Conformal Bootstrap Signatures of the Tricritical Ising Universality Class

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Abstract

We study the tricritical Ising universality class using conformal bootstrap techniques. By studying bootstrap constraints originating from multiple correlators on the CFT data of multiple OPEs, we are able to determine the scaling dimension of the spin field $\Delta_\sigma$ in various non-integer dimensions $2 \leq d \leq 3$. $\Delta_\sigma$ is connected to the critical exponent $\eta$ that governs the (tri-)critical behaviour of the two point function via the relation, $\eta = 2 - d + 2\Delta_\sigma$. Our results for $\Delta_\sigma$ match with the exactly known values in two and three dimensions and are a conjecture for non-integer dimensions. We also compare our CFT results for $\Delta_\sigma$ with $\epsilon$-expansion results, available up to $\epsilon^3$ order. Our techniques can be naturally extended to study higher-order multi-critical points.
The consequences of the conformal hypothesis which posits conformal invariance to the behaviour of physical systems at criticality in addition to scale invariance are most far-reaching in two dimensions, where the conformal symmetry is the infinite dimensional Virasoro algebra. The seminal work of Belavin, Polyakov and Zamolodchikov [1, 2] resulted in the discovery of a whole class of two dimensional conformal field theories (CFTs) viz. the Virasoro minimal models. Each of these models describes a universality class and the exact knowledge of the scaling dimensions of the operators amounts to a derivation of the critical exponents purely from conformal invariance. The Ising model, the tricritical Ising model, the three and four state Potts models in two dimensions were thus exactly solved thirty years ago. The infinitude of conformal symmetry made two dimensions rather special and the use of conformal field theories to study critical phenomena was restricted to two dimensions alone.

Recent times have seen a breakthrough in doing the same for three and other dimensions. Even though the conformal group is finite dimensional now, technical advances made in the explicit computation of (global) conformal blocks [3] resulted in astounding progress and has provided the most precise values [4–7] for the critical exponents of the three dimensional Ising model. Rychkov et al analyzed the restrictions imposed by conformal invariance and found that the conformal field theory corresponding to the Ising universality class sits on the boundary of the allowed region at a kink-like point in the space of scaling dimensions of the only two relevant operators. Furthermore, remarkably, they could extend this analysis to all non-integer dimensions between two and four and showed that even here, the theory corresponding to the Ising universality class is always located at a kink-like point [8]. Hitherto [9–14], only critical points have been studied using conformal field theory methods. Other critical points (tricritical, multi-critical) also have conformal symmetry and here we use conformal field theory techniques to study tricritical points. In this paper, we look at the tricritical Ising point in two and higher dimensions and show that just like the Ising critical point, the tricritical Ising point can also be recognised by its special signatures in the space of scaling dimensions of the appropriate operators.

A tricritical point is a fixed point where three critical lines meet. The scaling hypothesis [15] for the tricritical point gives the form of the singular part of the free energy to be:

\[ G_{\text{sing}}(t, g) \sim |t|^{2-\alpha_{t}} f^\pm (g/|t|^{\phi_{t}}) \] (1)
Here $t$ and $g$ are scaling fields which are a linear combination of the thermodynamic fields (temperature and external field) and $\phi_t$ is known as the crossover exponent. $|g||t|^{-\phi_t} \ll 1$ corresponds to the vicinity of the tricritical point; as $t$ decreases and $|g||t|^{-\phi_t}$ increases, a $\lambda$-line (the line of critical points in Ising universality class) is approached and thus corresponds to ordinary critical behaviour. Interestingly, these two critical behaviours have different upper critical dimensions; three for the tricritical point and four for the Ising critical point. Employing renormalisation group analysis is complicated as there can be operators other than the ones directly responsible for the crossover which are irrelevant for one fixed point but are relevant or marginal for the other fixed point [16]. Due to this, field theory and renormalization group based techniques such as the $\epsilon$-expansion have had limited success [17, 18]. But on the other hand, CFT techniques have the advantage that they avoid the flow and only study the fixed point which is where the extra scaling and conformal symmetries are present. Yet, so far, a CFT study of the tricritical point has not met with as much success as the critical point, in dimensions other than two.

A CFT is specified by a list of local primary operators a.k.a scaling operators (each primary operator is specified by its scaling dimension and spin). The scaling dimension of a local operator in an unitary CFT is bounded from below depending on it’s spin. A product of two such local operators is expandable in terms of all the local operators [19, 20]; this is known as the operator product expansion (OPE)

$$\phi_1(x_1)\phi_2(x_2) = \sum_\mathcal{O} \lambda_{12\mathcal{O}} C(x_{12}, \partial x_2) \mathcal{O}(x_2) ,$$

(2)

where $C(x_{12}, \partial x_2)$ is fully determined by conformal invariance and $\lambda_{12\mathcal{O}}$ is referred to as an OPE co-efficient, which is real in an unitary CFT. The set of local primary operators and the set of OPE co-efficients (one for every ordered triple of primary operators) are together referred to as CFT data. The critical exponents are encoded in the scaling dimensions of only a few low-lying primary operators (the relevant ones). There are only two relevant operators in the Ising CFT, while there are four relevant operators in the CFT of the tricritical Ising point [21]. Of the four, one needs only three to define the critical exponents of the tricritical Ising universality class. In two dimensions the values of the critical exponents of the tricritical Ising point are known exactly [1, 2]; the CFT is the second Virasaro minimal model consisting of six primary operators (while the Ising CFT is the first Virasaro minimal model consisting of three primary operators).
In a CFT, a four point function of four scalar fields, \( \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle \), can be reduced to a two point function after using OPEs, in two different ways, each giving an answer quadratic in two different subsets of the OPE coefficients of the theory, and these two answers are equal. Such equations that arise from every four point function in the theory are known as bootstrap equations or crossing symmetry constraints. The conformal bootstrap program employs these equations to search for CFTs by successively constraining the CFT data. These equations were shown to be tractable numerically starting with the seminal work [4] and leading up to [22], the main tools for the results of this paper.

In this paper, we study a class of CFT’s whose low-lying operator spectrum includes four relevant scalars \( \sigma, \epsilon, \sigma', \epsilon' \) in the increasing order of scaling dimensions. The two dimensional tricritical Ising model is one example; there the \( \sigma \)’s are \( \mathbb{Z}_2 \)-odd while the \( \epsilon \)’s are \( \mathbb{Z}_2 \)-even. The field theory associated to tricritical Ising is the \( \phi^6 \) scalar field theory; the four relevant scalars as \( \sigma \leftrightarrow \Phi, \epsilon \leftrightarrow \Phi^2, \sigma' \leftrightarrow \Phi^3, \epsilon' \leftrightarrow \Phi^4 \). There are ten OPEs that concern these four fields but we will focus only on two of them viz. \( \sigma \times \sigma \) and the \( \epsilon \times \epsilon \) OPEs. Furthermore, the class of CFT’s we study are those with the following particular OPEs:

\[
\sigma \times \sigma = 1 + \epsilon + \epsilon' + \ldots, \quad \epsilon \times \epsilon = 1 + \epsilon'
\]  

That is, the global conformal families of \( \epsilon \) and \( \epsilon' \) are present in the \( \sigma \times \sigma \) OPE and the global conformal family of \( \epsilon' \) is present in the \( \epsilon \times \epsilon \) OPE. \( \epsilon \) is the scalar operator with the lowest scaling dimension that is present in the \( \sigma \times \sigma \) OPE and will be referred to as the ‘lowest scalar’ of the OPE; similarly \( \epsilon' \) is the ‘next to lowest scalar’ of the OPE. Also note that \( \epsilon' \) is the lowest scalar of the \( \epsilon \times \epsilon \) OPE. Thus \( \epsilon' \) is the lowest scalar in one OPE and the next to lowest scalar in another. This is the problem we study here using the numerical conformal bootstrap [22]: CFTs (in all dimensions, 2 to 3 and beyond) with a low lying spectrum and OPE’s given by (3) and we will find that the bootstrap constraints reflect many aspects of tricritical phenomena.

First we recall the two results we need; both originate from the crossing symmetry constraints on the four point function \( \langle \phi \phi \phi \phi \rangle \) of a scalar operator. (i) The first result is the RRTV bound [4]: the operator in the \( \phi \times \phi \) OPE with the lowest scaling dimension is always a scalar and it’s scaling dimension is bounded from above. This bound is a function of the scaling dimension of \( \phi (\Delta_\phi) \) and is determined numerically in all dimensions. (ii) The second result is the Rychkov bound [5]. This is a bound on the next to lowest scalar in the \( \phi \times \phi \)
FIG. 1. The RRTV (solid) and the Rychkov (dashed) bounds in two dimensions. The two bounds cross at \( r_2(0.076 \pm 0.002) \).

OPE, if present, which unlike the lowest scalar is not required to always exist. The result is that there is an upper bound, determined numerically, for any set of values for \( \Delta_\phi \) and for the scaling dimension of the lowest scalar (which is already constrained by the RRTV bound). For a given value of \( \Delta_\sigma \), say \( s \), we first find the RRTV bound on the lowest scalar in the \( \epsilon \times \epsilon \) OPE which is a bound on \( \Delta_\epsilon, r_1(s) \). We plot the points \((s, r_1(s))\) in a figure where the x-axis is \( \Delta_\sigma \) and the y-axis is \( \Delta_\epsilon \). We refer to this graph as the RRTV bound on \( \Delta_\epsilon \). For two dimensions it is shown as a solid line in figure 1.

We can get another bound on \( \Delta_\epsilon \) in the following way. For a given value of \( \Delta_\sigma \), say \( r \), we first find the RRTV bound on the lowest scalar in the \( \sigma \times \sigma \) OPE, which is a bound on \( \Delta_\sigma, r_2(r) \). We plot the points \((r, r_2(r))\) in a figure where the x-axis is \( \Delta_\sigma \) and the y-axis is \( \Delta_\epsilon \). For two dimensions it is shown in the inset of figure 1. Then for the pair \((\Delta_\sigma = r, \Delta_\epsilon = r_2(r))\) we find the Rychkov bound on the next to lowest scalar in the \( \sigma \times \sigma \) OPE which is a bound on \( \Delta_\epsilon, r_3(r) \). We plot the points \((r_2(r), r_3(r))\) in the same figure as that of the RRTV bound. We refer to this graph as the Rychkov bound on \( \Delta_\epsilon \). In two dimensions it is shown as the dotted line in figure 1.

We compare the Rychkov and RRTV bounds on \( \Delta_\epsilon \). For smaller values of \( \Delta_\sigma (= r) \)
(which corresponds to smaller values of \(r_2(r)\), because \(r_2(r)\) is a monotonically increasing function, see inset of figure 1), we find that the Rychkov bound is bigger than the RRTV bound. For larger values of \(\Delta_\sigma\), the graph for the Rychkov bound crosses the graph for the RRTV bound so that to the left of the crossing point the Rychkov bound is larger than the RRTV bound and to the right of the crossing point the RRTV bound is larger. There is a specific value for \(\Delta_\sigma\), say \(\Delta^{cross}_\sigma\), for which this crossing happens; to be precise \(r_2(\Delta^{cross}_\sigma)\) is the value on the x-axis for the crossing point. In two dimensions, the graphs for the RRTV and Rychkov bounds are plotted in figure 1 and we find \(\Delta^{cross}_\sigma = 0.076 \pm 0.002\). This compares well with the known value for \(\Delta_\sigma\) in the two dimensional tricritical Ising CFT (the second Virasoro minimal model), 0.075 within error bars. Hence, we conclude that the value of \(\Delta^{cross}_\sigma\), determined by conformal bootstrap constraints as described above, is the value of \(\Delta_\sigma\) in the CFT.

The equations that encode the bootstrap constraints and in fact the whole formalism is analytic in the number of dimensions \(d\) [8] and thus provides a remarkable way to study CFTs in non-integer dimensions; the Ising model was studied in all dimensions \(2 \leq d \leq 4\). We numerically determine the RRTV and Rychkov bounds and look for the point, if any, where the bounds coincide and obtain \(\Delta^{cross}_\sigma\). We find that \(\Delta^{cross}_\sigma\) increases as \(d\) increases from 2 to 3 (see table 1). Note that \(\Delta^{cross}_\sigma\) is bigger than the unitarity bound, \(\frac{d-2}{d}\), for dimensions between 2 and 3 and this difference is maximum for \(d = 2\) and decreases with increasing \(d\) (see figure 2). We also find that \(\Delta^{cross}_\sigma\) tends to \(\frac{1}{2}\) for \(d = 3\). This matches with the known value for \(\Delta_\sigma\) in \(d = 3\); the upper critical dimension for the tricritical Ising model is 3 wherein the scaling dimension of \(\sigma\) is it’s classical value of \(\frac{1}{2}\).

Our numerical studies for \(3 < d < 4\) show that the two bounds do not cross. This observation is consistent with known facts. The tricritical Ising exponents for \(3 < d < 4\) should be the same as for three dimensions which is a violation of the unitarity bound (for \(\Delta_\sigma\)) and hence one is in the realm of non-unitary CFTs. Our analysis, following [4, 5] is an analysis of constraints on unitary CFTs.

Our surmise that \(\Delta^{cross}_\sigma\) gives the value for \(\Delta_\sigma\) in the CFT thus has passed non-trivial tests by reproducing the known values in \(d = 2, d = 3\) and in \(3 < d < 4\) and hence the computations for \(2 < d < 3\) constitute a prediction coming from conformal bootstrap analysis.

To test the hypothesis, we compare the value of \(\Delta^{cross}_\sigma\) obtained above with the best
TABLE I. Estimates of $\Delta_\sigma$ in dimensions 2 to 3 from CFT analysis and from $\epsilon$-expansion. The value is known exactly in two and three dimensions: 0.075 and 0.5 respectively. Our estimates are in agreement with the known values within error bars. The estimate from $\epsilon$-expansion in two dimensions is way off the actual value and matches in three dimensions. Hence the estimate from $\epsilon$-expansion gets worse as we go from three to two dimensions. We see that difference in our estimate and estimate from $\epsilon$ expansion broadens as we go from three to two, as expected.

| Dimension (d) | $\Delta_\sigma^{\text{cross}}$ from CFT | $\Delta_\sigma$ from $\epsilon$-expansion |
|---------------|---------------------------------------|---------------------------------------|
| 3.00          | $-$                                   | 0.50000                               |
| 2.90          | $-$                                   | 0.45002                               |
| 2.80          | 0.4002(1)                             | 0.40014                               |
| 2.70          | 0.3519(3)                             | 0.35043                               |
| 2.60          | 0.3061(1)                             | 0.30098                               |
| 2.50          | 0.26003(7)                            | 0.25184                               |
| 2.40          | 0.217(1)                              | 0.20311                               |
| 2.30          | 0.179(1)                              | 0.15486                               |
| 2.20          | 0.145(1)                              | 0.10716                               |
| 2.10          | 0.113(1)                              | 0.06010                               |
| 2.00          | 0.076(1)                              | 0.01374                               |

known estimate from $\epsilon$-expansion. Computations up to $\epsilon^3$-order were done in \cite{17,18}:

$$
\Delta_\sigma = \frac{1}{2} - \frac{\epsilon}{2} + \frac{\epsilon^2}{1000} + \frac{10125 \pi^2 + 91160}{15,000,000} \epsilon^3 + \ldots \tag{4}
$$

The comparison between the CFT results and $\epsilon$-expansion results is shown in table I. The results coincide in dimensions close to three, but deviate as one approaches two dimensions. This is because the study of the tricritical Ising point using $\epsilon$-expansion is known to give poor estimates in two dimensions. For example, the $\epsilon$-expansion result for the crossover exponent, $\phi_t$, is negative \cite{17,18}. One expects that modern approaches to $\epsilon$-expansion computations that incorporate conformal symmetry such as the Rychkov-Tan \cite{23} method and the Polyakov-Mellin bootstrap \cite{14} would give better results \cite{24}.

We now focus on another observation about two dimensions. We find that there are two plateaus in the plot of the Rychkov bound (dashed line of figure I). One is the plateau that
FIG. 2. The bold line shows the CFT computation for $\Delta_\sigma$ and the dashed line shows the unitarity bound for scalar operators, for $2 \leq d \leq 3$. As the spatial dimension increases $\Delta_\sigma$ becomes very close to the unitarity bound.

FIG. 3. The plots of the Rychkov bound for the $\sigma \times \sigma$ OPE in various non-integer dimensions, consist of two plateaus. The higher plateau is associated with the Ising universality class while the lower one with the tricritical Ising universality class. The lower plateau is smaller for higher dimensions and vanishes at four dimensions as expected.

starts around $\Delta_\epsilon = 1, \Delta_\epsilon' = 4$; this one has been well studied in [5], where it was argued that as one approaches the Ising CFT point, $\epsilon'$ becomes irrelevant, and hence this plateau is associated with the Ising universality class. There is a second plateau roughly for $\Delta_\epsilon < 1; \Delta_\epsilon' < 2$ throughout this plateau. Hence if a certain CFT is such that $\epsilon'$ can not be an irrelevant operator in it (such as the two dimensional tricritical Ising CFT) then that CFT has to exist in the region ($\Delta_\epsilon < 1$) of this second plateau. We can thus associate this second
plateau with the tricritical Ising universality class. The two plateaus in the plot of the Rychkov bound indicate that the conformal bootstrap constraints allow for the (possible) existence of two different universality classes: one where $\epsilon'$ is relevant and another where $\epsilon'$ is irrelevant or in other words the Ising and the tricritical Ising universality classes. This result from CFT analysis is consistent with our understanding from studies of $\phi^6$ field theory \[16\]: the $\epsilon'$ operator is the $\phi^4$ operator and it is known that the $\phi^4$ operator determines whether the flow is towards the tricritical point or the Ising point.

Motivated by the above observations on plateaus in two dimensions we plot the Rychkov bound in various non-integer dimensions in figure 3 and find that the two plateau structure exists there too. This is the conformal bootstrap signature for the existence of both the Ising and tricritical Ising universality classes in non-integer dimensions as well. On closer examination, we find that the width of the lower plateau decreases with increasing dimension and vanishes at $d = 4$. This result from CFT analysis is to be expected as the upper critical dimension of the Ising critical point is four where the classical values $\Delta_\sigma = 1$, $\Delta_\epsilon = 2$, $\Delta_{\epsilon'} = 4$ are attained but the unitarity bound on $\Delta_\sigma$ is also 1, hence the vanishing of the two plateau structure at $d = 4$.

So far we have studied the bootstrap constraints for only one four point function viz. the pure $\langle \sigma \sigma \sigma \sigma \rangle$ correlator and this involves only the subset of the CFT data that occur in the $\sigma \times \sigma$ OPE. We will now show that these constraints encode even more information, even of higher order critical points.

In two dimensions, when we fix $\Delta_\sigma$ and $\Delta_\epsilon$ to their exact values of 0.075 and 0.2 re-
spectively, the bootstrap equations from the $\langle \sigma \sigma \sigma \sigma \rangle$ correlator give an upper bound on $\Delta_{\prime''}$ ($\prime''$ is the operator that occurs in the $\sigma \times \sigma$ OPE and is the operator above $\prime'$ in the CFT spectrum) for a given value of $\Delta_{\prime'}$. This is plotted in figure 4(a) and one observes that the bound on $\Delta_{\prime''}$ shows a discontinuous jump to $\approx 3.2$ at $\Delta_{\prime'} = 1.2$ (the minimal model values for $\prime''$ and $\prime'$ are 3.0 and 1.2). In another computation, we fix $\Delta_{\sigma}$ and $\Delta_{\prime'}$ to their minimal model values. Constraints from the $\langle \sigma \sigma \sigma \sigma \rangle$ and the $\langle \epsilon \epsilon \epsilon \epsilon \rangle$ correlators give upper bounds on $\Delta_{\prime''}$ which are plotted in figure 4(b) as the solid and dotted lines respectively. When one traces the lower of the two upper bounds, one finds a jump at $\Delta_{\epsilon} = 0.2$ where $\Delta_{\prime''}$ is $\approx 3.1$. Hence we conclude that the tricritical Ising CFT can also be isolated by using a bootstrap analysis much like the Ising critical point.

It was shown [5] that the Ising CFT is characterised by the property that $\prime'$ goes from being relevant to irrelevant, as a function of $\Delta_{\epsilon}$, while keeping $\Delta_{\sigma}$ fixed. We have shown that the tricritical Ising CFT is characterised by the property that $\prime''$ goes to being irrelevant as a function of $\Delta_{\prime'}$, keeping both $\Delta_{\sigma}$ and $\Delta_{\epsilon}$ fixed. All these involved the bootstrap constraints on the CFT data encoded in the $\sigma \times \sigma$ OPE. Going on, one expects that information of higher order critical points, that is multi-critical points is also contained in the $\sigma \times \sigma$ OPE. The next higher order critical point would be where the $\Delta_{\prime'''}$ ($\prime'''$ is the operator above $\prime''$ in the CFT spectrum) would go from relevant to irrelevant as a function of $\Delta_{\prime''}$, keeping all of $\Delta_{\sigma}, \Delta_{\epsilon}$ and $\Delta_{\prime'}$ fixed.

The tricritical point is known to be unstable since the system can easily crossover to the ordinary critical point. We have shown how it can be studied using conformal bootstrap techniques. Such non-perturbative methods using CFTs are even more significant for the tricritical point, as the success with epsilon expansion has been limited. Using the bootstrap constraints coming from only two pure correlators on the CFT data contained in two OPEs, we have seen many signatures of tricritical physics and also obtained the precise value of one critical exponent. To get precise values for the other exponents one will have to consider crossing symmetry constraints coming from all correlators, pure and mixed, which would involve CFT data in other OPEs [24].

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