Search for periodic vacuum in QED$_2$

S. Nagy and K. Sailer

Department for Theoretical Physics, Kossuth Lajos University, Debrecen, Hungary

Abstract

It is shown that the vacuum of QED$_2$ in Minkowski spacetime does not favour a periodic electric mean field. The projected effective action exhibiting a genuine dependence on the non-vanishing background field has been introduced. The functional dependence of the energy density of the vacuum on the assumed periodic vacuum expectation value of the vector potential is determined from the component $T^{00}$ of the energy-momentum tensor at one-loop order. Treating the background field non-perturbatively, the energy of the vacuum in the presence of a periodic mean field is found not to be equal to the negative of the effective action.

PACS: 12.20.Ds

1 Introduction

It is well-known that the perturbative vacuum of QED is trivial, i.e. it is characterized by the vanishing expectation value of the electromagnetic vector potential. Our goal is to investigate the vacuum non-perturbatively in the framework of the continuum theory in Minkowski spacetime. As a working hypothesis we assume the presence of a periodic electric background field. The interaction of the fermions with the background field is treated exactly by integrating out the fermion fields, whereas the quantum fluctuations of the electromagnetic field are taken into account at one-loop order. The non-perturbative treatment of the interaction with the background introduces infinitely many non-renormalizable, i.e. irrelevant interaction vertices. It may happen that such vertices generate non-trivial, periodic (otherwise anti-ferromagnetic) vacuum structure. Similar examples are known for Yang-Mills theories on the lattice [1], where the existence of various anti-ferromagnetic vacua have been established due to irrelevant interaction terms.

Our problem setting can also be motivated by the following intuitive picture. In the presence of a background scalar potential $A_0 = a \cos(\ell x)$ with
overcritical amplitude \( a > m \) (with the rest mass of the electron \( m \)) localized electrons of the Dirac sea may tunnel into localized electron states of the same energy above the mass gap (see Fig. 1). Therefore, the creation of a certain amount of electron-positron pairs with the spatial separation of \( \pi/\ell \) is imaginable. This might stabilize itself with a periodic charge density (but zero net charge) and a periodic mean electric field. The expected effect might be destroyed due to the uncertainty principle. Localisation in an interval \( \Pi/2\ell \) means a momentum and, consequently, an energy spread of order \( \Delta E = \frac{p}{E} \Delta p \) and the effect may occur only if \( a >> m + \frac{\Delta E}{2} \).

Figure 1: Intuitive picture of pair creation in overcritical periodic external electric field

2 Projection method

It is not completely trivial to define an effective action depending on a background field \( \bar{A}^\mu(x) \). The simple shift of the integration variable \( A^\mu(x) \) to \( \alpha^\mu(x) = A^\mu(x) - \bar{A}^\mu(x) \) is not the answer requested, since a dependence on the background \( \bar{A}^\mu(x) \) shall only occur due to terminating the loop expansion.
Below we introduce an effective action exhibiting a genuine dependence on the background field.

Our method of defining the sector of QED for field configurations belonging to quantum fluctuations around a given vacuum expectation value \( \langle A^\mu(x) \rangle \) consists of the following steps:

1. *Projection to the sector with a given background field.* The generating functional of QED in Minkowski spacetime has the form:

\[
Z_{QED} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \exp\left\{ iS_{em}[A,\xi] + iS_D[A,\bar{\psi},\psi] \right\}
\]

with the vector potential \( A^\mu(x) (\mu = 0, 1, 2, 3) \) and the electron-positron field \( \psi(x) \), where \( S_{em}[A,\xi] \) and \( S_D[A,\bar{\psi},\psi] \) are the action of the electromagnetic field and the Dirac action, resp., in the covariant gauge with the gauge parameter \( \xi \). Let the vector \( n^\mu(x) \) be introduced in the space of the vector potential configurations. Multiply the integrand of the generating functional by the factor 1 written in the form

\[
1 = \int dc \delta\left( \int dx A_\mu n^\mu - c\Omega \right)
\]

with \( \Omega = TV \) the spacetime volume. Then, we find

\[
Z_{QED} = \int dcZ'_{QED}[n,c],
\]

where the functional

\[
Z'_{QED}[n,c] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \exp\left\{ iS_{em}[A,\xi] + iS_D[A,\bar{\psi},\psi] \right\} \delta\left( \int dx A_\mu n^\mu - c\Omega \right)
\]

can be conceived as the generating functional of the sector belonging to vector potential configurations in a hypersurface orthogonal to \( n^\mu(x) \). The projected generating functional \( Z'_{QED}[n,c] \) is gauge invariant for the choice of \( n^\mu(x) \) satisfying the condition \( \partial_\nu n^\mu(x) = 0 \). The projected effective action \( S_{eff}[n,c] \) is defined by

\[
Z'_{QED}[n,c] = \exp\{ iS_{eff}[n,c] \}.
\]
for the sector belonging to the given background $A^\mu(x)$. Introducing the shifted integration variable $\alpha^\mu(x) = A^\mu(x) - \bar{A}^\mu(x)$, the projected generating functional can be rewritten as

$$Z'_{QED}[\bar{A}, c] = \int d\lambda \int D\alpha \exp \left\{ iS_{em}[\bar{A} + \alpha, \xi] \right\} \exp \left\{ i\lambda \left( \int dx \bar{A}^\mu A_\mu + \int dx \bar{A}^\mu \alpha_\mu - c\Omega \right) \right\} \int D\bar{\psi}D\psi \exp \left\{ iS_D[\bar{A} + \alpha, \bar{\psi}, \psi] \right\}. \quad (6)$$

2. Identification of the background with the vacuum expectation value of the vector potential. As to the next, it is required that

$$\langle A^\mu(x) \rangle = \bar{A}^\mu(x), \quad \text{i.e.} \quad \langle \alpha^\mu \rangle = 0. \quad (7)$$

This condition is used to determine the constant $c$ as the functional of the vacuum expectation value of the vector potential $\langle A^\mu(x) \rangle$.

For later use it is useful to introduce the external sources $j^\mu(x), \bar{\eta}(x)$, and $\eta(x)$ coupled to the quantum fluctuation $\alpha^\mu(x)$ of the vector potential, and to the fermion fields $\psi(x)$ and $\bar{\psi}(x)$, resp. Then, we find instead of Eq. (6) the expression

$$Z'_{QED}[\bar{A}, c, j, \eta, \bar{\eta}] = \int d\lambda \int D\alpha \exp \left\{ iS_{\lambda}[\bar{A} + \alpha, \lambda, \xi, c] + \int dx j^\mu \alpha_\mu \right\} \frac{Z_F[A, \eta, \bar{\eta}]}{Z_F[A, \eta, \bar{\eta}]} \quad (8)$$

with

$$S_{\lambda}[\bar{A} + \alpha, \lambda, \xi, c] = S_{em}[\bar{A} + \alpha, \xi] + \lambda \left( \int dx \bar{A}^\mu A_\mu + \int dx \bar{A}^\mu \alpha_\mu - c\Omega \right) \quad (9)$$

and

$$Z_F[A, \eta, \bar{\eta}] = \int D\bar{\psi}D\psi \exp \left\{ iS_D[\bar{A} + \alpha, \bar{\psi}, \psi] \right\}. \quad (10)$$

3 Projected effective action

The projected generating functional (8) defines the projected effective action via Eq. (5). We determine it in one-loop approximation, i.e. we replace the full vector potential $A^\mu(x)$ in the Dirac action $S_D$ by the background field $\bar{A}^\mu(x)$. 


Thus, the generating functional is factored into an electromagnetic part and a fermionic part,

$$Z'_{QED}[\bar{A}, c, j, \eta, \bar{\eta}] = Z'_{em}[\bar{A}, c, j]Z_F[\bar{A}, \eta, \bar{\eta}].$$

The explicit forms of the actions are given as

$$S_{em}[\bar{A} + \alpha, \xi] = \frac{1}{2}(\bar{A}D^{-1}\bar{A}) + \frac{1}{2}(\alpha D^{-1}\alpha) + (\alpha D^{-1}\bar{A}),$$

and

$$S_D[\bar{A}] = (\bar{\psi}G^{-1}\psi)$$

with the inverse of the photon propagator in Lorentz gauge,

$$D^{-1}_{\mu\nu}(x, y) = \left[g_{\mu\nu}\Box_x + (\xi^{-1} - 1)\partial^\mu_x \partial^\nu_y \right] \delta(x - y),$$

and the inverse of the fermion propagator in the background field $\bar{A}^\mu(x)$,

$$G^{-1}(x, y) = \left(i\gamma^\mu(\partial^\mu_x - i\bar{A}_\mu(x)) - m\right) \delta(x - y)$$

with the Dirac matrices $\gamma^\mu$ ($\mu = 0, 1, 2, 3$). For the sake of simplicity, the notation $(fOg) = \int dx dy f^a(x)O_{ab}(x, y)g^b(y)$ is used, where $a$ and $b$ are either Lorentz or spinor summation indices. In the one-loop approximation the path integrals are Gaussian ones and can be performed explicitly, leading to

$$\ln Z'_{em}[\bar{A}, c, j] = -\frac{1}{2} \text{Tr} \ln D^{-1} - \frac{1}{2} \ln(AD\bar{A}) + \frac{i}{2}(jD_1j) - (j\bar{A}) + \frac{i}{2}c^2\Omega^2 + c\Omega\left\langle\frac{jD\bar{A}}{AD\bar{A}}\right\rangle,$$

and

$$\ln Z_F[\bar{A}, \eta, \bar{\eta}] = \text{Tr} \ln G^{-1} + i(\bar{\eta}G\eta)$$

with the modified photon propagator

$$D_1^{\mu\nu}(x, y) = D^{\mu\nu}(x, y) - \frac{\int du D^{\mu\rho}(y, u)\bar{A}_\rho(u) \int dv D^{\nu\sigma}(x, v)\bar{\rho}(v)}{(AD\bar{A})}.$$

Now we restrict our considerations to $1 + 1$ dimensional systems and time independent periodic backgrounds of the form

$$\bar{A}^\mu = \delta^\mu_0 a \cos(\ell x_1)$$
satisfying the Lorentz condition.

The constant $c$ can be determined by using the fact that

$$0 = \int D\alpha \int d\lambda \frac{\partial}{\partial \lambda} \exp \left\{ i S_\lambda [\bar{A} + \alpha, \xi, c] + \int dx j^\mu \alpha_\mu \right\}. \quad (20)$$

From this we find that

$$c_0 \Omega = \int dx \bar{A}_\mu A^\mu. \quad (21)$$

One establishes now for the one-loop effective action:

$$i S_{eff} = \ln Z'_{em} [\bar{A}, c_0, j = 0] + \ln Z_F [\bar{A}, \eta = 0, \bar{\eta} = 0]$$

$$= \text{Tr} \ln G^{-1} - \frac{1}{2} \text{Tr} \ln D^{-1} + \Omega \frac{1}{4} a^2 \ell^2 - \frac{1}{2} \ln \frac{a^2}{\ell^2}. \quad (22)$$

In the infinite volume limit one finds

$$-\Omega^{-1} S_{eff} \sim -V^{-1} \sum_k^> \epsilon_k + V^{-1} \sum_k \frac{1}{2} \omega_k - \frac{1}{4} a^2 \ell^2. \quad (23)$$

![Figure 2: Energy eigenvalues with changing momentum](image)
Here the first and the second terms represent the energy density of the Dirac vacuum and that of the free electromagnetic field, resp. (see e.g. [2]). \( \sum^> \) denotes the summation over all non-negative single fermion energies \( \epsilon_k \geq 0 \), being the energy eigenvalues of the Dirac equation in the external field \( \bar{A}^\mu(x) \).

The third term on the r.h.s. of Eq. (23) is just the negative of the energy density of the periodic electric background field. The last term of the effective action (22) gives a vanishing contribution to the action density in the infinite volume limit.

The meaning of the first two terms of Eq. (23) might lead one to the false conclusion that the negative of the action density is equal to the energy density of the vacuum, as it would happen if the background field were constant [2]. Determining the energy density from the energy-momentum tensor we will show that the energy density does not equal the negative of the action density for inhomogeneous background field.

Figure 3: The difference of the negative of the effective actions in the presence and in the absence of a periodic background field for various values of \( a \) and \( \ell \).

As 1+1 dimensional QED is a superrenormalizable theory, the action density is UV finite after subtracting the action density of the free vacuum \( S_{\text{eff,free}} \), i.e. that of the vacuum in the absence of the background field. This difference
\(-\Gamma = \Omega^{-1}(-S_{\text{eff}} + S_{\text{eff,free}})\) has been calculated numerically by solving the Dirac equation for the single-fermion energies.

These energy eigenvalues show the same band structure plotted against the momentum in fig. 2 as the electrons in the Kronig-Penney model because of the periodic potential. Fig. 3 shows how \(-\Gamma\) changes by modifying the amplitude \(a\) and the wavelength \(\ell\) of the potential. (The fermion rest mass is chosen for \(m=1\).) One can recapitulate from fig. 3 that the surface \(-\Gamma(a, \ell)\) has only a single stationary point at the origin of the parameter space \((a, \ell)\), i.e. the path integral defining the vacuum-vacuum transition amplitude is dominated by the trivial, identically vanishing field configuration \(A^\mu(x) = 0\).

## 4 Mean field equation

The vacuum expectation value of the current can be written at one-loop order in the following form:

\[
\langle \bar{\psi}(x) \gamma^\mu \psi(x) \rangle = \int d\lambda \int \mathcal{D}\alpha e^{iS_\lambda} \frac{\delta}{i\partial A_\mu(x)} Z_F / \int d\lambda \int \mathcal{D}\alpha e^{iS_\lambda} Z_F = e^{-iS_{\text{eff}}} \left[ \frac{\delta}{i\partial A_\mu(x)} e^{iS_{\text{eff}}} - Z_F \frac{\delta}{i\partial A_\mu(x)} Z'_{\text{em}} \right]. \tag{24}
\]

The first term on the r.h.s. of Eq. (24) is just the first functional derivative of the effective action \(\delta S_{\text{eff}}/\delta \bar{A}_\mu(x)\). In the infinite volume limit the second term gives:

\[
- \exp \left\{ -\frac{ic_0}{2(\bar{A}A)} \right\} \frac{\delta}{i\partial A_\mu(x)} \exp \left\{ \frac{ic_0}{2(\bar{A}A)} \right\} = -\frac{\bar{A}^\nu(x)}{(\bar{A}A)^\nu(x)} \left( \frac{\delta c_0}{\delta A_\mu(x)} - \bar{A}^\mu(x) \right) = -\ell^2 \bar{A}^\mu(x) = \partial_\nu \bar{F}^{\mu\nu} \tag{25}
\]

with the field strength tensor \(\bar{F}^{\mu\nu}\) evaluated from the background \(\bar{A}^\mu\). Here we used that

\[
\frac{\delta c_0}{\delta A_\mu(x)} = 2\bar{A}^\mu(x) \tag{26}
\]

due to Eq. (24). Thus, one obtains the equation

\[
\langle \bar{\psi}(x) \gamma^\mu \psi(x) \rangle = \frac{\delta S_{\text{eff}}}{\delta A_\mu(x)} + \partial_\nu \bar{F}^{\mu\nu}. \tag{27}
\]
For the background field configuration making the effective action extremum, one recovers the vacuum expectation value of the Maxwell equation. Thus, the effective action has an extremum at the background field configuration coinciding with the mean field solution. In our numerical search, this is the trivial extremum found at $a = \ell = 0$.

Satisfying the necessary condition of the extremum of the effective action, Eq. (27) results in the Poisson’s equation for the mean field $\bar{A}_0$, which must be considered together with coupled set of operator equations for the quantum fields $\psi(x)$, $\bar{\psi}(x)$ and $\alpha^\mu(x)$. The latter equations should be solved and the result substituted in the l.h.s. of Eq. (27), in order to make the charge density explicit.

5 Energy density of the vacuum

The symmetric energy momentum tensor is defined by [3]:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^{\kappa}}F^{\kappa}_{\nu} + \frac{\partial \mathcal{L}}{\partial (D_{\mu}\psi)}D^{\nu}\psi + D^{\mu*}\psi\frac{\partial \mathcal{L}}{\partial (D_{\mu}^*\psi)} - g^{\mu\nu}\mathcal{L}$$

with the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_{\mu}A^{\mu})^2 + \lambda A_{\mu}\bar{A}^{\mu} - \lambda c_0 + \frac{i}{2}\left(\bar{\psi}\gamma^{\mu}D_{\mu}\psi - D^{*}_{\mu}\bar{\psi}\gamma^{\mu}\psi\right) - m\bar{\psi}\psi$$

corresponding to the action $\int dx\mathcal{L} = S_\lambda + S_D$ and with the covariant derivative $D^{\mu} = \partial^{\mu} - iA^{\mu}$.

Substituting the ansatz (19) one obtains for the component $T^{00}$ of the energy-momentum tensor:

$$T^{00} = T^{00}_{em,2}[\alpha] + T^{00}_{em,1}[\bar{A}, \alpha, \lambda] + T^{00}_{em,0}[\bar{A}] + T^{00}_{F}[\bar{A} + \alpha, \bar{\psi}, \psi]$$

where $T^{00}_{em,a}$ $(a = 0, 1, 2)$ are the terms independent of the fermion field and being of the order $(\alpha)^a$. Due to the constraint (7), the expectation value of the first order term vanishes, therefore it can be neglected. The second order term

$$T^{00}_{em,2}[\alpha] = -\partial^0\alpha_\mu\partial^0\alpha^\mu + 2\partial^0\alpha_\mu\partial^\mu\alpha^0 - \partial^\mu\alpha_0\partial^0\alpha^\mu + \frac{1}{2}\partial^\mu\alpha_\nu\partial^\mu\alpha^\nu - \frac{1}{2}\partial_\mu\alpha_\nu\partial^\nu\alpha^\mu + \frac{1}{2\xi}\partial^0\alpha_\mu\partial_\nu\alpha^\nu$$

(31)
is the expression for the free electromagnetic field, whereas the zeroth order term is given by

\[
T_{00,0}[\bar{A}] = -\partial_\mu \bar{A}_\mu \bar{A}_0 + \frac{1}{2} \partial_\mu \bar{A}_\nu \partial^\mu \bar{A}^\nu \\
= \frac{1}{2} (\nabla_j \bar{A}^0)^2 = \frac{1}{2} \ell^2 a^2 \sin^2(\ell x_1)
\]

and represents the energy density of the periodic background field. Due to the projection to a particular sector of the theory, the additional term

\[
T_{00}^\lambda[\bar{A}, c] = (c_0 - \bar{A}^\mu(x) \bar{A}_\mu(x)) \lambda \\
= \left( (\Omega^{-1} \int du \bar{A}^\mu(u) \bar{A}_\mu(u) - \bar{A}^\mu(x) \bar{A}_\mu(x) \right) \lambda \\
= \frac{1}{2} a^2 (1 - 2 \cos^2(\ell x_1)) \lambda
\]

occurs. It is easy to see that this term does not contribute to the energy, but gives a non-vanishing periodic contribution to the energy density. Finally, the term

\[
T_{F0} = -\frac{i}{2} \partial^0 \left( \bar{\psi} \gamma^0 \psi \right) + \bar{\psi} \gamma^0 \tilde{H}_D(\bar{A} + \alpha) \psi
\]

represents the contribution of the fermions to the energy density. The first term on the r.h.s. gives vanishing contribution to the total energy if charge conservation is required, whereas the second term accounts for the kinetic energy of the fermion system. At one-loop order, we have to substitute \( \alpha^\mu = 0 \) in the ‘Hamilton operator’, i.e. write

\[
\tilde{H}_D(\bar{A}) = -i \gamma^0 \gamma^j \partial_j^x - \gamma^0 \gamma^j \bar{A}_j(x) + \gamma^0 m.
\]

Since \( \bar{A}_j(x) = 0 \) in our case, we obtain \( \tilde{H}_D(\bar{A}) = -i \gamma^0 \gamma^j \partial_j^x + \gamma^0 m \), i.e. the free ‘Hamilton operator’.

The vacuum expectation value of \( T_{00} \) in the presence of the background field \( \bar{A}^\mu \) is defined as

\[
\langle T_{00}(x) \rangle = \int d\lambda \int D\alpha D\bar{\psi} D\psi T_{00}(x) e^{i(S_\lambda + S_D)} / \int d\lambda \int D\alpha D\bar{\psi} D\psi e^{i(S_\lambda + S_D)}
\]

Similarly, the vacuum expectation value of \( T_{free}^{00} \) in the absence of the background field \( \langle T_{free}^{00}(x) \rangle_0 \) is given by substituting \( \bar{A}_\mu(x) = 0 \) in Eq.\( (30) \) and
replacing \( T^{00}(x) \) by \( T^{00}_{\text{free}}(x) \). Then, the Casimir energy of the vacuum due to the background field \( \bar{A}_\mu(x) \neq 0 \) is

\[
E_c = \int dx_1 \langle T^{00}(x_1) \rangle - \int dx_1 \langle T^{00}_{\text{free}}(x_1) \rangle
= \int dx_1 \left[ \left( \frac{\ell^2 a^2}{2} \sin^2(\ell x_1) + \langle \bar{\Psi} \gamma^0 H D_0 \Psi \rangle - \langle \bar{\Psi} \gamma^0 H D_0 \Psi \rangle_0 \right) \right] \tag{37}
\]

This expression of the Casimir-energy reminds one on the expression for the energy of a system of electric charges in classical electrodynamics. There, the energy is the sum of the energy of the electromagnetic field and the kinetic energy of the charges \([4]\). In our case the Casimir energy is the sum of the background electric field and the change of the relativistic kinetic energy (including the rest mass) of the Dirac-sea due to the presence of the background field.

Eq.\((36)\) can be rewritten by the help of the generating functional \( Z_{QED}[\bar{A}, c, j, \eta, \bar{\eta}] \) as

\[
\langle T^{00}(x) \rangle = \left( 1/Z_{QED}'[\bar{A}, c_0] \right) T^{00}_{\text{op}}(x) Z_{QED}'[\bar{A}, c, j, \eta, \bar{\eta}] \bigg|_{j, \bar{\eta}, \eta = 0; c = c_0} \tag{38}
\]

where \( T^{00}_{\text{op}}(x) \) denotes the operator obtained from \( T^{00} \) by replacing the fields \( \alpha^\mu(x), \psi(x), \) and \( \bar{\psi}(x) \) by the operators \( \delta/\delta j^\mu(x), \delta/\delta \bar{\eta}(x), \) and \( -\delta/\delta \eta(x) \) and the variable \( \lambda \) by \( i\Omega^{-1} \partial/\partial c \).

Then, we find

\[
\langle T^{00}_\lambda \rangle = \frac{1}{2} a^2 \ell^2 \left( 2 \cos^2(\ell x_1) - 1 \right). \tag{39}
\]

Furthermore, the expectation value \( \langle T^{00}_{em,2}[\alpha] \rangle \) is equal to the energy density of the free electromagnetic field. Indeed, it holds for the second derivatives

\[
\delta^2 Z_{em}' \bigg|_{j=0,c_0} = i D^{\mu\nu}_1(x, y) Z_{em}' \bigg|_{j=0,c_0} \tag{40}
\]

where \( D^{\mu\nu}_1(x, y) \) tends to the free propagator \( D^{\mu\nu}(x, y) \) in the infinite volume limit, since the last term on the r.h.s. of Eq. \((18)\) is of the order \( \Omega^{-1} \). Consequently, the pure electromagnetic contribution to the Casimir energy density is given by

\[
e_{em}(x) = \langle T^{00}_{em,0}[\bar{A}] + T^{00}_\lambda[\bar{A}, c_0] \rangle
= \frac{1}{2} a^2 \ell^2 \cos^2(\ell x_1). \tag{41}
\]
It is more cumbersome to evaluate the fermionic contribution \( e_F(x) = \langle T^{00}_F \rangle - \langle T^{00}_{\text{free}} \rangle_0 \) to the Casimir energy density, where the energy density of the free Dirac vacuum is subtracted. Since QED in dimensions 1 + 1 is superrenormalizable, the difference turns out to be UV finite without further renormalizations. We perform the evaluation of the fermionic part of the Casimir energy in the second quantized formalism. Then we can write \( e_F = \langle 0 | : T^{00}_F : | 0 \rangle \) where \( | 0 \rangle \) is the ‘interacting’ vacuum (in the presence of the periodic background field) and \( : \ldots : \) denotes normal ordering with respect to the normal vacuum (in the absence of the background field). The evaluation is performed in the following steps:

1. The fermion field \( \psi \) is expanded in terms of the eigenspinors \( f^{(ks)}(x_1) \) and \( g^{(ks)}(x_1) \) of the Dirac Hamiltonian \( H_D(\bar{A}) \) belonging to the energy eigenvalues \( \epsilon_{ks} \) and \( -\epsilon_{ks} \), resp. Here the quasi-momentum \( k \in [-\ell/2, \ell/2] \) and the integer \( s \geq 0 \) enumerating the bands are introduced.

2. Then the creation and annihilation operators \( a^\dagger_{ks}, b^\dagger_{ks} \) and \( a_{ks}, b_{ks} \) of these stationary single particle states are expressed as linear combinations of the creation and annihilation operators \( A^\dagger_{ks}, B^\dagger_{ks} \) and \( A_{ks}, B_{ks} \) of the free fermion states \( F^{(kr)}(x_1) \) and \( G^{(kr)}(x_1) \) of energies \( \epsilon_{kr}^{(0)} = \sqrt{m^2 + (k + \ell r)^2} \) and \( -\epsilon_{kr}^{(0)} \), resp. In terms of the latter, the normal ordering is performed.

3. Finally, the normal ordered operator is reexpressed in terms of the operators \( a^\dagger_{ks}, b^\dagger_{ks} \) and \( a_{ks}, b_{ks} \) and the vacuum expectation value with respect to the vacuum in the presence of the background field is taken.

Thus, one arrives to the following expression

\[
T^{00}_F(x) = e_F(x) + \sum_{\rho \rho'} \left\{ a^\dagger_\rho(t) a_\rho(t) \tilde{B}^\rho_+ \cdot B^\rho_- + b^\dagger_\rho(t) b_{\rho'}(t) \tilde{A}^\rho_+ \cdot A^\rho_- \\
+ b_{\rho'}(t) a_\rho(t) \tilde{A}^\rho_+ \cdot B^\rho_- + a^\dagger_{\rho'}(t) b^\dagger_\rho(t) \tilde{B}^\rho_+ \cdot A^\rho_- \right\}
\]

(42)

where, with \( \rho \equiv (ps) \), \( \rho' \equiv (p's') \),

\[
\tilde{A}^\rho_+ = \hat{\alpha}^\dagger_F \rho' + \hat{\alpha}^\dagger_G \rho', \quad \tilde{B}^\rho_+ = \hat{\beta}^\dagger_F \rho' + \hat{\beta}^\dagger_G \rho', \\
A^\rho_- = \alpha^\rho_F - \alpha^\rho_G, \quad B^\rho_- = \beta^\rho_F - \beta^\rho_G;
\]

(43)

and

\[
\alpha^\rho_F = \sum_{kr} \epsilon^0_{kr} \alpha^pks_F, \quad \alpha^\rho_G = \sum_{kr} \epsilon^0_{kr} \alpha^pks_G
\]

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\[
\beta_{F}^{ps} = \sum_{kr}^{0} \epsilon_{kr}^{0} \beta_{F}^{pksr} F_{kr}(x), \quad \beta_{G}^{ps} = \sum_{kr}^{0} \epsilon_{kr}^{0} \beta_{G}^{pksr} G_{kr}(x),
\]

\[
\bar{\alpha}_{F}^{ps} = \sum_{kr}^{0} \alpha_{-}^{pksr} F_{kr}^{*}(x), \quad \bar{\alpha}_{G}^{ps} = \sum_{kr}^{0} \alpha_{+}^{pksr} G_{kr}^{*}(x),
\]

\[
\tilde{\beta}_{F}^{ps} = \sum_{kr}^{0} \beta_{-}^{pksr} F_{kr}^{*}(x), \quad \tilde{\beta}_{G}^{ps} = \sum_{kr}^{0} \beta_{+}^{pksr} G_{kr}^{*}(x),
\]

with the constant coefficients

\[
\alpha_{-}^{pksr} = \int dx F^{*}_{kr}(x) g^{ps}(x), \quad \alpha_{+}^{pksr} = \int dx G^{*}_{kr}(x) g^{ps}(x),
\]

\[
\beta_{-}^{pksr} = \int dx F^{*}_{kr}(x) f^{ps}(x), \quad \beta_{+}^{pksr} = \int dx G^{*}_{kr}(x) f^{ps}(x).
\]

The latter are the overlap integrals of the eigenspinors in the presence of the background and those in the absence of the background. Furthermore, the time dependent creation-annihilation operators are introduced:

\[
a_{ps}(t) \equiv a_{ps} e^{-i\epsilon_{ps} t}, \quad b_{ps}(t) \equiv b_{ps} e^{-i\epsilon_{ps} t}.
\]

The constant term

\[
e_{F}(x) = \sum_{ps} \left( \tilde{\beta}_{G}^{ps} \cdot \beta_{G}^{ps} + \bar{\alpha}_{F}^{ps} \cdot \bar{\alpha}_{F}^{ps} + \alpha_{G}^{ps} \cdot \alpha_{G}^{ps} - \bar{\alpha}_{F}^{ps} \cdot \alpha_{G}^{ps} \right)
\]

represents the Casimir energy density of the fermion vacuum. The last two terms of Eq. (47) cancel.

Performing the calculation results in:

\[
e_{F}(x) = \sum_{k \in \left[-\frac{1}{2}, \frac{1}{2}\right], r \in \mathbb{N}} \left[ \epsilon_{kr}^{0} (\mathcal{F}_{1}^{(kr)}, \mathcal{F}_{2}^{(kr)} \cos((r - r')\ell x) + (\epsilon_{kr}^{0} - \epsilon_{kr}^{-1}) \mathcal{F}_{3}^{(kr)} \cos((r + r')\ell x) \right],
\]

where we introduced the following notations:

\[
\mathcal{F}_{1}^{(kr)} = \sum_{s \in \mathbb{Z}} U_{\alpha}^{(-kr)} U_{\beta}^{(-kr)} V_{\alpha}^{(-kr')} U_{\beta}^{(-kr')} V_{\gamma}^{(-r'k)} V_{\gamma}^{(-krk)}
\]

\[
\mathcal{F}_{2}^{(kr)} = \sum_{s \in \mathbb{Z}} V_{\alpha}^{(-kr)} V_{\beta}^{(-kr')} U_{\alpha}^{(-kr')} U_{\beta}^{(-krk)} V_{\gamma}^{(-krk)}
\]

\[
\mathcal{F}_{3}^{(kr)} = \sum_{s \in \mathbb{Z}} U_{\alpha}^{(-kr)} V_{\alpha}^{(kr)} V_{\beta}^{(-kr')} U_{\beta}^{(-r'k)} V_{\gamma}^{(kr)} V_{\gamma}^{(r'k)}.
\]

\[U^{(kr)} \text{ and } V^{(kr)} \text{ denotes the eigenspinors of the free Dirac-equation for the positive and negative energy eigenvalues, respectively. It is straightforward to} \]
establish from Eq. (48) that the volume integral of the energy density is not negative:

\[ E_F = \sum_{k, \alpha} \left\{ \epsilon_0 \left( |U^{(-kr)}_{\beta} v^{(rks)}_{\beta}|^2 + |V^{(-kr)}_{\beta} v^{(rks)}_{\beta}|^2 \right) \right\} \geq 0. \] (50)

By numerical calculations we were convinced (see Fig. 4) that this expression of the energy only vanishes in case of \( a = \ell = 0 \). This means that the Casimir energy \( E_c \) of the vacuum is always positive if a non-vanishing periodic electric background field is assumed. Thus, the vacuum of \( QED_2 \) does not favour a periodic mean field energetically with respect to the normal vacuum.

6 Necessary condition of energy minimum

The vacuum expectation value \( \bar{A}^0(x) \) is defined by the minimum of the energy functional

\[ TE[\bar{A}^0(x)] = \int dx \langle T^{00}(x) \rangle = \int dx \left\langle \left( T^{00}_{\text{em}}(x) + T^{00}_\chi(x) + T^{00}_F(x) \right) \right\rangle. \] (51)
The second term vanishes due to the explicit value of $c_0$. Thus, the necessary condition of the energy minimum takes the form

$$
\frac{\delta}{\delta A^0(x)} \int dy \left( T_{em,0}[\bar{A}^0(y)] + \langle T_F^{00}(y) \rangle \right) = 0.
$$

(52)

The functional derivative of the first term gives

$$
\frac{\delta}{\delta A^0(x)} \int dy \left( T_{em,0}[\bar{A}^0(y)] \right) = -\nabla^2 \bar{A}^0(x).
$$

(53)

At one-loop order the fermionic contribution to the energy can be rewritten as

$$
TE_F = \int dy \langle T_F^{00}(y) \rangle = Z_F^{-1}[\bar{A}, 0, 0] \int dy T_{F,op}^{00}(y) Z_F[\bar{A}, \eta, \bar{\eta}]|_{\eta=\bar{\eta}=0}.
$$

(54)

Taking its functional derivative we find

$$
\frac{\delta}{\delta A^0(x)} TE_F = -\frac{\delta \ln Z_F[\bar{A}, 0, 0]}{\delta A^0(x)} TE_F
$$

$$
+ Z_F^{-1}[\bar{A}, 0, 0] \int dy T_{F,op}^{00}(y) \frac{\delta Z_F[\bar{A}, \eta, \bar{\eta}]}{\delta A^0(x)} |_{\eta=\bar{\eta}=0}
$$

$$
= i \int dy \left[ \frac{i}{2} \frac{\delta^0}{\delta \eta(y)} \left( \frac{\delta \gamma^0}{\delta \bar{\eta}(y)} \right) \right]
$$

$$
- \frac{\delta}{\delta \eta(y)} \gamma^0 H_{0y} \frac{\delta}{\delta \bar{\eta}(y)} \right] \left( \bar{\eta} \frac{\delta G}{\delta A^0(x)} \eta \right) e^{i(y \eta + \bar{\eta})} \right|_0
$$

$$
= -i \text{tr} \left( \gamma^0 G(x, x) \right) + c^0(x)
$$

(55)

with

$$
c^0(x) = \frac{1}{2} \int dy \left[ \text{tr} \left( \gamma^0 G(y, x) \gamma^0 D^0_y G(x, y) \right) - \text{tr} \left( \gamma^0 G(x, y) \gamma^0 D^0_y G(y, x) \right) \right].
$$

(56)

Thus, we find the following equation for the field $\bar{A}^0(x)$ minimizing the energy:

$$
\partial_{\nu} F^{0\nu} = \langle \bar{\psi}(x) \gamma^0 \psi(x) \rangle + c^0(x).
$$

(57)

The first term on r.h.s. is the expectation value of the charge density $j^0 = -i \text{tr} (\gamma^0 G(x, x))$, determined via the propagator $G(x, x)$ as a given functional of $\bar{A}^0(x)$, and a similar statement holds for $c^0(x)$.
A similar equation appears in the classical case when a certain charge distribution moves in an electromagnetic field. Integrating out the effect of the charged particle distribution it will result in a polarisation charge density term beside the common charge density in the equation of motion for the scalar potential. In Eq. (57) the term \( c^0(x) \) corresponds to a polarisation charge density, so minimizing the energy with respect to \( \tilde{A}^0(x) \) leads to an equation similar to that of Poisson’s equation in a polarised medium. Thus Eq. (57) can be solved directly for \( \tilde{A}^0(x) \) in principle without the need for solving any other equations. On the contrary to this the mean field equation (27) obtained by extremizing the effective action does not take the polarisation of the vacuum into account, in order to do this we have to solve a system of operator equations as well.

7 Conclusions

The energy density of \( QED_2 \) in the presence of a periodic mean field is determined from the energy-momentum tensor. For this purpose a projection method is worked out which is applied to treat the mean electromagnetic field self-consistently. The projected effective action and the energy density of the vacuum are derived at one-loop order, whereas the interaction of the electron-positron field with the periodic mean field is treated exactly. It is established, that the negative of the effective action must not be regarded as the energy of the system if the background field is not constant. It was shown that the necessary conditions of the extremum of the effective action and the minimum of the energy lead to different equations for the vacuum expectation value of \( \tilde{A}^0 \). Eq. (27) obtained by extremizing the effective action is the vacuum expectation value of the Poisson’s equation that does not include the polarisation of the vacuum due to one-loop radiation corrections. Those are accounted for by separate operator equations for the quantum fluctuations of the electromagnetic field and for the Dirac field. On the other hand Eq. (57) obtained by minimizing the energy functional includes the polarisation effects of the vacuum.

The expectation value of the component \( T^{00} \) of the energy-momentum tensor was determined as the function of the amplitude \( a \) and wave number \( l \) of the static periodic scalar potential \( \tilde{A}^0(x_1) = a \cos(lx_1) \). It is found that the vacuum configuration with this periodic electric mean field is not favoured energetically compared to the normal vacuum. The volume integral of the energy density plotted against the amplitude \( a \) in Fig. [4] shows that the energy of the
system increases with increasing $a$ monotonically.

The result obtained contradicts to our naive expectation discussed in the Introduction. Possibly, the reason is that the naive picture mentioned there does not take into account the uncertainty principle i.e. for a certain $\ell$ the energy spread of a wave packet localized in an interval of $\sim 1/\ell$ could be much larger than the amplitude of the potential and of course in this case our naive picture is not valid any more. We set now forth our work looking for periodic ground states at finite chemical potential.

A Solution of the Dirac equation in periodic external field

A.1. Relativistic Bloch waves

To get the projected effective action at one-loop order in Eq. (22) we must find the fermionic single-particle energies, so we have to solve the Dirac equation in the presence of the potential (19):

$$(i\gamma^\mu(\partial_\mu - ie\vec{A}_\mu) - m)\psi = 0.$$  \hfill (58)

We look for the solution of Eq. (58) in the form of Bloch waves corresponding to the energy eigenvalues $E$:

$$\psi_\alpha = e^{-i\epsilon_{ps}t}e^{ipx} \sum_{n=-\infty}^{\infty} u_\alpha n e^{i\kappa x} = \begin{cases} e^{-i\epsilon_{ps}t}f^\alpha(x), & \text{if } E = \epsilon_{ps} > 0 \\ e^{i\epsilon_{ps}t}g^\alpha(x), & \text{if } E = -\epsilon_{ps} > 0 \end{cases} \hfill (59)$$

Inserting (59) into (58) we find

$$\sum_{n=-\infty}^{\infty} \left( u_{\alpha n}(\epsilon\gamma^0 - (p-n)\gamma^1 - m\mathbf{1}) + \frac{a}{2} \gamma^0(u_{\alpha n+1}+u_{\alpha n-1}) \right) e^{i(p+nl)x-iEt} = 0. \hfill (60)$$

We get non-trivial solutions when the determinant of the matrix appearing next to the Dirac-spinors equals zero. If we had solved the Schrödinger equation in the presence of this sinusoidal potential, the form of the matrix would have been tridiagonal which means that the diagonal and the neighbouring diagonal elements are nonzero. In relativistic case the matrix elements are replaced by $d$ dimensional $\gamma$ matrices but the structure of the matrix remains unchanged.


A.2. Eigenvalues, eigenspinors

We cannot get the eigenvalues analytically because of the complicated structure of the matrix but we can determine them as precisely as we wish. In numerical calculations we work with matrices with finite dimension. Using the well-known identity \[5\]

\[
\det[A] = \det\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det[P] \det[S - RP^{-1}Q] \tag{61}
\]

we can reduce the problem of calculating the determinants of these \(d(2n + 1) \times d(2n + 1)\) dimensional matrices to calculating four dimensional matrices by identifying the upper left matrix element of the matrix under investigation with \(P\) in Eq. \((51)\). \((n\) denotes the number of \(u_{\alpha,i}\)-s taken into account in Eq. \((60)\).\) Using identity \((61)\) \(2n\) times we get a product of \(2n\) determinants of four dimensional matrices. We computed the determinant of Eq. \((60)\) as the function of \(E\) and determined its zeros, corresponding to the energy eigenvalues \(E = \pm \epsilon\).

In order to evaluate \(-\Gamma\) we have to sum the eigenvalues \(-\epsilon_{ps} < 0\) and extract from it the sum of the negative eigenvalues of the free Dirac equation. This difference will depend on the accuracy of the eigenvalues, the number of the eigenvalues taken into account in the sum, the size of the chosen matrix and, of course, the parameters of the potential, the amplitude \(a\) and the wave number \(\ell\). We have to be convinced of the stability of the numerical calculation. We increased the number of members in the sum and the numerical accuracy of the determination of eigenspinors by choosing larger matrices as far as we have seen that the energy difference does not change significantly.

To get the expectation value of the component \(T^{00}\) of the energy-momentum tensor we also have to determine the eigenspinors of the Dirac equation \((58)\). We calculated them with the help of the eigenvalues by solving a system of homogeneous linear equations \((60)\) for the eigenspinors.

Acknowledgement

The authors would like to thank J. Polónyi for consulting this work and G. Plunien for the valuable discussions. S.N. thanks G. Soff for his kind hospitality. K.S. expresses his gratitude for the follow-up grant of the Alexander von Humboldt Foundation and W. Greiner for his kind hospitality. This work was supported by the projects OTKA T023844/97, DAAD-MÖB 27/1999 and NATO Grant PST.CLG.975722.
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