Abstract. We construct an integral representation of solutions of the Knizhnik-Zamolodchikov-Bernard equations, using the Wakimoto modules.
0. INTRODUCTION

The purpose of this paper is to construct an integral representation of solutions of the Knizhnik-Zamolodchikov-Bernard equations, using the Wakimoto modules.

Correlation functions of the (chiral) Wess-Zumino-Witten models satisfy a system of differential equations. In the genus zero case, it is the well-known Knizhnik-Zamolodchikov (KZ) equations [KZ], [TK]. Bernard [B1] found a system of equations for the genus one case, which is now called the Knizhnik-Zamolodchikov-Bernard (KZB) equations. In general, it is known (cf. [TUY]) that correlation functions are solutions of a holonomic system over the base space of a family of Riemann surfaces with marked points and principal bundles on them. See also [B2] and [F].

There are a vast amount of works on the KZ equations, among which are studies on integral representation of solutions. There are several different approaches to this subject. One is from the viewpoint of the theory of hypergeometric type integrals; e.g., [DJMM], [Mat], [SV1], [SV2]. Another approach comes from free field theories on Riemann surfaces and the Wakimoto realization of affine Lie algebras; e.g., [Mar], [GMMOS], [BeF], [ATY], [A]. The third one is the off-shell Bethe Ansatz developed in [BaF], the representation theoretical meaning of which was clarified by [FFR]. See also [RV] and [Ch].

The first approach in the genus one case was pursued by Felder and Varchenko in [FV], while Babujian et al. [BLP] study the off-shell Bethe Ansatz approach of the KZB equations (with an additional term). Our goal in this paper is to apply the second approach to the genus one case to obtain an integral representation of solutions of the KZB equations. In the genus zero case, an integral over a suitable (twisted) cycle of a matrix element of product of vertex operators and screening charges gives a solution of the KZ equations. In the genus one case, we take a twisted trace instead of a matrix element to give a solution of the KZB equation in the integral form. We apply the method of the screening current Ward identity used in [ATY] and [A] mutatis mutandis, and obtain an explicit formula for this integral representation. For the \textit{sl}(2) case Bernard and Felder found the same result in [BeF] by using Wakimoto modules.

The paper is organized as follows. In Section 1, mainly following [Ku], we review several fundamental techniques in the conformal field theory, especially the free field realization of the affine Lie algebra found by Wakimoto [W], Feigin and Frenkel [FF1], [FF2]. We state the problem in Section 2. Namely we formulate the Wess-Zumino-Witten model on elliptic curves, following [FW], and give a definition of N-point functions. The KZB equations are introduced as a system of equations satisfied by N-point functions. Section 3 is the main part of this paper, where we give an integral representation of an N-point function on elliptic curves (Theorem 3.4). If we restrict an N-point function to a certain submodule, it is a solution of the KZB equations. We thus give an integral representation of solutions of the KZB equations (Theorem 3.12). To write down the integrand explicitly, we use the screening current Ward identity on elliptic curves. Appendix A is a table of theta functions used in the paper. In Appendix B we review the method of coherent states well known in the string theory. We also compute the one-loop correlation function of vertex operators.

1. BOSONIZATION AND WAKIMOTO MODULES

In this section we review basic facts about the Wakimoto representations of the affine Lie algebras, following [Ku]. See also [W], [FF1], [FF2], [FFR].

1.1. Notations for the finite dimensional algebra. Here we recall fundamental facts about finite dimensional simple Lie algebras to fix the notations.

Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra of rank \( l \), \( \mathfrak{h} \) its Cartan subalgebra and

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}
\]

the root space decomposition, where \( \Delta \) is the set of roots. The Cartan-Killing form is denoted by \((\cdot | \cdot)\), through which we identify \( \mathfrak{h} \) and its dual space \( \mathfrak{h}^* \). We fix the simple roots \( \{\alpha_1, \ldots, \alpha_l\} \), Chevalley generators \( \{H_i, E_i, F_i\}_{i=1,\ldots,l} \) and a basis \( e_\alpha \) of \( \mathfrak{g}_{\alpha} \), such that \( e_{\alpha_i} = E_i \) for \( i = 1, \ldots, l \) and \( (e_\alpha | e_{-\alpha'}) = \delta_{\alpha,\alpha'} \). The set of positive and negative roots are denoted by \( \Delta_+ = \{\beta_1, \ldots, \beta_s\} \) and \( \Delta_- \) respectively. The Borel and nilpotent subalgebras corresponding to \( \Delta_0 \) are denoted by \( \mathfrak{b}_+, \mathfrak{n}_\pm \) as usual.
Let $G$, $B_\pm$ and $N_\pm$ be an algebraic group corresponding to $\mathfrak{g}$, the subgroups corresponding to $\mathfrak{b}_\pm$ and to $\mathfrak{n}_\pm$. As is well known, there exists a Lie algebra homomorphism $R_\lambda$ from $\mathfrak{g}$ to the sheaf of twisted differential operators $\mathcal{D}_\lambda$ on the flag variety $B_-\setminus G$ once one fixes a dual vector $\lambda \in \mathfrak{h}^*$. Denote the base point $[B_+] \in B_-\setminus G$ by $o$. With respect to the coordinate on $oN_+ \cong n_+$, a big cell of $B_-\setminus G$, introduced by the exponential map:

$$\mathbb{C}^{\Delta_+} \ni (x^\alpha)_{\alpha \in \Delta_+} \mapsto o \exp(x^{\beta_1}e_{\beta_1}) \cdots \exp(x^{\beta_\nu}e_{\beta_\nu}) \in oN_+,$$

the twisted differential operator $R_\lambda(X)$ for $X \in \mathfrak{g}$ is represented by a first order differential operator acting on the space of polynomials $\mathbb{C}[x^\alpha; \alpha \in \Delta_+]$:

$$R_\lambda(X) = R(X;x,\partial_x,\lambda),$$

where $\partial_x = (\partial/\partial x^\alpha)_{\alpha \in \Delta_+}$. The operator $R(X;x,\partial_x,\lambda)$ is a polynomial in $X$, $x$, $\partial_x$ and $\lambda(H_i)$ ($i = 1, \ldots, l$). More explicitly, there are polynomials $R_\alpha(X;x)$ in $x$ for $X \in \mathfrak{g}$, $\alpha \in \Delta_+$ such that

$$R_\lambda(E_i) = \sum_{\alpha \in \Delta_+} R_\alpha(E_i;x) \frac{\partial}{\partial x^\alpha},$$

$$R_\lambda(F_i) = \sum_{\alpha \in \Delta_+} R_\alpha(F_i;x) \frac{\partial}{\partial x^\alpha} + x_{\alpha\lambda}(H_i),$$

$$R_\lambda(H) = -\sum_{\alpha \in \Delta_+} \alpha(H)x^\alpha \frac{\partial}{\partial x^\alpha} + \lambda(H),$$

for Chevalley generators $E_i$, $F_i$ ($i = 1, \ldots, l$) and $H \in \mathfrak{h}$. Note that $R_\lambda(E_i)$ does not depend on $\lambda$. Hence we sometimes omit the suffix and denote it by $R(E_i)$.

The nilpotent subgroup $N_+$ acts on the big cell from the left as

$$n \cdot (oa) = ana \quad \text{for} \; n, a \in N_+.$$

The infinitesimal action of $n_+$ induced by this action is denoted by $\text{Scr}$:

$$n_+ \ni X \mapsto \text{Scr}(X;x,\partial_x) \in \mathbb{C}[x,\partial_x].$$

1.2. Ghosts and free bosons. Let us introduce the algebra of (bosonic) ghosts, $\widehat{\text{Gh}}(\mathfrak{g})$, and the algebra of free bosons, $\text{Bos}(\mathfrak{g})$.

The generators of the algebra $\widehat{\text{Gh}}(\mathfrak{g})$ are $\beta_\alpha[m]$ and $\gamma^\alpha[n]$ ($\alpha \in \Delta_+$, $m \in \mathbb{Z}$) satisfying the canonical commutation relations:

$$[\beta_\alpha[m], \gamma^{\alpha'}[n]] = \delta^\alpha_{\alpha'} \delta_{m+n,0} \cdot 1,$$

for $\alpha, \alpha' \in \Delta_+$ and $m,n \in \mathbb{Z}$. We call the formal generating functions of generators,

$$\beta_\alpha(z) = \sum_{m \in \mathbb{Z}} z^{-m-1}\beta_\alpha[m], \quad \gamma^\alpha(z) = \sum_{m \in \mathbb{Z}} z^{-m}\gamma^\alpha[m],$$

ghost fields. They satisfy the following operator product expansions:

$$\beta_\alpha(z)\gamma^{\alpha'}(w) \sim \frac{\delta^\alpha_{\alpha'}}{z-w}.$$

The ghost Fock space $\mathcal{F}^\mathfrak{h}$ is defined as a left $\widehat{\text{Gh}}(\mathfrak{g})$-module generated by the vacuum vector $|0\rangle^\mathfrak{h}$, satisfying

$$\beta_\alpha[m]|0\rangle^\mathfrak{h} = 0, \quad \gamma^{\alpha'}[n]|0\rangle^\mathfrak{h} = 0$$

for any $\alpha \in \Delta_+$, $m \geq 0$, $n > 0$.

The algebra $\text{Bos}(\mathfrak{g})$ is generated by $\phi_i[m]$ ($i = 1, \ldots, l$, $m \in \mathbb{Z}$), the defining relation of which is

$$[\phi_i[m], \phi_j[n]] = \kappa(H_i|H_j) m \delta_{m+n,0} \cdot 1,$$

where $\kappa$ is a non-zero complex parameter. We extend the algebra to $\hat{\text{Bos}}(\mathfrak{g})$ by adding the boost operator $e^p_i$ and its logarithm $p_i$ which satisfies the relation:

$$[\phi_i[m], e^{p_i}] = \kappa(H_i|H_j) m \delta_{m+n,0}, \quad [\phi_i[m], p_j] = \kappa(H_i|H_j) \delta_{m,0}. $$
Fields

\[
\phi_i(z) := \kappa p_i + \phi_i[0] \log z + \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{z^{-m}}{-m} \phi_i[m],
\]

(1.11)

\[
\partial \phi_i(z) := \sum_{m \in \mathbb{Z}} z^{-m-1} \phi_i[m]
\]

are important generating functions of generators of this algebra. The field \(\phi_i(z)\) is called the \textit{free boson field}.

For any \(H = \sum_{i=1}^{l} a_i H_i \in \mathfrak{h}\), we also use the notations like

\[
\phi[H; m] = \sum_{i=1}^{l} a_i \phi_i[m], \quad p[H] = \sum_{i=1}^{l} a_i p_i, \quad \phi(H; z) = \sum_{i=1}^{l} a_i \phi_i(z).
\]

The free boson fields satisfy the following operator product expansion.

\[
\phi(H; z)\phi(H'; w) \sim \kappa(H[H']) \log(z - w),
\]

(1.12)

\[
\partial \phi(H; z)\partial \phi(H'; w) \sim \frac{\kappa(H[H'])}{(z - w)^2},
\]

for any \(H, H' \in \mathfrak{h}\). For a dual vector \(\lambda \in \mathfrak{h}^*\), the \textit{boson Fock space} \(\mathcal{F}_\lambda^{bos}\) with momentum \(\lambda\) is defined as a left \(\text{Bos}(\mathfrak{g})\)-module generated by the vacuum vector \(|\lambda\rangle^{bos}\), satisfying

\[
\phi_i[m]|\lambda\rangle^{bos} = 0, \quad \phi_i[0]|\lambda\rangle^{bos} = \lambda(H_i)|\lambda\rangle^{bos}
\]

(1.13)

for any \(i = 1, \ldots, l, m > 0\). The boost operator \(e^{p_i}\) acts on the direct sum of Fock spaces \(\bigoplus \mathcal{F}_\lambda^{bos}\) by shifting the momentum:

\[
e^{p_i}|\lambda\rangle^{bos} = |\lambda + H_i\rangle^{bos}.
\]

(1.14)

The \textit{normal ordered product} \(\mathcal{P}\): of a monomial \(P\) of \(\beta_0[m]s, \gamma^\alpha[m]'s, \phi_i[m]'s\) and \(e^{p_i}'s\) is defined by putting annihilation operators of \(|0\rangle^{\mathfrak{g}'}\) and \(|\lambda\rangle^{bos}\) \(\beta_0[m] (m \geq 0), \gamma^\alpha[m] (m > 0), \phi_i[m] (m > 0)\) and \(\phi_i[0]\) appearing in \(P\) to the right side in the product.

For example, the \textit{bosonic vertex operator} is defined by

\[
V(\lambda; z) := \mathcal{P}: e^{V(\lambda, z)} := e^{z\phi(\lambda; z)} = \exp\left(\frac{1}{\kappa} \sum_{m < 0} \frac{z^{-m}}{-m} \phi[\lambda; m]\right) e^{p[\lambda] z \frac{z}{2} \phi[\lambda; 0]} \exp\left(\frac{1}{\kappa} \sum_{m > 0} \frac{z^{-m}}{-m} \phi[\lambda; m]\right).
\]

(1.15)

We introduce the following notations for later use.

\[
V(\lambda; z) = \tilde{V}(\lambda; z) V_0(\lambda; z),
\]

(1.16)

\[
\tilde{V}(\lambda; z) := \mathcal{P}: \tilde{e}^{\tilde{V}(\lambda, z)} := \exp\left(\frac{1}{\kappa} \sum_{m < 0} \frac{z^{-m}}{-m} \phi[\lambda; m]\right) \exp\left(\frac{1}{\kappa} \sum_{m > 0} \frac{z^{-m}}{-m} \phi[\lambda; m]\right),
\]

(1.17)

\[
V_0(\lambda; z) := e^{p[\lambda] z \frac{z}{2} \phi[\lambda; 0]},
\]

(1.18)

where \(\tilde{\phi}(\lambda; z) = \phi(\lambda; z) - \kappa p[\lambda] - \phi[\lambda; 0]\) is the non-zero mode part of \(\phi(\lambda; z)\).

1.3. \textbf{Bosonization and Wakimoto modules}. Bosonizing the differential operators \(R(X; x, \partial_x, \lambda)\) in Section 1.1 by ghosts and free bosons in Section 1.2 gives the Wakimoto realization of the affine Lie algebra \(\hat{\mathfrak{g}}\).
Define current operator $X(z)$ ($X \in \mathfrak{g}$) and the energy-momentum tensor $T(z)$ by

\begin{align}
(1.19) \quad X(z) &:= :R(X; \gamma(z), \beta(z), \partial \phi(z)): \quad \text{for } X = E_i, H_i \text{ and } i = 1, \ldots, l, \\
(1.20) \quad F_i(z) &:= :R(F_i; \gamma(z), \beta(z), \partial \phi(z)):+ c_i \partial \gamma^{\alpha_i}(z) \quad \text{for } i = 1, \ldots, l, \\
(1.21) \quad T(z) &:= T^{\h}(z) + T^{\phi}(z), \\
(1.22) \quad T^{\h}(z) &:= \sum_{\alpha \in \Delta_+} :\partial \gamma^{\alpha}(z)\beta_\alpha(z):, \\
(1.23) \quad T^{\phi}(z) &:= \frac{1}{2\kappa} \sum_{i=1}^l :\partial \phi(H_i; z)\partial \phi(H^i; z): - \frac{1}{2\kappa} \partial^2 \phi(2\rho; z),
\end{align}

where $\{H^i\}_{i=1, \ldots, l}$ is a basis of $\mathfrak{h}$ dual to $\{H_i\}$ with respect to $\langle \cdot | \cdot \rangle$, $\rho$ is the half sum of positive roots ($\mathfrak{h}$ and $\mathfrak{h}^*$ are identified via the inner product) and $\{c_i\}_{i=1, \ldots, l}$ is a set of constants to be determined. More explicitly, we have from (1.3)

\begin{align}
(1.24) \quad E_i(z) &:= \sum_{\alpha \in \Delta_+} :R_\alpha(E_i; \gamma(z))\beta_\alpha(z):, \\
(1.25) \quad F_i(z) &:= \sum_{\alpha \in \Delta_+} :R_\alpha(F_i; \gamma(z))\beta_\alpha(z): + \gamma_\alpha(z)\partial \phi_\alpha(z) + c_i \partial \gamma^{\alpha_i}(z), \\
(1.26) \quad H(z) &:= H^{\h}(z) + \partial \phi(H; z), \quad H^{\h}(z) := - \sum_{\alpha \in \Delta_+} \alpha(H)\gamma^{\alpha}(z)\beta_\alpha(z):
\end{align}

for Chevalley generators $E_i, F_i$ ($i = 1, \ldots, l$) and $H \in \mathfrak{h}$.

We expand these series in the following way: for $X \in \mathfrak{g}$,

\begin{align}
(1.27) \quad X(z) &= \sum_{m \in \mathbb{Z}} z^{-m-1}X[m], \quad T(z) = \sum_{m \in \mathbb{Z}} z^{-m-2}T[m].
\end{align}

The coefficients $X[m]$ and $T[m]$ belong to a certain completion of the algebra $\widehat{\mathfrak{g}} \otimes \widehat{\text{Bos}}(\mathfrak{g})$.

**Theorem 1.1.** [M], [FF1], [FF2], [K1]. There exists a unique set of constants $\{c_i\}_{i=1, \ldots, l}$, such that a Lie algebra homomorphism from the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{k}$ to a completion of $\widehat{\mathfrak{g}} \otimes \widehat{\text{Bos}}(\mathfrak{g})$ can be defined by

\begin{align}
(1.28) \quad \omega(X \otimes t^n) &= X[m], \quad \omega(\hat{k}) = \kappa - h^\vee,
\end{align}

for all $X \in \mathfrak{g}$, $m \in \mathbb{Z}$, where $\hat{k}$ is the center of $\hat{\mathfrak{g}}$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Moreover, the energy-momentum tensor $T(z)$ defined by (1.21) coincides with the image of $T_{\text{Sug}}(z)$ in $U\hat{\mathfrak{g}}$ defined by the Sugawara construction:

\begin{align}
(1.29) \quad T(z) &= \omega(T_{\text{Sug}}(z)), \quad T_{\text{Sug}}(z) := \frac{1}{2\kappa} \sum_{p=1}^{\dim \mathfrak{g}} \circ J_p(z) J^p(z) \circ,
\end{align}

and $T[m]$’s generate the Virasoro algebra Vir with the central charge $c_V = \dim \mathfrak{g} - 12(\rho | \rho)/\kappa = k \dim \mathfrak{g} / \kappa$. Here $J_p(z) = \sum_{m \in \mathbb{Z}} z^{-m-1}J_p \otimes t^m$, $\{J_p\}$ is a basis of $\mathfrak{g}$, $\{J^p\}$ is its dual basis with respect to $\langle \cdot | \cdot \rangle$ and the symbol $\circ \circ \circ$ is the normal ordered product in $U\hat{\mathfrak{g}}$.

Namely, $\omega$ can be extended to a Lie algebra homomorphism from $\hat{\mathfrak{g}} \oplus \text{Vir}$ to a completion of $\widehat{\mathfrak{g}} \oplus \widehat{\text{Bos}}(\mathfrak{g})$ such that

\begin{align}
(1.30) \quad T[m] &= \omega(T_{\text{Sug}}[m]).
\end{align}

Therefore Kac-Moody current operators satisfy the operator product expansions:

\begin{align}
(1.31) \quad X(z)Y(w) &\sim \frac{k(X|Y)}{(z-w)^2} + \frac{[X, Y](w)}{z-w},
\end{align}
where $X, Y \in \mathfrak{g}$ and $k = \kappa - h^\vee$, and the energy-momentum tensor satisfies

$$T(z)T(w) \sim \frac{c v^2 / 2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{1.32}$$

$$T(z)X(w) \sim \frac{X(w)}{(z-w)^2} + \frac{\partial X(w)}{z-w}. \tag{1.33}$$

We can regard $F^g \otimes F^\Lambda^\text{bos}$ as a representation of $\hat{g}$ of level $k = \kappa - h^\vee$ through $\omega$.

**Definition 1.2.** Denote $F^g \otimes F^\Lambda^\text{bos}$ by $\text{Wak}_{\lambda, k}$ and call it a *Wakimoto module of level* $k$, weight $\lambda$.

There is a $g$-submodule generated by $|0\rangle^g \otimes |\lambda\rangle^{\text{bos}}$, which is spanned by $\prod_{\alpha \in \Delta_+} \gamma_\alpha |0\rangle^g(\lambda) |0\rangle^g \otimes |\lambda\rangle^{\text{bos}}$ ($I(\alpha) \in \mathbb{N}$), and is isomorphic to the dual Verma module $M^*_\lambda$ of $g$ with the highest weight $\lambda$. (See Proposition 4.4 of [Ku].) We denote it by $\text{Wak}^0_{\lambda, k}$:

$$\text{Wak}^0_{\lambda, k} := \text{Span}_C \left\{ \prod_{\alpha \in \Delta_+} \gamma_\alpha |0\rangle^g(\lambda) |0\rangle^g \otimes |\lambda\rangle^{\text{bos}} \mid (I(\alpha))_{\alpha \in \Delta_+} \in \mathbb{N}^{\Delta_+} \right\} \cong M^*_\lambda. \tag{1.34}$$

It is easy to show that for any $m > 0$ and $X \in g$,

$$X[m]|\text{Wak}^0_{\lambda, k} = 0, \tag{1.35}$$

and the quadratic Casimir operator $C_2 = \sum p J_p J_p$ acts as a multiplication operator:

$$C_2|\text{Wak}^0_{\lambda, k} = (\lambda| \lambda + 2\rho)|\text{id}. \tag{1.36}$$

### 1.4. State-operator correspondence

Let us recall the state-operator correspondence in the two dimensional conformal field theories. A primary field generates a highest weight representation of the algebra of symmetries in the space of operator valued functions ("local operators"). In our case, the algebra of symmetries is $\widehat{G}(g) \oplus \widehat{\text{Bos}}(g)$ and the representation space is:

$$\mathcal{O}_\lambda := \text{Span}_C \{ x[m_1] \cdots x[m_n]V(\lambda;z)|n \in \mathbb{N}; x_i = \beta_\alpha, \gamma_\alpha \text{ or } \phi_i \text{ for certain } \alpha, j; m_i \in \mathbb{Z} \} \subset \widehat{G}(g) \oplus \widehat{\text{Bos}}(g)((z)),$$

where $V(\lambda;z)$ is defined by (1.13) and the action of $x[m], \text{ denoted by } \widehat{x[m]}$, is defined by

$$\widehat{x[m]}\Phi(z) := \text{Res}_{\zeta = z}(\zeta - z)^{m-h-1} x(\zeta)\Phi(z) \quad (x = \beta_\alpha, \gamma_\alpha),$$

$$\phi_i[m]\Phi(z) := \text{Res}_{\zeta = z}(\zeta - z)^m \partial \phi_i(\zeta)\Phi(z),$$

where $h$ is the conformal spin of the field $x(z) = \sum_{n \in \mathbb{Z}} x[z]z^{-n-h}$ (cf. (1.0)) and $\Phi(z) \in \mathcal{O}_\lambda$. An element of $\mathcal{O}_\lambda$ maps $\text{Wak}_{\mu, k}$ to $\text{Wak}_{\lambda + \mu, k}$ for any $\mu$:

$$\mathcal{O}_\lambda \subset \text{Hom}_C(\text{Wak}_{\mu, k}, \text{Wak}_{\lambda + \mu, k}). \tag{1.39}$$

Because of the operator product expansions, (1.12), (1.13), $\mathcal{O}_\lambda$ is a $\widehat{G}(g) \oplus \widehat{\text{Bos}}(g)$ module:

**Lemma 1.3.** (i) *For any fields* $x, y = \beta_\alpha, \gamma_\alpha, \phi_i$ *and any integers* $m, n \in \mathbb{Z}$, we have

$$[\widehat{x[m]}, y[n]] = [x[m], y[n]]^\wedge \in \text{End}_C(\mathcal{O}_\lambda).$$

(ii) $\mathcal{O}_\lambda$ *is generated by* $V(\lambda;z)$, *which satisfies*

$$\widehat{\beta_\alpha[m]}V(\lambda;z) = \gamma_\alpha[n]V(\lambda;z) = \phi_i[m]V(\lambda;z) = 0, \quad \phi_i[0]V(\lambda;z) = \lambda(H_i)V(\lambda;z),$$

*for* $\alpha \in \Delta_+, i = 1, \ldots, l, m \geq 0, n > 0$.

The second statement is due to the operator product expansions:

$$\beta_\alpha(z)V(\lambda;w) \sim 0, \quad \gamma_\alpha(z)V(\lambda;w) \sim 0, \quad \phi_i(z)V(\lambda;w) \sim \frac{\lambda(H_i)}{z-w}V(\lambda;w). \tag{1.41}$$

Hence, the universality of Fock representations implies
Corollary 1.4. There exists a unique surjective homomorphism
\begin{equation}
\Phi_\lambda: \text{Wak}_\lambda \rightarrow \mathcal{O}_\lambda,
\end{equation}
which maps $|0\rangle_{\text{gh}} \otimes |\lambda\rangle_{\text{bos}}$ to $V(\lambda; z)$.

It follows immediately from (1.38) that the $\mathfrak{g}$-submodule $\text{Wak}^0_{\lambda,k}$ defined by (1.34) is isomorphically mapped to
\begin{equation}
\mathcal{P}_\lambda := \{ P(\gamma(z)) V(\lambda; z) | P(x) \text{ is a polynomial of } x = (x^a)_{a \in \Delta_+} \}
\end{equation}
by $\Phi_\lambda$:
\begin{equation}
\Phi_\lambda(P(\gamma(0))|0\rangle_{\text{gh}} \otimes |\lambda\rangle_{\text{bos}}) = P(\gamma(z)) V(\lambda; z).
\end{equation}

Since $\hat{\mathfrak{g}} \oplus \text{Vir}$ is realized in terms of ghosts and bosons through (1.28) and (1.30), we can define action of $X[m] \ (X \in \mathfrak{g}, m \in \mathbb{Z})$ and $T[m] \ (m \in \mathbb{Z})$ on $\mathcal{O}_\lambda$ by replacing $\beta_\alpha[n]$ etc. in $\omega(X[m])$, $\omega(T[m])$ with $\hat{\beta}_\alpha[n]$ etc. defined by (1.38) respectively. In fact, their actions are described more simply; thanks to the following lemma:

Assume that $B_i(z) = \sum_{m \in \mathbb{Z}} B_i[m] z^{-m-h_i} \ (i = 1, 2)$ are fields which have the operator product expansion
\begin{equation}
B_1(z) B_2(w) = \sum_{j=1}^N \frac{B_{12,j}(w)}{(z-w)^j} + :B_1(z)B_2(w):,
\end{equation}
where the normal order product $\cdots$ is defined by
\begin{equation}
:B_1[m] B_2[n]: = \begin{cases} B_1[m] B_2[n], & m \leq -h_1, \\ B_2[n] B_1[m], & m > -h_1 \end{cases}
\end{equation}
and the field $B_i(z) B_2(w)$: has no singularity at $z = w$. We denote its restriction to the diagonal $\{ z = w \}$ by $B_3(z)$:
\begin{equation}
B_3(z) := \sum_{m \in \mathbb{Z}} B_3[m] z^{-m-h_3}, \quad B_3[m] = \sum_{n \in \mathbb{Z}} :B_1[n] B_2[m-n]:,
\end{equation}
where $h_3 = h_1 + h_2$.

Lemma 1.5. For any $\Phi(z) \in \mathcal{O}_\lambda$, we have
\begin{equation}
\hat{B}_3[m] \Phi(z) = \sum_{n \in \mathbb{Z}} \hat{B}_1[n] \hat{B}_2[m-n] \hat{\Phi}(z),
\end{equation}
where
\begin{equation}
\hat{B}_1[m] \Phi(z) = \text{Res}_{\zeta = z}(\zeta - z)^{m+h_1-1} B_1(\zeta) \Phi(z),
\end{equation}
and the normal ordering $\hat{\circ}$ is defined by the same rule as in (1.46).

Corollary 1.6. For $X \in \mathfrak{g}, m \in \mathbb{Z}$ and $\Phi(z) \in \mathcal{O}_\lambda$, we have
\begin{equation}
\hat{X}[m] \Phi(z) = \text{Res}_{\zeta = z}(\zeta - z)^m X(\zeta) \Phi(z),
\end{equation}
\begin{equation}
\hat{T}[m] \Phi(z) = \text{Res}_{\zeta = z}(\zeta - z)^{m+1} T(\zeta) \Phi(z),
\end{equation}
where action in the left hand side is defined by replacing $\beta_\alpha[m]$ etc. in $\omega(X[m])$ and $\omega(T[m])$ by $\hat{\beta}_\alpha[m]$ etc. defined by (1.38).

Using the operator product expansions,
\begin{equation}
T(z) V(\lambda; w) \sim \frac{\partial V(\lambda; w)}{z-w} + \frac{\Delta_\lambda V(\lambda; w)}{(z-w)^2},
\end{equation}
where $\Delta_\lambda = (\lambda|\lambda + 2\rho)/2\kappa$ is the conformal weight of $V(\lambda; w)$, and the fact
\begin{equation}
[T[-1], x(z)] = \partial x(z),
\end{equation}
for $x = \beta_\alpha, \gamma^a$ or $\partial \phi_i$, which is a direct consequence of (1.21), we can prove the following formula by induction.

Lemma 1.7. For any $\Phi(z) \in \mathcal{O}_\lambda$, we have
\begin{equation}
\partial \Phi(z) = [T[-1], \Phi(z)] = \hat{T}[-1] \Phi(z).
\end{equation}
1.5. **Screening operators.** Bosonization of the operator $Scr(X; x, \partial_x)$ in (1.4) gives the screening operator. The ghost sector of the screening operator is defined for any positive root $\alpha \in \Delta_+$ as follows:

$$Scr_\alpha(z) := :Scr(e_\alpha; \gamma(z), \beta(z)):.$$  

The screening operator or the screening current is defined for simple roots $\alpha_i$ ($i = 1, \ldots, l$) as the product of $Scr_\alpha$, and a bosonic vertex operator (see (1.15)):

$$scr_i(z) := Scr_{\alpha_i}(z)V(-\alpha_i; z) \in \mathcal{O}_{-\alpha_i}.$$  

Important property of screening currents is the following operator product expansions.

$$X(z)scr_i(w) \sim 0,$$

$$F_j(z)scr_i(w) \sim -\kappa \delta_{ij} \frac{\partial}{\partial w} \frac{V(-\alpha_i; w)}{z - w},$$

$$T(z)scr_i(w) \sim \frac{\partial}{\partial w} \frac{scr_i(w)}{z - w},$$

where $X \in \mathfrak{h}_+$ and $i, j = 1, \ldots, l$. The ghost sector of the screening current has the following operator product expansion, which shall be used in computing explicit forms of the integral representations of solutions of the KZB equations:

$$H^h(z)Scr_\alpha(w) \sim \frac{\alpha(H)Scr_\alpha(w)}{z - w},$$

$$T^h(z)Scr_\alpha(w) \sim \frac{\partial Scr_\alpha(w)}{z - w} + \frac{Scr_\alpha(w)}{(z - w)^2},$$

$$Scr_\alpha(z)Scr_\beta(w) \sim \frac{f^{\alpha+\beta}}{\alpha, \beta} \frac{Scr_{\alpha+\beta}(w)}{z - w},$$

$$Scr_\alpha(z)P(\gamma(w)) \sim \frac{(Scr_\alpha P(\gamma(w)))}{z - w},$$

for any $P(x) \in \mathbb{C}[x]$ ($x = (x^\alpha)_{\alpha \in \Delta}$), where $f^{\alpha+\beta}$ is the structure constant of the Lie algebra $\mathfrak{n}_+$, $[e_\alpha, e_\beta] = f^{\alpha+\beta} e_{\alpha+\beta}$, and $(Scr_\alpha P)(x) \in \mathbb{C}[x]$ is the polynomial $(Scr_\alpha P)(x) = Scr(e_\alpha; x, \partial_x)P(x) \in \mathbb{C}[x].$

2. WZW models on elliptic curves.

In this section we recall the definition (or a characterization) of $N$-point functions on elliptic curves.

2.1. **Space of conformal coinvariants and space of conformal blocks.** $N$-point functions of the WZW model take values in the space of the conformal blocks which is the dual of the space of the conformal coinvariants. (Exactly speaking, $N$-point functions are sections of a vector bundle, a fiber of which is the space of conformal blocks. See Section 2.2.) To define the space of conformal coinvariants and conformal blocks, we first need a Lie algebra bundle over an elliptic curve with marked points.

For each $q \in \mathbb{C}^\times$, $|q| < 1$, we define an elliptic curve $X = X_q$ by

$$X_q := \mathbb{C}^\times / q^\mathbb{Z},$$

where $q^\mathbb{Z} = \{q^n \mid n \in \mathbb{Z}\}$ is a multiplicative group acting on $\mathbb{C}^\times$ by $z \mapsto q^n z$. A Lie algebra bundle $\mathfrak{g}^H$ is defined for each $H \in \mathfrak{h}$ by

$$\mathfrak{g}^H = \mathbb{C}^\times \times \mathfrak{g} / \sim,$$

where the equivalence relation $\sim$ is

$$(t, A) \sim (qt, e^{-ad_H}A).$$

This Lie algebra bundle $\mathfrak{g}^H$ has a natural connection, $\nabla_{A/dt} = td/dt$, and is decomposed into a direct sum of line bundles and a trivial bundle with fiber $\mathfrak{h}$:

$$\mathfrak{g}^H \cong \bigoplus_{\alpha \in \Delta} L_{\alpha(H)} \oplus (\mathfrak{h} \times X).$$
Here the line bundle $L_c$ ($c \in \mathbb{C}$) on $X$ is defined by
\begin{equation}
L_c := (\mathbb{C} \times \mathbb{C})/\sim_c,
\end{equation}
where $\sim_c$ is an equivalence relation defined by
\begin{equation}
(t, x) \sim_c (qt, e^{-c}x).
\end{equation}

As usual, the structure sheaf on $X = X_c$ is denoted by $\mathcal{O}_X$ and the sheaf of meromorphic functions on $X$ by $\mathcal{K}_X$. A stalk of a sheaf $\mathcal{F}$ on $X$ at a point $P \in X$ is denoted by $\mathcal{F}_P$. When $\mathcal{F}$ is an $\mathcal{O}_X$-module, we denote its fiber $\mathcal{F}_P/\mathcal{m}_P\mathcal{F}_P$ by $\mathcal{F}|_P$, where $\mathcal{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Denote by $\mathcal{F}_\mathbb{A}$ the $\mathcal{m}_P$-adic completion of $\mathcal{F}_P$.

We shall use the same symbol for a vector bundle and for a locally free coherent $\mathcal{O}_X$-module consisting of its local holomorphic sections. For instance, the invertible sheaf associated to the line bundle $L_c$ is also denoted by the same symbol $L_c$. Denote by $\Omega^1_X$ the sheaf of holomorphic 1-forms on $X$, which is isomorphic to $\mathcal{O}_X$ since $X$ is an elliptic curve. The fiberwise Lie algebra structure of the bundle $\mathfrak{g}^H$ induces that of the associated sheaf $\mathfrak{g}^H$ over $\mathcal{O}_X$. Define the invariant $\mathcal{O}_X$-inner product on $\mathfrak{g}^H$ by
\begin{equation}
\langle A|B \rangle := \frac{1}{2\hbar^2} \text{Tr}_{\mathfrak{g}^H} (\text{ad} A \text{ ad} B) \in \mathcal{O}_X \quad \text{for} \ A, B \in \mathfrak{g}^H,
\end{equation}
where the symbol $\text{ad}$ denotes the adjoint representation of the $\mathcal{O}_X$-Lie algebra $\mathfrak{g}$. Then the inner product on $\mathfrak{g}^H$ is invariant under the translations with respect to the connection $\nabla : \mathfrak{g}^H \to \mathfrak{g}^H \otimes_{\mathcal{O}_X} \Omega^1_X$:
\begin{equation}
d(A|B) = (\nabla A|B) + (A|\nabla B) \in \Omega^1_X \quad \text{for} \ A, B \in \mathfrak{g}^H.
\end{equation}

Under the trivialization of $\mathfrak{g}^H$ defined by the construction (2.2), the connection $\nabla$ coincides with the exterior derivative by $td/dt$.

The fiber of $\mathfrak{g}^H$ is isomorphic to $\mathfrak{g}$. For any point $P$ on $X$ with $t(P) = z$, we put
\begin{equation}
\mathfrak{g}^P := (\mathfrak{g}^H \otimes_{\mathcal{O}_X} \mathcal{K}_X)_{\hat{z}}^\mathbb{A},
\end{equation}
which is a topological Lie algebra non-canonically isomorphic to the loop algebra $\mathfrak{g}([t-z])$. Its subspace $\mathfrak{g}^P_{\hat{z}} := (\mathfrak{g}^H)_{\hat{z}}^\mathbb{A} \cong \mathfrak{g}[[t-z]]$ is a maximal linearly compact subalgebra of $\mathfrak{g}^P$ under the $(t-z)$-adic linear topology.

Let us fix mutually distinct points $P_1, \ldots, P_N$ on $X$ whose coordinates are $t = z_1, \ldots, z_N$ and put $D := \{P_1, \ldots, P_N\}$. We shall also regard $D$ as a divisor on $X$ (i.e., $D = P_1 + \cdots + P_N$). Denote $X \setminus D$ by $\breve{X}$. The Lie algebra $\mathfrak{g}^D := \bigoplus_{i=1}^N \mathfrak{g}^P_{\hat{z}_i}$ has the natural 2-cocycle defined by
\begin{equation}
c_a(A, B) := \sum_{i=1}^N \text{Res}_{t=z_i} (\nabla A_i|B_i),
\end{equation}
where $A = (A_i)_{i=1}^N, B = (B_i)_{i=1}^N \in \mathfrak{g}^D$ and $\text{Res}_{t=z}$ is the residue at $t = z$. (The symbol "$c_a$" stands for "Cocycle defining the Affine Lie algebra".) We denote the central extension of $\mathfrak{g}^D$ with respect to this cocycle by $\mathfrak{g}^D_{\hat{k}}$:
\begin{equation}
\mathfrak{g}^D_{\hat{k}} := \mathfrak{g}^D \oplus \mathbb{C}\hat{k},
\end{equation}
where $\hat{k}$ is a central element. Explicitly the bracket of $\mathfrak{g}^D_{\hat{k}}$ is represented as
\begin{equation}
[A, B] = ([A, B_i]|^0_{i=1} + c_a(A, B)\hat{k}) \quad \text{for} \ A, B \in \mathfrak{g}^D,
\end{equation}
where $[A, B]^0$ are the natural bracket in $\mathfrak{g}^P$. The Lie algebra $\mathfrak{g}^P$ for a point $P$ is non-canonically isomorphic to the affine Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^P_{\hat{z}}$ can be regarded as a subalgebra of $\mathfrak{g}^D$. Put $\mathfrak{g}^{P^i} := (\mathfrak{g}^H)_{\hat{z}_i}$ as above. Then $\mathfrak{g}^{P^i}$ can be also regarded as a subalgebra of $\mathfrak{g}^D$ and $\mathfrak{g}^{P^i}_{\hat{z}_i}$.

Let $\mathfrak{g}^{H,D}_X$ be the space of global meromorphic sections of $\mathfrak{g}^H$ which are holomorphic on $\breve{X}$:
\begin{equation}
\mathfrak{g}^{H,D}_X := \Gamma(X, \mathfrak{g}^H(*D)).
\end{equation}
There is a natural linear map from $\mathfrak{g}^{H,D}_X$ into $\mathfrak{g}^D$ which maps a meromorphic section of $\mathfrak{g}^H$ to its germ at $P_i$'s. The residue theorem implies that this linear map is extended to a Lie algebra injective homomorphism from $\mathfrak{g}^{H,D}_X$ into $\mathfrak{g}^D$, which allows us to regard $\mathfrak{g}^{H,D}_X$ as a subalgebra of $\mathfrak{g}^D$. 
Definition 2.1. The space of conformal coinvariants $CC_H(X_q, D, M)$ and that of conformal blocks $CB_H(X_q, D, M)$ associated to $\mathfrak{g}^P$-modules $M_i$ with the same level $\hat{k} = k$ are defined to be the space of coinvariants of $M := \bigotimes_{i=1}^N M_i$ with respect to $\mathfrak{g}_X^{H,D}$ and its dual:

\begin{equation}
CC_H(X_q, D, M) := M/\mathfrak{g}_X^{H,D}M, \quad CB_H(X_q, D, M) := (M/\mathfrak{g}_X^{H,D}M)^*.
\end{equation}

(In [FW], $CC_H(X_q, D, M)$ and $CB_H(X_q, D, M)$ are called the space of covacua and that of vacua respectively.)

In other words, the space of conformal coinvariants $CC_H(X_q, D, M)$ is generated by $M$ with relations

\begin{equation}
A_X v = 0 \quad \text{for all } A_X \in \mathfrak{g}_X^{H,D} \text{ and } v \in M,
\end{equation}

and a linear functional $\Phi$ on $M$ belongs to the space of conformal blocks $CB_H(X_q, D, M)$ if and only if it satisfies that

\begin{equation}
\Phi(A_X v) = 0 \quad \text{for all } A_X \in \mathfrak{g}_X^{H,D} \text{ and } v \in M.
\end{equation}

These equations (2.13) and (2.14) are called the Ward identities.

2.2. N-point functions. N-point functions are flat sections of a sheaf of conformal blocks over the base space $S$ of a family $\tilde{X}$ of pointed curves with marked points defined as follows:

\[ S := \{ (z; q; H) = (z_1, \ldots, z_N; q; H) \in (\mathbb{C}^\times)^N \times \mathbb{C}^\times \times \mathfrak{h} \mid z_i/z_j \notin \mathbb{Q}^\times \text{ if } i \neq j \}, \]

\[ \tilde{X} := S \times \mathbb{C}^\times. \]

Let $\tilde{\pi} = \pi_{\tilde{X}/S}$ be the projection from $\tilde{X}$ onto $S$ along $\mathbb{C}^\times$, $\pi(z; q; H; t) = (z; q; H)$, and $\tilde{p}_i$ the section of $\tilde{\pi}$ given by $z_i$:

\[ \tilde{p}_i(z; q; H; t) := (z; q; H; z_i) \in \tilde{X} \quad \text{for } (z; q; H) = (z_1, \ldots, z_N; q; H) \in S. \]

A family of $N$-pointed elliptic curves $\pi : \tilde{X} \rightarrow S$ is constructed as follows. Define the action of $\mathbb{Z}$ on $\tilde{X}$ by

\begin{equation}
m \cdot (z; q; H; t) := (z; q; H; q^m t) \quad \text{for } m \in \mathbb{Z}, (z; q; H; t) \in \tilde{X}.
\end{equation}

Let $X$ be the quotient space of $\tilde{X}$ by the action of $\mathbb{Z}$:

\begin{equation}
X := \tilde{X} / \mathbb{Z}.
\end{equation}

Let $\pi_{\tilde{X}/X}$ be the natural projection from $\tilde{X}$ onto $X$ and $\pi = \pi_{\tilde{X}/S}$ the projection from $\tilde{X}$ onto $S$ induced by $\tilde{\pi}$. We put

\[ p_i := \pi_{\tilde{X}/X} \circ \tilde{p}_i, \quad P_i := p_i(S), \quad D := \bigcup_{i=1}^N P_i, \quad \hat{X} := X \setminus D, \quad \hat{D} := \pi_{\tilde{X}/X}^{-1}(D). \]

Here $p_i$ is the section of $\pi$ induced by $\tilde{p}_i$ and $D$ is also regarded as a divisor $\sum_{i=1}^N P_i$ on $X$. The fiber of $\pi$ at $(z; q; H) = (z_1, \ldots, z_N; q; H) \in S$ is an elliptic curve with modulus $q$ and marked points $z_1, \ldots, z_N$.

We refer to [FW] for the construction of a sheaf of conformal blocks and its flat connection and, using their result, define the $N$-point functions as follows. (See also [3].)

We identify each fiber of $\mathfrak{g}^H$ with $\mathfrak{g}$ via the standard trivialization defined by the construction of $\mathfrak{g}_X^H$, (2.3). Then the algebra $\mathfrak{g}^P$ defined by (2.9) is identified with $\mathfrak{g}(t-z)$ where $t$ is the coordinate on the complex plane and $z$ is the coordinate of $P$. For $X \in \mathfrak{g}$, the element $X \otimes (t-z)^m$ of $\mathfrak{g}^P \cong \mathfrak{g}(t-z)$ is denoted by $X[m]$. The Virasoro generator defined by the Sugawara construction (1.29) is denoted by $T[m]$:

\begin{equation}
T[m] = \frac{1}{2\kappa} \sum_{p=1}^{\text{dim} \mathfrak{g}} \sum_{n \in \mathbb{Z}} J_p[m-n]J^p[n].
\end{equation}

Let us denote the representation of $\hat{\mathfrak{g}}^P$ on the $i$-th component of the tensor product $M = \bigotimes_{i=1}^N M_i$ by $\rho_i$ and its dual by $\rho_i^*$: for $v \in M$, $v^* \in M^*$, $A \in \hat{\mathfrak{g}}^P$,

\begin{equation}
\langle \rho_i^*(A)v^*, v \rangle = -\langle v^*, \rho_i(A)v \rangle
\end{equation}

where $\langle \cdot, \cdot \rangle$ is the pairing of $M^*$ and $M$. We assume that the Virasoro algebra with central charge $c_V$ acts on $M_i$ and $M_i^*$ through the Sugawara construction (2.17).
Then we can restrict an

Let us define a function \( \Pi(z) \) by

(2.19)

\[
\rho \left( T \left\{ f(t) \frac{d}{dt} \right\} \right) v := \sum_{i=1}^{N} \sum_{m \in \mathbb{Z}} f_{i,m+1} \rho_i(T[m]) v, \quad \rho^* \left( T \left\{ f(t) \frac{d}{dt} \right\} \right) v^* := \sum_{i=1}^{N} \sum_{m \in \mathbb{Z}} f_{i,m+1} \rho_i^*(T[m]) v^*,
\]

where \( f(t) = \sum_{m \in \mathbb{Z}} f_{i,m}(t - z_i)^m \) is the Laurent expansion of \( f(t) \) around \( t = z_i \).

Fix a meromorphic function \( Z(z; q; H; t) \) on \( \mathbb{F} \) with poles only at \( \{z_1, \ldots, z_N\} \) (mod \( q^Z \)) (namely \( Z(z; q; H; t) \in \Gamma(\mathbb{F}, \mathcal{O}_\mathbb{F}(\mathcal{D})) \)) satisfying

(2.21)

\[
Z(z; q; H; t) = Z(z; q; H; t) - 1.
\]

We abbreviate \( Z(q; z; H; t) \) as \( Z(t) \).

Example 2.2. We may take the following function as \( Z(t) = Z(q; z; H; t) \):

(2.22)

\[
Z(q; z; H; t) := \frac{1}{2\pi i z_{i_0}} \frac{d}{dt} \log \theta_{11}(t/\alpha_{i_0}; q),
\]

for \( i_0 \in \{1, \ldots, N\} \).

Let us take a coordinate system of \( \mathfrak{h} \) as \( \mathfrak{h} \ni H = \sum_{a=1}^{l} \xi_a H_a \), where \( \{H_a\}_{a=1}^{l} \) is an orthonormal basis of \( \mathfrak{h} \).

Definition 2.3. A multi-valued holomorphic function \( \Psi(z; q; H) \) on \( S \) with values in \( M^* \) is called an \( N \)-point function in genus one if it satisfies the following conditions (I), (II), (III) and (IV):

(I) For any \( (z, q, H) \in S \), \( \Psi(z; q; H) \in \text{CB}_H(X_q, D, M) \);

(II) For \( j = 1, \ldots, N \),

(2.23)

\[
\frac{\partial}{\partial z_j} \Psi(z; q; H) = \rho_j^*(T[-1])\Psi(z; q; H);
\]

(III)

(2.24)

\[
\left( q \frac{\partial}{\partial q} + \frac{C_{11}}{24} \right) \Psi(z; q; H) = \rho^* \left( T \left\{ Z(t) \frac{d}{dt} \right\} \right) \Psi(z; q; H);
\]

(IV) For \( r = 1, \ldots, l \),

(2.25)

\[
\frac{\partial}{\partial \xi_r} \Psi(z; q; H) = -\rho^*(H_r \{ Z(t) \}) \Psi(z; q; H).
\]

Assume that each \( M_i \) contains a \( \mathfrak{g} \)-submodule \( V_i \) such that for any \( m > 0 \) and \( X \in \mathfrak{g} \),

(2.26)

\[
X[m] V_i = 0,
\]

and the Casimir operator \( C_2 = \sum_{p} J_p J_p \) acts as a multiplication,

(2.27)

\[
\rho_i(C_2) = c_{2}^{(i)} \text{id}_{V_i}.
\]

Let us define a function \( \Pi(q; H) \) by

(2.28)

\[
\Pi(q; H) := q^{\dim \mathfrak{g}/24} (q; q)_t^{l} \prod_{\alpha \in \Delta^+} 2 \sinh(\alpha(H)/2) \prod_{\alpha \in \Delta} (q e^{\alpha(H)}; q)^{\infty}.
\]

Then we can restrict an \( N \)-point function \( \Psi(z; q; H) \) to \( V = \bigotimes_{i=1}^{N} V_i \):

(2.29)

\[
\Psi_V(z; q; H) := \Psi(z; q; H)|_V,
\]

and \( \Psi_V(z; q; H) = \Pi(q; H) \Psi_V(z; q; H) \) satisfies the following system, which is called the Knizhnik-Zamolodchikov-Bernard (KZB) equations first found by Bernard [1]. (See Theorem 4.1 of [FW]): The functions \( \sigma_c(z) \) and \( \zeta(z) \) below are defined by \( [A.3] \) and \( [A.8] \).
(I') For any $H \in \mathfrak{h}$,

$$
(2.30) \quad \sum_{i=1}^{N} \rho_i^*(H) \tilde{\Psi}_V(z; q; H) = 0;
$$

(II') For $j = 1, \ldots, N$,

$$
(2.31) \quad \kappa \left( z_j \frac{\partial}{\partial z_j} + \frac{c_j(j)}{2\kappa} \right) \tilde{\Psi}_V(z; q; H) = \sum_{r=1}^{l} \rho_j^*(H_r) \frac{\partial}{\partial \xi_r} \tilde{\Psi}_V(z; q; H) + \sum_{i \neq j} (\rho_i^* \otimes \rho_j^* \Omega(z_i, z_j)) \tilde{\Psi}_V(z; q; H),
$$

where

$$
(2.32) \quad \Omega(z, w) = \Omega(z, w; q; H) := -\sum_{\alpha \in \Delta} \sigma_{-\alpha(H)} \left( \frac{z}{w} \right) e_\alpha \otimes e_{-\alpha} - \sum_{r=1}^{l} \frac{1}{2} \left( \frac{z}{w} \right)^2 + \frac{z'}{w} \right) H_r \otimes H_r.
$$

(III')

$$
(2.33) \quad 2\kappa q \frac{\partial}{\partial q} \tilde{\Psi}_V(z; q; H) = \sum_{r=1}^{l} \left( \frac{\partial}{\partial \xi_r} \right)^2 \tilde{\Psi}_V(z; q; H) + \sum_{i,j=1}^{N} (\rho_i^* \otimes \rho_j^* H(z_i, z_j)) \tilde{\Psi}_V(z; q; H),
$$

where

$$
(2.34) \quad H(z, w) = H(z, w; q; H) := -\sum_{\alpha \in \Delta} \left( \zeta \left( \frac{e^{\alpha(H)} z}{w} \right) \right) \sigma_{-\alpha(H)} \left( \frac{z}{w} \right) e_\alpha \otimes e_{-\alpha} - \sum_{r=1}^{l} \frac{1}{2} \left( \frac{z}{w} \right)^2 + \frac{z'}{w} \right) H_r \otimes H_r.
$$

Conversely, restriction of an $N$-point function to $V$ is characterized by the KZB equations with additional conditions. For example, Felder and Wieczerkowski [FW] used the automorphic properties and asymptotic behavior of $\tilde{\Psi}_V$ as the additional conditions, while Suzuki [S] found a holonomic system characterizing $\tilde{\Psi}_V$ which includes the KZB equations.

3. $N$-POINT FUNCTIONS FROM WAKIMOTO MODULES

In this section we construct $N$-point functions for Wakimoto modules, $M_i = \text{Wak}_{\lambda_i, k}$, where $\lambda_i \in \mathfrak{h}^*$ and $\sum_{i=1}^{N} \lambda_i$ belongs to the positive root lattice of $\mathfrak{g}$.

Fix ordered sets of the simple roots of $\mathfrak{g}$, $\{\alpha_1, \ldots, \alpha_m\}$, such that

$$
(3.1) \quad \sum_{j=1}^{M} \alpha_{i_j} = \sum_{i=1}^{N} \lambda_i.
$$

Denote the following linear map by $\psi(z; t; q; H)$, where $z = (z_1, \ldots, z_N)$, $t = (t_1, \ldots, t_M)$ and $(z, t)$ belongs to (the universal covering of) $(\mathbb{C}^\times)^{N+M} \setminus \{\text{diagonals}\}$:

$$
(3.2) \quad \text{Wak}_{\lambda_i, k} \otimes \cdots \otimes \text{Wak}_{\lambda_N, k} \ni v_1 \otimes \cdots \otimes v_N \mapsto \text{Tr}_{\text{Wak}_{\lambda_i, k}}(\Phi_{\lambda_i}(v_1; z_1) \cdots \Phi_{\lambda_N}(v_N; z_N) \text{sc}_{r_1}(t_1) \cdots \text{sc}_{r_M}(t_M)) q^{-c/2} e^{H[0]} dt_1 \wedge \cdots \wedge dt_M.
$$

Note that, thanks to (3.39) and (3.1), the operator inside the bracket in the right hand side is an endomorphism of $\text{Wak}_{\lambda_i, k}$.

**Proposition 3.1.** There is a local system of rank one $\mathcal{L}$ on $\{ (z; t; q; H) \mid (z; t) = (z_1, \ldots, z_N; t_1, \ldots, t_M) \in X_{\mathfrak{g}, q}^{N+M} \setminus \{\text{diagonals}\}, q \in \mathbb{C}^\times, H \in \mathfrak{h} \}$ such that $\psi(z; t; q; H)(v)$ is a holomorphic section of $\mathcal{L}$ for any $v \in M$. Namely, the monodromy of $\psi(z; t; q; H)(v)$ is independent of $v$.

**Proof.** First note that $\Phi_{\lambda_i}(v_i; z_i) \in \mathcal{O}_{\lambda}$ is of the form,

$$
(3.3) \quad \Phi_{\lambda_i}(v_i; z_i) = \sum_{m_1}^{\infty} \cdots x_n[m_n] V(\lambda_i; z_i)
$$

$$
\int_{C_{\lambda_i}} d\zeta_{i1}(\zeta_{i1} - z_{i1})^{m_1 + h_i - 1} \cdots \int_{C_{\lambda_n}} d\zeta_{in}(\zeta_{in} - z_{in})^{m_n + h_n - 1} x_1(\zeta_{i1}) \cdots x_n(\zeta_{in}) V(\lambda_i; z_i),
$$
where $x_j$ is $\beta_\alpha$, $\gamma_\alpha$ or $\partial \phi_j$ and $m_j \in \mathbb{Z}$. The contour $C_{ij}$ encircles $z_i$, lies outside of $C_{ij'}$ ($j' > j$) and does not contain 0 and $z_{t'}$ ($i' \neq i$) inside it.

It follows from (3.4), (5.53) and this expression (3.3) that $\psi(z; t; q; H)(v_1 \otimes \cdots \otimes v_N)$ is sum of integrals of the form

$$\int_{C_{ij}} d\zeta \cdots \int_{C_{in}} d\zeta_n \text{(rational function of } \zeta_{ij}, \zeta_{ij'} \text{)} F^{\text{rational}}(\zeta_{ij}, t; q; H) F^{\text{rational}}(\zeta_{ij'}, z_{t'}; q; H) dt_1 \wedge \cdots \wedge dt_M,$$

(3.4)

$$F^{\text{bos}}(\zeta_{ij}, t; q; H) := \text{Tr}_{F^{\text{bos}}}(\text{(polynomial of } \beta_\alpha(\zeta_{ij}), \beta_\alpha(t), \gamma_\alpha(\zeta_{ij}), \gamma_\alpha(t)) q^{\text{bos}}(0) e^{H^{\text{bos}}(0)})$$

(3.5)

$$F^{\text{bos}}(\zeta_{ij}, z_{t}, t; q; H) := \text{Tr}_{F^{\text{bos}}}(\prod_j \phi_{r_{ij}}(\zeta_{ij}) V(\lambda_1; z_1) \cdots \prod_j \phi_{r_{Nj}}(\zeta_{Nj}) V(\lambda_N; z_N) \times$$

$$\times V(-\alpha_{i1}; t_1) \cdots V(-\alpha_{im}; t_m) q^{\text{bos}}(0) e^{i[H, 0]}$$

(3.6)

where “polynomial of $\beta_\alpha$, etc.” contain possibly normal ordered products coming from $\text{Scr}_{\alpha_{ij}}(t_j)$.

The first trace in the integrand in (3.3), $F^{\text{bos}}(\zeta_{ij}, t; q; H)$, has singularities at $\zeta_{ij} = \zeta_{i'j'}$ and $\zeta_{ij} = t_{i'}$, which are poles because of the operator product expansions (1.7).

The second trace in the integrand in (3.4), $F^{\text{bos}}(\zeta_{ij}, z_{t}, t; q; H)$, has singularities at (1) $\zeta_{ij} = \zeta_{i'j'}$; (2) $\zeta_{ij} = z_{t'}$ or $t_{i'}$; (3) $z_i = t_i$, $t_i = t_{i'}$. The first singularities (1) are poles because of the operator product expansion (1.12). The second singularities (2) are also rational by virtue of the third expansion in (1.41). The formula (cf. (3.2))

$$V(\lambda; z)V(\lambda'; w) = \lambda^{(\lambda')/\kappa}$$

implies that $F^{\text{bos}}$ has non-trivial monodromy around the singularities (3):

$$F^{\text{bos}}(\zeta; z; t; q; H) \rightarrow e^{2\pi i(\lambda_1)/\kappa} F^{\text{bos}}(\zeta; z; t; q; H), \text{ when } z_i \text{ goes around } z_j, \text{ (1 \leq i < j \leq N)},$$

$$F^{\text{bos}}(\zeta; z; t; q; H) \rightarrow e^{2\pi i(\lambda_1 - \alpha_{ij})/\kappa} F^{\text{bos}}(\zeta; z; t; q; H), \text{ when } z_i \text{ goes around } t_j, \text{ (1 \leq i \leq N, 1 \leq j \leq M)}),$$

$$F^{\text{bos}}(\zeta; z; t; q; H) \rightarrow e^{2\pi i(\alpha_{ij} - \alpha_{i'j'})/\kappa} F^{\text{bos}}(\zeta; z; t; q; H), \text{ when } t_{i_j} \text{ goes around } t_{i'j'}, \text{ (1 \leq j < j' \leq M)}.$$

(3.7)

Summarizing, we conclude that the integrand in (3.3) is rational function with respect to $\zeta_{ij}$’s and also rational with respect to $z_i$’s and $t_i$’s except at $z_i = t_{i'}$, $z_i = t_{i'}$, where it has the same monodromy as $F^{\text{bos}}$, (3.8).

As a next step, we show that $\psi(z; t; q; H)(v_1 \otimes \cdots \otimes v_N)$ has the same monodromy as the integrand in (3.4). This is proved by applying the following lemma iteratively.

**Lemma 3.2.** Assume that a function $F(\zeta, z, t)$ is rational with respect to $\zeta$ and has monodromy with respect to $z$ and $t$ around the diagonal $z = t$: $F(\zeta, z, t) \rightarrow cF(\zeta, z, t)$ when $z$ goes around $t$, where $c$ is a constant. Then

$$\psi(z, t) = \oint_{C(z)} F(\zeta, z, t)d\zeta$$

(3.9)

has the same monodromy around $z = t$, where $C(z)$ is a small contour surrounding $z$.

**Proof.** Fix a small circle $\gamma$ around $t$,

$$\gamma(\theta) = t + \varepsilon \exp(i\theta), \quad \theta \in [0, 2\pi].$$

When $z$ goes around $t$ along $\gamma$, $F(\zeta, z, t)$ is multiplied by $c$. Since $F$ is rational with respect to $\zeta$, the integration contour $C(z)$ in (3.3) can be replaced with a cycle $\gamma_+ - \gamma_-$, where

$$\gamma_+(\theta) = t + \varepsilon_+ \exp(i\theta), \quad \theta \in [0, 2\pi]$$

and $\varepsilon_\pm$ are suitable constants satisfying $\varepsilon_- < \varepsilon < \varepsilon_+$. Now that $\gamma_\pm$ do not depend on $z$ and do not intersect with $\gamma$, it is obvious that

$$\psi(z, t) = \oint_{\gamma_+} F(\zeta, z, t)d\zeta - \oint_{\gamma_-} F(\zeta, z, t)d\zeta$$

is multiplied by $c$ when $z$ goes around $t$ along $\gamma$. 


We can similarly prove that the monodromies of $\psi(z; t; q; H)(v_1 \otimes \cdots \otimes v_N)$ around the cycles of $X_\lambda$ (along the paths, $z_i = r \exp(2\pi i \theta)$ ($0 \leq \theta \leq 2\pi$) and $z_i \to q z_i$, and the same for $t_j$) do not depend on $v_1 \otimes \cdots \otimes v_N$. This completes the proof of the proposition. 

Remark 3.3. We can also write down an explicit expression of the second trace in the integrand in (3.4), $\text{Tr}_{F_{\nu \alpha}}((\prod_{i,j} \partial \phi_{t,j}(\zeta_{ij})V(\lambda; z_i))q^{T[0]}e^{\nu(0)}e^{\nu[H,0]})$, by using (3.13) and (3.17). In fact, since
\[
\frac{\partial}{\partial z} \frac{\partial}{\partial t} \left|_{t=0} \right. \hat{V}(\tau; z) = \partial \phi(\tau; z),
\]
applying differential operators of the form $\partial_z \partial_t \mid_{t=0}$ to (3.17) and combining the result with (3.15), we have a desired expression, which also shows that this trace is rational with respect to $\zeta_{ij}$ and has monodromy around the diagonals $z_i = z_i'$ etc.

Theorem 3.4. 
(3.10) 
$\Psi(z; q; H) = \int_{C(z,q,H)} \psi(z; t; q; H)$

is an $N$-point function with values in $(Wak_{\lambda_1,k} \otimes \cdots \otimes Wak_{\lambda_N,k})^*$, where $C(z, q, H)$ is a family of $M$-cycles with coefficients in the local system $\mathcal{L}$.

Remark 3.5. We refer to [AK] or [FV] for integrals over cycles with coefficients in the local system. Proposition 3.1 guarantees that the integration in the right hand side of (3.10) is well-defined and that the right hand sides of (3.23), (3.24) and (3.25) are meaningful.

Proof. This can be shown in almost the same way as Proposition 3.4.1 of [3].

The condition (I) of Definition 3.3 is checked as follows. Fix $(z, q, H) \in S$. The condition (I) means that for any $J(t) \in g_X^H$,

(3.11) 
$\sum_{i=1}^N \rho_i^*(J(t)) \Psi(z; q; H) = 0.$

Thanks to the decomposition (2.4), we may assume $J(t) = X \otimes f(t)$ where $X \in g_\alpha$ ($\alpha \in \Delta \cup \{0\}$, $g_0 := h$), $f(t) \in \Gamma(X_\lambda, L_{\alpha}(H)(sD))$ ($L_0 = \mathcal{O}_{X_\lambda}$). The left hand side of (3.11) is equal to

\[
\int_{C(z,q)} \sum_{i=1}^N \rho_i^*(X \otimes f(t))\psi(z; t; q; H)(v_1 \otimes \cdots \otimes v_N)
\]

(3.12) 
$= \int_{C(z,q)} - \sum_{i=1}^N \text{Tr}(\Phi_{\lambda_1}(v_1; z_1) \cdots (X \otimes f(t))(\zeta_{ij}) \Phi_{\lambda_i}(v_i; z_i) \cdots \text{scr}_{t_j}(t_j) \cdots q^{T[0] - cv/24} e^{H[0]})$

$= \int_{C(z,q)} - \sum_{i=1}^N \text{Res}_{x=z} \psi(\zeta; \text{Tr}(\Phi_{\lambda_1}(v_1; z_1) \cdots X(\zeta) \Phi_{\lambda_i}(v_i; z_i) \cdots \text{scr}_{t_j}(t_j) \cdots q^{T[0] - cv/24} e^{H[0]}))d\zeta,$

where $\text{Tr}$ is $\text{Tr}_{Wak_{\mu,k}}$ and $(X \otimes f(t))(t)_{t_1}^{t_2}$ is the Laurent expansion of $X \otimes f(t)$ at $t = t_j$. The last line is due to the following fact, which is easily checked by (1.48): for any $\Phi(z) \in \mathcal{O}_\lambda$ and $X \in g_\alpha$,

(3.13) 
$(X \otimes f(t))_{t_1}^{t_2} \Phi(z) = \text{Res}_{x=z} \psi(\zeta) X(\zeta) \Phi(z))d\zeta.$

By the commutativity of current $X(\zeta)$, vertex operator $\Phi_{\lambda}(v; z)$ and screening current $\text{scr}_i(z)$, we have

(3.14) 
$f(\zeta) \text{Tr}(\Phi_{\lambda_1}(v_1; z_1) \cdots X(\zeta) \Phi_{\lambda_i}(v_i; z_i) \cdots \text{scr}_{t_j}(t_j) \cdots q^{T[0] - cv/24} e^{H[0]}))d\zeta$

$= f(\zeta) \text{Tr}(X(\zeta) \Phi_{\lambda_1}(v_1; z_1) \cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \text{scr}_{t_j}(t_j) \cdots q^{T[0] - cv/24} e^{H[0]})d\zeta$

$= f(\zeta) \text{Tr}(\Phi_{\lambda_1}(v_1; z_1) \cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \text{scr}_{t_j}(t_j) \cdots q^{T[0] - cv/24} e^{H[0]} X(\zeta))d\zeta$

$= f(q\zeta) \text{Tr}(\Phi_{\lambda_1}(v_1; z_1) \cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \text{scr}_{t_j}(t_j) \cdots X(q\zeta) q^{T[0] - cv/24} e^{H[0]}))d\zeta,$

where we used

(3.15) 
$e^{H[0]} X(\zeta) = e^{\alpha(0)} X(\zeta) e^{H[0]}$, $q^{T[0]} X(\zeta) = q X(q\zeta) q^{T[0]}.$
and \( f(qt) = e^{-\alpha H} f(t) \) (cf. (2.3)). Therefore,
\[
f(\zeta) \text{Tr} \left( \Phi_{\lambda_i}(v_i; z_i) \cdots X(\zeta) \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots q^{T[0]} - cv/24 e^{H[0]} \right) d\zeta \in \Gamma(X_q, \Omega^1_X(+D)).
\]
Hence the sum of its residues at \( \zeta = z_j \) \( (j = 1, \ldots, N) \) is zero by the residue theorem. Thus (3.12) implies (3.11).

The condition (II) of Definition 2.3 is a direct consequence of (1.52).

We prove the condition (III), assuming \( |q| < |t_1| < \cdots < |t_1| < |z_1| < \cdots < |z_i| < 1 \). The general case follows from this case by the analytic continuation. By the same argument as (3.13), we have
\[
(3.16) \quad \rho^* \left( T \left\{ Z(t) t \frac{d}{dt} \right\} \right) \psi(z; q; H) = \\
= \sum_{i=1}^N \text{Res}_{\zeta = z_i} Z(\zeta) \text{Tr} (T(\zeta) \cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots q^{T[0]} - cv/24 e^{H[0]}) d_M t \otimes \zeta d\zeta.
\]
Deforming the integration contour, the right hand side of (3.16) is rewritten as
\[
(3.17) \quad \frac{1}{2\pi i} \left( \oint_{|\zeta| = 1} - \oint_{|\zeta| = |q|} - \sum_{j=1}^M \oint_{|\zeta - t_j|} \right) \zeta d\zeta Z(\zeta) \times \\
\times \text{Tr} (T(\zeta) \cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots q^{T[0]} - cv/24 e^{H[0]}) d_M t
\]
The first integral in (3.17) is equal to
\[
(3.18) \quad \frac{1}{2\pi i} \oint_{|\zeta| = 1} \zeta d\zeta Z(\zeta) \text{Tr} (\cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots q^{T[0]} - cv/24 e^{H[0]}) T(\zeta) d_M t \\
= \frac{1}{2\pi i} \oint_{|\zeta| = 1} q d(q\zeta) Z(\zeta) \text{Tr} (\cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots T(\zeta) q^{T[0]} - cv/24 e^{H[0]}) d_M t \\
= \frac{1}{2\pi i} \oint_{|\zeta| = |q|} \zeta d\zeta Z(q^{-1} \zeta) \text{Tr} (\cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots T(\zeta) q^{T[0]} - cv/24 e^{H[0]}) d_M t,
\]
where we used
\[
(3.19) \quad e^{H[0]} T(\zeta) = T(\zeta) e^{H[0]}, \quad q^{T[0]} T(\zeta) = q^2 T(q\zeta) q^{T[0]}.
\]
The second integral in (3.17) is equal to
\[
(3.20) \quad \frac{1}{2\pi i} \oint_{|\zeta| = |q|} \zeta d\zeta Z(\zeta) \text{Tr} (\cdots \Phi_{\lambda_i}(v_i; z_i) \cdots \varpi \zeta \zeta_t^j (t_j) \cdots T(\zeta) q^{T[0]} - cv/24 e^{H[0]}) d_M t
\]
The third integral in (3.17) turns into a term of the form \( \partial/\partial t_j (\cdots) \) by the operator product expansion (1.57), and therefore the sum of those terms is an exact \( M \)-form. Hence they do not contribute to the integral over \( C \). Thus by summing up (3.16), (3.18) and (3.20) and using the property of \( Z \), (2.21), we obtain
\[
(3.21) \quad \rho^* \left( T \left\{ Z(t) t \frac{d}{dt} \right\} \right) \Psi(z; q; H) = \\
\left( q \frac{\partial}{\partial q} + \frac{cv}{24} \right) \Psi(z; q; H),
\]
which proves the condition (III).

The condition (IV) is proved in the same way.

Recall that the Wakimoto module \( \text{Wak}_{\lambda k} \) contains a \( q \)-submodule \( \text{Wak}_{\lambda, k}^0 \) (3.14), which is isomorphic to the dual Verma module \( M_{\lambda}^* \) and satisfies (1.35) and (1.36). As is mentioned at the end of Section 2.2, the restriction of \( \Psi(z; q; H) \) to \( \bigotimes_{i=1}^N \text{Wak}_{\lambda, k}^0 \) satisfies the KZB equation. The simple structure of \( \text{Wak}_{\lambda, k}^0 \) makes it possible to write down the restriction of \( \psi(z; t; q; H) \) explicitly.
Each vector $v_i$ in $Wak^0_{\lambda_i,k}$ corresponds to a polynomial $P_i(x) \in \mathbb{C}[x]$ $(x = (x^a)_{a \in \Delta_+})$ and to an operator in $P_{\lambda_i}^{(1.43)}$ as
\begin{equation}
(3.22) \quad v_i = P_i(\gamma(0)|0)_{gh} \otimes |\lambda_i\rangle_{bos}^\text{bos}, \quad \Phi_{\lambda_i}(v_i; z) = P_i(\gamma(z))V(\lambda_i; z).
\end{equation}

See (1.44). Let us compute $\psi(z; t; q; H)$ for this $v_i$. Inserting the expression (3.22) into the definition (3.2), we obtain
\begin{equation}
(3.23) \quad \psi(z; t; q; H)(v_1 \otimes \cdots \otimes v_N)
= q^{-cV/24} \text{Tr}_{F^{bos}_{\mu}}(V(\lambda_1; z_1) \cdots V(\lambda_N; z_N)V(-\alpha_1; t_1) \cdots V(-\alpha_M; t_M)q^{T^a(0)|0}_{e^H|0}) \times
\times \text{Tr}_{F^{gh}}(P_1(\gamma(z_1)) \cdots P_N(\gamma(z_N)))Scr_{\alpha_1}(t_1) \cdots Scr_{\alpha_N}(t_N)q^{T^a(0)|0}_{e^{H^{gh}|0}}dt_1 \wedge \cdots \wedge dt_M,
\end{equation}
where $T^a(0)$, $T^a(0)$ and $H^{gh(0)}$ are zero mode part of $T^a(z)$ (1.23), $T^a(z)$ (1.22) and $H^{gh}(z)$ (1.26), respectively. Since the right hand side of (3.23) splits into the bosonic sector and the ghost sector, we can calculate each part separately.

The computation of the bosonic sector correlation function reduces to the following lemma. Denote the one-loop correlation function of any element $A$ of $\text{ Bos}(g)$ (Section 1.3) by
\begin{equation}
(3.24) \quad (A)_{\mu_i, q, H}^{bos} := \text{Tr}_{F^{bos}_{\mu}}(Aq^{T^a(0)|0}_{e^H|0}),
\end{equation}
when $A|\mu_i\rangle_{bos} \in F^{bos}_{\mu_i}$.

**Lemma 3.6.** Let $\mu_i$ $(i = 1 \ldots N)$ be weights in $\mathfrak{h}$ satisfying $\sum_{i=1}^N \mu_i = 0$. Then the one-loop correlation function of bosonic vertex operators (1.13) is
\begin{equation}
(3.25) \quad (V(\mu_1; z_1) \cdots V(\lambda_i; z_i))_{\mu_i, q, H}^{bos} = \ell_{\mu_1, \cdots, \mu_N; \mu}(z_1, \ldots, z_N; q; H) :=
= (q; q)_\infty^{-1} e^{\Delta_{\mu}/4} \prod_{i=1}^N \left( \prod_{j \neq i} \prod_{1 \leq i \leq N} \theta_{11}(z_i/z_j; q^{(\mu_i, \mu_j)/2\kappa}) \right).
\end{equation}
where $\Delta_{\mu} = (\mu + 2\rho)/2\kappa$ is the conformal weight and $\eta(g) = q^{1/24}(q; q)_\infty$ is the Dedekind eta function.

The proof is in Appendix B. This is shown by the standard method of coherent states (cf. for example, GSW).

The ghost sector can be computed in a similar way as Proposition 3.2 of ATY and Theorem I of A. Let us define the ghost sector one-loop correlation function by
\begin{equation}
(3.26) \quad (A)^{gh}_{q, H} := \text{Tr}_{F^{gh}}(Aq^{T^a(0)|0}_{e^{H^{gh}|0}}),
\end{equation}
for $A \in \widehat{\text{ Gh}}(g)$.

The important lemma is the following screening current Ward identity.

**Lemma 3.7.** For any $P_a(x) \in \mathbb{C}[x]$ $(a = 1, \ldots, n)$, a root $\alpha$ and a sequence of positive roots $\{\alpha(j)\}_{j=1}^m$, we have
\begin{equation}
(3.27) \quad \langle P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n))Scr_{\alpha}(t)Scr_{\alpha(1)}(t_1) \cdots Scr_{\alpha(m)}(t_m) \rangle_{q, H}^{gh} =
= \sum_{a=1}^n (-w_{a(H)}(t, z_a))(P_a(\gamma(z_1)) \cdots (Scr\alphaP_a)(\gamma(z_a)) \cdots P_n(\gamma(z_n)))Scr_{\alpha(1)}(t_1) \cdots Scr_{\alpha(m)}(t_m))_{q, H}^{gh} +
+ \sum_{j=1}^m (-w_{a(H)}(t, t_j))f_{\alpha, \alpha(j)}^{a+\alpha(j)}(P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)))Scr_{\alpha(1)}(t_1) \cdots Scr_{\alpha+\alpha(j)}(t_j) \cdots Scr_{\alpha(m)}(t_m))_{q, H}^{gh}.
\end{equation}

Here we used the notations in (1.60) and (1.61).
Proof. We may assume that \(|q| < \|t_m\| < \cdots < |t_1| < |t| < |z_n| < \cdots < |z_1| < 1\). The left hand side of (3.27) is rewritten as follows because of (A.3):

\[
\langle P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)) \text{Scr}_\alpha(t) \text{Scr}_\alpha(1)(t_1) \cdots \text{Scr}_\alpha(m)(t_m) \rangle^{gh}_{q,H} = \frac{1}{2\pi i} \oint_{\zeta=t} d\zeta w_{\alpha(H)}(t, \zeta) \langle P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)) \text{Scr}_\alpha(\zeta) \text{Scr}_\alpha(1)(t_1) \cdots \text{Scr}_\alpha(m)(t_m) \rangle^{gh}_{q,H} \\
\times \langle P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)) \text{Scr}_\alpha(\zeta) \text{Scr}_\alpha(1)(t_1) \cdots \text{Scr}_\alpha(m)(t_m) \rangle^{gh}_{q,H}. 
\]

The first integral in (3.28) is equal to

\[
\frac{1}{2\pi i} \oint_{|\zeta|=1} d\zeta w_{\alpha(H)}(t, \zeta) \text{Tr} e^{\alpha(H)} w_{\alpha(H)}(t, \zeta) \times \\
\times \text{Tr} e^{\alpha(H)} P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)) \text{Scr}_\alpha(1)(t_1) \cdots \text{Scr}_\alpha(m)(t_m) \text{Scr}_\alpha(\zeta) = 1.
\]

Therefore the property (A.4) of the function \(w_{\alpha(H)}(t, \zeta)\) and (3.29) imply that the first integral and the last integral in (3.28) cancel.

Using the operator product expansions (1.60) and (1.61), the second and third integrals in (3.28) are rewritten as

\[
\frac{1}{2\pi i} \oint_{|\zeta|=1} d\zeta w_{\alpha(H)}(t, \zeta) \langle P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)) \text{Scr}_\alpha(\zeta) \text{Scr}_\alpha(1)(t_1) \cdots \text{Scr}_\alpha(m)(t_m) \rangle^{gh}_{q,H} \\
= w_{\alpha(H)}(t, z_a) \langle P_1(\gamma(z_1)) \cdots (\text{Scr}_\alpha P_\alpha)(\gamma(z_a)) \cdots P_n(\gamma(z_n)) \text{Scr}_\alpha(1)(t_1) \cdots \text{Scr}_\alpha(m)(t_m) \rangle^{gh}_{q,H},
\]

which proves the lemma. \(
\)

Lemma 3.8. For any polynomial \(P(x) \in \mathbb{C}[x]\) with constant term \(c_P\),

\[
\langle P(\gamma(z)) \rangle^{gh}_{q,H} = c_P \text{ch}_{\text{Fock}}(q, H),
\]

where \(\text{ch}_{\text{Fock}}(q, H) = \langle 1 \rangle^{gh}_{q,H}\) is the character of the ghost Fock space. An explicit expression of the character is

\[
\text{ch}_{\text{Fock}}(q, H) = \prod_{\alpha \in \Delta_+} (e^{-\alpha(H)}; q)_\infty (q e^{\alpha(H)}; q)_{-\infty}^{-1}.
\]

Proof. It is sufficient to prove that

\[
\text{Tr}_{\text{Fock}}(\prod_{i=1}^{n} \gamma_{\beta_i} [n_i] q^{\beta_i(0)} e^{H(0)}) = 0,
\]
for any $n \in \mathbb{Z}_{>0}$, $\beta_i \in \Delta_+$, $n_i \in \mathbb{Z}$ ($i = 1, \ldots, n$). The Fock space $\mathcal{F}^{gh}$ has a basis consisting of vectors of the form

$$\prod \gamma_{\alpha(i)}[-m_i] \prod \beta_{\alpha'(j)}[-m'_j]|0\rangle^{gh},$$

where $\alpha(i), \alpha'(j) \in \Delta_+$, $m_i \in \mathbb{Z}_{>0}$, $m'_j \in \mathbb{Z}_{>0}$. The action of $T^{gh}[0]$ and $H^{gh}[0]$ is diagonal with respect to this basis. Hence what we must show is that the action of $\prod_{i=1}^n \gamma_{\beta_i}[n_i]$ does not have diagonal components with respect to this basis. This can be shown by elementary method which uses only the commutation relation $[\gamma, H] = 0$.

The character $\text{ch}_{\mathcal{F}^{gh}}(q; H)$ is calculated by factorizing the total Fock space into the Fock space $\mathcal{F}_{\alpha,m}$ generated by $\beta_{\alpha}[m]$ and $\gamma_{\alpha}[-m]$:

$$\mathcal{F}^{gh} = \bigotimes_{\alpha \in \Delta_+, m \in \mathbb{Z}} \mathcal{F}_{\alpha,m},$$

When $m \geq 0$, $\mathcal{F}_{\alpha,m} = \mathbb{C}[\gamma_{\alpha}[-m]]|0\rangle^{gh}$, and when $m < 0$, $\mathcal{F}_{\alpha,m} = \mathbb{C}[\beta_{\alpha}[m]]|0\rangle^{gh}$. The character of each space is:

$$\text{Tr}_{\mathcal{F}_{\alpha,m}}(q^{T^{gh}[0]} e^{H^{gh}[0]}) = \begin{cases} (1 - q^m e^{-\alpha(H)})^{-1}, & m \geq 0, \\ (1 - q^{-m} e^{\alpha(H)})^{-1}, & m < 0, \end{cases}$$

which follows from the commutation relations,

$$[T^{gh}[0], \beta_{\alpha}[m]] = -m \beta_{\alpha}[m], \quad [T^{gh}[0], \gamma_{\alpha}[m]] = -m \gamma_{\alpha}[m],$$

$$[H^{gh}[0], \beta_{\alpha}[m]] = \alpha(H) \beta_{\alpha}[m], \quad [H^{gh}[0], \gamma_{\alpha}[m]] = -\alpha(H) \gamma_{\alpha}[m],$$

and $T^{gh}[0]|0\rangle^{gh} = H^{gh}[0]|0\rangle^{gh} = 0$. Multiplying (3.33) over all $\alpha$ and $m$, we obtain (3.31). \hfill \Box

**Corollary 3.9.** For $P(x) \in \mathbb{C}[x]$ and a sequence of simple roots $\{\alpha_{ij}\}_{i=1}^n$, we have

$$\langle (\text{Scr}_{\alpha_{i(1)}} \cdots \text{Scr}_{\alpha_{i(n)}} P)(\gamma(z)) \rangle_{q,H}^{gh} = \text{ch}_{\mathcal{F}^{gh}}(q; H)(-1)^n j(E_{i(n)} \cdots E_{i(1)} P),$$

where $j : M^*_\lambda \cong \text{Wak}_\lambda \rightarrow \mathbb{C}$ is the pairing with the highest weight vector of the Verma module of $\mathfrak{g}$ with the highest weight $\lambda$, $M_\lambda$. (See (1.3).) Explicitly written,

$$j(E_{i(n)} \cdots E_{i(1)} P) = R(E_{i(n)}) \cdots R(E_{i(1)} P(x)|_{x=0}),$$

where $R(E_i)$ is the differential operator corresponding to the Chevalley generator $E_i$ given by (1.3). In particular, $j(E_{i(n)} \cdots E_{i(1)} P)$ does not depend on $\lambda$.

**Proof.** According to Lemma 3.3 of [ATY], the constant term of $(\text{Scr}_{\alpha_{i(1)}} \cdots \text{Scr}_{\alpha_{i(n)}} P)(z)$ is given by $(-1)^n j(E_{i(n)} \cdots E_{i(1)} P) = (-1)^n R(E_{i(n)}) \cdots R(E_{i(1)} P(x)|_{x=0}).$ \hfill \Box

**Lemma 3.10.** For any $\alpha(i) \in \Delta_+$ ($i = 1, \ldots, m$) and $P_a(x) \in \mathbb{C}[x]$ ($a = 1, \ldots, n$), we have

$$\langle P_1(\gamma(z_1)) \cdots P_n(\gamma(z_n)) \text{Scr}_{\alpha(1)}(t_1) \cdots \text{Scr}_{\alpha(m)}(t_m) \rangle_{q,H}^{gh} = \sum_{t_1, \ldots, t_m} \prod_{i=1}^n \frac{\langle P_a(\gamma(z_a)) \prod_{i \in I_a} \text{Scr}_{\alpha(i)}(t_i) \rangle_{q,H}^{gh}}{\text{ch}_{\mathcal{F}^{gh}}(q; H)}.$$ 

**Proof.** This is a purely combinatorial lemma. We can apply the inductive proof of (5.3) in [A], replacing the screening current Ward identity for genus 0 with that for genus 1, (3.27). The first step of the induction (the case $m = 0$) is assured by Lemma 3.8. \hfill \Box
Lemma 3.11. For any $P(x) \in \mathbb{C}[x]$ and roots $\alpha(i)$ ($i = 1, \ldots, m$), we have

\begin{equation}
\langle P(\gamma(z)) \text{Scr}_{\alpha(1)}(t_1) \cdots \text{Scr}_{\alpha(m)}(t_m) \rangle_{q,H}^{\text{gh}} = \\
= \sum_{\sigma \in S_m} (-w_{\alpha(\sigma(1))}(t_{\sigma(1)}), t_{\sigma(2)}))(-w_{\alpha(\sigma(1))+\alpha(\sigma(2))}(t_{\sigma(1)}, t_{\sigma(2)})) \cdots (-w_{\alpha(\sigma(1))+\cdots+\alpha(\sigma(m))}(t_{\sigma(m)}, z)) \times \\
\times \langle (\text{Scr}_{\alpha(\sigma(1))} \cdots \text{Scr}_{\alpha(\sigma(m))} P)(\gamma(z)) \rangle_{q,H}^{\text{gh}},
\end{equation}

where we write $w_{\alpha(H)}$ as $w_{\alpha}$ for short.

Proof. We prove this statement by induction on $m$, as in the proof of (5.4) in [A]. When $m = 0$, the statement is trivial and when $m = 1$, it is nothing but the screening current Ward identity (3.27).

Assume that (3.33) holds for all $m \leq n$. Let us regard the left and right hand side of (3.38) for $m = n + 1$ as functions of $t_0$:

\begin{equation}
F_1(t_0) := \langle P(\gamma(z)) \text{Scr}_{\alpha(0)}(t_0) \text{Scr}_{\alpha(1)}(t_1) \cdots \text{Scr}_{\alpha(n)}(t_n) \rangle_{q,H}^{\text{gh}},
\end{equation}

\begin{equation}
F_2(t_0) := \sum_{\sigma \in S_{n+1}} (-w_{\alpha(\sigma(0))}(t_{\sigma(0)}), t_{\sigma(1)}))(-w_{\alpha(\sigma(0))+\alpha(\sigma(1))}(t_{\sigma(1)}, t_{\sigma(2)})) \cdots (-w_{\alpha(\sigma(0))+\cdots+\alpha(\sigma(n))}(t_{\sigma(n)}, z)) \times \\
\times \langle (\text{Scr}_{\alpha(\sigma(1))} \cdots \text{Scr}_{\alpha(\sigma(n))} P)(\gamma(z)) \rangle_{q,H}^{\text{gh}},
\end{equation}

where $\sigma$ is a permutation, $\sigma : \{0, 1, \ldots, n\} \rightarrow \{0, 1, \ldots, n\}$. We now show that $F_1(t_0) = F_2(t_0)$.

First, note that both functions are meromorphic on $\mathbb{C}^\times$ and poles exist at $t_0 = t_i$ ($i = 1, \ldots, n$) and at $t_0 = z$.

(i) Both functions have the same quasi-periodicity,

\begin{equation}
f(qt_0) = e^{-\alpha(0)(H)} q^{-1} f(t_0).
\end{equation}

In fact, (3.41) is proved for $f = F_1$ similarly to (3.28). It follows from the property of the function $w$ (A.4) that $F_2(t_0)$ also satisfies the same periodicity property (3.41).

(ii) The principal parts of the pole at $t_0 = z$ are equal to

\begin{equation}
\frac{1}{t_0 - z} \langle (\text{Scr}_{\alpha(0)} P)(\gamma(z)) \text{Scr}_{\alpha(1)}(t_1) \cdots \text{Scr}_{\alpha(n)}(t_n) \rangle_{q,H}^{\text{gh}}.
\end{equation}

For $F_1(t_0)$, this is a direct consequence of the Ward identity (3.27). The pole of $F_2(t_0)$ at $t_0 = z$ comes from terms in (3.40) such that $\sigma(n) = 0$. Using (A.3) and the induction hypothesis, we can show that its principal part is of the form (3.42).

(iii) The principal parts of the pole at $t_0 = t_i$ are equal to

\begin{equation}
\frac{f_{\alpha(0)+\alpha(i)}}{t_0 - t_i} \langle P(\gamma(z)) \text{Scr}_{\alpha(1)}(t_1) \cdots \text{Scr}_{\alpha(0)+\alpha(i)}(t_i) \cdots \text{Scr}_{\alpha(n)}(t_n) \rangle_{q,H}^{\text{gh}}.
\end{equation}

The Ward identity (3.27) implies (3.43) for $F_1(t_0)$. The pole of $F_2(t_0)$ at $t_0 = t_i$ comes from terms in (3.40) such that $\sigma(0), \sigma(i) = (j - 1, j)$ or $\sigma(0), \sigma(i) = (j, j - 1)$ ($j = 1, \ldots, n$). The principal part becomes

\begin{equation}
\frac{1}{t_0 - t_i} \sum_{j = 1}^{n} \sum_{\sigma} (-w_{\alpha(\sigma(0))}(t_{\sigma(0)}, t_{\sigma(1)}) \cdots \\
\cdots (-w_{\alpha(\sigma(0))+\cdots+\alpha(\sigma(j-2))}(t_{\sigma(j-2)}, t_{\sigma(j-1)}))(-w_{\alpha(\sigma(0))+\cdots+\alpha(\sigma(j-2))+\alpha(i)}(t_{\sigma(j+1)}) \cdots \\
\langle (\text{Scr}_{\alpha(\sigma(0))} \cdots [\text{Scr}_{\alpha(0)}, \text{Scr}_{\alpha(i)}] \cdots \text{Scr}_{\alpha(\sigma(n))} P)(z) \rangle_{q,H}^{\text{gh}}.
\end{equation}

where $\sigma$ runs through the set of permutations $\sigma : \{0, \ldots, j - 2, j + 1, \ldots, n\} \rightarrow \{1, \ldots, i - 1, i + 1, \ldots, n\}$. Since $[\text{Scr}_{\alpha(0)}, \text{Scr}_{\alpha(i)}] = f_{\alpha(0)+\alpha(i)} \text{Scr}_{\alpha(0)+\alpha(i)}$, it follows from the induction hypothesis that (3.44) is equal to (3.43).

Comparing $F_1(t_0)$ and $F_2(t_0)$ by (i), (ii) and (iii), we conclude that $F_1(t_0) = F_2(t_0)$.

Putting together (3.29), Lemma 3.1, Lemma 3.10, Lemma 3.11, Corollary 3.9, and (3.31), we finally obtain the integral representation of a solution of the KZB equations.
Theorem 3.12. The following integral gives a solution of the KZ equations, (2.30), (2.31) with $c_z^{(j)} = (\lambda_j|\lambda_j + 2\rho)$ and (2.33):

\begin{equation}
\Psi^0(z; q; H) = \int_{\mathcal{L}(z,q,H)} \ell^{0-\alpha_1,\ldots,-\alpha_M,\lambda_1,\ldots,\lambda_N,\mu}(z; q; H) \psi^{\text{gh}}(t; z; q; H; P_1, \ldots, P_N),
\end{equation}

where $\mathcal{L}(z,q,H)$ is a family of $M$-cycles with coefficients in $\mathcal{L}^*$.

\begin{equation}
\ell_{\mu_1,\ldots,\mu_N,\mu}(z_1, \ldots, z_N; q; H) :=
q^{(\mu + \rho + \rho)/2}\ell_{\mu}(H) \left( \prod_{i=1}^{N} (\sqrt{-1}\eta(q)^3(\mu_i|\mu_i)/2\kappa) z_i^{(\mu_i|2\mu_i-\mu_i)/2\kappa} \right) \left( \prod_{1 \leq i < j \leq N} \theta_{11}(z_i/z_j; q)^{(\mu_i|\mu_j)/\kappa} \right)
\end{equation}

and the $M$-form $\psi^{\text{gh}}$ is defined as follows: $P_a(x)$ are polynomials in $x$,

\begin{equation}
\psi^{\text{gh}}(t; z; q; H; P_1, \ldots, P_N) = e^{\ell(H)} \sum_{t_1 \sqcup \cdots \sqcup t_N \{1, \ldots, M\}} \prod_{a=1}^{N} \frac{\langle P_a(\gamma(z_a)) \rangle_{\mathcal{L}, \langle \mathcal{L} \rangle}}{\text{ch}_{\mathcal{F} \alpha \langle \gamma \rangle}(q; H)} dt_1 \wedge \cdots \wedge dt_M,
\end{equation}

and the last factor in (3.47) for a $(1 \leq a \leq N)$ is

\begin{equation}
\frac{\langle P(\gamma(z)) \rangle \text{Scre}_{\alpha_{(1)}}(t_1) \cdots \text{Scre}_{\alpha_{(m)}}(t_m) \rangle_{q; H}}{\text{ch}_{\mathcal{F} \alpha \langle \gamma \rangle}(q; H)} =
\sum_{\sigma \in \Sigma_m} w_{\alpha_{(1)}(\sigma)}(t_{\sigma(1)}; t_{\sigma(2)}; t_{\sigma(3)}; \cdots; t_{\sigma(m)}) x \times \prod_{i \in I_a} \langle E_{i(\sigma_{(1)})} \cdots E_{i(\sigma_{(m)})} \rangle,
\end{equation}

if $\{i_j | j \in I_a\} = \{i(1), \ldots, i(m)\}$.

This result is an elliptic analogue of [SV1], [SV2], [ATY], [A] and a generalization of a result for $sl(2)$ in [BeF]. Felder and Varchenko have obtained a similar formula in [FV] from a different standpoint.

4. Concluding remarks

We found an integral representation of $N$-point functions of the WZW model on elliptic curves and gives an explicit expression for a solution of the KZ equations, using the Wakimoto realization. Let us list some of related problems.

1. Higher genus: Is there a similar integral representation of correlation functions of the Wess-Zumino-Witten models on higher genus Riemann surfaces? There are several works to this direction [GMMOS], [K]. Their formulations are, however, different from ours.

2. Twisted Wess-Zumino-Witten models: In [K] we formulated “another” Wess-Zumino-Witten model on elliptic curves which we named a “twisted WZW model”. Is it possible to give an integral representation of solutions of the KZ type equations for the correlation functions?

3. Critical level: Feigin, Frenkel and Reshetikhin [FR] found that the Bethe vector of a certain spin chain model is obtained from the Wakimoto realization of the Wess-Zumino-Witten model on the Riemann sphere at the critical level. How about the genus one case?

We shall study the last question in the forthcoming paper. In fact, Felder and Varchenko [FV] have found a relation of their integral representation of solutions of the KZB equations with a solution of a quantum $N$-body system. We shall take the conformal field theoretical approach to this problem.

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Appendix A. Theta functions

We denote the theta function with characteristic \((1/2,1/2)\) (cf. Chapter I of [Mu]) additively as
\[
\theta_{11}(x; \tau) = \sum_{n \in \mathbb{Z}} e^{(n+1/2)2\pi i x + 2\pi i(n+1/2)(x+1/2)}
\]
and multiplicatively as
\[
\theta_{11}(z; q) = \theta_{11}(x; \tau),
\]
where \(z = \exp(2\pi i x)\) and \(q = \exp(2\pi i \tau)\). The infinite product expansion
\[
\theta_{11}(z; q) = i(q; q)\infty q^{1/8}z^{1/2}(z^{-1}; q)\infty(qz)^{\infty}
\]
is also useful, where \((x; q)\infty = \prod_{n=0}^{\infty}(1 - xq^n)\).

We use a function \(w_c(w, z)\) on \(\mathbb{C}^\times \times \mathbb{C}^\times\) with parameter \(c \in \mathbb{C}^\times\) characterized by the following properties:
1. \(w_c(w, z)\) is a meromorphic function of \(z\) and \(w\).
2. \(w_c(w, z)\) has a following (quasi-)periodicity
\[
w_c(w, qz) = e^{c}w_c(w, z), \quad w_c(qw, z) = q^{-1}e^{-c}w_c(w, z).
\]
3. \(w_c(w, z)\) has only one simple pole on the elliptic curve \(\mathbb{C}^\times/q^2\) at \(z = w\) as a function of \(z\). Its Laurent expansion around \(z = w\) is:
\[
w_c(w, z) = \frac{1}{z - w} + \text{regular}.
\]

An explicit form of \(w_c(w, z)\) is as follows:
\[
w_c(w, z) = \frac{\theta_1'(1; q)}{\theta_1(1; q)} \frac{\theta_{11}(e^{-c}z/w; q)}{\theta_{11}(e^{-c}; q)} \frac{w\theta_{11}(z/w; q)}{\theta_{11}(z; q)},
\]
where \(\theta_1'(z; q) = d/dz\theta_1(z; q)\).

To write down the Knizhnik-Zamolodchikov-Bernard equations, we need following functions:
\[
\sigma_c(z) := \frac{\theta_1'(1; q)}{\theta_1(1; q)} \frac{\theta_{11}(e^{-c}z; q)}{\theta_{11}(z; q)}
\]
\[
\zeta(z) := \frac{\theta_1'(z; q)}{\theta_1(z; q)}.
\]

Appendix B. Method of coherent states and one-loop correlation functions

In this appendix we review the method of coherent states, following Chapter 7.A and 8.1 of [GSW] and compute one-loop correlation functions of vertex operators of the free bosons, which proves Lemma 3.6.

Let \(a\) and \(a^\dagger\) be generators of a Heisenberg algebra \(\mathcal{H}\):
\[
[a, a^\dagger] = 1,
\]
and \(\langle 0 \rangle\) and \(\langle 0 \rangle\) be the generating vector of the Fock space representation of \(\mathcal{H}\) and that of its (restricted) dual, respectively:
\[
\mathcal{F} := \mathcal{H}\langle 0 \rangle, \quad a\langle 0 \rangle = 0,
\]
\[
\mathcal{F}^\ast := \langle 0 \rangle\mathcal{H}, \quad \langle 0 \rangle a^\dagger = 0.
\]
There are natural bases \(\{|n\rangle\}_{n \in \mathbb{N}}\) of \(\mathcal{F}\) and \(\{\langle n \rangle\}_{n \in \mathbb{N}}\) of \(\mathcal{F}^\ast\), consisting of eigenvectors of the number counting operator \(N_a = a^\dagger a\):
\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad \langle n| = \langle 0| \frac{a^n}{\sqrt{n!}}
\]
\[
N_a|n\rangle = n|n\rangle, \quad \langle n|N_a = \langle n|n, \quad \langle m|n\rangle = \delta_{mn}.
\]
The coherent states are defined by

$|\lambda\rangle := \exp(\lambda a^\dagger)|0\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle$,

(B.5)

$(\lambda) := \langle 0|\exp(\bar{\lambda}a) = \sum_{n=0}^{\infty} \langle n|\bar{\lambda}^n\sqrt{n!}$

for $\lambda \in \mathbb{C}$. Here $\bar{\lambda}$ is the complex conjugate of $\lambda$. In particular, $|0\rangle = |0\rangle$ and $(0) = (0)$. They have the following properties:

(B.6) $a|\lambda\rangle = \lambda|\lambda\rangle$, \hspace{1em} $(\lambda)a^\dagger = (\lambda)|\bar{\lambda}\rangle$

(B.7) $(\mu|\lambda\rangle = e^{\bar{\mu}\bar{\lambda}}$

(B.8) $q^{|\lambda\rangle} = |q\rangle$

where $q \in \mathbb{C}^\times$. The trace of an operator $A \in \operatorname{End}_\mathbb{C}(\mathcal{F})$ is computed by the following integral:

(B.9) $\operatorname{Tr}_{\mathcal{F}}(A) = \frac{1}{\pi} \int_\mathbb{C} d^2\lambda e^{-|\lambda|^2} (\lambda|A|\lambda)$.

Using these formulae, we calculate the one-loop correlation function (3.24) of vertex operators of the free boson fields $V(\mu_i; z_i)$ (B.17):

(B.10) $\langle V(\mu_1; z_1) \cdots V(\mu_N; z_N) \rangle^{\text{bos}}_{\mu, q, H} = \operatorname{Tr}_{\mathcal{F}^{\text{bos}}} (V(\mu_1; z_1) \cdots V(\mu_N; z_N) q^{T^0[0]} e^{\phi[H;0]})$

where $\sum_{i=1}^N \mu_i = 0$.

Let us fix an orthonormal basis $\{H_r\}_{r=1}^l$ of $\mathfrak{h}$. The boson Fock space $\mathcal{F}^{\text{bos}}_\mu$ is factorized as

(B.11) $\mathcal{F}^{\text{bos}}_\mu = \mathcal{F}_{0,\mu} \otimes \bigotimes_{r=1}^l \bigotimes_{n=1}^\infty \mathcal{F}_{r,n}$

where $\mathcal{F}_{0,\mu}$ is the zero-mode space $\mathbb{C}|\mu\rangle^{\text{bos}}$ and $\mathcal{F}_{r,n}$ is a non-zero-mode Fock space generated by $\phi_r[\pm m] := \phi[H_r; \pm n]$. Note that $\phi_r[\pm m]$ and $\phi_r[\pm n] (m, n \in \mathbb{Z}_{>0}, r, r' = 1, \ldots, l)$ commute with each other unless $r = r'$ and $m = n$. Hence, to compute the value of (B.24), we have to compute the trace of $V(\mu_1; z_1) \cdots V(\mu_N; z_N) q^{T^0[0]} e^{\phi[H;0]}$ over $\mathcal{F}_{0,\mu}$ and $\mathcal{F}_{r,n}$ and multiply all of them. The vertex operators $V(\mu_i; z_i)$ are factorized into product of the zero mode part $V_0(\mu_1; z_1)$ (B.18) and the non-zero mode part $\tilde{V}(\mu_i; z_i)$ (B.17), while the operator $T^0(\mu)$ is decomposed into sum of the zero mode and the non-zero mode parts as

(B.12) $T^0(z) = T_0^0(z) + \tilde{T}^0(z)$

(B.13) $T_0^0(z) := \frac{1}{2\kappa} \sum_{i=1}^l \phi[H_i; 0] \phi[H_i; 0] + \frac{1}{2\kappa} \phi[2\rho; 0]$

(B.14) $\tilde{T}^0(z) := \frac{1}{\kappa} \sum_{r=1}^l \sum_{n=1}^\infty \phi_r[-n] \phi_r[n]$

(I) Zero-mode: It is easy to see that the zero mode part of $V(\mu_1; z_1) \cdots V(\mu_N; z_N) q^{T_0^0[0]} e^{\phi[H;0]}$ acts on $|\mu\rangle^{\text{bos}}$ as

(B.15) $V_0(\mu_1; z_1) \cdots V_0(\mu_N; z_N) q^{T_0^0[0]} e^{\phi[H;0]} = \prod_{1 \leq i < j \leq N} z_{ij}^{(\mu_i|\mu_j)/\kappa} \prod_{i=1}^N z_i^{(\mu_i|\mu_i)/\kappa} q^{(\mu_i+2\rho)|\mu_i}/2\kappa e^{\mu_i(H)} |\mu\rangle^{\text{bos}}$

(II) Non-zero mode: The algebra $\mathcal{H}_{r,n}$ generated by $\phi_r[\pm n]$ is isomorphic to $\mathcal{H}$ through the isomorphism defined by

$a = \frac{\phi_r[n]}{\sqrt{\kappa n!}}, \quad a^\dagger = \frac{\phi_r[-n]}{\sqrt{\kappa n!}}$. 
The $\mathcal{H}_{r,n}$ part of $V(\mu_1; z_1) \cdots V(\mu_N; z_N) q^{-\frac{r}{\kappa} \phi_r|n|}$ is

$$\exp \left( \frac{\mu_r^r}{\kappa} \phi_r|n| \right) \exp \left( \frac{\mu_1^1}{\kappa} \phi_1|n| \right) \cdots \exp \left( \frac{\mu_N^N}{\kappa} \phi_N|n| \right)$$

where $\mu_i = \sum_{r=1}^i \mu_i^r H_r$. Its trace over $\mathcal{F}_{r,n}$ is computed by means of the formula (3.9). The result is

$$(B.16) \quad \frac{1}{1 - q^n} \exp \left( \frac{-1}{\kappa n (1 - q^n)} \sum_{1 \leq i < j \leq N} \mu_i^r \mu_j^r \left( \frac{z_i}{z_j} \right)^n + \frac{-q^n}{\kappa n (1 - q^n)} \sum_{1 \leq i < j \leq N} \mu_i^r \mu_j^r \left( \frac{z_i}{z_j} \right)^n \right).$$

Multiplying (B.16) for all $r$ and $n$, we have

$$(B.17) \quad \text{Tr}_{\mathcal{F}_{r,n}} (V(\mu_1; z_1) \cdots V(\mu_N; z_N) q^{-\frac{r}{\kappa} \phi_r|n|}) =$$

$$= (q; q)_\infty^{-l} \prod_{i=1}^N (q; q)^{(\mu_i|\mu_i)/\kappa} \prod_{1 \leq i < j \leq N} ((z_i/z_j; q)_\infty (q z_i/z_j; q)_\infty)^{(\mu_i|\mu_i)/\kappa}.$$

Putting (B.15) and (B.17) together, we obtain the final result (3.25), using the infinite product expansion of the theta function (A.3). This completes the proof of Lemma 3.6.

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