Scaling Regimes in the Distribution of Galaxies

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Abstract If we treat the galaxies in published redshift catalogues as point sets, we may determine the generalized dimensions of such sets by standard means, outlined here. For galaxy separations up to about 5 Mpc, we find the dimensions of the galaxy set to be about 1.2, with not a strong indication of multifractality. For larger scales, out to about 30 Mpc, there is also good scaling with a dimension of about 1.8. For even larger scales, the data seem too sparse to be conclusive, but we find that the dimension is climbing as the scales increase. We report simulations that suggest a rationalization of such measurements, namely that in the intermediate range the scaling behavior is dominated by flat structures (pancakes) and that the results on the smallest scales are a reflection of the formation of density singularities.

1. Introduction

An interesting question in modern cosmology is how to reconcile the very inhomogeneous distribution of galaxies with the very isotropic distribution of the cosmic background radiation. The conventional picture is that the marginally detectable anisotropies of the radiation field are the seeds of the impressive structures seen in the galaxy distribution. The transition from weak initial perturbation to strong clustering is supposed to be initiated by gravitational instability. This process, which is weak in an expanding medium,
is nevertheless called upon to take the perturbations into a nonlinear collapse phase that produces well defined structures that are reflected in the lumpy distribution of galaxies.

In the present work, we try to make sense of the complicated galaxy distribution in terms of simple — perhaps overly simple — pictures. We are suggesting that the data on the galaxy distribution imply that there are two principal scales of structure formation each with its own characteristics. The aim of this informal general discussion is explain why we believe that we can detect these regimes in the analysis of publically available galaxy catalogues. The existence of a possible third regime at the largest scales may also be discernable. The final clarification of course awaits the coming of the tidal wave of data that should soon engulf us and which may perhaps wash away all our present conceptions.

Before we describe how the galaxy distribution is quantified and the regimes are detected, we briefly describe the simulations of gravitational clustering that have produced the preconceptions that we bring to the analysis. Then we outline one of the main methods of quantification of structure that has been used in this problem. Finally, we offer a preliminary interpretation of the observed structures.

2. Density singularities
Gravitational instability is usually assumed to be the primary mechanism that shapes the galaxy distribution. There may be other interactions that play a role in the process, depending on the era and on what matter and fields are present, but we ignore those here. Most of the models on the market are based on a strong preponderance of dark matter, and this is consistent with many lines of evidence. It is not clear what this matter is, but we shall here suppose that it is rather cool and describe a CDM (cold dark matter) scenario.

Early analytic studies have indicated that gravitational collapse may generate virialized mass concentrations, with approximately isothermal density profiles (Gunn and Gott 1972, Fillmore and Goldreich 1984, Bertschinger 1985, Gurevich and Zybin 1988), with details depending on the initial density perturbations and on the value of $\Omega$. A study by Henriksen and Widrow (1995) based on Vlasov equation in expanding space produces local
singular density distributions with density proportional to $r^{-\alpha}$ where $1.5 \leq \alpha \leq 3$ and $r$ is the distance from the singularity.

The means for simulating structure formation are diverse. Some effort has gone into modeling with the Vlasov equations complemented by the Poisson equation. This latter would in principle be a good way to proceed if one could really carry it out. In fact, it is difficult to follow in detail the development of plasma turbulence through the collective effects that it contains. Such collective effects perhaps make the gravitational problem more like a fluid dynamical problem than a particle dynamics problem.

More typically, numerical simulation of the structure formation process has been pursued by N-body techniques, where point masses, representing fluid elements, move under the influence of their own gravitational interaction in an expanding background. On the scales of galaxies and galaxy clusters, these simulations generally confirm the presence of strong density concentrations with a local density growing algebraically, at least in a certain range of scales (Navarro, Frenk and White 1996, 1997).

Many simulations have been performed and we may mention the recent report of large scale calculations of this kind by Zurek and Warren (1994). Our own simulations, which we describe in this section, were performed with an adaptation of Couchman’s (1991) public AP3M code for present purposes. These simulations have $N_p = 128^3$ massive particles on a Cartesian grid of $N_g = 128^3$. For the initial conditions, we introduce, as is usual, a small perturbation superposed onto a homogeneous density field that is a solution to the Friedmann-Lemaitre equations. The average density has been taken to be equal to the critical density $\rho_{cr}$, so that $\Omega = \rho_{av}/\rho_{cr} = 1$; this makes it easier for structures to form than do the observationally suggested values which are noticeably less than unity. The initial conditions have Gaussian, scale-free density perturbation spectra with random Fourier phases and density power spectrum

$$P(k) \propto k^n.$$ (2.1)

During the computed evolution, strong density concentrations form. For each particle cluster, the center of mass has been determined and the radial density profile has been
evaluated. Each profile has been normalized to the half-mass radius (the radius within which half of the mass in a cluster is contained) and the average density profile has been calculated.

Figure 1 shows the averaged dark halo profiles for the four initial conditions considered, at the time when the variance of the density field, coarse-grained on a scale of \(1/16\) of the domain size is unity. At scales smaller than the half-mass radii, the dark halos have power-law profiles for all the four initial conditions considered. These profiles are slightly steeper for larger \(n\), showing also a tendency to display a larger range of power-law behavior for larger values of \(n\).

What is the reason for the difference between the four average profiles, and what is their time evolution? Are they evolving in time or do they have a logarithmic slope which is constant in time?

Figure 2 shows the average profiles of collapsed objects, obtained from the three initial conditions with \(n = 1, 0, -1\), at different times. This figure shows that, independently of the initial conditions, the slope of the profile is a function of the central density of the collapsed object, the profile being steeper for larger central density. The average profiles of objects with similar central density, produced by simulations with different initial conditions, are the same. For the three spectral indices shown in Figure 3, the profile steepens, and comes closer to a power-law, as time increases.

A different behavior is observed for \(n = -2\). In this case, shown in Figure 3, the profiles at different times do not evolve; rather, they keep the same shape and logarithmic slope. The initial conditions for the CDM scenario are often assumed to have a power spectrum with \(n \approx -2\) on small scales. This suggests that dark halos for CDM initial conditions do not evolve in time, once the transients have decayed.

We turn now to a discussion of the evolved structures. In doing this we must leave several questions unanswered: why do the \(n = -2\) initial conditions behave so differently from the other three cases considered? For \(n = 1, 0, -1\), do the dark halo profiles evolve in time forever, with an ever-increasing steepness, or do they reach a limiting slope after which
they do not evolve further? To answer these questions, one should properly understand
the behavior of gravitational collapse at small scales. On more general grounds, we recall
that the results reported here (and in several other numerical studies) refer to cold initial
conditions. Does the addition of random dispersion velocities significantly modify the
picture? And, finally, what happens when one includes hydrodynamical effects, such that
the collapse is not purely gravitational any longer?

3. Quantifying Cosmic Scaling

Early analyses of the galaxy distribution were based upon the the correlation function
(Groth and Peebles 1977, Peebles 1980) but, in recent years, the correlation integral (Grass-
berger and Procaccia 1983, Paladin and Vulpiani 1984, Borgani 1995) has also been used.
It seems that if the galaxy distribution is a fractal, the latter is advantageous since the de-
inition of the correlation involves the density of galaxies. Though the notion of density is
intuitively reasonable, it is not well defined for a fractal (Coleman, Pietronero and Sanders
1988, Coleman and Pietronero 1992) and this makes the use of correlations problematic.
At any rate, in the study of analytically understood point sets, the correlation integral is a
more reliable tool of analysis (Thieberger, Spiegel and Smith 1990), so it seems worthwhile
to describe it briefly here. We treat the galaxies as a point set for this purpose.

Let \( N_i(r) \), be the number of galaxies of the set lying within a distance \( r \) of the typical
(the \( i^{th} \)) galaxy. This number is then averaged over the galaxies in the set to give us the
number of galaxies within a distance \( r \) of a typical galaxy. If there are a lot of galaxies
around, we may replace the sum in the calculation of \( N \) by an integral and write

\[
N(r) = 4\pi \int_0^r n(s)s^2ds .
\]  

(3.1)

The radial distribution function \( n \) is not a good density of galaxies since it does not in
general tend to a finite limit at extremes of \( r \).

If the galaxies in the set are found in a finite volume, we may divide the total number
of them by this volume and call this \( n_0 \). The correlation used in the study of the galaxy
distribution may then be written as \( \xi = n/n_0 - 1 \). (The choice of \( n_0 \) is not crucial and
it may be defined otherwise than here.) For a uniform Poisson distribution, \( \xi = 0 \), the galaxies are uncorrelated and \( n = n_0 \). If \( \xi \) is positive, the galaxy positions are correlated.

When \( r \) goes to zero, so must \( N(r) \) and the simplest way to achieve this is to let \( n \) go as a power, say, \( n = n_0(r/r_0)^{-\gamma} \), where \( r_0 \) is a conveniently chosen length. For small \( r/r_0 \), we see that \( \xi \propto r^{-\gamma} \), so there seems little to choose between the two approaches. But when the data are few, the tendency is to let \( r/r_0 \) get close to unity, and the differences appear in the results. Enough has been written about these differences so that we have simply chosen one approach, which we now outline.

The correlation integral (Grassberger and Procaccia 1983) can be defined as

\[
C_2(r) = \frac{1}{N(N-1)} \sum_i \sum_{j \neq i} \Theta(r - |X_i - X_j|)
\]

where \( \Theta \) is the Heaviside function. The summations are over the whole set of \( N-1 \) galaxies with coordinates \( X_j, j \neq i \). (Modifications of this formula to allow for the effects of the finiteness of the sample have been used in dealing with the data. We omit these technicalities here but see Murante et al. (1997).) We see from the formula that \( C_2(r) = N(r)/N \). The quantity \( C_2(r) \) is thus proportional to the volume integral of \( n(r) \). As \( r \) gets small, \( C_2 \) must go to zero and for general distributions, in the limit \( r \to 0 \), and we express this as \( C_2 \propto r^{D_2} \). The exponent \( D_2 \) is called the correlation dimension and we see that \( D_2 = 3 - \gamma \) (Provenzale 1991).

As with the correlations, it is possible study higher order statistics. To generalize the correlation integral formalism (Grassberger and Procaccia 1983) one introduces correlation integrals defined as

\[
C_q(r) = \left( \frac{1}{N} \sum_i \left[ \frac{1}{N-1} \sum_{j \neq i} \Theta(r - |X_i - X_j|) \right]^{q-1} \right)^{-\frac{1}{q-1}}
\]

where \( q \) is a parameter that defines the order of the moment. For \( q = 2 \) a two-point probability (second order moment) is evaluated and the standard correlation integral is recovered. More generally, for any integer value of \( q \), \( C_q(r) \) is the fraction of of \( q \)-tuples in the set whose members lie within a distance \( r \) of one another. For sufficiently small...
\( r \), \( C_q \) will go to zero for \( q > 1 \) and, for a typically well-behaved set, it will vanish like \( r^{D_q} \). The index \( D_q \) is called a generalized or Renyi dimension (Renyi 1970, Halsey et al. 1986). A fractal for which the dimensions are all the same (\( D_q \) independent of \( q \)) is called a homogeneous fractal, or a monofractal. The more general cases with \( D_q \) depending on \( q \) are called multifractals. Some authors reserve the general term fractal for the special case of a monofractal, but here we retain the general sense of the term fractal, with the multifractal as a particular case.

4. Lacunarity

Another feature of the scaling analysis may ultimately prove interesting for the study of the galaxy distribution once the data are sufficiently abundant. As yet we have only some preliminary results on this, but we find them intriguing enough to mention here.

For the typical case, the correlation integrals go to zero like a power of the separation. This behavior may be considered as the first term in an asymptotic expansion of \( \log C_q \) in \( \log r \). More generally, we may seek an expansion of the form

\[
\log C_q(\log r) = D_q \log r + E_q + \frac{F_q}{\log r} + \ldots .
\]  \hspace{1cm} (4.1)

Higher terms are hard to detect when there are few data, but it may be possible to get an estimate of the second term. When only two terms are kept in the expansion, we have the representation

\[
C_q = \Lambda_q r^{D_q} .
\]  \hspace{1cm} (4.2)

Mandelbrot (1982) has named \( \Lambda \) the lacunarity.

A weak dependence of \( E_q \) on \( r \) is compatible with this representation. This is seen easily for the case of the homogeneous fractal, for which the generalized dimensions, \( D_q \), are all the same, \( D_q = D \), say. We may expect in that case that any statistical moment \( C(r) \) satisfies the scaling law

\[
C(r) = aC(br)
\]  \hspace{1cm} (4.3)

with constant \( a \) and \( b \). This functional equation has solutions of the form (4.2) with

\[
D_q \equiv D = \frac{\log a}{\log b} .
\]  \hspace{1cm} (4.4)
As we see, (4.2) and (4.4) satisfy the functional equation (4.2) even when \( \Lambda \) depends on \( \log r \) provided that \( \Lambda(\log r - \log b) = \Lambda(\log r) \). In this case, we call \( \Lambda \) the lacunarity function, or LF, and we may identify \( \log \Lambda \) with \( E_q \) in the asymptotic development, so long as \( E_q \) remains of order unity. The appearance of a periodic dependence of \( E_q \) on \( \log r \) is typically seen in monofractals, when there are enough data. Solis and Tao (1997) have seen this in theoretical multifractals, though more weakly. In the case of the theoretical fractals, the period of the LF is a remnant of the decimation procedure that produced the fractal.

5. Cosmic Scaling Regimes

We concentrate here on the analysis of redshift catalogues, so we shall not go into the issues that the study of position catalogues entail such as the effect of projection of the spatial distribution onto the celestial sphere. A full discussion of this issue would involve the basic notions developed in the study of stellar statistics and the analysis of radio source counts. In the latter, a dependence on \( r \) was often overlooked and anomalies in the data were rationalized by assuming a dependence on time. We shall not go that far back into the history of the subject.

The first analyses of the distribution of the galaxies on the celestial sphere to gain wide acceptance were based on the correlation function and they found an approximate scaling regime on scales smaller than about \( 5 \, h^{-1} \) Mpc, where \( h \) is the Hubble constant in units of 100 km/sec Mpc\(^{-1} \) (Groth and Peebles 1977, Peebles 1980). Out to these scales, the two-point galaxy correlation function has an approximate power-law shape with \( \xi(r) \propto r^{-\gamma} \) where \( \gamma \approx 1.8 \) (Peebles 1980). In the fractal vernacular, this would mean that the correlation dimension of the galaxy distribution is \( D_2 \approx 1.2 \) out to an outer scale of \( 5 \, h^{-1} \) Mpc.

When the data were later analyzed using correlation integrals, the result \( D_2 \approx 1.8 \) was found (Thieberger, Spiegel and Smith 1990, Martinez and Jones 1990, Borgani 1995). The explanation for this discrepancy with the earlier work, at least the one that we adopt, is that the scaling law on scales under 5 Mpc is different from those at larger scales (Guzzo
et al. 1991, Murante et al. 1996). That is, we have $0.8 < D_q < 1.4$ for $q \geq 2$ at scales below $5 \, h^{-1} \text{Mpc}$, whereas, on intermediate scales, the value of $D_2$ is just under 2; we find 1.8. On the very largest scales, above about $30Mpc$, we find that $D_2$ has begun another rise.

To understand the meaning of $D_2 = 3 - \gamma \approx 1.2$ for scales less than 5 Mpc, we have computed the dimension of a random distribution of density singularities with power law density distributions. Once strong clustering has occurred, simulations of the kind we discussed in the section 2 suggest that the collapse into clusters continues into the formation of local structures with approximate power-law density profile around the center of the collapsed distribution. Within such distributions, there is no fractal structure in the sense that this term is now normally used, though one may broaden that sense. When there are local smooth structures, the usual notion of density may be used, even if its definition is still not so clear. As mentioned in the discussion on dynamics, roughly spherical structures with density distributions with densities varying like $r^{-\alpha}$ form with $r$ being the distance from the singularity center. The value $\alpha = 2$, which is appropriate for the isothermal sphere, is just the one that produces flat rotation curves in a disk around the singularity center. We have found that a random distribution of such power-law density singularities can give global scaling properties like those seen on scales seen for sizes under about five Mpc, for a density exponent, $\alpha$, of order two.

On the other hand, simulations show that the first structures to form are highly flattened, as anticipated by Zeldovich (1970). Such structures produce a fractal dimension of about 2. The natural interpretation is that the initial formation of pancakes of this kind then leads to the formation of substructures which collapse down to singularities and the two separate regimes produce the two observed correlation dimensions. The value of $D_2 \approx 1.8$ that now seems to hold in the scale range of 5-50 Mpc is compatible with the pancake regime.

As we go to larger scales, $D_2$ seems to be increasing, but the data are too sparse as yet for a definite conclusion. Many believe that there has not been enough time as yet to
have allowed the formation of well defined structures on scales of hundreds of megaparsecs. On the other hand, Pietronero and collaborators defend the de Vaucouleur idea that there is structure at all scales in terms of these same data. That vision may call for a ‘spooky action at a distance’ in Einstein’s famous phrase and so test our notions of causality.

In conclusion, we believe that the considerations we have outlined here, rationalize the observed exponents but they do not tell us why the scaling regimes are what is observed. One way to go further would be to look at finer details. To do this, we have made an attempt to detect the lacunarity function of the galaxy distribution. For this purpose, we have used a sample of 30000 galaxies from the CfA-ZCAT galaxy catalogue (Huchra et al. 1993). The reductions are described elsewhere (Provenzale, Spiegel and Thieberger 1997).

A plot of

\[ \log \Lambda = \log C_q - D_q \log r \]  

(5.1)

vs. \( \log r \) for \( q = 2, 3, 4 \) is shown for the northern galaxies, see Figure 4. We see perhaps one period of the anticipated oscillation.

Since the amplitude of the oscillation in the LF is so small, we must of course be wary in accepting it as real. Nevertheless, there are some features that do make it seem worth pursuing this matter further. First, there is the apparent constancy of what we may optimistically call the period for the three values of \( q \) that we have examined. This is suggestive of monofractality. Second, although the LF for the southern hemisphere is not nearly so well defined as for the northern hemisphere, its ‘period’ is the same as in the northern hemisphere. If this result holds up, it is striking evidence of statistical homogeneity of the galaxy distribution on large scales. Since we do not see this kind of oscillatory LF for the smallest scales the LF may also be in keeping with our interpretation of the dimensions on those scales.
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Figure Captions.

Figure 1.
Average density profiles of the dark halos for the four scale-free initial conditions considered in the text, at the time when the density variance is 1 at 1/16th of the simulation box.

Figure 2.
Average profiles of the dark halos produced by cosmological N-body simulations for three scale-free initial conditions with spectral indeces $n = 1, 0, -1$, at different evolutive times.

Figure 3.
Average profiles of the dark halos produced by a cosmological N-body simulation with scale-free initial conditions and spectral index $n = -2$, at different evolutive times.

Figure 4.
Lacunarity function for $q = 2, 4, 6$ for the CfA galaxies in the Northern galactic emisphere.
Figure 1
Average density

$R/HMR$

Figure 2
Figure 3
