Transportation Proof of an inequality by Anantharam, Jog and Nair

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Abstract
Anantharam, Jog and Nair recently put forth an entropic inequality which simultaneously
generalizes the Shannon-Stam entropy power inequality and the Brascamp-Lieb inequality in
entropic form. We give a brief proof of their result based on optimal transport.

1 Introduction
Let \((c_1, \ldots, c_k)\) and \((d_1, \ldots, d_m)\) be nonnegative numbers, and let \((A_1, \ldots, A_m)\) be a collection
of surjective linear transformations identified as matrices, satisfying \(A_j : \mathbb{R}^n \rightarrow \mathbb{R}^n\) for \(j = 1, \ldots, m\). Let \(S^+ (\mathbb{R}^n)\) denote the set of \(n \times n\) real symmetric, positive definite matrices, and define

\[
M_g := \sup_{B_1, \ldots, B_k \in S^+ (\mathbb{R}^{r_i})} \frac{1}{2} \sum_{i=1}^k c_i \log \det (B_i) - \frac{1}{2} \sum_{j=1}^m d_j \log \det (A_j B A_j^T),
\]

where \(B := \text{diag} (B_1, \ldots, B_k)\). For a random vector \(X\) in \(\mathbb{R}^n\) with density \(f\) with respect to Lebesgue
measure, we define the Shannon (differential) entropy according to

\[
h(X) := - \int_{\mathbb{R}^n} f(x) \log f(x) dx,
\]

and say that the entropy exists if the defining integral exists in the Lebesgue sense and is finite.

Anantharam, Jog and Nair recently established the following result:

**Theorem 1** ([1, Theorem 3]). Let \(X\) be a random vector in \(\mathbb{R}^n\) that can be partitioned into \(k\) mutually independent components \(X = (X_1, \ldots, X_k)\), where each \(X_i\) is a random vector in \(\mathbb{R}^{r_i}\), and \(\sum_{i=1}^k r_i = n\). If \(X\) has finite entropy and second moments, then letting the above notation prevail,

\[
\sum_{i=1}^k c_i h(X_i) - \sum_{j=1}^m d_j h(A_j X) \leq M_g.
\]

As discussed in detail in [1], this result contains as special cases both the Shannon-Stam entropy
power inequality [2, 3] (in Lieb’s form [4]), and the Brascamp-Lieb inequality (in entropic form, due to Carlen and Cordero-Erasquin [5]).

Anantharam et al.’s proof of Theorem 1 is based on a doubling argument applied to information
measures, following the scheme developed in [6] by Geng and Nair. This doubling-trick for proving
Gaussian optimality goes back at least to Lieb’s original proof of the Brascamp-Lieb inequality \cite{lieb1976}, but the Geng-Nair interpretation in the context of information measures has enjoyed recent popularity in information theory (e.g., \cite{geng2001, nair2003, nair2004}). The contribution in the present note is to give a brief proof of Theorem 1 based on optimal transport. It has the advantage of being considerably shorter than the doubling proof in \cite{zaslavsky2009}. Interestingly, the proof here also seems to be simpler than Barthe’s transport proof of the Brascamp-Lieb inequality \cite{barthe2001}. However, Barthe’s argument and the proof contained herein are not truly comparable on account of the following caveats: (i) Theorem 1 implies the entropic form of the Brascamp-Lieb inequality, so some work is required to recover the functional form; and (ii) Barthe’s argument simultaneously establishes a reverse form of the Brascamp-Lieb inequality (i.e., Barthe’s inequality), and further gives a precise relationship between best constants in the forward and reverse inequalities.

\section{Proof of Theorem 1}

The key lemma is the following change-of-variables estimate, inspired by Rioul and Zamir’s recent proof \cite{rioul2009} of the Zamir-Feder entropy power inequality \cite{zamir2009} (which also follows from Theorem 1 as noted in \cite{zaslavsky2009}). We remark that other applications of optimal transport to entropy power inequalities can be found in \cite{geng2001, nair2003, nair2004}. Readers are referred to \cite{zaslavsky2009} for background on optimal transport.

\textbf{Lemma 1.} Let $\tilde{Z} \sim N(0, I)$ be a standard normal random variable in $\mathbb{R}^n$, and let $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a surjective linear map. Let $X$ be a random vector in $\mathbb{R}^n$, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the Brenier map sending $\tilde{Z}$ to $X$. If $T$ is differentiable with pointwise positive definite Jacobian $\nabla T$, then

\[ h(A(\tilde{Z})) \geq h(Z) + \frac{1}{2} \mathbb{E} \log \det(A(\nabla T(\tilde{Z}))^2A^T), \]

where $Z$ is standard normal on $\mathbb{R}^m$.

\textbf{Proof.} Consider the QR decomposition of $A^T = QR = [Q_1, Q_2][R_1^T, 0]^T$, where $Q$ is an orthogonal $n \times n$ matrix, and $R_1$ is an upper triangular $m \times m$ matrix, with positive entries on the diagonal. Let $Z'$ be standard normal on $\mathbb{R}^{n-m}$, independent of $Z$, and note that $\tilde{Z} = Q_1Z + Q_2Z'$ is a valid coupling. Now, for fixed $z'$, the map $z \in \mathbb{R}^m \rightarrow AT(Q_1z + Q_2z') \in \mathbb{R}^m$ is invertible and differentiable. Differentiability follows from our assumption on $T$, and invertibility follows by writing $T = \nabla \varphi$ for strictly convex $\varphi$ (Brenier’s theorem with the positivity assumption), and noting that the map $z \rightarrow Q_1^T \nabla \varphi(Q_1z + Q_2z')$ is the gradient of the strictly convex function $z \rightarrow \varphi(Q_1z + Q_2z')$, and is therefore invertible. So, we have

\begin{align*}
\hfill (1) \quad h(AX) &= h(AT(\tilde{Z})) = h(AT(Q_1Z + Q_2Z')) \\
&\geq h(AT(Q_1Z + Q_2Z')|Z') \\
&\geq h(Z) + \frac{1}{2} \mathbb{E} \log \det(Q_1^T \nabla T(\tilde{Z})Q_1) + \log \det R_1 \hfill (2) \\
&= h(Z) + \frac{1}{2} \mathbb{E} \log \left(\det(Q_1^T \nabla T(\tilde{Z})Q_1)\right)^2 + \log \det R_1 \hfill (3) \\
&= h(Z) + \frac{1}{2} \mathbb{E} \log \det(Q_1^T(\nabla T(\tilde{Z}))^2Q_1) + \log \det R_1 \hfill (4) \\
&= h(Z) + \frac{1}{2} \mathbb{E} \log \det(A(\nabla T(\tilde{Z}))^2A^T).
\end{align*}
Above, (1) follows since $X = T(\hat{Z})$ in distribution; (2) follows from the fact that conditioning reduces entropy; (3) is the change of variables formula for entropy; and (4) follows since the squared spectrum of $Q_1^T \nabla T(\hat{z}) Q_1$ is equal to the spectrum of $Q_1^T (\nabla T(\hat{z}))^2 Q_1$ for each $\hat{z}$ by symmetry of $\nabla T$ (an easy exercise, e.g., seen by diagonalizing $\nabla T(\hat{z})$).

Now, we begin the proof of Theorem 1. Without loss of generality, we may assume the density of each $X_i$ is smooth, bounded and strictly positive. Indeed, if this is not the case, then we first regularize the density of $X$ via convolution with a Gaussian density. The general claim then follows by continuity in the limit of vanishing regularization, which is valid provided entropies and second moments are finite (e.g., [18, Lemma 1.2]).

By dimensional analysis, a necessary condition for $M_g < \infty$ is that $\sum_{i=1}^{k} c_i r_i = \sum_{j=1}^{m} d_j n_j$. So, we make this assumption henceforth. Now, let $Z = (Z_1, \ldots, Z_k)$ be independent, standard normal random vectors with $Z_i \in \mathbb{R}^{r_i}$, and let $T_i : \mathbb{R}^{r_i} \to \mathbb{R}^{r_i}$ be the Brenier map sending $Z_i$ to $X_i$. Define $T = (T_1, \ldots, T_k)$, which is the Brenier map transporting $Z$ to $X$ by the independence assumption. We remark that each $T_i$ is differentiable, with $\nabla T_i$ being pointwise symmetric and positive definite. This follows from Brenier’s Theorem [19] and regularity estimates for the Monge-Ampère equation under our assumption that the densities of the $X_i$’s are smooth with full support [17, Remark 4.15].

So, by Lemma 1, we have

$$
\sum_{j=1}^{m} d_j h(A_j X) \geq \sum_{j=1}^{m} d_j h(Z'_j) + \frac{1}{2} \sum_{j=1}^{m} d_j \mathbb{E} \log \det(A_j (\nabla T(Z))^2 A_j^T),
$$

where $Z'_j$ is standard normal on $\mathbb{R}^{n_j}$. By the change of variables formula, $h(X_i) = h(Z_i) + \mathbb{E} \log \det(\nabla T_i(Z_i))$ for each $i = 1, \ldots, k$, so summing terms gives

$$
\sum_{i=1}^{k} c_i h(X_i) = \sum_{i=1}^{k} c_i h(Z_i) + \frac{1}{2} \sum_{i=1}^{k} c_i \mathbb{E} \log \det((\nabla T_i(Z_i))^2).
$$

By the relation $\sum_{i=1}^{k} c_i r_i = \sum_{j=1}^{m} d_j n_j$, we have $\sum_{i=1}^{k} c_i h(Z_i) = \sum_{j=1}^{m} d_j h(Z'_j)$. Hence, on combining the above estimates, we have

$$
\sum_{i=1}^{k} c_i h(X_i) - \sum_{j=1}^{m} d_j h(A_j X) \leq \frac{1}{2} \sum_{i=1}^{k} c_i \mathbb{E} \log \det((\nabla T_i(Z_i))^2) - \frac{1}{2} \sum_{j=1}^{m} d_j \mathbb{E} \log \det(A_j (\nabla T(Z))^2 A_j^T) \leq M_g,
$$

where the last line follows by definition of $M_g$ (applied pointwise inside the expectation).

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