OPTIMAL CONTROL PROBLEM OF VARIABLE-ORDER DELAY SYSTEM OF ADVERTISING PROCEDURE: NUMERICAL TREATMENT

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Abstract. This paper presents an optimal control problem of the general variable-order fractional delay model of advertising procedure. The problem describes the flow of the clients from the unaware people group to the conscious or bought band. The new formulation generalizes the model that proposed by Muller. Two control variables are considered to increase the number of customers who purchased the products. An efficient nonstandard difference approach is used to study numerically the behavior of the solution of the mentioned problem. Properties of the proposed system were introduced analytically and numerically. The proposed difference schema maintains the properties of the analytic solutions as boundedness and the positivity. Numerical examples, for testing the applicability of the utilized method and to show the simplicity, accuracy and efficiency of this approximation approach, are presented with some comprising with standard difference methods.

1. Introduction. It is widely known, that the goal of the advertisements is to persuade the people to buy specific goods, which consists on the focusing on the necessity of the goods in generic and by appearing the difference between a particular kind over the other goods for motivating the audiences to purchases the product. There are many strategies to modify the thought of audience on goods or services. The advertising messages are one of these methods. These missions may be through magazines, newspapers, television and radio, which known as the body media. The missions, also, maybe through soft media like letters, text websites, and speech [15].

Usually, the effect of the advertising is delayed in time, thus we need to incorporate the memory to systems of advertisement. So, to describe the strategies of the advertisement, the models in which the present status follows all of its prior

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statuses are more suitable than those in which the present status follows only its first previous one.

It is well known, the delay differential equations form a class of mathematical models which allow the systems rate of change to depend on its past history. The most important difference between ordinary differential equations and delay differential equations is that the derivative at any time on the delay differential equation depends on the solution at prior times. Systems of delay differential equations now occupy a place of central importance in all areas of science because it give more realistic distributed assumptions.

Also, calculus of fractional order has obtained major fame and gravity depending on its attractive applications as a new mathematical modeling which labor in engineering and scientific fields, like thermoelasticity ([1], [28], [40]) viscoelasticity [16], hydrology [7], fractional dynamics ([45], [41]) and system control ([27], [6] and [5]). Differential equations which have fractional order is consider to be the best method to depict the fractional systems. Unfortunately, rarely and in some easy cases we can take out the proper solutions of fractional differential equations. Therefore, it is very important to use numerical approaches to find approximation solutions of those models, these methods require great efforts.

On the other hand, a lot of differential operators in huge number of dynamic phenomena are fractional operator and have a dynamical properties, that means their order of derivative is field-changeable, that can flux with time and/or space. Thereafter, many scientists have studied the properties of the fractional derivatives of variable-order. Samko ([30], [31]), starting from 1993, with his team have proposed this interesting concept by updating the constant fractional order of fractional derivatives into fractional derivatives with order as a function of at least one variable. After that, a lot of authors have introduced many concepts of fractional derivative with variable-order. Lorenzo et al. in [19] discussed the relation between the derivative operators of variable-order and introduced some definitions of variable-order derivatives. A novel definition for fractional operator of variable-order was proposed by Coimbra [12]. In this operator, he used the Laplace transformation of the Caputo concept for fractional derivative. Many mathematical, engineering, physical, mechanical, viscoelasticity and financial, phenomena have been described using differential operators of variable order, see ([4], [11], [35], [42]-[44]). Also,

So, in this paper we analysis numerically an updated general model of advertising phenomena with variable-order fractional derivatives and delay on time.

The analytic exact solution of like differential models of fractional variable-order is very difficult to be obtained [3]. So on, to approximate the solutions of these models, it is important to develop some numerical method. The finite difference methods (FDMs) have been mostly updated to approximate the solution of the variable-order fractional differential equations (see [10], [42], [44], [47], [34] and [33]). Though it is easy to implement FDMs, their local character and limited accuracy are the main challenges in FDMs, whereas the fractional derivatives are well known to be global character. Therefore, modified techniques like NSFDM are more appropriate to discretizing differential operators of fractional order. Being in mind that most of the biological systems are stiff, therefore, it is necessary to use efficient numerical methods for obtaining good results when approximate the solution of those systems.

It is very important to study the strategies of advertisement which raise the rate of sales and to obtain superior earn for the companies. Therefore, building
and studying suitable dynamic advertising models for describing the sales, which depends on the time and depends on the public population \[9\]. There are many proposed systems for describing the issue of the advertisement that study these issues from viewpoint of economic, operations management and marketing (\[15\] and \[46\]), such that analyzing of the advertisement policies are done over time using dynamical systems (\[25\] and \[13\]). Most of these dynamical systems are considered to be differential models, such that the market subpopulations, sales, share and the all significant status variables are supposed to be continuously changeable with respect to the time. The advertisement models will be constructed depending on the purposes of the advertising. Kinds of these models are aimed at the comparison between more than two products. Other may be aimed to inject a new product to the market.

Furthermore, it is well known that the optimal control problem is the procedure of determining both the state variables and the control trajectories for the studied dynamical model during interval of time to minimize or maximize an attitude index. The state variable (or function) is the set of variables (functions) that utilized for describing the mathematical status of the model. Historically, optimal control is an amplification of the calculus of variations. To solve the basic optimal control problems, some set of the necessary conditions should be satisfied. In the 1950’s, Pontryagin, a Russian mathematician and his co-authors introduced these conditions. They proposed the adjoint function to place the studied differential equation to a function which called an objective functional. As the Lagrange multipliers in multivariable calculus, these functions do an identical target.

In the recent years, fractional optimal control problems (FOCPs) refers to the minimization (maximization) of an objective functional with dynamic constraints, on state and control variables, such that this conditions have derivative of fractional order. Some numerical methods to find approximation solutions of some types of FOCPs were recorded (\[37\]-\[39\]) and the references cited therein.

The purpose of this article is: to study numerically the approximation solutions of the following variable-order fractional optimal control problem of the advertising model with time delay memory \(\tau\):

\[
J(u, v) = \int_0^{t_{final}} e^{-rt} \left[ cz(t) - \frac{B_1}{2}(1 - u(t))^2 - \frac{B_2}{2}(1 - v(t))^2 \right] dt, \tag{1}
\]

subject to variable order fractional differential equations:

\[
\begin{align*}
\frac{c_0}{\tau} D_t^\alpha x(t) &= -u(t)x(t - \tau) - \frac{k^\alpha(t)}{N(t)}x(t)(N(t) - x(t)) + \mu_b^\alpha N(t) - \mu_d^\alpha x(t), \\
\frac{c_0}{\tau} D_t^\alpha z(t) &= (a^\alpha(t) + v(t))(N(t) - x(t - \tau) - z(t - \tau)) - \delta^\alpha(t)z(t) - \mu_d^\alpha z(t). \tag{2}
\end{align*}
\]

The problem here is to seek the control variables \(u^*(t)\) and \(v^*(t)\) such as to maximize the objective functional \(J\), i.e,

\[
J(u^*, v^*) = \max \left\{ J(u, v), u, v \in U \right\}.
\]

Here, the admissible controls set \(U\) is defined as the following

\[
U = \{ u : 0 \leq u, v \leq b < 1, t \in [0, t_f], u and v are lebesgue measurable functions \}
\]

This study will depend on developing a straightforward and efficient method based on the NSFD technique. System (2) is generalization of the system which was
Table 1. Notations in the proposed model (1)-(2) with their definition

| Symbol | Definition |
|--------|------------|
| $N(t)$ | The whole number of the population, $N(t) = x(t) + y(t) + z(t)$; (summation of all unknowns) |
| $\frac{D^{\alpha(t)}}{D^t}$ | Fractional variable order derivative operator in Caputo sense. |
| $\alpha(t)$ | The order of variable fractional derivative. |
| $x(t)$ | $t \geq 0$, time. |
| $u$ | The cardinality of the set of persons who did not realize anything about the goods. |
| $v$ | Trial advertisement, which switches the people from the prospective one $y(t)$ by letting them know the goods. |
| $y(t)$ | The cardinality of the set of persons who realize the goods but they did not purchase it till now. |
| $z(t)$ | The cardinality of the set of individuals who really bought the goods. |
| $a$ | First purchase, (Trial rate). |
| $r$ | Contact rate. |
| $q$ | Discount rate. |
| $s$ | Switching rate. |
| $c$ | $c = q(r + s + g)$. |
| $p$ | Net price. |
| $g$ | Repeat purchase. |
| $\mu_b$ | Birth rate. |
| $\rho_d$ | Death rate. |

proposed, in case $\alpha(t) = 1$, by Muller [25] as the following:

$$
x'(t) = -ux(t) \frac{k}{N(t)} x(t)(N(t) - x(t)),
$$

$$
y'(t) = ux(t) + \frac{k}{N(t)} x(t)(N(t) - x(t)) - (a + v)y(t) + \delta z(t),
$$

$$
z'(t) = (a + v)y(t) - \delta z(t).
$$

(3)

where $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ and $N(t) = x(t) + y(t) + z(t)$ is the total population number at time $t$. In addition, we assume, for financial reasons, that $x(\theta)$, $y(\theta)$ and $z(\theta)$ are nonnegative continuous functions for $\theta \in [-\tau, 0]$.

The system describes how the consumers flows from group into different one. Where the whole number of persons $ux(t)$ flow to the potential set $y(t)$ depending on advertisement. Furthermore, $(N(t) - x(t))$ consumers whom know about the output, inform and contact the whole of $k(N(t) - x(t))$, out of them only the quotient $x(t)/N(t)$ are latterly acquainted. In our previous work [43] we have approximated the solution of the fractional system of this phenomena without delay. Also, Benito et al. [9] solved a fractional case of this model without delay using a numerical technique.

Definitions of the parameters and variables in the above equations are given in the table (1). It is necessary to mention here that all the parameters in the new system follow $\alpha(t)$, the fractional derivative order. We will omit the symbol $\alpha(t)$ from the above of the parameters to make it easier, in the notation in the successions of the current article.

We concern in the case of derivative of fractional variable-order because the action of advertisements is not immediate, then combining the memory is so necessary to understand and explain the advertisement with both elements: trial and awareness advertisement. We refer to ([9], [25] and [13]), to understand deeply the elaboration of this model.

The main contribution of this paper is utilizing an efficient method to approximate the solution of a variable-order fractional nonlinear optimal control model with delay. This system describes the advertising procedure that aims to increase
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the income of the companies. All systems like the studied one have no explicit exact solutions and it is very rare to find in literature a paper that studies a similar model. This motivated us to deal with this optimal control problem using the proposed stable nonstandard discretization technique.

The article is structured as follows: In the following section, we introduce the preliminaries of NSFDM and recollect pertinent definitions on fractional variable order derivative. In section 3 we discuss some characteristics of the solutions of the proposed model. Necessary optimality conditions for the proposed model was introduced in Section 4. In Section 5, the nonstandard finite difference scheme (NSFDS) for system (2) was proposed then we demonstrated that this schema preserves the boundedness and the positivity of the actual solutions of the proposed model, we also, studied the stability of the proposed discretization scheme. In Section 6, we simulated numerically the proposed model and we stated the results to display the applicability and the efficiency of NSFDM. Finally, the conclusion of the paper is offered in the last section.

2. Preliminaries and notations. Here, we will mention a few necessary preparations for posterior debates. Firstly, we define the NSFDM. Secondly, we mention few helpful definitions and mathematical preliminaries of fractional calculus.

2.1. The nonstandard finite difference technique. Mickens ([22], [24]) was the first mathematician who modified the finite difference method to the NSFDM approach. The adaptive method depends on construction a modulated discretize numerical schema for ordinary differential equations (ODEs) or partial differential equations (PDEs). This method can maintain the attributes of the analytic exact solution of the studies differential model depending on some specific steps:

1. The terms which are not linear should be approximated using a nonlocal manner.
2. The denominator of the approximated derivatives should be a function of the step size.
3. The NSFD schema must not have a solution that does not consort to the solutions of the original differential equation.
4. Any specific properties that the solutions of the differential equations have should be also properties for the schema of the nonstandard finite difference method.
5. The actual derivatives which appear in the studied differential equations and the corresponding approximation derivatives should have the same order.

Briefly, using Euler’s method to approximate \( \frac{dy}{dt} \) instead of using \( \frac{y(t + h) - y(t)}{h} \), we utilize \( \frac{y(t + h) - y(t)}{\phi(h)} \), such that \( \phi(h) \) is a function of the step size \( h \) which must be continuous function, and when \( h \rightarrow 0 \) then

\[
\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1.
\]

Further more to this replacement, the appeared nonlinear terms in the studied differential equation, if it exists, we may change it to a linear term by approximating
2.2. Definitions of fractional variable-order derivatives. In literatures, many definitions of the operators of variable-order fractional derivatives are introduced (see e.g., [18], [8], [35], [36]). The Grünwald Letnikov, Riemann Liouville, or Caputo definitions, are usually used to define the time variable-order fractional derivatives.

Describing the derivative in terms of Caputo variable-order fractional operators in the initial value problems has a lot of advantages. The most useful characteristic of the variable-order Caputo’s operator is that the conditions at the initial value for the differential equations with the variable-order Caputo operator are usually taken to be the same as in the case of standard derivative of integer order. So, the Caputo variable-order operator are often taken to define the time variable-order fractional derivatives.

Definition 2.1. The Caputo’s definition of derivative of variable order fractional \( \alpha(t) \), \( \alpha(t) \in \mathbb{R}^+ \), is

\[
\left( \overset{\circ}{0}D_t^{\alpha(t)} f \right)(t) = \frac{1}{\Gamma(n - \alpha(t))} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{1-n+\alpha(t)}} dx, \quad t > 0,
\]

such that \( n = \lfloor \alpha(t) \rfloor + 1 \), \( f(x) \in C^n[0, \infty] \).

Depending on the above operator definition we notice that the derivative of any constant depending on Caputo’s definition will be zero, and

\[
\overset{\circ}{0}D_t^{\alpha(t)} t^\beta = \begin{cases} 
0, & \text{for } \beta < \alpha(t), \\
\frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha(t))} t^{\beta - \alpha(t)}, & \text{for } \beta \geq \alpha(t).
\end{cases}
\]

Furthermore, as to the standard derivative of integer-order, the Caputo’s variable order fractional operator satisfy the linearity property, i.e.

\[
\overset{\circ}{0}D_t^{\alpha(t)}(\lambda f(t) + \gamma g(t)) = \lambda \overset{\circ}{0}D_t^{\alpha(t)} f(t) + \gamma \overset{\circ}{0}D_t^{\alpha(t)} g(t).
\]

Notice, if \( \alpha(t) \in \mathbb{N} \) then Caputo’s derivative will matches with the standard derivative of an integer order.

3. Properties of the solution of the advertising model.

3.1. Stability.

Theorem 3.1. [21] Let \( g(t) \) is the function on right hand side of the model (2), the equilibrium points of this system are carried out by finding \( t \) which satisfy: \( g(t) = 0 \).

The obtained equilibrium points will be asymptotically local stable when \( \lambda_i \) are satisfy the condition: \( |\arg(\lambda_i)| > \frac{\alpha(t)\pi}{2} \), where \( \lambda_i \) the eigenvalues of the matrix \( J = \frac{\partial g}{\partial t} \) that calculated at these points. The associated matrix \( J = \frac{\partial g}{\partial t} \) called Jacobian matrix.

The equilibrium points of system (2) is \( x = 0, \ z = \frac{N(a+\eta)}{a+v+\delta} \). The Jacobian matrix \( J \) for this system that calculated at the obtained equilibrium point is:

\[
J = \begin{pmatrix} 
-u - k & 0 \\
-a - v & -a - v - \delta 
\end{pmatrix},
\]
where the eigenvalues are found to be as the following
\[ \sigma_1 = -k - u, \quad \sigma_2 = -a - \delta - v. \]
So, all equilibrium points for the model are locally asymptotically stable, for all \( t. \)

3.2. **Positivity.** It is well known that the populations must always be non-negative then we can prove that the solutions of the proposed model are positive. For proving the theorem of the positivity we will use the following Lemma (which call generalized mean value theorem):

**Lemma 3.2.** [26] If the function \( g(t) \in C[a, b] \) and \( \mathbb{D}_t^\alpha g(t) \in C[a, b], \) when \( 0 < \alpha(t) \leq 1. \) Therefore:
\[
g(t) = g(a) + \mathbb{D}_t^\alpha g(\xi) \frac{(t-a)^\alpha}{\Gamma(\alpha(t))},
\]
with \( 0 \leq \xi \leq t. \)

So, If the function \( g \) satisfies the following conditions: \( g(t) \in C[0, b], \mathbb{D}_t^\alpha g(t) \in C[0, b] \) and also \( \mathbb{D}_t^\alpha g(t) \geq 0 \) then the function \( g \) is nondecreasing.

**Theorem 3.3.** There is a unique solution for system (2) and the solution is positive.

**Proof.** The existence and uniqueness of the solutions of (2) follow from the results given in [17]. Depending on the above lemma and because we have:
\[
\mathbb{D}_t^\alpha x(t)|_{x=0} \geq 0,
\]
\[
\mathbb{D}_t^\alpha z(t)|_{z=0} \geq 0.
\]
So \( x(t) \geq 0 \) and \( z(t) \geq 0 \) for any \( t. \)

4. **Prerequisite optimality conditions for the studied problem.** Depending on the objective functional (1) and the constraints equations (2), the Hamiltonian function of the studied fractional optimal control problem is introduced, using a Lagrange multiplier technique, in the following form:
\[
H = e^{-\tau t}[cz(t) - \frac{B_1}{2} (1-u(t))^2 - \frac{B_2}{2} (1-v(t))^2] + \lambda_1 [-u(t)x(t) - \frac{k}{N(t)} x(t)(N(t) - x(t)) + \mu_b N(t) - \mu_d x(t)] + \lambda_2 [(a + v(t))(N(t) - x(t) - z(t - \tau)) - \delta z(t) - \mu_d z(t)].
\]
Where \( \lambda_1 \) and \( \lambda_2 \) are the co-state variables (Lagrange multipliers). Agrawal [2] derived the necessary optimality conditions for the fractional optimal control problem when the derivatives are defined using Caputo fractional operator. Therefore, the necessary optimality conditions of the proposed model are derived as following:

Let the optimal controls \( u^*, v^* \) and the solutions of the corresponding state system (2) \( x \) and \( z \) then there are co-state variables \( \lambda_1 \) and \( \lambda_2 \) such that:

- **State equations**
\[
\mathbb{D}_t^\alpha x(t) = -u(t)x(t) - \frac{k}{N(t)} x(t)(N(t) - x(t)) + \mu_b N(t) - \mu_d x(t), \quad x(0) = N,
\]
\[
\mathbb{D}_t^\alpha z(t) = (a + v(t))(N(t) - x(t - \tau) - z(t - \tau)) - \delta z(t) - \mu_d z(t), \quad z(0) = 0.
\]
Adjoint variable order fractional differential equations

\[ \frac{d}{dt} \int_{t_*}^{t_f} \lambda_1(t) = \frac{\partial H}{\partial x} + \chi_{[0,t_f-t]} \frac{\partial H(t + \tau)}{\partial x(t - \tau)}, \]

\[ \frac{d}{dt} \int_{t_*}^{t_f} \lambda_2(t) = \frac{\partial H}{\partial z} + \chi_{[0,t_f-t]} \frac{\partial H(t + \tau)}{\partial z(t - \tau)}. \]

Such that

\[ \chi_{[0,t_f-t]} = \begin{cases} 1, & t \in [0,t_f-t] \\ 0, & \text{otherwise}. \end{cases} \]

Those co-state equations condition which obtained similarly to the procedure of Pontryagin’s maximum principle with delay can be formulated as the following

\[ \frac{d}{dt} \int_{t_*}^{t_f} \lambda_1(t) = \lambda_1(t) \left( -\frac{k}{N(t)(N(t) - x(t))} + \frac{k}{N(t)} x(t) - \mu_d \right) - \chi_{[0,t_f-t]} u(t + \tau) \lambda_1(t + \tau), \]

\[ \frac{d}{dt} \int_{t_*}^{t_f} \lambda_2(t) = \lambda_2(t) \left( -\delta - \mu_d \right) - \chi_{[0,t_f-t]} (a + v(t + \tau)) \lambda_2(t + \tau). \]

Stationarity conditions

\[ \frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0, \]

i.e.,

\[ u^* = \min \left\{ \max \left\{ 1 - \frac{\lambda_1 x}{B_1 e^{-\rho \tau}}, 0 \right\}, 1 \right\}, \quad v^* = \min \left\{ \max \left\{ 1 + \frac{\lambda_2 (N - x - z)}{B_2 e^{-\rho \tau}}, 0 \right\}, 1 \right\}. \]

Transversality conditions

\[ \lambda_1(t_{final}) = 0, \quad \lambda_2(t_{final}) = 0. \]

The constraints system (8) coupled with the co-state system (9) and with the transversality conditions (12) together with the relationship (11) construct the optimality system.

Existence of an optimal solution is guaranteed depending on using a result by Fleming and Rishel in [14] such that the following statements are satisfied:

- The sets of state variables and controls variable are nonempty.
- The control set \( U \) is closed and convex.
- The right-hand sides of the state system are bounded by linear combinations of the control variables and state variables.
- The integrand in the objective functional \( J \) is convex on \( U \).
- There are \( c_1, c_2 > 0 \), and \( \rho > 1 \), constants, where the integrand \( L(u, v, z) \) of the objective functional \( J \) has the following property

\[ L(u, v, z) \geq c_2 + c_1 (u^2 + v^2)^{\rho/2}. \]

Condition 1 is satisfied depending on the existence of solution of system (2) with bounded coefficients that follow from the existence result by Lukes [20]. The set of the control variables is closed and convex depending on its definition. concerning the state system is linear in \( u, v \), then the right hand side of (2) satisfies condition 3, that depending on the boundedness of the solution. The integrand in the objective functional (1) is convex on \( U \). Farther more, one can easily prove that there are positive numbers \( c_1 \) and \( c_2 \) and a constant \( \rho > 1 \) such that

\[ L(u, v, z) \geq c_2 + c_1 (u^2 + v^2)^{\rho/2}. \]
5. **Numerical technique.** Here, we use NSFDM to built an explicit discretization formula of the optimality system. To construct the new schema, an appropriate non-local approximation is applied to discretize the nonlinear terms and approximations of the derivatives is used with an appropriate function in the denominator.

Let the coordinate of the mesh points as following:

\[ t_n = n \Delta t, \quad n = 0, 1, 2, \ldots, N_n, \]

such that \( N_n \) is a specific positive integer number and

\[ h := \Delta t = \frac{t_{j+1}}{N_n}. \]

Also, \( \alpha_n, x_n, z_n, u_n \) and \( v_n \) are approximation values of \( \alpha, x, z, u, \) and \( v \), respectively, at the mesh points \( (t_n) \). The approximation of Caputo derivative operator is built depending on the Grünwald-Letnikov method using nonstandard technique:

\[
\left. \frac{\partial^\alpha}{\partial t^\alpha} x(t) \right|_{t=x_n} = \frac{1}{(\varphi(\Delta t))^\alpha(n)} (x_{n+1} - \sum_{i=1}^{n+1} w_i^n x_{n+1-i} - r_{n+1} x_0), \quad (13)
\]

where

\[ w_i^n = (-1)^{i-1} \left( \begin{array}{c} \alpha_n \\ i \end{array} \right), \quad w_1^n = \alpha_n, \]

\[ r_i = \frac{i^{-\alpha_n}}{\Gamma(1-\alpha_n)}, \quad i = 1, 2, \ldots, n + 1. \]

**Theorem 5.1.** [32] Let \( \alpha(t) \in (0,1) \), then the following two relations:

\[ 0 < r_{i+1} < r_i < \ldots < r_1 = \frac{1}{\Gamma(1-\alpha_n)}, \quad (14) \]

\[ 0 < w_{i+1}^n < w_i^n < \ldots < w_1^n = \alpha_n < 1. \quad (15) \]

are satisfied for the coefficients \( r_i \) and \( w_i \) where \( i \geq 1 \).

**Proof.** see [32].

Using relation (13) and the nonstandard technique with \( \tau = qh \), then the explicit nonstandard schema of the model (8) is:

\[
x_{n+1} - \sum_{i=1}^{n+1} w_i^n x_{n+1-i} - r_{n+1} x_0 = (\varphi(h))^{\alpha_n} (-u_n x_{n-q} - \frac{k}{N} x_{n+1}(N - x_n) + \mu_k N - \mu_d x_{n+1}),
\]

\[
z_{n+1} - \sum_{i=1}^{n+1} w_i^n z_{n+1-i} - r_{n+1} z_0 = (\varphi(h))^{\alpha_n} [(a + v_n)(N - x_{n-q} - z_{n-q}) - \delta z_{n+1} - \mu_d z_{n+1}).
\]

Since each of these equations is linear in \( x_{n+1} \) and \( z_{n+1} \) so, some calculations give us the following expressions, which are explicit:

\[
x_{n+1} = \frac{1}{1 + \varphi(h))^{\alpha_n} \left( \frac{k}{N} (N - x_n) + \mu_d \right) \sum_{i=1}^{n+1} w_i^n x_{n+1-i} + r_{n+1} x_0 + (\varphi(h))^{\alpha_n} (\mu_k N - u_n x_{n-q})],
\]

\[
z_{n+1} = \frac{1}{1 + \varphi(h))^{\alpha_n} \left( \frac{\delta}{\mu_d} \right) \sum_{i=1}^{n+1} w_i^n z_{n+1-i} + r_{n+1} z_0 + (\varphi(h))^{\alpha_n} [a + v_n)(N - x_{n-q} - z_{n-q})].
\]

(16)
5.1. Stability of NSFDS. We consider the model test problem of linear fractional delay differential equation \[ (\frac{D_t^\alpha}{\Gamma(1-\alpha)}f)(t) = \rho_0 f(t) + \rho_1 f(t-\tau), \quad t \geq 0, \quad 0 < \alpha \leq 1, \quad (17) \]
\[ f(t) = \Psi(t), \quad t \in [-\tau,0], \quad f(0) = f_0, \]
such that \( 0 < \alpha(t) \leq 1, \rho_0 < 0, \rho_1 < \rho_0 \) and \( \Psi(t) \) is continuous and bounded function.

Let \( f(t_n) = f_n = \xi_n \) is the approximate solution of this equation.

**Theorem 5.2.** The NSFD technique is consistent and stable when was used to discretize the test problem (17) for all positive \( t \).

**Proof.** Let the approximate solution of (17) has the form \( f(t_n) \approx f^n \equiv \xi_n \), then (17) can be rewrite as the following:
\[
\frac{1}{\Gamma(1-\alpha_n)} \left( \xi_{n+1} - \sum_{i=1}^{n+1} w^n_i \xi_{n+1-i} - r_{n+1} \xi_0 \right) = \rho_0 \xi_n + \rho_1 \xi_{n-\tau},
\]
or
\[
\xi_{n+1} = h^{\alpha_n} \rho_0 \xi_n + h^{\alpha_n} \rho_1 \xi_{n-\tau} + \sum_{i=1}^{n+1} w^n_i \xi_{n+1-i} + r_{n+1} \xi_0, \quad n \geq 1.
\]
Since \( w_1^n < 1, \rho_0 < 0 \) and \( 0 < r_{n+1} < r_{n} < ... < r_1 = \frac{1}{\Gamma(1-\alpha_n)}, \) then
\[
\xi_1 \leq \xi_0, \quad \xi_{n+1} \leq h^{\alpha(t)} \rho_0 \xi_n + h^{\alpha(t)} \rho_1 \xi_{n-\tau} + \xi_n + \sum_{i=1}^{n+1} w^n_i \xi_{n+1-i}, \quad n \geq 1. \quad (19)
\]
Thus, for \( n = 1 \), the inequality (19) implies
\[
\xi_2 \leq \alpha(t) \rho_0 \xi_1 + h^{\alpha_n} \rho_1 \xi_{n-\tau} + w^n_1 \xi_1 + w^n_2 \xi_0 \leq (\alpha_n \rho_0 + w^n_1) \xi_1 + w^n_2 \xi_0.
\]

Using the relation (18) and the positivity of the coefficients \( \rho_0, w^n_1, w^n_2 \) and \( \alpha(t) \) we get
\[
\xi_2 \leq \xi_1. \quad (20)
\]
Repeating the process, we have from (19)
\[
\xi_{n+1} \leq h^{\alpha_n} \rho_0 \xi_n + h^{\alpha_n} \rho_1 \xi_{n-\tau} + \sum_{i=1}^{n+1} w^n_i \xi_{n+1-i} \leq \xi_n.
\]
Thus,
\[
\xi_{n+1} \leq \xi_n \leq \xi_{n-1} \leq \xi_{n-2} \leq ... \leq \xi_0
\]
with the assumption that
\[
\xi_{n+1} = ||f^{n+1}| \leq \xi_0 = ||f^0||, \quad \text{which entails} \quad ||f^{n+1}|| \leq ||f^0||, \quad \text{so that we have stability.}
\]
5.2. Properties of the NSFD solutions. In the current subsection, we study some properties of the introduced scheme (16).

**Theorem 5.3. (Positivity).** Let \( x_0 \geq 0, \ z_0 \geq 0 \), then \( x_n > 0, \ z_n > 0 \) for all \( n = 1, 2, \ldots \) is satisfied and all the parameter of this system are positive.

**Proof.** Using induction. Let \( n = 0 \), then from system (16) we have:

\[
\begin{align*}
x_1 &= \frac{1}{1 + (\varphi(h))^{\alpha_n}(\frac{k}{N}(N-x_0) + \mu_d)}[w_0^0 x_0 + r_1 x_0 + (\varphi(h))^{\alpha_n}(\mu_b N - u_0 x_{-q})] \geq 0, \\
z_1 &= \frac{1}{1 + (\varphi(h))^{\alpha_n}(\delta + \mu_d)}[w_1^0 z_0 + r_1 z_0 + (\varphi(h))^{\alpha_n}(a + v_0)(N - x_{-q} - z_{-q})] \geq 0. \quad (21)
\end{align*}
\]

Supposing, for all \( n < n + 1 \), that \( x_n \geq 0 \), and \( z_n \geq 0 \), then depending on this hypothesis and (5.1) we have

\[
\begin{align*}
&x_{n+1} = \frac{1}{1 + (\varphi(h))^{\alpha_n}(\frac{k}{N}(N-x_n) + \mu_d)}\left[\sum_{i=1}^{n+1} w_i^n x_{n+1-i} + r_{n+1} x_0 + (\varphi(h))^{\alpha_n}(\mu_b N - u_n x_{-q})\right] \geq 0, \\
&z_{n+1} = \frac{1}{1 + (\varphi(h))^{\alpha_n}(\delta + \mu_d)}\left[\sum_{i=1}^{n+1} w_i^n z_{n+1-i} + r_{n+1} z_0 + (\varphi(h))^{\alpha_n}(a + v_n)(N - x_{-q} - z_{-q})\right] \geq 0. 
\end{align*}
\]

(22)

**Theorem 5.4. (Boundedness).** Let \( x_0 = N, \ z_0 = 0 \) are the initial conditions such that \( x_0 + z_0 = N \), then for all \( n = 1, 2, \ldots \) we have \( x_n \) and \( z_n \) to be bounded.

**Proof.** By multiplying each single equation of the system (16) by its denominator we get:

\[
\begin{align*}
x_{n+1}(1 + (\varphi(h))^{\alpha_n}(a_n + \frac{k}{N}(N-x_n) + \mu_d)) + z_{n+1}(1 + (\varphi(h))^{\alpha_n}(\delta + \mu_d)) \\
&= \sum_{i=1}^{n+1} w_i^n (x_{n+1-i} + z_{n+1-i}) + r_{n+1} N + (\varphi(h))^{\alpha_n}(\mu_b N - u_n x_{-q}) + (a + v_n)(N - x_{-q} - z_{-q}), \\
&\leq \sum_{i=1}^{n+1} w_i^n (x_{n+1-i} + z_{n+1-i}) + r_{n+1} N + N(\varphi(h))^{\alpha_n}(\mu_b + a + v_0), \quad (23)
\end{align*}
\]

using induction, let \( n = 0 \), then:

\[
\begin{align*}
x_1(1 + (\varphi(h))^{\alpha_n}(u_0 + \mu_d)) + z_1(1 + (\varphi(h))^{\alpha_n}(\delta + \mu_d)) \\
&\leq w_0^0 N + r_1 N + N(\varphi(h))^{\alpha_n}(\mu_b + a + v_0), \\
&\leq N\left(1 + \frac{1}{\Gamma(1 - \alpha_n)} + m_0\right), \\
&= NM_0, \quad (24)
\end{align*}
\]

such that \( m_0 = \mu_b + a + v_0 \) and \( M_0 = 1 + \frac{1}{\Gamma(1 - \alpha_n)} + m_0 \).

So, we have

\[
x_1 \leq \frac{NM_0}{(1 + (\varphi(h))^{\alpha_n}(u_0 + \mu_d))}, \quad z_1 \leq \frac{NM_0}{(1 + (\varphi(h))^{\alpha_n}(\delta + \mu_d))},
\]

i.e.,

\[
x_1 \leq NM_0, \quad z_1 \leq NM_0.
\]
For $n = 1$, we have:

\[
x_2(1 + (\varphi(h))^{\alpha_1}(u_1 + \frac{k}{N}(N - x_1) + \mu_d)) + z_2(1 + (\varphi(h))^{\alpha_1}(\delta + \mu_d)) \\
\leq w_1^1(x_1 + z_1) + w_2^1(x_0 + z_0) + r_2N + N(\varphi(h))^{\alpha_1}(\mu_b + a + v_1), \\
\leq 2NM_0 + N + r_1N + N(\mu_b + a + v_1), \\
= N(1 + 2M_0 + \frac{1}{\Gamma(1 - \alpha_n)} + m_1), \\
= NM_1,
\]

such that $m_1 = \mu_b + a + v_1$ and $M_1 = 1 + 2M_0 + \frac{1}{\Gamma(1 - \alpha_n)} + m_1$.
So,

\[
x_2 \leq NM_1, \quad z_2 \leq NM_1.
\]

For $n = 2$, we have:

\[
x_3(1 + (\varphi(h))^{\alpha_2}(u_2 + \frac{k}{N}(N - x_2) + \mu_d)) + z_3(1 + (\varphi(h))^{\alpha_2}(\delta + \mu_d)) \\
\leq w_1^2(x_2 + z_2) + w_2^2(x_1 + z_1) + w_3^2(x_0 + z_0) + r_3N + N(\varphi(h))^{\alpha_2}(\mu_b + a + v_2), \\
\leq 2NM_1 + 2NM_0 + N + r_1N + N(\mu_b + a + v_2), \\
= N(1 + 2M_0 + 2M_1 + \frac{1}{\Gamma(1 - \alpha_n)} + m_2), \\
= NM_2,
\]

such that $m_2 = \mu_b + a + v_2$ and $M_2 = 1 + 2M_0 + 2M_1 + \frac{1}{\Gamma(1 - \alpha_n)} + m_2$.
So,

\[
x_3 \leq NM_2, \quad z_3 \leq NM_2.
\]

Now we suppose that

\[
x_n \leq NM_{n-1}, \quad z_n \leq NM_{n-1},
\]

where,

\[
m_{n-1} = \mu_b + a + v_{n-1},
\]

and

\[
M_{n-1} = 1 + 2M_0 + 2M_1 + 2M_2 + ... + 2M_{n-2} + \frac{1}{\Gamma(1 - \alpha_n)} + m_{n-1}.
\]

Now, we will proof

\[
x_{n+1} \leq NM_n, \quad z_{n+1} \leq NM_n,
\]

where

\[
M_n = 1 + 2M_0 + 2M_1 + 2M_2 + ... + 2M_{n-1} + \frac{1}{\Gamma(1 - \alpha_n)} + m_n.
\]

and

\[
m_n = \mu_b + a + v_n.
\]
From Eq. (23) we have
\[
x_{n+1}(1 + (\varphi(h))^{\alpha}(u_n + k(N - x_n) + \mu_d)) + z_{n+1}(1 + (\varphi(h))^{\alpha}(\delta + \mu_d)) \\
\leq w_1^n(x_n + z_n) + \ldots + w_n^2(x_n + z_n) + r_{n+1}N + N(\varphi(h))^{\alpha}(\mu_b + a + v_n). \\
\leq 2N\lambda_n^{b-1} + \ldots + 2N\lambda_1^2 + 2NM_0 + N + r_iN + N(\mu_b + a + v_n), \\
= N(1 + 2M_0 + 2M_1 + 2M_2 + \ldots + 2M_{n-1} + \frac{1}{\Gamma(1 - \alpha_n)} + m_n), \\
= NM_n, \tag{27}
\]
so,
\[
x_{n+1} \leq NM_n, \quad z_{n+1} \leq NM_n.
\]

5.3. Stable solutions.

**Definition 5.5.** The proposed schema (16) is called asymptotically stable, if \( L_1 \) and \( L_2 \) are exist as constants when \( \alpha(t) \rightarrow 1 \), such that
\[
x_{n+1} \leq L_1 \text{and} \quad z_{n+1} \leq L_2,
\]
satisfy for \( 0 < x_0 + z_0 = N \), the arbitrary initial values.

Depending on the boundedness theorem we deduce that the introduced NSFDS (16) is asymptotically stable.

In the following algorithm we summarize the process for obtaining the optimal solution:

5.4. **Algorithm of the procedure.** This algorithm has the following steps, in its specific order:

1. **Step 1.** Let the tolerance \( \epsilon > 0 \), and \( u^* \) and \( v^* \) are the initial control values, \( (x_0^*, z_0^*) \) are the starting point of state system (16) and for the adjoint system \( (\lambda_0^*, \mu_0^*), i = 1, 2 \).
2. **Step 2.** Use \( (x_0^*, z_0^*) \) to solve the state system (16) for \( (x^*, z^*) \) forward in time and use \( \varphi(h) = e^h - 1 \).
3. **Step 3.** Using the transversality conditions \( \lambda_i^*(t_f) = 0, i = 1, 2 \), solve the adjoint system (10) backward in time and use \( \varphi(h) = e^h - 1 \).
4. **Step 4.** Calculate the new control variables depending on equations (11) to find \( u^* \) and \( v^* \).
5. **Step 5.** If \( |u_j^* - u_{j+1}^*| < \epsilon \), and \( |v_j^* - v_{j+1}^*| < \epsilon \), ten stop the procedure, else put \( j = j + 1 \), and go to the step number 1.

6. **Numerical simulation.** In order to perform the numerical simulations in the potential section, we adopted NSFDM to solve the fractional optimization system numerically with the transversality conditions. NSFDM avoids a long computational’s time. For the following numerical treatment we choose the denominator of the discrete approximations of the derivative to be \( \varphi(h) = \frac{1 - e^{-2h}}{2} \).

The parameters of our numerical simulations are taken to be \( \delta = 0.2 \) the switching rate, first purchase rate \( a = 0.02 \), advertising trial rate \( r = 0.1 \), \( g = 0.1 \), \( p = 0.2 \), \( \mu_b = \mu_d = 0 \) and \( k = 0.01 \), where \( B_1 = B_2 = 500 \) and \( N = 100 \) total population. The conditions at the first time are \( x(0) = N \) and \( z(0) = 0 \). The role of the positive parameters \( B_1, B_2 \) is only to balance the terms size in the equations.
Figure (1) shows the approximations of control variables $u, v$ (obtained by NSFDM) when $\alpha(t) = 1$ and $\tau = 0$.

Figure (2) comparison between the solutions of $x, z$ with control and without control when $\alpha(t)$ has different constant values.

Figure (3) compares the solutions utilizing NSFDM when $\tau$ and $\alpha(t)$ have different values at the final time is 5.

Figure (4) compares the solutions utilizing NSFDM when $\tau$ and $\alpha(t)$ have different values at the final time is 10.

Figure (5) simulates the solutions of $x, z$ when $\tau$ has different values, $\alpha(t) = 0.5 + 0.5e^{-(t)^2-1}$ and the final time is 5.

Figure (6) simulates the solutions of $x, z$ when $\tau$ has different values, $\alpha(t) = \frac{5 + \cos^2(t)}{10}$ and the final time is 5.

Figure (7) simulates the solutions of $x, z$ when $\tau$ has different values, $\alpha(t) = 0.5 + 0.5e^{-(t)^2-1}$ and the final time is 10.

Figure (8) simulates the solutions of $x, z$ when $\tau$ has different values, $\alpha(t) = \frac{5 + \cos^2(t)}{10}$ and the final time is 10.

Figure (9) depicts the relationship between the variables $x(t)$ and $x(t - \tau)$ for different values of $\tau$ and $\alpha(t)$.

Table (2) contains the value of the cost functional and the final values of $x$ and $z$, which obtained using NSFDM, for different $\alpha(t)$ when final time is 10 and $\tau = 0.2$.

![Figure 1](image1.png) ![Figure 2](image2.png)

**Figure 1.** Approximations of the control variables with different final time.

Depending on the shown diagrams, we notice that the amount of consumers who are not informed about the existence of the product is decreased rapidly when the control variables are used and the amount of individuals who have purchased the product is higher when we use control variables than without using these control variables. Also, comparing between changing of the populations in the advertisement model with derivative of integer order and the inhabitants in the advertisement system with variable order derivative we notice that the variations are slower in the model with the variable order fractional derivatives. That is explain action of memory. Furthermore, It is clear from the figures that $x(t)$ and $x(t - \tau)$ will be identical after long time, similar to $z(t)$ and $z(t - \tau)$. 
Figure 2. Comparison between the solutions of $x$, $z$ with control and without control when $\alpha(t)$ takes different constant values.

Table 2. Final values of the states variables and the values of objective functional using NSFDM and SFDM when $t_{\text{final}} = 10$ and different $\alpha(t)$.

| $\alpha(t)$                         | NSFDM          |
|-------------------------------------|----------------|
|                                     | $J$  | $x$  | $z$  |
| 1                                   | 286.14 | 0    | 834  |
| 0.9                                 | 267.09 | 17   | 804  |
| $0.5 + 0.5e^{-(t)^2-1}$             | 248.98 | 42   | 762  |
| $\frac{5+\cos^2(t)}{10}$           | 256.74 | 77   | 709  |

7. Conclusion. Our work in this article contributes to a growing literature on applying variable-order fractional optimal control techniques to advertisement procedure. The studied nonlinear optimal model has two control variables and delay on the time was analyzed numerically to explain the awareness and trial advertising phenomena. The proposed variable order fractional advertisement system with delay on the time, like the most models of fractional variable order derivative, has solutions which are more convenient than the models with the derivatives of integer-order, that due to the advertising proceedings have a souvenir impact on the individuals. The utilized method to study numerically the proposed problem was NSFDM. This approximation method is preserve the properties of the analytic solutions like positivity and boundedness. Numerical outcomes are introduced to show the validity and applicability of the proposed scheme. By utilizing the
Figure 3. Solutions of $x$ and $z$ when $\tau$ and $\alpha(t)$ have different values.

NSFD scheme, numerical instabilities and the false solutions can be taken away, and accurate numerical solutions are accomplished for each time-step size.

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REFERENCES

[1] A. I. Abbas, On a Thermoelastic Fractional Order Model, *Journal of Physics*, 1 (2012), 24–30.
[2] O. P. Agrawal, A formulation and numerical scheme for fractional optimal control problems, *J. Vib. Control*, 14 (2008), 1291–1299.
[3] I. Area, J. J. Nieto and J. Losada, A note on the fractional logistic equation, *Physica A*, 444 (2016), 182–187.
[4] A. Atangana and A. H. Cloot, Stability and convergence of the space fractional variable-order Schrödinger equation, *Adv. Difference Equ.*, 2013 (2013).
[5] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus, Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, 3, Springer Science and Business Media LLC, 2012.
[6] D. Baleanu, J. A. T. Machado and A. C. J. Luo, *Fractional Dynamics and Control*, Springer, New York, 2012.
[7] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, Application of a fractional advection-dispersion equation, *Water Resour. Res.*, 36 (2000), 1403–1412.
[8] A. V. Chechkin, R. Gorenflo and I. M. Sokolov, Fractional diffusion in inhomogeneous media, *J. Phys. A: Math. Gen.*, 38 (2005), L679–L684.
[9] B. Chen-Charpentier, G. González-Parra and A. J. Arenas, Fractional order financial models for awareness and trial advertising decisions, *Comput. Econ.*, 48 (2016), 555–568.
Figure 4. Solutions of $x$ and $z$ when $\tau$ and $\alpha(t)$ have different values.

[10] C. Chen, F. Liu, K. Burrage and Y. Chen, Numerical methods of the variable-order Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative, *IMA J. Appl. Math.*, 78 (2013), 924–944.

[11] C. M. Chen, F. Liu, V. Anh and I. Turner, Numerical simulation for the variable-order Galilei invariant advection diffusion equation with a nonlinear source term, *Appl. Math. Comput.*, 217 (2011), 5729–5742.

[12] C. F. M. Coimbra, Mechanics with variable-order differential operators, *Ann. Phys.*, 12 (2003), 692–703.

[13] A. J. Dodson and E. Muller, Models of new product diffusion through advertising and word-of-mouth, *Management Science*, 24 (1978), 1557–1676.

[14] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, NY, USA, 1975.

[15] J. Huang, M. Leng and L. Liang, Recent developments in dynamic advertising research, *European Journal of Operational Research*, 220 (2012), 591–609.

[16] R. C. Koeller, Application of fractional calculus to the theory of viscoelasticity, *J. Appl. Mech.*, 51 (1984), 229–307.

[17] W. Lin, Global existence theory and chaos control of fractional differential equations, *J. Math. Anal. Appl.*, 332 (2007), 709–726.

[18] C. F. Lorenzo and T. T. Hartley, Variable order and distributed order fractional operators, *Nonlinear Dyn.*, 29 (2002), 57–98.

[19] C. F. Lorenzo and T. T. Hartley, Initialization, conceptualization, and application in the generalized fractional calculus, *Critical Reviews in Biomedical Engineering*, 5 (2007), 447–553.

[20] D. L. Lukes, *Differential Equations: Classical to controlled*, Mathematics in Science and Engineering, 162, Academic Press, New York, NY, USA, 1982.
Figure 5. Solutions of $x$ and $z$ when $\tau$ has different values.

[21] D. Matignon, Stability result on fractional differential equations with applications to control processing, Computational Engineering in Systems Applications, 2 (1996), 963–968.
[22] R. E. Mickens, Nonstandard Finite Difference Model of Differential Equations, World Scientific, Singapore, 1994.
[23] R. E. Mickens, Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis, Numerical Methods for Partial Differential Equations, 5 (1989), 313–325.
[24] R. E. Mickens, Nonstandard finite difference schemes for differential equations, Journal of Difference Equations and Applications, 8 (2002), 823–847.
[25] E. Muller, Trial/awareness advertising decisions: A control problem with phase diagrams with non-stationary boundaries, Journal of Economic Dynamics and Control, 6 (1983), 333–350.
[26] Z. M. Odibat and N. T. Shawagfeh, Generalized taylor’s formula, Applied Mathematics and Computation, 186 (2007), 286–293.
[27] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[28] Y. Povstenko, Fractional Thermoelasticity, Solid Mechanics and Its Applications, Springer International Publishing Switzerland, 2015.
[29] F. A. Rihan, S. Lakshmanan, A. H. Hashish, R. Rakkiyappan and E. Ahmed, Fractional-order delayed predator-prey systems with Holling type-II functional response, Nonlinear Dyn., 80 (2015), 777–789.
[30] S. G. Samko and B. Ross, Integration and differentiation to a variable fractional order, Integral Transform and Special Functions, 1 (1993), 277–300.
[31] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, New York Gordon and Breach Science Publishers, 1993.
[32] R. Scherer, S. Kalla, Y. Tang and J. Huang, The Grünwald-Letnikov method for fractional differential equations, Comput. Math. Appl., 62 (2011), 902–917.
Figure 6. Solutions of $x$ and $z$ when $\tau$ has different values.

Figure 7. Solutions of $x$ and $z$ when $\tau$ has different values.
Figure 8. Solutions of $x$ and $z$ when $\tau$ has different values.

Figure 9. Relation between the variables $x(t)$ and $x(t-\tau)$. 
[33] S. Shen, F. Liu, V. Anh, I. Turner and J. Chen, A characteristic difference method for the variable-order fractional advection-diffusion equation, *J. Appl. Math. Comput.*, 42 (2013), 371–386.

[34] S. Shen, F. Liu, J. Chen, I. Turner and V. Anh, Numerical techniques for the variable order time fractional diffusion equation, *Appl. Math. Comput.*, 218 (2012), 10861–10870.

[35] H. G. Sun, W. Chen, H. Wei and Y. Q. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, *Eur. Phys. J. Spec. Top.*, 193 (2011), 185–192.

[36] H. G. Sun, A. Chang, Y. Zhang and W. Chen, A review on variable-order fractional differential equations: Mathematical foundations, physical models, numerical methods and applications, *Fract. Calc. Appl. Anal.*, 22 (2019), 27–59.

[37] N. H. Sweilam and S. M. AL-Mekhlafi, Optimal control for a time delay multi-strain tuberculosis fractional model: A numerical approach, *IMA Journal of Mathematical Control and Information*, 36 (2019), 317–340.

[38] N. H. Sweilam and S. M. AL-Mekhlafi, On the optimal control for fractional multi-strain TB model, *Optimal Control Applications and Methods*, 37 (2016), 1355–1374.

[39] N. H. Sweilam and S. M. AL-Mekhlafi, Legendre spectral-collocation method for solving fractional optimal control of HIV infection of Cd4+T cells mathematical model, *The Journal of Defense Modeling and Simulation*, 14 (2017), 273–284.

[40] N. H. Sweilam and M. M. Abou Hasan, Numerical solutions of a general coupled nonlinear system of parabolic and hyperbolic equations of thermoelasticity, *Eur. Phys. J. Plus*, 132 (2017).

[41] N. H. Sweilam and M. M. Abou Hasan, Numerical approximation of Lévy-Feller fractional diffusion equation via Chebyshev-Legendre collocation method, *Eur. Phys. J. Plus*, 131 (2016).

[42] N. H. Sweilam and M. M. Abou Hasan, Numerical simulation for the variable-order fractional Schrödinger equation with the quantum Riesz-Feller derivative, *Adv. Appl. Math. Mech.*, 9 (2017), 990–1011.

[43] N. H. Sweilam, M. M. Abou Hasan and D. Baleanu, New studies for general fractional financial models of awareness and trial advertising decisions, *Chaos, Solitons and Fractals*, 104 (2017), 772–784.

[44] N. H. Sweilam and M. M. Abou Hasan, An improved method for nonlinear variable order Lévy-Feller advection-dispersion equation, *Bull. Malays. Math. Sci. Soc.*, 42 (2019), 3021–3046.

[45] V. E. Tarasov, *Fractional Dynamics Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer Science and Business Media, 2011.

[46] M. Wang, Q. Gou, C. Wu and L. Liang, An aggregate advertising response model based on consumer population dynamics, *International Journal of Applied Management Science*, 5 (2013), 22–38.

[47] P. Zhuang, F. Liu, V. Anh and I. Turner, Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term, *SIAM J. Numer. Anal.*, 47 (2009), 1760–1781.

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