MATHEMATICAL SEMANTICS OF INTUITIONISTIC LOGIC

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Abstract. This is an elementary introduction to intuitionistic logic, assuming a modest literacy in mathematics (such as topological spaces and posets) but no training in formal logic.

We adopt and develop Kolmogorov’s understanding of intuitionistic logic as the logic of schemes of solutions of mathematical problems. Here intuitionistic logic is viewed as an extension package that upgrades classical logic without removing it (in contrast to the standard conception of Brouwer and Heyting, which regards intuitionistic logic as an alternative to classical logic that criminalizes some of its principles). The main purpose of the upgrade comes, for us, from Hilbert’s idea of equivalence between proofs of a given theorem, and from the intuition of this equivalence relation as capable of being nontrivial.

Mathematically, this idea of “proof-relevance” amounts to categorification. Accordingly, we construct sheaf-valued models of intuitionistic logic, in which conjunction and disjunction are interpreted by product and disjoint union (of sheaves of sets); these can be seen as a categorification of the familiar (since Leibniz, Euler and Venn models of classical logic, in which conjunction and disjunction are interpreted by intersection and union (of sets). Our sheaf-valued models (not to be confused with the usual open set-valued “sheaf models”) turn out to be a special case of Palmgren’s categorical models. We prove first-order intuitionistic logic to be complete with respect to our sheaf-valued semantics.

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1. Introduction

The title of this paper is not only provocative but also ambiguous. There are two ways to understand the expression “mathematical semantics”, which reflect two different facets of the present work.

1.1. Mathematical “semantics”

In the first reading, “semantics” is understood informally, as a conceptual explanation; and “mathematics” is understood on a par with other fields of study. Traditional ways to understand intuitionistic logic (semantically) have been rooted either in philosophy — with emphasis on the development of knowledge (Brouwer, Heyting, Kripke) or in computer science — with emphasis on effective operations (Kleene, Markov, Curry–Howard). We develop Kolmogorov’s approach [76], which emphasizes the right order of quantifiers, and is rooted in the mathematical tradition, dating back to the Antiquity, of solving of problems (such as geometric construction problems) and in the mathematical ideas of method (of solution or proof) and canonicity (of a mathematical object). The approach of Kolmogorov has enjoyed some limited appreciation in Europe in the 1930s, and had a deeper and more lasting impact on the Moscow school of logic¹ — but nowadays is widely seen as a mere historical curiosity. One important reason for that² could have been the lack of a sufficiently clear formulation and of any comprehensive exposition of intuitionistic logic based on this idea.

The present paper attempts to remedy this defect. Thus we introduce (i) a tiny revision of Kolmogorov’s interpretation³ of intuitionistic logic (§3.9), which attempts to clarify matters from the viewpoint of traditional mathematical intuitions, and to achieve the independence from “any special, e.g. intuitionistic, epistemological presuppositions” that Kolmogorov has announced, but arguably not quite achieved in his own paper; (ii) more substantially, an extension and a modification of the clarified Kolmogorov interpretation (§4.7 and §4.8), which additionally include an interpretation of the syntactic consequence relation, and are compatible with several classes of models of intuitionistic logic.

¹In the introduction to her survey of logic in the USSR up to 1957, S. A. Yanovskaya emphasizes “The difference of the viewpoints of the ‘Moscow’ school of students and followers of P. S. Novikov and A. N. Kolmogorov and the ‘Leningrad’ school of students of A. A. Markov on the issue of the meaning of non-constructive objects and methods”, where the former viewpoint “consists in allowing in one’s work not only constructive, but also classical methods of mathematics and mathematical logic” [152]. In the West, the principles of the ‘Leningrad’ school are better known as “Russian constructivism” (cf. [145]).

²Besides the lack of an English translation of Kolmogorov’s paper [76] until the 1990s, and the lack of a word in English to express the notion of a “mathematical task” (German: Aufgabe, Russian: задача) as distinguished from the notion of an “open problem” (German: Problem, Russian: проблема). Let us note, in particular, that Kolmogorov himself, when writing in French, took care to mention the German word and so spoke of “problèmes (Aufgaben)” [77].

³To be precise, here by “Kolmogorov’s interpretation” we mean its slightly improved version, whose equivalent in the language of Brouwer and Heyting has come to be known as the “BHK interpretation”.
1.2. “Mathematical” semantics

In the second reading of the title, “semantics” is understood formally, in the sense of (mathematical) models; and “mathematics” is understood in the sense of the outlook of the so-called “working mathematician”, who professes ignorance in the foundations and has little interest in notions motivated by the foundations (such as Heyting algebras or Kripke frames). In this view, “real mathematics” is mostly concerned with notions that have a “universal” significance, as judged by their appeal to general mathematical audiences and their presence in several branches of mathematics, and somewhat disdains more peripheral activities. Even if this standpoint involves too much arrogance and too little concern for diversity, one cannot simply ignore it; if intuitionistic logic has seen much more applications and enthusiasm in computer science and in philosophical logic than in non-foundational mathematics, it could be attributed in part to the scarcity of logicians’ interest in “real mathematical” models of intuitionistic logic.

Our introduction to some basic themes of intuitionistic logic (§5) is in the language of its usual topological models, in which parameterless formulas are interpreted by open subsets of a topological space. These were discovered independently by Stone, Tang and Tarski in the 1930s but largely went out of fashion since the 60s. (However, in the propositional case these models include — by considering finite topological spaces and other Alexandroff spaces — all Kripke models, which have had far better luck with fashion.) A methodological novelty is a motivation of these topological models through a model in “Euler diagrams” of a classical first-order theory extracted from the clauses of Kolmogorov’s interpretation (§5.1).

We then introduce “proof-relevant” topological models (§6) of intuitionistic logic, in which parameterless formulas are interpreted by sheaves of sets over a space $B$; these should not be confused with the usual “sheaf models” (see [9] and references there), in which parameterless formulas are interpreted by open subsets of $B$. Quite expectedly, our models turned out to be a special case of something well-known: the categorical semantics (cf. [8]), and more specifically Palmgren’s models of intuitionistic logic in locally cartesian closed categories with finite sums [11]. Quite unexpectedly, this very simple and natural special case with obvious relevance for the informal semantics of intuitionistic logic does not seem to have been considered per se (apart from what amounts to the $\land, \to$ fragment [7]). We prove completeness of intuitionistic logic with respect to our sheaf-valued models by showing that in a certain rather special case (sheaves over zero-dimensional separable metrizable spaces) they interpret the usual topological models over the same space. Normally (e.g. for sheaves over Euclidean spaces or Alexandroff spaces) this is far from being so. On the other hand, our sheaf-valued models also interpret Medvedev–Skvortsov models (§3.11).

Disclaimer

This unconventional introduction to intuitionistic logic should suffice for the purposes of the author’s papers [1], [2], but as such it does not pretend to be complete or even
finished. It is hoped that a future version will cover certain additional topics and will be accompanied by an unconventional introduction to homotopy type theory.

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2. Motivation: homotopies between proofs

2.1. Hilbert’s 24th Problem

In 1900, Hilbert published (and partly presented in his lecture at the International Congress of Mathematicians in Paris) a list of 23 open problems in mathematics [66]. Shortly before the 100th anniversary of the list, the historian Rüdiger Thiele discovered a would-be 24th problem in Hilbert’s notebooks [138] (for a photocopy see [74] and [139]). We don’t know why Hilbert refrained from publicizing it, but one obvious possibility is that he simply did not see how to state it precisely enough.

“The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs. Given two routes, it is not right to take either of these two or to look for a third; it is necessary to investigate the area lying between the two routes. Attempts at judging the simplicity of a proof are in my examination of syzygies and syzygies between syzygies (see [[65]], Lectures XXXII–XXXIX). The use or the knowledge of a syzygy simplifies in an essential way a proof that a certain identity is true. Because any process of addition [is] an application of the commutative law of addition etc. [and because] this always corresponds to geometric theorems or logical conclusions, one can count these [processes], and, for instance, in proving certain theorems of elementary geometry (the Pythagoras theorem, [theorems] on remarkable points of triangles), one can very well decide which of the proofs is the simplest.”

A rather narrow understanding of the problem, as merely asking for “a criterion of simplicity in mathematical proofs and the development of a proof theory with the power to prove that a given proof is the simplest possible” has became widespread in the literature (see e.g. [14; p. 38], [59], [138]). However, Hilbert’s original text above is unambiguously posing a totally different problem as well (in the two sentences starting with “Quite generally”): to develop a proof theory that can tell whether two given proofs of a theorem are “essentially same”, or homotopic, in the sense that one can derive each from the other (whatever that means). It is this aspect of the 24th problem that we shall be concerned with.

Moreover, Hilbert’s reference to syzygies between syzygies is obviously provoking one to think of a hypothetical space of proofs of a given theorem and of whether it should be acyclic or contractible (so that proofs can be simplified in a canonical way) under a given set of conditions.

Certainly, Hilbert was not alone in being concerned with (formal) questions of this kind. For instance, a sample of attempts by mathematicians other than proof theorists at judging “essential sameness” of certain proofs (of the irrationality of $\sqrt{2}$ and of Gauss’ quadratic reciprocity law) can be found in T. Gowers’ blog [51]. A more systematic study of the question of identity of proofs has taken place in intuitionistic proof theory, starting in the 70s (Prawitz [116], Kreisel [79], Martin-Löf [89]; see a brief survey in [32; §2]) and continued in the context of higher categories and higher $\lambda$-calculus [86], [126], [60], [69]. The more recent approach of homotopy type theory [70] is to a large extent a development of this proof-theoretic tradition. We should also point out some very different approaches to somewhat similar questions, which cannot be discussed here in detail: [85] (and references there), [123], [28], [14], [103; §8.2], [17].

So how to judge if two proofs of a theorem are “essentially same”? Hilbert’s idea that there might be the simplest proof of a theorem $\Theta$ and his mention of the use of variant conditions (and aids) in proving $\Theta$ establish a context where it seems natural to identify

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4Translation by Thiele, who also comments that “a part of the last sentence is not only barely legible in Hilbert’s notebook but also grammatically incorrect”, and points out that “corrections and insertions that Hilbert made in this entry show that he wrote down the problem in haste”.

5Quoted from Wikipedia’s “Hilbert’s 24th problem” (English) as of 11/11/11

6Let us also note the connection between (higher) syzygies and (higher) Martin-Löf identity types. In Voevodsky’s model of Martin-Löf’s type theory, types are interpreted as Kan complexes (see [70]), so one kind of a type is an abelian simplicial group, which by the Dold–Kan correspondence (see [50]) can be identified with a pointed chain complex $C$ of abelian groups: $b \in C_{-1} \xleftarrow{\partial_0} C_0 \xrightarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xrightarrow{\partial_3} \ldots$. Write $t : C$ to mean $t \in \partial_0^{-1}(b)$, and $t = u : C$ to mean that $t : C$ and $u : C$ are homologous relative to $\partial_0(t) = \partial_0(u)$, that is, $t - u \in \partial_1(C_1)$. Then the identity type $Id_C(t, u)$ will be the following pointed chain complex: $t - u \in C_0 \xleftarrow{\partial_1} C_1 \xrightarrow{\partial_2} C_2 \xleftarrow{\partial_3} \ldots$.

7Perhaps we should also note the connection between (higher) identity types and (higher) “homotopical syzygies” (see [88]), which goes back to J. H. C. Whitehead’s classical work on identities among relations (see [3]). The role of the Dold–Kan correspondence is played here by the not so well understood connection (see [104]) between extended group presentations of [24], where higher relators correspond to higher cells of a CW complex, and free simplicial groups, such as the Kan loop group (see [50]) of a simplicial set.
a proof $p$ of $\Theta$ with every its simplification, that is, a proof of $\Theta$ obtained from $p$ by crossing out some redundant steps. Of course, the notion of a “redundant step” (or a step that is not “essentially used” in the proof) needs an accurate definition; we will return to this issue later.

On the other hand, Hilbert’s words (“keep going until you have derived each from the other”) may seem to hint at a formal theory of proofs containing a number of specified elementary transformations between proofs of the same assertion. In this case the relation of homotopy between proofs of the same assertion can be defined as the equivalence relation generated by elementary transformations.

Such formal theories endowed with elementary transformations between proofs are best known in the case of constructive proofs (in particular, the natural deduction system of Gentzen–Prawitz and the corresponding $\lambda$-calculus, see [48]). Before turning to constructive proofs, let us review the situation with usual, non-constructive ones.

2.2. The collapse of non-constructive proofs

A standard theory of derivations based on classical logic is Gentzen’s sequent calculus [49] (see [48]). In particular, the elementary inferences

- $(\text{Cut})$ from $A \lor C$ and $C \rightarrow B$ infer $A \lor B$;
- $(C_R)$ from $A \lor A$ infer $A$;
- $(W_{R1})$ from $A$ infer $A \lor B$;
- $(W_{R2})$ from $A$ infer $B \lor A$;
- $(W_L)$ from $B$ infer $C \rightarrow B$

can be interpreted as instances of the inference rules of sequent calculus. A number of elementary transformations between derivations of the same proposition in sequent calculus can be found in the proof of Gentzen’s Hauptsatz on cut elimination (see [48], [49]). These $\Gamma$-transformations include, in particular, the following $(\Gamma_L)$ and $(\Gamma_R)$.

- $(\Gamma_L)$ Suppose that $q$ is a proof of $C \rightarrow B$. Then the following fragment of a proof:
  - $A$ implies $A \lor C$ by $(W_{R1})$;
  - $C \rightarrow B$ by $q$;
  - hence $A \lor B$ by $(\text{Cut})$

transforms into
  - $A$ implies $A \lor B$ by $(W_{R1})$.

We note that $(\Gamma_L)$ can be seen as a simplifying transformation: it eliminates the proof $q$ of the redundant lemma $C \rightarrow B$ at the cost of replacing one weakening ($A$ implies $A \lor C$) by another ($A$ implies $A \lor B$).

\footnote{Namely, $(\text{Cut})$ can be interpreted as an instance of the cut rule, $(C_R)$ as an instance of the right contraction rule, $(W_{R1})$ and $(W_L)$ as instances of the right and left weakening rules, and $(W_{R2})$ as $(W_{R1})$ followed by an instance of the exchange rule.}
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(\(\Gamma_R\)) Suppose that \(p\) is a proof of \(A \lor C\). Then the following fragment of a proof:

- \(B\) implies \(C \rightarrow B\) by \((W_L)\);
- \(A \lor C\) holds by \(p\);
- hence \(A \lor B\) by \((Cut)\)

transforms into

- \(B\) implies \(A \lor B\) by \((W_R2)\).

The symmetry between \((\Gamma_L)\) and \((\Gamma_R)\) is highlighted in Gentzen’s original notation [49; III.3.113.1-2]. We can see it explicitly by rewriting \(C \rightarrow B\) and \(A \lor C\) respectively as \(B \lor \neg C\) and \(\neg C \rightarrow A\), using tautologies of classical logic.

Let us also consider two additional \(\Lambda\)-transformations:

- \((\Lambda_1)\), \((\Lambda_2)\): The following fragment of a proof can be eliminated:
  - \(A\) implies \(A \lor A\) by \((W_R1)\) or by \((W_R2)\);
  - \(A \lor A\) implies \(A\) by \((C_R)\).

**Theorem 2.1** (Lafont [48; Appendix B.1]; see also [32; §7]). All proofs of any given theorem are equivalent modulo \(\Gamma\)- and \(\Lambda\)-transformations.

Lafont’s own summary of his result is that “classical logic is inconsistent, not from the logical viewpoint ([falsity] is not provable) but from an algorithmic one.”

**Proof.** Let \(p\) and \(q\) be proofs of a theorem \(\Theta\). They can be augmented to yield a proof \(p + W_{R1}\) of \(\Theta \lor \Phi\), where \(\Phi\) is some fixed assertion, and a proof \(q + W_L\) of the implication \(\Phi \rightarrow \Theta\). Combining the latter two proofs, we obtain a proof \((p + W_{R1}) + (q + W_L) + Cut\) of the assertion \(\Theta \lor \Theta\). Hence we get a third proof, \((p + W_{R1}) + (q + W_L) + Cut + C_R\), of \(\Theta\), which we will denote \(p \oplus q\) for brevity.\(^9\)

It remains to show that \(p \oplus q\) can be reduced to \(p\), as well as to \(q\) by \(\Gamma\)- and \(\Lambda\)-transformations. Indeed, \(\Gamma_L\) reduces \(p \oplus q\) to the proof \(p + W_{R1} + C_R\) of \(\Theta\) (which augments \(p\) by saying that \(\Theta\) implies \(\Theta \lor \Theta\), which in turn implies \(\Theta\)). The latter proof is reduced to \(p\) by \(\Lambda_1\). Similarly, \(\Gamma_R\) reduces \(p \oplus q\) to the proof \(q + W_{R2} + C_R\) of \(\Theta\), which in turn is reduced to \(q\) by \(\Lambda_2\). \(\blacksquare\)

It should be noted that, despite Lafont’s theorem, some authors, including Došen [32] (see details in [33]; see also [34] and references there) and Guglielmi [55], [56] (see also [57] and references there) have been able to find inequivalent proofs in the framework of classical logic.

**Remark 2.2.** One issue with \(\Gamma\)-transformations is that \((\Gamma_L)\) does not merely eliminate \(q\), but only does so at the expense of changing a weakening. So it is not immediately clear that the lemma \(C \rightarrow B\) is redundant in the initial proof of \((\Gamma_L)\), in the strict sense of not being used at all in that proof.

\(^9\)A version of this \(\oplus\) is also found in Artemov’s system LP (Logic of Proofs) [5].
To address this concern, let us revisit Lafont’s construction.\(^{10}\) Recall that we were free to choose any assertion as \(\Phi\). We can set \(\Phi\), for instance, to be the negation \(\neg \Theta\) (any weaker assertion, such as \(\Theta \lor \neg \Theta\), would also do). Then \(\Theta \lor \Phi\) holds in classical logic regardless of validity of \(\Theta\). Therefore the step \(W_{R1}\) of the proof \(p \oplus q\) replaces the proved theorem \(\Theta\) with the tautology \(\Theta \lor \Phi\). The effect of this step is that we forget all that we have learned from the proof \(p\). Thus \(p \oplus q\) does not essentially use \(p\), so \(p\) must be redundant even in the stricter sense.

On the other hand, we could take \(\Phi\) to be \(\Theta\) itself (any stronger assertion, for instance \(\Theta \land \neg \Theta\), would also do). Then the step \(W^l\) replaces the proved theorem \(\Theta\) by the tautology \(\Phi \rightarrow \Theta\). The effect of this step is that we forget all that we have learned from \(q\), and we similarly conclude that \(q\) must be redundant. However, it appears that the meaning of the proof \(p \oplus q\) should not vary depending on the strength of \(\Phi\), since for the purposes of this proof we did not need to know anything at all about \(\Phi\).

Our conclusion, that neither \(p\) nor \(q\) is “essentially used” in \(p \oplus q\), shows that the notion of a proof “essentially using” its part cannot be sensibly formalized in the framework of classical logic. It turns out, however, that this notion can be consistently defined once we abandon classical logic in favor of intuitionistic.

### 2.3. About constructive proofs

In contrast to classical logic, where one can sometimes prove that a certain object exists without giving its explicit construction, proofs in intuitionistic logic can be interpreted, at least roughly, as algorithms (the Curry–Howard correspondence, see [48], [132]). For example, if \(S\) is a set, an intuitionistic proof of the assertion “\(S\) is nonempty” can be roughly thought of as a program (executed on a Turing machine, say) that returns a specific element of \(S\). (These issues will be discussed more accurately in §??.)

In writing a computer program, it is not enough to know that some subsidiary computation is possible in principle; for the program to run, every step must be actually implemented. In fact, the value of every particular variable can be chased through all dependent subroutines using a debugger. This is in sharp contrast with classical mathematics, where the particular way that a lemma is proved has absolutely no effect on the proof of the theorem modulo that lemma. In intuitionistic logic, there can be no dependence between statements other than a dependence between their specific proofs. In fact, one can think of a proposition in intuitionistic logic as being identified with the set of its proofs. (We will also return to this issue in §??.)

With the above in mind, although \(\Theta \lor \Phi\) is intuitionistically a tautology as long \(\Phi\) is (for instance, \(\Phi\) can be taken to be \(\Theta \rightarrow \Theta\)), the intuitionistic step \(W_{R1}\), replacing the proved theorem \(\Theta\) with the tautology \(\Theta \lor \Phi\) does not have the effect of forgetting how \(\Theta\) was proved. On the contrary, it has the effect of specifying that \(\Theta \lor \Phi\) is to be proved “via \(\Theta\)” and not “via \(\Phi\)”, and extending a given proof of \(\Theta\) to a specific proof of \(\Theta \lor \Phi\). Since the latter proof does not depend on any proof of \(\Phi\), the intuitionistic step

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\(^{10}\)What follows is only an informal discussion, perhaps raising more issues than it attempts to settle.
Cut promotes it to a proof of $\exists \forall \exists$ that does not depend on any proof assuming $\Phi$ as a premise.

It follows that if $p$ and $q$ are constructive, then $p \oplus q$, understood constructively, essentially uses $p$ and does not essentially use $q$. In agreement with this, the intuitionistic version of the sequent calculus (see [48]) admits only asymmetric interpretations of $p \oplus q$, which can be reduced by the intuitionistic analogues of $\Gamma$- and $\Lambda$-transformations to $p$, but not to $q$.

3. What is intuitionistic logic?

3.1. Semantics of classical logic

In classical logic, the meaning of propositions is determined by answering the question: *When is a given proposition true?* A standard answer to this question is given by Tarski’s definition of truth. The definition is by induction on the number of logical connectives:

- which primitive propositions are true is assumed to be known from context.

For instance, in the context of arithmetic, a proposition of the form $t = s$ (with $t$ and $s$ being arithmetical expressions) is true whenever $t$ and $s$ rewrite to the same numeral.

- $P \lor Q$ is true if $P$ is true or $Q$ is true;
- $\neg P$ is true if $P$ is not true;
- $\exists x R(x)$ is true if $R(x_0)$ is true for some $x_0 \in D$.

Here $P, Q$ are propositions, possibly with parameters, and $R(x)$ is a proposition with parameter $x$ and possibly other parameters. In first-order logic, all parameters are understood to run over a fixed set $D$, which is called the *domain of discourse*.

The other classical connectives $\land, \rightarrow, \forall$ do not need to be mentioned since they are expressible in terms of $\lor, \neg$ and $\exists$. For the record, we get for them, as a consequence of the above:

- $P \land Q$ is true if $P$ is true and $Q$ is true;
- $P \rightarrow Q$ is true if $Q$ is true or $P$ is not true;
- $\forall x R(x)$ is true if $R(x_0)$ is true for all $x_0 \in D$.

We can also understand the latter six clauses the other way round, as an “explanation” of classical connectives in terms of truth of propositions.

In this paper we always assume our meta-logic to be classical; that is, when discussing intuitionistic and other non-classical logics, we always mean *this discussion* to take place in classical logic.

Intuitionistic logic can be approached in two ways.

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11In logic, parameters are usually called “free variables”, and propositions with parameters are usually called “predicates”. We will not use this terminology.
3.2. Truth vs. knowledge of proofs

The tradition initiated by Brouwer in 1907 and continued by his school in Amsterdam (Heyting, Troelstra, et al.) and elsewhere till the present day — perhaps more successfully in computer science than in mathematics — presupposes a deep rethink of the conventional understanding of what mathematics is and what it is not. In the words of Troelstra and van Dalen [145; p. 4]:

“It does not make sense to think of truth or falsity of a mathematical statement independently of our knowledge concerning the statement. A statement is true if we have [a constructive] proof of it, and false if we can show that the assumption that there is a [constructive] proof for the statement leads to a contradiction. For an arbitrary statement we can therefore not assert that it is either true or false.”

Thus, in the tradition of Brouwer and Heyting, propositions are considered to have a different — intuitionistic — meaning, which is determined by answering the question: When do we know a constructive proof of a given proposition? We will return to this question in §3.6, once it will have been restated in §3.3 so as to avoid the terminological clash with the classical meaning of propositions.

Whether the classical meaning of propositions “makes sense” or not is a philosophical question which we will try to avoid — so as to focus on answering mathematical questions. But at the very least, the author is determined to keep good relations with the 99% of mathematicians who are convinced that the classical meaning of propositions does make sense; and so considers the terminology of Brouwer and Heyting, which is in direct conflict with the convictions of the 99% of mathematicians, to be unfortunate. Indeed, a useful terminology would provoke mathematical, and not philosophical questions.

The depth of the rethink of mathematics promoted by the battlefield terminology of Brouwer and Heyting should not be underestimated. In the above quote, we are asked to abstain not only from using the principle of excluded middle (in the form asserting that every statement is either true or false), but also from understanding mathematical objects with the customary mental aid of Platonism (i.e. Plato’s vision of mathematical objects as ideal entities existing independently of our knowledge about them). To be sure, Heyting has stated it boldly enough [62]:

“Faith in transcendental existence, unsupported by concepts, must be rejected as a means of mathematical proof, [...] which] is the reason for doubting the law of excluded middle.”

The present author is no proponent of Platonism, but, on the contrary, feels that properly understood mathematics should not depend on any particular philosophy (for reasons related to Occam’s Razor). Heyting certainly has a point in that when we mathematicians speak of “existence” of mathematical objects, it is not at all clear what we mean by this; the implicit hypothesis that this talk “makes sense” at all can certainly be considered as a religious dogma. However, from the viewpoint of Intuitionism, it is trivial to understand classical mathematics: one just needs to explicitly mention the principle of excluded middle in the hypothesis of every classical theorem, thus transforming it into a conditional assertion.
Moreover, as shown and emphasized by Kolmogorov [75], the use of this principle alone cannot be the source of a contradiction (see §5.6); nor it can lead to a contradiction when combined with the axioms of first-order arithmetic (Gödel–Gentzen, cf. [92]). The same is true of the second-order arithmetic (Kreisel, cf. [27]) and, with some additional effort, also of the set theory ZF (without the axiom of choice) [44], [114]—albeit this requires impredicative intuitionism, which is something that not all constructivists accept [92], [27]. However, in presence of the countable axiom of choice (which suffices for classical analysis) in its standard form, interpreted intuitionistically according to Kolmogorov, the addition of the principle of excluded middle generally increases consistency strength even with respect to impredicative intuitionism [15] (see also [108]). In fact, the issue of precise formulation of the axiom of choice is important here; see [27; 2.2.4] and [91].

This is entirely parallel to the situation within classical mathematics. Some principles of classical set theory, including the continuum hypothesis and the axiom of choice, are internally provable to be consistent with ZF; in particular, their addition to ZF does not increase its consistency strength—or in other words, their use alone cannot be the source of a contradiction. Other principles, like large cardinal axioms or the axiom of determinacy are known or conjectured to increase consistency strength when added to ZF; yet there are various good reasons to believe that resulting systems are still consistent, perhaps not significantly different from the reasons for one’s belief in the consistency of ZF. Thus classical mathematics with or without the axiom of choice should be, from the viewpoint of an intuitionist, on a par with areas of classical mathematics that depend on the said principles as seen by a classical mathematician who does not find these principles to be intuitively justified. Conversely, it would be beneficial to understand intuitionistic logic and its relations with classical logic in a way that is compatible with the plain Platonic intuition arguably shared by most “working mathematicians”.

First attempts at meaning explanations of intuitionistic logic in terms of ordinary practices of classical mathematics can be found in a 1928 paper by Orlov [109; §§6,7] (see also [30]) and in the 1930 and 1931 papers by Heyting [61], [62]. Both Orlov and Heyting gave explanations (see [1; §5.1] for a detailed review)—in terms of provability of propositions (“propositions” and “provability” being understood in a sense compatible with classical logic) which directly anticipate Gödel’s 1933 provability translation—to be reviewed in §5.9. Gödel’s translation certainly yields an accurate explanation in classical terms of the form of intuitionistic logic, as captured by Heyting’s axioms and inference rules. But it misses the spirit of intuitionistic logic (as understood in the present paper), in that it does not explain how one proposition can have essentially different proofs.
3.3. Solutions of problems

The second approach to intuitionistic logic, initiated by Kolmogorov and continued to some extent by his school in Moscow aimed at being less demanding of one’s philosophical preferences. In the words of Kolmogorov [76]:

“On a par with theoretical logic, which systematizes schemes of proofs of theoretical truths, one can systematize schemes of solutions of problems — for example, of geometric construction problems. For instance, similarly to the principle of syllogism we have the following principle here: If we can reduce the task of solving \( c \) to that of \( b \), and the task of solving \( b \) to that of \( a \), then we can also reduce the task of solving of \( c \) to that of \( a \). Upon introducing appropriate notation, one can specify the rules of a formal calculus that yield a symbolic construction of a system of such problem solving schemes. Thus, in addition to theoretical logic, a certain new calculus of problems arises. In this setting there is no need for any special, e.g. intuitionistic, epistemological presuppositions.

The following striking fact holds: The calculus of problems coincides in form with Brouwer’s intuitionistic logic, as recently formalized by Mr. Heyting.

In [the other] section, accepting general intuitionistic presuppositions, we undertake a critical analysis of intuitionistic logic; and observe that it should be replaced with the calculus of problems, since its objects are in reality not theoretical propositions but rather problems.”

In this paper, we use the word *problem* solely in the sense of a request (or desire) to find a construction meeting specified criteria on output and permitted means (as in “chess problem”, “geometric construction problem”, “initial value problem”). With *problem* understood in this sense, Kolmogorov’s interpretation is, in the words of Martin-Löf [90; p. 6],

“very close to programming. ‘\( a \) is a method [of solving the problem (doing the task) \( A \)] can be read as ‘\( a \) is a program ...’. Since programming languages have a formal notation for the program \( a \), but not for \( A \), we complete the sentence with ‘... which meets the specification \( A' \). In Kolmogorov’s interpretation, the word *problem* refers to something to be done and the word *solution* to how to do it.”

As a matter of convention, we will always understand a *solution* of a problem positively, in that a proof that a problem \( \Gamma \) has no solutions is not considered to be a solution of \( \Gamma \). Instead, we consider it to be a solution of a different problem, denoted \( \neg \Gamma \); and the problem of either finding a solution of \( \Gamma \) or proving that one does not exist is denoted \( \Gamma \lor \neg \Gamma \). We say that \( \Gamma \) is *solvable* if \( \Gamma \) has a solution, and *decidable* if \( \Gamma \lor \neg \Gamma \) has a solution.

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12 Most translations quoted in this paper are lightly edited by the author in order to improve syntactic and semantic faithfulness to the original source. On the other hand, we occasionally take the liberty of renaming variables used in quoted text in order to facilitate its understanding in our context.

13 Martin-Löf’s use of the word *program* here appears to be a misprint.
3.4. The Hilbert–Brouwer controversy

A problem of the form $\Gamma \lor \neg\Gamma$ may in general be very hard. There is nothing surprising here, of course, since the law of excluded middle applies to propositions, whereas $\Gamma$ is a problem. Nevertheless, the principle of decidability of every mathematical problem has been stressed forcefully by Hilbert in the preface to his famous list of problems [66]:

“Occasionally it happens that we seek the solution [of a mathematical problem] under insufficient hypotheses or in an incorrect sense, and for this reason do not succeed. The problem then arises: to show the impossibility of the solution under the given hypotheses, or in the sense contemplated. Such proofs of impossibility were effected by the ancients, for instance when they showed that the ratio of the hypotenuse to the side of an isosceles right triangle is irrational. In later mathematics, the question as to the impossibility of certain solutions plays a preeminent part, and we perceive in this way that old and difficult problems, such as the proof of the axiom of parallels, the squaring of the circle, or the solution of equations of the fifth degree by radicals have finally found fully satisfactory and rigorous solutions, although in another sense than that originally intended. It is probably this important fact along with other philosophical reasons that gives rise to the conviction (which every mathematician shares, but which no one has as yet supported by a proof) that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts. Take any definite unsolved problem, such as the question of the irrationality of the Euler–Mascheroni constant $C$, or of the existence of an infinite number of prime numbers of the form $2n + 1$. However unapproachable these problems may seem to us and however helpless we stand before them, we have, nevertheless, the firm conviction that their solution must follow by a finite number of purely logical processes.”

A counterexample to Hilbert’s ascription of his conviction to all mathematicians was soon provided by Brouwer, who himself fiercely opposed it [20; Statement XXI], [21], [22]. Yet one can hardly deny that the task of settlement, in the sense of Hilbert, of mathematical problems (including, indeed, those in Hilbert’s list), i.e., the task of solving problems of the form $\Gamma \lor \neg\Gamma$, is something that we mathematicians all constantly undertake, with great efforts — including those 99% of us who consider the law of excluded middle to be a triviality, which can be used freely whenever the need arises.

We thus come to distinguish the law of excluded middle (for propositions) and the principle of decidability (for problems). Yet the two have been systematically conflated by Heyting (see [62] and [144; p. 235]) and especially by Brouwer; in fact, their conflation was at the very source of Brouwer’s opposition to the law of excluded middle, whose first published record [21] reads:
Firstly, the *syllogism* infers from the embedding of a system $b$ into a system $c$, along with the embedding of a system $a$ into the system $b$, a direct embedding of the system $a$ into the system $c$. This is nothing more than a tautology. Likewise the principle of *non-contradiction* is indisputable: The accomplishment of the embedding of a system $a$ into a system $b$ in a prescribed manner, and the obstruction showing the impossibility of such an embedding, exclude each other.

Now consider the principle of *excluded middle*: It claims that every hypothesis is either true or false; in mathematics this means that for every hypothetical embedding of one system in another, satisfying certain given conditions, we can either accomplish such an embedding by a construction, or we can construct the obstruction showing its impossibility. It follows that the question of the validity of the principle of excluded middle is equivalent to the question whether undecidable mathematical problems can exist. There is no indication of a justification for the occasionally pronounced (see [Hilbert’s problem list](#)) conviction that there exist no undecidable mathematical problems.

This all is remarkably similar to what one could say about the *problem* of “embedding of one system in another” (whatever that means), except for the alleged connection with the law of excluded middle. Twenty years later, Brouwer has even complained that after his repeated declarations that the law of excluded middle is the same as the principle of decidability, Hilbert still fails to recognize their alleged identity [22]. Unfortunately, neither this complaint nor the repeated declarations cited in it seem to bring in any arguments as to why the two principles should be equated (apart from the insistence that they obviously should). In fact, in the opinion of Kolmogorov [78],

“The principle of excluded middle is, according to Brouwer, inapplicable only to judgments of a special kind, in which a theoretical proposition is inseparably linked with construction of the object of the proposition. Therefore, one may suppose that Brouwer’s ideas are actually not at all in contradiction with the traditional logic, which has in fact never dealt with judgments of this kind.”

### 3.5. Problems vs. conjectures

Obviously, Brouwer committed what most mathematicians still do today: he conflated a problem (understood as a request for finding, for instance, a proof of a certain assertion, according to a certain specification of what counts as a proof) with a conjecture (understood as a judgement that a certain assertion is true). Indeed, most of Hilbert’s

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14This is the first occurrence of this term in the text, but it was explained in Brouwer’s dissertation [20; p. 77], where he apparently rejects the notion of a subset defined by a property of its elements in favor of an explicitly constructed subset: “Often it is quite easy to construct inside such a structure, independently of how it originated, new structures, whose elements are taken from the elements of the original structure or systems of these, arranged in a new way, but bearing in mind their original arrangement. The so-called ‘properties’ of a system express the possibility of constructing such new systems having a certain connection with the given system. And it is exactly this *embedding* of new systems *in a given system* that plays an important part in building up mathematics, often in the form of an inquiry into the possibility or impossibility of an embedding satisfying certain conditions, and in the case of possibility into the ways in which it is possible.”
problems consist in proving a conjecture. Let us take, for instance, his first problem, which asks to prove the Continuum Hypothesis (C): *Every infinite subset of \( \mathbb{R} \) is of the same cardinality as \( \mathbb{R} \) or \( \mathbb{N} \).*

Is it the case that C is either true or false? Yes: this is a theorem of ZFC — a direct consequence of the law of excluded middle.

So, is C true? Not really: C cannot be proved in ZFC, as shown by P. Cohen.

So, is C false? Not really: C cannot be disproved in ZFC, as shown by Gödel.

This is why in modern classical logic, *truth* is considered to be a semantic notion, depending on the choice of a two-valued model of the theory (cf. §3.1; *two-valued* means that every formula without parameters is either true or false). For instance, C is true in Gödel’s constructible universe, and false in Cohen’s models of ZFC.

In contrast, the following problem (!C): *Prove C in ZFC* — has no solutions (by Cohen); and the problem (!-C): *Disprove C in ZFC* — also has no solutions (by Gödel). And we do not need to fix a model of ZFC for this to make sense. Thus we should not equate the problem !C with any question pertaining to *truth*. Then, if we want to be precise, Hilbert’s first problem should indeed be seen as a *problem* — arguably, best formalized as !C — and not a *yes/no question.* But had Hilbert used a *whether* question in his first problem, it could as well be formalized as the problem !C ∨ !-C, i.e., *Prove or disprove C in ZFC.* Note, incidentally, that insolubility of the latter problem (which follows from Gödel *and* Cohen) does not yield a counterexample to Hilbert’s principle of decidability of all mathematical problems. Indeed, !C is decidable (by Cohen), !-C is decidable (by Gödel), and even !C ∨ !-C is decidable (by Gödel & Cohen). But the decidability of these problems is not, of course, merely a consequence of the law of excluded middle.

In general, a problem — as understood in the present paper — should not be conflated with any yes/no question. In particular, the problem *Prove P,* where P is a proposition, is closely related to two distinct questions: *Is P true?* and *Does there exist a proof of P?* Truth should not be conflated with provability, since by Gödel’s incompleteness theorem most theories are not complete with respect to any model. On the other hand, if G and H are groups, the question *Is G isomorphic to H?* is closely related to two distinct problems: *Prove that G is isomorphic to H* and *Find an isomorphism between G and H.* Proofs should not be conflated with constructions of isomorphisms: one proof might correspond to several isomorphisms or not correspond to any specific isomorphism.

\(^{15}\)This is even reasonably consistent with Hilbert’s original wording [66]: “The investigations of Cantor on such assemblages of points suggest a very plausible theorem, which nevertheless, in spite of the most strenuous efforts, no one has succeeded in proving. This is the theorem: Every system of infinitely many real numbers, i.e., every assemblage of numbers (or points), is either equivalent to the assemblage of natural integers, 1, 2, 3, . . . or to the assemblage of all real numbers”. Alongside the Continuum Hypothesis, Hilbert also poses the problem of well-ordering of the continuum: “It appears to me most desirable to obtain a direct proof of this remarkable statement of Cantor’s, perhaps by actually giving an arrangement of numbers such that in every partial system a first number can be pointed out.”
3.6. The BHK interpretation

The mathematical meaning of problems is determined by answering the question: What does it mean to find a solution to a given problem? A standard but informal answer to this question was given essentially in Kolmogorov’s 1932 paper [76] and can be also seen as an interpretation of the intuitionistic connectives in terms of solutions of problems. In fact, Kolmogorov’s paper was the first one to give a systematic interpretation of the intuitionistic connectives. (An interpretation of $\exists$ in a slightly different language is found in the other section of Kolmogorov’s paper.) Independently of Kolmogorov, Heyting gave a rather similar interpretation of some intuitionistic connectives in terms of proofs of propositions ($\vdash$, $\neg$, $\lor$ in [62] and $\rightarrow$ in a 1930 letter to Freudenthal [150], [147] and in print in a 1934 book [63]).

A formal answer to the question What does it mean to find a solution to a given problem? (or What does it mean to give a constructive proof to a given proposition?) was given, in particular, by Gentzen’s 1935 natural deduction calculus [49]. (An independent but essentially equivalent answer is given by the $\lambda$-calculus, see [48], [71], [84], [27].) This answer can of course be read syntactically, on a par with, say, the sequent calculus for classical logic. Yet the introduction rules of the natural deduction calculus can be also read as a semantic interpretation of the intuitionistic connectives (excluding $\vdash$), which turns out to be closely related to Kolmogorov’s and Heyting’s interpretations (see [41]).

Initially, Heyting has switched to Kolmogorov’s language; in particular, the introduction into intuitionistic logic found in his 1934 book [63] is based on Kolmogorov’s understanding of the connectives (apart from $\vdash$, which Heyting interpreted differently). More influential, however, was his later return to the ideologically loaded language of Brouwer, most notably in his 1956 book [64]. This in turn inspired the modern tradition of the so-called “BHK interpretation”, named so by Troelstra after Brouwer, Heyting and Kreisel [140], with “K” later reassigned to Kolmogorov by Troelstra and van Dalen [145]. It is also known as the “standard interpretation”, and even the “intended interpretation” of intuitionistic logic (see [142], [132], [134], [48], [148], [120], [151]). Compared to Kolmogorov’s original interpretation, the BHK interpretation (i) omits Kolmogorov’s interpretation of $\vdash$; (ii) treats the quantifiers in a more systematic way; (iii) gives a more detailed interpretation of $\rightarrow$ due to Heyting (which is similar to Kolmogorov’s interpretation of $\forall$). We will return to the issue of interpretation of $\vdash$ in §3.9 and revisit it in §4.8.

Most presentations of the BHK interpretation are in Brouwer’s language, but here it is in the language of Kolmogorov.

• what are solutions of primitive problems is assumed to be known from context.

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16A note inserted at proof-reading in Kolmogorov’s paper [76] acknowledged the connection with Heyting’s [62]. A remark found in the body of [76]: “This explanation of the meaning of the sign $\vdash$ is very different from that by Heyting, although it leads to the same formal calculus” must refer to Heyting’s syntactic interpretation of $\vdash$ and indicates that at the time of writing, Kolmogorov was unaware of Heyting’s semantic interpretation of $\vdash$ in [62], which is similar to Kolmogorov’s.
For instance, Euclid’s first three postulates are the following primitive problems:\textsuperscript{17}

1. draw a straight line segment from a given point to a given point;
2. extend any given straight line segment continuously to a longer one;
3. draw a circle with a given center and a given radius.

We may thus stipulate that each of the primitive problems (1) and (3) has a unique solution, and give some description of the possible solutions of (2).

- a solution of $\Gamma \land \Delta$ consists of a solution of $\Gamma$ and a solution of $\Delta$;
- a solution of $\Gamma \lor \Delta$ consists of an explicit choice between $\Gamma$ and $\Delta$ along with a solution of the chosen problem;
- a solution of $\Gamma \rightarrow \Delta$ is a reduction of $\Delta$ to $\Gamma$; that is, a general method of solving $\Delta$ on the basis of any given solution of $\Gamma$.
- the absurdity $\bot$ has no solutions; $\neg \Gamma$ is an abbreviation for $\Gamma \rightarrow \bot$.
- a solution of $\exists x \Gamma(x)$ is a solution of $\Gamma(x_0)$ for some explicitly chosen $x_0 \in D$;
- a solution of $\forall x \Gamma(x)$ is a general method of solving $\Gamma(x_0)$ for all $x_0 \in D$.

Kolmogorov’s notion of a “general method” roughly corresponds to the notion of a “construction” advocated by Brouwer and Heyting, but is perhaps less rhetorical in that it puts a central emphasis on the tangible issue of the right order of quantifiers. According to Kolmogorov, if $\Gamma(X)$ is a problem depending on the parameter $X$ “of any sort”, then “to present a general method of solving $\Gamma(X)$ for every particular value of $X$” should be understood as “to be able to solve $\Gamma(X_0)$ for every given specific value of $X_0$ of the variable $X$ by a finite sequence of steps, known in advance (i.e. before the choice of $X_0$)”. This does not pretend to be a fully unambiguous definition, but “hopefully cannot lead to confusion in specific areas of mathematics”.

3.7. Understanding the connectives

The seemingly innocuous interpretations of $\lor$ and $\exists$ are already in sharp contrast with classical logic. Let us consider the following classical proof that $x^y$ can be rational for irrational $x$ and $y$: if $\sqrt{2}^{\sqrt{2}}$ is rational, then $x = y = \sqrt{2}$ will do; else let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Although we have proved that the proposition $P(x, y) = \text{"}x^y\text{ is rational\}"$ holds either for $(x, y) = (\sqrt{2}, \sqrt{2})$ or for $(x, y) = (\sqrt{2}^{\sqrt{2}}, \sqrt{2})$, this method is not going to prove it for any specific pair. Thus if $!Q$ denotes the problem Prove the proposition $Q$, then we have solved the problem $!\exists x \exists y P(x, y)$ (with $x$ and $y$ ranging over all irrational numbers) but we have no clue about solving the problem $\exists x \exists y !P(x, y)$.\textsuperscript{18}

An important clarification to the BHK interpretation, emphasized by Kreisel (with a focus on the $\rightarrow$ and $\forall$ clauses; see details and further discussion in [29] and [13; §5.1]), is that a solution of a problem must not only consist of a construction (such as a general method) of a prescribed type, but also include a proof that the proposed

\textsuperscript{17}See [2; §1, §4.3] for a thorough discussion of postulates vs. axioms.

\textsuperscript{18}Irrationality of $\sqrt{2}^{\sqrt{2}}$ follows from the Gelfond–Schneider theorem (which answered a part of Hilbert’s 7th problem): $a^b$ is transcendental if $a$ and $b$ are algebraic, $a \neq 0, 1$, and $b$ is not rational.
construction does solve the problem. In fact, this point is found already in Proclus’ ancient commentary on Euclid’s Elements (see [2]). The significance of this provision is illustrated by the following Schwichtenberg’s paradox. Let us consider the problem \( P(x, y, z, n) \) of proving the following proposition \( P(x, y, z, n) \):

\[
x^n + y^n = z^n \rightarrow n \leq 2.
\]

Of course, it is trivial to devise a general method \( m \) that will verify the inequality \( x^n + y^n \neq z^n \) and thus solve \( P(x, y, z, n) \) for every particular choice of positive integers \( x, y, z \) and \( n > 2 \) for which it can be solved. What is hard is to prove that \( m \) actually succeeds on all inputs. Thus it is only due to Kreisel’s clarification that the problem \( \forall x \forall y \forall z \forall n \ P(x, y, z, n) \) (with \( x, y, z, n \) ranging over all positive integers) is nontrivial. In fact, due to the existence of \( m \), the latter problem is equivalent to the problem \( \forall x \forall y \forall z \forall n \ P(x, y, z, n) \) of proving Fermat’s last theorem.\(^{19}\)

Let us now discuss the meaning of the negation, \( \neg \Gamma \). Let \( \neg \Gamma \) denote the proposition: There exists a solution of \( \Gamma \). Using \( ! \) as before, and writing \( \neg \) also for the classical negation of propositions, we get the problem \( ! \neg \neg \Gamma \), which reads: Prove that \( \Gamma \) has no solutions. (As before, this refers to proofs in classical logic.) We will now argue that this problem is equivalent to \( \neg \Gamma \) on the BHK interpretation (with our classical meta-logic), by describing solutions of the problems \( \neg \Gamma \rightarrow ! \neg \neg \Gamma \) and \( ! \neg \neg \Gamma \rightarrow \neg \neg \Gamma \).

Indeed, suppose that we have a solution of \( \neg \Gamma \), that is, a general method \( m \) of obtaining a solution of \( \bot \) based on any given solution of \( \Gamma \), including a proof that this method works. But since \( \bot \) has no solutions, we get the following proof \( p_m \) that \( \Gamma \) has no solutions: “If \( s \) is a solution of \( \Gamma \), then \( m \) applied to \( s \) yields a non-existent object, which is a contradiction; thus \( \Gamma \) has no solutions.” Moreover, it is clear that the procedure associating \( p_m \) to \( m \) is a general method, which works regardless of the nature of \( m \).

Conversely, let \( p \) be a proof that \( \Gamma \) has no solutions. If \( s \) is a solution of \( \Gamma \), then \( p \) yields a proof that \( s \) is itself also a solution of \( \bot \); indeed, \( p \) proves that \( s \) does not exist, but everything is true of a non-existent object. Thus, given a \( p \), we get a general method \( m_p \) of obtaining a solution of \( \bot \) on the basis of any given solution \( s \) of \( \Gamma \); namely, \( m_p \) returns \( s \) itself and includes a verification that \( s \) is a solution of \( \bot \) (which is given by \( p \)). Thus \( m_p \) is a solution of \( \neg \Gamma \). Moreover, it is clear that the procedure associating \( m_p \) to \( p \) is a general method, which works regardless of the nature of \( p \).

Curiously, this analysis is at odds with Kolmogorov’s remark [76]:

“Let us note that \( \neg \Gamma \) should not be understood as the problem ‘prove the unsolvability of \( \Gamma \)’. In general, if one considers ‘unsolvability of \( \Gamma \)’ as a well-defined notion, then one only obtains the theorem that \( \neg \Gamma \) implies the unsolvability of \( \Gamma \), but not vice versa. If, for example, it were proved that the well-ordering of the continuum surpasses our capabilities, one could still not claim that the existence of such a well-ordering leads to a contradiction.”

\(^{19}\)A similar remark is found in Kolmogorov’s first letter to Heyting (see [1; §5.2]), which indicates that he did implicitly assume general methods to include verification of their own correctness and seems to have been aware of the significance of this provision. In his paper [76], he also emphasized universal acceptability of the validity of solutions as an inherent property of logical and mathematical problems.
Could one make any sense out of the last sentence, mathematically? It could be referring, for instance, to the fact that the existence of a well-ordering of the continuum might be (i) not provable in some formal theory $F$ (such as ZF), yet (ii) consistent with $F$. (For example, this is how Kolmogorov’s words are interpreted by Coquand [27].) If (i) is to be interpreted as unsolvability of some problem $\Gamma$, then $\Gamma$ would have to be the problem of deriving the existence of the well-ordering of the continuum from the axioms of $F$; a solution of $\Gamma$ would be such a derivation, and the proof of the unsolvability of $\Gamma$ would be, naturally, in the meta-theory of $F$. But then $\neg\Gamma$ was solved by Cohen, too: assuming such a derivation, he did get a contradiction (by a method whose correctness is verified, naturally, in the meta-theory of $F$). Of course, in view of (ii), we would not get a contradiction merely from constructing (using tools exceeding those available in $F$) a model of $F$ and a well-ordering of the continuum in this model. But this means only that $\neg\Delta$ has no solution, where $\Delta$ is an entirely different problem, which was solved by Gödel: construct a well-ordering of the continuum in some model of $F$. Thus it is possible that Kolmogorov might have simply conflated $\Gamma$ and $\Delta$, thinking of a single informal problem, “Find a well-ordering of the continuum”.

On the other hand, Kolmogorov’s wording (“$\neg\Gamma$ implies the unsolvability of $\Gamma$”; “the existence of such a well-ordering leads to a contradiction”) certainly conflates implications between problems with implications between propositions; this makes us suspect that in writing of what must have been the former, he might have actually been thinking of the latter (in this particular remark about the well-ordering of the continuum). Indeed, if $C$ is the proposition *The continuum is well-orderable*, then the unsolvability of the problem of proving it, or $\neg?!C$ in the above notation, certainly does not imply $\neg C$, the proposition that the existence of a well-ordering of the continuum leads to a contradiction. In this connection, let us also not forget Kolmogorov’s initial rejection of the explosion principle $\bot \Rightarrow \alpha$ [75], [150].

It should be noted that Heyting, while using Kolmogorov’s terminology (and being informed of Kolmogorov’s remark quoted above) has explicitly identified $\neg\Gamma$ with $!\neg?\Gamma$, in his comment on the axiom scheme $\neg\gamma \rightarrow (\gamma \rightarrow \delta)$ [63]:

“It is appropriate to interpret the notion of ‘reduction’ in such a way that the proof of the impossibility of solving $\Gamma$ at the same time reduces the solution of any problem whatsoever to that of $\Gamma$.”

Two more precautions on interpreting the BHK interpretation:

- The domain $D$ must be a “simple domain” such as the set $\mathbb{N}$ of natural numbers; a subset of $\mathbb{N}$ defined with aid of quantifiers would require a more elaborate version of the BHK interpretation (see [29]). In fact, Martin-Löf’s type theory [90] is based on extending the BHK interpretation to more general domains.
- The status of all primitive problems (i.e. if they are considered to have a solution) is supposed to be fixed before the logical connectives can be explained, and cannot be updated as we learn the meaning of the connectives (Prawitz; see [122]).
3.8. Something is missing here

Troelstra and van Dalen note of the BHK interpretation [145]:

“This explanation is quite informal and rests itself on our understanding of the notion of construction and, implicitly, the notion of mapping; it is not hard to show that, on a very ‘classical’ interpretation of construction and mapping, [the six clauses of the BHK interpretation] justify the principles of two-valued (classical) logic.”

“[Thus] the BHK-interpretation in itself has no ‘explanatory power’: the possibility of recognizing a classically valid logical schema as being constructively unacceptable depends entirely on our interpretation of ‘construction’, ‘function’, ‘operation’.”

Indeed (see also [145; pp. 32–33] and [124]), let $|\Gamma|$ denote the set of solutions of the problem $\Gamma$. Then the BHK interpretation guarantees that:

- $|\Gamma \land \Delta|$ is the product $|\Gamma| \times |\Delta|$;
- $|\Gamma \lor \Delta|$ is the disjoint union $|\Gamma| \sqcup |\Delta|$;
- there is a map $\mathfrak{F}: |\Gamma| \to |\Delta| \to \text{Hom}(|\Gamma|, |\Delta|)$ into the set of all maps;
- $|\bot| = \emptyset$;
- $|\exists x \Gamma(x)|$ is the disjoint union $\bigsqcup_{d \in D} |\Gamma(d)|$;
- there is a map $\mathfrak{G}: |\forall x \Gamma(x)| \to \prod_{d \in D} |\Gamma(d)|$ into the product.

If we force $\mathfrak{F}$ to be the identity map, we obtain what might be called the “classical BHK”. Indeed, $|\Gamma \lor \neg \Gamma| = |\Gamma| \sqcup \text{Hom}(|\Gamma|, \emptyset)$ is never empty; thus $\Gamma \lor \neg \Gamma$ has a solution for each problem $\Gamma$. Note that we get the same result as long as $\mathfrak{F}$ is surjective.

Thus the BHK interpretation alone fails to capture the essence of intuitionistic logic. The way to deal with this issue in the intuitionistic tradition has been, unfortunately, just to sweep it under the carpet. Thus, elsewhere in the same book by Troelstra and van Dalen [145], we find an analysis of “the principle of excluded middle” (or rather, in our terminology, of the principle of decidability) essentially repeating Heyting’s one (in [62]), including a characteristic puzzling appeal to the empirical fact that we don’t yet know whether there are infinitely many twin primes, with a striking conclusion: “Thus we cannot accept [the Principle of Excluded Middle] as a universally valid principle on the BHK-interpretation.” Never mind the apparent contradiction with the previous quote from the same book, where the same BHK interpretation was found to “justify the principles of two-valued (classical) logic”!

The only sign of hope for understanding what is going on here is the clause “as a universally valid principle”, whose meaning is not really explained in either [145] or [62]. (Other intuitionistic literature known to the present author does not seem to be more helpful on this issue, as it traditionally tends to back the apparently mathematical point by apparently philosophical considerations.)

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\[\text{The author suspects that it is this particular sweeping under the carpet that is largely responsible for the fact that most mathematicians are aware of intuitionistic logic, yet fail to understand it.}\]
3.9. Clarified BHK interpretation

If we return for the moment to classical logic, the law of excluded middle is perhaps most clearly expressed as
\[(\forall P) \ P \lor \neg P\]
if we want to emphasize it as a universally valid principle. Here the quantification is over all propositions \(P\), such as “0 \(=\) 1”, “the sum of the angles of every triangle is \(\pi\), and perhaps even “all ravens are black”; and the meaning of the expression \((\forall P) \ P \lor \neg P\) is, in accordance with Tarski’s semantics for \(\forall\), that \(P \lor \neg P\) is true for all propositions \(P\). Of course, there is no hope of making this quantification formal; so some literature, including [145], just ignores it, whereas other literature, including [62], has this written instead:
\[\vdash p \lor \neg p.\]

The modern reading of the latter expression is that it represents the judgement that the schema \(p \lor \neg p\) is derivable from the axiom schemata of classical logic (in fact, it is often taken as one of the axiom schemata). Here \(p\) denotes an arbitrary formula in the language of, say, first-order logic, so \(p \lor \neg p\) is strictly speaking not a single formula but an infinite scheme of formulas; and classical logic is implicit in the sign \(\vdash\), whose more detailed version would be \(\vdash_{CL}\).

If we now fix a model \(M\) of classical logic — for instance, consisting of all propositions about natural numbers (such as \(2 + 2 = 4\) and \(0 = 1\)) — then the law of excluded middle should hold in \(M\), in particular. In symbols,
\[\models p \lor \neg p.\]

The meaning of this expression (where \(M\) is implicit in the sign \(\models\), whose more detailed version would be \(\models_{M}\)) is that it represents the judgement that for every proposition \(P\) about natural numbers, \(P \lor \neg P\) holds. This is getting closer to our original expression \((\forall P) \ P \lor \neg P\) — which can therefore be seen as an informal interpretation of \(\vdash p \lor \neg p\), in the same sense in which \(\models p \lor \neg p\) is its formal interpretation.

Let us now try to do the same with intuitionistic logic and with the principle of decidability. The judgement
\[\vdash \gamma \lor \neg \gamma\]
now says that the schema \(\gamma \lor \neg \gamma\) is derivable from the axioms in intuitionistic logic, where \(\gamma\) again denotes an arbitrary formula of first-order logic. At this point, we are completely ignorant about the axioms of intuitionistic logic, and \(a fortiori\) about its models. But at least we could hope that an informal interpretation of our judgement is
\[\forall \Gamma \ (\Gamma \lor \neg \Gamma).\]

What is this? As an interpretation of a judgement, it should probably be a judgement; after all, we would like to judge whether the principle of decidability holds on our informal interpretation of intuitionistic logic. (Heyting [63] also thinks that a problem preceded by the \(\vdash\) sign should be a judgement and not a problem.) On the other hand, we
can try to emulate the $\forall$ clause in the BHK interpretation, which gives us the problem whose solution is a general method of solving the problem $\Gamma \lor \neg \Gamma$ for every particular problem $\Gamma$. (And this is precisely Kolmogorov's original interpretation\textsuperscript{21} of $\vdash \gamma \lor \neg \gamma$ [76].) As an obvious compromise, we settle for the judgement that there exists a general method of solving the problem $\Gamma \lor \neg \Gamma$ for every particular problem $\Gamma$ as our informal interpretation of $\vdash \gamma \lor \neg \gamma$. (And this is reminiscent of Brouwer’s “Third insight. The identification of the principle of excluded middle with the principle of decidability of every mathematical problem.”\textsuperscript{22} [22; §1.]) In general, our informal interpretation of $\vdash \Sigma(\gamma_1, \ldots, \gamma_n)$, where each $\gamma_i$ stands for a formula of first-order logic, possibly with parameters, and $\Sigma(\gamma_1, \ldots, \gamma_n)$ is a schema in the metavariables $\gamma_i$, will be

the judgement that there exists a general method of solving the problem $\Sigma(\Gamma_1, \ldots, \Gamma_n)$ for every particular problems $\Gamma_1, \ldots, \Gamma_n$ of appropriate arities.

To fix a name, we will call this the clarified BHK interpretation of intuitionistic logic, where the ordinary intuitionistic connectives $\land$, $\lor$, $\rightarrow$, $\bot$, $\exists$, $\forall$ are interpreted according to the usual BHK interpretation (as in §3.6). This suffices in the case where the problem $\Sigma(\Gamma_1, \ldots, \Gamma_n)$ has no parameters (i.e., “free variables”); and if it does, we simply replace it with its universally quantified parameterless closure. For instance, we interpret the parametric problem $\Gamma(s) \rightarrow \exists t \Gamma(t)$ according to usual the BHK interpretation of the parameterless problem $\forall s (\Gamma(s) \rightarrow \exists t \Gamma(t))$.

How does this interpretation fit with our practical intuition about problem solving? Suppose $\Gamma$ asks to divide an angle of $\frac{\pi}{3}$ into three equal parts with (unmarked) ruler and compass. The problem $\Gamma \lor \neg \Gamma$: \textit{Trisect an angle of $\frac{\pi}{3}$ or show that it is impossible to do so} is well-known from the history to be highly non-trivial (even though a solution

\textsuperscript{21}According to Kolmogorov [76], the problem $\vdash \alpha \lor \neg \alpha$ “reads as follows: to give a general method that for every problem $\alpha$ either finds a solution of $\alpha$ or deduces a contradiction on the basis of such a solution! In particular, when the problem $\alpha$ consists in proving a proposition, one must possess a general method to either prove or reduce to a contradiction any proposition. If the reader does not consider himself to be omniscient, he will probably determine that $\vdash \alpha \lor \neg \alpha$ cannot be on the list of problems solved by him.” Let us note that Kolmogorov effectively appeals to some assumption regarding the knowledge of the reader here. Elsewhere in [76] he introduces a list of axioms of intuitionistic logic by stating that “we must assume that the solutions of some elementary problems are already known.” Thus Kolmogorov’s announcement that in his setting “there is no need for any special, e.g. intuitionistic, epistemological presuppositions” (cf. his quote in §3.3) does not seem to be accurate. In fact, his assumption that solutions of certain problems beginning with $\vdash \alpha$ are already known is not so different from Heyting’s epistemological interpretation of $\vdash \alpha$ [61]: “To satisfy the intuitionistic demands, the assertion must be the observation of an empirical fact, that is, of the realization of the expectation expressed by the proposition $p$. Here, then, is the Brouwerian assertion of $p$: It is known how to prove $p$. We will denote this by $\vdash p$. The words ‘to prove’ must be taken in the sense of ‘to prove by construction’.”

\textsuperscript{22}Brouwer’s four “insights” intended to summarize his work in intuitionism; the third is the shortest, and essentially reiterates after 20 years the identification discussed in §3.5 — except that this “Third insight” taken alone can be seen as a definition of the principle of excluded middle. The principle in question, originally formulated by Hilbert (see §3.3), hardly intended to assert a general method for settlement of all mathematical problems. However, it is likely that Brouwer, with his philosophy that all mathematical statements should be read constructively, did actually mean this principle to assert a general method — at least in the later paper [22]. See, for instance, footnote 6 in [22], which guarantees this sort of reading for another principle.
is not so hard to describe once you know it). In contrast, a problem such as $\Gamma \rightarrow \Gamma \land \Gamma$ is clearly trivial. It seems, indeed, that what makes the latter problem trivial is the existence of a general method$^{23}$ of solving this problem that works regardless of the content of the problem $\Gamma$; and what makes the former problem hard is precisely the apparent lack of such a general method. But if intuitionistic logic is to be the logic of schemes of solutions of problems, as Kolmogorov put it, then clearly such schemes must be precisely the general methods of solving composite problems that work regardless of the content of their constituent elementary problems. For example, the solutions of the problems $\neg\Gamma \rightarrow \neg\neg\neg\neg\Gamma$ and $\neg\neg\neg\neg\neg\rightarrow \neg\Gamma$ described above are general methods that work regardless of the content of $\Gamma$; however this does not give rise to an intuitionistic validity since $\neg$ and $\neg$ are not intuitionistic connectives. (Many intuitionistic validities will be explained by the clarified BHK interpretation in §3.12.)

3.10. The principle of decidability

So what about the principle of decidability, then? Does there exist a general method$^{24}$ of solving the problem $\Gamma \lor \neg\Gamma$ for every particular problem $\Gamma$? If $G_p$ is a group given by a finite presentation $p$ (i.e. a finite list of generators and relations), let $\Gamma_p$ be the problem, Find an isomorphism between $G_p$ and the trivial group $1$. Then from any solution of $\Gamma_p$ we can extract a proof that $G_p \simeq 1$ (by Kreisel’s clarification), whereas from any solution of $\neg\Gamma_p$ we can extract a proof that $G_p \not\simeq 1$ (by our solution of $\neg\Gamma_p \rightarrow \neg\neg\neg\neg\Gamma_p$).

Hence any solution of $\Gamma_p \lor \neg\Gamma_p$ would tell us, in particular, whether $G_p$ is isomorphic to $1$ or not (since any solution of $\Gamma \lor \Delta$ involves, in the first place, an explicit choice between $\Gamma$ and $\Delta$).

Now if we have a general method of solving all problems of the form $\Gamma \lor \neg\Gamma$, then in particular we have a general method of solving the problem $\Gamma_p \lor \neg\Gamma_p$ for all values of $p$. If the latter general method is interpreted as an algorithm (in the sense of Turing machines) with input $p$ (which is perfectly consistent with Kolmogorov’s explanation of a general method), then such a general method would yield an algorithm deciding whether $G_p$ is isomorphic to $1$. But it is well-known that there exists no such algorithm.$^{25}$ This shows that the principle of decidability is not valid under the clarified BHK interpretation:

$\not\vdash \gamma \lor \neg\gamma$.

---

$^{23}$An obvious one: take the given solution $s$ of $\Gamma$ and produce from it the solution $(s, s)$ of $\Gamma \land \Gamma$.

$^{24}$One can also wonder about Hilbert’s conviction, that $\Gamma \lor \neg\Gamma$ admits a solution (possibly not by a general method) for every particular problem $\Gamma$. On the clarified BHK interpretation, this judgement falls outside of the scope of intuitionistic logic. Indeed, in the above notation it can be expressed as $\vdash ?(\gamma \lor \neg\gamma)$, where $?$ takes us out of intuitionistic logic. An extension of both intuitionistic and classical logic, which includes the connectives $?$ and $\neg$ (and no further connectives) is studied in the author’s papers [1], [2]. Hilbert’s conviction amounts to an independent principle in this logic [1; §2.7].

$^{25}$The geometrically-minded reader might prefer the modification of this example based of S. P. Novikov’s theorem (improving on an earlier result of A. A. Markov, Jr.), that for a certain sequence of finite simplicial complexes $K_1, K_2, \ldots$ there exists no algorithm to decide, for any positive integer input $n$, whether $K_n$ is piecewise-linearly homeomorphic to the 5-dimensional sphere (see [25]).
In §3.11 we will review a much more elementary example, which shows incidentally that it is not really necessary to interpret general methods constructively, as algorithms.

Let us pause to note that we have not established validity, under the clarified BHK interpretation, of the negation of the principle of decidability. For that we would need a demonstration (=a general method to prove) that each problem of the form $\Gamma \lor \neg \Gamma$ has no solutions. But there exists no such demonstration, since, in fact, any problem of the form $(\Delta \to \Delta) \lor \neg(\Delta \to \Delta)$ does have a solution. Thus we see, quite trivially, that

$$\not\vdash \neg(\gamma \lor \neg \gamma)$$

on the clarified BHK interpretation. In fact, a stronger assertion turns out to hold:

$$\not\vdash \neg\neg(\gamma \lor \neg \gamma);$$

thus on the clarified BHK interpretation, there exists a demonstration (we will describe it explicitly in §3.12, (26)) that for each problem $\Gamma$ there exists no proof that $\Gamma \lor \neg \Gamma$ has no solutions. Yet on the other hand, we have just seen a proof that the problem $\forall \gamma (\Gamma, \neg \Gamma)$ has no solutions. In particular, our former argument works also to establish

$$\not\vdash \neg\neg \forall x (\gamma(x) \lor \neg \gamma(x))$$

on the clarified BHK interpretation; this was originally observed by Brouwer, who presented “counterexamples to the freedom from contradiction of the Multiple Principle of Excluded Middle of the second kind” [22; §2], and now is better known as independence of the Double Negation Shift principle (see §5.7).

3.11. Medvedev–Skvortsov problems

Let us fix a set $X$, and consider the class of problems $\Gamma_f$ of the form: Solve the equation $f(x) = 0$, where $f : X \to \{0, 1\}$ is an arbitrary (set-theoretic) function. Thus a solution of $\Gamma_f$ is any $x \in X$ such that $f(x) = 0$. Of course, $f$ is determined by the pair of sets $(X, \Gamma_f)$, where $\Gamma_f = f^{-1}(0)$ is the set of solutions of $\Gamma_f$. Thus we may write $[X, Y]$ to denote the problem $\Gamma_f$, where $f : X \to \{0, 1\}$ is such that $f^{-1}(0) = Y$. It is this language of pairs that was used by Medvedev [97], [98], Skvortsov [129] and Läuchli [84]. If $D$ is a fixed set (the domain of discourse), we may also consider parametric problems of the form $[X, Y_{d_1, \ldots, d_n}]$ where $n \in \mathbb{N}$ and each $d_i \in D$. Such problems (including those with $n = 0$) will be called Medvedev–Skvortsov problems with domain $X$.

Let us write $\Gamma = \Delta$ if the problems $\Gamma$ and $\Delta$ have the same set of solutions. (In fact, in the following clauses they will have the same “initial data” as well, except that it is not really clear what is the “initial data” of $\bot$.) Then it is not hard to see that:

- $[X, Y] \lor [X', Y'] = [X \cup X', Y \cup Y']$;
- $[X, Y] \land [X', Y'] = [X \times X', Y \times Y']$;
- $[X, Y] \to [X', Y'] = [\text{Hom}(X, X'), \{f : X \to X' | f(Y) \subseteq Y\}]$;
- $[X, \emptyset] = \bot$;
- $\exists[X, Y] = [D \times X, \{(d, x) \in D \times X | x \in Y_d\}]$;
- $\forall[X, Y] = [\text{Hom}(D, X), \{f : D \to X \mid \forall d \in D f(d) \in Y_d\}]$. 


It follows that Medvedev–Skortsov problems satisfy the BHK interpretation (with $\mathcal{O}$ being the identity, and $\mathfrak{F}$ being surjective).

Now let us fix a set $S$, and let us regard Medvedev–Skvortsov problems with domain $S$ as the primitive problems. Given a formula $\Sigma(\gamma_1, \ldots, \gamma_n)$ in the metavariables $\gamma_i$, for any primitive problems $[S, T_1], \ldots, [S, T_n]$ of appropriate arities, the problem $\Sigma([S, T_1], \ldots, [S, T_n])$ is equal (in the above sense) to a Medvedev–Skvortsov problem $[X, Y]$, where $X$ depends only on $\Sigma$, and $Y$ additionally depends on $T_1, \ldots, T_n$. We now fix $\Sigma$ and hence $X$, and write $Y = Y_{T_1, \ldots, T_n}$. A solution of the problem $[X, Y_{T_1, \ldots, T_n}]$ is an element of $Y_{T_1, \ldots, T_n} \subseteq X$; in particular, all solutions of all problems of the form $\Sigma([S, T_1], \ldots, [S, T_n])$ are elements of $X$. If a solution of the problem $[X, Y_{T_1, \ldots, T_n}]$ is given by a general method that works for arbitrary $T_1, \ldots, T_n \subseteq S$, then it should belong to $Y_{T_1, \ldots, T_n}$ regardless of the choice of $T_1, \ldots, T_n$; in other words, it should belong to the intersection of all subsets $Y_{T_1, \ldots, T_n} \subseteq X$ where each $T_i(t_1, \ldots, t_n)$ runs, for any fixed $t_1, \ldots, t_n$, over all subsets of $S$.

Now suppose that there exists a general method of solving all problems of the form $[S, T] \lor \lnot [S, T]$.

Then a solution given by this general method belongs to the intersection of all $Y_T \subseteq X = S \sqcup \text{Hom}(S, S)$, where $T$ runs over all subsets of $S$. Now $Y_S = S \sqcup \emptyset$ and $Y_{\emptyset} = \emptyset \sqcup \text{Hom}(S, S)$. Their intersection is empty, which is a contradiction. Thus the principle of decidability does not hold for Medvedev–Skvortsov problems on (a reasonable reading of) the clarified BHK interpretation.

### 3.12. Some intuitionistic validities

The point of the BHK interpretation is that it provides a reasonable explanation of intuitionistic validities, without relying on any formal system of axioms and inference rules. Although this explanation is highly informal, it works — and is often more helpful than any formal calculus if one needs to quickly verify whether a given formula is an intuitionistic validity.\(^{26}\)

In what follows, we abbreviate $\vdash \alpha \Rightarrow \beta$ by $\alpha \Rightarrow \beta$. We also write $\alpha \leftrightarrow \beta$ to mean $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$; and $\alpha \leftrightarrow \beta$ to mean $\vdash \alpha \leftrightarrow \beta$. We may write e.g. $\alpha \leftrightarrow \beta \Rightarrow \gamma$ to mean "$\alpha \leftrightarrow \beta$ and $\beta \Rightarrow \gamma$", etc.

The order of precedence of connectives is (in groups of equal priority, starting with higher priority/stronger binding): 1) $\lnot$; 2) $\exists$ and $\forall$; 2) $\land$ and $\lor$; 3) $\rightarrow$ and $\leftrightarrow$; 4) $\Rightarrow$ and

---

\(^{26}\)Why do these informal BHK-arguments work? The author believes that they do just because they are in fact sketches of proofs that the schemata in question are satisfied in the sheaf-valued models of §6. Intuitionistic logic is shown to be complete with respect to this class of models in §6, so by formalizing these sketches one would indeed establish that the schemata in question are intuitionistic validities. Alternatively, one can interpret the informal BHK-arguments as textual representations of $\lambda$-terms, or, equivalently, as sketches of proofs that the schemata in question are satisfied in Läuchli’s models [84], [71]. In fact, there seems to be no significant difference between the two alternatives (see [7]).
We will omit brackets according to this convention only when the order of precedence anyway seems clear from context.

Yet another convention: whenever a formula $\varphi$ that displays no explicit dependence on a parameter $x$ occurs under the scope of a quantifier over $x$ (for instance, as in $\forall x(\varphi \lor \psi(x))$), then it is tacitly assumed that $\varphi$ indeed does not depend on $x$ (i.e., $x$ is not a “free variable” in $\varphi$); this agrees with the more general conventions of §4.1.

(1) $\gamma \Rightarrow \neg \neg \gamma$

Given a problem $\Gamma$ and a solution $s$ of $\Gamma$, we need to produce a solution of $\neg \neg \Gamma$ by a general method. Indeed, given a solution of $\neg \Gamma$, that is, a method turning solutions of $\Gamma$ into solutions of $\bot$, we simply apply this method to $s$ and get a contradiction.

(2) (contrapositive) $\gamma \rightarrow \delta \implies \neg \delta \rightarrow \neg \gamma$

Indeed, from $\Gamma \rightarrow \Delta$ and $\Delta \rightarrow \bot$ we infer $\Gamma \rightarrow \bot$.

(3) $\neg \gamma \iff \neg \neg \neg \gamma$

This follows from (1): $\rightarrow$ by substitution, and $\leftarrow$ via (2).

(4) (explosion) $\bot \Rightarrow \gamma$

If $s$ is a solution of $\bot$, then $s$ does not exist; in particular, $s$ is itself also a solution of any given problem $\Gamma$.

(5) $\gamma \lor \neg \delta \implies \delta \rightarrow \gamma$

Given a solution of $\Gamma \lor \neg \Delta$ and a solution of $\Delta$, we get a solution of $\Gamma \lor \bot$, hence a solution of $\Gamma \lor \Gamma$. This yields a solution of $\Gamma$ by considering two cases.

(6) (decidability implies stability) $\gamma \lor \neg \gamma \implies \neg \neg \gamma \rightarrow \gamma$

This follows from (5) with $\delta = \neg \neg \gamma$ using (3).

(7) $\exists x \forall y \gamma(x, y) \implies \forall y \exists x \gamma(x, y)$

Indeed, if we found an $x_0$ and a method to turn every $y$ into a solution of $\Gamma(x_0, y)$, then we have a method to produce for every $y$ an $x$ and a solution of $\Gamma(x, y)$.

Here is a good place to note that, like in classical logic, $\exists$ and $\forall$ can be thought of as generalizations of $\lor$ and $\land$. Thus similarly to (7) we get

(8) $\exists x (\gamma(x) \land \delta(x)) \implies \exists x \gamma(x) \land \exists x \delta(x)$;
(9) $\forall y (\gamma(y) \lor \forall y \delta(y)) \implies \forall y (\gamma(y) \lor \delta(y))$.

Intuitionistic logic features an additional connection: $\exists$ and $\forall$ behave as if they were generalizations of $\land$, and $\rightarrow$, respectively. (This connection is made precise in dependent type theory, where no distinction is made between the domain of a variable, like $D$ is the BHK clauses for $\exists$ and $\forall$, and the set of solutions of a problem, like that of $\Gamma$ in the BHK clauses for $\land$ and $\rightarrow$.) Thus similarly to (7) we also get

(10) $\exists x (\vartheta \rightarrow \gamma(x)) \implies \vartheta \rightarrow \exists x \gamma(x)$.

Moreover, similarly to the obvious validities: $\forall x \forall y \gamma(x, y) \iff \forall y \forall x \gamma(x, y)$ and $\exists y \exists x \gamma(x, y) \iff \exists x \exists y \gamma(x, y)$ we get something otherwise not so obvious:

(11) $\forall x (\vartheta \rightarrow \gamma(x)) \iff \vartheta \rightarrow \forall x \gamma(x)$;
(12) $\vartheta \land \exists x \gamma(x) \iff \exists x (\vartheta \land \gamma(x))$. 
In addition, by specializing (9) to the case where \( \delta(y) \) does not depend on \( y \), we get

\[
\tag{13} \vartheta \lor \forall x \gamma(x) \implies \forall x (\vartheta \lor \gamma(x)).
\]

There are two more validities of the above sort. Firstly,

\[
\tag{14} \forall x (\gamma(x) \to \vartheta) \iff \exists x (\gamma(x)) \to \vartheta.
\]

Indeed, a solution of \( \forall x (\Gamma(x) \to \Theta) \) turns each \( x \) into a solution of \( \Gamma(x) \to \Theta \). A solution of \( (\exists x \Gamma(x)) \to \Theta \) turns any given \( x_0 \) and solution of \( \Gamma(x_0) \) into a solution of \( \Theta \). These are obviously reducible to each other.

Secondly,

\[
\tag{15} \exists x (\gamma(x) \to \vartheta) \implies (\forall x \gamma(x)) \to \vartheta.
\]

Indeed, given an \( x_0 \) and a solution of \( \Gamma(x_0) \to \Theta \), and assuming a method of solving \( \Gamma(x) \) for every \( x \), we apply this method to \( x = x_0 \) to get a solution of \( \Gamma(x_0) \) and consequently a solution of \( \Theta \).

\[
\tag{16} \neg \exists x \gamma(x) \iff \forall x \neg \gamma(x)
\]

\[
\tag{17} \exists x \neg \gamma(x) \implies \neg \forall x \gamma(x)
\]

These are special cases of (14) and (15).

\[
\tag{18} \vartheta \land (\gamma \lor \delta) \iff (\vartheta \land \gamma) \lor (\vartheta \land \delta)
\]

\[
\tag{19} \vartheta \lor (\gamma \land \delta) \iff (\vartheta \lor \gamma) \land (\vartheta \lor \delta)
\]

Here (18) and the “\( \Rightarrow \)” in (19) follow similarly to (12) and (13).

The remaining implication says that if we have (i) either a solution of \( \Theta \), or a solution of \( \Gamma \), and (ii) either a solution of \( \Theta \), or a solution of \( \Delta \), then we can produce, by a general method, (iii) either a solution of \( \Theta \), or solutions of \( \Gamma \) and \( \Delta \). Clearly, there is such a general method which proceeds by a finite analysis of cases; in fact, there are two such distinct methods \( m_i \), \( i = 1, 2 \), which in the case that we have two solutions of \( \Theta \) select the \( i \)th one.

\[
\tag{20} (\vartheta \to \gamma) \land (\vartheta \to \delta) \iff \vartheta \to (\gamma \land \delta)
\]

\[
\tag{21} (\vartheta \to \gamma) \lor (\vartheta \to \delta) \implies \vartheta \to (\gamma \lor \delta)
\]

These follow similarly to (11) and (9).

\[
\tag{22} (\gamma \to \vartheta) \land (\delta \to \vartheta) \iff (\gamma \lor \delta) \to \vartheta
\]

\[
\tag{23} (\gamma \to \vartheta) \lor (\delta \to \vartheta) \implies (\gamma \land \delta) \to \vartheta
\]

These follow similarly to (14) and (15).

\[
\tag{24} (\text{de Morgan law}) \neg \gamma \land \neg \delta \iff \neg (\gamma \lor \delta)
\]

\[
\tag{25} (\text{de Morgan law}) \neg \gamma \lor \neg \delta \implies \neg (\gamma \land \delta)
\]

These are special cases of (22) and (23).

\[
\tag{26} \vdash \neg \neg (\gamma \lor \neg \gamma)
\]

Indeed, a solution of \( \neg (\Gamma \lor \neg \Gamma) \) yields by (24) a solution of \( \neg \Gamma \land \neg \neg \Gamma \), that is, a solution of \( \neg \Gamma \) together with a method to turn such a solution into a contradiction.

\[
\tag{27} (\text{exponential law}) (\gamma \land \delta) \to \gamma \iff \gamma \to (\delta \to \gamma)
\]

\[
\tag{28} \gamma \land (\delta \to \gamma) \implies (\gamma \to \delta) \to \gamma
\]
These are again similar to (14) and (15).

\[(29) \neg(\gamma \land \delta) \iff \gamma \rightarrow \neg \delta\]

This follows from (27).

\[(30) \gamma \rightarrow \neg \delta \iff \delta \rightarrow \neg \gamma\]

This follows from (29).

\[(31) \neg\neg\gamma \rightarrow \neg\neg\delta \iff \neg\delta \rightarrow \neg \gamma\]

This “idempotence of contrapositive” follows from (3) and (30).

\[(32) \neg\neg\gamma \rightarrow \neg \delta \iff \delta \rightarrow \neg \gamma\]

This follows from (30) and (3), or alternatively from (1), (2) and (3).

Note that (28) specializes to \(\gamma \land \neg \delta \implies \neg(\gamma \rightarrow \delta)\). But a stronger assertion holds:

\[(33) \neg\neg\gamma \land \neg\neg\delta \iff \neg(\gamma \rightarrow \delta)\]

Indeed, the \(\leftarrow\) direction follows from (24) and the contrapositive of (5).

Conversely, suppose we are given a solution of \(\neg\neg\Gamma \land \neg\Delta\) and a solution of \(\Gamma \rightarrow \Delta\). Then we have solutions of \(\neg\Delta\), of \(\neg\Gamma \rightarrow \bot\), and of \(\neg\Delta \rightarrow \neg\Gamma\) (the contrapositive). Applying the latter to the solution of \(\neg\Delta\), we get a solution of \(\neg\Gamma\), and hence a contradiction.

\[(34) \neg(\gamma \lor \neg\delta) \iff \neg(\delta \rightarrow \gamma)\]

This improvement on the contrapositive of (5) follows from (33) and (24).

\[(35) \neg(\neg\gamma \lor \neg\delta) \iff \neg\neg(\gamma \land \delta)\]

This improvement on the contrapositive of (25) follows from (34) and (29).

\[(36) \neg\neg\gamma \land \neg\neg\delta \iff \neg\neg(\gamma \land \delta)\]

This follows from (35) and (24).

\[(37) \neg\neg\gamma \rightarrow \neg\neg\delta \iff \neg\neg(\gamma \rightarrow \delta)\]

This follows from (33) and (29).

\[(38) \neg\neg\gamma \lor \neg\neg\delta \implies \neg\neg(\gamma \lor \delta)\]

This follows from (35) using (24).

\[(39) \neg\forall x \gamma(x) \implies \forall x \neg\neg\gamma(x)\]

\[(40) \exists x \neg\neg\gamma(x) \implies \neg\exists x \gamma(x)\]

Each of these follows from (17) using (16).

4. **First-order logic and the consequence relation**

Traditional first-order logic attempts to formalize the practice of proving theorems on the basis of axioms, which is found in mathematical texts since Euclid; thus its basic notions (such as individual variable, term, predicate, formula) are intended to be formal abstractions from this informal *mathematical* practice. On the other hand, the informal *meta-mathematical* practice of elucidating general logical schemes of reasoning, which is found in texts on first-order logics, classical and non-classical, naturally has somewhat different basic notions. In practice, these are usually left at an informal level. Thus,
most textbooks on first-order logics treat metavariables (or “syntactic variables” in the terminology of Church [26]) informally. For instance, in the words of Kleene [73; §34],

“The use of a formal substitution rule can be largely avoided by stating results in the schematic form, with metamathematical letters used instead of particular predicate letters. We then substitute informally in applying the results with a change in the signification of the metamathematical letters, but this substitution does not constitute application of a formal substitution rule. We have been doing this continually, from the very beginning of our study of the formal system.”

One could expect that having said this, Kleene would proceed to formalize the informal substitution for schemata that he indeed uses all the time; but instead he only gives one brief example of what he means by it, and then proceeds to define his “formal substitution rule” for formulas, with the only practical result of his discussion of the informal substitution for schemata being the warning that “the reader may therefore, if he wishes, omit the detailed treatment of substitution” for formulas given afterwards.

Apparently people see this discrepancy between the official syntax of a first-order logic and the actual syntax found in the literature on first-order logics as only a minor inconvenience and perhaps a small price to pay for the perceived separation of ‘syntax’ from ‘semantics’. However the price rises when one tries to argue formally about arbitrary axiom schemes, theorem schemes or inference rules of a first-order logic; indeed, it is not so clear exactly what they are, since the official language includes neither the syntax for their side conditions such as “provided that $x$ is not free in $\alpha$” or “provided that $t$ is free for $x$ in $\alpha(x)$” nor any formal rules for dealing with these side conditions.

In order to clarify this matter, we will now sketch a simple formalization of metavariables along the lines of [125; §1 and §2.2], including a formalization of Kleene’s informal substitution for schemata. The special case where the only terms of the language are individual variables (this case mostly suffices for the purposes of the present paper) is essentially the same as in [107] (see also [121; §3.7]) and so is well understood from the semantical viewpoint.

### 4.1. Language. Metavars

A first-order language $\Lambda$ is supposed to be a formal means to speak about individuals, which are elements of an unspecified set $D$, called the domain of discourse. (For instance, if $\Lambda$ is the usual language of arithmetic, a good choice for $D$ would be the set of natural numbers.) Now these are semantic objects, and $\Lambda$ itself is about syntax only. Thus $\Lambda$ contains individual variables $x_1, x_2, \ldots$, thought of as denoting individuals (for reasons of readability we will also write $x, y, z$ for $x_1, x_2, x_3$), and may also contain countably many individual operators (such as $+$ and $*$), each standing for an operation (such as addition and multiplication of natural numbers) that inputs an $n$-tuple of individuals (for some specified $n \geq 0$) and outputs one individual. Individual operators of arity 0 are called individual constants and stand for specific individuals (such as the natural numbers 0 and 1). Individual variables and individual constants are atomic terms; general terms
of $\Lambda$ are defined inductively, as built out of atomic terms using individual operators. (A
sample term in the case of arithmetic is $(x + 1) \ast y + x.$)

In logic, instead of specific terms one usually deals with term variables, which are
placeholders for terms, and which are a sort of metavariables in that they are not con-
sidered to be a part of $\Lambda$. (In fact, these are just variables in the ordinary mathematical
sense of the word; whereas individual variables are rather like coordinates in analytic
geometry, i.e. a predefined stack of indeterminates that are set to vary independently of
each other.) Term variables and individual constants are atomic term schemes; general
term schemes are defined inductively, as built out of atomic term schemes using the
individual operators of $\Lambda$. Thus if $t$ and $u$ are term variables, the arithmetical term scheme
$(t + t) \ast (u + 1)$ may represent either of the terms $(x + x) \ast (x + 1), \ (1 + 1) \ast (y \ast z + 1),
(x \ast x + x \ast x) \ast (x + 1)$. A term-with-input is a term scheme where some term variables
have been instantiated by individual variables; it can be regarded as a function of the
remaining term variables, which inputs a tuple of terms and outputs one term.

In fact, $\Lambda$ is not supposed to be a means to speak directly about individuals, but
rather about logical functions of the individuals (such as propositions or problems about
individuals); in fact the domain of discourse is also known as the range of quantification.
But again these are semantic objects, and $\Lambda$ itself is about syntax only. Thus $\Lambda$ may
contain logical variables

$$\pi_1, \pi_2, \ldots; \ \pi_{11}(t_1), \pi_{12}(t_1), \ldots; \ \pi_{21}(t_1, t_2), \pi_{22}(t_1, t_2), \ldots; \ldots,$$

which are indeterminate functions of tuples of term variables; they are thought of as
denoting logical functions of individuals. For brevity, we will also write $\alpha, \beta, \gamma, \delta$ for
$\pi_1, \pi_2, \pi_3, \pi_4$ and $\pi(t), \rho(t)$ for $\pi_{11}(t), \pi_{12}(t)$. (In logic, logical variables are usually
called propositional and predicate variables, but in intuitionistic logic, Kolmogorov calls
them indeterminate problems. To have at least both of these choices available, we re-
main agnostic at this level of generality about the interpretation of logical variables.)

Also, $\Lambda$ may contain countably many logical operators (such as $\land$ or $\rightarrow$), each standing
for a logical operation (such as conjunction of propositions or reduction between prob-
lems) that inputs an $m$-tuple of logical functions (for some specified $m \geq 0$) of sets of
parameters $S_1, \ldots, S_m$ and outputs one logical function of a specified set of parameters
$S = F(S_1, \ldots, S_m)$. The function $F$ is called the type of the logical operator. A logi-
cal operator of type $S_1, \ldots, S_m \Rightarrow S_1 \cup \cdots \cup S_m$ is called an $m$-ary connective, and a
logical operator of type $S \Rightarrow S \setminus \{x\}$ is called a quantifier over $x$. A logical operator
with empty input and $n$-element output will be called a logical constant of arity $n$. (In
classical logic, logical constants of arity 0 are the truth values: truth, $\top$, and falsity,
$\bot$; and logical constants of positive arity are predicates, such as the binary predicate of
equality, $x = y.$)

An atomic formula is a logical variable or a logical constant of $\Lambda$, whose arguments (i.e.
term variables) are instantiated by particular terms. The individual variables involved
in these terms are called the parameters (=free variables) of the atomic formula. A
general formula of $\Lambda$ is defined inductively along with its parameters, as built out of
atomic formulas using the logical operators of \( \Lambda \), where all quantifiers are over individual variables. A formula-with-input (called “name form” by Kleene [73]) is defined similarly, where an atomic formula-with-input is a logical variable or a logical constant of \( \Lambda \), whose arguments are instantiated by terms-with-input. (A sample arithmetical formula-with-input is \( \exists x t = 4 \cdot x + 3 \).) Thus a formula-with-input can be regarded as a function that inputs a tuple of terms and outputs a formula of \( \Lambda \).

In logic, instead of specific formulas or even specific formulas-with-input one usually deals with formula variables, which are placeholders for formulas-with-input. Thus a formula variable can be regarded as an indeterminate function \( \alpha(t_1, \ldots, t_k) \) of a specified number of term variables; formula variables of arity 0 are placeholders for formulas. Formula variables of arity > 0 are, strictly speaking, metametavariabes, as they only denote entities that themselves denote formulas of \( \Lambda \). An atomic formula scheme is a formula variable or a logical constant whose arguments (i.e. term variables) are instantiated by term schemes. The term variables involved in these term schemes are called the parameters of the atomic formula scheme. General formula schemes, or simply schemata are defined inductively along with their parameters, as built out of atomic formula schemes using the logical operators of \( \Lambda \), where all quantifiers are over term variables. Note that every formula variable \( \varphi(t_1, \ldots, t_n) \) is itself also a schema; but not conversely, e.g. the schema \( \varphi \lor \lnot \varphi \) is not a formula variable.

Thus, Fermat’s theorem is a formula, like most other mathematical theorems, whereas the law of excluded middle is a schema, like most other logical laws. (This concerns only first-order statements.) On the border of mathematics and logic, there are first-order theories (such as the Peano arithmetic and the Zermelo–Fraenkel set theory) where most axioms are single formulas, but some infinite groups of axioms are schemata.

### 4.2. Instances and special cases of schemata

We say that a formula of \( \Lambda \) is an instance of a schema if the latter evaluates to the former upon the following steps:

1. Instantiate each formula variable by a formula-with-input of the same arity.\(^{27}\)
2. Instantiate, in some order, each term variable \( x \) that occurs as a variable of quantification by an individual variable \( x_i \) such that \( x_i \) is not a parameter of any subformula that is prefixed by a quantifier over \( x \), and \( x \) is not a parameter of any subformula that is prefixed by a quantifier over \( x_i \).
3. Instantiate every remaining term variable \( t \) by a term involving no individual variables \( x_i \) such that \( t \) occurs in a subformula prefixed by a quantifier over \( x_i \).

More generally, a tuple of formulas of \( \Lambda \) is an instance of a tuple of schemata if the latter ones evaluate to the former ones upon the same three steps, where the instantiation is made uniformly, that is, all occurrences of any variable are instantiated by the same

---

\(^{27}\)This step does not involve any side conditions because all the “anonymous” variables, to use Kleene’s terminology [73; §34], are individual variables and so are a priori distinct from the “explicit” variables, which at this stage are term variables.
expression throughout all of the formulas. For example, let us consider the following pair of arithmetical schemata

\[ \forall t \varphi(t, u, 0), \quad \exists u \neg 3 \ast w = 5. \]

We can instantiate \( \varphi(t, u, v) \) for example by \( \exists x t = x \ast u + y \ast v \). Given this, we cannot instantiate \( t \) by \( x \) (because \( \exists x \) would then bind \( t \)), nor by \( y \) (because \( \forall t \) would then bind \( y \), even though \( v = 0 \)), but we can instantiate \( t \) by \( z \), say. Given that, we cannot instantiate \( u \) by \( x \) or \( z \), but we can instantiate it, for example, by \( y \). We do not need to instantiate \( v \) by anything, because it does not enter our pair of schemata. Finally, assuming this choice, we cannot instantiate \( w \) by \( y + 1 \), but we can instantiate it, for example, by \( x + z \). To summarize, one instance of the given pair of schemata is the pair of arithmetical formulas

\[ \forall z \exists x z = x \ast y + y \ast 0, \quad \exists y \neg 3 \ast (x + z) = 5. \]

We say that a tuple of formulas of \( \Lambda \) is a trivial instance of a tuple of schemata if the latter evaluates to the former upon instantiating uniformly every formula variable with a logical variable and every term variable with an individual variable, so that distinct variables remain distinct upon the instantiation. It is easy to see that if a formula \( \varphi \) is a trivial instance of a schema \( \Phi \), then arbitrary instances of \( \Phi \) are precisely those formulas \( \psi \) that can be obtained from \( \varphi \) by free (=capture-avoiding) substitution (as discussed, for instance, by Church [26; §35] or Kleene [73; §34]).

If, conversely, a formula \( \psi \) is an instance of a schema \( \Phi \) and a trivial instance of a schema \( \Psi \), we will say that \( \Psi \) is a special case of \( \Phi \). More generally, a tuple of schemata \( \Psi_1, \ldots, \Psi_k \) is a special case of a tuple of schemata \( \Phi_1, \ldots, \Phi_k \) if some instance of the latter is a trivial instance of the former. (This specialization is our formalization of Kleene’s informal substitution as discussed in his quote above.)

4.3. Logic. Derivable and admissible rules

By a (structural) rule we mean a tuple of schemata, written as

\[
\frac{\Phi_1, \ldots, \Phi_k}{\Psi},
\]

or in a more concise form as \( \Phi_1, \ldots, \Phi_k \rightarrow \Psi \). The list of premisses of a rule may be empty, in which case the rule amounts to a single schema. A logic \( L \) is determined (in a way to be specified in a moment) by its deductive system \( R \), which consists of a set of rules; the rules with no premisses are called the primary laws of \( R \) (or axiom schemes in classical logic, or postulate schemes in intuitionistic logic), and the rules with at least one premise are, called the inference rules of \( R \). Here \( \Lambda \) is fixed, and is called the language of the logic \( L \).

By a law of \( L \) (or a theorem scheme in classical logic, or a soluble problem scheme in intuitionistic logic) we mean a schema \( \Phi \) such that either \( \Phi \) is a special case of an primary law of \( R \), or there are laws \( \Phi_1, \ldots, \Phi_k \) such that \( \Phi_1, \ldots, \Phi_k \rightarrow \Phi \) is a special case of an
inference rule of $R$. More generally, a schema $\Psi$ is said to be a \textit{syntactic consequence} of schemata $\Phi_1, \ldots, \Phi_k$ if $\Psi$ is either one of the $\Phi_i$'s\textsuperscript{28} or a special case of an primary law of $R$, or there are syntactic consequences $\Psi_1, \ldots, \Psi_l$ of $\Phi_1, \ldots, \Phi_k$ such that $\Psi_1, \ldots, \Psi_l \vdash \Psi$ is a special case of an inference rule of $R$.

The judgement that $\Psi$ is a syntactic consequence of $\Phi_1, \ldots, \Phi_k$ is denoted by $\Phi_1, \ldots, \Phi_k \vdash \Psi$, or in more detail $\Phi_1, \ldots, \Phi_k \vdash_L \Psi$; in particular, the judgement that $\Phi$ is a law of $L$ is denoted by $\vdash L \Phi$, or in more detail $\vdash_L L \Phi$. A rule $\Phi_1, \ldots, \Phi_k / \Psi$ is called \textit{derivable} if $\Phi_1, \ldots, \Phi_k \vdash \Psi$. It is not hard to see that every special case of a derivable rule is derivable (cf. [73; §34]); in particular, every special case of a law is a law (cf. [73; §35]). Clearly, adding a derivable rule to a deductive system (as a new inference rule) does not change the syntactic consequence relation.

Some authors give a slightly different definition of $\vdash$ in the case of a non-empty list of premisses, imposing certain restrictions on the use of rules that involve quantifiers (see for instance, [26] and [143]) or modalities (see [58] and references there). They obviously do so in order to save the deduction theorem (of zero-order classical logic) in its original, unrestricted form — at the cost of what is arguably an acute conflict with semantics (see §4.5 and §5.9). Thus, the unrestricted deduction theorem holds in first-order classical logic as understood e.g. by Church [26], but fails in first-order classical logic as understood e.g. by Mendelson [102]. The two approaches to defining syntactic consequence are discussed and compared — with the opposite conclusions reached — in [18] and [58]. Some authors, including Kleene [73], use both approaches at the same time; we will be content with the above definition (which is in agreement with Mendelson), since the more elaborate ones are irrelevant for our purposes.

We say that two deductive systems $R$ and $R'$ determine the same \textit{logic} $L$, if they have the same derivable rules (hence, in particular, the same laws). This amounts to saying that the primary laws of $R$ are laws of $R'$ and the inference rules of $R$ are derivable rules of $R'$ — and conversely. In the terminology of Wójcicki and Blok–Pigozzi [18] this is the “inferential” conception of logic, as opposed to the “formulaic” one, in which a logic is identified with the collection of its laws. The two conceptions ascribe somewhat different meanings to the notion of an inference rule, which is related to the difference between derivable and admissible rules.

A rule $r$ is called \textit{admissible} for the logic determined by a deductive system $R$ if every law of the logic determined by the extended system $R \cup \{r\}$ is also a law of the original logic. Thus adding an admissible rule to a deductive system does not change the set of laws, but unless this rule is derivable, it changes the logic (i.e. the syntactic consequence relation). This can happen even with classical logic, as we will see in §4.5. Clearly, a rule $r$ is admissible if and only if for every its special case $\Gamma_1, \ldots, \Gamma_k \vdash_L \Delta$ such that each $\Gamma_i$ is a law of $L$, so is $\Delta$. For this reason, admissible rules of the form $\Phi_1, \Phi_2, \ldots / \Psi$ are

\textsuperscript{28}But not just a special case of one of the $\Phi_i$'s. Thus the premisses $\Phi_1, \ldots, \Phi_k$ are not treated just as if they were primary laws.
sometimes stated in the literature in the form “if ⊢ \( \Phi_1, \vdash \Phi_2 \), etc., then \( \vdash \Psi \)” (where the quantification over all special cases is meant but not pronounced, in order to confuse the uninitiated). Indeed, these judgements are not internally expressible in \( L \) and so belong to the meta-theory of \( L \). In older literature, however, one often finds *inference rules* stated in this language (which certainly indicates the author’s acceptance of the “formulaic” conception of logic).

Since a syntactic consequence of laws is a law, every derivable rule is admissible. Moreover, a rule \( r \) is derivable if and only if for every its special case \( \Gamma_1, \ldots, \Gamma_k / \Delta \) and for every schemata \( \Theta_1, \ldots, \Theta_l \) such that \( \Theta_1, \ldots, \Theta_l \vdash \Gamma_i \) for each \( i \), we also have \( \Theta_1, \ldots, \Theta_l \vdash \Delta \). (This follows by considering the case \( l = k, \Theta_i = \Gamma_i \).) Let us note that if a new primary law is added to a deductive system, previously derivable rules do not cease to be derivable, but previously admissible rules may cease to be admissible.

We can also do first-order *theories* based on a logic \( L \). Such a theory \( T \) is given by a collection of formulas of \( \Lambda \), called the *axioms* of \( T \). A formula \( \varphi \) of \( \Lambda \) is a *theorem* of \( T \) if it is either an axiom of \( T \), or an instance of a law of \( L \), or there exist theorems \( \varphi_1, \ldots, \varphi_k \) of \( T \) such that \( \varphi_1, \ldots, \varphi_k / \varphi \) is an instance of a derived rule of \( L \). Note that \( \Phi_1, \ldots, \Phi_k / \vdash \Psi \) if and only if the trivial instance \( \psi \) of \( \Psi \) is a theorem of the theory \( \{ \varphi_1, \ldots, \varphi_k \} \), where the \( \varphi_i \) are the trivial instances of the \( \Phi_i \). A theory \( T \) is called *consistent* if its theorems do not include all formulas of \( \Lambda \).

### 4.4. Intuitionistic and classical logics

The language of intuitionistic logic can be taken to be the same as that of classical logic (by the price of some redundancy). This language contains no operation symbols, and in particular no constants, so all its terms are the individual variables \( \pi_i \). It also contains all the logical variables \( \pi_{ni}(t_1, \ldots, t_n) \) and the following logical operators: 0-ary connectives \( \perp \) and \( \top \), unary connective \( \neg \), binary connectives \( \land, \lor, \rightarrow \) and \( \leftrightarrow \), and quantifiers \( \exists \) and \( \forall \). Here \( \leftrightarrow \), \( \neg \) and \( \top \) are “syntactic sugar”: \( \alpha \leftrightarrow \beta \) is defined as \( (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha) \); \( \neg \alpha \) is defined as \( \alpha \rightarrow \perp \); and \( \top \) is defined as \( \neg \bot \). In classical logic there is, of course, additional redundancy: \( p \land q \) can be defined as \( \neg(\neg p \lor \neg q) \), \( p \rightarrow q \) can be defined as \( \neg p \lor q \), and \( \forall \pi \varphi(x) \) can be defined as \( \neg \exists \pi \neg \varphi(x) \).

In the intuitionistic case, there is no further redundancy [115; p. 59]. In particular, due to the validities (36), (37), (38), (39) and (40), \( \forall \) is the only connective through which \( \neg \neg \) cannot be pushed inside (see §5.7), and \( \exists \) is the only connective through which \( \neg \neg \) cannot be pushed outside in the stable and decidable cases (see §5.4); thus neither \( \exists \) nor \( \forall \) is expressible in terms of the other connectives. Of the propositional connectives, only \( \rightarrow \) is not preserved under the \( \Box \)-translation (in the McKinsey–Tarski form, see §5.9) and only \( \lor \) is not preserved under the \( \neg \neg \)-translation (in Gentzen’s form, see §5.6); thus neither \( \rightarrow \) nor \( \lor \) is expressible in terms of the other propositional connectives.

A formalization of intuitionistic predicate logic \( QH \) in terms of a deductive system was found by Heyting (1931), extending earlier partial formalizations by Glivenko (1929); the
zero-order fragment) and Kolmogorov [75] (1925; the →, ¬, ∃, ∀ fragment with the omission of ⊥ → α and with inadvertent omission of [∀x α(x)] → α(y), see [112]). Here is an equivalent formulation due to Spector (see [142], [143]), with a minor modification (to be discussed in a moment).

(I) (modus ponens rule) \( \frac{\alpha, \alpha \rightarrow \beta}{\beta} \)

(II) \( \alpha \rightarrow \alpha \)

(III) \( \frac{\alpha \rightarrow \beta, \beta \rightarrow \gamma}{\alpha \rightarrow \gamma} \)

(IV) \( \alpha \land \beta \rightarrow \alpha \) and \( \alpha \land \beta \rightarrow \beta \)

(V) \( \alpha \rightarrow \alpha \lor \beta \) and \( \beta \rightarrow \alpha \lor \beta \)

(VI) \( \frac{\alpha \rightarrow \beta, \alpha \rightarrow \gamma}{\alpha \rightarrow \beta \land \gamma} \)

(VII) \( \frac{\alpha \rightarrow \gamma, \beta \rightarrow \gamma}{\alpha \lor \beta \rightarrow \gamma} \)

(VIII) (exponential law) \( \frac{(\alpha \land \beta) \rightarrow \gamma}{\alpha \rightarrow (\beta \rightarrow \gamma)} \) and \( \frac{\alpha \rightarrow (\beta \rightarrow \gamma)}{(\alpha \land \beta) \rightarrow \gamma} \)

(IX) (explosion) \( \bot \rightarrow \alpha \)

(X) \( (\forall x \alpha(x)) \rightarrow \alpha(t) \)

(XI) \( \alpha(t) \rightarrow \exists x \alpha(x) \)

(XII) (generalization rule) \( \frac{\alpha(x)}{\forall x \alpha(x)} \)

(XIII) \( \frac{\forall x (\beta \rightarrow \alpha(x))}{\beta \rightarrow \forall x \alpha(x)} \)

(XIV) \( \frac{\forall x (\alpha(x) \rightarrow \beta)}{(\exists x \alpha(x)) \rightarrow \beta} \)

Classical predicate logic QC is obtained by adding just one more primary law:

(XV) (law of excluded middle) \( \alpha \lor \neg \alpha \)

\( ^{29} \)Kolmogorov’s point of departure was Hilbert’s deductive system for classical logic, which did not use other connectives. Presumably Kolmogorov was also aware of Russell’s definition of ∧, ∨ and ∃ in terms of → and the second-order ∀, which is valid intuitionistically, cf. [4].

\( ^{30} \)In (X) and (XI), \( \alpha(t) \) and \( \alpha(x) \) are, of course, the results of feeding two different term variables into the same formula variable of arity 1. Note that by our conventions every special case of, say, (XI) is of the form \( \Gamma(T) \rightarrow \exists y \Gamma(y) \), where \( \Gamma \) is a formula scheme, \( y \) is a term variable, which may have multiple occurrences in \( \Gamma \), and \( T \) denotes a term scheme which is “free for \( y \) in \( \Gamma(y) \)”.

\( ^{31} \)By our conventions, in a special case of (XIII) and (XIV), \( x \) can only specialize to a term variable, and this variable cannot occur in the specialization of \( \beta \).
Clearly, in the presence of *modus ponens*, \( \vdash \alpha \rightarrow \beta \) implies \( \alpha \vdash \beta \), and more generally \( \gamma_1, \ldots, \gamma_k \vdash \alpha \rightarrow \beta \) implies \( \gamma_1, \ldots, \gamma_k, \alpha \vdash \beta \). If \( \alpha \) as well as \( \gamma_1, \ldots, \gamma_k \) have no parameters, the converse implication holds in classical and intuitionistic logics by the *deduction theorem* (see [143], where the deduction theorem for intuitionistic logic is proved for Spector’s system).

By combining this with the exponential law, we also obtain that \( \alpha, \beta \vdash \gamma \) is equivalent to \( \alpha \land \beta \vdash \gamma \), as long as \( \alpha \) and \( \beta \) have no parameters. In fact, the latter assumption is superfluous here, since the same without this assumption follows from (V) and (VII), using additionally that \( \top \rightarrow \alpha \vdash \alpha \) by (I) and (II), and \( \alpha \vdash \top \rightarrow \alpha \) by (IV) and (VIII).

Thus, should we wish to have fewer inference rules, we may well replace the inference rules (III), (VI), (VII), (VIII), (XIII) and (XIV) by primary laws (in which the horizontal bar is replaced by \( \rightarrow \), and the comma by \( \land \)). The generalization rule is, however, strictly weaker than the corresponding schema, \( \alpha(x) \rightarrow \forall x \alpha(x) \) (see §4.5).

**Proposition 4.1.** The deductive system given by (I)–(XIV) is equivalent to Spector’s system.

This equivalence is not precisely the coincidence of the resulting logics as defined above — for the sole reason that Spector’s system does not exactly fit with our definition of a deductive system; this is why we have to modify it.

**Proof.** The original system of Spector (as in [143]) has instead of the rules (XIII), (XIV) and (XII) the following two non-structural rules, where it is assumed that “\( x \) does not occur freely in \( \beta \)”, that is, only those instances are considered in which the individual variable instantiating \( x \) is not a parameter of the formula instantiating \( \beta \):

\[
\begin{align*}
(X') & \quad \beta \rightarrow \alpha(x) \\
& \quad \beta \rightarrow \forall x \alpha(x) \\
(XI') & \quad \alpha(x) \rightarrow \beta \\
& \quad (\exists x \alpha(x)) \rightarrow \beta
\end{align*}
\]

Using the generalization rule (XII) and its converse, which follows from (X), we can clearly exchange (XIII) for (X’) and (XIV) for (XI’). Since \( \top \rightarrow \alpha \) is equivalent to \( \alpha \), the generalization rule follows from (X’). \( \square \)

**4.5. Models. Semantic consequence**

Let us fix a first-order language \( \Lambda \).

A \( \Lambda \)-structure \( M \) consists of a choice of a nonempty domain of discourse \( D = D_M \), of a set \( \Omega = \Omega_M \) of *truth values*, along with its distinguished subset of *truths*, and of an *interpretation* of every \( n \)-ary operation symbol \( s \) in \( \Lambda \) by a map \( s^M : D^n \rightarrow D \) (in particular, constants of \( \Lambda \) are interpreted by elements of \( D \)), of every \( n \)-ary logical variable \( \pi_{ni} \) by a map \( \pi^M_{ni} : D^n \rightarrow \Omega \), and of every logical operator \( c \) of type \( S_1, \ldots, S_m \rightarrow S \) by an operation \( c^M \) of the same type on such maps (in particular, logical constants of arity \( n \) are interpreted by maps \( D^n \rightarrow \Omega \)). In addition, a *variable assignment*, which is
not considered to be a part of \( M \), associates elements of \( D \) to individual variables \( x_i \) of \( \Lambda \).

If a variable assignment \( \mu \) is given, any formula \( \varphi(x_{i_1}, \ldots, x_{i_n}) \) of \( \Lambda \) with \( n \) parameters is interpreted by an element \( \varphi^\mu \in \Omega \); and when no variable assignment is given, by a map \( \varphi^M : D^n \to \Omega \), so that \( \varphi^M(\mu(x_{i_1}), \ldots, \mu(x_{i_n})) = \varphi^\mu \). Note that if \( \varphi \) has no parameters, \( \varphi^\mu \) does not depend on \( \mu \).

A formula \( \varphi \) of \( \Lambda \) is called valid in \( M \) if \( \varphi^M \) is a truth for every variable assignment \( \mu \). We say that a schema \( \Phi \) is satisfied in \( M \) and write \( \models^M \Phi \) if every instance of \( \Phi \) is valid in \( M \). More generally, we say that a rule \( \Phi_1, \ldots, \Phi_k \vdash \Psi \) is satisfied in \( M \) and write

\[
\Phi_1, \ldots, \Phi_k \models^M \Psi
\]

if for every its instance \( \varphi_1, \ldots, \varphi_k / \psi \) such that each \( \varphi_i \) is valid in \( M \), \( \psi \) is also valid in \( M \). In this case we also say that the schema \( \Psi_1 \) is a semantic consequence in \( M \) of the schemata \( \Phi_1, \ldots, \Phi_k \).\(^3\)

Let us now fix a logic \( L \) determined by a deductive system \( R \) whose language is \( \Lambda \). By a model of \( L \) we mean a \( \Lambda \)-structure \( M \) such that the primary laws of \( R \) are satisfied in \( M \), and the conclusion of each inference rule of \( R \) is a semantic consequence in \( M \) of its premises. Then clearly all laws of \( L \) are satisfied in \( M \), and more generally, it is easy to see that

\[
\Phi_1, \ldots, \Phi_k \models^L \Psi \implies \Phi_1, \ldots, \Phi_k \models^M \Psi.
\]

For many logics one also has the converse, where \( M \) runs over all models of \( L \) (see [121; 1.4.11]). In fact, a logic \( L \) is said to be complete with respect to a family of its models \( M(t) \) (possibly consisting of just one model), if \( \models^M(t) \Phi \) for all \( t \) implies \( \models^L \Phi \), and strongly complete if \( \Phi_1, \ldots, \Phi_k \models^M(t) \Psi \) for all \( t \) implies \( \Phi_1, \ldots, \Phi_k \models^L \Psi \).

A model of a theory \( T \) based on \( L \) is a model \( M \) of \( L \) such that all axioms of \( T \) are valid in \( M \). Clearly, all theorems of \( T \) are also valid in such an \( M \); if no other formulas of \( L \) are valid in \( M \), \( T \) is said to be complete with respect to \( M \). If every consistent theory \( T \) based on \( L \) is complete with respect to some its model \( M_T \), then \( L \) is strongly complete with respect to the entire family \( M_T \). Indeed, if \( \Phi_1, \ldots, \Phi_k \not\models^L \Psi \), let \( T = \{ \varphi_1, \ldots, \varphi_k \} \), where the formulas \( \varphi_i \) are the trivial instances of the schemata \( \Phi_i \). Then the trivial instance \( \psi \) of \( \Psi \) is not a theorem of \( T \), so in particular \( T \) is consistent; thus \( \psi \) is not valid in \( M_T \). But each \( \varphi_i \) is valid in \( M_T \), so \( \Phi_1, \ldots, \Phi_k \not\models^M_T \Psi \).

In two-valued models of classical logic, the set \( \Omega \) of truth values consists of two elements, \( \{ \top, \bot \} \), of which only \( \top \) is a truth, and the interpretation of the connectives is defined according to the standard truth tables (see §3.1). It is not hard to check that, indeed, the primary laws and the inference rules of classical logic (see §4.4) hold under this interpretation. In fact, every consistent first-order theory based on classical logic is complete with respect to a two-valued model with countable domain [118; VIII.5.3].

\(^3\)This definition of semantic consequence is generally credited to Tarski [136]. A slight variation is found in the literature (see [58]), but we will not discuss the alternative version since it does not even validate the (unrestricted) generalization rule.
Thus classical logic is strongly complete with respect to the class of all its two-valued models with countable domain.

**Remark 4.2.** Strong completeness is closely related to Dragalin’s composite theories [35], which besides axioms also contain a non-empty list of what might be called *taboos*. A formula $\varphi$ of $\Lambda$ is a *theorem* of a composite theory $T$ over the logic $L$ if either $\varphi$ or some taboo of $T$ is a theorem of the ordinary theory with the axioms of $T$; and a *contradiction* of $T$ if some taboo of $T$ is a theorem of the ordinary theory whose axioms are $\varphi$ and the axioms of $T$. A composite theory $T$ is called *consistent* if not every formula is a theorem, or equivalently not every formula is a contradiction. Thus ordinary theories can be identified with composite theories whose only taboo is $\bot$. In classical logic, a composite theory $T$ amounts to the ordinary theory whose axioms are the axioms of $T$ along with the negations of the taboos of $T$.

A model of a composite theory $T$ is a model $M$ of the logic $L$ such that the axioms of $T$ are valid in $M$, and no finite disjunction of the taboos of $T$ is valid in $M$. If every consistent composite theory $T$ over $L$ has a model $M_T$, then $L$ is strongly complete with respect to the family $M_T$. Indeed, if $\psi$ is not a consequence of $\varphi_1, \ldots, \varphi_n$, then the composite theory $T$ with axioms $\varphi_1, \ldots, \varphi_n$ and the only taboo $\psi$ is consistent; thus $\varphi_1, \ldots, \varphi_n$ are valid in $M_T$, whereas $\psi$ is not.

**Example 4.3.** The judgements $\vdash \forall x \, p(x)$ and $\vdash p(t) \rightarrow \forall x \, p(x)$ are not equivalent already in the case of classical logic. Indeed, in a two-valued model $M$ of classical logic with $D_M = \{0, 1\}$ and with $\pi^M(0) = \bot$ and $\pi^M(1) = \top$, the interpretation of $\pi(x) \rightarrow \forall x \, \pi(x)$ is truth if $\mu(x) = 0$ and falsity if $\mu(x) = 1$. Hence $\pi(x) \rightarrow \forall x \, \pi(x)$ is not valid in this model, and in particular $\not\vdash_M p(t) \rightarrow \forall x \, p(x)$. But then also $\not\vdash p(t) \rightarrow \forall x \, p(x)$.

On the other hand, $\vdash \forall x \, p(x)$ by the generalization rule (XII), and therefore also $p(x) \models_M \forall x \, p(x)$. In particular, the implication “if $\pi(x)$ is valid in $M$, then $\forall x \, \pi(x)$ is valid in $M$” holds; indeed, neither $\pi(x)$ nor $\forall x \, \pi(x)$ is valid in $M$.

**Example 4.4.** In classical logic, the rule

$$
\begin{array}{c}
\exists x \, p(x), \, \exists x \, \neg p(x) \\
\hline
\bot
\end{array}
$$

is admissible but not derivable. To see the latter, we can use the same model $M$ as in the previous example. Then $\exists x \, \pi(x)$ and $\exists x \, \neg \pi(x)$ are both valid in $M$, yet $\bot$ is not valid. Thus $\exists x \, p(x)$, $\exists x \, \neg p(x) \not\models_M \bot$, whence $\exists x \, p(x)$, $\exists x \, \neg p(x) \not\models \bot$.

On the other hand, suppose that for some special case $Q(x)$ of the schema $p(x)$ we have both $\vdash \exists x \, Q(x)$ and $\vdash \exists x \, \neg Q(x)$. Let $N$ be a two-valued model of classical logic with the domain consisting of a single element, $D_N = \{\ast\}$. If $q(x)$ is an instance of $Q(x)$, then $\exists x \, q(x)$ and $\exists x \, \neg q(x)$ must both be valid in $N$. Since $\ast$ is the only element in $D$, this means that both $q^N(\ast)$ and $\neg q^N(\ast)$ equal $\top$, which is a contradiction. Thus no special case of the two premisses of the rule $\exists x \, p(x)$, $\exists x \, \neg p(x)$ / $\bot$ can be derivable simultaneously. Thus the rule is admissible.
In fact, this example is characteristic of non-derivable admissible rules in classical logic. They are all such that the conjunction of their premisses admits no special case that is a law — nor even satisfied in any two-valued model with 1-element domain [36].

### 4.6. Euler’s models of classical logic

It was known already to Leibniz, and has been well-known since Euler that classical logic can be interpreted by set-theoretic operations (see [12]). One starts with a set \( S \) and a domain \( D \). Truth values will be arbitrary subsets of \( S \), of which only the entire \( S \) is considered true. Thus, every propositional variable \( \pi_i \) is interpreted by a subset \( \lambda_i^V \) of \( S \), and every \( n \)-ary predicate variable \( \pi_{ni} \) is interpreted by a family of subsets of \( S \) indexed by \( D^n \). Disjunction of propositions corresponds to union of subsets, conjunction to intersection, and negation to complement (within \( S \)); universal quantification corresponds to union, and existential to intersection. Implication is, of course, determined by disjunction and negation. The logical constants \( \bot \) and \( \top \), thought of as the empty disjunction and the empty conjunction, are interpreted by \( \emptyset \) and \( S \).

We call this interpretation an Euler model of classical logic; indeed, it is not hard to check that the primary laws and the inference rules of classical logic (see §4.4) hold under this interpretation. (For instance, for a formula \( q(x) \) to be valid in such a model \( V \), we must have \( q^V(d) = S \) for each \( d = \mu(x) \in D \); and for \( \forall x q(x) \) to be valid in \( V \), we must have \( \bigcap_{d \in D} q^V(d) = S \); this establishes the generalization rule.)

In the case where \( S \) is a singleton, \( \{\ast\} \), Euler models reduce to the two-valued models of the previous section. Two-valued models suffice to discern classical truth; in particular, a proposition that is (not) true in some Euler model is also (not) true in some two-valued model. Namely, to get such a two-valued model from an Euler model, pick any \( s \in S \), and send a propositional variable \( \pi_i \) to \( \{\ast\} \) if \( s \in \pi_i^V \) and to \( \emptyset \) otherwise; similarly for predicate variables.

On the other hand, classical logic is complete with respect to some Euler model with countable domain [118; VIII.3.3] but of course not with respect to any two-valued model — not even in the propositional case. Also, a formula \( q(x) \) is satisfiable in a two-valued model \( M \) (that is, becomes valid in \( M \) under some assignment of variables) if and only if \( \exists x q(x) \) is satisfiable in \( M \); of course, this strange property does not hold in Euler models.

Nevertheless, Euler models such that \( S = D^k \) for some \( k \) can be derived from two-valued models, by introducing \( k \) implicit parameters. Thus we fix a domain \( D \), and to interpret an \( n \)-ary predicate variable \( \pi_{ni} \) in an Euler model \( V \) with \( S = D^k \) we look at the interpretation of the \( (n+k) \)-ary predicate variable \( \pi_{n+k,i} \) in a two-valued model \( M \) with the same domain \( D \), and set

\[
\pi_{n,i}^V(x_1, \ldots , x_n) = \{(x_{n+1}, \ldots , x_{n+k}) \in D^k \mid \pi_{n+k,i}^M(x_1, \ldots , x_{n+k}) = \top\}.
\]

**Example 4.5.** By the deduction theorem, either in classical or intuitionistic logic, \( \varphi \vdash \psi \) is equivalent to \( \vdash \varphi \to \psi \) as long as \( \varphi \) has no parameters. However, already in an Euler
model $V$ of classical logic, $\varphi \models_V \psi$ need not be equivalent to $\models_V \varphi \rightarrow \psi$ where $\psi = \bot$, and $\varphi = \exists x \, p(x) \land \exists x \, \neg p(x)$ has no parameters.

Indeed, let $M$ and $N$ be the two-valued models described in the previous two examples, with $D_M = \{0, 1\}$, $\pi^M(0) = \bot$ and $\pi^M(1) = \top$, and with $D_N = \{\ast\}$ and $\pi^N(\ast) = \top$. Let $V$ be an Euler model with $S = \{m, n\}$, with $D = \{0, 1\}$ and with $m \in q^V(t)$ if and only if $q^M(t) = \top$ and with $n \in q^V(t)$ if and only if $q^N(\ast) = \top$, for any predicate $q(t)$.

Then, in particular, $\pi^V(0) = \{n\}$ and $\pi^V(1) = \{m, n\}$. Consequently we have $[\exists x \pi(x)]^V = \{m, n\}$ and $[\exists x \neg \pi(x)]^V = \{m\}$. Hence $[\exists x \pi(x) \land \exists x \neg \pi(x) \rightarrow \bot]^V = \{n\}$, and so $\not\models_V \exists x \, p(x) \land \exists x \, \neg p(x) \rightarrow \bot$.

On the other hand, by construction, $p^V(0)$ contains $n$ if and only if $p^V(1)$ does so. Hence $[\exists x \, p(x) \land \exists x \, \neg p(x)]^V$ never contains $n$. Thus $\not\models_V \exists x \, p(x) \land \exists x \, \neg p(x)$.

4.7. Extended BHK interpretation

The clarified BHK interpretation as presented in §3.9 is an informal interpretation of only the “formulaic” aspect of intuitionistic logic. In particular, it gives an interpretation of $\vdash \alpha_1 \land \ldots \land \alpha_n \rightarrow \beta$ and of admissible rules, “$\vdash \alpha_1, \ldots, \vdash \alpha_n$ imply $\vdash \beta$”. In fact, it is the latter that Kolmogorov presents as his interpretation of a version of Heyting’s inference rules (except that he treats them as problems and not judgements) [76].

To extend this interpretation to the “inferential” conception of intuitionistic logic, we also need an interpretation of syntactic consequence, $\alpha_1, \ldots, \alpha_n \vdash \beta$.

As a preliminary step, let us first reformulate the clarified BHK interpretation so that when instantiating the metavariabes $\gamma_i$ of the given schema $\Sigma(\gamma_1, \ldots, \gamma_n)$ by particular problems $\Gamma_i$, we allow the $\Gamma_i$ to depend on additional parameters (apart from those specified explicitly in the schema), provided that the resulting problem $\Sigma(\Gamma_1, \ldots, \Gamma_n)$ retains each occurrence of each additional parameter as a parameter (i.e. it is not bound by any quantifier). Such an instantiation is similar to the instantiation in the sense of §4.2; to distinguish it from direct instantiation as used in §3.9, we will call it instantiation with anonymous parameters (the parameters being “anonymous” in the sense of Kleene [73; §34]). As discussed in §3.9, to interpret the resulting problem $\Sigma(\Gamma_1, \ldots, \Gamma_n)$ according to the BHK clauses, we first replace it by its parameterless universally quantified closure (now quantified over both explicit and anonymous parameters), which we will now denote $UC[\Sigma(\Gamma_1, \ldots, \Gamma_n)]$.

For example, the schema $\gamma \lor \neg \gamma$ can now be instantiated not only by parameterless problems of the form $\Gamma \lor \neg \Gamma$, but also by the parametric problem $\Delta(t) \lor \neg \Delta(t)$, say, where $\Delta(t)$ asks to trisect any given angle $t$, rather than some particular fixed angle. As already observed in §3.10, this comes at no loss of generality; for if we have a general method of solving all problems of the form $\Gamma \lor \neg \Gamma$, then it applies, in particular, to all our instantiated problems $\Delta(t_0) \lor \neg \Delta(t_0)$, and thus constitutes a general method of solving the latter for each $t_0$ — that is, a solution of $\forall t [\Delta(t) \lor \neg \Delta(t)]$.

We are now ready to present what we call the extended BHK interpretation of intuitionistic logic (in the inferential conception). Given schemata $\Sigma_i(\gamma_1, \ldots, \gamma_n)$, $i = 0, \ldots, m$,
in the metavariables $\gamma_i$, we interpret
$$\Sigma_1(\gamma_1, \ldots, \gamma_n), \ldots, \Sigma_m(\gamma_1, \ldots, \gamma_n) \vdash \Sigma_0(\gamma_1, \ldots, \gamma_n)$$
by the judgement that There exists a general method of solving the problem
$$UC[\Sigma_1(\Gamma_1, \ldots, \Gamma_n) \land \cdots \land \Sigma_m(\Gamma_1, \ldots, \Gamma_n)] \to UC[\Sigma_0(\Gamma_1, \ldots, \Gamma_n)]$$
for every instantiation with anonymous parameters of the metavariables $\gamma_i$ by particular problems $\Gamma_i$. What it means to be a solution of the resulting problem $UC[\ldots] \to UC[\ldots]$ is, of course, determined by the usual BHK interpretation (see §3.6).

There is a similar informal interpretation of the consequence relation in classical logic, which does not seem to be well-known. Given schemata $\Sigma_i(p_1, \ldots, p_n), i = 0, \ldots, m$, in the metavariables $p_i$, we interpret
$$\Sigma_1(p_1, \ldots, p_n), \ldots, \Sigma_m(p_1, \ldots, p_n) \vdash \Sigma_0(p_1, \ldots, p_n)$$
by the judgement that The proposition
$$UC[\Sigma_1(P_1, \ldots, P_n) \land \cdots \land \Sigma_m(P_1, \ldots, P_n)] \to UC[\Sigma_0(P_1, \ldots, P_n)]$$
is true for every instantiation with anonymous parameters of the metavariables $p_i$ by particular propositions $P_i$. What it means for the resulting proposition $UC[\ldots] \to UC[\ldots]$ to be true is, of course, determined by the usual truth tables (see §3.1). To fix a name, we will call this simply the extended interpretation of classical logic (in the inferential conception).

Let us note that this informal interpretation of the classical syntactic consequence relation is, in fact, closely related to its formal interpretation by the semantic consequence relation in Euler models. Indeed, as discussed in §4.6, the anonymous parameters amount essentially to the ground set of an Euler model.

Let us also note that there is a straightforward generalization of the extended interpretation of classical logic to the modal logic QS4 (see §5.8), which is similarly compatible with topological models of QS4. Thus it can be said that the extended interpretation provides a satisfactory “explanation” not only of the generalization rule of classical logic, but also of the necessitation rule of QS4.

### 4.8. Modified BHK interpretation

The informal judgement For every particular proposition $P$, the proposition $P \lor \neg P$ is true contains quantification over all propositions, which might sound weird; however, it essentially amounts to a double quantification in more familiar terms: first over all models of classical logic, and then over all propositions within a model. In contrast, There exists a general method of solving the problem $\Gamma \lor \neg \Gamma$ for every particular problem $\Gamma$ contains constructive quantification over all problems, which does not seem to be compatible with usual model theory. (Presumably, this is one reason for the difficulties with explaining $\vdash$ in the literature that deals with the BHK interpretation.) The “modified
BHK interpretation” overcomes this difficulty by replacing this unbounded constructive quantification with a usual unbounded quantification plus a bounded constructive quantification. Related ideas are also found in [87; Definition 4.2] (see also [117; Ch. 7]).

Given schemata \( \Sigma_i(\gamma_1, \ldots, \gamma_n) \), \( i = 0, \ldots, m \), in the metavariables \( \gamma_i \), the modified BHK interpretation interprets

\[
\Sigma_1(\gamma_1, \ldots, \gamma_n), \ldots, \Sigma_m(\gamma_1, \ldots, \gamma_n) \vdash \Sigma_0(\gamma_1, \ldots, \gamma_n)
\]

by the judgement that For every instantiation with anonymous parameters of the metavariables \( \gamma_i \) by particular problems \( \Gamma_i \) there exists a solution the problem

\[
UC[\Sigma_1(\Gamma_1, \ldots, \Gamma_n) \land \cdots \land \Sigma_m(\Gamma_1, \ldots, \Gamma_n)] \rightarrow UC[\Sigma_0(\Gamma_1, \ldots, \Gamma_n)]
\]

What it means to be such a solution is, of course, determined by the usual BHK interpretation (see §3.6).

For example, \( \vdash \gamma \lor \neg \gamma \) is interpreted by the judgement that For every particular parametric problem \( \Gamma(t) \) there exists a solution of \( \forall t [\Gamma(t) \lor \neg \Gamma(t)] \); that is, a general method of solving the problem \( \Gamma(t) \lor \neg \Gamma(t) \) for every particular value of \( t \). (Strictly speaking, we should have allowed multi-parameter problems \( \Gamma(t_1, \ldots, t_r) \), but since we put no restrictions on the ranges of the parameters, the tuple of parameters boils down to a single vector parameter \( t = (t_1, \ldots, t_r) \).) Let us note that the arguments of §3.10 and §3.11 still work to show that \( \not\vdash \gamma \lor \neg \gamma \) under this modified BHK interpretation.

It should be noted that unless we are content with a computational interpretation of general methods as algorithms (as in §3.10), the anonymous parameters would typically range over uncountable domains (as in §3.11), which may well be thought of as topological spaces (compare §5 and §6). Of course, problems with continuous parameters abound in mathematical practice: for instance, to find a root of a parametric algebraic equation (see Example 6.2), or to find a solution of an initial value problem (where either the initial value or some coefficient of the equation is treated as a parameter). On the other hand, the explicit parameters of the schemata \( \Sigma_i(\gamma_1, \ldots, \gamma_n) \) are normally understood in first-order logic to range over the domain of discourse, which is traditionally thought of as countable (and “discrete”). While this issue is entirely informal, it deserves to be taken into account when dealing with the (anyway informal) modified BHK interpretation.

An advantage of the modified BHK interpretation is its still better compatibility with the spirit of classical mathematics; a disadvantage is its notable divergence from Kolmogorov’s conception of intuitionistic logic as the logic of problem solving schemes, which is certainly better captured by the clarified/extended BHK interpretation.

5. Topology, translations and principles

Our next goal is to naturally arrive at Tarski’s topological models of intuitionistic logic by means of a symbolic analysis of the BHK interpretation.
5.1. Deriving a model from BHK

Given a problem $\Gamma$, let $\exists \Gamma$ denote, as in §3.7, the proposition $A$ solution of $\Gamma$ exists. In the notation of §3.8, $\exists \Gamma$ can be also expressed as $|\Gamma| \neq \emptyset$, where $|\Gamma|$ denotes the set of solutions of $\Gamma$. Validity of propositions of the form $\exists \Gamma$ is understood to be determined, at least to some extent, by the BHK interpretation (in particular, it should be known from context for primitive $\Gamma$). Then the BHK entails, in particular, the following judgements, which involve both intuitionistic and classical connectives:

1. $\exists (\Gamma \land \Delta)$ if and only if $\exists \Gamma \land \exists \Delta$;
2. $\exists (\Gamma \lor \Delta)$ if and only if $\exists \Gamma \lor \exists \Delta$;
3. $\exists (\Gamma \to \Delta)$ implies $\exists \Gamma \to \exists \Delta$;
4. $\neg \exists \bot$;
5. $\exists x \exists \Gamma(x)$ if and only if $\exists x \exists \Gamma(x)$;
6. $\forall x \exists \Gamma(x)$ implies $\forall x \exists \Gamma(x)$.

Judgements (1)–(6) are perhaps best understood as immediate consequences of the six assertions in §3.8, which were in turn obtained directly from the BHK interpretation. For instance, from $|\Gamma \lor \Delta| = |\Gamma| \cup |\Delta|$ we immediately obtain (2), since $S \cup T \neq \emptyset$ if and only if $S \neq \emptyset \lor T \neq \emptyset$. On the other hand, since $\bot = \bot \to \bot$ by definition, judgement (7) can be considered as a special case of either $\exists (\Gamma \to \Gamma)$ or $\neg (\bot \to \Gamma)$. Both are clearly true on the BHK interpretation: (i) given a solution of $\Gamma$, by doing nothing to it we get a solution of $\Gamma$, which is a general method hence a solution of $\Gamma \to \Gamma$; (ii) $\bot \to \Gamma$ was already discussed in §3.12, (4).

Given a formula $\varphi$ with $n$ parameters, which is understood to belong to intuitionistic logic and thus to denote a problem of the form $\Phi(x_1, \ldots, x_n)$, we may introduce an $n$-ary predicate variable $\exists \varphi$, to be treated as an object of classical logic, which is understood to denote propositions of the form $\exists \Phi(x_1, \ldots, x_n)$. Then the assertions in the above list can be seen, just like in §3.9, as informal interpretations of the following judgements about schemata:

1. $\vdash ?(\varphi \land \psi) \iff \varphi \land ?\psi$;
2. $\vdash ?(\varphi \lor \psi) \iff \varphi \lor ?\psi$;
3. $\vdash ?(\varphi \to \psi) \to (?\varphi \to ?\psi)$;
4. $\vdash \neg ?\bot$;
5. $\vdash ?\exists x \varphi(x) \iff \exists x ?\varphi(x)$;
6. $\vdash ?\forall x \varphi(x) \to \forall x ?\varphi(x)$;
7. $\vdash ?\top$.

The schemata to the right of the $\vdash$ symbols can be understood as axiom schemes of a first-order theory $T_{BHK}$ based on classical logic.

Suppose now that we have an Euler model $V$ of this theory in subsets of a set $S$, where every element of the domain $D$ interprets some constant of $T_{BHK}$ (we are thus assuming the language of $T_{BHK}$ to contain enough constants). Let us call a subset of $S$
open if it is of the form \((?\varphi)^V\) for some formula \(\varphi\) with no parameters. Since elements of \(D\) interpret constants, subsets of the form \((?\psi)^V_\mu\) are also open, for formulas \(\psi\) with any number of parameters, and for each valuation \(\mu\). Then \(\emptyset\) is open by (4), and \(S\) is open by (7). Also, the intersection of any two open subsets is open by (1), and the union of any two open subsets, or the union of a family of open subsets that interprets some predicate, is open by (2) and (5). For reasons pertaining to cardinality we cannot be sure that the union of any family of open subsets is open, but otherwise this would be just a topology on \(S\).

Let us then see if we can get something interesting under somewhat stronger hypotheses — not entirely covered by the BHK interpretation. Firstly, suppose that a topology on \(S\) is given (for we almost have it anyway), and that our Euler model \(V\) is open set-valued, that is, subsets of \(S\) of the form \((?\psi)^V_\mu\) are open (in other words, the given topology agrees with what we anyway have of a topology). Secondly, let us strengthen the axiom schemes (3) and (6), which we have not used at all as yet, so as compensate for the vagueness of the BHK clauses for \(\to\) and \(\forall\):

\[
(3') \vdash ?\chi \to (?\varphi \to ?\psi) \quad \iff \quad (?\chi) \to (?\varphi \to ?\psi); \\
(6') \vdash (?\chi) \to ?\forall x ?\varphi(x) \quad \iff \quad (?\chi) \to ?\forall x ?\varphi(x).
\]

Indeed, (3') implies (3) by considering \(\chi = \varphi \to \psi\), and (6') implies (6) by considering \(\chi = \forall x ?\varphi(x)\). Let us emphasize that there is nothing in the BHK interpretation that would suggest strengthening (3) and (6) in this particular way — although some strengthening of (3) and (6) is certainly present in the BHK interpretation.

In terms of our Euler model, (3’) guarantees that \([(?\varphi \to ?\psi)^V]\) is the largest open set contained in \([(?\varphi \to ?\psi)^V]\), and (6’) guarantees that \([(?\forall x ?\varphi(x))^V]\) is the largest open set contained in \([(?\forall x ?\varphi(x))^V]\). Symbolically, \([(?\varphi \to ?\psi)^V] = \text{Int}[(S \setminus [(?\varphi)^V] \cup [(?\psi)^V])]\) and \([(?\forall x ?\varphi(x))^V] = \text{Int} \bigcap_{d \in D} [(?\varphi)^V(d)]\), where \(\text{Int}\) denotes topological interior.

Finally, if we understand the open set \((?\varphi)^V\) as an interpretation of the “problem” \(\varphi\) (and not just of the “proposition” \(?\varphi\)), we will get a model of intuitionistic logic.

5.2. Tarski models

Let us summarize in closed terms the models of intuitionistic logic that we have just obtained. In the zero-order case, they were found independently by Stone [133], Tang [135] (see also §5.9) and Tarski [137] in the 1930s, with Tarski having also established completeness of zero-order intuitionistic logic with respect to this class of models. The models were extended to the first-order case by Mostowski, with completeness established by Rasiowa and Sikorski (see §5.3).

We fix a topological space \(X\) and a domain \(D\). The set \(\Omega\) of truth values will consist of all open subsets of \(X\). Only the entire space \(X\) will be a truth. Thus a formula \(\alpha\) with no parameters is represented by an open subset of \(X\), to be denoted \(|\alpha|\), and in general a formula with \(n\) parameters is represented by a \(D^n\)-indexed family of open subsets of \(X\). Validity in the model, \(\models \alpha\), means that \(|\alpha| = X\) if \(\alpha\) has no parameters, and that \(|\alpha|(t_1, \ldots, t_n) = X\) for all \((t_1, \ldots, t_n) \in D^n\), if \(\alpha\) has \(n\) parameters.
Disjunction of formulas corresponds to union of subsets, conjunction to intersection, and \( \bot \) to the empty set. Implication is interpreted by \(|\alpha \rightarrow \beta| = \text{Int}((X \setminus |\alpha|) \cup |\beta|)\). In other words, \(|\alpha \rightarrow \beta|\) is the union of all open sets \( U \) such that \( U \cap |\alpha| \subset U \cap |\beta|\). Universal quantification corresponds to \( D \)-indexed union, and existential to the interior of the \( D \)-indexed intersection.

In particular, we have \(|\neg \alpha| = \text{Int}(X \setminus |\alpha|) = X \setminus \text{Cl} |\alpha|\) and \(|\neg \neg \alpha| = \text{Int} (\text{Cl} |\alpha|)\). Thus \textit{decidable} formulas (i.e. \( \alpha \) such that \( \vdash \alpha \lor \neg \alpha \)) are represented by clopen (=closed and open) sets; and \textit{stable} formulas (i.e. \( \alpha \) such that \( \vdash \neg \neg \alpha \rightarrow \alpha \)) are represented by regular open sets (i.e. sets equal to the interior of their closure). Stable formulas coincide (by \((3)\) and \((1)\) in §3.12) with those that are equivalent to a negated formula. Decidable formulas are stable by \((6)\). On the other hand, if \( \alpha \lor \neg \alpha \) is stable, then \( \alpha \) is decidable by \((26)\).

Tarski models yield simple and intuitive proofs that many classically valid principles are not derivable in intuitionistic logic. Here are a few examples.

\textbf{Example 5.1.} The converse to \((13)\), known as the \textit{Constant Domain Principle} (for reasons related to Kripke models):

\[ \forall x (\alpha \lor \pi(x)) \rightarrow \alpha \lor \forall x \pi(x) \]

is not an intuitionistic law, as witnessed by a Tarski model with \( X = \mathbb{R}, D = \mathbb{N}, |\pi| = \{(x,y) | x > 0, y \leq 0\} \) and \( |\alpha| = \mathbb{R} \setminus \{0\} \).\textsuperscript{33} A simple modification of this model, which we purposely leave to the reader, disproves the \textit{Negative Constant Domain Principle},

\[ \forall x (\neg \alpha \lor \pi(x)) \rightarrow \neg \alpha \lor \forall x \pi(x) \].

\textbf{Example 5.2.} The converse to a special case of \((10)\), the principle of \textit{Independence of Premise}:

\[ (\neg \alpha \rightarrow \exists x \pi(x)) \rightarrow \exists x (\neg \alpha \rightarrow \pi(x)) \]

is not derivable in intuutionistic logic. In fact, even if \( \exists \) is “specialized” to \( \lor \), the resulting formula, known as the \textit{Kreisel–Putnam Principle}:

\[ (\neg \alpha \rightarrow \beta \lor \gamma) \rightarrow (\neg \alpha \rightarrow \beta) \lor (\neg \alpha \rightarrow \gamma) \]

is not an intuitionistic law. Indeed, let \( X = \mathbb{R}^2 \),

\[ |\beta| = \{(x,y) | x > 0\}, \]
\[ |\gamma| = \{(x,y) | y > 0\} \]

and \( |\alpha| = -(\beta \lor \gamma) \), so that \( |\neg \alpha| = \{(x,y) | x > 0 \lor y > 0\} \). Then \( \neg \alpha \rightarrow \beta \lor \gamma = \mathbb{R}^2 \).

\textsuperscript{33}Hereafter it is tacitly suggested to obtain the desired counterexample by instantiating the metavariables \( \alpha, \pi(x) \), etc. by their boldface cousins, the logical variables \( \alpha, \pi(x) \), etc. One should not forget, however, of the difference: the logical variables \( \alpha, \pi(x) \) have no parameters, resp. only one parameter; whereas the indeterminate formulas \( \alpha, \pi(x) \) may have any number of parameters, resp. at least one parameter.
whereas

\[ |\neg \alpha \to \beta| = \{(x, y) \mid x > 0 \lor y < 0\}; \]

\[ |\neg \alpha \to \gamma| = \{(x, y) \mid x < 0 \lor y > 0\}. \]

Hence \(|\neg \alpha \to \beta| \cup |\neg \alpha \to \gamma| = \mathbb{R}^2 \setminus \{(0, 0)\}\).

This Tarski model shows also that Harrop’s Rule:

\[
\frac{\neg \alpha \to \beta \lor \gamma}{\neg \alpha \to \beta \lor (\neg \alpha \to \gamma)}
\]

is not derivable in intuitionistic logic. However, it is known to be an admissible rule for intuitionistic logic [??]. The stronger rule:

\[
\frac{\alpha \to \beta \lor \gamma}{(\alpha \to \beta) \lor (\alpha \to \gamma)}
\]

is not even admissible, by considering its special case with \(\alpha = \beta \lor \gamma\). In this case the premiss is an intuitionistic law, but the conclusion,

\[(\beta \lor \gamma \to \beta) \lor (\beta \lor \gamma \to \gamma)\]

is not — in fact the previous example in \(\mathbb{R}^2\) works to show this.

5.3. On completeness

Rasiowa and Sikorski proved that every consistent first-order theory \(T\) based on intuitionistic logic is complete with respect to a certain Tarski model with countable \(D\) and with \(X\) a subspace of the Baire space (that is, \(\mathbb{N}^\mathbb{N}\) with the product topology; this space is well-known to be homeomorphic to the set of irrational reals) [118; X.3.2]. In particular, intuitionistic logic is complete with respect to one such Tarski model, and strongly complete with respect to the class of all such Tarski models. The latter result was improved by Dragalin, who showed that intuitionistic logic is strongly complete with respect to all Tarski models with countable domain and with \(X\) the Baire space itself [35] (see also Remark 4.2). It is easy to see that intuitionistic logic is not strongly complete for Tarski models in any connected space, including Euclidean spaces (see §5.5).

Originally, Tarski [137] (see also [96]; for a sketch, see [132; 2.4.1], and for simplified proofs see [130], [80], [81] and references there) proved that if \(X\) is a metrizable space with no isolated points (for instance, \(\mathbb{R}^n\) for any \(n > 0\)), then zero-order intuitionistic logic is complete with respect to the class of Tarski models in \(X\) with countable domain. For instance, the Kreisel–Putnam Principle (and, in fact, even Harrop’s Rule) is also disproved by the model with \(X = \mathbb{R}\), \(|\neg \alpha| = (0, 1), |\beta| = \bigcup_{i=1}^{\infty} (\frac{1}{2i+2}, \frac{1}{2i})\) and \(|\gamma| = \bigcup_{i=1}^{\infty} (\frac{1}{2i+1}, \frac{1}{2i-1})\). In general, it should be noted that in general open subsets of \(\mathbb{R}\) can be quite complicated (see [105; §III.4, Example 8])

\[34\text{Thus if } \alpha \text{ is not a theorem of } T, \text{ then } \alpha \text{ is not valid in some such Tarski model of } T. \text{ In the case } \alpha = \neg \beta, \text{ Rasiowa and Sikorski further show that } \beta \text{ is satisfiable in some (other) such Tarski model [118; X.3.4].}\]
Tarski’s result was only recently extended to the full intuitionistic logic by Kremer [83], who showed it to be complete for Tarski models with countable domain in zero-dimensional separable metrizable spaces with no isolated points. These include the Cantor set, the Baire space, and the set of rational numbers. On the other hand, intuitionistic logic is not complete for Tarski models in any locally connected space, including open subsets of Euclidean spaces [83].

It is interesting to note that the judgement \( \vdash \beta \lor \neg \beta \) then \( \vdash \beta \) or \( \vdash \neg \beta \), provided that \( \beta \) has no parameters is valid in every connected space \( X \) (since its only clopen sets are \( \emptyset \) and \( X \)), and in particular in all topological models with \( X = \mathbb{R} \). It is clearly not valid in some topological models with disconnected \( X \), and in particular it is not true that for any two schemata \( \Phi \) and \( \Psi \) with no parameters, either \( \Phi \lor \Psi \vdash \Phi \) or \( \Phi \lor \Psi \vdash \Psi \). However, the corresponding external judgement (which can be regarded as a non-structural multiple-conclusion admissible rule): For any two schemata \( \Phi \) and \( \Psi \) with no parameters, \( \vdash \Phi \lor \Psi \) implies either \( \vdash \Phi \) or \( \vdash \Psi \) — is true [??], and is known as the Disjunction Property of intuitionistic logic.

McKinsey and Tarski also proved that zero-order intuitionistic logic is complete with respect to the class of Tarski models with finite \( X \). Finite topological spaces are included in the class of Alexandroff spaces, where the intersection of any (possibly infinite) family of open subsets of \( X \) is open. The relation \( x \in \text{Cl}(y) \) on the points \( x, y \) of an Alexandroff space is a preorder, i.e. it is reflexive and transitive. Conversely, any preordered set \( P \) can be endowed with its Alexandroff topology, where a subset of \( P \) is defined to be closed if it is an order ideal. This gives a one-to-one correspondence between Alexandroff spaces and preordered sets, under which \( T_0 \) spaces correspond precisely to posets. It is immediate from this that in the case of zero-order formulas, Tarski models in Alexandrov spaces are essentially the same as the so-called Kripke models of intuitionistic logic (see ?? concerning the first-order case). Intuitionistic logic is well-known to be complete with respect to its Kripke models over posets. Note that the Alexandroff topology on a poset can only be \( T_1 \) if it is discrete. Note also that every Alexandroff space \( X \) admits a continuous open surjection onto the \( T_0 \) Alexandroff space obtained as the quotient of \( X \) by the equivalence relation whose equivalence classes are the minimal nonempty open sets of \( X \).

If \( K \) is a simplicial complex (or more generally a cell complex or a cone complex, see [100]), viewed as a topological space (with the metric topology, see [101]), let \( X(K) \) be the face poset of \( K \), viewed as a \( T_0 \) Alexandroff space. By sending a point of \( K \) to the minimal simplex (or cell or cone) that contains it, we obtain a continuous open surjection \( K \rightarrow X(K) \). Conversely, given a \( T_0 \) Alexandroff space \( X \), viewed as a poset, we have its order complex, whose simplices can be arranged into cones of a cone complex \( K(X) \) such that \( X(K(X)) = X \) (see [100]). Thus we have a continuous open surjection

\[^{35}\text{That is, spaces where at least one out of any two distinct points is contained in an open set not containing the other point.}\]

\[^{36}\text{A topological space is called } T_1 \text{ if for every pair of distinct points, each is contained in an open set not containing the other one; or equivalently if all singleton subsets are closed.}\]
Thus a Tarski model in an Alexandroff space \( X \) gives rise to a Tarski model in the polyhedron \( K(X) \), and all completeness results can be transferred accordingly.

### 5.4. Markov’s principle

A striking omission in our list of intuitionistic laws in §3.12 is the converse to (17), which is sometimes referred to as the Generalized Markov Principle:

\[
\neg \forall x \pi(x) \implies \exists x \neg \pi(x). \quad \text{(GMP)}
\]

This is far from being an intuitionistic law, as it fails even if \( \exists \) and \( \forall \) are “specialized” to \( \lor \) and \( \land \). Indeed, to see that \( \neg (\pi_1 \land \pi_2) \not\equiv \neg \pi_1 \lor \neg \pi_2 \) it suffices to take \( \pi_2 = \neg \pi_1 \), where \( \pi_1 \) is stable but not decidable. Also, (GMP) would fail if \( \exists \) and \( \forall \) are “specialized” to \( \land \) and \( \to \). Indeed, \( \neg (\chi \to \pi) \not\equiv (\chi \land \neg \pi) \) already for \( \pi = \bot \) and any non-stable \( \chi \).

For stable problems \( \pi(x) \leftrightarrow \neg \rho(x) \), (GMP) is equivalent, via (16), to the following Strong Markov Principle

\[
\neg \exists x \rho(x) \implies \exists x \neg \rho(x) \quad \text{(SMP)}
\]

whose converse is the intuitionistic law (40). This principle says, in particular, that if \( \rho(x) \) is stable, then so is \( \exists x \rho(x) \). Note that \( \forall x \pi(x) \) is always stable if \( \pi(x) \) is, due to the intuitionistic law (39).

If \( \exists \) is “specialized” to \( \land \) in (SMP):

\[
\neg \exists (\chi \land \rho) \implies (\chi \land \neg \rho)
\]

this still fails in general (e.g. for \( \rho = \bot \to \bot \) and any non-stable \( \chi \)), but holds for stable \( \chi \) and \( \rho \) by (35) and (24). If \( \exists \) is “specialized” to \( \lor \) in (SMP):

\[
\neg \exists (\rho_1 \lor \rho_2) \implies \neg \rho_1 \lor \neg \rho_2
\]

this still fails for stable \( \rho_i \) (the above example works), but holds for decidable \( \rho_i \), since in that case \( \rho_1 \lor \rho_2 \) is also decidable, by (24).

The following Inferential Markov Principle asserts that (SMP) holds for all decidable problems \( \rho(x) \); or equivalently that (GMP) holds for all decidable \( \pi(x) \):

\[
\forall x (\pi(x) \lor \neg \pi(x)) \implies \neg \forall x \pi(x) \to \exists x \neg \pi(x). \quad \text{(IMP)}
\]

To see that (IMP) is not derivable in QH, consider a Tarski model with \( D = \mathbb{N} \) and \( X = 2^\mathbb{N} \), the Cantor set of all functions \( \mathbb{N} \to \{0, 1\} \), and for a given \( n \in \mathbb{N} \) let \( |\pi|(n) \) be the subset of \( 2^\mathbb{N} \) consisting of all functions \( f(n) = 1 \). Then each \( |\pi|(n) \) is clopen, so that \( \rho(x) = \neg \pi(x) \) may be decidable; on the other hand, \( \bigcup_{n \in \mathbb{N}} |\rho|(n) = 2^\mathbb{N} \setminus \{(1, 1, \ldots)\} \) is not regular open. Another Tarski model where (IMP) fails has \( D = \mathbb{N} \) and \( X = \mathbb{N}_+ \), the one-point compactification of the countable discrete space \( \mathbb{N} \), and \( |\rho|(n) = \{n\} \). Note that in these models even the following Markov Rule fails:

\[
\forall x (\pi(x) \lor \neg \pi(x)), \neg \forall x \pi(x) \implies \exists x \neg \pi(x).
\]

Of course, by the deduction theorem, both (IMP) and the Markov Rule are syntactically equivalent to what some call the Markov Principle:

\[
\forall x (\pi(x) \lor \neg \pi(x)) \implies \neg \forall x \pi(x) \to \exists x \neg \pi(x),
\]
in the sense that they determine the same extension of intuitionistic logic. However they are semantically distinct in that a model of intuitionistic logic satisfying the Markov Rule might fail to satisfy (IMP), and one satisfying (IMP) might fail to satisfy the Markov Principle.

Markov’s original formulation was concerned with the case where $x$ ranges over $\mathbb{N}$.\(^{37}\) A decidable problem $\pi(n)$ is equivalent to the problem of verifying that $\pi_n = 1$, where each $\pi_n \in \{0, 1\}$ is defined by $\pi_n = 1$ if $\pi(n)$ is soluble, and $\pi_n = 0$ if $\neg \pi(n)$ is soluble. Under the BHK interpretation, the problem $\exists n \neg \pi(n)$ asks essentially to find a zero in the sequence $(\pi_i)$, and the problem $\forall n \pi(n)$ asks essentially to prove (taking into account Kreisel’s remark) that each entry in the sequence is one. Markov’s Principle is then asserting that there exists a general method that, given a sequence of zeroes and ones, and a proof that this sequence cannot be proved to be all ones, produces a zero somewhere in this sequence. Of course, we are not really told how to find this zero by constructive means, so this principle is not justified by the clarified BHK interpretation.

However, on a very computational reading of the BHK interpretation, Markov’s principle is validated. Indeed, suppose we have a Turing machine that reads a sequence of zeroes and ones until it finds a zero. If this machine continues forever on some sequence, this can (perhaps) be considered a proof that the sequence is all ones. Now, given a proof that the machine does not continue forever on some sequence, by actually running it on this sequence we get a general method of finding a zero in the sequence. In this connection Markov’s constructivist school considered Markov’s Principle to be constructively acceptable (cf. [145]).

5.5. Principle of Omniscience

The Inferential Markov Principle should not be confused with the stronger Rule of Omniscience:\(^{38}\)

$$\forall x (\pi(x) \lor \neg \pi(x)) \quad \text{(RO)}$$

which implies (IMP) by (5). Using (6) and (16), one can reformulate (RO) in terms of $\rho(x) = \neg \pi(x)$:

$$\forall x (\rho(x) \lor \neg \rho(x)) \quad \text{(RO)}$$

Thus (RO) asserts precisely that the problem $\exists x \rho(x)$ is decidable whenever $\rho(x)$ is. In particular, (RO) is satisfied in spaces where arbitrary union of clopen sets is clopen. These include all Alexandroff spaces and all connected spaces. In particular, Tarski models in Euclidean spaces satisfy (RO), and so intuitionistic logic is not strongly complete with respect to this class of models.

\(^{37}\)In some form, Markov’s Principle was discussed long before A. A. Markov, Jr. (who himself called it the “Leningrad Principle”), in particular by P. S. Novikov [106] and, according to Hilbert [67; p. 268], already by Kronecker.

\(^{38}\)This is a form of Bishop’s “Principle of Omniscience” [16].
The symmetric assertion, that \( \forall x \rho(x) \) is decidable whenever \( \rho(x) \) is, can be called the \textit{Weak Rule of Omniscience}:

\[
\forall x (\pi(x) \lor \neg \pi(x)) \\
\forall x \pi(x) \lor \neg \forall x \pi(x)
\]

(WRO)

because it follows from (RO) by (17). Also, (WRO) together with (IMP) obviously imply (RO). Thus in fact

\[\text{RO} \iff \text{WRO} \land \text{IMP}.\]

When \( x \) ranges over \( \mathbb{N} \), the three principles can be recast under the BHK interpretation in terms of the Cantor set \( 2^\mathbb{N} \) of all functions \( \mathbb{N} \to \{0, 1\} \). If \( \infty \) denotes the constant function \( n \mapsto 1 \), and \( p \) is understood to range over \( 2^\mathbb{N} \), then the three principles assert the existence of general methods for solving the following problems:

- Weak Rule of Omniscience: \( p = \infty \lor \neg(p = \infty) \);
- Inferential Markov Principle: \( \neg(p = \infty) \to p \neq \infty \);
- Rule of Omniscience: \( p = \infty \lor p \neq \infty \),

where \( p = q \) denotes the problem, \textit{Prove that} \( p(n) = q(n) \) for all \( n \in \mathbb{N} \), and \( p \neq q \) denotes the problem, \textit{Find an} \( n \in \mathbb{N} \) \textit{such that} \( p(n) \neq q(n) \). As observed by Escardo [40], there is no loss of generality in permitting \( p \) to range just over the subset \( \mathbb{N}_+ = \mathbb{N} \cup \{\infty\} \), where \( \mathbb{N} \) is a copy of \( \mathbb{N} \). This follows from the existence of a constructive retraction \( r \) of \( 2^\mathbb{N} \) onto \( \mathbb{N}_+ \), defined by \( r(p)(n) = \min\{p(k) \mid k \leq n\} \).

5.6. \( \neg
\neg \)-Translation

A translation of classical predicate logic QC into intuitionistic predicate logic QH was discovered essentially by Kolmogorov (even though he only had his formalization of a fragment of QH available at the moment) [75]. Given a formula \( p \) in the language of classical logic, insert the double negation \( \neg\neg \) in front of every subformula; then the resulting formula \( \neg\neg p \) will be derivable in intuitionistic logic if and only if \( p \) is derivable in classical logic. Kolmogorov’s translation was later rediscovered by Gödel and Gentzen, who gave proofs in the setup of QH and also observed that prefixing \( \land, \forall \) and (in the case of Gentzen) \( \to \) by \( \neg\neg \) is superfluous:

\textbf{Proposition 5.3.} The following holds in intuitionistic logic:

\( (a) \neg\neg \alpha \land \neg\neg \beta \iff \neg\neg(\neg\neg \alpha \land \neg\neg \beta); \)

\( (b) \neg\neg \alpha \to \neg\neg \beta \iff \neg\neg(\neg\neg \alpha \to \neg\neg \beta); \)

\( (c) \forall x \neg\neg \alpha(x) \iff \neg\neg \forall x \neg\neg \alpha(x). \)

\textit{Proof.} By (24) and (29), if \( \gamma \) and \( \delta \) are stable, then so are \( \gamma \land \delta \) and \( \gamma \to \delta \). Also, by (16), if \( \gamma(x) \) is stable, then so is \( \forall x \gamma(x) \). But by (3), \( \neg\neg \varepsilon \iff \varepsilon \) holds for stable \( \varepsilon \). \( \Box \)

As for the remaining connectives \( \lor \) and \( \exists \), note that one can eliminate them altogether using the intuitionistic de Morgan laws (24), (16):

\[ \neg\neg(\alpha \lor \beta) \iff \neg(\neg\alpha \land \neg\beta), \]
\(\neg\neg\exists x\alpha(x) \iff \forall x\neg\alpha(x)\).

To summarize, the \(\neg\neg\)-translation of classical logic into intuitionistic can be presented as follows:

- atomic subformulas (i.e. predicate variables) are interpreted as stable problems;
- classical \(\land\), \(\forall\) and \(\rightarrow\) (and hence \(\neg\)) are interpreted as the intuitionistic ones;
- classical \(\lor\) and \(\exists\) are interpreted as the \(\neg\)-conjugates of intuitionistic \(\land\) and \(\forall\).

Topologically, we have interpreted classical logic as the logic of regular open sets, where \(\land\), \(\forall\) and \(\rightarrow\) (and hence \(\neg\)) are interpreted as in Tarski models, whereas \(\lor\) and \(\exists\) are interpreted via the Tarski interpretations of \(\land\) and \(\forall\).

In embedding classical logic into intuitionistic in a type theoretic context it may be essential to avoid the prefixing of atomic subformulas (see [23]). This is achieved by the Kuroda embedding, which places \(\neg\neg\) before the entire formula and also after each universal quantifier. It is still equivalent to Kolmogorov’s embedding.\(^{39}\)

**Proposition 5.4.** The following holds in intuitionistic logic:

(a) \(\neg\neg(\alpha \land \beta) \iff \neg\neg(\neg\neg\alpha \land \neg\neg\beta)\);
(b) \(\neg\neg(\alpha \lor \beta) \iff \neg\neg(\neg\neg\alpha \lor \neg\neg\beta)\);
(c) \(\neg\neg(\alpha \rightarrow \beta) \iff \neg\neg(\neg\neg\alpha \rightarrow \neg\neg\beta)\);
(d) \(\neg\neg\exists x\alpha(x) \iff \neg\neg\exists x\neg\neg\alpha(x)\).

**Proof.** Using respectively (35), (24), (33) and (16), the left hand sides can be equivalently rewritten in a form where each atomic subformula is prefixed by a \(\neg\). In this form, each atomic subformula can be replaced by its double negation, due to (3). \(\square\)

### 5.7. \(\neg\neg\)-Shift

Prefixing only the entire formula with \(\neg\neg\), which is known as Glivenko’s transformation, does not work in the presence of universal quantifiers. Indeed, let \(\Phi\) denote the schema \(\forall x\pi(x) \lor \exists x\neg\pi(x)\), which we encountered in the Principle of Omniscience and which represents a classical law. If Glivenko’s transformation were taking classical laws to intuitionistic laws, then the double negation of \(\Phi\),

\[\neg\neg[\forall x\pi(x) \lor \exists x\neg\pi(x)],\]

would be an intuitionistic law. The latter schema is equivalent, via (24) and (29), to

\[\neg\exists x\neg\pi(x) \rightarrow \neg\neg\forall x\pi(x),\]

which is nothing but the contrapositive of (GMP), Generalized Markov’s Principle. By (16), the latter schema is in turn equivalent to the following \(\neg\neg\)-Shift Principle:

\[\forall x\neg\neg\pi(x) \rightarrow \neg\neg\forall x\pi(x).\]  \(^{(DNS)}\)

Note that the converse to the \(\neg\neg\)-Shift Principle is nothing but the intuitionistic law (39). If \(\forall\) is “specialized” to \(\land\), the \(\neg\neg\)-Shift Principle itself turns into the intuitionistic law

\(^{39}\)Other variants of the \(\neg\neg\)-translation achieve still more economy in the use of \(\neg\) (see [42]).
and if $\forall$ is “specialized” to $\rightarrow$, the resulting schema $(\chi \rightarrow \neg\neg\pi) \rightarrow \neg\neg(\chi \rightarrow \pi)$ is again an intuitionistic law (by (37) and (32)).

To see that the $\neg\neg$-Shift Principle fails in general, consider a Tarski model with $X = \mathbb{R}$, $D = \mathbb{Q}$ and $|\pi|(n) = \mathbb{R} \setminus \{n\}$. Then $|\forall x \neg\neg\pi(x)| = \mathbb{R}$, whereas $|\neg\neg\forall x \pi(x)| = \emptyset$.

Another Tarski model can be obtained by observing that the poset of integers (ordered in the usual way) with its Alexandroff topology contains only countably many open sets, whose intersection is empty, and no regular open sets other than $\emptyset$ and the whole space.

By 5.3, the $\neg\neg$-Shift Principle can be equivalently rewritten as

$$\neg\neg\forall x \neg\neg\pi(x) \rightarrow \neg\neg\forall x \pi(x).$$

It follows that the $\neg\neg$-Shift Principle is satisfied by all stable formulas. In other words, the contrapositive of (SMP), the Strong Markov Principle, is an intuitionistic law.

Yet another equivalent formulation of the $\neg\neg$-Shift Principle was already discussed in §3.10:

$$\neg\neg\forall x (\pi(x) \lor \neg\pi(x)).$$

To see that (DNS) implies (DNS") it suffices to note that by (26), $\forall x \neg\neg(\pi(x) \lor \neg\pi(x))$ is an intuitionistic law. Conversely, (DNS") implies (DNS') using the intuitionistic law

$$\neg\neg\alpha \land (\alpha \lor \neg\alpha) \rightarrow \alpha.$$

On the other hand, the $\neg\neg$-Shift Principle holds in any completely metrizable space $X$ if the domain $D$ is countable. Indeed, $|\alpha \lor \neg\alpha|$ is dense in $X$ due to $\vdash \neg\neg(\alpha \lor \neg\alpha)$ (see (26)). Then by Baire’s Theorem, $|\forall x (\pi(x) \lor \neg\pi(x))|$ is also dense in $X$. Hence (DNS") is satisfied in the model.

5.8. Modal logic QS4

Suppose that the connectives of classical logic are modelled by set-theoretic operations on subsets of a set $T$ (as in §4.6), and that $T$ also happens to be a topological space. As long as the Euler model is closed under the topological interior operator $\text{Int}$, this operator can be regarded as modelling an operator $\Box$ on formulas. In an Euler model, equivalent formulas are always interpreted by the same subset; thus such a $\Box$ must certainly be well-defined on equivalence classes of formulas, in the sense that if $p \leftrightarrow q$ is a classical law, then so is $\Box p \leftrightarrow \Box q$. This requirement along with the usual axioms of a topological space $T$ in terms of interior,

- $\text{Int} P \subset P$;
- $\text{Int} P \subset \text{Int}(\text{Int} P)$;
- $T \subset \text{Int} T$;
- $\text{Int} P \cap \text{Int} Q = \text{Int}(P \cap Q),$

correspond to the following properties of $\Box$, which suggest the intended reading of $\Box p$ as “there exists a proof of $p$”.\footnote{It is remarkable that Orlov, who first introduced these axioms in 1928, did so in order to give a provability explanation of propositional intuitionistic logic [109; §6.7] (see [1; §5.1]). His work remained virtually unknown, while the same axioms were rediscovered in a few years by Becker and Gödel [53].}
• (reflection) $\Box p \rightarrow p$;
• (proof checking) $\Box p \rightarrow \Box \Box p$;
• (modus ponens) $\Box p \land \Box (p \rightarrow q) \rightarrow \Box q$.
• (necessitation) $\frac{p}{\Box p}$;

Indeed, the modus ponens schema follows from:

(i) $\Box p \land \Box q \iff \Box (p \land q)$;
(ii) $p \leftrightarrow q \iff \Box p \leftrightarrow \Box q$.

using the classical law $p \land (p \rightarrow q) \iff p \land q$. Conversely, the modus ponens schema implies

(iii) $\Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

by the exponential law (regarded as a classical law). Now (iii) and necessitation imply (ii) and (i). In more detail, the $\leftarrow$ implication in (i) is proved using the classical laws $p \land q \rightarrow p$ and $p \land q \rightarrow q$; and the $\rightarrow$ implication in (i), transformed by the exponential law, is proved using the classical law $p \rightarrow (q \rightarrow (p \land q))$.

Rather than viewing $\Box$ as an operator on formulas of classical logic, we can think of it as a new connective, added on top of classical logic. The four properties of $\Box$ (reflection, proof checking, necessitation and modus ponens), read as three schemata and one rule, can then be supplemented by the usual primary laws and inference rules of classical logic (see §4.4), with understanding that they now apply to all formulas, possibly containing the new connective $\Box$. The resulting logic is known as QS4.

By design, an Euler model of classical logic in subsets of a topological space extends to a topological model of QS4 by interpreting $\Box$ by the interior operator. It is well-known that QS4 is complete with respect to its topological models [118] (see also [82]). In fact, if we have an Euler model $V$ of QS4 as a “theory” over classical logic,41 in subsets of a set $S$, then the interior of a subset $P \subset S$ can be defined as the union of all sets of the form $(\Box \varphi)^V$, where $\varphi^V \subset S$. It is easy to check that the so defined interior operator indeed satisfies the four axioms above, and hence determines a topology on $S$. Thus if we interpret $\Box$ by this interior operator, we get a topological model $V^+$ of QS4. In particular, if QS4 happens to be complete with respect to $V$, then so it is with respect to $V^+$, since as we have just seen, the axioms of a topology add nothing new to the laws of QS4.

In what sense does $\Box$ correspond to provability? It certainly fails to accurately represent provability in any theory containing Peano Arithmetic. Indeed, reflection and necessitation imply $\vdash \Box (\Box \perp \rightarrow \perp)$; this is saying that the consistency of the theory can be proved within the theory. Thus Gödel spoke of derivability “understood not in a particular system, but in the absolute sense (that is, one can make it evident)” and suggested an intended reading of $\Box$ as “is provable in the absolute sense” [52].

41 QS4 is not a literally a theory over classical logic because it contains an additional inference rule. So the result of existence of Euler models for consistent theories over classical logic does not apply.
Clearly, provability in the sense of the “classical BHK” (see §3.8) satisfies the axioms and rules of QS4 — though in a trivial way, with □F ⇔ F. An interesting provability operator □ satisfying the primary laws and inference rules of QS4 and compatible with the BHK interpretation (where “to solve” is read as “to prove”) was constructed by Artemov, see [5]; it represents existence of proofs in Peano Arithmetic, where “proofs” have the usual meaning of formal proofs, but “existence” is understood in an explicit sense, not expressible internally in Peano Arithmetic. Another approach is that □p should not be read as “p is provable” but interpreted as p∧¬□p instead, where it is ¬□p that should be read as “p is provable”. Indeed, this □ is closely related to provability within Peano Arithmetic (see [141] and [38]). Topologically, ⊤□¬ corresponds to Cantor’s derivative operation (see [38]).

5.9. Provability translation

Tarski models yield the following embedding of intuitionistic logic into QS4: given a formula α in the language of intuitionistic logic, insert a box in front of every atomic subformula (because atomic formulas correspond to open sets in Tarski models) and in front of every implication clause and every universal quantifier (because of their special treatment in Tarski models). The resulting formula α′ is derivable in QS4 if and only if α is derivable in intuitionistic logic. This is a version of Gödel’s translation due to McKinsey–Tarski, as extended to the first-order case by Rasiowa–Sikorski, Maehara and Prawitz–Malmnäs (see [141], [105], [43]).

Of course, we could as well insert a box in front of every subformula of α; the resulting formula α□ will be equivalent to α′ in QS4:

**Proposition 5.5.** The following are laws of QS4:

(a) (□p ∧ □q) ⇔ □(□p ∧ □q);
(b) (□p ∨ □q) ⇔ □(□p ∨ □q);
(c) ∃xp(x) ⇔ □∃xp(x).

The topological counterpart of this proposition is saying essentially that finite intersections and arbitrary unions of open sets are open. Any proof of this fact from the axioms of a topological space in terms of Int should translate into a proof in QS4.

Gödel’s original translation, as extended to the first-order case (cf. [141]) is as follows: given a formula α, insert boxes after each instance of ∨, → and ∃. Then the resulting formula α'' is a law of QS4 if and only if α is an intuitionistic law. Indeed, α'' is a law of QS4 if and only if □α'' is. On the other hand, α□ ⇔ □α'' is a law of QS4:

**Proposition 5.6.** The following are laws of QS4:

(a) □(p ∧ q) ⇔ □(□p ∧ □q);
(b) □∀xp(x) ⇔ □∀xp(x).

Here each of the two laws is proved similarly to their topological counterpart, Int ∩ i S_i = Int ∩ Int S_i. Indeed, here ⊃ follows from S_i ⊃ Int S_i. Conversely, we have ∩ i S_i ⊂ S_n.
for each \( n \), whence \( \text{Int} \bigcap_i S_i \subset \text{Int} S_n \). Since this holds for each \( n \), we get \( \text{Int} \bigcap_i S_i \subset \bigcap_i \text{Int} S_i \). Applying Int to both sides completes the proof.

6. Sheaf-valued models

The first three sections contain, in particular, a review of basic sheaf theory. One reference for sheaf theory is [19].

6.1. Sheaves vs. presheaves

We recall that a sheaf (of sets) on a topological space \( B \) is a map \( F: E \to B \) that is a local embedding (i.e. each \( e \in E \) has a neighborhood \( O \) in \( E \) such that \( F|_O: O \to F(O) \) is a homeomorphism). The set \( F_b = F^{-1}(b) \) is called the stalk of the sheaf \( F \) at \( b \). A section of \( F \) over a subset \( U \subset B \) is a continuous map \( s: U \to E \) such that \( Fs = \text{id}_U \).

A section \( s: U \to E \) is said to extend a section \( t: V \to E \) if \( V \subset U \) and \( t = s|_V \). Clearly, every section is a homeomorphism onto its image, and \( F \) is an open map; in particular, \( F(E) \) is always open in \( B \). It is easy to see that a base of topology on \( E \) is given by the images of all sections of \( F \) (over all open subsets of \( B \)).

A morphism \( F \to F' \) between sheaves \( F: E \to B \) and \( F': E' \to B \) is a continuous map \( f: E \to E' \) such that \( Ff = F' \). Clearly, such a map is itself a local embedding.

A constant sheaf is the projection \( B \times \Xi \to B \), where \( \Xi \) is a discrete space. A sheaf is locally constant if it is a covering map, that is, its restriction over a sufficiently small neighborhood of every point of \( B \) is isomorphic to a constant sheaf. There are two basic reasons why a sheaf can fail to be locally constant:

- Even locally, sections need not extend, as is the case for the characteristic sheaf \( \chi_U: U \to B \), where \( U \) is an open subset of \( B \) and \( \chi_U \) is the inclusion map.
- Even locally, sections may extend non-uniquely, as is the case for the amalgamated union \( \chi_U \cup_{\chi_W} \chi_V: U \cup_W V \to B \), where \( U \) and \( V \) are open subsets of \( B \) and \( W \) is an open subset of \( U \cap V \). Here \( U \cup_W V \) is the quotient space of the disjoint union \( U \cup V = U \times \{0\} \cup V \times \{1\} \subset B \times \{0, 1\} \) by the equivalence relation \( (b, 0) \sim (b, 1) \) if \( b \in W \), and \( \chi_U \cup_{\chi_W} \chi_V \) sends the class of \( (b, i) \) to \( b \).

Note that \( U \cap_W V \) is non-Hausdorff, as long as some \( b \in (U \cap V) \setminus W \) lies in the closure of \( W \) — in this case \( (b, 0) \) and \( (b, 1) \) have no disjoint neighborhoods in \( U \cap_W V \). In general, if \( s: U \to E \) and \( t: V \to E \) are sections of a sheaf \( F: E \to B \) over open sets, then \( \{b \in U \cap V \mid s(b) = t(b)\} \) is open. If \( E \) happens to be Hausdorff, then this set must also be closed in \( U \cap V \).

A presheaf (of sets) on a topological space \( B \) is a functor from the category of open sets of \( B \) and their inclusions into the category of sets. A morphism \( \varphi: F \to G \) of presheaves on \( B \) is a natural transformation of functors, that is, a collection of maps \( \varphi_U: F(U) \to G(U) \) that commute with the restriction maps \( F(j): F(U) \to F(V) \) and \( G(j): G(U) \to G(V) \).

Given a continuous map \( f: X \to B \), its presheaf of sections \( \sigma f \) assigns to an open subset \( U \subset B \) the set \((\sigma f)(U)\) of all continuous sections of \( f|_{f^{-1}(U)}: f^{-1}(U) \to U \),
and to an inclusion $j: V \hookrightarrow U$ of open subsets of $B$ the map $(\sigma f)(U) \rightarrow (\sigma f)(V)$ given by usual restriction of sections. In general, for an arbitrary presheaf $F$, elements of each $F(U)$ are called sections over $U$, and the image of a section $s \in F(U)$ under $F(j): F(U) \rightarrow F(V)$ is called the restriction of $s$ and is denoted $s|_V$. If $U$ is an open subset of $B$, the presheaf $\text{Char}U := \sigma \chi_U$ can be described by $(\text{Char}U)(V) = \{\text{id}_V\}$ if $V \subseteq U$ and $(\text{Char}U)(V) = \emptyset$ otherwise.

Conversely, a presheaf $F$ on a space $B$ gives rise to the sheaf of germs (or sheafafication) $\gamma F$, whose stalk $(\gamma F)_b$ at $b$ is the direct limit (also known as colimit or inductive limit) $F_b := \lim F(U)$ over all open neighborhoods $U$ of $b$ ordered by inclusion. Thus an element of $F_b$ is a germ of sections at $b$; that is, an equivalence class of local sections $s_U \in F(U)$ over open neighborhoods $U$ of $b$, where $s_U \sim t_V$ if $U \cap V$ contains an open neighborhood $W$ of $b$ such that $s_U|_W = t_V|_W$. In particular, $F_b = \emptyset$ if and only if $F(U) = \emptyset$ for all open neighborhoods $U$ of $b$. A base of topology on $E = \bigcup_{b \in B} F_b$ consists of sets of the form $O_{s,U} = \{\varphi_U,b(s) \mid b \in U\}$, where $U \subseteq B$ is open and $\varphi_U,b: F(U) \rightarrow F_b$ is the natural map. The open set $\text{Supp}F := (\gamma F)(E)$ can be described as the union of all open subsets $U$ of $B$ such that $F(U) \neq \emptyset$. We will also denote it by $\text{Supp}F$.

Given a sheaf $F$, it is easy to see that $\gamma \sigma F$ is isomorphic to $F$. Given a presheaf $F$, we have an obvious morphism $F \rightarrow \sigma \gamma F$, whose component $F(U) \rightarrow \sigma \gamma F(U)$ is a bijection if and only if the following two sheaf conditions hold:

- If $s, t \in F(U)$ are such that every $b \in U$ has an open neighborhood $V \subseteq U$ such that $s|_V = t|_V$, then $s = t$.
- If $\{U_\alpha\}$ is an open cover of $U$, and $s_\alpha \in F(U_\alpha)$ are such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta$, then there exists an $s \in F(U)$ such that each $s_\alpha = s|_{U_\alpha}$.

Thus sheaves on $B$ are in one-to-one correspondence with presheaves on $B$ satisfying the sheaf conditions for every open $U \subseteq B$. For example, a presheaf of the form $\sigma F$ always satisfies the sheaf conditions, so is isomorphic to $\sigma F$, where $F = \gamma \sigma f$ is called the sheaf of germs of sections of $f$. Moreover, every sheaf morphism $F \rightarrow G$ clearly determines a presheaf morphism $\sigma F \rightarrow \sigma G$; and every presheaf morphism $F \rightarrow G$ clearly determines a sheaf morphism $\gamma F \rightarrow \gamma G$. It follows that the category of sheaves on $B$ can be identified with a full subcategory of the category of presheaves on $B$.

**Example 6.1.** If $\pi: B \times X \rightarrow B$ is the projection, then $\gamma \pi$ is the sheaf of germs of continuous functions $B \rightarrow X$. Note that the total space $E$ of this sheaf is non-Hausdorff already for $B = X = \mathbb{R}$, for the germs at 0 of the constant function $f(b) = 0$ and the function $g(b) = \max(b,0)$ have no disjoint neighborhoods.

**Example 6.2.** Let $f: X \rightarrow B$ be a continuous map (for instance, this could be a real polynomial $f: \mathbb{R} \rightarrow \mathbb{R}$ or a complex polynomial $f: \mathbb{C} \rightarrow \mathbb{C}$, $f(x) = a_n x^n + \cdots + a_0$), and consider the following parametric problem $\Gamma_f(b), b \in B$:

Find a solution of the equation $f(x) = b$.

Thus each point-inverse $f^{-1}(b)$ is nothing but the set of solutions of $\Gamma_f(b)$. 
If the parameter $b$ represents experimental data (which inevitably contains noise), then any talk about the exact value of $b$ (e.g. whether $b$ is a rational or irrational number when $B = \mathbb{R}$) is pointless. It would be not so unreasonable, however, to assume that the value of $b$ can in principle be determined up to arbitrary precision. Indeed, the point of continuous maps is that if a point in the domain is known up to a sufficient accuracy, it still makes sense to speak of its image — it will be known up to a desired accuracy. This leads us to consider only stable solutions of the equation $f(x) = b$, that is, such $x_0 \in X$ that $f(x_0) = b$ and there exists a neighborhood $U$ of $b$ in the space $B$ over which $f$ has a section. Indeed, if the parameter $b$ is only known to us up to a certain degree of precision, we should only stick with a solution $x_0$ of $f(x) = b$ as long as we can be certain that it would not disappear when our knowledge of $b$ improves.\footnote{This may remind the reader of Brouwer’s idea that “all functions are continuous”, as well as some of Poincaré’s writings; a more rigorous exposition of these ideas can be found in M. Escardo’s notes [39] and in P. S. Novikov’s physical interpretation of intuitionistic logic in terms of weighting of masses [105; §III.4].}

Thus we will consider the following parametric problem $\Delta_b$, $b \in B$:

$$\text{Find a stable solution of the equation } f(x) = b.$$

If is easy to see that the set $\mathcal{F}_b \subset f^{-1}(b)$ of all stable solutions of $f(x) = b$ is nothing but the stalk of the sheaf $\mathcal{F} = \gamma \sigma f$ of germs of sections of the map $f$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is a polynomial with real coefficients, then stable solutions of $f(x) = b$ are all the non-repeated real roots of the polynomial $f(x) - b$, along with its other roots of odd multiplicity. Indeed, $f$ is a local homeomorphism at each of those roots, but has a local extremum at every root $x_0$ of even multiplicity. Hence the perturbed equation $f(x) = b \pm \varepsilon$ (with + chosen in the case of a local maximum, and − in the case of a local minimum) has no real solutions near $x_0$ as long as $\varepsilon > 0$ is sufficiently small.

If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial with complex coefficients, then stable solutions of the equation $f(z) = b$, where $b \in \mathbb{C}$, are precisely all the non-repeated complex roots of the polynomial $f(z) - b$. Indeed, at a root $z_0$ of multiplicity $d > 1$ we can write $f(z) = b + k(z - z_0)^d + o((z - z_0)^d)$. So $f$ is locally a $d$-fold branched covering, and therefore admits no section over any neighborhood of $b = f(z_0)$. In other words, even though the perturbed equation $f(z) = b + \varepsilon$ does have solutions near $z_0$ whenever $|\varepsilon|$ is sufficiently small, we cannot choose such a solution continuously depending on $\varepsilon$; but with discontinuous choice, we can never be sure that our chosen explicit solution will persist when our knowledge of $b$ improves.

Similar behaviour occurs if we consider the problem of finding a root of a complex polynomial $f_b = a_n z^n + \cdots + a_0$ as a function of all its coefficients $b = (a_0, \ldots, a_n)$. Set $\Sigma_n = \{(a_0, \ldots, a_n, x) : \mathbb{C}^n+2 \mid a_n x^n + \cdots + a_0 = 0\}$, and let $\varphi : \Sigma_n \to \mathbb{C}^{n+1}$ be the projection along the first coordinate. Then $\varphi^{-1}(b)$ is the set of roots of $f_b$, and the stalk $(\sigma \varphi)_b$ is the set of its stable roots. By varying just $a_0$ like above we see that no multiple root is stable; clearly, all non-repeated roots are stable.
Example 6.3. Let \( f : E \to B \) be a monotone map between posets. If we endow \( E \) and \( B \) with the Alexandrov topology, then \( f \) is continuous. Every poset endowed with the Alexandrov topology is weakly homotopy equivalent to its order complex [93], so singular (co)homology of a poset, absolute or relative, may be thought of as (co)homology of its order complex. For each \( i = 0, 1, \ldots \) define presheaves \( L^i \) and \( L_i \) on \( B \) by \( L^i(U) = H^i(f^{-1}(U); \mathbb{Z}) \) and \( L_i(U) = H_i(E, E \setminus f^{-1}(U); \mathbb{Z}) \); the restriction maps \( L^i(U) \to L^i(V) \) and \( L_i(U) \to L_i(V) \) are the usual forgetful homomorphisms.

The sheaf conditions on an arbitrary presheaf \( F \) with values in abelian groups are easily seen to be equivalent to exactness of the sequence

\[
0 \to F(U) \xrightarrow{\varphi} \prod_{\alpha} F(U_\alpha) \xrightarrow{\psi} \prod_{(\alpha, \beta)} F(U_\alpha \cap U_\beta)
\]

for any open subset \( U \subset B \) and any open cover \( \{U_\alpha\} \) of \( U \), where \( \varphi(s) = (\alpha \mapsto s|_{U_\alpha}) \) and \( \psi(\alpha \mapsto s_\alpha) = ((\alpha, \beta) \mapsto s_\alpha|_{U_\alpha \cap U_\beta} - s_\beta|_{U_\alpha \cap U_\beta}) \). By the Mayer–Vietoris spectral sequence, this sequence is indeed exact for \( F = L^0 \) and, if the order complex of \( E \) is of dimension \( n < \infty \), then also for \( F = L_n \). Thus \( L^0 \) and \( L_n \) satisfy the sheaf conditions.

In general, the sheaf of germs \( \mathcal{H}^i(f) = \gamma L^i \) is known as the \( i \)th Leray sheaf of \( f \), and when \( f = \text{id}_X : X \to X \), the sheaf of germs \( \mathcal{H}_i(X) = \gamma L_i \) is known as the \( i \)th local homology sheaf of \( X \). If \( X \) is the face poset of a graph, then \( \mathcal{H}_i(X) \) has stalk \( \mathbb{Z} \) at every edge and stalk \( \mathbb{Z}^{d-1} \) at every vertex of degree \( d \). If \( X \) is the face poset of a triangulation of a closed \( n \)-manifold, then \( \mathcal{H}_n(X) \) is locally constant, with stalks \( \mathbb{Z} \); it will be constant precisely when the manifold is orientable. If \( B \) and \( E \) are the face posets of simplicial complexes, and \( f \) comes from a simplicial map that triangulates a bundle with fiber a closed orientable \( k \)-manifold, then \( \mathcal{H}^k(f) \) is locally constant, with stalks \( \mathbb{Z} \); it will be constant precisely when the bundle is orientable.

6.2. Product and coproduct

Given a collection of presheaves \( F_i, i \in I \), their product \( \prod_i F_i \) and coproduct \( \bigsqcup_i F_i \) are defined respectively by \( U \mapsto \prod_i F_i(U) \) and by \( U \mapsto \bigsqcup_i F_i(U) \), with the obvious restriction maps. If the \( F_i \) satisfy the sheaf conditions, then clearly so does \( \prod_i F_i \). On the other hand, \( \bigsqcup_i F_i \) generally does not satisfy the second sheaf condition, since for a disconnected open set \( U \cup V \) the sheaf conditions for each \( F_i \) imply \( F_i(U \cup V) \simeq F_i(U) \times F_i(V) \), so that \( (\bigsqcup_i F_i)(U \cup V) \simeq \bigsqcup_i F_i(U) \times F_i(V) \), which is not the same as \( (\prod_i F_i)(U) \times (\prod_i F_i)(V) = (\prod_i F_i(U)) \times (\prod_i F_i(V)) \). Thus the product \( \prod_i \mathcal{F}_i = \gamma \prod_i F_i \) of the sheaves \( \mathcal{F}_i \) satisfies \( \sigma \prod_i \mathcal{F}_i \simeq \prod_i \sigma \mathcal{F}_i \); their coproduct is defined by \( \bigsqcup \mathcal{F}_i = \gamma \bigsqcup \sigma \mathcal{F}_i \). It is not hard to see that the introduced operations are precisely the product and the coproduct in the categories of sheaves and presheaves.

The coproduct of sheaves \( \mathcal{F}_i : E_i \to B \) can be alternatively described as the sheaf \( \bigsqcup \mathcal{F}_i : \bigsqcup E_i \to B \), where \( \bigsqcup E_i \) is the disjoint union of spaces, and \( \bigsqcup \mathcal{F}_i \) is defined by \( \mathcal{F}_i \) on the \( i \)th summand. Thus \( (\bigsqcup \mathcal{F}_i)_b \simeq \bigsqcup (\mathcal{F}_i)_b \) (isomorphism in the category of sets) for
each \( b \in B \). More generally, \((\bigcup F_i)_b \simeq \bigcup (F_i)_b\) for any presheaves \( F_i \) since direct limit commutes with coproducts.

The product of two sheaves \( F : E \to B \) and \( F' : E' \to B \) can be alternatively described as the fiberwise product \( F \times F' : E \times_B E' \to B \), where \( E \times_B E' \) is the subspace of the Cartesian product \( E \times E' \) consisting of all pairs \((e, e')\) such that \( F(e) = F'(e') \). Thus \((F \times F')_b \simeq F_b \times F'_b\). More generally, \((F \times F')_b \simeq F_b \times F'_b\) for any presheaves \( F \) and \( F' \) since direct limit commutes with finite products.

The fiberwise product \((\prod_B E_i)_b \to B\) of the sheaves \( F_i : E_i \to B \) need not be a sheaf when \( I \) is infinite. For, although every \( e_i \in (F_i)_b \) has a neighborhood in \( E_i \) that maps homeomorphically onto a neighborhood \( U_i \) of \( b \) in \( B \), the fiberwise product may fail to be a local homeomorphism at the point \((i \mapsto x_i)\) if \( \bigcap_i U_i \) is not a neighborhood of \( b \).

For each \( n \in I \), the projection \( \prod_i F_i \to F_n \) induces a map \((\prod_i F_i)_b \to (F_n)_b\) for each \( b \in B \). These in turn yield a map

\[ \mathfrak{G}_b : \left( \prod F_i \right)_b \to \prod (F_i)_b. \]

An element of the right hand side is a collection of germs of local sections \( s_i : U_i \to E_i \), whereas an element of the left hand side is a germ of a collection of local sections \( U \to E_i \). The germs of \( s_i \)'s may fail to define such a single germ if \( \bigcap U_i \) is not a neighborhood of \( b \); thus \( \mathfrak{G}_b \) is not surjective in general. On the other hand, given two collections of local sections, \( \sigma_i : U \to E_i \) and \( \tau_i : V \to E_i \) such that each \( \sigma_i \) equals \( \tau_i \) on some open set \( W_i \subset U \cap V \), there might be no single open set \( W \subset U \cap V \) such that each \( \sigma_i|_W = \tau_i|_W \); thus \( \mathfrak{G}_b \) is not injective in general.

It is clear, however, that \( \mathfrak{G}_b \) is a bijection when \( B \) is an Alexandroff space.

Note that the existence of \( \mathfrak{G}_b \) implies that if \((\prod F_i)_b \neq \emptyset\), then each \((F_i)_b \neq \emptyset\). The converse fails:

**Proposition 6.4.** There exist sheaves \( F_1, F_2, \ldots \) on \( \mathbb{R} \) such that each \((F_i)_b \neq \emptyset\) for each \( b \in \mathbb{R} \) but \( \prod F_i = \chi_{\mathbb{R}} \).

**Proof.** It suffices to consider, for each \( i \), the cover \( C_i = \{(r - \frac{1}{i}, r + \frac{1}{i}) \mid r \in \mathbb{R}\} \) of \( \mathbb{R} \) by all open intervals of length \( 2/i \), and the sheaf \( F_i = \bigcup_r \chi_{(r-1/i, r+1/i)} \). \( \square \)

A topological space \( X \) is of **covering dimension zero** if every open cover of \( X \) has a refinement consisting of disjoint open sets; and of **inductive dimension zero** if it has a base of topology consisting of clopen sets. For \( T_1 \) spaces (i.e. \( T_1 \) spaces where disjoint closed subsets are separated by neighborhoods), covering dimension zero implies inductive dimension zero (see [37; 6.2.5, 6.2.6]). The converse holds for spaces whose topology has a countable base (see [37; 6.2.7]). In particular, the two notions are equivalent for separable metrizable spaces, so we may speak of **zero-dimensional separable metrizable spaces**. Examples of such spaces include the Cantor set and the Baire space, as well their arbitrary subspaces. Indeed, it is easy to see that if a space is of inductive dimension zero, then so is every its subspace.
Lemma 6.5. If $\mathcal{F}$ is a sheaf on a zero-dimensional separable metrizable space $B$, and $U$ is an open subset of $B$, then $\mathcal{F}$ has a section over $U$ if and only if $U \subseteq \text{Supp } \mathcal{F}$.

Proof. The “only if” implication follows from the definitions. Given an open $U \subseteq \text{Supp } \mathcal{F}$, each $x \in U$ is contained in an open set $V_x$ over which $\mathcal{F}$ has a section. Without loss of generality $V_x \subseteq U$. Since $U$ is zero-dimensional, the cover $\{V_x \mid x \in U\}$ of $U$ has a refinement $\{U_\alpha\}$ consisting of disjoint open sets. Since $\mathcal{F}$ has a section over each $U_\alpha$ and is a sheaf (not just a presheaf), by combining these sections we get a section over $U$. $\Box$

Proposition 6.6. If $F_i$ are presheaves over $B$, then $\text{Supp } \prod F_i \subseteq \text{Int } \bigcap \text{Supp } F_i$. The reverse inclusion holds in each of the following cases:

(a) $B$ is an Alexandroff space;

(b) $B$ is a zero-dimensional separable metrizable space, and $F_i = \sigma F_i$ for some $F_i$.

Proof. The inclusion follows from the existence of the map $\mathfrak{S}_b$ and from the fact that the left hand side is an open set. The reverse inclusion for holds for Alexandroff spaces since their $\mathfrak{S}_b$ is bijective. If $B$ is a zero-dimensional separable metrizable space and $x$ lies in an open set $U$ contained in $\text{Supp } F_i$ for each $i$, then by 6.5 each $F_i$ has a section over $U$. Hence so does their product, whose support therefore contains $x$. $\Box$

6.3. The Hom-sheaf

If $F$ is a presheaf on $B$, and $A \subseteq B$ is open, then open subsets of $Y$ are open in $A$, so $F$ determines the restriction presheaf $F|_A$ on $A$. In the case of a sheaf $\mathcal{F}: E \to B$, the corresponding construction generalizes to the case of an arbitrary continuous map $f: A \to B$, which induces the inverse image $f^*\mathcal{F}: f^*E \to A$, where $f^*E = \{(a,e) \in A \times E \mid f(a) = \mathcal{F}(e)\}$ and $(f^*\mathcal{F})(a,e) = a$.

If $F$ and $G$ are presheaves on $B$, let us consider the set $\text{Hom}(F,G)(U)$ of morphisms $F|_U \to G|_U$. If $V$ is an open subset of $U$, every such morphism $\varphi$ restricts to a morphism $\varphi|_V: F|_V \to G|_V$, and thus we get the exponential presheaf $\text{Hom}(F,G)$. If $G$ satisfies the sheaf conditions, then clearly so does $\text{Hom}(F,G)$. This enables one to define the sheaf $\text{Hom}(\mathcal{F},\mathcal{G}) = \gamma \text{Hom}(\sigma \mathcal{F},\sigma \mathcal{G})$, called the sheaf of germs of morphisms of the sheaves $\mathcal{F}$ and $\mathcal{G}$. In general, for any presheaves $F$ and $G$, the stalk $\text{Hom}(F,G)_b$ consists of germs of morphisms $F \to G$, that is, of equivalence classes of local morphisms $\varphi_U: F|_U \to G|_U$ over open neighborhoods $U$ of $b$, where $\varphi_U \sim \varphi_V$ if $U \cap V$ contains an open neighborhood $W$ of $b$ such that $\varphi_U|_W = \varphi_V|_W$. A germ of morphisms determines a map of germs, so we get a map $\mathfrak{S}_b: \text{Hom}(F,G)_b \to \text{Hom}(F_b,G_b)$, where $\text{Hom}(S,T)$ denotes the set of all maps between the sets $S$ and $T$. This map is neither injective nor surjective in general:

Proposition 6.7. For any set $S$ there exist sheaves $\mathcal{F}$, $\mathcal{G}$ on \( \mathbb{R} \) such that $\mathcal{F}_0 = \mathcal{G}_0 = \emptyset$ and $\text{Hom}(\mathcal{F},\mathcal{G})_0 \simeq S$. 
Proof. Let \( U = \mathbb{R} \setminus \{0\} \) and let \( \mathcal{F} = \chi_U \). Let \( \mathcal{G} \) be the disjoint union of copies of \( \mathcal{F} \) indexed by \( S \). Then for every open neighborhood \( V \) of \( 0 \) the set of sheaf morphisms \( \mathcal{F}|_V \to \mathcal{G}|_V \) is in a bijection with \( S \), and its elements persist under restriction.

Let us mention two more peculiar examples:

**Example 6.8.** There exist sheaves \( \mathcal{F}, \mathcal{G} \) on \( \mathbb{R}^2 \) such that \( \text{Supp} \mathcal{F} = \text{Supp} \mathcal{G} = \mathbb{R}^2 \setminus \{(0,0)\} \), yet \( \text{Hom}(\mathcal{F}, \mathcal{G})(0,0) = \emptyset \). Indeed, let \( U = \mathbb{R}^2 \setminus \{0\} \) and let \( \mathcal{F} = \chi_U \). Let \( \mathcal{G} : \tilde{U} \to U \subset \mathbb{R}^2 \) be the nontrivial double covering over \( U \). Then for every open neighborhood \( V \) of \( 0 \) there exist no sheaf morphisms \( \mathcal{F}|_V \to \mathcal{G}|_V \).

**Example 6.9.** There exist sheaves \( \mathcal{F}_n \) and \( \mathcal{G} \) on \( \mathbb{R} \), \( n \in \mathbb{N} \), such that each \( \text{Hom}(\mathcal{F}_n, \mathcal{G}) \neq \chi_\emptyset \) but \( \text{Hom}(\bigsqcup_n \mathcal{F}_n, \mathcal{G}) = \chi_\emptyset \). Indeed, let \( \mathcal{F}_n = \chi_{(-\infty, -1/n) \cup (1/n, \infty)} \) and let \( \mathcal{G} = \chi_\emptyset \). Then \( \text{Hom}(\mathcal{F}_n, \mathcal{G}) \simeq \chi_{(-1/n, 1/n)} \). It follows that \( \text{Hom}(\bigsqcup_n \mathcal{F}_n, \mathcal{G}) \simeq \prod_{n \in \mathbb{N}} \chi_{(-1/n, 1/n)} \simeq \chi_\emptyset \).

**Proposition 6.10.** \( \text{Supp} \text{Hom}(F,G) \subset \text{Int}(\text{Supp} G \cup (B \setminus \text{Supp} F)) \) for presheaves \( F, G \) over a space \( B \), and the reverse inclusion holds in each of the following cases:

(a) \( G \) is the characteristic presheaf of an open set, in which case so is \( \text{Hom}(F,G) \);

(b) \( B \) is a zero-dimensional separable metrizable space, and \( G = \sigma \mathcal{G} \) for some \( \mathcal{G} \).

*Proof.* The inclusion follows from the existence of the map \( \mathfrak{F}_b \) and from the fact that the left hand side is an open set. Suppose that \( x \) lies in an open set \( U \) contained in \( \text{Supp} G \cup (B \setminus \text{Supp} F) \). Thus if \( y \in U \) and \( y \in \text{Supp} G \), then \( y \in \text{Supp} G \); that is, \( U \cap \text{Supp} F \subset U \cap \text{Supp} G \). Then for each open \( V \subset U \) such that \( V \not\subset \text{Supp} G \) we have \( V \not\subset \text{Supp} F \), hence \( F(V) = \emptyset \) and there is a unique map \( F(V) \to G(V) \). Let us now consider an open \( V \subset U \) such that \( V \subset \text{Supp} G \).

If \( G = \text{Char}(\text{Supp} G) \), then \( G(V) = \{ \text{id}_V \} \) and there is a unique map \( F(V) \to G(V) \).

If \( B \) is a zero-dimensional separable metrizable space, and \( G = \sigma \mathcal{G} \), then by 6.5 \( \mathcal{G} \) has a section \( s \) over \( U \cap \text{Supp} G \), and we have the constant map \( F(V) \to G(V) \) onto \( \{ s|_V \} \).

In either case, the constructed maps \( F(V) \to G(V) \) commute with the restriction maps \( G(V) \to G(W) \) and \( F(V) \to F(W) \) for each open \( W \subset V \subset U \), and so determine a morphism of presheaves \( F|_U \to G|_U \). The germ at \( x \) of this morphism is an element of the stalk \( \text{Hom}(F,G)_x \), so \( x \) is contained in \( \text{Supp} \text{Hom}(F,G) \).

Moreover, in the case (a) we have shown that for every open \( U \) contained in \( \text{Supp} G \cup (B \setminus \text{Supp} F) \), we have a unique morphism \( F|_U \to G|_U \); thus \( \text{Hom}(F,G)(U) \) contains precisely one element. If \( U \) is not contained in \( \text{Supp} G \cup (B \cup \text{Supp} F) \), then \( U \) contains a point \( y \) such that \( y \not\in \text{Supp} G \) and \( y \in \text{Supp} F \). The latter implies that \( y \) lies in an open set \( V \) such that \( F(V) \neq \emptyset \). Without loss of generality \( V \subset U \). On the other hand, since \( y \not\in \text{Supp} G \), we have \( G(V) = \emptyset \). Hence there exists no map \( F(V) \to G(V) \), and consequently no morphism \( F|_U \to G|_U \). Thus \( \text{Hom}(F,G)(U) = \emptyset \). It then follows that \( \text{Hom}(F,G) \) is isomorphic to \( \text{Char}(\text{Supp} G \cup (B \cup \text{Supp} F)) \).

**Corollary 6.11.** \( \text{Hom}(F, \text{Char} \emptyset) \simeq \text{Char}(\text{Int}(B \setminus \text{Supp} F)) \).

The inclusion in 6.10 cannot be reversed for sheaves over an Alexandroff space:
Proposition 6.12. There exist sheaves $\mathcal{F}$, $\mathcal{G}$ over an Alexandroff space $B$ such that $\text{Supp} \mathcal{F} = \text{Supp} \mathcal{G}$, yet $\text{Supp} \text{Hom}(\mathcal{F}, \mathcal{G}) \neq B$.

For future purposes we fix some terminology regarding sheaves on Alexandroff spaces. If $B$ in an Alexandroff space, and for $p \in B$ let $(p) = \{q \in B \mid q \geq p\}$, the minimal open set containing $p$. Given a sheaf $\mathcal{F} : E \to B$, for each $p, q \in B$ with $p \leq q$ let $\mathcal{F}_{pq} : \mathcal{F}_p \to \mathcal{F}_q$ denote the restriction map $(\sigma \mathcal{F})(j) : (\sigma \mathcal{F})(\langle p \rangle) \to (\sigma \mathcal{F})(\langle q \rangle)$ corresponding to the inclusion $j : \langle q \rangle \hookrightarrow \langle p \rangle$. Note that the topology on $E$ is an Alexandroff topology; its corresponding preorder is given by $x \leq y$ if $x \in \mathcal{F}_p$, $y \in \mathcal{F}_q$, $p \leq q$ and $y = \mathcal{F}_{pq}(x)$.

Proof. Let $B$ be the product of two copies of the poset $0 < 1$. Let $\mathcal{F}_{(0,0)} = \emptyset$ and $\mathcal{F}_{(1,0)} = \mathcal{F}_{(0,1)} = \{z\}$. Let $\mathcal{G}_{(0,0)} = \emptyset$, $\mathcal{G}_{(1,0)} = \{x\}$, $\mathcal{G}_{(0,1)} = \{y\}$ and $\mathcal{G}_{(1,1)} = \{x, y\}$, where the maps $\mathcal{G}_{pq}$ are treated as inclusions. Clearly, $\text{Hom}(\mathcal{F}, \mathcal{G})_{(0,0)} = \emptyset$. \qed

In the general case, the inclusion in 6.10 can be partially reversed:

Proposition 6.13. $\text{Supp} G \cup \text{Int}(B \setminus \text{Supp} F) \subset \text{Supp} \text{Hom}(F, G)$.

Proof. If $b \in \text{Supp} G$, then $G(U)$ is nonempty for some open neighborhood $U$ of $b$. Then we can pick some $s_U \in G(U)$ and define a morphism $F|_U \to G|_U$ by sending every $F(V), V \subset U$, onto $\{s_U|_V\} \subset G(V)$.

If $B \setminus \text{Supp} F$ contains an open neighborhood $U$ of $b$, then $F(V) = \emptyset$ for all open $V \subset U$. Hence a morphism $F|_U \to G|_U$ is defined by observing that all objects in the domain are empty. \qed

6.4. Sheaf-valued models

Let us summarize the behavior of stalks under operations on presheaves:

Proposition 6.14. If $F$, $G$ and $F_i$, $i \in I$, are presheaves on $B$, and $b \in B$, then

- $(F \times G)_b \simeq F_b \times G_b$;
- $(F \sqcup G)_b \simeq F_b \sqcup G_b$;
- there is a map $\mathfrak{S}_b : \text{Hom}(F,G)_b \to \text{Hom}(F_b,G_b)$;
- $(\text{Char} \emptyset)_b = \emptyset$;
- $(\bigsqcup F_i)_b \simeq \bigsqcup (F_i)_b$;
- there is a map $\mathfrak{G}_b : (\prod F_i)_b \to \prod (F_i)_b$.

This looks suspiciously similar to the BHK interpretation (see, in particular, §3.8), which suggests that sheaves might have something to do with intuitionistic logic.

Let $B$ be a topological space and $D$ a set. The set $\Omega$ of truth values will consist of all sheaves over $B$; a sheaf $\mathcal{F}$ is considered a truth if it has a global section, i.e. $(\sigma \mathcal{F})(B) \neq \emptyset$. In other words, we assign to each formula $\alpha$ with $n$ parameters a sheaf $|\alpha|$ over $B$ depending on $n$ variables ranging over $D$; and $\vdash \alpha$ means that $|\alpha|(t_1, \ldots, t_n)$ has a global section for each tuple $t_1, \ldots, t_n \in D$.

Intuitionistic connectives are interpreted by the basic operations on sheaves:

- $|\alpha \lor \beta| = |\alpha| \sqcup |\beta|$;
- $|\alpha \land \beta| = |\alpha| \times |\beta|$;
• \(|\alpha \to \beta| = \text{Hom}(|\alpha|, |\beta|);\)
• \(|\bot| = \text{Char} \emptyset;\)
• \(\exists x \alpha(x) = \bigsqcup_{d \in D} |\alpha|(d);\)
• \(\forall x \beta(x) = \prod_{d \in D} |\alpha|(d).\)

In exactly the same way we define presheaf structures, the only essential difference being that the coproduct of presheaves \(F_i\) that happen to be of the form \(\sigma F_i\) is not the same as the presheaf of sections of the coproduct of the sheaves \(F_i\).

**Theorem 6.15.** These (sheaf and presheaf structures) are models of intuitionistic logic.

Since (pre)sheaves form a topos, this is a special case of Palmgren’s models of intuitionistic logic in locally cartesian closed categories with finite sums [111].

**Proof.** It suffices to show that all primary laws and inference rules of the modified Specter’s deductive system in §4.4 hold in the two kinds of structures.

(I) (modus ponens). We are given a global section \(s\) of \(|\alpha|\) and a global section of \(|\alpha \to \beta|\), that is, a presheaf morphism \(F: |\alpha| \to |\beta|\). Then \(F(B): |\alpha|(B) \to |\beta|(B)\) sends \(s\) to a global section of \(|\beta|\).

(II), (III) Indeed, the identity is a natural transformation, and composition of natural transformations is a natural transformation.

(IV), (V) Indeed, the projections of the product of presheaves onto its factors as well as the inclusions of summands into the coproduct of presheaves are presheaf morphisms. Also, the inclusions of summands into the coproduct of sheaves are sheaf morphisms.

(VI), (VII) Indeed, (co)product of presheaves is their (co)product in the category of presheaves. Also, coproduct of sheaves is their coproduct in the category of sheaves.

(VIII) (exponential law) Given a presheaf morphism \(F: |\alpha| \times |\beta| \to |\gamma|\), we need to construct a presheaf morphism \(G: |\alpha| \to \text{Hom}(|\beta|, |\gamma|)\). Let \(G(U)\) send an \(s \in |\alpha|(U)\) to the following presheaf morphism \(H: |\beta||V| \to |\gamma||V'|\). If \(V \subset U\), define \(H(V): |\beta|(V) \to |\gamma|(V)\) by \(t \mapsto F(V)(s|_V, t)\). If \(V' \subset V\), then \(F(V')(s|_{V'}, t|_{V'}) = F(V)(s|_V, t)|_{V'}\) since \(s|_{V'} = (s|_V)|_{V'}\) and \(F\) is a natural transformation. Hence \(H = G(U)(s)\) is a natural transformation.

Conversely, given a \(G\), we need to construct an \(F\). Let \(F(U): |\alpha|(U) \times |\beta|(U) \to |\gamma|(U)\) send a pair \((s, t)\) to \(H(U)(t)\), where \(H = G(U)(s)\). If \(U' \subset U\), then \(F(U')(s|_{U'}, t|_{U'}) = H'(U')(t|_{U'})\), where \(H' = G(U')(s|_{U'})\). Since \(G\) is a natural transformation, \(H' = H|_{U'}\) as presheaf morphisms \(|\beta||U'| \to |\gamma||U'|\), where again \(H = G(U)(s)\). In particular, \(H'(U') = H(U')\) as maps \(|\beta|(U') \to |\gamma|(U')\).

(X) (explosion) Indeed, the empty sheaf is the initial object in the category of sheaves.

(XI) We need to prove that \(\vdash (\forall x \alpha(x)) \to \alpha(t)\) and \(\vdash \alpha(t) \to \exists x \alpha(x)\).
Indeed, pick a variable assignment $\mu$. Then $t$ evaluates to a fixed element $c \in D$. The projection onto the $c$th factor is a presheaf morphism $\prod_{d \in D} |\alpha|(d) \to |\alpha|(c)$, and the inclusion onto the $c$th summand is a (pre)sheaf morphism $|\alpha|(c) \to \bigsqcup_{d \in D} |\alpha|(d)$.

(XII) (generalization rule) We need to show that if $\alpha(x)$ is valid in a (pre)sheaf structure for every variable assignment, then $\forall x \alpha(x)$ is valid in the structure.

Indeed, the hypothesis says that for each $d \in D$ we have a global section of $|\alpha|(d)$. These yield a global section of the presheaf $\prod_{d \in D} |\alpha|(d)$.

(XIII), (XIV) We need to prove that if $\forall x (\beta \to \alpha(x))$ is valid in a (pre)sheaf structure, then so is $\beta \to \forall x \alpha(x)$; and if $\forall x (\alpha(x) \to \beta)$ is valid, then so is $(\exists x \alpha(x)) \to \beta$.

The hypothesis implies that for each $d \in D$ we have a presheaf morphism $|\beta| \to |\alpha|(d)$ (resp. $|\alpha|(d) \to |\beta|$). Since $(\co)$product of presheaves is their $(\co)$product in the category of presheaves, we get a presheaf morphism $|\beta| \to \prod_{d \in D} |\alpha|(d)$ (resp. $\bigsqcup_{d \in D} |\alpha|(d) \to |\beta|$). This works similarly for coproduct of sheaves.

Sheaf-valued models can be seen as an extension of Medvedev–Skvortsov models:

**Example 6.16.** Let $B$ be the two-element poset $0 < 1$ endowed with its Alexandroff topology. A presheaf over $B$ is always a sheaf; a sheaf $\mathcal{F}$ over $B$ consists of two stalks $\mathcal{F}_0$ and $\mathcal{F}_1$ and a map $\mathcal{F}_{01} : \mathcal{F}_0 \to \mathcal{F}_1$. We have $\text{Hom}(\mathcal{F}, \mathcal{G})_1 = \text{Hom}(\mathcal{F}_1, \mathcal{G}_1)$, whereas $\text{Hom}(\mathcal{F}, \mathcal{G})_0$ consists of commutative diagrams

$$
\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{\mathcal{F}_{01}} & \mathcal{F}_1 \\
\downarrow & & \downarrow \\
\mathcal{G}_0 & \xrightarrow{\mathcal{G}_{01}} & \mathcal{G}_1.
\end{array}
$$

If $\mathcal{F}_{01}$ and $\mathcal{G}_{01}$ are inclusions, such a diagram amounts to a map $\varphi : \mathcal{F}_0 \to \mathcal{G}_0$ such that $\varphi(\mathcal{F}_1) \subset \mathcal{G}_1$.

In the case where the restriction maps of the sheaves are inclusions, by 6.14 we further have $(\bigsqcup_i (\mathcal{F}_i)_1 : (\bigsqcup_i (\mathcal{F}_i)_0) = (\bigsqcup_i (\mathcal{F}_i)_1, \bigsqcup_i (\mathcal{F}_i)_0)$ and, using that $B$ is an Alexandroff space, also $((\bigsqcup_i (\mathcal{F}_i)_1 : (\bigsqcup_i (\mathcal{F}_i)_0) = (\bigsqcup_i (\mathcal{F}_i)_1, \bigsqcup_i (\mathcal{F}_i)_0)$.

We thus recover Medvedev–Skvortsov problems (see §3.11). To get an interpretation of $\vdash$, let us fix a set $S$, a domain $D$ and the arities $m_1, \ldots, m_n$ of all logical variables entering the formula to be interpreted. Let $X = \text{Hom}(D^{m_1}, 2^S) \times \ldots \times \text{Hom}(D^{m_n}, 2^S)$, where $2^S$ is the set of all subsets of $S$, and $\text{Hom}(Y, Z)$ is the set of all maps $Y \to Z$. Let us consider the Alexandroff space $X^+ = X \cup \{\hat{1}\}$, where $X$ is regarded as a discrete space, or alternatively as a poset with no comparable pair of elements, and $\hat{1}$ is an additional element that is greater than all elements of $X$. Thus every $n$-tuple of subsets $T_1, \ldots, T_n \subset S$, where each $T_i = T_i(t_1, \ldots, t_{m_i})$, corresponds to the two-element subspace $B_{T_1, \ldots, T_n} = \{(T_1, \ldots, T_n), \hat{1}\}$ of $X^+$, which is homeomorphic to $B$. Now let $\mathcal{F}_i(t_1, \ldots, t_{m_i})$ be the sheaf over $X^+$ with $\mathcal{F}_1 = X$, $\mathcal{F}_{(T_1, \ldots, T_n)} = T_i$ and $\mathcal{F}_{(T_1, \ldots, T_n)\{i\}} : T_i \to X$ the inclusion map. If we model every atomic formula by such a sheaf $\mathcal{F}_i$ of an appropriate arity, then it is easy to see that the resulting sheaf has a global section if and only if the corresponding Medvedev–Skvortsov problem is soluble by a “general method”, in the sense of §3.11.
6.5. Completeness, examples

We note the following consequence of 6.14.

**Proposition 6.17.** If $F$, $G$ and $F_i$, $i \in I$, are presheaves on $B$, then

1. $\text{Supp}(F \times G) = \text{Supp} F \cap \text{Supp} G$;
2. $\text{Supp}(F \sqcup G) = \text{Supp} F \cup \text{Supp} G$;
3. $\text{Supp} \text{Hom}(F,G) \subset \text{Int}(\text{Supp} G \cup (B \setminus \text{Supp} F))$;
4. $\text{Supp}(\text{Char} \emptyset) = \emptyset$;
5. $\text{Supp} \left( \bigsqcup F_i \right) = \bigcup \text{Supp} F_i$;
6. $\text{Supp} \left( \prod F_i \right) \subset \text{Int} \left( \bigcap \text{Supp} F_i \right)$.

This looks suspiciously similar to Tarski models, except that we have seen in the previous sections that the inclusions in (3) and (6) are strict in general, with (3) being strict even for some Alexandroff spaces.

However, by 6.6 and 6.10, the inclusions in (3) and (6) can be reversed if the presheaves satisfy the sheaf conditions, and $B$ is a zero-dimensional separable metrizable space. Also, by 6.5 a sheaf over such a space $B$ has a global section if and only if its support is the entire $B$. Thus from the completeness results for Tarski models (see §5.3) we obtain

**Theorem 6.18.** Intuitionistic logic is strongly complete with respect to its sheaf-valued models with countable domain over the Baire space $\mathbb{N}^\mathbb{N}$, and complete with respect to its sheaf-valued models with countable domain over any given zero-dimensional separable metrizable space with no isolated points.

Although over zero-dimensional separable metrizable spaces sheaf-valued models can be seen as merely a proof-relevant conservative extension of Tarski models, in general they tend to be very different from Tarski models.

In particular, from 6.4 it is easy to get a Tarski model over $\mathbb{R}$ that is not obtainable by taking supports from any sheaf-valued model over $\mathbb{R}$.

**Proposition 6.19.** There exists a Tarski model over an Alexandroff space $B$ that is not obtainable by taking supports from any sheaf-valued model over $B$.

**Proof.** Let $B = \{x, y, xy, yx, \hat{1}\}$ with partial order generated by $x, y < xy; x, y < yx$ and $xy, yx < \hat{1}$. Then $\langle x \rangle \cap \langle y \rangle = \langle xy \rangle \cup \langle yx \rangle$, and therefore if $\alpha, \beta, \gamma, \delta$ are interpreted respectively by $\langle x \rangle, \langle y \rangle, \langle xy \rangle, \langle yx \rangle$, then $(\alpha \land \beta) \rightarrow (\gamma \lor \delta)$ is interpreted by $B$.

On the other hand, if $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$, $\mathcal{G}'$ are sheaves with supports $\langle x \rangle, \langle y \rangle, \langle xy \rangle$ and $\langle yx \rangle$ respectively, then we have sheaf sections $\text{Char} \langle x \rangle \rightarrow \mathcal{F}$ and $\text{Char} \langle y \rangle \rightarrow \mathcal{F}'$ and sheaf morphisms $\mathcal{G} \rightarrow \text{Char} \langle xy \rangle$ and $\mathcal{G}' \rightarrow \text{Char} \langle yx \rangle$. Consequently we have a sheaf morphism $\text{Hom}(\mathcal{F} \times \mathcal{F}', \mathcal{G} \sqcup \mathcal{G}') \rightarrow \text{Hom}(\text{Char} \langle x \rangle \times \text{Char} \langle y \rangle, \text{Char} \langle xy \rangle \sqcup \text{Char} \langle yx \rangle)$. Therefore from 6.12, $\text{Hom}(\mathcal{F} \times \mathcal{F}', \mathcal{G} \sqcup \mathcal{G}')_x = \emptyset = \text{Hom}(\mathcal{F} \times \mathcal{F}', \mathcal{G} \sqcup \mathcal{G}')_y$. Thus if $\alpha, \beta, \gamma, \delta$ are interpreted respectively by $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$, $\mathcal{G}'$, then $(\alpha \land \beta) \rightarrow (\gamma \lor \delta)$ is interpreted by a sheaf with support $\{xy, yx, \hat{1}\}$ and not $B$. □
Example 6.20. Let us discuss the Kreisel–Putnam principle and Harrop’s rule in the light of sheaf-valued models. Let us set $|\beta| = \chi_{U_1}, |\gamma| = \chi_{U_2}$ and $|\neg \alpha| = \chi_U$. Then by 6.10(a), each $\text{Hom}(\chi_U, \chi_{U_i}) \simeq \chi_{V_i}$, where $V_i = \text{Int}(U_i \cup (X \setminus U))$. On the other hand, $|\beta \lor \gamma| \simeq \chi_{U_1} \cup \chi_{U_2}$.

In our previous discussion of the Kreisel–Putnam principle and Harrop’s rule, which was based on Tarski models (Example 5.2), we had $B = \mathbb{R}^2$, $U_1 = \{(x, y) \mid x > 0\}$, $U_2 = \{(x, y) \mid y > 0\}$ and $U = U_1 \cup U_2$. It is not hard to see that in this case, $|\neg \alpha \lor \beta| = \text{Hom}(\chi_U, \chi_{U_1} \cup \chi_{U_2})$ is isomorphic to the amalgamated union $\chi_{V_1} \cup_{\chi_V} \chi_{V_2}$, where $V = \{(x, y) \mid x < 0 \land y < 0\}$. The support of this amalgamated union is $\mathbb{R}^2 \setminus \{(0, 0)\}$, so it does not have a global section (in fact, it does not even have a section over its support). So this example no longer works to refute Harrop’s rule.

To refute Harrop’s rule with a sheaf-valued model we can, however, take $B = \mathbb{R}^2$, $U_1 = \{(x, y) \mid x > 0 \land y > 0\}$, $U_2 = \{(x, y) \mid x < 0 \land y < 0\}$ and $U = U_1 \cup U_2$. Then $V_1 = \{(x, y) \mid x < 0 \lor y < 0\}$ and $V_2 = \{(x, y) \mid x > 0 \lor y > 0\}$, so that the support of $|\neg \alpha \lor \beta| \lor (\neg \alpha \lor \gamma|)$ is $V_1 \cup V_2 = \mathbb{R}^2 \setminus \{(0, 0)\}$. On the other hand, $|\neg \alpha \lor \beta| \lor (\neg \alpha \lor \gamma)$.

Let us now analyze the difference between the two models just considered. If we write $U_0 = \text{Int}((U_1 \cup U_2) \setminus (U_1 \cap U_2))$, then the sheaf morphism $\chi_{U_0} \to \chi_{U_1} \cup \chi_{U_2}$ induces a sheaf morphism $\text{Hom}(\chi_U, \chi_{U_0}) \to \text{Hom}(\chi_U, \chi_{U_1} \cup \chi_{U_2})$. By 6.10(a), $\text{Hom}(\chi_U, \chi_{U_0}) \simeq \chi_{V_0}$, where $V_0 = \text{Int}(U_0 \cup (X \setminus U))$. Thus $|\neg \alpha \lor \beta| \lor (\neg \alpha \lor \gamma)$ will have a global section as long as $V_0$ is the entire space, and in general, $\text{Supp} \neg \alpha \lor \beta \lor \gamma$ contains $V_0$. However, $V_0$ need not be contained in $\text{Supp} |\neg \alpha \lor \beta| \lor (\neg \alpha \lor \gamma)$ as shown by the second example. On the other hand, if $V_0 \subset V_1 \cup V_2$, then $\text{Hom}(\chi_U, \chi_{U_1} \cup \chi_{U_2})$ is isomorphic to the amalgamated union $\chi_{V_1} \cup_{\chi_V} \chi_{V_2}$, where $V = \text{Int}(X \setminus U)$. In general, one can show that $\text{Hom}(\chi_U, \chi_{U_1} \cup \chi_{U_2}) \simeq (\chi_{V_1} \cup_{\chi_V} \chi_{V_2}) \cup_{\chi_W} \chi_{V_0}$, where $W = V_0 \cap (V_1 \cup V_2)$.

Example 6.21. Let us construct a sheaf-valued model where the Negative Constant Domain principle fails. Let $D = \mathbb{N}$ and let $B = (-\infty, 0] \cup \mathbb{N} \subset \mathbb{R}$, where $\mathbb{N} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\mathbb{N}$ is a regular open set in $B$, so $F = \chi_{\mathbb{N}}$ could model a negated problem. Let $F_i = \chi_{B \setminus \{1/i\}}$. Then $\prod_i F_i \simeq \chi_{(-\infty, 0]}$, so $F \sqcup \prod_i F_i \simeq \chi_{B \setminus \{0\}}$, which has no global sections. On the other hand, each $F \sqcup F_i$ has a global section, hence so does their product. Thus there exist no sheaf morphisms $\prod_i (F \sqcup F_i) \to F \sqcup \prod_i F_i$.

6.6. Generalizations
Presheaf-valued models are not as good as sheaf-valued models:

Proposition 6.22. Presheaf-valued models satisfy the Negative Constant Domain Principle.
Proof. We need to construct a morphism of presheaves \( \prod_i (F_i \sqcup F_i) \to F \sqcup \prod_i F_i \), where \( F \) is of the form \(|\neg \alpha|\). Given an open set \( U \) and a collection \( s = (s_i) \) of sections \( s_i \in F(U) \sqcup F_i(U) \), we consider two cases. If none of the \( s_i \) actually belongs to \( F(U) \), then we send \( s \) via the inclusion \( \prod_i F_i(U) \subset F(U) \sqcup \prod_i F_i(U) \). If some \( s_i \) belongs to \( F(U) \), we send \( s \) to the image of this \( s_i \) under the inclusion \( F(U) \subset F(U) \sqcup \prod_i F_i(U) \). Since \( F = |\neg \alpha| \) is the characteristic presheaf of some open set (in fact, a regular one), \( F(U) \) contains at most one element, so the choice of \( i \) is irrelevant. This construction is clearly natural in \( U \). \qed

**Proposition 6.23.** Presheaf-valued models satisfy the Rule of Omniscience and the Disjunction Property.

In sheaf-valued models, every formula interpreted by the characteristic sheaf of a clopen set is decidable. Hence the Disjunction Property fails if \( B \) is disconnected; and the Rule of Omniscience can be refuted similarly to §5.4.

Proof. To prove that the Disjunction Property holds in presheaf-valued models it suffices to note that \( F \sqcup G \) has a global section if and only if one of the presheaves \( F, G \) has a global section.

In particular, if \( \alpha \) is a formula that is decidable in the model, then either \(|\alpha|\) has a global section or \(|\neg \alpha|\) has a global section; the latter amounts to saying that \(|\alpha| = \text{Char} \emptyset \). If each \(|\beta(n)|\), \( n \in \mathbb{N} \), coincides with \( \text{Char} \emptyset \), then so does \(|\exists n \beta(n)| = \bigsqcup |\beta(n)|\); and if some \(|\beta(n)|\) has a global section, then so does \( \bigsqcup |\beta(n)| \). Thus \( \exists n \beta(n) \) is decidable in the model as long as each \( \beta(n) \) is. \qed

It is clear from the proofs of 6.22 and 6.23 that when the space \( B \) is connected, sheaf-valued models also satisfy the Negative Constant Domain Principle and the Disjunction Property — in contrast to Tarski models.

As a cheap trick to construct sheaf-type models that do not satisfy the Disjunction Property over a connected space \( B \), one could try to redefine \( \models \alpha \) as \( \text{Supp} |\alpha|(t_1, \ldots, t_n) = B \) for all \( t_1, \ldots, t_n \in D \). However, this fails to yield a model of intuitionistic logic, since by Proposition 6.4, \( \models \alpha(t) \) will not be equivalent to \( \models \forall x \alpha(x) \).

Surprisingly, a little modification of this actually works. Thus, to define a local sheaf structure, we repeat the definition of a sheaf-valued structure with just one amendment:

- \( \models \alpha \) means that every \( b \in B \) has a neighborhood \( U \) such that \(|\alpha|_b \) has a section over \( U \) for each valuation \( \mu \).

If \( \alpha \) has no parameters, this is equivalent to saying that the stalk \(|\alpha|_b \neq \emptyset \) for all \( b \in B \), that is, \( \text{Supp} |\alpha| = B \).

There is also a more general class of models, which in a sense extrapolates between the usual sheaf-valued models and the local ones. Thus, to define a uniform sheaf structure, we repeat the definition of a sheaf-valued structure with just two amendments (see [72] or [99] concerning uniform spaces):

- \( B \) is not just a topological space but a uniform space;
• $\models \alpha$ means that there exists a uniform cover $\{U_\iota\}$ of $B$ such that $|\alpha|_\mu$ has a section over each $U_\iota$ for each valuation $\mu$.

Remark 6.24. For example, let us consider the usual uniformity on $B = (0, \infty)$, induced by the Euclidean metric on $\mathbb{R}$, and the sheaf $\mathcal{F} = \bigsqcup_{n \in \mathbb{N}} \chi([\ln(n-1), \ln(n+1)])$. Its support is the entire $(0, \infty)$, but since $\ln(n+1) - \ln(n-1) \to 0$ as $n \to \infty$, there exists no uniform cover $\{U_\alpha\}$ such that $\mathcal{F}$ has a section over each $U_\alpha$.

Theorem 6.25. (a) Every local (pre)sheaf structure is a model of intuitionistic logic.

(b) Every uniform (pre)sheaf structure is a model of intuitionistic logic.

Part (a) can be proved similarly to (b) when $B$ is a metrizable space, since in this case uniform (pre)sheaf models include local (pre)sheaf models by considering the fine uniformity, that is, the finest uniform structure that induces the given topology. For metrizable spaces, the uniform covers of the fine uniformity are precisely those covers that can be refined by open covers.

Proof of (b). If $\Phi$ is an primary law of intuitionistic logic, by Theorem 6.15 $\Phi$ is satisfied in sheaf-valued structures. Then a fortiori $\Phi$ is satisfied in uniform structures. So it suffices to check the satisfaction of the inference rules.

For the modus ponens rule, we are given uniform covers $\{U_\iota\}$ and $\{V_\kappa\}$ of $B$ and sections $s_\mu \in |\alpha|_\mu(U_\iota)$ and $t_\mu \in |\alpha \to \beta|_\mu(V_\kappa)$ for each $\mu$, $\iota$ and $\kappa$. Thus $t_\mu$ is a morphism of sheaves $|\alpha|_\mu|_{V_\kappa} \to |\beta|_\mu|_{V_\kappa}$, which in particular includes a map $t_\mu(W_\lambda) : |\alpha|_\mu(W_\lambda) \to |\beta|_\mu(W_\lambda)$, where $\lambda = (\iota, \kappa)$ and $W_\lambda = U_\iota \cap V_\kappa$. Then $t_\mu(W_\lambda)(s_\mu|_{W_\lambda}) \in |\beta|_\mu(W_\lambda)$. Thus $|\beta|_\mu(W_\lambda) \neq \emptyset$ for each $\lambda$, where $\{W_\lambda\}$ is a uniform cover of $B$, which does not depend on $\mu$.

For the generalization rule, we are given a uniform cover $U_\iota$ of $B$ and a section $s_\mu \in |\alpha|_\mu(U_\iota)$ for each $\iota$ and each valuation $\mu$. Each valuation $\mu$ is determined by $\mu(x)$ along with the restriction $\mu^-$ of $\mu$ to all variables except $x$. We may write $|\alpha(x)|_\mu = |\alpha|_\mu^-(\mu(x))$, where $|\alpha|_\mu^-$ denotes $|\alpha|$ evaluated via $\mu^-$ at all variables except $x$. Then $\forall x \alpha(x)|_\mu = \prod_{d \in D} |\alpha|_\mu^-(d)$. The given sections $s_{(\mu^-, x \mapsto d)} \in |\alpha|_\mu^-(d)(U_\iota)$, $d \in D$, then yield a section $s_{\mu^-} \in \forall x \alpha(x)|(U_\iota)$.

The proof of Theorem 6.18 works to establish similar completeness results for local sheaf models and, in particular, uniform sheaf models.

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