ON BIANCHI’S BÄCKLUND TRANSFORMATION OF QUADRICS

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Abstract. We investigate basic features of Bianchi’s Bäcklund transformation of quadrics to see if it can be obtained under weaker assumptions and if it can be generalized to deformations of other surfaces.

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1. Introduction

It has been an open question ever since the classical times to find all surfaces for which an interesting theory of deformation (procedures of generating more or less explicit examples depending on arbitrarily many constants; for example Bäcklund (B) transformation) can be build, just as for quadrics (see § 5.17 of R. Calapso’s introduction to Vol 4, part 1 of Bianchi’s Opere).

Based on Bianchi’s [3] (also appearing as ([5], Vol 4, (143))) characterization of the (isotropic) singular B transformation of quadrics with auxiliary surface plane, we consider basic features of Bianchi’s Bäcklund (B) transformation of quadrics to see if it can be obtained under weaker assumptions and if it can be generalized to deformations (isometric deformations) of other (classes of) surfaces.

Due to the frequent use of certain keywords we shall define and use abbreviated notations.

All computations are local and assumed to be valid on their open domain of validity without further details; all functions have the assumed order of differentiability and are assumed to be invertible, non-zero, etc when required (for all practical purposes we can assume all functions to be analytic).

By necessity (all quadrics in the real 3-dimensional Euclidean space are equivalent from a complex projective point of view) we shall consider the complexification

\[(\mathbb{C}^3, < \cdot, \cdot >), \quad < x, y > := x^T y, \quad |x|^2 := x^T x, \quad x, y \in \mathbb{C}^3\]

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of the real 3-dimensional Euclidean space.

Isotropic (null) vectors are those vectors of length 0; since most vectors are not isotropic we call a vector simply vector and we shall emphasize isotropic for isotropic vectors. The same denomination will apply in other settings: for example we call quadric a non-degenerate quadric (a quadric projectively equivalent to the complex unit sphere).

We call surface any sub-manifold of $C^3 \simeq R^9$ of real dimension 2, 3 or 4 such that all its complexified tangent spaces (called henceforth tangent spaces) have complex dimension 2. In this case the distribution $T \cap iT$ formed by intersecting real tangent spaces $T$ with $iT$ has constant real dimension (which must respectively be 0, 2 or 4). This distribution is integrable (by its own definition); by an application of Frobenius one obtains leaves; on leaves the Nijenhuis condition (with the almost complex structure being induced by the surrounding space) holds and by Nirenberg-Newlander this is equivalent (locally) to prescribing $x : D \rightarrow C^4$, where $D$ is a domain of $R^2$ or $C \times R$ or $C^2$ such that $dx \wedge dx \neq 0$ (the $\wedge$ applies to the vector structure and the $\wedge$ to the exterior form structure, so the order does not matter: $\wedge \wedge = \wedge \wedge$); from a practical point of view we need to consider only surfaces that are real 2- or 4-dimensional.

For any two curves there is a developable circumscribed to them; isotropic developables are surfaces circumscribed to a curve and to the circle $C(\infty)$ at $\infty$ (Cayley’s absolute); they are the only surfaces with degenerate 2-dimensional linear element. Thus we assume a surface not to be isotropic developable unless otherwise stated; note however that the importance of isotropic developables cannot be cast aside, as most isotropic developables circumscribed to (isotropic) (singular) conics (the singular part of a doubly ruled (isotropic) plane image of the unit sphere or of the equilateral paraboloid under an affine transformation of $C^3$ with 1-dimensional kernel (which is different from the axis of the equilateral paraboloid in this case)) generate the confocal family of quadrics one of whose (isotropic) singular quadrics (including in a general sense (isotropic) plane of certain pencils of (isotropic) planes) contains as a singular set the given conic: any quadric of the confocal family cuts the conic at 4 points (Bézout) and the isotropic rulings of the isotropic developable passing through those points are all the isotropic rulings of the quadric; three rulings of a ruling family of a quadric are enough to determine the quadric (although there are quadrics with less than three isotropic rulings in both of the ruling families, by inspection all quadrics except (pseudo-)spheres are uniquely determined by their finitely many isotropic rulings).

If we let the spectral parameter $z$ vary in the family of quadrics confocal to a given one, then we get an affine correspondence between confocal quadrics (called the Ivory affinity) with good metric properties.

In particular it preserves lengths of rulings, so it takes isotropic rulings and umbilics (their finite intersections; the remaining 4 points of intersection of isotropic rulings are situated on $C(\infty)$ and for either of them having multiplicity we have quadrics of revolution or Darboux quadrics) to isotropic rulings and umbilics; thus as $z$ varies an umbilic describes a singular conic of a(n isotropic) singular confocal quadric and an isotropic ruling for each ruling family describes the isotropic developable circumscribed to the singular conic.

Standard geometric formulae for surfaces in $R^3$ remain valid (with their usual denomination) for all surfaces and outside the locus of isotropic normal directions. Since the Gauß-Weingarten (GW) and the Gauß-Codazzi-Mainardi-Peterson (G-CMP) equations for a surface $x \subset R^3$ assume only the non-degeneracy of the linear element of $x$ (and thus the existence of an orthonormal normal), they are still valid and sufficient to describe the geometry of surfaces in $C^3$ almost everywhere; for example the Lorentz space of signature $(2, 1)$ can be realized as the totally real subspace $R^2 \times iR \subset C^3$ (the induced scalar product is real (valued) and non-degenerate).

By the Gauß-Bonnet-Peterson fundamental theorem of surfaces the GW and G-CMP equations suffice to describe the geometry of all surfaces almost everywhere and surfaces such defined are the natural completion of the usual real surfaces, although the usual coordinates used in geometry (asymptotic, conjugate systems, orthogonal, principal, etc) may not be the ones which clearly split into purely real and purely complex (for example asymptotic coordinates on a real surface of positive
Gauß curvature). The change of coordinates for real 3-dimensional surfaces still holds outside bi-holomorphic change of the complex coordinate and diffeomorphism of the real one, but if the real and complex coordinates are mixed, they lose the character of being purely real or complex and the new parametrization may not be holomorphic in some variables. For example if we have a surface $x = x(z, t) \subset \mathbb{C}^3$, $(z, t) \in D \subset \mathbb{C} \times \mathbb{R}$ and consider two complex functions $u = u(z, t)$, $v = v(z, t)$ of $z$ and $t$ with $J := \frac{du}{dz} \frac{dv}{dt} - \frac{du}{dt} \frac{dv}{dz} \neq 0$, then one can invert $z = z(u, v)$, $t = t(u, v)$ at least formally and at the infinitesimal level by the usual calculus rule of taking the inverse of the Jacobian: $dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$, $dt = \frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv$. Although the holomorphic dependence on a variable is lost, the 2-dimensionality character remains clear. After using formal coordinates of geometric meaning to simplify certain computations, one can return to the purely real and complex variable is lost, the 2-dimensionality character remains clear. After using formal coordinates of geometric meaning to simplify certain computations, one can return to the purely real and complex coordinates, just as the use of $z, \bar{z}$ parametrization on real surfaces. Note that although on such surfaces the vector fields $\partial_z, \partial_{\bar{z}}$ do not admit integral curves (lines of coordinates), statements about infinitesimal behavior of such lines remain valid, so one can assume such lines to exist and derive corresponding results; thus for all practical purposes we can assume that coordinates descend upon lines of coordinates on the surface.

Consider Lie’s viewpoint: one can replace a surface $x \subset \mathbb{C}^3$ with a 2-dimensional distribution of facets (pairs of points and planes passing through those points): the collection of its tangent planes (with the points of tangency highlighted); thus a facet is the infinitesimal version of a surface (the integral element $\{x, dz\}_{pt}$ of the surface). Conversely, a 2-dimensional distribution of facets is not always the collection of the tangent planes of a surface (with the points of tangency highlighted), but the condition that a 2-dimensional distribution of facets is integrable (that is it is the collection of the tangent planes of a leaf (sub-manifold)) does not distinguish between the cases when this sub-manifold is a surface, curve or point, thus allowing the collapsing of the leaf.

A 3-dimensional distribution of facets is integrable if it is the collection of the tangent planes of an 1-dimensional family of leaves.

Two rollable (applicable) surfaces can be rolled (applied) one onto the other such that at any instant they meet tangentially and with the same differential at the tangency point. The rolling $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$, $(R, t) \subset O_3(\mathbb{C}) \times \mathbb{C}^3$ of two applicable surfaces $x_0$, $x$ (that is $|dx_0|^2 = |dx|^2$) introduces the flat connection form (it encodes the difference of the second fundamental forms of $x_0, x$ and it being flat encodes the difference of the G-CMP equations of $x_0, x$).

In rolling various geometric objects (points, curves, surfaces, facets, etc or families of these) can be rigidly attached to each point of the rolling surface $x_0$, thus producing rolling objects (rolling congruences (2-dimensional families of objects, presumed lines unless otherwise stated), rolling distributions of facets, etc). The study of the rolling problem for surfaces as it was carried out by the classical geometers (see references Bianchi, Darboux, Eisenhart) boils down mainly to finding the features of these rolling objects independent of the shape of the surface of rolling $x$ (for example envelopes of a rolling congruence of curves or of a sphere congruence centered on $x_0$, integrable rolling distributions of facets (IRDF), cyclic systems (3-dimensional IRDF such that facets are centered on circles in the tangent planes of $x_0$ and normal to the circles)). Note that due to the indeterminacy (in the normal bundle) of the rolling (one can roll a surface on both sides of one of its deformations), these rolling objects come in two families that reflect in the tangent bundle of the rolling surface.

Conversely, existence (integrability) conditions for such an object for a particular deformation $x_0$ depend usually on the linear element of $x_0$ and linearly on the second fundamental form of $x_0$ (the terms of the second fundamental form appearing quadratically in the existence conditions become, via Gauß’s theorem, dependent of the linear element of $x_0$); to formulate the conditions for the existence of the rolled objects it is sufficient to individuate for cancelling the coefficients of the linearly appearing second fundamental form (note that in the reflected object these coefficients have changed signs).
Since by infinitesimal rolling in an arbitrary tangential infinitesimal direction \( \delta \) an initial facet \( F \) which is common tangent plane to two applicable surfaces is replaced with an infinitesimally close facet \( F' \) having in common with \( F \) the direction \( \delta \), in the actual rolling problem we have facets centered on each other (the symmetric tangency configuration (TC)) and facets centered on another one \( F \) reflect in \( F \); note that we assumed a finite law of a general nature as a consequence of an infinitesimal law via discretization (the converse is clear); see Bianchi [3].

Thus for a theory of deformation of surfaces with the assumption above we are led to consider, via rolling, certain 4-dimensional distributions of facets centered on the tangent planes of the considered surface \( x_0 \) and passing through the origin of the tangent planes (each point of each tangent plane is the center of finitely many facets) and their rolling counterparts on the applicable surface \( x \).

After a thorough study of infinitesimal laws (in particular infinitesimal deformations) and their iterations the classical geometers were led to consider the B transformation (a finite law of a general nature) as a consequence of an infinitesimal law via discretization.

The B transformation in the deformation problem naturally appears by splitting the 4-dimensional distribution of facets into an 1-dimensional family of 3-dimensional IRDF, thus introducing a spectral parameter \( z \) \( (B_z) \) (each 3-dimensional distribution of facets is integrable (with leaves \( x^1 \)) regardless of the shape of the seed surface \( x^0 \); note that there are 4-dimensional IRDF that do not split into an 1-dimensional family of 3-dimensional IRDF: for example to each point of \( x_0 \) we associate the tangent planes of a surface).

By imposing the initial collapsing ansatz of leaves of the 4-dimensional IRDF to be a 2-dimensional family of curves (this ansatz individuates the defining surface \( x_0 \)) the 2-dimensional family of curves naturally splits into an 1-dimensional family of auxiliary surfaces \( x_z \) (1-dimensional family of curves) such that the facets centered on these auxiliary surfaces form a 3-dimensional IRDF whose integrability is independent of the shape of the seed surface \( x^0 \) applicable to the surface \( x_0^0 \subset x_0 \). For \( x_0 \) quadric the auxiliary surfaces \( x_z \) are ((isotropic) singular) confocal quadrics doubly ruled by collapsed leaves and with \( z \) being the spectral parameter of the confocal family (this includes in a general sense the (isotropic) planes of the pencil of (isotropic) planes through the axis of revolution for \( x_0 \) quadric of revolution (excluding (pseudo-)spheres) (in this case the actual isotropic singular quadric of the confocal family is the two isotropic planes of this pencil) or (isotropic) planes of the pencil of (isotropic) planes through an isotropic line (which is the actual isotropic singular quadric of its confocal family) for \( x_0 \) Darboux quadric with tangency of order 3 with \( C(\infty) \)).

The case of the singular quadric of the family of confocal (pseudo-)spheres (the isotropic cone) falls into a different pattern: the only real cyclic systems with the symmetry of the TC are obtained for \( x_0 \) having constant Gauß curvature (CGC) \((\leq \) homothety) \( \leq 1 \), when the leaves also have the same curvature (Riabouch, 1870); the transformation from \( x_0 \) to one of the leaves is Bianchi’s complementary transformation introduced by him in his Ph. D. thesis in 1879 (thus Bianchi is credited with the idea of using transformations in the study of (deformations) of surfaces).

In 1883 Bäcklund introduced his original B transformation of surfaces of CGC \( \leq 1 \) by allowing the facets (still centered on circles in and centered at the center of tangent planes of the seed) to have constant inclination to the tangent planes of the seed (thus introducing the spectral parameter).

Lie studied both Bianchi’s complementary transformation and Bäcklund’s transformation as arrangements of facets; he proved their inversion (thus the B transformation can be iterated only by quadratures, since a Ricatti equation becomes linear once we know a solution; with the Bianchi Permutability Theorem (BPT) of 1892 the B transformation can be iterated using only algebraic computations and derivatives once we know all B transforms of the seed) and found the spectral family of a CGC \( \leq 1 \) surface (similar to Bonnet’s spectral family of CGC \( \leq 1 \) and constant mean curvature (CMC) surfaces); the B transformation of CGC \( \leq 1 \) surfaces is a conjugation of Bianchi’s complementary transformation with Lie’s transformation.

Note that although the B transformation of the pseudo-sphere does not differ much from Bianchi’s complementary transformation from an analytic point of view, it is essentially different from a geometric point of view, as for the later the reflected distributions of facets coincide (the collapsed
leaves are the rulings of the isotropic cone), while for the former the reflected distributions of facets are different (the collapsed leaves are the rulings of the two ruling families of a confocal pseudo-sphere).

This collapsing ansatz allows us to simplify the denomination \textit{B transformation of surfaces applicable to quadrics} to \textit{B transformation of quadrics}; Bianchi’s B transformation of quadrics is just the metric-projective generalization of Lie’s point of view on the B transformation of the pseudo-sphere (one replaces ‘pseudo-sphere’ with ‘quadric’ and ‘circle’ with ‘conic’).

Bianchi considered the most general form of a B transformation as the focal surfaces (one transform of the other) of a \textit{Weingarten} (W) congruence (congruence upon whose two focal surfaces the asymptotic directions correspond; equivalently the second fundamental forms are proportional). Note that although the correspondence provided by the W congruence does not give the applicability correspondence, the B transformation is the tool best suited to attack the deformation problem via transformation, since it provides correspondence of the characteristics of the deformation problem (according to Darboux these are the asymptotic directions), it is directly linked to the infinitesimal deformation problem (Darboux proved that infinitesimal deformations generate W congruences and Guichard proved the converse: there is an infinitesimal deformation of a focal surface of a W congruence in the direction normal to the other focal surface) and it admits a version of the BPT for its second iteration.

First we consider general 3-dimensional IRDF and obtain the \textit{integrability condition} (IC). There are several basic facts of the \textit{Bz} transformation of quadrics that can be considered separately or partially grouped for 3-dimensional IRDF: the TC requirement (tangential distributions of facets) that facets are centered on tangent planes, the symmetric TC requirement (facets further pass through the origin of the tangent planes), W congruence property (which assumes the symmetric TC), \textit{applicability correspondence of leaves of a general nature} (ACLGN) (independent of the shape of the seed), collapsing ansatz of leaves, etc; for example Bianchi \cite{3} obtained the (isotropic) singular \textit{Bz} transformation of quadrics with $x_z$ plane by assuming the symmetric TC with collapsing to curves ansatz and $x_z$ plane (note that the leaves are applicable to a quadric different from that of the applicability of the seed for the isotropic singular \textit{Bz} transformation of quadrics except when $x_z$ is an isotropic plane of a pencil of (isotropic) planes).

We derive the IC of a 3-dimensional IRDF with the symmetric TC and collapsing to curves ansatz (Bianchi’s original assumption from \cite{3} except for $x_z$ being a plane); in this case the IC is a relation which depends from $x_0$ on $x_0$ and its normal $N_0$ and from the auxiliary surface $x_z$ on $x_z$ and its first and second mixed derivatives in the special coordinates that give the degenerate leaves and must be a consequence of the TC (since the TC establishes a functional relationship between the four parameters of $x_0$ and $x_z$, leaving only three of them independent to generate the 3-dimensional IRDF, any other needed functional relationship between these four parameters must be a consequence of the TC); a simple analysis of the prerequisite showed that the symmetric assumption can be dropped.

In this case we found that the ACLGN does not impose any other condition (it is a consequence of the IC) and thus all B transformations with defining surface are pertinent to the deformation problem (only the W congruence property required by Bianchi’s definition of a B transformation remained to be proved).

By assuming that $x_z$ is a quadric doubly ruled by leaves the influence from $x_z$ in the IC dissolves completely with the TC used only to symmetrize the IC; the remaining IC depends only on $x_0$ and $N_0$ and leads to $x_0$ being a quadric confocal to $x_z$ (again isotropic developables circumscribed to conics appear as singular cases in our discussion).

Conversely, when the TC only symmetrizes the IC we get $x_z$ ((isotropic) singular) quadric (including in a general sense (isotropic) plane of certain pencils of (isotropic) planes) doubly ruled by leaves, $x_0$ quadric confocal to $x_z$ and we capture all ((isotropic) singular) finite B transformations of quadrics; thus we complete Bianchi’s approach \cite{3} to isotropic planes (Bianchi may have been
aware of these isotropic cases, since some of the B transformations in these cases are inverses of his B transformations).

In trying to prove the rigidity of the B transformation of quadrics we were led to discuss the case \( x_0 \) quadric, in which case \( x_z \) must be as in the case when the TC only symmetrizes the IC.

Our IC of 3-dimensional tangential IRDF with collapsing to curves ansatz is not suitable to making hypothesis on the auxiliary surface \( x_z \) for \( x_z \) (isotropic) plane or quadric, since it is very strong in assuming also the curves given by degenerate leaves.

Bianchi’s approach from [3] assumes collapsing ansatz to \( x_z \) plane only to introduce a (linear) parameter \( w \) in a particular 3-dimensional IRDF with the symmetric TC in order to derive its IC; he was able to prove the more general statement of rigidity of the (isotropic) singular B transformation with \( x_z \) plane by assuming that \( x_z \) is a plane (with no hypothesis made on the degenerate leaves).

Having Bianchi’s approach in mind, we impose the TC on the IC of the general 3-dimensional IRDF and obtain some simplification of the IC; by imposing the symmetric TC we get further simplifications of the IC similar to those of Bianchi [3].

In trying to see what conditions are obtained by imposing the W congruence property (a necessary and sufficient condition to obtain B transformation according to Bianchi’s definition; the only remaining step to obtain B transformation pertinent to the deformation problem is to impose the ACLGN) on a 3-dimensional IRDF we found that there are no such conditions: the W congruence property is a consequence of the IC of a 3-dimensional IRDF with the symmetric TC (in fact the W congruence property individuates the first compatibility condition of such an IC). According to a theorem of Ribacour \( x^0 \) and \( x^1 \) are the focal surfaces of a W congruence if and only if \( K(x^0)K(x^1) = \sin^4(\beta) \), where \( \beta \) (respectively \( d \)) is the angle (the distance) between the tangent planes at the corresponding points (the corresponding points); all these quantities except \( K(x^1) \) are preserved for different rolled leaves, so we get same Gauß curvature correspondence of leaves of a general nature, which is close to ACLGN.

Thus we obtain an important theorem:

**Theorem 1.1.** The necessary and sufficient condition for a 2-dimensional linear element to admit Bäcklund transformation of its deformations* is [3].

By imposing post-priori the collapsing to curves ansatz on the IC of a 3-dimensional IRDF with the symmetric TC we were able to show that \( x_0 \) must be a quadric confocal to \( x_z \) when \( x_z \) is a quadric; thus by the previous argument \( x_z \) must be doubly ruled by degenerate leaves (the case \( x_z \) plane is due to Bianchi [3]; the same argument applies to the isotropic case and from Bianchi’s argument we only need the part up to obtaining \( x_z \) doubly ruled by degenerate leaves).

Using both approaches we were finally able to prove:

**Theorem 1.2.** The only Bäcklund transformation with defining surface is Bianchi’s Bäcklund transformation of quadrics.

It remains to consider in detail the B transformation of deformations of abstract linear elements (without defining surface)and with ACLGN.

By requiring that the leaves are applicable to certain surfaces with ACLGN (independent of the shape of \( x \)) and since the distribution of facets into leaves changes with the shape of \( x \) we get the fact that all leaves of all rolled distributions of facets must be applicable to the same surface \( y \); further we get the TC and the necessary algebraic conditions for such a configuration to exist (it depends on three arbitrary constants) are satisfied for the symmetric TC.

By imposing compatibility conditions on these algebraic conditions we must get the space of solutions depending on two functions of a variable for the deformations of \( y \); again these conditions may be satisfied by 3-dimensional IRDF with the symmetric TC or further compatibility conditions

* Bianchi quotes in [3] two articles of Bäcklund’s from 1914 and 1916 on Bianchi’s B transformation of quadrics, so Bäcklund himself may have some other results on the deformation problem via the B transformation, but unfortunately we are not familiar with German.
may be imposed on the linear element of \( x \) which may make the family of such surfaces depending on constants or on a function of one variable and constants.

The change from a facet \( F \) to an infinitesimally close facet \( F' \) via infinitesimal rolling in an arbitrary common tangential infinitesimal direction \( \delta \) can be reversed by considering \(-\delta\) acting on \( F'\); thus this infinitesimal law corresponds via discretization to the prescription of the finite ACLGN (correspondence between the leaf \( x^1 \) and a surface \( x_0^0 \subset x_0 \) or between \( x^1 \) and a surface \( x_0^0 \subset y_0 \) with \( y_0 \) another (defining) surface; according to Bianchi there are B transformations such that the seed and the leaf are applicable to different quadrics) and the inversion of the B transformation (the seed and the leaf exchange places; note that in this case the leaf applicable to the (defining) surface \( x_0 \) is required to belong to an 1-dimensional family, which is certainly the case if the original leaf can play the rôle of seed (that is it admits B transformation)).

In dealing with iterations of infinitesimal rollings in arbitrary tangential infinitesimal directions we are led to consider their (partial) discretizations; these are realized by Möbius configurations.

A tetrahedron consists of 2² points and 2² planes, each point (plane) belonging to (containing) 2 + 1 planes (points). Möbius considered configurations \( M_2 \) of two tetrahedra inscribed one into the other, that is configurations of 2³ points and 2³ planes, each point (plane) belonging to (containing) 3 + 1 planes (points); two \( M_3 \) configurations inscribed one into the other give raise to a configuration of 2⁴ points and 2⁴ planes, each point (plane) belonging to (containing) 4 + 1 planes (points). Therefore Bianchi (5, Vol 5, (117)) calls a configuration of 2ⁿ points and 2ⁿ planes such that any point (plane) belongs to (contains) \( n + 1 \) planes (points) a Möbius configuration \( M_n \).

For the discussion up to the third iteration of the tangency configuration (TICC) we consider only the (singular) B transformation of quadrics, when the seed and the leaf are applicable via the Ivory affinity to the same quadric.

If we compose the inverse of the rolling of \( x_0^1 \subset x_0 \) on the leaf \( x^1 \) with the rolling of \( x_0^0 \) on the seed \( x_0^0 \), then we get a rigid motion at the level of the static picture with the defining surface \( x_0 \) and auxiliary surface \( x_z \) (the rigid motion provided by the Ivory affinity (RMPIA); this led Bianchi to discover the applicability correspondence provided by the Ivory affinity (ACPIA) in 11): thus the prescription of the applicability correspondence becomes of a general nature (independent of the shape of the seed \( x_0^0 \)) and becomes, when the leaves collapse, a correspondence between \( x_z \) and \( x_0 \). In this correspondence the special coordinates on \( x_z \) (rulings given by the collapsed leaves) correspond to special coordinates on \( x_0 \), thus providing special coordinates for the whole problem.

We have the symmetry of the TC: four 1-Möbius configurations \( M_1 \), one taken to two others via a reflection in a facet or a RMPIA; the symmetry of the TC implies the inversion of the B transformation and existence of RMPIA in the TC (note that a-priori the RMPIA does not require the TC) implies prescription of the applicability correspondence as a law of a general nature (independent of the shape of the seed).

\[
(R_j, t_j)(x_0^j, dx_0^j) := (R_j x_0^j + t_j, R_j dx_0^j) = (x^j, dx^j), (R_j, t_j) \subset O_3(\mathbb{C}) \times \mathbb{C}^3, j = 0, 1,
\]

\[
(R_0^1, t_0^1) = (R_1, t_1)^{-1}(R_0, t_0), V_0^1 := x_1^1 - x_0^1, V_0^1 := x_2^1 - x_0^1, (V_0^1)^T N_0^1 = 0 \Rightarrow (V_0^1)^T N_0^1 = 0,\]

\[
R_0^1 [V_0^1 \partial_{x_1}, x_2^0 \partial_{x_0}, x_0^0] = [-V_0^1 \partial_{x_1}, x_2^0 \partial_{x_0}, x_0^0], (V_0^1)^T N_0^1 = 0 \Rightarrow R_0^1 (j_1 - 2 N_0^1 (N_0^1)^T) \partial e_1 x_2^1 = \partial e_1 x_2^1.
\]

For these arguments we need only a spectral parameter \( z \) (a 3-dimensional IRDF); for the remaining arguments we need \( z \) to vary.

The issue of discretizing the commutation of the composition of infinitesimal rolling in an arbitrary tangential direction \( \delta \) followed by infinitesimal rolling in an arbitrary tangential direction
\(\delta' : \delta \delta' = \delta' \delta\) (equivalent to the symmetry of the difference of the second fundamental forms of the rolling surfaces) in one of \(\delta, \delta'\) (in which case the other remains infinitesimal) leads to finding the differential system subjacent to the B transformation (Ricatti equation) and proving the applicability correspondence; in both of \(\delta, \delta'\) leads to algebraic computations of the second iterated tangency configuration (SITC) (four 2-Möbius configurations \(\mathcal{M}_2\), each taken to two others via a RMPIA in the TC and to the fourth via a RMPIA without the TC): for \(x_{z_1}, x_{z_2}^3 \in T_{x_0}x_0\) in the pencil of planes containing \(x_{z_2}, x_{z_1}^3\) there are two planes tangent to \(x_0\) at \(x_{z_0}^3\); thus we have two choices of \(x_{z_0}^3\) according to the rulings of \(x_{z_1}^3, x_{z_2}^3\) belonging to the same or different ruling families.

Once the SITC is established to be valid, one can let one of \(\delta, \delta'\) be infinitesimal and obtain the differential system subjacent to the B transformation and proof of the applicability correspondence, so the SITC is sufficient to imply the differential system subjacent the B transformation and the proof of the applicability correspondence (that is the first moving Möbius configuration). Note that the SITC is equivalent to the rulings at \(x_{z_1}^3, x_{z_2}^3\) cutting the segment \([x_0(\mathcal{R}^0_3, t_3^0)]x_0] \) with cross-ratio \(\Delta\) and it boils down to a homography between four rulings and having the symmetries of the square (the Bianchi quadrilateral); the rulings of opposite vertices belong simultaneously to the same or different ruling families.

\[
\begin{align*}
(R_j, t_j)(x_0^j, dx_0^j) &= (x^j, dx^j), \quad j = 0, 1, 2, 3, \\
(R^k_j, t^k_j) &= (R_k, t_k)^{-1}(R_j, t_j), \quad (j, k) = (0, 1), (0, 2), (1, 3), (2, 3), \\
(R_0^1, t_0^1)(R_0^2, t_0^2) &= (R_0^3, t_0^3), \quad (R_0^1, t_0^1)(R_3^1, t_3^1) = (R_2^1, t_2^1) = (R_2^0, t_0^2)(R_2^3, t_2^3).
\end{align*}
\]

The issue of discretizing the commutation of the composition of infinitesimal rolling in an arbitrary tangential direction \(\delta\), followed by infinitesimal rolling in an arbitrary tangential direction \(\delta'\), followed by infinitesimal rolling in an arbitrary tangential direction \(\delta''\): \(\delta \delta' \delta'' = \delta'' \delta' \delta\) (equivalent to the difference of the G-CMP equations of the rolling surfaces or the flatness of the flat connection form) in one of \(\delta, \delta', \delta''\) (in which case the other two remain infinitesimal) leads to the existence of the B transformation (equivalently the complete integrability of the differential system subjacent
to the B transformation or the integrability of the considered 3-dimensional rolled distributions of facets); in two of δ, δ', δ'' (in which case the other remains infinitesimal) leads to the BPT and in all three δ, δ', δ'' leads to the algebraic computations of the TITC (eight 3-Möbius configurations \(M_3\), each taken to three others via a RPPIA in the TC and to three others via a RPPIA without the TC). Once the TITC is established to be valid, one can let one of δ, δ', δ'' be infinitesimal and obtain the BPT (that is the second moving Möbius configuration) or one can let two of δ, δ', δ'' be infinitesimal and obtain the full theory (existence of B transformation). Note that the TITC uses the cross-ratio properties with \(\frac{z_1}{z_2}, \frac{z_2}{z_3}, \frac{z_3}{z_1}\) and the Menelaos theorem; it is equivalent to a homography between four rulings of corresponding sameness ruling families and having the symmetries of the cube; the four rulings involved are from the vertices of a regular tetrahedron and thus are in symmetric relationship one to the other if they belong to the same ruling family.

Existence of 3-Möbius configurations \(M_3\) as in the TITC implies existence of arbitrary \(n\)-Möbius configurations \(M_n\) for arbitrary \(n\) iterations of the TC, as expected (the G-CMP equations involve 3 derivatives and are necessary and sufficient for the characterization of a surface); thus the algebraic computations of the TITC are necessary and sufficient for a complete description of the B transformation (some interesting exterior algebra type formulae but no relevant information can be found beyond the TITC).

Note also that the SITC and the TITC behave well with respect to totally real considerations, which implies the B transformation of totally real quadrics.

These considerations can be extended to include the isotropic singular B transformations (the RPPIA is replaced with a suitable rigid motion; one cannot construct the isotropic singular B transformation using the Ivory affinity since one cannot take square roots of symmetric matrices with isotropic kernels and for quadrics of revolution the Ivory affinity has image only the singular part of the singular isotropic confocal quadric (the axis of revolution)); one gets the SITC and the TITC (and thus the BPT and higher moving Möbius configurations) with certain iterations exchanging the quadric of applicability (most isotropic singular B transformations have seed and leaf applicable to different quadrics).

There is another application of isotropic developables circumscribed to conics, which leads to the Hazzidakis (H) transformation of quadrics (an involutory algebraic transformation of deformations of quadrics which commutes with the B transformation): since homographies preserve tangency and its order, they preserve asymptotes (whose osculating planes are the tangent planes of the surface), so it takes developables (on which the two asymptotic directions coincide) to developables; the homography \(H\) takes the isotropic developable circumscribed to \(H^{-1}(C(\infty))\) to the isotropic developable circumscribed to \(H(C(\infty))\); for most homographies \(H\) the isotropic developable circumscribed to \(H^{-1}(C(\infty))\) generates a family of confocal quadrics.

Now by the Chasles-Jacobi result (Jacobi proved that the tangents of a geodesic on a quadric remain tangent to a confocal quadric and Chasles the converse: the congruence of common tangents
to two confocal quadrics is normal and envelopes geodesics on the two quadrics) \( H \) takes geodesics on quadrics to geodesics on quadrics, so it provides pairs of quadrics \textit{conjugate in deformation} (two non-flat non-homothetic surfaces with a correspondence of asymptotes and of all virtual asymptotes (coordinates that become asymptotes on a deformation); it is enough to have correspondence of asymptotes and geodesics); only quadrics can be conjugate in deformation (Bianchi proved the statement for surfaces of revolution and Servant the general statement).

By use of the H transformation singular finite B transformations are taken to singular finite B transformations (Bianchi) or Calapso’s singular \( B_{\infty} \) transformation of \textit{quadrics with center} (QC) (see [2]; note that in this case one knows the \( B_{\infty} \) transform only infinitesimally (the first and second fundamental forms of the leaf) and we don’t have the W congruence); this behavior may be generalized to isotropic singular B transformations taken to isotropic singular B transformations or \( B_{\infty} \) transformations of \textit{(isotropic) quadrics without center} (I)QWC (there is some partial evidence in favor of existence of such \( B_{\infty} \) transformations).

2. Rolling Surfaces

The study of the rolling problem was initiated by Ribacour and has been extensively pursued in Bianchi [3], [4], Vol 7 and Darboux [7]. Let \((u, v) \equiv D\) with \( D \) a domain in \( \mathbb{R}^2 \), \( C \times \mathbb{R} \) or \( \mathbb{C}^2 \) and \( x : D \mapsto \mathbb{C}^3 \) be a surface. For \( \omega_1, \omega_2 : C^3 \)-valued 1-forms on \( D \) and \( a, b \in C^3 \) we have

\[
a^T \omega_1 \wedge b^T \omega_2 = ((a \times b) \times \omega_1 + b^T \omega_1 a)^T \wedge \omega_2 = (a \times b)^T (\omega_1 \times \omega_2) + b^T \omega_1 \wedge a^T \omega_2;
\]

(2.1)

in particular \( a^T \omega \wedge b^T \omega = \frac{1}{2} (a \times b)^T (\omega \times \omega) \).

Since both \( \times \) and \( \wedge \) are skew-symmetric, we have \( 2\omega_1 \times \wedge \omega_2 = \omega_1 \times \wedge \omega_2 + \omega_2 \times \wedge \omega_1 = 2\omega_2 \times \wedge \omega_1 \).

Consider the scalar product \( < , > \) on \( M_3(\mathbb{C}) \): \( < X, Y > := \frac{1}{2} \text{tr}(X^T Y) \). We have the isometry

\[
\alpha : \mathbb{C}^3 \rightarrow \mathfrak{so}_3(\mathbb{C}), \quad \alpha(\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}) = \begin{bmatrix} 0 & x^3 & -x^1 \\ -x^3 & 0 & x^1 \\ x^1 & -x^2 & 0 \end{bmatrix}, \quad x^T y = < \alpha(x), \alpha(y) > = \frac{1}{2} \text{tr}(\alpha(x)^T \alpha(y)),
\]

\[
\alpha(x \times y) = [\alpha(x), \alpha(y)] = \alpha(x) \alpha(y) = yx^T - xy^T, \quad \alpha(Rx) = R\alpha(x)R^{-1}, \quad x, y \in \mathbb{C}^3, \quad R \in \mathfrak{so}_3(\mathbb{C}).
\]

Let \( x \in \mathbb{C}^3 \) be a surface non-rigidly applicable to a surface \( x_0 \subset \mathbb{C}^3 \):

\[
(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0),
\]

where \((R, t)\) is sub-manifold in \( \mathfrak{so}_3(\mathbb{C}) \times \mathbb{C}^3 \) (in general surface, but it is a curve if \( x_0, x \) are ruled and the rulings correspond under the applicability or a point if \( x_0, x \) differ by a rigid motion). The sub-manifold \( R \) gives the rolling of \( x_0 \) on \( x \), that is if we rigidly roll \( x_0 \) on \( x \) such that points corresponding under the applicability will have the same differentials, \( R \) will dictate the rotation of \( x_0 \); the translation \( t \) will satisfy \( dt = -dRx_0 \).

For \((u, v)\) parametrization on \( x_0, x \) and outside the locus of isotropic (degenerate) linear element of \( x_0, x \) we have \( N_0 := \frac{\partial_x x_0 \times \partial_x x_0}{|\partial_x x_0 \times \partial_x x_0|}, \quad N := \frac{\partial_x x \times \partial_x x}{|\partial_x x \times \partial_x x|} \) respectively positively oriented unit normal fields of \( x_0, x \) and \( R \) is determined by \( R = [\partial_x x \partial_x x \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0 \partial_x x_0] \) we take \( R \) with \( \det(R) = 1 \); thus the rotation of the rolling with the other face of \( x_0 \) (or on the other face of \( x \)) is \( R' := R(I - 2NN^T) = (I - 2NN^T)R, \quad \det(R') = -1 \).

Therefore \( O_3(\mathbb{C}) \times \mathbb{C}^3 \) acts on \( 2\)-dimensional integral distributions of facets \((x_0, dx_0)\) in \( T^*(\mathbb{C}^3) \) as: \((R, t)(x_0, dx_0) = (Rx_0 + t, Rdx_0)\); a rolling is a sub-manifold \((R, t) \subset O_3(\mathbb{C}) \times \mathbb{C}^3 \) such that \((R, t)(x_0, dx_0)\) is still integrable.

We have:

\[
R^{-1}dR N_0 = R^{-1}dN - dN_0.
\]

In order to preserve the classical notation \( d^2 \) for the tensorial (symmetric) second derivative we shall use \( d^\wedge \) for the exterior (antisymmetric) derivative. Applying the compatibility condition \( d^\wedge \)
to (2.2) we get:

\[
R^{-1}dR \wedge dx_0 = 0, \quad dRR^{-1} \wedge dx = 0.
\]

Applying \(R^{-1}d\) to (2.2) we get

\[
R^{-1}d^2x = R^{-1}dRdx_0 + d^2x_0.
\]

Since \(R^{-1}dR\) is skew-symmetric and using (2.4) we have

\[
dx_0^T R^{-1}dRdx_0 = 0.
\]

From (2.6) for \(a \in \mathbb{C}^3\) we get \(R^{-1}dRa = R^{-1}dR(a^+ + a^T) = a^T N_0 R^{-1}dR N_0 - a^T R^{-1}dR N_0 N_0 = \omega \times a, \quad \omega := N_0 \times R^{-1}dR N_0 = \frac{|\det R|}{\omega} R^{-1}(N \times dN) - N_0 \times dN_0 = R^{-1}(N \times dN) - N_0 \times dN_0.
\)

Thus \(R^{-1}dR = \alpha(\omega)\) and \(\omega\) is flat connection form in \(T^*x_0\):

\[
d \wedge \omega + \frac{1}{2} \omega \wedge \omega = 0, \quad \omega \wedge dx_0 = 0, \quad (\omega)^{\perp} = 0.
\]

With \(s := N_0^T (R^{-1}d^2x - d^2x_0) = s_{11}du^2 + s_{12}dudv + s_{21}dvdv + s_{22}dv^2\) the difference of the second fundamental forms of \(x, \ x_0\) we have

\[
\omega = \frac{s_{12}\partial_a x_0 - s_{11}\partial_c x_0}{|\partial_a x_0 \times \partial_c x_0|}du + \frac{s_{22}\partial_a x_0 - s_{21}\partial_c x_0}{|\partial_a x_0 \times \partial_c x_0|}dv;
\]

\((\omega \times dx_0 = 0\) is equivalent to \(s_{12} = s_{21}; \quad (d \wedge \omega)^{\perp} + \frac{1}{2} \omega \times \omega = 0, \quad (d \wedge \omega)^T = 0\) respectively encode the difference of the G-CMP equations of \(x_0\) and \(x\).

Using \(\frac{1}{2}dN_0 \times \wedge dN_0 = K[\partial_a x_0 \times \partial_c x_0]N_0 du \times dv, K\) being the Gauß curvature we get \(dN_0 \times \wedge dN_0 = R^{-1}(dN \times dN) = \omega \times N_0 + dN_0 \times (\omega \times N_0 + dN_0) = dN_0 \times \wedge dN_0 + 2(\omega \times N_0) \times \wedge dN_0 \times \omega \times \omega; \) thus

\[
\frac{1}{2} \omega \times \omega = dN_0^T \wedge \omega N_0.
\]

Note also

\[
\omega' = N_0 \times R^{-1}dR N_0 = -\omega - 2N_0 \times dN_0
\]

and

\[
a^T \wedge \omega = 0, \quad \forall \omega \text{ satisfying (2.7) for } a \mid 1 \to \text{ form } a^T \odot dx_0 := \frac{a^T dx_0 + dx_0^T a}{2} = 0.
\]

Note that the converse \(a^T \odot dx_0 = 0, \quad a \mid 1 \to \text{ form } a^T \wedge \omega = 0, \quad \forall \omega \text{ satisfying (2.7) is also true.}

2.1. Isotropic developables. For any two curves \(c_1(v), c_2(v)\) the developable circumscribed to them is \((u, v) \mapsto c_1(v) + u[c_2(f(v)) - c_1(v)],\) where \(f(v)\) is determined from \(|c_2(f(v)) - c_1(v)|^T c_1^T (v) \times c_2^T (f(v)) = 0\), that is \(c_2(f(v)) - c_1(v)\) belongs to tangent planes of \(c_1\) and \(c_2;\) isotropic developables are the developables circumscribed to a finite curve and \(C(\infty).\)

With \(e_1, e_2, e_3, \ e_k^T e_k = \delta_{jk}\) the standard basis of \(\mathbb{C}^3\) and \(f_1 := \frac{\text{e}_2^T \text{e}_3}{\sqrt{2}}\) the standard isotropic vector we have \(Y(v) := -v^2 f_1 + 2f_2 + 2v e_3\) the standard parametrization of the rulings of the isotropic cone and the isotropic developable circumscribed to \(c(v)\) is \((u, v) \mapsto uY(f(v)) + c(v), \ c'(v)Y(f(v)) = 0\) (there are two choices of \(f(v)\) except for \(c'(v)\) isotropic).

If \(x_0 \subset \mathbb{C}^3\) is a surface with degenerate linear element, then its tangent planes are isotropic and we can take the curves whose tangents are isotropic as the curves \(v = ct;\) thus \(\partial_a x_0 = a(u, v)Y(w(u, v)), \ \partial_c x_0 = b(u, v)Y(w(u, v)) + c(u, v)Y'(w(u, v));\) after a change of coordinates we get \(\partial_a x_0 = a(u, v)Y(v)\) (so after another change of coordinates \(x_0(u, v) = uY(v) + c(v), \ Y(v)^T c'(v) = 0\) or \(\partial_a x_0 = a(u, v)Y(u), \ \partial_c x_0 = b(u, v)Y(u) + c(u, v)Y'(u)\) (in this case from \(\partial_a x_0 = \partial_a aY(u)\) we get \(b = c = 0, \) a contradiction) or \(x_0 = uY(w) + vY'(w), \ w = ct.\)
3. 3-DIMENSIONAL INTEGRABLE ROLLING DISTRIBUTIONS OF FACETS

Assume that we have a 3-dimensional distribution of facets

\[(p, P) = (p(u, v, w), P(u, v, w)), \quad p \in P, \quad du \wedge dv \wedge dw \neq 0\]

in \(\mathbb{C}^3\) with normal fields \(m = m(u, v, w) \subset \mathbb{C}^3 \setminus \{0\}, \quad m \perp P\).

With \(d\hat{w} := \partial_u \cdot du + \partial_v \cdot dv + \partial_w \cdot dw = d\cdot + \partial_w \cdot dw\) if the distribution of facets is integrable, then along the leaves we have

\[(3.1) \quad 0 = m^T \tilde{dp} = m^T (\partial_w pdv + \partial_v pdw + \partial_w p \cdot dw) = m^T (dp + \partial_w p \cdot dw).\]

Assuming \(m^T \partial_w p \neq 0\), applying the compatibility condition \(m^T \partial_w pd\wedge\) to (3.1) and using the equation itself we get the IC \(m^T \partial_w p \neq 0\), \((\partial_w m^T dp - dm^T \partial_w p) \wedge m^T dp = m^T dp + m^T \partial_w pdm^T \wedge dp = 0\), or:

\[(3.2) m^T \partial_w p \neq 0, \quad (dp \times \partial_w p)^T \wedge (m \times dm) + \frac{1}{2}(\partial_w m \times m)^T (dp \times \wedge dp) = 0, \quad du \wedge dv \wedge dw \neq 0\]

(in order to get the 1-dimensional family of leaves \(c = ct\) from the integration of \(3.1\) \((w = w(u, v, c)\) for \(m^T \partial_w pm^T \partial_w p \neq 0\) or \(w = w(u, c)\) for \(m^T \partial_u p = 0, \quad m^T \partial_v p = 0\) or \(w = w(c)\) for \(m^T \partial_w p = m^T \partial_v p = 0\), \((3.2)\) must be identically satisfied (without imposing a functional relationship between \(u, v\) and \(w\); note that the scaling of \(m\) is irrelevant, but \(m\) may be isotropic, in which case the leaves are isotropic developables)).

Since along the leaves we have \(\frac{1}{2} dp \wedge \wedge \partial_w p = [(I_3 - \frac{\partial_w pm^T}{m^T m} \partial_w p) v] \wedge m \wedge \wedge \partial_w p = (\partial_w p^T \partial_w p) du \wedge \wedge dv = \text{ker}(I_3 - \frac{\partial_w pm^T}{m^T m} \partial_w p)^T = \text{Im}(I_3 - \frac{\partial_w pm^T}{m^T m} \partial_w p)^T = (\partial_w p^T)\), the leaves are 2-dimensional unless \(\partial_w p^T \partial_w p = 0\) (note that along the leaves we need \(du \wedge dv \neq 0\) in order to preserve the 3-dimensionality of the distribution of facets).

By symmetry the variables \((u, v, w)\) the only remaining singular case to discuss is when along the leaves we have \(m^T dp = m^T \partial_w p = 0, \quad du \wedge dv \neq 0\), in which case \(\partial_w p^T dp \wedge \wedge \partial_w p = 0\), so the centers of facets are situated on a surface, curve or point. Using \(d(m^T \partial_w p) = \partial_w (m^T dp) = 0\) and applying the compatibility condition \(d\wedge \wedge 0 = m^T dp = m^T dp\) we get the IC \(0 = dm^T \wedge dp + (dm^T \partial_w p - \partial_w m^T dp)^T \wedge dv = dm^T \wedge dp, \quad or\)

\[(3.3) \quad m^T \partial_w p = 0, \quad m^T dp = 0, \quad dm^T \wedge dp = 0, \quad du \wedge dv \wedge dw \neq 0\]

In this case the 3-dimensional integrable distribution of facets is just a 2-dimensional integrable distribution of facets (the tangent planes of a surface, curve or point), each facet being counted with the simple \(\infty\) multiplicity of \(w\) (the dependence on \(w\) is irrelevant to our problem).

We can distribute the distribution of facets along the surface \(x_0 = x_0(u, v)\) (the parameters \((u, v)\) and \(w\) are indistinguishable such that to each point of \(x_0\) corresponds an 1-dimensional family of facets of the distribution depending on \(w\); thus one must discuss singular cases according only to the symmetry \(u \leftrightarrow v\); a-priori the distribution of facets has no other relation to \(x_0\).

By referring \(V := p-x_0, \quad m\) to \(dx_0\) and \(N_0 \{3.2\}\) becomes an equation involving the geometry of \(x_0\) (depending on the linear element and linearly on the second fundamental form; by an application of the Gauß theorem the terms of the second fundamental form appearing quadratically group together to give dependence on the linear element).

The natural question thus appears wether the IC \(\{3.2\}\) depends only on the linear element of \(x_0\) (that is we require the cancellation of the coefficients of the (linearly appearing) second fundamental form); equivalently if we roll \(x_0\) on an applicable surface \((x, dx) = (Rx_0 + t, Rdx_0)\), then \(\{3.2\}\) is still satisfied if we replace \(V, m, x_0\) with \(RV, Rm, x\) (note that by referring \(V, m\) to \(dx_0\) and \(N_0\), their coefficients may depend on \(dx_0\) and \(N_0\), so the derivatives of these coefficients may depend on the second fundamental form of \(x_0\); we ignore this dependence in our considerations since the coefficients themselves are preserved by rolling).

Note \(N_0^T VdN_0 + d(I_3 - N_0 N_0^T) V]^T N_0 N_0 = [N_0 \times dN_0] \times V\) is the part of \(dV\) depending (linearly) on the second fundamental form of \(x_0\), so in \(\{3.2\}\) we need to consider the condition that the terms that do not depend linearly on \(N_0 \times dN_0\) cancel and individuate those that depend linearly on
\[ N_0 \times dN_0 \text{ to be cancelled separately by replacing } (...)^T \wedge (N_0 \times dN_0) = 0 \text{ with } (...)^T \odot dx_0 = 0 \]

(Thus the use of \( r = R^{-1}(N \times dN) - N_0 \times dN_0 \) and (2.11) becomes clear).

Equation (3.2) becomes \([|d(V + x_0) - (N_0 \times dN_0) \times V| + m \times ((dm - (N_0 \times dN_0) \times m) + \frac{1}{2}(\partial_{uv} m \times m)^T)(|d(V + x_0) - (N_0 \times dN_0) \times V| + (N_0 \times dN_0) \times V) \wedge (d(V + x_0) - (N_0 \times dN_0) \times V) = 0 \] (note that in the reflected distribution of facets \( V' := (I_3 - 2N_0N_0^T)V, m' := (I_3 - 2N_0N_0^T)m \) the terms depending linearly on the second fundamental form have changed signs); by separating the terms as explained we get the IC

\[ m^T \partial_u V \neq 0, 2[(d(V + x_0) - (N_0 \times dN_0) \times V) \times \partial_u V]^T N_0^T \wedge N_0^T [m \times (dm - m^T N_0 N_0^T) n] + \]

\( (\partial_{uv} m \times m)^T (|d(V + x_0) - (N_0 \times dN_0) \times V| \wedge (d(V + x_0) - (N_0 \times dN_0) \times V)] + [\partial_u (m \times V) \times \partial_u V^T m \times V]^T N_0^T [m \times (dm - (N_0 \times dN_0) \times m)] \)

\( + \partial_u (m \times V) \times \partial_u V^T m \times V]^T N_0^T [m \times (dm - (N_0 \times dN_0) \times m)] \times V \]

\begin{align*}
\text{(3.4)} & \quad - m^T \partial_u V^T [m \times (dm - (N_0 \times dN_0) \times m)] \times V \times (\partial_u V^T m \times V) = 0 ,
\end{align*}

Thus we have a 3-dimensional IRDF provided (3.3) is identically satisfied (without imposing a functional relationship between \( u, v \) and \( w \)).

In the case of tangential distributions of facets \( V^T N_0 = 0 \) (3.1) for the rolled distribution of facets becomes \( m^T [-V^T (\omega \times N_0)] N_0 + d(V + x_0) \times \partial_v V dw = 0 \); imposing the compatibility condition \( m^T \partial_u V^T, \partial_u V, \) and the equation itself we get \( 0 = m^T \partial_u V^T [d(V + x_0) - (N_0 \times dN_0) \times V] \wedge (\omega \times N_0) + dm^T \wedge (d(V + x_0)] + \partial_u (m \times V) \times \partial_u V^T m \times V]^T N_0^T \wedge (dN_0 \times \wedge dN_0) = 0 \).

Thus if further we have the symmetric TC \( m^T V = 0 \), then we can take \( m =: V \times N_0 + m N_0 \) and (3.5) becomes

\[ V \times \partial_u V \neq 0, \quad 2[\partial_u V \times d(V + x_0)]^T \wedge (V \times dx_0) + (m^2 + |V|^2) K = 0 , \]

\begin{align*}
\text{(3.6)} & \quad dm = -m^T [V \times (V \times dx_0)] + m^T [V \times (V \times dx_0)], \quad du \wedge dv \wedge dw \neq 0 .
\end{align*}

If \( m = 0 \), then \( (m \times V) \times N_0 = 0 \) and as we shall see later the 3-dimensional IRDF must be a 2-dimensional IRDF counted with the simple \( \infty \) of \( w \) such that \( x_0, x_0 + V \) are the focal surfaces of a normal congruence, which contradicts \( V \times \partial_u V \neq 0 \).

Thus \( m \neq 0 \); excluding the case \( x_0 \) developable we can take \( m^2 \) from the first equation of (3.6) and replace it into the second one; applying the compatibility condition \( d \wedge \) to this equation and using the equation itself we get \( 0 = d \wedge dm = d \wedge (-m^T \partial_u V^T [d(V + x_0)] + m^T [\partial_u V \times dV]) \triangleq 2.1 \)

\begin{align*}
\text{(2.1)} & \quad -m^T [\partial_u V \times dV] + m^2 N_0^T [\partial_u V \times dV] + N_0^T [\partial_u V \times dV] = 0 ,
\end{align*}

\[ -m^T N_0^T [\partial_u V \times dV] + m^2 N_0^T [\partial_u V \times dV] = 0 \]

\[ N_0^T (\partial_u V \times dV) = 2^T N_0^T (dN_0 \times \wedge dN_0) . \]
Applying $\partial_w$ to the first equation of (3.6) we get
\[\left(\frac{N_0^T(\partial_wV \times V)}{N_d^a(\partial_wV \times V)} + \frac{N_0^T(V \times dV)}{N_d^a(\partial_wV \times V)} + \frac{N_0^T(\partial_wV \times dV)}{N_d^a(\partial_wV \times V)} \right) \wedge \left(\frac{N_0^T(V \times dV)}{N_d^a(\partial_wV \times V)} + \frac{N_0^T(\partial_wV \times dV)}{N_d^a(\partial_wV \times V)} \right) \wedge \left(\frac{N_0^T(V \times dV)}{N_d^a(\partial_wV \times V)} + \frac{N_0^T(\partial_wV \times dV)}{N_d^a(\partial_wV \times V)} \right) = 0;\]

hence the previous relation becomes
\[
\begin{align*}
(\partial_w V \times dV) & = (\partial_w V \times dV)^	op \wedge (\partial_w V \times dV) + (m \partial_w m + V^T \partial_w V)K = 0, \quad du \wedge dv \wedge dw \neq 0
\end{align*}
\]

and (3.6) becomes
\[
\begin{align*}
\partial_w N_0^T(\partial_w V \times d(V + x_0)) & \wedge N_0^T(V \times d(x_0)) - \frac{1}{2} \frac{d(2(\partial_w V \times d(V + x_0))^T \wedge (V \times d(x_0)))}{K(\partial_w V \times V) + |V|^2} + |V|^2 (\partial_w V \times d(V + x_0))^T \wedge (V \times d(x_0)) + |V|^2 N_0^T(\partial_w V \times dV) \nonumber
\end{align*}
\]

(3.8)

\[
\begin{align*}
du \wedge dv \wedge dw \neq 0;
\end{align*}
\]

applying $\partial_w$ to the second equation of (3.8) and using the first one we get another second order equation in $V$.

3.1. The Weingarten congruence property. We consider a 3-dimensional tangential IRDF with the symmetry of the TC (thus $m = V \times N_0 + m_0 N_0$ and $V, m$ satisfy (3.9) and inquire in what case any deformation $(x, dx) = (R x_0 + t, Rd x_0)$ of $x_0$ and any leaf are the focal surfaces of a W congruence.

By the Darboux-Guichard’s this is equivalent to the requirement that for any leaf there is an infinitesimal deformation of $x$ in the direction $Rm$ normal to the leaf, that is $0 = \frac{1}{2} \frac{d}{dx} |\mathbf{e}_0|^2 R^{-1} d(x + \varepsilon p Rm)^2 = dx_0^T \circ (d \log \rho m + \omega \times m + dm - m_T(\omega \times \omega V \times (V \times dx_0)) \circ \partial_w m) = dx_0^T (V \times N_0) \circ (d \log \rho + \frac{N_0^T(\partial_w V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} + m \partial_w V^T (\omega \times N_0 + dN_0))$, or

\[
\begin{align*}
d \log \rho = - \frac{N_0^T(\partial_w V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} - m \partial_w V^T (\omega \times N_0 + dN_0).
\end{align*}
\]

Imposing the compatibility condition $d \wedge$ on (3.9), using $dw = \frac{N_0^T(V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} + m V^T (\omega \times N_0 + dN_0)$ (note that for $w = w(u, v, c)$, $c = c(x)$ we have $d \log \rho = -d \log \rho + ...$ so up to a certain scaling $\rho^{-1}$ can be interpreted as an infinitesimal deformation of $w$), $(\omega \times N_0 + dN_0) \times (\omega \times N_0 + dN_0) = dN_0 \times dN_0$ and replacing $dm$ from the second equation of (3.6) we get $0 = -d \wedge \frac{N_0^T(\partial_w V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} + \partial_w N_0^T(\omega \times N_0 + dN_0) + \partial_w N_0^T(\partial_w V \times (V \times dx_0)) + m \partial_w V^T (\omega \times N_0 + dN_0)$

\[
\begin{align*}
\partial_w N_0^T(\partial_w V \times (V \times dx_0)) & \wedge N_0^T(V \times dV) + m \partial_w V^T (\omega \times N_0 + dN_0) + \partial_w N_0^T(\partial_w V \times (V \times dx_0)) \wedge N_0^T(V \times dV) + m \partial_w V^T (\omega \times N_0 + dN_0),
\end{align*}
\]

the last term cancels and the remaining first three terms boil down to (3.7).

Note also that the IC (3.6) can be easier obtained by applying the compatibility condition $d \wedge$ to $dw = \frac{N_0^T(V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} + m V^T (\omega \times N_0 + dN_0)$ and using the equation itself: $0 = d \wedge (\frac{N_0^T(V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} + m V^T (\omega \times N_0 + dN_0))$

\[
\begin{align*}
d \wedge \frac{N_0^T(V \times d(V + x_0))}{N_d^a(\partial_w V \times V)} + m V^T (\omega \times N_0 + dN_0) = 0.
\end{align*}
\]
3.2. Applicability correspondence of leaves of a general nature. We consider the question whether the leaves are deformations of surfaces; we exclude the case \( m \) isotropic because all leaves are isotropic developables with degenerate 2-dimensional linear element.

The leaves for a particular deformation \( x_0 \) are applicable to the same surface or to an 1-dimensional family of surfaces.

Since we already have a correspondence between the facets of leaves given by rolling, it is natural to require that the applicability correspondence is independent of the shape of \( x \).

However, for \( (m \times V) \times N_0 \neq 0 \) the distribution of facets into leaves changes with the shape of \( x \); thus all leaves of all rolled distributions of facets must be applicable to the same surface (and we have a submersion from the 3-dimensional IRDF to the distribution of tangent planes of the fixed surface), or \( (m \times V) \times N_0 = 0 \) and we have to discuss only a 2-dimensional IRDF with the leaf having linear element independent of the shape of \( x \).

Consider first the general case \( m^T \partial_w V (m \times V) \times N_0 \neq 0 \) and all leaves of all rolled distributions of facets are applicable to the same surface \( y = y(u_1, v_1) = \{u_1(u, v, w), v_1(u, v, w)\} \); if for a particular deformation of \( x_0 \) one knows the applicability correspondence of all leaves to the surface \( y \), then one finds the applicability correspondence to \( y \) of all rolled leaves (including the case of leaves with degenerate linear element) by composing the rolling of the particular leaves on \( y \) with the inverse of the rolling of \( x_0 \) on \( x \) for \( u_1, v_1 = c \).

Thus excluding \( x_0 \) we have \( \Omega \) changing; however in order to be determined by the rolled \( x_0 \), we have \( \Omega \) changing with the shape of \( x \).

The leaves for a particular deformation \( x_0 \) have \( \Omega \) independent of the shape of \( x \) and outside \( \Omega \) changing; however in order to be determined by the rolled \( x_0 \), we have \( \Omega \) changing with the shape of \( x \).

The leaves are applicable to different regions of \( y \) because the constant \( c \) in \( w = w(u, v, c) \) changes for \( \omega \) fixed and for different \( \omega \) changes; however in order to be determined by the rolled \( x_0 \), \( w \) is not allowed to be linked to \( \omega \) by any other relation, either functional (as a-priori \( 3.10 \) is) or differential; thus in \( 3.10 \) \( \omega \) cancels independently of \( w \) and outside \( \omega \) we can replace \( \omega \) with any other \( \omega \) satisfying \( 2.7 \).

With \( \mathcal{M} := (I_3 - \mathring{m}_T \mathring{V}_m m^T)^T (I_3 - \mathring{m}_T \mathring{V}_m m^T) - \mathring{m}_T \mathring{V}_m m, N := \mathring{V}_m m \) we have \( d(V + x_0)^T \circ [\mathcal{M}(V + x_0) + 2N] - |dy|^2 = (\omega \times V)^T \circ [\mathcal{M}(\omega \times V + 2d(V + x_0)) + 2N] = 0, \forall \omega \) satisfying \( 2.7 \); in particular it is true if we replace \( \omega := -2N_0 \times dN_0, -2N_0 \times dN_0; \) with \( \Omega_0 := (N_0 \times dN_0) \times V \) we have \( \Omega_0 \circ [\mathcal{M}(\Omega_0 - d(V + x_0)) - N] = (\omega \times V)^T \circ [\mathcal{M}(\Omega_0 - d(V + x_0)) - N] = (\omega \times V)^T \circ \mathcal{M}(\omega \times V + \Omega_0) = 0, \forall \omega \) satisfying \( 2.7 \) and \( \delta \omega \) infinitesimal deformation of \( \omega \) (that is \( \delta \omega \) is a 0, \( d \times \delta \omega = \delta \omega \times \omega = 0, \delta \omega \times \delta \omega = 0 \)) from the last relation we get \( (\delta \omega \times V)^T \circ \mathcal{M}(\omega \times V + \Omega_0) = 0, \forall \omega \) satisfying \( 2.7 \) and \( \delta \omega \) infinitesimal deformation of \( \omega \).

Thus excluding \( x_0 \) developable we need \( d(V + x_0)^T \circ [\mathcal{M}(V + x_0) + 2N] - |dy|^2 = 0, \forall \omega \times \mathcal{M}(dx_0 \times V) \times N_0 = [\mathcal{M}(dx_0 \times (V + x_0)) + \mathcal{N}] \times N_0 = 0 \).

If \( V^T N_0 \neq 0 \), then we need \( V \times \mathcal{M}(dx_0 \times V) = V \times [\mathcal{M}(dx_0 + x_0) + \mathcal{N}] = 0 \); in particular we get \( 0 = (m \times V)^T \mathcal{M}(dx_0 \times V) = (m \times V)^T (I_3 - \mathring{m}_T \mathring{V}_m m)(dx_0 \times V), \) so \( (I_3 - \mathring{m}_T \mathring{V}_m m)(dx_0 \times V) \times \mathcal{M}(dx_0 \times V) \). If \( (I_3 - \mathring{m}_T \mathring{V}_m m)^T (dx_0 \times V) \) is a multiple of \( m \times V \), since \( \ker [(I_3 - \mathring{m}_T \mathring{V}_m m)^T] = \text{Im}[(I_3 - \mathring{m}_T \mathring{V}_m m)^T] = \mathcal{M}(dx_0 \times V) \times \mathcal{M}(dx_0 \times V) \), and excluding \( m \) isotropic we get \( V^T \partial_w V = 0 \) and further \( \partial_w V = v \times V \), \( V^T N_0 = 0 \); similarly we get \( \partial_w V^T [d(V + x_0) \times \omega] = 0 \), that is the leaves are (isotopic) curves or points. But the choice of surface \( x_0 \) is irrelevant, so this property must be true if we replace \( x_0 \) with \( x \) and \( V \) with \( RV \), that is \( 0 = \partial_w V^T [d(V + x_0) \times \omega] = 0 \), and \( V^T d(V + x_0) \), \( \forall \omega \) satisfying \( 2.7 \), or \( N_0^T (dx_0 \times v) \times V^T d(V + x_0) = 0 \); such a configuration, even if it existed, does not apply to our considerations since we are looking for leaves which are surfaces.
Thus we have the TC $V^TN_0 = 0$ and we need $d(V + x_0)^T \odot [Md(V + x_0) + 2N] - |dy|^2 = 0$, $N_0^T[Md(V + x_0) + N] = 0$, so $|\partial_w y|^2 = \partial_w V^T(I_3 + \frac{mn}{(m^2 + N_0^2)})\partial_w V, \partial_w y^T dy = d(V + x_0)^T(I_3 - N_0 N_0^T)I_j + \frac{mn}{(m^2 + N_0^2)}|dy|^2 = d(V + x_0)^T(I_3 - N_0 N_0^T)(I_3 - \frac{mn}{(m^2 + N_0^2)})d(V + x_0)$ (note $m^T N_0 = 0 \Rightarrow (m \times V) \times N_0 = 0$); thus $|\tilde{dy}|^2 = \tilde{d}(V + x_0)^T(I_3 - N_0 N_0^T)(I_3 + \frac{mn}{(m^2 + N_0^2)})\tilde{d}(V + x_0) = (I_3 - \frac{mn}{(m^2 + N_0^2)})(I_3 - N_0 N_0^T)\tilde{d}(V + x_0) = (I_3 - N_0 N_0^T)\tilde{d}(V + x_0) = (3.11) \tilde{dy} = R_1(I_3 - \frac{N_0 m^T}{m^T N_0})\tilde{d}(V + x_0), \quad R_1 \subset O_3(\mathbb{C}), \quad du \wedge dv \wedge dw \neq 0.$

Note that since along the leaves we have $m^T \tilde{d}(V + x_0) = 0$, $R_1$ is the rotation of the rolling of the leaves on $y$; if we replace $x_0$ with an applicable surface $x = Rx_0 + t$, then $R_1$ is replaced with $R_1R^{-1}$.

Imposing the compatibility condition $R_1^{-1}\tilde{d}\wedge$ on (3.11) we get $0 = [R_1^{-1}\tilde{d}R_1(I_3 - \frac{N_0 m^T}{m^T N_0}) - \tilde{d}(\frac{N_0 m^T}{m^T N_0})] \wedge \tilde{d}(V + x_0)$: with $R_1^{-1}\tilde{d}R_1 = \Omega_1 du + \Omega_2 dv + \Omega_3 dw$ this constitutes a linear system of 9 equations on the 9 entries of $\Omega_j, \ j = 1, 2, 3$ with the rank of the matrix of the system being 6, so the rank of the augmented matrix of the system must be also 6 and the solution $R_1^{-1}\tilde{d}R_1$ must also satisfy the compatibility condition $d \wedge (R_1^{-1}\tilde{d}R_1) + \frac{1}{2}[R_1^{-1}\tilde{d}R_1 \wedge R_1^{-1}\tilde{d}R_1] = 0$; these are the necessary and sufficient conditions on the 3-dimensional tangential IRDF in order to obtain applicability correspondence of leaves of a general nature.

By applying, if necessary, a change of variable $w = w(\bar{w}, u, v)$, we have $N_0^T[d(V + x_0) \times \wedge d(V + x_0)] \neq 0$ and the above considered linear system is consistent for

$$d(V + x_0)^T(I_3 - \frac{N_0 m^T}{m^T N_0}) \odot [d(\frac{N_0 m^T}{m^T N_0})\partial_w V - \partial_w (\frac{N_0 m^T}{m^T N_0})d(V + x_0) + 2N_0^T[\partial_w V \times d(V + x_0)]] N_0^T [d(V + x_0) \wedge d(V + x_0)] = 0, \quad du \wedge dv \wedge dw \neq 0;$$

if we further assume the symmetric TC $m^TV = 0$, then by using (3.6) (3.12) is satisfied.

3.3. The singular cases. The law (3.1) of distribution of facets into leaves is independent of rolling if $0 = m^T(\omega \times V + d(V + x_0) + \partial_w V dw)$ is independent of $\omega$, $\forall \omega$ satisfying (2.7), that is for $(m \times V) \times N_0 = 0 \leftrightarrow V^TN_0 = m^TN_0 = 0 \forall m = V$, in which case we have an arbitrary 1-dimensional family of 2-dimensional IRDF ($w$ can be prescribed in any continuous manner).

In the case of 2-dimensional IRDF we have $0 = m^T(\omega \times V + d(V + x_0)), \forall \omega$ satisfying (2.7), so $(m \times V) \times N_0 = 0$ and for the cancelling of the coefficients of the linearly appearing second fundamental form we get the vacuous $0 = [d(m \times V) - (N_0 \times dN_0) \times (m \times V)] \wedge (N_0 \times dN_0); \quad$ thus we need only

$$(m \times V) \times N_0 = 0, \quad 2(dm - (N_0 \times dN_0) \times m)^T \wedge (d(V + x_0) - (N_0 \times dN_0) \times V) + (3.13) \quad (m \times V)^T (dN_0 \times \wedge dN_0) = 0.$$  

In the case $m = V$ we have $V^Td(x_0 + V) = 0$, so $|V|^2 \neq 0$ and $x_0 + sV$ forms a normal congruence; if $V^TN_0 \neq 0$, then we have envelopes of sphere congruences (facets are tangent to spheres centered on $x_0$).

Dupin and Malus studied normal congruences; they remain normal after reflections and refractions in surfaces and this property is independent of the shape of the surface (that is if we transversely capture a normal congruence in $x_0$, deform $x_0$ and release the congruence after a constant angle refraction law, it remains normal), which explains the fact that envelopes of sphere congruences centered on a surface are independent of the shape of the surface (Beltrami); conversely Levi-Civita proved that any two normal congruences can be transformed one into the other by two reflections or refractions in surfaces.

If we tangentially capture a normal congruence in $x_0$ ($m = V, \ V^TN_0 = 0$), then $x_0$ is a focal surface of the normal congruence. The developables of the normal congruence are generated by
varying the normals on a normal surface along the lines of curvature; they envelope an 1-dimensional family of curves on each of the focal surfaces; thus this system of curves give a conjugate system on both focal surfaces. Since the tangent planes of the two focal surfaces are generated by the normals of a normal surface and the tangents of one of its lines of curvature, the osculating plane of a curve on a focal surface (whose tangent surface is one of the developables of the normal congruence) is normal to the tangent plane of the respective focal surface and thus the curve is a geodesic (see Eisenhart ([9], §74)).

If we deform \( x_0 \) and release the congruence, then it remains normal with the same focal surfaces (but the linear element of the other focal surface and the other curves of the conjugate system change except for normal W congruences (see Bianchi ([5], Vol 4, (202))): take an 1-dimensional family of geodesics \( v = \text{ct} \) on \( x_0 \) and their orthogonal trajectories \( u = \text{ct} \); the normal congruence is formed by the tangents to geodesics. We have the linear element \( |dx_0|^2 = du^2 + G(u, v)dv^2 \) such that the curves \( v = \text{ct} \) are geodesics on \( x_0 \). From \( p = x_0 + s(u, v)\partial_s x_0 \), \( 0 = \partial_s x_0^T(dp \wedge \times dp) = s(\partial_u x_0 \times \partial_V x_0)^T(\partial_s x_0 + s\partial_{uu}^2 x_0)du \wedge dv \) we get \( p = x_0 - (\partial_u x_0 \times \partial_V^2 x_0)^T \partial_s x_0 \partial_x x_0 = x_0 - 2G \partial_u G \partial_u x_0 \) (we exclude the case of developables \( \partial_s G = 0 \), when such a surface \( p \) does not exist); the normal surfaces are given by \( x_0 + (c - u)\partial_u x_0 \), \( c = \text{ct} \), so \( u \) is, up to addition with a constant, a principal curvature (see Eisenhart ([9], §76)). In particular since by Chasles’s theorem the common tangents to 2 confocal quadrics form a normal congruence that envelopes geodesics on the 2 confocal quadrics, if we capture this normal congruence in a quadric, deform the quadric and release the congruence, then it remains normal with the same focal surfaces (but the linear element of the other focal surface changes). The linear element of the other focal surface is independent of the shape of \( x \) only for normal W congruences, when both focal surfaces are applicable to surfaces of revolution, with the geodesics on \( x_0 \) corresponding to meridians.

In the case \( V^T N_0 = m^T N_0 = 0 \), \(|m|^2 \neq 0 \) the configuration is obtained by taking \( x_0 \) the envelope of a family of normal planes to the leaf \( p \) (planes containing the direction \( m \); to each point of \( p \) corresponds a plane); the facets and the surface \( x_0 \) are thus individuated and if we deform \( x_0 \), then the rolled distribution of facets is still integrable. If the particular leaf \( p \) is a curve, then we take the intersection of tangent planes of \( x_0 \) with \( p \) and the facet is tangent to \( p \) and normal to the tangent plane of \( x_0 \); if the particular leaf \( p \) is a point, then \( x_0 \) must be a cone passing through \( p \).

If one of the isotropic directions in each facet of the distribution can be brought for a particular deformation \( x_0 \) to coincide for all points of \( x_0 \), then the leaf in this particular position is an isotropic line (since it cannot be isotropic developable). In this case we can take any isotropic developable containing this isotropic line, intersect the tangent planes of \( x_0 \) with it to obtain curves \( c = c(u, v, w) \) in \( T x_0 \) such that the 3-dimensional distribution formed by the facets normal planes of \( c \) is integrable for all deformations \( x \) of \( x_0 \) (Ribaucour and Darboux; see Darboux ([7], §762)). Such is the case for cyclic systems (the generating isotropic developable is a null cone and the curves are circles in tangent planes of \( x \)). If \( m^T V = 0 \), then \( V \) is a normal congruence (captured tangentially in \( x_0 \)) with \( V + x_0 \) being the other focal surface; if further this is a normal W congruence, then the construction above is possible and we get the deformation of a surface of revolution to an isotropic line (see Darboux ([8],§169)).

In what concerns ACLGN assume that we have all rolled leaves applicable without collapsing ansatz:

\[
(\omega \times V + d(V + x_0))^T (\omega \times V + d(V + x_0)) = d(V + x_0)^T d(V + x_0) \text{ non–degenerate,}
\]

(3.14)

\[\forall \omega \text{ satisfying } \mathcal{L}_{\omega} \]

This becomes \((\omega \times V)^T \odot (\omega \times V + 2d(V + x_0)) = 0, \forall \omega \text{ satisfying } \mathcal{L}_{\omega} \); in particular it is true for \( \omega := -2N_0 \times dN_0 - \omega - 2N_0 \times dN_0 \); with \( \Omega_0 := (N_0 \times dN_0) \times V \) we have \( \Omega_0^T \odot (\Omega_0 - d(V + x_0)) = 0, (\omega \times V)^T \odot (\omega \times V + 2\Omega_0) = 0, \forall \omega \text{ satisfying } \mathcal{L}_{\omega} \). For \( \omega + \epsilon \delta \omega \) infinitesimal deformation of \( \omega \) (that is \((\delta \omega)^T = 0, d \wedge \delta \omega + \delta \omega \wedge \wedge \omega = 0, \delta \omega \times \wedge dx_0 = 0\)) we get \((\delta \omega \times V)^T \odot (\omega \times V + \Omega_0) = 0, \forall \omega \text{ satisfying } \mathcal{L}_{\omega} \) and \( \delta \omega \) infinitesimal deformation of \( \omega \), a contradiction.
Thus (3.14) must be refined to assume collapsing ansatz of leaves: for a particular deformation of $x_0$ (which can be taken to be $x_0$) the leaf has degenerate 2-dimensional element and must be a(n isotropic) curve or a point.

For $n = V$ the sphere congruence is defined so that the spheres centered on $x_0$ are tangent to the given (isotropic) curve or point; the linear element of the leaves depends however on the shape of $x$.

Note however that envelopes of sphere congruences enjoy other properties; it is worth mentioning here the sequence of events that led Bianchi to his discovery of the B transformation of quadrics: in 1899 (and based on an earlier result of 1897) Guichard discovered that when a quadric with(out) center and of revolution around the focal axis rolls on one of its deformations, its foci (focus) describe CMC (minimal) surfaces (thus generalizing an earlier result of Bonnet on the rolling of the unit sphere on a CGC 1 surface; since a surface of revolution can be rolled on the axis of revolution with the arc-length of the meridian corresponding to the arc-length of the axis, this is also a generalization of Delaunay’s generation of CMC (minimal) surfaces of revolution with the meridian being described by a focus of a conic as the conic rolls on the axis in a meridian plane) and the same result for the intersections of the isotropic rulings of the quadric with the tangent planes of the quadric (in this case according to Darboux the CMC (minimal) surfaces are the rolled foci (focus) as the quadric rolls on the complementary transform of the considered deformation).

Building on Guichard’s result Darboux reduced the deformations of the Darboux quadrics with center to the deformations of the (pseudo-)sphere (Goursat had integrated earlier the equations for the deformations of certain Darboux paraboloids): if we intersect the tangent planes of a surface with a(n isotropic) plane and consider the resulting congruence of lines as the surface rolls on one of its deformations, then the focal surfaces of this congruence are obtained from the intersection of the common conjugate directions with the lines and the conjugate system induced by the developables of the congruence on the focal surfaces corresponds to the conjugate system common to the surface and its deformation; if the plane is isotropic, then the congruence is normal and the parallel surfaces are given by the intersection of the isotropic lines of the isotropic plane with the line of the congruence (thus they are envelopes of rolling congruences of the isotropic lines of the isotropic plane).

In Guichard’s result with 4 isotropic rulings (situated in two isotropic planes) the 4 points in the tangent plane are situated on the rulings at the tangency point; since the asymptotes and any conjugate system are harmonically conjugate we get two normal congruences each having two parallel surfaces in harmonic ratio with the focal surfaces and this configuration is possible only if the two parallel surfaces have CMC (this configuration is preserved under conformal changes of the space that preserve geodesics, so we get parallel CMC surfaces in space forms by changing Cayley’s absolute; see Bianchi ([3], Vol 4,(108))).

Thus Guichard’s for isotropic rulings remains valid if we consider only two parallel isotropic rulings in an isotropic plane, that is a Darboux quadric with center.

Darboux inquired what becomes of Guichard’s result if we consider the intersection of the 8 isotropic rulings on the general quadric with tangent planes; the resulting surfaces (envelopes of rolling congruences of isotropic rulings of the quadric) are isothermic (surfaces with isothermal lines of curvature) with the lines of curvature corresponding to the conjugate system common to the quadric and its deformation (thus in conformal correspondence); however, since the deformation problem depends on two functions of a variable and the space of isothermic surfaces in conformal correspondence is much larger, Darboux sought to find the properties that individuate the special isothermic surfaces obtained above.

The surfaces corresponding to parallel isotropic rulings have same normal direction and form a harmonic ratio with the focal surfaces of the congruence of their joins (thus giving the involutory Christoffel transformation of isothermic surfaces) and the surfaces corresponding to isotropic rulings that intersect (at umbilics) are leaves of the cyclic system generated by the isotropic cones at umbilics; thus they are envelopes of a sphere congruence.
Conversely, Darboux proved that any isothermic surface appears as envelope of a 4-dimensional family of sphere congruences whose other envelopes are isothermic surfaces in conformal correspondence with the given isothermic surface; thus introducing the Darboux (D) transformation of isothermic surfaces (a particular case of Ribaucour transformation (envelopes of sphere congruences with correspondence of lines of curvature; note however that this property is independent of the shape of the surface of centers only if it is applicable to a quadric of revolution), which in turn corresponds via Lie’s contact transformation (which exchanges the points and planes of facets) to the B transformation of the focal surfaces of a W congruences); later on he found the sought characterization of special isothermic surfaces and the fact that in the 4-dimensional space of D transforms of a special isothermic surface a 3-dimensional sub-space are special isothermic associated to the same quadric; thus the space of special isothermic surfaces associated to the same quadric remains closed under the iteration of the D transformation, which provided the first theory of deformations of general quadrics depending on arbitrarily many constants.

Building on Guichard’s and Darboux’s results Bianchi proved in 1899 the inversion of Guichard’s result (all CMC (minimal) surfaces can be realized in a 2-dimensional fashion as in Guichard’s result and if a point rigidly attached to a surface generates CMC (minimal) surfaces when the surface rolls on its deformations, then the surface and point must be as in Guichard’s result); in 1904 he proved the BPT for the D transformation of isothermic surfaces (the iteration of the D transformation can be realized using only algebraic and differential computations) and provided a simple geometric interpretation of Darboux’s characterization of special isothermic surfaces: they are the closure of CMC surfaces under conformal transformations of the space (see Bianchi ([5], Vol.4,108))).

In trying to see what becomes of Darboux’s result for CMC \( \frac{1}{4} \) surfaces he realized that the transformation induced for CGC 1 surfaces is not a fundamental one, but rather the composition of two such ones; thus he found in 1899 the B transformation of the sphere (the reason this B transformation was discovered much later than that of the pseudo-sphere is that at the first iteration it gives complex leaves; one needs the iteration of two complex conjugate B transformations (via the BPT) to get back real surfaces), which he later generalized to quadrics of revolution and paraboloids (1905); once he realized that the applicability law of the leaves is given at the level of confocal quadrics by the Ivory affinity he generalized the B transformation to all quadrics [1].

For \( m^T N_0 = V^T N_0 = 0 \) consider the defining surface \( x_0 \) such that the leaf is a curve \( c = c(s) \): the tangent planes of \( x_0 \) cut \( c \) at points \( p \) and at those points \( m = N_0 \times c'(s) \). We have

\[
0 = N_0^T (c(s) - x_0), \quad \text{so } ds^2 = \frac{dN_0^T V}{N_0^T c'(s)^2} \quad \text{and the linear element } |\omega \times V - \frac{dN_0^T V}{N_0^T c'(s)^2} c'(s)|^2 = |V^T (\omega \times N_0 + dN_0)|^2 + \frac{(dN_0^T V)^2}{(N_0^T c'(s))^2} |c'(s) \times N_0|^2 \quad \text{of the rolled leaf must be non-degenerate independent of } \omega \quad \text{for most } \omega \neq 0, -2N_0 \times dN_0 \text{ satisfying } \omega \text{.}
\]

Using (2.3) this becomes \( |(N_0 \times V)^T R^{-1} dN|^2 \) being independent of the shape of \( x \) for most deformations \( x \) of \( x_0 \); if we refer \( x_0 \) to a system of coordinates formed by the curves \( u = ct \) envelopes of the tangent field \( N_0 \times V \) and their orthogonal trajectories \( v = ct \), then we need \( (N^T d\partial_u x)^2 \) to be independent of shape of \( x \) for most deformations \( x \) of \( x_0 \), so we get a normal W congruence.

If the original leaf is a point \( p \), then \( N_0^T (p - x_0) = 0, \ dN_0^T (p - x_0) = 0 \) and \( x_0 \) is developable with rulings passing through \( p \).

For \( (m \times V) \times N_0 \neq 0, \ m^T \partial_u V = 0 \) to each point of \( x_0 \) we associate an 1-dimensional family of facets centered on a curve (or at a point) in a leaf. If the leaves of the original distribution are an 1-dimensional family of points, curves or surfaces, then we can take the parametrization \( (u, v) \) on \( x_0 \) such that as \( u \) varies the curve of facets varies in its leaf and as \( v \) varies the curve of facets varies in different leaves; if the original distribution is just the facets tangent to a curve, each facet being counted with simple \( \infty \) and such that the facets associated to a point of \( x_0 \) are the simple \( \infty \) of facets centered at a point (counted with the simple \( \infty \) of \( w \)), then again we can take the parametrization \( (u, v) \) on \( x_0 \) such that when \( u \) varies we get the simple \( \infty \) multiplicity of the facet and when \( v \) varies we get facets centered at different points; the remaining case is when the original
distribution is just the facets tangent to a curve or surface, each facet being counted with simple \( \infty \) multiplicity, but such that the facets associated to a point of \( x_0 \) are centered on a curve.

However, only the case of the symmetric TC is pertinent to our considerations and we shall not discuss it here.

4. 3-DIMENSIONAL INTEGRABLE TANGENTIAL ROLLING DISTRIBUTIONS OF FACETS WITH COLLAPSING ANSATZ

In the case of collapsing ansatz of leaves (for a particular deformation of \( x_0 \) (which can be taken to be \( x_0 \)) the leaves of the 3-dimensional IRDF collapse to an 1-dimensional family of points or curves) without the TC one can derive the IC by requiring that the reflected original distribution is also integrable with the cancelling of the (linearly appearing) terms containing the second fundamental form of \( x_0 \).

However, since we are interested in ACLGN this IC does not apply to our considerations, so we have to further assume the TC.

Assume that we have a 3-dimensional tangential distribution of facets

\[
(p, P) = (p(u_0, v_0, w), P(u_0, v_0, w)), \ p \in P
\]
distributed in the tangent planes of the surface \( x_0 = x_0(u_0, v_0) : V := p - x_0, V^T N_0 = 0 \).

Further assume that the IC of this distribution depends only on the linear element of \( x_0 \), that is if we roll \( x_0^0 \subset x_0 \) on one if its deformations \( (x^0, dx^0) = (R_0 x_0^0 + t_0, d x_0^0) \), then the rolled distribution of facets \( (R_0 p + t_0, R P) = (R_0 V + x^0, R_0 P) \) remains integrable with leaves \( x^1 \) (and normal fields \( R_0 m_1^0 \)).

Further assume that for one of the rolled distributions of facets \( (R_0 p + t_0, R_0 P) \) the leaves collapse from surfaces to curves (if they collapse to points, then the rolled leaves are always curves); we can take that defining distribution to be the initial one and thus \( x_0 \) is the defining surface of the IRDF.

The leaves of the defining distribution of facets are an 1-dimensional family of curves that generate an auxiliary surface \( x_z \) (assume that we don’t have a curve counted with simple \( \infty \) multiplicity, in which case the rolled leaves are always curves).

If we reflect the distribution of facets in the tangent bundle of \( x_0 \) (that is we roll \( x_0 \) on its other side), then we get a 3-dimensional integrable distribution of facets centered on \( x_z \), so the leaves are also curves that generate \( x_z \).

Excluding the case \( (N_0^0)^T m_1^0 = 0 \) (when the two families of curves coincide and we get an 1-dimensional family of 2-dimensional IRDF) and the case when \( x_0 = x_z \) is a ruled surface with the leaves of the distribution being the rulings of \( x_0 \) (note that we also have to exclude \( x_0 \) developable, in which case by deforming it to a plane we get the rolled \( x_z \) (captured in the 1-dimensional family of tangent planes of \( x_0 \)) the same plane, so the plane itself is defining and auxiliary surface, but the planes of the facets (which must be tangent planes of curves in this plane) must leave the plane, so we are in the previous situation), the two families of curves give a parametrization \((u_1, v_1) \) of \( x_z \) suited to our purposes; \( (V_0^1)^T N_0^0 = 0 \), \( V_0^1 := x_z^1 - x_0^1 \) imposes a functional relationship between the four independent variables \( u_0, v_0, u_1, v_1 \), leaving only three of them independent and any other needed functional relationship between the four independent variables \( u_0, v_0, u_1, v_1 \) must be a consequence of \( (V_0^1)^T N_0^0 = 0 \).

We have the reflection property

\[
(m_1^0)^T \partial_{u_1} x_1^1 = (m_1^0)^T (I_3 - 2 N_0^0 (N_0^0)^T) \partial_{u_1} x_1^1 = 0
\]

and

\[
R_0^{-1} dx^1 = d (V_0^1 + x_0^0) + R_0^{-1} d R_0 V_0^1 = \tilde{d} x^1 + \omega_0 \times V_0^1, \ \omega_0 := N_0^0 \times R_0^{-1} d R_0 N_0^0. \text{ But } (\omega_0)^{-1} = 0
\]

and

\[
\tilde{d} x^1 = \partial_{u_1} x_1^1 du_1 + \partial_{v_1} x_1^1 dv_1, \text{ so } 0 = (R_0 m_1^0)^T d x^1 \text{ (and similarly } 0 = (R_0 m_1^0)^T d x^1 \text{ for } R_0 = R_0 (I_3 - 2 N_0^0 (N_0^0)^T), \ \omega_0 = -\omega_0 - 2 N_0^0 \times d N_0^0 \text{ becomes:}
\]

\[
-(V_0^1)^T (\omega_0 \times N_0^0) (m_1^0)^T N_0^0 + (m_1^0)^T \partial_{u_1} x_1^1 dv_1 = 0
\]

\[
(\Leftrightarrow -(V_0^1)^T (\omega_0 \times N_0^0) (m_1^0)^T N_0^0 + (m_1^0)^T \partial_{u_1} x_1^1 du_1 = 0) \Leftrightarrow B_z \text{ transformation.}
\]

20
We get the transformation $B'_z$ for the reflected rolled distribution of facets by the change $(u_1, m_0) \leftrightarrow (v_1, m_0')$ or $\omega_0 \leftrightarrow \omega'_0$. Since the distributions of facets $\mathcal{D}, \mathcal{D}'$ with normal fields $m_0', m_0$ reflect in $Tx_0$, the rolled distributions of facets $(R_0, t_0)\mathcal{D}, (R_0, t_0)\mathcal{D}'$ reflect in $Tx_0$, so $B'_z(x^0)$ is just $B_2(x^0)$ when $x^0$ rolls on the other face of $x^0$.

Using (4.1) becomes:

$$(4.3) - (V^1_0)^T (\omega_0 \times N^0_0) + 2(\partial_{u_1} x^1_1)^T N^0_0 du_1 = 0 \Rightarrow -(V^1_0)^T (\omega_0 \times N^0_0) + 2(\partial_{u_1} x^1_1)^T N^0_0 du_1 = 0.$$ 

(because $0 = \tilde{d}((V^1_0)^T N^0_0) = (\partial_{v_1} x^1_1)^T N^0_0 dv_1 + (\partial_{u_1} x^1_1)^T N^0_0 du_1 + (V^1_0)^T dN^0_0$ the equivalency of the equations of (4.3) is clear).

Imposing the compatibility condition $\tilde{d} \land \mathbf{1}$ and using the equation itself we get the IC

$$0 = -(\partial_{u_1} x^1_1)^T N^0_0 \land \omega_0 \times N^0_0 + 2(\partial^2_{u_1 v_1} x^1_1)^T N^0_0 du_1 \land dv_1 + 2(\partial_{u_1} x^1_1)^T N^0_0 dv_1 = 2 \omega_0 - \left( \partial^2_{u_1 v_1} x^1_1 \right)^T N^0_0 (dN^0_0)^T \land \omega_0 + (dN^0_0)^T V^1_0 [N^0_0 \times \left( \frac{\partial_{u_1} x^1_1}{(\partial_{v_1} x^1_1)^T N^0_0} + \frac{\partial_{u_1} x^1_1}{(\partial_{v_1} x^1_1)^T N^0_0} \right)]^T \land \omega_0 - (\partial^2_{u_1 v_1} x^1_1)^T N^0_0 (dN^0_0)^T \land \omega_0 + (dN^0_0)^T V^1_0 [N^0_0 \times \left( \frac{\partial_{u_1} x^1_1}{(\partial_{v_1} x^1_1)^T N^0_0} + \frac{\partial_{u_1} x^1_1}{(\partial_{v_1} x^1_1)^T N^0_0} \right)]^T \land \omega_0 = 0.$$ 

Referring $x^0$ to its lines of curvature parametrization we get (excluding $x^0$) the IC

$$(V^1_0)^T N^0_0 = 0 \Rightarrow (I - N^0_0 (N^0_0)^T) [\partial_{u_1} x^1_1 (\partial_{v_1} x^1_1)^T + \partial_{u_1} x^1_1 (\partial_{v_1} x^1_1)^T] = (\partial^2_{u_1 v_1} x^1_1)^T N^0_0.$$ 

(4.4)

If $\partial^2_{u_1 v_1} x^1_1 = 0$, then $x^1_1 (u_1, v_1) = f(u_1) + g(v_1)$, $f^1(u_1) \times g^1(v_1) \neq 0$ and $(V^1_0)^T N^0_0 = 0 \Rightarrow f^1(u_1) N^0_0 g^1(v_1) N^0_0 \neq 0$, $f^1(u_1) N^0_0 g^1(v_1) N^0_0 \neq 0$, $f^1(u_1) N^0_0 g^1(v_1) N^0_0 = 2 N^0_0$, $f^1(u_1) N^0_0 g^1(v_1) N^0_0 = 0$; in the first case we have $N^0_0 = \frac{f^1(u_1) N^0_0 g^1(v_1) N^0_0}{[f^1(u_1) N^0_0 g^1(v_1) N^0_0]}$, but along the curve of tangency of the tangent cone of $x^0$ from $x_2 (u_1, v_1)$ and $x_0$ is developable and in the second case $N^0_0$ along the curve of tangency of the tangent cone of $x^0$ from $x_2 (u_1, v_1)$ must span a line or a plane; since the tangent cone of $x^0$ from $x_2 (u_1, v_1)$ cannot be a cylinder, $x_0$ is again developable.

Note also $(\partial_{u_1} x^1_1) \times V^1_0 \times (I - 2 N^0_0 (N^0_0)^T) \partial_{v_1} x^1_1 = -[V^1_0 \times N^0_0] \times (\partial_{u_1} x^1_1) \times (\partial_{v_1} x^1_1)^T + \partial_{u_1} x^1_1 (\partial_{v_1} x^1_1)^T N^0_0 + (V^1_0)^T N^0_0 (\partial_{u_1} x^1_1 \times (\partial_{v_1} x^1_1)^T) N^0_0 \partial_{u_1} x^1_1$, so we are actually in the symmetric TC case $(V^1_0)^T N^0_0 = 0$: we can take

$$(4.5) m_1^0 := B_1 \partial_{u_1} x^1_1 \times V^1_0, \quad m_1^v := B_1 \partial_{v_1} x^1_1 \times V^1_0, \quad B_1 := -z[(\partial_{u_1} x^1_1)^T \partial_{v_1} x^1_1 (\partial_{v_1} x^1_1)^T N^0_0]^{-1}.$$ 

Multiplying the first equation of (4.3) with $B_1 (\partial_{u_1} x^1_1)^T N^0_0$ and using $-B_1 (\partial_{u_1} x^1_1)^T N^0_0 V^1_0 = -B_1 (\partial_{u_1} x^1_1 \times V^1_0) \times N^0_0 = m_0^1 N^0_0$ we get

$$(m_1^0)^T \omega_0 + 2zdu_1 = 0 \Leftrightarrow (m_1^0)^T \omega'_0 + 2zdu_1 = 0 \Leftrightarrow B_z \text{ transformation},$$

(6.6) $$(m_1^0)^T \omega'_0 + 2zdu_1 = 0 \Leftrightarrow (m_1^0)^T \omega_0 + 2zdu_1 = 0 \Leftrightarrow B_z' \text{ transformation}.$$ 

Using $R_0^{-1} dx^1 = dx^1_1 + \omega_0 \times V^1_0$ and (4.3) we get $R_0^{-1} dx^1 = dx^1_1 - 2(\partial_{v_1} x^1_1)^T N^0_0 \partial_{u_1} x^1_1$ and

$$(4.7) \quad |dx^1|^2 = |dx^1_1|^2 - 4(\partial_{v_1} x^1_1)^T N^0_0 (\partial_{u_1} x^1_1)^T N^0_0 du_1 \land dv_1,$$

so the linear element of the leaf $x^1_1$ does not depend on the shape of the seed $x^0$ (we have ACLGN provided $(V^1_0)^T N^0_0 = 0 \Rightarrow B_1 = B_1 (u_1, v_1)$ (is $dB_1 = 0)$ (that is $B_1$ does not vary when $x^0$ varies on the tangent cone of $x_0$ from $x_2$ for $x_2$ fixed). But for $x_2$ fixed and $x^0_0$ varying in the TC $(V^1_0)^T N^0_0 = 0$ we have $(V^1_0)^T N^0_0 = 0$ and $d \log(B_1) = z^{-1} B_1 (dN^0_0)^T (\partial_{u_1} x^1_1 \times (\partial_{v_1} x^1_1)^T + \partial_{u_1} x^1_1 \times (\partial_{v_1} x^1_1)^T) N^0_0 = (4.3)$.

Thus all degenerate leaves $u_1 \leftrightarrow v_1 \leftrightarrow v_1$ are applicable to a surface $y_0 = y_0 (u_1, v_1)$, the simple infinite multiplicity of facets of each leaf at a point $x^2$ corresponding to the tangent planes of $y_0$ at a point counted with simple infinite multiplicity and with $(\partial_{u_1} x^1_1 \times (\partial_{v_1} x^1_1)^T) (\partial_{v_1} x^1_1, \partial_{v_1} x^1_1)$ taken to the same direction $\partial_{v_1} y_0$ ($\partial_{v_1} y_0$): by reflection in $Tx_0$ (along the points of $x^0_0$ on the tangent cone of $x_0$ from $x_2$) we get also the fact that $(I_3 - 2 N^0_0 (N^0_0)^T) \partial_{u_1} x^1_1 \times ((I_3 - 2 N^0_0 (N^0_0)^T) \partial_{u_1} x^1_1)$ are taken to the same direction, since
the angle between these directions and the initial ones $\partial_{u_i}x^1_z$ ($\partial_{u_i}x^2_z$) is independent of $(u_0, v_0)$; thus these last directions must be taken to $\partial_{u_i}y_0$ ($\partial_{u_i}y_0$).

For quadrics with $u_1, u_1 = ct$ being the ruling families on the quadric $x_z$ confocal to $x_0$ (4.3) (which gives integrability and applicability correspondence of a general nature) is satisfied, but $m^1_0$ (respectively $m^2_0$) are defined by (4.3) with a-priori $B_1 = B_0(u_0, v_1)$ such that they depend only on $x^0_0, v_1$ (respectively $x^0_0, u_1$) (they depend quadratically on $v_1$ (respectively $u_1$), which makes (4.0) a Riccati equation). In this case by requiring that $m^1_0, m^2_0$ as defined by (4.3) satisfy the reflection property (4.1) we get the needed part of (4.3) (without the coefficients of $V^1_0, N^0_0$) by purely algebraic manipulations and the direct proof of the integrability of (4.0) is simpler and enforces the definition of $B_1$ from (4.3): imposing the compatibility condition $2zd\langle 0 \rangle$ on the first equation of (4.0) and using the equation itself we get the IC $0 = 2z(\partial_{u_i}m^1_0dv_1 + B_1dx^0_0 \times \partial_{u_i}x^1_z) \wedge \omega_0 + 2z(m^1_0)^2d \wedge \omega_0 = (2zT) - (N^0_0)^T(2zm^1_0 + m^1_0 \times \partial_{u_i}m^1_0)(N^0_0)^T \wedge \omega_0$; using (4.1), (4.4) and (4.5) this becomes

$$
0 = (N^0_0)^T(2zm^1_0 + m^1_0 \times \partial_{u_i}m^1_0) = \frac{z(m^1_0)^2}{(N^0_0)^T \partial_{u_i}x^1_z} - B_1(\partial_{u_i}x^1_z)^T N^0_0 (V^1_0)^T \partial_{u_i}m^1_0 = \frac{z(v^1_0)^T(\partial_{u_i}x^1_z \times \partial_{u_i}x^1_z + \partial_{u_i}(B_1 \partial_{u_i}x^1_z \times V^1_0))}{(N^0_0)^T \partial_{u_i}x^1_z},
$$

which is straightforward; replacing $(m^1_0, v_1)$ with $(m^1_0, v_1)$ we get a similar relation.

### 4.1. The tangency configuration only symmetrizes the integrability condition.

Consider first the case when the TC $(V^0_0)^T N^0_0 = 0$ is used only to symmetrize (4.3):

$$
M\partial^2_{u_i u_j} x_z - x_z = ct(= 0), \quad (I_3 - N_0 N^0_0)(M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z)) - x_z x^T = 0 \quad N^0_0 = 0 \quad \forall
$$

$$
M\partial^2_{u_i u_j} x_z - x_z \neq ct, \quad (I_3 - N_0 N^0_0)(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - \partial^2_{u_i u_j} x_z x^T = 0
$$

(4.8)

In the first case with $M = M(u_0, v_1) := M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - x_z x^T)$ we have $(I_3 - N_0 N^0_0)(M(u_0, v_1) - M(u_0, v_1))dN_0 = 0$ with $u_0, v_0, u_1, v_1, u_1, v_1$ independent variables; applying $d$ we get $[M(u_0, v_1) - M(u_0, v_1)]dN_0 = (I_3 - N_0 N^0_0)(M(u_0, v_1) - M(u_0, v_1))dN_0$, so $M(u_0, v_1) = M(u_0, v_1) + N_0 N^0_0 (M(u_0, v_1) - M(u_0, v_1))dN_0$; differentiating this with respect to $u_0, v_0$ we get $\partial_{u_i}(M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - x_z x^T)) = 0; \quad M = \partial_{u_i}(M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - x_z x^T))$.

In the second case if $M\partial^2_{u_i u_j} x_z \neq w, \quad w = ct \in C^3 \setminus \{0\}$, then we shall see later that the condition imposed on $x_0$ is over-determined.

Thus $M\partial^2_{u_i u_j} x_z = w, \quad w = ct \in C^3 \setminus \{0\}$; with $M := M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - x_z x^T)$ and as above we get $\partial_{u_i}(M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - x_z x^T)) = 0$. Hence $M = \partial_{u_i}(M(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z - x_z x^T))$.

(1)QWC doubly ruled by leaves (the image of the equilateral paraboloid $Z = Z(u_0, v_1) := u_0 f_1 + v_1 f_1 + u_0 v_1 e_3$ under an affine (which can be taken to be linear) transformation of $C^3$ with kernel $\neq C^3$ at most 1-dimensional).

With $J_2 := f_1 f_1^T, \quad J_3 := f_1 e_3 + e_3 f_1^T$ being the symmetric Jordan (SJ) blocks of dimension 2 and 3 any $A = A^T \in M_3(C)$ can be brought, via conjugation with a complex rotation, to a form with diagonal blocks, each block being a number or $aJ_2 + J_2$ or $aJ_3 + J_3, \quad a \in \mathbb{C}$ and, since one can take square roots of any SJ matrix without isotropic kernel, any $A \in GL_3(C)$ has singular value decomposition (SVD) $A = R_1 D R_2^T, \quad R_1, R_2 \in O_3(C), \quad D^2 SJ$.

We have

$$
-\frac{1}{2}(u - v)^2(\partial_{u_i}x_z \partial_{u_j}x^T_z + \partial_{u_i}x_z \partial_{u_j}x^T_z) - X X^T = -I_3, \quad \partial_{u_i}x_z X = \frac{2}{(u - v)^2} X, \quad X = X(u, v)
$$

(4.9)

$$
\partial_{u_i}x_z \partial_{u_i}x^T_z + \partial_{u_i}x_z \partial_{u_i}x^T_z - \partial^2_{u_i u_i} z Z^T = f_1 f_1^T + f_1 f_1^T, \quad Z = Z(u, v).
$$
so \((4.8)\) becomes
\[
(I_3 - N_0N_0^T)\begin{pmatrix} -AA^T + x_0^T \end{pmatrix}N_0 = 0, \dim(\ker(A)) \leq 1 \text{ for } x_0 = AX_1 \text{ (isotropic singular) QC,}
\]
\[
(I_3 - N_0N_0^T)(A(f_1^T + \bar{f}_1f_1^T)A^T + x_0(Ae_3)^T + Ae_3x_0^T)N_0 = 0, \dim(\ker(A)) \leq 1,
\]
(4.10)
\[Ae_3 \neq 0 \text{ for } x_2 = AZ_1 \text{ (isotropic singular) (I)QWC;}
\]
for \(x_2\) (isotropic singular) QC one can multiply \(A\) on the right with a rotation \(R_2 \in O_3(\mathbb{C})\) (which has the effect of changing each of \(u_1, v_1\) by the same Möbius transformation); for \(x_2\) (isotropic singular) (I)QWC one can multiply \(A\) on the right with \(e^{i\epsilon_2(\epsilon_4^2f_1f_1^T + e^{-i\epsilon_2}f_1f_1^T) + e^{2\epsilon_3}e_3e_3^T}\) (which has the effect \((u_1, v_1) \to (e^{i\epsilon_2}e_3^2u_1, e^{-i\epsilon_2}e_3^2v_1)\) or with the reflection \(e_2 \mapsto -e_2\) (which has the effect \(u_1 \leftrightarrow v_1\); in both cases one can multiply \(A\) on the left with an arbitrary rotation (which must also apply to \(x_0\)).

For \(x_1\) (isotropic singular) QC apply this rotation to bring \(AA^T\) to the SJ canonical form; for \(\ker(AA^T)\) containing the isotropic direction \(f_1\) we have \(AA^T = J_2 + ae_3e_3^T, a \in \mathbb{C} \text{ or } J_3\), so\(Ae_3e_3^T\) is a root of the cubic equation \(X^3 + (\bar{w} + aw)X^2 + \frac{1}{a}(\bar{w}^2 + 2aw\bar{w} - 4aw^2 + 2a^2)X - \frac{1}{a}w^2\bar{w} = 0\) in \(X\); the condition that this equation has a triple root is over-determined (it imposes two functionally independent conditions on \(w, \bar{w}, z\)) and the condition of it having two distinct roots (with \(\partial_w z\) double root) leads to \(x_0\) being the isotropic developable circumscribed to one (and thus to the other) of the conics \(\frac{w^2}{c_1} + \frac{1}{a}z = 1 \land \bar{w} = 0, -\frac{1}{a} \bar{w}^2 - \frac{1}{a}w^2 \bar{w} = 1 \land z = 0\) which extended with the isotropic plane \(f_1^T x = 0\) determines the confocal quadrics below.

Otherwise applying \(\partial_w z\) to this cubic equation we get \(\partial^2_w z = 0, z = wF(\bar{w}) + G(\bar{w})\) and we need \(0 = [\bar{w} + 2F(\bar{w})]([wF'(\bar{w}) - F(\bar{w})] = 2aF(\bar{w}) + [\bar{w} + 2F(\bar{w})][wG'(\bar{w}) - 2G(\bar{w})] = [1 + 2F(\bar{w})][wF'(\bar{w}) - F(\bar{w})] = 2aF(\bar{w}) + [1 + 2F(\bar{w})][wG'(\bar{w}) - 2G(\bar{w})] + G(\bar{w})][wF'(\bar{w}) - F(\bar{w})] = -2F(\bar{w}) + G(\bar{w})]2a + wG'(\bar{w}) - 2G(\bar{w})\), so \(F(\bar{w}) = cw\), \(c \in \mathbb{C} \text{ \{1, -1, -\bar{w}^2\}}\), \(G(\bar{w}) = \frac{aw}{1 + c\bar{w}^2} + c^{(1+2c)}\bar{w}^2\); this leads to \(x_0\) being one of the confocal Darboux quadrics \(x_0^TAR^{-1}x_0 = 1, A := (a_1I_3 + J_2 + ae_3e_3^T)^{-1}, a_1 \in \mathbb{C} \text{ \{-0, -a\}}, R \in \mathbb{C} \text{ \{a_1, a_2 \leftrightarrow a_1 + a\}}\); since \(AR_{x_0}^{-1} = ((a_1 - w)I_3 + J_2 + ae_3e_3^T)^{-1}\) we can take \(w = 0\) for \(x_0\). In this case \(A := f_1(\frac{f_1}{2}f_1^T + \sqrt{ae_3^2}, z \in \mathbb{C} \text{ \{0\}}\) and \(x_0 = x_0^T \bar{x}_0 \cdot f_1 + \sqrt{\frac{x_0^T \bar{x}_0}{x_0^T \bar{x}_0}} \cdot e_3, z \in \mathbb{C} \text{ \{0\}}\) is the isotropic plane \(f_1^T x = 0\) with rulings the lines tangent to the conic \((f_1^T x)^2 + \frac{1}{a}(e_3^2x_0^2) = 1\) obtained for \(z^2 + \bar{u}_1 = 0\) (this is the same conic as the one in the plane \(\bar{w} = 0\), generating the isotropic above); this is the isotropic singular quadric of the above considered family of confocal quadrics (obtained for \(w = a_1\)).

For \(x_0^T \subset x_0\) from \((1.3)\) and \((x_1)^T N_0^0 = (x_0^T)^T N_0^0\) we get \(4(\partial_{u_1} x_1)^T N_0^0(\partial_{u_1} x_1)^T N_0^0 du_1 \otimes dv_1 = -4a_{1u} du_1 \otimes dv_1 = -a_1|dX_1|^2\) and from \((4.7)\) the linear element of the leaves is \(|dx| = |dx_1|^2 + a_1|dx_1|^2\) and \(dx_1^T (A^T A = I_3 + I_3) dx_1 = dx_1^T (a_1 I_3 + ae_3e_3^T) dx_1\), so the leaves are applicable to a quadric with center and of revolution.

For \(a = 0\) \(w \partial_{w} z + \bar{w} \partial_{\bar{w}} z - 2z = 0\) leads to \(x_0\) being the plane passing through the origin and having unit normal \(N_0 = c f_1 + e_3, c \in \mathbb{C} \text{ and } \bar{w} + 2\partial_{w} z = 0\) leads to \(z + \frac{\bar{w}w}{c} = F(\bar{w}), F(\bar{w})^2 - 2\frac{\bar{w}w}{c} F(\bar{w}) + \frac{1}{c} = 0\) with solutions \(F(\bar{w}) = \pm \frac{\bar{w}w}{c}\), which leads to \(x_0\) being the isotropic cones centered at \(\pm f_1\) (the isotropic developable which extended with the isotropic plane \(f_1^T x = 0\) generates the confocal quadrics below) and \(F(\bar{w}) = c \bar{w}w + \frac{1}{c}, c \in \mathbb{C} \text{ \{0\}}\), which leads to \(x_0\) being one of the confocal Darboux quadrics \(x_0^TAR_{x_0}^{-1} = 1, A := (a_1I_3 + J_2)^{-1}, a_1 \in \mathbb{C} \text{ \{0\}}\), \(w \in \mathbb{C} \text{ \{a_1\}}\) (the only quadrics having contact of order 3 with \(C(\infty)\)); we can take \(w = 0\) for \(x_0\). In this case \(A := f_1e_3^T + (-\frac{a}{u_1} f_1 + \bar{f}_1 + ze_3) f_1^T, x_0 = \frac{aw}{u_1 - c_1} f_1 + \frac{2}{u_1-c_1}(-\frac{a}{u_1} f_1 + \bar{f}_1 + ze_3), z \in \mathbb{C} \text{, } A_{\infty} = \)
$J_3$, $x_\infty = \frac{u_1+u_3}{u_1-u_3} f_1 + \frac{2}{u_1-u_3} e_3$ (after multiplication with a rotation $R_2 \in O_3(\mathbb{C})$ on the right one can make $A_3 = f_1 e_3^T + (f_1 + z e_3) f_3^T$ and thus the definition of $A_3^\infty$ is clear) is the pencil of (isotropic) planes containing $f_1$ and the rulings are lines passing through $\pm f_1$ in their respective planes (these foci are obtained from $x_2(u_1, \infty)$, $x_3(\infty, v_1)$); the only finite (isotropic) singular quadric of the confocal family of $x_0$ is the line $x_{u_1} = \mathbb{C} f_1$ which is also singular set of $x_z$, $z \in \mathbb{C} \cup \{\infty\}$; however for convenience one can also include the pencil of (isotropic) planes containing $f_1$ in this isotropic singular quadric.

From (4.10) and $(x_1^T)^2 N_0^0 = (x_0^T)^2 N_0^0$ we get $4(\partial_{\bar{u}_1} x_0^2)^* T N_0^0 \partial_{\bar{u}_1} x_0^* T N_0^0 d\bar{u}_1 \cdot d\bar{v}_1 = -a_{21} |dX_1|^2$ and from (4.17) the linear element of the leaves is $|dx|^2 = |dx_1|^2 + a_1 |dX_1|^2 = dX_1^T (A_2^T A_2 + a_1 I_3) dX_1$; since $A_2^T A_2 = J_3$ for $z \in \mathbb{C}$, $= J_3^2 = J_2$ for $z = \infty$, the leaves are applicable to another region $x_0^0$ of $x_0$ for $z = \infty$ and for $z \in \mathbb{C}$ they are applicable to a Darboux quadric with $A^{-1} = a_1 I_3 + J_3$ (thus different from the type of $x_0$). In this case (4.10) are linear ($m_1^0, m_1^T$ depend linearly on $u_1, u_2$), so this B transformation can be found by quadratures.

For $A A T = J_3$ with $x_0 = \sqrt{\bar{w}} f_1 + \sqrt{\bar{w}} f_1 + z e_3$ we have $dx_0 = \frac{1}{\sqrt{\bar{w}}}(\partial_w w f_1 + \bar{f}_1) d\bar{w} + (\frac{\partial_w}{\sqrt{\bar{w}}} f_1 + e_3) d\bar{z}$, $N_0 = \sqrt{(\partial_w w f_1 + \bar{f}_1) + (\partial_w w f_1 + e_3)}$, and we need $-\sqrt{\bar{w}} w (\partial_w w f_1 + \bar{f}_1) + \bar{w} \partial_w w f_1 + z \partial_w w = 2 \sqrt{\bar{w}} (2 z + \bar{w} \partial_w w z \partial_2 w) = 0$, $w = \partial_w w = 0$. We have $z = \sqrt{\bar{w}} w = 0$, $w = -2 z = \sqrt{\bar{w}} w$, $z = \sqrt{\bar{w}} w$, and $\partial_w w$ solution of the cubic equation $x^3 + 2 \sqrt{\bar{w}} w x^2 + 4 w x + 4 \sqrt{\bar{w}} w x^2 = 0$ in $x$; the condition that this cubic equation has a triple root is over-determined and the condition that of it having two distinct roots (with $\partial_w w$ double root) leads to $x_0$ being the isotropic developable circumscribed to the cone $\sqrt{\bar{w}} w z = 1 \wedge \dot{\bar{w}} = 0$ which extended with the isotropic plane $f_1^T x = 0$ determines the confocal quadrics below.

Otherwise applying $\partial_w$ to this equation we get $\partial_w^2 w = -\frac{2}{\sqrt{\bar{w}}}$, so $w = -\sqrt{\bar{w}} + z F'(\bar{w}) + G(\bar{w})$ and we need $0 = F'(\bar{w}) + F'(\bar{w}) = 2 \sqrt{\bar{w}} + c \sqrt{\bar{w}} G'(\bar{w}) - G(\bar{w})) = -\sqrt{\bar{w}} (c + \sqrt{\bar{w}} G'(\bar{w}) - G(\bar{w}))$, so $c = 0$ and $G(w) = -\frac{2}{2} \sqrt{\bar{w}} + \frac{2}{2} \sqrt{\bar{w}}$; this leads to $x_0$ being one of the confocal Darboux quadrics $x_0^0 T A R_1^0 x_0^0 = 1$, $A := (a_1 I_3 + J_3)^{-1}$, $a_1 \in \mathbb{C} \setminus \{0\}$, we can take $w = 0$ for $x_0$. In this case $A_2 = f_1 (1 - \frac{2}{2} f_1 + \bar{f}_1 + \bar{z}_3 e_3)^T + e_3 f_1^T$, $x_z = -\frac{u_1-v_1}{u_1-v_1} f_1 + \frac{2}{u_1-v_1} e_3$, $z \in \mathbb{C}$ is the isotropic plane $f_1^T x_0 = 0$ with rulings the lines tangent to the conic $2 f_1 x_0 - 3 x_0 = 1$ obtained for $(u_1-z) + (u_1-z) = 0$; this is the isotropic singular quadric of the above considered family of confocal quadrics (obtained for $w = a_1$); as above the leaves are applicable to the Darboux quadric with $A^{-1} = a_1 I_3 + J_3$ discussed above, so this B transformation is the inversion of the previous one.

For $A A T = a_1 e_3 f_1^T$, $a_1 \in \mathbb{C} \setminus \{0\}$ with $x_0 := \sqrt{\bar{w}} e_1 + \sqrt{\bar{w}} e_2 + \sqrt{\bar{w}} e_3$ we need $x_0 \partial_w z + y \partial_y z - z = a_{12} \partial_w \partial_y x_z = a_{12} \partial_y x_z$, so $\partial_w x_z = \partial_y x_z$, $z = x(x+y)$ and $\partial_y x_z$ is a root of the quadratic equation $(x+y+z-a)^2 = 4 a (x+y) (x+y+a (\sqrt{\bar{w}} + \sqrt{\bar{w}})) = 0$, that is the isotropic cones centered at $\pm \sqrt{\bar{w}} e_3$ circumscribed to the circle $x+y + a = 0 \wedge z = 0$, which extended with the isotropic planes $x+y = 0$ for which the above quadratic equation becomes linear) determines the confocal quadrics of revolution below.

Otherwise applying $\partial_w$ to this equation we get $\partial_w^2 x_z = 0$, so $z(x+y) = c(1 + \frac{2}{2} x + y)$, $c \in \mathbb{C} \setminus \{0\}$, which gives the family of confocal quadrics of revolution $x_0^0 T A R_1^0 x_0^0 = 1$, $A := (a_1 I_3 + a_1 e_3 f_1^T)^{-1}$, $a_1 \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C} \setminus \{a_1, a_2 := a_1 + a\}$. In this case $A_2 = (\tilde{\bar{f}}_1 f_1 + \bar{f}_1) f_1^T + \sqrt{\bar{w}} e_3$, $x_z = \frac{1}{u_1-v_1} (x f_1 + \frac{2}{u_1-v_1} f_1) + \sqrt{\bar{w}} e_3$, $z \in \mathbb{C} \setminus \{0\}$, $A_0 = \tilde{f}_1 f_1^T + \sqrt{\bar{w}} e_3$, $x_z = \frac{2}{u_1-v_1} f_1 + \sqrt{\bar{w}} e_3$, $x_0 = \frac{2}{u_1-v_1} f_1 + \sqrt{\bar{w}} e_3$, $x_z = \frac{2}{u_1-v_1} f_1 + \sqrt{\bar{w}} e_3$, $x_z = \frac{2}{u_1-v_1} f_1 + \sqrt{\bar{w}} e_3$ is the pencil of (isotropic) planes containing $e_3$ and the rulings are lines passing through $\pm \sqrt{\bar{w}} e_3$ in their respective planes (these foci are obtained from $x_z(u_1, \infty)$, $x_z(\infty, v_1)$); in this case the isotropic singular quadric of the confocal family is the isotropic planes $f_1^T x_0 = 0$, $x_1^T x_0 = 0$: its singular part is the $e_3$ axis which appears as the singular set of $x_z$, $z \in \mathbb{C} \cup \{\infty\}$ and gives the complementary transformation; however for convenience one can also include the pencil of planes containing $e_3$ in this isotropic
singular quadric (they form a triply orthogonal system with the remaining quadrics of the confocal family).

We have $A^2 \mathcal{A}_x = J_2 + ae_3 e_3^T$, $z \in \mathbb{C} \setminus \{0\}$, $A_0^2 \mathcal{A}_x = A^2 \mathcal{A}_\infty = ae_3 e_3^T$; thus as above for $z = 0, \infty$ the leaves are applicable to another region $x_0$ of $x_0$ and for $z \in \mathbb{C} \setminus \{0\}$ the leaves are applicable to the Darboux quadric with $A^{-1} = a_1 I_3 + J_2 + ae_3 e_3^T$, $a_1 \in \mathbb{C} \setminus \{0, -a\}$ discussed above; thus this B transformation is the inversion of the previous one.

For $\ker(AA^T) = \mathcal{C}_3$ we have $AA^T = a(f_1 f_1^T + f_1 f_1^T) + J_2$, $a \in \mathbb{C} \setminus \{0\}$ or $ae_1 e_1^T + be_2 e_2^T$, $a, b \in \mathbb{C} \setminus \{0\}$.

Since by adding a multiple of $I_3$ to $AA^T$ does not change the procedure, the first case and the second one for $a = b$ have already been discussed for $x_0$; the only change is for $x_z$: we obtain the singular $B_z$ transformation, when the applicability correspondence is given by the Ivory affinity between confocal quadrics $(\sqrt{A})^{-1}X \rightarrow \sqrt{\mathcal{R}_3(\sqrt{A})^{-1}X}$ (which can be extended to an affine correspondence between $x_0$ and the singular $x_z$ since one can take square roots of symmetric matrices with non-isotropic kernels).

For $a \neq b$ with $x_0 =: \sqrt{X} e_1 + \sqrt{Y} e_2 + \sqrt{Z} e_3$ we need $\partial_x z = \frac{a \partial_z z}{b - (a-b) \partial_x z}$ and $\partial_x z$ solution of the cubic equation $(1 + X)(b - (a-b)X) [X + y (\frac{aX}{a-b} - z - a \frac{X}{a-b})] = 0$ in $X$; the condition that this cubic equation has a triple root is over-determined and the condition of it having two distinct roots (with $\partial_x z$ double root) leads to $x_0$ being the isotropic developable circumscribed to one (and thus to all) of the conics $X + y \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C} \setminus \{a_1 := a + a_3, a_2 := b + a_3; a_3 \in \mathbb{C} \setminus \{0, -a, -b\}\}$ we can take $w = 0$ for $x_0$. All confocal quadrics cut the above isotropic developable along 4 isotropic rulings of each ruling family and passing through $(\frac{X}{a+a_3} + \frac{Y}{b+a_3} = 1, z = 0)$ (any 3 rulings of a family individuate the quadric) which intersect in the 12 finite umbilics and in 4 points situated on $C(\infty)$ (Darboux).

We have $A = \sqrt{X} e_1 e_1^T + \sqrt{Y} e_2 e_2^T$, $x_z$ is the singular quadric $x_{a_3}$ of the confocal family (the plane $e_3^T x = 0$) and we have the singular $B_{a_3}$ transformation.

For $\ker(AA^T) = 0$ all cases of quadrics except the (pseudo-)sphere have been covered for $x_0$; using the SVD of $A$ we get $x_z$ quadric confocal to $x_0$ and the usual $B_z$ transformation; the case of the (pseudo-)sphere follows from $dx_0^T x_0 = 0$.

For $x_z$ (isotropic singular) (1)QWC with $A(f_1 f_1^T + f_1 f_1^T)A^T = 0$ we have $A f_1 = 0$; multiplying $A$ on the left with a rotation we get $A e_3 = e_3, a \in \mathbb{C} \setminus \{0\}$ or $f_1$.

In the first case with $x_0 =: \sqrt{X} e_1 + \sqrt{Y} e_2 + ze_3$ we need $x \partial_x z + y \partial_y z - z = \frac{1 \partial z}{\partial_y z} = \frac{1 \partial z}{\partial x z}$, so $z = z(x + y)$ and $\partial_x z$ is solution of the quadratic equation $4(x+y)X^2 - 4zX - 1 - 0$ in $X$; the equation has a double root for the isotropic cone $x + y + z^2 = 0$ which extended with the isotropic planes $x + y = 0$ (for which the above quadratic equation becomes linear) determines the confocal quadrics below.

Otherwise applying $\partial_x$ to this equation we get $\partial_z^2 z = 0$, so $x_0$ is one of the confocal paraboloids of revolution $\frac{x + y}{a_1 - w} = 2z + (a_1 - w), a_1 \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C} \setminus \{a_1\}$; we can take $w = 0$ for $x_0$ (note that to obtain the canonical form for $x_0$ we need to further apply the translation $\frac{z}{\partial z}(e_3)$. In this case we have $A = \sqrt{z} f_1^T + ae_3 e_3^T$, $\sqrt{z} x e_3 \neq 0$, $x_z = v_1(v + au_1e_3)$; by a transformation of $u_1, v_1$ into linear functions of themselves we can make $A = (\frac{z}{a_1} f_1 + \frac{z}{a_1} f_1) f_1 + e_3 e_3^T$, $z = v_1(z f_1 + \frac{z}{a_1} f_1 + u_1 e_3)$, $z \in \mathbb{C} \setminus \{0\}$, $A_0 = f_1 f_1^T + e_3 e_3^T$, $z = 0 = v_1(f_1 + u_1 e_3)$, $A_\infty = f_1 f_1^T + e_3 e_3^T$, $x = 0 = v_1(f_1 + u_1 e_3)$ the pencil of (isotropic) planes containing $e_3$ and the rulings are lines passing through 0 and $\infty e_3$ in their respective planes; in this case the isotropic singular quadric of the confocal family is the isotropic planes $f_1^T x = 0$, $f_1^T x = 0$; its singular part is the $e_3$ axis which appears as the singular set of $x_z, z \in \mathbb{C} \cup \{\infty\}$ and gives the complementary transformation; however for convenience one
can also include the pencil of planes containing $e_3$ in this isotropic singular quadric (they form a triply orthogonal system with the remaining quadrics of the confocal family).

For $x^2 | x_0$ from (1.9) and $(x^2)^T N_0 = (x^2)^T N_0$ we get $4(\partial_{x_1} x^2)^T N_0 (\partial_{x_1} x^2)^T N_0 u_1 \circ dv_1 = -2a_1 du_1 \circ dv_1 = -a_1((f_1 f_1^T + f_1 f_1^T) dZ_1)^2$ and from (1.7) the linear element of the leaves is $|dx_1|^2 = |dx_1|^2 + a_1(f_1 f_1^T + f_1 f_1^T) dZ_1^2 = dZ_1^2 [A_2^T A_2 + a_1(f_1 f_1^T + f_1 f_1^T)] dZ_1^2$; since $A_2 A_2 = J_0 + e_3 e_3^T$ for $z \in \mathbb{C} \setminus \{0\}$ and $e_3 e_3^T$ otherwise the leaves are applicable to a Darboux quadric without center for $z \in \mathbb{C} \setminus \{0\}$ and to another region of $x_0$ otherwise.

The case $A_{e_3} = f_1$ leads to $x_0$ being the plane passing through the origin and having unit normal $N_0 = e_1 f_1 + e_3$, $e_3 \in \mathbb{C}$ or the confocal cone $|x_0|^2 = 0$.

The remaining cases of IQWC should follow by similar computations.

4.2. Rigidity of the Bäcklund transformation of quadrics. Given the defining surface $x_0$ being a quadric, we should get the auxiliary surface $x_z$ as in (4.5).

For $x_0 = (\sqrt{A})^{-1} X(u_0, v_0)$ canonical QC with $Y(v) := -e_1 f_1 + 2f_1 + 2e_3$ the standard parametrization of the rulings of the isotropic cone we have $X(u, v) = \frac{1}{\sigma(Y(v))} Y(v) + \frac{1}{\sigma(Y(v))} Y'(v) = \frac{1}{\sigma(Y(v))} Y(v) - \frac{1}{2} Y'(v), Y(v) \times Y'(v) = 2Y(v)$ and with $y := \sqrt{A} x_z$ from the TC $y^T X(u_0, v_0) = 1$ we get $X = X(u_0, v_0) = (Y(v_0) + Y(v_0) \times y)^T y^T (Y(v_0) + Y(v_0) \times y)$; replacing this into (4.3) and with $M := \partial_u y_0 y_0 y_0^T + \partial_v y_0 y_0^T + y_0^2 \partial_{y_0} y_0^T$ we get $0 = (Y(v_0) + Y(v_0) \times y)^T (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y)^T (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y)^T (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y)) = (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y)) = (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y)) = (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y)) = (y^T Y(v_0) + M + (Y(v_0) + Y(v_0) \times y))$.}

That is two polynomials of degree 6 in $v_0$ are identically 0; this imposes a linear homogeneous system of 14 equations in the variables of $M$ and $\partial_{y_0} y_0^T$ (optionally one can consider $M := \partial_u y_0 y_0 y_0^T + \partial_v y_0 y_0^T + y_0^2 \partial_{y_0} y_0^T$ and we have only 9 variables) with obvious solution $\partial_{y_0} y_0^T = M y, y \in \mathbb{C}$ discussed in (4.4) for $x_z$ is ((isotropic) singular) quadric doubly ruled by degenerate leaves and confocal to $x_0$ (requiring a space of solutions at least 3-dimensional leads to over-determinate conditions on $y$).

For $x_0$ canonical (I)QC we have $x_0 = L Z(u_0, v_0), L := (\sqrt{a_1^{-1}})^{-1} e_1 e_1^T + (\sqrt{a_2^{-1}})^{-1} e_2 e_2^T + e_3 e_3^T, a_1, a_2 \in \mathbb{C} \setminus \{0\}$ for QWC or $x_0 = L e_1 e_1^T + f_1 f_1^T + \frac{1}{2a_1} J_2 + e_3 e_3^T, a_1 \in \mathbb{C} \setminus \{0\}$ for IQWC. In all cases $N_0$ is a multiple of $(L^T)^{-1} (u_0 f_1 + v_0 f_1 - e_3)$ and with $y := \sqrt{L} x_0$ from the TC $y^T (u_0 f_1 + v_0 f_1 - e_3) = u_0 v_0$ we get $Z_0 = Z(u_0, v_0) = (y^T (e_3 - v_0 f_1))(f_1 + v_0 e_3) + v_0 f_1 = (y^T (e_3 - v_0 f_1))(f_1 + v_0 e_3) + v_0 f_1$; replacing this (4.4) with $M := \partial_u y_0 y_0 y_0^T + \partial_v y_0 y_0^T + y_0^2 \partial_{y_0} y_0^T$ we get $0 = (f_1^T v_0 + e_3) T L^T \left[y^T (v_0 f_1 - e_3) + v_0 f_1 \right] + \partial_{y_0} y_0^T \left[y^T (e_3 - v_0 f_1)(f_1 + v_0 e_3)\partial_{y_0} y_0^T + (y^T (e_3 - v_0 f_1)(f_1 + v_0 e_3)\partial_{y_0} y_0^T)ight] + \partial_{y_0} y_0^T \left[y^T (e_3 - v_0 f_1)(f_1 + v_0 e_3)\partial_{y_0} y_0^T + (y^T (e_3 - v_0 f_1)(f_1 + v_0 e_3)\partial_{y_0} y_0^T)\right]\left[y^T (e_3 - v_0 f_1) + (y^T (e_3 - v_0 f_1))\right] + \partial_{y_0} y_0^T \left[y^T (e_3 - v_0 f_1)(f_1 + v_0 e_3)\partial_{y_0} y_0^T + (y^T (e_3 - v_0 f_1)(f_1 + v_0 e_3)\partial_{y_0} y_0^T)\right]\left[y^T (e_3 - v_0 f_1) + (y^T (e_3 - v_0 f_1))\right]$ in $y_0$ and $v_0 \in \mathbb{C}$, that is two polynomials of degree 3 respectively in $u_0, v_0$ are identically 0: this imposes a linear homogeneous system of 12 equations in the 9 entries of $M$ and $\partial_{y_0} y_0^T$ (optionally one can consider $M := \partial_u y_0 y_0 y_0^T + \partial_v y_0 y_0^T + y_0^2 \partial_{y_0} y_0^T$ with obvious solution $\partial_{y_0} y_0^T = M e_3, M := (f_1^T f_1 + f_1^T f_1 - z (L^T)^{-1})$, $z \in \mathbb{C}$ discussed in (4.4) so $x_z$ is ((isotropic) singular) quadric doubly ruled by degenerate leaves and confocal to $x_0$ (requiring a space of solutions at least 3-dimensional leads to over-determinate conditions on $y$).

When making hypothesis on the auxiliary surface $x_z$ being (isotropic) plane or quadric one cannot use directly (4.4), since it is strongly rigid (it assumes the curves given by collapsed leaves).

If $x_z$ is (a) isotropic plane, then we can take $x_1 = 0$ (this case is due to Bianchi [3] or $f_1^T x_2 = 0$. We shall reproduce here Bianchi’s argument, which will also apply to $x_2$ isotropic plane.

Consider $x_0 = x_0(x, y) = x_1 + y_2 + e_3, N_0 := \partial_y z e_1 + \partial_y z e_2 - e_3$ for $e_1^T x_2 = 0$ or $x_0 = y f_1 + x f_1 + e_3, N_0 := \partial_y z f_1 + \partial_y z f_1 - e_3$ for $f_1^T x_2 = 0$; in both cases we have $V := x_z - x_0 = -x \partial_y x_0 + w d y x_0$ and with $m := V \times N_0 + x \Theta N_0$ from (3.6) we get $\Theta^2$ polynomial of
degree at most two in $w$. If $\Theta^2$ is a perfect square (including $a$ priori the case when it does not depend of $w$), then as $w$ varies the facets envelope two lines passing through $x_0$ and two foci in $x_z$ (one of them possibly situated at $\infty$), so the leaves are the lines passing through the two foci in $x_z$; if $\Theta^2$ is not a perfect square (including $a$ priori the case when it depends linearly on $w$), then as $w$ varies the facets envelope a quadratic cone centered at $x_0$ (to see this from the cone $(t, w) \to x_0 + t(c(w) - x_0)$, $c(w) = c_1(w)c_2 + c_2(w)c_3$ or $c_1(w)f_1 + c_2(w)c_3$ having normal fields $m = wv_1 + v_2 + \sqrt{A}v_3 + 2Bw + Cv_3$, $B^2 - AC \neq 0$, $v_1, v_2, v_3$ linearly independent we get $c_1(w) = \frac{(x_0x_0)(m \times \partial_{\mu_0})}{e_1^2(m \times \partial_{\mu_0})}$, $c_2(w) = \frac{(x_0x_0)(m \times \partial_{\mu_0})}{e_1^2(m \times \partial_{\mu_0})}$ or $c_1(w) = i\frac{(x_0x_0)(m \times \partial_{\mu_0})}{f_1^2(m \times \partial_{\mu_0})}$, $c_2(w) = i\frac{(f_1f_2)(m \times \partial_{\mu_0})}{f_1^2(m \times \partial_{\mu_0})}$; in both cases $c(w)$ is a conic), so the leaves are the tangents to a conic in $x_z$; in both cases we have $x_z$ (isotropic) singular quadric doubly ruled by degenerate leaves, so from (4.3) $x_0$ is a confocal quadric to $x_z$ (except for the case when $x_0$ does not admit confocal family).

Given the auxiliary surface $x_z$ being a quadric, we should get the defining surface $x_0$ a confocal quadratic (and thus by the previous argument $x_z$ doubly ruled by degenerate leaves).

More generally given the auxiliary surface $x_z = x_z(s, w)$ one solves the TC for $s = s(x_0, N_0, w)$; replacing this into $x_z$ we get $V = V(x_0, N_0, w)$ and by implicit differentiation we get $\partial_w s = -\frac{\partial x_0^T N_0}{\partial x_0^T N_0}$; as $dV = -\frac{\partial x_0^T N_0}{\partial x_0^T N_0} d\partial_w s$, we get $\partial_w V = \frac{dV}{d\partial_w s}$; from the equation of (4.3) becomes 0 $\partial_w [V^2 dN_0 N_0^{T}(\partial x_0^T N_0 \partial x_0)] + V^T N_0^{T}(\partial x_0^T N_0 \partial x_0) \partial_w V = 0$.

For $x_z = (\sqrt{A})^{-1}X(s, w)$ or $X(w, s)$ canonical QC we have $c_2(w) = (\sqrt{A})^{-1}Y(w)$.

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we get

\[
\frac{\partial^3_{u_1v_1v_1} x^T N_0 - \partial^3_{u_1u_1v_1} x^T N_0 - \partial^3_{u_1x_1v_1} x^T N_0 - (\partial_{v_1} x^T N_0)^2 \partial^2_{u_1x_1} x^T N_0 =}
\]

Differentiating (4.11) with respect to \( u_1 \) we get

\[
0 = (I_3 - N_0 (\partial_{u_1} x^T N_0) \partial_{u_1} x^T N_0(N_0 (\partial_{u_1} x^T N_0) + \partial_{v_1} x^T N_0) -
\]

\[
(3 - N_0 \partial_{u_1} x^T N_0 (\partial_{u_1} x^T N_0) + \partial_{v_1} x^T N_0) - \partial_{v_1} x^T N_0 (\partial_{u_1} x^T N_0) + \partial_{v_1} x^T N_0) - \partial_{v_1} x^T N_0 \partial_{v_1} x^T N_0 - (\partial_{v_1} x^T N_0)^2 \partial^2_{u_1x_1} x^T N_0
\]

gives

\[
V^T N_0 = 0 \Rightarrow \frac{\partial^2_{u_1} x \times \partial_{u_1} x}{(\partial_{u_1} x^T N_0)^3} - \frac{\partial^2_{v_1} x \times \partial_{v_1} x}{(\partial_{v_1} x^T N_0)^3} \times N_0 = 0.
\]

If \( \frac{\partial^2_{u_1} x \times \partial_{u_1} x}{(\partial_{u_1} x^T N_0)^3} - \frac{\partial^2_{v_1} x \times \partial_{v_1} x}{(\partial_{v_1} x^T N_0)^3} \neq 0 \), then along the curve of tangency of the tangent cone of \( x_0 \) from \( x_0(u_1, v_1) N_0 \) spans a line or a plane; since the tangent cone cannot be a cylinder \( x_0 \) must be developable.

Thus \( \frac{\partial^2_{u_1} x \times \partial_{u_1} x}{(\partial_{u_1} x^T N_0)^3} - \frac{\partial^2_{v_1} x \times \partial_{v_1} x}{(\partial_{v_1} x^T N_0)^3} = 0 \); since along the curve of tangency of the tangent cone of \( x_0 \) from \( x_0(u_1, v_1) \) for \( u_1, v_1 \) fixed \( \partial_{u_1} x^T N_0 / \partial_{v_1} x^T N_0 \) is constant, the ratio \( \partial_{u_1} x^T N_0 / \partial_{v_1} x^T N_0 \) cannot be constant, so \( \partial_{u_1} x^T N_0 / \partial_{v_1} x^T N_0 \) is doubly ruled by degenerate leaves.

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