Reconstructing Classical Spacetimes from the S-Matrix in Twistor Space

Alfredo Guevara

1Center for the Fundamental Laws of Nature, Society of Fellows, & Black Hole Initiative, Harvard University, Cambridge, MA 02138, USA

We present a holographic construction of solutions to the gravitational wave equation starting from QFT scattering amplitudes. The construction amounts to a change of basis from momentum to (2, 2) twistor space, together with a recently introduced analytic continuation between (2, 2) and (1, 3) spacetimes. We test the transform for three and four-point amplitudes in a classical limit, recovering both stationary and dynamical solutions in GR as parametrized by their tower of multipole moments, including the Kerr black hole. As a corollary, this provides a link between the Kerr-Schild classical double copy and the QFT double copy.

INTRODUCTION

Can classical spacetimes be understood as fundamental particles? Recently, a novel body of research has shown that dynamical observables for compact bodies in GR, such as black holes, can be derived from massive QFT particles interacting with gravitons, see e.g. [1–14]. This entails that such spacetimes can be treated as perturbations of flat space sourced by these particles: Perturbative precision is attained by evaluating the classical limit of scattering amplitudes to the desired order in G. Pushing the state-of-the-art accuracy in GR, the correspondence has mainly emerged through the evaluation of observables such as radiated momentum, waveforms, or on-shell effective actions [15–64].

The correspondence suggests that asymptotically flat spacetimes can be reconstructed to some extent from a flat space gravitational S-Matrix. However, even in the classical sense, reconstructing a bulk solution from scattering amplitudes is only natural for the radiative modes reaching null infinity. This excludes so-called Coulomb or potential modes which can only explore spatial infinity [138]. To sort this out, in this work we find motivation in the analytic continuation recently studied in [65]: In (2, 2) signature Coulomb modes can probe null infinity and hence be captured by (2, 2) scattering amplitudes. We will provide evidence that these Kleinian spacetimes can indeed be reconstructed from the analytic S-Matrix, after which they can be Wick rotated back to (1, 3) signature. We obtain explicitly stationary and time-dependent solutions even outside their radiation zone, with Schwarzschild and Kerr spacetimes being perhaps our prime applications. Crucially, we rely on the observation that the three- and four-point S-Matrices for massive particles have a simple classical limit through a multipole expansion [14, 38, 47]. As solutions to the wave equation for compact sources can be described exactly via an infinite tower of multipoles, we reconstruct them by matching to the multipoles of the amplitude.

The main geometrical tool of this note is twistor theory. Two complementary perspectives play a role: On the one hand, given that twistors beautifully parametrize null geodesics, they can be directly used to reconstruct purely radiative fields from their asymptotic data at future null infinity, through the Kirchoff-d’Adhémar formula [66–69]. This application emerges naturally for the (1, 3) S-Matrix. On the other hand, twistors were historically used in (2, 2) signature to generate exact solutions to the massless equation, via the Penrose-Ward transform [70–76]. We find that this connects naturally to the (2, 2) S-Matrix and enables us to recover even non-radiative modes. As our massive states are classical, the transform only acts on massless particles, in precisely the sense introduced by Witten’s twistor string [77].

For clarity of exposition we will first be concerned with a massless scalar, the relation to the gravitational field is explained subsequently.

TWISTOR THEORY

Let us give a pragmatal introduction to the basic ingredients of twistor theory, see [69, 78–80] for recent reviews. As it turns out, the theory is especially suitable for both (2, 2) and (1, 3) signatures. It is then customary to have in mind that the coordinates inhabit a complexified spacetime \{x^\mu\} ∈ \mathbb{C}^4. Now, a twistor Z^\alpha ∈ TP^4 is the projective four-component object

\[ Z^\alpha = (\lambda^A, \eta_\lambda) ∈ \mathbb{C}^4 \text{, } A, \dot{A} = 1, 2, \]

with λ ≠ 0. Thus there is a natural fibration Z^\alpha → λ^A over \mathbb{CP}^1, which in (1,3) naturally corresponds to the celestial sphere. More generally, to connect twistor space with flat space we parametrize the latter by introducing

\[ x_{\dot{A}B} = x_\mu^{\dot{A}B} \sigma^\mu_{\dot{A}B} = \begin{pmatrix} t - x & z - y \\ -z - y & t + x \end{pmatrix}, \]

(we use the conventions of App. E of [65]; both x_\dot{A}B and Z^\alpha will be real in (2, 2) signature). Then, the condition

\[ \rho : \eta^\dot{A} = x^\dot{A}B \lambda_B, \]

is known as the incidence relation. Through it, twistors parametrize null geodesics: For a fixed Z^\alpha the set of points satisfying this condition form a null ray.
We will use $\rho_x[f(Z^\alpha)]$ to mean that we evaluate $f(Z^\alpha)$ on the support of the incidence map. Consider now a holomorphic function $f_{-2}(Z^\alpha)$ which will encode some dynamical data. For a closed contour $C$ in the Riemann sphere of $\lambda^A$, the Penrose transform is

$$\phi_f(x) = \oint_C \frac{\langle\lambda d\lambda\rangle}{2\pi} \times \rho_x[f_{-2}(Z^\alpha)]. \tag{4}$$

It is essentially an integral over null directions passing through $x$. The notation $f_{-2}$ means the function is homogeneous of degree $-2$, as required by the identification $Z^\alpha \sim tZ^\alpha$. In practice, we can fix the scale by taking $\lambda^A = (1 \, z)$.

We shall assume that $f_{-2}(Z)$ has finite many poles, encoding physical data, and that the contour $\mathcal{C}$ defines two hemispheres that contain at least one pole each. The Penrose transform then constructs non-trivial exact solutions of the wave equation: As the function $f_{-2}$ is holomorphic, i.e. independent of $\tilde{\lambda}$, we have that

$$\frac{\partial}{\partial x^A} \rho_x[f(Z^\alpha)] = \lambda_A \rho_x \frac{\partial f(Z^\alpha)}{\partial \tilde{\eta}^A}, \tag{5}$$

from which it follows immediately that (4) satisfies

$$\partial_{\tilde{A}A} \partial^{\tilde{A}A} \phi_f = 0 . \tag{6}$$

Even though the field appears free, singularities of the integral (4) in $x^\mu$ encode compact sources when interpreted in (3, 1) signature. We anticipate that $\phi_f$ will be identified with gravitational perturbations as in [65].

**ANALYTIC CONTINUATION AND RECONSTRUCTION**

We initiate our discussion of physics in (1, 3) before continuing to (2, 2). The analytic continuation picture will suggest that a twistor S-Matrix based on Feynman (F) rather than retarded (R) propagators can indeed be used to fully reconstruct exact solutions. To understand the difference we consider both cases simultaneously, using $c = F, R$ respectively.

We start with perturbations in flat space, as described by a scalar $\phi(x)$ sourced by an effective current $\mathcal{J}(k)$. It shall become clear that the scalar is related to a gravitational perturbation to a desired accuracy in $G$. The two aforementioned solutions are

$$\Phi_c(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{\mathcal{J}(k)}{(k^0)^2 - \tilde{k}^2 + i\mu(c)}, \quad c = F, R \tag{7}$$

which solely differ by the $ic$ prescription, $\mu(F) = \epsilon$ whereas $\mu(R) = \epsilon k^0$. They correspond to different boundary conditions, in the Feynman case the result is a time-symmetric solution and hence boundary conditions are required at both past and future null infinity.

Both fields contain Coulomb or stationary modes $k^0 \approx 0$ directed towards spatial infinity. On the other hand, the modes directed towards null infinity have $k^2 = 0$ and correspond to radiation/free modes. In (1, 3) such momentum regions do not overlap reflecting that stationary solutions do not radiate. Now, to connect to the S-Matrix we first aim to extract the radiative modes from the field. It turns out that this can be done easily if we parametrize the integration via spinors

$$k_{A\bar{A}} = \omega \lambda_{A} \tilde{\lambda}_{\bar{A}} + \xi q_{A\bar{A}}, \tag{8}$$

where $|\lambda\rangle, |\tilde{\lambda}\rangle$ are conjugate coordinates on $\mathbb{CP}^1$, which we shall think of as a celestial sphere. Also $\omega, \xi \in \mathbb{R}$ and $q$ is a null reference vector, which we can take such that $\langle \lambda | q | \tilde{\lambda} \rangle > 0$. After some computation we obtain

$$\Phi_c(x) = \int_{\mathbb{CP}^1} \frac{\langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}]}{(2\pi)^2} \int_\mathbb{R} \frac{\omega d\omega e^{i\frac{\omega}{2} k^0}}{4(2\pi)^2} \int_\mathbb{R} \frac{d\xi e^{i\xi k^0}}{\xi - i\mu(c)} \mathcal{J}(k),$$

$$\bar{\mu}(F) = \epsilon \omega, \quad \bar{\mu}(R) = \epsilon , \tag{9}$$

The way to extract the radiation modes with $k^2 = 0$ is now clear: We simply pick up the pole at $\xi = 0$. Thus we drop, in principle, singularities in the source $\mathcal{J}(k(\xi))$ which may lead to extra (e.g. Coulomb) modes. But we will see that such information is still there.

Assuming w.l.o.g. that $q \cdot x > 0$ we can close the $\xi$ contour upwards. Crucially, because of different $ic$ prescriptions in (9), we note that in the case $c=F$ the pole at $\xi=0$ only contributes for $\omega > 0$, i.e. the integral is restricted to positive frequencies. The two results are

$$\phi_c(x) = \int_{\mathbb{CP}^1} \frac{\langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}]}{(2\pi)^2} \int_\mathbb{C} \frac{\omega d\omega e^{i\frac{\omega}{2} k^0}}{4(2\pi)^2} \mathcal{J}(k),$$

$$L(F) = 0 , \quad L(R) = -\infty . \tag{10}$$

In fact, we have just recovered the textbook wisdom that the Feynman propagator only incorporates a particle $\omega > 0$, whereas the retarded case incorporates both particle ($\omega > 0$) and antiparticle ($\omega < 0$). By performing the full integral over $\omega \in \mathbb{R}$ the Coulomb modes cancel out between particle/antiparticle; such a signal only propagates in the light cone. In contrast, by restricting to $\omega > 0$, i.e. the Feynman prescription, the modes indeed have support outside the light-cone; hence there is hope to recover Coulomb components from amplitudes!

To interpret the formula (10) consider first $c=R$. We then identify the $\omega$ integral as $\psi(\ell_x, \ell^\mu)$, where the radiative data at null infinity is described by

$$\psi(u, \ell^\mu) := \frac{1}{8\pi^2} \int_{-\infty}^{\infty} i\omega d\omega e^{i\omega u} \mathcal{J}(\omega \ell) , \tag{11}$$

see appendix I. Here $\ell = |\lambda\rangle |\tilde{\lambda}\rangle$ parametrizes the celestial sphere and $u$ is a retarded time. In this case, our equation (10) is nothing but a (free) bulk field reconstructed...
from its radiative data $\psi$, by inverting the retarded propagation. Now, quite remarkably, it has been argued that the modes of $\psi(u)$, being purely on-shell, are given by a coherent-classical limit of scattering amplitudes \[81–84\]. For a massless particle of momentum $\omega|\lambda|/|\bar{\lambda}|$, 

$$J(k) = M(\omega|\lambda|/|\bar{\lambda}|) \quad \text{at} \quad k^2 = 0,$$  

(12)  

(other particles are implicit in the RHS). Now, what about the other case $c=F^2$? It does not yield such a simple interpretation in (1, 3) signature. It turns out, the picture becomes strikingly clear in (2, 2) where the scattering amplitudes have natural support. Let us then continue (10) and (12) by imposing that the product $\langle \lambda|x|\bar{\lambda}\rangle$ is invariant in form; this entails that $\langle \lambda d\lambda|\bar{\lambda}d\bar{\lambda}\rangle \sim i(\lambda d\lambda)(\bar{\lambda}d\bar{\lambda})$ and that $|\lambda|$ and $|\bar{\lambda}|$ are real and independent, i.e. $\mathbb{C}P^1 \sim \mathbb{R} \times \mathbb{R}$. Introducing $|\bar{\mu}| := \omega|\lambda|$ we get

$$\phi_c(x) = \int_\mathbb{R} \frac{\langle \lambda d\lambda \rangle}{2\pi} \rho_x \mathfrak{M}^c(\lambda, \bar{\eta}) \quad \text{with}$$ 

$$\mathfrak{M}^c(\lambda, \bar{\eta}) = \int_{\mathbb{R} \times L(c)} \frac{d^2 \bar{\mu}}{(4\pi)^2} e^{i[\bar{\eta]/2} \psi(\bar{\mu}) M(\lambda) |\bar{\mu}|) ,$$ 

(13) 

where the first line is precisely a Penrose transform (4)!. The relation to $\mathfrak{M}^c$, the classical amplitude in (2, 2) twistor space, constitutes the main result of this note. For $c = R$ the integration covers the full $\bar{\eta}$ plane; this is the usual definition of the twistor S-Matrix in the context of $N = 4$ SYM \[77, 85, 86\]. In contrast, the Feynman case yields a new transform that excludes the `antiparticles' \[139\] and captures the Coulomb modes, where $|\bar{\mu}| = \omega(1 + z)$ and $\omega > 0$. Here the energy integration simply becomes a Laplace transform.

Next we provide applications of the new formula for both stationary and non-stationary cases. The usual downside of the Feynman prescription, namely that it requires boundary data at both past and future null infinity, disappears in this case since (2, 2) Kleinian space has only a single null infinity \[65\].

THREE-POINT AMPLITUDES: STATIONARY SOLUTIONS

We have recently argued that linearized stationary black holes can be decomposed purely in terms of on-shell modes when continued to (2, 2) signature \[65\]. Although implicit, the continuation done there is equivalent to the Feynman prescription \[140\]. We shall see that these results also emerge naturally from twistor three-point amplitudes (see appendix IV for comparison).

Consider a $p_1 \rightarrow k + p_2$ process involving a massive particle, representing the source, and massless particle of momentum $k^\mu$. The process is classical if the massive particles are very heavy. Thus the massive momenta define a classical time direction via 

$$p_1^\mu \approx p_2^\mu \approx Mu^\mu,$$ 

(14)  

Momentum conservation yields $p_2^\mu - p_1^\mu \approx 2Mu \cdot k$ so that the corresponding classical source must have the form

$$J(k) = 2\pi\delta(u \cdot k)j(k),$$ 

(15)  

which means it is time independent and associated to Coulomb modes \[141\]. To connect this to a three-point S-Matrix we need to make sure that its kinematics, i.e. $u \cdot k = k^2 = 0$, have solutions. As anticipated this is impossible in (1, 3) signature, but happens naturally in (2, 2) signature! Plugging (15) into (13) localizes $\bar{\mu} \rightarrow \omega u^{AB}\lambda B$, giving

$$\mathfrak{M}_3(\lambda, \bar{\eta}) = \int_0^\infty \frac{d\omega}{4\pi} e^{-\bar{\pi}\bar{\eta}u|\lambda|} M_3(\omega) |\lambda| |u| , \quad c=F,$$ 

(16)  

where $M_3(k) = j(k)$ at $k^2 = 0$, while $c = R$ will lead to the anticipated vanishing result \[142\]. Now, what is the most general form of a stationary current? In QFT the currents for massless quanta are described in terms of form factors and multipoles. General multipoles do not have an intrinsic definition, having been developed via many different approaches both on QFT and classical GR fronts \[87–95\]. Luckily, for our amplitudes there is an intrinsic definition: For the massive particle, the spin degrees of freedom lead to a classical Pauli-Lubanski vector $a^\mu$ \[14, 36, 38, 46, 96\], which satisfies 

$$a \cdot u = 0,$$ 

(17)  

It then essentially follows from its on-shell conditions that the general amplitude becomes, in the classical limit, 

$$M_3(k) = \sum_n \frac{c_n}{n!}(a \cdot k)^n,$$ 

(18)  

for constants $c_n$. We derive and elaborate on this result for our scalar mode in Appendices II and III, both in vector and twistor pictures. The crucial take-away is that the operators $k^\mu_1 \cdots k^\mu_n$ are symmetric trace-free (STF) and transverse due to the conditions $u \cdot k = k^2 = 0$. Thus they project the operators $a^\mu_1 \cdots a^\mu_n$ into their STF piece, i.e. irreps of the massive little-group $sl(2, \mathbb{R})$. This is precisely the defining property of the multipole expansion, which here just follows from the on-shell kinematics of the (2, 2) three-point amplitude.

Indeed, the above becomes explicit in converting \(18\) to twistor space. Inserting it in (16) (see appendix IV for convergence) the result is

$$\mathfrak{M}_3(|\lambda|, x|\lambda|) = \sum_n \frac{c_n}{2\pi} \frac{\langle |\lambda| A|\lambda|^n}{\langle \lambda|x|\lambda|)^{n+1}},$$ 

(19)  

where we have introduced the spatial projections

$$X_{AB} = x_{\langle ABu} \bar{B}^{\rangle} ; A_{AB} = a_{\langle ABu} \bar{B}^{\rangle},$$ 

(20)  

corresponding to $x_\mu u_\nu \sigma^{\mu\nu}$ and $a_\mu u_\nu \sigma^{\mu\nu}$. From (2) we note that we can define spatial distance as

$$r^2 := \text{det}(X) = z^2 - x^2 - y^2.$$  

(21)
Feeding this into the Penrose transform (4), using (67), gives the sought classical solution
\[
\phi(x) = \frac{1}{4\pi} \sum_n \frac{(-1)^n}{n!} Q_{A_1 \cdots A_{2n}} \frac{\partial}{\partial X^{A_1 A_2}} \cdots \frac{\partial}{\partial X^{A_{2n-1} A_{2n}}} \frac{1}{r^n},
\]
where the multipole moments are \( Q_{A_1 \cdots A_{2n}} = c_n A_{(A_1 A_2} \cdots A_{A_{2n} A_{2n})} \) in agreement with the twistor multipoles of [79, 97]. Here we have used the residue theorem
\[
\int \frac{1}{2\pi} \frac{\langle \lambda d\lambda \rangle}{\langle \lambda |X|\lambda \rangle} = \frac{1}{2\sqrt{\text{det}(X)}} = \frac{1}{2r},
\]
by noticing that the contour \( C = \mathbb{R} \) separates the two conjugate roots of \( \langle \lambda |X|\lambda \rangle \), as required for the Penrose transform. An advantage of this language is that the implicit STF tensors have explicitly become the fully symmetric \( s(l,2,\mathbb{R}) \) irreps appearing in (22) or (19). They correspond to \( m = 0 \) harmonics as shown in appendix III.

To rotate back to (1, 3) signature we can use \( i(x, y) \to i(x, y) \) in (21). We see that the terms in (22) scale as \( 1/r^n \), and are thus hidden from the \( 1/r \) order that defines radiation at null infinity. This reveals that we have recovered the full tower of Coulomb modes and that the field is not free in (1, 3). A perhaps more familiar form of (22) is
\[
\phi(x) = \sum_n \frac{(-1)^n}{4\pi} Q^{i_1 \cdots i_n} \times \frac{\partial}{\partial x^i_1} \cdots \frac{\partial}{\partial x^i_n} \left( \frac{1}{r} \right),
\]
where \( Q^{i_1 \cdots i_n} = c_n a^{i_1} \cdots a^{i_n} \) as expected. This is the multipole form of the general stationary solution for compact sources. It is remarkable that we have reconstructed it from an a priori radiative three-point S-Matrix. Indeed, the key ingredients in the characterization of the solution, given in (20), are here directly obtained from linear and angular momentum of QFT massive particles.

### GRAVITATIONAL FIELD FROM A SCALAR POTENTIAL

It is easy to extend the above to gravity or gauge theory and connect it with the Penrose transform for higher helicity. For a vacuum solution the gravitational field is naturally associated to higher multipole moments e.g. \[79, 97\]. Too see how this relation emerges here, consider first a linearized \( O(G) \) metric \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \) that is stationary with respect to a time direction \( u^\mu \).

Following [65], we can decompose it into (anti)self-dual components \( h^+_{\mu\nu} \) and \( h^-_{\mu\nu} \), and then consider the scalar function
\[
\phi^\pm = u^\mu u^\nu h^\pm_{\mu\nu} = h_{00}^\pm.
\]
Being the norm of a Killing direction, \( \phi^\pm \) are scalar invariants in the linearized theory. In fact, these two scalars are nothing but the two on-shell degrees of freedom of a free massless particle as explained in appendix II.

The two self-dual pieces of the curvature tensor can be obtained in spinor components. Recalling that \( \sigma^\pm_{AB} \) is self-dual in its Lorentz indices, the projection is given by
\[
C^+_{ABCD} = 2\sigma^+_{AB} \sigma^-_{CD} \partial_\mu \partial_\nu h^00 = \frac{2}{\sqrt{\text{det}(\partial X AB \partial X CD)}} \phi^+(x),
\]
where \( \sigma^0_{AB} x^i = \frac{1}{2} X_{AB}, \sigma^1_{AB} \partial_{AB} = \frac{1}{2} \partial_\mu \). The conjugate relation holds for \( C^-_{ABCD} \). Now, we argued in [65] that \( \phi^\pm \) are in correspondence with two 3-point amplitudes \( M^\pm_3 \), representing graviton emission from a spinning particle. Using this, we can insert the Penrose transform (13), (16) into equation (26), leading to the formulae
\[
C^+_{ABCD} = \frac{1}{2\pi} \int_C \langle \lambda d\lambda \rangle \times \lambda \lambda_{AB} \lambda \lambda_{CD} \rho_\epsilon [\mathfrak{M}^+_3 \langle Z^\alpha \rangle],
\]
\[
C^-_{ABCD} = \frac{1}{2\pi} \int_C \langle \lambda d\lambda \rangle \times \rho_\epsilon [\tilde{\partial}^4 \tilde{\eta}^A \tilde{\eta}^B \tilde{\eta}^C \tilde{\eta}^D [\mathfrak{M}^-_3 \langle Z^\alpha \rangle],
\]
where we have followed analogous steps for \( \phi^- \). The homogeneous functions (of degrees \(-6 \) and \(+2 \)) are
\[
\mathfrak{M}^+_3 (Z^\alpha) = \frac{1}{4} \int_0^\infty \frac{d\omega}{2\pi} e^{-\frac{-3}{2} |\eta| |u|} M^+_3 (\sqrt{\omega}|\lambda|, \sqrt{\omega}|\lambda| u)
\]
\[
\mathfrak{M}^-_3 (Z^\alpha) = 4 \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-\frac{-3}{2} |\eta| |u|} M^-_3 (\sqrt{\omega}|\lambda|, \sqrt{\omega}|\lambda| u)
\]
(\( \eta \to 0 \)). Formulæ (27)-(28) are indeed the Penrose transform for the curvature tensor \[68, 98\]. Here we have connected it to three-point amplitudes through (29) and (30). Similar formulæ hold at higher orders in \( G \), as well as non-stationary metrics as detailed somewhere else \[99\].

To present a direct application of the construction, let us focus on the Kerr black hole and compute its curvature tensors (27)-(28). We start with the leading order in \( G \); Remarkably, it corresponds to the \emph{minimally coupled} 3-point amplitudes introduced in \[7, 14, 45\]. They can be obtained from our generic multipole expansion (18) by fixing the coefficients as \( c_n^\pm = 4\pi GM(\pm)^n \), i.e.
\[
M^+_3 (\sqrt{\omega}|\lambda|, \sqrt{\omega}|\lambda| u) = 4\pi GM e^{+\frac{3}{2} \frac{\lambda}{|\lambda|} |\lambda| A^1 |\lambda|},
\]
where we used (20). Inserting this into (29) we get
\[
\mathfrak{M}^+_3 (|\lambda|, x|\lambda|) = \frac{GM}{2} \int_0^\infty d\omega \omega^2 e^{-\frac{3}{2} |\eta| |\lambda|} J_+ |\lambda| = \frac{8GM}{|\lambda| J_+ |\lambda|^3}.
\]
whereas $\mathfrak{M}_3^4(Z^\alpha)$ is a function of $J_{AB}^-$. Here
\[ J_{AB}^+ = X_{AB} \pm A_{AB} = 2(x_\mu u_\nu) \pm a_\mu u_\nu)\sigma^\mu\nu_{AB}. \quad (33) \]

We recognize them as the angular momentum and Killing-Yano tensors of the Kerr metric [143]. The relative sign signals that the spin $a_\mu$ is a pseudovector. They have an important geometric meaning: The Penrose transform of (32) is a simple quadrupolar integral (68) and gives
\[ C_{ABCD}^+ = \frac{3GM/2}{[(z + a)^2 - x^2 - y^2]^{3/2}} J_{(AB}^+ J_{CD)}^+. \quad (34) \]

Now, any symmetric spinor can be written in terms of its roots $J_{AB}^\pm = \mu(AK_B)$ satisfying $\langle \mu | \kappa \rangle = \langle \mu | \kappa \rangle$ [144]. We normalize them by introducing $\langle \kappa | : = | \kappa \rangle / \langle \mu | \kappa \rangle$. Further continuing the coordinates to (1, 3) signature as in (65), we obtain
\[ C_{ABCD}^{+,(1,3)} = -\frac{6GM}{(r + i\alpha \cos \theta)^5} \mu(A\hat{k}_B \mu C \hat{k}_D), \quad (35) \]

i.e. the known form of the curvature tensor of Kerr [100]. The decay $\sim 1/r^3$ again confirms Coulomb modes well hidden from the usual radiative expansion. Moreover, the Penrose transform has given us an exact result in $G$: This will occur for the classical spacetimes satisfying the so-called Kerr-Schild/Weyl double copy of [101, 102]. By definition, the spinors $\mu_A$ and $\hat{k}_A$ form the principal null directions (PNDs) of the spacetime. Even more, (33) is nothing but the sum of orbital and intrinsic angular momenta of the Kerr black hole [68]! It is remarkable that here it originates directly from the QFT momentum and spin vector of massive particles.

**FOUR-POINT AMPLITUDES: RADIATIVE MOMENTS**

As a last example we consider time-dependent perturbations of spacetimes, which emerge naturally in their dynamics. One can for instance consider a scalar perturbation $\phi$ of a Schwarzschild background controlled by the Regge-Wheeler equation: such perturbations impinge the source from past null infinity and radiate to future null infinity [103]. The extension to Gravitational Waves is straightforward. The QFT description of such processes in terms of a four-point S-Matrix in (1, 3) was recently studied in [83, 99].

To see how this emerges here, we extend the previous case by considering an additional wave of momentum $k'$ scattering in the source, i.e. $p_1 + k' \rightarrow p_2 + k$. The corresponding on-shell condition is $p_2^2 - p_1^2 \approx 2Mu - (k - k')$ and hence
\[ J(k) = 2\pi \delta(u \cdot k - u \cdot k')j(k). \quad (36) \]

In (1, 3) signature, this is radiative because the massive source is accelerated by an incoming wave. Identifying the source with a linearized black hole, it was found in [83] that the Regge-Wheeler amplitudes $j(k) = M_4(k)$ solely depend on the direction of $k^\mu = \omega \ell^\mu$ in the long-wavelength regime. Thus they can be easily written in terms of spherical harmonics, see appendix III,
\[ M_4(k) = m_0 + m_1 \ell \cdot d / \ell \cdot u + \ldots = m_0 + m_1 \langle \lambda | d | \lambda \rangle + \ldots \quad (37) \]

where the dipole vector satisfies $\cdot u = 0$ and we have parametrized $\ell = |\lambda| |\lambda|$. The numerical moments $\{m_i\}$ can be computed from QFT.

We can reconstruct a bulk solution from the radiative Regge-Wheeler amplitude by procuring via (2, 2) signature again. We illustrate this with the two terms in (37). Plugging (36) into the Feynman Twistor transform (13), we find $\mathfrak{M}_4(|\lambda|, x |\lambda|)$ to be given by
\[ e^{-\omega'(r-t)} \left( \frac{m_0}{\langle \lambda | X | \lambda \rangle} + \frac{im_1}{\omega'} \left( \langle \lambda | X | \lambda \rangle^2 + \langle \lambda | D | \xi \rangle \langle \xi | X | \lambda \rangle \right) \right) \quad (38) \]

where $\omega' = u \cdot k'$ and $D_{AB} = d_{(AB}^\tilde{k}_{B)}$. Here $\xi$ is a reference spinor. The first two terms are again Coulomb multipoles in twistor space, c.f. (19). The last term is a new, radiative contribution $\sim 1/r$. The solution is time dependent, as can be seen by using (20) to decompose $x_{A\hat{B}}u^A = X_{AB} + \epsilon_{AB}t$ in the exponent. Plugging this into the Penrose transform, analogously to (22), we obtain
\[ \phi_{\omega'} = \frac{e^{-i\omega'(r-t)}}{4\pi} \left( \frac{m_0}{r} - \frac{im_1}{\omega'} \frac{d \cdot x}{r^2 + i\omega' t} \right). \quad (39) \]

We have recovered an exact solution of the (2, 2) wave equation, including both Coulomb and radiative modes. At face value it is neither an ‘in’ or ‘out’ solution but resembles a quasinormal mode decaying in time. However, one can obtain a (1, 3) solution simply by rotating $t \rightarrow -it$. It corresponds to an outgoing mode, which at null infinity yields the radiative data
\[ \lim_{r \rightarrow \infty} r \phi_{\omega'} = \frac{e^{i\omega' u}}{4\pi} \left( m_0 + m_1 \frac{d \cdot \ell}{u \cdot \ell} \right). \quad (40) \]

Here the retarded time $u : t-r$ is fixed and $\ell$ now corresponds to the celestial sphere. As in [83], this translates directly into our starting data (37), see also appendix I. The interpretation of (39) is now clear: It is the outgoing mode associated to a monopole-dipole excitation of frequency $\omega'$. Indeed, smearing over $\omega'$ frequencies yields a more familiar form of such solution:
\[ \int d\omega' \tilde{a}(\omega') \partial_u \phi_{\omega'} = \frac{ma \partial_u a(u)}{r} + m_1 d \cdot \hat{x} \left[ \frac{a(u)}{r^2} + \frac{\partial_u a(u)}{r} \right] \quad (41) \]
where \( \tilde{a}(\omega') \) and \( a(u) \) are Fourier conjugates. Finally note that in deriving the form (36) we have also assumed that an incoming solution of frequency \( \omega' \) exists. In (1, 3) the incoming solution is needed by physical considerations to cancel out outgoing radiation from the black hole [103]. However, from the perspective of the (2, 2) space, the sole outgoing solution appears consistent since there is no black hole source at \( r = 0 \) [65].

**OPEN QUESTIONS**

Via twistor theory, we have just scratched the surface of using the analytic S-Matrix to reconstruct solutions of gravitational wave equations. Remarkably, both radiative and Coulomb modes arise on the same footing in (2, 2), and in a sense correspond to purely outgoing plane waves. In turn, from our last discussion it would seem that we lost the information of (1, 3) incoming radiation, although it may be possible to use crossing symmetry of the S-Matrix to recover it.

On the other hand, the fact that null infinity has only one connected component in (2, 2) strongly suggests that the reconstruction is suitable for realistic scenarios with no-incoming radiation [95, 104], e.g. the generation of gravitational waves. The associated amplitudes may have more matter sources but only one outgoing graviton [6, 47, 51, 105, 106], making them suitable for the techniques presented (see [34] for a recent discussion of retarded vs Feynman propagators in this context). They are controlled to some extent by soft factorization [46]: Indeed the amplitude \( M_3 \) treated here is the simplest instance of such, where the Coulomb poles of the soft factor 
\[
\frac{1}{u - k}
\]
are equivalent to the delta function \( \delta(u \cdot k) \).

Our twistor transform (13) amounts to a change of basis of the gravitational S-Matrix. Similar transforms have unveiled new symmetries of the gauge-theory S-Matrix to all loop orders, see e.g. [107, 108]. It would be interesting to explore this vein for our amplitudes. Indeed, our construction relies on the relation between classical spacetimes and gravitational amplitudes, which has been tested to high loop precision [17, 33, 50, 84, 109]. It is a pressing question to understand how non-linear effects interrelate both sides, especially in view that the gravitational field is not free asymptotically in the non-linear theory, which is reflected as an IR dressing in the perturbative theory.

For certain scenarios such as Kerr-Schild (KS) spacetimes the linearized examples we have discussed should suffice. In fact, [78, 79] have argued that the Kerr-Schild double copy structure [101, 102, 110–117] follows from the Penrose transform. Due to this, it can be readily checked that our formula (13) links the KS double copy directly to the double copy between gauge and gravity massive amplitudes of [46], also developed in [47, 105, 106, 118–127].

On the other hand, in [72] Penrose argued against perturbation theory. His seminal work indeed showed that the Penrose transform is an exact formula for self-dual solutions [70–74, 76, 128]. Here we have relied on perturbation theory. Still, there is hope that both approaches can be reconciled through the connection with amplitudes. For instance, amplitudes in self-dual gravity are known to be 1-loop exact [129], which suggests a close connection to the exact Penrose transform (see [69, 80, 130, 131] for related recent works).

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*aguevaragonzalez@fas.harvard.edu

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parametrizes the celestial sphere. Using the radiative field in the bulk (zero modes correspond to $J$ the residue at $\xi$) advanced solutions, such that Coulomb modes cancel out us characterize radiative solutions in the somewhat more standard fashion.

There is no intrinsic definition of antiparticles in (2, 2) as $s(2, \mathbb{R})$ invariant (this may be related to the discussion in [85]). This is the one of the reasons we resort to analytic continuation.

The related work [81] has examined the retarded (2, 2) solution $k_x \rightarrow \infty$, with $k^0 = 0$ in (9), with $\mu = 0$ everywhere and hence can be expanded in free modes. Radiation is given exclusively by the presence of the step functions shows that we include both particle/antiparticle phase space as expected for the retarded conditions as opposed to Feynman’s. On the other hand, for Coulomb solutions with $k^0 = 0$ we find that $\phi_R = \Phi_A$ and hence the integral vanishes.

We usually characterize null (radiation) data as follows. As $r \rightarrow \infty$, with $u := t - r$ fixed, the saddle point approximation can be used to localize the oscillatory integrals in (42). They will only receive contributions from the null directions $k_\mu = \omega \ell_\mu(x)$ that reach a particular point $x$, where

$$\ell_\mu(x) = \left(1, \hat{n} = \frac{x}{r}\right),$$

parametrizes the celestial sphere. Using $u = \ell(x) \cdot x$ the radiative data is then given by the asymptotic mode expansion [132]

$$\tilde{\phi}(u, \ell) := \lim_{r \rightarrow \infty} r\Phi_R(x) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} d\omega \times e^{i\omega u} J(\omega \ell^\mu) .$$

This field fulfils $\partial^2 \phi = 0$ everywhere and hence can be expanded in free modes. Radiation is given exclusively by the leading $1/r$ component of the fields at future null infinity. In this regime, $\phi(x)$ contains the same data as the retarded solution $\phi_R(x)$, since the advanced solution vanishes. Thus (42) confirms the correspondence between radiative data and the on-shell modes $k^2 = 0$. The presence of the step functions shows that we include both particle/antiparticle data as follows. As $r \rightarrow \infty$, with $u := t - r$ fixed, the saddle point approximation can be used to localize the oscillatory integrals in (42). They will only receive contributions from the null directions $k_\mu = \omega \ell_\mu(x)$ that reach a particular point $x$, where

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We are now in a good position to connect to twistor theory. Indeed, a simple way of resolving the bulk integration (42) has been already provided in the main text: Note that the delta function in the integrand amounts to extract the residue at $\xi = 0$ in (9), with $c = R$. Defining the ‘curvature’ $\psi(u) := \partial_u \phi$ the result is then, as stated

$$\phi(x) = \int_{\mathbb{CP}^1} \frac{\lambda d\lambda}{(2\pi)^2} \psi(\ell \cdot x, \ell) , \ell = |\lambda| |\tilde{\lambda}|$$

(45)

(3)

the field $\psi$ is the scalar analog of the Weyl radiative component $\psi_4 = C_{xuuz}$ in Bondi coordinates. Up to zero modes of $\tilde{\phi}(u)$ that do not enter in the curvature, namely terms that survive as $u \rightarrow \pm \infty$, this result completely reconstructs the radiative field in the bulk (zero modes correspond to $\mathcal{J}(k) = \delta(k^0) \times \text{constant}$; in such case the field $\phi$ vanishes but its radiative data (44) does not). Indeed this is nothing but the Kirchoff-d’Adhémar formula introduced by Penrose in [66, 68]. Our derivation in momentum space allowed us to make direct contact with the S-Matrix. In contrast,

Appendices

I. Twistors in (1, 3) Spacetime and Radiative Solutions

In a beautiful geometrical construction, Penrose derived a formula that enables direct bulk reconstruction of free radiative solutions [68]. It is based on a map from twistors to null infinity. Before we revisit it in this appendix, let us characterize radiative solutions in the somewhat more standard fashion.

Continuing with the scalar model (7), the purely radiative solution is constructed by subtracting the retarded and advanced solutions, such that Coulomb modes cancel out

$$\phi(x) := \Phi_R - \Phi_A = \int \frac{d^3k}{(2\pi)^3} e^{ikx} [\theta(k^0) + \theta(-k^0)] \delta(k^2) \mathcal{J}(k) , \Phi_A = \Phi_R^* .$$

(42)

This field fulfils $\partial^2 \phi = 0$ everywhere and hence can be expanded in free modes. Radiation is given exclusively by the leading $1/r$ component of the fields at future null infinity. In this regime, $\phi(x)$ contains the same data as the retarded solution $\phi_R(x)$, since the advanced solution vanishes. Thus (42) confirms the correspondence between radiative data and the on-shell modes $k^2 = 0$. The presence of the step functions shows that we include both particle/antiparticle phase space as expected for the retarded conditions as opposed to Feynman’s. On the other hand, for Coulomb solutions with $k^0 = 0$ we find that $\phi_R = \phi_A$ and hence the integral vanishes.

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$$\ell_\mu(x) = \left(1, \hat{n} = \frac{x}{r}\right) ,$$

parametrizes the celestial sphere. Using $u = \ell(x) \cdot x$ the radiative data is then given by the asymptotic mode expansion [132]

$$\tilde{\phi}(u, \ell) := \lim_{r \rightarrow \infty} r\Phi_R(x) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} d\omega \times e^{i\omega u} J(\omega \ell^\mu).$$

(44)
his proof followed a completely different argument, using the following twistor interpretation: In (1,3) signature the celestial sphere coordinates $\lambda$ and $\bar{\lambda}$ are complex conjugates. One can thus define the map from $\mathbb{T}P$ to future null infinity $\mathcal{I}^+$ as

$$Z = (\lambda, \bar{\mu}) \rightarrow (u = \frac{1}{2}[\bar{\mu}\bar{\lambda}], \lambda, \bar{\lambda}) \in \mathcal{I}^+ \tag{46}$$

Via such map, the radiative data (44) becomes a function in twistor space $\tilde{\phi}(Z, \bar{Z})$. The integrand in (45), namely $\rho_x \partial_u \tilde{\phi}$, is not holomorphic in contrast with the Penrose transform. However, it is still straightforward that (45) satisfies

$$\partial^A \partial_A \phi(x) = 0. \tag{47}$$

Under the incidence relation (3) the retarded time $u = \frac{1}{2}[\bar{\mu}\bar{\lambda}]$ admits the geometrical interpretation of being the point where the ray $\ell = |\lambda|\bar{\lambda}$ emanating along the null cone of $x$ intersects future null infinity. The celestial sphere parametrized by $\lambda, \bar{\lambda}$ then corresponds to a cross section of such null cone. Penrose's proof relies on the observation that the integral (45) is nicely independent of such cross section.

II. Potential Modes from the Worldline EFT

Equation (18) corresponds to a formula for three-point amplitudes associated to a scalar field. As discussed throughout the text, such potentials correspond to on-shell free modes of photons or gravitons. Here we delive into this correspondence for the case of gauge-theory, the graviton situation being analogous but notationally inconvenient. For illustration purposes we follow a complementary picture to that of the scattering amplitudes in [38, 46], provided by worldline EFT models [133–136]. We consider particles carrying both electric and magnetic charges, the latter corresponding to the NUT parameter in the gravitational case [136].

A compact object in Maxwell theory can be described by a classical worldline action carrying an infinite tower of Wilson coefficients. The interacting piece of the worldline action is

$$S_{\text{int}} = \int (q_0 A_\mu + i\tilde{q}_0 A^*_\mu) u^\mu d\tau + i \sum_{n=0}^{\infty} \frac{1}{(n + 1)!} \int d\tau \left[ q_{n+1} (ia \cdot \partial)^n F_{\mu\nu} u^\mu a^\nu + i\tilde{q}_{n+1} (ia \cdot \partial)^n F_{\mu\nu}^* u^\mu a^\nu \right] \tag{48}$$

Here $F_{\mu\nu}^*$ is the Hodge dual of $F_{\mu\nu}$, which can be written as a closed form $F_{\mu\nu}^* = 2\partial_{[\mu} A_{\nu]}^*$. This action is closely related to the one presented in [136] but differs in that it carries magnetic charges. In fact, note that because $a^\mu$ and $A^*_\mu$ are pseudovectors, a parity transformation acts as $a^\mu \rightarrow -a^\mu$, $A^*_\mu \rightarrow -A^*_\mu$, which implies that the Wilson coefficients must have a parity odd piece, i.e. magnetic charges. At leading order in spin it is given by the coupling $\tilde{q}_0$ in the first term.

Now, since we are interested in vacuum configurations it is convenient to introduce the self-dual components

$$A^\pm_\mu = \frac{1}{\sqrt{2}} (A_\mu \pm iA^*_\mu) \tag{49}$$

which parametrize the two degrees of freedom of a photon. They are independent modes with associated currents

$$j^{\pm\mu}(x) = \frac{\delta S_{\text{int}}}{\delta A^\pm_\mu(x)} = \int d\tau \left[ c^\pm_n u^\mu + \sum_n \frac{2c_{n+1}^- (ia \cdot \partial)^n}{(n + 1)!} iu^\mu a^\nu \partial_\nu \right] \delta^4(x - x(\tau)) \tag{50},$$

$$c^\pm_n = q_n \pm \tilde{q}_n \sqrt{2}. \tag{50}$$

Assuming no incoming radiation, the trajectories of the compact object are given by constant vectors $u^\mu(\tau) = u^\mu$ and $a^\mu(\tau) = a^\mu$, hence we can take $x^\mu(\tau) = u^\mu \tau$. Further dropping the total derivative term $u^\mu \partial_\mu = \frac{d}{d\tau}$ (the analog of three-point kinematics $u \cdot k=0$) a short computation shows that in Fourier space (50) becomes

$$j^{\pm\mu}(k) = u^\mu 2\pi \delta(u \cdot k) \times \rho^\pm(\hat{k}) \text{ , with } \rho^\pm(\hat{k}) = \sum_n \frac{c^\pm_n (\hat{a} \cdot \hat{k})^n}{n!} \tag{51}$$
The charge densities $\rho^\pm$ then yield a multipole expansion in momentum space, and indeed correspond to the three-point amplitudes of the main text. To connect with the more familiar definition of multipole moments we introduce position-space densities via

$$
\rho^\pm(x) = \int d^3x e^{i \vec{k} \cdot \vec{x}} \rho^\pm(x)
$$

$$
= \sum_n \frac{i^n \vec{k}^{n \cdot 1} \cdots \vec{k}^{n \cdot n}}{n!} \int d^3x \vec{x}^{n \cdot 1} \cdots \vec{x}^{n \cdot n} \rho^\pm(\vec{x})
$$

$$
=: \sum_n \frac{i^n \vec{k}^{n \cdot 1} \cdots \vec{k}^{n \cdot n} Q^\pm_{i_1 \cdots i_n}}{n!}
$$

(52)

where we have expanded in $\vec{k}$, after which we defined the multipole moments of a charge distribution as usual. By comparing to (51) we obtain

$$
Q^\pm_{i_1 \cdots i_n} = (-i)^n c_n^{a_1 \cdots a_i_n} \rho^\pm_{i_1 \cdots i_n}
$$

(53)

as in the main text. These multipole moments are transverse with respect to $u^\mu$ but fail to be trace-free. However, this distinction is irrelevant as we now show.

As argued in the text, the two free degrees of freedom can be described by self-dual potentials. This is because in a self-dual configuration the magnetic and electric fields are proportional to $E_i^\pm = \partial_i \phi^\pm$ (or their ‘gravitoelectric’ versions in the gravity case). The potential follows from (51)

$$
\phi^\pm(x) = \int \frac{d^4k}{(2\pi)^4} \frac{2\pi \delta(u \cdot k) \times \rho^\pm(\vec{k})}{k^2} = \int \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \rho^\pm(\vec{k}) = \sum_n \frac{i^n Q^\pm_{i_1 \cdots i_n}}{n!} \int \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \vec{k}^{n \cdot 1} \cdots \vec{k}^{n \cdot n} \frac{1}{k^2}
$$

(54)

The structures under the integral are now the so(3) STF tensors, given by (see next section for a spinorial analog)

$$
\int \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \vec{k}^{n \cdot 1} \cdots \vec{k}^{n \cdot n} \frac{1}{k^2} = \partial_{i_1} \cdots \partial_{i_n} \int \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \frac{1}{k^2} = \partial_{i_1} \cdots \partial_{i_n} \frac{1}{4\pi r} = \frac{(2n - 1)!!}{(1)^n (n+1)!!} \hat{x}^{i_1} \cdots \hat{x}^{i_n} - \text{traces}
$$

(55)

They are STF because tracing two derivatives leads to $\partial^2 1/r = 0$ away from $r = 0$. This entails that traces of $Q_{i_1 \cdots i_n}$ in (54) will not contribute to the potential. In other words, we are free to evaluate the current $\rho^\pm(\vec{k})$ by dropping contractions $\vec{k}^2$ in momentum space, since they lead to ultra-local terms. Indeed, in (2, 2) signature we can evaluate $\rho^\pm(\vec{k})$ at strict $\vec{k}^2 = 0$ with $\vec{k} \neq 0$, where it becomes a three-point amplitude! This complexified interpretation of the potential was indeed the starting observation of [7].

Finally, we note that the physical interpretation of the tensors in (54) follows by examining the “minimal-coupling” case $c_n = 1$,

$$
\phi^\pm(x) = \sum_n \frac{i^n a_{i_1} \cdots a_{i_n}}{n!} \partial_{i_1} \cdots \partial_{i_n} \frac{1}{4\pi r} = \frac{1}{4\pi |\vec{x} + ia|}
$$

(56)

which is the generating function of $m = 0$ spherical harmonics if $a$ is aligned with the $z$ axis. Thus the stationary multipole moments appearing in the series (54) correspond to $m = 0$ spherical harmonics. General $m$ is obtained from general, non-stationary, multipoles as we now outline.

III. Multipoles from Vectors to Twistors

Throughout the text we use both vector and spinorial/twistor language to describe multipoles. We outline here how to relate both forms.

In its most basic form multipoles can be defined as conjugates to spherical harmonics. These form irreducible representations of the little group associated that preserves certain time direction. Let $u^\mu$ be such direction: We can thought it as a four-velocity of a massive particle. A spin-$s$ representations is then given by $2s + 1$ polarizations

$$
u_{\mu_1}^{\nu_1} \cdots \nu_s^{\nu_s} = \eta_{\mu_1 \mu_2}^{\nu_1 \nu_2} \cdots \nu_s = 0 , -s \leq m \leq s .
$$

(57)
i.e. by completely symmetric, trace-free (STF) tensors, yielding an irreducible representation of the massive little group, i.e. so(3) in (1, 3) signature.

Consider now powers of the direction vector \( \hat{x}^\mu = (0, \hat{x}(\theta, \phi), \hat{x}) \) given by \( \hat{x}_{\mu_1} \cdots \hat{x}_{\mu_s} \). Since these products have non-vanishing traces they are not STF. Instead we can extract their \( 2s + 1 \) irreducible components simply by projecting with the complete set of states (57), giving

\[
Y_{sm}(\theta, \phi) = \epsilon_m^{\mu_1 \cdots \mu_s} \hat{x}_{\mu_1} \cdots \hat{x}_{\mu_s}
\]

(58)
i.e. the standard spherical harmonics. The so(3) action of polarization tensors then acts naturally on these functions.

Alternatively, a manifestly STF version of the tensors is obtained from the generating function 1

\[
Y_{sm}(\theta, \phi) = \frac{(-1)^s r^{s+1}}{(2s - 1)!!} \times \epsilon_m^{\mu_1 \cdots \mu_s} \partial_{\mu_1} \cdots \partial_{\mu_s} 1/r
\]

(59)
In this form we can replace, for the case of stationary multipoles, \( \epsilon_m^{\mu_1 \cdots \mu_s} \to \tilde{\alpha}^{\mu_1} \cdots \tilde{\alpha}^{\mu_s} \) since all the traces are projected out. For general \( m \) in the non-stationary case, each of the polarization tensors (57) corresponds to an independent multipole structure, i.e. functions on the sphere can be expanded as

\[
\sum_{sm} a_{sm} Y_{sm}(\theta, \phi) = \sum_{s} Q_{i_1 \cdots i_s} \hat{x}^{i_1} \cdots \hat{x}^{i_s}, \quad Q_{i_1 \cdots i_s} := \sum_{m} a_{sm} \epsilon_m^{i_1 \cdots i_s}
\]

(60)
Further note that due to the condition (57) we can replace the direction vector by the null direction \( \ell_\mu = (1, \hat{x}) \) used in the main text, giving

\[
Y_{sm}(\theta, \phi) = \epsilon_m^{\mu_1 \cdots \mu_s} \ell_{\mu_1} (\theta, \phi) \cdots \ell_{\mu_s} (\theta, \phi),
\]

(61)
Let us now translate the discussion to spinors. Their advantage is that STF tensors can be constructed trivially. Indeed, as exploited in [137], representations of \( su(2) \approx \text{so}(3) \) are completely symmetric spinors rather than STF tensors. To see this, we use that the momentum vector defines a canonical basis of spinors

\[
u_{A\bar{A}} = \sigma_0 = |1_a\rangle[1^a] = |1_+\rangle[1_] - |1_-\rangle[1_+]
\]

(62)
where \( a = 1, 2 \) are little-group indices. Note that the orthogonal vectors to \( u^{\mu} \) are simply the three symmetric components \( \varepsilon_{A\bar{A}}^{ab} = |1^a\rangle[1^b] \). It is customary to take the spin direction as \( u_{A\bar{A}} = a\varepsilon_{A\bar{A}}^{+} = a\sigma_z \). Now, the STF tensors (57) are simply, up to a normalization,

\[
\varepsilon_{A_1\bar{A}_1 \cdots A_s\bar{A}_s} = |1^{a_1}\rangle[1^{b_1}] \cdots |1^{a_s}\rangle[1^{b_s}], \quad a_i, b_j \in \{0, 1\}
\]

(63)
Crucially, this is not simply a tensor product of \( \varepsilon_{A\bar{A}}^{ab} = |1^a\rangle[1^b] \) vectors: Due to the full symmetrizations of little-group indices, one can check that the traces vanish by contracting both sides with \( \eta^{\mu_1 \mu_2} \to \epsilon_{A_1\bar{A}_1 \cdots A_s\bar{A}_s}, X_{A_1A_2 \cdots A_{2s}} \). We may omit the explicit little-group indices hereafter. Further contracting both sides with \( u_{A\bar{A}} \) shows that they are indeed transverse and hence fulfill (57). Using the form (63), the harmonics (58) now read

\[
Y_{sm}(\theta, \phi) = \frac{1}{2^{s}(1|\hat{x}|1) \cdots (1|\hat{x}|1)}
\]

\[=
\frac{1}{2^{2s}(1\lambda)[\ell_\lambda|1\lambda]} \langle 1\lambda| \tilde{\lambda} \cdot 1\lambda \rangle [\tilde{\lambda} 1]
\]

The second line is the spinor version of (61), where \( \ell = 1/2 \sigma \) parametrizes \( CP^1 \). Now, using that \( |1^b\rangle = \tilde{u}|1^b\rangle \) and the definition (20) for the spatial vector \( \tilde{x} = X_{AB} \), we can write

\[
Y_{sm}(\theta, \phi) = \frac{1}{(2r)^s} (1|X|1) \cdots (1|X|1) = \frac{1}{(2r)^s} \varepsilon_{A_1\bar{A}_1 \cdots A_s\bar{A}_s} X_{A_1A_2 \cdots A_{2s}} X_{A_1A_2 \cdots A_{2s}} \]

(65)
Following the twistor construction [97] we defined the multipole components as

\[
\varepsilon_{A_1B_1A_2 \cdots B_s} := \varepsilon_{A_1A_2 \cdots A_s} u_{A_1B_1} \cdots u_{A_sB_s}
\]

(66)
Note that the particular case of $m = 0$, corresponding to stationary multipoles, is again obtained by replacement $\varepsilon_{A_1 A_2 \ldots A_\ell} \rightarrow A_{(A_1 B_1 \cdot \cdot \cdot A_\ell B_\ell)}$ as in the main text. Actually, symmetrization here is not needed if we use the twistor version of (59) which we now derive.

Let us focus on $(2, 2)$ signature, where the little group is $so(2, 1) \approx sl(2, \mathbb{R})$. Just as the identity (55) leads to their momentum representation, the spherical harmonics also fulfill an analogous identity leading to their twistor representation

$$\int_C \frac{\langle ndn \rangle}{2\pi} \frac{\eta_{A_1} \cdot \eta_{A_{2\ell}}}{(\eta^A X_{AB} \eta^B)^{s+1}} = \frac{(1-s)^s}{s!} \frac{\partial}{\partial X_{A_1 A_2}} \cdots \frac{\partial}{\partial X_{A_{2\ell-1} A_{2\ell}}} \int_C \frac{\langle ndn \rangle}{2\pi} \frac{1}{\eta^A X_{AB} \eta^B}$$

$$= \frac{(-1)^s}{s!} \frac{\partial}{\partial X_{A_1 A_2}} \cdots \frac{\partial}{\partial X_{A_{2\ell-1} A_{2\ell}}} \left( \frac{1}{2\sqrt{|X|}} \right)$$

(67)

Also analogously to the momentum integral (55), it is direct that this generates irreducible representations of the little group. Indeed, one can alternatively use $sl(2, \mathbb{R})$ covariance to guess that the result must be a completely symmetric tensor which scales as $\sim 1/X^{s+1}$, i.e.

$$\int_C \frac{\langle ndn \rangle}{2\pi} \frac{\eta_{A_1} \cdot \eta_{A_{2\ell}}}{(\eta^A X_{AB} \eta^B)^{s+1}} = \frac{(2s-1)!!}{s! \sqrt{2}} \frac{X_{(A_1 A_2} \cdots X_{A_{2\ell-1} A_{2\ell})}}{(X_{AB} X^{AB})^{s+1/2}}$$

(68)

with $X_{AB} X^{AB} = 2|X|$. The numerical prefactor is fixed by projection. This holds for a general symmetric tensor $X_{AB}$. Note that using (64), (67) and (68) we can write the spinorial version of (59):

$$Y_{\lambda \mu}(\theta, \phi) = \frac{(-1)^s r^{s+1}}{(2s-1)!!} \left( \frac{1}{r} \right)^s \sum_{A_1, A_{2\ell}} \varepsilon_{A_1 A_2 \cdot \cdot \cdot A_{2\ell}} \frac{\partial}{\partial X_{A_1 A_2}} \cdots \frac{\partial}{\partial X_{A_{2\ell-1} A_{2\ell}}} \left( \frac{1}{r} \right).$$

(69)

IV. Convergence and Relation to Previous Formulae

An integral formulae was already given in [65] for stationary metric perturbations and their curvature in (2, 2), starting from three-point amplitudes. Convergence of the integration was analyzed only for the Taub-NUT solution and played a key role in constructing its linearized version in (2, 2) Klein space.

The goal of this appendix is then two-fold. For one, we analyze the convergence of our integral formulae for the most general stationary solution. Second, we will recover the integral formulae of [65] thereby completing its convergence analysis.

First consider the Penrose-Ward formula

$$\phi =: \int_{\mathbb{R}} \frac{\langle ndn \rangle}{2\pi} \rho_{\mathcal{M}_3}(\eta, \bar{\eta}),$$

(70)

for the most general stationary amplitude. Using (16), (18), together with the definition (20) this is

$$\mathcal{M}_3(\eta, \bar{\eta}) = \int_0^\infty \frac{d\omega}{4\pi} \omega e^{\frac{\omega}{2} |\bar{\eta}| u |\eta|} \times \sum_n c_n \frac{\langle |\eta| A |\eta| \omega \rangle^n}{n!}.$$  

(71)

That this is a well-defined Laplace transform is because three-point amplitudes, as opposed to higher-point amplitudes, are analytic functions of the energy $\omega$. As explained in [14], such soft expansion is in correspondence with the multipole expansion of the source. Because of this we assume, as in textbooks, that it is uniformly convergent in the long-wavelength regime. This is just to say that the integral can be evaluated term by term in the series.

Consider (2, 2) signature where the twistor $Z = (\eta, \bar{\eta})$ is real valued. The integral defined in (71) only converges on a region of real twistor space, given by

$$|\bar{\eta}| u |\eta| = Z^\alpha \Sigma_{\alpha \beta} Z^\beta > 0$$

(72)

Through the incidence relation, this region corresponds to the following region of flat space

$$\rho_{\mathcal{M}_3}[Z^\alpha \Sigma_{\alpha \beta} Z^\beta] = \langle |\eta| \bar{\eta} |\eta| \rangle = \eta^A X_{AB} \eta^B > 0 \text{ for all } \eta^A.$$  

(73)

The above implies $\det(X_{AB}) = z^2 - x^2 - y^2 > 0$ which is the so-called Rindler wedge. This extends the analysis done in [65] for Taub-NUT to the case of general stationary solutions. Note that the Laplace transform (71) can also be analytically extended to complex twistors $Z^\alpha$, in particular the rotation.
\[ |\tilde{\eta}| \to e^{i\alpha} |\tilde{\eta}| \quad 0 < \alpha < \pi/2 \]  

(74)

can be compensated by a deformation of the \( \omega \) contour in the complex plane. The end result is that as long as \( \text{Re}[|\tilde{\eta}|u|\eta|] > 0 \) (i.e. \( \alpha < \pi/2 \)) we obtain

\[ \mathfrak{M}_3(\eta, \tilde{\eta}) = \sum_n \frac{c_n}{2\pi} \frac{(\eta | A | \eta)^n}{|\tilde{\eta}|u|\eta|^{n+1}} \]

(75)

The continuation for (74) is the reason we could replace \( i|\tilde{\eta}| \) by \( |\tilde{\eta}| \) in our original formula (13) (the original formula corresponds to \( \alpha = \pi/2 \), which requires an \( i\epsilon \) prescription). Note that through the incidence relation this is the same as \( \alpha \to x \), which preserves the \((2,2)\) signature.

We briefly comment on the convergence of the Penrose-Ward transform (70). Since the transform is projective in \( \eta \), it can be evaluated simply via contour deformation. The fact that the real form (73) is positive-valued automatically implies that its two roots (i.e. the poles of the integrand) are complex and conjugate. Hence the real contour \( \mathcal{C} = \mathbb{R} \) in (70) splits them as we required. On the other hand, when \( \det(X_{AB}) = 0 \) the form (73) admits a real solution in \( \eta \). This effectively pinches the contour and generates a singularity in (70) Thus we recover the conclusion pointed out in [65]: Singularities arise lie at the so-called Rindler horizon \( z = \sqrt{x^2 + y^2} \).

We can now compare our integral formulae with the expressions given in [65]. Let us consider for instance, the Gaussian formula

\[ \mathcal{C}_{ABCD} = \int_{\mathbb{R}} \frac{d\tilde{\lambda}}{(4\pi)^2} e^{-\frac{1}{2} |\lambda| x u |\lambda|} \tilde{\lambda}_A \tilde{\lambda}_B \tilde{\lambda}_C \tilde{\lambda}_D M_3^- (|\lambda| = |\tilde{\lambda}| u, |\tilde{\lambda}|) \]

(76)

given there for the anti-self dual components of the curvature (we have adjusted the conventions via \( M_3^\text{here} = M_3^-/8\pi G \), so that the three-point amplitudes agree with (31) for Kerr). Given that the classical three-point amplitudes (18) are functions of \( k = |\lambda| |\tilde{\lambda}| u \), hence even in \( |\tilde{\lambda}| \), we can restrict the integration to \( |\tilde{\lambda}|^2 > 0 \) by including a factor of 2. This makes contact with our Feynman prescription, as it means we can parametrize \( |\tilde{\lambda}| = \sqrt{\omega} |\eta| u \) with \( \omega > 0 \) and a real projective \( \eta = (1 \eta) \). The result is

\[ \mathcal{C}_{ABCD} = \int_{\mathbb{R}} \frac{(\eta d\eta)}{2\pi} u_{\dot{A}A} \eta^A \ldots u_{\dot{D}D} \eta^D \int_{0}^{\infty} \frac{d\omega}{8\pi} e^{-\frac{1}{2} (\eta | X | \eta)} M_3^+ (\sqrt{\omega} |\eta|, \sqrt{\omega} |\eta| u) \]

(77)

It is then straightforward to check that this agrees with our results (30)-(28). Similar analysis can be done for the potential functions \( \phi^\pm \). The advantage of the Penrose transform, over the Gaussian integration (76) presented in [65], is that the former can be used to also evaluate the linearized metric \( h_{\dot{A}A\dot{B}B} \). In such case, the polarization tensors \( \epsilon_{\mu} \epsilon_{\nu} \to \frac{\tilde{\lambda}_A \tilde{\lambda}_B \epsilon_{\mu\nu}}{\eta |\lambda|} \) yield poles in the Gaussian integration, but can be resolved easily in the Penrose contour integral. The result is the linearized metric in Bondi gauge as in e.g. [131].