Induction and restriction functors for cellular categories

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Abstract

Cellular categories are a generalization of cellular algebras, which include a number of important categories such as (affine)Temperley-Lieb categories, Brauer diagram categories, partition categories, the categories of invariant tensors for certain quantised enveloping algebras and their highest weight representations, Hecke categories and so on. The common feather is that, for most of the examples, the endomorphism algebras of the categories form a tower of algebras. In this paper, we give an axiomatic framework for the cellular categories related to the quasi-hereditary tower and then study the representations in terms of induction and restriction. In particular, a criteria for the semisimplicity of cellular categories is given by using the cohomology groups of cell modules. Moreover, we investigate the algebraic structures on Grothendieck groups of cellular categories and provide a diagrammatic approach to compute the multiplication in the Grothendieck groups of Temperley-Lieb categories.

1 Introduction

Cellular categories were defined by Westbury in [46] as a generalization of cellular algebras, which were first introduced by Graham and Lehrer in [20]. In a cellular category, the hom-space of any two objects is spanned by a distinguished basis, so-called cellular basis. Therefore, an endomorphism algebra in a cellular category is cellular. In particular, if we can regard an algebra as a category with one object, then a cellular algebra is indeed a cellular category. It was shown that many important classes of algebras arising in representation theory, invariant theory, knot theory, subfactors and statistical mechanics are cellular (see e.g. [4, 17, 20, 24, 40, 49, 50]), and most of their categorical analogues are also cellular, such as Temperley-Lieb categories [48], Brauer diagram categories [30], partition categories [23, 33], the categories of invariant tensors for certain quantised enveloping algebras and their highest weight representations [46, 47], the categories of Soergel bimodules [16] and other more general Hecke categories (a strictly object-adapted cellular category due

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to [15]). As is known, if an algebra admits a cellular structure, one will have a prac-
ticable way to describe the representations and homological properties of the algebra
[20, 8]. In this article, we shall investigate the properties of cellular categories.

Lots of important examples of cellular algebras actually occur in towers $A_0 \subset A_1 \subset A_2 \subset \cdots$ with an intense interplay between each other in terms of induction and restriction. These include Temperley-Lieb algebras [31] and their cyclotomic analogues [33], Brauer algebras [6, 7, 38, 39], blob algebras [36, 37], partition algebras [5, 21, 33, 34] and so on. The tower method was first developed by Jones [22] and Wenzl [45] for semi-simple case. Further, for the general case, Cox, Martin, Parker and Xi established a framework of towers of quasi-hereditary algebras by combining the ideas from the tower formalism in [18] with the notion of recollement in [9]. Then, influenced by the work of König and Xi [25] as well as by the work of Cox et al [12], the analogous for cellularity were given by Goodman and Graber [19].

There are in fact a number of cellular categories with the endomorphism algebras forming a tower. Further, note that the hom-space of two objects in a cellular category is a natural nexus for their endomorphism algebras.

Those motivate us to introduce a class of cellular categories by an axiomatic manner, so-called cellular tower category, or CTC (see Section 3.2), in which the endomorphism algebras are required to be quasi-hereditary, and herein the induction and restriction behave well in the tower. In this setting, the homological aspects of representation theory are computed efficiently by using induction and restriction.

Precisely, let $K$ be a field, and $\mathcal{A}$ be a cellular tower category (see Definition 3.6) with its object set the set of natural numbers $\mathbb{N}$. By definition, the endomorphism algebra $A_n$ of an object $n$ is a cellular algebra with an index set $\Lambda_n$. Suppose that $\Delta_n(\lambda)$ denotes the cell module of $A_n$ corresponding to the index $\lambda \in \Lambda_n$.

Our main result can be stated as the following theorem.

**Theorem 1.1** Let $\mathcal{A}$ be a cellular tower category. Suppose that for all $n \in \mathbb{N}$ and pairs of indices $\lambda \in \Lambda_n \setminus \Lambda_{n-2}$ and $\mu \in \Lambda_n \setminus \Lambda_{n-4}$ we have

$$\text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0,$$

Then each of the endomorphism algebras $A_n$ in $\mathcal{A}$ is semi-simple.

In [8], Cao provided a criteria of semi-simplicity for a cellular algebra by checking the first cohomology groups of cell modules for all indices. Rather, Theorem 1.1 tells us that, the verification needs only some of the indices in a cellular tower category.

The Grothendieck groups of the categories of finitely generated modules and finitely generated projective modules over a tower of algebras can be endowed with algebra and coalgebras structures. Many cases of interest, such as symmetric group algebras, Hecke algebras and other deformations, give rise to a dual pair of Hopf algebras, which are realization of some classical algebras in the theory of symmetric function [51, 28, 2, 44]. The common feature is that the examples admit Mackey’s formula, which implies that the comultiplication is an algebra homomorphism, that is, making both Grothendieck groups into bialgebra.

In [3], Bergeron and Li gave an analogue of Mackey’s formula by an axiom and introduced a general notation of a tower of algebras, which ensures that the
Grothendieck groups of a tower of algebras can be a pair of graded dual Hopf algebras. For a cellular category with a quasi-hereditary tower, we shall see that the analogue of Mackey’s formula never holds. The Grothendieck groups, however, have algebra and coalgebra structures under certain conditions. Further, we shall study the algebraic structures on the Grothendieck groups of Temperley-Lieb categories.

This paper is organized as follows. In Section 2, we shall recall some notations and basic facts. In Section 3, we first study Morita contexts of endomorphism algebras in a cellular category in 3.1 and then we introduce the cellular tower categories and prove Theorem 1.1 in 3.2. Section 4 studies the algebraic structures on Grothendieck groups of cellular categories. In particular, we give a method to compute the multiplication in the Grothendieck group of a Temperley-Lieb category in 4.2.

2 Preliminaries

In this section, we shall recall some basic definitions and facts needed in our later proofs.

Throughout the paper, all algebras are finite-dimensional algebras over a fixed field $K$. All modules are finitely generated unitary left modules. For an algebra $A$, the category of $A$-modules is denoted by $A\text{-mod}$. Let $A$ always be a small $K$-linear category with finite dimensional hom-spaces, that is, the class of objects is a set and every hom-set is a finite dimensional $K$-vector space and the composition map of morphisms is bilinear. For an object $n$ in $A$, the endomorphism algebra $\text{End}(n)$ of $n$ is simply denoted by $A_n$.

2.1 Cellular categories

We now recall the definition of cellular categories, which are generalised by Westbury [46] from the cellular algebras.

**Definition 2.1** [46] Let $A$ be a $K$-linear category with an anti-involution $*$ (that is, $*$ is a dual functor on $A$ with $(-)^{**} = \text{id}_A$). Then cell datum for $A$ consists of a partially ordered set $\Lambda$, a finite set $M(m, \lambda)$ for each $\lambda \in \Lambda$ and each object $n$ of $A$, and for $\lambda \in \Lambda$ and $m, n$ any two objects $A$ we have an inclusion

$$C : M(m, \lambda) \times M(n, \lambda) \to \text{Hom}_A(m, n)$$

$$C : (S, T) \mapsto C_{S,T}^\lambda.$$

The conditions for this datum are required to:

(C1) For all objects $m, n$ in $A$, the image of the map

$$C : \prod_{\lambda \in \Lambda} M(m, \lambda) \times M(n, \lambda) \to \text{Hom}_A(m, n)$$

is a basis for $\text{Hom}_A(m, n)$ as a $K$-space.

(C2) For all objects $m, n$, all $\lambda \in \Lambda$ and $S \in M(m, \lambda)$, $T \in M(n, \lambda)$ we have

$$(C_{S,T}^\lambda)^* = C_{T,S}^\lambda.$$
(C3) For all objects $p, m, n$, all $\lambda \in \Lambda$ and all $a \in \text{Hom}_A(p, m)$, $S \in M(m, \lambda)$, $T \in M(n, \lambda)$ we have
\[
aC^\lambda_{S, T} = \sum_{S' \in M(p, \lambda)} r_a(S, S')C^\lambda_{S', T} \mod A(< \lambda),
\]
where $r_a(S', S) \in K$ is independent of $T$ and $A(< \lambda)$ is the $K$-span of
\[\{C^\mu_{S, T} \mid \mu < \lambda; \ S \in M(p, \mu), T \in M(n, \mu)\}.\]

Remark 2.1 If we regard any $K$-algebra as a $K$-linear category with one object, then this is a generalisation of the definition of a cellular algebra. On the other hand, suppose $A$ is a cellular category. For any object $n$, let $\Lambda_n = \{ \lambda \in \Lambda \mid M(n, \lambda) \neq \emptyset \}$ and $M_n := \bigcup_{\lambda \in \Lambda_n} M(n, \lambda)$, then the endomorphism algebra $A_n$ of $n$ is a cellular algebra with cell datum $(\Lambda_n, M_n, C, \ast)$, where $C$ and $\ast$ are restrictions on $A_n$. The basis $\{C^\lambda_{S, T} \mid S, T \in M_n, \lambda \in \Lambda_n\}$ is called a cellular basis for $A_n$.

In [20], Graham and Lehrer introduced the definition of cell modules of a cellular algebra by using the cellular basis. For cellular categories, we can define cell modules of the endomorphism algebra of an object in the same way.

Definition 2.2 Let $\mathcal{A}$ be a cellular category. For each object $n$, let $\Lambda_n$ be the index set of the cellular algebra $A_n$ (see Remark 2.1 above). For each $\lambda \in \Lambda_n$ define the left $A_n$-module $\Delta_n(\lambda)$ as follows: $\Delta_n(\lambda)$ is a $K$-space with basis $C^{(n, \lambda)} = \{C_X^{(n, \lambda)} \mid X \in M(n, \lambda)\}$ and $A_n$-action defined by
\[
aC^{(n, \lambda)}_X = \sum_{Y \in M(n, \lambda)} r_a(X, Y)C^{(n, \lambda)}_Y, \quad (a \in A_n, \ X \in M(n, \lambda))
\]
where $r_a(X, Y)$ is the element of $K$ defined in (C3). $\Delta_n(\lambda)$ is called the cell module of $A_n$ corresponding to $\lambda$.

Applying ‘$\ast$’ on (C3) in Definition 2.1 we obtain
\[(C3') \quad C^\lambda_{T, S} a^* = \sum_{S' \in M(p, \lambda)} r_a(S, S')C^\lambda_{T, S'} \mod A(< \lambda),\]

where $r_a(S', S) \in K$ is independent of $T$.

The following lemma is a direct generalization of [20] Lemma 1.7 from cellular algebras to cellular categories.

Lemma 2.3 Let $\mathcal{A}$ be a cellular category, $m, n$ and $q$ be objects in $\mathcal{A}$, and let $\lambda \in \Lambda$. Then for any elements $U \in M(m, \lambda)$, $T, X \in M(n, \lambda)$ and $Y \in M(p, \lambda)$, we have
\[
C^\lambda_{U, T}C^\lambda_{X, Y} = \phi_{(n, \lambda)}(T, X)C^\lambda_{U, Y} \mod A(< \lambda),
\]
where $\phi_{(n, \lambda)}$ is a map from $M(n, \lambda) \times M(n, \lambda)$ to $K$. 

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Proof. By (C3), we have
\[ C_{U,T}^\lambda C_{X,Y}^\lambda \equiv \sum_{Z \in \mathcal{M}(m,\lambda)} r_{C_{U,T}}^\lambda (X,Z)C_{Z,Y}^\lambda \mod \mathcal{A}(\lambda), \]
and by (C3'), it follows
\[ C_{U,T}^\lambda C_{X,Y}^\lambda \equiv \sum_{V \in \mathcal{M}(p,\lambda)} r_{C_{V,X}}^\lambda (V,T)C_{U,V}^\lambda \mod \mathcal{A}(\lambda). \]

Comparing the previous equations follows
\[ r_{C_{U,T}}^\lambda (X,U)C_{U,Y}^\lambda = r_{C_{Y,X}}^\lambda (Y,T)C_{U,Y}^\lambda. \]
Because \( r_{C_{U,T}}^\lambda (X,U) \) is indep of \( Y \) by (C3) and \( r_{C_{Y,X}}^\lambda (Y,T) \) is indep of \( U \) by (C3'), we may write
\[ \phi_{(n,\lambda)}(T,X) := r_{C_{U,T}}^\lambda (X,U) = r_{C_{Y,X}}^\lambda (Y,T). \]

Hence
\[ C_{U,T}^\lambda C_{X,Y}^\lambda \equiv \phi_{(n,\lambda)}(T,X)C_{U,Y}^\lambda \mod \mathcal{A}(\lambda). \]

Thus we can define a bilinear form \( \phi_{(n,\lambda)} : \Delta_n(\lambda) \times \Delta_n(\lambda) \to K \) by
\[ \phi_{(n,\lambda)}(C_U^{(n,\lambda)}, C_V^{(n,\lambda)}) = \phi_{(n,\lambda)}(U,V), \]
where \( C_U^{(n,\lambda)}, C_V^{(n,\lambda)} \in C^{(n,\lambda)} \) with \( U, V \in \mathcal{M}(n,\lambda) \), extended \( \phi_{(n,\lambda)} \) bilinearly.

The following lemma collects some known facts for the bilinear form \( \phi_{(n,\lambda)} \).

**Lemma 2.4** [20] *Prop 2.4* Keep the notation above. Then:
1. \( \phi_{(n,\lambda)} \) is symmetric, that is, \( x, y \in \Delta_n(\lambda), \phi_{(n,\lambda)}(x,y) = \phi_{(n,\lambda)}(y,x); \)
2. For \( x, y \in \Delta_n(\lambda) \) and \( a \in A_n \), we have \( \phi_{(n,\lambda)}(ax,y) = \phi_{(n,\lambda)}(x,ay); \)
3. For \( C_{S,T}^\lambda \in A_n \) and \( C_{U}^{(n,\lambda)} \in C^{(n,\lambda)} \), we have \( C_{S,T}^\lambda \cdot C_{U}^{(n,\lambda)} = \phi_{(n,\lambda)}(T,U)C_{S}^{(n,\lambda)}. \)

Define
\[ \text{rad}_n(\lambda) := \{ x \in \Delta_n(\lambda) \mid \phi_{(n,\lambda)}(x,y) = 0 \text{ for all } y \in \Delta_n(\lambda) \}. \]
If \( \phi_{(n,\lambda)} \neq 0 \), then \( \text{rad}_n(\lambda) \) is the radical of the \( A_n \)-module \( \Delta_n(\lambda) \).

Let \( \Lambda^0_n := \{ \lambda \in \Lambda_n \mid \phi_{(n,\lambda)} \neq 0 \} \). The following result shows that this set parameterizes the simple modules, which was proved by Graham and Lehrer for cellular algebras.

**Theorem 2.5** [27] *Prop 3.4* Let \( A \) be a cellular \( K \)-algebra with the cell datum \((\Lambda, M, C, \ast)\). Suppose \( \Delta(\lambda) \) and \( \phi_{\lambda} \) are the cell module and the bilinear form, respectively, corresponding to \( \lambda \in \Lambda \). Let \( \Lambda^0 := \{ \lambda \in \Lambda \mid \phi_{\lambda} \neq 0 \} \). Then the set \( \{ L(\lambda) := \Delta(\lambda)/\text{rad}(\lambda) \mid \phi_{\lambda} \neq 0 \} \) is a complete set of non-isomorphic absolutely simple \( A \)-module.
The following theorem says that the issue of semi-simplicity reduces to the computation of the discriminants of bilinear forms associated to cell modules.

**Theorem 2.6 [20, Prop 3.8]** Let $A$ be a cellular $K$-algebra as above. Then the following are equivalent:

1. The algebra $A$ is semi-simple;
2. All cell modules are simple and pairwise non-isomorphic;
3. The bilinear form $\phi_\lambda$ is non-degenerate (that is, $\text{rad}(\lambda) = 0$) for each $\lambda \in \Lambda$.

In [26, 27], Konig and Xi investigated the relationships between cellular algebras and quasi-hereditary algebras in terms of comparing the so-called cell chains with hereditary chains (for quasi-hereditary algebras we refer to [9]), they gave some criterions for a cellular algebra to be quasi-hereditary.

The following is equivalent to [26, Theorem 3.1].

**Theorem 2.7** Let $A$ be a cellular algebra with the cell datum $(\Lambda, M, C, *)$, and $\Lambda^0$ be as above. Then $A$ is quasi-heredity if and only if $\Lambda = \Lambda^0$.

This lemma says that a cellular algebra is quasi-heredity if and only if the poset $\Lambda$ coincides with $\Lambda^0$, that is, $\phi_\lambda \neq 0$ for all $\lambda \in \Lambda$. Thus in the case the set $\Lambda$ parameterizes the simple modules.

For our purpose in this paper, we need that the endomorphism algebras of a cellular category are quasi-hereditary, so we define:

**Definition 2.8** Let $\mathcal{A}$ be a cellular category. Suppose that

1. the object set of $\mathcal{A}$ is the set of natural numbers $\mathbb{N}$,
2. for all $m, n \in \mathbb{N}$ satisfying $m \leq n$, if $\text{Hom}_\mathcal{A}(m, n) \neq 0$ (equivalently, $\text{Hom}_\mathcal{A}(n, m) \neq 0$ by anti-involution $*$), then $\Lambda_m$ is a saturated ordered subset of $\Lambda_n$, that is, $\Lambda_m \subseteq \Lambda_n$ preserving ordering, and if $\lambda < \mu$ with $\mu \in \Lambda_m$ and $\lambda \in \Lambda_n$, then $\lambda \in \Lambda_m$.
3. each endomorphism algebra $A_n$ is quasi-hereditary.

Then $\mathcal{A}$ is called a hereditary cellular category.

**Remark 2.2** (A) In Definition 2.8 condition (2) says that index sets $\Lambda_n$ preserve the ordering of natural numbers.

(B) Let $\mathcal{A}$ be a hereditary cellular category with cell datum $(\Lambda, M, C, *)$. For all $m, n \in \mathbb{N}$, denote by $\Lambda_{(m,n)} := \{ \lambda \in \Lambda \mid M(m, \lambda) \neq \emptyset \text{ and } M(n, \lambda) \neq \emptyset \}$, and thus $\Lambda_{(n,n)}$ is just $\Lambda_n$.

By condition (2), if $\text{Hom}_\mathcal{A}(m, n) \neq 0$ with $m \leq n$ and $M(m, \lambda) \neq \emptyset$, then $\lambda \in \Lambda_m \subseteq \Lambda_n$. Therefore this implies $M(n, \lambda) \neq \emptyset$ and $\Lambda_{(m,n)} = \Lambda_m$.

**3 Induction and restriction functors for cellular categories**

Let $\mathcal{A}$ be a $K$-linear category, and let $m, n$ be objects in $\mathcal{A}$. As is known, the hom-space $\text{Hom}(m, n)$ is a natural left $A_m$-right $A_n$-bimodule, and hence we can consider
the functors $\text{Hom}_{A}(m, n) \otimes_{A_n} -$ and $- \otimes_{A_m}$ $\text{Hom}_{A}(m, n)$. In this section, we first study those functors in Section 3.1.

In Section 3.2, We first give the definition of cellular tower category by an axiomatic manner and then give a proof of our main result Theorem 1.1.

3.1 Morita contexts

Let $A$ be a hereditary cellular category with cell datum $(\Lambda, M, C, \ast)$. For each $m, n \in \mathbb{N}$, $\text{Hom}_{A}(m, n) \otimes_{A_n} -$ is a functor from category $A_{n}\text{-mod}$ to category $A_{m}\text{-mod}$. For each $\lambda \in \Lambda_{(m,n)} = \{ \lambda \in \Lambda \mid M(m, \lambda) \neq \emptyset \text{ and } M(n, \lambda) \neq \emptyset \}$, let $\Delta_{m}(\lambda)$ and $\Delta_{n}(\lambda)$ be the cell modules of $A_{m}$ and $A_{n}$ corresponding to $\lambda$, respectively.

It is easy to see that the following map is an $A_{m}$-module homomorphism:

$$\alpha : \text{Hom}_{A}(m, n) \otimes_{A_n} \Delta_{n}(\lambda) \rightarrow \Delta_{m}(\lambda)$$

$$a \otimes C_{X}^{(n, \lambda)} \mapsto \sum_{Z \in M(m, \lambda)} r_{a}(Z, X) C_{Z}^{(m, \lambda)}$$

where $a \in \text{Hom}_{A}(m, n), C_{X}^{(n, \lambda)} \in C^{(n, \lambda)}, r_{a}(Z, X)$ is given by definition 2.1(C3).

The following lemma show that it is further an $A_{m}$-module isomorphism.

\textbf{Lemma 3.1} Let $A$ be a hereditary cellular category. For each $m, n \in \mathbb{N}$ and $\lambda \in \Lambda_{(m,n)}$, we have

$$\text{Hom}_{A}(m, n) \otimes_{A_n} \Delta_{n}(\lambda) \simeq \Delta_{m}(\lambda),$$

as an $A_{m}$-isomorphism.

\textbf{Proof.} Because $A_{n}$ is quasi-hereditary, it follows $\phi_{(n, \lambda)} \neq 0$ by Theorem 2.7, hence there exist $U_{0}, T_{0} \in M(n, \lambda)$ such that $\phi_{(n, \lambda)}(U_{0}, T_{0}) \neq 0$. We then fix $U_{0}, T_{0}$.

For any $C_{Y}^{(m, \lambda)} \in C^{(m, \lambda)}$ with $Y \in M(m, \lambda)$, we have $\frac{1}{\phi_{(n, \lambda)}(U_{0}, T_{0})} \alpha(C_{Y,U_{0}}^{\lambda} \otimes C_{T_{0}}^{(n, \lambda)}) = C_{Y}^{(m, \lambda)}$. It follows that $\alpha$ is surjective and

$$\dim_{K}(\text{Hom}_{A}(m, n) \otimes_{A_n} \Delta_{n}(\lambda)) \geq \dim_{K}(\Delta_{m}(\lambda)) = \# M(m, \lambda).$$

To prove previous inequality is actually an equality, fixed $U_{0}, T_{0}$ as above, it is sufficient to show that $\{C_{S,U_{0}}^{\lambda} \otimes C_{T_{0}}^{(n, \lambda)} \mid S \in M(m, \lambda)\}$ is a spanning set of the $K$-space $\text{Hom}_{A}(m, n) \otimes_{A_n} \Delta_{n}(\lambda)$. Thus, for any $a \in \text{Hom}_{A}(m, n)$ and $C_{X}^{(n, \lambda)} \in C^{(n, \lambda)}$, we have

$$a \otimes C_{X}^{(n, \lambda)} = \frac{1}{\phi_{(n, \lambda)}(U_{0}, T_{0})} a \otimes C_{X,U_{0}}^{\lambda} \cdot C_{T_{0}}^{(n, \lambda)} = \frac{1}{\phi_{(n, \lambda)}(U_{0}, T_{0})} a C_{X,U_{0}}^{\lambda} \otimes C_{T_{0}}^{(n, \lambda)} = \frac{1}{\phi_{(n, \lambda)}(U_{0}, T_{0})} \left( \sum_{S \in M(m, \lambda)} r_{a}(X, S) C_{S,U_{0}}^{\lambda} \otimes C_{T_{0}}^{(n, \lambda)} + b \otimes C_{T_{0}}^{(n, \lambda)} \right),$$

where $b \in A(< \lambda)$. We next show $b \otimes C_{T_{0}}^{(n, \lambda)} = 0$. 


Without loss of generality, consider a basis element \( b = C^\mu_{W,V} \) with \( \mu < \lambda \), \( W \in M(m, \mu) \) and \( V \in M(n, \mu) \). According to the quasi-heredity, we have \( \phi_{(n, \mu)} \neq 0 \). Hence, there exist \( U', T' \in M(n, \mu) \) such that \( \phi_{(n, \mu)}(U', T') \neq 0 \). It follows

\[
C^\mu_{W,V} \otimes C^{(n, \lambda)}_{T_0} = \frac{1}{\phi_{(n, \mu)}(U', T')} C^\mu_{W,U'} C^\mu_{T', V} \otimes C^{(n, \lambda)}_{T_0} = \frac{1}{\phi_{(n, \mu)}(U', T')} C^\mu_{W,U'} \otimes C^\mu_{T', V} \cdot C^{(n, \lambda)}_{T_0} = 0.
\]

Hence \( \alpha \) is injective. This finishes the proof.

This lemma says that the functor \( \text{Hom}_A(m, n) \otimes A_n \) sends a cellular module to a cellular module with the same index. In fact, we shall show that this functor is an idempotent embedding functors [1]. We first recall the definition of Morita context.

**Definition 3.2** [29] Let \( A, B \) be two \( K \)-algebras, and let \( _B P_A, _A Q_B \) be bimodules, and \( \theta, \phi \) be a pair of bimodule homomorphisms

\[
\theta : P \otimes_A Q \rightarrow_B B_B, \quad \phi : Q \otimes_B P \rightarrow_A A_A
\]

such that for all \( x, y \in P \) and \( f, g \in Q \),

\[
\theta(x \otimes f)g = x\phi(f \otimes y), \quad f\theta(x \otimes g) = \phi(f \otimes x)g.
\]

Then the tuple \((A, B, P_A, A_B, \theta, \phi)\) is called a Morita context.

If \( \theta \) is surjective in the previous definition, we have the following lemma.

**Lemma 3.3** [29] Prop 18.17] Keep the notation above.

Suppose \( \theta \) is surjective. Then

1. \( \theta \) is an isomorphism;
2. \( _A P_A \) and \( _A Q_A \) are finitely generated projective.
3. There are \( K \)-algebra isomorphism \( B \cong \text{End}(P_A) \cong \text{End}(A_Q) \).

Let \( \mathcal{A} \) be a hereditary cellular category. For objects \( m, n \in \mathbb{N} \), the tuple

\[
(A_m, _A A, _m A_m, _A A, \eta, \rho)
\]

is a Morita context, where \( \text{Hom}(m, n) \) is a \( A_m\text{-}A_n \)-bimodule, \( \text{Hom}(n, m) \) is a \( A_n\text{-}A_m \)-module. The map

\[
\eta : \text{Hom}_A(m, n) \otimes A_n \text{Hom}_A(n, m) \rightarrow A_m
\]

defined by the composition of morphisms, is an \( A_m \)-bimodule homomorphism;

Similarly, the map

\[
\rho : \text{Hom}_A(n, m) \otimes A_m \text{Hom}_A(m, n) \rightarrow A_n
\]

defined by the composition of morphisms, is an \( A_n \)-bimodule homomorphism.
Lemma 3.4  Keep the notation above. Let \( m, n \in \mathbb{N} \) satisfying \( m \leq n \). Suppose \( \text{Hom}_A(m, n) \neq 0 \). Then \( \rho \) is surjective.

Proof. Let \( C_{S,T}^\lambda \) be a cellular basis element of \( A_n \) with \( \lambda \in \Lambda_n \) and \( S, T \in M_n(\lambda) \). By Definition 2.8 and Remark 2.2(B), we have \( \Lambda_n = \Lambda_{(n,m)} \subseteq \Lambda_m \). Hence \( \lambda \in \Lambda_m \).

Since \( A_m \) is quasi-hereditary, it follows \( \phi(m,\lambda)(U_0, V_0) \neq 0 \) by Theorem 2.7, therefore there exist \( U_0, V_0 \in M_n(m, \lambda) \) such that \( \phi(m,\lambda)(U_0, V_0) \neq 0 \). This implies

\[
\rho\left( \frac{1}{\phi(n,\lambda)(U_0, V_0)} C_{S,U_0} \otimes C_{V_0,T}^\lambda \right) = C_{S,T}^\lambda,
\]

Because \( C_{S,T}^\lambda \) is arbitrary, \( \rho \) is surjective.

As an immediate consequence of lemma 3.3 and 3.4 we get the following.

Proposition 3.5  Let \( A \) be a hereditary cellular category. Suppose \( m, n \in \mathbb{N} \) satisfying \( m \leq n \) and \( \text{Hom}_A(m, n) \neq 0 \). Then

1. \( \rho \) is an \( A_n \)-bimodule isomorphism;
2. There exist idempotents \( e \) and \( f \) of \( A_m \) such that \( \text{Hom}_A(m, n) \simeq (A_m)e \) as \( A_m \)-module isomorphism, and \( \text{Hom}_A(n, m) \simeq f(A_m) \) as right \( A_m \)-module isomorphism, that is, \( \text{Hom}_A(m, n) \otimes_{A_n} - \) and \( - \otimes_{A_n} \text{Hom}_A(n, m) \) are idempotent embedding functors;
3. There are \( K \)-algebra isomorphism \( A_n \simeq e(A_m)e \simeq f(A_m)f \).

3.2 Cellular tower categories

Inspired by the theory of quasi-heredity towers of recollement studied by Cox, Martin, Parker and Xi, we first introduce the definition of cellular tower category. Under the framework, the homological aspects of representation theory are computed efficiently by using induction and restriction. We then give a criteria for a cellular tower category to be semi-simple.

Definition 3.6  A hereditary cellular category is called a cellular tower category (or CTC) if it satisfies (A1) – (A4) as follow:

(A1) For each \( n \geq 0 \) the endomorphism algebra \( A_n \) can be identified with a subalgebra of \( A_{n+1} \) which preserves the identities.

For an \( A_{n+1} \)-module \( M \), it has a natural \( A_n \)-module structure. Furthermore, we have the restriction functor:

\[
\text{Res}_{n+1} : \ A_{n+1} \text{-mod} \rightarrow A_n \text{-mod}
\]

\( A_{n+1}M \mapsto A_nM = \text{Hom}_{A_{n+1}}(A_{n+1}A_n, M) \).

We also have the induction functor:

\[
\text{Ind}_n : \ A_n \text{-mod} \rightarrow A_{n+1} \text{-mod}
\]
we have successive quotients isomorphic to some cell modules \( \Delta \) support of \( A \) we have \( \text{Hom}(n, n - 2) \cong_A A_{n-1} A_{n-2} \).

By our assumption that \( A \) is quasi-hereditary, due to lemma \[3.1\] for any \( \lambda \in \Lambda_n \) we have \( \text{Hom}(n+2, n) \otimes \Delta_n(\lambda) \cong \Delta_{n+2}(\lambda) \). Furthermore, by using (A1) and (A2), we have

\[
\text{Ind}_n(\Delta_n(\lambda)) = \text{Res}_{(n-1)}(\text{Hom}(n+2, n) \otimes_A \Delta_n(\lambda)).
\] (1)

If an \( A_n \)-module \( M \) in \( A_n \text{-mod} \) has a \( \Delta_n \)-filtration, that is, a filtration with successive quotients isomorphic to some cell modules \( \Delta_n(\lambda) \)'s, then we define the support of \( M \), denoted by \( \text{supp}_n(M) \), to be the set of labels \( \lambda \) for which \( \Delta_n(\lambda) \) occurs in this filtration.

(A3) For all \( m, n \in \mathbb{N} \) satisfying \( m \leq n \) and that \( n - m \) is even, and for all \( \lambda \in \Lambda_m \setminus \Lambda_{m-2} \), we have that \( \text{Res}_n(\Delta_n(\lambda)) \) has a \( \Delta_{(n-1)} \)-filtration and

\[
\text{supp}_{n-1}(\text{Res}_n(\Delta_n(\lambda))) \subseteq (\Lambda_{m-1} \setminus \Lambda_{m-3}) \cup (\Lambda_{m+1} \setminus \Lambda_{m-1}) \subseteq \Lambda_{n-1}.
\]

For (A3), using \( \text{Hom}(n+2, n) \otimes \Delta_n(\lambda) \cong \Delta_{n+2}(\lambda) \) and equation (1), we deduce that for all \( m, n \in \mathbb{N} \) satisfying \( m \leq n \) and that \( n - m \) is even, and for all \( \lambda \in \Lambda_m \setminus \Lambda_{m-2} \), we have that \( \text{Ind}_n(\Delta_n(\lambda)) \) has a \( \Delta_{(n+1)} \)-filtration and

\[
\text{supp}_{n+1}(\text{Ind}_n(\Delta_n(\lambda))) \subseteq (\Lambda_{m-1} \setminus \Lambda_{m-3}) \cup (\Lambda_{m+1} \setminus \Lambda_{m-1}) \subseteq \Lambda_{n+1}.
\]

(A4) Let \( n \in \mathbb{N} \). For each \( \lambda \in \Lambda_n \setminus \Lambda_{n-2} \) there exists \( \mu \in \Lambda_{n-1} \setminus \Lambda_{n-3} \) such that

\[
\lambda \in \text{supp}_n(\text{Ind}_{n-1}(\Delta_{n-1}(\mu))).
\]

By using (A3), (A4) is equivalent to

(A4') Let \( n \in \mathbb{N} \). For each \( \lambda \in \Lambda_n \setminus \Lambda_{n-2} \), there exists \( \mu \in \Lambda_{n-1} \setminus \Lambda_{n-3} \) such that

\[
\lambda \in \text{supp}_n(\text{Res}_{n+1}(\Delta_{n+1}(\mu))).
\]

For a quasi-hereditary algebra we have that \( \text{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0 \) implies that \( \mu < \lambda \). Therefore (A4) is also equivalent to:

Let \( n \in \mathbb{N} \). For each \( \lambda \in \Lambda_n \) there exists \( \mu \in \Lambda_{n-1} \) such that there is a surjection

\[
\text{Ind}_{n-1}(\Delta_{n-1}(\mu)) \to \Delta_n(\lambda) \to 0.
\]

Due to Cao \[8\], the following theorem provides some homological characterizations of the semi-simplicity of cellular algebra.

**Theorem 3.7** \[8\] Thm. 1.2] Let \( A \) be a cellular \( K \)-algebra with respect to an involution \( i \) and a poset \( (\Lambda, \leq) \). Let \( \Lambda_0 \) be the subset of \( \Lambda \), which parametrizes the isomorphism classes of simple \( A \)-modules. Then the following statements are equivalent:

\[
\Delta_n(\lambda) \mapsto A_{n+1} \otimes_A M.
\]
(a) The algebra $A$ is semi-simple.

(b) $\text{Ext}_{A}^{1}(\Delta(\lambda), S(\mu)) = 0$ for any $\lambda, \mu \in \Lambda_0$ satisfying $\mu \leq \lambda$, where $S(\mu)$ is the simple module with respect to $\mu$.

(c) $\text{Ext}_{A}^{1}(\Delta(\lambda), \Delta(\mu)) = 0$ for any $\lambda, \mu \in \Lambda_0$ satisfying $\mu \leq \lambda$.

(c') $\text{Ext}_{A}^{1}(\Delta(\lambda), \Delta(\mu)) = 0$ for any $\lambda, \mu \in \Lambda$ satisfying $\mu \leq \lambda$.

(c'') $\text{Ext}_{A}^{1}(\Delta(\lambda), \Delta(\mu)) = 0$ for all $\lambda, \mu \in \Lambda$.

The main result of this note can be stated as the following Theorem 3.8.

**Theorem 3.8** Let $A$ be a cellular tower category. Suppose that for all $n \in \mathbb{N}$ and pairs of indices $\lambda \in \Lambda_n \setminus \Lambda_{n-2}$ and $\mu \in \Lambda_n \setminus \Lambda_{n-4}$ we have

$$\text{Ext}_{A}^{1}(\Delta_n(\lambda), \Delta_n(\mu)) = 0,$$

Then each of the endomorphism algebras $A_n$ in $A$ is semi-simple.

To prove Theorem 3.8, we first need the following lemmas.

**Lemma 3.9** Let $A$ be a $K$-algebra. Suppose a $K$-algebra $B$ is a semi-simple subalgebra of $A$. Then for any $A$-module $M$ and $B$-module $N$, we have

$$\text{Ext}_{A}^{i}(\text{Ind}(N), M) \begin{cases} \simeq \text{Hom}_{B}(N, \text{Res}(M)) & \text{if } i = 0, \\ = 0 & \text{if } i > 0. \end{cases}$$

**Proof.** Let $M \to Q^{\bullet}$ be an injective resolution of $M$ in $A\text{-mod}$. Applying Res, due to $B$ is semi-simple, it follows that $\text{Res}M \to (\text{Res}(Q))^{\bullet}$ is an injective resolution of $\text{Res}(M)$ in $B\text{-mod}$ and

$$(\text{Ext}_{B}^{i}(N, \text{Res}(M)) = H^{i}(\text{Hom}_{B}(N, (\text{Res}(Q))^{\bullet}))$$

$$= H^{i}(\text{Hom}_{B}(N, \text{Hom}(A_B, Q^{\bullet})))$$

$$\simeq H^{i}(\text{Hom}_{A}(A \otimes_{B} N, Q^{\bullet}))$$

$$= \text{Ext}_{A}^{i}(\text{Ind}(N), M).$$

Since $B$ is semi-simple, we have $\text{Ext}_{B}^{i}(N, \text{Res}(M)) = 0$ for $i > 0$. The lemma follows. \hfill \Box

Martin and Woodcock [37] studied standard modules of quasi-hereditary algebras in terms of induction and restriction, the following result is a generalization of [37, Proposition 3.5].

**Lemma 3.10** Let $A$ be a hereditary cellular category. Suppose $m, n \in \mathbb{N}$ satisfying $\text{Hom}_{A}(m, n) \neq 0$. Then for each $\lambda, \mu \in \Lambda_{(m,n)}$ and $i \geq 0$, we have

$$\text{Ext}_{A_{m}}^{i}(\Delta_{m}(\lambda), \Delta_{m}(\mu)) \simeq \text{Ext}_{A_{n}}^{i}(\Delta_{n}(\lambda), \Delta_{n}(\mu)).$$

**Proof.** Without loss of generality, we may suppose $n \leq m$. Let $P^{\bullet} \to \Delta_{n}(\lambda)$ be a projective resolution of $\Delta_{n}(\lambda)$ in $A_n\text{-mod}$. Denoted by $F : = \text{Hom}_{A}(m, n) \otimes_{A_{n}} -$ , due to Proposition 3.5 $F$ is an idempotent embedding functor. Because by assumption $\Lambda_n$ is a saturated subset of $\Lambda_m$, due to [13, Proposition A 3.2], for
any \( i > 0 \), we have \( L_i F \Delta_n(\lambda) = 0 \), where \( L_i F \) denotes the \( i \)th left derived functor. Moreover, due to Lemma 3.1 it follows that \( \text{Hom}_A(m, n) \otimes P^* \to \Delta_m(\lambda) \) is a projective resolution of \( \Delta_m(\lambda) \) in \( A_m\text{-mod} \). Let \( e \) be an idempotent of \( A_m \) satisfying \( A_m \text{Hom}_A(m, n) \simeq (A_m)e \), we have

\[
\text{Ext}^i_{A_m}(\Delta_m(\lambda), \Delta_m(\mu)) = H^i(\text{Hom}_{A_m}((\text{Hom}_A(m, n) \otimes P^*), \Delta_m(\mu))) \\
\simeq H^i(\text{Hom}_{A_m}((A_m)e \otimes P^*), \Delta_m(\mu))) \\
\simeq H^i(\text{Hom}_{A_m}(P^*, e(\Delta_m(\mu)))) \\
= \text{Ext}^i_{A_m}(\Delta_n(\lambda), \Delta_n(\mu)).
\]

We are now in the position to give a proof of Theorem 3.8.

**Proof.** By the assumption that \( A_n \) is quasi-hereditary and due to Theorem 3.7 we need to prove, for all \( \lambda, \mu \in \Lambda_n \) with \( \mu < \lambda \), we have

\[
\text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0.
\]

Thus we always assume that \( \mu < \lambda \).

We use induction on \( n \). Assume that \( \lambda \in \Lambda_{n-2} \), then it is clear that \( \mu \in \Lambda_{n-2} \). Due to Lemma 3.10,

\[
\text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) \simeq \text{Ext}^1_{A_{n-2}}(\Delta_{n-2}(\lambda), \Delta_{n-2}(\mu)).
\]

According to the induction hypothesis, the right side of the above vanishes. Hence so is the left.

Assume that \( \lambda \in \Lambda_n \setminus \Lambda_{n-2} \), \( \mu \in \Lambda_n \setminus \Lambda_{n-4} \), it follows directly from the assumptions that

\[
\text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0.
\]

Assume now that \( \lambda \in \Lambda_n \setminus \Lambda_{n-2} \), \( \mu \in \Lambda_n \setminus \Lambda_{n-4} \), by axiom (A4), there exists \( \tau \in \Lambda_{n-1} \setminus \Lambda_{n-3} \) such that there is an exact sequence

\[
0 \to K \to \text{Ind}_{n-1}(\Delta_{n-1}(\tau)) \to \Delta_n(\lambda) \to 0. \tag{2}
\]

and

\[
\text{supp}(\text{Ind}_{n-1}(\Delta_{n-1}(\tau))) \subseteq (\Lambda_{n-2} \setminus \Lambda_{n-4}) \cup (\Lambda_n \setminus \Lambda_{n-2}).
\]

By (2), we have the exact sequence

\[
0 \to \text{Hom}_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) \\
\to \text{Hom}_{A_n}(\text{Ind}_{n-1}(\Delta_{n-1}(\tau)), \Delta_n(\mu)) \to \text{Hom}_{A_n}(K, \Delta_n(\mu)) \tag{3} \\
\to \text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) \to \text{Ext}^1_{A_n}(\text{Ind}_{n-1}(\Delta_{n-1}(\tau)), \Delta_n(\mu)).
\]

Due to (A3), it follows that \( \text{supp}(\text{Res}_n(\Delta_n(\mu))) \subseteq \Lambda_{n-3} \). Furthermore, by the induction hypothesis, \( A_{n-1} \) is semi-simple. Hence \( \text{Res}_n(\Delta_n(\mu)) \simeq \bigoplus_i \Delta_{n-1}(v_i) \) with some \( v_i \in \Lambda_{n-3} \). According to Lemma 3.9 we also have

\[
\text{Ext}^1_{A_n}(\text{Ind}_{n-1}(\Delta_{n-1}(\tau)), \Delta_n(\mu)) = 0.
\]
and
\[
\text{Hom}_{A_n} (\text{Ind}_{n-1}(\Delta_{n-1}(\tau)), \Delta_n(\mu)) \\
= \text{Hom}_{A_{n-1}} (\Delta_{n-1}(\tau), \text{Res}_n(\Delta_n(\mu))) \\
= \text{Hom}_{A_{n-1}} (\Delta_{n-1}(\tau), \bigoplus_i \Delta_{n-1}(v_i)) \\
= 0.
\]

Consequently, in the long sequence (3), for \( \lambda \in \Lambda_n \setminus \Lambda_{n-2}, \mu \in \Lambda_{n-4}, \text{Hom}_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0 \).

To prove \( \text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0 \), it is sufficient to show \( \text{Hom}_{A_n}(K, \Delta_n(\mu)) = 0 \) in the long sequence (3).

Firstly, for \( \lambda \in \Lambda_{n-2} \setminus \Lambda_{n-4}, \mu \in \Lambda_{n-4} \), it is clear that
\[
\text{Hom}_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = \text{Hom}_{A_{n-2}}(\Delta_{n-2}(\lambda), \Delta_{n-2}(\mu)) = 0.
\]

Thus, for all \( \lambda \in \Lambda_n \setminus \Lambda_{n-4}, \mu \in \Lambda_{n-4} \) we have \( \text{Hom}_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0 \).

Moreover, because \( \text{supp}_n(K) \subseteq \Lambda_n \setminus \Lambda_{n-4} \), this implies \( \text{Hom}_{A_n}(K, \Delta_n(\mu)) = 0 \).

Hence \( \text{Ext}^1_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0 \). This finishes the proof. \( \blacksquare \)

There are a large number of concrete algebras in our axiom scheme. We briefly list some as follows.

**Example 1** (1) The Temperley-Lieb category \( T\mathcal{L}(\delta) \) with \( \delta \) not a root of unity. See Section 4.2 for the definition. For \( n \in \mathbb{N} \), the poset is \( \Lambda_n = \{ 0 \text{ or } 1, \ldots, n-4, n-2, n \} \). For each \( 0 \leq i < n \), there is a short exact sequence
\[
0 \to \Delta_{n+1}(i-1) \to \text{Ind}_n(\Delta_n(i)) \to \Delta_{n+1}(i+1) \to 0.
\]

For more details on restriction and induction for Temperley-Lieb algebras, the reader is referred to [48, 42].

(2) The Brauer diagram category \( B(\delta) \) with \( \delta \neq 0 \). For the definition we refer the reader to [30] for details.

For \( n \in \mathbb{N} \), the endomorphism algebra of \( n \) is the Brauer algebra \( B_n(\delta) \), due to Cox [11], it is quasi-hereditary whenever it is characteristic zero or characteristic zero \( p > n \).

By [14] Theorem 4.1 and Corollary 6.4] and [11] Proposition 2.7], in arbitrary characteristic, for each \( \lambda \in \Lambda_n \), there is a short exact sequence
\[
0 \to \bigoplus_{\tau < \lambda} \Delta_{n+1}(\tau) \to \text{Ind}_n(\Delta_n(\lambda)) \to \bigoplus_{\mu \triangleright \lambda} \Delta_{n+1}(\mu) \to 0,
\]

where \( \Lambda_n \) is the indexing set of \( B_n(\delta) \), \( \tau \triangleleft \lambda \) represents that the young tableau \( \lambda \) is obtained by adding a box from the young tableau \( \tau \) and \( \mu \triangleright \lambda \) represents that the young tableau \( \lambda \) is obtained by removing a box from the young tableau \( \mu \).

For similar results on partition algebras and blob algebras and their categorical analogues, we refer the reader to [10, 32, 35, 37].
4 Algebraic structures on Grothendieck groups of tower categories

A number of diagram categories of great interest, such as Temperley-Lieb categories, Brauer diagram categories and so on, are cellular. The common feature of these categories is that they admit a natural tensor structure, the ‘juxtaposition’ of two diagrams, that is, the tensor product $A \otimes B$ of two diagrams is obtained by putting the diagram of $A$ on the left of the diagram of $B$. Another feature is that $A_0 \cong K \cong A_1$. So we consider the following axiom in our framework and equip the hereditary cellular category with a tensor product.

$\quad (A_1') \ A_0 \cong K, \ A_1 \cong K$ and there is an external multiplication $\rho_{m,n} : A_m \otimes A_n \to A_{m+n}$, for all $m, n \geq 0$, such that

(a) For all $m$ and $n$, $\rho_{m,n}$ is an injective homomorphism of algebras, sending $1_m \otimes 1_n$ to $1_{m+n}$;

(b) $\rho$ is associative, that is, $\rho_{l+m,n} \cdot (\rho_{l,m} \otimes 1_n) = \rho_{l,m+n} \cdot (1_n \otimes \rho_{m,n})$, for all $l, m, n$.

Obviously, $(A1)$, in Definition 3.6, is the special case for $m = 1$ in $(A1')$. The following definition is obtained by replacing $(A1)$ with $(A1')$.

**Definition 4.1** A hereditary cellular category is called a tensor cellular tower category (TCTC) if it satisfies $(A1')$ and $(A2)-(A4)$.

Let $\mathcal{A}$ be a TCTC. Denote by $G_0(A_n)$ the Grothendieck group of the endomorphism algebra of object $n$. Let $G_0(\mathcal{A}) := \bigoplus_{n \geq 0} G_0(A_n)$. The second reason allowing tensor product is that one can consider the algebraic structures on Grothendieck groups. In [3], Bergeron and Li introduced a general notation of a tower of algebras by way of axioms, which guarantee that the Grothendieck groups of a tower of algebras $\bigoplus_{n \geq 0} G_0(A_n)$ can be a pair of graded dual Hopf algebras. The tower of symmetric group algebras and their quantum deformations (such as Hecke algebras and Hecke-Clifford algebras and so on) are typical examples in their notation. The common feature of those examples is that they admit Mackey’s formula, which is just the compatibility relation between the multiplication and comultiplication on Grothendieck groups. Then they gave an analogue of Mackey’s formula (see equality (5) below) as an axiom in their framework. We shall see that the analogue of Mackey’s formula never holds in the quasi-hereditary case. The Grothendieck groups, however, have algebra and coalgebra structures under certain conditions.

In this section, the Grothendieck groups for TCTC is studied in 4.1. As a typical example of TCTC, we study Temperley-Lieb categories and obtain the structure constants of the multiplication on the Grothendieck groups in 4.2.

4.1 Induction and restriction functors on $G_0(\mathcal{A})$

Let $\mathcal{A}$ be a TCTC, we next consider the induction and restriction on $G_0(\mathcal{A})$ in terms of tensor products.

For $m, n \in \mathbb{N}$, let $M$ be a left $A_m$-module, and $N$ be a left $A_n$-module. Recall that the tensor product $M \otimes_K N$ is a left $A_m \otimes_K A_n$-module with the action $(a \otimes b) \cdot (w \otimes u) = aw \otimes bu$ for $a \in A_m, b \in A_n, w \in M$ and $u \in N$.  

Define the inductions on $G_0(A)$ as follows:

$$i_{m,n} : G_0(A_m) \otimes \mathbb{Z} G_0(A_n) \to G_0(A_{m+n})$$

$$[M] \otimes [N] \mapsto \text{Ind}_{A_m \otimes A_n}^{A_{m+n}} (M \otimes N),$$

where

$$\text{Ind}_{A_m \otimes A_n}^{A_{m+n}} (M \otimes N) = A_{m+n} \otimes_{A_m \otimes A_n} (M \otimes N)$$

$$= \frac{A_{m+n} \otimes (M \otimes N)}{< a \otimes [(b \otimes c)(w \otimes u)] - [a \rho_{m,n}(b \otimes c)] \otimes w \otimes u >}$$

for $a \in A_{m+n}$, $b \in A_m$, $c \in A_n$, $w \in M$ and $u \in N$.

Also define

$$r_{k,l} : G_0(A_n) \to G_0(A_k) \otimes_{\mathbb{Z}} G_0(A_l) \text{ with } k + l = n$$

$$[N] \mapsto [\text{Res}_{A_k \otimes A_l}^{A_n} (N)],$$

where $\text{Res}_{A_k \otimes A_l}^{A_n} (N) = \text{Hom}_{A_n} (A_n, N)$ is an $A_k \otimes A_l$-module with the action defined by $((b \otimes c) \cdot f)(a) = f(a \rho_{k,l}(b \otimes c))$ for $a \in A_n$, $b \in A_k$, $c \in A_l$ and $f \in \text{Hom}_{A_n} (A_n, N)$.

In $\mathbb{R}$, the following condition is as an axiom to insure that the maps $i$ and $r$ are well defined on for a tower of algebras:

$A_{m+n}$ is a two-sided projective $A_m \otimes A_n$-module with the action defined by $a \cdot (b \otimes c) = a \rho_{m,n}(b \otimes c)$ and $(b \otimes c) \cdot a = \rho_{m,n}(b \otimes c)a$ for all $m, n \geq 0, a \in A_{m+n}$, $b \in A_m$, $c \in A_n$.

Thus, we immediately have

**Proposition 4.2** Let $\mathcal{A}$ be a TCTC. Suppose $A_{m+n}$ is a two-sided projective $A_m \otimes A_n$-module for any $m, n \in \mathbb{N}$. Then the maps $i$ and $r$ are well defined on $G_0(\mathcal{A})$.

Further, we construct the multiplication and comultiplication by $i$ and $r$ and define the unit and counit on $G_0(\mathcal{A})$ as follows:

$$\pi : G_0(\mathcal{A}) \otimes_{\mathbb{Z}} G_0(\mathcal{A}) \to G_0(\mathcal{A})$$

where $\pi|_{G_0(A_k) \otimes G_0(A_l)} = i_{k,l}$

$$\Delta : G_0(\mathcal{A}) \to G_0(\mathcal{A}) \otimes_{\mathbb{Z}} G_0(\mathcal{A})$$

where $\Delta|_{G_0(A_n)} = \sum_{k+l=n} r_{k,l}$

$$\mu : \mathbb{Z} \to G_0(\mathcal{A})$$

where $\mu(a) = a[K] \in G_0(A_0)$, for $a \in \mathbb{Z}$

$$\epsilon : G_0(\mathcal{A}) \to \mathbb{Z}$$

where $\epsilon([M]) = \begin{cases} a & \text{if } [M] = a[K] \in G_0(A_0), \text{ for } a \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$
Also, Bergeron and Li [3] Theorem 3.5 proved that the associativity of $\pi$, the unitary property of $\mu$, the coassociativity of $\Delta$ and the counitary property of $\epsilon$, which imply that $(G_0(\mathcal{A}), \pi, \mu)$ is an algebra and $(G_0(\mathcal{A}), \Delta, \epsilon)$ is a coalgebra. Thus, we have

**Theorem 4.3** Let $\mathcal{A}$ be a TCTC. Suppose $A_{m+n}$ is a two-sided projective $A_m \otimes A_n$-module for any $m, n \in \mathbb{N}$. Then $(G_0(\mathcal{A}), \pi, \mu)$ is an algebra and $(G_0(\mathcal{A}), \Delta, \epsilon)$ is a coalgebra.

As a special case in Theorem 4.3, it is easy to see that if each $A_n$ is semi-simple then $(G_0(\mathcal{A}), \pi, \mu)$ is an algebra and $(G_0(\mathcal{A}), \Delta, \epsilon)$ is a coalgebra.

It is natural to ask whether $G_0(\mathcal{A})$ is a bialgebra? This means that whether the equality

$$\Delta(\pi([M] \otimes [N])) = \pi(\Delta([M]) \otimes \Delta([N])) \tag{4}$$

holds, that is, $\Delta$ is an algebra homomorphism.

To check this, however, it first needs to define a reasonable multiplication on $\Delta([M]) \otimes \Delta([N])$. To do this, note that there is a natural manner in terms of the twisted tensor product which is called the twisted induction in [3].

Let $k = t + s$, define

$$G_0(A_t \otimes A_{m-t}) \otimes G_0(A_s \otimes A_{n-s}) \rightarrow G_0(A_k \otimes A_{m+n-k})$$

$$[M_1 \otimes M_2] \otimes [N_1 \otimes N_2] \rightarrow \widetilde{\text{Ind}}_{A_t \otimes A_m}^A (M_1 \otimes M_2) \otimes (N_1 \otimes N_2)),$$

where

$$\widetilde{\text{Ind}}_{A_t \otimes A_{m-t}}^{A} (M_1 \otimes M_2) \otimes (N_1 \otimes N_2)) = (A_k \otimes A_{m+n-k}) \otimes_{A_t \otimes A_{m-t} \otimes A_s \otimes A_{n-s}} (M_1 \otimes M_2) \otimes (N_1 \otimes N_2))$$

This means

$$(a \otimes b) \otimes [(c_1 \otimes c_2) \cdot (w_1 \cdot w_2) \otimes (d_1 \otimes d_2) \cdot (u_1 \cdot u_2)]$$

$$\equiv [Ap_{t,s}(c_1 \otimes d_1) \otimes b_{m-t,n-s}(c_2 \otimes d_2)] \otimes (w_1 \otimes u_1 \otimes w_2 \otimes w_2).$$

Therefore equality (4) can be interpreted as the following equality:

$$[\text{Res}_{A_k \otimes A_{m+n-k}}^A \text{Ind}_{A_n \otimes A_m}^A (M \otimes N)]$$

$$= \sum_{t+s=k} \widetilde{\text{Ind}}_{A_t \otimes A_{m-t} \otimes A_s \otimes A_{n-s}} (\text{Res}_{A_k \otimes A_{m+n-k}}^A (M) \otimes \text{Res}_{A_s \otimes A_{n-s}}^A (N)) \tag{5}$$

for all $0 < k < m + n$, an $A_m$-module $M$ and an $A_n$-module $N$.

In the rest of the subsection, we verify that equality (5) is not true for Temperley-Lieb categories. Let $\mathcal{T} \mathcal{L}(\delta)$ be a $\mathcal{T} \mathcal{L}$-category with $\delta$ not a root of unity, and let $\Delta_n(r)$ be the cell module of the endomorphism algebra $TL_n$ corresponding to the index $r$. 


It is known that (see [42, Corollary 4.6 and Corollary 6.6])

\[ \text{Res}^{A_n}_{A_{n-1}}(\Delta_n(p)) \cong \Delta_{n-1}(p) \oplus \Delta_{n-1}(p-1) \quad \text{(as a } TL_{n-1} \text{ - module)} \]

and

\[ \text{Ind}^{A_{n+1}}_{A_n}(\Delta_n(p)) \cong \Delta_{n+1}(p+1) \oplus \Delta_{n+1}(p) \quad \text{(as a } TL_{n+1} \text{ - module)}. \]

Then, for cell modules \( \Delta_n(p) \) and \( \Delta_1(0) \), the left side of (5) equals:

\[
\begin{align*}
\text{Res}^{A_n}_{A_{n-1}} \text{Ind}^{A_n}_{A_{n-1}} (\Delta_n(p) \otimes \Delta_1(0)) &= \text{Res}^{A_n}_{A_{n-1}} (\Delta_n(p+1) \otimes \Delta_1(0) \oplus (\Delta_{n+1}(p) \otimes \Delta_1(0))) \\
&+ \text{Res}^{A_n}_{A_{n-1}} (\Delta_n(p+1) \otimes \Delta_1(0)) + \text{Res}^{A_n}_{A_{n-1}} (\Delta_{n+1}(p) \otimes \Delta_1(0)) \\
&= [\Delta_n(p+1)] + 2[\Delta_n(p)] + [\Delta_n(p-1)].
\end{align*}
\]

On the other hand, the right side of (5) is:

\[
\begin{align*}
\sum_{t+s=1} \text{Ind}^{A_1 \otimes A_n}_{A_0 \otimes A_1 \otimes A_{n-1} \otimes A_{1-s}} (\text{Res}^{A_n}_{A_{n-1}}(\Delta_n(p)) \otimes \text{Res}^{A_1}_{A_0}(\Delta_1(0))) \\
&= [\text{Ind}^{A_1 \otimes A_n}_{A_0 \otimes A_1 \otimes A_0} (\text{Res}^{A_n}_{A_0}(\Delta_n(p)) \otimes \text{Res}^{A_1}_{A_1}(\Delta_1(0))) \\
&+ [\text{Ind}^{A_1 \otimes A_{n-1}}_{A_1 \otimes A_{n-1}} (\text{Res}^{A_n}_{A_{n-1}}(\Delta_n(p)) \otimes \text{Res}^{A_1}_{A_1}(\Delta_1(0))) \\
&= [\text{Ind}^{A_n}_{A_{n-1}}(\Delta_n(p) \otimes \Delta_1(0))] + [\text{Ind}^{A_{n-1}}_{A_{n-1}}(\Delta_{n-1}(p) \oplus \Delta_{n-1}(p-1)) \\
&= [\Delta_n(p)] + [\Delta_n(p+1) \oplus \Delta_n(p) \oplus \Delta_n(p) \oplus \Delta_n(p-1)] \\
&= [\Delta_n(p)] + 3[\Delta_n(p)] + [\Delta_n(p-1)].
\end{align*}
\]

Consequently, the equality (5) do not hold. Hence \((G_0(\mathcal{T}L(\delta)), \pi, \mu, \Delta, \epsilon)\) is not a bialgebra.

### 4.2 Algebraic structures on the Grothendieck groups of \( \mathcal{T}L \)-categories

In this section, we shall study the multiplication in the Grothendieck groups of semi-simple \( \mathcal{T}L \)-categories, and have the following result.

**Theorem 4.4** Let \( \mathcal{T}L(\delta) \) be a \( \mathcal{T}L \)-category with \( \delta \) not a root of unity. Then:

For each \( m, n \in \mathbb{N} \),

\[
i_{m,n}[\Delta_m(p) \otimes \Delta_n(q)] = \sum_{0 \leq r \leq (m+n)/2} a_{(p,q,r)}^{(m|n)}[\Delta_{m+n}(r)],
\]

where

\[
a_{(p,q,r)}^{(m|n)} = \begin{cases} 1 & \text{if } p + q \leq r \text{ and } m - s \geq 2p \text{ and } m - s \geq 2q, \\ 0 & \text{others}, \end{cases}
\]

with \( s = r - (p + q) \) and \( \Delta_k(l), 0 \leq l \leq [k/2] \), the cell module of \( TL_k \).
It is known that Temperley-Lieb algebra $T_L^n$ is semi-simple if $\delta$ is not a root of unity, and hence all cell modules of $T_L^n$ form a complete set of non-isomorphism simple $T_L^n$-modules by Theorem 2.6. Moreover, all $\Delta_m(p) \otimes_K \Delta_n(q)$ for $0 \leq p \leq \lfloor m/2 \rfloor$ and $0 \leq q \leq \lfloor n/2 \rfloor$ form a complete set of non-isomorphism simple modules for semi-simple algebra $T_L^m \otimes_K T_L^n$.

Recall that an $n$-diagram consist of two rows of $n$ dots in which each dot is joined to just one other dot and none of the joins intersect when drawn in the rectangle defined by the two rows of $n$ dots. All such $n$-diagrams form a cellular basis of $T_L^n$ and the multiplication of the basis is given by concatenating two $n$-diagrams, that is, stacking the first diagram on top of the second diagram, matching the relevant bottom and top vertices, and replacing all closed cycles by a factor $\delta^r$ with $r$ the number of closed cycles. See Figure 1 for an example for $n = 3$.

\[ \begin{array}{c}
\begin{array}{c}
\bullet \cdot \bullet \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\bullet \cdot \bullet \\
\cdot \cdot \cdot \\
\end{array}
\end{array} \quad = \quad \delta \cdot \begin{array}{c}
\begin{array}{c}
\bullet \cdot \bullet \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot
\end{array}
\end{array} \]

Figure 1: the multiplication of 3-diagrams.

Due to [20], the cell module $\Delta_n(r)$ of $T_L^n$ is spanned by all $(n, r)$-cap diagrams, which is the ‘half-diagrams’ obtained from an $n$-diagram by cutting horizontally down the middle. Figure 2(2) illustrates an (11,4)-cap diagram in which the arcs $\{3, 8\}, \{4, 5\}, \{6, 7\}$ and $\{10, 11\}$ are caps and vertices 1, 2, 9 are called single points.

For further informations on the diagrammatic and algebraic definitions of Temperley-Lieb algebras and their cell modules, we refer the reader to [20, 42, 48].

To prove Theorem 4.4, we first regard a $(m+n)$-diagram (and a cap diagram) as a ‘walled’ diagram in Definition 4.5. By using such ‘walled’ diagrams, we then give a composition series of $T_L^m \otimes T_L^n \Delta_m(n)\Delta_m(n)$ in Proposition 4.6. Finally, Theorem 4.4 holds true directly from the following equalities.

\[
i_{m,n}[\Delta_m(p) \otimes \Delta_n(q)] = \text{Ind}_{A_m \otimes A_n}^{A_{m+n}}[\Delta_m(p) \otimes \Delta_n(q)] = \sum_r a_{(m,n)}^{(m+n)}(\Delta_{m+n}(r)),
\]

and

\[
a_{(m,n)}^{(m+n)} = \dim \text{Hom}[\text{Ind}_{A_m \otimes A_n}^{A_{m+n}}[\Delta_m(p) \otimes \Delta_n(q)], \Delta_{m+n}(r)]
\]

\[
= \dim \text{Hom}[\Delta_{m+n}(p) \otimes \Delta_n(q), A_m \otimes A_n \Delta_{m+n}(r)]. \quad (6)
\]

The following observations is the key to our proof. Indeed, the juxtaposition (see the beginning of the section 4) of an $m$-diagram and an $n$-diagram can be viewed as a ‘walled’ diagram if we imagine that there is a wall between the two diagrams, and we herein call such diagram an $(m|n)$-diagram. It is clear that all such diagrams form a basis of $T_L^m \otimes T_L^n$. Figure 2(1) demonstrates the juxtaposition of an 6-diagram in $T_L^6$ and an 5-diagram in $T_L^5$. 
Similarly, an \((m + n, r)\)-cap diagram in \(\mathcal{T}L_m \otimes \mathcal{T}L_n \Delta_{m+n}(r)\) can be viewed as a ‘walled’ cap diagram if we imagine that there is a wall which separates the vertices \(m, m + 1\). Figure 2(2) illustrates that an \((11,4)\)-cap diagram can be viewed as a ‘walled’ diagram when \(\Delta_{11}(4)\) is restricted as a \(\mathcal{T}L_6 \otimes \mathcal{T}L_5\)-module.

Due to the above observations, we define

**Definition 4.5** Let \(\mathcal{T}\mathcal{L}(\delta)\) be a \(\mathcal{T}\mathcal{L}\)-category, and let \(\Delta_{m+n}(r)\) be the cell module of \(\mathcal{T}L_n\) corresponding to index \(r\). Then we call \(\mathcal{T}L_m \otimes \mathcal{T}L_n \Delta_{m+n}(r)\), the restriction of \(\Delta_{m+n}(r)\) as a \(\mathcal{T}L_m \otimes\mathcal{T}L_n\)-module, an \((m \mid n, r)\)-walled module as well as call an \((m + n, r)\)-cap diagram in \(\mathcal{T}L_m \otimes \mathcal{T}L_n \Delta_{m+n}(r)\) an \((m \mid n, r)\)-walled cap diagram.

For an \((m \mid n, r)\)-walled cap diagram, we call the arcs crossing the wall *through strings* and call the caps on the left side of the wall *left caps*, the caps on the right side of the wall *right caps*. Furthermore, for an \((m \mid n, r)\)-walled cap diagram, we can correspond to a triple \((s, l_m, l_n)\), where \(s\) is the number of the through strings, \(l_m\) and \(l_n\) are the number of the left caps and the right caps, respectively. It is clear that \(s + l_m + l_n = r\).

For instance, Figure 2(2) show that an \((11,4)\)-cap diagram in \(\Delta_{11}(4)\) can be viewed as a \((6 \mid 5, 4)\)-walled cap diagram in \(\mathcal{T}L_6 \otimes \mathcal{T}L_5 \Delta_{m+n}(4)\). Here, \(\{3, 8\}\), \(\{6, 7\}\) are through strings, \(\{4, 5\}\) is a left cap, \(\{10, 11\}\) is a right cap, and hence it correspond to the triple \((2, 1, 1)\).

Moreover, denote by \(I\) the index set consisting of the triples \((s, l_m, l_n)\) of all the \((m \mid n)\)-walled cap diagrams, and the triples \((s, l_m, l_n)\) is ordered lexicographically with ‘\(s\)’ using natural numbers ordering, \(l_m\) and \(l_n\) using inverse ordering of natural numbers.

Let \(W_{m \mid n}(s, l_m, l_n)\) be the \(K\)-subspace of \(\mathcal{T}L_m \otimes \mathcal{T}L_n \Delta_{m+n}(r)\) spanned by all walled \((m \mid n, r)\)-walled cap diagrams with indices no more than \((s, l_m, l_n)\).

We then get a chain in terms of the total ordering on \(I\)

\[
0 \subset \cdots \subset W_{m \mid n}(s', l'_m, l'_n) \subset W_{m \mid n}(s, l_m, l_n) \subset \cdots \subset \mathcal{T}L_m \otimes \mathcal{T}L_n \Delta_{m+n}(r). \quad (a)
\]

Now, we claim:
Proposition 4.6 Keep the notation as above. Then:

(1) Set
\[ \Delta_{m|n}(s, l_m, l_n) := W_{m|n}(s, l_m, l_n) / W_{m|n}(s', l'_m, l'_n). \]
Then we have
\[ \Delta_{m|n}(s, l_m, l_n) \cong \Delta_m(l_m) \otimes \Delta_n(l_n) \]
as a TL\(_m\otimes\)TL\(_n\)-module;

(2) The chain (a) is a composition series of TL\(_m\otimes\)TL\(_n\)\(\Delta_{m+n}(r)\).

To prove this, we first show that \(W(s, l_m, l_n)\) is a submodule of TL\(_m\otimes\)TL\(_n\)\(\Delta_{m+n}(r)\). Secondly, we prove Proposition 4.6(1), and then Proposition 4.6(2) follows immediately from 4.6(1).

The key to prove Proposition 4.6 is the following observations.

In fact, we notice that an \((m|n, r)\)-walled cap diagram with the triple \((s, l_m, l_n)\) also can be viewed as a so-called \((m, n)\)-diagram, which is a diagram with \(m\) vertices on the top row and \(n\) vertices on the bottom row labeling the vertices in a clockwise direction and the arc \(\{i, j\}\) in the \((m, n)\)-diagram is the same as in the \((m|n, r)\)-walled cap diagram. (see Figure 3 for an example.)

![Figure 3: a (6|5, 4)-walled cap diagram represented as a (6, 5)-diagram](image)

In Figure 3 the triple \((s, l_m, l_n)\) corresponding to the left diagram is \((2, 1, 1)\). The triple in the right diagram can be read as follows: \(s = 2\) is the number of vertical arcs (therefore called the through strings), \(l_m = 1\) is the number of the top horizontal arcs and \(l_n = 1\) is the number of the bottom horizontal arcs.

Another observation is that, TL\(_m\otimes\)TL\(_n\)-action on an \((m|n, r)\)-walled cap diagram can be viewed as a left TL\(_m\)-right TL\(_n^{op}\)-action on the corresponding \((m, n)\)-diagram. Here, an \(n\)-diagram in TL\(_n^{op}\) are obtained by rotating the \(n\)-diagram in TL\(_n\) by a half turn. Figure 4 is an example.

With the above observations and by the properties of the concatenation of two Temperley-Lieb diagrams [42], the following lemma holds immediately.

Lemma 4.7 Let \(x\) be an \((m|n, r)\)-walled cap diagram with the triple \((s, l_m, l_n)\), and let \(d\) be an \((m|n)\)-diagrams in TL\(_m\otimes\)TL\(_n\). Suppose the result diagram of the TL\(_m\otimes\)TL\(_n\)-action \(d \cdot x\) has the triple \((s', l'_m, l'_n)\). Then we have \(s' \leq s, l_m \geq l'_m\) and \(l_n \geq l'_n\), that is, the number of the through strings never increases and the number of the caps never decreases.

Due to above lemma, \(W_{m|n}(s, l_m, l_n)\) is a TL\(_m\otimes\)TL\(_n\)-module. Hence, the chain(a) is a chain of modules.

Before proving Proposition 4.6(1), we also need the following result.
Lemma 4.8 Let $u$ be an $(m, l_m)$-cap diagram in $\Delta_m(l_m)$, and let $v$ be an $(n, l_n)$-diagram in $\Delta_n(l_n)$ as well as let $u \otimes v$, an $(m|n, l_m + l_n)$-walled cap diagram, be the juxtaposition of $u$ and $v$. Suppose one can add $s$ through strings on $u \otimes v$. Then there is a unique $(m|n, l_m + l_n + s)$-walled cap diagram obtained by adding $s$ through strings on $u \otimes v$.

Proof. The result diagram by adding $s$ through strings on $u \otimes v$ can be obtained by the following steps (see Figure 5 below). Firstly, all caps are removed directly from the diagram $u \otimes v$, and thus this leaves exactly vertices, say $i_1 < i_2 < \cdots < i_{m+n-2l_m} < i'_1 < i'_2 < \cdots < i'_{m+n-2l_n}$, where $i_j$ is on the left of the wall and $i'_k$ is on the right of the wall. It is an $(m - l_m|n - l_n, 0)$-walled cap diagram.

Secondly, we add $s$ through strings on the $(m - l_m|n - l_n, 0)$-walled cap diagram. Because the through strings cross the wall and the $(m - l_m|n - l_n, 0)$-walled cap diagram is a half-diagram of an $(m + n - l_m - l_n)$-diagram. This means, for any added through string $\{i_j, i'_k\}$, there is no single vertex $q$ with $i_j < q < i'_k$ and moreover, for any two added through strings $\{i_{j_1}, i'_{k_1}\}, \{i_{j_2}, i'_{k_2}\}$ with $i_{j_1} < i_{j_2}$, since $i_{j_2} < i'_{k_1}$ and the strings cannot cross each other in a diagram, this implies $i_{j_1} < i_{j_2} < i'_{k_2} < i'_{k_1}$.

Therefore, the $s$ through strings must be $\{i_{m+n-2l_m}, i'_{1}\}$, $\{i_{m+n-2l_m-1}, i'_{2}\}$, $\cdots$, $\{i_{m+n-2l_m-s}, i'_{s}\}$.

Finally, we get the result diagram by recovering all caps which we removed. \qed

We are now in the position to give a proof of Proposition 4.6(1).

Proof. Since a $TL_m \otimes TL_n$-module can be viewed as a left $TL_m$-right $TL_n^{op}$-bimodule, it is sufficient to prove $\Delta_{m|n}(s, l_m, l_n) \cong \Delta_m(l_m) \otimes \Delta_n(l_n)$ as a $TL_m$-module.
Firstly, we give an one-to-one map from $\Delta_m(l_m) \otimes \Delta_n(l_n)$ to $\Delta_{m|n}(s, l_m, l_n)$. Due to Lemma 4.8 given an $(m|n, l_m + l_n)$-cap diagram $u \otimes v$ in $\Delta_m(l_m) \otimes \Delta_n(l_n)$, there is a unique diagram obtained by adding $s$ through strings on $u \otimes v$ in $\Delta_{m|n}(s, l_m, l_n)$. Consequently, let such diagram correspond to $u \otimes v$. We then denote this map by $\sigma$ and extend linearly.

We next show that $\sigma$ preserves the left $T L_m$-action.

Precisely, let $u = C^{l_m}_{S_1}$ and $v = C^{l_n}_{S_2}$ be two cap diagrams in $\Delta_m(l_m)$ and $\Delta_n(l_n)$ respectively. For any $m$-diagram $C_{S,T}^l$ in $T L_m$(since all $m$-diagram form a cellular basis), if $l > l_m$, then $C_{S,T}^l \cdot (C^{l_m}_{S_1} \otimes C^{l_n}_{S_2}) = (C_{S,T}^l \cdot C^{l_m}_{S_1}) \otimes C^{l_n}_{S_2} = 0$. On the other hand, the number of the left cups of walled cap diagram $C_{S,T}^l \cdot \sigma(C^{l_m}_{S_1} \otimes C^{l_n}_{S_2})$ is increases and the number of the through strings is never increase due to Lemma 4.7, hence it is equal to 0 in $\Delta_{m|n}(s, l_m, l_n)$.

If $l \leq l_m$, due to previous observation in Figure 4, the diagram $C_{S,T}^l \cdot \sigma(C^{l_m}_{S_1} \otimes C^{l_n}_{S_2})$ can be obtained by following steps (see Figure 6 below): Firstly, cutting the diagram $C_{S,T}^l \cdot \sigma(C^{l_m}_{S_1} \otimes C^{l_n}_{S_2})$ along the wall, the diagram becomes the juxtaposition of $C_{S,T}^l \cdot C^{l_m}_{S_1}$ and $C^{l_n}_{S_2}$, say $(C_{S,T}^l \cdot C^{l_m}_{S_1}) \otimes C^{l_n}_{S_2}$. It remains to add $s$ through strings on $(C_{S,T}^l \cdot C^{l_m}_{S_1}) \otimes C^{l_n}_{S_2}$. It is, however, just $\sigma(C_{S,T}^l \cdot (C^{l_m}_{S_1} \otimes C^{l_n}_{S_2}))$ by Lemma 4.8.

\[\square\]
Due to Proposition 4.6(2) and equality (6), the structure constant \(a_{(p,q,r)}^{(m|n)}\) in Theorem 4.4 is the multiplicity of the simple module \(\Delta_m(p) \otimes \Delta_n(q)\) occurred as a composition factor in the chain (a), a composition series of \(T_L m \otimes T_L n \Delta_{m+n}(r)\).

Moreover, by Proposition 4.6(1), suppose \(\Delta_{m|n}(s,p,q) \cong \Delta_m(p) \otimes \Delta_n(q)\) is a composition factor in the chain (a). Then the triple \((s,p,q)\) must be satisfied:

\[
\begin{align*}
&s + p + q = r \quad \text{(total r arcs)}, \\
&m - s \geq 2p \quad \text{(p left caps)}, \\
&n - s \geq 2q \quad \text{(q right caps)}. \\
\end{align*}
\]

Conversely, if \(\Delta_m(p) \otimes \Delta_n(q)\) occur as a composition factor in the chain (a), then \(\Delta_{m|n}(s,p,q)\), with \(s = r - p - q\), is the unique \((m|n,r)\)-walled cap module isomorphic to \(\Delta_m(p) \otimes \Delta_n(q)\). Therefore, the multiplicity equals 1.

Consequently, due to above facts, Theorem 4.4 follows.

Now, we end this paper by a simple examples to illustrate our results.

**Example 2** For \(T_L 4 \otimes T_L 3 \Delta_7(2)\), we display a composition series as follows:

\[
0 \subset W_{4|3}(0,2,0) \subset W_{4|3}(0,1,1) \subset W_{4|3}(1,1,0) \subset W_{4|3}(1,0,1) \subset W_{4|3}(2,0,0) = T_L 4 \otimes T_L 3 \Delta_7(2).
\]

\[\Delta_{4|3}(2,0,0) \cong \Delta_4(0) \otimes \Delta_3(0)\]

\[\Delta_{4|3}(1,0,1) \cong \Delta_4(0) \otimes \Delta_3(1)\]

\[\Delta_{4|3}(1,1,0) \cong \Delta_4(1) \otimes \Delta_3(0)\]
\( \Delta_{4|3}(0, 1, 1) \cong \Delta_4(1) \otimes \Delta_3(1) \)

\( \Delta_{4|3}(0, 2, 0) \cong \Delta_4(2) \otimes \Delta_3(0) \)

Figure 7: bases of the walled modules

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References

[1] I. Assem, D. Simson, A. Skowroński, Elements of Representation Theory of Associative Algebras, vol. 1: Techniques of Representation Theory, Cambridge University Press, Cambridge, 2006.

[2] N. Bergeron, F. Hivert, J.Y. Thibon, The peak algebra and the Hecke-Clifford algebras at \( q = 0 \), J.Comb.Theory 107(1)(2004)1–19.

[3] N. Bergeron, H. Li, Algebraic structures on Grothendieck groups of a tower of algebras, J.Algebra 321(8)(2006)2068–2084.

[4] C. Bowman, Brauer algebras of type C are cellularly stratified, Math.Proc.Camb.Phil.Soc. 153(01)(2012)1–7.

[5] M. Bloss, G-colored partition algebras as centralizer algebras of wreath products, J.Algebra 265(2)(2003)690–710.

[6] R. Brauer, On algebras which are connected with the semisimple continuous groups, Ann.Math. 38(4)(1937)857–872.

[7] W.P. Brown, An algebra related to the orthogonal group, Mich.Math.J. 3(1)(1956)1–22.
[8] Y.Z. Cao, On the quasi-heredity and the semi-simplicity of cellular algebras, J.Algebra 267(2003) 323–341.

[9] E. Cline, B. Parshall, L. Scott, Finite-dimensional algebras and highest weight categories, J.reine angew.Math. 391(1988)85–99.

[10] A.G. Cox, J.J. Graham, P.P. Martin, The blob algebra in positive characteristic, J. Algebra 266(2003)584–635.

[11] A.G. Cox, M. Visscher, P.P. Martin, The blocks of the Brauer algebra in characteristic zero, Represent.Theory 13(2009)272–308.

[12] A.G. Cox, P.P. Martin, A.E. Parker, C.C. Xi, Representation theory of towers of recollement: theory, notes and examples, J.Algebra 302(2006)340–360.

[13] S. Donkin, The q-Schur Algebra, Cambridge University Press, Cambridge, 1998.

[14] W.F. Doran, D.B. Wales, P.J. Hanlon, On the semisimplicity of the Brauer centralizer algebras, J.Algebra 211(1999)647–685.

[15] B. Elias, A.D. Lauda, Trace decategorification of the Hecke category, J. Algebra 449(2015)615–634.

[16] B. Elias, G. Williamson, Soergel calculus, Represent. Theory 20(2016)295–374.

[17] M. Geck, Hecke algebras of finite type are cellular, Invent.Math. 169(3)(2007)501–517.

[18] F.M. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, 1989.

[19] F.M. Goodman, J. Graber, Cellularity and the Jones basic construction. Adv.Appl.Math. 46(1-4)(2009)312–362.

[20] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent.Math. 123(1996)1–34.

[21] R.M. Green. Generalized Temperley-Lieb algebras and decorated tangles, J.Knot Theory Ramif. 07(2)(2011)155–171.

[22] V.F.R. Jones, Index for Subfactors, Invent.math. 72(1)(1983)1–25.

[23] V. Jones, The Potts model and the symmetric group, in: Subfactors, World Sci.Publishing, River Edge, NJ, 1994, pp. 259–267.

[24] S. Koenig, C.C. Xi, A characteristic free approach to Brauer algebras, Trans.Amer.Math.Soc. 353 (2001)1489–1505.

[25] S. Koenig, C.C. Xi, Cellular algebras: inflations and Morita equivalences, J.London Math.Soc. 60(2)(1999)700–722.

[26] S. Koenig, C.C. Xi, Cellular algebras and quasi-hereditary algebras:a comparison, Electron.Res.Announc.Amer.Math.Soc. 5(1999)71–75.
[27] S. Koenig, C.C. Xi, When is a cellular algebra quasi-hereditary, Math.Ann. 315(2)(1999)281–293.

[28] D. Krob, J.Y. Thibon, Noncommutative symmetric functions IV: quantum linear groups and Hecke algebras at $q = 0$. J.Alg.Comb. 6(4)(1997)339–376.

[29] T.Y. Lam, Lectures on Modules and Rings, Springer-Verlag, New York, 1999.

[30] G.I. Lehrer, R.B. Zhang, The Brauer category and invariant theory, J.Eur.Math.Soc. 62(10)(2012)863–70.

[31] P.P. Martin, Potts Models and Related Problems in Statistical Mechanics, World Scientific, Singapore, 1991.

[32] P.P. Martin, The partition algebra and the Potts model transfer matrix spectrum in high dimensions, J. Phys.A: Math.Gen. 33(2000)3669.

[33] P.P. Martin, Temperley-Lieb algebras for non-planar statistical mechanics-the partition algebra construction, J. Knot Theory Ramif. 3(1)(1994)51–82.

[34] P.P. Martin, A, Elgamal, Ramified partition algebras, Math.Z. 246(3)(2004)473–500.

[35] P.P. Martin, S. Ryom-Hansen, Virtual algebraic Lie theory: Tilting modules and Ringel duals for blob algebras, Proc.London Math.Soc. 89(2004)655–675.

[36] P.P. Martin, H. Saleur, The blob algebra and the periodic Temperley-Lieb algebra, Lett.Math.Phys. 30(3)(1994)189–206.

[37] P.P. Martin, D. Woodcock, On the structure of the blob algebras, J. Algebra 255(2000)957–988.

[38] V. Mazorchuk, On the structure of Brauer semigroup and its partial analogue, Problems in Algebra, 13(1998)29–45.

[39] V. Mazorchuk, Endomorphisms of $\mathfrak{B}_n$, $\mathcal{P}\mathfrak{B}_n$, and $\mathfrak{C}_n$, Comm. Algebra 30 (2002) 3489–3513.

[40] G.E. Murphy, The representations of Hecke algebras of type An, J. Algebra 173(1995)97–121.

[41] D.T. Nguyen, Cellular structure of $q$-Brauer algebras, Algebra.Represent.Th. 17(5)(2014)1359–1400.

[42] D. Ridout, Y. Saint-Aubin, Standard modules, induction and the structure of the Temperley-Lieb algebra, Adv.Theor.Math.Phys. 18(2014)957–1041.

[43] H. Rui, C.C. Xi, The representation theory of cyclotomic Temperley-Lieb algebras, Comment.Math.Helv. 79(2)(2004)427-450.

[44] A.N. Sergeev, The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{sl}(n,m)$, and $Q(n)$, Math.USSR, Sbornik 51(2)(1984)422–430.
[45] H. Wenzl, On the structure of Brauer's centralizer algebras. Ann.Math. 128(1)(1988)173–193.

[46] B.W. Westbury, Invariant tensors and cellular categories, J.Algebra 321(2009)3563–3567.

[47] B.W. Westbury, Invariant tensors for the spin representation of so(7), Math.Proc.Cambridge Philos. Soc. 114(1)(2008)217–240.

[48] B.W. Westbury, The representation theory of the Temperley-Lieb algebras, Math.Z. 219(4)(1995)539–565.

[49] C.C. Xi, On the quasi-heredity of Birman-Wenzl algebras, Adv.Math. 154(2000)280–298.

[50] C.C. Xi, Partition algebras are cellular, Compos.Math. 119(1)(1999)99–109.

[51] A.V. Zelevinsky, Representations of Finite Classical Groups. A Hopf Algebra Approach, Lecture Notes in Math., vol. 869, Springer-Verlag, Berlin, 1981.

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