ZX-calculus over arbitrary commutative rings and semirings (extended abstract)

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Abstract
In this extended abstract we give an axiomatisation of ZX-calculus over arbitrary commutative rings and semirings respectively. By a normal form inspired from matrix elementary operations such as row addition and row multiplication, we can obtain that these versions of ZX-calculus are still universal and complete.

1 Introduction

The ZX-calculus was introduced by Coecke and Duncan [2] as a graphical language for quantum computing, especially for quantum circuits [5]. The core part of ZX-calculus is a pair of spiders (based on the quantum Z observable and X observable) with strong complementarity [3]. As a graphical language with string diagrams, the ZX-calculus is quite intuitive. Moreover, it is also mathematically strict: the ZX-calculus dwells in a compact closed category as well as being a PROP [9], thus it is usually presented in terms of generators and rewriting rules.

Until now, the ZX-calculus has been focused on the particular algebraic object $\mathbb{C}$: rewriting of ZX-calculus diagrams corresponds to algebraic operations on matrices over the field of complex numbers. On the other hand, the ZW-calculus, another graphical language for quantum computing [6], has been generalised to arbitrary commutative rings by Amar Hadzihasanovic [7] [8]. This generalisation has brought forth applications in the proof of completeness of ZX-calculus for some fragments of quantum computing [11] [10]. Therefore, it is natural to ask whether ZX-calculus can also be generalised to arbitrary commutative rings, or even more broadly, to arbitrary commutative semirings.

In the case of the ZX-calculus over an arbitrary commutative ring $\mathcal{R}$, we cannot have the same Hadamard node such as used in [2], due to having no element like $\frac{1}{\sqrt{2}}$ in a general commutative ring. Similarly, we cannot have the exactly same red spider as usual for arbitrary commutative rings. In the case of the ZX-calculus over an arbitrary commutative semiring $\mathcal{S}$, we cannot have a Hadamard node, since there is no negative element now. For the same reason, we do not have an inverse of the triangle node. The red spider for the semiring case is the same as that of the ring case.
In this extended abstract, based on the framework as given in [13], we describe
generalisations of ZX-calculus over arbitrary commutative rings and arbitrary com-
mutative semirings respectively, by giving the corresponding generators and rewriting
rules. The ZX-calculus over any commutative ring (semiring) is universal and complete
with respect to the given rules. The key idea here is to use a normal form presented in
[12] base on elementary matrix operations. All the details of proofs will be given in a
forthcoming paper.

2 ZX-calculus over commutative rings

The ZX-calculus is based on a compact closed PROP [9], which is a strict symmetric
monoidal category whose objects are generated by one object, with a compact structure
[4] as well. Each PROP can be described as a presentation in terms of generators and
relations [1].

Since now we are working over an arbitrary commutative ring $\mathcal{R}$, we won’t expect
to have the same Hadamard node such as used in [2]. Instead, we work in the frame-
work as presented in [13], and use a Hadamard node whose corresponding matrix is
scalar-free (each element is either 1 or $-1$ in the Hadamard matrix). For the same rea-
son, we use a scalar-free red spider as a generator, with all the coefficients of the terms
being 1 in the summation which represents the corresponding map of the red spider.
Already having this, we can give the generators of ZX-calculus over $\mathcal{R}$ in the following
table. Note that through out this abstract all the diagrams should be read from top to
top.
Table 1: Generators of ZX-calculus, where $m, n \in \mathbb{N}$, $a \in \mathbb{R}$, and $e$ represents an empty diagram.

For simplicity, we make the following conventions:

There is a standard interpretation $\llbracket \cdot \rrbracket$ for the ZX diagrams over $\mathcal{R}$:

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$
where \( b_i, a_i \in \{0, 1\} \) for all \( i, s, t \); there are an even number of 1’s in each vector \((b_{i_1}, \cdots, b_{i_m}, a_{i_1}, \cdots, a_{i_n})\). Let \(|b_{i_1} \cdots b_{i_m}\rangle \langle a_{i_1} \cdots a_{i_n}| = 1\) if \( m = 0 \); let \(|a_{i_1} \cdots a_{i_n}| = 1\) if \( n = 0 \).

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

Remark 2.1 If \( \mathcal{R} = \mathbb{C} \), then the interpretation of the red spider we defined here is just the normal red spider \([2]\) written in terms of computational basis with all the coefficients being 1. To see this, one just need to notice that the red spider can be generated by the monoid pair (and its flipped version) \( \bullet \) corresponding to matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
\end{bmatrix}
\]

respectively, which means the red spider defined in this way is the same as the normal red spider (see e.g. \([2]\) ) up to a scalar depending on the number of inputs and outputs of the spider.

Now we give rules for ZX-calculus over \( \mathcal{R} \).
Figure 1: ZX rules I, over an arbitrary commutative ring $\mathcal{R}, m \geq 0, a, b \in \mathcal{R}$. 
Figure 2: ZX rules II, over an arbitrary commutative ring $\mathcal{R}, a, b \in \mathcal{R}$. 

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Figure 3: ZX rules III, over an arbitrary commutative ring $\mathcal{R}, a, b \in \mathcal{R}$.

Because of rule (Piph), we can make the following denotation:

\[
\begin{array}{c}
\ldots \\
\vdots \\
\end{array}
:=
\begin{array}{c}
\ldots \\
\vdots \\
\end{array}
\]

Further with the rule (S4) and (Ivo), we can formulate the following property:

\[
\begin{array}{c}
\ldots \\
\vdots \\
\end{array}
= (S4')
\]
where $\alpha, \beta \in \{0, \pi\}$, $+$ is a modulo $2\pi$ addition. It is a routine check that these rules are sound in the sense that they still hold under the standard interpretation $\llbracket \cdot \rrbracket$.

3 Normal form, universality and completeness over commutative rings

Given an arbitrary commutative ring $\mathcal{R}$, any vector $(a_0, a_1, \cdots, a_{2^{m-1}})^T$ with $a_i \in \mathcal{R}$ can be uniquely represented by the following normal form:

where $a_i$ connects to wires by red nodes depending on $i$, and all possible connections are included in the normal form. We point out that the diagram

represents a matrix row addition from the bottom row, and the diagram

represents a matrix row multiplication on the bottom row multiplied by $a_{2^{m-1}}$. Note that

$$
\begin{bmatrix}
\vdots \\
0 \\
0 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
$$
the normal form (1) is actually obtained via the following processes:

\[
\begin{bmatrix}
0 \\
\vdots \\
1
\end{bmatrix}
\xrightarrow{\text{row addition}}
\begin{bmatrix}
a_0 \\
\vdots \\
a_{2^m-1}
\end{bmatrix}
\xrightarrow{\text{row addition}}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{2^m-2} \\
a_{2^m-1}
\end{bmatrix}
\xrightarrow{\text{row multiplication}}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{2^m-2} \\
a_{2^m-1}
\end{bmatrix}
\]

By the map-state duality, we obtain the universality of ZX-calculus over an arbitrary commutative ring.

For proof of completeness, we need to prove the following statements:

- the juxtaposition of any two diagrams in normal form can be rewritten into a normal form.
- a self-plugging on a diagram in normal form can be rewritten into a normal form.
- all generators can be rewritten into normal forms.

We will give the proof in a fully expanded version of this extended abstract, following ideas and proofs from [12].

4 ZX-calculus over commutative semirings

Given an arbitrary commutative semiring \( S \), it is clear that one could not not have a Hadamard node or an inverse of the triangle node any more, due to a short of negative elements. Bearing this in mind, we give the generators of ZX-calculus over \( S \) in the following table.
Table 2: Generators of ZX-calculus, where $m, n \in \mathbb{N}$, $a \in S$, and $e$ represents an empty diagram.

For simplicity, we make the following conventions:

$|0\rangle \otimes |0\rangle = |0\rangle |0\rangle$, $a|1\rangle \otimes |1\rangle = a|1\rangle |1\rangle$, where

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$. 


where \( b_i, a_i \in \{0, 1\} \) for all \( i, s, t \); there are an even number of 1’s in each vector \((b_{i_1}, \ldots, b_{i_m}, a_{i_1}, \ldots, a_{i_n})\). Let \(|b_{i_1} \cdots b_{i_m}\rangle\langle a_{i_1} \cdots a_{i_n}| = 1\), if \( m = 0\); let \(|a_{i_1} \cdots a_{i_n}| = 1\), if \( n = 0\).

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
\vdots & \cdots
\end{bmatrix} = 1, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
\bullet & \bullet
\end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
\circ & \circ
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix}
\circ & \circ
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}.
\]

\[
[D_1 \otimes D_2] = [D_1] \otimes [D_2], \quad [D_1 \circ D_2] = [D_1] \circ [D_2].
\]

**Remark 4.1** If \( S = \mathbb{C} \), then the interpretation of the red spider we defined here is just the normal red spider \([2]\) written in terms of computational basis with all the coefficients being 1. To see this, one just need to notice that the red spider can be generated by the monoid pair (and its flipped version) \( \bullet \), \( \bullet \) corresponding to matrices
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]
and \( \begin{bmatrix} 1 & 0 \end{bmatrix} \) respectively, which means the red spider defined in this way is the same as the normal red spider (see e.g. \([2]\) ) up to a scalar depending on the number of inputs and outputs of the spider.

Now we give rules for ZX-calculus over an arbitrary commutative semiring \( S \).
Figure 4: ZX rules I, over an arbitrary commutative semiring $S, m \geq 0, a, b \in S$. 
Figure 5: ZX rules II, over an arbitrary commutative semiring \( S, a, b \in S \).
Figure 6: ZX rules III, over an arbitrary commutative semiring $S, a, b \in S$. 

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Because of rule (Piph), we can make the following denotation:

\[
\pi \vdash \pi
\]

Further with the rule (S4) and (Ivo), we can formulate the following property:

\[
\vdash (S'4)
\]

where \(\alpha, \beta \in \{0, \pi\}\), \(\pi\) is a modulo \(2\pi\) addition.

Also due to the associative rule (AAs) for the AND gate, the following notation is well defined and used in Figure 6:

\[
\land \vdash \land
\]

It is a routine check that these rules are sound in the sense that they still hold under the standard interpretation \([\cdot]\).

5 Normal form, universality and completeness over commutative semirings

Given an arbitrary commutative semiring \(S\), any vector \((a_0, a_1, \ldots, a_{2^n-1})^T\) with \(a_i \in S\) can be uniquely represented by the following normal form:

\[
(2)
\]
where \( a_i \) connects to wires by red nodes depending on \( i \), and all possible connections are included in the normal form. We point out that the diagram

represents a matrix row addition from the bottom row, and the diagram

represents a matrix row multiplication on the bottom row multiplied by \( a_{2^n-1} \). Note that

the normal form (2) is actually obtained via the following processes:

By the map-state duality, we obtain the universality of ZX-calculus over an arbitrary commutative semiring.

For proof of completeness, we need to prove the following statements:

- the juxtaposition of any two diagrams in normal form can be rewritten into a normal form.
- a self-plugging on a diagram in normal form can be rewritten into a normal form.
- all generators can be rewritten into normal forms.

We will give the proof in a fully expanded version of this extended abstract, following ideas and proofs from [12].
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