Index and secondary index theory for flat bundles with duality

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Abstract

We discuss some aspects of index and secondary index theory for flat bundles with duality. This theory was first developed by Lott [9]. Our main purpose in the present paper is provide a modification with better functorial properties.

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1 The functor $L$

1.1 Definition of $L$

We define a contravariant functor $L$ from the category $\mathcal{T}op$ of topological spaces and continuous maps to $\mathbb{Z}_2$-graded rings.

Let $X$ be a topological space and $\epsilon \in \mathbb{Z}_2$. If $\mathcal{F}$ is a locally constant sheaf of finite-dimensional $\mathbb{R}$-modules over $X$, then an $\epsilon$-symmetric duality structure on $\mathcal{F}$ is an isomorphism of sheaves $q : \mathcal{F} \to \mathcal{F}^*$ satisfying $q^* = (-1)^\epsilon q$, where $\mathcal{F}^* := \text{Hom}(\mathcal{F}, \mathbb{R})$.

We first define a semigroup $\hat{L}_\epsilon(X)$ and then obtain $L_\epsilon(X)$ by introducing a relation. A generator of the semigroup $\hat{L}_\epsilon(X)$ is a pair $(\mathcal{F}, q)$ consisting of a locally constant sheaf of finite-dimensional $\mathbb{R}$-modules and an $\epsilon$-symmetric duality structure $q$. The operation in $\hat{L}_\epsilon(X)$ is given by direct sum

$$(\mathcal{F}, q) + (\mathcal{F}', q') := (\mathcal{F} \oplus \mathcal{F}', q \oplus q').$$

The group $L_\epsilon(X)$ is obtained from $\hat{L}_\epsilon(X)$ by introducing the relation lagrangian reduction.

We require $(\mathcal{F}, q) \sim 0$ if there exists a locally constant lagrangian subsheaf $i : \mathcal{L} \hookrightarrow \mathcal{F}$, i.e. the composition $\mathcal{L} \overset{i}{\to} \mathcal{F} \overset{q}{\to} \mathcal{F}^* \overset{i^*}{\to} \mathcal{L}^*$ vanishes and $\mathcal{L} = \mathcal{L}^\perp := \ker(i^* \circ q)$.

The class of $(\mathcal{F}, q)$ in $L_\epsilon(X)$ will be denoted by $[\mathcal{F}, q]$. The ring structure $L_\epsilon(X) \otimes L_{\epsilon'}(X) \to L_{\epsilon+\epsilon'}(X)$ is induced by the tensor product:

$$[\mathcal{F}, q][\mathcal{F}', q'] := \sqrt{(-1)^\epsilon} \sqrt{(-1)^{\epsilon'}} [\mathcal{F} \otimes \mathcal{F}', q \otimes q'].$$

The sign-convention is made such that later we have a natural transformation of rings from $L$ to complex $K$-theory $K^0$.

If $f : Y \to X$ is a morphism in $\mathcal{T}op$, then $f^* : L(X) \to L(Y)$ is defined by $f^*[\mathcal{F}, q] = [f^*\mathcal{F}, f^*q]$. The map $f^*$ only depends on the homotopy class of $f$.

Remark: A version $L(X)^{\text{Lott}}$ of this ring was first introduced by Lott [9]. His definition differs from ours since our relation "lagrangian reduction" is repaced by "hyperbolic reduction" in the definition of Lott. Here a generator $(\mathcal{F}, q)$ is called hyperbolic if there is an lagrangian subsheaf $\mathcal{L} \subset \mathcal{F}$ which admits a complement inside $\mathcal{F}$. In particular, $L(X)$ is a quotient of $L(X)^{\text{Lott}}$.

1.2 Computation of $L(X)$

Assume that $X$ is path connected and fix a base point $x$. Let $G := \pi_1(X, x)$ be the fundamental group. We consider the semigroup $\hat{L}_\epsilon(G)$ of all tuples $(\mathcal{F}, q, \rho)$, where $\mathcal{F}$ is a finite-dimensional real vector space, $q : \mathcal{F} \to \mathcal{F}^*$ is an $\epsilon$-symmetric duality structure, and $\rho : G \to \text{Aut}(\mathcal{F}, q)$ is a representation of $G$. The semigroup operation is given by direct sum. We then define the
group \( L_e(G) \) by introducing the relation \((F, q, \rho) \sim 0\) if there exists an invariant lagrangian subspace \( L \subset F \). Taking the holonomy representation on the fibre over \( x \) we obtain a bijection \( L(X) \cong L(G) \).

Let \( \hat{G} := G \times \mathbb{Z}_2 \). If \((F, q, \rho) \in \hat{L}(G)\), then we form the representation \((\hat{F}, \hat{\rho})\) of \( \hat{G} \) by \( \hat{F} := F \oplus F^* \), \( \hat{\rho}((g, 0)) := \rho(g) \oplus \rho(g^{-1})^* \), \( \hat{\rho}(1, 1) := \begin{pmatrix} 0 & q^{-1} \\ q & 0 \end{pmatrix} \).

If \((F, q, \rho)\) is an irreducible representation of \( G \), then \((\hat{F}, \hat{\rho})\) is an irreducible representation of \( \hat{G} \). An irreducible representation \((V, \sigma)\) of \( \hat{G} \) is called real, complex, or quaternionic, if \( \text{End}_{\hat{G}}(V) \cong \mathbb{R}, \cong \mathbb{C} \) or \( \cong \mathbb{H} \), respectively.

For each irreducible representation \((F, \rho)\) of \( G \) we define a \( \mathbb{Z}_2 \)-graded group \( A(F, \rho) \) as follows. If \((F, \rho)\) admits an invariant \( \epsilon \)-symmetric form \( q \), then we define \( A_e(F, \rho) := \mathbb{Z} \), iff \((\hat{F}, \hat{\rho})\) is real or quaternionic, \( A_e(F, \rho) := \mathbb{Z}_2 \), iff \((\hat{F}, \hat{\rho})\) is complex, and we set \( A_e(F, \rho) := 0 \), if \((F, \rho)\) does not admit an invariant \( \epsilon \)-symmetric form. If \( A(F, \rho) \neq 0 \), then we fix an invariant \( \epsilon \)-symmetric form \( q_{F, \rho} \).

Let \( \text{Rep}(G) \) denote the set of isomorphism classes of irreducible representations of \( G \).

**Theorem 1.1** There is a natural isomorphism of \( \mathbb{Z}_2 \)-graded groups

\[
L(G) = \bigoplus_{(F, \rho) \in \text{Rep}(G)} A(F, \rho)
\]

fixed by the condition that \((F, q, q_{F, q}) \mapsto 1_{(F, q)} \).

### 1.3 The natural transformation to \( K \)-theory

By \( \mathcal{T}op_{\text{para}} \) we denote the full subcategory of \( \mathcal{T}op \) of paracompact topological spaces. Let \( K^0(X) \) be the complex \( K \)-theory functor. We construct a natural transformation \( b : L \rightarrow K^0 \) of functors from \( \mathcal{T}op_{\text{para}} \) to rings.

A locally constant sheaf of finite-dimensional \( \mathbb{R} \)-modules on \( X \) gives rise to a locally trivial real vector bundle \( \text{bundle}(F) \) in a natural way. The correspondence \( \text{bundle} \) is functorial and compatible with direct sum, tensor product, and duality. Thus applying the bundle construction to \((F, q)\) we obtain a pair \((F, Q)\) consisting of a finite-dimensional real vector bundle and an isomorphism \( Q : F \rightarrow F^* \).

Let \((F, Q)\) be a real vector bundle with an isomorphism \( Q : F \rightarrow F^* \) such that \( Q^* = (-1)^\epsilon Q \) for \( \epsilon \in \mathbb{Z}_2 \). Following the language introduced by Lott we call an isomorphism \( J : F \rightarrow F \) a metric structure, if

1. \( J^* \circ Q \) defines a scalar product on \( F \),
2. $J^2 = (-1)^c \text{id}_F$,
3. $J^* \circ Q = (-1)^r Q \circ J$.

Since we assume that $X$ is paracompact it admits partitions of unity. This implies that metric structures exist, and that the space of all metric structures is contractible.

Given $(F, Q)$ as above we choose a metric structure $J$. Let $F_C$ be the complexification of $F$. Then $\frac{1}{\sqrt{(-1)^c}} J$ is a $\mathbb{Z}_2$-grading of $F_C$, and the pair $(F_C, \frac{1}{\sqrt{(-1)^r}} J)$ represents an element of $K^0(X)$ which does not depend on the choice of $J$.

The transformation $b : L \to K^0$ is obtained by composing the latter construction with bundle.

1.4 Secondary $L$-groups. The $\mathbb{R}/\mathbb{Z}$-variant

We first recall the definition of the 2-periodic cohomology theory $K_{\mathbb{R}/\mathbb{Z}}$ introduced by [7], [8]. Let $BU$ be the classifying space of complex $K$-theory. The Chern character (with real coefficients) is induced by a map $\text{ch} : BU \to \prod_{n=1}^\infty K(\mathbb{R}, 2n)$. The homotopy fibre of this map classifies $K_{\mathbb{R}/\mathbb{Z}}$. In particular, for any paracompact space $X$ there is a natural exact sequence of $K^0(X)$-modules

$$
\begin{align*}
K^{-1}(X) \xrightarrow{\text{ch}} H^{\text{odd}}(X, \mathbb{R}) &\to K_{\mathbb{R}/\mathbb{Z}}^{-1}(X) \xrightarrow{\partial} K^0(X) \xrightarrow{\text{ch}} H^{\text{ev}}(X, \mathbb{R}) \to ,
\end{align*}
$$

where $K^0(X)$ acts on cohomology via the Chern character.

We now define the functor $X \mapsto \tilde{L}_{\mathbb{R}/\mathbb{Z}}(X)$ from paracompact topological spaces to $\mathbb{Z}_2$-graded groups by the following pull-back diagram:

$$
\begin{array}{ccc}
L^{-1}(X) & \xrightarrow{\text{ch}} & H^{\text{odd}}(X, \mathbb{R}) \\
\downarrow & & \downarrow b \\
K_{\mathbb{R}/\mathbb{Z}}^{-1}(X) & \xrightarrow{\partial} & K^0(X)
\end{array}
$$

The grading is induced from that of $L(X)$. On morphisms the functor $L_{\mathbb{R}/\mathbb{Z}}$ only depends on homotopy classes. Note that $K^0(X)$ and $K_{\mathbb{R}/\mathbb{Z}}^{-1}(X)$ are $L(X)$-modules via $b$. This induces a graded $L(X)$-module structure on $\tilde{L}_{\mathbb{R}/\mathbb{Z}}(X)$. We have the following natural exact sequence of $L(X)$-modules

$$
\begin{align*}
K^{-1}(X) \xrightarrow{\text{ch}} H^{\text{odd}}(X, \mathbb{R}) &\to \tilde{L}_{\mathbb{R}/\mathbb{Z}}(X) \to L(X) \xrightarrow{\text{ch} \circ b} H^{\text{ev}}(X, \mathbb{R}) .
\end{align*}
$$

1.5 Index and secondary index

Let $X \to B$ be a smooth locally trivial fibre bundle over a compact base $B$ with compact even-dimensional fibres such that the vertical bundle $TX/B$ is orientable. In this case we have the following maps:
\begin{itemize}
  \item $\pi^{\text{sign}}_*: H^*(X, \mathbb{R}) \to H^*(B, \mathbb{R})$ defined by $\pi_* (\omega) = \int_{X/B} \omega \cap L(TX/B)$, where $\int_{X/B}$ is integration over the fibre and $L(TX/B)$ denotes the Hirzebruch $L$-class of the vertical bundle.
  
  \item $\pi^{\text{sign}}_1: K^0(X) \to K^0(B)$ defined by $\pi^{\text{sign}}_1([E]) = \text{index}(D^{\text{sign}}_E)$, where $D^{\text{sign}}_E$ is the fibrewise signature operator twisted by $E$ and $\text{index}(D^{\text{sign}}_E) \in K^0(B)$ denotes the class of the index bundle.
  
  \item $\pi^L_*: L(X) \to L(B)$ defined by $\pi^L_*[\mathcal{F}, q] = [H^* R\pi_* \mathcal{F}, \pi_*(q)]$, where $H^* R\pi_* \mathcal{F} = \bigoplus_{i=0}^{\infty} R^i \pi_*(\mathcal{F})$ is the direct sum of higher derived direct images of $\mathcal{F}$. The orientation of $TX/B$ provides an isomorphism $p : R^{\dim(TX/B)} \pi_* \mathbb{R} \xrightarrow{\sim} \mathbb{R}$. It induces isomorphisms $R^i \pi_* \mathcal{F} \xrightarrow{\sim} (R^{\dim(TX/B)}\pi_*)^{-i} \pi_* \mathcal{F}^*$, and $\pi_*(q)$ is the composition of the sum of these maps with the sum of $R^i \pi_*(q) : R^i \pi_* \mathcal{F} \xrightarrow{\sim} R^i \pi_* \mathcal{F}^*$.
\end{itemize}

It is an exercise in spectral sequences to show that $\pi^L_*$ is well-defined.

\textbf{Remark:} In [Lott] Lott defines $\pi^{L,Lott}_*: L^{\text{Lott}}(X) \to L^{\text{Lott}}(B)$ which induces $\pi^L_*$ by passing to quotients.

By the index theorem for families and fibrewise Hodge theory the following diagram commutes

\[
\begin{array}{ccc}
L(X) & \to & K^0(X) & \to & H^{ev}(X, \mathbb{R}) \\
\pi_*^L \downarrow & & \pi^{\text{sign}}_* \downarrow & & \pi^{\text{sign}}_* \downarrow \\
L(B) & \to & K^0(B) & \to & H^{ev}(B, \mathbb{R})
\end{array}
\]

In order to define an index map for $K_{\mathbb{R}/\mathbb{Z}}$ we need the further assumption that $\pi$ is $K$-oriented. Thus assume that $TX/B$ has a $\text{Spin}^c$-structure. Then there are maps $\pi_1^{\text{Spin}^c}: K^0(X) \to K^0(B)$ and $\pi_1^{\text{Spin}^c,\mathbb{R}/\mathbb{Z}} : K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$ (compare e.g. [Lott]), such that the following diagram commutes:

\[
\begin{array}{ccc}
H^{\text{odd}}(X, \mathbb{R}) & \to & K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) & \to & K^0(X) & \to & H^{ev}(X, \mathbb{R}) \\
\pi_1^{\text{Spin}^c} \downarrow & & \pi_1^{\text{Spin}^c,\mathbb{R}/\mathbb{Z}} \downarrow & & \pi_1^{\text{Spin}^c} \downarrow & & \pi_1^{\text{Spin}^c} \downarrow \\
H^{\text{odd}}(B, \mathbb{R}) & \to & K^{-1}_{\mathbb{R}/\mathbb{Z}}(B) & \to & K^0(B) & \to & H^{ev}(B, \mathbb{R})
\end{array}
\]

where $\pi_1^{\text{Spin}^c}(\omega) = \int_{X/B} \hat{A}(TX/B) \cap e^{c_1/2} \cap \omega$ and $c_1$ is the first Chern class determined by the $\text{Spin}^c$-structure. There is an unique element $E_{\text{sign}} \in K_0(X)$ such that $\pi^{\text{sign}}_1(x) = \pi_1^{\text{Spin}^c}(E_{\text{sign}} \bullet x)$. Note that $\text{ch}(E_{\text{sign}}) \cap \hat{A}(TX/B) \cap e^{c_1/2} = L(TX/B)$.

\item $\pi^{\text{sign},\mathbb{R}/\mathbb{Z}}_*: K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$ is defined by $\pi^{\text{sign},\mathbb{R}/\mathbb{Z}}_*(x) := \pi_1^{\text{Spin}^c,\mathbb{R}/\mathbb{Z}}(E_{\text{sign}} \bullet x)$ so that

\[
\begin{array}{ccc}
H^{\text{odd}}(X, \mathbb{R}) & \to & K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) & \to & K^0(X) & \to & H^{ev}(X, \mathbb{R}) \\
\pi^{\text{sign}}_* \downarrow & & \pi^{\text{sign},\mathbb{R}/\mathbb{Z}}_* \downarrow & & \pi^{\text{sign}}_* \downarrow & & \pi^{\text{sign}}_* \downarrow \\
H^{\text{odd}}(B, \mathbb{R}) & \to & K^{-1}_{\mathbb{R}/\mathbb{Z}}(B) & \to & K^0(B) & \to & H^{ev}(B, \mathbb{R})
\end{array}
\]
The motivation of introducing the quotient bundle $T X/B$ is the map induced by $\pi_1^{\text{sign}, \mathbb{R}/\mathbb{Z}}$ and $\pi_*^L$. The following diagram commutes:

$$
\begin{array}{ccc}
H^\text{odd}(X, \mathbb{R}) & \rightarrow & \tilde{L}^{\mathbb{R}/\mathbb{Z}}(X) \\
\pi_*^\text{sign} \downarrow & & \downarrow \\
\pi_*^{\mathbb{L}_{\mathbb{R}/\mathbb{Z}}} & \rightarrow & \pi_*^L \\
H^\text{odd}(B, \mathbb{R}) & \rightarrow & \tilde{L}^{\mathbb{R}/\mathbb{Z}}(B) \\
\pi_*^\text{sign} \downarrow & & \downarrow \\
\pi_*^{\mathbb{L}_2} & \rightarrow & \pi_*^L \\
\end{array}
$$

All index maps are natural with respect to pull-back of fibre bundles.

Let $\pi_1 : X \rightarrow X_1$ and $\pi_2 : X_1 \rightarrow B$ be locally trivial smooth fibre bundles with closed even-dimensional fibres and compact base. Further assume that the vertical bundles $TX_1/X_2$ and $TX_2/B$ carry $\text{Spin}_c$-structures (and are therefore oriented). Then the composition $\pi = \pi_2 \circ \pi_1 : X \rightarrow B$ is a locally trivial fibre bundle with closed even-dimensional fibres, and the vertical bundle $TX/B$ carries an induced $\text{Spin}_c$-structure. In this situation the index maps on complex $K$-theory, $K_{\mathbb{R}/\mathbb{Z}}$-theory, and in cohomology behave functorially with respect to the iterated fibre bundle. It is again an exercise in spectral sequences to show that $\pi_*^L$ is also functorial. As a consequence, $\pi_*^{\mathbb{L}_{\mathbb{R}/\mathbb{Z}}}$ is functorial, i.e. $\pi_*^{\mathbb{L}_{\mathbb{R}/\mathbb{Z}}} = (\pi_*^{\mathbb{L}_2})_* \circ (\pi_*^L)_*$.

Remark: The motivation of introducing the quotient $L(X)$ of Lott’s group $L^{\text{Lott}}(X)$ is to implement this functoriality for $\pi_*^L$, which is not true for Lott’s definition.

## 2 Geometry and extended secondary $L$-groups

### 2.1 Definition of $\tilde{L}$

The functor $X \mapsto \tilde{L}^{\mathbb{R}/\mathbb{Z}}(X)$ from $\text{Top}_{\text{para}}$ to $\mathbb{Z}_2$-graded $L(X)$-modules was defined by a purely homotopy-theoretic construction as an extension of the functor

$$
X \mapsto \ker (\text{ch} \circ b : L(X) \rightarrow H^\text{ev}(X, \mathbb{R}))
$$

by $X \mapsto H^\text{odd}(X, \mathbb{R})/\text{ch}(K^{-1}(X))$.

Let $\text{Top}_{\text{smooth}}$ denote the full subcategory of topological spaces $\text{Top}_{\text{para}}$ which are homotopy equivalent to smooth manifolds. In the present section we use a differential geometric construction in order to define on $\text{Top}_{\text{smooth}}$ a functor $X \mapsto \tilde{L}(X)$ to graded $L(X)$-modules which extends $X \mapsto \ker(\text{ch} \circ b)$ by $X \mapsto H^\text{odd}(X, \mathbb{R})$. First we define $\tilde{L}(M)$ for a smooth manifold $M$. Again we first define a semigroup $\tilde{L}(M)$. A generator of $\tilde{L}(M)$ is a tuple $(\mathcal{F}, q, J, \rho)$, where $(\mathcal{F}, q)$ is a generator of $\hat{L}(M)$, $J$ is a metric structure on $F := \text{bundle}(\mathcal{F})$, and $\rho \in \Omega^{k*-(-1)^r}(M)/\text{im}(d)$ satisfies $d\rho = p(\nabla^F, J)$. Here $(\Omega^*(M), d)$ is the real deRham complex of $M$ and $\nabla^F$ is the canonical flat connection on $F$ (such that $\ker(\nabla^F) \cong \mathcal{F}$). The metric structure $J$ induces a $\mathbb{Z}_2$-grading of $F_C$, and if $\nabla^{F_C} J$ denotes the even part of the extension of $\nabla^F$ to
If $F$, then $p(\nabla^F, J) := \text{ch}(\nabla^{F \oplus J}) = \text{tr}_s \exp(-R^{F \oplus J}/2\pi i)$ is the characteristic form representing \text{ch} \circ b([F, q])$. The semigroup operation is given by

$$(\mathcal{F}, q, J, \rho) + (\mathcal{F}', q', J', \rho') := (\mathcal{F} \oplus \mathcal{F}', q \oplus q', J \oplus J', \rho + \rho') .$$

We form $\bar{\mathcal{L}}(M)$ by introducing the relation lagrangian reduction. Let $\mathcal{L} \subset \mathcal{F}$ be a locally constant lagrangian subsheaf and $L := \text{bundle}(\mathcal{L})$. Then we have a decomposition $F = L \oplus J(L)$. Let $\nabla^\oplus$ be the part of $\nabla^{F \oplus J}$ which preserves this decomposition, and let $\tilde{\text{ch}}(\nabla^\oplus, \nabla^{F \oplus J})$ be the transgression Chern form such that $d\tilde{\text{ch}}(\nabla^\oplus, \nabla^{F \oplus J}) = \text{ch}(\nabla^\oplus) - \text{ch}(\nabla^{F \oplus J})$. Note that $\text{ch}(\nabla^\oplus) = 0$. In $\bar{\mathcal{L}}(M)$ we require

$$(\mathcal{F}, q, J, \rho) \sim (0, 0, 0, \rho + \tilde{\text{ch}}(\nabla^\oplus, \nabla^{F \oplus J})) .$$

By $[\mathcal{F}, q, J, \rho]$ we denote the class in $\bar{\mathcal{L}}$ represented by $(\mathcal{F}, q, J, \rho)$.

Remark: In [9] similar functors $\bar{L}^{\text{Lott}}$ were defined replacing the relation ”lagrangian reduction” by ”hyperbolic reduction” and ”change of metric structure”. Lott’s relation is smaller than ours, and we have a natural surjective map $\bar{L}^{\text{Lott}}(M) \rightarrow \bar{L}(M)$.

The graded module structure of $\bar{L}(X)$ over $L(X)$ is defined by

$$[\mathcal{F}, q, J, \rho] \bullet [\mathcal{E}, p] := \frac{\sqrt{(-1)^{c} \sqrt{(-1)^{c}}} \cdot \sqrt{(-1)^{c}}}{\sqrt{(-1)^{c+c}}} [\mathcal{F} \otimes \mathcal{E}, q \otimes p, J \otimes J^E, \rho \wedge \text{ch}(\nabla^{E \oplus J})] ,$$

where $J^E$ is any metric structure on $\text{bundle}(\mathcal{E})$.

### 2.2 Functorial properties

If $f : M \rightarrow N$ is a smooth map of manifolds, then we obtain an induced map $f^* : \bar{L}(N) \rightarrow \bar{L}(M)$, which is given on generators by pull-back of structures. $f^*$ only depends on the smooth homotopy class of $f$ and is compatible with the $L(M)$-module structures.

We define natural maps $H^{od}(M, \mathbb{R}) \rightarrow \bar{L}(M)$ and $\bar{L}(M) \rightarrow L(M)$ by $[\rho] \mapsto [0, 0, 0, \rho]$ and $[\mathcal{F}, q, J, \rho] \mapsto [\mathcal{F}, q]$. Then we have the following exact sequence of $L(M)$-modules

$$H^{od}(M, \mathbb{R}) \rightarrow \bar{L}(M) \rightarrow L(M) \rightarrow H^{ev}(M, \mathbb{R})$$

(see [9], Prop. 21, for a similar argument).

**Proposition 2.1** The map $H^{od}(M, \mathbb{R}) \rightarrow \bar{L}(M)$ is injective.

Indeed, let $\omega \in \Omega^{od}(M)$ be a closed form. If $(0, 0, 0, \omega) \sim 0$ in $\bar{L}(M)$, there exists $(\mathcal{F}, q, J, \rho)$ together with two lagrangian subsheaves $\mathcal{L}_0, \mathcal{L}_1$ such that

$$[\omega] = [\tilde{\text{ch}}(\nabla^\oplus, \nabla^{F \oplus J}) - \tilde{\text{ch}}(\nabla^\oplus, \nabla^{F \oplus J})] = [\tilde{\text{ch}}(\nabla^\oplus, \nabla^{F \oplus J})]$$
2.2 Functorial properties

in $H^{\text{odd}}(M)$, where $\nabla^{\oplus_i}, i = 0, 1$ are defined using the decompositions $F = L_i \oplus J(L_i)$. The right-hand side belongs to $\text{ch}(K^{-1}(X))$ and is therefore rational. On the other hand it depends continuously on $J$. Therefore it is independent of $J$. First reducing to the case that $L_0 \cap L_1 = \{0\}$ and then choosing $J$ such that $JL_i = L_{1-i}, i = 0, 1$, we see that $[\omega] = 0$.

We now construct a natural morphism of $L(M)$-modules $\bar{\gamma} : \bar{L}(M) \to K^{-1}_{R/Z}(M)$. Here we use the definition of $K^{-1}_{R/Z}(M)$ in terms of generators and relations given in [8], Def. 5 + 6. A generator of $K^{-1}_{R/Z}(M)$ is a tuple $(E, h^E, \nabla^E, \rho)$, where $E$ is a $\mathbb{Z}_2$-graded complex vector bundle of virtual dimension zero, $h^E$ is a hermitian metric and $\nabla^E$ is a metric connection, both being compatible with the grading, and $\rho \in \Omega^{\text{odd}}(M)/\text{im}(d)$ satisfies $d\rho = \text{ch}(\nabla^E)$. The relations of $K^{-1}_{R/Z}(M)$ are generated by

1. **isomorphism** $(E, h^E, \nabla^E, \rho) \sim (E', h^{E'}, \nabla^{E'}, \rho)$ if there exists an isomorphism from $E$ to $E'$ which is compatible with metrics and connections.

2. **direct sum** $(E, h^E, \nabla^E, \rho) + (E', h^{E'}, \nabla^{E'}, \rho') = (E \oplus E', h^{E \oplus E'}, \nabla^{E \oplus E'}, \rho + \rho')$

3. **change of connections** $(E, h^E, \nabla, \rho) \sim (E, h^E, \nabla', \rho')$ if $\rho' = \rho + \tilde{\text{ch}}(\nabla', \nabla)$.

4. **trivial elements** If $(E, h^E, \nabla^E)$ is a $\mathbb{Z}_2$-graded hermitian vector bundle with connection, then $(E \oplus E^{op}, h^{E \oplus E}, \nabla^{E \oplus E}, 0) \sim 0$, where $E^{op}$ denotes $E$ with the opposite grading.

Let $[E, h^E, \nabla^E, \rho]$ denote the class of $(E, h^E, \nabla^E, \rho)$ in $K^{-1}_{R/Z}(M)$.

We define $\bar{L}(M) \to K^{-1}_{R/Z}(M)$ by $\bar{\gamma} [\mathcal{F}, q, J, \rho] = [F_C, h^{F_C}, \nabla^{F_C}, J, \rho]$, where $h^{F_C}$ is the hermitian extension of the metric $J^* \circ Q$ on $F$.

The following diagram commutes

$$
\begin{array}{ccc}
\bar{L}(M) & \to & L(M) \\
\downarrow & & \downarrow \\
K^{-1}_{R/Z}(M) & \to & K^0(M)
\end{array}
$$

We therefore obtain a natural map $\bar{L}(M) \to \bar{L}^{R/Z}(M)$ which is in fact surjective.

We now extend the functor $\bar{L}$ to $\mathcal{T}_{op,\text{smooth}}$ by setting

$$\bar{L}(X) := \lim_{\to} \bar{L}(M),$$

where the limit is taken over the category of manifolds over $X$. This extension has all functorial properties discussed above.
3 Eta invariants and index maps

3.1 The $\eta$-invariant

Using the $\eta$-invariant of the twisted signature operator for a closed odd-dimensional oriented manifold $M$ in [9] Lott constructed a group homomorphism $\eta^{\text{Lott}} : \hat{L}^{\text{Lott}}(M) \to \mathbb{R}$. Its reduction modulo $\mathbb{Z}$ factors over the homomorphism $\eta^{\mathbb{R}/\mathbb{Z}} : K_{\mathbb{R}/\mathbb{Z}}^{-1}(M) \to \mathbb{R}/\mathbb{Z}$ constructed in [8], which is given by the pairing with the $K$-homology class induced by the odd signature operator. Unfortunately, the homomorphism $\eta^{\text{Lott}}$ does not factor over the quotient $\hat{L}/\mathbb{Z}$. In order to fix this for a given closed odd-dimensional oriented manifold $M$ we define an extension

$$0 \to \mathbb{Z} \to \hat{L}^e(M) \to L_e(M) \to 0,$$

such that

$$\hat{\eta} : \hat{L}^e(M) \to \hat{L}_e(M) \to K_{\mathbb{R}/\mathbb{Z}}^{-1}(M) \xrightarrow{\eta^{\mathbb{R}/\mathbb{Z}}} \mathbb{R}/\mathbb{Z}$$

lifts to $\eta : \hat{L}^e(M) \to \mathbb{R}$. Note that $\hat{L}^e(M)$ is not a functor on $M$. Here $n := \dim(M)$, $\epsilon_n := \left[\frac{n(n-1)}{2}\right] \in \mathbb{Z}_2$, and we assume that $\epsilon = 1 - \epsilon_n$.

Let $(\mathcal{F}, q)$ be a generator of $L_e(M)$ and $\mathcal{L}$ be a lagrangian locally constant subsheaf of $\mathcal{F}$. We define an integer $\tau(\mathcal{F}, q, \mathcal{L})$ by the following construction. We consider the complex of sheaves

$$K^\bullet : \mathcal{L} \xrightarrow{i} \mathcal{F} \xrightarrow{i^*} \mathcal{L}^*.$$

Let $(E_r^{\bullet\bullet}, d_r)$, $r \geq 1$ be the associated hyper cohomology spectral sequence. The duality $q$ and the pairing between $\mathcal{L}$ and $\mathcal{L}^*$ induce a duality $q_\mathcal{K} : \mathcal{K} \xrightarrow{i} \mathcal{K}^*[-2]$ (the argument "$[-2]$" indicates that $q_\mathcal{K}$ is a map of degree $-2$). The orientation of $M$ and $q$ induce a duality $q_{E_r} : E_r \xrightarrow{i} E_r^*[−2, −\dim(M)]$ (i.e. of bidegree (−2, −dim(M))). Let $N_r$ denote the total grading on $E_r$. Then we define the symmetric form $Q_r$ on $E_r$ by $Q_r(v, w) = q_{E_r}((−1)^{(N_r(N_r−1))}v)(d_rw)$. The integer $\tau(\mathcal{F}, q, \mathcal{L})$ is now given by

$$\tau(\mathcal{F}, q, \mathcal{L}) := 2(\text{sign}(Q_1) + \text{sign}(Q_2)).$$

We first define an extension $L^e_r(M)$ of $L_e(M)$. A generator of the semigroup group $\hat{L}^e_r(M)$ is a triple $(\mathcal{F}, q, z)$ consisting of a locally constant sheaf of finite-dimensional $\mathbb{R}$-modules, an $\epsilon$-symmetric duality structure $q$, and an integer $z$. The semigroup operation is given by

$$(\mathcal{F}, q, z) + (\mathcal{F}', q', z') := (\mathcal{F} \oplus \mathcal{F}', q \oplus q', z + z').$$

Then we form $L^e_r(M)$ by introducing the relation lagrangian reduction

$$(\mathcal{F}, q, z) \sim (0, 0, z + \tau(\mathcal{F}, q, \mathcal{L})).$$
if $\mathcal{F}$ admits a locally constant lagrangian subsheaf $\mathcal{L}$.

There is an exact sequence

$$0 \to \mathbb{Z} \to L^\text{ex}_c(M) \to L_c(M) \to 0,$$

where the maps are the obvious inclusion and projection. We now define $\bar{L}^\text{ex}_c(M)$ by the following pull-back diagram:

$$\bar{L}^\text{ex}_c(M) \to L^\text{ex}_c(M) \downarrow \downarrow \bar{L}_c(M) \to L_c(M)$$

Then we have exact sequences

$$0 \to \mathbb{Z} \to \bar{L}^\text{ex}_c(M) \to \bar{L}_c(M) \to 0$$

and

$$H^{4s-\epsilon}(M, \mathbb{R}) \to \bar{L}^\text{ex}_c(M) \to L^\text{ex}_c(M) \to H^{4s+1-\epsilon}(M, \mathbb{R})$$

in a natural way.

An element of $\bar{L}^\text{ex}_c(M)$ can be represented by a tuple $(\mathcal{F}, q, J^F, \rho, z)$. We obtain a class $[\mathcal{F}, q, J^F, \rho] \in \bar{L}_c^{\text{Lott}}(M)$. If we set

$$\eta(\mathcal{F}, q, J^F, \rho, z) := \eta^{\text{Lott}}([\mathcal{F}, q, J^F, \rho]) - z,$$

then we have the following result.

**Theorem 3.1** The map $(\mathcal{F}, q, J^F, \rho, z) \mapsto \eta(\mathcal{F}, q, J^F, \rho, z)$ induces a well-defined homomorphism $\eta : \bar{L}^\text{ex}_c(M) \to \mathbb{R}$ which lifts $\bar{\eta}$.

We must show that $\eta$ is well-defined with respect to lagrangian reduction. The idea is to consider the $\eta$-invariant of the signature operator on $M$ twisted with the complex $K = \text{bundle}(\mathcal{K})$ and to rescale its differential. The corresponding adiabatic limits can be understood by the methods developed in [3] and [8] without investing essentially new ideas.

### 3.2 The secondary index map

Let $M \to B$ be a smooth locally trivial fibre bundle with even-dimensional closed fibres over a compact base $B$ such that the vertical bundle $TM/B$ is oriented. We set $n := \dim(TM/B)$.

In [2] Lott constructed a secondary index map $\pi^L_{\text{Lott}} : \bar{L}^\text{Lott}_c(M) \to L^{\text{Lott}}_{c+\epsilon_n}(B)$ which fits into the commutative diagram

$$
\begin{align*}
H^{4s-\epsilon}(M, \mathbb{R}) & \to \bar{L}^\text{Lott}_c(M) \to \bar{L}^{\mathbb{R}/\mathbb{Z}}_c(M) \\
\pi^\text{sign} \downarrow & \hspace{1cm} \pi^\text{Lott} \downarrow & \hspace{1cm} \pi^{\mathbb{R}/\mathbb{Z}} \downarrow \\
H^{4s-\epsilon+\epsilon_n}(M, \mathbb{R}) & \to \bar{L}^\text{Lott}_{c+\epsilon_n}(B) \to \bar{L}^{\mathbb{R}/\mathbb{Z}}_{c+\epsilon_n}(B)
\end{align*}
$$
Theorem 3.2  By passing to quotients the map $\pi_*^{L,Lott}$ induces a well-defined secondary index map

$$\pi^\bar{L}_* : \bar{L}_{\epsilon}(M) \to \bar{L}_{\epsilon+\epsilon_n}(B).$$

The idea of the proof consists in investigating the adiabatic limits of the $\eta$-form of the fibrewise signature operator twisted with the complex $K$ under rescaling the differential. The arguments are similar to the case of analytic torsion forms. It suffices to adapt the methods developed in [4], [1], [2], [10], [11], [12], [13].

The following diagram commutes

$$H^{4*}(-1)^\epsilon(M,\mathbb{R}) \xrightarrow{\pi_*^{sign}} \bar{L}_{\epsilon}(M) \xrightarrow{\pi^\bar{L}_*} L_{\epsilon}(M)$$

$$H^{4*}(-1)^{\epsilon+\epsilon_n}(M,\mathbb{R}) \xrightarrow{\pi_*} \bar{L}_{\epsilon+\epsilon_n}(B) \xrightarrow{\pi^L_*} L_{\epsilon+\epsilon_n}(B).$$

The secondary index map is natural with respect to pull-back of fibre bundles. Furthermore, if $\pi_1 : X \to X_1$ and $\pi_2 : X_2 \to B$ is an iterated bundle with oriented vertical bundles, and $\pi = \pi_2 \circ \pi_1$, then

Theorem 3.3  $\pi^\bar{L}_* = (\pi^L_2)_* \circ (\pi^L_1)_*$

This follows from the functoriality of the $\eta$-form studied in [11] in a similar way as the functoriality of the secondary index in [5] was deduced from the functoriality of the higher analytic torsion form [10].

3.3 The index map for $L^{ex}$ and $\bar{L}^{ex}$

Let $\pi : M \to B$ as above and assume that $B$ is oriented, closed, of dimension $m$. Furthermore, assume that $\epsilon = 1 - \epsilon_n - \epsilon_m$. Let $(\mathcal{F}, q)$ be a generator of $L_{\epsilon}(M)$. We define an integer $\tau(\mathcal{F}, q, M \xrightarrow{\pi} B)$ by the following construction. Let $(E^{**}, d_r)$ be the Leray-Serre spectral sequence converging to sheaf cohomology $H^*(M, \mathcal{F})$ with $E_r^{p,q} = H^p(B, R^q\pi_*\mathcal{F})$. The orientations of $TM/B$ and $B$ and the duality $q$ induce dualities $q_{Er} : E_r \to E_r^*[\epsilon - m, -n]$ which give the Poincaré duality on the limit. We define the symmetric form $Q_r$ on $E_r$ by $Q_r(v, w) = q_{Er}((-1)^{Ne_r(Nr-1)/2})v)(d_r w)$, where $N_r$ denotes the total degree. Then we set

$$\tau(\mathcal{F}, q, M \xrightarrow{\pi} B) := 2\sum_{r \geq 2} \text{sign}(Q_r).$$

Theorem 3.4  1. The prescription

$$\pi^{L^{ex}}_*[\mathcal{F}, q, z] := [H^*R\pi_*\mathcal{F}, \pi_*(q), z - \tau(\mathcal{F}, q, M \xrightarrow{\pi} B)].$$

defines an extended index map $\pi^{L^{ex}}_* : L^{ex}_\epsilon(M) \to L^{ex}_{\epsilon+\epsilon_n}(B)$. 

2. The extended index map is functorial with respect to iterated fibre bundles.

The following diagram

\[
\begin{array}{c}
\tilde{L}_\epsilon(M) \rightarrow L_\epsilon(M) \leftarrow L^\epsilon_{ex}(M) \\
\pi^L_* \downarrow \quad \pi^L_* \downarrow \quad \pi^L_* \downarrow \\
\tilde{L}_{\epsilon+\epsilon_n}(B) \rightarrow L_{\epsilon+\epsilon_n}(B) \leftarrow L^\epsilon_{ex+\epsilon_n}(B)
\end{array}
\]

commutes and induces an extended secondary index map \( \pi^L_* : \tilde{L}^\epsilon_{ex}(M) \rightarrow \tilde{L}^\epsilon_{ex+\epsilon_n}(B) \) which is functorial with respect to iterated fibre bundles. We have the following compatibility of this extended secondary index map with the \( \eta \)-homomorphism.

**Theorem 3.5**

\[
\begin{array}{c}
\tilde{L}^\epsilon_{ex}(M) \xrightarrow{\eta} \mathbb{R} \\
\pi^L_* \downarrow \quad \| \\
L^\epsilon_{ex+\epsilon_n}(B) \xrightarrow{\eta} \mathbb{R}
\end{array}
\]

This theorem essentially follows from Dai’s adabatic limit formula for the \( \eta \)-invariant \( \mathbb{F} \).

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