Two quantum Ising algorithms for the Shortest Vector Problem: one for now and one for later

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Quantum computers are expected to break today’s public key cryptography within a few decades. New cryptosystems are being designed and standardised for the post-quantum era, and a significant proportion of these rely on the hardness of problems like the Shortest Vector Problem to a quantum adversary. In this paper we describe two variants of a quantum Ising algorithm to solve this problem. One variant is spatially efficient, requiring only $O(N \log N)$ qubits where $N$ is the lattice dimension, while the other variant is more robust to noise. Analysis of the algorithms’ performance on a quantum annealer and in numerical simulations show that the more qubit-efficient variant will outperform in the long run, while the other variant is more suitable for near-term implementation.

I. INTRODUCTION

The concept of quantum computing (QC) was first conceived of in the early 1980s [1–3] and has slowly grown to become a major field within modern computer science and physics. Utilising intrinsic properties of quantum mechanics allows some computations to be sped up beyond what is classically possible. Some offer exponential speed-up, such as integer factorisation and the discrete logarithm [4], whereas others offer polynomial, but still impressive, improvements like Grover’s algorithm for searching unsorted lists [5].

As the field of quantum computing has blossomed, multiple paradigms and families of algorithms have emerged. The gate model of quantum computing most closely resembles classical computers, with directly programmable qubit architectures, whereas quantum annealer-style algorithms can be seen as an analogue version, whereby after system initialisation in some eigenstate of a Hamiltonian, the Hamiltonian is gradually altered, until at completion the system is measured to be in some eigenstate of a new Hamiltonian which offers a solution to the problem under consideration.

While some forms of quantum computing, such as the gate model and adiabatic quantum computing (AQC) [6, 7] are universal, near-term quantum annealing devices are likely to be more suited to specific problem types. Nonetheless, these near term devices have compelling use cases—from simulating quantum chemistry to developing medicines [8, 9]—though some are more practical in the near future than others. In 2020 the QC community finds itself at a turning point, with the first credible claim to quantum supremacy [10] having been made, though performing useful computations of this size is still some way off for quantum computers.

A. Post-quantum cryptography

One area subject to much disruption is that of cryptography. Classical cryptography will be a victim—once quantum hardware reaches maturity—of the exponential speed-up due to Shor’s algorithm for integer factorisation and discrete logarithm computation. This is because the security of public key cryptography relies upon the existence of a problem that is intractable without a certain piece of information (known as a key) but is efficiently computable when in possession of the key. An example of such a problem is factorisation of a large semi-prime number $n = pq$, which has only two non-trivial prime factors $p$ and $q$. If either factor is known, one can divide $n$ by that factor to ascertain the other. If, however, one knows neither factor, then one must resort to a much more computationally expensive approach, such as attempting to divide $n$ by every integer up to $\sqrt{n}$ (naive) or applying one of the family of number field sieves [11], which in the best cases take super-polynomial time.

The security of the RSA cryptosystem [12] relies on integer factorization, while the security of Diffie-Hellman key exchange, ElGamal and
more rely on closely related problems [13, 14], all efficiently computable in the QC domain. To preserve security of communications and information storage moving forward into a post-quantum world, a new set of cryptographic primitives must be developed and demonstrated to be invulnerable to quantum attacks. This new field is known as post-quantum cryptography (PQC).

The search for quantum-safe primitives centres around five families of problems, as outlined in [14]: lattice-based cryptography (LBC), code-based cryptography, isogenies, multivariate-based and hash-based cryptography. Candidate systems are being assessed in the NIST Post-Quantum Cryptography Standardization process and there are 26 systems being analysed in round two of the process, of which 12 derive from lattice-based primitives.

LBC has spawned the celebrated learning with errors (LWE) problem [15], later adapted for efficiency (at the risk of as-yet unknown security tradeoffs) into Ring-LWE whereby computations are performed in algebraic number-fields, and also Module-LWE. LWE has even served as the basis for the first fully homomorphic encryption scheme (FHE) [16]. Other notable lattice-based cryptosystems include NTRU [17] and GGH [18]. The central problems in LBC tend to revolve around minimisation of distances in high-dimensional spaces.

Two closely related problems are finding the shortest distance between two points in a lattice, known as the shortest vector problem (SVP) and finding the closest lattice point to any given vector in the ambient space (CVP). Under a guarantee that said point is at most a certain distance from the nearest lattice point CVP becomes a bounded distance decoding problem (BDD) upon which the security proof of LWE is based.

B. Quantum algorithms for LBC

Up to this point most quantum algorithms for lattice problems focus on application of a preexisting algorithm to gain a quantum speed up in a primarily classical approach. The two main approaches to lattice problems are enumeration and sieving. Sieving takes as input a selection of vectors from some distribution over the lattice and iteratively combines them in order to output short solutions probabilistically [19]. On the other hand, enumeration literally enumerates all vectors within a certain ball around the origin, which if picked carefully is guaranteed to contain the best possible solution [20, 21], though recently significant speed-ups have been obtained by moving into the probabilistic domain (using a technique called extreme pruning), randomizing the input, and repeating many times [22].

Grover’s algorithm has been applied to sieving and saturation algorithms to achieve speedups of roughly 25% in the exponent [23]. It seems unlikely that Grover’s algorithm can be applied to lattice enumeration, but building on developments for quantum tree algorithms [24, 25] a quadratic speed-up has been obtained [26]. The quantum Fourier transform (QFT) plays a part in many quantum algorithms, such as those for solving variants of the hidden subgroup problem (HSP) —Shor’s is an example. The dihedral coset problem is another type of HSP; a relaxed form, the extrapolated dihedral coset problem, has been shown to be equivalent to LWE [27]. Lattice problems in certain algebraic number fields can be solved using quantum HSP algorithms that compute unit groups [28] and principal ideals [29]. A recent work [30] proposes a new approach to finding short vectors, encoding vector norms into a Hamiltonian of a system of ultra-cold bosons trapped in a potential landscape. Whilst broadly in the adiabatic quantum optimisation regime, sweeps are performed subadiabatically to obtain results from a distribution over low energy eigenstates (consequently, ‘short’ vectors).

C. Contribution

Section II contains preliminaries, then in Section III we detail a derivative quantum shortest vector algorithm based on the quantum Ising model. In particular, two variants are presented with provable asymptotic space requirements. In Section IV these two algorithms are analysed in a noiseless setting numerically, and are also implemented on the D-Wave quantum annealer [31], providing a fully quantum analysis of lattice problems in up to 7-dimensional analysis utilising 56 logical qubits (and over 1000 physical qubits).

The two variants of the algorithm presented relate to different ways of encoding qudits in the
quantum Ising model, and we offer an analysis of both implementations, including the circumstances in which each is superior. This last contribution has wider relevance for the QC community, especially when looking at algorithms to optimise over integral combinations of more general vectors.

II. PRELIMINARIES

Vectors and matrices are denoted by boldface lower and upper-case letters respectively, while Hamiltonians are denoted by $H$. Throughout the paper two vector norms are of interest: the $l^2$ (or Euclidean) norm of a $d$-dimensional vector $x$ is described by $\|x\| = x_1^2 + \ldots + x_d^2$, and the infinity norm is $\|x\|_\infty = \max\{|x_i|, 1 \leq i \leq d\}$. The length of the shortest vector is denoted $\lambda_1$; there are at least two vectors of this length in a lattice, as any vector can be reflected about the origin to produce another of identical length. Where log is used, it is in base 2.

A. Lattices

Lattices are simply a repeating pattern of points in $N$-dimensional space. Fig 1 shows an example of a lattice in two dimensions. Lattices have two attractive mathematical properties: they all contain the origin, and adding any two lattice vectors together with integer coefficients gives a point that is also in the lattice. The concept can be formalised as follows:

Definition II.1. A lattice $\mathcal{L} \in \mathbb{R}^N$ is the discrete set of all integer combinations of a set of $N$ linearly independent basis vectors $B = \{b_0, \ldots, b_N\}$:

$$\mathcal{L} = \left\{ \sum_{i=1}^{N} x_i b_i \right\} = \{B \cdot x | x \in \mathbb{Z}^N\}.$$ 

Every lattice contains the zero vector, denoted $0$ and this is considered a trivial lattice vector, as $0 \cdot B = 0$. A set of $N$ linearly independent basis vectors together are known as ‘a basis’. Each lattice has infinitely many bases, and every basis can be mapped to every other basis by a unimodular transformation. There is a notion of good and bad bases, where good means that the basis vectors are quite orthogonal and quite short, where ‘quite’ varies according to the context, but can be defined in terms of bounds on the angles between basis vectors or ratios of their lengths. Knowledge of the specifics of these conditions will not be important to the work, but it is essential to appreciate the concept that short and close-to-orthogonal is good.

Necessarily, there are only a small number of good bases (a finite number by any definition of good), and infinitely many bad bases. The subject of turning arbitrary bad lattice bases into ‘good enough’ bases is a widely studied one, and is at the centre of lattice cryptanalysis [32-34].

![FIG. 1: An example of a two-dimensional lattice with lattice points depicted in black. Red arrows represent a bad basis with long vectors that are far from orthogonal (though still linearly-independent); green arrows are a good basis for the same lattice, with short and highly orthogonal vectors. Both bases (by definition) span the lattice.](image)

Definition II.2. The fundamental parallelepiped $\mathcal{P}(\mathcal{L})$ of a lattice described by a basis $B$ is the set of points in $\mathbb{R}^N$:

$$\mathcal{P}(\mathcal{L}) = \{x \cdot B | x \in [0, 1)^N\},$$

and the covolume of $\mathcal{L}$ is defined to be the volume of this $N$-dimensional polyhedron.

The shape of the fundamental parallelepiped depends on the geometry of the particular basis, but they will all have the same covolume, and provide a tiling on the ambient space $\mathbb{R}^N$. One can compute the covolume of a lattice by taking its determinant, which is obtained by taking the determinant of the basis $\det(\mathcal{L}) = \det(B)$.

One important family of bases for this work is that of Hermite Normal Form (HNF) bases. The reason these are of interest are that:
1. They can be described (and therefore communicated) efficiently;
2. Each lattice has a unique HNF;
3. The HNF of a lattice can be efficiently computed from any provided basis.

Definition II.3. Hermite Normal Form (row-basis version) of a full rank lattice $L \subset \mathbb{Z}^N$ is an upper-triangular matrix $H$ that satisfies:
- $H_{ij} = 0$ for $i > j$;
- The first nonzero term from the left (the pivot) is positive and strictly to the right of the first nonzero term of the row above;
- Elements directly below the pivot are zero and those directly above are reduced modulo the pivot.

The HNF of a lattice is described as optimal if there exists only one non-unit pivot. This means (for a row basis HNF) that there is only one column of non-zero values. In this work, we examine only full rank integer lattices, so we take the $b_i \in \mathbb{Z}^N$ from this point onward. In general, the HNF of a lattice is bad basis by any canonical measure, though it is an efficient means of representing the lattice as it contains fewer non-zero entries than a general basis.

The lattice problem that is the focus of this work is the shortest vector problem.

Definition II.4. Shortest Vector Problem: given a basis $B = \{b_1, ..., b_n\}$ describing a lattice $L$ find the closest (non-zero) lattice point to the origin,

$$\lambda_1(L) = \min\{||x|| : x \in L\setminus\{\mathbf{0}\}\}.$$ 

This can alternatively be stated as finding the shortest distance between any two distinct points in the lattice. A major stepping-stone in the field was the reduction from GAPSVP (a close relative to SVP) to the Learning With Errors (LWE) cryptosystem [15].

B. Continuous-time Quantum Computing

Continuous-time quantum computing (CTQC) refers to a group of quantum computational strategies in which a specially engineered Hamiltonian $H(t)$ is applied to a physical system in order to drive it from an initial state toward a state from which the solution to a problem can be read. Typically, the Hamiltonian $H(t)$ is a linear combination (which may or may not be time-dependent) of a problem-independent driver Hamiltonian $H_0$ (often referred to as an initial Hamiltonian when appropriate), and a problem Hamiltonian $H_P$ which encodes the problem to be solved. The eigenstates of the problem Hamiltonian $H_P$ correspond to solutions to a problem, (often the ground state is desired), and targeting these eigenstates is where the crux of CTQC lies.

CTQC encompasses a number of quantum algorithmic families, for example adiabatic quantum computation [6] (AQC), quantum annealing [35] (QA) and continuous-time quantum walk (QW) computing [36, 37] and others. Furthermore, the quantum approximate optimisation algorithm [38] (QAOA) in the discrete-time gate model is inspired by continuous-time methods.

For the purposes of this work, we will consider systems which are initialised in the ground state of the initial Hamiltonian, and are evolved according to a linear time-sweep of length $T$, leaving the system in a Hamiltonian which at any time $t \in [0, T]$ can be described as

$$H(t) = \left(1 - \frac{t}{T}\right)H_0 + \left(\frac{t}{T}\right)H_P. \quad (1)$$

In the following section, the initial Hamiltonian $H_0$, the problem Hamiltonian $H_P$, and consequently the full Hamiltonian $H(t)$ are defined in the context of the quantum Ising model. The key to constructing the algorithm is in defining the problem Hamiltonian $H_P$ such that low energy eigenstates relate to good solutions, which reduces in the Ising model to the setting of appropriate fields on and coupling between spins.

C. Quantum annealing in the Ising Model

The Ising model originated as a tool for modelling ferromagnetism in materials. In a given material each magnetic domain has a dipole, or spin — denoted $s_i$. These spins interact with their neighbours in a manner dependent on the properties of the material. The system achieves its lowest energy state when the spins align so as to minimise the total interaction energies. The energy of such a system is described by the Hamiltonian $H = -\sum_{i,j} J_{ij}s_is_j - \sum_i h_is_i$, where
where $J_{ij}$ are coupling coefficients and $h_i$ are field strengths.

The transverse Ising model was introduced to quantum computing back in 1998 [35]. A transverse magnetic field can represent temperature, and reducing the strength of this field brings about ‘quantum cooling’. The ground state of a system modelled by an Ising Hamiltonian can be found if the system is cooled sufficiently slowly. In a material, the coefficients $J_{ij}$ and $h_i$ of the Ising model are determined by the properties of the material. In a quantum computing device that implements the Ising model, however, the programmer chooses the coefficients in order to encode their problem. The programmer sets the material. In a quantum computing device consisting of coupled qudits rather than qubits; that is, if the device implemented a generalised Ising model Hamiltonian with the spins $s_i = \pm 1$ generalised to take integer values. Since this is not the case, it is necessary to construct an ersatz qudit by combining multiple qubits.

This can be visualised—as in Fig 2—as a grid of spins, where each column represents a qudit for a different basis vector.

FIG. 2: Representation of qudits as a collection of several qubits. Each column corresponds to a basis vector, and the qubits in that column are interpreted as an integer, according to the qudit definitions, which are set out in Sections III A, III B. A binary-encoded qudit in the above can take one of 16 values, whereas a Hamming-encoded qudit can take one of 4 values.

### III. QUANTUM ISING-SVP

In this section we describe how the encoding of the SVP problem into the Ising coefficients $J_{ij}, h_i$ that define the energy of the system is performed. The coefficients are derived directly from the input basis, following a similar process as for the Bose-Hubbard quantum SVP algorithm [30]. Any lattice point can be written as the vector

$$\mathbf{v} = \mathbf{x} \cdot \mathbf{B} = x_1 \mathbf{b}_1 + \ldots + x_N \mathbf{b}_N.$$  

The $l_2$ norm of the vector $\mathbf{v}$ can be written

$$||\mathbf{v}||^2 = \sum_{i,j} x_i x_j \mathbf{b}_i \cdot \mathbf{b}_j.$$  

The aim is to find the integer combination $\mathbf{x} = (x_1, \ldots, x_N)$ that minimises this sum over $i,j = 1, \ldots, N$, given the fixed scalar values $\mathbf{b}_i, \mathbf{b}_j$, which are determined by the lattice basis with which the algorithm is run. It would be rather straightforward to map this minimisation into a device consisting of coupled qudits rather than qubits; that is, if the device implemented a generalised Ising model Hamiltonian with the spins $s_i = \pm 1$ generalised to take integer values.
Assuming such qudits are available, we can write \( Q^{(j)} \) to mean the qudit operator acting on qudit \( j \). Then, following [30], the problem Hamiltonian can be written as

\[
H_P = \sum_{i,j} \hat{Q}^{(i)} \hat{Q}^{(j)} G_{ij},
\]

where \( G_{ij} = b_i \cdot b_j \) is the \((i,j)\)th element of the Gram matrix for the lattice basis.

The eigenstates of \( H_P \) in Eq. 9 all correspond to vectors in the lattice for which every component \( x_j \) is expressible within the range of values taken by the qudits (how big this range should be is a separate question, which we address later in this section). The corresponding eigenvalues are simply the squared Euclidean length of those vectors. Thus, the ground-state of the problem Hamiltonian \( H_P \) will correspond to the uninteresting zero vector, while the first-excited manifold will consist of states that correspond to vectors with length \( \lambda_1(L) \) (the shortest vectors); there are usually at least two such shortest vectors, since applying the transformation \( x_j \to -x_j \) to each vector coordinate of \( \mathbf{v} \) produces \( -\mathbf{v} \), which has the same length. Solving SVP thus becomes equivalent to finding a state in the first-excited manifold of the problem Hamiltonian \( H_P \).

In the following, we describe two different ways to encode the qudits into these column bit-strings, as well as present bounds describing how big the range of qudit values must be, and therefore how many physical qubits are needed, to ensure that the problem Hamiltonian \( H_P \) actually expresses at least one shortest vector.

A. Hamming-weight-encoded Qudits

This qudit mapping is extremely simple and is not optimal in terms of space. This is because it leads to redundancies, with multiple spin configurations corresponding to the same qudit value in \([-2^k, 2^k]\). The reason for presenting this mapping is that it is more robust to noise (explained in Appendix A) than the more spatially efficient approach we present subsequently. This is demonstrated in Section IV. While it is not expected that even today’s cryptosystems will be broken in the NISQ era, for the foreseeable future the quantum computing community must contend with poor quality qubits. Consequently, this simple qudit may become a useful tool in the near term. For the purposes of the analysis in Section IV, the D-Wave 2000Q provides ample qubits to compensate for the inefficiencies of this mapping, and so what follows is the quantum Ising SVP algorithm for now.

Here, we define a qudit operator by a simple sum of qubit operators,

\[
\hat{Q}_{\text{Ham}}^{(j)} = \sum_{p=0}^{2^k-1} \frac{\hat{Z}_{pj}}{2}.
\]

This qudit operator assigns to each computational basis state a value by counting the number of qubits in the the +1 state, its \textit{Hamming weight}, shifted so that the possible values are symmetric about zero. From here on, when referring to the quantum Ising SVP algorithm with Hamming-weight-encoded qudits, we will use the term \textit{Ham}.

B. Binary-encoded qudits

This qudit mapping assigns values to the states of its qubit register by combining the constituent qubits into a binary number. This is maximally efficient in space as each Ising spin configuration results in a distinct coefficient vector. This optimal efficiency, however, necessitates high quality qubits as we will show in Section IV, and so this is the quantum Ising SVP algorithm for later.

The first step is to define the mapping from \( s_{ij} = \{\pm 1\} \) to \( w_{ij} = \{0, 1\} \), by the operator

\[
\frac{1 - \hat{Z}}{2}.
\]

An (unsigned) qudit operator (acting on the \( j^{th} \) qudit) then takes the form

\[
\hat{Q}_{\text{unsigned}}^{(j)} = \sum_{p=0}^{2^k-1} 2^p \left( \frac{1 - \hat{Z}_{p}}{2} \right),
\]

the eigenvalues of which are positive integers integer in the range \([0, 2^k - 1]\). However, for the purposes of solving SVP, we required a range that is symmetric about 0, for example, \([-2^k, 2^k - 1]\), and so a signed integer is required. To achieve this, introduce an extra qubit (doubling the range) and shift downwards by \( 2^k \).
The HNF basis of \( L \) basis vectors, is vector, written as a linear combination of the diagonal entries. The form of a general lattice \( L \) looks as such:

\[
Q_{\text{Bin}} = \sum_{p=0}^{k} 2^p \left( \frac{a - \hat{Z}_p}{2} \right) - 2^k \mathbb{1}
\]

(13)

From here on, when referring to the quantum Ising SVP algorithm with binary-encoded qudits, we will use the term \( \text{Bin} \).

Examination of the space requirements for implementation of this quantum Ising SVP algorithm leads to the following theorem:

**Theorem III.1.** For any \( N \)-dimensional lattice \( L \) with covolume \( \det(L) = D \) and optimal Hermite Normal Form, there exists a quantum SVP algorithm that can be run on a system of size at most \( \left( \frac{3N}{2} \log N + N + \log D \right) \) qubits.

**Proof.** The point of the proof is to ascertain how many qubits are required per qubit in order to guarantee that the shortest vector in \( L \) is represented in the Hilbert space explored by the algorithm. This question can be reduced to finding a bound on the range in which to search for coefficients for the basis vectors. That is, the algorithm seeks to minimise the length of vectors \( \mathbf{v} = \mathbf{x} \cdot \mathbf{B} \) where \(-2^k \leq x_i < 2^k\) for \( 1 \leq i \leq N \).

The HNF basis of \( L \) looks as such:

\[
\begin{pmatrix}
\mathbf{b}_1 \\
\mathbf{b}_2 \\
\vdots \\
\mathbf{b}_N
\end{pmatrix} = \begin{pmatrix}
1 & \cdots & b_1 \\
\cdots & \cdots & \cdots \\
1 & \cdots & b_{N-1} \\
D
\end{pmatrix}.
\]

(14)

The covolume of \( L \) is \( D \) as, for bases of this form, the determinant is just the product of the diagonal entries. The form of a general lattice vector, written as a linear combination of the basis vectors, is

\[
[x_1 \ldots x_N] \cdot \begin{pmatrix}
1 & \cdots & b_1 \\
\cdots & \cdots & \cdots \\
1 & \cdots & b_{N-1} \\
D
\end{pmatrix} = \begin{pmatrix}
x_1 \\
\vdots \\
x_{N-1} \\
x_1 b_1 + \ldots + x_N D
\end{pmatrix}.
\]

(15)

Minkowski’s theorem \([39]\) provides the bound

\[
\lambda_1(L) \leq \sqrt{N} \cdot D^{1/N}
\]

(16)

on the length of the shortest vector, and we will use a weaker version of this to bound the coefficients on the right hand side of Eq (15).

Minkowski’s bound prescribes a sphere about the origin, with radius equal to the bound in Eq (16), in which to search for the shortest vector. Relaxing this constraint slightly, it can be asserted that every coordinate \( x_i \) of a must be less than or equal to \( \sqrt{N} \cdot D^{1/N} \), which now prescribes a (larger) \( N \)-cube around the origin in which to search.

It follows that

\[
|x_j|, \ldots, |x_{N-1}| \leq \sqrt{N} \cdot D^{1/N}.
\]

(17)

For the final coordinate \( x_N \), we have that

\[
|x_1 b_1 + \ldots + x_N D| \leq \sqrt{N} \cdot D^{1/N}
\]

(18)

and so

\[
|x_N D| \leq \sqrt{N} \cdot D^{1/N} + |x_1 b_1 + \ldots + x_{N-1} b_{N-1}|.
\]

(19)

Applying the triangle inequality,

\[
|x_N D| \leq |x_1 b_1| + \ldots + |x_{N-1} b_{N-1}| + \sqrt{N} \cdot D^{1/N},
\]

(20)

and because the \( b_i \) are reduced modulo \( D \),

\[
|x_N D| \leq |x_1 D| + \ldots + |x_{N-1} D| + \sqrt{N} \cdot D^{1/N}.
\]

(21)

Then by applying Eq (17)

\[
|x_N D| \leq (N - 1)\sqrt{N} \cdot D^{1/N} \cdot D + \sqrt{N} \cdot D^{1/N} \leq N^{3/2} \cdot D^{1+1/N}.
\]

(22)

This gives an interval to search for \( x_N \) bounded in size by

\[
|\mathbf{x}_N| \leq 2N^{3/2} \cdot D^{1/N},
\]

(23)

and an interval to search for \( x_i \) (1 \leq i < N) bounded in size (from Eq (17) by \( (2N^{1/2} \cdot D^{1/N}) \). The algorithm therefore requires

\[
\log \left( 2N^{3/2} D^{1/N} \right) = 1 + \frac{3}{2} \log N + \frac{1}{N} \log D
\]

(24)

The total number of qubits per qudit, meaning the total system requires \( O(N \log N + \log D) \) qubits. \( \square \)
Corollary III.1.1. Ham can be implemented on any $N$-dimensional lattice $\mathcal{L}$ with cocomplex $\det(\mathcal{L}) = D$ and optimal Hermite Normal Form, using a system of at most $2N^{5/2} \cdot D^{1/N}$ qubits. The first excited eigenstate, modulo degeneracies, of the system at completion solves SVP on $\mathcal{L}$.

Sketch of proof: Recycling working from the proof of Theorem III.1, the number of qubits in the grid is determined by the dimension of the lattice ($N$), and the interval over which to search for coefficients, which is given in Eq (23) as $2N^{3/2} \cdot D^{1/N}$, leaving total qubit scaling as $2N^{5/2} \cdot D^{1/N}$.

The bounds given in Theorem III.1 and Corollary III.1.1 are upper bounds that guarantee the existence of a vector with length $\lambda_1(\mathcal{L})$ (a shortest vector) within the Hilbert space explored by the quantum Ising algorithms described in III B, III A. In practice, an attacker may choose to reduce the complexity of the system by reducing the qubits-per-qudit parameter in order to give stronger probabilities of finding ‘short’ vectors, at the expense of the assurance of being able to find a shortest vector.

It is also the case that Theorem III.1 and Corollary III.1.1 give upper bounds for HNF input bases. Using different, (classically) better-reduced bases is also likely to require lower qubits-per-qudit values and so yield better results, but giving a bound for these is not straightforward, and the HNF has the advantage that each lattice has a unique HNF that can be efficiently obtained from any other basis.

IV. RESULTS

To analyse the performance of the two algorithms described in Section III, Ham and Bin were implemented on the D-Wave 2000Q quantum annealer, to shed light on what is possible in the NISQ era of quantum computing. We also performed numerical simulations of ideal, closed-system versions of these algorithms to give an indication of what may be possible in the future — though we were limited to smaller experiment sizes due to the computational constraints of simulating quantum systems on classical hardware.

In our experiments we generated random full-rank integer lattices, and obtained a ‘bad’ input basis by post-multiplying with randomly generated unimodular matrices. These bad bases were not in HNF, and we did not use the upper bounds given in III.1, III.1.1, but instead fixed our qudits to the ranges $[-4,4]$ and $[-4,3]$ for Ham, Bin respectively. This choice was made in order to improve overall results by a range of metrics discussed in this section, while also fixing a degree of freedom to make the analysis easier to perform.

This means that it is not guaranteed that a shortest vector (of length $\lambda_1(\mathcal{L})$) is represented in the Hilbert space for every instance. In fact, a shortest vector is represented in all of the three-dimensional lattice instances we used, and approximately half of the instances in each of the higher dimensions. This is reflected in the results presented in this section for finding a shortest vector; we have not post-selected on instances where shortest vectors are obtainable with coefficients in the specified range.

A. Numerical Simulation

We numerically simulated results for both Ham and Bin algorithms for three-dimensional lattices for a range of sweep times to investigate what performance could be expected in the limit of perfect hardware.

The numerical simulations in Fig 3 demonstrate that in the limit of fast sweeps, the outputs are of little use: their distribution will be close to uniform across all eigenstates. In the limit of slow sweeps, also known as the adiabatic limit, the ground state is observed with certainty. As the ground-state is a the trivial zero-vector $\mathbf{0}$, this is of similarly little use. However, what both sets of results also demonstrate is a ‘Goldilocks zone’, as first identified in [30].

When a sweep time $T$ is picked that is long enough that (at completion) a significant proportion of probability density is concentrated on low-energy eigenstates, but short enough that it is not adiabatic (with probability density concentrated on the ground state), eigenstates corresponding to short vectors are observed with good probability. Both versions, Ham and Bin, of the Ising SVP algorithm suggest that sweeps which are neither ‘too slow’ nor ‘too fast’ are optimal for achieving short non-zero lattice vec-
FIG. 3: Numerical experiments carried out on 20 3D lattices. The lines show mean final probabilities for measuring the system to be in the ground (blue), first (red) and second (green) excited eigenstates —grouped by degeneracy —at completion for sweep lengths \( T \), increasing in powers of 2, for the quantum Ising algorithm implemented with (a) Hamming-weight-encoded qudits (Ham) and (b) binary-encoded qudits (Bin). The Ham algorithm nears the adiabatic limit at \( T = 2^8 \), after the shortest vector success probability peak at around \( T = 2^2 \), which is considerably lower than the peak time for Bin. The Bin algorithm nears the adiabatic limit past the rightmost data point but not before an encouraging peak in shortest vector success probability at \( T = 2^9 \). Standard errors across the 20 samples are shown in both plots by vertical error bars.

B. D-Wave Quantum Annealer

The algorithms Bin and Ham, due to their Ising formulation, can be performed on the D-Wave 2000Q quantum annealer [31]. We examined the performance of the algorithms presented here experimentally using the D-Wave quantum processor. Interpreting these results requires a subtle understanding of the interplay between features of the algorithm and the mechanics of the QPU, discussed in Appendix A. In particular, we used the default embedding provided with the API to map from the fully connected qubit graph specified by the theoretical model described in Section III (the logical qubits), and the QPU qubit graph (the physical qubits). In general, this embedding incurs a quadratic cost (logical qubits to physical qubits), resulting in a maximum system size for the experiments listed of 56 logical qubits, which maps (non-deterministically) to over 1000 physical qubits.

The left (a,c) and right-hand (b,d) plots of Fig 4 display the same information, but on the right the x-axis is logarithmic in order to make it easier to see where the bulk of the probability density lies. Top row (a,b) is Bin on a representative six-dimensional lattice example, and the bottom row (c,d) is Ham on the same example.

In this case, Ham appears to return shorter vectors and higher rates of occurrence, with some coefficient vectors occurring over 25 times, whereas Bin only records a maximum of 4 occurrences for any combination. It should also be noted that Ham returns many more vectors in the ‘somewhat useful’ range (any nonzero vector
shorter than the longest input-basis vector). In Fig. 4, this region lies between the green marker ($\|\lambda_1(L)\|^2$, $\log(\lambda_1(L))^2$) for left and right plots respectively, and the farthest right red marker ($\max_i(\|b_i\|^2), \max_i(\|\log(b_i)\|^2)$) for left and right plots respectively.

While Fig 4 helps to understand what the quantum annealing results look like for an individual example, as well as to qualitatively compare Ham with Bin, for rigorous performance analysis other visualisations of results are appropriate, such as in Fig 5.

Fig 5 shows a summary of results obtained for 19 lattices in dimensions 3 to 7 inclusive, according to four key figures of merit (from left to right), and performed using two timesweep lengths: 1\,\mu s, 64\,\mu s for top and bottom rows respectively. The 1\,\mu s sweep was the fastest run, and the most successful sweep was at 64\,\mu s which was found at the peak of a broad global maximum. The red lines show the performance of Ham, the green ones are Bin, and baselines from uniform random sampling of the Hilbert spaces are shown in black solid and dotted lines (Ham, Bin have different, but asymptotically close, baselines due to the slight difference in coefficients searched over). Immediately one can see that the red lines sit higher than the green lines indicating higher probabilities of success for Ham on the quantum annealing hardware. We will now break down the results of Fig 5 so as to understand fully what these results imply.

The leftmost plots show probability of the annealer returning the zero vector 0. This may seem to be a useless figure of merit (FoM) because the zero vector is of no use computation-
FIG. 5: $T = 1\mu s$ (a-d) and $64\mu s$ (e-h) for Ham (red) and Bin (green) algorithms performed on 19 randomly generated 'bad' lattice bases in dimensions 3 to 7 inclusive, and for 4 different figures of merit: ground state ((a,e) not desired), shortest vector ((b,f) ideal), shorter than minimum input basis vector (c,g), and shorter than median input basis vector (d,h). The black lines are baselines indicating the results from uniform random sampling from the searched-over Hilbert space (dotted for Ham, solid for Bin), i.e. what proportion of lattice vectors in the solution space are shorter than minimum (c,g), median (d,h) input basis vectors.

ally. It is best understood, however, in relation to other figures of merit and the performance of the algorithm overall, as 0 results can be taken as a measure of 'adiabaticity'. Through this lens, it can be seen that Ham is far superior to Bin in returning ground states on the noisy quantum annealing hardware. The second-leftmost FoM is the probability to obtain the shortest vector, which is the most important. On this FoM it is interesting to observe that both Ham and Bin perform remarkably similarly (with Ham again slightly outshining Bin) despite a significant difference for the ground state FoM. As expected, probabilities for both 0 and $\lambda_1(L)$ decay as lattice dimension $N$ —and, accordingly, Hilbert space size —increases. To verify the rate of decay would require quantum hardware that is beyond present capabilities.

It is therefore easier to analyse ensembles of vectors as opposed to individual results, as these give probabilities which can be more easily analysed within the constraints of 900 repetitions per lattice sample per algorithm per time sweep. This is where the rightmost two FoM offer insight. The centre-right plots (c,g) give the probabilities observed of returned vectors having length shorter than the shortest basis vector, and the rightmost (d,h) give probabilities of vectors having length shorter than the longest basis vector. Both FoM provide the opportunity to improve on the bad basis provided as input to the algorithm.

The basis vectors themselves are trivial results (given by coefficient vectors of the form $(0,\ldots,0,1,0\ldots,0)$), but only represent a very small proportion of the outputs. It is encouraging to note that probabilities do not tail off as for 0 and $\lambda_1(L)$ at dimensions 6 and 7, and in fact appear to increase for the more relaxed FoM probabilities in (d,h).

It is not important that probability of obtaining $\lambda_1$ tends to zero as dimension increases (this is inevitable), but how fast this occurs is of interest. A polynomial decay, for example, would be catastrophic for LBC, whereas an exponential decay could even match the current state of the art for lattice algorithms. At this stage, not enough data points are available to heuristically estimate the rate of decay. A key takeaway is that Ham outperforms Bin at almost
every data point across the 8 subplots, demonstrating a significant improvement on $Bin$ for the present regime (many qubits available but not high quality), cementing it as the choice for now.

V. CONCLUSION

The algorithms described in Section III go a way to establishing the vector optimisation framework first proposed in [30], and the work described in Section IV signals emphatically that quantum cryptanalysis is drawing into the empirical realm, and is no longer purely a theoretical endeavour. With an ever increasing pace of development in quantum hardware, coupled with worst-case asymptotic scaling of $O(N \log N)$ for $Bin$, it is foreseeable that in the near future much larger experiments can be carried out to put quantum lattice algorithms to the test, be it on annealers such as D-Wave’s as described here or gate architectures.

AQC usually suffers poor time-scaling due to its dependence on $1/\Delta$ (where $\Delta$ is the minimum spectral gap between the ground and first excited eigenstates along the Hamiltonian path) which can grow very quickly as system size increases. This is necessary in order to preserve the system in its ground state throughout the evolution. The identification of the ‘Goldilocks zone’ well away from the adiabatic limit in this work is encouraging as it hints that algorithms such as the ones described in Section III may achieve much more appealing time-scaling.

In fact, this raises the curious question of how to approach extracting asymptotic time scaling for an algorithm where success is defined to be measuring the system in its first excited eigenstate. More generally, the approximate form of SVP, $\text{SVP}_\gamma$, only requires an attacker to find a vector of length polynomially ($\text{poly}(N)$) larger than $\lambda_1(\mathcal{L})$. This of course means that $\text{poly}(N)$ excitations are admissible during the evolution, which could potentially be traded off against significant speed-ups. Answering this question would doubtless be of interest well beyond the post-quantum cryptography, as it would unlock solutions to many approximate optimisation problems in QC.

One key area of progress bearing significance for QC is that of improving the fidelity of qubits both in the annealer and gate architectures.

In the meantime, as demonstrated above, it is down to theorists to think carefully about how their mathematical constructions might best make use of the hardware available to them as the significant improvements we extracted from $Ham$ qudits were nearly dropped from consideration due to the asymptotic inefficiency in space of $O(N^2)$ versus $O(N \log N)$ for $Bin$ qudits. In this way, it is possible for experimental and theoretical work to meet in the middle.

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Appendix A: D-Wave

The quantum Ising model assumes full qubit-qubit connectivity, whereas D-Wave 2000Q is constructed according to a Chimera topology, whereby each qubit is connected to a handful of nearby qubits that form a sort of cluster, and each cluster is connected to a few others. Each cluster is represented as the diamond formation of dots in Fig 6. A model requiring full connectivity can be mapped into D-Wave’s Chimera topology incurring a quadratic cost in the number of qubits required. This is done by creating qubit ‘chains’, which are illustrated in Fig 6. A model requiring full connectivity can be mapped into D-Wave’s Chimera topology incurring a quadratic cost in the number of qubits required. This is done by creating qubit ‘chains’, which are illustrated in Fig 6. In a qubit chain, all qubits are strongly incentivised to return the same value by assigning qubits in the same chain with stronger qubit-qubit ferromagnetic interactions than between qubits in different chains. By way of error correction, when not all qubits in a chain return the same value (called a chain break), a simple majority vote is taken to decide on the final value.
FIG. 6: Diagram, from [40] shows logical qubits embedded as qubit chains (of physical qubits) into the Chimera topology, where each qubit sharing the same colour has strong ferromagnetic interactions with other qubits which it is connected to in the same chain.

1. Ham qudit advantage

The reason the Ham qudits trump the Bin qudits despite the larger system size required to search the same solution space is that they allow for greater utilisation of the energy spectrum available. In the Bin setting, the strength of qubit-qubit interactions decrease on average by a factor of 2 for each step away from the most significant qubit. This means that despite the instability of SVP solutions (by which we mean that a small error in the coefficient vector makes a large difference to the output eigenenergy) the least significant qudits have relatively very weak interactions with the rest of the system, as all $J, h$ interaction values must be scaled down to the energy spectrum provided. Crucially, this also means that errors in the interaction energies become much larger relative to the problem Hamiltonian, in effect meaning that it is much more likely that the qpu is solving the wrong problem [41], leading to poorer performance relative to Ham.

In Ham, however, each qubit contributes the same amount to the output of the qudit, and so small differences ($\pm 1$ in the value of a coefficient) are effected by stronger forces, increasing likelihood of attaining low-energy solutions. We believe this effect to be quite significant, but tempered somewhat by the effects of having a larger system size: longer chains are required, which leads to higher chain-break probabilities, and thus more errors occur.

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