A Complete Algebraic Transformational Solution for the Optimal Dynamic Policy in Inventory Rationing across Two Demand Classes

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Abstract

In this paper, we apply the sensitivity-based optimization to propose and develop a complete algebraic transformational solution for the optimal dynamic rationing policy in inventory rationing across two demand classes. Our results provide a unified framework to set up a new transformational threshold type structure for the optimal dynamic rationing policy. Based on this, we can provide a complete description that the optimal dynamic rationing policy is either of critical rationing level (i.e. threshold type or a static rationing policy) or of no critical rationing level. Also, two basic classifications can be described by means of our algebraic transformational solution. To this end, we first establish a policy-based birth-death process and set up a more general reward (or cost) function with respect to both states and policies of the birth-death process, hence this gives our policy optimal problem. Then we set up a policy-based Poisson equation, which, together with a performance difference equation, characterizes monotonicity and optimality of the long-run average profit of the rationing inventory system. Finally, we apply the sensitivity-based optimization to construct a threshold type policy to further study the rationing inventory system. Furthermore, we use some numerical experiments to verify our theoretic results and computational
validity, and specifically, compare the optimal dynamic rationing policy with the optimal threshold type rationing policy from two different policy spaces. We hope that the methodology and results developed in this paper can shed light to the study of rationing inventory systems, and will open a series of potentially promising research by means of the sensitivity-based optimization and our algebraic transformational solution.

**Keywords:** Inventory rationing; Multiple demand classes; Critical rationing level; Optimal dynamic rationing policy; Sensitivity-based optimization; Markov decision process; Algebraic transformational solution.

1 Introduction

During the last six decades considerable attention has been paid to studying inventory rationing across multiple demand classes. The practice of inventory rationing is to allocate on-hand inventory among different demand classes according to an intuitive discipline: Supplying high priority demand but delaying demand fulfillment for low priority demand, that is, inventory rationing can reserve low on-hand inventory for future potential important demand due to some practical factors, such as different service level requirements, supply priorities, product or service prices, profit margins, lost sale costs and so on. Thus inventory rationing is an increasingly important tool for matching limited inventory supply with uncertain demands. Since the introduction of the concept of inventory rationing by Veinott Jr [80], research on inventory rationing across multiple demand classes has been greatly motivated by many practical applications such as military operations by Kaplan [48], airline by Lee and Hersh [49], maritime by John [47], hotel by Bitran and Mondschein [5], manufacturing by Zhao et al. [91], machine failures by Cheng [13], health care by Papastavrou et al. [67], rental business by Papier and Thonemann [68,69] and Jain et al. [46] and so forth.

For inventory rationing across multiple demand classes, the inventory rationing policies always respond to different supply priorities or service level requirements, such that so called a *critical rationing level* can be intuitively imagined by early research: When on-hand inventory falls below the critical level, a low priority demand is rejected or back-ordered; while the left on-hand inventory is reserved to supply future high priority demands. Through summarizing the main current research on inventory rationing across multiple demand classes, this paper is interested in
(a) Does such a critical rationing level exist?
(b) If yes, then what is a sufficient (or necessary) condition of the existence?
(c) If no, then which useful characteristics can be found in order to further describe and analyze the optimal rationing policy?

In fact, it is always very difficult and challenging to answer the above three problems, and specifically, one of the major challenges encountered in beginning our study of rationing inventory systems is how to set up a basic research perspective or mathematical background, such that we can establish a mathematical modeling and analysis for inventory rationing problems.

To answer the above three problems, in Section 5 we luckily find such an interesting research perspective and mathematical background by means of the unique solution $P_i^{(d)}$ (depending on the penalty cost $P$) of the linear equation $G^{(d)}(i) + b = 0$ with $b = R + C_{2,2} - P$ for $1 \leq i \leq K$ and with $G^{(d)}(i) = \lambda^{-i} \left[ f \left(0 \right) - \eta^d \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ f^{(d)}(r) - \eta^d \right] \prod_{k=r+1}^{i-1} v(d_k)$, in which the explicit expression of this solution is given by

$$P_i^{(d)} = \frac{R + C_{2,2} + \lambda^{-i} \left[ B_0 - D^{(d)} \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ B^{(d)}_r - D^{(d)} \right] \prod_{k=r+1}^{i-1} v(d_k)}{1 + \lambda^{-i} \left[ A_0 - F^{(d)} \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ A^{(d)}_r - F^{(d)} \right] \prod_{k=r+1}^{i-1} v(d_k)}$$

for $1 \leq i \leq K$. Furthermore, we show that the unique solution $P_i^{(d)}$ for $1 \leq i \leq K$ plays a key role in our later study to solve the above three problems (a), (b) and (c). See Theorems 2 to 6 in Section 7 for more details. Based on this, this paper uses the unique solution $P_i^{(d)}$ for $1 \leq i \leq K$ to develop a complete algebraic transformational solution for the optimal dynamic rationing policy in inventory rationing across multiple demand classes.

To explain the reasons why the solution $P_i^{(d)}$ for $1 \leq i \leq K$ is crucial in solving the above three problems, we need to consider two basic points:

(a) **Modeling extension and generalization.** This paper puts a more general cost or reward structure into rationing inventory systems than those in the literature. Our algebraic transformational solution can work well because of two basic research points: Setting up a policy-based birth-death process whose stationary probability vector is given an explicit expression; and providing a more general reward (or cost) function with respect
to both states and policies of the birth-death process. Based on this, our optimization problem is established by means of the policy-based stationary probability vector and the policy-based reward (or cost) function.

(b) Optimal policy realization. Our optimization problem can be discussed by using a performance difference equation \( \eta^{d'} - \eta^d = \mu_2 \pi^{(d')}(i) \left[ G^{(d')}(i) + b \right] \), whose sign is determined by that of \( G^{(d')}(i) + b \); while the sign of \( G^{(d')}(i) + b \) is deduced by means of the solution \( \mathcal{P}_i^{(d)} \) for \( 1 \leq i \leq K \) according to three different areas of the penalty cost \( P \) (see Proposition 1) in Section 6. Therefore, our algebraic transformational solution provides a new and more extensive research perspective and mathematical background in optimal dynamic control of inventory rationing across multiple demand classes. This will lead to our final complete solution to the above three problems (a), (b) and (c).

To the best of our knowledge, this paper is the first to apply the sensitivity-based optimization to study rationing inventory systems with multiple classes demand, in which the role played by the solution \( \mathcal{P}_i^{(d)} \) for \( 1 \leq i \leq K \) is given a detailed analysis on our algebraic transformational solution for the optimal dynamic rationing policy. On such a research line, this paper also has several close works in the recent literature, e.g., Ma et al. [56,57] for studying energy-efficient data centers and Xia et al. [88] for analyzing optimal dynamic control of group-server queues. Different from [56,57] and [88], this paper provides a new algebraic technique to a basic case: \( P_L < P < P_H \), which can not be well solved in the recent literature yet except that the bang-bang control was found and established. For the basic case, this paper shows that the optimal rationing policy must be of transformational threshold type even if it is not of threshold type under an ordinary setting yet (see Theorem 5 and Remark 2) in Section 7. Therefore, our transformational threshold type is a stronger result than the bang-bang control in analyzing the optimal dynamic rationing policy.

So far much research has applied the Markov decision processes (MDPs) to discuss inventory rationing across multiple demand classes, e.g., see Ha [36,38], Gayon et al. [34], Benjaafar and ElHafsi [3], Nadar et al. [63] and so on. When applying the MDPs to study inventory rationing, it is key that we must first investigate the structure of the optimal policy by identifying a set of structured value functions that is preserved under an optimal operator. In general, it is not easy (and even very difficult) to analyze the structure properties of the optimal policy. To do this, sometimes it has to introduce stronger conditions or model constraint to guarantee the structure properties of the optimal policy;
otherwise we cannot obtain any useful result by using the MDPs. For this, it is necessary and useful to extend and generalize the MDPs and their methodologies to increase ability of the MDP applications. This motivates us in this paper to develop the sensitivity-based optimization for discussing inventory rationing across multiple demand classes, and to provide a complete algebraic transformational solution for the optimal dynamic rationing policy by means of the solution $\Psi_i^{(d)}$ for $1 \leq i \leq K$.

Based on the above analysis, we summarize the main contributions of this paper as follows:

1. To the best of our knowledge, this paper is the first to apply the sensitivity-based optimization to deal with inventory rationing across multiple demand classes, and provides a complete algebraic transformational solution for the optimal dynamic rationing policy by means of the solution $\Psi_i^{(d)}$ for $1 \leq i \leq K$.

2. This paper extends and generalizes model assumptions of the rationing inventory system, for example, introducing many practical cost, price and reward; and subjectively setting up a penalty cost field $\{1, 2, \ldots, K\}$ whose purpose may supply low priority demand but must have some penalty cost through observing actual markets and enterprise operations. Although model assumptions are extended and generalized, this paper indicates that such an extension and generalization of inventory rationing will not increase any difficulty under our algebraic solution framework.

3. This paper develops a unified framework of applying the sensitivity-based optimization to inventory rationing problems through using three concrete steps: (a) Setting up a policy-based Markov process, construct a more general reward (or cost) function with respect to both states and policies of the Markov process, and establish the optimal problem of dynamic rationing policies. (b) Setting up a policy-based Poisson equation, which, together with a performance difference equation, characterizes monotonicity and optimality of the long-run average profit of the rationing inventory system. (3) Determining the optimal dynamic rationing policy, and showing that the optimal dynamic rationing policy must be of transpositional threshold type (see Theorems 5 and 6 in Section 7). This provides a more convenient computational technique to deal with practical inventory rationing problems.

4. This paper also applies the sensitivity-based optimization to discuss the rationing
inventory system under a static rationing policy, in which numerical computation is developed effectively (see Section 8). Based on this, this paper uses some numerical examples to indicate that the optimal threshold type rationing policy is not the optimal dynamic rationing policy (see Section 9). This indicates that the optimal dynamic rationing policy really has a more complicated structure. See Theorems 5 and 6 and Remark 2 for more details.

The remainder of this paper is organized as follows. Section 2 provides a literature review. Section 3 gives model description for a rationing inventory system. Section 4 establishes a policy-based birth-death process, defines a more general reward function with respect to both states and policies of the birth-death process, and establishes an optimization problem to find the optimal dynamic rationing policy. Section 5 sets up a policy-based Poisson equation. Section 6 provides an explicit expression for the perturbation realization factor $G^{(d)}(i)$ by means of the policy-based Poisson equation, and discusses how the sign of the critical item $G^{(d)}(i) + b$ depends on the penalty cost $P$. Section 7 discusses monotonicity and optimality of the long-run average profit of the rationing inventory system from three different areas of the penalty cost, and determines the optimal dynamic rationing policy. Section 8 applies the sensitivity-based optimization to analyze the rationing inventory under a threshold type rationing policy. Section 9 uses numerical examples to study some useful relations between the optimal dynamic rationing policy and the optimal threshold type rationing policy, and provides some examples to show that the optimal threshold type rationing policy is not the optimal dynamic rationing policy. Finally, some concluding remarks are given in Section 10. In addition, an appendix is used to discuss the general solution to the Poisson equation, and crucially, finding the two free constant structure of this general solution.

2 Literature Review

Our current research is related to four literature streams of inventory rationing across multiple demand classes. The first is the research on a critical rationing level and its MDP proof. The second and third are on a static rationing policy and a dynamic rationing policy, respectively. The fourth is on a simple introduction to the sensitivity-based optimization with applications to queues and networks.
Inventory rationing across multiple demand classes was first analyzed by Veinott Jr [80] in the context of inventory control theory. From then on, many authors have discussed the inventory rationing problems. Readers may refer to a book by Möllering [60]; survey papers by Kleijn and Dekker [50] and Li et al. [55]; and a research classification by Teunter and Haneveld [75], Möllering and Thonemann [61], Van Foreest and Wijngaard [78] and Alfieri et al. [1].

(a) A critical rationing level and its MDP proof

In a rationing inventory system, from observing two demand classes with different priorities, a critical rationing level may be imagined from early research and practical experience. Fortunately, Veinott Jr [80] first proposed such a critical level while Topkis [76] proved that the critical rationing level really exists and it is an optimal policy. Further, similar results to that in Topkis [76] were developed for two demand classes by Evans [30] and Kaplan [48].

It may be a most basic problem how to mathematically prove whether a rationing inventory system exists such a critical rationing level. On this research line, Ha [36] made a breakthrough by applying the MDPs to analysis of a production and inventory system with exponential production times, Poisson demand arrivals, lost sales and multiple demand classes. He proved that the optimal rationing policy is of critical rationing levels, and shown that not only do the optimal critical rationing levels exist, but also they are monotone and stationary. Therefore, the optimal rationing policy can be characterized as a monotone constant sequence of critical rationing levels corresponding to the multiple demand classes.

Since the seminal work of Ha [36], it has been interesting to extend and generalize how to apply the MDPs to deal with rationing inventory systems. Important examples include the Erlang production times by Ha [38] and Gayon et al. [34]; the backorders with two demand classes by Ha [37] and with multiple demand classes by de Véricourt [16,17]; the parallel production channels by Bulut and Fadıloğlu [6]; the batch ordering by Huang and Iravani [42], and the batch production by Pang et al. [66]; the utilization of information by Gayon et al. [33] and ElHafsi et al. [20]; an assemble-to-order production system by Benjaafar and ElHafsi [3], ElHafsi [21], ElHafsi et al. [25,27], Benjaafar et al. [4] and Nadar et al. [63]; a two-stage tandem production system by Xu [89], supply chain by Huang and Iravani [41] and van Wijk et al. [79]; periodic review by Frank et al. [32] and Chen et al. [12]; dynamic price by Ding et al. [21,22], Schulte and Pibernik [71] and so forth.
In the inventory rationing literature, there exist two kinds of rationing policies: The static rationing policy, and the dynamic rationing policy. Note that the dynamic rationing policy allows a threshold rationing level to be able to change in time, which depends on the number and ages of outstanding orders. On the other hand, if there exist multiple replenishment opportunities, then the ordering policies are taken as different types: Periodic review and continuous review. Therefore, our literature analysis for inventory rationing will focus on four different classes through combining the rationing policy (static vs. dynamic) with the inventory review (continuous vs. periodic): Static-continuous, static-periodic, dynamic-continuous, and dynamic-periodic.

(b) The static rationing policy (periodic vs. continuous)

The periodic review: Veinott Jr [80] is the first to introduce an inventory rationing across different demand classes and to propose a critical rationing level (i.e., the static rationing policy) in a periodic review inventory system with backorders. Following the basic results given in Veinott Jr [80], subsequent research further investigated the periodic review inventory system with multiple demand classes, for example, the \((s,S)\) policy by Cohen et al. [15] and Tempelmeier [74]; the \((S−1,S)\) policy by de Véricourt [17], Ha [36,37]; the lost sales by Dekkeret et al. [18]; the backorders by Møllering and Thonemann [61]; and the anticipated critical levels by Wang et al. [83].

The continuous review: Nahmias and Demmy [64] is the first to propose and develop a constant critical level \((Q,r,C)\) policy in a continuous review inventory model with two demand classes, where \(Q\) is the fixed batch size, \(r\) is the reorder point and \(C = (C_1,C_2,\ldots,C_{n−1})\) is a set of critical rationing levels for \(n\) demand classes. From that time on, some authors have discussed the constant critical level \((Q,r,C)\) policy in continuous review inventory systems. Readers may refer to recent publications for details, among which are Melchiors et al. [59], Dekkeret et al. [19], Deshpande et al. [20], Isotupa [45], Arslan et al. [2], Möllering and Thonemann [61,62] and Escalona et al. [28,29]. In addition, the \((S−1,S,C)\) inventory system was discussed by Dekkeret et al. [18], Kranenburg and van Houtum [54] and so on.

(c) The dynamic rationing policy (continuous vs. periodic)

In general, the static rationing policy is possible to miss some chance to further improve system performance, while the dynamic rationing decision should reflect better various continuously updated information so that system performance can be dynamically improved. Deshpande et al. [20] indicated that the optimal dynamic rationing policy may
significantly reduce the inventory cost compared with the static rationing policy.

The continuous review: Topkis [76] is the first to analyze the dynamic rationing policy, and he indicated that the optimal rationing policy is a dynamic policy. Evans [30] and Kaplan [48] obtained similar results to that in Topkis [76] for two demand classes. Melchiors [58] considered a dynamic rationing policy in a \((s, Q)\) inventory system with a key assumption that there was at most one outstanding order. Teunter and Haneveld [75] developed a time continuous approach to determine the dynamic rationing policy for two Poisson demand classes, analyzed the marginal cost to determine the optimal remaining time for each rationing level, and expressed the optimal threshold policy through a schematic diagram or a lookup table. Fadıloğlu and Bulut [31] proposed a dynamic rationing policy: Rationing with Exponential Replenishment Flow (RERF), for continuous review inventory systems with either backorders or lost sales. Wang et al. [82] developed a dynamic threshold mechanism to allocate backorders when multiple outstanding orders for different demand classes exist for the \((Q, R)\) inventory system.

The periodic review: For the dynamic rationing policy in a periodic review inventory system, readers may refer to, such as two demand classes by Sobel and Zhang [72], Frank et al. [32] and Tan et al. [73]; dynamic critical levels and lost sales by Haynsworth and Price [39]; multiple demand classes by Hung and Hsiao [44]; two backorder classes by Chew et al. [14]; general demand processes by Hung et al. [43]; mixed backorders and lost sales by Wang and Tang [81]; uncertain demand and production rates by Turgay et al. [77]; and incremental upgrading demands by You [90].

(d) The sensitivity-based optimization with applications to queueing systems

In the early 1980s, Ho and Cao [40] proposed and developed the infinitesimal perturbation method for discrete event dynamic systems (DEDSs), which is a new research direction for online simulation optimization of DEDSs. Further excellent books include Cao [7], Glasserman [35] and Cassandras and Lafortune [11].

Cao et al. [10] and Cao and Chen [9] published a seminal work that transforms the infinitesimal perturbation, together with MDPs and reinforcement learning, into the so-called sensitivity-based optimization. On this research line, the excellent book by Cao [8] summarized the main results in the study of sensitivity-based optimization. Li and Liu [54] and Chapter 11 in Li [53] further extended and generalized the sensitivity-based optimization to a more general framework of perturbed Markov processes. In addition,
the sensitivity-based optimization can be effectively developed by means of the matrix-analytic method by Neuts [65], Latouche and Ramaswami [52], and the RG-factorizations of block-structured Markov processes by Li [53]. We would like to refer readers to Ma et al. [56, 57] for some detailed discussion.

So far some research has applied the sensitivity-based optimization to analyze the MDPs of queues and networks, e.g., see Xia and Cao [84], Xia and Shihada [87], Xia and Jia [86], Xia et al. [85, 88] and Ma et al. [56,57].

3 Model Description

In this section, we describe an inventory rationing across two priority demand classes with a single class of products to stock. Also, we provide system structure, operational mode and mathematical notations.

**An inventory system:** The inventory system has the maximal capacity \(N\) to stock a single type of products, where each product requires a holding cost \(C_1\) per unit time. There are two classes of demands to order the products, in which the demands of Class 1 have a higher priority than that of Class 2, so that demands of Class 1 can always be satisfied in any non-zero inventory; while demands of Class 2 may be either satisfied or refused based on the inventory level of the products.

**An arrival process:** The arrivals of products at the inventory system are a Poisson process with arrival rate \(\lambda\), where the price of per product is \(C_3\). If the inventory system is full by the products, then any new arriving product has to be lost, and also the inventory system will have a lost sale cost \(C_4\) per lost product.

**The service processes:** The service times provided by the inventory system to satisfy demands of Classes 1 and 2 are i.i.d. and exponential with service rates \(\mu_1\) and \(\mu_2\), respectively. The service disciplines for the two classes of demands are all First Come First Serve (FCFS). We assume that the inventory system can obtain a fixed service reward (i.e. service price) \(R\) from each satisfied demand of Class 1 or 2.

**A more general rationing rule:** For the inventory rationing across two priority demand classes, each demand of Class 1 is always satisfied in any non-zero inventory; while for the demands of Class 2, we need to consider three different cases as follows:

**Case one:** The inventory level is zero. In this case, there is no product in the inventory system, thus any new arriving demand has to be rejected immediately. This leads to
opportunity costs \( C_{2,1} \) and \( C_{2,2} \) per unit time for each lost demands of Classes 1 and 2, respectively. We assume that \( C_{2,1} > C_{2,2} \), which is used to guarantee a priority serving the demands of Class 1 to those of Class 2.

*Case two: The inventory level is low.* In this case, the number of products in the inventory system is not more than a threshold \( K \), where the threshold is subjective with some experience. The inventory system will not provide any product to demands of Class 2 if the number of products in the inventory is not more than \( K \), this ensures that the demands of Class 1 can be satisfied from the non-zero inventory because the demands of Class 1 have a higher priority to receive the products than those of Class 2.

On the contrast, if such a service priority is violated (i.e., the demands of Class 2 are satisfied from a low level inventory), then the inventory system must pay a penalty cost \( P \) per product to any demand of Class 1. Note that the penalty cost \( P \) measures different priority levels to provide the products to the two classes of demands.

*Case three: The inventory level is high.* In this case, the number of products in the system is more than the threshold \( K \). Thus each demand of Classes 1 and 2 can be satisfied by many enough products in the inventory system.

**Independence:** We assume that all the random variables defined above are independent of each other.

Finally, for convenience of readers, Figure 1 provides a simple physical structure to understand the rationing inventory system. Also a rationing rule and some mathematical notations are depicted here.

![Figure 1: An inventory rationing across two demand classes](image)

\[ I(t): \text{The total number of products in the inventory system at time } t \]

In addition, we use Table 1 to summarize some notations. This may be helpful for readers’ understanding in our later study.
4 Optimization Model Formulation

In this section, for the rationing inventory system, we first establish a continuous-time policy-based Markov process, and show that its infinitesimal generator has the simple structure of a birth-death process with finite states. Then we define a more general reward (cost) function with respect to both states and policies of the policy-based birth-death process. Note that this section will be necessary for applying an MDP to find the optimal dynamic rationing policy of the inventory system.

In the rationing inventory system, there are two different classes of demands and a single type of products to stock. To study such a system, we first need to define both ‘states’ and ‘policies’ to express stochastic dynamics of this inventory system.

Let \( I(t) \) be the number of products in the inventory system at time \( t \). Then it is regarded as the state of this system at time \( t \). Obviously, all the cases of State \( I(t) \) form a state space as follows:

\[
\Omega = \{0, 1, 2, \ldots, N\}.
\]

Also, we regard a specific State \( i \in \Omega \) as an inventory level of this system.

Different from the states, the policies are defined with a little bit complication. Let \( d_i \) be a rationing policy related to State \( i \in \Omega \), and it expresses whether or not the inventory system prefers to supply some products to the demands of Class 2 under a basic condition that when the inventory level is not more than the threshold \( K \) for \( 0 < K \leq N \), that is, \[
d_i = \begin{cases} 
0, & i = 0, \\
0,1, & i = 1,2,\ldots,K, \\
1, & i = K+1,K+2,\ldots,N,
\end{cases}
\]
where \( d_i = 0 \) and \( 1 \) represent that the inventory system rejects and satisfies the demands of Class 2, respectively. Obviously, not only does the dynamic rationing policy depend on State \( i \in \Omega \), but also it is controlled by the threshold \( K \). Note that the positive integer \( K \) is determined by means of some subjective consideration and real experience. If \( K = N \), then \( d_i \in \{0, 1\} \) for \( 1 \leq i \leq N \). This indicates that the inventory system is completely dynamic for the rationing policy.

Corresponding to each state in \( \Omega \), we define a time-homogeneous rationing policy as

\[
d = (d_0; d_1, d_2, \ldots, d_K; d_{K+1}, d_{K+2}, \ldots, d_N).
\]

It follows from (1) that

\[
d = (0; d_1, d_2, \ldots, d_K; 1, 1, \ldots, 1).
\] (2)

Thus the rationing policy vector \( d \) depends on the rationing policy \( d_i \) related to State \( i \) for \( 1 \leq i \leq K \). Let all the possible policy vectors given in (2) compose a policy space as follows:

\[
D = \{d : d = (0; d_1, d_2, \ldots, d_K; 1, 1, \ldots, 1), d_i \in \{0, 1\}, 1 \leq i \leq K\}.
\]

**Remark 1** In general, the threshold \( K \) is subjective and is designed by some experience of the inventory manager. If \( K \geq N \), then the rationing policy is given by

\[
d = (0; d_1, d_2, \ldots, d_N).
\]

Thus our \( K \)-based rationing policy is more general than those in the literature.

Let \( I^{(d)}(t) \) be the state of the rationing inventory system at time \( t \) under a rationing policy \( d \in D \). Then \( \{I^{(d)}(t) : t \geq 0\} \) is a continuous-time policy-based Markov process on the state space \( \Omega \) whose state transition relations are depicted in Figure 2.

![Figure 2: State transition relations of the policy-based Markov process](image-url)
It is clear to see from Figure 2 that \( \{I^{(d)}(t) : t \geq 0\} \) is a policy-based birth-death process. Based on this, the infinitesimal generator of the policy-based birth-death process \( \{I^{(d)}(t) : t \geq 0\} \) is given by

\[
B^{(d)} = \begin{pmatrix}
-\lambda & \lambda & & & \\
\vdots & \ddots & \ddots & \ddots & \\
v(d_K) & -[\lambda + v(d_1)] & \lambda & & \\
v(1) & -[\lambda + v(1)] & \lambda & & \\
v(1) & \ldots & \ldots & \ldots & \lambda
\end{pmatrix},
\]

where \( v(d_i) = \mu_1 + d_i \mu_2 \) for \( i = 1, 2, \ldots, K \), and \( v(1) = \mu_1 + \mu_2 \). It is clear that \( v(d_i) > 0 \) for \( i = 1, 2, \ldots, K \). Thus the policy-based birth-death process \( B^{(d)} \) is irreducible, aperiodic and positive recurrent for any given rationing policy \( d \in D \). In this case, we write the stationary probability vector of the policy-based birth-death process \( \{I^{(d)}(t) : t \geq 0\} \) as

\[
\pi^{(d)} = \left( \pi^{(d)}(0); \pi^{(d)}(1), \ldots, \pi^{(d)}(K); \pi^{(d)}(K + 1), \ldots, \pi^{(d)}(N) \right).
\]

Obviously, the stationary probability vector \( \pi^{(d)} \) is the unique solution to the system of linear equations: \( \pi^{(d)} B^{(d)} = 0 \) and \( \pi^{(d)} e = 1 \), where \( e \) is a column vector of ones with a suitable dimension. We write

\[
\xi_0 = 1, \quad i = 0,
\]

\[
\xi^{(d)}_i = \begin{cases}
\frac{\lambda^i}{\prod_{j=1}^{i} v(d_j)}, & i = 1, 2, \ldots, K, \\
\frac{\lambda^i}{(\mu_1 + \mu_2)^K \prod_{j=1}^{i} v(d_j)}, & i = K + 1, K + 2, \ldots, N,
\end{cases}
\]

and

\[
h^{(d)} = 1 + \sum_{i=1}^{N} \xi^{(d)}_i.
\]

It follows from Subsection 1.1.4 of Chapter 1 in Li [53] that

\[
\pi^{(d)}(i) = \begin{cases}
\frac{1}{h^{(d)}}, & i = 0 \\
\frac{1}{h^{(d)}}, & i = 1, 2, \ldots, N.
\end{cases}
\]
By using the policy-based birth-death process $B^{(d)}$, now we define a more general reward (cost) function in the rationing inventory system. It is seen from Table 1 that a reward function with respect to both states and policies is defined as a profit rate (i.e. the total revenues minus the total costs per unit time). By observing the impact of the dynamic rationing policy vector $d$ on the profit rate, the reward function at State $i$ under the rationing policy $d$ is given by:

$$f^{(d)}(i) = R(\mu_1 1_{\{i>0\}} + \mu_2 d_i) - C_1 i - C_2,1\mu_1 1_{\{i=0\}} - C_2,2\mu_2 (1 - d_i)$$
$$- C_3 \lambda 1_{\{i<N\}} - C_4 \lambda 1_{\{i=N\}} - P \mu_2 d_i 1_{\{1 \leq i \leq K\}},$$

where $1_{\{\cdot\}}$ represents an indicator function whose value is 1 when the event occurs; otherwise it is zero; satisfying and rejecting the demands of Class 1 are given by $1_{\{i>0\}}$ and $1_{\{i=0\}}$, respectively; while the outside products enter or are lost from the inventory system according to $1_{\{i<N\}}$ and $1_{\{i=N\}}$, respectively; paying a penalty cost is under the condition: $1_{\{1 \leq i \leq K\}}$.

For convenience of readers, it may be useful and necessary to explain the reward (cost) function from four different cases as follows:

**Case (a):** For $i = 0$,

$$f(0) = -C_2,1\mu_1 - C_2,2\mu_2 - C_3 \lambda.$$  

Note that in Case (a), there is no product in the inventory system, thus it has to reject each demand of Classes 1 and 2.

**Case (b):** For $1 \leq i \leq K$,

$$f^{(d)}(i) = R(\mu_1 + \mu_2 d_i) - C_1 i - C_2,2\mu_2 (1 - d_i) - C_3 \lambda - P \mu_2 d_i.$$  

In Case (b), since the inventory level is low for $1 \leq i \leq K$, the penalty cost is necessary in supplying the products to the demands of Class 2. Note that the rationing policy $d_i$ will play a key role in finding the optimal dynamic rationing policy in order to get the maximal long-run average profit of the rationing inventory system.

Different from Cases (a) and (b), in the following Cases (c) and (d), the inventory level is high for $K + 1 \leq i \leq N$, thus it can synchronously satisfy the demands of Classes 1 and 2.

**Case (c):** For $K + 1 \leq i \leq N - 1$,

$$f(i) = R(\mu_1 + \mu_2) - C_1 i - C_3 \lambda.$$  

Note that in Case (c), there is no product in the inventory system, thus it has to reject each demand of Classes 1 and 2.
**Case (d):** For \( i = N \),

\[
f(N) = R(\mu_1 + \mu_2) - C_1 N - C_4 \lambda. \quad (11)
\]

Note that \( C_3 \) is the price of per product paid by the inventory system to the product supplier; while \( C_4 \) is the lost sale cost per product rejected into the inventory system.

Based on the above analysis, we define an \((N + 1)\)-dimensional column vector composed of the elements \( f(0) \), \( f^{(d)}(i) \) for \( 1 \leq i \leq K \) and \( f(j) \) for \( K + 1 \leq j \leq N \) as follows:

\[
f^{(d)} = \left( f(0) ; f^{(d)}(1) ; f^{(d)}(2) ; \ldots , f^{(d)}(K) ; f(K + 1) , f(K + 2) , \ldots , f(N) \right)^T. \quad (12)
\]

In the remainder of this section, the long-run average profit of the rationing inventory system (or the continuous-time policy-based birth-death process \( \{I^{(d)}(t) : t \geq 0\} \)) under a dynamic rationing policy \( d \) is given by

\[
\eta^d = \lim_{T \to \infty} E \left\{ \frac{1}{T} \int_0^T f^{(d)}(I^{(d)}(t)) \, dt \right\} = \pi^{(d)} f^{(d)}, \quad (13)
\]

where \( \pi^{(d)} \) and \( f^{(d)} \) are given by (4) and (12), respectively. Also see Section 1 of Chapter 9 of Li [53] and Li and Cao [54] for more details.

We observed that when the inventory level is low, supplying the products to the demands of Class 2 leads to that both the total revenues and the total costs increase synchronously, vice versa. Thus there is a tradeoff between the total revenues and the total costs. This motivates us to find an optimal dynamic rationing policy such that the inventory system has the maximal profit. Therefore, our objective is to find an optimal dynamic rationing policy \( d^* \) such that the long-run average profit \( \eta^d \) is maximal, that is,

\[
d^* = \arg \max_{d \in D} \left\{ \eta^d \right\}. \quad (14)
\]

In fact, it is more difficult and challenging not only to analyze some interesting structure properties of the rationing policies \( d \) in order to find the optimal one, but also to provide an effective algorithm for computing the optimal dynamic rationing policy \( d^* \).

In the remainder of this paper, we will introduce and apply the sensitivity-based optimization to study the optimal dynamic rationing policy problem.
5 A Poisson Equation

In this section, for the rationing inventory system, we set up a Poisson equation which is derived by means of the law of total probability and some analysis on stopping times of the policy-based birth-death process \{I^{(d)}(t), t \geq 0\}. It is worth noting that the Poisson equation provides a useful relation between the sensitivity-based optimization and the MDPs.

For \(d \in D\), it follows from Chapter 2 in Cao [8] that for the policy-based continuous-time birth-death process \{I^{(d)}(t), t \geq 0\}, we define the performance potential as

\[
g^{(d)}(i) = E \{ \int_0^{+\infty} \left[ f^{(d)}(I^{(d)}(t)) - \eta^d \right] dt \mid I^{(d)}(0) = i \},
\]

where \(\eta^d\) is defined in (13). It is seen from Cao [8] that for any rationing policy \(d \in D\), \(g^{(d)}(i)\) quantifies the contribution of the initial State \(i\) to the long-run average profit of the inventory system. Here, \(g^{(d)}(i)\) is also called the relative value function or the bias in the traditional MDP theory, see, e.g. Puterman [70]. We further define a column vector \(g^{(d)}\) with the elements \(g^{(d)}(i)\) for \(i \in \Omega\) as

\[
g^{(d)} = \left( g^{(d)}(0); g^{(d)}(1); \ldots; g^{(d)}(K); g^{(d)}(K + 1); \ldots; g^{(d)}(N) \right)^T.
\]

To compute the vector \(g^{(d)}\), we define the first departure time from State \(i\) as

\[\tau = \inf \left\{ t \geq 0 : I^{(d)}(t) \neq i \right\},\]

where \(I^{(d)}(0) = i\). Clearly, \(\tau\) is a stopping time of the policy-based birth-death process \(\{I^{(d)}(t) : t \geq 0\}\). Based on this, it is seen that if \(i = 0\), then it is seen from (3) that State 0 is a boundary state of the policy-based birth-death process \(B^{(d)}\), hence \(I^{(d)}(\tau) = 1\). Similarly, for each State \(i \in \Omega\), we get a basic relation as follows:

\[
I^{(d)}(\tau) = \begin{cases} 
1, & i = 0, \\
i - 1 \text{ or } i + 1, & i = 1, 2, \ldots, N - 1, \\
N - 1, & i = N.
\end{cases}
\]

Now, we derive a Poisson equation to compute the column vector \(g^{(d)}\) in terms of the stopping time \(\tau\) and the basic relation (17). By a similar computation to that in Li and Cao [51], our analysis for setting up the poisson equation is decomposed into four parts as follows:
**Part (a):** For $i = 0$, we have

$$g^{(d)}(0) = E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)}(t) \right) - \eta^d \right] dt \left| I^{(d)}(0) = 0 \right. \right\}$$

$$= E \left\{ \tau \left| I^{(d)}(0) = 0 \right. \right\} \left[ f(0) - \eta^d \right] + E \left\{ \int_\tau^{+\infty} \left[ f^{(d)} \left( I^{(d)}(t) \right) - \eta^d \right] dt \left| I^{(d)}(\tau) \right. \right\}$$

$$= \frac{1}{\lambda} \left[ f(0) - \eta^d \right] + E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)}(t) \right) - \eta^d \right] dt \left| I^{(d)}(0) = 1 \right. \right\}$$

$$= \frac{1}{\lambda} \left[ f(0) - \eta^d \right] + g^{(d)}(1),$$

where for the policy-based birth-death process $\left\{ I^{(d)}(t) : t \geq 0 \right\}$, it is easy to see from Figure 2 that by using $I^{(d)}(t) = 0$ for $0 \leq t < \tau$,

$$\int_0^\tau \left[ f^{(d)} \left( I^{(d)}(t) \right) - \eta^d \right] dt = \tau \left[ f(0) - \eta^d \right],$$

$$E \left\{ \tau \left| I^{(d)}(0) = 0 \right. \right\} = \frac{1}{\lambda}.$$

We obtain

$$- \lambda g^{(d)}(0) + \lambda g^{(d)}(1) = \eta^d - f(0). \quad (18)$$

**Part (b):** For $i = 1, 2, \ldots, K$, it is easy to see from Figure 2 that

$$g^{(d)}(i) = E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)}(t) \right) - \eta^d \right] dt \left| I^{(d)}(0) = i \right. \right\}$$

$$= E \left\{ \tau \left| I^{(d)}(0) = i \right. \right\} \left[ f^{(d)}(i) - \eta^d \right] + E \left\{ \int_\tau^{+\infty} \left[ f^{(d)} \left( I^{(d)}(t) \right) - \eta^d \right] dt \left| I^{(d)}(\tau) \right. \right\}$$

$$= \frac{1}{v(d_i) + \lambda} \left[ f^{(d)}(i) - \eta^d \right]$$

$$+ \frac{\lambda}{v(d_i) + \lambda} E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)}(0) = i + 1 \right) \right] dt \left| I^{(d)}(0) = i + 1 \right. \right\}$$

$$+ \frac{\lambda}{v(d_i) + \lambda} E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)}(0) = i - 1 \right) \right] dt \left| I^{(d)}(0) = i - 1 \right. \right\}$$

$$= \frac{1}{v(d_i) + \lambda} \left[ f^{(d)}(i) - \eta^d \right] + \frac{\lambda}{v(d_i) + \lambda} g^{(d)}(i + 1) + \frac{v(d_i)}{v(d_i) + \lambda} g^{(d)}(i - 1),$$

where

$$E \left\{ \tau \left| I^{(d)}(0) = i \right. \right\} = \frac{1}{v(d_i) + \lambda}.$$

We obtain

$$v(d_i) g^{(d)}(i - 1) - \left[ v(d_i) + \lambda \right] g^{(d)}(i) + \lambda g^{(d)}(i + 1) = \eta^d - f^{(d)}(i). \quad (19)$$
Part (c): For \( i = K + 1, K + 2, \ldots, N - 1 \), by using Figure 2 we have

\[
g^{(d)} (i) = E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (0) = i \right. \right\}
\]

\[
= E \left\{ \tau \left| I^{(d)} (0) = i \right. \right\} \left[ f (i) - \eta^d \right] + E \left\{ \int_\tau^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (\tau) \right. \right\}
\]

\[
= \frac{1}{\mu_1 + \mu_2 + \lambda} \left[ f (i) - \eta^d \right] + \frac{\lambda}{\mu_1 + \mu_2 + \lambda} E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (0) = i + 1 \right. \right\}
\]

\[
+ \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \lambda} E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (0) = i - 1 \right. \right\}
\]

\[
= \frac{1}{\mu_1 + \mu_2 + \lambda} \left[ f (i) - \eta^d \right] + \frac{\lambda}{\mu_1 + \mu_2 + \lambda} g^{(d)} (i + 1) + \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \lambda} g^{(d)} (i - 1),
\]

where

\[
E \left\{ \tau \left| I^{(d)} (t) = i \right. \right\} = \frac{1}{\mu_1 + \mu_2 + \lambda}.
\]

We obtain

\[
(\mu_1 + \mu_2) g^{(d)} (i - 1) - (\mu_1 + \mu_2 + \lambda) g^{(d)} (i) + \lambda g^{(d)} (i + 1) = \eta^d - f (i). \quad (20)
\]

Part (d): For \( i = N \), by using Figure 2 we have

\[
g^{(d)} (N) = E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (0) = N \right. \right\}
\]

\[
= E \left\{ \tau \left| I^{(d)} (0) = N \right. \right\} \left[ f (N) - \eta^d \right] + E \left\{ \int_\tau^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (\tau) \right. \right\}
\]

\[
= \frac{1}{\mu_1 + \mu_2} \left[ f (N) - \eta^d \right] + E \left\{ \int_0^{+\infty} \left[ f^{(d)} \left( I^{(d)} (t) \right) - \eta^d \right] dt \left| I^{(d)} (0) = N - 1 \right. \right\}
\]

\[
= \frac{1}{\mu_1 + \mu_2} \left[ f (N) - \eta^d \right] + g^{(d)} (N - 1),
\]

where

\[
E \left\{ \tau \left| I^{(d)} (t) = N \right. \right\} = \frac{1}{\mu_1 + \mu_2}.
\]

We obtain

\[
(\mu_1 + \mu_2) g^{(d)} (N - 1) - (\mu_1 + \mu_2 + \lambda) g^{(d)} (N) = \eta^d - f (N). \quad (21)
\]

Thus it follows from (18), (19), (20) and (21) that

\[
-B^{(d)} g^{(d)} = f^{(d)} - \eta^d e,
\]

where \( B^{(d)}, f^{(d)} \) and \( \eta^d \) are given in (3), (12) and (13), respectively.
Note that the solution to the poisson equation will play a key role in applying the sensitivity-based optimization to the study of rationing inventory systems, thus we use Appendix A to provide a detailed analysis for structure and expression of this solution.

6 Impact of the Penalty Cost

In this section, we define a perturbation realization factor of the policy-based birth-death process, and provide an explicit expression for the perturbation realization factor. Based on this, we analyze how the penalty cost has an impact on the perturbation realization factor. Note that the results given in this section will be utilized to establish the optimal dynamic rationing policy of the inventory system in the later sections.

We define a perturbation realization factor as

\[ G^{(d)} (i) \overset{\text{def}}{=} g^{(d)} (i - 1) - g^{(d)} (i), i = 1, 2, \ldots, N. \]

It is easy to see that \( G^{(d)} (i) \) quantifies the difference among two adjacent performance potentials \( g^{(d)} (i) \) and \( g^{(d)} (i - 1) \), and measures the long-run effect on the average profit of the inventory system when the system state is changed from \( i - 1 \) to \( i \). This also indicates the occurrence of a demand satisfied event. By using the Poisson equation given in Section 5, we can derive a new system of linear equations which is used to compute and express the perturbation realization factor \( G^{(d)} (i) \) for \( i = 1, 2, \ldots, N \).

For \( i = 1 \), it follows from (18) that

\[ -\lambda \left[ g^{(d)} (0) - g^{(d)} (1) \right] = -\lambda G^{(d)} (1), \]

this gives

\[ \lambda G^{(d)} (1) = f (0) - \eta^d. \] (23)

For \( i = 2, 3, \ldots, K \), it follows from (19) that

\[ v (d_i) \left[ g^{(d)} (i - 1) - g^{(d)} (i) \right] - \lambda \left[ g^{(d)} (i) - g^{(d)} (i + 1) \right] \]

\[ = v (d_i) G^{(d)} (i) - \lambda G^{(d)} (i + 1), \]

this gives

\[ \lambda G^{(d)} (i + 1) = v (d_i) G^{(d)} (i) + f^{(d)} (i) - \eta^d. \] (24)
For $i = K + 1, K + 2, \ldots, N - 1$, it follows from (20) that
\[
(\mu_1 + \mu_2) \left[ g^{(d)}(i - 1) - g^{(d)}(i) \right] - \lambda \left[ g^{(d)}(i) - g^{(d)}(i + 1) \right] = (\mu_1 + \mu_2) G^{(d)}(i) - \lambda G^{(d)}(i + 1),
\]
this gives
\[
\lambda G^{(d)}(i + 1) = (\mu_1 + \mu_2) G^{(d)}(i) + f(i) - \eta^d. \tag{25}
\]
For $i = N$, it follows from (21) that
\[
(\mu_1 + \mu_2) G^{(d)}(N) = \eta^d - f(N). \tag{26}
\]
Thus by using (23), (24), (25) and (26) we obtain a new system of linear equations satisfied by $G^{(d)}(i)$ as follows:
\[
\begin{cases}
\lambda G^{(d)}(1) = f(0) - \eta^d, & i = 1, \\
\lambda G^{(d)}(i + 1) = v(d_i) G^{(d)}(i) + f^{(d)}(i) - \eta^d, & i = 2, 3, \ldots, K, \\
\lambda G^{(d)}(i + 1) = (\mu_1 + \mu_2) G^{(d)}(i) + f(i) - \eta^d, & i = K + 1, K + 2, \ldots, N - 1, \\
(\mu_1 + \mu_2) G^{(d)}(N) = \eta^d - f(N), & i = N.
\end{cases} \tag{27}
\]

Fortunately, the following theorem provides an explicit expression for the perturbation realization factor $G^{(d)}(i)$ for $1 \leq i \leq N$, while the case with $K + 1 \leq i \leq N$ will not be useful in our later study.

**Theorem 1** For any dynamic rationing policy $d$, the perturbation realization factor $G^{(d)}(i)$ for $i = 1, 2, \ldots, N$ is given by

(a) for $1 \leq i \leq K$,
\[
G^{(d)}(i) = \lambda^{-i} \left[ f(0) - \eta^d \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ f^{(d)}(r) - \eta^d \right] \prod_{k=r+1}^{i-1} v(d_k); \tag{28}
\]
and (b) for $K + 1 \leq i \leq N$,
\[
G^{(d)}(i) = \lambda^{-i} \left[ f(0) - \eta^d \right] \prod_{k=1}^{K} v(d_k) [v(1)]^{i-K-1} + \sum_{r=1}^{K-1} \lambda^{r-K} \left[ f^{(d)}(r) - \eta^d \right] \prod_{k=r+1}^{K} v(d_k) + \sum_{r=K}^{i-1} \lambda^{r-i} \left[ f(r) - \eta^d \right] [v(1)]^{i-r-2}.
\]
Proof: We only prove (a), while the proof of (b) is similar.

It follows from (27) that
\[ G^{(d)}(1) = \frac{f(0) - \eta d}{\lambda}. \]

Similarly, we obtain
\[ G^{(d)}(i + 1) = \frac{v(d_i) G^{(d)}(i)}{\lambda} + \frac{f^{(d)}(i) - \eta d}{\lambda}, \quad i = 1, 2, \ldots, K. \]

By using (1.2.4) in Chapter 1 of Elaydi [23], for \( i = 1, 2, \ldots, K \), we can obtain the explicit expression of the perturbation realization factor as follows:
\[ G^{(d)}(i) = \lambda^{-i} \left[ f(0) - \eta d \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ f^{(d)}(r) - \eta d \right] \prod_{k=r+1}^{i-1} v(d_k). \]

This completes the proof. \( \blacksquare \)

Now, we express the perturbation realization factor \( G^{(d)}(i) \) by means of the penalty cost \( P \). To do this, we write that for \( i = 0 \),
\[ A_0 = 0, \quad B_0 = -C_{2,1}\mu_1 - C_{2,2}\mu_2 - C_3\lambda; \]
for \( i = 1, 2, \ldots, K \),
\[ A_i^{(d)} = \mu_2d_i, \quad B_i^{(d)} = R(\mu_1 + \mu_2d_i) - C_1i - C_{2,2}\mu_2(1 - d_i) - C_3\lambda; \]
for \( i = K + 1, K + 2, \ldots, N - 1 \),
\[ A_i = 0, \quad B_i = R(\mu_1 + \mu_2) - C_1i - C_3\lambda; \]
for \( i = N \),
\[ A_i = 0, \quad B_N = R(\mu_1 + \mu_2) - C_1N - C_4\lambda. \]

Then it follows from (8) to (11) that for \( i = 0 \),
\[ f(0) = B_0; \quad (29) \]
for \( i = 1, 2, \ldots, K \),
\[ f^{(d)}(i) = B_i^{(d)} - PA_i^{(d)}; \quad (30) \]
for \( i = K + 1, K + 2, \ldots, N \),
\[ f(i) = B_i. \quad (31) \]
It follows from (6) and (29) to (31) that

$$\eta^d = \pi^{(d)} f^{(d)}$$

$$= \pi^{(d)} (0) f (0) + \sum_{i=1}^{K} \pi^{(d)} (i) f^{(d)} (i) + \sum_{i=K+1}^{N} \pi^{(d)} (i) f (i)$$

$$= D^{(d)} - PF^{(d)},$$

where

$$D^{(d)} = \pi^{(d)} (0) B_0 + \sum_{i=1}^{K} \pi^{(d)} (i) B_i^{(d)} + \sum_{i=K+1}^{N} \pi^{(d)} (i) B_i,$$

and

$$F^{(d)} = \sum_{i=1}^{K} \pi^{(d)} (i) A_i^{(d)}.$$

When the inventory level is low, if the service priority is violated (i.e. the demands of Class 2 are served at a low inventory), then the inventory system has to pay the penalty cost $P$ for each product supplied to the demands of Class 2. Now, we study the influence of the penalty cost $P$ on the perturbation realization factor $G^{(d)} (i)$. From our later discussion in Section 7, it is easy to see that $G^{(d)} (i)$ plays a fundamental role in performance optimization of the inventory system, and crucially, the sign of $G^{(d)} (i) + b$ directly determines some selection of decision actions. To this end, we analyze how the penalty cost $P$ impacts on the sign of $G^{(d)} (i) + b$, where $b = R + C_{2,2} - P$.

By using (28), substitute (29), (30) and (31) into the linear equation $G^{(d)} (i) + b = 0$, we obtain that for $1 \leq i \leq K$,

$$P \left\{ 1 + \lambda^{-i} \left[ A_0 - F^{(d)} \right] \prod_{k=1}^{i-1} v (d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ A_r^{(d)} - F^{(d)} \right] \prod_{k=r+1}^{i-1} v (d_k) \right\}$$

$$= R + C_{2,2} + \lambda^{-i} \left[ B_0 - D^{(d)} \right] \prod_{k=1}^{i-1} v (d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ B_r^{(d)} - D^{(d)} \right] \prod_{k=r+1}^{i-1} v (d_k), \quad (32)$$

thus the unique solution of the penalty cost $P$ to Equation (32) is given by

$$P_i^{(d)} = \frac{R + C_{2,2} + \lambda^{-i} \left[ B_0 - D^{(d)} \right] \prod_{k=1}^{i-1} v (d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ B_r^{(d)} - D^{(d)} \right] \prod_{k=r+1}^{i-1} v (d_k)}{1 + \lambda^{-i} \left[ A_0 - F^{(d)} \right] \prod_{k=1}^{i-1} v (d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ A_r^{(d)} - F^{(d)} \right] \prod_{k=r+1}^{i-1} v (d_k)}. \quad (33)$$
It’s easy to see that if $0 \leq P \leq \Phi^{(d)}_i$, then $G^{(d)}(i) + b \geq 0$; and if $P \geq \Phi^{(d)}_i$, then $G^{(d)}(i) + b \leq 0$.

To understand the solution $\Phi^{(d)}_i$ for $1 \leq i \leq K$, we provide a numerical example in Table 2 as follows. At the same time, we observe three rationing policies:

$$d_1 = (0; 1, 1, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1),$$

$$d_2 = (0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 1, 1, 1, 1, 1),$$

$$d_3 = (0; 0, 0, 0, 0, 1, 1, 1, 1; 1, 1, 1, 1, 1).$$

Table 2: A numerical analysis for the solution depending on the rationing policies

| $\Phi^{(d)}_i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|------------|----|----|----|----|----|----|----|----|----|----|----|
| $d_1$      | 11 | 11 | 38.3 | 20.7 | 21.4 | 21.3 | 20.9 | 20.7 | 20.4 | 20.1 | 19.8 |
| $d_2$      | 11 | 11 | -6.5 | 14.3 | 88.4 | 384.9 | 1.5e3 | 5.6e3 | 2.1e4 | 7.5e4 | 2.6e5 |
| $d_3$      | 11 | 11 | -7.1 | 23.0 | -181.4 | -80.4 | -68.6 | 15.0 | 13.7 | 13.3 | 13.1 |

In the rationing inventory system, we define two critical values related to the penalty cost $P$ as

$$P_H = \max_{d \in D} \left\{ 0, \Phi^{(d)}_1, \Phi^{(d)}_2, \ldots, \Phi^{(d)}_K \right\},$$

and

$$P_L = \min_{d \in D} \left\{ \Phi^{(d)}_1, \Phi^{(d)}_2, \ldots, \Phi^{(d)}_K \right\}.$$  

Note that it is possible to have $P_L < 0$ by using Table 2.

The following proposition uses the two critical values $P_H$ and $P_L$, related to the penalty cost $P$, to provide some conditions for how to determine the sign of $G^{(d)}(i) + b$. This is very necessary for us to establish a sensitivity-based optimization framework to analyze the rationing inventory system in our later study.

**Proposition 1**  
(1) If $P \geq P_H$, then for any rationing policy $d \in D$ and for each $i = 1, 2, \ldots, K$, we have

$$G^{(d)}(i) + b \leq 0.$$  

(36)
(2) If $0 \leq P \leq P_L$, then for any rationing policy $d \in D$ and for each $i = 1, 2, \ldots, K$, we have
\[ G^{(d)}(i) + b \geq 0. \]  

Proof: (1) For any rationing policy $d \in D$, if $P \geq P_H$, then it follows from (34) that for each $i = 1, 2, \ldots, K$,
\[ P \geq \Phi_i^{(d)}, \]
which clearly makes that $G^{(d)}(i) + b \leq 0$.

(2) For any rationing policy $d \in D$, if $0 \leq P \leq P_L$, then from (35) we get that for each $i = 1, 2, \ldots, K$,
\[ 0 \leq P \leq \Phi_i^{(d)}, \]
this gives that $G^{(d)}(i) + b \geq 0$. This completes the proof.

Obviously, when $P_L < P < P_H$, it is not easy to determine the sign of $G^{(d)}(i) + b$ for a given number $i = 1, 2, \ldots, K$. For this case, we need to use the other mathematical methods in our later study.

7 Monotonicity and Optimality

In this section, we use the solution of the Poisson equation to derive a useful performance difference equation, which is used to establish some key comparison between any two different rationing policies. Based on this, we can give the optimal dynamic rationing policy under three different areas of the penalty cost, and also compute the maximal long-run average profit of this system corresponding to each of the three optimal dynamic rationing policies.

For any given rationing policy $d \in D$, the long-run average profit of the inventory system is given by
\[ \eta^d = \pi^{(d)} f^{(d)}. \]

Also, the Poisson equation is written as
\[ B^{(d)} g^{(d)} = \eta^d e - f^{(d)}. \]

It is seen from (3) that the dynamic rationing policy $d$ directly affects not only the elements of the infinitesimal generator $B^{(d)}$ but also the reward function $f^{(d)}$. Based on
this, if the dynamic rationing policy changes from $d$ to $d'$, then the infinitesimal generator $B^{(d)}$ and the reward function $f^{(d)}$ can have their corresponding changes $B^{(d')}$ and $f^{(d')}$, respectively.

The following lemma provides a useful equation for the difference $\eta^{d'} - \eta^d$ between the two long-run average performances $\eta^{d'}$ and $\eta^d$ corresponding any two rationing policies $d, d' \in D$. Here, we only restate the difference $\eta^{d'} - \eta^d$ without proof; while readers may refer to Cao [8] or Ma et al. [57] for more details.

**Lemma 1** For any two dynamic rationing policies $d, d' \in D$, we have

$$\eta^{d'} - \eta^d = \pi^{(d')} \left[ \left( B^{(d')} - B^{(d)} \right) g^{(d)} + \left( f^{(d')} - f^{(d)} \right) \right].$$  \hspace{1cm} (38)

Now, we describe the first role played by the performance difference equation (38), in which we set up a partial order relation in the policy set $D$ so that the optimal dynamic rationing policy can be found in the finite set $D$ by means of finite comparisons. Based on the performance difference $\eta^{d'} - \eta^d$ corresponding to any two dynamic rationing policies $d, d' \in D$, we can set up a partial order relation in the policy set $D$ as follows:

We write that $d' \succ d$ if $\eta^{d'} > \eta^d$; $d \approx d'$ if $\eta^{d'} = \eta^d$; and $d' \prec d$ if $\eta^{d'} < \eta^d$. Also, we write that $d' \succeq d$ if $\eta^{d'} \geq \eta^d$; and $d' \preceq d$ if $\eta^{d'} \leq \eta^d$. By using this partial order relation, our research target is to find an optimal dynamic rationing policy $d^* \in D$ such that $d^* \succeq d$ for any dynamic rationing policy $d \in D$, or

$$d^* = \arg \max_{d \in D} \left\{ \eta^d \right\}.$$  

Note that the policy set $D$ and the state set $\Omega$ are all finite, thus an enumeration method is feasible for finding the optimal dynamic rationing policy $d^*$ in the policy set $D$.

To find the optimal rationing policy $d^*$, we consider two dynamic rationing policies with an interrelated structure as follows:

$$d = \left( 0; d_1, d_2, \ldots, d_{i-1}, \underline{d_i}, d_{i+1}, \ldots, d_K; 1, 1, \ldots, 1 \right),$$  

$$d' = \left( 0; d_1, d_2, \ldots, d_{i-1}, \underline{d'_i}, d_{i+1}, \ldots, d_K; 1, 1, \ldots, 1 \right),$$
where $d'_i, d_i \in \{0, 1\}$. It is easy to check from (3) that

$$B^{(d')} - B^{(d)} = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(d'_i - d_i) \mu_2 - (d'_i - d_i) \mu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}. \quad (39)$$

On the other hand, from the reward function given in (9), it is seen that for $i = 1, 2, \ldots, K$, and $d_i, d'_i \in \{0, 1\}$,

$$f^{(d)}(i) = (R + C_{2, 2} - P) \mu_2 d_i + R \mu_1 - C_1 i - C_{2, 2} \mu_2 - C_3 \lambda$$
and

$$f^{(d')}(i) = (R + C_{2, 2} - P) \mu_2 d'_i + R \mu_1 - C_1 i - C_{2, 2} \mu_2 - C_3 \lambda.$$ 

Hence, we have

$$f^{(d')} - f^{(d)} = (0, 0, \ldots, 0, b \mu_2 (d'_i - d_i), 0, \ldots, 0)^T. \quad (40)$$

By using (39) and (40), in what follows our analysis for comparing any two dynamic rationing policies is classified as three different cases related to the penalty cost. These are described in the following three subsections.

7.1 The penalty cost $P \geq P_H$

In this subsection, we give a comparison between any two different dynamic rationing policies, this leads to finding the optimal dynamic rationing policy. Based on this, we can compute the maximal long-run average profit of this system.

For each $i = 1, 2, \ldots, K$, we consider two dynamic rationing policies with an interrelated structure as follows:

$$d = (0; d_1, d_2, \ldots, d_{i-1}, \underline{d_i}, d_{i+1}, \ldots, d_K; 1, 1, \ldots, 1),$$
$$d' = (0; d_1, d_2, \ldots, d_{i-1}, \underline{d'_i}, d_{i+1}, \ldots, d_K; 1, 1, \ldots, 1),$$

where $d'_i = 1 > d_i = 0$ for each $i = 1, 2, \ldots, K$. 

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Theorem 2 If $P \geq P_H$, then for any two dynamic rationing policies $d, d' \in D$ with $d'_i = 1 > d_i = 0$ for each $i = 1, 2, \ldots, K$,

$$\eta^{d'} \leq \eta^d.$$ 

Thus the optimal dynamic rationing policy is given by

$$d^* = (0; 0, 0, \ldots, 0; 1, 1, \ldots, 1),$$

that is, if the penalty cost is higher with $P \geq P_H$, then the rationing inventory system can not supply any product to the demands of Class 2.

**Proof:** By using Lemma 1, it follows from (39) and (40) that for each $i = 1, 2, \ldots, K$,

$$\eta^{d'} - \eta^d = \pi(d') \left[ \left( B(d') - B(d) \right) g(d) + \left( f(d') - f(d) \right) \right]$$

$$= \mu_2 \pi(d') (i) \left( d'_i - d_i \right) \left[ g(d) (i - 1) - g(d) (i) + b \right]$$

$$= \mu_2 \pi(d') (i) \left[ G(d) (i) + b \right], \quad (41)$$

where $d'_i - d_i = 1$ and $G(d) (i) = g(d) (i - 1) - g(d) (i)$. If $P \geq P_H$, then it is seen from Proposition 1 that $G(d) (i) + b \leq 0$. Thus for any two dynamic rationing policies $d, d' \in D$, with $d'_i = 1 > d_i = 0$ for each $i = 1, 2, \ldots, K$, we obtain

$$\eta^{d'} \leq \eta^d.$$ 

This gives

$$d^* = (0; 0, 0, \ldots, 0; 1, 1, \ldots, 1).$$

This completes the proof. $\blacksquare$

When $P \geq P_H$, the optimal dynamic rationing policy is given by

$$d^* = (0; 0, 0, \ldots, 0; 1, 1, \ldots, 1),$$

thus it follows from (5) that

$$\xi_0 = 1, \quad i = 0,$$

$$\xi_i(d^*) = \begin{cases} 
\alpha^i, & i = 1, 2, \ldots, K, \\
\left( \frac{\alpha}{\beta} \right)^K \beta^i, & i = K + 1, K + 2, \ldots, N,
\end{cases}$$

and

$$h(d^*) = 1 + \sum_{i=1}^{N} \xi_i(d^*) = 1 + \frac{\alpha (1 - \alpha^K)}{1 - \alpha} + \left( \frac{\alpha}{\beta} \right)^K \frac{\beta^{K+1} (1 - \beta^{N-K})}{1 - \beta}.$$
where $\alpha = \lambda / \mu_1$ and $\beta = \lambda / (\mu_1 + \mu_2)$. It follows from (6) that

$$
\pi^{(d^*)}(i) = \begin{cases} 
\frac{1}{h(d^*)}, & i = 0, \\
\frac{1}{h(d^*)} \xi_i^{(d^*)}, & i = 1, 2, \ldots, N. 
\end{cases}
$$

At the same time, it follows from (8) to (11) that

$$
f(0) = -C_{2,1}\mu_1 - C_{2,2}\mu_2 - C_3\lambda, \quad i = 0; \\
f^{(d^*)}(i) = R\mu_1 - C_1 i - C_{2,2}\mu_2 - C_3\lambda, \quad 1 \leq i \leq K; \\
f(i) = R (\mu_1 + \mu_2) - C_1 i - C_{3}\lambda 1_{\{i < N\}} - C_4\lambda 1_{\{i = N\}}, \quad K + 1 \leq i \leq N.
$$

Since

$$
\eta^{d^*} = \sum_{i=0}^{N} \pi^{(d^*)}(i) f^{(d^*)}(i),
$$

we obtain

$$
\eta^{d^*} = \frac{1}{h(d^*)} \left\{ - (C_{2,1}\mu_1 + C_{2,2}\mu_2 + C_3\lambda) + \sum_{i=0}^{K} (R\mu_1 - C_1 i - C_{2,2}\mu_2 - C_3\lambda) \alpha^i \\
+ \sum_{i=K+1}^{N} [R (\mu_1 + \mu_2) - C_1 i - C_{3}\lambda 1_{\{i < N\}} - C_4\lambda 1_{\{i = N\}}] \left( \frac{\alpha}{\beta} \right)^i \frac{K}{\beta^i} \right\}
$$

$$
= \frac{1}{h(d^*)} \left\{ - \gamma_1 + \frac{\alpha}{1 - \alpha} (1 - \alpha^K) - C_1 \left[ \frac{\alpha (1 - \alpha^K)}{(1 - \alpha)^2} - \frac{K \alpha^{K+1}}{1 - \alpha} \right] \\
+ \left( \frac{\alpha}{\beta} \right)^K \frac{\beta^{K+1} (1 - \beta^{N-K})}{1 - \beta} - \left( \frac{\alpha}{\beta} \right)^K C_1 \left[ \frac{K \beta^{K+1} - N \beta^{N+1}}{1 - \beta} + \frac{\beta^{K+1} (1 - \beta^{N-K})}{(1 - \beta)^2} \right] \right\},
$$

where

$$
\gamma_1 = C_{2,1}\mu_1 + C_{2,2}\mu_2 + C_3\lambda, \\
\gamma_2 = R\mu_1 - C_{2,2}\mu_2 - C_3\lambda, \\
\gamma_3 = R (\mu_1 + \mu_2) - C_{3}\lambda 1_{\{i < N\}} - C_4\lambda 1_{\{i = N\}}.
$$

**7.2 The penalty cost $0 \leq P \leq P_L$**

By a similar analysis to that in Subsection 6.1 with the case $P \geq P_H$, here we simply provide a comparison between any two different dynamic rationing policies, and then find the optimal dynamic rationing policy and compute the maximal long-run average profit of this system.
For each \( i = 1, 2, \ldots, K \), we consider any two dynamic rationing policies with an interrelated structure as follows:

\[
\begin{align*}
d &= (0; d_1, d_2, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_K; 1, 1, \ldots, 1), \\
d' &= (0; d_1, d_2, \ldots, d_{i-1}, d'_i, d_{i+1}, \ldots, d_K; 1, 1, \ldots, 1),
\end{align*}
\]

where \( d'_i = 1 > d_i = 0 \) for each \( i = 1, 2, \ldots, K \).

**Theorem 3** If \( 0 \leq P \leq P_L \), then for any two policies \( d, d' \in \mathcal{D} \) with \( d'_i = 1 > d_i = 0 \) for each \( i = 1, 2, \ldots, K \),

\[
\eta^{d'} \geq \eta^d.
\]

Here the optimal dynamic rationing policy is given by

\[
d^* = (0; 1, 1, \ldots, 1; 1, 1, \ldots, 1),
\]

that is, if the penalty cost is lower with \( 0 \leq P \leq P_L \), then the inventory system can supply the products to the demands of Class 2.

**Proof:** This proof is similar to that of Theorem 2. It is clear that for each \( i = 1, 2, \ldots, K \),

\[
\eta^{d'} - \eta^d = \mu_2 \pi^{(d')}(i) \left[ G^{(d)}(i) + b \right].
\]

If \( 0 \leq P \leq P_L \), then it is easy to see from Proposition 1 that for any \( d \in \mathcal{D} \) and for each \( i = 1, 2, \ldots, K \), \( G^{(d)}(i) + b \geq 0 \). Thus for the two dynamic rationing policies \( d, d' \in \mathcal{D} \) with \( d'_i = 1 > d_i = 0 \) for each \( i = 1, 2, \ldots, K \), we obtain

\[
\eta^{d'} \geq \eta^d.
\]

This gives

\[
d^* = (0; 1, 1, \ldots, 1; 1, 1, \ldots, 1).
\]

This completes the proof. 

When \( 0 \leq P \leq P_L \), the optimal dynamic rationing policy is given by

\[
d^* = (0; 1, 1, \ldots, 1; 1, 1, \ldots, 1).
\]

By a similar analysis to that in (42), we obtain

\[
\xi_0 = 1, \quad i = 0, \\
\xi_i^{(d^*)} = \beta^i, \quad i = 1, 2, \ldots, N,
\]

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and

\[ h(d) = 1 + \sum_{i=1}^{N} \xi_i(d) = 1 + \frac{\beta(1 - \beta^N)}{1 - \beta}. \]

It follows from Subsection 1.1.4 of Chapter 1 in Li [53] that

\[ \pi_i(d) = \begin{cases} \frac{1}{h(d)}, & i = 0, \\ \frac{\beta_i}{h(d)}, & i = 1, 2, \ldots, N, \end{cases} \]

At the same time, it follows from (8) to (11) that

\[ f(0) = -C_{2,1}\mu_1 - C_{2,2}\mu_2 - C_3\lambda, \quad i = 0; \]
\[ f_i(d) = R(\mu_1 + \mu_2) - C_1i - C_3\lambda - P\mu_2, \quad 1 \leq i \leq K; \]
\[ f_K(d) = R(\mu_1 + \mu_2) - C_1i - C_3\lambda_{1{\{i<N\}}} - C_4\lambda_{1{\{i=N\}}}, \quad K + 1 \leq i \leq N. \]

Thus we obtain

\[ \eta^d = \frac{1}{h(d)} \left\{ - (C_{2,1}\mu_1 + C_{2,2}\mu_2 + C_3\lambda) + \sum_{i=1}^{K} [R(\mu_1 + \mu_2) - C_1i - C_3\lambda - P\mu_2] \beta^i \right. \]
\[ + \sum_{i=K+1}^{N} \left[ R(\mu_1 + \mu_2) - C_1i - C_3\lambda_{1{\{i<N\}}} - C_4\lambda_{1{\{i=N\}}} \right] \beta^i \right\} \]
\[ = \frac{1}{h(d)} \left\{ -\gamma_1 + \gamma_2 \frac{\beta(1 - \beta^K)}{1 - \beta} - C_1 \left[ \frac{\beta(1 - \beta^K)}{(1 - \beta)^2} - \frac{K\beta^{K+1}}{1 - \beta} \right] \right. \]
\[ + \gamma_3 \frac{\beta^{K+1}(1 - \beta^{N-K})}{1 - \beta} - C_1 \left[ \frac{K\beta^{K+1} - N\beta^{N+1}}{1 - \beta} + \frac{\beta^{K+1}(1 - \beta^{N-K})}{(1 - \beta)^2} \right] \}, \]

where

\[ \gamma_1 = C_{2,1}\mu_1 + C_{2,2}\mu_2 + C_3\lambda, \]
\[ \gamma_2 = R\mu_1 - C_{2,2}\mu_2 - C_3\lambda, \]
\[ \gamma_3 = R(\mu_1 + \mu_2) - C_3\lambda_{1{\{i<N\}}} - C_4\lambda_{1{\{i=N\}}}, \]
\[ \gamma_4 = R(\mu_1 + \mu_2) - C_3\lambda - P\mu_2. \]

7.3 The penalty cost \( P_L < P < P_H \)

In this subsection, we discuss a basic case with the penalty cost \( P_L < P < P_H \), while its analysis is more complicated than the above two cases. To this end, we propose a new method to find the optimal dynamic rationing policy, and to develop a complete algebraic transformational solution to show that the optimal dynamic rationing policy is of transformational threshold type.
Now, we recall two useful indices given in Section 6 as follows:

\[ P_H = \max_{d \in D} \left\{ 0, P_1^{(d)}, P_2^{(d)}, \ldots, P_K^{(d)} \right\} \]

and

\[ P_L = \min_{d \in D} \left\{ P_1^{(d)}, P_2^{(d)}, \ldots, P_K^{(d)} \right\}, \]

where

\[
Q_i^{(d)} = \frac{R + C_{2.2} + \lambda^{-i} \left[ B_0 - D^{(d)} \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ B_r^{(d)} - D^{(d)} \right] \prod_{k=r+1}^{i-1} v(d_k)}{1 + \lambda^{-i} \left[ A_0 - F^{(d)} \right] \prod_{k=1}^{i-1} v(d_k) + \sum_{r=1}^{i-1} \lambda^{r-i} \left[ A_r^{(d)} - F^{(d)} \right] \prod_{k=r+1}^{i-1} v(d_k)},
\]

for \(1 \leq i \leq K\).

For convenience of description, we introduce a convention: If \(P_{n-1}^{(d)} < P_n^{(d)} = \cdots = P_{n+i}^{(d)}\) and \(P = P_n^{(d)}\), then we write

\[ P_{n-1}^{(d)} < P = P_n^{(d)} = P_{n+1}^{(d)} = \cdots = P_{n+i}^{(d)}, \]

that is, the penalty cost \(P\) must be written in front of all the equal elements in the sequence \(\left\{ Q_k^{(d)} : n \leq k \leq n + i \right\}\).

For the sequence \(\left\{ Q_k^{(d)} : 1 \leq k \leq K \right\}\), now we set up an order from the smallest to the largest as follows:

\[ Q_{i_1}^{(d)} \leq Q_{i_2}^{(d)} \leq \cdots \leq Q_{i_{K-1}}^{(d)} \leq Q_{i_K}^{(d)}, \]

it is clear that \(Q_{i_1}^{(d)} = P_L\) and \(Q_{i_K}^{(d)} = P_H\). In this case, we write a subscript vector of the incremental sequence \(\left\{ Q_{i_j}^{(d)} : 1 \leq j \leq K \right\}\) as \((i_1, i_2, \ldots, i_{K-1}, i_K)\).

The following lemma determines the location of the penalty cost \(P\) in the sequence \(\left\{ Q_k^{(d)} : 1 \leq k \leq K \right\}\). This is very useful and necessary in our later study.

**Lemma 2** When \(P_L < P < P_H\), there exists a unique positive integer \(n_0 \in \{1, 2, \ldots, K\}\) such that either

\[ Q_{i_{n_0}}^{(d)} < P = Q_{i_{n_0+1}}^{(d)} \]

or

\[ Q_{i_{n_0}}^{(d)} < P < Q_{i_{n_0+1}}^{(d)}. \]
Proof: By using the condition \( P_L < P < P_H \), it is easy to see that there exists a unique positive integer \( n_0 \in \{1, 2, \ldots, K - 1, K\} \) such that

\[
P_L \leq \mathcal{P}_{i_{n_0}}^{(d)} < P \leq \mathcal{P}_{i_{n_0+1}}^{(d)} < P_H,
\]

hence this shows that either for \( P = \mathcal{P}_{i_{n_0+1}}^{(d)} \),

\[
\mathcal{P}_{i_{n_0}}^{(d)} < P = \mathcal{P}_{i_{n_0+1}}^{(d)},
\]

or for \( P < \mathcal{P}_{i_{n_0+1}}^{(d)} \),

\[
\mathcal{P}_{i_{n_0}}^{(d)} < P < \mathcal{P}_{i_{n_0+1}}^{(d)}.
\]

This completes the proof.

In what follows we focus on how to use the useful information:

\[
P_L = \mathcal{P}_1^{(d)} \leq \mathcal{P}_2^{(d)} \leq \cdots \leq \mathcal{P}_{K-1}^{(d)} \leq \mathcal{P}_K^{(d)} = P_H;
\]

and \( \mathcal{P}_{i_{n_0}}^{(d)} < P \leq \mathcal{P}_{i_{n_0+1}}^{(d)} \), in which \( n_0 \) is the unique positive integer in the set \( \{1, 2, \ldots, K - 1, K\} \).

Based on this, our analysis is classified as two different cases: A simple case and a general case.

Case one: A simple case with

\[
P_L = \mathcal{P}_1^{(d)} \leq \mathcal{P}_2^{(d)} \leq \cdots \leq \mathcal{P}_{n_0-1}^{(d)} < P \leq \mathcal{P}_{n_0}^{(d)} \leq \cdots \leq \mathcal{P}_K^{(d)} = P_H.
\]

That is, the subscript vector is given by \((1, 2, 3, \ldots, K - 1, K)\).

In the simple case, if \( P_L < P < P_H \), then there exists a unique positive integer \( n_0 \in \{1, 2, \ldots, K - 1, K\} \) such that

\[
P_L = \mathcal{P}_1^{(d)} \leq \cdots \leq \mathcal{P}_{n_0-1}^{(d)} < P \leq \mathcal{P}_{n_0}^{(d)} \leq \cdots \leq \mathcal{P}_K^{(d)} = P_H.
\]

Based on this, we take two different sets

\[
\Lambda_1 = \{ \mathcal{P}_1^{(d)}, \mathcal{P}_2^{(d)}, \ldots, \mathcal{P}_{n_0-1}^{(d)} \}
\]

and

\[
\Lambda_2 = \{ \mathcal{P}_{n_0}^{(d)}, \mathcal{P}_{n_0+1}^{(d)}, \ldots, \mathcal{P}_K^{(d)} \}.
\]

By using the two sets \( \Lambda_1 \) and \( \Lambda_2 \), we write

\[
\mathcal{P}_H(1 \rightarrow n_0 - 1) = \max_{1 \leq i \leq n_0 - 1} \{ \mathcal{P}_i^{(d)} \}\]
and

\[
\mathcal{T}_L(n_0 \to K) = \min_{n_0 \leq j \leq K} \left\{ \mathcal{P}_j^{(d)} \right\}.
\]

The following theorem provides expression for the optimal dynamic rationing policy \(d^*\) in this simple case.

**Theorem 4** For the simple case with \(P_L < P < P_H\), if there exists a unique positive integer \(n_0 \in \{1, 2, \ldots, K-1, K\}\) such that

\[
P_L = \mathcal{P}_1^{(d)} \leq \cdots \leq \mathcal{P}_{n_0-1}^{(d)} < P \leq \mathcal{P}_{n_0}^{(d)} \leq \cdots \leq \mathcal{P}_K^{(d)} = P_H,
\]

then the optimal dynamic rationing policy is given by

\[
d^* = \left(\begin{array}{cccc}
0; 0, 0, \ldots, 0, & 1, 1, \ldots, 1; 1, 1, \ldots, 1
\end{array}\right).
\]

**Proof:** On the one hand, in the set \(\Lambda_1\), it is easy to see that \(P > \mathcal{T}_H(1 \to n_0 - 1)\) from the two facts that

\[
P > \mathcal{P}_{n_0-1}^{(d)}
\]

and

\[
\mathcal{T}_H = \max_{1 \leq i \leq n_0-1} \left\{ \mathcal{P}_i^{(d)} \right\} = \mathcal{P}_{n_0-1}^{(d)},
\]

noting that \(\mathcal{P}_1^{(d)} \leq \mathcal{P}_2^{(d)} \leq \cdots \leq \mathcal{P}_{n_0-1}^{(d)}\). Based on this, our aim is to focus on such a dynamic rationing sub-policy

\[
d = (0; d_1, d_2, \ldots, d_{n_0-1}, *, *, \ldots, *, 1, 1, \ldots, 1).
\]

When only observing the policy vector \((d_1, d_2, \ldots, d_{n_0-1})\) related to \(P > \mathcal{T}_H(1 \to n_0 - 1)\), it is easy to see from Theorem 2 that the optimal dynamic rationing sub-policy is given by

\[
d_{sa} = (0; 0, 0, \ldots, 0, *, *, \ldots, *, 1, 1, \ldots, 1).
\]

On the other hand, it is seen from the set \(\Lambda_2\) that \(0 \leq P \leq \mathcal{T}_L(n_0 \to K)\) due to the fact that \(P \leq \mathcal{P}_{n_0}^{(d)}\) and \(\mathcal{T}_L(n_0 \to k) = \mathcal{P}_k^{(d)}\) in terms of \(\mathcal{P}_{n_0}^{(d)} \leq \mathcal{P}_{n_0+1}^{(d)} \cdots \leq \mathcal{P}_K^{(d)}\). Based on this, our aim is to focus on such a dynamic rationing sub-policy

\[
d = (0; *, *, \ldots, *, d_{n_0}, d_{n_0+1}, \ldots, d_K; 1, 1, \ldots, 1).
\]
When only observing the vector \((d_{n_0}, d_{n_0+1}, \ldots, d_K)\) related to \(0 \leq P \leq P_L(n_0 \rightarrow K)\), it is easy to see from Theorem 3 that the optimal dynamic rationing sub-policy is given by
\[
d_{*b} = (0; *, *, \ldots, 1, 1, \ldots, 1; 1, 1, \ldots, 1).
\]

Based on the above two different discussions, from the total set \(\Lambda_1 \cup \Lambda_2\), by observing the vector \((d_1, d_2, \ldots, d_{n_0-1}, d_{n_0}, d_{n_0+1}, \ldots, d_K)\) related to \(P_H(1 \rightarrow n_0 - 1) < P \leq P_L(n_0 \rightarrow K)\), the optimal dynamic rationing policy is given by
\[
d^* = (d_{*a})_{*b} = (d_{*b})_{*a} = \left(0; 0, 0, \ldots, 1, 1, \ldots, 1; 1, 1, \ldots, 1\right).
\]

This completes the proof. ■

**Case two**: A general case with
\[
P_L = P_{i_1}^{(d)} \leq P_{i_2}^{(d)} \leq \cdots \leq P_{i_{n_0-1}}^{(d)} \leq P_{i_{n_0}}^{(d)} = P_H.
\]
That is, the subscript vector is given by \((i_1, i_2, \ldots, i_{K-1}, i_K)\).

If \(P_L < P < P_H\), then there exists a unique positive integer \(n_0 \in \{1, 2, \ldots, K-1, K\}\) such that
\[
P_L = P_{i_1}^{(d)} \leq \cdots \leq P_{i_{n_0}}^{(d)} < P \leq P_{i_{n_0}}^{(d)} \leq \cdots \leq P_{i_{K}}^{(d)} = P_H.
\]

Based on this, we take two sets
\[
\Lambda_1^G = \left\{P_{i_1}^{(d)}, P_{i_2}^{(d)}, \ldots, P_{i_{n_0}-1}^{(d)}\right\}
\]
and
\[
\Lambda_2^G = \left\{P_{i_{n_0}}^{(d)}, P_{i_{n_0}+1}^{(d)}, \ldots, P_{i_K}^{(d)}\right\}.
\]

For the two sets \(\Lambda_1^G\) and \(\Lambda_2^G\), we write
\[
\overline{P}_H^G(1 \rightarrow n_0 - 1) = \max_{1 \leq k \leq n_0-1} \left\{P_{i_k}^{(d)}\right\}
\]
and
\[
\overline{P}_L^G(n_0 \rightarrow K) = \min_{n_0 \leq k \leq K} \left\{P_{i_k}^{(d)}\right\}.
\]

By corresponding to the order of the incremental sequence \(\left\{P_{i_k}^{(d)} : 1 \leq k \leq K\right\}\), we transfer the rationing policy
\[
d = (0; d_1, d_2, \ldots, d_{n_0-1}, d_{n_0}, d_{n_0+1}, \ldots, d_K; 1, 1, \ldots, 1)
\]
into a transformational version

\[
d (\text{Transfer}) = (0; d_{i_1}, d_{i_2}, \ldots, d_{i_{n_0-1}}, d_{i_{n_0}}, d_{i_{n_0+1}}, \ldots, d_{i_K}; 1, 1, \ldots, 1).
\]

called transformational dynamic rationing policy.

The following theorem provides an explicit expression for the optimal transformational dynamic rationing policy \( d^* (\text{Transfer}) \) in the rationing inventory system.

**Theorem 5** For the general case with \( P_L < P < P_H \), if there exists a unique positive integer \( n_0 \in \{1, 2, \ldots, K-1, K\} \) such that

\[
P_L = \mathcal{P}^{(d)}_{i_1} \leq \cdots \leq \mathcal{P}^{(d)}_{i_{n_0-1}} < P \leq \mathcal{P}^{(d)}_{i_{n_0}} \leq \cdots \leq \mathcal{P}^{(d)}_{i_K} = P_H,
\]

then the optimal transformational dynamic rationing policy is given by

\[
d^* (\text{Transfer}) = \left(0; 0, 0, \ldots, 0 \underbrace{1, 1, \ldots, 1}_{n_0-1 \text{ zeros}}, \underbrace{1, 1, \ldots, 1}_{K-n_0+1 \text{ ones}}; 1, 1, \ldots, 1\right).
\]

**Proof:** Note that \( P > \mathcal{P}^G_{H} (1 \to n_0-1) \) in the set \( \Lambda^G_1 \), our analysis is to focus on the transformational rationing sub-policy

\[
d (\text{Transfer}) = (0; d_{i_1}, d_{i_2}, \ldots, d_{i_{n_0-1}}, *, *, \ldots, *; 1, 1, \ldots, 1).
\]

By observing the vector \((d_{i_1}, d_{i_2}, \ldots, d_{i_{n_0-1}})\) related to \( P > \mathcal{P}^G_{H} (1 \to n_0-1) \), it is easy to see from the proof of Theorem 4 that the optimal transformational dynamic rationing sub-policy is given by

\[
d_{sa} (\text{Transfer}) = (0; 0, 0, \ldots, 0, *, *, \ldots, *; 1, 1, \ldots, 1).
\]

Similarly, from \( 0 \leq P \leq \mathcal{P}^G_{L} (n_0 \to K) \) in the set \( \Lambda^G_2 \), we discuss the transformational dynamic rationing sub-policy

\[
d (\text{Transfer}) = (0; *, *, \ldots, *, d_{i_{n_0}}, d_{i_{n_0+1}}, \ldots, d_{i_K}; 1, 1, \ldots, 1).
\]

From observing the vector \((d_{i_{n_0}}, d_{i_{n_0+1}}, \ldots, d_{i_K})\) related to \( 0 \leq P \leq \mathcal{P}^G_{L} (n_0 \to K) \), it is easy to see from the proof of Theorem 4 that the optimal transformational dynamic rationing sub-policy is given by

\[
d_{sb} (\text{Transfer}) = (0; *, *, \ldots, *, 1, 1, \ldots, 1; 1, 1, \ldots, 1).
\]
Therefore, by observing the total vector \((d_{i1}, d_{i2}, \ldots, d_{i_{n_0-1}}, d_{i_{n_0}}, d_{i_{n_0+1}}, \ldots, d_{i_K})\) in the total set \(\Lambda^G_1 \cup \Lambda^G_2\), which is related to

\[P^G_H (1 \rightarrow n_0 - 1) < P \leq P^G_L (n_0 \rightarrow K),\]

the optimal transformational dynamic rationing policy is given by

\[d^* \text{ (Transfer)} = (d^* \text{ (Transfer)})_{sa} = (d^* \text{ (Transfer)})_{sb} = \left(0; 0, 0, \ldots, 0, 1, 1, 1, \ldots, 1 \right).\]

This completes the proof. 

**Remark 2** For the general case in the rationing inventory system, the optimal transformational dynamic rationing policy is given by a beautiful transformational form

\[d^* \text{ (Transfer)} = \left(0; 0, 0, \ldots, 0, 1, 1, 1, \ldots, 1 \right).\]

Based on this, we use the inverse transformation of \(d^* \text{ (Transfer)}\) to be able to restore the original optimal dynamic rationing policy \(d^*\). However, the element structure of \(d^*\) is not of threshold type, for example, the optimal dynamic rationing policy

\[d^* = (0; 0, 1, 0, 0, 1, 0, 1; 1, 1, 1)\]

may correspond to a case

\[\mathcal{P}_1^{(d)} \leq \mathcal{P}_3^{(d)} \leq \mathcal{P}_4^{(d)} \leq \mathcal{P}_7^{(d)} < P \leq \mathcal{P}_2^{(d)} \leq \mathcal{P}_5^{(d)} \leq \mathcal{P}_6^{(d)} \leq \mathcal{P}_8^{(d)}\]

with the subscript vector \((1, 3, 4, 7, 2, 5, 6, 8)\) or to another case

\[\mathcal{P}_3^{(d)} \leq \mathcal{P}_4^{(d)} \leq \mathcal{P}_1^{(d)} \leq \mathcal{P}_7^{(d)} < P \leq \mathcal{P}_5^{(d)} \leq \mathcal{P}_8^{(d)} \leq \mathcal{P}_2^{(d)} \leq \mathcal{P}_6^{(d)}\]

with the subscript vector \((3, 4, 1, 7, 5, 8, 2, 6)\). It is obvious that the inverse \(d^*\) of the optimal transformational dynamic rationing policy \(d^* \text{ (Transfer)}\) may exist multiple different solutions of the subscript vector \((i_1, i_2, \ldots, i_{K-1}, i_K)\).

**Remark 3** The optimal transformational version \(d^* \text{ (Transfer)}\) of the optimal dynamic rationing policy \(d^*\) plays a key role in applications of the sensitivity-based optimization to the study of rationing inventory systems. At the same time, the RG-factorization of block-structured Markov processes will further extend and generalize establishing the transformational version \(d^* \text{ (Transfer)}\) of the optimal dynamic rationing policy \(d^*\).
Remark 4  Note that the bang-bang control (e.g., see Ma et al. [56, 57]) is an effective method to partly find the optimal dynamic rationing policy; while our optimal transformational dynamic rationing policy $\mathbf{d}^* (\text{Transfer})$ provides a more basic result, which can restore the original optimal dynamic rationing policy $\mathbf{d}^*$ from the subscript vector $(i_1, i_2, \ldots, i_{K-1}, i_K)$. Therefore, our optimal transformational dynamic rationing policy is superior to the bang-bang control from the information of the optimal dynamic rationing policy.

The following theorem provides a necessary summarization for Theorems 2 to 5, while its proof is clear and omitted here. It is easy to see from this theorem that we provide a complete algebraic transformational solution to the optimal dynamic rationing policy.

**Theorem 6** For the inventory rationing across two demand classes, there must exist an optimal transformational dynamic rationing policy $\mathbf{d}^* (\text{Transfer})$. Further, we still have the following two basic results:

(a) The optimal policy is of critical rationing level under satisfying every one of the three conditions: (i) $P \geq P_H$; (ii) $0 \leq P \leq P_L$; and (iii) $P_L < P < P_H$ with the subscript vector $(1, 2, \ldots, K-1, K)$.

(b) The optimal policy is not of critical rationing level if $P_L < P < P_H$ with the subscript vector $(i_1, i_2, \ldots, i_{K-1}, i_K) \neq (1, 2, \ldots, K-1, K)$.

In the remainder of this section, we show that our algebraic transformational solution can easily be extended and generalized to deal with inventory rationing across multiple demand classes. Here, we only provide a solution idea as follows:

If there are $n$ demand classes with different priorities, called Class 1, Class 2, \ldots, Class $n-1$ and Class $n$. We assume that the priority of Class $i$ is strictly higher than that of Class $j$ for $i < j$. In this case, we can obtain the critical rationing level (if only) between any two demand classes.

(1) Comparing \{Class 1\} with \{Classes 2 to $n$\}, we can give the critical rationing level $L_1$.

(2) Comparing \{Classes 1 and 2\} with \{Classes 3 to $n$\}, we can give the critical rationing level $L_2$. 
Comparing \{\text{Classes }1 \text{ and } i\} \text{ with } \{\text{Classes }i+1 \text{ to } n\}, \text{ we can give the critical rationing level } L_i \text{ for } 3 \leq i \leq n.

Based on this, we can obtain \( n \) critical rationing levels with \( L_1 \leq L_2 \leq \cdots \leq L_n \).

It is necessary useful to explain how to use the critical rationing level \( L_i \) for \( 1 \leq i \leq n \).

(a) If the on-hand inventory is lower than \( L_1 \), then we supply the products to only the demands of Class 1.

(b) If the on-hand inventory is lower than \( L_i \), then we supply the products to only the demands of Classes 1 to \( i \) for \( 2 \leq i \leq n \).

8 \textbf{The threshold type rationing policies}

Note that the threshold type policies are simple and useful in many real areas, thus they can be widely adopted in control of many practical systems. In this section, we focus on such threshold type rationing policies, although their optimality is not proved rigorously in our problem yet. Finally, we provide a necessary condition under which a threshold type rationing policy is optimal.

Now, we introduce an interesting subset of the policy set \( \mathcal{D} \) as follows. For \( \theta = 1, 2, \ldots, K, K+1 \), we write \( d_{\triangle, \theta} \) as a rationing policy \( d \) with \( d_i = 0 \) if \( 1 \leq i \leq \theta - 1 \) and \( d_i = 1 \) if \( \theta \leq i \leq K \). It is easy to see that if \( \theta = 1 \), then

\[
d_{\triangle, 1} = (0; 1, 1, \ldots, 1; 1, 1, \ldots, 1);
\]

if \( \theta = K \), then

\[
d_{\triangle, K} = (0; 0, 0, \ldots, 0, 1; 1, 1, \ldots, 1);
\]

and if \( \theta = K+1 \), then

\[
d_{\triangle, K+1} = (0; 0, 0, \ldots, 0; 1, 1, \ldots, 1).
\]

Let

\[
\mathcal{D}^{\triangle} = \{d_{\triangle, \theta} : \theta = 1, 2, \ldots, K, K+1\}.
\]

Then

\[
\mathcal{D}^{\triangle} = \left\{ \left( \underbrace{0; 0, 0, \ldots, 0}_{\theta-1 \text{ zeros}}, 1, 1, \ldots, 1; 1, 1, \ldots, 1} \right) : \theta = 1, 2, \ldots, K, K+1 \right\}.
\]
It is easy to see that $\Delta \subset D$.

For a rationing policy $d_{\Delta, \theta} = \left(0; 0, 0, \ldots, 0, 1, 1, \ldots, 1; 1, 1, \ldots, 1\right)$ with $\theta = 1, 2, \ldots, K, K+1$, it is clear that if $1 \leq i \leq \theta - 1$, then $d_i = 0$; and if $\theta \leq i \leq K$, then $d_i = 1$. In this case, it follows from (5) that

$$\xi_0 = 1,$$

$$\xi_i^{(d_{\Delta, \theta})} = \begin{cases} \alpha_i, & i = 1, 2, \ldots, \theta - 1; \\ \left(\frac{\alpha}{\beta}\right)^{\theta-1} \beta^i, & i = \theta, \theta + 1, \ldots, N. \end{cases}$$

and

$$h^{(d_{\Delta, \theta})} = 1 + \sum_{i=1}^{N} \xi_i^{(d_{\Delta, \theta})}$$

$$= 1 + \alpha \left(1 - \alpha^{\theta-1}\right) \frac{\beta^\theta \left(1 - \beta^{N-\theta+1}\right)}{1 - \beta}.$$

It follows from (6) that

$$\pi^{(d_{\Delta, \theta})} (i) = \begin{cases} \frac{1}{h^{(d_{\Delta, \theta})}}, & i = 0; \\ \frac{1}{h^{(d_{\Delta, \theta})}} \alpha_i, & i = 1, 2, \ldots, \theta - 1; \\ \frac{1}{h^{(d_{\Delta, \theta})}} \left(\frac{\alpha}{\beta}\right)^{\theta-1} \beta^i, & i = \theta, \theta + 1, \ldots, N. \end{cases}$$

On the other hand, it follows from (8) to (11) that for $i = 0$

$$f (0) = -C_{2,1} \mu_1 - C_{2,2} \mu_2 - C_3 \lambda;$$

for $i = 1, 2, \ldots, \theta - 1$,

$$f^{(d_{\Delta, \theta})} (i) = R \mu_1 - C_1 i - C_{2,2} \mu_2 - C_3 \lambda;$$

for $i = \theta, \theta + 1, \ldots, K$,

$$f^{(d_{\Delta, \theta})} (i) = R \left(\mu_1 + \mu_2\right) - C_1 i - C_3 \lambda - P \mu_2;$$

and for $i = K + 1, K + 2, \ldots, N$,

$$f (i) = R \left(\mu_1 + \mu_2\right) - C_1 i - C_3 \lambda 1_{\{i < N\}} - C_4 \lambda 1_{\{i = N\}}.$$

Note that

$$\eta^{d_{\Delta, \theta}} = \pi^{(d_{\Delta, \theta})} (0) f (0) + \sum_{i=1}^{\theta-1} \pi^{(d_{\Delta, \theta})} (i) f^{(d_{\Delta, \theta})} (i)$$

$$+ \sum_{i=\theta}^{K} \pi^{(d_{\Delta, \theta})} (i) f^{(d_{\Delta, \theta})} (i) + \sum_{i=K+1}^{N} \pi^{(d_{\Delta, \theta})} (i) f (i),$$

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we obtain an explicit expression for the long-run average profit of the inventory rationing systems under the threshold type rationing policy $d_{\Delta, \theta}$ as follows:

$$\eta^{d_{\Delta, \theta}} = \frac{1}{h(d_{\Delta, \theta})} \left\{ - (C_{2,1} \mu_1 + C_{2,2} \mu_2 + C_{3,3} \lambda) + \sum_{i=1}^{\theta-1} \alpha^i (R \mu_1 - C_{1,i} - C_{2,2} \mu_2 - C_{3,3} \lambda) \\
+ K \left( \frac{\alpha}{\beta} \right)^{\theta-1} \beta^i [R (\mu_1 + \mu_2) - C_{1,i} - C_{3,3} \lambda - P \mu_2] \\
+ \sum_{i=K+1}^{N} \left( \frac{\alpha}{\beta} \right)^{\theta-1} \beta^i \left[ R (\mu_1 + \mu_2) - C_{1,i} - C_{3,3} \lambda i_{i< N} - C_{4,4} \lambda i_{i=N} \right] \right\}$$

$$= \frac{1}{h(d_{\Delta, \theta})} \left\{ - \gamma_1 + \gamma_2 \frac{\alpha (1 - \alpha^{\theta-1})}{1 - \alpha} - C_1 \left[ \frac{\alpha (1 - \alpha^{\theta-1})}{(1 - \alpha)^2} - \frac{\theta (1 - \alpha^{\theta-1})}{1 - \alpha} \right] \\
- \left( \frac{\alpha}{\beta} \right)^\theta \frac{\beta^\theta N^{N+1}}{1 - \beta} + \left( \frac{\alpha}{\beta} \right)^\theta \frac{\beta^\theta N^{N+1}}{(1 - \beta)^2} \\
+ \left( \frac{\alpha}{\beta} \right)^\theta \frac{\beta^\theta (1 - \beta^{N+1})}{1 - \beta} + \left( \frac{\alpha}{\beta} \right)^\theta \frac{\beta^\theta (1 - \beta^{N+1})}{1 - \beta} \right\}.$$ 

Let

$$d'_{\Delta, \theta} = \arg\max_{d_{\Delta, \theta} \in D} \{ \eta^{d_{\Delta, \theta}} \}$$

or

$$d_{\Delta, \theta^*} = \arg\max_{1 \leq \theta \leq K+1} \{ \eta^{d_{\Delta, \theta}} \}.$$ 

Then $d'_{\Delta, \theta} = d_{\Delta, \theta^*}$. Hence we call $d'_{\Delta, \theta}$ (or $d_{\Delta, \theta^*}$) the optimal threshold type rationing policy in the policy set $D$. Since $D \subset D$, the partially ordered set $D$ shows that $D$ is also a partially ordered set. Based on this, it is easy to see from the two partially ordered sets $D$ and $D$ that

$$\eta^{d_{\Delta, \theta^*}} \leq \eta^{d^*}.$$ 

If $\eta^{d_{\Delta, \theta^*}} = \eta^{d^*}$, then we call $d_{\Delta, \theta}$ the optimal rationing policy in the policy set $D$; while if $\eta^{d_{\Delta, \theta^*}} < \eta^{d^*}$, then we call $d_{\Delta, \theta}$ the optimal threshold type rationing policy in the policy subset $D$ a suboptimal rationing policy in the policy set $D$.

In the remainder of this section, we simply analyze how the optimal threshold type rationing policy $d_{\Delta, \theta}^*$ influences the sign of the perturbation realization factor $G^{(d_{\Delta, \theta})}(\theta)$.

It is easy to see that there must exist a minimal positive integer $\theta^* \in \{1, 2, \ldots, K, K + 1\}$ such that

$$d_{\Delta, \theta^*} = d_{\Delta, \theta^*} = \left( 0; 0, 0, \ldots, 0, 1, 1, \ldots, 1; 1, 1, \ldots, 1 \right).$$
By using the optimal threshold type rationing policy \( d_{\Delta, \theta}^* \) (or \( d_{\Delta, \theta^*}^* \)), the following theorem determines the positive or negative of the function \( G(d_{\Delta, \theta}) (\theta) + b \) in the three different points: \( \theta = \theta^* - 1, \theta^*, \theta^* + 1 \), although the explicit expression of the perturbation realization factor \( G(d_{\Delta, \theta}) (\theta) \) is not given yet. This may be useful for us to understand the role played by Proposition 1 in finding the optimality of the long-run average profit of this system. Furthermore, we also derive a necessary condition of the optimal threshold type rationing policy.

**Theorem 7** In the threshold type rationing policies of the inventory system, the optimal threshold type policy \( d_{\Delta, \theta^*}^* \) satisfies the following condition:

\[
G(d_{\Delta, \theta^*-1}) (\theta^* - 1) + b \leq 0, \quad G(d_{\Delta, \theta^*}) (\theta^*) + b \geq 0, \quad G(d_{\Delta, \theta^*+1}) (\theta^* + 1) + b \geq 0.
\]

**Proof:** We consider three threshold type rationing policies with an interrelated structure as follows:

\[
d_{\Delta, \theta^*-1} = \begin{pmatrix}
0; 0, \ldots, 0, 1, 1, \ldots, 1; 1, 1, \ldots, 1
\end{pmatrix},
\]

\[
d_{\Delta, \theta^*} = \begin{pmatrix}
0; 0, 0, \ldots, 0, 1, 1, \ldots, 1; 1, 1, \ldots, 1
\end{pmatrix},
\]

\[
d_{\Delta, \theta^*+1} = \begin{pmatrix}
0; 0, 0, \ldots, 0, 0, 1, \ldots, 1; 1, 1, \ldots, 1
\end{pmatrix}.
\]

Note that \( d_{\Delta, \theta^*} \) is the optimal threshold type rationing policy, it is clear that \( d_{\Delta, \theta^*} \geq d_{\Delta, \theta^*-1} \) and \( d_{\Delta, \theta^*} \geq d_{\Delta, \theta^*+1} \). Thus it follows from Lemma 1 that

\[
\eta^{d_{\Delta, \theta^*+1}} - \eta^{d_{\Delta, \theta^*}} = -\mu_2 \pi(d_{\Delta, \theta^*+1}) (\theta^* + 1) \left[G(d_{\Delta, \theta^*}) (\theta^*) + b\right],
\]

which, together with \( \eta^{d_{\Delta, \theta^*+1}} - \eta^{d_{\Delta, \theta^*}} \leq 0 \), leads to

\[
G(d_{\Delta, \theta^*}) (\theta^*) + b \geq 0.
\]

Similarly, by using \( \eta^{d_{\Delta, \theta^*}} \geq \eta^{d_{\Delta, \theta^*+1}} \) and

\[
\eta^{d_{\Delta, \theta^*}} - \eta^{d_{\Delta, \theta^*+1}} = \mu_2 \pi(d_{\Delta, \theta^*}) (\theta^*) \left[G(d_{\Delta, \theta^*+1}) (\theta^* + 1) + b\right],
\]

we obtain

\[
G(d_{\Delta, \theta^*+1}) (\theta^* + 1) + b \geq 0.
\]

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On the other hand, by means of $\eta^{d_\Delta, \theta^*} \geq \eta^{d_\Delta, \theta^* - 1}$ and

$$\eta^{d_\Delta, \theta^*} - \eta^{d_\Delta, \theta^* - 1} = -\mu_2 \pi(d_\Delta, \theta^*) (\theta^* - 1) + b,$$

we have

$$G(d_\Delta, \theta^* - 1) (\theta^* - 1) + b \leq 0.$$

This completes the proof. \(\blacksquare\)

9 Numerical Experiments

In this section, by observing different penalty costs, we use some numerical experiments to gain insights into understanding the rationing inventory system, and especially, caring for the optimal dynamic rationing policy.

In Examples 1 to 4, we take some common parameters in the rationing inventory system as follows: $C_1 = 1, C_{2,1} = 4, C_{2,2} = 1, C_3 = 5, C_4 = 1, R = 15$ and $N = 100$.

Example 1. We give some comparison between any two different rationing policies under some different penalty costs, our aim is to verify the optimality in Theorems 2 and 3, respectively. To this end, we take that $\mu_1 = 4, \mu_2 = 2$, $\lambda = 3$, $K = 15$ and $1 \leq i \leq 15$.

(1) A higher penalty cost

We take a higher penalty cost $P = 10$. In Theorem 2, if $d_i^* = 0$ for $1 \leq i \leq 15$, then the optimal dynamic rationing policy $d^* = (0; 0, 0, \ldots, 1, 1, \ldots, 1)$. This gives $\eta^{d^*} = 22.3$. On the other hand, if $d_i^* = 1$ for $1 \leq i \leq 15$, then the optimal dynamic rationing policy $d^* = (0; 1, 1, \ldots, 1, 1, \ldots, 1)$. This gives $\eta^{d^*} = 13$. By such a comparison $\eta^{d^*} = 22.3$ with $\eta^{d^*} = 13$, if the penalty cost is chosen to be suitably high, then the optimal dynamic rationing policy should be $d^* = (0; 0, 0, \ldots, 0, 1, 1, \ldots, 1)$. Table 2 demonstrates how the optimal rationing policy $d^*$ is determined by means of $\eta^{d^*}$.

| Penalty cost | $P = 10$ |
|--------------|---------|
| Rationing policy | $d_i^* = 0, 1 \leq i \leq 15$ | $d_i^* = 1, 1 \leq i \leq 15$ |
| Long-run average profit | $\eta^{d^*} = 22.3$ | $\eta^{d^*} = 13$ |

(2) A lower penalty cost
Now, we choose a lower penalty cost $P = 0.1$. In Theorem 3, if $d_i^* = 1$, then the optimal dynamic rationing policy $d^* = (0; 1, 1, \ldots, 1; 1, \ldots, 1)$. This gives $\eta^{d^*} = 22.9$. On the other hand, if $d_i^* = 0$, then the optimal dynamic rationing policy $d^* = (0; 0, 0, \ldots, 0; 1, 1, \ldots, 1)$. This gives $\eta^{d^*} = 22.3$. Obviously, when the penalty cost is suitably lower, the optimal dynamic rationing policy should be $d^* = (0; 1, 1, \ldots, 1; 1, 1, \ldots, 1)$ by using a comparison between $\eta^{d^*}$ and $\eta^{d^*}$. Table 3 demonstrates how the rationing policy $d^*$ is chosen by means of $\eta^{d^*}$.

Table 4: The optimal long-run average profit under a lower penalty cost

| Penalty cost | $P = 0.1$ |
|--------------|------------|
| Rationing policy | $d_i^* = 1, 1 \leq i \leq 15$ | $d_i^* = 0, 1 \leq i \leq 15$ |
| Long-run average profit | $\eta^{d^*} = 22.9$ | $\eta^{d^*} = 22.3$ |

Example 2. We analyze how the optimal long-run average profit depends on some crucial parameters of the rationing inventory system. Here, our observation still takes the higher penalty cost $P = 10$ and the lower penalty cost $P = 0.1$, respectively. Further, we take the common parameters: $\mu_1 = 30$, $\mu_2 = 40$ under several different inventory levels: $K = 5, 6, 10$.

(1) A higher penalty cost

We analyze how the optimal long-run average profit $\eta^{d^*}$ depends on $\lambda$ for $\lambda \in (47, 50)$. From Figure 3, it is seen that the optimal long-run average profit $\eta^{d^*}$ increases as $\lambda$ increases. On the other hand, if the design threshold $K$ increases, then the optimal long-run average profit $\eta^{d^*}$ increases slower as $\lambda$ increases.

(2) A lower penalty cost

We discuss how the optimal long-run average profit $\eta^{d^*}$ depends on $\lambda$ for $\lambda \in (65, 80)$. From Figure 4, it is seen that the optimal long-run average profit $\eta^{d^*}$ increases as $\lambda$ increases. On the other hand, if the design threshold $K$ increases, then the optimal long-run average profit $\eta^{d^*}$ increases slower as $\lambda$ increases.

Example 3. We verify the optimal threshold type rationing policy is suboptimal in the policy set $D$. In what follow we take some common parameters: $\lambda = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $K = 15$, $1 \leq \theta \leq 15$. Our observation is to set the higher penalty cost $P = 10$ and the lower penalty cost $P = 0.1$, respectively.

(1) A higher penalty cost
Figure 3: $\eta^{dr}$ vs. $\lambda$ under three different design thresholds $K$

Figure 4: $\eta^{dr}$ vs. $\lambda$ under three different design thresholds $K$
We observe how the optimal long-run average profit $\eta^d$ depends on the threshold from $\theta = 1$ to $\theta = 15$. From Figure 5, it is seen that the optimal threshold is $\theta^* = 9$ and $\eta^{d_{\Delta,\theta^*}} = 21.4$. By using (1) of Example 1 with $\eta^d = 22.3$, thus we obtain that $\eta^{d_{\Delta,\theta^*}} = 21.4 < \eta^d = 22.3$. This shows that the optimal threshold type rationing policy is suboptimal in the policy set $\mathcal{D}$.

![Figure 5: The optimal long-run average profit $\eta^d$ vs. the threshold $\theta$](image)

(2) A lower penalty cost

From Figure 6, it is seen that the optimal threshold is $\theta^* = 3$ and $\eta^{d_{\Delta,\theta^*}} = 22.9$. In (2) of Example 1, we obtained $\eta^d = 22.9$. Hence, $\eta^{d_{\Delta,\theta^*}} = \eta^d = 22.9$, so that the optimal threshold type rationing policy is really the optimal rationing policy.

**Example 4.** Our observation is to focus on how the penalty cost $P$ influences the optimal long-run average profit $\eta^d$. We assume that $P \in (0,50)$, $\mu_1 = 4$, $\mu_2 = 2$, $\lambda = 3$, and $K = 15$. When $P$ is suitably high, the optimal dynamic rationing policy $d^* = (0;0,0,\ldots,0;1,1,\ldots,1)$, and it is easy to see from (7) that $P$ will influence $\eta^d$. Based on this we only need to consider the case with a lower $P$. From Figure 7, it is seen that the optimal long-run average profit $\eta^d$ linearly decreases as $P$ increases.

**10 Concluding**

In this paper, we highlight understanding on the optimal rationing policy of an inventory system by applying the sensitivity-based optimization. It is seen that the dynamic rationing policy is more important and necessary in the study of rationing inventory
Figure 6: The optimal long-run average profit $\eta^{d^*}$ vs. the threshold $\theta$

Figure 7: The optimal long-run average profit $\eta^{d^*}$ vs. the penalty cost $P$
systems, but it largely makes optimal analysis of inventory management more interesting, difficult and challenging. To find the optimal dynamic rationing policy, we set up a policy-based Poisson equation and provide an explicit expression for its solution. Based on this, we provide some comparison between any two different policies, and further compute the maximal long-run average profit under some different violation penalty costs. We also study the threshold type rationing policy and derive the necessary condition of the optimal threshold type policy. Different from previous works in the literature on applying the traditional MDP theory to the dynamic control of inventory systems, the sensitivity-based optimization used in this paper is easier and more convenient in the study of rationing inventory systems. This sensitivity-based optimization may open a new avenue to find the optimal dynamic rationing policy for more complicated inventory systems.

Along such a line, there are a number of interesting directions for potential future research, for example:

- Extending to the rationing inventory system with multiple classes of demands, multiple types of products, backorders, batch order, batch production and so on;
  - analyzing non-Poisson inputs such as Markovian arrival processes (MAPs) and/or non-exponential service times, e.g. the PH distributions;
  - discussing the long-run performance is influenced by some concave or convex reward (or cost) functions;
  - studying individual or social optimization for dynamic rationing inventory systems from a perspective of game theory.

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Appendix: Solution to the Poisson Equation

In this appendix, we provide an effective method to solve the Poisson equation. This is necessary and useful for dealing with a more general Poisson equation developed in the study of rationing inventory systems.

To solve the system of linear equations (22), it is easy to see that \( \text{rank}(B^{(d)}) = N \) and \( \det(B^{(d)}) = 0 \) due to the fact that the size of the matrix \( B^{(d)} \) is \( N + 1 \). Hence, this system of linear equations (22) exists infinitely-many solutions with a free constant of an additive term.

Let \( B \) be a matrix obtained through omitting the first row and the first column vectors of the matrix \( B^{(d)} \). Then,

\[
B = \begin{pmatrix}
-\lambda + \nu(d_1) & \lambda \\
\nu(d_2) & -\lambda + \nu(d_2) & \lambda \\
\vdots & \ddots & \ddots & \ddots \\
\nu(d_K) & -\lambda + \nu(d_K) & \lambda & \nu(1) & -\lambda + \nu(1) \\
\nu(1) & \vdots & \lambda \\
\nu(1) & -\lambda + \nu(1) & \lambda & \nu(1) & -\nu(1)
\end{pmatrix}
\]

Obviously, \( \text{rank}(B) = N \). Since the size of the matrix \( B \) is \( N \), the matrix \( B \) is invertible, and \( (-B)^{-1} > 0 \).

Let \( H^{(d)} \) and \( \varphi^{(d)} \) be two column vectors of size \( N \) obtained through omitting the first element of the two column vectors \( f^{(d)} - \eta^d e \) and \( g^{(d)} \) with size \( N \), respectively. Then,

\[
H^{(d)} = \begin{pmatrix}
H_1^{(d)} \\
H_2^{(d)} \\
\vdots \\
H_K^{(d)} \\
H_{K+1}^{(d)} \\
\vdots \\
H_N^{(d)}
\end{pmatrix} = \begin{pmatrix}
f^{(d)}(1) - \eta^d \\
f^{(d)}(2) - \eta^d \\
\vdots \\
f^{(d)}(K) - \eta^d \\
f(K + 1) - \eta^d \\
\vdots \\
f(N) - \eta^d
\end{pmatrix} = \begin{pmatrix}
B_1^{(d)} - D^{(d)} - P [A_1^{(d)} - F^{(d)}] \\
B_2^{(d)} - D^{(d)} - P [A_2^{(d)} - F^{(d)}] \\
\vdots \\
B_K^{(d)} - D^{(d)} - P [A_K^{(d)} - F^{(d)}] \\
B_{K+1}^{(d)} - D^{(d)} - P [A_{K+1}^{(d)} - F^{(d)}] \\
\vdots \\
B_N^{(d)} - D^{(d)} - P [A_N^{(d)} - F^{(d)}]
\end{pmatrix}
\]

and

\[
\varphi^{(d)} = \begin{pmatrix} g^{(d)}(1), g^{(d)}(2), \ldots, g^{(d)}(K); g^{(d)}(K + 1), g^{(d)}(K + 2), \ldots, g^{(d)}(N) \end{pmatrix}^T.
\]

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Therefore, it follows from (22) that
\[ -B\varphi^{(d)} = H^{(d)} + \nu (d_1) e_1 g^{(d)} (0), \]
where \( e_1 \) is a column vector with the first element be one and all the others be zero. Note that the matrix \(-B\) is invertible and \((-B)^{-1} > 0\), thus the system of linear equations (43) always exists one unique solution
\[ \varphi^{(d)} = (-B)^{-1} H^{(d)} + \nu (d_1) (-B)^{-1} e_1 \cdot \Im, \]
where \( g^{(d)} (0) = \Im \) is any given constant. Thus we have
\[ g^{(d)} = \left( 0, (-B)^{-1} H^{(d)} \right)^T + \left( 1, \nu (d_1) (-B)^{-1} e_1 \right)^T \Im, \]
(45)
Note that \( B^{(d)} e = 0 \), thus a general solution to the Poisson equation is further given by
\[ g^{(d)} = \left( 0, (-B)^{-1} H^{(d)} \right)^T + \left( 1, \nu (d_1) (-B)^{-1} e_1 \right)^T \Im + \xi e, \]
where \( \Im \) and \( \xi \) are two free constants.

Based on the above analysis, we summarize the general solution of the Poisson equation, which is the first to be able to have two different free constants.

**Theorem 8** For the Poisson equation \(-B^{(d)} g^{(d)} = f^{(d)} - \eta d e\), it exists a key special solution \( g_{Sp}^{(d)} = \left( 0, (-B)^{-1} H^{(d)} \right)^T \), and its general solution are related to two free constants such that
\[ g^{(d)} = g_{Sp}^{(d)} + \left( 1, \nu (d_1) (-B)^{-1} e_1 \right)^T \Im + \xi e, \]
where \( \xi \) is regarded as a potential displacement constant, and \( \Im \) is a solution-free constant.

**Remark 5** In an ordinary discussion, note that \( \pi^{(d)} g^{(d)} = \eta d \) and the matrix \(-B^{(d)} + e \pi^{(d)} \) is invertible, thus the Poisson equation \(-B^{(d)} g^{(d)} = f^{(d)} - \eta d e\) can become
\[ \left( -B^{(d)} + e \pi^{(d)} \right) g^{(d)} = f^{(d)}. \]
This gives a solution of the Poisson equation as follows:
\[ g^{(d)} = \left( -B^{(d)} + e \pi^{(d)} \right)^{-1} f^{(d)} + \xi e, \]
which is still a special solution of the Poisson equation.
To provide an explicit expression for the solution to the system of linear equations (43), it is easy to see from (45) that we need to first establish an explicit expression for the invertible matrix \((-B)^{-1}\). While the explicit expression of the invertible matrix \((-B)^{-1}\) can be obtained by means of the RG-factorization, which is given in Li and Cao [54] for QBD processes, and more generally, Li [53] for general Markov processes.

To express the invertible matrix \((-B)^{-1}\), we can compute the UL-type \(U\)-measure as follows:

\[
U_N = -\nu (1), \\
U_k = \begin{cases} 
-\nu (1), & k = N - 1, N - 2, \ldots, K + 1, \\
-\nu (d_k), & k = K, K - 1, \ldots, 2, 1,
\end{cases}
\]

hence the UL-type \(R\)- and \(G\)-measures are respectively given by

\[
R_k = \begin{cases} 
\lambda \nu^{-1} (d_{k+1}), & k = 1, 2, \ldots, K, \\
\lambda \nu^{-1} (1), & k = K + 1, K + 2, \ldots, N - 1,
\end{cases}
\]

and

\[
G_k = 1, \quad k = 1, 2, \ldots, N.
\]

Thus the UL-type RG-factorization of the birth-death process \(B\) is given by

\[
B = (I - R_U) U_D (I - G_L),
\]

where

\[
R_U = \begin{pmatrix} 
0 & R_1 \\
0 & R_2 \\
& \ddots & \ddots \\
0 & R_{N-1} & 0
\end{pmatrix}, \quad G_L = \begin{pmatrix} 
0 & 0 \\
G_2 & 0 \\
& \ddots & \ddots \\
& & & G_N & 0
\end{pmatrix},
\]

and

\[
U_D = \text{diag} (U_1, U_2, \ldots, U_N).
\]

Therefore, we obtain

\[
(-B)^{-1} = (I - G_L)^{-1} (-U_D)^{-1} (I - R_U)^{-1}.
\]
Let

\[
(-B)^{-1} = \begin{pmatrix}
  x_{1,1} & x_{1,2} & \cdots & x_{1,N-1} & x_{1,N} \\
  x_{2,1} & x_{2,2} & \cdots & x_{2,N-1} & x_{2,N} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{N-1,1} & x_{N-1,2} & \cdots & x_{N-1,N-1} & x_{N-1,N} \\
  x_{N,1} & x_{N,2} & \cdots & x_{N,N-1} & x_{N,N}
\end{pmatrix}
\]

Then the element \(x_{n,m}\) is given by

(a) for \(n < m\)

\[
x_{n,m} = \begin{cases}
  \sum_{k=1}^{n} \frac{\lambda^{n-k}}{v(d_j)} & 1 \leq n < m \leq K, \\
  \sum_{k=1}^{n} \frac{\lambda^{n-k}}{(\mu_1+\mu_2)^{n-K} \prod_{j=k}^{K} v(d_j)} & 1 + K \leq n < m \leq N;
\end{cases}
\]

(b) for \(n = m\)

\[
x_{n,n} = \begin{cases}
  \sum_{k=1}^{n} \frac{\lambda^{n-k}}{v(d_j)} & 1 \leq n \leq K, \\
  \sum_{k=1}^{n} \frac{\lambda^{n-k}}{(\mu_1+\mu_2)^{n-K} \prod_{j=k}^{K} v(d_j)} & 1 + K \leq n \leq N;
\end{cases}
\]

(c) for \(n > m\)

\[
x_{n,m} = x_{n-1,n-1}, \quad 2 \leq m < n \leq N - 1.
\]

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