q-DIFFERENTIAL OPERATOR REPRESENTATION OF
THE QUANTUM SUPERALGEBRA $U_q(\mathfrak{sl}(M + 1|N + 1))$

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ABSTRACT. A representation of the quantum superalgebra $U_q(\mathfrak{sl}(M + 1|N + 1))$ is
constructed based on the $q$-differential operators acting on the coherent states parameterized by coordinates. These coordinates correspond to the local ones of the flag manifold. This realization provides us with a guide to construct the free field realization for the quantum affine superalgebra $U_q(\hat{\mathfrak{sl}}(M + 1|N + 1))$ at arbitrary level.

1. Introduction.
It has been established that affine quantum algebras play an important role in studies on the mathematical physics and solvable systems. It has also been cleared that the representation theory of these algebras, particularly, the free field realization\cite{1-3} provides us a powerful tools. In order to construct the free field realization for the quantum algebra $U_q(\hat{\mathfrak{g}})$\cite{4-9}, it gives us a guide that the corresponding Lie algebras $\hat{\mathfrak{g}}$ is realized in terms of the $q$-differential operators, in other words, the $q$-deformed harmonic oscillators\cite{10-13}.

On the other hand, affine superalgebras have also attracted attention in relation to superconformal algebras and superstrings and topological field theories. There is possibility for the $q$-deformed models of these models to have physical meanings. So it is worthwhile to investigate the representation of affine quantum superalgebras\cite{14-17}. In the previous work\cite{18} we have constructed a level-one representation of the quantum affine superalgebra $U_q(\hat{\mathfrak{sl}}(M + 1|N + 1))$ and the vertex operators associated with representations on the line of the Frenkel and Jing’s construction\cite{19}. In order to extend the representation to the case of arbitrary level, we have to consult the forms of the Wakimoto construction with ghost systems\cite{1}. It is well known that there is the close relation between the Wakimoto construction and the realization by the differential operators on the flag manifold.

The aim of this paper is to construct the realization based on the $q$-differential operators acting on the space of analytic functions on the coordinates of the coherent states corresponding to the local coordinates of the flag manifold. There have been already the papers to construct the representations of quantum Lie superalgebras in terms of $q$-oscillators or $q$-differential operators\cite{20,21}. We construct the representation of the quantum Lie superalgebra $U_q(\mathfrak{sl}(M + 1|N + 1))$ on the same method in the paper\cite{21}. This realization will help us to construct the free field
realization for the quantum affine algebra \( U_q(\hat{\mathfrak{sl}}(M+1|N+1)) \) at arbitrary level, which is the \( q \)-deformation of the Wakimoto construction.

2. Differential operator realization of \( \mathfrak{sl}(M+1|N+1) \).

The Cartan matrix of the Lie superalgebra \( \mathfrak{sl}(M+1|N+1) \) is written as \( a_{ij} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i \delta_{i,j+1} - \nu_{i+1} \delta_{i+1,j} \) \((i, j = 1, 2 \cdots, M + N + 1)\), where \( \nu_j = 1 \) for \( j = 1, \cdots, M + 1 \) and \( \nu_j = -1 \) for \( j = M + 2, \cdots, M + N + 2 \). The Lie superalgebra \( \mathfrak{sl}(M+1|N+1) \) is defined by the Chevalley generators \( h_i, e_i, f_i(i = 1, \cdots, M+N+1) \). The \( Z_2 \) - grading \( | \cdot | \to Z_2 \) of the generators are \(|e_{M+1}| = |f_{M+1}| = 1 \) and zero otherwise. The relations among these generators are

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_{ij},
\]

\[
[e_j, [e_j, e_i]] = 0 \quad \text{for} \quad |a_{ij}| = 1, j \neq M + 1,
\]

\[
[f_j, [f_j, f_i]] = 0
\]

\[
[e_{M+1}, [e_{M+2}, [e_{M+1}, e_{M}]]] = 0,
\]

\[
[f_{M+1}, [f_{M+2}, [f_{M+1}, f_{M}]]] = 0,
\]

where we have used the notations \([X, Y] = XY - (-1)^{|X||Y|}YX\). For the odd generator \( e_{M+1} \), the Serre relations are changed.

Let us introduce an orthonormal basis \( \{ \epsilon_i| i = 1, \cdots, M + N + 1 \} \) with the bilinear form \( (\epsilon_i|\epsilon_j) = \nu_j \delta_{i,j} \). The simple roots are written as \( \alpha_i = \nu_i \epsilon_i - \nu_{i+1} \epsilon_{i+1} \) and \( (\alpha_i|\alpha_j) = a_{ij} \) is satisfied. Let \( V_\Lambda \) be the Verma module over \( \mathfrak{sl}(M+1|N+1) \) generated by the highest weight vector \(|\Lambda>\) which satisfies the highest weight conditions: \( e_i|\Lambda> = 0, h_i|\Lambda> = (\alpha_i|\Lambda)|\Lambda>\). Weight vectors are written as

\[
|\Lambda; n_{lm}>(X_{lm}^-)^{n_{lm}}|\Lambda>(1 \leq l \leq m \leq M + N + 1),
\]

where are the generators correspond to the roots \( \alpha_{lm} = \alpha_l + \alpha_{l+1} + \cdots + \alpha_m \) \((l < m)\). The dual module \( V_\Lambda^* \) is generated by \( <\Lambda| \) such that \( <\Lambda|f_i = 0, <\Lambda|h_i = (\alpha_i|\Lambda) <\Lambda|, \) whose dual weight vectors are \( <\Lambda; n_{lm}| = <\Lambda; n_{lm}|(X_{lm})^{n_{lm}}(1 \leq l \leq m \leq M + N + 1) \). Let’s introduce the coherent state:

\[
|\mathbf{x}>= |x_{11}, x_{12}, \cdots, x_{M+N+1M+N+1}> = \prod_{l \leq m} \sum_{n_{lm}=0}^{\infty} \frac{x_{n_{lm}}^{n_{lm}}}{n_{lm}!}|\Lambda; n_{lm}> = \prod_{l \leq m} e^{x_{lm}X_{lm}}|\Lambda>.
\]

Here \( x_{lm}(1 \leq l \leq m \leq M + N + 1) \) are the coordinates defined as

\[
x_{lm} = \begin{cases} z_{lm}(1 \leq l \leq m \leq M; M + 2 \leq l \leq m \leq M + N + 1) \\ \theta_{lm}(1 \leq l \leq M + 1, M + 1 \leq m \leq M + N + 1) \end{cases}
\]

where \( z_{lm} \) are complex variables and \( \theta_{lm} \) are Grassmann odd ones which satisfy the relations \( \theta_{lm}\theta_{lm} = 0, \theta_{lm}\theta_{ij} = -\theta_{ij}\theta_{lm}(l, m) \neq (i, j) \). The bilinear form \( V_\Lambda \otimes
confusion, we adopt the same notations in the case of the quantum Lie superalgebra $U_q(\mathfrak{sl}(M+1|N+1))$ and $X \in \mathfrak{sl}(M+1|N+1)$. We can construct the representation in the space of analytic functions on the coordinates $\{x_{ij}\}$ through the bilinear form. The coordinates $x_{ij}, x_{ij}^{-1}$ and the differential operators $M_{lm} = x_{lm} \frac{\partial}{\partial x_{lm}} (1 \leq l \leq m \leq M + N + 1)$ generate the Heisenberg algebra $\mathcal{H}_L$ with relations: $[M_{ij}, x_{lm}^\pm] = \pm \delta_{l,i} \delta_{j,m} x_{lm}^\pm$.

Proposition 1. The superalgebra $\mathfrak{sl}(M+1|N+1)$ is realized by the Heisenberg algebra $\mathcal{H}_L$. 

\[ h_i = - \sum_{m=1}^{M+N+1} \sum_{l=1}^{m} x \sum_{r=l}^{m} a_{ir} M_{lm} + (\alpha_i|\Lambda) \] (11)

\[ e_i = x_{i1}^{-1} M_{ii} + \sum_{l=1}^{i-1} x_{li}^{-1} M_{li} \] (12)

\[ f_i = \nu_i \sum_{l=1}^{i-1} x_{li} x_{i1}^{-1} M_{li-1} - \nu_{i+1} \sum_{l=i+1}^{M+N+1} x_{ii} x_{i1}^{-1} M_{i+1l} + x_{ii} \left( -\frac{a_{ii}}{2} M_{ii} - \sum_{m=i+1}^{M+N+1} \sum_{l=1}^{m} x \sum_{r=l}^{m} a_{ir} M_{lm} + (\alpha_i|\Lambda) \right) \] (13)

proof.

We can determine the form of the generators using the following commutation relations:

\[ [h_i, e^{x_{lm}} X_{lm}] = - \sum_{r=1}^{m} a_{ir} M_{rm} e^{x_{lm}} X_{lm}, \] (14)

\[ [e_i, e^{x_{ij}} X_{j1}] |\Lambda> = \delta_{i,j} x_{ii} \left( -\frac{a_{ii}}{2} M_{ii} + (\alpha_i|\Lambda) \right) e^{x_{ij}} X_{j1} |\Lambda>, \] (15)

\[ [e_i, X_{jj}] = \nu_i X_{j1}, \] (16)

\[ [e_i, X_{ij}] = -\nu_{i+1} X_{i1+j}, \] (17)

\[ [f_i, X_{j1}] = X_{j1}, \] (18)

\[ [f_i, X_{ji}] = 0, \quad [f_j, X_{ji}] = 0, \quad [f_k, X_{ji}] = 0 \quad (j + 1 \leq k \leq i - 1). \] (19)

We also check the results by the explicit calculation for the commutations of the generators.

3. $q$-Differential operator realization of $U_q(\mathfrak{sl}(M+1|N+1))$.

Now we are the position to extend the representation on the Heisenberg algebra to the case of the quantum Lie superalgebra $U_q(\mathfrak{sl}(M+1|N+1))$. If there is no confusion, we adopt the same notations in the case of $\mathfrak{sl}(M+1|N+1)$. The relations

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1$\theta_{ij}^{-1}$ is defined as a differential operator, $\partial_{\theta_{ij}}$
among the Chevalley generators $t_i, e_i, f_i (i = 1, \cdots, M + 1)$ are written by
\begin{align}
t_i t_j &= t_j t_i, \\
t_i e_j t_i^{-1} &= q^{a_{ij}} e_j, \\
t_i f_j t_i^{-1} &= q^{-a_{ij}} e_j, \\
[e_i, f_j] &= \delta_{ij} t_i t_i^{-1} - t_i t_i^{-1} q - q^{-1},
\end{align}
where we have used the notations \( [X, Y]_\xi = XY - (\xi Y X) \) and \( q \) is complex number such that \( |q| \neq 1 \). The deformation of the differential operator is defined by
\begin{align}
D_{x_{ij}} f(x) = \frac{f(q x_{ij}) - f(q^{-1} x_{ij})}{q - q^{-1}} = q^{M_{ij}} - q^{-M_{ij}} x_{ij}(q - q^{-1}) f(x) = x_{ij}^{-1} [M_{ij}] f(x),
\end{align}
where \([n]\) is a standard notation \( \frac{q^n - q^{-n}}{q - q^{-1}} \). The quantum affine super algebra \( U_q(\mathfrak{sl}(M + 1|N + 1)) \) can be endowed with the graded Hopf algebra structure. We take the following coproduct
\begin{align}
\Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1},
\end{align}
and the antipode
\begin{align}
a(e_i) &= -t_i^{-1} e_i, \quad a(f_i) = -f_i t_i, \quad a(t_i^{\pm 1}) = t_i^{\pm 1}.
\end{align}
The coproduct is an algebra automorphism \( \Delta(xy) = \Delta(x)\Delta(y) \) and the antipode is a graded algebra anti-automorphism \( a(xy) = (-1)^{|x||y|} a(y)a(x) \) for \( x, y \in U_q(\mathfrak{sl}(M + 1|N + 1)) \).

Let \( V_\Lambda \) be the Verma module over \( U_q(\mathfrak{sl}(M + 1|N + 1)) \) generated by the highest weight vector \( |\Lambda > \) such that \( e_i |\Lambda > = 0, t_i |\Lambda > = q^{(\alpha_i, |\Lambda |)} |\Lambda > \). Weight vectors are written as \(|\Lambda; n_{lm} > = (X_{lm})^{n_{lm}} |\Lambda >; n_{lm} > (1 \leq l \leq m \leq M + N + 1)\), where \( X_{lm}^- = f_{l}^- \) and \( X_{lm}^- (l \leq m) \) are the generators corresponding to the non-simple roots \( \alpha_l + \alpha_{l+1} + \cdots + \alpha_m \):
\begin{align}
X_{lm}^- = [f_m, [f_{m-1}, [f_{m-2}, \cdots, [f_{l+1}, f_l]_{q^{-\nu+1}}]_{q^{-\nu+2}} \cdots]_{q^{-\nu}}.
\end{align}
The dual module \( V_\Lambda^* \) is generated by \( <\Lambda | \) such that \( <\Lambda | f_i = 0, <\Lambda | t_i = q^{(\alpha_i, |\Lambda |)} <\Lambda | \). A \( q \)-deformed coherent state is defined by
\begin{align}
|\mathbf{x} > &= |x_1, x_2, \cdots, x_{M+N+1}|_{M+N+1} > = \prod_{l \leq m} \sum_{n_{lm} = 0}^{\infty} \frac{x^{n_{lm}}_{lm}}{[n_{lm}]!} |\Lambda; n_{lm} > = \prod_{l \leq m} e_q^{x_{lm} X_{lm}^-} |\Lambda >,
\end{align}
where \( e_q^n = \sum_{n=0}^{\infty} \frac{n!}{q^n} \). The bilinear form \( V_\Lambda^* \otimes V_\Lambda \rightarrow \mathbb{C} \) is defined by
\begin{align}
<\Lambda | \Lambda > = 1 \text{ and } <\mathbf{x} | X | \mathbf{y} > = <\mathbf{x} | (X | \mathbf{y} >) \text{ for any } <\mathbf{x} | \in V_\Lambda^* \text{, } | \mathbf{y} > \in V_\Lambda \text{ and } X \in U_q(\mathfrak{sl}(M + 1|N + 1)) \). We can construct the representation in the space of analytic functions on the coordinates \( \{x_{ij}\} \) through the bilinear form.\footnote{There is no change in the case of Grassmann coordinates, \( D_{\theta_{ij}} = \theta_{ij}^{-1} [M_{ij}] = \theta_{ij}^{-1} M_{ij} = \theta_{ij} \).}
Proposition 2. The quantum superalgebra $U_q(\mathfrak{sl}(M+1|N+1))$ is realized by the Heisenberg algebra $\mathcal{H}_L$.

\[ t_i = q^{-\sum_{m=1}^{M+1} a_{ir} M_{im} + (\alpha_i | \Lambda)} \]

\[ e_i = q^{\sum_{k=1}^{i-1} (\nu_i M_{ki} - \nu_i M_{ki-1}) M_{ii}} + q^{\sum_{k=1}^{i-1} (\nu_i M_{ki} - \nu_i M_{ki-1}) x_{ii-1}^{-1} M_{ii}}, \]

\[ f_i = \nu_i \sum_{j=1}^{i-1} q^{\rho_{ij} x_{ji-1}} M_{ji} - \nu_i \sum_{j=1}^{M+1} q^{-\eta_{ij}} x_{ij} x_{i+1j}^{-1} M_{i+1j} \]

\[ + x_{ii} \left( -\frac{a_{ii}}{2} M_{ii} - \sum_{m=1}^{M+N+1} \sum_{l=1}^{m} a_{ir} M_{im} + (\alpha_i | \Lambda) \right), \]

where

\[ \rho_{ij} = -\sum_{m=1}^{M+N+1} \sum_{l=1}^{m} a_{ir} M_{im} - \sum_{l=j+1}^{i} a_{ir} M_{ij} + \sum_{l=j+1}^{i-1} \nu_i M_{li-1} + (\alpha_i | \Lambda), \]

\[ \eta_{ij} = -\sum_{m=1}^{M+N+1} \sum_{l=1}^{m} a_{ir} M_{im} - \sum_{j=1}^{M+N+1} a_{ij} M_{ij} - \sum_{l=j+1}^{j} a_{il} - \sum_{l=j+1}^{j} a_{il} + (\alpha_i | \Lambda). \]

proof.

We derive the forms by acting the generators $t_i, e_i, f_i (i = 1, \cdots, M + N + 1)$ on the $q$-deformed coherent states (31) and using the following commutation relations:\(^3\)

\[ [h_i, e_q x_{im} X_{im}^{-}] = -\sum_{r=1}^{m} a_{ir} M_{im} e_q x_{im} X_{im}^{-}, \]

\[ [e_i, e_q x_{ij} x_{ij}^{-}] \Lambda > = \delta_{i,j} x_{ij} \left( -\frac{a_{ii}}{2} M_{ii} + (\alpha_i | \Lambda) \right) e_q x_{ii} X_{ii}^{-} \Lambda >, \]

\[ [e_i, X_{ji}] = \nu_i X_{ji-1}^{-1} t_i, \]

\[ [e_i, X_{ij}] = -\nu_i + 1 t_i^{-1} X_{jj+1i}^{-1}, \]

\[ f_i (X_{ji-1}^{-})^n = (X_{ji-1}^{-} q^{-\nu_i})^n f_i + [n](X_{ji-1}^{-})^{n-1} X_{ji}^{-}, \]

\[ [f_i, X_{ji}] q^{\nu_i} = 0, \quad [f_j, X_{ji}] q^{-\nu_j} = 0, \quad [f_k, X_{ji}] = 0 \quad (j + 1 \leq k \leq i - 1). \]

Substituting the explicit Cartan matrix $a_{ij}$, we obtain the following forms in terms

\(^3\)Last three relations are derived from the Serre relations, the first two use only one Serre relation, while the third one uses two Serre relations. The detail calculations are given in the paper[22].
of $\nu_i, M_{ij}$ and $x_{ij}$:

\[
\sum_{m=1}^{M+N+1} \sum_{l=1}^{m} \sum_{r=l}^{m} a_{ir} M_{lm} = \sum_{l=1}^{i-1} (\nu_{i+1} M_{li} - \nu_i M_{li-1}) + \sum_{l=i+1}^{M+N+1} (\nu_i M_{il} - \nu_{i+1} M_{i+1l}) + (\nu_i + \nu_{i+1}) M_{ii},
\]

(43)

\[
\sum_{m=i+1}^{M+N+1} \sum_{l=1}^{m} \sum_{r=l}^{m} a_{ir} M_{lm} = \sum_{l=i+1}^{M+N+1} (\nu_i M_{il} - \nu_{i+1} M_{i+1l}),
\]

(44)

\[
\sum_{l=j+1}^{i} \sum_{r=l}^{i} a_{ir} M_{li} = \sum_{l=j+1}^{i-1} \nu_{i+1} M_{li} + (\nu_i + \nu_{i+1}) M_{ii},
\]

(45)

\[
\sum_{l=i}^{j} \sum_{r=l}^{j} a_{ir} M_{ij} = \nu_i M_{ij} - \nu_{i+1} M_{i+1j},
\]

(46)

\[
\sum_{l=i}^{j} a_{il} = \sum_{l=i+1}^{j} a_{il} = \nu_i + \nu_{i+1}.
\]

(47)

**Proposition 3.** The superalgebra $\mathfrak{sl}(M+1|N+1)$ is realized in terms of $M_{ij}$, $x_{ij}$, $D_{ij}$ and $\nu_{ij}$.

\[
t_i = q^{-\sum_{l=1}^{i-1} (\nu_{i+1} M_{li} - \nu_i M_{li-1}) - \sum_{l=i+1}^{M+N+1} (\nu_i M_{il} - \nu_{i+1} M_{i+1l}) - (\nu_i + \nu_{i+1}) M_{ii} + (\alpha_i | \Lambda)},
\]

(48)

\[
e_i = q^{\sum_{k=1}^{i-1} (\nu_{i+1} M_{ki} - \nu_i M_{ki-1})} D_{x_{ii}} + \sum_{l=1}^{i-1} q^{\sum_{k=1}^{l-1} (\nu_{i+1} M_{ki} - \nu_i M_{ki-1})} D_{x_{ii-l}} D_{x_{ii}},
\]

(49)

\[
f_i = \nu_i \sum_{j=1}^{i-1} q^{\alpha_{ij}} x_{ij} D_{x_{ji-1}} - \nu_{i+1} \sum_{j=i+1}^{M+N+1} q^{-\alpha_{ij}} x_{ij} D_{x_{i+1j}}
\]

\[
+ x_{ii} \left[ -\frac{\nu_i + \nu_{i+1}}{2} M_{ii} - \sum_{l=i+1}^{M+N+1} (\nu_i M_{il} - \nu_{i+1} M_{i+1l}) + (\alpha_i | \Lambda) \right],
\]

(50)

where

\[
\rho_{ij} = -\sum_{l=j+1}^{i-1} (\nu_{i+1} M_{li} - \nu_i M_{li-1}) - \sum_{l=i+1}^{M+N+1} (\nu_i M_{il} - \nu_{i+1} M_{i+1l})
\]

\[
- (\nu_i + \nu_{i+1}) M_{ii} + (\alpha_i | \Lambda),
\]

(51)

\[
\eta_{ij} = -\sum_{l=j}^{M+N+1} (\nu_i M_{il} - \nu_{i+1} M_{i+1l}) + (\nu_i + \nu_{i+1}) + (\alpha_i | \Lambda).
\]

(52)

proof.

It can be checked that the generators satisfy the relations (20-26) by the explicit
calculation of the commutation relations using the following relations:

\[
[M_{ij}]x_{ij} = x_{ij}[M_{ij} + 1],
\]
\[
[M_{ij}]D_{ij} = D_{ij}[M_{ij} - 1],
\]
\[
[D_{ij}, z_{lm}] = \delta_{i,l}\delta_{j,m}\{[M_{lm} + 1] - [M_{lm}]\},
\]
\[
[D_{ij}, \theta_{lm}] = \delta_{i,l}\delta_{j,m}\{1 - [M_{lm}] + [M_{lm}]\} = \delta_{i,l}\delta_{j,m}.
\]

We show the example of \(U_q(sl(2|1))\).

\[
t_1 = q^{-2M_{11} - M_{12} + M_{22} + (\alpha_1|\Lambda)},
\]
\[
t_2 = q^{M_{11} + M_{12} + (\alpha_2|\Lambda)},
\]
\[
e_1 = D_{x_{11}},
\]
\[
e_2 = q^{-M_{11} - M_{22}}\partial_{\theta_{22}} + x_{11}\partial_{\theta_{12}},
\]
\[
f_1 = -q^{M_{12} - M_{22} - (\alpha_1|\Lambda) - 2\theta_{12}\partial_{\theta_{22}} + x_{11}[-M_{11} - M_{12} + M_{22} + (\alpha_1|\Lambda)]},
\]
\[
f_2 = q^{(\alpha_2|\Lambda)}\theta_{12}D_{x_{11}} + \theta_{22}[(\alpha_2|\Lambda)].
\]

We give one concrete check.

\[
[e_2, f_2] = [q^{-M_{11} - M_{22}}\partial_{\theta_{22}}, \theta_{22}[(\alpha_2|\Lambda)] + [x_{11}\partial_{\theta_{12}}, q^{(\alpha_2|\Lambda)}\theta_{12}D_{x_{11}}]
\]
\[
= q^{-M_{11} - M_{22}}[(\alpha_2|\Lambda)] + q^{(\alpha_2|\Lambda)}[M_{11} + M_{12}] = \frac{t_2 - t_2^{-1}}{q - q^{-1}}.
\]
References

[1] Wakimoto M., Fock representations of the affine Lie algebra $A_1^{(1)}$, Commun. Math. Phys. 104 (1986), 605-609.
[2] Feigin B.L. and Frenkel E.V., Affine Kac-Moody algebras and semi-infinite flag manifolds, Commun. Math. Phys. 128 (1990), 161-189.
[3] Bershadsky M. and Ooguri H., Hidden $SL(N)$ symmetry in conformal field theories, Commun. Math. Phys. 126 (1989), 49-83.
[4] Kimura K., On Free boson representation of the quantum affine algebra $U_q(\hat{sl}_2)$, RIMS-910 (Dec. 1992).
[5] Shiraishi J., Free boson representation of $U_q(\hat{sl}_2)$, Phys. Lett. A171 (1992), 243-248.
[6] Matsuo A., Free field representation of the quantum affine algebra $U_q(\hat{sl}_2)$, Phys. Lett. B308 (1993), 260-265.
[7] Abada A., Bougourzi A.H. and EL Gradechi, Deformation of the Wakimoto construction, Mod. Phys. Lett. A8 (1993), 715-723.
[8] Awata H., Odake S. and Shiraishi J., Free boson representation of $U_q(\hat{sl}_3)$, Lett. Math. Phys. 30 (1994), 209-216.
[9] Biedenharn L.C., The quantum group $SU_q(2)$ and a q-analogue of the boson operators, J. Phys. A: Math. Gen. 22 (1989), L873-L878.
[10] Macfarlane, On q-analogues of the quantum harmonic oscillator and quantum group $SU_q(2)$, J. Phys. A: Math. Gen. 22 (1989), 4581-4588.
[11] Awata H., Noumi M., and Odake S., Heisenberg realization for $U_q(sl_n)$ on the flag manifold, Lett. Math. Phys. 30 (1993), 35-44.
[12] Shafiekhani A., $U_q(sl(n))$ difference operator realization, Mod. Phys. Lett. A9 (1994), 3273-3283.
[13] Bouwknegt P., Ceresole A., McCarthy J.G. and Nieuwenhuizen P. van, Extended Sugawara construction for the superalgebras $SU(M+1|N+1)$. I. Free-field representation and bosonization of super Kac-Moody currents, Physical Review D 39 (1989), 2971-2987.
[14] Kac V., Representations of Classical Lie Superalgebras Lecture Notes in Mathematics, vol. 676, Springer-Verlag, Berlin, 1978, pp. 597.
[15] Frappat D., Sciarrino A. and Sorba P., Structure of basic Lie superalgebras and of their affine extensions, Commun. Math. Phys. 121 (1989), 457-500.
[16] Yamane H., On defining relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras, preprint (1996), [q-alg/9603015].
[17] Kimura K., Shiraishi J. and Uchiyama J., A level-one representation of the quantum affine superalgebra $U_q(\hat{sl}(M+1|N+1))$, preprint (1996), [q-alg/9605047].
[18] Frenkel I. and Jing N., Vertex representations of quantum affine algebras, Proc. Nat’l. Acad. Sci. USA 85 (1988), 9373-9377.
[19] Chaichian M. and Kulish P., Quantum Lie superalgebras and q-oscillators, Phys. Lett. B234 (1990), 72-80.
[20] Chung W.-S. and Shafiekhani A., Free field representation of $osp(2\mid 1)$ and $U_q(osp(2\mid 1))$ and $N=1$ (q-) superstring correlation functions, Phys. Lett. B381 (1996), 68-72.
[21] Jing N., On Drinfeld realization of quantum affine algebras, preprint (1996), [q-alg/9610035].