Isospin Effect in Three-Body Kaonic Clusters

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Abstract

The kaonic clusters $K^-K^-p$ and $ppK^-$ are described based on the configuration space Faddeev equations for $AAB$ system. The $AB$ interaction is given by isospin-dependent potentials. For this isospin model, we show that the relation $|E_3(V_{AA} = 0)| < 2|E_2|$ is satisfied when $E_2$ is the binding energy of the $AB$ subsystem and $E_3(V_{AA} = 0)$ is the three-body binding energy when interaction between identical particles is omitted, $V_{AA} = 0$. For the $NN\bar{K}$ system, taking into account weak attraction of $NN$ interaction the relation leads to the evaluation $|E_3| \leq 2|E_2|$. The ”isospinless model” for the kaonic clusters based on the isospin averaged $N\bar{K}$ potential demonstrates the opposite relation $|E_3(V_{AA} = 0)| > 2|E_2|$. The isospin ”given charge formalism” is presented for $NN\bar{K}$ cluster. This formalism is related to isospin model by unitary transformation of the isospin basis. An interpretation of the ”particle representation” for $NN\bar{K}$ system is proposed.

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I. INTRODUCTION

The quasi-bound states in the kaonic clusters $NN\bar{K}$ and $\bar{K}\bar{K}N$ are intensively debated during the last years. The main problem is that theoretical evaluations for the binding energy are in significant disagreement with the values derived from existing experimental data \[1\]. The properties of the kaonic clusters are defined by $\bar{K}\bar{K}$ interaction, having significant difference for the isospin singlet and triplet channels. The isospin singlet component of the $\bar{K}N$ potential generates a quasi-bound state corresponding to the $\Lambda(1405)$ resonance below the $K^-p$ threshold \[2\]. The resonance has the double state nature due to the $\bar{K}N$ quasi-bound state and $\pi\Sigma$ resonance \[3-5\]. One can point out two models for the $\bar{K}N$ quasi-bound state to be used within three-body calculations. The AY $\bar{K}N$ potential effectively taking into account the $\pi\Sigma$ coupling has been proposed in Ref. \[3\]. The effective $\bar{K}N$ interactions have the strong attraction in the singlet $I = 0$ channel and the weak attraction in the triplet $I = 1$ channel. The $ppK^-$ binding energy obtained within this model is $|E_{NN\bar{K}}| = 48$ MeV. Two-body threshold is close to the bound state energy of $\Lambda(1405)$ as $K^-p$ bound pair (about 30 MeV). Similar results have been obtained within more complex models \[6-11\]. This value is much smaller than the experimentally motivated value of about 100 MeV for the $ppK^-$ "deeply bound state" \[12-14\]. Alternatively, the chiral model for the potential has been proposed (see \[15-17\]) for the $\bar{K}N$ interaction. The model reduces the singlet component of $\bar{K}N$ potential due to the strong coupling $\bar{K}N$ and $\pi\Sigma$ channels which give contribution $3/1$ in three-body amplitudes, respectively. The value about 20 MeV for $|E_{NN\bar{K}}|$ was obtained with the two-body threshold about 12 MeV. This model is accompanied by the energy dependence of $\bar{K}N$ interaction and includes the $p$- wave component of $\bar{K}N$ potential into consideration. The coupling between the channels $NN\bar{K}(S_{NN} = 0)$ and $NN\bar{K}(S_{NN} = 1)$ is also taken into account. Note that these factors affect three-body binding energy in 1-7 MeV and contribute with different signs. The effect of the energy dependence is discussed \[18\].

Discussion about the experimental background and theoretical interpretations can be found in Ref. \[1, 19-21\].

In the presented work we consider the lower bounds for the ground state energy of three-body kaonic clusters appeared due to strong isospin dependence of $\bar{K}N$ potential when $\bar{K}N$ pair is deeply bound and $NN$ potential is relatively weaker. For the $NN\bar{K}$ cluster, we
show that the "AY-like" models \cite{22} cannot lead to "ppK− deeply bound state" which was assumed to be taking into account in the existing experimental data treatments.

Our study is based on the Faddeev equations in configuration space \cite{23}. The Faddeev equations allow to separate components of the total wave function corresponding to the different particle rearrangements and evaluate the contribution of each configuration.

We consider the $NN\bar{K}$ and $\bar{K}KN$ clusters as three-body $AAB$ systems that include two identical particles $AA$ and study the relation between the ground state energies of the $AB$ subsystem, $E_2$, and three-body system, $E_3(V_{AA} = 0)$, when the interaction between the identical particles is omitted. The kaonic clusters are the systems having isospin dependent $AB$ interaction. The triplet and singlet components of $N\bar{K}$ interaction are essentially different. We show, that for such systems, the relation $|E_3(V_{AA} = 0)| < |2E_2|$ takes place. Based on this relation, we can formulate more strong statement: $|E_3| < |2E_2|$, taking into account weak attractive (or weak repulsive) $AA$ potentials. The last relation shows that $NN\bar{K}$ "AY-like" calculations for $|E_3|$ have to result in the values smaller than 60 MeV.

The relation between $E_2$ and $E_3(V_{AA} = 0)$ has been previously considered for bosonic-like $AAB$ systems. The mass polarization term of the three-body kinetic-energy operator is important for evaluation of the $AA$ interaction strength\cite{24}. For a bosonic-like system, when the particle masses are related as $m_B \sim m_A$, the contribution of the mass polarization term to the three-body energy can be evaluated as $\Delta = 2E_2 - E_3(V_{AA} = 0)$, where $\Delta > 0$\cite{24,25}. It means that $|E_3(V_{AA} = 0)| > |2E_2|$. For the $AAB$ systems having isospin dependent $AB$ interaction we have obtained the opposite relation.

It has to be noted that, the $NN\bar{K}$ system can be described by the "bosonic-like" model in which the isospin-dependence of $N\bar{K}$ interaction is removed by an averaging of the $N\bar{K}$ potential over isospin variables \cite{16,17,26,27}. Resulted Faddeev equations describe a three-body system where the interaction between non-identical particles is given by isospinless potential presented by a superposition of the singlet and triplet components of the $N\bar{K}$ potential. The averaged potential can be defined by an algebraic transformation \cite{28} for the Faddeev equations. We consider some properties of this averaged potential model and compare with the isospin model.

An alternative for the isospin formalism based model was proposed in the papers \cite{29–31} as "particle representation" for the kaonic cluster $NN\bar{K}(S_{NN} = 1)$. In this model, the $N\bar{K}$ subsystem is considered to be isospinless one to separate $pK^−$ and $n\bar{K}^0$ channels.
The channel coupling was obtained by unitary transformation for the Schrödinger equation described the \( N\bar{K} \) pair. Motivated by the model we develop the "given charge formalism" for \( NN\bar{K} \) system to find a relation to the "particle representation".

II. FORMALISM

A. Faddeev equations for \( AAB \) system

The kaonic clusters \( ppK^- \) and \( K^-K^-p \) represent the three-body \( AAB \) systems with two identical particles. The total wave function of the \( AAB \) system is decomposed into the sum of the Faddeev components \( U \) and \( W \) corresponding to the \((AA)B\) and \(A(AB)\) types of rearrangements: \( \Psi = U + W \pm PW \), where \( P \) is the permutation operator for two identical particles. In the expression for \( \Psi \), the sign "+" corresponds to two identical bosons, while the sign "−" corresponds to two identical fermions, respectively. Each component is expressed by corresponding Jacobi coordinates. For a three-body system with two identical particles the set of the Faddeev equations is presented as a set of two equations for the components \( U \) and \( W \) [32]:

\[
(H_0 + V_{AA} - E)U = -V_{AA}(W \pm PW), \\
(H_0 + V_{AB} - E)W = -V_{AB}(U \pm PW),
\]

where again the signs "+" and "−" correspond to two identical bosons and fermions, respectively and \( H_0 \) is the kinetic energy operator presented in the Jacobi coordinates for corresponding rearrangement. The wave function of the system \( AAB \) is symmetrized with respect to two identical bosons, while it is antisymmetrized with respect to two identical fermions. In the presented work, we consider the \( s \)-wave approach for the \( AAB \) systems. The total angular momentum \( L = 0 \) and angular momenta in the subsystem \((AA)B\) and \(A(AB)\) are equal zero.

B. Isospin formalism for kaonic clusters

In Eq. (1), the Faddeev components \( U \) and \( W \) of the total wave-function are expressed in terms of spin and isospin spaces. The \( NN\bar{K} \) system is a system with two identical particles described by Eq. (1). In Eq. (1), the Faddeev component \( U \) (and \( W \)) of the total
wave-function is expressed in terms of spin and isospin spaces:

\[ U = \mathcal{U} \chi_{\text{spin}} \eta_{\text{isospin}}. \]

The \( NN \bar{K} \) system with the triplet isospin state of the pair of nucleons \( I_{NN} = 1 \) is considered. The isospin basis for \( NN \bar{K} \) system in the the state \( I = 1/2 \) and \( I^3 = 1/2 \) can be written using the isospin functions: \( \eta_{++} = \eta_{+}(1)\eta_{-}(2)\eta_{+}(3) \), \( \eta_{--} = \eta_{-}(1)\eta_{+}(2)\eta_{+}(3) \), \( \eta_{+-} = \eta_{+}(1)\eta_{+}(2)\eta_{-}(3) \). Where, for example, \( \eta_{-}(k) \) is eigenfunction of the isospin of \( k \)-th particle with projection of \(-\frac{1}{2}\). The three-body isospin basis for the configuration \((1+2)+3\) includes two elements with different isospins (single or triplet) of the \((1+2)\) pair written as

\[ \eta_1 = \frac{1}{\sqrt{2}}(\eta_{++} - \eta_{--}), \quad \text{singlet}, \]
\[ \eta_2 = \sqrt{\frac{2}{3}}(\eta_{++} - \frac{1}{2}\eta_{+-} - \frac{1}{2}\eta_{--}), \quad \text{triplet}, \]

The basis for the configurations \((3 + 1) + 2 \) (and \((2 + 3) + 1\)) can be obtained from (2) by cyclical permutations of the isospin projections.

The spin states of the \( NN \bar{K} \) system can be described by spin states of nucleon pair which can be spin singlet or spin triplet. The singlet spin function \( \chi^{s=0}(NN) \) is an asymmetrical function relatively the permutation of nucleons: \( \chi^{s=0}(NN) = \frac{1}{\sqrt{2}}(\chi_{++} - \chi_{--}) \), that provides the sign ”+” in Eq. [1]. The triplet spin function \( \chi^{s=1}(NN) \) is symmetric, that gives the sign ”−” in Eq. [1].

We employ the s-wave spin/isospin dependent \( V_{AA} \) and \( V_{AB} \) potentials. For \( NN \bar{K} \) system, the separation of spin-isospin variables leads to the following form of the Faddeev equations:

\[ (H_0 + V_{AA} - E)U = -V_{AA}D(1 + p)W, \]
\[ (H_0 + V_{AB} - E)W = -V_{AB}(D^TU + GpW), \]

where \( W \) is a column matrix with the singlet and triplet coordinate dependent parts of the Faddeev component \( W \), and the exchange operator \( p \) acts on the particle coordinates. The component \( U \) is presented by single part \( U \) corresponding to isospin triplet state of \( AA \) pair. The sign before operator \( p \) in Eq. [1] depends on spin state of the pair. Within s-wave approach, the coordinate dependent part of \( U \) corresponding to isospin singlet state is dropped out from consideration due to the operator \((1 - p)\), which is appeared in right-hand side of Eq. [1].

For \( \bar{K} \bar{K}N \) and \( NN \bar{K} \), despite of the fact that there are two identical bosons and two identical fermions, respectively, due to the symmetry of the spin-isospin configurations in
the kaonic clusters, the $D$ and $G$ matrices in (3) are the same and have the following form \[^33\]:

$$D = \left( -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right), G = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad W = \begin{pmatrix} W^s \\ W^t \end{pmatrix}, \quad U = U^t. \quad (4)$$

The superscripts $s$ and $t$ in (4) denote the isospin singlet and isospin triplet coordinate dependent parts of the components $U$ and $W$. For the $\bar{K}KN$ kaonic cluster, $V_{AA} = v_{\bar{K}K}^t$ is the $\bar{K}K$ potential in the triplet isospin state. For the $NN\bar{K}$ cluster, $V_{AA} = v_{NN}^s$ is the $NN$ potential in the singlet spin state. For both systems, one has to take $V_{AB} = \text{diag}\{v_{\bar{N}\bar{K}}^s, v_{\bar{N}\bar{K}}^t\}$.

In the presented work, we used the $s$-wave Akaishi-Yamazaki (AY) \[^3\] and the simulating Hyodo-Weise (sHW) effective potentials \[^34\] for $\bar{K}K$ and $N\bar{K}$ interactions, which include the coupled-channel dynamics into a single channel $N\bar{K}$ interaction.

The AY and sHW $N\bar{K}$ potentials are written in the form of one range Gaussian:

$$V_{N\bar{K}}^{s(t)}(r) = V_0^{s(t)} \exp((r/b)^2), \quad \text{(5)}$$

where for the AY potentials: $V_0^s = -595.0$ MeV, $V_0^t = -175.0$ MeV, $b = 0.66$ fm, and for the sHW potentials: $V_0^s = -908.0$ MeV, $V_0^t = -415.0$ MeV, $b = 0.47$ fm \[^34\]. The isospin triplet $\bar{K}K$ potential is defined by the set: $V_0^t = 104.0$ MeV, $b = 0.66$ fm for the AY potential, and $V_0^t = 313.0$ MeV, $b = 0.47$ fm \[^35\] for the sHW potential. To describe the spin singlet nucleon-nucleon interaction ($I = 1$) we use the semi-realistic Malfliet-Tjon MT I-III \[^36\] potential with the modification from Ref. \[^37\] :

$$V_{NN}(r) = (-513.968 \exp(-1.55r) + 1438.72 \exp(-3.11r))/r,$$

where the strength parameters are given in MeV and the range parameters are in fm$^{-1}$.

**C. Effect of isospin splitting of $AB$ potential**

Let us consider the $s$-wave approach for the Faddeev equations \[^3\] for the $AAB$ system when particles $A$ and $B$ interact via the isospin dependent $V_{AB}$ potential, assuming that the interaction between two identical particles is omitted, therefore $V_{AA} = 0$.

For the $NN\bar{K}$ and $\bar{K}KN$ systems, Eq. \[^3\] takes the form:

$$\begin{align*}
(H_0 + v_{N\bar{K}}^s - E)W^s &= -v_{N\bar{K}}^s \left( \frac{1}{2}pW^s + \frac{\sqrt{3}}{2}pW^t \right), \\
(H_0 + v_{N\bar{K}}^t - E)W^t &= -v_{N\bar{K}}^t \left( \frac{\sqrt{3}}{2}pW^s - \frac{1}{2}pW^t \right).
\end{align*} \quad \text{(6)}$$
Here, we have to take into account the significant difference between the isospin singlet and triplet components of the $N\bar{K}$ potential. The isospin singlet component generates a deep bound state, while there is no bound state with the triplet component. We can formulate that $|v_{N\bar{K}}^t| < |v_{N\bar{K}}^s|$. The strengths of the isospin components for the AY and sHW potentials are given in [5].

The Faddeev component $W^s$ of the total wave function of the kaonic clusters related to the isospin singlet interaction is dominant [28]. We rewrite the equations (6) to a simple form ignoring the small contribution coming from the isospin triplet component $W^t$:

$$(H_0 + v_{N\bar{K}}^s - E) W^s = -v_{N\bar{K}}^s \left(\frac{1}{2}p\right) W^s$$

or

$$(H_0 + v_{N\bar{K}}^s + v_{N\bar{K}}^s p + v_{N\bar{K}}^s (-\frac{1}{2}p) - E) W^s = 0. \tag{7}$$

Here, taking into account the coefficient $\frac{1}{2}$ we assume that the term $\frac{1}{2}v_{N\bar{K}}^s p$ is not a large perturbation. Ignoring this term, the Eq. (7) is rewritten as

$$(H_0 + v_{N\bar{K}}^s + v_{N\bar{K}}^s p - E) W = 0. \tag{8}$$

It has been shown in Ref. [28], that the last equation leads to the relation:

$$2E_2 - \Delta - E_3(V_{AA} = 0) = 0, \tag{9}$$

where $\Delta > 0$ is evaluation for the mass polarization in three-body system [24]. The mass polarization effect expressed by Eq. (9) can be described by the mass polarization term $T_{mp}$, which is clearly seen in the Schrödinger equation for the $AAB$ in the coordinate system of non-identical particles [24]. The $\Delta$ is an approximation for the averaged value $<T_{mp}>$: $\Delta \approx <T_{mp}>$ when the contribution of the mass polarization term to the three-body energy $E_3(V_{AA} = 0)$ is small: $\Delta/|E_3(V_{AA} = 0)| < 1$. In the limit $m_B/m_A >> 1$, the term can be neglected and $2E_2 = E_3(V_{AA} = 0)$.

Taking into account Eqs. (8) and (9), we obtain an approximation for Eq. (7):

$$2E_2 - \Delta - \frac{1}{2} <v_{N\bar{K}}^s p> -E_3(V_{AA} = 0) = 0. \tag{10}$$

The matrix element $\frac{1}{2} <v_{N\bar{K}}^s p>$ has negative value due to attractive $v_{KN}^s$ potential. Thus, in Eq. (10), the terms $-\Delta$ and $-\frac{1}{2} <v_{N\bar{K}}^s p>$ have opposite sings. The interplay of the terms
results in the two possible correlations $|E_3(V_{AA} = 0)| < 2|E_2|$ or $|E_3(V_{AA} = 0)| > 2|E_2|$.

Below, we show that the relation

$$|E_3(V_{AA} = 0)| < 2|E_2|$$  \quad (11)

takes place for the kaonic clusters. It can be explained by strong attraction of the isospin singlet $N\bar{K}$ potential having a deep bound state. In this case, the Faddeev component $W^s$ is well factorized by the wave functions of the bound pairs $A_1B$ and $A_2B$ [28]. Here, the indexes 1 and 2 distinguish the identical particles $AA$. At the same time, from the Eqs. (8)-(9) it is clear that the relation $|E_3(V_{AA} = 0)| > 2|E_2|$ is satisfied for bosonic-like systems.

It has to be noted here that, the relation (11) is not generally guaranteed (see Ref. [44]) for case $m_B < m_A$, in the systems complicated by spins or isospins. It should be again noted that the relation (11) has been obtained under the condition $m_B/m_A > 1$ and within the first order of the perturbation theory. The condition of significant "spin/isospin splitting" of the $AB$ potential is also necessary to be satisfied.

D. Averaged potential model

In this section, we define the effective potential obtained by averaging of the initial potential over the isospin variables. This averaging produces the "isospinless" model for the kaonic clusters. We will apply the model for the $NN\bar{K}$ system when $S_{NN} = 1$.

The isospin averaged potential $V_{KN}^{av}$ is defined as:

$$V_{KN}^{av} = \frac{3}{4}v_{KN}^s + \frac{1}{4}v_{KN}^t.$$  \quad (12)

Here, we use the isospin single and triplet components $v_{KN}^s$ and $v_{KN}^t$ of the $AY\bar{K}N$ potential. This potential has a moderate attraction in comparison with the strong attraction in the $I = 0$ channel. The two-body threshold is changed to lower one and is not related to the $K^-p$ bound state as $\Lambda(1405)$.

Using the isospin averaging, Eqs. (3) can be reduced to the scalar form by an algebraic transformation. Taking into account that $W = DW$, $V_{AB}^{av} = DV_{AB}DT$ and $DD^T = 1$, $DV_{AB}GD^T = V_{AB}^{av}$ one obtains

$$\begin{align*}
(H_0^U + V_{AA} - E)U &= -V_{AA}(1 + p)W, \\
(H_0^W + V_{AB}^{av} - E)W &= -V_{AB}^{av}(U + pW). \\
\end{align*}$$  \quad (13)
In this case, one can evaluate the mass polarization in the three-body system as \( \Delta = 2E_{av}^2 - E_{av}^3 (V_{AA} = 0) \). Here, \( E_{av}^2 \) is the two-body energy for the \( AB \) pair with the averaged potential and \( E_{av}^3 (V_{AA} = 0) \) is the three-body energy with the averaged potential when the \( AA \) interaction is omitted. The value of \( \Delta \) is positive one \(^{28} \). The averaged potential model was previously used in Refs. \(^{16, 17, 26, 27} \) for \( NN\bar{K} \) calculations when \( S_{NN} = 1 \) and \( S_{NN} = 0 \). For the last case, the averaged potential is defined by Eq. \(^{12} \) where the superscripts \( s \) and \( t \) are exchanged.

E. Isospin given charge formalism

The systems \( NN\bar{K} (I_{NN} = 1) \) and \( NN\bar{K} (s_{NN} = 0) \) is separate when the \( N\bar{K} \) interaction does not include an isospin mixing component. The isospin bases are orthogonal due to chose the total isospin projections which is motivated by the ”isospin charge” set of particles: \( ppK^- \) and \( pnK^- \): \( ++- \) and \( +-- \). The isospin functions \( \eta_{+-+}, \eta_{-++} \) and \( \eta_{+++} \) represent new isospin basis \( \tau \) with the elements \( \tau_1, \tau_2, \tau_3 \), respectively. The \( \tau \)-basis elements not relate to the fixed isospin of pair. We will describe these basis as ”given charge basis”. To obtain matrix of transformation between both bases we have to add an additional element to \( \eta \)-basis \(^{2} \). The element relates to the isospin state of the \( NN\bar{K} \) system with total isospin equal to 3/2 and projection 1/2 (or \(-1/2\)). For the configuration \((1+2)+3\), this basis element is written as

\[
\eta_3 = \frac{1}{\sqrt{3}} (\eta_{+-+} + \eta_{-++} + \eta_{+++}), \text{ triplet. (14)}
\]

The pair potentials \( NN \) and \( \bar{K}N \) have diagonal representation in the basis \(^{2}, (14): \)

\[
V = \text{diag}\{v^s, v^t, v^f\}. \text{ (15)}
\]

The matrix of transformation of the \( \eta \) and \( \tau \) bases is given by flowing relation:

\[
\eta = S \tau, \text{ (16)}
\]

where

\[
\tau = (\tau_1, \tau_2, \tau_3)^T, \quad \tau_1 = \eta_{+-+}, \quad \tau_2 = \eta_{-++}, \quad \tau_3 = \eta_{+++}, \text{ (17)}
\]

and

\[
S = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \text{ (18)}
\]
The matrix $S$ is unitary: $S^T S = I$.

In the "given charge" basis \([17]\), the matrix representation for potentials has non-diagonal elements:

\[
S^T V S = \begin{pmatrix}
\frac{1}{2} (v^t + v^s) & \frac{1}{2} (v^t - v^s) & 0 \\
\frac{1}{2} (v^t - v^s) & \frac{1}{2} (v^t + v^s) & 0 \\
0 & 0 & v^t
\end{pmatrix} = \begin{pmatrix}
V^+ & V^- & 0 \\
V^- & V^+ & 0 \\
0 & 0 & v^t
\end{pmatrix}, \tag{19}
\]

where $V^+ = \frac{1}{2} (v^t + v^s)$ and $V^- = \frac{1}{2} (v^t - v^s)$.

Let us to define the cyclical permutation operators $P_c$. The $\eta$-bases related to the configuration $(3 + 1) + 2$ and $(2 + 3) + 1$ are $\tilde{\eta} = P_c \eta$ and $\tilde{\tilde{\eta}} = P_c P_c \eta$. Taking into account the Eq. (1), we write the matrix representation of operators $I$ (or $P_c$) and $P$ in the bases $\eta$ and $\tilde{\eta}$ as following

\[
<\eta | I | \tilde{\eta} > = I^{(1,2)} = \begin{pmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}, <\eta | P | \tilde{\eta} > = P^{(1,2)} = \begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}, \tag{20}
\]

\[
<\tilde{\eta} | I | \eta > = I^{(2,1)} = \begin{pmatrix}
-\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}, <\tilde{\eta} | P | \tilde{\eta} > = P^{(2,2)} = \begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}. \tag{21}
\]

The unitary transformation given by $S$ matrix of Eq. \([18]\) leads to the matrices:

\[
S^T V I^{(1,2)} S = \begin{pmatrix}
0 & V^+ & V^- \\
0 & V^- & V^+ \\
v^t & 0 & 0
\end{pmatrix}, \quad S^T V P^{(1,2)} S = \begin{pmatrix}
0 & V^- & V^+ \\
0 & V^+ & V^- \\
v^t & 0 & 0
\end{pmatrix}, \tag{22}
\]

\[
S^T V I^{(2,1)} S = \begin{pmatrix}
V^- & 0 & V^+ \\
V^+ & 0 & V^- \\
0 & v^t & 0
\end{pmatrix}, \quad S^T V P^{(2,2)} S = \begin{pmatrix}
V^+ & 0 & V^- \\
V^- & 0 & V^+ \\
0 & v^t & 0
\end{pmatrix}. \tag{23}
\]

Thus, the unitary transformation $S$ results in new set of the Faddeev equations instead Eq. \([3]\). New set includes an additional equation with isospin triplet potential and relates to expansion of the $\eta$-basis by the isospin channel $I = 3/2$.

Similar transformation have been proposed in Ref. \cite{38} within "particle representation" for $NNK(s_{NN} = 1)$ system. Within the model, the author of \cite{38} represented elements of the given charge basis \([17]\) as the physical channels $\bar{K}^0 nn, K^- pn, K^- np$ taking into account...
possible particle transition. Within such interpretation, the non-diagonal elements $V^-$ of the matrix representation (19) of the $N\bar{K}$ potentials in $\tau$-basis were considered as a ”channel coupling”. Obviously, the ”channel coupling” appeared after unitary transformation of the two-body equation for $N\bar{K}$ is not related with a new physical effect.

The similar channel interpretation one can found in Ref. [39, 40]. The coupled channel Schrödinger equation was written as

$$\left(H_0 - E + \begin{pmatrix} V_+ & V_- \\ V_- & V_+ \end{pmatrix}\right)\phi = 0,$$

where $\phi = (\phi_1, \phi_2)^T$ and $\phi_1$ ($\phi_2$) corresponds to $K^-p$ ($\bar{K}^0n$) state of $N\bar{K}$, $V_+ = V^+ + V^-$, $V_- = V^+ - V^-$. The unitary transformation $t = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ separates the channels as following

$$\left(H_0 - E + \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix}\right)\tilde{\phi} = 0.$$

The last equations mean a redefinition for singlet and triplet components of $N\bar{K}$ potential. The potentials $v^s$ and $v^t$ did not clarify in this model due to the components $\phi_1$ and $\phi_2$ are represented as $(-+) \, \text{and} \, (-+)$ and the components $\tilde{\phi}_1$ and $\tilde{\phi}_1$ as symmetric and antisymmetric combinations of $\phi_1$ and $\phi_2$.

Let us to consider the $V_{AA}$ and $V_{AB}$ potentials without the isospin dependence. We can assume that $v^s = v^t = v$. The non-diagonal elements in Eqs. (19), (22), (23) will be equal to zero. The corresponding set of the Faddeev equations can be reduced using the matrix transformation: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, and we obtain the ”isopin-less” model, like the model given by Eq. (13). Such approach was employed in Ref. [41] to describe the $nnp$ and $nnp$ systems as an ”isospinless” systems. The final equations were obtained by using a spin basis and taking into account the spin-splitting nucleon-nucleon potential.

One can make one more unitary transformation $T$ for the Eq. (22) and (23). The $T$ is defined by the matrix

$$T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (24)$$
The new matrix representation of the potentials has the same form (15) as it had for the $\eta$-basis. The operators $I$ and $P$ are presented as

$$
I^{(1,2)} = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\
-\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}, \quad P^{(1,2)} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\
-\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix},
$$

$$
I^{(2,1)} = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}, \quad P^{(2,2)} = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}.
$$

In this case, the right-hand side of the Faddeev equations mixes the components related to the singlet and triplet potentials. The type of symmetry of spin wave function of nucleon pair defines the set of the Faddeev equations. For singlet spin state with antisymmetric spin function, the equation with isospin singlet $NN$ potential is dropped out due to the factor $(1 - p)$ in the right side of the Faddeev equations for $U$ component. The symmetric spin function corresponding to $s_{NN} = 1$ drops out the second equation for $U$ component, which is associated to the isospin triplet $NN$ potential. In this case, the first equation includes the isospin singlet/spin triplet $NN$ state, realized as the deuteron.

The quantum numbers of $NNK$ system are determined as $s_{NN} = 0$, $I = 1/2$, $I_z = 1/2$ for $ppK^-$ and as $s_{NN} = 1$, $I = 1/2$, $I_z = -1/2$ for $npK^-$. The states can be described separately because the $\tau$-bases are orthogonal: $(\{+ +\}, \{- -\}) = 0, \ldots$. Obviously, the condition that $NK$ potential does not mix the corresponding bases is necessary. With this condition, $< ppK^- | V_{AB} | npK^- > = 0$ due to the orthogonality. In opposite to the isospin model, the model [16] with the averaged $NK$ potential yields non-zero matrix element. This coupling effect takes place in the averaged potential model as well as in an "isospinless" model.

One more difference of the model [16] and the isospin model is appropriate to noted here. One can consider the $NNK$ system in these two states as $ppK^- / npK^-$ two-level system as it was proposed in Ref. [16]. The level anti-crossing formalism requests that two levels have to be bound states. However, the system $npK^-$ is unbound within the isospin formalism and is bound in the model developed in Ref. [16]. The similar calculations [42] used continuous spectrum function in the $npK^-$ channel within the "sharp resonance" approximation. The contribution of this channel has been evaluated as negligible.
TABLE I: Ground state energies $E_3$ of the $\bar{K}KN$ and $NN\bar{K}$ systems with the AY and sHW potentials for the $\bar{K}N$ and $\bar{K}\bar{K}$ interactions and the MT I-III potential for the $NN$ interaction. The difference $\delta$ of the two-body $2E_2$ and three-body $E_3$ energies, $\delta = 2E_2 - E_3$, is presented. The energies are given in MeV.

| System    | Potentials | $E_2$  | $E_3$  | $\delta$ |
|-----------|------------|--------|--------|----------|
| $K^-K^-p$ | $V_{\bar{K}\bar{K}} = 0$, AY | -30.30 | -35.2  | -25.3    |
|           | AY, AY    |        |        |          |
|           | sHW, sHW  |        |        | -11.16 unbound |
|           | $V_{\bar{K}\bar{K}} = 0$, sHW |        | -12.2  | -10.1    |
| $ppK^-$   | $V_{NN} = 0$, AY | -30.30 | -42.9  | -17.6    |
|           | MT I-III, AY |        | -46.0  | -14.6    |
|           | $V_{NN} = 0$, sHW | -11.16 | -17.1  | -5.20    |
|           | MT I-III, sHW | -21.0  | -1.3   |          |

III. NUMERICAL RESULTS

We have calculated the ground state energy for $\bar{K}KN$ and $NN\bar{K}$ systems for a complete set of the potentials and under the condition $V_{AA} = 0$. The Faddeev equations (3) were numerically solved using the cluster reduction method [43]. The results are presented in Table I. For the both potentials AY and sHW, the relation $2E_2 - E_3(V_{AA} = 0) < 0$ is satisfied. The three-body binding energy $|E_3|$ is smaller than the value $|E_3(V_{AA} = 0)|$ for the $\bar{K}KN$ system and is larger for $NN\bar{K}$, due to the repulsive and weak attractive $V_{AA}$ potential, respectively. For the $NN\bar{K}$ system $|E_3|$ increases from the $|E_3(V_{AA} = 0)|$ value for 3 MeV, when the AY potential is used, that is about 24% in the binding energy $|E_3|$ measured relatively to the two-body threshold. Obtained results for the $\bar{K}KN$ and $NN\bar{K}$ systems are comparable with the results of calculations performed within different approaches [28]. For example, calculated values $|E_3|$ reported in Ref. [10] are 47–54 MeV for the phenomenological $KN$ potentials. Including the possible physical channels into consideration may result in a small increase of the binding energy $|E_3|$ relatively to $|E_3(V_{AA} = 0)|$ (see for example [6]). However, such approaches cannot reach the values above 60 MeV.

To illustrate the relation $2E_2 - E_3(V_{AA} = 0) < 0$, in Fig. 1 we show the evolution
FIG. 1: The evolution of $NN\bar{K}$ to $\bar{K}KN$ through the mass transformation $m_N \rightarrow m_{\bar{K}}$ and $m_{\bar{K}} \rightarrow m_N$ when $m_N + m_{\bar{K}} = \text{const}$. The energies $2E_2$ (solid line) and $E_3(V_{AA} = 0)$ (dashed line) are shown as functions of the ratio $m_B/m_A$ for AY and sHW $\bar{K}N$ potentials. The $NN\bar{K}$ ($\bar{K}KN$) system corresponds to the value of 0.526 (1.90) for the ratio $m_B/m_A$.

of the $NN\bar{K}$ to $\bar{K}KN$ through the mass transformation $m_N \rightarrow m_{\bar{K}}$ and $m_{\bar{K}} \rightarrow m_N$ when $m_N + m_{\bar{K}} = \text{const}$. A parametric representation for the mass change is given by the formula:

$$m_N^{\xi} = (1 - \frac{m_{\bar{K}}}{m_N}\xi)m_N, \quad m_{\bar{K}}^{\xi} = (1 + \xi)m_{\bar{K}},$$

(27)

where $0 < \xi < m_N/m_{\bar{K}}$. The ratio $m_B/m_A$ has the value of 0.526 for $NN\bar{K}$ system and the value of 1.90 for $\bar{K}KN$ system. The relation (11) is well satisfied for AY potential when $m_B/m_A < 1$. For the sHW potential which is weaker, the relation is still satisfied. It is clear that subsequently weaker $KN$ potential could violate the relation (11). Thus, the existence of deep bound state of nucleon and kaon is necessary for the relation (11). For $AAB$ systems with weak spin/isospin dependent $AB$ potential the relation (11) is not guaranteed when $m_B/m_A < 1$. This conclusion is also supported by the calculations for the $NN\Xi$ and $\Xi\Xi N$ systems presented in Ref. [44]. The relation (11) will be satisfied for any $\bar{K}N$ potential when $m_B/m_A > 1$, since the contribution of the mass polarization energy decreases to zero when $m_B/m_A >> 1$.

We illustrate the existence the lower bounds for the ground state energy of the $NN\bar{K}$ system in Fig. 2 and 3 using the AY, sHW and averaged (av) potentials for $\bar{K}N$ interaction. The energies $E_2$, $2E_2$ and $E_3$ are shown as functions of the scaling factor $\alpha$ which controls the strength of interaction between non-identical particles: $V_{AB} \rightarrow \alpha V_{AB}$. The case, when
the $AA$ potential acting between identical particles is neglected, $V_{AA} = 0$, is presented in Fig. 2. One can see that the relation (11) is well satisfied for both models with the AY and sHW potentials. The isospin-less model with averaged (av) potential demonstrates opposite relation. Here, the mass polarization depends weakly on strength of the $AB$ potential and $2E_2 - E_3(V_{AA} = 0) \approx const$.

The situation is slightly altered when the $AA$ interaction is included to the calculations as it is shown in Fig. 3. The attractive $NN$ interaction affects the $E_3$ curve which becomes lower than is one in Fig. 2. The relation (11) is well satisfied for the large values of the two-body ground state energy, $E_2 > 10$ MeV. In the sector of weak $AB$ potential, the competition between the terms $-\Delta$ and $-\frac{1}{2} < v_{NKp}^s >$ of Eq. (10) leads to domination of the first one due to adding the weak $NN$ attraction to the $-\Delta$ and the opposite relation $|E_3(V_{AA} = 0)| > 2|E_2|$ is satisfied.

Note here that for the model with the averaged (av) potential, the $E_3$ becomes to closer to $2E_2$ in the sector of large strength of $AB$ potential. It can be explained by the core effect of the $NN$ potential which is only appeared for the isospin-less model. The repulsion of the core plays a role when three-body system is very compact. Strong $N\bar{K}$ interaction provides the repulsive effect of $NN$ potential.

**TABLE II:** Ground state energy $E_3$ of the $NN\bar{K}$ system calculated within the averaged potential and isospin models. The energies are given in MeV.

| Model         | [27] | [17] | Our  | [3] | [33] |
|---------------|------|------|------|-----|------|
| KWW AY+T     | AY+MT | AY+T | AY+AV14 |
| Averaged potential | -35.5 | -39.1 | -33.6 |
| Isospin       | -46.0 | -48  | -47.34 |

The deference between calculations for the $NN\bar{K}$ binding energy within isospin and averaged potential models is shown in Table. II. We compare the results of the different authors from Refs. [17, 27] for the averaged AY potential model and ones obtained in the isospin model [3, 33]. Our results correspond to the calculations with the AY and MT-I-III potentials for both models. In Ref. [27], the KWW potential set was used. The AY and Tamagaki potentials were applied in Refs. [17] and [3]. The AY and AV14 potentials were used in [33]. The energy $E_3$ of the averaged potential model is always larger relatively one
FIG. 2: $NN\bar{K}$ system: the energies $E_2$ (dashed line), $2E_2$ (dot-dashed line) and $E_3$ (solid line) are shown as functions of the scaling factor $\alpha$: a) for $AY$ and averaged (av) $AY \bar{K}N$ potentials, b) for $sHW$ and averaged (av) $AY \bar{K}N$ potentials. The $AA$ potential acting between identical particles is neglected, $V_{AA} = 0$.

FIG. 3: $NN\bar{K}$ system: the energies $E_2$ (dashed line), $2E_2$ (dot-dashed line) and $E_3$ (solid line) are shown as functions of the scaling factor $\alpha$: a) for $AY$ and averaged (av) $AY \bar{K}N$ potentials, b) for $sHW$ and averaged (av) $AY \bar{K}N$ potentials.

calculated in the isospin model. This could be expected due to more higher position of two-body threshold $E_2$ of the averaged potential model.
IV. CONCLUSIONS

The kaonic clusters \(\bar{K}K\bar{N}\) and \(NN\bar{K}\) are examples of three-body \(AAB\) system with an isospin dependent \(AB\) interaction. The relation between three, \(E_3\), and two, \(E_2\), - body ground state energies was studied for the kaonic clusters. It was shown that the ”isospin splitting” of the \(AB\) interaction leads to the relation \(|E_3(V_{AA} = 0)| < |2E_2|\).

Based on the Faddeev calculations for the kaonic clusters, we have found that this relation is satisfied for the case of the \(AY\) \(N\bar{K}\) potentials having one range Gaussian form when mass ration is \(0.5 < m_B/m_A < 2\). Thus, we have obtained the lower bound for \(E_3(V_{AA} = 0)\) which can be reached by using this phenomenological isospin-dependent potential. For \(NN\bar{K}\) cluster, we evaluated \(|E_3(V_{NN} = 0)| \approx 43\) MeV. \(|E_3|\) has to be larger than \(|E_3(V_{NN} = 0)|\), due to the weak attraction of the \(NN\) force. However, calculated value of \(|E_3|\) is smaller than \(2E_2\) (\(\sim 60\) MeV) and is significantly smaller the ”experimentally motivated value” of 100 MeV.

The model based on the isospin averaged \(N\bar{K}\) potential was considered. This ”isospinless” model demonstrates the opposite relation: \(|E_3(V_{AA} = 0)| > 2|E_2|\). The averaged potential changes two-body threshold \(|E_2|\) to a smaller value. Thus, three-body binding energy \(|E_3|\) is significantly different comparing to one calculated within the isospin model. The coupling between the states \(s_{NN} = 0\) and \(s_{NN} = 1\) of the \(NN\bar{K}\) system occurred in the averaged potential model \([16]\) looks as artificial. Generally, one can conclude, that the averaged potential model is a rough approximation for the isospin model.

The isospin ”given charge formalism” for \(NN\bar{K}\) cluster was proposed. This formalism was motivated by the ”particle representation” which has been developed in a number of papers. We shown that the ”channel interpretation” of the particle model is not appropriate to describe a possible particle transition. The ”channel coupling” is appeared as a result of unitary transformation of the \(\eta\) - isospin basis.

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