Modified clock inequalities and modified black hole lifetime

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Based on a generalized uncertainty principle, Salecker-Wigner inequalities are modified. When applied to black holes, they give a modified black hole lifetime: \( T_{MB} \sim \frac{M}{m_p^3} (1 - m_p^2/M^2) t_p \), and the number of bits required to specify the information content of the black hole as the event horizon area in Planck units: \( N \sim \frac{M^2}{m_p^3} (1 - m_p^2/M^2) \).

Key Words: modified clock inequalities, modified black hole lifetime, generalized uncertainity principle

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I. INTRODUCTION

The conventional derivation of the Hawking lifetime uses the Heisenberg’s uncertainty principle on the event horizon scale \( R_g \) to determine a temperature for the black hole which, under the assumption that the black hole is a black body, then allows one to use the Stefan-Boltzmann law to calculate the lifetime of the black hole for complete evaporation (see, e.g., [1, 2]).

By applying Salecker-Wigner’s clock inequalities to black holes, Barrow obtained the same result [3]. The heuristic way is as follows: According to Heisenberg’s uncertainty principle: \( \Delta p \sim \hbar/\Delta x \), if a clock of mass \( M \) has quantum position uncertainty \( \Delta x \), then its moment uncertainty is \( \hbar \Delta x^{-1} \). The clock to be considered should have an accuracy \( \tau \) (the minimum time interval that the clock is capable of resolving) and be able to measure time intervals up to a maximum \( T \). After a time \( t \), the uncertainty in position of the clock will grow to \( \Delta x' = \Delta x + \hbar M^{-1} \Delta x^{-1} \). If the effects on mass are neglected, then this will be a minimum when \( \Delta x = \sqrt{\hbar/t/M} \). Hence, to keep the clock accurate over the total running time \( T \), its linear spread \( \lambda \) must be limited:

\[
\lambda \geq 2 \sqrt{\hbar T/M},
\]

the same order of magnitude of the position uncertainty, meaning that the size of the clock must be larger than the uncertainty in its position. This is Salecker-Wigner’s first clock inequality [4]. To give time to within an accuracy \( \tau \), the quantum position uncertainty must not be larger than the minimum wavelength of the quanta striking it (in order to read the time); that is, \( \Delta x' \leq c \tau \). The use of a signal with nonzero rest mass would give a more rigorous limit. This condition gives a bound on the minimum mass of the clock:

\[
M \geq \frac{4 \hbar}{c^3 \tau} \left( \frac{T}{\tau} \right).
\]

This is Salecker-Wigner’s second clock inequality [4]. This inequality is more restrictive than that imposed by Heisenberg’s energy-time uncertainty principle because it requires that a clock still show proper time after being read: the quantum uncertainty in its position must not introduce significant inaccuracies in its measurement of time over the total running time. To derive Salecker-Wigner’s clock inequalities [1] and [2], it assumes unsqueezed, unentangled, and Gaussian wave packets without any detailed phase information; they are valid only for single analog clocks (black holes can be seen as analog clocks [5]), not for digital quantum clocks.

Barrow applied Salecker-Wigner’s size limit [1] to a black hole, assuming that the minimum clock size is the Schwarzschild radius \( R_g = 2GM/c^2 \) and found the maximum running time of the black hole is [6]:

\[
T \sim \frac{G^2 M^3}{\hbar c^4} = \frac{M^3}{m_p^3} t_p,
\]

where \( t_p = \sqrt{\hbar c^5} \) and \( m_p = \sqrt{\hbar c/G} \) are the Planck time and mass. The maximum running time of a black hole is the Hawking lifetime [6]. If we had not known of the existence of black hole evaporation, Eq. (3) would have implied that there is a maximum lifetime for a black hole state. Compared with the conventional method, the application of the Salecker-Wigner inequality [1] to the event horizon scale predicts the Hawking lifetime [4] without the assumption that the black hole is a black body radiator.

But, one may suggest, when considering black holes, the effect of gravity may be taken into account. In this...
work, we obtain modified clock inequalities based on a
generalized uncertainty principle that takes into account
some properties of black holes, and find a modified black
hole lifetime which may throw light on quantum gravity
at the Planck scale.

II. MODIFIED CLOCK INEQUALITIES

Salecker-Wigner’s clock inequalities are based on the
Heisenberg’s position-momentum uncertainty principle:
\( p \sim \hbar / \Delta x \). But, if we combine quantum theory and
some basic concepts of gravity, Heisenberg’s position-
momentum uncertainty principle may be modified \[7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24\],
and so do Salecker-Wigner’s clock inequalities. Using
Heisenberg’s uncertainty principle and some properties
of black holes, Scardigli had shown how a generalized
uncertainty principle (GUP) can be derived from a mea-
sure gedanken experiment \[23\]:
\[
\Delta x \geq \frac{\hbar}{\Delta p} + t_p^2 \frac{\Delta p}{\hbar},
\]
where \( t_p^2 = \sqrt{\frac{G\hbar}{c^3}} \) is the Planck distance. As Scardigli
argued, this GUP is independent from particular versions
of quantum gravity. This GUP also arises from quantum fluctuation of the background space-time met-
tion and many authors considered various problems in
the framework of GUP, such as Refs. \[28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60\]. Note,
however, it should be kept in mind that this GUP is de-
erived based upon only heuristic arguments, and is thus
far from proven.

Basing on the GUP \[4\], Adler et al. obtained a modi-
fied black hole lifetime with the conventional method \[2\].
\[
T_{\text{ACS}} = \frac{1}{16} \left(\frac{8}{3} \left(\frac{M}{m_p}\right)^3 - 8 \frac{M}{m_p} - \frac{m_p}{M} + \frac{8}{3} \left(\frac{M}{m_p}\right)^2 - 1 \right)^{3/2} - 4 \sqrt{\frac{M}{m_p} - 1 + 4 \arccos \left(\frac{m_p}{M} + \frac{19}{3}\right)} t_ch
\]
where \( t_ch = 16^2 \times 60 \tau t_p \). To derive this black hole life-
time, Adler et al. also assume that the black hole is a
black body radiator and the dispersion relation \( E = pc \)
holds. But if the uncertainty principle is modified, the
dispersion relation may also be modified (see, e.g., \[61\]).

Because the space-time fluctuation will be significant
when the measured length scale approaches to the Planck
distance, it is reasonable to expect that the linear spread
of a clock must be less than the Planck distance. In fact, the GUP \[4\] implies a minimum length: \( 2 t_p \),
which can be considered as a limit on the linear spread
of a clock. This limit can be improved, as we see below.

From Eq. \[4\], if a clock of mass \( M \) has quantum position
uncertainty \( \Delta x \), then its momentum uncertainty will be
\[
\Delta p \sim \frac{\Delta x}{2t_p} \left[ 1 - \sqrt{1 - 4t_p^2/\Delta x^2} \right] \[2\].
\]
Following the steps to derive the Salecker-Wigner’s clock inequalities, Eq. \[1\]
is modified as (see Appendix)
\[
\lambda \geq 2t_p \sqrt{1 + \frac{hT}{M l_p^2}},
\]
stronger than limit \[11\] and come back to limit \[1\] for
\( hT \gg M l_p^2 \). Here we also require that the position un-
certainty created by the measurement of time must not
be larger than the minimum wavelength of the quanta
used to read the clock. Then Salecker-Wigner’s second
clock inequality \[2\] is modified as:
\[
M \geq \frac{4hT}{c^2\tau^2} \frac{1}{1 - 4t_p^2/\tau^2},
\]
This inequality links the mass, total running time, ac-
curacy of the clock, and the Planck time together, and
may links together our concepts of gravity and quan-
tum uncertainty. Obviously, it firstly gives a limit on the
accuracy of the clock \( \tau > 2t_p \). Like Salecker-Wigner
inequalities \[11\] and \[2\], Eqs. \[6\] and \[7\] are valid for
single analog clocks, not for digital quantum clocks.

III. MODIFIED BLACK HOLE LIFETIME

Now applying modified clock inequality \[4\] to black
holes and assuming that the minimum clock size is the
Schwarzschild radius \( R_g = 2GM/c^2 \), one may find the
maximum running time of the black hole is modified as:
\[
T_{\text{MB}} \sim \frac{M R_g^2}{4h}(1 - 4t_p^2/ R_g^2) = \frac{M^3}{m_p^3} \left(1 - m_p^2/M^2\right)t_p, \tag{8}
\]
which has a term \( M t_p / m_p \) different from the Hawking
lifetime \[3\] and holds for \( M \geq m_p \). This difference may
throw light on quantum gravity in some sense at Planck
scale. Using the GUP \[4\], Adler et al. found that the
thermal radiation of the black hole will stop at the Planck
distance, and the black hole becomes an inert remnant,
possessing only gravitational interaction \[2\], consistent
the results obtained in modified clock inequalities back-
ground. Aside from about a factor of \( 16^2 \times 60 \pi \), the first
two terms of the Adler-Chen-Santiago lifetime \( T_{\text{ACS}} \) is
consistent with the modified black hole lifetime \( T_{\text{MB}} \). The
comparison among the Hawking lifetime \( T_H \), the modi-
ified black hole lifetime \( T_{\text{MB}} \), and Adler-Chen-Santiago
lifetime \( T_{\text{ACS}} \) are shown in Fig. 1.

The minimum interval that the black hole can be used
to measure is given by the light travel time across the
black hole \[3,5\]: \( \tau \sim 2GM/c^3 = R_g/c \). Thus we are led
to view the black hole as an information-processing
system in which the number of computational steps is

\[ N = \frac{T_{MB}}{\tau} \sim \frac{M^2}{m_p^2}(1 - m_p^2/M^2). \]  

As expected from the identification of a black hole entropy \[62\] or holographic principle \[63, 64\], this gives the number of bits required to specify the information content of the black hole as the event horizon area in Planck units.

\[ \text{FIG. 1: Comparison among the Hawking lifetime } T_H, \text{ modified clock inequality lifetime } T_{MB}, \text{ and Adler-Chen-Santiago lifetime } T_{ACS}, \text{ aside from a numerical factor } 16^2 \times 60\pi. \]

IV. SUMMARY

To summarize, based on a generalized uncertainty principle, we obtain modified clock inequalities, which give bounds on the size and the accuracy of the analog clock that must be larger than 2 times the Planck distance \( l_p \) and time \( t_p \) respectively. As an application, we discussed the case of black holes, and obtained a modified black hole lifetime \( T_{MB} \sim \frac{M^3}{m_p^2}t_p(1 - m_p^2/M^2) \), which is different from Hawking lifetime and give a limit on the mass of black holes naturally. Viewing a black hole as an information-processing system, we also find the number of bits required to specify the information content of the black hole as the event horizon area in Planck units \( N \sim \frac{M^2}{m_p^2}(1 - m_p^2/M^2) \). These results reinforce the central importance of black holes as the simplest and most fundamental constructs of space-time, linking together our concept of gravity, information, and quantum uncertainty. Note, however, applying clock inequalities to obtain the lifetimes of other type black holes is still an open interesting problem, work is in progress in this direction.

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Appendix

According to the generalized uncertainty principle

\[ \Delta x \geq \frac{\hbar}{\Delta p} + l_p^2 \frac{\Delta p}{\hbar}, \]  

where \( l_p^2 = \sqrt{G\hbar/c^3} \) is the Planck distance, if a clock with mass \( M \) has quantum position uncertainty \( \Delta x \), then its momentum uncertainty will be

\[ \Delta p \sim \frac{\Delta x h}{2l_p^2} \left[ 1 - \sqrt{1 - 4l_p^2/\Delta x^2} \right]. \]  

After a time \( t \) the uncertainty in position of the clock becomes

\[ \Delta x' = \Delta x + \frac{\Delta x h t}{2Ml_p^2} \left[ 1 - \sqrt{1 - 4l_p^2/\Delta x^2} \right]. \]  

To obtain the minimal value of \( \Delta x' \) in this case, using the condition

\[ 0 = \frac{d\Delta x'}{d\Delta x} \]  

\[ = 1 + \frac{ht}{2Ml_p^2} \left[ 1 - \sqrt{1 - 4l_p^2/\Delta x^2} \right] - \frac{2ht}{M\Delta x^2 \sqrt{1 - 4l_p^2/\Delta x^2}}, \]

we get \( \Delta x = [2Ml_p^2 + th]/\sqrt{M(Ml_p^2 + th)} \). By inserting this value into Eq. \[12\], we obtain the minimal value of \( \Delta x' \)

\[ \Delta x'_{\min} = 2l_p \sqrt{1 + \frac{th}{Ml_p^2}}. \]  

By taking \( t \) as the total running time \( T \) during which the clock can remain accurate, and consider the condition that the linear spread of clock \( \lambda \) must not be less than the uncertainty in position \( \Delta x' \), that’s \( \lambda \geq \Delta x' \geq \Delta x'_{\min} \), we obtain Eq. \[13\].

[1] P.K. Townsend, arXiv:gr-qc/9707012
[2] R. J. Adler, P. Chen, and D. I. Santiago, Gen. Rel. Grav.
