Regularity of a double null coordinate system for Kerr–Newman–de Sitter spacetimes

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Abstract
We construct a double null coordinate system \((u,v,\theta^\ast,\phi^\ast)\) for Kerr–Newman–de Sitter spacetimes and prove that the two-spheres given by the intersection of the hypersurfaces \(u = \text{constant}\) and \(v = \text{constant}\) are \(C^\infty\) in Boyer–Lindquist coordinates (including at the “poles”). The null coordinates allow one to immediately extend some results previously proven for Kerr. As an example, we illustrate how Sbierski’s result in [29], for the wave equation on the black hole interior, for Reissner–Nordström and Kerr spacetimes, applies to Kerr–Newman–de Sitter spacetimes.

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1 Introduction

The Kerr–Newman–de Sitter (KNdS) metric is a solution of the Einstein–Maxwell equations with a positive cosmological constant $\Lambda$:

$$\begin{align*}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} &= 2T_{\mu\nu}, \\
dF &= d\star F = 0, \\
T_{\mu\nu} &= F_{\mu\alpha}F^\alpha_{\nu} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}.
\end{align*}$$

Here $R_{\mu\nu}$ are the components of the Ricci tensor of the spacetime metric $g$, $R$ is the scalar curvature, $\star$ is the Hodge star operator, and $F_{\mu\nu}$ is the Faraday electromagnetic 2-form. So, the KNdS metric is an electrovacuum solution of the Einstein field equations, i.e. it is a solution in which the only nongravitational mass-energy present is an electromagnetic field. The spacetime is the four-dimensional manifold $\mathbb{R}^2 \times S^2$ with metric given by

$$g = \frac{\rho^2}{\Delta_r} \, dt^2 + \frac{\rho^2}{\Delta_\theta} \, d\theta^2 + \frac{1}{\rho^2} (a^2 \sin^2 \theta \Delta_\theta - \Delta_r) \, dr^2 + \frac{\sin^2 \theta}{\Xi \rho^2} ((r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r) \, d\phi^2 - \frac{2a^2 \sin^2 \theta}{\Xi \rho^2} ((r^2 + a^2) \Delta_\theta - \Delta_r) \, d\phi \, dt \quad (1)$$

in Boyer–Lindquist coordinates, where

$$\begin{align*}
\Delta_r &= (r^2 + a^2) \left(1 - \frac{\Lambda}{3} r^2\right) - 2Mr + e^2, \\
\Delta_\theta &= 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta, \\
\rho^2 &= r^2 + a^2 \cos^2 \theta, \\
\Xi &= 1 + \frac{\Lambda}{3} a^2,
\end{align*}$$

$\theta$ is the colatitude, with

$$0 \leq \theta \leq \pi,$$

and $\phi$ is the longitude, with

$$\phi \in \mathbb{S}^1,$$

(see Carter [6], and Akcay and Matzner [1] and Kraniotis [23], for example). Here $M$ and $a$ are mass and angular momentum parameters, respectively, both nonzero, and $e$ is a charge parameter (which may be zero). Without loss of generality, we assume that the magnetic charge is zero and $a$ is positive. This metric is supposed to represent a rotating black hole, with charge, in a universe which is expanding at an accelerated rate (as ours is).

We refer to Appendix C, where we calculate Komar integrals over the event horizon, for the relation between these parameters and the physical quantities of the black hole. We wish to consider subextremal metrics, meaning that $\Delta_r$ has four distinct real roots

$$r_n < 0 < r_- < r_+ < r_c.$$

The event horizon $\mathcal{H}$ corresponds to the hypersurface where $r = r_+$, the Cauchy horizon $\mathcal{CH}$ corresponds to the hypersurface where $r = r_-$, and the cosmological horizon corresponds to the hypersurface $r = r_c$. We are interested in studying solutions of the wave equation in the black hole region, $r_- < r < r_+$, and we wish to work in double null coordinates.

This article follows closely the strategy and tools developed in [9]. Double null coordinate systems were constructed by Pretorius and Israel [28] for Kerr spacetimes (in the local setting), by Balushi and Mann [3] for Kerr–(anti) de Sitter spacetimes, and by Imseis, Balushi and Mann [19] for Kerr–Newman–(anti) de Sitter spacetimes. In [3] and [19] the authors also study the formation of caustics. In Section 2 we construct a double null coordinate system for Kerr–Newman–de Sitter spacetimes. This construction only differs from the one in [3] and [19] (that we were unaware of until the completion of this work) in the choice of $\lambda$ (our choice $\lambda = \sin^2 \theta_*$ is identical to the one in [9] and [28]). Consider the transformation $(t, r, \theta, \phi) \mapsto (t, r_*, \theta_*, \phi)$ in the black hole region $r_- < r < r_+$, where $\Delta_r < 0$. The coordinate $\theta_*$ is defined implicitly as the solution of $F(r, \theta, \theta_*) = 0$, where $F$ is given by

$$F(r, \theta, \theta_*) = \int_{\theta_*}^{\theta} \frac{d\theta'}{a \sin^2 \theta_* \Delta_{\theta'} - \sin^2 \theta'} + \int_{r}^{r_+} \frac{dr'}{\sqrt{((r')^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta_r}}.$$
The coordinate \( r_* \) is defined by 
\[
g(r, \theta, \sin^2 \theta_*(r, \theta)) = \int_{r_0}^{r} \left( \frac{r'^2 + a^2}{\Delta_{r'}} \right) \, dr' + \int_{r}^{r_+} \left( \frac{r'^2 + a^2}{\Delta_{r'}} - \sqrt{\left( \frac{r'^2}{\Delta_{r'}} \right)^2 + a^2 \lambda r' \Delta_{r'}} \right) \, dr'
\]
for some fixed \( r_0 \in (r_-, r_+) \), and where \( f \) is the function which satisfies \( f(0) = 0 \) and
\[
f'(\lambda) = - \int_0^{\arcsin \frac{\sqrt{\lambda}}{\Delta_0}} \frac{a \sqrt{\lambda} \Delta_{r'} - \sin^2 \theta'}{\Delta_{r'}} \, d\theta'
\]
(note the difference, \( g \) versus \( \rho \) in (1)).

The regularity of transformation of coordinates \( (r_*, \theta_*) \rightarrow (r, \theta) \) for Kerr spacetimes, namely at \( \theta_* = 0 \), was shown by Dafermos and Luk [9]. We adapt their work to the setting of Kerr–Newman–de Sitter spacetimes. We check that \( \theta_* \) is well defined and continuous at \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) with
\[
\frac{\sin \theta_*}{\sin \theta} + \frac{\cos \theta}{\cos \theta_*} \lesssim 1.
\]  
(2)
The proof of (2) requires that we use conditions characterizing subextremal black holes which are deduced in Appendix B. Namely, we use
\[
\Xi < 1 + \frac{1}{3} \left( \frac{r_+}{r_-}, \Lambda e^2 \right),
\]
where \( l \) is given by (137). This implies the inequality
\[
\Xi < \csc^2 \left( \arctan \left( \frac{r_+}{r_-} \right) + \arctan \left( \frac{r_-}{r_+} \right) \right),
\]
which in turn implies (2). We use the fact that \( \Lambda \) is nonnegative so that our computations are not immediately applicable to the setting of Kerr–AdS.

We also show that \( r \) and \( \theta \) are smooth functions of \( r_* \) and \( \theta_* \). When the cosmological constant is equal to zero and \( e = 0 \), \( \Delta_0 \) is equal to 1 and our formulas reduce to the ones for the Kerr spacetime in [9]. The trigonometric identity
\[
\sin^2(2\theta_*)\Delta_0 = - \sin(2\theta) \partial_0 D(\theta, \theta_*) + 2(\cos(2\theta) + \cos(2\theta_*)) D(\theta, \theta_*),
\]
for \( D(\theta, \theta_*) = \sin^2 \theta_\* \Delta_0 - \sin^2 \theta \) is the key to the calculation of \( \partial_\theta \theta_* \) and \( \partial_0 \theta_* \), as well as the successful completion of some new identities, such as (53) and (58), which we need in order to calculate the derivatives of \( r \) and \( \theta \) with respect to \( r_* \) and \( \theta_* \). We would like to emphasize that our calculations are successful because the dependence of \( \Delta_0 \) on \( \theta \) occurs through \( \sin^2 \theta \), and not through \( \sin \theta \), or on any other non-smooth function of \( \sin \theta \). (This is a reflection of the fact that the Kerr–Newman–de Sitter metric is regular on a manifold diffeomorphic to \( \mathbb{R}^2 \times S^2_{\theta, \phi} \), i.e. it is regular on the full Boyer–Lindquist spheres of constant time coordinate \( t \) and radial coordinate \( r \).) We also obtain bounds on the derivatives of \( r \) and \( \theta \) which we need later on. These bounds parallel the ones in [9].

Using
\[
u = \frac{r_* - t}{2} \quad \text{and} \quad \nu = \frac{t + r_*}{2},
\]
the final transformation
\[
\phi_* = \phi - h(r_*, \phi_*),
\]
with \( h \) given by
\[
\partial_{\nu} h(r_*, \theta_*) = - \frac{\Xi a (r^2 + a^2) \Delta_0 - \Delta_*}{(r^2 + a^2)^2 \Delta_0 - a^2 \sin^2 \theta \Delta_{r'}}, \quad \text{with} \quad h(0, \theta_*) = 0,
\]
allows one to bring the metric to the double null form
\[
g = -2\Omega^2 (du \otimes dv + dv \otimes du) + g_{\theta, \theta} (d\theta_* \otimes d\theta_* + (d\phi_* - b \phi^\nu dv) \otimes d\theta_* + (d\phi_* - b \phi^\nu dv) \otimes (d\phi_* - b \phi^\nu dv),
\]
which is particularly adequate for carrying out energy estimates.
Neither the Boyer–Lindquist coordinate system \((t, r, \theta, \phi)\) nor the double null coordinate system \((u, v, \theta_*, \phi_*)\) cover the axis \(\theta_*=0\), obviously. But they can be naturally extended to an atlas that does cover \(\theta_*=0\) using

\[
\begin{align*}
    x &= \sin \theta_* \cos \phi, \\
    y &= \sin \theta_* \sin \phi, \\
    \tilde{x} &= \sin \theta \cos \phi, \\
    \tilde{y} &= \sin \theta \sin \phi.
\end{align*}
\]

In Subsection 2.2.6, we prove that the two atlases \(A_{BL} = \{(t, r, \theta, \phi), (\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})\}\) and \(A_{DN} = \{(u, v, \theta_*, \phi_*)\}\) are compatible (which would be clear if we were to exclude the points where \(\theta = \theta_* = 0\) and \(\theta = \theta_* = \pi\) from our manifold). This implies that the two-spheres given by the intersection of the hypersurfaces \(u = \text{constant}\) and \(v = \text{constant}\) are \(C^\infty\) with respect to \(A_{BL}\) (see Theorem 2.23).

Following [9], we analyze the decay of \(\Omega^2\) at the future event and Cauchy horizons, \(\mathcal{H}^+\) and \(\mathcal{CH}^+\). Finally, we give regular coordinates at \(\mathcal{H}^+\) and \(\mathcal{CH}^+\). The Christoffel symbols of \(g\) in the double null coordinates \((u, v, \theta_*, \phi_*)\) are given in Appendix A, along with some covariant derivatives that are needed to carry out energy estimates.

In Section 3, using the vector field method, we study, in the black hole interior, the energy of solutions of the wave equation which have compact support on \(\mathcal{H}^+\). We apply the form (4) of the metric to construct certain blue-shift and red-shift vector fields and to calculate their covariant derivatives. We obtain the usual inequalities relating the vector and scalar currents associated to these vector fields. This allows us to illustrate how Sbierski’s result in [10] extends to KNdS.

An alternative approach towards extending the results of [7] to the Kerr–Newman–de Sitter setting would be to work in Boyer–Lindquist coordinates as is done in the work [25] on Kerr black hole interiors.

In Appendix B, we characterize subextremal Kerr–Newman–de Sitter black hole mass in terms of \((r_-, r_+, \Lambda a^2, \Lambda c^2)\), proving (3), in particular, as mentioned above. The subset of \(\mathbb{R}^3\) where one can choose \(\left(\frac{r_+}{r_-}, \Lambda a^2, \Lambda c^2\right)\) is sketched in Figure 3 on page 44. We make additional remarks concerning alternative choices of parameters, namely \((\Lambda, \frac{2}{\Lambda}, a, c)\) or \((\Lambda, M, a, c)\). Related characterizations of the parameters of subextremal Kerr–de Sitter solutions, for the case when there is no charge, can be found in Lake and Zannias [24] and Borthwick [4].

Hintz and Vasy give a uniform analysis of linear waves up to the Cauchy horizon using methods from scattering theory and microlocal analysis in [18]. Moreover, Hintz proves non-linear stability of the Kerr–Newman–de Sitter family of charged black holes in [17].

2 A double null coordinate system

2.1 Construction of the double null coordinate system

2.1.1 The coordinates \(r_*\) and \(\theta_*\)

We look for a function \(r_*\) such that the axisymmetric hypersurface,

\[v(t, r, \theta) = t \pm r_*(r, \theta) = \text{constant}\]

(ingoing when the plus sign is chosen, and outgoing when the minus sign is chosen), is lightlike. Then, the function \(v\) must satisfy the eikonal equation

\[g^{\alpha \beta}(\partial_\alpha v)(\partial_\beta v) = \frac{1}{\rho^2} \left[ \Delta_r (\partial_r r_*)^2 + \Delta_\theta (\partial_\theta r_*)^2 - \frac{1}{\Delta_r \Delta_\theta} \left( (r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r \right) \right] = 0.\]

We follow [28] and construct particular separable solutions of the eikonal equation. We define \(P\) and \(Q\) by

\[P(\theta, \theta_*) = a \sqrt{\sin^2 \theta_* \Delta_\theta - \sin^2 \theta} \]

\[Q(r, \theta_*) = \sqrt{(r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta_r}.\]

Note that

\[\frac{1}{\Delta_r \Delta_\theta} \left( (r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r \right) = \frac{Q^2}{\Delta_r} + \frac{P^2}{\Delta_\theta} \]

(5)
and so the eikonal equation becomes
\[ \Delta_r (\partial_r r_\star)^2 + \Delta_\theta (\partial_\theta r_\star)^2 = \frac{Q^2}{\Delta_r} + \frac{P^2}{\Delta_\theta}. \]

As \( P \) is independent of \( r \), and \( Q \) is independent of \( \theta \), we look for special solutions \( r_\star \) of this equation, where
\[ \partial_r r_\star = \frac{Q}{\Delta_r} \quad \text{and} \quad \partial_\theta r_\star = \frac{P}{\Delta_\theta}, \quad \text{(6)} \]
so that
\[ dr_\star = \frac{Q}{\Delta_r} \, dr + \frac{P}{\Delta_\theta} \, d\theta. \quad \text{(7)} \]

Both \( P \) and \( Q \) depend on (what is so far the parameter) \( \theta_\star \), which arises because of the degree of freedom one has in breaking up the left-hand side of (5) to a sum. Indeed, to the left-hand side of (5) we subtracted and added the quantity \( a^2 \sin^2 \theta_\star \) (which is independent of both \( r \) and \( \theta \)) and then we decomposed the resulting expression into a sum of a function depending solely on \( r \) and a function depending solely on \( \theta \). We integrate (7) and obtain
\[ r_\star = \int_{r_\star}^{r} \frac{Q(r', \theta_\star)}{\Delta_{r'}} \, dr' + \int_{0}^{\theta} \frac{P(\theta', \theta_\star)}{\Delta_{\theta'}} \, d\theta' + \frac{a^2}{2} f(\sin^2 \theta_\star), \]
where the function \( f \) accounts for an integration constant. Thus we have
\[ r_\star = g(r, \theta, \lambda), \quad \text{(8)} \]
with
\[ g(r, \theta, \lambda) = \int_{r_\star}^{r} \frac{(r'^2 + a^2)}{\Delta_{r'}} \, dr' + \int_{r}^{r_\star} \frac{(r'^2 + a^2) - Q(r', \lambda)}{\Delta_{r'}} \, dr' + \int_{0}^{\theta} \frac{P(\theta', \lambda)}{\Delta_{\theta'}} \, d\theta' + \frac{a^2}{2} f(\lambda), \]
for some fixed \( r_0 \in (r_-, r_+) \), and
\[ \lambda = \sin^2 \theta_\star. \]

The expression for \( g \) is written so that the second integral converges. Here
\[ \hat{P}^2(\theta, \lambda) = a^2 (\lambda \Delta_\theta - \sin^2 \theta), \]
\[ \hat{Q}^2(r, \lambda) = ((r^2 + a^2)^2 - a^2 \lambda \Delta_r), \]
so that \( \hat{P}(\theta, \lambda) = P(\theta, \arcsin \sqrt{\lambda}) = P(\theta, \theta_\star) \) and \( \hat{Q}(r, \lambda) = Q(r, \arcsin \sqrt{\lambda}) = Q(r, \theta_\star) \).

For each fixed \( \lambda \), (8) is a solution of (7). We now proceed to obtain another solution of (7). Calculating the differential of \( g \), we obtain
\[ dg = \frac{Q}{\Delta_r} \, dr + \frac{P}{\Delta_\theta} \, d\theta + \partial_\lambda g \, d\lambda, \]
where
\[ \partial_\lambda g = \frac{a^2}{2} \left( \int_{0}^{\theta} \frac{d\theta'}{P(\theta', \lambda)} + \int_{r}^{r_\star} \frac{dr'}{Q(r', \lambda)} + f'(\lambda) \right). \]

Define the function
\[ F(r, \theta, \theta_\star) := \int_{0}^{\theta} \frac{d\theta'}{P(\theta', \theta_\star)} + \int_{r}^{r_\star} \frac{dr'}{Q(r', \theta_\star)} + f'(\sin^2 \theta_\star), \]
Note that the function \( f \) is still free. We choose \( f : [0, 1] \to \mathbb{R} \) to be the function which satisfies \( f(0) = 0 \) and
\[ f'(\lambda) = -\int_{0}^{\arcsin \sqrt{\lambda}} \frac{d\theta'}{P(\theta', \lambda)}, \quad \text{i.e.} \quad f'(\sin^2 \theta_\star) = -\int_{0}^{\theta_\star} \frac{d\theta'}{P(\theta', \theta_\star)}. \]

The function \( f \) is bounded. Then the expression for \( F \) becomes
\[ F(r, \theta, \theta_\star) = \int_{0}^{\theta} \frac{d\theta'}{P(\theta', \theta_\star)} + \int_{r}^{r_\star} \frac{dr'}{Q(r', \theta_\star)}, \quad \text{(9)} \]

Adapting the construction of [9] to our case, for \( r \in [r_-, r_+] \) and \( \theta \in (0, \pi) \), we define \( \theta_\star \in \left[ \frac{\pi}{2}, \pi \right) \) implicitly to be the solution of
\[ F(r, \theta, \theta_\star) = 0. \]
Also, let
\[ \theta_\ast(r, 0) = 0, \quad \theta_\ast \left( r, \frac{\pi}{2} \right) = \frac{\pi}{2} \]
and
\[ \theta_\ast(r, \theta) = \pi - \theta_\ast(r, \pi - \theta) \text{ for } \theta \in \left( \frac{\pi}{2}, \pi \right). \] (10)

Then, the function
\[ r_\ast = \varrho(r, \theta, \sin^2 \theta_\ast(r, \theta)) \]
is another solution of (7). The functions
\[ u = \frac{r_\ast(r, \theta) - t}{2} \quad \text{and} \quad v = \frac{t + r_\ast(r, \theta)}{2} \] (11)
are solutions of the eikonal equation. Just as in the case of Kerr, it turns out that \( \theta_\ast \) is an appropriate angle coordinate. This can be understood starting with the construction of [28]; when \( \Lambda = M = a = c = 0 \) (so that we are reduced to the Minkowski spacetime), \( \theta_\ast \) is the spherical polar angle. Moreover, for \( r \) close to \( r_+ \), \( \theta_\ast \) is close to \( \theta \). The function \( \theta_\ast \) is interpreted as the spherical polar angle and the hypersurfaces where \( u \) and \( v \) are constant are called quasi-spherical light cones. From
\[ \left| \frac{\partial u}{\partial r} \right| = \frac{1}{\varrho^2} \Delta_r \Delta_\theta (\partial_r r_\ast)^2 + \Delta_\theta (\partial_\theta r_\ast)^2 = \frac{1}{\varrho^2} \Delta_r \Delta_\theta (\Delta_r \varrho^2 + \Delta_\theta Q^2) \]
\[ = \frac{1}{\varrho^2} \Delta_r \Delta_\theta (r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_\theta < 0 \]
(recall that \( \Delta_\theta < 0 \)), a hypersurface where \( r_\ast \) equals a constant is spacelike.

**Remark 2.1.** Note that \( r_\ast \) ranges between \(-\infty \) and \(+\infty \), as \( r \) ranges between \( r_+ \) and \( r_- \). More precisely, given \( L > 0 \), there exists \( \delta > 0 \) such that \( r_\ast(r, \theta) > L \) for all \( (r, \theta) \in (r-, r_+ + \delta) \times [0, \pi] \). Moreover, given \( \delta > 0 \), there exists \( L > 0 \) such that \( r_\ast(r, \theta) > L \) implies that \( r \in (r-, r_+ + \delta) \). In Lemma 2.9 we will prove that \( (r, \theta) \rightarrow (r_\ast, \theta_\ast) \) is invertible. So, we are observing that \( \lim_{r \rightarrow r_-^+} r_\ast(r, \theta) = +\infty \) and that \( \lim_{r \rightarrow r_+^-} r_\ast(r, \theta) = r_- \), and that these limits are uniform in \( \theta \) and in \( \theta_\ast \), respectively, for \( \theta \) and \( \theta_\ast \) in \( [0, \pi] \). Analogous statements can be made for the other endpoint.

**Remark 2.2.** Because of the symmetry in (10), our statements about \( \theta \) and \( \theta_\ast \) will refer to the interval \( (0, \frac{\pi}{2}) \) and it is understood that corresponding statements will hold in \( \left( \frac{\pi}{2}, \pi \right) \).

### 2.1.2 The metric in \( (t, r_\ast, \theta_\ast, \phi) \) coordinates

Denote
\[ \Upsilon := \Delta_r \varrho^2 + \Delta_\theta Q^2 = (r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r, \]
\[ G := \partial_\theta F \]
and
\[ L := -G P Q. \] (13)

Differentiating both sides of (9) with respect to \( r \) and \( \theta \) yields
\[ \partial_r \theta_\ast = \frac{1}{G Q} \Delta_r \varrho^2 \Delta_\theta (\partial_r r_\ast)^2 + \Delta_\theta \Delta_r (\partial_\theta r_\ast)^2 = \frac{1}{G P} \Delta_r \Delta_\theta L, \] (14)

Since
\[ \left| \begin{array}{ccc} \partial_r \ast & \partial_\theta \ast & \partial r_\ast \\ \partial_r \ast & \partial_\theta \ast & \partial_\theta \ast \end{array} \right| = \frac{\Upsilon}{\Delta_r \Delta_\theta L}, \]

the differentials of \( r \) and \( \theta \) are given by
\[ dr = \frac{\Delta_r \Delta_\theta Q}{\Upsilon} dr_\ast - \frac{\Delta_r L P}{\Upsilon} d\theta_\ast, \] (15)
\[ d\theta = \frac{\Delta_r \Delta_\theta P}{\Upsilon} dr_\ast + \frac{\Delta_\theta L Q}{\Upsilon} d\theta_\ast. \] (16)

To write the metric in \( (t, r_\ast, \theta_\ast, \phi) \) coordinates one uses (15) and (16) in (1) and obtains
\[ g = \frac{\varrho^2}{\Upsilon} \Delta_r \varrho^2 dr_\ast^2 + \frac{\varrho^2}{\Upsilon} L^2 dt^2 + \frac{1}{\varrho^2} (\Delta_\theta \varrho^2 + \Delta_r) dt^2 + \frac{\sin^2 \theta}{\varrho^2} \Upsilon d\phi^2 \]
\[ - \frac{2 a \sin^2 \theta}{\varrho^2} ((r^2 + a^2) \Delta_\theta - \Delta_r) d\phi dt \]
\[ = g_{t \ast r_\ast} (dt \otimes dt \otimes dt) + g_{r_\ast r_\ast} d\theta_\ast \otimes d\theta_\ast + g_{\phi \phi} (d\phi - B dt) \otimes (d\phi - B dt), \]

\[ 6 \]
with
\[ B = - \frac{g_{\phi \phi}}{g_{\phi \phi}} \frac{\Xi a((r^2 + a^2)\Delta_\theta - \Delta_r)}{\Delta}. \]

### 2.1.3 Definition of \( \phi_* \) and the metric in double null coordinates \((u, v, \theta_*; \phi_*)\)

From (11), one gets
\[
\begin{align*}
\int dr_* &= du + dv, \\
\int dt &= dv - du.
\end{align*}
\]

Now introduce a new coordinate \( \phi_* \), defined by
\[ \phi_* = \phi - h(r_*, \theta_*), \]

where
\[ \partial_{r_*} h(r_*, \theta_*) = -B = -\frac{\Xi a((r^2 + a^2)\Delta_\theta - \Delta_r)}{\Delta}, \quad \text{with } h(0, \theta_*) = 0. \tag{17} \]

For a general function \( f \), one has
\[ f(\hat{u}, \hat{v}, \hat{\theta}_*, \phi) = f(\hat{u}, \hat{v}, \hat{\theta}_*, \phi_* + h(\hat{u} + \hat{v}, \hat{\theta}_*)) = f(u, v, \theta_*, \phi_*), \]

where \( f \) is the function \( f \) written in the \((\hat{u}, \hat{v}, \hat{\theta}_*, \phi)\) coordinate system and \( \hat{u} = u, \hat{v} = v, \hat{\theta}_* = \theta_* \). So, it follows that
\[
\begin{align*}
\partial_u &= \partial_{u_*} + \partial_{r_*} h \partial_{\phi_*}, \\
\partial_v &= \partial_{v_*} + \partial_{r_*} h \partial_{\phi_*}, \\
\partial_{\theta_*} &= \partial_{\theta_*} + \partial_{r_*} h \partial_{\phi_*}, \\
\partial_{\phi_*} &= \partial_{\phi_*}.
\end{align*}
\]

These equations help us with the geometric interpretation of the change of coordinates operated by passing from \( \phi \) to \( \phi_* \). Of course, for functions \( f \) that do not depend on \( \phi \), like the coefficients of our metric, \( \partial_u f = \partial_{u_*} f, \partial_v f = \partial_{v_*} f \) and \( \partial_{\theta_*} f = \partial_{\theta_*} f \). Defining
\[
\Omega^2 = -g_{r_* r_*} = -\frac{\rho^2 \Delta_{r_*} \Delta_\theta}{\Delta}, \tag{18}
\]
\[ b^{\phi_*} = 2B = 2\frac{\Xi a((r^2 + a^2)\Delta_\theta - \Delta_r)}{\Delta}, \tag{19} \]

the expression for the metric becomes
\[
g = -2\Omega^2 (du \otimes dv + dv \otimes du) + g_{\theta_* \theta_*} \otimes d\theta_* + g_{\phi_* \phi_*} \otimes (d\phi_* - b^{\phi_*} dv) + g_{\phi_* \phi_*} \otimes (d\phi_* - b^{\phi_*} dv). \tag{20} \]

For each pair \((u, v)\), \( \hat{g} \) is a metric defined on a two-sphere. The calculation above shows that the coefficients of this metric are
\[
\begin{align*}
\hat{g}_{\theta_* \theta_*} &= g_{\theta_* \theta_*} + g_{\phi \phi}(\partial_{\theta_*} h)^2 = \frac{\rho^2 L^2}{\Delta} + \frac{\sin^2 \theta}{\Xi^2} \hat{\Omega}(\partial_{\theta_*} h)^2, \tag{21} \\
\hat{g}_{\theta_* \phi_*} &= g_{\phi \phi}(\partial_{\theta_*} h) = \frac{\sin^2 \theta}{\Xi^2} \hat{\Omega}(\partial_{\theta_*} h), \tag{22} \\
\hat{g}_{\phi_* \phi_*} &= g_{\phi \phi} = \frac{\sin^2 \theta}{\Xi^2} \hat{\Omega}. \tag{23} 
\end{align*}
\]

The determinants of the metrics \( \hat{g} \) and \( g \) are
\[
\det \hat{g} = \frac{L^2 \sin^2 \theta}{\Xi^2}, \tag{24}
\]
and
\[
\det g = -4\Omega^4 L^2 \sin^2 \theta \Xi^2.
\]
and the inverse of the metric $g$ is

$$g^{-1} = -\frac{1}{2\Omega^2} \left( \partial_u \otimes (\partial_v + b^\phi \partial_{\phi_*}) + (\partial_v + b^\phi \partial_{\phi_*}) \otimes \partial_u \right)$$

$$+ g^{\theta, \phi} \partial_{\theta_*} \otimes \partial_{\theta_*} + g^{\theta, \phi} \partial_{\theta} \otimes \partial_{\phi_*} + g^{\phi, \phi} \partial_{\phi} \otimes \partial_{\phi_*} + g^{\phi, \phi} \partial_{\phi} \otimes \partial_{\phi_*},$$

with coefficients given by

$$\left[ g^{\theta, \phi}, g^{\theta, \phi} \right] = \left[ -\frac{\gamma}{L^2 r^2} (\partial_\theta, h), -\frac{\gamma}{L^2 r^2} (\partial_\theta, h) \right].$$

### 2.1.4 Normals to hypersurfaces and volume elements

We finish this subsection by writing down the volume elements of hypersurfaces

\[ \Sigma_{r_*} = \{ r_* = \text{constant} \}, \quad \mathcal{L}_r = \{ v = \text{constant} \} \quad \text{and} \quad \mathcal{C}_u = \{ u = \text{constant} \} \]

of our spacetime, corresponding to constant $r_*$, $v$ and $u$. As

\[ 2\Omega^2 (du \otimes dv + dv \otimes du) = \Omega^2 (dr_* \otimes dr_* - dt \otimes dt), \]

recalling (24) for the determinant of $g$, the volume element for $\Sigma_{r_*}$ is

\[ d\text{Vol}_{\Sigma_{r_*}} = \frac{L}{\Xi} \sin \theta d\theta_* d\phi_* dt. \]

Our choice for the normals to constant $v$ and $u$ hypersurfaces are

\[ n_{\mathcal{L}_r} = \partial_u \quad \text{and} \quad n_{\mathcal{C}_u} = \partial_v + b^\phi \partial_{\phi_*}. \]

We have that

\[ g \left( \partial_v, \frac{1}{2L^2} \partial_v \right) = -1 \quad \text{and} \quad g \left( \partial_v + b^\phi \partial_{\phi_*}, \frac{1}{2L^2} \partial_v \right) = -1 \]

and so, since the volume element associated to the metric $g$ is

\[ d\text{Vol} = 2\Omega \frac{L}{\Xi} \sin \theta du dv d\theta_* d\phi_* , \]

the volume elements associated to constant $v$ and $u$ hypersurfaces are

\[ d\text{Vol}_{\mathcal{L}_r} = \frac{L}{\Xi} \sin \theta du d\theta_* d\phi_* \]

and

\[ d\text{Vol}_{\mathcal{C}_u} = \frac{L}{\Xi} dv d\theta_* d\phi_* . \]

As $r_* = u + v$, the tangent space to a hypersurface $\Sigma_{r_*}$, where $r_*$ is constant, is spanned by $\partial_v - \partial_u$, $\partial_{\theta}$, and $\partial_{\phi_*}$, and

\[ n_{\Sigma_{r_*}} = \partial_u + \partial_v + b^\phi \partial_{\phi_*} = -\frac{\text{grad} r_*}{2\Omega \left( -g(\text{grad} r_*, \text{grad} r_*) \right)^{1/2}}. \]

**Remark 2.3.** $\partial_{\phi_*}$ is equal to zero when $r_*$ is either 0 or $\pi$.

Indeed, the vector field $\partial_{\phi_*}$ is tangent to the spheres $u = \text{constant}$ and $v = \text{constant}$ which are contained in the spacelike hypersurfaces $r_* = \text{constant}$ and

\[ \lim_{\theta_* \to 0} g(\partial_{\phi_*}, \partial_{\phi_*}) = \lim_{\theta_* \to 0} g^{\phi, \phi} = 0. \]

### 2.2 Regularity of the change of coordinates

The regularity of transformation of coordinates $(r_*, \theta_*) \mapsto (r, \theta)$ for Kerr spacetimes, namely at $\theta_* = 0$ and $\theta_* = \frac{\pi}{2}$, was shown by Dafermos and Luk [9]. In this subsection we adapt their work to the setting of Kerr–Newman–de Sitter spacetimes.
2.2.1 The coordinate $\theta_*$ is well defined and continuous

**Lemma 2.4.** $\theta_*$ is well defined.

**Proof.** Clearly 

$$F(r, \theta, \theta) \geq 0.$$ 

Moreover

$$\int_{\theta}^{\theta_*} \frac{d\theta'}{P(\theta', \theta_*)} \geq \int_{0}^{\theta_*} \frac{d\theta'}{a\sqrt{2\theta' - \sin^2 \theta'}} = \int_{0}^{\theta_*} \frac{d\theta'}{2a} \cos \theta' \to +\infty \text{ as } \theta_* \to \pi/2,$$

which implies that 

$$\lim_{\theta_* \to \pi/2} F(r, \theta, \theta_*) = -\infty.$$ 

The continuity of $F$ implies that for each pair $(r, \theta)$, with $\theta \in (0, \pi/2)$, there exists $\theta_* = \theta_*(r, \theta) \in (\theta, \pi/2)$ such that $F(r, \theta, \theta_*) = 0$.

To show that $F$ is decreasing in $\theta_*$, note that 

$$\theta_* \to \int_{0}^{\theta_*} \frac{d\theta'}{\sqrt{2\theta_* \Delta \theta' - \sin^2 \theta'}}$$

is strictly increasing. Indeed, suppose $\theta_* < \tilde{\theta}_*$. Setting $\sin \tilde{\theta}' = \frac{\sin \tilde{\theta}_*}{\sin \theta_*} \sin \theta'$, as $\theta' < \tilde{\theta}'$, we have $\Delta \theta' > \Delta \tilde{\theta}'$ and

$$\int_{0}^{\tilde{\theta}_*} \frac{d\tilde{\theta}'}{\sqrt{2\tilde{\theta}_* \Delta \tilde{\theta}_' - \sin^2 \tilde{\theta}'}} - \int_{0}^{\theta_*} \frac{d\theta'}{\sqrt{2\theta_* \Delta \theta' - \sin^2 \theta'}} \geq \int_{0}^{\theta_*} \frac{d\theta'}{\sqrt{2\theta_* \Delta \theta' - \sin^2 \theta'}} \cdot \left( \cos \theta' - 1 \right) \geq 0.$$ 

\[\blacksquare\]

**Lemma 2.5.** $\theta_*$ is continuous at $\theta = 0$ and $\theta = \pi/2$ with

$$\frac{\sin \theta_*}{\sin \theta} + \frac{\cos \theta}{\cos \theta_*} \lesssim 1.$$ 

(27)

**Proof.** The second integral in (9) is bounded above by

$$\int_{r}^{r\pm} \frac{dr'}{Q(r', \theta_*)} \leq \int_{r\pm}^{r\pm} \frac{dr'}{r'^2 + a^2} = \frac{1}{a} \left( \arctan \left( \frac{r_+}{a} \right) - \arctan \left( \frac{r_-}{a} \right) \right) = \frac{1}{a} \left( \frac{\pi}{2} - c_{\Lambda,M,a,e} \right),$$

for some positive $c_{\Lambda,M,a,e}$. On the other hand, using the substitution $\sin \theta' = \sin \theta_\ast \sin \tilde{\theta}$ we see that the negative
of the first integral in (9) is bounded below by

\[ \int_0^\theta \frac{d\theta'}{P(\theta', \theta)} = \frac{1}{a} \int_\theta^\pi \frac{d\theta'}{\sqrt{\sin^2 \theta \Delta \theta' - \sin^2 \theta'}} \]

\[ = \frac{1}{a} \int_{\arcsin(\sin \theta')}^{\pi} \frac{\cos \theta' d\theta}{\cos \theta' \sqrt{\Delta \theta' - \sin^2 \theta'}} \]

\[ = \frac{1}{a} \int_{\arcsin(\sin \theta')}^{\pi} \sqrt{\left(1 + \frac{A}{3} a^2\right) - \left(1 + \frac{A}{3} a^2 \sin^2 \theta'\right) \sin^2 \theta} \]

\[ \geq \frac{1}{a} \int_{\sin \theta'}^\pi \frac{dx}{\sqrt{1 - x^2}} \]

\[ = \frac{1}{a} \int_{\arcsin(\sin \theta')}^{\pi} \frac{dy}{\sqrt{1 - y^2}} \]

\[ = \frac{1}{a} \left( \arcsin\left(\frac{1}{\sqrt{\Xi}}\right) - \arcsin\left(\frac{\sin \theta}{\sqrt{\Xi} \sin \theta'}\right) \right) \]

Since \( F(r, \theta, \theta *) \) is equal zero, we obtain the estimate

\[ \arcsin\left(\frac{\sin \theta}{\sin \theta'}\right) \geq \arcsin\left(\frac{1}{\sqrt{\Xi}}\right) + c_{A, M, a, e} - \frac{\pi}{2}, \]

provided the right-hand side is positive, i.e.

\[ \Xi < \csc^2\left(\frac{\pi}{2} - c_{A, M, a, e}\right) \]

\[ = \csc^2\left(\arctan\left(\frac{r_+}{a}\right) - \arctan\left(\frac{r_-}{a}\right)\right). \]

For fixed \( r_- \) and \( r_+ \), the minimum of the last expression is attained when \( a = \sqrt{r_- r_+} \). So the last inequality is implied by the stronger restriction

\[ \Xi < \csc^2\left(\arctan\left(\sqrt{\alpha}\right) - \arctan\left(\frac{1}{\sqrt{\alpha}}\right)\right), \quad \alpha = \frac{r_+}{r_-}. \]

According to Lemma B.1, for all subextremal black holes, we have

\[ \Xi < 1 + \frac{l(\alpha, \Lambda e^2)}{3} \leq 1 + \frac{l(\alpha, 0)}{3}, \]

where \( l \) is given by (137), so that

\[ l(\alpha, 0) := \frac{3}{2a^2(1 + 2a)} \left(1 + 4a + 12a^2 + 16a^3 + 9a^4\right) \]

\[- \left(1 + 3a + 5a^2 + 3a^3\right) \sqrt{1 + 2a + 9a^2}\].

We claim that

\[ 1 + \frac{l(\alpha, 0)}{3} < \csc^2\left(\arctan\left(\sqrt{\alpha}\right) - \arctan\left(\frac{1}{\sqrt{\alpha}}\right)\right), \]

for all \( \alpha \) greater than 1. To show this, we define the function

\[ \tilde{\rho}(\alpha) := \frac{\csc^2\left(\arctan\left(\sqrt{\alpha}\right) - \arctan\left(\frac{1}{\sqrt{\alpha}}\right)\right) - 1}{\frac{l(\alpha, 0)}{3}}, \]

which we wish to check is always greater than one. Since

\[ \lim_{\alpha \to 1} \csc^2\left(\arctan\left(\sqrt{\alpha}\right) - \arctan\left(\frac{1}{\sqrt{\alpha}}\right)\right) = +\infty, \]
and \( l(1,0) = 21 - 12\sqrt{3} \), we have
\[
\lim_{\alpha \searrow 1} \hat{\rho}(\alpha) = +\infty.
\]
Moreover,
\[
\lim_{\alpha \to +\infty} \frac{csc^2 \left( \arctan \sqrt{\alpha} - \arctan \left( \frac{1}{\sqrt{\alpha}} \right) \right) - 1}{\alpha} = 1
\]
and
\[
\lim_{\alpha \to +\infty} \frac{l(\alpha,0)}{\frac{1}{\alpha}} = 1
\]
(see (139)) imply that
\[
\lim_{\alpha \to +\infty} \hat{\rho}(\alpha) = 18.
\]
So certainly \( \hat{\rho}(\alpha) \) is greater than one for \( \alpha \) close to 1 and \( \alpha \) sufficiently large. Estimating, one can determine
\( 1 < \alpha_m < \alpha_M \) such that \( \hat{\rho}(\alpha) > 36 \) for all \( 1 < \alpha < \alpha_m \), and \( \hat{\rho}(\alpha) > 9 \) for all \( \alpha > \alpha_M \). Then one can plot the graph of \( \hat{\rho} \) in the interval \([\alpha_m, \alpha_M]\) and check that it is also greater than one in that interval. This is illustrated in Figure 1. The claim is proven.

![Figure 1: The fact that \( \hat{\rho} > 1 \) implies that we can guarantee (33) for all subextremal black holes.](image)

The previous paragraph implies that (32) is satisfied, and so the right-hand side of (30) is positive. Then, we have
\[
\frac{\sin \theta}{\sin \theta_\star} \geq c > 0. \tag{33}
\]
If we use the substitution \( \cos \theta' = \cos \theta_\star \sec \bar{\theta} \) in (29) we get
\[
\int_0^\theta \frac{d\theta'}{P(\theta', \theta_\star)} = \frac{1}{a} \int_0^\theta \frac{d\theta'}{\sqrt{\sin^2 \theta_\star \Delta \theta - \sin^2 \theta'}} = \frac{1}{a} \int_0^{\text{arccsc} \left( \frac{\cos \theta}{\sin \theta_\star} \right)} \frac{\sec \bar{\theta} \cos \theta_\star \tan \bar{\theta} d\bar{\theta}}{\sin \theta' \sqrt{\Delta \theta' - 1 + \cos^2 \theta_\star \sec^2 \bar{\theta} - \Delta \theta' \cos^2 \theta_\star}} = \frac{1}{a} \int_0^{\text{arccsc} \left( \frac{\cos \theta}{\sin \theta_\star} \right)} \frac{\sec \bar{\theta} \tan \bar{\theta} d\bar{\theta}}{\sin \theta' \sqrt{\frac{1}{3} a^2 \sec^2 \bar{\theta} \sin^2 \theta_\star \tan^2 \bar{\theta}}} = \frac{1}{a} \int_0^{\text{arccsc} \left( \frac{\cos \theta}{\sin \theta_\star} \right)} \frac{\sec \bar{\theta} d\bar{\theta}}{\sin \theta' \sqrt{1 + \frac{3}{2} a^2 \sec^2 \bar{\theta} \sin^2 \theta_\star}}.
\]
If \( \frac{\cos \theta}{\cos \theta_\star} \leq \sqrt{2} \), we are done. Otherwise, as
\[
\frac{1}{\sin^2(\text{arccsc} x)} = \frac{1}{1 - \frac{1}{x^2}}
\]
and \( \sin^2 \theta_* \leq 1 \), we have that

\[
\int_\theta^\theta' \frac{d\theta'}{P(\theta', \theta_*)} \geq \frac{1}{a} \int_{\frac{\pi}{2}}^{\arccsc \left( \frac{\sin \theta}{\sin \theta_*} \right)} \frac{\sec \hat{\theta} d\hat{\theta}}{\sqrt{1 + \frac{2 \Lambda}{a^2}}} \\
\geq \frac{1}{\sqrt{2 \pm a}} \ln(\sec x + \tan x) \bigg|_{\frac{\pi}{2}}^{\arccsc \left( \frac{\sin \theta}{\sin \theta_*} \right)} \\
= \frac{1}{\sqrt{2 \pm a}} \left[ \ln \left( \frac{\cos \theta}{\cos \theta_*} + \sqrt{\frac{\cos^2 \theta}{\cos^2 \theta_*} - 1} \right) - \ln(1 + \sqrt{2}) \right].
\]

The result follows since (28) implies that the last expression is bounded above. \( \square \)

2.2.2 The derivative of the function defining \( \theta_* \)

Remark 2.6. Let us define

\[
D(\theta, \theta_*) := \sin^2 \theta_* \Delta_\theta - \sin^2 \theta,
\]

so that \( D(\theta, \theta_*) = P^2(\theta, \theta_*)/a^2 \). Clearly,

\[
\partial_\theta D(\theta, \theta_*) = \partial_\theta \left( \sin^2 \theta_* \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right) - \sin^2 \theta \right) = -\sin(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right)
\]

holds. The trigonometric equality

\[
\sin^2(2\theta_*) \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right) = \sin^2(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) + 2(\cos(2\theta) + \cos(2\theta_*) \left( \sin^2 \theta_* \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right) - \sin^2 \theta \right)
\]

allows us to conclude that

\[
\sin^2(2\theta_*) \Delta_\theta = -\sin(2\theta) \left[ \partial_\theta \left( \sin^2 \theta_* \Delta_\theta - \sin^2 \theta \right) \right] + 2(\cos(2\theta) + \cos(2\theta_*) \left( \sin^2 \theta_* \Delta_\theta - \sin^2 \theta \right) \\
= -\sin(2\theta) \partial_\theta D(\theta, \theta_*) + 2(\cos(2\theta) + \cos(2\theta_*) D(\theta, \theta_*). \tag{34}
\]

We will have to use the identity (34) repeatedly in our calculations.

Lemma 2.7. For \( \theta_* \in \left(0, \frac{\pi}{2}\right) \), we have that

\[
G(r, \theta, \theta_*) = \partial_\theta \circ F(r, \theta, \theta_*)
\]

\[
= -\frac{\csc(2\theta_*) \sin(2\theta)}{a\sqrt{\sin^2 \theta_* \Delta_\theta - \sin^2 \theta}} - \frac{2 \csc(2\theta_*)}{a} \int_\theta^\theta' \frac{(\sin^2 \theta_* - \sin^2 \theta')}{\sqrt{\sin^2 \theta_* \Delta_\theta - \sin^2 \theta'}} d\theta' \\
+ \int_r^{r*} \frac{a^2 \sin(2\theta_*) \Delta_{r'}}{2 \left( (r'^2 + a^2) - a^2 \sin^2 \theta_* \Delta_{r'} \right)^2} dr'. \tag{35}
\]

Proof. We start from the definition of \( F \) in (9). Since \( \theta_* \in \left(0, \frac{\pi}{2}\right) \), for \( \theta' \in [\theta, \theta_*] \), we get \( \Delta_{\theta'} \geq 1 + \frac{\Lambda}{a^2} a^2 \cos^2 \theta_* \), and so \( P(\theta', \theta_*) \geq \sqrt{\frac{\Lambda}{a^2} a^2} \sin(2\theta_*) > 0 \). So, we do not have to worry about vanishing denominators. The result
follows from the calculation

\[ \frac{a}{\partial \theta^*} \bigg|_{\theta \text{ fixed}} \int_0^{\theta_*} \frac{d\theta'}{P(\theta', \theta_*)} = \frac{\partial}{\partial \theta_*} \bigg|_{\theta \text{ fixed}} \int_0^{\theta_*} \frac{d\theta'}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} \]

\[ = \frac{2}{\sqrt{2} a \sin(2\theta_*)} - \int_0^{\theta_*} \frac{\sin(2\theta_*) \Delta \theta'}{2 \sqrt{(D(\theta', \theta_*))^3}} d\theta' \]

\[ \text{(34)} \]

\[ = \frac{2}{\sqrt{2} a \sin(2\theta_*)} + \int_0^{\theta_*} \frac{\csc(2\theta_*) \sin(2\theta') \partial \theta' D(\theta', \theta_*)}{2 \sqrt{(D(\theta', \theta_*))^3}} d\theta' \]

\[ - \int_0^{\theta_*} \frac{2 \csc(2\theta_*) \cos(2\theta')}{\sqrt{D(\theta', \theta_*)}} d\theta' + \int_0^{\theta_*} \csc(2\theta_*) \left( \cos(2\theta') - \cos(2\theta_*) \right) d\theta' \]

\[ = \frac{2}{\sqrt{2} a \sin(2\theta_*)} - \int_0^{\theta_*} \frac{\partial}{\partial \theta'} \frac{\csc(2\theta_*) \sin(2\theta')}{\sqrt{D(\theta', \theta_*)}} d\theta' \]

\[ + \int_0^{\theta_*} \frac{2 \csc(2\theta_*) (\sin^2 \theta_* - \sin^2 \theta')}{\sqrt{D(\theta', \theta_*)}} d\theta' \]

\[ = \frac{\csc(2\theta_*) \sin(2\theta)}{\sqrt{D(\theta', \theta_*)}} + 2 \csc(2\theta_*) \int_0^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')}{\sqrt{D(\theta', \theta_*)}} d\theta'. \]

For the second equality we used (34). \( \square \)

The expression for \( G \) and the definition of \( L \) in (13) yield

\[ L = \sqrt{(r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta \theta} \left( \frac{\sin(2\theta)}{\sin(2\theta_*)} \right) \]

\[ + 2 \sin^2(2\theta_*) \sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta} \left( \frac{1}{\sin^2(2\theta_*)} \int_0^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta' \right) \]

\[ - \sin^2(2\theta_*) \sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta} \int_0^{\theta_*} \frac{a^3 \Delta \theta'}{2 \left( (r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta \theta' \right)^{3/2}} d\theta'. \]

\[ (36) \]

**Lemma 2.8.** We have the following estimate for \( G \):

\[ - \frac{C_{\Lambda,M,a,c}}{\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}} \leq G(r, \theta, \theta_*(r, \theta)) \leq - \frac{\csc(2\theta_*) \sin(2\theta)}{a \sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}}. \]

\[ (37) \]

for some constant \( C_{\Lambda,M,a,c} > 0 \) depending on \( \Lambda, M, a \) and \( c \).

**Proof.** We use expression (35). Note that

\[ \frac{2 \csc(2\theta_*)}{a} \int_0^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta' \leq \frac{2 \csc(2\theta_*)}{a} \int_0^{\theta_*} \sqrt{\sin^2 \theta_* - \sin^2 \theta'} \cos \theta' d\theta' \]

For \( \theta_* \) close to zero, the right-hand side is bounded above by

\[ \frac{\pi/2}{a \cos \theta_*}, \]

whereas for \( \theta_* \) close to \( \frac{\pi}{2} \), the right-hand side is bounded above by

\[ \frac{2 \csc(2\theta_*)}{a} \int_0^{\theta_*} \cos \theta' d\theta' = \frac{2 \csc(2\theta_*)}{a} (\sin \theta_* - \sin \theta) \]

\[ \leq \frac{2 \csc(2\theta_*)}{a} (\theta_* - \theta) \cos \theta \]

\[ \leq \frac{\pi/2}{a \sin \theta_* \cos \theta_*}. \]
Recalling (27), we conclude that the second term on the right-hand side of (35) is uniformly bounded below by a negative constant. Clearly, the same is true for the third term on the right-hand side of (35). Since both terms are negative, \( \csc(2\theta_*) \sin(2\theta) \) is bounded, and \( 1/\sin^2 \theta_* \Delta_\theta - \sin^2 \theta \) is bounded below by a positive constant, we obtain (37).

\[ \square \]

### 2.2.3 The coordinates \( r \) and \( \theta \) as functions of \( r_* \) and \( \theta_* \)

**Lemma 2.9.** The mapping \((r, \theta) \rightarrow (r_*, \theta_*)\) is globally invertible.

**Proof.** Fix \((r_*, \theta_*) \in \mathbb{R}^2\). For each fixed \( \theta \in (0, \frac{\pi}{2}) \), let \( r_\theta(\theta) \) be the unique (because \( \partial_r r_* < 0 \)) value of \( r \in (r_-, r_+) \) such that

\[ r_*(\theta, \theta) = (r_*)_0. \]

The Implicit Function Theorem guarantees that \((r(\cdot), \cdot)\) is a \( C^1 \) curve and that

\[ \dot{r} = -\frac{\partial r_*}{\partial r} = -\frac{P}{Q} \frac{\partial \Delta}{\partial \theta}. \]

To see how \( \theta_* \) varies along this curve we calculate

\[ \frac{d}{d\theta} \theta_*(r(\theta), \theta) = \dot{r} \partial_r \theta_* + \partial_\theta \theta_* = \frac{1}{GQ} \left( -\frac{P \Delta_r}{Q \Delta_\theta} \right) - \frac{1}{GP} \frac{Y}{GPQ^2} > 0. \]

So, \( \theta_* \) is strictly increasing along the curve \((r(\cdot), \cdot)\). Moreover, (27) shows that \( \lim_{\theta \rightarrow 0} \theta_* (r(\theta), \theta) = \frac{\pi}{2} \). Thus, given \((r_*(0), (\theta_*)_0) \in (-\infty, +\infty) \times (0, \frac{\pi}{2})\) there exists one and only one \((r, \theta)\) such that \((r_*(r, \theta), \theta_*(r, \theta)) = ((r_*)_0, (\theta_*)_0)\). We conclude that the mapping

\[ (r_*, \theta_*) : (r_-, r_+) \times [0, \frac{\pi}{2}] \rightarrow (-\infty, +\infty) \times [0, \frac{\pi}{2}] \]

is one-to-one and onto.

\[ \square \]

### 2.2.4 First partial derivatives

**Lemma 2.10.** The partial derivatives of \((r, \theta)\) with respect to \((r_*, \theta_*)\) are given by

\[ \frac{\partial r}{\partial r_*} = \frac{\Delta_r \Delta_\theta Q}{Y} = \frac{\Delta_\theta \Delta_\theta \sqrt{(r_1 + a^2)^2 - a^2 \Delta_r \sin^2 \theta_*}}{(r_1 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta}, \]

\[ \frac{\partial r}{\partial \theta_*} = - \frac{\Delta_r L P}{Y} = \frac{\Delta_\theta GPQ^2}{Y} = \frac{a^2 \Delta_r \left( \sin^2 \theta_* \Delta_\theta \sin \theta \right)}{(r_1 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta}, \]

\[ \frac{\partial \theta}{\partial r_*} = \frac{\Delta \Delta \theta P}{Y} = \frac{\sqrt{\sin^2 \theta_*, \Delta_\theta \sin \theta}}{\sin(2\theta_*)} \frac{a^2 \Delta_\theta}{(r_1 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta} \sin(2\theta_*), \]

\[ \frac{\partial \theta}{\partial \theta_*} = \frac{\Delta \Delta \theta Q}{Y} = \frac{-\Delta_\theta GPQ^2}{Y} = -G \frac{a^2 \Delta_\theta \left( \sin^2 \theta_* \Delta_\theta \sin \theta \right)}{(r_1 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta} \sin(2\theta_*), \]

\[ \frac{\partial \theta}{\partial \theta_*} = \frac{\Delta \Delta \theta Q}{Y} = \frac{-\Delta_\theta GPQ^2}{Y} = -G \frac{a^2 \Delta_\theta \left( \sin^2 \theta_* \Delta_\theta \sin \theta \right)}{(r_1 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta} \sin(2\theta_*). \]
In the region $r_- < r < r_+$, we have
\[
\left| \frac{\partial r}{\partial r_*} \right| \lesssim |\Delta_r|, \quad \left| \frac{\partial r}{\partial \theta_*} \right| \lesssim |\Delta_r| \sin(2\theta_*),
\]
\[
\left| \frac{\partial \theta}{\partial r_*} \right| \lesssim |\Delta_r| \sin(2\theta_*), \quad \left| \frac{\partial \theta}{\partial \theta_*} \right| \lesssim 1.
\] (42)

Proof. We start from (15) and (16). We use (35) to obtain (39) and (41). The estimates for the derivatives of $r$ and $\theta$ follow from the estimates
\[
\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta} \lesssim \sin \theta_* \sqrt{\Delta \theta} \lesssim \sin \theta_* \lesssim \sin(2\theta_*), \text{ for } \theta_* \text{ close to } 0,
\]
\[
\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta} = \cos \theta_* \sqrt{\left( \frac{\cos^2 \theta}{\cos^2 \theta_*} - \Delta \theta \right)} \lesssim \cos \theta_* \lesssim \sin(2\theta_*), \text{ for } \theta_* \text{ close to } \frac{\pi}{2},
\]
which together imply that
\[
\frac{\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}}{\sin(2\theta_#)} \lesssim 1.
\]

$$\Box$$

2.2.5 Higher order derivatives

Lemma 2.11. The functions $r$ and $\frac{\partial \theta}{\partial r_*}$ are $C^\infty$. For every $k \geq 2$, we have that
\[
\sum_{1 \leq k_1 + k_2 \leq k} \left| \left( \frac{\partial r_*}{\partial r} \right)^{k_1} \left( \frac{1}{\sin(2\theta_#)} \frac{\partial \theta}{\partial \theta} \right)^{k_2} \right| r \lesssim |\Delta_r|, \quad (43)
\]
\[
\sum_{1 \leq k_1 \leq k} \left| \left( \frac{\partial r_*}{\partial \theta} \right)^{k_1} \right| \lesssim |\Delta_r| \sin(2\theta_*), \quad (44)
\]
\[
\sum_{0 \leq k_1 \leq k-1} \left| \left( \frac{1}{\sin(2\theta_#)} \frac{\partial \theta}{\partial \theta} \right)^{k_1} \right| \lesssim 1, \quad (45)
\]
\[
\sum_{1 \leq k_1 + k_2 \leq k-1 \atop k_2 \geq 1} \left| \left( \frac{\partial r_*}{\partial r} \right)^{k_1} \left( \frac{1}{\sin(2\theta_#)} \frac{\partial \theta}{\partial \theta} \right)^{k_2} \right| \lesssim |\Delta_r|. \quad (46)
\]

Moreover, the derivatives of the function $L$ given in (13) are bounded as follows:
\[
\sum_{0 \leq k_1 \leq k-1} \left| \left( \frac{1}{\sin(2\theta_#)} \frac{\partial \theta}{\partial \theta} \right)^{k_1} \right| L \lesssim 1, \quad (47)
\]
\[
\sum_{1 \leq k_1 + k_2 \leq k-1 \atop k_2 \geq 1} \left| \left( \frac{\partial r_*}{\partial r} \right)^{k_1} \left( \frac{1}{\sin(2\theta_#)} \frac{\partial \theta}{\partial \theta} \right)^{k_2} \right| \lesssim |\Delta_r|. \quad (48)
\]

Remark 2.12. We say that a function depending on $r_*$ and $\theta_*$ is $C^\infty$ if it has derivatives of all orders with respect of $\partial_{r_*}$ and $\frac{1}{\sin(2\theta_#)} \frac{\partial \theta}{\partial \theta}$. Refer to Remarks (2.19), (2.20) and (2.21) for an explanation about the reason for introducing the factor $\frac{1}{\sin(2\theta_#)}$ behind $\partial_{\theta}$. The function $\sin^2 \theta$ is $C^\infty$.

To prove Lemma 2.11 we need to know the derivatives of
\[
R_1 = \frac{1}{\sin^{2n}(2\theta_#)} \int_0^{2\theta_#} \frac{\left( \sin^2 \theta_* - \sin^2 \theta \right)^n}{\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}} \, d\theta,
\]
\[
R_2 = \frac{\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}}{\sin(2\theta_#)},
\]
\[
R_3 = \frac{\sin^2 \theta_* - \sin^2 \theta}{\sin^2(2\theta_#)},
\]
\[
R_4 = \frac{\sin(2\theta)}{\sin(2\theta_#)}.
\]
Note that the dependence of \( \frac{\partial r}{\partial r_*}, \frac{1}{\sin(2\theta_*)} \frac{\partial r}{\partial \theta_*}, \frac{\partial \theta}{\partial \theta_*} \) and \( L \) on \( \theta \) and \( \theta_* \) is done through \( \sin^2 \theta, \sin^2 \theta_*, \Delta \theta \) (which is a function of \( \sin^2 \theta \), and this is crucial for our argument to work), \( \sin^2(2\theta_*), \) and \( R_1 \) to \( R_4 \). The same is true for \( \frac{\partial \theta}{\partial \theta_*} \), which however also has a factor \( \sin(2\theta_*) \). Moreover, note that

\[
\begin{align*}
\partial_r, (\sin^2 \theta) &= \sin(\theta) \frac{\partial \theta}{\partial \theta_*}, \quad \text{this has a factor } \sin(2\theta) \sin(2\theta_*), \\
\partial_\theta, (\sin(2\theta) \sin(2\theta_*)) &= \cos(\theta) \frac{\partial \theta}{\partial \theta_*} \sin(2\theta_*), \quad \text{this has factors } \cos(\theta) \text{ and } \sin^2(2\theta_*), \\
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*}, (\sin(2\theta) \sin(2\theta_*)) &= 2 \cos(\theta) \frac{\partial \theta}{\partial \theta_*} + 2 \frac{\sin(\theta)}{\sin(2\theta_*)} \cos(2\theta) \\
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*}, (\sin^2 \theta) &= \frac{\sin(\theta)}{\sin(2\theta_*)} \theta, \\
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*}, (\cos^2 \theta) &= \cos(2\theta_*) \\
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*}, (\cos(2\theta_*)) &= -2 \frac{\sin(\theta)}{\sin(2\theta_*)} \theta, \\
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*}, (\cos(2\theta_*)) &= -2,
\end{align*}
\]

or, alternatively, \( \cos(\theta) = 1 - 2 \sin^2 \theta \). If the factor \( \sin(2\theta) \) were to appear by itself in our formulas, or if the factor \( \sin(2\theta_*) \) were to appear by itself, then our argument would not go through because the derivatives \( \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} \) of each of these are not bounded. The structure of our problem is such that when one of these factors is present, then the other one is also present, and their product has derivatives of all orders with respect to \( \partial_r \) and \( \partial_{\theta_*} \), which are bounded. Furthermore, as we will see below, the derivatives of \( R_1 \) to \( R_4 \) are sums whose summands are products of factors that are either \( R_{1, 2}, R_{3, 4} \), or, when this is not the case, others that are clearly smooth (since the denominators that will appear do not vanish for \( \theta \)). Hence, it turns out that to prove that \( r \) and \( \theta \) are smooth, we just have to check that \( R_1 \) to \( R_4 \) are \( C^1 \) and that their first derivatives have the aforementioned property. Next we calculate the first derivatives of \( R_1 \) to \( R_4 \).

As a small note, let \( \theta_* \in (0, \frac{\pi}{2}) \). Consider the function

\[
\frac{1}{\sin(2\theta_*)} \int_\theta^{\theta_*} \partial \theta = \frac{\theta_* - \theta}{\sin(2\theta_*)}.
\]

This is clearly bounded for \( \theta_* \) close to zero. It is also bounded for \( \theta_* \) close to \( \frac{\pi}{2} \). Indeed, in this situation, we have

\[
\frac{\theta_* - \theta}{\sin(2\theta_*)} \leq \frac{\frac{\pi}{2} - \theta}{2 \sin(\theta_*) \cos(\theta_*)} \leq \frac{C}{2} \frac{\frac{\pi}{2} - \theta}{\cos(\theta)} \leq \frac{C}{2} \frac{\frac{\pi}{2} - \theta}{\theta - \theta} \leq C,
\]

because \( \frac{\cos \theta}{\cos \theta_*} \) is bounded. So the function is bounded for \( \theta_* \in (0, \frac{\pi}{2}) \).

**Remark 2.13.** The reader will notice, using the expressions below, that

\[
|\partial_r, R_i| \leq \Delta_i, \quad \text{(50)}
\]

\[
|\frac{1}{\sin(2\theta_*)} \partial_{\theta_*}, R_i| \leq 1, \quad \text{(51)}
\]

for each \( i \) between 1 and 4.

**Derivatives of \( R_1 \).** For any integer \( n \geq 1 \), the following identities hold:

\[
\begin{align*}
\partial_r \left( \frac{1}{\sin^{2n}(2\theta_*)} \int_\theta^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} \partial \theta' \right) &= \frac{(\sin^2 \theta_* - \sin^2 \theta)^n}{\sin^{2n}(2\theta_*)} - \frac{\Delta \theta \Delta^2 \theta}{(r^2 + a^2) \Delta \theta - a^2 \sin^2 \theta \Delta \theta}.
\end{align*}
\]
\[
\left(\frac{1}{\sin(2\theta_*) \partial \theta_*}\right) \left( \frac{1}{\sin^{2n}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta' \right)
= \frac{2(2n + 1)}{\sin^{2(n+1)}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^{n+1}}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta'
- \frac{2}{\sin^2(2\theta_*)} \int_{\theta}^{\theta_*} \frac{\sin^2 \theta_* - \sin^2 \theta'}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta'
- \int_{r}^{r_*} \frac{a^2 \Delta \theta'}{2((r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta \theta')}^{3/2} d\theta'
\times \frac{\Delta \theta}{(r^2 + a^2)^2 \Delta \theta - a^2 \sin^2 \theta_\Delta \theta} \frac{(\sin^2 \theta_* - \sin^2 \theta)^n}{\sin(2\theta_*) \sin(2\theta_*) \sin^{2n}(2\theta_*)}.
\]  

(53)

**Proof.** The proof of (52) is immediate using (40), as
\[
\partial_{\theta_*} \left( \frac{1}{\sin^{2n}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta' \right)
= - \frac{1}{\sin^{2n}(2\theta_*)} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\partial \theta} \frac{\partial \theta}{\partial \theta_*} \sin^{2n}(2\theta_*) \sqrt{D(\theta', \theta_*)}
\]

By differentiation we obtain
\[
\partial_{\theta_*} \left( \frac{1}{\sin^{2n}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{\sin^2 \theta_* \Delta \theta' - \sin^2 \theta'}} d\theta' \right)
= - \frac{4n \cos(2\theta_*)}{\sin^{2n+1}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{D(\theta', \theta_*)}} d\theta'
+ \frac{n}{\sin^{2n-1}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^{n-1}}{\sqrt{D(\theta', \theta_*)}} d\theta'
- \frac{1}{2 \sin^{2n-1}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\cos(\theta' - \theta_*) - \cos(\theta' + \theta_*) - \cos(2\theta_*) (\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{D(\theta', \theta_*)}} d\theta'
\]

(54)

We keep the two first integrals unchanged and use (34) on the integral marked (54) to obtain
\[
- \frac{4n \cos(2\theta_*)}{\sin^{2n+1}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{D(\theta', \theta_*)}} d\theta'
+ \frac{n}{\sin^{2n-1}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^{n-1}}{\sqrt{D(\theta', \theta_*)}} d\theta'
- \frac{1}{2 \sin^{2n-1}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\cos(\theta' + \cos(2\theta_*) (\sin^2 \theta_* - \sin^2 \theta')^n}{\sqrt{D(\theta', \theta_*)}} d\theta'
- \frac{1}{\sin^{2n}(2\theta_*)} \int_{\theta}^{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sin(2\theta_*) \partial \theta} \frac{1}{\sqrt{D(\theta', \theta_*)}} d\theta'
- \partial_{\theta_*} \frac{(\sin^2 \theta_* - \sin^2 \theta')^n}{\sin^{2n}(2\theta_*) \sqrt{D(\theta', \theta_*)}}
\]

Integrating the last integral by parts and using the fact that
\[
-4n \cos(2\theta_*) (\sin^2 \theta_* - \sin^2 \theta) + n \sin^2(2\theta_*) - (\cos(2\theta_*) + \cos(2\theta_*)) (\sin^2 \theta_* - \sin^2 \theta) + 2 \cos(2\theta_*) (\sin^2 \theta_* - \sin^2 \theta) - n \sin^2(2\theta_*) = 2(2n + 1)(\sin^2 \theta_* - \sin^2 \theta)^2,
\]

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we obtain

\[
\partial\varphi_\star \left( \frac{1}{\sin^{2n}(2\varphi)} \int_0^{\partial\varphi_\star} \frac{(\sin^2 \varphi_\star - \sin^2 \varphi)^n}{\sqrt{D(\varphi', \varphi_\star)}} \, d\varphi' \right) = 2(2n + 1) \int_0^{\partial\varphi_\star} \frac{(\sin^2 \varphi_\star - \sin^2 \varphi)^{n+1}}{\sqrt{D(\varphi', \varphi_\star)}} \, d\varphi' + \left( \frac{\sin(2\varphi)}{\sin(2\varphi_\star)} - \frac{\partial\varphi_\star}{\partial\varphi_\star} \right) \frac{(\sin^2 \varphi_\star - \sin^2 \varphi)^n}{\sin^{2n}(2\varphi_\star) \sqrt{D(\varphi', \varphi_\star)}}.
\]

(55)

From equality (41), it follows that

\[
\frac{\partial\varphi_\star}{\partial\varphi_\star} = \frac{\sin(2\varphi)}{\sin(2\varphi_\star)} - \frac{\partial\varphi_\star}{\partial\varphi_\star} \frac{(\sin^2 \varphi_\star - \sin^2 \varphi)^n}{\sin^{2n}(2\varphi_\star) \sqrt{D(\varphi', \varphi_\star)}} \tag{56}\]

Using the last equality in (55) and dividing by \(\sin(2\varphi_\star)\), we obtain (53). \(\square\)

Derivatives of \(R_\Delta\). The following identities hold:

\[
\partial r_\star \frac{\sqrt{\sin^2 \varphi_\star \Delta\varphi - \sin^2 \theta}}{\sin(2\varphi_\star)} = -\frac{a}{2 \sin(2\varphi_\star)} \frac{\Delta\varphi}{(r^2 + 2a^2)^2 \Delta\varphi - a^2 \sin^2 \varphi_\star \Delta r}, \tag{57}\]

\[
\partial \Delta \frac{\sqrt{\sin^2 \varphi_\star \Delta\varphi - \sin^2 \theta}}{\sin(2\varphi_\star)} = 2 \left( \frac{\sin^2 \varphi_\star - \sin^2 \theta}{\sin^2(2\varphi_\star)} \frac{\sin^2 \varphi_\star \Delta\varphi - \sin^2 \theta}{\sin(2\varphi_\star)} \right) \frac{1}{2 \sin^2(2\varphi_\star)} \frac{\sin \varphi_\star}{\sin(2\varphi_\star)} \frac{\Delta \varphi - a^2 \sin^2 \varphi_\star}{(r^2 + 2a^2)^2 \Delta\varphi - a^2 \sin^2 \varphi_\star \Delta r} \times \left( \int_0^{\partial\varphi_\star} \frac{\sin^2 \varphi_\star - \sin^2 \theta'}{\sin^2 \varphi_\star \Delta\varphi - \sin^2 \theta'} \, d\varphi' \right) \frac{1}{2} \frac{a^2 \Delta r}{(r^2 + 2a^2)^2 \Delta\varphi - a^2 \sin^2 \varphi_\star \Delta r}. \tag{58}\]

Proof. From (40), we get (57). To start the proof of (58), note that

\[
\frac{1}{\sin(2\varphi_\star)} \frac{\sqrt{\sin^2 \varphi_\star \Delta\varphi - \sin^2 \theta}}{\sin(2\varphi_\star)} = -\frac{2 \cos(2\varphi_\star) \sqrt{D(\varphi, \varphi_\star)}}{\sin(2\varphi_\star)} + \frac{1}{2 \sin(2\varphi_\star)} \frac{\Delta \varphi}{\sqrt{D(\varphi, \varphi_\star)}} - \frac{1}{2} \frac{\sin(2\varphi)}{\sin(2\varphi_\star)} \frac{1}{\sqrt{D(\varphi, \varphi_\star)}} \frac{\partial\varphi_\star}{\partial\varphi_\star}.
\]
For any \( A \), the last expression is equal to
\[
\begin{align*}
= & \frac{1}{2 \sin^3(2\theta_*) \sqrt{D(\theta, \theta_*)}} \left( -4 \cos(2\theta_*) D(\theta, \theta_*) \\
+ & \sin^2(2\theta_*) \Delta_\theta - \sin^2(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \\
- & (A - 1) \sin^2(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \\
+ & \sin(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \sin(2\theta_*) \left( A \frac{\sin(2\theta)}{\sin(2\theta_*)} - \frac{\partial \theta}{\partial \theta_*} \right)
\end{align*}
\]
because the terms with \( A \) cancel out and the terms with \( \sin^2(2\theta) \) cancel out. The value of \( A \) will be chosen taking (41) into account, that is we choose \( A = \Delta_\theta \frac{(r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r} \). The reason we made a term with \( A - 1 \) appear is that such a term is proportional to \( \Delta_r \). In fact,
\[
A - 1 = \Delta_\theta \frac{(r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r} - 1 = - \frac{a^2 \Delta_r (\sin^2 \theta_* \Delta_\theta - \sin^2 \theta)}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r}.
\]
According to (34), we have that
\[
-4 \cos(2\theta_*) D(\theta, \theta_*) + \sin^2(2\theta_*) \Delta_\theta - \sin^2(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \\
= & -4 \cos(2\theta_*) D(\theta, \theta_*) + \sin^2(2\theta_*) \Delta_\theta + \sin(2\theta) \partial \theta_\theta D(\theta, \theta_*) \\
= & -4 \cos(2\theta_*) D(\theta, \theta_*) + 2(\cos(2\theta) + \cos(2\theta_*)) D(\theta, \theta_*) \\
= & 2(\cos(2\theta) - \cos(2\theta_*)) D(\theta, \theta_*) \\
= & 4 \left( \sin^2 \theta_* - \sin^2 \theta \right) D(\theta, \theta_*).
\]
Hence, the expression above is equal to
\[
\begin{align*}
= & \frac{1}{2 \sin^3(2\theta_*) \sqrt{D(\theta, \theta_*)}} \left( 4 \left( \sin^2 \theta_* - \sin^2 \theta \right) D(\theta, \theta_*) \\
- & (A - 1) \sin^2(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \\
- & \sin(2\theta) \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \sin(2\theta_*) \left( \frac{\partial \theta}{\partial \theta_*} - A \frac{\sin(2\theta)}{\sin(2\theta_*)} \right)
\end{align*}
\]
\[
= 2 \left( \sin^2 \theta_* - \sin^2 \theta \right) \frac{\sqrt{D(\theta, \theta_*)}}{\sin^2(2\theta_*)} \\
+ & \frac{1}{2} \frac{\sin^2(2\theta)}{\sin^2(2\theta_*)} \frac{\partial \theta}{\partial \theta_*} \left( 1 + \frac{\Lambda}{3} a^2 \sin^2 \theta_* \right) \frac{a \Delta_r \Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r} \\
- & \frac{1}{2} \frac{\sin(2\theta) \sqrt{\sin^2 \theta_* \Delta_\theta - \sin^2 \theta}}{\sin^2(2\theta_*)} \frac{a \Delta_r \Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r}.
\]
The final expression (58) is obtained using (41).

\[
\square
\]

**Derivatives of \( R_3 \).** The following identities hold:
\[
\frac{\partial}{\partial \theta_*} \left( \sin^2 \theta_* - \sin^2 \theta \right) \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \frac{a \Delta_r \Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r},
\]
(59)
One concludes by once again applying (41). 

**Proof.** From (40), we get (59). Arguing as in the proof of (58), we obtain

\[
\frac{1}{\sin(2\theta_*)} \frac{\partial}{\partial \theta_*} \left( \frac{\sin^2 \theta_* - \sin^2 \theta}{\sin^2(2\theta_*)} \right) = -\frac{4 \cos(2\theta_*) (\sin^2 \theta_* - \sin^2 \theta)}{\sin^4(2\theta_*)} + \frac{1}{\sin^4(2\theta_*)} \frac{\partial}{\partial \theta_*} \left( \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \right) - a^2 \Delta_r.
\]

Moreover

\[
\frac{1}{\sin(2\theta_*)} \frac{\partial}{\partial \theta_*} \left( \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \right) = \frac{2 \sin(2\theta)}{\sin^2(2\theta_*)} \left( \frac{\sin^2 \theta_*, \Delta_\theta - \sin^2 \theta}{\sin^2(2\theta_*)} \right) - \frac{a^2 \Delta_r}{\sin^2(2\theta_*)} \left( \frac{\partial}{\partial \theta_*} - \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \right) \frac{2 \left( (r^2 + a^2)^2 - a^2 \sin^2 \theta_*, \Delta_{\theta'} \right)^{3/2}}{dr'}.
\]

**Derivatives of** \(R_3\). The following identities hold:

\[
\frac{1}{\sin(2\theta_*)} \frac{\partial}{\partial \theta_*} \left( \frac{\sin(2\theta)}{\sin(2\theta_*)} \right) = \frac{\sqrt{\sin^2 \theta_*, \Delta_\theta - \sin^2 \theta}}{\sin(2\theta_*)} \left( \frac{2a \Delta_r, \Delta_\theta \cos(2\theta)}{((r^2 + a^2)^2 - a^2 \sin^2 \theta_*, \Delta_{\theta'})} \right)^{1/2}.
\]

**Proof.** From (40), we immediately get (60). Moreover

\[
\frac{1}{\sin(2\theta_*)} \frac{\partial}{\partial \theta_*} \left( \frac{\sin(2\theta)}{\sin(2\theta_*)} \right) = -\frac{2 \sin(2\theta) \cos(2\theta_*)}{\sin^3(2\theta_*)} + \frac{2 \cos(2\theta)}{\sin^2(2\theta_*)} \frac{\partial}{\partial \theta_*} \left( \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \right) - \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \left( \frac{\partial}{\partial \theta_*} - \frac{\sin(2\theta)}{\sin^2(2\theta_*)} \right) \frac{2 \left( (r^2 + a^2)^2 - a^2 \sin^2 \theta_*, \Delta_{\theta'} \right)^{3/2}}{dr'}.
\]
Equality (61) is obtained using (56).

Proof of Lemma 2.11.

Proof of (43). We have seen in (42) that \( \left| \frac{\partial r}{\partial r_*} \right| \lesssim |\Delta_r| \) (this is (43) with \( k_1 = 1 \) and \( k_2 = 0 \)) and that \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial r}{\partial r_*} \right| \lesssim |\Delta_r| \) (this is (43) with \( k_1 = 0 \) and \( k_2 = 1 \)). Thus, inequalities (43) follow from (i) (50) holds,

(ii) (51) holds,

(iii) \( \left| \frac{\partial r}{\partial r_*} \right| \lesssim |\Delta_r| \), which in particular implies that \( \left| \frac{\partial \Delta}{\partial r_*} \right| \lesssim |\Delta_r| \),

(iv) \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial r}{\partial r_*} \right| \lesssim |\Delta_r| \), which in particular implies that \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial \Delta}{\partial r_*} \right| \lesssim |\Delta_r| \),

(v) \( \left| \frac{\partial \theta}{\partial r_*} \right| \lesssim |\Delta_r| \sin(2\theta_*) \lesssim |\Delta_r| \lesssim 1 \),

(vi) \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial r}{\partial r_*} \right| \lesssim 1, \left| \frac{1}{\sin(2\theta_*)} \frac{\partial (\sin^2 \theta_*)}{\partial r_*} \right| \lesssim 1 \), \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial (\sin^2 \theta_*)}{\partial r_*} \right| = 1 \), \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial (\cos^2 \theta_*)}{\partial r_*} \right| \lesssim 1 \), \( \left| \frac{1}{\sin(2\theta_*)} \frac{\partial (\cos^2 \theta_*)}{\partial r_*} \right| = 2 \).

Proof of (44). It was shown in (42) that \( \left| \frac{\partial \theta}{\partial r_*} \right| \lesssim |\Delta_r| \sin(2\theta_*) \) (this is (44) with \( k_1 = 1 \)). Hence, inequalities (44) follow from (i), (iii) and (v) (with the bound 1).

Proof of (45). We have seen in (42) that \( \left| \frac{\partial \theta}{\partial r_*} \right| \lesssim 1 \) (this is (45) with \( k_1 = 0 \)). Thus, inequalities (45) are a consequence of (ii), (iv) and (vi).

Proof of (46). Inequalities (46) result from (45), (i), (iii) and (v) (with the bound \( |\Delta_r| \)).

Proof of (47) and (48). These inequalities ensue from (36), (43) to (46) and (49).

This completes the proof of Lemma 2.11.

Remark 2.14. Note that (47) and (48) hold for any smooth function of \( r_* \) and \( \sin^2 \theta_* \).

Lemma 2.15. The functions \( \frac{1}{\sin(2\theta_*)} \frac{\partial \theta}{\partial r_*}, \frac{\partial h}{\partial r_*}, \frac{\theta_\phi \phi}{\sin^2 \theta_* \sin(2\theta_*)}, \frac{\phi_\theta \phi}{\sin^2 \phi_*} \) and \( \frac{1}{\sin^2 \phi_*} (\theta_\phi \phi - \phi_\theta \phi) \) are smooth.

Proof. Since \( h(0, \theta_*) = 0 \), we have that \( \partial_\theta h(0, \theta_*) = 0 \). So, as \( \partial_r h = -B \), it follows that

\[
\partial_\theta h(r_*, \theta_*) = \int_0^{r_*} (\partial_r \partial_\theta h)(r', \theta_*) \, dr' = \int_0^{r_*} (\partial_\theta h) \, (r', \theta_*) \, dr'
\]

\[
= - \int_0^{r_*} ((\partial_\theta r)(\partial_\theta B) + (\partial_\theta \theta)(\partial_\theta B))(r', \theta_*) \, dr'
\]

\[
= - \sin(2\theta_*) \int_0^{r_*} \left( \frac{1}{\sin(2\theta_*)} \partial_\theta r \right) \, (\partial_\theta B) + (\partial_\theta \theta) \left( \frac{1}{\sin(2\theta_*)} \partial_\theta B \right) \, (r', \theta_*) \, dr'.
\]

Recall from (39) that \( \partial_\theta r \) has a factor \( \Delta_r \sin(2\theta_*) \) and observe that

\[
\frac{1}{\sin(2\theta_*)} \partial_\theta B = \frac{\Xi a^3 \Delta_r (r^2 + a^2) (1 - \frac{3}{2} r^2) - \Delta_r}{\mathcal{Y}^2}.
\]
This shows that \( \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} h \) has derivatives of all orders with respect to \( \partial_{r_*} \) and \( \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} \). Moreover, as \( b^{\phi_*} = 2B \), we obtain
\[
\left| \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} b^{\phi_*} \right| \lesssim |\Delta_r|
\]
and \( \hat{\theta}_{\theta_*} \) is smooth.

Note that \( \frac{\sin^2 \theta}{\sin^2 \theta_*} \) has derivatives of all orders with respect to \( r_* \) and \( \sin^2 \theta_* \). Indeed, we have
\[
\partial_{r_*} \left( \frac{\sin^2 \theta}{\sin^2 \theta_*} \right) = \frac{\sin(2\theta)}{\sin^2 \theta_*} \frac{\partial \theta}{\partial r_*}.
\]
As \( \frac{\partial \theta}{\partial r_*} \) has a factor \( \sin(2\theta_*) \), the right-hand side has a factor \( \sin(2\theta) \sin(2\theta_*) \). Taking (49) into account, differentiability at \( \theta_* \) equal to \( \frac{\pi}{2} \) is not a problem. Neither is there a problem at \( \theta_* \) equal to zero because
\[
\frac{\sin(2\theta) \sin(2\theta_*)}{\sin^2 \theta_*} = 4\frac{\sin(2\theta)}{\sin(2\theta_*)} (1 - \sin^2 \theta_*).
\]
Differentiability with respect to \( \sin^2 \theta_* \) holds at \( \theta_* \) equal to zero because
\[
\frac{\sin^2 \theta}{\sin^2 \theta_*} = \left( \frac{\sin(2\theta)}{\sin(2\theta_*)} \right)^2 \frac{1 - \sin^2 \theta_*}{1 - \sin^2 \theta}
\]
and differentiability holds at \( \theta_* \) equal to \( \frac{\pi}{2} \) because
\[
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*} \left( \frac{\sin^2 \theta}{\sin^2 \theta_*} \right) = \frac{\sin(2\theta)}{\sin(2\theta_*)} \frac{\partial \theta}{\partial r_*} \frac{1}{\sin^2 \theta_*} \frac{\partial \theta}{\partial r_*} - \frac{\sin^2 \theta}{\sin^2 \theta_*} \frac{1}{\sin^2 \theta_*} \frac{1}{\sin \theta_*}.
\]
So, we see that
\[
\hat{\theta}_{\theta_*} \quad \text{and} \quad \hat{\phi}_{\phi_*}
\]
are smooth. Using (12) and (36), we get
\[
\frac{1}{\sin^2 \theta_*} \left( \hat{\theta}_{\theta_*} - \hat{\phi}_{\phi_*} \right) \quad \text{(62)}
\]

The expression inside the square parenthesis is a polynomial \( p \) in \( \alpha := \sin^2 \theta \) and \( \beta := \sin^2 \theta_* \) satisfying \( p(0, 0) = 0 \). So, \( p(\alpha, \beta) = D_1 p(0, 0) \alpha + D_2 p(0, 0) \beta + \) higher order terms. Thus, it is clear that \( \frac{p(\sin^2 \theta, \sin^2 \theta_*)}{\sin^2 \theta_*} \) is smooth, and so it is clear that (62) is smooth at zero. Differentiability at \( \frac{\pi}{2} \) is not an issue since
\[
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*} \frac{1}{\sin^2 \theta_*} = -\frac{1}{\sin^4 \theta_*}.
\]

**Remark 2.16.** The Christoffel symbols of the metric \( \hat{g} \) are bounded with the exception of \( \Gamma^\phi_{\theta_*, \theta_*} \) which blows up precisely like \( \frac{1}{\sin \theta_*} \).
Proof. Because $\dot{g}$ behaves like the round metric on $S^2$, $g^{-1}$ behaves like the inverse of the round metric on $S^2$, in that $g^0 \circ \dot{g}$ blows up at $\theta_*$ like $\frac{1}{\sin^2 \theta_*}$. Hence, the Christoffel symbols of the metric $g$ also behave like the ones of the round metric on $S^2$, namely, they are all smooth with the exception of $\Gamma^\phi_{\theta, \phi_*}$,

$$\Gamma^\phi_{\theta, \phi_*} = \frac{1}{2} g^{\phi \alpha} (\partial_\phi g_{\alpha \phi} + \partial_\alpha g_{\phi \phi} - \partial_\alpha g_{\phi \phi})$$

= smooth $+ \frac{1}{2} g^{\phi \alpha} (\partial_\phi g_{\alpha \phi} + \partial_\alpha g_{\phi \phi} - \partial_\alpha g_{\phi \phi})$

= smooth $\frac{1}{2} g^{\phi \alpha} \partial_\phi \dot{g}_{\phi \phi} +$ smooth

= $\frac{1}{2} \Xi^2 \rho^2 \frac{1}{\sin^2 \theta_*} \partial_\phi \left( \sin^2 \theta \Phi \right) +$ smooth

= $\frac{1}{2} \sin^2 \theta \partial_\phi ((\sin^2 \theta) +$ smooth

= $\cos \theta \partial_\theta +$ smooth

= $\Delta g (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta g - (r^2 + a^2)^2 \Delta g \frac{1}{\sin^2 \theta} \sin \theta +$ smooth,

which blows up precisely like $\frac{1}{\sin \theta_*}$. \qed

2.2.6 Regularity at $\theta_* = 0$ and $\theta_* = \pi$

Note that the coordinates that are constructed are not just locally defined but are well-defined globally on a manifold diffeomorphic to $\mathbb{R}^2 \times S^2$.

Regularity of the metric. We check the regularity of the metric at $\theta_* = 0$ using coordinates

$$x = \sin \theta_* \cos \phi_*$$
$$y = \cos \theta_* \sin \phi_*$$

(63)

Lemma 2.17. The metric is smooth at $\theta_* = 0$ and

$$\lim_{\theta_* \to 0} g = -2\Omega^2 (du \otimes dv + dv \otimes du) + \frac{r^2 + a^2}{\Xi} \left( \frac{\partial \theta}{\partial \theta_*} \right)^2 (r_*, 0) (dx \otimes dx + dy \otimes dy).$$

(64)

Proof. The relations

$$d\theta_* = \frac{\cos \phi_*}{\cos \theta_*} dx + \frac{\sin \phi_*}{\cos \theta_*} dy,$$
$$d\phi_* = -\frac{\sin \phi_*}{\sin \theta_*} dx + \frac{\cos \phi_*}{\sin \theta_*} dy$$

imply that

$$\dot{g}_{\theta, \theta_*} = \frac{\cos \phi_*}{\cos \theta_*} dx \otimes dx + \frac{\sin \phi_*}{\cos \theta_*} dy \otimes dy + \frac{\sin (2\phi_*)}{2 \sin (2\theta_*)} dy \otimes dx + \frac{\sin (2\phi_*)}{2 \cos^2 \theta_*} dy \otimes dy$$

$$+ \frac{\sin (2\phi_*)}{2 \sin (2\theta_*)} dx \otimes dx + \frac{\cos (2\phi_*)}{2 \sin (2\theta_*)} dx \otimes dy + \frac{\cos (2\phi_*)}{2 \sin (2\theta_*)} dy \otimes dx + \frac{\cos (2\phi_*)}{2 \cos^2 \theta_*} dy \otimes dy$$

Using (12), (21), (23) and (36), we have

$$\lim_{\theta_* \to 0} \dot{g}_{\theta, \theta_*} = \frac{r^2 + a^2}{\Xi} \left( \frac{\partial \theta}{\partial \theta_*} \right)^2 (r_*, 0),$$
$$\lim_{\theta_* \to 0} \dot{g}_{\phi, \phi_*} = \frac{r^2 + a^2}{\Xi} \left( \frac{\partial \theta}{\partial \theta_*} \right)^2 (r_*, 0).$$
As, by (22),

$$\lim_{\theta_0 \to 0} \frac{g_{\text{ext}}}{\sin(2\theta_0)} = 0,$$

we obtain

$$\lim_{\theta_0 \to 0} g = \frac{r^2 + a^2}{\Xi} \left( \frac{\partial \theta}{\partial \theta_0} \right)^2 (r_*, 0) \left( dx \otimes dx + dy \otimes dy \right).$$

The trigonometric functions of \( \phi_* \), which would make the metric discontinuous at \( \theta_* = 0 \), have disappeared. Moreover, since

$$\lim_{\theta_* \to 0} b_{\phi_*} \theta_* \, d\theta_* \, dv = 0, \quad \lim_{\theta_* \to 0} b_{\phi_*} \phi_* \, dv \otimes d\theta_* = 0$$

and

$$\lim_{\theta_* \to 0} g_{\phi_*\phi_*} = 0,$$

we conclude that (64) holds. This shows that the metric is continuous at \( \theta_* = 0 \). Lemma 2.15 implies that the extension of the metric is smooth.

**Calculation of \( \partial_r \theta_0(r_*, 0) \).**

**Lemma 2.18.** We have that

$$\frac{\partial \theta}{\partial \theta_*}(r_*, 0) = \frac{1}{\sin \theta_0(r_0)} = \sqrt{-1} \sin \left( \arcsin \left( \frac{1}{\sqrt{-1}} \right) + \arctan \left( \frac{r}{a} \right) - \arctan \left( \frac{r + a}{a} \right) \right). \quad (65)$$

**Proof.** Using the definition of \( F \) in (9), the equation \( F(r, \theta, \theta_0(r, \theta)) = 0 \) can be written as

$$\int_\phi^{\phi'} \frac{d\phi'}{P(\phi', \theta_0)} = \int_r^{r_+} \frac{dr'}{Q(r', \theta_0)}. \quad (66)$$

We will take the limit of both sides of (66) as \( \theta \) goes to 0. As \( \lim_{\theta_0 \to 0} \theta_0(r; \theta) = 0 \), uniformly in \( r \),

$$\lim_{\theta_0 \to 0} \int_r^{r_+} \frac{dr'}{Q(r', \theta_0)} = \int_r^{r_+} \frac{dr'}{(r')^2 + a^2} = \frac{1}{a} \arctan \left( \frac{r + a}{a} \right) - \frac{1}{a} \arctan \left( \frac{r}{a} \right).$$

Using the substitution \( s = \frac{\sin \theta_0}{\sin \theta_0(r_0)} \),

$$\int_0^{\phi_0} \frac{d\theta}{P(\theta', \theta_0)} = \frac{1}{a} \int_0^{\sin \theta_0(r_0)} \frac{1}{\cos \theta_0 \sqrt{(1 + \frac{4}{a^2} \cos^2 \theta_0) - s^2}} \, ds,$$

with \( \cos \theta_0 = \sqrt{1 - s^2 \sin^2 \theta_0} \). The last integrand converges uniformly to

$$\frac{1}{\sqrt{-1} - s^2}$$

as \( \theta \) goes to zero. Thus, we get

$$\lim_{\theta_0 \to 0} \int_\phi^{\phi'} \frac{d\theta}{P(\theta', \theta_0)} = \frac{1}{a} \int_1^{\sin \theta_0(r_0)} \frac{ds}{\sqrt{(1 + \frac{4}{a^2}) - s^2}} = \frac{1}{a} \arcsin \left( \frac{s}{\sqrt{3}} \right)^{1/2} \left( \frac{\sin \theta_0(r, 0)}{\sin \theta_0(r_0)} \right) = \frac{1}{a} \arcsin \left( \frac{1}{\sqrt{3}} \right) - \frac{1}{a} \arcsin \left( \frac{1}{\sqrt{3} \sin \theta_0(r_0)} \right).$$

So, from (66) we conclude that

$$\arcsin \left( \frac{1}{\sqrt{3} \sin \theta_0(r, 0)} \right) = \arcsin \left( \frac{1}{\sqrt{3}} \right) + \arctan \left( \frac{r}{a} \right) - \arctan \left( \frac{r + a}{a} \right).$$

We know that the right-hand side of this equality is positive because we guaranteed that (31) holds. And the right-hand side is obviously smaller than \( \frac{\pi}{2} \). Therefore, we have

$$\frac{\partial \theta_*}{\partial \theta}(r_*, 0) = \frac{1}{\sqrt{3} \csc \left( \arcsin \left( \frac{1}{\sqrt{3}} \right) + \arctan \left( \frac{r}{a} \right) - \arctan \left( \frac{r + a}{a} \right) \right).$$

This is strictly greater than 1 (and goes to 1 as \( r \to r_* \)). According to (6) and (14), the derivative of the map \( (r, \theta) \mapsto (r_*, \theta_0) \) at \( (r, 0) \) is represented by the matrix

$$\begin{bmatrix} \frac{r^2 + a^2}{\Delta_r} & 0 \\ 0 & \frac{\partial \theta}{\partial \theta_0}(r, 0) \end{bmatrix}.$$

Equality (65) follows. □
Regularity of functions at the poles. Using the change of coordinates (63), the variable \( \theta_* \) is written in terms of \( x \) and \( y \) as

\[
\theta_* = \arcsin \sqrt{x^2 + y^2}.
\]

Given a function \( f \) that transforms pairs \((r_*, \theta_*)\), we want to study the differentiability of

\[
(r_*, x, y) \mapsto f \left( r_*, \arcsin \sqrt{x^2 + y^2} \right).
\]

Remark 2.19. Let \( f : (\infty, +\infty) \times [0, \pi] \to \mathbb{R} \) be \( C^1 \) such that \( \frac{1}{\sin(2\theta_*)} (\partial_{\theta_*} f) \) has a limit when \( \theta_* = 0 \). Then \( \hat{f} \) is \( C^1 \).

Proof. The derivative of \( \hat{f} \) with respect to \( x \) is

\[
\partial_x \hat{f} = (\partial_{\theta_*} f) \frac{1}{\cos \theta_* \sin \theta_*} \frac{x}{\sqrt{1 - (x^2 + y^2)^2}} \frac{1}{\sqrt{x^2 + y^2}}.
\]  \hspace{1cm} (67)

We observe that, although the map \((x, y) \mapsto \sqrt{x^2 + y^2} \) is not differentiable at the origin, when \( x = y = 0 \) the quotient \( \frac{x}{\sqrt{x^2 + y^2}} \) in (67) is either +1 or -1, according to the calculation of a right or a left derivative. If we assume that \( \frac{1}{\sin(2\theta_*)} (\partial_{\theta_*} f) \) has a limit at \((r_0, 0, 0)\), then \( \partial_{\theta_*} f \) has a finite limit at \((r_0, 0, 0)\), so there is no indetermination in (67), and

\[
\partial_x \hat{f}(r_*, 0, 0) = 0.
\]

The function \( \hat{f} \) has continuous partial derivatives with respect to \( r_*, x \) and \( y \) in a neighborhood of each point \((r_0, 0, 0)\), and so it is differentiable. \( \square \)

Note that we can write (67) as

\[
\partial_x \hat{f} = \frac{1}{\cos \theta_* \sin \theta_*} \frac{x}{\sqrt{1 - (x^2 + y^2)^2}} \frac{1}{\sqrt{x^2 + y^2}} (\partial_{\theta_*} f).
\]  \hspace{1cm} (68)

This is the derivative of \( f \) with respect to \( \sin^2 \theta_* = x^2 + y^2 \) multiplied by the derivative of \( \sin^2 \theta_* \) with respect to \( x \) \). Expression (68) shows that if the quotient \( \frac{1}{\sin(2\theta_*)} (\partial_{\theta_*} f) \) were unbounded, then one might have problems with the differentiability when \((x, y) = (0, 0)\). For example, if this quotient were to behave like \( \frac{1}{\sin \theta_*} \) around \((r_0, 0, 0)\), then \( \partial_x \hat{f} \) would behave like \( \cot \theta_* \), so that \( \partial_x \hat{f}(r_0, 0, 0) \) would not exist.

Remark 2.20. Let \( f : (\infty, +\infty) \times [0, \pi] \to \mathbb{R} \) be \( C^\infty \) in the sense of Remark 2.12. Then \( \hat{f} \) is \( C^\infty \).

Proof. It is clear that \( \partial_{r_*} \hat{f}, \partial_x \partial_{r_*} \hat{f}, \partial_y \partial_{r_*} \hat{f} \) are continuous. The continuity of \( \partial^2_{r_*} \hat{f} \) is a consequence of

\[
\partial^2_{r_*} \hat{f} = \partial_x \left( 4 \frac{x}{\sin(2\theta_*)} \frac{1}{\sqrt{x^2 + y^2}} \partial_{\theta_*} f \right) = 2 \frac{x}{\sin(2\theta_*)} \partial_{\theta_*} f + 4 x^2 \left( \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} f \right)^2.
\]

The continuity of \( \partial_x \partial_y \hat{f} \) and \( \partial^2_y \hat{f} \) follow in the same way. So \( \hat{f} \) is \( C^2 \). One proves by induction that \( \hat{f} \) is \( C^\infty \). \( \square \)

Remark 2.21. A differentiable function such that \( f(r_*, \theta_*) = f(r_*, \pi - \theta_*) \) satisfies \( \partial_{\theta_*} f(r_*, \frac{\pi}{2}) = 0 \). The quotient \( \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} f \) has a finite limit at \((r_*, \frac{\pi}{2})\), provided that the numerator is analytic.

Indeed, both the numerator and the denominator vanish at \((r_*, \frac{\pi}{2})\) and the denominator has a first order zero there.

Remark 2.22. Suppose that \( f : (\infty, +\infty) \times [0, \pi] \times S^1 \to \mathbb{R} \) is smooth and define

\[
f(r_*, x, y) = f \left( r_*, \arcsin \sqrt{x^2 + y^2}, \arg(x + iy) \right),
\]

where \( \phi_* = \arg(x + iy) = \arctan \frac{y}{x} \) for \( x > 0 \), and otherwise \( \arg(x + iy) = \arctan \frac{y}{x} \) with an appropriate constant added. Then

\[
\begin{align*}
\partial_x \hat{f} &= \frac{x}{\sqrt{1 - (x^2 + y^2)^2} \sqrt{x^2 + y^2}} \partial_{\theta_*} f - \frac{y}{x^2 + y^2} \partial_{\theta_*} f, \\
&= \frac{\cos \phi_*}{\cos \theta_*} \partial_{\theta_*} f - \frac{\sin \phi_*}{\sin \theta_*} \partial_{\theta_*} f, \\
\partial_y \hat{f} &= \frac{y}{\sqrt{1 - (x^2 + y^2)^2} \sqrt{x^2 + y^2}} \partial_{\theta_*} f + \frac{x}{x^2 + y^2} \partial_{\theta_*} f, \\
&= \frac{\sin \phi_*}{\cos \theta_*} \partial_{\theta_*} f + \frac{\cos \phi_*}{\sin \theta_*} \partial_{\theta_*} f.
\end{align*}
\]
Regularity of the change of coordinates \((t, r, \theta, \phi) \mapsto (t, r_*, \theta_*, \phi)\). Define
\[
\hat{x} = \sin \hat{\theta}_* \cos \phi, \quad \hat{y} = \sin \hat{\theta}_* \sin \phi,
\]
\[
\check{x} = \sin \check{\theta} \cos \phi, \quad \check{y} = \sin \check{\theta} \sin \phi.
\]

(i) The map \((t, r_*, \hat{x}, \hat{y}) \mapsto (t, r, \check{x}, \check{y})\) is smooth. The derivative \(\partial_{r*} r\) is given by (38). Moreover, using (68), we get
\[
\partial_{\hat{x}} r = 2\check{x} \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} r, \quad \text{(see (39) for } \partial_{\theta_*} r),
\]
\[
\partial_{\hat{y}} r = 2\check{y} \frac{1}{\sin(2\theta_*)} \partial_{\theta_*} r,
\]
\[
\partial_{\hat{r}_*} \sin \theta = \cos \theta \frac{\partial \theta}{\partial r_*} = 2 \frac{\sin(2\theta_*)}{\sin(2\theta)} \sin \theta \cos^2 \theta \left( \frac{1}{\sin(2\theta_*)} \partial_{r_*} \theta \right), \quad \text{(see (40) for } \partial_{r_*} \theta).
\]

Applying Remark 2.22, we obtain
\[
\partial_{\hat{x}} \hat{x} = \frac{\cos \theta}{\cos \theta_*} \cos^2 \phi \partial_{\theta_*} \theta + \frac{\sin \theta}{\sin \theta_*} \sin^2 \phi,
\]
\[
\partial_{\hat{y}} \hat{x} = \frac{\cos \theta}{\cos \theta_*} \sin \phi \cos \phi \partial_{\theta_*} \theta - \frac{\sin \theta}{\sin \theta_*} \sin \phi \cos \phi,
\]
\[
\partial_{\hat{x}} \hat{y} = \frac{\cos \theta}{\cos \theta_*} \sin \phi \cos \phi \partial_{\theta_*} \theta - \frac{\sin \theta}{\sin \theta_*} \sin \phi \cos \phi,
\]
\[
\partial_{\hat{y}} \hat{y} = \frac{\cos \theta}{\cos \theta_*} \sin^2 \phi \partial_{\theta_*} \theta + \frac{\sin \theta}{\sin \theta_*} \sin \phi \cos \phi.
\]

Notice that when \(\theta = 0\), we have
\[
\partial_{\hat{x}} \hat{x} = \partial_{\theta_*} \theta, \quad \partial_{\hat{y}} \hat{x} = 0, \quad \partial_{\hat{r}_*} \hat{y} = 0, \quad \partial_{\hat{y}} \hat{y} = \partial_{\theta_*} \theta.
\]

The quotient \(\frac{\cos \theta}{\cos \theta_*}\) is smooth because
\[
\frac{1}{\sin(2\theta_*)} \partial_{\theta_*} \left( \frac{\cos \theta}{\cos \theta_*} \right) = - \frac{1}{2 \sin(2\theta_*)} \cos \theta_* \frac{\partial \theta}{\partial \theta_*} + \frac{1}{2 \cos \theta_* \sin^2 \theta_*}.
\] (69)

The quotient \(\frac{\sin \theta}{\sin \theta_*}\) is smooth because
\[
\frac{\sin \theta}{\sin \theta_*} = \frac{\sin(2\theta_*) \cos \theta_*}{\sin(2\theta) \cos \theta}.
\]

Thus \(r, \hat{x}\) and \(\hat{y}\) are \(C^\infty\) functions of \(r_*, \hat{x}\) and \(\hat{y}\).

(ii) The map \((t, r, \hat{x}, \hat{y}) \mapsto (t, r_*, \check{x}, \check{y})\) is smooth. Recall that \(\partial_{r_*} r_*\) is given in (6), and
\[
\partial_{r_*} \sin \theta_* = \cos \theta_* \partial_{r_*} \theta_* = \cos \theta_* \frac{1}{GQ} = 2 \frac{\sin(2\theta)}{\sin(2\theta_*)} \sin \theta_* \cos^2 \theta_* \left( \frac{1}{\sin(2\theta)} \frac{1}{GQ} \right),
\]
as \(\partial_{r_*} \theta_*\) is given in (14). Inequalities (37) yield
\[
\lim_{\theta \to 0} \frac{1}{G(r, \theta, \theta_*(r, \theta))} = 0.
\]

One other consequence of (6) and (14) is
\[
\partial_{\hat{x}} r_* = 2\hat{x} \frac{1}{\sin(2\theta)} \partial_{\theta_*} r_* = 2a \frac{\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}}{\Delta \theta \sin(2\theta)},
\]
\[
\partial_{\hat{y}} r_* = 2\hat{y} \frac{1}{\sin(2\theta)} \partial_{\theta_*} r_* = 2a \frac{\sqrt{\sin^2 \theta_* \Delta \theta - \sin^2 \theta}}{\Delta \theta \sin(2\theta)}.
\]

The formulas for \(\partial_{\hat{x}} x, \partial_{\hat{y}} x, \partial_{\hat{x}} y\), and \(\partial_{\hat{y}} y\), are similar to the ones for \(\partial_{\hat{x}} x, \partial_{\hat{y}} x, \partial_{\hat{x}} y\), and \(\partial_{\hat{y}} y\) (interchange \(\theta\) and \(\theta_*\)). Recall that \(\partial_{\theta} \theta_* = -\frac{1}{GQ}\). It follows that \(r_*, \hat{x}\) and \(\hat{y}\) are \(C^\infty\) functions of \(r, \hat{x}\) and \(\hat{y}\).
Regularity of the change of coordinates \((t, r_*, \theta_*, \phi) \mapsto (t, r_*, \theta_*, \phi_*)\). Recall that

\[ \phi_* = \phi - h(r_*, \theta_*). \]

The fact that both \((t, r_*, \theta_*, \phi) \mapsto (t, r_*, \theta_*, \phi_*)\) and \((t, r_*, \theta_*, \phi) \mapsto (t, r_*, \theta_*, \phi)\) are \(C^\infty\) is a simple consequence of (17), which gives \(\partial_r h\), and Lemma 2.15, which gives \(\frac{\sin(2\theta_*)}{2\sin(2\theta_*)} \partial_{\theta_*} h\).

We remark that

\[
\begin{align*}
\partial_x &= \frac{\cos \phi_*}{\cos \theta_*} \partial_{\theta_*} - \frac{\sin \phi_*}{\sin \theta_*} \partial_{\phi_*}, \\
\partial_y &= \frac{\sin \phi_*}{\cos \theta_*} \partial_{\theta_*} + \frac{\cos \phi_*}{\sin \theta_*} \partial_{\phi_*}.
\end{align*}
\]

(70)

(71)

Hence, the differentiability of \((t, r_*, x, y) \mapsto (t, r_*, \hat{x}, \hat{y})\) follows from

\[
\begin{align*}
\partial_x \hat{x} &= -\sin \theta_* \sin \phi \partial_{r_*} h, \\
\partial_x \hat{y} &= \cos(h(r_*, \theta_*)) - \frac{\sin \theta_*}{\cos \theta_*} \cos \phi_* \sin \phi \partial_{\theta_*} h, \\
\partial_y \hat{x} &= -\sin(h(r_*, \theta_*)) - \frac{\sin \theta_*}{\cos \theta_*} \sin \phi_* \sin \phi \partial_{\theta_*} h, \\
\partial_y \hat{y} &= \sin(h(r_*, \theta_*)) + \frac{\sin \theta_*}{\cos \theta_*} \cos \phi_* \cos \phi \partial_{\theta_*} h.
\end{align*}
\]

Note that these are smooth functions and that at \(\theta_* = 0\) they are independent of \(\phi\) and \(\phi_*\). The differentiability of \((t, r_*, \hat{x}, \hat{y}) \mapsto (t, r_*, x, y)\) follows in a similar way.

Regularity of the spheres given by the intersection of hypersurfaces \(u = \text{constant}\) and \(v = \text{constant}\). It is obvious that the two atlases \(\{ (t, r_*, \theta_*, \phi) \}, (t, r_*, x, y) \}\) and \(A_{\text{DN}} = \{ (u, v, \theta_*, \phi_*), (u, v, x, y) \}\) are compatible. So, combining the conclusions of the previous paragraphs, the two atlases \(A_{\text{ml}} = \{ (t, r, \theta, \phi), (t, r, \hat{x}, \hat{y}) \}\) and \(A_{\text{DN}}\) are compatible. The spheres given by the intersection of hypersurfaces \(u = \text{constant}\) and \(v = \text{constant}\) are smooth in the atlas \(A_{\text{DN}}\) by definition. Therefore, they are smooth in the atlas \(A_{\text{ml}}\). This proves

Theorem 2.23. The topological two-spheres given by the intersection of hypersurfaces \(u = \text{constant}\) and \(v = \text{constant}\) are \(C^\infty\) in the Broyer–Lindquist coordinates.

2.3 Coordinates at the horizons

2.3.1 The decay of \(\Omega^2\) at the horizons

Recall that the surface gravities of the horizons are given by

\[
\begin{align*}
\kappa_- &= -\frac{\Lambda (r_- - r_*)(r_- - r_+)(r_- - r_n)}{6 \left( r_-^2 + a^2 \right)}, \\
\kappa_+ &= -\frac{\Lambda (r_+ - r_*)(r_+ - r_-)(r_+ - r_n)}{6 \left( r_+^2 + a^2 \right)}, \\
\kappa_c &= -\frac{\Lambda (r_c - r_*)(r_c - r_-)(r_c - r_n)}{6 \left( r_c^2 + a^2 \right)}
\end{align*}
\]

(72)

(confirm the formula for \(\kappa_-\) with Example A.3).

Lemma 2.24. Given \(C_R \in \mathbb{R}\), there exist \(c, C > 0\) such that

\[
\begin{align*}
ce^{2\kappa_- r_*} &\leq \Omega^2 \leq Ce^{2\kappa_- r_*} \text{ for } r_* \geq C_R, \\
ce^{2\kappa_+ r_*} &\leq \Omega^2 \leq Ce^{2\kappa_+ r_*} \text{ for } r_* \leq C_R.
\end{align*}
\]

(73)

(74)

Proof. The formula

\[
\frac{\partial r}{\partial r_*} = \frac{\Delta \Delta \sqrt{(r^2 + a^2)^2 - a^2 \sin^2 \theta_* \Delta}}{\sqrt{r^2 + a^2}}
\]
According to Remark 2.1, there exists $C > 0$ and $(r_*)_0$ such that for $r_* \geq (r_*)_0$ we have

$$(1 - C(r_* - r_0))(1 + O(r_* - r)) \leq 1 \leq (1 + C(r_* - r_0))(1 + O(r_* - r)).$$

So, for $r \geq (r_*)_0$ we obtain

$$\frac{1}{r - r_*} - C \leq \frac{1}{(r - r_0)} \frac{1}{1 + O(r_* - r_0)} \leq \frac{1}{r - r_*} + C.$$

Remembering that $\kappa_- < 0$, it follows that

$$\left(\frac{1}{r - r_*} + C\right) \frac{\partial r}{\partial r_*} \leq \frac{1}{r - r_0} \frac{1}{1 + O(r_* - r_0)} \frac{\partial r}{\partial r_*} = 2\kappa_- \leq \left(\frac{1}{r - r_*} - C\right) \frac{\partial r}{\partial r_*}.$$

Integrating from $(r_*)_0$ to $r_*$ yields

$$\log (r(r_*, \theta_*)) - r_* - \log ((r_*)_0, \theta_*) - r_0 + C (r(r_*, \theta_*)) - r((r_*)_0, \theta_*)) \leq 2\kappa_- (r_* - (r_*)_0) \leq \log (r(r_*, \theta_*)) - r_* - \log ((r_*)_0, \theta_*) - r_0 - C (r(r_*, \theta_*)) - r((r_*)_0, \theta_*)).$$

These inequalities can be rearranged to

$$2\kappa_- (r_* - (r_*)_0) + \log (r((r_*)_0, \theta_*)) - r_* - \log ((r_*)_0, \theta_*) - r_0 + C (r(r_*, \theta_*)) - r((r_*)_0, \theta_*)) \leq 2\kappa_- (r_* - (r_*)_0) + \log (r((r_*)_0, \theta_*)) - r_* - \log ((r_*)_0, \theta_*) - r_0 - C (r(r_*, \theta_*)) - r((r_*)_0, \theta_*)).$$

This shows that there exists $D > 0$ and $(r_*)_0$ such that

$$2\kappa_- (r_* - (r_*)_0) - D \leq \log (r(r_*, \theta_*)) - r_* - \log ((r_*)_0, \theta_*) - r_0 - 2\kappa_- (r_* - (r_*)_0) + D$$

for $r_* \geq (r_*)_0$. Thus, there exist $c, C > 0$, such that $r_* \geq (r_*)_0$ implies

$$ce^{2\kappa_- r_*} \leq r - r_* \leq Ce^{2\kappa_- r_*}.$$  

Let $C_R$ be a real number. Decreasing $c$ and increasing $C$ if necessary, one sees that

$$ce^{2\kappa_- r_*} \leq r - r_* \leq Ce^{2\kappa_- r_*}$$

for $C_R \leq r_* \leq (r_*)_0$. Combining the previous two inequalities, they hold for $r_* \geq C_R$. Moreover, in this region,

$$c(r - r_0) - \Delta_r \leq C(r - r_*),$$
for other appropriate constants $c, C > 0$. Hence, given $C_R \in \mathbb{R}$, there exist $c, C > 0$ such that
\[ ce^{2\kappa - r_*} \leq -\Delta_r \leq Ce^{2\kappa - r_*}. \] (75)
for $r_* \geq C_R$. As $\Omega^2$ is comparable to $|\Delta_r|$, we conclude that $\Omega^2$ is comparable to $e^{2\kappa - r_*}$ in the region $r_* \geq C_R$. This proves (73). The proof of (74) is analogous.

**Remark 2.25.** The function $\Omega^2$ given in (18) is obviously smooth on the spheres where $u$ and $v$ are simultaneously constant.

### 2.3.2 Coordinates at the Cauchy horizon

Let us recall how one may define coordinates to cover the Cauchy horizon. We consider a new smooth coordinate $v_{CH^+}(v)$, with positive derivative, equal to $v$ for $v \leq -1$, satisfying $v_{CH^+} \to 0$ as $v \to +\infty$, and satisfying
\[ dv = e^{-2\kappa - v} dv_{CH^+} \] (76)
for $v \geq 0$. Moreover, we define
\[ \phi_* = \phi - b^{\phi*}|_{r=r_-}v. \]

**Remark 2.26.** $v_{CH^+}$ is also a smooth function of $v$, and so the change of coordinates $(v, \phi_*) \leftrightarrow (v_{CH^+}, \phi_*, \mathcal{C}_H^+)$ is smooth.

From (19), we see that
\[ b^{\phi*}|_{r=r_-} = 2 \frac{\Xi a((r^2 + a^2)\Delta_\theta - \Delta_r)}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r} \bigg|_{r=r_-} = \frac{2a(\Xi)}{r^2 + a^2}. \]

For $v \geq 0$, the differentials of $\phi_*$ and $\phi_*, \mathcal{C}_H^+$ are related by
\[ d\phi_* = d\phi_*, \mathcal{C}_H^+ + b^{\phi*}|_{r=r_-} e^{-2\kappa - v} dv_{CH^+}. \]

For $v \geq 0$, the expression of the metric (20) in the new coordinates is
\[ g = -2\Omega^2_{CH^+} (du \otimes dv_{CH^+} + dv_{CH^+} \otimes du) \]
\[ + \mathcal{g}_{\theta, \theta_*} d\theta_* \otimes d\theta_* + \mathcal{g}_{\theta, \phi_*} d\theta_* \otimes (d\phi_*, \mathcal{C}_H^+ - b^{\phi*}_{CH^+} dv_{CH^+}) + \mathcal{g}_{\theta, \phi_*} (d\phi_*, \mathcal{C}_H^+ - b^{\phi*}_{CH^+} dv_{CH^+}) \otimes d\theta_* \]
\[ + \mathcal{g}_{\phi, \phi_*} (d\phi_*, \mathcal{C}_H^+ - b^{\phi*}_{CH^+} dv_{CH^+}) \otimes (d\phi_*, \mathcal{C}_H^+ - b^{\phi*}_{CH^+} dv_{CH^+}), \]
with
\[ \Omega^2_{CH^+} = \Omega^2 e^{-2\kappa - v}, \]
\[ b^{\phi*}_{CH^+} = \left( b^{\phi*} - b^{\phi_*}|_{r=r_-} \right) e^{-2\kappa - v}. \] (77)

Henceforth we will assume that we are working in the region $v \geq 0$, our formulas will always refer to this region. To estimate $b^{\phi*}_{CH^+}$ we calculate
\[ b^{\phi*} - b^{\phi_*}|_{r=r_-} = -2a(\Xi) \frac{(r^2 + a^2 \cos^2 \theta)\Delta_\theta + (r^2 + a^2)(r + r_*)(r - r_-)\Delta_\theta}{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r}. \]

Using (11) and (75), we estimate
\[ \left| b^{\phi*} - b^{\phi_*}|_{r=r_-} \right| \lesssim e^{2\kappa_- (u+v)}, \]
for $u + v \geq C_R$. Thus, inequalities (73) implies the following bounds for $\Omega^2_{CH^+}$ and $b^{\phi*}_{CH^+}$, when $u + v \geq C_R$:
\[ e^{2\kappa_- u} \lesssim \Omega^2_{CH^+} \lesssim e^{2\kappa_- u}, \]
\[ |b^{\phi*}_{CH^+}| \lesssim e^{2\kappa_- u}. \]

For a general function $f$, we have
\[ f(u, v, \theta_*, \phi_*) = f \left( u, v, \theta_*, \phi_*, \mathcal{C}_H^+ + b^{\phi*}|_{r=r_-} v \right) = f(u, v, \theta_*, \phi_*, \mathcal{C}_H^+). \]
where $\tilde{f}$ is the function $f$ written in the coordinates $(u, \tilde{v}, \theta_*, \phi_*, \mathcal{C}_H^+)$ and $\tilde{v} = v$. So
\[ \partial_\tilde{v} = \partial_v + b^{\phi_*} \big|_{r=r_-} \partial_{\phi_*}. \] (78)

We define
\[ \sigma = e^{-2\kappa_- v}. \] (79)

This, (76) and $\partial_{\phi_*} = \partial_{\phi_*, H^+}$ imply that
\[
\partial_{\phi_*, H^+} + b^{\phi_*}_{c,H^+} \partial_{\phi_*} = e^{-2\kappa_- v} \left( \partial_{\tilde{v}} + \left( b^{\phi_*} - b^{\phi_*} \big|_{r=r_-} \right) \partial_{\phi_*} \right) = e^{-2\kappa_- v} \left( \partial_{\tilde{v}} + b^{\phi_*} \partial_{\phi_*} \right) = \sigma \left( \partial_{\tilde{v}} + b^{\phi_*} \partial_{\phi_*} \right). \]

We can write the vector field $\partial_t$ using the coordinates at the Cauchy horizon as
\[
\partial_t = \frac{1}{2} \left( \partial_{\tilde{v}} - b^{\phi_*} \big|_{r=r_-} \partial_{\phi_*} \right) - \frac{1}{2} \partial_u = \frac{e^{-2\kappa_- v}}{2} \left( \partial_{\tilde{v}} + b^{\phi_*}_{c,H^+} \partial_{\phi_*} \right) - \frac{1}{2} b^{\phi_*} \partial_{\phi_*} - \frac{1}{2} \partial_u. \]

The vector field $\partial_t$ is not null on the Cauchy horizon. A Killing vector field which is null on the Cauchy horizon is
\[
Z = -\partial_t - \frac{1}{2} b^{\phi_*} \big|_{r=r_-} \partial_{\phi_*}, \quad \frac{1}{2} \partial_u + \frac{1}{2} \left( b^{\phi_*} - b^{\phi_*} \big|_{r=r_-} \right) \partial_{\phi_*} = -\frac{e^{2\kappa_- v}}{2} \left( \partial_{\tilde{v}} + b^{\phi_*}_{c,H^+} \partial_{\phi_*} \right) = \frac{1}{2} \partial_u + \frac{1}{2} \left( b^{\phi_*} - b^{\phi_*} \big|_{r=r_-} \right) \partial_{\phi_*} = \frac{\Omega^2 \partial_{\tilde{v}} + b^{\phi_*}_{c,H^+} \partial_{\phi_*}}{2 \Omega^2_{c,H^+}}, \] (80)

### 2.3.3 Coordinates at the event horizon

We consider a new smooth coordinate $u_{H^+}(u)$, with positive derivative, equal to $u$ for $u \geq 1$, satisfying $u_{H^+} \rightarrow 0$ as $u \rightarrow -\infty$, and satisfying
\[ du = e^{-2\kappa_+ u} \, du_{H^+} \]
for $u \leq 0$. Moreover, we define $v_{H^+} = v$ and
\[ \phi_{*, H^+} = \phi_* - b^{\phi_*} \big|_{r=r_+} v. \]

**Remark 2.27.** $u_{H^+}$ is also a smooth function of $u$, and the change of coordinates $(v, \phi_*) \leftrightarrow (v_{H^+}, \phi_{*, H^+})$ is smooth.

Note that
\[ b^{\phi_*} \big|_{r=r_+} = \frac{2n\Xi}{r_+^2 + a^2}. \]

For $u \leq 0$, we may write the metric as
\[
g = -2 \Omega^2_{H^+} (du_{H^+} \otimes dv_{H^+} + dv_{H^+} \otimes du_{H^+}) + \hat{g}_{\theta_*, \theta_*} \, d\theta_* \otimes d\theta_* + \hat{g}_{\phi_*, \phi_*} \, d\phi_* \otimes (d\phi_{*, H^+} - b^{\phi_*}_{H^+} \, dv_{H^+}) + \hat{g}_{\theta_*, \phi_*} (d\phi_{*, H^+} - b^{\phi_*}_{H^+} \, dv_{H^+}) \otimes d\theta_* + \hat{g}_{\phi_*, \phi_*} (d\phi_{*, H^+} - b^{\phi_*}_{H^+} \, dv_{H^+}) \otimes (d\phi_{*, H^+} - b^{\phi_*}_{H^+} \, dv_{H^+}), \]
with
\[ \Omega^2_{H^+} = \Omega^2 e^{-2\kappa_+ u}, \]
\[ b^{\phi_*}_{H^+} = b^{\phi_*} \big|_{r=r_+}. \]

Henceforth we will assume that we are working in the region $u \leq 0$, our formulas will always refer to this region.

For a general function $f$, we have
\[ f(u, v, \theta_*, \phi_*) = f \left( u(u_{H^+}), v, \theta_*, \phi_{*, H^+} + b^{\phi_*} \big|_{r=r_+} v \right) = \tilde{f}(u_{H^+}, v_{H^+}, \theta_*, \phi_{*, H^+}), \]

30
where $\tilde{f}$ is the function $f$ written in the coordinates $(u_{H^+}, v_{H^+}, \theta_*, \phi_{*, H^+})$. So, defining
\[
\varsigma = e^{-2u_{H^+}},
\]we get
\[
\partial u_{H^+} = e^{-2u_{H^+}} \partial_u = \varsigma \partial_u,
\]
\[
\partial v_{H^+} = \partial_v + b_{\phi_*} \bigg|_{r=r_*} \partial_{\phi_*}.
\]
This, (76) and $\partial_{\phi_*} = \partial_{\phi_{*, H^+}}$ imply that
\[
\partial_v + b_{\phi_*} \partial_{\phi_*} = \partial_{v_{H^+}} + b_{\phi_{*, H^+}} \partial_{\phi_*}.
\]
A Killing vector field which is null on the event horizon is
\[
W = \partial_t + \frac{1}{2} b_{\phi_*} \bigg|_{r=r_*} \partial_{\phi_*} = \frac{1}{2} \partial_{v_{H^+}} - \frac{e^{2u_{H^+}}}{2} \partial_{u_{H^+}}.
\]The value
\[
\frac{1}{2} b_{\phi_*} \bigg|_{r=r_*} = \frac{a \Xi}{r_*^2 + a^2}
\]is the angular velocity $\Omega_{H^+}$ on the event horizon.

3 The energy of the solutions of the wave equation

We will use the vector field method to study the energy of solutions of the wave equation which have compact support on $H^+$. As is well known, the method, used by Morawetz [26], John [20], Klainerman [21, 22], Dafermos [8, 9, 10, 11, 12] and Rodnianski [11, 22], among many others, consists in applying the Divergence Theorem to some currents obtained by contracting the energy-momentum tensor $T_{\mu \nu}$ with appropriate vector field multipliers constructed specifically according to each region of spacetime.

We refer to the region close to the Cauchy horizon as the blue-shift region, and the region close to the event horizon as the red-shift region. We call the intermediate region the no-shift region. A very general construction of red-shift vector fields on general spacetimes which contain Killing horizons with positive surface gravity is carried out in the lecture notes [11]. Here we perform the computations explicitly in double null coordinates.

The blue-shift vector field (84) is constructed using the vector field $Y$ in Lemma 3.1, and the red-shift vector field (95) is constructed using the vector field $V$ in Lemma 3.2. We go on to calculate the covariant derivative of $Y$, $\nabla^\mu Y^\nu$, and the scalar current associated to $Y$, $T_{\mu \nu} \nabla^\mu Y^\nu$. We obtain the usual inequalities for the currents associated to the blue-shift vector field, and for the currents associated to the red-shift vector field. We finish with Theorem 3.5, which is Sierecki’s result, for the Reissner–Nordström and Kerr spacetimes, applied to Kerr–Newman–de Sitter spacetimes.

3.1 The blue-shift and red-shift vector fields

3.1.1 Construction

Here the blue-shift vector field is defined to be
\[
N_b = Y + Z,
\]
where $Y$ is given in

Lemma 3.1. Let $t \in \mathbb{R}^+$ be given. The initial value problem
\[
\nabla_Y Y = -t(Y + Z),
\]
\[
Y \big|_{t=H^+} = \frac{\partial_{u_{H^+}} + b_{\phi_*} \partial_{\phi_*}}{\Omega_{H^+}}
\]
(Z as in (80)) has a unique time invariant solution, $Y$, defined in a neighborhood of the Cauchy horizon, i.e. defined for $r_*$ sufficiently large.

Proof.
(a) *(Y time invariant.)* The vector field \( \frac{1}{\Omega_{CN}^2} (\partial_{\psi_{CN}^+} + b_{CN}^\phi \partial_{\phi_0}) \) commutes with \( \partial_t \). In fact, we have
\[
\left[ \partial_t, \frac{1}{\Omega_{CN}^2} \partial_{\psi_{CN}^+} \right] = \left[ \partial_t, \frac{1}{\Omega_{CN}^2} \partial_\theta \right] = \frac{1}{\Omega_{CN}^2} \left[ \frac{1}{2} \partial_\theta - \frac{1}{2} \partial_{\psi_{CN}^+} \right] (\partial_t + b_{CN}^\phi |_{r=r_0} \partial_{\psi_{CN}^+}) = 0
\]
and
\[
b_{CN}^\phi \partial_{\psi_{CN}^+} = \frac{b_{CN}^\phi - b_{CN}^\phi |_{r=r_0}}{\Omega_{CN}^2} \Rightarrow \left[ \partial_t, \frac{b_{CN}^\phi}{\Omega_{CN}^2} \partial_{\psi_{CN}^+} \right] = 0.
\]
So, if we choose \( Y \) of the form
\[
Y = \tilde{f} \partial_t + \tilde{g} \frac{1}{\Omega_{CN}^2} \left( \partial_{\psi_{CN}^+} + b_{CN}^\phi \partial_{\phi_0} \right) + \tilde{h} \partial_{\phi_0} + \tilde{j} \partial_{\phi_0},
\]
with
\[
\tilde{f} = \tilde{f}(r, \theta), \quad \tilde{g} = \tilde{g}(r, \theta), \quad \tilde{h} = \tilde{h}(r, \theta), \quad \tilde{j} = \tilde{j}(r, \theta),
\]
then, \( Y \) commutes with \( \partial_t \).

(b) *(The differential equation.)* Expanding the left-hand side of (85), we get
\[
\nabla_Y Y = \tilde{f} \partial_r, \tilde{f} \partial_\theta + \tilde{f} \partial_r, \tilde{g} \frac{1}{\Omega_{CN}^2} \left( \partial_{\psi_{CN}^+} + b_{CN}^\phi \partial_{\phi_0} \right) + \tilde{f} \partial_r, \tilde{h} \partial_{\phi_0} + \tilde{f} \partial_r, \tilde{j} \partial_{\phi_0},
\]
\[
+ \tilde{g} \frac{\sigma}{\Omega_{CN}^2} \partial_r \partial_\theta \partial_\phi + \tilde{g} \frac{\sigma}{\Omega_{CN}^2} \partial_r \partial_\phi \partial_\phi + \tilde{g} \frac{\sigma}{\Omega_{CN}^2} \partial_r, \tilde{h} \partial_{\phi_0} + \tilde{g} \frac{\sigma}{\Omega_{CN}^2} \partial_r, \tilde{j} \partial_{\phi_0},
\]
\[
+ \tilde{h} \partial_\phi \partial_\theta \partial_\phi + \tilde{h} \partial_\theta \partial_\phi \partial_\phi + \tilde{h} \partial_\phi \partial_{\phi_0} \partial_{\phi_0} + \tilde{j} \partial_\phi \partial_{\phi_0} \partial_{\phi_0},
\]
\[
+ \tilde{j} \partial_\phi \partial_{\phi_0} \partial_{\phi_0} + \tilde{j} \partial_\theta \partial_{\phi_0} \partial_{\phi_0} + \tilde{j} \partial_{\phi_0} \partial_{\phi_0} \partial_{\phi_0} + \tilde{j} \partial_{\phi_0} \partial_{\phi_0} \partial_{\phi_0}.
\]
We used (108) to eliminate the term in \( \tilde{g}^2 \). This initial value problem (85), (86) is equivalent to a system of four equations, for the \( \partial_\phi \frac{1}{\Omega_{CN}^2} \left( \partial_{\psi_{CN}^+} + b_{CN}^\phi \partial_{\phi_0} \right), \partial_{\phi_0} \) and \( \partial_{\phi_0} \) components of each side, for the four unknowns \( \tilde{f}, \tilde{g}, \tilde{h} \), and \( \tilde{j} \), with
\[
\tilde{f}(r, \theta) = 0, \quad \tilde{g}(r, \theta) = 1, \quad \tilde{h}(r, \theta) = 0, \quad \tilde{j}(r, \theta) = 0.
\]
(87)

The system reads
\[
\tilde{f} \partial_r, \tilde{f} \partial_\theta + \tilde{g} \frac{\sigma}{\Omega_{CN}^2} \partial_r, \tilde{f} + \tilde{h} \partial_\phi, \tilde{f} = \ldots,
\]
\[
\tilde{f} \partial_r, \tilde{g} \frac{\sigma}{\Omega_{CN}^2} \partial_\theta, \tilde{g} + \tilde{h} \partial_\phi, \tilde{g} = \ldots,
\]
\[
\tilde{f} \partial_r, \tilde{h} \frac{\sigma}{\Omega_{CN}^2} \partial_\phi, \tilde{h} + \tilde{h} \partial_\phi, \tilde{h} = \ldots,
\]
\[
\tilde{f} \partial_r, \tilde{j} \frac{\sigma}{\Omega_{CN}^2} \partial_\phi, \tilde{j} + \tilde{h} \partial_\phi, \tilde{j} = \ldots,
\]
where the right-hand sides involve the Christoffel symbols of the metric, \( \partial_r b_{CN}^\phi, \partial_\theta b_{CN}^\phi, \) and \( \tilde{f}, \tilde{g}, \tilde{h} \) and \( \tilde{j} \), but do not involve any derivatives of these last four functions. The system may be written as
\[
\psi \cdot \tilde{f} = \ldots, \quad \psi \cdot \tilde{g} = \ldots, \quad \psi \cdot \tilde{h} = \ldots, \quad \psi \cdot \tilde{j} = \ldots,
\]
(88) (89) (90) (91)
where
\[
\psi = \left( \frac{1}{\Delta_r} \frac{\partial \phi}{\partial \phi} \right) \left( \tilde{f} \Delta_r + \tilde{g} \left( -\frac{\chi}{\rho^2 \Delta \phi} \right) \right) + \tilde{h} \frac{\partial \phi}{\partial \phi} \frac{\partial \phi}{\partial \phi} + \left( \frac{1}{\Delta_r} \frac{\partial \theta}{\partial \phi} \right) \left( \tilde{f} \Delta_r + \tilde{g} \left( -\frac{\chi}{\rho^2 \Delta \phi} \right) \right) + \tilde{h} \frac{\partial \theta}{\partial \phi} \frac{\partial \phi}{\partial \phi}.
\]
32
We used
\[ \frac{\sigma \Delta_r}{\Omega_{2n}^2} = - \frac{Y}{\rho^2 \Delta \phi}, \]
which we obtain from (18), (77) and (79).

(c) **(An auxiliary calculation.)** In the next step we will use the following identity. We mention that it implies that \( h \) is not identically equal to zero. Using (110) and (111), we have
\[
d\theta^* \left( \nabla \partial_{r} \left( \frac{1}{\Omega_{2n}^2} \left( \partial_{r} + b^\phi \partial_{\phi} \right) \right) + \nabla \partial_{\phi} \left( \frac{1}{\Omega_{2n}^2} \left( \partial_{r} + b^\phi \partial_{\phi} \right) \right) \partial_{u} + \right) = 2 g^\phi \partial_{\phi} \frac{\Omega_{2n}^2}{\Omega_{2n}^2}. \tag{92}
\]

(d) **(The characteristics do not cross the boundary.)** We now check that
\[
\tilde{h}(r, 0) = 0 \quad \text{and} \quad \tilde{h} \left( \frac{\pi}{2} \right) = 0. \tag{93}
\]
This implies
\[
\nu^\theta \left( r, 0 \right) = 0 \quad \text{and} \quad \nu^\theta \left( \frac{\pi}{2} \right) = 0, \tag{94}
\]
and guarantees that the characteristics of our differential equations do not leave the region, \([r_-, r_+] \times \left[0, \frac{\pi}{2}\right]\), where we want to solve our system. It also guarantees that \( Y \) is well defined when \( \theta = 0 \), notwithstanding \( \partial_{\theta} \), not being well defined when \( \theta = 0 \). The vanishing of \( \tilde{h} \) at \( \theta = \frac{\pi}{2} \) can also be seen as a consequence of the symmetry of our problem under the reflection \( \theta \mapsto \pi - \theta \), which implies that \( Y \) should not have any component in the \( \partial_{\theta} \) direction at the equators of the spheres where \( u \) and \( v \) are both constant. The right hand side of (90) consists of a sum of terms which we divide into two parts. The first part consists of sum of the eight summands that have \( h \) as a factor. The term
\[
d\theta \left( -i(Y + Z) \right) = -i d\theta(Y) = -i \tilde{h}
\]
is proportional to \( \tilde{h} \). The second part consists of the sum of the remaining eight summands, which are \( d\theta \), applied to
\[
- \tilde{f}^2 \nabla \partial_u \partial_u - \tilde{f} \tilde{g} \nabla \partial_u \partial_{u} \left( \partial_{r} + b^\phi \partial_{\phi} \right) - 2 \tilde{f} \tilde{g} \nabla \partial_u \partial_{\phi},
\]
\[
- \tilde{f} \tilde{g} \nabla \partial_u \left( \frac{1}{\Omega_{2n}^2} \left( \partial_{r} + b^\phi \partial_{\phi} \right) \right) \partial_u - 2 \tilde{g} \tilde{g} \nabla \partial_u \left( \frac{1}{\Omega_{2n}^2} \left( \partial_{r} + b^\phi \partial_{\phi} \right) \right) \partial_{\phi} - \tilde{f} \nabla \partial_{\phi} \partial_{\phi}.
\]
Taking into account (92), the second part is
\[
- \tilde{f}^2 \tilde{u}^{\theta} \partial_u - 4 \tilde{f} \tilde{g} \tilde{u}^{\theta} \partial_{\theta} \Omega_{2n}^2 - 2 \tilde{f} \tilde{g} \tilde{u}^{\phi}, \Omega_{2n}^2
\]
\[
- 2 \tilde{g} \Omega_{2n}^2 \left( \Gamma_{\phi,\phi} \partial_{r} + b^\phi \partial_{\phi} \right) \phi \Gamma_{\phi,\phi} \phi, \Omega_{2n}^2.
\]
At \( \theta_* = 0 \) and \( \theta_* = \frac{\pi}{2} \) this sum is zero because each of the terms is equal to zero. Indeed, all terms are a product of a differentiable function by \( \sin(2\theta_*) \). This is easy to check. Let us exemplify this assertion with one of the less immediate terms to analyze, the one which contains
\[
\Gamma_{\phi,\phi} \partial_{r} \partial_{r} \tilde{h} - \tilde{h} \partial_{r} \partial_{r} \tilde{h} + \tilde{h} \times \text{smooth function} = \tilde{h} \times \text{smooth function}.
\]
As
\[
\tilde{h}(r, 0) = 0 \quad \text{and} \quad \tilde{h} \left( \frac{\pi}{2} \right) = 0
\]
and \( \nu^r \) is not zero (at least initially at \( r_+ \), see below), we conclude that, if there exists a solution to our initial value problem, then it must satisfy (93). Using the fact that \( \partial_{\theta} \) contains a factor \( \sin(2\theta_*) \) once again, we obtain (94).
(e) (Existence and uniqueness of solution.) Using (12) and (38), we have
\[ V(r, \theta) = \frac{1}{\Delta_r} \frac{\partial r}{\partial r_*} - \frac{\nabla}{\rho^2} = \frac{1}{(r_*^2 + a^2)^2} \left( \frac{(r_*^2 + a^2)}{\rho^2} \right) = - \frac{(r_*^2 + a^2)}{r_*^2 + a^2 \cos^2 \theta}. \]
This shows that the segment \( \{r_-\} \times [0, \frac{\pi}{2}] \) is noncharacteristic for our system of first-order quasilinear partial differential equations.

We observe that when the Christoffel symbols \( \Gamma^\phi_{\theta \phi_0} \) and \( \Gamma^\phi_{\theta_0 \phi} \), which blow up at \( \theta_* = 0 \) like \( \frac{1}{\sin \theta_*} \) (see Corollary A.2), appear in the system above, then they appear multiplied by \( \tilde{h} \), which has to vanish to first order at \( (r, 0) \). So that the summands where these Christoffel appear are continuous functions.

By a standard existence and uniqueness theorem for non characteristic first order quasilinear partial differential equations, we know that our initial value problem has a solution for \( (r, \theta) \in [r_, r_- + \delta] \times [0, \pi] \), for some positive \( \delta \). Recall that constructing the solution involves solving the system of ordinary differential equations
\[
\begin{aligned}
\dot{r} &= V^r \\
\dot{\theta} &= V^\theta \\
\dot{\tilde{f}} &= \text{right-hand side of (88)} \\
\dot{\tilde{g}} &= \text{right-hand side of (89)} \\
\dot{\tilde{h}} &= \text{right-hand side of (90)} \\
\dot{\tilde{j}} &= \text{right-hand side of (91)}
\end{aligned}
\quad \text{with} \quad
\begin{aligned}
(r(0) &= r_- \\
\theta(0) &= \theta_0 \\
\tilde{f}(0) &= 0 \\
\tilde{g}(0) &= 1 \\
\tilde{h}(0) &= 0 \\
\tilde{j}(0) &= 0
\end{aligned}
\]
We know from Remark 2.1 that, for \( r_* \) sufficiently large, \( r \) is close to \( r_- \). So we have the existence of a solution for large \( r_* \).

\[ N_r = V + W, \quad (95) \]

Here the red-shift vector field is defined to be

**Lemma 3.2.** Let \( \iota \in \mathbb{R}^+ \) be given. The initial value problem
\[ \nabla_V V = -\iota(V + W), \]
\[ V|_{\mathbb{H}^+} = \frac{\partial u_{\mathbb{H}^+}}{\Omega^2_{\mathbb{H}^+}} \]
(W as in (82)) has a unique time invariant solution, \( V \), defined in a neighborhood of the event horizon, i.e. defined for \( r_* \) sufficiently negative (i.e. for \( r_* < -C \) with \( C \) sufficiently large).

**Proof.** Choose \( V \) of the form
\[ V = \tilde{f} \frac{1}{\Omega_{\mathbb{H}^+}} \partial u_{\mathbb{H}^+} + \tilde{g} \left( \partial u_{\mathbb{H}^+} + b^\phi \partial \phi_0 \right) + \tilde{h} \partial \theta_* + \tilde{j} \partial \phi_0, \]
with
\[ \tilde{f} = \tilde{f}(r, \theta), \quad \tilde{g} = \tilde{g}(r, \theta), \quad \tilde{h} = \tilde{h}(r, \theta), \quad \tilde{j} = \tilde{j}(r, \theta), \]
\[ \tilde{f}(r_+, \theta) = 1, \quad \tilde{g}(r_+, \theta) = 0, \quad \tilde{h}(r_+, \theta) = 0, \quad \tilde{j}(r_+, \theta) = 0. \]

\[ 3.1.2 \quad \text{Covariant derivative} \]
One could consider working in the frame \( (Z, Y, \partial \theta_0, \partial \phi_0) \) but this is not a good choice because the energy-momentum tensor does not have a simple expression in this frame. So, instead, we define
\[ T = \frac{1}{2} \partial u \quad \text{and} \quad \tilde{Y} = \frac{1}{\Omega_{\mathbb{H}^+}} \left( \partial u_{\mathbb{H}^+} + b^\phi \partial \phi_0 \right) = \frac{\partial u + b^\phi \partial \phi_0}{\Omega^2_{\mathbb{H}^+}}, \]
and work in the frame
\[ (X_\tilde{Y}, X_{\tilde{Y}}, X_{\phi_0}, X_{\phi_0}) := (T, \tilde{Y}, \partial \theta_0, \partial \phi_0). \]
We see that
\[ Y = 2 \tilde{Y} + g \tilde{Y} + \tilde{h} \partial \theta_0 + \tilde{j} \partial \phi_0. \]
The dual frame is
\[ (\omega_T, \omega_T, \omega_\theta, \omega_{\phi_0}) = \left( 2 \partial u, \Omega^2_{\mathbb{H}^+} \partial v_{\mathbb{H}^+}, \partial \theta_0, \partial \phi_0, \partial v_{\mathbb{H}^+} \right) = \left( 2 \partial u, \Omega^2 \partial v, \partial \theta_0, \partial \phi_0 - b^\phi \partial v \right). \]
Calculation of the covariant derivative of \( Y \). The covariant derivative of \( Y \) is
\[
\nabla Y = \omega^T \otimes \nabla F + \omega^T \otimes \nabla F + \omega^T \otimes \nabla \theta, Y + \omega^T \otimes \nabla \theta, Y.
\]
One readily checks that the metric dual basis to \((\omega^T, \omega^T, \omega^T)\) is
\[
\left( \omega^T, \omega^T, \omega^T \right) = (-Y, -T, \theta), \quad \text{and} \quad \left( \omega^T, \omega^T, \omega^T \right) = \left( -Y, -T, \theta^*, \theta^* \right).
\]
Using the fact that \(2T = \partial_u = \partial_u = (\partial_u, r) \partial_u + (\partial_u, \theta) \partial_u, (87) \) and \((111)\), we obtain
\[
\nabla F Y = \nabla F (2fT + T^Y + h \partial_u + j \partial_u)
= \nabla F Y + \nabla F (r - r)
= -\kappa Y + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u + \hat{O}(r - r),
\]
where
\[
\hat{O}(r - r) = O(r - r) T + O(r - r) Y + O(r - r) \partial_u + O(r - r) \partial_u.
\]
The values of \( a \dot{\theta} \) and \( a \dot{\theta} \) can be read off from \((111)\). Using \((85)\) and the formulas in Appendix A, we get
\[
\nabla F Y = \nabla F Y - \nabla F Y
= \nabla F Y - \nabla F (2fT + T^Y + h \partial_u + j \partial_u)
= -\kappa \partial_u + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u + \hat{O}(r - r),
\]
where
\[
\hat{O}(r - r) = O(r - r) T + O(r - r) Y + O(r - r) \partial_u + O(r - r) \partial_u \frac{1}{\sin \theta} \partial_u.
\]
The factor \( \frac{1}{\sin \theta} \) in front of \( \partial_u \) in the last summand arises from \( \Gamma^\phi_{\theta, \theta, \theta} \) and \( \Gamma^\phi_{\theta, \theta, \theta} \). Using the fact that \( \partial_u = (\partial_u, r) \partial_u + (\partial_u, \theta) \partial_u \) and \((112)\), we obtain
\[
\nabla \partial_u Y = \nabla \partial_u (2fT + T^Y + h \partial_u + j \partial_u)
= \nabla \partial_u Y + \nabla \partial_u (r - r)
= -a \partial_u + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u + \hat{O}(r - r),
\]
The values of \( a \dot{\theta} \) and \( a \dot{\theta} \) can be read off from \((112)\). Finally, we have
\[
\nabla \partial_u Y = \nabla \partial_u (2fT + T^Y + h \partial_u + j \partial_u)
= \nabla \partial_u Y + \nabla \partial_u (r - r)
= -a \partial_u + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u + \hat{O}(r - r).
\]
The values of \( a \dot{\theta} \) and \( a \dot{\theta} \) can be read off from \((113)\). Here
\[
\hat{O}(r - r) = O(r - r) \sin \theta T + O(r - r) \sin \theta Y + O(r - r) \sin \theta \partial_u + O(r - r) \sin \theta \partial_u.
\]
Note that although \( \Gamma^\phi_{\theta, \theta} \) does contain the factor \( \frac{1}{\sin \theta} \), in the last calculation this Christoffel symbol appears multiplied by \( \hat{h} \) which vanishes to first order at \( \theta = 0 \) and \( \theta = \pi \). So, the last equality is a consequence of Lemma A.1 and of the fact that \( f, g, \hat{h}, \) and \( \hat{j} \) do not depend on \( \phi \). Indeed, the components of \( \nabla \partial_u X_1 \) in \( T, Y \) and \( \partial_u \) all contain the factor \( \sin \theta \), for \( \hat{f} \in \{ T, Y, \partial_u, \phi \} \). The expressions \((96)-(99)\) above correspond to \([11,(19)-(22)]\). Combining the previous results, we can write the covariant derivative of \( Y \) as
\[
\nabla Y = -Y \otimes (-\kappa Y + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u) + Y \otimes \hat{O}(r - r)
- Y \otimes (-\kappa Y + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u) + Y \otimes \hat{O}(r - r)
+ \theta \dot{\theta} \otimes (-a \partial_u + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u) + \theta \dot{\theta} \otimes \hat{O}(r - r)
+ \theta \dot{\theta} \otimes (-a \partial_u + a \dot{\theta} \partial_u + a \dot{\theta} \partial_u) + \theta \dot{\theta} \otimes \hat{O}(r - r)
= \kappa - Y \otimes Y - Y \otimes (a \dot{\theta} \partial_u) - Y \otimes (a \dot{\theta} \partial_u)
+ \iota \otimes Y + \iota \otimes Y
- (a \dot{\theta} \partial_u) \otimes Y - (a \dot{\theta} \partial_u) \otimes Y
\]
We may write the error term as
\[
\overline{O}(r - r_-) = \sum_{\ell, \lambda} \mathcal{O}(r - r_-) \omega_\ell^\lambda \otimes X_\lambda \\
+ O(r - r_-) Y \otimes \frac{1}{\sin \theta} \partial_{\phi_*} + O(r - r_-) \frac{1}{\sin \theta} \partial_{\phi_*} \otimes Y \\
+ O(r - r_-) T \otimes \frac{1}{\sin \theta} \partial_{\phi_*} + O(r - r_-) \frac{1}{\sin \theta} \partial_{\phi_*} \otimes T \\
+ O(r - r_-) \partial_{\phi_*} \otimes \frac{1}{\sin \theta} \partial_{\phi_*} + O(r - r_-) \frac{1}{\sin \theta} \partial_{\phi_*} \otimes \partial_{\phi_*} \\
+ O(r - r_-) \frac{1}{\sin^2 \theta} \partial_{\phi_*} \otimes \partial_{\phi_*}.
\]

because \(\omega_\ell^\lambda \equiv \partial^{\phi_*}\) behaves like \(\hat{g}^{\phi, \phi_* \cdot, \phi_*}\), which in turn behaves like \(\frac{1}{\sin \theta} \partial_{\phi_*}\).

### 3.1.3 Currents

**The energy momentum tensor of a massless scalar field and the vector current.** The energy momentum tensor is given by
\[
T_{\mu \nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu \nu} \partial^\rho \partial_\rho \psi,
\]
and one readily checks that
\[
\partial^\rho \psi \partial_\rho \psi = \frac{\partial (\partial_\mu \psi)(\partial_\nu \psi + b^\phi \partial_\rho \psi)}{\Omega} (3 \omega_\nu \otimes \omega_\rho + b^\phi \psi) = \frac{\partial (\partial_\mu \psi)(\partial_\nu \psi)(1 + b^\phi \partial_\rho \psi) + 2 b^\phi \psi} {\Omega} \psi.
\]

The energy-momentum tensor is written as
\[
T = \frac{1}{4} \left( (\partial_\mu \psi)^2 \omega_\nu \otimes \omega_\nu + \frac{1}{2} \frac{|\nabla \psi|^2}{\rho} \left( \omega_\nu \otimes \omega_\nu + \omega_\nu \otimes \omega_\nu + \frac{1}{2} \right) \left( \partial_\mu \psi + b_\nu^\phi \partial_\rho \psi \right)^2 \omega_\nu \otimes \omega_\nu
+ \frac{1}{2} \left( \partial_\mu \psi \right) \left( \partial_\mu \psi + b_\nu^\phi \partial_\rho \psi \right) \left( \partial_\nu \psi \right) \left( \omega_\nu \otimes \omega_\nu + \omega_\nu \otimes \omega_\nu + \frac{1}{2} \right) \right)
+ \left( \partial_\nu \psi \right)^2 \left( \frac{\theta}{2} - \frac{1}{\Omega} \left( \partial_\nu \psi \right) \left( \partial_\mu \psi + b_\nu^\phi \partial_\rho \psi \right) \left( \omega_\nu \otimes \omega_\nu + \omega_\nu \otimes \omega_\nu + \frac{1}{2} \right) \right)
+ \left( \partial_\nu \psi \right)^2 \left( \frac{\theta}{2} \right) \left( \omega_\nu \otimes \omega_\nu + \omega_\nu \otimes \omega_\nu + \frac{1}{2} \right)
\]

The vector currents associated to the blue-shift and red-shift vector fields are
\[
J_{\mu}^N = T_{\mu \nu} N_\nu^b \quad \text{and} \quad J_{\mu}^N = T_{\mu \nu} N_\nu^r,
\]
respectively.

**The scalar current associated to the blue-shift vector field.** The scalar current associated to \(N_{\phi_*} = K_{\phi_*} \nabla^\mu N_{\phi_*}^\mu\). Since \(Z\) is a Killing vector field this is equal to \(K_\mu\), which is
\[
K_\mu = T_{\mu \nu} \nabla^\nu \nabla Y
= \kappa^{-1} T(Y, \nabla Y) - 2 T(Y, a^\phi \partial_\rho \psi) - 2 T(Y, a^\phi \partial_\rho \psi)
+ \partial T(Y, \nabla Y) + \partial \mu T(Y, \nabla Y)
+ h^\phi \partial_\rho \psi \partial_\rho \psi \partial_\rho \psi + h^\phi \partial_\rho \psi \partial_\rho \psi + h^\phi \partial_\rho \psi \partial_\rho \psi
+ O(r - r_-) (T(T, T) + T(T, T) + T(Y, Y)).
\]
We estimate \( K^Y \). Suppose we are given \( \delta \in (0, \frac{1}{4}) \). Let \( \bar{r} = 2(-\kappa_-)\delta \). Choose \( \epsilon = c + \bar{r} + 1 \), where \( c = c_5 \) is the constant below (which is independent of \( Y \)). There exists \( r_0 > r_- \) such that \( r \in (r_-, r_0) \) implies that

\[
K^Y \geq \kappa_- T(\bar{Y}, \bar{Y}) + \epsilon T(\bar{T}, \bar{Y}) + \epsilon T(\bar{T}, \bar{T}) - cT(\bar{T}, \bar{T}) - O(r - r_-)T(\bar{T}, \bar{T}) + O(r - r_-)(\bar{T}, \bar{T}) + O(T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T})).
\]

We have used the fact that

\[
y^\phi, \phi, (\partial_{\phi}, \psi)^2 \leq |\nabla \psi|^2_y.
\]

**Inequalities relating the currents.** The blue-shift vector field satisfies

**Lemma 3.3.** Let \( 0 < \delta < \frac{1}{4} \). If \( (r_*)_0 \) is chosen sufficiently large (see Remark 2.1), then

\[
\int_{u < c < u_1} K^{N_u} dVol \geq 2\kappa_-(1 + 3\delta) \int_{[u_0, u_1]} \left( \int_{u < (r_*)_0 - u} J^N_{\mu} \eta^{\mu}_{\nu} dVol \right) du.
\]

**Proof.** Using (26) and (100), we obtain

\[
\int_{u < c < u_1} K^N dVol \geq \int_{u < (r_*)_0 - u} 2\kappa_-(1 + \delta) T(\bar{Y}, \bar{Y}) + 2\epsilon T(\bar{T}, \bar{T}) + O(\epsilon T(\bar{T}, \bar{T}) - \kappa_- (1 + \delta) T(\bar{Y}, \bar{Y}) + O(T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T})).
\]

Using (25), we see that

\[
T \left( N_{\mu} \eta_{\mu}^{\Omega^2} \right) = T(N_{\mu} \bar{Y}) = T(\bar{T} + \bar{Y}, \bar{Y}) + \epsilon T(\bar{T} + \bar{Y}, \bar{Y}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}).
\]

For each \( 0 < \delta < \frac{1}{4} \), there exists \( r_0 > r_- \) such that \( r \in (r_-, r_0) \) implies that

\[
\int_{u < (r_*)_0 - u} T(\bar{N}_{\mu} \bar{\eta}_{\mu}^{\Omega^2}) \Omega^2 dVol \geq \int_{u < (r_*)_0 - u} (1 - \delta) (T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T})) - \delta T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}).
\]

Multiplying both sides by \( 2\kappa_-(1 + \delta)/(-1 - \delta) \), we get

\[
2\kappa_-(1 + \delta) \int_{u < (r_*)_0 - u} \left( (T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T})) - \delta T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) + T(\bar{T}, \bar{T}) \right) \Omega^2 dVol \geq 2\kappa_-(1 + \delta) \int_{u < (r_*)_0 - u} T(\bar{N}_{\mu} \bar{\eta}_{\mu}^{\Omega^2}) \Omega^2 dVol.
\]

This implies (101) because \( \frac{1 + \delta}{1 - \delta} < 1 + 3\delta \) and \( -2\kappa_-(1 + \delta) < -4\kappa_- = 2\bar{r} \). \( \square \)

Similarly to (101), the red-shift vector field satisfies

**Lemma 3.4.** Let \( \delta > 0 \). For \( (r_*)_0 \) sufficiently negative (see Remark 2.1), we have

\[
\int_{\bar{r} < c < \bar{r}_1} K^{N_{\mu}} dVol \geq 2\kappa_+(1 + \delta) \int_{[\bar{u}_0, \bar{u}_1]} \left( \int_{u < (r_*)_0 - u} J^N_{\mu} \eta^{\mu}_{\nu} dVol \right) du.
\]

**Proof.** Work in the frame \( (\bar{\nabla}, \bar{T}, \partial_{\phi}, \partial_{\phi}) \), with

\[
\bar{\nabla} = \frac{1}{\eta_{\mu}^{\Omega^2}} \partial_{u_{\mu}^+}, \quad \bar{T} = \frac{1}{2} \left( \partial_{\phi_{\mu}^+} + \bar{b}_{\mu}^+ \partial_{\phi} \right).
\]

\[\square\]
3.2 Energy estimates

We are interested in solutions of the wave equation which are regular up to, and including, \( \mathcal{H}^+ \), and which have compact support on \( \mathcal{H}^+ \), i.e. we are interested in functions belonging to the space

\[
\mathcal{F} := \left\{ \psi \in C^\infty(\mathcal{M} \cup \mathcal{H}^+) : \nabla \psi = 0 \right\}.
\]

The following theorem, established by Sbierski in his thesis [20] for Reissner–Nordström and Kerr black holes, applies to Kerr–Newman–de Sitter black holes.

**Theorem 3.5.** Let \( 2\kappa_+ > -\kappa_- \) and \( \psi \in \mathcal{F} \). Then, for any \( u_0, v_0 \in \mathbb{R} \), we have

\[
\int_{C^{-\kappa_+}(\infty, u_0)} J^N_\mu n^\mu_{\Sigma(r_*)0} \text{dVol}_{\mathcal{C}H} + \int_{C^{-\kappa_+}(\infty, v_0)} J^N_\mu n^\mu_{\Sigma(r_*)0} \text{dVol}_{\mathcal{C}H} < +\infty.
\]

**Proof.** We sketch the proof and refer to [20] for further details.

(a) Estimates in the red-shift region. For \( \kappa < \kappa_+ \), define \( N_\kappa = e^{2\kappa u} N_r \). Choose the \( \delta \) in (102) such that \( \kappa < \kappa_+ (1 - \delta) \). Since \( K_\kappa \geq 0 \), the Divergence Theorem implies that we have

\[
\int_{\Sigma \cap \{ r < r_0 \}} \left\{ \begin{array}{l}
\frac{\partial}{\partial r} \nabla \psi + \frac{1}{2} \left( \nabla \psi \right)^2 + \frac{\Omega^2}{\Omega_{\mathcal{H}^+}} L \sin \theta \left( \tilde{\omega} \right) \end{array} \right\} \text{dVol}_{\mathcal{H}^+} \\
\leq C \int_{\Sigma \cap \{ r < r_0 \}} e^{2\kappa u(r)} e^{-2\kappa u(r_+) \delta} \text{dVol}_{\mathcal{H}^+}.
\]

The left-hand side can be bounded below by

\[
\int_{\Sigma \cap \{ r < r_0 \}} \frac{\partial}{\partial r} \nabla \psi + \frac{1}{2} \left( \nabla \psi \right)^2 + \frac{\Omega^2}{\Omega_{\mathcal{H}^+}} L \sin \theta \left( \tilde{\omega} \right) \text{dVol}_{\mathcal{H}^+} \\
\leq C u_{\mathcal{H}^+} = C e^{2\kappa u(r_+)}.
\]

(b) Estimates in the no-shift region. We recall [20, Lemma 4.5.6]: Given \( (r_*)_1 > (r_*)_0 \) and a smooth future directed timelike time invariant vector field \( N \), there exists a constant \( C > 0 \) such that

\[
\int_{\Sigma \cap \{ r < r_0 \}} \frac{\partial}{\partial r} \nabla \psi + \frac{1}{2} \left( \nabla \psi \right)^2 + \frac{\Omega^2}{\Omega_{\mathcal{H}^+}} L \sin \theta \left( \tilde{\omega} \right) \text{dVol}_{\mathcal{H}^+} \\
\leq C e^{4\kappa u(r_+)}.
\]

(c) Estimates in the blue-shift region. We also recall [7, Lemma 4.5]: Let \( f : [t_0, \infty] \rightarrow \mathbb{R} \) and assume that for some \( \alpha_1, C > 0 \), and for all \( t \geq t_0 \),

\[
\int_{t_0}^\infty f(s) \, ds \leq C e^{-\alpha_1 t}.
\]

Then, for all \( 0 < \alpha_2 < \alpha_1 \) and \( t \geq t_0 \), we have

\[
\int_{t_0}^\infty e^{\alpha_2 s} f(s) \, ds \leq C e^{-\alpha_1 (1 - \alpha_2) t}.
\]
By assumption $2\kappa_+ > -\kappa_-$. If we choose $\kappa < \kappa_+$ sufficiently close to $\kappa_+$ and and $\mathfrak{r} < \kappa_-$ sufficiently close to $\kappa_-$, then $2\kappa > -\mathfrak{r}$. Using (103), (104) and the fact that

$$
\int_{\Sigma_{(r^+)1}} J^{N_{\mu}}_\mu n^{\Xi}(r^+) \, d\text{Vol}_{\Sigma(r^+)} \quad \text{and} \quad \int_{\Sigma_{(r^+)1}} J^{N_{\mu}}_\mu n^{\Xi}(r^+) \, d\text{Vol}_{\Sigma(r^+)}
$$

are comparable, we conclude that

$$
\int_{(\infty, \mathfrak{r}) \cap (-\infty, \mathfrak{r}_C)} e^{-2\mathfrak{r}_u} J^{N_{\mu}}_\mu n^{\Xi}(r^+) \, d\text{Vol}_{\Sigma(r^+)} < +\infty.
$$

Define $N_b = e^{-2\mathfrak{r}_u} N_b$. Choose the $\delta$ in (101) such that $\mathfrak{r} < \kappa_-(1 + 3\delta)$. Since $K_{\mathfrak{r}^2} \geq 0$, the Divergence Theorem implies that

$$
\int_{(\infty, \mathfrak{r}) \cap (-\infty, \mathfrak{r}_C)} e^{-2\mathfrak{r}_u} J^{N_{\mu}}_\mu n^{\Xi}(r^+) \, d\text{Vol}_{\Sigma(r^+)} + \int_{\Sigma_{\infty}} e^{-2\mathfrak{r}_u} J^{N_{\mu}}_\mu n^{\Xi}(r^+) \, d\text{Vol}_{\Sigma_{\infty}}
$$

$$
\leq \int_{(\infty, \mathfrak{r}) \cap (-\infty, \mathfrak{r}_C)} e^{-2\mathfrak{r}_u} J^{N_{\mu}}_\mu n^{\Xi}(r^+) \, d\text{Vol}_{\Sigma(r^+)}. \tag{\text{A1}}
$$

This finishes the proof. \(\square\)

## A The Christoffel symbols

### A.1 $(u, v, \theta_*, \phi_*)$ coordinates

Using the notation

$$
b_{\theta_*} := b_{\phi_*} \theta_{*, \phi_*}, \quad b_{\phi_*} := b_{\phi_*} \theta_{*, \phi_*}, \quad \|b_{\phi_*}\|^2 := (b_{\phi_*})^2 \theta_{*, \phi_*} = b_{\phi_*} b_{\phi_*},
$$

we write the Christoffel symbols of the metric $g$. If we use coordinates $(u, v, \theta_*, \phi_*)$, the values of both $\sigma$ and $\zeta$ are equal to 1. We have that

$$
\Gamma^u_{uu} = \zeta \frac{\partial_u (\Omega^2)}{\Omega^2}, \quad \Gamma^v_{uv} = \Gamma^\theta_{u\theta} = \Gamma^\phi_{u\phi} = 0;
$$

$$
\Gamma^u_{uv} = -\frac{b_{\phi_*}}{4\Omega^2} \partial_u \theta_{*, \phi_*}, \quad \Gamma^v_{uv} = 0,
$$

$$
\Gamma^\theta_{uw} = -\frac{2}{2} g^{\theta_* \phi_*} \partial_u \theta_{*, \phi_*} - \frac{2}{2} g^{\theta_* \phi_*} \partial_u \phi_{*, \phi_*} + g^{\theta_* \theta_*} \partial_u (\Omega^2),
$$

$$
\Gamma^\phi_{uw} = -\frac{2}{2} g^{\phi_* \phi_*} \partial_u \phi_{*, \phi_*} - \frac{2}{2} g^{\theta_* \phi_*} \partial_u \phi_{*, \phi_*} + g^{\phi_* \phi_*} \partial_u (\Omega^2) + \frac{\zeta}{2} \partial_u b_{\phi_*};
$$

$$
\Gamma^\theta_{vv} = \frac{(b_{\phi_*})^2}{4\Omega^2} \partial_v \theta_{*, \phi_*},
$$

$$
\Gamma^\phi_{vv} = \zeta \frac{1}{\Omega^2} \partial_v (\|b_{\phi_*}\|^2) + \frac{1}{\Omega^2} \partial_v (\Omega^2),
$$

$$
\Gamma_{uv} = -\sigma g^{\theta_* \phi_*} \partial_u \theta_{*, \phi_*} - \sigma g^{\theta_* \phi_*} b_{\phi_*} \partial_u \phi_{*, \phi_*} - \frac{2}{2} g^{\theta_* \phi_*} \partial_u (\|b_{\phi_*}\|^2),
$$

$$
\Gamma_{uv} = -\sigma g^{\phi_* \phi_*} \partial_u \phi_{*, \phi_*} - \sigma g^{\phi_* \phi_*} b_{\phi_*} \partial_u \phi_{*, \phi_*} - \frac{2}{2} g^{\phi_* \phi_*} \partial_u (\|b_{\phi_*}\|^2)
$$

$$
+ \frac{b_{\phi_*}}{\Omega^2} \partial_u (\Omega^2) + \frac{b_{\phi_*}}{4\Omega^2} \partial_u (\|b_{\phi_*}\|^2) - \sigma \partial_u b_{\phi_*};
$$

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\[ \Gamma_{u\theta_\ast}^u = \frac{\dot{g}_{\theta,\phi^*}}{4\Omega^2} \partial_r \mathcal{b}_{\theta^*} + \frac{1}{2\Omega^2} \partial_{\theta^*} (\Omega^2), \]
\[ \Gamma_{u\phi_\ast}^u = 0, \]
\[ \Gamma_{\theta_\ast u}^\theta = \frac{g^{\theta,\phi^*}}{2} \partial_r \mathcal{g}_{\theta,\phi^*} + \frac{g^{\theta,\phi^*}}{2} \partial_{\theta^*} \mathcal{g}_{\theta,\phi^*}, \]
\[ \Gamma_{\phi_\ast u}^\phi = \frac{g^{\theta,\phi^*}}{2} \partial_r \mathcal{g}_{\theta,\phi^*} + \frac{g^{\theta,\phi^*}}{2} \partial_{\theta^*} \mathcal{g}_{\theta,\phi^*}. \]
\[\Gamma^u_{\theta,\phi} = \frac{1}{4\Omega^2} \partial_u \phi_\theta, \phi, + \frac{\phi_{\theta_u,\phi}}{4\Omega^2} \partial_u \phi, \]
\[\Gamma^v_{\theta,\phi} = \frac{\phi_{\theta_v,\phi}}{4\Omega} \partial_v \phi, \phi, \]
\[\Gamma^\phi_{\theta,\phi} = \frac{\phi_{\theta,\phi}}{2} \partial_0 \phi, \phi, \]
\[\Gamma^\phi_{\theta,\phi} = \frac{\phi_{\theta,\phi}}{4\Omega^2} \partial_0 \phi, \phi, + \frac{\phi_{\theta,\phi}}{2} \partial_u \phi, \phi,. \]

(107)

Recall that \( \phi_{\theta,\phi} \) behaves like \( \frac{1}{\sin^2 \theta} \). In the Christoffel symbols above, this metric coefficient appears multiplied by

\[\partial_u \phi, \phi, \partial_u \phi, \phi, \partial_u \phi, \phi, \partial_v \phi, \phi, \partial_0 \phi, \phi,. \]

Lemma A.1. The derivatives \( \partial_u \phi, \phi, \partial_v \phi, \phi, \partial_0 \phi, \phi, \partial_0 \phi, \phi,. \) behave like \( \sin^2 \theta, \) and the derivatives \( \partial_u \phi, \phi, \partial_v \phi, \phi, \partial_0 \phi, \phi,. \) behave like \( \sin(2\theta). \)

Lemma A.1 implies

Corollary A.2. The Christoffel symbols of the metric \( g \) are all bounded, except for

\[\Gamma^\phi_{v,\theta,} \quad \text{and} \quad \Gamma^\phi_{\theta,\phi,.} \]

which blow up like \( \frac{1}{\sin \theta} \).

In the formulas above, the terms that blow up appear in (106) and (107).

Proof of Lemma A.1. Using (23), \( \phi_{\theta,\phi,.} \), and thus \( \phi_{\theta,\phi,.} \) is of the form \( A \sin^2 \theta \) for a regular function \( A \). We have

\[ \partial_u (A \sin^2 \theta) = \sin^2 \theta \partial_u A + A \sin(2\theta) \frac{\partial \theta}{\partial v}. \]

Using (40), we see that \( \frac{\partial \theta}{\partial v} \) contains a factor \( \sin(2\theta) \). Moreover,

\[ \frac{1}{\sin(2\theta)} \partial_\theta (A \sin^2 \theta) = \sin^2 \theta \left( \frac{\partial \theta}{\sin(2\theta)} \partial_\theta A \right) + A \sin(2\theta) \frac{\partial \theta}{\sin(2\theta)} \partial_\theta . \]

We recall that, according to (42), \( \frac{\partial \theta}{\partial v} \) is bounded. Thus, recalling (27), \( \partial_v (A \sin^2 \theta) \) behaves like \( \sin^2 \theta \) and \( \partial_\theta (A \sin^2 \theta) \) behaves like \( \sin(2\theta) \). On the other hand, according to Lemma 2.15, \( \phi_{\theta,\phi,.} = A \sin^2 \theta, \phi, \theta(2\theta), \) for another smooth function \( A \). Therefore, \( \partial_\theta, \phi, \phi,. \) behaves like \( \sin^2 \theta \).

Note that the vector fields \( \frac{\partial_\theta}{\partial v} \) and \( \frac{1}{\Omega^2} (\partial_\theta + b^\phi \cdot \partial_\phi) \) are geodesic:

\[ \nabla \frac{\partial_\theta}{\partial v} \partial_\theta = 0, \]
\[ \nabla \left( \frac{1}{\Omega^2} (\partial_\theta + b^\phi \cdot \partial_\phi) \right) \partial_\theta = 0. \]

(108)

Since we work in the frame

\[ \left( \frac{\partial_\theta}{2}, \frac{\partial_\theta + b^\phi \cdot \partial_\phi}{\Omega^2}, \partial_\theta, \partial_\phi, \right) = \left( \frac{\partial_\theta}{2}, \frac{\partial_\theta + b^\theta \cdot \partial_\theta + b^\phi \cdot \partial_\phi}{\Omega^2}, \partial_\theta, \partial_\phi \right), \]

(109)
it is also convenient to have the following covariant derivatives:

\[
\nabla \frac{1}{\Omega^2} (\partial_\nu + b_\nu \partial_{\phi_*}) \partial_u = \left( g^{\rho, \phi} \frac{\partial_\nu \Omega^2}{\Omega^2} \partial_{\phi_*} + g^{\rho, \phi} \frac{\partial_\rho \Omega^2}{\Omega^2} \frac{\partial_\nu}{\partial \phi_*} - \frac{\partial_\nu}{\partial \phi_*} \right) \phi_\nu, \tag{110}
\]

\[
\nabla \frac{1}{\Omega^2} (\partial_\nu + b_\nu \partial_{\phi_*}) \partial_\nu = \left( g^{\rho, \phi} \frac{\partial_\nu \Omega^2}{\Omega^2} \partial_{\phi_*} + g^{\rho, \phi} \frac{\partial_\rho \Omega^2}{\Omega^2} \frac{\partial_\nu}{\partial \phi_*} - \frac{\partial_\nu}{\partial \phi_*} \right) \partial_\nu, \tag{111}
\]

\[
\nabla \frac{1}{\Omega^2} (\partial_\nu + b_\nu \partial_{\phi_*}) \partial_\phi = \nabla \frac{1}{\Omega^2} \left( \partial_\nu + b_\nu \partial_{\phi_*} \right) \phi_\phi.
\]

\[
\nabla \frac{1}{\Omega^2} (\partial_\nu + b_\nu \partial_{\phi_*}) \phi_\phi = \left( g^{\rho, \phi} \frac{\partial_\nu \Omega^2}{\Omega^2} \partial_{\phi_*} + g^{\rho, \phi} \frac{\partial_\rho \Omega^2}{\Omega^2} \frac{\partial_\nu}{\partial \phi_*} - \frac{\partial_\nu}{\partial \phi_*} \right) \partial_\nu, \tag{112}
\]

\[
\nabla \frac{1}{\Omega^2} (\partial_\nu + b_\nu \partial_{\phi_*}) \phi_\phi = \left( g^{\rho, \phi} \frac{\partial_\nu \Omega^2}{\Omega^2} \partial_{\phi_*} + g^{\rho, \phi} \frac{\partial_\rho \Omega^2}{\Omega^2} \frac{\partial_\nu}{\partial \phi_*} - \frac{\partial_\nu}{\partial \phi_*} \right) \phi_\phi. \tag{113}
\]

**A.2 (u, v_{CN^+}, \theta_*, \phi_*, \chi^+ coordinates)**

If we use coordinates \((u, v_{CN^{+}}, \theta_*, \phi_*, \chi^+)\) then, in the formulas above, the value of \(\zeta\) continues to be 1, but one has to replace \(\Omega^2\) by \(\Omega_{CN^+}^2\), \(b_{\phi_*}\) by \(b_{\phi_*}^{CN^+}\), and the value of \(\sigma\) has to be as in \((79)\). Indeed, the expressions \(\sigma \partial_\nu\) arise when taking derivatives with respect to \(v_{CN^+}\). Now, using \((76)\) and \((78)\), we have

\[
\partial_{v_{CN^+}} = e^{-2\kappa - v} \partial_\nu = e^{-2\kappa - v} \left( \partial_\nu + b_{\phi_*}^{CN^+} \right) \partial_\nu \quad \text{and} \quad \partial_\nu = \partial_{\nu^*} + \partial_t.
\]

But the coefficients of the metric do not depend either on \(\phi_*\) or on \(t\). So, when computing the Christoffel symbols, and differentiating functions that depend exclusively on \(r\) and \(\theta\), the derivative \(\partial_{v_{CN^+}}\) may be replaced by

\[
\partial_{v_{CN^+}} = \sigma \partial_{\nu^*} = e^{-2\kappa - v} \partial_{\nu^*} = e^{-2\kappa - v} \left( \partial_{\nu^*} \right) \partial_\nu + \partial \theta \partial_\nu \partial_\theta = e^{-2\kappa - v} \Delta_r \left( \frac{1}{\Delta_r} \partial \theta \partial_\nu \partial_\theta \right) \partial_\nu + \left( \frac{1}{\Delta_r} \partial \theta \partial_\nu \partial_\theta \right) \partial_\nu.
\]

The expressions for the innermost parenthesis are obtained from \((38)\) and \((40)\). Note that they are well defined on the Cauchy horizon, notwithstanding the coordinate \(r_\nu\) not being defined there. The factor \(e^{-2\kappa - v} \Delta_r\) is bounded above and below according to \((75)\). So, given \(C_R \in \mathbb{R}\), there exist constants \(c, C > 0\) such that

\[
c e^{2\kappa - u} \leq -e^{-2\kappa - v} \Delta_r \leq C e^{2\kappa - u},
\]

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for $u + v \geq CR$. Special care has to be taken when differentiating $b^\phi_{c_n+}$ and $\Omega^r_{c_n+}$ with respect to $v_{c_n+}$ because these functions also depend on $v$. This occurs when calculating $\Gamma^u_{c_n+ v_{c_n+}}$:

$$\Gamma^u_{c_n+ v_{c_n+}} = \frac{\sigma (b^\phi_{c_n+})^2}{4 \Omega^2_{c_n+}} \partial_r \, \hat{g}_{\phi, \phi},$$

$$\Gamma^v_{c_n+ v_{c_n+}} = \frac{\hat{g}_{\phi, \phi} b^\phi_{c_n+}}{2 \Omega^2_{c_n+}} \partial_r + \frac{(b^\phi_{c_n+})^2}{4 \Omega^2_{c_n+}} \partial_r, \hat{g}_{\phi, \phi} + \sigma \left( \frac{\partial_r (\Omega^2)}{4 \Omega^2} - 2 \kappa_\ast \right),$$

$$\Gamma^\theta_{c_n+ v_{c_n+}} = -\sigma \hat{g}^\theta \phi_{c_n+} \partial_r, \hat{g}_{\phi, \phi} - \sigma \hat{g}^\phi \phi_{c_n+} \partial_r, \hat{g}_{\phi, \phi} = \frac{\hat{g}^\theta \phi_{c_n+}}{2 \partial_r, \hat{g}_{\phi, \phi} (||b^\phi_{c_n+}||^2)},$$

$$\Gamma^\phi_{c_n+ v_{c_n+}} = -\sigma \hat{g}^\theta \phi_{c_n+} \partial_r, \hat{g}_{\phi, \phi} - \sigma \hat{g}^\phi \phi_{c_n+} \partial_r, \hat{g}_{\phi, \phi} = \frac{(b^\phi_{c_n+})^2}{2 \Omega^2_{c_n+}} \partial_r, b^\phi_{c_n+} + \frac{(b^\phi_{c_n+})^3}{4 \Omega^2_{c_n+}} \partial_r, \hat{g}_{\phi, \phi}.$$

Notice that all terms are bounded except the last two summands of $\Gamma^\phi_{c_n+ v_{c_n+}}$ that add up to

$$2 \kappa_\ast \sigma b^\phi_{c_n+} - \sigma^2 \partial_r, b^\phi_{c_n+} = - \frac{\partial b^\phi_{c_n+}}{\partial v_{c_n+}}.$$

We remark that no terms that blow up at the Cauchy horizon appear when calculating $\nabla X_1 X_1$, for $X_1$ and $X_1$ elements of the frame (109). For example, we have that

$$\Gamma^u_{c_n+ v_{c_n+}} = \frac{(b^\phi_{c_n+})^2}{4 \Omega^2_{c_n+}} (e^{-2k_v} \Delta_r) \left( \frac{1}{\Delta_r} \partial_r \hat{g}_{\phi, \phi}, + \frac{1}{\Delta_r} \partial_r \hat{g}_{\phi, \phi} \right).$$

**Example A.3.** As a simple example, we calculate directly the surface gravity of the Cauchy horizon. Recall that the surface gravity $\kappa_\ast$ is given by

$$\nabla Z = \kappa_\ast Z,$$

where $Z$ is the Killing vector field given by (80). On the Cauchy horizon $\nabla \omega_\ast \omega_\ast = \nabla \omega_\ast \omega_\ast = 0$ apply, because $\Gamma^u_{\omega_\ast \omega_\ast c_n+} = \Gamma^v_{\omega_\ast \omega_\ast c_n+} = \Gamma^\theta_{\omega_\ast \omega_\ast c_n+} = \Gamma^\phi_{\omega_\ast \omega_\ast c_n+} = 0$ on the Cauchy horizon, and $\Gamma^v_{\omega_\ast \omega_\ast c_n+}$ is identically zero. Obviously,

$$\partial_u (b^\phi_{c_n+} - b^\phi_{c_n+}) = 0$$

and

$$\partial_r (b^\phi_{c_n+} - b^\phi_{c_n+}) = 0$$

hold because the vector fields $\partial_u$ and $\partial_r$ are tangent to the Cauchy horizon. Moreover, we know that

$$\partial_r e^{2k_v} = 2 \kappa_\ast e^{2k_v} \partial_r = 2 \kappa_\ast.$$

So, we conclude that

$$\nabla Z = \left[ \frac{1}{4} \nabla \omega_\ast \omega_\ast \partial_u + \frac{1}{4} \partial_u (b^\phi_{c_n+} - b^\phi_{c_n+}) \partial_r, \omega_\ast \omega_\ast + \frac{1}{4} (b^\phi_{c_n+} - b^\phi_{c_n+}) \partial_r, \omega_\ast \omega_\ast \right],$$

$$= \frac{1}{4} \nabla \omega_\ast \omega_\ast \partial_u + \frac{1}{4} \partial_u (b^\phi_{c_n+} - b^\phi_{c_n+}) \partial_r, \omega_\ast \omega_\ast + \frac{1}{2} (b^\phi_{c_n+} - b^\phi_{c_n+}) \nabla \omega_\ast \omega_\ast Z = \frac{e^{2k_v} - 2 \kappa_\ast}{2} \nabla \omega_\ast \omega_\ast + \frac{b^\phi_{c_n+} - b^\phi_{c_n+}}{2} \partial_r \omega_\ast \omega_\ast Z.$$

We used (77), then (18) and finally (38). This shows that

$$\kappa_\ast = \frac{\partial_r \omega_\ast \omega_\ast}{2 \Omega^2} = \frac{(\partial_r \Delta_r)_{r=r_\ast}}{2 (r_\ast^2 + a^2)},$$

which is equality (72).
A.3 \((u_{H+}, v_{H+}, \theta_*, \phi_*, H+)\) coordinates

If we use coordinates \((u_{H+}, v_{H+}, \theta_*, \phi_*, H+)\) then, in the formulas above, the value of \(\sigma\) is 1, but one has to replace \(\Omega^2\) by \(\Omega_{H+}^2\), \(b^\phi\) by \(b_{H+}^\phi\), and the value of \(\varsigma\) has to be as in (81). Special care has to be taken when calculating \(\Gamma_{u_{H+} u_{H+}}\) because in this case one has to differentiate \(\Omega_{H+}^2\) with respect to \(u\):

\[
\Gamma_{u_{H+} u_{H+}} = \varsigma \left( \frac{\partial_r (\Omega^2)}{\Omega^2} - 2\kappa \right),
\]

\[
\Gamma_{v_{H+} u_{H+}} = \Gamma_{\theta_{H+} u_{H+}} = \Gamma_{\phi_{H+} u_{H+}} = 0.
\]

B Characterization of the parameters of subextremal \(\text{KN}dS\) spacetimes

The main result of this appendix is summarized in Lemma B.1, which characterizes subextremal Kerr–Newman–de Sitter black holes in terms of \((r_-, r_+, \Lambda \alpha^2, \Lambda \epsilon^2)\). In Remarks B.2 and B.3, we consider alternative choices of parameters, namely \((\Lambda, \frac{r_+ - r_-}{2}, a, e)\) or \((\Lambda, M, a, e)\). As mentioned in the Introduction, in the case that \(e = 0\), related results can be found in Lake and Zannias [24] and Borthwick [4].

**Lemma B.1.** Each Kerr–Newman–de Sitter subextremal solution is determined by a quadruple \((r_-, r_+, \Lambda \alpha^2, \Lambda \epsilon^2)\) satisfying \(0 < r_- < r_+\),

\[
0 \leq \Lambda \epsilon^2 < \frac{3\alpha^2(1 + 2\alpha)}{(1 + 2\alpha + 3\alpha^2)^2} < \frac{1}{4}
\]

and

\[
0 < \Lambda \alpha^2 < l(\alpha, \Lambda \epsilon^2),
\]

with

\[
\alpha = \frac{r_+}{r_-},
\]

where \(l\) is the function given by (137). The graph of \(l\) is sketched in Figure 3. The graphs of \(l(\cdot, \Lambda \epsilon^2)\) for several values of \(\Lambda \epsilon^2 \in \left[0, \frac{1}{4}\right]\) are sketched in Figure 4. For a choice of parameters on the graph of \(l\) we have \(r_+ = r_c\) (see Figure 5). The value of \(\Lambda\) is given by

![Figure 3](image-url)

Figure 3: The graph of \(l\) bounds the region where the parameters \(\alpha = \frac{r_+}{r_-}\), \(\epsilon = \Lambda \alpha^2\) and \(\gamma = \Lambda \epsilon^2\) can vary.
Figure 4: Sketch of the graphs of \( l(\cdot, 0) \), \( l(\cdot, 0.05) \), \( l(\cdot, 0.1) \), \( l(\cdot, 0.15) \), \( l(\cdot, 0.20) \), \( l(\cdot, 0.24) \).

Figure 5: The graph of \( l(\cdot, 1/8) \).

\[
\Lambda = \frac{(3 - \Lambda a^2) - 3 \sqrt{(1 - \frac{\Lambda a^2}{3})^2 - \frac{4(\Lambda a^2 + \Lambda e^2)(r_+^2 + r_+^2 + r_-)}{3r_--r_+}}}{2(r_+^2 + r_+^2 + r_-)} , \tag{115}
\]

and the value of \( M \) is given by

\[
M = \frac{\Lambda}{6} (2r_-r_+r_c + r_+^2r_c + r_-^2r_c + r_+^2r_- + r_+^2r_c + r_+^2r_- + r_+^2r_c) , \tag{116}
\]

with \( r_c \) given by

\[
r_c = -\frac{r_- + r_+}{2} + \sqrt{\left(\frac{r_- + r_+}{2}\right)^2 + \frac{3(a^2 + e^2)}{\Lambda r_-} - 12} . \tag{117}
\]

Alternative formulas for the mass are

\[
\sqrt{\Lambda}M = \frac{(1 + \alpha)(\alpha^2 (\sqrt{\Lambda r_-})^4 + 3(\Lambda a^2 + \Lambda e^2))}{6\alpha(\sqrt{\Lambda r_-})} , \tag{118}
\]

\[
= \frac{-(\sqrt{\Lambda r_-})^4 + (3 - \Lambda a^2)(\sqrt{\Lambda r_-})^2 + 3(\Lambda a^2 + \Lambda e^2)}{6(\sqrt{\Lambda r_-})} . \tag{119}
\]

**Remark B.2.** We can write (115) in the form

\[
r_-^2 = \frac{3 - \Lambda a^2 - \sqrt{9a - 6(2 + 3a + 2e^2)\Lambda a^2 + 3(\Lambda a^2)^2 - 12(1 + \alpha + a^2)\Lambda e^2}}{2\Lambda (1 + \alpha + a^2)} . \tag{120}
\]

This formula is useful if we start with a parameter set

\((\Lambda, \alpha, a, e)\), with \(0 \leq \Lambda e^2 < \frac{1}{4}\), \(0 < \Lambda a^2 \leq 21 - 6\sqrt{12 + \Lambda e^2}\) and \(1 < \alpha < l^{-1} (\Lambda a^2, \Lambda e^2)\)
(see Remark B.6 for a clarification of the meaning of $l^{-1}$). Fixing $\Lambda, e$ and $a$, consider the function $r_-(\Lambda, \cdot, a, e)$. We have

$$\partial_\alpha r_-(\Lambda, 1, a, e) = -\frac{r}{2}(\Lambda, 1, a, e) = -\partial_\alpha r_+ (\Lambda, 1, a, e),$$

$$\partial_\alpha r_-(\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e) = 0,$$

$$\partial_\alpha r_e (\Lambda, 1, a, e) = 0,$$

$$\partial_\alpha r_e (\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e) = -\partial_\alpha r_+ (\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e).$$

Note that the functions $\sqrt{\Lambda r_\#}$ depend solely on $(\alpha, \Lambda a^2, \Lambda e^2)$. In Figure 6, we sketch their graphs for $\Lambda e^2 = \gamma = 1/8$ and $\Lambda a^2 = \epsilon = (21 - 6\sqrt{12 + \gamma})/2$. (See (141) for a formula for $l^{-1}(\Lambda a^2, \Lambda e^2)$.) This implies that the mass $M$ (whose expression is given in (116), (118) and (119)) satisfies

$$\partial_\alpha M_-(\Lambda, 1, a, e) = \partial_\alpha M(\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e) = 0.$$

The function $M(\Lambda, \cdot, a, e)$ is strictly increasing in the interval $[1, l^{-1}(\Lambda a^2, \Lambda e^2)]$. The function $\sqrt{\Lambda M}$ also depends solely on $(\alpha, \Lambda a^2, \Lambda e^2)$. In Figure (7), we sketch the graph of $\sqrt{\Lambda M}$ for $\Lambda e^2 = 1/8$ (recall that $\Lambda e^2 \in [0, \frac{1}{4})$).

We can use $\partial_\alpha r_-(\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e) = 0$ to show that $\partial_\alpha M(\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e) = 0$; this together with (116) allows us to deduce that $\partial_\alpha r_e (\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e) = -\partial_\alpha r_+ (\Lambda, l^{-1}(\Lambda a^2, \Lambda e^2), a, e)$. The other assertions follow by direct calculation.

**Remark B.3.** In Lemma B.1 we can think that our subextremal solution is characterized by $(r_-, \alpha, \Lambda a^2, \Lambda e^2)$ and in Remark B.2 we can think that our subextremal solution is characterized by $(\Lambda, \alpha, \Lambda a^2, \Lambda e^2)$. Formula (120) relates $r_-$ with $\Lambda$. In this paper, the preferred viewpoint is that each spacetime is characterized
by the three parameters \((\alpha, \epsilon, \gamma) = \left(\frac{\sqrt{\Lambda} + c}{\sqrt{\Lambda}} - 1, \frac{\sqrt{\Lambda} a}{2}, \frac{\sqrt{\Lambda} e}{2}\right)\). These determine the other quantities, like \(\sqrt{\Lambda} r\) or \(\sqrt{\Lambda} M\). In this sense, the cosmological constant appears as a scale parameter.

**Remark B.4.** The reader interested in the characterization of Kerr–de Sitter spacetimes in terms of the parameters \((\Lambda, M, a)\) (which is physically more natural, although not what we need in Section 2) should consult Borthwick [4], where extreme and fast Kerr–de Sitter are also considered.

**Remark B.5.** In summary, the region where the parameters \((\alpha, \epsilon, \gamma)\) can vary (which is sketched in Figure 3) is defined by

\[
\{(\alpha, \epsilon, \gamma) \in \mathbb{R}^3 : 1 < \alpha < \infty, 0 \leq \gamma < E(\alpha), 0 < \epsilon < l(\alpha, \gamma)\},
\]

where \(E\) is given in (136) and \(l\) is given in (137); alternatively, by

\[
\{(\alpha, \epsilon, \gamma) \in \mathbb{R}^3 : 0 \leq \gamma < \frac{1}{4}, 1 < \alpha < \Gamma(\gamma), 0 < \epsilon < l(\alpha, \gamma)\},
\]

where \(\Gamma\) is given in (138); or still by

\[
\{(\alpha, \epsilon, \gamma) \in \mathbb{R}^3 : 0 \leq \gamma < \frac{1}{4}, 0 < \epsilon < 21 - 6\sqrt{12 + \gamma}, 1 < \alpha < l^{-1}(\epsilon, \gamma)\},
\]

where \(l^{-1}\) is given in (141).

**Proof of Lemma B.1.**

1) \((\Delta_r)\) The relationship between \((M, \Lambda, a, e)\) and \((r_-, r_+, \Lambda a^2, \Lambda e^2)\) is obtained via the polynomial \(\Delta_r\). On the one hand this polynomial is given by

\[
\Delta_r = (r^2 + a^2) \left(1 - \frac{\Lambda}{3} r^2\right) - 2Mr + e^2
\]

\[= -\frac{\Lambda}{3} r^4 + \left(1 - \frac{\Lambda}{3} r^2\right) r^2 - 2Mr + a^2 + e^2,
\]

and on the other hand it is given by

\[
\Delta_r = -\frac{\Lambda}{3} (r - r_n)(r - r_-)(r - r_+)(r - r_c),
\]

since, according to our subextremal hypothesis, \(\Delta_r\) has one negative root \(r_n\), and three positive roots \(r_- < r_+ < r_c\). Expanding the last expression yields

\[
\Delta_r = -\frac{\Lambda}{3} r^4 + \frac{\Lambda}{3} (r_n + r_- + r_+ + r_c) r^3
\]

\[-\frac{\Lambda}{3} (r_n r_- + r_n r_+ + r_n r_c + r_- r_+ + r_- r_c + r_+ r_c) r^2
\]

\[-\frac{\Lambda}{3} (-r_n r_- r_+ - r_n r_- r_c - r_n r_+ r_c - r_- r_+ r_c) r
\]

\[-\frac{\Lambda}{3} r_n r_- r_+ r_c.
\]

We will now compare the coefficients of \(\Delta_r\) in \(r^0, r^1, r^2\) and \(r^3\); the coefficient in \(r^4\) is already matched.

1a) **(Coefficient in \(r^3\))** We start with the coefficient of \(\Delta_r\) in \(r^3\). As it is equal to 0, we have

\[
r_n = -(r_- + r_+ + r_c).
\]

(121)

1b) **(Coefficient in \(r^0\))** Equating the coefficients of \(\Delta_r\) in \(r^0\), we get, using (121),

\[
\frac{\Lambda}{3} (r_- + r_+ + r_c) r_- r_+ r_c = a^2 + e^2.
\]

(122)

We regard this as a quadratic equation for \(r_c\), which will be satisfied by choosing an appropriate \(r_c\). Defining

\[
x := \frac{3(a^2 + e^2)}{\Lambda r_- r_+},
\]

(123)

(122) can be rewritten as

\[
x = r_c (r_c + r_- + r_+).
\]

(124)
1c) (Coefficient in $r^1$) Equating the coefficients of $\Delta r$ in $r^1$ we arrive at the value of $M$ given in (116). The expression (118) is obtained from (116) using $r_+ = \alpha r_-$ and (117) (deduced below). The formula (119) is the statement that $r_-$ is a root of $\Delta r$. Of course, in (119) we may replace $r_-$ by $r_+$ or by $r_c$.

1d) (Coefficient in $r^2$) Finally, we compare the coefficients of $\Delta r$ in $r^2$. Using again (121), leads to

$$\frac{\Lambda}{3}(r^2 + r_+^2 + r_- r_+ + x) = 1 - \frac{\Lambda}{3} a^2.$$  \hfill (125)

Defining

$$b := r^2 + r_+^2 + r_- r_+,$$
$$\epsilon := \Lambda a^2,$$
$$\gamma := \Lambda c^2,$$

and using (123), equation (125) is equivalent to

$$\frac{\Lambda}{3}b + \frac{\epsilon + \gamma}{\Lambda r_- r_+} = 1 - \frac{\epsilon}{3}.$$

or

$$\frac{b}{3}\Lambda^2 - \left(1 - \frac{\epsilon}{3}\right) \Lambda + \frac{\epsilon + \gamma}{r_- r_+} = 0.$$  \hfill (126)

This is a quadratic equation which will determine $\Lambda$.

2) (\Lambda) As we are considering de Sitter black holes, $\Lambda$ is positive, and therefore $\epsilon$ is positive and $\gamma$ is nonnegative. So, the product of the roots of (126) is positive, and the two roots have the same sign. Since $\Lambda$ is positive, we must have

$$0 < \epsilon < 3.$$  \hfill (127)

Solving (126), we get

$$\Lambda = \frac{(1 - \frac{\epsilon}{3}) \pm \sqrt{(1 - \frac{\epsilon}{3})^2 - 4(1 + \frac{1}{\alpha})(\epsilon + \gamma)}}{2\frac{1}{\alpha}}.$$  \hfill (128)

We denote these values of $\Lambda$ by $\Lambda_0$ and $\Lambda_1$. $\Lambda_0$ and $\Lambda_1$ will have the minus sign and the plus sign in front of the square root, respectively, so that $\Lambda_0 \leq \Lambda_1$. We will see below that only the value of $\Lambda_0$ is admissible.

2a) (\Lambda real) Define

$$\alpha := \frac{r_+}{r_-} > 1,$$

so that

$$\frac{b}{r_- r_+} = 1 + \alpha + \frac{1}{\alpha}.$$

The discriminant in (128) should be nonnegative, that is

$$\left(1 - \frac{\epsilon}{3}\right)^2 - 4(1 + \alpha + \alpha^2)(\epsilon + \gamma) \geq 0.$$  \hfill (129)

Since $\epsilon$ is positive, we see that $\gamma$ has to satisfy

$$0 \leq \gamma < \frac{3}{4(1 + \alpha + \frac{1}{\alpha})} \leq \frac{1}{4}.$$  \hfill (130)

This is not our final restriction on $\epsilon$ though. By requiring that $r_c > r_+$ a further restriction on $\epsilon$ will arise, as well as the requirement that $\Lambda = \Lambda_0$.

3) ($r_c$) To obtain $r_c$, we rewrite (124) as

$$r_c^2 + (r_- + r_+)r_c - x = 0.$$  \hfill (124)

Hence,

$$r_c = -\frac{r_- + r_+}{2} + \sqrt{\frac{(r_- + r_+)^2}{4} + x}.$$  \hfill (124)

This is (117). The condition $r_c > r_+$ is equivalent to

$$(r_- + r_+)^2 + 4x > (r_- + 3r_+)^2,$$

or

$$x > 2r_+^2 + r_- r_+.$$  \hfill (117)
Using the definition of $x$ in (123), this is the same as
\[
\frac{3(a^2 + e^2)}{\Lambda r_{-} r_{+}} > 2r_{+}^2 + r_{-} r_{+},
\]
or
\[
\frac{a^2 + e^2}{\Lambda} > \frac{1}{3} r_{-} r_{+}^2 (r_{-} + 2 r_{+}) =: k.
\]
(131)

Now we use the definition of $\epsilon$ and the assumption that $\Lambda$ is positive which, as we have seen, implies $0 < \epsilon < 3$.

For $\epsilon$ in this range, (131) is equivalent to
\[
\epsilon + \gamma > k \Lambda^2 = \frac{3k}{b} (1 - \frac{\epsilon}{3}) \Lambda - \frac{3k}{b r_{-} r_{+}} (\epsilon + \gamma),
\]
where, for the last equality, we used the quadratic equation (126) for $\Lambda$. Our last task is to guarantee (132).

\textbf{3a) } ($\Lambda = \Lambda_0$) First we take $\Lambda = \Lambda_0$ in (132). Then
\[
\left( \frac{b}{3k} + \frac{1}{r_{-} r_{+}} \right) \frac{\epsilon + \gamma}{1 - \frac{\epsilon}{3}} > \Lambda_0 = \left( 1 - \frac{4}{9} \right) - \sqrt{\left( 1 - \frac{4}{9} \right)^2 - \frac{4(\epsilon + \gamma) b}{3 r_{-} r_{+}}},
\]
which can be rewritten as
\[
\sqrt{\left( 1 - \frac{\epsilon}{3} \right)^2 - \frac{4(\epsilon + \gamma) b}{3 r_{-} r_{+}}} > \left( 1 - \frac{\epsilon}{3} \right) - \frac{2}{3} \left( \frac{b}{3k} + \frac{1}{r_{-} r_{+}} \right) \frac{\epsilon + \gamma}{1 - \frac{\epsilon}{3}}
\]
(133)

We need to examine the sign of the right-hand side of (133). This can be written as
\[
\left( 1 - \frac{\epsilon}{3} \right) - \frac{2}{3} \left( 1 + \alpha + \frac{1}{\alpha} \right) \left( 1 + 2 \alpha + 3 \alpha^2 \right) \left( \epsilon + \gamma \right)
\]
(134)

Above we guaranteed (129). Since we have that
\[
\frac{4(1 + \alpha + \alpha^2)}{3 \alpha^2 (1 + 2 \alpha)} \geq \frac{2 \alpha (1 + 2 \alpha)}{1 + 2 \alpha + 3 \alpha^2} \geq 1
\]
for $\alpha \geq 1$, the right-hand side of (133) is nonnegative. Therefore, (133) is equivalent to
\[
\left( 1 - \frac{\epsilon}{3} \right)^2 - \frac{2}{3} \left( 1 + \alpha + \frac{1}{\alpha} \right) \left( \epsilon + \gamma \right)
\]
\[
- \left[ \left( 1 - \frac{\epsilon}{3} \right) - \frac{2}{3} \left( 1 + \alpha + \alpha^2 \right) \left( 1 + 2 \alpha + 3 \alpha^2 \right) \left( \epsilon + \gamma \right) \right]^2 > 0.
\]
(135)

Let us define
\[
c = \frac{4}{3} \left( 1 + \alpha + \frac{1}{\alpha} \right),
\]
\[
d = \frac{2}{3} \left( 1 + \alpha + \alpha^2 \right) \left( 1 + 2 \alpha + 3 \alpha^2 \right)
\]
Then, expanding the left-hand side of (134), we get
\[
\left( 1 - \frac{\epsilon}{3} \right)^2 (2d - c) - d^2 (\epsilon + \gamma) > 0,
\]
or
\[
\left( 1 - \frac{\epsilon}{3} \right)^2 - \frac{d^2}{2d - c} (\epsilon + \gamma) > 0,
\]
because
\[
2d - c = \frac{4(1 + \alpha + \alpha^2)^2}{3 \alpha^2 (1 + 2 \alpha)}
\]
is positive. Substituting $c$ and $d$ by their expressions in terms of $\alpha$, we obtain
\[
\left( 1 - \frac{\epsilon}{3} \right)^2 - \frac{(1 + 2 \alpha + 3 \alpha^2)^2}{3 \alpha^2 (1 + 2 \alpha)} (\epsilon + \gamma) > 0.
\]
The inequality (135) has solutions for $\epsilon$ positive if and only if
\[ \gamma < E(\alpha) := \frac{3\alpha^2(1 + 2\alpha)}{(1 + 2\alpha + 3\alpha^2)^2} \leq \frac{1}{4}. \tag{136} \]

This inequality for $\gamma$ is more restrictive than (130). We remark that
\[ \frac{(1 + 2\alpha + 3\alpha^2)^2}{4(1 + 2\alpha)} = \frac{(1 + 2\alpha + 3\alpha^2)^2}{4\alpha(1 + 2\alpha)(1 + \alpha + \alpha^2)} \geq 1 \text{ for } \alpha \geq 1 \]
(this function is 1 at $\alpha = 1$ and grows to $\frac{3}{2}$ as $\alpha$ goes to $+\infty$). Therefore, all solutions of (135) are solutions of (129). The inequality (135) is equivalent to
\[ 0 < \epsilon < l(\alpha, \gamma) \text{ or } \epsilon > \tilde{r}(\alpha, \gamma), \]
where
\[ l(\alpha, \gamma) := \frac{3}{2\alpha^2(1 + 2\alpha)} \left( \left(1 + 4\alpha + 12\alpha^2 + 16\alpha^3 + 9\alpha^4\right) - \left(1 + 2\alpha + 3\alpha^2\right) \sqrt{(1 + \alpha)^2(1 + 2\alpha + 9\alpha^2) + \frac{4\gamma}{3}\alpha^2(1 + 2\alpha)} \right), \tag{137} \]
and $\tilde{r}(\alpha, \gamma)$ is defined by the same expression, with the minus sign replaced by a plus sign. The function $\tilde{r}$ satisfies
\[ \tilde{r}(\alpha, \gamma) > \frac{3}{2\alpha^2(1 + 2\alpha)} (12\alpha^2 + 16\alpha^3) > 3. \]

So, the inequality $\epsilon > \tilde{r}(\alpha, \gamma)$ is incompatible with (127). Fix a $\gamma$ belonging to the interval $[0, \frac{1}{4})$. The function $l(\cdot, \gamma)$ is strictly decreasing and satisfies
\[ l(1, \gamma) = 21 - 6\sqrt{12 + \gamma}, \quad \partial_\gamma l(1, \gamma) = 0. \]

In Figure 4, we sketch the graphs of $l(\cdot, 0)$, $l(\cdot, 0.05)$, $l(\cdot, 0.1)$, $l(\cdot, 0.15)$, $l(\cdot, 0.20)$ and $l(\cdot, 0.24)$. The function $l(\cdot, \gamma)$ is nonnegative for $\alpha \in (1, \Gamma(\gamma))$, and
\[ l(\Gamma(\gamma), \gamma) = 0, \]
where $\Gamma(\gamma)$ is the inverse of the function $E$ in (136), i.e. is the solution greater than one of
\[ \gamma = \frac{3\Gamma^2(\gamma)(1 + 2\Gamma(\gamma))}{(1 + 2\Gamma(\gamma) + 3\Gamma^2(\gamma))^2}, \tag{138} \]

namely
\[ \Gamma(\gamma) := \frac{1 - 2\gamma + \sqrt{1 - 4\gamma + \sqrt{2 - 1 - 4\gamma - 4\gamma^2 + (1 + \gamma)\sqrt{1 - 4\gamma}}}}{6\gamma} \]
for $\gamma \in (0, \frac{1}{4})$, $\Gamma(0) = +\infty$ (in which case $\alpha \in (1, +\infty)$). Note that
\[ \lim_{\gamma \to \frac{1}{4}} \Gamma'(\gamma) = -\infty, \quad \lim_{\gamma \to 0} \frac{\Gamma(\gamma)}{2\gamma^3} = 1. \]

The graph of $\Gamma$ is sketched in Figure 8. We remark that
\[ \lim_{\alpha \to +\infty} \frac{l(\alpha, 0)}{2\alpha^3} = 1. \tag{139} \]

In summary, for $\Lambda = \Lambda_0$ inequality (132) is satisfied for
\[ 0 < \epsilon < l(\alpha, \gamma). \]

3b) ($\Lambda = \Lambda_1$) Now we take $\Lambda = \Lambda_1$ in (132). Then, instead of (133), we obtain
\[ \frac{2b}{3} \left( \frac{b}{3k} + \frac{1}{r-r_+} \right) \epsilon + \gamma \left(1 - \frac{\epsilon}{3}\right) < \left(1 - \frac{\epsilon}{3}\right)^2 - \frac{4(\epsilon + \gamma)b}{3r-r_+}. \tag{140} \]
Figure 8: Sketch the graph of $\Gamma$. The shaded region is the projection of the parameter region in the $(\alpha, 0, \gamma)$ plane.

Note that the left-hand side of (140) is the symmetric of the right-hand side of (133). We have shown above that the right-hand side of (133) is nonnegative. Hence, inequality (140) never holds.

We conclude that we must choose $\Lambda = \Lambda_0$ in (128). This establishes expression (115) of Lemma B.1.

Define

$$s(\epsilon, \gamma) := \sqrt{(\epsilon - (21 - 6\sqrt{12} + \gamma))(\epsilon - (21 + 6\sqrt{12} + \gamma))} = \sqrt{9 - 42\epsilon + \epsilon^2 - 36\gamma}.$$

**Remark B.6.** The inverse of the function $l$, in the sense that

$$l^{-1}(l(\alpha, \gamma), \gamma) = \alpha \quad \text{and} \quad l(l^{-1}(\epsilon, \gamma), \gamma) = \epsilon,$$

is

$$l^{-1}(\epsilon, \gamma) = \frac{1}{54(\epsilon + \gamma)} \left( (3 - \epsilon)s(\epsilon, \gamma) + 9 - 24\epsilon + \epsilon^2 - 18\gamma \right. \left. + \sqrt{2 \left( (27 - \epsilon^3 + 9\gamma(3 - \epsilon))s(\epsilon, \gamma) + (81 - 189\epsilon - 216\epsilon^2 - 21\epsilon^3 + \epsilon^4) - 9\gamma(9 + 36\gamma + 66\epsilon + \epsilon^2) \right)} \right),$$

for $0 \leq \gamma < \frac{1}{4}$ and $\epsilon \in [0, 21 - 6\sqrt{12} + \gamma]$. In particular, $l^{-1}(21 - 6\sqrt{12} + \gamma, \gamma) = l^{-1}(l(1, \gamma), \gamma) = 1$.

**Remark B.7.** If we start with a parameter set $(\Lambda, M, a, e)$, we can regard (115) and (118) as a system for $r_-$ and $\alpha$. In this case, it is natural to consider the quantities

$$A := \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}},$$

$$R := \sqrt{\Lambda \alpha r_-} = \sqrt{\Lambda r_- r_+}.$$

In terms of these quantities, (115) and (118) can be written as

$$R^2 = \frac{3 - \epsilon - 3\sqrt{(1 - \frac{\epsilon}{4})^2 - \frac{4(A^2 - 1)(\epsilon + \gamma)}{3}}}{2(A^2 - 1)},$$

$$\sqrt{\Lambda M} = \frac{A R^4 + 3(\epsilon + \gamma)}{R^2}.$$  

The equation (142) can be solved for $A$, yielding

$$A = \frac{\sqrt{R^4 + (3 - \epsilon)R^2 - 3(\epsilon + \gamma)}}{R^2}.$$  

The system (143)–(144) allows us to determine $A$ and $R$ from $\sqrt{\Lambda M}$, $\Lambda a^2$ and $\Lambda e^2$. Of course, another way to do this would be by calculating the roots of the polynomial $\Delta_r$. 

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C Komar integrals

The purpose of this appendix is to obtain expression (149), for the mass of the black hole at the event horizon. Besides the expected term $\frac{\sqrt{\Omega}}{16\pi}$, the mass has one correction term which is a constant multiple of the product of the cosmological constant by the Parikh volume of the black hole, and has another term which reflects the energy due to the electric field. In this way we recall the physical interpretation of the parameters of our metric. Let $K := \partial_a$. Consider the 2-dimensional surface on the event horizon $S = \{(u_{H+}, v_{H+}, \theta_*, \phi_*, H): u_{H+} = 0, v_{H+} = (v_{H+})_0\}$, where $v_{H+} = (v_{H+})_0$, for some fixed $(v_{H+})_0$. Let us calculate the Komar integral

$$J Q = \frac{1}{16\pi} \int_S \ast d\tilde{K}_s.$$  

($J Q$ is not the angular momentum if $e \neq 0$. See [15, Subsection III.B] for the definition of the angular momentum.)

Remark C.1. In the calculation below we use the coordinates $(u, v, \theta, \phi)$ of Subsection 2.1.3, notwithstanding the fact that $S$ is contained in the event horizon, where these coordinates are not defined. Our computations may be easily justified by considering a sequence $S_n$ of surfaces which approximate $S$ from within the black hole, which is covered by the coordinates, computing the integrals over $S_n$, and then passing to the limit.

We have that

$$K = -b^\phi \ast \hat{g}_{\phi, \phi}, dv + \hat{g}_{\theta, \phi}, d\theta + \hat{g}_{\phi, \phi}, d\phi$$

and

$$(\ast \ast d\tilde{K}_s)_{\alpha \beta} = \frac{1}{2}(d\tilde{K}_s)^{\mu\nu} \mathrm{d} \mathrm{Vol}_{\alpha \beta}.$$  

As the vectors on the tangent space of $S$ do not have neither components in $\partial_u$ nor components in $\partial_v$, only the components $(d\tilde{K}_s)^{uv}$ and $(d\tilde{K}_s)^{vu}$ are relevant for the calculation of $J Q$. On the other hand, as

$$(du)^2 = -\frac{1}{24\Omega^2} \partial_u + \frac{b^\phi}{2\Omega^2} \partial_\phi,,$$

$$(dv)^2 = -\frac{1}{24\Omega^2} \partial_u,$$

$$(d\theta)^2 = \hat{g}^{\theta, \theta} \partial_u + \hat{g}^{\theta, \phi} \partial_\phi,$$

$$(d\phi)^2 = -\frac{b^\phi}{2\Omega^2} \partial_u + \hat{g}^{\phi, \phi} \partial_\phi,$$

the terms $(d\tilde{K}_s)_{\theta, \theta}$ and $(d\tilde{K}_s)_{\phi, \phi}$ do not contribute to $J Q$. Notice that $(d\tilde{K}_s)_{v, \phi}$ and $(d\tilde{K}_s)_{\phi, v}$ do not enter into the calculation of $(d\tilde{K}_s)^{uv}$ and $(d\tilde{K}_s)^{vu}$ either. Thus, we write

$$d\tilde{K}_s = -\partial_u (b^\phi \ast \hat{g}_{\phi, \phi}) du \wedge dv + \partial_u \hat{g}_{\phi, \phi}, du \wedge d\phi + \ldots,$$

and

$$\ast \ast d\tilde{K}_s = \frac{1}{24\Omega^2} \partial_u \hat{g}_{\phi, \phi}, \partial_\phi, b^\phi \frac{L \sin \theta}{\Xi} d\theta \wedge d\phi + \ldots.$$  

Remembering that $b^\phi$ is constant on the event horizon, we have

$$\frac{\partial_r b^\phi}{\Omega^2} = \frac{\partial_r \frac{b^\phi}{\Omega^2}}{\Omega^2} = \frac{\partial_r b^\phi}{\Omega^2} \frac{\Delta_r}{\Delta_r} = -\partial_r b^\phi \frac{\Upsilon}{\rho^2 \Delta_r} \frac{1}{r^2 + a^2} = -\partial_r b^\phi \frac{r^2 + a^2}{\rho^2}$$

on the event horizon, because $\Upsilon = 4(r^2 + a^2)\Delta_r$ there. Since

$$\partial_r b^\phi = \frac{4\Xi a \partial_r \Delta_r}{\Upsilon} - \frac{2\Xi a \Delta_r}{\Upsilon} - \frac{2\Xi a (r^2 + a^2) \Delta_r}{\Upsilon^2} (4(r^2 + a^2)\Delta_r - a^2 \sin^2 \theta \partial_r \Delta_r)$$

$$= \frac{2\Xi a \rho^2 \partial_r \Delta_r}{\Upsilon(r^2 + a^2)}$$

and

$$\hat{g}_{\phi, \phi} = \frac{\sin^2 \theta}{\Xi^2 \rho^2} \Upsilon,$$

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it follows that
\[
* dK = \frac{a}{\Xi} \left( 2(r^2 + a^2) \left( \frac{\Delta_0 \sin^2 \theta}{(r^2 + a^2 \cos^2 \theta)^2} + \partial_r \Delta_r \frac{\sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \right) \right) L \sin \frac{\theta}{\Xi} \, d\theta_0 \wedge d\phi_* + \ldots
\]
\[
= \frac{a}{\Xi} \left( 2(r^2 + a^2) \left( \frac{\Delta_0 \sin^3 \theta}{(r^2 + a^2 \cos^2 \theta)^2} + \partial_r \Delta_r \frac{\sin^3 \theta}{r^2 + a^2 \cos^2 \theta} \right) \right) \frac{r^2 + a^2 \theta \theta \theta_0}{\Xi} \, d\theta_0 \wedge d\phi_* + \ldots,
\]
because \( L = (r^2 + a^2) \partial_\theta \theta \) on the event horizon. Now, we have
\[
\int_0^\pi \frac{(1 + \frac{1}{3} a^2 \cos^2 \theta) \sin^3 \theta}{(r^2 + a^2 \cos^2 \theta)^2} \, d\theta = \int_0^1 \frac{(1 + \frac{1}{3} a^2 x^2)(1 - x^2)}{(r^2 + a^2 x^2)^2} \, dx
\]
\[
= -3a r \left( \frac{r^2 - 1}{a^2} + 3 a \left( \frac{r^2 - 1}{a^2} \right) \right) \arctan \left( \frac{x}{2} \right),
\]
\[
\int_0^\pi \frac{\sin^3 \theta}{r^2 + a^2 \cos^2 \theta} \, d\theta = \int_0^1 \frac{1 - x^2}{r^2 + a^2 x^2} \, dx
\]
\[
= \frac{-a r + (r^2 + a^2) \arctan \left( \frac{x}{2} \right)}{a^3 r}.
\]
Taking into account that
\[
\partial_r \Delta_r = -\frac{2}{3} a r \left( r^2 + a^2 \right) - 2 M + 2 r \left( 1 - \frac{\Delta r^2}{3} \right)
\]
and
\[
M = \frac{(r^2 + a^2) \left( 1 - \frac{\Delta r^2}{3} \right) + e^2}{2 r},
\]
we arrive at
\[
2 r(r^2 + a^2) \int_0^\pi \frac{\Delta_0 \sin^2 \theta}{(r^2 + a^2 \cos^2 \theta)^2} \, d\theta + \partial_r \Delta_r \int_0^\pi \frac{\sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \, d\theta
\]
\[
= -4a^2 r \left( \frac{r^2 - 3}{a^2} + 6 e^2 (a r - (r^2 + a^2) \arctan \left( \frac{x}{2} \right)) \right)
\]
\[
= \frac{8 M}{r^2 + a^2} - \frac{2e^2}{a(r^2 + a^2)} \frac{a^3 r^2}{\frac{a}{r}}
\]
\[
= \frac{8 M}{r^2 + a^2} - \frac{2e^2}{a(r^2 + a^2)} BT \left( \frac{a}{r} \right).
\]
Here
\[
BT(x) = (x^2 + 2) \arctan x + \frac{\arctan x - x + x^3}{x^2},
\]
an expression that makes clear the behavior of the function BT, as the first two terms of the Taylor expansion of \( x \mapsto \arctan x \) around zero are \( x - \frac{x^3}{3} \). In particular, the function BT is strictly increasing, satisfies BT(0) = 0, BT(0) = \( \frac{1}{2} \), and \( \lim_{x \to \infty} \left( BT(x) / \left( \frac{x^2}{2} \right) \right) = 1 \). This finally yields
\[
JQ = \frac{Ma}{\Xi^2} - \frac{e^2}{4 \Xi^2} BT \left( \frac{a}{r_v} \right) = \frac{Ma}{\Xi^2} - \frac{Q^2}{4} BT \left( \frac{a}{r_v} \right),
\]
with
\[
Q = \frac{e}{\Xi}
\]
the total charge (see [5, (19)] and [17, beginning of Subsection 3.2 on p. 32]).

Now we consider \( K \) to be \( K = \frac{\partial}{\partial r} \) and we turn to the calculation of the Komar integral
\[
\mathcal{M} = -\frac{1}{8 \pi} \int_S * dK_\gamma.
\]
This is the mass calculated at the event horizon. To justify this choice of \( K \) we refer to [16, (2.12) and Subsection 2.2] (see also [5, (18)]). As \( \partial_t = \frac{1}{2} \partial_t - \frac{1}{2} \partial_\theta \), we have that
\[
\Xi K_\gamma = -\Omega^2 du + \Omega^2 dv + \frac{(b^\phi)^2 g_{\phi^* \phi}}{2} \, dv - \frac{b^\phi \cdot g_{\phi^* \phi}}{2} \, d\theta_* - \frac{b^\phi \cdot g_{\phi^* \phi}}{2} \, d\phi_*,
\]
53
\[ \Xi dK_s = 2 \partial_r \Omega^2 \, du + \partial_r ((b^s) d\phi_s) du + \frac{1}{2} \partial_r ((b^s) (\partial_r \phi_s)) du + \frac{1}{2} \partial_r ((b^s) (\partial_r \phi_s)) du + \ldots, \]

\[ \Xi(dK_s)^2 = -\frac{1}{\Omega^2} \left( 2 \partial_r \Omega^2 + \frac{1}{2} \partial_r ((b^s) (\partial_r \phi_s)) (\partial_u \partial_r - \partial_v \partial_u) + \ldots \right) \]

\[ = -\frac{1}{\Omega^2} \left( 2 \partial_r \Omega^2 + \frac{b^s}{2} (\partial_r \phi_s) (\partial_u \partial_r - \partial_v \partial_u) + \ldots \right) \]

and

\[-\Xi \times dK_s = \left( \frac{\partial_r \Omega^2}{\Omega^2} + \frac{b^s}{2} \frac{1}{\Omega^2} (\partial_r \phi_s) (\partial_r \phi_s) \right) \frac{L \sin \theta}{\Xi} \, d\theta_s \wedge d\phi_s + \ldots. \]

Using (114), (145), (146) and (147), this yields

\[ M = \frac{1}{8 \pi \Xi} \int_S \left( \frac{\partial_r \Omega^2}{\Omega^2} - \frac{\Xi}{2} \frac{\Xi}{\Xi} \right) d\theta_s \wedge d\phi_s + 2 \frac{\Xi}{r^2 + a^2} \Delta JQ \]

\[ = \frac{1}{8 \pi \Xi} \int_S \left( 2 \kappa_+ \frac{r^2 + a^2}{\Xi} \sin \theta \frac{\partial \theta}{\partial \theta_s} \right) d\theta_s \wedge d\phi_s + 2 \frac{\Xi}{r^2 + a^2} \Delta JQ \]

\[ = \frac{1}{2 \Xi^2} \left( \frac{1}{2} \left( -\frac{a^2 (\Lambda r^2 + 3)}{3r} - \Lambda r^3 + r - \frac{e^2}{r} \right) + 2 M a^2 \right) - \frac{e^2 \left( (r^2 + a^2)^2 \arctan \left( \frac{\Xi}{2} \right) - ar (r^2 - a^2) \right)}{2 r^2 (r^2 + a^2)} \]

\[ = \frac{1}{2 \Xi^2} \left( \frac{1}{2} \left( -\frac{(r^2 + a^2) \left( ar (r^2 - 1) + e^2 \arctan \left( \frac{\Xi}{2} \right) \right)}{2 r^2} \right) \right) \]

\[ = \frac{1}{2 \Xi^2} \left( M - \frac{\Lambda}{3} \frac{r^2 + a^2}{2 \Xi} \right) - \frac{e^2}{2 \Xi} \frac{\Lambda}{2} \arctan \left( \frac{a}{r^2} \right) \]

\[ = \frac{M}{2 \Xi^2} - \frac{\Xi}{2 \Xi} \frac{a^2}{2 \Xi} \frac{\Xi}{2 \Xi} - \frac{Q^2}{2 a} \frac{\Xi}{2 \Xi} \frac{a}{r^2} \]

(148)

(149)

with

\[ \text{AT}(x) = x + (1 + x^2) \arctan x. \]

**Remark C.2.** The quantity

\[ \frac{4 \pi r_+ (r_+^2 + a^2)}{3} \frac{\Xi}{2 \Xi} \frac{\Xi}{2 \Xi} \frac{a^2}{2 \Xi} \frac{\Xi}{2 \Xi} \frac{a}{r^2} \]

is the Parikh [27, (10)] volume of the black hole (see also [2, (91)]).

Of course, (148) is Smarr’s formula (recall (83))

\[ M = \frac{k_+}{4 \pi \Xi} \left( \frac{4 \pi r_+ (r_+^2 + a^2)}{3} \right) + 2 \Omega H JQ = \frac{k_+}{4 \pi \Xi} A + 2 \Omega H JQ \]

\[ = \frac{k_+}{4 \pi \Xi} A + 2 \Omega H \frac{M a}{\Xi^2} - \frac{Q^2}{2} \frac{\Xi}{r^2 + a^2} \frac{\Xi}{2 \Xi} \frac{a}{r^2} \]

(see, for example, [5, (9)] for the value of \( A \), and see [13] for much more on Smarr’s formula).

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