Bayesian mixed-effect models for independent dynamic social network data

Fabio Vieira¹  Roger Leenders²,³  Daniel McFarland⁴  Joris Mulder¹

¹Department of Methodology and Statistics, Tilburg University
²Department of Organization Studies, Tilburg University
³Jheronimus Academy of Data Science
⁴Department of Education, Stanford University

April 25, 2022

Abstract

Relational event or time-stamped social network data have become increasingly available over the years. Accordingly, statistical methods for such data have also surfaced. These techniques are based on log-linear models of the rates of interactions in a social network via actor covariates and network statistics. Particularly, the use of survival analysis concepts has stimulated the development of powerful methods over the past decade. These models mainly focus on the analysis of single networks. To date, there are few models that can deal with multiple relational event networks jointly. In this paper, we propose a new Bayesian hierarchical model for multiple relational event sequences. This approach allows inferences at the actor level, which are useful in understanding which effects guide actors’ preferences in social interactions. We also present Bayes factors for hypothesis testing in this class of models. In addition, a new Bayes factor to test random-effect structures is developed. In this test, we let the prior be determined by the data, alleviating the issue of employing improper priors in Bayes factors and thus preventing the use of ad-hoc choices in absence of prior information. We use data of classroom interactions among high school students to illustrate the proposed methods.

Keywords: Relational event history, Bayesian inference, Multilevel analysis, Social networks
1 Introduction

The spreading of wearable technologies within organizations has opened the possibility for scientists to gain a broader understanding about the intricacies behind social dynamics (Eagle & Pentland 2003). Nowadays, most people carry in their pockets devices that are more powerful than early desktop computers. These technological advancements, combined with constant development of new communication applications, has originated huge amounts of data on social interactions. This has enabled researchers to build and test increasingly rich and complex models of human interaction using time-stamped data of people interacting with each other in continuous time. The field of social network analysis is among the fields that have benefited the most from having access to rich temporal interaction data. In this paper, we specifically focus on the Relational Event network model (Butts & Marcum 2017; Stadtfeld & Block 2017; Mulder & Leenders 2019).

A relational event network is comprised of multiple interactions among a finite set of actors. Observation units are called "relational events", which are defined as discrete instances of interaction among social entities along a timescale (Butts & Marcum 2017). In a directed network, events contain a clear indication of sender and receiver. Thus, every observation unit displays information on which actor was the sender, which actor was the receiver, and at which point in time the interaction occurred.

One of the traditional approaches to social network analysis is based on Markovian random graphs theory and involves the aggregation of events into a graph (Van Der Hofstad 2009). Transitions on the graph structure are then modeled via sufficient statistics, which are functions of the observed network (Frank & Strauss 1986; Hanneke, Fu, & Xing 2010; Lusher, Koskinen, & Robins 2013). A different approach was taken by Butts (2008), who employed survival analysis concepts and introduced the relational event model which profits from the temporal structure of relational events. In this framework, the main goal is to model the rates of communication among sender and receivers via a log-linear function, without requiring the data to be aggregated. This is done by employing sufficient statistics that capture important social patterns together with actor-specific covariates.

The relational event model has then gained popularity and received multiple extensions. For instance, Vu et al. (2011) introduced a model with time-varying parameters, based on the additive Aalen model (Aalen 1989), and developed an algorithm for online inference in social networks. DuBois et al. (2013) introduced a Bayesian hierarchical model to estimate population effects in multiple social networks. Perry & Wolfe (2013) used a partial likelihood approach to modeling who the receiver will be for a given sender. Vu et al. (2015) implemented case-control sampling to decrease the number of computations in estimating relational event models. Later, the control-case sampling approach was further explored by Lerner & Lomi (2020) to estimate relational event models in large networks. Stadtfeld, Hollway, & Block (2017) and Stadtfeld & Block (2017) introduced a two-step model, where the first step consists of modeling the activity rate of the sender and the second one features the choice of the receiver conditional.
on the sender. Mulder & Leenders (2019) proposed a way to analyze the temporal evolution of effects in the social network, by estimating effects in different subsets of the data determined by overlapping intervals, which they called moving windows.

In relational event modeling, the stream of events in a given network (ordered over time) is often referred to as a relational event sequence. The common approach is to deploy a relation event model to a single relational event sequence, so as to estimate parameters about how that specific network evolved. In contrast, in this paper, we focus on modeling multiple relational event sequences. The example we will use to illustrate the methods is about social interactions in fifteen classrooms in the same high school. Rather than fitting a relational event model to each classroom separately, we fit the parameters of the fifteen classrooms jointly, using a hierarchical modeling approach, and derive metrics that allow researchers to then test a wide variety of substantively interesting effects across these classrooms. We achieve this by modeling the relational event sequences in a Bayesian hierarchical fashion via Markov chain Monte Carlo (MCMC) methods. DuBois et al. (2013) previously extended the standard relational event model to incorporate multiple sequences, proposing a hierarchical model and focusing their work mainly on the estimation process. Here, we go a step further and propose inferential methods to test population parameters in hierarchical relational event models. In addition, we propose a new hierarchical extension for the dynamic actor-oriented model (Stadtfeld et al. 2017). This approach consists of modeling two different rates: the sender activity rate and the receiver choice rate. Both rates are represented in log-linear form and can accommodate actor specific covariates and network endogenous statistics. Hence, this model allows us to perform inferences at the actor level, making it possible to determine which effects affect the activity level of the sender and the choice of the receiver. Our testing approach utilizes Bayes Factors, which enable researchers to consistently quantify the evidence of multiple hypotheses against each other (Jeffreys 1961).

Moreover, we propose a new approach to test the adequacy of random-effect structures. For example, we may be interested whether the effect of the teacher on the interaction rates varies across classrooms or whether it can be considered as fixed. If we have $K$ sequences of relational events (in our example, we have $K = fifteen$ classrooms) and we define a random effect, say $\beta_k$, in each one of those sequences as in a hierarchical model, i.e. $\beta_k \sim N(\mu, \sigma^2)$, $k = 1, \ldots, K$, where $\mu$ quantifies the global mean effect and $\sigma^2$ quantifies the variability of effects across the independent groups around the global mean. The test we propose focuses on whether we can assume all $\beta_k$ as being equal ($\beta_1 = \beta_2 = \cdots = \beta_K$) while using the second level, $\beta_k \sim N(\mu, \sigma^2)$ as a prior. This allows us to compute a Bayes factor in the absence of prior information. We will show that our proposal is easier to compute than other approaches (e.g., Mulder & Fox 2019) as we avoid the calculation of marginal likelihoods which can be computationally expensive.

Furthermore, we present Bayes factors for testing equality and ordered constraint hypotheses. Regarding means and fixed effects, those tests are based on the approximated fractional Bayes factor approach (Gu, Mulder, & Hoijtink 2018), which consists of approximating a Bayes factor using Gaussian
distributions and does not require the computation of marginal likelihoods. In addition, we present a method to test ordered constraint hypotheses of variance parameters, which can be used to formulate hypotheses of variability expectations in effects when differences among the groups are more pronounced (Böing-Messing & Mulder 2020).

Our methods have been implemented in R (R Core Team 2017), interfacing with Stan through the rstan package (Stan Development Team 2018). We provide the full code to facilitate the fitting of other relational event models (see https://github.com/Fabio-Vieira/bayesian_dynamic_network).

The remainder of this paper is divided as follows: in Section 2 we describe the relational events framework and present the hierarchical actor-oriented relational event model, this Section also introduces the Bayesian specification of the model by presenting prior distributions and briefly discussing a reparametrization that allows more efficient sampling; we details the hypothesis testing methods in Section 3 and conduct the empirical illustration in Sections 4 (outline) and 5 (results). The paper ends with a discussion in Section 6, where we go through some limitations of the model and possible routes for future research.

2 The relational event framework

The relational event model (REM) is used to model the rate of interactions in dynamic social networks (Butts 2008). In this framework, events happen among actors at particular points in time, being represented by tuples in the form $e_t = (s, r, t)$, where $s$ is the sender, $r$ is the receiver, and $t$ is the time point of the interaction. Thus $e_t$ is called a relational event. Therefore, in a hierarchical setting (DuBois et al. 2013), we have $K$ clusters (in this paper we will use cluster, groups, relational event sequence and network interchangeably), each of which with $N_1, \ldots, N_K$ social actors allowed to be senders or receivers at any given time. At time $t$, in cluster $k$, sender $s \in \{1, 2, \ldots, N_k\}$ interacts with receiver $r \in \{1, 2, \ldots, N_k \mid r \neq s\}$, forming a dyad $(s, r) \in R_k(t)$, where $R_k(t) = \{(s, r) : s, r \in \{1, 2, \ldots, N_k\}, s \neq r\}$ is called the risk set, which comprises all possible dyads $(s, r)$ at a particular point in time.

The main idea consists of assuming that, for each cluster $k$, for $k = 1, \ldots, K$, we are able to observe an ordered sequence of $M_k$ dyadic events among $N_k$ individuals on the time window $[0, \tau_k) \in \mathbb{R}^+$. Thus a relational event sequence for cluster $k$ is formally defined as

$$E_k = \{e_{tm} = (s_m, r_m, t_m) : (s_m, r_m) \in R_k(t_m), 0 < t_1 < \cdots < t_{M_k} < \tau_k\},$$

where $t_m$ is the time at which the $m^{th}$ event occurred, in cluster $k$. Then, following Butts (2008), who borrows concepts from survival analysis, the relational event history $E_k$ is modeled as a stochastic process, where the rate of events from sender $s$ to receiver $r$, with $(s, r) \in R_k(t)$, is given by the intensity $\lambda_{sr}(t|E_k)$. This intensity function has the form of a Cox proportional hazards model (Cox 1972). Moreover, the intensity is assumed constant between subsequent events and the waiting times
conditioned exponentially distributed. This amounts to the well know piecewise constant exponential model for survival data (Friedman 1982). Thus, the survival function is given by $S_{sr}(t_m - t_{m-1}|E_k) = \exp \{- (t_m - t_{m-1}) \lambda_{sr}(t_m|E_k)\}$.

2.1 A hierarchical extension of the actor-oriented model

In this paper, we focus on an alternative relational event model that was proposed by Stadtfeld et al. (2017) and Stadtfeld & Block (2017). Their approach models the receiver given the sender, similar to Perry & Wolfe (2013). Conceptually, it builds on the same tradition as the stochastic actor-oriented model (Snijders 1996), where the evolution of the network is assumed to be a product of actors’ individual behaviors as they constantly seek to maximize their own utilities. The basic framework consists of a two-step approach based on log-linear predictors. First, the waiting time until an actor becomes active is modeled. After that, a multinomial choice model (McFadden 1973) is employed to determine the choice of the receiver by the active sender actor. These two steps are assumed to be conditionally independent given the available information about the past up to that event.

The Bayesian hierarchical approach we will develop for multiple relational event sequences (denoted as “clusters”) will build on this model. At each point in time, in cluster $k$, sender $s \in \{1, 2, \ldots, N_k\}$ starts an interaction with intensity $\lambda_s(t|E_k)$. This intensity is directly proportional to the probability of a given actor $s$ to be the next sender. This probability is given by $P(s_m = s|E_k) = \lambda_s(t|E_k)/\sum_{h \in E_k} \lambda_h(t|E_k), \forall h \in \{1, 2, \ldots, N_k\}$ and $m \in \{1, 2, \ldots, M_k\}$, where $M_k$ is the number of events in cluster $k$. Next, this sender chooses the receiver $r$, forming the dyad $(s, r) \in R_k(t)$, with intensity $\lambda_{sr}(t|s, E_k)$ and $r \in \{1, 2, \ldots, N_k|r \neq s\}$. The receiver intensity represents the rate at which actor $s$ chooses actor $r$ to form a dyad, which is proportional to the probability of observing dyad $(s, r) \in R_k(t)$ as the next one in the sequence. This probability is given by $P(r_m = r|s_m = s, E_k) = \lambda_{sr}(t|s, E_k)/\sum_{h \in E_k} \lambda_{rh}(t|s, E_k), \forall h \in \{1, 2, \ldots, N_k|h \neq s\}$ and $m \in \{1, 2, \ldots, M_k\}$. Then, these intensities for cluster $k$, for the sender and receiver steps of this model, will be given by the following log-linear functions:

$$\lambda_s(t|E_k) = \exp \{\phi^t z_s(t) + \gamma_s x_s(t)\},$$

$$\lambda_{sr}(t|s, E_k) = \exp \{\psi^t z_{sr}(t) + \beta x_{sr}(t)\},$$

where $\phi$ and $\psi$ are vectors of fixed-effect parameters, $\gamma_k$ and $\beta_k$ are vectors of random-effect parameters for cluster $k$, with $k = 1, \ldots, K$. The vectors of statistics $z_s(t)$ and $x_s(t)$ are associated with actor $s \in \{1, 2, \ldots, N_k\}$, whereas $z_{sr}(t)$ and $x_{sr}(t)$ are vectors of statistics associated with dyad $(s, r) \in R_k(t)$.

Assuming this log-linear form will allow us to conduct inference at the actor level unveiling effects that make actors more (or less) prone to start interactions or more (or less) likely to be chosen as the
next receiver. Also, this idea helps us limit the size of the set of possible dyads that need to be analyzed at each point in time: if actor $s$ is the one starting an interaction at time $t$, then all dyads where $s$ is not the sender become impossible to happen. Thus, for the actor-oriented model, the risk set, $R_k(t)$, will have size $N_k - 1$ (given the sender), whereas for a dyadic model, such as in DuBois et al. (2013), the risk set has size $(N_k - 1)N_k$, which can easily become massive for medium to large networks.

The likelihood for the actor-oriented model is given by

$$p(E|\theta, Z, X) = \prod_{k=1}^{K} \prod_{m=1}^{M_k} \left[ \left( \lambda_{sm} \left( t_m | E_k \right) \prod_{s \in k} S_s \left( t_m - t_{m-1} | E_k \right) \right) \times \frac{\lambda_{sm|s} \left( t_m | s_m, E_k \right)}{\sum_{r \in k} \lambda_{r|s} \left( t_m | s_m, E_k \right)} \right] \times \prod_{s \in k} S_s \left( \tau_k - t_{M_k} | E_k \right),$$

where $\theta$ is the vector containing all parameters and $Z$ and $X$ are matrices with fixed and random effects covariates, respectively. The time of the last observed event in cluster $k$ is denoted by $t_{M_k}$ and $\tau_k$ the end of the observation period. In most empirical applications, it is assumed that $t_{M_k} = \tau_k$. This way the last part of the likelihood is equal to 1, since due to the piecewise constant exponential assumption we have $S_{sm} \left( t_m - t_{m-1} | E_k \right) = \exp \left\{ - (t_m - t_{m-1}) \lambda_s \left( t_m | E_k \right) \right\}$.

Finally, since the likelihood of the actor-oriented model is simply a multiplication of the piece-wise exponential likelihood with the likelihood of a multinomial regression model, one can generate data from this model using the following steps:

1. For cluster $k$ in 1 to $K$;
2. Initialize endogenous statistics values;
3. Compute $\lambda_s \left( t_m | E_k \right) = \exp \{ \phi' z_s(t) + \gamma'_s x_s(t) \}, \forall s \in k$;
4. Sample inter-event time $t_m - t_{m-1} \sim \text{Exp} \left( \sum_{h \in k} \lambda_h \left( t_m | E_k \right) \right)$;
5. Sample the sender $P \left( s_m = s | E_k \right) = \frac{\lambda_s \left( t_m | E_k \right)}{\sum_{h \in k} \lambda_h \left( t_m | E_k \right)}, \forall s \in k$;
6. Compute $\lambda_{r|s} \left( t_m | s_m, E_k \right) = \exp \{ \psi' z_{sr}(t) + \beta'_r x_{sr}(t) \}, \forall r \in k$;
7. Sample dyad (i.e., the receiver, for the given sender): $P \left( r_m = r | s_m = s, E_k \right) = \frac{\lambda_{r|s} \left( t_m | s_m, E_k \right)}{\sum_{h \in k} \lambda_h \left( t_m | s_m, E_k \right)}, \forall r \in k$;
8. Update endogenous statistics given the sampled event;
9. Repeat steps 3 to 8 $M_k$ times.

In this setting, at time $t_m$, $P \left( s = s_m | E_k \right)$ is the probability of sender $s$ being active; the probability of actor $r$ being the receiver given that $s$ is the sender is given by $P \left( r_m = r | s_m = s, E_k \right)$.
2.2 Prior distributions

In this paper we implement a Bayesian framework, employing Markov chain Monte Carlo (MCMC) sampling methods. Therefore, we need to combine the likelihood in equation (2) with prior distributions in order to fully specify the models and derive posterior distributions. We start by focusing on the receiver model.

The hierarchical structure of the data requires that we specify a second level in the model as $\beta_k \sim \mathcal{N}(\mu_\beta, \Sigma_\beta)$, $k = 1, \ldots, K$, where $\mu_\beta$ is the mean vector and $\Sigma_\beta$ is a covariance matrix. From a Bayesian perspective this second level is seen as a prior for the random effects. Assuming we have $Q$ fixed effects, $P$ random effects and $K$ clusters, we use the following priors

- $\psi_q \sim \mathcal{N}(0, \sigma_\psi^2)$, $q = 1, \ldots, Q$,
- $\mu_\beta_p \sim \mathcal{N}(0, \sigma_\mu^2)$, $p = 1, \ldots, P$,

where $\psi$ are fixed-effect parameters and $\mu_\beta \in \mathbb{R}^P$ is the grand mean vector of the hierarchical prior for the random effects $\beta_k$. If a sequence of events does not have enough information about a particular random-effect, that component of $\beta_k$ is shrunk towards its grand mean in $\mu_\beta$ (DuBois et al. 2013). This allows us to borrow strength from other sequences where the random-effect could be better estimated, making it possible to conduct inference even when a sequence has a small sample size (Gelman & Hill 2006).

Completing the prior specification of $\beta_k$, let a $\Sigma_\beta$ be a $P \times P$ covariance matrix. We can decompose this matrix into

$$
\Sigma_\beta = \tau \times \Omega \times \tau,
$$

where $\tau := \{\text{diag}(\tau_1, \tau_2, \ldots, \tau_P)\}$ is a diagonal matrix of standard deviations and $\Omega$ is a correlation matrix (Gelman & Hill 2006). Thus, following Carpenter et al. (2017), $\Omega$ will have a Lewandowski-Kurowicka-Joe (LKJ) prior and $\tau$ a half-Cauchy prior as follows

- $\tau_p \sim \text{half-Cauchy}(0, \sigma_\tau)$, $p = 1, \ldots, P$,
- $\Omega \sim \text{LKJCorr}(\eta)$.

The LKJ prior is defined as $\text{LKJCorr}(\Omega|\eta) \propto \det(\Omega)^{\eta-1}$, with $\eta \in \mathbb{R}^+$. This distribution allows us to sample uniformly from the space of positive definite correlation matrices and has a behavior similar to the beta distribution (Wang, Wu, & Chu 2018; Lewandowski, Kurowicka, & Joe 2009). For example, when $\eta = 1$ it has a uniform behavior, when $\eta < 1$ it favors stronger correlation, whereas when $\eta > 1$ it favors weaker correlation.

The priors above fully specify the receiver model, but we also need priors for the regression parameters of the sender model. Using the same idea as before, with $\gamma_k \sim \mathcal{N}(\zeta_\gamma, \Sigma_\gamma)$, $k = 1, \ldots, K$, and assuming we have $U$ fixed and $V$ random effects,
• $\phi_u \sim \mathcal{N}(0, \sigma_{\phi}^2)$, $u = 1, \ldots, U$.

• $(\zeta_v) \sim \mathcal{N}(0, \sigma_{\zeta}^2)$, $v = 1, \ldots, V$.

where $\phi$ are fixed-effect parameters and $\zeta_v \in \mathbb{R}^V$ is the grand mean of the random-effects parameter vector $\gamma_k$. The decomposition of the covariance matrix, $\Sigma$, receives an LKJ-half-Cauchy prior as well. Also, the same reasoning for using the multivariate normal specification discussed above for $\beta$ is valid for $\gamma$.

2.3 Reparameterization and computational issues

Due to the hierarchical structure of our data, the random-effects parameters $\beta_k$ and $\gamma_k$ are highly correlated with the population parameters $\mu$, $\zeta$ and $\Sigma$. This introduces severe inefficiencies into the sampling process. When the data are sparse, which is a characteristic of most social network data, the geometry of the posterior distribution makes it very difficult to sample from the highest posterior density areas. Betancourt & Girolami (2015) called these issues pathologies of the hierarchical model. Therefore, to ease the burden on the sampler, we take advantage of the multivariate normal structure of the random-effects and apply a non-centered linear transformation to those parameters.

**Lemma 1.** Let $\beta \sim \mathcal{N}(\mu, \Sigma)$, where $\beta \in \mathbb{R}^p$. Then, with $\mu \in \mathbb{R}^p$ and $A$ being a $p \times p$ matrix, such that $AA' = \Sigma$, one can write $\beta = AZ + \mu$, where $Z \in \mathbb{R}^p$ and $Z \sim \mathcal{N}(0, I)$, where $I$ is a $p \times p$ identity matrix.

This transformation can be applied to both $\beta$ and $\gamma$. A natural candidate for matrix $A$ is the Cholesky factorization of the covariance matrix $\Sigma$. For details see appendix A.

This reparameterization is more efficient for two reasons. First, it reduces the dependency between the random effects parameters and the population parameters by sampling from independent standard normal distributions. This simplifies the geometry of the posterior and avoids inverting $\Sigma$ at every evaluation of the multivariate normal density (Carpenter et al. 2017). Therefore, we can safely and efficiently transform the random-effects parameters without causing any change in the prior specification of population parameters.

We implement our approach in Stan due to its simplicity, which only requires a specification of the likelihood and the prior distributions. Stan is a probabilistic programming language that employs Hamiltonian Monte Carlo (HMC) algorithms to sample from posterior distributions. The advantage of HMC methods is that they avoid the random walk behavior and the sensitivity to posterior correlations that plague many Bayesian applications (Hoffman & Gelman 2014), including hierarchical models. This particularities allow HMC to generally converge to high dimensional distributions much faster than Metropolis-Hastings or Gibbs sampler methods (Hoffman & Gelman 2014, Betancourt & Girolami 2015).
3 Bayesian hypothesis testing for mixed-effects relational event models

In multilevel analysis, researchers are usually interested in testing which theories receive the most support from the observed data, so inferences about the population can be conducted. This kind of analysis is carried out through a process called hypothesis testing. From a Bayesian perspective those tests are usually performed by the computation of Bayes factors (BF) \cite{Jeffreys1961}. The BF is given by the ratio of marginal likelihoods under the parameter space of competing hypotheses. For instance, let \( E \) be the observed data, \( \theta \) a vector of parameters in the space \( \Theta \), and \( H_0 \in \Theta_0 \) be a hypothesis that will be tested against \( H_1 \in \Theta_1 \), then \( \text{BF}_{01} \) is expressed as

\[
\text{BF}_{01} = \frac{m(E|H_0)}{m(E|H_1)},
\]

where \( m(E|H_i) = \int_{\theta_i \in \Theta_i} p(\theta_i|E)p(\theta_i)\,d\theta_i \), for \( i = 0, 1 \), with \( p(\theta_i|E) \) being the likelihood and \( p(\theta_i) \) the prior. Also, \( \Theta_0 \cap \Theta_1 = \emptyset \), with both \( \Theta_0 \) and \( \Theta_1 \) being subsets of \( \Theta \). \cite{KassRaftery1995} provide a rule-of-thumb for Bayes Factors interpretation. In their setting, the evidence provided by the \( \text{BF}_{01} \) in favor \( H_0 \) can be seen as "insufficient" if \( 1 < \text{BF}_{01} < 3 \), "positive" if \( \text{BF}_{01} > 3 \), "strong" if \( \text{BF}_{01} > 20 \), and "very strong" if \( \text{BF}_{01} > 150 \). These are rough guidelines to aid the interpretation of Bayes factors and should not be used as strict cut-off values.

3.1 Testing for the presence of random effects

When testing whether the random effect structure is needed, our goal is to determine whether the variance in the second level \( \beta_k \sim \mathcal{N}(\mu, \sigma^2) \), for \( k = 1, \ldots, K \), is significantly different from zero. In other words, we want to determine whether the effect is homogeneous across clusters. Generally, a researcher’s interest is in testing \( H_0 : \sigma^2 = 0 \) against \( H_1 : \sigma^2 > 0 \). If \( H_0 \) is not supported by the data, this means that we do not have one \( \beta_k \) for each group, but only one parameter fixed across all clusters. From a statistical perspective, this test is appealing because fixing an effect across groups can result in a drastic dimensionality reduction in the model. This yields more precise estimates and better inference, since the data from all groups would be used to estimate a single parameter. Moreover, it is substantively interesting to understand whether the effect of a covariate varies across clusters. For example, when aiming to understand the effect of gender on the interaction intensity in classrooms, it is important to test whether gender can be seen as a fixed effect, having equal strength across classrooms in a school, or whether the effect of gender varies between classrooms and should therefore be considered random.

Unfortunately, since \( \sigma^2 > 0 \) should hold under the hierarchical model, testing whether \( \sigma^2 = 0 \) amounts to testing whether the parameter lies in a region that is outside its admissible space. Thus, we formulate this test in a slightly different, but equivalent way as \( H_0 : \beta_1 = \cdots = \beta_K \) versus \( H_1 : \text{not } H_0 \).
In order to perform this hypothesis test using the Bayes factor the challenge is how to specify the prior distribution of the $\beta_k$’s (in particular the prior variance). We solve this by using the fact that the second level of a multilevel model (which defines the distribution of the random effects) can be viewed as a prior for the $\beta_k$’s. By estimating the hyperparameters of the random effects distribution we obtain a prior under the alternative hypothesis.

First, we run the sampler to obtain posterior samples for $\mu$, $\sigma^2$, and $\beta_k$, for $k = 1, \ldots, K$. Based on this, we compute the posterior estimates for $\bar{\mu}$ and $\bar{\sigma}^2$. We approximate the posterior distribution of $\beta_k$ by a Normal distribution (Walker 1969) by computing the sample mean, $\bar{\beta}_k$, and the sample variance, $\bar{\tau}_k^2$, of the posterior distribution of $\beta_k$. Thus, for $k = 1, \ldots, K$ we can write,

$$\text{prior: } \beta_k \sim \mathcal{N}(\bar{\mu}, \bar{\sigma}^2)$$

$$\text{posterior: } \beta_k | E_k \sim \mathcal{N}(\bar{\beta}_k, \bar{\tau}_k^2). (5)$$

Thus, we can test the hypothesis of whether all $\beta_k$, for $k = 1, \ldots, K$, are equal by writing $\xi_j = \beta_{j+1} - \beta_j$, for $j = 1, \ldots, K - 1$:

$$H_0 : \xi_1 = \ldots = \xi_{K-1} = 0$$

$$H_1 : \text{not } H_0$$

Then, we have to derive the prior and posterior of $\xi = (\xi_1, \ldots, \xi_{K-1})$. These distributions can be approximated by multivariate Normals with dimension $K - 1$:

$$\text{prior: } \xi \sim \mathcal{N}(\bar{\mu}_\xi, \bar{\Lambda}_\xi)$$

$$\text{posterior: } \xi | E \sim \mathcal{N}(\bar{\mu}_\xi | E, \bar{\Sigma}_\xi)$$

where, the $j^{th}$ component of $\bar{\mu}_\xi$ is equal to $\bar{\beta}_{j+1} - \bar{\beta}_j$. For the diagonal of matrix $\bar{\Sigma}_\xi$ we have $\text{Var}(\xi_j | E) = \bar{\tau}_j^2 + \bar{\tau}_j^2$. Matrix $\bar{\Lambda}_\xi$ follows the same reasoning, but with diagonal $\text{Var}(\xi_j) = \bar{\sigma}^2 + \bar{\sigma}^2$. Off-diagonal we have:

$$\text{Cov}(\xi_j, \xi_{j+h}) = \begin{cases} -\text{Var}(\beta_{j+1}), & \text{if } h = 1 \\ 0, & \text{if } h > 1. \end{cases}$$

For the prior we therefore have $\text{Cov}(\xi_j, \xi_{j+h}) = -\bar{\sigma}^2$ and for the posterior $\text{Cov}(\xi_j, \xi_{j+h} | E) = -\bar{\tau}_{j+1}^2$, if $h = 1$ (and 0 otherwise). Details of the derivation of these parameters are available in Appendix C. Then we can test $H_0$ against $H_1$ by computing the Savage-Dickey density ratio (Dickey 1971) between the posterior and prior (see appendix B) as follows

$$\text{BF}_{01} = \frac{\pi(\xi_1, \ldots, \xi_{K-1} = 0 | E)}{\pi(\xi_1, \ldots, \xi_{K-1} = 0)},$$
where \( \pi(\cdot) \) represents the probability density function.

One interesting aspect of this Bayes factor is that the prior is fully determined from the data, similar to empirical Bayes estimation of hierarchical models. This property is especially useful here, as Bayes factors are known to be sensitive to the choice of the prior. As far as we know this has not yet been proposed in the literature and it is quite generally applicable. Furthermore, the advantage is that no prior information is required and no other ad-hoc choices are needed and we get a nice interpretable outcome regarding the relative evidence in the data of whether a hierarchical structure is applicable or not given the observed data.

3.2 Testing order constraints on random-effect variances

After establishing which effects are random across clusters (using the test from the previous subsection) it is useful to investigate whether a specific effect contributes more to the differences (or heterogeneity) among sequences than does another effect. This comes down to testing whether a specific random effect varies more across clusters than another random effect, or, equivalently, whether one random effect’s variance is larger than another. More generally we could also test for a specific ordering of random effect variances. Substantively this test would be useful to get insights about the heterogeneity of drivers of social interaction behavior across groups. For example, one might be interested to test whether the effect of gender varies more strongly between classrooms than the effect of race. Or, extending the hypothesis, one might test whether teacher effect on interaction is even more heterogeneous across classes than gender, while race displays the least heterogeneity. Despite the substantive usefulness of such a test, we have not yet seen a statistical test for ordered random effect variances in the statistical literature.

This test is carried out by comparing the magnitude of the variances. Assuming that we have \( P \) random effects in our model, this expectation can be formulated as \( H_0 : \sigma_1 < \sigma_2 < \cdots < \sigma_p \). Following Boing-Messing & Mulder (2020), the Bayes factor of an ordered constraint hypothesis on variance parameters, such as \( H_0 \), against its complement (denoted by \( H_1 \)) is given by the ratio of the posterior and prior probabilities in the space of \( H_0 \) (see appendix B) divided by the same probability ratio in the space of \( H_1 \). Hence,

\[
BF_{01} = \frac{P(\sigma_1 < \sigma_2 < \cdots < \sigma_p | E)}{P(\sigma_1 < \sigma_2 < \cdots < \sigma_p)} \times \left( \frac{P(\text{not } H_0 | E)}{P(\text{not } H_0)} \right)^{-1}.
\]

If \( BF_{01} > 1 \) the evidence suggests that effect \( P \) has the largest impact on the differences in the rates of interaction among the relational event sequences. Computationally, the probability in the numerator for each hypothesis is approximated by the proportion of posterior draws that satisfy the hypothesis of interest, and the probability in the denominator is approximated by the proportion of prior draws in the same space (Boing-Messing & Mulder 2020).
3.3 Testing fixed effects and random-effects means

Since random-effect means and fixed effects have the same role in a mixed effects model, we discuss a set of tests for these parameters in this section. Testing these types of parameters is important in three instances: 1) when evaluating whether a covariate has an actual impact on the general interaction rate in the population, 2) when testing the direction of the effect (positive or negative) to determine whether the covariate increases or decreases interaction rates, and 3) when comparing effect sizes against one another in order to determine which effect has, on average, the largest influence in the interaction behavior. Thus, assuming we have \( P \) random effects such that \( \mu \in \mathbb{R}^P \), we could be interested in testing a constrained hypothesis against the non-constrained one, so that our constraint hypothesis would be of the form \( H_i : \mu \in M_i \), with \( M_i \subset \mathbb{R}^P \). This test could be carried out by computing a probability ratio between posterior and prior distributions (see appendix B), such as

\[
BF_{iu} = \frac{P(\mu \in M_i | E)}{P(\mu \in M)}.
\]

(8)

Since we seek to test a hyperparameter that belongs to the lowest level of the model, the approach (presented in subsection 3.1) does not work. Therefore, we will employ the approximate fractional Bayes factor approach as described in Gu et al. (2018). The fractional Bayes factor uses a small fraction \( b \) of the data to update a non-informative improper prior to obtain the proper default prior. If \( \pi(\mu) \) is a non-informative improper prior, then the proper default prior is given by the Bayes’ rule

\[
\pi^b(\mu | E) = \frac{\pi^b(E | \mu)\pi(\mu)}{\int \pi^b(E | \mu)\pi(\mu)\,d\mu},
\]

(9)

where \( \pi^b(\mu | E) \) is called the fractional prior (Mulder 2014). A multitude of hypothesis tests can be conducted by using this idea

\[
BF_{iu} = \frac{\int_{\mu \in M_i} \pi(\mu | E)\,d\mu}{\int_{\mu \in M} \pi^b(\mu | E)\,d\mu} \times \frac{P(\mu \in M_i | E)}{P^b(\mu \in M_i | E)}.
\]

(10)

The intuition is that if the posterior probability of \( \mu \in M_i \) is higher than the prior probability in the same space, then the evidence in the data support the hypothesis that \( \mu \) lies in \( M_i \). Moreover, if we wish to test an exact hypothesis, such as \( H_i : \mu = 0 \), equation (10) reduces to the Savage-Dickey density ratio

\[
BF_{iu} = \frac{\pi(\mu = 0 | E)}{\pi^b(\mu = 0 | E)}.
\]

(11)

If we wish to perform an ordered constraint hypothesis of random (or fixed) effects against each other, such as \( H_0 : \mu_1 < \mu_2 < \cdots < \mu_P \) against its complement (ie. \( H_1 \)), we get

\[
BF_{01} = \frac{P(\mu_1 < \mu_2 < \cdots < \mu_P | E)}{P^b(\mu_1 < \mu_2 < \cdots < \mu_P | E)} \times \left( \frac{P(\text{not } H_0 | E)}{P^b(\text{not } H_0 | E)} \right)^{-1}.
\]

(12)
Moreover, one may be interested in testing whether a dichotomous covariate has a larger effect on the interaction rates than another dichotomous covariate, even if we do not know which categories of these variables have a positive or negative impact on the event rate. For example, for the classrooms we might be interested in testing whether the effect of gender (male = 1, female = 0) on the interaction rates on average has an equal, smaller, or larger effect than race (Caucasian = 1, non-Caucasian = 0), without making any assumptions about the effects of the specific categories. We could test this according to

\[ H_1 : |\mu_{\text{gender}}| = |\mu_{\text{race}}| \]
\[ H_2 : |\mu_{\text{gender}}| < |\mu_{\text{race}}| \]
\[ H_3 : |\mu_{\text{gender}}| > |\mu_{\text{race}}| , \]

where \(|\mu|\) denotes the absolute value of parameters \(\mu\). To test these hypotheses we can compute the FBFs of each constrained hypothesis against the unconstrained hypothesis according to

\[ \text{FBF}_{1u} = \frac{\pi(|\mu_{\text{gender}}| = |\mu_{\text{race}}| \mid \mathbf{E})}{\pi(|\mu_{\text{gender}}| = |\mu_{\text{race}}| \mid \mathbf{E})} \]
\[ \text{FBF}_{2u} = \frac{P(|\mu_{\text{gender}}| < |\mu_{\text{race}}| \mid \mathbf{E})}{P(|\mu_{\text{gender}}| < |\mu_{\text{race}}| \mid \mathbf{E})} \]
\[ \text{FBF}_{3u} = \frac{P(|\mu_{\text{gender}}| > |\mu_{\text{race}}| \mid \mathbf{E})}{P(|\mu_{\text{gender}}| > |\mu_{\text{race}}| \mid \mathbf{E})} \]

and subsequently compute the Bayes factors across constrained hypotheses using the transitive property

\[ \text{FBF}_{12} = \frac{\text{FBF}_{1u}}{\text{FBF}_{2u}} \text{ (O'Hagan [1995]).} \]

We can utilize the fact that \(P(|\mu_{\text{gender}}| = |\mu_{\text{race}}| \mid \mathbf{E}) = P(\mu_{\text{gender}} = \mu_{\text{race}} | \mathbf{E}) + P(\mu_{\text{gender}} = -\mu_{\text{race}} | \mathbf{E})\), which can be approximated by normal distributions.

Finally, another useful test compares effects in the sender model with effects in the receiver model. For example, this allows us to test whether the teacher has a stronger effect on starting a conversation (i.e. the teacher effect in the sender model) than on the rate of receiving a conversation (i.e. the teacher effect in the receiver model). Let \(\phi_{\text{teacher}}\) be the effect of a variable in the sender model (e.g., teacher) and \(\psi_{\text{teacher}}\) the effect of that same variable in the receiver model. Then, testing whether that variable makes an actor more likely to send the next message than to receive the next event, amounts to test

\[ H_1 : \phi_{\text{teacher}} > \psi_{\text{teacher}} \text{ against } H_2 : \phi_{\text{teacher}} \leq \psi_{\text{teacher}}. \]

Since a ratio of Bayes factors is also a Bayes factor, we have

\[ \text{FBF}_{12} = \left( \frac{\text{FBF}_{1u}}{\text{FBF}_{2u}} \right) \times \left( \frac{P(\phi_{\text{teacher}} > \psi_{\text{teacher}} | \mathbf{E})}{P(\phi_{\text{teacher}} > \psi_{\text{teacher}} | \mathbf{E})} \right)^{-1}. \]

Applying this test to a random-effect mean is also straightforward. Even though the sender and receiver models are distinct in the dynamic actor-oriented model (as discussed in section 2), both the probability of being the next sender as well as the probability of being the next receiver are proportional to their
respective intensity functions. Because of this, with all else being kept equal, if we get more evidence for $H_1$ ($H_2$) that means that the effect is stronger in the sender (receiver) model.

Ultimately, large-sample theory allows us to approximate the distributions in equation (10), (11) and (12) by normal distributions. The distribution in the numerator could be approximated by $p(\mu|E) \approx N(\bar{\mu}, \bar{\Sigma}_\mu)$ and the denominator $p^b(\mu|E) \approx N(\bar{\mu}, \bar{\Sigma}_\mu/b)$, where $\bar{\mu}$ and $\bar{\Sigma}_\mu$ are posterior estimates for $\mu$ and $\Sigma_\mu$. The probability and density ratios can thus be trivially obtained. For more information on the choice of the fraction $b$, see O’Hagan (1995), Berger & Pericchi (1996) and Gu et al. (2018). In this paper we choose $b$ following the procedure suggested by Mulder & Fox (2019), which consists of selecting a minimal sample that is based on the ratio between the total number of parameters (excluding group specific effects) and the total number of observations. Since sender and receiver models are two different processes assumed to be conditionally independent given the past, we have to define two difference fractions, one for each model (see also Hoijtink, Gu, & Mulder 2019). In the sender model, we have $P$ random effects and $Q$ fixed effects, hence the fraction is $b_{Snd} = \frac{(P(P+1)/2) + (P+Q)}{\sum_{k=1}^K N_k}$ and in the receiver model we have $V$ random effects and $U$ fixed effects, resulting in the fraction $b_{Rec} = \frac{(V(V+1)/2) + (V+U)}{\sum_{k=1}^K N_k}$.

4 Empirical illustration

The data used in this illustration were collected by McFarland (2001) for a study to investigate student rebellion in the classroom. The data feature observations of interactions among high-school students in two different schools in the United States. For this illustration we consider 15 independent classrooms from Magnet High School during the 1996-1997 school year. The student body of this high school can be considered academically homogeneous. The data were collected through classroom observation in which the conversations within the classroom were coded. In each of these fifteen classes a teacher is present in addition to the students. The number of events (the number of times one person said something to another) range from 86 to 628 and the number of persons (the students plus the teacher) ranges between 19 and 30. The conversations happened in an orderly lecture-like fashion, so only was person was speaking at each time.

The aim of this illustration is to show how the Bayes Factor approach can be used in the modeling and testing of hierarchical relational event data. We do not aim to provide a full empirical analysis of these data but fully focus on the illustration of the methods.

4.1 Model specification

We fit the hierarchical actor-oriented relational event model (described on Section 2) to the data. Initially, all effects are considered random, so the need for a hierarchical structure can be evaluated using the Bayes factor presented in Section 3. This first step will allow us to determine whether some effects can
be treated as fixed and the model can hence be simplified.

We set the prior parameters to $\sigma^2_\phi = \sigma^2_\beta = \sigma^2_\psi = \sigma_\gamma = 10$, making the priors for the fixed and random effects relatively vague, and $\eta = 2$, slightly favoring smaller correlations. For our illustration of the proposed methods, we will include the following covariates into the model:

**Teacher:** A dummy variable that indicates whether the actor is the teacher (one if the actor is the teacher and zero otherwise).

**Gender:** A variable dummy that indicates the gender of the actor (one if the actor is male and zero otherwise).

**Race:** A dummy variable that indicates the race of the actor. McFarland (2001) notes that 50 percent of the Magnet High population is Caucasian. Hence, the variable is one if the actor is Caucasian and zero otherwise.

**Inertia:** Inertia captures the persistence of the communication, where past interaction is likely to be repeated (Leenders, Contractor, & DeChurch 2016). It is computed as the accumulated volume of past communication from a specific sender to a specific receiver. Because this statistic is essentially dyadic, it will be included only in the receiver model.

**Participation Shifts:** These statistics are used to reflect expectations of adherence of communication norms in small groups (Butts 2008). Gibson (2005) describes the framework for several types of participation shifts. In this application two distinct types of participation shift will be included. The first belongs to the group of "turn-receiving" and is represented by the event pattern ABBA: an interaction from person A to person B is immediately followed by an interaction from B to A. The second is ABAB, which can be considered as a special case of "turn-continuing": an interaction from person A to B is immediately followed by another interaction event from A to B.

The ABBA and ABAB participation shifts are concerned with dyads, therefore they will be included in the receiver model only. However, we do include adapted versions in the sender model. Here, the statistics become ABB (after A has spoken to B, the next event is B starting the conversation) and ABA (after A has spoken to B, A speaks again).

**Activity:** This set of covariates capture the effect of actor activity as a sender or as a receiver (Vu et al. 2017). The first is the outgoingness of a person, defined as the number of events sent by one actor up until a specific point in time. This captures the tendency of the person to start conversations (or just to talk). The second is the popularity of a person, given by the number of events received by one
actor up until an specific point in time. This captures the popularity of an individual as a receiver on conversation.

Relational event models are vulnerable to process explosion, which happens when $\lambda(t) \to \infty$ (often due to a feedback loop that may be caused by using statistics that are computed as cumulative sums (Aalen, Borgan, & Gjessing 2008)). This is particularly a problem for inertia and activity statistics. One way to alleviate this problem is via $z$-score standardization at every time point, which is defined as $z(t) = (x(t) - \bar{x}(t))/S_x(t)$, where $x(t)$ is the value of the statistic, $\bar{x}(t)$ is the sample mean, and $S_x(t)$ is the sample standard deviation at time $t$. We use the standardized statistics in this application.

5 Results

5.1 Testing random-effect structures

The first step is to fit the model with all effects considered as random. Table 1 ("Random Effects Model") shows the result of the models where all statistics are considered random. Next, we compute the Bayes factors to test which random-effects can be treated as fixed. As discussed in section 3, the hypotheses tested are

$H_0: \xi_1 = \xi_2 = \cdots = \xi_{K-1} = 0$, with $\xi_j = \beta_{j+1} - \beta_j$, for $j = 1, \ldots, K-1$

$H_1$ : not $H_0$.

Table 1 shows the results. We ran only 1000 MCMC iterations and discarded the first 500 samples as burn-in. Figure 1 shows trace plots. The chains seem to have converged. The number of posterior draws is not high, because the Hamiltonian dynamics in Stan’s algorithm allows for faster convergence than standard MCMC methods (Hoffman & Gelman 2014). Since a more parsimonious model is preferred, we reject that effects are equal across clusters only when $BF_{01} < 1/3$. In this case, the evidence in favor of an effect not being fixed (i.e. being random) is considerably larger than the evidence in favor of the effect being fixed.

Most of the Bayes Factors are zero (see Table 2), providing strong evidence for their random nature. However, the bold numbers in the table show that a few effects can indeed be treated as fixed (ABA, outgoingness, and popularity in the sender model and race in the receiver model). Therefore, we continue with a mixed-effect model, where in the sender model $z_s(t)$ contain ABA, outgoingness, and popularity as fixed effects, $x_s(t)$ contain intercept, teacher, gender, race, and ABB as random effects. Whereas in the receiver model the fixed-effect covariates, $z_{sr}(t)$, contain only race and the random-effect covariates, $x_{sr}(t)$, the remaining covariates (teacher, gender, inertia, ABBA, ABAB and popularity). For this final model, we run 2000 MCMC iterations, with a burn-in of 1000. Figure 2 shows the trace plots of a few parameters, the chains appear to have converged.
We can see that the Bayes factor for testing which effects are random worked correctly, because a similar fit to the data is obtained in the more parsimonious model (where some effects are fixed) compared to the larger (i.e. all random) model. We can see this is the case by computing the point-wise deviance residuals for both models. If a similar fit is indeed obtained from both models, the point-wise deviance residuals should lie roughly in a straight line, meaning that no model dominates the other in terms of fit. Figure 3 clearly shows that this is the case: most points lie on or close to the blue equality line, indicating that the fit provided by both models is very similar. Details of the point-wise deviance computation are provided in appendix D. We proceed illustrating the other hypothesis testing methods using this more parsimonious mixed-effects model. We start with the test of ordered constraints on random effects variances, then we illustrate the tests on random-effects means and fixed effects.

5.2 Testing ordered constraints on random-effect variances

We proceed to illustrate the hypothesis testing of ordered constraints on variance parameters. The random effects teacher, gender, and race in the sender model will be used to illustrate this test. Though we focus on the sender model here, the same test can also be applied to the variance parameters in the receiver model in an equally straightforward manner. The objective is to determine the amount of evidence regarding the level of heterogeneity among these effects. The higher the variance of an effect, the larger is the difference in that effect among the independent sequences. Let $\sigma^2_{\text{teacher}}$, $\sigma^2_{\text{gender}}$, and $\sigma^2_{\text{race}}$ be the variances of teacher, gender, and race effects, respectively, where these parameters are extracted from the diagonal of $\Sigma_\gamma$. Suppose we want to test every combination of ordered constraints hypothesis against one another, such that

- $H_1: \sigma^2_{\text{teacher}} > \sigma^2_{\text{gender}} > \sigma^2_{\text{race}}$
- $H_2: \sigma^2_{\text{teacher}} > \sigma^2_{\text{race}} > \sigma^2_{\text{gender}}$
- $H_3: \sigma^2_{\text{race}} > \sigma^2_{\text{teacher}} > \sigma^2_{\text{gender}}$
- $H_4: \sigma^2_{\text{race}} > \sigma^2_{\text{gender}} > \sigma^2_{\text{teacher}}$
- $H_5: \sigma^2_{\text{gender}} > \sigma^2_{\text{teacher}} > \sigma^2_{\text{race}}$
- $H_6: \sigma^2_{\text{gender}} > \sigma^2_{\text{race}} > \sigma^2_{\text{teacher}}$

We want to determine which effect has greater heterogeneity across clusters. Below the Bayes factors for each one of these hypotheses against one another is represented in the form of an evidence matrix.
Each cell of the matrix represent the comparison of the hypothesis in the rows against the hypothesis in the columns. So, for example, the evidence for $H_1$ against $H_2$ is equal to 1.5. The evidence for $H_2$ against $H_1$ is then $1/1.50 = 0.67$.

The evidence favoring hypotheses $H_1$ and $H_2$ is much larger than the evidence in favor of any other hypothesis, which indicates that the teacher effect has indeed more heterogeneity than the other two effects across the fifteen classrooms. This makes sense, since the posterior mean of these variance parameters are $\bar{\sigma}^2_{\text{teacher}} = 1.65$, $\bar{\sigma}^2_{\text{gender}} = 0.25$, and $\bar{\sigma}^2_{\text{race}} = 0.22$. In addition, substantively teachers have a strong effect on the flow of the conversation during a lecture, it makes sense that they are the strongest source of heterogeneity between the classrooms.

5.3 Testing fixed effects and random-effect means

In this subsection, we compare the strengths of the effects the covariates have in the model. We use the R package BFpack to compute the tests (Mulder et al. 2019). In the sender model, suppose that due to the special role of the teacher in controlling the flow of interactions during a lecture, we expect the teacher effect to have the strongest impact on the rate at which an actor starts an interaction, when compared to other personal-trait variables. we then think that gender is important (but less strongly than teacher), and we expect that the effect of race is the least strong of the three, although we assume that gender and race might have similar (or even the same) impact on the overall communication rate.

We do not care about the direction of an effect, but are interested in its strength relative to the others. Hence, let $|\zeta|$ be the absolute value of $\zeta$, where $\zeta$ is the random-effect mean in the sender model, we test

$$H_1 : |\zeta_{\text{teacher}}| > |\zeta_{\text{gender}}| > |\zeta_{\text{race}}|$$

$$H_2 : |\zeta_{\text{teacher}}| > |\zeta_{\text{gender}}| = |\zeta_{\text{race}}|$$

$$H_3 : \text{not } H_0,$$

where $H_3$ is the complement of $H_1$ and $H_2$, which means that if the data provides more support to $H_3$, we must seek an additional hypothesis. Once again, we present those fractional Bayes factors in an evidence matrix form.
This matrix shows very little evidence in favor of \( H_3 \), which means that our initial hypothesis about the role of the teacher is confirmed. This result makes perfect sense, because table 1 shows that \( \hat{\zeta}_{\text{teacher}} = 2.22 \), \( \hat{\zeta}_{\text{gender}} = 0.29 \), and \( \hat{\zeta}_{\text{race}} = -0.12 \). In addition, it is clear that \( H_2 \) dominates the other hypotheses, and that is due to the fact that there is a big overlap between the posterior intervals of gender (lower bound equals 0.01) and race (upper bound equals 0.11). Thus, the data provides more evidence for the equality of gender and race. Therefore, on average, the teacher increases the actor sender activity rate by 2.22. Being male increments the sender rate with 0.29, and being Caucasian decreases sender activity with \(-0.12\). We can conclude that the teacher effect has the strongest impact on the sender rate, followed by gender, and race, respectively.

In the receiver model we would not expect the teacher to be the strongest effect among the personal-trait covariates. Due to the fact that, since in a lecture, teachers would be doing most of the talking, they would not be able to do most of the listening simultaneously. But regardless of that, we will check whether this is actually true, by testing the same hypotheses for the receiver model as we did for the sender model

\[
\begin{array}{ccc}
H_1 & H_2 & H_3 \\
1.00 & 0.22 & 17.04 \\
4.55 & 1.00 & 77.58 \\
0.06 & 0.01 & 1.00
\end{array}
\]

This time the evidence we obtain in favor of the teacher effect being the strongest, is insufficient for both \( H_1 \) and \( H_2 \). Which makes sense, when looking at table 1, we see that \( \mu_{\text{teacher}} = 0.20 \), \( \mu_{\text{gender}} = -0.31 \) and \( \mu_{\text{race}} = -0.06 \), thus the gender effect has the largest mean (in absolute value) in the receiver model. Therefore, we now test the confirmatory hypothesis:

\[
\begin{align*}
H_1 &: |\mu_{\text{gender}}| > |\mu_{\text{teacher}}| > |\mu_{\text{race}}| \\
H_2 &: \text{not } H_0
\end{align*}
\]

where \(|\mu|\) is the absolute value of \( \mu \), which represents the global mean of the random effects in the receiver model. We also display the results in the form of an evidence matrix:

\[
\begin{array}{ccc}
H_1 & H_2 & H_3 \\
1.00 & 1.01 & 2.03 \\
0.98 & 1.00 & 1.99 \\
0.48 & 0.50 & 1.00
\end{array}
\]

This time the evidence we obtain in favor of the teacher effect being the strongest, is insufficient for both \( H_1 \) and \( H_2 \). Which makes sense, when looking at table 1, we see that \( \mu_{\text{teacher}} = 0.20 \), \( \mu_{\text{gender}} = -0.31 \) and \( \mu_{\text{race}} = -0.06 \), thus the gender effect has the largest mean (in absolute value) in the receiver model. Therefore, we now test the confirmatory hypothesis:

\[
\begin{align*}
H_1 &: |\mu_{\text{gender}}| > |\mu_{\text{teacher}}| > |\mu_{\text{race}}| \\
H_2 &: \text{not } H_0
\end{align*}
\]

19
which yields

\[
  \text{FBF}_{12} = \frac{P(|\mu_{\text{gender}}| > |\mu_{\text{race}}| > |\mu_{\text{teacher}}| \mid \mathbf{E})}{\tilde{P}(\mathbf{E})} \times \left( \frac{P(\text{not } H_1 \mid \mathbf{E})}{\tilde{P}(\text{not } H_1 \mid \mathbf{E})} \right)^{-1} = 8.15.
\]

Thus, this provides positive evidence that \textit{gender} is the strongest effect among the personal-trait covariates in the receiver model, followed by \textit{teacher} and \textit{race}. This result shows the usefulness of doing an actor-oriented analysis. The model is able to capture the processes that govern actors preferences in social interaction, even when different effects have larger influence in the sender activity and the receiver choice rates.

Next, we evaluate whether the fixed effects of popularity or outgoingness makes actors more likely to be the next sender. For this, we compare the fixed-effect of \textit{popularity} with that of \textit{outgoingness} in the sender model. We can see from table \[\text{I}\] that there is a small overlap in the posterior intervals of those effects–the upper limit of the 95% CI of \textit{outgoingness} is 0.13 and the lower limit of \textit{popularity} is 0.11. Table \[\text{I}\] also shows that the posterior mean of \textit{popularity} is almost 3 times larger than the posterior mean of \textit{outgoingness}, 0.17 and 0.07 respectively. Which effect is the strongest becomes clear from the table already, but we illustrate the formal test below.

Let \(\phi_{\text{popularity}}\) be the mean effect of \textit{popularity} and \(\phi_{\text{outgoingness}}\) be the mean \textit{outgoingness} effect, both in the sender model. We test all possible hypotheses against one another:

\[
  \begin{align*}
    H_1 : & \quad \phi_{\text{popularity}} = \phi_{\text{outgoingness}} \\
    H_2 : & \quad \phi_{\text{popularity}} < \phi_{\text{outgoingness}} \\
    H_3 : & \quad \phi_{\text{popularity}} > \phi_{\text{outgoingness}}.
  \end{align*}
\]

Having more than two hypotheses, we, once more, show the fractional Bayes factors in an evidence matrix form.

\[
\begin{pmatrix}
  H_1 & H_2 & H_3 \\
  H_1 & 1.00 & 48.03 & 0.63 \\
  H_2 & 0.02 & 1.00 & 0.01 \\
  H_3 & 2.12 & 101.72 & 1.00
\end{pmatrix}
\]

The evidence matrix gives a clear indication against \(H_2\): the mean effect of \textit{popularity} is smaller than that of \textit{outgoingness}. Also, there is more evidence in favor of \(H_3\) against \(H_1\). This reinforces the idea that the more popular a receiver of conversation has been, the more intensely this person will act as a sender of a future conversation event. Figure \[\text{I}\] shows this graphically: the posterior distribution of \(\phi_{\text{popularity}} - \phi_{\text{outgoingness}}\) practically only favors \(H_3\). Because \(\phi_{\text{popularity}} - \phi_{\text{outgoingness}} > 0\) means that \(\phi_{\text{popularity}} > \phi_{\text{outgoingness}}\).

We can do a similar thing comparing the population means of random effects (rather than fixed-effects above). To illustrate this test, we consider the means of the participation shifts \textit{ABBA} and
ABAB (both are random-effects) in the receiver model. Let $\mu_{abba}$ be the mean effect of ABBA and $\mu_{abab}$ be the mean effect of ABAB. This comparison captures the tension between an individual’s tendency to continue speaking and the societal norm of reciprocity. In polite conversation, we would expect to see more turn-switches between two individuals (A speaks to B and then B responds to A) than turn-continuing (where A directs a comment to B and then continues to speak without giving B the opportunity to respond first). Especially in a classroom setting, a conversation with back-and-forths is more likely than a monologue of one person to another (with the exception of the teacher addressing the group as a whole). Table 1 suggests that this is indeed the case, since the mean effect of ABBA is almost four times as large as the mean effect of ABAB, 4.27 and 1.18 respectively. The formal test works as follows. We again test all hypothesis against one another:

$$H_1: \mu_{abba} = \mu_{abab}$$
$$H_2: \mu_{abba} < \mu_{abab}$$
$$H_3: \mu_{abba} > \mu_{abab}.$$ 

This yields the following evidence matrix:

$$\begin{pmatrix}
H_1 & H_2 & H_3 \\
H_1 & 1.00 & 142.26 & 0.00 \\
H_2 & 0.00 & 1.00 & 0.00 \\
H_3 & 2.34 \times 10^{12} & 3.33 \times 10^{14} & 1.00 \\
\end{pmatrix}$$

The evidence favoring $H_3$ is overwhelmingly larger than the evidence for any other hypothesis, strongly supporting the hypothesis that the effect of immediate reciprocity is larger than that of immediate inertia across these fifteen classrooms at Magnet High school.

We already saw that the teacher effect has the largest impact on the sender rate. Our final illustrative test evaluates whether teacher has a larger effect on the rate of being a sender or on the rate of being chosen as receiver. Considering that the relational event sequences were collected from school classes during a lecture, we expected that the teacher effect is larger in the sender model than in the receiver model. Table 1 shows this to be the case, since $\zeta_{teacher} = 2.22$ and $\mu_{teacher} = 0.20$, where $\zeta_{teacher}$ and $\mu_{teacher}$ are the mean effects in the sender and receiver model respectively. This means that, keeping everything equal in both models and not taking into account the effect of the intercept in the sender model, the effect of being the teacher in the sender model is over ten times higher than the effect of being the teacher in the receiver model. As confirmation, we will test this effect formally. We formulate the hypotheses as follows:
H₁ : ζ_{teacher} = μ_{teacher}
H₂ : ζ_{teacher} < μ_{teacher}
H₃ : ζ_{teacher} > μ_{teacher}.

The resulting evidence matrix for the fractional Bayes factors is:

\[
\begin{pmatrix}
H_1 & H_2 & H_3 \\
H_1 & 1.00 & 90.40 & 0.00 \\
H_2 & 0.00 & 1.00 & 0.00 \\
H_3 & 3.42 \times 10^4 & 3.09 \times 10^6 & 1.00
\end{pmatrix}
\]

Indeed, the evidence matrix shows very strong evidence that the teacher effect is larger in the sender model than in the receiver model.

6 Discussion

In this paper we presented a Bayesian mixed-effects extension to the dynamic actor-oriented relational event model. This model allows for inferences at the actor level, thus opening the possibility of unveiling effects that make an actor more prone to send or receive an interaction in the population under study. Our results show that the model is able to capture the effects that have the largest impacts in actors preferences, even when those effects are different in the sender activity and receiver choice rates. The models can be estimated using the Stan programming language, the Stan code for the actor-oriented relational event model is available on Github [https://github.com/Fabio-Vieira/bayesian_dynamic_network](https://github.com/Fabio-Vieira/bayesian_dynamic_network).

We introduced hypothesis testing for this class of models, facilitating inferences on population parameters of the estimated effects. Another contribution of this paper is the development of a Bayes factor for testing random-effects structures, which works well in the absence of prior information and does not require ad-hoc choices. We showed that our Bayes factor is able to correctly identify which effects should be treated as fixed, hence providing a more parsimonious model of the data than a purely random effects model.

The computational issues originating from the inefficiencies that result from the hierarchical structure of the model are the main limitations in the estimation process. We have taken advantage of the multivariate normal structure of the hierarchical prior to induce a non-centered transformation in the random-effects parameters, which is more efficient in practice for the reasons aforementioned. An issue that we have not addressed in our paper is that sparsity in the data makes the geometry of the posterior distribution complex [Betancourt & Girolami (2015)]. Since social network data are often sparse, this
can also be an issue for estimating the models presented in this paper. The alien form of the likelihood of the relational event model also presents a challenge to the estimation of hierarchical relational event models. Therefore, our methods may be a first and useful step, we do see plenty of room (or even need) to improve the methods further. An especially attractive next step would be to construct alternative representations of the model so that posterior distributions in closed form could be derived, improving efficiency in the estimation process.

A further attractive future avenue is to pool independent estimates coming from different studies together in a random-effects model, as is common in meta-analysis (Borenstein et al. 2010). A Gibbs sampler could be derived using the point estimates from the independent relational event sequences as data. Our Bayes factors could then be built on top of that. Finally, other possible extensions could include conducting simulation experiments to determine the minimum sample size required to estimate population effects and allowing for time-varying coefficients in order to discover complex temporal structures in the effects.
A Linear transformation of multivariate normal distribution

Here we show that the transformation of the random-effects parameters is valid. Let $X$ be a $p$-dimensional random variable, with $X \sim N(\mu, \Sigma)$, where $\mu \in \mathbb{R}^p$ is the mean vector and $\Sigma$ is a $p \times p$ covariance matrix. Thus, the moment generating function of $X$ is given by

$$M_X(t) = \exp \left\{ t'\mu + \frac{1}{2} t'\Sigma t \right\}.$$ 

Now, assuming we can write $X = \mu + AZ$, where $A$ is a $p \times p$ matrix, with $\Sigma = A' A$, and $Z = (Z_1, Z_2, \ldots, Z_p)$ is an independent normal random vector, with $Z_i \sim N(0, 1)$, for $i = 1, \ldots, p$. Therefore, if we can show that $M_X(t) = M_{(\mu + AZ)}(t)$, then the transformation holds. So, we derive the moment generating function of the transformation as

**Proof.**

$$M_{(\mu + AZ)}(t) = E(e^{t'X}) = E(e^{t'\mu + t'AZ})$$

$$= e^{t'\mu} E(e^{t'Z}), \text{ where } l' = t'A$$

$$= e^{t'\mu} E(e^{\sum_{i=1}^{p} l_i Z_i})$$

$$= e^{t'\mu} \prod_{i=1}^{p} E(e^{l_i Z_i})$$

$$= e^{t'\mu} \prod_{i=1}^{p} e^{l_i^2/2}$$

$$= \exp \left\{ t'\mu + \sum_{i=1}^{p} \frac{l_i^2}{2} \right\}$$

$$= \exp \left\{ t'\mu + \frac{1}{2} t'l \right\} = \exp \left\{ t'\mu + \frac{1}{2} t'\Sigma t \right\}.$$

Thus $M_X(t) = M_{(\mu + AZ)}(t)$. Hence, we can transform the random-effects parameters in an MCMC algorithm by following the steps,

1. Obtain posterior samples of mean vector $\mu \in \mathbb{R}^p$;
2. Obtain posterior samples of the $p \times p$ covariance matrix $\Sigma$;
3. Compute the Cholesky factor $A$ of $\Sigma$;
4. Sample a $p$-dimensional vector of independent and identically distributed standard normal variables $Z$;
5. Compute the transformation $\beta = \mu + AZ$. 

24
Let us suppose that we wish to test an informative hypothesis $H_i : \theta \in \Theta_i$, where $\Theta_i \subset \Theta$. Thus, if we define the prior in the space $\Theta_i$ as a truncation of the prior in $\Theta$, such as $c = \int_{\theta \in \Theta_i} \pi(\theta) d\theta$, where $\pi(\theta)$ is the prior under the unconstrained space. Then, the Bayes factor $BF_{iu}$ of $H_i$ against $H_u$ will be given by

$$BF_{iu} = \frac{\pi(E|H_i)}{\pi(E|H_u)} = c^{-1} \int_{\theta \in \Theta_i} \pi(\theta|E) d\theta$$

Therefore, the Bayes factor of a constrained against an unconstrained hypothesis can be reduced to a ratio of posterior and prior probabilities in the space of the constrained hypothesis. In case of an exact hypothesis, such as $H : \theta = 0$, the ratio of probabilities becomes the Savage-Dickey density ratio (Dickey 1971).

In this appendix we show details on the derivation of the parameters in the Bayes factor for testing random-effect structures. Assuming we have $K$ social networks and each of them has its own $\beta_k$, for $i = 1, \ldots, K$, regression parameter. Where $\beta_k \sim N(\mu, \sigma^2)$. Thus, let $\bar{\mu}, \bar{\sigma}^2, \bar{\beta}_k$ be posterior estimates for $\mu, \sigma^2$ and $\beta_k, \forall k$. Also, let $\bar{\tau}_k^2$ be the point estimate for the variance of the posterior distribution of $\beta_k$.

Then we can write

$$\text{posterior: } \beta_k | E_k \sim N(\bar{\beta}_k, \bar{\tau}_k^2)$$
$$\text{prior: } \beta_k \sim N(\bar{\mu}, \bar{\sigma}^2).$$

Assuming we want to test $H_0 : \beta_1 = \cdots = \beta_K$ against $H_1 : \text{"at least one is different".}$. Thus we can do so by writing $\xi_j = \beta_{j+1} - \beta_j$, for $j = 1, \ldots, K - 1$. Therefore we can approximate the joint distribution of $\xi = (\xi_1, \ldots, \xi_{K-1})$ by a multivariate normal

$$\text{posterior: } \xi | E \sim N(\bar{\xi}, \bar{\Sigma})$$
$$\text{prior: } \xi \sim N(0, \bar{A}).$$
Thus we can derive the parameters of the distributions of every component $j = 1, \ldots, K - 1$ of $\xi$ as follows

\[
E(\xi_j | E) = E(\beta_{j+1} - \beta_j) = E(\beta_{j+1}) - E(\beta_j) = \bar{\beta}_{j+1} - \bar{\beta}_j,
\]

\[
\text{Var}(\xi_j | E) = \text{Var}(\beta_{j+1} - \beta_j) = \text{Var}(\beta_{j+1}) + \text{Var}(\beta_j) = \bar{\tau}^2_{j+1} + \bar{\tau}^2_j,
\]

\[
\text{Cov}(\xi_j, \xi_{j+h} | E) = E(\xi_j \xi_{j+h}) - E(\xi_j)E(\xi_{j+h}) = E\left(\left(\beta_{j+1} - \beta_j\right)\left(\beta_{j+h+1} - \beta_{j+h}\right)\right) - E\left(\left(\beta_{j+1} - \beta_j\right)\right)E\left(\left(\beta_{j+h+1} - \beta_{j+h}\right)\right)
= E(\beta_{j+1}\beta_{j+h+1}) - E(\beta_{j+h+1})E(\beta_{j+h} - E(\beta_{j+h+1}) + E(\beta_j)E(\beta_{j+h+1}) - E(\beta_j)E(\beta_{j+h})
= \begin{cases} 
-\text{Var}(\beta_{j+1}) = -\bar{\tau}^2_{j+1}, & \text{if } h = 1 \\
0, & \text{if } h > 1.
\end{cases}
\]

The parameters in the prior are derived in the same way with $E(\xi_j) = 0$, $\text{Var}(\xi_j) = 2\bar{\sigma}^2$ and $\text{Cov}(\xi_j, \xi_{j+h}) = -\bar{\sigma}^2$, if $h = 1$, and zero otherwise.

## D Point-wise deviance residuals

After estimation, it is common practice to check how the proposed model fits the data under study. In [DuBois et al. (2013)](https://doi.org/10.1007/s10683-013-0239-5), they used the posterior draws to evaluate the log likelihood at every data point to determine the dynamic adequacy of the model to every data point. Thus, for the actor-oriented relational event model, for $k = 1, \ldots, K$ and $m = 1, \ldots, M_k$, this quantity would be computed as

\[
\text{Res}_m = -2\left\{ \log \left( \lambda_{s_m}(t_m | E_k) \right) + \log \left( \lambda_{r_m | s_m}(t_m | s_m, E_k) \right) - \log \left( \sum_{s,r} \lambda_{r | s_m}(t_m | s_m, E_k) \right) - (t_m - t_{m-1}) \left( \sum_i \lambda_i(t_m | E_k) \right) \right\}.
\]

This measure is usually called residual deviance ([Collett 2015](https://doi.org/10.1007/978-1-4471-5185-3)), and it is useful to compare models and see which one fits better each data point by having smaller values of $\text{Res}_m$. 

26
References

Aalen, O. (1989). A linear regression model for the analysis of life times. *Statistics in medicine*, 8(8), 907–925.

Aalen, O., Borgan, O., & Gjessing, H. (2008). *Survival and event history analysis: a process point of view*. Springer Science & Business Media.

Berger, J. O., & Pericchi, L. R. (1996). The intrinsic bayes factor for model selection and prediction. *Journal of the American Statistical Association*, 91(433), 109–122.

Betancourt, M., & Girolami, M. (2015). Hamiltonian monte carlo for hierarchical models. *Current trends in Bayesian methodology with applications*, 79(30), 2–4.

Böing-Messing, F., & Mulder, J. (2020). Bayes factors for testing order constraints on variances of dependent outcomes. *The American Statistician*, 1–10.

Borenstein, M., Hedges, L. V., Higgins, J. P., & Rothstein, H. R. (2010). A basic introduction to fixed-effect and random-effects models for meta-analysis. *Research synthesis methods*, 1(2), 97–111.

Butts, C. T. (2008). A relational event framework for social action. *Sociological Methodology*, 38(1), 155–200.

Butts, C. T., & Marcum, C. S. (2017). A relational event approach to modeling behavioral dynamics. In *Group processes* (pp. 51–92). Springer.

Carpenter, B., Gelman, A., Hoffman, M. D., Lee, D., Goodrich, B., Betancourt, M., … Riddell, A. (2017). Stan: A probabilistic programming language. *Journal of statistical software*, 76(1).

Collett, D. (2015). *Modelling survival data in medical research*. CRC press.

Cox, D. R. (1972). Regression models and life-tables. *Journal of the Royal Statistical Society: Series B (Methodological)*, 34(2), 187–202.

Dickey, J. M. (1971). The weighted likelihood ratio, linear hypotheses on normal location parameters. *The Annals of Mathematical Statistics*, 204–223.

DuBois, C., Butts, C. T., McFarland, D., & Smyth, P. (2013). Hierarchical models for relational event sequences. *Journal of Mathematical Psychology*, 57(6), 297–309.
Eagle, N., & Pentland, A. S. (2003). Social network computing. In *International conference on ubiquitous computing* (pp. 289–296).

Frank, O., & Strauss, D. (1986). Markov graphs. *Journal of the american Statistical association, 81*(395), 832–842.

Friedman, M. (1982). Piecewise exponential models for survival data with covariates. *The Annals of Statistics, 10*(1), 101–113.

Gelman, A., & Hill, J. (2006). *Data analysis using regression and multilevel/hierarchical models*. Cambridge university press.

Gibson, D. R. (2005). Taking turns and talking ties: Networks and conversational interaction. *American journal of sociology, 110*(6), 1561–1597.

Gu, X., Mulder, J., & Hoijtink, H. (2018). Approximated adjusted fractional bayes factors: A general method for testing informative hypotheses. *British Journal of Mathematical and Statistical Psychology, 71*(2), 229–261.

Hanneke, S., Fu, W., & Xing, E. P. (2010). Discrete temporal models of social networks. *Electronic Journal of Statistics, 4*, 585–605.

Hoffman, M. D., & Gelman, A. (2014). The no-u-turn sampler: adaptively setting path lengths in hamiltonian monte carlo. *J. Mach. Learn. Res., 15*(1), 1593–1623.

Hoijtink, H., Gu, X., & Mulder, J. (2019). Bayesian evaluation of informative hypotheses for multiple populations. *British Journal of Mathematical and Statistical Psychology, 72*(2), 219–243.

Jeffreys, H. (1961). *Theory of probability, ed. 3* oxford university press. Oxford.[Google Scholar].

Kass, R. E., & Raftery, A. E. (1995). Bayes factors. *Journal of the american statistical association, 90*(430), 773–795.

Leenders, R. T. A., Contractor, N. S., & DeChurch, L. A. (2016). Once upon a time: Understanding team processes as relational event networks. *Organizational Psychology Review, 6*(1), 92–115.

Lerner, J., & Lomi, A. (2020). Reliability of relational event model estimates under sampling: How to fit a relational event model to 360 million dyadic events. *Network Science, 8*(1), 97–135.
Lewandowski, D., Kurowicka, D., & Joe, H. (2009). Generating random correlation matrices based on vines and extended onion method. *Journal of multivariate analysis, 100*(9), 1989–2001.

Lusher, D., Koskinen, J., & Robins, G. (2013). *Exponential random graph models for social networks: Theory, methods, and applications.* Cambridge University Press.

McFadden, D. (1973). Conditional logit analysis of qualitative choice behavior. Institute of Urban and Regional Development, University of California . . . .

McFadden, D. A. (2001). Student resistance: How the formal and informal organization of classrooms facilitate everyday forms of student defiance. *American journal of Sociology, 107*(3), 612–678.

Mulder, J. (2014). Prior adjusted default bayes factors for testing (in) equality constrained hypotheses. *Computational Statistics & Data Analysis, 71,* 448–463.

Mulder, J., & Fox, J.-P. (2019). Bayes factor testing of multiple intraclass correlations. *Bayesian Analysis, 14*(2), 521–552.

Mulder, J., Gu, X., Olsson-Collentine, A., Tomarken, A., Böing-Messing, F., Hoijtink, H., . . . others (2019). Bfpack: Flexible bayes factor testing of scientific theories in r. *arXiv preprint arXiv:1911.07728*.

Mulder, J., & Leenders, R. T. A. (2019). Modeling the evolution of interaction behavior in social networks: A dynamic relational event approach for real-time analysis. *Chaos, Solitons & Fractals, 119,* 73–85.

O’Hagan, A. (1995). Fractional bayes factors for model comparison. *Journal of the Royal Statistical Society: Series B (Methodological), 57*(1), 99–118.

Perry, P. O., & Wolfe, P. J. (2013). Point process modelling for directed interaction networks. *Journal of the Royal Statistical Society: Series B (Statistical Methodology), 75*(5), 821–849.

R Core Team. (2017). R: A language and environment for statistical computing [Computer software manual]. Vienna, Austria. Retrieved from [https://www.R-project.org/](https://www.R-project.org/)

Snijders, T. A. (1996). Stochastic actor-oriented models for network change. *Journal of mathematical sociology, 21*(1-2), 149–172.
Stadtfeld, C., & Block, P. (2017). Interactions, actors, and time: Dynamic network actor models for relational events. *Sociological Science, 4*, 318–352.

Stadtfeld, C., Hollway, J., & Block, P. (2017). Dynamic network actor models: Investigating coordination ties through time. *Sociological Methodology, 47*(1), 1–40.

Stan Development Team. (2018). *RStan: the R interface to Stan.* Retrieved from [http://mc-stan.org/](http://mc-stan.org/) (R package version 2.17.3)

Van Der Hofstad, R. (2009). Random graphs and complex networks. *Available on http://www.win.tue.nl/rhofstad/NotesRGCN.pdf, 11, 60.*

Vu, D., Hunter, D., Smyth, P., & Asuncion, A. U. (2011). Continuous-time regression models for longitudinal networks. In *Advances in neural information processing systems* (pp. 2492–2500).

Vu, D., Lomi, A., Mascia, D., & Pallotti, F. (2017). Relational event models for longitudinal network data with an application to interhospital patient transfers. *Statistics in medicine, 36*(14), 2265–2287.

Vu, D., Pattison, P., & Robins, G. (2015). Relational event models for social learning in moocs. *Social Networks, 43*, 121–135.

Walker, A. M. (1969). On the asymptotic behaviour of posterior distributions. *Journal of the Royal Statistical Society: Series B (Methodological), 31*(1), 80–88.

Wang, Z., Wu, Y., & Chu, H. (2018). On equivalence of the lkj distribution and the restricted wishart distribution. *arXiv preprint arXiv:1809.04746.*
Parameter Estimates of Mean Effects

| Effect       | Random-Effects Model |  | Mixed-Effects Model |  |
|--------------|----------------------|---------------------|---------------------|---------------------|
|              | Sender               | Receiver            | Sender               | Receiver            |
| intercept    | -1.75 (-2.08, -1.37) | -                   | -1.77 (-2.19, -1.32) | -                   |
| teacher      | 2.22 (1.56, 2.83)    | 0.21 (-0.25, 0.67)  | 2.22 (1.60, 2.97)    | 0.20 (-0.21, 0.65)  |
| gender       | 0.27 (0.01, 0.53)    | -0.30 (-0.51, -0.12)| 0.29 (0.02, 0.57)    | -0.31 (-0.49, -0.09)|
| race         | -0.12 (-0.36, 0.11)  | -0.07 (-0.18, 0.05) | -0.12 (-0.37, 0.14) | -0.06 (-0.17, 0.04) |
| inertia      | -                    | 0.42 (-0.38, 1.15)  | -                   | 0.36 (-0.46, 1.17)  |
| ABB (ABBA)   | 1.06 (0.67, 1.46)    | 4.30 (3.88, 4.72)   | 1.08 (0.67, 1.52)    | 4.27 (3.79, 4.70)   |
| ABA (ABAB)   | -0.06 (-0.52, 0.33)  | 1.29 (0.57, 1.84)   | -0.05 (-0.42, 0.31)  | 1.18 (0.53, 1.81)   |
| outgoingness | 0.05 (-0.08, 0.20)   | -                   | -0.05 (-0.42, 0.31)  | -                   |
| popularity   | 0.25 (0.12, 0.39)    | 0.58 (0.40, 0.77)   | 0.17 (0.11, 0.23)    | 0.59 (0.39, 0.78)   |

Table 1: Parameter estimates of the mean effects. Numbers in bold are effects that were treated as fixed, either $\phi$ or $\psi$, in the mixed-effects model. Not-bolded numbers represent random-effect means, $\mu$. Numbers between parentheses are 95% credible intervals. Dashes indicate that the covariate was not included in the respective model.
| Effect        | Sender | Receiver |
|--------------|--------|----------|
| intercept    | 0.0000 | -        |
| teacher      | 0.0000 | 0.0000   |
| gender       | 0.0000 | 0.0224   |
| race         | 0.0000 | 0.3993   |
| inertia      | -      | 0.0000   |
| ABB (ABBA)   | 0.0000 | 0.0000   |
| ABA (ABAB)   | 4.1018 | 0.0000   |
| outgoingness | 2.6149 | -        |
| popularity   | 2.4797 | 0.0000   |

Table 2: Results for the validation test of random effects. It was decided that only positive evidence for random effects would be kept random, e.g. BF_{01} < 1/3. Numbers in bold represent most evidence for a fixed effect. Dashes indicate that the covariate was not included in the respective model.
Figure 1: Trace plots of some of the random-effect parameters. The number 2 in the subscript shows that the parameters belong to cluster number two. The top (bottom) line shows parameters in the sender (receiver) model.
Figure 2: Trace plots of some of the population parameters in the mixed-effects model. The top (bottom) line show parameters in the sender (receiver) model. The figure display one fixed effect and one random-effect mean with its variance for each model.
Figure 3: Comparison of the point-wise deviance residuals between the random-effects (x-axis) and the mixed-effects (y-axis) models. The grey line is the equality line.
Figure 4: Posterior distribution of $\phi_{\text{popularity}}$ (solid line) and $\phi_{\text{outgoingness}}$ (traced line) and their difference (dotted line).