Solutions of Extended Supersymmetric Matrix Models for Arbitrary Gauge Groups

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Abstract

Energy eigenstates for \( N = 2 \) supersymmetric gauged quantum mechanics are found for the gauges groups \( SU(n) \) and \( U(n) \). The analysis is aided by the existence of an infinite number of conserved operators. The spectrum is continuous. Perturbative eigenstates for \( N > 2 \) are also presented, a case which is relevant for the conjectured description of M theory in the infinite momentum frame.
I. Introduction

A view has emerged that different superstring theories are various limits of one unifying theory. That theory is M theory, an eleven-dimensional system whose low-energy limit is $D = 11$ supergravity and whose compactifications to ten dimensions on a circle and an interval yield type IIA and $E_8 \times E_8$ heterotic superstrings. Dualities, as well as new string degrees of freedom such as D branes and supermembranes, have played a role in obtaining this unification. Supersymmetric matrix quantum mechanics has been useful in gaining insight into D-branes, supermembranes and M theory. Although a precise covariant formulation of M theory is still lacking, it has been conjectured that the $n \to \infty$ limit of an $SU(n)$-matrix quantum mechanics system with $N = 16$ supersymmetry describes M theory in the infinite momentum frame. Hence, any progress in understanding such systems is of interest.

Ground states of supersymmetry quantum mechanics are often of the form $\exp(\pm W)$, where $W$ is the superpotential. Excited energy states are not known with one exception: All such states of the $N = 2$ supersymmetric $SU(2)$ gauge theory have been found. In this letter, we obtain energy eigenstates for the $U(n)$ and $SU(n)$ systems for general $n$. Given the interest in the $SU(\infty)$ case, our results should be of use in future research. The $N = 16$ case, which is relevant for M theory is not solved. However, perturbative $N = 16$ energy eigenstates, for which the coupling constant is set to zero, are found; this is non-trivial due to the requirement of satisfying the Gauss law constraints. Our results are a first step toward a perturbative analysis of the M theory proposal of ref.

II. Particular Solutions for the $N = 2$ Case

The $N = 2$ quantum-mechanic gauge theory involves a real gauge potential $A_B$, a real scalar $\phi_B$, and a complex fermion $\psi_B$, all in the adjoint representation. Here, $B$, which is a gauge index, runs over the number $n_G$ of generators of the Lie group $G$, e.g. $n_G = n^2 - 1$ for $SU(n)$ and $n_G = n^2$ for $U(n)$. The lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2} (\mathcal{D}_t \phi)_A (\mathcal{D}_t \phi)_A + i \bar{\psi}_A (\mathcal{D}_t \psi)_A - ig f_{ABC} \bar{\psi}_A \phi_B \psi_C ,$$

where the covariant derivative $\mathcal{D}_t$ on any field $\varphi$ is $(\mathcal{D}_t \varphi)_A \equiv \partial_t \varphi_A - g f_{ABC} A_B \varphi_C$.
and where \( g \) is the gauge coupling. Here and elsewhere, the presence of a repeated index indicates summation. The \( f_{ABC} \) are structure constants: \([\sigma_A, \sigma_B] = if_{ABC}\sigma_C\), where \( \sigma_A \) are the generators of the Lie algebra of \( G \). We choose the \( \sigma_A \) to be matrices satisfying

\[
\text{Tr} (\sigma_A \sigma_B) = \delta_{AB}.
\] (2)

It is straightforward to quantize the gauge system governed by Eq.(1). The hamiltonian is

\[
H = \frac{1}{2} \pi_A \pi_A + ig f_{ABC} \bar{\psi}_A \phi_B \psi_C ,
\] (3)

where \( \phi_A \) and \( \pi_B \), as well as \( \psi_A \) and \( \bar{\psi}_B \) are conjugate variables satisfying \([\phi_A, \pi_B] = i\delta_{AB}\) and \( \{\psi_A, \bar{\psi}_B\} = \delta_{AB}\). States \(|s\rangle\) must satisfy the Gauss law constraints

\[
G_A |s\rangle = 0 ,
\] (4)

where

\[
G_A = f_{ABC} (\phi_B \pi_C - i\bar{\psi}_B \psi_C) .
\] (5)

In other words, states must be gauge invariant. The degrees of freedom \( A_B \) do not enter the hamiltonian – their role has been replaced by these Gauss law constraints.

Note that

\[
H = \frac{1}{2} \pi_A \pi_A + g \phi_A G_A ,
\] (6)

so that, on gauge-invariant states, \( H \) reduces to \( \frac{1}{2} \pi_A \pi_A \). If it were not for Eq.(1), the theory would be free. One needs only to find the gauge-invariant plane wave states \(|s\rangle\):

\[
\frac{1}{2} \pi_A \pi_A |s\rangle = E_s |s\rangle , \quad G_A |s\rangle = 0 .
\] (7)

The lagrangian in Eq.(1) is invariant under the supersymmetry transformations generated by \( Q = \psi_A \pi_A \) and \( \bar{Q} = \bar{\psi}_A \pi_A \). As usual, the anticommutator of \( Q \) and \( \bar{Q} \) yields the hamiltonian up to gauge transformations: \( \{Q, \bar{Q}\} = \pi_A \pi_A = 2H - 2g \phi_A G_A \).

States are classified according to their fermion number. In particular, one can define the fermion vacuum \(|+\rangle\) to be annihilated by all the \( \psi_A \):

\[
\psi_A |+\rangle = 0 .
\] (8)
Assign fermion number 0 to $| + \rangle$. All other fermionic sectors are obtained by repeatedly applying $\bar{\psi}_A$. Since there are $n_G$ such fermions, there are states with fermion number 0, 1, \ldots, $n_G$, and the total number of fermionic Fock-space states is $2^{n_G}$.

The case of $G = SU(2)$, for which $n_G = 3$, has been solved by M. Claudson and M. Halpern \[27\]. They found
\begin{equation}
| k \rangle_0 = \frac{\sin (kr)}{kr} | + \rangle ,
\end{equation}
\begin{equation}
| k \rangle_1 = \frac{Q}{k} | k \rangle_0 = \left[ \frac{\sin (kr)}{(kr)^2} - \frac{\cos (kr)}{kr} \right] \frac{1}{r} \phi_A \bar{\psi}_A | + \rangle ,
\end{equation}
\begin{equation}
| k \rangle_2 = \left[ \frac{\sin (kr)}{(kr)^2} - \frac{\cos (kr)}{kr} \right] \frac{1}{2r} \varepsilon_{ABC} \phi_A \bar{\psi}_B \bar{\psi}_C | + \rangle ,
\end{equation}
\begin{equation}
| k \rangle_3 = \frac{Q}{k} | k \rangle_2 = \frac{1}{k} \psi_1 \bar{\psi}_2 \bar{\psi}_3 | + \rangle ,
\end{equation}
where the subscript $p$ on $| k \rangle_p$ indicates the fermion number. Here, $r = \sqrt{\phi_A \phi_A}$, $k$ is any non-negative real number, and $\varepsilon_{ABC}$ is the completely antisymmetric tensor on three indices. The states in Eq. (9) are only plane-wave normalizable, as expected, since the spectrum is continuous.

The goal of this section is to obtain zero-fermion-number solutions for $G = U(n)$ and $G = SU(n)$. We first treat the $U(n)$ case. Let $\sigma_A$ be the matrix generators in the fundamental representation, with a normalization respecting Eq. (2). Let $\Phi = \phi_A \sigma_A$ be an $n \times n$ matrix of scalar fields. Since a gauge-invariant functional of the $\phi_A$ depends only on the eigenvalues of $\Phi$, write
\begin{equation}
\Phi = \phi_A \sigma_A = U^{-1}DU ,
\end{equation}
where $D$ is a diagonal matrix of the eigenvalues $\lambda_j$ of $\Phi$:
\begin{equation}
D = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \cdots & 0 \\
0 & \cdots & \lambda_n 
\end{pmatrix} ,
\end{equation}
and $U$ is some unitary transformation. Let us look for solutions of the form $f(\lambda_1, \ldots, \lambda_n) | + \rangle$.

Acting on such a state,
\begin{equation}
\pi_A \pi_A \to - \frac{1}{\mathcal{M}^2} \sum_{j=1}^n \frac{\partial}{\partial \lambda_j} \mathcal{M}^2 \frac{\partial}{\partial \lambda_j} ,
\end{equation}
where $\mathcal{M}$ is the mass of the scalar field.
where $M^2$, the Vandermonde determinant measure factor, is the square of

$$M = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j) \quad .$$

After some guesswork, we have found solutions to $\frac{1}{2} \pi_A \pi_A f = Ef$. They are given by

$$f = M^{-1} \exp \left[ i \sum_{j=1}^{n} k_j \lambda_j \right] \quad ,$$

where the “momenta” $k_j$ are arbitrary real numbers. The energy eigenvalue is

$$E = \frac{1}{2} \sum_{j=1}^{n} k_j^2 \quad .$$

In verifying $\frac{1}{2} \pi_A \pi_A f = Ef$, one needs to use the identity

$$\sum_{k \neq j \neq i} \frac{1}{\lambda_i - \lambda_j} \frac{1}{\lambda_i - \lambda_k} = 0 \quad ,$$

which, incidently, arises in obtaining covariant superstring amplitudes for fermionic scattering processes [28].

The solutions in Eq.(14) behave badly when any two eigenvalues approach each other. It is possible, however, to obtain regular solutions by taking linear combinations of Eq.(14). A unique class of regular solutions is achieved by antisymmetrization using the permutations $\sigma$ of the permutation group $S_n$ on $n$ elements:

$$| k \rangle_0 = N M^{-1} \sum_{\sigma \in S_n} (-1)^\sigma \exp \left[ i \sum_{j} k_{\sigma(j)} \lambda_j \right] | + \rangle \quad ,$$

where $N$ is a normalization factor and $(-1)^\sigma$ is $+1$ for even permutations and $-1$ for odd permutations. It is easy to verify that $| k \rangle_0$ is non-singular as $\lambda_j \to \lambda_i$.

When $G = SU(n)$, a solution is obtainable from the $G = U(n)$ case because the system is separable. Select the generator index for the diagonal $U(1)$ subgroup to be the last one, $n^2$. For convenience, relabel this index as 0. Hence, $\sigma_{n^2} = \sigma_0$ where $\sigma_0 = I_n / \sqrt{n}$ and $I_n$ is the $n \times n$ identity matrix. Since the laplacian on $U(n)$, as well as the hamiltonian, splits into a $U(1)$ part and an $SU(n)$ part as

$$\frac{1}{2} \pi_A \pi_A = \frac{1}{2} \pi_0 \pi_0 + \frac{1}{2} \sum_{A=1}^{n^2-1} \pi_A \pi_A = H_{U(1)} + H_{SU(n)} - \phi_A G_A \quad ,$$

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solutions factorize into a product of a $U(1)$ wave function $f_{U(1)}$ times an $SU(n)$ wave function $f_{SU(n)}$ via

$$f_{U(n)} = f_{U(1)} f_{SU(n)} \ .$$

(19)

Since $f_{U(1)}$ is a function of the sum of the eigenvalues and $f_{SU(n)}$ is a function of differences of eigenvalues, write

$$\lambda_j = (\lambda_j - \Sigma_{\lambda}) + \Sigma_{\lambda} , \quad \text{where} \quad \Sigma_{\lambda} = \frac{1}{n} \sum_{j=1}^{n} \lambda_j \ .$$

(20)

Performing the factorization in Eq.(19), one finds

$$f_{U(1)} = \exp \left[ i n \sum_{k} \Sigma_{\lambda} \right] ,$$

$$f_{SU(n)} = N' \mathcal{M}^{-1} \sum_{\sigma \in S_n} (-1)^{\sigma} \exp \left[ i \sum_{j=1}^{n} k_{\sigma(j)} (\lambda_j - \Sigma_{\lambda}) \right] \ .$$

(21)

In Eq.(21), $k_{\sigma(j)}$ can be replaced by $k_{\sigma(j)} - \Sigma_k$, where

$$\Sigma_k = \frac{1}{n} \sum_{j=1}^{n} k_j \ ,$$

(22)

due to $\sum_{j=1}^{n} (\lambda_j - \Sigma_{\lambda}) = 0$. Hence, the $SU(n)$ wave functions really depend on only $n-1$ momenta since $\sum_{j=1}^{n} (k_j - \Sigma_k) = 0$. The energy separates into a $U(1)$-part $E_{U(1)}$ and an $SU(n)$-part $E_{SU(n)}$:

$$E = \frac{1}{2} \sum_{j=1}^{n} k_j^2 = \frac{1}{2} \tilde{k}_0^2 + \frac{1}{2} \frac{n-1}{n} \sum_{j=1}^{n} \bar{k}_j^2 = E_{U(1)} + E_{SU(n)} \ ,$$

(23)

where the barred momenta are

$$\tilde{k}_0 \equiv \sqrt{n} \Sigma_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} k_j \ , \quad \bar{k}_j = \sqrt{\frac{n}{n-1}} (k_j - \Sigma_k) \ .$$

(24)

The $(n-1)/n$ in $E_{SU(n)}$ compensates for the one constraint on the $\bar{k}_j$ of $\sum_{j=1}^{n} \bar{k}_j = 0$.

III. The Construction of Other Solutions

It turns out that the $N = 2$ system has infinitely many operators $O$ that are conserved up to Gauss’s law. These operators can generate new solutions from old ones. Let $O$ be gauge invariant and not a functional of the $\phi_A$. Then if $| s \rangle$ satifies
\( H | s \rangle = E_s | s \rangle \), for \( H \) in Eq.(6) and if \( O | s \rangle = | s' \rangle \neq 0 \), then \( | s' \rangle \) is also a gauge-invariant eigenstate of \( H \) with the same energy: \( H | s' \rangle = E_s | s' \rangle \). The proof is straightforward. Constructing \( O \) satisfying these criteria is simple. It suffices to take \( O \) to be a trace or products of traces of the \( \Pi \equiv \sigma^A \pi_A \) and the \( \bar{\Psi} \equiv \sigma^A \bar{\psi}_A \). Examples of such operators are

\[
O_1 = Q = Tr (\bar{\Psi} \Pi) , \quad O_2 = Tr (\bar{\Psi} \bar{\Psi} \Pi) , \quad O_3 = Tr (\bar{\Psi} \bar{\Psi} \bar{\Psi}) = 1 \quad f_{ABC} \bar{\psi}_A \bar{\psi}_B \bar{\psi}_C , \quad (25)
\]

\( O' = Tr (\bar{\Psi} \Pi \Pi) Tr (\bar{\Psi} \bar{\Psi} \Pi) \), etc. When these operators act on the states \( | k \rangle_0 \) of Eq.(17), energy eigenstates are produced, although they might not be new states. For example, \( Tr (\Pi \Pi) \) produces the same eigenstate up to a factor of \( 2E_s \). Whether a new eigenstate arises also depends on the group \( G \): When \( G = SU(2) \), \( Tr (\Pi \Pi \Pi) \) gives zero because the symmetric “d symbol” vanishes; for \( G = SU(n) \) with \( n \geq 3 \), \( Tr (\Pi \Pi \Pi) \) yields a new state. Like the \( | k \rangle_0 \), the \( O \)-generated states are not normalizable because the spectrum is continuous.\(^1\) Although an infinite number of \( O \) can be constructed, only a finite number generate independent states. It may be that all eigenstates of \( H \) can be obtained by applying the \( O \) to the \( | k \rangle_0 \).

Let us verify this conjecture for \( G = SU(2) \). In doing so, we shall also illustrate the reduction procedure of Sect.II for going from \( U(n) \) to \( SU(n) \). Factorizing the wave function in Eq.(17) for the \( n = 2 \) case yields

\[
\frac{1}{\lambda_1 - \lambda_2} \left( \exp [ik_1 \lambda_1 + ik_2 \lambda_2] - \exp [ik_2 \lambda_1 + ik_1 \lambda_2] \right) = \\
\exp \left[ \frac{i}{2} (k_1 + k_2) (\lambda_1 + \lambda_2) \right] \times \\
\frac{1}{\lambda_1 - \lambda_2} \left( \exp \left[ \frac{i}{2} (k_1 - k_2) (\lambda_1 - \lambda_2) \right] - \exp \left[ -\frac{i}{2} (k_1 - k_2) (\lambda_1 - \lambda_2) \right] \right) \\
= i \sqrt{2k} \exp \left[ \frac{i}{2} (k_1 + k_2) (\lambda_1 + \lambda_2) \right] \frac{1}{k r} \sin \left( kr \right) , \quad (26)
\]

where

\[
\bar{k} \equiv \frac{k_1 - k_2}{\sqrt{2}} , \quad (27)
\]

\(^1\) Certain states might be badly non-normalizable. The issue of which states should be retained in the Hilbert space goes beyond the scope of the present work.
and
\[ r \equiv \sqrt{\phi_A \phi_A} \Rightarrow r = \frac{|\lambda_1 - \lambda_2|}{\sqrt{2}}, \tag{28} \]
which follows from \( \Phi = \sigma_A \phi_A \) after diagonalization: \( \Phi \rightarrow \phi_3 = (\lambda_1 - \lambda_2) / \sqrt{2} \) with \( \phi_1 = \phi_2 = 0 \). Letting \( N^{-1} = i \sqrt{2} \bar{k} \), one obtains the wave function in \( |k \rangle_0 \) in Eq.(9) since the factorized form in Eq.(26) leads to
\[ f_{U(1)} = \exp \left[ i \bar{k}_0 \lambda_0 \right], \quad f_{SU(2)} = \frac{1}{kr} \sin \left( \bar{k}r \right), \tag{29} \]
Here, \( \bar{k}_0 = (k_1 + k_2) / \sqrt{2} \) and \( \bar{\lambda}_0 = (\lambda_1 + \lambda_2) / \sqrt{2} \). The energy separates as in Eq.(23) with
\[ E_{U(1)} = \frac{1}{2} \bar{k}_0^2, \quad E_{SU(2)} = \frac{1}{2} \bar{k}^2. \tag{30} \]
Finally, a short calculation shows that when the \( O_i \) in Eq.(25) are applied to \( |k \rangle_0 = f_{SU(2)} |+\rangle \), the states \( |k \rangle_i \) in Eq.(9) are generated up to an overall normalization.

IV. Perturbative Solutions for \( N > 2 \)

When more than two supersymmetries are present, the matrix models no longer appear to be exactly solvable. For \( N > 2 \), the degrees of freedom are a real gauge potential \( A_B \), a set of real scalar \( \phi^m_B \), and a set of fermions \( \psi^\alpha_B \), where \( m \) and \( \alpha \) label the different sets. Quantization leads to hamiltonians of the form[27]
\[ H = \frac{1}{2} \pi_A^m \pi_A^m + \frac{1}{4} g^2 \left( f_{ABC} \phi^l_B \phi^m_C \right) \left( f_{ADE} \phi^l_D \phi^m_E \right) + ig f_{ABC} \bar{\psi}^\alpha_B \phi^m_B \Gamma^m_{\alpha \beta} \psi^\beta_C, \tag{31} \]
the Gauss constraints
\[ G_A = f_{ABC} \left( \phi^m_B \pi^m_C - i \bar{\psi}^\alpha_B \psi^\alpha_C \right), \tag{32} \]
and the commutation relations \( \left[ \phi^l_A, \pi^m_B \right] = i \delta^{lm} \delta_{AB} \) and \( \left\{ \psi^\alpha_A, \bar{\psi}^\beta_B \right\} = \delta^{\alpha \beta} \delta_{AB} \). In Eq.(31), the \( \Gamma^m_{\alpha \beta} \), for \( m = 1, 2, \ldots, p \), are matrix representations of an \( SO(p) \) Clifford algebra. For example, when \( N = 4 \), \( p = 3 \), \( \alpha = 1 \) or 2, and the \( \Gamma^m_{\alpha \beta} \) are \( 2 \times 2 \) Pauli matrices. When \( N = 16 \), \( p = 9 \), \( \alpha = 1, 2, \ldots, 16 \), the \( \Gamma^m_{\alpha \beta} \) are \( 16 \times 16 \) real matrices satisfying \( \left[ \Gamma^l, \Gamma^m \right] = 2 \delta^{lm} \), and the fermions are real. It is possible to organize the 16 real fermions into 8 complex ones at the cost of making less manifest group properties.

Because the potential-energy terms in Eq.(31) are no longer proportional to Gauss law constraints, the full effect of the interactions is felt so that the equation \( H \mid s \rangle =
$E_s | s \rangle$ is difficult to solve. A perturbative approach is possible. To begin a perturbative expansion, solutions to the $g = 0$ system must be known. Such solutions are obtainable using the methods of Sections II and III because, when $g$ is zero, the Hamiltonian is a sum of $p$ independent $N = 2$ Hamiltonians. Hence, the system factorizes. The analog of the state in Eq. (17) is

$$| k \rangle_0 = \mathcal{N} \prod_m \left[ \mathcal{M}_m^{-1} \sum_{\sigma \in S_n} (-1)^{\sigma} \exp \left( i \sum_j k_j^{(m)} \lambda_j^{(m)} \right) \right] | + \rangle,$$

where $| + \rangle$ is annihilated by all the $\psi_\alpha^m$, $k_j^{(m)}$ are momenta for the $m$th sector, and

$$\mathcal{M}_m^{-1} \equiv \prod_{i < j} \left( \lambda_i^{(m)} - \lambda_j^{(m)} \right)^{-1},$$

where the $\lambda_j^{(m)}$ are the eigenvalues of the matrix $\phi^m = \sigma^A \phi_A^m$. The energy is

$$E = \frac{1}{2} \sum_{j,m} \left( k_j^{(m)} \right)^2.$$ 

Additional eigenstates are generated by applying to $| k \rangle_0$ operators $O$ that are gauge invariant and that are functionals only of the $\pi_B^m$ and fermions. Such $O$ involve a trace or products of traces of the $\Pi^m \equiv \sigma^A \pi^m_A$ and the $\bar{\Psi}^\alpha \equiv \sigma^A \bar{\psi}^\alpha_A$. Examples of such operators are $Tr \left( \Pi^i \Pi^m \right), Tr \left( \bar{\Psi}^\alpha \bar{\Psi}^\beta \right)$ for $\alpha \neq \beta$, $Tr \left( \bar{\Psi}^\alpha \Pi^m \right) Tr \left( \bar{\Psi}^\alpha \bar{\Psi}^\beta \Pi^i \right)$, etc.

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