GLOBAL EFFECTIVE VERSIONS OF THE BRIANÇON-SKODA-HUNEKE THEOREM

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Abstract. We prove global effective versions of the Briançon-Skoda-Huneke theorem. Our results extend, to singular varieties, a result of Hickel on the membership problem in polynomial ideals in $\mathbb{C}^n$, and a related theorem of Ein and Lazarsfeld for smooth projective varieties. The proofs rely on known geometric estimates and new results on multivariable residue calculus.

1. Introduction

Let $V$ be a reduced $n$-dimensional subvariety of $\mathbb{C}^N$. If $F_1, \ldots, F_m$ are polynomials in $\mathbb{C}^N$ with no common zeros on $V$, then by the Nullstellensatz there are polynomials $Q_j$ such that $\sum F_j Q_j = 1$ on $V$. It was proved by Jelonek, [22], that if $F_j$ have degree at most $d$, then one can find $Q_j$ such that

$$\deg (F_j Q_j) \leq c_m d^\mu \deg V,$$

where $c_m = 1$ if $m \leq n$, $c_m = 2$ if $m > n$, and, throughout this paper,

$$\mu := \min(m, n).$$

Here $\deg V$ means the degree of the closure of $V$ in $\mathbb{P}^N$. This result generalizes Kollár’s theorem [19], [23], for $V = \mathbb{C}^n$ and does not require any smoothness assumptions on $V$. The bound is optimal when $m \leq n$ and almost optimal when $m > n$. However, in view of various known results in the case when $V = \mathbb{C}^n$, one can expect sharper degree estimates if the common zero set of the polynomials $F_j$ behaves nicely at infinity in $\mathbb{P}^N$.

More generally one can take arbitrary polynomials $F_j$ of degree at most $d$ and look for a solution $Q_j$ to

$$(1.1) \quad F_1 Q_1 + \cdots + F_m Q_m = \Phi$$

with good degree estimates, provided that the polynomial $\Phi$ belongs to the ideal $(F_j)$ generated by the $F_j$ on $V$. It follows from a result of Hermann, [19], that one can choose $Q_j$ such that $\deg (F_j Q_j) \leq \deg \Phi + C(d, N)$, where $C(d, N)$ is like $2(2d)^{2N-1}$ for large $d$, thus doubly exponential. It is shown in [27] that this estimate cannot be substantially improved. However, under additional hypotheses on $\Phi$ and the common zero set of the $F_j$, much sharper estimates are possible. In the extreme case when

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1In Kollár’s theorem $c_m = 1$ even for $m > n$ (unless $d = 2$, cf., [29]) and this estimate is optimal.

2In Kollár’s and Jelonek’s theorems, as well as in [29], there are more precise results that take into account different degree bounds $d_j$ of $F_j$, but for simplicity, in this paper we always keep all $d_j = d$. 
the polynomials \( F_j \) have no common zero set even at infinity, a classical result of Macaulay, [26], states that one can solve (1.1) with polynomials \( Q_j \) such that
\[ \deg F_j Q_j \leq \max (\deg \Phi, d(n + 1) - n). \]

By homogenization, this kind of effective results can be reformulated as geometric statements: Let \( z = (z_0, \ldots, z_N) \), \( z' = (z_1, \ldots, z_N) \), let \( f_i(z) := z_0^d F_i(z'/z_0) \) be the \( d \)-homogenizations of \( F_i \), and let \( \varphi(z) := z_0^{-\deg \Phi}(z'/z_0) \). Then there is a representation (1.1) on \( V \) with \( \deg (F_j Q_j) \leq \rho \) if and only if there are \((\rho - d)\)-homogeneous forms \( q_i \) on \( \mathbb{P}^N \) such that
\[ f_1 q_1 + \cdots + f_m q_m = z_0^{-\deg \Phi} \varphi \]
on the closure \( X \) of \( V \) in \( \mathbb{P}^N \). As usual, we can consider \( f_j \) as holomorphic sections of the (restriction to \( X \) of the) line bundle \( \mathcal{O}(d) \to \mathbb{P}^N \), \( z_0^{-\deg \Phi} \varphi \) as a section of \( \mathcal{O}(\rho) \), etc, so that (1.2) is a statement about sections of line bundles.

In this paper we present global effective versions of the Briançon-Skoda-Huneke theorem:

Given \( x \in V \) there is a number \( \mu_0 \) such that if \( F_1, \ldots, F_m, \Phi \) are any holomorphic functions at \( x \), \( \ell \geq 1 \), and \( |\Phi| \leq C |F|^{|\mu_0 + \ell - 1|} \) in a neighborhood of \( x \), where \( |F|^2 = |F_1|^2 + \cdots + |F_m|^2 \) and \( C \) is a positive constant, then \( \Phi \) belongs to the local ideal \( (F_j)_{\mathcal{O}}(x) \).

If \( x \) is a smooth point, then one can take \( \mu_0 = 0 \); this is the classical Briançon-Skoda theorem, [13]. The general case was proved by Huneke, [21], by purely algebraic methods. An analytic proof appeared in [7].

Given polynomials \( F_1, \ldots, F_m \) on \( X \), let \( f_j \) denote the corresponding sections of \( \mathcal{O}(d)|_X \), and let \( \mathcal{J}_f \) be the coherent analytic sheaf over \( X \) generated by \( f_j \). Furthermore, let \( c_{\infty} \) be the maximal codimension of the so-called distinguished varieties of the sheaf \( \mathcal{J}_f \), in the sense of Fulton-MacPherson, that are contained in \( X_{\infty} := X \setminus V \), see Section 3. If there are no distinguished varieties contained in \( X_{\infty} \), then we interpret \( c_{\infty} \) as \( -\infty \). It is well-known that the codimension of a distinguished variety cannot exceed the number \( m \), see, e.g., Proposition 2.6 in [15], and thus
\[ c_{\infty} \leq \mu. \]
We let \( Z^f \) denote the zero variety of \( \mathcal{J}_f \) in \( X \).

Our first result involves the so-called regularity, \( \text{reg} \), of \( X \subset \mathbb{P}^N \), see Section 2.7 for the definition.

**Theorem 1.1.** Assume that \( V \) is a reduced \( n \)-dimensional algebraic subvariety of \( \mathbb{C}^N \) and let \( X \) be its closure in \( \mathbb{P}^N \).

(i) There exists a number \( \mu_0 \) such that if \( F_1, \ldots, F_m \) are polynomials of degree \( \leq d \) and \( \Phi \) is a polynomial such that
\[ |\Phi| \leq C |F|^{\mu_0} \]
locally on \( V \), then one can solve (1.1) on \( V \) with
\[ \deg (F_j Q_j) \leq \max (\deg \Phi + (\mu + \mu_0)d^{\infty} \deg X, (d - 1) \min (m, n + 1) + \text{reg} X). \]
(ii) If $V$ is smooth, then there is a number $\mu'$ such that if $F_1, \ldots, F_m$ are polynomials of degree $\leq d$ and $\Phi$ is a polynomial such that
\begin{equation}
|\Phi| \leq C|F|^\mu
\end{equation}
locally on $V$, then one can solve (1.1) on $V$ with
\begin{equation}
\deg(F_j Q_j) \leq \max\left(\deg \Phi + \mu d^\infty \deg X + \mu', \left(d - 1\right) \min(m, n + 1) + \text{reg } X\right).
\end{equation}
If $X$ is smooth, then one can take $\mu' = 0$.

There are analogous results for powers $(F_j)^k$ of $(F_j)$, see Theorem 6.4. Notice that if there are no distinguished varieties contained in $X_\infty$, then $d^\infty = 0$. The number $\mu'$ in (ii) is related to the singularities of $X$ at infinity. If $V$ is arbitrary and $Z^j \cap X_{\text{sing}} = \emptyset$, then (i) holds with $\mu_0 = 0$; for a slightly stronger statement, see Remark 6.2.

**Example 1.2.** If we apply Theorem 1.1 to Nullstellensatz data, i.e., $F_j$ with no common zeros on $V$ and $\Phi = 1$, we get back the optimal result of Jelonek, except for the annoying factor $\mu + \mu_0$ in front of $d^\infty$. On the other hand, $(\mu + \mu_0)d^\infty < d^\mu$ if $c_\infty < \mu$ and $d$ is large enough.

**Example 1.3.** If $f_j$ have no common zeros on $X$, then we can find a solution to $F_1 Q_1 + \cdots + F_m Q_m = 1$ on $V$ such that
\[ \deg F_j Q_j \leq \max(\deg \Phi, (d - 1)(n + 1) + \text{reg } X). \]
If $X = \mathbb{P}^n$, then $\text{reg } X = 1$ and hence we get back the Macaulay theorem, cf., above.

**Remark 1.4.** If $X$ is Cohen-Macaulay, for instance $X$ is a complete intersection in $\mathbb{P}^N$ or even $X = \mathbb{P}^N$, and $m \leq n$, then the last entries in (1.5) and (1.7) can be omitted, i.e., we get the sharper estimates $\deg(F_j Q_j) \leq \deg \Phi + (m + \mu_0)d^\infty \deg X$ and $\deg(F_j Q_j) \leq \deg \Phi + md^\infty \deg X + \mu'$ in (i) and (ii), respectively. Moreover, if $X$ is Cohen-Macaulay and $(N$ is minimal), then $\text{reg } X \leq \deg X - (N - n)$, see [17, Corollary 4.15].

**Remark 1.5.** If $X$ is smooth, then
\[ \text{reg } X \leq (n + 1)(\deg X - 1) + 1; \]
this is Mumford’s bound, see [25, Example 1.8.48].

**Example 1.6.** For $V = \mathbb{C}^n$ Theorem 1.1 gives the estimate
\begin{equation}
\deg(F_j Q_j) \leq \max\left(\deg \Phi + \mu d^\infty, d \min(m, n + 1) - n\right).
\end{equation}
This estimate was proved by Hickel, [20], but with the term $\min(m, n + 1)d^\mu$ rather than our $\mu d^\infty$. The ideas in [20] are similar to the ones used in [15]. If one applies the geometric estimate in [15], rather than the (closely related) so-called refined Bezout estimate by Fulton-MacPherson that is used in [20], one can replace the exponent $\mu$ by $c_\infty$. This refinement was pointed out already in Example 1 in [15].

We have the following more abstract variant of Theorem 1.1. It is a generalization to nonsmooth varieties of the geometric effective Nullstellensatz of Ein-Lazarsfeld in [15] (Theorem 7.1 below). Let $X$ be a reduced projective variety. Recall that if $L \to X$ is an ample line bundle, then there is a (smallest) number $\nu_L$ such that
\[ H^i(X, L^s) = 0 \text{ for } i \geq 1 \text{ and } s \geq \nu_L, \text{ cf., [25, Ch. 1.2].} \]
When $X$ is smooth, by Kodaira’s vanishing theorem, $\nu_L$ is less than or equal to the least number $\sigma$ such
that $L^s \otimes K_X^{-1}$ is strictly positive, where $K_X$ is the canonical bundle. In particular, if $V = \mathbb{C}^n$, i.e., $X = \mathbb{P}^n$, then $\nu_{O(1)} = -n$.

**Theorem 1.7.** Let $X$ be a reduced projective variety of dimension $n$. There is a number $\mu_0$, only depending on $X$, such that the following holds: Let $f_1, \ldots, f_m$ be global holomorphic sections of an ample Hermitian line bundle $L \to X$, and let $\phi$ be a section of $L^{\otimes s}$, where $s \geq \nu_L + \min(m, n + 1)$. If

$$\left| \phi \right| \leq C|f|^{\mu+\mu_0},$$

then there are holomorphic sections $q_j$ of $L^{\otimes (s-1)}$ such that

$$f_1q_1 + \cdots + f_mq_m = \phi.$$

If $X$ is smooth we can choose $\mu_0 = 0$, cf., Theorem 1.4.

**Remark 1.8.** There is a close relation between $\text{reg} X$ and $\nu := \nu_{O(1)|_X}$ for $X \subset \mathbb{P}^N$, see, e.g., [17, Ch. 4D], so Theorem 1.1 can be formulated in terms of $\nu$ rather than $\text{reg} X$.

Let $\mathcal{J}_f$ be the ideal sheaf generated by $f_j$ and assume that the associated distinguished varieties $Z_k$ have multiplicities $r_k$, cf., Section 5. If we assume that $\phi$ is in $\cap_k \mathcal{J}(Z_k)_{r_k(\mu+\mu_0)}$, where $\mathcal{J}(Z_k)$ is the radical ideal associated with the distinguished variety $Z_k$, then (1.9) holds, and hence we have a representation (1.10).

The following example shows that $\mu_0$ can be arbitrarily large.

**Example 1.9.** Let $X$ be the cusp $\{z_1^2z_0^{p-2} - z_2^p = 0\} \subset \mathbb{P}^2$, where $p > 2$ is odd. Then the sections $f = z_2$ of $L := O(1)$ and $\phi = z_0^{p-1}z_1$ of $L^{\otimes s}$ satisfy $|\phi| \leq C|f|^{\frac{p-1}{2}}$ on $X$ as soon as $s \geq 2$. However, $\phi$ is not in $(f)$ on $X$ at the singular point $\{z_1 = z_2 = 0\}$ nor at $\{z_0 = z_2 = 0\}$ (unless $p = 3$).

One can check that the local Briançon-Skoda number at $\{z_1 = z_2 = 0\}$ is $\frac{p-1}{2}$, i.e., $\frac{p-1}{2}$ is the smallest number $\mu_0$ such that $|\psi| \leq C|g|^{1+\mu_0}$ implies that the germ $\psi$ is in the local ideal generated by $g$ on $X_{\{z_1 = z_2 = 0\}}$. Similarly the Briançon-Skoda number at $\{z_0 = z_2 = 0\}$ is $\left[ \frac{(p-3)(p-1)}{p-2} \right]$. It follows that $\mu_0 \geq \max(\frac{p-1}{2}, \left[ \frac{(p-3)(p-1)}{p-2} \right])$; in fact, $\mu_0 = \max(\frac{p-1}{2}, \left[ \frac{(p-3)(p-1)}{p-2} \right])$.

From Example 6.2 we know that $\nu_L \leq p - 2$. One can check that (1.9) (at $z_0 = z_2 = 0$) for our given $f$ and $\phi$ implies that $\nu_L + 1$, so that the hypothesis on $s$ is vacuous in this case.

Given $V \subset \mathbb{C}^N$, let $\mu_0(V)$ be the smallest number such that for any polynomials $F_j$ and $\Phi$, $|\Phi| \leq C|F|^{\mu_0(V)}$ locally on $V$ implies that $\Phi \in (F_j)$ on $V$, e.g., $\mu_0(\{z_1^2 - z_2^p = 0\}) = \frac{p-1}{2}$, cf., Example 1.9. As in the example the $\mu_0$ associated with the closure $X \subset \mathbb{P}^N$ has to be at least $\mu_0(V)$. Indeed, if $\Phi$ satisfies (1.4) on $V$, then the homogenization $\phi = z_0^{s-\deg \Phi} \Phi$ satisfies (1.9) if $s$ is large enough. Thus Theorem 1.7 gives a solution to (1.10) on $X$; in particular $\Phi \in (F_j)$.

The starting point for the proofs of Theorems 1.1 and 1.7 is the framework introduced in [2], and further developed in [5], for polynomial ideals in $\mathbb{C}^n$. In [30] and [31] these ideas are adapted to toric compactifications of $\mathbb{C}^n$ other than $\mathbb{P}^n$, which leads to “sparse” effective membership results. In our case, let us first assume that $X$ is a smooth projective variety and that $f_1, \ldots, f_m$ are sections of an ample line bundle $L \to X$ with common zero set $Z^f$. From the Koszul complex generated by the $f_j$ we define a current $R^f$ with support on $Z^f$ and taking values in (a direct
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Let \( X \) be a reduced projective variety of pure dimension \( n \). The sheaf \( \mathcal{C}_{\ell,k} \) of currents of bidegree \( (\ell,k) \) on \( X \) is by definition the dual of the sheaf \( \mathcal{E}_{n-\ell,n-k} \) of smooth \((n-\ell,n-k)\)-forms on \( X \). If \( i: X \to Y \) is an embedding in a smooth manifold \( Y \) of dimension \( N \), then \( \mathcal{E}_{n-\ell,n-k} \) can be identified with the quotient sheaf \( \mathcal{E}_{n-\ell,n-k}/\text{Ker } i^* \), where \( \text{Ker } i^* \) is the sheaf of forms \( \xi \) on \( Y \) such that \( i^*\xi \) vanish on \( X_{\text{reg}} \). It follows that the currents \( \tau \) in \( \mathcal{C}_{\ell,k} \) can be identified with currents \( \tau' = i_*\tau \) on \( Y \) of bidegree \((N-n+\ell,N-n+k)\) that vanish on \( \text{Ker } i^* \).

Given a holomorphic function \( f \) on \( X \), we have the principal value current \([1/f] \), defined for instance as the limit

\[
\lim_{\epsilon \to 0} \chi(|f|^2/\epsilon)^{1/2}/f,
\]

where \( \chi(t) \) is the characteristic function of the interval \([1, \infty)\) or a smooth approximand of it. The existence of this limit for a general \( f \) relies on Hironaka’s theorem that ensures that there is a modification \( \tilde{X} \to X \) such that \( \pi^* f \) is locally a monomial. It also follows that the function \( \lambda \to |f|^{2\lambda}/(1/f) \), a priori defined for \( \text{Re } \lambda > 0 \), has a current-valued analytic continuation to \( \text{Re } \lambda > -\epsilon \), and that the value at \( \lambda = 0 \) is precisely the current \([1/f] \), see, for instance, [11] or [12]. Although less natural at first sight, this latter definition via analytic continuation turns out to be much more convenient. The same idea will be used throughout this paper. For the rest of this paper we skip the brackets and write just \( 1/f \). It is readily checked that

\[
f^{1/f} = 1, \quad f\bar{\partial}^{1/f} = 0.
\]

2.2. Pseudomeromorphic currents. In [9] we introduced the sheaf \( \mathcal{P}\mathcal{M} \) of pseudomeromorphic currents on a smooth manifold \( X \). The definition when \( X \) is singular
is identical. In this paper we will use the slightly extended definition introduced in \cite{6}: We say that a current of the form

$$\frac{\xi}{s_1^{\alpha_1} \cdots s_n^{\alpha_n}} \wedge \partial \frac{1}{s_n^{\alpha_n}},$$

where \(s\) is a local coordinate system and \(\xi\) is a smooth form with compact support, is an elementary pseudomeromorphic current. The sheaf \(\mathcal{PM}\) consists of all possible (locally finite sums of) push-forwards under a sequence of maps \(X^m \rightarrow \cdots \rightarrow X^1 \rightarrow X\), of elementary pseudomeromorphic currents, where \(X^m\) is smooth, and each mapping is either a modification, a simple projection \(\hat{X} \times Y \rightarrow \hat{X}\), or an open inclusion, i.e., \(X^j\) is an open subset of \(X^{j-1}\).

The sheaf \(\mathcal{PM}\) is closed under \(\bar{\partial}\) (and \(\partial\)) and multiplication with smooth forms. If \(\tau\) is in \(\mathcal{PM}\) and has support on a subvariety \(V\) and \(\eta\) is a holomorphic form that vanishes on \(V\), then \(\eta \wedge \tau = 0\). We also have the

\textbf{Dimension principle}: If \(\tau\) is a pseudomeromorphic current on \(X\) of bidegree \((\ast, p)\) that has support on a variety \(V\) of codimension \(> p\), then \(\tau = 0\).

If \(\tau\) is in \(\mathcal{PM}\) and \(V\) is a subvariety of \(X\), then the natural restriction of \(\tau\) to the open set \(X \setminus V\) has a canonical extension as a principal value to a pseudomeromorphic current \(1_{X \setminus V}\) \(\tau\) on \(X\): Let \(h\) be a holomorphic tuple with common zero set \(V\). The current-valued function \(\lambda \mapsto |h|^{2\lambda} \tau\), a priori defined for \(\text{Re} \lambda > 0\), has an analytic continuation to \(\text{Re} \lambda > -\epsilon\) and its value at \(\lambda = 0\) is by definition \(1_{X \setminus V} \tau\), see, e.g., \cite{9}. One can also take a smooth approximand \(\chi\) of the characteristic function of the interval \([1, \infty)\) and obtain \(1_{X \setminus V} \tau\) as the limit of \(\chi(|h|^{2/\epsilon}) \tau\) when \(\epsilon \rightarrow 0\). It follows that \(1_{V} \tau := \tau - 1_{X \setminus V} \tau\) is pseudomeromorphic and has support on \(V\). Notice that if \(\alpha\) is a smooth form, then \(1_{V} \alpha \wedge \tau = \alpha \wedge 1_{V} \tau\). Moreover, If \(\pi: \hat{X} \rightarrow X\) is a modification, \(\hat{\tau}\) is in \(\mathcal{PM}(\hat{X})\), and \(\tau = \pi_{\ast} \hat{\tau}\), then

\begin{equation}
1_{V} \tau = \pi_{\ast} (1_{X \setminus V} \hat{\tau})
\end{equation}

for any subvariety \(V \subset X\). There is actually a reasonable definition of \(1_{W} \tau\) for any constructible set \(W\), and

\begin{equation}
1_{W} 1_{W^c} \tau = 1_{W \cap W^c} \tau.
\end{equation}

Recall that a current is \textit{semimeromorphic} if it is the quotient of a smooth form and a holomorphic function. We say that a current \(\tau\) is \textit{almost semimeromorphic} in \(X\) if there is a modification \(\pi: \hat{X} \rightarrow X\) and a semimeromorphic current \(\hat{\tau}\) such that \(\tau = \pi_{\ast} \hat{\tau}\). Analogously we say that \(\tau\) is \textit{almost smooth} if \(\tau = \pi_{\ast} \hat{\tau}\) and \(\hat{\tau}\) is smooth. Any almost semimeromorphic (or smooth) \(\tau\) is pseudomeromorphic.

\subsection{Residues associated with Hermitian complexes.} Assume that

\begin{equation}
0 \rightarrow E_{M} \xrightarrow{f_{M}} \cdots \xrightarrow{f_{3}} E_{2} \xrightarrow{f_{2}} E_{1} \xrightarrow{f_{1}} E_{0} \rightarrow 0
\end{equation}

is a generically exact complex of Hermitian vector bundles over \(X\) and let \(Z\) be the subvariety where \(\text{(2.3)}\) is not pointwise exact. The bundle \(E = E^{+} \oplus E^{-}\) gets a natural superbundle structure, i.e., a \(\mathbb{Z}_{2}\)-grading, \(E = E^{+} \oplus E^{-}\), \(E^{+}\) and \(E^{-}\) being the subspaces of even and odd elements, respectively, by letting \(E^{+} = \oplus_{2k} E_{k}\) and \(E^{-} = \oplus_{2k+1} E_{k}\). This extends to a \(\mathbb{Z}_{2}\)-grading of the sheaf \(\mathcal{C}_{\ast}(E)\) of \(E\)-valued currents, so that the degree of \(\xi \otimes e\) is the sum of the current degree of \(\xi\) and the degree of \(e\), modulo 2. An endomorphism on \(\mathcal{C}_{\ast}(E)\) is even if it preserves degree and odd if it switches degrees. The mappings \(f := \sum f_{j}\) and \(\bar{\partial}\) are then odd mappings on
\( C_\bullet(E) \). We introduce \( \nabla = \nabla_f = f - \bar{\partial} \); it is just (minus) the \((0,1)\)-part of Quillen’s superconnection \( D - \bar{\partial} \). Since the odd mappings \( f \) and \( \bar{\partial} \) anti-commute, \( \nabla^2 = 0 \). Moreover, \( \nabla \) extends to an odd mapping \( \nabla_{\text{End}} \) on \( C_\bullet(\text{End}E) \) so that
\[
(2.5) \quad \nabla(\alpha \xi) = \nabla_{\text{End}}\alpha \cdot \xi + (-1)^{\deg \alpha} \alpha \nabla \xi
\]
for sections \( \xi \) and \( \alpha \) of \( E \) and \( \text{End}E \), respectively, and then \( \nabla^2_{\text{End}} = 0 \). In \( X \setminus Z \) we define, following [8, Section 2], a smooth \( \text{End}E \)-valued form \( u \) such that
\[
\nabla_{\text{End}} u = I,
\]
where \( I = I_E \) is the identity endomorphism on \( E \). We have that
\[
u = \sum_\ell u_\ell = \sum_\ell \sum_{k \geq \ell + 1} u_\ell^k,
\]
where \( u_\ell^k \) is in \( \mathcal{E}_{0,k-\ell-1}(\text{Hom}(E_\ell, E_k)) \) over \( X \setminus Z \). Following [8] we define a pseudomeromorphic current extension \( U_u \) of \( u \) across \( Z \), as the value at \( \lambda = 0 \) of the current-valued analytic function
\[
\lambda \mapsto U^\lambda := |F|^2 \lambda u,
\]
a priori defined for \( \text{Re} \lambda >> 0 \), where \( F \) is the tuple \( f^1 \). In the same way we define the residue current \( R \) associated with \((2.4)\) as the value at \( \lambda = 0 \) of
\[
\lambda \mapsto R^\lambda := (1 - |F|^2 \lambda) I + \bar{\partial} |F|^2 \lambda \wedge u.
\]
This current clearly has its support on \( Z \), and
\[
R = \sum_\ell R^\ell = \sum_\ell \sum_{k \geq \ell + 1} R^k,
\]
where \( R^k \) is a \( \text{Hom}(E_\ell, E_k) \)-valued \((0, k - \ell)\)-current. The currents \( U^\ell \) and \( U^k \) are defined analogously. Notice that \( U \) has odd degree and \( R \) has even degree. By the dimension principle, \( R^k \) vanishes if \( k - \ell < \text{codim} Z \). In particular, \( R^0 = (1 - |F|^2 \lambda) I_{E_0} \big|_{\lambda = 0} \) is zero, unless some components \( W \) of \( Z \) has codimension 0, in which case \( R^0 \) is the characteristic form for \( W \) times the identity \( I_{E_0} \) on \( E_0 \). However, when we define products of currents later on, all components of \( R^\lambda \) may play a role.

Since \( \nabla_{\text{End}} U^\lambda = I - R^\lambda \) and \( \nabla_{\text{End}} R^\lambda = 0 \) when \( \text{Re} \lambda >> 0 \), we conclude that
\[
(2.6) \quad \nabla_{\text{End}} U = I - R, \quad \nabla_{\text{End}} R = 0.
\]
In particular, if \( \xi \) is a section of \( E \)
\[
\nabla(U \xi) = \xi - R \wedge \xi.
\]
Also, \( (2.6) \) means that, cf. \( (2.5) \),
\[
f^1 U^0_1 = I_{E_0}, \quad f^{k+1} U^0_{k+1} - \bar{\partial} U^0_k = R^0_k; \quad k \geq 1.
\]

Notice that when \( \phi \) is a section of \( E_0 \), then \( R^0 \phi = R \phi \) and \( U^0 \phi = U \phi \), and we will often skip the upper indices.

**Example 2.1 (The Koszul complex).** Let \( f_1, \ldots, f_m \) be holomorphic sections of a Hermitian line bundle \( L \to X \). Let \( E^j \) be disjoint trivial line bundles with basis elements \( e_j \) and define the rank \( m \) bundle
\[
E = L^{-1} \otimes E^1 \oplus \cdots \oplus L^{-1} \otimes E^m
\]
3The definition is the same when \( X \) is singular.
over $X$. Then $f := \sum f_j e_j^*$, where $e_j^*$ is the dual basis, is a section of the dual bundle $E^* = L \otimes (E^{1})^* \oplus \cdots \oplus L \otimes (E^{m})^*$. If $S \to X$ is a Hermitian line bundle we can form a complex \cite{23} with

$$E_0 = S, \quad E_k = S \otimes \Lambda^k E,$$

where all the mappings $f_k$ in \cite{23} are interior multiplication $\delta_j$ with the section $f$. Notice that

$$E_k = S \otimes L^{-k} \otimes \Lambda^k (E^1 \oplus \cdots \oplus E^m).$$

The superstructure of $\oplus_k E_k$ in this case coincides with the natural grading of the exterior algebra $\Lambda E$ of $E$ modulo 2.

Let us recall how the currents $U^0$ and $R^0$ are defined in this case. For simplicity we suppress the upper indices throughout this example. We have the natural norm

$$|f|^2 = \sum_j |f_j|^2_L$$
on $E^*$. Let $\sigma$ be the section of $E$ over $X \setminus Z$ of pointwise minimal norm such that $f \cdot \sigma = \delta_j \sigma = 1$, i.e.,

$$\sigma = \sum_j f_j^* e_j / |f|^2,$$

where $f_j^*$ are the sections of $L^{-1}$ of minimal norm such that $f_j^* f_j = |f_j|^2_L$.

Let us consider the exterior algebra over $E \oplus T^*(X)$ so that $d\bar{z}_j \wedge e_\ell = -e_\ell \wedge d\bar{z}_j$ etc. Then, e.g., $\partial \sigma$ is a form of positive degree. We have the smooth form $u = \sum u_k$, where $u_k = \sigma \wedge (\partial \sigma)^{k-1}$, in $X \setminus Z$, and it turns out that it admits a natural current extension $U$ across $Z$, e.g., defined as the analytic continuation of $U^\lambda = |f|^{2\lambda} u$ to $\lambda = 0$. Furthermore, the associated residue current $R$ is obtained as the evaluation at $\lambda = 0$ of

$$R^\lambda := 1 - |f|^{2\lambda} + \bar{\partial}|f|^{2\lambda} \wedge u = 1 - |f|^{2\lambda} + \bar{\partial}|f|^{2\lambda} \wedge u_1 + \cdots + \bar{\partial}|f|^{2\lambda} \wedge u_{\text{min}(m,n)} =: R_0^\lambda + R_1^\lambda + \cdots + R_{\text{min}(m,n)}^\lambda.$$

The existence of the analytic continuations follows from a suitable resolution $\tilde{X} \to X$, see [1], see also Section 5 below.

This current was introduced in [1] in this form, much inspired by [28] where the coefficients appeared. \hfill \square

2.3. **The associated sheaf complex.** Given the complex \cite{23} we have the associated complex of locally free sheaves

$$0 \to \mathcal{O}(E_s) \xrightarrow{f_1} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_3} \mathcal{O}(E_0).$$

In this paper $E_0$ is always a line bundle so that $J := \text{Im } f_1$ is a coherent ideal sheaf over $X$.

Consider the double sheaf complex $\mathcal{M}_{\ell,k} := C_{0,k}(E_\ell)$ with mappings $f$ and $\bar{\partial}$. We have the associated total complex

$$\cdots \xrightarrow{\nabla_{\ell-1}} \mathcal{M}_{j-1} \xrightarrow{\nabla_{\ell-1}} \mathcal{M}_j \xrightarrow{\nabla_{\ell}} \cdots,$$

where $\mathcal{M}_j = \oplus_{\ell-k} \mathcal{M}_{\ell,k}$. If $X$ is smooth, then $\mathcal{M}_{\ell,k}$ is exact in the $k$-direction except at $k = 0$, and the kernels there are $\mathcal{O}(E_\ell)$. Notice that if $\phi$ is in $\mathcal{O}(E_\ell)$ and $f^\ell \phi = 0$, then also $\nabla f \phi = 0$. We therefore have a natural mapping

$$H^j(\mathcal{O}(E_s)) \to H^j(\mathcal{M}_s).$$
By standard homological algebra, \(2.8\) is in fact an isomorphism. We can also consider the corresponding sheaf complexes \(\mathcal{M}_{\ell,k}^\bullet := \mathcal{E}_{0,k}(E_\ell)\), \(\mathcal{M}_J^\bullet\) of smooth sections, and the analogue of \(2.8\) is then an isomorphism as well.

**Lemma 2.2.** Assume that \(X\) is smooth. If \(\phi\) is a holomorphic section of \(E_0\) that annihilates \(R\), i.e., \(R\phi = 0\), then \(\phi\) is in \(J\).

**Proof.** In fact, by \(2.6\) we have that \(\nabla_f(U\phi) = \phi - R\phi = \phi\).

Since \(X\) is smooth, \(2.8\) is an isomorphism, and thus locally \(\phi = f^1\psi\) for some holomorphic \(\psi\), i.e., \(\phi\) is in \(J\).

The smoothness assumption is crucial, as the following example shows.

**Example 2.3.** Let \(f\) be one single function. Then the residue condition \(R\phi = 0\) means that \(\partial(\phi/f) = 0\). Thus \(\psi = \phi/f\) is in the Barlet-Henkin-Passare class, cf., [18] and [6]; however in general \(\psi\) is not (strongly) holomorphic, i.e., in general \(\phi\) is not in \(J = (f)\).

We shall now see that if \(X\) is smooth and there is a global current solution to \(\nabla W = \phi\), then there is also a smooth global solution. For further reference however we need a slightly more general statement about of the associated complex of global sections. Let \(\mathcal{M}_{\ell,k}(X)\) and \(\mathcal{M}_{J,k}(X)\) be the double complexes of global current valued and smooth sections, respectively, and let \(\mathcal{M}_\bullet(X)\) and \(\mathcal{M}_J^\bullet(X)\) be the associated total complexes. Notice that we have natural mappings

\[
(2.9) \quad H^j(\mathcal{M}_\bullet(X)) \to H^j(\mathcal{M}_J^\bullet(X)), \quad j \in \mathbb{Z}.
\]

**Proposition 2.4.** If \(X\) is smooth, then the mappings \(2.9\) are isomorphisms.

**Proof.** By the de Rham theorem, the natural mappings

\[
(2.10) \quad H^k(\mathcal{E}_{0,\bullet}(X,E_\ell)) \to H^k(\mathcal{C}_{0,\bullet}(X,E_\ell)), \quad k \in \mathbb{Z},
\]

are isomorphisms; these spaces are in fact naturally isomorphic to the cohomology groups \(H^k(X,\mathcal{O}(E_\ell))\). The short exact sequence

\[
0 \to \mathcal{M}_J^\bullet(X) \to \mathcal{M}_\bullet(X) \to \mathcal{M}_\bullet(X)/\mathcal{M}_J^\bullet(X) \to 0
\]

gives rise to, for each fixed \(\ell\), the long exact sequence

\[
\ldots \to H^{k-1}(\mathcal{E}_{0,\bullet}(X,E_\ell)) \to H^{k-1}(\mathcal{C}_{0,\bullet}(X,E_\ell)) \to H^{k-1}(\mathcal{C}_{0,\bullet}(X,E_\ell)/\mathcal{E}_{0,\bullet}(X,E_\ell)) \to H^k(\mathcal{E}_{0,\bullet}(X,E_\ell)) \to \ldots,
\]

and in view of \(2.10\) therefore the cohomology in the \(k\)-direction of \(\mathcal{M}_{J,k}(X)/\mathcal{M}_{J,k}^\bullet(X)\) is zero. By a simple homological algebra argument, using that the the double complexes involved are bounded it follows that

\[
H^k(\mathcal{M}_\bullet(X)/\mathcal{M}_J^\bullet(X)) = 0
\]

for each \(k\). The proposition now follows from the long exact sequence

\[
\ldots \to H^{k-1}(\mathcal{M}_J^\bullet(X)) \to H^{k-1}(\mathcal{M}_\bullet(X)) \to H^{k-1}(\mathcal{M}_\bullet(X)/\mathcal{M}_J^\bullet(X)) \to H^k(\mathcal{M}_\bullet(X)/\mathcal{M}_J^\bullet(X)) \to \ldots.
\]
2.4. BEF-varieties and duality principle. We now consider the case when the locally free complex (2.7) is exact, i.e., a resolution of the sheaf \( \mathcal{O}(E_0)/\mathcal{J} \). Let \( Z^{\text{def}}_k \) be the (algebraic) set where the mapping \( f^k \) does not have optimal rank. These sets \( Z^{\text{def}}_k \) are independent of the choice of resolution; we call them the BEF varieties. It follows from the Buchsbaum-Eisenbud theorem that \( \text{codim } Z^{\text{def}}_k \geq k \). If moreover \( \mathcal{J} \) has pure dimension, for instance \( \mathcal{J} \) is the radical ideal sheaf of a pure-dimensional subvariety, then \( \text{codim } Z^{\text{def}}_k \geq k + 1 \) for \( k \geq 1 + \text{codim } \mathcal{J} \), see [10].

We will refer to a (locally free) resolution \( \mathcal{O}(E_0)/\mathcal{J} \) together with a choice of Hermitian metrics on the corresponding vector bundles \( E_k \) as a Hermitian (locally free) resolution. Then by [8, Theorem 3.1], we have that \( R^\ell = 0 \) for each \( \ell \geq 1 \), i.e., \( R = R^0 \). Moreover, there are almost semimeromorphic \( \text{Hom}(E_k, E_{k+1}) \)-valued \((0,1)\)-forms \( \alpha_{k+1} \), that are smooth outside \( Z_{k+1} \), such that

\[
R_{k+1} = \alpha_{k+1} R_k
\]

there, see [8]. From [8] we also have the

**Duality principle:** If \( X \) is smooth and (2.7) is a resolution of the sheaf \( \mathcal{O}(E_0)/\mathcal{J} \), then \( \phi \in \mathcal{J} \) if and only if \( R\phi = 0 \).

That is, the annihilator ideal sheaf of the residue current \( R \) is precisely the ideal sheaf \( \mathcal{J} \) generated by \( f^1 \).

If for instance \( f^1 = (f_1, \ldots, f_m) \) defines a complete intersection, i.e, \( \text{codim } Z^f = m \), then the Koszul complex is a resolution of \( \mathcal{J} \) and hence the duality principle states that the annihilator of the residue current in Example 2.1 is the ideal itself.

2.5. Tensor products of complexes. Assume that \( E^g_k, g \) and \( E^h_k, h \) are Hermitian complexes. We can then define a complex \( E^f_k = E^g_k \otimes E^h_k, f \), where

\[
E^f_k = \bigoplus_{i+j=k} E^g_i \otimes E^h_j,
\]

and \( f = g + h \), or more formally \( f = g \otimes I_{E^h_k} + I_{E^g_k} \otimes h \), such that

\[
f(\xi \otimes \eta) = g\xi \otimes \eta + (-1)^{\text{deg } \xi} \eta h \eta.
\]

Notice that \( E^f_0 \) is the line bundle \( E^g_0 \otimes E^h_0 \). If \( g^1 \mathcal{O}(E^g_1) = \mathcal{J}_g \) and \( h^1 \mathcal{O}(E^h_1) = \mathcal{J}_h \), then \( f^1 \mathcal{O}(E^f_1) = \mathcal{J}_g + \mathcal{J}_h \). One extends (2.11) to current-valued sections \( \xi \) and \( \eta \), and \( \text{deg } \eta \) then means total degree. We write \( \xi \cdot \eta \), or sometimes \( \xi \wedge \eta \) to emphasize that the sections may be form- or current-valued, rather than \( \xi \otimes \eta \), and define

\[
\eta \cdot \xi = (-1)^{\text{deg } \xi} \eta \xi \cdot \eta.
\]

Notice that

\[
\nabla f(\xi \cdot \eta) = \nabla_g \xi \cdot \eta + (-1)^{\text{deg } \xi} \eta h \eta.
\]

Let \( u^g \) and \( u^h \) be the corresponding \( \text{End}(E^g) \)-valued and \( \text{End}(E^h) \)-valued forms, cf., Section 2.2. Then \( u^h \wedge u^g \) is an \( \text{End}(E^f) \)-valued form defined outside \( Z^g \cup Z^h \). Following the proof of Proposition 2.1 in [9] we can define \( \text{End}(E^f) \)-valued pseudomeromorphic currents

\[
U^h \wedge R^g := U^{h, \lambda} \wedge R^g|_{\lambda=0}, \quad R^h \wedge R^g := R^{h, \lambda} \wedge R^g|_{\lambda=0}.
\]
We have that, cf., (2.6) and [3] Section 4,
\[ \nabla_{\text{End}_f}(U^h \wedge R^g + U^g) = I_{E_f} - R^h \wedge R^g. \]
In general, the current \( R^h \wedge R^g \) will change if we interchange the roles of \( g \) and \( h \).

In particular we can form the product \( E^h_\bullet \otimes E^h_\bullet \) of \( E^h_\bullet \) by itself. In this case we consider (2.12) as an identification, so that, for instance,
\[ (E^h_\bullet \otimes E^h_\bullet)_1 = E^h_1 \otimes E^h_0, \quad (E^h_\bullet \otimes E^h_\bullet)_2 = E^h_2 \otimes E^h_0 + \Lambda^2 E^h_1, \]
etc, where \( \hat{\otimes} \) denotes symmetric tensor product. In general, \( \xi \cdot \xi = 0 \) if \( \xi \) has odd (total) degree.

We can just as well form a similar product of more than two complexes, and in particular, we can form the product \( (E^h)^{\otimes k} = E^h \otimes E^h \otimes \cdots \otimes E^h \) of a given complex by itself.

2.6. The structure form \( \omega \) on a singular variety. Let \( i: X \to Y \) be an embedding of \( X \) in a smooth projective manifold \( Y \) of dimension \( N \), with \( J_X \) the radical ideal sheaf associated with \( X \) in \( Y \), and let \( S \to Y \) be an ample Hermitian line bundle. Moreover, let \( E^i_k \) be disjoint trivial line bundles over \( Y \) with basis elements \( e_{k,j} \). There is a (possibly infinite) resolution, see, e.g., [25] Ch.1, Example 1.2.21,
\[ (2.13) \quad \cdots \xrightarrow{g^3} \mathcal{O}(E_2) \xrightarrow{g^2} \mathcal{O}(E_1) \xrightarrow{g^1} \mathcal{O}(E_0) \]
of \( \mathcal{O}(E_0)/J_X = \mathcal{O}^X \), where \( E_k \) is of the form
\[ E_k = (E^1_k \otimes S^{-d^1_k}) \oplus \cdots \oplus (E^r_k \otimes S^{-d^r_k}), \quad E_0 = E_0^0 \simeq \mathbb{C}, \]
and \( E^i_k \) are trivial line bundles, and
\[ g^k = \sum_{ij} \hat{g}^k_{ij} e_{k-1,i} \otimes e^*_{k,j} \]
are matrices of sections
\[ \hat{g}^k_{ij} \in \mathcal{O}(Y, S^{d^i_k}_{k-1}); \]
here \( e^*_{k,j} \) are the dual basis elements. There are natural induced norms on \( E_k \).

The associated residue current \( \hat{R} \) is annihilated by all smooth forms \( \xi \) such that \( i^* \xi = 0 \). Let \( \Omega \) be a global nonvanishing \( (\dim Y, 0) \)-form with values in \( K_Y^{-1} \). By [6], Proposition 16] there is a (unique) almost semimeromorphic current \( \omega \) on \( X \), smooth on \( X_{\text{reg}} \), such that
\[ (2.14) \quad i_* \omega = R \wedge \Omega. \]
We say that \( \omega \) is a structure form on \( X \).

As an immediate consequence of the existence of \( \omega \), the product \( \alpha \wedge R \) is well-defined for (sufficiently) smooth forms \( \alpha \) on \( X \). If \( \alpha = i^* a \), we let \( \alpha \wedge R := a \wedge R \).
This product only depends on \( \alpha \), since if \( i^* a = 0 \), then \( a \wedge R \wedge \Omega = i_* (i^* a \wedge \omega) = 0 \) and hence \( a \wedge R = 0 \) since \( \Omega \neq 0 \).

Let \( X_k \) be the BEF varieties of \( J_X \), and define
\[ (2.15) \quad X^0 = X_{\text{sing}}, \quad X^\ell = X_{N-n+\ell}, \quad \ell \geq 1. \]
Since \( J_X \) has pure dimension it follows that \( \text{codim } X^k \geq k + 1 \), and in particular, \( X^n = \emptyset \). These sets \( X^\ell \) are actually independent of the choice of embedding of \( X \), cf., the comment after Lemma 3.1 in Section 3.

\footnote{The fact that \( (2.13) \) may be infinite causes no problem, since, for degree reasons, \( U \) and \( R \) only contain a finite number of terms.}
Let $g_\ell$ be the restriction to $X$ of $g^{N-n+\ell}$, and let $\nabla^g = g_\ell - \bar{\partial}$ on $X$. Let $E^\ell = E^{N-n+\ell}|_X$. Then $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$, where $\omega_j$ is a $(n, \ell)$-form on $X$ taking values in $E^\ell$, and $\nabla^g \omega = 0$ on $X$. There are almost semimeromorphic $\text{Hom}(E^\ell, E^{\ell+1})$-valued $(0, 1)$-forms $\alpha^{\ell+1}$ such that

$$\omega_{\ell+1} = \alpha^{\ell+1} \omega_{\ell}$$

(2.16)

there. In fact, $\alpha^\ell$ is the pullback to $X$ of the form $\alpha_{N-n+\ell}$ associated with a resolution of $O^Y/J_X$ in $Y$, cf., Section 2.4.

Since $\omega$ is almost semi-meromorphic, it has the standard extension property, SEP on $X$, which means that $1_W \omega = 0$ for all varieties $W \subset X$ of positive codimension.

The singularities of a structure form $\omega$ only depend on $X$, in the following sense:

**Proposition 2.5.** Let $X$ be a projective variety. There is a smooth modification $\tau: \tilde{X} \to X$ and a holomorphic section $\eta$ of a line bundle $S \to \tilde{X}$ such that the following holds: If $X \to Y$ is an embedding of $X$ in a smooth manifold $Y$, $O(E^\bullet_\tau)$, $g$ is a Hermitian locally free resolution of $O^Y/J_X$, and $\omega$ is the associated structure form on $X$, then $\eta \tau^* \omega$ is smooth on $\tilde{X}$. We can choose $\eta$ to be nonvanishing in $\tilde{X} \setminus \tau^{-1}X_{\text{sing}}$.

After further resolving we may assume that $\eta$ is locally a monomial in $\tilde{X}$.

The proof is postponed to Section 3. Since $\omega$ is almost semi-meromorphic, the pullback $\tau^* \omega$ is well-defined; this follows from the proof below, cf., also the remark after Definition 12 in [6].

2.7. **Subvarieties of $\mathbb{P}^N$.** Let $X$ be a subvariety of $Y = \mathbb{P}^N$, $S = O(1)$, and let $O(E_\bullet), g$ be a resolution of $O(E_0)/J_X$ as in (2.13). Then, see [X, Section 6],

$$E_k = (E^1_k \otimes O(-d^1_k)) \oplus \cdots \oplus (E^r_k \otimes O(-d^r_k))$$

and $g^k = (g^k_{ij})$ are matrices of homogeneous forms with $\text{deg} g^k_{ij} = d_k^i - d_k^{i-1}$. We choose the Hermitian metric so that

$$|\xi(z)|_{E^k}^2 = \sum_{j=1}^{r_k} |\xi_j(z)|^2 |z|^{2d_k^j}$$

if $\xi = (\xi_1, \ldots, \xi_{r_k})$ is a section of $E_k$. Moreover,

$$\Omega = \text{const} \times \sum (-1)^i z_0^i \cdots z_N^i d_0^i \cdots d_N^i$$

in $\mathbb{P}^N$.

Let $J_X$ denote the homogeneous ideal in the graded ring $S = \mathbb{C}[z_0, \ldots, z_N]$ that is associated with $X$, and let $S(\ell)$ denote the module $S$ where all degrees are shifted by $\ell$. Then $O(E_\bullet), g$ corresponds to a free resolution

$$\ldots \to \oplus_i S(-d^i_k) \to \ldots \to \oplus_i S(-d^2_0) \to \oplus_i S(-d^1_0) \to S$$

of the module $S/J_X$. Conversely, any such free resolution corresponds to a sheaf resolution $O(E_\bullet), g$.

Notice that the ideal $J_X$ has pure dimension in $S$, so that in particular the ideal associated to the origin is not an associated prime ideal. From Theorem 20.14 in [10], applied to $S$, it follows that the BEF-variety of dimension zero must vanish, therefore the depth of $S/J_X$ is at least 1, and hence a minimal free resolution of $S/J_X$ has length $\leq N$. Recall that the regularity of a homogeneous module with
free graded resolution \( \{2.17\} \) is defined as \( \max_{k,i} (d_i^k - k) \), see, e.g., \([17\text{ Ch.}4]\). The regularity \( \text{reg} X \) of \( X \subset \mathbb{P}^N \) is defined as the regularity of the ideal \( J_X \), which is, cf., \([17\text{ Exercise} \ 4.3]\), equal to \( \text{reg} (S/J_X) + 1 \). If the minimal free resolution of \( S/J_X \) has length \( M \leq N \) we conclude that

\[
\text{reg} X = \max_{k \leq M} (d_k^i - k) + 1.
\]

The regularity of \( X \) is also equal to the \textit{regularity} of the sheaf \( \mathcal{I}_X \), see again \([17\text{ Exercise} \ 4.3]\).

### 2.8. Local division problems on a singular variety.

Still assume that we have the embedding \( i: X \to Y \), where \( Y \) is smooth, and the complex \( E^*_Y, g \) over \( Y \) corresponding to a Hermitian locally free resolution of \( \mathcal{O}^Y/J_X \). If \( E^*_Y, f \) is an arbitrary Hermitian complex over \( Y \) we have the complex \( E^F = E^f \otimes E^g \) with mappings \( F = f + g \) as in Section \([2.5]\). Let \( F^k = F|_{E^k} \). Since \( R^f \wedge R^g = R^f \lambda \wedge R^g |_{\lambda = 0} \) and \( U^f \wedge R^g = U^f \lambda \wedge R^g |_{\lambda = 0} \), cf., Section \([2.3]\), these currents only depend on the values of \( f \) on \( X \). From Section \([2.3]\) we also have that

\[
(2.19) \quad \nabla_{\text{End},F} U = I - R^f \wedge R^g
\]

if \( U = U^f \wedge R^g + U^g \). If \( \Phi \) is a (locally defined) holomorphic function in \( Y \) and \( R^f \wedge R^g \Phi = 0 \), then, following the proof of Lemma \([2.2]\), there is a local holomorphic solution \( v = v_f + v_g \) in \( E^F_1 = E^f_1 \otimes E^g_0 + E^g_0 \otimes E^f_1 \) to \( g^1v_f + f^1v_g = F^1v = \Phi \). Notice that in fact \( R^f \wedge R^g \Phi \) only depends on the class \( \phi \) of \( \Phi \) in \( \mathcal{O}^Y/J_X = \mathcal{O}^X \), so \( R^f \wedge R^g \phi \) is well-defined for \( \phi \) in \( \mathcal{O}^X \). We can define the intrinsic residue current

\[
R^f \wedge \omega := R^f \lambda \wedge \omega |_{\lambda = 0}
\]

on \( X \). Since \( i_* R^f \lambda \wedge \omega = R^f \lambda \wedge R^g \wedge \Omega \) when \( \text{Re} \lambda >> 0 \), we can conclude that

\[
i_* R^f \wedge \omega = R^f \wedge R^g \wedge \Omega.
\]

Since \( \Omega \) is nonvanishing, \( R^f \wedge \omega \phi = 0 \) implies that \( R^f \wedge R^g \phi = 0 \) and thus we have:

**Proposition 2.6.** Assume that \( E^*_Y, f \) is a Hermitian complex on \( X \). If \( \phi \) is a holomorphic section of \( E^g_0 \) on \( X \) such that \( R^f \wedge \omega \phi = 0 \), then locally \( \phi \) is in the image of \( f^1 \) on \( X \).

### 2.9. A fine resolution of \( \mathcal{O} \) on \( X \).

It was proved in \([3]\), see \([3\text{ Theorem} 2]\), that there exist sheaves \( \mathcal{A}_k \) of \((0,k)\)-currents on \( X \) with the following properties:

(i) \( \mathcal{A}_k \) is equal to \( \mathcal{E}_{0,k} \) on \( X_{\text{reg}} \),

(ii) \( \mathcal{A} = \oplus_k \mathcal{A}_k \) is closed under multiplication with smooth \((0,*)\)-forms,

(iii) \( \bar{\partial} \) maps \( \mathcal{A}_k \to \mathcal{A}_{k+1} \) and if \( E \) is any vector bundle over \( X \), then the sheaf complex

\[
0 \to \mathcal{O}(E) \to \mathcal{A}_0(E) \xrightarrow{\partial} \mathcal{A}_1(E) \xrightarrow{\partial} \mathcal{A}_2(E) \xrightarrow{\partial} \ldots
\]

is exact.

By standard sheaf theory we have canonical isomorphisms

\[
H^k(X, \mathcal{O}(E)) = \frac{\text{Ker} \left( \Gamma(X, \mathcal{A}_k(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{A}_{k+1}(E)) \right)}{\text{Im} \left( \Gamma(X, \mathcal{A}_{k-1}(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{A}_k(E)) \right)}, \quad k \geq 1.
\]
3. Singularities of the structure form

In this section we provide a proof of Proposition 2.5. Let \( i : X \to Y \) be an embedding where \( Y \) is projective and smooth of dimension \( N \). Recall that the \( k \)th Fitting ideal (sheaf) of \( \mathcal{O}^Y/X \), \( \text{Fitt}_k \), is the ideal generated by all \( r_k \)-minors of \((\text{the matrix}) g^k \) in a locally free resolution \( \mathcal{O}(E^k_\bullet), g \) of \( \mathcal{O}^Y/X \), where \( r_k \) is the generic rank of \( g^k \), see, e.g., [16]. It is well-known that these ideals are independent of the resolution \( \mathcal{O}(E^k_\bullet), g \); the zero variety of \( \text{Fitt}_k g^k \) is just the BEF-variety \( Z_k^\text{bef} \), cf., Section 2.4. Since \( X \) has pure dimension, \( \text{Fitt}_k g^k \) is trivial when \( k \geq N \), cf., (2.15). Let \( p = N - n \) be the codimension of \( X \) in \( Y \). For \( \ell = 1, \ldots, n - 1 \), let \( a_\ell \) be the pullback (restriction) of \( \text{Fitt}_0 g^{p+\ell} \) to \( X \). It follows that these ideals only depend on the embedding \( i : X \to Y \). We call them the structure ideals on \( X \) associated with the given embedding.

Given a Hermitian resolution \( \mathcal{O}(E^\bullet_\delta), g \) of \( \mathcal{O}^Y/X \), let \( \sigma_k \) be the pointwise minimal inverse of \( g^k \). If (after resolution of singularities) \( \text{Fitt}_0 g^k \) is principal, generated by the holomorphic section \( g \) inverse of \( g \), then \( \sigma_k \) is smooth. Thus \( i^* \sigma_{p+k} =: a^k \) is well-defined and semimeromorphic on \( X \).

**Lemma 3.1.** Assume that \( a_\ell \) and \( a'_\ell \) are the structure ideals associated with the embeddings \( i : X \to Y \) and \( i' : X \to Y' \), respectively. Then for each \( \ell \geq 1 \),

\[
\tag{3.1}
\forall \ell \geq 1,
\]

Since the zero set of \( a_{k+1} \) is contained in the zero set of \( a_k \) it follows that the zero set \( X^\ell \) of \( a_\ell \), cf., Section 2.6, coincides with the zero set of \( a'_\ell \). It follows that \( X^\ell \) is independent of the embedding \( i \).

**Proof.** Given \( i : X \to Y \) and a point \( x \in X \) there is a neighborhood \( V \subset X \) such that the restriction to \( V \) of \( i \) factorizes as

\[
\tag{3.2}
\nu = j \circ \hat{\Omega} \circ \hat{i} : \hat{\Omega} \times \mathbb{B}_M =: \Omega,
\]

where \( j \) is a minimal (and therefore basically unique) embedding at \( x \), \( \mathbb{B}_M 
subset \mathbb{C}^M \) is a ball centered at 0, \( \iota \) is the trivial embedding \( z \mapsto (z, 0) \) if \( z \) are coordinates in \( \hat{\Omega} \), and \( \Omega \) is a neighborhood of \( x \) in \( Y \). Let now \( \mathcal{O}(E^\bullet_\delta), \hat{g} \) be a Hermitian minimal resolution of \( \mathcal{O}\Omega/X_Y \) at \( x \in \hat{\Omega} \) and assume that \( \hat{p} \) is the codimension of \( V \) in \( \hat{\Omega} \). Thus \( p = \hat{p} + M \), where as before \( p \) is the codimension of \( X \) in \( Y \).

Let \( E^w, \delta_w \) be the Koszul complex generated by \( w = (w_1, \ldots, w_M) \), cf., Example 2.1. The sheaf complex associated with the product complex \( E^\delta \otimes E^w \) with mappings \( g = \hat{g}(z) + \delta_w \), cf., Section 2.5, provides a (minimal) resolution of \( \mathcal{O}\Omega/X_Y \) in \( \Omega \), see [11] Remark 8. Notice that \( \hat{g}^{p+\ell} \) is the mapping

\[
\tag{3.3}
\left(E^\hat{g}_{p+\ell} \otimes E^w_M \right) \oplus \left(E^\hat{g}_{p+\ell+1} \otimes E^w_{M-1} \right) \oplus \cdots \oplus \left(E^\hat{g}_{p+\ell+M} \otimes E^w_0 \right) \to \hat{g}(z) + \delta_w
\]

Since \( w = 0 \) on \( X \), the restriction of \( \hat{g}^{p+\ell} \) to \( X \) splits into the direct sum of the separate mappings

\[
\hat{g}^{p+\ell+1} : E^\hat{g}_{p+\ell+1} \otimes E^w_{M-j} \to E^\hat{g}_{p+\ell+1} \otimes E^w_{M-j}, \quad j = 0, 1, \ldots, M.
\]

Since the optimal rank \( r_{p+\ell} \) of \( \hat{g}^{p+\ell} \) is attained at every point on \( X_{\text{reg}} \), it follows that \( r_{p+\ell} = \hat{r}_{p+\ell} + \hat{r}_{p+\ell+1} + \cdots + \hat{r}_{p+M} \), where \( \hat{r}_k \) is the generic rank of \( \hat{g}^k \). Therefore, the
restriction to $X$ of $\text{Fitt}_0 g^{p+\ell}$ is equal to (the restriction to $X$ of) the product ideal $\text{Fitt}_0 g^{p+\ell} \cdot \text{Fitt}_0 g^{p+\ell+1} \cdots \text{Fitt}_0 g^{p+\ell+M}$.

Since $X$ has pure dimension, $\text{Fitt}_0 g^k$ is trivial for $k \geq p + n = \dim \Omega$, and thus if $\hat{a}_\ell$ are the structure ideals associated with $j : \mathcal{V} \to \Omega$, we see that (3.1) holds in a neighborhood of

$$\hat{a}_\ell = \hat{a}_\ell \cdots \hat{a}_{\min(n-1,\ell+M)}.$$ (3.3)

Hence

$$\hat{a}_\ell \cdots \hat{a}_{n-1} \subset \hat{a}_\ell \subset \hat{a}_\ell.$$ (3.4)

By the same argument, since $i'$ factorizes as $\mathcal{V} \xrightarrow{j} \hat{\Omega} \xrightarrow{i'} \hat{\Omega} \times \mathbb{B}_M$, at least if $\mathcal{V}$ is small enough, $a'_\ell = \hat{a}_\ell \cdots \hat{a}_{\min(n-1,\ell+M')}$, and so (3.3) holds at $x$ for $a'_\ell$ instead of $a_\ell$. Combining we see that (3.1) holds in a neighborhood of $x$. Since $x \in X$ is arbitrary, the inclusion holds globally on $X$. \qed

**Lemma 3.2.** There is a smooth resolution $\tau : \tilde{X} \to X$ and a holomorphic section $\eta_0$ of a line bundle $S_0 \to \tilde{X}$, which is nonvanishing in $\tilde{X} \setminus \tau^{-1} X_{\text{sing}}$, with the following properties: If $i : X \to Y$ is an embedding, $\dim Y = N$, $p = N - n$, and $\mathcal{O}(E^n)_g$ is a Hermitian locally free resolution of $\mathcal{O}^Y/J_X$, then:

(i) all the ideals $\tau^* a_\ell$, $\ell = 1, \ldots, n-1$, are principal,

(ii) the subbundles $\text{Im} \tau^* i^* g^{p+\ell} \subset \tau^* i^* E_{p+\ell-1}$, $\ell = 1, \ldots, n-1$, a priori defined over $X \setminus \tau^{-1} X^\ell$, all have holomorphic extensions to $\tilde{X}$,

(iii) if $\omega$ is the induced structure form, then $\eta_0 \tau^* \omega_0$ is smooth.

For the proof we will need the following, probably well-known, result.

**Lemma 3.3.** Let $E, Q$ be holomorphic vector bundles over $X$ and let $g : E \to Q$ be a holomorphic morphism. Let $Z \subset X$ be the analytic set where $g$ does not have optimal rank. There is a (smooth) resolution $\pi : \tilde{X} \to X$ such that the subbundle $\pi^* \text{Im} g \subset \pi^* Q$, a priori defined in $\tilde{X} \setminus \pi^{-1} Z$, has a holomorphic extension to $\tilde{X}$.

**Proof.** Let $G : Q \to F$ be a morphism such that $F$ is a direct sum of line bundles, say $S_1, \ldots, S_r$, and

$$\mathcal{O}(F^*) \xrightarrow{G^*} \mathcal{O}(Q^*) \xrightarrow{g^*} \mathcal{O}(E^*)$$

is exact, cf., [6] Proposition 3.3; we write $G = (G_1, \ldots, G_r)$, where $G_j : Q \to S_j$. It then follows that

$$E \xrightarrow{g^*+1} Q \xrightarrow{G_j} F$$

is pointwise exact in $X \setminus Z$. Therefore,

$$\text{Im} g = \ker G = \bigcap_j \ker (Q \xrightarrow{G_j} S_j)$$

on $X \setminus Z$.

To prove that $\ker G$ has a holomorphic extension, let us first assume that $F$ has rank 1, so that $G$ defines an ideal sheaf $\mathcal{J}_G \subset \mathcal{O}^X$. Also, let us assume that $X$ is connected; if not we just consider each connected component separately. If $G$ is identically zero we define $K := Q$. Otherwise let $\pi : \tilde{X} \to X$ be the blow-up of $X$ along $\mathcal{J}_G$; let $D$ be the corresponding divisor, and let $\mathcal{O}(-D)$ be the line bundle defined by $D$. Then (the pullback to $\tilde{X}$ of) $G$ is of the form $G^0 G'$, where $G'$ is a nonvanishing mapping $Q \to F \otimes \mathcal{O}(-D)$ and $G^0 : F \otimes \mathcal{O}(-D) \to F$ is generically
invertible. Thus $K := \text{Ker } G'$ is a holomorphic subbundle of $Q$, and it generically coincides with $\pi^* \text{Ker } G$.

For the general case, we proceed by induction: We let $K_1$ be an extension of $\text{Ker } G_1$ as above. Then we let $K_2 \subset K_1$ be an extension of the kernel of $G_2|_{K_1}: K_1 \to S_2$. Proceeding in this way we find subbundles $K_r \subset \cdots \subset K_1 \subset Q$, such that $K_j$ generically coincides with $\text{Ker } G_1 \cap \cdots \cap \text{Ker } G_j$ on $X$. In particular, $K_r$ coincides with $\text{Im } g$ generically on $X$, and so we have found a holomorphic extension of $\text{Im } g$. □

**Proof of Lemma 3.2.** Let $\tau: \tilde{X} \to X$ be a smooth resolution and $\eta_0$ a holomorphic section of a line bundle $S_0 \to \tilde{X}$. We claim that (i), (ii), and (iii) hold for a given embedding $i: X \to Y$ and Hermitian locally free resolution $\mathcal{O}(E^q)$, $g$ if and only they hold for any other embedding and Hermitian locally free resolution. Since this claim is local, it is enough to prove the claim at a fixed point $x$.

To prove the claim, let us first fix an embedding $i: X \to Y$. Let $\mathcal{O}(E^q)$, $g$ be a Hermitian locally free resolution of $\mathcal{O}^Y/X$ and let $\mathcal{O}(E^q')$, $g'$ be a minimal such resolution at $x$. It is well-known that $E^q_x, g'$ is a direct summand in $E^q_x, g$, i.e., there is a decomposition $E^q_x = E^q_x \oplus E^q''_x, g = g' \oplus g''_x$, where the complex $E^q''_x, g''_x$ is pointwise exact, see, e.g., [8, Section 4]. Since $\text{Im } g^{p+\ell} = \text{Im } (g')^{p+\ell} \oplus \text{Im } (g'')^{p+\ell}$ it follows that (the pullback to $\tilde{X}$ of) $\text{Im } g^{p+\ell}$ has a holomorphic extension if and only if $\text{Im } (g')^{p+\ell}$ has, for $\ell \geq 1$. From this decomposition it also follows immediately, as we already know, that $\text{Fitto } g^{p+\ell} = \text{Fitto } (g')^{p+\ell}$, so that the structure ideals $a_i$ are well-defined, i.e., independent of the Hermitian resolution.

The structure form $\omega'$ associated with $\mathcal{O}(E^q')$, $g'$ can be considered as a structure form associated with $\mathcal{O}(E^q_x), g$ but with a Hermitian metric that respects the direct sum, cf., [8, Section 4] and (2.14). Now $\omega_0 = \pi' \omega'_0$, where $\pi$ is the orthogonal projection onto the orthogonal complement (with respect to the first metric) of $\text{Im } g^{p+1}$ in $E_p|_X$ over $X \setminus X^1$, and $\omega_0 = \pi' \omega'_0$, where $\pi'$ is the orthogonal projection onto the orthogonal complement (with respect to the second metric) of $\text{Im } (g')^{p+1}$, cf., the proof of Theorem 4.4 in [8]. If (ii) holds for (at least one of) the resolutions, then $\tau^* \pi$ and $\tau^* \pi'$ are smooth and it follows that $\eta_0 \tau^* \omega'_0$ is smooth if and only if $\eta_0 \tau^* \omega'_0$ is. Since the minimal resolution $\mathcal{O}(E^q')$, $g'$ is unique (up to isomorphism) we conclude that if (i) – (iii) hold for one Hermitian resolution of $\mathcal{O}^Y/X$ at a given point $x \in X$, then they hold for any such resolution at $x$.

Next, pick an embedding $i: X \to Y$. In a neighborhood of $x$, $i$ factorizes as (3.2). Let $\mathcal{O}(E^q), \tilde{g}$ be a Hermitian minimal resolution of $\mathcal{O}^\Omega/X$. Then, using the notation from the proof of Lemma 3.1 $\mathcal{O}(E^q) \otimes E^w, \tilde{g} + \delta_w =: g$ is a (minimal) resolution of $\mathcal{O}^\Omega/X$. The associated residue current is equal to $R^{\Omega(z)} \wedge R^w$, see [4, Remark 4.6]. Since a product of local ideals is principal if and only each of its factor is principal it follows from (3.3) that $\tau^* a_{\ell}$ are principal for $\ell = 1, \ldots, n-1$ if and only if $\tau^* a_{\ell}$ are principal for $\ell = 1, \ldots, n-1$. Moreover, since the restriction of $g^{p+\ell}$ to $X$ is a direct sum of restrictions of $g^{p+\ell+1}$, cf., the proof of Lemma 3.1 it follows that (the pull-back to $\tilde{X}$ of) $\text{Im } g^{p+\ell}, \ell \geq 1$, have holomorphic extensions if and only if $\text{Im } g^{p+\ell}$, $\ell \geq 1$, have.

Since $w$ are just the coordinate functions in $\mathbb{C}^M$, the Bochner-Martinelli formula asserts that $R^M_M \wedge dw_1 \ldots \wedge dw_M = (2\pi i)^M [w = 0]$, where $[w = 0]$ is the current of integration over the affine set $\{w = 0\}$. Let $\tilde{N} = \text{dim } \Omega$, and let $\tilde{\omega}$ denote the
structure form in $\hat{\Omega}$ associated with $R^\delta(z)$, so that $j_*\hat{\omega} = R^\delta \wedge dz_1 \wedge \ldots \wedge dz_N$. Then,

$$i_*\hat{\omega} = i_* R^\delta \wedge dz_1 \wedge \ldots \wedge dz_N \cong R^\delta \wedge dz_1 \wedge \ldots \wedge dz_N \hat{\omega} [w = 0] \sim R^\delta \wedge R^\delta \wedge dw_1 \wedge \ldots \wedge dw_M \wedge dz_1 \wedge \ldots \wedge dz_N,$$

where $\sim$ denotes “equal to a nonzero constant times”. We conclude, cf., (2.14) that $\hat{\omega}$ is also a structure form associated with a Hermitian resolution of $\eta$. Above we know, provided that $(ii)$ holds. To this end let us fix an embedding $i: X \to Y$ and a Hermitian locally free resolution $O(E^\bullet)_Y$, $g$. To begin with, by resolution of singularities, we can find a smooth resolution $\tau: \tilde{X} \to X$ such that $\tau^*\omega$ is smooth. Thus, by Lemma 3.2 we can find a resolution $\tau: \tilde{X} \to \tilde{X}$ so that $\text{Im} \tau^* \tau h$ are holomorphic extensions.

Finally we consider $(iii)$. According to Proposition 3.3 in [6],

$$\omega_0 = \sigma_G h,$$

where $h$ is holomorphic in the Barlet-Henkin-Passare sense, i.e., $\ddbar h = 0$ on $X$, and $\sigma_G: F \to E_p$ is the inverse of $G$ in $X \setminus X^1$ with pointwise minimal norm, vanishing on the orthogonal complement of $\text{Im} G$. After further resolving we may assume that $\tau$ is chosen so that also (the pullback of) the ideal $\sigma_G$ is principal in $\tilde{X}$, say, generated by the section $s_G$. Then, by [8, Lemma 2.1], $s_G \sigma_G$ is smooth, cf., the text preceding Lemma 3.2. Since $h$ is meromorphic, there is a section $\eta_0$ of a line bundle $S_0 \to \tilde{X}$ such that $\eta_0 \tau^* \omega$ is smooth. We may also assume that $\tau^{-1}X_{\text{sing}}$ is a divisor, so that $\tau^* h$ is meromorphic with poles contained in $\tau^{-1}X_{\text{sing}}$. Since the variety of $\sigma_G$ is contained in $X_{\text{sing}}$, it follows that we can choose $\eta_0$ to be nonvanishing in $\tilde{X} \setminus \tau^{-1}X_{\text{sing}}$. $\square$

We can now conclude the proof of Proposition 2.5. Let $\tau: \tilde{X} \to X$ and $\eta_0$ be as in Lemma 3.2. Fix an embedding $i': X \to Y'$ and let $s_1', \ldots, s_{n-1}'$ be sections on $\tilde{X}$ defining (the pull-back to $\tilde{X}$ of) the ideals $a_1, \ldots, a_{n-1}$. Let $\eta = s_1' \cdots s_{n-1}'$, $\ell \geq 1$, and $\eta = \eta_0 \eta_1 \cdots \eta_{n-1}$. Note that $s_1'$ is nonvanishing outside $\tau^{-1}X_{\text{sing}}$ so that $\eta$ is nonvanishing in $\tilde{X} \setminus \tau^{-1}X_{\text{sing}}$ if $\eta_0$ is. We claim that $\tau^* \eta$ is smooth for any structure form $\omega$ on $X$.

To prove the claim, let $\omega$ be the structure form associated with an embedding $i: X \to Y$ and a Hermitian locally free resolution $O(E^\bullet)_Y$, $g$. Assume that (the pullbacks of) the corresponding structure ideals are defined by sections $s_1, \ldots, s_{n-1}$. Outside $X^\ell$, $\omega_\ell = \alpha^\ell \omega_{\ell-1}$, where $\alpha^\ell = 1_{X_{\text{sing}}} \ddbar \sigma^\ell$, cf., (2.10) and [8, Section 2]. By [8, Lemma 2.1], $s_\ell \tau^* \sigma^\ell$ is smooth in $\tilde{X}$. Thus, since $\omega_\ell$ has the SEP, $\eta_0 s_1 \cdots s_{n-1} \omega_\ell$ is smooth, and so $\eta_0 s_1 \cdots s_{n-1} \omega$ is smooth. By Lemma 3.2, $s_\ell$ divides $\eta_\ell$ and hence the claim follows.

4. Global division problems and residues

Let (2.4) be a generically exact Hermitian complex over a smooth variety $X$. Moreover, let $\phi$ be a global holomorphic section of $E_0$ such that $R^\phi \phi = 0$. As we have seen in Section 2 then $\nabla_j(U^j \phi) = \phi$. If the double complex $M_{\ell, k} = C_{0, k}(X, E_\ell)$
is exact in the \( k \)-direction except at \( k = 0 \), then it follows, cf., (2.8), that there is a global holomorphic solution to \( f^1 q = \phi \). Let us see more precisely what is needed. Notice that \( U_{\min(M,n+1)}^f \) is automatically \( \bar{\partial} \)-closed. Since \( X \) is smooth then by the Dolbeault isomorphism for currents it is possible to solve successively the equations

\[
\bar{\partial}w_{\min(M,n+1)} = U_{\min(M,n+1)}^f \phi, \quad \bar{\partial}w_k = U_k^f \phi - f^{k+1} w_{k+1}, \quad 1 \leq k < \min(M, n+1),
\]

if

\[
H^{k-1}(X, \mathcal{O}(E_k)) = 0, \quad 1 \leq k \leq \min(M, n+1).
\]

Then

\[
q := U_1^f \phi - f^2 w_2
\]
is a holomorphic solution to \( f^1 q = \phi \). To sum up we have

**Proposition 4.1.** Assume that \( X \) is smooth and \( \phi \) is a holomorphic section of \( E_0 \). If \( R^f \phi = 0 \) and (4.1) holds, then there is a global holomorphic section \( q \) of \( E_1 \) such that \( f^1 q = \phi \).

**Remark 4.2.** Assume that \( \phi \) belongs to the sheaf \( \mathcal{J}_f = \text{Im} f^1 \). This means that locally we have a holomorphic solution \( q \) to \( \nabla_f q = \phi \). However, this does not imply that there is a global (smooth or current) solution to \( \nabla_f v = \phi \), unless the complex \( \mathcal{O}(E_i^\bullet), f \) is exact.

For instance, take global sections \( f_j^1 \) of \( \mathcal{O}(d) \to \mathbb{P}^n \), i.e., homogeneous forms \( f_j^1 \) of degree \( d \) on \( \mathbb{C}^{n+1} \), and let \( \mathcal{O}(E^\bullet_f) \), \( f \) be the Koszul complex generated by \( f^1 := (f_1^1, \ldots, f_m^1) \), cf., Example 2.1, tensorized by \( \mathcal{O}(\rho) \). Assume that \( \phi \) is a section of \( \mathcal{O}(\rho) \) that is locally in the image of \( f^1 \), i.e., \( \phi \) is a global section of \( \mathcal{J}_f \otimes \mathcal{O}(\rho) \). If there is a global solution to \( \nabla_f v = \phi \) and \( \rho \geq (n+1)d - n \) so that (4.1) is fulfilled, then, cf., the proof of Theorem 1.1 below, there are holomorphic forms \( q_j \) such that \( \sum f_j^1 q_j = \phi \). However, in general the mapping

\[
\oplus \Gamma(\mathbb{P}^n, \mathcal{O}(\rho - d)) \to \Gamma(\mathbb{P}^n, \mathcal{J}_f \otimes \mathcal{O}(\rho))
\]

seems to be surjective only if \( \rho \) is much larger than \( (n+1)d - n \), see, e.g., [10] and [17], Proposition 4.16.

If \( \mathcal{O}(E^\bullet_f), f \) is exact, then by the duality principle, \( \phi \) annihilates the residue \( R^f \), and so we get a global solution to \( \nabla_f v = \phi \). One can also piece together local holomorphic solutions to a global smooth solution elementarily, using the exactness of \( \mathcal{O}(E^\bullet_f), f \).

We will now look for analogous results for a singular \( X \). Since we have no access to a \( \bar{\partial} \)-theory for currents on \( X \), we will choose an embedding \( i: X \to Y \) in a smooth (projective) variety \( Y \). To begin with we will assume that our complex \( E^\bullet_f, f \) on \( X \) extends to \( Y \) and that \( \phi \) is a global section on \( Y \). Let \( E^g, g \) be an exact Hermitian complex on \( Y \) associated to \( X \) as in Section 2.6.

If \( \phi \) annihilates \( R^f \wedge \omega \), then it annihilates the product residue \( R^f \wedge R^g \), cf., Section 2.8, and it follows that \( v = (U^g + U^f \wedge R^g) \phi \) is a global current solution to \( \nabla v = \phi \) in \( Y \), and if there are no cohomological obstructions we can solve a sequence of \( \bar{\partial} \)-equations in \( Y \) and obtain a global holomorphic solution to \( f \cdot q + g \cdot q' = \phi \) on \( Y \). In particular, we have the solution \( q \) to \( f \cdot q = \phi \) on \( X \).

**Example 4.3.** Assume that \( X \subset \mathbb{P}^N \) and that \( E^\bullet_f, f \) is the Koszul complex generated by homogeneous forms \( f_j \) of degree \( d \), i.e., global sections of \( \mathcal{O}(d) \), and let \( E^g, g \) be
a complex associated with $X$ as in Section 2.4. Moreover, assume that $\phi$ is a section of $O(\rho)$ over $\mathbb{P}^N$ such that $R^j \wedge R^g \phi = 0$. As above we have a global solution to $f \cdot q + g \cdot q' = \phi$ provided that all the occurring $\bar{\partial}$-equations on $\mathbb{P}^N$ are solvable. However, see, e.g., (14),

\begin{equation}
H^k(\mathbb{P}^N, O(\ell)) = 0 \quad \text{if} \quad \ell \geq -N \quad \text{or} \quad k < N
\end{equation}

so the only possible obstruction is the equation

\begin{equation}
\bar{\partial} W = U_{N+1} \phi,
\end{equation}

where $U = U^f \wedge R^g + U^g$. Since $E^\ell_q, g$ ends at level $N$, $U^g_{N+1} = 0$. Moreover, $R^g_k = 0$ for $k < N - n$ by the dimension principle, so

\begin{equation}
U_{N+1} = \sum_{k=1}^{\min(m, n+1)} U^f_k \wedge R^g_{N+1-k}.
\end{equation}

cf., Section 2.5. The term corresponding to $k$ takes values in a direct sum of line bundles $O(-dk - d_{N+1-k})$. In view of (4.2), one can solve (4.3) if $\rho \geq dk + d_{N+1-k} - N$ for all $i$ and $k = 1, 2, \ldots, \min(m, n+1)$. Notice that, cf., (2.18),

\begin{equation}
dk + d_{N+1-k} - N = dk + (d_{N+1-k} - (N + 1 - k)) + 1 - k \leq (d - 1)k + \text{reg } X.
\end{equation}

It follows that (4.3) is solvable as soon as

\begin{equation}
\rho \geq (d - 1) \min(m, n+1) + \text{reg } X.
\end{equation}

Summing up we have:

If $\rho$ satisfies (4.5) and $\phi$ is a section of $O(\rho)$ on $\mathbb{P}^N$ such that $R^j \wedge \omega \phi = 0$, then there are global sections $q_i$ of $O(\rho - d)$ such that $f_1 q_1 + \cdots + f_m q_m = \phi$ on $X$.

To be more precise, only terms where $N + 1 - k \leq M$ occur in (4.4), where $M$ is the length of $E^\ell_q, g$. If for instance $X$ is Cohen-Macaulay, i.e., the ring $S/J_X$ is Cohen-Macaulay, then $M = N - n$ so that $k \geq n + 1$. If in addition $m \leq n$ thus $U_{N+1}$ vanishes so there is no cohomological obstruction at all.

In our next result (Theorem 4.6) we still assume that the complex $E^\ell_q, f$ extends to the the smooth manifold $Y$ but we do not assume that $\phi$ has a global extension.

Remark 4.4. At least when $E^\ell_q, f$ is the Koszul complex generated by global sections of a line bundle $L \to X$ over $X$ one can get rid of the assumption that $L$ and $f$ extend to $Y$ to the cost of a slightly more complicated residue current to annihilate; see the proof of Theorem 1.7 below.

As a substitute for a holomorphic extension of $\phi$ we will use a $\nabla_g$-closed extension $\Phi$ of $\phi$ to $Y$.

Lemma 4.5. Let $i: X \to Y$ be an embedding of $X$ in a projective manifold $Y$, let $O(E^\ell_q), g$ be a Hermitian resolution of $O^Y / J_X$, and let $\phi$ be a global holomorphic section on $X$ of a line bundle $S \to Y$.

(i) There is a global smooth section $\Phi = \sum_{\ell \geq 0} \Phi_{\ell}$ of $\oplus_{\ell} E^\ell_0, g (E^\ell_q \otimes S)$ on $Y$ such that $\nabla_g \Phi = 0$ on $Y$ and $\Phi_0 = \phi$ on $X$, i.e., $i^* \Phi_0 = \phi$.

(ii) $\Phi$ is such an extension of $\phi$ if and only if

\begin{equation}
\Phi - R^g \phi = \nabla_g w
\end{equation}

for some current $w$. 

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Recall that $E_0^q \simeq \mathbb{C}$ is a trivial line bundle.

One can obtain a $\nabla_g$-closed extension $\Phi$ of $\phi$ quite elementarily by piecing together local holomorphic extensions, due to the exactness of $\mathcal{O}(E_0^q)$, $g$. However, we prefer an argument that also relates to residue calculus as in (ii), and we also think that Lemma 4.5 (ii) may be of independent interest.

**Proof.** As noted in Section 2.4, $R^g \phi$ is a well-defined $\nabla_g$-closed current in $Y$. In view of Proposition 2.4 there is a smooth $\nabla_g$-closed $\Phi$ such that (4.6) holds for some current $w$. Thus (i) follows from (ii).

Assume that $\Phi$ is a smooth extension of $\phi$ as in (i). From (2.6) we have that $\nabla_g(U^q \wedge \Phi) = \Phi - R^g \wedge \phi$. Since $\mathcal{O}(E_0^q)$, $g$ is exact, $(R^g)^\ell = 0$ for $\ell \geq 1$, cf., Section 2.4 and hence $R^g \wedge \phi = R^g \Phi_0 = R^g \phi$, since $\Phi_0 = \phi$ on $X$. Thus

$$\nabla_g(U^q \wedge \Phi) = \Phi - R^g \phi.$$  

Conversely, assume that $\Phi$ is smooth and (4.6) holds. Then clearly $\nabla_g \Phi = 0$. We have to prove that $\Phi_0 = \phi$ on $X$. Since this is a local statement, given a point on $X$ there is a neighborhood $U$ where we have holomorphic extension $\hat{\phi}$ of $\phi$. Then $\nabla_g(U^q \hat{\phi}) = \hat{\phi} - R^g \hat{\phi} = \hat{\phi} - R^g \phi$ in $U$. Thus $\nabla_g(w - U^q \hat{\phi}) = \Phi - \hat{\phi}$. By Proposition 2.4 there is a smooth $\xi$ such that $\nabla_g \xi = \Phi - \hat{\phi}$. It follows that $g^1 \xi_1 = \Phi_0 - \hat{\phi}$ and hence $\Phi_0 = \hat{\phi} = \phi$ in $U$. □

**Theorem 4.6.** Let $i: X \rightarrow Y$ be an embedding of $X$ in a projective manifold $Y$, let $\mathcal{O}(E_0^q)$, $g$ be a locally free Hermitian resolution of $\mathcal{O}^Y/\mathcal{J}_X$ in $Y$, and let $\omega$ be an associated structure form on $X$.

Let (2.4) be a Hermitian complex over (an open neighborhood $U$ of $X$ in) $Y$, and let $R^1 \wedge \omega$ be the associated residue current. Moreover let $\phi$ be a global section of $E_0$ on $X$.

(i) If $R^1 \wedge \omega \phi = 0$, then there is a global smooth solution $W$ on $X$ to

$$\nabla_f W = \phi. \tag{4.7}$$

(ii) If (4.7) has a global smooth solution on $X$ and (4.1) holds, then there is a global holomorphic section $q$ of $\mathcal{O}(E_1)$ such that $f^1 q = \phi$ on $X$.

With minor modifications of the proof we get the following more general version of Theorem 4.6.

With the general hypotheses of Theorem 4.6. assume that $\phi$ is a global holomorphic section of $E_\ell$ such that $f^\ell \phi = 0$.

(i) If $R^\ell \wedge \omega \phi = 0$ then there is a smooth global solution to (4.7).

(ii) If (4.7) has a smooth solution and

$$H^{0,k-1-\ell}(X, \mathcal{O}(E_k)) = 0, \quad \ell + 1 \leq k \leq \min(M, n + 1 + \ell),$$

then there is a global holomorphic section $q$ of $E_{\ell+1}$ such that $f^{\ell+1} q = \phi$.

**Remark 4.7.** If we just have a current solution to $\nabla_f V = \phi$ on $X$ it does not follow that there is a holomorphic solution, not even locally. In fact, if $X$ is nonnormal, there are holomorphic $f$ and $\phi$ such that $\partial(\phi/f) = 0$ but $U = \phi/f$ is not holomorphic. Thus $(f - \overline{f})U = \phi$ but $\phi$ is not in the ideal $(f)$. If $X$ is normal but nonsmooth, there are similar examples with more generators, see [24]. □
Proof of Theorem 4.6. Recall from Section 2.8 that $R^j \wedge \omega \phi = 0$ implies that $R^j \wedge R^q \phi = 0$. Let $\Phi$ be a $\nabla$-closed smooth extension of $\phi$, as in Lemma 4.3 (i), to $Y$. As in the proof of Lemma 4.5, $R^q \Phi = R^q \Phi_0 = R^q \phi$. It follows that $R^j \wedge R^q \wedge \Phi = R^j \wedge R^q \phi = 0$. Hence, from (2.19) we get, cf., Section 2.8
\[
\nabla_F[(U^j \wedge R^q + U^q) \wedge \Phi] = \Phi.
\]
By Proposition 2.4 we have a smooth solution $\Psi$ to $\nabla_F \Psi = \Phi$ in $Y$; i.e.,
\[
F^1 \Psi_1 = \Phi_0, \quad F^{k+1} \Psi_{k+1} - \partial \Psi_k = \Phi_k, \quad k \geq 1.
\]
If we let lower indices $(i, j)$ denote values in $E^j_i \otimes E^q_j$, and notice that $\Phi_k = \Phi_{0,k}$, we see that
\[
f^1 \Psi_{1,0} + g^1 \Psi_{0,1} = \Phi_0, \quad f^{k+1} \Psi_{k+1,0} + g^1 \Psi_{k,1} - \partial \Psi_{k,0} = 0, \quad k \geq 1.
\]
Since $\Psi$ is smooth we can define the forms $W_k = i^* \Psi_{k,0}$ on $X$, and (4.8) then means that
\[
f^1 W_1 = \phi, \quad f^{k+1} W_{k+1} - \partial W_k = 0, \quad k \geq 1.
\]
Thus (i) follows.

Now (ii) follows as in the case when $X$ is smooth, cf., the beginning of Section 4 (now $W$ plays the role of $U^j \phi$), but using the sheaves $A_k$ over $X$, rather than $C_{0,k}$, and the isomorphisms in Section 2.9.

It should be possible to express the $\nabla_F$-exactness of $\Phi$ in $Y$ by means of Čech cohomology, then make the restriction to $X$, and rely on the vanishing of the relevant Čech cohomology groups on $X$. In this way one could possibly avoid the reference to the sheaves $A_k$ over $X$.

5. INTEGRAL CLOSURE, DISTINGUISHED VARIETIES AND RESIDUES

Let $f_1, \ldots, f_m$ be global holomorphic sections of the ample Hermitian line bundle $L \to X$, and let $\mathcal{J}_f$ be the coherent ideal sheaf they generate. Let
\[
\nu: X_+ \to X
\]
be the normalization of the blow-up of $X$ along $\mathcal{J}_f$, and let $W = \sum r_j W_j$ be the exceptional divisor; here $W_j$ are irreducible Cartier divisors. The images $Z_j := \nu(W_j)$ are called the (Fulton-MacPherson) distinguished varieties associated with $\mathcal{J}_f$. If $f = (f_1, \ldots, f_m)$ is considered as a section of $E^* := \bigoplus_{i=1}^m L$, then $\nu^* f = f^0 f'$, where $f^0$ is a section of the line bundle $\mathcal{O}(−W)$ defined by $W$, and $f' = (f_1', \ldots, f_m')$ is a nonvanishing section of $\nu^* E^* \otimes \mathcal{O}(W)$, where $\mathcal{O}(W) = \mathcal{O}(-W)^{-1}$. Furthermore, $\omega_f := dd^c \log |f|^2$ is a smooth first Chern form for $\nu^* L \otimes \mathcal{O}(W)$.

Recall that (a germ of) a holomorphic function $\phi$ belongs to the integral closure $\overline{\mathcal{J}_f}_{x_0}$ of $\mathcal{J}_f_{x_0}$ at $x$ if $\nu^* \phi$ vanishes to order (at least) $r_j$ on $W_j$ for all $j$ such that $x \in Z_j$. This holds if and only if $|\nu^* \phi| \leq C|f^0|$ in a neighborhood of the relevant $W_j$, which in turn holds if and only if $|\phi| \leq C|f|$ in some neighborhood of $x$. Let $\overline{\mathcal{J}_f}$ denote the integral closure sheaf. It follows that
\[
|\phi| \leq C|f|^\ell \quad \text{if and only if} \quad \phi \in \overline{\mathcal{J}_f}.
\]
If $X$ is smooth it follows that $\phi$ is in the integral closure, if for each $j$, $\phi$ vanishes to order $r_j$ at a generic point on $Z_j$. See [25, Section 10.5] for more details (e.g., the proof of Lemma 10.5.2).
We will use the geometric estimate
\[
\sum r_j \deg L Z_j \leq \deg LX
\]
from [15], see also [25, (5.20)].

**Lemma 5.1.** There is a number \( \mu_0 \), only depending on \( X \), such that if
\[
|\phi| \leq C|f|^\mu + \mu_0,
\]
then \( R^f \wedge \phi = 0 \) if \( \omega \) is a structure form of \( X \) and \( R^f \) is the residue current obtained from the Koszul complex of \( f \). If \( X \) is smooth one can take \( \mu_0 = 0 \).

This proposition (and its proof) is analogous to Proposition 4.1 in [7]; the important novelty here is that \( \mu_0 \) can be chosen uniform in \( \omega \), which is ensured by Proposition 2.5. However, for the readers convenience and further reference we discuss the proof.

**Proof.** Let us first assume that \( X \) is smooth and \( \mu_0 = 0 \), and that \( \phi \) satisfies (5.3).

Let \( \omega \) is smooth so we have to show that \( R^f \phi = 0 \). If \( f \equiv 0 \) on (a component of) \( X \), then \( R^f \equiv 1 \) and \( \phi \equiv 0 \), and thus \( R^f \phi = 0 \). Let us now assume that \( \text{codim} Z^f \geq 1 \).

Then \( R^f_0 = 0 \) by the dimension principle. Let \( \nu : X_+ \to X \) be the normalization of the blow-up along \( Jf \) as above, so that \( \nu^* f = f_0 f' \). Using the notation in Example 2.1, then \( \nu^* \sigma = (1/f_0^0) \sigma' \), where \( 1/f_0^0 \) is a meromorphic section of \( O(W) \) and \( \sigma' \) is a smooth section of \( \nu^* E \otimes O(-W) \). It follows that
\[
\nu^* (\sigma \wedge (\bar{\partial}\sigma)^{k-1}) = \frac{1}{(f_0^0)^k} \sigma' \wedge (\bar{\partial}\sigma')^{k-1},
\]
and hence
\[
\nu^* R^f_+ = \bar{\partial} \frac{1}{(f_0^0)^k} \sigma' \wedge (\bar{\partial}\sigma')^{k-1},
\]
when \( k \geq 1 \). Since \( f' \) is nonvanishing, the value at \( \lambda = 0 \) is precisely, see, e.g., [1, Lemma 2.1],
\[
R^f_+ := \bar{\partial} \frac{1}{(f_0^0)^k} \sigma' \wedge (\bar{\partial}\sigma')^{k-1}.
\]

Notice that
\[
R^f_+ \nu = R^f_+.
\]

Since \( \phi \) satisfies (5.3) for \( \mu_0 = 0 \), \( |\nu^* \phi| \leq C|f|^\mu \) and, since \( X_+ \) is normal it follows that \( \nu^* \phi \) contains a factor \( (f_0^0)^\mu \). Therefore,
\[
|\nu^* \phi \bar{\partial} \frac{1}{(f_0^0)^k} = 0, \quad k \leq \mu,
\]
because of (2.1). Moreover, since \( \sigma' \wedge (\bar{\partial}\sigma')^{k-1} \) is smooth on \( X_+ \), it follows from (5.6) and (5.4) that \( R^f_+ \nu^* \phi = 0 \). Therefore, cf., (5.5), \( R^f_+ \phi = \nu^* (R^f_+ \nu^* \phi) = 0 \).

Notice that we could have used any normal modification \( \pi : \tilde{X} \to X \) such that \( \pi^* f \) is of the form \( f_0^0 f' \) in the proof so far.

Now consider a general \( X \). Let us take a smooth modification \( \tau : \tilde{X} \to X \) as in Proposition 2.5 so that, for each structure form \( \omega \) on \( X \), \( \tau^* \omega \) is semimeromorphic with a denominator that divides the section \( \eta \), and such that \( \tilde{X} \) so that \( \eta \) is locally a monomial in suitable coordinates \( s_j \).
Let \( \omega \) be a structure form on \( X \). In this proof it is convenient to use the regularization
\[
R^\ell = \lim_{\varepsilon \to 0} R^{\ell, \varepsilon},
\]
where \( R^{\ell, \varepsilon} := 1 - \chi(|f|^2/\varepsilon) + \partial \chi(|f|^2/\varepsilon) \wedge u \),
where \( u \) is the form from Example 2.4, and \( \chi \) is a smooth approximand of the characteristic function of \([1, \infty)\), cf., the beginning of Section 2, so that all the approximands \( R^{\ell, \varepsilon} \) are smooth. If \( f \equiv 0 \) on a component \( \tilde{X}_j \) of \( \tilde{X} \), then \( R^{\ell, \varepsilon} \equiv 1 \) on \( \tilde{X}_j \) and if \( \phi \) satisfies (5.3) for any \( \mu_0 \), then \( \phi \equiv 0 \) on \( \tilde{X}_j \); here we have suppressed the notation \( \tau^* \) for simplicity. Hence \( 1_{\tilde{X}_j} R^{\ell, \varepsilon} \wedge \omega \phi = 0 \) and so \( 1_{\tilde{X}_j} R^\ell \wedge \omega \phi = 0 \). We can therefore assume that \( f \not\equiv 0 \) on \( \tilde{X} \). Thus the action of \( R^{\ell, \varepsilon} \wedge \omega \phi \) on a test form is, via a partition of unity, a sum of integrals like
\[
\int \frac{ds_1 \wedge \ldots \wedge ds_n}{\eta} R^{\ell, \varepsilon} \phi \wedge \xi,
\]
where \( \alpha_j \) are nonnegative integers and \( \xi \) is a smooth form. Following [7, Section 3] one can integrate by parts such that \( \eta \equiv 1 \) and \( \phi \) satisfies (5.3), then by the smooth Briançon-Skoda theorem, locally in \( \tilde{X} \), \( \phi \) is in the ideal \( (f)^\mu + |\alpha| + 1 \). Therefore,
\[
|\partial^\alpha_{\eta} \phi| \leq C|f|^\mu + |\alpha| + 1.
\]
Following [7] Section 4 one finds that \( \partial^\alpha_{\eta} R^{\ell, \varepsilon} \) has a certain homogeneity with a singularity that increases with \( \ell \) if we take a smooth modification \( \tau : \tilde{X} \to \tilde{X} \) such that \( \tau^* f \) is principal, then \( \partial^\alpha_{\eta} R^{\ell, \varepsilon} \) is like \( 1/(f)^{\mu + |\alpha| + 1} + \text{reg} \) with support where \( |f|^2 \sim \varepsilon \). It follows by dominated convergence that (5.7) tends to zero when \( \varepsilon \to 0 \). Now if \( \mu_0 \geq \mu + |\alpha| + 1 \) for all local representations \( \eta = s_1^{\alpha_1} \ldots s_n^{\alpha_n} \) it follows that \( R^\ell \wedge \omega \phi = 0 \).

6. PROOFS OF THEOREM 1.1 AND VARIATIONS

For the proof of Theorem 1.1 besides the basic Lemma 5.1 we also need

**Lemma 6.1.** Assume that \( V \subset \mathbb{C}^N \) is smooth, and let \( \omega \) be a structure form on \( X \). Then there is a number \( \mu' \) such that \( \omega^{\mu'} \) is almost smooth on \( X \).

**Proof.** Let \( \tau : \tilde{X} \to X \) be as in Proposition 2.5. Then \( \tilde{\omega} := \tau^* \omega \) is a semimeromorphic form whose denominator locally is a monomial whose zeros are contained in \( \tau^{-1} X_{\text{sing}} \).
Since \( V \) is smooth, \( X_{\text{sing}} \subset X_{\infty} \subset \{z_0 = 0\} \), and it follows that \( \tau^*(z_0^{\mu'}) \tau^* \omega \) is smooth for some large enough number \( \mu' \). Hence \( \omega^{\mu'} \) is almost smooth.

**Proof of Theorem 1.1.** Let \( f_j \) be the \( d \)-homogenizations of \( F_j \), let \( R^\ell \) be the residue current constructed from the Koszul complex \( E^\ell \), \( \delta f \) generated by \( f_1, \ldots, f_m \), and let \( \phi \) be the \( \rho \)-homogenization of \( \Phi \), with
\[
\rho = \max(\deg \Phi + (\mu + \mu_0)d^\infty \deg X, (d - 1) \min(m, n + 1) + \text{reg} X),
\]
where \( \mu_0 \) is chosen as in Lemma 5.1 in particular, \( \mu_0 = 0 \) if \( X \) is smooth. Throughout this proof we will use the notation from Section 5.

The assumption (1.3) implies that \( \nu^* \phi \) vanishes to order \( (\mu + \mu_0)r_j \) on each \( W_j \) such that \( \nu(W_j) \) is not contained in \( X_{\infty} \). Now consider \( W_j \) such that \( \nu(W_j) \subset X_{\infty} \).
If $\Omega$ is a first Chern form for $O(1)|_X$, e.g., $\Omega = dd^c \log |z|^2$, then $d\Omega$ is a first Chern form for $L = O(d)|_X$ on $X$ (notice that $d$ denotes the degree and not the differential). By (5.2) we therefore have that

$$r_j \int_{Z_j} (d\Omega)^{\dim Z_j} \leq \int_X (d\Omega)^n,$$

which implies that

$$(6.2) \quad r_j \leq d^\codim Z_i \deg X.$$

By the choice (6.1) of $\rho$, $\phi$ is of the form $\frac{1}{\omega^0} (\mu + \mu_0) e^{d\infty} \deg X$ times a holomorphic section, and thus $\nu^* \phi$ vanishes to order at least $(\mu + \mu_0) r_j$ on $W_j$ for each $j$. Hence (5.3) holds, cf., (5.1), and it follows from Lemma 5.1 that $R^f \wedge \omega \phi = 0$.

Since $\rho \geq (d - 1) \min(m, n + 1) + \text{reg} X$ it follows from Example 1.3 that we have a global $q$ such that $f \cdot q = \phi$ on $X$. After dehomogenization we get a tuple of polynomials $Q_j$ such that $1.4$ holds and $\deg F_j Q_j \leq \rho$. Thus part (i) of Theorem 1.1 is proved.

For the second part choose $\rho = \max(\deg \Phi + \mu d^{\infty} \deg X + \mu', (d - 1) \min(m, n + 1) + \text{reg} X)$, where $\mu'$ is chosen as in Lemma 6.1 and let $\phi$ and $\phi'$ be the $\rho$- and $(\deg \Phi + \mu d^{\infty} \deg X)$-homogenizations of $\Phi$, respectively. Then, by Lemma 6.1

$$R^f \wedge \omega \phi = R^f \wedge \psi \phi',$$

where $\beta$ is almost smooth, and by (1.6) and (6.2),

$$(6.3) \quad \|\phi'\| \leq C |f|^\mu.$$

Now take a smooth modification $\pi: \tilde{X} \to X$ such that $\beta = \pi_* \tilde{\beta}$, where $\tilde{\beta}$ is smooth, and $f = f^0 f'$, where $f^0$ is a section of a line bundle and $f'$ is nonvanishing. Then $R^f \wedge \omega \phi$ is the push-forward under $\pi$ of a finite sum of currents like

$$(\pi^* \phi') \tilde{\beta} \frac{1}{(f^0)^\mu} \text{smooth},$$

cf., (6.1), (6.3), and in view of (6.3) they must vanish. Thus $R^f \wedge \omega \phi = 0$ and (ii) is proved as (i). If $X$ is smooth even at infinity, then $\omega$ is smooth on $X$ so that we can choose $\mu' = 0$ in Lemma 6.1.

The statement in Remark 1.4 follows from the last remark in Example 4.3. \hfill \Box

Remark 6.2. If

$$(6.4) \quad \text{codim} (Z^f \cap X^f) \geq \mu + \ell + 1, \quad \ell \geq 0,$$

where $X^f$ are as in Section 2.3 (thus either $X_\text{sing} \cap Z^f = \emptyset$ or $m < n$), then Theorem 1.1 (i) holds with $\mu_0 = 0$. To see this, take $\rho \geq \deg \Phi + \mu d^{\infty} \deg X$ in the proof of Theorem 1.1. Then $R^f \wedge \phi = 0$ on $X_{\text{reg}}$, and thus $R^f \wedge \omega \phi$ has support on $Z^f \cap X^0$. Since $R^f \wedge \omega_0 \phi$ has bidegree at most $(n, \mu)$ and $\text{codim} (Z^f \cap X^0) \geq \mu + 1$ by (6.4), it follows from the dimension principle that $R^f \wedge \omega_0 \phi = 0$. Thus $R^f \wedge \omega_1 \phi = R^f \wedge \omega_0 \phi$ vanishes outside $X^1$, so again by (6.4) and the dimension principle we find that $R^f \wedge \omega_1 \phi$ vanishes identically. By induction, $R^f \wedge \omega \phi = 0$. \hfill \Box

Example 6.3. In light of Example 2.3 of Kollár in [23] one can see that the power $c_\infty$ in Theorem 1.1 cannot be improved: Let $X = \mathbb{P}^m$ and let $m$ be an integer with $2 \leq m \leq n$. Consider the $m$ polynomials

$$z_1^d, z_1 z_2^{d-1} - z_2^d, \ldots, z_m - z_m^{d-1}, z_m - z_m^{d-1} - 1,$$
in $\mathbb{C}^n$. The associated projective variety $\{ z_0 = z_1 = \cdots = z_{m-1} = 0 \} \subset X_\infty$ has codimension $m$, and hence $c_\infty = m$, cf., (1.3). It follows from Theorem 1.1 that we have a representation (1.1) with $\Phi = 1$ and $\deg F_j Q_j \leq md^m$ (if $d$ is not too small). However, if $Q_j$ are any polynomials so that (1.1) holds with $\Phi = 1$, then by considering the curve

$$t \mapsto (t^{d^m-1}, t^{d^m-2}, \ldots, t^{d^m-1}, 1/t, 0, \ldots, 0),$$

one can conclude that $Q_1$ must have degree at least $d^m$ so that $\deg F_1 Q_1 \geq d^m$. □

In [3] is used a slight generalization of the Koszul complex to deal with a positive power $J^\ell f$ of $J f$, cf. [15, p. 439]. The first mapping in the complex is the natural mapping $E \otimes \ell \rightarrow C$ induced by the $f_j$. The associated residue current is the push-forward of currents like

$$\bar{\partial} (f^k) \wedge \text{smooth}$$

for $\ell \leq k \leq \mu + \ell - 1$. By an analogous proof we get the following generalization of Theorem 1.1.

**Theorem 6.4.** With the notation in Theorem 1.1 if

$$|\Phi| \leq C|f|^\mu + \mu_0 + \ell - 1$$

locally on $V$, then $\Phi \in (F_j)^\ell$ and there are polynomials $Q_I$ such that

$$\Phi = \sum_{I_1 + \cdots + I_m = \ell} F_{I_1} \cdots F_{I_m} Q_I$$

and

$$\deg (F_{I_1} \cdots F_{I_m} Q_I) \leq \max \{ \deg \Phi + (\mu + \mu_0 + \ell - 1) d^\infty \deg X, d(\min(m, n+1) + \ell - 1) - \min(m, n+1) + \text{reg } X \}.$$

There is also an analogous generalization of part (ii) of Theorem 1.1.

**7. Proofs of Theorem 1.7 and variations**

We first look at the case when $X$ is smooth, which is due to Ein-Lazarsfeld [15].

**Theorem 7.1.** Let $X$ be a smooth projective variety, let $L \rightarrow X$ be an ample line bundle, and let $A \rightarrow X$ be a line bundle that is either ample or big and nef. Moreover, let $f_1, \ldots, f_m$ be global holomorphic sections of $L$, and let $\phi$ be a global section of

$$L^\otimes s \otimes K_X \otimes A,$$

where $s \geq \min(m, n+1)$. If

$$(7.1) \quad |\phi| \leq C|f|^\mu$$

on $X$, then there are holomorphic sections $q_j$ of $L^\otimes (s-1) \otimes K_X \otimes A$ such that

$$(7.2) \quad f_1 q_1 + \cdots + f_m q_m = \phi.$$

Let $J f$ be the ideal sheaf generated by $f_j$ and assume that the associated distinguished varieties $Z_k$ have multiplicities $r_k$, cf., Section 5. If $\phi$ vanishes to (at least) order $r_k \mu$ at a generic point on $Z_k$ for each $k$, then (7.1) holds, cf., Section 5 and thus we have

**Corollary 7.2.** If $\phi$ vanishes to order $r_k \mu$ at a generic point on $Z_k$, for each $k$, then we have a representation (7.2).
This corollary is precisely part (iii) of the main theorem in [15], p. 430, except for that we have \(\mu_k\) rather than \((n+1)r_k\), cf., the discussion in Remark 1.6. Recall from Section 5 that one can estimate the multiplicities \(r_k\); for instance \(r_k \leq \deg LX\), see (6.2).

One can also have a mixed hypothesis, and for instance assume that (7.1) holds outside a hypersurface \(H\) and that \(\phi\) vanishes to order \(\mu r_k\) on each distinguished variety \(Z_k\) contained in \(H\); this would lead to an “abstract” Huckleberry theorem.

**Proof of Theorem 7.7** Equip \(L\) with a Hermitian metric, let \(E^\bullet_h, \delta_f\) be the Koszul complex generated by \(f_1, \ldots, f_m\), as in Example 2.1, tensorized with \(L^{\otimes s} \otimes A \otimes K_X\), and let \(R^f\) be the associated residue current on \(X\). From the hypothesis (7.1) and Lemma 5.1 we conclude that \(R^f \phi = 0\). The bundle \(E_h\) is a direct sum of line bundles \(L^{\otimes (s-k)} \otimes A \otimes K_X\) and so all the relevant cohomology groups (4.1) vanish by Kodaira’s vanishing theorem, or, at the top degree, by the Kawamata-Viehweg vanishing theorem if \(A\) is nef and big. Thus Theorem 7.1 follows from Proposition 4.1. □

**Proof of Theorem 1.7** Let \(E^\bullet_h, \delta_f\) be the Koszul complex generated by \(f_1, \ldots, f_m\) tensorized with \(L^{\otimes s}\). The choice of \(s\) guarantees that (4.1) is satisfied and thus by the same arguments as in the proof of Theorem 1.6 (ii) we get the desired holomorphic solution to (1.10) as soon as we have a smooth solution to (7.3)

\[
\nabla_f W = \phi
\]

on \(X\). Hence to prove the theorem it suffices to show that there is a \(\mu_0\) such that we can find a smooth solution to (7.3) for each global section \(\phi\) of \(L^{\otimes s}\) that satisfies (1.9). The strategy will be to follow and further elaborate the proof of Theorem 1.6 (i). Note that we cannot apply Theorem 1.6 (i) directly since a priori \(L\) and the sections \(f_j\) are only defined on \(X\).

We first claim that there is an embedding \(i: X \rightarrow Y\) into a smooth projective manifold \(Y\) and a line bundle \(L \rightarrow Y\) such that \(L = L|_X\), i.e., \(L = i^*L\). In fact, if \(M\) is large enough, there are embeddings \(i_j: X \rightarrow \mathbb{P}^{N_j}, \ j = 1, 2\), such that \(\mathcal{O}(1)_{\mathbb{P}^{N_j}}|_X = L^M\) and \(\mathcal{O}(1)_{\mathbb{P}^{N_j}} = L^{M+1}\). If \(\pi_j: \mathbb{P}^{N_j} \times \mathbb{P}^{N_j} \rightarrow \mathbb{P}^{N_j}\), then \(L := \pi_2^*\mathcal{O}(1)_{\mathbb{P}^{N_j}} \otimes \pi_1^*\mathcal{O}(-1)_{\mathbb{P}^{N_j}}\) is a line bundle over \(Y := \mathbb{P}^{N_j} \times \mathbb{P}^{N_j}\) and its restriction to \(X \cong \Delta_{X \times X} \subset Y\) is precisely \(L\). This argument was recently communicated to us by R. Lazarsfeld. Let \(\mathcal{O}(E^\bullet_h, g)\) be a Hermitian resolution of \(\mathcal{O}^Y/J_X\) in \(Y\) as in Section 2.4.

In general we cannot assume that the \(f_j\) extend holomorphically to \(Y\) nor even to a neighborhood of \(X\) in \(Y\). However, let \(E^\bullet_h, h\) be a complex that is isomorphic to but disjoint from \(E^\bullet_h, g\). In view of Lemma 1.3 we can choose smooth \(\nabla_{h, L}\)-closed extensions \(\tilde{f}_j \in \oplus \mathcal{E}_{0,j}(E^\bullet_h \otimes L)\) of \(f_j\) to \(Y\), as defined in Section 4. Let \(E^1, \ldots, E^m\) be trivial line bundles as in Example 2.4 with basis elements \(e_1, \ldots, e_m\), respectively, and let \(\tilde{f}\) be the section \(\tilde{f} := \sum_{j=1}^m \lambda_j \otimes E_j\), where \(|\lambda_j| = 1\) and \(E_j\) are the dual basis elements. Note that each \(\tilde{f}_j\) has even degree so that \(\tilde{f}\) has odd degree.

Inspired by Example 2.4 we want to construct a Koszul complex of \(\tilde{f}\) as an extension of \(E^\bullet_h, \delta_f\) and an associated residue current. In order to do this we will need to take products of sections of \(E^\bullet_h\). We therefore introduce \(E^\bullet_h^H := \bigcup_{k \geq 1} (E^\bullet_h)^{\otimes_k}\). Since \(E^\bullet_h^0\) is the trivial line bundle, \((E^\bullet_h)^{\otimes_k}\) is a natural subcomplex of \((E^\bullet_h)^{(k+1)}\) and thus the definition makes sense. In fact, our objects will all take values in \((E^\bullet_h)^{\otimes_n}\). Next let \(\delta_f: E^\bullet_h \otimes \Lambda^k E \rightarrow E^\bullet_h \otimes \Lambda^{k-1} E\) be contraction with \(\tilde{f}\), i.e., for a section
\( \xi = \sum_{l=\{i_1, \ldots, i_k\}} \xi_l \wedge e_l \), with \( \xi_l \in E^*_l \) and \( e_l = e_{i_1} \wedge \cdots \wedge e_{i_k} \), of \( E^*_l \otimes \Lambda^k E \),

\[ \delta_f \xi = \sum_{l=\{i_1, \ldots, i_k\}} (-1)^{d_{\xi_l} - 1} \xi_l \sum_j \delta f_j^* \wedge e_{i_j} \wedge \sigma_{i_j} . \]

As long as we restrict to \( X \) we can write \( f = f - f' \), where \( f := \sum f_j e_j^* \) and \( f' \) has positive form degree. Let \( \delta_f' \) be defined analogously to \( \delta_f \) and let \( \sigma \) be the section of \( E \) over \( X \setminus Z \) of pointwise minimal norm such that \( \delta_f \sigma = 1 \) there, cf. Example 2.4.

Then

\[ \delta_f \sigma = \delta_f - \delta_f' \sigma = 1 - \delta_f' \sigma \]
on \( X \setminus Z \). Notice that \( \delta_f' \sigma \) has even degree, and form bidegree at least \((0,1)\), so that

\[ \frac{1}{1 - \delta_f' \sigma} = 1 + \delta_f' \sigma + (\delta_f' \sigma)^2 + \cdots + (\delta_f' \sigma)^n \]
is a form on \( X \setminus Z \) with values in \( E^*_l \otimes \Lambda^* E \). Let \( \tilde{\sigma} := \sigma / (1 - \delta_f' \sigma) \) on \( X \setminus Z \); then \( \delta_f \tilde{\sigma} = 1 \) on \( X \setminus Z \). Next, let

\[ \tilde{\omega} = \tilde{\sigma} \wedge (\nabla \delta_f + \nabla h) \tilde{\sigma} \]
cf., Example 2.1 and [1]. Note that \( \delta_f \) anti-commutes with \((\text{the extension to } E^*_l \otimes \Lambda^* E \text{ of}) \nabla h\), i.e., \( \delta_f \nabla h = -\nabla h \delta_f \). It follows that \((\delta_f + \nabla h)^2 = 0 \) and so

\[ (\delta_f + \nabla h) \tilde{\omega} = 1 \]
on \( X \setminus Z \), cf. Section 2.2.

Let \( R^g \) be the residue current associated with the resolution \( O(E^*_l) \), and let \( \omega \) be an associated structure form. Recall from Section 2.6 that if \( \alpha \) is a sufficiently smooth form on \( X \), then \( \alpha \wedge R^g \) is a well-defined in \( Y \); in particular, \( \chi(\xi f / |f|) \tilde{\omega} \wedge R^g \) is a well-defined current in \( Y \) with values in \( E^*_l \otimes \Lambda^* E \otimes E^*_l \). Letting

\[ \nabla = g + \delta_f + \nabla h = g + \delta_f + h - \tilde{\partial}, \]

note that

\[ \nabla(\chi(\xi f / |f|) \tilde{\omega} \wedge R^g + U^g) = I - \tilde{R} \wedge R^g, \]
where \( \tilde{R}^\epsilon = I - \chi(\xi f / |f|) I + \tilde{\partial} \chi(\xi f / |f|) \wedge \tilde{\omega} \).

We claim that \( \chi(\xi f / |f|) \tilde{\omega} \wedge R^g \) has a limit when \( \epsilon \to 0 \). To see this, recall from Section 2.6 using the notation from that section, that \( \chi(\xi f / |f|) \tilde{\omega} \wedge R^g \wedge \Omega = I_\ast(\chi(\xi f / |f|) \tilde{\omega} \wedge \omega) \). Next, notice that

\[ \tilde{\sigma} \wedge (\nabla h \tilde{\sigma})^{k-1} = \sigma \wedge (\tilde{\partial} \sigma)^{k-1} \wedge \sum_{j=0}^n c_j(\delta_f \sigma)^j, \]

for some numbers \( c_j \), since \( \sigma \wedge \sigma = 0 \) and \( \sigma \) has degree 0 in \( E^*_l \). Let \( \pi : \tilde{X} \to X \) be a smooth modification such that \( \pi \omega \) is semimeromorphic and \( \pi^* \sigma \) is of the form \( \sigma / f^0 \), cf. Section 5. Then \( \pi^* \tilde{\omega} \) is a finite sum of terms \( \gamma_k / (f^0)^k \), where \( \gamma_k \) are smooth, and hence \( \lim_{\epsilon \to 0} \pi^* \chi(\xi f / |f|) \tilde{\omega} \wedge \omega \) exists, see, e.g., [12]. Since \( \Omega \) is nonvanishing it follows that the limit of \( \chi(\xi f / |f|) \tilde{\omega} \wedge R^g \) exists.

Let

\[ \tilde{U} \wedge R^g = \lim_{\epsilon \to 0} \chi(\xi f / |f|) \tilde{\omega} \wedge R^g, \quad \tilde{R} \wedge R^g = \lim_{\epsilon \to 0} \tilde{R}^\epsilon \wedge R^g. \]

Then

\[ \nabla(\tilde{U} \wedge R^g + U^g) = I - \tilde{R} \wedge R^g, \]
and if \( \Phi \) is a smooth \( \nabla \)-closed extension of \( \phi \) as in Lemma 1.9 (regarded as a section of \( L^{\otimes s} \otimes E^H_0 \otimes \Lambda^*E \otimes E^2_0 \)), it follows that

\[
(7.6) \quad \nabla \left( (\hat{U} \wedge R^g + U^g) \wedge \Phi \right) = \Phi
\]
in \( Y \) as soon as

\[
(7.7) \quad \bar{R} \wedge R^g \phi = 0,
\]
since, as was noted in the proof of Lemma 1.5, \( R^g \wedge \Phi = R^g \phi \).

We claim that there is a \( \mu_0 \), only depending on \( X \), such that (7.4) holds as soon as \( \phi \) satisfies (1.9). Note that (7.7) is equivalent to that \( \bar{R} \wedge R^g \wedge \Omega \phi = \lim_{t \to 0} i_*(\bar{R}^c \wedge \omega \phi) \) vanishes. Let \( \tau : \tilde{X} \to X \) be a smooth modification as in Proposition 2.5, so that locally \( \tau^\ast \omega = \frac{\text{smooth}}{\alpha + 1} \), where \( s^{\alpha + 1} \) is a local representation of the section \( \eta \), as in the proof of Lemma 5.1. Following that proof, the action of \( \bar{R}^c \wedge \omega \phi \) on a test form is a sum of integrals like (suppressing \( \tau^\ast \) for simplicity)

\[
(7.8) \quad \int_{\tilde{X}} \frac{ds_1 \cdots ds_n}{s_1 \cdots s_n} \partial_{\bar{s}} \left( \bar{R}^c \phi \wedge \xi \right),
\]
where \( \xi \) is smooth. As in the proof of Lemma 5.1 it is enough to consider components of \( \tilde{X} \) where \( f \) does not vanish identically. Also, as in the proof of Lemma 5.1, \( \partial_{\bar{s}} \bar{R}^c \) has a certain homogeneity that increases with \( \ell \). Indeed, in view of (7.5), \( \bar{R}^c \) is a finite sum of terms like

\[
\partial_{\bar{s}}(|f|^2/\epsilon) \wedge \sigma \wedge (\partial_{\bar{s}} \sigma)^{k-1} \wedge (f' \sigma)^j,
\]
where \( k + j \leq n \) for degree reasons; recall that \( f' \) has form degree at least \((0,1)\). Now, if we take a smooth modification \( \pi : \tilde{X} \to X \) such that \( \pi^\ast f \) is principal, then \( \pi^\ast \sigma = \text{smooth}/f^0 \), \( \pi^\ast \left( \frac{\partial}{\partial s} \sigma \right) = \text{smooth}/(f^0)^2 \) and following [7, Section 4] we get that \( \pi^\ast (\partial_{\bar{s}} \bar{R}^c) \) is like \( 1/(f^0)^{n+|\ell|+1} \) with support where \( |f|^2 \sim \epsilon \), cf., the proof of Lemma 5.1. As in that proof, choose \( \mu_0 \geq n + |\alpha| + 1 \) for all local representations \( \eta = s^{\alpha + 1} \); assume that \( \phi \) satisfies (1.9). Then \( \partial_{\bar{s}} \phi \leq C |f|^{n+|\alpha|-|\ell|+1} \), cf. the proof of Lemma 5.1. Now by dominated convergence (7.8) tends to zero when \( \epsilon \to 0 \), and since the choice of \( \mu_0 \) only depends on the section \( \eta \) and \( n \) the claim follows.

To sum up so far, there is a \( \mu_0 \) such that if \( \phi \) satisfies (1.9), we have a current solution (7.6) to \( \nabla \Psi = \Phi \). By a slight modification of Proposition 2.6 we also have a smooth solution. To see this, let \( E^F_\bullet = \Lambda^*E \otimes E^2 \) and let \( M_\bullet \) and \( M^\ell_\bullet \) be defined as in Section 2.3 but for the complex \( E^H_\bullet \) instead of \( E^F_\bullet \). Then we have the double complex

\[
B_{\ell,k} = \oplus_j C_0(j, E^H_{j+k} \otimes E^F_{\ell}) =: M_k(E^F_{\ell})
\]
with mappings \( \nabla_h : B_{\ell,k} \to B_{\ell,k-1} \) and \( F := g + \delta_f : B_{\ell,k} \to B_{\ell-1,k} \); indeed note that

\[
\nabla_h \circ F = -F \circ \nabla_h.
\]
If \( B_j := \bigoplus_{k+j} B_{\ell,k} \) we get the associated total complex

\[
\cdots \nabla_{B_j} \nabla_{B_j} \nabla_{B_{j-1}} \nabla \cdots,
\]
with \( \nabla \) as in (7.4). Analogously let \( B^\ell_{\bullet,k} = \oplus_j C_0(j, E^H_{j+k} \otimes E^F_{\ell}) =: M^\ell_k(E^F_{\ell}) \) with total complex \( B^\ell_\bullet \). Moreover, let \( B_\bullet(X) \) and \( B^\ell_\bullet(X) \) be the associated complexes of global sections. Note that we have natural mappings

\[
(7.9) \quad H^j(B^{\ell}_\bullet(X)) \to H^j(B_\bullet(X)), \quad j \in \mathbb{Z}.
\]
Proposition 2.6 implies that the natural mappings \( H^k(M^\ell_\bullet(X, E^F_\bullet)) \to H^k(M_\bullet(X, E^F_\bullet)) \) are isomorphisms. Now, by repeating the proof of Proposition 2.6 with \( M_\bullet, M^\ell_\bullet \),
C_0, and E_0 replaced by \(B_\bullet, \mathcal{B}_{\ell,k}^0, \mathcal{M}_{\bullet}, \) and \(\mathcal{M}_{\ell,k}^0,\) respectively, using that the double complex \(B_{\ell,k}\) is bounded in the \(\ell\)-direction, we can therefore prove that the mappings \((7.9)\) are in fact isomorphisms. Hence the current solution \((7.6)\) gives a smooth solution to \(\nabla \Phi = \Phi\). Let lower indices \((i,j,k)\) denote components in \(L^{s*} \otimes E^i H \otimes \Lambda^j E \otimes E^k_{\nu}\). Then \(\Phi = \Phi_{0,0,0} + \cdots + \Phi_{0,0,n}\), where \(\Phi_{0,0,k}\) has form bidegree \((0,k)\). Notice that we have the decomposition \(\bar{f} = f_0 - f'\) in \(Y\), where \(f_0\) denotes the 0-component of \(\bar{f}\) and hence is a smooth extension of \(f\) to \(Y\). It follows that

\[
(7.10) \quad h \Psi_{1,0,0} + \delta f_0 \Psi_{0,1,0} + g \Psi_{0,0,1} = \Phi_0,
\]

\[
h \Psi_{1,j,0} + \delta f_0 \Psi_{0,j+1,0} + g \Psi_{0,j,1} - \partial \Psi_{0,j,0} = 0, \quad j \geq 1.
\]

Indeed, note that \(\delta f' \Psi_{i,j,k}\) has positive degree in \(E^i H\) for all nonvanishing \(\Psi_{i,j,k}\). Since \(\Psi\) is smooth, we can define the smooth forms \(W_j := i^* \Psi_{0,j,0}\) on \(X\). Note that \(L^{s*} \otimes \Lambda^j E|_X = E^j f\), so that \(W_j\) takes values in \(E^j f\). Since \(g \Psi_{0,j,1} = g \Psi_{0,j,1}\) and \(h \Psi_{1,j,0} = h \Psi_{1,j,0}\) are in \(\mathcal{J}_X, (7.10)\) implies

\[
\delta f W_1 = \phi, \quad \delta f W_{j+1} - \partial W_j = 0, \quad j \geq 1.
\]

Thus we have shown that there is a \(\mu_0\) such that if \(\phi\) satisfies \((1.9)\), then we get a smooth solution to \((7.3)\); this concludes the proof. \(\Box\)

**Remark 7.3.** If \(E^h\) is a Koszul complex, then we just simply take \(E^H = E^h\), since he desired "product" already exists within \(E^h\). \(\Box\)

In analogy with Theorem 6.4 we also have the following generalizations of Theorems 7.1 and 1.7.

**Theorem 7.4.** With the notation in Theorem 7.1, if \(\phi\) is a section of \(L^{s*} \otimes K_X \otimes A\), where \(s \geq \min(m, n+1) + \ell - 1\), and

\[
|\phi| \leq C |f|^{\mu + \ell - 1},
\]

there are holomorphic sections \(q_I\) of \(L^{s*} \otimes K_X \otimes A\), such that

\[
(7.11) \quad \phi = \sum f_1^{i_1} \cdots f_m^{i_m} q_I.
\]

With the notation in Theorem 1.7, if \(\phi\) is a section of \(L^{s*}\) with \(s \geq \nu + \min(m, n+1) + \ell - 1\) such that

\[
|\phi| \leq C |f|^{\mu_0 + \mu \ell - 1},
\]

then there are holomorphic sections \(q_I\) of \(L^{s*} \otimes K_X \otimes A\) such that \((7.11)\) holds.

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