No classical limit of quantum decay for broad states

N G Kelkar and M Nowakowski

Departamento de Fisica, Universidad de los Andes, Cra.1E No 18A-10, Santafe de Bogotá, Colombia

E-mail: nkelkar@uniandes.edu.co

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Abstract

Though the classical treatment of spontaneous decay leads to an exponential decay law, it is well known that this is an approximation of the quantum mechanical result which is non-exponential at very small and large times for narrow states. The non-exponential nature at large times is, however, hard to establish from experiments. A method to recover the time evolution of unstable states from a parametrization of the amplitude fitted to data is presented. We apply the method to a realistic example of a very broad state, the $\sigma$ meson and reveal that an exponential decay is not a valid approximation at any time for this state. This example derived from experiment shows the unique nature of broad resonances.

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1. Introduction

It is well known by now that the exponential nature of the decay law which appears practically in every field of physics and which follows from classical physics is an approximation [1] and deviations from the exponential are expected at extremely short and large times. Apart from the power law behaviour at small times, the quantum mechanical survival probability of an unstable state typically displays three regions: an exponential decay law followed by an oscillatory behaviour corresponding to the transition region and finally a power law behaviour (the non-exponential (NE) tail). Over the years there have been several unsuccessful attempts in particle and nuclear physics [2] to verify the NE tail experimentally. The failure of such experiments is due to the fact that the critical time for the transition from the exponential to the NE depends on the width of the state. For narrow states (i.e. small width and hence long lived) the critical time is large (up to several years for heavy nuclei) and the exponentially decaying sample would physically diminish to an unmeasurable amount. For broad states the critical time is small. However, the dominant decay law $\exp(-\Gamma t)$ at small times (following
the extremely short time region) reduces the sample rapidly due to the large value of $\Gamma_1$ in the exponential. In essence, one could say that nature conspires to hide the NE tail.

In the present work, we investigate the survival probabilities of broad states which display a peculiar behaviour in contrast to the above-mentioned standard picture. The scattering data in reactions where an unstable intermediate state is formed can be used to evaluate the survival probability of the intermediate state at all times [3]. Using this method which leads to an indirect measurement of the NE decay at large times, we show that for a very broad state, the decay law is never exponential. The classical limit confirmed so often in the laboratory for narrow states does not exist for broad unstable states. An understanding of the time evolution of unstable (sometimes referred to as metastable or resonant) states is of fundamental importance for every branch of physics where decaying states appear. As a matter of fact, except for the electron and proton every other elementary particle is unstable and decays spontaneously.

Since the main objective of the present work is to present a semi-empirical method to evaluate survival probabilities and then discuss the special case of broad resonances, in what follows, we first introduce the formalism used to evaluate the survival probability. We then present a realistic parametrization of pion–pion elastic scattering data and use it to demonstrate the above-mentioned result for broad resonances.

2. Formalism

The quantum mechanical survival probability of a decaying state without any approximation is given as

$$ P(t) = |A(t)|^2 = |\langle \Psi_1 | e^{-iHt} | \Psi_1 \rangle|^2. \tag{1} $$

At very small times, it can be shown to be $P(t) \simeq 1 - \frac{1}{2} (\Delta_\Psi H)^2 t^2$, where $\Delta_\Psi H$ is the uncertainty in energy. A direct deviation from the exponential decay law at short times has been experimentally verified [4]. This deviation is expected at extremely short times and is not the topic of concern in the present work. The exponential fall of $P(t)$ which follows is the behaviour most commonly verified in the laboratory. With the exception of the experiment with organic materials [5], the large time behaviour has however not been measured and there exist different theoretical approaches for the evaluation of $P(t)$ at large times [1]. An interesting discussion on the difficulties and possibilities of measuring the NE tail can be found in [6].

2.1. Fock–Krylov method

In the present work we use the Fock–Krylov method [3, 7] which relies on basic results in quantum mechanics and is briefly presented below. Given the fact that an unstable state $|\Psi_1\rangle$ cannot be an eigenstate to the Hamiltonian ($H |\Psi_1\rangle \neq E |\Psi_1\rangle$ otherwise $A(t) = \langle \Psi_1 | e^{-iHt} | \Psi_1 \rangle = e^{-iEt}$ and $P(t) = 1$ implying that the state never decays) one can expand $|\Psi_1\rangle$ as

$$ |\Psi_1\rangle = \int_\text{Spect}(H) \, dE \, a(E) \, |E\rangle, \tag{2} $$

where $H |E\rangle = E |E\rangle$. Using $\langle E' | E \rangle = \delta(E' - E)$, we arrive at the result that

$$ \rho(E) = \frac{d \text{Prob}_\Psi(E)}{dE} = |\langle E | \Psi_1 \rangle|^2 = |a(E)|^2 \tag{3} $$

is a probability density (and as such positive definite) [3] to find the states with energy $E$ in the resonance. One can now evaluate the survival amplitude

$$ A(t) = \int_\text{Spect}(H) \, dE \, \rho(E) \, e^{-iEt} = \int_{E_a}^{\infty} \, dE \, \rho(E) \, e^{-iEt}, \tag{4} $$

2
which turns out to be a Fourier transform of the spectral function \( \rho(E) \). Here \( E_{\text{th}} \) is the sum of the masses of the decay products. The general form of \( \rho(E) = (\text{threshold factor}) \times (\text{pole}) \times (\text{form factor}) \), i.e.

\[
\rho(E) = (E - E_{\text{th}})^\gamma \times P(E) \times F(E).
\]

(5)

\( P(E) \) has a simple pole at \( z_R = E_R - i\Gamma_R/2 \) which leads to the exponential decay law. \( F(E) \) is a smooth function which should tend to zero for large \( E \). Going over to the complex plane and performing the integral as described in [3], the survival amplitude \( A(t) \) is given as

\[
A(t) = \text{Res}[\rho(z), z_R] + e^{-iE_{\text{th}}t} (-i)^{\gamma+1} \int_0^\infty dx \ P(-ix + E_{\text{th}}) F(-ix + E_{\text{th}}) x^{\gamma} e^{-xt}
\]

(6)

with \( A_{\text{exp}}(t) \) and \( A_L(t) \) representing the exponential and the remaining part of the amplitude respectively. We will now proceed to evaluate \( A(t) \) for a particular choice of \( \rho(E) \) which connects it to scattering data.

2.2. Density of states

The connection between scattering data and \( \rho(E) \) as noticed in [3, 8] is briefly repeated here for clarity. While calculating the virial coefficients in the equation of an ideal gas, Beth and Uhlenbeck [9] found that the difference between the density of states (of scattered particles) with the interaction \( d_0 l(E)/dE \) and without \( d_0^{(0)} l(E)/dE \) is

\[
\frac{dn}{dE} = \frac{dn_l(E)}{dE} - \frac{dn_l^{(0)}(E)}{dE} = \frac{2l + 1}{\pi} \frac{d \delta_l(E)}{dE}.
\]

(7)

Here \( \delta_l(E) \) is the scattering phase shift for the \( l \)th partial wave in elastic scattering. For an intermediate unstable state occurring in the scattering of two particles, this is the density of states of the unstable state (or resonance) in terms of the decay products. Thus, the spectral function \( \rho(E) = \frac{d \text{Prob}_\rho(E)}{dE} \propto \frac{dn}{dE} \) can be expressed in terms of the derivative of the scattering phase shift (which in turn is related to the scattering amplitude \( T_l \) as \( T_l = \exp(2i \delta_l) - 1)/2i \)). This phase shift derivative was in fact found to be the delay time (or phase time delay) in scattering by Wigner [10] and also used to characterize resonances in hadron scattering [11]. This interpretation works well for all \( l \)-values except for the \( s \)-wave \( (l = 0) \), because in this case \( d \delta_l/dE \propto (E - E_{\text{th}})^{-1/2} \) and we encounter a threshold singularity\(^1\).

The problem can however be resolved by defining rather a dwell time delay (which has also been shown to be a density of states [12]) as proposed in [13]. Thus, \( E_{\text{th}} \) can be expressed as

\[
\left( \frac{dn}{dE} \right)_{\text{new}} = \text{dwell time delay} = \frac{2 \delta_l}{dE} - \frac{2 \Re e(T)}{\sqrt{s}} \frac{\sqrt{s}}{s - E_{\text{th}}^2}
\]

(8)

which is the relativistic version of the expression found in [13]. Here, \( s = E^2 \) and \( E_{\text{th}} \) is the sum of the masses of the decay products. With (8) as \( \rho(E) \) one can check that one gets the standard threshold behaviour and replacing this \( \rho(E) \) in (4), the correct power law as also found in [14].

\(^1\) For a particle with an incident energy, \( E = \hbar^2 k^2/2\mu \), a relation between the phase and the dwell time \( (\tau_p(E) \) and \( \tau_d(E) \), respectively) is easily obtained after some rearrangement of the Schrödinger equation and is given by \( \tau_p(E) = \tau_d(E) - \hbar (\text{Im}(R)/k) d\delta/dE \). The second term on the right-hand side of this equation is the self-interference term which arises due to the overlap of the incident and reflected waves in front of the barrier. This term is important at low energies and becomes singular as \( E \to 0 \). A similar relation occurs in scattering problems (see [13]) where \( R \) is related to the scattering amplitude. At high energies, the phase and dwell times are the same.
3. The case of a broad state: $\sigma$ meson

Being equipped with the theoretical framework for evaluating $A(t)$ and hence the survival probability $P(t) = |A(t)|^2$, we now proceed to calculate $P(t)$ for a realistic broad unstable state. The choice we make is that of the scalar meson $\sigma$ formed in pion–pion ($\pi\pi$) elastic scattering. The very short lived $\sigma$ has been and is still one of the most controversial problems among particle physicists. It was removed from the particle data listing in 1974 and reappeared there much later. It is sometimes claimed that this meson behaves differently in different physical situations [15], i.e. displaying different masses and lifetimes. Theoretically, it can be viewed as a Higgs particle in the context of the linear sigma model [16] after the spontaneous breaking of chiral symmetry. It can also be looked upon as a low energy manifestation of the scale invariance breaking in the strong interaction [17]. Some recent discussions on this enigmatic scalar meson can be found in [18].

3.1. Parametrization of the amplitude using $\pi\pi$ scattering data

To evaluate the density of states for the $\sigma$ meson, we use a parametrization of the scattering phase shift given in [19] which is obtained from a consistent fit to the production and elastic $\pi\pi$ scattering data and includes the effects of the Adler zeros which are important in the context of these analyses. Within this parametrization and using (8),

$$\frac{dn}{dE} = (E - 2m_\pi)^{1/2} P_\sigma(E) F_\sigma(E) = \rho(E),$$

where

$$P_\sigma(E) = 4 M b_2 / [(M^2 - s)^2 + M^2 \Gamma^2(s)]$$

with $s = E^2$, and

$$\Gamma^2(s) = \frac{s - 4m_\pi^2}{s} \left( \frac{s - s_A}{M^2 - s_A} \right)^2 (b_1 + b_2 s)^2 \exp \left[ -\frac{2(s - M^2)}{A} \right],$$

$$F_\sigma(E) = \sqrt{E + 2m_\pi(s - s_A)(M^2 - s)} e^{\text{im}_1}$$

$$\times \left\{ 1 + \frac{b_1 + b_2 s}{b_2} \left( \frac{1}{M^2 - s} + \frac{1}{s - s_A} - \frac{1}{2s} - \frac{1}{A} \right) \right\}.$$  

Replacing this parametrization of $\rho(E)$ (with the parameters $M, A, s_A, b_1$ and $b_2$ fitted to data taken from [19]) in (6), the survival probability is evaluated numerically and is plotted in figure 1. The different curves in figure 1(a) display the contributions of the exponential term in (6), the remaining term (which leads to the power law at large times) and the interference of the two terms in the amplitude which appear in $P(t)$. According to [19], the $\sigma$ meson here has a mass $E_R = 542$ MeV and a width $\Gamma_R = 498$ MeV. Though there exist other predictions [20–22] of the $\sigma$ mass, they all agree on a large width. It is clear from the figure that there is a sizable contribution from all terms up to about 15 lifetimes when $P(t)$ completely approaches the power law. Figure 1(b) makes it clear that the decay law can never be approximated by an exponential decay in the case of the $\sigma$ resonance.

Before proceeding, some remarks regarding the use of such a parametrization are in order. Firstly, the calculation of the survival amplitude $A(t)$ requires the analytic continuation of the amplitude up to the negative imaginary axis in the lower-right complex energy plane (corresponding to the second Riemann sheet in the Mandelstam variable $s$). This means that the knowledge of the amplitude far outside the experimental region is required. It is known that there are many parametrizations ([21], [23] and references therein) that describe...
Figure 1. Survival probability of the $\sigma$ meson with a width, $\Gamma_R = 498$ MeV as obtained from a parametrization of experimental data [19]. (a) The solid line is the full survival probability $P(t)$, the dashed line is the contribution of the exponential term, the dashed dotted is the remaining part and the dotted line is the magnitude of the oscillatory interference term. (b) Comparison of the full $P(t)$ with a pure exponential decay law on a log scale. The inlay displays the curves in (a) on a linear scale.

the data equally well, but are very different when continued in the complex plane. The large uncertainties in the determination of the pole position of the sigma resonance are due precisely to this ‘instability’ of analytic continuation. This issue has been discussed in detail in [22]. Secondly, the parametrization of Bugg [19] is valid only up to a region of about 1 GeV. As a result of this fact, one encounters several poles in the parametrization at high energies which have no physical meaning. Clearly, the occurrence of these poles is an artefact of the parametrization and should not be considered in a calculation of the survival amplitude of the $\sigma$ meson. This is clear alone from the fact that such additional poles do not correspond to any known resonant states. Hence, relying on the long energy tail of the parametrization, we simply neglect the residues due to these poles at high energies. In principle, we could have used another parametrization which does not have the drawback of such unwanted poles. To clarify this issue in a more detailed way, we refer to the Breit–Wigner (B-W) model where we find that the survival probability calculated from the B-W model is qualitatively not very different from that obtained using the parametrization in [19] (see figure 3 to be discussed later). This justifies the neglect of the poles present in the parametrization of [19] at high energies.
3.2. Breit–Wigner amplitude

To get a comparative feeling of the results in figure 1 with those of longer lived states, we perform some simple model calculations for unstable states with varying lifetimes. We choose \( \rho(E) \) to have the standard B-W form with a threshold factor and an exponentially falling form factor \( F(E) \). Thus,

\[
\rho_{\text{B-W}}(E) = \frac{(E - E_{\text{th}})}{2} \times \frac{1}{[(E - E_R)^2 + \Gamma_R^2/4]} \times e^{-E/E_0},
\]

where \( E_0 = 1.1 \text{ GeV} \) has been adjusted to match the tail of a realistic parametrization. In figure 2 we show the plots for unstable states with different ratios \( R = \Gamma_R/(E_R - E_{\text{th}}) \) which depend on the width as well as the position of the resonance from threshold. It can be seen that for narrow states there is a very well-defined oscillatory region of transition from the exponential to the NE decay law. The oscillatory region shifts to smaller times as \( R \) increases and for very broad states, the classical approximation of an exponential decay law does not hold good at any time. This is essentially similar to the result shown in figure 1 for the realistic case of the \( \sigma \) meson in \( \pi \pi \) scattering. Indeed there is no distinct oscillatory region of a transition from the exponential to the power law. A similar discussion based on a very simplistic model and using a different approach than the one used in this work can be found in [23]. In figure 3(a), we compare the full survival probability \( P(t) \) evaluated using the
B-W model (with pole values obtained in [19]) with that using the parametrization of [19]. Qualitatively, there is not much difference between the two results, implying that the present calculation of the survival probability may not be sensitive to the details of the parametrization used. This follows for instance from the fact that the main result of the present work agrees with purely theoretical expectations found in [23]. We therefore have to conclude that the overall behaviour of the survival probability as constructed from the amplitude is insensitive to the choice of the parametrization. We note that the extra poles in Bugg’s parametrization are unphysical and an artefact of the parametrization (caused probably by the fact that the parametrization is valid only up to a certain energy). The comparison with the B-W model leads us to the conclusion that we can safely neglect these poles which, however, does not imply that both calculations (B-W and the actual result) are equivalent. That Bugg’s model is not a simple B-W can be seen from the expressions.

Since the B-W model seems reliable once the pole value is given, we can use it to examine other parametrizations. Indeed, the survival probability is more sensitive to the pole value of the unstable state. This can be seen in figure 3(b) where we use the pole values given in [22]. In [22] Caprini performed a detailed analysis of 16 different parametrizations and provided certain average best fit pole values for the $\sigma$ meson with the corresponding error bars. The various curves presented in figure 3(b) correspond to the pole values from [22] within error bars. It is interesting to note that though the result of a NE decay law for the sigma at all times still remains, the behaviour of $P(t)$ in the region where the power law sets in depends on the
ratio of the width to the mass of the $\sigma$ meson. For widths bigger than the resonance mass, the survival probability shows a dip at the onset of the power law region.

4. Summary

To conclude, we summarize the findings of the present work.

1. A semi-empirical method to determine the survival probability of an unstable state from scattering data is demonstrated with a realistic example of a very broad resonance from meson–meson scattering. Though we have found an empirical method of recovering the time evolution from experiment, it does depend on the theoretical input of the parametrization and the uncertainties associated with it. The power law behaviour at large times which is hard to find experimentally is verified for the $\sigma$ resonance. However, the more interesting finding is that the decay law for this realistic case of a broad state is never close to an exponential.

2. In order to get a more general view of the behaviour of survival probabilities ($P(t)$), a study using the B-W model was performed and led to the following findings.
   (a) Investigations on the dependence of the critical times (for the transition from an exponential to the NE decay) on the positions and widths of the unstable states reveal the following.
      (i) For narrow states there exist three distinct regions, namely an exponential decay, an oscillatory transition region and a NE power law. The critical times depend on the width as well as the position of the resonance mass from the threshold. They shift to smaller values with increasing values of $R = \Gamma_R/(E_R - E_{th})$.
      (ii) For very broad states, the decay law does not approach the classical result of an exponential decay law at any time. Hence, a well-separated exponential followed by an oscillatory region does not exist.
   (b) A comparison of $P(t)$ using the BW model and the parametrization in [19] for the $\sigma$ meson shows that overall, $P(t)$ is not sensitive to the details of the parametrization. The transition region in $P(t)$ where the power law sets in is sensitive to the ratio of the width to the mass of the unstable state. Since the BW result is sensitive only in the transition region to the pole values used, the main result that the decay is not exponential at any time still remains valid. However, it seems worthwhile to come back in future to the transition region which is sensitive to the pole values and use different parametrizations to investigate it.

3. Our method to extract the survival probability has a further significance. The survival probability presented in this work is valid only if the system evolving is isolated according to its intrinsic dynamics. Interactions with the environment including (repeated) measurements yield a different picture all together [14]. To quote [14]: ‘The experimentally observed survival probability law is exponential at all times. This is due to repeated measurements provided $\lambda \tau \gg 1$ where $\lambda$ is the frequency of the measurements and $\tau$ the lifetime (for the exact definition of the latter see [14]). If this is the case, the direct measurement of the survival probability defined alone through its intrinsic dynamics can be hampered’. However, our semi-empirical extraction of this quantity is indirect and does not require a reduction of the state.

These findings should be relevant to most branches of physics where unstable states occur. In particular, the example of the $\sigma$ meson presented provides yet another way of investigating this elusive scalar meson which has remained a topic of controversy over the years. We note that the $\sigma$ meson, which does not have an exponential decay law at any time is an exception
among hadron resonances. For all known hadron resonances, the survival probability displays an exponential behaviour before the onset of the power law at large times.

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