TWISTED GLOBAL SECTION FUNCTOR FOR D-MODULES ON AFFINE GRASSMANNIAN

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ABSTRACT. For each integral dominant weight \( \lambda \), we construct a twisted global section functor \( \Gamma^\lambda \) from the category of critical twisted \( D \)-modules on affine Grassmannian to the category of \( \lambda \)-regular modules of affine Lie algebra at critical level. We proved that \( \Gamma^\lambda \) is exact and faithful. This generalized the work of Frenkel and Gaitsgory [FG] in the case when \( \lambda = 0 \).

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INTRODUCTION

0.1. Let \( \mathfrak{g} \) be a simple Lie algebra over the complex numbers, and \( G \) be the corresponding algebraic group of adjoint type. Let \( \kappa \) be an invariant non-degenerate bilinear form

\[
\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}.
\]

Let \( \hat{\mathfrak{g}}_\kappa \) be the affine Kac-Moody algebra given as the central extension of the loop algebra \( \mathfrak{g}(t) \)

\[
(0.1.1) \quad 0 \to \mathbb{C} \mathbb{1} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}(t) \to 0,
\]

with bracket given by

\[
[af(t), bg(t)] = [a, b]f(t)g(t) + \kappa(a, b) \text{Res}(fg) \cdot \mathbb{1},
\]

where \( a \) and \( b \) are elements in \( \mathfrak{g} \), and \( \mathbb{1} \) is the central element.

Denote by \( \hat{\mathfrak{g}}_{\text{crit}} \) the affine Kac-Moody algebra corresponding to \( \kappa = -1/2\kappa_{\text{kill}} =: \kappa_{\text{crit}} \), where \( \kappa_{\text{kill}} \) denotes the Killing form, and let \( \hat{\mathfrak{g}}_{\text{crit}} \) be the center of appropriately completed twisted enveloping algebra \( U'_{\text{crit}} \) of \( \hat{\mathfrak{g}}_{\text{crit}} \). We are mostly interested in the category \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \) of continuous \( U'_{\text{crit}} \)-modules. These are the same as discrete \( \hat{\mathfrak{g}}_{\text{crit}}\text{-modules} \) on which the central element \( \mathbb{1} \) acts as the identity.
0.2. Let $\text{Gr}_G = G((t))/G[[t]]$ be the affine Grassmannian of $G$. Denote by $D_{\text{crit}}$-$\text{mod}(\text{Gr}_G)$ the category of critical-twisted $D$-modules on $\text{Gr}_G$ as introduced in [BD2]. We have the functor of global sections

$$\Gamma : D_{\text{crit}}$-$\text{mod}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}, \ F \mapsto \Gamma(\text{Gr}_G, F).$$

In [FG], it is shown that the above functor $\Gamma$ is exact and faithful. This functor plays an important role in the study of the Geometric Langlands. More precisely, Let $V^\lambda$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda$ and set $\mathbb{V} = \text{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}}(V^\mu)$. Denote by $\mathfrak{z}_{\text{crit}, \lambda}$ the algebra $\mathfrak{z}_{\text{crit}, \lambda} := \text{End}(\mathbb{V})$. It can be shown that $\mathfrak{z}_{\text{crit}, \lambda}$ is in fact commutative, and the map $\hat{\mathfrak{z}}_{\text{crit}} \to \mathfrak{z}_{\text{crit}, \lambda}$ is a surjection

$$\hat{\mathfrak{z}}_{\text{crit}} \twoheadrightarrow \mathfrak{z}_{\text{crit}, \lambda}.$$
Denote by $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}, \lambda}$ the subcategory of $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}$ consisting of modules such that the action of the center $\hat{\mathfrak{z}}_{\text{crit}}$ factors through $\mathfrak{z}_{\text{crit}, \lambda}$. For $\lambda = 0$ we will simply denote $\mathfrak{z}_{\text{crit}, 0}$ and $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}, 0}$ by $\mathfrak{z}_{\text{crit}}$ and $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}}$ respectively.

It can be shown that the functor $\Gamma : D_{\text{crit}}$-$\text{mod}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}$, in fact, lands in the subcategory $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}}$. Moreover, as it is shown in [FG5], the exactness and faithfulness of the above functor, allows to show the equivalence of categories

$$(0.2.1) \quad D_{\text{Hecke}}^{\text{crit}}$-$\text{mod}(\text{Gr}_G)^{I_0} \simeq \hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}}^{I_0}.$$

In the above expression, $D_{\text{Hecke}}^{\text{crit}}$-$\text{mod}(\text{Gr}_G)$ denotes the Hecke category

$$D_{\text{Hecke}}^{\text{crit}}$-$\text{mod}(\text{Gr}_G) := D_{\text{crit}}$-$\text{mod}(\text{Gr}_G) \times \text{Spec}(\mathfrak{z}_{\text{crit}}),$$

and $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}}^{I_0}$ (resp. $D_{\text{Hecke}}^{\text{crit}}$-$\text{mod}(\text{Gr}_G)^{I_0}$) denotes the subcategory of $\hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}}$ (resp. of $D_{\text{Hecke}}^{\text{crit}}$-$\text{mod}(\text{Gr}_G)$) consisting of modules which are $I_0$-integrable (resp. $I_0$-equivariant ), where $I_0$ denotes the unipotent radical of the Iwahori subgroup $I \subset G[[t]]$.

0.3. The main purpose of this paper is to construct a different functor $\Gamma^\lambda$

$$\Gamma^\lambda : D_{\text{crit}}$-$\text{mod}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}, \tau(\lambda)},$$

where $\tau$ is the involution of the Dynkin diagram that sends a weight $\lambda$ to $-w_0(\lambda)$, for $w_0$ longest element in the Weyl group. We will show that $\Gamma^\lambda$, for $\lambda$ dominant, is exact and faithful. Following [FG5], this will be the point of departure for a following-up paper where an equivalence similar to $[0.2.1]$ will be shown. More precisely, the functor $\Gamma^\lambda$ can be used to construct a different functor $\Gamma^\lambda_{\text{Hecke}}$ yielding an equivalence

$$D_{\text{Hecke}}^{\text{crit}}$-$\text{mod}(\text{Gr}_G)^{I_0} \simeq \hat{\mathfrak{g}}_{\text{crit}}$-$\text{mod}_{\text{reg}, \tau(\lambda)}^{I_0}.$$

0.4. In order to explain how the functor $\Gamma^\lambda$ arises, it is important to recall the construction of the functor of global sections given in [FG]. For this, recall the chiral algebra $\mathcal{D}_{\text{crit}}$ of critically twisted chiral differential operators on the loop group $G((t))$, introduced in [AG]. As it is shown in loc. cit., it admits two embeddings

$$A_{\text{crit}} \overset{I}{\to} \mathcal{D}_{\text{crit}} \overset{r}{\leftarrow} A_{\text{crit}},$$

where $A_{\text{crit}}$ is the chiral algebra attached to the Lie*-algebra $L = \mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X$ at the critical level as explained in [BD]. If we restrict these two embeddings to the center $\mathfrak{z}_{\text{crit}}$...
of $A_{\text{crit}}$, as it is explained in [FG] Theorem 5.4, we have

\[ l(3_{\text{crit}}) = l(A_{\text{crit}}) \cap r(A_{\text{crit}}) = r(3_{\text{crit}}). \]

Moreover the two compositions

\[ 3_{\text{crit}} \hookrightarrow A_{\text{crit}} \xrightarrow{l} D_{\text{crit}} \xrightarrow{r} A_{\text{crit}} \hookleftarrow 3_{\text{crit}} \tag{0.4.1} \]

are intertwined by the automorphism $\tau : 3_{\text{crit}} \to 3_{\text{crit}}$ mentioned before.

One would expect to have some sort of functor between $D$-modules on the affine Grassmanian and $D$-modules on the loop group $G[[t]]$ that are $G[[t]]$-equivariant. However, the difficulties in defined the category of $D$-modules on the loop group, make the existence of such functor vague. However, as it is explained in [AG], it is natural to relate $D_{\text{crit}}$-mod$(\text{Gr}_G)$ with the category $D_{\text{crit}}$-mod$^{G[[t]]}$ of $G[[t]]$-equivariant $D_{\text{crit}}$-modules supported at $x \in X$. In fact the following is true.

**Theorem 0.4.1.** There exist a canonical equivalence of categories

\[ D_{\text{crit}}\text{-mod}(\text{Gr}_G) \cong D_{\text{crit}}\text{-mod}^{G[[t]]}. \]

Under the above equivalence, the functor $\Gamma$ is given by the composition

\[ D_{\text{crit}}\text{-mod}(\text{Gr}_G) \cong D_{\text{crit}}\text{-mod}^{G[[t]]} \xrightarrow{\text{For}} (\mathfrak{g}_{\text{crit}} \times \mathfrak{g}_{\text{crit}})\text{-mod}^{G[[t]]} \xrightarrow{\text{Hom}(\mathbb{V}_\ell, \cdot)} \mathfrak{g}_{\text{crit}}\text{-mod}, \]

where the forgetful functor to $(\mathfrak{g}_{\text{crit}} \times \mathfrak{g}_{\text{crit}})\text{-mod}^{G[[t]]}$ is given by the embeddings $l$ and $r$ and by the equivalence between $A_{\text{crit}}$-modules supported at $x$ and $\mathfrak{g}_{\text{crit}}\text{-mod}$. In other words, given a $D_{\text{crit}}$(Gr$_G$)-module $\mathcal{F}$, if we denote by $M_\mathcal{F}$ the corresponding $D_{\text{crit}}$-module, then $\Gamma(\text{Gr}_G, \mathcal{F})$ is given by $(M_\mathcal{F})^{\mathbb{V}_\ell} = \text{Hom}(\mathbb{V}_\ell, M_\mathcal{F})$.

**0.5. Statement of the Main Theorem.** The above construction suggests a way of defining a different functor $\Gamma^\lambda$ from $D_{\text{crit}}$-mod$(\text{Gr}_G)$ to the category $\mathfrak{g}_{\text{crit}}$-mod$_{\text{reg}, \tau(\lambda)}$ introduced earlier. In fact we can define $\Gamma^\lambda$ as the composition

\[ D_{\text{crit}}\text{-mod}(\text{Gr}_G) \cong D_{\text{crit}}\text{-mod}^{G[[t]]} \xrightarrow{\text{For}} (\mathfrak{g}_{\text{crit}} \times \mathfrak{g}_{\text{crit}})\text{-mod}^{G[[t]]} \xrightarrow{\text{Hom}(\mathbb{V}_\ell, \cdot)} \mathfrak{g}_{\text{crit}}\text{-mod}_{\text{reg}, \tau(\lambda)}. \tag{0.5.1} \]

The main theorem of this paper is that, for dominant weights $\lambda$’s, the functor $\Gamma^\lambda$ remains exact and faithful.

**Theorem 0.5.1.** For any dominant weight $\lambda$, the functor

\[ \Gamma^\lambda : D_{\text{crit}}(\text{Gr}_G)\text{-mod} \to \mathfrak{g}_{\text{crit}}\text{-mod}_{\text{reg}, \tau(\lambda)} \]

is exact and faithful.

The proof of Theorem 0.5.1 follows the line of [FG] and can be divided into two parts:

- Showing that for $\mathfrak{g}_{\text{crit}}$-modules $M_\mathcal{F}$ corresponding to $\mathcal{F} \in D_{\text{crit}}$-mod$(\text{Gr}_G)$ under Theorem 0.4.1, the functor of taking maximal submodule of $M_\mathcal{F}$ which is supported on $\mathfrak{g}_{\text{crit}}$ is exact.
- The functor $\text{Hom}(\mathbb{V}^\lambda, \cdot)$ from $\mathfrak{g}_{\text{crit}}$-mod$_{\text{reg}, \lambda}$ to the category of vector spaces Vect is exact.

In fact, part (2) follows from a result of [FG] that says that $\mathbb{V}^\lambda$ is projective in $\mathfrak{g}_{\text{crit}}$-mod$_{\text{reg}, \lambda}$. Therefore, our job in the present note is to prove the claim in part (1). This will be done using the construction of ”modification at a point” (Proposition 1.5.1) and a chiral-version of Kashiwara lemma (Proposition 1.2.2). In the case when $\lambda = 0$, which is
0.6. The paper is constructed as follows. In Section 1 we recall the main results about the center \( Z_{\text{crit}} \), the space of operas on a curve \( X \) and we introduce the construction of "modification at a point" for general chiral algebras. In section 2 we study Lie*-algebroids and chiral algebroids arising from \( Z_{\text{crit}} \). In section 3 we reduce the exactness of the functor \( \Gamma^\lambda \) to a chiral-version of Kashiwara lemma. In Section 4 we prove the Kashiwara lemma. In section 5 we prove the faithfulness of \( \Gamma^\lambda \).

0.7. Conventions. Our basic tool in this paper is the theory of chiral algebras. We will assume the reader is familiar with the foundational work [BD] on this subject. However we will briefly recall some basic definitions and notations.

Throughout this paper \( \Delta : X \hookrightarrow X \times X \) will denote the diagonal embedding and \( j : U \rightarrow X \times X \) its complement, where \( U = (X \times X) - \Delta(X) \).

For any two sheaves \( M \) and \( N \) denote by \( M \boxtimes N \) the external tensor product \( \pi_1^* M \otimes_{\Omega^*_X} \pi_2^* N \), where \( \pi_1 \) and \( \pi_2 \) are the two projections from \( X \times X \) to \( X \). For a right \( D_X \)-module \( M \) define the extension \( \Delta_!(M) \) as

\[
\Delta_!(M) = j_* j^*(\Omega_X \boxtimes M)/\Omega_X \boxtimes M.
\]

Sections of \( \Delta_!(M) \) can be thought as distributions on \( X \times X \) with support on the diagonal and with values on \( M \). If \( M \) and \( N \) are two right \( D_X \)-modules, we will denote by \( M \boxtimes N \) the right \( D_X \)-module \( M \otimes N \otimes \Omega^*_X \).

A chiral algebra over \( X \) is a right \( D_X \)-module \( A \) endowed with a chiral bracket, i.e. with a map of \( D_X \)-modules

\[
\mu : j_* j^*(A \boxtimes A) \rightarrow \Delta_!(A)
\]

which is antisymmetric and satisfies the Jacobi identity.

We will denote by \( [\ , \ ]_A \) the restriction of \( \mu \) to \( A \boxtimes A \hookrightarrow j_* j^*(A \boxtimes A) \), and we will refer to it as the induced Lie*-bracket.

By a commutative chiral algebra we mean a chiral algebra \( R \) such that \( [\ , \ ]_R \) vanishes. In other words it is a chiral algebra such that the chiral bracket \( \mu \) factors as

\[
j_* j^*(R \boxtimes R) \rightarrow \Delta_!(R \boxtimes R) \rightarrow \Delta_!(R).
\]

Equivalently, \( R \) can be described as a right \( D_X \)-module with a commutative product on the corresponding left \( D_X \)-module \( R^l := R \otimes \Omega^*_X \).

A chiral \( A \)-module is a right \( D_X \)-module \( M \) endowed with a map

\[
\mu_{A,M} : j_* j^*(A \boxtimes M) \rightarrow \Delta_!(M),
\]

satisfying certain properties. We call the restriction of \( \mu_{A,M} \) to \( A \boxtimes M \hookrightarrow j_* j^*(A \boxtimes R) \) the induced Lie*-action.

For a commutative chiral algebra \( R \), and a chiral \( R \)-module \( M \), we say that \( M \) is central if the induced Lie*-action of \( \mu_{R,M} \) vanishes. Equivalently, \( M \) is a chiral \( R \)-module such that the chiral action \( \mu_{R,M} \) factors as

\[
j_* j^*(R \boxtimes M) \rightarrow \Delta_!(R \boxtimes M) \rightarrow \Delta_!(M).
\]
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1. **The center at the critical level and the space of opers.**

Recall that, for the critical level $\kappa = \kappa_{\text{crit}}$, we denote by $A_{\text{crit}}$ the chiral attached to the Lie*-algebra $L = \mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X$ as explained in [BD], and by $\mathfrak{z}_{\text{crit}}$ its center. Recall that $\mathfrak{z}_{\text{crit}}$ is a commutative chiral algebra whose fiber $\mathfrak{z}_{\text{crit}} := \langle \mathfrak{z}_{\text{crit}} \rangle_x$ at any $x \in X$ is isomorphic to

$\mathfrak{z}_{\text{crit}} \simeq \text{End}(\mathbb{V}^0),$

where $\mathbb{V}^0$ denotes the vacuum module for $\mathfrak{g}_{\text{crit}}$ given as

$\mathbb{V}^0 := \text{Ind}_{\hat{\mathfrak{g}}_{\text{crit}}}^{\hat{\mathfrak{g}}} \mathbb{C}.$

Let $\hat{\mathfrak{z}}_{\text{crit}}$ be the topological associative algebra attached to $\mathfrak{z}_{\text{crit}}$, introduced in [BD], 3.6.18. One can show that $\hat{\mathfrak{z}}_{\text{crit}}$ is in fact isomorphic to the center of the appropriately completed twisted enveloping algebra $U_{\text{crit}}'$ of $\mathfrak{g}_{\text{crit}}$, where $U_{\text{crit}}'$ denotes the quotient $U(\mathfrak{g}_{\text{crit}})/(1 - 1)$, (here 1 denotes the identity element in $U(\mathfrak{g}_{\text{crit}})$).

Denote by $\hat{\mathfrak{g}}$ the Langlands dual Lie algebra to $\mathfrak{g}$ and let $\text{Op}_{\hat{\mathfrak{g}},X}$ be the $\mathcal{D}_X$-scheme of $\hat{\mathfrak{g}}$-opers on $X$ introduced in [BD2]. For every point $x \in X$, and coordinate $t$ around $x$, denote by $\text{Op}_{\hat{\mathfrak{g}}}(D_x^x)$ the ind-scheme of opers on the punctured disc $D_x^x = \text{Spec}((\mathbb{C}[[t]])$, and by $\text{Op}_{\hat{\mathfrak{g}}}(D_x)$ the space of regular opers. Explicitly, an oper $\nabla \in \text{Op}_{\hat{\mathfrak{g}}}(D_x^x)$ is the equivalence class, under the Gauge action of $\hat{\mathfrak{g}}(t)$, of elements of the form

\begin{equation}
\nabla = \nabla_0 + p_{-1} dt + v(t) dt,
\end{equation}

where $v(t) \in \hat{\mathfrak{g}}((t))$ and $p_{-1}$ denotes the element $p_{-1} = \sum_{i \in I} f_i$, for $I$ index set of simple roots. The condition that the oper is regular, i.e. it belongs to $\text{Op}_{\hat{\mathfrak{g}}}(D_x)$, is given by $v(t) \in \hat{\mathfrak{g}}[[t]].$

Recall now the following Theorem, established in [BT].

**Theorem 1.0.1.** There exist a canonical isomorphism of $\mathcal{D}_X$-algebras

$\mathfrak{z}_{\text{crit}} \simeq \text{Fun}(\text{Op}_{\hat{\mathfrak{g}},X}),$

in particular we have an isomorphism of commutative algebras $\mathfrak{z}_{\text{crit}} \simeq \text{Fun}(\text{Op}_{\hat{\mathfrak{g}}}(D_x))$ and of commutative topological algebras $\hat{\mathfrak{z}}_{\text{crit}} \simeq \text{Fun}(\text{Op}_{\hat{\mathfrak{g}}}(D_x^x)).$

1.1. **Poisson structure on the space of opers.** Recall now the Poisson structure on $\text{Op}_{\hat{\mathfrak{g}}}(D_x^x)$. Consider the space $\text{Conn}_G(D_x^x)$ of all connections on the trivial $G$-bundle on $D_x^x$, i.e. operators of the form

$\nabla_0 + \phi(t) dt,$

where $\phi(t) \in \hat{\mathfrak{g}} \otimes \Omega_{D_x^x}$.

If we denote by $\hat{\mathfrak{g}}_{\text{crit}}^*$ the topological dual of $\hat{\mathfrak{g}}_{\text{crit}}$, then we can identify $\text{Conn}_G(D_x^x)$ with the hyperplane in $\hat{\mathfrak{g}}_{\text{crit}}^*$ consisting of all functions $h : \hat{\mathfrak{g}}_{\text{crit}} \to \mathbb{C}$ such that the restriction of $h$ to the center of $\hat{\mathfrak{g}}_{\text{crit}}$ is the identity. Under this identification, the coadjoint action of $\hat{G}(t)$ on $\hat{\mathfrak{g}}_{\text{crit}}$ corresponds to the gauge action of $\hat{G}(t)$ on $\text{Conn}_G(D_x^x)$. The space $\hat{\mathfrak{g}}_{\text{crit}}^*$ carries a natural Poisson structure that induces a Poisson structure on $\text{Conn}_G(D_x^x)$.

Consider now the action of $\hat{\mathfrak{n}}(t)$ on $\hat{\mathfrak{g}}_{\text{crit}}$. The map $\mu : \hat{\mathfrak{g}}_{\text{crit}} \to (\hat{\mathfrak{n}}(t))^* \simeq \hat{\mathfrak{g}} / \hat{\mathfrak{b}} \otimes \Omega_{D_x^x}$ is
a moment map for this action and in particular on $\text{Conn}_G(D_x^\times)$. Moreover, from [BD2, §3.7.14], we have an identification

$$\text{Op}_{\g}^\times(D_x^\times) \simeq (\mu^{-1}(t))/\hat{N}(t).$$

for any non-degenerate characters $l$ of $\hat{n}(t)$. In other words we can construct $\text{Op}_{\g}^\times(D_x^\times)$ as the Hamiltonian reduction of $\text{Conn}_G(D_x^\times)$ along $\mu$. In particular $\text{Op}_{\g}^\times(D_x^\times)$ acquires a Poisson structure

$$\{\cdot,\cdot\} : \text{Fun}(\text{Op}_{\g}^\times(D_x^\times)) \otimes \text{Fun}(\text{Op}_{\g}^\times(D_x^\times)) \to \text{Fun}(\text{Op}_{\g}^\times(D_x^\times)).$$

The Poisson structure on $\text{Op}_{\g}^\times(D_x^\times)$ gives $\Omega(\text{Op}_{\g}^\times(D_x^\times))$ a structure of Lie algebroid over it. We will denote by $\omega$ the resulting anchor map $\omega : \Omega(\text{Op}_{\g}^\times(D_x^\times)) \to T(\text{Op}_{\g}^\times(D_x^\times))$, where $T(\text{Op}_{\g}^\times(D_x^\times))$ is the tangent sheaf to $\text{Op}_{\g}^\times(D_x^\times)$.

Denote by $I_0$ the ideal corresponding to $\text{Op}_{\g}(D_x) \subset \text{Op}_{\g}^\times(D_x^\times)$. Recall that, in [FG3] it is shown that $I_0$ is co-isotropic, i.e. that $\{I_0, I_0\} \subset I_0$. In particular $I_0/I_0^2$ is a Lie algebroid over $\text{Op}_{\g}^\times(D_x)$ and we have the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & I_0/I_0^2 & \longrightarrow & \Omega(\text{Op}_{\g}^\times(D_x^\times))|_{\text{Op}_{\g}(D_x)} & \longrightarrow & \Omega(\text{Op}_{\g}(D_x)) & \longrightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T(\text{Op}_{\g}(D_x)) & \longrightarrow & T(\text{Op}_{\g}^\times(D_x^\times))|_{\text{Op}_{\g}(D_x)} & \longrightarrow & N_{\text{Op}_{\g}(D_x)/\text{Op}_{\g}(D_x^\times)} & \longrightarrow & 0
\end{array}
$$

where $N_{\text{Op}_{\g}(D_x)/\text{Op}_{\g}(D_x^\times)}$ denotes the normal bundle of $\text{Op}_{\g}(D_x) \subset \text{Op}_{\g}^\times(D_x^\times)$ (in particular $N_{\text{Op}_{\g}(D_x)/\text{Op}_{\g}(D_x^\times)} = (I_0/I_0^2)$).

1.2. Recall now the sub-scheme $\text{Op}_{\g}^{\lambda,\text{reg}} \subset \text{Op}_{\g}^\times(D_x^\times)$ of $\lambda$-regular opers, for $\lambda$ dominant coweight of $\g$. This sub-scheme consists of equivalence classes of connections of the form

$$\nabla = \nabla_0 + \left( \sum_i t^{<\alpha_i,\lambda>} f_i \right) dt + v(t) dt, \quad \text{for } v(t) \in \hat{b}[[t]]
$$

under the action of $\hat{N}[[t]]$. In particular $\text{Op}_{\g}(D_x) = \text{Op}_{\g}^{\lambda,\text{reg}}$.

Note that the space $\text{Op}_{\g}(D_x^\times)$ of opers on the punctures disc, can be constructed as the Hamiltonian reduction along the map $\mu : \text{Conn}_G(D_x) \to \hat{n}(t)^* \simeq \hat{g}/\hat{b} \otimes D_x^\times$ by choosing the fiber of the character $\left( \sum_i t^{<\alpha_i,\lambda>} f_i \right) dt \in \hat{n}(t)^*$,

$$\text{Op}_{\g}(D_x^\times) \simeq (\mu^{-1}\left( \sum t^{<\alpha_i,\lambda>} f_i \right) dt)/\hat{N}(t).$$

Denote by $I_\lambda$ the ideal corresponding to $\text{Op}_{\g}^{\lambda,\text{reg}} \subset \text{Op}_{\g}(D_x^\times)$. We have the following lemma:

**Lemma 1.2.1.** The sub-scheme $\text{Op}_{\g}^{\lambda,\text{reg}}$ is co-isotropic, i.e. $\{I_\lambda, I_\lambda\} \subset I_\lambda$.

**Proof.** We follow the argument in [FG3, Lemma 4.4.1]. Let $\text{Conn}^{\text{reg}} \subset \text{Conn}_G(D_x^\times)$ be the subscheme of regular connections, i.e., connections of the form $\nabla = d + f(t) dt$ where
For a dominant weight $\lambda$ associative algebra $\hat{\mathfrak{g}}^{1.4}$. Deformation of the commutative chiral algebra

Proposition 1.3.1. Following [BD2] and [FG3] we have the following.

1.3. Let $\mathcal{O}_{\text{reg}}^{\lambda}$ be a cyclic $\mathcal{A}$ such that

$$\lambda,\text{reg}$$

where $\lambda,\text{reg}$ is injective, in particular we have

Proposition 1.5.1. The natural map $\Omega(\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}}) \rightarrow N_{\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}}/\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}}}$ is injective, in particular we have

$$\Omega(\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}}) \simeq N_{\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}}/\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}}}.$$

1.4. Deformation of the commutative chiral algebra $3_{\text{crit}}$. Recall the topological associative algebra $\mathcal{F}_{\lambda,\text{reg}} \simeq \text{Fun}(\mathcal{O}_{\mathfrak{g}}^{\lambda,\text{reg}})$ attached to $3_{\text{crit}}$ at the point $x \in X$.

For a dominant weight $\lambda$, recall that we denote by $\mathbb{V}^{\lambda}$ the $\mathfrak{g}_{\text{crit}}$-module

$$\mathbb{V}^{\lambda} := \text{Ind}_{\mathfrak{g}_{\text{crit}} \oplus \mathbb{C}}^{\mathfrak{g}} V^{\lambda},$$

where $V^{\lambda}$ denotes the irreducible finite dimensional representation of $\mathfrak{g}$ of highest weight $\lambda$. In particular, recall that the fiber of $3_{\text{crit}}$ at $x$ is isomorphic to $3_{\text{crit}} := \text{End}(\mathbb{V}^{\lambda})$.

We will now construct a different commutative chiral algebra $3_{\text{crit},\lambda}$, isomorphic to $3_{\text{crit}}$ on $X - x$ and such that $3_{\text{crit},\lambda} := (3_{\text{crit},\lambda})_x \simeq \text{End}(\mathbb{V}^{\lambda})$.

Note that, in particular, the isomorphism $3_{\text{crit}}|_{X - x} \simeq 3_{\text{crit},\lambda}|_{X - x}$ guarantees that the topological associative algebra $3_{\text{crit},\lambda}$ attached to $3_{\text{crit},\lambda}$ will be isomorphic to $3_{\text{crit}}$.

1.5. Consider the following general set-up. Let $\mathcal{A}$ be a chiral algebra on $X$, and let $M$ be a cyclic $\mathcal{A}$-module supported at $x \in X$, i.e. a module generated by a single element $v \in M$. Then the following is true.

Proposition 1.5.1. Under the above assumptions, there exist a chiral algebra $\mathcal{A}'$ on $X$ such that $\mathcal{A}|_{X - x} \simeq \mathcal{A}'|_{X - x}$ and such that the fiber $i^{1}_{x}(\mathcal{A})[1]$ is isomorphic to $M$. 

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Proof. The existence of a cyclic vector \( v \in M \) ensures the surjectivity of the map
\[ j_*j^*(A) \to i_*(M), \]
obtained by composing the action map with the map \( j_*j^*(A) \to j_*j^*(A) \otimes M \) given by \( a \mapsto a \otimes v \). We define \( \mathcal{A}' \) as the kernel of the above map, i.e. we have
\[ 0 \to \mathcal{A}' \to j_*j^*(A) \to i_*(M) \to 0. \]
It is easy to see that \( \mathcal{A}' \) satisfies the required properties. \( \square \)

By applying the above proposition to the case \( \mathcal{B} = \mathcal{F}_{\text{crit,}} \) and \( M = \mathcal{B}'_x = \mathcal{F}_{\text{crit,};x} \), we obtain the desired chiral algebra \( \mathcal{F}_{\text{crit,};x} \).

Let now \( \text{Op}_{\mathcal{F}_{\text{crit,}}X}^\lambda \) be the \( \mathcal{D}_X \)-scheme of \( \mathcal{F}_{\text{crit,}} \)-opers on \( X \) that have \( \lambda \)-regularity at \( x \in X \), i.e. that can be written as
\[ \nabla = \nabla_0 + \left( \sum_i t^{<\alpha_i,\lambda>} \cdot f_i \right) dt + v(t)dt, \text{ for } v(t) \in \mathfrak{b}(t) \]
around the point \( x \in X \), with coordinate \( t \). We have the following.

**Theorem 1.5.2.** There exist a canonical isomorphism of \( \mathcal{D}_X \)-algebras
\[ \mathcal{F}_{\text{crit,}} \simeq \text{Fun}(\text{Op}_{\mathcal{F}_{\text{crit,}}X}^\lambda), \]
in particular we have an isomorphism of commutative algebras \( \mathcal{F}_{\text{crit,}} \simeq \text{Fun}(\text{Op}_{\mathcal{F}_{\text{crit,}}X}^\lambda,\text{reg})\) and of commutative topological algebras \( \tilde{\mathcal{F}}_{\text{crit,}} \simeq \tilde{\mathcal{F}}_{\text{crit}} \simeq \text{Fun}(\text{Op}_{\mathcal{F}_{\text{crit,}}X}^\lambda(D^\lambda_x)). \)

1.6. Poisson structure on \( \mathcal{F}_{\text{crit}} \) and \( \mathcal{F}_{\text{crit,}} \).

1.7. We will now recall the Poisson structure on \( \mathcal{F}_{\text{crit}} \) coming from the chiral algebra \( \mathcal{A}_{\text{crit}} \).

Consider the \( \mathbb{C}[[h]] \)-family of chiral algebras \( \mathcal{A}_h \), corresponding to the bilinear form \( \kappa = \kappa_{\text{crit}} + h \cdot \kappa_{\text{kill}} \), and denote by \([\cdot,\cdot]_{\mathcal{A}_h}\) the Lie*- bracket induced by the chiral bracket on \( \mathcal{A}_h \).

For two sections \( z, w \in \mathcal{F}_{\text{crit}} \), consider two arbitrary sections \( z_h, w_h \) whose values, modulo \( h \), are \( z \) and \( w \) respectively. Then we can define a Poisson bracket \( \{z, w\} \) in the following way:
\[ \{z, w\} := \frac{1}{h} [z_h, w_h]_{\mathcal{A}_h} \pmod{h}. \]
The above expression makes sense since \([z_h, w_h]_{\mathcal{A}_h}\) vanishes \( \pmod{h} \), and it is easy to see that the element \( \{z, w\} \) is indeed in \( \Delta_! (\mathcal{F}_{\text{crit}}) \). The above bracket defines a Poisson structure on \( \mathcal{F}_{\text{crit}} \).

The isomorphism stated in Theorem 1.0.1 is compatible with this Poisson bracket.

1.8. Similarly we will now define a Poisson bracket on \( \mathcal{F}_{\text{crit,}} \) that will be compatible with the isomorphism in Theorem 1.5.2. To do this, we need to introduce a different chiral algebra \( \mathcal{A}_{\text{crit,}} \) with the property that \( \mathcal{A}_{\text{crit}}|_{X-x} \simeq \mathcal{A}_{\text{crit,}}|_{X-x} \) and with fiber at \( x \) isomorphic to \( \nabla^\lambda \). In particular the above conditions will guarantee that the center of \( \mathcal{A}_{\text{crit,}} \) will be the commutative chiral algebra \( \mathcal{F}_{\text{crit,}} \) previously defined.

The construction of \( \mathcal{A}_{\text{crit,}} \) follows from Proposition 1.5.1 by taking \( \mathcal{A} \) to be \( \mathcal{A}_{\text{crit}} \), the module \( M \) to be \( \nabla^\lambda \) and \( v \in \nabla^\lambda \) any highest weight vector in \( V^\lambda \). Now we can proceed as in the previous subsection and define a Poisson structure on \( \mathcal{F}_{\text{crit,}} \). We simply consider the \( \mathbb{C}[[h]] \)-family of chiral algebras \( \mathcal{A}_h' \), given by modifying \( \mathcal{A}_h \).
in the same way as we did for $A_{\text{crit}}$, and define the bracket of two elements in $3_{\text{crit}, \lambda}$ by the same formula. The fact that $3_{\text{crit}, \lambda}$ is the center of $A_{\text{crit}, \lambda}$ guarantees that the above expression still makes sense and in fact defines a Lie$^*$-bracket on $3_{\text{crit}, \lambda}$. Moreover the isomorphism from Theorem 1.5.2 can be upgraded to an isomorphism of Poisson algebras.

1.9. Consider now the chiral algebra $3_{\text{crit}, \lambda}$ and recall the Lie algebroid $I_{\lambda}/I_{\lambda}^2$ introduced in 1.2. Recall the diagram 1.2.2. Because of the Poisson structure on $3_{\text{crit}, \lambda}$, the $3_{\text{crit}, \lambda}$-module $\Omega(3_{\text{crit}, \lambda})$ acquires a structure of Lie$^*$-algebroid. Denote by

$$\omega : \Omega(3_{\text{crit}, \lambda}) \to T(3_{\text{crit}, \lambda})$$

the anchor map, where $T(3_{\text{crit}, \lambda})$ denotes the Lie$^*$-algebroid of vector fields on $3_{\text{crit}, \lambda}$.

Recall that, for a $D_X$-scheme $B$, geometric points of $\text{Spec}(B_x)$ are the same as horizontal sections of $\text{Spec}(B)$ over the formal disc $\text{Spec}(\mathbb{C}[\![t]\!]$, for $t$ a coordinate around $x \in X$. Let now $z$ be a geometric point of $\text{Spec}(B_x)$, corresponding to an horizontal section $\phi_z : \mathbb{C}[\![t]\!] \to B$. As it is explained in [FG] Sect. 3.5, we have

$$(1.9.1) \quad \phi_z^*(T(B))_x \simeq \left( N_{B_x/B_x} \right)_z,$$

where $\hat{B}_x$ denotes the topological algebra attached to the commutative chiral algebra $B$ and $\left( N_{B_x/B_x} \right)_z$ denotes the normal bundle at $z$ to $\text{Spec}(B_x)$ inside $\text{Spec}(\hat{B}_x)$.

In particular, if we take $B = 3_{\text{crit}, \lambda}$, and take the fiber at $x$ of the map $\omega$, we obtain a map

$$\omega : \Omega(3_{\text{crit}, \lambda})_x \simeq \Omega(3_{\text{crit}, \lambda}) \to N_{3_{\text{crit}, \lambda}/3_{\text{crit}, \lambda}}.$$

Under the isomorphism with the space of opers given by Theorem 1.5.2 and the isomorphism from Proposition 1.3.1, this map corresponds to the injective map from 1.3.

2. Lie$^*$-algebroids and chiral algebroids arising from $3_{\text{crit}, \lambda}$

In this section we will assume the reader is familiar with the notion of Lie$^*$-algebroids, chiral extensions of Lie$^*$-algebroids and chiral envelope of such, as introduced in [BD].

Recall the functor $\Gamma^\lambda$, regarded as a functor

$$\Gamma^\lambda : D_{\text{crit}}\text{-mod}(\text{Gr}_G)^{\mathbb{C}[\![t]\!]_{G'}} \to \mathfrak{g}_{\text{crit}}\text{-mod}_{\text{reg}, \tau(\lambda)}.$$

To show the exactness of the above functor, we need to study in more details the category on the left, which involves some constructions from [FG] that we will now recall.

2.1. Recall the Lie$^*$-algebroids $A^\lambda_{\text{crit}}$ and the chiral algebroid $A^{\text{ren}, \tau}_{\text{crit}}$ introduced in [FG] Sect. 4.

Recall That $A^\lambda_{\text{crit}}$ is a Lie$^*$-algebroid over $3_{\text{crit}}$ which fits into the exact sequence

$$0 \to A_{\text{crit}}/3_{\text{crit}} \to A^\lambda_{\text{crit}} \to \Omega(3_{\text{crit}}) \to 0,$$

and it is constructed using the Poisson bracket on the center $3_{\text{crit}}$ together with the chiral algebras $A_h$ introduced in 1.7. Recall, in fact, that If we denote by $A^\#_h$

$$A^\#_h = \left\{ a_h \mid a \in A_h, \ a \ (\text{mod}\ h) \in 3_{\text{crit}} \right\}.$$
then the Lie*-action of $\mathcal{A}_h^\# / \mathfrak{z}_{\text{crit}}$ on $\mathfrak{z}_{\text{crit}}$ via the projection $\mathcal{A}_h^\# \to \mathfrak{z}_{\text{crit}}$ followed by the
poisson bracket on $\mathfrak{z}_{\text{crit}}$, gives the tensor product $\mathfrak{z}_{\text{crit}} \otimes \mathcal{A}_h^\# / \mathfrak{z}_{\text{crit}}$ a Lie*-algebroid structure over $\mathfrak{z}_{\text{crit}}$. The algebroid $\mathcal{A}_{\text{crit}}^\#$ is defined in [FG] as a certain quotient of this tensor
product.

Recall the category $\mathcal{A}_{\text{crit}}^\# / \mathfrak{z}_{\text{crit}}/\mod$ of $\mathcal{A}_{\text{crit}}^\# / \mathfrak{z}_{\text{crit}}$-modules supported at $x \in X$. It consists of $\mathcal{M} \in \mathcal{A}_{\text{crit}}^\# / \mathfrak{z}_{\text{crit}}/\mod$, with an additional action of the Lie*-algebroid $\mathcal{A}_{\text{crit}}^\#$ (see [BD] sect. 2.5.16) such that:

1. As a chiral module over $\mathfrak{z}_{\text{crit}} \hookrightarrow \mathcal{A}_{\text{crit}}$ it is central.

2. The two induced Lie*-actions of $\mathcal{A}_{\text{crit}}/\mathfrak{z}_{\text{crit}}/\mod$ coincide.

3. The chiral action of $\mathcal{A}_{\text{crit}}$ and the Lie*-action of $\mathcal{A}_{\text{crit}}^\#$ on $\mathcal{M}$ are compatible with
the Lie*-action of $\mathcal{A}_{\text{crit}}^\#$ on $\mathcal{A}_{\text{crit}}$.

2.2. Recall now the chiral algebroid $\mathcal{A}_{\text{crit}}^{\text{ren},\tau}$. This is a chiral alegbroid (as defined in [BD]
sect. 3.9.) fitting into

$$0 \to \mathcal{A}_{\text{crit}} \otimes \mathfrak{z}_{\text{crit}} \to \mathcal{A}_{\text{crit}}^{\text{ren},\tau} \to \Omega(\mathfrak{z}_{\text{crit}}) \to 0,$$

constructed using the action on $\mathfrak{z}_{\text{crit}}$ given by the Poisson bracket, of a certain Lie*-algebra
mapping to it.

Recall that the category $\mathcal{A}_{\text{crit}}^{\text{ren},\tau} / \mathfrak{z}_{\text{crit}}\mod^{\text{ch}}$ of chiral modules supported at $x$, consists of $\mathcal{A}_{\text{crit}} \otimes \mathfrak{z}_{\text{crit}}\mod$, with an additional chiral action of $\mathcal{A}_{\text{crit}}^{\text{ren},\tau}$ such that the former
action coincides with the latter when restricted to $\mathcal{A}_{\text{crit}} \otimes \mathfrak{z}_{\text{crit}} \subset \mathcal{A}_{\text{crit}}^{\text{ren},\tau}$. Moreover,
following [BD] Sect 3.9.24. $\mathcal{A}_{\text{crit}}^{\text{ren},\tau} / \mathfrak{z}_{\text{crit}}\mod^{\text{ch}}$ is equivalent to the category of chiral modules
over a chiral algebra $U(\mathcal{A}_{\text{crit}}^{\text{ren},\tau})$ called the chiral envelope of $\mathcal{A}_{\text{crit}}^{\text{ren},\tau}$.

According to [FG] Theorem 5.4. the chiral algebra $U(\mathcal{A}_{\text{crit}}^{\text{ren},\tau})$ is isomorphic to the 0-th
part of the sheaf $\mathcal{D}_{\text{crit}}$ of chiral differential operators on the loop group $G(t)$ (see the
introduction for the definition of the latter).

More precisely, as a bimodule over $\mathfrak{g}_{\text{crit}}$, the fiber at any point $x \in X$ of $\mathcal{D}_{\text{crit}}$ is $G[[t]]$-
integrable with respect to both actions, and we have two direct sum decompositions of $(\mathcal{D}_{\text{crit}})_x$ corresponding to the left and right action of $\mathfrak{g}_{\text{crit}}$. These decompositions coincide
up to the involution $\tau$ and we have

$$(\mathcal{D}_{\text{crit}})_x = \bigoplus_{\lambda \text{ dominant}} (\mathcal{D}_{\text{crit}})^{\lambda}_x,$$

where $(\mathcal{D}_{\text{crit}})^{\lambda}_x$ is the direct summand supported on the formal completion of $\text{Spec}(\mathfrak{g}_{\text{crit}}^{\lambda})$.
The 0-th part of the sheaf $\mathcal{D}_{\text{crit}}$, is defined as the chiral algebra corresponding to $(\mathcal{D}_{\text{crit}})_x^0$.
We will denote such a chiral algebra by $\mathcal{D}_{\text{crit}}^0$. Hence we have

$$\mathcal{A}_{\text{crit}}^{\text{ren},\tau} / \mathfrak{z}_{\text{crit}}\mod^{\text{ch}} \simeq \mathcal{D}_{\text{crit}}^0 \mod.$$

2.3. Recall now the Lie*-algebroid $\mathfrak{a}_{\text{crit}}^{\#} := \mathcal{A}_{\text{crit}}^{\text{ren},\tau} / \mathfrak{z}_{\text{crit}}$. An $\mathfrak{a}_{\text{crit}}^{\#}$-module $\mathcal{M}$ supported
at the point $x$ is a $\mathcal{A}_{\text{crit}} \otimes \mathfrak{z}_{\text{crit}}\mod$ with an additional action of the Lie*-algebroid $\mathcal{A}_{\text{crit}}^{\#}$ such that:
(1) As a chiral module over $\mathfrak{z}_{\text{crit}} \hookrightarrow \mathcal{A}_{\text{crit}}$ it is central.

(2) The two induced Lie* -actions of $\mathcal{A}_{\text{crit}} \otimes \mathfrak{z}_{\text{crit}}$ coincide.

(3) The chiral action of $\mathcal{A}_{\text{crit}} \otimes \mathfrak{z}_{\text{crit}}$ and the Lie* -action of $\mathcal{A}_{\text{crit}}^{\lambda, \tau}$ on $M$ are compatible with the Lie* -action of $\mathcal{A}_{\text{crit}}^{\lambda, \tau}$ on $\mathcal{A}_{\text{crit}}$.

2.4. Using the chiral algebra $\mathcal{A}_{\text{crit}, \lambda}$ given by proposition 1.5.1 and its center $\mathfrak{z}_{\text{crit}, \lambda}$ with the Poisson structure explained in [FG], we can construct a Lie* -algebroid $\mathcal{A}_{\text{crit}, \lambda}^b$, a chiral algebroid $\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}$, and define the Lie* -algebroid $\mathcal{A}_{\text{crit}, \lambda}^{b, \tau}$ to be $\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau} / \mathfrak{z}_{\text{crit}, \lambda}$. The construction is a word by word repetition of what it is done in [FG], and we will omit it. The first two objects $\mathcal{A}_{\text{crit}, \lambda}^b$ and $\mathcal{A}_{\text{crit}, \lambda}^{b, \tau}$ fit into the following exact sequences

$$0 \to \mathcal{A}_{\text{crit}, \lambda} / \mathfrak{z}_{\text{crit}, \lambda} \to \mathcal{A}_{\text{crit}, \lambda}^b \to \Omega(\mathfrak{z}_{\text{crit}, \lambda}) \to 0,$$

$$0 \to \mathcal{A}_{\text{crit}, \lambda} \otimes \mathfrak{z}_{\text{crit}, \lambda} \mathcal{A}_{\text{crit}, \lambda} \to \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau} / \mathfrak{z}_{\text{crit}, \lambda} \to \Omega(\mathfrak{z}_{\text{crit}, \lambda}) \to 0$$

respectively. As before we define the appropriate category of modules over them in the following way.

The category $\mathcal{A}_{\text{crit}, \lambda}^b$-mod of modules supported at $x$ consists of $M \in \mathcal{A}_{\text{crit}, \lambda}$-mod, with an additional action of the Lie* -algebroid $\mathcal{A}_{\text{crit}, \lambda}^b$ such that:

(1) As a chiral module over $\mathfrak{z}_{\text{crit}, \lambda} \hookrightarrow \mathcal{A}_{\text{crit}, \lambda}$ it is central.

(2) The two induced Lie* -actions of $\mathcal{A}_{\text{crit}, \lambda} / \mathfrak{z}_{\text{crit}, \lambda}$ coincide.

(3) The chiral action of $\mathcal{A}_{\text{crit}, \lambda}$ and the Lie* -action of $\mathcal{A}_{\text{crit}, \lambda}^b$ on $M$ are compatible with the Lie* -action of $\mathcal{A}_{\text{crit}, \lambda}^b$ on $\mathcal{A}_{\text{crit}, \lambda}$.

The category $\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}$-mod* of chiral modules supported at $x$ consists of $\mathcal{A}_{\text{crit}, \lambda} \otimes \mathfrak{z}_{\text{crit}, \lambda}$-modules, with an additional chiral action of $\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}$ such that the former action coincides with the latter when restricted to $\mathcal{A}_{\text{crit}, \lambda} \otimes \mathfrak{z}_{\text{crit}, \lambda} \subset \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}$.

The category $\mathcal{A}_{\text{crit}, \lambda}^{b, \tau}$-mod of modules supported at $x$ consists of a $M \in \mathcal{A}_{\text{crit}, \lambda} \otimes \mathcal{A}_{\text{crit}, \lambda}$-module with an additional action of the Lie* -algebroid $\mathcal{A}_{\text{crit}, \lambda}^{b, \tau}$ such that:

(1) As a chiral module over $\mathfrak{z}_{\text{crit}, \lambda} \hookrightarrow \mathcal{A}_{\text{crit}, \lambda}$ it is central.

(2) The two induced Lie* -actions of $\mathcal{A}_{\text{crit}, \lambda} \otimes \mathfrak{z}_{\text{crit}, \lambda}$ coincide.

(3) The chiral action of $\mathcal{A}_{\text{crit}, \lambda} \otimes \mathcal{A}_{\text{crit}, \lambda}$ and the Lie* -action of $\mathcal{A}_{\text{crit}, \lambda}^{b, \tau}$ on $M$ are compatible with the Lie* -action of $\mathcal{A}_{\text{crit}, \lambda}^{b, \tau}$ on $\mathcal{A}_{\text{crit}, \lambda}$.

**Remark 2.4.1.** Note that a central $\mathfrak{z}_{\text{crit}}$-module $M$ supported at the point $x$, is the same as a $\mathfrak{z}_{\text{crit}}$-module on which the action of $\mathfrak{z}_{\text{crit}}$ factors through $\mathfrak{z}_{\text{crit}}$. Similarly, central $\mathfrak{z}_{\text{crit}, \lambda}$-modules are the same as $\mathfrak{z}_{\text{crit}}$-modules on which the action factors through $\mathfrak{z}_{\text{crit}, \lambda}$.
2.5. The unramified center. Recall the commutative chiral algebra \( \mathfrak{z}_{\text{crit}} \), its modification \( \mathfrak{z}_{\text{crit},\lambda} \), and the topological associative algebra \( \hat{\mathfrak{z}}_{\text{crit}} \) attached to them, for every \( x \in X \). Recall moreover the commutative algebras \( \mathfrak{z}_{\text{crit}} \) and \( \mathfrak{z}_{\text{crit},\lambda} \), and the ideals \( I_0 \) and \( I_\lambda \) given by the kernel of the projections \( \hat{\mathfrak{z}}_{\text{crit}} \twoheadrightarrow \mathfrak{z}_{\text{crit}} \) and \( \mathfrak{z}_{\text{crit}} \twoheadrightarrow \mathfrak{z}_{\text{crit},\lambda} \) respectively.

Denote by \( \mathfrak{z}_{\text{reg},\lambda} \) the formal completion of \( \mathfrak{z}_{\text{crit}} \) with respect to the ideal \( I_\lambda \), and by \( \text{Spec}(\mathfrak{z}_{\text{unr}}) \) the subscheme of \( \text{Spec}(\mathfrak{z}_{\text{crit}}) \) corresponding to \( \text{Op}_{\mathfrak{z}_{\text{crit}}} \), i.e. operate that are monodromy free as local systems, under the isomorphism from Theorem 1.1. We clearly have:

\[
\bigsqcup_\lambda \text{Spec}(\mathfrak{z}_{\text{crit},\lambda}) \subseteq \text{Spec}(\mathfrak{z}_{\text{unr}}) \subseteq \bigsqcup_\lambda \text{Spec}(\mathfrak{z}_{\text{reg},\lambda}).
\]

Consider the functor \( i^1_\lambda : \hat{\mathfrak{z}}_{\text{crit}}\text{-mod} \to \mathfrak{z}_{\text{crit},\lambda}\text{-mod} \) that takes a \( \hat{\mathfrak{z}}_{\text{crit}}\text{-mod} \) module to its maximal submodule scheme-theoretically supported on \( \text{Spec}(\mathfrak{z}_{\text{crit},\lambda}) \), i.e. for any module \( M \), \( i^1_\lambda(M) \) consists of sections that are annihilated by \( I_\lambda = \text{Ker}(\mathfrak{z}_{\text{crit}} \twoheadrightarrow \mathfrak{z}_{\text{crit},\lambda}) \).

Similarly, let us define \( i^\lambda : \hat{\mathfrak{z}}_{\text{crit}}\text{-mod} \to \mathfrak{z}_{\text{reg},\lambda}\text{-mod} \) as the functor that assigns to \( \hat{\mathfrak{z}}_{\text{crit}}\text{-mod} \) its maximal submodule set-theoretically supported on \( \text{Spec}(\mathfrak{z}_{\text{crit},\lambda}) \) (i.e. supported on \( \text{Spec}(\mathfrak{z}_{\text{reg},\lambda}) \)). In other words \( i^\lambda(M) \) admits a filtration whose sub-quotients are annihilated by \( I_\lambda \).

Let \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg},\lambda} \) be the subcategory of \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \) of modules on which the action of \( \hat{\mathfrak{z}}_{\text{crit}} \) factors through \( \mathfrak{z}_{\text{crit},\lambda} \), and let \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg},\lambda} \) be the subcategory of \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \) consisting of modules that, when regarded as \( \hat{\mathfrak{z}}_{\text{crit}}\text{-mod} \) modules, are set-theoretically supported on \( \mathfrak{z}_{\text{crit},\lambda} \).

Denote in the same way the corresponding functors

\[
\hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \xrightarrow{i_\lambda} \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg},\lambda} \subset \mathfrak{z}_{\text{reg},\lambda}\text{-mod}, \quad \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]]} \xrightarrow{i^\lambda} \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg},\lambda}^{G[[t]]} \subset \mathfrak{z}_{\text{reg},\lambda}\text{-mod}.
\]

It is well known that if a \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \) module \( M \) is \( G[[t]] \) integrable, i.e. if \( M \in \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]]} \), then it is supported on the disjoint union of some \( \text{Spec}(\mathfrak{z}_{\text{reg},\lambda}) \)'s. However, something more can be said. In fact we have the following proposition proved in [FG3] Theorem 1.10.

**Proposition 2.5.1.** The support in \( \text{Spec}(\mathfrak{z}_{\text{crit}}) \) of a \( G[[t]] \)-integrable module \( M \) is contained in \( \text{Spec}(\mathfrak{z}_{\text{unr}}) \).

In other words, the above proposition tells us that the image of the functor

\[
\hat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]]} \xrightarrow{i^\lambda} \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg},\lambda}^{G[[t]]},
\]

which is a priori contained in \( \mathfrak{z}_{\text{reg},\lambda}\text{-mod} \), belongs to \( \mathfrak{z}_{\text{unr}}\text{-mod} \). We will denote by \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{unr},\lambda} \) the category consisting of \( \hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \) modules, whose support in contained in the \( \lambda \) part of \( \mathfrak{z}_{\text{unr}} \).

Consider now the functor \( i^\lambda_\lambda \) as a functor

\[
i^\lambda_\lambda : \hat{\mathfrak{g}}_{\text{crit}}\text{-mod} \to \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg},\lambda}.
\]

Clearly we have \( i^\lambda_\lambda = i^1_\lambda \circ i^\lambda \). Moreover we have the following:

**Proposition 2.5.2.** The functor \( i^\lambda_\lambda : \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]]} \to \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{unr},\lambda}^{G[[t]]} \) is exact.

**Proof.** The proof follows from the following Lemma. □
Lemma 2.5.3. For any two positive weights \( \lambda \) and \( \mu \), we have \( \text{Spec}(\mathfrak{Z}_{\text{crit},\lambda}) \cap \text{Spec}(\mathfrak{Z}_{\text{crit},\mu}) = \emptyset \)

Proof. Recall from [FG6] that, for any dominant weight \( \lambda \), we have \( \text{Spec}(\mathfrak{Z}_{\text{crit},\lambda}) \cong \text{Op}_{\mathfrak{g}}^{\lambda,\text{reg}} \).
Moreover, it is clear from the description of the latter, that for \( \lambda \) and \( \mu \) dominant weights, we have \( \text{Op}_{\mathfrak{g}}^{\lambda,\text{reg}} \cap \text{Op}_{\mathfrak{g}}^{\mu,\text{reg}} = \emptyset \).

3. Proof of the Main Theorem

Recall now the functor \( \Gamma^\lambda : D_{\text{crit-mod}}(\text{Gr}_G) \to \mathfrak{g}_{\text{crit-mod,reg,}\tau(\lambda)} \), defined as the composition
\[
D_{\text{crit-mod}}(\text{Gr}_G) \cong D_{\text{crit-mod}}G[[t]] \xrightarrow{\text{For}} (\mathfrak{g}_{\text{crit}} \times \mathfrak{g}_{\text{crit}})-\text{mod}G[[t]] \xrightarrow{\text{Hom}(\mathbb{V}^\lambda, \cdot)} \mathfrak{g}_{\text{crit-mod,reg,}\tau(\lambda)}.
\]
To show the exactness of the above functor, we will compose \( \Gamma^\lambda \) with the forgetful functor to \( \text{Vect} \), and show that the composition
\[
\Gamma^\lambda : D_{\text{crit}}(\text{Gr}_G)-\text{mod} \to \text{Vect}
\]
is exact.

3.1 By formula (3.0.1), the functor of global sections \( \Gamma^\lambda : D_{\text{crit-mod}}(\text{Gr}_G) \to \text{Vect} \) can be viewed as a functor \( D_{\text{crit-mod}}G[[t]] \to \text{Vect} \) given by \( M \to \text{Hom}_{\mathfrak{g}_{\text{crit}}}(\mathbb{V}^\lambda, M) \). Recall that the module \( \mathbb{V}^\lambda \) is supported on \( \text{Spec}(\mathfrak{Z}_{\text{crit},\lambda}) \), and moreover, according to [BD2] Sect 8, is a projective generator for the category \( \mathfrak{g}_{\text{crit-mod,reg,}\lambda} \).

The fact that the support of \( \mathbb{V}^\lambda \) is \( \text{Spec}(\mathfrak{Z}_{\text{crit},\lambda}) \), makes the functor \( \Gamma^\lambda \) factor as
\[
D_{\text{crit-mod}}G[[t]] \xrightarrow{\text{For}} \mathfrak{g}_{\text{crit-mod}}G[[t]] \xrightarrow{i^!} \mathfrak{g}_{\text{crit-mod,reg,}\lambda} \xrightarrow{i_\lambda^!} \text{Hom}_{\mathfrak{g}_{\text{crit-mod,reg,}\lambda}}(\mathbb{V}, i_\lambda^!(M)) \to \text{Vect}.
\]
However, because of the projectivity result just recalled, the last functor is exact. Therefore, Theorem 0.5.1 is equivalent to the following:

Theorem 3.1.1. The composition
\[
D_{\text{crit-mod}}G[[t]] \to \mathfrak{g}_{\text{crit-mod}}G[[t]] \xrightarrow{i^!} \mathfrak{g}_{\text{crit-mod,reg,}\lambda}
\]
is exact.

3.2 Recall the categories \( A^{\text{reg,}\tau}_{\text{crit-mod}} \cong D^0_{\text{crit-mod}} \) and \( A^{\text{b,}\tau}_{\text{crit-mod}} \) introduced in [2.4].
Denote by \( D^0_{\text{crit-mod,reg,}\lambda}, D^0_{\text{crit-mod,}\text{reg},\text{unr,}\lambda} \) and \( D^0_{\text{crit-mod,}\text{reg,}\text{unr,}\lambda} \) the pre-image of \( \mathfrak{g}_{\text{crit-mod,reg,}\lambda} \), \( \mathfrak{g}_{\text{crit-mod,}\text{reg,}\lambda} \) and \( \mathfrak{g}_{\text{crit-mod,}\text{reg,}\lambda} \) under the forgetful functor
\[
\text{For} : D^0_{\text{crit-mod}} \to A^{\text{reg,}\tau}_{\text{crit-mod}} \to \mathfrak{g}_{\text{crit-mod}}
\]
respectively. Recall that a module \( M \) in \( A^{\text{b,}\tau}_{\text{crit-mod}} \) is, in particular, a central \( \mathfrak{Z}_{\text{crit,}\lambda} \)-modules, and that this is equivalent to the fact that the underlying vector space \( M \), viewed as a \( \mathfrak{Z}_{\text{crit}} \)-module, is supported on \( \text{Spec}(\mathfrak{Z}_{\text{crit},\lambda}) \), being \( \mathfrak{Z}_{\text{crit},\lambda} \) the fiber of \( \mathfrak{Z}_{\text{crit}} \) at \( x \).

Note that, moreover, we have an inclusion
\[
D^0_{\text{crit-mod,reg,}\lambda} \subset A^{\text{b,}\tau}_{\text{crit-mod}}.
\]
In fact, a module $M$ in $\mathcal{D}_{\text{crit},\lambda}^0$-mod can be seen as a module for $A^{\text{ren,}\tau}_{\text{crit,}\lambda}$ with a central action of $\mathfrak{g}_{\text{crit,}\lambda}$. In other words it receives a Lie*-action of $A^{\text{ren,}\tau}_{\text{crit,}\lambda}$ that factors through $A^{\text{ren,}\tau}_{\text{crit,}\lambda}/\mathfrak{g}_{\text{crit,}\lambda} = A^{\text{reg,}\tau}_{\text{crit,}\lambda}$.

Consider the functors $i^!_\lambda : \mathcal{D}_{\text{crit,}\lambda} \to \mathfrak{g}_{\text{crit,}\lambda}$-mod and $i^!_\lambda : \mathcal{D}_{\text{crit}} \to \mathfrak{g}_{\text{reg,}\lambda}$-mod. Denote by the same letters the functors $D$. In other words it receives a Lie*-action of $A^{\text{reg,}\tau}_{\text{crit,}\lambda}$ that factors through $A^{\text{reg,}\tau}_{\text{crit,}\lambda}/\mathfrak{g}_{\text{reg,}\lambda} = A^{\text{reg,}\tau}_{\text{crit,}\lambda}$.

Proposition 3.3.1. We can now proceed to the proof of the Theorem. Recall that we have a homeomorphism of chiral algebras

$$U(A^{\text{ren,}\tau}_{\text{crit,}\lambda}) \simeq \mathcal{D}_{\text{crit}}^0 \to \mathcal{D}_{\text{crit}}.$$

hence the forgetful functor $\mathcal{D}_{\text{crit}}$-mod$^{G[[t]]}$ factors as

$$\mathcal{D}_{\text{crit}}$-mod$^{G[[t]]} \to \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]} \to \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]}$.

We have a commutative diagram of functors

$$\begin{array}{ccc}
\mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} & \xrightarrow{i^!_\lambda} & A^{\text{reg,}\tau}_{\text{crit,}\lambda}$-mod$^{G[[t]]} \\
\mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]} & \xrightarrow{i^!_\lambda} & \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]}
\end{array}$$

Thus to prove Theorem 3.1.1 it is sufficient to show that $i^!_\lambda : \mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} \to A^{\text{reg,}\tau}_{\text{crit,}\lambda}$-mod$^{G[[t]]}$ is exact. By the discussion in §3.2 the functor $i^!_\lambda$ factors as

$$\begin{array}{ccc}
\mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} & \xrightarrow{i^!_\lambda} & \mathcal{D}_{\text{crit}}$-mod$^{G[[t]]} \\
\mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]} & \xrightarrow{i^!_\lambda} & \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]}
\end{array}$$

We can now proceed to the proof of the Theorem. Recall that we have a homeomorphism of chiral algebras

$$U(A^{\text{ren,}\tau}_{\text{crit,}\lambda}) \simeq \mathcal{D}_{\text{crit}}^0 \to \mathcal{D}_{\text{crit}}.$$

hence the forgetful functor $\mathcal{D}_{\text{crit}}$-mod$^{G[[t]]}$ factors as

$$\mathcal{D}_{\text{crit}}$-mod$^{G[[t]]} \to \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]} \to \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]}$.

By Proposition 2.5.2 the first functor is exact. Moreover, by Proposition 2.5.1 its image is contained in $\mathcal{D}_{\text{crit}}$-mod$^{G[[t]]}$. Therefore the functor $i^!_\lambda : \mathcal{D}_{\text{crit}}$-mod$^{G[[t]]} \to A^{\text{reg,}\tau}_{\text{crit,}\lambda}$-mod$^{G[[t]]}$ is the composition of the following two functors

$$i^!_\lambda : \mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} \to \mathcal{D}_{\text{crit}}$-mod$^{G[[t]]}$

$$(i^!_\lambda)_{\text{crit}} : \mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} \to A^{\text{reg,}\tau}_{\text{crit,}\lambda}$-mod$^{G[[t]]}.$

Here $(i^!_\lambda)_{\text{crit}}$ is the restriction of $i^!_\lambda$ to $\mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]}$. Hence

$$\mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} \xrightarrow{i^!_\lambda} \mathcal{D}_{\text{crit}}$-mod$^{G[[t]]} \xrightarrow{(i^!_\lambda)_{\text{crit}}} A^{\text{reg,}\tau}_{\text{crit,}\lambda}$-mod$^{G[[t]]}.$

3.3. **Proof of Theorem 3.1.1.** We can now proceed to the proof of the Theorem. Recall that we want to show that the composition

$$\mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} \to \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]} \xrightarrow{i^!_\lambda} \mathfrak{g}_{\text{crit}}$-mod$^{G[[t]]}$

is exact. This follows from the following proposition:

**Proposition 3.3.1.** The functor

$$(i^!_\lambda)_{\text{crit}} : \mathcal{D}_{\text{crit}}^0$-mod$^{G[[t]]} \to A^{\text{reg,}\tau}_{\text{crit,}\lambda}$-mod$^{G[[t]]}$

is an equivalence of category.

**Proof of Theorem 3.1.1.** Recall that we have a homeomorphism of chiral algebras

$$U(A^{\text{ren,}\tau}_{\text{crit,}\lambda}) \simeq \mathcal{D}_{\text{crit}}^0 \to \mathcal{D}_{\text{crit}}.$$
The first arrow is exact by Proposition \[2.5.2\] and the second arrow is exact by Proposition \[3.3.1\] This finished the proof.

4. Proof of Proposition \[3.3.1\]

In this section we will prove Proposition \[3.3.1\] The proof is based on Proposition \[4.2.2\] a version of Kashiwara lemma for chiral modules for a chiral Lie algebroid.

4.1. We will first prove the following proposition, that can be regarded as a different version of Kashiwara’s Theorem, where instead of the action of differential operators, we have an action of an algebroid.

**Proposition 4.1.1.** Let $Z$ and $Y$ be smooth closed subschemes of a smooth scheme $X$, such that $Z \subset Y \subset X$. Let $L$ be a Lie algebroid on $X$ preserving $Y$, such that $L \to N_{Z/Y}$. Denote by $L\text{-mod}_{Z}^{Y}$ the category of $L$-modules set-theoretically supported on $Z$ and scheme theoretically supported on $Y$. Then this category is generated modules scheme theoretically supported on $Z$.

The above proposition basically says that, if we denote by $I$ the sheaf of ideals defining $Z \subset X$, a section $m$ of $M \in L\text{-mod}_{Z}^{Y}$ can always be written as $\sum_{i} \eta_{i} \cdot m_{i}$, where $I \cdot m_{i} = 0$, and $\eta_{i} \in L$.

**Proof.** Let $I$ be the ideal sheaf of $Z$ in $X$. Choose a basis $\{l_{i}\}_{i=1, \ldots, k}$ of $N_{Z/Y}$ and let $\{l_{i}\}$ be a lifting to $L$ (It is possible since $L$ maps onto $N_{Z/Y}$). Choose $\{f_{i}\} \in I$ such that their image $\{f_{i}\}$ in $I/I^{2}$ form a dual of $\{l_{i}\}$, i.e., under the natural paring

$$
<,> : N_{Z/Y} \otimes I/I^{2} \to N_{Z/X} \otimes I/I^{2} \to \mathcal{O}_{Z}
$$

we have $< l_{i}, f_{j} >= \delta_{i,j}$. It implies $l_{i}(f_{j}) = \delta_{i,j}$ mod $I$, where $l_{i}(f_{j})$ is the natural action of $L$ on $I$.

Let $M \in L\text{-mod}_{Z}^{Y}$. We define $M^{i} = \ker(I^{i})$. Since $M$ is set theoretically supported on $Z$ we have $\bigcup M^{i} = M$. Thus it is enough to show that $M^{i}$ is generated by $M^{1}$. By induction, we can assume $M^{i-1}$ is generated by $M^{1}$. Let us show that $M^{i}$ is generated by $M^{1}$. Since $f_{i}$ maps $M^{i}$ to $M^{i-1}$ and $l_{i}$ maps $M^{i-1}$ to $M^{i}$, we have a well defined map $
abla = \sum_{i} l_{i}f_{i} : M^{i}/M^{i-1} \to M^{0}/M^{i-1}$ and it is enough to show that $\nabla$ is surjective.

We show that $\nabla$ is equal to $(i - 1)\text{id}$ on $M^{i}/M^{i-1}$. Let $x \in M^{i}$. We need to show that

$$(\nabla - (i - 1)) \cdot x \in M^{i-1}$$

and it is equivalent to show that for any $f \in I$ we have

$$f \cdot (\nabla - (i - 1)) \cdot x \in M^{i-2}.$$ 

Since $M$ is supported on $Y$, it implies the annihilator $N_{Z/Y}^{\perp} \subset I/I^{2}$ of $N_{Z/Y}$ in $I/I^{2}$ acts trivially on $M^{i}/M^{i-1}$. Since $N_{Z/Y}^{\perp}$ and $\{f_{i}\}$ span $I/I^{2}$ it reduce to show that

$$f_{j} \cdot (\nabla - (i - 1)) \cdot x \in M^{i-2}$$

for $j = 1, \ldots, k$.

Write $f_{j} \cdot (\nabla - (i - 1)) \cdot x = f_{j} \cdot (\sum l_{i}f_{i} - (i - 1)) \cdot x = \left\{ \sum l_{i}f_{i}(f_{j} \cdot x) - (i - 2)(f_{j} \cdot x) \right\} + \left\{ l_{j}(f_{j})(f_{j} \cdot x) - (f_{j} \cdot x) \right\} + \left\{ \sum_{i \neq j} l_{i}(f_{i})(f_{j} \cdot x) \right\}$. 

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In the above expression, the first term is in $M^{i-2}$ by induction. The second term is in $M^{i-2}$ because $l_j(f_j) - 1 \in I$ and $f_j \cdot x \in M^{i-1}$. The third term is in $M^{i-2}$ because $l_i(f_j) \in I$ for $i \neq j$ and $f_j \cdot x \in M^{i-1}$. The proof is finished. □

4.2. In this section we generalize Proposition 4.1.1 to chiral Lie algebroids. We begin with the review of the result of [BD, §3.9]. Let $(R, B, \mathcal{L}, \mathcal{L}^\flat)$ be a chiral Lie algebroids (cf. [BD, §3.9.6]). We assume that $R, B$ and $\mathcal{L}$ are $R$-flat. Following [BD, §3.9.24] we denote by $\mathcal{L}^\flat$-mod the category of chiral $(B, \mathcal{L}^\flat)$-module and $\mathcal{L}^\flat$-mod the category of $(B, \mathcal{L}^\flat)$-module. We have a natural adjoint pair:

$$Ind : \mathcal{L}^\flat\text{-mod} \rightleftarrows \mathcal{L}^\flat\text{-mod}^ch : For$$

where $For$ is the forgetful functor and $Ind$ is its left adjoint.

For a $M \in \mathcal{L}$-mod we define the PBW filtration on $Ind(M)$ as the image by the chiral action $j_*j^*U(B, \mathcal{L}^\flat) \boxtimes M$. If $M$ is a central $R$-module, then we have a natural isomorphism

$$M \otimes_R \text{Sym}_R \mathcal{L} \cong \text{gr}Ind(M).$$

We denote by $\mathcal{Z} = \text{Spec}(\hat{R}_e)$ and $\mathcal{Y} = \text{Spec}(R_e)$. Let $\hat{\mathcal{Y}}$ be the formal neighborhood of $\mathcal{Y}$ in $\mathcal{Z}$, and let $\mathcal{Y}'$ be an ind-subscheme of $\mathcal{Y}$ containing $\mathcal{Y}$ stable under the action of $\mathcal{L}$. Let $\mathcal{L}^\flat$-mod, be the category of chiral $\mathcal{L}^\flat$-module supported on $\mathcal{Y}'$. Let $\mathcal{L}^\flat$-mod_cent $\subset \mathcal{L}^\flat$-mod be the subcategory $(B, \mathcal{L}^\flat)$-modules which are central as $R$-module.

Assume now that $\mathcal{L}$ is elliptic, i.e., the arch map $\omega : \mathcal{L} \to \Theta(R)$ is injective and the the co-kernel of $\omega$ is projective $R$-module of finite rank. We further assume the image of $\omega_x : \mathcal{L}_x \hookrightarrow \Theta(R)_x \simeq N_{\mathcal{Y}/\mathcal{Z}}$ is equal to $N_{\mathcal{Y}/\mathcal{Z}}$.

**Lemma 4.2.1.** For any $M \in \mathcal{L}^\flat$-mod_cent, we have

a) $\text{Ind}(M) \in \mathcal{L}^\flat$-mod$^ch$.

b) $F^i(\text{Ind}(M)) = \text{Ker}(F^i : \text{Ind}(M) \to \text{Ind}(M))$, where $I$ is the ideal sheaf of $\mathcal{Y}$ in $\mathcal{Z}$.

**Proof.** Let us first prove a). We prove it by induction on the PBW filtration on $Ind(M)$. For simplicity, we denote by $F^i := F^iInd(M)$, the $i$-the filtration. By definition, $F^1$ is the image of the canonical map $M \to Ind(M)$, therefore, is supported on $\mathcal{Y} \subset \mathcal{Y}'$. Now by induction, we assume that $F^{i-1}Ind(M)$ is supported on $\mathcal{Y}'$. Since $F^{i-1}Ind(M)$ is stable under $B$ and $(j_*j^*(\mathcal{L}^\flat) \boxtimes F^{i-1}Ind(M))$ maps surjectively onto $F^iInd(M)$, taking into account that $\mathcal{L}^\flat/B \simeq \mathcal{L}$ and $\mathcal{Y}'$ is stable under the action of $\mathcal{L}$, we obtain $F^iInd(M)$ is supported on $\mathcal{Y}'$.

Proof of b). Since $\text{gr}Ind(M)$ is central $R$-module, it implies $F^i \subset \text{Ker}(F^i)$. To see the other inclusion, considering

$$f^i : I/I^2 \otimes_{\mathcal{Y}} (F^{i+1}/F^{i}) \to (F^{i}/F^{i-1}).$$

We claim that for any nonzero element $\tilde{x} \in F^{i+1}/F^i$ the image of $\tilde{x}$ under $f^i$ is nonzero. The claim will imply the inclusion. Indeed, for any $x \neq 0 \in \text{Ker}(F^i)$, let $k$ be the smallest non-negative integer such that $x \in F^{i+k}$. If $k = 0$, we are done. If not, we have $\tilde{x} \neq 0 \in F^{i+k}/F^{i+k-1}$. Considering the map

$$f : (I/I^2)^{\otimes i} \boxtimes F^{i+k}/F^{i+k-1} \to F^k/F^{k-1}.$$ 

Since $x \in \text{Ker}(F^i)$, the image of $\tilde{x}$ under $f$ is zero. On the other hand, the claim implies the image is nonzero. Contradiction.

Proof of the claim. It is enough to show that the dual of $f^i$

$$(f^i)^\vee : F^{i+1}/F^i \to F^i/F^{i-1} \otimes_{\mathcal{Y}} N_{\mathcal{Y}/\mathcal{Z}}$$

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is injective. But by Proposition 4.2.1, above map is obtained by tensoring with \( \mathcal{M} \) from
\[
\text{Sym}^y_\mathcal{L}(\mathcal{L}_x) \to \text{Sym}^y_\mathcal{L}(\mathcal{L}_x) \otimes \mathcal{L}_x \to \text{Sym}^y_\mathcal{L}(\mathcal{L}_x) \otimes N_{\mathcal{Y}/\mathcal{X}}
\]
which is injective since \( \mathcal{L} \) is elliptic. \( \square \)

By above Lemma, we obtain a functor \( i_1 : \mathcal{L}^b\text{-mod}_{\text{cent}} \to \mathcal{L}^b\text{-mod}^{ch}_{\mathcal{Y}/\mathcal{X}} \), by restriction of \( \text{Ind} \) to \( \mathcal{L}^b\text{-mod}_{\text{cent}} \). Let \( i_! : \mathcal{L}^b\text{-mod}^{ch}_{\mathcal{Y}/\mathcal{X}} \to \mathcal{L}^b\text{-mod}_{\text{cent}} \) be the functor of taking maximal submodule supported on \( \mathcal{Y} \). By construction and Lemma 4.2.1 it is easy to see that \( i_! \) is the right adjoint of \( i_1 \) and the adjunction map \( \mathcal{M} \to i_! i_1(\mathcal{M}) \) is an isomorphism.

**Proposition 4.2.2.** The functor \( i_1 : \mathcal{L}^b\text{-mod}_{\text{cent}} \to \mathcal{L}^b\text{-mod}^{ch}_{\mathcal{Y}/\mathcal{X}} \) is an equivalence of categories with inverse given by \( i_! \).

4.3. Proof of Proposition 4.2.2. We start with the following easy Lemma:

**Lemma 4.3.1.** Let \( C_1, C_2 \) be two abelian categories and let
\[
F : C_1 \rightleftharpoons C_2 : G
\]
be a pair of adjoint functors. We assume that \( G \) is conservative. If for any \( \mathcal{M} \in C_1, \mathcal{M} \to G \circ F(\mathcal{M}) \) is an isomorphism and for any \( \mathcal{N} \in C_2, F \circ G(\mathcal{N}) \to \mathcal{N} \) is surjective, then \( F \) is an equivalence of categories with inverse \( G \).

We apply above Lemma to our setting, i.e. \( C_1 = \mathcal{L}^b\text{-mod}_{\text{cent}}, C_2 = \mathcal{L}^b\text{-mod}^{ch}_{\mathcal{Y}/\mathcal{X}}, F = i_1 \) and \( G = i_! \). We already showed that \( \mathcal{M} \to i_! i_1(\mathcal{M}) \) is an isomorphism. Since \( \mathcal{Y} \subset \mathcal{X} \), it implies \( i_! \) is conservative. Therefore, it remains to prove the following:

**Lemma 4.3.2.** For any \( \mathcal{N} \in \mathcal{L}^b\text{-mod}^{ch}_{\mathcal{Y}/\mathcal{X}} \), the adjunction map \( i_1(\mathcal{N}) \to \mathcal{N} \) is surjective.

**Proof.** We applied Proposition 4.1.1 to \( \mathcal{Y} \subset \mathcal{X} \subset \mathcal{Y} \) and the algebroid \( L = H^0_{\mathcal{Y}}(D_{x*}, \mathcal{L}) \). By assumption, the morphism \( L \to H^0_{\mathcal{Y}}(D_{x*}, \mathcal{L}) \to \mathcal{L}_x \simeq N_{\mathcal{Y}/\mathcal{X}} \) is surjective, thus Proposition implies \( L \otimes i_1(\mathcal{N}) \) maps surjectively onto \( \mathcal{N} \). Since the image of \( i_1(\mathcal{N}) \to \mathcal{N} \) contains the image of \( L \otimes i_1(\mathcal{N}) \), it implies \( i_! \circ i_1(\mathcal{M}) \to \mathcal{N} \) is surjective. \( \square \)

4.4. Proof of Proposition 4.3.3.1. Let us apply above discussion to the case \( \mathcal{L}^b = \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}, \mathcal{X} = \text{Op}_{\mathcal{Y}}(D_{x}^\mathcal{X}), \mathcal{Y} = \text{Op}_{\mathcal{Y}}^{\lambda, \text{reg}} \text{ and } \mathcal{Y}' = \text{Op}_{\mathcal{Y}}^{\text{unr}}. \) By definition, we have an equivalence of categories
\[
\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\text{cent}} \simeq \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}.
\]
On the other hand, since \( \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau} \) and \( \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau} \) are isomorphic on \( X - x \), there is an equivalence of categories \( \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\mathcal{X}/\mathcal{Y}_{\mathcal{X}}} \simeq \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}^{ch}. \) Moreover, above equivalence induces an equivalence of sub-categories
\[
\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\mathcal{Y}/\mathcal{X}} \simeq \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\text{unr}, \lambda}
\]
and we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\mathcal{Y}/\mathcal{X}} & \xrightarrow{i_1} & \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\text{unr}, \lambda} \\
\downarrow{i_1} & & \downarrow{(i_1)^{-1}} \\
\mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}_{\text{cent}} & \xrightarrow{i_1} & \mathcal{A}_{\text{crit}, \lambda}^{\text{ren}, \tau}\text{-mod}
\end{array}
\]
Thus, Proposition 4.2.2 implies the following:
Theorem 4.4.1 (Proposition 3.3.1). The functor
\[(\lambda)_{\text{unr}} : \mathcal{D}_{\text{crit}}{-}\text{mod}_{\text{unr}, \lambda} \simeq \mathcal{A}_{\text{crit}}{\tau}{\text{-mod}}_{\text{unr}, \lambda} \to \mathcal{A}_{\text{crit}, \lambda}{\cdot}\text{-mod}\]
is an equivalence of categories.

5. Faithfulness

In this section we show that the functor \(\Gamma^\lambda : \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G) \to \hat{\mathcal{C}}_{\text{crit}}{-}\text{mod}\) is faithful. The proof is similar to the case \(\lambda = 0\) using ideal of Harish-Chandra action of \([BD2, \S 7.14]\). Namely, let \(K \subset G[[t]]\) be a compact open subgroup. Let \(\mathcal{D}_{\text{crit}}{\cdot}\text{-mod}(\text{Gr}_G)^K\) be the category of \(K\)-equivariant \(\mathcal{D}_{\text{crit}}\)-module on \(\text{Gr}_G\) and let \(\hat{\mathcal{C}}_{\text{crit}}{\cdot}\text{-mod}^K\) be the category of \(K\)-integrable \(\hat{\mathcal{C}}_{\text{crit}}\)-modules. It is clear that \(\Gamma^\lambda\) maps \(\mathcal{D}_{\text{crit}}{\cdot}\text{-mod}(\text{Gr}_G)^K\) to \(\hat{\mathcal{C}}_{\text{crit}}{\cdot}\text{-mod}^K\).

Let \(D^b(\mathcal{D}_{\text{crit}}(G((t))/K))\) be the bounded derived category of \(\mathcal{D}_{\text{crit}}\)-modules on \(G((t))/K\). We have the convolution functor
\[
\star : D^b(\mathcal{D}_{\text{crit}}(G((t))/K)) \times \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^K \to D^b(\mathcal{D}_{\text{crit}}(\text{Gr}_G)).
\]
Moreover, we have the action functor
\[
\star : D^b(\mathcal{D}_{\text{crit}}(G((t))/K)) \times \hat{\mathcal{C}}_{\text{crit}}{\cdot}\text{-mod}^K \to D^b(\hat{\mathcal{C}}_{\text{crit}}{-}\text{-mod}).
\]

Lemma 5.0.2. The functor \(\Gamma^\lambda = R\Gamma^\lambda : D^b(\mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^K) \to D^b(\hat{\mathcal{C}}_{\text{crit}}{\cdot}\text{-mod}^K)\) intertwines the \(D^b(\mathcal{D}_{\text{crit}}(G((t))/K))\)-action.

Proof. Let \(\mathcal{F} \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)\) and let \(M_\mathcal{F} \in \mathcal{D}_{\text{crit}}{-}\text{mod}^{G[[t]]}\) be the corresponding chiral \(\mathcal{D}_{\text{crit}}\)-module under Theorem 4.4.1. By the result in [FG, Section 1.12], we have
\[
M_\mathcal{F} = \mathcal{F} \star \mathcal{D}_{\text{crit}, x}
\]
where \(\mathcal{D}_{\text{crit}, x}\) is the fiber of \(\mathcal{D}_{\text{crit}}\) at \(x\) which is a \((\hat{\mathcal{C}}_{\text{crit}}, G[[t]])\)-bimodule. Thus for any \(\mathcal{F}_1 \in D^b(\mathcal{D}_{\text{crit}}(G((t))/K))\) and \(\mathcal{F}_2 \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^K\), we have
\[
\Gamma^\lambda(\text{Gr}_G, \mathcal{F}_1 \star \mathcal{F}_2) = \text{Hom}_{\hat{\mathcal{C}}_{\text{crit}}}(\mathcal{V}_\lambda, M_\mathcal{F}_1 \star \mathcal{F}_2 \star \mathcal{D}_{\text{crit}, x}) = \mathcal{F}_1 \star \text{Hom}_{\hat{\mathcal{C}}_{\text{crit}}}(\mathcal{V}_\lambda, \mathcal{F}_2 \star \mathcal{D}_{\text{crit}, x}) = \mathcal{F}_1 \star \Gamma^\lambda(\text{Gr}_G, \mathcal{F}_2).
\]

Proposition 5.0.3. The functor \(\Gamma^\lambda\) is faithfull, i.e., for any \(M \neq 0 \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)\) we have \(\Gamma^\lambda(\text{Gr}_G, M) \neq 0\).

Proof. We follow [FG, Section 9.10]. It is shown in [FG, Lemma 9.11], for any \(M \neq 0 \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^K\) there exists \(M' \in \mathcal{D}_{\text{crit}}{-}\text{mod}(G((t))/K)^{G[[t]]}\) such that \(M' \star M \neq 0 \in D^b(\mathcal{D}_{\text{crit}}(\text{Gr}_G)^{G[[t]]})\). Therefore, by Lemma 5.0.2, it is enough to show that \(\Gamma^\lambda(M) \neq 0\) for \(M \neq 0 \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^{G[[t]]}\). Recall that the convolution functor \(\star\) on \(\mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^{G[[t]]}\) is exact and geometric Satake identify the monoidal category \((\mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^{G[[t]]}, \star)\) with the category of representation of the dual group. In particular, it implies for any non-zero \(M \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^{G[[t]]}\) we can find \(M' \in \mathcal{D}_{\text{crit}}{-}\text{mod}(\text{Gr}_G)^{G[[t]]}\) such that there is an injection \(\delta_e \hookrightarrow M' \star M\), where \(\delta_e\) is the delta \(D\)-module at the unit \(e \in \text{Gr}_G\). Since \(\Gamma^\lambda\) is exact, we obtain an injection
\[
0 \neq V^\tau(\lambda) = \Gamma^\lambda(\delta_e) \hookrightarrow \Gamma^\lambda(M' \star M) = M' \star \Gamma^\lambda(M)
\]
and it implies \(\Gamma^\lambda(M) \neq 0\). \(\square\)
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