CHARACTERIZING GENERALIZED DERIVATIVES OF SET-VALUED MAPS: EXTENDING THE AUBIN AND MORDUKHOVICH CRITERIONS

C.H. JEFFREY PANG

Abstract. For a set-valued map, we characterize, in terms of its (unconvexified or convexified) graphical derivatives near the point of interest, positively homogeneous maps that are generalized derivatives in the sense of [17]. This result generalizes the Aubin criterion in [8]. A second characterization of these generalized derivatives is easier to check in practice, especially in the finite dimensional case. Finally, the third characterization in terms of limiting normal cones and coderivatives generalizes the Mordukhovich criterion in the finite dimensional case. The convexified coderivative has a bijective relationship with the set of possible generalized derivatives. We conclude by using the above results to describe the generalized differentiability properties of constraint systems, which includes the case of mixed equalities and inequalities satisfying the Mangasarian-Fromovitz constraint qualification as a particular example.

1. INTRODUCTION

1. Introduction
2. A first characterization: Generalizing the Aubin criterion
3. A second characterization: Limits of graphical derivatives
4. Examples
5. A third characterization: Generalizing the Mordukhovich criterion
6. Application to constraint mappings
7. Acknowledgements
References

1. Introduction

We say that $S$ is a set-valued map or a multifunction, denoted by $S : X \rightrightarrows Y$, if $S(x) \subseteq Y$ for all $x \in X$. There are many examples of set-valued maps in optimization and related areas. For example, the generalized derivatives of a nonsmooth function, the feasible set of a parametric optimization problem, and the set of optimizers to a parametric optimization problem may be profitably viewed as set-valued maps.

The Lipschitz analysis of a set-valued map is equivalent to the metric regularity of its inverse. There are now many examples of how such a Lipschitz analysis can be used to derive stability conditions for problems in nonsmooth optimization, feasibility and equilibrium. Such a Lipschitz analysis can identify when a problem is ill conditioned and cannot be easily resolved by any numerical method, explaining the presence of different types of constraint qualification conditions in numerical algorithms. The tools for such Lipschitz analysis...
analysis usually involve the Mordukhovich criterion or the Aubin criterion, which we will discuss later in more detail.

As an example, consider the problem $P(u, v)$ defined by

$$
P(u, v) := \inf_{x \in S(u)} v^T x,
$$

where $S : U \rightrightarrows X$ is a set-valued map. A profitable way of analyzing $P(u, v)$ is by studying the set-valued map $S$. The Lipschitz continuity of $P(\cdot, v)$ is established if $S$ satisfies a local Lipschitz property, which can be easily checked when $S$ has a closed convex graph through the Robinson-Ursescu Theorem. See for example [7, 9].

It is natural to ask whether a first order analysis of set-valued maps can be effective for the first order analysis of these problems, but we need to first build the basic tools. This paper studies how a first order analysis of set-valued maps may be obtained from the geometrical properties of its graph, generalizing the Aubin and Mordukhovich criterions. We will apply our results to study the set-valued map of feasible points satisfying a set of equalities and inequalities in Proposition [6, 1].

The Aubin criterion as presented in [8] characterizes the Lipschitz properties of a set-valued map $S : X \rightrightarrows Y$ using the tangent cones of its graph $\text{gph}(S)$. Here, the graph $\text{gph}(S)$ is the set $\{(x, y) \mid y \in S(x)\} \subset X \times Y$. One contribution of this paper is to characterize the generalized derivatives, introduced in [17], of the set-valued map $S$ in terms of the tangent cones of its graph. It is usually easier to obtain information on the tangent cones of $\text{gph}(S)$ rather than the generalized derivatives, so our result will play the role the Mordukhovich criterion and the Aubin criterion currently have in Lipschitz analysis. We now recall some standard definitions necessary to proceed. The ball with center $\bar{x}$ and radius $\bar{\epsilon}$ is denoted by $B_{\bar{x}}(\bar{\epsilon})$, and $B_{1}(0)$ is written simply as $B$.

**Definition 1.1.** (Positive homogeneity) Let $X$ and $Y$ be linear spaces. A set-valued map $H : X \rightrightarrows Y$ is positively homogeneous if

$$
H(\bar{0}) \text{ is a cone, and } H(kw) = kH(w) \text{ for all } k > 0 \text{ and } w \in X.
$$

A positively homogeneous map is also called a process. The positively homogeneous map $(H + \delta) : X \rightrightarrows Y$, where $H : X \rightrightarrows Y$ is positively homogeneous and $\delta > 0$ is a real number, is defined by

$$(H + \delta)(w) := H(w) + \delta\|w\|B.$$

Here is the definition of generalized differentiability of set-valued maps introduced in [17].

**Definition 1.2.** [17] (Generalized differentiability) Let $X$ and $Y$ be normed linear spaces. Let $S : X \rightrightarrows Y$ be such that $(\bar{x}, \bar{y}) \in \text{gph}(S)$, and let $H : X \rightrightarrows Y$ be positively homogeneous. The map $S$ is pseudo strictly $H$-differentiable at $(\bar{x}, \bar{y})$ if for any $\delta > 0$, there are neighborhoods $U_{\delta}$ of $\bar{x}$ and $V_{\delta}$ of $\bar{y}$ such that

$$S(x) \cap V_{\delta} \subset S(x') + (H + \delta)(x - x') \text{ for all } x, x' \in U_{\delta}.$$

If $S$ is pseudo strictly $H$-differentiable for some $H$ defined by $H(w) = \kappa\|w\|B$, where $\kappa \geq 0$, then $S$ satisfies the Aubin property, also referred to as the pseudo-Lipschitz property. The graphical modulus is the infimum of all such $\kappa$, and is denoted by $\text{lip}_S(\bar{x} \mid \bar{y})$.

We had used $T : X \rightrightarrows Y$ to denote the positively homogeneous map in [17], but we now use $H$ to denote the positively homogeneous map instead. We reserve $T$ to denote the tangent cone, defined as follows.
Definition 1.3. (Tangent cones) Let $X$ be a normed linear space. A vector $w \in X$ is tangent to a set $C \subset X$ at a point $\bar{x} \in C$, written $w \in T_C(\bar{x})$, if
\[
x_i - \bar{x} \rightarrow w \text{ for some } x_i \rightarrow \bar{x}, \ x_i, \ \tau_i \downarrow 0.
\]
The set $T_C(\bar{x})$ is referred to as the tangent cone (also called the contingent cone) to $C$ at $\bar{x}$.

We say that $S: X \rightrightarrows Y$ is locally closed at $(\bar{x}, \bar{y}) \in \text{gph}(S)$ if $\text{gph}(S) \cap B_\varepsilon((\bar{x}, \bar{y}))$ is a closed set for some $\varepsilon > 0$.

Definition 1.4. (Graphical derivative) Let $X$ and $Y$ be normed linear spaces. For a set-valued map $S: X \rightrightarrows Y$ locally closed at $(\bar{x}, \bar{y}) \in \text{gph}(S)$, the graphical derivative, also known as the contingent derivative, is denoted by $DS(\bar{x} \mid \bar{y}): X \rightrightarrows Y$ and defined by
\[
\text{gph}(DS(\bar{x} \mid \bar{y})) = T_{\text{gph}(S)}(\bar{x}, \bar{y}).
\]
The convexified graphical derivative is denoted by $D^\star\star S(x \mid y)$ and is defined by
\[
\text{gph}(D^\star\star S(x \mid y)) = \text{cl} \text{co}T_{\text{gph}(S)}(x, y),
\]
i.e., the closed convex hull of $T_{\text{gph}(S)}(x, y)$.

The study of the relationship between the graphical derivative and the graphical modulus can be traced back to the papers of Aubin and his co-authors [5, 4, Theorem 7.5.4] and [6, Theorem 5.4.3]. The main result in [8] characterizes $\text{lip}(\bar{x} \mid \bar{y})$ in terms of the graphical derivatives, and was named the Aubin criterion to recognize the efforts of Aubin and his coauthors. Their result will be stated as Theorem 2.4. Their proof was motivated by the proof of [2, Theorem 3.2.4] due to Frankowska. Another paper of interest on the Aubin criterion is [3], where a proof of part of the result in [8] was obtained using viability theory.

In Asplund spaces, a different characterization of $\text{lip}(\bar{x} \mid \bar{y})$ can be obtained in terms of (limiting) coderivatives. Coderivatives are defined in terms of the (limiting) normals of $\text{gph}(S)$ at $(\bar{x}, \bar{y})$, so this approach can be considered as the dual approach to the Aubin criterion. This characterization known as the Mordukhovich criterion in [18]. We refer to [3, 15, 16, 18] for more on the history of this result, where the contributions of Ioffe are also highlighted. The Mordukhovich criterion has been frequently applied to analyze many problems in nonsmooth optimization, feasibility and equilibria. Quoting [8], we note that when $X$ is any Banach space and $Y$ is finite dimensional, a necessary and sufficient for $S: X \rightrightarrows Y$ to have the Aubin property is given in terms of the Ioffe approximate coderivative in [12]. We also show how our result generalizes the Mordukhovich criterion in the finite dimensional case, and that the convexified coderivative has a bijective relationship with the set of possible generalized derivatives.

The original context of the Aubin criterion was metric regularity, while the original context of the Mordukhovich criterion was linear openness. Metric regularity gives a description of solutions sets to nonsmooth problems, which is one of the themes of the recent books [9, 13]. Further references of metric regularity and linear openness are [11, 16, 18]. The equivalence between the Aubin property, metric regularity and linear openness is well known. For readers interested in applying the results in this paper in the context of metric regularity or linear openness, we refer to [17] Section 7, where a similar equivalence for generalized differentiability of set-valued maps, generalized metric regularity, and generalized linear openness is obtained.
1.1. Contributions of this paper. We present three sets of theorems to characterize the
generalized derivatives of a set-valued map using the tangent and normal cones. The first
set of theorems are presented in Section 2. Theorem 2.4 extends the Aubin criterion in the
sense of generalized derivatives in Definition 1.2 and has a simple proof.

We present a second characterization of the generalized derivatives in Section 3 that is
easier to check in practice. The proof of this set of theorems depend on the first characteri-
zation in Section 2. More specifically, consider \( S : X \rightrightarrows Y \) locally closed at \((\bar{x}, \bar{y}) \in \text{gph}(S)\)
and a positively homogeneous map \( H : X \rightrightarrows Y \). Consider also \( \{G_i\}_{i \in I} \), where \( G_i : X \rightrightarrows Y \)
are positively homogeneous and \( I \) is some index set. We impose further conditions so that
\( S \) is pseudo strictly \( H \)-differentiable at \((\bar{x}, \bar{y}) \) if and only if

\[
G_i(p) \cap [-H(-p)] \neq \emptyset \quad \text{for all } i \in I \text{ and } p \in X \setminus \{0\}.
\]

These conditions are easier to check in the finite dimensional case, and the Clarke regular
case leads to further simplifications.

Finally, a third characterization is expressed in terms of the limiting normal cones for
the finite dimensional case in Theorem 5.4, generalizing the Mordukhovich criterion. This
characterization depends on the second characterization in Section 3. The convexified
coderivative will be shown to have a bijective relationship with the set of possible general-
ized derivatives in Theorem 5.8.

We apply the results above in Proposition 6.1 to study the generalized differentiability
properties of constraint systems, which includes the case of mixed equalities and inequali-
ties satisfying the Mangasarian-Fromovitz constraint qualification as a particular example.

1.2. Preliminaries and notation. We recall other definitions in set-valued analysis needed
for the rest of this paper. We say that \( S \) is closed-valued if \( S(x) \) is closed for all \( x \in X \), and
the definitions for compact-valuedness and convex-valuedness are similar. For set-valued
maps \( S_1 : X \rightrightarrows Y \) and \( S_2 : X \rightrightarrows Y \), we use \( S_1 \subset S_2 \) to denote \( S_1(x) \subset S_2(x) \) for all \( x \in X \),
which also corresponds to \( \text{gph}(S_1) \subset \text{gph}(S_2) \).

The outer and inner norms of positively homogeneous maps will be needed later.

**Definition 1.5.** (Outer and inner norms) The outer norm \( \|H\|^+ \) and inner norm \( \|H\|^- \) of a
positively homogeneous map \( H : X \rightrightarrows Y \) are defined by

\[
\|H\|^+ := \sup_{\|w\| \leq 1} \sup_{z \in H(w)} \|z\|
\]

and

\[
\|H\|^- := \sup_{\|w\| \leq 1} \inf_{z \in H(w)} \|z\|.
\]

The positively homogeneous maps defined as fans and prefans in [10] will be used
frequently in the rest of this paper, and we recall their definitions below.

**Definition 1.6.** [10] (Fans and prefans) We say that \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is a fan if

1. \( H(p) \) is nonempty, convex and compact for all \( p \in \mathbb{R}^n \),
2. \( H \) is positive homogeneous, and
3. \( \|H\|^+ \) is finite.

If in addition, \( H(p_1 + p_2) \subset H(p_1) + H(p_2) \) for all \( p_1, p_2 \in \mathbb{R}^n \), then we say that \( H \) is a
fan.

Fans feature heavily in the nonsmooth analysis of single-valued maps in [10]. The
additional assumption needed for fans turns out to be restrictive for the first order analysis
of set-valued maps, as illustrated in Example 1.7 below. We shall only study prefans in this
paper.
Example 1.7. (Prefans over fans) Consider $S: \mathbb{R} \rightrightarrows \mathbb{R}$ and $H: \mathbb{R} \rightrightarrows \mathbb{R}$ defined by
\[ S(x) := (-\infty, x] \quad \text{and} \quad H(x) := \max\{0, x\}. \]
The set-valued map $S$ is pseudo strictly $H$-differentiable at $(0, 0)$, as can be easily checked from definitions or by applying Corollary 3.6 later. But $H$ is a prefan and not a fan.

We recall one possible definition of inner semicontinuity. Note that this definition may not be standard when $X$ and $Y$ are infinite dimensional.

**Definition 1.8.** (Inner and outer semicontinuity) For a closed-valued mapping $S: C \rightrightarrows Y$ and a point $\bar{x} \in C \subset X$, $S$ is inner semicontinuous (written as isc) with respect to $C$ at $\bar{x}$ if for every $\rho > 0$ and $\varepsilon > 0$, there exists a neighborhood $V$ of $\bar{x}$ such that
\[ S(\bar{x}) \cap \rho B \subset S(x) + \varepsilon B \quad \text{for all} \quad x \in C \cap V. \]

We say that $S$ is outer semicontinuous (written osc) with respect to $C$ at $\bar{x}$ if for every $\rho > 0$ and $\varepsilon > 0$, there exists a neighborhood $V$ of $\bar{x}$ such that
\[ S(x) \cap \rho B \subset S(\bar{x}) + \varepsilon B \quad \text{for all} \quad x \in C \cap V. \]

We say that a set $C \subset \mathbb{R}^n$ is Clarke regular at $\bar{x} \in C$ if the tangent map $T_C : C \rightrightarrows \mathbb{R}^n$ is inner semicontinuous at $\bar{x}$. We shall only look at Clarke regularity of sets in finite dimensions, in part because our definition of inner semicontinuity is nonstandard in infinite dimensions. We say that $S$ is graphically regular at $(\bar{x}, \bar{y}) \in \text{gph}(S)$ if gph$(S)$ is Clarke regular at $(\bar{x}, \bar{y})$.

We recall the definition of the outer limit of sets. For $\{x_i\}_{i=1}^\infty \subset X$, $\bar{x} \in X$ and $C \subset X$, the notation $x_i \xrightarrow{C} \bar{x}$ means $x_i \in C$ for all $i$ and $x_i \to \bar{x}$.

**Definition 1.9.** (Outer limits) Let $C \subset X$. For a set-valued map $S: C \rightrightarrows Y$, the outer limit of $S$ at $\bar{x} \in C$, is defined by
\[ \limsup_{x \to \bar{x}} S(x) := \{ y \mid \text{there exists} \; x_i \xrightarrow{C} \bar{x}, \; y_i \in S(x_i) \; \text{s.t.} \; y_i \to y \}. \]

Lastly, for $K \subset X$, the negative polar cone of $K$ is denoted by $K^0$, and is defined by $K^0 = \{ v \mid \langle v, x \rangle \leq 0 \; \text{for all} \; x \in K \}$.

2. A FIRST CHARACTERIZATION: GENERALIZING THE AUBIN CRITERION

The main result of this section is Theorem 2.2 where we generalize the Aubin criterion. We also mention that Lemma 2.3 will be used for much of the paper later.

We list assumptions that will be used often in the rest of the paper.

**Assumption 2.1.** Let $X$ and $Y$ be normed linear spaces, and assume further that $Y$ is complete (i.e., $Y$ is a Banach space). Let $S: X \rightrightarrows Y$ be locally closed at $(\bar{x}, \bar{y}) \in \text{gph}(S)$, and let $H: X \rightrightarrows Y$ be positively homogeneous.

The following is our first characterization of the generalized derivatives of $S$.

**Theorem 2.2.** (Generalized Aubin criterion) Suppose Assumption 2.1 holds. Consider the statements:

1. $S$ is pseudo strictly $H$-differentiable at $(\bar{x}, \bar{y})$.
2. For all $\delta > 0$, there are neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that
\[ DS(x | y)(p) \cap [-\{H + \delta)(-p)\] \neq \emptyset \]
for all $(x, y) \in \text{gph}(S) \cap [U \times V]$ and $p \in X \setminus \{0\}.$
We then have the following:

(a) If \( Y \) is finite dimensional and \( H \) is compact-valued, then (1) implies (2).
(b) If \( \|H\|_{+} < \infty \) and \( H \) is convex-valued, then (2) implies (1).
(c) If both \( X \) and \( Y \) are finite dimensional and \( H \) is a prefi,, then (1), (2) and (2′) are equivalent.

We begin with the proof of Theorem 2.2(a), which is the simplest.

Proof. [Theorem 2.2(a)] Suppose \( S \) is pseudo strictly \( T \)-differentiable at \((\bar{x}, \bar{y})\). Then for any \( \delta > 0 \), there are neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) such that if \( (x, y) \in [U \times V] \cap \text{gph}(S) \), \( p \in \{0\} \) and \( t \) is small enough so that \( x + tp \in U \), then

\[
S(x) \cap V \subset S(x + tp) + (H + \delta)(-tp).
\]

(2.3)

(This can be seen as a lower generalized differentiation property, which resembles the lower Lipschitz or Lipschitz lower semicontinuous property in \([13,14]\).) Since \( y \) lies in the LHS of (2.3), there exists some \( y(t) \in S(x + tp) \) such that \( y \in y(t) + (H + \delta)(-tp) \). Then

\[
\frac{y(t) - y}{t} \in -(H + \delta)(-p).
\]

Let \( \hat{y} \) be a cluster point of \( \{y(t) - y\} \) as \( t \to 0 \), which exists since \( Y \) is finite dimensional and \(- (H + \delta)(-p) \) is compact. We have \( \hat{y} \in DS(\bar{x}, \bar{y})(p) \), so \( DS(\bar{x}, \bar{y})(p) \cap -(H + \delta)(-p) \neq \emptyset \) as needed.

To prove Theorem 2.2(b), we need the following lemma.

Lemma 2.3. (Estimates of generalized differentiability from tangent cones) Suppose Assumption 2.7 holds, and assume further that \( H \) is convex-valued and \( \|H\|_{+} < \infty \). Let \( \delta > 0 \). Suppose there are neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) such that whenever \( (x, y) \in \{U \times V\} \cap \text{gph}(S) \) and \( p \in \{0\} \), there are \( (p', q') \) such that

\[
(p', q') \in T_{\text{gph}(S)}(x, y), \quad \|p - p'\| < \delta\|p\| \quad \text{and} \quad q' \in -(H + \delta)(-p).
\]

Then provided \( \varepsilon > 0 \) is such that \( 0 < \varepsilon + \delta < 1 \), there are neighborhoods \( U_{\varepsilon} \) of \( \bar{x} \) and \( V_{\varepsilon} \) of \( \bar{y} \) such that \( x^*, x^0 \in U_{\varepsilon} \) implies

\[
S(x^*) \cap V_{\varepsilon} \subset S(x^0) + \left(H + \delta + \varepsilon + \|[H]^{+} + \delta + \varepsilon\| \frac{\delta + \varepsilon}{1 - \delta - \varepsilon}\right) (x^* - x^0).
\]

Proof. Let \( U_{\varepsilon} \) and \( V_{\varepsilon} \) be neighborhoods of \( \bar{x} \) and \( \bar{y} \) respectively such that

\[
[x^*, x^0] + (\delta + \varepsilon)\|x^* - x^0\| \subset U, \quad (2.5a)
\]

\[
x^0 + [(\delta + \varepsilon) + (\delta + \varepsilon)^2]\|x^* - x^0\| \subset U, \quad (2.5b)
\]

\[
y^0 + \frac{1}{1 - \delta - \varepsilon}\|x^* - x^0\|\|[H]^{+} + \delta + \varepsilon\| \subset V \quad \text{for all (}x^*, y^0\text{) in } \{U_{\varepsilon} \times V_{\varepsilon}\} \cap \text{gph}(S) \text{ and } x^0 \in U_{\varepsilon}. \quad (2.5c)
\]

for all \((x^*, y^0) \in [U_{\varepsilon} \times V_{\varepsilon}] \cap \text{gph}(S)\) and \( x^0 \in U_{\varepsilon} \). Here, \([x^*, x^0]\) is the line segment connecting \( x^* \) and \( x^0 \). Figure 2.1 may be helpful in understanding the steps of the proof.

Step 1: There are \((x', y') \in \text{gph}(S)\) such that

\[
\|x' - x^0\| < (\delta + \varepsilon)\|x^* - x^0\|
\]

and \( y' - y^0 \in -(H + \delta + \varepsilon)(x^* - x^0) \).
Step 1

$$x^* \xrightarrow{\mathcal{T}} \tilde{x}_1 \xrightarrow{\mathcal{T}_1(x^O - x^*)} \tilde{x}_2 \cdots x' \xrightarrow{\mathcal{T}_2(x^O - x^*)} x^O$$

Step 2: Wrapping up

To simplify notation, let $\bar{\rho} := x^* - x^*$. Let $\bar{\tau}$ be the supremum of all $\tau \in [0,1]$ such that there exists $(x',y') \in \text{gph}(S)$ satisfying

$$\|x' - x^*\| < (\delta + \varepsilon)\bar{\tau}\bar{\rho}$$

and

$$y' - y^* \in -\tau(H + \delta + \varepsilon)(-\bar{\rho}). \tag{2.6}$$

Given $(x,y) \in [U \times V] \cap \text{gph}(S)$ and a direction $\bar{\rho} \in X \setminus \{0\}$, there are $\rho' \in X$ and $q' \in -(H + \delta)(-\bar{\rho})$ such that $(\rho',q') \in T_{\text{gph}(S)}(x,y)$. By the definition of tangent cones, for any $\lambda \in (0,\varepsilon)$, there is some $(x_\lambda,y_\lambda) \in \text{gph}(S)$ such that $\|x_\lambda - x_\lambda - t\rho'\| \leq \lambda t\|\bar{\rho}\|$ and $\|y_\lambda - y_\lambda - t\epsilon\| \leq \lambda t\|\bar{\rho}\|$ for some $t > 0$. We thus have

$$\|(x_\lambda,y_\lambda) - (x,y)\| < \lambda,$$

$$\|x_\lambda - x\| - t\rho' \leq (\delta + \lambda)(t\|\bar{\rho}\|),$$

and

$$y_\lambda - y \in t[-(H + \delta)(-\bar{\rho}) + \lambda\|\bar{\rho}\||B],$$

Taking $(x,y) = (x^*,y^*)$ gives us $\bar{\tau} > 0$. Let $(\tilde{x}_1,\tilde{y}_1) \in \text{gph}(S)$ and $\tau_1 \in (0,1]$ be such that $\mathcal{T}_1$ holds for $(x',y') = (\tilde{x}_1,\tilde{y}_1)$ and $\tau = \tau_1$. If $\tau_1 < 1$, we can use the existence of some $(\tilde{p}',\tilde{q}') \in \text{gph}(DS(\tilde{x}_1 | \tilde{y}_1))$ and obtain the existence of $(\tilde{x}_2,\tilde{y}_2) \in \text{gph}(S)$ and $\tau_2 \in (\tau_1,1]$ such that

$$\|\tilde{x}_2 - \tilde{x}_1\| - (\tau_2 - \tau_1)\bar{\rho} \leq (\delta + \epsilon)(\tau_2 - \tau_1)\|\bar{\rho}\|$$

and

$$\tilde{y}_2 - \tilde{y}_1 \in -(\tau_2 - \tau_1)(H + \delta + \epsilon)(-\bar{\rho}).$$

The implication

$$\tilde{y}_1 - y^* \in -\tau_1(H + \delta + \epsilon)(-\bar{\rho})$$

and

$$\tilde{y}_2 - \tilde{y}_1 \in -(\tau_2 - \tau_1)(H + \delta + \epsilon)(-\bar{\rho})$$

implies $y_2 - y^* \in -(\tau_2 - \tau_1)(H + \delta + \epsilon)(-\bar{\rho})$. This requires the convexity of $(H + \delta + \epsilon)(-\bar{\rho})$. The conditions imply that $(2.6)$ holds for $(x',y') = (\tilde{x}_2,\tilde{y}_2)$ and $\tau = \tau_2$. Similarly, we can obtain a Cauchy sequence $\{(x_i,y_i)\}$ with limit $(\tilde{x},\tilde{y})$ and $\tau_i \searrow \tau$ such that $(2.6)$ holds for $(x',y') = (\tilde{x}_i,\tilde{y}_i)$ and $\tau = \tau_i$.

Either $\bar{\tau} = 1$ or $\bar{\tau} < 1$. If $\bar{\tau} < 1$, then it is elementary from the fact that $\{(x_i,y_i)\}$ is Cauchy and \text{gph}(S) is locally closed, that $(2.6)$ holds for $(x',y') = (\tilde{x},\tilde{y})$ and $\tau = \bar{\tau}$. This argument also tells us that we can find some $(x',y')$ so that $(2.6)$ is satisfied for $\tau = \bar{\tau}$. With $(\tilde{x},\tilde{y})$, we can proceed as before to find $(x',y')$ satisfying $(2.6)$ for $\tau = 1$.

Step 2: Wrapping up
So far, we have shown that for all \((x^*,y^*) \in [U_e \times V_e] \cap \text{gph}(S)\) and \(x^0 \in U_e\), we can find \((x',y') \in [U \times V] \cap \text{gph}(S)\) such that
\[
\|x' - x^0\| < (\delta + \varepsilon)\|x^0 - x^*\|
\]
and
\[
y^* - y' \in (H + \delta + \varepsilon)(x^* - x^0).
\]

Write \((x'_1, y'_1) = (x', y')\), and \(\bar{p}_1 = x^0 - x'_1\). Using a similar process as outlined in step 1 and the fact that for we can find \((p'_1, q'_1) \in T_{\text{gph}(S)}(x'_1, y'_1)\) such that \(\|\bar{p}_1 - p'_1\| \leq \delta \|\bar{p}_1\|\) and \(q'_1 \in -(H + \delta)(-\bar{p}_1)\), we can find \((x'_2, y'_2) \in \text{gph}(S)\) such that
\[
\|x'_2 - x^0\| < (\delta + \varepsilon)\|x^0 - x^*\|
\]
and
\[
y'_1 - y'_2 \in (H + \delta + \varepsilon)(x'_1 - x^0)
\]
(2.7)
Note that \(\|x^0 - x'_1\| < (\varepsilon + \delta)\|x^0 - x^*\|\). The condition (2.5b) was defined so that step 1 can be applied here to find \((x'_2, y'_2)\). Formula (2.7) implies
\[
\|x'_2 - x^0\| < (\delta + \varepsilon)^2\|x^0 - x^*\|,
\]
and
\[
y'_1 - y'_2 \subset [(H^+) + \delta + \varepsilon]\|x^0 - x'_1\|B
\]
\[
\subset (\varepsilon + \delta)[(H^+) + \delta + \varepsilon]\|x^0 - x^*\|B.
\]
Likewise, we can find \((x'_i, y'_i) \in \text{gph}(S)\) inductively such that
\[
\|x'_i - x^0\| < (\delta + \varepsilon)^i\|x^0 - x^*\|,
\]
and
\[
y'_i - y'_{i-1} \subset (\delta + \varepsilon)^{i-1}[(H^+) + \delta + \varepsilon]\|x^0 - x^*\|B.
\]

The sequence \(\{(x'_i, y'_i)\}\) is Cauchy, and hence converges to a limit in the closed set \(\text{gph}(S) \cap [U \times V]\). The \(x\) coordinate of this limit is \(x^0\). Let the \(y\)-coordinate of this limit be \(y^0\). Since \(0 < \delta + \varepsilon < 1\), we have
\[
y^* - y^0 = y^* - y'_1 + \sum_{i=1}^{\infty} [y'_i - y'_{i+1}]
\]
\[
\in (H + \delta + \varepsilon)(x^* - x^0) + [(H^+) + \delta + \varepsilon] \frac{\delta + \varepsilon}{1 - \delta - \varepsilon}\|x^0 - x^*\|B.
\]
This gives
\[
y^* \in y^0 + (H + \delta + \varepsilon + [(H^+) + \delta + \varepsilon] \frac{\delta + \varepsilon}{1 - \delta - \varepsilon})(x^* - x^0)
\]
\[
\subset S(x^0) + (H + \delta + \varepsilon + [(H^+) + \delta + \varepsilon] \frac{\delta + \varepsilon}{1 - \delta - \varepsilon})(x^* - x^0).
\]
Since \((x^*, y^*)\) is arbitrarily chosen in \([U_e \times V_e] \cap \text{gph}(S)\) and \(x^0\) is arbitrarily chosen in \(U_e\), we are done.

We now continue with the proof of Theorem 2.2(b).

**Proof.** [Theorem 2.2(b)] Since \(S\) is locally closed at \((\bar{x}, \bar{y})\), we can always reduce the neighborhoods \(U\) and \(V\) if necessary so that \([U \times V] \cap \text{gph}(S)\) is closed. The condition \(DS(x)(y)(p) \cap -(H + \delta)(-p) \neq \emptyset\) easily implies the existence of \((p', q')\) satisfying (2.4). (In fact, the vector \(p'\) in (2.4) can be chosen to be \(p\).) Therefore the conditions in Lemma 2.3 are satisfied. Since the \(\varepsilon\) and \(\delta\) are in the statement of Lemma 2.3 are arbitrary, we have the pseudo strict \(H\)-differentiability of \(S\) as needed.
The proof of Theorem 2.2(c) follows with minor modifications from the methods in [8], who were in turn motivated by the proof of [2] Theorem 3.2.4 due to Frankowska.

**Proof.** [Theorem 2.2(c)] Condition (2′) is identical to Condition (2) except for the use of the convexified graphical derivative $D^*(x \mid y)$. We show that Conditions (2′) and (2) are equivalent under the added conditions. It is clear that (2) $\Rightarrow$ (2′), so we only need to prove the opposite direction. Our proof is a slight amendment of Step 3 in the proof of [8] Theorem 1.2.

Suppose Condition (2′) holds. Fix some $\delta > 0$. For any sets $A,B \subset X \times Y$, denote by $d(A,B) := \inf \{ \|a-b\| \mid a \in A,b \in B \}$. Let us fix $(x,y) \in \text{gph}(S) \cap [U \times V]$ and $p \in X \setminus \{0\}$. Let $w \in -(H + \delta)(-p)$ and $(p^*,q^*) \in \text{gph}(DS(x \mid y))$ be such that

$$\| (p,w) - (p^*,q^*) \| = d(\{p\} \times -(H + \delta)(-p), \text{gph}(DS(x \mid y))).$$

Observe that the point $(p^*,q^*)$ is the unique projection of any point in the open segment $\langle (p^*,q^*), (p,w) \rangle$ on $\text{gph}(DS(x \mid y))$. We will prove that $\langle p^*, q^* \rangle = \langle p, w \rangle$ and this will prove that $w \in DS(x \mid y)(p) \cap -(H + \delta)(-p)$.

By the definition of the graphical derivative, there exists sequences $t_n \searrow 0$, $p_n \to p^*$, $q_n \to q^*$ such that $y + t_nq_n \in S(x + t_np_n)$ for all $n$. Let $(x_n,y_n)$ be a point in $\text{cl gph}(S)$ which is closest to $(x,y) + \frac{t_n}{2}(p^* + q^* + w)$ (a projection, not necessarily unique, of the latter point on the closure of gph(S)). Since $(x,y) \in \text{gph}(S)$ we have

$$\left\| (x,y) + \frac{t_n}{2}(p^* + p, q^* + w) - (x_n,y_n) \right\| \leq \frac{t_n}{2} \| (p^* + p, q^* + w) \|,$$

and hence

$$\| (x,y) - (x_n,y_n) \| \leq \left\| (x,y) + \frac{t_n}{2}(p^* + p, q^* + w) - (x_n,y_n) \right\| + \frac{t_n}{2} \| (p^* + p, q^* + w) \| \leq t_n \| (p^* + p, q^* + w) \|.$$

Thus for $n$ sufficiently large, we have $(x_n,y_n) \in U \times V$ and hence $(x_n,y_n) \in \text{gph}(S) \cap [U \times V]$. Setting $(\tilde{p}_n,\tilde{q}_n) = (x_n - x, y_n - y)/t_n$, we deduce by the usual property of a projection that

$$\frac{1}{2}(p^* + q^* + w) - (\tilde{p}_n,\tilde{q}_n) \in [T_{\text{gph}(S)}(x_n,y_n)]^0 = [\text{gph}(D^*(x \mid y))]^0.$$

By the assumptions in (2′), there exists $w_n \in D^*(x_n \mid y_n)(p) \cap -(H + \delta)(-p)$ and we have from the above relation

$$\left< \left( \frac{p^* + p}{2} - \tilde{p}_n, p \right) + \left( \frac{q^* + w}{2} - \tilde{q}_n, w_n \right) \right> \leq 0. \quad (2.8)$$

We claim that $(\tilde{p}_n,\tilde{q}_n)$ converges to $(p^*,q^*)$ as $n \to \infty$. Indeed,

$$\left\| \left( \frac{p^* + p}{2}, \frac{q^* + w}{2} \right) - (\tilde{p}_n,\tilde{q}_n) \right\| \leq \frac{1}{t_n} \left\| (x,y) + \frac{t_n}{2} \left( \frac{p^* + p}{2}, \frac{q^* + w}{2} \right) - (x_n,y_n) \right\| \leq \frac{1}{t_n} \left\| (x,y) + t_n \left( \frac{p^* + p}{2}, \frac{q^* + w}{2} \right) - (x,y) - t_n(p_n,q_n) \right\| \leq \left\| \left( \frac{p^* + p}{2}, \frac{q^* + w}{2} \right) - (p_n,q_n) \right\|. $$
Proof. Since \( \{\bar{p}_n, \bar{q}_n\} \) is a bounded sequence and then, since \( y_n = y + t_n \bar{q}_n \in S(x_n) = S(x + t_n \bar{p}_n) \), every cluster point \((\bar{p}, \bar{q})\) of it belongs to \( \text{gph}(DS(x \mid y)) \). Moreover, \((\bar{p}, \bar{q})\) satisfies

\[
\left\| \frac{p^* + p}{2}, \frac{q^* + w}{2} \right\| - (\bar{p}, \bar{q}) \right\| \leq \left\| \frac{p^* + p}{2}, \frac{q^* + w}{2} \right\| - (p^*, q^*) \right\|.
\]

The above inequality together with the fact that \((p^*, q^*)\) is the unique closest point to \( \frac{1}{2}(p^* + p, q^* + w) \) in \( \text{gph}(DS(x \mid y)) \) implies that \((\bar{p}, \bar{q}) = (p^*, q^*)\). Our claim is proved.

Up to a subsequence, \( w_n \) satisfying (2.8) converges to some \( \bar{w} \in -(H + \delta)(-p) \). Passing to the limit in (2.8) one obtains

\[
\langle p - p^*, p \rangle + \langle w - q^*, \bar{w} \rangle \leq 0.
\]

(2.9)

Since \((p, w)\) is the unique closest point of \((p^*, q^*)\) to the closed convex set \{\( p \) \times \([-\{H + \delta\}(-p)\]\), we have

\[
\langle w - q^*, w - \bar{w} \rangle \leq 0.
\]

(2.10)

Finally, since \((p^*, q^*)\) is the unique closest point to \( \frac{1}{2}(p^* + p, q^* + w) \) in \( \text{gph}(DS(x \mid y)) \) which is a closed cone, we get

\[
\langle p - p^*, p^* \rangle + \langle w - q^*, q^* \rangle = 0.
\]

(2.11)

In view of (2.9), (2.10) and (2.11), we obtain

\[
\| (p, w) - (p^*, q^*) \|^2
\]

\[
= \langle w - q^*, w - \bar{w} \rangle + (\langle p - p^*, p \rangle + \langle w - q^*, \bar{w} \rangle) - (\langle p - p^*, p^* \rangle + \langle w - q^*, q^* \rangle) \leq 0.
\]

Hence \( p = p^* \) and \( w = q^* \). We have \( DS(x \mid y)(p) \cap [-\{H + \delta\}(p)] \) containing at least the element \( w \), so it cannot be empty. Since \( \delta > 0 \), \((x, y) \in gph(S) \cap \{U \times V\} \) and \( p \in X \setminus \{0\} \) are arbitrary, we have Condition (2) in Theorem 2.2 as needed.

As a corollary to Theorem 2.2, we obtain the Aubin criterion as proved in [3].

**Theorem 2.4. (Aubin Criterion) Suppose Assumption (2.7) holds. Let**

\[
\alpha := \limsup_{(x,y) \to (\bar{x}, \bar{y}) \in \text{gph}(S)} \|DS(x \mid y)\|^{-}. \]

(a) We have \( \text{lip}S(\bar{x} \mid \bar{y}) \leq \alpha \), and equality holds if \( Y \) is finite dimensional.

(b) If both \( X \) and \( Y \) are finite dimensional, then

\[
\text{lip}S(\bar{x} \mid \bar{y}) = \limsup_{(x,y) \to (\bar{x}, \bar{y}) \in \text{gph}(S)} \|D^**S(x \mid y)\|^{-}. \]

**Proof.** Recall that \( S \) has the Aubin property at \((\bar{x}, \bar{y}) \in \text{gph}(S)\) if and only if it is pseudostrictly \( H \)-differentiable there for \( H \) defined by \( H(w) := \kappa \|w\| \mathbb{B} \). For a given \((x, y) \in \text{gph}(S)\), the smallest value of \( \kappa \geq 0 \) such that \( DS(x \mid y)(p) \cap \kappa \|p\| \mathbb{B} \neq \emptyset \) for all \( p \neq 0 \) is \( \|DS(x \mid y)\|^{-}. \)

We apply these observations to Theorem 2.2. In (a), if \( \alpha < \infty \), the condition \( \text{lip}S(\bar{x} \mid \bar{y}) \leq \alpha \) holds by Theorem 2.2(b). The statement is trivially true if \( \alpha = \infty \). If \( \text{lip}S(\bar{x} \mid \bar{y}) = \infty \), then \( \text{lip}S(\bar{x} \mid \bar{y}) \leq \alpha \) from before gives \( \alpha = \infty \). When \( Y \) is finite dimensional and \( \text{lip}S(\bar{x} \mid \bar{y}) \) is finite, it follows from Theorem 2.2(a) that \( \text{lip}S(\bar{x} \mid \bar{y}) = \alpha \). For (b), the proof is similar.

**Theorem 2.4** (b) was also proved with viability theory in [3].

We remark on the similarities between Lemma 2.3 and Aubin’s original results. For a set-valued map \( S : X \rightrightarrows Y \), the inverse \( S^{-1} : Y \rightrightarrows X \) is defined by \( S^{-1}(y) := \{x \mid y \in S(x)\} \), and satisfies \( \text{gph}(S^{-1}) = \{(y,x) \mid (x,y) \in \text{gph}(S)\} \).
Remark 2.5. (Comparison to Aubin’s original results) Let \( H : X \rightrightarrows Y \) be defined by \( H(w) := \kappa \|w\| \mathbb{B} \). In [4, Theorem 7.5.4] and [6, Theorem 5.4.3], the necessary condition in both results (up to some rephrasing) is that there are neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) such that for all \( p \in X \setminus \{0\} \) and \( (x, y) \in gph(S) \cap [U \times V] \), there exists \( q' \) and \( w \) such that

\[
p \in [DS(x \, | \, y)]^{-1}(q') + w, \quad q' \in -H(-p) \text{ and } \|w\| < \delta \|p\|.
\]

Let \( p' = p - w \). Then \( \|p - p'\| < \delta \|p\| \) and \( (p', q') \in T_{gph(S)}(x, y) \), so the condition in (2.4) is satisfied.

3. A second characterization: Limits of graphical derivatives

While conditions (2) and (2') in Theorem 2.2 classify the generalized derivative, the presence of the term \( \delta \) may make these conditions difficult to check in practice. In Subsection 3.2, we present another characterization on the generalized derivatives \( H : X \rightrightarrows Y \) that may be easier to check than Theorem 2.2, especially in the finite dimensional case.

More specifically, consider \( \{G_i\}_{i \in I} \), where \( G_i : X \rightrightarrows Y \) are positively homogeneous and \( I \) is some index set. We impose further conditions so that \( S \) is pseudo strictly \( H \)-differential at \((\bar{x}, \bar{y})\) if and only if

\[
G_i(p) \cap [-H(-p)] \neq \emptyset \text{ for all } i \in I \text{ and } p \in X \setminus \{0\}.
\]

3.1. A generalized inner semicontinuity condition. Before we move on to the next subsection for a second characterization of generalized derivatives \( H : X \rightrightarrows Y \) such that \( S : X \rightrightarrows Y \) is pseudo strictly \( H \)-differentiable at \((\bar{x}, \bar{y})\), we propose a generalized notion of lower semicontinuity to simplify the results in Subsection 3.2.

We say that \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is a piecewise polyhedral map if \( gph(S) \subset \mathbb{R}^n \times \mathbb{R}^m \) is a piecewise polyhedral set, i.e., expressible as the union of finitely many polyhedral sets. The tangent cones on any two elements of a face of a polyhedron are the same, which leads us to the following result.

Proposition 3.1. (Finitely many tangent cones for piecewise polyhedral maps) Suppose \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is piecewise polyhedral. At any point \((\bar{x}, \bar{y}) \in gph(S)\), there is a neighborhood of \((\bar{x}, \bar{y})\) and a finite set \( \{T_i\}_{i \in I} \subset \mathbb{R}^n \times \mathbb{R}^m \) such that \( T_{gph(S)}(x, y) = T_i \) for some \( i \in I \).

We next state a piecewise polyhedral example.

Example 3.2. (Piecewise polyhedral \( S_1 : \mathbb{R} \rightrightarrows \mathbb{R} \)) Consider the piecewise polyhedral set-valued map \( S_1 : \mathbb{R} \rightrightarrows \mathbb{R} \) defined by

\[
S_1(x) := (-\infty, -|x|] \cup [|x|, \infty).
\]

See Figure 4.1 for a diagram of \( S_1 \). The possibilities for \( T_{gph(S_1)}(x, y) \), where \((x, y) \in gph(S_1)\) and \((x, y)\) is close to \((0, 0)\) are \( gph(G_i) \) for \( i \in \{1, \ldots, 6\} \), where \( G_i : \mathbb{R} \rightrightarrows \mathbb{R} \) are defined by

\[
\begin{align*}
G_1(x) & = [x, \infty) \\
G_2(x) & = (-\infty, x] \\
G_3(x) & = [-x, \infty) \\
G_4(x) & = (-\infty, -x] \\
G_5(x) & = \mathbb{R} \\
G_6(x) & = S_1(x).
\end{align*}
\]
Notice that for the map \( S_1 \) defined in (3.1), while \( T_{\text{gph}(S_1)} : \text{gph}(S_1) \Rightarrow \mathbb{R} \times \mathbb{R} \) is not inner semicontinuous at \((0,0)\), the possible limits for \( \{ T_{\text{gph}(S_1)}(x_j,y_j) \} \), where \((x_j,y_j) \to (0,0)\), take on only a finite number of possibilities as stated in Proposition 3.1. We now define a generalized inner semicontinuity that gets around this difficulty.

**Definition 3.3.** (Generalized inner semicontinuity) Let \( \{ T_i \}_{i \in I} \subset Y \), where \( I \) is some index set, and let \( C \subset X \). For a closed-valued mapping \( S : C \Rightarrow Y \) and a point \( \bar{x} \in C \subset X \), \( S \) is said to be \( \{ T_i \}_{i \in I} \)-inner semicontinuous (or \( \{ T_i \}_{i \in I} \)-isc) with respect to \( C \) if for all \( \rho > 0 \) and \( \epsilon > 0 \), there exists a neighborhood \( V \) of \( \bar{x} \) such that for all \( x \in C \cap V \), there is some \( i \in I \) such that \( T_i \bar{\rho} \mathbb{B} \subset S(x) + \epsilon \mathbb{B} \).

In the case where \(|I| = 1\) and \( T_1 = S(\bar{x}) \), \( \{ T_i \}_{i \in I} \)-inner semicontinuity reduces to the definition of inner semicontinuity. Going back to the map \( S_1 \) in (3.1), we note that the map \( T_{\text{gph}(S_1)} : \text{gph}(S_1) \Rightarrow \mathbb{R} \times \mathbb{R} \) is \( \{ \text{gph}(G_i) \}_{i \in I} \)-isc at \((0,0)\), where \( G_i : \mathbb{R} \Rightarrow \mathbb{R} \) are defined in (3.2). The choice of \( \{ G_i \}_{i \in I} \) is not unique. We can instead define \( I = \{1,2\} \) and \( G_i \) by

\[
G_1(x) = x \quad \text{and} \quad G_2(x) = -x, \tag{3.3}
\]

and \( T_{\text{gph}(S_1)} : \text{gph}(S_1) \Rightarrow \mathbb{R} \times \mathbb{R} \) will still be \( \{ \text{gph}(G_i) \}_{i \in I} \)-isc at \((0,0)\). Of course, the \( \{ \text{gph}(G_i) \}_{i \in I} \) defined in (3.3) are not possible limits of \( T_{\text{gph}(S_1)}(x,y) \) as \((x,y) \xrightarrow{\text{gph}(S_1)} (0,0)\).

See Lemma 3.7 for a criteria in finite dimensions.

The index set \( I \) need not be finite, as the following examples show.

**Example 3.4.** (Infinite index set \( I \)) We give two examples where the index set \( I \) is necessarily infinite.

(a) Consider the function \( f_1 : \mathbb{R} \to \mathbb{R} \) defined by

\[
f_1(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(1/x) & \text{otherwise}, \end{cases}
\]

which has Fréchet derivative

\[
f_1'(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2x \sin(1/x) - \cos(1/x) & \text{otherwise.} \end{cases}
\]

The map \( T_{\text{gph}(f_1)} : \text{gph}(f_1) \Rightarrow \mathbb{R} \times \mathbb{R} \) is \( \{ \text{gph}(G_\lambda) \}_{\lambda \in [-1,1]} \)-isc at \((0,0)\), where \( G_\lambda : \mathbb{R} \to \mathbb{R} \) is the linear map with gradient \( \lambda \).

(b) Next, consider the function \( f_2 : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f_2(x) = ||x||_2 \). The map \( T_{\text{gph}(f_2)} : \text{gph}(f_2) \Rightarrow \mathbb{R}^2 \times \mathbb{R} \) is \( \{ \text{gph}(G_\lambda) \}_{||\lambda||_1 = 1} \cup T_{\text{gph}(f_2)}(0,0) \)-isc at \((0,0)\), where \( G_\lambda : \mathbb{R}^2 \to \mathbb{R} \) is the linear map with gradient \( \lambda \in \mathbb{R}^2 \). The map \( f_2 \) shows that the index set \( I \) in Definition 3.3 can be infinite even when the function is single-valued and semi-algebraic.

### 3.2. A second characterization of generalized derivatives

For Theorem 3.5 below, assume that the norm in \( X \times Y \) is defined by \( ||(p,q)||_{X \times Y} : = \left(||p||_X,||q||_Y\right)||_2 \), where \( \cdot ||_2 \) is a norm in \( \mathbb{R}^2 \).

**Theorem 3.5.** (Characterization of generalized derivative) Suppose Assumption 2.1 holds. Let \( I \) be some index set, and \( G_i : X \Rightarrow Y \) be positively homogeneous maps for all \( i \in I \). Consider the conditions

1. \( S \) is pseudo strictly \( H \)-differentiable at \((\bar{x},\bar{y})\).
2. \( G_i(p) \cap [-H(-p)] \neq \emptyset \) for all \( p \in X \setminus \{0\} \) and \( i \in I \).

Then the following hold:
(a) Suppose $X$ is finite dimensional and $H$ is compact-valued. If for all $i \in I$, there exists $\{(x_j, y_j)\} \subset \text{gph}(S)$ such that $(x_j, y_j) \to (\bar{x}, \bar{y})$ and
\[
\limsup_{j \to \infty} T_{\text{gph}(S)}(x_j, y_j) \subset \text{gph}(G_i),
\]
then (1) implies (2).

(b) Suppose $\|H\|^+ \leq \infty$, $H$ is convex-valued, and the mapping $T_{\text{gph}(S)} : \text{gph}(S) \rightrightarrows X \times Y$ is $\{\text{gph}(G_i)\}_{i \in I}$-isc at $(\bar{x}, \bar{y})$. Then (2) implies (1).

Furthermore, if $X$ and $Y$ are finite dimensional, and the mapping $T_{\text{gph}(S)} : \text{gph}(S) \rightrightarrows X \times Y$ was replaced by the mapping $\text{clco} T_{\text{gph}(S)} : \text{gph}(S) \rightrightarrows X \times Y$ to the closed convex hull of the tangent cone instead in (a) and (b), then the modified statements hold:

(a') Suppose $X$ and $Y$ are finite dimensional, and $H$ is compact-valued. If for all $i \in I$, there exists $\{(x_j, y_j)\} \subset \text{gph}(S)$ such that $(x_j, y_j) \to (\bar{x}, \bar{y})$ and
\[
\limsup_{j \to \infty} \text{clco} T_{\text{gph}(S)}(x_j, y_j) \subset \text{gph}(G_i),
\]
then (1) implies (2).

(b') Suppose $X$ and $Y$ are finite dimensional, and $H$ is a prefan. If the mapping $\text{clco} T_{\text{gph}(S)} : \text{gph}(S) \rightrightarrows X \times Y$ is $\{\text{gph}(G_i)\}_{i \in I}$-isc at $(\bar{x}, \bar{y})$, then (2) implies (1).

**Proof.** (a) Suppose $S$ is pseudo strictly $H$-differentiable at $(\bar{x}, \bar{y})$. For each $G_i$, there is a sequence $\{(x_j, y_j)\} \subset \text{gph}(S)$ converging to $(\bar{x}, \bar{y})$ such that $\limsup_{j \to \infty} T_{\text{gph}(S)}(x_j, y_j) \subset \text{gph}(G_i)$. Furthermore, by Theorem 2.2(a), there is some $\delta_j \searrow 0$ such that
\[
\text{DS}(x_j \mid y_j)(p) \cap [-H + \delta_j](-p) \neq \emptyset.
\]
Let $q_j$ be in the LHS of the above, and let $\bar{q}$ be a cluster point of $\{q_j\}^\infty_{j=1}$, which exists by the compactness of $H(-p)$. Since $\text{gph}(G_i) \supset \limsup_{j \to \infty} T_{\text{gph}(S)}(x_j, y_j)$, we have $(p, \bar{q}) \in \text{gph}(G_i)$, and so $G_i(p) \cap [-H(-p)]$ contains $\bar{q}$. Hence $G_i(p) \cap [-H(-p)] \neq \emptyset$.

(b) Given $\gamma > 0$, we have neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that for all $(x, y) \in [U \times V] \cap \text{gph}(S)$, we have
\[
\text{gph}(G_i) \cap \mathbb{B}_{X \times Y} \subset \left[T_{\text{gph}(S)}(x, y) \cap \mathbb{B}_{X \times Y}\right] + \gamma \mathbb{B}_{X \times Y} \tag{3.5}
\]
for some $i \in I$, where $\mathbb{B}_{X \times Y}$ is the unit ball in $X \times Y$. Let $(x^*, y^*) \in \text{gph}(S) \cap [U \times V]$, and let $i^*$ be such that (3.5) holds for $(x, y) = (x^*, y^*)$ and $i = i^*$. Choose $p \in X \setminus \{0\}$. Since $G_{r^*}(p) \cap [-H(-p)] \neq \emptyset$, choose $q \in G_{r^*}(p) \cap [-H(-p)]$. Then $(p, q) \in \text{gph}(G_{r^*})$. Since $\text{gph}(G_{r^*})$ is a cone, we may rescale $(p, q)$ so that $\|p, q\| = 1$. From the fact that $\|H\|^+$ is finite, and the equivalence of finite dimensional norms, there is some $\kappa > 0$ such that
\[
1 = \|(p, q)\| \\
\leq \kappa(\|p\| + \|q\|) \\
\leq \kappa(\|p\| + \|H\|^+\|p\|) \\
= \kappa\|p\|(1 + \|H\|^+).
\]
Recall that the choice of $\kappa$ in view of the generalized inner semicontinuity property gives us some $(p', q') \in T_{\text{gph}(S)}(x^*, y^*)$ such that
\[
\|(p', q') - (p, q)\| < \gamma.
\]
We have
\[
\|p - p'\| < \gamma \leq \kappa\gamma(1 + \|H\|^+\|p\|) \quad \text{and} \quad \|q - q'\| < \gamma \leq \kappa\gamma(1 + \|H\|^+\|p\|).
\]
The formula involving \( q \) implies that \( q' \in -[H + \kappa \gamma(1 + \|H\|^+)](-p) \). If \( \gamma \) is chosen so that \( \kappa \gamma(1 + \|H\|^+) < \delta \), then the conditions in Lemma 2.3 are satisfied, which easily implies that \( S \) is pseudo strictly \( H \)-differentiable at \((\bar{x}, \bar{y})\).

(a') The proof for this statement carries through with minor changes from that of (a).

(b') The proof for this statement requires added details from that of (b). Using the methods in the proof of (b), given \( \gamma > 0 \), we have neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) such that for all \((x^*, y^*) \in [U \times V] \cap \text{gph}(S)\) and \( p \in X \setminus \{0\} \), there exists \((p', q') \in \text{clco} T_{\text{gph}(S)}(x^*, y^*)\) such that

\[
\|p - p'\| < \gamma \leq \kappa \gamma(1 + \|H\|^+)\|p\| \quad \text{and} \quad q' \in -[H + \kappa \gamma(1 + \|H\|^+)](-p).
\]

The current proof now departs from that in (b). We first claim that we can reduce \( U \) and \( V \) if necessary so that \( \|D^{**}S(x | y)\| - < 1 + \|H\|^+ \) for all \((x, y) \in [U \times V] \cap \text{gph}(S)\). Seeking a proof by contradiction to the claim, suppose there exists a sequence \( \{(x_j, y_j)\}_j \subset \text{gph}(S) \) such that \((x_j, y_j) \to (\bar{x}, \bar{y})\) and \( \|D^{**}S(x_j | y_j)\| - \geq (1 + \|H\|^+) \) for all \( j \). Then by [18, Theorem 4.18], \( \{|\text{gph}(D^{**}S(x_j | y_j))\}|_{j=1}^{\infty} \) has a subsequence that converges in the set-valued sense to \( \text{gph}(G) \), where \( G : X \rightrightarrows Y \) is positively homogeneous and \( \|G\| - \geq (1 + \|H\|^+) \). We must then have \( G(p) \cap [-H(-p)] = \emptyset \) for some \( p \in X \setminus \{0\} \). By the generalized inner semicontinuity property, there is some \( \bar{r} \in I \) such that \( G_{\bar{r}} \subset G \), giving us \( G_{\bar{r}}(p) \cap [-H(-p)] = \emptyset \), which is a violation of the assumption in (2).

From the claim we just proved, we have \( \|D^{**}S(x^* | y^*)\| - < 1 + \|H\|^+ \). Note also that \( D^{**}S(x^* | y^*) \) is graphically convex and positively homogeneous. By the Aubin criterion in Theorem 2.4, \( D^{**}S(x^* | y^*) \) has the Aubin property with \( \text{lip}D^{**}S(x^* | y^*) \leq 1 + \|H\|^+ \). We have

\[
q' \in D^{**}S(x^* | y^*)(p') \subset D^{**}S(x^* | y^*)(p) + (1 + \|H\|^+)\|p - p'\|\|B\|.
\]

This means that there exists \( q'' \) such that

\[
\|q'' - q'\| \leq (1 + \|H\|^+)\|p - p'\| \leq \kappa \gamma(1 + \|H\|^+)\|p\|,
\]

and \( q'' \in D^{**}S(x^* | y^*)(p) \). So \( q'' \in -([H + \kappa \gamma(2 + 3\|H\|^+) + (\|H\|^+)](-p) \), which implies

\[
D^{**}S(x^* | y^*)(p) \cap [-([H + \kappa \gamma(2 + 3\|H\|^+) + (\|H\|^+)](-p) \) \neq \emptyset.
\]

Since \( \gamma > 0 \) can be made arbitrarily small, \((x^*, y^*) \) is arbitrary in \([U \times V] \cap \text{gph}(S)\) and \( p \) is arbitrary in \( X \setminus \{0\} \), we can apply Theorem 2.2(c) and prove what we need. □

We have the following simplification in the Clarke regular case.

**Corollary 3.6.** (Clarke regular, finite dimensional case) Suppose \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is locally closed and \( \text{gph}(S) \) is Clarke regular at \((\bar{x}, \bar{y}) \in \text{gph}(S)\). Let \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a proper fan. Then \( S \) is pseudo strictly \( H \)-differentiable at \((\bar{x}, \bar{y}) \) if and only if

\[
DS(\bar{x} | \bar{y})(p) \cap [-H(-p)] \neq \emptyset \text{ for all } p \in \mathbb{R}^n \setminus \{0\}.
\]

**Proof.** In finite dimensions, the Clarke regularity of \( \text{gph}(S) \) is defined by the inner semicontinuity of \( T_{\text{gph}(S)} : \text{gph}(S) \rightrightarrows X \times Y \). Apply Theorem 3.3(a) and (b) for \( I = \{1\} \) and \( G_1 \equiv DS(\bar{x} | \bar{y}) \). □

An easy consequence of the Clarke regularity of \( \text{gph}(S) \) is that the positively homogeneous map \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) can be chosen to be single-valued.

For our next result, we need to take a different view of the tangent cone mapping \( T_{\text{gph}(S)} : \text{gph}(S) \rightrightarrows X \times Y \). Denote the family of closed sets in \( X \times Y \) to be \( \mathcal{C}(X \times Y) \). We can
write the set-valued map $T_{\text{gph}(S)}$ as $T_{\text{gph}(S)} : \text{gph}(S) \to \mathcal{C}(X \times Y)$, as is sometimes done in standard texts in set-valued analysis. If $X$ and $Y$ are finite dimensional, then it is known that $\mathcal{C}(X \times Y)$ is a metric space. We need to consider the outer limit of the map $T_{\text{gph}(S)} : \text{gph}(S) \to \mathcal{C}(X \times Y)$. To avoid confusion, we will use capitals to denote the outer limit in the set-valued sense, that is

$$
\text{LIMSUP}_{(x,y) \to (\bar{x},\bar{y})} T_{\text{gph}(S)}(x,y)
$$

:= $\{C \subset X \times Y \mid \exists (x_j,y_j) \to (\bar{x},\bar{y}) \text{ s.t. } T_{\text{gph}(S)}(x_j,y_j) \to C\}.$

In general, for $S : X \rightrightarrows Y$, \cite{18} Theorem 4.19] gives

$$
\limsup_{x \to \bar{x}} S(x) = \bigcup\{C \mid C \in \text{LIMSUP}_{x \to \bar{x}} S(x)\},
$$

and \liminf_{x \to \bar{x}} S(x) = \bigcap\{C \mid C \in \text{LIMSUP}_{x \to \bar{x}} S(x)\}.$$

We now compare the conditions in Theorem \ref{thm:finite-dimensional} with what we can get from the outer limit LIMSUP.

**Lemma 3.7.** (Finite dimensional LIMSUP) Suppose $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is closed-valued and \{C_i\}_{i \in I} \subset \mathcal{C}(Y). Then

(a) For all $i \in I$, there exists $\{x_i\} \subset X$ such that $x_i \to \bar{x}$ and \limsup_{k \to \infty} S(x_k) \subset C_i$ if and only if for all $i \in I$, there exists some $D \in \text{LIMSUP}_{x \to \bar{x}} S(x)$ such that $D \subset C_i$.

(b) If for all $D \in \text{LIMSUP}_{x \to \bar{x}} S(x)$, there exists $i \in I$ such that $C_i \subset D$, then $S$ is \{C_i\}_{i \in I}-isc at $\bar{x}$.

**Proof.** (a) The forward direction follows immediately from the fact that for $x_k \to \bar{x}$, we can find a subsequence if necessary so that $\lim_{k \to \infty} S(x_k)$ exists and converges to some $D \in \text{LIMSUP}_{x \to \bar{x}} S(x)$ by \cite{18} Theorem 4.18]. The reverse direction is straightforward.

(b) We prove this by contradiction. Suppose that $S$ is not \{C_i\}_{i \in I}-isc at $\bar{x}$. That is, there exists $\varepsilon > 0$ and $\rho > 0$ and a sequence $\{x_k\}_{k=1}^\infty$ such that $x_k \to \bar{x}$ and

$$
C_i \cap \rho B \not\subset S(x_k) + \varepsilon B \text{ for all } i \in I.
$$

We may choose a subsequence of $\{x_k\}_{k=1}^\infty$ if necessary so that $\lim_{k \to \infty} S(x_k)$ exists. Fix $i^* \in I$. A straightforward application of \cite{18} Theorem 4.10(a) shows that

$$
C_{i^*} \not\subset \lim_{k \to \infty} S(x_k) = \limsup_{x \to \bar{x}} S(x).
$$

Since $i^*$ is arbitrary and $\lim_{k \to \infty} S(x_k) \in \text{LIMSUP}_{x \to \bar{x}} S(x)$, we have a contradiction, and our proof is complete.

**Theorem 3.8.** (Finite dimensional classification of generalized derivatives) Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be such that $S$ is locally closed at $(\bar{x},\bar{y}) \in \text{gph}(S)$, and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a prefan. Then $S$ is pseudo strictly $H$-differentiable at $(\bar{x},\bar{y})$ if and only if

$$
G(p) \cap [-H(-p)] \neq \emptyset
$$

whenever $p \in \mathbb{R}^n \setminus \{0\}$ and $gph(G) \in \text{LIMSUP}_{(x,y) \to (\bar{x},\bar{y})} T_{\text{gph}(S)}(x,y)$.

(3.7)

The above continues to hold if the term $\alpha$ in (3.7) is replaced by $\text{clco} T_{\text{gph}(S)}$, the closed convex hull of the tangent cone.

**Proof.** Let $\text{LIMSUP}_{(x,y) \to (\bar{x},\bar{y})} T_{\text{gph}(S)}(x,y) = \{gph(G_i)\}_{i \in I}$ and use Theorem \ref{thm:finite-dimensional} and Lemma \ref{lem:finite-dimensional}. The case of the closed convex hull is similar.  \qed
4. Examples

We illustrate how the results in Subsection 3.2 can be used to characterize the generalized derivatives $H : X \rightrightarrows Y$ in various cases.

Example 4.1. (Characterizing generalized derivatives) We apply the results in Subsection 3.2 to characterize the prefixed maps $H : X \rightrightarrows Y$ that are the generalized derivatives in several functions defined earlier.

(1) Consider the map $S_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $S_1(x) = (-\infty, -|x|] \cup [|x|, \infty)$. See Figure 4.1. Let $I = \{1, \ldots, 6\}$, define $G_i : \mathbb{R} \rightrightarrows \mathbb{R}$ as in (3.2), and check that $\{G_i\}_{i \in I}$ are such that $\{ \text{gph}(G_i) \}_{i \in I}$ equals \text{LIMSUP}_{(x,y) \in \text{gph}(S_1)} T_{\text{gph}(S_1)}(x,y)$. Hence the conditions in Theorem 3.8 are satisfied at $(0,0)$. We observe that

$$G_1(1) \cap [-H(-1)] \neq \emptyset \quad \text{and} \quad G_3(1) \cap [-H(-1)] \neq \emptyset$$

implies $[-1,1] \subset H(-1)$,

and $G_2(-1) \cap [-H(1)] \neq \emptyset \quad \text{and} \quad G_4(-1) \cap [-H(1)] \neq \emptyset$

implies $[-1,1] \subset H(1)$.

So $S_1$ is pseudo strictly $H$-differentiable at $(0,0)$ if and only if $[-|p|, |p|] \subset H(p)$ for all $p \in \mathbb{R}$.

Note that $G_6(x)$ does not set any restriction on $H$. Observe that we only used $G_i$ for $i = 1,2,3,4$ in (3.2). We can also apply Theorem 3.8 with the fact that $\{ \text{gph}(G_i) \}_{i \in \{1, \ldots, 5\}}$ equals \text{LIMSUP}_{(x,y) \in \text{gph}(S_1)} clco \text{gph}(S_1)(x,y)$ to see that $G_6$ is not needed in characterizing the generalized derivative $H$.

(2) Consider the map $S_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $S_2(x) := \{x\} \cup \{-x\}$. See Figure 4.1. Let $I = \{1,2,3\}$, and define $G_i : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$G_1(x) = \quad x$$

$$G_2(x) = \quad -x$$

$$G_3(x) = \quad S_2(x).$$

Then $\{G_i\}_{i \in I}$ are such that $\{ \text{gph}(G_i) \}_{i \in I}$ equals \text{LIMSUP}_{(x,y) \in \text{gph}(S_2)} T_{\text{gph}(S_2)}(x,y)$, and hence satisfies the conditions in Theorem 3.8 at $(0,0)$. We can easily to check that $S_1$ is pseudo strictly $H$-differentiable at $(0,0)$ if and only if $[-1,1] \subset H(p)$ for $p = 1,-1$, though the proof is slightly different from that in $S_1$.

(3) Consider the maps $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ in Example 3.4. With additional work, we get $f_i$ is pseudo strictly $H$-differentiable at $(0,0)$ if and only if $\mathbb{B} \subset H(p)$ for all $\|p\| = 1$ for both $i = 1,2$.

We remark on the assumption that $H$ is convex-valued in many of the results in this paper.

Remark 4.2. (Convex-valuedness of $H$) Consider the set-valued maps $S_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ and $S_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$S_1(x) := (-\infty, -|x|] \cup [|x|, \infty) \quad \text{and} \quad S_2(x) = \{-x\} \cup \{x\}.$$

Also define $S_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$S_3(x) = \begin{cases} 
\{x\} \cup \{-x\} & \text{if } x \leq 0 \\
[-x,x] & \text{if } x \geq 0.
\end{cases}$$
Define the map \( H' : \mathbb{R} \rightrightarrows \mathbb{R} \) by \( H' \equiv S_2 \). See Figure 4.1. Note that \( H' \) is not convex-valued, but satisfies all other requirements in Theorems 2.2 and 3.5 for both \( S_1 \) and \( S_3 \). While \( S_1 \) is pseudo strictly \( H' \)-differentiable at \((0,0)\), \( S_3 \) is not pseudo strictly \( H' \)-differentiable at \((0,0)\).

5. A third characterization: Generalizing the Mordukhovich criterion

For \( S : X \rightrightarrows Y \), the Mordukhovich criterion expresses \( \text{lip}S(\bar{x} \mid \bar{y}) \) in terms of the limiting normal cone. In this section, we make use of previous results to show how the limiting normal cone can give a characterization of the generalized derivative \( H : X \rightrightarrows Y \) when \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^m \). We also show that the convexified coderivatives have a bijective relationship with the set of possible generalized derivatives.

We start by defining the limiting normal cone.

**Definition 5.1.** (Normal cones) For a set \( C \subseteq \mathbb{R}^n \), the regular normal cone at \( \bar{x} \) is defined as

\[
\hat{N}_C(\bar{x}) := \{ y \mid \langle y, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|) \text{ for all } x \in C. \}
\]

The limiting (or Mordukhovich) normal cone \( N_C(\bar{x}) \) is defined as \( \limsup_{\epsilon \to 0} \epsilon^{-1} \hat{N}_C(x) \), or as

\[
N_C(\bar{x}) = \{ y \mid \text{there exists } x_i \to \bar{x}, y_i \in \hat{N}_C(x_i) \text{ such that } y_i \to y \}. \]

In Lemma 5.2 and Theorem 5.3 below, let \( \mathbb{R}_+ = [0, \infty) \) so that for \( v \neq 0 \), \( \mathbb{R}_+ \{v\} \) is the cone generated by \( v \). We shall refer to positively homogeneous maps that have convex graphs as convex processes, as is commonly done in the literature.

**Lemma 5.2.** (Polar cone criteria) Let \( G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a convex process, and \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a prefan. For each \((u,v) \in [\text{gph}(G)]^0 \subseteq \mathbb{R}^n \times \mathbb{R}^m\), define \( \tilde{G}_{(u,v)} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) by \( \text{gph}(\tilde{G}_{(u,v)}) = [\mathbb{R}_+ \{(u,v)\}]^0 \). Then

\[
G(p) \cap [-H(-p)] \neq \emptyset \text{ for all } p \in \mathbb{R}^n \setminus \{0\}
\]

if and only if \( \tilde{G}_{(u,v)}(p) \cap [-H(-p)] \neq \emptyset \) for all \( p \in \mathbb{R}^n \setminus \{0\} \) and \((u,v) \in [\text{gph}(G)]^0\).

**Proof.** It is clear that for each \( v \in [\text{gph}(G)]^0 \), the \( G \subset \tilde{G}_v \), so the forward direction is easy. We now prove the reverse direction by contradiction.

Suppose \( G(\tilde{p}) \cap [-H(-\tilde{p})] = \emptyset \) for some \( \tilde{p} \neq 0 \). This means that the convex sets \( \text{gph}(G) \) and \( \{\tilde{p}\} \times [-H(-\tilde{p})] \) do not intersect, so there exists some \((\tilde{u}, \tilde{v}) \in \mathbb{R}^n \times \mathbb{R}^m\) and \( \alpha \in \mathbb{R}\). Figure 4.1. The maps \( S_i : \mathbb{R} \rightrightarrows \mathbb{R} \) for \( i = 1, 2, 3 \) are used in Remark 4.2 and in Examples 3.2 and 4.1.
such that
\[ \langle (\bar{u}, \bar{v}), (x, y) \rangle < \alpha \text{ for all } (x, y) \in \text{gph}(G) \]
and
\[ \langle (\bar{u}, \bar{v}), (\bar{p}, y) \rangle > \alpha \text{ for all } y \in [-H(-\bar{p})]. \]

Since \((0, 0) \in \text{gph}(G)\), \(\alpha\) must be positive. Furthermore, since \(\text{gph}(G)\) is a cone, we have
\[ \langle (\bar{u}, \bar{v}), (x, y) \rangle \leq 0 \text{ for all } (x, y) \in \text{gph}(G) \quad (5.1a) \]
and
\[ \langle (\bar{u}, \bar{v}), (\bar{p}, y) \rangle > 0 \text{ for all } y \in [-H(-\bar{p})]. \quad (5.1b) \]

Note that (5.1a) implies that \((\bar{u}, \bar{v}) \in [\text{gph}(G)]^0\), and that \(\bar{G}_{(\bar{u}, \bar{v})} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m\) is defined by \(\bar{G}_{(\bar{u}, \bar{v})}(p) := \{y \mid \langle (\bar{u}, \bar{v}), (p, y) \rangle \leq 0\}\). By the definition of \(\bar{G}_{(\bar{u}, \bar{v})}\) and (5.1b), we have \(\bar{G}_{(\bar{u}, \bar{v})}(\bar{p}) \cap [-H(-\bar{p})] = \emptyset\), which is what we need.

We now recall the definition of coderivatives.

**Definition 5.3.** (Coderivatives) For a set-valued map \(S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m\) and \((\bar{x}, \bar{y}) \in \text{gph}(S)\), the
regular coderivative at \((\bar{x}, \bar{y})\), denoted by \(D^* S(\bar{x} \mid \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n\), is defined by
\[ v \in D^* S(\bar{x} \mid \bar{y})(u) \iff \langle v, -u \rangle \in N_{\text{gph}(S)}(\bar{x}, \bar{y}) \]
\[ \iff \langle (v, -u), (x, y) - (\bar{x}, \bar{y}) \rangle \leq o(\|x, y - (\bar{x}, \bar{y})\|) \]
for all \((x, y) \in \text{gph}(S)\).

The limiting coderivative (or Mordukhovich coderivative) at \((\bar{x}, \bar{y}) \in \text{gph}(S)\) is denoted by
\(D^* S(\bar{x} \mid \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n\) and is defined by
\[ v \in D^* S(\bar{x} \mid \bar{y})(u) \iff \langle v, -u \rangle \in N_{\text{gph}(S)}(\bar{x}, \bar{y}). \]

In the definitions of both the regular and limiting coderivatives, the minus sign before \(u\) is necessary so that if \(f : \mathbb{R}^n \rightarrow \mathbb{R}^m\) is \(C^1\) at \(\bar{x}\), then
\[ D^* f(\bar{x} \mid f(\bar{x}))(y) = \nabla f(\bar{x})^* y \text{ for all } y \in \mathbb{R}^m. \]

We can now state the main result of this section.

**Theorem 5.4.** (Generalized Mordukhovich criterion) Let \(S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m\) be locally closed at \((\bar{x}, \bar{y}) \in \text{gph}(S)\) and let \(H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m\) be a prefan. Then \(S\) is pseudo strictly \(H\)-differentiable at \((\bar{x}, \bar{y})\) if and only if any of the following equivalent conditions hold:

(a) For all \(p \in \mathbb{R}^n \setminus \{0\}\) and \((u, v) \in N_{\text{gph}(S)}(\bar{x}, \bar{y})\), there exists \(y \in H(p)\) s.t.
\[ \langle v, y \rangle \geq -\langle u, p \rangle. \]
(b) For all \(p \in \mathbb{R}^n \setminus \{0\}\) and \(u \in \mathbb{R}^m\),
\[ \min_{y \in H(p)} \langle u, y \rangle \leq \min_{v \in D^* S(\bar{x} \mid \bar{y})(u)} \langle v, p \rangle. \]
(c) For all \(p \in \mathbb{R}^n \setminus \{0\}\) and \(u \in \mathbb{R}^m\),
\[ \min_{y \in H(p)} \langle u, y \rangle \leq \min_{v \in \text{clco} D^* S(\bar{x} \mid \bar{y})(u)} \langle v, p \rangle. \]

**Proof.** To simplify notation, let \(\{\text{gph}(G)\}_{i \in I} = \text{LIMSUP}_{\text{gph}(S)}(x, y)\rightarrow(\bar{x}, \bar{y})\text{clco} T_{\text{gph}(S)}(x, y)\).

We also use the definition of \(\bar{G}_{(\bar{u}, \bar{v})} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m\) defined in the statement of Lemma 5.2 and the following equivalent formulations of (b) and (c):

(b') For all \(p \in \mathbb{R}^n \setminus \{0\}\) and \(v \in D^* S(\bar{x} \mid \bar{y})(u)\), there exists \(y \in H(p)\) s.t. \(\langle u, y \rangle \leq \langle v, p \rangle\).
(c') For all \(p \in \mathbb{R}^n \setminus \{0\}\) and \(v \in \text{clco} D^* S(\bar{x} \mid \bar{y})(u)\), there exists \(y \in H(p)\) s.t. \(\langle u, y \rangle \leq \langle v, p \rangle\).
Using [18 Corollary 11.35(b)] (which states that a closed convex cone converges if and only if its polar converges), we have

\[
\limsup_{(x,y) \to \overline{\text{co}}(x,y)} N_{\overline{\text{co}}(S)}(x,y) = \limsup_{(x,y) \to \overline{\text{co}}(x,y)} \left[ \text{cl} \text{co} T_{\overline{\text{co}}(S)}(x,y) \right]^0
\]

= \{ [\text{gph}(G_i)]^0 \}_{i \in I}

By the observation in (3.6a), which recalls [18 Proposition 4.19], we have

\[
N_{\overline{\text{co}}(S)}(\bar{x}, \bar{y}) = \limsup_{(x,y) \to \overline{\text{co}}(x,y)} N_{\overline{\text{co}}(S)}(x,y) = \bigcup_{i \in I} \text{gph}(G_i)^0.
\]

By Lemma 5.2

\[
G(p) \cap [-H(-p)] \neq \emptyset \text{ for all } p \in \mathbb{R}^n \setminus \{0\}
\]

if and only if \( \tilde{G}_{(u,v)}(p) \cap [-H(-p)] \neq \emptyset \) for all \( p \in \mathbb{R}^n \setminus \{0\} \) and \((u,v) \in [\text{gph}(G)]^0 \).

By Theorem 5.3 \( S \) is pseudo strictly \( H \)-differentiable at \((\bar{x}, \bar{y})\) if and only if

\[
G_i(p) \cap [-H(-p)] \neq \emptyset \text{ for all } p \in \mathbb{R}^n \setminus \{0\} \text{ and } i \in I,
\]

or equivalently, \( \tilde{G}_{(u,v)}(p) \cap [-H(-p)] \neq \emptyset \) for all \( p \in \mathbb{R}^n \setminus \{0\} \) and \((u,v) \in \bigcup_{i \in I} [\text{gph}(G_i)]^0 \).

We can substitute \( \bigcup_{i \in I} [\text{gph}(G_i)]^0 \) in the above formula. Unrolling the definition of \( \tilde{G}_{(u,v)} \) gives: For all \( p \in \mathbb{R}^n \setminus \{0\} \) and \((u,v) \in N_{\overline{\text{co}}(S)}(\bar{x}, \bar{y})\), there exists some \( y \in [-H(-p)] \) such that \( \langle (u,v), (p,y) \rangle \leq 0 \), which is easily seen to be condition (a). Condition (b') rephrases condition (a) in terms of the limiting coderivatives, while condition (c) is equivalent to condition (b) by elementary properties of convexity. 

Characterizing the generalized derivatives \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) in terms \( N_{\overline{\text{co}}(S)}(\bar{x}, \bar{y}) \) or \( D^*S(\bar{x} \mid \bar{y}) \) instead of the tangent cones not only enjoys a simpler statement, it also enables one to use tools for normal cones that may not be present for tangent cones. For example, estimates of the coderivatives of the composition of two set-valued maps are more easily available than corresponding results in terms of tangent cones.

In the particular case of considering the Aubin property, we obtain the classical Mordukhovich criterion.

**Remark 5.5.** (Mordukhovich criterion) Using Lemma 5.2 and Theorem 5.4, if \((u,v) \in N_{\overline{\text{co}}(S)}(\bar{x}, \bar{y})\), the smallest \( \kappa \) such that \( \tilde{G}_{(u,v)}(p) \cap \kappa \|p\| \neq \emptyset \) for all \( p \in \mathbb{R}^n \setminus \{0\} \) is \( \|u\|/\|v\| \).

This gives

\[
\text{lip} S(\bar{x} \mid \bar{y}) = \sup \{ \|u\|/\|v\| \mid (u,v) \in N_{\overline{\text{co}}(S)}(\bar{x}, \bar{y}) \}
\]

= \( \|D^*S(\bar{x} \mid \bar{y})\|^+ \)

= \( \|\text{cl} \text{co} D^*S(\bar{x} \mid \bar{y})\|^+ \),

which is precisely the conclusion of the Mordukhovich criterion in the finite dimensional case.

Theorem 5.4 (c) shows that \( \text{cl} \text{co} D^*S(\bar{x} \mid \bar{y}) \) characterizes all possible generalized derivatives \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \). As Theorem 5.8 shows, the reverse holds as well.

**Lemma 5.6.** (Outer semicontinuity of convexified maps) Suppose \( D : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) is osc, and is locally bounded at \( \bar{x} \). Then the map \( \text{co} D : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \), which maps \( x \) to the convex hull of \( D(x) \), is osc at \( \bar{x} \).
Proof. It suffices to show that if $y_i \in \text{co} D(x_i)$, $y_i \to \bar{y}$ and $x_i \to \bar{x}$, then $\bar{y} \in D(\bar{x})$. By Caratheodory’s theorem, we can write $y_i$ as a convex combination of $z_{i,1}$, $z_{i,2}$, ..., $z_{i,n+1}$. By taking a subsequence if necessary, we can assume that $z_{i,1}$ converges to some $\bar{z}_i$ in $D(\bar{x})$. Doing this $n + 1$ times allows us to assume that for any $j \in \{1, \ldots, n+1\}$, $\{z_{i,j}\}_{i=1}^\infty$ converges to some $\bar{z}_j \in D(\bar{x})$. It is elementary that $\bar{y}$ is in the convex hull of $\{\bar{z}_1, \ldots, \bar{z}_{n+1}\}$, which gives $\bar{y} \in D(\bar{x})$ as needed. \\

For $D : \mathbb{R}^m \to \mathbb{R}^n$ such that $D$ is positively homogeneous and $\|D\|^+$ is finite, define $\mathcal{H}(D)$ by

$$
\mathcal{H}(D) := \{ H : \mathbb{R}^n \to \mathbb{R}^m : H \text{ is a prefan,} \}
$$

and for all $p \in \mathbb{R}^n \setminus \{0\}$ and $u \in \mathbb{R}^m$,

$$
\min_{y \in H(p)} \langle u, y \rangle \leq \min_{v \in \text{clco}D(u)} \langle v, p \rangle.
$$

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ is locally closed at $(\bar{z}, \bar{v})$. By Theorem 5.4, $\mathcal{H}(D,S(\bar{z} \mid \bar{v}))$ is the set of all possible $H$ with the relevant properties such that $S$ is pseudo strictly $H$-differentiable at $(\bar{z}, \bar{v})$. We now state another lemma.

Lemma 5.7. (Convexified coderivatives and generalized derivatives) Suppose $D_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that $D_1$ is positively homogeneous, osc. and $\|D_1\|^+$ is finite for $i = 1, 2$. Then the following hold.

1. $\text{clco}D_1 \subset \text{clco}D_2$ implies $\mathcal{H}(D_1) \subset \mathcal{H}(D_2)$.

2. $\text{clco}D_1 \neq \text{clco}D_2$ implies $\mathcal{H}(D_1) \neq \mathcal{H}(D_2)$.

3. $\text{clco}D_1 \subset \text{clco}D_2$ implies $\mathcal{H}(D_1) \supset \mathcal{H}(D_2)$.

4. $\mathcal{H}(D_1) = \mathcal{H}(D_2)$ implies $\text{clco}D_1 = \text{clco}D_2$.

Proof. Property (1) follows easily from the definitions, property (4) is equivalent to property (2), and property (3) follows easily from property (1) and (2). We thus concentrate on proving property (4). We shall assume throughout that $D_1$ and $D_2$ are convex-valued to cut down on notation. Assume $\mathcal{H}(D_1) = \mathcal{H}(D_2)$. By the classical Mordukhovich criterion (in Remark 5.5), we must have $\|D_i\|^+ = \|D_i\|^+$.

Suppose on the contrary that $D_1 \neq D_2$. There must be some $u$ and $\bar{v}$ such that without loss of generality, $\bar{v} \notin D_1(\bar{u})$ but $\bar{v} \in D_2(\bar{u})$. Since $D_1(\bar{u})$ is convex, there is some $\bar{w} \neq 0$ such that $\langle \bar{w}, \bar{v} \rangle < \alpha$ and $\langle \bar{w}, v \rangle > \alpha$ for all $v \in D_1(\bar{u})$.

By the outer semicontinuity of $D_1$, there is a neighborhood $\mathbb{B}_\varepsilon(\bar{u})$ of $\bar{u}$ such that $D_1(u) \subset \{v \mid \langle \bar{w}, v \rangle > \alpha\}$ for all $u \in \mathbb{B}_\varepsilon(\bar{u})$. Let the variable $\theta > 0$ be such that $2 \sin(\theta/2) = \varepsilon$.

We can assume that $\|\bar{w}\| = \|\bar{u}\| = 1$. Define $H : \mathbb{R}^n \to \mathbb{R}^m$ to be

$$
H(x) := \begin{cases} 
\bar{u} + \frac{1}{2}[(\alpha + \langle \bar{u}, \bar{w} \rangle) \leq \langle \bar{u}, y \rangle \leq \|D_1\|^+, 
\|y\|^2 - \langle \bar{u}, y \rangle^2 \leq L^2 
\lambda H(\bar{w}) 
\|D_1\|^+ \|x\| \mathbb{B} 
\end{cases} 
$$

if $x = \bar{w}$

Otherwise,

$$
\|D_1\|^+ \|x\| \mathbb{B}
$$

where $L > \frac{1}{\sin\theta}\|D_1\|^+ + \max(\alpha, 0) \cos \theta$. See Figure 5.1 for an illustration of $H(\bar{w})$ when $\alpha > 0$.

Clearly, $H$ is convex-valued, compact-valued and positively homogeneous, and $\|H\|^+$ is finite. Once the claim below is proved, we will establish the result at hand.
reducing the possible pairs

Figure 5.1 shows the two dimensional space in $y$ point such that the case when $\in u$ check that for all $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{w}$, $\bar{v}$.

Claim: The map $H$ satisfies the conditions in Theorem 5.4(c) for the map $D_1 : \mathbb{R}^m \rightarrow \mathbb{R}^n$.

Next, we show that $D_1$ for $D_1$ but not for $D_2$.

Since $\bar{v} \in D_2(\bar{u})$, Theorem 5.4(c) says that we can find a $y \in H(\bar{w})$ such that $\langle \bar{u}, y \rangle \leq \langle \bar{v}, \bar{w} \rangle$. But this is not the case since for all $y \in H(\bar{w})$, we have $\langle \bar{u}, y \rangle = \frac{1}{2} \langle \alpha + \langle \bar{v}, \bar{w} \rangle \rangle \geq \langle \bar{v}, \bar{w} \rangle$.

Next, we show that $H$ satisfies the conditions in Theorem 5.4(c) for $D_1$. We need to check that for all $p \in \mathbb{R}^n \setminus \{0\}$ and $(u, v) \in \text{gph}(D_1)$, we can find a $y \in H(p)$ such that $\langle u, y \rangle \leq \langle v, p \rangle$. Since $D_1$ is positively homogeneous, we can check only $(u, v) \in \text{gph}(D_1)$ such that $\|u\| = 1$. We can further assume that $\|p\| = 1$. By the classical Mordukhovich criterion, we see that $p \neq \bar{w}$ poses no problems.

Let $\partial B := \{u \in \mathbb{R}^n \| u \| = 1\}$. By our earlier discussion on the outer semicontinuity of $D_1$ and the fact that $\|D_1\|^{\dagger}$ is finite, we have

$$\partial B \times \mathbb{R}^n \cap \text{gph}(D_1) \subset \left(\partial B \cap B_{2\sin(\theta/2)}(\bar{u})\right) \times \{D_1 \cap B \cap \{v \| \langle \bar{w}, v \rangle \geq \alpha \}\}
\cup \left(\partial B \cap B_{2\sin(\theta/2)}(\bar{u})\right) \times \{D_1 \cap B \cap \{v \| \langle \bar{w}, v \rangle \geq \alpha \}\},$$

reducing the possible pairs $(u, v) \in \text{gph}(D_1)$ that we need to check.

Case 1: If $(u, v) \in \partial B \cap B_{2\sin(\theta/2)}(\bar{u}) \times \{D_1 \cap B \cap \{v \| \langle \bar{w}, v \rangle \geq \alpha \}\}$, then we can find $y \in H(\bar{w})$ s.t. $(u, y) \leq (v, \bar{w})$.

The case when $u = \bar{u}$ is easy and will be covered later. Consider the $u \in \partial B \cap B_{2\sin(\theta/2)}(\bar{u}) \setminus \{\bar{u}\}$.

Figure 5.1 shows the two dimensional space in $\mathbb{R}^n$ containing the points $0, u$ and $\bar{u}$ in the case when $\alpha > 0$. The intersection of $H(\bar{w})$ is also illustrated. The condition that $u \in \partial B \cap B_{2\sin(\theta/2)}(\bar{u})$ implies that the angle $\theta$ in Figure 5.1 is in the interval $[0, \theta]$. The point $y_1$ is formally defined as the point lying in the two dimensional subspace spanned by $u$ and $\bar{u}$, and satisfies $\langle \bar{u}, y_1 \rangle = \frac{1}{2} \langle \alpha + \langle \bar{v}, \bar{w} \rangle \rangle$ and $\|y_1\|^2 = \langle \bar{u}, y_1 \rangle^2 = L^2$. By restricting the
maximum angle $\hat{\theta}$ if necessary when $\alpha < 0$ and using elementary geometry, we have
\[
\langle u, y_1 \rangle \leq \langle \bar{u}, y_1 \rangle
\]
\[
= \frac{1}{2} (\alpha + \langle \bar{v}, \bar{w} \rangle)
\]
\[
< \alpha
\]
\[
\leq \langle v, \bar{w} \rangle,
\]
which gives us what we need.

**Case 2:** If $(u, v) \in [\partial B \setminus 2 \sin(\hat{\theta}/2)] \times \|D_1\| B$, then we can find $y \in H(\bar{w})$ s.t. $\langle u, y \rangle \leq \langle v, \bar{w} \rangle$.

In this case, we need to show that for all $(u, v)$ given, we can find $y \in H(\bar{w})$ such that $\langle u, y \rangle \leq -\|D_1\|^+$. The fact that $-\|D_1\|^+ \leq \langle v, \bar{w} \rangle$ will give us what we need. We split this case into two subcases.

**Subcase 2a:** $\alpha \geq 0$.

For the choice of $y_1$, we have
\[
\langle u, y_1 \rangle = -L \sin \theta + \alpha \cos \theta.
\]
We first consider $\theta \in [\hat{\theta}, \pi/2]$. The RHS of the above attains its maximum when $\theta = \hat{\theta}$. The condition $L \geq \frac{1}{\sin \theta} (\|D_1\|^+ + \alpha \cos \hat{\theta})$ implies that $\langle u, y \rangle \leq -\|D_1\|^+$ as needed.

We now treat the case where $\theta \in [\pi/2, \pi]$. The point $y_2$ is defined similarly as in $y_1$, except that $\langle \bar{u}, y_2 \rangle = \|D_1\|^+$. We have
\[
\langle u, y_2 \rangle = -L \sin \theta + \|D_1\|^+ \cos \theta.
\]
From Figure 5.1 we can see that the RHS of the above attains its maximum when $\theta = \pi$, which gives $\langle u, y_2 \rangle \leq -\|D_1\|^+$ as needed.

**Subcase 2b:** $\alpha < 0$.

Repeat the arguments for when $\theta \in [\hat{\theta}, \pi/2]$, but replace all occurrences of $y_1$ by $y_3$ as marked in Figure 5.1 (The point $y_3$ satisfies $\langle \bar{u}, y_3 \rangle = 0$ will lie in $H(\bar{w})$. The case when $\theta \in [\pi/2, \pi]$ is also similar. This concludes the proof of the claim, and establishes (4). \qed

With the above lemma, we state a theorem on the relationship between convexified coderivatives and the generalized derivatives.

**Theorem 5.8.** (Convexified coderivatives from generalized derivatives) Suppose $S : R^n \rightrightarrows R^m$ is locally closed at $(\bar{x}, \bar{y}) \in \text{gph}(S)$ and has the Aubin property there. Then the convexified coderivative $\text{clco} D^* S(\bar{x} | \bar{y}) : R^m \rightrightarrows R^n$ is uniquely determined by the set of all prefans $H : R^n \rightrightarrows R^m$ such that $S$ is pseudo strictly $H$-differentiable at $(\bar{x}, \bar{y})$.

**Proof.** The map $D^* S(\bar{x} | \bar{y})$ is osc, and by Lemma 5.6 so is $\text{clco} D^* S(\bar{x} | \bar{y})$. Apply Lemma 5.7 to get the result. \qed

6. **Application to Constraint Mappings**

We conclude this paper by illustrating how the results presented in this paper give stronger results to constraint mappings. One particular example is the set-valued map defined by mixed equalities and inequalities that is usually studied with the Mangasarian-Fromovitz constraint qualification. The relationship between the generalized differentiability of set-valued maps and optimization problems is clear from the example in [11].

In Proposition 6.1 below, we shall only treat the case where $D$ is Clarke regular and apply Corollary 5.6 to illustrate the spirit of our results. While stronger conditions for the
case where \( D \) is not Clarke regular can be deduced from the characterizations in Sections 3 and 5, the extra calculations do not give additional insight.

**Proposition 6.1.** (Constraint mappings, adapted from [18]) Let \( S(x) = F(x) - D \) for smooth \( F : \mathbb{R}^n \to \mathbb{R}^m \) and a closed, Clarke regular set \( D \subseteq \mathbb{R}^m \); then \( S^{-1}(u) \) consists of all \( x \) satisfying the constraint system \( F(x) - u \in D \), with \( u \) as a parameter.

Suppose \( H : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) is a prefan such that for all \( p \in \mathbb{R}^m \setminus \{0\} \), there exists \( q \in -H(-p) \) such that \( \nabla F(\bar{x})q - p \in T_D(F(\bar{x}) - \bar{u}) \). Then \( S^{-1} \) is pseudo strictly \( H \)-differentiable at \((\bar{u}, \bar{x})\).

**Proof.** The set \( \text{gph}(S) \) is specified by \( F_0(x, u) = F(x) - u \). The mapping \( F_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) is smooth, and its Jacobian \( \nabla F_0(\bar{x}, \bar{u}) = [\nabla F(\bar{x}), -I] \) has full rank \( m \). Applying the rule in [18, Exercise 6.7], we see that

\[
\nabla \text{gph}(S)(\bar{x}, \bar{u}) = \{ (v, -y) \mid y \in \nabla D(F(\bar{x}) - \bar{u}), v = \nabla F(\bar{x})^T y \} = \{ (v, -y) \mid y \in D(F(\bar{x}) - \bar{u}), v = \nabla F(\bar{x})^T y \} = N_{\text{gph}(S)}(\bar{x}, \bar{u}).
\]

Therefore, \( \text{gph}(S) \) is Clarke regular at \((\bar{x}, \bar{u})\). From [18, Exercise 6.7] again, we see that \( T_{\text{gph}(S)}(\bar{x}, \bar{u}) = \{ (q, p) \in \mathbb{R}^n \times \mathbb{R}^m \mid \nabla F(\bar{x})q - p \in T_D(F(\bar{x}) - \bar{u}) \} \), so \( T_{\text{gph}(S^{-1})}(\bar{u}, \bar{x}) = \{ (p, q) \in \mathbb{R}^m \times \mathbb{R}^n \mid \nabla F(\bar{x})q - p \in T_D(F(\bar{x}) - \bar{u}) \} \).

The formula for \( T_{\text{gph}(S^{-1})} \), together with Corollary 3.6 gives the conclusion needed. \( \square \)

If the constraint qualification

\[
y \in D(F(\bar{x}) - \bar{u}), \nabla F(\bar{x})^T y = 0 \implies y = 0 \tag{6.1}
\]

holds in Proposition 6.1, then [18, Exercise 9.44] states that \( S^{-1} \) has the Aubin property with modulus

\[
\max_{\|y\| = 1} \frac{1}{\|\nabla F(\bar{x})^T y\|},
\]

so an \( H : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) satisfying the stated conditions can be found.

The case where \( D = \{0\}^r \times \mathbb{R}^m - r \) in Proposition 6.1 gives

\[
S^{-1}(u) := \{ x : F_i(x) = u_i \text{ for } i = 1, \ldots, r \text{ and } F_i(x) \leq u_i \text{ for } i = r + 1, \ldots, m \}.
\]

In this case, the constraint qualification (6.1) is equivalent to the Mangasarian-Fromovitz constraint qualification defined by the existence of \( w \in \mathbb{R}^r \) satisfying

\[
\nabla F_i(\bar{x})w < 0 \text{ for all } i \in \{r + 1, \ldots, m\} \text{ s.t. } F_i(\bar{x}) = 0,
\]

and

\[
\nabla F_i(\bar{x})w = 0 \text{ for all } i \in \{1, \ldots, r\}.
\]

The corresponding conclusion in Proposition 6.1 can be easily deduced.

Finally, we remark that the Aubin property of the constraint mapping \( S^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) at \((\bar{u}, \bar{x})\) is also equivalently studied as the metric regularity of \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) at \((\bar{u}, \bar{x})\). One may refer to standard references [13, 16, 18] for more on metric regularity and its relationship with the Aubin property. The equivalence between pseudo strict \( H \)-differentiability and generalized metric regularity is discussed in [17, Section 7].

7. ACKNOWLEDGEMENTS

I thank Alexander Ioffe, Adrian Lewis, Dmitriy Drusvyatskiy and ShanShan Zhang for conversations that prompted the addition of Section 6 and also to Alexander Ioffe for probing how the results here can be stated in terms of fans, which simplified some of the statements in this paper.
CHARACTERIZING GENERALIZED DERIVATIVES OF SET-VALUED MAPS

REFERENCES

1. J.-P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, Adv. Math., Suppl. Stud. 7A (1981), 159–229.
2. ________, Viability theory, Birkhäuser, Boston, 1991, Republished as a Modern Birkhäuser Classic, 2009.
3. ________, A viability approach to the inverse set-valued map theorem, Journal of Evolution Equations 6 (2006), 419–432.
4. J.-P. Aubin and I. Ekeland, Applied nonlinear analysis, Wiley, New York, 1984, Reprinted by Dover 2006.
5. J.-P. Aubin and H. Frankowska, On the inverse function theorem for set-valued maps, J. Math. Pures Appl. 66 (1987), 71–89.
6. ________, Set-valued analysis, Birkhäuser, Boston, 1990, Republished as a Modern Birkhäuser Classic, 2009.
7. F.H. Clarke, Optimization and nonsmooth analysis, Wiley, Philadelphia, 1983, Republished as a SIAM Classic in Applied Mathematics, 1990.
8. A.L. Dontchev, M. Quincampoix, and N. Zlateva, Aubin criterion for metric regularity, Journal of Convex Analysis 3 (2006), 45–63.
9. A.L. Dontchev and R.T. Rockafellar, Implicit functions and solution mappings: A view from variational analysis, Springer, New York, 2009, Springer Monographs in Mathematics.
10. A.D. Ioffe, Nonsmooth analysis: differential calculus of non-differentiable mappings, Trans. Amer. Math. Soc. 266 (1981), 1–56.
11. ________, Metric regularity and subdifferential calculus, Russian Math. Surveys 55:3 (2000), 501–558.
12. A. Jourani and L. Thibault, coderivatives of multivalued mappings, locally compact cones and metric regularity, Nonlinear Analysis, Theory Methods Appl. 35(7) (1999), 925–945.
13. D. Klatte and B. Kummer, Nonsmooth equations in optimization: Regularity, calculus, methods and applications, Kluwer, Dordrecht, the Netherlands, 2002.
14. ________, Stability of inclusions: characterizations via suitable lipschitz functions and algorithms, Optimization 5-6 (2006), 627–660.
15. B.S. Mordukhovich, Complete characterization of openness, metric regularity and Lipschitzian properties of multifunctions, Trans. Amer. Math. Soc. 34 (1993), 1–35.
16. ________, Variational analysis and generalized differentiation I and II, Springer, Berlin, 2006, Grundlehren der mathematischen Wissenschaften, Vols 330 and 331.
17. C.H.J. Pang, Generalized differentiation with positively homogeneous maps: Applications in set-valued analysis and metric regularity, Math. Oper. Res. (2011), Accepted.
18. R.T. Rockafellar and R.J.-B. Wets, Variational analysis, Springer, Berlin, 1998, Grundlehren der mathematischen Wissenschaften, Vol 317.

Current address: Massachusetts Institute of Technology, Department of Mathematics, 2-334, 77 Massachusetts Avenue, Cambridge MA 02139-4307.
E-mail address: chj2pang@mit.edu