Monotonicity of $p$-norms of multiple operators via unitary swivels

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Abstract

Following the various statements of [DW16] to their logical conclusion, this note explicitly argues the following statement, implicit in [DW16]: for positive semi-definite operators $C_1, \ldots, C_L$, a unitary $V_{C_i}$ commuting with $C_i$, and $p \geq 1$, the quantity

$$\max_{V_{C_1}, \ldots, V_{C_L}} \left\{ \| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \|_p^p \right\}$$

is monotone non-increasing with respect to $p$. The idea from [DW16] is that by allowing unitary swivels connecting a long chain of positive semi-definite operators together, we can establish such a statement, which might not hold generally without the presence of the unitary swivels. Other related statements follow directly from [DW16] as well, being implicit there, and are given explicitly in this note.

1 Introduction

In this short note, I conduct the exercise of combining the various statements given in [DW16] and taking them to their logical conclusion. The result is a monotonicity inequality regarding $p$-norms of multiple operators strung together in a sequence. The only modification I make to the prior statements from [DW16] is to substitute density operators with general positive semidefinite operators. In [DW16], my coauthor and I were motivated by concerns in quantum information theory, and so there we worked exclusively with density operators (positive semi-definite operators with trace equal to one); however, it is obvious that all of the inequalities established there extend to the more general case when the operators are positive semi-definite with no restriction on their trace.

One of the main messages of [DW16] is that it is possible to establish non-trivial orderings of generalized Rényi entropies formed by connecting the marginals of density operators together in a product under a Schatten $p$-norm, while at the same time allowing for “unitary swivels” between these operators. In [DW16], my coauthor and I used the phrase “unitary swivels” to describe the method for arriving at the aforementioned inequalities, because, in spite of the fact that straightforward multi-operator extensions of the statements do not appear to be generally true, we showed how they hold if allowing for unitary swivels interleaved in a large chain of operators connected together. The bedrock upon which these results rested is the powerful method of complex

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interpolation \[BL76\], which has found a number of applications in a variety of areas in mathematics and physics.

To begin with, let us recall the following explicit statement from \[DW16\] Proposition 18, as specialized in \[DW16\] Corollary 19:\[^1\]

\[
\max_{V_{BC}} \left\| \frac{1}{p} \rho_{AC} V_{BC} \rho_{BC} \right\|_p^{1/p} \text{ is monotone non-increasing for } p \geq 2,
\]

(1)

where \( \rho_{ABC} \) is a density operator acting on a Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \), \( \rho_{BC} = \text{Tr}_A(\rho_{ABC}) \), \( \rho_{AC} = \text{Tr}_B(\rho_{ABC}) \), and \( \rho_C = \text{Tr}_A(\rho_{ABC}) \) are its marginals, \( V_{BC} \) is a unitary commuting with \( \rho_C \), and \( \|X\|_p \equiv [\text{Tr}\{\|X\|^p\}]^{1/p} \) is the Schatten \( p \)-norm of an operator \( X \). In \[DW16\] Section 6, it was discussed how one can chain together various density operators acting on tensor-product Hilbert spaces and obtain results similar to those given in the rest of the paper \[DW16\]. Carrying this through, the conclusion is that the following statement holds

\[
\max_{V_{C_1}, \ldots, V_{C_L}} \left\| C_1^{1/p} V_{C_1} C_2^{1/p} V_{C_2} \cdots C_L^{1/p} V_{C_L} \right\|_p \text{ is monotone non-increasing for } p \geq 2,
\]

(2)

for \( C_1, \ldots, C_L \) density operators and \( V_{C_i} \), a unitary commuting with \( C_i \). In \[DW16\] Remark 12, it was mentioned how the optimizations over commuting unitaries can be replaced with more explicit bounds found by applying the Stein–Hirschman operator interpolation theorem \[Ste56\] \[Hir52\]. Carrying this statement through as well, the conclusion is that the following inequality holds for \( 2 \leq q \leq p \):

\[
\log \left\| C_1^{1/p} C_2^{1/p} \cdots C_L^{1/p} \right\|_p^p \leq \int_{-\infty}^{\infty} dt \beta_{q/p}(t) \log \left\| C_1^{(1+i)t/q} C_2^{(1+i)t/q} \cdots C_L^{(1+i)t/q} \right\|_q^q,
\]

(3)

if \( C_1, \ldots, C_L \) are density operators and \( \beta_{q}(t) \equiv \sin(\pi \theta)/(2\theta [\cosh(\pi t) + \cos(\pi \theta)]) \), a probability distribution over \( t \in \mathbb{R} \) and with a parameter \( \theta \in [0, 1] \). In \[DW16\] Section 6, it was also discussed how one can obtain limits of the inequalities presented in the paper by applying the well known Lie-Trotter product formula. Carrying this through (i.e., taking the limit \( p \to \infty \)), the conclusion is that the following inequality holds

\[
\log \text{Tr} \{ \exp \{ \log C_1 + \cdots + \log C_L \} \} \leq \int_{-\infty}^{\infty} dt \beta_0(t) \log \left\| C_1^{(1+i)t/q} C_2^{(1+i)t/q} \cdots C_L^{(1+i)t/q} \right\|_q^q,
\]

(4)

where \( \beta_0(t) \equiv \lim_{\theta \to 0} \beta_{q}(t) = \pi/(2\theta [\cosh(\pi \theta) + 1]) \). By inspection of the proof given in \[DW16\] Proposition 18, it is clear that the inequalities in (3)-(4) hold for positive semi-definite operators as well. We can also see from that proof that (3) holds more generally for \( 1 \leq q \leq p \) and (4) for \( 1 \leq q \).

### 2 Explicit Proofs of (2)-(4)

In the rest of this note, I give explicit proofs of (2)-(4) for the benefit of the reader, following the steps outlined in \[DW16\] line by line.

\[^1\]Here and throughout, I am following the labeling in the arXiv post for \[DW16\].
Theorem 1 Let $C_1, \ldots, C_L$ be positive semi-definite operators, let $V_i$ denote a unitary commuting with $C_i$ for all $i \in \{1, \ldots, L\}$, and let $p \geq 1$. Then the following quantity is monotone non-increasing with respect to $p$:

$$\max_{V_1, \ldots, V_L} \left\| C_1^{1/p} V_1 \cdots C_L^{1/p} V_L \right\|^p_{p}.$$  \hfill (5)

Proof. The proof of this statement is essentially identical to the proof of [DW16, Proposition 18]. It is a consequence of a well known complex interpolation theorem recalled as Lemma 3 below. Let $V_1, \ldots, V_L$ denote a set of fixed unitaries, where $V_i$ commutes with $C_i$. Let $q$ be such that $1 \leq q < p$ (there is nothing to prove if $q = p$). For $z \in \mathbb{C}$, pick

$$G(z) = C_1^{1/q} V_1 \cdots C_L^{1/q} V_L,$$

$$p_0 = \infty,$$

$$p_1 = q,$$

$$\theta = q/p,$$

the choices above being identical to those in [DW16, Eq. (7.6)-(7.9)]. This implies that $p_\theta = p$. Applying Lemma 4 gives

$$\|G(\theta)\|_p \leq \sup_{t \in \mathbb{R}} \|G(it)\|_\infty^{1-\theta} \sup_{t \in \mathbb{R}} \|G(1 + it)\|^\theta_q.$$  \hfill (10)

Consider that

$$\|G(\theta)\|_p = \left\| C_1^{1/p} V_1 \cdots C_L^{1/p} V_L \right\|_p,$$

$$\|G(it)\|_\infty = \left\| C_1^{it/q} V_1 \cdots C_L^{it/q} V_L \right\|_\infty \leq 1,$$

$$\|G(1 + it)\|_q = \left\| C_1^{(1+it)/q} V_1 \cdots C_L^{(1+it)/q} V_L \right\|_q,$$

$$\leq \max_{W_1, \ldots, W_L} \left\| C_1^{1/q} V_1 \cdots C_L^{1/q} W_1 \cdots C_L^{1/q} W_L \right\|_q,$$

which are conclusions identical to those in [DW16, Eq. (7.11)-(7.17)]. Putting everything together, we find that, for all $V_1, \ldots, V_L$, the following inequality holds

$$\left\| C_1^{1/p} V_1 \cdots C_L^{1/p} V_L \right\|_p \leq \max_{W_1, \ldots, W_L} \left\| C_1^{1/q} W_1 \cdots C_L^{1/q} W_L \right\|^\theta_q,$$

which is equivalent to

$$\left\| C_1^{1/p} V_1 \cdots C_L^{1/p} V_L \right\|^p_p \leq \max_{W_1, \ldots, W_L} \left\| C_1^{1/q} W_1 \cdots C_L^{1/q} W_L \right\|^q_q.$$

(15)

Since (17) holds for all $V_1, \ldots, V_L$, the statement of the theorem follows. \hfill \n
Theorem 2 Let $C_1, \ldots, C_L$ be positive semi-definite operators, and let $p > q \geq 1$. Then the following inequality holds:

$$\log \left\| C_1^{1/p} C_2^{1/p} \cdots C_L^{1/p} \right\|^p_p \leq \int_{-\infty}^\infty dt \, \beta_q/p(t) \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \cdots C_L^{(1+it)/q} \right\|^q_q.$$  \hfill (18)
Proof. Here we directly follow the suggestion from [DW16, Remark 12]. Pick $G(z)$, $p_0$, $p_1$, and $\theta$ as in (6)–(9), with $V_{C_1} = \cdots = V_{C_L} = I$. Applying Lemma 5 below, we find that

$$\log \|G(\theta)\|_{p_0} \leq \int_{-\infty}^{\infty} dt \alpha_\theta(t) \log \|G(it)\|_{p_0}^{1-\theta} + \beta_\theta(t) \log \|G(1 + it)\|_{p_1}^\theta. \tag{19}$$

After using that

$$\log \|G(it)\|_{p_0}^{1-\theta} = \log \left\|C_1^{it/q} \cdots C_L^{it/q}\right\|_{\infty}^{1-\theta} \leq 0, \tag{20}$$

as recalled above, we are left with

$$\log \|G(\theta)\|_{p_0} \leq \int_{-\infty}^{\infty} dt \beta_\theta(t) \log \|G(1 + it)\|_{p_1}^\theta. \tag{21}$$

This is then equivalent to the statement of the theorem. $\blacksquare$

Corollary 3 Let $C_1, \ldots, C_L$ be positive definite operators, and let $q \geq 1$. Then the following inequality holds:

$$\log \text{Tr} \left\{ \exp \left\{ \log C_1 + \cdots + \log C_L \right\} \right\} \leq \int_{-\infty}^{\infty} dt \beta_0(t) \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \cdots C_L^{(1+it)/q} \right\|_q^q. \tag{22}$$

Proof. Consider that

$$\left\| C_1^{1/2p} C_2^{1/2p} \cdots C_L^{1/2p} \right\|_{2p} = \text{Tr} \left\{ \left[ C_1^{1/2p} C_2^{1/2p} \cdots C_L^{1/2p} \right]^p \right\}. \tag{23}$$

Then by the multioperator Lie–Trotter product formula [Suz85], we have that

$$\lim_{p \to \infty} \text{Tr} \left\{ \left[ C_1^{1/2p} C_2^{1/2p} \cdots C_L^{1/2p} \right]^p \right\} = \text{Tr} \left\{ \exp \left\{ \log C_1 + \cdots + \log C_L \right\} \right\}. \tag{24}$$

The inequality in the statement of the corollary is then a direct consequence of Theorem 2 and the above. $\blacksquare$

Lemma 4 Let $S \equiv \{ z \in \mathbb{C} : 0 \leq \text{Re} \{ z \} \leq 1 \}$, and let $L(\mathcal{H})$ be the space of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Let $G : S \to L(\mathcal{H})$ be a bounded map that is holomorphic on the interior of $S$ and continuous on the boundary.\(^2\) Let $\theta \in (0, 1)$ and define $p_\theta$ by

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \tag{25}$$

where $p_0, p_1 \in [1, \infty]$. For $k = 0, 1$ define

$$M_k = \sup_{t \in \mathbb{R}} \|G(k + it)\|_{p_k}. \tag{26}$$

Then

$$\|G(\theta)\|_{p_0} \leq M_0^{1-\theta} M_1^\theta. \tag{27}$$

\(^2\)A map $G : S \to L(\mathcal{H})$ is holomorphic (continuous, bounded) if the corresponding functions to matrix entries are holomorphic (continuous, bounded).
The following lemma is based on Hirschman's improvement of the Hadamard three-line theorem [Hir52].

Lemma 5 (Stein–Hirschman) Let \( S \equiv \{ z \in \mathbb{C} : 0 \leq \text{Re}\{z\} \leq 1 \} \) and let \( G : S \to L(\mathcal{H}) \) be a bounded map that is holomorphic on the interior of \( S \) and continuous on the boundary. Let \( \theta \in (0, 1) \) and define \( p_\theta \) by

\[
\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]

where \( p_0, p_1 \in [1, \infty] \). Then the following bound holds

\[
\log \| G(\theta) \|_{p_\theta} \leq \int_{-\infty}^{\infty} dt \alpha_\theta(t) \log \| G(it) \|_{1-\theta}^{1-\theta} + \beta_\theta(t) \log \| G(1 + it) \|_{\theta},
\]

where \( \alpha_\theta(t) \) and \( \beta_\theta(t) \) are defined by

\[
\alpha_\theta(t) = \frac{\sin(\pi \theta)}{2(1 - \theta) \cosh(\pi t) - \cos(\pi \theta)},
\]

\[
\beta_\theta(t) = \frac{\sin(\pi \theta)}{2\theta \cosh(\pi t) + \cos(\pi \theta)}.
\]

Remark 6 Fix \( \theta \in (0, 1) \). Observe that \( \alpha_\theta(t), \beta_\theta(t) \geq 0 \) for all \( t \in \mathbb{R} \) and we have

\[
\int_{-\infty}^{\infty} dt \alpha_\theta(t) = \int_{-\infty}^{\infty} dt \beta_\theta(t) = 1,
\]

(see, e.g., [Gra08, Exercise 1.3.8]) so that \( \alpha_\theta(t) \) and \( \beta_\theta(t) \) can be interpreted as probability density functions. Furthermore, the following limit holds

\[
\lim_{\theta \to 0} \beta_\theta(t) = \frac{\pi}{2|\cosh(\pi t) + 1|} = \beta_0(t),
\]

where \( \beta_0 \) is also a probability density function on \( \mathbb{R} \).

References

[BL76] J. Bergh and Jorgen L"ofstr"om. Interpolation Spaces. Springer-Verlag Berlin Heidelberg, 1976.

[DW16] Frédéric Dupuis and Mark M. Wilde. Swiveled Rényi entropies. Quantum Information Processing, 15(3):1309–1345, March 2016. arXiv:1506.00981.

[Gra08] Loukas Grafakos. Classical Fourier Analysis. Springer, second edition, 2008.

[Hir52] Isidore Isaac Hirschman. A convexity theorem for certain groups of transformations. Journal d’Analyse Mathématique, 2(2):209–218, December 1952.

[Ste56] Elias M. Stein. Interpolation of linear operators. Transactions of the American Mathematical Society, 83(2):482–492, November 1956.

[Suz85] Masuo Suzuki. Transfer-matrix method and Monte Carlo simulation in quantum spin systems. Physical Review B, 31(5):2957, March 1985.