Fibre bundle framework for unitary quantum fault tolerance

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Motivations

 Fault tolerance $\rightarrow$ robust computer (major obstacle):
  - Classical fault tolerance – e.g. repetition code
  - Quantum fault tolerance – e.g. transversal gates with ancilla constructions, topological fault tolerance

 We know of various protocols of fault tolerance, we want to understand them in some unified framework.

 Achieved:
  - Developed conjecture of a global and geometric picture of unitary quantum fault tolerance.
  - Proof of conjecture for transversal gates
  - Proof of conjecture for a family of topological codes, including the toric code

 Hope: new insights, new fault tolerant protocols . . .
Conjecture

Correspondence for appropriate fibre bundles $F$, with base space $M$:

- **Unitary fault tolerance**
- **Fault-tolerant logical gates**
- **Fibre bundle $F$ with flat proj. connection**
- **Monodromy rep. of $\pi_1(M)$**

The conjecture ($\rightarrow$) is proven for the cases of:

- transversal gates and
- generalized string operators for a family of topological codes.
Ingredients of a fault-tolerant protocol

Here, we focus on only the QECCs and the FT operations.
Example 1: Transversal gates definition

- Code blocks (of equal size): qudits represented by same colour
- Transversal gates: Interact the $i^{th}$ qudit of each block

A transversal gate on multiple blocks of a QECC can be considered as a transversal gate on a single block of a QECC with larger physical qudits. We group together qudits in the same column to make the larger qudits.
Example 2: Modified toric codes and String operators

- Original toric code by Kitaev in arXiv:quant-ph/9707021
- Modified toric code Hamiltonian (primal defects at $S_v$, dual at $S_f$):

$$H(S_v, S_f) = - \sum_{v \in V \setminus S_v} A_v - \sum_{f \in F \setminus S_f} B_f + \sum_{v \in S_v} A_v + \sum_{f \in S_f} B_f.$$ 

String operators transport defects.
Fibre bundle – The Möbius band

- Constituents: total space, base space, fibre, structure group
- An example:

A nontrivial fibre bundle over the base space $S^1$ (in red) with fiber $\mathbb{R}$ (fiber at one point shown in blue). Structure group is $\mathbb{Z}_2$ in this case.
Base space: Codes as the Grassmannian manifold

Over the next couple of slides, we build up the “big vector bundle” for our picture, from mathematical objects natural for QEC. First,

- Base space is the Grassmannian (a set of codes):
  - An \(((n, K))\) qudit code is a \(K\)-dimensional subspace in \(\mathbb{C}^N\) where \(N = d^n\) (\(n\)-qudit Hilbert space).
  - \(\text{Gr}(K, N) = \{\text{The set of } K\text{-dimensional subspaces in } \mathbb{C}^N\}\)
    - Example: \(\mathbb{C}P^1 = \text{Gr}(1, 2)\)
    - Known as the Grassmannian.
  - Clearly, for \(N = d^n\),
    \[
    \text{Gr}(K, N) = \{\text{The set of } ((n, K)) \text{ qudit codes}\}.
    \]
Vector bundle: Codewords as the tautological vector bundle $\xi(K, N)$

- Total space is the tautological vector bundle (a set of codewords):
  - A codeword in an $((n, K))$ qudit code is a pair $(C, w)$ where $C$ is an $((n, K))$ qudit code and $w \in C$ is a vector.
  - $\xi(K, N)$ is a vector bundle with:
    - Base space is $\text{Gr}(K, N)$, consisting of subspaces $W$
    - Fibre over $W$ is $W$ itself, i.e. the elements are vectors $w \in W$
    - Known as the tautological vector bundle

- Similarly, for $N = d^n$, we have the natural mathematical-QEC correspondence:

$$\xi(K, N) = \{\text{Codewords in some } ((n, K)) \text{ qudit code}\}. \quad (1)$$
Some correspondences between the theory of QECCs and that of fibre bundles

A summary:

| Quantum information objects | Mathematical objects |
|-----------------------------|----------------------|
| Space of ((n, K)) qudit codes | Grassmannian Gr(K, N) where $N = d^n$ |
| Space of the *codewords* $(C, w)$ | tautological vector bundle $\xi(K, N)$ |
| Space of the *encodings* or orthonormal $K$-frames $\beta$ in $\mathbb{C}^N$ | tautological principle $\mathcal{U}(K)$-bundle $P(K, N)$ |
Dynamics in unitary fault tolerance (or unitary QM)

Definition

A *unitary evolution* is a one-parameter family $U(t)$ of unitary operators such that, at time 0, $U(0) = I$, and as time passes, $U(t)$ evolves smoothly (or piecewise smoothly) with time, until at time 1, it accomplishes some target unitary $U(1) = U$.

- Modelling unitary evolutions in our geometric picture
  - Task 1: Unitary evolutions of the codewords (states)
  - Task 2: Unitary evolutions of the QECC (subspaces)
“Dynamics” in the “big vector bundle”

- Given a unitary evolution $U(t)$ and a code $C$, we obtain:
  - a path in the bundle (evolution of codewords)
  - a path in the base space (evolution of codes)

- Resembles a parallel transport/connection (*pre-connection*)
  - Problem: The lift $\tilde{\gamma}(t)$ of $\gamma(t)$ might not be unique.

\[ U(N) \quad \xi(K, N) \]

\[ U(t) \quad \gamma(t) \quad \Gr(K, N) \]

\[ \tilde{\gamma}(t) \]

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Fibre bundle framework for unitary quantum fault tolerance
Restricting bundle to \( \mathcal{M} \subset \text{Gr}(K, N) \) and \( \mathcal{F} \subset \mathcal{U}(N) \)

Schematic illustration of the restrictions:

Conjecture (fault tolerance magic)

For appropriate restrictions (depending on FT protocol), \( \text{FT} \Rightarrow \) the natural (proj.) pre-connection becomes an flat (proj.) connection.
Examples: $\mathcal{F}$ and $\mathcal{M}$

- **Example 1:** Distance $\geq 2$ code with transversal gates
  - $\mathcal{C}$ any code with distance $\geq 2$.
  - $\mathcal{F} = \{\text{Transversal gates}\} \subset \mathcal{U}(N)$
  - $\mathcal{M} = \mathcal{F}(\mathcal{C}) \subset \text{Gr}(K, N)$
  - Flatness results follow from arXiv:0811.4262 (Eastin and Knill)

- **Example 2:** Toric code with string operators
  - $\mathcal{F}_{\text{discr}}$, $\mathcal{F}_{\text{graph}}$, $\mathcal{F}_{\text{ext}}$
  - $\mathcal{C}_K^{HC,(n_v,n_f)} \subset \mathcal{M} \subset \mathcal{M} \subset \text{Gr}(K, N)$

  - $\mathcal{M} \cong \text{defect configuration space (fixed number of defects, hardcore condition)}$; **There is freedom in the choice of $\mathcal{M}$**.
  - Flatness results in arXiv:1309.7062 (Gottesman and Zhang)
**Corollary: Monodromy action**

Fault-tolerant logical gates \[\Rightarrow\] monodromies

A cartoon of $\mathcal{M}$ for single-block transversal gates for the 5-qubit code.
Imperfection of the current conjecture:
- Multiple valid choices of $M$ for the same protocol
- Lacks concrete instructions to construct $M$

Improve conjecture: incorporate error model, propose *canonical* construction of $M$ for each fault-tolerant protocol.
- Stricter correspondence between FT protocols (with error models etc.) and fibre bundles with flat (proj.) connection
- Will enable us to read off new FT protocols from a “nice” bundle construction with flat (proj.) connection
- Proof of improved conjecture

Extend to *full* fault tolerance: e.g. ancilla constructions (appending extra degrees of freedom), measurements

Other applications of the this geometric picture, e.g. TQFTs and topological phases

Thank you!