A dynamic model for the two-parameter Dirichlet process

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Abstract Let \( \alpha = 1/2, \theta > -1/2, \) and \( \nu_0 \) be a probability measure on a type space \( S \). In this paper, we investigate the stochastic dynamic model for the two-parameter Dirichlet process \( \Pi_{\alpha,\theta,\nu_0} \). If \( S = \mathbb{N} \), we show that the bilinear form

\[
\mathcal{E}(F,G) = \frac{1}{2} \int_{\mathcal{P}_1(\mathbb{N})} \langle \nabla F(\mu), \nabla G(\mu) \rangle_{\mu} \Pi_{\alpha,\theta,\nu_0}(d\mu), \quad F, G \in \mathcal{F},
\]

\[
\mathcal{F} = \{ F(\mu) = f(\mu(1), \ldots, \mu(d)) : f \in C^\infty(\mathbb{R}^d), d \geq 1 \}
\]

is closable on \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha,\theta,\nu_0}) \) and its closure \( (\mathcal{E}, D(\mathcal{E})) \) is a quasi-regular Dirichlet form. Hence \( (\mathcal{E}, D(\mathcal{E})) \) is associated with a diffusion process in \( \mathcal{P}_1(\mathbb{N}) \) which is time-reversible with the stationary distribution \( \Pi_{\alpha,\theta,\nu_0} \). If \( S \) is a general locally compact, separable metric space, we discuss properties of the model

\[
\mathcal{E}(F,G) = \frac{1}{2} \int_{\mathcal{P}_1(S)} \langle \nabla F(\mu), \nabla G(\mu) \rangle_{\mu} \Pi_{\alpha,\theta,\nu_0}(d\mu), \quad F, G \in \mathcal{F},
\]

\[
\mathcal{F} = \{ F(\mu) = f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_d, \mu \rangle) : \phi_i \in B_b(S), 1 \leq i \leq d, f \in C^\infty(\mathbb{R}^d), d \geq 1 \}.
\]

In particular, we prove the Mosco convergence of its projection forms.

Keywords Two-parameter Dirichlet process, dynamic model, Dirichlet form, closability, Mosco convergence.
1 Introduction

For any $0 \leq \alpha < 1$ and $\theta > -\alpha$, let $U_k$, $k = 1, 2, \ldots$, be a sequence of independent random variables such that $U_k$ has $Beta(1 - \alpha, \theta + k\alpha)$ distribution. Set

$$V_1^{\alpha, \theta} = U_1, \quad V_n^{\alpha, \theta} = (1 - U_1) \cdots (1 - U_{n-1})U_n, \quad n \geq 2,$$

and let $P(\alpha, \theta) = (P_1(\alpha, \theta), P_2(\alpha, \theta), \ldots)$ denote $(V_1^{\alpha, \theta}, V_2^{\alpha, \theta}, \ldots)$ in descending order. The distribution of $(V_1^{\alpha, \theta}, V_2^{\alpha, \theta}, \ldots)$ is called the two-parameter GEM distribution, denoted by $GEM(\alpha, \theta)$.

The law of $P(\alpha, \theta)$ is called the two-parameter Poisson-Dirichlet distribution, denoted by $PD(\alpha, \theta)$ ([11]). For a locally compact, separable metric space $S$, and a sequence of i.i.d. $S$-valued random variables $\xi_k$, $k = 1, 2, \ldots$, with common distribution $\nu_0$ on $S$, let

$$\Theta_{\alpha, \theta, \nu_0} = \sum_{k=1}^{\infty} P_k(\alpha, \theta) \delta_{\xi_k}.$$

Hereafter, we denote by $\delta_x$ the Dirac delta measure at $x$ for $x \in S$. The distribution of $\Theta_{\alpha, \theta, \nu_0}$, denoted by $Dirichlet(\alpha, \theta, \nu_0)$ or $\Pi_{\alpha, \theta, \nu_0}$, is called the two-parameter Dirichlet process. Both $GEM(\alpha, \theta)$ and $PD(\alpha, \theta)$ carry the information on proportions only while $\Pi_{\alpha, \theta, \nu_0}$ contains information on both proportions and types or labels.

The two-parameter models are natural generalizations to the case $\alpha = 0$. Specifically $PD(0, \theta)$, $GEM(0, \theta)$ and $\Pi_{0, \theta, \nu_0}$ correspond to the well known Poisson-Dirichlet distribution, the GEM distribution and the Dirichlet process, respectively. The Poisson-Dirichlet distribution $PD(0, \theta)$ was introduced by Kingman in [11] to describe the distribution of gene frequencies in a large neutral population at a particular locus. The component $P_k(\theta)$ represents the proportion of the $k$-th most frequent allele. The age-ordered proportions follow the GEM distribution. The Dirichlet process $\Pi_{0, \theta, \nu_0}$ first appeared in [8] in the context of Bayesian statistics. It is a pure atomic random measure with masses distributed according to $PD(0, \theta)$. In the context of population genetics, both the Poisson-Dirichlet distribution and the Dirichlet process appear as approximations to the equilibrium behavior of certain large populations evolving under the influence of mutation and random genetic drift.

Let

$$\nabla_\infty := \left\{ (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} x_i = 1 \right\},$$

denote the infinite dimensional ordered simplex and

$$\nabla_\infty := \left\{ (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}$$

be the closure of $\nabla_\infty$ in the product space $[0, 1]^\infty$. In [4] an infinite dimensional diffusion process, the unlabeled infinitely-many-neutral-alleles model, is constructed on $\nabla_\infty$ with generator

$$A_\theta = \frac{1}{2} \left\{ \sum_{i,j=1}^{\infty} x_i (\delta_{ij} - x_j) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{\infty} \theta x_i \frac{\partial}{\partial x_i} \right\},$$

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defined on an appropriate domain. The reversible measure of this process is shown to be $PD(0, \theta)$.

Let $d \geq 1$, $\phi_i \in B_b(S)$, $1 \leq i \leq d$, $f \in C^\infty(\mathbb{R}^d)$ and $F(\mu) = f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_d, \mu \rangle)$ for $\mu \in \mathcal{P}_1(S)$. Hereafter, we denote by $B_b(S)$ the set of all bounded Borel measurable functions on $S$, $C^\infty(\mathbb{R}^d)$ the set of all infinitely differentiable functions on $\mathbb{R}^d$, and $\mathcal{P}_1(S)$ the space of all probability measures on the Borel $\sigma$-algebra $\mathcal{B}(S)$ in $S$. For $x \in S$ and $\mu \in \mathcal{P}_1(S)$, we define

$$\nabla x F(\mu) := \left. \frac{dF}{ds}(\mu + s\delta_x) \right|_{s=0} = \sum_{i=1}^d \partial_i f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_d, \mu \rangle) \phi_i(x).$$

We write $\nabla F(\mu)$ for the function $x \to \nabla x F(\mu)$. For $\phi, \psi \in B_b(S)$, define

$$\langle \phi, \psi \rangle_\mu := \int_S \phi \psi d\mu - \left( \int_S \phi d\mu \right) \left( \int_S \psi d\mu \right).$$

Given $\nu_0 \in \mathcal{P}_1(S)$ we consider the operator $A$ of the form

$$Ag(x) = \theta \int (g(y) - g(x)) \nu_0(dy), \quad g \in B_b(S).$$

Then, the Fleming-Viot process (cf. [9] and [5]) with neutral parent independent mutation or the labeled infinitely-many-neutral-alleles model is a pure atomic measure-valued Markov process with generator

$$L_\theta F(\mu) = \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_d, \mu \rangle) \langle \phi_i, \phi_j \rangle_\mu + \langle A \nabla F(\mu)(\cdot), \mu \rangle.$$

For compact space $S$ and diffuse probability $\nu_0$, i.e., $\nu_0(x) = 0$ for every $x$ in $S$, it is known ([3]) that the labeled infinitely-many-neutral-alleles model is time-reversible with reversible measure $\Pi_{0, \theta, \nu_0}$.

It is natural to ask whether these diffusion processes have two-parameter analogues when $\alpha$ is positive. Many progresses have been made in this direction over the last decade. In [7], a class of infinite dimensional reversible diffusions is constructed and the reversible measure is $GEM(\alpha, \theta)$. The unlabeled infinitely-many-neutral-alleles model in [4] is generalized to the two-parameter setting in [15] where the generator of the process on appropriate domain has the form

$$A_{\alpha, \theta} = \frac{1}{2} \sum_{i,j=1}^\infty x_i (\delta_{ij} - x_j) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^\infty (\alpha + \theta x_i) \frac{\partial}{\partial x_i},$$

and the reversible measure turns out to be $PD(\alpha, \theta)$. The process, called Petrov diffusion, is derived as the continuum limit of a family of up-down Markov chains involving the Chinese restaurant process. Connections to Bayesian statistics and ecology are explored in [18] and [19]. Going back to the context of population genetics, the Petrov diffusion is constructed recently.
in [2] from a family of the Wright-Fisher diffusions with special selection scheme. In [10], two interval partition-valued diffusions are constructed and the corresponding stationary distributions are \( PD(1/2, 0) \) and \( PD(1/2, 1/2) \), the two cases that are connected to the excursion intervals of Brownian motion and Brownian bridge ([14], [16]).

The situation is more complex in the construction of the labelled diffusion processes in the two-parameter setting. The only model we know of is the one in [6] where the type space consists of two types. In the case \( \alpha = 0 \), the Dirichlet process \( \Pi_{0, \theta, \nu_0} \) has the partition property, i.e., projection of \( \Pi_{0, \theta, \nu_0} \) on any finite partition of the type space \( S \) is a Dirichlet distribution. Exploring the connection between the Wright-Fisher diffusion and the Dirichlet distribution one can naturally construct the Fleming-Viot process from the finite-dimensional Wright-Fisher diffusions. When \( \alpha \) is positive, the projection \( \Pi_{\alpha, \theta, \nu_0} \) on any finite partition of \( S \) has a complicated distribution in general, and finite dimensional diffusion models are no longer available.

The main objective of this paper is to find a labelled reversible diffusion process with \( \Pi_{\alpha, \theta, \nu_0} \) as the reversible measure for certain positive \( \alpha \). This can be viewed as a two-parameter generalization of the Fleming-Viot process with parent independent mutation. The range of parameters we consider throughout the paper is \( \alpha = 1/2 \) and \( \theta > -1/2 \).

In Section 2, we construct the process when the base measure \( \nu_0 \) has countable support. Since the partition property does not hold, we will explore the partition structure through Dirichlet forms. This allows us to avoid certain exceptional sets that cause problems in the representation of generators. In Section 3, we consider the general type space with diffuse base measure. We first show that cylindrical functions do not belong to the domain of the pre-generator of the classical bilinear form. To establish the closability, we consider the relaxation of the bilinear form. The process is then constructed, when \( S \) is a compact Polish space, by taking the Mosco limit.

## 2 Dynamic model with atomic base distribution

Throughout this section, let \( S = \mathbb{N} \), the set of all natural numbers. We consider the bilinear form

\[
\mathcal{E}(F, G) = \frac{1}{2} \int_{\mathcal{P}_1(\mathbb{N})} \langle \nabla F(\mu), \nabla G(\mu) \rangle \mu \Pi_{\alpha, \theta, \nu_0}(d\mu), \quad F, G \in \mathcal{F},
\]

\[
\mathcal{F} = \{ F(\mu) = f(\mu(1), \ldots, \mu(d)) : f \in C^\infty(\mathbb{R}^d), d \geq 1 \}.
\]

**Theorem 2.1** The bilinear form \((\mathcal{E}, \mathcal{F})\) is closable on \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0}) \) and its closure \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular Dirichlet form. The diffusion process associated with \((\mathcal{E}, D(\mathcal{E}))\) is time-reversible with the stationary distribution \( \Pi_{\alpha, \theta, \nu_0} \).

Before proving Theorem 2.1, we make some preparation.

For \( d \geq 1 \), we define

\[
\Delta_d := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ and } \sum_{i=1}^d x_i \leq 1 \right\}.
\]
Denote \( p_i = \nu_0(i), 1 \leq i \leq d \), and \( p_{d+1} = 1 - p_1 - \cdots - p_d \). For \((x_1, \ldots, x_d) \in \Delta_d\), we define
\[
\rho_d(x_1, \ldots, x_d) := \frac{p_1 \cdots p_{d+1} \Gamma(\theta + \frac{d+1}{2})}{\pi^{d/2} \Gamma(\theta + \frac{1}{2})} \frac{x_1^{-3/2} \cdots x_d^{-3/2}}{(p_1^2 x_1 + \cdots + p_{d+1}^2 x_{d+1})^\theta + \frac{d+1}{2}} 1_{\{x_1 + \cdots + x_{d+1} = 1\}}.
\]

Denote \( S_{d+1} = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_i \geq 0 \) and \( \sum_{i=1}^{d+1} x_i = 1\}. \) Following the argument of [1] proof of Lemma 3.1, we get
\[
\int_{\Delta_d} \frac{p_1^4}{x_1^4} \cdots + \frac{p_{d+1}^2}{x_{d+1}^2 + \frac{p_{d+1}^2}{1-x_1-\cdots-x_{d+1}}} 2 \rho_d(x_1, \ldots, x_d) dx_1 \cdots dx_d = \frac{p_1^4 \Gamma(\theta + \frac{d+1}{2})}{\pi^{d/2} \Gamma(\theta + \frac{1}{2})} \frac{p_1 \cdots p_{d+1}}{\Gamma(2 + \theta + \frac{d+1}{2}) 2^{2+\theta+d+1}} \cdot \int_0^\infty \cdots \int_0^\infty (s_1 + \cdots + s_{d+1})^{-\theta} s_1^{-7/2} \prod_{j=1}^{d+1} s_j^{-3/2} \prod_{j=1}^{d+1} e^{-\frac{p_j^2}{2} s_j} ds_1 \cdots ds_{d+1} \leq \frac{C(\theta, p_1)}{(d + 1 + 2\theta)(d + 3 + 2\theta)},
\]

where \( C(\theta, p_1) \) is a positive constant depending only on \( \theta \) and \( p_1 \). Further, we obtain by symmetry that for each \( 1 \leq i \leq p \),
\[
\int_{\Delta_d} \frac{p_i^4}{x_i^4} \cdots + \frac{p_{d+1}^2}{x_{d+1}^2 + \frac{p_{d+1}^2}{1-x_1-\cdots-x_{d+1}}} 2 \rho_d(x_1, \ldots, x_d) dx_1 \cdots dx_d \leq \frac{C(\theta, p_i)}{(d + 1 + 2\theta)(d + 3 + 2\theta)}, \tag{2.1}
\]

where \( C(\theta, p_i) \) is a positive constant depending only on \( \theta \) and \( p_i \).

Denote \( C^\infty(\Delta_d) := \{f|_{\Delta_d} : f \in C^\infty(\mathbb{R}^d)\} \) and \( \Delta_d^0 := \) the interior of \( \Delta_d \). For \( f \in C^\infty(\Delta_d) \), we define
\[
L^{(d)} f(x) = \begin{cases} \frac{1}{2} \sum_{i=1}^d x_i \partial_i^2 f(x) - \frac{1}{2} \sum_{i,j=1}^d x_i x_j \partial_i \partial_j f(x) \\ + \frac{1}{2} \sum_{i=1}^d \left[ -\frac{1}{2} - \theta x_i + \frac{(\theta + d + 1)x_i^2}{p_i^2 x_i + \cdots + p_{d+1}^2 x_{d+1} + \frac{p_{d+1}^2}{1-x_1-\cdots-x_{d+1}}} \right] \partial_i f(x), & x \in \Delta_d^0, \\ 0, & x \in \Delta_d \setminus \Delta_d^0. \end{cases} \tag{2.2}
\]
If $f \in C^\infty(\mathbb{R}^p)$ for some $p \leq d$, we regard $f$ as a function in $C^\infty(\mathbb{R}^d)$ by setting $f(x) = f(x_1, \ldots, x_p)$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. By [2.2], there exists a constant $C(p, f) > 0$, which depends on $p$, $f$ and is independent of $d$, such that for any $x \in \Delta^\circ_d$,

$$|L^{(d)}f(x)| \leq C(p, f) \left[1 + (d + 1) \sum_{i=1}^{p} \frac{\rho_i^2}{x_i} + \cdots + \frac{\rho_d^2}{x_d} + \frac{\rho_{d+1}^2}{1-x_1-\cdots-x_d}\right]. \tag{2.3}$$

By (2.2) and (2.3), we get

$$\int_{\Delta_d} |L^{(d)}f(x)|^2 \rho_d(x) dx \leq C^*(\theta, \nu_0(1), \ldots, \nu_0(p), p, f), \tag{2.4}$$

where $C^*(\theta, \nu_0(1), \ldots, \nu_0(p), p, f)$ is a positive constant depending only on $\theta, \nu_0(1), \ldots, \nu_0(p), p$, and $f$.

**Proof of Theorem 2.1.** Let $d \geq 1$. We consider the map

$$\Upsilon_d: \mathcal{P}_1(\mathbb{N}) \rightarrow \Delta_d,$$

$$\mu \rightarrow \Upsilon_d(\mu) = (\mu(1), \ldots, \mu(d)).$$

By [1, Theorem 3.1], we have $\Pi_{\alpha, \theta, \nu_0} \circ \Upsilon_d^{-1} = \rho_d(x_1, \ldots, x_d)dx_1 \cdots dx_d$. The induced bilinear form of $(\mathcal{E}, \mathcal{F})$ by the map $\Upsilon_d$ is given by

$$\mathcal{E}^{(d)}(f, g) = \frac{1}{2} \sum_{i,j=1}^{d} \int_{\Delta_d} x_i(\delta_{ij} - x_j) \partial_i f(x) \partial_j g(x) \rho_d(x) dx, \quad f, g \in C^\infty(\Delta_d). \tag{2.5}$$

For $x = (x_1, \ldots, x_d) \in \Delta_d$ and $1 \leq j \leq d$, we define

$$V_j(x) = \left(\sum_{i=1}^{d} x_i(\delta_{ij} - x_j) \partial_i f(x)\right) \rho_d(x) g(x),$$

and

$$V = (V_1, \ldots, V_d).$$

Denote by $\partial \Delta_d$ the boundary of $\Delta_d$, $\mathbf{n}$ the outward pointing unit normal field of $\partial \Delta_d$, and $dS_d$ the induced volume form on the surface $\partial \Delta_d$. For the face $\{x = (x_1, \ldots, x_d) \in \Delta_d : x_j = 0\}$, $1 \leq j \leq d$, we have

$$V \cdot \mathbf{n} = V_j$$

$$= \left(x_j(1-x_j) \partial_j f(x) - \sum_{i \neq j} x_i x_j \partial_i f(x)\right) \rho_d(x) g(x)$$

$$= 0.$$
and for the face \( \{ x = (x_1, \ldots, x_d) \in \Delta_d : \sum_{j=1}^d x_j = 1 \} \), we have

\[
V \cdot n = \frac{1}{\sqrt{d}} \sum_{j=1}^d V_j
= \frac{1}{\sqrt{d}} \left( \sum_{i=1}^d x_i \partial_i f(x) \right) \left( 1 - \sum_{j=1}^d x_j \right) \rho_d(x) g(x)
= 0.
\]

Hence \( V \cdot n = 0 \) on \( \partial \Delta_d \). Then, we obtain by (2.4) and the divergence theorem that

\[
\mathcal{E}^{(d)}(f, g) + \int_{\Delta_d} L^{(d)} f(x) g(x) \rho_d(x) dx
= \int_{\Delta_d} \sum_{j=1}^d \partial_j \left[ \left( \sum_{i=1}^d x_i \delta_{ij} - x_j \right) \partial_i f(x_1, \ldots, x_d) \right] \rho_d(x_1, \ldots, x_d) g(x_1, \ldots, x_d) dx_1 \cdots dx_d
= \int_{\Delta_d} \text{div} V dx_1 \cdots dx_d
= \int_{\partial \Delta_d} V \cdot n dS_d
= 0.
\]

Thus, we have

\[
\mathcal{E}^{(d)}(f, g) = \int_{\Delta_d} -L^{(d)} f(x) g(x) \rho_d(x) dx. \tag{2.6}
\]

Now we use the estimate (2.4) to show that \( (\mathcal{E}, \mathcal{F}) \) is closable on \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0}) \). To this end, let \( \{ F_n \in \mathcal{F} \} \) be a sequence satisfying

\[
\lim_{n \to \infty} \| F_n \|_{L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0})} = 0 \quad \text{and} \quad \lim_{n,m \to \infty} \mathcal{E}(F_n - F_m, F_n - F_m) = 0.
\]

Note that

\[
\mathcal{E}(F_n, F_n) = \mathcal{E}(F_n - F_k, F_n) + \mathcal{E}(F_k, F_n)
\leq \mathcal{E}^{1/2}(F_n - F_k, F_n - F_k) \mathcal{E}^{1/2}(F_n, F_n) + \mathcal{E}(F_k, F_n).
\]

To show \( \lim_{n \to \infty} \mathcal{E}(F_n, F_n) = 0 \), we need only show that for any fixed \( k \),

\[
\lim_{n \to \infty} \mathcal{E}(F_k, F_n) = 0.
\]

Suppose that \( F_n(\mu) = f^{(n)}(\mu(1), \ldots, \mu(p^{(n)})) \) with \( f^{(n)} \in C^\infty(\mathbb{R}^{p^{(n)}}) \) and \( p^{(n)}, n \in \mathbb{N} \). By (2.4) and (2.6), we get

\[
|\mathcal{E}(F_k, F_n)| = |\mathcal{E}(p^{(k)} \cup p^{(n)})(f^{(k)}, f^{(n)})|
\]
Proof. Let \( f, g \) be any functions. Thus, \((E, F)\) is closable on \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0})\).

Following the argument of [20] Proposition 5.11 and Lemma 7.5, we can show that the closure \((\mathcal{E}, D(\mathcal{E}))\) of \((\mathcal{E}, \mathcal{F})\) is a quasi-regular, symmetric, local Dirichlet form on \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0})\). Therefore, there exists an associated diffusion process in \( \mathcal{P}_1(\mathbb{N}) \) which is time-reversible with the stationary distribution \( \Pi_{\alpha, \theta, \nu_0} \).

Denote by \((L, D(L))\) the generator of \((\mathcal{E}, D(\mathcal{E}))\) on \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0})\). In the following, we will give an explicit expression for \( L \).

**Theorem 2.2** (i) \( \mathcal{F} \subset D(L) \).

(ii) For each \( i \in \mathbb{N} \),

\[
\lim_{d \to \infty} \frac{(d+1)^{\nu(i)^2}}{\mu(1)} \sum_{i=1}^{d} \frac{\nu(j)^2}{\mu(d)} + \frac{\nu(d)^2}{\mu(1) \cdots \mu(d)} \quad \text{exists in } L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0}). \tag{2.7}
\]

(iii) For \( i \in \mathbb{N} \), denote by \( B_i(\mu) \) the \( L^2 \)-limit given in (2.7). Let \( F(\mu) = f(\mu(1), \ldots, \mu(d)) \) with \( f \in C^\infty(\mathbb{R}^d) \) and \( d \geq 1 \). We have

\[
LF(\mu) = \frac{1}{2} \sum_{i=1}^{d} \mu(i) \partial_i^2 f(\mu(1), \ldots, \mu(d)) - \frac{1}{2} \sum_{i,j=1}^{d} \mu(i) \mu(j) \partial_i \partial_j f(\mu(1), \ldots, \mu(d)) \\
+ \frac{1}{2} \sum_{i=1}^{d} \left[ -\frac{1}{2} - \theta \mu(i) + \frac{1}{2} B_i(\mu) \right] \partial_i f(\mu(1), \ldots, \mu(d)). \tag{2.8}
\]

**Proof.** Let \( F(\mu) = f(\mu(1), \ldots, \mu(d)) \) for some \( f \in C^\infty(\mathbb{R}^d) \) and \( d \geq 1 \). For \( G(\mu) = g(\mu(1), \ldots, \mu(d')) \) with \( g \in C^\infty(\mathbb{R}^{d'}) \) and \( d' \geq 1 \), we obtain by (2.7) that

\[
|\mathcal{E}(F, G)| = |\mathcal{E}^{(d\times d')}(f, g)| \leq (C^\ast(\theta, \nu_0(1), \ldots, \nu_0(d), d, f))^{1/2} \|G\|_{L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0})}.
\]

Since \( G \) is arbitrary, we conclude that \( F \in D(L) \) by [12] Chapter I, Proposition 2.16.

For \( n > d \), we regard \( f \) as a function in \( C^\infty(\mathbb{R}^n) \) by setting \( f(x) = f(x_1, \ldots, x_d) \) for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). We claim that

\[
LF = \lim_{n \to \infty} (L^{(n)} f) \circ \Upsilon_n \quad \text{in } L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0}). \tag{2.9}
\]

In fact, it is easy to see that

\[
P_n(LF) = (L^{(n)} f) \circ \Upsilon_n \quad \text{for } n \geq d,
\]
where \( P_n \) is the orthogonal projection of \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0}) \) onto the closure of \( \{ G(\mu) = g(\mu(1), \ldots, \mu(n)) : g \in C^\infty(\mathbb{R}^n) \} \). Since \( \mathcal{F} \) is dense in \( L^2(\mathcal{P}_1(\mathbb{N}); \Pi_{\alpha, \theta, \nu_0}) \), we obtain (2.9).

For \( i \in \mathbb{N} \), let \( F_i(\mu) = \mu(i) \) for \( \mu \in \mathcal{P}_1(\mathbb{N}) \). By (2.9), we get

\[
LF_i(\mu) = \frac{1}{2} \lim_{n \to \infty} \frac{\theta + \frac{n+1}{2} \frac{\nu_0(i)^2}{\mu(i)} + \cdots + \frac{\nu_0(n)^2}{\mu(n)} + \frac{\nu_0(n+1)^2}{1-\mu(1) \cdots \mu(n)}}{\mu(i)}.
\]

Hence, (2.7) holds and \( B_i = 4LF_i \). Therefore, we obtain (2.8) by (2.7) and (2.9).

**Remark 2.3** For \( f \in C^\infty(\mathbb{R}) \) and \( x \in (0, 1) \), we have

\[
L^{(1)} f(x) = \frac{1}{2} x(1-x) f''(x) + \frac{1}{2} \left( \theta + \frac{1}{2} - (2\theta + 1)x \right) f'(x).
\]

The eigenvalues of \( L^{(1)} \) are \(-i(i+2\theta)/2\) with multiplicity 1, \( i \in \mathbb{N} \). It deserves further investigation to characterize the eigenvalues of \( L^{(d)} \) for \( d > 1 \).

### 3 Dynamic model with diffuse base distribution

In this section, let \( S \) be a general locally compact, separable metric space and \( \nu_0 \) a diffuse probability measure on \( S \). We consider the classical bilinear form

\[
\begin{align*}
\mathcal{E}(F, G) &= \frac{1}{2} \int_{\mathcal{P}_1(S)} \langle \nabla F(\mu), \nabla G(\mu) \rangle_{\mu, \Pi_{\alpha, \theta, \nu_0}(d\mu)}, \quad F, G \in \mathcal{F},\
\mathcal{F} &= \{ F(\mu) = f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_d, \mu \rangle) : \phi_i \in B_i(S), 1 \leq i \leq d, f \in C^\infty(\mathbb{R}^d), d \geq 1 \}.
\end{align*}
\]

If \( (\mathcal{E}, \mathcal{F}) \) is closable on \( L^2(\mathcal{P}_1(S); \Pi_{\alpha, \theta, \nu_0}) \), then following the argument of [20, Proposition 5.11 and Lemma 7.5], we can show that the closure of \( (\mathcal{E}, \mathcal{F}) \) is a quasi-regular, symmetric, local Dirichlet form on \( L^2(\mathcal{P}_1(S); \Pi_{\alpha, \theta, \nu_0}) \). Therefore, there exists an associated diffusion process in \( \mathcal{P}_1(S) \) which is time-reversible with the stationary distribution \( \Pi_{\alpha, \theta, \nu_0} \).

Up to now we still cannot prove that \( (\mathcal{E}, \mathcal{F}) \) is closable on \( L^2(\mathcal{P}_1(S); \Pi_{\alpha, \theta, \nu_0}) \). In the following, we will discuss properties of the model (3.1). We fix a sequence \( \{(B_1^k, \ldots, B_{2^k})\}_{k=1}^\infty \) of partitions of \( S \) satisfying the following conditions:

1. \( \nu_0(B_j^k) = 1/2^k, 1 \leq j \leq 2^k, k \in \mathbb{N} \).
2. \( B_j^k = B_{2j-1}^{k+1} \cup B_{2j}^{k+1}, 1 \leq j \leq 2^k, k \in \mathbb{N} \).

#### 3.1 \( \mathcal{F} \not\subset D(L) \)

Denote by \( (L, D(L)) \) the pre-generator of \( (\mathcal{E}, \mathcal{F}) \) on \( L^2(\mathcal{P}_1(S); \Pi_{\alpha, \theta, \nu_0}) \). A special feature of the model (3.1) is that \( \mathcal{F} \not\subset D(L) \). More precisely, we have the following result.
**Proposition 3.1** Suppose that \( \theta = 0 \). Let \( H(\mu) = \langle 1_{B_1^i}, \mu \rangle \). There does not exist \( LH \in L^2(M_1(S); \Pi_{\alpha, \theta, \rho_0}) \) such that

\[
\mathcal{E}(H, G) = \int_{\mathcal{P}_1(S)} -LH(\mu)G(\mu)\Pi_{\alpha, \theta, \rho_0}(d\mu), \quad \forall G \in \mathcal{F}.
\] (3.2)

**Proof.** Let \( d \geq 1 \) and \( p \leq d \). For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we define

\[
f(x) = x_1 + \cdots + x_p.
\]

Set \( p_i = 1/(d+1) \), \( 1 \leq i \leq d+1 \). We define \( L^{(d)} \) as in [22]. Then, we have

\[
L^{(d)} f(x) = \frac{1}{4} \left( -p + (d+1) \sum_{i=1}^p \frac{1}{x_i} + \frac{1}{x_d} + \frac{1}{1-x_1-\cdots-x_d} \right).
\] (3.3)

Following the argument of [11; proof of Lemma 3.1], we get

\[
\int_{\Delta_d} \frac{1}{x_1 + \cdots + x_d + 1} \rho_d(x_1 \ldots x_d)dx_1 \ldots dx_d
\]

\[
= \frac{\Gamma(\frac{d+1}{2})}{\pi^{d/2} \Gamma(\frac{1}{2})} \int_{S_{d+1}} \frac{x_1 \cdots x_d}{x_1 + \cdots + x_d} \frac{x_1^{3/2} \cdots x_{d+1}^{3/2}}{2^{d+1}} dx_1 \cdots dx_{d+1}
\]

\[
= \frac{\Gamma(\frac{d+1}{2})}{(d+1)^{3/2} \Gamma(\frac{1}{2}) (2 + \frac{d+1}{2})2^{2+\frac{d+1}{2}}} \int_0^\infty \int_0^\infty \cdots \int_0^\infty s_1^{-\frac{7}{2}} s_3^{-\frac{3}{2}} \cdots s_{d+1}^{-\frac{3}{2}} e^{-p_i^2 s_j} ds_1 \cdots ds_{d+1}
\]

\[
= \frac{3}{(d+1)(d+3)}.
\]

and

\[
\int_{\Delta_d} \frac{1}{x_1 + \cdots + x_d + 1} \rho_d(x_1 \ldots x_d)dx_1 \cdots dx_d
\]

\[
= \frac{\Gamma(\frac{d+1}{2})}{\pi^{d/2} \Gamma(\frac{1}{2})} \int_{S_{d+1}} \frac{x_1 \cdots x_d}{x_1 + \cdots + x_d} \frac{x_1^{-5/2} \cdots x_{d+1}^{-5/2}}{2^{d+1}} dx_1 \cdots dx_{d+1}
\]

\[
= \frac{1}{(d+1)(d+3)}.
\]

Further, we obtain by symmetry that

\[
\int_{\Delta_d} \frac{1}{x_1 + \cdots + x_d + 1} \rho_d(x_1 \ldots x_d)dx_1 \cdots dx_d = \frac{3}{(d+1)(d+3)}, \quad 1 \leq i \leq d, \quad (3.4)
\]
and
\[
\int_{\Delta_d} \frac{1}{x_i x_j} \rho_d(x_1 \ldots x_d) dx_1 \ldots dx_d = \frac{1}{(d+1)(d+3)}, \quad 1 \leq i < j \leq d.
\]

By (3.3)–(3.5), we get
\[
16 \int_{\Delta_d} |L^{(d)} f(x)|^2 \rho_d(x) dx
= \int_{\Delta_d} \left( -p + (d + 1) \sum_{i=1}^{p} \frac{1}{x_i} + \frac{1}{x_d} + \frac{1}{1-x_1 \ldots x_d} \right)^2 \rho_d(x_1 \ldots x_d) dx_1 \ldots dx_d
= -p^2 + (d + 1)^2 p \cdot \frac{3}{(d + 1)(d + 3)} + (d + 1)^2 (p^2 - p) \cdot \frac{1}{(d + 1)(d + 3)}
= \frac{2p(d + 1 - p)}{d + 3}.
\]

Suppose there exists \( LH \in L^2(M_1(S); \Pi_{\alpha,\theta,\nu_0}) \) such that (3.2) holds. For \( k \in \mathbb{N} \), we define
\[
F_k = \{ F(\mu) = f(1_{B_1^k} \mu), \ldots, (1_{B_{2k-1}^k} \mu) : f \in C^\infty(\mathbb{R}^{2k-1}) \},
\]
and
\[
f_k(x_1, \ldots, x_{2k-1}) = x_1 + \cdots + x_{2k-1}, \quad x_i \in \mathbb{R}, \quad 1 \leq i \leq 2k - 1.
\]
Let \( G(\mu) = g(1_{B_1^k} \mu), \ldots, (1_{B_{2k-1}^k} \mu) \) for some \( g \in C^\infty(\mathbb{R}^{2k-1}) \). By (3.2), we get
\[
\int_{\mathcal{P}_1(S)} -LH(\mu) G(\mu) \Pi_{\alpha,\theta,\nu_0}(d\mu) = E(H,G)
= E^{(2k-1)}(f_k, g)
= \int_{\Delta_{2k-1}} -L^{(2k-1)} f_k(x) g(x) \rho_{2k-1}(x) dx.
\]

Since \( G \in F_k \) is arbitrary, we obtain by (3.6) and (3.8) that
\[
\| LH \|_{L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})} \geq \| L^{(2k-1)} f_k \|_{L^2(\Delta_{2k-1}; \Pi_{\alpha,\theta,\nu_0} Y_{2k-1})}^2
= \frac{2k-1(2^k - 2k-1)}{8(2^k + 2)}.
\]

Since \( k \in \mathbb{N} \) is arbitrary, there is a contradiction. Therefore, there does not exist \( LH \in L^2(M_1(S); \Pi_{\alpha,\theta,\nu_0}) \) such that (3.2) holds.

\[\square\]

### 3.2 Mosco convergence of projection forms

Since we do not know if \((\mathcal{E}, \mathcal{F})\) is closable on \( L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0}) \), we consider the relaxation of \((\mathcal{E}, \mathcal{F})\). By [13 page 373], there exists a greatest lower semicontinuous bilinear form on
\(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\) which is a minorant of \((\mathcal{E}, \mathcal{F})\). This unique determined closed form is called the relaxation of \((\mathcal{E}, \mathcal{F})\), denoted by \((\Xi, D(\Xi))\). We have that \(\mathcal{F} \subset D(\Xi)\) and \(\Xi(F, F) \leq \mathcal{E}(F, F)\) for any \(F \in \mathcal{F}\), and for every \(F \in D(\Xi)\),

\[
\Xi(F, F) = \min \left\{ \liminf_{n \to \infty} \mathcal{E}(F_n, F_n) : F_n \in \mathcal{F} \text{ for } n \in \mathbb{N} \right\},
\]

and
\[
\lim_{n \to \infty} F_n = F \text{ in } L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\}
\]

(3.9)

Note that if \((\mathcal{E}, \mathcal{F})\) is closable, then \((\Xi, D(\Xi))\) is just the closure of \((\mathcal{E}, \mathcal{F})\) on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\).

By Corollary 2.8.2, \((\Xi, D(\Xi))\) is a Dirichlet form on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\). Further, if \(S\) is a compact Polish space, then \((\Xi, D(\Xi))\) is a regular Dirichlet form. Hence \((\Xi, D(\Xi))\) is associated with a Markov process in \(\mathcal{P}_1(S)\) which is time-reversible with the stationary distribution \(\Pi_{\alpha,\theta,\nu_0}\). Let \(\mathcal{F}_k\) be defined as in (3.7) for \(k \in \mathbb{N}\). In this subsection, we will show that the limit of the sequence \(\{(\mathcal{E}, \mathcal{F}_k)\}\) is given by \((\Xi, D(\Xi))\).

For \(k \in \mathbb{N}\), we consider the map \(\Gamma_k : \mathcal{P}_1(S) \rightarrow \Delta_{2^k-1}\),

\[
\mu \mapsto \Gamma_k(\mu) = (\mu(B_1^k), \ldots, \mu(B_{2^k-1}^k)).
\]

For \(F(\mu) = f(\langle 1_{B_1^k}, \mu \rangle, \ldots, \langle 1_{B_{2^k-1}^k}, \mu \rangle)\) with \(f \in C^\infty(\mathbb{R}^{2^k-1})\) and \(G(\mu) = g(\langle 1_{B_1^k}, \mu \rangle, \ldots, \langle 1_{B_{2^k-1}^k}, \mu \rangle)\) with \(g \in C^\infty(\mathbb{R}^{2^k-1})\), we obtain by (2.6) that

\[
\mathcal{E}(F, G) = \mathcal{E}^{2^k-1}(f, g)
\]

\[
= \int_{\Delta_{2^k-1}} -L^{2^{k}-1} f(x) g(x) \rho_{2^{k}-1}(x) dx
\]

\[
= ((-L^{2^{k}-1} f) \circ \Gamma_k, G)_{L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})}.
\]

Hence \((\mathcal{E}, \mathcal{F}_k)\) is closable on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\) by Corollary 2.8.2. Denote by \((\mathcal{E}, D(\mathcal{E}))\) the closure of \((\mathcal{E}, \mathcal{F}_k)\). We have \(D(\mathcal{E})_1 \subset D(\mathcal{E})_2 \subset \cdots \subset L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\) and \((\mathcal{E}, D(\mathcal{E}))_k\) is an extension of \((\mathcal{E}, D(\mathcal{E}))_k\) for each \(k \in \mathbb{N}\).

For \(k \in \mathbb{N}\), we define the resolvent \((G_{\beta}^k)_{\beta>0}\) of \((\mathcal{E}, D(\mathcal{E}))_k\) by

\[
\mathcal{E}_\beta(G_{\beta}^k F, G) = (F, G)_{L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})}, \quad \forall G \in D(\mathcal{E})_k,
\]

(3.10)

where \(\mathcal{E}_\beta(F, G) := \mathcal{E}(F, G) + \beta (F, G)_{L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})}\) for \(F, G \in D(\mathcal{E})_k\). Given \(F \in L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\), the existence and uniqueness of \(G_{\beta}^k F \in D(\mathcal{E})_k\) satisfying (3.10) follows from the Riesz representation theorem.

Denote by \((G_{\beta})_{\beta>0}\) the strongly continuous contraction resolvent associated with the Dirichlet form \((\Xi, D(\Xi))\) on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\). We have the following characterization of \((G_{\beta})_{\beta>0}\) by virtue of \((G_{\beta}^k)_{\beta>0}\).

**Theorem 3.2** For every \(\beta > 0\), the sequence of resolvent operators \(\{G_{\beta}^k\}\) converges to \(G_{\beta}\) in the strong operator topology.
Proof. We first show that for any subsequence \( \{k'\} \) of \( \{k\} \), there exists a subsequence \( \{k''\} \) of \( \{k'\} \) such that for every \( \beta > 0 \) the sequence \( \{G^k_\beta\} \) converges to a resolvent operator.

For \( k \in \mathbb{N} \), we define

\[
\mathcal{F}_k^* = \{ F(\mu) = f(\langle 1_{B^k_1}^k, \mu \rangle, \ldots, \langle 1_{B^k_{2^k-1}}^k, \mu \rangle) : f \text{ is a polynomial on } \mathbb{R}^{2^k-1} \text{ with rational coefficients} \}.
\]

Denote

\[
\mathcal{F}^* = \bigcup_{k=1}^{\infty} \mathcal{F}_k^*.
\]

Let \( Q_+ \) be the set of all positive rational numbers. Note that

\[
\|G^k_\beta F\|_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0})} \leq \frac{1}{\beta} \|F\|_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0})}, \quad \forall F \in \mathcal{F}_k, l \geq k, \beta > 0.
\]

(3.12)

By the diagonal argument, there exists a subsequence \( \{k''\} \) of \( \{k'\} \) such that

\[
G^k_\beta F \text{ converges weakly in } L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0}) \text{ as } k'' \to \infty, \quad \forall F \in \mathcal{F}, \beta \in Q_+.
\]

We fix such a subsequence \( \{k''\} \) and define

\[
G^*_\beta F := w - \lim_{k'' \to \infty} G^{k''}_\beta F \text{ in } L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0}), \quad F \in \mathcal{F}, \beta \in Q_+.
\]

Let \( k'' \leq l'' \). For \( F \in \mathcal{F} \) and \( \beta \in Q_+ \), we have

\[
E_\beta(G^{k''}_\beta F - G^{l''}_\beta F, G^{k''}_\beta F - G^{l''}_\beta F) = E_\beta(G^{k''}_\beta F - G^{l''}_\beta F, G^{k''}_\beta F) - E_\beta(G^{k''}_\beta F - G^{l''}_\beta F, G^{l''}_\beta F) = \{(F, G^{k''}_\beta F) - (F, G^{l''}_\beta F)\} - (G^{k''}_\beta F - G^{l''}_\beta F).
\]

(3.13)

Hence

\[
\lim_{k'' \to \infty} \lim_{l'' \to \infty} E_\beta(G^{k''}_\beta F - G^{l''}_\beta F, G^{k''}_\beta F - G^{l''}_\beta F) = 0.
\]

Thus,

\[
G^*_\beta F = \lim_{k'' \to \infty} G^{k''}_\beta F \text{ in } L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0}), \quad F \in \mathcal{F}, \beta \in Q_+.
\]

(3.14)

By \( 3.12 \) and \( 3.14 \), we get

\[
\|\beta G^*_\beta F\|_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0})} \leq \|F\|_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0})}, \quad \forall F \in \mathcal{F}, \beta \in Q_+.
\]

(3.15)

For every \( \beta \in Q_+ \), by \( 3.15 \), we can extend \( \beta G^*_\beta \) to a continuous contraction operator on \( L^2(P_1(S); \Pi_{\alpha,\theta,\nu_0}) \). Further, by \( 3.14 \), the resolvent equations for \( \{G^{k''}_\beta\} \), and the density of \( \mathcal{F}^* \) in
Let $F \in \mathcal{F}^*$ and $\beta \in \mathbb{Q}_+$. By (3.13), we find that $\{(\beta G_\beta^k F, F)_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})}\}_{k''=1}^\infty$ is an increasing sequence. Then, we obtain by (3.14) that
\[
\liminf_{\beta \in \mathbb{Q}_+, \beta \to \infty} (\beta G_\beta^* F, F)_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})} \geq \limsup_{k'' \to \infty} \lim_{\beta \in \mathbb{Q}_+, \beta \to \infty} (\beta G_\beta^{k''} F, F)_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})} = \|F\|_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})}.
\]
By (i) and (3.16), we get
\[
\lim_{\beta \in \mathbb{Q}_+, \beta \to \infty} (\beta G_\beta^* F, F)_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})} = \|F\|_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})}.
\]
Thus, we obtain by (i), (iii), and the density of $\mathcal{F}^*$ in $L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})$ that
\[(iv) \lim_{\beta \to \infty} \|\beta G_\beta^* F - F\|_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})} = 0, \ \forall F \in L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0}).\]

By (i), (iii), and (iv), we know that $(G_\beta^*)_{\beta > 0}$ is a strongly continuous contraction resolvent on $L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})$ (cf. [12, Chapter I, Definition 1.4]). Then, there exists a unique symmetric Dirichlet form $(\Lambda, D(\Lambda))$ on $L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})$ such that its resolvent is given by $(G_\beta^*)_{\beta > 0}$, i.e.,
\[\Lambda(G_\beta^* F, G) + \beta(G_\beta^* F, G) = (F, G)_{L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})}, \ \forall F \in L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0}), G \in D(\Lambda).\]
By (ii) and [13, Theorem 2.4.1], we find that $(\mathcal{E}, D(\mathcal{E})_{k''})$ converges to $(\Lambda, D(\Lambda))$ in the sense of Mosco convergence as $k'' \to \infty$, i.e.,

(a) For every $F_{k''} \in D(\mathcal{E})_{k''}$ converging weakly to $F \in D(\Lambda)$ in $L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})$,
\[
\liminf_{k'' \to \infty} \mathcal{E}(F_{k''}, F_{k''}) \geq \Lambda(F, F).
\]

(b) For every $F \in D(\Lambda)$, there exists $F_{k''} \in D(\mathcal{E})_{k''}$ converging strongly to $F \in D(\Lambda)$ in $L^2(\mathcal{P}_1(S); \Pi_{a,\theta,v_0})$, such that
\[
\limsup_{k'' \to \infty} \mathcal{E}(F_{k''}, F_{k''}) \leq \Lambda(F, F).
\]
By (a), we know that \((\Lambda, D(\Lambda))\) is a minorant of \((\mathcal{E}, \mathcal{F})\). By (b) and \([3.9]\), we obtain that 
\[ D(\Lambda) \subset D(\Xi) \text{ and } \Xi(F, F) \leq \Lambda(F, F) \text{ for } F \in D(\Lambda). \]
Since \((\Xi, D(\Xi))\) is the greatest closed form on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\) which is a minorant of \((\mathcal{E}, \mathcal{F})\), we get \((\Lambda, D(\Lambda)) = (\Xi, D(\Xi))\). Then, we obtain by (ii) that
\[
G_\beta F = G_\beta^* F = \lim_{k' \to \infty} G_\beta^{k'} F, \quad \forall F \in L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0}), \beta > 0.
\]
Since the subsequence \(\{k'\}\) of \(\{k\}\) is arbitrary, we get
\[
G_\beta F = \lim_{k \to \infty} G_\beta^k F, \quad \forall F \in L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0}), \beta > 0. \tag{3.17}
\]

As a direct consequence of Theorem \(3.2\) and \([13\text{, Theorem 2.4.1}]\), we obtain the Mosco convergence of projection forms of the model \(3.1\).

**Corollary 3.3** The sequence of bilinear forms \((\mathcal{E}, \mathcal{F}_k)\) converges to \((\Xi, D(\Xi))\) in the sense of Mosco convergence.

For \(k \in \mathbb{N}\), we define the bilinear form \(\mathcal{E}^{(2k-1)}\) as in \([2.5]\). By \([2.6]\), we know that \((\mathcal{E}^{(2k-1)}, C^\infty(\Delta_{2k-1}))\) is closable on \(L^2(\Delta_{2k-1}, \Pi_{\alpha,\theta,\nu_0} \circ \Gamma_k^{-1})\). Denote by \((\mathcal{E}^{(2k-1)}, D(\mathcal{E}^{(2k-1)}))\) the closure of \((\mathcal{E}^{(2k-1)}, C^\infty(\Delta_{2k-1}))\), \((T_t^{(2k-1)})_{t \geq 0}\) the semigroup associated with \((\mathcal{E}^{(2k-1)}, D(\mathcal{E}^{(2k-1)}))\) on \(L^2(\Delta_{2k-1}, \Pi_{\alpha,\theta,\nu_0} \circ \Gamma_k^{-1})\), and \(Q_k\) the orthogonal projection of \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\) onto the closure of \(\mathcal{F}_k\). For \(F \in L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\) and \(t \geq 0\), we define
\[
T_t^{k} F = (T_t^{(2k-1)}((Q_k F) \circ \Gamma_k^{-1}) \circ \Gamma_k).
\]
Then, \((T_t^{k})_{t \geq 0}\) is the semigroup associated with the bilinear form \((\mathcal{E}, D(\mathcal{E})_k)\) on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\).

Denote by \((T_t)_{t \geq 0}\) the strongly continuous contraction semigroup associated with the Dirichlet form \((\Xi, D(\Xi))\) on \(L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0})\). We have the following characterization of \((T_t)_{t \geq 0}\) by virtue of \((T_t^{k})_{t \geq 0}\).

**Theorem 3.4** For every \(t \geq 0\), the sequence of semigroup operators \(\{T_t^{k}\}\) converges to \(T_t\) in the strong operator topology.

**Proof.** Let \(\{k'\}\) be a subsequence of \(\{k\}\). By the diagonal argument, there exists a subsequence \(\{k''\}\) of \(\{k'\}\) such that
\[
w - \lim_{k'' \to \infty} T_t^{k''} F \text{ exists in } L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0}), \quad \forall F \in \mathcal{F}^*, t \in \mathbb{Q}_+,
\]
where \(\mathcal{F}^*\) is defined as in \([3.11]\). We define
\[
T_t^\Delta F := w - \lim_{k'' \to \infty} T_t^{k''} F \text{ in } L^2(\mathcal{P}_1(S); \Pi_{\alpha,\theta,\nu_0}), \quad F \in \mathcal{F}^*, t \in \mathbb{Q}_+.
\]
By the density of $F^*$ in $L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})$ and the contraction of the semigroup operators $\{T_t^k\}$, we can extend $(T_t^k)_{t \in \mathbb{Q}_+}$ to a collection of contraction linear operators on $L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})$ such that

$$T_t^\Delta F = w - \lim_{k' \to \infty} T_t^{k''} F \text{ in } L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0}), \quad \forall F \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0}), \quad t \in \mathbb{Q}_+. \quad (3.18)$$

By (3.18), we find that

$$t \to (T_t^\Delta F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \text{ is decreasing on } \mathbb{Q}_+, \quad \forall F \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0}). \quad (3.19)$$

Hence, for any $t \geq 0$ and $F, G \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})$, we have

$$\lim_{s \in \mathbb{Q}_+, s \downarrow t} \frac{1}{4} \lim_{s \in \mathbb{Q}_+, s \downarrow t} \left\{ (T_t^\Delta (F + G), (F + G))_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} - (T_t^\Delta (F - G), (F - G))_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \right\} \exists. \quad (3.20)$$

By (3.19) and (3.20), we know that

$$T_t^* F := w - \lim_{s \in \mathbb{Q}_+, s \downarrow t} T_s^\Delta F, \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0}) \quad (3.21)$$

is well-defined. Moreover, by (3.21), we can show that $(T_t^*)_{t \geq 0}$ is a collection of contraction linear operators on $L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})$.

By (3.19) and (3.21), we find that

$$t \to (T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \text{ is decreasing on } [0, \infty), \quad \forall F \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0}). \quad (3.22)$$

Hence, there exists a collection of subsets $\{E_F\}_{F \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})}$ of $[0, \infty)$ such that

$$t \to (T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \text{ is continuous on } [0, \infty) \setminus E_F, \quad \forall F \in L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0}).$$

For $t \geq 0$, we obtain by (3.18) and (3.21) that

$$(T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} = \lim_{s \in \mathbb{Q}_+, s \downarrow t} (T_s^\Delta F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} = \lim_{s \in \mathbb{Q}_+, s \downarrow t} \lim_{k'' \to \infty} (T_s^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \leq \lim_{k'' \to \infty} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})}. \quad (3.22)$$

For $t \in (0, \infty) \setminus E_F$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \geq (T_s^* F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} - \varepsilon, \quad \forall s \in ((t - \delta) \vee 0, t). \quad (3.23)$$

By (3.18), (3.21), and (3.23), we know that there exists $t^* \in (0, t) \cap \mathbb{Q}_+$ such that

$$(T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} \geq \lim_{k'' \to \infty} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} - 2\varepsilon \geq \lim_{k'' \to \infty} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha, \theta, \nu_0})} - 2\varepsilon.$$
Since $\varepsilon$ is arbitrary, we get
\[(T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} \geq \lim_{k'' \to \infty} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}, \quad \forall t \in (0, \infty) \setminus E_F. \quad (3.24)\]

By (3.22) and (3.24), we get
\[(T_t F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} = \lim_{k'' \to \infty} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}, \quad \forall t \in (0, \infty) \setminus E_F, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}). \quad (3.25)\]

For $\beta > 0$ and $F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu})$, we obtain by (3.17), (3.25), and the dominated convergence theorem that
\[(G_{\beta} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} = \lim_{k'' \to \infty} (G_{\beta}^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} = \lim_{k'' \to \infty} \int_0^\infty e^{-\beta t} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} \, dt = \int_0^\infty e^{-\beta t} (T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} \, dt. \quad (3.26)\]

By (3.20), the right continuity of the function $t \to (T_t F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}$ on $[0, \infty)$, and the uniqueness of the Laplace transform, we find that
\[(T_t F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} = (T_t^* F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}, \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}), \]
which implies that
\[T_t F = T_t^* F, \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}). \quad (3.27)\]

By (3.25), (3.27), the fact that the function $t \to (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}$ is decreasing on $[0, \infty)$, and the continuity of the function $t \to (T_t F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}$ on $[0, \infty)$, we get
\[(T_t F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})} = \lim_{k'' \to \infty} (T_t^{k''} F, F)_{L^2(P_1(S); \Pi_{\alpha,\theta,\nu})}, \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}), \]
which implies that
\[T_t F = w - \lim_{k'' \to \infty} T_t^{k''} F \text{ in } L^2(P_1(S); \Pi_{\alpha,\theta,\nu}), \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}). \]

Further, we obtain by the semigroup property that
\[T_t F = \lim_{k'' \to \infty} T_t^{k''} F \text{ in } L^2(P_1(S); \Pi_{\alpha,\theta,\nu}), \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}). \]

Since the subsequence $\{k''\}$ of $\{k\}$ is arbitrary, we get
\[T_t F = \lim_{k'' \to \infty} T_t^{k''} F \text{ in } L^2(P_1(S); \Pi_{\alpha,\theta,\nu}), \quad \forall t \geq 0, F \in L^2(P_1(S); \Pi_{\alpha,\theta,\nu}). \]

\[\square\]
References

[1] M.A. Carlton (2002). A family of densities derived from the three-parameter Dirichlet process. J. Appl. Prob. 39 764-774.

[2] C. Costantini, P. De Blasi, S.N. Ethier, M. Ruggiero, and D. Spanò (2017). Wright-Fisher construction of the two-parameter Poisson-Dirichlet diffusion. To appear in Ann. Appl. Probab.

[3] S.N. Ethier (1990). The infinitely-many-neutral-alleles diffusion model with ages. Adv. Appl. Prob. 22 1-24.

[4] S.N. Ethier and T.G. Kurtz (1981). The infinitely-many-neutral-alleles diffusion model. Adv. Appl. Prob. 13 429-452.

[5] S.N. Ethier and T.G. Kurtz (1993). Fleming–Viot processes in population genetics. SIAM J. Control Optimization 31 345–386.

[6] S. Feng and W. Sun (2010). Some diffusion processes associated with two parameter Poisson-Dirichlet distribution and Dirichlet process. Probab. Theory Relat. Fields 148 501–525.

[7] S. Feng and F.Y. Wang (2007). A class of infinite-dimensional diffusion processes with connection to population genetics. J. Appl. Prob. 44 938-949.

[8] T.S. Ferguson (1973). A Bayesian analysis of some nonparametric problems. Ann. Stat. 1 209-230.

[9] W.H. Fleming and M. Viot (1979). Some measure-valued Markov processes in population genetics theory. Indiana Univ. Math. J. 28 817–843.

[10] N. Forman, S. Pal, D. Rizzolo, and M. Winkel (2017). Diffusions on a space of interval partitions with Poisson-Dirichlet stationary distributions. Preprint at https://arxiv.org/abs/1609.06706.

[11] J.C.F. Kingman (1975). Random discrete distributions. J. Roy. Statist. Soc. B. 37 1-22.

[12] Z.M. Ma and M. Röckner (1992). Introduction to the theory of (non-symmetric) Dirichlet forms. Springer-Verlag.

[13] U. Mosco (1994). Composite media and asymptotic Dirichlet forms. J. Funct. Anal. 123 368-421.

[14] Perman, M., Pitman, J. and Yor, M.: Size-biased sampling of Poisson point processes and excursions. Probab. Theory Relat. Fields, 92, (1992), 21–39.

[15] L.A. Petrov (2009). Two-parameter family of infinite-dimensional diffusions on the Kingman simplex. Funct. Anal. Appl. 43 279–296.
[16] Pitman, J. and Yor, M.: Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc.*, 3, 65, (1992), 326–356.

[17] J. Pitman and M. Yor (1997). The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* 25 855-900.

[18] M. Ruggiero (2014). Species dynamics in the two-parameter Poisson-Dirichlet diffusion model. *J. Appl. Probab.* 51 174–190.

[19] M. Ruggiero and S.G. Walker (2009). Countable representation for infinite dimensional diffusions derived from the two-parameter Poisson-Dirichlet process. *Electron. Commun. Probab.* 14 501–517.

[20] B. Schmuland (1995). Lecture Notes on Dirichlet Forms. University of Alberta. [http://www.stat.ualberta.ca/people/schmu/preprints/yonsei.pdf](http://www.stat.ualberta.ca/people/schmu/preprints/yonsei.pdf)