DONOHO-LOGAN LARGE SIEVE PRINCIPLES FOR THE
WAVELET TRANSFORM

LUÍS DANIEL ABREU AND MICHAEL SPECKBACHER

ABSTRACT. In this paper we formulate Donoho and Logan’s large sieve principle for
the wavelet transform on the Hardy space, adapting the concept of maximum Nyquist
density to the hyperbolic geometry of the underlying space. The results provide de-
terministic guarantees for $L_1$-minimization methods and hold for a class of mother
wavelets that constitutes an orthonormal basis of the Hardy space and can be associ-
ated with higher hyperbolic Landau levels. Explicit calculations of the basis functions
reveal a connection with the Zernike polynomials. We prove a novel local reproducing
formula for the spaces in consideration and use it to derive concentration estimates
of the large sieve type for the corresponding wavelet transforms. We conclude with
a discussion of optimality of localization and Lieb inequalities in the analytic case by
building on recent results of Kulikov, Ramos and Tilli based on the groundbreaking
methods of Nicola and Tilli. This leads to a sharp uncertainty principle and a local
Lieb inequality for the wavelet transform.

MSC2020: 42C40, 46E15, 30H10, 42C15, 11N36

Keywords: large sieve principle, wavelet transform, concentration estimates, maximum
Nyquist rate, wavelet coorbit spaces, uncertainty principles

1. Introduction

Let $d\mu^+(z) = \text{Im}(z)^{-2} \, d\mu_{\mathbb{C}^+}(z)$, where $d\mu_{\mathbb{C}^+}(z)$ is the Lebesgue measure in $\mathbb{C}^+$. We
denote the hyperbolic measure of a nonempty set $\Delta \subset \mathbb{C}^+$ by $|\Delta|_h := \int_{\Delta} d\mu^+(z)$ and de-
fine the weighted Lebesgue spaces on $\mathbb{C}^+$ via the norms $\| F \|^p_{L^p(\mathbb{C}^+)} = \int_{\mathbb{C}^+} |F(z)|^p d\mu^+(z)$. Let $0 < R < 1$, and $1 \leq p < \infty$. We will be concerned with inequalities of the type

$$\nu_p(\Delta) = \inf_{f \in \text{Co}L^p(\mathbb{C}^+)} \frac{\int_{\Delta} |W_\psi f(z)|^p d\mu^+(z)}{\int_{\mathbb{C}^+} |W_\psi f(z)|^p d\mu^+(z)} \leq \frac{\rho(\Delta, R)}{C_\psi(R)}.$$  

Here, $W_\psi f$ denotes the continuous analytic wavelet transform of a function $f$ with re-
spect to a wavelet $\psi$, $\text{Co}L^p(\mathbb{C}^+)$ is the wavelet coorbit space with the norm $\| f \|_{\text{Co}L^p(\mathbb{C}^+)} = \| W_\psi f \|_{L^p(\mathbb{C}^+)}$ (see the preliminaries section for precise definitions) and $\rho(\Delta, R)$ is the maximum Nyquist density given by

$$\rho(\Delta, R) := \sup_{z \in \mathbb{C}^+} |\Delta \cap \mathcal{D}_R(z)|_h,$$
where $D_R(z)$ is a disc with respect to the pseudohyperbolic distance on $\mathbb{C}^+$. Sets such that $\rho(\Delta, R) \ll |\Delta|^h$ will be said to be $R$-sparse in the hyperbolic measure.

The problem consists of obtaining the constant $C_\psi(R)$, which is independent of $\Delta$ and grows with $R$. Given such a constant, the estimate (1) shows its full potential for sets $\Delta$ which have low concentration of measure in any hyperbolic disk $D_R(z)$, allowing the use of $L_1$-minimization methods for signal recovery as in [18, 17, 6]. See also [9, 52, 30, 33, 43] for similar concentration estimates.

The inequality (1) will be proven for an orthonormal basis $\{\psi_n^\alpha\}_{n \geq 0}$ of the Hardy space $H^2(\mathbb{C}^+)$ defined via the Fourier transform as

$$\hat{\psi}_n^\alpha(t) := \sqrt{\frac{2^{\alpha+2n}n!}{\Gamma(n+\alpha+1)}} t^{\frac{n}{2}} e^{-t} L_n^\alpha(2t), \quad t > 0,$$

where $L_n^\alpha$, $\alpha > 0$, denotes the generalized Laguerre polynomial of degree $n$,

$$L_n^\alpha(t) = \sum_{k=0}^{n} \frac{(-1)^{k} k!}{n-k} \binom{n+\alpha}{n-k} t^k, \quad t > 0.$$

Note that the normalizing constant for $\psi_n^\alpha$ is chosen such that $\|\hat{\psi}_n^\alpha\|_{L^2(\mathbb{R}_+, t^{-1})}^2 = 4\pi/\alpha$, and $\|\psi_n^\alpha\|_2^2 = 1$. This family was also the central object of [7] where conditions for frames generated by orbits of Fuchsian groups in terms of a Nyquist-rate were obtained. The family $\{\psi_n^\alpha\}_{n \geq 0}$ is connected to the eigenspaces of the Maass Laplacian (see Section 2.2), a Schrödinger operator that is of central importance in number theory due to its role in the theory of Maass forms [39], and Selberg trace formulas [47], as well as in physics, where the pure point spectrum of the Maass Laplacian admits an interpretation as a hyperbolic analogue of the Euclidean Landau levels [14]. It has been shown in [7] that these wavelet eigenspaces are $PSL(2, \mathbb{R})$ invariant (a convenient property to define point processes [2]) and also that, assuming reasonably mild restrictions on $\psi$ (see Theorem 3 in [7]), this is essentially the only choice with such property (see Theorem 3 in [7]). The sequence $\{\psi_n^\alpha\}_{n \geq 0}$ also arises as the best localized functions of Daubechies and Paul’s localization problem in the wavelet domain [16], and have been used as time-scale tapers in spectral estimation [10] under the name of Morse wavelets. It should be noticed that the slightly different choice of wavelets whose Fourier transforms are a constant multiple of $t^{\frac{\alpha+1}{2}} e^{-t} L_n^\alpha(2t)$ (note the different exponent of $t$) leads to the polyanalytic Bergman structure discovered by Vasilevski [53] (see [31, 1, 12] for these special choices in connection to poly-Bergman spaces, which are also fundamental in the theory of the affine Wigner distribution [11] and in commutative algebras of Toeplitz operators [54]). What we would like to emphasize is that, in the STFT case, Hermite functions lead both to the polyanalytic structure and the Euclidean Landau levels [6], but, for the wavelet transform, the choices leading to Vasilevski’s polyanalytic decomposition...
structure [53] are different from those leading to the $\text{PSL}(2,\mathbb{R})$ invariant hyperbolic Landau level eigenspaces.

Our proof of (1) starts with a general Schur-type argument, as in [6]. Then the explicitly defined constant $C_{\psi_\alpha_n}(R) = C_{\alpha_n}(R)$ is computed using the following local reproducing formula for the wavelet coefficients:

$$W_{\psi_\alpha_n} f(z) = \frac{1}{C_{\alpha_n}(R)} \int_{D_R(z)} W_{\psi_\alpha_n} f(w) K_{\psi_\alpha_n}(z, w) d\mu(w), \quad z \in \mathbb{C}^+.$$  

Proving this formula will constitute a significant part of the paper. In [6], a local reproducing formula for the short-time Fourier transform (STFT) with Hermite windows was proven using the well-known correspondence between the STFT and complex Hermite polynomials, which are orthogonal in concentric discs of the plane. Since we could not find a similar correspondence for wavelets in the literature, we have computed the wavelet transforms $W_{\psi_\alpha_n} \psi_\alpha_m$, and the result yields a correspondence (up to a conformal map) to the well known Zernike polynomials in the disc [55], defined in terms of Jacobi polynomials in a fashion reminiscent of the definition of the complex Hermite polynomials using Laguerre functions [22, 32]. This allows us to show the following orthogonality in concentric pseudohyperbolic discs:

$$\int_{D_R} W_{\psi_\alpha_n} \psi_\alpha_m(z) W_{\psi_\alpha_n} \psi_\alpha_k(z) d\mu(z) = C_{\alpha_n}(R) \delta_{m-k}.$$  

The case $n = m$ of this orthogonality relation is then sufficient to show (3). This is a fundamental step in our derivation of the estimates of the maximal Nyquist rate, where the radial nature of the basis functions plays a role. Localization problems have been considered for other transforms (e.g., [41, 40]), but it is not clear how to obtain local reproducing formulas in such settings.

For $p = 1$, $\nu_1(\Delta) < 1/2$ implies that

$$\|W_\psi f \cdot \chi_\Delta\|_{L^1(\mathbb{C}^+)} < \frac{1}{2} \|W_\psi f\|_{L^1(\mathbb{C}^+)},$$

meaning that for every signal $f \in \text{Co} L^1(\mathbb{C}^+)$, $W_\psi f(z)$ is sparse (low concentration on $\Delta$). By generalizing an observation of Donoho and Stark for bandlimited discrete functions [18, Theorem 9], we obtain an interesting reconstruction result. In the absence of noise, if $\nu_1(\Delta) < 1/2$ and we only sense the projection of a general $W_\psi f(z)$, $f \in \text{Co} L^1(\mathbb{C}^+)$ on $\Delta^c$, then $W_\psi f(z)$ can be perfectly reconstructed as the solution of the $L^1$-minimization problem

$$W_\psi f = \arg\min_{h \in \text{Co} L^1(\mathbb{C}^+)} \|W_\psi h\|_{L^1(\mathbb{C}^+)}, \quad \text{subject to } W_\psi h|_{\Delta^c} = W_\psi f|_{\Delta^c}.$$  

Besides this result, condition (4) allows several signal approximation and recovery scenarios by using $L_1$-minimization [6].
A simple example shows that good estimates for \( \nu_p(\Delta) \) can be obtained if \( \Delta \) is sparse in the hyperbolic measure (low concentration in any disc \( \mathcal{D}_R(z) \)). Consider first \( \Delta_1 = \mathcal{D}_R \). Then \( \rho(\Delta_1, R) = |\mathcal{D}_R|_h \). Now if \( \Delta_2 = \mathcal{D}_R \cup (\mathcal{D}_{R+2\delta} - \mathcal{D}_{R+\delta}) \), with \( 0 < \delta < R < 1/2 \), then \( \rho(\Delta_2, R + \delta) = |\mathcal{D}_R|_h \). This gives the example of two sets \( \Delta_1, \Delta_2 \) with \( |\Delta_1|_h < |\Delta_2|_h \) but with an estimate for \( \nu_p(\Delta_2) \) better than the estimate for \( \nu_p(\Delta_1) \), since \( \rho(\Delta_2, R + \delta)/C_\psi(R + \delta) < \rho(\Delta_1, R)/C_\psi(R) \) (because, as we will see, in the estimates of the form (1), \( C_\psi(R) \) is independent of \( \Delta \) and grows with \( R \)). This elementary observation already hints in the direction that motivated Donoho and Logan, as well as our previous work \[6\]: if a set is poorly concentrated in discs of big enough fixed radius \( R \) (‘\( R \)-sparse’), then \( \nu_p(\Delta) < \frac{1}{2} \).

Another interesting setting where the maximum Nyquist rate can be estimated is the hyperbolic Cantor set \[4\]. This has been done in \[37\], Section 3.1 using our concept from \[6\], and a similar calculation applies to the wavelet case. This approach may help in dealing with more general hyperbolic fractal sets, adapting the concept of porous sets from the time-frequency plane \[36\].

The problem of estimating \( \nu_p(\Delta) \) is closely connected to the study of localization operators first introduced by Daubechies \[15\] in the time-frequency domain and by Daubechies and Paul in the wavelet domain \[16\]. An analysis similar to the one in \[6\] shows that our estimates fall short from being sharp. However, in the analytic case \( n = 0 \) it was recently shown by Ramos and Tilli \[48\] that pseudohyperbolic disks maximize \( \nu_2(\Delta) \) among all sets \( \Delta \) of a given hyperbolic measure, using methods from the breakthrough paper \[44\], where the analogue result for the time-frequency case has been obtained. For a discussion of optimal norm bounds for localization operators with non-binary symbols, see \[45, 50\]. We include a section where we derive some direct consequences of the results in \[48\], including a local Lieb inequality, an analogue of Theorem 5.2 in \[44\]. This requires a Lieb-type inequality for the Wavelet transform, that, as we shall see, has recently been proved in a slightly disguised form by Kulikov \[38\], Theorem 1.2.

The outline of the paper is as follows. In Section 2, we gather all the concepts required to follow the paper in a relatively self-contained manner. In Section 3 we illustrate the connection between the basis functions \( W_{\psi_m}^{\alpha_n} \psi_m^\alpha \) and the Zernike polynomials. Then the double orthogonality of these basis functions in concentric pseudohyperbolic discs and the resulting local reproducing formula are proved. With these formulas at hand, we show our large sieve estimates in Section 4. Section 5 is devoted to optimality results in the analytic setting, concluding with a local Lieb uncertainty principle for analytic wavelets and a short discussion about general Lieb inequalities. The computation of the
LARGE SIEVE PRINCIPLES FOR THE WAVELET TRANSFORM

basis functions $W_{\psi}^{a,b}$, the explicit formulas for the reproducing kernels, and a proof of the integrability of the mother wavelets are left to the Appendix.

2. Preliminaries

We use the following convention for the Fourier transform: for $f \in L^1(\mathbb{R})$, its Fourier transform $\hat{f}$ is defined as

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}} f(t)e^{-i\xi t} dt, \quad \xi \in \mathbb{R}.$$  

By standard arguments, the Fourier transform extends to $L^2(\mathbb{R})$ and Plancherel’s formula holds

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \langle \hat{f}_1, \hat{f}_2 \rangle, \quad \text{for } f_1, f_2 \in L^2(\mathbb{R}).$$

We use the basic notation for $H^2(\mathbb{C}^+)$, the Hardy space on the upper half plane, of analytic functions in $\mathbb{C}^+$ with the norm

$$\|f\|_{H^2(\mathbb{C}^+)} = \sup_{0 < s < \infty} \int_{-\infty}^{\infty} |f(x + is)|^2 dx < \infty.$$  

To simplify the computations it is often convenient to use the equivalent definition (since the Paley-Wiener theorem [19] gives $\mathcal{F}(H^2(\mathbb{C}^+)) = L^2(\mathbb{R}^+)$$)

$$H^2(\mathbb{C}^+) = \{ f \in L^2(\mathbb{R}) : (\mathcal{F}f)(\xi) = 0 \text{ for almost all } \xi < 0 \}.$$  

2.1. The Continuous Wavelet Transform.

2.1.1. The affine group. Consider the $ax + b$ group (see [25, Chapter 10] for the listed properties) $G \sim \mathbb{R} \times \mathbb{R}^+ \sim \mathbb{C}^+$ with the multiplication

$$(x, s) \cdot (x', s') = (sx + x', ss').$$

The identification $G \sim \mathbb{C}^+$ is done by setting $(x, s) \sim x + is$. The neutral element of the group is $(0, 1) \sim i$ and the inverse element is given by $(x, s)^{-1} = (-\frac{x}{s}, \frac{1}{s}) \sim -\frac{x}{s} + is$. The $ax + b$ group is not unimodular, since the left Haar measure on $G$ is $\frac{dxds}{s^2}$ and the right Haar measure on $G$ is $dxdx$. Throughout this paper we will follow the convention that for $z, w \in \mathbb{C}^+$, $zw$ denotes the product of two complex numbers and $z \cdot w$ denotes the above group multiplication induced by the affine group. For a set $\Delta \subset \mathbb{C}^+$ we define $\Delta^{-1} := \{ z \in \mathbb{C}^+ : z^{-1} \in \Delta \}$. The left Haar measure of a set $\Delta \subseteq G$

$$|\Delta| = \int_{\Delta} \frac{dxds}{s^2},$$

equals, under the identification of the $ax + b$ group with $\mathbb{C}^+$, the hyperbolic measure

$$|\Delta| = |\Delta|_h := \int_{\Delta} \text{Im}(z)^{-2} d\mu_{\mathbb{C}^+}(z),$$
where $d\mu_{\mathbb{C}^+}(z)$ is the Lebesgue measure in $\mathbb{C}^+$. We will write
\begin{equation}
(5)
d\mu^+(z) = \text{Im}(z)^{-2}d\mu_{\mathbb{C}^+}(z).
\end{equation}

2.1.2. The continuous wavelet transform. For every $x \in \mathbb{R}$ and $s \in \mathbb{R}^+$, define the translation operator $T_x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T_x f(t) = f(t-x)$, and the dilation operator $D_s : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $D_s f(t) = \frac{1}{\sqrt{s}} f(t/s)$. Let $z = x + is \in \mathbb{C}^+$ and define the unitary representation $\pi : G \to \mathcal{U}(H^2(\mathbb{C}^+))$
\begin{equation}
(6)
\pi(z)\psi(t) := T_x D_s \psi(t) = s^{-\frac{1}{2}} \psi(s^{-1}(t-x)), \quad \psi \in H^2(\mathbb{C}^+).
\end{equation}
To properly define the wavelet transform, one needs to take into account that the $ax+b$ group is not unimodular which requires the following additional condition on the integrability of $\psi \in H^2(\mathbb{C}^+)$
\begin{equation}
(7)
0 < \|\mathcal{F}\psi\|_{L^2(\mathbb{R}^+,dt)}^2 =: C_\psi < \infty.
\end{equation}
Functions satisfying (7) are called admissible and the constant $C_\psi$ is the admissibility constant. The continuous analytic wavelet transform of a function $f$ with respect to a wavelet $\psi$ is defined as
\begin{equation}
(8)
W_\psi f(z) = \langle f, \pi(z)\psi \rangle_{H^2(\mathbb{C}^+)} = \frac{1}{\sqrt{s}} \int_\mathbb{R} \tilde{f}(t) \tilde{\psi}\left(\frac{t-x}{s}\right) dt, \quad z = x + is \in \mathbb{C}^+,
\end{equation}
where $\tilde{f}(t) = \lim_{y \to 0^+} f(t + iy)$. Using $\mathcal{F}(H^2(\mathbb{C}^+)) = L^2(\mathbb{R}^+)$, this can also be written (and we will do it as a rule to simplify the calculations) as
\begin{equation}
(9)
W_\psi f(z) = \frac{\sqrt{s}}{2\pi} \int_{\mathbb{R}^+} (\mathcal{F}f)(\xi) e^{ix\xi}(\mathcal{F}\psi)(s\xi) d\xi.
\end{equation}
As proven recently in [29], $W_\psi f(z)$ only leads to analytic (Bergman) phase spaces for a very special choice of $\psi$, but it is common practice to also call it continuous analytic wavelet transform for general wavelets $\psi$. The orthogonality relation
\begin{equation}
(10)
\int_{\mathbb{C}^+} W_{\psi_1} f_1(z) \overline{W_{\psi_2} f_2(z)} d\mu^+(z) = \langle \mathcal{F}\psi_1, \mathcal{F}\psi_2 \rangle_{L^2(\mathbb{R}^+,dt)} \langle f_1, f_2 \rangle_{H^2(\mathbb{C}^+)},
\end{equation}
is valid for all $f_1, f_2 \in H^2(\mathbb{C}^+)$ and $\psi_1, \psi_2 \in H^2(\mathbb{C}^+)$ admissible. Setting $\psi_1 = \psi_2 = \psi$ and $f_1 = f_2$ in (10) then gives
\begin{equation}
(11)
\int_{\mathbb{C}^+} |W_\psi f(z)|^2 d\mu^+(z) = C_\psi \|f\|^2_{H^2(\mathbb{C}^+)},
\end{equation}
that is, the continuous wavelet transform is a multiple of an isometric inclusion $W_\psi : H^2(\mathbb{C}^+) \to L^2(\mathbb{C}^+)$. Setting $\psi_1 = \psi_2 = \psi$ and $f_2 = \pi(z)\psi$ in (10) also shows that for
that we can choose equipped with the natural norm \( \| \cdot \| \).

We will show in Appendix 2.2. Hyperbolic Landau Level Spaces.

\[ W_\psi (H^2 (\mathbb{C}^+)) := \{ F \in L^2 (\mathbb{C}^+, \mu^+) : F = W_\psi f, \ f \in H^2 (\mathbb{C}^+) \} \]

is a reproducing kernel subspace of \( L^2 (\mathbb{C}^+) \) with kernel

\[ K_\psi (z, w) = \frac{1}{C_\psi} \langle \pi (w) \psi, \pi (z) \psi \rangle_{H^2 (\mathbb{C}^+)} = \frac{1}{C_\psi} W_\psi (w^{-1} \cdot z), \]

and \( K_\psi (z, z) = \| \psi \|_2^2 / C_\psi. \) The Fourier transform \( \mathcal{F} : H^2 (\mathbb{C}^+) \to L^2 (\mathbb{R}^+) \) can be used to simplify computations, since for \( z = x + is \) and \( w = x' + is' \)

\[ \langle \pi (w) \psi, \pi (z) \psi \rangle_{H^2 (\mathbb{C}^+)} = \frac{1}{2\pi} \left\langle \pi (w) \psi, \pi (z) \psi \right\rangle_{L^2 (\mathbb{R}^+)} \]

\[ = \frac{1}{2\pi} (ss')^{1/2} \int_{\mathbb{R}^+} \hat{\psi} (s^2 \xi) \hat{\psi} (s \xi) e^{i(x-x')\xi} d\xi. \]

2.1.3. Wavelet coorbit spaces. It is commonly known that the representation \( \pi \) is also integrable, that is, there exist admissible mother wavelets \( \psi \neq 0 \) such that

\[ \int_{\mathbb{C}^+} |\langle \psi, \pi (z) \psi \rangle| d\mu^+ (z) < \infty. \]

We will show in Appendix 6.2 that we can choose \( \psi = \psi_n^\alpha, \alpha > 1, \) here. For such a mother wavelet, we define the space of test functions

\[ \mathcal{H}_1 := \{ f \in H^2 (\mathbb{C}^+) : W_\psi f \in L^1 (\mathbb{C}^+) \}, \]

and denote by \( \mathcal{H}_1' \) its anti-dual space, i.e., the space of antilinear continuous functionals on \( \mathcal{H}_1. \) Then the coorbit space with respect to \( L^p (\mathbb{C}^+), 1 \leq p \leq \infty, \) is defined as

\[ Co L^p (\mathbb{C}^+) := \{ f \in \mathcal{H}_1' : W_\psi f \in L^p (\mathbb{C}^+) \}, \]

equipped with the natural norm \( \| f \|_{Co L^p (\mathbb{C}^+)} := \int_{\mathbb{C}^+} |W_\psi f (z)|^p d\mu^+ (z), 1 \leq p < \infty, \) and the usual modification if \( p = \infty. \) See, e.g., [20, 21, 24] for more information on coorbit space theory. The spaces \( Co L^p (\mathbb{C}^+) \) are Banach spaces for \( 1 \leq p \leq \infty, \) they are independent of the particular choice of mother wavelet \( \psi \in \mathcal{H}_1, \) the reproducing formula (12) extends to \( Co L^p (\mathbb{C}^+), \) and

\[ Co L^1 (\mathbb{C}^+) = \mathcal{H}_1 \subset Co L^2 (\mathbb{C}^+) = H^2 (\mathbb{C}^+) \subset Co L^\infty (\mathbb{C}^+) = \mathcal{H}_1'. \]

2.2. Hyperbolic Landau Level Spaces. We first make an important remark on the transforms \( W_{\psi^n}. \) The spaces \( W_{\psi^n} (H^2 (\mathbb{C}^+)), n \geq 0, \) are not orthogonal for a general choice of \( \alpha \) because the wavelets \( \psi_n^\alpha \) are orthogonal in the Hardy space and not in the
space of admissible wavelets. But if we make the particular choice \( \alpha = 2B - 2n - 1 \), they become orthogonal, that is, for \( n, m \in \{ 0, 1, \ldots, \lfloor B - \frac{1}{2} \rfloor \} \),

\[
\langle W_{\psi_n^{2B-2n-1} f}, W_{\psi_m^{2B-2m-1} g} \rangle_{L^2(C^+)} = \frac{4\pi}{2B - 2n - 1} (f, g)_{H^2(C^+)} \delta_{n,m}, \quad f, g \in H^2(C^+).
\]

This is related to the identification of the spaces \( W_{\psi_n^{2B-2n-1}}(H^2(C^+)) \) as the eigenspaces of the Maass-Landau levels operator with a constant magnetic field \( B \) originally studied in number theory in the theory of Maass forms [39], and the Selberg trace formula [47]. We provide a brief account of this Schrödinger operator from a physical perspective.

The *Schrödinger operator* describing the dynamics of a charged particle moving in \( C^+ \) under the action of the constant magnetic field \( B \) [14] is given by

\[
H_B = -s^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right) + 2iBs \frac{\partial}{\partial x}.
\]

\( H_B \) is an elliptic, and densely defined operator on \( L^2(C^+) \). Its spectrum consists of a continuous part \([1/4, \infty)\) corresponding to *scattering states* and a finite number of eigenvalues

\[
\lambda_n^B = (B - n)(B - n - 1), \quad n = 0, 1, \ldots, \lfloor B - \frac{1}{2} \rfloor.
\]

The eigenvalues exist provided that \( 2B > 1 \), and each eigenvalue corresponds to an infinite dimensional reproducing kernel Hilbert space \( E_n^B(C^+) \). From a physical viewpoint, this condition guarantees that the magnetic field is strong enough to capture particles in a closed orbit, see [26, p. 189]. The eigenspaces corresponding to \( \lambda_n^B \) are called *hyperbolic Landau levels* and we define them by

\[
E_n^B(C^+) := \{ F \in L^2(C^+) : H_B F = \lambda_n^B F \}.
\]

In [2, Section 4.5] it is shown in detail (this connection was first observed by Mouayn [42]) that

\[
W_{\psi_n^{2B-2n-1}}(H^2(C^+)) = E_n^B(C^+).
\]

This way we obtain an orthogonal basis for all the spaces \( E_n^B(C^+) \), by noting that for \( n, m \in \{ 0, 1, \ldots, \lfloor B - \frac{1}{2} \rfloor \} \), (10) gives

\[
\langle W_{\psi_n^{2B-2n-1} \psi_k^0}, W_{\psi_m^{2B-2m-1} \psi_l^0} \rangle_{L^2(C^+)} = \frac{2}{2B - 2n - 1} \delta_{n,m} \delta_{k,l}.
\]

Our large sieve inequalities for this case provide conditions allowing for the recovery of a continuous coherent state in higher hyperbolic Landau levels, from partial information, using \( L_1 \)-minimization, as outlined in the introduction.
2.3. Pseudohyperbolic Metric and Möbius Transformation. The pseudohyperbolic distance on $\mathbb{C}^+$ is given by

$$\rho(z, w) := \left| \frac{z - w}{z - iw} \right|, \quad z, w \in \mathbb{C}^+, \quad 0 < |z|^2 < 1,$$

and the pseudohyperbolic disk of radius $R > 0$ centered at $z \in \mathbb{C}^+$ is defined by $D_R(z) := \{\omega \in \mathbb{C}^+ : \rho(z, \omega) < R\}$. We write for short $D_R = D_R(i)$. Note that $\rho$ only takes values in the half open interval $[0, 1)$. It can be checked that the pseudohyperbolic distance is symmetric and invariant under the action of the affine group. This leads to several useful properties, like $\rho(z, w) = \rho(z - i \cdot w, iz)$ and $\rho(u, z \cdot w) = \rho(z^{-1} \cdot u, w)$.

Let $D_R$ denote the disk of radius $R > 0$ in $\mathbb{C}$. We write for short $D := D_1$ to denote the unit disk. The mapping $T : D \to \mathbb{C}^+$, $T(u) := \frac{1 + u}{1 - u}$, maps the pseudohyperbolic distance in $\mathbb{C}^+$ to the pseudohyperbolic distance in $D$:

$$\rho(T(z), T(w)) = \left| \frac{w - z}{1 - zw} \right| = \rho_D(z, w).$$

The mapping $T$ will be used to map some integrals from the upper half plane to the unit disk according to the change of variables formula

$$\int_{\mathbb{C}^+} F(z) s^\beta d\mu_{\mathbb{C}^+}(z) = \int_D F(T(u)) \frac{4(1 - |u|^2)^\beta}{|1 - u|^2 \beta + 4} du, \quad \beta \in \mathbb{R}, \quad z = x + is.$$

3. Double Orthogonality and the Local Reproducing Formula

The following family of polynomials defined on $[0, 1]$ is closely related to the Zernike polynomials [55], a family of orthogonal polynomials in $(\alpha, \beta)$ defined on $D$,

$$Z_{n,m}^{\alpha}(t) := \sqrt{\frac{\Gamma(\max(n,m) + \alpha + 1) \min(n,m)!}{\Gamma(\min(n,m) + \alpha + 1) \max(n,m)!}} (-t)^{-\min(n,m)} P^{(n-m, \alpha)}_{\min(n,m)} (1 - 2t),$$

where $P^{(\alpha, \beta)}_n$ denotes the Jacobi polynomials which are given explicitly in (40). We will see in this section that

$$Z_{n,m}^{\alpha} \left( \left| \frac{z - i}{z + i} \right|^2 \right), \quad z \in \mathbb{C}^+, \quad \alpha > 0,$$

plays a role in wavelet analysis similar to the one of complex Hermite polynomials in time-frequency analysis [6]. The following result is proved in the Appendix.

**Proposition 1.** Let $\alpha > 0$, and $z = x + is$ and $w = x' + is'$ be in $\mathbb{C}^+$. For every $n \in \mathbb{N}_0$, one has

$$C_{\psi_n^\alpha} = \frac{4\pi}{\alpha},$$
and
\begin{equation}
K_{\psi}(z, w) = \frac{\alpha}{4\pi} \left( \frac{w - z}{\overline{w} - \overline{z}} \right)^n \left( \frac{2\sqrt{ss'}}{i(w - \overline{z})} \right)^{\alpha + 1} \frac{P_n^{(0, \alpha)}}{(1 - 2 \left| \frac{z - w}{\overline{w} - \overline{z}} \right|^2)}.
\end{equation}
Moreover, for \( n, m \in \mathbb{N}_0 \), it holds
\begin{equation}
W_{\psi}(z) = \left( \frac{z - i}{z + i} \right)^m \left( \frac{z + i}{z - i} \right)^n \left( \frac{2\sqrt{s}}{1 - iz} \right)^{\alpha + 1} \frac{Z_{n,m}^\alpha \left( \frac{z - i}{z + i} \right)^2}{(1 - 2 \left| \frac{z - w}{\overline{w} - \overline{z}} \right|^2)}.
\end{equation}

Remark 2. For \( n = 0 \), i.e. the case of the Cauchy wavelet, one has that the Jacobi polynomial \( P_n^{(0, \alpha)} \) is constantly one. Consequently,
\begin{equation}
\Phi_m(z) := \sqrt{\frac{\Gamma(m + \alpha + 1)}{\Gamma(\alpha + 1)m!}} \left( \frac{z - i}{z + i} \right)^m \left( \frac{2\sqrt{s}}{1 - iz} \right)^{\alpha + 1} Z_{n,m}^\alpha \left( \frac{z - i}{z + i} \right)^2,
\end{equation}
which are (up to a constant factor due to different normalizations) the basis functions of the Bergman space \( A_{\alpha}^2(\mathbb{C}^+) \), where
\[ A_{\alpha}^p(\mathbb{C}^+) := \{ F \text{ holomorphic}: \|F\|_{A_{\alpha}^p(\mathbb{C}^+)} < \infty \}, \]
and
\begin{equation}
\|F\|_{A_{\alpha}^p(\mathbb{C}^+)}^p := \int_{\mathbb{C}^+} |F(z)|^p s^\alpha d\mu_{\mathbb{C}^+}(z),
\end{equation}
see, e.g., [3, Section 4.4].

In the following, we show a local orthogonality relation for the family \( \{W_{\psi}(z)\}_{m \in \mathbb{N}_0} \) and subsequently derive a local reproducing formula.

**Theorem 3.** Let \( \alpha > 0 \), \( 0 < R < 1 \), and \( n, m, k \in \mathbb{N}_0 \). Then it holds
\begin{equation}
\int_{D_R} W_{\psi}(z) W_{\psi}(z) d\mu^+(z) = C_{m,n}^\alpha (R) \delta_{m-k},
\end{equation}
with
\[ C_{m,n}^\alpha (R) = 4\pi \int_0^{R^2} r^{n+m} (1 - r)^{\alpha - 1} Z_{n,m}^\alpha (r^2) dr. \]
If \( n = m \), we write \( C_n^\alpha (R) = C_{n,n}^\alpha (R) \) which is given explicitly by
\begin{equation}
C_n^\alpha (R) = 4\pi \int_0^{R^2} (1 - r)^{\alpha - 1} P_n^{(0, \alpha)} (1 - 2r)^2 dr.
\end{equation}

**Proof:** First we note that putting \( z = T(u) \) in (21) yields
\begin{equation}
W_{\psi}(T(u)) = (-1)^{n+m+\frac{\alpha+1}{2}} u^{n+m} \left( \frac{1 - u}{|1 - u|} \right)^{2n+\alpha+1} (1 - |u|^2)^{\frac{\alpha+1}{2}} Z_{n,m}^\alpha (|u|^2).
\end{equation}
Using (17) for \( \beta = -2 \) then leads us to
\[
\int_{D_R} W_{\psi_n}^{\alpha}(z) W_{\psi_n}^{\alpha}(z) d\mu^+(z) = \int_{D_R} W_{\psi_n}^{\alpha}(T(u)) W_{\psi_n}^{\alpha}(T(u)) \frac{4du}{(1 + |u|^2)^2} = 4 \int_{D_R} u^m |u|2n(1 + |u|^2)^{-1} Z_{n,m}^{\alpha}(|u|^2) Z_{n,k}^{\alpha}(|u|^2) du = \odot.
\]
Integration using polar coordinates then yields
\[
\odot = 4 \int_0^{2\pi} e^{i(m-k)\varphi} r^{m+k+2n+1} (1 - r^2)^{-1} Z_{n,m}^{\alpha}(r^2) Z_{n,k}^{\alpha}(r^2) dr d\varphi
= 8\pi \delta_{m-k} \int_0^R r^{2(n+m)+1} (1 - r^2)^{-1} Z_{n,m}^{\alpha}(r^2) dr
= 4\pi \delta_{m-k} \int_0^R r^{n+m}(1 - r)^{-1} Z_{n,m}^{\alpha}(r^2) dr,
\]
which concludes the proof once we recall the definition of \( Z_{n,m}^{\alpha} \) in (18).

Let us shortly consider the case \( n = 0 \):
\[
C_0^{\alpha}(R) = 4\pi \int_0^R (1 - r)^{-1} = \frac{4\pi}{\alpha} (1 - (1 - R^2)^{-\alpha}),
\]
which converges to \( C_0^{\psi_0} = 4\pi/\alpha \) (as \( R \to 1 \)) as one expects from (10).

**Theorem 4.** Let \( n \in \mathbb{N}_0, \alpha > 1, \) and \( 0 < R < 1 \). The following identity holds in the weak sense
\[
\int_{D_R} W_{\psi_n}^{\alpha}(z) \pi(z) \psi_n^{\alpha} d\mu^+(z) = C_n^{\alpha}(R) \psi_n^{\alpha}.
\]
If \( f \in Co L^\infty(C^+) \), then the wavelet coefficients can locally be reconstructed via
\[
W_{\psi_n} f(z) = \frac{4\pi}{\alpha C_n^{\alpha}(R)} \int_{D_R(z)} W_{\psi_n} f(w) K_{\psi_n}(z, w) d\mu^+(w), \quad z \in C^+.
\]
**Proof:** Setting \( m = n \) in (24) shows that (27) holds weakly. If we apply \( \pi(w) \) on both sides of (27) and subsequently take the inner product with \( f \), we obtain
\[
W_{\psi_n} f(z) = \langle f, \pi(z) \psi_n^{\alpha} \rangle
= \frac{1}{C_n^{\alpha}(R)} \int_{D_R} \langle f, \pi(z \cdot w) \psi_n^{\alpha} \rangle \langle \pi(w) \psi_n^{\alpha}, \psi_n^{\alpha} \rangle d\mu^+(w)
= \frac{1}{C_n^{\alpha}(R)} \int_{D_R(z)} \langle f, \pi(w) \psi_n^{\alpha} \rangle \langle \pi(z^{-1} \cdot w) \psi_n^{\alpha}, \psi_n^{\alpha} \rangle d\mu^+(w)
\]
\[
\frac{1}{C_n(R)} \int_{D_R(z)} \langle f, \pi(w)\psi_n, \pi(z)\psi_n \rangle d\mu^+(w) \\
= \frac{C_{\psi_n}}{C_n(R)} \int_{D_R(z)} W \psi_n f(w)K_{\psi_n}(z,w)d\mu^+(w),
\]
where we used \(g(z, z \cdot w) = g(i, w),\) and the left invariance of the Haar measure. The second equality holds since the integral and the duality pairing may be interchanged knowing that \(f \in C_oL^\infty(C^+),\) and \(K_{\psi_n}(z, \cdot) \in L^1(C^+).\) The result follows once we recall that \(C_{\psi_n} = \frac{4\pi}{\alpha}.\)

\[\square\]

4. LARGE SIEVE ESTIMATES

4.1. A Schur-type estimate. Let \((X, \mu), (X, \nu)\) be measure spaces and \(B_1 \subset L^1(X, \nu)\) a Banach space. In [6, Proposition 1], a bound on the embedding \((B_1, \| \cdot \|_{L^1_\nu}) \hookrightarrow (B_1, \| \cdot \|_{L^1_\mu})\) was derived using an argument similar to Schur’s test which we shortly recall here.

**Lemma 5.** Let \(\mu\) be a positive \(\sigma\)-finite measure on \(X, B_1 \subset L^1(X, \nu)\) be a Banach space, and \(K : X \times X \to \mathbb{C}\) be such that \(K : B_1 \to B_1, KF(x) := \int_X F(y)K(x, y)\) is bounded and boundedly invertible on \(B_1.\) Then, for every \(F \in B_1,\) we have

\[
\frac{\int_X |F|d\mu}{\int_X |F|d\nu} \leq \theta(K) \sup_{y \in X} \int_X |K(x, y)|d\mu(x),
\]
where

\[
\theta(K) := \sup_{H \in B_1} \left( \frac{\|H\|_{L^1_\nu}}{\|KH\|_{L^1_\mu}} \right).
\]

4.2. Estimates with explicit constants. Recall that the maximum Nyquist density is given by

\[
\rho(\Delta, R) := \sup_{z \in C^+} |\Delta \cap D_R(z)|_h,
\]
where \(|\Delta|_h := \int_\Delta d\mu^+(z)\) denotes hyperbolic measure of \(\Delta.\) We will also use a second notion of density given by

\[
D_n^\alpha(\Delta, R) := \sup_{z \in C^+} \int_{\Delta \cap D_R(z)} |\langle \pi(w)\psi_n, \pi(z)\psi_n \rangle|d\mu^+(w).
\]
Note that \(D_n^\alpha(\Delta, R) \leq \rho(\Delta, R),\) as

\[
|\langle \pi(w)\psi_n, \pi(z)\psi_n \rangle| \leq \|\psi_n\|^2_{H^2(C^+)} = 1.
\]
We have gathered all the ingredients to formulate and prove our main result.
Theorem 6. Let $\Delta \subset \mathbb{C}^+$, $f \in CoL^p(\mathbb{C}^+)$, $1 \leq p < \infty$, and $\alpha > 1$. For every $0 < R < 1$, it holds
\[
\frac{\|W_\psi f\|_{L^p(\mathbb{C}^+)}^p}{\|W_\psi f\|_{L^p(\mathbb{C}^+)}^p} \leq \frac{D^\alpha_n(\Delta, R)}{C^\alpha_n(R)} \leq \frac{\rho(\Delta, R)}{C^\alpha_n(R)}.
\]

Proof: We take $K(z, w) := \langle \pi(w)\psi_\alpha^\alpha, \pi(z)\psi_\alpha^\alpha \rangle_{C_D R(z)}(w)$ and $B_1 = W_\psi (CoL^1(\mathbb{C}^+))$ in Lemma 5. Then by Theorem 4 we get
\[
\theta(K) = \sup_{f \in CoL^1(\mathbb{C}^+)} \left( \frac{\|W_\psi f\|_{L^1(\mathbb{C}^+)}^p}{\|W_\psi f\|_{L^1(\mathbb{C}^+)}^p} \right) = 1.
\]
Thus, if $d\mu(z) = \chi_{\Delta}(z)d\mu^+(z)$, we have by (29) and (31)
\[
\frac{\|W_\psi f\|_{L^1(\mathbb{C}^+)}^p}{\|W_\psi f\|_{L^1(\mathbb{C}^+)}^p} \leq \frac{1}{C^\alpha_n(R)} \sup_{z \in \mathbb{C}^+} \int_{\Delta \cap D_R(z)} |\langle \pi(w)\psi_\alpha^\alpha, \pi(z)\psi_\alpha^\alpha \rangle_{C_D R(z)}(w)| d\mu^+(w) \leq \frac{\rho(\Delta, R)}{C^\alpha_n(R)}.
\]
The result thus holds for $p = 1$. For $p = \infty$ one trivially has
\[
\sup_{f \in CoL^\infty(\mathbb{C}^+)} \frac{\|W_\psi f\|_{L^\infty(\mathbb{C}^+)}^p}{\|W_\psi f\|_{L^\infty(\mathbb{C}^+)}^p} = 1.
\]
Since the coorbit space $CoL^p(\mathbb{C}^+)$, $1 < p < \infty$, is the complex interpolation space $(CoL^\infty(\mathbb{C}^+), CoL^1(\mathbb{C}^+))_{1/p}$, see [20, Theorem 4.7], the result for $1 < p < \infty$ follows from an application of [13, Theorem 4.1.2].

Corollary 7. Let $\alpha > 1$, and suppose that $f \in CoL^p(\mathbb{C}^+)$, $1 \leq p < \infty$, satisfies $\|W_\psi f\|_{L^p(\mathbb{C}^+)} = 1$, and that $W_\psi f$ is $\varepsilon$-concentrated on $\Delta \subset \mathbb{C}^+$, i.e.,
\[
1 - \varepsilon \leq \int_{\Delta} |W_\psi f(z)|^p d\mu^+(z).
\]
Then
\[
1 - \varepsilon \leq \inf_{0 < R < 1} \left( \frac{\rho(\Delta, R)}{C^\alpha_n(R)} \right) \leq |\Delta|_h.
\]

Remark 8. Let us introduce the Bergman spaces on the unit disk by
\[
A_w^p(\mathbb{D}) := \left\{ f \text{ analytic} : \|f\|_{A_w^p}^p := \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dz < \infty \right\}.
\]
We derived estimates similar to Theorem 6 for the Bergman space $A_w^{p-2}(\mathbb{D})$ in [5, Theorem 2]. There Seip’s double orthogonality result [51] was applied to obtain the concentration estimate.

5. Optimal bounds in the analytic setting

The following result was proved recently for the case $p = 2$ in [48]. We note here that the case of general $p$ works with exactly the same arguments as Ramos and Tilli used.
with one extra information: \( \text{Co} L^p(\mathbb{C}^+) \) can be mapped isometrically and surjectively onto a certain Bergman space.

**Theorem 9** (Ramos-Tilli). Let \( 1 < p < \infty, \Delta \subset \mathbb{C}^+ \) be a set of finite hyperbolic measure \( A > 0 \), and \( \alpha > 1 \). Then

\[
\sup_{|\Delta|_h = A} \sup_{f \in \text{Co} L^p(\Delta)} \frac{\|W_{\psi_0} f\|_{L^p(\Delta)}}{\|W_{\psi_0} f\|_{L^p(\Delta)}}
\]

is attained if and only if \( \Delta \) is a pseudohyperbolic disc centered at some \( z^* \in \mathbb{C}^+ \), up to perturbations of Lebesgue measure zero, and \( f = \pi(z^*)\psi_0^\alpha \). In particular,

\[
\int_\Delta |W_{\psi_0} f(z)|^p d\mu^+(z) \leq \left(1 - \left(1 + \frac{|\Delta|_h}{4\pi}\right)^{1-(\alpha+1)p/2}\right) \|W_{\psi_0} f\|_{L^p(\Delta)}^p,
\]

To prove the result for general \( p \), it just takes a simple observation. As in [48] one can map the space \( A^p_{(\alpha+1)p/2-2}(\mathbb{C}^+) \) to \( A^p_{(\alpha+1)p/2-2}(\mathbb{D}) \) (see definitions (23) and (33)) by virtue of the transformation

\[
T_\alpha F(u) = F\left(\frac{1+u}{1-u}\right) \left(\frac{1}{1-u}\right)^{\alpha+1}, \quad u \in \mathbb{D},
\]

which, by (17), is a multiple of an isometry. Thus, the composition \( T_\alpha \circ s^{-\frac{\alpha+1}{2}}W_{\psi_0} : \text{Co} L^p(\mathbb{C}^+) \to A^p_{(\alpha+1)p/2-2}(\mathbb{D}) \)

\[
\text{Co} L^p(\mathbb{C}^+) \xrightarrow{s^{-\frac{\alpha+1}{2}}W_{\psi_0}} A^p_{(\alpha+1)p/2-2}(\mathbb{C}^+) \xrightarrow{T_\alpha} A^p_{(\alpha+1)p/2-2}(\mathbb{D}),
\]

is surjective. To see this, we note that by (22) the sequence \( \{\psi_m^\alpha\}_{m \in \mathbb{N}_0} \) (which is complete in \( \text{Co} L^p(\mathbb{C}^+) \)) is first mapped to the sequence

\[
s^{-\frac{\alpha+1}{2}}W_{\psi_0}\psi_m^\alpha(z) = \sqrt{\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)m!}} \left(\frac{z-i}{z+i}\right)^m \left(\frac{2}{1-iz}\right)^{\alpha+1}, \quad m \in \mathbb{N}_0, \quad z = x + is,
\]

which is in turn mapped to the sequence of monomials. The completeness of the monomials in \( A^p_{(\alpha+1)p/2-2}(\mathbb{D}) \), \( 1 < p < \infty \), is proven, for example, in [56, Corollary 4 and Theorem 16]. Therefore, just as in [48, Theorem 3.1], the problem of optimal concentration in (34) reduces to optimal concentration on the Bergman space \( A^p_{(\alpha+1)p/2-2}(\mathbb{D}) \). In fact, the result can be proven for \( A^p_\beta(\mathbb{D}), \beta > 0 \), without any dependence of \( \beta \) on \( p \). To do so, one may follow the proof of [48, Theorem 3.1] step by step. We do not repeat the argument here as this would exceed the scope of this paper. There are however only two steps that need to be adapted. First, the definition of the auxiliary function \( u \) (see [48, p. 8]) needs to be changed to \( u(z) := |F(z)|^p(1-|z|^2)^{\beta+2}, F \in A^p_\beta(\mathbb{D}) \). Second, the
proper analogue of equation (3.9) in [48] can be found in [27, Lemma 3.2]:

\[ u(z) \leq \frac{\beta + 1}{\pi} \| F \|_{A^p_\beta}(\mathbb{D}), \quad z \in \mathbb{D}. \]

Note that we use a different normalization of the area measure on \( \mathbb{D} \) than [27]. The optimal concentration bound is attained if equality in (35) holds for some \( z^* \in \mathbb{D} \).

Using the correspondence between wavelet coorbit spaces and Bergman spaces as above (choosing \( \beta = (\alpha + 1)p/2 - 2 \)), we define

\[ F_{z^*}(z) = (1 - |z|^2)^{-\beta+2} (\pi(T(z^*))\psi_0^{(\beta+1)2/p-1}, \pi(T(z))\psi_0^{(\beta+1)2/p-1}) \in A^p_\beta(\mathbb{D}) \]

which satisfies

\[
\|u\|_\infty = \sup_{z \in \mathbb{D}} |F_{z^*}(z)| P(1 - |z|^2)^{\beta+2} = \sup_{z \in \mathbb{D}} |\langle \pi(T(z^*))\psi_0^{(\beta+1)2/p-1}, \pi(T(z))\psi_0^{(\beta+1)2/p-1} \rangle|^p \\
= 1 = \frac{\beta + 1}{\pi} \| F_{z^*} \|_{A^p_\beta}(\mathbb{D}),
\]

where we used that

\[
\frac{\pi}{\beta + 1} = 4^{-1} \| \langle \pi(T(z^*))\psi_0^{(\beta+2)2/p-1}, \pi(\cdot)\psi_0^{(\beta+2)2/p-1} \rangle \|_{L^p(\mathbb{C}^+)} = \| F_{z^*} \|_{A^p_\beta}(\mathbb{D}),
\]

which follows from left-invariance of the Haar measure, (17), and (42). This concludes the proof.

If \( W_{\psi^*_0} f \) is \( \varepsilon \)-concentrated on \( \Delta \subset \mathbb{C}^+ \), i.e.,

\[
(1 - \varepsilon)\| W_{\psi^*_0} f \|_{L^p(\mathbb{C}^+)} \leq \int_\Delta |W_{\psi^*_0} f(z)|^p d\mu^+(z),
\]

then (34) yields

\[
1 - \varepsilon \leq 1 - \left( 1 + \frac{|\Delta|}{4\pi} \right)^{1-(\alpha+1)p/2}.
\]

This leads to the following sharp uncertainty principle for the wavelet transform.

**Theorem 10.** Let \( \alpha > 1 \), \( \varepsilon \in (0,1) \) and \( 1 < p < \infty \). Suppose that \( f \in Co L^p(\mathbb{C}^+) \) satisfies \( \| W_{\psi^*_0} f \|_{L^p(\mathbb{C}^+)} = 1 \), and that \( W_{\psi^*_0} f \) is \( \varepsilon \)-concentrated with respect to the \( p \)-norm in \( \Delta \subset \mathbb{C}^+ \). Then

\[
4\pi \left( e^{2-(\alpha+1)p} - 1 \right) \leq |\Delta| h.
\]

**Remark 11.** Extending these optimal results to the general family of wavelets \( \psi^*_n \) remains an open problem (as well as the corresponding optimal bound for general Hermite functions in [6]). The methods of [44] and [48, Theorem 3.1] are extremely dependent on the analytic structure of the case \( n = 0 \), and for \( n > 0 \) the transforms are not related to analytic functions. However, since a derivative of an analytic function is still analytic, the methods of [44] and [48, Theorem 3.1] can be adapted to obtain sharp inequalities for weighted \( L^p \)-norms of derivatives of holomorphic functions. In [34], combined with sharp
pointwise inequalities for derivatives in Fock spaces (involving Laguerre functions with complex argument), this has been used to obtain a contraction inequality for derivatives in Fock spaces. It is reasonable to expect a related inequality for derivatives in Bergman spaces. This requires sharp pointwise inequalities for derivatives in Bergman spaces, which possibly involve Jacobi polynomials with complex argument.

5.1. Lieb’s uncertainty principle. The following result is due to Kulikov [38, Theorem 1.2]. Note that the normalization of the measure on $\mathbb{D}$ is chosen such that $\|1\|_{A_\alpha^p(\mathbb{D})} = \frac{\pi}{\alpha+1}$ (as defined in (33)).

**Theorem 12** (Kulikov). Let $G : [0, \infty) \to \mathbb{R}^+$ be a convex function. Then for every $f \in A_{\alpha-2}^p(\mathbb{D})$ with $\|f\|_{A_{\alpha-2}^p(\mathbb{D})} = \frac{\pi}{\alpha+1}$ it holds

$$\int_{\mathbb{D}} G \left(|f(z)|^p(1-|z|^2)^\alpha \right)(1-|z|^2)^{-2}dz \leq \int_{\mathbb{D}} G \left((1-|z|^2)^\alpha \right)(1-|z|^2)^{-2}dz,$$

and equality is attained for $f \equiv 1$.

The next theorem is simply a restatement of Kulikov’s theorem for a particular choice of $G$, but we would like to highlight this case since this is the first instance of a counterpart of Lieb’s inequality for the short-time Fourier transform. From the experience in time-frequency analysis, this inequality should have many applications (as noticed in the comments after Corollary 1.2 in [45], Lieb’s inequality is equivalent to sharp norms for localization operators; for applications in signal analysis see [8] and [49]). It will also be essential in the derivation of the local Lieb formula in the next section.

**Theorem 13** (Lieb’s inequality for analytic wavelets). Let $\alpha > 1$, $p \geq 2$, and $f \in H^2(\mathbb{C}^+)$. Then

$$(36) \quad \|W_{\psi_0^\beta}^\alpha f\|_{L^p(\mathbb{C}^+)}^p \leq \frac{8\pi}{(\alpha+1)p - 2}\|f\|_{H^2(\mathbb{C}^+)}^p,$$

with the inequality being sharp.

**Proof:** If we set $\alpha = \beta + 1$ and define

$$F(z) = s^{-(\beta/2+1)} \left(\|W_{\psi_0^\beta}^\alpha f\|_{L^2(\mathbb{C}^+)}\right)^{-1} W_{\psi_0^\beta}^\alpha f(z),$$

then $F$ is analytic and $\|F\|_{A_2^\beta(\mathbb{C}^+)} = 1$. Moreover, taking $G(t) = t^{p/2}$ (which is convex for $p \geq 2$) in Theorem 12 and applying (17) shows

$$\frac{\|W_{\psi_0^\beta}^\alpha f\|_{L^p(\mathbb{C}^+)}^p}{\|W_{\psi_0^\beta}^\alpha f\|_{L^2(\mathbb{C}^+)}^p} = \int_{\mathbb{C}^+} |F(z)|^p s^{p/2-p-2}d\mu_{\mathbb{C}^+}(z)$$

$$= 4 \left(\frac{\beta+1}{4\pi}\right)^{\frac{\beta}{2}} \int_{\mathbb{D}} \left(\frac{4\pi}{\beta+1}|F(T(u))|^2|1-u|^{-(2\beta+4)}(1-|u|^2)^{\beta+2}\right)^{\frac{p}{2}} (1-|u|^2)^{-2}du.$$
\[
\leq 4 \left( \frac{\beta + 1}{4\pi} \right)^{\frac{p}{2}} \int_\mathbb{D} (1 - |u|^2)^{\beta p/2 + p - 2} du = 4\pi \left( \frac{\beta + 1}{4\pi} \right)^{\frac{p}{2}} \int_0^1 r^{\beta p/2 + p - 2} dr \\
= \frac{4\pi}{p(\beta/2 + 1) - 1} \left( \frac{\beta + 1}{4\pi} \right)^{\frac{p}{2}}.
\]

Note that we were allowed to apply Theorem 12 as
\[
\left\| \frac{4\pi}{\beta + 1} F(T(u))^2 (1 - u)^{-(2\beta + 4)} \right\|_{A^1_\beta(\mathbb{D})} = \frac{4\pi}{\beta + 1} \int_\mathbb{D} |F(T(u))|^2 |1 - u|^{-(2\beta + 4)} (1 - |u|^2)^{\beta} du \\
= \frac{\pi}{\beta + 1} \int_{\mathbb{C}^+} |F(z)|^2 s^{\beta} dz = \frac{\pi}{\beta + 1}.
\]
Substituting back \( \alpha = \beta + 1 \) and using \( ||W_\psi f||^2_{L^2(\mathbb{C}^+)} = ||f||^2_{H^2(\mathbb{C}^+)} ||\hat{\psi}||^2_{L^2(\mathbb{R}^+,t^{-1})} \) yields
\[
||W_{\psi_0} f||^p_{L^p(\mathbb{C}^+)} \leq \frac{4\pi}{p(\alpha + 1)/2 - 1} \left( \frac{\alpha}{4\pi} \right)^{p/2} ||f||^p_{H^2(\mathbb{C}^+)} ||\hat{\psi}_0||^p_{L^2(\mathbb{R}^+,t^{-1})} \\
= \frac{4\pi}{p(\alpha + 1)/2 - 1} ||f||^p_{H^2(\mathbb{C}^+)}.
\]
where we used that \( ||\hat{\psi}_0||^2_{L^2(\mathbb{R}^+,t^{-1})} = 4\pi/\alpha \).

5.2. Local Lieb’s uncertainty principle. We conclude our considerations on sharp concentration inequalities with the following immediate consequence of the results of the last two subsections. This is the wavelet analogue of Theorem 5.2 in [44], but we provide a more direct proof.

**Theorem 14** (Local Lieb’s inequality for analytic wavelets). Let \( \Delta \subset \mathbb{C}^+ \) be a set of finite measure, \( \alpha > 1 \), and \( 2 \leq p < \infty \). For \( f \in H^2(\mathbb{C}^+) \), it holds
\[
||W_{\psi_0} f||^p_{L^p(\mathbb{C}^+)} \leq \frac{8\pi}{(\alpha + 1)p - 2} \left( 1 - \left( 1 + \frac{|\Delta| h}{4\pi} \right)^{1-(\alpha+1)p/2} \right).
\]

**Proof:** Apply first (34) and then Theorem 13.

5.3. Discussion of general Lieb inequalities. The currently available methods do not seem to provide a Lieb inequality with sharp constants for a general window \( \psi \). The problem can be stated as providing the best constant in the inequality
\[
||W_\psi f||^p_{L^p(\mathbb{C}^+)} \leq C(\psi,p)||f||^p_{H^2(\mathbb{C}^+)}.
\]
This seems to be a non-trivial question. Kulikov’s methods are dependent on analytic functions, which, by the results of [29], restricts the window to a weight times \( \psi_0^0 \). On the other side, Lieb’s methods depend on the optimal constants of Young inequality,
which is not possible to conveniently use in this context. An intermediate result would be to obtain (37) for the family $\psi^n$, which still keeps some properties of the analytic case. We will obtain an inequality for $\|W_{\psi^0}f\|_{L^p(\mathbb{C}^+)}$ using the information from Theorem 13, but our estimates are quite rough, and serve only to give an idea of a significant obstruction we face, namely the non-unimodularity of the affine group, which does not allow to apply Young’s inequality in a conventional form (there are versions of Young’s inequality for the affine group, see Section 4 in [35], but they do not seem to apply to the problem at hand). For locally compact Abelian groups, Lieb’s uncertainty principle is well-established, and its optimizers were recently characterized in [43].

For any non-unimodular locally compact group $\mathcal{G}$ with left-invariant Haar measure $\mu_\mathcal{G}$, we show the following version of Young’s convolution inequality. First, the convolution of two functions $F, G : \mathcal{G} \to \mathbb{C}$ is defined as

$$(F * G)(z) = \int_\mathcal{G} F(w)G(w^{-1} \cdot z)d\mu_\mathcal{G}(w).$$

In addition, we define reflection operator $\mathcal{R}F(z) = F(z^{-1})$.

**Lemma 15.** Let $\mathcal{G}$ be a locally compact group with left Haar measure $\mu_\mathcal{G}$, and $1 \leq p, q, r \leq \infty$ be such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $F \in L^p(\mathcal{G})$, and $G, \mathcal{R}G \in L^q(\mathcal{G})$, then

$$\|F * G\|_{L^r(\mathcal{G})} \leq \|F\|_{L^p(\mathcal{G})} \cdot \max\{\|G\|_{L^q(\mathcal{G})}, \|\mathcal{R}G\|_{L^q(\mathcal{G})}\}.$$  

**Proof:** We adopt the classical proof of Young’s inequality for unimodular groups via the Riesz-Thorin theorem applied to the operator $T_GF = F * G$. Let us assume that $G, \mathcal{R}G \in L^q(\mathcal{G})$. First, if $r = \infty$ we note that by Hölder’s inequality

$$\|T_GF\|_{L^\infty(\mathcal{G})} \leq \sup_{z \in \mathcal{G}} \int_{\mathcal{G}} |F(w)\mathcal{R}G(z^{-1} \cdot w)|d\mu_\mathcal{G}(w) \leq \|F\|_{L^p(\mathcal{G})}\|\mathcal{R}G\|_{L^q(\mathcal{G})}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$  

On the other hand, for $r = q$, we have that $\|T_GF\|_{L^q(\mathcal{G})} \leq \|F\|_{L^q(\mathcal{G})}\|G\|_{L^q(\mathcal{G})}$ by [28, (20.14)]. An application of the Riesz-Thorin theorem then completes the proof. \hfill \Box

Together with Theorem 13 this leads to the following.

**Proposition 16.** Let $f \in H^2(\mathbb{C}^2), \psi \in \mathcal{C}oL^1(\mathbb{C}^+)$ be an admissible wavelet. For every $\alpha > 1$, and every $2 \leq p < \infty$, it holds

$$\frac{\|W_{\psi^0}f\|_{L^p(\mathbb{C}^+)}^p}{\|f\|_{H^2(\mathbb{C}^+)}^p} \leq \frac{8\pi}{(\alpha + 1)p - 2} \left(\frac{\alpha}{4\pi}\right)^p \max\left\{\|W_{\psi^0}\|_{L^1(\mathbb{C}^+)}^p, \|W_{\psi^0}\|_{L^1(\mathbb{C}^+)}^p\right\}.$$  

**Proof:** Let us apply consecutively (10), Lemma 15, and Theorem 13 to show

$$\|W_{\psi^0}f\|_{L^p(\mathbb{C}^+)}^p = \|\psi^0\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^+)}^{2p} \int_{\mathbb{C}^+} \left|\int_{\mathbb{C}^+} W_{\psi^0} f(w)\langle \psi^0, \pi(w^{-1} \cdot z)\psi\rangle d\mu^+(w)\right|^p d\mu^+(z)$$
\[
\int_0^\infty e^{-bt} t^\alpha L_n^\alpha(\lambda t)L_m^\alpha(\mu t) dt = \frac{\Gamma(m+n+\alpha+1)(b-\lambda)^n(b-\mu)^m}{b^{n+m+\alpha+1}} \times F\left(-m, -n; -m-n-\alpha; \frac{b(b-\lambda)-\mu}{(b-\mu)(b-\lambda)}\right),
\]

where \( \text{Re}(\alpha) > -1, \text{Re}(b) > 0, \) and \( F = 2F_1 \) denotes the Gauss hypergeometric function defined by

\[
F(a, b; c; z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k k!} z^k,
\]

and \((x)_k\) denotes Pochhammer’s rising factorial. Moreover, we need the following explicit representation of the Jacobi polynomials \([46, p. 442, (18.5.7)]\)

\[
P_n^{(\alpha, \beta)}(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)} \left(\frac{z-1}{2}\right)^k.
\]

This formula implies the following connection between the hypergeometric function and the Jacobi polynomials.

**Lemma 17.** For \( n, m \in \mathbb{N}_0, \) it holds

\[
F(-n, -m; -n-m-\alpha; z) = \min(n+m)!\Gamma(\max(n,m) + \alpha + 1)(-z)^{\min(n,m)} P_{\min(n,m)}^{(n-m, \alpha)} \left(1 - \frac{2}{z}\right).
\]

**Proof:** As \( F(-n, -m; -n-m-\alpha; z) = F(-m, -n; -n-m-\alpha; z), \) we may assume that \( n \geq m. \) The definition of \( F \) and \((-x)_k = (-1)^k (x-k+1)_k\) lead us to

\[
F(-n, -m; -n-m-\alpha; z) = \sum_{k=0}^\infty \frac{(-n)_k(-m)_k}{(-n-m-\alpha)_k k!} z^k
\]
\[
= \sum_{k=0}^{m} \frac{(n-k+1)k(m-k+1)_k}{(n+m+\alpha-k+1)_k} (\text{w})^k
\]
\[
= \sum_{k=0}^{m} \frac{\Gamma(n+1)\Gamma(m+1)\Gamma(n+m+\alpha-k+1)}{\Gamma(n-k+1)\Gamma(n-m+k+1)\Gamma(n+m+\alpha+1)_k} (-\text{w})^k
\]
\[
= \frac{n!}{\Gamma(n+m+\alpha+1)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+m+\alpha-k+1)}{\Gamma(n-k+1)} (-\text{w})^k
\]
\[
= \frac{n!}{\Gamma(n+m+\alpha+1)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n-m+k+1)} (-\text{w})^{m-k}
\]
\[
= \frac{\Gamma(n+1)}{\Gamma(n+m+\alpha+1)} (-\text{w})^{m} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n-m+k+1)} \left( -\frac{1}{\text{w}} \right)^k
\]
\[
= \frac{m!\Gamma(n+m+\alpha+1)}{\Gamma(n+m+\alpha+1)} (-\text{w})^{m} \left( \frac{n-m,\alpha}{1 - \frac{2}{\text{w}}} \right).
\]

**Proof of Proposition 1:** Let \( z = x + is \) and \( w = x' + is' \). By definition of \( \psi_n^\alpha \), one has
\[
\frac{1}{2^{\alpha+2}} \sqrt{\frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{n!m!}} \langle \pi(w)\psi_n^\alpha, \pi(z)\psi_m^\alpha \rangle
\]
\[
= \frac{1}{2\pi} \int_0^\infty (s')^{-\alpha-1} \int_0^\infty (s')^{\alpha+1} e^{s't-iz't} L_m^\alpha(s's't) L_n^\alpha(st) \, ds' dt
\]
\[
= \frac{1}{2\pi} \sum_{n'=0}^\infty \left( \frac{1}{n!} \right)^2 \frac{\Gamma(m+n+\alpha+1)}{\Gamma(m+n+\alpha+1)} \left( \frac{n-m,\alpha}{1 - \frac{2}{\text{w}}} \right)^{n}
\]
\[
\times F \left( -m-n; -m-n-\alpha; \frac{z-w}{z-w} \right).
\]

Hence, setting \( b = -i(z-w) \), \( \lambda = 2s \), and \( \mu = 2s' \) in equation (38) yields \( b - \lambda = -i(z-w) \), \( b - \mu = -i(z-w) \), \( b - \lambda - \mu = -i(z-w) \), and consequently
\[
\sum_{n'=0}^\infty \left( \frac{1}{n!} \right)^2 \frac{\Gamma(m+n+\alpha+1)}{\Gamma(m+n+\alpha+1)} \left( \frac{n-m,\alpha}{1 - \frac{2}{\text{w}}} \right)^{n}
\]
\[
\times F \left( -m-n; -m-n-\alpha; \frac{z-w}{z-w} \right).
\]

Setting \( n = m \) gives (20) by Lemma 17. Taking \( w = i \) and applying Lemma 17 again yields (21). Finally, (19) follows from the basic identity \( L_n^{\alpha+1} = \sum_{k=0}^n L_k^\alpha \) and the orthogonality relation for generalized Laguerre polynomials (see, e.g., [46])
where the last identity can be easily shown by an inductive argument.

6.2. Integrability. It remains for us to show that the mother wavelets \( \psi_n^\alpha \) are admissible for the integrable representation \( \pi \) of the \( ax+b \) group for appropriate choices of \( \alpha \).

**Proposition 18.** Let \( n \in \mathbb{N}_0 \). If \( \alpha > 1 \), then

\[
\int_{\mathbb{C}^+} |\langle \psi_n^\alpha, \pi(z) \psi_n^\alpha \rangle| \, d\mu^+(z) < \infty.
\]

Moreover, if \( (\alpha + 1)p - 2 > 0 \) then

\[
\int_{\mathbb{C}^+} |W_{\psi_n^\alpha} \psi_n^\alpha(z)|^p \frac{d\mu_{\mathbb{C}^+}(z)}{\Im(z)^2} = \frac{8\pi}{(\alpha + 1)p - 2}.
\]

**Proof:** By (26), we have

\[
|W_{\psi_n^\alpha} \psi_n^\alpha(T(u))| = |u|^{2n} \left( 1 - \frac{|u|^2}{4} \right)^{\frac{\alpha+1}{2}} |Z_n^\alpha(|u|^2)|.
\]

Consequently, by (17) and (18)

\[
\int_{\mathbb{C}^+} |W_{\psi_n^\alpha} \psi_n^\alpha(z)| \frac{d\mu_{\mathbb{C}^+}(z)}{\Im(z)^2} = \int_{\mathbb{D}} |W_{\psi_n^\alpha} \psi_n^\alpha(T(u))| \frac{4du}{(1 - |u|^2)^2}
\]

\[
= 4^{-\frac{\alpha+1}{2}} \int_{\mathbb{D}} |u|^{2n} |Z_n^\alpha(|u|^2)| \left( 1 - |u|^2 \right)^{\frac{\alpha+3}{2}} du
\]

\[
\leq \int_{\mathbb{D}} (1 - |u|^2)^{\frac{\alpha+3}{2}} du = \pi \int_0^1 r^{\frac{\alpha+3}{2}} dr.
\]

The last integral is finite if and only if \( \alpha > 1 \). For \( n = 0 \) and \( p \geq 1 \) we argue similarly to obtain

\[
\int_{\mathbb{C}^+} |W_{\psi_n^\alpha} \psi_n^\alpha(z)|^p \frac{d\mu_{\mathbb{C}^+}(z)}{\Im(z)^2} = \int_{\mathbb{D}} |W_{\psi_0^\alpha} \psi_0^\alpha(T(u))|^p \frac{4du}{(1 - |u|^2)^2}
\]

\[
= 4 \int_{\mathbb{D}} (1 - |u|^2)^{\frac{(\alpha+1)p}{2} - 2} du
\]

\[
= 4\pi \int_0^1 (1 - t)^{\frac{(\alpha+1)p}{2} - 2} dt
\]

\[
= 4\pi \frac{1}{\frac{(\alpha+1)p}{2} - 1} = \frac{8\pi}{(\alpha + 1)p - 2}.
\]

\( \square \)
Acknowledgements

We would like to thank the anonymous reviewers. Their valuable input helped to substantially improve this article. The authors acknowledge the support of the Austrian Science Fund (FWF) through the projects 10.55776/P31225 (L.D. Abreu) and 10.55776/J4254, and 10.55776/Y1199 (M. Speckbacher).

References

[1] L. D. Abreu. Superframes and polyanalytic wavelets. *J. Fourier Anal. Appl.*, 23:1–20, 2017.
[2] L. D. Abreu, P. Balazs, and S. Jakšic. The affine ensemble: determinantal point processes associated with the $ax + b$ group. *J. Math. Soc. Japan*, in press, 2022.
[3] L. D. Abreu and M. Dörfler. An inverse problem for localization operators. *Inverse Problems*, 28(11):115001, 2012.
[4] L. D. Abreu, Z. Mouayn, and F. Voigtlaender. A fractal uncertainty principle for Bergman spaces and analytic wavelets. *J. Math. Anal. Appl.*, 519(1):126699, 2023.
[5] L. D. Abreu and M. Speckbacher. Deterministic guarantees for $L^1$-reconstruction: A large sieve approach with flexible geometry. *Proceedings of SampTA 19*, 2019.
[6] L. D. Abreu and M. Speckbacher. Donoho-Logan large sieve principles for modulation and polyanalytic Fock spaces. *Bull. Sci. Math.*, 171, 2021.
[7] L. D. Abreu and M. Speckbacher. Affine density, von Neumann dimension and a problem of Perelomov. *Adv. Math.*, 407:108564, 2022.
[8] R. G. Baraniuk, P. Flandrin, A. J. E. M. Janssen, and O. J. J. Michel. Measuring time-frequency information content using the Rényi entropies. *IEEE Trans. Inform. The.*, 47(4):1391–1409, 2001.
[9] A. Baranova, P. Jaming, K. Kellay, and M. Speckbacher. Oversampling and Donoho–Logan type theorems in model spaces. *Ann. Fenn. Math.*, 49(1):167–182, 2024.
[10] M. Bayram and R. G. Baraniuk. Multiple window time-frequency analysis. In *Proceedings of Third International Symposium on Time-Frequency and Time-Scale Analysis (TFTS-96)*, pages 173–176. IEEE, 1996.
[11] E. Berge, S. M. Berge, and F. Luef. The affine Wigner distribution. *Appl. Comp. Harm. Anal.*, 56:150–175, 2022.
[12] E. Berge, S. M. Berge, F. Luef, and E. Skrettingland. Affine quantum harmonic analysis. *J. Funct. Anal.*, 282(4):109327, 2022.
[13] J. Bergh and J. Lőrström. *Interpolation Spaces. An Introduction*, volume 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin - New York, 1976.
[14] A. Comtet. On the Landau levels on the hyperbolic plane. *Ann. Physics*, 173(1):185–209, 1987.
[15] I. Daubechies. Time-frequency localization operators: A geometric phase space approach. *IEEE Trans. Inform. Theory*, 34(4):605–612, 1988.
[16] I. Daubechies and T. Paul. Time-frequency localization operators - a geometric phase space approach: II. The use of dilations. *Inverse Problems*, 4:661–680, 1988.
[17] D. L. Donoho and B. F. Logan. Signal recovery and the large sieve. *SIAM J. Appl. Math.*, 52(2):577–591, 1992.
[18] D. L. Donoho and P. B. Stark. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.*, 49(3):906–931, 1989.
[19] P. Duren, E. A. Gallardo-Gutiérrez, and A. Montes-Rodríguez. A Paley–Wiener theorem for Bergman spaces with application to invariant subspaces. *Bull. London Math. Soc.*, 39(3):459–466, 2007.

[20] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions I. *J. Funct. Anal.*, 86:307–340, 1989.

[21] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions II. *Monatsh. Math.*, 108:129–148, 1989.

[22] A. Ghanmi. A class of generalized complex Hermite polynomials. *J. Math. Anal. Appl.*, 340(2):1395–1406, 2008.

[23] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products*. Academic Press, 7th edition, 2007.

[24] K. Gröchenig. Describing functions: Atomic decomposition versus frames. *Monatsh. Math.*, 112:1–41, 1991.

[25] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, 2001.

[26] C. Grosche. The path integral on the Poincaré upper half-plane with a magnetic field and for the Morse potential. *Ann. Phys.*, 187(1):110–134, 1988.

[27] H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman Spaces.*, volume 199 of *Graduate Texts in Mathematics*. New York: Springer Verlag, 2000.

[28] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I*. Springer New York, 1963.

[29] N. Holighaus, G. Koliander, Z. Průša, and L. D. Abreu. Characterization of analytic wavelet transforms and a new phaseless reconstruction algorithm. *IEEE Trans. Sign. Proc.*, 67(15):3894–3908, 2019.

[30] S. Husain and F. Littmann. Concentration estimates for the Paley-Wiener spaces. *arXiv:2210.10029v2*, 2022.

[31] O. Hutník. Wavelets from Laguerre polynomials and Toeplitz-type operators. *Int. Eq. Oper. Theor.*, 71(3):357–388, 2011.

[32] M. Ismail. Analytic properties of complex Hermite polynomials. *Trans. Amer. Math. Soc.*, 368(2):1189–1210, 2016.

[33] P. Jamming and M. Speckbacher. Concentration estimates for finite expansions of spherical harmonics on two-point homogeneous spaces via the large sieve principle. *Sampl. Theor. Signal Process. Data Anal.*, 19(9), 2021.

[34] D. Kalaj. Contraction property of differential operator on Fock space. *Comput. Meth. Funct. Theory*, 1–20, 2023.

[35] A. Klein and B. Russo. Sharp inequalities for Weyl operators and Heisenberg groups. *Math. Ann.*, 235(2):175–194, 1978.

[36] H. Knutsen. Daubechies’ time-frequency localization operator on Cantor type sets II. *J. Funct. Anal.*, 282(9):109412, 2022.

[37] H. Knutsen. A fractal uncertainty principle for the short-time Fourier transform and Gabor multipliers. *Appl. Comp. Harm. Anal.*, 62:365–389, 2023.

[38] A. Kulikov. Functionals with extrema at reproducing kernels. *Geom. Funct. Anal. 32*, 938–949, 2022.

[39] H. Maass. Über eine neue Art von nichtanalytischen automorphe Funktionen und die Bestimmung Dirichletcher Reihen durch Funktionalgleichungen. *Math. Ann.*, 121(1):141–183, 1949.
[40] H. Mejjaoli and S. Omri. Spectral theorems associated with the directional short-time Fourier transform. *J. Pseudo Diff. Oper. Appl.*, 11(1):15–54, 2020.

[41] H. Mejjaoli and K. Trimeche. Localization operators and scalogram associated with the generalized continuous wavelet transform on $\mathbb{R}^d$ for the Heckman–Opdam theory. *Rev. Unión Mat. Arg.*, 2020.

[42] Z. Mouayn. Characterization of hyperbolic Landau states by coherent state transforms. *J. Phys. A: Math. Gen.*, 36(29):8071, 2003.

[43] F. Nicola. Maximally localized Gabor orthonormal bases on locally compact Abelian groups. *arXiv:2305.02738*, 2023.

[44] F. Nicola and P. Tilli. The Faber–Krahn inequality for the short-time Fourier transform. *Invent. Math.*, 230:1–30, 2022.

[45] F. Nicola and P. Tilli. The norm of time-frequency localization operators. *Trans. Amer. Math. Soc.*, 376 (10), 7353-7375, 2023.

[46] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 1st edition, 2010.

[47] S. J. Patterson. The Laplacian operator on a Riemann surface. *Compos. Math.*, 31(1):83–107, 1975.

[48] J. Ramos and P. Tilli. A Faber–Krahn inequality for wavelet transforms. *Bull. London Math. Soc.*, 55(4):2018–2034, 2023.

[49] B. Ricaud and B. Terrésani. A survey of uncertainty principles and some signal processing applications. *Adv. Comp. Math.*, 40(3):629–650, 2014.

[50] F. Riccardi. A new optimal estimate for the norm of time-frequency localization operators. *arXiv:2311.06525*, 2023.

[51] K. Seip. Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. *SIAM J. Math. Anal.*, 22(3):856–876, 1991.

[52] M. Speckbacher and T. Hrycak. Concentration estimates for band-limited spherical harmonics expansions via the large sieve principle. *J. Fourier Anal. Appl.*, 38, 2020.

[53] N. L. Vasilevski. On the structure of Bergman and poly-Bergman spaces. *Integr. Equ. Oper. Theory*, 33:471–488, 1999.

[54] N. L. Vasilevski. *Commutative Algebras of Toeplitz Operators on the Bergman Space*, volume 185. Springer Science & Business Media, 2008.

[55] A. Wünsche. Generalized Zernike or disc polynomials. *J. Comp. Appl. Math.*, 174:135–163, 2005.

[56] K. H. Zhu. Duality of Bloch spaces and norm convergence of Taylor series. *Michigan. Math. J.*, 38:89–101, 1991.

(L. D. A.) FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA

Email address: abreul22@univie.ac.at

(M. S.) ACOUSTICS RESEARCH INSTITUTE, AUSTRIAN ACADEMY OF SCIENCES, DOMINIKANERBASTEI 16, A-1010 VIENNA, AUSTRIA

Email address: michael.speckbacher@oeaw.ac.at