ON THE (NON-)STABILITY OF THE SHEAF-FUNCTION CORRESPONDENCE

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Abstract. The sheaf-function correspondence identifies the group of constructible functions on a real analytic manifold $M$ with the Grothendieck group of constructible sheaves on $M$. When $M$ is a finite dimensional real vector space, Kashiwara-Schapira have recently introduced the convolution distance between sheaves of $k$-vector spaces on $M$. In this paper, we characterize distances on the group of constructible functions on a real finite dimensional vector space that can be controlled by the convolution distance through the sheaf-function correspondence. Our main result asserts that such distances are almost trivial: they vanish as soon as two constructible functions have the same Euler integral.

1. Introduction

Inspired by persistence theory from Topological Data Analysis (TDA) [26, 16], Kashiwara and Schapira have recently introduced the convolution distance between (derived) sheaves on a finite dimensional real normed vector space [21]. This construction has found important applications, both in TDA –where it allows us to express stability of certain constructions with respect to noise in datasets– [5, 6, 7, 8] and in symplectic topology [1, 2, 17]. A challenging research direction, of interest to these two fields, is to associate numerical invariants to a sheaf on a vector space, which satisfy a certain form of stability with respect to the convolution distance.

To do so, the TDA community has been mostly using module-theoretic notions, such as the rank-invariant [10, 11], the Hilbert function or the graded Betti numbers [18, 1, 25, 24]. From a sheaf-theoretic perspective, a natural numerical invariant to consider is the local Euler characteristic, which is a constructible function that encodes exactly the class of a sheaf in the Grothendieck group, by a result of Kashiwara [20]. This is usually called the sheaf-function correspondence.

The group of constructible functions is well-understood and has the surprisingly nice property that the formalism of Grothendieck’s six operations descend to it through the sheaf-function correspondence [29]. In particular, this allows one’s to introduce well-behaved transforms of constructible functions, such as the Radon or hybrid transforms [28, 3, 23, 22]. Constructible functions have already been successfully applied in several domains, such as target enumeration for sensor networks, image and shape analysis [3, 15], though the question of their stability

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with respect to noise in the input data remain poorly understood [14, Chapter 16]. For instance, in the context of predicting clinical outcomes in glioblastoma [13], the authors overcome numerical instability by introducing an ad-hoc smoothed version of the Euler Characteristic Transform (ECT) [15], that is empirically more stable than the standard ECT, though no theoretical stability result is provided.

In this context, a natural question is to understand the stability of the sheaf-function correspondence. The convolution distance is already considered as a meaningful measurement of dissimilarity between sheaves, both in applied and pure contexts. Therefore, we propose in this work to characterize the pseudo-extended metrics on the group of constructible functions on a vector space, which are controlled in an appropriate sense by the convolution distance through the sheaf-function correspondence. Our main result (Theorem 3.10) asserts that these metrics are almost trivial: they vanish as soon as two constructible functions have the same Euler integral.

We acknowledge that similar results have been obtained independently by Biran, Cornea and Zhang in [9], in the specific case of constructible functions over a one-dimensional vector space, with the aim to study $K$-theoretical invariants of triangulated persistence categories.

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2. Sheaves and Constructible Functions

In this section, we introduce the necessary background and terminology on constructible sheaves and constructible functions.

2.1. Sheaf-Function correspondence. Throughout this paper, $\mathbb{k}$ denotes a field of characteristic 0. For a topological space $X$, we denote by Mod($\mathbb{k}_X$) the category of sheaves of $\mathbb{k}$-vector spaces on $X$, and $D^b(\mathbb{k}_X)$ its bounded derived category. Let $M$ be a real analytic manifold. The definitions and results of this section are exposed in detail in [19, Chapters 8 & 9.7].

Definition 2.1. A sheaf $F \in \text{Mod}(\mathbb{k}_M)$ is $\mathbb{R}$-constructible (or constructible for simplicity), if there exists a locally finite covering of $M$ by subanalytic subsets $M = \bigcup_{\alpha} M_{\alpha}$ such that for all $M_{\alpha}$ and all $j \in \mathbb{Z}$, the restriction $F|_{M_{\alpha}}$ is locally constant and of finite rank.

We denote by Mod$_{\text{Rec}}(\mathbb{k}_M)$ the full subcategory of Mod($\mathbb{k}_M$) consisting of constructible sheaves and by $D^b_{\text{Rec}}(\mathbb{k}_M)$ the full subcategory of $D^b(\mathbb{k}_M)$ whose objects are sheaves $F \in D^b(\mathbb{k}_M)$ such that $H^j(F) \in \text{Mod}_{\text{Rec}}(\mathbb{k}_M)$ for $j \in \mathbb{Z}$. It is well-known [19, Th. 8.4.5] that the functor $D^b(\text{Mod}_{\text{Rec}}(\mathbb{k}_M)) \longrightarrow D^b_{\text{Rec}}(\mathbb{k}_M)$ is an equivalence. The objects of $D^b_{\text{Rec}}(\mathbb{k}_M)$ are still called constructible sheaves.
Definition 2.2. A constructible function on $M$ is a map $\varphi : M \rightarrow \mathbb{Z}$ such that the fibers $\varphi^{-1}(m)$ are subanalytic subsets, and the family $\{\varphi^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite in $M$.

We denote by $\text{CF}(M)$ the group of constructible functions on $M$. All the remaining results of the section are contained in [19, Chapter 9.7].

Theorem 2.3. Let $\varphi \in \text{CF}(M)$, there exists a locally finite family of compact contractible subanalytic subsets $\{X_\alpha\}$ such that $\varphi = \sum_\alpha C_\alpha \cdot 1_{X_\alpha}$, with $C_\alpha \in \mathbb{Z} - \{0\}$.

Proposition 2.4. Let $\varphi \in \text{CF}(M)$ with compact support. For any finite sum decomposition $\varphi = \sum_\alpha C_\alpha \cdot 1_{X_\alpha}$, where the $X_\alpha$’s are subanalytic compact and contractible, the quantity $\sum_\alpha C_\alpha$ only depends on $\varphi$.

Definition 2.5. With the above notations, we define $\int \varphi := \sum_\alpha C_\alpha$.

To any constructible sheaf $F \in \text{D}^b_{\text{Rc}}(k_M)$, it is possible to associate a constructible function $\chi(F) \in \text{CF}(M)$, called the local Euler characteristic of $F$, and defined by:

$$\chi(F)(x) = \chi(F_x) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k (H^i(F)_x).$$

It is clear that for any distinguished triangle $F' \rightarrow F \rightarrow F'' \rightarrow^1$ in $\text{D}^b_{\text{Rc}}(k_M)$, one has $\chi(F) = \chi(F') + \chi(F'')$. Therefore, $\chi$ factorizes through the Grothendieck group $K(\text{D}^b_{\text{Rc}}(k_M))$ and there is a well-defined morphism of groups $K(\text{D}^b_{\text{Rc}}(k_M)) \rightarrow \text{CF}(M)$ mapping $[F]$ to $\chi(F)$.

Theorem 2.6 (Sheaf-function correspondence). The morphism $K(\text{D}^b_{\text{Rc}}(k_M)) \rightarrow \text{CF}(M)$ is an isomorphism of groups.

Lemma 2.7. Let $F \in \text{D}^b_{\text{Rc}}(k_M)$ with compact support, then:

$$\int \chi(F) = \chi(\text{R}\Gamma(M; F)) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \left( H^i(M; F) \right).$$

We review briefly the construction of the direct image operation for constructible functions. Let $f : X \rightarrow Y$ be a morphism of real analytic manifolds and $\varphi \in \text{CF}(X)$ such that $f$ is proper on $\text{supp}(\varphi)$. Then, for each $y \in Y$, $\varphi \cdot 1_{f^{-1}(y)}$ is constructible and has compact support.

Definition 2.8. Keeping the above notations, one defines the function $f_* \varphi : Y \rightarrow \mathbb{Z}$ by:

$$(f_* \varphi)(y) := \int \varphi \cdot 1_{f^{-1}(y)}.$$ 

Remark 2.9. With $a_X : X \rightarrow \{\text{pt}\}$, one has $a_X \varphi = f \varphi \cdot 1_{\{\text{pt}\}}$. 


Theorem 2.10.  
(1) Let $\varphi \in \text{CF}(X)$ and $f : X \rightarrow Y$ be a morphism of real analytic manifolds such that $f$ is proper on $\text{supp}(\varphi)$. Then $f_*\varphi$ is constructible on $Y$.
(2) Let $F \in D^b_{\mathbb{Rc}}(k_X)$ such that $\chi(F) = \varphi$. Then $\chi(Rp_*F) = p_*\chi(F) = p_*\varphi$.
(3) Let $g : Y \rightarrow Z$ be another morphism of real analytic manifold, such that $g \circ f$ is proper on $\text{supp}(g \circ f)$. Then:

$$(g \circ f)_*\varphi = g_*f_*\varphi.$$ 

2.2. Convolution distance. We consider a finite dimensional real vector space $V$ endowed with a norm $\|\cdot\|$. We equip $V$ with the usual topology. Following [21], we briefly present the convolution distance, which is inspired from the interleaving distance between persistence modules [12]. We introduce the following notations:

$$s : V \times V \rightarrow V, \quad s(x, y) = x + y$$

$$p_i : V \times V \rightarrow V \ (i = 1, 2) \quad p_1(x, y) = x, \quad p_2(x, y) = y.$$ 

The convolution bifunctor $- \ast - : D^b(k_V) \times D^b(k_V) \rightarrow D^b(k_V)$ is defined as follows. For $F, G \in D^b(k_V)$, we set

$$F \ast G := R\mathbb{S}_!(F \boxtimes G).$$ 

For $r \geq 0$ and $x \in V$, let $B(x, r) = \{v \in V \mid \|x - v\| \leq r\}$ and $K_r := k_{B(0,r)}$. For $r < 0$, we set $K_r := k_{\{x \in V \mid \|x\| < -r\}[n]}$ (where $n$ is the dimension of $V$).

The following proposition is proved in [21].

Proposition 2.11. Let $\varepsilon, \varepsilon' \in \mathbb{R}$ and $F \in D^b(k_V)$. There are functorial isomorphisms

$$(K_{\varepsilon} \ast K_{\varepsilon'}) \ast F \simeq K_{\varepsilon + \varepsilon'} \ast F \text{ and } K_0 \ast F \simeq F.$$ 

If $\varepsilon \geq \varepsilon' \geq 0$, there is a canonical morphism $\chi_{\varepsilon, \varepsilon'} : K_{\varepsilon} \rightarrow K_{\varepsilon'}$ in $D^b(k_V)$. It induces a canonical morphism $\chi_{\varepsilon, \varepsilon'} \ast F : K_{\varepsilon} \ast F \rightarrow K_{\varepsilon'} \ast F$. In particular when $\varepsilon' = 0$, we get

$$(2.1) \quad \chi_{\varepsilon, 0} \ast F : K_{\varepsilon} \ast F \rightarrow F.$$ 

Following [21], we recall the notion of $\varepsilon$-isomorphic sheaves.

Definition 2.12. Let $F, G \in D^b(k_V)$ and let $\varepsilon \geq 0$. The sheaves $F$ and $G$ are $\varepsilon$-isomorphic if there are morphisms $f : K_{\varepsilon} \ast F \rightarrow G$ and $g : K_{\varepsilon} \ast G \rightarrow F$ such that the diagrams
are commutative. The pair of morphisms \((f, g)\) is called a pair of \(\varepsilon\)-isomorphisms.

**Definition 2.13.** For \(F, G \in D^b(k_V)\), their convolution distance is

\[
d_C(F, G) := \inf(\{\varepsilon \geq 0 \mid F \text{ and } G \text{ are } \varepsilon \text{-isomorphic}\} \cup \{\infty\}).
\]

**Definition 2.14.** A pseudo-extended metric on a set \(X\) is a map \(\delta : X \times X \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}\) satisfying for all \(x, y, z \in X\):

\[
\delta(x, y) \leq \delta(x, z) + \delta(z, y).
\]

We use the convention \(s + (+\infty) = +\infty\), for all \(s \in \mathbb{R}\).

It is proved in [21] that the convolution is, indeed, a pseudo-extended metric, that is, it satisfies the triangular inequality. Having isomorphic global sections is a necessary condition for two sheaves to be at finite convolution distance, as expressed in the following proposition, which can be found as [21, Remark 2.5 (i)].

**Proposition 2.15.** Let \(F, G \in D^b(k_V)\) such that \(d_C(F, G) < +\infty\). Then:

\[
R\Gamma(V; F) \simeq R\Gamma(V; G).
\]

Moreover, it satisfies the following important stability property.

**Theorem 2.16.** Let \(u, v : X \rightarrow V\) be continuous maps, and let \(F \in D^b(k_V)\).

Then,

\[
d_C(Ru_* F, Rv_* F) \leq \sup_{x \in X} \|u(x) - v(x)\|.
\]

We will often make use of the following result, that we call the additivity of interleavings, which is a direct consequence of the additivity of the convolution functor.

**Proposition 2.17** (Additivity of interleavings). Let \((F_i)_{i \in I}\) and \((G_j)_{j \in J}\) be two finite families of \(D^b(k_V)\). Let \(I' \subset I\) and \(J' \subset J\) together with a bijection \(\sigma : I' \rightarrow J'\). Then,

\[
d_C(\bigoplus_{i \in I} F_i, \bigoplus_{j \in J} G_j) \leq \max \left(\max_{i \in I'} d_C(F_i, G_{\sigma(i)}), \max_{i \in I \setminus I'} d_C(F_i, 0), \max_{j \in J \setminus J'} d_C(G_j, 0)\right).
\]
2.3. **PL-sheaves and functions.** We consider a finite dimensional real vector space $\mathbb{V}$ endowed with a norm $\| \cdot \|$. We equip $\mathbb{V}$ with the topology induced by the norm $\| \cdot \|$, and $D^b(k\mathbb{V})$ with the convolution distance $d_C$ associated to $\| \cdot \|$. The notion of Piecewise-Linear sheaves was introduced by Kashiwara-Schapira in [21].

**Definition 2.18.** A sheaf $F \in D^b(k\mathbb{V})$ is Piecewise Linear (PL) if there exists a locally-finite family $(P_a)_{a \in A}$ of locally closed convex polytopes covering $\mathbb{V}$, such that $F|_{P_a}$ is locally constant and of finite rank for all $a \in A$.

We shall denote by $D^b_{PL}(k\mathbb{V})$ the full subcategory of $D^b(k\mathbb{V})$ consisting of PL sheaves. The following approximation theorem is proved in [21].

**Theorem 2.19.** Let $F \in D^b_{RC}(k\mathbb{V})$, for every $\varepsilon > 0$, there exists a sheaf $F_\varepsilon \in D^b_{PL}(k\mathbb{V})$ satisfying:

1. $d_C(F, F_\varepsilon) \leq \varepsilon$,
2. $\text{supp}(F_\varepsilon) \subset \text{supp}(F) + B(0, \varepsilon)$.

Following [23], we introduce the PL counterpart of constructible functions.

**Definition 2.20.** A function $\varphi : \mathbb{V} \to \mathbb{R}$ is PL-constructible, if there exists a locally-finite covering $\mathbb{V} = \bigcup_{a \in A} P_a$ by locally closed convex polytopes, such that $\varphi$ is constant on each $P_a$.

We denote by $\text{CF}_{PL}(\mathbb{V})$ the group of PL-constructible functions on $\mathbb{V}$.

**Proposition 2.21** ([23]). Any $\varphi \in \text{CF}_{PL}(\mathbb{V})$ with compact support can be written as a finite sum $\varphi = \sum_{\alpha} C_\alpha \cdot 1_{X_\alpha}$, where $X_\alpha$ is a compact convex polytope, and $C_\alpha \in \mathbb{Z} - \{0\}$.

3. **Main result**

Let $(\mathbb{V}, \| \cdot \|)$ be a finite dimensional normed real vector space. We endow $D^b(k\mathbb{V})$ with the associated convolution distance $d_C$ [21].

Let $\delta$ be a pseudo-extended metric on $\text{CF}(\mathbb{V})$.

**Definition 3.1.** The pseudo-extended metric $\delta$ is said to be $d_C$-dominated if there exists a map $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\Phi(x) \to 0$ such that for all $F, G \in D^b_{RC}(k\mathbb{V})$ with compact support such that $d_C(F, G) < +\infty$,

$$\delta(\chi(F), \chi(G)) \leq \Phi(d_C(F, G)).$$

It shall be noted that by Proposition 2.15 and Lemma 2.7, the condition $d_C(F, G) < +\infty$ implies that $\int \chi(F) = \int \chi(G)$. Our aim is to characterize all $d_C$-dominated pseudo-extended metrics on $\text{CF}(\mathbb{V})$. This will be achieved in Theorem 3.10. In all this section, $\delta$ designates a $d_C$-dominated pseudo-extended metric on $\text{CF}(\mathbb{V})$, with respect to the function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. 
Our strategy is to prove that for any $\varphi \in CF(V)$ with compact support, it is possible to concentrate the "mass" of $\varphi$ on one single point, that is $\delta(\varphi, \int \varphi \cdot 1_{[0]} ) = 0$. To do so, we first assume that $\varphi$ is PL-constructible, which allows us to use rather straightforward arguments instead of sophisticated one from subanalytic geometry. We then generalize to any stratification thanks to Kashiwara-Schapira’s approximation Theorem \[2.19\]

In Section \[3.1\] we introduce the notion of \(\varepsilon\)-flag, which is a nested sequence of convex compact sets, and allows us to successively concentrate the mass of an indicator PL-function onto one single point. This allows us to treat the PL-case in Section \[3.2\] and the general one in Section \[3.3\].

### 3.1. Convolution distance of the difference of compact convex subsets.

Recall that for $x \in V$ and $\varepsilon \geq 0$, we denote by $B(x, \varepsilon)$ the closed ball of radius $\varepsilon$ centered at $x$.

**Lemma 3.2.** Let $F \in D^b(k_V)$ with compact support, and $\varepsilon \geq 0$. If for all $x \in \text{supp}(F)$ one has $R\Gamma(B(x, \varepsilon); F) \simeq 0$, then $F$ is $\frac{\varepsilon}{2}$-isomorphic to 0.

**Proof.** Let $F \in D^b(k_V)$ with compact support, and $\varepsilon \geq 0$ such that $R\Gamma(B(x, \varepsilon); F) \simeq 0$, for all $x \in \text{supp}(F)$. By definition of interleavings, it is sufficient to prove that the canonical map $F \star K_\varepsilon \longrightarrow F$ is zero. Let $x \in V$. If $x \notin \text{supp}(F)$, it is clear that the induced morphism $(F \star K_\varepsilon)_x \longrightarrow F_x$ is zero. Let us assume that $x \in \text{supp}(F)$. By equation (2.12) in \[27\], one has

$$
(F \star K_\varepsilon)_x \simeq R\Gamma(B(x, \varepsilon); F) \simeq 0.
$$

Therefore the morphism $(F \star K_\varepsilon)_x \longrightarrow F_x$ is zero in every case, which implies that $F \star K_\varepsilon \longrightarrow F$ is also zero.

**Definition 3.3.** Given $X \subset V$ and $\varepsilon \geq 0$, the $\varepsilon$-thickening of $X$ is defined by:

$$
T_\varepsilon(X) := \{ v \in V \mid d(v, X) \leq \varepsilon \}.
$$

**Lemma 3.4.** Let $X \subset Y$ be compact convex subsets of $V$, and assume that there exists $\varepsilon \geq 0$ such that $Y \subset T_\varepsilon(X)$. Then $d_C(k_Y \setminus X, 0) \leq \frac{\varepsilon}{2}$.

**Proof.** For $y \in Y$ and $\varepsilon' > \varepsilon$, one has the following distinguished triangle:

$$
R\Gamma(B(y, \varepsilon'); k_Y) \longrightarrow R\Gamma(B(y, \varepsilon'); k_X) \longrightarrow R\Gamma(B(y, \varepsilon'; k_{Y \setminus X}) \overset{+1}{\longrightarrow}.
$$

By hypothesis, $B(y, \varepsilon') \cap Y \cap X$ is non-empty and convex. Since $X$ and $Y$ are closed convex subsets, we deduce that the map $R\Gamma(B(y, \varepsilon'); k_Y) \longrightarrow R\Gamma(B(y, \varepsilon'; k_X)$ is an isomorphism. Therefore, $R\Gamma(B(y, \varepsilon'; k_{Y \setminus X}) \simeq 0$, for all $y \in \text{supp}(k_{Y \setminus X}) \subset Y$ and $\varepsilon' > \varepsilon$. Lemma \[3.2\] implies that $d_C(k_{Y \setminus X}, 0) \leq \frac{\varepsilon}{2}$.

**Definition 3.5.** Let $\varepsilon \geq 0$. An $\varepsilon$-flag is a sequence of nested subsets $X^0 \subset X^1 \subset \ldots \subset X^n$ of $V$ satisfying:

1. $X^i$ is a compact convex subset of $V$, for all $i$;
(2) $X^0 = \{x_0\}$ is a single point;
(3) $X^i \subset T_\varepsilon(X^{i-1})$ for all $i$.

We designate this data by $X^\bullet$.

Given an $\varepsilon$-flag $X^\bullet = (X^i)_{i=0...n}$, and $i \in \llbracket 0, n \rrbracket$, we define the spaces $\text{Gr}_i(X^\bullet)$ by

$$\text{Gr}_0(X^\bullet) := X^0,$$
$$\text{Gr}_i(X^\bullet) := X^i \setminus X^{i-1} \text{ for all } i \geq 1.$$

It is immediate to verify that $\text{Gr}_i(X^\bullet)$ is locally closed for all $i \in \llbracket 0, n \rrbracket$, and that one has $X^n = \bigsqcup_i \text{Gr}_i(X^\bullet)$. Moreover, we set:

$$S(X^\bullet) := \bigoplus_{i=0}^n k_{\text{Gr}_i(X^\bullet)} \in D^b_{\text{Reg}}(k_{\mathcal{V}}).$$

**Proposition 3.6.** Let $X^\bullet = (X^i)_{i=0...n}$ be an $\varepsilon$-flag. Then one has:

1. $\chi(S(X^\bullet)) = \chi(k_X^n)$;
2. $d_C(S(X^\bullet), k_X^0) \leq \frac{\varepsilon}{2}$.

**Proof.** (1) This is a direct consequence of the fact that $X^n = \bigsqcup_i \text{Gr}_i(X^\bullet)$.

(2) For $i \geq 1$, the definition of $\varepsilon$-flag implies that the pair $(X^{i-1}, X^i)$ satisfy the hypothesis of Lemma 3.4. Therefore, $d_C(k_{\text{Gr}_i(X^\bullet)}, 0) \leq \frac{\varepsilon}{2}$. By additivity of interleaveings, one deduces:

$$d_C(S(X^\bullet), k_X^0) = d_C(k_X^0 \oplus \bigoplus_{i=1}^n k_{\text{Gr}_i(X^\bullet)}, k_X^0)$$
$$\leq \max \left( d_C(k_X^0, k_X^0), \max_{i=1...n} d_C(k_{\text{Gr}_i(X^\bullet)}, 0) \right) \quad \text{(Proposition 2.17)}$$
$$= \max_{i=1...n} d_C(k_{\text{Gr}_i(X^\bullet)}, 0)$$
$$\leq \frac{\varepsilon}{2}.$$

\[ \square \]

### 3.2. PL-case.

The first step of our proof is the following concentration lemma in the Piecewise-Linear (PL) case, that we will extend later on to arbitrary stratification by density of PL-sheaves with respect to the convolution distance.

**Lemma 3.7.** Let $\delta$ be a $d_C$-dominated pseudo-extended metric on $\text{CF}(\mathbb{V})$ and let $\varphi \in \text{CF}_{\text{PL}}(\mathbb{V})$ with compact support, such that $\varphi = \sum_{\alpha \in A} C_\alpha \cdot 1_{x_\alpha}$, with $A$ finite and $X_\alpha$ compact and convex polytopes. For $\alpha \in A$, let $x_\alpha \in X_\alpha$. Then one has

$$\delta \left( \varphi, \sum_{\alpha \in A} C_\alpha \cdot 1_{\{x_\alpha\}} \right) = 0.$$

**Proof.** We consider the linear deformation retraction $F_\alpha : X_\alpha \times [0, 1] \to X_\alpha$ from $\{x_\alpha\}$ to $X_\alpha$ defined by:
Figure 1. Illustration of the $\eta$-flag $X_\alpha^\bullet$

\[ F_\alpha(x,t) = (1 - t) \cdot x_\alpha + t \cdot x. \]

We set $\ell_\alpha = \max\{\|x - x_\alpha\| \mid x \in X_\alpha\}$ and $\ell = \max_\alpha \ell_\alpha$. Let $\varepsilon > 0$, and $\eta > 0$ be such that for all $|t| \leq \eta$, $|\Phi(x)| \leq \varepsilon$. Consider an integer $n > \frac{\ell}{\eta}$. We define for $i \in [0, n]$ the sequence of subsets $X_\alpha^i := F_\alpha(X_\alpha \times [0, \frac{i}{n}])$. By construction, $X_\alpha^\bullet = (X_\alpha^i)_{i=0...n}$ is an $\eta$-flag. We depict an illustration of $X_\alpha^\bullet$ in figure 1.

Let us define the following sheaves:

\[ G_\eta = \bigoplus_{\alpha \in A} S(X_\alpha^\bullet)^{|C_\alpha|}[1 - \operatorname{sgn}(C_\alpha)]/2, \]

\[ G = \bigoplus_{\alpha \in A} k^{|C_\alpha|}[1 - \operatorname{sgn}(C_\alpha)]/2. \]

Then one has:

\[ \chi(G^n) = \chi \left( \bigoplus_{\alpha \in A} S(X_\alpha^\bullet)^{|C_\alpha|}[1 - \operatorname{sgn}(C_\alpha)]/2 \right) \]

\[ = \sum_{\alpha \in A} C_{\alpha} \cdot \chi(S(X_\alpha^\bullet)) \]

\[ = \varphi \quad \text{(Proposition 3.6(1))}. \]

Similarly:
\[ \chi(G) = \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}}. \]

Moreover, one has by additivity of interleavings (Proposition 2.17):

\[ d_{C}(G^{\eta}, G) \leq \max_{\alpha \in A} d_{C} \left( S(X_{\alpha}^{\star}) |_{[0, 1]} \left[ (1 - \text{sgn}(C_{\alpha}))/2 \right], k_{\{x_{\alpha}\}} |_{[0, 1]} \left[ (1 - \text{sgn}(C_{\alpha}))/2 \right] \right) \]
\[ = \max_{\alpha \in A} d_{C} \left( S(X_{\alpha}^{\star}), k_{\{x_{\alpha}\}} \right) \]
\[ \leq \frac{\eta}{2} \leq \eta \quad (\text{Proposition 3.6-(2)}). \]

Therefore:

\[ \delta \left( \varphi, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}} \right) = \delta \left( \chi(G^{\eta}), \chi(G) \right) \leq \Phi(d_{C}(G^{\eta}, G)) \leq \varepsilon. \]

The above being true for all \( \varepsilon > 0 \), we conclude that

\[ \delta \left( \varphi, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}} \right) = 0. \]

\[ \square \]

**Proposition 3.8.** Let \( \varphi \in \text{CF}_{\text{PL}}(V) \) with compact support, and let \( x \in V \). Then one has

\[ \delta \left( \varphi, \int \varphi \cdot 1_{\{x\}} \right) = 0. \]

**Proof.** Given \( u, v \in V \), we set \( [u, v] = \{ t \cdot u + (1 - t) \cdot v \mid t \in [0, 1] \} \). Let us write \( \varphi = \sum_{\alpha \in A} C_{\alpha} \cdot 1_{X_{\alpha}} \), with \( A \) finite, \( C_{\alpha} \in \mathbb{Z} - \{0\} \) and \( X_{\alpha} \) compact and convex polytopes. For \( \alpha \in A \), let \( x_{\alpha} \in X_{\alpha} \). Then by Lemma 3.7 applied to \( \psi = \sum_{\alpha \in A} C_{\alpha} \cdot 1_{[x_{\alpha}, x]} \), one has:

\[ \delta \left( \psi, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}} \right) = 0 = \delta \left( \psi, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x\}} \right). \]

Therefore:

\[ \delta \left( \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}}, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x\}} \right) = 0. \]

We now apply Lemma 3.7 to \( \varphi \):
δ \left( \varphi, \int \varphi \cdot 1_{\{x\}} \right) = \delta \left( \varphi, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x\}} \right)
\leq \delta \left( \varphi, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}} \right) + \delta \left( \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x\}}, \sum_{\alpha \in A} C_{\alpha} \cdot 1_{\{x_{\alpha}\}} \right)
= 0.

3.3. General case. In this final section, we generalize the previous results to arbitrary stratifications, by piecewise linear approximation (Theorem 2.19).

Lemma 3.9. Let \varphi \in CF(\mathbb{V}) with compact support, and let x \in \mathbb{V}. Then one has

\delta \left( \varphi, \int \varphi \cdot 1_{\{x\}} \right) = 0.

Proof. Let F \in D_{b,Re}^{b}(k_{\mathbb{V}}) with compact support such that \varphi = \chi(F). According to Theorem 2.19, for all n \in \mathbb{Z}_{>0}, there exists \bar{F} \in D_{PL}^{b}(k_{\mathbb{V}}) such that d_{C}(F, \bar{F}) \leq \frac{1}{n} and supp(\bar{F}) \subset T_{\frac{1}{n}}(supp(F)). In particular, \bar{F} has compact support for all n \geq 1. Moreover by Proposition 2.15, one has for all n \geq 1,

\int \chi(\bar{F}_{n}) = \int \varphi \text{ according to Lemma 2.7. Consequently, for all } n > 0:

\delta \left( \varphi, \int \varphi \cdot 1_{\{x\}} \right) \leq \delta \left( \varphi, \chi(F_{n}) \right) + \delta \left( \chi(F_{n}), \int \varphi \cdot 1_{\{x\}} \right)
= \delta \left( \varphi, \chi(F_{n}) \right) + \delta \left( \chi(F_{n}), \int \chi(F_{n}) \cdot 1_{\{x\}} \right)
= \delta \left( \varphi, \chi(F_{n}) \right) \text{ (Proposition 3.8)}
\leq \Phi \left( d_{C}(F, F_{n}) \right).

We conclude by making n go to +\infty. \square

Theorem 3.10. Let \delta be a d_{C}-dominated pseudo-extended metric on CF(\mathbb{V}), and let \varphi, \psi \in CF(\mathbb{V}) with compact supports be such that \int \varphi = \int \psi. Then:

\delta(\varphi, \psi) = 0.

Proof. By the above lemma,

\delta(\varphi, \psi) \leq \delta \left( \varphi, \int \varphi \cdot 1_{\{0\}} \right) + \delta \left( \int \psi \cdot 1_{\{0\}}, \psi \right)
= 0 \text{ (Lemma 3.9).} \square
Corollary 3.11. Let $F, G \in D^{b}_{Rc}(k_V)$ with compact support, such that $d_C(F, G) < +\infty$. Then:

$$\delta(\chi(F), \chi(G)) = 0.$$ 

Corollary 3.12. Let $X$ be a real analytic manifold, and let $\varphi \in CF(X)$ with compact support. Also, consider $f, g : X \to V$ some morphisms of real analytic manifolds proper on $\text{supp}(\varphi)$. Then:

$$\delta(f_*\varphi, g_*\varphi) = 0.$$ 

Proof. By [29, Theorem 2.3], $f_*\varphi$ and $g_*\varphi$ are indeed constructible and have compact support because the images of $f$ and $g$ are compact. Let $a_X : X \to \{\text{pt}\}$ and $a_V : V \to \{\text{pt}\}$ be the constant maps. Then by [29, Section 2], one has:

$$\int f_*\varphi = a_V(f_*\varphi) = (a_V \circ f)_*\varphi \quad \text{(Theorem 2.10-3)}$$
$$= a_X_*\varphi$$
$$= \int \varphi.$$ 

Similarly, $\int g_*\varphi = \int \varphi = \int f_*\varphi$. Since both $f_*\varphi$ and $g_*\varphi$ have compact support, we conclude the proof by applying Theorem 3.10. \hfill \Box

4. Discussion and further work

Our main results Theorem 3.10 and Corollary 3.11 show that any distance on the group of constructible that can be controlled by the convolution distance—in the sense of domination—vanishes as soon as two compactly supported constructible functions $\varphi, \psi$ have the same Euler integral, a condition that is satisfied whenever there exists two sheaves $F, G \in D^{b}_{Rc}(k_V)$ satisfying $d_C(F, G) < +\infty$ and such that $\varphi = \chi(F)$ and $\psi = \chi(G)$. The convolution distance is usually interpreted as a $\ell_\infty$ type metric, because of the form of stability (Theorem 2.16) it satisfies. Our results therefore give a strong negative incentive on the possibility of obtaining a $\ell_\infty$-control of the pushforward operation on constructible functions.

Schapira recently introduced the concept of constructible functions up to infinity [30], that allows one’s to define Euler integration of constructible functions without compact support. We conjecture that Theorem 3.10 holds when replacing with compact support by constructible up to infinity, though we do not know how to adapt our $\varepsilon$-flag technique to this setting.

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