Non-locality of experimental qutrit pairs

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Abstract
The insight due to John Bell that the joint behavior of individually measured entangled quantum systems cannot be explained by shared information remains a mystery to this day. We describe an experiment, and its analysis, displaying non-locality of entangled qutrit pairs. The non-locality of such systems, as compared to qubit pairs, is of particular interest since it potentially opens the door for tests of bipartite non-local behavior independent of probabilistic Bell inequalities, but of deterministic nature.

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(Some figures may appear in colour only in the online journal)

1. Non-locality in theory...

The discovery of John Bell that nature was non-local and quantum physics cannot be embedded into a local realistic theory in which the outcomes of experiments are completely determined by values located where the experiment actually takes place, and no far-away variables, is profound. It probably poses more questions than it has solved: if shared classical information [4] as well as hidden communication [3] both fall short of explaining non-local correlations. What ‘mechanism’ could possibly be behind this strange effect? What can we learn from Bell’s insight—and from the series of experiments carried out in the aftermath [2, 18, 28, 30]—about the nature of space and time? What can we conclude about free choices (randomness) and, more generally, the role of information in physics and in our understanding of natural laws? In fact, Bell’s theorem is an information-theoretic statement: it characterizes what kind of joint input–output behavior...
can in principle be explained by shared (classical) information, i.e., is of the form

\[ P(ab \mid xy), \]

\[ \sum_r \rho(r) P(a \mid xr) P(b \mid yr), \]

and which cannot. The relationship to (quantum) physics is given by the fact that the joint behavior—where the inputs are given by the choice of the measurement settings and the output is the measurement outcome—of two (or more) parts of some entangled system are non-local, i.e., violate a Bell inequality. Bell inequalities are linear constraints that all local systems satisfy and that, in their totality, define the local polytope in the space of non-signaling systems, the latter being all behaviors not allowing for the transmission of information.

In this article, an experiment is described in which entangled three-dimensional systems are generated and measured, and it is analyzed whether non-local behavior can be observed. More precisely, the experiment, and its analysis, deals with maximally as well as partially entangled qutrit pairs. One of the questions of interest is whether maximal entanglement also means maximal non-locality. Hereby, the strength of non-locality is measured not only by the extent of a Bell inequality violation, but also in terms of the distance to the local polytope, as well as a novel method based on the amount of communication required for classically simulating the behavior observed experimentally.

The experimental realization of entangled qutrit pairs as opposed to qubit pairs is of interest for several reasons. One of them is that in principle, non-local correlations directly based on impossible colorings, i.e., the Kochen/Specker theorem [22], become possible [24]. This type of non-local behavior is conceptually simpler since it does not depend on probabilities, but is deterministic in nature.

2. ... And experiment

Due to the extremely weak interaction with their environment, photons exhibit strong resistance against decoherence and can, therefore, be transmitted over large distances without altering their state. Photonic quantum states are thus ideal systems to carry out tests of non-locality with entangled states. Moreover, photons entangled in their polarization, momentum, or energy-time degrees of freedom can be experimentally generated by nonlinear processes and coherently controlled and modified through linear optics. In the context of non-locality tests, the experimental study of entangled photonic states started in 1972 with the observation of non-classical correlations in the polarization of two photons emitted in an atomic cascade from a calcium atomic effusive beam [18]. This type of source was further improved and an experiment was performed in 1982 where the changes of the analyzers’ orientation and the detection on each side were separated by a spacelike interval [2]. This experiment showed a statistically significant violation of Bell’s inequality, demonstrating the non-locality of nature at its fundamental level. Since then, non-locality has been tested by numerous experiments which all confirmed the predictions of quantum mechanics under different conditions. Theories reconciliating the quantum collapse and Lorentz invariance have been tested [27]. Further, boundaries on the speed of a potential superluminal influence have been experimentally set [26]. Entanglement between photons was shown to be conserved over distances on the order of kilometers in optical fibers [28] or in free-space [29]. Ongoing research is oriented towards experiments between ground station and low-orbit satellites.
Much effort is nowadays put into the closure of possible loopholes which can distort the outcomes of Bell-test measurements. The presence of loopholes allows to explain the measurement results by a local theory even if a measured violation of a Bell inequality pretends the existence of non-classical correlations. Apart from space-time separation between the measurement devices (the communication loophole), the most studied loophole is the so-called detection loophole. This loophole can be closed using detectors with sufficiently high efficiencies. Massive particles like ions or atoms can be detected with an efficiency of almost unity [25], however, have not yet been detected in a spacelike separated configuration [20]. Photons are nowadays detected with high efficiency in Bell experiments using superconducting single photon detectors [19].

The locality condition is very difficult to be fully satisfied without any additional hypothesis on the theory to be tested. It requires that the choices of the measurement settings have to be realized without any possible local causal relation between them. The most independent choices would be that Alice and Bob freely chose the settings themselves. The same applies for the detection of the measurement results. Because the measurement problem is still not solved in quantum mechanics, the detection should be considered to occur at the instant when Alice and Bob have conscience of the outcome result. In both cases the reaction times of Alice and Bob would require to perform the experiment over distances larger than the earth’s diameter. Most of the experiments mentioned here were carried out with photon pairs in entangled qubit states. Their non-locality was revealed by violating a Bell inequality, for instance the CHSH inequality [12], given by

\[ I_2 \leq 2. \] (1)

In the following, we let \(A_x\), with \(x = 1, 2\), be measurements performed by Alice and \(B_y\), with \(y = 1, 2\), be measurements performed by Bob. The Bell parameter \(I_2\) is then expressed as

\[
I_2 = \left| P(A_1 = B_1) - P(A_1 \neq B_1) + P(A_1 = B_2) - P(A_1 \neq B_2) 
- P(A_2 = B_1) + P(A_2 \neq B_1) + P(A_2 = B_2) - P(A_2 \neq B_2) \right|, \tag{2}
\]

and the outputs \(a\) and \(b\) are restricted to \(a, b = 0, 1\). The Bell parameter itself is a sum of probabilities \(P\) which are related to the measurement outputs. The maximal violation of equation (1), together with the corresponding measurement settings, can be fully determined by means of the two-qubit density matrix and the algorithm provided in [21]. The maximal violation of the CHSH inequality is achieved for a maximally entangled state given by

\[
|\psi\rangle^{(2)} = \frac{1}{\sqrt{2}} \left( |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right). \tag{3}
\]

However, nothing restrains us from using pairs of photons entangled in more than two-dimensional spaces. While the polarization of light only carries two orthogonal states, the other modes of light, energy, and transverse momentum, can in principle encode much larger dimensions. Whereas the behavior of entangled qubits is fully characterized, their extensions to higher dimensions, denoted as qudits, are far from having been fully studied. A first approach to examine the non-classical correlations between two entangled qudits is to consider these states in the context of a new family of Bell inequalities introduced by Collins et al. (hereafter referred to as CGLMP) in [13]. Violation of the inequality has been experimentally shown for various photonic entangled states [7] and references therein.
2.1. Bell inequality for entangled qutrits

In the following, we focus on a two-qutrit state

$$\psi(\gamma) = \frac{1}{\sqrt{2 + \gamma^2}} (|0\rangle_A |0\rangle_B + \gamma |1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B)$$

with a variable degree of entanglement $\gamma$.

In order to experimentally investigate the non-local properties of the state in equation (4) we make use of the CGLMP inequality for $d = 3$ given by

$$I_1 \leq 2$$

with the corresponding Bell parameter

$$I_3 = |P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) - P(A_1 = B_1 - 1) - P(B_1 = A_2) - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1)|.$$  

For $d = 3$, we have measurement outcomes $a, b = 0, 1, 2$. At first sight, it seems to be intuitive that a maximally entangled state leads to the highest violation of a Bell inequality. However, for entangled qudits with $d > 2$, there exists theoretical evidence that non-maximally entangled qudits reach higher violations of the CGLMP inequality than maximally entangled states $[1, 33]$. In fact, it was shown that equation (5) is maximally violated for $\gamma = \gamma_{\text{max}} \approx 0.792$. To experimentally determine $I_3(\gamma)$, we exploit energy-time entangled photons to encode entangled two-qutrit states according to equation (4).

2.2. Experiment

We use an experimental setup in which energy-time entangled photons are generated by means of a spontaneous parametric down-conversion (SPDC) process induced by a...
quasi-monochromatic pump laser. It can be shown with numerical methods \[32\] that in this case, the entanglement content in the generated two-photon state is potentially very high by calculating an entropy of entanglement of \( E = (21.1 \pm 0.2) \) ebits.

The experimental arrangement shown in figure 1 is subdivided into three parts: the SPDC crystal prepares entangled two-photon states whose spectrum is discretized with a spatial light modulator (SLM) to generate entangled qudits. The ability of the SLM to individually control the amplitude and phase of selected spectral components then allows to manipulate the qudit states. Finally, the photons are detected in coincidence through sum-frequency generation (SFG) in a second nonlinear crystal.

To prepare entangled photon pairs degenerated at 1064 nm, we pump a \( L_{DC} = 11.5 \) mm long and periodically poled KTiOPO\(_4\) (PPTKP) crystal with a poling periodicity of \( G_{DC} = 9 \) \( \mu \)m. The pump laser is a quasi-monochromatic Nd:YVO\(_4\) (Verdi) laser centered at 532 nm featuring a narrow spectral full width half maximum of about 5 MHz. The collinear pump beam is focused into the middle of the PPKTP crystal (\( \Sigma_0 \)) with a power of 5 W. The down-conversion crystal is mounted in a copper block whose temperature is stabilized to \( \pm 0.1^\circ \)C. The operating temperature of the PPKTP crystal is chosen for collinear emission and, according to type-0 SPDC, all involved photons are identically polarized. This configuration leads to a spectral width of \( \Delta \lambda_{DC} \approx 105 \) nm around 1064 nm which implies a spectral mode density of \( n \approx 0.2 \) entangled pairs per mode \[16\]. Therefore, we operate below the single photon limit, and thus in the quantum regime, where the entangled pairs are temporally well separated, i.e., the mean time separation between two successive pairs is larger than the coherence time of the individual photons. The power of the SPDC photons is measured to scale linearly with the power of the pump field. The corresponding two-photon state can be derived by first-order perturbation theory and, under the assumption of a quasi-monochromatic pump field, is described by

\[
|\psi\rangle = \int_{-\infty}^{\infty} d\omega \Lambda(\omega) \hat{a}^\dagger_s(\omega) \hat{a}^\dagger_i(-\omega) |0\rangle_s |0\rangle_i,
\]

where we omit the leading order vacuum state. Signal and idler photons are created with corresponding relative frequency \( \omega \) by acting with the operators \( \hat{a}^\dagger_s(\omega) \) on the composite vacuum state \( |0\rangle_s |0\rangle_i \). The joint spectral amplitude reads

\[
\Lambda(\omega) \propto \text{sinc} \left( \frac{\Delta k_{DC}(\omega) + \frac{2\pi}{\omega c} L_{DC}}{2} \right).
\]

where the phase mismatch \( \Delta k_{DC}(\omega) \) is responsible for the efficiency of the SPDC process and includes the dispersion properties of the down-conversion crystal through its corresponding Sellmeier equations.

To manipulate their spectrum, the entangled photons are imaged from the SPDC crystal through a four-prism compressor (P1–P4) to the plane \( \Sigma_2 \). The prism compressor consists of four equilateral N-SF11 prisms arranged in minimum deviation geometry. The lens L1 images the plane \( \Sigma_0 \) to \( \Sigma_1 \) such that the spectral components are spatially dispersed in order to form a quasi-Fourier plane. Placing a SLM (Jenoptik SLM-S640d) at \( \Sigma_1 \) then allows to individually manipulate the spectral amplitude and phase of the entangled photons similar to pulse shaping techniques applied to classical femtosecond laser pulses \[31\]. The effect of the SLM on each photon is described by a complex transfer function \( M^{\text{tr}}(\omega) \) which transforms the joint spectral amplitude of equation (8) according to
\[ \hat{A}(\omega) = M(\omega)\Lambda(\omega), \] 

with

\[ M(\omega) = M'(\omega)M^*(-\omega), \] 

where \( M(\omega) \) is restricted by time-stationarity and \( |M'(\omega)| \ll 1 \). Instead of using two spatially separated single-photon detectors, coincidences of the entangled photon pairs are measured by SFG in a second PPKTP crystal with length \( L_{\text{SFG}} = 11.5 \text{ mm} \) and poling period \( G_{\text{SFG}} = 9 \mu \text{m} \). This ensures an ultrafast coincidence window with femtosecond temporal resolution needed to observe a coherent superposition of entangled photons with different energies. To take account of the detection crystals acceptance bandwidth, we define the modified joint spectral amplitude

\[ \Gamma(\omega) \propto \Lambda(\omega)\Phi(\omega), \] 

with

\[ \Phi(\omega) = \text{sinc} \left[ \frac{\Delta k_{\text{SFG}}(\omega) - \frac{2\pi}{G_{\text{SFG}}} L_{\text{SFG}}}{2} \right], \] 

and \( \Delta k_{\text{SFG}}(\omega) \) now being the phase mismatch responsible for the efficiency of the SFG process. The recombined 532 nm photons are then imaged onto the photosensitive area of a SPCM (IDQ id100–50). The efficiency of the SFG detection is such that approximately each 10^9th photon pair is up-converted in the detection crystal and we measure a coincidence rate of about 900 counts per second if no transfer function is applied on the SLM. The dark count rate of the single photon counter is about 12 Hz. In order to exclude the detection of residual infrared photons we mounted a bandpass filter in front of lens L4. The SFG signal is in general proportional to

\[ S \propto \int_{-\infty}^{\infty} d\omega \Gamma(\omega)M(\omega)^2. \] 

However, diffraction effects due to the finite aperture of the lenses and prisms lead to a point-to-spot image from \( \Sigma_0 \) to \( \Sigma_1 \). Therefore, a given frequency component is blurred over several pixels of the SLM. This effect is taken into account by a convolution of \( \Gamma(\omega) \) with a Gaussian-shaped point spread function PSF(\( \omega \)), i.e.,

\[ \Gamma(\omega) \rightarrow \Gamma_{\text{PSF}}(\omega) \propto (\Gamma \otimes \text{PSF})(\omega). \] 

We encode qudits in the frequency domain by projecting the continuous state \( |\psi\rangle \) of equation (7) into a discrete \( d^2 \)-dimensional subspace spanned by frequency-bin states \( |j\rangle|k\rangle \), with \( |j\rangle|k\rangle \equiv \int_{-\infty}^{\infty} d\omega f^{j,k}_\omega(\omega)|\hat{a}_\omega^\dagger(\omega)|0\rangle|j\rangle|k\rangle \) and \( j = 0, \ldots, d - 1 \). The frequency bins itself are defined according to

\[ f^{j,k}_\omega(\omega) = \begin{cases} \frac{1}{\sqrt{\Delta\omega_j}} & \text{for } |\omega - \omega_j| < \frac{\Delta\omega_j}{2}, \\ 0 & \text{otherwise}, \end{cases} \] 

and are represented in figure 2 together with a measured SPDC spectrum.

Furthermore, imposing the constraint \( |\omega_j - \omega_k| > (\Delta\omega_j + \Delta\omega_k)/2 \) for all \( j, k \) ensures that adjacent bins do not overlap. Under the restriction of a monochromatic pump field, the projected state is expressed as
\[ \sum \psi_j = -c_j \text{ for } j, (16) \]

with amplitudes \( c_j = \int_{-\infty}^{\infty} \langle \omega \Gamma_j \rangle f_j^\dagger(\omega)f_j^s(-\omega) \). The frequency-bin structure of equation (15) is applied on the SLM by the transfer function

\[ M_{\pm s}^j(\omega) = \sum_{j=0}^{d-1} u_j^{\pm s} f_j^{\pm s}(\omega) = \sum_{j=0}^{d-1} u_j^{\pm s} e^{i\phi_j^{\pm s}} f_j^{\pm s}(\omega), \tag{17} \]

where the amplitude \( u_j^{\pm s} \) and phase \( \phi_j^{\pm s} \) of bin \( j \) are controlled independently. Since in our experiment there is no spatial separation between idler and signal modes, we address each photon individually by assigning \( M_j^s(\omega) \) to the lower-frequency part and \( M_j^i(\omega) \) to the higher-frequency part of the spectrum. In fact, due to the PSF, we do have a small overlap between the spectral components of the idler and the signal domain such that the two spaces cannot be considered completely independent of one another. The measured signal of equation (13) is equivalent to the projection

\[ S = \langle \langle \psi \rangle^{(d)} \rangle^2 = \sum_{j=0}^{d-1} u_j^{\pm s} u_j^s c_j, \tag{18} \]

for a direct product state

\[ |\psi\rangle = \left( \sum_{j=0}^{d-1} u_j^{\pm s}|j\rangle^s \right) \left( \sum_{j=0}^{d-1} u_j^s|j\rangle^i \right), \tag{19} \]

and a decomposition of the transfer function according to equation (17). The combination of the SLM together with an SFG coincidence detection therefore realizes a projective measurement. Different quantum information protocols can thus be implemented through a judicious choice of \( |\psi\rangle \) together with equation (17). The state of equation (19) can, for instance, be chosen in the form of a tomographically complete set to perform quantum state reconstruction of maximally entangled qudits up to \( d = 4 \) as demonstrated in [7]. Here, equation (19) serves to perform Bell measurements for maximally and non-maximally
entangled qutrits. If we identify $i \leftrightarrow \alpha$ and $s \leftrightarrow B$, the Bell parameter of equation (6) is described by a combination of projective measurements onto the state of equation (19) expressed as

$$|\psi\rangle_A = |\psi\rangle_A^A |\psi\rangle_A^B,$$

and defined by

$$|\psi\rangle_A^1 = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} \exp\left(i \frac{2\pi}{3} j (a + \alpha_s) - b + \beta_s\right) |j\rangle_A^1,$$

$$|\psi\rangle_A^2 = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} \exp\left(i \frac{2\pi}{3} j (a + \alpha_s) - b + \beta_s\right) |j\rangle_A^2,$$

with $a, b = 0, 1, 2$ for a specific choice of detection settings $\alpha_s = 0, \alpha_s = 1/2, \beta_s = 1/4$, and $\beta_s = -1/4$ according to [13]. Note, that these settings are only optimal in the case of $\gamma = 1$ and $\gamma = \gamma_{\text{max}} [1]$. Allowing for the state of equation (4), the individual joint probabilities become a function of $\gamma$. In accordance with equation (18), a measured coincidence signal is then given by

$$F_s \left(A_s = a, B_s = b \right) \propto |\langle \chi_{\alpha_s, \beta_s} |\psi(\gamma)\rangle^{(5)}|^2$$

with the projection states of equations (20) and (21). Starting from a maximally entangled state through Procrustean filtering [6], the reduced entanglement, i.e., $\gamma < 1$, is obtained by decreasing the transmission amplitudes $|\eta_{11}|^2$ and $|\eta_{11}|^2$ of the bins associated with $|1\rangle_A |1\rangle_B$ using the SLM. The experimental data $I_{\text{exp}} \left(\gamma\right)$ (red diamonds) are depicted in figure 3 together with the theoretical result for $I_{\text{cl}} \left(\gamma\right)$ (solid red line). The latter is scaled to the measured data using a symmetric noise model $\tilde{\rho}^{(d)} = \lambda_d |\psi^{(d)} (\gamma)| + (1 - \lambda_d) \mathbb{1}_{d^2} / d^2$, where the mixing parameter $\lambda_d$ quantifies the deviations from a pure state and $\mathbb{1}_{d^2}$ denotes the $d^2$-dimensional identity operator. Under the given measurement settings together with white noise and possible misalignments in the experimental setup figure (3) demonstrates a violation of equation (5) for $\gamma \geq 0.5$. The measured count rates of the single projections vary between 0 and 8 counts per second while the measurement times are chosen between 300 s and 1200 s.
3. Measures of non-locality

3.1. Systems and correlations

The goal of the experiment discussed in this article is to understand and quantify correlations that occur when two parts of a physical system in a common state are measured. More precisely, this type of correlation displays itself in the behavior of a system, i.e., a joint input–output behavior.

**Definition 1.** A two-party input–output behavior or system is a conditional probability distribution $P(ab|xy)$, where $x$ and $y$ are the respective inputs of the two parties and $a$ and $b$ are their outputs.

A two-partite quantum state is a natural way to obtain a system: given a set of possible alternative measurements ($x$ and $y$) to be carried out by the two parties, their outcomes ($a$ and $b$) correspond to the outputs of the system. We establish a general framework for studying such systems, and for quantifying their non-locality, independently of whether they come from a quantum state or not.

Classically speaking, the two players not allowed to communicate can, for obtaining a given joint behavior, apply local strategies. Generally, such a strategy can employ local randomness and even shared (classical) information.

**Definition 2.** Two players’ local strategies are ways of determining their respective outputs as a function of their inputs. A local strategy (for the first player) is of the form $a = f(x, L, R)$, where $f$ is a function, $L$ is a classical random variable and $R$ is a shared (i.e., also known to the other player) random variable. As a special case, a strategy is local deterministic if it is of the form $a = f(x)$ for some fixed function $f$. A system that cannot be simulated by any pair of local strategies is called non-local.

Non-local systems are, in classical terms, only explainable by communication. It is important to note, however, that this does not mean that any non-local system allows for such communication. Actually, our focus is precisely on the systems which are non-local yet non-signaling at the same time.

**Definition 3.** A two-partite system is local if it can be simulated by two players using a pair of local strategies. It is non-signaling if the output of a party, given its input, is independent of the input of the other party: we have $P(a|xy) = P(a|x)$, and vice versa.

It is the goal of the rest of this section to identify criteria for testing whether a (non-signaling) system is local or not and, if not, to give a measure for the strength or amount of non-locality it displays. Whereas the violation of a fixed Bell inequality is a sufficient criterion for non-locality, it is not necessary. In the same sense, the extent by which a given inequality is violated does not directly measure non-locality. We describe two criteria and measures: The first corresponds, in a sense, to taking all Bell inequalities into account, whereas the second is of different nature and based on the price, measured in terms of classical communication, one has to pay for establishing a correlation.

3.2. Distance to the polytope of local correlations

Within the space of non-signaling two-party systems, the local systems form a subset or, more precisely, a convex polytope, somewhat abusively called the local polytope. Whereas Bell
inequalities characterize this polytope in terms of its facets, we can also do the same through its extremal points, which correspond to the local deterministic strategies; every local strategy is a convex combination of local deterministic strategies. We use this description as a locality test as well as a quantification of the non-local content of a given, e.g., experimentally observed, system.

For a system \( P(ab|xy) \), we denote by \( |a|, |b|, |x|, \) and \( |y| \) the sizes of the ranges of the corresponding random variables. A pair of strategies can be represented in the \((|a|+|b|+|x|+|y|)\)-dimensional real space. The number of pairs of local deterministic strategies is \(|a||b||x||y|\). Now, any local (probabilistic) strategy is a (non-trivial) convex combination of local deterministic strategies. We check whether an observed behavior lies inside this polytope, and if not, we calculate the distance to this polytope in the \(L^1\) norm, using linear programming \([15, 17]\), by minimizing the difference between the given behavior and a local approximation of it. In other words, the distance to the local polytope is defined as the minimum of the distance between the given behavior and the closest point in the local polytope. The larger this distance, the stronger a behavior can be considered non-local. More specifically, the fact that a behavior has non-zero distance to the polytope can indicate that it violates a Bell inequality or that it is signaling (or both).

### 3.3. The amount of communication required for establishing correlations

A non-locality measure alternative to Bell-inequality or polytope-distance based measures uses the fact that the local correlations are exactly the ones that can, classically speaking, be explained by shared information and require no communication. Hence, a non-locality measure is given by the minimal amount of classical communication required to simulate the correlations. In \([23]\), this measure has been called non-local capacity. There, it was proven that the non-local capacity is the minimum of a convex functional over a suitable space of probability distributions, so that its computation is a convex-optimization problem. We review that method, which we apply for calculating the non-local capacity from the experimental data.

#### 3.3.1. Communication cost and non-local capacity

A general classical simulation of correlations \( P(ab|xy) \) employing a one-way communication between the two parties is as follows. A party, say Alice, chooses the input \( x \) corresponding to the measurement she wants to simulate. She generates a variable \( k \) and the measurement outcome \( a \) according to a conditional probability distribution \( P(ak|x,r) \) depending on the input \( x \) and a random variable \( R \) shared with the other party, say Bob, and generated with probability distribution \( \rho(R) \). Then, she sends \( k \) to Bob. Finally, Bob chooses the measurement he wants to simulate, labelled by the index \( y \), and generates an outcome \( B \) according to a conditional probability distribution \( \rho(b|ykr) \). The protocol exactly simulates the correlations \( P(ab|1xy) \) if

\[
\sum_k \int \text{d}r P(b|ykr)P(ak|x,r)\rho(r) = P(ab|1xy).
\]

There are different definitions of communication cost of a simulation. We could define the communication cost as the number of required bits in the worst case, or as the average number of bits. As done in \([23]\), we employ the entropic definition and we define the communication cost, say \( C \), as the maximum, over the space of distributions \( P(a) \), of the conditional Shannon entropy \( H(K|R) \equiv -\int \text{d}\rho(r)\sum_k P(k|r) \log_2 P(k|r) \). That is
We define the non-local capacity, denoted by $C_{nl}$, of the correlations $P(ab|xy)$ as the minimal amount of communication $C$ required for an exact simulation of them. More generally, we can perform a parallel simulation of $N$ distinct pairs of entangled systems. We define the asymptotic communication cost, denoted by $C_{asym}$, as the communication cost of the simulation divided by $N$, in the limit $N \to \infty$ (see [23] for a more detailed definition). The asymptotic non-local capacity, denoted by $C_{nl}^{asym}$, is defined as the minimum of $C_{asym}$ over the class of parallel simulations.

3.3.2. A convex-optimization problem. In the following, we assume that Bob can choose a measurement among a finite set of possibilities. The index $y$ can take values from 1 to $M$. In [23], it was shown that the asymptotic non-local capacity is the minimum of a convex functional over a suitable space $\mathcal{V}$ of probability distributions, defined as follows.

**Definition 4.** Given a non-signaling system $P(ab|xy)$, the set $\mathcal{V}$ contains any conditional probability $\rho(ab|x)$ over $a$ and the sequence $b = (b_1, \ldots, b_M)$ whose marginal distribution of $a$ and the $m$-th variable is the distribution $P(ab|x, y = m)$. In other words, $\mathcal{V}$ contains any $\rho(ab|x)$ satisfying the constraints

$$\sum_{b, h_a = b} \rho(ab|x) = P(ab|x, y = m),$$

where

$$\sum_{b, h_a = b} \rightarrow \sum_{b_1, \ldots, h_a = b_1, \ldots, b_M},$$

is the summation over every index in $b$ but the $m$-th one, which is set equal to $b$.

In [23], we proved that

$$C_{nl}^{asym} = \min_{\rho(ab|x) \in \mathcal{V}} C(x \to b),$$

where

$$C(x \to b) \equiv \max_{\rho(x)} I(b; x),$$

is the capacity of the channel $\rho(b|x)$. Let us recall that the capacity of a channel $x \to b$ is the maximum of the mutual information $I(b; x)$ between the input and the output over the space of input probability distributions $\rho(x)$ [14]. As the mutual information is convex and the maximum over a set of convex functions is still convex [8], the asymptotic communication complexity is the minimum of a convex function over the space $\mathcal{V}$. Furthermore, the equality constraints defining the set $\mathcal{V}$ are linear. This implies that the minimization problem is convex and can be numerically solved with standard methods [8].

4. Analysis of experimental data

4.1. The raw data display signaling

We analyze the experimentally obtained two-partite system according to equation (22). A first remarkable result of our analysis of this system is that it happens to be, actually, not only non-local but even signaling. From the point of view of how the qutrits were experimentally
realized (not spatially separated), this is not overly surprising and, in particular, not in violation of relativity. In order to be able to apply the non-locality quantification, designed for non-signaling systems, to the data, we free them of their signaling part, replacing them by the closest respective non-signaling systems. Figure 4 demonstrates that for $\gamma < 0.2$, the resulting non-signaling systems end up being local, whereas for $\gamma \geq 0.5$, there is no significant difference between the raw and the corrected data.

4.2. Removal of the signaling part

Removing the signaling part in the described way can be seen as some kind of error correction; after all, we know that quantum theory is, despite the phenomena of entanglement and non-locality, non-signaling. The computation of the non-signaling part is performed similarly to the computation of the distance to the polytope, where in this case, the non-signaling polytope instead of the local polytope is used. Our method of getting rid of experimental errors is similar to a procedure discussed in the literature [11] that proposes a maximum-likelihood estimation from the set of all quantum states. We adapt the method for non-signaling systems instead of physical states. Note that while the removal of the signaling part can greatly influence the measures for non-locality used here, it has almost no influence on Bell-inequality violations (figure 3).

4.3. Comparison between the two methods

First, we observe that the two non-locality measures behave very similarly for the investigated measurement data (figure 5). Second, we can conclude from our analysis that the non-monotonicity of the measured non-locality as compared to the entanglement—a phenomenon already reported previously in the context of non-maximally entangled qubit pairs [10]—is not merely an effect that is respective to a single (strange) Bell inequality, but that persists also with respect to the non-locality measures.
In this article, dedicated to the celebration of the 50th anniversary of John Stewart Bell’s famous and seminal theorem, we have described an experiment for measuring the non-locality of entangled qutrit pairs. We have analyzed the resulting experimental data, and in particular the strength of their non-locality, with two non-standard methods, namely the distance to the local polytope (Distlp), whereas the green squares with the green scale on the right show the asymptotic non-local capacity ($C_{nl}^{asym}$). The corresponding lines (dot-dashed blue, dashed green) are guides to the eyes. Theoretical values for Distlp and $C_{nl}^{asym}$ are indicated with solid lines (upper blue, lower green).

5. Concluding remarks

In this article, dedicated to the celebration of the 50th anniversary of John Stewart Bell’s famous and seminal theorem, we have described an experiment for measuring the non-locality of entangled qutrit pairs. We have analyzed the resulting experimental data, and in particular the strength of their non-locality, with two non-standard methods, namely the distance to the local polytope (in the space of non-signaling behaviors) as well as the communication cost for classically simulating the correlations.

The phenomenon of non-locality, John Bell’s profound discovery, continues to put into question the way we traditionally view space and time. When one tries to understand —just as Bell himself did in a number of articles [5]—non-locality in the context of different interpretations of quantum theory, the conclusion is always the same: it does not fit. In collapse theories, how can spontaneously generated information be identical in different spatial positions? In modern deterministic theories such as the ‘church of the larger Hilbert space’, parallel universes or ‘parallel lives’ [9], no mechanism that could explain the correlations has been described. It appears that the interpretation which passes the ‘Bell test’ and is suitable for embedding, in a non-artificial way, non-local correlations is yet to be found. It may have the property that space and time, in particular spatial distance and spacelike separation, do not exist prior to, but only emerge together with the (correlated) classical information.

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