Complete sets need not be reduced in Minkowski spaces

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Abstract It is well known that in n-dimensional Euclidean space \((n \geq 2)\) the classes of (diametrically) complete sets and of bodies of constant width coincide. Due to this, they both form a proper subfamily of the class of reduced bodies. For \(n\)-dimensional Minkowski spaces, this coincidence is no longer true if \(n \geq 3\). Thus, the question occurs whether for \(n \geq 3\) any complete set is reduced. Answering this in the negative for \(n \geq 3\), we construct \((2^k - 1)\)-dimensional \((k \geq 2)\) complete sets which are not reduced.

Keywords bodies of constant width · (diametrically) complete sets · Hadamard matrix · Minkowski Geometry · reduced body · Walsh matrix

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1 Introduction

We denote by \(X = (\mathbb{R}^n, \|\cdot\|)\) an \(n\)-dimensional normed or **Minkowski space**, i.e., an \(n\)-dimensional real Banach space with origin \(o\). We will use the common abbreviations aff and conv for affine and convex hull, respectively. A **convex body** in \(X\) is a compact, convex set with interior points, and the norm \(\|\cdot\|\) is induced by the \(o\)-symmetric convex body \(B_X\), called the **unit ball** of \(X\), via

\[
\|x\| = \min\{\lambda \geq 0 : x \in \lambda B_X\}, \quad \forall x \in \mathbb{R}^n.
\]

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An interesting part of the geometry of Minkowski spaces (see [9], [4], and [3]) refers to natural extensions of notions from classical convexity to normed spaces. Such notions are, e.g., special classes of convex bodies, like those of constant width, complete sets, and reduced bodies. It is well known that also in Minkowski spaces the width of a convex body $K$ in direction $u$ is the (Minkowskian) distance of the two supporting hyperplanes $H_1, H_2$ of $K$ which are both parallel to the $(n - 1)$-dimensional subspace orthogonal (in the Euclidean sense) to $u$. The minimum and the maximum of the width function of $K$ are called the thickness $\Delta(K)$ and the diameter $\delta(K)$ of $K$, respectively. In any Minkowski space $X$, bodies of constant width are precisely determined by the condition $\Delta(K) = \delta(K)$, and a convex body is called (diametrically) complete in $X$ if it is not properly contained in a compact set of the same diameter. In a sense dually, a convex body $R$ is said to be reduced in $X$ if $\Delta(C) < \Delta(R)$ holds for any convex body $C$ properly contained in $R$. There are interesting open problems about complete and reduced bodies in Minkowski spaces (see the survey [2]), and even for the Euclidean norm surprisingly elementary questions are still unsolved (cf. [1]). E.g., it is not known whether in Euclidean $n$-space $E^n$ ($n \geq 3$) there are reduced convex $n$-polytopes.

It is also interesting that the notion of reduced body, although it somehow dualizes that of completeness, yields a proper superset of the class of complete sets in $E^n$. The reason is that in $E^n$ the notions of constant width and completeness coincide, and that one can easily construct reduced bodies that are not of constant width. In Minkowski spaces, this coincidence of constant width and completeness is only assured for $n = 2$ (Meissner’s theorem; see p. 98 in [3]), and therefore in any Minkowski plane the class of reduced bodies clearly contains that of complete sets. Thus, the question remains whether in $n$-dimensional Minkowski spaces ($n \geq 3$) any complete set is reduced. A wrong short formulation was given in the survey [2] (cf. line 10 of the Abstract and, in repeated form, line 13 of page 406), which would imply that in Minkowski spaces any complete set is reduced. To correct this, we want to construct non-planar complete sets that are not reduced.

\section*{2 Results}

Let $X$ be the Minkowski space $l_1^3$, i.e., the Banach space on $\mathbb{R}^3$ endowed with the taxicab norm. Then $B_X$ is the crosspolytope whose vertices are

$$\pm(1,0,0), \pm(0,1,0), \text{ and } \pm(0,0,1).$$

Put $K$ to be the convex hull of the following four points (see Figure 1):

$$a_1 = (-1,-1,-1), \quad a_2 = (1,1,-1), \quad a_3 = (1,-1,1), \quad \text{and} \quad a_4 = (-1,1,1).$$

Clearly,

$$\|a_i - a_j\| = 4, \quad \forall\{i,j\} \subseteq \{1,2,3,4\}. \quad (1)$$

Claim 1: The tetrahedron $K$ is a complete set whose diameter is 4.

Proof We only need to show that, for each boundary point $x$ of $K$, there exists a point $y \in K$ such that $\|x - y\| = \delta(K)$, which follows directly from (1) and the trivial fact that

$$\left\| a_i - \frac{1}{3}(a_j + a_k + a_l) \right\| = 4, \quad \forall\{i,j,k,l\} \subseteq \{1,2,3,4\}.$$
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Fig. 1 The tetrahedron $K$ is not reduced.

\[\Box\]

Claim 2: $\Delta(K) = 2$.

Proof From the equalities
\[
\begin{align*}
\frac{1}{2}(a_2 + a_3) &= (1, 0, 0), \\
\frac{1}{2}(a_1 + a_2) &= (0, 0, -1), \\
\frac{1}{2}(a_2 + a_4) &= (0, 1, 0), \\
\frac{1}{2}(a_1 + a_3) &= (0, -1, 0), \\
\frac{1}{2}(a_3 + a_4) &= (0, 0, 1), \\
\frac{1}{2}(a_1 + a_4) &= (-1, 0, 0),
\end{align*}
\]

it follows that $B_X \subseteq K$. Therefore $\Delta(K) \geq 2$. Since the hyperplanes
\[\{(\alpha, \beta, \gamma) : \gamma = 1\}\] and \[\{(\alpha, \beta, \gamma) : \gamma = -1\}\]
are two parallel supporting hyperplanes of $K$ and the distance between them is $2$, $\Delta(K) \leq 2$. Therefore, $\Delta(K) = 2$. \[\Box\]

Now we are ready to show that $K$ is not reduced. Namely, the thickness of the convex body
\[K \cap \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma \geq -1\},\]
which is a proper subset of $K$, is still 2.

Now we try to extend this example to higher dimensions. Our main tool is that of Walsh matrices $H(2^k)$ (cf. [10, Chapters I and II] and [8]) which can be constructed by the following procedure:

\[H(2^1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\]
\[H(2^2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix},\]
\[H(2^k) = \begin{pmatrix} H(2^{k-1}) & H(2^{k-1}) \\ H(2^{k-1}) & -H(2^{k-1}) \end{pmatrix}, \forall k \in \mathbb{N}, k \geq 2.\]
Walsh matrices and the related Hadamard matrices can be a helpful tool for modelling and solving geometric problems in Banach spaces and high dimensions (see, e.g., [7] and [11, p. 51]). Since any Walsh matrix is a Hadamard matrix, we have

\[ H(2^n)H(2^n)^T = 2^n I_{2^n}, \forall n \in \mathbb{N}, n \geq 1. \] (2)

For each integer \( n \geq 2 \), let \((\alpha_{i,j})_{0 \leq i,j \leq 2^n-1}\) be the matrix \( H(2^n) \). For each integer \( 0 \leq i \leq 2^n - 1 \), put

\[ b_i = (\alpha_{i,j})_{j=0}^{2^n-1} \text{ and } a_i = (\alpha_{i,j})_{j=1}^{2^n-1}. \]

Then

\[ V = \{ a_i : 0 \leq i \leq 2^n - 1 \} \]

is a set of \( 2^n \) points in \( \mathbb{R}^{2n-1} \). By (2), we have that

\[ \{ b_i : 0 \leq i \leq 2^n - 1 \} \]

is linearly independent and, therefore, \( V \) is affinely independent. Thus \( S = \text{conv}V \) is a simplex in \( \mathbb{R}^{2n-1} \).

**Proposition 3** Let \( X = l_1^{2^n-1} \), and \( \| \cdot \| \) be the norm of \( X \). Then

1. \( \delta (S) = 2^n \);
2. for each vertex \( v \) of \( S \) and each point \( w \) in the facet of \( S \) opposite to \( v \), \( \| v - w \| = 2^n \); moreover, the distance from \( v \) to \( \text{aff} F_v \) is \( 2^n \);
3. \( B_X \subseteq S \);
4. \( \Delta (S) = 2 \);
5. \( S \) is not reduced.

**Proof** 1. By (2), for each set \( \{ i, j \} \subset \{ k : k \in \mathbb{Z}, 0 \leq k \leq 2^n - 1 \} \) the inner product of \( b_i \) and \( b_j \) is 0. Therefore \( b_i \) and \( b_j \) are different in precisely \( 2^{n-1} \) coordinates. This implies that

\[ \| a_i - a_j \| = 2^n. \]

Thus, the distance between each pair of the vertices of \( S \) is \( 2^n \), which implies that \( \delta (S) = 2^n \).

2. Being a Hadamard matrix, \( H(2^n) \) must have the following property: each column of \( H(2^n) \), except for the first one, is evenly divided between 1 and \(-1\). Thus

\[ \sum_{i=0}^{2^n-1} a_i = 0. \]

Let \( v \) be an arbitrary vertex of \( S \), and \( F_v \) be the facet of \( S \) opposite to \( v \). Then \( F_v = \text{conv}(V \setminus \{ v \}) \), and the point

\[ w = \frac{1}{2^n - 1} \sum_{z \in V \setminus \{ v \}} z \]
is a relatively interior point of $F_v$. Moreover,

$$
\|v - w\| = \left\|v - \frac{1}{2^n - 1} \sum_{z \in V \setminus \{v\}} z\right\|
= \left\|v + \frac{1}{2^n - 1} v\right\|
= 2^n.
$$

Since the functional

$$
f : F_v \mapsto \mathbb{R}
\quad z \mapsto \|v - z\|
$$

is convex,

$$
\|v - z\| = 2^n, \forall z \in F_v.
$$

The second part of the result follows from the fact that the functional

$$
f : \text{aff} F_v \mapsto \mathbb{R}
\quad z \mapsto \|v - z\|
$$

is convex and constant in $F_v$.

3. For each vertex $v$ of $S$, denote by $H_v$ the supporting halfspace of $S$ determined by $F_v$. Then

$$
S = \bigcap_{v \in V} H_v.
$$

Thus we only need to show that, for each $v \in V$, $H_v$ is also a supporting halfspace of $B_X$. Since $o \in S$, as can be easily verified, we only need to show that each hyperplane determined by $F_v$, which is aff$F_v$, is a supporting hyperplane of $B_X$ at the point $-\left(1/(2^n - 1)\right)v$. Clearly,

$$
-\frac{1}{2^n - 1} v \in F_v \cap B_X.
$$

We only deal with the case when $v = a_0$; the other cases can be dealt with in a similar way. For each point $w \in \text{aff} F_v$, there exist numbers $\lambda_1, \cdots, \lambda_{2^n - 1}$ such that

$$
w = \sum_{i=1}^{2^n - 1} \lambda_i a_i \quad \text{and} \quad \sum_{i=1}^{2^n - 1} \lambda_i = 1.
$$

Therefore,

$$
\|w\| = \frac{1}{2^n} \|2^n w + v - v\|
\geq \frac{1}{2^n} \|2^n \left(w + (2^n - 1) \left(w + \frac{1}{2^n - 1} v\right) - v\right) - v\|
\geq \frac{1}{2^n} 2^n = 1,
$$

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and the inequality holds since the distance from $v$ to $\text{aff} F_v$ is $2^n$ and

$$w + (2^n - 1) \left( w + \frac{1}{2^n - 1} v \right) = w + (2^n - 1) \left( w - \frac{1}{2^n - 1} v \right) \in \text{aff} F_v.$$ 

Thus $\text{aff} F_v$ is a supporting hyperplane of $B_X$. It follows that $B_X \subseteq S$.

4. Put

$$H_1 = \{ (\eta_i)_{i=1}^{2^n-1} : \eta_{2^n-1} = 1 \}, \quad H_2 = \{ (\eta_i)_{i=1}^{2^n-1} : \eta_{2^n-1} = -1 \}.$$ 

We have $B_X \subseteq S$. Thus, to show that $\Delta(S) = 2$, it suffices to show that $H_1$ and $H_2$ are supporting hyperplanes of $S$, which is clear since each of $H_1$ and $H_2$ supports both $B_X$ and its polar body

$$Q = \{ (\eta_i)_{i=1}^{2^n-1} : \max\{ |\eta_i| : 1 \leq i \leq 2^n - 1 \} = 1 \},$$

and $B_X \subseteq S \subseteq Q$.

5. Let

$$H = \left\{ (\eta_i)_{i=1}^{2^n-1} : \sum_{i=1}^{2^n-1} \eta_i \leq 1 \right\}.$$ 

Then $S \cap H$, a proper subset of $S$, is a convex body containing $B_X$ whose thickness is 2. Thus $S$ is not reduced.

Proposition 3 also shows that, for any positive number $\varepsilon$, there exists a positive integer $n$, an $n$-dimensional Minkowski space $X$, and a complete set $K$ in $X$ such that the ratio of $\Delta(K)$ to $\delta(K)$ is less than $\varepsilon$. Anyway, since each non-trivial complete set (i.e., a complete set containing at least two distinct points) in a Minkowski space has an interior point, its thickness is strictly greater than 0.

The situation in infinite dimensional Banach spaces is quite different: it is possible that a non-trivial complete set is contained in a hyperplane (cf. [5, Example 4.6]), and the thickness of such a complete set is clearly 0.

We finish this paper with a very natural question, clearly not posed in [2] and referring to all dimensions $n \geq 3$ (by Meissner’s theorem the planar case is clear; see [3, p. 98]).

**Problem:** In which Minkowski spaces is any complete set also reduced? Try to give geometric characterizations of their unit balls!

In some sense vice versa, it might be interesting to collect (e.g., as partial results in that direction) typical geometric properties shared by those Minkowski spaces which contain complete sets that are not reduced. All this is directly motivated by our construction.

More generally, it would be nice to give a complete clarification of the containment and intersection relations between the families of reduced, complete, and constant width bodies (and between suitable subfamilies of them) in Minkowski Geometry, for example via a completely described Venn diagram. This is of course closely related to recent research on those Minkowski spaces in which completeness and constant width are equivalent notions; see, e.g., [6].
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