M5 on a torus and the three brane

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Abstract

We examine the D-3 brane from the point of view of the double dimensionally reduced M theory 5 brane on a torus. M-theory, IIB identifications are explicitly constructed and a possible reformulation of the D-3 brane is discussed. The duality transformation of the reduced 3-brane necessary to make the identification is discussed in detail.
Introduction

One of the most interesting aspects of M-theory is the five brane [1,2,3,4,5,6]. Its six dimensional world volume supports a two form self-dual gauge field and the three form gauge potential of 11-dimensional supergravity. From this point of view the 5-brane is like a D-brane for the membrane in 11-dimensions in that the membrane may end on the 5-brane.

Importantly, the M-theory origin of many D-branes in string theory is the 5-brane. For example, the 5-brane double dimensionally reduced on a circle produces the world volume dual of the D-4 brane of IIA string theory [7]. (Double dimensional reduction implies that some of the coordinates of the 5-brane world volume are identified with the coordinates of the compact part of space-time. Hence, double dimensional reduction may be interpreted as wrapping the object around the compact dimensions). The M-theory 5-brane has also been related to the heterotic string. A double dimensional reduction of the 5-brane on $K3$ has been identified with the heterotic string compactified on $T^4$ [8]. This also produced a reformulation of the heterotic string in which the Narain duality was manifest in the action.

This paper will involve the relationship between M-theory and IIB string theory. Reducing M-theory on a torus ought to be identified with the IIB theory reduced on a circle. For example the 11-dimensional membrane wrapped around one cycle of the torus will be identified with the IIB fundamental string and the membrane wrapped around the other cycle will be identified with the D-string. As such, the IIB $SL(2,\mathbb{Z})$ duality which mixes Ramond Ramond and Neveu-Schwarz sectors may be seen as a geometrical consequence of the torus in the M-theory picture. More concretely, under the $SL(2,\mathbb{Z})$ transformations, the R-R and NS-NS two forms transform as an $SL(2,\mathbb{Z})$ doublet while the axion-dilaton undergoes an $SL(2,\mathbb{Z})$ fractional linear transformation and the R-R 4-form is left invariant.

The IIB string theory also possesses other branes apart from the fundamental string and D-string. The theory also contains a self-dual D-3 brane, a D-5 brane and a solitonic 5-brane. The self-dual three brane, so called because it couples to the self-dual, $SL(2,\mathbb{Z})$ inert, Ramond Ramond 4-form, will be the main topic of this paper. In particular, we will investigate its relationship to the M-theory 5-brane. For completeness we state that the D-5 and solitonic 5-brane couple magnetically to the R-R, NS-NS two forms respectively and so should transform
into each other under SL(2,Z). It would be interesting to see how these five branes are related to the M-theory 5-brane and how their duality properties appear. (However, we will not do so here).

Given the relationship between M-theory and IIB, we expect the M-theory 5-brane wrapped on the torus to be identified with the world volume dual of the direct reduction of the IIB self-dual three brane [9]. (By direct reduction we imply that the brane’s world volume is not reduced). The duality properties of the 3-brane should then arise as a consequence of the modular symmetry of the torus in the M-theory picture.

In [10] this identification was carried out for the Born-Infeld action ie. in the absence of R-R fields and without reference to the background space-time. Here we will include the R-R fields as well as the embedding in a superspace background and make the identification in 9-dimensions. This identification of M theory and IIB string theory has been discussed in detail for the low energy effective theories in [11] and with a view to extended objects in [12].

The structure of the paper will be as follows. First we will introduce our notation and describe the M5-brane action. No efforts will be made to compare our results with the interesting and indeed powerful 5-brane approach [3] based solely on the equations of motion. We will then carry out the double dimensional reduction on $T^2$. Following this we will describe the the direct reduction of the IIB three brane on $S^1$. To compare the two actions it will be necessary to make world volume duality transformations of some of the fields on the brane.

This duality procedure, for the case given above is far from trivial. We will make a variety of truncations that will enable us to construct the dual actions for the truncated cases. These duality transformations are of a similar type as those described in some detail in [7,14,15].

The point of the transformations is that we will be able to identify the dualized reduced 5-brane with the reduced D-3 brane. In doing so we will be able to explicitly identify the fields and construct the SL(2,Z) duality properties of the IIB theory from the M-theory picture. In particular, the SL(2,Z) transformation of the three brane will arise out of a gauge choice made on the 5-brane world volume.

**The 5-brane**

The kappa symmetric action for the 5-brane [1,2] is as follows. We work
with a flat Minkowski background, using a metric, \( \eta = \text{diag}(-1,1,1,..) \). The \( \theta \) coordinates are 32 component Majorana spinors and \( X^M \) are 11-dimensional space-time coordinates \( (M,N = 0..9,11) \). We will follow [1] and use the convention where the Clifford algebra for the \( \Gamma \) matrices is \( \{ \Gamma^M \Gamma^N \} = 2 \eta^{MN} \). The global supersymmetry transformations may be written as \( \delta \theta = \epsilon \), \( \delta X^M = \bar{\epsilon} \Gamma^M \theta \). The action is written in terms of the supersymmetric invariant one forms \( d\theta \) and \( \Pi^M = dX^M + \bar{\theta} \Gamma^M d\theta \). Where \( d = d\sigma^\mu \partial^\mu; \) the exterior derivative pulled back to the brane. \( \sigma^\mu \) are the coordinates of the brane, \( \hat{\mu} = 0..5 \). (We use the convention that \( d\sigma^\mu \) is odd with respect to the grassmann variables so that \( d\theta = d\sigma^\mu \partial^\mu \theta = -\partial^\mu \theta d\sigma^\mu \)).

The action will also contain a world volume self dual two form gauge field, \( B \) whose field strength is as usual given by \( H = dB \). In order to ensure supersymmetry this is extended as follows; \( \mathcal{H} = H - b_3 \) where \( b_3 \) is the 11 dimensional 3-form potential pulled back to the brane defined as follows: \( b_3 = \frac{1}{2} \bar{\theta} \Gamma_{MN} d\theta (dX^M dX^N + dX^M \bar{\theta} \Gamma^N d\theta + \frac{1}{3} \bar{\theta} \Gamma^M d\theta \Gamma^N d\theta) \).

We are implicitly assuming wedge products for forms unless stated otherwise. The action for the 5-brane will be written as follows:

\[
S = - \int_{M^6} d^6x \sqrt{-\det(G_{\hat{\mu} \hat{\nu}} + i \frac{\mathcal{H}_{\hat{\mu} \hat{\nu}}}{\sqrt{\partial^6 \sigma^\hat{\mu}}}) - \frac{\sqrt{-G \mathcal{H}^\hat{\nu} \mathcal{H}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}}}{4v^2}} + S_{WZ} \tag{1}
\]

where:

\[
\mathcal{H}_{\hat{\mu} \hat{\nu}} = \frac{1}{6} G_{\hat{\mu} \hat{\alpha}} G_{\hat{\nu} \hat{\beta}} \epsilon^{\hat{\alpha} \hat{\beta} \hat{\gamma} \hat{\rho} \hat{\sigma}} \sqrt{G} H_{\hat{\gamma} \hat{\rho} \hat{\sigma}} v^\gamma \tag{2}
\]

and

\[
G_{\hat{\mu} \hat{\nu}} = \Pi^M_{\hat{\mu}} \Pi^N_{\hat{\nu}} \eta_{MN}
\]

\( G = \det G_{\hat{\mu} \hat{\nu}}; \) \( v \) is a completely auxiliary closed one form field introduced to allow the self-duality condition to be imposed in the action while maintaining Lorentz invariance \(^1\). See the references [2] for a discussion on this Lorentz invariant formulation. The \( S_{WZ} \) is the so called Wess Zumino part of the action that is introduced to ensure the kappa symmetry of the action and in analogy with the usual Wess-Zumino type action may be written more conveniently as an exact form over a manifold whose boundary corresponds to the five brane world volume. That is:

\[
S_{WZ} = \int_{M^7} I_7
\]

\(^1\) Usually the action (1) is written with \( v = da \); however this is only locally correct as \( v \) is constrained to be closed but not necessarily exact.
where $dI_7 = 0$ and $\partial M^7 = M^6$ which implies locally we may write $I_7 = d\Omega_6$ and so we can write $S_{WZ}$ as an integral over the world volume giving $S_{WZ} = \int_{M^6} \Omega_6$.

\[ I_7 = -\frac{1}{4} \mathcal{H} d\bar{\theta} \psi\sqrt{\gamma} \sqrt{\gamma} d\bar{\theta} - \frac{1}{120} d\bar{\theta} \psi^5 d\bar{\theta} \quad (4a) \]

where $\psi = \Gamma_M \Pi^M$ the induced Gamma matrix. Integrating we find:

\[ \Omega_6 = C_6 + \mathcal{H} \wedge b_3 \quad (4b) \]

where $b_3$ is the same form that appears in combination with $H$ above. (We will not need an explicit form for $C_6$). This action has been shown to have all the properties required of the 5-brane [1]. Apart from the usual gauge symmetries associated with the gauge potential $B$ and the background field $C$, this action has additional local, so called PST symmetries one of which we will use later to eliminate half the degrees of freedom of the two form gauge field.

\[ \delta B = \chi \wedge v \quad (5) \]

This will be the action that we will double dimensionally reduce on $T^2$. And so we send, $M^6 \to M^4 \times T^2$ and $M^{11} \to M^9 \times T^2$. We will identify $(X^{11}, X^9) = (\sigma^4, \sigma^5) = (y^1, y^2)$ Where $(y^1, y^2)$ are the coordinates on the space-time torus. In these coordinates we will identify $y^1 = y^1 + 1$ and $y^2 = y^2 + 1$. Despite reducing to 9 dimensions we will not decompose the spinors as it will be convenient in what follows to leave them. We will drop all functional dependence of the fields on the compact coordinates, that is taking only the zero modes. $m, n = 0..8$ will be the non compact space-time indices, $i, j = 1, 2$ will be torus coordinate indices and $\mu, \nu = 0..3$ will be the coordinates of the non-wrapped 5-brane world volume. The space-time metric will be written as

\[ \eta_{MN} \to \eta_{mn} \oplus \eta_{ij} \quad (6) \]

This truncates the space-time Kaluza Klein fields associated with the torus. This is because we are only interested in the M-5 brane/D-3 relationship. Such Kaluza-Klein fields are associated with the wrapped D and fundamental string in IIB. We will take $\eta_{mn}$ to be flat Minkowski metric and take the metric on the torus to be given by

\[ \eta_{ij} dy^i \otimes dy^j = \frac{V}{\tau_2} (dy^1 \otimes dy^1 + \tau_1 dy^2 \otimes dy^1 + \tau_1 dy^1 \otimes dy^2 + |\tau|^2 dy^2 \otimes dy^2) \quad (7) \]
\( \tau = \tau_1 + i \tau_2 \) is the complex structure of the torus and \( V \) is the area of the torus.

The reduction of the brane metric \( G \) from (3) follows.

\[
G_{\hat{\mu} \hat{\nu}} d\hat{\sigma}^\hat{\mu} \otimes d\hat{\sigma}^\hat{\nu} = (\Pi^m_{\hat{\mu}} \Pi^\nu_{\hat{\nu}} \eta_{mn} + C^i_{\hat{\mu}} \Pi^\nu_{\hat{\nu}} \eta_{ij}) d\sigma^\mu \otimes d\sigma^\nu + C_{ji} d\sigma^\mu \otimes dy^j + C_{j\nu} dy^j \otimes d\sigma^\nu + \eta_{ij} dy^i \otimes dy^j \tag{8}
\]

Where

\[
C^i_{\mu} = -\bar{\theta} \Gamma^i T \partial_{\mu} \theta \tag{9}
\]

\( \Gamma_T \) are the Gamma matrices on the torus. As we have identified the space-time coordinates \((X^1, X^9)\) with \((y^1, y^2)\) the torus coordinates we have \((\Gamma^1_T, \Gamma^2_T) = (\Gamma^1, \Gamma^9)\). Now the background three form potential will reduce as follows:

\[
b_3 = b_3(3) + b_3(2) \wedge dy^i + b_3(1) dy^1 \wedge dy^2 \tag{10}\]

Where

\[
b_3(3) = \frac{1}{2} \bar{\theta} \Gamma_{mn} \theta (dX^m dX^n + dX^m \bar{\theta} \Gamma^n d\theta + \frac{1}{3} \bar{\theta} \Gamma^m \theta \theta \Gamma^n d\theta)
\]

\[
+ \frac{1}{2} \bar{\theta} \Gamma_m \Gamma_{T, i} \theta (dX^m \bar{\theta} \Gamma_i d\theta + \frac{2}{3} \bar{\theta} \Gamma^m \theta \theta \Gamma_i d\theta) + \frac{1}{6} \bar{\theta} \Gamma_{T, ij} \theta \theta \Gamma_i \Gamma_j d\theta \tag{11a}\]

\[
b_3(2) = \frac{1}{2} \bar{\theta} \Gamma_{T, i} \theta (2dX^n + \bar{\theta} \Gamma^n d\theta) + \frac{1}{2} \bar{\theta} \Gamma_{T, ij} \theta \theta \Gamma_i \Gamma_j d\theta \tag{11b}\]

\[
b_3(1) = \theta \Gamma_{T, 12} d\theta \tag{11c}\]

As usual \( \Gamma_{pq} \) implies \( \Gamma_{[p} \Gamma_{q]} \), where square brackets on the indices mean antisymmetrisation. Similarly, we reduce the world volume gauge field as follows:

\[
B = B_{(0)} dy^1 \wedge dy^2 + B_{(1)i} \wedge dy^i + B_{(2)} \tag{12}\]

so that the we may write for \( \mathcal{H} = H - b \)

\[
\mathcal{H} = \mathcal{J} + \mathcal{F}_i \wedge dy^i + \mathcal{L} dy^1 \wedge dy^2 \tag{13a}\]

Where we have defined:

\[
\mathcal{J} = dB_{(2)} - b_3(3) \quad \mathcal{F}_i = dB_{(1)i} - b_3(2) \quad \mathcal{L} = dB_{(0)} - b_3(1) \tag{13b}\]

We now need to determine whether the auxiliary one form will be in \( T^2 \) only or in \( M^4 \) only. The two choices are physically equivalent. The restriction simply corresponds to a partial gauge fixing. In what follows we will take \( v \) to be a member of the first cohomology on \( T^2 \). We will consider the specific choices
$v = dy^1$ and $v = dy^2$. These two independent gauge choices are what will eventually generate the S-duality on the 3-brane. Should we put $v$ in $M^4$, for example $v = dt$ then the SL(2,Z) symmetry of the 3-brane will become manifest in the action but we will lose manifest Lorentz invariance. (This will give an action of type given in [16].) The relationship between the formulation of the reduced action and the different gauge choices is discussed in [10]. For now we will take the torus to be have $\tau = 1$ and $V = 1$; we will reinstate the dependence on $V$ and $\tau$ when required. So with the specific gauge choice

$$v = dy^2$$

this implies:

$$\tilde{H}^\mu{}^\nu = (F^{\mu\nu}, J^{\mu1})$$

Therefore,

$$\tilde{H}_{\mu\nu} = * F_{\mu\nu} + C_\mu C_\rho * F^\rho{}\nu + * F_\mu C_\rho C_\nu + C_\mu C_\sigma C_\nu C_\rho * F^{\rho\nu} - C_1[\mu * J_\nu]$$  \hspace{1cm} (14a)

$$\tilde{H}_{\mu i} = \eta_{i1}(* J^\mu + C_\mu C_\rho * F^\rho) - C_{i\rho} * F_\mu - C_{i\rho} * F^{\rho\sigma} C_\sigma C_\nu - C_{i\rho} * J^\rho C_1\mu$$ \hspace{1cm} (14b)

$$\tilde{H}_{ij} = C_{i\mu} C_{j\nu} * F^{\mu\nu} - C_{2\rho} * J^\rho$$ \hspace{1cm} (14c)

$$v^2 = 1 + (C_2)^2$$ \hspace{1cm} (14d)

Where we use the notation $C_\mu C_\nu = C_\mu C_\nu$ and * is the Hodge dual in 4 dimensions. Combining the above equations with the reduced metric (8) we have for, $M$, the matrix inside the determinant of action (1):

$$M = (G_{\mu\nu} + C_\mu C_\nu + \frac{i\tilde{H}_{\mu\nu}}{\sqrt{1 + (C_2)^2}})d\sigma^\mu \otimes d\sigma^\nu + (C_{i\nu} - \frac{i\tilde{H}_{i\nu}}{\sqrt{1 + (C_2)^2}})d\sigma^i \otimes d\sigma^\nu + (C_{\mu j} + \frac{i\tilde{H}_{\mu j}}{\sqrt{1 + (C_2)^2}})d\sigma^\mu \otimes d\sigma^i$$
\[(\eta_{ij} + \frac{i\hat{H}_{ij}}{\sqrt{1 + (C_2)^2}}) d\sigma^i \otimes d\sigma^j \] (15)

Importantly, we remark that $M$ occurs in the action only in the determinant and so we are allowed to manipulate $M$ in anyway that leaves the determinant invariant. Our goal will be to compare with the D-3 brane, hence it is natural to express the above as a four dimensional determinant. Using the well known identities:

\[
det \left( \begin{array}{cc} L & P \\ Q & J \end{array} \right) = \det \left( \begin{array}{cc} L - Q^T J^{-1} P \\ 0 & J \end{array} \right)
\]

and

\[
det(A \oplus B) = \det(A)\det(B)
\]

We have

\[
det M = det(M_{ij})det(M_{\mu\nu} - M^T_{\mu i}(M^{-1})^{ij} M_{\nu j})
\]

which gives after numerous cancellations:

\[
det(M_{\mu\nu}) = det(M_{ij})det \left( G_{\mu\nu} + \frac{i^* F_{\mu\nu}}{\sqrt{1 + (C_2)^2}} + \frac{P_\mu C_{2\rho}^* F_{\rho\mu} C_{2\alpha} P^\alpha (1 + (C_2)^2)}{1 + (C_2)^2 - (C_2 P^\rho)^2} \right.
\]

\[
\left. - \frac{(P_\mu P_\nu + C_{2\rho}^* F_{\rho\mu} C_{2\sigma}^* F_{\sigma\nu})}{1 + (C_2)^2 - (C_2 P^\rho)^2} \right)
\]

(17)

Where $P_\mu = \star J_\mu - C_1 P^\mu$ and explicitly, $detM_{ij} = \frac{1 + (C_2)^2 - (C_2 P^\rho)^2}{1 + (C_2)^2}$

We will now turn to reducing the Wess-Zumino term. First, we note that

\[
\psi \rightarrow (\psi, \Gamma_{T_1} C^i, \Gamma_{T_1} dy^i)
\]

Using this and the reduction for $\hat{H}$ we calculate the reduced WZ terms by substituting these into $I_7$. Doing the reduction for $I_7$ is equivalent to doing the reduction for $\Omega_6$ provided that the compact space has no boundary, which is of course the case for a torus. We produce for $I_5$ where $S_{WZ^5} = \int_{M^5} I_5$ and $\partial M^5 = M^4$.

Taking care with factors this produces:

\[
I_5 = -\frac{1}{3!} d\bar{\theta} \psi^3 \Gamma_{T_{12}} d\theta - \frac{1}{2} F_{[\mu} (d\bar{\theta} \psi \Gamma_{T_{\rho\theta}} d\theta + d\bar{\theta} \Gamma_{T_1} C^i \Gamma_{T_{i}} d\theta) - \frac{1}{4} F_{ij} d\bar{\theta} \Gamma_{T_{12}} d\theta
\]

\[
+ \frac{1}{4} b_{(1)} d\bar{\theta} (\psi^2 + \psi \Gamma_{T_1} C^i + \Gamma_{T_k} C^k \Gamma_{T_m} C^m) d\theta
\]

(18)
Next, we will examine the PST term, the second term in action \(1\). Upon dimensional reduction this term naturally splits into a sum of two parts. The first part \(I^{(1)}_{\text{PST}}\), consists of terms that look like terms in the Wess-Zumino term and a total derivative (corresponding to the theta term). The second part, \(I^{(2)}_{\text{PST}}\) is distinct and will be associated with a term arising from dualizing the \(J\) field.

\[
I^{(1)}_{\text{PST}} = \int_{M^4} \frac{1}{2} (\mathcal{F}_i \wedge \mathcal{F}_j + \mathcal{J} \wedge \mathcal{L}_{ij}) \gamma^{ij}(v) \tag{19}
\]

\[
I^{(2)}_{\text{PST}} = -P^\mu \mathcal{F}_{\mu\nu(i)} C^{\nu(\ell)} \frac{v_j\epsilon^{il}v_l}{v^2} \tag{20}
\]

where \(\gamma^{ij}(v) = \frac{1}{v^2} \epsilon^{ij}v_jG^{lm}v_m\). 

\(F \wedge F\) is a theta type term that may contribute. In fact it is this term that we will later identify with the axion coupling in the 3-brane.

For a specific choice of \(v = dy^i\), we may gauge away \(L\) and \(F_i\) but this will not gauge away the fields \(b_{(1)}\) and \(b_{(2)}\) that must be kept. And so we integrate the Wess-Zumino terms and combine them with the relevant PST terms using, 

\(d(\Omega - I^{(1)}_{\text{PST}}) = I_5\). And so in terms of fields given in \(11\) this gives the interaction term for the reduced action: For choice \(dy^1\):  

\[
\Omega = b_{(4)} + b_{(2)}1 \wedge F - *P \wedge b_{(1)} - \frac{1}{2} \frac{\tau_1}{|\tau|^2} F \wedge F \tag{21}
\]

For choice \(v = dy^2\):  

\[
\Omega = b_{(4)} - b_{(2)}2 \wedge F - *P \wedge b_{(1)} + \frac{1}{2} \frac{\tau_1}{\tau_2} F \wedge F \tag{22}
\]

Here we remark that the index \(i\) is associated with the torus coordinates \(\{y^i\}\), see equation \((11b)\). Now, so that we may compare with the D3-brane we will rewrite the above expression in terms of orthonormal coordinates \(\bar{y}^\ell\) on the torus. Using the equation,

\[
b_{(2)}i = e_{\bar{\ell}} \bar{b}_{(2)\bar{i}} \tag{23}
\]

where

\[
e_{\bar{\ell}} = \sqrt{\frac{V}{\tau}} \left( \begin{array}{c} 1 \\ \tau_1 \\ \tau_2 \end{array} \right)
\]

is the zweibien of the torus whose metric is given by \((7)\). We then carry out a space time, Weyl scaling

\[X' = X \eta^{1/8} \quad \theta' = \theta \eta^{1/16} \tag{24}\]
We will discuss the relevance of this scaling later. And so when we substitute this into the above, we find: For $v = dy^1$:

$$\Omega = \tilde{b}_4 - \tilde{b}_{(2)i} \wedge \tilde{b}_{(2)i} - \frac{1}{2} \tau_2 \tilde{b}_{(2)i} \wedge \tilde{b}_{(2)i} + \frac{1}{\sqrt{\tau_2}} \tilde{b}_{(2)i} \wedge F - \eta^\perp P \wedge \tilde{b}_{(1)} - \frac{1}{2} \tau_1 F \wedge F$$  \hspace{1cm} (25)

and

$$F = F + \frac{\tau_1}{\sqrt{\tau_2}} \tilde{b}_{(2)i} + \sqrt{\tau_2} \tilde{b}_{(2)i}$$

For $v = dy^2$

$$\Omega = \tilde{b}_4 - \sqrt{\tau_2} \tilde{b}_{(2)i} \wedge F - \eta^\perp P \wedge \tilde{b}_{(1)} + \frac{1}{2} \tau_1 F \wedge F$$

and

$$F = F - \frac{1}{\sqrt{\tau_2}} \tilde{b}_{(2)i}$$  \hspace{1cm} (26)

We remark that all the terms in $\Omega$ depend on either $\tau$ or $\eta$ so they form essentially independent couplings. This will be true when we consider the first part of the action, see below. We also have the extra term, $I^{(2)}_{PSST}$ which becomes for the choice $v = dy^1$:

$$I^{(2)}_{PSST} = \frac{-1}{1 + (C^i) P\mu \nu \nu(i)}$$  \hspace{1cm} (27)

To begin with, we will consider the truncation where we set $\theta = 0$. (This is a consistent truncation). We will also explicitly reinstate the general metric $\eta_{ij}$ of the torus and leave the auxiliary field $v$ unspecified. (Apart from the fact that it is a closed one form on the torus.) This gives for the first part of the action:

$$S_{5-2} = -\int_{T^2} \int_M \sqrt{-\det(G_{\mu\nu} + i\alpha(v) F_{(i)\mu\nu} - \beta(v) J_{\mu}, J_{\nu})} + \frac{1}{2} F_i \wedge F_j + \gamma^{ij}(v)$$  \hspace{1cm} (28)

where $\alpha(v)$ and $\beta(v)$ and $\gamma(v)^{ij}$ are constants that remain to be evaluated and will be dependent on our choice of $v$.

However, before evaluating them we will put the $\sqrt{\eta}$ inside the determinant. This becomes $\eta^\perp$ inside the determinant. We will then carry out a Weyl scaling as before, see equation (24) so that we absorb this factor into the rescaled metric. That is

$$G'_{\mu\nu} = G_{\mu\nu} \eta^\perp$$  \hspace{1cm} (29)
We then rewrite the action in this rescaled metric taking care with factors of $\eta$. The $T^2$ integral is trivial.

We will use the symmetry given by equation (5) to eliminate half the degrees of freedom contained in the gauge fields. For the choice $v = dy^L$ we gauge away $F_{(L)}$ and $L_{12}$. This leaves only one vector gauge field in the action, with field strength $F$, and one two form gauge field, with field strength $J$. The PST part of the action will then contribute a total derivative that we shall be able to identify it with an axion coupling. We will now write the action in its final form as follows:

$$S_{5-2} = - \int_{M^4} \sqrt{-\det(G_{\mu\nu} + i\alpha(v)^* F_{\mu\nu} - \beta^* J_{\mu}^* J_\nu)} + \frac{1}{2} F_i \wedge F_j \gamma^{ij}(v)$$

(30)

We now consider the two natural independent gauge choices for $v$ and evaluate the coefficients, $\alpha$, $\beta$, and $\gamma$.

For $v = dy^1$:

$$\alpha = \sqrt{\tau_2} = \sqrt{\tau_2}^{|\tau|^2} \quad \beta = \eta^{3/4} \quad \gamma = -\frac{\tau_1}{|\tau|^2}$$

(31a)

for $v = dy^2$:

$$\alpha = \sqrt{\tau_2} \quad \beta = \eta^{3/4} \quad \gamma = \tau_1$$

(31b)

Note that the vector fields couple only to the complex structure of the torus. That is the couplings are completely determined by the shape of the torus and are independent of its size. Different choices of $v$ give different couplings. The opposite is true for the two form fields. The coupling for the two form field is independent of the choice of $v$ and is dependent only on the area of the torus.

Combining $\tau = \tau_1 + i\tau_2$ we see the different choices of $v$ generate the transformation $\tau \rightarrow \frac{1}{\tau}$ in the vector field couplings. This corresponds to one of the generators of $SL(2,\mathbb{Z})$ the modular group of the torus. The other generator will arise from an integral shift in $\tau_1$ which will cause a trivial shift in the total derivative term. Later when we compare with the 3 brane on $S^1$, we will identify the complex structure of the torus with the axion-dilaton and the area of the torus will be related to the radius of the compact dimension as given in [12].

**D-3 brane**
Starting with the 10 dimensional IIB three brane action in 10 dimensions \([7,9]\) we will directly reduce the action on a circle. We have two space-time spinors, \(\theta^\alpha, \alpha = 1, 2\). These are Majorana, Weyl spinors in 10 dimensions with the same chirality. The natural group acting this index is \(\text{SL}(2,\mathbb{R})\). In the actions below, following the conventions in \([7]\), we will combine these spinors using the Pauli matrices \(\tau_3\) and \(\tau_1\). The indices labeling the different spinors will be suppressed (as will the actual spinor indices). We will also take \(2\pi\alpha' = 1\). The action (in the Einstein frame) is written:

\[
S_3 = -\int d^4\sigma \sqrt{-\det(G_{\mu\nu} + e^{-\phi/2} F_{\mu\nu})} + \int_{M^5} I_5 \tag{32}
\]

where \(F = F - e^{\phi/2} b\) where \(b = -\bar{\theta}\tau_1 \Gamma_m d\theta (dX^m + \frac{i}{2} \theta \Gamma^m d\theta)\) and \(F\) is the field strength of an abelian vector field \(A\). As before, \(G_{\mu\nu} = \Pi^m_{\mu} \Pi^n_{\nu} g_{mn}\). The Wess-Zumino term is:

\[
I_5 = \frac{1}{6} d\bar{\theta} \tau_3 \tau_1 \psi^3 d\theta + d\bar{\theta} \tau_1 F \psi d\theta = d(C_4 + e^{-\phi/2} C_2 \wedge F) \tag{33}
\]

and we may add a term coupling it to the axion as follows:

\[
I_{td} = \frac{1}{2} C_0 F \wedge F \tag{34}
\]

We will reduce this action directly implying we will not identify any of the brane coordinates with the compact dimension. Hence, we will write \(X^9 = X^9 + 1 = \phi\) and so decompose the background metric \(g_{mn} \rightarrow g_{mn} \oplus R^2\) where \(R\) is the circumference of the compact dimension. That is as before we truncate out the space time Kaluza Klien field. (On the M-theory side this corresponds to truncating the wrapped membrane). Therefore, \(\Pi^m_{\mu} = (\Pi^m_{\mu}, \Pi_9^m)\) where \(\Pi_9^m = \partial_\mu \phi + C'_\mu\) and \(C'_\mu = -\bar{\theta} \Gamma^9 \partial_\mu \theta\). This gives for the induced world volume metric:

\[
G_{\mu\nu} \rightarrow G_{\mu\nu} + R^2 (\partial_\mu \phi + C'_\mu) (\partial_\nu \phi + C'_\nu) \tag{35}
\]

The world volume gauge field is left invariant. The NS 2 form \(b \rightarrow b - \bar{\theta} \tau_3 \Gamma_9 d\theta (d\phi + \frac{i}{2} \bar{\theta} \Gamma^9 d\theta)\) which we will write as \(b \rightarrow b + b^R \wedge d\phi\) where \(b^R\) corresponds to the NS two form reduced to a one form in 9 dimensions. It is this field that a wrapped fundamental string would couple to. The Wess-Zumino part becomes:

\[
I_5 = \frac{1}{6} d\bar{\theta} \tau_3 \tau_1 \psi^3 d\theta + d\theta \tau_1 F \psi d\theta + \frac{1}{2} d\bar{\theta} \tau_3 \tau_1 \psi^2 \chi d\theta + d\bar{\theta} \tau_1 F \chi d\theta \tag{36}
\]

where \(\chi = (d\phi + C') \Gamma_9\) So the final reduced action for the three brane becomes:
\[ S_{3,(S')} = -\int d^4\sigma \sqrt{-\det(G_{\mu\nu} + e^{-2\phi} F_{\mu\nu} - b R \partial_{\mu} \phi + R^2 (\partial_\mu \phi + C_\mu') (\partial_\nu \phi + C_\nu')) + \frac{1}{2} C_0 F \wedge F} \]

\[ + \int_{M^4} C_4 + e^{-\frac{2\phi}{2}} C_2 \wedge F + R^2 (C_3 + C R \wedge F) \wedge d\phi \quad (37) \]

We wish to compare the wrapped 5-brane with different choices of \( v \) with the 3-brane and its S-dual. The S-dual 3-brane is determined by dualizing the vector field on the brane using the same method as described below for dualizing the scalar field. This has been carried out in [7], hence we simply quote the result:

\[ S = -\int d^4\sigma \sqrt{-\det(G_{\mu\nu} + \frac{e^{-\frac{2\phi}{2}}}{C_0^2 + e^{-2\phi}} F_{\mu\nu})} \]

\[ + \int_{M^4} C_{(4)} - C_{(2)} \wedge b - \frac{1}{2} C_0 e^{\frac{\phi}{2}} b \wedge b + e^{\frac{3\phi}{2}} b \wedge F - \frac{C_0}{2(C_0^2 + e^{-2\phi})} F \wedge F \quad (38) \]

and \( F = (F + e^{-\frac{\phi}{2}} C_{(2)} + e^{\frac{3\phi}{2}} C_0 b) \)

The direct reduction would follow as before. The items to note are the, as expected, inversion of the the coupling \( \lambda \to -\frac{1}{\lambda} \) where \( \lambda = C_0 + i e^{-\phi} \) and the slightly altered form of \( F \) and the Wess Zumino terms.

In order to exactly identify the reduced 3-brane action with the 5-brane wrapped action we will first need to do a world volume duality transformation on the field \( \phi \). This is in the spirit of [14] whereby world volume dual actions are associated with the M-theory picture of the brane. To do this we follow the techniques of [7,14,15].

We will first deal with the bosonic truncation before moving on to consider the more general case. This gives the standard Dirac Born-Infeld action:

\[ S = -\int d^4\sigma \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu} + R^2 \partial_\mu \phi \partial_\nu \phi)} \quad (39) \]

We will dualize the scalar field \( \phi \) by replacing its field strength \( d\phi \) with \( l \) and then adding an additional constraint term to the action \( S_c = H \wedge (d\phi - l) \). \( H \) is a lagrange multiplier ensuring that \( l = d\phi \). To find the dual we first find the equations of motion for \( \phi \) and solve. This implies \( dH = 0 \) which means we may locally write \( H = dB \). Then we must find the equations of motion for \( l \) and solve
in terms of $H$. We simplify the problem by working in the frame in which $F$ is in Jordan form with eigenvalues $f_1$ and $f_2$. $l_i$ are the components of $l$ and $h_i$ are the components of the dual of $H$. The equations of motion for $l$ are:

$$h_1 = \frac{(1 + f_2^2)}{\sqrt{-\det M}} l_1 R^2 \quad h_2 = \frac{(1 + f_2^2)}{\sqrt{-\det M}} l_2 R^2$$

$$h_3 = \frac{(1 - f_1^2)}{\sqrt{-\det M}} l_3 R^2 \quad h_4 = \frac{(1 - f_1^2)}{\sqrt{-\det M}} l_4 R^2$$

where $M_{\mu\nu} = G_{\mu\nu} + F_{\mu\nu} + R^2 l_{\mu} l_{\nu}$. We then invert these equations to solve for $l_i$. The solutions are:

$$l_1 = \frac{(f_2^2 - 1)}{\sqrt{-\det M} R^2} h_1 \quad l_2 = \frac{-(f_2^2 - 1)}{\sqrt{-\det M} R^2} h_2$$

$$l_3 = \frac{(1 + f_2^2)}{\sqrt{-\det M} R^2} h_3 \quad l_4 = \frac{(1 + f_2^2)}{\sqrt{-\det M} R^2} h_4$$

Where $\tilde{M}_{\mu\nu} = G_{\mu\nu} + i^* F_{\mu\nu} - \frac{1}{R^2} (\ast H)_{\mu} (\ast H)_{\nu}$

When we substitute these equations into the action we find, reinstating dilaton dependence and the axion term:

$$S_D = - \int d^4\sigma \sqrt{-\det \left( G_{\mu\nu} + i e^{\frac{\phi}{2}} \ast F_{\mu\nu} - \frac{1}{R^2} (\ast H)_{\mu} (\ast H)_{\nu} \right) + \frac{1}{2} C_0 F \wedge F}$$

The axion term goes through untouched. Note how the radius which acts as a coupling for the scalar field is inverted in the dual action. We are now in a position to compare the dualized, directly reduced on $S^1$, IIB D-3 brane action with the double dimensionally reduced on $T^2$, M5 brane action.

In fact, we shall compare the reduced three brane with the with the vector fields dualized and non dualized with the wrapped 5-brane with the two different gauge choices described above. And so we compare equations (42,38) with (30),(31a,b) given above.

In doing so must identify the fields and the moduli of the two theories appropriately. When we compare with the usual M-theory predictions given in [12] concerning the relationship between the moduli of the IIB theory in 9 dimensions with the geometrical properties of the torus used in the M-theory compactification we have agreement. The scaling of the metric given in equation (29) implies

$$G_{\mu\nu}^B = \text{Area}(T^2) \frac{1}{2} G_{\mu\nu}^M$$

$$\text{(43)}$$
From both the coefficient in front of $F$ in the determinant and the coefficient in front of the $F \wedge F$ term, we identify the axion-dilaton of the IIB theory (in the 10 dimensional Einstein frame) with the complex structure of the torus.

$$\lambda = C_0 + ie^{-\phi} = \tau \quad (44)$$

From comparing the coefficient in front of $*H$, the radius of the the 10th dimension in IIB becomes:

$$R_B = Area(T^2)^{-\frac{1}{2}} \quad (45)$$

Where have identified the gauge field on the reduced 5-brane with the gauge field on the reduced D-3. The dualized scalar on the D-3 brane becomes identified with the three form on the reduced M-5 brane.

We will reinstate the truncated fields and attempt to identify these fields between the dual pictures. The duality transformation now becomes a great deal more complicated; it is essentially the terms involving $b^R$ that prevents us from dualizing the 3-brane action as above. We could however take advantage of the fact that the dualized action ought to be our reduced 5-brane action by carrying out the following consistency check. We can obtain an algebraic expression for $H$ from the equations of motion of $L_\mu$ from the reduced three brane. Instead of inverting these equations to obtain an expression for $L$ we may simply insert our expression for $H$ into the reduced 5-brane action and check that this action is the same as the original three brane action. This is essentially the method used in [7] to check the relationship between the 5-brane and 4-brane. This is algebraically extremely involved in this case and does not provide much insight. However, for the case in which the $b^R = 0$ can be dealt with directly. Recall, the integrated Wess-Zumino term:

$$\int_{M^4} C_{(4)} + C_{(2)} \wedge F + (C_{(3)} + C^R \wedge F) \wedge d\phi \quad (46)$$

With the $b^R$ term vanishing from the determinant in $S_{3,1(S^1)}$ we can see that the first term in the action is of the same form as that for the case $\theta = 0$ already considered. As already described, we replace $d\phi$ in the action with a generic one form $L$ and add the constraint $H \wedge (d\phi - L)$. Then integrating out $\phi$ implies $H$ is closed and we are left with the term $-H \wedge L$. Before we simply integrated out $L$ leaving an action in terms of $H$. Now we will combine the terms outside the square root that are linear in $L$ as follows:

$$S = -(H - C_{(3)} - C^R \wedge F) \wedge (L + C') - (H - C_{(3)} - C^R \wedge F) \wedge C' \quad (47)$$
We can now integrate out the combination \( L + C' \) which appears in the action in favour of \( H \equiv (H - C_{(3)} - C^R \wedge F) \) using equations (41). This gives the following dual action, (reinstating \( R \) dependence):

\[
S = -\sqrt{-\det(G_{\mu\nu} + i^*F_{\mu\nu} - \frac{1}{R^2}H_{\mu}^*H_{\nu}) + C_{(4)} + C_{(2)} \wedge F - \frac{1}{R}H \wedge C'}
\]

By comparing (48) with (26), corresponding to the case \( v = dy^2 \), we make the following identifications to equate this action with the reduced 5-brane action. Writing IIB fields on the left and M-fields after scaling and converting to orthonormal frame, see (23,24), on the right:

\[
\begin{align*}
{b(4)} &= C_{(4)} \\
{b(3)} &= C_{(3)} \\
{b(2)} &= b \\
{b(2)_2} &= C_{(2)} \\
C^1 &= C^R \\
{b(1)} &= C' \\
J &= H \\
F &= F
\end{align*}
\]

(49)

To make these identifications which are very natural we have set \( C^2 = 0 \) on the 5-brane side, this significantly simplifies the 5-brane action.

For the case \( v = dy^1 \) we compare with the S dual action (38) after reduction and set \( C^1 = 0 \) on the 5-brane side to make the corresponding simplification required in order to dualize the scalar field. See equations (25), (31) and (38). The identifications required to equate this action are the same as above with \( C^2 = b^R \). This is a requirement of consistency.

We now wish to consider cases where the duality transformation of the scalar field differs from above because of the interaction term with the \( b^R \) field (or \( C^R \) in the S-dual case) inside the determinant. Using the technique described above, once we know how the the Dirac Born Infeld part in the brane action transforms under duality we can recover how the full brane action including the Wess-Zumino terms transforms. Hence in what follows we drop the Wess-Zumino terms as the duality transformation to include them follows immediately. (This is essentially because adding terms that are linear in dualizing field does not change the form of the dual action.)

First, we consider the approximation whereby the Born-Infeld term is replaced with a Yang-Mills term. This gives, keeping only the scalar corresponding to compact direction:

\[
S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\phi\partial_\mu\phi\partial'^\mu\phi - \frac{1}{2}F_{\mu\nu}b^R_{(\nu}b_{\mu)\phi} - \frac{1}{4}b^R_{[\nu}\partial'_{\mu]}\phi b^R_{\rho\sigma]}\phi - \frac{1}{4}b^R_{[\nu}\partial_{\mu]}\phi b^R_{\rho\sigma]}\phi
\]

(50)
We now dualize \( \phi \) following the same procedure as before to obtain the following dual action:

\[
S_D = \frac{1}{(1 + (b^R)^2)} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} H_\mu H^\mu - \frac{1}{2} b^R_{\mu} F^{\mu \nu} b^R_{\nu} - \frac{1}{2} (H_\mu b^R_{\nu})^2 - H_\mu F^{\mu \nu} b^R_{\nu} \right) \tag{51}
\]

Should we make the same approximation to the 5-brane action, i.e. replacing the first term by a field strength squared term, we find that we recover directly the above action. Note the peculiar factor \( \frac{1}{(1 + (b^R)^2)} \) in front of the action which comes in the 5-brane case from the \( \frac{1}{v^2} \) factor is a result of dualizing the scalar field in the D3 brane. The final term in the action is identified with \( I^{(2)}_{PST} \).

Constructing the dual action directly for the full DBI action (37) is difficult as discussed above. However, with the rather specific case of vanishing \( F \) we can construct the dual theory exactly.

And so for the reduced D3 brane, writing out the determinant exactly we have:

\[
S_1 = -\int \sqrt{1 + \partial_\mu \phi \partial^\mu \phi - (b^R_\mu \partial^\mu \phi)^2 + (b^R_\mu \partial^\mu \phi)^2} \tag{52}
\]

Adding the usual constraint term and and integrating out \( \phi \) we have the following equations of motion for \( l_\mu \):

\[
*H_\mu = \frac{l_\mu (1 - b^R_\mu (b \cdot l) + (b^R)^2)}{\sqrt{1 + l^2 - (b^R \cdot l)^2 + (b^R)^2 l^2}} \tag{53}
\]

which we can invert to give an expression for \( l_\mu \):

\[
l_\mu = \frac{(*H_\mu + (*H \cdot b^R) b^R_\mu)}{\sqrt{(1 + (b^R)^2)(1 + (b^R)^2 - H^2 - (b^R \cdot *H)^2)}} \tag{54}
\]

Inserting this in the action (52) provides the dual:

\[
S_D = -\int \sqrt{1 + (b^R)^2 - *H^2 - (*H \cdot b^R)^2} \tag{55}
\]

which we may write as follows:

\[
S = -\int Q \sqrt{\det \left( G_{\mu \nu} - \frac{*H_\mu *H_\nu}{1 + (b^R)^2 - (b^R \cdot *H)^2} \right)} \tag{56}
\]
where
\[ Q = \sqrt{\frac{1 + (bR)^2 - (bR \cdot *H)^2}{1 + (bR)^2}} } \]

This is identical to the reduced 5-brane action with \( F \) set to zero, see equation (17), once we make the following identifications:
\[ P_\mu = * H_\mu \quad C_{2\mu} = b_\mu^R \]

This again is consistent with (49).

**Conclusions**

We have shown that the action (1) for the M theory five brane, under double dimensional reduction on a torus produces the self-dual three brane of IIB directly reduced on a circle. The S-duality of IIB becomes transparent as the modular symmetry of the torus. The different gauge choices for \( v \in H^1(T^2) \) correspond to different S-dual formulations of the 3-brane. The identification of the moduli and the fields of the two theories has been shown to be in agreement with work considering the ambient supergravity [11] and the identification of the string with the partially wrapped membrane [12]. In order to make this identification it was necessary to dualize the scalar corresponding to fluctuations in the compact direction. This duality transformation acts non-trivially on the action. In fact, in the most general case the dual action is extremely difficult to construct explicitly; even proving the equivalence with the reduced 5-brane which ought to be an algebraic exercise proves to be difficult due to the complexity of the duality transformation. However, by making approximations to the Born-Infeld part or by truncating fields we explicitly construct dual actions to the reduced three brane in these cases. It should be noted that the results are essentially classical with a very specific choice of world volume topology for the 5-brane, hence we do not encounter the problems reported in [6].

Recently, there has been an attempt to rewrite the 5-brane action with an auxiliary metric as one does for the string so as to make the action linear [17]. This essentially shifts the complexity of the action into the equations of motion for the auxiliary metric. Again the duality transformation becomes difficult to implement exactly.
One of the aspects not explored explicitly in this paper is the role in which the five brane may have in a reformulation of the three brane in which the S-duality of IIB is manifest, as reported in the recent work [18]. In [10], by taking $v$ to be a one form in $M^4$ instead of $T^2$ an action was produced that has the S-duality manifest [16,19]. The disadvantage with this approach is that the Lorentz invariance is then not manifest. It is not clear if a connection can be made between these two approaches. It would be interesting if one could give some physical interpretation to the auxiliary field $v$ which plays a crucial role in encoding the self-duality condition in the action. We remark that other relevant work regarding the five brane in an action formulation and its relationship to duality is given in [20,21,22].

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