A sharp sparse domination of pseudodifferential operators

Ryosuke Yamamoto*

Abstract

In this paper, we give a sharp sparse domination of pseudodifferential operators associated with symbols belonging to the Hörmander class, and fundamental solutions of dispersive equations. Furthermore, we give boundedness results of these operators on weighted Besov spaces by using the sparse domination.

1 Introduction and results

For any $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$, the Hörmander class $S^m_{\rho, \delta}$ is defined as the set of all $a \in C^\infty(\mathbb{R}^n)$ such that

$$|\partial^\beta_{\xi}^\alpha a(x, \xi)| \lesssim (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for any $(x, \xi) \in \mathbb{R}^n$. Here, $A \lesssim B$ means $A \leq CB$ with a positive constant $C > 0$. For given $a \in S^m_{\rho, \delta}$, we define the pseudodifferential operator $a(x, D)$ by

$$a(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi) d\xi,$$

where $f \in \mathcal{S}$ and $\hat{f}$ denotes the Fourier transform of $f$. Pseudodifferential operator is a useful tool for study of partial differential equations, and the many boundedness results are known. The most basic result is the $L^p$-boundedness given by Hörmander [14] and Fefferman [13]. Hörmander [14] showed that $m \leq -n(1-\rho)|1/2 - 1/p|$ is necessary for $a(x, D)$ with $a \in S^m_{\rho, \delta}$ to be $L^p$-bounded. Conversely, the $L^p$-boundedness of $a(x, D)$ with $a \in S^m_{\rho, \delta}$ and $m = -n(1-\rho)|1/2 - 1/p|$ was established by Fefferman [13]. As for the boundedness on Lebesgue spaces weighted by $\omega \in A_p$ which is so called Muckenhoupt weight, Miller [26] established the $L^p(\omega)$-boundedness of $a(x, D)$ with $a \in S^0_{1,0}$. For general $a \in S^m_{\rho, \delta}$, Michalowski, Rule and Staubach [28] showed the $L^p(\omega)$-boundedness of $a(x, D)$ with $a \in S^{-n(1-\rho)}_{\rho, \delta}$ and $\omega \in A_p$. Chanillo and Torchinsky [9] showed it for a larger class $a \in S^{-n(1-\rho)/2}_{\rho, \delta}$ (0 ≤ $\rho \leq 1$) and a smaller class $\omega \in A_{p/2}$, and Michalowski, Rule and Staubach [27] showed the same result for 0 < $\delta < \rho < 1$. It should be mentioned here that Beltran [11] showed it for $a \in S^m_{\rho, \delta}$ with $-n(1-\rho)/2 < m < -n(1-\rho)|1/2 - 1/p|$ and $\omega \in A_{p/2} \cap RH(2p'/p')$, where 2 ≤ $p < 2t'$ and $t'$ is the conjugate exponent of $t = -n(1-\rho)/(2m)$. We remark that there is no such $p$ that satisfies $2 \leq p < 2t'$ for the critical exponent $m = -n(1-\rho)|1/2 - 1/p|$. An important idea to deduce weighted estimates is to show pointwise estimates. For example, Chanillo and Torchinsky [9] established pointwise estimate

$$|(a(x, D)f)^*(x)| \lesssim M_2 f(x)$$

for $a \in S^{-n(1-\rho)/2}_{\rho, \delta}$ (0 ≤ $\rho \leq 1$), where $(a(x, D)f)^*$ denotes the sharp maximal function of $a(x, D)f$.

Recently as a refinement of pointwise estimates, the theory of sparse domination of operators was developed by Lerner [20]. For operators $T$ on function spaces, the sparse domination means the inequalities:

$$|Tf(x)| \lesssim \Lambda_{s, r} f(x) \quad \text{and} \quad |(Tf, g)| \lesssim \Lambda_{s, r, s'}(f, g).$$

In particular, we call the first one sparse bounds and the second one sparse form bounds. See below for the definition of $\Lambda_{s, r}$ and $\Lambda_{s, r, s'}$.

Definition 1.1. Let $\eta \in (0, 1)$. A collection $S$ of cubes in $\mathbb{R}^n$ is $\eta$-sparse family if there are pairwise disjoint subsets $\{E_Q\}_{Q \in S}$ such that $E_Q \subset Q$, and $|E_Q| > \eta|Q|$.

*Department of Mathematical Sciences, Nagoya University, Aichi, 464-8601, Japan.

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We often just say sparse instead of $\eta$-sparse whenever there is no confusion. For any cube $Q$ and $p \in [1, \infty)$, we define \( \langle f \rangle_{p,Q} := |Q|^{-\frac{1}{p}} \|f\|_{L^p(Q)} \). For a sparse collection $\mathcal{S}$ and $r, s \in [1, \infty)$, the $(r,s)$-sparse form operator $\Lambda_{\mathcal{S},r,s}$ and $r$-sparse operator $\Lambda_{\mathcal{S},r}$ are defined by

$$
\Lambda_{\mathcal{S},r,f}(x) := \sum_{Q \in \mathcal{S}} \langle f \rangle_{r,Q} 1_Q(x), \quad \Lambda_{\mathcal{S},r,s}(f,g) := \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{r,Q} (g)_{s,Q}
$$

for all $f, g \in L^1_{\text{loc}}$. If $r < p < s$, we have

$$
\Lambda_{\mathcal{S},r,s}(f,g) \lesssim \|f\|_p \|g\|_{p'}.
$$

This inequality is easily obtained from the $L^p$-boundedness of $r$-Hardy Littlewood maximal operator $M_r$ which is defined by $M_r f(x) = \sup_{Q \ni x} \langle f \rangle_{r,Q}$. Furthermore, weighted inequality with Muckenhoupt weights is deduced from sparse domination. Bernicot, Frey and Petermichl [3] showed

$$
\Lambda_{\mathcal{S},r,s}(f,g) \lesssim (|\omega|_{A_p/r} |\omega|_{\text{RH}_{(r/p')}^\alpha})^{\alpha} \|f\|_{L^p(\omega')} \|g\|_{L^{p'}(\omega)},
$$

where $\alpha = \max(\frac{1}{p-r}, \frac{s-1}{r})$, $|\omega|_{A_p} = \sup_{Q} |\omega|_{1,Q} |\omega|_1^{-\chi}$ and $|\omega|_{\text{RH}_q} = \sup_{Q} |\omega|_{1,Q}^{-1} |\omega|_{q,Q}$ for any $1 < q < \infty$. From these observations, sparse domination is used to study the weighted boundedness of operators, and Lerner [20] gave the simple proof of $A_2$ conjecture which means

$$
\|Tf\|_{L^2(\omega)} \lesssim |\omega|_{A_2} \|f\|_{L^2(\omega)},
$$

where $T$ denotes the Calderón-Zygmund operators. The $A_2$ conjecture was studied by many researchers. For example, Petermichl [31, 32] solved the $A_2$ conjecture for Hilbert transform and Riesz transform, and Perez, Treil and Volberg [30] gave

$$
\|Tf\|_{L^2(\omega)} \lesssim |\omega|_{A_2} \log(1 + |\omega|_{A_2}) \|f\|_{L^2(\omega)}
$$

for general Calderón-Zygmund operators. Finally, $A_2$ conjecture was completely solved by Hytönen [16]. Lerner [20] gave another proof by establishing

$$
\|Tf\|_X \lesssim \sup_{\mathcal{S}} \|\Lambda_{\mathcal{S},1}f\|_X
$$

for any Banach function space $X$, and it was improved to the pointwise estimate

$$
|Tf(x)| \lesssim \Lambda_{\mathcal{S},1} f(x)
$$

by Lerner [21], Lerner and Nazarov [23]. There are also results of sparse domination with other operators. Sparse form bounds of rough singular integral operators and Bochner-Riesz multipliers were shown by Conde-Alonso, Culic, Plinio and Ou [7], and Lacey, Mena and Reguera [25] respectively.

Beltran and Cladek [2] discussed the sparse domination of pseudodifferential operators with symbols in $S^m_{p,\delta}$, and they established

$$
|a(x,D)f(x)| \lesssim \Lambda_{\mathcal{S},r} f(x),
$$

with $a \in S^m_{p,\delta}$ and $1 < r < \infty$ which implies the weighted boundedness result of [28], that is the $L^p(\omega)$-boundedness with $\omega \in A_p$. We establish a pointwise estimate of $a(x,D)$ with larger class $a \in S^m_{p,\delta}$ by introducing another type of sparse bounds:

**Theorem 1.1.** Let $a \in S^m_{p,\delta}$ with $0 < \rho < 1$ and $m \in \mathbb{R}$. Then, for any $f \in L^\infty_c$, there exist the collection of finitely sparse families $\{\mathcal{F}_j\}_{j=1}^\infty$ such that

$$
|a(x,D)f(x)| \lesssim \sum_j \sum_{Q \in \mathcal{F}_j} \langle f \rangle_{2,Q} \sum_{R \subset Q, R \in \mathcal{F}_j} 1_R(x)
$$

if and only if

$$
m \leq -n(1-\rho)/2.
$$

Then as a corollary, we recover the weighted boundedness result which was showed by Michalowski, Rule and Staubach [27], that is the $L^p(\omega)$-boundedness with $a \in S^m_{p,\delta}$ and $\omega \in A_{p/2}$. Furthermore, as a benefit of our new sparse bounds, we have the boundedness of pseudodifferential operators and also the time evolution $e^{i(t-\Delta)^{\alpha/2}}$ with $0 < \alpha \leq 2$ of dispersive equations on weighted Besov spaces (Theorem 3.1 Corollary 3.3 Theorem 3.4 Corollary 3.5). We have also the following Coifman-Fefferman estimate for $a(x,D)$ by the same argument used in the proof of Theorem 1.1.
Theorem 1.2. Let \( a \in S_{p,p}^{-n(1-p)/2} \) with \( 0 < p < 1 \). Then, for any \( \omega \in A_\infty \) and \( 0 < p < \infty \), we have
\[
\|a(x,D)f\|_{LP(\omega)} \lesssim [\omega]_{A_\infty} \|M_2f\|_{LP(\omega)}.
\]

This paper is organized as follows. In the next section, we prove Theorem 1.1 and Theorem 1.2 by using Lerner and Nazarov’s method. The Section 3 is devoted to establish a sparse form bounds and the boundedness on weighted Besov spaces for \( a(x,D) \) and \( e^{it(-\Delta)^{\alpha}/2} \). Furthermore, we give some results about the sharpness of weighted boundedness of these operators.

2 Sparse bounds for pseudodifferential operators

2.1 The pointwise estimate for pseudodifferential operators

To establish Theorem 1.1, we use the following definition of dyadic lattice and sparse decomposition of measurable functions given by Lerner and Nazarov [23].

**Definition 2.1.** A Dyadic lattice \( \mathcal{D} \) in \( \mathbb{R}^n \) is any collection of cubes such that

- \( \text{(D-1)} \) if \( Q \in \mathcal{D} \), then each child of \( Q \) is in \( \mathcal{D} \),
- \( \text{(D-2)} \) every two cubes in \( \mathcal{D} \) have a common ancestor in \( \mathcal{D} \),
- \( \text{(D-3)} \) \( \mathcal{D} \) is regular, i.e., for any compact set \( K \) in \( \mathbb{R}^n \), there exists \( Q \in \mathcal{D} \) such that \( K \subset Q \).

**Theorem 2.1** ([23]). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be any measurable almost everywhere finite function such that for every \( \varepsilon > 0 \),
\[
\lim_{R \rightarrow \infty} R^{-n}|\{ x \in [-R,R]^n ; |f(x)| > \varepsilon \}| = 0.
\]
Then, for any dyadic lattice \( \mathcal{D} \) and any \( \lambda \in (0,2^{-n-2}) \), there exists the sparse family \( \mathcal{S} \subset \mathcal{D} \) such that
\[
|f(x)| \lesssim \sum_{Q \in \mathcal{S}} \omega_\lambda(f; Q) 1_Q(x),
\]
where
\[
\omega_\lambda(f; Q) = \inf_{E \subset Q} \sup_{|E| > (1-\lambda)|Q|} \sup_{x,x' \in E} |f(x) - f(x')|.
\]

By using Theorem 2.1, we have a pointwise estimate of \( a(x,D) \) with \( a \in S_{p,p}^{-n(1-p)/2} \):

**Lemma 2.1.** Let \( a \in S_{p,p}^{-n(1-p)/2} \) with \( 0 < p < 1 \). Then, for any \( f \in L^\infty_c \), there exist the sparse family \( \mathcal{S} \) so that
\[
|a(x,D)f(x)| \lesssim \sum_{k \geq 0} 2^{-\varepsilon k} \sum_{Q \in \mathcal{S}, |Q| \geq 3^{-n/\rho}} |f|_{2,2^{k+1}Q} 1_Q(x) + \sum_{k \geq 0} 2^{-\varepsilon k} \sum_{Q \in \mathcal{S}, |Q| < 3^{-n/\rho}} |f|_{2,2^{k+1}Q} 1_Q(x),
\]
where \( \varepsilon = [n/2] - n/2 + 1 \).

To prove the lemma, we give a partition of unity. Take \( \hat{\psi} \in C_0^\infty(\mathbb{R}^n) \) such that \( \text{supp} \hat{\psi} \subset B(0,2) \), \( \hat{\psi} = 1 \) on \( B(0,1) \) and \( \psi \geq 0 \), and denote \( \hat{\psi}_j(\xi) := \hat{\psi}(2^{-j} \xi) - \hat{\psi}(2^{-(j+1)} \xi) \) for \( j \in \mathbb{Z} \),
\[
\hat{\phi}_j = \begin{cases} \psi_j & j \in \mathbb{N} \\ \sum_{i \leq 0} \psi_1 & j = 0 \end{cases}.
\]
Then, \( a(x,D) \) is decomposed as
\[
a(x,D) = \sum_{j=0}^\infty a_j(x,D),
\]
where \( a(x,\xi) = a(x,\xi) \hat{\phi}_j(\xi) \). Furthermore, we use these notations in the following sections. Let us prove Lemma 2.1.
Proof. From Theorem 2.1, we have
\[ |a(x, D)f(x)| \leq \sum_{Q \in S} \omega_{\lambda}([a(x, D)f(x)]; Q)1_{Q}(x). \]

First, we consider the case \(|Q| < 3^{-\frac{n}{4\rho}}\). Let \(\alpha > 0\) and
\[ E = \{ x \in Q : |a(x, D)(f1_{2Q})| \leq \alpha \}. \]

Then, \(L^2 \to L^{2/\rho}\) boundness of \(a(x, D)\) yields
\[ |E^\alpha|^{\rho/2} \leq \alpha^{-1}||a(x, D)(f1_{2Q})||_{L^{2/\rho}} \leq \alpha^{-1}||a(x, D)||_{L^{2} \to L^{2/\rho}}||f||_{L^{2}(2Q)}. \]

By taking \(\alpha = 2^{n\lambda^{-\rho/2}}||a(x, D)||_{L^{2} \to L^{2/\rho}}(f)_{2,2Q^{\rho}}\), one has \(|E^\alpha| \leq \lambda|Q|\) and \(|E| \geq (1 - \lambda)|Q|\). Therefore, we have
\[ |a(x, D)f(x) - a(x, D)f(x')| \lesssim (f)_{2,2Q^{\rho}} + |a(x, D)(f1_{2Q^{\rho}})(x) - a(x, D)(f1_{2Q^{\rho}})(x')| \]
for any \(x, x' \in E\). We estimate the second term. Let \(a_j(x, \xi) := a(x, \xi)\hat{\phi}_j(\xi)\) and
\[ K_j(x, y) = \int e^{i(x-y)\xi}a_j(x, \xi)d\xi. \]

We integrate by parts in \(\xi\) to obtain
\[ |K(x, y)| \lesssim |x-y|^{-N} \sum_{|\alpha|=N} \left| \int e^{i(x-y)\xi}\partial^\alpha \hat{a}_j(x, \xi)d\xi \right| \]
for any \(n \in \mathbb{N}\). Hence, we have
\[ |a(x, D)(f1_{2Q^{\rho}})(x)| \leq \sum_{|\alpha|=N} \left| \int |x-y|^{-N}|f(y)|1_{(2Q^{\rho})^c}(y) \int e^{i(x-y)\xi}\partial^\alpha \hat{a}_j(x, \xi)d\xi \right| dy \]
\[ = \sum_{|\alpha|=N} \sup_{|g|_{L^\infty}} \left| \int |x-y|^{-N}f(y)1_{(2Q^{\rho})^c}(y)g(y) \int e^{i(x-y)\xi}\partial^\alpha \hat{a}_j(x, \xi)d\xi dy \right| \]
\[ \leq \sum_{|\alpha|=N} \sup_{|g|_{L^\infty}} \left( \int |\partial^\alpha \hat{a}_j(x, \xi)|^2 \right)^{1/2} ||F|||x-y|^{-N}f1_{(2Q^{\rho})^c}g||_{L^2} \]
\[ \lesssim 2^{jm/2-j\rho N} \left( \int_{(2Q^{\rho})^c} |x-y|^{-2N} |f(y)|^2 dy \right)^{1/2} \]
\[ \lesssim 2^{jm/2-j\rho N} \sum_{k \geq 1} \left( \int_{(2k+1)Q^{\rho}\setminus 2kQ^{\rho}} |x-y|^{-2N} |f(y)|^2 dy \right)^{1/2} \]
\[ \lesssim 2^{jm/2-j\rho N} \ell(Q)^{-\rho N+jn/2} \sum_{k \geq 1} 2^{-kn/2}(f)_{2,2k+1Q^{\rho}}. \]

By taking \(N > n/2\), one has
\[ \sum_{2^{-j} \leq \ell(Q)} |a(x, D)(f1_{2Q^{\rho}})(x)| \lesssim \sum_{k \geq 1} 2^{-kn/2-n/2}(f)_{2,2k+1Q^{\rho}}. \]

On the other hand, it holds that
\[ (x-y)^{\alpha}\{K_j(x, y) - K_j(x', y)\} \]
\[ = (x-y)^{\alpha} \int e^{i(x-y)\xi}(1 - e^{-i(x-x')\xi})a_j(x, \xi)d\xi + (x-y)^{\alpha} \int e^{i(x-y)\xi}(a_j(x, \xi) - a_j(x', \xi))d\xi \]
\[ = \int e^{i(x-y)\xi}\partial^\alpha \{1 - e^{-i(x-x')\xi}\}a_j(x, \xi)d\xi + \int e^{i(x-y)\xi}\partial^\alpha (a_j(x, \xi) - a_j(x', \xi))d\xi. \]

For any \(j\) such that \(2^{-j} > \ell(Q)\), Taylor’s formula yields
\[ |\partial^\alpha \{1 - e^{-i(x-x')\xi}\}a_j(x, \xi)| \lesssim \ell(Q)2^{-jn(1-\rho)/2+j-j\rho|\alpha|}, \]
and
\[ |\partial_\xi^a (a_j(x, \xi) - a_j(x', \xi))| = \left| \partial_\xi^a \int_0^1 (x - x') \cdot (\nabla_x a_j)(x' + t(x - x'), \xi) dt \right| \lesssim \ell(Q) 2^{-jn(1-\rho)/2 - j\rho |a| + j\rho}.
\]

From these results, we obtain
\[ \sum_{2^{-j} > \ell(Q)} |a_j(x, D)(f1_{2Q})^c(x) - a_j(x, D)(f1_{2Q})^c(x')| \lesssim \sum_{2^{-j} > \ell(Q)} 2^{jn/2 + j\rho N} \ell(Q)^{1-\rho + \rho n/2} \sum_{k \geq 1} 2^{-kN + kn/2} \langle f \rangle_{2,2k+1} Q^\alpha
\]
\[ \lesssim \sum_{k \geq 1} 2^{-kN + kn/2} \langle f \rangle_{2,2k+1} Q^\alpha
\]
by taking \( N = \lfloor n/2 \rfloor + 1 \). In the case \(|Q| \geq 3^{-3/\rho}\), the desired estimate is easily checked in the same way as above by setting
\[ E = \{ x \in Q : |a(x, D)(f1_{2Q})| \leq \alpha \}, \quad \alpha = 2^{n\lambda - 1/2} ||a(x, D)||_{L^2 \rightarrow L^2(f)}.
\]

2.2 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 by using Lemma 2.1 and Lerner and Nazarov’s technique [23].

Definition 2.2. Let \( \mathcal{P} \) denotes a map from \( \{(Q', Q) \in \mathcal{D} \times \mathcal{D} : Q' \subset Q\} \) to \{true, false\} such that \( \mathcal{P}(Q, Q) = true \) for any \( Q \in \mathcal{D} \). Then, we call that \((Q, Q')\) is one step if \( \mathcal{P}(Q, Q') = false \) and \( \mathcal{P}(Q, R) = true \) for any \( Q \subseteq R \subset Q \), and we call that \((Q, Q')\) is finite step if there exist \( m \in \mathbb{N} \) and sequence \( Q' = Q_0 \subset Q_1 \subset \cdots \subset Q_m = Q \) such that each \((Q_{j+1}, Q_j)\) is one step. Furthermore, we set
\[ stop(Q, \mathcal{P}) = \{ Q' \in \mathcal{D} : (Q, Q') \text{ is finite step} \}.
\]

Let \( \mathcal{S} \subset \mathcal{D} \) denotes a sparse family and that with every cube \( Q \in \mathcal{S} \) some family \( \mathcal{F}(Q) \subset \mathcal{D} \) of child of \( Q \) is associated so that \( Q \in \mathcal{F}(Q) \). Then, we define the family of cubes \( \hat{\mathcal{S}} \) by
\[ \hat{\mathcal{S}} := \bigcup_{Q \in \mathcal{S}} \hat{\mathcal{F}}(Q),
\]
\[ \hat{\mathcal{F}}(Q) := \{ P \in \mathcal{F}(Q) : P \notin \mathcal{F}(R) \text{ for any } Q \subseteq R \},
\]
and we call the argumentation of \( \mathcal{S} \) by \( \mathcal{F}(Q) \). In [23], Lerner and Nazarov proved \( \hat{\mathcal{S}} \) be a sparse family if \( \mathcal{F}(Q) \) are sparse families. In particular, they proved the following result in the same paper.

Proposition 2.1. Let \( \mathcal{S} \) be a sparse family and assume that
\[ \sum_j |Q_j| < \frac{1}{2} |Q|
\]
for any \( Q \in \mathcal{S} \) and finitely pairwise disjoint cubes \( \{Q_j\}_j \) included \( Q \) such that \( \mathcal{P}(Q, Q_j) = false \). Then, the augmentation of \( \mathcal{S} \) by \( stop(Q, \mathcal{P}) \) is a sparse family.

Let us prove Theorem 1.1.

Proof. In view of the three lattice theorem in [23], there exists the family of dyadic lattices \( \{\mathcal{D}_j\}_{j=1, 2, \ldots, 3^{2n}} \) so that \( Q^\sigma \subset R_Q \in \mathcal{D}_j, 2^k Q^\sigma \subset R \in \mathcal{D}_j \) and \(|Q^\sigma| \sim |R_Q|, |2^k Q^\sigma| \sim |R| \) with some \( j \). From this, we have
\[ \sum_{k \geq 0} 2^{-\varepsilon k} \sum_{Q \in \mathcal{S}} \langle f \rangle_{2,2k+1} Q 1_Q(x) \lesssim \sum_j \sum_{Q \in \mathcal{S}} \sum_{R \subset Q \subseteq R_Q} \left( \frac{|R_Q|}{|R|} \right)^{\varepsilon} \langle f \rangle_{2, R} 1_Q(x).
\]
Furthermore, we take \( \overline{Q} \in \mathcal{D}_j \) such that \( Q \subset \overline{Q} \) and \( |\overline{Q}| = 3^{2n}|Q| \) for any \( j \), and set \( \mathcal{S}_j = \{ \overline{Q} : Q \in \mathcal{S} \}, \mathcal{S}_j' = \{ \overline{Q} : Q \in \mathcal{S}, |Q| < 3^{-3m} \}, \) of course \( \mathcal{S}_j \) be a regular sparse collection. Since \( Q \rightarrow \overline{Q} \) is a injective map, we can define the \( R_{\overline{Q}} := R_Q \). Here, the assumption \(|Q| < 3^{-3m} \) gives
\[ |\overline{Q}| = 3^{2n}|Q| < |Q|^\rho \leq |R_Q|,
\]
which yields $\overline{Q} \subset R_Q$. From these results, for any regular sparse family $\mathcal{S}_j$ so that $S_j \subset \mathcal{S}_j \subset \mathcal{D}_j$, we obtain

$$
\begin{align*}
\sum_j \sum_{Q \in \mathcal{S}} \sum_{R \in \mathcal{S}_j} \left( \frac{|Q|}{|R|} \right)^{1/2} (f)_{2,R} 1_Q(x) &
\lesssim \sum_j \sum_{Q \in \mathcal{S}_j} \sum_{R \in \mathcal{S}_j} \left( \frac{|Q|}{|R|} \right)^{1/2} (f)_{2,R} 1_Q(x) \\
&= \sum_j \sum_{U \in \mathcal{S}_j} \sum_{Q \in \mathcal{S}_j} \left( \frac{|Q|}{|R|} \right)^{1/2} (f)_{2,R} 1_Q(x) \\
&\leq \sum_j \sum_{U \in \mathcal{S}_j} \sum_{R \in H_{\mathcal{S}_j}(U)} \left( \frac{|Q|}{|R|} \right)^{1/2} (f)_{2,R} 1_Q(x)
\end{align*}
$$

where

$$H_{\mathcal{S}_j}(U) := \{ R \in \mathcal{S}_j ; R \subset U, \text{ there is no cube } P \in \mathcal{S}_j \text{ so that } R \subset P \subset U \}.$$ 

Since

$$\sum_{R \in H_{\mathcal{S}_j}(U)} \left( \frac{|Q|}{|R|} \right)^{1/2} \lesssim 1,$$

one has

$$\sum_{k \geq 0} 2^{-2k} \sum_{Q \in \mathcal{S}} \langle f \rangle_{2,2^{k+1}Q} 1_Q(x) \lesssim \sum_j \sum_{U \in \mathcal{S}_j} \sum_{R \in H_{\mathcal{S}_j}(U)} \langle f \rangle_{2,R} 1_Q(x).$$

If

$$\sup_{R \in H_{\mathcal{S}_j}(U)} \langle f \rangle_{2,R} \lesssim \langle f \rangle_{2,U}$$

holds, the proof will be completed. We define the map $\mathcal{P}$ by

$$\mathcal{P}(U, R) = \left\{ \begin{array}{ll}
\text{true} & \langle f \rangle_{2,R} \leq \sqrt{2} \langle f \rangle_{2,U} \\
\text{false} & \text{other}
\end{array} \right.$$

Let $\{ R_j \}_j$ be a pairwise disjoint dyadic child of $U$ such that $\mathcal{P}(U, R_j) = \text{false}$, then we have

$$\sum_j |R_j| \leq \frac{1}{2} |U| \sum_j \| f \|^2_{L^2(R_j)} \| | f |^2_{L^2(U)} \leq \frac{1}{2} |U|.$$

Hence, the argumentation of $S_j$ by $\text{stop}(U, \mathcal{P})$ be a regular sparse family and set $\mathcal{S}_j$. We assume that there exists the $R \in H_{\mathcal{S}_j}(U)$ so that $\mathcal{P}(U, R) = \text{false}$. From the definition of $H_{\mathcal{S}_j}(U)$, we obtain $R \notin \mathcal{S}_j$ which yields $R \notin \text{stop}(U, \mathcal{P})$. We take $R \subset R_1 \subset U$ such that $\mathcal{P}(U, R_1) = \text{false}$. If $R_1 \neq U$, we can take $R_1 \subset R_2 \subset U$ such that $\mathcal{P}(U, R_2) = \text{false}$ again. By repeating this work, we have $\mathcal{P}(U, U) = \text{false}$ which contradict the definition of $\mathcal{P}$. Hence, we have $\mathcal{P}(Q, R) = \text{true}$ and

$$\sup_{R \in H_{\mathcal{S}_j}(Q)} \langle f \rangle_{2,R} \lesssim \langle f \rangle_{2,Q}.$$

\[ \square \]

### 2.3 Weighted $L^p$ bounds for pseudodifferential operators

This subsection is devoted to prove Theorem 12. The class $A_\infty$ denotes the set of all nonnegative locally integrable function $\omega$ such that

$$[\omega]_{A_\infty} := \sup_Q \frac{1}{\omega(Q)} \int_Q M(\omega 1_Q) < \infty.$$

The sharp reverse Hölder inequality of $A_\infty$ weights was shown by Hytönen and Pérez 18.
**Theorem 2.2 (LN).** Let $\omega \in A_\infty$. Then, there exists a constant $c_n$ depends on dimension $n$ such that

$$\left(\frac{1}{|Q|} \int_Q \omega^\delta \right)^{1/\delta} \leq \frac{2}{|Q|} \omega(Q)$$

for any $Q$ where $\delta = 1 + c_n[\omega]_{A_\infty}^{-1}$.

From this theorem, we remark that

$$\int_Q |f| \omega \leq \left(\int_Q |f|^{\delta'} \right)^{1/\delta'} \left(\int_Q \omega^\delta \right)^{1/\delta} \leq \frac{2}{|Q|} \omega(Q) \left(\int_Q |f|^{\delta'} \right)^{1/\delta'}$$

for each nonnegative locally integrable function $f$. In particular, for any measurable subset $E \subset Q$, we have

$$\omega(E) \leq 2 \left(\frac{|E|}{|Q|}\right)^{1/\delta'} \omega(Q)$$

by taking $f = 1_E$. To establish Theorem 2.2, it suffices to prove following estimate which is shown by using Cejas, Li, Pérez and Rivera-RÍos’s idea in [8].

**Lemma 2.2.** Let $X : \{\text{cube} \} \to \{\text{cube} \}$ be a map such that $Q \subset X(Q)$ for any cube $Q$, and let

$$\Lambda_{S,r,X} f(x) := \sum_{Q \in S} \langle f \rangle_{r,X(Q)} 1_Q(x)$$

for any sparse family $S$ and $1 \leq r < \infty$. Then, for any $\omega \in A_\infty$ and $p \in (0, \infty)$, one has

$$\|\Lambda_{S,r,X} f\|_{L_p(\omega)} \lesssim [\omega]_{A_\infty} \|M_r f\|_{L_p(\omega)}$$

for any $f \in L^\infty_p$.

**Proof.** Let $\gamma > 0$ and we have

$$\|\Lambda_{S,r,X} f\|_{L_p(\omega)}^p \lesssim \sum_{k \in \mathbb{Z}} 2^{kp\omega(\{\Lambda_{S,r,X} f > 2^k\})}$$

$$\leq \sum_{k \in \mathbb{Z}} 2^{kp\omega(\{\Lambda_{S,r,X} f > 2^k, M_r f \leq \gamma 2^k\})} + \sum_{k \in \mathbb{Z}} 2^{kp\omega(\{M_r f > \gamma 2^k\})}$$

$$\lesssim \sum_{k \in \mathbb{Z}} 2^{kp\omega(\{\Lambda_{S,r,X} f > 2^k, M_r f \leq \gamma 2^k\})} + \gamma^{-p}\|M_r f\|_{L_p(\omega)}^p.$$}

Here, we set

$$S_m = \{Q \in S : 2^m \leq \langle f \rangle_{r,X(Q)} < 2^{m+1}\},$$

$$S_m^* = \{Q \in S_m : Q \text{ is maximal with inclusion}\}$$

for any $m \in \mathbb{Z}$. If $2^m > \gamma 2^k$, we obtain $M_r f(x) > \gamma 2^k$ for any $x \in Q \in S_m$ from the assumption $Q \subset X(Q)$. Hence, one obtains

$$\omega(\{\Lambda_{S,r,X} f > 2^k, M_r f \leq 2^k\}) = \omega \left(\left\{ \sum_{2^m \leq 2^{2k}} \Lambda_{S_m,r,X} f > 2^k, M_r f \leq \gamma 2^k \right\} \right)$$

$$\leq \sum_{2^m \leq 2^{2k}} \omega \left(\left\{ \sum_{Q \in S_m} 1_Q > \gamma^{-1/2} 2^{(m+k)/2-1} \right\} \right)$$

$$\leq \sum_{2^m \leq 2^{2k}} \omega \left(\left\{ x \in U : \sum_{Q \in S_m, Q \subset U} 1_Q(x) > \gamma^{-1/2} 2^{(-m+k)/2-2} \right\} \right)$$

$$= \sum_{2^m \leq 2^{2k}} \sum_{U \in S_m^*} \omega(E).$$
For any \( s \in (1, \infty) \), the sparseness of \( S_m \) gives
\[
\left\| \sum_{Q \in S_m, Q \subset U} 1_Q \right\|_{L^s} \leq \sup_{\|g\|_{L^{s'}} = 1} \sum_{Q \in S_m, Q \subset U} \int_Q g \leq \sup_{\|g\|_{L^{s'}} = 1} \int_Q Mg \leq \sup_{\|g\|_{L^{s'}} = 1} |Q|^{1/s} |Mg|_{L^{s'}} \leq s|Q|^{1/s},
\]
which yields
\[
|E| \leq 2^{(m-k)s/2+2s\gamma-s^s/2} |U|.
\]
From this and Theorem 2.2 we obtain
\[
\sum_{k \in \mathbb{Z}} 2^{kp} \omega(\{A_{S,r,X} f > 2^k, M_r f \leq 2^k\}) \lesssim \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{m \in \mathbb{Z}} 2^{(m-k)s/(2\delta') + 2s/\delta'} \omega_m(U) \leq 2^{2s/\delta'} \sum_{m \in \mathbb{Z}} 2^{ms/(2\delta')} \omega_m(\{M_r f > 2^m\}) \leq 2^{2s/\delta'} \sup_{\|g\|_{L^{s'}} = 1} |M_r f|_{L^p}^p
\]
for any \( s/(2\delta') > p \). Since
\[
\delta' = \frac{1 + c_n [\omega]_{A}^{-1}}{c_n [\omega]_{A}^{-1}} \sim [\omega]_{A^c},
\]
we obtain the desired inequality by taking \( \gamma = [\omega]_{A^c}^{-1} \) and \( s = c [\omega]_{A^c} \) with some large constant \( c > 0 \) depends on only \( n \) and \( p \).

3 Sparse form bounds for Pseudodifferential operators

3.1 Besov-type sparse form bounds

Beltran and Cladek [2] established sparse form bounds of pseudodifferential operators
\[
|\langle a(x, D) f, g \rangle| \lesssim A_{r,s'}(f, g)
\]
with \( a \in S^m_{\rho, \rho} \) and \( m < m(r, s) \) where
\[
m(r, s) = \begin{cases} -n(1-\rho)(1/r - 1/2) & 1 \leq r \leq s \leq 2 \\ -n(1-\rho)(1/r - 1/s) & 1 \leq r \leq 2 \leq s \leq s' \end{cases}.
\]
It is natural to ask whether the such bounds hold or not when \( m = m(r, s) \). However, we do not know how to settle this problem. Therefore, we treat the case \( m = m(r, s) \) by using Besov type sparse form bounds
\[
|\langle a(x, D) f, g \rangle| \lesssim \sum_{j \geq 0} 2^{jk} A_{S, j, r, s'}(\phi_j * f, g)
\]
with suitable \( k \in \mathbb{R} \). By using Beltran and Cladek’s idea, it is not hard to see
\[
|\langle a(x, D) f, g \rangle| \lesssim \sum_{j \geq 0} 2^{jm - jm(r, s) + js} A_{S, j, r, s'}(\phi_j * f, g)
\]
for any \( \varepsilon > 0 \). Our purpose is to eliminate \( \varepsilon \) in the above inequality. More generally, we use
\[
A^\varepsilon_{S, r, s}(f, g) := \left( \sum_{Q \in S} |Q|^\alpha \langle f |_r, Q(g) |_{s, Q} \rangle \right)^{1/\alpha}
\]
to obtain the following results:
Theorem 3.1. Let $2 \leq s \leq \infty$ and $2/3 < \alpha \leq 1$, and $a \in S_{p,\rho}^m$ with $m \leq 0$, $0 < \rho < 1$. Then for any $f, g \in \mathcal{S}$, there exist the sequence of sparse families $\{S_j\}_{j=0,1,\ldots}$ such that

$$|\langle a(x, D)f, g \rangle| \lesssim \liminf_{R \to \infty} \sum_{j \geq 0} 2^{j\kappa_1} \Lambda_{j,2,s'}^\alpha((\phi_j * f)_1 Q_R, g),$$

where $\kappa_1 = m + n(1-\rho)(1-1/s) + \rho n(1/\alpha-1)$. Here, $Q_R$ denotes the cube whose center is origin and side length is $R$.

Theorem 3.2. (i) Let $2 \leq s \leq \infty$ and $s'/2 < \alpha \leq 1$, and $a \in S_{p,\rho}^m$ with $m \leq 0$, $0 < \rho < 1$. Then for any $f, g \in \mathcal{S}$, there exist the sequence of sparse families $\{S_j\}_{j=0,1,\ldots}$ such that

$$|\langle a(x, D)f, g \rangle| \lesssim \liminf_{R \to \infty} \sum_{j \geq 0} 2^{j\kappa_2} \Lambda_{j,s',s'}^\alpha((\phi_j * f)_1 Q_R, g),$$

where $\kappa_2 = m + n(1-\rho)(1-2/s) + \rho n(1/\alpha-1)$.

(ii) Let $1 \leq s' \leq r \leq 2 \leq s \leq \infty$ and $a \in S_{p,\rho}^m$ with $m \leq 0$, $0 < \rho < 1$. Then for any $f, g \in \mathcal{S}$, there exist the sequence of sparse families $\{S_j\}_{j=0,1,\ldots}$ such that

$$|\langle a(x, D)f, g \rangle| \lesssim \liminf_{R \to \infty} \sum_{j \geq 0} 2^{j\kappa_3} \Lambda_{j,r,s'}^\alpha((\phi_j * f)_1 Q_R, g),$$

where $\kappa_3 = m + n(1-\rho)(1/r-1/s)$.

To prove Theorem 3.1 and Theorem 3.2 we introduce maximal operators $M_{T,s}$ defined by

$$M_{T,s}f(x) := \sup_{Q \ni x} \left| Q \right|^{-1/s} \|T_1 3Q\|_{L^s(Q)}$$

for each linear operators $T$ and $s \in [1, \infty]$.

Proposition 3.1. Let $1 \leq r \leq s \leq \infty$ and $0 < \alpha \leq 1$, and $T$ denotes the linear operators on function spaces. We assume weak-type $(r,p)$ of $T$ and $M_{T,s}$ with

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{\alpha} + 1.$$

Then, for any $f \in L_c^\infty$ and $g \in \mathcal{S}$, there exists the sparse family $\mathcal{S}$ such that

$$\|Tf, g\| \lesssim (\|T\|_{L^r \to L^{p,\infty}} + \|M_{T,s}\|_{L^r \to L^{p,\infty}}) \Lambda_{r,s}^\alpha(f, g).$$

Proposition 3.1 with $\alpha = 1$ was proved by Lerner in [22]. The proposition with general $\alpha$ is proved in a similar manner, but we give the proof for reader’s convenience.

Lemma 3.1. Let $1 \leq r < s \leq \infty$ and $0 < \alpha \leq 1$, $f \in L_c^\infty$ and $g \in \mathcal{S}$, and $T$ denotes the linear operators on function spaces. We assume that for any cubes $Q \subset \mathbb{R}^n$ there exists some family $\mathcal{F}(Q)$ of dyadic child of $Q$ such that

(F-1) $\mathcal{F}_Q$ is pairwise disjoint,

(F-2) $\sum_{p \in \mathcal{F}_Q} |p| \leq \frac{1}{2}|Q|$, 

(F-3) $\int_Q T(f 3Q) g dx \leq \sum_{p \in \mathcal{F}_Q} \left| \int_p T(f 3p) g dx \right|$. 

Then, there exists the sparse family $\mathcal{S}$ such that

$$|\langle Tf, g \rangle| \leq C \Lambda_{r,s}^\alpha(f, g).$$

Proof. Pick up a cube $Q_0$ in $\mathbb{R}^n$ containing supports of $f$. Then, we construct $\{\mathcal{F}_k\}_{k=0,1,2,\ldots}$ by

$$\mathcal{F}_0 = \{Q_0\} \quad , \quad \mathcal{F}_{k+1} = \bigcup_{p \in \mathcal{F}_k} \mathcal{F}_p,$$
and set $S_k(Q_0) := S_k := \bigcup_{i=0}^k F_i$, $S(Q_0) := S = \bigcup_k S_k$. From the assumption (F-1), $F_k$ be a pairwise disjoint family. The assumption (F-3) gives
\[
\left| \int_{Q_0} T(f 1_{3Q_0}) g dx \right| \leq C \sum_{P \in \mathcal{S}_k} |P|^{1/\alpha} (f)_{r, 3P} (g)_{s, P} + \sum_{P \in \mathcal{F}_{k+1}} \left| \int_P T(f 1_{3P}) g dx \right|
\]
for any $k \in \mathbb{N}$. From
\[
\sum_{P \in \mathcal{F}_{k+1}} |P| \leq \sum_{L \in \mathcal{F}_k} \sum_{P \in \mathcal{F}_L} |P| \leq \frac{1}{2} \sum_{L \in \mathcal{F}_k} |L| \leq \cdots \leq 2^{-k+1} |Q_0|,
\]
we have
\[
\sum_{P \in \mathcal{F}_{k+1}} \left| \int_P T(f 1_{3P}) g dx \right| \to 0 \text{ as } k \to \infty.
\]
Therefore, one obtains
\[
\left| \int_{Q_0} T(f 1_{3Q_0}) g dx \right| \leq C A_{S, r, s}(f, g).
\]
We prove the sparseness of $S$. Let $Q$ be an any dyadic child of $Q_0$. For any $k$, we have
\[
\sum_{P \in \mathcal{F}_{k+1}, P \subset Q} |P| \leq \sum_{L \in \mathcal{F}_k} \sum_{P \in \mathcal{F}_L, P \subset Q} |P| \leq \frac{1}{2} \sum_{L \in \mathcal{F}_k} |L| + \sum_{L \in \mathcal{F}_k} \sum_{P \in \mathcal{F}_L, P \subset Q} |P| =: a_k + b_k.
\]
Here, if $b_k \neq 0$ for some $k$, it holds that $b_i = 0$ for any $i > k$. Actually, $b_k \neq 0$ means that there are $L \in \mathcal{F}_k$ and $P \in \mathcal{F}_L \subset \mathcal{F}_{k+1}$ so that $L \supset Q$ and $P \subset Q$. From the pairwise disjointness of $\mathcal{F}_{k+1}$, any cube in $\bigcup_{i=k}^{\infty} \mathcal{F}_i$ do not contain $Q$. Hence, we have $b_i = 0$ with $i > k$, and
\[
\sum_{k \geq 0} b_k \leq |Q|.
\]
From these results, one has
\[
\sum_{k \geq 0} a_k \leq \frac{1}{2} \sum_{k \geq 0} a_k + |Q| \quad \text{and} \quad \sum_{k \geq 0} a_k \leq 2|Q|,
\]
which means $S$ be a Carleson family, and therefore $S$ be a sparse family. To complete the proof, we take the pairwise disjoint family of cubes $\{Q_j\}_{j=0, 1, 2, \ldots}$ so that any $3Q_j$ contain the support of $f$ and the union of $Q_j$ coincides $\mathbb{R}^n$. Then, $S := \bigcup_{j=0}^{\infty} S(Q_j)$ be a sparse family, and we obtain the desired sparse form bound. 

Let us prove Proposition 5.1

Proof. For any cube $Q$ in $\mathbb{R}^n$ and $\lambda > 0$, set
\[
E = \{ x \in Q ; T(f 1_{3Q}) > \lambda |Q|^{1/\alpha - 1} (f)_{r, 3Q} \} \cup \{ x \in Q ; M_{T, s}(f 1_{3Q}) > \lambda |Q|^{1/\alpha - 1} (f)_{r, 3Q} \}.
\]
From weak-type boundedness of $T$ and $M_{T,s}$, we obtain
\[
|\{x \in Q : T(f1_{3Q}) > \lambda |Q|^{1/\alpha - 1} \langle f, 1_{3Q} \rangle\}|^{1/p} \leq \lambda^{-1} |Q|^{1/\alpha - 1} \langle f, 1_{3Q} \rangle^{1/p} \leq \lambda^{-1} |Q|^{1/p},
\]
and
\[
|\{x \in Q : M_{T,s}(f1_{3Q}) > \lambda |Q|^{1/\alpha - 1} \langle f, 1_{3Q} \rangle\}|^{1/q} \leq \lambda^{-1} |Q|^{1/\alpha + 1} \langle f, 1_{3Q} \rangle^{1/q} \leq \lambda^{-1} |Q|^{1/p}.
\]

We apply the Calderon-Zygmund decomposition to $1_E$ to construct the family $\{P_j\}_j$ of pairwise disjoint dyadic child of $Q$ so that
\[
\begin{cases}
2^{-n-1} |P_j| < |P_j \cap E| \leq 2^{-1} |P_j|, \\
|E \setminus P| = 0,
\end{cases}
\]
where $P = \bigcup P_j$. Here, the pairwise disjointness of $\{P_j\}_j$ gives
\[
\left| \int_Q T(f1_{3Q}) g dx \right| \leq \left| \int_{Q \setminus P} T(f1_{3Q}) g dx \right| + \sum_j \left| \int_{P_j} T(f1_{3Q \setminus 3P_j}) g dx \right| + \sum_j \left| \int_{P_j} T(f1_{3P_j}) g dx \right| =: I_1 + I_2 + I_3.
\]
Since $|E \setminus P| = 0$, one obtains
\[
I_1 \leq \int_{Q \setminus E} |T(f1_{3Q})||g| dx \lesssim \lambda |Q|^{1/\alpha - 1} \langle f, 1_{3Q} \rangle \int_Q |g| \leq \lambda |Q|^{1/\alpha} \langle f, 1_{3Q} \rangle \langle g, angle_Q.
\]
On the other hands,
\[
I_2 \leq \sum_j \|T(f1_{3Q \setminus 3P_j})\|_{L^r(P_j)} \|g\|_{L^{r'}(Q)} \lesssim \left( \sum_j \|T(f1_{3Q \setminus 3P_j})\|^s_{L^r(P_j)} \right)^{1/s} \|g\|_{L^{r'}(Q)} \lesssim \left( \sum_j |P_j| \right)^{1/s} \lambda |Q|^{1/\alpha - 1} \langle f, 1_{3Q} \rangle \|g\|_{L^{r'}(Q)} \lesssim \lambda |Q|^{1/\alpha} \langle f, 1_{3Q} \rangle \langle g, \rangle_Q.
\]
From these results with sufficient large $\lambda \sim (\|T\|_{L^r \rightarrow L^{p, \infty}} + \|M_{T,s}\|_{L^r \rightarrow L^{p, \infty}})$ and Lemma 3.1, we obtain $\sum |P_j| < 2^{-1} |Q|$ and complete the proof. \hfill $\square$

**Remark 3.1.** From Lebesgue’s differentiation theorem, we obtain
\[
|Tf(x)| = \lim_{|Q| \to 0} \left( \frac{1}{|Q|} \int_Q |Tf|^s \right)^{1/s} \leq M_{T,s} f(x) + \liminf_{|Q| \to 0} \left( \frac{1}{|Q|} \int_Q |T(f1_{3Q})|^s \right)^{1/s}.
\]

If $T$ is a bounded operator from $L^{s-\varepsilon}$ to $L^s$ with some $\varepsilon > 0$, then we have
\[
\liminf_{|Q| \to 0} \left( \frac{1}{|Q|} \int_Q |T(f1_{3Q})|^s \right)^{1/s} \lesssim \liminf_{|Q| \to 0} \left( \frac{1}{|Q|} \int_{3Q} |f|^{s-\varepsilon} \right)^{1/(s-\varepsilon)} \lesssim \liminf_{|Q| \to 0} \left( \frac{1}{|3Q|} \int_{3Q} |f|^{s-\varepsilon} \right)^{1/(s-\varepsilon)} = 0.
\]

Hence, we have $|Tf(x)| \leq M_{T,s} f(x)$ and $\|T\|_{L^r \rightarrow L^{p, \infty}} \leq \|M_{T,s}\|_{L^r \rightarrow L^{p, \infty}}$. 

The Proposition 3.1 gives some interpolation theorem.

**Corollary 3.1.** Let $1 \leq r \leq s_0, s_1, p_0, p_1 \leq \infty$. We assume linear operator $T$ satisfies
\[
\|M_{T,s_0} f\|_{L^{p_0,\infty}} \leq C_0 \|f\|_{L^r}, \\
\|M_{T,s_1} f\|_{L^{p_1,\infty}} \leq C_1 \|f\|_{L^r}.
\]
Then, for any $\theta \in (0,1)$, we have
\[
\|M_{T,s}\|_{L^r \rightarrow L^{p,\infty}} \lesssim C_0^{1-\theta} C_1^\theta
\]
where $1/s = (1-\theta)/s_0 + \theta/s_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. In particularly, we have
\[
|\langle Tf, g \rangle| \lesssim (\|T\|_{L^r \rightarrow L^{p,\infty}} + C_0^{1-\theta} C_1^\theta) \Lambda_{S,r,s'}(f,g),
\]
where
\[
\frac{1}{\alpha} = \frac{1}{r} + \frac{1}{p'}
\]

**Proof.** Let $Q$ and $x \in Q$. For any simple functions $f, g$ so that $\|g\|_{p'} = 1$, we define the analytic function $F$ on the open strip by
\[
F(z) = \int_Q T(f(1_{(3Q)^c})(x))g_z(x)dx,
\]
where
\[
g_z = \text{sgn}(g)|g|^{s/(1-s)/s_0+z/s_1}.
\]
Then, it holds that
\[
|F(iy)| \leq \int_Q |T(f(1_{(3Q)^c}))||g|^{s'/s_0'}
\]
\[
\leq \|T(f(1_{(3Q)^c}))\|_{L^{s_0}(Q)}
\]
\[
\leq |Q|^{1/s_0} M_{T,s_0} f(x).
\]
On the other hands, one has
\[
|F(1+iy)| \leq \int_Q |T(f(1_{(3Q)^c}))||g|^{s'/s_1'}
\]
\[
\leq \|T(f(1_{(3Q)^c}))\|_{L^{s_1}(Q)}
\]
\[
\leq |Q|^{1/s_1} M_{T,s_1} f(x).
\]
By using Hadamard’s three lines lemma, we have
\[
|F(\theta)| \leq |Q|^{(1-\theta)/s_0+\theta/s_1} M_{T,s_0} f(x)^{1-\theta} M_{T,s_1} f(x)^\theta
\]
\[
= |Q|^{1/s} M_{T,s_0} f(x)^{1-\theta} M_{T,s_1} f(x)^\theta,
\]
which yields
\[
M_{T,s} f(x) \leq M_{T,s_0} f(x)^{1-\theta} M_{T,s_1} f(x)^\theta.
\]
By Hölder’s inequality, we have
\[
\|(M_{T,s_0} f)^{1-\theta}(M_{T,s_1} f)\|_{L^{p,\infty}} \lesssim \|M_{T,s_0} f\|_{L^{p_0,\infty}}\|M_{T,s_1} f\|_{L^{p_1,\infty}}^{\theta}
\]
\[
\leq C_0^{1-\theta} C_1^\theta \|f\|_{L^r}
\]
for $1/p = (1-\theta)/p_0 + \theta/p_1$. By this and Proposition 3.1 we have $\|M_{T,s}\|_{L^r \rightarrow L^{p,\infty}} \lesssim C_0^{1-\theta} C_1^\theta$ and the desired sparse form bounds for $T$. 

\[\square\]
Corollary 3.2. Let $1 \leq r_0, r_1 \leq s_0, s_1, p_0, p_1 \leq \infty$. We assume linear operator $T$ satisfies

$$
\| M_{T,s_0}f \|_{L^{p_0},\infty} \leq C_0 \| f \|_{L^{r_0}},
\| M_{T,s_1}f \|_{L^{p_1},\infty} \leq C_1 \| f \|_{L^{r_1}}.
$$

and

$$
|Tf(x)| \leq T(\|f\|)(x) \quad \text{a.e. } x \in \mathbb{R}^n.
$$

Then, for any $\theta \in (0,1)$, we have

$$
\| M_{T,s} \|_{L^r \rightarrow L^p,\infty} \lesssim C_0^{1-\theta} C_1^\theta,
$$

where $1/s = (1-\theta)/s_0 + \theta/s_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. In particular, we have

$$
|\langle Tf, g \rangle| \lesssim (\|T\|_{L^r \rightarrow L^p,\infty} + C_0^{1-\theta} C_1^\theta) \Lambda_{q,s,\theta}^\alpha(f,g),
$$

where

$$
\frac{1}{\alpha} = \frac{1}{r} + \frac{1}{p'}.
$$

Proof. The proof is similar to that of Corollary 3.1. Let $Q$ and $x \in Q$. For any simple functions $f, g$ so that $\|g\|_{q'} = 1$, we define the analytic function $F$ on the open strip by

$$
F(z) = \int_Q T(fz1_{(3Q)^c})(x)g(x)dx,
$$

where

$$
f_z = \text{sgn}(f)|f|^{1/(1-z)/s_0 + z/r_1},
g_z = \text{sgn}(g)|g|^{1/(1-z)/s_1 + z/r_1}.
$$

From $|Tf| \leq T(\|f\|)$, we have

$$
|F(iy)| \leq \int_Q |T(fz1_{(3Q)^c})||g|^{s'/s_0}
\leq \|T(fz1_{(3Q)^c})\|_{L^{r_0}}(Q)
\leq |Q|^{1/s_0} M_{T,s_0}(\|f\|^{r_0})(x),
$$

and

$$
|F(1+iy)| \leq \int_Q |T(fz1_{(3Q)^c})||g|^{s'/s_1}
\leq \|T(fz1_{(3Q)^c})\|_{L^{r_1}}(Q)
\leq |Q|^{1/s_1} M_{T,s_1}(\|f\|^{r_1})(x).
$$

Hence, one obtains

$$
M_{T,s}f(x) \leq M_{T,s_0}(\|f\|^{r_0})(x)^{1-\theta} M_{T,s_1}(\|f\|^{r_1})(x)^{\theta}.
$$

By using Hölder’s inequality, we have

$$
\| (M_{T,s_0}(\|f\|^{r_0}))^{1-\theta} (M_{T,s_1}(\|f\|^{r_1}))^\theta \|_{L^{p',\infty}} \lesssim \|M_{T,s_0}(\|f\|^{r_0})\|_{L^{p_0},\infty}^{1-\theta} \|M_{T,s_1}(\|f\|^{r_1})\|_{L^{p_1},\infty}^\theta \leq C_0^{1-\theta} C_1^\theta \|f\|_{L^r}.
$$

We give a proof of Theorem 3.1.
Proof. We recall the dyadic decomposition in subsection 2.1. Since \( \phi_j \ast f = (\phi_{j-1} + \phi_j + \phi_{j+1}) \ast \phi_j \ast f \), we have

\[
|\langle a(x, D)f, g \rangle| = \sum_{j=0}^{j+1} \sum_{i=j-1}^{j+1} |\langle a_i(x, D)(\phi_j \ast f), g \rangle|
\]

\[
= \sum_{j=0}^{j+1} \sum_{i=j-1}^{j+1} |\langle a_i(x, D)(\lim_{R \to \infty} 1_{Q_R} \phi_j \ast f), g |\rangle|
\]

\[
\leq \lim_{R \to \infty} \sum_{j=0}^{j+1} \sum_{i=j-1}^{j+1} |\langle a_i(x, D)((\phi_j \ast f)1_{Q_R}), g \rangle|.
\]

Therefore, it is enough to prove

\[
|(a_j(x, D)f, g)| \lesssim 2^{jn} \Lambda_5^\alpha_s \Lambda_{\nu, \lambda, s'}(f, g)
\]

for any \( f \in L^\infty_c \) and \( g \in \mathcal{S} \). For any \( x, z \in Q \) and \( \gamma \in [0, 1] \), we integrate by parts \( N \in \mathbb{N} \) times to obtain

\[
|a_j(x, D)(f 1_{3Q})^N(z)| \lesssim 2^m+j/2 \left\{ \int (3Q)^\epsilon \left( 1 + 2^j |2^j N|^{2N} \right)^{-2} |f(y)|^2 \, dy \right\}^{1/2}
\]

\[
\lesssim 2^m+j/2 \left\{ \int (1 + 2^j |2^j N|^{2N} \right)^{-2} |f(y)|^2 \, dy \right\}^{1/2}
\]

\[
\lesssim 2^m+j/2 \sum_{k \in \mathbb{Z}} \left\{ \int |x-y|^{2-2j^2} \left( 1 + 2^{jN} \right)^{-2} |f(y)|^2 \, dy \right\}^{1/2}
\]

\[
\lesssim 2^m+j/2 \left( 1 -(1-\gamma)/2 \right) M^\gamma(\|f\|^2)(x)^{1/2},
\]

where \( M^\gamma h(x) := \sup_{Q \ni x} |Q|^{\gamma} \langle h \rangle_{Q} \). Hence, we obtain

\[
M_{a_j(x, D), \infty} \lesssim 2^m+j/2 \left( 1 -(1-\gamma)/2 \right) M^\gamma(\|f\|^2)(x)^{1/2}.
\]

By weak-type boundedness of \( M^\gamma \), for any \( p_0 \geq 2 \), one has

\[
\|M_{a_j(x, D), \infty}\|_{L^p_{\infty}} \lesssim 2^m+j/2 \left( 1 -(1-\gamma)/2 + j p N(1/2-1/p_0) \right)\|f\|_{L^2}
\]

by taking \( \gamma = 1 - 2/p_0 \). On the other hands, we have

\[
\|a_j(x, D)f\|_{L^p} \lesssim 2^m+j/2 \left( 1 -(1-\gamma)/2 \right) \|f\|_{L^2}
\]

for any \( p_1 \geq 2 \). From this and

\[
M_{a_j(x, D), p_1} f(x) \lesssim M_{p_1} (a_j(x, D)f)(x) + 2^m+j/2 \left( 1 -(1-\gamma)/2 \right) M_{p_1} \left( \|f\|^2 \right)(x)^{1/2},
\]

we obtain

\[
\|M_{a_j(x, D), p_1} f\|_{L_{p_1, \infty}} \lesssim 2^m+j/2 \left( 1 -(1-\gamma)/2 \right) \|f\|_{L^2}.
\]

Therefore, Corollary 3.1 gives

\[
|\langle a_j(x, D)f, g \rangle| \lesssim 2^m \omega j(1-\gamma)/2 - j \omega (1-\gamma)/2 - 1/p_0 - \theta/p_1 + 1 < 3/2.
\]

By simple calculation as following,

\[
(1-\theta)(1-\rho)/2 + \rho(1-\theta)(1/2-1/p_0) + \theta(1/2-1/p_1) = -\rho(1-\theta)/p_0 + 1/2 - \theta/p_1
\]

\[
= -\rho(1-1/\omega + 1/2 - 1/s) + 1/2 - 1/s
\]

\[
= (1-\rho)(1/2-1/s) + \rho(1/\omega - 1),
\]

we have the desired sparse bounds.

\( \square \)

To establish Theorem 3.2 by the interpolation argument as Corollary 3.2 we need the condition \( |a(x, D)f| \leq a(x, D)(|f|) \). Unfortunately, it fails in general and we need the following alternative argument:
Lemma 3.2. Let $0 \leq \gamma < 1$. We assume linear operator $T$ satisfies

$$||T||_{L^2 \rightarrow L^2} \leq C_0,$$

$$M_{T, \infty} f(x) \leq C_1 M^\gamma f(x) \text{ a.e. } x \in \mathbb{R}^n.$$

Then, for any $\theta \in (0, 1)$, we have

$$||M_{T,r^\gamma}||_{L^r \rightarrow L^{r, \infty}} \lesssim C_0 1^{1-\theta} C_1^\theta,$$

where $1/r = (1-\theta)/2 + \theta$ and $1/p = (1-\gamma)\theta + 1/r'$. In particularly, we have

$$|\langle T f, g \rangle| \lesssim (||T||_{L^{r, \infty} \rightarrow L^{r, \infty}} + C_0 1^{1-\theta} C_1^\theta) \Lambda_{S,r,r'}^\alpha(f, g),$$

where

$$\frac{1}{\alpha} = 1 + \left(\frac{2}{r} - 1\right) \gamma.$$

Proof. We put $E = \{M_{T,r^\gamma} f > \lambda\}$ for any $\lambda > 0$. For each $\delta > 0$, we have

$$|E| \leq |\{M_{T,r^\gamma} f > \lambda, M^\gamma f \leq \delta \lambda \}| + |\{M^\gamma f > \delta \lambda \}| =: |E_0| + |E_1|,$$

where $M^\gamma f = M^\gamma(|f|)^{1/r}$. Weak-type boundedness of $M^\gamma$ gives

$$|E_1|^{1/q} \lesssim \delta^{-1} \lambda^{-1} ||f||_{L^r},$$

with $1/q = 1/r - \gamma/r$. We need to estimate the $|E_0|$. For any $x \in E_0$, there exists a cube $Q_x$ such that

$$|Q_x| \leq \lambda^\gamma ||T(f 1_{(3Q_x)^c})||_{L^{r'}(Q_x)}. $$

Let $K \subset E_0$ be an any compact set, then we can select finite pairwise disjoint subcollection $\{3Q_j\}_j \subset \{3Q_x\}_{x \in E}$ such that

$$|K| \lesssim \sum_j |Q_j|.$$ 

From the duality of $\ell^{r'}(\mathbb{N}; L^{r'})$, we obtain

$$|K|^{1/r'} \leq \lambda^{-1} \left(\sum_j ||T(f 1_{(3Q_j)^c})||_{L^{r'}(Q_j)}^{r'}/r'\right)^{1/r'}$$

$$= \lambda^{-1} \sup_{\{g_j\}} \left|\sum_j \int_{Q_j} T(f 1_{(3Q_j)^c}) g_j \right|.$$

Here, the supremum is taken all over the $g = \{g_j\}_j$ such that $\|g\|_{\ell^{r'}(\mathbb{N}; L^{r'})} \leq 1$. We define the analytic function $F$ on the open strip by

$$F(z) = \sum_j \int_{Q_j} T(f 1_{(3Q_j)^c})(x)g_{z,j}(x)dx,$$

where

$$f_z = \text{sgn}(f)|f|^{1/(1-z/2+z)},$$

$$g_{z,j} = \text{sgn}(g_j)|g_j|^{1/(1-z/2+z)}.$$

By $L^2$ boundness of $T$, one has

$$|F(iy)| \leq \sum_j ||TF_z||_{L^2(Q_j)} ||g_j||_{L^2}^{r'/2} + C_0 \sum_j ||f_z||_{L^2(3Q_j)} ||g_j||_{L^2}^{r'/2}$$

$$\leq ||TF_z||_{L^2} + C_0 ||f_z||_{L^2}$$

$$\lesssim C_0 ||f||_{L^2}^{r'/2}.$$
Since $Q_j \cap E_0 \neq \emptyset$, we obtain
\[
|F(1 + iy)| \leq \sum_j \inf_{x \in Q_j} M_{T, \infty} f_z(x)\|g_j\|_{L^r}^r
\leq C_1 \sum_j \inf_{x \in Q_j} M^\alpha_x f(x)^r\|g_j\|_{L^r}^r
\leq C_1 \delta^r \lambda^r.
\]
By these results, we have
\[
|K| \leq (C_0^{1-\theta} C_1^{\theta})^{r'} \delta^{-r\theta} \lambda^{r'\theta - r'} \|f\|_{L^{r'}}^{r'(1-\theta)/2}
= (C_0^{1-\theta} C_1^{\theta})^{r'} \delta^{-r\theta} \lambda^{-r} \|f\|_{L^r}^{r'}. 
\]
and
\[
|E| \leq \delta^{-\theta} \lambda^{-q} \|f\|_{L^r}^q + (C_0^{1-\theta} C_1^{\theta})^{r'} \delta^{-r\theta} \lambda^{-r} \|f\|_{L^r}^{r'}.
\]
Here, we optimize for $\delta$ to obtain
\[
|E|^{1/p} \leq \lambda^{-1} \|f\|_{L^r},
\]
where $1/p = r\theta/q + 1/r' = (1 - \gamma)\theta + 1/r'$. Hence, $M_{T, r'}$ be a weak-type $(r, p)$ operator which yields
\[
|(Tf, g)| \lesssim (\|T\|_{L^p \to L^\infty} + C_0^{1-\theta} C_1^\theta) \Lambda_{S, r, r}(f, g).
\]

Let us prove the Theorem \ref{thm:3.2}

Proof. The theorem follows from the pointwise estimate
\[
M_{a_j(x, D), \infty} f(x) \lesssim 2^{jm + jn(1 - \rho(1 - \gamma))} M^\gamma f(x),
\]
Lemma \ref{lem:3.2} and Marchinkiewicz interpolation theorem. Indeed, this estimate and Lemma \ref{lem:3.2} yield
\[
\|M_{a_j(x, D), s}\|_{L^{r'} \to L^{r'/\infty}} \lesssim 2^{jm + jn(1 - \rho)(1 - 2/s) + jmn(\alpha - 1)}
\]
by taking $1/\alpha = 1 + (2/r - 1)\gamma$. Moreover, by interpolating this with $\alpha = 1$ and $\|M_{a_j(x, D), s}\|_{L^2 \to L^\infty} \lesssim 2^{jm + jn(1 - \rho)(1/2 - 1/s)}$, we have
\[
\|M_{a_j(x, D), s}\|_{L^{r-\infty} \to L^{r'/\infty}} \lesssim 2^{jm + jn(1 - \rho)(1/r - 1/s)}.
\]
Thus, we obtain the desired sparse form bounds. Now, we prove the above pointwise estimate. For any $x, z \in Q$ and $\gamma \in [0, 1)$, we integrate by parts $N \in \mathbb{N}$ times to obtain
\[
|a_j(x, D)(f 1_{(3Q)^c})(z)| \lesssim 2^{jm + jn} \int_{(3Q)^c} (1 + 2^{2j} p_N |z - y|^{2N})^{-1} |f(y)|dy 
\leq 2^{jm + jn} \int_{(3Q)^c} (1 + 2^{2j} p_N |x - y|^{2N})^{-1} |f(y)|dy 
\leq 2^{jm + jn(1 - \rho(1 - 1))} M^\gamma f(x).
\]
Hence, we obtain
\[
M_{a_j(x, D), \infty}(x) \lesssim 2^{jm + jn(1 - \rho(1 - \gamma))} M^\gamma f(x),
\]
and complete the proof. \qed
3.2 Application to the boundedness on weighted Besov spaces

This section is devoted to obtain the boundedness on weighted Besov space of pseudodifferential operators. To do this, we establish the weighted bounds for $\Lambda^w_{S,r,s'}$ by using Bernicot, Frey and Petermichl’s idea in [3].

**Proposition 3.2.** Let $1 \leq r < q < p < s \leq \infty$ and $1/\alpha = 1/p' + 1/q$. We assume the weight $\omega$ satisfy $\omega^q \in A_{q/r} \cap RH_{(p/q)(s/p')}$. Then, for any sparse family $\mathcal{S} \subset \mathcal{D}$ with some dyadic lattice $\mathcal{D}$, we have

$$\Lambda^w_{S,r,s'}(f,g) \lesssim (\sum_{Q \in \mathcal{S}} |Q| |f|^\alpha_{Q} |g|^{\alpha/s'}_{Q} F_Q^\alpha G_Q^\alpha)^{1/\alpha}$$

where

$$\delta = \max \left\{ \frac{1}{q-r}, \frac{p(s-1)}{q(s-p)} \right\}.$$

**Proof.** We set

$$\mu = \omega^{-q/(q-r)} \text{ and } \nu = \omega^{p'/s'/(p'-s')}.$$ 

Furthermore, let us define

$$F_Q = \left( \frac{1}{\mu(Q)} \int_Q |f|^r \right)^{1/r} \text{ and } G_Q = \left( \frac{1}{\nu(Q)} \int_Q |g|^{s'} \right)^{1/s'}.$$ 

Then, we have

$$\Lambda^w_{S,r,s'}(f,g) \leq \left( \sum_{Q \in \mathcal{S}} |Q||\mu|^\alpha_{Q} |\nu|^{\alpha/s'}_{Q} F_Q^\alpha G_Q^\alpha \right)^{1/\alpha}.$$ 

We estimate $|Q||\mu|^\alpha_{Q} |\nu|^{\alpha/s'}_{Q}$. By taking

$$\beta = 1 + \frac{1}{r} - \frac{1}{q} - \frac{1}{p} - \frac{1}{s'},$$

we obtain $\mu = \nu^{1-\beta}$ and $|Q|^{\alpha/s'}_{Q} \leq |Q|^{\alpha}_{A_{q/r}}$. Here, we assume

$$\frac{1}{q-r} \leq \frac{p(s-1)}{q(s-p)},$$

which gives $\gamma := 1/r - (\beta - 1)/s' \leq 0$. From this assumption and the sparseness of $\mathcal{S}$, one obtains

$$|Q||\mu|^\alpha_{Q} |\nu|^{\alpha/s'}_{Q} \leq |\nu|^{\alpha/s'}_{A_{q/r}} |Q||\mu|^\alpha_{Q}.$$

On the other hand, it holds that $\mu^{-\gamma} \mu^{1/q} \nu^{1/p'} = 1$ since $\nu = \mu^{1-\beta}$. Hence, by setting $1/t = 1/q + 1/p' - \gamma = 1/\alpha - \gamma$ and using Hölder’s inequality, we have

$$|E_Q|^{1/t} = \left( \mu^{-\gamma} \mu^{1/q} \nu^{1/p'} \right)_{L^t(E_Q)} \leq \mu_{L^t(E_Q)}^{-\gamma} \mu_{L^t(E_Q)}^{1/q} \nu_{L^t(E_Q)}^{1/p'},$$

which yields

$$|Q||\mu|^\alpha_{Q} |\nu|^{\alpha/s'}_{Q} \leq |\nu|^{\alpha/s'}_{A_{q/r}} \mu_{L^t(E_Q)}^{1/q} \nu_{L^t(E_Q)}^{1/p'}.$$ 

From these results, we obtain

$$\Lambda^w_{S,r,s'}(f,g) \leq |\nu|^{1/s'}_{A_{q/r}} \left( \sum_{Q \in \mathcal{S}} (F_Q \mu_{L^t(E_Q)}^{1/q} G_Q \nu_{L^t(E_Q)}^{1/p'})^\alpha \right)^{1/\alpha}.$$

$$\leq |\nu|^{1/s'}_{A_{q/r}} \left( \sum_{Q \in \mathcal{S}} F_Q^{\alpha} \mu_{L^t(E_Q)} \right)^{1/q} \left( \sum_{Q \in \mathcal{S}} G_Q^{\alpha} \nu_{L^t(E_Q)} \right)^{1/p'} \leq |\nu|^{1/s'}_{A_{q/r}} \left( \int |M_{\nu^{-1}}(f)\nu^{1/r})|^q d\mu \right)^{1/q} \left( \int |M_{\nu^{-1}}(g\nu^{1/s})|^{q/p'} d\nu \right)^{1/p'} \lesssim |\nu|^{1/s'}_{A_{q/r}} ||f||_{L^t(\omega)} ||g||_{L^{p'}(\omega^{-s'})}.$$
In another case, by using

\[
|Q|\langle \mu \rangle_{Q}^{\alpha/r} \langle \nu \rangle_{Q}^{\alpha/s'} \leq [\nu]_{A_{\kappa}}^{\alpha/(r(\beta-1))}|Q|\langle \nu \rangle_{Q}^{\alpha/s'-\alpha/(r(\beta-1))} \\
\leq [\nu]_{A_{\kappa}}^{\alpha/(r(\beta-1))} |E_{Q}|^{1-\alpha} \left( \int_{E_{Q}} \mu \right)^{\alpha},
\]

and the same discussion as above, we have

\[
\Lambda_{\alpha}^{s}(f,g) \leq [\nu]_{A_{\kappa}}^{\alpha/(r(\beta-1))} \|f\|_{L^{\kappa}(\omega)} \|g\|_{L^{\rho}(\omega)}. \]  

Concluding these results, we have

\[
\Lambda_{\alpha}^{s}(f,g) \leq [\nu]_{A_{\kappa}}^{\delta} \|f\|_{L^{\kappa}(\omega)} \|g\|_{L^{\rho}(\omega)},
\]

where

\[
\delta = \max \left\{ \frac{q(s-p)}{ps(q-r)}, \frac{s-1}{s} \right\}.
\]

To complete the proof, we need to estimate \([\nu]_{A_{\kappa}}^{s}\). However, it is deduced from the simple calculation. The detail is the following:

\[
\langle \nu \rangle_{Q} \langle \mu \rangle_{Q}^{\beta-1} = \omega^{q(p/s)/p'}/q \omega^{q(\nu/(q-r))} p(s/p')/q (q/r-1) \\
\leq (\omega^{q(p/s)/p'}/q \omega^{q(\nu/(q-r))} p(s/p')/q) \\
\leq (\omega^{q(p/s)/p'}/q \omega^{q(\nu/(q-r))} p(s/p')/q).
\]

We define the weighted Besov spaces according to Bui [4]. Suppose \(0 < p, \sigma < \infty\) and \(\kappa \in \mathbb{R}\), then weighted Besov spaces \(B_{p,\kappa}^{\sigma}(\omega)\) are defined by

\[
B_{p,\kappa}^{\sigma}(\omega) = \left\{ f \in \mathcal{S}'; \| f \|_{B_{p,\kappa}^{\sigma}(\omega)} < \infty \right\},
\]

\[
\| f \|_{B_{p,\kappa}^{\sigma}(\omega)} = \left( \sum_{j \geq 0} 2^{j\kappa\sigma} \| \phi_{j} * f \|_{L^{p}(\omega)} \right)^{1/\sigma}
\]

for any \(\omega \in A_{\infty}\). Bui showed that \(\mathcal{S}'\) is dense subset of \(B_{p,\kappa}^{\sigma}(\omega)\). Hence, the Theorem 8.1 and Theorem 8.2, and Proposition 8.2 give the following results about boundedness of pseudodifferential operators on weighted Besov spaces.

**Corollary 3.3.** Let \(a \in S_{p,\delta}^{m}\) with \(m \in \mathbb{R}\) and \(0 \leq \delta < p \leq 1\). Then, we have the following bounds.

(i) Let \(2 < q \leq p < \infty\) and \(\omega^{\kappa} \in A_{q/2} \cap RH_{(p/q)(s/p')}\) with some \(s \in (p, \infty]\). Then, for any \(\kappa \in \mathbb{R}\) and \(0 < \sigma < \infty\), \(a(x,D)\) be a bounded operators from \(B_{q,\kappa}^{\alpha+\hat{\kappa}_{1}}(\omega^{\kappa})\) to \(B_{p,\kappa}^{\sigma}(\omega^{\kappa})\) where

\[
\hat{\kappa}_{1} = m + n(1-\rho) (1/2 - 1/s) + mn (1/q - 1/p).
\]

(ii) Let \(1 < q \leq 2 \leq p \leq q' < \infty\) and \(\omega^{\kappa} \in A_{q} \cap RH_{(p/q)(r'/p')}\) with some \(r \in [1,q]\). Then, for any \(\kappa \in \mathbb{R}\) and \(0 < \sigma \leq \infty\), \(a(x,D)\) be a bounded operators from \(B_{q,\kappa}^{\alpha+\hat{\kappa}_{2}}(\omega^{\kappa})\) to \(B_{p,\kappa}^{\sigma}(\omega^{\kappa})\) where

\[
\hat{\kappa}_{2} = m + n(1-\rho) (1/2 - 1/r) + mn (1/q - 1/p).
\]

**Remark 3.2.** The Corollary 3.3 contains the following known boundedness results of \(a(x,D)\) with \(a \in S_{p,\delta}^{m}\).

(i) By taking \(\omega = 1\), \(p = q\) and suitable \(s\) in neighborhood of \(p\) in (i) of Corollary 3.3, we have the \(L^{p}\)-boundedness with \(m < -n(1-\rho)[1/p - 1/2]\) which was estimated by Fefferman [13].

(ii) By taking \(p = q\) and sufficiently large \(s\) in (i) of Corollary 3.3, we have the \(L^{p}(\omega)\)-boundedness with \(m < -n(1-\rho)/2\) and \(\omega \in A_{p/2}\) which was established by Chanillo and Torchinsky [9].

(iii) By taking \(r = 1\) and \(p = q\) in (ii) of Corollary 3.3, we have the \(L^{p}(\omega)\)-boundedness with \(m < -n(1-\rho)\) and \(\omega \in A_{p}\) which was established by Michalowski, Rule and Staubach [7].
Proof. First, we assume $1 \leq \sigma < \infty$. For any $\ell \in \mathbb{Z}$, there exists $b_\ell \in S^{m+\ell}_{\rho,\delta}$ so that $b(x, D) = \langle D \rangle^{j} a(x, D)$ since $\delta < \rho$. From this, we have
\[
||\phi_k * a(x, D)(\phi_j * f) || = ||\langle D \rangle^{-\ell} \phi_k * \langle D \rangle^{\ell} a(x, D)(\phi_j * f) || \\
= ||\langle b_\ell(x, D) f, \langle D \rangle^{-\ell} \phi_k(-\cdot) * g || \\
\lesssim 2^{j+\ell+\tilde{p}(\rho, q)} \liminf_{|R| \to \infty} \Lambda_0^\alpha_{r(p, q), s(p, q)} ||(\phi_j * f) 1_{Q_R}, \langle D \rangle^{-\ell} \phi_k(-\cdot) * g || \\
\lesssim 2^{j+\ell+\tilde{p}(\rho, q)} ||\phi_j * f ||_{L^{\sigma}(\omega)} ||\langle D \rangle^{-\ell} \phi_k(-\cdot) * g ||_{L^{p'}(\omega')},
\]
where
\[
(\tilde{\kappa}(p, q), r(p, q), s(p, q)) = \begin{cases} 
(1, 2, s') & 2 < q \leq p < \infty \\
(1, 2, 2) & 1 < q \leq p \leq q' < \infty
\end{cases}
\]
and $\alpha = 1/p' + 1/q$. After this, we write $\tilde{\kappa} = \tilde{\kappa}(p, q)$. Now, we obtain
\[
||\langle D \rangle^{\ell} \phi_k(-\cdot) * g(x) || \lesssim ||\langle D \rangle^{-\ell} \phi_k ||_{L^1} M g(x) \lesssim 2^{-k\ell} M f(x).
\]
Combining this and $\omega^{-p'} \in A_{p'}$, we obtain
\[
||\phi_k * a(x, D)(\phi_j * f) ||_{L^p(\omega')} \lesssim 2^{-k\ell} 2^{j+\ell+\tilde{p}(\rho, q)} ||\phi_j * f ||_{L^{\sigma}(\omega')}. 
\]
The Besov norm of $a(x, D)$ is handled by
\[
I_1 + I_2 := \left( \sum_{k \geq 0} 2^{k\kappa \sigma} \left( \sum_{0 \leq j \leq k} \| \phi_k * a(x, D)(\phi_j * f) \|_{L^p(\omega')} \right) \right)^{\sigma} + \left( \sum_{k \geq 0} 2^{k\kappa \sigma} \left( \sum_{k < j} \| \phi_k * a(x, D)(\phi_j * f) \|_{L^p(\omega')} \right) \right)^{\sigma}.
\]
Our purpose is to control $I_1$ and $I_2$ by $\| f \|_{B^{\kappa+\kappa}(\omega')^s}$. First, we give an estimation of $I_1$. From the observation above, we obtain
\[
I_1 \lesssim \left( \sum_{k \geq 0} 2^{k\kappa \sigma - k\ell \sigma} \left( \sum_{0 \leq j \leq k} 2^{j+\ell+\tilde{\kappa}} \| \phi_j * f \|_{L^\sigma(\omega')} \right) \right)^{\sigma} \\
\lesssim \left( \sum_{k \geq 0} 2^{k\kappa \sigma - k\ell \sigma} \left( \sum_{-k \leq j \leq 0} 2^{(j+k)\ell + (j+k)\tilde{\kappa}} \| \phi_{j+k} * f \|_{L^\sigma(\omega')} \right) \right)^{\sigma} \\
\lesssim \sum_{j \leq 0} 2^{j+\ell+\tilde{\kappa}} \left( \sum_{k \geq j} 2^{k(\kappa+\tilde{\kappa})} \| \phi_{j+k} * f \|_{L^\sigma(\omega')} \right)^{1/\sigma} \\
= \sum_{j \leq 0} \| f \|_{B^{\kappa+\kappa}(\omega')} \\
\lesssim \| f \|_{B^{\kappa+\kappa}(\omega')},
\]
by taking sufficiently large $\ell$. On the other hands, the same calculation gives
\[
I_2 \lesssim \left( \sum_{k \geq 0} 2^{k\kappa \sigma - k\ell \sigma} \left( \sum_{k < j} 2^{j+\ell+\tilde{\kappa}} \| \phi_j * f \|_{L^\sigma(\omega')} \right) \right)^{\sigma} \\
\lesssim \left( \sum_{k \geq 0} 2^{k\kappa \sigma - k\ell \sigma} \left( \sum_{j > 0} 2^{(j+k)\ell + (j+k)\tilde{\kappa}} \| \phi_{j+k} * f \|_{L^\sigma(\omega')} \right) \right)^{\sigma} \\
\lesssim \sum_{j > 0} 2^{j+\ell+\tilde{\kappa}} \left( \sum_{k \geq 0} 2^{k(\kappa+\tilde{\kappa})} \| \phi_{j+k} * f \|_{L^\sigma(\omega')} \right)^{1/\sigma} \\
= \sum_{j > 0} \| f \|_{B^{\kappa+\kappa}(\omega')} \\
\lesssim \| f \|_{B^{\kappa+\kappa}(\omega')}.
\]
by taking $\ell \ll -1$. Hence, we complete the proof in the case of $1 \leq \sigma < \infty$. To complete the proof, we treat the case of $0 < \sigma < 1$. However, this case is proved in a same manner by using $\sigma$-triangle inequality on $\ell^\sigma$. \hfill \blacksquare

### 3.3 The special case of pseudodifferential operators

For a given $-1 \leq \rho < 1$, $U_\rho f$ denotes the solution of

\[
\begin{cases}
   i\partial_t u + (-\Delta)^{(1-\rho)/2} u = 0, \\
   u(0) = f.
\end{cases}
\]

$U_\rho$ with $0 \leq \rho < 1$ can be regarded as a pseudodifferential operators associated $S^0_{\rho,0}$, and therefore gives sparse bounds in Theorem 3.1 and Theorem 3.2. However, we can improve the above results:

**Theorem 3.3.** Let $1 \leq r \leq 2$ and $-1 \leq \rho < 1$.

(i) Given $\rho \neq 0$, $1/r + 1/2 < 1/\alpha < 2/r$ and $f, g \in \mathcal{S}$, there exist the sequence of sparse families $\{S_j\}_{j=0,1,\ldots}$ such that

\[
|\langle U_\rho f(t), g \rangle| \lesssim t^n(1/r+1/2-(1/\alpha-1)) \liminf_{R \to \infty} \sum_{j \geq 0} 2^{j\kappa_4} \Lambda^\alpha_{S_j}(f \ast g) 1_{Q_R}(g),
\]

where $\kappa_4 = n(1-\rho)(1/r - 1/2) + \rho n(1/\alpha - 1)$.

(ii) Given $\alpha \in \mathbb{R}$ such that

\[
\frac{n+1}{rn} + \frac{n-1}{2n} < \frac{1}{\alpha} < \frac{2}{r},
\]

and $f, g \in \mathcal{S}$, there exist the sequence of sparse families $\{S_j\}_{j=0,1,\ldots}$ such that

\[
|\langle U_0 f(t), g \rangle| \lesssim f(n+1)(1/r-1/2-n(1/\alpha-1)) \liminf_{R \to \infty} \sum_{j \geq 0} 2^{j\kappa_5} \Lambda^\alpha_{S_j}(f \ast g) 1_{Q_R}(g),
\]

where $\kappa_5 = (n+1)(1/r - 1/2)$.

**Proof.** (i) It suffices to prove the pointwise estimate

\[
M_{U_\rho,j,\alpha}(f(t,x)) \lesssim t^{-n(\gamma-1/2)} 2^{jn(1-\rho)/2+jn\rho\gamma} M^\gamma f(x)
\]

for any $1/2 < \gamma < 1$ where $U_\rho,j f = U_\rho(\phi_j * f)$. Take any cube $Q$ and any $x, z \in Q$. First, we consider the case $j \geq 1$ and $2^{-j(1-\rho)} \leq t$. We integrate by parts $N \in \mathbb{N}$ times to obtain

\[
|U_\rho,j(f1_{(3Q)^c})(t,z)| \lesssim \int_{(3Q)^c} (1 + 2^j \rho N t^{-2N} |z-y|^{2N})^{-1} |f(y)| \left| \int e^{iz \cdot \xi}(1 + 2^j \rho N t^{-2N} \Delta^N)(e^{it|\xi|^{1-\rho}} \hat{\phi}_j(\xi)) d\xi \right| dy
\]

\[
\lesssim 2^{-j\rho n(1-\gamma)} t^{-n(1-\gamma)} M^\gamma f(x) \left| \int e^{iz \cdot \xi}(1 + 2^j \rho N t^{-2N} \Delta^N)(e^{it|\xi|^{1-\rho}} \hat{\phi}_j(\xi)) d\xi dy \right|.
\]

To obtain desired pointwise estimate, we need to prove

\[
\sup \left| \int e^{iw \cdot \xi}(1 + 2^j \rho N t^{-2N} \Delta^N)(e^{it|\xi|^{1-\rho}} \hat{\phi}_j(\xi)) d\xi dy \right| \lesssim 2^{jn(1+\rho)/2} t^{-n/2}.
\]

By the Leibniz formula, we have

\[
\Delta^N((e^{it|\xi|^{1-\rho}} \hat{\phi}_j(\xi))) = \sum_{|\alpha| = 2N} \sum_{\beta \leq \alpha} (\partial^{n-\beta} e^{it|\xi|^{1-\rho}})(\partial^\beta \hat{\phi}_j(\xi))
\]

\[
= 2^{jn(1-\rho)} \sum_{|\alpha| = 2N} \sum_{\beta \leq \alpha} (P_{\alpha,\beta}(\xi))(\partial^\beta \hat{\phi}_j(\xi)),
\]

where $P_{\alpha,\beta}$ denotes the functions such that

\[
\|\partial^\beta P_{\alpha,\beta}(2^j \cdot)\|_{L^\infty} \lesssim (2^{-j} + t2^{-j}\rho)^{2N-|\beta|}
\]
on support of $\hat{\psi}$ for any $\sigma \in \mathbb{N}^n$. By using Littman’s lemma, we have

$$\left| \int e^{iw\xi} \left( 1 + 2^{j+1} \phi \right) (e^{it|\xi|^{-\rho}} \hat{\phi}(\xi)) d\xi dy \right| \lesssim \left| \int e^{iw\xi+it|\xi|^{1-\rho}} \hat{\phi}(\xi) d\xi \right| + 2^{2j} t^{-2N} \sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} \int e^{iw\xi+it|\xi|^{1-\rho}} P_{\alpha,\beta}(\xi) \partial^\beta \hat{\phi}(\xi) d\xi$$

$$\lesssim 2^n \left| \int e^{iw\xi+it|2^{(1-\rho)}\xi|^{1-\rho}} \hat{\phi}(\xi) d\xi \right| + 2^n \sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} \int e^{iw\xi+it2^{(1-\rho)}|\xi|^{1-\rho}} P_{\alpha,\beta}(\xi) \partial^\beta \hat{\phi}(\xi) d\xi$$

$$\lesssim 2^n (\gamma|2^{1/2} t^{-n/2} + 2^n \sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|})$$

$$\lesssim 2^n (\gamma|2^{1/2} t^{-n/2}$$

Here, the last inequality follows from

$$\sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|} \lesssim t^2 2^{N-2j+1}.$$

The case of $j \geq 1$ and $2^{1-\rho} \leq t^{-1}$ is obtained from

$$|U_{\rho,\gamma}(f1_{(3Q)^c})(t, z)| \lesssim \int_{(3Q)^c} (1 + 2^{2jN} |z - y|^{2N})^{-1} |f(y)| \int e^{i(z-y)\xi} (1 + 2^{2jN} \Delta^N) (e^{it|\xi|^{1-\rho}} \hat{\phi}(\xi)) d\xi dy$$

$$\lesssim 2^{j} (\gamma \sum_{\gamma = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|})$$

$$\lesssim 2^{j} (\gamma \sum_{\gamma = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|})$$

Here, we use the condition $\gamma \geq 2$ to obtain

$$2^{jn} \gamma \lesssim 2^{jn} (\gamma |2^{1/2} t^{-n/2} + 2^n \sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|}) \lesssim t^n (\gamma |2^{1/2} t^{-n/2} + 2^n \sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|})$$

When $j = 0$, we recall $\phi_0 = \sum_{\xi \leq 0} \psi_\xi$ and obtain

$$|U_{\rho,\gamma}(f1_{(3Q)^c})(t, z)| \lesssim \sum_{\xi \leq 0} \left| \int e^{i(z-y)\xi} \hat{\psi}_\xi(\xi) f(y) 1_{(3Q)^c}(y) dy d\xi \right|$$

$$\lesssim \sum_{\xi \leq 0} \left| \int e^{i(z-y)\xi} \hat{\psi}_\xi(\xi) f(y) 1_{(3Q)^c}(y) dy d\xi \right|$$

where $\tau = \max\{1, \gamma\}$. Since $||\partial^\gamma P_{\alpha,\beta}(2^\gamma)\| \lesssim \tau^{1-|\gamma|} 2^{-\ell(|\alpha| - |\beta|)}$, one has

$$|U_{\rho,\gamma}(f1_{(3Q)^c})(t, z)| \lesssim \left( \sum_{\xi \leq 0} 2^{n(\gamma - 1/2)} \right) t^n (\gamma |2^{1/2} t^{-n/2} + 2^n \sum_{|\alpha| = 2N} \sum_{|\beta| \leq \alpha} 2^{-j|\beta|} (2^{-j} + t^2) 2^{N-|\beta|})$$

(ii) It suffices to prove the pointwise estimate

$$M_{\rho,\infty} f(t, x) \lesssim t^{-n(\gamma - 1/2) + 1/2} 2^{n(1+1/2) M^\gamma f(x)}$$

for any $(n + 1)/2n < \gamma < 1$. Take any cube $Q$ and any $x, z \in Q$. First, we consider the case $j \geq 1$ and $2^{-j} \leq t$. We integrate by parts $N \in \mathbb{N}$ times to obtain

$$|U_{\rho,\gamma}(f1_{(3Q)^c})(t, z)| \lesssim t^n (\gamma M^\gamma f(x) \int e^{i(z-y)\xi} (1 + t^{-2N} \Delta^N) (e^{it|\xi|^{1-\rho}} \hat{\phi}(\xi)) d\xi dy |.$$
By using Littman’s lemma, we have
\[
\left| \int e^{i\omega \xi} (1 + t^{-2N} \Delta^N)(e^{it\xi} \hat{\phi}_j(\xi)) d\xi d\tau \right| \\
\lesssim \left| \int e^{i\omega \xi+it\xi} \hat{\psi}(\xi) d\xi \right| + t^{-2N} \sum_{|\alpha|=2N} \sum_{\beta \leq \alpha} \left| \int e^{i\omega \xi+it\xi} P_{\alpha,\beta}(\xi) \partial^\beta \hat{\phi}_j(\xi) d\xi \right|
\]
\[
\lesssim 2^{jn} \left| \int e^{i\omega \xi+it2^j\xi} \hat{\psi}(\xi) d\xi \right| + 2^{jn} t^{-2N} \sum_{|\alpha|=2N} \sum_{\beta \leq \alpha} 2^{-j|\beta|} \left| \int e^{i\omega \xi+it2^j\xi} P_{\alpha,\beta}(2^j\xi) \partial^\beta \hat{\psi}(\xi) d\xi \right|
\]
\[
\lesssim 2^n (n+1)/2 t^{-(n-1)/2} + 2^{j(n+1)/2} t^{-(n-1)/2-2N} \sum_{|\alpha|=2N} \sum_{\beta \leq \alpha} 2^{-j|\beta|} (2^{-j} + t)^{2N-|\beta|}
\]
\[
\lesssim 2^{j(n+1)/2} t^{-(n-1)/2}
\]
for any \( w \in \mathbb{R}^n \). The case of \( j \geq 1 \) and \( 2^j \leq t^{-1} \) is obtained from
\[
|U_{0,j}(f 1_{(3Q)^c})(t, z)| \lesssim \int_{(3Q)^c} (1 + 2^{2jN} |z - y|^{2N})^{-1} f(y) \left| \int e^{i(z-y)\xi} (1 + 2^{2jN} \Delta^N)(e^{it\xi} \hat{\phi}_j(\xi)) d\xi \right| dy
\]
\[
\lesssim 2^{-jn(1-\gamma)} M^\gamma f(x) \left| \int e^{i(z-y)\xi} (1 + 2^{2jN} \Delta^N)(e^{it\xi} \hat{\phi}_j(\xi)) d\xi dy \right|
\]
\[
\lesssim 2^{jn\gamma} M^\gamma f(x)
\]
\[
\lesssim t^{-(n-2\gamma)+1/2} 2^{j(n+1)/2} M^\gamma f(x).
\]
Here, we use the condition \( \gamma \geq (n+1)/2n \) to obtain
\[
2^{jn\gamma} = 2^{-j(n+1)/2+jn\gamma} 2^{j(n+1)/2} \lesssim t^{-(n-1)/2} 2^{j(n+1)/2}.
\]
When \( j = 0 \), we recall \( \phi_0 = \sum_{\ell \leq 0} \psi_{\ell} \) and obtain
\[
|U_{0,0}(f 1_{(3Q)^c})(t, z)| \lesssim \sum_{\ell \leq 0} \int e^{i(z-y)\xi} a(z, \xi) \hat{\psi}(\xi) f(y) 1_{(3Q)^c} (y) dy d\xi
\]
\[
\lesssim \sum_{\ell \leq 0} \int_{(3Q)^c} (1 + 2^{2N} \tau^{-2N} |z - y|^{2N})^{-1} f(y) \left| \int e^{i(z-y)\xi} (1 + 2^{2N} \tau^{-2N} \Delta^N)(e^{it\xi} \hat{\psi}(\xi)) d\xi dy \right|
\]
\[
\lesssim \sum_{\ell \leq 0} 2^{-fn(1-\gamma)\tau^{-n(1-\gamma)/2} M^\gamma f(x) \left| \int e^{i(z-y)\xi} (1 + 2^{2N} \tau^{-2N} \Delta^N)(e^{it\xi} \hat{\psi}(\xi)) d\xi dy \right|,
\]
where \( \tau = \max\{1, t\} \). Since \( \|\partial^\alpha P_{\alpha,\beta}(2^j \cdot)\|_{L^\infty} \lesssim \tau^{-|\alpha|-|\beta|} 2^{-\ell(|\alpha|-|\beta|)} \), one has
\[
|U_{0,0}(f 1_{(3Q)^c})(t, z)| \lesssim \left( \sum_{\ell \leq 0} 2^{fn(1-\gamma)-(n+1)/2} \right) \tau^{n(1-\gamma)-(n+1)/2} M^\gamma f(x)
\]
\[
\lesssim t^{-(n-1)/2} M^\gamma f(x).
\]

\[\square\]

Theorem 3.3 and Proposition 3.2 give the boundedness of \( U_\rho \) on weighted Besov spaces.

**Corollary 3.4.** Let \(-1 \leq \rho < 1, 1 < q \leq 2 \leq p \leq q' < \infty \) and \( \omega^q \in A_{q/r} \cap RH_{(p/q)(r'/p')} \) with some \( r \in [1, q) \).

(i) If \( \rho \neq 0 \) and
\[
\frac{1}{r} - \frac{1}{2} \leq \frac{1}{q} - \frac{1}{p},
\]
then for any \( \kappa \in \mathbb{R} \) and \( 0 < \sigma \leq \infty \), \( U_\rho(t) \) be a bounded operators from \( B_{q,\sigma}^{\kappa+\bar{\kappa}}(\omega^q) \) to \( B_{p,\sigma}^\kappa(\omega^p) \) where
\[
\bar{\kappa}_1 = n(1-\rho) \left( \frac{1}{r} - \frac{1}{2} \right) + \rho m \left( \frac{1}{q} - \frac{1}{p} \right).
\]

Furthermore, we have
\[
\|U_\rho(t)\|_{B_{q,\sigma}^{\kappa+\bar{\kappa}}(\omega^q) \to B_{p,\sigma}^\kappa(\omega^p)} \lesssim t^{-n(1/q-1/p)-(1/r-1/2)} \omega^q_{A_q/r} \omega^p_{RH_{(p/q)(r'/p')}}^\delta.
\]
with \( \delta \) in Proposition 3.3.

(ii) If

\[
\frac{n+1}{n} \left(\frac{1}{r} - \frac{1}{2}\right) \leq \frac{1}{q} - \frac{1}{p},
\]

then for any \( \kappa \in \mathbb{R} \) and \( 0 < \sigma \leq \infty \), \( U_\rho(t) \) be a bounded operators from \( B^\kappa_{q,\sigma}(\omega^q) \) to \( B^\kappa_{p,\sigma}(\omega^p) \) where

\[
\tilde{r}_5 = (n+1) \left(\frac{1}{r} - \frac{1}{2}\right).
\]

Furthermore, we have

\[
\|U_\rho(t)\|_{\tilde{B}^\kappa_{q,\sigma}(\omega^q) \to \tilde{B}^\kappa_{p,\sigma}(\omega^p)} \leq t^{-(n(1/q-1/p)-(n+1)(1/r-1/2))} (\|\omega^q\|_{A_{q/r},[\omega^q]} \|RH_{(p/q)(r'/p)}\|)^{\delta}.
\]

### 3.4 A sharpness of weighted boundedness of pseudodifferential operators

In previous sections and subsections, we obtain some weighted inequalities for pseudodifferential operators and the time evolution \( U_\rho(t) \) of dispersive equations. In this subsection, we insure a sharpness of some of these inequalities as follows:

**Proposition 3.3.** Let \( 1 < q \leq p \leq q' < \infty \) and \( \gamma \in [1, \infty) \), and \( a(\xi) = e^{i|\xi|^1_{q'}} |\xi|^m \) with \( m \in \mathbb{R} \) and \( 0 < \rho \leq 1 \). If we have \( L^q(\cdot; |^q^{p'})-L^p(\cdot; |^p^{p'}) \) boundedness of \( a(D) \) for any \( s \in (-n/\gamma, 0) \), then we have

\[
m \leq -n(1-\rho) \left(\frac{1}{2} - \frac{1}{p}\right) - \rho n \left(\frac{1}{q} - \frac{1}{p}\right) - \frac{(1-\rho)}{\gamma}.
\]

In particular, if we have \( L^q(\omega^q)-L^p(\omega^p) \) boundedness of \( a(D) \) with any \( \omega^q \in RH_{(p/q)(r'/p)} \), for some \( r \in [1, q) \), then we have

\[
m \leq -n(1-\rho) \left(\frac{1}{2} - \frac{1}{p}\right) - \rho n \left(\frac{1}{q} - \frac{1}{p}\right).
\]

**Proof.** Our assumption gives

\[
|\langle a(D)f, g \rangle| \lesssim \|f\|_{L^q(\cdot; \omega^q)} \|g\|_{L^p(\cdot; \omega^p)}
\]

for any \( s \in (-n/\gamma, 0) \). We take a nonnegative function \( \phi \in C_0^\infty \) such that \( \text{supp} \phi \subset \{1/4 \leq |\xi| \leq 2\} \) and \( \phi = 1 \) on \( \{1/2 \leq |\xi| \leq 1\} \), and let

\[
\hat{f}(\xi) = e^{-i|\xi|^{1-\rho}} \phi(\xi/R),
\]

and

\[
\hat{g}(\xi) = \phi(\xi/R)
\]

for any \( R > 0 \). Then, we have

\[
|\langle a(D)f, g \rangle| = \left| \int |\xi|^m \phi(\xi/R) \phi(\xi/R) d\xi \right| \sim R^{m+n}.
\]

On the other hands, we have

\[
|f(x)| \lesssim \min \{ R^{n(1+\rho)/2}, R^{n(1+\rho)/2-2nN} |x|^{-2N} \}
\]

for any \( N \in \mathbb{N} \). In fact, Littman’s lemma gives

\[
|f(x)| = \left| \int e^{ix\xi - |\xi|^{1-\rho}} \phi(\xi/R) d\xi \right| \leq R^n \sup_x \left| \int e^{ix\xi + iR^{1-\rho} |\xi|^{1-\rho}} \phi(\xi) d\xi \right| \lesssim R^n R^{-n(1-\rho)/2} = R^n(1+\rho)/2.
\]
As for second estimates, we have
\[
|f(x)| = |x|^{-2N} \left| \int \Delta_x^N e^{ix\xi - i|\xi|^{1-p}} \phi(\xi/R) d\xi \right| \\
= |x|^{-2N} \left| \int e^{ix\xi} \Delta_x^N (e^{-i|\xi|^{1-p}} \phi(\xi/R)) d\xi \right| \\
\leq |x|^{-2N} \sum_{|\alpha|=2N} \sum_{\beta \leq \alpha} R^{-|\beta|} \left| \int e^{ix\xi - i|\xi|^{1-p}} (P_{\alpha,\beta}(\xi)) (\partial^{\beta} \phi)(\xi/R) d\xi \right|
\]
where \(P_{\alpha,\beta}\) denotes the functions such that
\[
||\partial^{\sigma} P_{\alpha,\beta}(R \cdot)||_{L^\infty} \lesssim R^{-2\rho N + \rho|\beta|}
\]
on support of \(\phi\) for any \(\sigma \in \mathbb{N}^n\). From this and Littman’s lemma, we obtain desired estimate. Therefore, we have
\[
||f||_{L^q(|\cdot|^r)} \leq \left( \int_{|x| \leq R^{-\rho}} |f(x)|^q |x|^q \frac{dx}{|x|^s} \right)^{1/q} + \left( \int_{|x| \geq R^{-\rho}} |f(x)|^q |x|^q \frac{dx}{|x|^s} \right)^{1/q} \\
\leq R^{n(1+\rho)/2} \left( \int_{|x| \leq R^{-\rho}} |x|^q \frac{dx}{|x|^s} \right)^{1/q} + R^{n(1+\rho)/2-2\rho N} \left( \int_{|x| \geq R^{-\rho}} |x|^{q^*-2\rho N} \frac{dx}{|x|^s} \right)^{1/q} \\
\leq R^{n(1+\rho)/2-\rho m/q-\rho s},
\]
and
\[
||g||_{L^{q'}(|\cdot|^{-r'})} = R^n \left( \int |\hat{g}(Rx)|^{q'} |x|^{-r'} dx \right)^{1/q'} \\
\leq R^{n-n/p'+s} = R^{n/p+s}.
\]
From these observations, we obtain
\[
R^{m+n} \lesssim R^{n(1+\rho)/2-\rho m/q-\rho s} R^{n/p+s} \lesssim R^{n(1+\rho)/2-\rho m/q-\rho s} R^{n/p+s} \\
R^m \lesssim R^{n(1/2-1/p)+\rho(1/2-1/q)+s(1-\rho)} R^{n(1/2-1/p)+\rho(1/2-1/q)+s(1-\rho)} \\
R^m \lesssim R^{\rho(1/2-1/p)-\rho(1/q-1/p)+s(1-\rho)} R^{n(1/2-1/p)-\rho(1/q-1/p)+s(1-\rho)} \\
m \leq -n(1-\rho)(1/2-1/p)-\rho(1/q-1/p)-n(1-\rho)/\gamma
\]
Here, we take the infimum all over the \(s \in (-n/\gamma, 0)\) to obtain the final inequality. In particular, we have \(|\cdot|^q \in RH_{(p/q)(r'/p)'}\) with \(s \in (-n/p(r'/p)', 0)\), that means
\[
m \leq -n(1-\rho)(1/2-1/p)-\rho(1/q-1/p)-n(1-\rho)/p(r'/p)' \\
= -n(1-\rho)(1/2-1/p)-\rho(1/q-1/p)-n(1-\rho)(1-p/r')/p \\
= -n(1-\rho)(1/2-1/p)-\rho(1/q-1/p).
\]
by taking \(\gamma = p(r'/p)'\).

**Remark 3.3.** Since \(e^{i|\xi|^{1-p}} |\xi|^m \notin S^m_{p,0}\), Proposition 3.3 cannot be applied to the pseudodifferential operators associated with symbols belonging to the Hörmander class directly. However, by the same proof of the proposition, it holds with \(a \in S^m_{p,0}\) such that \(a(\xi) = e^{i|\xi|^{1-p}} |\xi|^m\) for any \(|\xi| > 1\), that means a sharpness of weighted inequalities in Theorem 3.3 and (i) of Corollary 3.3.

### A Appendix A

To see the proof of Corollary 3.3, the operator norms of \(a(x, D)\) on weighted Besov spaces are controlled by
\[
([\omega^q]_{A_{n/r}} [\omega^q]_{RH_{(p/q)(r'/p)'}})^\delta [\omega^{-p}]_{A_{p'}}.
\]
However, we can eliminate the factor \([\omega^{-p}]_{A_{p'}}\) by having the sparse form bounds \(\phi_k * a(x, D)(\phi_j * \cdot)\) directly.
Proposition A.1. (i) Let $2 \leq s \leq \infty$ and $2/3 < \alpha \leq 1$, and $a \in S^m_{p,\delta}$ with $m \leq 0$, $0 \leq \delta < \rho \leq 1$. Then for any $f, g \in \mathcal{S}$ and $j, k \in \mathbb{Z}_{\geq 0}$, there exists the sparse family $\mathcal{S}$ such that

$$
|\langle \phi_k * a(x, D)(\phi_j * f), g \rangle| \lesssim 2^{-k \ell} 2^{j \kappa_1} \lim inf_{R \to \infty} A_{S,2}^a((\phi_j * f)1_{QR}, g).
$$

(ii) Let $2 \leq s \leq \infty$ and $s' < \alpha \leq 1$, and $a \in S^m_{p,\delta}$ with $m \leq 0$, $0 \leq \delta < \rho \leq 1$. Then for any $f, g \in \mathcal{S}$, there exists the sparse family $\mathcal{S}$ such that

$$
|\langle \phi_k * a(x, D)(\phi_j * f), g \rangle| \lesssim 2^{-k \ell} 2^{j \kappa_2} \lim inf_{R \to \infty} A_{S,s'}((\phi_j * f)1_{QR}, g).
$$

(iii) Let $1 \leq s' < r \leq 2 \leq s \leq \infty$ and $a \in S^m_{p,\delta}$ with $m \leq 0$, $0 \leq \delta < \rho \leq 1$. Then for any $f, g \in \mathcal{S}$, there exists the sparse family $\mathcal{S}$ such that

$$
|\langle \phi_k * a(x, D)(\phi_j * f), g \rangle| \lesssim 2^{-k \ell} 2^{j \kappa_2} \lim inf_{R \to \infty} A_{S,r,s'}((\phi_j * f)1_{QR}, g).
$$

Proof. We put

$$
T_{j,k}f := \phi_k * a(x, D)(\phi_j * f).
$$

Here, we remark that

$$
T_{j,k}f = (\langle D \rangle^{-\ell} \phi_k) * (\langle D \rangle^\ell a(x, D)(\phi_j * f)) = (\langle D \rangle^{-\ell} \phi_k) * b_{\ell}(x, D)(\phi_j * f),
$$

with some $b_{\ell} \in S^{m+\ell}_{p,\delta}$. For any cube $Q$ and $x \in Q$, one has

$$
\|T_{j,k}f1_{Q}\|_{L^\infty(Q)} \leq \|(\langle D \rangle^{-\ell} \phi_k) * [1_{Q}] b_{\ell}(x, D)(\phi_j * (f1_{Q}))\|_{L^\infty(Q)} + 2^{-k\ell}\|b_{\ell}(x, D)(\phi_j * (f1_{Q}))\|_{L^\infty(Q)}
$$

$$
= f_0(x) + f_1(x),
$$

where

$$
f_0(x) := \sup_{Q \ni x} \|(\langle D \rangle^{-\ell} \phi_k) * [1_{Q}] b_{\ell}(x, D)(\phi_j * (f1_{Q}))\|_{L^\infty(Q)},
$$

$$
f_1(x) := \sup_{Q \ni x} 2^{-k\ell}\|b_{\ell}(x, D)(\phi_j * (f1_{Q}))\|_{L^\infty(Q)}.
$$

(i) Now, we have

$$
f_0(x) \leq 2^{-k\ell} M_{2} b_{\ell}(x, D)(\phi_j * (f1_{Q}))\|_{L^\infty(Q)} \leq 2^{-k\ell} 2^{jm+jn(1-\rho(1-\gamma))/2} M_{2} f(x) \leq 2^{-k\ell} 2^{jm+jn(1-\rho(1-\gamma))/2} M_{2} f(x) \leq 2^{-k\ell} 2^{jm+jn(1-\rho(1-\gamma))/2} M_{2} f(x).
$$

Therefore, we obtain

$$
\|MT_{j,k}f\|_{L^\infty_{-\rho(1-\gamma)/2+1/p_0}} \leq 2^{-k\ell} 2^{jm+jn(1-\rho(1-\gamma))/2+1/p_0}
$$

for any $p_0 \geq 2$. On the other hand, we have

$$
\|T_{j,k}\|_{L^\infty_{-\rho(1-\gamma)/2+1/p_1}} \leq 2^{-k\ell} 2^{jm+jn(1/2-1/p)} \text{ and } \|MT_{j,k}f\|_{L^\infty_{-\rho(1-\gamma)/2+1/p_1}} \leq 2^{-k\ell} 2^{jm+jn(1/2-1/p_1)}
$$

for any $p_1 \geq 2$. By interpolating them, we have desired sparse bounds.

(ii), (iii) It suffices to prove the pointwise estimate

$$
f_0(x) + f_1(x) \lesssim 2^{-k\ell} 2^{jm+jn(1-\rho(1-\gamma))} M_f(x).
$$

We just handle the $f_0(x)$ since the estimate of $f_1(x)$ be obtained immediately from the proof of Theorem 3.2.

For any $N \in \mathbb{N}$ and $h \in L^1$, we have

$$
\langle D \rangle^{-\ell} \phi_k \ast h(z) \lesssim 2^{-k\ell+kn} \int (1 + 2^{kn}|z-y|^{2N})^{-1} |h(y)|dy,
$$

$$
|b_{\ell}(x, D)(\phi_j * (f1_{Q}))\|_{L^\infty(Q)} \lesssim 2^{j\ell+jm+n} \int (1 + 2^{jn}|y-w|^{2N})^{-1} |f(w)|1_{Q} \cdot (w)dw.
$$
Hence, we obtain
\[ f_0(x) \lesssim 2^{-k\ell+kn}2^\ell+jm+jn \sup_{Q \ni x} \| \Phi \ast (|f|1(3Q)^c) \|_{L^\infty(Q)}, \]
where \( \Phi \) denotes the radial function
\[ \Phi(z) = \int \frac{1}{1+2^{2j\rho}N|z-y|^{2N}} \cdot \frac{1}{1+2^{2kN}|y|^{2N}} dy. \]

To complete the proof, we decompose the integral region:
\[ \Phi(z) = \int_{|y|<|z|} + \int_{|z|\leq|y|} + \int_{|z|/2\leq|y|<2|z|}. \]

Since \( |z-y| \gtrsim |z| \) under the \( 2|y|<|z| \) or \( 2|z| \leq |y| \), one has
\[ \int_{|y|<|z|} + \int_{|z|\leq|y|} \lesssim \frac{2^{-kn}}{1+2^{2j\rho}N|z|^{2N}}. \]

Furthermore, it is not hard to see that
\[ \int_{|z|/2\leq|y|<2|z|} \lesssim \min \left\{ \frac{2^{-jm}}{1+2^{2kN}|z|^{2N}}, \frac{|z|^n}{1+2^{2kN}|z|^{2N}} \right\}. \]

From them, for any \( k \leq j\rho \), we have
\[ f_0(x) \lesssim 2^{-k\ell+jm+jn(1-\rho)}(2^{-jmn} + 2^{kn\gamma})M^\gamma f(x) \leq 2^{-k\ell+jm+jn(1-\rho)(1-\gamma)}M^\gamma f(x). \]

We assume \( k > j\rho \). Then, we have
\[ \sup_{z \in Q} \int_{(3Q)^c} \frac{|z-w|^n}{1+2^{2kN}|z-w|^{2N}} |f(w)|dw \lesssim \int \frac{|x-w|^n}{1+2^{2kN}|x-w|^{2N}} |f(w)|dw \leq \sum_{i \in Z} \int_{|x-w|=|2^{-k2^{i}}}} \frac{|x-w|^n}{1+2^{2kN}|x-w|^{2N}} |f(w)|dw \lesssim 2^{-kn}2^{-k(1-\gamma)}M^\gamma f(x) \lesssim 2^{-kn}2^{-jm(1-\gamma)}M^\gamma f(x), \]
which completes the proof. \( \square \)

## B Appendix B

From Proposition 3.1, the weak-type boundedness of \( M_{T,s} \) is a sufficient condition to have the sparse domination. It is natural to ask whether such condition be a necessary condition or not. However, it seems that the answer of this question is negative from following observations.

**Proposition B.1.** (i) Let \( 1 \leq r < \infty \). Then, there exist \( f \in L^\infty_c \) and collection of sparse families \( \{ S(Q) \}_{Q: \text{cube}} \), and measurable set \( K \) which has a non-zero measure, such that
\[ \sup_{Q \ni x} \| \Lambda_{S(Q),r}(f1(3Q)^c) \|_{L^\infty(Q)} = \infty \]
for any \( x \in K \).

(ii) Let \( 1 \leq r < s \leq \infty \). Then, there exist \( f \in L^\infty_c \) and collection of sparse families \( \{ S(Q) \}_{Q: \text{cube}} \), and measurable set \( K \) which has a non-zero measure, such that
\[ \sup_{Q \ni x} \sup_{\| g \|_{L^{s'}(Q)} = 1} |Q|^{1/s} \Lambda_{S(Q),r,s'}(f1_{R\setminus3Q},g) = \infty \]
for any \( x \in K \).
Proof. (i) Let fix a cube $Q_0$ and let $f = 1_{Q_0}$. Furthermore, we define sparse collection $\mathcal{S}(Q)$ for any cube $Q$ by

$$\mathcal{S}(Q) = \{3^k Q : k = 1, 2, 3, \ldots \}.$$  

For any cube $3Q \subset Q_0$ and $z \in Q$, we choose $N \in \mathbb{N}$ such that $3^{N+1}Q \cap Q_0^c \neq \emptyset$ and $3^N Q \subset Q_0$. Then, we have

$$\Lambda_{\mathcal{S}(Q), r}(f 1_{\mathbb{R}^n \setminus 3Q})(z) 1_Q(z) = \sum_{k=1}^{\infty} \langle 1_{Q_0 \setminus 3Q} \rangle_{r, 3^k Q} 1_Q(z) \geq \sum_{k=1}^{N} \left( \frac{|3^k Q \setminus 3Q|}{|3^k Q|} \right)^{1/r} 1_Q(z) \gtrsim N 1_Q(z),$$

which yields

$$\|\Lambda_{\mathcal{S}(Q), r}(f 1_{\mathbb{R}^n \setminus 3Q})\|_{L^\infty(Q)} \gtrsim N.$$

Since $N \to \infty$ at $|Q| \to 0$, we have

$$\sup_{Q \ni x} \|\Lambda_{\mathcal{S}(Q), r}(f 1_{\mathbb{R}^n \setminus 3Q})\|_{L^\infty(Q)} = \infty$$

for any $x \in Q_0$.

(ii) By taking $f$ and $\mathcal{S}(Q)$ as above, we have

$$\sup_{\|g\|_{L^{s'}(Q)} = 1} \Lambda_{\mathcal{S}(Q), r, s'}(f 1_{\mathbb{R}^n \setminus 3Q}, g) = \sup_{\|g\|_{L^{s'}(Q)} = 1} \sum_{k=1}^{\infty} |3^k Q| \langle 1_{Q_0 \setminus 3Q} \rangle_{r, 3^k Q} \langle g \rangle_{s', 3^k Q} \geq \sum_{k=1}^{N} |3^k Q| \left( \frac{|3^k Q \setminus 3Q|}{|3^k Q|} \right)^{1/r} |Q|^{-1/s'} \left( \frac{|Q|}{|3^k Q|} \right)^{1/s'} \geq |Q|^{1/s} 3^{Nn/s},$$

which complete the proof. \qed

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