Quantum parameter estimation with optimal control

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A pivotal task in quantum metrology, and quantum parameter estimation in general, is to design schemes that achieve the highest precision with given resources. Standard models of quantum metrology usually assume the dynamics is fixed, the highest precision is achieved by preparing the optimal probe states and performing optimal measurements. However, in many practical experimental settings, additional controls are usually available to alter the dynamics. Here we propose to use optimal control methods for further improvement on the precision limit of quantum parameter estimation. We show that by exploring the additional degree of freedom offered by the controls higher precision limit can be achieved. In particular we show that the precision limit under the controlled schemes can go beyond the constraints put by the coherent time, which is in contrast to the standard scheme where the precision limit is always bounded by the coherent time.

PACS numbers: 03.67.-a, 03.65.Yz, 03.65.-w.

I. INTRODUCTION

Quantum metrology, which exploits quantum mechanical effects to achieve high precision, has gained increased attention in recent years [1–25]. A typical metrological procedure is to first encode the interested parameter \( x \) on a probe state \( \rho_0 \) via a parameter dependent dynamics \( \mathcal{E}_x \), i.e., \( \rho_0 \xrightarrow{\mathcal{E}_x} \rho_x \), then perform a set of Positive Operator Valued Measurements (POVM) on \( \rho_x \). Based on the measurement results an estimation \( \hat{x} \) can then be obtained. It is known that for unbiased estimation quantum Cramér-Rao bound sets a lower bound on the precision \( \delta \hat{x} \) that \( \delta \hat{x} \geq 1/\sqrt{F} \), where \( \delta \hat{x} \) is the standard deviation and \( F \) is the quantum Fisher information (QFI). If the procedure is repeated \( n \) times, then \( \delta \hat{x} \geq 1/\sqrt{nF} \) where the bound can be achieved in the asymptotical limit.

In this standard procedure, the dynamics \( \mathcal{E}_x \) is usually assumed to be fixed, and the highest precision is achieved by preparing the optimal probe state and performing the optimal POVM that saturates the quantum Cramér-Rao bound. The obtained precision is often regarded as the ultimate precision. However, in many experimental settings, additional controls are usually available to alter the dynamics for further improvement of the precision limit, this provides another degree of freedom for optimization.

The parallel scheme and the sequential scheme, as shown in Fig. 1, are two standard schemes considered in quantum parameter estimation. It is known that if the dynamics is unitary and the Hamiltonian takes the multiplication form of the parameter, i.e., if \( \mathcal{E}_x = e^{-ixH} \), then the two schemes are equivalent [30]; while for general unitary dynamics \( \mathcal{E}_x = e^{-iH(x)} \), the parallel scheme is equivalent to the controlled sequential scheme [17]. For noisy quantum parameter estimation, special controlled schemes, such as quantum error correction and dynamical decoupling, have been used to improve the precision limit [31–40]. The controlled sequential scheme is more implementable on current experimental settings than the parallel scheme, as high-fidelity controls on small systems can now be routinely done while preparing large entangled states for the parallel scheme is still very challenging. The controlled sequential scheme thus starts to gain attention recently [36–41]. Existing controlled schemes that use quantum error correction or dynamical decoupling either need additional resources such as auxiliary systems that are completely immune to noises or require the underlying dynamics possessing certain symmetries, which restrict the scope of the applications. Systematic methods that can design controls to improve the precision limit for general dynamics are highly desired in practice.

In this paper we propose to employ optimal quantum control methods, in particular the GRadient Ascent Pulse Engineering (GRAPE) [42], to design controls for the improvement of the precision limit in quantum parameter estimation. Such methods can be used to automatically obtain the optimal controls for the improvement of the precision limit for general dynamics and can easily incorporate practical constraints on the controls. It thus provides a general method to design the controlled schemes in quantum metrology. With this method we
will show that the optimally controlled schemes can obtain precision limits beyond the coherent time, which is in contrast to the conventional schemes where the precision limit is always bounded by the coherent time.

II. METHODOLOGY

In this article we consider the system whose dynamics can be described by the master equation

$$\partial_t \rho(t) = \mathcal{L}[\rho(t)],$$  \hspace{1cm} (1)

where $\mathcal{L}$ is a super-operator. For unitary evolution $\mathcal{L} = -iH^\times$ where $H^\times(\rho) = [H, \rho]^\dagger$; for noisy evolution $\mathcal{L} = -iH^\times + \Gamma$ where $\Gamma$ denotes the super-operator for the noisy process. The Hamiltonian of a controlled system can be written as \cite{42, 44}

$$H = H_0(x) + \sum_{k=1}^{p} V_k(t)H_k,$$ \hspace{1cm} (2)

where $H_0(x)$ is the free evolution Hamiltonian, $x$ is the interested parameter, $\sum_{k=1}^{p} V_k(t)H_k$ are control Hamiltonians with $V_k(t)$ representing the amplitude of $k$th control field. Here we assume the correlation in the environment decays much faster than the evolution of the system under Eq. (1), and the Markovian approximation is still valid at the presence of controls \cite{43}. For example, in Nuclear Magnetic Resonance, the correlation time of the environment is around $10^{-8}$s and the coherent time is around $0.1 \sim 1$ s \cite{44}, if the time scale of the control is around $10^{-3}$s, then the Markovian approximation is valid, and the controls are fast enough to generate the desired operations. We also assume the controls do not change the noisy operators, this holds under some physical settings \cite{45-47} but not in general. The situations that noisy operators are affected by controls will be addressed in another work.

To implement the GRAPE we will divide the evolution time $T$ into small time steps, and within each time step $\Delta t$ the controls will be approximated as constants. The final state at time $T$ can thus be written as $\rho(T) = \Pi_{t=1}^{m} \exp(\Delta t \mathcal{L}_t)\rho(0)$, here $m = T/\Delta t$ is the number of time steps and $\mathcal{L}_t$ is the super-operator for the $t$th time step. The multiplication in $\rho(T)$ is taken from right to left.

GRAPE can obtain controls that optimize a given objective function. In this article we focus on the local precision limit for the measurement of small shifts around certain known values. Such local precision limit can be quantified by the QFI, we will thus take the QFI as the objective function. The QFI is defined as

$$F(T) = \text{Tr} \left[ \rho(T)L_s^2(T) \right],$$ \hspace{1cm} (3)

where $L_s(T)$ denotes the symmetric logarithmic derivative (SLD) which is the solution to the equation $\partial_t \rho(T) = [\rho(T)L_s(T) + L_s(T)\rho(T)]/2$. The flow of the algorithm is shown in Fig. 2 (detailed description is in appendix A). Some steps of the algorithm may require the knowledge of $x$, which is a-priori unknown, in that case an estimated value $\hat{x}$ will be used and the controls will be updated adaptively. This, however, does not affect the precision limit asymptotically.

In practical experiments, the measurements that can be taken are restricted. It is thus also of practical importance to find the optimal controls that can lead to the highest precision under a fixed measurement, which is quantified by the classical Fisher information (CFI) $F_{cl}$ under the particular measurement, instead of the QFI. This can also be treated via GRAPE. Given a set of
POVM measurement \( \{ E(y) \} \) with \( \sum_y E(y) = 1 \), the probability of getting the measurement result \( y \) is given by \( p_{y|x} = \text{Tr}(\rho(T)E(y)) \), and the CFI is given by
\[
F_{\text{cl}}(T) = \sum_y \left( \frac{\partial_x p_{y|x}}{p_{y|x}} \right)^2.
\]

III. APPLICATION

We first apply the algorithm to the phase estimation with a two-level system under dephasing dynamics. The dynamics is given by [43]
\[
\partial_t \rho = -i [H, \rho] + \frac{\gamma}{2} (\sigma_d \rho \sigma_d - \rho),
\]
here the system Hamiltonian is \( H = \frac{1}{2} \omega_0 \sigma_3 + \vec{V}(t) \cdot \vec{\sigma} \) with \( \vec{V}(t) = \{ V_1(t), V_2(t), V_3(t) \} \), \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \). \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are Pauli matrices. The dephasing is along \( \vec{n} \cdot \vec{\sigma} \) with \( \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). Here \( \theta \in [0, \pi] \), \( \phi \in [0, 2\pi] \). \( \omega_0 \) is the parameter to be estimated. Here we assume the controls can be performed along all three directions, however the results hold as long as the controls span \( su(2) \).

In Fig. 3 we plotted the QFIs with different dephasing dynamics for \( T = 5 \) (the unit is taken in the order of \( \omega_0^{-1} \)). The different dephasing dynamics are characterized by the angle \( \theta \) (\( \phi \) is taken as zero, as we can always make a rotation along \( \sigma_3 \) direction to make \( \phi \) equal to zero and such rotation does not affect the precision). From the figure we can see that the highest enhancement, compared to the uncontrolled schemes, occurs at \( \theta = \pi/2 \) where the noise is transverse to the direction of the parameter, and the enhancement reduces when \( \theta \) goes to zero (parallel noises). We note that here no ancillary systems are used, which is different from previous studies using quantum error correction where ancillary systems are necessary [36–40]. Besides, in this case, the highest precision does not strongly depend on the probe state, which can be seen in Fig. 3. \( |0\rangle \) (purple squares) and \( |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \) (dash-dotted green line) provide almost the same precision under the optimal controlled scheme.

We next provide some analysis on the controlled scheme under different noises to give some physical intuitions on how the controls actually helped improving the precision limit.

A. Transverse dephasing

The improvement of the controlled scheme with transverse noises is shown in Fig. 4(a). When the probe state is taken as \( |+\rangle \), the obtained optimal controls are \( V_1(t) = V_2(t) = 0 \) and \( V_3(t) = -0.5\omega_0 \), shown in Fig. 4(b). Such controls essentially keep the probe state at \( |+\rangle \), where it is not affected by the noises. The QFI under such controls is given by (see appendix for detailed derivation)
\[
F(T) = \frac{2}{\gamma^2} (e^{-\gamma T} + \gamma T - 1).
\]

It always increases with \( T \) as \( F'(T) > 0 \). The precision limit thus is not constrained by the coherent time. In contrast, without controls there is an optimal time \( T_{\text{opt}} \) (which is determined by the decay rate, for example when \( \omega_0 \gg \gamma \), \( T_{\text{opt}} \approx 2/\gamma \)), at which the precision reaches the maximum, and beyond \( T_{\text{opt}} \) the QFI starts to decrease with time, which can be seen in Fig. 4(a).

We note in this case the control \( V_3(t) = -0.5\omega_0 \) depends on the true value which is a-priori unknown, in practice an estimated value \( \hat{\omega}_0 \) need to be used and the controls need to be updated adaptively according to the estimated value as \( V_3(t) = -0.5\hat{\omega}_0 \). In Fig. 4(c) we plotted the improvement provided by the controls with different estimation error, it can be seen that the improvement is quite robust. For example assume the true value \( \omega_0 = 1 \) and \( T = 20 \), then as long as \( \hat{\omega}_0 \in [0.8, 1.2] \) the controlled scheme outperforms the uncontrolled scheme, and when \( \hat{\omega}_0 \in [0.9, 1.1] \), the QFI under the controlled scheme is more than 10 times larger than the value without controls, thus even with a 10% estimation error the controlled scheme still provides significant improvement over the uncontrolled schemes.

If the measurement is fixed, for example the measurement is taken as \( \{|+\rangle\langle+|, |\rangle\langle-|\} \) (here \( |\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2} \)), we can also use the optimal control to improve the precision. From Fig. 4(a) we can see that with the optimal controls the CFI can actually achieve the maximal QFI, indicating that the precision limit under the optimal controlled scheme is insensitive to the measurement performed on the final state, as long as it is projective. This is because the optimal measurement (which is a projective measurement) can always be rotated to the fixed measurement which corresponds to a counter rotation on the probe state that can be achieved via controls. As a comparison the precision without controls is also plotted, in this case the CFI oscillate with time and can only reach the QFI for some specific time points, indicating this measurement scheme is only optimal for some specific time points. From Fig. 4(c) it can be seen that the precision obtained is also very robust against the estimation error.

B. Parallel dephasing

We now provide some analysis for the case with parallel dephasing, which is usually a more dominant noise for many physical systems [49, 50], and cannot be corrected by quantum error correction techniques even with ancillary systems [36–40].

The QFI under parallel dephasing are shown in Fig. 5(a). It can be seen that the QFI under optimal
control continues to increase beyond the coherent time, while in contrast the QFI without control starts to decrease beyond the coherent time.

To gain some intuition on how controls improved the precision we consider a simple control strategy: we first prepare the probe state as $|+\rangle$ and let it evolve under the natural evolution (without controls) for a period of $t_0$, then apply a $\pi/2$-pulse along $y$-direction, after that let the state evolve for another period of $T - t_0$ under the natural evolution. To analyze the effect of this strategy we write the state with the Bloch representation as $\rho = (\mathbb{1} + \vec{r} \cdot \vec{\sigma})/2$, the initial state $|+\rangle$ thus corresponds to $(r_1(0), r_2(0), r_3(0)) = (1, 0, 0)$. Under the free evolution the state evolves as

$$r_1(t) = e^{-\gamma t} \left[ \sin(\omega_0 t) r_2(0) + \cos(\omega_0 t) r_1(0) \right],$$

$$r_2(t) = e^{-\gamma t} \left[ \cos(\omega_0 t) r_2(0) - \sin(\omega_0 t) r_1(0) \right],$$

$$r_3(t) = r_3(0),$$

which gives \( \vec{r}(t) = e^{-\gamma t}(\cos(\omega_0 t), -\sin(\omega_0 t), 0) \). If no controls are added, the QFI for $\omega_0$ can be easily computed using the following formula [51]

$$F(t) = |\partial_{\omega_0} \vec{r}(t)|^2 + \frac{(\vec{r}(t) \cdot \partial_{\omega_0} \vec{r}(t))^2}{1 - |\vec{r}(t)|^2},$$

which gives \( F(t) = t^2 e^{-2\gamma t} \), the maximum is achieved at the coherent time $T_{opt} = 1/\gamma$.

Now assume the target time is $T$ and we perform the rotation

$$R_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

at some time point $t_0 < T$. The quantum state after the $R_y$-rotation is $e^{-\gamma t_0}(0, \sin(\omega_0 t_0), \cos(\omega_0 t_0))$, which, after another free evolution with a period of $\Delta t = T - t_0$, leads to the final state $\vec{r}(T) = (r_1(T), r_2(T), r_3(T))$ where

$$r_1(T) = e^{-\gamma T} \sin(\omega_0 \Delta t) \sin(\omega_0 t_0),$$

$$r_2(T) = e^{-\gamma T} \cos(\omega_0 \Delta t) \sin(\omega_0 t_0),$$

$$r_3(T) = e^{-\gamma t_0} \cos(\omega_0 t_0).$$

The QFI can again be calculated from Eq. (10), with $t_0$ as a variable that can be changed to maximize $F(T)$ (the explicit form of $F(T)$ is in the appendix).

Figure 5(b) shows the QFI as a function of $t_0$, one can see that for $T = 5$, no matter when the rotation is performed, the QFI cannot be higher than the QFI without rotations. However, for $T = 15$, as long as the rotation is performed at a proper time point, we can obtain an improved QFI. It can also be seen that the QFI usually has multiple peaks with the variation of $t_0$, the maximum peak may differ for different $T$. The maximum QFI thus may not be smooth with respect to $T$.

In Fig. 5(a) we plotted the maximum QFI that can be achieved with this simple control strategy. It can be seen that this strategy does not help improving the QFI when $T$ is smaller than some time $T^*$ (which is approximately the coherent time $\gamma^{-1} = 10$ in this case, same behaviour are found for other values of $\gamma$), however when $T$ gets big, a control pulse at a proper time $t_0$ improves the QFI. The intuition of this simple strategy is that although states in the $x$-$y$ plane have a fast rate of parametrization under the Hamiltonian $\omega_0 \sigma_3$, they are also affected most by the parallel noise, when $T$ gets large the effect of noise overrides the parametrization, applying pulses at proper time that rotate the states away from $x$-$y$ plane help mitigate the noise effect thus improve the precision. More rotations can further improve the precision and GRAPE essentially provides a systematical way to find these rotations.
This is contrary to the conventional belief that coherent time sets the limit on the achievable precision, which is particular useful for those systems where the preparation of the probe states and the measurements are costly and one would like to extract more information for each measurement. Note that for the local precision limit which measures small shifts around certain known value, the phase can still be distinguished even under a long evolution time. For completely unknown phase, one needs to first evolve for a short time to avoid the possible ambiguity as the phase may wrap around the $2\pi$ interval. However after a rough estimation, the evolution time can get longer.

If the cost for the preparation and measurement is negligible, we should compare the QFI per unit of time, which is called the normalized QFI. As shown in Fig 6(a), with controls the maximum value of the normalized QFI is not improved compared to the values without controls(dashed black line), which indicates the normalized precision limit is still bounded by the coherent time under the parallel dephasing. However, when a fixed measurement is considered, for example the projective measurement $\{|+,|+;|-,|-\rangle\rangle$, the advantage of control shows up. With optimal controls the CFI is very close to the maximum QFI, indicating that the measurement $\{|+,|+;|-,|-\rangle\rangle$ is almost optimal under the controlled scheme, while without controls the CFI oscillates with time and is usually far from the maximum QFI.

In Fig. 6(b), the optimal controls obtained from the GRAPE are plotted. Generally the optimal control is not unique and the appearance of the controls in Fig. 6(b) is due to the algorithm. Such kind of controls seem complicated, but have been routinely implemented on physical systems, such as Nuclear Magnetic Resonance [52–56]. Various techniques have also been developed to smooth the controls [57, 58]. And as shown in Fig. 6(c) the controls obtained are again quite robust against the estimation error (in this figure we first obtain the controls with $\omega_0 = 1$, then apply the controls to dynamics with different $\omega_0$). It can be seen that the controlled scheme gains over the uncontrolled scheme with a quite broad range ($\sim 10\%$) of estimation error.

### C. Spontaneous emission

We give some analysis for the controlled scheme at the presence of the spontaneous emission, which is another major noise for many practical systems. We consider the general master equation

$$\frac{d}{dt}\rho(t) = -i[H, \rho] + \gamma_+ \left[ \sigma_+\rho(t)\sigma_- - \frac{1}{2} \{\sigma_-\sigma_+, \rho(t)\} \right] + \gamma_- \left[ \sigma_-\rho(t)\sigma_+ - \frac{1}{2} \{\sigma_+\sigma_-, \rho(t)\} \right],$$

(15)

where $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$ is a ladder operator, $H = \frac{1}{2}\omega_0\sigma_3 + \vec{V}(t) \cdot \vec{\sigma}$.

The effects of the controls are shown in Figure 7(a).

In this case the normalized QFI under the controlled scheme shows significant improvement over the value without controls. And similar to the dephasing case, under the measurement $\{|+,|+;|-,|-\rangle\rangle$, the normalized CFI (dashed blue line) not only achieves the maximum value, but also ceases to oscillate.

Again we use a simple control strategy with only one rotation to provide some intuition on how controls helped improving the precision. For simplicity, we assume $\gamma_+ = 0$ and $\gamma_- = \gamma$. In the Bloch representation, without the controls the states evolves as

$$r_1(t) = e^{-\frac{1}{2}\gamma t} \left[ \cos(\omega_0 t)r_1(0) - \sin(\omega_0 t)r_2(0) \right],$$

(16)

$$r_2(t) = e^{-\frac{1}{2}\gamma t} \left[ \cos(\omega_0 t)r_2(0) + \sin(\omega_0 t)r_1(0) \right],$$

(17)

$$r_3(t) = -1 + e^{-\gamma t} + e^{-\gamma t}r_3(0).$$

(18)

If the initial state is taken as $|+\rangle$, then the QFI at time $t$ is $F = e^{-\gamma t/2}$ and the maximum is achieved at $T_{\text{opt}} = 2/\gamma$.

Now consider a simple control strategy: we first let the initial state, which is $|+\rangle$, evolve for some time $t_0$ under
Figure 6. (Color online) Parallel dephasing: (a) The evolution of normalized QFI (by $T$) with and without controls. The red dots and solid black line represent the normalized QFI with and without controls, respectively. The solid yellow line represent the maximum normalized QFI with single $\pi/2$ pulse along $y$-axis at a proper time. The green stars and dash-dotted green lines represent the normalized CFI with and without controls, respectively. The measurement for CFI is $\{|+\rangle\langle+|,|-\rangle\langle-|\}$. (b) The controls obtained from GRAPE for the dynamics with $T = 20$. The initial guessing is randomly. (c) The normalized QFI (by $T^2$) for different $\omega_0$. The controls are obtained from the GRAPE for $\omega_0 = 1$. It can be seen that the QFI of the controlled scheme is higher than the QFI of the uncontrolled scheme as long as $|\omega_0 - 1|$ is not too big, i.e., as long as the estimated value is reasonably good. The true values of $\omega_0$ are assumed to be 1 and decay rate $\gamma = 0.1$ in all panels.

Figure 7. (Color online) Spontaneous emission: (a) normalized QFI (by $T$) as a function of $T$. The red dots and solid black lines represent the QFI with and without controls, respectively. And the dashed blue and dash-dotted green lines represent the CFI with and without controls, respectively. Here $\gamma_+ = 0$, $\gamma_- = 0.1$. The true value of $\omega_0$ is 1. The measurement for CFI is $\{|+\rangle\langle+|,|-\rangle\langle-|\}$. (b) The controls obtained from GRAPE for the dynamics with $T = 20$. The initial guessing is all zero. (c) The QFI and CFI (normalized by $T^2$) as a function of $\omega_0$. The solid blue and dash-dotted green lines represent the QFI and CFI under the controls given by GRPAE with $\hat{\omega}_0 = 1$, respectively. The dashed black line is the QFI without controls. (d) QFI as a function of $t_0$. The solid blue and red lines represent the QFI with rotation at $t_0$, for $T = 16$ and $T = 8$, respectively. The dashed and dash-dotted black lines represent the QFIs without rotation.
the free evolution (in this case the free evolution will drive the state away from the $x-y$ plane of the Bloch sphere), we then apply a control to rotate the state back to the $x-y$ plane, let it evolves for another period, $T-t_0$. The derivation of the QFI under this simple control strategy is given in appendix F.

In Fig. 7(d), we plotted the QFI under this simple control strategy as a function of $t_0$ for $T = 8, 16$. Here the control strategy is different from the parallel dephasing case. Under the parallel dephasing (the effects of the parallel dephasing on the states are shown in Fig. 8(a)), the states in the $x-y$ plane are affected most by the dephasing noise although they also undergo the fastest parametrization. For a long evolution time, the dephasing noise can override the parametrization, thus applying a control rotating the state away from $x-y$ plane is beneficial under the parallel dephasing. While under the spontaneous emission (with the effect on the states shown in Fig. 8(b)), the initial state is in the $x-y$ plane, which has the fastest parameterizations, but the free evolution quickly drives the states away from the $x-y$ plane before the noises override the parametrization. It is thus beneficial to apply a control rotating the state back to the $x-y$ plane for a fast parametrization. Also for the spontaneous emission the states in the $x-y$ plane are not the states affected most by the noise.

D. Energy cost

We provide some estimation on the energy cost for the optimal controls implemented in some of the examples. In Fig. 9 we plot the energy cost $E(t) = \sum_k \int_{t_0}^T V_k^2(\tau) d\tau$ of the optimal controls as a function of time within the total time $T = 10$. It can be seen that for the examples, the implementations of optimal controls do not require much energy and thus it will not be an obstacle in practice. Note that in the case of the parallel dephasing, there is a sudden change near the coherent time. This is due to the fact that under the parallel dephasing controls do not have much effect far before the coherent time, therefore it needs little controls in this regime. Strong controls only starts to appear near the coherent time.

IV. SUMMARY

For many current experimental settings the controlled sequential scheme is more implementable than the parallel scheme since high-fidelity controls can now be routinely done on many physical systems, such as nuclear magnetic resonance [52–56], nitrogen-vacancy centers [59–61] and cold atoms [62]. GRAPE provides a general method to obtain optimal controls for the improvement of the precision limit, which is expected to find wide applications for many practical quantum parameter estimation tasks.

As a demonstration we applied the method to the frequency estimation with different noises and showed that GRAPE can improve the precision limit beyond the limit set by the coherent time, which is contrary to the conventional belief that coherent time sets the limit on the achievable precision. This is particular useful for those systems where the measurements are costly and one would like to extract more information for each measurement. For the dephasing cases, we showed that the gain of the controlled scheme is most eminent when the dephasing noise is orthogonal to the Hamiltonian, while when the noise is parallel to the Hamiltonian, the controls do not increase the precision one can obtain per unit of time. Future research includes characterizing the dynamics and noises for which the controls are (un)useful.

The optimal control method can also be used for non-
Markovian dynamics [63–65] and easily incorporate various practical constraints on pulse shape [57, 58, 66–68], thus providing a versatile tool for designing controlled schemes for various quantum parameter estimation tasks.

ACKNOWLEDGMENTS

H.Yuan acknowledges partial financial support from RGC of Hong Kong with Grant No. 538213.

Appendix A: Algorithm description

GRAPE can obtain controls that optimize a given objective function. In this article we focus on the local precision limit for the measurement of small shifts around certain known values. Such local precision limit can be quantified by the QFI, we will thus take the QFI as the objective function. The QFI is given by $F(T) = \text{Tr} [\rho(T) L^2_{x}(T)]$ where $L_x(T)$ denotes the symmetric logarithmic derivative (SLD) which is the solution to the equation $\partial_{x} \rho(T) = [\rho(T) L_x(T) + L_x(T) \rho(T)]/2$.

The flow of the algorithm is as following:

1. guess initial values of $V_k(j)$ (here $V_k(j)$ denotes the $k$th control at the $j$th time step);
2. evolve the dynamics and obtain a trajectory of the system;
3. calculate the QFI at the target time;
4. calculate the gradient $\frac{\delta F(T)}{\delta V_k(j)}$;
5. update $V_k(j)$ to $V_k(j) + \epsilon \frac{\delta F(T)}{\delta V_k(j)}$;
6. restart from step 2 using the updated $V_k(j)$ until the QFI converges.

The detailed calculation of $\frac{\delta F(T)}{\delta V_k(j)}$ is in appendix B. The gradient of the QFI can be written as

$$\frac{\delta F(T)}{\delta V_k(j)} = \Delta t \text{Tr} \left[ L^2_x(T) M_j^{(1)} \right] - 2\Delta^2 t \text{Tr} \left[ L_x(T) (M_j^{(2)} + M_j^{(3)}) \right], \quad (A1)$$

where $M_j^{(1)}$, $M_j^{(2)}$, and $M_j^{(3)}$ are Hermitian operators and in the form below

$$M_j^{(1)} = i D^m_{j+1} H_k^\times (\rho_i),$$

$$M_j^{(2)} = \sum_{i=1}^m D^m_{j+1} H_k^\times D^i_{i+1} H_0^\times (\rho_i), \quad (A2)$$

$$M_j^{(3)} = (1 - \delta m) \sum_{i=j+1}^m D^m_{i+1} H_0^\times D^j_{j+1} H_k^\times (\rho_i),$$

Here $\delta_{jm}$ is Kronecker delta function. $L_s(T)$ is the SLD of $\rho(T)$. $D^m_{j+1} := \prod_{i=j+1}^m \exp(\Delta t L_i)$ is the propagating superoperator from the $j$th time point to the target time, $j < m$ (we will let $D^0_i = I$ when $i > i'$). $\rho_i = D^i_{j}(\rho(0))$ is the state at the $j$th time point. $H_k^\times = [H_k, \cdot]$ and $H_0^\times = [\partial_x H_0, \cdot]$.

In practical experiments, the measurements that can be taken are restricted. It is thus also of practical importance to find the optimal controls that can lead to the highest precision under a fixed measurement, which is quantified by the CFI $F_{cl}$ under the particular measurement, instead of the QFI. This can also be treated via GRAPE. Given a set of POVM measurement $\{E(y)\}$ with $\sum_y E(y) = I$, the probability of getting the measurement result $y$ is given by $p_{y|x} = \text{Tr}(\rho(T)E(y))$, and the CFI is given by

$$F_{cl}(T) = \sum_y (\frac{\partial_x p_{y|x}}{p_{y|x}})^2.$$

The gradient of $F_{cl}$ can be similarly obtained as Eq. (A1), which is

$$\frac{\delta F_{cl}(T)}{\delta V_k(j)} = \Delta t \text{Tr} \left[ \hat{L}_2 M_j^{(1)} \right] - 2\Delta^2 t \text{Tr} \left[ \hat{L}_1 (M_j^{(2)} + M_j^{(3)}) \right], \quad (A3)$$

where

$$\hat{L}_1 = \sum_y (\partial_x \ln p_y) E(y), \quad \hat{L}_2 = \sum_y (\partial_x \ln p_y)^2 E(y). \quad (A4)$$

Here $M_j^{(1,2,3)}$ takes the same form as in Eq. (A2) (see appendix C for detailed derivation).

Appendix B: Gradient for QFI

The dynamics for the density matrix of a system can be described by the following general master equation

$$\partial_t \rho(t) = \mathcal{L}[\rho(t)], \quad (B1)$$

where $\mathcal{L}$ is a super-operator and $\mathcal{L} = -i H^\times + \Gamma$. $H^\times = [H, \cdot]$ and $\Gamma$ is the super-operator for the noise part. The Hamiltonian here is

$$H = H_0(x) + \sum_{k=1}^p V_k(t) H_k, \quad (B2)$$

where $H_0(x)$ is the free evolution Hamiltonian, $x$ is the interested parameter, $\sum_{k=1}^p V_k(t) H_k$ are control Hamiltonians with $V_k(t)$ representing the amplitude of $k$th control field.

Our objective function is the QFI and the goal is to find the optimal control to obtain the maximum QFI. The QFI is defined as

$$F(T) = \text{Tr}(L^2_x(T) \rho(T)),$$

where $L_s(T)$ is the SLD operator at target time $T$ and is determined by the equation $2\partial_x \rho(T) = \rho(T) L_s(T) + L_s(T) \rho(T)$. 

Before utilizing GRAPE to obtain the optimal control, it is necessary to know the corresponding gradient for QFI on the control coefficients, i.e., \( \frac{\delta F(T)}{\delta V_k(j)} \) based on the general master equation. In this section we will show the detailed calculation for the general dynamics given by Eq. (B1).

Since \( F(T) = \text{Tr} \left[ L^2_k(T) D_{j+1}^m \rho_j \right] = \text{Tr} \left[ \lambda_j \rho_j \right] \), here \( L_k(T) \) is the symmetric logarithm derivative of \( \rho(T) \), \( D_{j+1}^m := \prod_{i=j+1}^m \exp(\Delta t \mathcal{L}_i) \) is the propagating superoperator from the \( j \)th time point to the target time point, \( j < m \) (we will let \( D_j^0 = 1 \) when \( i > j \)), \( \rho_j = D_1^j(0) \) is the state at the \( j \)th time point and \( \lambda_j = L^2_k(T) D_{j+1}^m \). The gradient of the QFI with respect to the controls at the \( j \)th time step \( \frac{\delta F(T)}{\delta V_k(j)} \) can then be computed

\[
\frac{\delta F(T)}{\delta V_k(j)} = \text{Tr} \left( \frac{\delta \lambda_j}{\delta V_k(j)} \rho_j \right) + \text{Tr} \left( \lambda_j \frac{\delta \rho_j}{\delta V_k(j)} \right), \quad (B4)
\]

we calculate both terms in the following.

1) First we calculate \( \frac{\delta \rho_j}{\delta V_k(j)} \). Here the only term contains \( V_k(j) \) is the propagator at the \( j \)th time point, \( e^{\Delta t \mathcal{L}_j} \). Since \( \rho_j = e^{\Delta t \mathcal{L}_j} \rho_{j-1} \), we have

\[
\frac{\delta \rho_j}{\delta V_k(j)} = \frac{\delta e^{\Delta t \mathcal{L}_j}}{\delta V_k(j)} \rho_{j-1}. \quad (B5)
\]

It is known that the derivative of an exponential operator is \( \partial_x e^{A(x)} = \int_0^1 e^{x A(\partial_x A)} ds \). Thus,

\[
\frac{\delta e^{\Delta t \mathcal{L}_j}}{\delta V_k(j)} = \int_0^1 e^{\tau \Delta t \mathcal{L}_j} \left( \Delta t \frac{\delta \mathcal{L}_j}{\delta V_k(j)} \right) e^{-\tau \Delta t \mathcal{L}_j} d\tau e^{\Delta t \mathcal{L}_j}. \quad (B6)
\]

Since \( \mathcal{L}_j(\cdot) = -i[H_0 + \sum_k V_k(j) H_k, \cdot] + \Gamma(\cdot) \), we have \( \frac{\delta \mathcal{L}_j}{\delta V_k(j)} = -i H_k^\times \) where \( H_k^\times \) represents the commutation superoperator, i.e., \( H_k^\times A = [H_k, A] \). We thus have

\[
\frac{\delta e^{\Delta t \mathcal{L}_j}}{\delta V_k(j)} = -i \Delta t \int_0^1 e^{\tau \Delta t \mathcal{L}_j^\times} H_k^\times e^{-\tau \Delta t \mathcal{L}_j} d\tau e^{\Delta t \mathcal{L}_j}, \quad (B7)
\]

which can be rewritten as

\[
\frac{\delta e^{\Delta t \mathcal{L}_j}}{\delta V_k(j)} = -i \Delta t \int_0^1 e^{\tau \Delta t \mathcal{L}_j^\times} H_k^\times d\tau e^{\Delta t \mathcal{L}_j}. \quad (B8)
\]

Expand it with the Taylor series,

\[
\frac{\delta e^{\Delta t \mathcal{L}_j}}{\delta V_k(j)} = -i \Delta t \int_0^1 \sum_{n=0}^\infty \frac{(\Delta t)^n}{n!} \left( \mathcal{L}_j^\times \right)^n H_k^\times e^{\Delta t \mathcal{L}_j},
\]

\[
= -i \Delta t \sum_{n=0}^\infty \frac{(\Delta t)^{n+1}}{(n+1)!} \left( \mathcal{L}_j^\times \right)^n H_k^\times e^{\Delta t \mathcal{L}_j},
\]

where the last equation we used the first order approximation. Thus

\[
\frac{\delta \rho_j}{\delta V_k(j)} = -i \Delta t H_k^\times \rho_j. \quad (B10)
\]

2) Next we calculate \( \text{Tr}[\rho \delta \lambda_j / \delta V_k(j)] \). We first consider the cases when \( j < m \). Since \( \lambda_j = L^2_k(T) D_{j+1}^m \), and \( D_{j+1}^m \) does not contain \( V_k(j) \), thus \( \frac{\delta \lambda_j}{\delta V_k(j)} = \frac{\delta L^2_k(T)}{\delta V_k(j)} D_{j+1}^m \), we then have

\[
\text{Tr} \left( \frac{\delta \lambda_j}{\delta V_k(j)} \rho_j \right) = \text{Tr} \left( \frac{\delta L^2_k(T)}{\delta V_k(j)} L_m(T) \rho_j \right) + \text{Tr} \left( L_m(T) \frac{\delta L^2_k(T)}{\delta V_k(j)} \rho_j \right). \quad (B11)
\]

where we used the fact that \( D_{j+1}^m \rho_j = \rho(T) \).

Now take the functional derivative at both sides of the equation \( \partial_x \rho(T) = \left[ \rho(T) L_m(T) + L_m(T) \rho(T) \right]/2 \), then multiply \( L_m(T) \) and take the trace, we get

\[
\text{Tr} \left[ \frac{\delta(L_m(T)))}{\delta V_k(j)} \right] = \text{Tr} \left[ \frac{\delta L^2_k(T)}{\delta V_k(j)} L_m(T) \right] + \frac{1}{2} \text{Tr} \left( L_m(T) \frac{\delta L^2_k(T)}{\delta V_k(j)} \rho(T) \right). \quad (B12)
\]

Compare with Eq. (B11), we then have

\[
\text{Tr} \left[ \frac{\delta L^2_k(T)}{\delta V_k(j)} L_m(T) \right] = \frac{1}{2} \text{Tr} \left( L_m(T) \frac{\delta L^2_k(T)}{\delta V_k(j)} \rho(T) \right), \quad (B13)
\]

which means

\[
\text{Tr} \left( \frac{\delta \lambda_j}{\delta V_k(j)} \rho_j \right) = 2 \text{Tr} \left[ \frac{\delta L^2_k(T)}{\delta V_k(j)} L_m(T) \right] - 2 \text{Tr} \left( L_m(T) \frac{\delta L^2_k(T)}{\delta V_k(j)} \rho(T) \right). \quad (B14)
\]

Since \( \text{Tr} \left[ \frac{\delta L^2_k(T)}{\delta V_k(j)} L_m(T) \right] = \text{Tr} \left[ L^2_k(T) D_{j+1}^m \frac{\delta \rho_j}{\delta V_k(j)} \right] \), we thus have

\[
\text{Tr} \left( \frac{\delta L^2_k(T)}{\delta V_k(j)} \rho_j \right) = 2 \text{Tr} \left[ \frac{\delta L^2_k(T)}{\delta V_k(j)} L_m(T) \right] - 2 \text{Tr} \left( L_m(T) \frac{\delta L^2_k(T)}{\delta V_k(j)} \rho(T) \right). \quad (B15)
\]

where the last equality we assume the functional derivative and partial differentiation can be exchanged. Substitute Eq. (B15) into Eq. (B4), we can obtain the expression for the gradient, which is

\[
\frac{\delta F(T)}{\delta V_k(j)} = 2 \text{Tr} \left[ \partial_x \left( \frac{\delta \rho(T)}{\delta V_k(j)} \right) L_m(T) \right] + i \Delta t \text{Tr} \left( \lambda_j H_k^\times \rho_j \right). \quad (B16)
\]
We now derive \( \partial_x \left( \frac{\delta p(T)}{\delta V_{k(J)}} \right) \). Since \( \frac{\delta p(T)}{\delta V_{k(J)}} = D^m_{j+1} \frac{\delta p_j}{\delta V_{k(J)}} \), we have

\[
\partial_x \left( \frac{\delta p(T)}{\delta V_{k(J)}} \right) = \Delta t \sum_{i=j+1}^m D^m_{i+1} (\partial_x L_i) D^j_{j+1} \frac{\delta p_j}{\delta V_{k(J)}}
+ D^m_{j+1} \partial_x \left( \frac{\delta p_j}{\delta V_{k(J)}} \right), \tag{B17}
\]

where in the first term we used the fact that in the first order

\[
\partial_x e^{\Delta t l_i} = \Delta t (\partial_x l_i) e^{\Delta t l_i} \tag{B18}
\]

From Eq. (B10), we then have \( \partial_x \left( \frac{\delta p_j}{\delta V_{k(J)}} \right) = \partial_x (-i \Delta t H_k^x \rho_j) = -i \Delta t H_k^x \partial_x \rho_j \). Now as

\[
\partial_x \rho_j = \Delta t \sum_{i=1}^j D^j_{i+1} (\partial_x L_i) \rho_i,
\]

one have \( \partial_x \left( \frac{\delta p_j}{\delta V_{k(J)}} \right) = -i \Delta t^2 \sum_{i=1}^j H_k^x D^j_{i+1} (\partial_x L_i) \rho_i \). With this expression, we have

\[
D^m_{j+1} \partial_x \left( \frac{\delta p_j}{\delta V_{k(J)}} \right) = -i \Delta t^2 \sum_{i=1}^j \left[ D^m_{i+1} (\partial_x L_i) D^j_{i+1} H_k^x \rho_i \right] . \tag{B19}
\]

Thus

\[
\partial_x \left( \frac{\delta p(T)}{\delta V_{k(J)}} \right) = -i \Delta t^2 \sum_{i=j+1}^m D^m_{i+1} (\partial_x L_i) D^j_{j+1} H_k^x \rho_j
- i \Delta t^2 \sum_{i=1}^j D^m_{i+1} H_k^x D^j_{i+1} (\partial_x L_i) \rho_i . \tag{B20}
\]

Note that here we cannot discard \( \Delta t^2 \) as the second order, as we have a summation which can effectively add up to cancel one order of \( \Delta t \) (for example \( \Delta t^2 \sum_{i=1}^m 1 = m \Delta t = T \Delta t \)).

Furthermore, as \( L_i(\cdot) = -i [H_0 + \sum_k V_k(i) H_k \cdot \cdot + \Gamma(\cdot) \), and only the free Hamiltonian \( H_0 \) contains \( x \), we have \( \partial_x L_i = -i [\partial_x H_0, \cdot \cdot] = -i (\partial_x H_0)^x \). Thus, Eq. (B20) can be expressed by

\[
\partial_x \left( \frac{\delta p(T)}{\delta V_{k(J)}} \right) = -i \Delta t^2 \sum_{i=j+1}^m D^m_{i+1} (\partial_x H_0) (\partial_x L_i) D^j_{j+1} H_k^x \rho_j
- i \Delta t^2 \sum_{i=1}^j D^m_{i+1} H_k^x D^j_{i+1} (\partial_x H_0) \rho_i \tag{B21}
\]

Multiplying \( L_s(T) \) on both sides of the equation above and taking the trace gives

\[
\text{Tr} \left[ \partial_x \left( \frac{\delta p(T)}{\delta V_{k(J)}} \right) L_s(T) \right] = -\Delta t^2 \sum_{i=j+1}^m \text{Tr} \left[ L_s(T) D^m_{i+1} (\partial_x H_0)^x D^j_{j+1} H_k^x \rho_j \right]
- \Delta t^2 \sum_{i=1}^j \text{Tr} \left[ L_s(T) D^m_{i+1} H_k^x D^j_{i+1} (\partial_x H_0)^x \rho_i \right] . \tag{B22}
\]

Utilizing above equation, one can obtain the final expression for the gradient, which is

\[
\frac{\delta F(T)}{\delta V_{k(J)}} = -2 \Delta t^2 \sum_{i=j+1}^m \text{Tr} \left[ L_s(T) D^m_{i+1} (\partial_x H_0)^x D^j_{j+1} H_k^x \rho_j \right]
- 2 \Delta t^2 \sum_{i=1}^j \text{Tr} \left[ L_s(T) D^m_{i+1} H_k^x D^j_{i+1} (\partial_x H_0)^x \rho_i \right]
+i \Delta t \text{Tr} \left[ L_j^2(T) D^m_{j+1} H_k^x \rho_j \right] . \tag{B23}
\]

For the case when \( j = m \), the gradient is

\[
\frac{\delta F(T)}{\delta V_{k(J)}} = 2 \text{Tr} \left[ \left( \frac{\delta p_m}{\delta V_{k(J)}} \right) L_s(T) \right] + i \text{Tr} \left[ L_j^2(T) \Delta t H_k^x \rho_m \right] . \tag{B24}
\]

In this case, \( \delta p_m / \delta V_{k(J)} = -i \Delta t H_k^x \rho_m \), thus

\[
\text{Tr} \left[ \left( \frac{\delta p_m}{\delta V_{k(J)}} \right) L_s(T) \right] = - \Delta t^2 \sum_{i=1}^m \text{Tr} \left[ L_s(T) H_k^x D^m_{i+1} (\partial_x H_0)^x \rho_i \right] \tag{B25}
\]

Combine this equation with Eq. (B23), the gradient of the QFI can be written compactly as the from in the main text.

\[\text{Appendix C: Gradient for CFI}\]

It is known that CFI is

\[F_{\text{CFI}}(T) = \sum_y \frac{(\partial y p_y)^2}{p_y}, \tag{C1}\]

where \( p_y = \text{Tr}(\rho(x,T)E(y)) \). Here \( E(y) \) is a POVM measurement which satisfying \( \sum_y E(y) = \mathbb{I} \). To calculate the gradient, we need to know

\[
\frac{\delta p_y}{\delta V_{k(J)}} = \text{Tr} \left[ \frac{\delta p(T)}{\delta V_{k(J)}} E(y) \right] = \text{Tr} \left[ D^m_{j+1} \frac{\delta p_j}{\delta V_{k(J)}} E(y) \right]
= -i \Delta t \text{Tr} \left[ E(y) D^m_{j+1} H_k^x \rho_j \right] \tag{C2}
= -i \Delta t \text{Tr} \left[ E(y) M^{(1)}_j \right].
\]
Then we have
\[
\frac{\delta (\partial_x p_y)}{\delta V_k(j)} = -i \Delta t \text{Tr} \left[ E(y) \partial_x \left( D^{m}_{j+1} H^x_k \rho_j \right) \right]
\]
\[
= -i \Delta t \text{Tr} \left\{ E(y) \left[ (\partial_x D^{m}_{j+1}) H^x_k \rho_j + D^{m}_{j+1} H^x_k \partial_x \rho_j \right] \right\}.
\] (C3)

From previous calculations, we know
\[
\partial_x D^{m}_{j+1} = \Delta t \sum_{i=j+1}^{m} D^{m}_{i+1} (\partial_x L_i) D^{i}_{j+1},
\] (C4)
\[
\partial_x \rho_j = \Delta t \sum_{i=1}^{j} D^{i}_{j+1} (\partial_x L_i) \rho_i, \quad \text{(C5)}
\]
then for \( j \neq m, \)
\[
\frac{\delta (\partial_x p_y)}{\delta V_k(j)} = -\Delta^2 t \text{Tr} \left[ \left( E(y) \sum_{i=j+1}^{m} D^{m}_{i+1} H^x_k D^{i}_{j+1} H^x_k \partial_x \rho_j \right) + \sum_{i=1}^{j} D^{m}_{j+1} H^x_k D^{i}_{j+1} H^x_k \partial_x \rho_i \right].
\] (C6)

for \( j = m, \) there is
\[
\frac{\delta (\partial_x p_y)}{\delta V_k(m)} = -i \Delta t \text{Tr} \left[ E(y) H^x_k \partial_x \rho_m \right]
\]
\[
= -\Delta^2 t \text{Tr} \left[ E(y) H^x_k \sum_{i=1}^{m} D^{m}_{i+1} H^x_k \partial_x \rho_i \right].
\]

Thus, combined above equations, we have
\[
\frac{\delta (\partial_x p_y)}{\delta V_k(j)} = -\Delta^2 t \text{Tr} \left\{ E(y) \left[ \mathcal{M}^{(2)}_j + \mathcal{M}^{(3)}_j \right] \right\}.
\]

Finally, the gradient is
\[
\frac{\delta F_{\text{cl}}(T)}{\delta V_k(j)} = \sum_y \frac{\delta}{\delta V_k(j)} \left( \frac{(\partial_x p_y)^2}{p_T(y|x)} \right)
\]
\[
= \sum_y 2 \frac{\partial_x p_y}{p_y} \left[ \frac{\delta (\partial_x p_y)}{\delta V_k(j)} \right] - \left( \frac{\partial_x p_y}{p_y} \right)^2 \frac{\delta p_y}{\delta V_k(j)}
\]
\[
= \sum_y -2 \Delta^2 t \frac{\partial_x p_y}{p_y} \text{Tr} \left[ E(y) \left( \mathcal{M}^{(2)}_j + \mathcal{M}^{(3)}_j \right) \right]
\]
\[
+ \Delta t \left( \frac{\partial_x p_y}{p_y} \right)^2 \text{Tr} \left( E(y) \mathcal{M}^{(1)}_j \right).
\] (C7)

The gradient for CFI is then obtained.

\textbf{Appendix D: Analytical solution for transverse dephasing noise}

For the dynamics with transverse dephasing noises, the controls obtained from the GRAPE are shown in Fig. 4(b), which are \( V_x = 0, \) \( V_y = 0 \) and \( V_z = -\omega_0/2. \) The QFI under such control is the same as the QFI under free evolution for \( \omega_0 = 0, \) which can actually be computed analytically. In the following we give a detailed calculation.

Under the Bloch representation \( \rho = \frac{1}{2} \left( \mathbb{1} + \vec{r} \cdot \vec{\sigma} \right), \) the initial state \( |+\rangle \) can be expressed as \( \vec{r}(0) = (1, 0, 0). \) From the master equation we can obtain the differential equations for the Bloch vector as
\[
\partial_t r_1(t) = \omega_0 r_2(t),
\] (D1)
\[
\partial_t r_2(t) = -\gamma r_2(t) - \omega_0 r_1(t),
\] (D2)
\[
\partial_t r_3(t) = -\gamma r_3(t).
\] (D3)

The solution of these equations are
\[
r_1(t) = e^{-\frac{1}{2} \gamma t} \left[ \gamma \sinh \left( \frac{1}{2} at \right) + \cosh \left( \frac{1}{2} at \right) \right],
\] (D4)
\[
r_2(t) = -\frac{2 \omega_0}{a} e^{-\frac{1}{2} \gamma t} \sinh \left( \frac{1}{2} at \right),
\] (D5)
\[
r_3(t) = 0,
\] (D6)

where \( a = \sqrt{\gamma^2 - 4 \omega_0^2}. \) We will now compute \( f(\rho_1, \rho_2, \omega_0) \) where \( f(\rho_1, \rho_2) = \text{Tr} \sqrt{\rho_1 \rho_2 \rho_1} \) denotes the fidelity, the QFI can then be obtained from the second order expansion of \( f(\rho_1, \rho_2, \omega_0) \) for \( \omega_0 = 0. \) It is easy to see that \( \vec{r}(t)|_{\omega_0=0} = (1, 0, 0), \) which is \( |+\rangle = \sqrt{1/2} (|0\rangle + |1\rangle) \) and the fidelity between \( |+\rangle \) and an evolved state with a general \( \omega_0 \) is
\[
f = \sqrt{\langle + | \left( \frac{1}{2} \mathbb{1} + \vec{r}(t) \cdot \vec{\sigma} \right) |+\rangle} = \sqrt{\frac{1}{2} + \frac{1}{2} r_1(t)}.
\] (D7)

For a small \( \delta \omega_0, \) up to the second order we have \( a = \gamma - \frac{\delta^2 \omega_0}{\gamma} \) and \( \frac{\gamma}{a} = 1 + \frac{1}{\gamma} \delta^2 \omega_0, \) then
\[
r_1(t) = e^{-\frac{1}{2} \gamma t} \left[ \frac{\gamma}{a} \sinh \left( \frac{1}{2} at \right) + \cosh \left( \frac{1}{2} at \right) \right]
\]
\[
= e^{-\frac{1}{2} \delta^2 \omega_0} + \frac{2}{\gamma^2} \delta^2 \omega_0 - e^{-\gamma t} e^{\frac{t}{2} \delta^2 \omega_0}
\]
\[
= 1 - \frac{1}{\gamma^2} \left( e^{-\gamma t} + \gamma t - 1 \right) \delta^2 \omega_0.
\] (D8)

Thus the fidelity is
\[
f(\rho_0, \rho_{\delta \omega_0}) = \sqrt{1 - \frac{1}{2} \gamma^2 (e^{-\gamma t} + \gamma t - 1) \delta^2 \omega_0}
\]
\[
= 1 - \frac{1}{4} \gamma^2 (e^{-\gamma t} + \gamma t - 1) \delta^2 \omega_0.
\] (D9)

The QFI can then be obtained from the second order term as
\[
F(t) = \frac{2}{\gamma^2} (e^{-\gamma t} + \gamma t - 1). \quad \text{(D10)}
\]

Now we consider the classical Fisher information under optimal controls. Taking the measurement as \( \{|+\rangle\langle+|,-\rangle\langle-|\}, \) then the probabilities are \( p_+(t) = (1+ \ldots \)
The classical Fisher information is
\[ F_{\text{classical}}(t) = \frac{(\partial_{\omega_0} p^+)^2}{p^+ p^-} = \frac{(\partial_{\omega_0} r_1)^2}{1 - r_1^2(t)}. \] (D11)

For the the controls \( \omega_0 \sigma_3 / 2 \), where \( \omega_0 \) is very close to \( \omega_0 \), based on Eq. (D8), one can see \( 1 - r_1^2(t) = F(t) \delta \omega_0 \) and \( \partial_{\omega_0} r_1(t) = -F(t) \delta \omega_0 \), which indicates \( F_{\text{classical}}(t) = F(t) \), i.e., the measurement \( \{|+\rangle\langle+|, |-\rangle\langle-|\} \) is the optimal measurement to access the quantum Fisher information.

**Appendix E: Parallel dephasing**

Here we consider the simple control strategy for the dynamics with parallel dephasing noises. Recall that the strategy is to first prepare the probe state at \( |+\rangle \) and let it evolve under the natural evolution (without controls) for a period of \( t_0 \), then apply a \( \pi/2 \)-pulse along \( y \)-direction and let it evolve for another period of \( T - t_0 \). As shown in the main text the final state at \( T \) in the Bloch representation is given by \( \tilde{r}(T) = (r_1(T), r_2(T), r_3(T)) \) with
\[
\begin{align*}
   r_1(T) &= e^{-\gamma T} \sin(\omega_0 t_0) \sin(\omega_0 t_0), & (E1) \\
   r_2(T) &= e^{-\gamma T} \cos(\omega_0 t_0) \sin(\omega_0 t_0), & (E2) \\
   r_3(T) &= e^{-\gamma t_0} \cos(\omega_0 t_0), & (E3)
\end{align*}
\]
from which we can obtain the QFI using the following formula [51]
\[
F(T) = |\partial_{\omega_0} \tilde{r}(T)|^2 + \frac{(\tilde{r}(T) \cdot \partial_{\omega_0} \tilde{r}(T))^2}{1 - |\tilde{r}(T)|^2},
\] (E4)
specifically,
\[
F(T) = e^{-2\gamma t_0} t_0^2 \sin^2(\omega_0 t_0) + e^{-2\gamma T} [t_0^2 + T(T - 2t_0) \sin^2(\omega_0 t_0)]
+ t_0^2(e^{-2\gamma T} - e^{-2\gamma t_0} \sin^2(\omega_0 t_0) \cos^2(\omega_0 t_0))
+ \frac{t_0^2}{1 - e^{-2\gamma T} \sin^2(\omega_0 t_0) - e^{-2\gamma t_0} \cos^2(\omega_0 t_0)}.
\]

**Appendix F: Spontaneous emission**

For the non-controlled scheme, recall the Hamiltonian of this example is \( H = \frac{1}{2} \omega_0 \sigma_3 \) and the master equation for the spontaneous emission is
\[
\partial_t \rho(t) = -i[H, \rho] + \gamma_+ \left[ \sigma_+ \rho(t) \sigma_- - \frac{1}{2} \{\sigma_- \sigma_+, \rho(t)\} \right] \\
+ \gamma_- \left[ \sigma_- \rho(t) \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho(t)\} \right],
\] (F1)
The solution for the master equation is
\[
\begin{align*}
   \rho_{00}(t) &= e^{-\gamma_+ \gamma_- t} \rho_{00}(0) + \gamma_+ \gamma_- \left[ 1 - e^{-\gamma_+ \gamma_- t} \right], \\
   \rho_{01}(t) &= e^{-i\omega t - \frac{1}{2}(\gamma_+ + \gamma_-)t} \rho_{01}(0).
\end{align*}
\]
In the Bloch representation, we have
\[
\begin{align*}
   r_1(t) &= e^{-\frac{1}{2}(\gamma_+ + \gamma_-)t} \{ \sin(\omega_0 t)r_1(0) - \sin(\omega_0 t)r_2(0) \}, \\
   r_2(t) &= e^{-\frac{1}{2}(\gamma_+ + \gamma_-)t} \{ \cos(\omega_0 t)r_2(0) + \sin(\omega_0 t)r_1(0) \}, \\
   r_3(t) &= \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \left[ 1 - e^{-\gamma_+ \gamma_- t} \right] + e^{-\gamma_+ \gamma_- t} r_3(0).
\end{align*}
\]
For the initial state \( |+\rangle \), the evolved Bloch vector reads
\[
\begin{align*}
   r_1(t) &= e^{-\frac{1}{2}(\gamma_+ + \gamma_-)t} \cos(\omega_0 t), & (F2) \\
   r_2(t) &= e^{-\frac{1}{2}(\gamma_+ + \gamma_-)t} \sin(\omega_0 t), & (F3) \\
   r_3(t) &= \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \left[ 1 - e^{-\gamma_+ \gamma_- t} \right]. & (F4)
\end{align*}
\]
From these expressions, we have \( |\partial_{\omega_0} \tilde{r}(T)|^2 = e^{-(\gamma_+ + \gamma_-)T^2} \) and \( \tilde{r}(T) \cdot \partial_{\omega_0} \tilde{r}(T) = 0 \), thus, the QFI at target time for non-controlled scheme is
\[
F = e^{-(\gamma_+ + \gamma_-)T^2},
\] (F5)

Next we perform a single rotation strategy as an intuitive mechanism for the effect of control: the Bloch vector is rotated by the control to \( x - y \) plane along the \( y \) axis at time \( t_0 \). Before the rotation, the Bloch vector is
\[
\begin{align*}
   r_1(t_0) &= e^{-\frac{1}{2}(\gamma_+ + \gamma_-)t_0} \cos(\omega_0 t_0), & (F6) \\
   r_2(t_0) &= e^{-\frac{1}{2}(\gamma_+ + \gamma_-)t_0} \sin(\omega_0 t_0), & (F7) \\
   r_3(t_0) &= \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \left[ 1 - e^{-\gamma_+ \gamma_- t_0} \right]. & (F8)
\end{align*}
\]
In the following we assume \( \gamma_+ = 0 \) and \( \gamma_- = \gamma \), above expressions then reduce to
\[
\begin{align*}
   r_1(t_0) &= e^{-\gamma t_0} \cos(\omega_0 t_0), & (F9) \\
   r_2(t_0) &= e^{-\gamma t_0} \sin(\omega_0 t_0), & (F10) \\
   r_3(t_0) &= -1 + e^{-\gamma t_0}. & (F11)
\end{align*}
\]
Now we perform the rotation \( R_{y,1} \) in the form below
\[
\begin{pmatrix}
   r_1(t_0) |_{\omega_0 = \tilde{\omega}_0} & 0 & r_3(t_0) |_{\omega_0 = \tilde{\omega}_0} \\
   \sqrt{r_1^2(t_0)} |_{\omega_0 = \omega_0 + r_2^2(t_0)} & 1 & 0 \\
   -r_2(t_0) |_{\omega_0 = \omega_0 + r_2^2(t_0)} & 0 & \sqrt{r_2^2(t_0)} |_{\omega_0 = \omega_0 + r_2^2(t_0)}
\end{pmatrix},
\]
where \( \tilde{\omega}_0 \) is the true value of \( \omega_0 \). After the rotation, we have
\[
R_{y,1} \tilde{r}(t_0) = \begin{pmatrix}
   r_1(t_0) |_{\omega_0 = \omega_0} + r_1(t_0) |_{\omega_0 = \tilde{\omega}_0} r_3(t_0) |_{\omega_0 = \omega_0} \\
   \sqrt{r_1^2(t_0)} |_{\omega_0 = \omega_0 + r_2^2(t_0)} & r_2(t_0) \\
   -r_2(t_0) |_{\omega_0 = \omega_0 + r_2^2(t_0)} & 0 & \sqrt{r_2^2(t_0)} |_{\omega_0 = \omega_0 + r_2^2(t_0)}
\end{pmatrix}.
\] (F12)
Then the Bloch vector at target time \( T \) reads
\[ r_1(T) = e^{-\frac{1}{2} \gamma(T-t_0)} \left\{ \cos[\omega_0(T-t_0)] \frac{r_1(t_0)|_{\omega_0=\bar{\omega}_0} r_1(t_0) + r_3(t_0)|_{\omega_0=\bar{\omega}_0} r_3(t_0)}{\sqrt{r_1^2(t_0)|_{\omega_0=\bar{\omega}_0} + r_3^2(t_0)|_{\omega_0=\bar{\omega}_0}}} - \sin[\omega_0(T-t_0)] r_2(t_0) \right\}, \]  
\[ r_2(T) = e^{-\frac{1}{2} \gamma(T-t_0)} \left\{ \cos[\omega_0(T-t_0)] r_2(t_0) + \sin[\omega_0(T-t_0)] \frac{r_1(t_0)|_{\omega_0=\bar{\omega}_0} r_1(t_0) + r_3(t_0)|_{\omega_0=\bar{\omega}_0} r_3(t_0)}{\sqrt{r_1^2(t_0)|_{\omega_0=\bar{\omega}_0} + r_3^2(t_0)|_{\omega_0=\bar{\omega}_0}}} \right\}, \]  
\[ r_3(T) = -1 + e^{-\gamma(T-t_0)} + e^{-\gamma(T-t_0)} - r_3(t_0)|_{\omega_0=\bar{\omega}_0} r_3(t_0) + r_1(t_0)|_{\omega_0=\bar{\omega}_0} r_1(t_0) \frac{\sqrt{r_1^2(t_0)|_{\omega_0=\bar{\omega}_0} + r_3^2(t_0)|_{\omega_0=\bar{\omega}_0}}} \right\}. \]  

At the point \( \omega_0 = \bar{\omega}_0 \), the Bloch vector is
\[ r_1(T)|_{\omega_0=\bar{\omega}_0} = e^{-\frac{1}{2} \gamma T} \left\{ - (T-t_0) \sin[\omega_0(T-t_0)] \sqrt{\cos^2(\bar{\omega}_0 t_0)} + e^{\gamma t_0} - 2 + e^{-\gamma t_0} + t_0 \sin[\omega_0(2t_0 - T)] \right\}, \]  
\[ - \cos[\omega_0(T-t_0)] \sin(\bar{\omega}_0 t_0) \left( \frac{t_0 \cos(\bar{\omega}_0 t_0)}{\sqrt{\cos^2(\bar{\omega}_0 t_0) + e^{\gamma t_0} - 2 + e^{-\gamma t_0}}} - T \right) \}, \]  
\[ [\partial_{\omega_0} r_2(T)]|_{\omega_0=\bar{\omega}_0} = e^{-\frac{1}{2} \gamma T} \left\{ (T-t_0) \cos[\omega_0(T-t_0)] \sqrt{\cos^2(\bar{\omega}_0 t_0)} + e^{\gamma t_0} - 2 + e^{-\gamma t_0} + t_0 \cos[\omega_0(2t_0 - T)] \right\}, \]  
\[ - \sin[\omega_0(T-t_0)] \sin(\bar{\omega}_0 t_0) \left( \frac{t_0 \cos(\bar{\omega}_0 t_0)}{\sqrt{\cos^2(\bar{\omega}_0 t_0) + e^{\gamma t_0} - 2 + e^{-\gamma t_0}}} + T \right) \}, \]  
\[ [\partial_{\omega_0} r_3(T)]|_{\omega_0=\bar{\omega}_0} = \frac{e^{-\gamma T} - e^{-\gamma(T-t_0)}}{\sqrt{\cos^2(\bar{\omega}_0 t_0) + e^{\gamma t_0} - 2 + e^{-\gamma t_0}}} t_0 \sin(\bar{\omega}_0 t_0). \]  

The QFI can then be obtained via Eq. (E4).

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