Adaptive Graph-based Total Variation for Tomographic Reconstructions

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Abstract—Sparsity exploiting image reconstruction (SER) methods have been extensively used with Total Variation (TV) regularization for tomographic reconstructions. Local TV methods fail to preserve texture details and often create additional artifacts due to over-smoothing. Non-Local TV (NLTV) has been proposed as a solution to this but lacks continuous update and is computationally complex. In this paper we propose Adaptive Graph-based TV (AGT). Similar to NLTV our proposed method goes beyond spatial similarity between different regions of an image being reconstructed by establishing a connection between similar regions in the image regardless of spatial distance. However, it is computationally more efficient and involves updating the graph prior during every iteration making the connection between similar regions stronger. Since TV is a special case of graph TV the proposed method can also be seen as a generalization of SER and TV methods. It promotes sparsity in the wavelet graph and gradient domains. Extensive experimentation shows that when compared to other methods we achieve a better result with AGT in every case.

Index Terms—Tomography, Total Variation, Graphs, Iterative Image Reconstruction, Non-local Total Variation

I. INTRODUCTION

RECONSTRUCTING tomographic densities from low-dose electron tomography (ET) or computed tomography (CT) data is an ill-posed inverse problem. Low-dose is a constraint to prevent sample degradation in ET [1,2] and to reduce exposure to ionizing radiation in CT [3,4]. Such requirements are often met by collecting limited or low-contrast data which renders noisy and erroneous reconstructions. Iterative Image Reconstruction (IIR) methods [6-11] have proved to be more effective in handling noise when compared to analytical methods [12-14]. However, such methods are computationally inefficient. Initial IIR methods were algebraic in nature [15-20]. More recently sparsity exploiting reconstructions have been extensively used for image reconstruction. Such methods are often used with Total Variation (TV) regularization [21-26]. We refer to the joint CS and TV setup as CSTV in the sequel. Recently, non-local TV (NLTV) [27] has been shown to be much more efficient for inverse problems [28-32]. In contrast to simple TV, which takes into account the similarity of a region with only its neighboring regions, NLTV overcomes this limitation by associating a similarity measure of every region of an image with all other regions.

A primary short-coming of NLTV is that the similarity matrix constructed in the beginning from the initial estimate or prior is not updated throughout the algorithm [30,31]. It is kept fixed because NLTV-based method suffers from a high cost of associating a similarity measure between every pair of regions in an image. For an \( n \times n \) image, NLTV costs \( O(n^4) \). NLTV also requires a threshold parameter which, based on the euclidean distance, can decide if the similarity is strong enough to be non-zero. This parameter depends on the scale of pairwise distances. Although, the results obtained by NLTV and its variants are state-of-the-art, the final reconstruction would be more faithful to the data if the similarity matrix is regularly updated during every iteration throughout the algorithm.

Introduction to Graphs: Recently, graphs have emerged as a very powerful tool for signal modeling [33]. A graph is represented as a tuple \( G = \{V,E,W\} \) where \( V \) is a set of vertices, \( E \) a set of edges, and \( W : V \times V \rightarrow \mathbb{R}_+ \) a weight function. We assume that the vertices are indexed from 1, \ldots, |V|. The weight matrix \( W \) is assumed to be non-negative, symmetric, and with a zero diagonal. Each entry of the weight matrix \( W = \mathbb{R}_+^{\{V\} \times \{V\}} \) corresponds to the weight of the edge connecting the corresponding vertices: \( W[i,j] = W(v_i,v_j) \) and if there is no edge between two vertices, the weight is set to 0. A node \( v_i \) connected to \( v_j \) is denoted by \( i \leftrightarrow j \). For a vertex \( v_j \in V \), the degree \( d(i) \) is defined as the sum of the weights of incident edges: \( d(i) = \sum_{j=1}^{|V|} W_{i,j} \). Let \( D \) be the diagonal degree matrix with diagonal entries \( D_{ii} = d(i) \), then the graph Laplacian \( L \) is defined as the difference of the weight matrix \( W \) from the degree matrix \( D \), thus \( L = D - W \), which is referred to as combinatorial Laplacian.

Contributions: In this letter we propose Adaptive Graph Total Variation (AGTV) as a method for simultaneous reconstruction and denoising of tomographic data. Our method is a more sophisticated, faster and scalable form of NLTV and enjoys a relatively lower computational complexity. We promote the use of \( k \)-nearest neighbor graphs in Graph Total Variation (GTV), where \( k \) is fixed and unlike NLTV, does not depend on the scale of the pairwise distances. Our proposed method involves simultaneous denoising and reconstruction of tomographic sample by modeling its sparsity in two domains.
II. ADAPTIVE GRAPH TOTAL VARIATION (AGTV)

Let $S \in \mathbb{R}^{p \times q}$ be the sinogram corresponding to the projections of the sample $X \in \mathbb{R}^{n \times n}$ being imaged, where $p$ is the number of rays passing through $X$ and $q$ is the number of angular variations at which $X$ has been imaged. Let $b \in \mathbb{R}^{p\times q}$ be the vectorized measurements or projections $(b = vec(S))$, where $vec(\cdot)$ denotes the vectorization operation and $A \in \mathbb{R}^{p\times n^2}$ be the sparse projection operator. Then, the goal in a typical CT or ET based reconstruction method is to recover the vectorized sample $x = vec(X)$ from the projections $b$. We propose:

$$\min_x \|Ax - b\|_2^2 + \lambda \|\Phi^* (x)\|_1 + \gamma \|\nabla G(x)\|_1,$$

where $\Phi$ is the wavelet operator and $\Phi^* (x)$, where $*$ represents the adjoint operation, denotes the wavelet transform of $x$ and $\|\nabla G(x)\|_1$ denotes the total variation of $x$ w.r.t graph $G$. The first two terms of the objective function above comprise the sparse reconstruction (GT) part of our method and model the sparsity of the wavelet coefficients. The second term, to which we refer as the graph total variation (GTV) regularizer acts as an additional prior for denoising and smoothing. It can be expanded as:

$$\|\nabla G(x)\|_1 = \sum_i \|\nabla G x_i\|_1 = \sum_i \sum_{j \in N_i} \sqrt{W_{ij}} \|x_i - x_j\|_1,$$

where the second sum runs over all the neighbors of $i$, denoted by $N_i$. The above expression clearly states that GTV involves the minimization of the sum of the gradients of the signals on the nodes of the graphs. In our case, we assume that the elements of the vector $x$ lie on the nodes of the graph $G$ which are connected with the edges whose weights are $W_{ij}$. Thus, the minimization of the GTV would ensure that $x_i$ and $x_j$ possess similar values if $W_{ij}$ is high and dissimilar values if $W_{ij}$ is small or zero. As compared to the standard TV, the structure of the sample $x$ is taken into account for the reconstruction purpose. It is a fact well known that $l_1$ norm promotes sparsity, so the GTV can also be viewed as a regularization which promotes sparse graph gradients. This corresponds to enforcing a piecewise smoothness of the signal $x$ w.r.t graph $G$.

The proposed method with GTV can be seen as a generalization of the compressed sensing and total variation based method studied in [25]. While, the standard TV minimizes the gradients of the signal $x$ w.r.t its spatial neighbors only, the GTV does so in a region which is not restricted only to the neighbors of the elements in $x$. Thus, the standard TV can be viewed as a specific case of the GTV, where the graph $G_{grid}$ is a grid graph. In a grid graph $G_{grid}$ of a sample $x$, the pixels are only connected to its spatial neighbors (upper, lower, left and right) via unity weights.

A. Graph Construction for Total Variation

An important step for our method is to construct a graph $G$ for GTV regularization. Ideally, $G$ should be representative of the reconstructed sample $x$, however, this is unknown before the reconstruction. To cater this problem, we propose to construct $G$ from the patches of an initial naive estimate of the sample $x_{fbp}$ using filtered back projection (FBP) method. In the first step $x_{fbp} \in \mathbb{R}^{n \times n}$ is divided into $n^2$ overlapping patches. Let $s_i$ be the patch of size $l \times l$ centered at the $i^{th}$ pixel of $x_{fbp}$ and assume that all patches are vectorized, i.e., $s_i \in \mathbb{R}^{n^2}$. In the second step the search for the closest neighbors for all vectorized patches is performed using the Euclidean distance metric. Each $s_i$ is connected to its $K = 10$ nearest neighbors $s_j$, resulting in $|E|$ number of connections. In the third step the graph weight matrix $W$ is computed using the Gaussian kernel weighting scheme, for which the parameter $\sigma$ is set experimentally as the average distance of the connected samples. Finally, the combinatorial Laplacian is computed.

B. Adaptive Graph Total Variation Regularization

The above description refers only to the non-adaptive part, where the graph $G$ is fixed. It is important to point out that the initial estimate of the graph $G$, obtained via the filtered back projection $x_{fbp}$ is not very faithful to the final solution $x$. As $x$ is being refined in every iteration, it is natural to update the graph $G$ as well in every iteration. This simultaneous update of the graph $G$ corresponds to the adaptive part of the proposed algorithm and its significance will be explained in detail in the supplementary Section A of the paper.

III. OPTIMIZATION SOLUTION

In the spirit of similar non-graph methods such as [25], we refer to eq. (1) without the graph update as “Compressed Sensing & Graph Total Variation (CSGT)”. We make use of “forward backward based primal dual method” [35] [36] to solve CSGT and then update the graph from the obtained sample in every iteration, until convergence. The complete algorithm with graph updates is called AGTV (we discard CS for the simplicity of the name). The main steps of this algorithm are visualized in Fig. 1.

The first term of (1), $f : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ is a convex differentiable function defined as $f(x) = \|Ax - b\|_2^2$. This function has a $\beta$-Lipschitz continuous gradient $\nabla f(x) = 2 A^T (Ax - b)$. Note that $\beta = 2 \|A\|_2$ where $\|A\|_2$ is the spectral norm (or maximum
Fig. 1: The complete methodology for AGTV. The input sinogram \( f \) projections \( b \in \mathbb{R}^{p \times q} \) first used to obtained a filtered back projection (FBP) \( x_{fbp} \in \mathbb{R}^{n \times n} \). It is then used to construct the initial patch graph \( G \) to be used by the CSGTV method. The output of CSGTV is used to refine / reconstruct the graph and this process is repeated until convergence.

The third term in eq. (1) \( g : \mathbb{R}^{[E]} \rightarrow \mathbb{R} \), where \(|E|\) denotes the cardinality of \( E \) the set of edges in \( G \), is a convex function defined as \( g(D) = \gamma_1\|D\|_1 \). The proximal operator is:

\[
\text{prox}_{\gamma_2}(D) = \text{sgn}(D) \circ \max(|D| - \lambda \gamma_1, 0),
\]

where \( \circ \) denotes the Hadamard product and \( D = \nabla g x \). The proximal operator of the function \( h = \lambda\|\Phi^{-1}(x)\|_1 \) is the \( \ell_1 \) soft-thresholding given by the elementwise operations.

\[
\text{prox}_{\gamma_3}(B(x)) = \text{sgn}(B(x)) \circ \max(|B(x)| - \lambda, 0),
\]

where \( B = \Phi^* \) denotes the adjoint wavelet operator and \( B(x) \) denotes the wavelet transform of \( x \).

Using these tools, we can use the forward backward based primal dual approach presented in [35] for AGTV, to define Algorithm 1 where \( \gamma_1, \gamma_2, \gamma_3 \) are convergence parameters \( \epsilon \) the stopping tolerance and \( I, J \) the maximum number of iterations. \( \delta \) is a very small number to avoid a possible division by zero. Since we use Unlocbox for solving the optimization problem, the convergence parameters \( \gamma_1, \gamma_2, \gamma_3 \) are set automatically according to the specified \( \beta \). \( U_j \) corresponds to the primal and \( V_j \) to the dual variable in Algorithm 1.

**Complexity:** As mentioned earlier, we use the Fast Approximate Nearest Neighbors search algorithm (FLANN) [34]. The computational complexity of the FLANN algorithm for \( n^2 \) patches of size \( i^2 \) each and fixed \( K \) is \( O(n^2 \log(n^2)) \). Note that \( i^2 \) and \( K \) do not appear in the complexity because they are constants. Furthermore, \( n^2 \) is the size of the sample under consideration so the computational complexity is much lower as compared to the NLTV [23] based methods. Let \( J \) denote the number of iterations for the algorithm (for loop in Algorithm 1) to converge, and \( J \) the number of outer iterations (step 4 of Algorithm 1), then the computational cost of our algorithm is \( O(J |E| |I|) \), where \(|E|\) denotes the number of non-zeros edges in the graph \( G \). For a \( K \)-nearest neighbors graph \(|E| \approx Kn^2 \) the computational complexity of our algorithm is linear in the size of the data sample \( n^2 \), i.e. \( O(JKn^2 |I|) \).

Algorithm 1 Forward-backward primal dual for AGTV

\[
x_0 = x_{fbp}
\]

1. **INPUT:** \( U_0 = x_0, V_0 = \nabla g x_0, \epsilon > 0 \)

   for \( j = 0, \ldots, J - 1 \) do

   a. \( P_j = \text{prox}_{\gamma_1 h}(\Phi^*(U_j) - \tau_3 \Phi^*(\nabla f(U_j) + \nabla g V_j)) \)

   b. \( T_j = V_j + \tau_2 \nabla g (2P_j - U_j) \)

   c. \( Q_j = T_j - \tau_3 \text{prox}_{\gamma_3} \left( \frac{1}{\tau_3} T_j \right) \)

   d. \( U_{j+1} = U_j + \tau_1 ((P_j, Q_j) - (U_j, V_j)) \)

   if \( \|U_{j+1} - U_j\|_2^2 < \epsilon \) and \( \|V_{j+1} - V_j\|_2^2 < \epsilon \) then BREAK

   end if

   end for

2. \( x = U_j + 1 \)

3. Construct patch graph \( G \) from \( x \)

Repeat steps 1 to 3 for \( I \) iterations

**OUTPUT:** \( x = U_j + 1 \)

The complexity of our algorithm is \( O(JKn^2 |I|) \) and the graph \( G \) is \( O(n^2 \log(n^2)) \). The graph \( G \) needs to be updated once in every outer iteration of the algorithm \( I \), thus the overall complexity of the proposed AGTV method is \( O(I(JKn^2 + n^2 \log(n^2))) \).

**IV. EXPERIMENTAL RESULTS**

To test the performance of our AGTV method, we perform reconstructions for many different types of phantoms from different number of projections with varying levels of Poisson noise, using GSPBox [35] UNLocBox [37]. Reconstructions were judged on an \( \ell_2 \) reconstruction error metric. We compare the performance of AGTV with many state-of-the-art iterative and convex optimization based algorithms, which include FBP, ART (Kaczmarz), SIRT (Cimmino), CS, CSTV and CSGTV.

Each of these methods has its own model parameters, which need to be set or tuned in an appropriate manner. ART (Kaczmarz) and SIRT (Cimmino) were performed using FBP as a priori. The stopping criteria for ART and SIRT was set to 100 iterations and the relaxation parameter \( \gamma \) was tuned to achieve the best result. For the graph based reconstruction (CSGT and AGTV) a graph prior \( G \) was generated by dividing the result from FBP into patches as explained in Section II-A. For example, for a Shepp-Logan phantom of size 64 \times 64, the graph was constructed by dividing it into 64 \times 64 = 4096 overlapping patches of size 3 \times 3, \( K = 15 \) and setting \( \sigma \) for the weight matrix to the average distance of the 15-nearest neighbors. For Algorithm 1, we set \( I = J = 50 \) and the convergence parameters \( \gamma_1, \gamma_2, \gamma_3 \) were set automatically by UnlocBox. It is worth mentioning here that our GTV based adaptive graph regularization is a faster method of implementing NLTV by using \( K \)-nearest neighbors graph. Thus the GTV and NLTV based regularization are equivalent in performance. Therefore, we did not include comparisons with the NLTV based method.

To explain the performance of our model in detail we reconstructed a 64 \times 64 Shepp-Logan [39] phantom from 36 erroneous projections. A sinogram \( S \) was built by projecting the phantom using Radon transform and 36 equally spaced projections were collected from 0 to 180 degrees. The sinogram was then corrupted with 10% Poisson noise. Fig. [3]
Fig. 2: Comparative analysis of reconstructing Shepp-Logan using various reconstruction methods. The sinogram of a $64 \times 64$ Shepp-Logan phantom corrupted with 10% Poisson noise was reconstructed using FBP (Linearly interpolated, Cropped Ram-Lak filter); CSTV ($\lambda = 0.5$, $\gamma = 0.1$, Prior: FBP, Stopping Criteria = 100 iterations); CSGTV ($\lambda = 0.5$, $\gamma = 0.2$, Prior: Patch Graph from FBP, Stopping Criteria = 100 iterations); AGTV ($\lambda = 0.5$, $\gamma = 1$, Prior: Patch Graph from FBP updated every iteration, $I$ and $J$ in Algorithm 1 set to 30). AGTV clearly gives a better intensity profile as compared to all other methods while preserving the edges.

Fig. 3: Comparative analysis of reconstructing a Shepp-Logan phantom using various reconstruction methods at 5% and 10% Poisson noise. FBP (Linearly interpolated, Cropped Ram-Lak filter); ART (Kaczmarz/Randomized Kaczmarz, Relaxation Parameter ($\eta$) = 0.25, Prior: FBP, Stopping Criteria = 100 iterations); SIRT (Cimmino/SART, ($\eta$) = 0.25, Prior: FBP, Stopping Criteria = 100 iterations); CS (500 Iterations, Prior: FBP); CSTV ($\lambda = 0.5$, $\gamma = 0.1$, Prior: FBP, Stopping Criteria = 100 iterations); CSGTV ($\lambda = 0.5$, $\gamma = 0.2$, Prior: Patch Graph from FBP, Stopping Criteria = 100 iterations); AGTV ($\lambda = 0.5$, $\gamma = 1$, Prior: Patch Graph from FBP updated every iteration, $I$ and $J$ in Algorithm 1 set to 30).

V. CONCLUSIONS

Similar to NLTV our proposed method (AGTV) goes beyond spatial similarity between different regions of an image being reconstructed by establishing a connection between similar regions in the image regardless of spatial distance. However, our approach is much more scalable and computationally efficient because it uses the approximate nearest neighbor search algorithm for graph construction, making it much more likely to be adapted in a clinical setting. Beyond NLTV, our proposed approach is adaptive. The non-local graph prior is updated every iteration making the connection between similar regions stronger. Thus improving the overall reconstruction quality as demonstrated by experiments. Since TV is a special case of graph TV the proposed method can be seen as a generalization of CS and TV methods and can promote future application specific studies for using CS for tomographic reconstruction from limited data.
REFERENCES

[1] J. Frank, *Electron tomography: methods for three-dimensional visualization of structures in the cell*. Springer Science & Business Media, 2008.

[2] A. Leis, M. Beck, M. Gruska, C. Best, R. Hegerl, and J. Leis, “Cryo-electron tomography of biological specimens,” *IEEE Signal Processing Magazine*, vol. 23, no. 3, pp. 95–103, May 2006.

[3] A. Berrington de González, “Projected Cancer Risks From Computed Tomographic Scans Performed in the United States in 2007,” *Archives of Internal Medicine*, vol. 169, no. 22, p. 2071, Dec. 2009.

[4] D. J. Brenner and E. J. Hall, “Computed tomography—an increasing source of radiation exposure,” *N Engl J Med*, vol. 357, pp. 2277–84, 2007.

[5] M. S. Pearce, J. A. Salotti, M. P. Little, K. McHugh, C. Lee, K. P. Kim, N. L. Howe, C. M. Ronkers, P. Rajaraman, A. W. Craft et al., “Radiation exposure from ct scans in childhood and subsequent risk of leukemia and brain tumours: a retrospective cohort study,” *The Lancet*, vol. 380, no. 9840, pp. 499–505, 2012.

[6] J. A. Fessler, “Statistical image reconstruction methods for transmission tomography,” *Handbook of medical imaging*, vol. 2, pp. 1–70, 2000.

[7] Y. Censor, “Finite series-expansion reconstruction methods,” *Proceedings of the IEEE*, vol. 71, no. 3, pp. 409–419, 1983.

[8] J. A. Fessler, “Row-action methods for huge and sparse systems and their applications,” *SIAM review*, vol. 23, no. 4, pp. 444–466, 1981.

[9] J. Qi and R. M. Leahy, “Iterative reconstruction techniques in emission computed tomography,” *Physics in medicine and biology*, vol. 51, no. 5, p. R541, 2006.

[10] U. Skoglund, L.-G. Öfverstedt, R. M. Burnett, and G. Bricogne, “Maximum-Entropy Three-Dimensional Reconstruction with Deconvolution of the Contrast Transfer Function: A Test Application with Adenovirus,” *Journal of Structural Biology*, vol. 117, no. 3, pp. 173–188, Nov. 1996.

[11] H. Rullgård, O. Öktem, and U. Skoglund, “A componentwise iterated relative entropy regularization method with updated prior and regularization parameter,” *Inverse Problems*, vol. 23, no. 5, pp. 2121–2139, Oct. 2007.

[12] F. Natterer, *The mathematics of computerized tomography*. Siam, 1986, vol. 32.

[13] E. T. Quinto, U. Skoglund, and O. Öktem, “Electron lambdatomography,” *Proceedings of the National Academy of Sciences*, vol. 106, no. 51, pp. 21842–21847, 2009.

[14] J. Hsieh, “Computed tomography: principles, design, artifacts, and recent advances.” SPIE Bellingham, WA, 2009.

[15] R. Gordon, R. Bender, and G. T. Herman, “Algebraic Reconstruction Techniques (ART) for three-dimensional electron microscopy and X-ray photography,” *Journal of Theoretical Biology*, vol. 29, no. 3, pp. 471–481, Dec. 1970.

[16] G. Cimmino and C. N. delle Ricerche, *Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari*. Istituto per le applicazioni del calcolo, 1938.

[17] P. C. Hansen and M. Saxild-Hansen, “AIR tools—a MATLAB package of algebraic iterative reconstruction methods,” *Journal of Computational and Applied Mathematics*, vol. 236, no. 8, pp. 2167–2178, 2012.

[18] L. Landweber, “An iteration formula for Fredholm integral equations of the first kind,” *American journal of mathematics*, vol. 73, no. 3, pp. 615–624, 1951.

[19] A. Brandt, “Algebraic multigrid theory: The symmetric case,” *Applied mathematics and computation*, vol. 19, no. 1, pp. 23–56, 1986.

[20] T. Srolovitz and R. Vershynin, “A randomized kaczmarz algorithm with exponential convergence,” *Journal of Fourier Analysis and Applications*, vol. 15, no. 2, pp. 262–278, 2009.

[21] C. G. Graff and E. Y. Sidky, “Compressive sensing in medical imaging,” *Applied optics*, vol. 54, no. 8, pp. C23–C44, 2015.

[22] G.-H. Chen, J. Tang, and S. Leng, “Prior image constrained compressed sensing (piccs): a method to accurately reconstruct dynamic ct images from highly undersampled projection data sets,” *Medical physics*, vol. 35, no. 2, pp. 660–663, 2008.

[23] J. Song, Q. H. Liu, G. A. Johnson, and C. T. Badea, “Sparseness prior based iterative image reconstruction for retrospectively gated cardiac micro-c,” *Medical physics*, vol. 34, no. 11, pp. 4476–4483, 2007.

[24] L. Ritschl, F. Bergner, C. Fleischmann, and M. Kachelrieß, “Improved total variation-based ct image reconstruction applied to clinical data,” *Physica in medicine and biology*, vol. 56, no. 6, p. 1545, 2011.

[25] J. Tang, B. E. Nett, and G.-H. Chen, “Performance comparison between total variation (TV)-based compressed sensing and statistical iterative reconstruction algorithms,” *Physics in Medicine and Biology*, vol. 54, no. 19, pp. 5781–5804, Oct. 2009.

[26] Z. Tian, X. Jia, K. Yuan, T. Pan, and S. B. Jiang, “Low-dose ct reconstruction via edge-preserving total variation regularization,” *Physics in medicine and biology*, vol. 56, no. 18, p. 5949, 2011.

[27] Y. Lou, X. Zhang, S. Osher, and A. Bertozzi, “Image recovery via nonlocal operators,” *Journal of Scientific Computing*, vol. 42, no. 2, pp. 185–197, 2010.

[28] G. Peyré, S. Bougleux, and L. Cohen, “Non-local regularization of inverse problems,” in *European Conference on Computer Vision*. Springer, 2008, pp. 57–68.

[29] G. Gilboa and S. Osher, “Nonlocal operators with applications to image processing,” *Multiscale Modeling & Simulation*, vol. 7, no. 3, pp. 1005–1028, 2008.

[30] J. Huang and F. Yang, “Compressed magnetic resonance imaging based on wavelet sparsity and nonlocal total variation,” in 2012 9th IEEE International Symposium on Biomedical Imaging (ISBI). IEEE, 2012, pp. 968–971.

[31] J. Liu, H. Ding, S. Molloi, X. Zhang, and H. Gao, “Ticmr: Total image constrained material reconstruction via nonlocal total variation regularization for spectral ct,” 2016.

[32] X. Jia, Y. Lou, B. Dong, Z. Tian, and S. Jiang, “4d computed tomography reconstruction from few-projection data via temporal non-local regularization,” in *International Conference on Medical Image Computing and Computer-Assisted Intervention*. Springer, 2010, pp. 143–150.

[33] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, “The Emerging Field of Signal Processing on Graphs: Extending High-Dimensional Data Analysis to Networks and Other Irregular Domains,” arXiv preprint arXiv:1211.0053, 2012.

[34] M. Muja and D. G. Lowe, “Scalable nearest neighbor algorithms for high dimensional data,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 36, no. 11, pp. 2227–2240, 2014.

[35] N. Komodakis and J.-C. Pesquet, “Playing with duality: An overview of recent primal? dual approaches for solving large-scale optimization problems,” *IEEE Signal Processing Magazine*, vol. 32, no. 6, pp. 31–54, 2015.

[36] P. L. Combettes and J.-C. Pesquet, “Proximal splitting methods in signal processing,” in *Fixed-point algorithms for inverse problems in science and engineering*. Springer, 2011, pp. 185–212.

[37] N. Perraudin, D. Shuman, G. Puy, and P. Vandergheynst, “Unlbox a matlab convex optimization toolbox using proximal splitting methods,” arXiv preprint arXiv:1402.0779, 2014.

[38] N. Perraudin, J. Paratte, D. Shuman, V. Kalofolias, P. Vandergheynst, and D. K. Hammond, “Gspbox: A toolbox for signal processing on graphs,” arXiv preprint arXiv:1408.5781, 2014.

[39] L. A. Shepp and B. F. Logan, “The Fourier reconstruction of a head section,” *IEEE Transactions on Nuclear Science*, vol. 21, no. 3, pp. 21–43, Jun. 1974.
A. Working Explanation of AGTV

We present a simple example to motivate the use of AGTV rather than simple CSGTV and CSTV. Clearly, the compressed sensing part of all these methods is responsible for retrieving the sample \( x \) from the projections \( b \). Thus, our comparison study is focused on the two regularizers, i.e., Adaptive Graph Total Variation (AGTV) and Total Variation (TV). Our two step exposition is: 1) CSGTV is better than CSTV, 2) Adaptive Graph Total Variation (AGTV) is better than CSGTV. Consider the example of a Shepp-Logan Phantom as shown in top leftmost plot of Fig. 5. The goal is to recover this phantom from its noisy projections so that the recovered sample is faithful to its original clean version. The CSTV method requires a total variation prior to recover the sample while the CSGTV method requires a graph total variation prior for the recovery. Both methods need an initial estimate for the construction of this prior, therefore, for the ease of demonstration we use the filtered back projection (FBP) as an initial estimate of the sample. Recall that our proposed method decomposes the FBP into \( n \times n \) patches of size \( l \times l \) each. Let \((i, j)\) denote the (horizontal, vertical) position of the center of each patch then:

- For the total variation, each patch \( s_{i,j} \) is connected to its spatial neighbors only, i.e., \( s_{i+1,j}, s_{i-1,j}, s_{i,j+1}, s_{i,j-1} \), as shown in Fig. 4. These connections are fixed throughout the algorithm.
- For the graph total variation, each patch \( s_{i,j} \) is only connected to the patches which are among the \( K \) nearest neighbors.

Note that unlike TV the connected patches can be spatially far from each other.

Now let us take the example of two patches ‘a’ and ‘b’ as labeled in the FBP of Fig. 4. Comparing with the clean phantom in Fig. 5, it is obvious that these patches should possess the same texture at the end of the reconstruction algorithm. Therefore, an intelligent regularizer should take into account the inherent similarity between these patches. To explain the difference between the TV and GTV priors we use a point model as shown in Fig. 4, where each point corresponds to a patch in the FBP. Since ‘a’ and ‘b’ are not spatially co-located, the total variation prior does not establish any connection between these patches. Thus, TV fails to exploit the similarity between these patches throughout the algorithm. This leads to slightly different textures for the two patches, as shown in the 3rd row of Fig. 5.

Now consider the case of GTV. Even though the initial estimate of graph \( G \) is obtained from the noisy estimate of sample, i.e., the FBP, patches ‘a’ and ‘b’ still possess enough structural resemblance to be connected together by an edge (even if it is weak) in the graph. Now, if the graph is kept fixed which is the case of CSGTV, one still obtains a better result as compared to CSTV, as shown in the 4th row of Fig. 5. This is due to the fact that the important connections are established by the graph \( G \) and similarity of patches is not restricted to spatially co-located patches only. This is also obvious from the intensity profile analysis in the 4th row of Fig. 5. Finally, we discuss the case of AGTV, where the graph \( G \) is updated in every iteration of the algorithm. Obviously, every iteration of the algorithm leads to a cleaner sample and updating the graph \( G \) is only going to make the connection between the patches ‘a’ and ‘b’ stronger. This leads to significantly better result than CSTV and CSGTV as shown in Fig. 5. Note that the patches ‘a’ and ‘b’ possess almost the same structure at the end of AGTV.

![Fig. 4: A comparison of the Total Variation (TV) and Adaptive Graph Total Variation (AGTV) priors for the methods CSGT and AGTV. The TV prior does not connect patches ‘a’ and ‘b’ which possess structural similarity, whereas the GTV prior connects them because the \( K \)-nearest neighbor graph is not restricted to spatial neighbors only. Furthermore, this connection keeps getting stronger due to iterative removal of noise and graph updates in every iteration.](image)

It is possible to appreciate this visually as the phantom obtained via AGTV is very similar to the original phantom. Furthermore, a comparison of the intensity profiles of the two phantoms also reveals the same fact. The next best result is obtained by CSGT. Algorithmically, the only difference between CSGT and AGTV is the regular graph update step in the latter, which tends to make the final reconstruction more faithful to the original phantom. CSTV also obtains a reasonable reconstruction, though worse than AGTV. CS alone however, has a poor performance. This is not surprising, as for the tomography applications, CS has been mostly used in combination with TV, as it alone does not preserve the GMI.

It is also interesting to note that the performance of AGTV saturates after 90 projections for each of the three cases, i.e., the reconstruction error does not improve if the number of projections are increased. Furthermore, for each of the three noise cases one can observe that the drop in the reconstruction error from 50 to 90 projections is not significant. Although, the same observation can be made about CSGT, the error is always higher than AGTV. All the other methods, perform far worse than
Fig. 5: Comparative analysis of reconstructing Shepp-Logan using various reconstruction methods. The sinogram of a $64 \times 64$ Shepp-Logan phantom corrupted with 10% Poisson noise was reconstructed using FBP (linearly interpolated, cropped Ram-Lak filter); ART (Kaczmarz/Randomized Kaczmarz, Relaxation Parameter ($\eta$) = 0.25, Prior: FBP, Stopping Criteria = 100 iterations); SIRT (Cimmino/SART, ($\eta$) = 0.25, Prior: FBP, Stopping Criteria = 100 iterations); CS (500 Iterations, Prior: FBP); CSTV ($\lambda = 0.5$, $\gamma = 0.1$, Prior: FBP, Stopping Criteria = 100 iterations); CSGTV ($\lambda = 0.5$, $\gamma = 0.2$, Prior: Patch Graph from FBP, Stopping Criteria = 100 iterations); AGTV ($\lambda = 0.5$, $\gamma = 1$, Prior: Patch Graph from FBP updated every iteration, $I$ and $J$ in Algorithm 1 set to 30). AGTV clearly gives a better intensity profile as compared to all other methods while preserving the edges.

Fig. 6: Comparative analysis of reconstructing a Torso phantom using various reconstruction methods. The sinogram of a $128 \times 128$ Torso phantom corrupted with 5% Gaussian Random noise was reconstructed using FBP (linearly interpolated, cropped Ram-Lak filter); CSTV ($\lambda = 0.5$, $\gamma = 0.1$, Prior: FBP, Stopping Criteria = 100 iterations); CSGTV ($\lambda = 0.5$, $\gamma = 0.2$, Prior: Patch Graph from FBP, Stopping Criteria = 100 iterations); AGTV ($\lambda = 0.5$, $\gamma = 1$, Prior: Patch Graph from FBP updated every iteration, $I$ and $J$ in Algorithm 1 set to 30).
AGTV. These tables clearly, lead to the conclusion that AGTV is a step towards getting very fine reconstructions from a very small number of projections, via a scalable method.

B. Hyperparameter tuning

Our model has two hyper-parameters, $\lambda$ for tuning the sparsity of CS based reconstruction and $\gamma$ to tune the amount of smoothing and denoising in the reconstruction. While, these are model hyper-parameters and need tuning, the graph parameter $K$, i.e., the number of nearest neighbors is quite easy to set for our application. This is shown in Fig. 7, where we perform a small experiment corresponding to the reconstruction of a $32 \times 32$ Shepp-Logan phantom from 36 projections $b \in \mathbb{R}^{36}$ using the pre-tuned parameters $\lambda = 0.1, \gamma = 5$ for different values of $K$ ranging from 5 to 50. The results clearly show that the reconstruction is quite robust to the choice of $K$, with a small error variation. Thus, $K$ is easy to set for our application. As the complexity of our proposed algorithm scales with the number of edges $|\mathcal{E}|$ in the graph $\mathcal{G}$ and $|\mathcal{E}| \approx Kn^2$, it is recommended to set $K$ as small as possible. However, a very small $K$ might lead to many disconnected components in the graph $\mathcal{G}$. On the other hand, a very large $K$ might increase the time required for the algorithm to converge and reduce the computational advantage we have over the NLTV method. Therefore, we choose to set $K = 15$ for our experiments.

In order to show the variation of reconstruction error with $(\lambda, \gamma)$ grid, we perform another experiment for the reconstruction of the Shepp-Logan phantom of size $32 \times 32$ from 36 projections. For this experiment we keep $K = 15$ and perform the reconstruction for every pair of parameter values in the tuple $(\lambda, \gamma)$, where $\lambda \in (0.1, 1)$ and $\gamma \in (0.1, 10)$. The reconstruction error grid is shown in Fig. 7. The minimum error 0.11 occurs at $\lambda = 0.2, \gamma = 0.1$. It is also interesting to note that the error increases gradually with an increase in the parameter values.

C. Shortcomings & Future Directions

The proposed AGTV method has proven to produce much better reconstructions as compared to the state-of-the-art CSTV method. Although, the proposed method is computationally far less cumbersome than NLTV, it still suffers from a few problems which we discuss in this section. The computational complexity of the proposed method is $O(I(JKn^2 + n^2 \log(n^2)))$. As already presented in Algorithm 1, the method requires a double loop, the outer with $I$ iterations and the inner with $J$ iterations. For our experiments we set $I = J = 50$. The main computational burden is offered by the graph construction, which needs to be performed every $J$ iterations. Thus, the method still suffers from a high complexity because of the double loop and regular graph updates. The complexity of graph construction can be reduced by using a parallel implementation of FLANN which is provided by the authors [33]. The degree of parallelism can be increased at the cost of increasing approximation in the estimation of nearest neighbors. As a result of this the graph $\mathcal{G}$ will be different every time the FLANN algorithm is run. However, this does not effect the quality of the graph and for tomographic applications, negligible loss in the performance was observed. It is obviously of interest to reduce the number of inner iterations $J$ and the complexity of the operations in the for loop. Our future work will therefore focus on introducing some approximations in the proposed algorithm to make it faster.

Tuning the hyperparameters is another short-coming of the proposed method. It is reasonable to set the number of $K$-nearest neighbors to 10 or 15, however, the sparsity parameter $\lambda$ and the GTV parameter $\gamma$ need to be tuned properly and are not known beforehand. The results of the validation experiment from Fig. 7 show that the error increases gradually with the parameter values. Our future work will thus also focus on finding smart methods to set these parameters automatically for specific tomographic applications.