ON A TRANSMISSION EIGENVALUE PROBLEM FOR A SPHERICALLY STRATIFIED COATED DIELECTRIC

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ABSTRACT. Suppose that the boundary of the unit ball in $\mathbb{R}^3$ is coated with a very thin layer of a highly conductive material and the refractive index $n(x)$ inside the ball is spherically stratified. We show that in this case the set of transmission eigenvalues behave quite differently than in the previous studied case of an uncoated ball. In particular, if the index of refraction varies smoothly across the boundary of the unit ball we show that complex eigenvalues always exist and accumulate on the real axis and that the real and complex eigenvalues uniquely determine the index of refraction without any restriction on its magnitude.

1. Introduction. The transmission eigenvalue problem has received considerable attention in recent years due to the central role it plays in the qualitative approach to inverse scattering theory (c.f. [4], [7]). Of particular interest is the question of whether or not complex transmission eigenvalues exist and what information about the index of refraction can be obtained from a knowledge of either the real or complex eigenvalues. Not surprisingly, the most complete answer to the general question is for the special case of a spherically stratified medium where conditions have been given for when complex eigenvalues exist and the inverse spectral problem has been studied (c.f. [1], [2], [8], [9], [12], [13], [15] and [16]). The results obtained in this investigation of the transmission eigenvalues for spherically stratified media are quite surprising. In particular, complex eigenvalues can exist sometimes and not exist at other times. When they exist, they always lie in a strip if the index refraction $n(r)$ is discontinuous across the boundary of the medium but if $n(r)$ varies smoothly across the boundary they fail to lie in a strip if the second derivative of $n(r)$ vanishes at the boundary. Finally, it has been shown that if $n(r)$ varies smoothly across the boundary then the real and complex eigenvalues uniquely determine $n(r)$ only if $0 < n(r) < 1$ (unless norming constants are given [16]).

In this paper we consider the case when the spherically stratified medium is coated by a very thin layer of highly conducting material (c.f. Section 8.4 of [4]). In this case, we will show that the above diverse behavior of the uncoated media becomes, by contrast, remarkably simple. In particular, we will show that, for the
case when \( n(r) \) varies smoothly across the boundary, complex eigenvalues always
exist and cluster toward the real axis. Furthermore, \( n(r) \) is uniquely determined
by a knowledge of the real and complex eigenvalues without any restriction on the
magnitude of \( n(r) \) or a knowledge of norming constants.

To fix our ideas, let \( B := \{ x : |x| < 1 \} \) be the unit ball in \( \mathbb{R}^3 \). Then the (isotropic)
transmission eigenvalue problem for a spherically stratified medium with support
\( B \) coated with a very thin layer of a homogeneous highly conductive material with
surface conductivity \( \eta \) is modeled by

\[
\begin{align*}
\Delta w + k^2 n(r) w &= 0 \quad \text{in } B, \\
\Delta v + k^2 v &= 0 \quad \text{in } B, \\
v - i\eta \frac{\partial w}{\partial r} &= w \quad \text{in } \partial B, \\
\frac{\partial w}{\partial r} &= \frac{\partial v}{\partial r} \quad \text{on } \partial B.
\end{align*}
\]

where \( \eta > 0 \) is a constant, \( n \in C^3[0,1] \) is positive and \( r = |x| \). Values of the wave
number \( k \) for which a nontrivial solution to (1)-(4) exists are called transmission
eigenvalues. Assuming that the eigenfunctions of (1)-(4) are spherically symmetric,
we can set

\[
\begin{align*}
w(r) &= a_o \frac{y(r)}{r}, \\
v(r) &= b_o \frac{\sin(kr)}{kr}
\end{align*}
\]

where \( y(r) \) is normalized such that \( y(0) = 0, y'(0) = 1 \) to arrive at the eigenvalue
problem

\[
\begin{align*}
y_{rr} + k^2 n(r)y &= 0, \\
-\frac{a_o}{k} \sin(k) - a_o i \eta (y(1) - y(1)) &= a_o y(1), \\
b_o (\cos(k) - \frac{\sin(k)}{k}) &= a_o (y'(1) - y(1)).
\end{align*}
\]

In particular the transmission eigenvalues correspond to the zeros of the determinant
\( \Delta(k) \) defined by

\[
\Delta(k) = \begin{vmatrix}
\sin(k) & -y(1) + i \eta (y(1) - y'(1)) \\
-k \cos(k) + \frac{\sin(k)}{k} & y(1) - y'(1)
\end{vmatrix}
\]

\[
= y(1) \cos(k) - y'(1) \sin(k) \\
- i \eta (y(1) - y'(1)) \left( \cos(k) - \frac{\sin(k)}{k} \right). 
\]

Note that the uncoated ball previously studied in the above cited references corre-
spond to the case \( \eta = 0 \).

2. **Existence of real eigenvalues.** Our first goal is to look at the situation where
real eigenvalues \( k \) could exist. We then show that for a fixed \( \eta > 0 \), the complex
eigenvalues approach the real line as \( \text{Re}(k) \) gets large. Since \( y(r) \) is a solution to a
Sturm-Liouville type differential equation (5)-(7), the function \( y(1) \), as a function of
\( k \), is an entire function of exponential type.
If $k$ is real, then both functions $y(1)$ and $y'(1)$ are real valued. From (8), we see that $\Delta(k) = 0$ at a real $k$ for a $\eta > 0$ only when both $y(1) = y'(1)$ and $\cos(k) = \sin(k)/k$.

**Example A.** When $n(r) = 4/(1 + r)^4$, called Borg’s potential, an infinite set of real eigenvalues occur. Here, one can calculate that $y(1) = \sin(k)/k$ and $y'(1) = 1/2 (\cos(k) + \sin(k)/k)$. Thus $y(1) = y'(1)$ is equivalent to $\sin(k) = k \cos(k)$. The determinant $\Delta(k)$ is equal to

$$\frac{i}{2k^2} (\sin(k) - k \cos(k)) \cdot [\eta(k \cos(k) - \sin(k)) + i \sin(k)].$$

This function has an infinite set of real eigenvalues determined by $\sin(k) - k \cos(k) = 0$. The entire set of eigenvalues is shown in Figure 1 with $\eta = 1$. The second factor above has the form $g(k) := \eta(k \cos(k) - \sin(k)) + i \sin(k)$. It is of interest to note that $\eta(k \cos(k) - \sin(k))$, being the real part of $g(k)$, has zeros which are the critical points of $\sin(k)/k$. Hence the zeros of the real part of $g(k)$ interlace those of the imaginary part. Thus by a generalized version of Hermite-Biehler Theorem (\cite{11} Theorem 2, Section 27.3), the complex zeros of $g(k)$ stay entirely on either the upper or lower half of the complex plane.

This $n(r)$ is the only index we could find to produce an infinite set of real eigenvalues. We show at the end of this section the existence of an infinite set of real eigenvalues implies that $\int_0^1 \sqrt{n(t)} \, dt = 1$. Thus an arbitrary constant index $n$ cannot produce an infinite set of real eigenvalues. For $n(r)$ being a constant, the next example suggests that exactly one real eigenvalue can exist. If $n$ is the square of a rational, then we will show that all eigenvalues have to be complex.

**Example B.** We let $0 < k_1 < k_2$ be any two real roots of $\sin(k) - k \cos(k) = 0$. Define $n(r) \equiv \mu^2$ where $\mu = k_2/k_1$. In this case, $y(r) = \sin(\mu kr)/(\mu k)$. Hence $y(1) =$
Figure 2. A plot of the transmission eigenvalues for Example B when $\eta = 1$ and $\mu = 1.29$

$$\sin(\mu k)/(\mu k) \text{ and } y'(1) = \cos(\mu k).$$ At $k = k_1$, $y(1) = \sin(k_2)/k_2$ and $y'(1) = \cos(k_2)$.

Thus $\Delta(k_1) = 0$. It seems that $\Delta(k) = 0$ does not have another real root. However we do not have a proof of it.

The distribution of the eigenvalues is shown in Figure 2 with $\eta = 1$ and $\mu = 14.0662/10.9041 = 1.2899$, a quotient of the 4th and 3rd positive real roots of $\tan(k) = k$. There is a real root $\Delta(k)$ located at $k_1 = 10.9041$.

To come up with an example of all complex eigenvalues, we set up the following proposition which has another use later on computing the refractive index $n(r)$.

**Proposition 2.1.** For a fixed $\mu > 0$, there is only one positive integer solution $x$ to

$$x\mu \cos(x\mu) = \sin(x\mu).$$

**Proof.** Suppose there are two different integer solutions $p$ and $q$, i.e.

$$p\mu \cos(p\mu) = \sin(p\mu) \text{ and } q\mu \cos(q\mu) = \sin(q\mu).$$

Then

$$p \sin(q\mu) \cos(p\mu) = q \sin(p\mu) \cos(q\mu).$$

Let $z = \cos(\mu)$. We recall that the two sets of classical Chebychev polynomials of the first and second kinds can be written respectively as $T_p(z) = \cos(p\mu)$ and $U_p(z) = \sin((p + 1)\mu)/\sin(\mu)$. Thus the equation (10) is equivalent to

$$pU_{q-1}(z)T_p(z) = qU_{p-1}(z)T_q(z).$$
Any root \( z = \cos(\mu) \) of this equation is an algebraic number. Consequently, the values \( \sin(\mu), T_j(z) \) and \( U_j(z) \) are also algebraic numbers for any integer \( j > 0 \). In particular, the value \( p\mu = \sin(p\mu)/\cos(p\mu) \) is algebraic. However \( p\mu = \tan(p\mu) \) has to be transcendental according to a theorem of Hermite-Landemann (c.f. Theorem 1.4 in [3]). We thus have proved our assertion that there could not be two integer solutions.

From this proposition, we can come up with an example of \( n(r) \) yielding all complex eigenvalues.

**Example C.** Let \( n(r) \) be the square of a rational number. We claim that no real eigenvalues exist. Suppose \( n = \mu^2 \) with \( \mu = p/q \) a quotient of two different positive integers. Then the system of equations \( \sin(k) = k\cos(k) \) and \( \sin(pk/q) = pk/q \cdot \cos(kp/q) \) has to hold at a real \( k \). Let \( k = qx \). The two equations become

\[
\sin(qx) = qx\cos(qx) \quad \text{and} \quad \sin(px) = px\cos(px).
\]

These two equations cannot hold unless \( p = q \) by the proposition above. So we have shown that \( \sin(k) = k\cos(k) \) and \( \sin(\mu k) = \mu k\cos(\mu k) \) cannot hold for any rational \( \mu \).

We conclude this section by showing the only situation where an infinite number of real eigenvalues can exist is when \( \int_0^1 \sqrt{n(t)} \, dt = 1 \). This follows from the observation that if \( k \) is a real eigenvalue then

\[
(11) \quad y'' + k^2 n(r)y = 0, \quad y(0) = 0, \quad y(1) - y'(1) = 0, \quad \text{and} \quad \tan(k) = k.
\]

(12)

This follows from (8) since if \( k \) is real we can assume that \( y(r) \) is real valued and hence if \( \Delta(k) = 0 \) then from \( \text{Im} \, \Delta(k) = 0 \), we know that \( y(1) = y'(1) \) or \( \tan(k) = k \). Then since \( \text{Re} \, \Delta(k) = 0 \) we can conclude from (8) that both \( y(1) = y'(1) \) and \( \tan(k) = k \).

So if \( k_j \)'s form a sub-sequence of eigenvalues satisfying (11), then according to a result on the asymptotic behavior of eigenvalues of the Sturm-Liouville problem (c.f. Courant-Hilbert [6], p.415), we find that

\[
\lim_{j \to \infty} k_j^2 j^2 = \frac{1}{\pi^2} \left( \int_0^1 \sqrt{n(t)} \, dt \right)^2
\]

and since \( k_j = \tan(k_j) \), we have that

\[
\lim_{j \to \infty} \frac{k_j}{j} = \pi.
\]

We can now conclude that \( \int_0^1 \sqrt{n(t)} \, dt = 1 \). We can do a careful analysis on the system of equations \( y(1) = y'(1) \) and \( v(1) = v'(1) \) to derive the same conclusion plus an additional necessary condition on \( n(r) \) at \( r = 1 \). This will be done at the end of the following section.
3. Distributions of eigenvalues. For a fixed \( \eta > 0 \), we will show that the zeros of \( \Delta(k) \) approach the real line for large \( \text{Re}(k) \). We recall that

\[
\Delta(k) = y(1) \cos(k) - y'(1) \frac{\sin(k)}{k} - i\eta (y(1) - y'(1)) \left( \cos(k) - \frac{\sin(k)}{k} \right)
\]

\[
= y(1) \left( 1 - i\eta \right) \cos(k) + i\eta \sin(k) \frac{k}{k} + y'(1) \left( i\eta \cos(k) - (1 + i\eta) \frac{\sin(k)}{k} \right).
\]

We first review a set of formulas arrived at from the Levitan-Gelfand transformation of the Sturm-Liouville problem as shown in [9]. Let \( \xi := \int_0^r \sqrt{n(t)} dt \) and \( \delta := \int_0^1 \sqrt{n(t)} dt \). By introducing \( z(\xi) := n(1) \frac{y(r)}{1 + i\eta \frac{n'(r)}{n(r)}} \), equation (5) with \( \alpha := n(0)^{1/4} \) together with its initial values become

\[
z''(\xi) + (k^2 - p(\xi))z(\xi) = 0 \quad (13)
\]

\[
z(0) = 0, \quad z'(0) = 1/\alpha, \quad (14)
\]

where

\[
p(\xi) := \frac{n''(r)}{4n(r)^2} = \frac{5}{16} \frac{n'(r)^2}{n(r)^3}.
\]

The solution \( z(\xi) \) is expressed as (c.f. [4])

\[
z(\xi) = \frac{1}{\alpha k} \left( \sin(k\xi) + \int_0^\xi K(\xi, t) \sin(kt) \, dt \right), \quad 0 \leq \xi \leq \delta,
\]

where \( K(\xi, t) \) is the unique solution of the Goursat problem

\[
K_{\xi t} - K_{tt} - p(\xi) K(\xi, t) = 0, \quad 0 < t < \xi \leq \delta,
\]

\[
K(\xi, 0) = 0, \quad 0 \leq \xi \leq \delta,
\]

\[
K(\xi, \xi) = \frac{1}{2} \int_0^\xi p(s) \, ds, \quad 0 \leq \xi \leq \delta.
\]

To simplify our derivations, we assume that \( n(1) = 1 \) and \( n'(1) = 0 \) as done in [8]. So \( y(1) = z(\delta) \), \( y'(1) = z'(\delta) \). Consequently,

\[
y(1) = z(\delta) = \frac{1}{\alpha k} \left[ \sin(k\delta) + \int_0^\delta K(\delta, t) \sin(kt) \, dt \right] + O(1/k^2), \quad (15)
\]

\[
y'(1) = z'(\delta) = \frac{1}{\alpha k} \left[ k \cos(k\delta) + K(\delta, \delta) \sin(k\delta) + \int_0^\delta K(\delta, t) \sin(kt) \, dt \right] + O(1/k), \quad (16)
\]

as \( k \to \infty \) along the real line. Hence

\[
\Delta(k) = y(1) \cos(k) - y'(1) \frac{\sin(k)}{k} - i\eta (y(1) - y'(1)) \left( \cos(k) - \frac{\sin(k)}{k} \right)
\]

\[
= i\eta \left( \frac{\cos(k\delta)}{\alpha} \right) \cos(k) + O(1/k), \quad (17)
\]

as \( k \) goes to infinity along the positive real axis. The entire functions \( y(1) \) and \( y'(1) \) are of exponential type with type both equal to \( \delta \). As noted in Corollary 2.7 in [9], the entire function \( \Delta(k) \) is of exponential type with type equal to \( \delta + 1 \).
Theorem 3.1. Assume that \( n(1) = 1 \) and \( n'(1) = 0 \). Then complex eigenvalues always exist. If \( \eta > 0 \), the zeros of \( \Delta(k) \) cluster toward the real axis as \( \text{Re}(k) \) gets large.

Proof. An illustration of a typical distribution of complex eigenvalues is provided by Figure 2.

The type of the entire function \( \Delta(k) \) is \( \delta + 1 \). It follows from a result of Levinson (c.f. Theorem 2.5 in [9]) that the density of the zeros of \( \Delta(k) \) in the right half plane is \( (\delta + 1)/\pi \). As shown in the previous section, all the positive zeros of \( \Delta(k) \) are also zeros of \( k\cos(k) - \sin(k) \). This entire function is of type 1. So the density of the positive zeros, if there are infinitely many of them, would be at most \( 1/\pi \). Consequently, the density of complex zeros is always positive.

As shown in the proof of Corollary 2.9 in [9], for a real entire function \( h(z) \) in the Paley-Wiener class of positive type \( \tau \) and of growth \( O(1/x) \) along the real line,

\[
|h(x + iy)| \leq M \frac{\cosh(\tau y)}{|z|},
\]

for some positive constant \( M \). This estimate is due to Duffin and Schaeffer [10]. We now use Rouche’s Theorem to show that the zeros of \( \Delta(k) \) are controlled by the term \( k\cos(k) - \sin(k) \) when \( |k| \) is large. From (17) and Duffin and Schaeffer’s estimate, we have that

\[
\left| \frac{\alpha \Delta(k)}{i\eta} - \cos(k\delta) \cos(k) \right| \leq M \frac{\cosh((\delta + 1)y)}{|k|}
\]

for some positive constant \( M \) and \( k := x + iy \). The right hand side of the above expression goes to zero as \( |k| \) gets large for a fixed \( y \). For this fixed value of \( y \),

\[
|\cos(k\delta) \cos(k)|^2 = (\cos(\delta x)^2 + \sinh(\delta y)^2) \cdot (\cos(x)^2 + \sinh(y)^2) > \sinh(\delta y)^2 \cdot \sinh(y)^2,
\]

which stays positive. Hence we can conclude that when \( \text{Re}(k) \) is large with \( y \) fixed,

\[
\left| \frac{\alpha \Delta(k)}{i\eta} - \cos(k\delta) \cos(k) \right| < |\cos(k\delta) \cos(k)|.
\]

Consequently the zeros of \( \Delta(k) \) lie close to that of \( \cos(\delta k) \cos(k) \) for large \( \text{Re}(k) \) and these zeros are purely real. \( \Box \)

We recall that

\[
\Delta(k) = (y(1) \cos(k) - y'((1)^2 + i\eta(y(1) - y'(1))(\cos(k) - \frac{\sin(k)}{k})).
\]

Suppose we have an infinite set of real eigenvalues \( k \). We saw earlier that a necessary condition for this to happen is that \( \delta = \int_0^1 \sqrt{n(t)} \, dt = 1 \). The first term \( y(1) \cos(k) - y'(1) \sin(k)/k \) in \( \Delta(k) \), under the assumptions that \( n(1) = 1, n'(1) = 0 \), is shown in [9] to have the asymptotic form

\[
\frac{1}{\alpha k} \left( -\frac{K(1, 1)}{k} + O\left( \frac{1}{k^2} \right) \right).
\]

Thus along an infinite subsequence of \( k \)'s where \( \Delta(k) = 0 \) and \( \cos(k) = \sin(k)/k \), we must have \( K(1, 1) = 0 \). But \( K(1, 1) = 1/2 \int_0^1 p(t) \, dt \). So the average value of the Liouville transform of \( n(r) \) has to be 0. One can develop the \( O(1/k^2) \) term further to get further necessary conditions on \( K(\xi, t) \) to have an infinite number of real eigenvalues. The Borg potential used in Example A above gives \( p(\xi) \equiv 0 \). Hence
\( K(1,1) = 0 \). At the moment, we could not find another example of \( n(r) \) producing more than one real eigenvalue.

4. **Determination of** \( n(r) \). In this last section, we show that the knowledge of \( \eta \) and all the transmission eigenvalues will determine \( n(r) \). To help simplify our calculations, we add an extra assumption on the non-constant refractive index that either \( n(r) \leq 1 \) or \( n(r) \geq 1 \) on \([0,1]\). Under this additional constraint, we claim the order of the zero of \( \Delta(k) \) at the origin is 2. The uncoated case \( \eta = 0 \) corresponds to the determinant studied in [5], [9] and [12] which is \( y(1) \cos(k) - y'(1) \sin(k)/k \).

The formulas (2.14) and (2.15) in [5] show that this expression has an asymptotic expansion \( c_2 k^2 + O(k^3) \) for small \( k \) with

\[
c_2 = \frac{\pi}{2 \Gamma(3/2)} \int_0^1 r^2 (1 - n(r)) dr.
\]

The second term in \( \Delta(k) \) is \( -i \eta (y(1) - y'(1))(\cos(k) - \sin(k)/k) \). It has a zero of order at least 2 at the origin. Hence we can conclude that \( \Delta(k) \) has a zero of order 2 at the origin. Now let

\[ c = \lim_{k \to 0} \frac{\Delta(k)}{k^2} \neq 0. \]

By Hadamard’s factorization theorem,

\[
\Delta(k) = y(1,k) \cos(k) - y'(1,k) \frac{\sin(k)}{k}
\]

\[ -i \eta (y(1,k) - y'(1,k)) \left( \cos(k) - \frac{\sin(k)}{k} \right) \]

\[ = c k^2 \prod_{j=1}^{\infty} \left( 1 - \frac{k}{k_j} \right), \]

where \( k_j \)'s are the zeros of \( \Delta(k) \) (including multiplicity).

Now keep \( k \) real and take the imaginary part of \( \alpha \Delta(k) \). Assuming \( \frac{\sin(k)}{k} - \cos(k) \neq 0 \), we let

\[ F := \alpha (y(1,k) - y'(1,k)). \]

From (15) and (16), we see that the factor \( F \) defined above has the asymptotic expansion

\[ F = \cos(k \delta) - \sin(k \delta) \frac{1 - K(\delta, \delta)}{k} + O(1/k^2). \]

Thus the value of \( \delta \) can be obtained from the sequence of the real zeros of \( F \).

Similarly, taking the real part of \( \alpha \Delta(k) \) gives us

\[ \alpha \left( y(1,k) \cos(k) - y'(1,k) \frac{\sin(k)}{k} \right). \]

From Proposition 2.1, \( k \neq \tan(k) \) for \( k = \frac{\ell \pi}{\delta} \) for \( \ell = 1,2,\cdots \) except for at most one value of \( \ell \). Hence from (20) and (21), we have that \( \alpha y(1, \frac{\ell \pi}{\delta}) \) and \( \alpha y'(1, \frac{\ell \pi}{\delta}) \) are known for \( \ell = 1,2,\cdots \) with at most one value of \( \ell \) excluded since

\[
\begin{vmatrix}
1 - \cos(k) & -1 \\
- \cos(k) & \frac{\sin(k)}{k}
\end{vmatrix} = \frac{\sin(k)}{k} - \cos(k).
\]
Using (15) and (16) to represent \( y(1, \ell \pi / \delta) \) and \( y'(1, \ell \pi / \delta) \) we have ( [4], p.273)
\[
\frac{\ell \pi}{\delta} \alpha y(1, \frac{\ell \pi}{\delta}) = \int_0^\delta K(\delta, t) \sin(\frac{\ell \pi t}{\delta}) \, dt \tag{22}
\]
and
\[
\frac{\ell \pi}{\delta} \alpha y'(1, \frac{\ell \pi}{\delta}) = (-1)^\ell + \int_0^\delta K_\xi(\delta, t) \sin(\frac{\ell \pi t}{\delta}) \, dt. \tag{23}
\]
But \( \{ \sin(\frac{\ell \pi t}{\delta}) \} \) is a complete set in \( L^2[0, \delta] \) even if one value of \( \ell \) is omitted (see Young [17], p. 97). Hence \( K(\delta, t) \) and \( K_\xi(\delta, t) \) are known and by a result of Rundell-Sacks ( [14], p.269), \( p(\xi) \) can be computed for \( 0 \leq \xi \leq \delta \) hence so is \( n(r) \) ( [4], p.276).

Our conclusion is:

**Theorem 4.1.** Assume that \( n \in C^3[0, 1] \), \( n(1) = 1 \), \( n'(1) = 0 \), the coefficient \( c_2 \) in (18) is non-zero and \( \eta \) is known. Then the transmission eigenvalues (including multiplicities) uniquely determine \( n(r) \).

Note that in the above theorem it is not necessary to know \( n(0) \), i.e. the set of transmission eigenvalues uniquely determine \( \alpha \Delta(k) \). This same observation also applies to [8], i.e. in Theorem 3 of [8] it is not necessary to know \( n(0) \).

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