Mixed Gaussian processes: a filtering approach

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Outline

1. Introduction
   - Problems statement and questions around
   - Historical survey

2. Results
   - From $L^2$ to $L^1$
   - Even from $L^1$, but in a partial case
   - Semimartingale Structure of $X$

3. The Proofs
   - Integro-Differential Equation
   - Diffusion type representation, Equivalence of measures

4. Concluding Remarks
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4 Concluding Remarks
Motivation and the Challenge of this talk

Challenge
To present a new approach to analysis of mixed Gaussian processes based on the linear filtering theory.

Motivation
To construct the likelihood type estimates for mixed Gaussian noises systems:

\[ Y_t = \int_0^t f(\theta, s) \, ds + X_t, \quad 0 \leq t \leq T, \]
Mixed Gaussian process

where

\[ X_t = B_t + G_t, \quad t \in [0, T], \quad T > 0, \]

with

- \( B_t \) — the standard Brownian motion
- \( G_t \) — an independent centered Gaussian process

with the covariance function (frequently) in the form:

\[ \Gamma(s, t) = \mathbb{E}G_t G_s = \int_0^t \int_0^s K(u, v)du dv \]
Object of studies

Mixed Gaussian process

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\[
\Gamma(s, t) = \mathbb{E}G_t G_s = \int_0^t \int_0^s K(u, v) \, du \, dv
\]
Two Partial Cases

Fractional type Mixed Noises

- $G = B^H$, $H \in (0, 1)$ — a fractional Brownian motion
  \[ \Gamma(s, t) = \mathbb{E} B_t^H B_s^H = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right) \]

- $G = B^{L,H}$, $H \in (0, 1)$ — a Riemann-Liouville process
  \[ B_t^{L,H} = \int_0^t (t - s)^{H-1/2} dB_s \]
Two Partial Cases

Fractional type Mixed Noises

- $G = B^H, \ H \in (0, 1)$ — a fractional Brownian motion
  \[
  \Gamma(s, t) = \mathbb{E}B_t^H B_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)
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- $G = B^{L,H}, \ H \in (0, 1)$ — a Riemann-Liouville process
  \[
  B_t^{L,H} = \int_0^t (t-s)^{H-1/2} dB_s
  \]
Questions around

- Stochastic analysis of $X$:
  - **canonical representations**: multiplicity of the innovations, structure of the fundamental martingale:

$$X_t = \int_0^t G(s, t) dM_s \quad M_t = \int_0^t g(s, t) dX_s$$

with an innovation martingale $M$; equivalence of the filtrations $F^X_t = F^M_t$

- the semimartingale representation
- the density with respect to the standard and fractional Wiener measures

- Stochastic analysis of $Y$: the density with respect to measure $\mu^X$
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4. Concluding Remarks
50 years ago; independently; at the same time

L. Shepp

- $\mu^X \sim \mu^B$ if and only if $K \in L^2([0, T]^{2})$
- the density $d\mu^X / d\mu^B$ involves Carleman-Fredholm determinant and resolvent kernel of the covariance operator associated with $K$
- it doesn’t immediately reveal the innovation structure
50 years ago; independently; at the same time

I. Gohberg and M. Krein

- Factorization theory of Fredholm operators in Hilbert spaces:

\[(Id + K)^{-1} = (Id + L_+)(Id + L_-)\]

with right and left Volterra kernels L.

- Resolvent identity for a continuous kernel K

\[L(s, t) + \int_0^t L(r, t)K(r, s)dr = -K(s, t), \quad 0 \leq s \leq t \leq T.\]

- The crucial role of the equation

\[g(s, t) \int_0^t g(r, t)K(r, s)dr = 1, \quad g(s, s) \neq 0.\]
50 years ago; independently; at the same time

T.Kailath

- the relevance of I. Gohberg and M. Krein theory:
- Shepp’s density formula is rewritten in the form

\[
\frac{d\mu^X}{d\mu^B}(X) = \exp \left( - \int_0^T \varphi_t(X) \, dX_t - \frac{1}{2} \int_0^T \varphi_t^2(X) \, dt \right),
\]

where

\[ \varphi_t(X) = \int_0^t L(s, t) \, dX_s \]

with \( L \in L^2([0, T]^2) \) being the unique solution of the Wiener-Hopf integral equation

\[
L(s, t) + \int_0^t L(r, t)K(r, s)\, dr = -K(s, t), \quad 0 \leq s \leq t \leq T.
\]
M. Hitsuda

- \( \mu^X \sim \mu^B \) if and only if \( X \) can be represented as

\[
X_t = \overline{B}_t - \int_0^t \int_0^s \ell(r, s) d\overline{B}_r ds,
\]

with some Brownian motion \( \overline{B} \) and some Volterra kernel \( \ell \in L^2([0, T]^2) \).

- Actually the kernel \( \ell \) solves the Riccati-Volterra equation:

\[
\ell(s, t) = K(s, t) - \int_0^{t \wedge s} \ell(s, r) \ell(t, r) dr.
\]
Recovered 10 years ago

Important references

- 2000 P.Cheridito

\[
\sup_{\tau} \sum_{j=0}^{n-1} \mathbb{E} \left| \mathbb{E}(X_{t_{j+1}} - X_{t_j} | F_{t_j}^X) \right| < \infty
\]

- 2003 F.Boduin and D.Nualart

\[
X := B + V, \quad \partial^2 K/\partial s \partial t \in L^2([0, T]^2)
\]

Hida-Hitsuda criterion.

- 2007 H.van Zanten equivalence of \( \xi = \sum_{k=1}^{n} \alpha_k B^{H_k} \) of \( n \) independent fBm’s to a single fBm. (Spectral techniques for processes with stationary increments).
Important references

- 2000 P. Cheridito

\[
\sup_{\tau} \sum_{j=0}^{n-1} \mathbb{E} \left[ \mathbb{E} \left( X_{t_j+1} - X_{t_j} \mid F_{t_j}^X \right) \right] < \infty
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  \[ \sup_{\tau} \sum_{j=0}^{n-1} \mathbb{E} \left| \mathbb{E} \left( X_{t_{j+1}} - X_{t_j} \middle| \mathcal{F}_{t_j} \right) \right| < \infty \]

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Where we are

What is missed?

- Different objects: general point of view
- Probabilistic interpretation
- What can we do for non $L^2$ kernels?

The week point: to consider only lower/upper triangular kernels.
What should we do: To forget the factorisation theory and try to find an other point of view
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Example

Multiplicity Greater than One

\[ X_t = B_t + \xi \int_0^t \frac{1}{\sqrt{|1-s|}} ds, \]

where \( \xi \sim N(0, 1) \) is independent of \( B \).

- \( \xi \) can be recovered precisely from \( \mathbb{F}_t^X \) for all \( t \geq 1 \).
- \( \mathbb{F}_t^X \) is discontinuous at \( t = 1 \),
- with \( \mathbb{F}_t^X \not\subset \mathbb{F}_t^X = \mathbb{F}_t^B \vee \sigma\{\xi\} \) for all \( t \geq 1 \).
From $L^2$ to $L^1$

Assumptions

- \[ X_t = B_t + G_t, \quad t \in [0, T], \quad T > 0, \]

- \[ \Gamma(s, t) = \mathbb{E}G_t G_s = \int_0^t \int_0^s K(u, v) \, du \, dv \]

- \[ K(s, t) = |s - t|^{-\alpha} M(s, t), \quad 0 \leq \alpha < 1, \]

where $M \in C([0, T]^2)$. 

Even from $L^1$, but in a partial case

Semimartingale Structure of $X$
Equations and interpretations

Equations

\[
L(s, t) + \int_0^t L(r, t)K(r, s)dr = -K(s, t), \quad 0 \leq s, t \leq T.
\]

\[
q(s, t) + \int_0^t q(r, t)K(r, s)dr = \phi(s), \quad 0 \leq s, t \leq T.
\]

with \( \phi_s = 1 - \int_0^s L(r, s)dr \).
Canonical Representations

Theorem

The process

\[ \overline{B}_t = \mathbb{E} \left( \int_0^t \phi_s dB_s \middle| \mathcal{F}_t^X \right) \]

is a Brownian motion, satisfying

\[ \overline{B}_t = \int_0^t q(s, t) dX_s, \]

The representation

\[ X_t = \int_0^t \hat{q}(s, t) d\overline{B}_s \]

with \( \hat{q}(s, t) = -\frac{\partial}{\partial s} \int_s^t q(r, s) dr \), is canonical, i.e. \( \mathcal{F}_t^X = \mathcal{F}_t^B \).
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Mixed fractional Brownian motion

We are back to the Krein’s remark:

\[ g(s, t) + \int_0^t g(r, t)K(r, s)\, dr = 1 \]

Notations: \( \mathbb{F}^X = (\mathbb{F}^X_t) \) and \( \mathbb{F} = (\mathbb{F}_t), \ t \in [0, T] \) —the natural filtrations of \( X \) and \( (B, B^H) \) respectively.

**Fundamental Martingale**

\[ M_t = \mathbb{E}(B_t|\mathbb{F}^X_t), \quad t \in [0, T]. \]

\( M \) encodes many of the essential features of the process \( X \), making its structure particularly transparent.
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Fundamental Martingale

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Fundamental Martingale Representation via $X$

Fundamental Martingale Representation

$$M_t = \int_0^t g(s, t) dX_s, \quad \langle M \rangle_t = \int_0^t g(s, t) ds, \quad t \geq 0.$$  

The kernel $g(s, t)$ solves integro-differential equation:

Equation for the Kernel

$$g(s, t) + H \frac{d}{ds} \int_0^t g(r, t) |s-r|^{2H-1} \text{sign}(s-r) dr = 1, 0 < s < t \leq T.$$  

The family of functions $\{g(s, t), 0 \leq s \leq t \leq T\}$ plays the key role in our approach to analysis of the mixed fBm.
**Fundamental Martingale Representation via $X$**

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**Equation for the Kernel**

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The family of functions $\{g(s, t), 0 \leq s \leq t \leq T\}$ plays the key role in our approach to analysis of the mixed fBm.
$X$ is a stochastic integral w.r.t $M$

**Representation of $X$ via $M$**

The following representation holds:

$$X_t = \int_0^t G(s, t) \, dM_s, \quad t \in [0, T],$$

where

$$G(s, t) := 1 - \frac{d}{d\langle M \rangle_s} \int_0^t g(\tau, s) \, d\tau, \quad 0 \leq s \leq t \leq T.$$  

and, in particular, $\mathbb{F}_t^X = \mathbb{F}_t^M$, $P$-a.s. for all $t \in [0, T]$.

Note that $s > \tau$ is possible.
$X$ is a stochastic integral w.r.t $M$

**Representation of $X$ via $M$**

The following representation holds:

$$X_t = \int_0^t G(s, t) dM_s, \quad t \in [0, T],$$

where

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and, in particular, $F_X^t = F_M^t$, $P$-a.s. for all $t \in [0, T]$. Note that $s > \tau$ is possible.
Let $Y = (Y_t)$ defined by

$$Y_t = \int_0^t f(s)ds + X_t, \quad t \in [0, T],$$

Then $Y$ admits the representation

$$Y_t = \int_0^t G(s, t)dZ_s$$

The fundamental semimartingale $Z = (Z_t)$

$$Z_t = \int_0^t g(s, t)dY_s = M_t + \int_0^t \Phi(s)d\langle M \rangle_s,$$

and

$$\Phi(t) = \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t)f(s)ds.$$
The measures $\mu^X$ and $\mu^Y$

In particular, $\mathbb{F}_t^Y = \mathbb{F}_t^Z$, $P$-a.s. for all $t \in [0, T]$ and, if

$$E \exp \left\{ - \int_0^T \Phi(t) dM_t - \frac{1}{2} \int_0^T \Phi^2(t) d\langle M \rangle_t \right\} = 1,$$

then the measures $\mu^X$ and $\mu^Y$ are equivalent and the corresponding Radon-Nikodym density is given by

$$\frac{d\mu^Y}{d\mu^X}(Y) = \exp \left\{ \int_0^T \hat{\Phi}(t) dZ_t - \frac{1}{2} \int_0^T \hat{\Phi}^2(t) d\langle M \rangle_t \right\},$$

where $\hat{\Phi}(t) = E(\Phi(t) | \mathbb{F}_t^Y)$. 
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4. Concluding Remarks
Let $H \in \left(\frac{3}{4}, 1\right]$. Then $X$ is a **diffusion** type process:

$$X_t = W_t - \int_0^t \varphi_s(X) ds, \quad W_t = \int_0^t \frac{dM_s}{g(s, s)},$$

$W$ is an $F^X$-Brownian motion; $\varphi_t(X) = \int_0^t \frac{\dot{g}(s, t) dX_s}{g(t, t)}$.
The density w.r.t $\mu^W$

Moreover, the measures $\mu^X$ and $\mu^W$ are equivalent and

$$
\frac{d\mu^X}{d\mu^W}(X) = \exp \left\{ - \int_0^T \varphi_t(X) dX_t - \frac{1}{2} \int_0^T \varphi_t^2(X) dt \right\}.
$$
For $H \in (0, \frac{1}{4})$, $X$ is a fractional diffusion type process

$$X_t = B_H^t - \int_0^t \rho(s, t) \varphi_s(X) ds,$$

where $B_H^t$ is fBm with $F_t^{B_H} = F_t^X$, $\varphi_t(X) = \int_0^t L(s, t) dX_s$ and

$$L(s, t) := \frac{\partial}{\partial t} g(s, t) \sqrt{\frac{d}{dt} \langle M \rangle_t - \frac{\partial}{\partial t} \tilde{\rho}(s, t)},$$
The density w.r.t \( \mu^{BH} \)

The measures \( \mu^X \) and \( \mu^{BH} \) are equivalent if and only if \( H \in (0, \frac{1}{4}) \) and

\[
\frac{d\mu^X}{d\mu^{BH}}(X) = \exp \left\{ - \int_0^T \varphi_t(X) d\tilde{X}_t - \frac{1}{2} \int_0^T \varphi_t^2(X) dt \right\},
\]

where \( \tilde{X}_t = \int_0^t \tilde{\rho}(s, t) dX_s \).
Mixed Riemann–Liouville process

\[ X_t = B_t + B_{t}^{L,H} \]

Everything remains valid with \( g(s, t) \) solving the equation

\[
    g(s, t) - \frac{\partial}{\partial s} \int_0^t \Gamma(r, s) \frac{\partial}{\partial r} g(r, t) dr + g(t, t) \frac{\partial}{\partial s} \Gamma(s, t) = 1,
\]

where \( 0 < s, t \leq T \), and \( \Gamma(s, t) \) is the covariance function of \( B_{t}^{L,H} \).
The kernel $g(s, t)$ is the unique continuous solution of the following equations:

- for $H \in (0, 1]$, the integro-differential equation:
  \[
g(s, t) + H \frac{d}{ds} \int_0^t g(r, t) |s - r|^{2H-1} \text{sign}(s - r) dr = 1.
  \]

- for $H \in (\frac{1}{2}, 1]$, the weakly singular integral equation:
  \[
g(s, t) + H(2H - 1) \int_0^t g(r, t) |s - r|^{2H-2} dr = 1.
  \]
for $H \in (0, \frac{1}{2})$, the **weakly singular integral** equation

$$
\begin{align*}
g(s, t) + \beta H t^{-2H} \int_0^t g(r, t) \tilde{\kappa} \left( \frac{r}{t}, \frac{s}{t} \right) dr = \\
c_H s^{1/2-H} (t - s)^{1/2-H},
\end{align*}
$$

with the kernel

$$
\tilde{\kappa}(u, v) = |u - v|^{-2H} N(u, v),
$$

where $N \in C([0, 1]^2)$. 
Integral Equation with $H > 1/2$, I

Properties of $g(s, t)$ on the diagonal

The function $g(t, t)$, $t \in [0, T]$ satisfies the properties:

- $g(t, t)$ is continuous on $[0, T]$ with $g(0, 0) := \lim_{t \to 0} g(t, t) = 1$
- $g(t, t) > 0$ for all $t \in [0, T]$
- $\int_0^t g(s, t) ds = \int_0^t g^2(s, s) ds$. 
Integral Equation with $H > 1/2$, II

Properties of $\dot{g}(s, t) = \frac{\partial}{\partial t} g(s, t)$

The kernel $g(s, t)$ satisfies the following properties

- $g(s, t)$ is continuously differentiable at $t \in (0, T]$ for any $s > 0, s \neq t$;

- the derivative $\dot{g}(s, t) := \frac{\partial}{\partial t} g(s, t)$ satisfies the equation

$$
\dot{g}(s, t) + H(2H - 1) \int_{0}^{t} \dot{g}(r, t) |r - s|^{2H-2} dr = 
$$

$$
-H(2H - 1)g(t, t) |r - s|^{2H-2}, \quad s \in (0, t).
$$

- $\dot{g}(\cdot, t) \in L^2([0, t])$ for $H > 3/4$
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| 4 | Concluding Remarks | |
$H > \frac{3}{4}$

**X is a diffusion type process**

\[
M_t = \int_0^t g(s, t) dX_s = \int_0^t g(s, s) dX_s + \int_0^t \left( g(r, t) - g(r, r) \right) dX_r \\
\int_0^t g(s, s) dX_s + \int_0^t \int_r^t \dot{g}(r, s) dsdX_r = \\
\int_0^t g(s, s) dX_s + \int_0^t \int_0^s \dot{g}(r, s) dX_r ds,
\]

where the last equality holds since $\dot{g}(\cdot, s) \in L^2([0, s])$. 

**Mixed Gaussian processes**

Cai, Chigansky, Kleptsyna

**Introduction**

Problems statement and questions around

Historical survey

**Results**

From $L^2$ to $L^1$

Even from $L^1$, but in a partial case

Semimartingale

Structure of $X$

**The Proofs**

Integro-Differential Equation

Diffusion type representation, Equivalence of measures

**Concluding Remarks**
$H > 3/4, \Pi$

**X is a diffusion type process, \Pi**

Hence

$$W_t = \int_0^t \frac{1}{g(s, s)} dM_s = X_t + \int_0^t \int_0^s \frac{g(r, s)}{g(s, s)} dX_r ds =:$$

$$X_t + \int_0^t \varphi_s(X) ds.$$
An interesting toy

Singular perturbations

Fix $\varepsilon > 0$ and let $g_\varepsilon$ be the solution of the equation:

$$\varepsilon g_\varepsilon(u) + \frac{d}{du} \int_0^1 g_\varepsilon(v)|u - v|^{2H-1} \text{sign}(u - v)dv = 1, \ u \in [0, 1],$$

A simple question

What can we say about $g_\varepsilon$ when $\varepsilon \to 0$?
An interesting toy

Singular perturbations

Fix \( \varepsilon > 0 \) and let \( g_\varepsilon \) be the solution of the equation:

\[
\varepsilon g_\varepsilon (u) + \frac{d}{du} \int_0^1 g_\varepsilon (v) |u - v|^{2H-1} \text{sign}(u - v)dv = 1, \quad u \in [0, 1],
\]

A simple question

What can we say about \( g_\varepsilon \) when \( \varepsilon \to 0 \)?
Some answers

The main tool

An asymptotic formula for the eigenfunctions of the corresponding integro-differential operator

About convergence

- Weak convergence with rate $\varepsilon$
- $L^2$ convergence with depending on $H$ rate
- Boundary layer construction with $\frac{1}{\sqrt{\varepsilon}}$ rate
Mixed Gaussian processes

Cai, Chigansky, Kleptsyna

Introduction
Problems statement and questions around
Historical survey

Results
From $L^2$ to $L^1$
Even from $L^1$, but in a partial case
Semimartingale
Structure of $X$

The Proofs
Integro-Differential Equation
Diffusion type representation,
Equivalence of measures

Concluding Remarks

Spectrum of the integro-differential operator

**Integro-differential operator**

$$(Kf)(u) = \frac{d}{du} \int_{-1}^{1} f(v)|u - v|^{1-\alpha} \text{sign}(u - v) dv, \quad \alpha = 2 - 2H$$

**Eigenvalues**

$$\lambda_n := \frac{\pi(1 - \alpha)}{\Gamma(\alpha) \sin \frac{1}{2}(1 - \alpha)\pi} \nu_n^{\alpha-1},$$

where

$$\nu_n = \frac{1}{2}(n - 1)\pi + \frac{1}{8}(1 + \alpha)\pi + O(n^{-1}), \quad n \to \infty.$$
Spectrum of the integro-differential operator

**Eigenfunctions, \( \alpha \in (0, 1) \)**

The normalized eigenfunctions of \( K \) with \( \alpha \in (0, 1) \) satisfy

\[
\phi_n(x) = \cos \left( \nu_n(x + 1) - \frac{1 + \alpha}{8} \pi \right) + \frac{1}{c} \frac{1}{\pi} \int_0^\infty D(\tau) \left( e^{(x-1)\nu_n\tau} - (-1)^n e^{-(x+1)\nu_n\tau} \right) d\tau + r_n(x),
\]

where the residual term satisfies \( |r_n(x)| \leq Cn^{-1} \) with a constant \( C \), depending only on \( \alpha \).
For \( \alpha \in (1, 2) \) the normalized eigenfunctions satisfy

\[
\varphi_n(x) = -\cos \left( \nu_n(x + 1) - \frac{1 + \alpha}{8} \pi \right) + \frac{\alpha - 1}{2c} \frac{1}{\pi} \int_0^\infty \! D(\tau) \left( e^{-(x+1)\tau \nu_n} - (-1)^n e^{(x-1)\tau \nu_n} \right) \, d\tau + r_n(x),
\]

where the residual term satisfies \( |r_n(x)| \leq Cn^{-1} \) with a constant \( C \), depending only on \( \alpha \).