{V}_2, \quad P_1 = P_2 + \Pi, \quad \vartheta_1 = \vartheta_2, \quad \varphi_1 = \varphi_2 \quad \text{in} \quad \Omega \times [0, +\infty),
\]

with \(\Pi = \Pi(t)\) an arbitrary function.
Proof. Let \((V, \Pi, \vartheta, \varphi)\) denote the difference between the two solutions \((V_i, P_i, \vartheta_i, \varphi_i)\) \(i = 1, 2\) of the problem \((6), (2), (3)\). Then the following system is satisfied

\[
\begin{align*}
\frac{\partial V}{\partial t} + V_1 \cdot \nabla V_1 - V_2 \cdot \nabla V_2 &= -\nabla \Pi + \Delta V + (R_T \vartheta - R_\Phi \varphi)e_3, \\
\nabla \cdot V &= 0, \\
Pr \left( \frac{\partial \vartheta}{\partial t} + V_1 \cdot \nabla \vartheta_1 - V_2 \cdot \nabla \vartheta_2 \right) &= \Delta \vartheta \quad \text{with} \quad L \geq \frac{1}{Pr} \left[ \nabla \vartheta_1 \cdot \nabla \varphi_1 - \nabla \vartheta_2 \cdot \nabla \varphi_2 + \frac{1}{N_{BT}} (|\nabla \vartheta_1|^2 - |\nabla \vartheta_2|^2) \right], \\
Sc \left( \frac{\partial \varphi}{\partial t} + V_1 \cdot \nabla \varphi_1 - V_2 \cdot \nabla \varphi_2 \right) &= \Delta \varphi + \frac{1}{N_{BT}} \Delta \vartheta \quad \text{with} \quad L \geq \frac{1}{Sc} \left[ \nabla \varphi_1 \cdot \nabla \varphi_1 - \nabla \varphi_2 \cdot \nabla \varphi_2 + \frac{1}{N_{BT}} (|\nabla \varphi_1|^2 - |\nabla \varphi_2|^2) \right], \\
\| \nabla \vartheta_1 \|_2^2 - \| \nabla \vartheta_2 \|_2^2 &= \vartheta \nabla \left[ (\nabla \varphi_1 + \nabla \varphi_2) \cdot \nabla \varphi_1 \right], \\
\frac{1}{2} \left| \nabla \varphi_1 \right|^2 - \frac{1}{2} \left| \nabla \varphi_2 \right|^2 &= \varphi \nabla \left[ (\nabla \varphi_1 + \nabla \varphi_2) \cdot \nabla \varphi_1 \right].
\end{align*}
\] (8)

together with homogeneous initial and boundary conditions for \(V, \vartheta, \varphi\).

Let us multiply (8) by \(V\), (8) by \(\vartheta\) and (8) by \(A\varphi\) with \(A\) some dimensionless positive constant to be specified and sum the resulting equations.

We now develop separately some terms:

\[
\begin{align*}
(V_1 \cdot \nabla V_1 - V_2 \cdot \nabla V_2) \cdot V &= V \cdot \nabla V_1 \cdot V + \frac{1}{2} V_2 \cdot \nabla V^2, \\
(V_1 \cdot \nabla \vartheta_1 - V_2 \cdot \nabla \vartheta_2) \vartheta &= (V \cdot \nabla \vartheta_1) \vartheta + \frac{1}{2} V_2 \cdot \nabla \vartheta^2, \\
(\nabla \vartheta_1 \cdot \nabla \varphi_1 - \nabla \vartheta_2 \cdot \nabla \varphi_2) \vartheta &= (\nabla \vartheta_1 \cdot \nabla \varphi + \nabla \vartheta \cdot \nabla \varphi_1) \vartheta, \\
(|\nabla \vartheta_1|^2 - |\nabla \vartheta_2|^2) \vartheta &= \vartheta \nabla (\vartheta_1 + \vartheta_2) \cdot \nabla \vartheta, \\
(V_1 \cdot \nabla \varphi_1 - V_2 \cdot \nabla \varphi_2) \varphi &= \varphi V \cdot \nabla \varphi_1 + \frac{1}{2} V_2 \cdot \nabla \varphi^2.
\end{align*}
\] (9)

At this point we fix \(T \in (0, +\infty)\) and integrate by parts over \(\Omega \quad \forall t \in [0, T]\).
Taking into account (8)\_2, (9), the divergence theorem and the homogeneous boundary conditions (3), after some calculations we arrive at

\[
\frac{d}{dt} \frac{1}{2} \int_{\Omega} \left( V^2 + P_r \vartheta^2 + ASc\varphi^2 \right) d\Omega \\
= \int_{\Omega} \left[ -V \cdot \nabla V_1 \cdot V - |\nabla V|^2 + (R_T \vartheta - R_\Phi \varphi)V_3 \right] d\Omega \\
+ \int_{\Omega} \left[ -P_r \vartheta \nabla \vartheta_1 \cdot V - |\nabla \vartheta|^2 + \frac{1}{Le} \vartheta (|\nabla \vartheta_2 \cdot \nabla \varphi + \nabla \vartheta \cdot \nabla \varphi_1) \right] \frac{1}{L e N_{BT}} \vartheta \nabla (\vartheta_1 + \vartheta_2) \cdot \nabla \vartheta \right] d\Omega
\]
UNIQUENESS AND DECAY RESULTS...

\[ + \int_{\Omega} -A \left( Sc\varphi \nabla \varphi_1 \cdot \mathbf{V} + |\nabla \varphi|^2 + \frac{1}{N_{BT}} \nabla \vartheta \cdot \nabla \varphi \right) d\Omega. \] (10)

In order to choose the constant \( A \) we consider the quadratic form

\[ \omega(\xi) = \sum_{i=1}^{3} \xi_i^2 + A \sum_{i=4}^{6} \xi_i^2 + \frac{A}{N_{BT}} (\xi_1 \xi_4 + \xi_2 \xi_5 + \xi_3 \xi_6), \quad \xi \in \mathbb{R}^6. \]

As it is easy to verify, \( \omega \) is positive definite if and only if

\[ \frac{A}{4N_{BT}^2} < 1. \] (11)

By virtue of this inequality, we deduce that if the material parameter \( N_{BT} > \frac{1}{2} \), then \( A = 1 \) assures that \( \omega \) is positive definite; while if \( N_{BT} \leq \frac{1}{2} \) then \( A \) must be chosen satisfying the inequality \( A < 4N_{BT}^2 \).

In any case we have that

\[ \omega(\xi) \geq \lambda_{\text{min}} |\xi|^2, \] (12)

where \( \lambda_{\text{min}} \) is the smallest eigenvalue of the matrix associated to \( \omega \) given by

\[ \lambda_{\text{min}} = \frac{N_{BT}(A + 1) - \sqrt{N_{BT}^2(A - 1)^2 + A^2}}{2N_{BT}}. \]

These considerations allow us to obtain the inequality

\[ -|\nabla \vartheta|^2 - A|\nabla \varphi|^2 - A \frac{1}{N_{BT}} \nabla \vartheta \cdot \nabla \varphi \leq -\lambda_{\text{min}} (|\nabla \vartheta|^2 + |\nabla \varphi|^2). \] (13)

We can now estimate some terms at the right hand side of (10) on taking into account that the involved fields are bounded in \( \Omega \times [0, \bar{t}] \).

Putting

\[ M_1 = \max_{\Omega \times [0, \bar{t}]} |\nabla \varphi_1|, \quad M_2 = \max_{\Omega \times [0, \bar{t}]} |\nabla \vartheta_1|, \]

\[ M_3 = \max_{\Omega \times [0, \bar{t}]} |\nabla \vartheta_2|, \quad M_4 = \max_{\Omega \times [0, \bar{t}]} |\nabla \varphi_1|, \]

we arrive at

\[ - \mathbf{V} \cdot \nabla \varphi_1 \cdot \mathbf{V} - |\nabla \varphi|^2 + (R_{T\vartheta} - R_{\Phi \varphi})V_3 - P_{\vartheta} \nabla \vartheta_1 \cdot \mathbf{V} - |\nabla \vartheta|^2 \]

\[ + \frac{1}{Le} \nabla \vartheta_2 \cdot \nabla \varphi + \nabla \vartheta \cdot \nabla \varphi_1 \]

\[ + \frac{1}{LeN_{BT}} \nabla (\vartheta_1 + \vartheta_2) \cdot \nabla \vartheta \]
\[- A \left( Sc\varphi \nabla \varphi_1 \cdot \mathbf{V} + |\nabla \varphi|^2 + \frac{1}{N_{BT}} \nabla \vartheta \cdot \nabla \varphi \right) \]
\[\leq \frac{1}{2} (2M_1 + R_T - R_\Phi + P_r M_2 + A Sc M_4) \mathbf{V}^2 \]
\[+ \frac{1}{2} \left( R_T + M_2 + \frac{M_3}{Le\epsilon_1} + \frac{M_4}{Le\epsilon_2} + \frac{M_2 + M_3}{Le N_{BT} \epsilon_2} \right) \vartheta^2 \]
\[+ \frac{1}{2} (A Sc M_4 - R_\Phi) \varphi^2 + \left[ -\lambda_{\text{min}} + \frac{M_4 \epsilon_2}{2Le} + \frac{(M_2 + M_3) \epsilon_2}{2Le N_{BT}} \right] |\nabla \vartheta|^2 \]
\[+ \left( -\lambda_{\text{min}} + \frac{M_3 \epsilon_1}{2Le} \right) |\nabla \varphi|^2, \] (14)

where \( \epsilon_1, \epsilon_2 \) are arbitrary positive constants and we used inequality (13).

If we choose
\[ \epsilon_1 = \frac{2Le \lambda_{\text{min}}}{M_3}, \quad \epsilon_2 = \frac{2Le \lambda_{\text{min}} N_{BT}}{M_2 + M_3 + M_4 N_{BT}}, \] (15)
then (10) and (14) lead to
\[ \frac{d}{dt} \frac{1}{2} \int_\Omega (\mathbf{V}^2 + P_r \vartheta^2 + A Sc \varphi^2) \, d\Omega \]
\[\leq \frac{1}{2} \int_\Omega (B_1 \mathbf{V}^2 + B_2 P_r \vartheta^2 + B_3 A Sc \varphi^2) \, d\Omega, \] (16)
where the coefficients \( B_i, \ i = 1, 2, 3, \) can be easily computed.

Putting
\[ B = \max\{B_1, B_2, B_3\}, \quad \mathcal{E}_A = \frac{1}{2} \int_\Omega (\mathbf{V}^2 + P_r \vartheta^2 + A Sc \varphi^2) \, d\Omega, \]
we arrive at
\[ \frac{d}{dt} \mathcal{E}_A(t) \leq B \mathcal{E}_A(t), \quad \forall t \in [0, \bar{t}]. \] (17)

Therefore we deduce
\[ \mathcal{E}_A(\bar{t}) \leq \mathcal{E}_A(0) e^{B\bar{t}}. \]

By virtue of the initial conditions we have that \( \mathcal{E}_A(0) = 0, \) so that
\[ \mathbf{V}(x, \bar{t}) = 0, \quad \vartheta(x, \bar{t}) = 0, \quad \varphi(x, \bar{t}) = 0, \quad \forall x \in \overline{\Omega}. \]

For the arbitrariness of \( \bar{t}, \) we get
\[ V_1 = V_2, \quad \vartheta_1 = \vartheta_2, \quad \varphi_1 = \varphi_2 \text{ in } \Omega \times [0, +\infty). \]

Finally, from (10)_1 we have
\[ \nabla \Pi = 0 \Rightarrow \Pi \equiv \Pi(t), \]
which completes the proof. \( \square \)

**Remark 1.** As it is easy to verify, the uniqueness theorem continues to hold even if we replace the boundary conditions (3) with
\[ \frac{\partial \vartheta}{\partial n}|_{\partial \Omega \times [0, +\infty)} = q_{\vartheta}^*, \quad \frac{\partial \varphi}{\partial n}|_{\partial \Omega \times [0, +\infty)} = q_\varphi^*, \]
or we maintain (3)_2 and replace (3)_3 with
\[ \frac{1}{N_{BT}} \frac{\partial \vartheta}{\partial n}|_{\partial \Omega \times [0, +\infty)} + \frac{\partial \varphi}{\partial n}|_{\partial \Omega \times [0, +\infty)} = j^*, \]
where \( n \) is the unit outward normal to \( \partial \Omega \) and \( q_{\vartheta}^*, q_\varphi^*, j^* \) are sufficiently smooth prescribed fields (see for example [12], [21]).

Actually, we get again an inequality analogous to (16): we have to modify in a suitable way the expressions of \( B_2 \) and \( B_3 \) in the first case and of \( B_3 \) in the latter one by inserting \( \max_{\Omega \times [0, \bar{t}]} |V_2| \).

**Remark 2.** The function \( E_A \) represents a suitable (dimensionless) energy of the nanofluid.

4. **Decay in time of the energy \( E_A \)**

In this section we prove the exponential decay in time of the energy \( E_A \) in \([0, \bar{t}] \) with \( \bar{t} \in (0, \infty) \) for solutions of problem (6), (2), (3) with homogeneous boundary conditions provided that some suitable inequalities hold.

**Theorem 2.** Let \((V, P, \vartheta, \varphi)\) be the solution of the problem (6), (2), (3) with homogeneous boundary conditions and \( \bar{t} \in (0, \infty) \). If the following inequalities
\[ R_T + R_\Phi < \frac{2}{\beta}, \quad \lambda_{\text{min}} > \frac{\beta}{2} \max\{R_T, R_\Phi\}, \]

\[ M_T < 2 \frac{Le}{\beta} \min \left\{ \frac{\sqrt{\beta}N_{BT}}{\sqrt{\beta}N_{BT} + 2} \left( \lambda_{\min} - \frac{\beta R_T}{2} \right), \lambda_{\min} - \frac{\beta R_\Phi}{2} \right\}, \]  

(18)

are verified, then there exists a positive constant \( C \), depending on the material parameters, Poincaré constant of \( \Omega \) and \( M_T = \max_{\Omega \times [0,T]} |\nabla \vartheta| \), such that

\[ \mathcal{E}_A(t) \leq \mathcal{E}_A(0) e^{-Ct}, \quad \forall t \in [0,T]. \]  

(19)

**Proof.** Fix \( \bar{t} \) and let \((V, P, \vartheta, \varphi)\) be the solution of the problem (6), (2), (3) with homogeneous boundary conditions. The equations (6)\(_1\), (6)\(_3\) and (6)\(_4\) multiplied by \(V, \vartheta\) and \(A\varphi\) respectively are summed and the resulting equation is integrated over \(\Omega\) to see that, thanks to integration by parts, we have

\[ \frac{d}{dt} \mathcal{E}_A = \int_{\Omega} \left[ -|\nabla V|^2 + (R_T \vartheta - R_\Phi \varphi)V_3 - |\nabla \vartheta|^2 \right] d\Omega \]

+ \[ \int_{\Omega} \left[ \frac{1}{Le} \vartheta \left( \nabla \vartheta \cdot \nabla \varphi + \frac{1}{N_{BT}} |\nabla \vartheta|^2 \right) - A \left( |\nabla \varphi|^2 + \frac{1}{N_{BT}} \nabla \vartheta \cdot \nabla \varphi \right) \right] d\Omega. \]

(20)

Now, by using Cauchy and Poincaré inequalities, we get

\[ \int_{\Omega} (R_T \vartheta - R_\Phi \varphi)V_3 d\Omega \]

\[ \leq \frac{\beta}{2} \int_{\Omega} \left[ R_T (|\nabla V|^2 + |\nabla \vartheta|^2) + R_\Phi (|\nabla V|^2 + |\nabla \varphi|^2) \right] d\Omega, \]

(21)

where \( \beta \) is the Poincaré constant for \( \Omega \).

Applying Cauchy, Schwarz and Poincaré inequalities, we obtain

\[ \int_{\Omega} \vartheta \left( \nabla \vartheta \cdot \nabla \varphi + \frac{1}{N_{BT}} |\nabla \vartheta|^2 \right) d\Omega \]

\[ \leq M_T \int_{\Omega} \left[ \left( \frac{\beta}{2} + \frac{\sqrt{\beta}}{N_{BT}} \right) |\nabla \vartheta|^2 + \frac{\beta}{2} |\nabla \varphi|^2 \right] d\Omega. \]

(22)

Taking into account inequalities (13), (21) and (22) we deduce in \([0,\bar{t}]\)

\[ \frac{d}{dt} \mathcal{E}_A \leq \int_{\Omega} \left[ \frac{\beta}{2} (R_T + R_\Phi) - 1 \right] |\nabla V|^2 d\Omega \]

+ \[ \int_{\Omega} \left( \frac{\beta R_T}{2Le} + \frac{\sqrt{\beta} M_T}{Le N_{BT}} - \lambda_{\min} \right) |\nabla \vartheta|^2 d\Omega \]

+ \[ \int_{\Omega} \left( \frac{\beta M_T}{2Le} - \frac{\beta R_\Phi}{2} - \lambda_{\min} \right) |\nabla \varphi|^2 d\Omega. \]

(23)
In order to get the thesis, the coefficients of $|\nabla V|^2$, $|\nabla \vartheta|^2$, $|\nabla \varphi|^2$ must be negative, i.e. inequalities (18) must be verified where it is convenient to choose in $\lambda_{\text{min}}$ the value of $A$ that makes it maximum. This value of $\lambda_{\text{min}}$ is given by $\frac{N_2^{BT}}{1+N_2^{BT}}$.

If Poincaré inequality is again applied, one gets

$$
\frac{d}{dt} \mathcal{E}_A(t) \leq -C \mathcal{E}_A(t), \quad \forall t \in [0, \overline{t}],
$$

(24)

where $C = \frac{1}{\beta} \min \{C_1, \frac{C_2}{P}, \frac{C_3}{\Lambda_{\text{SE}}}\}$, being $C_i$, $i = 1, 2, 3$ the opposite of the coefficients of the three gradients in (23). From (24) follows that

$$
\mathcal{E}_A(t) \leq \mathcal{E}_A(0)e^{-Ct}, \quad \forall t \in [0, \overline{t}],
$$

which is the desired conclusion.

**Remark 3.** If inequality (18)$_3$ holds for any $\overline{t} > 0$, then $\mathcal{E}_A$ decays exponentially in $[0, +\infty)$. This result may be regarded as a universal stability result for the basic flow $v = 0$, $T = T_R$, $\Phi = \Phi_R$ and the basic flow is asymptotically stable in the mean. Finally, we notice that if we do not consider the Boussinesq approximation ($\alpha_T = 0$, $\alpha_\Phi = 0$) then the previous theorems continue to hold with the suitable simplifications.

## 5. Extension of the results to the magnetohydrodynamic case

In this section we suppose that the nanofluid is embedded in a magnetic field and denote by $\mathbf{H}$ the total magnetic field. If the Hall and the displacement currents are disregarded, in order to formulate the new initial-boundary value problem we have to modify system (1) by adding in (1)$_1$ the term $\mu_e(\nabla \times \mathbf{H}) \times \mathbf{H}$ and the equations

$$
\nabla \cdot \mathbf{H} = 0,
$$

$$
\frac{\partial \mathbf{H}}{\partial t} = \eta \Delta \mathbf{H} + \nabla \times (\mathbf{v} \times \mathbf{H}),
$$

(25)

where $\mu_e$ is the magnetic permeability and $\eta$ the magnetic diffusivity given by

$$
\eta = \frac{1}{\sigma \mu_e}, \quad \sigma = \text{electrical conductivity}.
$$
Of course we must consider the initial and boundary conditions for the magnetic field:

$$H(x,0) = H_0(x), \quad \forall x \in \overline{\Omega},$$
$$H|_{\partial \Omega \times [0, +\infty)} = H^*.$$

(26)

As it is usual, we modify $p$ by adding the term $\mu_e H^2$ and continue to denote by $p$ the new expression of the modified pressure.

Then, in order to write the problem in dimensionless form we use the following transformation for the magnetic field:

$$h = \frac{H}{V \sqrt{\mu \sigma}}.$$

After some calculations, the dimensionless equations that govern the magnetohydrodynamic flow of the nanofluid become

$$\frac{\partial V}{\partial t} + V \cdot \nabla V = -\nabla P + \Delta V + (R_T \vartheta - R_\Phi \varphi)e_3 + R_m h \cdot \nabla h,$$
$$\nabla \cdot V = 0,$$
$$P_r \left( \frac{\partial \vartheta}{\partial t} + V \cdot \nabla \vartheta \right) = \Delta \vartheta + \frac{1}{Le} \left( \nabla \vartheta \cdot \nabla \varphi + \frac{1}{N_{BT}} |\nabla \vartheta|^2 \right),$$
$$Sc \left( \frac{\partial \varphi}{\partial t} + V \cdot \nabla \varphi \right) = \Delta \varphi + \frac{1}{N_{BT}} \Delta \vartheta,$$
$$\nabla \cdot h = 0,$$
$$R_m \frac{\partial h}{\partial t} = \Delta h + R_m (h \cdot \nabla V - V \cdot \nabla h),$$

(27)

where $R_m = \frac{\nu}{\eta}$ is the Reynolds magnetic number and $h$ has the same regularity properties as $V$.

By proceeding as in the previous sections we can state a uniqueness theorem and a decay result for the energy

$$\mathcal{E}_{Am} = \frac{1}{2} \int_\Omega \left( V^2 + P_r \vartheta^2 + ASc \varphi^2 + R_m h^2 \right) d\Omega.$$

(28)

**Theorem 3.** Let $(V_i, P_i, \vartheta_i, \varphi_i, h_i), \ i = 1, 2,$ be two solutions of the problem (27), (2), (3), (26). Then

$$V_1 = V_2, \quad P_1 = P_2 + \Pi, \quad \vartheta_1 = \vartheta_2, \quad \varphi_1 = \varphi_2,$$
$$h_1 = h_2 \quad \text{in} \ \Omega \times [0, +\infty)$$

with $\Pi = \Pi(t)$ an arbitrary function.
Proof. The theorem is proved by similar arguments as for Theorem 1; it is sufficient to consider only the new terms involving the magnetic field.

If we proceed as in the proof of Theorem 1 and set $h = h_1 - h_2$, we obtain an equation deducible from (10) by replacing at the left hand side $\mathcal{E}_A$ with $\mathcal{E}_{Am}$ and adding at the right hand side the following integral

$$\int_{\Omega} \left[ -|\nabla h|^2 + R_m (h \cdot \nabla h_1 \cdot V - V \cdot \nabla h_1 \cdot h + h \cdot \nabla V_1 \cdot h) \right] d\Omega.$$

After putting $M_5 = \max_{\Omega \times [0, \bar{t}]} |\nabla h_1|$, the function under the previous integral can be estimated as

$$- |\nabla h|^2 + R_m (h \cdot \nabla h_1 \cdot V - V \cdot \nabla h_1 \cdot h + h \cdot \nabla V_1 \cdot h) \leq R_m \left[ M_5 V^2 + (M_1 + M_5) h^2 \right].$$

From the equation (10) modified, (13), (14), (15) and (29) we get

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \left( V^2 + P r \vartheta^2 + A Sc \varphi^2 + R_m h^2 \right) d\Omega$$

$$\leq \frac{1}{2} \int_{\Omega} \left( B'_1 V^2 + B_2 P r \vartheta^2 + B_3 A Sc \varphi^2 + B_4 R_m h^2 \right) d\Omega,$$

where the coefficients $B_2, B_3$ are the same as in (16) while $B'_1$ is obtained from $B_1$ by means of a suitable modification and $B_4$ is new because of the presence of the magnetic field.

Inequality (30) leads to

$$\frac{d}{dt} \mathcal{E}_{Am}(t) \leq B \mathcal{E}_{Am}(t), \quad \forall t \in [0, \bar{t}],$$

where $B = \max \{B'_1, B_2, B_3, B_4\}$. By using (31) it is easy to complete the proof of the theorem.

**Theorem 4.** Let $(V, P, \vartheta, \varphi, h)$ be the solution of the problem (27), (2), (3), (26) with homogeneous boundary conditions and $\bar{t} \in (0, \infty)$. If the inequalities (18) are verified, then there exists a positive constant $C_m$, depending on the material parameters, Poincaré constant of $\Omega$ and $M_T = \max_{\Omega \times [0, \bar{t}]} |\nabla \vartheta|$, such that

$$\mathcal{E}_{Am}(t) \leq \mathcal{E}_{Am}(0) e^{-C_m t}, \quad \forall t \in [0, \bar{t}].$$

Proof. By the same arguments used in the proof of Theorem 2, it is easily to show that
\[
\frac{d}{dt} E_{Am} = \int_{\Omega} \left[ -|\nabla V|^2 + (R_T \vartheta - R_F \varphi) V_3 - |\nabla \vartheta|^2 - |\nabla h|^2 \right] d\Omega + \int_{\Omega} \left[ \frac{1}{Le} \vartheta \left( \nabla \vartheta \cdot \nabla \varphi + \frac{1}{N_{BT}} |\nabla \vartheta|^2 \right) - A \left( |\nabla \varphi|^2 + \frac{1}{N_{BT}} \nabla \vartheta \cdot \nabla \varphi \right) \right] d\Omega,
\]

(33)
because the other terms involving the magnetic field do not make any contribution.

An application of Poincaré inequality to \( h \) and the arguments of the proof of Theorem 2 lead us to conclude that under conditions (18)

\[ E_{Am}(t) \leq E_{Am}(0) e^{-C_m t}, \quad \forall t \in [0, 7], \]

where \( C_m = \frac{1}{\beta} \min\{C_1, \frac{C_2}{R^*}, \frac{C_3}{ASc}, \frac{1}{R_m}\} \), which completes the proof.

\[ \square \]

**Remark 4.** We underline that the presence of the magnetic field may only modify the decay constant of the energy by means of \( R_m \).

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