NUMERICAL INVESTIGATION OF THE SMALLEST EIGENVALUES OF THE $p$-LAPLACE OPERATOR ON PLANAR DOMAINS

JIŘÍ HORÁK

Abstract. The eigenvalue problem for the $p$-Laplace operator with $p > 1$ on planar domains with the zero Dirichlet boundary condition is considered. The Constrained Descent Method and the Constrained Mountain Pass Algorithm are used in the Sobolev space setting to numerically investigate the dependence of the two smallest eigenvalues on $p$. Computations are conducted for values of $p$ between 1.1 and 10. Symmetry properties of the second eigenfunction are also examined numerically. While for the disk an odd symmetry about the nodal line dividing the disk in halves is maintained for all the considered values of $p$, for rectangles and triangles symmetry changes as $p$ varies. Based on the numerical evidence the change of symmetry in this case occurs at a certain value $p_0$ which depends on the domain.

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$ and a parameter $p \in (1, \infty)$ consider the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

(1)

to be solved for a real function $u : \Omega \to \mathbb{R}$ and a parameter $\lambda \in \mathbb{R}$. The operator $\Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$ is called the $p$-Laplace operator. If for a certain $\lambda$ a nontrivial weak solution $u \in W^{1,p}_0(\Omega)$ of (1) exists, we call $\lambda$ and $u$ Dirichlet eigenvalue and eigenfunction of the $p$-Laplace operator, respectively. Problem (1) is homogeneous but in general not additive in $u$.

From [1, 20, 21, 4] and others it is well known that there exists the smallest eigenvalue $\lambda_1$ and that it is positive, isolated and simple (i.e., the corresponding eigenfunction $u_1$ is unique up to multiplication by a constant). Moreover, for any eigenfunction $u$ it holds: $u$ corresponds to $\lambda_1$ if and only if it does not change its sign on $\Omega$. In [12] using a variational approach the authors constructed a nondecreasing sequence of eigenvalues accumulating at infinity. Since between $\lambda_1$ and the next member of this sequence there are no other eigenvalues, as it was shown in [2], we call this second smallest eigenvalue $\lambda_2$ and a corresponding eigenfunction $u_2$. In general, however, it is not known yet whether this sequence contains all the eigenvalues. Nodal domains of variational eigenfunctions were studied in [11]. The regularity results of [10] imply that any eigenfunction (perhaps redefined on a class of measure zero) is of class $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$.

An early attempt at computing several eigenpairs of the $p$-Laplace operator on a planar domain ($N = 2$) numerically is due to Brown and Reichel [7]. Under the assumption of radial symmetry they used a shooting method for the resulting ordinary differential equation. The first genuinely two-dimensional approach was taken by Yao and Zhou [24] using their local minimax method based on a variational formulation. For a square $\Omega = \{(x_1, x_2) | x_1, x_2 \in (0, 2)\}$ and $p \in \{1.75, 2.5, 3.0\}$ the authors computed approximations to seven eigenvalues and corresponding eigenfunctions. They observed that the found eigenfunction $u_2$ has an odd symmetry about $x_1 = 1$ for $p < 2$ and about $x_1 = x_2$ for $p > 2$.

The goal of the current work is to apply the numerical variational methods of [14] to compute approximations of the two smallest eigenvalues and to visualize the corresponding eigenfunctions on a planar domain. In particular the focus is
to extend the Constrained Mountain Pass Algorithm from the Hilbert space setting (as described in [8] and [11]) to the Banach space $W^{1,p}_0(\Omega)$; to verify that this algorithm is suitable even for computations with $p$ “far” from 2;
• to observe the behavior of the eigenpairs for a large range of $p$ and compare it with the known theoretical results about the asymptotics for $p \to 1$ and $p \to \infty$;
• to observe changes in symmetry of $u_2$ on various domains.

In Section 2 we review known results about the variational properties of $\lambda_1$ and $\lambda_2$ and their asymptotic behavior for $p$ close to 1 and $p$ large.

2.1. Variational characterization of $\lambda_1$ and $\lambda_2$. Define two continuously Fréchet differentiable functionals $I, J \in C^1(W^{1,p}_0(\Omega), \mathbb{R})$:

\begin{equation}
I(u) := \int_{\Omega} |\nabla u|^p \, dx, \quad J(u) := \int_{\Omega} |u|^p \, dx.
\end{equation}

Their Fréchet derivatives $I'(u), J'(u)$ are members of the dual space of $W^{1,p}_0(\Omega)$ which we denote by $W^{-1,\frac{1}{p}}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and are given by

\begin{equation}
(I'(u), \phi) = p \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx, \quad (J'(u), \phi) = p \int_{\Omega} |u|^{p-2} u \phi \, dx.
\end{equation}

Two observation can be made: 1. After testing $I$ with $\phi \in W^{1,p}_0(\Omega)$ and integrating by parts it becomes clear that (11) is the Euler-Lagrange equation $I'(u) - \lambda J'(u) = 0$ (up to the factor $p$) which all critical points of $I$ with respect to the constraint

\begin{equation}
S := \{ u \in W^{1,p}_0(\Omega) \mid J(u) = 1 \}
\end{equation}

must satisfy for some value of the Lagrange multiplier $\lambda$.

2. If $(\lambda, u)$ is an eigenpair and we test (11) with $u$, we obtain the Rayleigh quotient

\begin{equation}
\lambda = \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx}.
\end{equation}

Since both its numerator and denominator are homogeneous of the same degree in $u$, finding the smallest eigenvalue $\lambda$ is the same as minimizing $I$ on $S$:

\begin{equation}
\lambda_1 = \min_{u \in S} I(u).
\end{equation}

A variational minimax characterization of the second eigenvalue $\lambda_2$ based on the Krasnosel’skii genus was given in [12]. Alternatively, since for $u_1 \in S$ both $u_1$ and $-u_1$ are local minimizers of $I$ on $S$, a mountain pass characterization of $\lambda_2$ is also possible [9]:

\begin{equation}
\lambda_2 = \inf_{\gamma \in \Gamma} \max_{\gamma([0,1])} I(u),
\end{equation}

where $\Gamma = \{ \gamma \in C([0,1], S) \mid \gamma(0) = u_1, \gamma(1) = -u_1 \}$ is the family of all paths in $S$ connecting the two local minimizers. Hence for the numerical computations we have the setting required by the Constrained Mountain Pass Algorithm of [13].

2. Background material

In the Introduction we mentioned the existence of the first two eigenvalues $\lambda_1$ and $\lambda_2$. Now we review some known results about their variational characterization based on the above references and their asymptotic behavior for $p$ close to 1 and $p$ large.
2.2. Asymptotic behavior of $\lambda_1$ and $\lambda_2$ as $p \to 1$. To make the dependence of an eigenvalue on the domain $\Omega$ and the parameter $p$ explicit in our notation we will write $\lambda(\Omega; p)$ if necessary (and similarly for eigenfunctions). The main result of [17] implies that for $\Omega$ with a Lipschitz boundary

$$\lim_{p \to 1} \lambda_1(\Omega; p) = h_1(\Omega),$$

where $h_1(\Omega) := \min_{D \subseteq \Omega} \frac{\text{Per}(D)}{|D|}$

is called Cheeger constant, $\text{Per}(D)$ denotes the perimeter of $D$ measured with respect to $\mathbb{R}^N$ and $|D|$ its $N$-dimensional Lebesgue measure. A minimizer in the definition of $h_1(\Omega)$ is called a Cheeger set of $\Omega$. Furthermore, any convex planar domain $\Omega$ possesses a unique Cheeger set $C_\Omega$ and

$$\lim_{p \to 1} u_1(\Omega; p) = \chi_{C_\Omega} \text{ in } L^1$$

along a subsequence.

Here the eigenfunctions $u_1(\Omega; p)$ have been normalized to 1 in the $L^\infty$-norm, $\chi_{C_\Omega}$ is the indicator function of $C_\Omega$.

A detailed description of how to find the Cheeger set $C_\Omega$ for a convex planar domain $\Omega$ is given in [18]. Its main property is

$$C_\Omega = \bigcup \left\{ B \subset \Omega \bigm| B \text{ is a ball of radius } \frac{1}{h_1(\Omega)} \right\}.$$ 

In [22] it was shown that for $\Omega$ with a Lipschitz boundary it holds:

$$\lim_{p \to 1} \lambda_2(\Omega; p) = h_2(\Omega),$$

where

$$h_2(\Omega) := \min \left\{ \mu \in \mathbb{R} \bigm| \exists D_1, D_2 \subset \Omega, D_1 \cap D_2 = \emptyset \text{ and } \max_{i=1,2} \frac{\text{Per}(D_i)}{|D_i|} \leq \mu \right\}$$

is called the second Cheeger constant and the convention $\text{Per}(D)/|D| = \infty$ is used if $|D| = 0$. Any two sets $D_1, D_2$ for which the minimum in the definition of $h_2(\Omega)$ is achieved are called coupled Cheeger sets of $\Omega$. For a result about the $L^1$-convergence of the second eigenfunctions we refer to [22] Thm. 5.11.

2.3. Asymptotic behavior of $\lambda_1$ and $\lambda_2$ as $p \to \infty$. For a bounded domain $\Omega$ of $\mathbb{R}^N$ a limit problem of $[10]$ as $p \to \infty$ is studied in [10] [15] for an unknown function $u$ and an unknown real parameter $\Lambda$ (see [13] Definition 2.1). The smallest $\Lambda$ for which this limit problem admits a nontrivial viscosity solution is called the first $\infty$-eigenvalue and denoted $\Lambda_1$. For $\Lambda_1$ there exists a positive viscosity solution and it holds:

$$\lim_{p \to \infty} \left( \lambda_1(\Omega; p) \right)^{1/p} = \Lambda_1(\Omega),$$

$$\Lambda_1(\Omega) = \frac{1}{r_1},$$

where $r_1 := \sup \{ r > 0 \mid \exists \text{ an open ball } B \subset \Omega \text{ of radius } r \}$,

$$\Lambda_1(\Omega) = \min \left\{ \|\nabla u\|_{L^\infty(\Omega)} \bigm| u \in W^{1,\infty}_0(\Omega) \setminus \{0\} \right\}.$$ 

The characterization (15) is an analogy of (13) and (14). Furthermore, for any sequence $\{u_i(\Omega; p_i)\}_{i=1}^\infty$ with $p_i \to \infty$ and $\|u_i(\Omega; p_i)\|_{L^{p_i}(\Omega)} = 1$ there exists a subsequence converging uniformly to a viscosity solution of the limit problem for $\Lambda_1(\Omega)$.

The smallest $\Lambda$ for which the limit problem admits a viscosity solution with at least two nodal domains is called the second $\infty$-eigenvalue and denoted $\Lambda_2$. From the definition it follows that $\Lambda_1 \leq \Lambda_2$. If $\Lambda_1 < \Lambda_2$, then for $\Lambda \in (\Lambda_1, \Lambda_2)$ zero is the only solution of the limit problem. It holds:

$$\lim_{p \to \infty} \left( \lambda_2(\Omega; p) \right)^{1/p} = \Lambda_2(\Omega),$$

$$\Lambda_2(\Omega) = \frac{1}{r_2},$$

where $r_2 := \sup \{ r > 0 \mid \exists \text{ an open ball } B \subset \Omega \text{ of radius } r \}$,
(17) \[ \Lambda_2(\Omega) = \frac{1}{r_2}, \] where \( r_2 := \sup\{r > 0 \mid \exists \text{ disjoint open balls } B_1, B_2 \subset \Omega \text{ of radius } r \}, \)

(18) \[ \Lambda_2(\Omega) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \| \nabla u \|_{L^\infty(\Omega)}, \]

where \( \Gamma \) is defined as in (7), \( u_1 \) is any first \( \infty \)-eigenfunction and \( S := \{ u \in W^{1,\infty}_0(\Omega) \mid \| u \|_{L^\infty(\Omega)} = 1 \} \). Furthermore, for any sequence \( \{ u_2(\Omega; p_i) \}_{i=1}^\infty \) with \( p_i \to \infty \) and \( \| u_2(\Omega; p_i) \|_{L^{p_i}(\Omega)} = 1 \) there exists a subsequence converging uniformly to a viscosity solution of the limit problem for \( \Lambda_2(\Omega) \) which has at least two nodal domains.

3. Numerical methods

An overview of the numerical methods used to compute approximations of the first and the second Dirichlet eigenpair of the \( p \)-Laplace operator is given in Fig. 1. In this section we will describe these methods. Our goal is to find \( u_1 \) as a minimizer of \( I \) on \( S \) according to (6) and \( u_2 \) as a mountain pass point of \( I \) on \( S \) according to (7). We first discretize the planar domain \( \Omega \) using a mesh of triangles and apply the finite element method to approximate \( W^{1,p}_0(\Omega) \) by a finite dimensional subspace. Then we fix \( p \in (1, \infty) \) and use a variant of the Constrained Steepest Descent Method (CDM) to find the first eigenpair, and the Constrained Mountain Pass Algorithm (CMPA) to find the second eigenpair. We implement both methods based on [14]. There are, however, several important issues arising from the fact that we work in a Banach space and not a Hilbert space as in [14]. How to deal with these issues will also be explained in this section. For the computation of the descent direction the Augmented Lagrangian Method of [13] is applied.

3.1. Finite element method. A finite element approximation of the \( p \)-Laplacian was studied in [3]. We adopt this approach for our computations. The planar domain \( \Omega \) is approximated by a polygonal domain \( \Omega_h \) which is partitioned into a finite number of triangles of diameter at most \( h \).

Let \( \{ a_i \}_{i=1}^k \) be the set of those triangle vertices which lie in the interior of \( \Omega_h \). Functions \( \{ \phi_i \}_{i=1}^k \) forming a basis of the \( k \)-dimensional subspace \( V_h^0(\Omega_h) \) of \( W^{1,p}_0(\Omega) \) are chosen linear on each triangle with \( \phi_i(a_j) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta, and zero on \( \partial \Omega_h \). The space \( V_h^0 \) is our finite element approximation of the Sobolev space \( W^{1,p}_0(\Omega) \).

In [3] a detailed description of this method was given for the boundary value problem

\[
-\Delta_p u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega
\]

with the right-hand side \( f \in L^2(\Omega) \). Since it is a straightforward task to adapt it to our problem (4) with \( \lambda |u|^{p-2}u \) on the right-hand side we will not show the details here.

We will however mention one additional technical detail involved. The evaluation of the functionals given in (2) and (3) for functions from \( V_h^0 \) amounts to adding up the contributions of the
individual triangles that make up $\Omega^h$. For example, for $I(u)$ with $u \in V^h_0$ one merely needs to integrate a constant on every triangle. The situation is different for $J(u)$. Let $T \subset \Omega^h$ be a triangle with area $|T|$ and vertices $A$, $B$, and $C$ and let $u$ be a linear function on $T$ with values $u_A$, $u_B$, and $u_C$ at these vertices, respectively. If these values are mutually different, then the following formula holds:

\[(20) \quad \int_T |u|^p \, dx = \frac{2 |T|}{(p + 1)(p + 2)(u_C - u_A)} \left( \frac{|u_C|^{p+2} - |u_B|^{p+2}}{u_C - u_B} - \frac{|u_A|^{p+2} - |u_B|^{p+2}}{u_A - u_B} \right).\]

By inspecting this formula we see that great care must be taken when implementing it to avoid numerical cancellations. This is crucial for the success of our method. A similar situation occurs for the integration of $I'(u)$ (i.e., the gradient) and $P_u$ the orthogonal projection on the tangent space of $S$ at $u \in \Omega$. Then

\[(21) \quad w_u = -P_u \nabla I(u), \quad u \in S\]

gives the steepest descent direction of $I$ at $u$ with respect to $S$.

Because of the lack of orthogonality in the Banach space $W^{1,p}_0(\Omega)$ we need to take a different approach. Let

\[(22) \quad T_u S := \left\{ v \in W^{1,p}_0(\Omega) \mid \langle J'(u), v \rangle = 0 \right\}, \quad \|v\| := \left( \int_{\Omega} |\nabla v|^p \, dx \right)^{1/p}\]

denote the tangent space of $S$ at $u \in S$ and the norm of $v \in W^{1,p}_0(\Omega)$, respectively. The problem of finding the steepest descent direction of $I$ with respect to $S$ can be written as follows: for a given $u \in S$ which is not a critical point of $I$ with respect to $S$,

\[(23) \quad \text{minimize } \langle I'(u), w \rangle \quad \text{subject to } \quad w \in \{ v \in T_u S \mid \|v\| = 1 \}.\]

It has a unique solution as Lemma A.1 in the Appendix shows. The Euler-Lagrange equation that this solution must satisfy can be written in the form

\[(24) \quad -\Delta_p w = \beta (-\Delta_p u - \alpha |u|^{p-2} u),\]

where $\alpha, \beta \in \mathbb{R}$ are unknown. The coefficient $\beta$ comes from the requirement $\|w\| = 1$. After testing (24) by $w$ it can be seen that $\beta < 0$ since the minimum in (23) is negative. For $p \neq 2$ finding the right $\alpha$ is not an easy problem.

We will try to find a different convenient descent direction instead, not necessarily the steepest one. A simple calculation shows that under no constraints the steepest descent direction of $I$ at $u \in B$ is given by $-u$. For $u \in S$ we consider the point $w_u \in T_u S$ closest to $-u$, i.e., the unique solution of the minimization problem

\[(25) \quad \text{minimize } \|w + u\| \quad \text{subject to } \quad w \in T_u S.\]

The minimizer must satisfy the Euler-Lagrange equation

\[(26) \quad -\Delta_p (w + u) = \alpha |u|^{p-2} u\]

for some $\alpha \in \mathbb{R}$. Unlike (23), this equation can be solved easily for $w$:

\[(27) \quad w_u = -u + \frac{1}{\int_{\Omega} |u|^{p-2} u \, dx} \, v_u, \quad \text{where } v_u := (-\Delta_p)^{-1} \left( |u|^{p-2} u \right).\]

The operator $(-\Delta_p)^{-1}$ is discussed later in Sec. 3.3. Lemma A.2 in the Appendix shows that $w_u$ is, indeed, a descent direction of $I$ with respect to $S$. This descent direction is used in our implementation of the variational numerical methods CDM and CMPA.
Remark. 1. Observe that if \(-\Delta_{p}\) were linear, equations (24) and (26) would coincide (after setting \(\beta = -1\)). Hence in case \(p = 2\) they yield the same descent direction (which is the one given by (21)).

2. With \(\beta = -1\) in (24), both equations (24) and (26) yield a zero solution if and only if \(u\) is a critical point of \(I\) with respect to \(S\).

3.3. Inverse of the \(p\)-Laplace operator. A classical result (see, e.g., [23, Theorem 1.3]) says that for any \(f \in W^{-1,q}(\Omega)\), the dual of \(W^{1,p}_{0}(\Omega)\) with \(\frac{1}{p} + \frac{1}{q} = 1\), the problem
\[
-\Delta_{p}u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega
\]
has a unique weak solution in \(W^{1,p}_{0}(\Omega)\). This means that the operator
\[
-\Delta_{p} : W^{1,p}_{0}(\Omega) \to W^{-1,q}(\Omega)
\]
given by
\[
\langle -\Delta_{p}u, v \rangle = \int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla v \, dx
\]
is invertible. We denote its inverse by \((-\Delta_{p})^{-1}\).

In order to use the descent direction given by (27) we need to compute \(v_{u} \) first, i.e., we need to solve problem (28) numerically. For that we apply the Augmented Lagrangian Method of [13]. Here we give a brief description of this method. Let \(V_{0}^{h}\) be again the subspace of continuous functions of \(W^{1,p}_{0}(\Omega^{h})\) which are linear on every triangle of a triangulation of \(\Omega^{h}\), \(D^{h}\) the space of functions with values in \(\mathbb{R}^{2}\) defined on \(\Omega^{h}\) which are constant on each triangle, and \(r > 0\) a parameter. For the Augmented Lagrangian
\[
\mathcal{L}_{r}(v, t, \mu) = \frac{1}{p} \int_{\Omega} |t|^{p} \, dx - \int_{\Omega} \langle f, v \rangle + \frac{r}{2} \int_{\Omega} |\nabla v - t|^{2} \, dx + \int_{\Omega} \mu \cdot (\nabla v - t) \, dx,
\]
where \(v \in V_{0}^{h}\) and \(t, \mu \in D^{h}\), a saddle point \((u, s, \eta)\) is searched for such that
\[
\mathcal{L}_{r}(u, s, \mu) \leq \mathcal{L}_{r}(u, s, \eta) \leq \mathcal{L}_{r}(v, t, \mu) \quad \forall (v, t, \mu) \in V_{0}^{h} \times D^{h} \times D^{h}.
\]
A sequence \((u^{(n)}, s^{(n)}, \eta^{(n)})\) approximating \((u, s, \eta)\) is constructed as follows: choose \((s^{(0)}, \eta^{(1)})\) in \(D^{h} \times D^{h}\) and for \(n \in \mathbb{N}\) solve
\[
-\tau \Delta u^{(n)} = f + \eta^{(n)} \cdot \nabla - r s^{(n-1)} \cdot \nabla \quad \text{in } \Omega, \\
u^{(n)} = 0 \quad \text{on } \partial\Omega, \\
|s^{(n)}|^{p-2} s^{(n)} + r s^{(n)} = r \nabla u^{(n)} + \eta^{(n)}, \\
\eta^{(n+1)} = \eta^{(n)} + r (\nabla u^{(n)} - s^{(n)}).
\]

For given \(s^{(n-1)}\) and \(\eta^{(n)}\) the boundary value problem (32) can be solved for \(u^{(n)}\). For this one just needs some standard algorithm for finding the inverse of the Laplace operator with Dirichlet boundary conditions. The equation in (32) is understood in the weak sense: for example the term \(\eta^{(n)} \cdot \nabla\) is evaluated as
\[
\int_{\Omega} \eta^{(n)} \cdot \nabla \phi \, dx
\]
for a test function \(\phi \in V_{0}^{h}\).

\[
|s^{(n)}|^{p-1} + r |s^{(n)}| = |r \nabla u^{(n)} + \eta^{(n)}|,
\]
which on each triangle is just a scalar nonlinear equation with one unknown. For each triangle it can be solved, e.g., by Newton’s method. After \(s^{(n)}\) has been obtained, \(s^{(n)}\) can be computed immediately from (33).

At the end \(\eta\) is updated according to (34) and a new iteration step can be started.

Next, equation (33) is used to find \(s^{(n)}\). The \(\mathbb{R}^{2}\)-norm of \(s^{(n)}\) must satisfy
\[
|s^{(n)}|^{p-1} + r |s^{(n)}| = |r \nabla u^{(n)} + \eta^{(n)}|,
\]
which on each triangle is just a scalar nonlinear equation with one unknown. For each triangle it can be solved, e.g., by Newton’s method. After \(s^{(n)}\) has been obtained, \(s^{(n)}\) can be computed immediately from (33).

At the end \(\eta\) is updated according to (34) and a new iteration step can be started.

The convergence of this method was studied in [13]. We use the norm of \(\nabla u^{(n)} - s^{(n)}\) to measure the convergence.
3.4. Constrained Descent Method. The Constrained Descent Method (CDM) is applied to find the first eigenpair of the p-Laplace operator: \( u_1 \) is found as the minimizer of \( I \) with respect to \( S \), \( \lambda_1 = I(u_1) \). As mentioned above, it differs from the Constrained Steepest Descent Methods of [14] in the way the descent direction is chosen.

The method solves numerically the following initial value problem:

\[
\frac{dw}{dt} = w_u(t), \quad w(0) = e_0 \in S,
\]

where \( w_u \) is given by (27) in the Appendix. Proposition A.3 in the Appendix states that this problem has a unique solution \( w(t) \in S \) for \( t \in (0, \infty) \) and that \( w(t) \) gets arbitrarily close to a critical point of \( I \) with respect to \( S \) as \( t \to \infty \).

After choosing the starting point \( e_0 \in S \) and setting \( w^{(0)} := e_0 \) the initial value problem is solved by repeating the following two steps: First (Euler’s step), given \( u^{(n-1)} \) find \( \bar{u}^{(n)} = u^{(n-1)} + \Delta t^{(n)} w_{u^{(n-1)}} \) with some small value \( \Delta t^{(n)} > 0 \). Second (scaling), define \( u^{(n)} = c \bar{u}^{(n)} \), where the coefficient \( c \in \mathbb{R} \) is chosen such that \( u^{(n)} \in S \). In case \( I(u^{(n)}) > I(u^{(n-1)}) \), halve the step \( \Delta t^{(n)} \) and compute \( \bar{u}^{(n)} \) and \( u^{(n)} \) again. If this halving has to be repeated and \( \Delta t^{(n)} \) becomes very small (smaller than a prescribed threshold value), stop the algorithm. The norm of the descent direction \( \|u^{(n-1)}\| \) is used to measure convergence of \( u^{(n)} \) to an eigenfunction \( u \). When computing \( w_u \) according to (27) the integral \( \nu := \int_I |u|^{p-2} u \, u_\nu \, dx \) has to be evaluated. If \( \|w_{u^{(n-1)}}\| \) is small, then \( (1/\nu^{(n-1)})^{p-1} \) approximates the eigenvalue.

We note that at every step of CDM the Augmented Lagrangian Method of Sec. 3.3 has to be applied to compute the descent direction \( w_{u^{(n-1)}} \).

3.5. Constrained Mountain Pass Algorithm. Suppose that an approximation of the first eigenvalue \( \lambda_1 \) and eigenfunction \( u_1 \) of the p-Laplace operator have been computed. Constrained Mountain Pass Algorithm (CMPA) is applied to find the second eigenpair: \( u_2 \) is found as a mountain pass point of \( I \) on \( S \) lying “between” the two local minimizers \( u_1 \) and \(-u_1 \), \( \lambda_2 = I(u_2) \).

After finding a path \( z_{\text{max}} \) at which \( I \) is maximal along the path, move this point a small distance in the tangent space to \( S \) at \( z_{\text{max}} \) in the descent direction \( w_{u_{\text{max}}} \) and then scale it (as in CDM) to come back to \( S \). Thus the path has been deformed on \( S \) and the maximum of \( I \) lowered. Repeat this deforming of the path until the maximum along the path cannot be lowered anymore: a mountain pass point of \( I \) with respect to \( S \) has been reached.

To construct the initial path connecting \( u_1 \) and \(-u_1 \) in \( S \) we choose an intermediate point \( e_M \in S \setminus \{\pm u_1\} \), set \( k := \lfloor P/2 \rfloor \) and define:

\[
\bar{z}_j := u_1 + \frac{j}{k} (e_M - u_1) \quad \text{for } j \in \{0, \ldots, k\},
\]

\[
\tilde{z}_j := e_M + \frac{j - k}{P - k} (-u_1 - e_M) \quad \text{for } j \in \{k, \ldots, P\},
\]

\[
z_j := e_j \tilde{z}_j \in S \quad \text{(scaling to } S \text{ as in Sec. 3.4)} \quad \text{for } j \in \{0, \ldots, P\}.
\]

Connecting \( u_1 \) and \(-u_1 \) by a line segment without the intermediate point \( e_M \) would not work. Such a line segment passes through \( 0 \) and hence cannot be scaled to get to \( S \).

Finally, as in CDM, \( \|w_{u_{\text{max}}}\| \) is used to measure convergence to an eigenfunction \( u \). The corresponding eigenvalue \( \lambda \) is computed as in CDM, too. At every step of CMPA the Augmented Lagrangian Method of Sec. 3.3 has to be applied to compute the descent direction \( w_{u_{\text{max}}} \).

4. Numerical results

In this section numerical results will be given for the following planar domains: the unit disk, the square with side length 2, the rectangle with sides 2 and \( 7/4 \), the isosceles triangle with base and height 1, the isosceles triangle with base 1 and height \( 3/4 \), and the equilateral triangle with side 1. Unless explicitly stated otherwise the computed eigenfunctions will be plotted as a surface.
over the domain with heights given by the function values and as a contour plot of these values (like, e.g., in Fig. 2). In order to better compare the shapes the eigenfunctions in these figures have been scaled to have the same maximum value. We do not explicitly differentiate between two eigenfunctions \( u \) and \( \tilde{u} \) if \( \tilde{u}(x) = cu(Tx) \), where \( c \in \mathbb{R} \) is a scaling coefficient and \( T: \Omega \rightarrow \Omega \) is some symmetry transformation of \( \Omega \) (e.g., for a square a rotation by \( \pi/2 \) about the center of the square).

4.1. Unit Disk. Let

\[
\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}.
\]

Before presenting the numerical results we make a remark about the radially symmetric case. It is known that the first eigenfunction for the disk is radially symmetric. One important question about the second eigenfunction for the disk has been whether it is radially symmetric, too. In \([22, 5]\) the authors proved that for \( p \) close to 1 the answer is no. The eigenvalue problem (1) under the assumption of radial symmetry \( u = u(r) \), \( r \in (0, 1) \) becomes

\[
-\left(ru''(r)^{p-2}u'(r)\right)' = \lambda r|u|^{p-2}u, \quad u'(0) = 0, \quad u(1) = 0.
\]  \hfill (37)

This and a related problem are treated, for example, in \([7]\) and \([5]\), where numerical approaches play an important role. For our numerical investigation we adapt the genuine 2D method of Sec. 3 in the following ways:

- all integrals are one-dimensional,
- the weight \( r \) is introduced,
- the natural boundary condition is implemented at \( r = 0 \) (the zero boundary condition stays at \( r = 1 \)).

Since these modifications are rather elementary, we will not describe them in more detail. We will refer to this method as radial 2D method.

For the computations carried out by the genuine 2D method the domain \( \Omega \) was approximated by a polygon and discretized using \( 68,608 \) triangles. For the computations carried out by the radial 2D method the interval \((0, 1)\) was divided into \( 1,000 \) subintervals of the same length.

Figures 2 and 3 show the eigenfunctions \( u_1 \) and \( u_2 \) computed by the genuine 2D method for several values of \( p \), respectively. The corresponding eigenvalues \( \lambda_1 \) and \( \lambda_2 \) for these and other values of \( p \) are listed in Table 1(a). Figure 5(a) shows the shape of the intermediate point \( e_M \) on the initial path connecting \( u_1 \) and \( -u_1 \) we used for CMPA to find \( u_2 \) for all the listed values of \( p \). The function \( u_2 \) found this way seems to possess an odd symmetry with respect to its nodal line. The slope of this nodal line in the coordinate system \((x_1, x_2)\) depends on the computation. For the depiction in Fig. 3 we rotated \( \Omega \) in each case to make the slope appear the same. CMPA needed between 120 and 600 iterations to converge.

Figure 5(b) shows an alternative shape of \( e_M \). With such an initial path CMPA converged for \( p = 1.1 \) and \( p = 1.2 \) to a radially symmetric function we call \( u_2^{\text{rad}} \) (but for higher values of \( p \) to the oddly symmetric function \( u_2 \)). Figure 4(a) shows \( u_2^{\text{rad}} \) for \( p = 1.1 \).

Figure 5(c) shows yet another choice of \( e_M \) (radially symmetric). With this intermediate point of the initial path and for \( p = 1.3 \) and \( p = 1.4 \) (but not larger) CMPA seems to converge to a radially symmetric function first but after many iterations the path slips down and the algorithm converges eventually to the oddly symmetric \( u_2 \). The graph in Fig. 5(d) shows how the maximum value of the Dirichlet functional \( I \) along the path develops during the run of the algorithm (for \( p = 1.3 \)). The horizontal axis shows the number of iterations. The flat part between iterations 70 and 260 indicates that the path is staying close to a critical point. When now the norm of the descent direction \( u_{2,\text{max}} \) given in (27) computed at the “highest” point \( z_{\text{max}} \) of the path gets small enough, we stop the algorithm and save this highest point. Since it displays a radial symmetry, we call it \( u_2^{\text{rad}} \) again.

The eigenvalues \( \lambda_2^{\text{rad}} \) corresponding to the found \( u_2^{\text{rad}} \) are also listed in Table 1(a).

Figure 4(b) shows profiles of the eigenfunction \( u_2^{\text{rad}} \) computed by the radial 2D method for several values of \( p \). The eigenvalues \( \lambda_1 \) and \( \lambda_2^{\text{rad}} \) computed by this method for these and other
NUMERICAL INVESTIGATION OF EIGENVALUES OF THE $p$-LAPLACE OPERATOR

Figure 2. The numerically computed first eigenfunction $u_1$ for the disk.

Figure 3. The numerically computed second eigenfunction $u_2$ for the disk.

Figure 4. The numerically computed radially symmetric second eigenfunction $u_2^{\text{rad}}$: (a) using the genuine 2D method; (b) using the radial 2D method. The profile of $u_2^{\text{rad}}$ for the radial coordinate $r \in (0, 1)$ is shown, the scaling along the vertical axis is chosen such that $\|u_2^{\text{rad}}\|_p = 1$. 
Table 1. Eigenvalues for the disk computed numerically by: (a) the genuine 2D method, (b) the radial 2D method.

|   | $p$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2^\text{rad}$ |
|---|-----|---------------|---------------|----------------------------|
| 1.1 | 2.5690 | 4.2008 | 5.6809 |
| 1.2 | 2.9654 | 5.0707 | 7.2277 |
| 1.3 | 3.3263 | 5.9604 | 8.9302 |
| 1.4 | 3.6740 | 6.9072 | 10.861 |
| 1.5 | 4.0179 | 7.9310 | |
| 1.6 | 4.3623 | 9.0465 | |
| 1.7 | 4.7097 | 10.266 | |
| 1.8 | 5.0618 | 11.604 | |
| 1.9 | 5.4194 | 13.072 | |
| 2.0 | 5.7834 | 14.683 | |
| 2.1 | 6.1542 | 16.452 | |
| 2.2 | 6.5320 | 18.395 | |
| 2.3 | 6.9173 | 20.527 | |
| 2.4 | 7.3102 | 22.866 | |
| 2.5 | 7.7107 | 25.432 | |
| 3.0 | 9.8323 | 42.460 | |
| 4.0 | 14.683 | 110.71 | |
| 5.0 | 20.351 | 273.00 | |
| 6.0 | 26.832 | 649.47 | |
| 8.0 | 42.210 | 3,430.1 | |
| 10.0 | 60.784 | 17.071 | |

Figure 5. (a)–(c) Intermediate point $e_M$ of the initial path used in CMPA. (d) Maximum value of the Dirichlet functional $I$ along the path during the run of CMPA with $e_M$ shown in figure (c) for $p = 1.3$.moil

values of $p$ are listed in Table 1(b). The convergence of CMPA does not seem to be sensitive to the choice of $e_M$ in this case.

By comparing the values of $\lambda_1$ and $\lambda_2^\text{rad}$ in Table 1(a) with those in Table 1(b) which were computed by the two different numerical methods we observe that their first three digits coincide in almost all the cases. Also, the profiles of $u_1$, $u_2^\text{rad}$ are very close for both methods, respectively (cf. Fig 4(a) and the top left graph in (b) for $u_2^\text{rad}$ and $p = 1.1$). We conclude that these are numerical approximations of the same eigenvalue-eigenfunction pairs.
Figure 6. Dependence of the numerically computed eigenvalues for the disk on $p$. The three cross symbols in the graph on the right mark the values $h_1(\Omega) = 2$, $h_2(\Omega) \approx 3.1543$, and $h_{\text{rad}}^2(\Omega) = 4$.

Figure 7. Dependence of the numerically computed eigenvalues for the disk raised to $1/p$ on $p$.

The behavior of CMPA suggests that although $u_{\text{rad}}^2$ is a constrained mountain pass point of $I$ among radially symmetric functions, it is not a constrained mountain pass point with no assumption on the symmetry (cf. Fig. 5(d)). The case of $p = 1.1$ and $p = 1.2$ when CMPA with $\epsilon_M$ from Fig. 5(b) converged to a radially symmetric function and the path did not slip off to asymmetric functions with lower values of $I$ seems to contradict this. However, we assume that this was caused by the “flat” shape of the landscape of $I$ close to $u_{\text{rad}}^2$ for $p$ close to 1 and by numerical inaccuracies.

The dependence of $\lambda_1$, $\lambda_2$, and $\lambda_{\text{rad}}^2$ on $p$ is presented in Figs. 6 and 7. First of all we observe that for all the values of $p$ considered the inequality $\lambda_2 < \lambda_{\text{rad}}^2$ holds. Hence this is a numerical evidence that the second eigenfunction for the disk is not radially symmetric not only for small $p$ but for a large range of $p$.

Second, we can observe the following asymptotic behavior:

| $p$ | $\lambda_1$ | $\lambda_2$ | $\lambda_{\text{rad}}^2$ |
|-----|--------------|--------------|--------------------------|
| $p \to 1^+$ | 2 | 3.1543 | 4 |
| $p \to \infty$ | 1 | 2 | 3 |

Theoretical results for $\lambda_1$ and $\lambda_2$ were summarize in Sec. 2. The values $h_1(\Omega)$, $\Lambda_1(\Omega)$, and $\Lambda_2(\Omega)$ for the disk are easy to compute. In [22] it was proved that $h_2(\Omega)$ for the disk equals the first Cheeger constant for the half-disk which is approximately 3.1543. We can observe (Fig. 2) that $u_1$ converges to 1 for $p \to 1$ as explained in [17] and to the distance function to the boundary for
\( p \to \infty \) as explained in [19]. In Fig. 4 we observe that for \( p \to 1 \) the function \( u_2 \) is getting close to the indicator function of the Cheeger set for the half-disk on each nodal domain.

In [5] a numerical evidence is given leading to the conjecture for the asymptotic behavior of \( \lambda_2^{rad} \) given in the above table. Our numerical results (at least for \( p \to 1 \)) support this conjecture. To motivate these values and the profiles of \( u_2^{rad} \) in Fig. 4 we make the following two remarks:

**Remark.** 1. For \( r \in (0, 1) \) let \( D(r) \) be the disk of radius \( r \) centered at the origin, \( A(r) = \Omega \setminus D(r) \) an annulus. It is easy to show that for \( r = 1/2 \) both \( D \) and \( A \) have the same Cheeger constant \( h_r^{rad}(\Omega) := h_1(D(1/2)) = h_1(A(1/2)) = 4 \) (see, e.g., [19] for a result about the Cheeger constant of an annulus). The function \( u_2^{rad} \) with its profile shown in Fig. 4 seems to get close to the indicator function of \( D(1/2) \) and \( A(1/2) \) on each nodal domain for \( p \to 1 \).

2. Under the assumption of radial symmetry two largest disjoint disks of the same radius inscribed in \( \Omega \) have radius \( 1/3 \). Hence we define \( \Lambda_3^{rad} = \frac{1}{17} = 3 \). The function \( u_2^{rad} \) with its profile shown in Fig. 4 seems to get close on each nodal domain to a multiple of the function giving the distance to the boundary on \( D(1/3) \) and \( A(1/3) \) for large \( p \).

### 4.2. Square. Let \( \Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \in (0, 2)\} \).

This domain was discretized using 83,968 triangles. Figures 5 and 6 show the eigenfunctions \( u_1 \) and \( u_2 \) computed for several values of \( p \), respectively. Table 2 lists the corresponding values of \( \lambda_1 \) and \( \lambda_2 \).

| \( p \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \lambda_6 \) | \( \lambda_7 \) | \( \lambda_8 \) | \( \lambda_9 \) | \( \lambda_{10} \) | \( \lambda_{11} \) | \( \lambda_{12} \) | \( \lambda_{13} \) | \( \lambda_{14} \) | \( \lambda_{15} \) | \( \lambda_{16} \) | \( \lambda_{17} \) | \( \lambda_{18} \) | \( \lambda_{19} \) | \( \lambda_{20} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 2.3649 | 3.7586 | 5.3702 | 1.2 | 2.0934 | 4.5012 | 5.6179 | 1.3 | 2.9986 | 5.2600 | 5.3715 | 1.4 | 3.2834 | 6.0385 | 6.1621 | 1.5 | 3.5611 | 6.8835 | 7.0053 | 1.6 | 3.8356 | 7.7971 | 7.9118 | 1.7 | 4.1092 | 8.7897 | 8.8903 | 1.8 | 4.3830 | 9.8708 | 9.9490 | 1.9 | 4.6581 | 11.050 | 11.095 | 2.0 | 4.9349 | 12.338 | 12.338 |
| 2.0 | 4.9349 | 12.338 | 12.338 | 2.1 | 5.2139 | 13.684 | 13.744 | 2.2 | 5.4952 | 15.144 | 15.282 | 2.3 | 5.7791 | 16.725 | 16.961 | 2.4 | 6.0688 | 18.438 | 18.791 | 2.5 | 6.3552 | 20.293 | 20.802 | 3.0 | 7.8452 | 32.107 | 33.956 | 4.0 | 11.038 | 74.757 | 85.447 | 5.0 | 14.497 | 163.59 | 205.08 | 6.0 | 18.194 | 343.77 | 477.60 | 8.0 | 26.221 | 1.402 | 2.443.4 | 10.0 | 34.990 | 5.339 | 11.888 |

**Table 2.** Eigenvalues for the square.
values of $\lambda_{S_1}$ can be computed using CDM on $\Omega$ with additional boundary conditions $u(1, x_2) = 0$ for $x_2 \in (0, 2)$ or $u(x, x) = 0$ for $x \in (0, 2)$, respectively, or as the first eigenvalue on the half-domain $\Omega^{\text{half}}_i$, where

$$
\Omega^{\text{half}}_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, 1), x_2 \in (0, 2)\},
$$

$$
\Omega^{\text{half}}_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, 2), x_2 \in (0, x_1)\}.
$$

Our numerical observations regarding these eigenvalues and $\lambda_2$ are summarized in Table 3(a) and the computed values are listed in Table 2. We stress that $\lambda_2$ was computed with no a priori assumptions on symmetry. The dependence of the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_{S_1}$ and $\lambda_{S_2}$ on $p$ is further plotted in Figures 10 and 11. These figures and Table 3(b) also explain the asymptotic behavior as $p \to 1$ and $p \to \infty$. While the table shows the limit values as given by the theory, the graphs indicate convergence to these values (at least for $p \to 1$; for $p \to \infty$ it seems a larger range of $p$ would be needed). The Cheeger constants $h_1$ shown in the first row of Table 3(b) have been computed according to [17, 18] by

$$
(38) \quad h_1((0, a) \times (0, b)) = \frac{4 - \pi}{a + b - \sqrt{(a - b)^2 + \pi ab}} \quad \text{for } a, b > 0.
$$

The evaluation of $\Lambda_1$ in the second row is straightforward.
Table 3. The smallest eigenvalues $\lambda_{S_1}$ and $\lambda_{S_2}$ under symmetry assumptions for the square. (a) Numerical comparison with $\lambda_2$. (b) Asymptotic behavior: the first row shows values of $h_1$, the second row values of $\Lambda_1$ for $\Omega$, $\Omega_1^{\text{half}}$, and $\Omega_2^{\text{half}}$, respectively.

| $p$ | $\lambda_{S_1}$ | $\lambda_{S_2}$ |
|-----|-----------------|-----------------|
| $p < 2$ | $= \lambda_2$ | $> \lambda_2$ |
| $2 < p$ | $> \lambda_2$ | $= \lambda_2$ |

\[
\lim_{p \to 1+} \lambda = 1 + \frac{1}{2} \sqrt{\pi}\]
\[
\lim_{p \to \infty} \lambda_{1/p} = 1 + \sqrt{2} \sqrt{\pi} - \frac{\pi}{4} - \sqrt{1 + 2 \pi} + \frac{1}{2} \sqrt{2} \pi
\]

4.3. Rectangle. Let

\[
\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, 2), x_2 \in (0, 1.75)\}.
\]

This domain was discretized using 77,312 triangles. The shape of the first eigenfunction $u_1$ and the graph of the first eigenvalue $\lambda_1(\Omega; p)$ are similar to those for the square. However, the symmetry properties of the second eigenfunction $u_2$ are different: According to our numerical observations, for $p \leq 3.6$ the eigenfunction $u_2$ preserves an odd symmetry about $x_1 = 1$ and an even symmetry about $x_2 = 0.875$ (which we call $S_1$ as in the case of the square). For $p \geq 3.7$ this symmetry is lost and $u_2$ maintains an odd symmetry with respect to $(1, 0.875)$, the center of $\Omega$. The contour lines of $u_2$ for several values of $p$ are shown in Fig. 12.
Figure 12. The numerically computed second eigenfunction $u_2$ for the rectangle.

Figure 13. Comparison of the numerically computed second eigenvalue $\lambda_2$ and the smallest eigenvalue $\lambda_{S_1}$ under the symmetry $S_1$ for the rectangle $\Omega$ ($\lambda_1$ is shown for reference).

Table 4. Comparison of the numerically computed second eigenvalue $\lambda_2$ and the smallest eigenvalue $\lambda_{S_1}$ under the symmetry $S_1$ for other rectangles.

For $p \geq 3.7$ the smallest eigenvalue $\lambda_{S_1}$ corresponding to an eigenfunction with symmetry $S_1$ is larger than $\lambda_2$ (cf. Fig. 13). This eigenpair can be computed by CDM on $\Omega$ with an additional boundary condition $u(1,x_2) = 0$ for $x_2 \in (0,0.75)$ or as the first eigenpair on the half-rectangle $\Omega_{\text{half}} = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1), x_2 \in (0,1.75)\}$.

Our conjecture is that for a rectangle $R = (0,a) \times (0,b)$ with $0 < b < a$ there exists $p_0 > 2$ such that $u_2$ has two nodal domains which for $p < p_0$ are rectangles with sides $a/2$ and $b$. For $p > p_0$ the nodal domains are not rectangular and $u_2$ has only an odd symmetry with respect to the center of $R$. According to our numerical observations, $p_0$ gets larger the larger the ratio $a/b$: Besides $\Omega$ we ran the computation for two other rectangles. For $R = (0,2) \times (0,1.9)$ the loss of symmetry $S_1$ of $u_2$ is observable approximately between $p = 2.44$ and $p = 2.48$ and for $R = (0,2) \times (0,1.6)$ between $p = 5.6$ and $p = 6.0$ (cf. Table 4). As $p$ grows and crosses $p_0$, the nodal line which is straight for $p < p_0$ gets distorted. This distortion is faster for smaller ratios $a/b$ and slower for larger ratios.
4.4. Triangle with height 1. Let

\[ \Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1), |x_2| < \frac{1}{2}(1-x_1) \right\} \]

be an isosceles triangle with base 1 and height 1. It was discretized using 38,912 triangles. Figures [4] and [5] show the eigenfunctions \( u_1 \) and \( u_2 \) for several values of \( p \), respectively. Table 5 lists the corresponding values of \( \lambda_1 \) and \( \lambda_2 \).

Various intermediate path points \( e_M \) were used to compute \( u_2 \). However, the function that CMPA converged to did not depend on this choice. The symmetry properties of the computed \( u_2 \) depend only on the value \( p \). For \( p \leq 2.6 \) it is even in \( x_2 \), i.e., it belongs to

\[ S_E := \{ u : \Omega \to \mathbb{R} \mid u(x_1, x_2) = u(x_1, -x_2) \}. \]

For \( p \geq 2.7 \) this symmetry is lost by \( u_2 \) as the graphs in Fig. [5] show.

For \( p = 2.6 \) the computation was repeated with intermediate path points \( e_M \) without symmetry \( S_E \) but CMPA always converged to the function shown in Fig. [5] which displays symmetry \( S_E \).

For \( p = 2.7 \) a symmetric \( e_M \in S_E \) was chosen. The graph in Fig. [6] shows how the maximum of the Dirichlet functional \( E \) along the path evolved during this run of CMPA. The path connecting \( u_1 \) with \( -u_1 \) which gets deformed at every step of CMPA seems to stay close to some critical point having symmetry \( S_E \) during the first 1000 steps but then it slips down to lower values of \( I \) and stays close to another critical point. This is the asymmetric \( u_2 \) which the algorithm eventually converges to.

Even beyond \( p = 2.6 \) there exist eigenfunctions with symmetry \( S_E \). Let \( u_{2,S_E} \) denote a sign-changing eigenfunction of the \( p \)-Laplace operator on \( \Omega \) which lies in \( S_E \) and has the smallest eigenvalue (which we denote \( \lambda_{2,S_E} \) ). As mentioned above, for \( p \leq 2.6 \) we observed that \( u_2 = u_{2,S_E} \) (up to scaling). To compute \( u_{2,S_E} \) for \( p \geq 2.7 \) consider the following eigenvalue problem:

\[
\begin{align*}
-\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } \Omega^{\text{half}}, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_1, \\
u &= 0 \quad \text{on } \Gamma_2,
\end{align*}
\]

where

\[
\begin{align*}
\Omega^{\text{half}} &= \{ (x_1, x_2) \in \Omega \mid x_2 > 0 \}, \\
\Gamma_1 &= \{ (x_1, x_2) \in \partial \Omega^{\text{half}} \mid x_2 = 0 \}, \\
\Gamma_2 &= \partial \Omega^{\text{half}} \setminus \Gamma_1.
\end{align*}
\]

| \( p \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_{2,S_E} \) |
|---|---|---|---|
| 1.1 | 8.0143 | 12.188 | |
| 1.2 | 10.208 | 16.211 | |
| 1.3 | 12.673 | 21.009 | |
| 1.4 | 15.515 | 26.847 | |
| 1.5 | 18.822 | 33.998 | |
| 1.6 | 22.083 | 42.774 | |
| 1.7 | 27.196 | 53.546 | |
| 1.8 | 32.471 | 66.762 | |
| 1.9 | 38.634 | 82.963 | |
| 2.0 | 45.831 | 102.80 | |
| 2.1 | 54.228 | 127.06 | |
| 2.2 | 64.016 | 156.72 | |
| 2.3 | 75.415 | 192.92 | |
| 2.4 | 88.681 | 237.08 | |
| 2.5 | 104.10 | 290.87 | |

Table 5. Eigenvalues for the triangle with height 1.
NUMERICAL INVESTIGATION OF EIGENVALUES OF THE $p$-LAPLACE OPERATOR

Figure 14. The numerically computed first eigenfunction $u_1$ for the triangle with height 1.

Figure 15. The numerically computed second eigenfunction $u_2$ for the triangle with height 1.

Figure 16. Triangle with height 1: (a) Maximum value of the Dirichlet functional $I$ along the path during the run of CMPA for $p = 2.7$ and $e_M \in S_E$. (b) The computed eigenfunction $u_{2,S_E}$ for $p = 3.5, 6.0,$ and $10.0$. 
Any eigenfunction solving this problem can be extended to an eigenfunction on the whole \( \Omega \) by even symmetry about \( x_2 = 0 \). Since the first eigenfunction of the original problem \( (1) \) for the triangle \( \Omega \) belongs to \( S_{\Omega} \), its restriction to \( \Omega^{\text{half}} \) is the first eigenfunction for \( (40) \). Hence to compute \( u_{2,S_{\Omega}} \) we just need to apply CMPA to problem \( (40) \) with paths which again connect \( u_1 \) and \( -u_1 \). The modification of the finite element method to take into account the natural boundary condition on \( \Gamma_1 \) is straightforward. The computed values of \( \lambda_{2,S_{\Omega}} \) are listed in Table 6. Figure 16(b) shows the corresponding eigenfunction \( u_{2,S_{\Omega}} \) for selected values of \( p \).

The dependence of the eigenvalues \( \lambda_1, \lambda_2, \) and \( \lambda_{2,S_{\Omega}} \) on \( p \) is further plotted in Fig. 17. The figure also shows the limits of \( \lambda_1 \) for \( p \to 1 \) and \( \infty \) and of \( \lambda_2 \) for \( p \to \infty \) which can be computed explicitly. As mentioned, for example, in [18], the Cheeger constant of a triangle is given by

\[
\lambda_1(\Omega) = 1 + \frac{\text{Per}(\Omega)}{2h_1(\Omega)} = 1 + \frac{\sqrt{\pi}}{\sqrt{\text{Area}(\Omega)}}.
\]

Hence in our case \( \lambda_1(\Omega) = 1 + \sqrt{5} + \sqrt{2\pi} \approx 5.7427 \). Simple computations yield \( \lambda_2(\Omega) = 1 + \frac{\sqrt{5}}{\sqrt{\pi}} \approx 3.2361 \) and \( \lambda_2(\Omega) = 1 + 9/\sqrt{\pi} \approx 5.0249 \).

4.5. Triangle with height 3/4. Let

\[
\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, \frac{1}{2}), |x_2| < \frac{3}{4} (\frac{1}{2} - x_1) \}
\]

be an isosceles triangle with base 1 and height 3/4. It was discretized using 28,672 triangles. Figures 18 and 19 show the eigenfunctions \( u_1 \) and \( u_2 \) for several values of \( p \), respectively. Table 6 lists the corresponding values of \( \lambda_1 \) and \( \lambda_2 \).

The symmetry properties of the computed \( u_2 \) change again with \( p \). For this triangle, however, \( u_2 \) gains more symmetry as \( p \) increases (unlike for the triangle with height 1 where \( u_2 \) lost symmetry). For \( p \leq 1.6 \) the nodal line of \( u_2 \) connects the base of the triangle with one of its other sides. For \( p \geq 1.7 \) this nodal line connects the base with the vertex above the base and \( u_2 \) is odd in \( x_2 \), i.e., it belongs to

\[
S_O := \{ u : \Omega \to \mathbb{R} \mid u(x_1, x_2) = -u(x_1, -x_2) \}
\]

### Table 6. Eigenvalues for the triangle with height 3/4.

| \( p \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_{S_{\Omega}} \) |
|-------|----------|----------|----------------|
| 1.1   | 9.389   | 14.38    | 14.50          |
| 1.2   | 12.13   | 19.41    | 19.52          |
| 1.3   | 15.28   | 25.53    | 25.62          |
| 1.4   | 18.97   | 33.10    | 33.17          |
| 1.5   | 23.35   | 42.52    | 42.55          |
| 1.6   | 28.55   | 54.22    | 54.22          |
| 1.7   | 34.73   | 68.76    | 68.76          |

| \( p \) | \( \lambda_1 \) | \( \lambda_2 \) |
|-------|----------|----------|
| 1.1   | 14.07    | 86.84    |
| 1.2   | 19.58    | 109.2    |
| 1.3   | 21.11    | 137.1    |
| 1.4   | 24.17    | 171.6    |
| 1.5   | 28.85    | 214.4    |
| 1.6   | 33.22    | 267.2    |
| 1.7   | 38.25    | 332.5    |

| \( p \) | \( \lambda_1 \) | \( \lambda_2 \) |
|-------|----------|----------|
| 2.5   | 149.1    | 413.0    |
| 3.0   | 350.0    | 1,196    |
| 4.0   | 1,789    | 9,351    |
| 5.0   | 5,491    | 6,871 \cdot 10^4 |
| 6.0   | 3,958 \cdot 10^4 | 4,849 \cdot 10^4 |
| 8.0   | 7,522 \cdot 10^4 | 2,232 \cdot 10^4 |
| 10.0  | 1,416 \cdot 10^5 | 9,602 \cdot 10^5 |
Figure 18. The numerically computed first eigenfunction $u_1$ for the triangle with height $3/4$.

Figure 19. The numerically computed second eigenfunction $u_2$ for the triangle with height $3/4$.

Figure 20. Higher eigenfunctions for the triangle with height $3/4$ for $p = 1.5$: $u_2$ and $u(2, S_{SE})$ computed as constrained local mountain pass points by CMPA with no a priori assumptions on symmetry, $u_{S_C}$ computed by CDM enforcing symmetry $S_C$. 

$$\lambda_2 = 42.52$$
$$\lambda_{S_C} = 42.55$$
$$\lambda_{(2, S_{SE})} = 44.42$$
as can be seen in Fig. 19. It is because of lack of resolution of the numerical method close to the vertex (where \( u_2 \) is flat) that the zero contour line in the figure for \( p = 1.7 \) and \( p = 8.0 \) does not exactly reach the vertex.

For \( p \leq 1.6 \) there also exist eigenfunctions with symmetry \( S_O \). Let \( \lambda_{S_O} \) denote the smallest eigenvalue with an eigenfunction belonging to \( S_O \) (denoted \( u_{S_O} \)). Using the notation defined in (11) this eigenvalue can be computed using CDM on \( \Omega \) with an additional boundary condition \( u = 0 \) on \( \Gamma_1 \) or as the first eigenvalue on the half-domain \( \Omega^{\text{half}} \). The computed values of \( \lambda_{S_O} \) are also listed in Table IV. For \( p \geq 1.7 \) the values of \( \lambda_2 \) and \( \lambda_{S_O} \) coincide. For \( p = 1.6 \) they differ in the sixth digit.

As in the previous computations, various choices of the intermediate path point \( \epsilon_M \) were used to compute \( u_2 \) on \( \Omega \) with no a priori assumptions on symmetry. In some cases CMPA converged to different functions depending on this choice (different local mountain passes). For example, for \( p = 1.5 \) two eigenfunctions were found: one with a nodal line connecting the base of the triangle with one of its sides, and another one with a nodal line connecting the two sides and having an even symmetry in \( x_2 \). Both eigenfunctions are (numerically) local mountain pass points of \( I \) with respect to the constraint \( S \). The first one is called \( u_2 \) since it has the smallest eigenvalue, the second one is called \( u_{(2,S_O)} \) because of its symmetry (it could also be understood as a solution of (10) formulated in a similar way for the triangle with height 3/4). Both eigenfunctions are shown in Fig. 20 together with \( u_{S_O} \) for comparison.

The dependence of the eigenvalues \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_{S_O} \) on \( p \) is further plotted in Fig. 21. The following limits of \( \lambda_1 \) and \( \lambda_{S_O} \) as \( p \to 1 \) and those of \( \lambda_1 \) and \( \lambda_2 \) as \( p \to \infty \) are also marked in the figure and have these respective values:

\[
\begin{align*}
\lambda_1(\Omega) &= \frac{3}{2} (2 + \sqrt{13} + 2\sqrt{3} \pi) \approx 6.631, \\
\lambda_{S_O}(\Omega) &= \frac{3}{2} (2 + \sqrt{13}) \approx 3.737, \\
\lambda_1(\Omega^{\text{half}}) &= \frac{3}{2} (5 + \sqrt{13} + 2\sqrt{3} \pi) \approx 9.830, \\
\lambda_{S_O}(\Omega^{\text{half}}) &= \frac{3}{2} (5 + \sqrt{13}) \approx 5.737.
\end{align*}
\]

4.6. Equilateral triangle. For isosceles triangles close but not equal to an equilateral triangle a similar observation has been made as for rectangles close but not equal to the square: the symmetry properties of the second eigenfunction \( u_2 \) change at a certain value \( p \neq 2 \). According to the following computations, for an equilateral triangle this change occurs at \( p = 2 \) (as it does for the square).

Let \( \Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \left( 0, \frac{\sqrt{3}}{2} \right), |x_2| < \frac{1}{\sqrt{3}} \left( \frac{\sqrt{3}}{2} - x_1 \right) \right\} \) be an equilateral triangle with side 1. It was discretized using 32,256 triangles. With the notation introduced in (11) we can define \( \lambda_{2,S_E} \) and \( u_{2,S_E} \) as in Sec. 4.4 and \( \lambda_{S_O} \) and \( u_{S_O} \) as in Sec. 4.5.

Our numerical observations are summarized in Table IV and Fig. 22. For \( p < 2 \) the second eigenfunction \( u_2 \) is even in \( x_2 \) while for \( p > 2 \) it is odd (up to a rotation of the triangle by \( \pm 2\pi/3 \)). We note that the values \( \lambda_2 \) listed in the table were computed with no a priori assumptions on
Table 7. Eigenvalues for the equilateral triangle with side 1.

| p   | $\lambda_1$ | $\lambda_2 (= \lambda_{2,Se})$ | $\lambda_{So}$ |
|-----|--------------|-------------------------------|---------------|
| 1.1 | 8.653        | 13.37                         | 13.61         |
| 1.9 | 44.07        | 98.20                         | 98.40         |
| 2.0 | 52.64        | 122.8                         | 122.8         |

| p   | $\lambda_1$ | $\lambda_2 (= \lambda_{2,So})$ | $\lambda_{2,Se}$ |
|-----|--------------|---------------------------------|------------------|
| 2.0 | 52.64        | 122.8                           | 122.8            |
| 2.1 | 62.71        | 152.9                           | 153.2            |
| 8.0 | 4.240 \cdot 10^5 | 1.483 \cdot 10^5 | 1.668 \cdot 10^5 |

Figure 22. The numerically computed eigenfunctions $u_2$, $u_{2,Se}$ and $u_{So}$ for the equilateral triangle.

Remarks on the numerics

5.1. Dependence on the mesh parameter $h$. Let $T^h$ denote the set of all the triangles of a triangulation of $\Omega^h$. The mesh parameter $h$ was introduced in Sec. 3.1 as the (smallest) upper bound on the diameter of the circumscribed circle for triangles of $T^h$. In this section the dependence of the computed values of $\lambda_1$ and $\lambda_2$ on $h$ is investigated. The investigation is conducted for one particular domain $\Omega$—the rectangular domain used for computations in Sec. 4.3. The finest triangulation $T^{0.011}$ is used as a reference.

Table 8 lists details about these discretizations ordered by the number of triangles. Essentially, a finer mesh was obtained from a coarser one by placing a new vertex in the middle of each triangle side of the old triangulation, in effect dividing each triangle in four.

Four discretizations of this domain are used. Table 8 lists details about these discretizations ordered by the number of triangles. Essentially, a finer mesh was obtained from a coarser one by placing a new vertex in the middle of each triangle side of the old triangulation, in effect dividing each triangle in four.

Table 8 shows the values of $\lambda_1$ and $\lambda_2$ computed for the four triangulations characterized by $h$ and for selected values of $p$. Figure 23 gives perhaps a more telling picture: for each $p$ it displays relative differences $\left(\frac{\lambda(T^h) - \lambda(T^{0.011})}{\lambda(T^{0.011})}\right)$, where $\lambda(T^h)$ denotes the eigenvalue computed on the triangulation $T^h$. The finest triangulation $T^{0.011}$ is used as a reference. We observe that the largest differences occur for large $p$ (here $p = 8.0$) and smallest differences for $p$ close to 2. The differences are about twice as large for $\lambda_2$ computed by CMPA compared to $\lambda_1$ computed by CDM.
Table 8. Triangulations used to discretize the rectangular domain $\Omega$.

| $\mathcal{T}^h$ | $h$ | number of triangles |
|-----------------|-----|---------------------|
| course          | 0.079 | 4,832               |
|                 | 0.044 | 19,328              |
|                 | 0.022 | 77,312              |
| fine            | 0.011 | 309,248             |

Table 9. Values of $\lambda_1$ and $\lambda_2$ computed for four different triangulations of the rectangular domain $\Omega$ and $p = 1.1, 1.6, 4.0, 8.0$.

| $h$  | $p = 1.1$  | $p = 1.6$  | $p = 4.0$  | $p = 8.0$  |
|------|------------|------------|------------|------------|
|      | $\lambda_1$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ |
| 0.079 | 2.5586  | 3.9507  | 4.2965  | 8.2462  | 14.884  | 91.306  | 50.005  | 2,234.8 |
| 0.044 | 2.5544  | 3.9376  | 4.2945  | 8.2381  | 14.859  | 90.962  | 49.625  | 2,202.7 |
| 0.022 | 2.5533  | 3.9342  | 4.2940  | 8.2361  | 14.853  | 90.881  | 49.533  | 2,193.6 |
| 0.011 | 2.5531  | 3.9334  | 4.2939  | 8.2356  | 14.851  | 90.861  | 49.510  | 2,191.0 |

Figure 23. Relative difference $\frac{\lambda(\mathcal{T}^h) - \lambda(\mathcal{T}^{0.011})}{\lambda(\mathcal{T}^{0.011})} \cdot 100\%$ for $\lambda_1$ (left) and $\lambda_2$ (right) computed on a triangulation $\mathcal{T}^h$ with respect to the finest triangulation $\mathcal{T}^{0.011}$. For $\lambda_1$ the relative differences for $p = 1.1$ and $p = 4.0$ almost coincide and cannot be distinguished in the graph.

5.2. The Augmented Lagrangian Method. As described in Sec. 3.3 this method is used to solve (28) iteratively for a given right-hand side. The Augmented Lagrangian $L_r$ defined in (30) depends on a parameter $r > 0$. As observed by the authors of [13] the algorithm is not very sensitive to the choice of $r$ but the analysis of the influence of $r$ on the behavior of the algorithm is complicated.

The choice of $r$ has an influence on the speed of convergence of the algorithm and at the same time on how precise the found numerical solutions can be. In general, for larger $r$ the algorithm seems to converge faster but it is able to find only less precise approximations of the solution. For our computations we tried various values of $r$ first and then chose the one which seemed to give a reasonable speed of convergence together with acceptable residual. This value depended strongly on $p$ and also on the particular domain $\Omega$. Table 10 shows the dependence of $r$ and of the number of iterations that the algorithm needed on $p$. For each domain one value of $r$ was chosen from the given range. Similarly, the number of iterations lay in the given range. We can observe that for a small $p$ a large $r$ was needed, for a large $p$ a smaller $r$. For $p$ close to 2 we could choose $r \approx 1$. The number of iterations needed turned larger for $p$ farther from 2.
Table 10. The dependence of the approximate values of $p$ and numbers of iterations in the Augmented Lagrangian Method on $p$.

| $p$  | range of $r$   | $\#$ of iterations |
|------|----------------|---------------------|
| 1.1  | $10^3 - 10^7$  | 700 – 2,000         |
| 1.2  | 500 – 2,500    | 500 – 1,000         |
| 1.8  | 1 – 1.5        | 80 – 90             |
| 3.0  | 0.3 – 0.4      | 200 – 300           |
| 10.0 | 0.03 – 0.1     | 1,200 – 3,000       |

We note that for values of $p$ smaller than 1.1 and larger than 10 (the particular value also depended on the domain) we were not able to find $r$ giving satisfactory results for our implementation of the Augmented Lagrangian Method in conjunction with CMPA.

5.3. CDM and CMPA. In both the Constrained Descent Method and the Constrained Mountain Pass Algorithm the measure of convergence is $\lVert w_u \rVert$, the $W_0^{1,p}(\Omega)$-norm of the descent direction evaluated at the approximation $u$ of the eigenfunction which is being computed. The smallest achieved value depended on the algorithm, on $p$, and in case of CMPA also on the fact whether there lies another critical point not far from $u$. For CDM the order of this value was between $10^{-5}$ and $10^{-8}$, for CMPA between $10^{-3}$ and $10^{-7}$. The number of iteration of CDM was approximately between 10 and 30. The number of iterations of CMPA varied, it depended on the shape of the initial path and on $p$, and was anywhere between 100 and 3,000.

6. Conclusion

In this work a concrete application of the variational numerical methods of [14] in a Banach space was presented. In particular, one possible choice of the descent direction required by these methods was proposed, implemented and tried in computations in the setting of the Sobolev space $W_0^{1,p}(\Omega)$. The computations yielded approximations of the smallest two Dirichlet eigenvalues and the corresponding eigenfunctions of the $p$-Laplace operator on several planar domains for $p$ ranging from 1.1 to 10. This relatively large range made it possible to study the change of symmetry of the second eigenfunction with varying $p$ on different domains which was first observed in [24] for the square and $p$ not far from 2. The computed eigenvalues seem to agree with the asymptotic behavior known from theory for $p \to 1$. Our range of $p$ seems to be too small, however, in order to clearly observe the asymptotic behavior of the eigenpairs as $p \to \infty$.

Numerical experiments were conducted for the following domains: the disk, rectangles, and isosceles triangles. We summarize the main observations about the symmetry of the second eigenfunction $u_2$. For the disk it was observed that $u_2$ has a straight nodal line dividing the disk into halves for the whole range of $p$.

For rectangles which are not a square and for small $p$ the second eigenfunction is odd about its nodal line which is straight and connects the midpoints of the longer sides. After $p$ crosses some value $p_0 > 2$ there are two second eigenfunctions which are mirror images of each other. Their nodal line is not straight but still connects the two longer sides.

For the square and $p \neq 2$ there are two second eigenfunctions which are images of each other under rotations by $\pi/2$ about the center of the square. For $p < 2$ their nodal line is straight and connects the midpoints of the opposite sides. For $p > 2$ the nodal line is a diagonal of the square. For $p = 2$ there are two linearly independent second eigenfunctions.

The symmetry observations for triangles are based on the family of isosceles triangles with vertices $(0, -1/2)$, $(0, 1/2)$, and $(\ell, 0)$ with base 1 and height $\ell > 0$ which are symmetric about the $x_1$-axis. For those shorter than the equilateral triangle and for small $p$ there are two asymmetric second eigenfunctions (up to scaling) which are symmetry images of each other. Their nodal line connects the base with one side of the triangle. After $p$ crosses some value $p_0 < 2$ there is only one eigenfunction $u_2$. It is odd about its nodal line which is straight and connects the middle of the base with the opposite vertex (symmetry $S_0$).
For triangles longer than the equilateral triangle and for small \( p \) there is one eigenfunction \( u_2 \). For larger \( p \) there are two asymmetric second eigenfunctions which are images of each other. Their nodal line still connects the two sides of the triangle.

For the equilateral triangle and \( p \neq 2 \) there are three second eigenfunctions which are images of each other under rotations of the triangle about its midpoint by \( \pm 2\pi/3 \). For \( p < 2 \) their nodal line connects two sides of the triangle and they have even symmetry about the height coming from the third side. For \( p > 2 \) the nodal line of the second eigenfunctions follows a height of the triangle and the eigenfunctions have odd symmetry about this height. For \( p = 2 \) there are two linearly independent second eigenfunctions.

Figure 20 indicates that our numerical methods could be used for finding some higher eigenfunctions and perhaps for a continuation in \( \ell \) to observe the connection between these eigenfunctions and the second eigenfunctions for the equilateral triangle. This lies however beyond the scope of this paper.

**Appendix A.**

Here we give a proof of some claims used in Sections \( \S 3.2 \) and \( \S 3.3 \). A subindex notation will be used for general sequences and does not refer to the enumeration of eigenfunctions and eigenvalues in this section.

**Lemma A.1.** Let \((B, \|\cdot\|)\) be a reflexive Banach space with a strictly convex norm, \( I, J \in C^1(B, \mathbb{R})\) be two continuously Fréchet differentiable functionals, and \( u \) be a point in \( B \) with \( J(u) = 1 \) which is not a critical point of \( I \) with respect to \( S := \{ v \in B | J(v) = 1 \} \). Then the problem

\[
\text{minimize } L(w) := \langle I'(u), w \rangle \text{ subject to } w \in C := \{ v \in B | \langle J'(u), v \rangle = 0 \text{ and } \|v\| = 1 \}
\]

has a unique solution.

**Proof.** First, we show that \( L \) has a negative infimum on \( C \): \( L \) is bounded below on \( C \) by \(-\|I'(u)\|\). It attains negative values on \( C \) if there exists \( w \in C \) such that \( L(w) \neq 0 \). But if \( L \equiv 0 \) on \( C \), then we would have

\[
\langle J'(u), w \rangle = 0 \quad \Rightarrow \quad \langle I'(u), w \rangle = 0 \quad \forall w \in B
\]

which would imply existence of \( \alpha \in \mathbb{R} \) such that \( I'(u) - \alpha J'(u) = 0 \). This is not possible since \( u \) is not a critical point of \( I \) with respect to \( S \).

Let \( \{w_n\} \subset C \) be a minimizing sequence of \( L \), i.e.,

\[
L(w_n) \to \inf_L L \in (-\infty, 0) \quad \text{as } n \to \infty.
\]

Since this sequence is bounded by 1, the reflexivity of \( B \) implies existence of a subsequence (still denoted \( \{w_n\} \)) which converges weakly to some \( w \in B \) such that \( \|w\| \leq 1 \). Since \( I'(u) \) and \( J'(u) \) are continuous linear functionals, we obtain

\[
L(w_n) \to L(w) \quad \text{and} \quad \langle J'(u), w_n \rangle \to \langle J'(u), w \rangle \quad \text{as } n \to \infty.
\]

This means that \( L(w) = \inf_C L \) and \( \langle J'(u), w \rangle = 0 \).

To prove that \( w \) is a minimizer it remains to show that \( \|w\| = 1 \). If \( \|w\| < 1 \), then \( \bar{w} := w/\|w\| \) belongs to \( C \) and

\[
L(\bar{w}) = \frac{L(w)}{\|w\|} < L(w)
\]

because \( L(w) < 0 \). But this is a contradiction with the minimality of \( L(w) \).

To show uniqueness let \( w_1 \) and \( w_2 \) be both minimizers. For \( \bar{w} := \frac{1}{2}(w_1 + w_2) / \|\frac{1}{2}(w_1 + w_2)\| \) we obtain

\[
\bar{w} \in C \quad \text{and} \quad \min_C L \leq L(\bar{w}) = \frac{\min_C L}{\|\frac{1}{2}(w_1 + w_2)\|}.
\]

Since the minimum is negative, this and the triangle inequality imply

\[
1 \leq \|\frac{1}{2}w_1 + \frac{1}{2}w_2\| \leq \frac{1}{2}\|w_1\| + \frac{1}{2}\|w_2\| = 1.
\]
Hence equality holds in the above inequalities and the strict convexity of the norm implies \( w_1 = w_2 \).

**Lemma A.2.** Let \( I \) and \( J \) be defined by (23) and \( S \) by (4). Further, let \( u \in S \) and

\[
(43) \quad w_u := -u + \frac{1}{\int \Omega |u|^{p-2}u \, v_u \, dx} \, v_u, \quad \text{where} \ v_u := (-\Delta_p)^{-1} (|u|^{p-2}u). 
\]

Then \( (I'(u), w_u) \leq 0 \). Equality holds if and only if \( u \) is a critical point of \( I \) with respect to \( S \) which is the case if and only if \( w_u = 0 \).

The proof of this lemma is based on the following inequality which is a direct consequence of the Cauchy-Schwarz and Hölder inequalities. Its proof is therefore omitted.

**Auxiliary Lemma.** Let \( f, g \in W_0^{1,p}(\Omega), f \neq 0 \). Then

\[
\int_{\Omega} |\nabla f|^p - 2 \nabla f \nabla g \, dx \leq \|f\|^{p-1} \|g\|. 
\]

Equality holds if and only if there exists \( \nu \geq 0 \) such that \( \nu f = g \).

**Proof of Lemma A.2.** We observe that

\[
(44) \quad \int_{\Omega} |u|^{p-2}u \, v_u \, dx = \int_{\Omega} (-\Delta_p v_u) \, v_u \, dx = \|v_u\|^p. 
\]

By the definition of \( w_u \), (43) and the auxiliary lemma we obtain

\[
\langle I'(u), w_u \rangle = -\|u\|^p + \frac{1}{\|v_u\|^p} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v_u \, dx 
\]

\[
\leq (\star) \quad -\|u\|^p + \frac{1}{\|v_u\|^p} \|u\|^{p-1} \|v_u\| = \left( -1 + \frac{1}{\|v_u\|^{p-1} \|u\|} \right) \|u\|^p. 
\]

Using \( u \in S \), testing the equation \( -\Delta_p v_u = |u|^{p-2}u \) by \( u \), and applying the auxiliary lemma yields

\[
(45) \quad 1 = \int_{\Omega} |u|^p \, dx = \int_{\Omega} |\nabla v_u|^{p-2} \nabla v_u \nabla u \, dx \leq \|v_u\|^{p-1} \|u\|. 
\]

By combining (45) and (46) we conclude that \( \langle I'(u), w_u \rangle \leq 0 \). Equality holds if and only if equality holds in (\( \star \)) and (\( \star \)). According to the auxiliary lemma this is the case if and only if \( \nu u = v_u \) for some \( \nu > 0 \). Finally, we argue that the following are equivalent:

(a) \( \nu u = v_u \) for some \( \nu > 0 \),
(b) \( u \) is a critical point of \( I \) with respect to \( S \),
(c) \( w_u = 0 \).

Statement (a) is equivalent to \( \nu^{p-1}(-\Delta_p u) = |u|^{p-2}u \) and hence to (b). If (a) holds, then \( \int_{\Omega} |u|^{p-2}u \, v_u \, dx = \nu \) because \( u \in S \). Hence \( w_u = -u + \frac{1}{\nu} v_u = 0 \) and (c) holds, too. It is obvious that (c) implies (a). \( \square \)

Before stating the next proposition we recall some known results (let \( p, q \in (0, \infty), \frac{1}{p} + \frac{1}{q} = 1 \)):

(i) The \( p \)-Laplace operator \(-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega) \) is uniformly continuous on bounded sets.
(ii) The mapping \( u \mapsto |u|^{p-2}u : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega) \) is compact and uniformly continuous on bounded sets.
(iii) The inverse \( p \)-Laplace operator \((-\Delta_p)^{-1} : W^{-1,q}(\Omega) \to W_0^{1,p}(\Omega) \) is uniformly continuous on bounded sets.

Both claims (i) and (ii) follow from standard inequalities found, e.g., in [13] Lemmas 5.3 and 5.4]. The compactness in (ii) follows from the compact embedding of \( W_0^{1,p}(\Omega) \) in \( L^p(\Omega) \). Claim (iii) follows from standard inequalities found, i.e., in [13] Propositions 5.1 and 5.2.
Proposition A.3. Let $I$ and $J$ be defined by (2) and $S$ by (4). The initial value problem

$$
\frac{d}{dt}u(t) = w_u(t), \quad u(0) = e_0 \in S
$$

with $w_u$ defined in (25) has a unique solution $u(t) \in S$ defined for $t \in (0, \infty)$. There exists a critical point $u \in S$ of $I$ with respect to $S$ and a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n \to \infty} t_n = \infty$ and $\lim_{n \to \infty} u(t_n) = u$ in $W^{1,p}_0(\Omega)$.

Proof. The proof of existence of a solution and its uniqueness follows the same lines as the proof of Lemma 5 in [14]. Hence we focus on establishing the existence of the sequence $\{t_n\}$. Since $0 \leq I(u(T)) = I(e_0) + \int_0^T \langle I'(u(t)), w_u(t) \rangle \, dt$ for $T > 0$ and the integrand is non-positive, we obtain $\int_0^\infty |\langle I'(u(t)), w_u(t) \rangle| \, dt \leq I(e_0)$. Hence there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} t_n = \infty$ such that for $u_n := u(t_n)$ and $w_n := w_u(t_n)$ it holds:

$$
\langle I'(u_n), w_n \rangle \to 0 \quad \text{for} \quad n \to \infty.
$$

We recall that by (43) and (44) we have

$$
w_n = -u_n + \frac{1}{\|v_n\|^p}v_n, \quad \text{where} \quad v_n := (-\Delta_p)^{-1} \left( |u_n|^{p-2} u_n \right).
$$

We observe that $\{u_n\}$ is a bounded sequence, hence it converges weakly to some $u \in W^{1,p}_0(\Omega)$ along a subsequence which we again denote $\{u_n\}$. From the compactness of the map $u \mapsto |u|^{p-2}u$ and the continuity of the inverse $p$-Laplacian it follows that

$$
v_n \rightharpoonup v := (-\Delta_p)^{-1} \left( |u|^{p-2} u \right) \quad \text{strongly in} \quad W^{1,p}_0(\Omega) \quad \text{as} \quad n \to \infty.
$$

Equation (49) and the fact that $w_n \in T_{u_n} S$ imply

$$
\langle -\Delta_p (w_n + u_n), w_n \rangle = \frac{1}{\|v_n\|^p} \int_{\Omega} |u_n|^{p-2} u_n w_n \, dx = 0.
$$

Combining (48) and (50) yields

$$
\int_{\Omega} (|\nabla (w_n + u_n)|^{p-2} \nabla (w_n + u_n) - |\nabla u_n|^{p-2} \nabla u_n) \nabla w_n \, dx \to 0 \quad \text{for} \quad n \to \infty.
$$

On the other hand standard estimates [13] Propositions 5.1 and 5.2] state that

$$
\int_{\Omega} (|\nabla (w_n + u_n)|^{p-2} \nabla (w_n + u_n) - |\nabla u_n|^{p-2} \nabla u_n) \nabla w_n \, dx \geq \delta \frac{\|w_n\|^2}{\left( \|w_n + u_n\| + \|u_n\| \right)^2} \quad \text{for} \quad 1 < p \leq 2,
$$

$$
\geq \frac{1}{2^{p-2} \|w_n\|^p} \quad \text{for} \quad 2 \leq p,
$$

where $\delta > 0$ is a constant which does not depend on $w_n$ and $u_n$. These inequalities and (52) imply

$$
w_n \to 0 \quad \text{strongly in} \quad W^{1,p}_0(\Omega) \quad \text{as} \quad n \to \infty.
$$

This and (50) in turn imply that $\{u_n\}$ converges strongly to $u$ and that

$$
u = \frac{1}{\|v\|^p} (-\Delta_p)^{-1} \left( |u|^{p-2} u \right),
$$

which means that $u$ is a critical point of $I$ with respect to $S$. \hfill \square

Remark. To better understand the implications of the choice of the descent direction we remark how the proof of the proposition would change if we used the steepest descent direction instead of the descent direction given by (24). Up to normalization the steepest descent direction $w$ defined by (23) can be written as the solution of

$$-\Delta_p w = \Delta_p u + \alpha |u|^{p-2} u$$

with $\alpha$ a constant.
for a suitable $\alpha$. Testing this equation by $w$ and using $w \in T_u S$ yields $\|w\|^p = -\frac{1}{p} \langle I'(u), w \rangle$. Hence equation (48) would directly imply $w_n \to 0$. We can write

$$0 \leftarrow \|w_n\|^{p-1} = \| -\Delta_p w_n - \alpha_n |w_n|^{p-2} w_n\| = \frac{1}{p} \| I'(u_n) - \alpha_n I'(u_n)\|,$$

where $\| \cdot \|$ denotes the norm in the dual space $W^{-1, q}(\Omega)$. If we define $\| I'|_{S_u} := \inf_{\alpha \in \mathbb{R}} \| I'(u) - \alpha I'(u)\|$, as in [6], [14], then we would obtain $\| I'|_{S_u} \to 0$. The Palais-Smale condition under constraints which was formulated in [6] states in its simplified form that if $\{I'(u_n)\}$ is bounded and $\| I'|_{S_u} \to 0$ then $\{u_n\}$ possesses a convergent subsequence. In [9] it was shown that this condition holds in our setting. Hence the choice of the steepest descent direction would yield a more “classical” proof of the proposition.

References

1. A. Anane, Simplicité et isolation de la première valeur propre du $p$-laplacien avec poids, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 16, 725–728.
2. A. Anane and N. Tsouli, On the second eigenvalue of the $p$-Laplacian, Nonlinear partial differential equations (Fès, 1994), Pitman Res. Notes Math. Ser., vol. 343, Longman, Harlow, 1996, pp. 1–9.
3. J. W. Barrett and W.-B. Liu, A direct uniqueness proof for equations involving the $p$-Laplacian, J. Comput. Appl. Math. 148 (2002), no. 1, 183–211.
4. P. Belloni and B. Kawohl, A direct uniqueness proof for equations involving the $p$-Laplace operator, Manuscripta Math. 109 (2002), no. 2, 229–231.
5. J. Benítek, P. Drábek, and P. Girg, The second eigenfunction of the $p$-Laplacian on the disk is not radial, Preprint, 2010.
6. A. Bonnet, A deformation lemma on a $C^1$ manifold, Manuscripta Math. 81 (1993), no. 3-4, 339–359.
7. M. B. Brown and W. Reichel, Computing eigenvalues and $\varphi$-Fucik-spectrum of the radially symmetric $p$-Laplacian, J. Comput. Appl. Math. 148 (2002), no. 1, 183–211.
8. Y. S. Choi and P. J. McKenna, A mountain pass method for the numerical solution of semilinear elliptic problems, Nonlinear Anal. 20 (1993), no. 4, 417–437.
9. M. Cuesta, D. de Figueiredo, and J.-P. Gossez, Existence and nonuniqueness for the $p$-Laplacian: nonlinear eigenvalues, Comm. Partial Differential Equations 159 (1999), no. 1, 212–238.
10. E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983), no. 8, 827–850.
11. P. Drábek and S. B. Robinson, On the generalization of the Courant nodal domain theorem, J. Differential Equations 181 (2002), no. 1, 58–71.
12. J. P. García Azorero and I. Peral Alonso, Existence and nonuniqueness for the $p$-Laplacian: nonlinear eigenvalues, Comm. Partial Differential Equations 12 (1987), no. 12, 1389–1430.
13. R. Glowinski and A. Marroco, Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité, d’une classe de problèmes de Dirichlet non linéaires, RAIRO Analyse Numérique 9 (1975), no. R-2, 41–76.
14. J. Horák, Constrained mountain pass algorithm for the numerical solution of semilinear elliptic problems, Numer. Math. 98 (2004), no. 2, 251–275.
15. P. Juutinen and P. Lindqvist, On the higher eigenvalues for the $\infty$-eigenvalue problem, Calc. Var. Partial Differential Equations 23 (2005), no. 2, 169–192.
16. P. Juutinen, P. Lindqvist, and J. J. Manfredi, The $\infty$-eigenvalue problem, Arch. Ration. Mech. Anal. 148 (1999), no. 2, 89–105.
17. B. Kawohl and V. Fridman, Isoperimetric estimates for the first eigenvalue of the $p$-Laplace operator and the Cheeger constant, Comment. Math. Univ. Carolin. 44 (2003), no. 4, 659–667.
18. B. Kawohl and T. Lachand-Robert, Characterization of Cheeger sets for convex subsets of the plane, Pacific J. Math. 225 (2006), no. 1, 105–118. MR 2233727 (2007e:52002)
19. D. Krejčířík and A. Pratelli, The Cheeger constant of curved strips, arXiv:1011.3490v1, 2010.
20. P. Lindqvist, On the equation $\text{div}(\|\nabla u\|^{p-2} \nabla u) + \lambda \|u\|^{p-2} u = 0$, Proc. Amer. Math. Soc. 109 (1990), no. 1, 157–164.
21. _____, Addendum: “On the equation $\text{div}(\|\nabla u\|^{p-2} \nabla u) + \lambda \|u\|^{p-2} u = 0$”, Proc. Amer. Math. Soc. 116 (1992), no. 2, 583–584.
22. E. Parini, The second eigenvalue of the $p$-Laplacian as $p$ goes to 1, Int. J. Differ. Equ. (2010), Art. ID 984671, 23 pages.
23. M. Struwe, Variational methods, Springer-Verlag, Berlin, 1990.
24. X. Yao and J. Zhou, Numerical methods for computing nonlinear eigenpairs. I. Iso-homogeneous cases, SIAM J. Sci. Comput. 29 (2007), no. 4, 1355–1374.

E-mail address: jhorak@math.uni-koeln.de