Mechanism design for large scale systems

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Abstract

In this paper, we consider infinite number of non atomic self-interested agents with private valuation of a divisible good. We design a pricing mechanism that is easy to implement, is individually rational, weakly budget balanced and incentive compatible. In this mechanism, agents send reports of their types, based on which, the designer solves a constrained optimization problem through Lagrange’s mechanism. The resulting optimal allocation and Lagrange’s multiplier is sent as the allocation and prices to the respective agent. We show that reporting one’s type truthfully is a dominant strategy of the players in this mechanism. We then extend this idea to the dynamic case, when player’s types are dynamically evolving as a controlled Markov process. In this case, in each time period, reporting one’s type is a dominant strategy of the players.

1 Introduction

Internet is ubiquitous these days with ever increasing number of smartphones and computers. There is a large and very complex interaction of people with each other, with the government and the firms. Thus it is quite an important problem to design large scale systems such that when interacted upon by people, who are self-interested agents and try to maximize their own utilities, the outcome is desirable to the planner. One of the common formulations is to maximize sum of the utilities of all the agents, also called social welfare or network utility [1].

In practice, a system designer does not have complete information to maximize the social welfare directly, while agents may have incentive to misreport such information so as to manipulate the system outcomes to their own advantages. Consequently, mechanism design has been proposed as an engineering side of game theory which aims to design games such that when played by the self-interested agents while the induced game-theoretic equilibrium leads

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to social welfare maximization in an efficient allocation. One such celebrated mechanism is Vickery Clark Grove (VCG) Mechanism \cite{2,3,4} that ensures that truthfully revealing each agent’s information is a dominant strategy. However, there are two main drawbacks that prevent VCG Mechanism from being practically implemented in large-scale systems, namely unaffordable communications and computation overheads. In particular, VCG Mechanism may incur significantly communication overheads since it requires each agent to submit its entire utility function carrying infinitely many messages. Related research efforts (e.g. \cite{6,7,8,9,10,11,12,13}) have been proposed towards resolving such an issue, in which case users are only required to submit limited messages while the social welfare is maximized at the induced equilibria. The second main issue of VCG Mechanism that this work considers is the computation issue: as the number of agents becomes large, it becomes difficult to compute the optimal action and implement VCG mechanism. Authors in \cite{5} consider a dynamic mean field model for large scale auctions, where it showed that each agent reports truthfully according to her conjoint valuation.

In this paper, we consider the problem of mechanism design where there are a large number of homogeneous players. Each player has a type which is her private information. She takes an action to report of her type to the system designer who, based on everyone’s type, allocates good and prices to the individuals. We present an incentive compatible (IC) mechanism where truthfully reporting one’s type is a dominant strategy of a player. Moreover, the mechanism is individually rational (IR) and also weakly balances budget (BB). We then consider where types of the players are dynamically evolving through a controlled Markov process. In this case, we extend our previous mechanism to this dynamic setting such that revealing one’s type in each time period is a dominant strategy of the players. As before, the mechanism is IC, IR and weakly BB.

The paper is structured as follows. In Section 2, we present the model and revise the VCG mechanism. In Section 3, we present the mechanism for the static setting. In Section 4, we consider the case when player’s types are dynamically evolving and extend our previous mechanism to that setting. We conclude in Section 5.

2 System Model

Consider a system with a set of $I = \{1, \ldots, I\}$ agents. Each agent $i$ has a type $\theta_i \in \mathcal{T}$, where |\mathcal{T}| is finite. Type-$\theta_i$ user’s received allocation is $x_{\theta_i}$. The resource allocation constraint is given by $x \in \mathcal{X}$. Type-$\theta_1$ users’ utilities: $U(\theta_i, x_i)$.

In the following, we revise the VCG mechanism.

**Mechanism 1** (VCG Mechanism). Each user sends a report of its type $\tilde{\theta}_i$ and auctioneer selects the allocation $\tilde{x}^*(\tilde{\theta})$ such that

$$\tilde{x}^*(\tilde{\theta}) = \arg\max_{x \in \mathcal{X}} \sum_{i \in I} U(\tilde{\theta}_i, x_i).$$

(1)
DSIC: Each user’s (effective) payoff satisfies

\[ U(\theta_i, x_i^*(\tilde{\theta})) + \sum_{j \neq i} U(\tilde{t}_i, x_j^*(\tilde{\theta})) \leq \max_{x \in X} \left[ U(\theta_i, x_i) + \sum_{j \neq i} U(\tilde{t}_j, x_j) \right]. \tag{2} \]

Note that, from (1), \( \max_{x \in X} \left[ U(\theta_i, x_i) + \sum_{j \neq i} U(\tilde{t}_j, x_j) \right] \) is achieved when type-\( \theta_i \) users reveal their truthful type, i.e., \( \tilde{t}_i = \theta_i \). Hence, truthful reporting is each type-\( \theta_i \) user’s optimal strategy, regardless of the decisions of all other types of users.

3 Large-Scale Mechanism Design

Let \( \rho_j, \forall j \in T \) be the portion of type \( j \) users i.e. \( \rho_j = \sum_{i=1}^{I} \frac{1_{(\theta_i = \theta_j)}}{I} \).

**Lemma 1.** The VCG Payment for user \( i \) converges to \( \lambda^* x_i \) when \( I \to \infty \), where \( \lambda^* \) is the optimal dual variable corresponding to the constraint of the following social welfare maximization problem:

\[
\begin{align*}
\text{(3a)} & \quad \max_x \sum_{i \in I} U_i(x_i) \\
\text{(3b)} & \quad \text{s.t. } \sum_{i \in I} x_i \leq C.
\end{align*}
\]

Proof: Let

\[ U_{-i}(x_{-i}) \triangleq \sum_{j \neq i} U_j(\tilde{t}_j, x_j) \]

\[ \bar{x}_{-i} = \arg \max_{x_{-i} \in X_{-i}} U_{-i}(x_{-i}). \tag{4} \]

The VCG payment for user \( i \) is given by:

\[ h_i(\tilde{t}_i, \tilde{\theta}_{-i}) = -\sum_{j \neq i} U_j(\tilde{t}_j, x_j^*(\tilde{\theta})) + \max_{x_i \in X_i} \sum_{j \neq i} U_j(\tilde{t}_j, x_j) \]

\[ = \int_{x_{-i}^*}^{\bar{x}_{-i}} \nabla_{x_{-i}} U_{-i}(x_{-i}) dx_{-i} \]

\[ = \int_{0}^{1} \nabla_{x_{-i}} U_{-i}((1-t)x_{-i}^* + t\bar{x}_{-i})[(1-t)x_{-i}^* + t\bar{x}_{-i}] dt. \tag{5} \]

Note that, as \( I \to \infty \),

\[ U_{-i}((1-t)x_{-i}^* + t\bar{x}_{-i}) = U_{-i}(x_{-i}^*), \forall t \in [0, 1]. \tag{6} \]

Hence, from (5), we have

\[ h_i(\tilde{t}_i, \tilde{\theta}_{-i}) = \lambda^* x_i^*. \tag{7} \]
Lemma 2. There exists a function $f_i = \lambda^* x_i^*$ such that truthfully reporting one’s type is a dominant strategy for a player.

To show this, consider the following mechanism:

**Mechanism 2 (Large-Scale VCG Mechanism).**

- **Message Space:** each agent $i$ reports its type $\tilde{\theta}_i \in \mathcal{T}$;
- **Allocation Outcome:** let $\rho^*_j(\tilde{\theta})$ be the portion of reported type-$j$ agents for all $j \in \mathcal{T}$. The auctioneer solves the following problem:

$$\max \sum_j \rho^*_j(\tilde{\theta}) U(j, z_j)$$

subject to:

$$\sum_j \rho^*_j(\tilde{\theta}) z_j \leq C.$$  

Let $z^*(\tilde{\theta})$ be the optimal primal solution and $p^*(\tilde{\theta})$ be the optimal dual solution. Each agent $i$ receives an allocation $x_i^*(\tilde{\theta}) = z^*_i(\tilde{\theta})$, and needs to pay $h_i^*(\tilde{\theta}) = p^*(\tilde{\theta}) z_i^*(\tilde{\theta})$.

We next prove such a mechanism achieves dominant-strategy incentive compatibility.

**Proof.** Given all other agents’ reports’ $\tilde{\theta}_{-i}$, each agent’s payoff maximization problem is:

$$\max_{\tilde{t}_i \in \mathcal{T}} P_i(\tilde{t}_i, \tilde{\theta}_{-i}) \triangleq U_i(x_i^*(\tilde{t}_i, \tilde{\theta}_{-i})) - p^*(\tilde{t}_i, \tilde{\theta}_{-i}) x_i^*(\tilde{t}_i, \tilde{\theta}_{-i}).$$

Note that $p^*(\cdot)$ is determined in (24). As $I \to \infty$, the value of $\rho^*_j(\tilde{\theta})$ remains the same even if agent $i$ solely deviates from reporting $\tilde{t}_i$, so does $p^*(\tilde{\theta})$. Hence, we can safely remove $\tilde{t}_i$ from $p^*(\cdot)$.

Note that each agent’s payoff in (27) is bounded by

$$P_i(\tilde{t}_i, \tilde{\theta}_{-i}) \leq \max_{x_i \geq 0} U_i(x_i) - p^*(\tilde{\theta}_{-i}) x_i,$$

which can be readily shown by contradiction. In addition, the maximizer $x_i^*(\tilde{\theta}_{-i})$ of $\max_{x_i \geq 0} U_i(x_i) - p^*(\tilde{\theta}_{-i}) x_i$ satisfies

$$\frac{\partial U_i(x_i^*(\tilde{\theta}_{-i}))}{x_i} = p^*(\tilde{\theta}_{-i}), \forall i \in \mathcal{I}.$$  

By analyzing the KKT conditions of (24), $x_i^*(\tilde{\theta}_{-i})$ is exactly achieved when agent $i$ reports $\tilde{t}_i = \theta_i$, i.e., $x_i^*(\tilde{t}_i, \tilde{\theta}_{-i}) = x_i^*(\tilde{t}_i, \tilde{\theta}_{-i})$, and, from (24),

$$P_i(\theta_i, \tilde{\theta}_{-i}) = U_i(x_i^*(\tilde{\theta}_{-i})) - p^*(\tilde{\theta}_{-i}) x_i^*(\tilde{\theta}_{-i}) = \max_{x_i \geq 0} U_i(x_i) - p^*(\tilde{\theta}_{-i}) x_i.$$
From \(28\), we see that the truthful report \(\tilde{t}_i = \theta_i\) maximizes agent \(i\)'s payoff. Therefore, the Large-Scale VCG Mechanism achieves dominant-strategy incentive compatibility.

We next prove the remaining two properties, namely Individual Rationality (IR) and weakly Budget Balance (BB).

**Proposition 1.** The Large-Scale VCG Mechanism satisfies IR and weakly BB.

**Proof.** The dual variable \(p^*(\tilde{\theta})\) is positive, and so is every user \(i\)'s payment \(h_i(\tilde{\theta})\). This shows that weakly BB is satisfied.

From (14), each agent’s payoff when truthful reporting satisfies

\[
P_i(\theta_i, \tilde{\theta}_{-i}) = \max_{x_i \geq 0} U_i(x_i) - p^*(\tilde{\theta}_i)x_i \geq U_i(0) - p^*(\tilde{\theta}_i) \cdot 0 = 0,
\]

which shows that IR is achieved.

### 4 Dynamic Large-Scale VCG Mechanism

In this section, we generalize the idea of the large-scale mechanism into dynamic environments.

The utility function \(u_i\) of agent \(i\) is

\[
P_i(x_i, h_i, \theta_i) = u_i(z_i, \theta_i) - h_i,
\]

where \(\theta_i\) for agent \(i\) is a general Markov process, where the Markov processes of agents are independent across agents.

\[
P(\theta_{i,t+1}|\theta_{i,t}, z_{i,t}) = Q(\theta_{i,t+1}|\theta_{i,t}, z_{i,t})
\]

All agents discount the future with a common discount factor \(\delta \in (0, 1)\). The socially efficient policy is obtained by maximizing the expected discounted utilities.

The socially optimal program starting in period \(t\) at state \(\rho_t\) can be written as

\[
W(\rho_t) \triangleq \max_{z_t \in \mathcal{Z}} \left[ \sum_{j \in \mathcal{J}} \rho_{j,t}u_i(z_{i,t}, \theta_{j,t}) + \mathbb{E} \left[ \sum_{s=t+1}^{\infty} \delta^{s-t-1} \sum_{j \in \mathcal{J}} \rho_{j,s}u_i(z_{i,s}, \theta_{j,s})|\rho_t, z_t \right] \right]
\]

\[
= \max_{z_t \in \mathcal{Z}} \left[ \sum_{j \in \mathcal{J}} \rho_{j,t}u_i(z_{i,t}, \theta_{j,t}) + \delta \mathbb{E}[W(\rho_{t+1})|\rho_t, z_t] \right]
\]

Define an optimum policy of the designer \(z_t^o\) as

\[
z_t^o(\rho_t) = \arg\max_{z_t \in \mathcal{Z}} \left[ \sum_{j \in \mathcal{J}} \rho_{j,t}u_i(z_{i,t}, \theta_{j,t}) + \delta \mathbb{E}[W(\rho_{t+1})|\rho_t, z_t] \right],
\]

5
Also define for a Markovian policy σ for a user, her utility to go \( U_i^\sigma(z_{i,t}, \theta_{i,t}, \rho_t) \) such that, for \( z_{i,t} = \sigma(\theta_{i,t}, \rho_t) \),

\[
U_i^\sigma(z_{i,t}, \theta_{i,t}, \rho_t) = u_i(z_{i,t}, \theta_{i,t}) + \delta \mathbb{E} \left[ \sum_{j \in J} \theta_{j,t+1} U_i^\sigma(z_{i,t+1}, \theta_{j,t+1}, \rho_{t+1}) \bigg| \theta_{i,t}, z_{i,t}, \rho_t \right]
\]

(21)

\[
W(\rho_t) \text{ can be rewritten as }
\]

\[
W(\rho_t) \triangleq \max_{z_t \in Z} \sum_{j \in J} \rho_{j,t} u_i(z_{j,t}, \theta_{j,t}) + \delta \mathbb{E} \left[ W(\rho_{t+1}) \big| \rho_t, z_t \right]
\]

\[
= \max_{z_t} \sum_{j \in J} \rho_{j,t} u_i(z_{j,t}, \theta_{j,t}, \rho_t)
\]

(22)

**Mechanism 3** (Dynamic Large-Scale VCG Mechanism).

- **Message Space:** each agent \( i \) reports its type \( \tilde{\theta}_{i,t} \in T \);

- **Allocation Outcome:** let \( \rho_j^*(\tilde{\theta}_t) \) be the portion of reported type-\( j \) agents for all \( j \in T \). The auctioneer solves the following problem:

\[
\max_{z_t \in Z} \sum_{j \in J} \rho_j^*(\tilde{\theta}_t) U_j(z_j, \theta_{j,t}, \rho_t)
\]

(24a)

\[
\text{s.t. } \sum_{j \in J} \rho_j^*(\tilde{\theta}_t) z_j \leq C.
\]

(24b)

Let \( z_t^*(\tilde{\theta}_t) \) be the optimal primal solution and \( p_t^*(\tilde{\theta}_t) \) be the optimal dual solution. Each agent \( i \) receives an allocation

\[
x_{i,t}^*(\tilde{\theta}_t) = z_{\tilde{\theta}_i}^*(\tilde{\theta}_t),
\]

(25)

and needs to pay

\[
h_{i,t}^*(\tilde{\theta}_t) = p_t^*(\tilde{\theta}_t) z_{\tilde{\theta}_i}^*(\tilde{\theta}_t).
\]

(26)

Given all other agents’ reports’ \( \tilde{\theta}_{-i} \), each agent’s payoff to go maximization problem is:

\[
\max_{\tilde{\theta}_{i,t} \in T} P_{i,t}(\tilde{\theta}_{i,t}, \tilde{\theta}_{-i}, \rho_t) \triangleq \sum_{j \in J} \rho_j^*(\tilde{\theta}_t) U_i(x_{i,t}^*(\tilde{\theta}_{i,t}, \tilde{\theta}_{-i,t}), \tilde{\theta}_{i,t}, \rho_t) - p_t^*(\tilde{\theta}_{i,t}, \tilde{\theta}_{-i,t}) x_{i,t}^*(\tilde{\theta}_{i,t}, \tilde{\theta}_{-i,t}).
\]

(27)

Note that \( p_t^*(\cdot) \) is determined in (24). As \( I \to \infty \), the value of \( \rho_j^*(\tilde{\theta}_t) \) remains the same even if agent \( i \) solely deviates from reporting \( t_i \), so does \( p_t^*(\tilde{\theta}_t) \). Hence, we can safely remove \( t_i \) from \( p^*(\cdot) \).
Note that each agent’s payoff in (27) is bounded by
\[ P_{i,t}(\tilde{\theta}_t, \rho_t) \leq \max_{z_{i,t} \geq 0} U(\theta_{i,t}, z_{i,t}, \rho_t) - p^*_t(\tilde{\theta}_t)z_{i,t}, \] (28)
which can be readily shown by contradiction. In addition, the maximizer \( x^o_{i,t}(\tilde{\theta}_t) \) of \( \max_{x_{i,t} \geq 0} U(\theta_{i,t}, z_{i,t}, \rho_t) - p^*_t(\tilde{\theta}_t)z_{i,t} \) satisfies
\[ \frac{\partial U(\theta_{i,t}, z_{i,t}, \rho_t)}{z_{i,t}} = p^*_t(\tilde{\theta}_t), \quad \forall i \in I. \] (29)
By analyzing the KKT conditions of (24), \( x^o_{i,t}(\tilde{\theta}_t) \) is exactly achieved when agent \( i \) reports \( \tilde{\theta}_{i,t} = \theta_{i,t} \), i.e., \( x^o_{i,t}(\tilde{\theta}_t) = x^*_{i,t}(\theta_{i,t}, \tilde{\theta}_t) \), and, from (27),
\[ P_{i,t}(\theta_{i,t}, \tilde{\theta}_t) = U_{i,t}(x^o_{i,t}(\tilde{\theta}_t)) - p^*_t(\tilde{\theta}_t)x^o_{i,t}(\tilde{\theta}_t) = \max_{x_{i,t} \geq 0} U_{i,t}(x_{i,t}) - p^*_t(\tilde{\theta}_t)x_{i,t}. \]
From (28), we see that the truthful report \( \tilde{\theta}_{i,t} = \theta_{i,t} \) maximizes agent \( i \)’s payoff. Therefore, the Dynamic Large-Scale VCG Mechanism achieves dominant-strategy incentive compatibility.

**Proposition 2.** The Dynamic Large-Scale VCG Mechanism satisfies IR and weakly BB.

**Proof.** The dual variable \( p^*_t(\tilde{\theta}_t) \) is positive, and so is every user \( i \)’s payment \( h_{i,t}(\tilde{\theta}_t) \). This shows that weakly BB is satisfied.

From (14), each agent’s payoff when truthful reporting satisfies
\[ P_{i,t}(\theta_{i,t}, \tilde{\theta}_t) = \max_{x_{i,t} \geq 0} U_{i,t}(x_{i,t}) - p^*_t(\tilde{\theta}_t)x_{i,t} \geq U_{i,t}(0) - p^*_t(\tilde{\theta}_t) \cdot 0 \geq 0, \] (30)
which shows that IR is achieved.

\[ \square \]

5 Conclusion

In this paper, we considered both static and dynamic models where players have private types that parameterize their utilities. We presented mechanisms that incentivize the users to reveal their private types, which are incentive compatible, individually rational and weakly budget balanced.

References

[1] D. P. Palomar and M. Chiang, “A tutorial on decomposition methods for network utility maximization,” in *IEEE JSAC*, vol. 24, no. 8, pp. 1439-1451, Aug. 2006.

[2] W. Vickery, “Counterspeculation, auctions, and competitive sealed tenders,” *J. Finance*, vol. 16, no. 1, 1961.
[3] E. H. Clarke, “Multipart pricing of public goods,” *Public choice*, pp.17-33, 1971.

[4] T. Groves, “Incentives in teams,” *Econometrica*, 1973.

[5] Iyer, Krishnamurthy, Ramesh Johari, and Mukund Sundararajan. ”Mean field equilibria of dynamic auctions with learning.” *Management Science*, 60.12 (2014): 2949-2970.

[6] S. Yang and B. Hajek, “VCG-Kelly mechanisms for allocation of divisible goods: Adapting VCG mechanisms to one-dimensional signals,” *IEEE J. Sel. Areas Commun.*, 2007.

[7] R. Johari and J. N. Tsitsiklis, “Efficiency of scalar-parameterized mechanisms,” *Operations Research*, 2009.

[8] R. Jain and J. Walrand, “An efficient Nash-implementation mechanism for network resource allocation.” *Automatica*, 2010.

[9] A. Kakhbod and D. Teneketzis, “An efficient game form for multi-rate multicast service provisioning,” *IEEE J. Sel. Areas Commun.*, 2012.

[10] S. Sharma and D. Teneketzis, “Local public good provisioning in networks: A Nash implementation mechanism,” *IEEE J. Sel. Areas Commun.*, 2012.

[11] A. Sinha and A. Anastasopoulos, “Distributed mechanism design with learning guarantees,” in Proc. IEEE CDC, 2017.

[12] H. Ge and R. A. Berry, “Dominant strategy allocation of divisible network resources with limited information exchange”, in Proc. IEEE INFOCOM, 2018.

[13] M. Zhang and J. Huang, “Efficient Network Sharing with Asymmetric Constraint Information”, *IEEE J. Sel. Areas Commun.*, 2019.