This tutorial review provides a guiding reference to researchers who want to have an overview of the large body of literature about graph spanners. It reviews the current literature covering various research streams about graph spanners, such as different formulations, sparsity and lightness results, computational complexity, dynamic algorithms, and applications. As an additional contribution, we offer a list of open problems on graph spanners.

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1 INTRODUCTION

Given a graph $G$, a graph spanner is, informally, a subgraph that preserves lengths of shortest paths in $G$ up to some amount of distortion (typically multiplicative and/or additive). Computing sparse or low weight graph spanners has theoretical and practical applications in various network design problems, including those in distributed computing and communication networks [15, 29]. Additionally, many graph spanner problems can be seen as generalizations of other well-known problems (e.g., the minimum spanning tree or minimum Steiner tree problems are special cases where the distortion is allowed to be arbitrarily large). This document provides a survey on the current literature of graph spanners, including different problem variants, hardness results, common techniques, and related open problems. The focus is on a method-based tutorial style which will allow those unfamiliar with the area to survey the problems considered and typical techniques used in the literature to compute spanners.

1.1 Notations

Graphs will be denoted $G = (V, E)$ containing $|V| = n$ vertices and $|E| = m$ edges. Graphs are assumed to be undirected and connected unless stated otherwise. We denote by $w_e = w_{u,v}$ the weight of edge $e$ (for weighted graphs), and we denote by $d_G(u, v)$ the weight of a minimum-weight $u$-$v$ path in $G$ (or, the number of edges in a shortest path if $G$ is unweighted). Given a (not necessarily proper) subset of edges, $E \subseteq E$, we denote its weight by $W(E) := \sum_{e \in E} w_e$. Thus we may write $d_G(u, v) := \min\{W(\bar{E}) : \bar{E} \text{ forms a path from } u \text{ to } v\}$. For any discrete set, $S$, $|S|$ will denote its size, or number of elements. Given two sets $A, B \subseteq V$, we define the distance from $A$ to $B$ in $G$ via $\text{dist}_G(A, B) = \min\{d_G(u, v) : u \in A, v \in B\}$, and if $A = \{v\}$ for a single $v$, we reduce the notation to $\text{dist}_G(v, B)$. The diameter of $G$ is given by $\text{diam}(G) = \max_{u,v \in V} d_G(u, v)$.

Recall that $f(n) = O(g(n))$ if there exist constants $C, n_0 > 0$ such that $f(n) \leq Cg(n)$ for $n \geq n_0$. Further, $f = \tilde{O}(g)$ if $f = O(g \text{ polylog } g)$. Additionally, $f = O_{\epsilon}(g)$ if $f = O(\text{poly}(\epsilon)g)$, where $\text{poly}(\epsilon)$ is a polynomial in $\epsilon$; this indicates that $f = O(g)$ for fixed $\epsilon$, which is not necessarily the case if $\epsilon$ is allowed to vary.

1.2 Layout

Throughout this survey, we use an annotated bibliography style in which we put important references to each subsection within the main text; as there are a great many references, this is done to enhance the readability of the survey and to make it easier for the reader to find references quickly on a given topic without having to go back and forth to the end of the paper. Section 2 gives a precise definition of the different types of spanners considered in the literature as well as classical results and relationships to well-known quantities such as minimum spanning trees and Erdős’ Girth Conjecture. Sections 3 and 4 detail the complexity and hardness of approximation for various spanner problems. Sections 5–9 comprise the bulk of the survey and describe the main techniques brought to bear on spanner problems to date including greedy algorithms, clustering + path buying algorithms, probabilistic constructions, and exact ILP formulations. The emphasis of these sections is on illustrating the methods and inclusion of proofs to give the main ideas, as well as on posing open problems related to the works discussed. Section 10 concerns distributed algorithms. Section 11 details the lower bounds established on size of spanners. Section 12 briefly shows other kinds of spanners considered in the literature, while Section 13 discusses spanners when the underlying graph is allowed to change. Section 14 discusses spanners for restricted classes of graphs, and Section 15 ends the paper with some of the numerous applications

\footnote{polylog $n$ denotes a polynomial in terms of $\log n$, i.e., $a_k (\log n)^k + \ldots + a_1 \log n + a_0$.}
of graph spanners. At the end of the paper, there are several summary tables for guarantees on the size and weight of spanners based on the type of problem.

2 GRAPH SPANNERS

Given a graph \( G \), possibly with edge weights, a graph spanner is a subgraph \( G' \) that preserves lengths of shortest paths in \( G \) up to some error (e.g., additive or multiplicative error). There are several definitions and formulations of graph spanners. Generically, a spanner has two parameters: a distortion function \( f \) which characterizes how much distances from the original graph are allowed to be distorted, and a subset \( P \subseteq V \times V \) which prescribes pairs of vertices of the initial graph for which the distances need to be preserved. That is, a subgraph \( G' = (V', E') \) is a spanner with distortion \( f \) for \( P \subseteq V \times V \) if

\[
d_{G'}(u, v) \leq f(d_G(u, v)), \quad (u, v) \in P.
\]

It is important to note that it is always true that \( d_G(v, u) = d_G(u, v) \) for all vertices \( u \) and \( v \) and every subgraph \( G' \); it follows that we cannot allow any distortion satisfying \( f(x) < x \) for any \( x \).

The typical formulations are as follows; first we illustrate typical choices of distortions \( f \) (from most to least general).

- **Multiplicative** (or \( t \)): distortion given by \( f(x) = tx \) for \( t \geq 1 \). That is, distances are stretched by no more than a factor of \( t \). A multiplicative \( t \)-spanner is a \((t, 0)\)-spanner. Note that some sources use \( k \) to denote multiplicative stretch.

- **Additive**: distortion given by \( f(x) = x + \beta \). In other words, distances in additive graph spanners are not elongated more than \( \beta \) units. These subgraphs preserve long distances with ratio close to 1. Additive spanners are sometimes called \(+\beta\)-spanners, and are evidently \((1, \beta)\)-spanners.

- **Linear** (or \((\alpha, \beta))\): distortion given by \( f(x) = \alpha x + \beta \) for \( \alpha, \beta \geq 0 \).

- **Sublinear**: distortion given by \( f(x) = x + o(x) \). In particular, distortions of the form \( f(x) = x + O(x^{1-\frac{1}{k}}) \) for positive integers \( k \) are of particular interest. Roughly, this is because this distortion function arises as an information-theoretic barrier to compressing graph distances; that is, the most space-efficient data structures that approximate graph distances have essentially these error functions (but it is still open whether they are right for spanners; see [4] or Section 12.1 for further discussion). Like linear error, sublinear error can be viewed as a compromise between multiplicative and additive distortion, as it stretches small distances up to a constant multiplicative factor while the stretch factor for long distances approaches 1.

- **Distance Preservers**: no distortion, i.e. \( f(x) = x \). These are \((1, 0)\)-spanners.

Next, we illustrate different terminologies based on the subset \( P \subseteq V \times V \) chosen (from most to least general).

- **Pairwise Spanners**: the given spanner condition holds only for specific pairs of nodes \( P \subseteq V \times V \) (note that \( P \) does not have to be symmetric).

- **Sourcewise Spanners**: the given spanner condition holds for \( P = S \times V \) for some subset \( S \subseteq V \).

- **S–T (Source–Target) Spanners**: the given spanner condition holds for \( P = S \times T \) for subsets \( S, T \subseteq V \) (possibly disjoint).

- **Subsetwise Spanners**: the given spanner condition holds between all vertices in a fixed subset \( S \subseteq V \). That is, subsetwise spanners are pairwise spanners in the case that \( P = S \times S \subseteq V \times V \).
**Spanners:** with no additional terminology, typically it is implied that \( P = V \times V \), i.e. distances are approximately preserved over the entire graph.

For almost all spanner problems, we may specify both the distortion parameters \((\alpha, \beta)\) and the pairs of nodes for the spanner condition in the manner prescribed above. To make notation concise, we typically write \((\alpha, \beta, P)\)–spanner, or \((f, P)\)–spanner for general distortions.

Note also that while all variants of spanners are well-defined for weighted and unweighted graphs, it is somewhat more natural to consider multiplicative spanners when the graph is edge-weighted, and additive spanners when the graph is unweighted. Indeed, if a graph is weighted but with large edge weights, then it may be that for many values of \( \beta \), there is no additive \( \beta \)–spanner for the graph.

### 2.1 Classical Results

[116] David Peleg and Alejandro A Schäffer. Graph spanners. *Journal of graph theory*, 13(1):99–116, 1989

[10] Ingo Althöfer, Gautam Das, David Dobkin, Deborah Joseph, and José Soares. On sparse spanners of weighted graphs. *Discrete & Computational Geometry*, 9(1):81–100, 1993

[104] Guy Kortsarz and David Peleg. Generating sparse 2-spanners. *J. Algorithms*, 17(2):222–236, 1994

The classical, and most common, spanner problem is the multiplicative \( t \)–spanner problem, which may be written as follows.

**Problem 1** (\( t \)–spanner Problem). Given a connected graph \( G = (V, E) \) and a fixed \( t \geq 1 \), find a subset \( E' \subset E \) such that the subgraph \( G' = (V, E') \) of \( G \) satisfies

\[
d_{G'}(u, v) \leq t \cdot d_G(u, v), \quad \text{for all } u, v \in V.
\]

The notion of a \( t \)–spanner was introduced by Peleg and Schäffer in [116] (see also Peleg and Ullman [117]), though the idea also appeared implicitly in earlier work of Awerbuch [15] and Chew [57]. Here, the parameter \( t \) is called the **stretch factor** of the spanner. Peleg and Schäffer [116] show that for unweighted graphs, determining if a \( t \)–spanner of \( G \) containing at most \( m \) edges exists is NP–complete. They also discuss at length a reduction of the \( t \)–spanner problem to particular classes of graphs – chordal graphs – and show that the generic lower bounds for the number of edges required to form a \( t \)–spanner for an arbitrary graph may be significantly improved for restricted graph classes.

One important note is that verifying that \( G' \) is a \( t \)–spanner of \( G \) does not require that one checks (1) for all pairs \( u, v \in V \), but only that one verifies the inequality for every edge \((u, v) \in E \), which is the content of the following proposition.

**Proposition 2.1** ([116, Lemma 2.1]). \( G' \) is a \( t \)–spanner of \( G \) if and only if \( d_{G'}(u, v) \leq t \cdot d_G(u, v) \) for all \((u, v) \in E \).

**Proof.** The forward direction is obvious. For the reverse direction, let \( u \) and \( v \) be distinct vertices in \( V \), and let \( u_0 u_1 \ldots u_m \) be a shortest \( u \)-\( v \) path in \( G \), where \( u_0 = u \) and \( u_m = v \). Then note that by assumption,

\[
d_{G'}(u, v) \leq \sum_{i=0}^{m-1} d_{G'}(u_i, u_{i+1}) \leq t \cdot \sum_{i=0}^{m-1} d_G(u_i, u_{i+1}) = t \cdot d_G(u, v).
\]

Thus, one can check the spanner inequality (1) for \(|E| \) pairs of vertices rather than for all \( \binom{|V|}{2} \) vertex pairs to verify that a given subgraph is a \( t \)–spanner.
2.2 Sparsity and Lightness

Althöfer et al. [10] discuss a specific aspect of the $t$–spanner problem, namely sparsity. Implicit in the formulation of the problem is the idea that a good $t$–spanner for a graph should contain very few edges while still preserving distances in the manner prescribed. Thus the sparsity of a spanner may be quantified by

$$\text{sparsity}(G') = \frac{|E'|}{|E|}.$$  

The authors of [10] prove many lower and upper bounds for general graphs on the sparsity of a $t$–spanner, and also give a simple polynomial time algorithm for producing such a sparse spanner. One advantage of their method is that the algorithm presented provides a $t$–spanner not only with few edges, but also whose weight is comparable to the weight of the minimum spanning tree (in the case that $G$ is a weighted graph). We will further discuss their results in Section 5.

For weighted graphs, a more natural consideration is the lightness of a spanner, which is the total weight of the spanner. We may thus define

$$\text{lightness}(G') = \frac{W(E')}{W(E)}.$$  

Evidently for any subgraph $G'$ of $G$,

$$0 \leq \text{sparsity}(G'), \text{lightness}(G') \leq 1.$$  

Often we are interested in finding the sparsest or lightest\(^2\) spanner of a given graph:

**Problem 2** (Sparsest/Lightest Spanner Problem). Given a graph $G = (V, E)$, distortion $f$, and $P \subset V \times V$, find $G' = (V', E')$ such that

1. $G'$ is a $(f, P)$–spanner for $G$, and
2. $\text{sparsity}(G') \leq \text{sparsity}(H)$ for all other $(f, P)$–spanners $H$ if $G$ is unweighted, or
3. $\text{lightness}(G') \leq \text{lightness}(H)$ for all other $(f, P)$–spanners $H$ if $G$ is weighted.

Kortsarz and Peleg [104] show that the problem of finding the sparsest 2–spanner of an unweighted graph $G = (V, E)$ admits a polynomial time $\log \left( \frac{|E|}{|W|} \right)$–approximation. Note that this problem is equivalent to finding an edge set $E' \subset E$ such that for every edge $e \in E$ not in the spanner, there is a triangle (3–cycle) in $G$ containing $e$ whose remaining two edges belong to the edge set $E'$.

To describe their approximation algorithm, note that given $U \subset V$, the density of $U$, denoted $\rho_G(U)$, is defined by $\rho_G(U) = \frac{|E(U)|}{|U|}$, where $(U, E(U))$ is the subgraph of $G$ induced by $U$. The maximum density problem is to find vertex subset $U \subset V$ such that $\rho_G(U)$ is maximized, which can be solved in polynomial time.

The algorithm maintains three sets of edges: $H^s$, the set of edges in the spanner; $H^c$, the set of “covered edges” (edges either in the spanner, or edges on a triangle containing two edges in $H^s$); $H^u$, the remaining unspanned edges. Given a set $H^u$ of unspanned edges and vertex $v$, the authors denote by $N(H^u, v)$ the subgraph of $G$ whose vertex set $N(v)$\(^3\), and whose edge set is the set of edges in $G$ induced by $N(v)$, which are also unspanned (i.e., $E(N(v)) \cap H^u$). The authors denote by $\rho(H^u, v)$ the maximum density of this neighborhood graph over all vertices $v \in G$.

The idea of the algorithm is: while $\rho(H^u, v) > 1$, find a vertex $v \in G$ such that $\rho(H^u, v)$ is maximum. Then within this restricted neighborhood graph, solve the maximum density problem to find a corresponding dense subset $U_v$ of neighbors of $v$. Add the edges from $v$ to each $u \in U(v)$

\(^2\)Note that some of the literature defines lightness($G'$) := $\frac{W(E')}{W(MST(G'))}$.

\(^3\)the set of neighbors of $v$ in $G$
to the set of spanner edges $H^s$. The edges “covered” by the newly-added edges in $H^s$ are added to $H^c$, and $H^s$ also updates accordingly.

Since at least one edge is added to $H^c$ at each iteration, the number of iterations is at most $m$. The authors show that, using the maximum density problem as a subroutine, a sparse 2-spanner with approximation ratio $O(\log (\frac{|V|}{m})) = O(\log n)$ can be found in $O(m^2n^2 \log (\frac{\Delta^2}{m}))$ time.

### 2.3 Relation to Minimum Spanning Trees

Recall that $G'$ is a spanning tree of $G$ if $G'$ is connected, acyclic, and spans all vertices of $G$. The minimum spanning tree (MST) problem is to compute a spanning tree of minimum total weight. This can easily be done in polynomial time (e.g., using Kruskal’s or Prim’s algorithm). We use $W(MST(G))$ to denote the weight of an MST of $G$.

Note that the MST of a graph can be an arbitrarily poor spanner; for example, consider an unweighted graph $G$ which contains a Hamiltonian path. Any spanning tree of $G$ is an MST of $G$, including the edges of the Hamiltonian path. However, this path is only a $(|V| - 1)$-spanner. The MST problem can be interpreted as a special case of the $t$-spanner problem where $t$ is arbitrarily large. Some lightness bounds, e.g., those given in [10] for the greedy spanner, are in terms of $W(MST(G))$.

### 2.4 Erdős’ Girth Conjecture

[86] Paul Erdős. Extremal problems in graph theory. In Proceedings of the Symposium on Theory of Graphs and its Applications, page 2936, 1963

[85] P. Erdős and M. Simonovits. Some extremal problems in graph theory. In Combinatorial theory and its applications, I (Proc. Colloq., Balatonfüred, 1969), pages 377–390. North-Holland, Amsterdam, 1970

[110] Merav Parter. Bypassing Erdős’ girth conjecture: Hybrid stretch and sourcewise spanners. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, Automata, Languages, and Programming, pages 608–619, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg

Recall that the girth of a graph $G$ is defined as the length of the smallest cycle in $G$, and is infinity if $G$ is acyclic. Letting $\gamma(n, k)$ denote the maximum possible number of edges in an $n$-node graph with girth $> k$, the authors in [10] give a simple algorithm that, for any $n$-node (undirected, possibly weighted) input graph and positive integer $t$, produces a $t$-spanner on $|E'| \leq \gamma(n, t + 1)$ edges. This is best possible in the following sense:

**Proposition 2.2.** An undirected unweighted graph $G = (V, E)$ of girth $> t + 1$ has no proper subgraph that is a $t$-spanner.

**Proof.** Consider any edge $(u, v) \in E$. If $(u, v)$ is removed from $G$, then $d_G(u, v)$ changes from 1 to one less than the length of the shortest cycle containing $(u, v)$ in the original graph. Hence, if $G$ has girth greater than $t + 1$, then $d_{G'}(u, v) > t$, so the spanner property is not satisfied for $(u, v)$ after $(u, v)$ is removed.

Thus, if we consider an $n$-node graph $G = (V, E)$ with girth greater than $t + 1$ and $|E| = \gamma(n, t + 1)$ edges, the only $t$-spanner of $G$ is $G$ itself. It follows that no algorithm can improve in general on the density bound of [10] with, say, a bound of $|E'| \leq \gamma(n, t + 1) - 1$.

It still remains a major open problem to determine $\gamma(n, k)$, even asymptotically, though upper bounds called the Moore Bounds are given by a folklore counting argument:

**Proposition 2.3.** $\gamma(n, k) = O\left(n^{1 + \frac{1}{\log^2 k}}\right)$. 
**Sketch of Proof.** See [10] for full detail. Let $G = (V, E)$ be an $n$-node graph of girth $> k$ and let $d$ be its average node degree. There are two steps in the proof. First, we find a nonempty subgraph $H \subset G$ of minimum degree $\Omega(d)$. We generate $H$ by iteratively deleting any node in $G$ of degree $\leq \frac{d}{2}$; one counts that at most $\frac{|E|}{2}$ edges are deleted in total, so $H$ remains nonempty. Second, pick an arbitrary node $v$ in $H$ and consider a BFS Tree, $T$, rooted at $v$ to depth $\lfloor \frac{k}{2} \rfloor$. Since $H$ has girth $> k$, each node in $T$ can have only one edge to a node in the same or previous layer of the tree (else we would have a cycle of length $\leq k$). Since every node in $T$ has degree $\Omega(d)$, one counts that there are $\Omega(d)^{\lfloor \frac{k}{2} \rfloor}$ total nodes in $T$. Thus $d = O(n^{\frac{1}{2\sqrt{k+\gamma}}})$, so $|E| = O\left(n^{1+\frac{1}{2\sqrt{k+\gamma}}}\right)$. □

Erdős’ Girth Conjecture is the statement that the Moore Bounds are tight; that is, $\gamma(n, k) = \Omega(n^{\frac{1}{k-\frac{1}{2}}} \frac{1}{\sqrt{k+1}})$. The conjecture generally plays two important roles in the literature on spanners. First, in some applications of spanners it is useful to know the number of edges the spanner could possibly have, and this currently requires the Girth Conjecture. Second, there are some known constructions of $t$-spanners on $O(n^{1+\frac{1}{(2t+1)t+1}})$ edges with certain desirable properties relative to [10] (for example, Baswana and Sen [24] give a linear time algorithm to produce spanners of this quality). Whereas [10] produces spanners with optimal size/distortion tradeoff regardless of the truth of the Girth Conjecture, these other constructions have optimal tradeoff only if the Girth Conjecture is assumed. Thus, many natural algorithmic questions about spanners (such as linear time computation of spanners with optimal size/distortion tradeoff) are currently closed only if the Girth Conjecture is assumed.

So far, we have discussed the Girth Conjecture with respect to multiplicative spanners, but it more generally constrains spanners with additive or mixed error in the same way. Specifically, assuming the Girth Conjecture, any construction of $(\alpha, \beta)$-spanners of size $O(n^{1+\frac{1}{1+\beta}})$ must have $\alpha + \beta \geq 2k - 1$. The proof of this is essentially the same as that of Proposition 2.2. Independent of the veracity of the girth conjecture, Woodruff [134] proved this fact in the setting $\alpha = 1$: for any $k \in \mathbb{N}$, there exists a graph on $n$ nodes for which any $(1, 2k-2)$-spanner has $\Omega(k^{-1}n^{1+\frac{1}{k-1}})$ edges.

### 2.5 Open Problems

1. Considering discontinuous distortion functions would also be of interest, e.g., one could allow small distances to be distorted more than large ones, or require distance preservation of close vertices, but small distortion for far away ones.

2. Erdős’ Girth Conjecture is known to be true for $k = 1, 2, 3, \text{ or } 5$ [132, 133]. Is it true for other values of $k$?

### 3 Complexity

[116] David Peleg and Alejandro A Schäffer. Graph spanners. *Journal of graph theory*, 13(1):99–116, 1989

[107] Arthur Liestman and Thomas Shermer. Additive graph spanners. *Networks*, 23:343 – 363, 07 1993

[45] Leizhen Cai. NP-completeness of minimum spanner problems. *Discrete Applied Mathematics*, 48(2):187–194, 1994

[42] Ulrik Brandes and Dagmar Handke. NP-completeness results for minimum planar spanners. *Discrete Mathematics and Theoretical Computer Science*, 3(1), 1998

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4The conjecture comes from [86], page 5, where Erdős wrote "It seems likely that" a certain equation holds (7) which is equivalent to the statement $\gamma(n, k) = \Omega_k(n^{1+\frac{1}{k-\frac{1}{2}}} \frac{1}{\sqrt{k+1}})$. Most modern papers quote a strengthened version of this statement where the implicit constant may not even depend on $k$. 

3.1 NP–Hardness of the Sparsest $t$–spanner Problem

Peleg and Schäffer [116] show that, given an unweighted graph $G$, and integers $t, m \geq 1$, determining if $G$ has a $t$–spanner containing $m$ or fewer edges is NP–complete, even when $t$ is fixed to be 2. The reduction is from the edge dominating set (EDS) on bipartite graphs, defined as follows: given a bipartite graph $H = (V, E)$ with bipartition $(X, Y)$, and an integer $K \geq 1$, determine if $H$ has an edge dominating set consisting of at most $K$ edges. The reduction can be summarized as follows: given an instance $\langle H = (V, E), K \rangle$ to EDS where $H$ has bipartition $(X, Y)$, $X = \{x_1, \ldots, x_n\}$, and $Y = \{y_1, \ldots, y_m\}$, let $G(H)$ be constructed by adding a vertex $x_{ij}$ for each $1 \leq i < j \leq n_x$, and similarly for $Y$. Let $V(G(H)) = X \cup Y \cup \{x_{ij} : 1 \leq i < j \leq n_x\} \cup \{y_{ij} : 1 \leq i < j \leq n_y\}$.

For each $1 \leq i < j \leq n_x$, add edges $x_ix_j, x_ix_{ij}$, and $x_jx_{ij}$, and do so similarly for $Y$. The authors define $E_{XY}$ as the union of all of these added edges. Let $E(G(H)) = E \cup E_{XY}$, $t = 2$, and $m = K + n_x(n_x - 1) + n_y(n_y - 1)$. One can show that $\langle H, K \rangle$ is accepted iff $\langle G(H), t, m \rangle$ is accepted.

Cai [45] shows that for any fixed $t \geq 2$, the minimum $t$–spanner problem is NP–hard, and for $t \geq 3$, the problem is NP–hard even when the input is restricted to bipartite graphs. The reduction is from the 3–SAT problem.

Kobayashi [102] considers the minimum additive $(1, \beta)$–spanner problem which is known to be NP–hard. He formulates a parameterized version of the problem in which the number of removed edges is regarded as a parameter, and gives a fixed-parameter algorithm for it. The main result is that there exists a fixed-parameter algorithm for Parameterized Minimum Additive $(1, \beta)$–spanner problem that runs in $(\beta + 1)^{O(k^2 + \beta k)}|V||E|$ time with running time $2^{O(k^2)}|V||E|$, for fixed positive constant $\beta$. He then generalizes these results to $(\alpha, \beta)$–spanners.

3.2 NP–Hardness for Planar Graphs

Brandes and Handke [42] show that the lightest $t$–spanner problem is NP–complete for fixed $t \geq 5$ when the input graph is planar and unweighted, and is also NP–complete for fixed $t \geq 3$ when the input graph is planar and weighted. For unweighted graphs, the reduction is from planar 3–SAT, a variant of 3–SAT where the underlying bipartite graph induced by the variables and clauses, where variable vertices are connected to clause vertices if such a variable appears in that clause, is necessarily planar. Kobayashi [101] recently showed that the minimum $t$–spanner problem is NP–hard on planar graphs for $t \in \{2, 3, 4\}$.

3.3 NP–Hardness of Additive Spanners

Liestman and Shermer [107] show that the sparsest additive spanner problem (given an unweighted graph $G$, integer $\beta \geq 1$, and integer $m$, does $G$ contain an additive $(1, \beta)$–spanner with $m$ or fewer edges?) is NP–hard via a reduction from the edge dominating set (EDS) problem.

3.4 Open Problems

(1) Kobayashi [101] leaves as an open question whether the minimum $t$–spanner problem on bounded-degree graphs of degree at most $k$ is NP–hard for certain fixed $t$ and $k$, namely $(t, k) = (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)$.

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5 An edge dominating set of a graph $H = (V, E)$ is a subset of edges $E' \subset E$ such that every edge $e \in E \setminus E'$ is adjacent to at least one edge in $E'$. 
4 HARDNESS OF APPROXIMATION

[79] Michael Elkin and David Peleg. Strong inapproximability of the basic k-spanner problem. In *International Colloquium on Automata, Languages, and Programming*, pages 636–648. Springer, 2000

[81] Michael Elkin and David Peleg. Approximating k-spanner problems for $k > 2$. *Theoretical Computer Science*, 337(1-3):249–277, 2005

[103] G. Kortsarz. On the hardness of approximating spanners. *Algorithmica*, 30(3):432–450, Jan 2001

[82] Michael Elkin and David Peleg. The hardness of approximating spanner problems. *Theory of Computing Systems*, 41(4):691–729, Dec 2007

[58] Eden Chlamtác and Michael Dinitz. Lowest-degree $k$-spanner: Approximation and hardness. *Theory of Computing*, 12(15):1–29, 2016

[68] Michael Dinitz, Guy Kortsarz, and Ran Raz. Label cover instances with large girth and the hardness of approximating basic k-spanner. *ACM Transactions on Algorithms (TALG)*, 12(2):25, 2016

[48] Keren Censor-Hillel and Michal Dory. Distributed spanner approximation. In *Proceedings of the 2018 ACM Symposium on Principles of Distributed Computing*, pages 139–148. ACM, 2018

[59] Eden Chlamtác, Michael Dinitz, Guy Kortsarz, and Bundit Laekhanukit. Approximating spanners and directed steiner forest: Upper and lower bounds. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 534–553. SIAM, 2017

Elkin and Peleg [79], and Kortsarz [103] consider the hardness of approximating the basic (sparsest) $k$-spanner on undirected graphs, and show that this problem cannot be approximated with ratio $O(2^{\log^* n})$ unless $NP \subset DTIME(n^{\text{polylog } n})$ for fixed $k > 2$ (often referred to as *strong inapproximability*). However, when $k = 2$, the basic $k$-spanner problem admits an $O(\log n)$ approximation ratio, and study certain classes of graphs for which logarithmic approximation is feasible. Elkin and Peleg [82] extend the hardness results to other spanner problems, and also show strong inapproximability for the directed unweighted $k$-spanner problem. Dinitz et al. [68] provided a proof of strong hardness for the $k$-spanner problem. More specifically, they proved the super-logarithmic hardness for this problem by showing that for every $k \geq 3$ and every constant $\epsilon > 0$ it is hard to approximate the basic $k$-spanner problem within a factor better than $2^{\log^{1-\epsilon} n}/k$. Their main technique is subsampling the edges of 2-query probabilistically checkable proofs (PCPs).

In [48], Censor-Hillel and Dory address hardness of approximation for spanner problems in the distributed setting. In particular, they show that an $\alpha$-approximation for the minimum directed $k$-spanner problem (where $k \geq 5$), requires $\Omega(\frac{n}{\sqrt{\alpha \log n}})$ rounds using any deterministic algorithm, or $\Omega(\frac{\sqrt{n}}{\sqrt{\alpha \log n}})$ rounds using randomized algorithms, under the CONGEST model of distributed computing.

The article [59] provides an $O(n^{1/2 + \epsilon})$-approximation for distance preservers and pairwise spanners. This is the first nontrivial upper bound for either problem, both of which are as hard to approximate as Label Cover. The authors also connect their analysis to Steiner trees in the following way: the proposed technique yields a $O(n^{1/2 + \epsilon})$-approximation for the Directed Steiner Forest problem, but only when all edges have uniform costs. This is an improvement over the previous best $O(n^{1/2 + \epsilon})$-approximation due to Berman et al. [27]. On the other hand, the approximation due to Berman et al. is more general and holds for edge-weighted graphs.
4.1 Minimum Degree Spanner
Chlamtáč and Dinitz [58] study the problem of computing a $k$–spanner of a given unweighted graph with the objective of minimizing the maximum degree of the spanner. They show that for any integer $k \geq 3$, this problem admits no polynomial time approximation with ratio $\Delta^{\Omega(1/k)}$ unless $\text{NP} \subseteq \text{BPTIME}(2^{\text{polylog}(n)})$, where $\Delta$ is the maximum degree of the input graph.

4.2 Open Problems
Elkin and Peleg [82] leave the following open problems:

1. Is the basic $k$–spanner problem strongly inapproximable?
2. Are any variants of the $k$–spanner problem $\Omega(n^\delta)$–approximable for some $\delta > 0$?
3. Does the additive spanner problem possess the ratio degradation\(^6\) property?
4. Are there approximation results for the additive 1–spanner problem? $((\alpha, \beta) = (1, 1))$

Chlamtáč and Dinitz [58] leave the following open problem:

5. Can the approximability gap for the low-degree $k$–spanner problem be closed? Is the problem easier or harder if $k$ increases?

5 THE GREEDY ALGORITHM FOR MULTIPLICATIVE SPANNERS
[10] Ingo Althöfer, Gautam Das, David Dobkin, Deborah Joseph, and José Soares. On sparse spanners of weighted graphs. *Discrete & Computational Geometry*, 9(1):81–100, 1993

[87] Arnold Filtser and Shay Solomon. The greedy spanner is existentially optimal. In *Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing*, pages 9–17. ACM, 2016

One of the original methods for constructing spanners is to use a greedy algorithm. The essential idea is to first sort the edges in nondecreasing order by weight, choose the edge with the smallest weight first, and then each subsequent edge is chosen or not according to some criteria which guarantees that the end result is a spanner of the desired type. Despite being the oldest construction of spanners, greedy algorithms remain one of the most utilized and best methods for achieving this task.

5.1 Kruskal’s Algorithm for Computing Minimum Spanning Trees
There are many algorithms for computing MSTs, the classical ones being Borůvka’s Algorithm, Prim’s Algorithm, and Kruskal’s Algorithm. All of these algorithms are examples of greedy algorithms. Since it is the basis for the greedy algorithm to construct $t$–spanners, we present Kruskal’s Algorithm here.

**Algorithm 1** MST($G = (V, E)$); Kruskal’s MST Algorithm

```
Sort edges in nondecreasing order of weight
$G' = (V, E' \leftarrow \emptyset)$
for $(u, v) \in E$ do
    if $d_{G'}(u, v) = \infty$ (i.e. $u, v$ are previously disconnected in $G'$) then
        $E' \leftarrow E' \cup \{(u, v)\}$
end if
end for
return $G'$
```

\(^6\)Informally, the property that when the stretch requirement is relaxed, then the approximability ratio decreases exponentially.
5.2 The Greedy Algorithm for Multiplicative Spanners

Althöfer et al. [10] proposed and analyzed the first greedy algorithm for computing a sparse $t$-spanner (i.e. a multiplicative one) for a weighted graph as follows.

\begin{algorithm}
\caption{GreedySpanner($G = (V, E), t$); Greedy $t$-spanner Algorithm}
\begin{algorithmic}
\STATE Sort edges in nondecreasing order of weight
\STATE $G'$ = ($V, E'$ $\leftarrow$ $\emptyset$)
\FOR {$(u, v) \in E$}
\IF {\(d_{G'}(u, v) > t \cdot w_{(u, v)}\)}
\STATE $E'$ $\leftarrow$ $E'$ $\cup$ \{$(u, v)$\}
\ENDIF
\ENDFOR
\STATE \textbf{return} $G'$
\end{algorithmic}
\end{algorithm}

Note first, that Algorithm 2 guarantees that the lowest weighted edge is chosen. Likewise if at any stage in the for loop there is no path in $G'$ from $u$ to $v$, then the edge $(u, v)$ is chosen. This algorithm is based on Kruskal’s Algorithm, but the key difference is that Algorithm 2 does not enforce the condition that there are no cycles in the graph. Indeed, suppose edges $(u_1, u_2)$ and $(u_1, u_3)$ have been chosen already to be in $E'$, and that $(u_2, u_3)$ is an edge in the original graph $G$, and the shortest path in $G$ from $u_2$ to $u_3$ is the two-edge path going from $u_2$ to $u_1$ to $u_3$. Kruskal’s Algorithm would reject the new edge because its endpoints are already connected. However, provided $t \cdot w_{(u_2, u_3)} < w_{(u_2, u_1)} + w_{(u_1, u_3)}$, this edge would be added to $E'$.

That said, an analogous cycle-free property holds for the greedy algorithm: a graph returned by Algorithm 2 with parameter $t$ will not have any cycles on $\leq t + 1$ edges. To see this, let $C$ be a cycle on $\leq t + 1$ edges in the input graph $G$, and let $(u, v) \in C$ be the last edge considered by the greedy algorithm. When $(u, v)$ is considered, either another edge in $C$ has been discarded, or else (due to the edge ordering) there is a $u$-$v$ path through $C$ of length $\leq t \cdot w_{(u, v)}$, and thus we will discard $(u, v)$. In either case, $C$ does not survive in the output graph $G'$. Thus, one can reasonably view Kruskal’s Algorithm as the special case of Algorithm 2 with $t = \infty$.

Another note is that it is not altogether obvious that the condition checked in the if statement leads to the constructed subgraph $G'$ being a $t$-spanner for $G$. Althöfer et al. prove that any $G'$ constructed from Algorithm 2 is a $t$-spanner for $G$. However, we note here that their proof is actually not dependent upon the greedy reordering of the edges, which is an interesting fact in its own right.

**Proposition 5.1 ([10]).** Algorithm 2 yields a $t$-spanner for $G$ regardless of the ordering of the edges in the first step.

**Proof.** First, we may assume without loss of generality that for each edge $(u, v)$ in the input graph $G$, we have $d_G(u, v) = w_{(u, v)}$. Otherwise, we may remove $(u, v)$ from $G$ without changing its shortest path metric at all, and thus any spanner of the remaining graph is also a spanner of $G$ itself.

For each $(u, v) \in E$, when we consider $(u, v)$ in the greedy algorithm, we either have

$$d_{G'}(u, v) \leq t \cdot w_{(u, v)} = t \cdot d_G(u, v),$$

or else we add $(u, v)$ to $G'$ and thus have $d_{G'}(u, v) = d_G(u, v)$. In either case, $(u, v)$ satisfies the spanner property. The proposition then follows from Proposition 2.1. □
Despite the fact that Algorithm 2 yields a $t$–spanner for any ordering of the edges, the rest of the analysis of Althöfer et al. crucially hinges upon the greedy edge ordering. In [10, Lemma 2], it is shown that the output $G'$ of Algorithm 2 is such that its girth is at least $t + 1$. However, this need not hold if we allow the algorithm to run with a different edge ordering as the example in Figure 1 demonstrates.

![Graph Example](image)

**Fig. 1.** Ordering the edges by their weights is pivotal for the girth property of the greedy $t$–spanner. The rightmost graph would be output of the algorithm for $t = 3$ if we allowed non-ordered edges by picking the $(c, d)$ edge with weight 10. This graph has girth 3, which is smaller than $(t + 1) = 4$.

**Remark 5.1.** It should be noted that the greedy algorithm presented here essentially relies on Proposition 2.1, which allows one to only enforce the $t$–spanner condition on edges in the initial graph $G$. This proposition does not hold for additive spanners (for example), and so to find a greedy algorithm to produce an additive (or mixed) spanner, something else would need to be done. Indeed, [99] might be viewed as a greedy algorithm for additive spanners, but for this reason it requires an extra preprocessing step before the greedy part, and thus is not a direct analog of the multiplicative greedy algorithm discussed here.

### 5.3 Existential Optimality of Greedy $t$–spanners

Garay, Kutten, and Peleg [89] (see also [115, Chapter 24]) distinguish between different notions of optimality of a given algorithmic output. The essential difference is one of a single quantifier: namely one is a “for all” statement and the other is a “there exists” statement. An algorithmic solution is *universally optimal* for a given class of graphs $\mathcal{G}$ if for every graph $G \in \mathcal{G}$, the algorithm gives the optimal solution. On the other hand, an algorithm is *existentially optimal* for a class of graphs $\mathcal{G}$ provided there exists a graph $G \in \mathcal{G}$ for which the algorithm constructs the optimal solution.

Filtser and Solomon [87], following [10], prove existential optimality of the greedy $t$–spanner; although they prove something somewhat stronger than the notion of Garay, Kutten, and Peleg described above. They prove that the greedy $t$–spanner for a graph $G$ in a given (fixed) class of graphs $\mathcal{G}$ is never worse than the worst-case optimal solution for the whole class $\mathcal{G}$. For example, given a class of graphs $\mathcal{G}$ on $n$ vertices, the worst-case number of edges in an optimal $t$–spanner is given by

$$m(t) := \sup_{G \in \mathcal{G}} \inf_{G' = (V, E'), G' \text{ is a } t\text{-spanner for } G} |E'|.$$
Then the greedy $t$-spanner algorithm is such that for every $G \in \mathcal{G}$, the algorithm’s output is a $t$-spanner with at most $m(t)$ edges. In particular, this implies that the greedy $t$-spanner is existentially optimal because there exists a graph $G$ whose optimal $t$-spanner has $m(t)$ edges, and its greedy $t$-spanner has at most $m(t)$ edges as well, and hence is an optimal solution.

In our setting, optimality of a $t$-spanner construction will be considered via two parameters, the number of edges in the spanner, and its total weight. Filtser and Solomon prove that the greedy $t$-spanner algorithm is existentially optimal. We state their result in a stronger way here, but in such a way that the proof becomes much simpler than the original. To begin, we state a crucial lemma which is interesting in its own right, which states that the only $t$-spanner of a greedy $t$-spanner is itself.

**Lemma 5.1 ([87, Lemma 3]).** Let $t \geq 1$ be fixed, and let $G$ be any weighted connected graph with $n$ vertices. Let $G'$ be the greedy $t$-spanner of $G$. If $G''$ is a $t$-spanner of $G'$, then $G'' = G'$.

**Proof.** Suppose $G' = (V, E')$ and $G'' = (V, E'')$. By way of contradiction, suppose that there exists an edge $e = (u, v) \in E' \setminus E''$. Let $P$ be a shortest path in $G''$ connecting $u$ and $v$ (and note that $e \notin P$). Consider the last edge in $P \cup \{e\}$, say $\hat{e}$, examined by the greedy algorithm when forming $G'$. Since the edges are sorted, we have $w_\hat{e} \leq w_e$. Since $P \cup \{e\}$ lies in $G'$, and each of these edges has weight at most that of $\hat{e}$, it follows that all edges in $P \cup \{e\} \setminus \{\hat{e}\}$ have already been added to $E'$ by the time the greedy algorithm examines $\hat{e}$. Thus this set forms a path connecting the endpoints of $\hat{e}$, and we have

$$W(P) - w_\hat{e} + w_e \leq W(P) \leq tw_e \leq tw_\hat{e},$$

which implies that the greedy algorithm will not add edge $\hat{e}$ to $E'$, which is a contradiction. Hence no such $e$ exists, and $E'' = E'$.

**Theorem 5.1 ([87, Theorem 4]).** Suppose that $\mathcal{G}$ is a class of graphs on $n$ vertices which is closed under edge deletion. Let $t > 1$ be fixed. Let

$$m(t, n) := \sup_{G \in \mathcal{G}} \inf_{G' = (V, E'), \quad G' \text{ is a } t\text{-spanner for } G} |E'|,$$

and

$$\ell(t, n) := \sup_{G \in \mathcal{G}} \inf_{G' = (V, E'), \quad G' \text{ is a } t\text{-spanner for } G} W(E').$$

Then for every $G \in \mathcal{G}$, the greedy $t$-spanner, $G'$ of $G$ has at most $m(t, n)$ edges and weight at most $\ell(t, n)$.

**Proof.** The key ingredient is that since a $t$-spanner of $G \in \mathcal{G}$ can be obtained by edge deletion, it must be in $\mathcal{G}$ as well. Combining this observation with Lemma 5.1 yields the desired bounds immediately. Indeed, let $G \in \mathcal{G}$ be arbitrary, and let $G'$ be its greedy $t$-spanner. Since $G'$ can be obtained from $G$ by edge deletion, it is in $\mathcal{G}$; hence by assumption $G'$ has a $t$-spanner, say $G''$, which has at most $m(t, n)$ edges and weight at most $\ell(t, n)$. But since $G'' = G'$, the proof is complete.

For a proof of optimality of the greedy algorithm for geometric graphs, see [40].

### 5.4 Arbitrarily Bad Greedy $t$-spanners

Universal optimality of course implies existential optimality, but the reverse is patently untrue. Here, we note that not only is the greedy $t$-spanner not universally optimal, but moreover there is a whole family of graphs for which the greedy $t$-spanner is as far away as possible from the optimal $t$-spanner. This family of examples is inspired by the one given by Filtser and Solomon.
based on the Petersen graph. Consider a complete bipartite graph on $n$ edges (seen in Figure 2) where each edge has weight 1. Subsequently, we add an extra vertex, which is connected to every vertex in the original bipartite graph by an edge with weight $1 + \epsilon$. Suppose that $2 < t < 3$ is fixed; then if $\epsilon$ is suitably small, the greedy $t$–spanner of $G$ is the the bipartite graph and half of the edges connecting the additional vertex. On the other hand, the optimal $t$–spanner is the spider graph that sits atop of the original bipartite graph.

In this case, if $G' = (V, E')$ is the greedy $t$–spanner, and $G_{\text{OPT}} = (V, E_{\text{OPT}})$ is the optimal $t$–spanner, we find that

$$|E_{\text{OPT}}| = n, \quad |E'| = n^2 + n,$$

and

$$W(E_{\text{OPT}}) = n(1 + \epsilon), \quad W(E') = n^2 + n(1 + \epsilon).$$

In other words,

$$\frac{|E'|}{|E_{\text{OPT}}|} = O(n), \quad \frac{W(E')}{W(E_{\text{OPT}})} = O(n).$$

Thus, greedy $t$–spanners can be arbitrarily worse in terms of both sparsity and lightness than optimal $t$–spanners.

### 5.5 Sparsity and Lightness Guarantees for Greedy $t$–spanners

[52] Barun Chandra, Gautam Das, Giri Narasimhan, and José Soares. New sparseness results on graph spanners. In Proceedings of the eighth annual symposium on Computational geometry, pages 192–201. ACM, 1992

[78] Michael Elkin, Ofer Neiman, and Shay Solomon. Light spanners. In International Colloquium on Automata, Languages, and Programming, pages 442–452. Springer, 2014

[56] Shiri Chechik and Christian Wulff-Nilsen. Near-optimal light spanners. ACM Transactions on Algorithms (TALG), 14(3):33, 2018
Implicit in the formulation of the $t$–spanner problem is the idea that a good $t$–spanner for a graph should contain very few edges while still preserving distances in the manner prescribed. While Algorithm 2 does not require the greedy ordering of the edges to produce a $t$–spanner, this ordering is crucial to constructing a sparse $t$–spanner, i.e. one that has few edges and/or small total edge weight. The original bounds given by Althöfer et al. are the following.

**Theorem 5.2 ([10, Theorem 1]).** Let $t > 0$ and $G' = (V, E') = \text{GREEDYSPANNER}(G, 2t + 1)$. Then $G'$ is a $(2t + 1)$–spanner of $G$ and

1. $|E'| < n\lceil\frac{n}{t}\rceil$,
2. $W(E') < (1 + \frac{n}{t})W(\text{MST}(G))$.

However, in modern papers a slightly strengthened version of this theorem is usually quoted:

**Theorem 5.3 ([10, Theorem 1, Strengthened]).** Let $k$ be a positive integer and let $G' = (V, E') = \text{GREEDYSPANNER}(G, 2k - 1)$. Then $G'$ is a $(2k - 1)$–spanner of $G$ and

1. $|E'| = O\left(n^{1+\frac{1}{k}}\right)$,
2. $W(E') = O(1 + \frac{2}{k})W(\text{MST}(G))$.

**Proof of Theorem 5.3, (1).** As discussed in Section 5.2 above, the greedy spanner with parameter $2k - 1$ has no cycles on $\leq 2k$ edges; that is, it has girth $> 2k$. By the Moore Bounds (Proposition 2.3), it thus has $O\left(n^{1+\frac{1}{k}}\right)$ edges. $\square$

An important observation implicit in the stronger phrasing is that it is without loss of generality to consider only odd integer stretch parameters for multiplicative spanners, at least with respect to extremal spanner size. This essentially follows from two graph-theoretic facts. First is that (as discussed in Section 5.2) the extremal sparsity of a $t$–spanner is the same as the extremal sparsity $\gamma(n, t)$ of a graph with girth $> t$, and girth is an integer parameter. Thus we have (say) $\gamma(n, 5) = \gamma(n, 5.5)$, and so the extremal sparsity of a $5$–spanner is the same as the extremal sparsity of a $5.5$–spanner, even though the latter is strictly more accurate. Thus the size bound for $5$–spanners is a strictly stronger result than the size bound for $5.5$–spanners.

The second fact is that $\gamma(n, 2k) = \Theta(n, 2k + 1)$, this holds for the following reason: given a graph $G$ with girth $> 2k$, one can find a bipartite subgraph $H$ by placing each node on the left or right side of the bipartition with probability $\frac{1}{2}$, and then keeping only edges crossing the divide. One computes that each edge survives in $H$ with probability $\frac{1}{2}$, and thus in expectation the size of $H$ is within a constant fraction of the size of $G$. On the other hand, all cycles in $H$ are even, so $H$ has girth $> 2k + 1$. Together, these facts that the extremal size of (say) a $5$–spanner is the same as the extremal size of a $6.99$–spanner, up to constant factors. Thus if we hide constant factors, as in Theorem 5.3, the size bounds for odd integer stretch subsume all other possible stretch values.

The size bound in Theorem 5.3 is tight up to a constant factor in the exponent of $n$ (assuming the Girth Conjecture); however, the weight bound is not tight. Indeed, Chandra et al. [52] improved the bound to $O_{\epsilon}(k \cdot n^{\frac{2}{k}})$ for the greedy $(2k - 1)(1 + \epsilon)$–spanner. Elkin et al. [78] further improved the bound to $O_{\epsilon}(\frac{k}{\log k} \cdot n^{\frac{2}{k}})$. Recently, Chechik and Wulff-Nilsen [56] proposed a new spanner construction with lightness $O_{\epsilon}(n^{\frac{2}{k}})$, completely removing dependency of the factor on $k$. According to the existential optimality of the greedy spanner as shown by Filtser and Solomon [87], the same sparseness bound holds for the original greedy spanner algorithm. This result is optimal up to a $(1 + \epsilon)$ factor in the stretch provided the girth conjecture is true.

### 5.6 Computational Space
[41] Prosenjit Bose, Paz Carmi, Mohammad Farshi, Anil Maheshwari, and Michiel H. M. Smid. Computing the greedy spanner in near-quadratic time. Algorithmica, 58(3):711–729, 2010
[9] Sander PA Alewijnse, Quirijn W Bouts, P Alex, and Kevin Buchin. Computing the greedy spanner in linear space. Algorithmica, 73(3):589–606, 2015

The greedy spanner algorithm of [10] uses $O(n^2)$ memory space to store a sorted list of minimum distances between each pair of vertices, which makes it impractical for large sets of vertices. Bose et al. [41] give a way to compute the greedy spanner in $O(n^2 \log n)$ time for geometric graphs. In [9], the authors give an algorithm which computes the greedy spanner for a geometric graph in linear memory instead of quadratic.

### 5.7 Summary of Greedy Algorithm Guarantees

|                | Stretched ($t$) | Size: $O(|E'|)$ | Weight: $O(W(E'))$ | Time                          | Reference |
|----------------|-----------------|-----------------|-------------------|-------------------------------|-----------|
| $2k - 1$       | $n^{1+\frac{1}{t}}$ | $(1 + \frac{2n}{k})W(MST(G))$ | $m(n^{1+\frac{1}{t}} + n \log n)$ | [10]          |
| $(2k - 1)(1 + \epsilon)$ | $n^{1+\frac{1}{t}}$ | $kn^{1+\frac{1}{t}}$ | $(\frac{1}{\epsilon})^{1+\frac{1}{t}}$ | $m(n^{1+\frac{1}{t}} + n \log n)$ | [52]      |
| $(2k - 1)(1 + \epsilon)$ | $n^{1+\frac{1}{t}}$ | $kn^{1+\frac{1}{t}}$ | $(\frac{1}{\epsilon})^{1+\frac{1}{t}}$ | $n^{\frac{1}{1+\frac{1}{t}}} + \frac{k}{\epsilon^{1+\frac{1}{t}} \log k}$ | [78]      |

Table 1. Spanners given by the Greedy Algorithm. Here $n = |V|$ and $m = |E|$.

### 5.8 Open Questions

1. Is there a natural reverse Greedy algorithm for constructing a $t$–spanner? That is, one which begins with the full edge set $E$ ordered in nonincreasing order, and deletes edges according to some criteria to arrive at a $t$–spanner.

2. Is there a Greedy algorithm to produce additive spanners, $(\alpha, \beta)$–spanners, subsetwise spanners, or more generally pairwise spanners? (The additive spanner construction of [99] has a greedy step; can it be brought completely in line with the multiplicative greedy algorithm, or understood by a similar analysis?)

### 6 CLUSTERING AND PATH BUYING METHODS

Another prominent set of techniques for computing graph spanners is what we call clustering and path buying techniques. In the clustering phase, we start from a set of vertices of a given graph as initial clusters and then expand each to produce a clustering of the graph. Typically at the same time, we are adding edges within the clusters to the candidate spanner. In the Path Buying phase, we will then add (buy) cheap edges to achieve the desired spanner. Let us stress here that the clustering we are discussing is much different than common graph clustering algorithms such as $k$–means or spectral clustering whose aim is to partition the graph into disjoint clusters.

#### 6.1 General Approach

Here we provide a general definition for clustering in the context of spanners.

**Definition 6.1.** A clustering $\mathcal{E} = \{C_1, \ldots, C_k\}$ of the graph $G = (V, E)$ is a collection of vertex sets $C_i \subset V$.

While this definition is quite general, typically one imposes extra conditions on the clustering based on the specific problem at hand. For instance, in some cases we require $\bigcup_i C_i = V$, and sometimes we require clusters to be pairwise disjoint.
Intuitively, in the clustering step we want to attack the spanner problem locally, and in the final step we make global adjustments to find the appropriate spanner. As an example of this global adjustment, one can iterate through a given set of vertex pairs \((u, v)\) and make sure to have a path with desired stretch factor from \(u\) to \(v\).

To better illustrate common variants of this technique, we first list some useful general concepts to keep in mind. Variants of each of these will be used in the different algorithms discussed in the sequel.

- **Value of a Path**: The value of the path is the overall improvement of the stretch factors that we gain by adding the path to the spanner under construction. For example, for a given path \(\rho\) we can measure the number of clusters that will be closer together after adding \(\rho\) to the spanner.

- **Cost of a Path**: In the path buying phase of the algorithm, we may decide to add some edges to the spanner to maintain a given path in \(G\). The cost of a path typically is the weight of the new edge set that we are adding to the spanner.

Most clustering + path buying algorithms run in two stages, which we summarize in the following proto-algorithms. Typical clustering algorithms begin with singleton vertices as clusters, which are then grown according to some rule which we simply call Rule 1, and subsequently edges are added to the spanner based on different criteria (called Rule 2) depending on if they connect vertices within a cluster or not.

### Algorithm 3 Proto-Clustering \((G, \text{Rule 1, Rule 2})\)

```plaintext
Initialize \(E' \leftarrow \emptyset\)
Choose initial clusters \(C_1, \ldots, C_k\) (deterministically or randomly)
for \(v \in V\) do
    Add nearby vertices to a cluster, and add nearby edges to \(E'\) according to Rule 1
end for
If an edge is not nearby a cluster, add it to \(E'\) according to Rule 2
return \(E'\)
```

To specify a proto-algorithm for the path buying stage, one first chooses a notion of the cost and value of a path which will be denoted \(\text{cost}(\rho)\) and \(\text{value}(\rho)\) for a given path \(\rho\), respectively. With these as parameters, the path-buying proto algorithm may be stated as follows (the set of all paths in \(G\) will be denoted by \(\mathcal{P}\)).

### Algorithm 4 Proto-Path Buying \((G, \text{cost}(\rho), \text{value}(\rho), \alpha, \text{Rule 1, Rule 2})\)

```plaintext
Initialize \(E' \leftarrow \text{Proto-Clustering}(G, \text{Rule 1, Rule 2})\)
for \(\rho \in \mathcal{P}\) do
    if \(\text{cost}(\rho) \leq \alpha \cdot \text{value}(\rho)\) then
        \(E' \leftarrow E' \cup \{e \in \rho\}\)
    end if
end for
return \(E'\)
```

The proto-algorithms given here have many degrees of flexibility: namely the initialization of clusters, the rules for adding clustered and unclustered edges to \(E'\), the definitions of cost and value of a path, and the relation of the final two quantities. As a general rule, the clustering phase is the
cheap one in terms of run-time complexity, whereas the path buying phase is more expensive due to the fact that one typically runs through all possible paths in $G$.

Let us also note that several of the clustering algorithms utilize a random edge selection step; however, the guarantees for the resulting spanners are deterministic. For algorithms which produce spanners only with high probability, see Section 7.

### 6.2 Illustrating Example

[22] Surender Baswana, Telikepalli Kavitha, Kurt Mehlhorn, and Seth Pettie. Additive spanners and $(\alpha, \beta)$-spanners. *ACM Transactions on Algorithms (TALG)*, 7(1):5, 2010

Let us begin with one of the simpler algorithms in the vein described above given by Baswana et al. [22] which computes a $(1, 6, V \times V)$–spanner (i.e. an additive 6–spanner) for an unweighted graph. We will describe only the terminology and rules required to describe their algorithm in terms of the proto-algorithms of the previous section.

**Initialization of Clusters:** first, $|V|^{\frac{1}{3}}$ cluster centers are chosen uniformly randomly from $V$. That is, we have $C_1 = \{v_1\}, \ldots, C_k = \{v_k\}$ for $k = |V|^{\frac{1}{3}}$.

**Rule 1:** if $v$ is adjacent to a cluster center, it joins an arbitrary cluster that it is adjacent to, and the corresponding edge is added to $E'$.

**Rule 2:** if $v$ has no adjacent cluster center, then it is left unclustered and all edges incident to $v$ are added to $E'$.

For the path-buying phase of this algorithm, the parameters involved are defined as follows:

$$\text{cost}(\rho) := |\rho \setminus E'|,$$

that is, the number of edges in $\rho$ that are not already in the spanner, and

$$\text{value}(\rho) := \left| \{(C_1, C_2) \in \mathcal{C} \times \mathcal{C} : \rho \text{ intersects } C_1, C_2, \text{ and } \text{dist}_{G'}(C_1, C_2) \text{ decreases after adding } \rho \text{ to } G'\} \right|,$$

$\alpha = 2$,

where $G'$ is the current spanner and $\text{dist}_{G'}(C_1, C_2)$ denotes the length of the shortest path in $G'$ between terminal vertices that lie in $C_1, C_2$ respectively.

**Theorem 6.1 ([22, Theorem 2.7]).** Given the cluster initialization, Rule 1, 2, and definitions of $\alpha$, cost and value of paths above, the algorithm PROTO-PATH BUYING returns a $(1, 6, V \times V)$–spanner of $G$.

**Sketch of Proof.** For the sake of illustration, we sketch the proof that the algorithm described in Theorem 6.1 produces a $(1, 6)$–spanner. Consider the special case of two clustered nodes $u$ and $v$. For a shortest path $\rho$ from $u$ to $v$, we look at a sequence of clusters $C_1, \ldots, C_\ell$ intersecting $\rho$ with $u \in C_1$ and $v \in C_\ell$. We will prove that after the path-buying phase, the spanner $G'$ satisfies the following property.

**Intermediate Cluster Property (ICP):** A shortest path $\rho$ satisfies the ICP if there exists a cluster $C_i$ ($1 < i < \ell$) along $\rho$ such that

$$\text{dist}_{G'}(C_1, C_i) \leq \text{dist}_\rho(C_1, C_i), \quad \text{dist}_{G'}(C_i, C_\ell) \leq \text{dist}_\rho(C_i, C_\ell).$$

Here, $\text{dist}_\rho(C_1, C_j)$ is the distance from $C_1$ to $C_j$ along the prescribed path $\rho$. Note that in the description of Rule 2, $v$ joins only one of the neighbor clusters and hence there are two types of missing edges in $\rho$: those with both endpoints in the same cluster (intracluster) and those with endpoints belonging to different clusters (intercluster). Now, assuming that the ICP is true for some shortest path $\rho$ and counting intracluster and intercluster missing edges, we can apply the
triangle inequality and use the fact that the diameter of each cluster is two, to get the desired 
\((1,6)\)-spanner condition.

Now, our goal is to prove the ICP; to do so, we first define the following sets:

\[ A = \{ (C_i, C_j) : i = 1 \text{ or } j = \ell \} \]

\[ A_0 = \{ (C_i, C_j) \in A : \text{dist}_{G'}(C_i, C_j) > \text{dist}_\rho(C_i, C_j) \} \]

\[ A_1 = A \setminus A_0. \]

Recall the definition for the value of a path:

\[ \text{value}(\rho) = |\{(C_i, C_j) : \text{dist}_{G'}(C_i, C_j) > \text{dist}_\rho(C_i, C_j)\}|, \]

hence we have \(|A_0| \leq \text{value}(\rho)|.

Claim: \(\text{value}(\rho) < \ell - 2\).

If the claim is true then \(|A_1| > \ell - 1\), and using the pigeonhole principle there exists some \(C_i\) satisfying the ICP. With a simple counting argument, and if shortest path ties are broken properly (roughly, shortest paths should stay in their current cluster as long as possible), given that \(\ell\) is the number of clusters intersecting \(\rho\), the number of missing intercluster edges in \(G'\) is at most \(\ell - 1\) and the number of missing intracluster edges is at most \(\ell - 2\). Therefore, \(\text{cost}(\rho) < 2\ell - 3\) and using the fact that \(2\text{value}(\rho) < \text{cost}(\rho)\), we get \(\text{value}(\rho) < \ell - 2\) which finishes the proof. \(\Box\)

6.3 Theme and Variations

[24] Surender Baswana and Sandeep Sen. A simple linear time algorithm for computing a \((2k - 1)\)-spanner of \(O(n^{1+1/k})\) size in weighted graphs. In Jos C. M. Baeten, Jan Karel Lenstra, Joachim Parrow, and Gerhard J. Woeginger, editors, Automata, Languages and Programming, pages 384–396, Berlin, Heidelberg, 2003. Springer Berlin Heidelberg

[64] Marek Cygan, Fabrizio Grandoni, and Telikepalli Kavitha. On Pairwise Spanners. In Natacha Portier and Thomas Wilke, editors, 30th International Symposium on Theoretical Aspects of Computer Science (STACS 2013), volume 20 of Leibniz International Proceedings in Informatics (LIPIcs), pages 209–220, Dagstuhl, Germany, 2013. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik

[97] Telikepalli Kavitha and Nithin M Varma. Small stretch pairwise spanners. In International Colloquium on Automata, Languages, and Programming, pages 601–612. Springer, 2013

[96] Telikepalli Kavitha. New pairwise spanners. Theory of Computing Systems, 61(4):1011–1036, 2017

[99] Mathias Bæk Tejs Knudsen. Additive spanners: A simple construction. In Scandinavian Workshop on Algorithm Theory, pages 277–281. Springer, 2014

[119] Seth Pettie. Low distortion spanners. ACM Transactions on Algorithms (TALG), 6(1):7, 2009

Cygan et al. [64] created a polynomial time clustering + path buying algorithm for computing \((\alpha, \beta)\) pairwise, subsetwise, and sourcewise spanners. Their clustering step is essentially a deterministic version of the one used by Baswana et al. [22]; this derandomization is orthogonal to the change from all-pairs to pairwise spanners, but we will include it here for completeness.

Clustering Phase. This follows Algorithm 3 with the following rules. In the sequel, let \(d\) be an integer parameter of the construction that we will choose later.

Choice of Clusters: Unmark all nodes, and then while there is a (possibly marked) node \(v \in V\) with \(\geq d\) unmarked neighbors, we choose a set \(C\) of exactly \(d\) of its neighbors and add \(C\) as a new cluster. The node \(v\) is called its center (note that \(v \notin C\)). We then mark all nodes in \(C\), and repeat until we can do so no longer.
Rule 1: All edges with both endpoints contained in the same cluster are included in the spanner. All edges between a cluster center and a node in its cluster are included in the spanner.

Rule 2: For each node $v$ not contained in a cluster, include all of its incident edges in the spanner.

The essential properties of this clustering step are as follows.

**Lemma 6.1 ([64]).** Algorithm 3, with the above parameters, provides a clustering $\mathcal{C}$ of $G$ and a subgraph $G' = (V, E')$ with the following properties:

1. The size of each cluster $C \in \mathcal{C}$ is exactly $|C| = d$.
2. The total number of clusters is $|\mathcal{C}| \leq n/d$.
3. Any two nodes in the same cluster have a common neighbor in $G'$ and hence the diameter of any cluster is at most 2.
4. $|E'| = O(nd)$
5. If an edge $(u, v)$ is absent in $E'$, then $u$ and $v$ belong to two different clusters (In particular, they cannot be unclustered nodes).

**Proof.** Items (1), (3) and (5) follow directly from the algorithm. (2) follows directly from (1) since clusters are node-disjoint. Finally, we argue (4) as follows. In Rule 1, we add at most $(d^2)$ edges per cluster, which is $O(nd)$ edges in total. For each edge $(u, v)$ added in Rule 2, we note that one (or both) endpoints are unmarked. Since every node has $<d$ unmarked neighbors, by a union bound over the $n$ nodes we add $O(nd)$ edges in this step as well.

**Path Buying Phase.** Cygan et al. give several variants of the path buying phase to obtain different types of spanners. All begin by using the above Clustering algorithm, but vary the definition of value of paths and $\alpha$ as in the Proto-path buying algorithm. The first variant is to define cost as in (2), and value as

$$\text{value}(\rho) := |\{(x, C) \in S \times \mathcal{C} : \rho \text{ intersects } x, C, \text{ and } \text{dist}_{G'}(x, C) \text{ decreases after adding } \rho \text{ to } G'\}|.$$ (3)

With these definitions of cost and value of a path, the clustering + path-buying algorithm yields an additive subsetwise spanner with additive stretch 2 by setting $\alpha = 2$.

**Theorem 6.2 ([64, Theorem 1.3]; also [73, 119]).** For any $S \subset V$, given the above cluster initialization with parameter $d = \sqrt{|S|}$, cost and value of paths specified by (2), (3), and $\alpha = 2$, the algorithm PROTO-PATH BUYING returns a $(1, 2, S \times S)$–spanner of $G$ of size $O(n\sqrt{|S|})$.

We remark that any choice of $d$ in the CLUSTER step will produce a $(1, 2, S \times S)$–spanner, but the choice of parameter $d = \sqrt{|S|}$ is needed to minimize the size of the final spanner. In particular, the clustering phase costs $O(nd)$ edges and the path buying step costs $O(|S|n/d)$ edges, and these balance at the choice $d = \sqrt{|S|}$.

The second variant of the path-buying algorithm uses path-buying as a preprocess in a larger spanner construction. This is more complicated to prove, but the statement is as follows:

**Theorem 6.3 ([64, Theorem 1.1]).** For any $\epsilon > 0$, positive integer $k$, and any $P \subset V \times V$, given the above cluster initialization with parameter

$$d = n^{2k}(2k + 3)|S|^{\frac{k}{2k + 1}},$$

cost and value of a path specified in Section 6.2, and

$$\alpha = \frac{12 \log n}{\epsilon},$$
Algorithm 5 Randomized \((k, k - 1)\)-Spanner Construction(\(G\))

Initialize \(E' \leftarrow \emptyset\) and \(C_0 \leftarrow \{\{v\} : v \in V\}\)

for \(i\) from 1 to \(k\) do

Let \(C_i\) be sampled from \(C_{i-1}\) with probability \(n^{-\frac{1}{k}}\) (If \(i = k\), then \(C_k = \emptyset\))

for \(u\) not belonging to a cluster in \(C_i\) do

If \(u\) is adjacent to some \(C \in C_i\), add \(u\) to \(C\) and add some edge of \(E(u, C)\) to \(E'\)

Otherwise, add to \(E'\) some edge from \(E(u, C)\), for each \(C \in C_{i-1}\) adjacent to \(u\)

end for

end for

Add to \(E'\) one edge from \(E(C, C')\) for each adjacent pair \(C \in C_i\) and \(C' \in C_{k-1-i}\) for \(i\) from 0 to \(k - 1\)

Add to \(E'\) one edge from \(E(C, C')\) for each adjacent pair \(C \in C_i\) and \(C' \in C_{i-1}\) for \(i\) from \(\lceil \frac{k}{2} \rceil\) to \(k - 1\)

return \(E'\)

Kavitha and Varma [97] give yet another clustering + path buying algorithm for computing \((\alpha, \beta, P)\)-spanners which utilizes breadth-first search (BFS) trees [63] in the path-buying phase. Recently, Kavitha [96] presents a modified version of the BFS strategy to compute additive pairwise spanners. The broad idea of both is that BFS trees provide highways along which distances are preserved, and they are used to connect clustered nodes together.

Kavitha [96] has provided algorithms similar in spirit to those of Cygan et al. to compute additive pairwise spanners.

Baswana and Sen [24] use a similar clustering algorithm to Algorithm 5 but a different path-buying phase than [22] to yield a \((2k - 1)\)-spanner with \(O(n^{1\frac{1}{k}})\) edges in expected linear time \(O(kn)\). This gives a spanner with optimal number of edges (up to a constant and assuming the Girth Conjecture) in optimal time.

Pettie [119] proposes a modular scheme for constructing \((\alpha, \beta)\)-spanners through utilizing connection schemes, many of which are variants of this clustering + path buying approach. Through assembling connection schemes properly, most existing results can be generated. Also, substantially improved results for almost additive spanners are obtained. In particular, it is shown that linear size \(|E'| = O(n)|\) spanners can be constructed with good stretch including \((5 + \epsilon, \text{polylog}(n))\)-spanners, and \((1, \tilde{O}(n^{\frac{1}{\epsilon}}))\)-spanners.

6.4 Further Reading

[80] Michael Elkin and David Peleg. \((1+\epsilon, \beta)\)-spanner constructions for general graphs. SIAM Journal on Computing, 33(3):608–631, 2004
[21] Surender Baswana, Telikepalli Kavitha, Kurt Mehlhorn, and Seth Pettie. New constructions of \((\alpha, \beta)\)-spanners and purely additive spanners. In Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2005, Vancouver, British Columbia, Canada, January 23-25, 2005, pages 672–681, 2005

[26] Surender Baswana and Sandeep Sen. A simple and linear time randomized algorithm for computing sparse spanners in weighted graphs. Random Structures & Algorithms, 30(4):532–563, 2007

[54] Shiri Chechik. New additive spanners. In Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms, pages 498–512. Society for Industrial and Applied Mathematics, 2013

Baswana et al. [21] give two important results on additive spanners: an additive 6–spanner of size \(O(n^{\frac{4}{3}})\), and a linear time construction of \((k, k-1)\)-spanners with size \(O(n^{1+\frac{1}{k}})\). In Baswana and Sen [26], the first linear time randomized algorithm that computes a \(t\)-spanner of a given weighted graph was given. Recall that this size/error tradeoff is optimal assuming the Girth Conjecture (see Section 2.4).

Chechik [54] produces a new purely additive 4–spanner of size \(\tilde{O}(n^{\frac{7}{5}})\). In addition, she presents a construction of additive spanners with \(\tilde{O}(n^{1+\frac{2}{3}})\) edges and additive stretch of \(\tilde{O}(\sqrt{d(u,v)})\) for each pair \(u, v \in V\).

Elkin and Peleg [80] show that the multiplicative factor can be made arbitrarily close to 1 while keeping the spanner size arbitrarily close to \(O(n)\) at the cost of allowing the additive term to be a sufficiently large constant. In other words, they show that for any constant \(\lambda > 0\) there exists a constant \(\beta = \beta(\epsilon, \lambda)\) such that for every \(n\)-vertex graph there is an efficiently constructible \((1 + \epsilon, \beta)\)-spanner of size \(O(n^{1+\lambda})\).

6.5 Open Problems

(1) Suppose one runs a traditional graph clustering algorithm (in the unsupervised learning sense; e.g. Normalized Cuts, Spectral Clustering, or Hierarchical Clustering) which produces clusters which are disjoint and cover the whole graph. Can one use this clustering to give rise to a good spanner of \(G\)?

7 PROBABILISTIC METHODS

[108] Gary L Miller, Richard Peng, Adrian Vladu, and Shen Chen Xu. Improved parallel algorithms for spanners and hopsets. In Proceedings of the 27th ACM symposium on Parallelism in Algorithms and Architectures, pages 192–201. ACM, 2015

[77] Michael Elkin and Ofer Neiman. Efficient algorithms for constructing very sparse spanners and emulators. ACM Transactions on Algorithms (TALG), 15(1):4, 2018

Miller et al. [108] construct the first randomized algorithm for producing a spanner with high probability (other algorithms using randomization give deterministic guarantees). The main result therein (Theorem 1.1) yields a multiplicative \(O(k)\)–spanner with high probability of expected size \(O(n^{1+\frac{k}{k} \log k})\) in expected time \(O(m)\).

Elkin and Neiman [77] improved on the work of Miller et al. [108] by giving a randomized construction which computes a \((2k - 1)\)-spanner with \(O(n^{1+\frac{k}{k} \epsilon^{-1}})\) edges with probability \(1 - \epsilon\).
Moreover, they give a runtime and edge count analysis for both PRAM and CONGEST computational models. To illustrate this recent technique, we give a partial proof of the main theorem in [77].

7.1 Probabilistic \((2k − 1)\)-spanners

To begin, note that given a parameter \(\lambda > 0\), the exponential distribution is defined by the probability density function (pdf)

\[
p_\lambda(x) = \begin{cases} 
    \lambda e^{-\lambda x} & x \geq 0 \\
    0 & x < 0.
\end{cases}
\]

We denote a random vector \(x\) drawn from this distribution by \(x \sim \text{Exp}(\lambda)\).

Algorithm 6 **Rand Exp \((2k − 1)\)-spanner** \((G, k, c)\)

```plaintext
\[
\lambda \leftarrow \frac{\ln(cn)}{k}
\]
```

**for** \(u \in V\) **do**

- Draw \(r_u \sim \text{Exp}(\lambda)\)
- \(u\) broadcasts \(r_u\) to all vertices \(v\) within distance \(k\)

  **if** \(x\) receives a message from \(u\) **then**
  - \(x\) stores the value \(m_u(x) = r_u - d_G(u, x)\)
  - \(x\) also stores a vertex \(p_u(x)\) along a shortest path from \(u\) to \(x\) (if there is more than one possibility, one is selected randomly)

**end if**

**end for**

\(m(x) \leftarrow \max_{u \in V} m_u(x)\)

**for** \(x \in V\) **do**

- \(E' \leftarrow E' \cup C(x) = \{(x, p_u(x)) : m_u(x) \geq m(x) - 1\}\)

**end for**

**return** \(E'\)

The key reason for using the exponential distribution to determine the messages passed from each vertex is its memoryless property, which is the fact that if \(x \sim \text{Exp}(\lambda)\) and \(s, t \in \mathbb{R}\), then \(\mathbb{P}(x > s + t \mid x > t) = \mathbb{P}(x > s)\), i.e. does not depend on the value of \(t\). The main result of [77] is the following.

**Theorem 7.1 ([77, Theorem 1]).** Let \(G = (V, E)\), \(k \in \mathbb{N}\), \(c > 3\), and \(\delta > 0\) be fixed. Then with probability at least \((1 − \frac{1}{c})^{\frac{k}{1 + \frac{1}{c}}\delta}\), Algorithm 6 computes a \((2k − 1)\)-spanner of \(G\) which has at most

\[
(1 + \delta)\frac{(cn)^{1+\frac{1}{c}}}{c - 1} - \delta(n - 1)
\]

edges.

**Sketch of Proof.** Choose \(\lambda = \frac{\ln(cn)}{k}\), and let \(X\) be the event \(\{\forall u \in V, r_u < k\}\). Note that \(\mathbb{P}(r_u \geq k) = e^{-\lambda k} = \frac{1}{cn}\); hence the union bound implies that \(\mathbb{P}(\exists u \in V \text{ s.t. } r_u \geq k) \leq \frac{1}{c}\). Hence \(\mathbb{P}(X) \geq 1 - \frac{1}{c}\).

The key observation made in [77] is that if the event \(X\) holds, then the subgraph \(G' = (V, E')\) output by Algorithm 6 is a spanner with stretch at most \(2k - 1\). Additionally, since \(G'\) is a spanner, it must have at least \(n - 1\) edges, and so if \(Y\) is the random variable \(|E'| − (n − 1)|\), then \(Y\) is positive. Moreover, \(\mathbb{E}(Y) \leq n(cn)^{\frac{1}{c}} - (n - 1)\) (Lemmas 1 and 2 of [77]), and so

\[
\mathbb{E}(Y \mid X) \leq \frac{\mathbb{E}(Y)}{\mathbb{P}(X)} \leq \frac{1}{1 - \frac{1}{c}} \left( n(cn)^{\frac{1}{c}} - (n - 1) \right).
\]
To turn this into a concrete bound, note that Markov’s inequality implies that for the given $\delta$,
\[
P(Y \geq (1 + \delta)\mathbb{E}(Y | X)) \leq \frac{1}{1 + \delta},
\]
whereby we have
\[
P((Y < (1 + \delta)\mathbb{E}(Y | X)) \cap X) \geq \left(1 - \frac{1}{c}\right) \frac{\delta}{1 + \delta}.
\]
Consequently, if these events hold, then $|E'| = Y + n - 1$ is at most $(1 + \delta)\mathbb{E}(Y | X)$, which is at most the quantity in the statement of the theorem by combining the above estimates. 

## 7.2 Open Problems

1. Can one optimize the construction of Elkin and Neiman over the pdf to get a better guarantee for the size of a $(2k - 1)$–spanner?
2. Can one design a probability distribution over the edges of a graph from which random sampling yields a spanner with high probability?

## 8 Subsetwise, Sourcewise, and Pairwise Spanners

### 8.1 Pairwise Distance Preservers

[38] Béla Bollobás, Don Coppersmith, and Michael Elkin. Sparse distance preservers and additive spanners. *SIAM Journal on Discrete Mathematics*, 19(4):1029–1055, 2005

[62] Don Coppersmith and Michael Elkin. Sparse sourcewise and pairwise distance preservers. *SIAM Journal on Discrete Mathematics*, 20(2):463–501, 2006

[37] Greg Bodwin and Virginia Vassilevska Williams. Better distance preservers and additive spanners. In *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, pages 855–872. Society for Industrial and Applied Mathematics, 2016

[1] Amir Abboud and Greg Bodwin. Error amplification for pairwise spanner lower bounds. In *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, pages 841–854. SIAM, 2016

[31] Greg Bodwin. Linear size distance preservers. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 600–615. Society for Industrial and Applied Mathematics, 2017

[32] Greg Bodwin. On the structure of unique shortest paths in graphs. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2071–2089. Society for Industrial and Applied Mathematics, 2019

[88] Kshitij Gajjar and Jaikumar Radhakrishnan. Distance-preserving subgraphs of interval graphs. In *25th Annual European Symposium on Algorithms (ESA 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017

A special case of the spanner problem is that of finding *distance preservers*, which requires that distances are preserved exactly between specified pairs of vertices of $G$. This variant of the problem is important given that some spanner constructions use distance preservers as a subroutine. Distance preservers were first introduced and studied in [38] and [62]. Coppersmith and Elkin [62] proved the existence of a linear size pairwise distance preserver for any pair set $P \subset V \times V$ with $|P| = O(\sqrt{n})$. Further work has been done in [31], [37], and [88].

Recall we may consider subsetwise distance preservers when $P = S \times S$ and sourcewise distance preserver when $P = S \times V$. A fundamental question about distance preservers is the following:

**Problem 3.** For a fixed $n$ and $P \subset V \times V$, what can be said about the number of edges, $|E'|$, in a distance preserver of $G$ over $P$?
For the moment, consider the simplest case where $P$ has the form $\{s\} \times V$. Here, the distance preserver is a tree with $O(n)$ edges; this fact is known as the Shortest Path Tree Lemma. Note that without further assumptions on the structure of $G$ or $P$, one cannot improve this construction in general. Indeed, if $G$ is a single path from $u$ to $v$ and $P = \{(u, v)\}$, the preserver must have $n - 1$ edges. On the other hand, if $G$ is a clique, there must be $|P|$ edges in the preserver, and hence in general, $|E'| = \Omega(n + |P|)$.

Here, we present two theorems on the existence and size of distance preservers along with an outline of one of the proofs. The original result of [62] is the following.

**Theorem 8.1 ([62]).** Given an undirected (weighted or unweighted) graph $G$ and a set of node pairs $P$, there exists a distance preserver of size $|E'| = O(n + n^{\frac{3}{2}}|P|)$.

The next theorem is a slightly worse in size complexity, but is more general in that it holds for a larger class of graphs.

**Theorem 8.2 ([31]).** Given a (directed or undirected, weighted or unweighted) graph $G$ and a set of node pairs $P$, there exists a distance preserver of size $|E'| = O(n + n^{\frac{3}{2}}|P|)$.

### 8.2 Sketch of Proof of Theorem 8.2

**Definition 8.1 (Tiebreaking Scheme).** A tiebreaking scheme is a map $\pi$ that sends each node pair $(u, v)$ to a shortest path from $u$ to $v$.

**Definition 8.2.** A consistent tiebreaking is a tiebreaking scheme $\pi$ such that for every $w, x, y, z \in V$, if $x, y \in \pi(w, z)$ then $\pi(x, y)$ is a subpath of $\pi(w, z)$.

**Lemma 8.1.** Every graph has a consistent tiebreaking.

**Proof.** Add a small random number to each edge so that we have a unique shortest path between any pair $(u, v)$; one can see that the output is a consistent tiebreaking. □

**Definition 8.3 (Branching Triple).** A branching triple is a set of three distinct directed edges $(u_1, v), (u_2, v), (u_3, v)$ in $E$ that all enter the same node.

Branching triples are based on the closely related notion of branching events, used in the proof of Theorem 8.1. We will prove a bound for the edge size of the distance preserver using a bound for the number of branching triples.

**Lemma 8.2.** Let $H = (V, \pi(P))$ where $\pi$ is any consistent tiebreaking. Then $H$ has at most $\binom{|P|}{3}$ branching triples.

**Proof.** Assign each edge $e$ to some pair $p = (v, v') \in P$ such that $e \in \pi(p)$ (i.e. edge $e$ belongs to the shortest path from $v$ to $v'$). We show that any three pairs in $P$ cannot share two or more branching triples. By way of contradiction, if three pairs $p_1, p_2, p_3$ share two branching triples $(e_1, e_2, e_3)$ and $(e'_1, e'_2, e'_3)$, then a situation similar to the following will always happen: the edge $e_1$ precedes $e'_1$ in $p_1$ and $e_2$ precedes $e'_2$ in $p_2$. Since $\pi$ is a consistent tiebreaking, one can check that $e'_1 = e'_2$ which is a contradiction. That means for every three pairs in $P$, there is at most one corresponding branching triple. Consequently the number of branching triples is at most $\binom{|P|}{3}$. □

The first two edges incident to any vertex $v$ will not create a branching triple but after that each new edge will contribute at least one new branching triple, so a graph with $O(n)$ branching triples has $O(n)$ edges. From the previous lemma we conclude that if $|P| = O(n^{\frac{3}{2}})$ then $|E(H)| = O(n)$. Now, given a graph $G$ and a set of node pairs $P$, we can partition $P$ into subsets of size $O(n^{\frac{3}{2}})$ and apply this fact, which gives the desired conclusion.
8.3 Further Reading

Bodwin and Williams [37] proved an upper bound of $O(n^{\frac{2}{3}}|P|^\frac{2}{3} + n|P|^{\frac{1}{3}})$ for undirected and unweighted graphs using a new type of tiebreaking scheme. They also rely on a clustering technique to prove new upper bounds for additive spanners.

Abboud and Bodwin [1] consider lower bounds for pairwise spanners. Most importantly, they prove that lower bounds for pairwise distance preservers imply lower bounds for pairwise spanners.

The existing trivial lower bounds show that in the worst case, the size of distance preservers is at least linear in $n$ and $p$. Bodwin [31] makes some progress on identifying the cases that these bounds are tight, i.e., when a linear size ($|E'| = O(n + p)$) distance preserver is guaranteed. For example, by the result discussed above, if the number of pairs is $O(n^{\frac{2}{3}})$ then one can always find a distance preserver with $O(n)$ edges.

Abboud and Bodwin [3] introduce and study reachability preservers, in which only reachability (typically in a directed graph) rather than distances between demand pairs must be preserved.

8.4 Multiplicative Pairwise Spanners

[75] Michael Elkin, Arnold Filtser, and Ofer Neiman. Terminal embeddings. *Theoretical Computer Science*, 697:1–36, 2017

The paper [75] consists of various embedding techniques of a metric space (or graph) into a normed space (or a family of graphs), bounding metric distortion for connecting nodes to terminals. As an example of the generic results in the paper, if one can embed any $m$–node graph into $\ell_p$ with distortion $\alpha(m)$ using $\gamma(m)$ dimensions, then there is an algorithm for embedding any graph with $k$ terminals into $\ell_p$, with distortion $2\alpha(k) + 1$, using $\gamma(k) + n - 1$ dimensions.

A terminal graph spanner is another name for a sourcewise spanner; that is, given a graph $G = (V, E)$ and a set of "terminal nodes" $K \subset V$, the spanner property must hold for all pairs in $K \times V$. This name is more common when dealing with metric embeddings rather than general graphs. Another result of [75] is the following construction of a $(4t - 1)$–terminal spanner with $O(n + \sqrt{n}|K|^{\frac{1}{1+t}})$ edges, for a graph $G$ with terminal set $K \subset G$:

- Create a $(2t - 1)$ metric spanner $H'$ of $G$, with $P = E(H')$ and $|P| < O(|K|^{\frac{1}{1+t}})$
- Applying a distance preserver algorithm to $P$, obtain $G' \subset G$ with $O(n + \sqrt{n}|P|)$ edges, preserving distances in $K$
- Create $H \supset G'$ by adding shortest path tree in $G$ with $K$ as root.

One reason that pairwise spanners with multiplicative error are perhaps less well studied than others types of error is that they are nontrivial only in a restricted range of parameters. Suppose we have $|P| = n^{1+c}$ demand pairs for some $c > 0$, and we want a multiplicative $t$–spanner. If $c$ is not too large relative to $t$, then at least $|P|$ edges may be needed in the spanner. On the other hand, $O(|P|)$ edges always suffice by the following construction (which is folklore). Preprocess the graph with the all-pairs mixed error spanner of [80], with parameters chosen such that the spanner has only $O(|P|)$ edges. One can then argue that all pairs in $P$ have been well spanned, except maybe for pairs at constant (depending only on $c, t$) distance in the original graph. Hence, we may add an exact shortest path for all remaining pairs, and these cost only $O(|P|)$ edges in total.

8.5 Open Problems

(1) What is the largest $p = p(n)$ so that, for any $|P| = p$ node pairs in an undirected unweighted $n$–node graph, there is a pairwise spanner of $P$ on $O(n)$ edges with $+c$ additive error? In
particular, it is known that any $|P| = O(n^2)$ pairs have a distance preserver (+0 error) on $O(n)$ edges, but it is not clear if one can handle more pairs with (say) a +2 error tolerance.

9 INTEGER LINEAR PROGRAMMING (ILP) FORMULATIONS

[124] Mikkel Sigurd and Martin Zachariasen. Construction of minimum-weight spanners. In Algorithms - ESA 2004, 12th Annual European Symposium, Bergen, Norway, September 14-17, 2004, Proceedings, pages 797–808, 2004

[6] Reyan Ahmed, Keaton Hamm, Mohammad Javad Latifi Jebelli, Stephen Kobourov, Faryad Darabi Sahneh, and Richard Spence. Approximation algorithms and an integer program for multilevel graph spanners. In Proceedings of the Special Event on Analysis of Experimental Algorithms, 2019

One method to compute an exact solution to the Lightest/Sparsest Spanner Problem is to formulate the problem as an ILP. Many real-world optimization problems can be modeled as ILPs, but solving an ILP is NP–hard in general, even if the variables are restricted to $\{0, 1\}$. Nonetheless, there are methods (for example, column generation [67]) for solving the ILP in smaller subproblems (which requires less computation time) and still obtaining an optimal solution.

There are relatively few ILP formulations for graph spanner problems. Sigurd and Zachariasen [124] give an ILP formulation for the pairwise spanner problem which uses paths as decision variables, and then evaluate the performance of the greedy spanner heuristic [10] against the optimal solution computed using the ILP formulation. The main practical drawback of this formulation is that the number of path variables is exponential.

9.1 Sigurd and Zachariasen’s ILP for Pairwise Multiplicative Spanners

Let $G = (V, E)$ be a graph (weighted or unweighted), let $P \subset V \times V$ be the desired pairs for which the spanner condition will hold, and $t \geq 1$ be a prescribed stretch factor. Recall that $w_e$ is the weight of an edge $e \in E$ (with $w_e = 1$ in the unweighted case). Let $x_e = 1$ if $e$ is selected in the spanner, and $x_e = 0$ otherwise. Further, let $P_{uv}$ denote the set of all $u-v$ paths of length at most $t \cdot d_G(u, v)$, let $P$ denote the union of all such paths ($P = \bigcup_{(u, v) \in P} P_{uv}$), and let $y_{\rho} = 1$ if path $\rho \in P$ is selected in the spanner, and 0 otherwise. Let $\delta_{\rho}^e = 1$ if edge $e$ is on path $\rho$, and 0 otherwise.

The ILP formulation by Sigurd and Zachariasen for the subsetwise spanner problem is as follows:

\[
\text{Minimize } \sum_{e \in E} w_e x_e \text{ subject to }
\]

\[
\sum_{\rho \in P_{uv}} y_{\rho} \delta_{\rho}^e \leq x_e \quad \forall e \in E; \forall (u, v) \in P
\]

(5)

\[
\sum_{\rho \in P_{uv}} y_{\rho} \geq 1 \quad \forall (u, v) \in P
\]

(6)

\[
x_e \in \{0, 1\} \quad \forall e \in E
\]

(7)

\[
y_{\rho} \in \{0, 1\} \quad \forall \rho \in P
\]

(8)

Constraint (5) enforces that if a path $\rho \in P_{uv}$ is selected in the spanner, then all edges on that path are selected as well. Constraint (6) enforces that every pair $(u, v) \in P$ has at least one selected $t$–spanner path. With this in mind we have the following (the proof is not given since we give a complete proof of another ILP in this section).

**Theorem 9.1** ([124]). Given a graph $G = (V, E)$, a subset $P \subset (V \times V)$, and stretch factor $t \geq 1$, the ILP (4)–(8) yields an optimal $(t, 0, P)$–spanner for $G$. 
This ILP has $O(|\mathcal{P}|)$ constraints, which can be enormous; however, the authors apply column generation to solve a restricted master problem (i.e. using only a subset of $\mathcal{P}$ for the constraints), and show that this procedure yields an optimal solution. The authors then use the ILP to show that the greedy algorithm of Althöfer et al. (see Section 5) performs optimally in many cases.

### 9.2 An ILP for Pairwise Spanners with General Distortion

Recently, Ahmed et al. [6, 7] gave an alternate compact ILP formulation for the pairwise spanner problem which can be used to compute a minimum weight spanner, even if the input graph is directed. Again let $P \subset V \times V$, and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a distortion function satisfying $f(x) \geq x$ for all $x$ (note the function need not be continuous). Then proceed as follows: first, direct the graph by replacing each edge $(u, v)$ with two directed edges $(u, v)$ and $(v, u)$ of equal weight. Let $E$ be the set of all directed edges, thus $|E| = 2|E|$. Given $(i, j) \in E$, and an unordered pair of vertices $(u, v) \in P$, let $\bar{x}_{ij} = 1$ if edge $(i, j)$ is included in the selected $u-v$ path in the spanner $G'$, and 0 otherwise.

Next we select a total order of all vertices so that the path constraints (11)–(13) are well-defined. Then proceed as follows: first, direct the graph by selecting an optimal pairwise spanner of $G$ (denote the minimum cost of the objective function in the ILP (9)). First we notice that $\bar{x}_{ij}$ is unweighted or weighted, respectively). Given $v \in V$, let $\text{In}(v)$ and $\text{Out}(v)$ denote the set of incoming and outgoing edges for $v$ in $E$. In (10)–(14) we assume $u < v$ in the total order, so spanner paths are from $u$ to $v$.

Minimize $\sum_{e \in E} w_e x_e$ subject to

$$\sum_{(i,j) \in E} x_{ij} w_e \leq f(d_G(u,v)) \quad \forall (u,v) \in P, u < v; e = (i,j) \in E$$

(10)

$$\sum_{(i,j) \in \text{Out}(i)} x_{ij} - \sum_{(j,i) \in \text{In}(i)} x_{ji} = \begin{cases} 1 & i = u \\ -1 & i = v \\ 0 & \text{else} \end{cases} \quad \forall (u,v) \in P, u < v; \forall i \in V$$

(11)

$$\sum_{(i,j) \in \text{Out}(i)} x_{ij} \leq 1 \quad \forall (u,v) \in P, u < v; \forall i \in V$$

(12)

$$x_{ij} + x_{ji} \leq x_e \quad \forall (u,v) \in P, u < v; \forall e = (i,j) \in E$$

(13)

$$x_e, x_{ij} \in \{0, 1\}$$

(14)

Ordering the edges induces $2|E||P|$ binary variables, or $2|E|(|V|) = |E||V|(|V| - 1)$ variables in the full spanner problem where $P = V \times V$. Note that if $u$ and $v$ are connected by multiple paths in $G'$ of length $\leq f(d_G(u,v))$, we need only set $x_{ij} = 1$ for edges along some path. The following is the main theorem for this ILP.

**Theorem 9.2 ([6, 7]).** Given a graph $G = (V, E)$, a subset $P \subset V \times V$, and any distortion function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies $f(x) \geq x$ for all $x$, the solution to the ILP given by (9)–(14) is an optimally light (or sparse if $G$ is unweighted) pairwise spanner for $G$ with distortion $f$.

**Proof.** Let $G^* = (V, E^*)$ denote an optimal pairwise spanner of $G$ with distortion $f$, and let $\text{OPT}$ denote the cost of $G^*$ (number of edges or total weight if $G$ is unweighted or weighted, respectively). Let $\text{OPT}_ILP$ denote the minimum cost of the objective function in the ILP (9). First we notice that from the minimum cost spanner $G^*$, a solution to the ILP can be constructed as follows: for each edge $e \in E^*$, set $x_e = 1$. Then for each unordered pair $(u, v) \in P$ with $u < v$, compute a shortest path $\rho_{uv}$ from $u$ to $v$ in $G^*$, and set $x_{ij} = 1$ for each edge along this path, and $x_{ij} = 0$ if $(i,j)$ is not on $\rho_{uv}$.

As each shortest path $\rho_{uv}$ necessarily has cost at most $f(d_G(u,v))$, constraint (10) is satisfied. Constraints (11)–(12) are satisfied as $\rho_{uv}$ is a simple $u-v$ path. Constraint (13) also holds as $\rho_{uv}$...
cannot traverse the same edge twice in opposite directions. In particular, every edge in $G^*$ appears on some shortest path; otherwise, removing such an edge yields a pairwise spanner of lower cost. Hence $OPT_{ILP} \leq OPT$.

Conversely, an optimal solution to the ILP induces a feasible pairwise spanner $G'$ with distortion $f$. Indeed, consider an unordered pair $(u, v) \in P$ with $u < v$, and the set of decision variables satisfying $x^u_{(i, j)} = 1$. By (11) and (12), these edges form a simple path from $u$ to $v$. The sum of the weights of these edges is at most $f(d_G(u, v))$ by (10). Then by (13), the chosen edges corresponding to $(u, v)$ appear in the spanner, which is induced by the set of edges $e$ with $x_e = 1$. Hence $OPT \leq OPT_{ILP}$.

Combining the above observations, we see that $OPT = OPT_{ILP}$, and the proof is complete. □

In the multiplicative spanner case, the number of ILP variables can be significantly reduced (see [6] for more details). These reductions are somewhat specific to multiplicative spanners, and so it would be interesting to determine if other simplifications are possible for more general distortion.

Note that the distortion $f$ does not have to be continuous, which allows for tremendous flexibility in the types of pairwise spanners the above ILP can produce.

10 DISTRIBUTED AND STREAMING ALGORITHMS

[74] Michael Elkin. Computing almost shortest paths. ACM Transactions on Algorithms (TALG), 1(2):283–323, 2005
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Elkin [74] gives a randomized distributed algorithm which produces a \((1 + \epsilon, \beta)\)-spanner with 
\[ \beta = \left( \frac{k}{\epsilon} \right)^{O(\log k)} \rho^{-\frac{1}{\beta}} \] of size \( \tilde{O}(\beta n^{1+\frac{1}{\beta}}) \) in time \( O(n^{1+\frac{1}{\beta}}) \). Elkin and Matar [76] study distributed algorithms computed in the CONGEST model for finding near-additive spanners, i.e. \((1+\epsilon, \beta)\)-spanners. The difference of their approach from previous works is that the algorithm is deterministic. The spanners constructed have three parameters \( \epsilon, k, \rho \) and require \( \beta = \left( \frac{O(\log kp+\rho^{-1})}{\rho \epsilon} \right)^{\log k\rho+\rho^{-1}+O(1)} \), and yield \((1+\epsilon, \beta)\)-spanners of size \( O(\beta n^{1+\frac{1}{\beta}}) \) with time complexity \( O(\beta n^\rho\rho^{-1}) \).

Pettie [118] describes algorithms for computing sparse low distortion spanners in distributed networks and provides some non-trivial lower bounds on the tradeoff between time, sparseness, and distortion. The algorithms assume a synchronized distributed network, where relatively short messages may be communicated in each time step. The first result is a fast distributed algorithm for finding an \( O(2^{\log^* n} \log n) \)-spanner with size \( O(n) \). The second result is a new class of efficiently constructible \((\alpha, \beta)\)-spanners called Fibonacci spanners whose distortion improves with the distance being approximated. At their sparsest, Fibonacci spanners can have nearly linear size, namely \( O(n(\log \log n)\Phi) \), where \( \Phi = \frac{1+\sqrt{5}}{2} \) is the golden ratio.

Lenzen and Peleg [105] show that additive \( 2 \)-spanners can be computed in the CONGEST model in \( O(n^{\frac{1}{2}} \log n + \text{diam}(G)) \) rounds. In [50], the authors give a sequential algorithm for computing an additive \( 6 \)-spanner for unweighted graphs which yields near-optimal sparsity \( O(n^{\frac{1}{2}} \log^{\frac{3}{2}} n) \) edges, but allows for a distributed construction algorithm (using the CONGEST model of computation) which runs via an efficient construction of weighted BFS trees in \( O(n^{\frac{1}{2}} \log^{\frac{3}{2}} n + \text{diam}(G)) \) rounds.

Elkin and Solomon [83] derive much faster algorithms for computing light multiplicative spanners. Previous works gave multiplicative \((2k-1)(1+\epsilon)\)-spanners but with high running times of order \( mn^{1+\epsilon} \); Elkin and Solomon’s algorithm yields spanners of the same stretch factor, but with slightly more edges and weight with the tradeoff of faster running time \( O(km+\min\{n \log n, ma(n)\}) \), where \( a(n) \) is the inverse Ackermann function. Chechik and Wulff-Nilsen improve on the results of [83] by removing the factor of \( k \) in the weight bounds therein while maintaining the small number of edges given in the classical greedy algorithm (i.e. \( O(n^{1+\frac{1}{\epsilon}}) \)).

In Derbel et al. [65] several deterministic distributed algorithms have been provided to compute different kinds of spanners. They give an algorithm to construct an \( O(k) \)-spanner of an unweighted graph with \( O(kn^{1+\frac{1}{k}}) \) edges in \( O(k^{-1} n) \) time. Also, they have shown that in \( O(\log n) \) time one can construct a \( O(n^{\frac{1}{k}}) \) edges that is both a \( 3 \)-spanner and a \( (1+\epsilon, 4) \)-spanner. Furthermore, they have shown that in \( n \) \( O\left(\frac{1}{\log n}\right) + O\left(\frac{1}{\epsilon}\right) \) time one can construct a spanner with \( O(n^{\frac{1}{k}}) \) edges which is both a \( 3 \)-spanner and a \( (1+\epsilon, 4) \)-spanner. In [66], the authors propose a deterministic, distributed algorithm that computes a \((2k-1, 0)\)-spanner in \( k \) rounds of computation (or \( 3k-2 \) rounds if \( n \) is not known) with \( O(kn^{1+\frac{1}{k}}) \) edges. Also, it is further shown that no (randomized) algorithm producing \( O(n^{1+\frac{1}{k}}) \) size spanners (possibly with additive stretch) can run distributively in sub-polynomial rounds of computation.

Baswana [19] proposed a streaming algorithm to find multiplicative spanners based on a multi-level clustering approach. The fundamental assumption in the streaming model is that the working memory is considerably smaller than the size of the entire input stream.

Kapralov and Woodruff [95] construct linear sketches of graphs that (approximately) preserve the spectral information of the graph in a few passes over the stream of graph data. They use

\[ \log^* n \] is the inverse tower function; essentially, it is the least height of a tower \( 2^{2^{\cdots}} \) whose value is \( > n \).
a multi-layer clustering approach to construct a multiplicative $2^k$–spanner using $O(n^{1+\frac{k}{2}})$ bits of space from a dynamic stream of inputs.

Censor-Hillel et al. [49] study the complexity of distributed constructions of purely additive spanners. The provided spanner algorithms have three general steps: first, each node tosses a coin to be a cluster center; second, each cluster center tosses another coin to be a BFS tree; third, add to the current graph edges that are part of certain short paths.

11 LOWER BOUNDS FOR SPANNERS

[134] David P Woodruff. Lower bounds for additive spanners, emulators, and more. In 2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS’06), pages 389–398. IEEE, 2006

[2] Amir Abboud and Greg Bodwin. The 4/3 additive spanner exponent is tight. Journal of the ACM (JACM), 64(4):28, 2017

[4] Amir Abboud, Greg Bodwin, and Seth Pettie. A hierarchy of lower bounds for sublinear additive spanners. SIAM Journal on Computing, 47(6):2203–2236, 2018

[93] Shang-En Huang and Seth Pettie. Lower bounds on sparse spanners, emulators, and diameter-reducing shortcuts. In 16th Scandinavian Symposium and Workshops on Algorithm Theory, 2018

If Erdős’ Girth Conjecture is true, a $(1, 2k−1)$–spanner will require $\Omega(n^{1+\frac{1}{k}})$ edges. Woodruff [134] constructs a graph such that any $(1, 2k−1)$–spanner with $k = O(\frac{\log n}{\log \log n})$ has $\Omega(\frac{n}{k} n^{1+\frac{1}{k}})$ edges. Notably, this result does not rely on correctness of the girth conjecture. Furthermore, he proves lower bounds of $\Omega(\frac{1}{k} |P| \min(|P|, \frac{1}{k} n^{1+\frac{1}{k}}))$ and $\Omega(\frac{1}{k} |S| \min(|S|, \frac{1}{k} n^{1+\frac{1}{k}}))$ on the size of pairwise and sourcewise $(1, 2k−1)$–spanners.

Abboud and Bodwin [2] study the sparsity versus stretch tradeoff for additive $(1, k)$–spanners. Previously, it was known that all graphs have $(1, 6)$–spanners containing $O(n^{\frac{1}{3}})$ edges [21]. The motivating open question is: given $0 < \epsilon < \frac{1}{3}$, is there a constant $k_\epsilon$ such that any graph has a $(1, k_\epsilon)$–spanner containing $O(n^{1+\epsilon})$ edges? The authors give a negative result to the above question; more generally, they show that for $0 < \epsilon < \frac{1}{3}$, one cannot even compress an input graph into $O(n^{1+\epsilon})$ bits, so that one can recover distance information for each pair of vertices within $n^{o(1)}$ additive error. This implies that one cannot construct an additive spanner using $O(n^{1+\epsilon})$ edges unless a polynomial amount of error is allowed.

A follow up paper [4] extends Abboud and Bodwin’s $\frac{1}{6}$ threshold for additive spanners and provides a hierarchy of lower bounds for other types of error. It was previously known [80] that sparser spanners are possible by tolerating some multiplicative error $(1+\epsilon$ for some $\epsilon > 0)$ alongside the additive error term $+\beta$; [4] proves limitations on the tradeoffs between $\epsilon, \beta$, and the spanner sparsity. See Section 12.1 below for more detail.

Huang and Pettie [93] improve lower bounds on additive spanners using ideas from shortcutting sets for directed graphs. Specifically they get additive stretch lower bounds of $\Omega(n^{\frac{1}{2k}})$ for $O(n)$–size spanners which is an improvement of the Abboud and Bodwin [2] bound of $\Omega(n^{\frac{1}{2k}})$.

12 SPECIAL TYPES OF SPANNERS

In this section, we describe some of the many variants of spanners that have been considered in the literature. We emphasize primarily the definitions and basic results for brevity, and readers are encouraged to consult the listed papers for more information.

12.1 Emulators
The word spanner generally implies a subgraph that approximates the distance metric of an input graph. When the approximating graph can instead be arbitrary (not necessarily a subgraph), it is called an emulator. Spanners are a special case of emulators. Emulators are typically allowed to be weighted and sometimes directed, even if the input graph is not.

Emulators hold an interesting place in the literature because basically optimal bounds are known for the size/error tradeoff of emulators, whereas the optimal bounds for spanners remain open. The first example of this is for emulators with purely additive error. In [71], the authors prove that every $n$–node undirected unweighted input graph has an emulator on $O(n^2)$ edges with 4–additive error. This is optimal in the sense that neither the edge bound nor the additive error in this result can be unilaterally improved (this follows from the Girth Conjecture which is confirmed in the relevant setting $k = 3$; see Section 2.4). However, it is only known that every graph has a spanner on $O(n^{3/2})$ edges and 6–additive error [22], and it is still open whether one can unilaterally improve the additive error to 4.

Essentially tight bounds for emulators with sublinear error are known as well. An important result in [131] is a simple construction of emulators with $O(n^{1+1/2^k-1})$ edges and error function $f(d) = d + O(d^{1-\beta})$, for any fixed positive integer $k$. The main result of [4] is that no emulator construction can improve polynomially on this size bound (e.g. to $O(n^{1+1/2^k-1-0.001})$ edges), or on the error function except maybe in the constant hidden by the $O$. (In fact, more generally, it is proved that no data structure that compresses graphs into small space, then approximately answers distance queries, can beat the emulators in [131].) It is again a major open question whether this size/error tradeoff can be achieved by spanners.

The same story holds for mixed error: one can alternately view error of type $f(d) = d + O(d^{1-\beta})$ by defining some tradeoff between parameters $\epsilon, \beta$ of the form $\epsilon = \Theta(\beta^{-\delta})$, and then this $f(d)$ sublinear error implies $(1 + \epsilon, \beta)$ mixed error for every possible choice of $\epsilon, \beta$ along this tradeoff curve simultaneously. Multiplicative error, however, is the exception. It is known that the best possible sparsity of a spanner, subject to some multiplicative error function, is exactly the same as the best possible sparsity of the corresponding multiplicative emulator.

Emulators have also been central to a line of research seeking fast approximation algorithms for all-pairs shortest paths (APSP). In [71], the authors give approximate APSP algorithms essentially by constructing a series of emulators and then running standard shortest path algorithms on these emulators, which are fast due to their sparsity. A related approach was taken in [53]: the authors prove that, for any undirected unweighted planar input graph $G = (V, E)$ and set of terminals $T \subset V$, there is an emulator on $\tilde{O}(\min\{|T|^2, \sqrt{n|T|})$ that exactly preserves the distances between node pairs in $T$ (called a distance emulator in analogy with distance preservers). With a similar technique to the above, the authors then show that one can compute distances between any $|T| = \tilde{O}(n^{1})$ terminals in an $n$–node planar graph in $\tilde{O}(n)$ time.

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[71] Dorit Dor, Shay Halperin, and Uri Zwick. All-pairs almost shortest paths. *SIAM Journal on Computing*, 29(5):1740–1759, 2000

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[53] Hsien-Chih Chang, Pawel Gawrychowski, Shay Mozes, and Oren Weimann. Near-Optimal Distance Emulator for Planar Graphs. In Yossi Azar, Hannah Bast, and Grzegorz Herman, editors, *26th Annual European Symposium on Algorithms (ESA 2018)*, volume 112 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:17, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik
12.2 Approximate Distance Oracles

[129] Mikkel Thorup and Uri Zwick. Approximate distance oracles. In Proceedings of the Thirty-third Annual ACM Symposium on Theory of Computing, STOC ’01, pages 183–192, New York, NY, USA, 2001. ACM

[121] Liam Roditty, Mikkel Thorup, and Uri Zwick. Deterministic constructions of approximate distance oracles and spanners. In Luís Caires, Giuseppe F. Italiano, Luís Monteiro, Catuscia Palamidessi, and Moti Yung, editors, Automata, Languages and Programming, pages 261–272, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg

[25] Surender Baswana and Sandeep Sen. Approximate distance oracles for unweighted graphs in expected $O(n^2)$ time. ACM Transactions on Algorithms (TALG), 2(4):557–577, 2006

[20] Surender Baswana, Akshay Gaur, Sandeep Sen, and Jayant Upadhyay. Distance oracles for unweighted graphs: Breaking the quadratic barrier with constant additive error. In International Colloquium on Automata, Languages, and Programming, pages 609–621. Springer, 2008

[126] Christian Sommer. Shortest-path queries in static networks. ACM Computing Surveys (CSUR), 46(4):45, 2014

Related to emulators are approximate distance oracles. Given a graph $G$, a distance oracle is a data structure that can answer exact distance queries between any pair of vertices $u, v \in V$. By preprocessing the graph using algorithms such as All-Pairs Shortest Path (APSP), these queries can be answered in constant time. However, algorithms like APSP are inefficient for this purpose as they have a runtime of $O(n^3)$ and space requirement of $O(n^2)$. If approximate distances are acceptable to the user, then the runtime and space bounds can be much improved. An approximate distance oracle is a data structure which approximately answers distance queries, and similar to a spanner, the answer of a query is often allowed to be at most $t$ times the actual distance between the two vertices. In this case, spanners are special cases of approximate distance oracles, but are more restrictive given that they need to be subgraphs. Related objects are emulators which are graphs $G^* = (V, E^*)$ on the same vertices as the original graph, but with different edge sets, but so that distances in $G^*$ approximate distance in $G$ [131].

Thorup and Zwick [129] proposed a randomized algorithm to compute an approximate distance oracle with a stretch $(2k - 1)$ such that for all $u, v \in V$, the oracle returns a distance $d_k(u, v)$ with $d_k(u, v) \leq (2k - 1)d_G(u, v)$. Note that the algorithm will not return a subgraph of $G$, and hence is not a spanner algorithm, but is rather returning an approximation to the length of the shortest paths in $G$. This algorithm has an improved $O(kmn^{\frac{1}{k}})$ runtime and $O(kn^{1+\frac{1}{k}})$ space requirement.

As a follow-up to [129], Roddity et al. [121] propose two extensions: restricting the approximation to a set of sources $S \subseteq V$, and derandomizing the algorithm. The former results in an algorithm with $O(km|S|^{\frac{1}{k}})$ runtime and $O(kn|S|^{\frac{1}{k}})$ space requirement, which becomes very useful when $|S| \ll n$. The deterministic variant of the algorithm leads to an increase in runtime by a logarithmic factor but retains the same space requirements as the randomized algorithm.

Finally, Roddity et al. extend the derandomization technique to the linear time algorithm by Baswana and Sen [24] to compute a $(2k - 1)$-spanner for a given graph $G$. The randomization in this algorithm involves the initial step of choosing cluster centers randomly with a probability of $|V|^2$. The authors propose doing this deterministically by building a bipartite graph of $B = (V_1, V_2, E)$ and applying the linear time algorithm for computing a closed dominating set on this graph to find the cluster centers.

Baswana and Sen [25] provide an algorithm to construct approximate distance oracles in expected $O(n^2)$ time if the graph is unweighted. One of the new ideas used in the algorithm also leads to the first expected linear time algorithm for computing an optimal size $(2, 1)$-spanner of
an unweighted graph, whereas existing algorithms had $\Theta(n^2)$ running time. In [20], they break this quadratic barrier at the expense of introducing a (small) constant additive error for unweighted graphs. In achieving this goal, they have been able to preserve the optimal size–stretch trade offs of the oracles. One of their algorithms can be extended to weighted graphs, where the additive error becomes $2w_{\text{max}}(u,v)$, where $w_{\text{max}}(u,v)$ is the heaviest edge in the shortest path between vertices $u$ and $v$.

In [71], a $\tilde{O}(\min\{n^{3/2}m^{1/2}, n^{7/3}\})$–time algorithm for computing all distances in a graph with an additive error of at most 2 has been described. For every even $k > 2$, they describe a $\tilde{O}(\min\{n^{2-\frac{2}{k+2}}m^{\frac{1}{k+2}}, n^{2+\frac{2}{(3k-2)}}\})$–time algorithm for computing all distances with an additive one-sided error of at most $k$. Then, they show that any weighted undirected graph on $n$ vertices has a multiplicative 3–spanner with $\tilde{O}(n^{4/3})$ edges and that such a 3–spanner can be built in $\tilde{O}(mn^{1/2})$ time.

12.3 Diameter and Eccentricity Spanners

[60] Keerti Choudhary and Omer Gold. Diameter spanner, eccentricity spanner, and approximating extremal graph distances: Static, dynamic, and fault tolerant. preprint, arXiv:1812.01602, 2018

In [60], the authors consider the following problem: given a directed graph $G = (V, E)$, compute a subgraph $H$ with the property that the graph diameter is preserved up to a multiplicative factor $t$, which the authors term a $t$–diameter spanner. The authors show that, given an unweighted directed graph $G$, there is a polynomial time algorithm that computes a 1.5–diameter spanner containing $O(n^2 \log n)$ edges, in time $\tilde{O}(m\sqrt{n})$ with high probability. They define the $t$–eccentricity spanner similarly, where the eccentricity of each vertex $v$ in $H$ should be no more than $t$ times the eccentricity of $v$ in $G$; recall that the eccentricity of a vertex $v$ is the maximum graph distance between $v$ and any other vertex. The authors show that given a weighted directed graph, there is an algorithm which computes a 2–eccentricity spanner containing $O(n \log^2 n)$ edges in expected time $O(m \log^2 n)$.

12.4 Ramsey Spanning Trees

[5] Ittai Abraham, Shiri Chechik, Michael Elkin, Arnold Filtser, and Ofer Neiman. Ramsey spanning trees and their applications. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1650–1664. SIAM, 2018

Abraham et al. [5] consider a generalization of the metric Ramsey problem to graphs. The problem is to find a subset $S \subset V$ of sources and a spanning tree $T$ which gives a $t$–spanner for small $t$; i.e. one wants to simultaneously find the sources and sourcewise spanner for a given graph. Their
main result is to find (in polynomial time) a subset $S$ of size at least $n^{1-\frac{1}{k}}$ and spanning tree $T$ which is an $(O(k \log \log n), 0, S \times V)$–spanner of $G$.

### 12.5 Slack Spanners

[51] T-H Hubert Chan, Michael Dinitz, and Anupam Gupta. Spanners with slack. In European Symposium on Algorithms, pages 196–207. Springer, 2006

In [51], the authors consider the problem of computing a subgraph $H$ that preserves shortest distances to all but an $\epsilon$ fraction of vertices. Specifically, $H = (V, E_H)$ is an $\epsilon$–slack spanner with distortion $D$ if for all $v \in V$, the distances in $H$ are preserved for the furthest $(1 - \epsilon)n$ vertices from $v$ up to distortion $D$. The authors show that given a graph $G$, one can find a subgraph $H$ which is an $O(\log \frac{1}{\epsilon})$–distortion $\epsilon$–slack spanner containing $O(n)$ edges, for every $\epsilon$.

### 12.6 Spanners in Sparse Graphs

[72] Feodor F Dragan, Fedor V Fomin, and Petr A Golovach. Spanners in sparse graphs. Journal of Computer and System Sciences, 77(6):1108–1119, 2011

Dragan et al. [72] show that the problem of computing a $t$–spanner of a planar graph $G$ of treewidth$^8$ at most $k$ is fixed parameter tractable parametrized by $k$ and $t$. They then show that the problem is also fixed parameter tractable on graphs that do not contain a fixed apex graph$^9$ as a minor.

### 12.7 Spectral Graph Sparsification

[127] Daniel A Spielman and Shang-Hua Teng. Spectral sparsification of graphs. SIAM Journal on Computing, 40(4):981–1025, 2011

Spielman and Teng [127] introduced the related notion of a spectral sparsifier of a graph. Rather than trying to achieve sparsification by removing edges, which is what spanners do, they search for graphs $\tilde{G}$ for which $x^T L_{\tilde{G}} x \approx x^T L_G x$ for all $x \in \mathbb{R}^n$, where $L_G$ is the unnormalized graph Laplacian of $G$, and $f \approx g$ if $c_1 f \leq g \leq c_2 f$ for some constants $c_1, c_2$.

### 12.8 Steiner Spanners

[10] Ingo Althöfer, Gautam Das, David Dobkin, Deborah Joseph, and José Soares. On sparse spanners of weighted graphs. Discrete & Computational Geometry, 9(1):81–100, 1993

[92] Dagmar Handke and Guy Kortsarz. Tree spanners for subgraphs and related tree covering problems. In International Workshop on Graph-Theoretic Concepts in Computer Science, pages 206–217. Springer, 2000

Given a graph $G = (R \cup S, E)$ where the induced graph of $R$ is connected, a subgraph $H$ of $G$ is a Steiner $t$–spanner of $G$ if $d_H(u, v) \leq t d_G(u, v)$, for all $u, v \in R$. Althöfer, et al. [10] deal with absolute lower bounds on the number of edges that an Steiner $t$–spanner can have. Handke and Korsatz [92] study the complexity of finding Steiner tree $t$–spanners for unweighted graphs.

### 12.9 Tree $t^*$–spanners

[46] Leizhen Cai and Derek G Corneil. Tree spanners. SIAM Journal on Discrete Mathematics, 8(3):359–387, 1995

[84] Yuval Emek and David Peleg. Approximating minimum max-stretch spanning trees on unweighted graphs. SIAM Journal on Computing, 38(5):1761–1781, 2008

$^8$The treewidth of a graph $G$ is the minimum width over all tree decompositions of $G$, where the width of a tree decomposition equals 1 less than the maximum size of a vertex set in the decomposition.

$^9$An apex graph is a graph that is planar if one vertex is removed.
[11] Eduardo Álvarez-Miranda and Markus Sinnl. Mixed-integer programming approaches for the tree $t^*$-spanner problem. *Optimization Letters*, pages 1–17, 2018

Cai and Conneil [46] propose the problem of computing a spanning tree $G'$ of a weighted graph with minimal stretch factor $t^*$. Thus, the problem seeks the minimum $t^* \geq 1$ for which there exists a spanning tree $G'$ that is also a $t^*$-spanner of $G$, i.e. to find

$$t^* := t^*(G) := \inf\{t \geq 1 : \text{there exists a spanning tree in } G \text{ which is a } t \text{-spanner of } G\}.$$ 

The problem is shown therein to be NP-hard.

Emek and Peleg [84] give an $O(\log n)$ approximation algorithm for computing the optimal $t^*$-spanner, as well as showing that the tree $t^*$-spanner problem is NP-hard to approximate within a factor of $\frac{5}{4}$. Álvarez-Miranda and Sinnl [11] give a compact, polynomial-size mixed integer program (MIP) for the tree $t^*$-spanner problem. They obtain a formulation with fewer variables by using a cut-based formulation to impose a spanning tree topology, then using Benders decomposition to get rid of path variables. For other approximation algorithms, see [125, 128].

13 SPANNERS FOR CHANGING GRAPHS

In recent years, there have been variants of spanners studied in the literature which typically involve changes allowed in the initial graph $G$, for example edge addition or deletion.

13.1 Fault Tolerant Spanners

[55] Shiri Chechik, Michael Langberg, David Peleg, and Liam Roditty. Fault tolerant spanners for general graphs. *SIAM Journal on Computing*, 39(7):3403–3423, 2010

[14] Giorgio Ausiello, Andrea Ribichini, Paolo G Franciosa, and Giuseppe F Italiano. Computing graph spanners in small memory: fault-tolerance and streaming. *Discrete Mathematics, Algorithms and Applications*, 2(04):591–605, 2010

[70] Michael Dinitz and Robert Krauthgamer. Fault-tolerant spanners: Better and simpler. In *Proceedings of the 30th Annual ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing*, PODC ’11, pages 169–178, New York, NY, USA, 2011. ACM

[43] Gilad Braunschvig, Shiri Chechik, David Peleg, and Adam Sealfon. Fault tolerant additive and $(\mu, \alpha)$-spanners. *Theoretical Computer Science*, 580:94–100, 2015

[30] Davide Bilò, Fabrizio Grandoni, Luciano Gualà, Stefano Leucci, and Guido Proietti. Improved purely additive fault-tolerant spanners. In *Algorithms-ESA 2015*, pages 167–178. Springer, 2015

[112] Merav Parter. Vertex fault tolerant additive spanners. *Distributed Computing*, 30(5):357–372, October 2017

[33] Greg Bodwin, Michael Dinitz, Merav Parter, and Virginia Vassilevska Williams. Optimal vertex fault tolerant spanners (for fixed stretch). In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1884–1900. Society for Industrial and Applied Mathematics, 2018

[36] Greg Bodwin and Shyamal Patel. A trivial yet optimal solution to vertex fault tolerant spanners. In *Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing*, pages 541–543. ACM, 2019

Chechik et al. [55] study fault tolerant spanners which were introduced originally for geometric graphs by Levcopoulos et al. [106]. Given a stretch parameter $t \geq 1$ and a fault parameter $f \in \mathbb{N}$, an $f$-vertex (resp. $f$-edge) fault tolerant multiplicative $t$-spanner is a subgraph $G' = (V, E')$ such that for any set $F \subset V$ (resp. $F \subset E$) with $|F| \leq f$, we have

$$d_{G' \setminus F}(u, v) \leq t \cdot d_G(u, v), \quad u, v \in V \setminus F \quad (\text{resp. } u, v \in V).$$
The main result of [55] is to provide a randomized algorithm which computes an $f$–vertex fault tolerant $(2k - 1)$–spanner with high probability containing $O(f^2k^{f+1}n^{1+\frac{1}{f}}\log^{1-\frac{1}{f}}n)$ edges in running time $O(f^2k^{f+1}n^{3+\frac{1}{f}}\log^{1-\frac{1}{f}}n)$. For the edge fault tolerant case, their algorithm produces a $(2k - 1)$–spanner with high probability containing $O(fn^{1+\frac{1}{f}})$ edges.

Ausiello et al. [14] give improved constructions of fault tolerant spanners both in terms of time and number of edges compared to [55]. They construct an $f$–vertex fault tolerant $(3, 2)$–spanner with $O(f^{\frac{1}{2}}n^{\frac{3}{2}})$ edges in $\tilde{O}(f^2m)$ time, and an $f$–vertex fault tolerant $(2, 1)$–spanner with $O(fn^{\frac{5}{4}})$ edges in $\tilde{O}(fm)$ time.

Dinitz and Krauthgamer [70] improve on the size of a fault tolerant spanner compared to Chechik et al. [55] by replacing the factor $k^{f+2}$ with $f^2$, thus giving a spanner polynomial in the fault size rather than exponential. The main results of [70] give a way to transform a spanner into a fault tolerant spanner. Specifically, if a $t$–spanner has $g(n)$ edges, then their algorithm produces an $f$–vertex fault tolerant $t$–spanner with $O(f^2 \log n)g(\frac{n}{2^t})$ edges. Applying this to the greedy $(2k - 1)$–spanner construction of [10] yields an $f$–vertex fault tolerant $(2k - 1)$–spanner with $\tilde{O}(f^{2}n^{\frac{3}{2}+\frac{1}{f}})$ edges. Additionally, in the unweighted case, they provide a $O(\log n)$–approximation algorithm for finding the minimal cost $f$–vertex fault $2$–spanner problem.

Bodwin et al. [33] give improved bounds on the size of fault tolerant spanners created by the direct analogue of the greedy algorithm of Althöfer et al. for fault tolerant spanners. They show that for any positive integer $f$, there is an $f$–vertex (resp. $f$–edge) fault tolerant $(2k - 1)$–spanner with $O(f^{1+\frac{1}{f}}n^{1+\frac{1}{f}}\exp(k))$ edges. They also demonstrate that this bound is tight, up to the $\exp(k)$ factor, in the setting of vertex faults assuming Erdős’ Girth Conjecture. Subsequently, Bodwin and Patel [36] greatly simplify the analysis of the so-called Fault Tolerant Greedy Algorithm, removing the $\exp(k)$ factor in the upper bound of [33], and proving existential optimality (up to constant factors) even if the Girth Conjecture fails.

Braunschvig et al. [43] study additive and linear fault tolerant spanners. They use a novel technique to combine a $(1, \beta)$–spanner construction with an $(\alpha, 0)$–spanner construction to obtain a vertex and edge fault tolerant $(\alpha, \beta)$–spanner. In particular, they prove that given an algorithm which produces an $(\alpha, \beta)$–spanner of size $O(n^{1+\delta})$, then for any $\epsilon > 0$, one can obtain an $f_{\epsilon}$–edge, $f_{\epsilon}$–vertex fault tolerant $(\alpha + \epsilon, \beta)$–spanner of size $O((f_{\epsilon} + f_{\epsilon}^{\beta})(\frac{\beta}{\epsilon})f_{\epsilon}^{1+\delta}n^{1+\delta} \log n)$ provided $\max\{f_{\epsilon}, f_{\epsilon}^{\beta}\} < \lfloor \frac{\beta}{\epsilon} \rfloor + 1$. As a corollary, they use existing spanner constructions of Elkin and Peleg [80] to give a randomized construction of an $f$–edge, $f$–vertex fault tolerant $(1 + \epsilon, \beta)$–spanner of size $O(f(\beta)(\frac{\beta}{\epsilon})f^{2}n^{1+\frac{1}{f}} \log n)$ with high probability. Additionally, applying their result to the construction of Baswana et al. [22] gives an $f$–edge, $f$–vertex fault tolerant $(k + \epsilon, k - 1)$–spanner with $O(f(\frac{k-1}{\epsilon})^{2}n^{1+\frac{1}{f}} \log n)$ edges with high probability.

Bilò et al. [30] improve the constructions of other additive fault tolerant spanners in certain cases. For $f = 1$, they produce edge fault tolerant additive $2$–spanners of size $O(n^{1+\frac{1}{f}})$, $4$–spanners of size $O(n^{1+\frac{1}{f}})$, $10$–spanners of size $O(n^{\frac{3}{2}})$ (with high probability), and $14$–spanners of size $O(n^{\frac{5}{2}})$.

Parter [112] studies clustering + path buying algorithms for producing vertex fault tolerant additive spanners and sourcewise spanners for a single fault vertex. The main results are to produce an additive $2$–spanner of size $O(n^{1+\frac{1}{f}})$, a $4$–additive sourcewise spanner of size $O(n|S| + (\frac{n}{|S|})^{3})$, an additive $6$–spanner of size $O(n^{1+\frac{1}{f}})$, and an $8$–additive sourcewise spanner of size $O(n^{1+\frac{1}{f}})$ provided $|S| = O(n^{1+\frac{1}{f}})$.

Just as (non-faulty) BFS trees are frequently useful towards building non-faulty spanners, a corresponding notion of fault tolerant BFS structures (FTBFS) is useful in many of the above constructions of fault tolerant spanners. An $f$–edge or $\alpha$–vertex FTBFS is defined as an $S \times V$ distance
preserver resilient to \( f \) edge or vertex faults, in the same sense as the above. These were introduced by Parter and Peleg [113], who showed that \( O(|S|^f n^{3/2}) \) edges are needed when \( f = 1 \), for edge or vertex faults, and this is tight in either setting. Subsequently, Parter [111] and Gupta and Khan [90] proved tight bounds of \( O(|S|^f n^{3/2}) \) for either setting with \( f = 2 \). For general \( f \), however, there remains a gap: Parter [111] proved a general lower bound of \( \Omega(|S|^{1/2} n^{3/2}) \) for edge or vertex faults, but the current best upper bound for general \( f \) is \( O(|S|^{1/2} n^{3/2}) \) by [34]. It is a significant open question in the area to close this gap. There is also a related area of fault tolerant reachability trees, which must preserve reachability between all pairs in \( S \times V \), not distance.

### 13.2 Resilient Spanners

[13] Giorgio Ausiello, Paolo Giulio Franciosa, Giuseppe F. Italiano, and Andrea Ribichini. On resilient graph spanners. *Algorithmica*, 74(4):1363–1385, 2016

Ausiello et al. [13] introduce the notion of resilience in graph spanners. Informally, a spanner is said to be resilient if the stretch factor is not increased much by deleting an edge from a spanner. Formally, if \( G \) is any graph, then the fragility of an edge \( e \) is defined by

\[
\text{frag}_G(e) := \max_{u,v \in V} \frac{d_G(u, v)}{d_G(x, y)}.
\]

Given a graph \( G \), \( \sigma \geq 1 \), \( t \geq 1 \), and a \( t \)-spanner \( G' \), an edge \( e \) is \( \sigma \)-fragile in \( G' \) if \( \text{frag}_{G'}(e) > \max\{\sigma, \text{frag}_G(e)\} \), and the \( t \)-spanner \( G' \) is \( \sigma \)-resilient if every edge is not \( \sigma \)-fragile. That is, \( G' \) is \( \sigma \)-resilient provided \( \text{frag}_{G'}(e) \leq \max\{\sigma, \text{frag}_G(e)\} \), \( e \in G' \).

Resilience is a strong property, as the authors show that there is an infinite family of dense graphs which do not admit any 2–resilient spanners other than the graphs themselves. Moreover, resilience is a stronger notion than fault tolerance, and the authors show that there are 1–edge fault tolerant \( t \)-spanners containing edges with fragility at least \( t^2 \) for any \( t \geq 3 \). Additionally, a polynomial time algorithm is given for producing a \( \sigma \)-resilient \((2k - 1)\)-spanner (for \( \sigma \geq 2k - 1 \) and \( k \geq 2 \)) with \( O(Wn^{3/2}) \) edges, where \( W = \frac{w_{\max}}{w_{\min}} \) with \( w_{\max} \) being the largest weighted edge of \( G \) and \( w_{\min} \) being the smallest weighted edge. The runtime for this algorithm is \( O(mn + n^2 \log n) \). The authors also demonstrate that \((\alpha, \beta)\)-spanners can be turned into \( \sigma \)-resilient spanners for any \( \sigma \geq \alpha + \beta \).

### 13.3 Dynamic Algorithms

[12] Giorgio Ausiello, Paolo Giulio Franciosa, and Giuseppe F. Italiano. Small stretch spanners on dynamic graphs. *Journal of Graph Algorithms and Applications*, 10(2):365–385, 2006. Announced at ESA’05

[18] Surender Baswana. Dynamic algorithms for graph spanners. In *Algorithms - ESA 2006, 14th Annual European Symposium, Zurich, Switzerland, September 11-13, 2006, Proceedings*, pages 76–87, 2006

[23] Surender Baswana, Sumeet Khurana, and Soumojit Sarkar. Fully dynamic algorithms for graph spanners. *ACM Transactions on Algorithms*, 8(4):35:1–35:51, 2012
Given a spanner for a graph, dynamic algorithms attempt to maintain the properties of the spanner while edges are being added to or deleted from the initial graph. Thus, edges may need to be added or deleted in the spanner to maintain the distortion property. Dynamic spanners have so far been studied with multiplicative error; it is unclear at present whether this is coincidental or if there is a hardness barrier to obtaining other types of error.

The initial work on the problem was in [12], where the authors present algorithms for maintaining a $3$- or $5$-spanner of an input graph with essentially optimal size and update time proportional to the maximum degree. This update time is amortized, meaning that it holds on average over a sequence of insertions and deletions, but individual updates could take much longer. In [23], the authors present two algorithms for maintaining a sparse (multiplicative) $t$-spanner of an unweighted graph, again with optimal size (assuming the Girth Conjecture). The first algorithm achieves $O(t^4)$ amortized update time (independent of the size of the graph $n$), and the second achieves $O(\text{polylog} n)$ amortized update time (independent of the stretch factor $t$).

Bodwin and Krinninger [35] address the problem of improving from amortized to worst-case update time, meaning that every individual update runs within the stated time with high probability. They provide randomized algorithms to maintain a $3$-spanner with $\tilde{O}(n^{1+\frac{1}{2}})$ edges with worst-case update time $\tilde{O}(n^{\frac{1}{2}})$, or a $5$-spanner with $\tilde{O}(n^{1+\frac{1}{4}})$ edges with worst-case update time $\tilde{O}(n^{\frac{1}{4}})$. Subsequently, [28] improved on these results by essentially converting the amortized construction of [23] to worst-case update time, with only minor changes to the construction parameters.

A notable open question in the area is to progress from oblivious to non-oblivious update time. That is: the update times in [35] and [28] hold with high probability against the randomness used in the algorithms, but only if the adversary choosing the graph updates is not allowed to see the random bits chosen by the algorithm. A non-oblivious construction would hold with high probability even if the adversary can base their updates on the random choices made by the construction. This is an important property because, if dynamic spanners are used as a subroutine in other graph algorithms, the next graph update may depend on the current state of the spanner, which thus requires non-obliviousness to keep the guarantees. The next step beyond non-obliviousness, of course, would be to obtain fully deterministic algorithms that maintain these spanners.

14 SPANNERS FOR SPECIAL CLASSES OF GRAPHS

While the sparse/light spanner problem is typically stated for generic graphs, it has also been studied when the class of input graphs is restricted. In many cases, much stronger guarantees can be made for spanners. Here we highlight some of the literature in this vein, but we highlight the results rather than the techniques except where appropriate.

14.1 Geometric Graphs

[109] Giri Narasimhan and Michiel Smid. Geometric Spanner Networks. Cambridge University Press, New York, NY, USA, 2007

The book [109] provides an extensive treatment of spanners for geometric graphs, and so we do not dwell on them here.
14.2 Directed Graphs

[120] Liam Roditty, Mikkel Thorup, and Uri Zwick. Roundtrip spanners and roundtrip routing in directed graphs. In Proceedings of the thirteenth annual ACM-SIAM symposium on Discrete algorithms, pages 844–851. Society for Industrial and Applied Mathematics, 2002

[69] Michael Dinitz and Robert Krauthgamer. Directed spanners via flow-based linear programs. In Proceedings of the forty-third annual ACM symposium on Theory of computing, pages 323–332. ACM, 2011

[27] Piotr Berman, Arnab Bhattacharyya, Konstantin Makarychev, Sofya Raskhodnikova, and Grigory Yaroslavtsev. Improved approximation for the directed spanner problem. In International Colloquium on Automata, Languages, and Programming, pages 1–12. Springer, 2011

[136] Chun Jiang Zhu and Kam-Yiu Lam. Source-wise round-trip spanners. Information Processing Letters, 124:42–45, 2017

[137] Chun Jiang Zhu and Kam-Yiu Lam. Deterministic improved round-trip spanners. Information Processing Letters, 129:57–60, 2018

Dinitz and Krauthgamer study multiplicative spanners for directed graphs, which is more subtle than the undirected case as one must maintain connectivity of the graph when computing a spanner. Berman et al. [27] propose a flow based linear program formulation of the directed $t$-spanner problem, and give an approximation algorithm to find sparsest $t$-spanner for a given directed graph. Roditty et al. [120] study roundtrip spanners, and Zhu and Lam [136, 137] introduce the notion of sourcewise roundtrip spanners for directed graphs.

14.3 Further Reading

[39] Glencora Borradaile, Hung Le, and Christian Wulff-Nilsen. Minor-free graphs have light spanners. In 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS), pages 767–778. IEEE, 2017

[61] Edith Cohen. Polylog-time and near-linear work approximation scheme for undirected shortest paths. Journal of the ACM (JACM), 47(1):132–166, 2000

[94] Shang-En Huang and Seth Pettie. Thorup–Zwick emulators are universally optimal hopsets. Information Processing Letters, 142:9–13, 2019

15 APPLICATIONS OF GRAPH SPANNERS

Lastly, we will survey some of the applications of spanners in both theory and practice to other parts of math and computer science.

Distributed Computing. A central challenge in distributed computing is to pass information between nodes in an efficient manner. Naively, nodes could constantly broadcast everything they know to all their neighbors – thus propagating information around the system fairly quickly – but this requires high information throughput and processing. Often, a better idea is to build a sparse spanner of the network, greatly reducing the demands of sharing information at the price of only minor latency in propagation. This paradigm appears, for example, in communication networks in parallel computing [29], synchronizers [15], broadcasting [114], arrow distributed queuing protocols [44], wireless sensor networks [123], online load balancing [16], and motion planning in robotics control optimization [47]. In all of these applications, the quality of the spanner that can be built controls the above tradeoff between latency and communication complexity.

Network Routing. Another class of applications arises in the task of passing messages of some kind around a network (this is related to distributed computing in some ways, but often generalized or abstracted differently in the literature). A classic example is to pass packets of information
around the internet in a way that quickly reaches their destination, but these “messages” can generally be construed quite broadly, e.g. as cars in a road network. The routing challenge is more involved than simply computing a spanner, as solutions must balance the efficiency of the chosen paths with the amount of information stored at each node and in the “packet header” being passed around the network. However, many modern routing algorithms exploit spanners as a useful step along the way. See [5, 50, 78, 120, 130] and references within for further information.

**Computational Biology.** To understand and model the history of organisms, Biologists have developed various ways to measure similarity between species, based either on their DNA sequence or on their level of interaction in an environment. But given a matrix of pairwise similarities, how can these be arranged into a graph that succinctly captures biological history? It turns out a good approach is to treat the matrix as a weighted graph (with all possible edges), and then build a spanner of the graph to determine its “most important” connections. This application has been used, for example, to measure genetic distance between contemporary species [17] and to visualize interactions between various proteins. [122]

**Theoretical Applications.** Since many graph algorithms have a runtime dependence on the number of edges in the input graph, one can often preprocess an input graph into a spanner in order to improve speed at the cost of a little accuracy. One area where this has been applied is in polynomial time approximation schemes (PTAS) for the traveling salesman problem [40]. Since improved spanners are often available on special graph classes, so too the PTAS can be improved. Some of these graph classes include planar graphs, bounded-genus graphs, unit disk graphs, and bounded path width graphs [78].

Another class of theoretical applications show that spanners efficiently capture the “backbone” of the network and thus often implicitly represent other important network properties besides distances [11]. This frequently includes the network spectrum, and accordingly spanners have been used to construct spectral sparsifiers [95].

Other miscellaneous applications of spanners include computing distances and shortest paths between points embedding in a geometric space [38, 52], testing graph properties (approximately) in sublinear time [58], the facility location problem [60], and key management in access control hierarchies [27]. Closer applications to spanners themselves include cycle covers of graphs [46], for which the extremal instances can often be decomposed into a union of tree spanners, and to labelling schemes in which the nodes of a graph are labelled in such a way that one can (approximately) recover the distance between nodes by inspecting only their labels [22].

**16 CONCLUSION**

Since their advent, graph spanners have become an important object of study and have found a broad range of applications as well as a rich theory. The goal of this survey was to introduce readers to the overarching techniques that have been employed to compute various types of spanners, and to tabulate the state-of-the-art algorithmic bounds in an accessible way. Additionally, we have posed several open problems along the way which may be of interest to experts and non-experts alike. The literature on spanners continues to grow at a rapid pace, and is unlikely to stop in the near future. Nonetheless, it is hoped that this survey will provide a guiding reference for the state of the field for some time.

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Table 2. Guarantees for Multiplicative Spanners. Here, $n = |V|$, $m = |E|$, $\alpha(n)$ is the inverse Ackermann function, and all weight bounds are multiplied by $W(MST(G))$. Complexities not given in the paper are denoted by **.

| Stretch ($t$) | Size: $O(|E'|)$ | Weight: $O(W(|E'|))$ | Time $O(\cdot)$ | Reference |
|--------------|----------------|----------------------|----------------|-----------|
| 3            | $O(n^2)$       | **                   | $\log n$      | [65]      |
| $2k-1$       | $n^{1+\frac{1}{k}}$ | $(1 + \frac{m}{2^{k+1}})$ | $m(n^{1+\frac{1}{k}} + n\log n)$ | [10]      |
| $2k-1$       | $n^{1+\frac{1}{k}}$ | $kn^{1+\frac{1}{k}}$ | $km$          | [24, 26]  |
| $2k-1$       | $O(kn^{1+\frac{1}{k}})$ | **                   | **            | [66]      |
| $2k-1$       | $O(\min(m, kn^{1+\frac{1}{k}}))$ | **                   | **            | [19]      |
| $(2k-1)(1+\epsilon)$ | $n^{1+\frac{1}{k}}$ | $kn^{\frac{1}{k}}(\frac{1}{\epsilon})^{1+\frac{1}{k}}$ | $m(n^{1+\frac{1}{k}} + n\log n)$ | [52]      |
| $(2k-1)(1+\epsilon)$ | $n^{1+\frac{1}{k}}$ | $kn^{\frac{1}{k}}(\frac{1}{\epsilon})^{1+\frac{1}{k}}$ | $n^{\frac{1}{k}}(1 + \frac{k}{\epsilon^{1+\frac{1}{k}}\log k})$ | [78]      |
| $(2k-1)(1+\epsilon)$ | $n^{1+\frac{1}{k}}(\frac{1}{\epsilon})^{2+\frac{1}{k}}$ | $kn^{\frac{1}{k}}(\frac{1}{\epsilon})^{2+\frac{1}{k}}$ | $kn^{2+\frac{1}{k}}$ | [83]      |
| $(2k-1)(1+\epsilon)$ | $n^{1+\frac{1}{k}}(k + \frac{1}{\epsilon})^{2+\frac{1}{k}}$ | $kn^{\frac{1}{k}}(\frac{1}{\epsilon})^{2+\frac{1}{k}}$ | $km + \min\{n\log n, m\alpha(n)\}$ | [83]      |
| $(2k-1)(1+\epsilon)$ | $n^{1+\frac{1}{k}}$ | $n^{\frac{1}{k}}(\frac{1}{\epsilon})^{2+\frac{1}{k}}$ | **            | [56]      |

Table 3. Guarantees for Additive Spanners. Here, $n = |V|$, $m = |E|$. Complexities not given in the paper are denoted by **. The term $D$ is the diameter of the graph.

| Additive Error ($\beta$) | Size: $O(|E'|)$ | Time $O(\cdot)$ | Reference |
|--------------------------|----------------|----------------|-----------|
| 2                        | $O(n^2)$       | $O(n^2\sqrt{\log n})$ | [8]       |
| 2                        | $n^{\frac{2}{3}}$ | $n^{\frac{2}{3}}$ | [8]       |
| 2                        | $n^{\frac{2}{3}}\log^{\frac{2}{3}}n$ | $n^{\frac{2}{3}}\log^2n$ | [71]      |
| 2                        | $n^{\frac{2}{3}}$ | $n^{2}$ | [100]     |
| 4                        | $O(n^{\frac{2}{3}}\log^{\frac{2}{3}}n))$ | $n^{\frac{2}{3}}\log^{\frac{2}{3}}n + D$ | [49]      |
| 4                        | $O(n^{\frac{2}{3}})$ | ** | [54]      |
| 6                        | $n^{\frac{2}{3}}$ | $mn^{\frac{2}{3}}$ | [22]       |
| 6                        | $n^{2}\log^3n$ | $n^{2}\log^2n$ | [135]      |
| 8                        | $n^{\frac{2}{3}}$ | $n^{2}$ | [135]     |
| $(1, 2k + 4\ell)$        | $\Gamma_k(G) + n^{1+\frac{1}{2}+\frac{1}{2}}$ | $mn^{1+\frac{1}{2}+\frac{1}{2}}$ | [22]      |
| $(1, O(\sqrt{d(u,v)}))$  | $O(n^{1+\frac{1}{2}})$ | ** | [54]      |

Table 4. Guarantees for $(\alpha, \beta)$–spanners. Here, $n = |V|$, $m = |E|$. In [74], $\rho > \frac{1}{2k}$ is required.
Table 5. Guarantees for pairwise and subsetwise spanners. In [98], an algorithm has been provided to compute multiplicative subsetwise spanner with constant approximation ratio for planar graphs.
| Fault $(f, E/V)$ | $(\alpha, \beta)$ | Size: $O(|E'|)$ | Time $O(\cdot)$ | Reference |
|-----------------|------------------|----------------|----------------|-----------|
| $f, V$          | $(2k - 1, 0)$    | $f^{2k^{\frac{f+1}{f-1}}}n^{1+\frac{1}{f}} \log^{1-\frac{1}{f}} n$ | $f^{2k^{\frac{f+2}{f-1}}}n^{3+\frac{1}{f}} \log^{1-\frac{1}{f}} n$ | [55]$^*$ |
| $f, E$          | $(2k - 1, 0)$    | $f^{n^{1+\frac{1}{f}}}$ | $f^{2k^{\frac{f+2}{f-1}}}n^{3+\frac{1}{f}} \log^{1-\frac{1}{f}} n$ | [55]$^*$ |
| $f, V$          | $(2k - 1, 0)$    | $f^{2-\frac{1}{f}}n^{1+\frac{1}{f}} \log n$ | $\cdot$ | [70] |
| $f, V$          | $(2k - 1, 0)$    | $f^{1-\frac{1}{f}}n^{1+\frac{1}{f}} 2^{O(k)}$ | $\cdot$ | [33] |
| $f, V$          | $(2k - 1, 0)$    | $f^{1-\frac{1}{f}}n^{1+\frac{1}{f}}$ | $\cdot$ | [36] |
| $f, V$          | $(3, 2)$         | $f^{\frac{n}{n+\frac{1}{f}}}$ | $O(f^2m)$ | [14] |
| $f, V$          | $(2, 1)$         | $f^{n^{\frac{2}{f}}}$ | $O(fm)$ | [14] |
| $f, E, V$       | $(1 + \varepsilon, \beta)$ | $f^{\beta(\frac{\beta}{k})^{2f}}n^{1+\frac{1}{f}} \log n$ | $\cdot$ | [43]$^*$ |
| $f, E, V$       | $(k + \varepsilon, k - 1)$ | $f^{(\frac{k-1}{f})^{2f}}n^{1+\frac{1}{f}} \log n$ | $\cdot$ | [43]$^*$ |
| $1, E$          | $(1, 2)$         | $n^{\frac{2}{f}}$ | $\cdot$ | [30] |
| $1, E$          | $(1, 4)$         | $n^{\frac{4}{f}}$ | $\cdot$ | [30] |
| $1, E$          | $(1, 10)$        | $\tilde{O}(n^{\frac{10}{f}})$ | $\cdot$ | [30]$^*$ |
| $1, E$          | $(1, 14)$        | $n^{\frac{14}{f}}$ | $\cdot$ | [30] |
| $1, V$          | $(1, 2)$         | $n^{\frac{2}{f}}$ | $\cdot$ | [112] |
| $1, V$          | $(1, 6)$         | $n^{\frac{6}{f}}$ | $\cdot$ | [112] |
| $f, E, V$       | $(2k - 1, 0)$    | $f^{1-\frac{1}{f}}n^{1+\frac{1}{f}} 2^{O(k)}$ | $\cdot$ | [33] |
| $f, E, V$       | $(2k - 1, 0)$    | $f^{1-\frac{1}{f}}n^{1+\frac{1}{f}}$ | $\cdot$ | [36] |

Table 6. Guarantees for Fault Tolerant Spanners. Here, $n = |V|$, $m = |E|$, $f$ is the number of faults allowed, and $E$ or $V$ in the first column denotes if the guarantee is for edge-fault or vertex-fault tolerance. Complexities not given in the paper are denoted by **. Spanners constructed with high probability are denoted by $^*$ next to the citation.