Boundary blow-up solutions to fractional elliptic equations in a measure framework

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Abstract

Let $\alpha \in (0,1)$, $\Omega$ be a bounded open domain in $\mathbb{R}^N$ ($N \geq 2$) with $C^2$ boundary $\partial \Omega$ and $\omega$ be the Hausdorff measure on $\partial \Omega$. We denote by $\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}$ a measure

$$\langle \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}, f \rangle = \int_{\partial \Omega} \frac{\partial^\alpha f(x)}{\partial \vec{n}^\alpha x} d\omega(x), \quad f \in C^1(\bar{\Omega}),$$

where $\vec{n}_x$ is the unit outward normal vector at point $x \in \partial \Omega$. In this paper, we prove that problem

$$(-\Delta)^\alpha u + g(u) = k \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha} \quad \text{in } \bar{\Omega},$$
$$u = 0 \quad \text{in } \Omega^c \quad (0.1)$$

admits a unique weak solution $u_k$ under the hypotheses that $k > 0$, $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (0,1)$ and $g$ is a nondecreasing function satisfying extra conditions. We prove that the weak solution of (0.1) is a classical solution of

$$(-\Delta)^\alpha u + g(u) = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\Omega},$$
$$\lim_{x \in \Omega, x \to \partial \Omega} u(x) = +\infty.$$

Key Words: Fractional Laplacian, Green kernel, Boundary blow-up solution.

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1 Introduction

1.1 motivation

Let $\Omega$ be a bounded open domain in $\mathbb{R}^N$ ($N \geq 2$) with $C^2$ boundary $\partial \Omega$. The pioneering works \cite{13,17} obtained that the nonlinear reaction diffusion equation

$$-\Delta u + h(u) = 0 \quad \text{in } \Omega,$$
$$u = +\infty \quad \text{on } \partial \Omega \quad (1.1)$$

admits a solution if $h$ is a locally Lipschitz continuous function which is increasing and satisfies Keller-Osserman condition

$$\int_1^{+\infty} \left[ \int_0^s h(t) dt \right]^{-\frac{1}{2}} ds < +\infty.$$

Great interests in existence, uniqueness and asymptotic behavior of boundary blow-up solution to (1.1) have been taken, see \cite{1,9,11,13,20,22,16}. It is well known that when

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\[ h(s) = s^p \] with \( p > 1 \), (1.11) has a unique solution with boundary asymptotic behavior
\[ \rho^{-\frac{p}{2-1}}(x), \text{ where } \rho(x) = \text{dist}(x, \partial \Omega). \]

Comparing with the Laplacian case, a much richer structure for the solutions set appears
for the non-local case. Recently, the authors in [5] obtained very different phenomena of the
boundary blow-up solutions to elliptic equations involving the fractional Laplacian, precisely,
\[ (-\Delta)^\alpha u + |u|^{p-1}u = 0 \text{ in } \Omega, \]
\[ u = 0 \text{ in } \Omega^c, \]
\[ \lim_{x \in \Omega, x \to \partial \Omega} u(x) = +\infty, \]
where \( p > 0 \) and the fractional Laplacian \((-\Delta)^\alpha\) with \( \alpha \in (0, 1) \) is defined by
\[ (-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)_{\epsilon}^\alpha u(x), \]
here for \( \epsilon > 0 \),
\[ (-\Delta)_{\epsilon}^\alpha u(x) = -\int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} dz. \]

The existence of boundary blow-up solution of (1.2) is derived by constructing appro-
priate super and sub-solutions and this construction involves the one dimensional truncated
Laplacian of power functions given by
\[ C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}(t)|1 - t|^{\tau} + (1 + t)^{\tau} - 2}{t^{1+2\alpha}} dt, \]  
where \( \tau \in (-1, 0) \) and \( \chi_{(0,1)} \) is the characteristic function of the interval \((0, 1)\). It is known
that there exists a unique zero point of (1.3) in \((-1, 0)\), denoting \( \tau_0(\alpha) \). Then

**Proposition 1.1** [5, Theorem 1.1] Assume that \( \Omega \) is an open, bounded and connected
domain of class \( C^2 \) and \( \alpha \in (0, 1) \). Then we have:

**Existence:** Assume that
\[ 1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}, \]
the equation (1.2) possesses at least one solution \( u \) satisfying
\[ 0 < \liminf_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty. \]  

**Uniqueness:** \( u \) is the unique solution of (1.2) satisfying (1.4).

**Nonexistence:** In the following three cases:

i) For any \( \tau \in (-1, 0) \setminus \{-\frac{2\alpha}{p-1}, \tau_0(\alpha)\} \) and
\[ 1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)} \]
or
\[ p \geq 1 - \frac{2\alpha}{\tau_0(\alpha)} \]

ii) For any \( \tau \in (-1, 0) \) and
\[ p \geq 1 - \frac{2\alpha}{\tau_0(\alpha)} \]
or

iii) For any \( \tau \in (-1, 0) \setminus \{\tau_0(\alpha)\} \) and
\[ 1 < p \leq 1 + 2\alpha, \]
problem (1.2) does not have a solution \( u \) satisfying
\[
0 < \liminf_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{-\tau} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{-\tau} < +\infty.
\] (1.5)

**Special existence for \( \tau = \tau_0(\alpha) \).** Assume that
\[
\max\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha)}{\tau_0(\alpha)} + 1\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}.
\]
Then for any \( t > 0 \), there is a positive solution \( u \) of equation (1.2) satisfying
\[
\lim_{x \in \Omega, x \to \partial \Omega} u(x)d(x)^{-\tau_0(\alpha)} = t.
\]

There are some challenging questions to ask:
1. Could \( \tau_0(\alpha) \) be expressed explicitly?
2. With what condition of general nonlinearity makes existence hold?
3. The uniqueness and nonexistence restricts in the class functions (1.4) and (1.3), so are there some solutions breaking the assumption (1.4)?

Our interest in this article is to introduce a new method to study the boundary blow-up solutions of semilinear fractional elliptic equations and answer above questions. The main idea is to find suitable type measure concentrated on the whole boundary and then by making basic estimates to prove that the corresponding weak solution solves (1.2). Our first result is stated as follows:

**Proposition 1.2** Let \( \alpha \in (0, 1) \) and \( \tau_0(\alpha) \) is the zero point of \( C(\cdot) \) when \( C(\cdot) \) given by (1.3), then
\[
\tau_0(\alpha) = \alpha - 1.
\]

We observe that the critical value \( 1 - \frac{2\alpha}{\tau_0(\alpha)} \) in Proposition 1.1 turns out to be \( \frac{1 + \alpha}{1 - \alpha} \). In what follows, we would like to show the details of our new method and answer the second and third questions in the following.

### 1.2 A new method and main results

Let \( \alpha \in (0, 1) \) and \( \omega \) be the Hausdorff measure on \( \partial \Omega \). We denote by \( \frac{\partial^\alpha \omega}{\partial n^\alpha} \) a measure
\[
\left( \frac{\partial^\alpha \omega}{\partial n^\alpha}, f \right) = \int_{\partial \Omega} \frac{\partial^\alpha f(x)}{\partial n_x^\alpha} d\omega(x), \quad f \in C^\alpha(\bar{\Omega}),
\]
where \( \vec{n}_x \) is the unit inward normal vector of \( \partial \Omega \) at point \( x \) and
\[
\frac{\partial^\alpha f(x)}{\partial n_x^\alpha} = \lim_{t \to 0^+} \frac{f(x + t\vec{n}_x) - f(x)}{t^\alpha}.
\]

In this paper, we are concerned with the existence and uniqueness of weak solution to the semilinear fractional elliptic problem
\[
(-\Delta)^\alpha u + g(u) = k \frac{\partial^\alpha \omega}{\partial n^\alpha} \text{ in } \bar{\Omega},
\]
\[
u = 0 \quad \text{in } \bar{\Omega}^c,
\] (1.6)
where \( k > 0 \) and \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous.

In [6], the authors studied problem (1.6) replaced \( \frac{\partial u}{\partial n} \) by \( \frac{\partial x}{\partial n} \) where \( \nu \) is a Radon measure concentrated on boundary measure. They proved that such a problem has a unique weak solution if \( g \) is a continuous nondecreasing function satisfying \( g(0) \geq 0 \) and

\[
\int_1^\infty g(s)s^{-1-\frac{N+\alpha}{N}}ds < +\infty.
\] (1.7)

Moreover, [6] analyzed the isolated singularity of weak solution of (1.6) in the case that \( \nu = \delta_{x_0} \) with \( x_0 \in \partial\Omega \). Our aim in this article is to investigate how the Hausdorff measure on \( \partial\Omega \) works on the weak solution of (1.6).

Before starting our main theorems we make precise the notion of weak solution used in this note.

**Definition 1.1** We say that \( u \) is a weak solution of (1.7), if \( u \in L^1(\Omega) \), \( g(u) \in L^1(\Omega, \rho^\alpha dx) \) and

\[
\int_{\Omega}(u(-\Delta)^{\alpha}\xi + g(u)\xi)dx = k\int_{\partial\Omega} \frac{\partial^{\alpha}\xi(x)}{\partial \nu^\alpha}d\omega(x), \quad \forall \xi \in \mathcal{X}_\alpha.
\]

where \( \rho(x) = \text{dist}(x, \partial\Omega) \) and \( \mathcal{X}_\alpha \subset C(\mathbb{R}^N) \) denotes the space of functions \( \xi \) satisfying:

(i) \( \text{supp}(\xi) \subset \bar{\Omega} \);

(ii) \( (-\Delta)^{\alpha}\xi(x) \) exists for all \( x \in \Omega \) and \( |(-\Delta)^{\alpha}\xi(x)| \leq C \) for some \( C > 0 \);

(iii) there exist \( \varphi \in L^1(\Omega, \rho^\alpha dx) \) and \( \varepsilon_0 > 0 \) such that \( |(-\Delta)^{\alpha}\xi| \leq \varphi \) a.e. in \( \Omega \) for all \( \varepsilon \in (0, \varepsilon_0] \).

Now we are ready to state our first result for problem (1.6).

**Theorem 1.1** Assume that \( k > 0 \), \( \rho(x) = \text{dist}(x, \partial\Omega) \) and \( g \) is a continuous nondecreasing function satisfying \( g(0) \geq 0 \) and

\[
\int_1^\infty g(s)s^{-1-\frac{N+\alpha}{N}}ds < +\infty.
\] (1.8)

Then (i) problem (1.6) admits a unique positive weak solution \( u_k \);

(ii) the mapping \( k \to u_k \) is increasing and there exists \( c_1 \geq 1 \) independent of \( k \) such that

\[
\frac{k}{c_1}\rho(x)^{\alpha-1} \leq u_k(x) \leq c_1k\rho(x)^{\alpha-1}, \quad \forall x \in \Omega;
\] (1.9)

(iii) if we assume additionally that \( g \) is \( C^\beta \) locally in \( \mathbb{R} \) with \( \beta > 0 \), then \( u_k \) is a classical solution of

\[
(-\Delta)^{\alpha}u + g(u) = 0 \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega},
\]

\[
\lim_{x \in \Omega, x \to \partial\Omega} u(x) = +\infty.
\] (1.10)

We remark that in Theorem 1.1 extends the special existence of boundary blow up solutions to fractional elliptic equation (1.10) with general nonlinearity \( g \) in integral subcritical case with the critical exponents \( \frac{N+\alpha}{N+\alpha-\beta} \), which is larger than \( \frac{N+\alpha}{N-\alpha} \). Specially, letting \( g \equiv 0 \), there exists infinitely many boundary blow up \( \alpha \)-harmonic functions.

Since \( \alpha - 1 > -\frac{2\alpha}{p-1} \), so we may call the solutions of (1.10) as the weak boundary blow-up solution from the asymptotic behavior (1.3). Our second interest is to consider the limit of weak boundary blow-up solutions.
**Theorem 1.2** Let \( g(s) = s^p \) with \( p \in (0, \frac{1+\alpha}{1-\alpha}) \) and \( u_k \) be the weak solution of (1.6), then

(i) if \( p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha}) \), then the limit of \( \{u_k\} \) as \( k \to \infty \) exists, denoting \( u_\infty \), which is a classical solution of (1.2). Moreover, \( u_\infty \) satisfies

\[
\frac{1}{c_2} \rho(x)^{-\frac{2\alpha}{p-1}} \leq u_\infty(x) \leq c_2 \rho(x)^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \Omega,
\]

where \( c_2 \geq 1 \).

(ii) if \( p \in (0, 1 + 2\alpha] \), then

\[
\lim_{k \to \infty} u_k(x) = +\infty, \quad \forall x \in \Omega.
\]

We notice that the limit of weak boundary blow-up solutions is the solution of (1.1) with behavior (1.4) when \( p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha}) \) stated in Proposition 1.1. As a consequence of Theorem 1.2 (ii), if \( p \in (0, 1 + 2\alpha] \), there is no solution \( u \) of (1.1) such that

\[
\lim_{x \to x_0, x \in \Omega} u(x) = 0 \text{ or } +\infty.
\]

From [6] and Theorem 1.1, the Dirac mass and Hausdorff measure have different contribution to the solution of

\[
(-\Delta)^\alpha u + g(u) = 0 \quad \text{in} \quad \Omega.
\]

Our interest is to understand what singularity of the solution to

\[
(-\Delta)^\alpha u + g(u) = \frac{\partial^\alpha(\omega + \delta_{x_0})}{\partial n^\alpha} \quad \text{in} \quad \bar{\Omega},
\]

\[
u = 0 \quad \text{in} \quad \bar{\Omega}^c,
\]

where \( x_0 \in \partial \Omega \) and \( \delta_{x_0} \) is the Dirac mass concentrated at \( x_0 \) on the boundary. Inspired by Definition 1.1, it is natural to give the definition of weak solution of (1.12) as following.

**Definition 1.2** We say that \( u \) is a weak solution of (1.12), if \( u \in L^1(\Omega) \), \( g(u) \in L^1(\Omega, \rho^\alpha dx) \) and

\[
\int_{\Omega} [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_{\partial \Omega} \frac{\partial^\alpha \xi(x)}{\partial n^\alpha_x} d\omega(x) + \frac{\partial^\alpha \xi(x_0)}{\partial n^\alpha_{x_0}}, \quad \forall \xi \in X_\alpha.
\]

**Theorem 1.3** Assume that \( x_0 \in \partial \Omega \), \( g \) is a continuous nondecreasing function satisfying \( g(0) \geq 0 \),

\[
\int_1^\infty g(s)s^{-1-\frac{1+\alpha}{\beta}} ds < +\infty
\]

and for some \( \lambda > 0 \),

\[
g(s + t) \leq \lambda[g(s) + g(t)], \quad \forall s, t > 0.
\]

Then problem (1.6) admits a unique positive weak solution \( v \) such that

\[
\frac{1}{c_3} \left[ \rho(x)^{\alpha-1} + \frac{\rho(x)^\alpha}{|x - x_0|^N} \right] \leq v(x) \leq c_3 \left[ \rho(x)^{\alpha-1} + \frac{\rho(x)^\alpha}{|x - x_0|^N} \right], \quad \forall x \in \Omega.
\]

Moreover, if assume additionally that \( g \) is \( C^\beta \) locally in \( \mathbb{R} \) with \( \beta > 0 \), then \( v \) is a classical solution of (1.10).
From Theorem 1.3, we find out a classical solution of (1.10) with explosive rate \( \rho(x)^{\alpha-1} + \frac{\partial x}{|x-x_0|^{\alpha}} \), this answers the question 3 in the first part of the introduction.

The boundary blow-up solutions of (1.10) could be searched for by making use of measure type data on boundary and the main difficulty is to do the estimate of \( G_\alpha \left[ \frac{\partial \omega}{\partial n} \right] \) and \( g(G_\alpha \left[ \frac{\partial \omega}{\partial n} \right]) \). Especially, it is dedicate to make the estimate of \( g(G_\alpha \left[ \frac{\partial \omega}{\partial n} \right]) \) near the boundary when the nonlinearity \( g \) is just integral-subcritical, i.e. (1.8).

This article is organized as follows. In Section §2 we present some preliminaries to the Marcinkiewicz type estimate for \( G_\alpha \left[ \frac{\partial \omega}{\partial n} \right] \) and present the existence and uniqueness of weak solution of (1.6) when \( g \) is bounded. Section §3, §4 are devoted to prove Theorem 1.1 and Theorem 1.2. Finally, we obtain one typical solution that blows up along the boundary with different power rate.

## 2 Preliminary

### 2.1 The Marcinkiewicz type estimate

In order to obtain the weak solution of (1.6) with integral subcritical nonlinearity, we have to introduce the Marcinkiewicz space and recall some related estimate.

**Definition 2.1** Let \( \Theta \subset \mathbb{R}^N \) be a domain and \( \omega \) be a positive Borel measure in \( \Theta \). For \( \kappa > 1, \kappa' = \kappa/(\kappa - 1) \) and \( u \in L^1_{\text{loc}}(\Theta, d\mu) \), we set

\[
\|u\|_{M^\kappa(\Theta, d\omega)} = \inf \left\{ c \in [0, \infty] : \int_E |u| d\omega \leq c \left( \int_E d\omega \right)^{\frac{1}{\kappa'}} , \forall E \subset \Theta, E \text{ Borel} \right\} \tag{2.1}
\]

and

\[
M^\kappa(\Theta, d\omega) = \{ u \in L^1_{\text{loc}}(\Theta, d\omega) : \|u\|_{M^\kappa(\Theta, d\omega)} < +\infty \} \tag{2.2}
\]

The space \( M^\kappa(\Theta, d\omega) \) is called the Marcinkiewicz space of exponent \( \kappa \), or weak \( L^\kappa \)-space and \( \|\cdot\|_{M^\kappa(\Theta, d\omega)} \) is a quasi-norm.

**Proposition 2.1** \([2, 7]\) Assume that \( 1 \leq q < \kappa < \infty \) and \( u \in L^1_{\text{loc}}(\Theta, d\omega) \). Then there exists \( c_4 > 0 \) dependent of \( q, \kappa \) such that

\[
\int_E |u|^q d\omega \leq c_4 \|u\|_{M^\kappa(\Theta, d\omega)} \left( \int_E d\omega \right)^{1 - q/\kappa} 
\]

for any Borel set \( E \) of \( \Theta \).

Denote by \( G_\alpha \) the Green kernel of \((-\Delta)^\alpha \) in \( \Omega \times \Omega \) and by \( G_\alpha[\cdot] \) the Green operator defined as

\[
G_\alpha[\frac{\partial \omega}{\partial n}] (x) = \lim_{t \to 0^+} \int_{\partial \Omega} G_\alpha(x, y + t\tilde{n}_y) t^{-\alpha} d\omega(y).
\]

Our purpose in this subsection is to do Marcinkiewicz type estimate for \( G_\alpha[\frac{\partial \omega}{\partial n}] \).

**Lemma 2.1** There exists \( c_5 \geq 1 \) such that for any \( x \in \Omega \),

\[
\frac{1}{c_5} \rho(x)^{\alpha - 1} \leq G_\alpha[\frac{\partial \omega}{\partial n}] (x) \leq c_5 \rho(x)^{\alpha - 1}. \tag{2.3}
\]
Proof. Since \( \partial\Omega \) is \( C^2 \), then there exists \( t_0 \in (0, \frac{1}{2}) \) such that for any \( x \in \Omega_t := \{ z \in \Omega, \rho(x) < t \} \) with \( t < t_0 \), there exists a unique \( x_\partial \in \partial\Omega \) such that

\[
|x - x_\partial| = \rho(x)
\]

and for \( t \in (0, t_0) \) letting

\[
C_t = \{ x \in \Omega : \rho(x) = t \},
\]

\( C_t \) is \( C^2 \) for \( t \in (0, t_0) \) and any Borel set \( E_t \) in \( C_t \) there exists unique set \( E \subset \partial\Omega \) such that for any \( x_t \in E_t \), there exists a unique \( x \in E \) such that

\[
|x_t - x| = t
\]  

(2.4)

and for any \( x \in E \), there exists a unique \( x_t \in E_t \) satisfying (2.4). Moreover, for \( x \in C_t \) with \( t \in (0, t_0) \), there exists a unique \( x_\partial \in \partial\Omega \) such that

\[
x = t\bar{n}_{x_\partial} + x_\partial \quad \text{and} \quad |x - x_\partial| = t = \rho(x).
\]

Denote by \( \omega_t \) a measure on \( C_t \) generated by \( \omega \) such that for \( t \in (0, t_0) \),

\[
\omega_t(E_t) = \omega(E) \quad \text{for any Borel set} \ E_t \subset C_t.
\]  

(2.5)

By compactness we only have to prove that (2.3) holds in a neighborhood of any point \( \bar{x} \in \partial\Omega \) and without loss of generality, we may assume that

\[
x_\partial = 0 \quad \text{and} \quad \bar{n}_{x_\partial} = e_N.
\]

From [6, Lemma 2.1], there exists \( c_6 > 0 \) such that

\[
\mathbb{G}_\alpha \left[ \frac{\partial^\alpha \omega}{\partial \bar{n}^\alpha} \right](x) \leq \frac{c_6}{\int_{\partial\Omega} \frac{1}{|x - y|^{N - \alpha}} d\omega(y)} \quad \forall x \in \Omega.
\]  

(2.6)

Let \( \phi : B_{t_0}'(0) \to \mathbb{R} \) such that \( (y', \phi(y')) \in \partial\Omega \), where \( B_{t_0}'(0) \) is the ball centered at origin with radium \( t_0 \) in \( \mathbb{R}^{N-1} \). We choose some \( s_0 \in (0, t_0) \) small enough, there exists \( c_7 \geq 1 \) such that for any Borel set \( E \subset B_{s_0}(0) \cap \partial\Omega \),

\[
\frac{1}{c_7}|E'| \leq \omega(E) \leq c_7|E'|,
\]

where

\[
E' = \{ y' \in \mathbb{R}^{N-1} : (y', \phi(y')) \in E \}.
\]

For \( s_0 > 0 \) small, there exists \( c_8 > 0 \) such that for \( y = (y', y_N) \in B_{s_0}(0) \cap \partial\Omega \),

\[
|te_N - y| \geq c_8|te_N - (y', 0)| = c_8\sqrt{t^2 + |y'|^2}
\]

Therefore,

\[
\int_{B_{s_0}(0) \cap \partial\Omega} \frac{1}{|te_N - y|^{N - \alpha}} d\omega(y) \leq c_9 \int_{B_{s_0}(0) \cap \partial\Omega} \frac{1}{(t^2 + |y'|^2)^{\frac{N - \alpha}{2}}} d\omega(y)
\]

\[
\leq c_9 \int_{B_{s_0}(0)} \frac{1}{(t^2 + |y'|^2)^{\frac{N - \alpha}{2}}} dy'
\]

\[
= c_{10} \int_0^{s_0} s^{N - 2} (t^2 + s^2)^{\frac{N - \alpha}{2}} ds
\]

\[
= c_{10} t^{\alpha - 1} \int_0^{s_0} s^{N - 2} \left( 1 + s^2 \right)^{\frac{N - \alpha}{2}} ds
\]

\[
\leq c_{11} t^{\alpha - 1},
\]
where \( c_9, c_{10} > 0 \) and \( c_{11} = \int_0^{+\infty} \frac{c_{10}}{(1+s)^{\frac{N}{2}}} ds < +\infty \) since \( \frac{2\alpha}{N} > \frac{1}{2} \). For \( y \in \partial \Omega \setminus B_{s_0}(0) \), there exists \( c_{12} > 0 \) such that \( |te_N - y| \geq c_{12}s_0 \), then

\[
\int_{\partial \Omega \setminus B_{s_0}(0)} \frac{1}{|te_N - y|^{N-\alpha}} d\omega(y) \leq c_{12}s_0^{\alpha-N} \int_{\partial \Omega \setminus B_{s_0}(0)} d\omega(y) \leq c_{12}s_0^{\alpha-N}\omega(\partial \Omega).
\]

Therefore, for \( t \in (0, t_0) \),

\[
\int_{\partial \Omega} \frac{1}{|te_N - y|^{N-\alpha}} d\omega(y) \leq c_{12}t^{\alpha-1}.
\]

For \( x \in \Omega \setminus \Omega_0 \) and \( y \in \partial \Omega \), we observe that \( |x - y| \geq t_0 \), then \( \int_{\partial \Omega} \frac{1}{|x - y|^{N-\alpha}} d\omega(y) \) is bounded by some constant dependent of \( t_0 \) and the diameter of \( \Omega \), thus, \( (2.3) \) holds.

We now prove that for \( t \in (0, t_0) \),

\[
G_\alpha[t^\alpha \omega(\partial \Omega), (te_N)] \geq \frac{1}{c_6} t^{\alpha-1}.
\]

(2.7)

For all \( s \in (0, \frac{t}{4}) \), we have that

\[
|te_N - y| > \frac{t}{2} \quad \text{for} \quad y \in \mathcal{C}_s \cap B_{\frac{t}{4}}(se_N).
\]

Therefore,

\[
|te_N - y| > \frac{t}{2} = \frac{1}{2} \max\{\rho(y), \rho(te_N)\}
\]

and apply [1] Theorem 1.2 to derive that there exists \( c_{13} > 0 \) such that for all \( s \in (0, \frac{t}{4}) \)

\[
G_\alpha(t e_N, y) \geq c_{13} \frac{\rho^\alpha(y) \rho^\alpha(te_N)}{|te_N - y|^N} = c_{13} \frac{t^\alpha s^\alpha}{|te_N - y|^N}, \quad y \in \mathcal{C}_s \cap B_{\frac{t}{4}}(se_N).
\]

(2.8)

Thus,

\[
G_\alpha[s^{-\alpha} \omega_s](te_N) \geq c_{13} \int_{\mathcal{C}_s \cap B_{\frac{t}{4}}(se_N)} \frac{t^\alpha}{|te_N - y|^N} d\omega_s(y).
\]

Denote

\[
D_{t,s} = \mathcal{C}_s \cap B_{\frac{t}{4}}(se_N) \quad \text{and} \quad D_t = \partial \Omega \cap B_{\frac{t}{4}}(0).
\]

We observe that

\[
|te_N - y| \leq c_{14}t, \quad \forall y \in D_{t,s}
\]

and

\[
\frac{1}{c_{14}} t^{N-1} \leq \omega_s(D_{t,s}) \leq c_{14} t^{N-1},
\]

where \( c_{14} > 1 \), then for any \( s \in (0, \frac{t}{4}) \)

\[
\int_{\mathcal{C}_s} \frac{t^\alpha}{|te_N - y|^N} d\omega_s(y) \geq \int_{D_{t,s}} \frac{t^\alpha}{|x - y|^N} d\omega_s(y) \geq c_{15} t^\alpha t^{-N} \omega_s(D_{t,s}) \geq c_{16} t^{\alpha-1},
\]

which implies \((2.7)\) by passing the limit of \( s \to 0^+ \). \( \square \)
Proposition 2.2 Let $\mathcal{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \partial^n}]$ given by (2.6) and $p^* = \frac{1+\alpha}{1-\alpha}$. Then there exists $c_{17} > 0$ such that
\[
\|\mathcal{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \partial^n}]\|_{M^p(\Omega,\rho^\alpha dx)} \leq c_{17}.
\] (2.9)

Proof. For any Borel set $E$ of $\Omega$ satisfying
\[
0 < |E| < |\Omega_t|,
\]
where $\Omega_r = \{x \in \Omega, \rho(x) < r\}$ for $r > 0$, there exists $t \in (0,t_0)$ such that
\[
|E| = |\Omega_t|.
\]
Then there exists $c_{18} > 0$ such that
\[
|\Omega_t| = \int_0^t \omega_t(C_t)dt \leq c_{18}t.
\]
We observe that
\[
|E \setminus \Omega_t| = |E| - |E \cap \Omega_t| = |\Omega_t \setminus E|
\]
and
\[
\rho(y) \leq \rho(z), \quad \forall y \in E \setminus \Omega_t, \forall z \in \Omega_t \setminus E.
\]
Then
\[
\int_E \rho^\alpha dx \geq \int_{\Omega_t} \rho^\alpha(x)dx = c_{19} \int_0^t \int_{C_s} s^\alpha d\omega_s ds \geq c_20 t^{\alpha+1}
\]
and together with (2.3), we deduce that
\[
\int_E \mathcal{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \partial^n}](x)^\rho^\alpha(x)dx \leq c_{21} \int_E \rho^{2\alpha-1}(x)dx \leq c_{21} \int_{\Omega_t} \rho^{2\alpha-1}(x)dx = c_{22} \int_0^t \int_{C_s} s^{2\alpha-1} d\omega_s ds = c_{23} t^{2\alpha}
\]
\[
\leq c_{23} (\int_E \rho^\alpha dx)^{\frac{2\alpha}{1+\alpha}},
\]
where $c_{22}, c_{23} > 0$. Therefore,
\[
\int_E \mathcal{G}_\alpha[\frac{\partial^\alpha \omega}{\partial \partial^n}](x)^\rho^\alpha(x)dx \leq c_{23} (\int_E \rho^\alpha dx)^{\frac{2\alpha}{1+\alpha}}.
\]
Together with
\[
\frac{2\alpha}{1+\alpha} = \frac{p^* - 1}{p^*},
\]
we derive (2.9). This completes the proof. □
2.2 Existence for bounded nonlinearity

We extend Hausdorff measure \( \omega \) to \( \bar{\Omega} \) by zero inside \( \Omega \), still denoting \( \omega \). For bounded \( C^2 \) domain, it follows [19, p 57] that \( \omega \) is a Radon measure in \( \bar{\Omega} \). In the approximating to weak solution of (1.6), we consider a sequence \( \{g_n\} \) of \( C^1 \) nonnegative functions defined on \( \mathbb{R}_+ \) such that 
\[
g_n(0) = g(0), \quad g_n \leq g_{n+1} \leq g, \quad \sup_{s \in \mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n \to \infty} \|g_n - g\|_{L^\infty_{loc}(\mathbb{R}_+)} = 0. \tag{2.10}
\]

**Proposition 2.3** Assume that \( \{g_n\} \) is given by (2.10). Then
\[
(\Delta)\alpha u + g_n(u) = k\frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \Omega^c \tag{2.11}
\]
admits a unique positive weak solution \( u_{k,n} \) satisfying

(i) the mapping \( k \to u_{k,n} \) is increasing, the mapping \( n \to u_{k,n} \) is decreasing
\[
k\mathcal{G}_\alpha[k \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}(x)] - k\mathcal{G}_\alpha[g_n(k \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha})](x) \leq u_{k,n}(x) \leq k\mathcal{G}_\alpha[k \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}(x)], \quad \forall x \in \Omega; \tag{2.12}
\]

(ii) \( u_{k,n} \) is a classical solution of
\[
(\Delta)\alpha u + g_n(u) = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}. \tag{2.13}
\]

**Proof.** Since \( \omega \) is a Radon measure in \( \bar{\Omega} \), we could apply [6, Theorem 1.1] to obtain that problem (2.11) admits a unique weak solution \( u_{k,n} \) satisfying that (i) and \( u_{k,n} \) is a classical solution of
\[
(\Delta)\alpha u + g_n(u) = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}. \tag{2.14}
\]

From Lemma 2.1 and (2.12), there exists \( c_{24} \geq 1 \) such that
\[
\frac{1}{c_{24}} \rho(x)^{\alpha-1} \leq u_{k,n}(x) \leq c_{24} \rho(x)^{\alpha-1}, \quad x \in \Omega. \tag{2.15}
\]

Therefore, \( u_{k,n} \) is a classical solution of (2.14). \( \square \)

In particular, let \( g_0 \equiv 0 \), we have that

**Corollary 2.1** \( \mathcal{G}_\alpha[k \frac{\partial^\alpha \omega}{\partial \vec{n}^\alpha}] \) is a classical solution of
\[
(\Delta)\alpha u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{\Omega}. \tag{2.16}
\]

With the help of Corollary 2.1, we are in the position to prove Proposition 1.2.

**Proof of Proposition 1.2** We first prove that \( \tau_0(\alpha) \leq \alpha - 1 \). Inversely, if \( \tau_0(\alpha) > \alpha - 1 \), then we have that
\[
1 - \frac{2\alpha}{\tau_0(\alpha)} > \frac{1 + \alpha}{1 - \alpha} > 1 + 2\alpha.
\]

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On the one hand, it follows by [5, Theorem 1.1] that for \( p = \frac{1 + \alpha}{1 - \alpha} \), problem

\[
(\Delta)^\alpha u + |u|^{p-1}u = 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \overline{\Omega},
\]

\[\lim_{x \in \Omega, x \to \partial \Omega} u(x) = +\infty\]  

admits a solution \( w \) such that

\[
\frac{1}{c_{25}} \leq w(x) \rho^{1-\alpha}(x) \leq c_{25}, \quad x \in \Omega,
\]

where \( c_{25} > 1 \) and

\[-\frac{2\alpha}{1-\alpha} - 1 = \alpha - 1.\]

On the other hand, form Corollary 2.1 we know that for any \( \mu > 0 \), \( \mu G_\alpha[\frac{\partial^\alpha \omega}{\partial \nu^\alpha}] \) is a super solution of problem (2.16). Furthermore, from Lemma 2.1

\[
\frac{1}{c_5} \leq G_\alpha[\frac{\partial^\alpha \omega}{\partial \nu^\alpha}](x) \rho^{1-\alpha} \leq c_5.
\]

Now choosing \( \mu_1 = \frac{1}{2c_{25}c_5} \) and \( \mu_2 = 2c_{25}c_5 \), we derive that

\[
\limsup_{x \in \Omega, x \to \partial \Omega} \mu_1 G_\alpha[\frac{\partial^\alpha \omega}{\partial \nu^\alpha}](x) \rho(x)^{1-\alpha} < \frac{1}{c_{25}}, \quad \liminf_{x \in \Omega, x \to \partial \Omega} \mu_2 G_\alpha[\frac{\partial^\alpha \omega}{\partial \nu^\alpha}](x) \rho(x)^{1-\alpha} > c_{25},
\]

thus, from [5, Proposition 6.1], there is no solution \( u \) such that

\[
\frac{1}{c_{25}} \leq \liminf_{x \in \Omega, x \to \partial \Omega} u(x) \rho(x)^{1-\alpha} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x) \rho(x)^{1-\alpha} \leq c_{25}.
\]

The contradiction is obvious.

We finally prove that \( \tau_0(\alpha) \geq \alpha - 1 \). Inversely, if \( \tau_0(\alpha) < \alpha - 1 \), then we have that

\[
1 - \frac{2\alpha}{\tau_0(\alpha)} < \frac{1 + \alpha}{1 - \alpha}.
\]

From [5, Theorem 1.1] nonexistence \((ii)\), there is no solution \( u \) of problem (2.16) with

\[
\max\{1 - \frac{2\alpha}{\tau_0(\alpha)}, \frac{2\alpha}{1 - \alpha}\} < p < \frac{1 + \alpha}{1 - \alpha}
\]

such that

\[
0 < \liminf_{x \in \Omega, x \to \partial \Omega} u(x) \rho(x)^{1-\alpha} \leq \limsup_{x \in \Omega, x \to \partial \Omega} u(x) \rho(x)^{1-\alpha} < +\infty.
\]

Let \( \bar{\tau} = 2\alpha - (1 - \alpha)p \), then

\[
\bar{\tau} > 2\alpha - (1 - \alpha)\frac{1 + \alpha}{1 - \alpha} = \alpha - 1
\]

and

\[
\bar{\tau} < 2\alpha - (1 - \alpha)\frac{2\alpha}{1 - \alpha} = 0.
\]
For \( t_0 > 0 \) small, \( \Omega_{t_0} = \{ x \in \Omega, \rho(x) < t_0 \} \) is \( C^2 \) and define

\[
V_1(x) = \begin{cases} 
  d(x)^p, & x \in \Omega_{t_0}, \\
  l(x), & x \in \Omega \setminus \Omega_{t_0}, \\
  0, & x \in \Omega^c,
\end{cases}
\]

(2.19)

where the function \( l \) is positive such that \( V_1 \) is \( C^2 \) in \( \Omega \). From [5, Proposition 3.2 (ii)], there exists \( \delta_1 \in (0, t_0] \) and \( c_{26} > 1 \) such that

\[
\frac{1}{c_{26}} \rho(x)^{-2\alpha} \leq (-\Delta)^\alpha V_1(x) \leq c_{26} \rho(x)^{-2\alpha}, \quad \forall x \in \Omega_{\delta_1}.
\]

We observe that \( G_\alpha \left( \frac{\partial^\alpha \omega}{\partial t^\alpha} \right) \) is a super solution of (2.16) with \( p \) in (2.17). Now we define

\[
W_\mu(x) = G_\alpha \left( \frac{\partial^\alpha \omega}{\partial t^\alpha} \right) - \mu V_\tau(x) - \mu^2 G_\alpha[1],
\]

where \( G_\alpha[1] \) is the solution of

\[
(-\Delta)^\alpha u = 1 \quad \text{in} \quad \Omega, \\
\rho = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.
\]

We see that \( \tau - 2\alpha = (\alpha - 1)p \), there exists \( \mu_1 > 0 \) such that for \( \mu \geq \mu_1 \) and \( x \in \Omega_{\delta_1} \),

\[
(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq -c_{26} \rho(x)^{-2\alpha} + c_5 \rho(x)^{(\alpha - 1)p} \leq 0
\]

and there exists \( \mu_2 > 0 \) such that for \( \mu \geq \mu_2 \) and \( x \in \Omega \setminus \Omega_{\delta_1} \),

\[
(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq c_{26} \rho \max_{\Omega \setminus \Omega_{\delta_1}} (-\Delta)^\alpha V_1 - \mu^2 + \left( \min_{\Omega \setminus \Omega_{\delta_1}} G_\alpha \left( \frac{\partial^\alpha \omega}{\partial t^\alpha} \right) \right)^\rho \leq 0.
\]

Therefore, for \( \mu = \max\{\mu_1, \mu_2\} \), \( W_\mu \) is a sub solution of (2.16) with \( p \) in (2.17) and

\[
c_5 \rho^{\alpha - 1} \geq G_\alpha \left( \frac{\partial^\alpha \omega}{\partial t^\alpha} \right) \geq W_\mu \quad \text{and} \quad \liminf_{x \rightarrow \partial \Omega, \ |x| \rightarrow 0} W_\mu(x) \rho^{1-\alpha}(x) \geq \frac{1}{c_5}.
\]

By [5, Theorem 2.6], there exists a solution \( u \) of (2.16) with \( p \) in (2.17) satisfying (2.18). A contradiction is obtained and the proof is complete.

**3 Proof of Theorem 1.1**

**Lemma 3.1** (i) Assume that \( g \) is a continuous nondecreasing function satisfying \( g(0) \geq 0 \) and (1.8). Then

\[
\lim_{\rho(x) \rightarrow 0} G_\alpha[g(G_\alpha[k \frac{\partial^\alpha \omega}{\partial t^\alpha}])](x) \rho(x)^{1-\alpha} = 0. \tag{3.1}
\]

(ii) Assume that \( p \in (0, \frac{1}{1-\alpha}) \), then there exists \( c_{27} > 0 \) such that for any \( x \in \Omega \) with \( t \in (0, t_0) \),

\[
G_\alpha[(G_\alpha[k \frac{\partial^\alpha \omega}{\partial t^\alpha}])^p](x) \leq c_{27} \rho(x)^{2\alpha - (1-\alpha)p} + c_{33}. \tag{3.2}
\]
Proof. (i) Without loss of generality, we may assume that

\[ 0 \in \partial \Omega, \quad \vec{n}_0 = e_N, \quad x_s = se_N \]

and we just need prove (3.2) and (3.1) for \( x \) with \( s \in (0, t_0) \). It follows by Lemma 2.1 that

\[ \frac{1}{c_5} \rho(x)^{\alpha - 1} \leq G_\alpha[\frac{\partial^{\alpha, \omega}}{\partial n^{\alpha}}](x) \leq c_5 \rho(x)^{\alpha - 1}, \quad \forall x \in \Omega. \]

(3.3)

Combining with monotonicity of \( g \), we have that

\[ G_\alpha[g(\alpha[k \frac{\partial^{\alpha, \omega}}{\partial n^{\alpha}}])](x)s^{1-\alpha} \leq \int_{\Omega} G_\alpha(x_s, z)g(c_5 \rho(z)^{\alpha - 1})dzs^{1-\alpha} \]

\[ \leq \int_{\Omega} \frac{c_5 \rho^\alpha(z)}{|x_s - z|^{N-\alpha}}g(c_5 \rho(z)^{\alpha - 1})dzs^{1-\alpha} \]

\[ = c_5 \left[ \int_{B_{t_0}(0) \cap \Omega} \frac{s^{1-\alpha} \rho^\alpha(z)}{|x_s - z|^{N-\alpha}}g(c_5 \rho(z)^{\alpha - 1})dz \right. \]

\[ + \left. \int_{\Omega \setminus B_{t_0}(0)} \frac{s^{1-\alpha} \rho^\alpha(z)}{|x_s - z|^{N-\alpha}}g(c_5 \rho(z)^{\alpha - 1})dz \right] \]

\[ := A_1(s) + A_2(s). \]

Let \( B_{t_0}(0) \) be the ball with radium \( t_0 \) and centered at the origin in \( \mathbb{R}^{N-1} \) and since \( \partial \Omega \) is \( C^2 \), there exists a \( C^2 \) function \( \psi : B_{t_0}(0) \to \mathbb{R} \) such that

\[ (z', \psi(z') \in \partial \Omega \quad \text{for any } z' \in B_{t_0}(0), \]

where \( \eta > 0 \). Denote

\[ \Psi(y) = (y', \psi(y')) + yN\vec{n}_0(y', \psi(y'))), \quad \forall y = (y', y_N) \in Q_\eta, \]

where

\[ Q_\eta = \{ z = (z', z_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, |z'| < \eta, 0 < z_N < \eta \}. \]

Thus, \( \Psi \) is a \( C^2 \) diffeomorphism mapping such that

\[ \Psi(y) = y, \quad \forall y = te_N, \quad t \in (0, \eta). \]

Therefore, if \( t_0 > 0 \) is chosen small enough, we have that \( \Omega \cap B_{t_0}(0) \subset \Psi(Q_\eta) \) and there exists \( c_{28} > 1 \) such that for \( z = \Psi(y) \in \Psi(Q_\eta), \)

\[ \rho(z) = y_N \quad \text{and} \quad \frac{1}{c_{28}} |se_N - y| \leq |se_N - z| \leq c_{28} |se_N - y|. \]

(3.4)

Then we have that

\[ A_1(s) = \int_{B_{t_0}(0) \cap \Omega} \frac{s^{1-\alpha} \rho^\alpha(z)}{|x_s - z|^{N-\alpha}}g(c_5 \rho(z)^{\alpha - 1})dz \]

\[ \leq c_{28} \int_{B_{t_0}(0) \cap \Omega} \frac{s^{1-\alpha} \rho^\alpha(z)}{|x_s - \Psi^{-1}(z)|^{N-\alpha}}g(c_5 \rho(z)^{\alpha - 1})dz \]

\[ \leq c_{28} \int_{Q_\eta} \frac{s^{1-\alpha} y_N^\alpha}{|x_s - y|^{N-\alpha}}g(c_5ky_N^{\alpha - 1})dy. \]
For \( s \in (0, \frac{\eta}{\sqrt{s}}) \), we decompose \( Q_\eta \) as following

\[
Q_{i,0} = \{ z = (z', z_N) \in Q_\eta : \frac{is}{2} \leq |z'| < \frac{(i+1)s}{2}, \ 0 < z_N < \frac{s}{2} \}
\]

and

\[
Q_{i,j} = \{ z = (z', z_N) \in Q_\eta : \frac{is}{2} \leq |z'| < \frac{(i+1)s}{2}, \ (j + 1)\frac{s}{2} \leq z_N < (j + \frac{3}{2})s \},
\]

where \( i = 0, 1, \cdots, N_s \), \( j = 1, \cdots, N_s \) and \( N_s \) is the largest integer number such that \( N_s \leq \frac{\eta}{\sqrt{s}} \).

For \( y \in Q_{0,1} \), we have that \( \frac{s}{2} \leq y_N \leq \frac{3s}{2} \) and

\[
\begin{align*}
\int_{Q_{0,1}} \frac{s^{1-\alpha}}{|x_s-y|^{N-\alpha}} d\gamma N_s c g(c_5 k y_N^{\alpha-1}) dy &\leq c_5 s^{1+\alpha} (c_2 k s^{\alpha-1}) \int_{B_{2/3}(e_N)} \frac{1}{|e_N-z|^{N-\alpha}} dz \\
&= c_5 r^{-\frac{\alpha+1}{\alpha}} g(c_2 k r) \int_{B_{2/3}(e_N)} \frac{1}{|v_n-z|^{N-\alpha}} dz \\
&\leq c_{30} r^{-\frac{\alpha+1}{\alpha}} g(c_2 k r),
\end{align*}
\]

where \( r = s^{\alpha-1} \) and \( c_29, c_{30} > 0 \).

For \( y \in Q_{i,0} \) with \( i = 0, \cdots, N_s \), we obtain that \( |x_s - y| \geq \frac{i+1}{4} s \) and

\[
\begin{align*}
\int_{Q_{i,0}} \frac{s^{1-\alpha}}{|x_s-y|^{N-\alpha}} d\gamma N_s c g(c_5 k y_N^{\alpha-1}) dy &\leq c_{31} s^{1-N} (1+i)^{N-\alpha} \int_{Q_{i,0}} y_N^{\alpha} g(c_5 k y_N^{\alpha-1}) dy \\
&= c_{32} (1+i)^{N-\alpha} \int_{0}^{\frac{\pi}{2}} t^{\alpha} (c_2 k t^{\alpha-1}) dt \\
&= c_{32} \left(1 - \frac{1}{\alpha}\right) (1+i)^{N-\alpha} \int_{s^{-\frac{1}{\alpha}}}^{\infty} \tau^{-\frac{1}{\alpha}+\frac{1}{\gamma}} g(c_5 k \tau) d\tau,
\end{align*}
\]

where \( r = s^{\alpha-1} \) and \( c_{31}, c_{32} > 0 \).

For \( y \in Q_{i,j} \) with \( i = 0, \cdots, N_s \), \( j = 1, \cdots, N_s \) and \( (i,j) \neq (0,1) \), we derive that \( |x_s - y| \geq \frac{i+j+1}{4} s \) and \( (j + \frac{1}{2}) s \leq y_N < (j + \frac{3}{2}) s \)

\[
\begin{align*}
\int_{Q_{i,j}} \frac{s^{1-\alpha}}{|x_s-y|^{N-\alpha}} d\gamma N_s c g(c_5 k y_N^{\alpha-1}) dy &\leq c_{33} s^{1-N} (i+j)^{N-\alpha} (j s)^{\alpha} g(c_5 k (j s)^{\alpha-1}) |Q_{i,j}| \\
&= c_{34} s^{\alpha} (i+j)^{N-\alpha} r^{-\frac{1}{\alpha}+\frac{1}{\gamma}} g(c_5 k j^{\alpha-1} r),
\end{align*}
\]

where \( r = s^{\alpha-1} \) and \( c_{33}, c_{34} > 0 \).

Therefore, there exist \( c_{35}, c_{36} > 0 \) such that

\[
A_1(s) \leq \sum_{i=0}^{N_s} c_{35} j^\alpha r^{-\frac{1+i}{\alpha}} g(c_5 k j^{\alpha-1} r) + \sum_{i=0}^{N_s} c_{36} (i+j)^{N-\alpha} \int_{s^{-\frac{1}{\gamma}}}^{\infty} \tau^{-\frac{1+i}{\alpha}} g(c_5 k \tau) d\tau.
\]

Since

\[
\sum_{i=0}^{N_s} c_{36} (i+j)^{N-\alpha} \int_{s^{-\frac{1}{\gamma}}}^{\infty} \tau^{-\frac{1+i}{\alpha}} g(c_5 k \tau) d\tau \leq c_{37} \int_{s^{-\frac{1}{\gamma}}}^{\infty} \tau^{-\frac{1+i}{\alpha}} g(c_5 k \tau) d\tau
\]

which tends to 0 as \( s \to 0^+ \) by hypothesis (1.8).
For any \( \epsilon > 0 \), there exists \( n_\epsilon > 1 \) such that

\[
\sum_{i,j=n_\epsilon}^{N} \frac{c_{35}}{(i + j + 1)^{N-\alpha}} \leq \epsilon,
\]

and since \( \{(j^{\alpha-1})^{-\frac{1}{\alpha}} g(c_{29} k j^{\alpha-1})\} \) is uniformly bounded, we imply that

\[
\sum_{i,j=n_\epsilon}^{N} \frac{c_{35} j^{\alpha}}{(i + j + 1)^{N-\alpha}} (j^{\alpha-1})^{-\frac{1}{\alpha}} g(c_{29} k j^{\alpha-1}) \leq \epsilon.
\]

For \( i, j < n_\epsilon \), there exists \( s_\epsilon \in (0, \eta) \) such that

\[
j^{\alpha} r^{-\frac{1}{\alpha}} g(c_{29} k j^{\alpha-1}) \leq \epsilon, \quad \text{for } r \geq s_\epsilon^{\alpha-1},
\]

thus, for any \( \epsilon > 0 \), there exists \( s_\epsilon \) such that for \( s \in (0, s_\epsilon) \)

\[
\sum_{i=0,j=0}^{n_\epsilon} \frac{c_{35} j^{\alpha}}{(i + j + 1)^{N-\alpha}} (j^{\alpha-1})^{-\frac{1}{\alpha}} g(c_{29} k j^{\alpha-1}) \leq \epsilon \sum_{i=0,j=0}^{n_\epsilon} \frac{c_{35} j^{\alpha}}{(i + j + 1)^{N-\alpha}} \leq \epsilon.
\]

Then we deduce that

\[
\lim_{s \to 0^+} A_1(s) = 0.
\]

Therefore,

\[
\lim_{s \to 0^+} G_\alpha[g(G_\alpha[k \frac{\partial^\alpha \omega}{\partial t^\alpha}])](x_s) s^{1-\alpha} = 0.
\]

Since \( |x_s - z| \geq c_{38} t_0 \) for \( \Omega \setminus B_{t_0}(0) \), therefore,

\[
A_2 \leq c_{39} s^{1-\alpha} \int_{\Omega \setminus B_{t_0}(0)} \rho^\alpha(z) g(c_{35} k \rho(z)^{\alpha-1}) dz
\]

\[
\leq c_{39} s^{1-\alpha} \omega(\partial \Omega) \int_0^{d_0} t^\alpha g(c_{35} k t^{\alpha-1}) dt
\]

\[
\leq c_{40} s^{1-\alpha},
\]

where \( c_{39}, c_{40} > 0 \) and \( d_0 = \max_{x \in \Omega} \rho(x) \). Thus, (3.1) holds.

(ii) When \( g(s) = s^p \) with \( p \in (0, \frac{1+\alpha}{\alpha-1}) \), we observe that (3.3) becomes that

\[
\int_{Q_{i+1}} \frac{s^{1-\alpha} g_N}{|x_s - y|^{N-\alpha}} (c_5 y_N^{\alpha-1})^p dy \leq c_5 s^{1+\alpha+(\alpha-1)p},
\]

(3.6) turns out to

\[
\int_{Q_{i+1}} \frac{s^{1-\alpha} g_N}{|x_s - y|^{N-\alpha}} (c_5 y_N^{\alpha-1})^p dy = \frac{c_{41}}{(1+i)^{N-\alpha}} \int_0^{\frac{z}{i}} t^{\alpha+(\alpha-1)p} dt
\]

\[
\leq \frac{c_{42}}{(1+i)^{N-\alpha}} s^{1+\alpha+(\alpha-1)p}
\]
and (3.7) becomes that
\[
\int_{Q_{i,j}} \frac{s^{1-\alpha}y_N}{|x_s-y|^{N-\alpha}} |(c_3y_N^{q-1})| \leq \frac{c_{43}}{(1+i+j)^{N-\alpha}s^{1+\alpha+(a-1)p}}.
\]
Therefore, we have that
\[
\int_{B_0(0) \cap \Omega} \frac{\rho(\gamma)}{|x_s-y|^{N-\alpha}} \rho(\gamma)^{(a-1)p} \, dy \leq c_{44}s^{2a+(a-1)p}.
\]
which, combining (3.8), implies (3.2). □

**Proof of Theorem 1.1.** To prove the existence of weak solution. Take \( \{g_n\} \) a sequence of \( C^1 \) nondecreasing functions defined on \( \mathbb{R} \) satisfying \( g_n(0) = g(0) \) and (2.10). By Proposition 2.3, problem (2.11) admits a unique weak solution \( u_{k,n} \) such that
\[
0 < u_{k,n} \leq G_{\alpha} \left[ \frac{\partial^\alpha \omega}{\partial \nu} \right] \quad \text{in} \quad \Omega
\]
and
\[
\int \Omega [u_{k,n}(-\Delta)^a \xi + g_n(u_{k,n})\xi] \, dx = k \int_{\partial \Omega} \frac{\partial^\alpha \xi(x)}{\partial \nu^2} d\omega(x), \quad \forall \xi \in X_\alpha. \quad (3.9)
\]
For any compact set \( K \subset \Omega \), we observe from \([6, \text{Lemma 3.2}]\) that for some \( \beta \in (0, \alpha) \),
\[
\|u_{k,n}\|_{C^\beta(K)} \leq c_{45}k.
\]
Therefore, up to some subsequence, there exists \( u_k \) such that
\[
\lim_{n \to \infty} u_{k,n} = u_k \quad \text{in} \quad \Omega.
\]
Then \( g_n(u_{k,n}) \) converge to \( g(u_k) \) in \( \Omega \) as \( n \to \infty \). By Proposition 2.2 and (3.19) in \([6]\), we have that
\[
u_{k,n} \to u_k \quad \text{in} \quad L^1(\Omega), \quad \|g_n(u_n)\|_{L^1(\Omega, \nu^\alpha \, dx)} \leq c_{46}\|G_{\alpha} \left[ \frac{\partial^\alpha \nu}{\partial \nu^2} \right]\|_{L^1(\Omega)}
\]
and
\[
m(\lambda) \leq c_{47} \lambda^{-\frac{1+n}{1-\alpha}} \quad \text{for} \quad \lambda > \lambda_0,
\]
where
\[
m(\lambda) = \int_{S_\lambda} \rho_{\partial \Omega}^\alpha(x) \, dx \quad \text{with} \quad S_\lambda = \{ x \in \Omega : G_{\alpha} \left[ \frac{\partial^\alpha \nu}{\partial \nu^2} \right] > \lambda \}.
\]
For any Borel set \( E \subset \Omega \), we have that
\[
\int_E |g_n(u_n)|\rho_{\partial \Omega}^\alpha(x) \, dx \leq \int_{E \cap S_{\frac{\lambda}{k}}} g \left( kG_{\alpha} \left[ \frac{\partial^\alpha \nu}{\partial \nu^2} \right] \right) \rho_{\partial \Omega}^\alpha(x) \, dx + \int_{E \cap S_{\frac{\lambda}{k}}} g \left( kG_{\alpha} \left[ \frac{\partial^\alpha \nu}{\partial \nu^2} \right] \right) \rho_{\partial \Omega}^\alpha(x) \, dx
\]
\[
\leq \tilde{g} \left( \frac{\lambda}{k} \right) \int_E \rho_{\partial \Omega}^\alpha(x) \, dx + \int_{S_{\frac{\lambda}{k}}} \tilde{g} \left( kG_{\alpha} \left[ \frac{\partial^\alpha \nu}{\partial \nu^2} \right] \right) \rho_{\partial \Omega}^\alpha(x) \, dx
\]
\[
\leq \tilde{g} \left( \frac{\lambda}{k} \right) \int_E \rho_{\partial \Omega}^\alpha(x) \, dx + \tilde{m} \left( \frac{\lambda}{k} \right) \tilde{g} \left( \frac{\lambda}{k} \right) + \int_{\frac{\lambda}{k}}^{\infty} \tilde{m}(s) \tilde{g}(s),
\]
where \( \tilde{g}(r) = g(|r|) - g(-|r|) \).
On the other hand,
\[ \int_{\frac{1}{k}}^{\infty} \tilde{g}(s)d\tilde{m}(s) = \lim_{T \to \infty} \int_{\frac{1}{k}}^{T} \tilde{g}(s)d\tilde{m}(s). \]

Thus,
\[ m \left( \frac{\lambda}{k} \right) \tilde{g} \left( \frac{\lambda}{k} \right) + \int_{\frac{1}{k}}^{T} \tilde{m}(s)d\tilde{g}(s) \leq c_{47} \tilde{g} \left( \frac{\lambda}{k} \right) \frac{1}{1-\alpha} + c_{47} \int_{\frac{1}{k}}^{T} s^{-\frac{1}{1-\alpha}} d\tilde{g}(s) \]
\[ \leq c_{48} T^{-\frac{1}{1-\alpha}} \tilde{g}(T) + \frac{c_{49}}{1-\alpha} + 1 \int_{\frac{1}{k}}^{T} s^{-\frac{1}{1-\alpha}} \tilde{g}(s) ds. \]

By assumption (1.8) and [6, Lemma 3.4] with \( p = \frac{1+\alpha}{1-\alpha} \), \( T \to \frac{1}{T} \tilde{g}(T) \to 0 \) when \( T \to \infty \), therefore,
\[ m \left( \frac{\lambda}{k} \right) \tilde{g} \left( \frac{\lambda}{k} \right) + \int_{\frac{1}{k}}^{\infty} \tilde{m}(s)d\tilde{g}(s) \leq \frac{c_{49}}{1-\alpha} + 1 \int_{\frac{1}{k}}^{\infty} s^{-\frac{1}{1-\alpha}} \tilde{g}(s) ds. \]

Notice that the above quantity on the right-hand side tends to 0 when \( \lambda \to \infty \). The conclusion follows: for any \( \epsilon > 0 \) there exists \( \lambda > 0 \) such that
\[ \frac{c_{49}}{1-\alpha} + 1 \int_{\frac{1}{k}}^{\infty} s^{-\frac{1}{1-\alpha}} \tilde{g}(s) ds \leq \frac{\epsilon}{2}. \]

For \( \lambda \) fixed, there exists \( \delta > 0 \) such that
\[ \int_{E} \rho_{\lambda\Omega}(x) dx \leq \delta \implies \tilde{g} \left( \frac{\lambda}{k} \right) \int_{E} \rho_{\lambda\Omega}(x) dx \leq \frac{\epsilon}{2}, \]
which implies that \( \{ g_{n} \circ u_{n} \} \) is uniformly integrable in \( L^{1}(\Omega, \rho_{\lambda\Omega} dx) \). Then \( g_{n} \circ u_{n} \to g \circ u_{\nu} \) in \( L^{1}(\Omega, \rho_{\lambda\Omega} dx) \) by Vitali convergence theorem.

Passing to the limit as \( n \to +\infty \) in the identity (3.9), it implies that
\[ \int_{\Omega} [u_{k}(\Delta)^{\alpha} \xi + g(u_{k})\xi] dx = k \int_{\partial\Omega} \frac{\partial^{\alpha} \xi}{\partial n_{\Omega}^{\alpha}} d\omega(x), \quad \forall \xi \in X_{\alpha}. \]

Then \( u_{k} \) is a weak solution of (1.6). Moreover, it follows by the fact
\[ k \mathcal{G}_{\alpha}\frac{\partial^{\alpha} \omega}{\partial n_{\Omega}^{\alpha}} - g(k \mathcal{G}_{\alpha}\frac{\partial^{\alpha} \omega}{\partial n_{\Omega}^{\alpha}}) \leq u_{k} \leq k \mathcal{G}_{\alpha}\frac{\partial^{\alpha} \omega}{\partial n_{\Omega}^{\alpha}} \quad \text{in } \Omega. \]

**Uniqueness of weak solution.** Let \( u_{1}, u_{2} \) be two weak solutions of (1.6) and \( w = u_{1} - u_{2}. \) Then \( (\Delta)^{\alpha} w = g_{n}(u_{2}) - g_{n}(u_{1}) \) and \( g_{n}(u_{2}) - g_{n}(u_{1}) \in L^{1}(\Omega, \rho_{\alpha} dx) \). By Kato’s inequality, see [8] Proposition 2.4, for \( \xi \in X_{\alpha}, \xi \geq 0 \), we have that
\[ \int_{\Omega} |w|(\Delta)^{\alpha} \xi dx + \int_{\Omega} [g_{n}(u_{1}) - g_{n}(u_{2})] \text{sign}(w)\xi dx \leq 0. \]

Combining with \( \int_{\Omega} [g_{n}(u_{1}) - g_{n}(u_{2})] \text{sign}(w)\xi dx \geq 0 \), then we have
\[ w = 0 \quad \text{a.e. in } \Omega. \]

**Regularity of \( u_{k,n} \) and \( u_{k} \).** Since \( g_{n} \) is \( C^{1} \) in \( \mathbb{R} \), then by [8] Lemma 3.2, we have
\[ \| u_{k,n} \|_{C^{2-\beta}(K)} \leq c_{50} k, \]
for any compact set \( K \) and some \( \beta \in (0, \alpha) \). Then \( u_{k,n} \) is \( C^{2-\beta} \) locally in \( \Omega \). Together with the fact that \( u_{n,k} \) is classical solution of (2.13), we derive by Theorem 2.2 in [5] that \( u_{k} \) is a classical solution of (2.13).

To prove (1.9) Plugging (3.1) and (3.3) into (3.10), we obtain that (1.9). \( \square \)
4 Proof of Theorem 1.2

4.1 Strong singularity for $p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha})$

In this subsection, we consider the limit of $\{u_k\}$ as $k \to \infty$, where $u_k$ is the weak solution of
\[
(-\Delta)^\alpha u + u^p = k \frac{\partial \omega}{\partial \vec{n}}^\alpha 
\text{ in } \bar{\Omega},
\]
\[
u = 0 \quad \text{ in } \bar{\Omega}^c,
\]
here $p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha})$. From Theorem 1.1, we know that $k \mapsto u_k$ is increasing and $u_k$ is a classical solution of (1.2).

In order to control the limit of $\{u_k\}$ as $k \to \infty$, we have to obtain barrier function, i.e. a suitable super solution of (1.2). To this end, we consider $C^2$ function $w_p$ satisfying
\[
w_p(x) = \begin{cases} 
\rho(x)^{-\frac{2\alpha}{p-1}}, & \text{for } x \in \Omega, \\
0, & \text{for } x \in \Omega^c.
\end{cases}
\] (4.1)

We see that $w_p \in L^1(\Omega)$ if $\frac{2\alpha}{p-1} < 1$, i.e. $p > 1 + 2\alpha$.

Lemma 4.1 Assume that $p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha})$ and $w_p$ is defined in (4.1). Then there exists $\lambda_0 > 0$ such that $\lambda_0 w_p$ is a super solution of (1.2).

**Proof.** For $p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha})$, we have that $-\frac{2\alpha}{p-1} \in (-1, 0)$ and from [5, Proposition 3.2], it shows that there exists $c(p) < 0$ such that
\[
(-\Delta)^\alpha w_p(x) \geq c(p) \rho(x)^{-\frac{2\alpha}{p-1} - 2\alpha}, \quad x \in \Omega.
\]
Thus, taking $\lambda_0 = |c(p)|^\frac{1}{p-1}$, we derive that
\[
(-\Delta)^\alpha (\lambda_0 w_p) + (\lambda_0 w_p)^p \geq 0 \quad \text{in } \Omega.
\]
Together with $\lambda_0 w_p = 0$ in $\Omega^c$, $\lambda_0 w_p$ is a super solution of (1.2). The proof ends. □

We observe that the super solution $\lambda_0 w_p$ constructed in Lemma 4.1 provide a upper bound for $u_\infty$.

**Proof of Theorem 1.2 (i).** For $p \in (1 + 2\alpha, \frac{1+\alpha}{1-\alpha})$, we have that
\[
-\frac{2\alpha}{p-1} \in (-1, -1 + \alpha)
\]
and it follows by (3.3) that
\[
u_k(x) \leq k \mathbb{G}_\alpha \left[ \frac{\partial \omega}{\partial \vec{n}}^\alpha \right](x) \leq \frac{c_5 k}{|x|^{1-\alpha}}, \quad x \in \Omega.
\]
Then $\lim_{x \in \Omega, \rho(x) \to 0} \frac{\nu_k(x)}{w_p(x)} = 0$ and we claim that
\[
u_k \leq \lambda_0 w_p \quad \text{in } \Omega.
\]
In fact, if it fails, then there exists $z_0 \in \Omega$ such that
\[
(\nu_k - \lambda_0 w_p)(z_0) = \inf_{\Omega} (\nu_k - \lambda_0 w_p) = \text{ess inf}_{\mathbb{R}^N} (\nu_k - \lambda_0 w_p) < 0.
\]
Then we have $(-\Delta)^{\alpha}(u_k - \lambda_0 w_p)(z_0) < 0$, which contradicts the fact that

$$(-\Delta)^{\alpha}(u_k - \lambda_0 w_p)(z_0) = \lambda_0 w_p^p(z_0) - u_k^p(z_0) > 0.$$ 

By monotonicity of the mapping $k \rightarrow u_k$, there holds

$$u_\infty(x) := \lim_{k \rightarrow \infty} u_k(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

which is a classical solution of (1.2) and

$$u_\infty(x) \leq \lambda_0 w_p(x) = \lambda_0 \rho(x)^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \Omega.$$ 

By applying Stability Theorem [5, Theorem 2.4], we obtain that $u_\infty$ is a classical solution of (1.2).

Finally, we claim that there exists $c_{51} > 0$ such that for $t \in (0, t_0)$,

$$u_\infty(x) \geq c_{51} t^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \mathcal{C}_t. \quad (4.2)$$

Indeed, let $t_k = (\sigma^{-1})^{\frac{p-1}{(1-\alpha)p-1}}$, where $\sigma > 0$ will be chosen later, then $k = \sigma t_k^{\frac{(1-\alpha)p-1}{p-1}}$ and for $x \in \mathcal{C}_t$, we apply Lemma [3.1] with $p \in (1 + 2\alpha, \frac{1 + \alpha}{1 - \alpha})$ that

$$u_k(x) \geq k[G_\alpha \frac{\partial^\alpha \omega}{\partial \eta^\alpha}(x) - k^p G_\alpha[(G_\alpha \frac{\partial^\alpha \omega}{\partial \eta^\alpha})^p](x)$$

$$\geq c_5 k t^{a-N} [1 - c_52 k^{p-1} (\alpha^{-1} + p + a+1)]$$

$$\geq c_5 \sigma t_k^{\frac{2\alpha}{p-1}} [1 - c_52 \sigma^{p-1} t_k^{p-1} (t_k/2)^{(\alpha-1)p+a+1}]$$

$$\geq c_5 \sigma t_k^{\frac{2\alpha}{p-1}} [1 - c_52 \sigma^{p-1} (1 - \alpha)p-a-1]$$

$$\geq \frac{c_5 \sigma}{2} \rho(x)^{-\frac{2\alpha}{p-1}},$$

where we choose $\sigma$ such that $c_52 \sigma^{p-1} (N-\alpha)p-a-N = \frac{1}{2}$. Then for any $x \in \Omega$, there exists $k > 0$ such that $x \in \Omega$ and then

$$u_\infty(x) \geq u_k(x) \geq \frac{c_5 \sigma}{2} \rho(x)^{-\frac{2\alpha}{p-1}}, \quad \forall x \in \Omega.$$

This ends the proof. □

### 4.2 The limit of $\{u_k\}$ blows up when $p \in (0, 1 + 2\alpha]$

In this subsection, we derive the blow-up behavior of the limit of $\{u_k\}$ when $p \in (0, 1 + 2\alpha]$. To this end, we have to do more estimate for $u_k$.

**Lemma 4.2** Assume that $g(s) = s^p$ with $p \in (1, 1 + 2\alpha]$ and $u_k$ is the solution of (1.6) obtained by Theorem [1.1]. Then there exist $c_{52} > 0$, $r_0 \in (0, \frac{1}{2})$ and $\{r_k\}_k \subset (0, r_0)$ satisfying $r_k \rightarrow 0$ as $k \rightarrow \infty$ such that

$$u_k(x) \geq \frac{c_{52}}{\rho(x)}, \quad \forall x \in \Omega_{r_0}. \quad (4.3)$$
Proof. We divide $\alpha, p$ into 4 cases:
Case I: $1 < \frac{2\alpha}{1-\alpha} < 1 + 2\alpha$ and $p \in \left[\frac{2\alpha}{1-\alpha}, 1 + 2\alpha\right]$; Case II: $1 < \frac{2\alpha}{1-\alpha} < 1 + 2\alpha$ and $p \in \left[1, \frac{2\alpha}{1-\alpha}\right]$;
Case III: $\frac{2\alpha}{1-\alpha} \leq 1$ and $p \in (1, 1 + 2\alpha]$; Case IV: $\frac{2\alpha}{1-\alpha} \geq 1 + 2\alpha$ and $p \in (1, 1 + 2\alpha]$.

To prove (4.3) in Case I and Case III. Let $j = \frac{j}{2}$ with $j \in (k_0, k)$, then $j = r_j^{-\alpha}$.
Applying Lemma 3.1 (ii), for $p \geq \frac{2\alpha}{1-\alpha}$, we have that for $x \in \Omega_{r_j} \setminus \Omega_{r_j}$,

$$u_j(x) \geq jG_\alpha[\frac{\partial^\omega}{\partial t^\omega}](x) - j^pG_\alpha[(G_\alpha[\frac{\partial^\omega}{\partial t^\omega}])^p](x) \geq c_\alpha^{-1}j^{-\alpha - 1} - c_\alpha j^p \rho(x)^{(1-\alpha)p+2\alpha} \geq c_\alpha^{-1}r_j^{-\alpha - 1} - c_\alpha r_j^{-\alpha - (1-\alpha)p+2\alpha} \geq \frac{1}{2c_\alpha} \rho(x)^{-1},$$

where the last inequality holds since $-\alpha p - (1-\alpha)p + 2\alpha > -1$ and $r_j \to 0$ as $j \to \infty$. Then for any $x \in \Omega_{t_0}$, there exists $j \in (k_0, k)$ such that $x \in \Omega_{t_j} \setminus \Omega_{t_j}$ and then

$$u_k(x) \geq u_j(x) \geq \frac{1}{2c_\alpha} \rho^{-1}(x), \quad \forall x \in \Omega_{t_0}.$$  

To prove (4.3) in Case II and Case IV. Let $r_j = j^{-\frac{1}{2}}$ with $j \in (k_0, k)$, then $j = r_j^{-\alpha}$ and for $x \in \Omega_{r_j} \setminus \Omega_{r_j}$, we have that

$$u_j(x) \geq jG_\alpha[\frac{\partial^\omega}{\partial t^\omega}](x) - j^pG_\alpha[(G_\alpha[\frac{\partial^\omega}{\partial t^\omega}])^p](x) \geq c_\alpha^{-1}j^{-\alpha - 1} - c_\alpha j^p r_j^{-\alpha - (1-\alpha)p+2\alpha} \geq \frac{1}{2c_\alpha} \rho(x)^{-1},$$

where the last inequality holds since $-\alpha p > -1$ and $r_j \to 0$ as $j \to \infty$. For any $x \in \Omega_{t_0}$, there exists $j \in (k_0, k)$ such that $x \in \Omega_{r_j} \setminus \Omega_{r_j}$ and then

$$u_k(x) \geq u_j(x) \geq \frac{1}{2c_\alpha} \rho(x)^{-1}, \quad \forall x \in \Omega_{t_0}.$$  

The proof ends. \hfill \Box

Proof of Theorem 1.2 (ii). It derives by Lemma 4.2 that

$$\pi_k := \int_{\Omega_{t_0} \setminus \Omega_{t_0}} u_k(x) \geq c_5 \int_{\Omega_{t_0} \setminus \Omega_{t_0}} \rho^{-1}(x) dx \to \infty \quad \text{as} \quad k \to \infty. \quad (4.4)$$

Then

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_{t_0},$$

$$u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,$$

$$u = u_k \quad \text{in} \quad \Omega_{t_0}$$

admits a unique solution $w_k$. By Comparison Principle,

$$u_k \geq w_k \quad \text{in} \quad B_{2\alpha}(y_0). \quad (4.6)$$
Let \( \tilde{w}_k = w_k - u_k \chi_{\Omega \setminus \Omega_0} \), then \( \tilde{w}_k = w_k \) in \( \Omega \setminus \Omega_0 \) and for \( x \in \Omega \setminus \Omega_0 \),

\[
(-\Delta)^a \tilde{w}_k(x) = -\lim_{\varepsilon \to 0^+} \int_{B_{\varepsilon_0}(y_0) \setminus B_r(x)} \frac{u_k(z) - u_k(x)}{|z-x|^{N+2a}} \, dz \\
+ \lim_{\varepsilon \to 0^+} \int_{B_{\varepsilon_0}(y_0) \setminus B_r(x)} \frac{u_k(x)}{|z-x|^{N+2a}} \, dz \\
= -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{u_k(z) - u_k(x)}{|z-x|^{N+2a}} \, dz + \int_{B_{\varepsilon_0}(0)} \frac{u_k(x)}{|z-x|^{N+2a}} \, dz \\
\geq (-\Delta)^a w_k(x) + c_{53} \pi_k,
\]

where \( c_{53} = (|y_0| + r_0)^{-N - 2a} \) and the last inequality follows by the fact of \( |z - x| \leq |x| + |z| \leq |y_0| + r_0 \) for \( \forall z \in B_{\frac{r_0}{\varepsilon}}(0), \ \forall x \in \Omega \setminus \Omega_0 \).

Therefore,

\[
(-\Delta)^a \tilde{w}_k(x) + \tilde{w}_k^p(x) \geq (-\Delta)^a w_k(x) + w_k^p(x) + c_{53} \pi_k \\
= c_{53} \pi_k, \quad \forall x \in \Omega \setminus \Omega_0,
\]

that is, \( \tilde{w}_k \) is a super solution of

\[
(-\Delta)^a u + u^p = c_{53} \pi_k \quad \text{in} \quad \Omega \setminus \Omega_0, \\
u = 0 \quad \text{in} \quad (\Omega \setminus \Omega_0)^c. \tag{4.7}
\]

Let \( \eta_1 \) be the solution of

\[
(-\Delta)^a u = 1 \quad \text{in} \quad \Omega \setminus \Omega_0, \\
u = 0 \quad \text{in} \quad (\Omega \setminus \Omega_0)^c.
\]

Then \( (c_{53} \pi_k)^{\frac{1}{p}} \frac{\eta_1(x)}{2 \max_{\mathbb{R}^N} \eta_1} \) is sub solution of (4.7) for \( k \) large enough. By Comparison Principle, we have that

\[
\tilde{w}_k(x) \geq (c_{53} \pi_k)^{\frac{1}{p}} \frac{\eta_1(x)}{2 \max_{\mathbb{R}^N} \eta_1}, \quad \forall x \in B_{\varepsilon_0}(y_0),
\]

which implies that

\[
w_k(y) \geq c_{54} (c_{53} \pi_k)^{\frac{1}{p}}, \quad \forall y \in \Omega \setminus \Omega_0,
\]

where \( c_{54} = \min_{x \in B_{\varepsilon_0}(y_0)} \frac{\eta_1(x)}{2 \max_{\mathbb{R}^N} \eta_1} \). Therefore, (4.6) and (4.4) imply that

\[
\lim_{k \to \infty} u_k(y) \geq \lim_{k \to \infty} w_k(y) = \infty, \quad \forall \Omega \setminus \Omega_0.
\]

Similarly, we can prove

\[
\lim_{k \to \infty} u_k(y) \geq \lim_{k \to \infty} w_k(y) = \infty, \quad \forall y \in \Omega.
\]

The proof ends. \( \square \)

5 \quad \textbf{Proof of Theorem 1.3}

In this section, we are devoted to consider the solution of (1.10) with different blow-up speeds. Without loss of generality, we assume that

\[
x_0 = 0 \in \partial \Omega.
\]

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For $k, j \in \mathbb{N}$, denote 
\[ \nu = \omega + \delta_0, \]  
and then  
\[ G_\alpha \left[ \frac{\partial^s \nu}{\partial \nu^s} \right] = G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right] + G_\alpha \left[ \frac{\partial^s \omega}{\partial \nu^s} \right], \]  
which is a $\alpha$–harmonic function of (2.15), since $G_\alpha \left[ \frac{\partial^s \omega}{\partial \nu^s} \right]$ is a $\alpha$–harmonic function of (2.15) and $G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right]$ is $\alpha$–harmonic.

**Proposition 5.1** Let $\alpha \in [0, 1]$ and $p^* = \frac{N+\alpha}{N-\alpha}$. Then there exists $c_{57} > 0$ such that  
\[ \|G_\alpha \left[ \frac{\partial^s \nu}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)} \leq c_{57}. \]  

**Proof.** Since  
\[ \|G_\alpha \left[ \frac{\partial^s \nu}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)} \leq k \|G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)} + j \|G_\alpha \left[ \frac{\partial^s \omega}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)}. \]  
From Proposition 2.2 and $p^* = \frac{1+\alpha}{1-\alpha} > \frac{N+\alpha}{N-\alpha}$, on the one hand, we have that  
\[ \|G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)} \leq \frac{c_{56}}{2} \|G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)} \leq c_{56} \]  
On the other hand, for $\lambda > 0$, denote  
\[ S_\lambda = \{ x \in \Omega : G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right](x) > \lambda \} \quad \text{and} \quad m(\lambda) = \int_{S_\lambda} \rho_\delta^\beta(x) dx \]  
and from [6] Lemma 3.3 with $\nu = \delta_0$, there exist $\lambda_0 > 1$ and $c_{57} > 0$ such that for any $\lambda \geq \lambda_0$,  
\[ \tilde{m}(\lambda) \leq c_{57} \lambda \frac{N+\alpha}{N-\alpha}. \]  
which implies that  
\[ \|G_\alpha \left[ \frac{\partial^s \delta_0}{\partial \nu^s} \right]\|_{M^p_N(\Omega, \rho^\omega dx)} \leq c_{55}. \]  
Thus, (5.2) follows by (5.4) and (5.3). Combining [6] Theorem 1.1 and the proof of Proposition 2.3 we have following result.

**Proposition 5.2** Assume that $\{g_n\}_n$ is given by (2.10). Then  
\[ (-\Delta)^{\alpha} u + g_n(u) = k \frac{\partial^s \nu}{\partial \nu^s} \quad \text{in} \quad \Omega, \]  
\[ u = 0 \quad \text{in} \quad \Omega^c \]  
admits a unique positive weak solution $u_{k,j,n}$ satisfying  
(i) the mappings $k \rightarrow u_{k,j,n}$, $j \rightarrow u_{k,j,n}$ are increasing, the mapping $n \rightarrow u_{k,j,n}$ is decreasing  
\[ G_\alpha \left[ \frac{\partial^s \nu}{\partial \nu^s} \right](x) - G_\alpha \left[ g_n \left( G_\alpha \left[ \frac{\partial^s \nu}{\partial \nu^s} \right] \right) \right](x) \leq u_{k,n}(x) \leq G_\alpha \left[ \frac{\partial^s \nu}{\partial \nu^s} \right](x), \quad \forall x \in \Omega; \]  
(ii) $u_{k,n}$ is a classical solution of (2.13).
Lemma 5.1 There exists \( c_{58} > 1 \) such that
\[
\frac{1}{c_{58}} \rho^\alpha(x)|x|^{-N} \leq G_\alpha \left( \frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha} \right) \leq c_{58} \rho^\alpha(x)|x|^{-N}, \quad \forall x \in \Omega. \tag{5.8}
\]

Proof. For any \( x \in \Omega \), there exists \( s > 0 \) such that \( x \in C_s \), and then letting \( y_t = t\vec{n}_0 \) with \( t \in (0, s/2) \), we have that
\[
|y_t - x| = s - t > s \geq \frac{1}{2} \max\{\rho_{\partial\Omega}(y_t), \rho_{\partial\Omega}(x)\}.
\]
Thus, it follows by apply [4, Theorem 1.1, Theorem 1.2] that
\[
\frac{1}{c_{59}} \rho^\alpha(y_t)|y_t|^N \leq G_\alpha(x, y_t) \leq c_{59} \rho^\alpha(x)|x|^N, \quad \forall x \in \Omega.
\]
From
\[
G_\alpha \left( \frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha} \right)(x) = \lim_{t \to 0^+} G_\alpha(x, y_t), \quad \forall x \in \Omega, \tag{5.9}
\]
we deduce that
\[
\frac{1}{c_{60}} \rho^\alpha(x)|x|^{-N} \leq G_\alpha \left( \frac{\partial^\alpha \delta_0}{\partial \vec{n}^\alpha} \right) \leq c_{60} \rho^\alpha(x)|x|^{-N}, \quad \forall x \in \Omega.
\]

Lemma 5.2 Assume that \( g \) is a continuous nondecreasing function satisfying \( g(0) \geq 0 \), (1.13) and (1.14). Then
\[
\lim_{\rho(x) \to 0^+} \frac{G_\alpha[g(\alpha \rho_{\partial\Omega}^\alpha)](x)}{\rho(x)^{\alpha-1} + \rho^\alpha(x)|x|^{-N}} = 0. \tag{5.10}
\]

Proof. From Lemma 3.1 and (1.13),
\[
\lim_{\rho(x) \to 0^+} G_\alpha[g(\alpha \rho_{\partial\Omega}^\alpha)](x) \rho^{1-\alpha}(x) = 0. \tag{5.11}
\]
From [6, Lemma 4.1], we have that
\[
\lim_{s \to 0^+} G_\alpha[g(\alpha \rho_{\partial\Omega}^\alpha)](s\vec{n}_0)s^{N-\alpha} = 0. \tag{5.12}
\]
By (1.14),
\[
G_\alpha[g(\alpha \rho_{\partial\Omega}^\alpha)](x) \leq \lambda \left[ G_\alpha[g(\alpha \rho_{\partial\Omega}^\alpha)](x) + G_\alpha[g(\alpha \rho_{\partial\Omega}^\alpha)](x) \right],
\]
together with (5.11) and (5.12), we implies (5.10). \( \square \)

Proof of Theorem 1.3. The existence of weak solution just follows the procedure of the proof of Theorem 1.1 by using Proposition 5.1 and Proposition 5.2. It is the same to prove the uniqueness and regularity of weak solution. Finally, plugging (2.3), (5.8) and (5.10) into (5.7) replaced \( g_n \) by \( g \), we obtain (1.15). \( \square \)

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