New Type of Degenerate Poly-Frobenius-Euler Polynomials and Numbers

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Abstract. Motivated by Kim-Kim [19] introduced the new type of degenerate poly-Bernoulli polynomials by means of the degenerate polylogarithm function. In this paper, we define the degenerate poly-Frobenius-Euler polynomials, called the new type of degenerate poly-Frobenius-Euler polynomials, by means of the degenerate polylogarithm function. Then, we derive explicit expressions and some identities of those numbers and polynomials.

Keywords: polylogarithm function; Frobenius-Euler polynomials; new type of degenerate poly-Frobenius-Euler polynomials

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1. Introduction

Throughout this presentation, we use the following standard notions \( \mathbb{N} = \{1, 2, \cdots \} \), \( \mathbb{N}_0 = \{0, 1, 2, \cdots \} = \mathbb{N} \cup \{0\} \), \( \mathbb{Z} = \{-1, -2, \cdots \} \). Also as usual \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{C} \) denotes the set of complex numbers.

The classical Bernoulli \( B_n(x) \), Euler \( E_n(x) \) and Genocchi \( G_n(x) \) polynomial are defined by means of the following generating function as follows

\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad |t| < 2\pi,
\]

and

\[
\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad |t| < \pi,
\]

respectively.

For \( u \in \mathbb{C} \) with \( u \neq 1 \), the classical Frobenius-Euler polynomials \( H_n^{(\alpha)}(x; u) \) of order \( \alpha \) are defined by means of the following generating function

\[
\left( \frac{1 - u}{e^t - u} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u) \frac{t^n}{n!} \quad (\text{see } [1, 5, 11, 12]).
\]

In the special case when \( x = 0 \), \( H_n^{(\alpha)}(u) = H_n^{(\alpha)}(0; u) \) are called \( n^{th} \) Frobenius-Euler numbers of order \( \alpha \). For \( \alpha = 1 \) into (1.2), \( H_n^{(1)}(x, u) = H_n(x, u) \), are called the Frobenius-Euler polynomials and \( H_n^{(\alpha)}(0; u) = h_n^{(\alpha)}(u) \), are called the Frobenius-Euler numbers of order \( \alpha \). Substituting \( u = -1 \) into (1.2), \( H_n(x; -1) = E_n(x) \), are called the Euler polynomials, (see [8, 22, 24, 25]).

In (2017), Kurt [10] introduced the poly-Frobenius-Euler polynomials are given by

\[
\frac{(1 - u)\text{Li}_k(1 - e^{-t})}{t(e^t - u)} e^{xt} = \sum_{n=0}^{\infty} H_n^{(k)}(x; u) \frac{t^n}{n!}.
\]
In the case when $x = 0$, $H_n^{(k)}(u) = H_n^{(k)}(0; u)$ are called the poly-Frobenius-Euler numbers.

For any non-zero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by
$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \text{(see [13, 18, 19])}. \quad (1.4)$$

By binomial expansion, we get
$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \text{(see [14, 17])}. \quad (1.5)$$

where $(x)_{n,\lambda} = (x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), (n \geq 1)$.

Note that
$$\lim_{\lambda \to 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = e^{xt}.$$

In [2, 3] Carlitz introduced the degenerate Bernoulli and degenerate Euler polynomials are defined by
$$\frac{z}{e_\lambda(z) - 1} e_\lambda^x(z) = \sum_{j=0}^{\infty} B_{j,\lambda}(x) \frac{z^j}{j!}, \quad \frac{2}{e_\lambda(z) + 1} e_\lambda^x(z) = \sum_{j=0}^{\infty} E_{j,\lambda}(x) \frac{z^j}{j!}, \quad (1.6)$$
respectively.

In the case when $x = 0$, $B_{j,\lambda} = B_{j,\lambda}(0)$ are called the degenerate Bernoulli numbers and $x = 0$, $E_{j,\lambda} = E_{j,\lambda}(0)$ are called the degenerate Euler numbers.

Obviously
$$\lim_{\lambda \to 0} B_n(x; \lambda) = B_n(x), \lim_{\lambda \to 0} E_n(x; \lambda) = E_n(x).$$

Kim et al. [15] introduced the degenerate Frobenius-Euler polynomials are defined by means of the generating function as follows
$$\frac{1 - u}{(1 + \lambda t)^{\frac{x}{\lambda}} - u} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u) \frac{t^n}{n!}, \quad (1.7)$$

At the value $x = 0$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called the degenerate Frobenius-Euler numbers.

It is readily seen that
$$\lim_{\lambda \to 0} h_{n,\lambda}(x|u) = H_n(x|u), (n \geq 0).$$

For $s \in \mathbb{Z}$, the polylogarithm function is defined by a power series in $z$ as
$$\text{Li}_s(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s}, (|z| < 1), \text{(see, [4, 9])}. \quad (1.8)$$

It is notice that
$$\text{Li}_1(z) = \sum_{j=1}^{\infty} \frac{z^j}{j} = -\log(1 - z). \quad (1.9)$$
For $\lambda \in \mathbb{R}$, Kim-Kim [19] defined the degenerate version of the logarithm function, denoted by $\log_{\lambda}(1 + z)$ as follows:

$$\log_{\lambda}(1 + z) = \sum_{j=1}^{\infty} \lambda^{j-1}(1)_{j, 1/\lambda} \frac{z^j}{j!}, \text{ (see, [18])}$$

(1.10)

being the inverse of the degenerate version of the exponential function $e_{\lambda}(z)$ as has been shown below

$$e_{\lambda}(\log_{\lambda}(z)) = \log_{\lambda}(e_{\lambda}(z)) = z.$$ 

It is noteworthy to mention that

$$\lim_{\lambda \to 0} \log_{\lambda}(1 + z) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{j!} = \log(1 + z).$$

The degenerate polylogarithm function [19] is defined by Kim-Kim to be

$$l_{k,\lambda}(z) = \sum_{j=0}^{\infty} \frac{(-\lambda)^{j-1}(1)_{j, 1/\lambda}}{(j-1)! j^k} z^n, \quad (k \in \mathbb{Z}, \ |z| < 1).$$

(1.11)

It is clear that

$$\lim_{\lambda \to 0} l_{k,\lambda}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j^k} = \text{Li}_k(z), \text{ (see [4, 9]).}$$

From (1.10) and (1.11), we get

$$l_{1,\lambda}(z) = \sum_{j=1}^{\infty} \frac{(-\lambda)^{j-1}(1)_{j, 1/\lambda}}{j!} z^j = -\log_{\lambda}(1 - z).$$

(1.8)

Very recently, Kim-Kim [19] introduced the new type degenerate version of the Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows

$$\frac{l_{k,\lambda}(1 - e_{\lambda}(-z))}{1 - e_{\lambda}(-z)} e_{\lambda}^n(z) = \sum_{j=0}^{\infty} \beta_{j,\lambda}(n) \frac{z^j}{j!}.$$ 

(1.12)

In the special case $x = 0$, $\beta_{j,\lambda}^{(k)} = \beta_{j,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

It is well known that the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n, l) x^l, \text{ (see [24, 25])},$$

(1.13)

where $(x)_0 = 1$, and $(x)_n = x(x - 1) \cdots (x - n + 1), (n \geq 1)$. From (1.13), it is easily to see that

$$\frac{1}{k!} (\log(1 + t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \text{ (see [12, 13, 18]).}$$

(1.14)

In the inverse expression to (1.14), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n, l) (x)_l, \text{ (see [10, 12, 22])}.$$ 

(1.15)

From (1.15), it is easily to see that

$$\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \text{ (see [22, 24, 25]).}$$

(1.16)
For \( n \geq 0 \), the degenerate Stirling numbers of the second kind \([7, 8, 17]\) are defined by

\[
\frac{1}{n!}(e_\lambda(t) - 1)^n = \sum_{l=0}^{\infty} S_{l,\lambda}(l, n) \frac{t^l}{l!}, \quad (n \geq 0). \tag{1.17}
\]

In this paper, we construct the degenerate poly-Frobenius-Euler polynomials and numbers, called the new type of poly-Frobenius-Euler polynomials and numbers by using the degenerate polylogarithm function and derive several properties on the degenerate poly-Frobenius-Euler polynomials and numbers.

2. New type of degenerate poly-Frobenius-Euler polynomials

Let \( \lambda, u \in \mathbb{C} \) with \( u \neq 1 \) and \( k \in \mathbb{Z} \), by using the degenerate polylogarithm function, we define the new type of degenerate poly-Frobenius-Euler polynomials as follows

\[
\frac{(1 - u)}{t(e_\lambda(t) - u)} e_\lambda^n(t) = \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x; u) \frac{t^n}{n!}. \tag{2.1}
\]

In the special case, \( x = 0 \), \( H_{n,\lambda}^{(k)}(u) = H_{n,\lambda}^{(k)}(0; u) \) are called the new type of degenerate Frobenius-Euler polynomials.

For \( k = 1 \) in (2.1), we get

\[
\frac{1 - u}{e_\lambda(t) - u} e_\lambda^n(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x; u) \frac{t^n}{n!}, \quad \text{(see [15])} \tag{2.2}
\]

where \( h_{n,\lambda}(x; u) \) are called the degenerate Frobenius-Euler polynomials.

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
H_{n,\lambda}^{(k)}(x; u) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(u)(x)_m. 
\]

**Proof.** From (2.1), we have

\[
\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x; u) \frac{t^n}{n!} = \left( \frac{(1 - u)}{t(e_\lambda(t) - u)} \right) e_\lambda^n(t)
\]

\[
= \sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(u) \frac{t^n}{n!} \sum_{m=0}^{\infty} (x)_m \frac{t^m}{m!}
\]

\[
L.H.S. = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} H_{n-m,\lambda}^{(k)}(u)(x)_m \frac{t^n}{n!}. \tag{2.3}
\]

Therefore, by (2.1) and (2.3), we require at the desired result. \( \square \)

**Theorem 2.2.** For \( k \in \mathbb{Z} \) and \( n \geq 0 \), we have

\[
H_{n,\lambda}^{(k)}(x; u) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{(1)(m+1,1/\lambda)(-\lambda)^m(-1)^{1-m}S_{l,\lambda}(l+1,m+1)}{(m+1)^{k+1}l+1} h_{n-l,\lambda}(x; u).
\]
Proof. It is proved by using (1.7), (1.11) and (2.1) that

\[
\sum_{n=0}^{\infty} H_{n,\lambda}(x; u) \frac{t^n}{n!} = \left( \frac{(1-u) e_{\lambda}(t)}{t(e_{\lambda}(t) - u)} \right) L_{k,\lambda}(1 - e_{\lambda}(-t))
\]

\[
= \left( \frac{(1-u)e_{\lambda}(t)}{t(e_{\lambda}(t) - u)} \right) \sum_{m=1}^{\infty} \frac{(1)_{m,1,\lambda}(-\lambda)^{m-1}}{(m-1)! m^k} (1 - e_{\lambda}(-t))^m
\]

\[
= \left( \frac{(1-u)e_{\lambda}(t)}{t(e_{\lambda}(t) - u)} \right) \sum_{m=0}^{\infty} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m}{(m+1)! m^k-1} (1 - e_{\lambda}(-t))^{m+1}
\]

\[
= \left( \frac{(1-u)e_{\lambda}(t)}{t(e_{\lambda}(t) - u)} \right) \sum_{m=0}^{\infty} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m}{(m+1)!} \sum_{l=m+1}^{\infty} S_{2,\lambda}(l, m+1) (-1)^{l-m-1} \frac{t^l}{l!}
\]

\[
= \left( \frac{(1-u)e_{\lambda}(t)}{e_{\lambda}(t) - u} \right) \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(x; u)
\]

L.H.S. = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(x; u) \right) \frac{t^n}{n!} \quad (2.4)

By comparing the coefficients of \( \frac{x^n}{m^n} \) on both sides, we get the result. \( \square \)

Corollary 2.1. For \( k \in \mathbb{Z} \) and \( n \geq 0 \), we have

\[
H_{n,\lambda}^{(k)}(u) = \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(u) \right)
\]

Corollary 2.2. For \( n \geq 0 \), we have

\[
H_{n,\lambda}(x; u) = \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(x; u) \right)
\]

Corollary 2.3. For \( n \geq 0 \), we have

\[
E_{n,\lambda}(x) = \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(x) \right)
\]

Moreover,

\[
\sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(x) \right) = 0
\]

Theorem 2.3. For \( n \geq 0 \), we have

\[
\sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{(1)_{m+1,1,\lambda}(-\lambda)^m(-1)^{l-m} S_{2,\lambda}(l+1, m+1)}{(m+1)!} \frac{S_{2,\lambda}(l, m+1) t^l}{l+1} h_{n-l,\lambda}(x) \right)
\]

\[
= \frac{1}{1-u} \left[ \sum_{m=0}^{n} \frac{n}{m} H_{n-m,\lambda}^{(k)}(x; u) (1)_{m,\lambda} - u H_{n,\lambda}^{(k)}(x; u) \right].
\]
Proof. From (2.1), we have
\[
\frac{(1-u)l_{k,\lambda}(1-e_{\lambda}(-t))}{l} e^{\lambda}_x(t) = e^{\lambda}_x(t) \sum_{n=0}^{\infty} H_{n,\lambda}(x; u) \frac{t^n}{n!} - u \sum_{n=0}^{\infty} H_{n,\lambda}(x; u) \frac{t^n}{n!}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} H_{n,\lambda}(x; u) \frac{t^n}{n!} \right) \frac{(1)_{m,1/\lambda}(-\lambda)^{m-1}}{(m-1)!m^k} \left( 1 - e^{\lambda}_x(-t) \right)^m
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \sum_{\lambda=1}^{m} \binom{m+1,1/\lambda(-\lambda)^m}{m+1,k-1} \frac{S_{2,\lambda}(l+1,m+1) t^l}{l+1} \right) \frac{t^n}{n!}.
\]

On the other hand,
\[
\frac{(1-u)l_{k,\lambda}(1-e_{\lambda}(-t))}{l} e^{\lambda}_x(t)
\]
\[
= \left( \sum_{n=0}^{\infty} \frac{(x)_{m,\lambda} t^n}{n!} \right) \frac{(1-u) t}{l} \left( \sum_{\lambda=1}^{\infty} \frac{(1)_{m+1,1/\lambda(-\lambda)^m}}{(m+1)^{k-1}m!} \left( 1 - e^{\lambda}_x(-t) \right)^m \right)
\]
\[
= (1-u) \left( \sum_{n=0}^{\infty} \frac{(x)_{m,\lambda} t^n}{n!} \right) \left( \sum_{\lambda=1}^{\infty} \sum_{m=0}^{l} \frac{(1)_{m+1,1/\lambda(-\lambda)^m}}{(m+1)^{k-1}} \frac{S_{2,\lambda}(l+1,m+1) t^l}{l+1} \right) \left( x_{n-l,\lambda} \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we get the result.

\[\square\]

Theorem 2.4. For \( n \geq 0 \), we have
\[
H_{n,\lambda}(u) = \frac{(1-u)}{x(e^{\lambda}_x(u) - u)} \int_0^x \frac{e^{\lambda}_x(-t)}{1-e^{\lambda}_x(-t)} \int_0^t \frac{e^{\lambda}_x(-t)}{1-e^{\lambda}_x(-t)} \cdots \int_0^t \frac{e^{\lambda}_x(-t)}{1-e^{\lambda}_x(-t)} dt \cdot \cdot \cdot dt.
\]

\[\text{(k-2)-times}\]

Proof. Using (1.11), we first consider the following expression
\[
\frac{d}{dx} l_{k,\lambda}(1-e_{\lambda}(-x)) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(1)_{n,1/\lambda}}{(n+1)!n^k} (1 - e^{\lambda}_x(-x))^n
\]
\[
= \frac{1}{1 - e^{\lambda}_x(-x)} l_{k-1,\lambda}(1-e_{\lambda}(-x)).
\]

From (2.7), \( k \geq 2 \), we have
\[
\sum_{n=0}^{\infty} \frac{H_{n,\lambda}(u) x^n}{n!} = \frac{(1-u)}{x(e^{\lambda}_x(u) - u)} l_{k,\lambda}(1-e_{\lambda}(-x)).
\]
\[
= \frac{(1-u)}{x(e^{\lambda}_x(u) - u)} \int_0^x \frac{e^{\lambda}_x(-t)}{1-e^{\lambda}_x(-t)} \int_0^t \frac{e^{\lambda}_x(-t)}{1-e^{\lambda}_x(-t)} \cdots \int_0^t \frac{e^{\lambda}_x(-t)}{1-e^{\lambda}_x(-t)} dt \cdot \cdot \cdot dt.
\]

\[\text{(k-2)-times}\]
By (2.9), we obtain at the desired result. Thus, we complete the proof.

**Theorem 2.5.** Let \( n \geq 0 \). Then

\[
H_{n,\lambda}(u) = \sum_{m=0}^{\infty} \left( \frac{n}{m} \right) \frac{B_{m,\lambda}(1 - \lambda)}{m + 1} h_{n-m,\lambda}(u) = \sum_{m=0}^{n} \left( \frac{n}{m} \right) \frac{(1 - n - m) B_{n-m,\lambda}(1 - \lambda)}{n - m + 1} h_{m,\lambda}(u).
\]

**Proof.** By using (1.17) and Theorem 2.4, we get

\[
\sum_{n=0}^{\infty} H_{n,\lambda}(u) \frac{x^n}{n!} = \frac{1 - u}{x e^{\lambda(x - u)}} \int_0^x \frac{-e^{-\lambda(t)}}{e^{t}} \frac{e^{-\lambda(-t)}}{n!} dt
\]

\[
= \frac{1 - u}{x e^{\lambda(x - u)}} \int_0^x \sum_{n=0}^{\infty} \frac{B_{n,\lambda}(1 - \lambda)}{m + 1} \frac{(-t)^n}{n!} dt
\]

\[
= \frac{1 - u}{x e^{\lambda(x - u)}} \sum_{m=0}^{\infty} (-1)^m \frac{B_{m,\lambda}(1 - \lambda) x^m}{m + 1} \frac{1}{n!}.
\]

Therefore, by (2.10), we get the following theorem.

**Theorem 2.6.** Let \( k \in \mathbb{Z} \) and \( n \geq 0 \), we have

\[
H_{n,\lambda}^{(k)}(u) = \sum_{m=0}^{\infty} \left( \frac{n}{m} \right) \sum_{m_1 + m_2 + \cdots + m_{k-1} = m} \left( m_1, m_2, \ldots, m_k \right) h_{n-m,\lambda}(u)
\]

\[
\times \frac{\beta_{m_1,\lambda}(\lambda - 1) \beta_{m_2,\lambda}(\lambda - 1) \cdots \beta_{m_{k-1},\lambda}(\lambda - 1)}{m_1 + 1 \cdot m_2 + m_2 + 1 \cdots m_{k-1} + n_{k-1} + 1}.
\]

**Proof.** In general, from (2.9), we note that

\[
\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(u) \frac{x^n}{n!} = \frac{1 - u}{x e^{\lambda(x - u)}} \int_0^x \frac{e^{-\lambda(-t)}}{1 - e^{\lambda(-t)}} \int_0^t \frac{e^{-\lambda(-t)}}{1 - e^{\lambda(-t)}} \cdots \int_0^{t_1} \frac{e^{-\lambda(-t)}}{1 - e^{\lambda(-t)}} dt \cdots dt
\]

\[
= \frac{1 - u}{x e^{\lambda(x - u)}} \sum_{m=0}^{\infty} \sum_{m_1 + m_2 + \cdots + m_{k-1} = m} \left( m_1, m_2, \ldots, m_k \right) \frac{\beta_{m_1,\lambda}(\lambda - 1) \beta_{m_2,\lambda}(\lambda - 1) \cdots \beta_{m_{k-1},\lambda}(\lambda - 1)}{m_1 + 1 \cdot m_2 + m_2 + 1 \cdots m_{k-1} + n_{k-1} + 1} x^m
\]

\[
L.H.S = \sum_{n=0}^{\infty} \left( \frac{n}{m} \right) \sum_{m_1 + m_2 + \cdots + m_{k-1} = m} \left( m_1, m_2, \ldots, m_k \right) h_{n-m,\lambda}(u)
\]

\[
\times \frac{\beta_{m_1,\lambda}(\lambda - 1) \beta_{m_2,\lambda}(\lambda - 1) \cdots \beta_{m_{k-1},\lambda}(\lambda - 1)}{m_1 + 1 \cdot m_2 + m_2 + 1 \cdots m_{k-1} + n_{k-1} + 1} \frac{x^n}{n!}.
\]

Therefore, by comparing the coefficients of \( t^n \) on both sides, we obtain the result.
Theorem 2.7. For \( n \geq 0 \), we have
\[
H^{(k)}_{n,\lambda}(x + y; u) = \sum_{m=0}^{\infty} \binom{n}{m} H^{(k)}_{n-m,\lambda}(x; u)(y)_{m,\lambda}.
\]

Proof. From (2.1), we have
\[
\sum_{n=0}^{\infty} H^{(k)}_{n,\lambda}(x + y; u) \frac{t^n}{n!} = \frac{(1-u)k_{\lambda}(1-e_{\lambda}(-t))}{t(e_{\lambda}(t) - u)} e_{\lambda}^y(t)
\]
\[
= \left( \sum_{n=0}^{\infty} H^{(k)}_{n,\lambda}(x; u) \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} (y)_{m,\lambda} \frac{t^m}{m!} \right)
\]
\[
L.H.S = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} H^{(k)}_{n-m,\lambda}(x; u)(y)_{m,\lambda} \right) \frac{t^n}{n!}.
\]
Comparing the coefficients on both sides of (2.12), we get the result. \( \square \)

Theorem 2.8. For \( n \geq 0 \), we have
\[
H^{(k)}_{n,\lambda}(x + 1; u) = \sum_{m=0}^{n} \binom{n}{m} H^{(k)}_{n-m,\lambda}(x; u)(1)_{m,\lambda}.
\]

Proof. By (2.1), we observe that
\[
\sum_{n=0}^{\infty} \left[ H^{(k)}_{n,\lambda}(x + 1; u) - H^{(k)}_{n,\lambda}(x; u) \right] \frac{t^n}{n!} = \frac{(1-u)k_{\lambda}(1-e_{\lambda}(-t))}{t(e_{\lambda}(t) - u)} e_{\lambda}^x(t) [e_{\lambda}(t) - 1]
\]
\[
L.H.S = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} H^{(k)}_{n-m,\lambda}(x; u)(1)_{m,\lambda} \frac{t^n}{n!} - \sum_{n=0}^{\infty} H^{(k)}_{n,\lambda}(x; u) \frac{t^n}{n!}.
\]
Comparing the coefficients of \( t^n \) on both sides, we obtain the result. \( \square \)

Theorem 2.9. For \( n \geq 0 \), we have
\[
H^{(k)}_{n,\lambda}(x; u) = \sum_{m=0}^{n} \sum_{q=0}^{m} \binom{n}{m} (x)_q S^{(2)}_{\lambda}(m, q) H^{(k)}_{n-m,\lambda}(u).
\]

Proof. From (2.1), we have
\[
\sum_{n=0}^{\infty} H^{(k)}_{n,\lambda}(x; u) \frac{t^n}{n!} = \frac{(1-u)k_{\lambda}(1-e_{\lambda}(-x))}{t(e_{\lambda}(t) - u)} e_{\lambda}^x(t)
\]
\[
= \frac{(1-u)k_{\lambda}(1-e_{\lambda}(-x))}{t(e_{\lambda}(t) - u)} [e_{\lambda}(t) - 1 + 1]^x
\]
\[
= \left( \frac{(1-u)k_{\lambda}(1-e_{\lambda}(-x))}{t(e_{\lambda}(t) - u)} \right) \left( \sum_{q=0}^{\infty} (x)_q \frac{t^q}{q!} \right)
\]
\[
L.H.S = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \sum_{q=0}^{m} \binom{n}{m} (x)_q S^{(2)}_{\lambda}(m, q) H^{(k)}_{n-m,\lambda}(u) \right) \frac{t^n}{n!}.
\]
By comparing the coefficients of \( t^n \) on both sides, we get the result. \( \square \)

Theorem 2.10. For \( n \geq 0 \), we have
\[
H^{(k)}_{n,\lambda}(x + \alpha; u) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} x^m m! S_{\lambda}(l + \alpha, m + \alpha) H^{(k)}_{n-l,\lambda}(u).
\]
Proof. Replacing $x$ by $x + \alpha$ in (2.1), we have
\[
\sum_{n=0}^{\infty} H_{n,\lambda}^{(k)}(x + \alpha; u) \frac{t^n}{n!} = \left( \frac{1 - u}{t(e_{\lambda}(t) - u)} \right) e_{\lambda}^{x+\alpha}(t)
\]
\[
= \left( \frac{1 - u}{t(e_{\lambda}(t) - u)} \right) e_{\lambda}^{x}(t) \left( \sum_{m=0}^{\infty} x^m (e_{\lambda}(t) - 1)^m \right)
\]
\[
= \left( \frac{1 - u}{t(e_{\lambda}(t) - u)} \right) e_{\lambda}^{x}(t) \sum_{m=0}^{\infty} x^m m! \sum_{l=m}^{\infty} S_{2,\lambda}(l, m) \frac{t^l}{l!}
\]
\[
L.H.S = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \sum_{m=0}^{l} \left( \begin{array}{c} n \\ l \end{array} \right) x^m m! S_{2,\lambda}(l + \alpha, m + \alpha) H_{n,\lambda}^{(k)}(u) \right) \frac{t^n}{n!}.
\] (2.15)
Therefore, by (2.1) and (2.15), we obtain the result. \hfill \Box

4. Conclusions

Motivated by the definition of the degenerate poly-Bernoulli polynomials introduced by Kim-Kim [19], in the present paper, we have considered a class of new generating function for the degenerate Frobenius-Euler polynomials, called the new type of degenerate poly-Frobenius-Euler polynomials, by means of the degenerate polylogarithm function. Then, we have derived some useful relations and properties. We have showed that the new type of degenerate poly-Frobenius-Euler polynomials equal a linear combination of the degenerate Frobenius-Euler polynomials and Stirling numbers of the first and second kind. In a special case, we have given a relation between the new type of degenerate Frobenius-Euler polynomials and Bernoulli polynomials of order $n$.

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