SPECTRUM OF SEMISIMPLE LOCALLY SYMMETRIC SPACES
AND ADMISSIBILITY OF SPHERICAL REPRESENTATIONS

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Abstract. We consider compact locally symmetric spaces \( \Gamma \backslash G/H \) where \( G/H \) is a non-compact semisimple symmetric space and \( \Gamma \) is a discrete subgroup of \( G \). We discuss some features of the joint spectrum of the (commutative) algebra \( D(G/H) \) of invariant differential operators acting, as unbounded operators, on the Hilbert space \( L^2(\Gamma \backslash G/H) \) of square integrable complex functions on \( \Gamma \backslash G/H \). In the case of the Lorentzian symmetric space \( SO_0(2,2n)/SO_0(1,2n) \), the representation theoretic spectrum is described explicitly. The strategy is to consider connected reductive Lie groups \( L \) acting transitively and co-compactly on \( G/H \), a cocompact lattice \( \Gamma \subset L \), and study the spectrum of the algebra \( D(L/L \cap H) \) on \( L^2(\Gamma \backslash L/L \cap H) \). Though the group \( G \) does not act on \( L^2(\Gamma \backslash G/H) \), we explain how (not necessarily unitary) \( G \)-representations enter into the spectral decomposition of \( D(G/H) \) on \( L^2(\Gamma \backslash G/H) \) and why one should expect a continuous contribution to the spectrum in some cases. As a byproduct, we obtain a result on the \( L \)-admissibility of \( G \)-representations. These notes contain the statements of the main results, the proofs and the details will appear elsewhere.

1. Introduction.

Let \( G \) be a connected non-compact semisimple real Lie group and \( H \) a connected closed subgroup of \( G \) with complexified Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \) respectively. Suppose that \( G/H \) is a semisimple symmetric space of the non-compact type with respect to an involution \( \sigma \) and consider the associated decomposition

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{where} \quad \mathfrak{q} := \{ X \in \mathfrak{g} \mid \sigma(X) = -X \}
\]

Fix an invariant measure on \( G/H \) and consider the Hilbert space \( L^2(G/H) \) of square integrable complex functions on \( G/H \). The action by left translations of \( G \) on \( L^2(G/H) \) defines a unitary representation whose decomposition is known as the Plancherel formula for symmetric spaces \([2][3][7][20]\):

\[
L^2(G/H) \simeq \int_{\hat{G}_H} \left( W^H_{\rho, -\infty} \right)_0 \otimes W_{\rho} d\mu(\rho)
\]

\[
D(G/H) \leftarrow \mathcal{Z}(\mathfrak{g})
\]

where \( W_{\rho} \) is a unitary irreducible representation of \( G \), \( W_{\overline{\rho}} \) the dual representation, \( W^H_{\overline{\rho}, -\infty} \) the space of \( H \)-invariants distribution vectors, \( \left( W^H_{\overline{\rho}, -\infty} \right)_0 \) is a finite dimensional Hilbert space embedded in \( W^H_{\overline{\rho}, -\infty} \) whose dimension is the multiplicity of \( W_{\rho} \) in \( L^2(G/H) \), \( \hat{G}_H \) is the \( H \)-spherical dual \( \{ \rho \in \hat{G} \mid W^H_{\overline{\rho}, -\infty} \neq \{0\} \} \) and \( d\mu \) the Plancherel measure. The (commutative) algebra \( D(G/H) \) of \( G \)-invariant differential operators on \( G/H \) acts on \( \left( W^H_{\overline{\rho}, -\infty} \right)_0 \) and the center \( \mathcal{Z}(\mathfrak{g}) \) of the enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) of \( \mathfrak{g} \) acts via infinitesimal character on \( W_{\rho} \). Except for exceptional cases where \( G \)
is $E_6$, $E_7$ and $E_8$ which we will not consider here, there is a surjective map $\mathcal{Z}(g) \to D(G/H)$ [11]. In other words, harmonic analysis on $G/H$ is closely related to the spectrum of $D(G/H)$ on $L^2(G/H)$:

\[
\text{harmonic analysis on } G/H \iff \text{Spec}_{L^2(G/H)}(D(G/H))
\]

A compact locally symmetric space is a smooth compact manifold of the form $\Gamma \backslash G/H$ where $\Gamma$ is a discrete subgroup of $G$, in particular the natural map $G \to \Gamma \backslash G/H$ is smooth for the quotient topology. As above one may ask about the analysis on $\Gamma \backslash G/H$. Let us consider the following group case example.

**Example 1.1.** $G = G_1 \times G_1$, $H = \Delta(G_1 \times G_1)$ and $\Gamma = \Gamma_1 \times \{e\}$, where $G_1$ is a connected non-compact semisimple real Lie group, $\sigma$ is the permutation $G_1 \times G_1 \to G_1 \times G_1$, $(a, b) \mapsto (b, a)$ and $\Gamma_1$ a co-compact torsion free discrete subgroup of $G_1$. Then $L^2(\Gamma \backslash G/H) \simeq L^2(\Gamma_1 \backslash G_1) \simeq \bigoplus_{\pi \in \hat{G}_1} V_{\pi, -\infty}^{\Gamma_1} \otimes V_{\pi}$

where $\dim(V_{\pi, -\infty}^{\Gamma_1}) < \infty$, $\dim(V_{\pi}) = \infty$ (except for the trivial representation $\pi$) and the set $\{\pi \in \hat{G}_1 \mid V_{\pi, -\infty}^{\Gamma_1} \neq \{0\}\}$ is discrete. Moreover $D(G/H) \simeq \mathcal{Z}(g)$ acts scalarly on $\pi$ by infinitesimal characters. In particular, the spectrum of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ is discrete, consists of eigencharacters only, and (almost) all eigenspaces are infinite dimensional.

Already this example shows that analysis on $\Gamma \backslash G/H$ is often related to automorphic forms:

analysis on $\Gamma \backslash G/H \iff$ automorphic forms

**Problem 1.2.** Describe the spectrum $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$ of $D(G/H)$ on $L^2(\Gamma \backslash G/H)$ in general.

**Issues:**

1. When $H$ is not compact then a discrete subgroup $\Gamma \subset G$ such that $\Gamma \backslash G/H$ is a smooth compact manifold need not exist as it is illustrated by the Calabi-Markus phenomenon for $SO_0(1, n+1)/SO_0(1, n)$ [6]. Given such a $\Gamma$, the space $\Gamma \backslash G/H$ will be called a compact Clifford-Klein form of $G/H$. One can ask for deformations of $\Gamma$ such that the new quotient is still a Clifford-Klein form (see [10]).

2. The group $G$ does not act on $\Gamma \backslash G/H$, and it is not clear how representations of $G$ are involved in $\text{Spec}_{L^2(\Gamma \backslash G/H)}(D(G/H))$.

3. When $H$ is not compact, the coset $\Gamma \backslash G$ is not compact and it is not clear if $L^2(\Gamma \backslash G)$ and $L^2(\Gamma \backslash G/H)$ are related.

4. $D(G/H)$ acts on $C^\infty(\Gamma \backslash G/H)$ and $C^\infty(\Gamma \backslash G/H)$ is dense in $L^2(\Gamma \backslash G/H)$, so $D(G/H)$ acts via unbounded operators on $L^2(\Gamma \backslash G/H)$. Some care is required with the domains of the operators.

Problem [12] is related with a program developed by Kobayashi to study hidden symmetries (see [14], [15] and references therein). In the following we shall describe and state some of the main results we prove in [21].
2. Existence of compact Clifford-Klein forms and spherical triples.

When $H$ is compact, the existence of a discrete subgroup $\Gamma$ of $G$ such that $\Gamma \backslash G/H$ is a smooth compact manifold is equivalent to the existence of torsion free uniform lattices in $G$. It turns out that torsion free uniform lattices always exist in $G$ by a result of Borel and Harish-Chandra [5].

For $H$ noncompact, this gives a sufficient condition for the existence of $\Gamma$, employed extensively by Kobayashi starting with [18]:

existence of a closed connected Lie group $L \subset G$ such that

| (i) $L$ is reductively embedded in $G$ |
| (ii) $L$ acts transitively on $G/H$ |
| (iii) $L \cap H$ is compact |

⇒ existence of compact Clifford-Klein forms for $G/H$

Indeed, if $\Gamma$ is a co-compact lattice in $L$ then $\Gamma \backslash G/H$ is a compact locally symmetric space. Triples $(G, H, L)$ satisfying (i) and (ii) above were classified, when $G$ is simple, by Onishik in the late 1960’s [23] (see also [19]). We will refer to triples $(G, H, L)$ satisfying (i)(ii)(iii) as transitive triples.

In fact, the above implication remains true, if we weaken transitively in (ii) to cocompactly. However, we are able to show that this does not change anything: cocompactly implies transitively [21].

Next fix $P = MAN$ the Langlands decomposition of a minimal parabolic subgroup in $G$ and consider a principal series representation $\text{Ind}^G_P \tau \otimes e^\nu \otimes 1$ of $G$, where $\tau \in \widehat{M}$ is a finite dimensional irreducible representation of $M$ and $\nu$ is a linear form on the complexified Lie algebra of $A$. Then van den Ban proved in [1] that the space of $H$-invariants distributions vectors in $\text{Ind}^G_P \tau \otimes e^\nu \otimes 1$ is finite dimensional:

$$\dim \left( \text{Ind}^G_P \tau \otimes e^\nu \otimes 1 \right)_{H \cap H}^L < \infty$$

Now pick a transitive triple $(G, H, L)$ and let $P_L = M_L A_L N_L$ be the Langlands decomposition of a minimal parabolic in $L$. Let $\tau_L \in \widehat{M}_L$ be a finite dimensional irreducible representation of $M_L$ and $\nu_L$ a linear form on the complexified Lie algebra of $A_L$. Observe that $L$ is not $\sigma$-stable, $L/L \cap H$ need not be a symmetric space and the space of $L \cap H$-invariants distributions $\left( \text{Ind}^L_P \tau_L \otimes e^{\nu_L} \otimes 1 \right)_{L \cap H \cap H}$ could be infinite dimensional as it is the case for the triple $(G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})$. In fact we obtain the following criterion for the space of $L \cap H$-invariants to be finite dimensional.

**Proposition 2.1.** For any $\tau_L \in \widehat{M}_L$ a finite dimensional irreducible representation of $M_L$ and $\nu_L$ a linear form on the complexified Lie algebra of $A_L$, one has

$$\dim \left( \text{Ind}^L_P \tau_L \otimes e^{\nu_L} \otimes 1 \right)_{L \cap H \cap H}^L < \infty \Leftrightarrow P_L \text{ acts transitively on } L/L \cap H.$$
Example 2.2.

(1) \((G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\})\) is not spherical.

(2) \((G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1)\) is spherical (where \(K_1\) is maximal compact in \(G_1\)).

(3) \((SO_0(2, 2n), SO_0(1, 2n), U(1, n))\) is spherical (\(n \geq 2\)).

(4) \((SO_0(4, 3), SO_0(4, 1) \times SO(2), G_2(2))\) is not spherical.

3. L-admissibility.

Suppose \(G' \subset G\) is a connected closed reductive subgroup and \(\pi\) is an irreducible unitary representation of \(G\). From a general result of Mautner and Teleman, the restriction of \(\pi\) to \(G'\) can be decomposed as the direct integral sum of irreducible unitary representations:

\[
\pi|_{G'} \simeq \int_{\hat{G}} M_\pi(\tau) \hat{\otimes} V_\tau \, d\mu(\tau)
\]

where \(M_\pi(\tau)\) is the multiplicity (Hilbert) space.

Definition 3.1. \(\pi\) is \(G'\)-admissible if \(\pi|_{G'}\) decomposes discretely and \(\dim M_\pi(\tau) < \infty\).

Example 3.2.

(1) \(\hat{G} \ni \pi\) is \(K\)-admissible (Harish-Chandra admissibility theorem [8]).

(2) \(G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) and \(H = \Delta(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))\). Let \(\hat{G}_H \ni \pi = \pi_1 \otimes \pi_1\) where \(\pi_1\) is a holomorphic discrete series representation of \(SL(2, \mathbb{R})\). From [24], \(\pi|_H\) has continuous spectrum (but multiplicities are finite). Therefore \(\pi\) is not \(H\)-admissible.

There is a criterion for \(H\)-admissibility in terms of associated varieties due to Kobayashi [17]. We prove the following admissibility result for spherical representations associated with triples \((G, H, L)\).

Theorem 3.3. Suppose \((G, H, L)\) is a spherical triple. If \(\pi \in \hat{G}_H\) then \(\pi\) is \(L\)-admissible. Moreover, if \(\pi\) is not trivial then there are infinitely many summands in \(\pi|_L\) all having \(L \cap H\)-invariants.

4. Embedding of Casimir operators.

Fix a transitive triple \((G, H, L)\). Write \(\mathcal{U}(\mathfrak{g})^H\) for the \(H\)-invariant elements in the enveloping algebra of \(\mathfrak{g}\) and \(\mathcal{U}(\mathfrak{g})\mathfrak{h}\) the left \(\mathcal{U}(\mathfrak{g})\)-ideal generated by \(\mathfrak{h}\). The algebra \(D(G/H)\) of \(G\)-invariant differential operators on \(G/H\) is isomorphic to the following quotient algebra:

\[
D(G/H) \simeq \mathcal{U}(\mathfrak{g})^H / \mathcal{U}(\mathfrak{g})\mathfrak{h} \cap \mathcal{U}(\mathfrak{g})^H
\]

It is known that since \(G/H\) is a symmetric space, \(D(G/H)\) is commutative (see [10] and references therein).

By definition, if \((G, H, L)\) is a transitive triple then one has a diffeomorphism of homogeneous spaces

\[
G/H \simeq L/L \cap H
\]

and an isomorphism of algebra

\[
\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{g})\mathfrak{h}} \mathcal{U}(\mathfrak{l}).
\]
In particular, we get an embedding of algebras

\[ \iota : D(G/H) \rightarrow D(L/L \cap H) \].

The algebra \( D(L/L \cap H) \) need not be commutative, though it is for many transitive triples (see [12]). Of special interest in \( D(G/H) \) (resp. \( D(L/L \cap H) \), \( D(L \cap K/L \cap H) \)) is the Casimir element \( \Omega_G \) (resp. \( \Omega_L \), \( \Omega_{L \cap K} \)). The algebras \( D(L/L \cap H) \) and \( D(L \cap K/L \cap H) \) are defined in an analogous way to \( D(G/H) \). Observe that when \( G/H \) has rank one then \( D(G/H) \) is generated by the Casimir \( \Omega_G \).

**Proposition 4.1.** For each transitive triple \( (G, H, L) \) with \( G/H \) irreducible, the embedding \( \iota(\Omega_G) \) is computed explicitly in terms of the decomposition of \( 1/1 \cap h \) into \( \text{Ad}(L \cap H) \)-irreducibles.

In particular, based on the restriction to \( l \) and \( l \cap h \) of the Killing form of \( g \), we have:

**Example 4.2.**

1. \((G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times \{e\}): \iota(\Omega_G) = 2\Omega_L.
2. \((G_1 \times G_1, \Delta(G_1 \times G_1), G_1 \times K_1): \iota(\Omega_G) = 2\Omega_L - \Omega_{L \cap K}.
3. \((SO_0(2, 2n), SO_0(1, 2n), U(1, n)): \iota(\Omega_G) = 2\Omega_L - \Omega_{L\cap K}.
4. \((SO_0(4, 3), SO_0(4, 1) \times SO(2), G_{2(2)}): \iota(\Omega_G) = 3\Omega_L - \frac{3}{2}\Omega_{L \cap K} + 2\Omega_{U(1)\cap q}

In case (3), Schlichtkrull, Trapa and Vogan obtained a similar formula for \( O(2n)/U(n) \) [25].

**Remark 4.3.** By inspection, we observe that \( \iota(\Omega_G) \) involves ‘non-compact terms’ terms from \( l \cap q \cap s \) only when \( (G, H, L) \) is not spherical, except in the first group case (1).

5. **On the spectrum \( \text{Spec}_{L^2(\Gamma \setminus G/H)}(D(G/H)) \).**

Fix a transitive triple \( (G, H, L) \). We have the following sequence of isomorphisms:

\[ L^2(\Gamma \setminus G/H) \simeq L^2(\Gamma \setminus L/L \cap H) \simeq L^2(\Gamma \setminus L)^{L \cap H} \simeq \bigoplus_{\pi \in L} V^T_{\pi, -\infty} \otimes V^{L \cap H}_{\pi, -\infty} \] (Hilbert sum)

\[ \bigcup \bigcup D \in D(G/H) \]

\[ C^\infty(\Gamma \setminus G/H) \simeq C^\infty(\Gamma \setminus L/L \cap H) \simeq C^\infty(\Gamma \setminus L)^{L \cap H} \simeq \bigoplus_{\pi \in L} V^T_{\pi, -\infty} \otimes V^{L \cap H}_{\pi, -\infty} \] (closure of algebraic direct sum)

\[ \iota(D) \in D(L/L \cap H) \]

Suppose now that \( (G, H, L) \) is spherical, i.e \( \dim (\text{Ind}_{K_L}^L \tau_L \otimes e^{\nu_L} \otimes 1)^{L \cap H} < \infty \), for any \( \tau_L \in \hat{M}_L \) a finite dimensional irreducible representation of \( M_L \) and \( \nu_L \) a linear form on the complexified Lie algebra of \( A_L \). Then by Proposition 2.1 and Casselman embedding Theorem, we deduce that

**Proposition 5.1.** For any irreducible unitary representation \( (\pi, V_\pi) \) of \( L \), one has

\[ \dim V^{L \cap H}_{\pi, \infty} < \infty \]
This means that the spectrum of $D(G/H)$ on $C^\infty(\Gamma\backslash G/H)$ may be reduced to the diagonalization of the operator $\pi(\iota(D))$ on the (infinitely many) finite dimensional blocks $V^{L\cap H}_{\pi,\infty}$. Then we obtain the following description of the spectrum of $D(G/H)$ on $L^2(\Gamma\backslash G/H)$.

**Theorem 5.2.** Suppose $(G, H, L)$ is a spherical triple.

1. There is a direct sum decomposition of $C^\infty(\Gamma\backslash G/H)$ and $L^2(\Gamma\backslash G/H)$ into joint eigenspaces of $D(G/H)$. $\text{Spec}_{L^2(\Gamma\backslash G/H)}(D(G/H))$ consists of the corresponding eigencharacters and their possible accumulation points.

2. In the Lorentzian case $(SO_0(2,2n), SO_0(1,2n), U(1,n))$, $D(G/H)$ is generated by the Casimir $\Omega_G$, the set $\{\pi \in \hat{L} \mid V^{L\cap H}_{\pi,\infty} \neq \{0\}\}$ is computed explicitly and the representations $\pi$ are identified as well as their contributions to eigenvalues for $\iota(\Omega_G)$. More precisely, if $\Delta$ denotes the Laplace operator on $\Gamma\backslash G/H$ induced by $\Omega_G$, one has
   - $\text{spec}(\Delta) \cap (-\infty, -n^2]$ comes from unitary principal series and, at $-n^2$, also from limits of discrete series of $L$.
   - $\text{spec}(\Delta) \cap (-n^2, 0]$ consists of contributions from complementary series, ends of complementary series and non-integrable discrete series.
   - $\text{spec}(\Delta) \cap (0, \infty) = \bigcup_{\ell=n+1}^{\infty} \{\ell^2-n^2\}$ is the contribution of integrable discrete series. The corresponding eigenspaces are infinite dimensional.

The last assertion about the contribution of integrable discrete series with $L \cap H$-invariants generalizes to arbitrary spherical triples $(G, H, L)$. These contributions combine to infinite dimensional eigenspaces of $D(G/H)$ [21]. The shortest proof of this uses Theorem 3.3 instead of the embedding

$$\iota : D(G/H) \hookrightarrow D(L/L \cap H).$$

In view of Theorem 3.3 the result can be rephrased in terms of $G$-representations as follows. Each integrable discrete series for $G/H$ contributes an infinite dimensional eigenspace of $D(G/H)$ in $L^2(\Gamma\backslash G/H)$. Compare related results obtained in [13].

6. **Generalized matrix coefficients, unitarity and continuous spectrum.**

The Hilbert spaces $L^2(G/H)$ and $L^2(\Gamma\backslash G)$ are both unitary $G$-representations which decompose into a direct integral of irreducible ones. Though the Hilbert space $L^2(\Gamma\backslash G/H)$ is not a $G$-module, there is a natural way to involve representations of $G$.

For an admissible representation $(\rho, W_\rho)$ of $G$ of finite length, one may consider generalized matrix coefficients (see [9]):

$$W_{\tilde{\rho},-\infty} \otimes W_{\rho,-\infty} \to C^{-\infty}(G), \quad \tilde{w} \otimes w \mapsto c_{\tilde{w},w}.$$
with
\[ c_{\tilde{w},w}(f) = \langle \tilde{w}, \rho(f)w \rangle \]
\[ < \rho(f)w, v > = \int_G f(g) < \rho(g)w, v > dg \quad \text{for } f \in C_c^\infty(G), \ v \in W_{\rho,\infty} \]
Then there is a \( D(G/H) \)-equivariant map (which is injective if \( \rho \) is irreducible):
\[
W_{\rho,-\infty}^T \otimes W_{\rho,-\infty}^H \rightarrow C^\infty(\Gamma\backslash G/H) \simeq \bigoplus_{\pi \in \hat{L}} V_{\pi,-\infty}^T \otimes V_{\pi,-\infty}^{L/H} \]
\[
\bigoplus_{\pi \in L} \bigoplus_{\ell \in \mathbb{C}} D(\Gamma \backslash G/H) \otimes D(L/L \cap H) \]
A comparison of (6.1) with Theorem 5.2 leads, among other things, to the following funny observation.
On one hand, from Theorem 5.2, there are complementary series representations \( \pi \) of \( U(1,n) \) contributing to finite dimensional eigenspaces. On the other hand, if \( \rho \) were unitary then \( \rho|_L \) would contain, by Theorem 3.3, infinitely many unitary summands having \( L \cap H \)-invariants. Moreover, a result of Bergeron and Clozel [4], Thm. 6.5.1, tell us that there are indeed lattices \( \Gamma \subset U(1,n) \) such that \( V_{\Gamma,\rho,-\infty}^\infty \neq \{0\} \) for complementary series \( \pi \) with certain integral parameters. Therefore there must exist non-unitary representations \( \rho \) of \( G \) that are involved in \( L^2(\Gamma \backslash G/H) \) via (6.1).
One can also try to use (6.1) to produce families of eigendistributions on \( \Gamma \backslash G/H \) depending continuously on the spectral parameter. Let \( 0 \neq \tilde{v} \in V_{\pi,-\infty}^T \) and assume we have a continuous nonconstant family \( t \mapsto \rho_t \), \( t \in (a,b) \subset \mathbb{R} \) of \( H \)-spherical principal series of \( G \) with \( 0 \neq w_t \in W_{\rho_t,-\infty}^H \) and, the crucial ingredient, a continuous family of non-zero intertwining operators \( \Psi_t \in \text{Hom}_L(W_{\rho_t,\infty},V_{\pi,\infty}) \). Then \( \Psi_t^*v \in W_{\rho_t,-\infty}^T \) and
\[
t \mapsto c_{\Psi_t^*v,w_t}
\]
is the desired continuous family of eigendistributions. Note that the existence of such a family for \( \Gamma \backslash G/H \) excludes the existence of a discrete spectral decomposition as in Theorem 5.2. Therefore such families can only exist for non-spherical triples \( (G,H,L) \). One expects that such families are the building blocks for the continuous part of the spectral decomposition of \( L^2(\Gamma \backslash G/H) \). We are able to find such families of intertwiners for some non-spherical triples, and expect their existence for all non-spherical \( (G,H,L) \) except for the group case as in Example 2.2 (1). For instance, for the non-spherical triple \( (SO(8,\mathbb{C}), SO_0(1,7), Spin(7,\mathbb{C})) \) the existence of many families \( \Psi_t \) is ensured by the work of Jan Frahm [22].

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