ANALYTIC EQUIVALENCE OF GEOMETRIC TRANSITIONS

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Abstract. In this paper analytic equivalence of geometric transition is defined in such a way that equivalence classes of geometric transitions turn out to be the arrows of the Calabi–Yau web. Then it seems natural and useful, both from the mathematical and physical point of view, look for privileged arrows’ representatives, called canonical models, laying the foundations of an analytic classification of geometric transitions. At this purpose a numerical invariant, called bi–degree, summarizing the topological, geometric and physical changing properties of a geometric transition, is defined for a large class of geometric transitions.

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A geometric transition (g.t.) between two Calabi–Yau threefolds is the process obtained by “composing” a birational contraction to a normal threefold with a complex smoothing. If the normal intermediate threefold has only nodal singularities then the considered g.t. is called a conifold transition (see Definitions 1.15 and 1.17). Then a g.t. is a machinery connecting each other topologically distinct Calabi–Yau threefolds. This feature is probably the main source of interest in the study of g.t.’s both in mathematics and physics.

From the mathematical, actually algebraic geometric, point of view the story goes back to deep speculations due to H. Clemens [14], R. Friedman [24], F. Hirzebruch [39], J. Werner [79] and M. Reid [65]. In fact the huge multitude of known topologically distinct Calabi–Yau threefolds makes any concept of moduli space for them, both varying their complex and symplectic structures, immediately wildly reducible. This unpleasant fact dramatically clashes with well known irreducibility of moduli spaces of both elliptic curves and K3 surfaces, which are the lower dimensional analogue of Calabi–Yau threefolds. Actually Reid (op.cit.) underlines that, at the beginning (in the forties), the moduli space of (algebraic) K3 surfaces seemed to F. Enriques to be a 19–dimensional variety admitting a countable number of irreducible components $\mathcal{M}_g$, one for each value of the sectional genus $g \geq 3$ [22]. Twenty years later K. Kodaira [44] was able to recover a 20–dimensional irreducible moduli space $\mathcal{M}$ for K3 surfaces by leaving the algebraic category to work in the larger category of analytic, compact, Kähler varieties, and discovering that the generic K3 surface is an analytic non–algebraic complex variety: in particular the moduli space $\mathcal{M}^{alg} = \bigcup_g \mathcal{M}_g$ of algebraic K3’s embeds in $\mathcal{M}$ as a dense closed subset. The so called Reid’s fantasy suggests that an analogous situation could happen for Calabi–Yau threefolds where birational transformations and small g.t.’s (see Definition 1.23) could be the right instruments to reduce the parameterization of birational classes of Calabi–Yau threefolds to an irreducible moduli space of complex structures over suitable connected sums of complex hypertori.

In physics Calabi–Yau threefolds make their appearance, in the eighties [11], as the space spanned by the so called internal degrees of freedom of a certain $(1 + 1)$–dimensional world–sheet field theory describing superstrings’ propagation in $(3+1)$–dimensional Minkowski space–time. The main observables of the superstring model are then typically determined by this internal structure (the Calabi–Yau vacuum). Therefore, in spite of the almost uniqueness, via dualities (and, later, $\mathcal{M}$–theory), of consistent superstring theories, the superstring model remains undetermined due to the unavoidable uncertainty on the topology of the Calabi–Yau vacuum: this is the so called vacuum degeneracy problem. A solution to this problem was firstly proposed by P.S. Green and T. Hübsch [30], [31], who conjectured, motivated by the contemporary Reid’s fantasy, that topologically distinct Calabi–Yau vacua could be connected each other by means of conifold transitions which should induce a phase transition between corresponding superstring models. This latter fact, which is the physical counterpart of the mathematical process given by a conifold transition, was actually understood by A. Strominger as a condensation of massive black holes to massless ones [74].
The synthesis between Reid’s fantasy and Green–Hübsch vacuum degeneracy was definitely formalized ten years later by M. Gross [34] with the so called Calabi–Yau web connectedness conjecture, avoiding the more difficult (although suggestive) part of the Reid’s fantasy, by employing the more concrete physicists’ concept of Calabi–Yau web, to insist in staying in the projective category. Precisely this conjecture states that Calabi–Yau threefolds can be arranged in a giant connected web whose nodes are deformation classes (i.e. analytic flat families) of Calabi–Yau threefolds and connections between distinct nodes (so called arrows) are induced by g.t.’s. The parameterizations of deformation classes of Calabi–Yau threefolds is then reduced to consider all the nodes with no outgoing arrows: these are precisely deformation classes of Calabi–Yau threefolds, called primitive, with no birational contractions to normal projective threefolds, to stay in the projective category (e.g. whose Picard number is 1). In [13] a significative evidence for this conjecture is presented, announcing, by a computer check, that all the 7555 deformation classes of hypersurfaces in 4–dimensional weighted projective spaces can be connected each other by means of g.t.’s. An important point here is that the g.t.’s there employed can be very general and do not simply reduce to the class of conifold transitions, as often conjectured by physicists, or small g.t.’s, as those considered by Reid in connecting birational classes of Calabi–Yau threefolds with complex structures on suitable connected sums of hypertori.

A first natural question is then the following:

**Problem (I).** Assume that the connectedness conjecture is true. Is the Calabi–Yau web connectable only by means of

(a) conifold transitions?
(b) small geometric transitions?

A positive answer to the first question (I.a) would have a deep meaning both in mathematics and in physics.

In mathematics because of the Clemens interpretation of a local conifold transition as a *standard topological surgery* from $\mathbb{R}^4 \times S^2$ to $S^3 \times \mathbb{R}^3$ obtained by removing a tubular neighborhood of the *exceptional* $\mathbb{P}^1 \cong S^2$ and pasting in a tubular neighborhood of the *vanishing cycle* $S^3$ ([14] Lemma 1.11). Then a positive answer to (I.a) would give a simple local description of the changing in topology induced by a g.t. as a surgery replacing a number of exceptional $S^2$’s with vanishing $S^3$’s: the global changing in topology would then consequently described by relations on exceptional $S^2$’s and those on vanishing $S^3$’s.

In physics because of conifold transitions are among the few g.t.’s whose induced “phase transition” on the superstring model has been physically understood, as already underlined above (see e.g. [3], [12], [9] for further improvements).

Otherwise if (I.a) would not be true, a positive answer to (I.b) would considerably reduce the spectrum of singularities, exceptional fibres and vanishing cycles one have to carefully examine to obtain general information about changing in topology, geometry and physics induced by a generic g.t..

Unfortunately we were not able to resolve Problem (I) in such a general context but, by assuming a stronger version of the connectedness conjecture, which we call *direct connectedness conjecture of the Calabi–Yau web* (see subsection 5.1), we are able to state and prove the following
Theorem 5.11 + Example 5.12. There exist nodes of the Calabi–Yau web which cannot be directly connected by means of small g.t. ’s.

This fact gives a negative answer to Problem (I) under the stronger hypothesis of direct connectedness of the Calabi–Yau web. Anyway this negative result, being intrinsically related with the rigidity property of the exceptional divisor of a type II g.t. and then with the nature of the arising singularities on the normal intermediate threefold of this transition, stakes a negative claim on the possible solution of Problem (I).

By the way any attempt of giving an answer to Problem (I) has to deal with the following, one another deeply related, questions:

Questions. \hspace{1cm} (1) \textit{What is actually an arrow of the Calabi–Yau web?}

\hspace{1cm} (2) \textit{How many arrows may connect two distinct nodes?}

\hspace{1cm} (3) \textit{What is the relation between g.t.’s connecting smooth fibres of the same two distinct nodes?}

Answering the latter question opens the way to solve the former two questions. Since Calabi–Yau fibres of a flat analytic family, being smooth, have to be diffeomorphic, two g.t.’s $T_1, T_2$ connecting the same nodes $\mathcal{M}_1, \mathcal{M}_2$ necessarily induce \textit{the same change in topology} and they can simply obtained each other by varying the complex structures of the involved threefolds. It makes then sense to say that they are \textit{analytic equivalent} ($T_1 \sim T_2$), giving rise to the main concept of the present paper (see Definition 2.1 and equivalent versions given with Theorem 2.4, Corollary 2.5 and Remark 2.6). In particular it turns out that:

Theorem 2.4 $\Rightarrow$ Corollary 2.5. Two g.t.’s are analytically equivalent if and only if there exists a morphism deformation linking the associated birational contractions.

Then an arrow becomes naturally an \textit{equivalence class of g.t.’s} and it turns out that if there exists an arrow connecting $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e. if they are \textit{directly connected} (see Definition 5.10), then it is \textit{unique} (see Proposition 2.3).

It arises then a very natural question:

Problem (II). \textit{If an arrow is an analytic equivalence class of g.t.’s, does it admit a privileged representative g.t., precisely what is called a canonical model?}

The hope is that every arrow could be represented by a conifold transition meaning that a canonical model of a g.t. is always a conifold transition. This would also give a positive answer to Problem (I.a). But unfortunately this is not the case. In fact:

Proposition 4.2. A type II g.t. turns out to be never equivalent to a conifold transition.

More deeply, it is possible to exhibit small g.t.’s which are not equivalent to a conifold transition (see the Namikawa example 4.9 and considerations in Remark 5.10). Then the determination of possible canonical models for g.t.’s turns out to be less obvious than what one could a priori expect.

The present paper, after the definition and a first comprehension of main properties of analytic equivalence between g.t.’s, which is our “looking glass” for studying g.t.’s and their properties, concentrates on the \textit{research of canonical models}. 
Let us first of all underline that determining canonical models reveals of significative interest both in mathematics and in physics, since a canonical model resumes all the main features of g.t.’s representing the same arrow of the Calabi–Yau web. This means that the study of changing in topology, geometry and physics produced by a general g.t. can be reduced to consider a (finite?) number of canonical models.

The main feature of a canonical model is its rigidity (see Definition 4.12) which means that it is a g.t. with minimal singularities in the exceptional locus and in the intermediate normal threefold. We obtain the following:

Results.

- Conifold transitions are the canonical models of any small g.t. respecting the cohomological conditions of Theorem 4.20, in particular this is the case of the general small g.t. whose normal intermediate threefolds admit only Arnol’d’s simple singularities (Corollary 4.23).
- (Theorem 3.1) The canonical model of a type II g.t. (which is never of the previous kind) is obtained by smoothing the exceptional locus of the associated birational contraction; then we get exactly 8 kinds of canonical models accordingly with the possible degree of smooth Del Pezzo surface giving the exceptional locus.

Moreover it seems conceivable that:

- (Conjecture 5.7) a type I g.t. admits a type I conifold transitions as a canonical model,
- (Conjectures 5.8 and 5.9) the canonical model of a small rigid g.t. is the “composition” of a conifold transition with, in case, some further small g.t.’s,
- (§ 6.1(2)) the canonical model of a type III g.t. admitting as exceptional locus of the associated birational contraction a conic bundle over a smooth curve of genus $\geq 2$ is a type I conifold transitions.

Among primitive g.t.’s it remains to consider type III g.t.’s admitting as exceptional locus a conic bundle over a smooth curve of genus $\leq 1$. While the case $g = 0$ can probably settled as that of type II g.t.’s, since the exceptional conic bundle turns out to be the blow up of a rational scroll in a finite number of points, for what concerns the case $g = 1$ I have no idea of candidates for a canonical model.

For what concerns non–primitive g.t.’s, in the present paper we considered only small g.t.’s. More general g.t.’s may have so wild (canonical) singularities and very intricate exceptional locus that is inconceivable to think of directly finding any sort of canonical model. The hope is that of reducing, up to analytic equivalence, to a suitable composition of geometric transitions whose canonical model is known, as e.g. in Proposition 5.4. But to settle the general case there is still a long way to go.

In classifying canonical models and related arrows it has been useful to introduce numerical invariants, under analytic equivalence, giving an account of the change in topology, geometry and physics produced by the considered equivalence class of g.t.’s. At this purpose we introduced a degree for primitive g.t.’s and a bi–degree for more general g.t.’s; in particular the first entry of the bi–degree turns out to be the relative Picard number of the associated birational contraction in such a way that a bi–degree reduces to its second entry (and then to a degree) for primitive g.t.’s. The second entry of bi–degree of a g.t. (and then degree of primitive ones) summarizes
its changing properties: e.g. for a small g.t. it gives precisely \textit{the produced increasing on complex moduli} (see Definition 4.4). Observe that in this case the first entry of the bi–degree gives precisely \textit{the produced decreasing of Kähler moduli} (which is just the relative Picard number of the associated birational contraction). Moreover these numerical values are of significative interest also in physics to understand the size of the induced “phase transition” on the superstring model. In fact, in the physical Strominger’s explanation of a conifold transition they give precisely the decreasing of vector multiplets and the increasing of hypermultiplets related with the induced black hole condensation.

In defining the degree of a type III g.t. we tried to be coherent with the just defined degree of type I g.t.’s, by calibrating the definition on type III g.t.’s admitting a singular curve of genus $\geq 2$ (see 6.1(2) and Definition 6.3). On the contrary for type II we preferred to relate the g.t.’s degree with the degree of the del Pezzo exceptional divisor since the relation with the produced complex moduli increasing is not very evident (see Remark 3.5). Probably all these definitions of degree and bi–degree might need to be \textit{renormalized} when a clearer global context for g.t.’s would be known. But let us say that this is just no more than a detail.

Let us finally recall, as a conclusive remark, that more recently g.t.’s has been newly in the spotlight due to the R. Gopakumar and C. Vafa work [27] where the \textit{local} conifold transition is proposed as a geometrical setup for a duality between SU($N$) Chern–Simons gauge theories on $S^3$ and type II–A closed string theories “compactified” on the resolution $\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$, giving evidences of a long aged t’Hooft conjecture of equivalence between large $N$ limit of gauge theories and some kind of closed string theory. After E. Witten’s observation that a SU($N$) (or $U(N)$) Chern–Simons gauge theory is equivalent to an open type II–A string theory on the cotangent bundle $T^*S^3$ [52], the local conifold transition reveals then as a geometrical setup for a duality between open/closed A–model string theories (the interested reader may also consult the extensive survey on the argument [28]). After the pioneering work of Gopakumar and Vafa many further local g.t.’s have been studied as a geometric setup of open/closed A–model string duality (see e.g. [20] among many others). More recently the same has been done for B–model closed/open string dualities by applying Mirror Symmetry and looking at the so obtained \textit{reverse} geometric transition [1], [23], [17], [18], [19]. Our hope is that the \textit{analytic classification} of g.t.’s we are sketching in this paper could be of some utility also for the develop of this kind of open/closed string dualities.

This paper is organized as follows.

Section I is devoted to an extensive presentation of preliminaries with the purpose of making the present work as much self–contained as possible. Fundamental results and techniques for the following are the Douady–Grauert–Palamodov Theorem 1.4 guaranteeing the existence of a versal deformation space for a compact complex space, Z. Ran’s morphism deformation technique 1.4 and Theorem 1.26 giving a generalization to small g.t.’s of Clemens formulas for conifold transitions. We do not report here the proof of this latter result, remanding the interested reader to the preprint [65]: anyway it is completely topological and directly generalizing the proof of Clemens formulas given in [67] Theorem 3.3. Anyway for most applications of this result Remark (3.8) in [54] suffices.
Motivated by the Calabi–Yau web concept of an arrow between deformation families of Calabi–Yau threefolds, the central notion of analytic equivalence of g.t.’s is introduced in Section 2 and Theorem 2.4 explains its relation with the deformation of associated birational contractions.

The research of canonical models of g.t.’s starts from Section 3. In particular this section is dedicated to define canonical models for type II g.t.’s and consequently their degree.

The extensive Section 4 is devoted to understand what kind of g.t.’s may admit a conifold transition as a canonical model. After the immediate exclusion of type II g.t.’s the spotlight turns on small g.t.’s. But also in this case it is possible to exhibit examples of small g.t.’s not equivalent to a conifold transition: Example 4.9 is obtained by drawing on an example due to Y. Namikawa [53] and further developed in [69] by employing R. Friedman’s techniques [24]. The main results of this section are Theorem 4.19 with which a characterization for weighted homogeneous conifold transitions is given, and Theorem 4.20 giving cohomological conditions, called local and global non–rigidity, for a g.t. to be equivalent with a conifold transition. In particular these conditions turn out to be satisfied by small g.t.’s admitting only Arnol’d’s simple singularities: the section ends up with a local analysis on Kuranishi spaces of the latter singularities.

Motivated by the fact that the counterexample 4.9 is given by a small rigid non–primitive g.t., in Section 5 the Calabi–Yau web structure is taken up to understand the meaning of an arrows’ composition and then of composing g.t.’s. It turns out that the g.t. of Example 4.9 is a composition of conifold transition opening the way to suppose that this could be the structure of canonical models of small g.t.’s non equivalent to a conifold transition. The section ends up with a series of conjectures.

The last Section 6 is then dedicated to list some known result and conjecture on type III g.t.’s essentially developed in [80], [42] and [12]. We remand a deeper treatment of type III g.t.’s and their canonical models to a forthcoming paper.

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1. Preliminaries and notation

Definition 1.1 (Calabi–Yau 3–folds). A smooth, complex, projective 3–fold \( Y \) is called Calabi–Yau if

1. \( K_Y \cong \mathcal{O}_Y \),
2. \( h^{1,0}(Y) = h^{2,0}(Y) = 0 \).
Remark 1.2. There are a lot of more or less equivalent definitions of Calabi–Yau 3–folds e.g.: a Kähler complex, compact 3–fold admitting either (1) a Ricci flat metric (Calabi conjecture and Yau theorem), or (2) a flat, non–degenerate, holomorphic 3–form, or (3) holonomy group a subgroup of SU(3) (see [40] for a complete description of equivalences and implications).

In particular the given definition is equivalent to ask for the existence of a Kähler metric $h$ on $Y$ whose holonomy group is exactly SU(3).

In the algebraic context, the given definition gives the 3–dimensional analogue of smooth elliptic curves and smooth $K3$ surfaces.

Examples 1.3. (1) Smooth hypersurfaces of degree 5 in $\mathbb{P}^4$,

(2) the general element of the anti–canonical system of a sufficiently good 4–dimensional toric Fano variety (see [6]),

(3) suitable complete intersections.... (iterate the previous examples),

(4) the double covering of $\mathbb{P}^3$ ramified along a smooth surface of degree 8 in $\mathbb{P}^3$ (octic double solid).

1.1. The Picard number of a Calabi–Yau threefold. Let $Y$ be a Calabi–Yau threefold and consider the Picard group

$$\text{Pic}(Y) := \langle \text{Invertible Sheaves on } Y \rangle \cong H^1(Y, \mathcal{O}_Y^*) .$$

The Calabi–Yau conditions $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$, applied to the cohomology of the exponential sequence, give then the canonical isomorphism

$$\text{Pic}(Y) \cong H^2(Y, \mathbb{Z}) .$$

Therefore the Picard number of $Y$, which is defined as $\rho(Y) := \text{rk} \text{Pic}(Y)$, turns out to be

$$(1) \quad \rho(Y) = b_2(Y) = b_4(Y) = h^{1,1}(Y) .$$

Since the interior $K(Y)$ of the closed Kähler cone $\overline{K}(Y)$ of $Y$, generated in the Kleiman space $H^2(Y, \mathbb{R})$ by the classes of nef divisors, turns out to be the cone generated by the Kähler classes, the Picard number $\rho(Y) = h^{1,1}(Y)$ turns out to be the dimension of the Kähler moduli space of $Y$.

1.2. Deformations of Calabi–Yau threefolds. Let $X \to B$ be a flat, surjective map of complex spaces such that $B$ is connected and there exists a special point $o \in B$ whose fibre $X = f^{-1}(o)$ may be singular. Then $X$ is called a deformation family of $X$. If the fibre $X_b = f^{-1}(b)$ is smooth, for some $b \in B$, then $X_b$ is called a smoothing of $X$.

Let $\Omega_X$ be the sheaf of holomorphic differential forms on $X$ and consider the Lichtenbaum–Schlessinger cotangent sheaves [47] of $X$, $\Theta^i_X = \mathcal{E}xt^i(\Omega_X, \mathcal{O}_X)$. Then $\Theta^1_X = \text{Hom}(\Omega_X, \mathcal{O}_X) =: \Theta_X$ is the “tangent” sheaf of $X$ and $\Theta^i_X$ is supported over $\text{Sing}(X)$, for any $i > 0$. Consider the associated local and global deformation objects

$$T^i_X := H^0(X, \Theta^i_X) \, , \quad T^i_X := \mathcal{E}xt^i(\Omega^1_X, \mathcal{O}_X) \, , \quad i = 0, 1, 2 .$$

Then by the local to global spectral sequence relating the global Ext and sheaf $\mathcal{E}xt$ (see [85] and [26] II, 7.3.3) we get

$$E_2^{p,q} = H^p(X, \Theta^q_X) \Longrightarrow T^{p+q}_X$$
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Given that

\[ \mathbb{T}_X^0 \cong T_X^0 \cong H^0(X, \Theta_X), \]

(2)

if \( X \) is smooth then \( \mathbb{T}_X^i \cong H^i(X, \Theta_X) \),

(3)

if \( X \) is Stein then \( \mathbb{T}_X^i \cong T_X^i \).

(4)

Given a deformation family \( X_f \to B \) of \( X \) for each point \( b \in B \) there is a well defined linear (and functorial) map

\[ D_b f : T_b B \to \mathbb{T}_{X_b}^1 \] (Generalized Kodaira–Spencer map)

(see e.g. [56] Theorem 5.1). Recall that \( X_f \to B \) is called

- a versal deformation family of \( X \) if for any deformation family \( (Y, X) \to (C, 0) \) of \( X \) there exists a map of pointed complex spaces \( h : (U, 0) \to (B, o) \), defined on a neighborhood \( 0 \in U \subset C \), such that \( Y|_U \) is the pull–back of \( X \) by \( h \) i.e.

\[ Y|_U = U \times_B X \]

(versal deformation family)

- an effective versal deformation family of \( X \) if it is versal and the generalized Kodaira–Spencer map evaluated at \( o \in B \), \( D_o f : T_o B \to \mathbb{T}_{X}^1 \) is injective,

- a universal family if it is versal and \( h \) is defined on every point of \( C \) and is uniquely determined in a neighborhood \( 0 \in U \subset C \).

**Theorem 1.4** (Douady–Grauert–Palamodov [21], [29], [55] and [56] Theorems 5.4 and 5.6). Every compact complex space \( X \) has an effective versal deformation \( X_f \to B \) which is a proper map and a versal deformation of each of its fibers. Moreover the germ of analytic space \( (B, o) \) is isomorphic to the germ of analytic space \( (q^{-1}(0), 0) \), where \( q : T_X^1 \to T_X^2 \) is a suitable holomorphic map (the obstruction map) such that \( q(0) = 0 \).

In particular if \( q \equiv 0 \) (e.g. when \( T_X^2 = 0 \)) then \( (B, o) \) turns out to be isomorphic to the germ of a neighborhood of the origin in \( T_X^1 \).

**Definition 1.5** (Kuranishi space and number). The germ of analytic space

\[ \text{Def}(X) := (B, o) \]

as defined in Theorem 1.4 is called the *Kuranishi space of \( X \).* The *Kuranishi number* \( \text{def}(X) \) of \( X \) is then the maximum dimension of irreducible components of \( \text{Def}(X) \).

\( \text{Def}(X) \) is said to be unobstructed or smooth if the obstruction map \( q \) is the constant map \( q \equiv 0 \). In this case \( \text{def}(X) = \dim \mathbb{T}_X^1 \).

**Theorem 1.6** ([56] Theorem 5.5). If \( T_X^0 = 0 \) then the versal effective deformation of \( X \), given by Theorem 1.4, is actually universal for all the fibres close enough to \( X \).
Let us now consider the case of a Calabi–Yau threefold $Y$. By the Bogomolov–Tian–Todorov–Ran (in the following BTTR) Theorem, the Kuranishi space $\text{Def}(Y)$ is smooth and \cite{[3]} gives that

$$\text{def}(Y) = \dim_{\mathbb{C}} T^1_Y = h^1(Y, \Theta_Y) = h^{2,1}(Y)$$

where the last equality on the right is obtained by the Calabi–Yau condition $K_Y \cong \mathcal{O}_Y$. Applying the Calabi–Yau condition once again gives $h^0(\Theta_Y) = h^{2,0}(Y) = 0$. Therefore $\text{def}(Y)$ gives that $h^{2,1}(Y) =$\cite{[2]} and Theorem \cite{[1.6]} give the existence of a universal effective family of Calabi–Yau deformations of $Y$. In particular $h^{2,1}(Y)$ turns out to be the dimension of the complex moduli space of $Y$.

1.3. Analytic equivalence of Calabi–Yau varieties. As a consequence of the Douady–Grauert–Palamodov (in the following DGP) Theorem \cite{[1.4]} and of Theorem \cite{[1.6]} we can set the following equivalence relation between Calabi–Yau varieties.

**Definition 1.7** (Analytic equivalence of Calabi–Yau varieties). Two Calabi–Yau varieties $Y_i$, $i = 1, 2$, will be called analytically equivalent (or simply equivalent), write $Y_1 \sim Y_2$, if there exists a deformation family $Y \to B$ and points $b_i \in B$, $i = 1, 2$ such that $Y_i = f^{-1}(b_i)$.

Reflexive and symmetric properties of the previous equivalence between complex spaces are clear. The transitive property is a direct consequence of the DGP Theorem \cite{[1.4]} and Theorem \cite{[1.6]}. In fact if $Y_1 \sim Y_2$ and $Y_2 \sim Y_3$ then there exist deformation families $X \to B$, $Y \to C$ and points $b_1, b_2, c_2, c_3 \in C$ such that $f^{-1}(b_1) = Y_1$, $f^{-1}(b_2) = Y_2 = g^{-1}(c_2)$, $Y_3 = g^{-1}(c_3)$.

Look at the universal deformation family of $Y_2$ whose existence is guaranteed by Theorems \cite{[1.4]} and \cite{[1.6]} it is a flat, analytic family with special fibre

$$Y_2 \leftarrow \mathcal{V} \quad \begin{array}{ccc} & v \downarrow \bigcirc \in V \\ & \downarrow \end{array}$$

such that there exist morphisms $\beta : B \to V$ and $\gamma : C \to V$ giving $X$ and $Y$ as pull–back families of $\mathcal{V}$ i.e.

$$X = B \times_{\beta} \mathcal{V} \quad \begin{array}{ccc} & v \downarrow \bigcirc \in V \\ & \downarrow \end{array} \quad C \times_{\gamma} \mathcal{V} = \mathcal{V}$$

Then $Y_1 = v^{-1}(\beta(b_1))$ and $Y_3 = v^{-1}(\gamma(c_3))$ turns out to be connected by the deformation family $\mathcal{V}$, giving their equivalence $Y_1 \sim Y_2$.

**Remark 1.8.** Notice that the just defined equivalence relation extends also to singular fibers of deformation families of Calabi–Yau varieties since both the DGP Theorem \cite{[1.4]} and Theorem \cite{[1.6]} do not need smoothness of the central fibre $X$.

---

\footnote{The more general statement and proof of this theorem, actually holding for cohomologically Kähler variety, is in \cite{[76]}. Special cases were previously proven in \cite{[10]}. The proof given in \cite{[77]} uses the existence of a Ricci flat Kähler metric on $Y$. Finally an algebraic proof is given in \cite{[61]}.}
Moreover it may be extended to compact complex spaces $X$ whose DGP versal family is actually universal e.g. when $T^0_X = 0$.

**Remark 1.9.** Recall that a Calabi–Yau threefold $Y$ is projective, as required by Definition 1.1. In particular, by the cohomology of the exponential sequence, there is an ample line bundle $L$ on $Y$ whose first Chern class $c_1(L)$ has to be a positive class in $H^2(Y, \mathbb{Z}) \cap H^{1,1}(Y)$. Then, for any deformation family $\mathcal{Y} \to B$ of $Y$, the choice of $L$ induces a line bundle $\mathcal{L}$ on $\mathcal{Y}$ which is flat over $B$ and such that $\mathcal{L}|_{\mathcal{Y}} = L$. In particular every fibre $Y_s$ of $\mathcal{Y}$, even singular, is still projective since $c_1(\mathcal{L}|_{Y_s})$ is still an integer, positive class.

As a consequence the analytic equivalence relation between Calabi–Yau threefolds, even singular for what observed in the previous Remark 1.8, is an equivalence relation internal to the projective category.

1.4. Deformation of a morphism.

**Definition 1.10** (Deformation of a morphism, [59] Definition 1.1). Let $\phi : Y \to Y'$ be a morphism and $(S, s_0)$ a pointed analytic space. A deformation of $\phi$ parameterized by $(S, s_0)$ is a commutative diagram

\[ \begin{array}{ccc}
Y' & \rightarrow & X' \\
\phi \downarrow & & \Phi \downarrow \\
Y & \rightarrow & X \\
\phi \downarrow & & \Phi \downarrow \\
S & \ni & s_0 \\
\end{array} \]

where

- $\Phi$ and $\Phi'$ are flat,
- $Y' = \Phi^{-1}(s_0)$, $Y = \Phi^{-1}(s_0)$ and $\phi = \Phi|_{\Phi^{-1}(s_0)}$.

Given a morphism $f : X \to Y$ let us denote by $\text{Def}(X, f, Y)$ a germ of analytic space representing the deformation functor of the morphism $f$, according with the Definition 1.10. Then let

- $T^1_f$ be the tangent space to $\text{Def}(X, f, Y)$ parameterizing 1st-order deformations of $f$
- $T^2_f$ be the associated obstructions’ space

(see [59] Proposition 3.1). Let $\text{Ext}^i_f(\Omega^1_Y, \mathcal{O}_X)$ be the right derived functors of

\[ \text{Hom}_f(\Omega^1_Y, \mathcal{O}_X) := \text{Hom}_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, f_*\mathcal{O}_X) \]
whose elements are called \( f \)-linear homomorphisms. Then there is an exact sequence (59 (2.2))

\[
\begin{array}{cccccccc}
\text{Hom}(\Omega^1_X, \mathcal{O}_X) & \oplus & \text{Hom}(\Omega^1_Y, \mathcal{O}_Y) & \longrightarrow & \text{Hom}_f(\Omega^1_Y, \mathcal{O}_X) \\
T^1_f & \alpha_1 & T^1_X & \oplus & T^1_Y & \alpha_2 & \longrightarrow & \text{Ext}^1_f(\Omega^1_Y, \mathcal{O}_X) \\
T^2_f & \beta_1 & T^2_X & \oplus & T^2_Y & \beta_2 & \longrightarrow & \text{Ext}^2_f(\Omega^1_Y, \mathcal{O}_X)
\end{array}
\]

and a spectral sequence (59 (6))

\[
E_2^{p,q} = \text{Ext}^p(\Omega^1_Y, R^q f_* \mathcal{O}_X) \longrightarrow \text{Ext}^q_f(\Omega^1_Y, \mathcal{O}_X).
\]

If \( f_* \mathcal{O}_X \cong \mathcal{O}_Y \) then the latter spectral sequence gives rise to the following exact sequence

\[
0 \longrightarrow T^1_Y \longrightarrow \text{Ext}^1_f(\Omega^1_Y, \mathcal{O}_X) \longrightarrow \text{Hom}_f(\Omega^1_Y, R^1 f_* \mathcal{O}_Y) \longrightarrow T^2_Y \longrightarrow \text{Ext}^2_f(\Omega^1_Y, \mathcal{O}_X)
\]

1.5. Milnor and Tyurina numbers of isolated hypersurface singularities.

Let \( \mathcal{O}_0 \) be the local ring of germs of holomorphic function of \( \mathbb{C}^{n+1} \) at the origin, which is the localization of the polynomial ring \( \mathcal{O} := \mathbb{C}[x_1, \ldots, x_{n+1}] \) at the maximal ideal \( m_0 := (x_1, \ldots, x_{n+1}) \). By definition of holomorphic function and the identity principle we have that \( \mathcal{O}_0 \) is isomorphic to the ring of convergent power series \( \mathbb{C}\{x_1, \ldots, x_{n+1}\} \). A germ of hypersurface singularity is defined as the Stein complex space

\( U_0 := \text{Spec}(\mathcal{O}_{F,0}) \)

where \( \mathcal{O}_{F,0} := \mathcal{O}_0/(F) \) and \( F \) is the germ represented by a polynomial function.

**Definition 1.11.** The **Milnor number** of the hypersurface singularity \( 0 \in U_0 \) is defined as the multiplicity of \( 0 \) as solution of the system of partials of \( F \) (18 §7) which is

\[
\mu(0) := \dim_{\mathbb{C}}(\mathcal{O}_0/J_F) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/J_F)
\]

as a \( \mathbb{C} \)-vector space, where \( J_F \) is the jacobian ideal \( \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n+1}} \right) \).

The **Tyurina number** of the hypersurface singularity \( 0 \in U_0 \) is defined as the **Kuranishi number** \( \text{def}(U_0) \). Since \( U_0 \) is Stein, \( \text{Def}(U_0) \) is smooth and, by [1],

\[
\tau(0) := \text{def}(U_0) = \dim_{\mathbb{C}} T^1_{U_0} = \dim_{\mathbb{C}} T^1_{U_{U_0}} = h^0(U_0, \Theta^1_{U_0})
= \dim_{\mathbb{C}}(\mathcal{O}_{F,0}/J_F) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/((F) + J_F)).
\]

Then \( \tau(0) \leq \mu(0) \). According with [71], \( \tau(0) = \mu(0) \) if and only if \( F \) is the germ of a weighted homogeneous polynomial (w.h.p.).

An example of an isolated hypersurface singularity is given by a **compound Du Val (cDV)** singularity which is a 3-fold point \( p \) such that, for a hyperplane section \( H \) through \( p, p \in H \) is a Du Val surface singularity i.e. an A–D–E singular point (see
Then a cDV point $p$ is a germ of hypersurface singularity $U_p := \text{Spec}(\mathcal{O}_{F,p})$, where $F$ is the polynomial of the local equation
\begin{equation}
F(x, y, z, t) := x^2 + q(y, z) + t f(x, y, z, t) = 0,
\end{equation}
such that $f(x, y, z, t)$ is a generic element of the maximal ideal $m_0 := (x, y, z, t) \subset \mathcal{O} := \mathbb{C}[x, y, z, t]$ and
\begin{align*}
A_n & : q(y, z) := y^2 + z^{n+1} \quad \text{for } n \geq 1 \\
D_n & : q(y, z) := y^2z + z^{n-1} \quad \text{for } n \geq 4 \\
E_6 & : q(y, z) := y^3 + z^4 \\
E_7 & : q(y, z) := y^3 + yz^3 \\
E_8 & : q(y, z) := y^3 + z^5
\end{align*}
In particular if $f(x, y, z, t) = t$ then $F = 0$ in (10) is said to define an Arnold’s simple (threefold) singularity (3), §15 denoted $A_n, D_n, E_n$, respectively.

The index $(n, 6, 7, 8)$ turns out to be the Milnor number of the surface Du Val singularity $p \in U_p \cap \{t = 0\}$ or equivalently its Tyurina number, since a Du Val singular point always admits a weighted homogeneous local equation. If the defining polynomial $F$ is assumed w.h. of global weight $w(F) = 1$, let us denote by $w(x), w(y), w(z), w(t)$ the rational positive numbers giving the variables’ weights i.e. such that
\begin{equation}
a w(x) + b w(y) + c w(z) + d w(t) = 1
\end{equation}
for every monomial $x^a y^b z^c t^d$ in $F$. Notice that these weights have to be $\leq 1/2$, since $F \in m_0^3$. In particular we get
\begin{align*}
w(x) & = 1/2 \\
w(y) & = \begin{cases} 
1/2 & \text{if } p \text{ is } cA_n, \\
(n-2)/(2n-2) & \text{if } p \text{ is } cD_n, \\
1/3 & \text{if } p \text{ is } cE_{6,7,8}.
\end{cases} \\
w(z) & = \begin{cases} 
1/(n+1) & \text{if } p \text{ is } cA_n, \\
1/(n-1) & \text{if } p \text{ is } cD_n, \\
2/9 & \text{if } p \text{ is } cE_7, \\
1/5 & \text{if } p \text{ is } cE_8.
\end{cases}
\end{align*}

**Theorem 1.12** (Milibor–Orlik [49], Thm. 1). Let $F(z_1, \ldots, z_m)$ be a w.h.p., with variables’ weights $w(z_1), \ldots, w(z_m)$, having an isolated critical point at the origin. Then the Milnor number of the origin is given by
$$
\mu(0) = \left( w(z_1)^{-1} - 1 \right) \left( w(z_2)^{-1} - 1 \right) \cdots \left( w(z_n)^{-1} - 1 \right).
$$

By putting weights (13) in the previous Milnor–Orlik formula we get the following

**Corollary 1.13.** Le $p \in U_p$ be a w.h. cDV point of index $n$. Then
$$
\mu(p) = n \left( w(t)^{-1} - 1 \right)
$$
**Definition 1.14.** The least index of a cDV point \( p \in U_p \) is the minimum index of \( p \in U_p \cap H \) when \( H \) is a hyperplane section realizing \( p \) as a Du Val surface singularity.

For example the least index of Arnol’d’s simple singularities is 1 in the \( A_n \) case and \( \leq 2 \) in the \( E_{6,7,8} \) cases, as it is immediately verified by taking the hyperplane section \( H = \{ z = 0 \} \). For \( D_n \) Arnol’d’s simple singularities notice that, after the coordinates change \( z \mapsto y + z \), the local equation (10) can be rewritten as follows

\[
x^2 + t^2 + y^3(1 + y^{n-4}) + z(\cdots) = 0
\]

giving the germ of a \( cA_2 \) singular point. The least index is then \( \leq 2 \).

### 1.6. Geometric transitions.

**Definition 1.15 (Geometric transitions).** (cfr. [51], [16], [28], [67]) Let \( Y \) be a Calabi–Yau 3–fold and \( \phi : Y \rightarrow \bar{Y} \) be a birational contraction onto a normal variety. If there exists a complex deformation (smoothing) of \( \bar{Y} \) to a Calabi–Yau 3–fold \( \tilde{Y} \) (i.e. if \( \bar{Y} \sim \tilde{Y} \), recalling 1.3) then the process of going from \( Y \) to \( \tilde{Y} \) is called a geometric transition (for short transition or g.t.) and denoted by \( T(Y, \bar{Y}, \tilde{Y}) \) or by the diagram

\[
\begin{array}{c}
Y \\
\phi \downarrow \\
\bar{Y} \leftrightarrow \tilde{Y}
\end{array}
\]

A transition \( T(Y, \bar{Y}, \tilde{Y}) \) is called trivial if \( \tilde{Y} \) is a deformation of \( Y \) i.e. if \( Y \sim \tilde{Y} \).

**Remark 1.16.** Notice that asking for \( \tilde{Y} \) to be a Calabi–Yau threefold implies that \( \tilde{Y} \) is projective, by Definition 1.1, and that the normal threefold \( \bar{Y} \) has to be projective too, for what observed in Remark 1.9. Therefore, as defined, a geometric transition is a “connection process” inside the projective category.

Anyway it is always possible to realize a birational contraction from a Calabi–Yau threefold \( Y \) to a normal non–projective threefold e.g. when the Picard number \( \rho(Y) = 1 \). In this case we cannot get a Calabi–Yau smoothing \( \tilde{Y} \) of \( \bar{Y} \) and then a g.t. \( T(Y, \bar{Y}, \tilde{Y}) \), up to relax Definition 1.1 to include non–projective elements e.g. by admitting strict subgroups of \( SU(3) \) as holonomy group of the Kähler metric (see Remark 1.8). For this reason M.Gross [34] called primitive a Calabi–Yau threefold \( Y \) with \( \rho(Y) = 1 \) (or with no birational contractions to normal projective threefolds).

**Definition 1.17 (Conifold transitions).** A g.t.

\[
\begin{array}{c}
Y \\
\phi \downarrow \\
\bar{Y} \leftrightarrow \tilde{Y}
\end{array}
\]

is called conifold (c.t. for short) and denoted \( CT(Y, \bar{Y}, \tilde{Y}) \), if \( \bar{Y} \) admits only ordinary double points (nodes or o.d.p.) as singularities.

**Example 1.18 (cfr. [32]).** The following is a non–trivial c.t.. For details see [67], 1.3.

Let \( \bar{Y} \subset \mathbb{P}^4 \) be the generic quintic 3–fold containing the plane \( \pi : x_3 = x_4 = 0 \). Its equation is

\[
x_3g(x_0, \ldots, x_4) + x_4h(x_0, \ldots, x_4) = 0
\]
ANALYTIC EQUIVALENCE OF GEOMETRIC TRANSITION

where $g$ and $h$ are generic homogeneous polynomials of degree 4. $\overline{Y}$ is then singular and

$$\text{Sing}(\overline{Y}) = \{ [x] \in \mathbb{P}^4 | x_3 = x_4 = g(x) = h(x) = 0 \} = \{ 16 \text{ nodes} \}.$$ 

Blow up $\mathbb{P}^4$ along the plane $\pi$ and consider the proper transform $Y$ of $\overline{Y}$. Then:
- $Y$ is a smooth, Calabi–Yau 3-fold,
- the restriction to $Y$ of the blow up morphism gives a crepant resolution $\phi : Y \to \overline{Y}$.

The obvious smoothing of $Y$ given by the generic quintic 3-fold $\tilde{Y}$ completes the c.t. $CT(Y, \overline{Y}, \tilde{Y})$.

**Definition 1.19 (Primitive contractions and transitions).** A birational contraction from a Calabi–Yau 3-fold to a normal one is called primitive if it cannot be factored into birational morphisms of normal varieties. A g.t.

$$Y \xrightarrow{\phi} \overline{Y} \xleftarrow{\tilde{Y}}$$

is called primitive if the associated birational contraction $\phi$ is primitive.

**Proposition 1.20.** Let $T(Y, \overline{Y}, \tilde{Y})$ be a g.t. and $\phi : Y \to \overline{Y}$ the associated birational contraction. Then $\phi$ can always be factored into a composite of a finite number of primitive contractions.

**Proof.** The statement follows from the fact that any primitive contraction reduces by 1 the Picard number $\rho = \text{rk} (\text{Pic}(Y)) = h^{1,1}(Y)$. □

**Theorem 1.21 (Classification of primitive contractions and transitions) [80, 81].** Let $\phi : Y \to \overline{Y}$ be a primitive contraction from a Calabi–Yau threefold to a normal variety and let $E$ be the exceptional locus of $\phi$. Then one of the following is true:
- **type I:** $\phi$ is small which means that $E$ is composed of finitely many rational curves;
- **type II:** $\phi$ contracts a divisor down to a point; in this case $E$ is irreducible and in particular it is a (generalized) del Pezzo surface (see [62]);
- **type III:** $\phi$ contracts a divisor down to a curve $C$; in this case $E$ is still irreducible and it is a conic bundle over a smooth curve $C$.

**Definition 1.22 (Classification of primitive transitions).** A transition $T(Y, \overline{Y}, \tilde{Y})$ is called of type I, II or III if it is primitive and if the associated birational contraction $\phi : Y \to \overline{Y}$ is of type I, II or III, respectively.

**Definition 1.23 (Small geometric transition).** A geometric transition $Y \xrightarrow{T} \overline{Y} \xleftarrow{\tilde{Y}}$ will be called small if $\phi$ is the composition of primitive contractions of type I.

Let $T(Y, \overline{Y}, \tilde{Y})$ be a small g.t.. Then $\text{Sing}(\overline{Y})$ is composed of a finite number of terminal singularities of index 1 (see [62], §1, for the definition) since $\dim \phi^{-1}(p) = 1$, for any singular point $p \in \overline{Y}$. The singular locus $P := \text{Sing}(\overline{Y})$ and the exceptional locus $E := \phi^{-1}(P)$ have then a well known geometry reviewed by the following statement.
Theorem 1.24 ([63], [46], [58], [50], [24]). Given a small g.t. as in (14) then:

1. any \( p \in P \) is a cDV singularity,
2. for any \( p \in P \), \( E_p := \phi^{-1}(p) = \bigcup_{i=1}^{n_p} E_i \) is a connected union of rational curves meeting transversally, whose configuration is dually represented either by one of the following graphs:

\( A_n : \)
\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \]
\( n \geq 1 \) vertices

\( D_n : \)
\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \]
\( n \geq 4 \) vertices

\( E_n : \)
\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \]
\( n = 6, 7, 8 \) vertices

or, if \( p \) is a non–planar singularity, by one the following graphs

\( \tilde{D}_n : \)
\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \]
\( n \geq 3 \) vertices

\( \tilde{E}_n : \)
\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \]
\( n = 5, 6, 7 \) vertices

where triangles are dual graphs representing the transverse intersection of three rational curves at a single point,

3. the normal bundle of every irreducible rational component \( E_i \) in \( Y \) is given by \( N_{E_i|Y} \cong O_{P_1}(a) \oplus O_{P_1}(b) \) with either \((a, b) = (-1, -1)\) or \((a, b) = (0, -2)\).

Remark 1.25. Notice that, for any \( p \in \text{Sing}(\overline{Y}) \), the index \( n = n_p \), associated with the configuration dual graph of the exceptional fibre \( E_p = \phi^{-1}(p) \), is the number of irreducible components of \( E_p \) and turns out to be precisely the least index of the cDV point \( p \), as defined in Definition 1.14. Their sum

\[ n(\overline{Y}) = \sum_{p \in \text{Sing}(\overline{Y})} n_p \]

will be called the resolution number of \( \overline{Y} \) and gives the total number of irreducible components of the exceptional locus \( \overline{E} = \text{Exc}(\phi) \). It do not depends on the choice of the resolution \( Y \xrightarrow{\phi} \overline{Y} \), being the sum of least indexes of points in \( \text{Sing}(\overline{Y}) \).
**Theorem 1.26** (Generalized Clemens’ formulas \[14\], \[37\] Remark (3.8), \[68\]). Given a small g.t. \(T(Y, \overline{Y}, \overline{Y})\) there exist three non-negative integers \(k, c', c''\) such that

1. the resolution number of \(\overline{Y}\) is
   \[
   n := \sum_{p \in \text{Sing}(Y)} n_p = k + c',
   \]
2. the global Milnor number of \(\overline{Y}\) is
   \[
   m := \sum_{p \in \text{Sing}(Y)} \mu(p) = k + c'',
   \]
3. (Betti numbers) \(b_i(Y) = b_i(\overline{Y}) = b_i(\overline{Y})\) for \(i \neq 2, 3, 4,\) and
   \[
   \begin{align*}
   b_2(Y) &= b_2(\overline{Y}) + k = b_2(\overline{Y}) + k \\
   b_4(Y) &= b_4(\overline{Y}) = b_4(\overline{Y}) + k \\
   b_3(Y) &= b_3(\overline{Y}) - c' = b_3(\overline{Y}) - (c' + c'')
   \end{align*}
   \]
   where vertical equalities are given by Poincaré Duality,
4. (Hodge numbers)
   \[
   \begin{align*}
   h^{1,1}(\overline{Y}) &= h^{1,1}(Y) - k \\
   h^{2,1}(\overline{Y}) &= h^{2,1}(Y) + c
   \end{align*}
   \]
   where \(c := (c' + c'')/2,\)
5. (Euler–Poincaré characteristic)
   \[
   \chi(Y) = \chi(\overline{Y}) + n = \chi(\overline{Y}) + n + m.
   \]

**Remark 1.27.** In particular, if \(T(Y, \overline{Y}, \overline{Y})\) is a conifold transition, then \(c' = c'' = c\) and \(|\text{Sing}(\overline{Y})| = k + c = n = m\) giving the usual Clemens’ formulae (see \[67\] Theorem 3.3).

**Remark 1.28.** Integers \(k, c', c''\) and \(c\) admit topological, geometrical and physical interpretations. Topologically \(k\) turns out to be the dimension of the subspace \(\langle [\phi^{-1}(p)] \mid p \in \text{Sing}(Y) \rangle_{\mathbb{Q}}\) of \(H_2(Y, \mathbb{Q})\), generated by the 2–cycles composing the exceptional locus of the birational resolution \(\phi : Y \to \overline{Y}\). Moreover \(k\) turns out to be the number of independent linear relations linking the vanishing 3–cycles in \(H_3(\overline{Y}, \mathbb{Q})\). Degenerating \(\overline{Y}\) to \(Y\) shrinks the vanishing 3–cycles to the isolated singular points of \(\overline{Y}\) giving rise to \(k\) new independent 4–cycles in \(H_4(\overline{Y}, \mathbb{Q})\). In \[54\] Y. Namikawa and J. Steenbrink observe that

\[(*)\quad k = b_4(\overline{Y}) - b_2(\overline{Y}), \text{ called the defect of } \overline{Y}, \text{ turns out to give the rank of the quotient group } \text{Cl}(\overline{Y})/\text{CaCl}(\overline{Y})\]

of Weil divisors (mod. linear equivalence) with respect to the subgroup of Cartier divisors (mod. linear equivalence) (notation as in \[37\] II.6) meaning that the \(k\) new independent 4–cycles in \(H_4(\overline{Y}, \mathbb{Q})\) are the homology classes of Weil divisors on \(\overline{Y}\) which are not Cartier divisors. Since \(\overline{Y}\) is a reduced, irreducible and normal threefold, \(\text{CaCl}(\overline{Y}) \cong \text{Pic}(\overline{Y})\) (see \[37\] Propositions II.3.1 and II.6.15). Then statement \((*)\) and equations \[14\] allow to conclude that

\[(15)\quad \rho(\overline{Y}) = \rho(\overline{Y}) = h^{1,1}(\overline{Y}) = h^{1,1}(Y) - k = \rho(Y) - k \implies k = \rho(Y/\overline{Y}).\]
On the other hand \( c' \) gives the number of independent linear relations linking the 2–cycles of irreducible components of \( \text{Exc}(\phi) \) in \( H_2(Y, \mathbb{Q}) \). Since these 2–cycles are contracted by \( \phi \) down to the isolated singularities of \( Y' \), the previous relations generate \( c' \) new independent 3–cycles in \( H_3(Y', \mathbb{Q}) \). Moreover \( c'' \) is the dimension of the subspace generated in \( H_3(Y', \mathbb{Q}) \) by the vanishing 3–cycles. Geometrically equations (4) in the statement of Proposition [26] mean that a small g.t. decreases Kähler moduli by \( k \) and increases complex moduli by \( c \): this is a consequence of [1] and [1]. At last (but not least) a conifold transition has been physically understood by A. Strominger [74] as the process connecting two topologically distinct Calabi–Yau vacua by means of a condensation of massive black holes to massless ones inducing a decreasing of \( k \) vector multiplets and an increasing of \( c \) hypermultiplets.

2. Analytic equivalence of geometric transitions

Let us recall here the Calabi–Yau web structure as formalized by M. Gross in [33]. Web’s nodes are given by complex structure deformation classes of Calabi–Yau threefolds or better, recalling DGP Theorem [1.4] and Theorem [1.6] by universal families of Calabi–Yau threefolds. Two of such nodes, say \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \), are connected by an arrow \( \mathcal{M} \rightarrow \tilde{\mathcal{M}} \) if a smooth element of \( \mathcal{M} \) is connected with a smooth element of \( \tilde{\mathcal{M}} \) by means of a g.t.. The connectedness conjecture, summarizing part of the Reid’s Fantasy [63] and of the Green–Hübsh solution [30], [31], to the vacuum degeneracy problem, states that two distinct nodes of the web can always be connected by a finite chain of similar arrows, i.e. by a finite “composition” of g.t.’s (see Definition [5.2]).

Keeping in mind questions (1), (2), (3) in the introduction, consider the following

**Definition 2.1** (Analytical equivalence of geometric transitions). Let

\[
(16) \quad Y_1 \xrightarrow{\phi_1} \tilde{Y}_1 \sim \sim \tilde{Y}_1, \quad Y_2 \xrightarrow{\phi_2} \tilde{Y}_2 \sim \sim \tilde{Y}_2
\]

be two g.t.’s. We will say that \( T_1 \) and \( T_2 \) are analytically equivalent (\( T_1 \sim T_2 \)) if \( Y_1 \sim Y_2, \tilde{Y}_1 \sim \tilde{Y}_2 \) and \( \tilde{Y}_1 \sim \tilde{Y}_2 \), recalling analytic equivalence between Calabi–Yau varieties introduced in [1.3] i.e. if there exist two smooth families \( p : X \rightarrow B \), \( \tilde{p} : \tilde{X} \rightarrow \tilde{B} \) and a flat family \( \pi : \mathcal{X} \rightarrow \mathcal{B} \), over analytic spaces \( B, \tilde{B} \) and \( \mathcal{B} \), such that

\[
\begin{array}{c}
Y_1 \xrightarrow{\phi_1} \tilde{Y}_1 \sim \sim \tilde{Y}_1, \quad Y_2 \xrightarrow{\phi_2} \tilde{Y}_2 \sim \sim \tilde{Y}_2 \\
\end{array}
\]

We will denote the deformation by the following diagram

\[
(17)
\]

\[
\begin{array}{c}
Y_1 \xrightarrow{\phi_1} \tilde{Y}_1 \sim \sim \tilde{Y}_1, \quad Y_2 \xrightarrow{\phi_2} \tilde{Y}_2 \sim \sim \tilde{Y}_2 \\
\end{array}
\]
Remark 2.2. Definition 2.1 introduces an equivalence relation on the set $\mathcal{T}(\mathcal{M}, \tilde{\mathcal{M}})$ of geometric transitions connecting two given nodes $\mathcal{M}, \tilde{\mathcal{M}}$ of the Calabi–Yau web. Reflexive and symmetric properties are clear. The transitive property follows directly by the transitive property of analytic equivalence of Calabi–Yau manifolds, as defined in 1.3. In fact, given three g.t.’s $T_i(Y_i, \tilde{Y}_i, \bar{Y}_i) \in \mathcal{T}(\mathcal{M}, \tilde{\mathcal{M}})$, $i = 1, 2, 3$, such that $T_1 \sim T_2$ and $T_2 \sim T_3$, then $Y_1 \sim Y_2$ and $Y_2 \sim Y_3$ implying that $Y_1 \sim Y_3$. The same argument applies to $\tilde{Y}_i$ and, by Remark 1.8, to $Y_i$, $i = 1, 2, 3$. Therefore $T_1 \sim T_3$.

$\bullet$ It makes then sense to think an arrow connecting the nodes $\mathcal{M}$ and $\tilde{\mathcal{M}}$ in the Calabi–Yau web as an analytic equivalence class of g.t.’s. In particular the quotient set

$$A(\mathcal{M}, \tilde{\mathcal{M}}) := \mathcal{T}(\mathcal{M}, \tilde{\mathcal{M}})/\sim$$

is the set of all the arrows connecting $\mathcal{M}$ and $\tilde{\mathcal{M}}$.

Proposition 2.3. Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be distinct nodes of the Calabi–Yau web. Then

$$A(\mathcal{M}, \tilde{\mathcal{M}}) = \begin{cases} \emptyset, & \text{if there exists an arrow } \mathcal{M} \to \tilde{\mathcal{M}} \text{ it is unique.} \\ \{\ast\}, & \text{otherwise.} \end{cases}$$

Proof of Proposition 2.3. We have to prove that any two g.t.’s connecting elements of $\mathcal{M}$ with elements of $\tilde{\mathcal{M}}$ are equivalent. Precisely let $Y_i \in \mathcal{M}$ and $\tilde{Y}_i \in \tilde{\mathcal{M}}$ be smooth elements, for $i = 1, 2$, and let $T_i(Y_i, \tilde{Y}_i, \bar{Y}_i)$ be two g.t.’s like in (16). We have then to prove that $Y_1 \sim Y_2$. The situation is the following:

1. $Y_1 \sim Y_2$, belonging to the same node $\mathcal{M}$,
2. $\tilde{Y}_1 \sim \tilde{Y}_2$, belonging to the same node $\tilde{\mathcal{M}}$,
3. $\bar{Y}_i \sim Y_i$, for $i = 1, 2$, by the definition of g.t..

We can then conclude by applying the transitive property of analytic equivalence of Calabi–Yau varieties.

An equivalent definition of analytic equivalence for g.t.’s can be obtained by recalling the Definition 1.10 of a morphism deformation.

Theorem 2.4. Let $T_1 \sim T_2$ be two equivalent g.t.’s as in (17). Then the birational contractions $\phi_1$ and $\phi_2$ are connected by a morphism deformation, i.e. there exist an analytic space $S$, a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{X} \\ \downarrow p & & \downarrow \tilde{p} \\ S & \xleftarrow{\Phi} & \bar{S} \end{array}$$

and two points $s_1, s_2 \in S$, such that

- $p$ and $\tilde{p}$ are flat,
- $\bar{Y}_i = \tilde{p}^{-1}(s_i)$, $Y_i = p^{-1}(s_i)$ and $\phi_i = \Phi_{|p^{-1}(s_i)}$, for $i = 1, 2$.

The statement is actually a direct consequence of a Z. Ran result ([59] Theorem 3.3). A proof will be given in the following subsection 2.4.
Corollary 2.5 (Equivalent version of the Definition 2.1). Two given g.t.’s (16) are analytically equivalent if and only if the birational contractions $\phi_1$ and $\phi_2$ are connected by a morphism deformation.

Proof. Theorem 2.4 give the sufficient condition. For the necessary condition assume that $\phi_1$ and $\phi_2$ are connected by a morphism deformation parameterized by $(S,s_1,s_2)$ and set

$$B = S = \overline{B}$$

in the Definition 2.1. The proof that $\tilde{Y}_1 \sim \tilde{Y}_2$ follows by transitive property as in the proof of Proposition 2.3. □

Remark 2.6. As a consequence of Corollary 2.5, it is possible to choose (18) $B = \overline{B} = \tilde{B}$ in Definition 2.1. In fact if $T_1 \sim T_2$ then we can set $B = s = \overline{B}$ where $S$ is the parameter space of morphism deformation $\Phi$ linking the birational contraction $\phi_1$ and $\phi_2$ (notation as in Theorem 2.4). Then equality (18) follows by an iterated application of transitive property of analytic equivalence of Calabi–Yau threefolds, since $\tilde{Y}_1 \sim \tilde{Y}_1 \sim \tilde{Y}_2 \sim \tilde{Y}_2$. More precisely look at the universal family of $\tilde{Y}_1$ and then at the universal family of $\tilde{Y}_2$. By successive pull back of all the families to the latter one we get that $X$, $\tilde{X}$ and $\tilde{X}$ can be all parameterized by the same parameter space.

2.1. Proof of Theorem 2.4. Let us start with the equivalence diagram (17) and consider the birational contraction

$$\phi_1 : Y_1 \longrightarrow \tilde{Y}_1.$$ 

Since $\tilde{Y}_1$ is normal, $\codim_T(\text{Sing}(\tilde{Y}_1)) \geq 2$ and

$$\text{(19)} (\phi_1)_* \mathcal{O}_{Y_1} \cong \mathcal{O}_{\tilde{Y}_1}.$$ 

Moreover $\text{Sing}(\tilde{Y}_1)$ is composed by canonical singularities which are rational singularities (see [64] (3.8)) giving

$$\forall i > 0 \quad R^i (\phi_1)_* \mathcal{O}_{Y_1} \cong 0.$$ 

(19) and (20) give the hypothesis of Theorem 3.3 in [59]: in fact put them in the spectral sequence (8), with the exact sequence (9), to get

$$\text{(21)} \quad \text{Ext}^1_{\phi_1} \left( \Omega^1_{\tilde{Y}_1}, \mathcal{O}_{Y_1} \right) \cong \mathcal{T}^1_{\tilde{Y}_1},$$

$$\text{Ext}^2_{\phi_1} \left( \Omega^1_{\tilde{Y}_1}, \mathcal{O}_{Y_1} \right) \cong \mathcal{T}^2_{\tilde{Y}_1}.$$ 

The surjectivity in the second relation follows by observing that

$$\text{Ext}^2_{\phi_1} \left( \Omega^1_{\tilde{Y}_1}, \mathcal{O}_{Y_1} \right) \cong E^2_{\infty},$$

since $E^2_{\infty}$ is the direct limit of $E^p_{\infty}$ (see [26] § I.4.3) and

$$E^{0,2}_2 = \text{Hom} \left( \Omega^1_{Y_1}, R^2 (\phi_1)_* \mathcal{O}_{Y_1} \right) = 0 \Rightarrow E^{0,2}_{\infty} = 0.$$

This Remark answers a question of R. Donagi.
The Calabi–Yau condition for $Y_1$ gives
\[ \text{Hom}(\Omega^1_{Y_1}, \mathcal{O}_{Y_1}) \cong H^0(Y_1, \Theta_{Y_1}) \cong H^0(Y_1, \Omega^2_{Y_1}) \cong 0 \]
and, by Hartogs' Theorem (see the following Lemma 4.10) and the Leray spectral sequence, also \( \text{Hom}(\Omega^1_{Y_1}, \mathcal{O}_{Y_1}) \cong 0 \). Moreover, by (19),
\[ \text{Hom}_{\phi_1}(\Omega^1_{Y_1}, \mathcal{O}_{Y_1}) \cong \text{Hom}(\Omega^1_{Y_1}, \mathcal{O}_{Y_1}) \cong 0. \]
Therefore the exact sequence (7) can be rewritten as follows
\[
\begin{array}{c}
0 \xrightarrow{\partial} T^1_{\phi_1} \xrightarrow{\alpha_1} T^1_{Y_1} \oplus T^1_{Y_1} \xrightarrow{\alpha_2} \text{Ext}^1_{\phi_1}(\Omega^1_{Y_1}, \mathcal{O}_{Y_1}) \\
\end{array}
\]
Lemma 2.7. Let
\[ V \xrightarrow{\alpha=(\alpha_1, \alpha_2)} A_1 \oplus A_2 \xrightarrow{\beta=\beta_1+\beta_2} W \]
be an exact sequence of modules, with $\alpha_i \in \text{Hom}(V, A_i)$ and $\beta_i \in \text{Hom}(A_i, W)$, for $i = 1, 2$, and assume that $\beta_j$ is surjective. Then $\alpha_i$, with $i \neq j$, is surjective too.

Proof. Let us assume $i = 1$ and $j = 2$. For any $a \in A_1$ consider $w := -\beta_1(a)$. Since $\beta_2$ is surjective, there exists $b \in A_2$ such that $\beta_2(b) = w$, meaning that $(a, b) \in \ker \beta = \text{Im} \alpha$. Therefore there exists $v \in V$ such that
\[ \alpha(v) = (a, b) \implies a = \alpha_1(v). \]

Apply conditions (21) and Lemma 2.7 to the exact sequence (22). Then
\[
\text{ker}(\alpha_2) = \text{Im}(\alpha_1) \cong T^1_{Y_1},
\text{ker}(\beta_2) = \text{Im}(\beta_1) \cong T^2_{Y_1}.
\]
Moreover $\alpha_2$ is surjective implying that
\[
\text{Im}(\partial) = 0 = \ker(\beta_1).
\]
The first relation in (23) says that $T^1_{\phi_1} \cong T^1_{Y_1}$ which means that a 1st–order deformation of the resolution $Y_1$ always gives a 1st–order deformation of $\phi_1$ and viceversa. On the other hand (24) and the second relation in (23) mean that $T^2_{\phi_1} \cong T^2_{Y_1}$ hence every unobstructed deformation of the resolution $Y_1$ generates an unobstructed deformation of the birational contraction $\phi_1$, and viceversa. If we now consider the deformation connecting $Y_1$ and $Y_2$, it generates a deformation of $\phi_1$: more precisely there exists a pointed analytic space $(S, s_1)$ and a commutative diagram

\[
\begin{array}{c}
Y_1 \xrightarrow{\phi_1} X \\
Y_1 \xrightarrow{\phi_1} S \xleftarrow{s_1} X \\
\end{array}
\]
where
\begin{itemize}
  \item \(p\) and \(\overline{p}\) are flat,
  \item \(\overline{Y}_1 = \overline{p}^{-1}(s_1), Y_1 = p^{-1}(s_1)\) and \(\phi_1 = \Phi_{|p^{-1}(s_1)}\),
  \item \(\exists s_2 \in S\) such that \(Y_2 = p^{-1}(s_2)\).
\end{itemize}

Theorem \ref{thm:main} is proven if it is shown that:
\begin{enumerate}
  \item \(Y_2 = \overline{p}^{-1}(s_2)\),
  \item \(\phi_2 = \Phi_{|p^{-1}(s_2)}\).
\end{enumerate}

Clearly if (1) is true then (2) follows, since \(\Phi_{|p^{-1}(s_2)}\) and \(\phi_2\) are both simultaneous resolution of \(\text{Sing}(\overline{Y}_2)\) with the same domain \(Y_2\). To prove (1) observe that the BTTR Theorem \cite{76,10,77,61} ensures that \(\text{Def}(Y_1)\) and \(\text{Def}(Y_2)\) are smooth, since \(Y_1\) and \(Y_2\) are Calabi–Yau threefolds. Moreover

\[
\text{Def}(Y_1) \cong H^1(Y_1, \Theta_{Y_1}) \cong H^1(Y_2, \Theta_{Y_2}) \cong \text{Def}(Y_2),
\]

since \(Y_1\) and \(Y_2\) are analytically equivalent. Let then \(U\) be a germ of complex space representing \(\text{Def}(Y_1) \cong \text{Def}(Y_2)\) and

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{u} & Y_2 \\
\downarrow & & \downarrow \\
u_1 \in U & & u_2
\end{array}
\]

be the universal family of both \(Y_1\) and \(Y_2\). We can then assume the base \(S\) of the family \(X\) to be a simply connected subset of \(U\) and \(s_i = u_i, i = 1, 2\). On the other hand \(T_1 \sim T_2\) and there should exist an analytic, flat family connecting \(\overline{Y}_1\) and \(\overline{Y}_2\) as follows

\[
\begin{array}{ccc}
\overline{Y}_1 & \xrightarrow{v} & \overline{Y}_2 \\
\downarrow & & \downarrow \\
u_1 \in V & & u_2
\end{array}
\]

Since, for \(i = 1, 2\), the Kuranishi space \(\text{Def}(\overline{Y}_i)\) parameterizes the universal family of \(\overline{Y}_i\) we get a commutative diagram

\[
\begin{array}{ccc}
\text{Def}(\overline{Y}_1) & \xrightarrow{f} & V \\
\downarrow & & \downarrow \\
\text{Def}(\overline{Y}_2) & \xrightarrow{\beta_2} & V
\end{array}
\]

Recall now that there exist well defined morphisms \(\text{Def}(Y_i) \xrightarrow{D_i} \text{Def}(\overline{Y}_i), i = 1, 2\) (see \cite{78} Theorem 1.4.(c) for maps between deformation functors and \cite{45} Proposition (11.4) for maps between analytic representatives) completing \ref{eq:commute} with the
following commutative diagram

\[
\begin{array}{c}
\text{Def}(\overline{Y}_2) & \overset{\beta_1}{\rightarrow} & \text{Def}(\overline{Y}_1) \\
\text{Def}(\overline{Y}_2) & \overset{\beta_2}{\rightarrow} & \text{Def}(\overline{Y}_1) \\
U & \overset{f}{\rightarrow} & V \\
\end{array}
\]

Observe that \(p^{-1}(s_2)\) is the fibre over \(D_1(u_2) \in \text{Def}(\overline{Y}_1)\) in the universal family of \(\overline{Y}_1\). On the other hand \(\overline{Y}_2\) is the fibre over \(\beta_1(w_2) \in \text{Def}(\overline{Y}_1)\) in the same family. (1) would then be proven if \(D_1(u_2) = \beta_1(w_2)\). This latter equality actually follows by the commutativity of diagram (26). In fact

\[
D_1(u_2) = f \circ D_2(u_2), \quad \beta_1(w_2) = f \circ \beta_2(w_2)
\]

and \(D_2(u_2) = \beta_2(w_2)\) since they both give the origin in \(\text{Def}(\overline{Y}_2)\). \(\square\)

3. Canonical models of type II geometric transitions

Let us consider a type II g.t.

\[
Y \overset{\phi}{\longrightarrow} \overline{Y} \rightarrow \text{Def}(\overline{Y})
\]

Recalling Theorem 1.21 and Definition 1.22, \(\phi\) turns out to be a primitive contraction of an irreducible divisor \(E \subset Y\) down to a point \(p \in \overline{Y}\). Let us summarize what is known (for more details the interested reader is remanded to [38], [62], [67], [75], [66], [33], and [34]).

**about the singularity** \(p = \phi(E)\): it is a canonical singularity and in particular it is a rational Gorenstein singular point. The Reid invariant \(k\) of \(p\) (see [62]) can assume every value \(1 \leq k \leq 8\) and

- for \(k \leq 3\), \(p\) is an isolated hypersurface singularity;
- for \(k \geq 4\), \(p\) has multiplicity \(k\) and embedding dimension

\[
\dim(m_p/m_p^2) = k + 1
\]

in particular, for \(k = 4\), \(p\) is a complete intersection singularity and, for \(k \geq 5\), \(p\) is never a complete intersection singular point;

**about the exceptional locus** \(E\) of \(\phi\): it is a generalized del Pezzo surface (see [62], Proposition (2.13)) which is:

- either \(E\) is a normal del Pezzo surface of degree \(1 \leq d = K_E^2 \leq 8\) \(^3\);
- or \(E\) is a non-normal del Pezzo surface (see [60]) of degree \(K_E^2 = 7\).

\(^3\)The values 0 and 9 cannot be assumed by \(k = d\): the former since \(E\) has ample anti-canonical bundle, the latter because [62] is a transition while the contraction of a normal del Pezzo surface of degree 9 down to a point do not admit any smoothing: in this case \(E \cong \mathbb{F}^3\) and \((\overline{Y}, p)\) is rigid (see [2] and [72]).
Theorem 3.1. Let $T(Y, \overline{Y}, \tilde{Y})$ be a type II g.t. as in [27]. Then it is equivalent to a g.t. $T(Y', \overline{Y}', \tilde{Y}')$ such that $E' = \text{Exc}(\phi')$ is a smooth del Pezzo surface. Then $T'$ will be called a canonical model of $T$.

Proof. The statement is a direct consequence of a result obtained by M. Gross in proving Theorem 5.8 of [33]. In fact, by Corollary 2.1, it is enough to show that the primitive birational contraction $\phi'$ is a deformation of the primitive birational contraction $\phi$. At this purpose let us consider the morphism given by the inclusion $E \hookrightarrow Y$, where $E = \text{Exc}(\phi)$, and look at the 1st-order deformation of it, parameterized by the tangent space $T_i^1$ of $\text{Def}(E, i, Y)$, under the Z. Ran’s notation of [59]. Proposition 3.1, as already introduced in [14]. Then the exact sequence (7) gives the following:

\[
\begin{array}{cccccc}
\text{Hom}(\Omega^1_Y, \mathcal{O}_Y) \oplus \text{Hom}(\Omega^1_{\overline{Y}}, \mathcal{O}_{\overline{Y}}) & \rightarrow & \text{Hom}_i(\Omega^1_Y, \mathcal{O}_E) \\
T_i^1 & \rightarrow & T_E^1 \oplus T_i^1 & \rightarrow & \text{Ext}^1_i(\Omega^1_Y, \mathcal{O}_E) \\
T_i^2 & \rightarrow & T_E^2 \oplus T_i^2 & \rightarrow & \text{Ext}^2_i(\Omega^1_Y, \mathcal{O}_E)
\end{array}
\]

Since $Y$ is smooth, we get

$T_Y^1 \cong H^1(Y, \Theta_Y)$ and $T_Y^2 \cong H^2(Y, \Theta_Y)$

and moreover, since $\Omega^1_Y$ is a locally free sheaf of finite rank,

$\forall k > 0 \quad \mathcal{E}x t^k(\Omega^1_Y, i_* \mathcal{O}_E) \cong \mathcal{E}x t^k(\mathcal{O}_Y, i_* \mathcal{O}_E) \otimes \Theta_Y \cong 0$

(see [37] § III, Propositions 6.3 and 6.7). On the other hand the Theorem on Formal Functions (see e.g. [37], III.11) gives that $R^k i_* \mathcal{O}_E = 0$, for any $k > 0$ implying that, by the spectral sequence [38] applied to the inclusion $i$ and the local to global spectral sequence,

$\forall k \geq 0 \quad \text{Ext}_i^k(\Omega^1_Y, \mathcal{O}_E) \cong \text{Ext}^k(i_* \Omega^1_Y, \mathcal{O}_E) = H^k(\mathcal{H}om(i_* \Omega^1_Y, \mathcal{O}_E)) = H^k(\Theta_Y|_E)$. The fact that $Y$ is a Calabi–Yau threefold gives the vanishing

$\text{Hom}(\Omega^1_Y, \mathcal{O}_Y) \cong H^0(\Theta_Y) \cong H^0(\Omega^2_Y) \cong 0$.

\footnote{Since $i$ is a closed embedding, the higher direct images $R^k i_* \mathcal{O}_E$ actually reduce to the cohomology $H^k(E, \mathcal{O}_E)$. To avoid an unnecessary use of the Theorem on Formal Function one may then observe that $E$ is a generalized del Pezzo surface giving

$H^2(E, \mathcal{O}_E) \cong H^0(E, \omega_E) \cong 0$

$H^1(E, \mathcal{O}_E) \cong H^1(E, \omega_E) \cong 0$

as will be observed later. Viceversa the use of the Theorem on Formal Function gives a further proof of the latter vanishing in cohomology.}
Then the exact sequence (28) can be rewritten as follows

\[(29)\]

\[
\begin{array}{ccc}
H^0(E, \Theta_E) & \xrightarrow{f} & H^0(E, \Theta_Y|_E) \\
\downarrow \alpha_1 & & \downarrow \alpha_2 \\
T_1 & \xrightarrow{\alpha_1} & T_2 \oplus H^1(Y, \Theta_Y) \\
\downarrow \beta_1 & & \downarrow \beta_2 \\
T_2 & \xrightarrow{\beta_1} & T_2 \oplus H^2(Y, \Theta_Y) \rightarrow H^2(E, \Theta_Y|_E)
\end{array}
\]

where \(\Theta_E := \mathcal{H}om(\Omega^1_E, \mathcal{O}_E)\) is the “tangent” sheaf of \(E\) (recall that \(E\) may be singular and even non–normal).

**Claim (1).** The morphism \(f : H^0(E, \Theta_E) \rightarrow H^0(E, \Theta_Y|_E)\) in (29) is an isomorphism.

**Proof of Claim (1).** Let \(\mathcal{I}_E\) be the ideal sheaf of \(E \subset Y\) and recall the co–normal sheaf exact sequence

\[(30)\]

\[
\mathcal{I}_E/\mathcal{I}^2_E \rightarrow \Omega^1_Y \otimes \mathcal{O}_E \rightarrow \Omega^1_E \rightarrow 0.
\]

Since \(\mathcal{H}om(\cdot, \mathcal{O}_E)\) is a contravariant left exact functor we get

\[
0 \rightarrow \Theta_E \rightarrow \Theta_Y|_E \rightarrow \mathcal{N}_{E|Y}
\]

where \(\mathcal{N}_{E|Y} := \mathcal{H}om(\mathcal{I}_E/\mathcal{I}^2_E, \mathcal{O}_E)\) is the normal sheaf of \(E \subset Y\), which turns out to be the dualizing sheaf \(\omega_E\) because of \(K_Y \cong \mathcal{O}_Y\). Since \(E\) is a generalized del Pezzo surface, \(\omega_E \cong \mathcal{O}_E(-1)\) and \(H^0(E, \omega_E) \cong 0\), giving the claimed isomorphism. □

Consequently the morphism \(H^0(E, \Theta_Y|_E) \xrightarrow{\partial} T_1\) factorizes by 0 giving the following exact subsequence of (29)

\[(31)\]

\[
\begin{array}{ccc}
0 & \rightarrow & T_1 \\
\downarrow \alpha_1 & & \downarrow \alpha_2 \\
T_1 & \xrightarrow{\alpha_1} & T_2 \oplus H^1(Y, \Theta_Y) \\
\downarrow \beta_1 & & \downarrow \beta_2 \\
T_2 & \xrightarrow{\beta_1} & T_2 \oplus H^2(Y, \Theta_Y) \rightarrow H^2(E, \Theta_Y|_E)
\end{array}
\]

**Claim (2).** Consider the morphism \(T_1 \oplus H^1(Y, \Theta_Y) \xrightarrow{\alpha_2} H^1(E, \Theta_Y|_E)\) in (31) and write \(\alpha_2 = \alpha_{2,1} + \alpha_{2,2}\) with

\[
\begin{align*}
\alpha_{2,1} & \in \text{Hom}(T_1, H^1(E, \Theta_Y|_E)) \\
\alpha_{2,2} & \in \text{Hom}(H^1(Y, \Theta_Y), H^1(E, \Theta_Y|_E)).
\end{align*}
\]

Then \(\alpha_{2,1}\) is an isomorphism and \(\alpha_{2,2}\) is surjective.

**Sketch of proof of Claim (2).** Let us at first observe that \(T_1 = \text{Ext}^1(\Omega^1_E, \mathcal{O}_E)\) and the exact sequence (30) gives

\[(32)\]

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}(\Omega^1_E, \mathcal{O}_E) \\
\downarrow f & & \downarrow \cong \text{Hom}(\Omega^1_Y \otimes \mathcal{O}_E, \mathcal{O}_E) \\
\text{Hom}(\mathcal{I}_E/\mathcal{I}^2_E, \mathcal{O}_E) & \rightarrow & \text{Hom}(\mathcal{I}_E/\mathcal{I}^2_E, \mathcal{O}_E). \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}^1(\Omega^1_E, \mathcal{O}_E) & \rightarrow & \text{Ext}^1(\Omega^1_Y \otimes \mathcal{O}_E, \mathcal{O}_E) \\
\downarrow & & \downarrow \\
& \rightarrow & \text{Ext}^1(\mathcal{I}_E/\mathcal{I}^2_E, \mathcal{O}_E).
\end{array}
\]
where $f$ is just the morphism considered in Claim (1): it is an isomorphism since $\text{Hom}(\mathcal{T}_E/\mathcal{T}_E^2, \mathcal{O}_E) \cong H^0(\mathcal{N}_E|Y) \cong 0$. Moreover the local to global spectral sequences

$$E_2^{p,q} = H^p(\mathcal{E}x t^q(\Omega^1_Y \otimes \mathcal{O}_E, \mathcal{O}_E)) \implies \text{Ext}^n(\Omega^1_Y \otimes \mathcal{O}_E, \mathcal{O}_E)$$

allow to write down, by the associated lower terms exact sequences,

$$\text{Ext}^1(\Omega^1_Y \otimes \mathcal{O}_E, \mathcal{O}_E) \cong H^1(E, \Theta_Y|E)$$
$$\text{Ext}^1(\mathcal{I}_E/\mathcal{T}_E^2, \mathcal{O}_E) \cong H^1(E, \mathcal{N}_E|Y).$$

Then (32) reduces to the following exact subsequence

$$0 \to \text{Ext}^1(\Omega^1_Y \otimes \mathcal{O}_E, \mathcal{O}_E) \xrightarrow{\alpha_{2,1}} H^1(E, \Theta_Y|E) \to H^1(E, \mathcal{N}_E|Y).$$

in which $H^1(\mathcal{N}_E|Y) \cong H^1(\omega_E) \cong 0$, since $E$ is a generalized del Pezzo surface (see [38] Prop. 4.2 and [66] Cor. 4.10 or the previous footnote [4]).

On the other hand the surjectivity of $\alpha_{2,2}$ follows by considering the cohomology associated with the structure sequence of $E \subset Y$ twisted by $\Theta_Y$

$$0 \to \Theta_Y(-E) \to \Theta_Y \to \Theta_Y|E \to 0.$$

In fact M. Gross had shown that $H^2(E, \Theta_Y(-E)) \cong 0$ (the interested reader is remanded to [33], Claim 1 in the proof of Theorem 5.8 and Lemma 5.6). □

Consequently $\alpha_2$ is surjective and the morphism $H^1(E, \Theta_Y|E) \xrightarrow{\beta} T^2_Y$ factorizes by 0 giving the following splitting of the exact sequence (31)

$$0 \to T^1_E \xrightarrow{\alpha_1} T^1_E \oplus H^1(Y, \Theta_Y) \xrightarrow{\alpha_2} H^1(E, \Theta_Y|E) \to 0.$$

Applying Lemma 2.7 and the previous Claim (2) to the first sequence in (34) we get that $\alpha_1 = (\alpha_{1,1}, \alpha_{1,2})$ is composed by two isomorphisms and precisely

$$T^1_i \cong T^1_E, \quad T^1_i \cong H^1(Y, \Theta_Y) \cong T^1_Y.$$

Let now $\phi$ be the primitive type II birational contraction of the g.t. $T(Y, \tilde{Y}, \tilde{Y})$ and recall [23] and considerations which immediately follow. Then we get the following chain of isomorphisms

$$T^1_E \cong T^1_i \cong T^1_Y \cong T^1_\phi.$$

Recall now that $E$ is smoothable (see [33] Lemma 5.6 (iii)): in fact $H^2(E, \Theta_E) \cong 0$ and the lower terms exact sequence of the local to global spectral sequence converging at $\text{Ext}^n(\Omega^1_Y, \mathcal{O}_E)$ gives the surjection

$$T^1_E \to T^1_E,$$

meaning that any small deformation of a neighborhood of the singular points of $E$ is realized by a global first order deformation of $E$. At last the second exact sequence in (34) asserts that the morphism $\beta_2 = (\beta_{1,1}, \beta_{1,2})$ is injective, meaning that obstructions to extend a deformation of $i$ give actually obstructions to extend the induced (by (35)) deformations of $E$ and $Y$. But $Y$ is a
Calabi–Yau threefold and $\text{Def}(Y)$ turns out to be smooth, by the BTTR Theorem. This is enough to conclude that it is possible to recover flat families

\[ \begin{array}{c}
\mathcal{E} \\
\downarrow e \\
Y \\
\downarrow \Phi \\
\tilde{Y}
\end{array} \]

such that:

1. $Y$ is a smooth family,
2. there exists $0 \in B$ such that
   \[ E \cong \epsilon^{-1}(0), \; Y \cong \gamma^{-1}(0), \; \tilde{Y} \cong \tilde{y}^{-1}(0) \quad \text{and} \quad \iota|_E = i, \; \Phi|_Y = \phi, \]
3. for a generic $0 \neq t \in B$, $E_t := \epsilon^{-1}(t)$ is a smooth del Pezzo surface, $i_t := \iota|_{E_t}$ is the inclusion $E_t \hookrightarrow Y_t := y^{-1}(t)$ and $\phi_t := \Phi|_{Y_t} : Y_t \twoheadrightarrow \overline{Y}_t := \tilde{y}^{-1}(t)$ is a birational contraction such that $\text{Exc}(\phi_t) = E_t$.

The proof of Theorem 3.1 ends up by setting $E' = E_t, Y' = Y_t, \tilde{Y}' = \tilde{Y}_t$ and $\tilde{Y}'$ be a generic smoothing of $\tilde{Y}$.

3.1. **Degree of a type II geometric transition.**

**Definition 3.2** (Degree of a canonical model). A canonical model of a type II g.t. is said of degree $r$, with $1 \leq r \leq 8$, if the smooth exceptional del Pezzo surface has degree $r$.

**Corollary 3.3.** Let $T_1$ and $T_2$ be type II canonical models. Then

\[ T_1 \sim T_2 \implies \deg T_1 = \deg T_2. \]

**Proof.** The statement follows by (35) and the fact that smooth del Pezzo surfaces of different degree cannot be analytically equivalent since they have different second Betti numbers. \hfill \Box

**Definition 3.4** (Degree of a type II g.t.). The degree of a type II g.t. $T$ is the degree $\deg T'$ of a canonical model $T'$ such that $T \sim T'$. The previous Corollary 3.3 ensures that this definition is consistent.

**Remark 3.5.** Observe that if $T(Y, \overline{Y}, \tilde{Y})$ is a type II canonical model with $\deg T \leq 5$ then the difference

\[ c := h^{1,2}(\overline{Y}) - h^{1,2}(Y) \]

is univocally determined by $\deg T$. In fact, as observed in the recent [41] Theorem 3.3 and Remark 3.4,

\[
\begin{align*}
\deg T = 1 & \implies c = 29, \\
\deg T = 2 & \implies c = 17, \\
\deg T = 3 & \implies c = 11, \\
\deg T = 4 & \implies c = 7, \\
\deg T = 5 & \implies c = 5.
\end{align*}
\]

We remand the interested reader to [41] for further details and for several examples (of canonical models of different degree) of type II transitions.
Remark 3.6. Let us observe that isomorphisms \((35)\) reflect on the fact that a smooth del Pezzo surface \(E\) embedded in a Calabi–Yau threefold \(Y\) is a stable submanifold in the sense of Kodaira, since \(h^1(\mathcal{N}_{E|Y}) = 0\) (see \([43]\) Theorem 1).

In fact, by the Adjunction Formula, \(K_E \sim K_Y \otimes \mathcal{N}_{E|Y} \cong \mathcal{N}_{E|Y}\) and, by Serre duality, \(H^1(\mathcal{N}_{E|Y}) \cong H^1(\mathcal{O}_E) \cong 0\) since \(E\) is a blow up of \(\mathbb{P}^2\).

Remark 3.7. If \(E\) is a normal del Pezzo surface with \(\deg E = k \geq 5\), then the surjection \((36)\) turns out to be an isomorphism since \(H^1(E, \Theta_E) \cong 0\) (see \([75]\) §6). This fact actually means that locally the birational contraction \(Y \xrightarrow{\phi} \tilde{Y}\) is analytically rigid, which is it is locally analytically univocally determined by \(E = \text{Exc}(\phi)\) and the normal sheaf \(\mathcal{N}_{E|Y}\) (see \([75]\) Def. 2.2).

4. Conifold degeneration of geometric transitions

Conifold transitions are the easiest g.t.’s to understand both by the mathematical and the physical point of view. It is then natural try to understand which g.t.’s are analytically equivalent to a c.t.. These g.t.’s will be called conifoldable and it would seem quite natural that canonical models of conifoldable g.t.’s would be given by conifold transitions in the same equivalence class. Is this consistent with e.g. the choice of canonical models of type II g.t.’s given in section 3?

Definition 4.1 (Conifold degeneration of a geometric transition). A conifold transition

\[
\begin{align*}
X & \xrightarrow{\psi} \tilde{X} \\
Y & \xrightarrow{\phi} \tilde{Y}
\end{align*}
\]

is said to be a conifold degeneration of a g.t. \(T\) if \(T \sim CT\) (notation as in Definition 2.1). By Theorem 2.4, this actually means that \(\psi\) is a deformation of \(\phi\). A g.t. admitting a conifold degeneration will be called a conifoldable geometric transition (c.g.t.).

Proposition 4.2. A type II g.t. \(T(Y, \overline{Y}, \tilde{Y})\) cannot be equivalent to any small g.t.: in particular it is not conifoldable.

Proof. Observe that an equivalence between \(T\) and a small g.t. \(T'\) would give a deformation of the birational morphism \(\phi : Y \to \overline{Y}\) and then an unobstructed deformation of \(Y\) along which the closed embedding of the exceptional divisor \(E \xrightarrow{i} Y\) can’t be deformed. This fact contradicts the chain of isomorphisms \((35)\) and the injectivity of morphism \(\beta_1\) in the exact sequence \((34)\) implying that unobstructed deformations of \(E\) and \(Y\) generate unobstructed deformations of \(i\). \(\square\)

This proposition answers affirmatively the opening question of the present section. The following definition is then consistent.

Definition 4.3 (Canonical model of a c.g.t.). If a g.t. \(T\) admits a conifold degeneration \(CT \sim T\) then \(CT\) is called a canonical model of \(T\).
4.1. Bi-degree of small geometric transitions (and degree of type I g.t.’s).

By Theorem 1.26 it is possible to naturally attach a couple of integers to a small g.t.

**Definition 4.4** (Bi-degree of a small g.t.). Let \( T(Y, \overline{Y}, \tilde{Y}) \) be a small g.t. such that

\[
\begin{align*}
    k &= h^{1,1}(Y) - h^{1,1}(\overline{Y}) \\
    c &= h^{2,1}(\overline{Y}) - h^{2,1}(Y)
\end{align*}
\]

Then we will set bi–deg\(T := (k, c)\).

**Remark 4.5** (Degree of a type I g.t.). As already observed in Remark 1.28, the bi–degree of a small g.t. \( T(Y, \overline{Y}, \tilde{Y}) \) resumes a number of topological, geometric and physical properties of \( T \). In particular, by (15), \( k \) gives the relative Picard number \( \rho(Y/\overline{Y}) \) of the birational contraction \( Y \xrightarrow{\phi} \overline{Y} \). Therefore if \( T \) is primitive then it is a type I g.t. and bi–deg\(T = (1, c)\). It makes then sense to set

\[
\text{deg } T := c
\]

defining a *degree* for a further kind of primitive g.t.’s beyond the case of type II g.t.’s, already settled in Definition 3.4. Hence the general philosophy is that a bi–degree is needed in presence of a “composition” of primitive g.t.’s where the first entry \( k \) gives exactly the cardinality of such a “factorization” (see the following section).

Moreover observe that \( \text{deg } T \geq 1 \) for a type I g.t. \( T \) (analogously to the case of a type II g.t) since \( \text{deg } T(Y, \overline{Y}, \tilde{Y}) = 0 \) would give \( n = m = k = 1 \) meaning that \( \text{Sing}(\overline{Y}) \) is composed by a unique node. But this is a contradiction because in this case \( \overline{Y} \) cannot admit a Calabi–Yau smoothing \( \tilde{Y} \).

**Proposition 4.6.** Let \( T_1 \) and \( T_2 \) be small g.t.’s. Then

\[
T_1 \sim T_2 \implies \text{bi–deg } T_1 = \text{bi–deg } T_2.
\]

**Proof.** The statement follows immediately by Theorem 1.26 (3) by recalling that analytical equivalent Calabi–Yau threefolds have the same Betti numbers. \( \square \)

This proposition allows to extend to *every c.g.t.* the definition of bi–degree given above for a *small* g.t..

**Definition 4.7** (Bi–degree of a c.g.t.). If \( T \) is a g.t. admitting a conifold degeneration \( CT \sim T \) then set

\[
\text{bi–deg } T := \text{bi–deg } CT
\]

**Remark 4.8** (Degree of a primitive c.g.t.). If \( T \) is a primitive g.t. admitting a conifold degeneration \( CT \sim T \) then \( CT \) is a type I g.t. and its bi–degree is determined by \( \text{deg } CT \) as defined in Remark 4.5. Then we can naturally define

\[
\text{deg } T := \text{deg } CT
\]

further enlarging the class of primitive g.t.’s to which we are able to attach a *degree* \(( \geq 1 )\).
4.2. Conifold degeneration of small geometric transitions. As proved by Proposition 4.2, it is not true that every g.t. admits a conifold degeneration. Anyway the same question can be restricted to the class of small g.t.’s where the exceptional locus is composed by trees of rational curves meeting transversely. In this case the problem is that of understanding if there exists a deformation of the small resolution \( Y \xrightarrow{\phi} \overline{Y} \) such that each tree of rational curves split up into mutually disjoint rational curves whose normal bundle is \( O(-1) \oplus O(-1) \). In spite of what one could expect this is also not generally true, as shown by the following example due to Y.Namikawa.

Example 4.9 ([53] Example 1.11 and Remark 2.8, [69]). Let \( S \) be the rational elliptic surface with sections obtained as the Weierstrass fibration associated with the bundles homomorphism

\[
(0, B) : \quad \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} 
\]

\[(x, y, z) \quad \rightarrow \quad -x^2z + y^3 + B(\lambda) z^3 \]

for a generic \( B \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)) \) i.e. \( S \) is the zero locus of \((0, B)\) in the projectivized bundle \( \mathbb{P}(\mathcal{E}) \). Then:

1. the natural fibration \( S \rightarrow \mathbb{P}^1 \) has generic smooth fibre and 6 distinct cuspidal fibres,
2. the fiber product \( X := S \times_{\mathbb{P}^1} S \) is a threefold admitting 6 singularities of type \( II \times II \), in the standard Kodaira notation [44],
3. \( X \) admits a small resolution \( \tilde{X} \xrightarrow{\phi} X \) whose exceptional locus is composed by 6 disjoint couples of \((-1, -1)\)-curves intersecting in one point i.e. 6 disjoint \( A_2 \) exceptional trees in the notation of Theorem 1.24 ([53] §0.1, [69] Proposition 3.1),
4. \( X \) is a special fibre of the family of fiber products \( S_1 \times_{\mathbb{P}^1} S_2 \) of rational elliptic surfaces with sections: in particular for \( S_1 \) and \( S_2 \) sufficiently general \( \tilde{X} = S_1 \times_{\mathbb{P}^1} S_2 \) is a Calabi–Yau threefold giving a smoothing of \( X \) ([73] §2, [69] Proposition 2.1 (1)).

Since \( \phi \) is a small, crepant resolution, \( \tilde{X} \) turns out to be a Calabi–Yau threefold and

\[
\tilde{X} \xrightarrow{\phi} X \quad \xrightarrow{\sim} \quad \tilde{X}
\]

is a small non–conifold g.t.. Let \( p \) be one of the six singular points of \( X \), locally defined as a germ of singularity by the polynomial

\[
F := x^2 - z^2 - y^3 + w^3 \in \mathbb{C}[x, y, z, w]
\]

([69] Remark 2.9), and consider the localization near to \( p \)

\[
(39) \quad U_p := \phi^{-1}(U_p) \quad \xrightarrow{\phi} \quad \tilde{X} 
\]

\[
U_p := \text{Spec} \mathcal{O}_{F,p} \quad \xrightarrow{\phi} \quad X
\]
which induces, since \( p \) is a rational singularity, the following commutative diagram between Kuranishi spaces

\[
\begin{array}{ccc}
\text{Def}(\hat{\mathcal{X}}) & \overset{\tilde{t}_p}{\longrightarrow} & \text{Def}(\hat{U}_p) \\
\downarrow D & & \downarrow D_{\text{loc}} \\
\text{Def}(X) & \overset{t_p}{\longrightarrow} & \text{Def}(U_p) \cong T^1_{U_p}
\end{array}
\]

where the horizontal maps are the natural localization maps while the vertical maps are \textit{injective maps} induced by the resolution \( \phi \) (see [78] Propositions 1.8 and 1.12, [45] Proposition (11.4)). Then

\[
\begin{align*}
\text{def}(\hat{U}_p) = 1 \\
\text{Im}(t_p) \cap \text{Im}(D_{\text{loc}}) = 0 \\
\end{align*}
\]

([69] Theorem 3.6) meaning that

(5) no global deformation of \( \hat{\mathcal{X}} \) may induce a local non–trivial deformation of \( \hat{U}_p \); in particular \( \phi^{-1}(p) \) turns out to be a rigid \( A_2 \) exceptional tree and \( T(\hat{\mathcal{X}}, X, \tilde{\mathcal{X}}) \) cannot admit any conifold degeneration.

4.3. The Friedman diagram. Let \( \phi : Y \to \overline{Y} \) be a \textit{small birational contraction.} Then it is well defined the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & H^1(R^0\phi_*\Theta_Y) \\
\downarrow & & \downarrow \lambda \\
T^1_Y & \overset{d_1}{\longrightarrow} & H^0(R^1\phi_*\Theta_Y) \\
\downarrow & & \downarrow d_2 \\
H^2(R^0\phi_*\Theta_Y) & \longrightarrow & T^2_Y \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & H^1(\Theta_Y) \\
\downarrow & & \downarrow \delta_1 \\
T^1_Y & \overset{\delta_2}{\longrightarrow} & H^2(\Theta_Y) \\
\downarrow & & \downarrow \\
T^2_Y & \longrightarrow & 0 \\
\end{array}
\]

where:

- the first row is the lower terms exact sequence of the Leray spectral sequence of \( \phi_*\Theta_Y \), recalling 3 since \( Y \) is smooth,
- the second row is the lower terms exact sequence of the Local to Global spectral sequence converging to \( T^2_Y = \text{Ext}^n(\Omega^1_Y, O_Y) \),
- the vertical equalities come from the following application of Hartogs Theorem

\[
\text{Lemma 4.10 ([24] Lemma (3.1))} \quad R^0\phi_*\Theta_Y \cong \Theta_{\overline{Y}}.
\]

Since \( \text{Sing}(\overline{Y}) \) is entirely composed by rational singularities (actually cDV singular points, by Theorem 1.24), for any \( p \in \text{Sing}(\overline{Y}) \), a localization diagram similar to (39) is defined and induces a commutative diagram between Kuranishi spaces as in (40), which can be rewritten in the present context as follows

\[
\begin{array}{ccc}
\text{Def}(Y) & \overset{t_p}{\longrightarrow} & \text{Def}(U_p) \\
\downarrow D & & \downarrow D_{\text{loc}} \\
\text{Def}(Y) & \overset{\tilde{t}_p}{\longrightarrow} & \text{Def}(\overline{U}_p) \cong T^1_{U_p} \cong T^1_{\overline{U}_p}
\end{array}
\]

The last isomorphism comes from 41, since \( \overline{U}_p \) is Stein.
Proposition 4.11. The central part of the Friedman diagram (42) is the differential of diagram (43) extended to every point in Sing(Y). In particular, for any \( p \in P := \text{Sing}(\bar{Y}) \),

1. \( T_{U_p}^1 \cong H^1(U_p, \Theta_{U_p}) \cong H^0(R^1\phi_* \Theta_{U_p}) \) and \( H^0(R^1\phi_* \Theta_Y) \cong \bigoplus_{p \in P} T_{U_p}^1 \) is the tangent space to \( \prod_{p \in P} \text{Def}(U_p) \).

2. \( T_{U_p}^1 \cong T_{\bar{U}_p}^1 \) and \( T_{U_p}^1 \cong \bigoplus_{p \in P} T_{\bar{U}_p}^1 \) is the tangent space to \( \prod_{p \in P} \text{Def}(\bar{U}_p) \).

3. morphisms \( \delta_{loc} \) and \( \delta_1 \) in diagram (44) are injective.

Proof. To prove (1) observe that \( U_p \) is smooth and, by (3) in subsection 1.2, \( \bar{T}_{U_p}^1 \cong H^1(U_p, \Theta_{U_p}) \). Apply now the Leray spectral sequence for the cohomology of \( \Theta_{U_p} \), whose associated exact sequence gives

\[
0 \longrightarrow H^1(R^0\phi_* \Theta_{U_p}) \longrightarrow H^1(\Theta_{U_p}) \longrightarrow H^0(R^1\phi_* \Theta_{U_p}) \longrightarrow H^2(R^0\phi_* \Theta_{U_p}) \]

By Lemma 4.10 \( R^0\phi_* \Theta_{U_p} \cong \Theta_{\bar{U}_p} \) and \( H^i(\Theta_{\bar{U}_p}) = 0 \), for \( i > 0 \), since \( \bar{U}_p \) is Stein, giving the first isomorphism in (1).

On the other hand, to prove (2), the local to global spectral sequence converging to \( T_{\bar{U}_p}^1 \) gives

\[
0 \longrightarrow H^1(\Theta_{\bar{U}_p}) \longrightarrow T_{\bar{U}_p}^1 \longrightarrow T_{\bar{U}_p}^1 \longrightarrow H^2(\Theta_{\bar{U}_p}) \]

and we can conclude as before to get the first isomorphism in (2).

The complete statements of (1) and (2) can now be obtained as an easy application of the local cohomology sequence (44)

\[
0 \longrightarrow H^0(\bar{Y}, \mathcal{F}) \longrightarrow H^0(Y, \mathcal{F}) \longrightarrow H^0(\bar{Y}, \mathcal{F}) \longrightarrow H^1_p(Y, \mathcal{F}) \longrightarrow \cdots
\]

where \( \bar{Y} := Y \setminus P \) and \( \mathcal{F} \) is given either by \( R^1\phi_* \Theta_Y \) or by \( \Theta_{\bar{Y}} = \mathcal{E}x \cdot t^1(\Omega^1_{\bar{Y}}, \mathcal{O}_{\bar{Y}}) \).

Precisely, for \( \mathcal{F} = R^1\phi_* \Theta_Y \), by the Theorem on Formal Functions (37, III.11), we get

\[
H^0(\bar{Y}, R^1\phi_* \Theta_Y) = 0
\]

\( \forall p \in P \)

\[
H^0(\bar{U}_p, R^1\phi_* \Theta_{U_p}) = 0
\]

since \( \phi|_{Y \setminus E} \) is an isomorphism. Then

\[
H^0(\bar{Y}, R^1\phi_* \Theta_Y) \cong H^0_p(\bar{Y}, R^1\phi_* \Theta_Y) \cong \bigoplus_{p \in P} H^0_p(\bar{U}_p, R^1\phi_* \Theta_{U_p}) \cong \bigoplus_{p \in P} T_{U_p}^1
\]

where the first row is obtained by applying the first vanishing in (45), the second row is an application of the Excision Theorem (36, Prop. I.2.2) and the last row is obtained by the second vanishing in (45) and by localizing (44) near to \( p \).

On the other hand, for \( \mathcal{F} = \Theta_{\bar{Y}}^1 \), we get

\[
H^0(\bar{Y}, \Theta_{\bar{Y}}^1) = 0
\]

\( \forall p \in P \)

\[
H^0(\bar{U}_p, \Theta_{U_p}^1) = 0
\]
since the sheaf $\Theta^1_Y$ is supported on $P = \text{Sing}(\overline{Y})$. Then, as before, we get

$$T^1_{\overline{Y}} = H^0(\overline{Y}, \Theta^1_{\overline{Y}}) \cong H^0_P(\overline{Y}, \Theta^1_{\overline{Y}}) \cong \bigoplus_{p \in P} H^0(P, \Theta^1_U) \cong \bigoplus_{p \in P} H^0(\overline{U}_p, \Theta^1_U).$$

To prove statement (3) in Proposition 4.11, observe first of all that exactness of rows in diagram (42) shows easily that $\delta^{\text{loc}}$ injective implies $\delta^1$ injective. To prove that $\delta^{\text{loc}}$ is injective we will repeat here an argument of R. Friedmann (see [24] Proposition (2.1), (2)).

Let us first of all recall the following result of M. Schlessinger (see [72] Theorem 2)

$$T^1_{\overline{U}_p} \cong H^1(\overline{U}_p, \Theta^1_{\overline{U}_p})$$

where

$$\overline{U}_p := U_p \setminus \{p\} \quad \text{and} \quad U^*: = \left( \bigcup_{p \in P} U_p \right) \setminus E.$$

Then $U^* \xrightarrow{\phi} \overline{U}^*$ is an isomorphism and (46), with isomorphisms in (1) and (2), allows to conclude that

$$H^0(\overline{Y}, R^1\phi_* \Theta_Y) \cong \bigoplus_{p \in P} T^1_{\overline{U}_p} \cong H^1(U, \Theta_U)$$

and $\delta^{\text{loc}}$ can then be obtained by the local cohomology exact sequence

$$0 \to H^0_E(U, \Theta_U) \to H^0(U, \Theta_U) \to H^0(U^*, \Theta_U)$$

$$\xrightarrow{H^1_E(U, \Theta_U)} \to H^1(U, \Theta_U) \xrightarrow{\delta^{\text{loc}}} H^1(U^*, \Theta_U) \to \cdots$$

By Hartogs’ Theorem $H^0(U, \Theta_U) \cong H^0(U^*, \Theta_U)$: then

$$\ker(\delta^{\text{loc}}) = H^1_E(U, \Theta_U) = 0,$$

where the vanishing can be obtained by a depth argument, since $\text{codim}_U E = 2$. □

4.4. Local and global rigidity of small geometric transitions.

**Definition 4.12** (Local and global rigidity of a small g.t.). Let $T(Y, \overline{Y}, \overline{Y})$ be a small g.t. and observe that the first row in diagram (42) gives that $h^1(\Theta_Y) \leq h^1(\Theta_{\overline{Y}})$. In the following $T$ will be called **analytically rigid** (or simply **rigid**) if

$$h^1(\overline{Y}, \Theta_{\overline{Y}}) = h^1(Y, \Theta_Y) \quad \text{(i.e. if } \text{Im } \lambda = 0).$$

On the contrary $T$ will be called **non–rigid** if the previous condition is violated i.e. if $h^1(\Theta_{\overline{Y}}) < h^1(\Theta_Y)$ and $\text{Im } \lambda \neq 0$. Moreover, by Proposition 4.11 (1), it is possible
to decompose
\[ \text{Im} \lambda = \bigoplus_{p \in P} \text{Im} \lambda_p \]
where \( T^1_Y \xrightarrow{\lambda_p} T^1_{U_p} \) is the composition of \( \lambda \) with the projection \( H^0(R^1\phi_*\Theta_Y) \rightarrow T^1_{U_p} \).
Then \( T \) will be called \textit{analytically rigid} (or simply \textit{rigid}) at \( p \) if
\[ \text{Im} \lambda_p = 0 \]
On the contrary \( T \) will be called \textit{non–rigid} at \( p \) if the previous condition is violated i.e. if \( \dim(\text{Im} \lambda_p) > 0 \).

\textbf{Remark 4.13.} The following considerations are easily implied by the previous Definition \ref{definition12}.

1. \( T \) is rigid at \( p \) if \( \dim_C T^1_{U_p} = 0 \) i.e. if \( h^1(U_p, \Theta_{U_p}) = 0 \) as follows obviously by the smoothness of \( U_p \).
2. \( T(\bar{Y}, \bar{Y}, Y) \) is rigid if and only if \( T \) is rigid at any \( p \in \text{Sing}(\bar{Y}) \), since
\[ \text{Im} \lambda = \bigoplus \text{Im} \lambda_p = 0. \]
3. \( T \) is rigid if \( h^0(R^1\phi_*\Theta_Y) = 0 \).

Let us observe that the converse of (1) and (3) are not true meaning that, for any \( p \in \text{Sing}(\bar{Y}) \), \( h^1(U_p, \Theta_{U_p}) > 0 \) do not imply \( \dim(\text{Im} \lambda_p) > 0 \) and that \( h^0(R^1\phi_*\Theta_Y) > 0 \) do not imply \( \dim(\text{Im} \lambda) > 0 \). An example is given by the small g.t. \cite{38} obtained resolving and smoothing the Namikawa cuspidal fiber product \( X = S \times_{\pi^1} S \). In fact in this case the global and local rigidity conditions \cite{17} and \cite{38} follow by observing that \( h^1(\Theta_X) = 3 = h^1(\Theta_{\bar{S}}) \) although \( h^0(R^1\phi_*\Theta_{\bar{S}}) = 6 \) (see e.g. \cite{69} Remark 4.2 and diagram (68)): in this particular case the morphism \( \lambda \) in diagram \cite{12} and its localizations \( \lambda_p \)'s turn out to be trivial giving the differential version of \cite{11}. On the other hand \( h^1(\tilde{U}_p, \Theta_{\tilde{U}_p}) = 1 \) for any \( p \in \text{Sing}(X) \).

\textbf{Theorem 4.14.} Given a small birational contraction \( \phi : Y \rightarrow \bar{Y} \) the following facts are equivalent:
\begin{enumerate}
\item \( h^0(\bar{Y}, R^1\phi_*\Theta_Y) = 0 \).
\item \( N_{l|Y} \cong \mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(-1) \) for every irreducible component \( l \) of \( \text{Exc}(\phi) \).
\end{enumerate}

By Remark \ref{12} we get then the following:

\textbf{Corollary 4.15.} A conifold transition \( T(Y, \bar{Y}, \tilde{Y}) \) is a rigid small g.t..
(see [26] Remark 4.17.1) and

\[(52) \quad H^0(\overline{Y}, R^1\phi_*\Theta_Y) = \bigoplus_{p \in P} (R^1\phi_*\Theta_Y)_p \cong \bigoplus_{p \in P} H^1(E_p, \Theta_Y|_{E_p}) = H^1(E, \Theta_Y|_E).\]

Let us at first assume that \(E = \bigcup_{p \in P} E_p\) is a disjoint union of irreducible rational curves \(E_p \cong \mathbb{P}^1\), which is that \(\text{Sing}(\overline{Y})\) is composed by a finite number of \(c\Lambda_1\) singularities (recall their definition (10) and (11)). By Theorem 1.24(3)

\[\mathcal{N}_{E_\mathbb{P}^1|Y} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)\]

with either \((a, b) = (-1, -1)\) or \((0, -2)\).

On the other hand consider the cohomology long exact sequence associated with

\[(53) \quad 0 \rightarrow \Theta_{E_p} \rightarrow \Theta_Y|_{E_p} \rightarrow \mathcal{N}_{E_\mathbb{P}^1|Y} \rightarrow 0.\]

Observe that \(\Theta_{E_p} \cong \mathcal{K}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)\). Therefore if \((a, b) = (-1, -1)\) then

\[H^1(E_p, \Theta_Y|_{E_p}) \cong H^1(E_p, \Theta_{E_p})\]

since \(h^0(\mathcal{O}_{\mathbb{P}^1}(-1)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0\). Hence

\[H^1(E, \Theta_Y|_E) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = 0\]

On the other hand, if \((a, b) = (0, -2)\) then

\[h^1(E_p, \Theta_Y|_{E_p}) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 1 \Rightarrow h^1(E, \Theta_Y|_E) > 0\]

and (52) concludes the proof in this particular case.

We can now assume that there exists \(p \in P\) such that the fiber \(E_p = \phi^{-1}(p)\) is a connected union of at least two irreducible components meeting transversely at one point

\[E_p = l \cup E' \quad \text{and} \quad l \cap E' = \{q\}\]

where \(l \cong \mathbb{P}^1, E' \supseteq \mathbb{P}^1\). Let \(f : \tilde{Y} \longrightarrow Y\) be the blow up of \(Y\) at the intersection point \(q\) and look at the strict transform \(\tilde{E}\) of \(E_p\) which is the disjoint union \(\tilde{E} = \tilde{l} \cup \tilde{E}'\) of the strict transforms of \(l\) and \(E'\) respectively. Set \(\varphi := f|_{\tilde{E}}\) and \(\mathcal{F} := \Theta_{\tilde{Y}}|_{\tilde{E}}\) and consider the Leray spectral sequence of \(\mathcal{F}\) with respect to \(\varphi\). Then the associated lower terms exact sequence gives

\[0 \longrightarrow H^1(E_p, R^0\varphi_*\mathcal{F}) \longrightarrow H^1(\tilde{E}, \mathcal{F}) \longrightarrow H^0(E_p, R^1\varphi_*\mathcal{F}) \longrightarrow \cdots.\]

Observe that \(R^0\varphi_*\mathcal{F} = (f_*\Theta_{\tilde{Y}})|_{\tilde{E}} \cong \Theta_Y|_{E_p}\), by Lemma 4.10. On the other hand \(R^1\varphi_*\mathcal{F} = 0\), since \(\varphi\) has 0-dimensional fibers, giving

\[H^1(E_p, \Theta_Y|_{E_p}) \cong H^1(\tilde{E}, \mathcal{F}) \cong H^1(\tilde{l}, \Theta_{\tilde{Y}}|_{\tilde{l}}) \oplus H^1(\tilde{E}', \Theta_{\tilde{Y}}|_{\tilde{E}'})\]

But \(H^1(\tilde{l}, \Theta_{\tilde{Y}}|_{\tilde{l}}) \cong H^1(l, \Theta_Y|_l)\) and, by the previous step, it vanishes if and only if \(\mathcal{N}_l|_Y \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\). Recalling (52), the proof ends up by induction on the irreducible components of \(E'\).

**Proposition 4.16.** Given a small g.t. \(T(Y, \overline{Y}, \tilde{Y})\) let \(n, m, \tau\) be the resolution number, the global Milnor number and the global Tyurina number of \(\overline{Y}\), respectively, as defined in Remark 4.26 in Theorem 1.26(1) and (2) and in Definition 1.11. Then

\[(54) \quad \tau = \frac{m + n}{2} + h^0(\overline{Y}, R^1\phi_*\Theta_Y).\]

Consequently:

(a) \(n \leq \tau \leq m\),
(b) \( n = \tau \) if and only if \( n = m \): in particular \( \tau = m \) and \( h^0(\mathcal{Y}, R^1\phi_*\Theta_{\mathcal{Y}}) = 0 \),
(c) if \( m = \tau \) then facts (1) and (2) in Theorem 4.14 are equivalent to assert that \( n = m \).

Proof. First of all let us observe that \( \text{Sing}(\mathcal{Y}) \) is composed by cDV singularities which are terminal of index 1. In [52], Theorem A, Y. Namikawa proved an extension of the BTTR Theorem allowing to conclude that \( \text{Def}(\mathcal{Y}) \) is smooth also in this case. Therefore

\[
\dim T^1_Y =: \text{def}(\mathcal{Y}) = h^{2,1}(\overline{\mathcal{Y}}) = \text{def}(\mathcal{Y}) + \frac{m+n}{2} - k
\]

by Theorem 1.25. Moreover Remark 1.28 and equations (15) give

\[
\dim T^1_Y = \rho(\mathcal{Y}) = \rho(\mathcal{Y}) - k = \dim T^1_Y - k.
\]

The following relations are then obtained by the Friedman diagram (42):

\[
\begin{align*}
    h^1(\Theta_{\mathcal{Y}}) - \text{def}(\mathcal{Y}) + h^0(R^1\phi_*\Theta_{\mathcal{Y}}) - h^2(\Theta_{\mathcal{Y}}) + \rho(\mathcal{Y}) &= 0, \\
    h^1(\Theta_{\mathcal{Y}}) - \text{def}(\mathcal{Y}) + \tau(\overline{\mathcal{Y}}) - h^2(\Theta_{\mathcal{Y}}) + \rho(\mathcal{Y}) &= 0
\end{align*}
\]

where \( \tau(\overline{\mathcal{Y}}) = \dim T^1_Y \) is the global Tyurina number of \( \mathcal{Y} \) as follows by Definition 1.11 and Proposition 1.11(2). By (54) and (56) their difference gives precisely (54).

(a) is then obtained by recalling that \( \tau \leq m \) by Definition 1.11. Then (54) gives

\[
\begin{align*}
    m &\geq 2\tau - m = n + 2h^0(R^1\phi_*\Theta_{\mathcal{Y}}) \Rightarrow m \geq n \\
    \tau &= \frac{m+n}{2} + h^0(R^1\phi_*\Theta_{\mathcal{Y}}) \geq n + h^0(R^1\phi_*\Theta_{\mathcal{Y}}) \Rightarrow \tau \geq n.
\end{align*}
\]

While the necessary condition in (b) is obvious for the sufficient condition observe that by (54)

\[
\tau = n \Rightarrow 2n = 2\tau = m + n + 2h^0(R^1\phi_*\Theta_{\mathcal{Y}}) \Rightarrow \\
= m + 2h^0(R^1\phi_*\Theta_{\mathcal{Y}}) \Rightarrow n \geq m \Rightarrow n = m.
\]

At last, for (c), assume that \( m = \tau \). Then (54) gives

\[
2m = 2\tau = m + n + 2h^0(R^1\phi_*\Theta_{\mathcal{Y}}) \Rightarrow m = n + 2h^0(R^1\phi_*\Theta_{\mathcal{Y}}).
\]

Therefore \( m = n \) if and only if \( h^0(R^1\phi_*\Theta_{\mathcal{Y}}) = 0 \). \qed

Remark 4.17. Observe that \( \tau(\overline{\mathcal{Y}}) = m(\overline{\mathcal{Y}}) \) if and only if \( \tau(p) = \mu(p) \), for any \( p \in \text{Sing}(\mathcal{Y}) \), which holds, by [71], if and only if every singular point of \( \mathcal{Y} \) is a germ of singularity defined by a weighted homogeneous polynomial: this is the case e.g. when \( \mathcal{Y} \) is a (singular) hypersurface of a 4–dimensional weighted projective space.

In principle, Proposition 1.16 allows to generalize Corollary 1.13 as follows:

- Let \( T(\mathcal{Y}, \overline{\mathcal{Y}}, \tilde{\mathcal{Y}}) \) be a small g.t. such that \( n(\tilde{\phi}) = m(\overline{\mathcal{Y}}) \) (e.g. when \( T \) is a c.t.). Then it is a rigid small g.t.

Actually it is not an effective generalization due to the following consequence of the Milnor–Orlik Theorem 1.12 and its Corollary 1.13

**Proposition 4.18.** Assume that for a small g.t. \( T(\mathcal{Y}, \overline{\mathcal{Y}}, \tilde{\mathcal{Y}}) \) the resolution number \( n = n(\tilde{\phi}) \) equals the global Milnor number \( m = m(\overline{\mathcal{Y}}) \). Then \( T \) is a conifold transition.
Proof. Since \( n = m \) Proposition 11.10(a) gives \( \tau = m \). Therefore, as observed in the previous Remark 11.17 every singular point \( p \in \text{Sing}(Y) \) is a germ of singularity defined by a w.h. polynomial. By Corollary 11.13 and Remark 11.25 we get

\[
\forall p \in \text{Sing}(Y) \quad n_p = \mu(p) = n_p \left( w(t)^{-1} - 1 \right) \Rightarrow w(t) = \frac{1}{2}
\]

where \( n_p \) is the least index of \( p \) as defined in Definition 11.14. Then \( f(x, y, z, t) \) in the local equation (10) of \( p \) has weight \( w(f) = 1/2 \). Recalling (13) for the weights of \( x, y, z, t \), then \( f \) is a w.h. linear combination of a monomial basis of the \( C \)-vector space \( \mathbb{C}[x, y, z, t]/I \) where \( I = (x^2, y^2, t^2, xy, xz, xt, yz, yt, zt) \) giving

\[
f(x, y, z, t) = \begin{cases} 
  ax + by + ct + dz & \text{if } p \text{ is } cA_1, \\
  ax + by + c + \alpha z^k, \alpha \in \mathbb{C} & \text{if } p \text{ is } cA_{2k-1}, k \geq 2 \\
  ax + by + c & \text{if } p \text{ is } cA_{2k}, k \geq 1, \\
  ax + c + \alpha z^k, \alpha \in \mathbb{C} & \text{if } p \text{ is } cD_{2k+1}, k \geq 2, \\
  ax + c + \alpha z^2 & \text{if } p \text{ is } cE_6, \\
  ax + c & \text{if } p \text{ is } cE_7,8 \text{ or } cD_{2k}, k \geq 2.
\end{cases}
\]

By introducing in \( F \) those of the following change of variables

\[
x \mapsto x - \frac{a}{2} t, \quad y \mapsto y - \frac{b}{2} t, \quad z \mapsto z - \frac{d}{2} t
\]

which are w.h., the singularity \( p \) reduces to the following one

\[(57) \quad x^2 + q(y, z) + \beta t^2 + t g(z) = 0, \quad \beta \in \mathbb{C},
\]

with

\[
g(z) = \begin{cases} 
  0 & \text{if } p \text{ is either } cA_1, cA_{2k} \text{ with } k \geq 1, cE_7,8 \text{ or } cD_{2k} \text{ with } k \geq 2 \\
  \alpha z^k & \text{if } p \text{ is either } cA_{2k-1} \text{ or } cD_{2k+1}, k \geq 2, \\
  \alpha z^2 & \text{if } p \text{ is } cE_6.
\end{cases}
\]

Observe that, since \( p \) is an isolated singularity, it is possible to assume \( \beta \neq 0 \) otherwise points \((0, 0, 0, t)\) would be singular for any \( t \). Then, by replacing \( t \mapsto t/\sqrt{\gamma} \), (57) reduces to

\[(58) \quad x^2 + q(y, z) + t^2 + t g(z) = 0
\]

giving:

1. if \( p \) is \( cA_1 \) it turns out to be actually a node since \( g(z) = 0 \) and \( q(y, z) = y^2 + z^2 \) in (58),
2. by looking at the hyperplane section \( z = 0 \), (58) gives either a \( cA_1 \) singularity if \( p \) is \( cA_n, n \geq 2 \), or a \( cA_2 \) singularity if \( p \) is \( cE_{6,7,8} \),
3. by looking at the hyperplane section \( y = 0 \), (58) gives a \( cA_{2k-2} \) singularity if \( p \) is \( cD_{2k}, k \geq 2 \),
4. if \( p \) is \( cD_{2k+1}, k \geq 2 \), then \( w(z^k) = 1/2 \) and the transformation \( t \mapsto t - \frac{1}{2} z^k \) is weighted homogeneous and allows to rewrite (58) as follows

\[
x^2 + y^2 z + t^2 + \gamma z^{2k} = 0, \quad \gamma \in \mathbb{C}.
\]

As before it is possible to assume \( \gamma \neq 0 \) otherwise points \((0, 0, z, 0)\) would be singular for any \( z \), against the fact that \( p \) is an isolated singular point; by normalizing \( z \mapsto z/\sqrt{\gamma} \) and looking at the hyperplane section \( y = 0 \) we get then a \( cA_{2k-1} \) singular point.

Recall that \( n_p \) is the least index of \( p \) meaning that (58) cannot give a presentation of \( p \) with lower index: hence only case (1) can occur. \( \square \)
It is then possible to state the following characterization of a w.h. conifold transition:

**Theorem 4.19** (A characterization of w.h. conifold t.’s). Let \( T(Y, \overline{Y}, \tilde{Y}) \) be a small g.t. such that \( \text{Sing}(\overline{Y}) \) is entirely composed by germs of singularities locally defined by w.h. polynomial\(^5\) and let \( \phi : Y \to \overline{Y} \) be the associated small birational contraction. Then the following facts are equivalent:

1. \( T \) is a conifold transition,
2. the global Milnor number \( m(\overline{Y}) \) coincides with the resolution number \( n(\phi) \),
3. \( h^0(\overline{Y}, R^3_{\phi_*} \Theta_Y) = 0 \),
4. \( N_{l/l} = O_{P^1}(-1) \oplus O_{P^1}(-1) \) for every irreducible component \( l \subset \text{Exc}(\phi) \).

**Proof.** (1) \( \iff \) (2): If \( T \) is conifold then the global Milnor number \( m(\overline{Y}) \) and the resolution number \( n(\phi) \) both coincides with the cardinality \( N := |\text{Sing}(\overline{Y})| \). Vice versa if \( n = m \) then \( T \) is conifold by Proposition 4.18.

(2) \( \iff \) (3): It follows directly by Proposition 4.16(c) recalling that every \( p \in \text{Sing}(\overline{Y}) \) is w.h. giving \( \tau(p) = \mu(p) \).

(3) \( \iff \) (4): It is just Theorem 4.14. \( \square \)

### 4.5. Cohomological conditions for the existence of conifold degenerations.

**Theorem 4.20.** Let

\[
\begin{align*}
Y & \xrightarrow{\phi} \overline{Y} \xleftarrow{\psi} \tilde{Y}
\end{align*}
\]

be a small g.t. which is non–rigid at any \( p \in P' := \text{Sing}(\overline{Y}) \setminus \{ \text{nodes of } \overline{Y} \} \), i.e.

\[
\forall p \in P' \quad \dim(\text{Im } \lambda_p) > 0 .
\]

Then, in general, it admits a conifold degeneration i.e. there exists a conifold g.t.

\[
\begin{align*}
X & \xrightarrow{\psi} \overline{X} \xleftarrow{\phi} \tilde{X}
\end{align*}
\]

such that \( \psi \) is a deformation of \( \phi \).

**Proof.** If \( \text{Sing}(\overline{Y}) \) is entirely composed by nodes then \( T(Y, \overline{Y}, \tilde{Y}) \) is a c.t. and there is nothing to prove. It is then possible to assume that

\[
P' := \text{Sing}(\overline{Y}) \setminus \{ \text{nodes of } \overline{Y} \} \neq \emptyset .
\]

To prove that \( \psi \) is a deformation of \( \phi \) means to show that there exist an analytic space \( S \), a commutative diagram

\[
\begin{align*}
\mathcal{X} & \xrightarrow{\Phi} \overline{X} \\
& \xleftarrow{\psi} \tilde{X}
\end{align*}
\]

with \( p \) and \( q \) flat morphisms, and two points \( x, y \in S \) such that \( X = p^{-1}(x) \), \( Y = p^{-1}(y) \), \( \overline{X} = q^{-1}(x) \), \( \overline{Y} = q^{-1}(y) \), and \( \psi = \Phi|_{\overline{Y}} \), \( \phi = \Phi|_{\overline{Y}} \).

\(^5\)E.g. when \( \overline{Y} \) is a hypersurface of a 4–dimensional weighted projective space.
In the present context, the exact sequence (7) looks as follows
\[
0 \xrightarrow{} T^1_{\phi} \xrightarrow{\alpha_1} T^1_Y \oplus T^2_{\phi} \xrightarrow{\alpha_2} \Ext^1_{\phi}(\Omega^1_{\Theta Y}, O_Y) \xrightarrow{\partial} T^2_{\phi} \xrightarrow{\beta_1} T^2_Y \oplus T^2_{\phi} \xrightarrow{\beta_2} \Ext^2_{\phi}(\Omega^1_{\Theta Y}, O_Y)
\]
Since \(Y\) is a Calabi–Yau threefold, hypothesis of Theorem 3.3 in [59] still holds and we can proceed as in the proof of Theorem 2.4 reproducing steps (19), (20), (21), (23) and (24). Precisely we get
\[
\phi_* O_Y \cong O_Y \quad \text{and} \quad R^1 \phi_* O_Y = R^2 \phi_* O_Y = 0 \tag{61},
\]
giving, by the spectral sequence (8),
\[
\Ext^1_{\phi}(\Omega^1_{\Theta Y}, O_Y) \cong T^1_Y \quad \text{and} \quad \Ext^2_{\phi}(\Omega^1_{\Theta Y}, O_Y) \cong T^2_Y.
\]
Then
\[
\ker(\alpha_2) = \Im(\alpha_1) \cong T^1_Y, \quad \ker(\beta_2) = \Im(\beta_1) \cong T^2_Y, \quad \Im(\partial) = 0 = \ker(\beta_1)
\]
and
\[
T^1_{\phi} \cong T^1_Y \quad \text{i.e. a 1st-order deformation of } Y \text{ always gives a 1st-order deformation of } \phi \text{ and viceversa},
\]
\[
T^2_{\phi} \cong T^2_Y \quad \text{i.e. every unobstructed deformation of } Y \text{ generates an unobstructed deformation of } \phi \text{ and viceversa}.
\]
Hypothesis (61) allows to choice a non–trivial class \(\vartheta_p \in \Im \lambda_p \subset T^1_{U_p}\), for any \(p \in P'\). Then, by Proposition 4.11(1) and hypothesis (60), there exists a non–trivial class \(\vartheta \in H^1(Y, \Theta_Y) \cong T^1_Y\) such that \(\lambda(\vartheta) = \sum_{p \in P'} \vartheta_p \neq 0 \in H^0(R^1 \phi_* \Theta_Y)\).
In particular \(\vartheta\) induces a non–trivial 1st-order deformation of \(Y\) and, by (61), a non–trivial 1st-order deformation of \(\phi\). Proposition 4.11(3) ensures that morphisms \(\delta_1\) and \(\delta_{loc}\) in diagram (62) are injective. Then \(\delta_1(\vartheta) \in T^1_Y\) gives a non–trivial 1st-order deformation of \(Y\) and, by Proposition 4.11(2), \(\delta_{loc} \circ \lambda(\vartheta) = \sum_{p \in P'} \overline{\vartheta_p}\), where \(\overline{\vartheta_p} := \delta_{loc} \circ \lambda_p(\vartheta) \in T^1_{U_p}\) do not vanish for any \(p \in P'\). Since \(Y\) is a Calabi–Yau threefold, the BTTR Theorem (66) ensures that any 1st–order deformation of \(Y\) is unobstructed. In particular \(q(\vartheta) = 0 \in H^2(Y, \Theta_Y) \cong T^2_Y\), where \(q : T^1_Y \to T^2_Y\) is the obstruction map introduced in Theorem 1.3 Then, by (62), also the induced 1st–order deformation of \(\phi\) is unobstructed and there exists an effective deformation \(X \xrightarrow{\phi} Y\) generating an effective deformation of \(Y \xrightarrow{\phi} \overline{Y}\), which is a point \(y \in S\) and a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \overline{Y} \\
\downarrow p \downarrow \Phi & & \downarrow q \\
Y & = & \overline{Y}
\end{array}
\]
with \(p\) and \(q\) flat morphisms and
\[
Y = p^{-1}(y) \quad \overline{Y} = q^{-1}(y) \quad \text{and} \quad \phi = \Phi_{|p^{-1}(y)}.
\]
Proof of Lemma 4.21. Consider \( p \in P' := \text{Sing}(\overline{\mathcal{Y}}) \setminus \{\text{nodes}\} \) and let \( T_{U_p}^1 = H^0(\overline{U}_p, \Theta_{\overline{U}_p}^1) \) be the tangent space to the Kuranishi space \( \text{Def} \overline{U}_p \) of versal deformation of \( \overline{U}_p \) (notation as in Definition 4.20). Then the locus of 1st-order deformations deforming \( p \) to a node is described by an hypersurface \( \mathcal{L}_p \subset T_{U_p}^1 \) while the locus of 1st-order deformations deforming \( p \) to a worse singularity than a node is described by a closed sub-scheme \( \mathcal{V}_p \subset \mathcal{L}_p \).

Let us postpone the proof of this Lemma to conclude the proof of Theorem 4.20. At this purpose denote \( l_p \) the line through the origin generated in \( T_{U_p}^1 \) by the non-trivial class \( \overline{\mathcal{V}}_p \). Since \( \mathcal{L}_p \) is an hypersurface, for any \( p \in P' \) there is a point \( x_p \in l_p \cap \mathcal{L}_p \neq 0 \) such that in general

\[
x := \sum_{p \in P'} x_p \in T_{U_p}^1
\]

turns out to define a point of the parameter space of the 1st-order deformation of \( \overline{\mathcal{Y}} \) defined by \( \delta_1(\vartheta) \), whose associated fiber admits only nodal singularities. \( \square \)

Remark 4.22. In a non general case it may happen that \( l_p \cap \mathcal{L}_p \subseteq \mathcal{V}_p \), for some \( p \in P' \). Then:

- (a) either \( l_p \cap \mathcal{L}_p = l_p \cap \mathcal{V}_p = \{0\} \subset T_{U_p}^1 \), which is that the 1st-order deformation defined by \( \overline{\mathcal{V}}_p \) is trivial contradicting that \( \overline{\mathcal{V}}_p \neq 0 \),
- (b) or \( l_p \cap \mathcal{V}_p \cap \{0\} \neq \emptyset \), meaning that the 1st-order deformation defined by \( \overline{\mathcal{V}}_p \) deforms \( p \) to a milder adjacent singularity, in the sense of Arnold’s classification (3 §1.1.0).

Therefore the generality hypothesis in the statement of Theorem 4.21 could be dropped if it would be possible to guarantee non-rigidity hypothesis like (59) for milder singularities obtained in (b) giving the possibility of establishing an induction process. Unfortunately at the moment I wasn’t able to prove such iterated non-rigidity conditions.

Proof of Lemma 4.21 Recall that \( \overline{U}_p = \text{Spec}(\mathcal{O}_{F,p}) \) where the ideal generator \( F \) is described by (10) and (11). Let \( J_F := (\partial_x F, \partial_y F, \partial_z F, \partial_t F) \) be the Jacobian ideal of \( F \). Since \( p \) is an hypersurface singularity

\[
T_{U_p}^1 = \mathcal{O}_{F,p}/J_F = \mathcal{O}_0/(F, \partial_x F, \partial_y F, \partial_z F, \partial_t F)
\]

where \( \mathcal{O}_0 \) is the localization of \( \mathcal{O} := \mathbb{C}[x, y, z, t] \) at the maximal ideal \( m_0 = (x, y, z, t) \). Given \( \Lambda \in \mathcal{O} \) let \( Y_\Lambda \) be the deformed hypersurface associated with the class \( [\Lambda] \in T_{U_p}^1 \) and locally defined by

\[
Y_\Lambda = \{ F_\Lambda := F + \Lambda = 0 \}.
\]
To require $Y_{\Lambda}$ to be singular means to require the jacobian rank not to be maximum, looking for common solutions of the following conditions

\[(65) \quad \partial_x F + \partial_x \Lambda = \partial_y F + \partial_y \Lambda = \partial_z F + \partial_z \Lambda = \partial_t F + \partial_t \Lambda = 0\]

Let $p_{\Lambda} = (x_{\Lambda}, y_{\Lambda}, z_{\Lambda}, t_{\Lambda})$ be such a solution, which clearly always exists and depends on $[\Lambda] \in T^1_{U_p}$. Then $p_{\Lambda} \in \text{Sing}(Y_{\Lambda})$ if and only if $p_{\Lambda}$ is actually a point of $Y_{\Lambda}$, giving a condition on $[\Lambda] \in T^1_{U_p}$ defining a hypersurface $\mathcal{L}_p \subset T^1_{U_p}$ i.e.

\[(66) \quad p_{\Lambda} \in Y_{\Lambda} \iff [\Lambda] \in \mathcal{L}_p .\]

In general $p_{\Lambda}$ is a node of $Y_{\Lambda}$. In fact, after the translation $x \mapsto x' + x_{\Lambda}, y \mapsto y' + y_{\Lambda}, z \mapsto z' + z_{\Lambda}, t \mapsto t' + t_{\Lambda}$, to require that the origin is a worse singular point than a node means to impose conditions on the degree 2 monomials of $F_{\Lambda}(x', y', z', t')$ which are further conditions on $[\Lambda]$ defining closed sub–schemes of $\mathcal{L}_p$ i.e.

\[(67) \quad p_{\Lambda} \in \text{Sing}(Y_{\Lambda}) \setminus \{\text{nodes}\} \iff [\Lambda] \in V_p \subset \mathcal{L}_p , \quad \text{codim}_{T^1_{U_p}} V_p \geq 2 .\]

\[\square\]

4.6. Example: the case of Arnol’d’s simple singularities. A priori it is not easy to understand which concrete geometric example may satisfy the non–rigidity condition (59) in Theorem 4.20. At this purpose let us recall here that, by a result of Y. Namikawa ([52] Theorem B and Lemma (5.7)), the non–rigidity hypothesis (59) is satisfied by Arnol’d’s simple threefold singularities worse than nodes ([3], [4] §15). Therefore we get the following

**Corollary 4.23.** A general small g.t. $T(Y, \overline{Y}, \tilde{Y})$, such that $\text{Sing}(\overline{Y})$ is composed only by Arnol’d’s simple singularities, admits a conifold degeneration.

To give a more concrete idea of the meaning of word “general” in statements of Theorem 4.20 and Corollary 4.23 let us now describe conditions (66) and (67) in the case of Arnol’d’s simple threefold singularities.

In the following space (64) will be simply denoted by $T^1$, forgetting the subscript $U_p$. When needed either its dimension or the Tyurina and Milnor numbers of a “deformed Arnol’d’s singularity” $p_{\Lambda}$ will be computed by a Gröbner basis calculation to study its adjacency w.r.t. the original Arnol’d singular point. More precisely if $I \subset \mathbb{C}[x_1, \ldots, x_r]$ is an ideal and $HT(I)$ is the ideal generated by higher monomials of $F_{\Lambda}(x', y', z', t')$ with respect to a fixed term order, of elements in $I$ (notation as in [7] §5), it is a well known fact that the following quotient $\mathbb{C}$–vector spaces are isomorphic

\[(68) \quad \mathbb{C}[x_1, \ldots, x_r]/I \cong_{\text{as } \mathbb{C}-\text{vector spaces}} \mathbb{C}[x_1, \ldots, x_r]/HT(I) .\]

Therefore their dimension has to be equal and that on the right is quite easier to compute. The point is then determining a Gröbner basis of $I$ w.r.t. the fixed term order which can be realized e.g. by the “Groebner Package” of MAPLE: with L. Terracini we performed the MAPLE-11 subroutine [70] to compute the dimension of the second quotient $\mathbb{C}$–vector space in (68) when $I$ is either the Jacobian ideal $J_F$ or the ideal $(F) + J_F$ of a polynomial $F$. 
4.6.1. \( (A_n , \, n \geq 1) \) : \( F(x, \, y, \, z, \, t) = x^2 + y^2 + z^{n+1} + t^2 \).

Formula 64 gives
\[
T^1 \cong \mathbb{C}[x, \, y, \, z, \, t]/(x, \, y, \, z^n, \, t) \cong \langle 1, \, z, \ldots, \, z^{n-1} \rangle_{\mathbb{C}}.
\]

Given \( \Lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in T^1 \), the associated deformation of \( Y \) is
\[
Y_\Lambda = \{ F_\Lambda(x, \, y, \, z, \, t) := F(x, \, y, \, z, \, t) + \sum_{i=0}^{n-1} \lambda_i z^i = 0 \}.
\]

A solution of the jacobian system (65) is then given by \( p_\Lambda = (0, \, 0, \, z_\Lambda, \, 0) \) with
\[
(69) \quad z_\Lambda : (n + 1)z_\Lambda^n + \sum_{i=1}^{n-1} i\lambda_i z_\Lambda^{i-1} = 0.
\]

Condition (66) becomes then the following
\[
(70) \quad p_\Lambda \in Y_\Lambda \iff \Lambda \in \mathcal{L} := \{ z_\Lambda^{n+1} + \sum_{i=0}^{n-1} \lambda_i z_\Lambda^i = 0 \} \subset T^1.
\]

After translating \( z \mapsto z + z_\Lambda \), we get
\[
F_\Lambda(x, \, y, \, z + z_\Lambda, \, t) = F(x, \, y, \, z, \, t) + \left( z_\Lambda^{n+1} + \sum_{i=0}^{n-1} \lambda_i z_\Lambda^i \right) + \left( (n + 1)z_\Lambda^n + \sum_{i=1}^{n-1} i\lambda_i z_\Lambda^{i-1} \right) z + \sum_{k=2}^{n-1} \binom{n+1}{k} z_\Lambda^{n+1-k} + \sum_{i=k}^{n-1} \binom{i}{k} \lambda_i z_\Lambda^{i-k} \right) z^k + (n + 1)z_\Lambda z^n.
\]

Then by (70) and (69) the origin (and then \( p_\Lambda \in Y_\Lambda \)) is at least a node. It is actually a worse singularity if and only if one or more of the following further conditions on \( \Lambda \) hold
\[
(71) \quad \begin{align*}
\binom{n+1}{k} z_\Lambda^{n+1-k} + \sum_{i=k}^{n-1} \binom{i}{k} \lambda_i z_\Lambda^{i-k} &= 0, \quad k = 2, \ldots, n-1 \quad (\text{for } n \geq 3) \\
z_\Lambda &= 0.
\end{align*}
\]

Then the closed sub–scheme \( V \subset \mathcal{L} \) defined in (67) turns out to be, in the present \( A_n \) case, the following union of \( n - 1 \), codimension 1 sub–schemes of \( \mathcal{L} \)
\[
V = \bigcup_{k=2}^{n} V_k \quad \text{where}
\]
\[
V_k := \left\{ \binom{n+1}{k} z_\Lambda^{n+1-k} + \sum_{i=k}^{n-1} \binom{i}{k} \lambda_i z_\Lambda^{i-k} = 0 \right\}, \quad \text{for } k = 2, \ldots, n-1,
\]
\[
V_n := \{ z_\Lambda = 0 \}.
\]
Precisely \( p_\Lambda \) turns out to be a \( A_m \) \((2 \leq m \leq n)\) simple hypersurface singularity if and only if \( \Lambda \) is the generic element of \( \bigcap_{k=2}^n V_k \), respecting the Arnol’d’s adjacency diagram

\[
A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_m \leftrightarrow \cdots A_n .
\]

4.6.2. \((D_n, n \geq 4)\) : \( F(x, y, z, t) = x^2 + y^2z + z^{n-1} + t^2 \).

Formula (64) gives

\[ T^1 \equiv \mathbb{C}[y, z]/(yz, y^2 + (n-1)z^{n-2}) \cong (1, y, z, \ldots, z^{n-2})_{\mathbb{C}} . \]

Given \( \Lambda = (\lambda_0, \lambda, \lambda_1, \ldots, \lambda_{n-2}) \in T^1 \), the associated deformation of \( Y \) is

\[ Y_\Lambda = \{ F_\Lambda(x, y, z, t) := F(x, y, z, t) + \lambda y + \sum_{i=0}^{n-2} \lambda_i z^i = 0 \} . \]

A solution of the jacobian system (65) is then given by \( p_\Lambda = (0, y_\Lambda, z_\Lambda, 0) \) with

\[
y_\Lambda, z_\Lambda : \begin{cases} 
2y_\Lambda z_\Lambda + \lambda \equiv 0 \\
(n - 1)z_\Lambda^{n-2} + y_\Lambda^2 + \sum_{i=1}^{n-2} i\lambda_i z_\Lambda^{i-1} \equiv 0
\end{cases}
\]

Condition (66) becomes then the following

\[
p_\Lambda \in Y_\Lambda \iff \Lambda \in \mathcal{L} := \{ y_\Lambda^2 z_\Lambda + z_\Lambda^{n-1} + \lambda y_\Lambda + \sum_{i=0}^{n-2} \lambda_i z_\Lambda^i = 0 \} \subset T^1 .
\]

After translating \( y \mapsto y + y_\Lambda, z \mapsto z + z_\Lambda \), we get

\[
F_\Lambda(x, y + y_\Lambda, z + z_\Lambda, t) = F + \left( y_\Lambda^2 z_\Lambda + z_\Lambda^{n-1} + \lambda y_\Lambda + \sum_{i=0}^{n-2} \lambda_i z_\Lambda^i \right) \\
+ (2y_\Lambda z_\Lambda + \lambda) y \\
+ \left( (n - 1)z_\Lambda^{n-2} + y_\Lambda^2 + \sum_{i=1}^{n-2} i\lambda_i z_\Lambda^{i-1} \right) z \\
+ 2y_\Lambda yz + z_\Lambda y^2 \\
+ \sum_{k=2}^{n-2} \left( \binom{n-1}{k} z_\Lambda^{n-1-k} + \sum_{i=k}^{n-2} \binom{i}{k} \lambda_i z_\Lambda^{i-k} \right) z^k
\]

Then by (73) and (72) the origin (and then \( p_\Lambda \in Y_\Lambda \)) is at least a node. Let us define

\[
\nu_1 := 2y_\Lambda \\
\nu_k := \binom{n-1}{k} z_\Lambda^{n-1-k} + \sum_{i=k}^{n-2} \binom{i}{k} \lambda_i z_\Lambda^{i-k} , k = 2, \ldots, n - 2 \\
\nu_{n-1} := z_\Lambda
\]

and consider the associated codimension 1 sub–schemes of \( \mathcal{L} \)

\[
\mathcal{V}_k := \{ \nu_k = 0 \} , \quad 1 \leq k \leq n - 1 \\
\mathcal{V}' := \{ \nu_1^2 - \nu_2 \nu_{n-1} = 0 \} = \left\{ y_\Lambda^2 - \frac{(n - 1)}{2} z_\Lambda^{n-2} - \sum_{i=2}^{n-2} \binom{i}{2} \lambda_i z_\Lambda^{i-1} = 0 \right\} .
\]
By computing the Tyurina number\footnote{Apply \cite{70} in the origin for the polynomial} of $p_\Lambda$ one finds that the closed sub–scheme $\mathcal{V}$ of $L$ defined in (67) is reducible and admits two codimension 1 irreducible components as follows

$$\mathcal{V} = \mathcal{V}_{n-1} \cup \mathcal{V}' \subset L \subset T^1.$$ 

In particular we get

- $p_\Lambda$ is of type $A_2$ for $\Lambda$ generic in $\mathcal{V}_{n-1} \cup \mathcal{V}'$
- $p_\Lambda$ is of type $A_3$ for $\Lambda$ generic in $(\mathcal{V}_1 \cap \mathcal{V}_{n-1})$
- $p_\Lambda$ is of type $D_4$ for $\Lambda$ generic in $(\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_{n-1})$
- $p_\Lambda$ is of type $D_m$ for $\Lambda$ generic in $(\mathcal{V}_1 \cap \mathcal{V}_2 \cap \cdots \cap \mathcal{V}_{m-2} \cap \mathcal{V}_{n-1})$ (for $4 \leq m \leq n$)
- $p_\Lambda$ is of type $A_l$ for $\Lambda$ generic in $(\mathcal{V}_1 \cap \mathcal{V}_2 \cap \cdots \cap \mathcal{V}_l)$ (for $3 \leq l \leq n-2$)

respecting the following Arnol’d’s adjacency diagram

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$$

$$\cdots \leftarrow D_4 \leftarrow D_5 \leftarrow \cdots \leftarrow D_m \leftarrow \cdots \leftarrow D_n$$

4.6.3. $(E_6)$ : $F(x, y, z, t) = x^2 + y^3 + z^4 + t^2$.

Formula (64) gives

$$T^1 \cong \mathbb{C}[y, z]/(y^2, z^3) \cong \langle 1, y, z, z^2, yz^2 \rangle \mathbb{C}.$$

Given $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_5) \in T^1$, the associated deformation of $Y$ is

$$Y_\Lambda = \{ F_\Lambda(x, y, z, t) = 0 \} \quad \text{where}$$

$$F_\Lambda(x, y, z, t) := F(x, y, z, t) + \lambda_0 + \lambda_1 y + \lambda_2 z + \lambda_3 yz + \lambda_4 z^2 + \lambda_5 yz^2$$

A solution of the jacobian system (65) is then given by $p_\Lambda = (0, y_\Lambda, z_\Lambda, 0)$ with

$$y_\Lambda, z_\Lambda : \begin{cases} 3y^2_\Lambda + \lambda_1 + \lambda_3 z_\Lambda + \lambda_5 z^2_\Lambda \equiv 0 \\ 4z^3_\Lambda + 2\lambda_4 z_\Lambda + \lambda_2 + y_\Lambda(\lambda_3 + 2\lambda_5 z_\Lambda) \equiv 0 \end{cases}$$

Condition (66) becomes then the following

$$p_\Lambda \in Y_\Lambda \iff \Lambda \in \mathcal{L} \quad \text{where}$$

$$\mathcal{L} := \{ y^3_\Lambda + z^2_\Lambda + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda + \lambda_4 z^2_\Lambda + \lambda_5 y_\Lambda z^2_\Lambda = 0 \} \subset T^1.$$
After translating $y \mapsto y + y_A, z \mapsto z + z_A$, we get

$$F_A(x, y + y_A, z + z_A, t) = F(x, y, z, t) + \left( y_A^3 + z_A^3 + \lambda_0 + \lambda_1 y_A + \lambda_2 z_A + \lambda_3 y_A z_A + \lambda_4 z_A^2 + \lambda_5 y_A z_A^2 \right) + (3y_A^2 + \lambda_1 + \lambda_3 z_A + \lambda_5 z_A^2) y + (4z_A^3 + 2\lambda_4 z_A + \lambda_2 + y_A(\lambda_3 + 2\lambda_5 z_A)) z + (\lambda_3 + 2\lambda_5 z_A) y z + 3y_A y^2 + (6z_A^2 + \lambda_4 + 5y_A) z^2 + 4z_A^3 z + 5y_A y^2$$

Then by (75) and (74) the origin (and then $p_A \in Y_A$) is at least a node. By setting $\nu_1 = \lambda_3 + 2\lambda_5 z_A, \nu_2 = 6z_A^2 + \lambda_4 + 5y_A, \nu_3 = 3y_A, \nu_4 = 4z_A, \nu_5 = 5$ define the following codimension 1 sub–schemes of $\mathcal{L}$

$$\mathcal{V}_k := \{ \nu_k = 0 \}, \quad 1 \leq k \leq 5$$
$$\mathcal{V}' := \{ \nu_5^2 - 4\nu_3 = 0 \} = \{ \lambda_3^2 - 12y_A = 0 \}$$
$$\mathcal{V}'' := \{ 27\nu_4^2 + 4\nu_3 = 0 \} = \{ \lambda_3^2 + 108z_A^2 = 0 \}$$

By computing the Tyurina number of $p_A$, we get that the closed sub–scheme of $\mathcal{L}$ defined in (67) is given by

$$\mathcal{V} := \{ \nu_3^2 - 4\nu_2 \nu_3 = 0 \} = \{ 4\lambda_5^2 z_A^2 + 4\lambda_3 \lambda_5 z_A + \lambda_3^2 - 72z_A^2 y_A - 12\lambda_5 z_A^2 - 12\lambda_4 y_A = 0 \}.$$ 

Clearly $\bigcap_{k=1}^5 \mathcal{V}_k = \{ 0 \} \subset \mathcal{L} \subset T^1$ giving $p_A = 0$ and the trivial deformation. Moreover

- $p_A$ is of type $A_2$ for $\Lambda$ generic in $\mathcal{V} \supset (\mathcal{V}_1 \cap \mathcal{V}_2) \cup (\mathcal{V}_1 \cap \mathcal{V}_3)$
- $p_A$ is of type $A_3$ for $\Lambda$ generic in $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_4$
- $p_A$ is of type $D_4$ for $\Lambda$ generic in $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3$
- $p_A$ is of type $D_5$ for $\Lambda$ generic in $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \cap \mathcal{V}''$.

Notice that if $\Lambda$ is the generic element of $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \cap \mathcal{V}''$ then $p_A$ may be a singular point of type $A_4$, with $n > 3$, as follows by applying e.g. the standard local analysis described in [5] II.8. All these relations respect the following Arnol’d’s adjacency

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7Let us give a brief account of what happens in this case, since it admits an easy explicit description. The subvariety $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_4 \cap \mathcal{V}'' \subset \mathcal{L} \subset T^1 \cong \mathbb{C}^6$ is 1-dimensional and rational, admitting the following parameterization $\Lambda = (\frac{2\lambda_6}{27}, -\frac{\lambda_4}{3}, 0, 0, -\frac{2\lambda_3}{3}, 2\lambda)$, and $Y_\Lambda$ has the unique singular point $p_A = (0, \frac{2\lambda_4}{3}, 0, 0)$. After the translation $y \mapsto y + \frac{2\lambda_4}{3}$, sending $p_A$ to the origin, $Y_\Lambda$ turns out to be the 1–parameter deformation of $Y$ given by

$$F_A = x^2 + y^3 + z^3 + t^2 + 2 \lambda t z^2 + 2 \lambda y z^2 = 0.$$ 

For $\lambda \neq 0$, the Tyurina number of the singularity in the origin is

$$\dimc (\mathcal{O}_0/(F_A + J_{F_A})) = \dimc (\mathbb{C}[x, y, z]/(y^2, yz^2, z^3)) = \dimc (1, y, z, y^2, z^3) = 5.$$

since $(x, t, y^2, yz^2, z^3) = HT((F_A) + J_{F_A})$, being

$$\mathcal{G} = \{ t, x, 3y^2 + 2\lambda t y + 2\lambda y z, -2\lambda y z + 2\lambda z^2 z + 3yz^2 \}$$

a Gröbner basis of $(F_A) + J_{F_A}$ (apply [10]). Hence $p_A$ is of type $A_5$.

Alternatively the standard procedure of [5] II.8 can be applied to the singular curve $y^2 + x^4 + \lambda^2 y^2 + 2\lambda y z^2 = 0$, to get the normal form $y^2 + z^2 = 0$ and then, as above, for $\lambda \neq 0$, $p_A$ turns out to be a simple $A_5$ singularity.
4.6.4. \((E_7)\) : \(F(x, y, z, t) = x^2 + y^3 + yz^3 + t^2\).

Formula \((64)\) gives
\[
T^1 \cong \mathbb{C}[y, z]/(3y^2 + z^3, yz^2) \cong \langle 1, y, z, yz, z^3, y^4 \rangle_{\mathbb{C}}.
\]

Given \(\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_6) \in T^1\), the associated deformation of \( Y \) is
\[
Y_\Lambda = \{ F_\Lambda(x, y, z, t) = 0 \} \quad \text{where}
\]
\[
F_\Lambda(x, y, z, t) := F(x, y, z, t) + \lambda_0 + \lambda_1 y + \lambda_2 z + \lambda_3 yz + \lambda_4 z^2 + \lambda_5 z^3 + \lambda_6 z^4.
\]

A solution of the jacobian system \((65)\) is then given by \(p_\Lambda = (0, y_\Lambda, z_\Lambda, 0)\) with
\[
(76) \quad y_\Lambda, z_\Lambda : \begin{cases} 
3y_\Lambda^2 + z_\Lambda^3 + \lambda_1 + \lambda_3 z_\Lambda \equiv 0 \\
3y_\Lambda z_\Lambda^2 + \lambda_2 + 3\lambda_3 y_\Lambda + 2\lambda_4 z_\Lambda + 3\lambda_5 z_\Lambda^2 + 4\lambda_6 z_\Lambda^3 \equiv 0
\end{cases}
\]

Condition \((66)\) becomes then the following
\[
(77) \quad p_\Lambda \in Y_\Lambda \iff \Lambda \in \mathcal{L} \subset T^1 \quad \text{where}
\]
\[\mathcal{L} := \{ y_\Lambda^3 + y_\Lambda z_\Lambda^3 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + 3\lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 z_\Lambda^3 + \lambda_6 z_\Lambda^4 = 0 \}\]

After translating \( y \mapsto y + y_\Lambda, z \mapsto z + z_\Lambda\), we get
\[
F_\Lambda(x, y + y_\Lambda, z + z_\Lambda, t) =
\]
\[
F(x, y, z, t) + \left( y_\Lambda^3 + y_\Lambda z_\Lambda^3 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + 3\lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 z_\Lambda^3 + \lambda_6 z_\Lambda^4 \right)
+ \left( 3y_\Lambda^2 + z_\Lambda^3 + \lambda_1 + \lambda_3 z_\Lambda \right) y
+ \left( 3y_\Lambda z_\Lambda^2 + \lambda_2 + 3\lambda_3 y_\Lambda + 2\lambda_4 z_\Lambda + 3\lambda_5 z_\Lambda^2 + 4\lambda_6 z_\Lambda^3 \right) z
+ \left( 3z_\Lambda^3 + \lambda_3 \right) yz + \left( 3y_\Lambda z_\Lambda^2 + \lambda_4 + 3\lambda_5 z_\Lambda + 6\lambda_6 z_\Lambda^2 \right) z^2
+ 3z_\Lambda yz^2 + (\lambda_5 + 4\lambda_6 z_\Lambda + y_\Lambda) z^3 + \lambda_6 z^4.
\]

Then by \((77)\) and \((70)\) the origin (and then \( p_\Lambda \in Y_\Lambda \)) is at least a node. As in the previous \(E_6\) case, set
\[
\nu_1 = 3z_\Lambda^2 + \lambda_3 , \quad \nu_2 = 3y_\Lambda z_\Lambda + \lambda_4 + 3\lambda_5 z_\Lambda + 6\lambda_6 z_\Lambda^2 , \quad \nu_3 = 3y_\Lambda ,
\nu_4 = 3z_\Lambda , \quad \nu_5 = \lambda_5 + 4\lambda_6 z_\Lambda + y_\Lambda , \quad \nu_6 = \lambda_6 ,
\]
and define the following codimension 1 sub-schemes of \( \mathcal{L} \)
\[
\begin{array}{ll}
Y_k & := \{ \nu_k = 0 \} , \quad 1 \leq k \leq 6 \\
\mathcal{V}' & := \{ \nu_2^2 - 4\nu_3\nu_6 = 0 \} = \{ 3z_\Lambda^2 - 4\lambda_6 y_\Lambda = 0 \} \\
\mathcal{V}'' & := \{ 27\nu_5^2 + 4\nu_6^2 = 0 \}
\end{array}
\]
\[
\begin{array}{ll}
= \{ 108z_\Lambda^4 + 432\lambda_6 z_\Lambda^2 + 216\lambda_6 y_\Lambda z_\Lambda + 27y_\Lambda^2 + 216\lambda_5 \lambda_6 z_\Lambda + 54\lambda_5 y_\Lambda + 27\lambda_6^2 = 0 \}
\end{array}
\]
The closed sub–scheme of $\mathcal{L}$ defined in (67), is still given by

\[ V := \{ \nu_1^2 - 4\nu_2\nu_3 = 0 \} = \{ 9z^4 - 72\lambda_6y_\Lambda z_\Lambda^2 - 36y_\Lambda^2z_\Lambda + 6\lambda_3z_\Lambda^2 - 36\lambda_5y_\Lambda z_\Lambda - 12\lambda_4y_\Lambda + \lambda_3^2 = 0 \} . \]

Clearly $\bigcap_{k=1}^{6} V_k = \{0\} \subset \mathcal{L} \subset T^1$ giving $p_\Lambda = 0$ and the trivial deformation. Moreover

- $p_\Lambda$ is of type $A_2$ for $\Lambda$ generic in $V \supset (V_1 \cap V_2) \cup (V_1 \cap V_3)$
- $p_\Lambda$ is of type $A_3$ for $\Lambda$ generic in $V_1 \cap V_2 \cap V_5$
- $p_\Lambda$ is of type $A_4$ for $\Lambda$ generic in $V_1 \cap V_2 \cap V_5 \cap V'$
- $p_\Lambda$ is of type $D_4$ for $\Lambda$ generic in $V_1 \cap V_2 \cap V_3$
- $p_\Lambda$ is of type $D_5$ for $\Lambda$ generic in $V_1 \cap V_2 \cap V_3 \cap V''$
- $p_\Lambda$ is of type $E_6$ for $\Lambda$ generic in $V_1 \cap V_2 \cap V_3 \cap V_4 \cap V_5$.

All these relations respect the following Arnol’d’s adjacency diagram

```
A1 ←← A2 ←← A3 ←← ···
```

```
D4 ←← D5
```

```
E6 ←← E7
```

4.6.5. $(E_6)$ : $F(x, y, z, t) = x^2 + y^3 + z^5 + t^2$.

Formula (64) gives

\[ T^1 \cong C[y, z]/(y^2, z^4) \cong \langle 1, y, z, yz, z^2, y^2, z^3, yz^3 \rangle_C . \]

Given $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_7) \in T^1$, the associated deformation of $Y$ is

\[ Y_\Lambda = \{ F_\Lambda(x, y, z, t) = 0 \} \quad \text{where} \quad F_\Lambda(x, y, z, t) := F(x, y, z, t) + \lambda_0 + \lambda_1y + \lambda_2z + \lambda_3yz + \lambda_4z^2 + \lambda_5y^2 + \lambda_6z^3 + \lambda_7yz^3 \]

A solution of the jacobian system (65) is then given by $p_\Lambda = (0, y_\Lambda, z_\Lambda, 0)$ with

\[ y_\Lambda, z_\Lambda : \left\{ \begin{array}{l}
3y_\Lambda^3 + \lambda_1 + \lambda_3z_\Lambda + \lambda_5z_\Lambda^2 + \lambda_7z_\Lambda^3 \equiv 0 \\
5z_\Lambda^4 + \lambda_2 + 2\lambda_4z_\Lambda + 3\lambda_6z_\Lambda^2 + y_\Lambda(\lambda_3 + 2\lambda_5z_\Lambda + 3\lambda_7z_\Lambda^2) \equiv 0
\end{array} \right. \]

Condition (66) becomes then the following

\[ p_\Lambda \in Y_\Lambda \iff \Lambda \in \mathcal{L} \subset T^1 \quad \text{where} \quad \mathcal{L} := \{ y_\Lambda^3 + z_\Lambda^2 + \lambda_0 + \lambda_1y_\Lambda + \lambda_2z_\Lambda + \lambda_3y_\Lambda z_\Lambda + \lambda_4z_\Lambda^2 + \lambda_5y_\Lambda z_\Lambda + \lambda_6z_\Lambda^2 + \lambda_7y_\Lambda z_\Lambda = 0 \} \]
After translating \( y \mapsto y + y_A, z \mapsto z + z_A \), we get

\[
F_{\lambda}(x, y + y_A, z + z_A, t) = F(x, y, z, t) + \\
\left( y_1^3 + z_1^3 + \lambda_0 + \lambda_1 y_1 A + \lambda_2 z_1 + \lambda_3 y_1 z_1 + \lambda_4 z_1^2 + \lambda_5 y_1 z_1^2 + \lambda_6 z_1^3 + \lambda_7 y_1 z_1^3 \right) \\
+ \left( 3y_1^2 + \lambda_1 + \lambda_3 z_1 + \lambda_5 z_1^2 + \lambda_7 z_1^3 \right) y \\
+ \left( 5z_1^4 + \lambda_2 + 2\lambda_4 z_1 + 3\lambda_6 z_1^2 + y_A(\lambda_3 + 2\lambda_5 z_1 + 3\lambda_7 z_1^3) \right) z \\
+ \left( \lambda_3 + 2\lambda_5 z_1 + 3\lambda_7 z_1^3 \right) yz + y_A y^2 + (10z_1^4 + \lambda_4 + \lambda_5 y_A + 3\lambda_6 z_1 + 3\lambda_7 y_A z_1) z^2 \\
+ \left( \lambda_5 + 3\lambda_7 z_1 \right) yz^2 + (10z_1^4 + \lambda_6 + \lambda_7 y_A) z^3 + \lambda_7 y^3 + 5z_A z^4
\]

Then by (74) and (75) the origin (and then \( p_A \in Y_A \)) is at least a node. As usual set

\[
\nu_1 = \lambda_3 + 2\lambda_5 z_1 + 3\lambda_7 z_1^3, \quad \nu_2 = 10z_1^4 + \lambda_4 + \lambda_5 y_A + 3\lambda_6 z_1 + 3\lambda_7 y_A z_1, \\
\nu_3 = 3y_A, \quad \nu_4 = \lambda_5 + 3\lambda_7 z_1, \quad \nu_5 = 10z_1^4 + \lambda_6 + \lambda_7 y_A, \\
\nu_6 = \lambda_7, \quad \nu_7 = 5z_A,
\]

and define the following codimension 1 sub–schemes of \( \mathcal{L} \)

\[
\mathcal{V}_k := \{ \nu_k = 0 \}, \quad 1 \leq k \leq 7 \\
\mathcal{V}'' := \{ \nu_1^2 - 4\nu_3 \nu_7 = 0 \} = \{ 9\lambda_7^2 z_1^2 - 60y_A z_1 + 6\lambda_5 \lambda_7 z_1 + \lambda_5^2 = 0 \} \\
\mathcal{V}''' := \{ 27\nu_1^2 + 4\nu_3^2 = 0 \} \\
\quad = \{ 2700z_1^4 + 108\lambda_7^2 z_1^3 + 540\lambda_7 y_A z_1^2 + 108(5\lambda_6 + \lambda_5 \lambda_7^2) z_1^2 + 27\lambda_7 y_A^2 \\
\quad \quad + 36\lambda_5 \lambda_7 z_1 + 54\lambda_6 \lambda_7 y_A + 27\lambda_7^2 + 4\lambda_5^3 = 0 \}
\]

As before, the closed sub–scheme of \( \mathcal{L} \) defined in (67), is given by

\[
\mathcal{V} := \{ \nu_1^2 - 4\nu_2 \nu_3 = 0 \} \\
\quad = \{ 9\lambda_7^2 z_1^2 - 120y_A z_1^2 + 12\lambda_5 \lambda_7 z_1^3 - 36\lambda_7 y_A^2 z_1 + 2(2\lambda_5^2 + 3\lambda_3 \lambda_7) z_1^2 \\
\quad \quad - 36\lambda_6 y_A z_1 - 12\lambda_5 y_A^2 + 4\lambda_3 \lambda_5 z_1 - 12\lambda_4 y_A + \lambda_5^3 = 0 \}.
\]

Clearly \( \bigcap_{k=1}^7 \mathcal{V}_k = \{ 0 \} \subset \mathcal{L} \subset T^1 \) giving \( p_A = 0 \) and the trivial deformation. Moreover

\[
p_A \text{ is of type } A_2 \text{ for } \Lambda \text{ generic in } \mathcal{V} \supset (\mathcal{V}_1 \cap \mathcal{V}_2) \cup (\mathcal{V}_1 \cap \mathcal{V}_3) \\
p_A \text{ is of type } A_3 \text{ for } \Lambda \text{ generic in } \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_5 \\
(p_A \text{ is of type } A_4 \text{ for } \Lambda \text{ generic in } \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_5 \cap \mathcal{V}') \\
p_A \text{ is of type } D_4 \text{ for } \Lambda \text{ generic in } \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \\
p_A \text{ is of type } D_5 \text{ for } \Lambda \text{ generic in } \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_4 \cap \mathcal{V}' \\
(p_A \text{ is of type } D_6 \text{ for } \Lambda \text{ generic in } \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 \cap \mathcal{V}_6 \cap \mathcal{V}_7 \cap \mathcal{V}') \\
p_A \text{ is of type } E_8 \text{ for } \Lambda \text{ generic in } \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_4 \cap \mathcal{V}_5 \cap \mathcal{V}_7. 
\]
All these relations respect the following Arnol’d’s adjacency diagram

\[
\begin{array}{ccccccc}
A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & \cdots \\
\uparrow & & & & & & \\
D_4 & \leftarrow & D_5 & \leftarrow & \cdots \\
\uparrow & & & & & & \\
E_6 & \leftarrow & E_7 & \leftarrow & E_8 & \leftarrow & \cdots \\
\end{array}
\]

5. Composition of geometric transitions

**Proposition 5.1** (Arrows’ composition in the Calabi–Yau web). Let \( \mathcal{M}_1, \mathcal{M}, \mathcal{M}_2 \) be three distinct nodes of the Calabi–Yau web, i.e. three moduli spaces of topologically distinct Calabi–Yau threefolds, and assume there exist arrows \( \mathcal{M}_1 \overset{a_1}{\rightarrow} \mathcal{M} \) and \( \mathcal{M} \overset{a_2}{\rightarrow} \mathcal{M}_2 \). Then there exists an arrow \( \mathcal{M}_1 \overset{a}{\rightarrow} \mathcal{M}_2 \).

By **Proposition 2.3** it makes than sense to call \( a \) the composition of \( a_1 \) and \( a_2 \), write \( a = a_2 \circ a_1 \) and say commutative the following sub–graph of the Calabi–Yau web (81)

\[
\begin{array}{ccc}
\mathcal{M}_1 & \overset{a_1}{\rightarrow} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{M} & \overset{a_2}{\rightarrow} & \mathcal{M}_2
\end{array}
\]

Let us postpone the proof of the previous statement to set the following

**Definition 5.2** (Composition of geometric transitions). Let \( T_i, i = 1, 2 \), be g.t.’s whose equivalence classes \( a_i := [T_i]_\sim \) give consecutive arrows as in the sub–graph (81). Then any representative \( T \) of the arrow \( a = a_2 \circ a_1 \) is called a composition of \( T_1 \) and \( T_2 \) (write \( T = T_2 \circ T_1 \)).

**Remark 5.3.** By the previous **Definition 5.2** it follows that the composition of two given g.t.’s is not univocally determined but only defined up to analytic equivalence.

**Proof of Proposition 5.1.** Let

\[
Y_1 \overset{\phi_1}{\rightarrow} \tilde{Y}_1 \overset{\sim}{\leftarrow} Y_1 \\
\downarrow_{T_1} \quad \downarrow_{\sim} \quad \downarrow_{T_1}
\]

be a representative of \( a_i, i = 1, 2 \). First of all observe that \( \tilde{Y}_1 \) and \( Y_2 \) are smooth Calabi–Yau threefolds in \( \mathcal{M} \) giving \( \tilde{Y}_1 \sim Y_2 \) by recalling analytic equivalence (1.3). On the other hand \( \tilde{Y}_1 \sim Y_1 \) for the g.t. \( T_1 \). Transitivity then gives \( \tilde{Y}_1 \sim Y_2 \) meaning that there exists a deformation family \( Y \overset{f}{\rightarrow} B \) and points \( b_1, b_2 \in B \) such that \( \tilde{Y}_1 = f^{-1}(b_1) \) and \( Y_2 = f^{-1}(b_2) \). Recall then the isomorphism (61) to conclude that the deformation family \( Y \) naturally induces a morphism deformation \( \Phi \) of the birational contraction \( Y_2 \overset{\phi_2}{\rightarrow} \tilde{Y}_2 \) and a deformation family \( \tilde{Y} \overset{f}{\rightarrow} B \) fitting into the following commutative diagram

\[
\begin{array}{ccc}
Y & \overset{\Phi}{\rightarrow} & \tilde{Y} \\
\downarrow_{f} & & \downarrow_{f} \\
\sim & \rightarrow & \sim
\end{array}
\]
and such that \( f^{-1}(b_2) = \overline{Y}_2 \). Set \( \overline{Y} := f^{-1}(b_1) \) and \( \tilde{\phi}_2 := \Phi|_{\overline{Y}_1} : \overline{Y}_1 \rightarrow \overline{Y} \) and consider the composed birational contraction \( \phi := \tilde{\phi}_2 \circ \phi_1 : Y_1 \rightarrow \overline{Y} \). At last observe that \( \overline{Y} \sim \overline{Y}_2 \) by construction and \( \overline{Y}_2 \sim \tilde{\overline{Y}}_2 \) for the g.t. \( T_2 \). Then by transitivity we get \( \overline{Y} \sim \tilde{\overline{Y}}_2 \) and a g.t.

\[
\begin{array}{c}
Y_1 \xrightarrow{\phi} \overline{Y} \xleftarrow{\sim} \tilde{\overline{Y}}_2
\end{array}
\]

whose equivalence class gives an arrow \( a := [T]_{\sim} : M_1 \rightarrow M_2 \).

\[\square\]

**Proposition 5.4.** Let \( Y \xrightarrow{\phi} \overline{Y} \xleftarrow{\sim} \tilde{Y} \) be a g.t. such that for any factorization

\[
\begin{array}{c}
Y \xrightarrow{\gamma} Z \xrightarrow{\varphi} \overline{Y}
\end{array}
\]

the normal threefold \( Z \) is smoothable. Then \( T \) is the composition of a finite number of primitive g.t.'s.

**Proof.** The DGP Theorem 1.4 ensures the existence of versal families \( Y \) of \( Y \), \( Z \) of \( Z \) and \( \overline{Y} \) of \( \overline{Y} \), which are actually universal families by Theorem 1.6 and the fact that \( Y \) is a Calabi–Yau threefold, in fact

\[
\begin{align*}
T^0_Y &\cong H^0(Y, \Theta_Y) \cong H^0(Y, \Omega_Y^2) = 0 \\
T^0_Z &\cong H^0(Z, \Theta_Z^0) \cong H^0(Y, R^0\gamma_* \Theta_Y) \xrightarrow{\text{Leray}} H^0(Y, \Theta_Y) = 0 \\
T^0_{\overline{Y}} &\cong H^0(\overline{Y}, \Theta_{\overline{Y}}^0) \cong H^0(Y, R^0\phi_* \Theta_Y) \xrightarrow{\text{Leray}} H^0(Y, \Theta_Y) = 0.
\end{align*}
\]

Let \( \tilde{Z} \) be a generic smooth (hence Calabi–Yau) fibre of \( Z \) which exists since \( Z \) is smoothable: then \( T_1(Y, Z, \tilde{Z}) \) is a g.t.. Apply Theorem 3.3 of [59] to the morphism \( \varphi : Z \rightarrow \overline{Y} \) (as in proofs of Theorems 2.4 and 4.20). One gets the following isomorphisms

\[
T^i_Z \cong T^i_{\varphi}, \quad i = 1, 2
\]

analogous to isomorphisms (61) and (62). Therefore an effective deformation admitting fibres \( Z \) and \( \tilde{Z} \) generates an effective deformation \( \Phi \) of \( \varphi \). In particular

\[
\tilde{\varphi} := \Phi|_{\tilde{Z}} : \tilde{Z} \rightarrow \tilde{\overline{Y}}
\]

is a birational contraction with \( \tilde{\overline{Y}} \sim \tilde{Y} \sim \tilde{\overline{Y}} \). By transitivity \( T_2(\tilde{Z}, \tilde{\overline{Y}}, \tilde{\overline{Y}}) \) turns out to be a g.t. and \( T = T_2 \circ T_1 \). Since we may assume \( T_1 \) to be primitive, the statement follows by induction on the relative Picard number \( \rho(\phi) := \rho(Y/\overline{Y}) \).

\[\square\]

**Example 5.5 (Decomposition of the Namikawa g.t. [69 §3.2]).** Consider the Namikawa small and rigid g.t. [69] described in the Example 4.9

\[
\begin{array}{c}
\tilde{X} \xrightarrow{\phi} X \xleftarrow{\sim} \tilde{X}
\end{array}
\]
and consider the universal family $\mathcal{X} \xrightarrow{\sim} D$ of fiber products $S_1 \times_{P^1} S_2$ of rational elliptic surfaces with sections. Let $K \subset D$ be the closed subset parameterizing fiber self-products $S \times_{P^1} S$ of the same rational elliptic surface $S_1 = S = S_2$. For a general $k \in K$ the fiber self-product $X_k = S_k \times_{P^1} S_k$ admits a singular locus $\text{Sing}(X_k)$ composed by 12 nodes\(^8\) and a small resolution $Z_k \xrightarrow{\varphi_k} X_k$ obtained by choosing $Z_k$ as the strict transform of $X_k$ in the “blow up” of the “diagonal locus” of $X_k$ (see [73] Lemma (3.1) and [69] Proposition 2.1.(3) for more details). Let us then denote by $Z \xrightarrow{\sim} K$ the universal family of such small resolutions of fiber self-products of rational elliptic surfaces with sections. If $\tilde{X}$ is a generic smooth fiber of the family $\mathcal{X}$ then the small g.t.

$$Z_k \xrightarrow{\varphi_k} X_k \xrightarrow{\sim} \tilde{X}$$

is a representative of an arrow $a_2 := [T_2] : Z \longrightarrow \mathcal{X}$. Recall now that the Namikawa cuspidal fiber self–product $X$ of a singular rational elliptic surface whose Weierstrass representation is given by (37) can be thought as a special singular fibre

$$\xymatrix{ \overset{X}{\circ} \ar[rr] \ar[d] && \overset{X}{\circ} \ar[d] \ar@{.}[rr] \ar[d] \ar[r] & D \ar[d] \ar@{.}[rr] \ar[d] \ar[r] & K \ar@{.}[rr] \ar[d] \ar[r] & D }$$

such that $\text{Sing}(X)$ is composed by six $cA_2$ singularities, as described in (2) of Example [19]. The small resolution $\tilde{X} \xrightarrow{\phi} X$, described in (3) of Example [19] can then be factorized as follows

$$\xymatrix{ \overset{\tilde{X}}{\circ} \ar[rr] \ar[dr]^{\gamma} && X \ar[dl]_{\varphi_o} \ar[d] \ar[dr]^{\phi} \ar[d] \ar[r] & Z_o \ar[d] \ar[r] & \overset{Z}{\circ} \ar[dr]_{\gamma} \ar[d] \ar[r] & \overset{\tilde{Z}}{\circ} \ar[d] \ar[r] & Z \ar[d] }$$

If $\tilde{Z}$ is a generic smooth fibre of the family $Z$ then the small g.t.

$$\xymatrix{ \overset{\tilde{X}}{\circ} \ar[rr] \ar[r]_{a_1} & \overset{X}{\circ} \ar[r]_{a_2} & \overset{\mathcal{X}}{\circ} \ar[r] & \overset{Z}{\circ} \ar[r] & \overset{\tilde{Z}}{\circ} \ar[r] & Z }$$

is a representative of an arrow $a_1 := [T_1] : \overset{\tilde{X}}{\circ} \longrightarrow Z$ where $\overset{\tilde{X}}{\circ}$ is the universal family of small resolution of cuspidal fiber self–products $X = S \times_{P^1} S$ obtained by varying $B(\lambda) \in H^0(O_{P^1}(6))$ in the Weierstrass representation (37) of $S$. We have then the following commutative sub–graph of the Calabi–Yau web

$$\xymatrix{ \overset{\overset{\tilde{X}}{\circ}}{\circ} \ar[rr]_{a=[T]} \ar[dr]_{a_1} && \overset{X}{\circ} \ar[dl]^{a_2} \ar[dr] \ar[d] \ar[r] & \overset{\mathcal{X}}{\circ} \ar[r] & \overset{Z}{\circ} }$$

giving $a = a_2 \circ a_1$ and $T = T_2 \circ T_1$. In particular observe that:

\(^8\)Points of $\text{Sing}(X_k)$ are associated with roots of the degree 12 discriminant polynomial of the Weierstrass representation of $S_k$.\]
• both the small g.t.’s $T_1$ and $T_2$ are actually conifold transitions giving a decomposition of the Namikawa small and rigid g.t. $T$, with bi–deg$(T) = (2,16)$, as a composition of 2 conifold transitions $T_1$ and $T_2$ with

$$\text{bi–deg}(T_1) = (1,5) \quad \text{and} \quad \text{bi–deg}(T_2) = (1,11).$$

(Bi–degrees can be easily deduced by table (65) of Theorem 3.8 in [69]).

**Remark 5.6.** The previous Example 5.5 underlines that the Namikawa Example 4.9, which is a counterexample against the possible existence of a conifold degeneration of a small g.t., doesn’t work as a counterexample for primitive small g.t.’s, since the relative Picard number $\rho(\hat{X}/X) = 2 > 1$. Moreover the decomposition of the g.t. [35] as a composition of conifold transitions, which turn out to be rigid by Corollary 4.15, opens the way to a deeper understanding of the rigidity of Namikawa counterexample. In fact Theorem 4.4 in [69] asserts that:

\[ (**): \text{Let } X \text{ be a normal threefold with terminal isolated singularities and suppose there exists a commutative diagram} \]

\[
\begin{array}{ccc}
\hat{X} & \phi \rightarrow & X \\
\downarrow^\gamma & & \downarrow^\varphi \\
Z & & 
\end{array}
\]

of small (partial) resolutions of $X$ such that $\hat{X}$ is smooth and $Z$ admits only nodal singularities. Then the differential localization map

$$\lambda : T^1_{\hat{X}} \rightarrow H^0(X, R^1\varphi_*\Theta_{\hat{X}})$$

in the Friedman diagram [42], factorizes as follows

\[
\begin{array}{ccc}
T^1_{\hat{X}} & \lambda \rightarrow & H^0(X, R^1\varphi_*\Theta_{\hat{X}}) \\
\downarrow^\kappa & & \downarrow^\delta_{\text{loc}} \\
H^0(Z, R^1\gamma_*\Theta_{\hat{X}}) = 0 & & \rightarrow T^1_{\hat{X}}
\end{array}
\]

where $\delta$ is an injective map; therefore $\lambda$ is necessarily the 0 map meaning that the (small) exceptional locus $\text{Exc}(\phi)$ turns out to be rigid under global deformations of $\hat{X}$.

It seems then natural to ask if also the converse of the previous statement (***) is true.

We then conclude the present subsection with the following list of conjectures:

**Conjecture 5.7.** Every type I g.t. admits a conifold degeneration. In particular the canonical model of a type I g.t. is a primitive conifold transition.

**Conjecture 5.8** (Converse of (**)). Every small and rigid g.t. $T$ admits a factorization $T = T_2 \circ T_1$ whose first factor $T_1$ is a conifold transition.

A stronger version of both these conjecture is the following

**Conjecture 5.9.** Every small g.t. which does not admit any conifold degeneration is equivalent to a composition of conifold transitions. Then canonical models of small g.t.’s are given by composition of conifold transitions.
A first evidence for the latter conjecture is given by the conifold decomposition of the Namikawa small and rigid g.t. \[ (38) \] given in the Example 5.5.

5.1. **The Calabi–Yau web can’t be directly connected by small g.t.’s.** Let us observe that we can attach a *Picard number* to every node \( M \) of the Calabi–Yau web. In fact, by the Ehresmann fibration theorem, all the Calabi–Yau fibres of \( M \) are diffeomorphic since they are smooth elements of the same flat family \( M \). Therefore their Picard number, which coincides with their 2–nd Betti number (recall relations (11)), have to be constant in \( M \). In particular nodes with Picard number 1 admits only *primitive* (following M. Gross’ notation [34]) Calabi–Yau fibres. They are called *primitive nodes* and can’t admit outgoing arrows, if we want the Calabi–Yau web to stay inside the projective category.

**Definition 5.10** (Direct connection of nodes). Let \( M_1 \) and \( M_2 \) be nodes of the Calabi–Yau web admitting distinct Picard numbers, say \( \rho_1 > \rho_2 \) (≥ 1). Then \( M_1 \) and \( M_2 \) will be called directly connected if there exists an arrow \( M_1 \rightarrow M_2 \).

The following is a stronger version of the Calabi–Yau web connectedness conjecture.

**Conjecture (Direct connectedness of the Calabi–Yau web).** If \( M_1 \) and \( M_2 \) are nodes of the Calabi–Yau web with distinct Picard numbers \( \rho_1 > \rho_2 \), then there exists a (unique, by Proposition 2.3) arrow \( M_1 \rightarrow M_2 \).

**Theorem 5.11.** Two nodes \( M_1 \) and \( M_2 \) of the Calabi–Yau web which are connected by a type II g.t. can never be connected by any composition of small g.t.’s. In particular they can’t be connected by any composition of conifold transitions.

**Proof.** Since a composition of small g.t.’s is always a small g.t. the statement follows by Propositions 2.3 and 4.2.

**Example 5.12.** A consequence of Theorem 5.11 is that there exists nodes of the Calabi–Yau web which cannot be directly connected by means either of small g.t.’s or, even more, of conifold transitions. At this purpose it suffices to exhibit nodes connected by a type II g.t.. Let then \( Y \rightarrow \tilde{Y} \) be a quintic threefold with a unique triple point, \( \phi : Y ightarrow \tilde{Y} \) be the blow up of the triple point and \( \tilde{Y} \) be a generic smooth quintic threefold. Then \( T(Y, Y, \tilde{Y}) \) is a type II g.t. of degree \( \deg(T) = 3 \). Then universal families \( Y \) of \( Y \) and \( \tilde{Y} \) of \( \tilde{Y} \) are nodes of the Calabi–Yau web with

\[
\rho(Y) := \rho(Y) = b_2(Y) = 2 > 1 = b_2(\tilde{Y}) = \rho(\tilde{Y}) =: \rho(\tilde{Y})
\]

which cannot be directly connected by small g.t.’s. The interested reader in remanded to [41] for further examples of topologically distinct Calabi–Yau threefolds connected by type II g.t.’s.

6. **A few words about type III geometric transitions**

In the case of a type III g.t. \( T(Y, \tilde{Y}, \tilde{Y}) \) many of the techniques applied for type II and small g.t.’s fail to be useful since \( \tilde{Y} \) does not admit isolated singularities.

In the present section we will summarize some known result, essentially due to [80], [42] and [12], to outline a research program on this topic.

First of all recall that, by Theorem 1.21, the birational contraction \( \phi : Y ightarrow \tilde{Y} \) contracts an irreducible conic bundle \( E \subset Y \) down to the smooth base curve \( C \subset \tilde{Y} \).
Assume, from now on, the conic bundle $E$ to be smooth, although many of the following considerations can be extended to the case when $E$ has singular fibres (see \[80] Remark after Proposition 4.1, \[52] Proposition (6.5) and \[34] Theorem 1.3).

**Proposition 6.1.** Let $E \subset Y$ be a smooth conic bundle over a smooth curve $C$ of genus $g$. Then

$$H^1(E, \mathcal{N}_{E|Y}) \cong H^0(C, \mathcal{K}_C).$$

In particular if $C$ is not a rational curve then $E$ is not a stable submanifold of $Y$, in the sense of Kodaira \[43].

**Proof.** Since $Y$ is a Calabi–Yau threefold the Adjunction Formula gives $\mathcal{N}_{E|Y} \cong \mathcal{K}_E$, as already observed in Remark \[3.6]. Let $\pi : E \rightarrow C$ be the structural projection whose fibre is $l \cong \mathbb{P}^1$. Then

$$\text{Pic}(E) \cong \mathbb{Z}[C_0] \oplus \pi^* \text{Pic}(C)$$

where $C_0$ is a section of $E$ (\[37\] \S V Proposition 2.3). In particular the canonical divisor $K_E$ is numerical equivalent to $-2C_0 + (2g + 2)l$, giving $K_E \cdot l = -2$ (\[37\] \S V Corollary 2.11). Then $H^0(l, K_E \otimes O_l) \cong H^0(O_{\mathbb{P}^1}(-2)) = 0$ which globalizes to give $R^0\pi_* K_E = 0$. The Leray spectral sequence

$$E_2^{p,q} = H^p(C, R^q\pi_* K_E) \Longrightarrow H^{p+q}(E, K_E)$$

allows then to conclude that $H^1(E, K_E) \cong H^0(R^1\pi_* K_E)$. The proof ends up by observing that

$$R^1\pi_* K_E \cong (R^1\pi_* K_{E/C}) \otimes \mathcal{K}_C \cong \mathcal{K}_C$$

since $R^1\pi_* K_{E/C} \cong \pi_* \mathcal{O}_E \cong \mathcal{O}_C$ by duality theory. \hfill $\square$

By the BTTR Theorem \[10], \[76], \[77], \[61], the Kuranishi space $\text{Def}(Y)$ is smooth and then a polydisc in $\mathbb{T}_Y \cong H^1(Y, \Theta_Y)$.

**Theorem 6.2 (\[80\] Proposition 4.1).** The locus $\Gamma \subset \text{Def}(Y)$ of deformations of $Y$ for which $E$ deforms in the family is a complex submanifold of codimension $g = g(C)$. In particular $g(C) \leq h^{1,2}(X)$.

**Sketch of proof.** It is an application of Theorem 2.1 in \[60\]. In fact, given the immersion morphism $i : E \hookrightarrow Y$, observe that the space $\mathbb{T}_E^1$ defined in \[4.4] can be identified with $H^1(\Theta)$, where $\Theta$ is defined by the exact sequence

$$0 \rightarrow \Theta \rightarrow \Theta_Y \rightarrow \mathcal{N}_{E|Y} \rightarrow 0.$$  

Since $\text{Def}(Y)$ is a polydisc in $\mathbb{T}_Y \cong H^1(Y, \Theta_Y)$ the result then follows by \[84\] showing that the natural map $H^1(\Theta_Y) \rightarrow H^1(\mathcal{N}_{E|Y})$ is always surjective. For further details we remand the interested reader to the proof of Proposition 4.1 in \[80\]. \hfill $\square$

**6.1. Some deduction on conifold degeneration of a type III g.t.**

1. If $g(C) = 0$ then $E \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ and it is a Kodaira stable submanifold of $Y$; in particular the type III g.t. $T(Y, \overline{\Gamma}, \overline{\Gamma})$ cannot admit any conifold degeneration: moreover we can only have $E^3 \leq 6$ since for $E^3 = 7,8$ the normal threefold $\overline{\Gamma}$ turns out to be projectively un-smoothable (\[34\], Theorem 0.4).
(2) ([12] §9, [12] §3.4) If \( g(C) \geq 2 \) then it is possible to choose a non-trivial element \( s \in H^1(\mathcal{N}_{E/Y}) \cong H^0(K_C) \) which, by the natural surjection \( T^1_Y \cong H^1(\Theta_Y) \to H^1(\mathcal{N}_{E/Y}) \), generates a non-trivial first order deformation of \( Y \) along which \( E \) is obstructed. Let \( p_1, \ldots, p_{2g-2} \) be the \( 2g-2 \) zeros of the section \( s \in H^0(K_C) \). Then fibres \( l_i := \pi^{-1}(p_i) \), \( i = 1, \ldots, 2g-2 \) should deform along the chosen deformation of \( Y \) generated by \( s \). This fact should give rise to a primitive conifold degeneration \( CT(X, X, \tilde{Y}) \) of \( T(Y, Y, \tilde{Y}) \). In particular one should have bi–deg(\( CT \)) = \( (1, 2g-3) \), since \( CT \) is primitive and \( \text{Sing}(X) \) composed by \( N = 2g-2 \) nodes. It should then make sense to call \( CT \) a canonical model of the type III g.t. \( T \).

(3) If \( g(C) = 1 \) then every non–trivial section \( s \in H^0(K_C) \) is necessarily a never vanishing constant section. This should imply that \( T(Y, Y, \tilde{Y}) \) cannot admit a conifold degeneration.

6.2. Degree of a type III geometric transition. The above considerations drive naturally to set the following

**Definition 6.3** (Degree of a type III g.t.). Let \( T(Y, Y, \tilde{Y}) \) be a type III g.t. with \( E := \text{Exc}(Y \phi^*) \) a conic bundle over a smooth curve \( C \) of genus \( g \). Then set

\[
\text{deg}(T) := 2g - 3
\]

In particular \( T \) should admit a conifold degeneration if and only if \( \text{deg}(T) \geq 1 \): in this case the degree of \( T \) turns out to coincide with the degree of any canonical model \( CT \) of it, as defined in Definition 4.4.

This Definition ends up the program of attaching a degree to every primitive g.t. \( T \) in some sense describing the changing in topology induced by \( T \). The given definitions of degree (and bi–degree, for non–primitive g.t.’s) look as the “more natural” definitions one could give in the different topological and geometrical contexts. In fact, for type I and type III conifoldable g.t.’s, the degree turns out to be the number of complex moduli “added” by the considered primitive g.t.. This is actually not true for type II g.t.’s (see Remark 3.5) where we preferred to relate the g.t.’s degree with the degree of the del Pezzo exceptional divisor. Moreover the meaning of a non–conifoldable type III g.t.’s degree (it could be negative!) is not yet well understood.

**References**

[1] Aganagic M. and Vafa C. “\( G_2 \) manifolds, mirror symmetry and geometric engineering” [hep-th/0110171].

[2] Altmann K. “The versal deformation of an isolated toric Gorensteins singularity” *Invent. Math.* **128**(1997), 443–479; [math.AG/9403004].

[3] Arnol’d V. I. “Local normal forms of functions” *Invent. Math.* **35** (1976), 87–109.

[4] Arnol’d V. I., Gusein-Zade S. M. and Varchenko A. N. *Singularities of differentiable maps*, Vol. I, *The classification of critical points caustics and wave fronts*, Monographs in Mathematics **82**, Birkhäuser Boston, Inc., Boston, MA, 1988.

[5] Barth W., Peters C. and Van de Ven A. *Compact complex surfaces* vol. 4 E.M.G, Springer–Verlag (1984)

[6] Batyrev V. “Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties” *J. Alg. Geom.* **3** (1994), 493–535; [math.AG/9310003].

[7] Becker T. and Weispfenning V. *Gröbner bases. A computational approach to commutative algebra* Graduate Texts in Mathematics **141**, Springer–Verlag, New York, 1993.
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[8] Berglund P., Katz S. and Klemm A. “Mirror Symmetry and the moduli space for generic hypersurfaces in toric varieties” Nucl. Phys. B456 (1995), 153–204; hep-th/9506091
[9] Berglund P., Katz S., Klemm A. and Mayr P. “New Higgs transition between $N = 2$ string models” Nucl. Phys. B483 (1997), 209–228; hep-th/9605154
[10] Bogomolov F. “Hamiltonian Kähler manifolds” Dokl. Akad. Nauk. SSSR 243/5 (1978), 1101–1104.
[11] Candelas P., Horowitz G.T., Strominger A. and Witten E. “Vacuum configurations for superstrings” Nuclear Phys. B258(1) (1985), 46–74.
[12] Candelas P., de la Ossa X., Font A., Katz S. and Morrison D.R. “Mirror symmetry for two parameter models–I” in Mirror symmetry II 483–543, AMS/IP Stud. Adv. Math. 1, Amer. Math. Soc. Providence, RI, 1997.
[13] Chiang T.M., Greene B., Gross M. and Kanter Y. “Black Hole condensation and the web of Calabi–Yau manifolds” Nucl. Phys. Proc. Suppl. 46 (1996), 82–95; hep-th/9511204.
[14] Clemens C.H. “Double Solids” Adv. in Math. 47 (1983), 107–230.
[15] Clemens C.H. “Homological equivalence modulo algebraic equivalence is not finitely generated” Publ. Math. I.H.E.S. 58 (1983), 19–38.
[16] Cox A.D. and Katz S. Mirror Symmetry and Algebraic Geometry vol. 68 Math. Surveys and Monographs, Amer. Math. Soc., Providence RI (1999).
[17] Diaconescu D.E., Dijkgraaf R., Donagi R., Hofman C. and Pantev, T. “Geometric transitions and integrable systems” Nuclear Phys. B752 (2006) (3), 329–390.
[18] Diaconescu D.E., Donagi R. and Pantev T. “Geometric transitions and mixed Hodge structures” Adv. Theor. Math. Phys. 11 (1) (2007), 65–89.
[19] Diaconescu D.E., Donagi R. and Pantev T. “Intermediate Jacobians and ADE Hitchin systems” Math. Res. Lett. 14 (5) (2007), 745–756.
[20] Diaconescu D.E., Florea B. and Grassi A. “Geometric transitions and open string instantons” Adv. Theor. Math. Phys. 6 (2003), 619–642; hep-th/0205234.
[21] Douady A. “Le problème des modules locaux pour les espaces C–analytiques compacts” Ann. scient. Éc. Norm. Sup. 4e série, 7 569–602 (1974).
[22] Enriques F. Le superficie algebriche Zanichelli, Bologna (1946).
[23] Forbes B. “Computations on B-model geometric transitions” Modern Phys. Lett. A20 (35) (2005), 2685–2697.
[24] Friedman R. “Simultaneous resolution of threefold double points” Math. Ann. 247 (1986), 671–689.
[25] Friedman R. “On threefolds with trivial canonical bundles” in Complex geometry and Lie theory, Sundance, UT, 1989, Proc. Sympos. Pure Math. 53, Amer. Math. Soc. (1991), 103–134.
[26] Godement R. Topologie Algébrique et Théorie des Faisceaux, Hermann, Paris (1958).
[27] Gopakumar R. and Vafa C. “On the gauge theory/geometry correspondence” Adv. Theor. Math. Phys. 3 (1999), 1415–1443; hep-th/9811131.
[28] Grassi A. and Rossi M. “Large N–dualities and transitions in geometry” in Geometry and Physics of Branes, Como 2001 Series of H.E.P., Cosmology and Gravitation, IoP Bristol (2003), 210–278: math.AG/0209044.
[29] Grauert H. “Der Satz von Kuranishi für Kompakte Komplexe Räume” Invent. Math. 25, 107–142 (1974).
[30] Green P.S. and Hübsch T. “Possible phase transitions among Calabi–Yau compactifications” Phys. Rev. Lett. 61 (1988), 1163–1166.
[31] Green P.S. and Hübsch T. “Connetting moduli spaces of Calabi–Yau threefolds” Comm. Math. Phys. 119 (1988), 431–441.
[32] Greene B., Morrison D.R. and Strominger A. “Black hole condensation and the unification of string vacua” Nucl. Phys. B451 (1995), 109–120; hep-th/9504145.
[33] Gross M. “Deforming Calabi–Yau threefolds” Math. Ann. 308 (1997), 187–220; math.AG/9506022.
[34] Gross M. “Primitive Calabi–Yau threefolds” J. Diff. Geom. 45 (1997), 288–318; math.AG/9512002.
[35] Grothendieck A. “Sur quelques points d’algèbre homologique” Tôhoku Math. J. 9, 199–221 (1957).
[36] Grothendieck A. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2) Séminaire de Géométrie Algébrique du Bois Marie, 1962. Revised reprint
of the 1968 French original. Documents Mathématiques (Paris) Société Mathématique de France, Paris, 2005.

[37] Hartshorne R. *Algebraic Geometry* G.T.M. *52*, Springer–Verlag, Berlin–Hidelberg–New York (1977).

[38] Hidaka H. and Watanabe K.–I. “Normal Gorenstein surfaces with ample anti–canonical divisor” *Tokyo J. Math.* 4 (1981), 319–330.

[39] Hirzebruch F. “Some examples of threefolds with trivial canonical bundle” in *Collected papers*, vol. II, 757–770, Springer (1987).

[40] Joyce D. *Compact manifolds with Special Holonomy*, Oxford Science Publications, Oxford–New York (2000).

[41] Kapustka G. and M. “Primitive contractions of Calabi–Yau threefolds I,” arXiv:math/0703810 [math.AG].

[42] Katz S., Morrison D.R. and Plesser M. “Enhanced gauge symmetry in type II string theory” *Nucl. Phys.* B477 (1996), 105–140; hep-th/9601108.

[43] Kodaira K. “On stability of compact submanifolds of complex manifolds” *Amer. J. Math.* 85 (1963), 79–94.

[44] Kodaira K. “On the structure of compact complex analytic surfaces” *Am. J. Math.* 86 (1964), 751–798.

[45] Kollár J. and Mori S. “Classification of three–dimensional flips” *J. Amer. Math. Soc.* 5 (1992), 533–703.

[46] Laufer H. “On CP^1 as an exceptional set” in *Recente developments in several complex variables* Ann. Math. Stud. 100 (1981), 261–276.

[47] Lichtenbaum S. and Schlessinger M. “On the cotangent complex of a morphism” *Trans. A.M.S.* 128, 41–70 (1967).

[48] Milnor J. *Singular points of complex hypersurfaces*, Annals of Math. Studies 61, Princeton University Press, Princeton (1968).

[49] Milnor J. and Orlik P. “Isolated singularities defined by weighted homogeneous polynomials”, *Topology* 9 (1970), 385–393.

[50] Morrison D.R. “The birational geometry of surfaces with rational double points” *Math. Ann.* 271 (1985), 415–438.

[51] Morrison D.R. “Through the looking glass” in *Mirror Symmetry III*, American Mathematical Society and International Press (1999), 263–277; [math.AG/9705028](http://arxiv.org/abs/math.AG/9705028).

[52] Namikawa Y. “On deformations of Calabi-Yau 3-folds with terminal singularities” *Topology* 33(3) (1994), 429–446.

[53] Namikawa Y. “Stratified local moduli of Calabi–Yau 3–folds” *Topology* 41 (2002), 1219–1237.

[54] Namikawa Y. and Steenbrink J. “Global smoothing of Calabi–Yau 3-fold” *Invent. Math.* 122 (1995), 403–419.

[55] Palamodov V. P. “The existence of versal deformations of complex spaces” *Dokl. Akad. Nauk SSSR* 206 (1972), 538–541.

[56] Palamodov V. P. “Deformations of complex spaces” *Russian Math. Surveys* 31(3) (1976), 129–197; from russian *Uspekhi Mat. Nauk* 31(3) (1976), 129–194.

[57] Pinkham H. “Simple elliptic singularities, del Pezzo surfaces and Cremona transformations” in *Several Complex Variables* Proc. Symp. Pure Math. 30 (1977), 69–70.

[58] Pinkham H. “Factorization of birational maps in dimension three” in *Singularities* Proc. Symp. Pure Math. 40 (1981), 343–372.

[59] Ran Z. “Deformations of maps” in *Algebraic curves and projective geometry*, Ballico E. and Ciliberto C., Eds. LNM 1389, Springer–Verlag (1989).

[60] Ran Z. “Lifting of cohomology and unobstructedness of certain holomorphic maps” *Bull. Amer. Math. Soc. (N.S.*) 26 (1) (1992), 113–117.

[61] Ran Z. “Deformations of manifolds with torsion or negative canonical bundle” *J. Alg. Geom.* 1 (1992), 279–291.

[62] Reid M. “Canonical 3–folds” in *Journées de géométrie algébrique d’Angers*, Sijthoff & Nordhoff (1980), 671–689.

[63] Reid M. “Minimal model of canonical 3–folds” in *Algebraic varieties and analytic varieties*, Adv. Stud. Pure Math. 1, North–Holland (1983), 131–180.

[64] Reid M. “Young person’s guide to canonical singularities” in *Algebraic Geometry, Bowdoin 1985* vol. 1, Proc. Sym. Pure Math. 46, AMS (1987), 354–414.
[65] Reid M. “The moduli space of 3–folds with $K = 0$ may nevertheless be irreducible” Math. Ann. 287 (1987), 329–334.
[66] Reid M. “Non–normal del Pezzo surfaces” Math. Proc. RIMS 30 (1994), 695–727.
[67] Rossi M. “Geometric transitions” J. Geom. Phys. 56(9) (2006), 1940–1983.
[68] Rossi M. Homological type of geometric transitions Preprint.
[69] Rossi M. On a remark by Y. Namikawa arXiv:0807.4104 [math.AG], http://sites.google.com/a/unito.info/michelerossi/Home/nami.pdf
[70] Rossi M. and L. Terracini A MAPLE subroutine for computing Milnor and Tyurina numbers of hypersurface singularities, will appear in the next few days at http://sites.google.com/a/unito.info/michelerossi/Home/Milnurina.pdf
[71] Saïto K. “Quasihomogene isolierte Singularitäten von Hyperflächen” Invent. Math. 14 (1971), 123–142.
[72] Schlessinger M. “Rigidity of quotient singularities” Invent. Math. 14 (1971), 17–26.
[73] Schoen C. “On fiber products of rational elliptic surfaces with section” Math. Z. 197(2) (1988), 177–199.
[74] Strominger A. “Massless black holes and conifolds in string theory” Nucl. Phys. B451 (1995), 97–109; hep-th/9504145
[75] Tamás C. “On the classification of crepant analytically extremal contractions of smooth three–folds” Compos. Math. 140 (2004), 1561–1578; math.AG/0204015
[76] Tian G. “Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Weil–Petersson metric” in Mathematical aspects of string theory (S.-T. Yau, ed.) World Scientific, Singapore (1987), 629–646.
[77] Todorov A. “The Weil–Petersson geometry of the moduli space of SU(n ≥ 3) (Calabi–Yau ) manifolds” Comm. Math. Phys. 126 (1989), 325–346.
[78] Wahl, J.M. “Equivisigular deformatons of normal surface singularities, I” Ann. of Math. 104 (1976), 325–356.
[79] Werner J. “Kleine Auflösungen spezieller dreidimensionaler Varietäten” Bonn. Math. Schr. 186 (1987).
[80] Wilson P.M.H. “The Kähler cone on Calabi–Yau threefolds” Invent.Math. 107 (1992), 561–583.
[81] Wilson P.M.H. Erratum to “The Kähler cone on Calabi–Yau threefolds” Invent.Math. 114 (1993), 231–233.
[82] Witten E. “Chern–Simons gauge theory as a string theory” in The Floer memorial volume Birkhäuser (1995), 637–678; hep-th/9207094

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