Crossed Products and Entropy of Automorphisms

Ciprian Pop*

Institute of Mathematics of the Romanian Academy
C.P. 1–764
Bucharest, Romania

Roger R. Smith

Department of Mathematics
Texas A&M University
College Station, TX 77843–3368

Abstract

Let \( \mathcal{A} \) be an exact \( C^* \)-algebra, let \( G \) be a locally compact group, and let \((\mathcal{A}, G, \alpha)\) be a \( C^* \)-dynamical system. Each automorphism \( \alpha_g \) induces a spatial automorphism \( Ad_{\lambda_g} \) on the reduced crossed product \( \mathcal{A} \times_{\alpha} G \). In this paper we examine the question, first raised by E. Størner, of when the topological entropies of \( \alpha_g \) and \( Ad_{\lambda_g} \) coincide. This had been answered by N. Brown for the particular case of discrete abelian groups. Using different methods, we extend his result to preservation of entropy for \( \alpha_g \) when the subgroup of \( \text{Aut}(G) \) generated by the corresponding inner automorphism \( Ad_g \) has compact closure. This property is satisfied by all elements of a wide class of groups called locally \([FIA]^\text{−}\). This class includes all abelian groups, both discrete and continuous, as well as all compact groups.

* Current Address: Department of Mathematics Texas A&M University College Station, TX 77843

Email addresses: cpop@math.tamu.edu (Ciprian Pop), rsmith@math.tamu.edu (Roger R. Smith).

1 Partially supported by a grant from the National Science Foundation.
1 Introduction

In [22], Voiculescu introduced the topological entropy \( ht(\alpha) \) of an automorphism \( \alpha \) of a nuclear \( C^* \)-algebra, generalizing the classical notion for abelian \( C^* \)-algebras. The definition, which we recall in Section 2, was based on the theorem of Choi and Effros, [2], which characterized nuclear \( C^* \)-algebras as those which admit approximate point norm completely positive factorizations of the identity map through matrix algebras. Brown observed in [1] that the definition could be extended to exact \( C^* \)-algebras \( A \), by allowing the completely positive approximations to have range in any containing \( B(H) \) rather than \( A \) itself. This was based on two results. Wassermann, [23], had shown that the existence of such factorizations implies exactness and subsequently Kirchberg, [6], proved that this property characterizes exact \( C^* \)-algebras. Any automorphism \( \alpha \) of \( A \) induces an action of \( Z \) on \( A \), and then \( \alpha \) is implemented by an inner automorphism \( Ad_u \) on the crossed product \( A \times_\alpha Z \). In [20], Størmer had posed the question of whether passing to the crossed product preserves the entropy, the point being to replace an arbitrary automorphism by an inner one which would be more amenable to analysis. Brown, [1], answered this positively for exact \( C^* \)-algebras. Since exactness is preserved by crossed products of amenable groups, [5], Størmer’s question is relevant and interesting in this wider context. In this paper we show that an affirmative answer can be given for a large class of locally compact amenable groups.

Brown’s approach, stated for \( Z \) but valid for any discrete abelian group, was based on work in [17], where completely positive factorizations of \( A \) through \( A \times_\alpha G \) were constructed for discrete groups. We do not know whether such factorizations exist beyond the discrete case, but we were able to find completely contractive factorizations of \( A \) through \( A \times_\alpha G \) for general locally compact groups, [10]. In order to make use of this result, we have been led to introduce a new entropy \( ht'(\alpha) \) for an automorphism \( \alpha \) involving complete contractions rather than completely positive maps. The first goal of the paper is to show that \( ht'(\alpha) = ht(\alpha) \) (see Theorem 3.7), and then the point of defining \( ht'(\alpha) \) is to have an equivalent formulation of \( ht(\alpha) \) which is more suited to our context. The proof of equality of these two entropies relies, in part, on techniques from [18].

The second section reviews some background material and the third discusses our new definition of entropy. The results of Section 5 ultimately depend on non–abelian duality as developed in [4,21], and we state a modified version of the Imai–Takai duality theorem in Theorem 4.1, essentially due to Quigg, [16], in order to obtain extra information not available from [4]. Section 4 also shows how entropy changes as automorphisms are lifted from \( A \) to the crossed
product $\mathcal{A} \times_\alpha G$ and then to the double crossed product $(\mathcal{A} \times_\alpha G) \times_\alpha G$, using the theory of coactions.

When $\mathcal{A} = \mathbb{C}$, the action is trivial and we have $ht(\alpha_g) = 0$. The crossed product becomes $C^*_r(G)$ and $\lambda_g$ is the lifting of the corresponding inner automorphism of $G$ to this $C^*$-algebra. If entropy is to be preserved, then these automorphisms of $C^*_r(G)$ must have zero entropy. This is clearly true for abelian groups where the set of inner automorphisms reduces to a single point. A natural extension of this case is to impose a compactness requirement on the closure in $\text{Aut}(G)$ of the subgroup generated by the inner automorphism $Ad_g$ for a single group element $g \in G$. This leads us to consider some classes of groups which are standard in the literature, [11]. Our results are proved in the context of $[SIN]$ groups and $[FIA]^{-}$ groups (see [11] for a detailed survey of classes of locally compact groups). The first class is defined by the property of having small invariant neighborhoods of the identity element $e$ under the inner automorphisms, while the second class requires the closure of the set of inner automorphisms to be compact in $\text{Aut}(G)$ for its natural topology (see [15] for a nice characterization of this). Groups in the second class are amenable and both classes contain all abelian and all compact groups. In fact our results are valid for a much larger class of groups which are amenable and satisfy a condition which we call locally $[FIA]^{-}$. The definition only requires that the closure of the group generated by a single but arbitrary inner automorphism should be compact in $\text{Aut}(G)$. These matters are discussed in Section 5 which contains our main results.

Throughout the paper we restrict $C^*$-algebras to be exact since, as pointed out by Brown, [1], topological entropy only makes sense for this class. However, an exactness hypothesis is only necessary in Section 2, Theorems 3.5 and 3.7, and Section 5. All other results are valid in general.

We thank John Quigg for helpful correspondence concerning Theorem 4.1, and also the referee for some enlightening comments.
In this section we give a brief review of crossed products and entropy of automorphisms. We will recall the completely contractive factorization property (CCFP) from [10], and define a covariant version (CCCPF) which will be useful subsequently. We will also introduce a new version of topological entropy, using completely contractive maps rather than completely positive ones. We will show that this version coincides with the standard definition, and the point is to obtain a more flexible situation in which to work.

Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha : G \to \text{Aut}(A)$ be a homomorphism such that the map $t \mapsto \alpha_t(a)$ is norm continuous on $G$ for each $a \in A$. Then the triple $(A, G, \alpha)$ is called a $C^*$-dynamical system. The reduced crossed $A \times_\alpha r G$ is constructed by taking a faithful representation $\pi : A \to B(H)$ and then defining a new representation $\tilde{\pi} : A \to B(L^2(G, H))$ by

$$\tilde{\pi}(a)\xi(t) = \alpha_{t^{-1}}(a)(\xi(t)), \quad a \in A, \ \xi \in L^2(G, H). \quad (2.1)$$

There is an associated unitary representation of the group given by

$$(\lambda_s\xi)(t) = \xi(s^{-1}t), \quad s \in G, \ \xi \in L^2(G, H), \quad (2.2)$$

and in this representation the action $\alpha$ is spatially implemented, as can be seen from the easily established equation

$$\lambda_s\tilde{\pi}(a)\lambda_{s^{-1}} = \tilde{\pi}(\alpha_s(a)), \quad a \in A, \ s \in G. \quad (2.3)$$

The norm closed span of operators on $L^2(G, H)$ of the form

$$\int_G f(s)\tilde{\pi}(a)\lambda_s \, ds, \quad a \in A, \ f \in C_c(G), \quad (2.4)$$

for a fixed left Haar measure $ds$, is a $C^*$-algebra called the reduced crossed product. There is also a full crossed product, denoted $A \times_\alpha G$, which will not appear here. The groups considered will eventually be amenable, where the two notions coincide. For this reason we will abuse the standard notation and write $A \times_\alpha G$ for the reduced crossed product throughout. The construction is independent of the representation $\pi$, and so we may carry it out twice, if necessary, so that we may always assume that the action is spatially implemented by a unitary representation of $G$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras. If there exist nets of complete contractions

$$\mathcal{A} \xrightarrow{S_\lambda} \mathcal{B} \xrightarrow{T_\lambda} \mathcal{A} \quad (2.5)$$
so that
\[ \lim_{\lambda} \| T_\lambda(S_\lambda(a)) - a \| = 0, \quad a \in \mathcal{A}, \] (2.6)
then we say that the pair \((\mathcal{A}, \mathcal{B})\) has the completely contractive factorization property (CCFP), [10]. We will also be considering a pair of automorphisms \(\alpha\) and \(\beta\) of \(\mathcal{A}\) and \(\mathcal{B}\) respectively. If the above nets can be chosen to satisfy the additional requirement that
\[ S_\lambda \alpha = \beta S_\lambda, \quad T_\lambda \beta = \alpha T_\lambda, \] (2.7)
then the pair \((\mathcal{A}, \alpha), (\mathcal{B}, \beta)\) is said to have the covariant CCFP (CCCFP).

In [22], Voiculescu introduced the notion of topological entropy for an automorphism of a nuclear \(C^*\)-algebra, and this was extended by Brown to the case of an exact \(C^*\)-algebra, [1]. We assume that \(\mathcal{A}\) is concretely represented on a Hilbert space \(H\), and we fix an automorphism \(\alpha\) of \(\mathcal{A}\). Given a finite subset \(\omega \subseteq \mathcal{A}\) and \(\delta > 0\), \(rcc(\omega, \delta)\) denotes the smallest integer \(m\) for which complete contractions
\[ \mathcal{A} \xrightarrow{\phi} \mathbb{M}_m \xrightarrow{\psi} B(H) \] (2.8)
can be found to satisfy
\[ \| \psi(\phi(a)) - a \| < \delta, \quad a \in \omega. \] (2.9)

We then define successively
\[ h t'(\alpha, \omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log \left( rcc \left( \bigcup_{i=0}^{n-1} \alpha^i(\omega), \delta \right) \right), \] (2.10)
\[ h t'(\alpha, \omega) = \sup_{\delta > 0} h t'(\alpha, \omega, \delta), \] (2.11)
\[ h t'(\alpha) = \sup_{\omega} h t'(\alpha, \omega). \] (2.12)

This last quantity is our version of the topological entropy \(ht(\alpha)\) of \(\alpha\), originally defined as above using completely positive contractions throughout. The term \(rcc(\cdot)\) (derived from completely contractive rank) replaces the completely positive rank \(rcp(\cdot)\). Since positivity of maps is a more stringent requirement, the entropy arising from the completely contractive framework is immediately seen to satisfy
\[ h t'(\alpha) \leq h t(\alpha). \] (2.13)

We will prove equality in the next section, so \(ht'(\alpha)\) is a temporary notation soon to be replaced by \(ht(\alpha)\). When confusion might arise, we will use the notation \(ht_\mathcal{A}(\alpha)\) to indicate the algebra on which \(\alpha\) acts. If \(\alpha \in \text{Aut}(\mathcal{A})\) and \(\mathcal{B}\) is an \(\alpha\)-invariant \(C^*\)-subalgebra, it may be the case that \(ht_\mathcal{A}(\alpha) \neq ht_\mathcal{B}(\alpha)\).

The following lemmas will be useful subsequently in comparing the entropies of two automorphisms.
Lemma 2.1. Let $\alpha$ and $\beta$ be automorphisms of exact $C^*$-algebras $A$ and $B$ respectively.

(i) Suppose that, given an arbitrary finite subset $\omega \subseteq A$ and $\delta > 0$, there exist a finite subset $\omega' \subseteq B$ and $\delta' > 0$ so that

$$ht'_{A}(\alpha, \omega, \delta) \leq ht'_{B}(\beta, \omega', \delta').$$

(2.14)

Then $ht'_{A}(\alpha) \leq ht'_{B}(\beta)$.

(ii) Suppose that there exists a function $g: (0, \infty) \to (0, \infty)$ satisfying $\lim_{\delta \to 0} g(\delta) = 0$, and suppose that, given an arbitrary finite subset $\omega \subseteq A$ and $\delta > 0$, there exists a finite subset $\omega' \subseteq B$ such that

$$ht'_{A}(\alpha, \omega, g(\delta)) \leq ht'_{B}(\beta, \omega', \delta).$$

(2.15)

Then $ht'_{A}(\alpha) \leq ht'_{B}(\beta)$.

Proof. These are simple consequences of the definitions, after observing that $ht'(\alpha, \omega, \delta)$ is a monotonically decreasing function of $\delta$ for fixed $\alpha$ and $\omega$. $\square$

Lemma 2.2. Let $\alpha$ and $\beta$ be respectively automorphisms of the exact $C^*$-algebras $A \subseteq B(H)$ and $B \subseteq B(K)$, and suppose that the pair $(A, \alpha)$, $(B, \beta)$ has the CCCFP. Then $ht'_{A}(\alpha) \leq ht'_{B}(\beta)$.

Proof. Let

$$A \xrightarrow{S_{\lambda}} B \xrightarrow{T_{\lambda}} A$$

(2.16)

be nets of complete contractions satisfying (2.7). Fix a finite subset $\omega \subseteq A$, and $\delta > 0$. Choose $\lambda$ so that

$$\|T_{\lambda}(S_{\lambda}(a)) - a\| < \delta, \quad a \in \omega,$$

(2.17)

and let $\omega' = S_{\lambda}(\omega) \subseteq B$. For each integer $n$, let $r_{n}$ be $rcc\left(\bigcup_{i=0}^{n-1} \beta^{i}(\omega'), \delta\right)$. Then, by definition, there exist complete contractions

$$B \xrightarrow{\phi_{n}} M_{r_{n}} \xrightarrow{\psi_{n}} B(K)$$

(2.18)

such that

$$\|\psi_{n}\phi_{n}(b) - b\| < \delta, \quad b \in \beta^{i}(\omega'), \ 0 \leq i \leq n - 1.$$  

(2.19)

By injectivity of $B(H)$, $T_{\lambda}$ has a completely contractive extension $\bar{T}_{\lambda}: B(K) \to B(H)$. Consider the diagram

$$A \xrightarrow{S_{\lambda}} B \xrightarrow{\phi_{n}} M_{r_{n}} \xrightarrow{\psi_{n}} B(K) \xrightarrow{\bar{T}_{\lambda}} B(H).$$

(2.20)
If $a \in \omega$ then

$$S_\lambda \alpha^i(a) = \beta^i(S_\lambda(a)), \quad T_\lambda \beta^i(S_\lambda(a)) = \alpha^i(T_\lambda(S_\lambda(a))), \quad (2.21)$$

and a simple approximation argument shows that $rcc\left(\bigcup_{i=0}^{n-1} \alpha^i(\omega), 2\delta \right) \leq r_n$. It follows that

$$ht'(\alpha, \omega, 2\delta) \leq ht'(\beta, \omega', \delta). \quad (2.22)$$

The result follows from Lemma 2.1 (ii) with $g(\delta) = 2\delta$. \hfill \Box
3 Completely contractive topological entropy

The objective in this section is to show that the two entropies \( h_t(\alpha) \) and \( h_t'(\alpha) \) coincide. This will be accomplished in two stages. The first is to show preservation of \( h_t'(\cdot) \) when lifting an automorphism from \( \mathcal{A} \) to its unitization \( \tilde{\mathcal{A}} \). The second is to show equality for automorphisms of unital \( C^* \)-algebras. Throughout \( \mathcal{A} \) is assumed to be exact, and so \( \tilde{\mathcal{A}} \) is also exact, [6]. Various norm estimates and approximations will be needed and so we begin with some technical lemmas. Many of the complications stem from allowing \( \mathcal{A} \) to be non–unital. These are unavoidable since crossed products inevitably take us out of the category of unital \( C^* \)-algebras. Below, \( \mathcal{A}_1^+ \) denotes the positive part of the closed unit ball.

**Lemma 3.1.** Let \( a, p \in \mathcal{A}_1^+ \) satisfy \( \|a - ap\| \leq C \) for some \( C > 0 \). If \( 0 \leq b \leq a \), then \( \|b - bp\| \leq \sqrt{C} \).

**Proof.** This follows from the inequality

\[
0 \leq (1 - p)b^2(1 - p) \leq (1 - p)b(1 - p) \leq (1 - p)a(1 - p) \leq C1,
\]

which gives \( \|b(1 - p)\|^2 \leq C \). □

**Lemma 3.2.** If \( t \in B(H) \) satisfies

\[
\|1 + e^{i\theta}t\| \leq 1 + C, \quad \theta \in \mathbb{R},
\]

for some constant \( C > 0 \), then \( \|t\| \leq 2C \).

**Proof.** If \( \phi \) is a state on \( B(H) \), then

\[
|1 + e^{i\theta} \phi(t)| \leq 1 + C, \quad \theta \in \mathbb{R}.
\]

A suitable choice of \( \theta \) shows that \( |\phi(t)| \leq C \), so the numerical radius is at most \( C \), and the result follows. □

**Lemma 3.3.** Let \( \phi: \mathcal{A} \to B(H) \) be a complete contraction on a unital \( C^* \)-algebra, and let \( t \) be the self-adjoint part of \( \phi(1) \). Then there exists a completely positive contraction \( \psi: \mathcal{A} \to B(H) \) such that

\[
\|\phi - \psi\|_{cb} \leq \sqrt{2}\|1 - t\|.
\]

**Proof.** From the representation theorem for completely bounded maps, [13], there exist a representation \( \pi: \mathcal{A} \to B(K) \) and contractions \( V_1, V_2: H \to K \)
such that
\[ \phi(x) = V_1^* \pi(x) V_2, \quad x \in \mathcal{A}. \] (3.5)

Then
\[ 0 \leq (V_1 - V_2)^*(V_1 - V_2) = V_1^* V_1 + V_2^* V_2 - V_1^* V_2 - V_2^* V_1 \leq 2 - 2t \leq 2\|1 - t\|, \] (3.6)

so \( \|V_1 - V_2\| \leq \sqrt{2\|1 - t\|} \). If we define \( \psi \) to be \( V_1^* \pi(\cdot) V_1 \), then (3.4) is immediate.

\[ \text{Lemma 3.4. Let } \omega \text{ be a subset of } \mathcal{A}_{s.a.}, \text{ let } \varepsilon > 0, \text{ and let} \]
\[ \mathcal{A} \xrightarrow{\phi} \mathbb{M}_n \xrightarrow{\psi} B(H) \] (3.7)

be a diagram of complete contractions satisfying
\[ \|\psi(\phi(x)) - x\| < \varepsilon, \quad x \in \omega. \] (3.8)

Then there exist self-adjoint complete contractions
\[ \mathcal{A} \xrightarrow{\tilde{\phi}} \mathbb{M}_{2n} \xrightarrow{\tilde{\psi}} B(H) \] (3.9)

satisfying
\[ \|\tilde{\psi}(\tilde{\phi}(x)) - x\| < \varepsilon, \quad x \in \omega. \] (3.10)

\[ \text{Proof. Following the methods of Paulsen, [13], we define } \tilde{\phi}: \mathcal{A} \to \mathbb{M}_{2n} \text{ by} \]
\[ \tilde{\phi}(a) = \begin{pmatrix} 0 & \phi(a) \\ \phi(a^*)^* & 0 \end{pmatrix}, \quad a \in \mathcal{A}, \] (3.11)

which is a self-adjoint complete contraction. The map
\[ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}, \quad x, y, z, w \in \mathbb{M}_n, \] (3.12)

is a complete contraction and so \( \tilde{\psi}: \mathbb{M}_{2n} \to B(H) \), defined by
\[ \tilde{\psi} \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = (\psi(y) + \psi(z^*)^*)/2, \quad x, y, z, w \in \mathbb{M}_n, \] (3.13)

is a self-adjoint complete contraction. Then
\[ \tilde{\psi}(\tilde{\phi}(a)) = (\psi(\phi(a)) + \psi(\phi(a^*))^*)/2, \quad a \in \mathcal{A}, \] (3.14)

and (3.10) follows for the self-adjoint elements \( x \in \omega \). \qed
We now show that entropy lifts from a $C^*$-algebra to its unitization.

**Theorem 3.5.** Let $\mathcal{A}$ be a non-unital $C^*$-algebra. If $\alpha$ is an automorphism of $\mathcal{A}$, then $ht'_{\tilde{\mathcal{A}}}(\alpha) = ht'_{\tilde{\mathcal{A}}} (\tilde{\alpha})$, where $\tilde{\alpha}$ is the unique extension of $\alpha$ to the unitization $\tilde{\mathcal{A}}$.

**Proof.** If $\mathcal{A}$ is faithfully represented on a Hilbert space $H$, then so too is $\tilde{\mathcal{A}}$. The inequality $ht'_{\mathcal{A}}(\alpha) \leq ht'_{\tilde{\mathcal{A}}} (\tilde{\alpha})$ is then trivial. To prove the reverse inequality, we may restrict attention to a finite subset $\omega$ of the positive open unit ball. The group $\{\alpha^n\}_{n \in \mathbb{Z}}$ gives an action of $\mathbb{Z}$ on $\mathcal{A}$, so the crossed product construction of the previous section allows us to assume that $\alpha$ is spatially implemented. Thus we may regard $\tilde{\alpha}$ as an automorphism of $B(H)$, not just of $\tilde{\mathcal{A}}$.

Fix $\delta \in (0, 1)$ and choose $a \in \mathcal{A}^+, \|a\| < 1$, such that $x \leq a$ for $x \in \omega$, which is possible since the positive part of the open unit ball of any $C^*$-algebra is upward filtering, [9, Example 3.1.1]. Define $f_\delta : [0, 1] \to [0, 1]$ by $f_\delta(t) = t/\delta$ for $t \in [0, \delta]$, and $f_\delta(t) = 1$ elsewhere. Let $f = f_\delta(a) \in \mathcal{A}$, and let $p \in B(H)$ be the spectral projection of $a$ for the interval $[\delta, 1]$. The relations

$$fp = pf = p, \quad ap = pa, \quad \|a - ap\| \leq \delta$$

are immediate from the functional calculus. Now enlarge $\omega$ by defining $\omega' = \omega \cup \{f, a\}$, and fix an integer $N$. Then let $n$ denote $rcc \left( \bigcup_{i=0}^{N-1} \alpha^i(\omega'), \delta \right)$. By definition, there exist complete contractions

$$\mathcal{A} \xrightarrow{\phi} \mathbb{M}_n \xrightarrow{\psi} B(H)$$

such that

$$\|\psi(\phi(x)) - x\| < \delta, \quad x \in \bigcup_{i=0}^{N-1} \alpha^i(\omega').$$

By Lemma 3.4, we may assume that $\phi$ and $\psi$ are self-adjoint. Here we may ignore the doubling of dimension which does not affect the entropy. By injectivity of $\mathbb{M}_n$, we may extend $\phi$ to a self-adjoint complete contraction on $\tilde{\mathcal{A}}$ which we also denote by $\phi$. Then $t = \psi(\phi(1)) \in B(H)$ is self-adjoint, so we may define $q$ to be its spectral projection for the interval $[1 - 2\delta^{1/2}, 1]$. The functional calculus gives

$$1 - t \geq 2\delta^{1/2}(1 - q).$$

Since $\|1 - 2f\| \leq 1$, it follows that $\|t - 2\psi(\phi(f))\| \leq 1$, and thus

$$\|t - 2f\| \leq \|\psi(\phi(1 - 2f))\| + 2\|\psi(\phi(f)) - f\| \leq 1 + 2\delta.$$
rearrangement of the terms, is

\[ 2f - t - 1 \leq 2\delta. \quad (3.20) \]

We may now multiply on both sides by \( p \) to reach

\[ p - ptp \leq 2\delta p, \quad (3.21) \]

using (3.15). Then, from (3.18) and (3.21),

\[ 2\delta p \geq p(1 - t)p \geq 2\delta^{1/2}p(1 - q)p, \quad (3.22) \]

from which it follows that \( \|p(1 - q)\| \leq \delta^{1/4} \). This yields the estimate

\[ \|a - aq\| \leq \|a - ap\| + \|a(p - pq)\| + \|(ap - a)q\| \leq 2\delta + \delta^{1/4} \leq 3\delta^{1/4}, \quad (3.23) \]

using (3.15). By Lemma 3.1 and (3.23), if \( x \in \omega \) then

\[ \|x - qxq\| \leq \|x - qx\| + \|q(x - xq)\| \leq 2\|x - qxq\| \leq 4\delta^{1/8}. \quad (3.24) \]

By repeating this argument with \( \alpha^i(a), \alpha^i(f) \) and \( \tilde{\alpha}^i(p) \) replacing respectively \( a, f \) and \( p \), we see that (3.24) also holds for \( x \in \bigcup_{i=0}^{N-1} \alpha^i(\omega) \).

Since \( \mathcal{A} \) is a closed ideal in \( \tilde{\mathcal{A}} \), we may choose a state \( \mu \) on \( \tilde{\mathcal{A}} \) which annihilates \( \mathcal{A} \). Then define \( \tilde{\phi}: \tilde{\mathcal{A}} \to \mathbb{M}_{n+1} \) by

\[ \tilde{\phi}(x) = \begin{pmatrix} \phi(x) & 0 \\ 0 & \mu(x) \end{pmatrix}, \quad x \in \tilde{\mathcal{A}}, \quad (3.25) \]

and define \( \tilde{\psi}: \mathbb{M}_{n+1} \to B(H) \) to be any completely contractive extension of the map

\[ \begin{pmatrix} m & 0 \\ 0 & \lambda \end{pmatrix} \mapsto q\psi(m)q + \lambda(1 - q), \quad m \in \mathbb{M}_n, \quad \lambda \in \mathbb{C}. \quad (3.26) \]

If \( x \in \bigcup_{i=0}^{N-1} \alpha^i(\omega) \), then (3.24) implies that

\[ \|\tilde{\psi}(\tilde{\phi}(x)) - x\| \leq 4\delta^{1/8}. \quad (3.27) \]
From (3.26)
\[ \tilde{\psi}(\tilde{\phi}(1)) = qtq + 1 - q, \] (3.28)
so by the definition of \( q \),
\[ \| \tilde{\psi}(\tilde{\phi}(1)) - 1 \| \leq 2\delta^{1/2}. \] (3.29)

Passage from \( \mathbb{M}_n \) to \( \mathbb{M}_{n+1} \) does not affect the entropy, and so we have proved that
\[ ht'(\tilde{\alpha}, \{ 1 \} \cup \omega, 4\delta^{1/8}) \leq ht'(\alpha, \omega, \delta), \] (3.30)
and the result follows from Lemma 2.1 (ii) with \( g(\delta) = 4\delta^{1/8} \).

We now turn to the question of whether \( ht(\alpha) = ht'(\alpha) \).

**Proposition 3.6.** Let \( \mathcal{A} \) be a unital \( \mathcal{C}^* \)-algebra and let \( \omega \) be a finite subset of \( \mathcal{A}_1 \) containing the identity. For \( \delta \in (0, 1) \),
\[ rcc(\omega, \delta) \geq rcp(\omega, 12\delta^{1/8}). \] (3.31)

**Proof.** Let \( n = rcc(\omega, \delta) \) and choose complete contractions\n\[ \mathcal{A} \xrightarrow{\phi} \mathbb{M}_n \xrightarrow{\psi} B(H) \] (3.32)
so that
\[ \| \psi(\phi(x)) - x \| < \delta, \quad x \in \omega. \] (3.33)

By polar decomposition followed by diagonalization, there is a non-negative diagonal matrix \( D \) and two unitaries so that
\[ \phi(1) = UDV. \] (3.34)

By replacing \( \phi \) by \( U^*\phi(\cdot)V^* \) and \( \psi \) by \( \psi(U \cdot V) \), we may assume that \( \phi(1) = D \), and it is clear that \( \| D \| \in (1 - \delta, 1] \). Let \( E \) be the projection onto the space spanned by eigenvectors of \( D \) corresponding to eigenvalues in the interval \([1 - \delta^{1/2}, 1] \).

If \( X \in (1 - E)\mathbb{M}_n(1 - E) \), \( \| X \| \leq 1 \), then
\[ \| D + \delta^{1/2}e^{i\theta}X \| \leq 1, \quad \theta \in \mathbb{R}. \] (3.35)

Apply \( \psi \) to obtain
\[ \| 1 + \delta^{1/2}e^{i\theta}\psi(X) \| \leq \| \psi(D + \delta^{1/2}e^{i\theta}X) \| + \delta, \quad \theta \in \mathbb{R}, \] (3.36)
and it follows from Lemma 3.2 that
\[ \| \psi(X) \| \leq 2\delta^{1/2}. \] (3.37)
In particular, (3.37) is valid for $1 - E$ and $D(1 - E)$. Since $\|DE - E\| \leq \delta^{1/2}$, we obtain
\[
\|\psi(D) - \psi(E)\| \leq \|\psi(DE - E)\| + \|\psi(D(1 - E))\|
\leq 3\delta^{1/2},
\] (3.38)
from (3.37). Thus
\[
\|1 - \psi(E)\| \leq \delta + 3\delta^{1/2} < 4\delta^{1/2}.
\] (3.39)
Using (3.37) with $X = 1 - E$, this leads to
\[
\|1 - \psi(1)\| < 6\delta^{1/2}.
\] (3.40)
By Lemma 3.3, there is a completely positive contraction $\psi_1: M_n \to B(H)$ so that
\[
\|\psi - \psi_1\|_{cb} < 4\delta^{1/4}.
\] (3.41)
From (3.37),
\[
\|\psi_1(1 - E)\| < 6\delta^{1/4}.
\] (3.42)
Let $V^*\pi(\cdot)V$ be the Stinespring representation, [19], of $\psi_1$. Then (3.42) implies the inequality $\|V^*(1 - E)\| < 6^{1/2}\delta^{1/8}$, so
\[
\|\psi_1((1 - E)X)\| \leq 6^{1/2}\delta^{1/8}\|X\|,
\] (3.43)
with a similar estimate for $\psi_1(X(1 - E))$. Since
\[
Y - EYE = Y(1 - E) + (1 - E)YE, \quad Y \in M_n,
\] (3.44)
we obtain
\[
\|\psi_1(Y) - \psi_1(EYE)\| \leq 2 \cdot 6^{1/2}\delta^{1/8}\|Y\|
\] (3.45)
from (3.43). Let $m$ be the rank of $E$ and identify $EM_nE$ with $M_m$. Then let $\psi_2: M_m \to B(H)$ be the restriction of $\psi_1$ to $EM_nE$. Also define $\phi_1: A \to M_m$ by $\phi_1(\cdot) = E\phi(\cdot)E$. Then $\|\phi_1(1 - E)\| \leq \delta^{1/2}$, so by Lemma 3.3 there exists a completely positive contraction $\phi_2: A \to M_m$ such that $\|\phi_1 - \phi_2\|_{cb} \leq 2^{1/2}\delta^{1/4}$. Putting together the estimates, we obtain, for $x \in \omega$,
\[
\|x - \psi_2(\phi_2(x))\| \leq \|x - \psi_2(\phi_1(x))\| + 2^{1/2}\delta^{1/4}
= \|x - \psi(E\phi_1(x)E)\| + 2^{1/2}\delta^{1/4}
\leq \|x - \psi_1(\phi(x))\| + 2 \cdot 6^{1/2}\delta^{1/8} + 2^{1/2}\delta^{1/4}
\leq \|x - \psi_1(\phi(x))\| + (4 + 2^{1/2})\delta^{1/4} + 2 \cdot 6^{1/2}\delta^{1/8}
< 12\delta^{1/8},
\] (3.46)
using (3.45) and (3.33). This proves (3.31).

This completes the technical results needed to prove the following.
Theorem 3.7. Let $\alpha$ be an automorphism of $\mathcal{A}$. Then

$$ht(\alpha) = ht'(\alpha). \quad (3.47)$$

Proof. If $\mathcal{A}$ is unital then the non-trivial inequality $ht(\alpha) \leq ht'(\alpha)$ follows from Proposition 3.6. If $\mathcal{A}$ is non-unital then we also use Theorem 3.5 to conclude that

$$ht'_A(\alpha) = ht'_{\tilde{A}}(\tilde{\alpha}) = ht_{\tilde{A}}(\tilde{\alpha}) \geq ht_A(\alpha), \quad (3.48)$$

proving the result. \qed

Remark 3.8. The idea for Proposition 3.6 comes from [18], where it appears in a different context. Henceforth we will refer to entropy as $ht(\alpha)$, but use the definition of $ht'(\alpha)$. \qed
4 Duality

One of the most useful results in the theory of crossed products is the Imai–Takai duality theorem, [4], which generalizes to locally compact group actions the Takai duality theorem, [21], for abelian groups. This asserts the existence of a dual coaction $\hat{\alpha}$ on $A \times_\alpha G$ so that

$$ (A \times_\alpha G) \times_{\hat{\alpha}} G \approx A \otimes K(L^2(G)). \quad (4.1) $$

The new crossed product algebra $(A \times_\alpha G) \times_{\hat{\alpha}} G$ is formed by combining copies of $A \times_\alpha G$ and $C_0(G)$ acting on $L^2(G,H) \otimes L^2(G)$. In this representation, the typical generator $\int_G f(s)\tilde{\pi}(a)\lambda_s \, ds$ of the crossed product becomes

$$ \int_G f(s)\tilde{\pi}(a)\lambda_s \otimes l_s \, ds, \quad f \in C_c(G), \ a \in A, \quad (4.2) $$

where $l_s$ is left translation on $L^2(G)$ (and $r_s$ will denote the corresponding right regular representation). The algebra $C_0(G)$ is represented on the second copy of $L^2(G)$ as multiplication operators. Then $(A \times_\alpha G) \times_{\hat{\alpha}} G$ is the norm closed span of operators of the form

$$ \int_G f(s)\tilde{\pi}(a)\lambda_s \otimes M_F l_s \, ds, \quad f \in C_c(G), \ F \in C_0(G), \ a \in A. \quad (4.3) $$

In Theorem 4.1 we state a modified version of the Imai–Takai duality theorem, essentially due to Quigg, [16]. This is in order to obtain information which is very difficult to extract from the original argument of [4]. We then give some consequences to be used in Section 5.

In [10], the notation $f \cdot a$ (where $f \in C_c(G)$ and $a \in A$) was used to denote the function $t \mapsto f(t)a$ in $C_c(G,A) \subseteq L^1(G,A)$. The crossed product is the $C^*$–completion of a representation $\tilde{\pi} \times \lambda$ of $L^1(G,A)$, where

$$ \tilde{\pi} \times \lambda(f \cdot a) = \int_G f(s)\tilde{\pi}(a)\lambda_s \, ds. \quad (4.4) $$

Below we will simplify notation and use $f \cdot a$ when we really mean its image $\tilde{\pi} \times \lambda(f \cdot a) \in A \times_\alpha G$ under a faithful representation. It should also be noted that all maps considered throughout this section are well defined (see for example [10]).

**Theorem 4.1.** There exists an isomorphism

$$ \tau : (A \times_\alpha G) \times_{\hat{\alpha}} G \to A \otimes K(L^2(G)) \quad (4.5) $$
which induces a covariant isomorphism between the two $C^*$-dynamical systems

$$((\mathcal{A} \rtimes \alpha G) \times _{\partial} G, Ad_{\lambda \otimes l_g r_g} G) \text{ and } (\mathcal{A} \otimes \mathcal{K}(L^2(G)), \alpha_g \otimes Ad_{l_g r_g}, G).$$

Proof. This is a special case of [16, Theorem 3.1], with minor changes. The required isomorphism is given by

$$\tilde{\pi}(a) \lambda_s \otimes M_F l_s = (1 \otimes M_F) \tilde{\pi}(a) \lambda_s, \quad (4.6)$$

for any $a \in \mathcal{A}$, $f \in C_c(G)$ and $F \in C_c(G)$. \qed

We now begin to construct complete contractions which are covariant with respect to certain automorphisms, and we note that they are variants of maps considered in [10]. Below, $\Delta(\cdot)$ denotes the modular function on a group $G$.

**Proposition 4.2.** Let $g \in G$ satisfy $\Delta(g) = 1$ and $f \in C_c(G)$ be such that $f(t) = f(gt g^{-1})$ for all $t \in G$. Define the map $S_f : \mathcal{A} \rightarrow \mathcal{A} \rtimes \alpha G$ by $S_f(a) = \int_G f(s) \tilde{\pi}(a) \lambda_s ds$.

Then $\|S_f\|_{cb} \leq \|f\|_1$ and the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{A} \\
\downarrow \alpha_g \\
\mathcal{A} \rtimes \alpha G
\end{array} \\
\begin{array}{ccc}
\mathcal{A} \\
\downarrow \lambda_g \otimes l_g \\
\mathcal{A} \rtimes \alpha G
\end{array}$$

Then $\|S_f\|_{cb} \leq \|f\|_1$ and the following diagram is commutative:

(4.7)

Proof. Fix an arbitrary element $a \in \mathcal{A}$. Then

$$\lambda_g S_f(a) \lambda_{g^{-1}} = \int_G f(s) \lambda_g \tilde{\pi}(a) \lambda_s \lambda_{g^{-1}} ds$$

$$= \int_G f(s) \tilde{\pi}(\alpha_g(a)) \lambda_{g^{-1}} ds$$

$$= \int_G f(g^{-1} sg) \tilde{\pi}(\alpha_g(a)) \lambda_s ds$$

$$= S_f(\alpha_g(a)). \quad (4.8)$$

The inequality $\|S_f\|_{cb} \leq \|f\|_1$ was proved in [10]. \qed

**Proposition 4.3.** Let $g \in G$ satisfy $\Delta(g) = 1$ and suppose that $\eta \in L^2(G)$ is such that $\|\eta\|_2 = 1$ and $\eta(gt g^{-1}) = \eta(t)$ for almost all $t \in G$. Define $T_\eta$ to be the right slice map by $\omega_\eta$ on $B(L^2(G, H))$, restricted to $\mathcal{A} \rtimes \alpha G$. Then
∥Tη∥cb = 1 and the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{A} \times_\alpha G & \xrightarrow{T_\eta} & \mathcal{A} \\
\downarrow_{\text{Ad}_g} & & \downarrow_{\alpha_g} \\
\mathcal{A} \times_\alpha G & \xrightarrow{T_\eta} & \mathcal{A}
\end{array}
\]

(4.9)

Proof. Let \( a \in \mathcal{A} \) and \( f \in C_c(G) \). Then

\[
T_\eta(f \cdot a) = \int_G F_{f,\eta}(t) \alpha_{t^{-1}}(a) \, dt
\]

(4.10)

where \( F_{f,\eta} \in C_c(G) \) is defined by

\[
F_{f,\eta}(t) = \int_G f(s) \eta(s^{-1} t) \eta(t) \, ds.
\]

(4.11)

Then

\[
\alpha_g(T_\eta(f \cdot a)) = \int_G F_{f,\eta}(t) \alpha_{gt^{-1}}(a) \, dt
\]

(4.12)

On the other hand, define a function \( h \in C_c(G) \) by \( h(t) = f(g^{-1}tg) \) for all \( t \in G \). Then

\[
\lambda_g(f \cdot a) \lambda_{g^{-1}} = \int_G f(s) \tilde{\pi}(\alpha_g(a)) \lambda_{gs^{-1}} \, ds
\]

\[
= \int_G h(s) \tilde{\pi}(\alpha_g(a)) \lambda_s \, ds
\]

\[
= h \cdot \alpha_g(a).
\]

(4.13)

Furthermore,

\[
T_\eta(h \cdot \alpha_g(a)) = \int_G F_{h,\eta}(t) \alpha_{t^{-2}}(a) \, dt
\]

\[
= \int_G F_{h,\eta}(gtg^{-1}) \alpha_{gt^{-1}}(a) \, dt
\]

(4.14)

where \( F_{h,\eta} \) is defined as before. To conclude the proof it only remains to show that, for any \( t \in G \), \( F_{h,\eta}(tg^{-1}) = F_{f,\eta}(t) \). Indeed,

\[
F_{h,\eta}(tg^{-1}) = \int_G h(s) \eta(s^{-1}tg^{-1}) \eta(tg^{-1}) \, ds
\]

\[
= \int_G f(g^{-1}sg) \eta(g^{-1}s^{-1}gt) \eta(t) \, ds
\]

\[
= \int_G f(s) \eta(s^{-1}t) \eta(t) \, ds
\]

\[
= F_{f,\eta}(t),
\]

(4.15)

as required. \( \square \)
Proposition 4.4. Let $g \in G$. Let $F \in C_0(G)$ be such that, for any $t \in G$, $F(gtg^{-1}) = F(t)$. Define the map $S_F : A \times_\alpha G \to (A \times_\alpha G) \times \hat{\alpha} G$ by

$$S_F(f \cdot a) = \int_G f(s)\bar{\pi}(a)\lambda_s \otimes M_{Fl_s} ds$$

(4.16)

for all $f \in C_*(G)$ and $a \in A$. Then $\|S_F\|_{cb} \leq \|F\|_{\infty}$ and the following diagram is commutative:

$$\begin{array}{ccc}
A \times_\alpha G & \xrightarrow{S_F} & (A \times_\alpha G) \times \hat{\alpha} G \\
\|_{cb} & \text{Ad}_{\lambda g} & \|_{cb} \\
A \times_\alpha G & \xrightarrow{S_F} & (A \times_\alpha G) \times \hat{\alpha} G
\end{array}$$

(4.17)

Proof. Fix elements $a \in A$ and $f \in C_*(G)$. Then

$$S_F(\lambda_g(f \cdot a)\lambda_{g^{-1}}) = \int_G f(s)\bar{\pi}(\alpha_g(a))\lambda_{g_{sg^{-1}}} \otimes M_{F_{l_{sg_{g^{-1}}}}} ds$$

(4.18)

On the other hand,

$$\text{Ad}_{\lambda g \otimes l_{rg}}(S_F(f \cdot a)) = \int_G f(s)\bar{\pi}(\alpha_g(a))\lambda_{g_{sg^{-1}}} \otimes l_{rg} M_{F_{l_{g_{sg^{-1}}}}} l_{g^{-1}} ds$$

$$= \int_G f(s)\bar{\pi}(\alpha_g(a))\lambda_{g_{sg^{-1}}} \otimes M_{F_{l_{sg_{g^{-1}}}}} ds,$$

(4.19)

so these equations prove commutativity of the diagram. \qed

Proposition 4.5. Let $g \in G$ be such that $\Delta(g) = 1$. Let $\eta \in L^2(G)$, $\|\eta\|_2 = 1$ be such that, for almost all $t \in G$, $\eta(gtg^{-1}) = \eta(t)$. Let $T_\eta : (A \times_\alpha G) \times \hat{\alpha} G \to A \times_\alpha G$ be the restriction of the right slice map by $\omega_\eta$. Then $\|T_\eta\|_{cb} = 1$ and the following diagram is commutative:

$$\begin{array}{ccc}
(A \times_\alpha G) \times \hat{\alpha} G & \xrightarrow{T_\eta} & A \times_\alpha G \\
\|_{cb} & \text{Ad}_{\lambda g \otimes l_{rg}} & \|_{cb} \\
(A \times_\alpha G) \times \hat{\alpha} G & \xrightarrow{T_\eta} & A \times_\alpha G
\end{array}$$

(4.20)

Proof. For simplicity put $T = T_\eta$. Let

$$y = \int_G f(s)\bar{\pi}(a)\lambda_s \otimes M_{F l_s} ds$$

(4.21)

be an element of $(A \times_\alpha G) \times \hat{\alpha} G$, where $f, F \in C_*(G)$. Then
\[ \lambda_g T(y) \lambda_{g^{-1}} = \int_G f(s) \omega_\eta(M_F l_s) \tilde{\pi}(\alpha_g(a)) \lambda_{gs^{-1}} \, ds \]
\[ = \int_G f(g^{-1} sg) \omega_\eta(M_F l_{g^{-1}sg}) \tilde{\pi}(\alpha_g(a)) \lambda_s \, ds. \]  
(4.22)

On the other hand let \( \tilde{F} \in C_c(G) \) be defined by \( \tilde{F}(t) = F(g^{-1}tg) \) for all \( t \in G \).

Since \( l_g r_g M_F l_s r_g^{-1} l_g^{-1} = M_F l_{gs^{-1}} \), we also have

\[ (\lambda_g \otimes l_g r_g) y(\lambda_{g^{-1}} \otimes r_g^{-1} l_g^{-1}) = \int_G f(s) \tilde{\pi}(\alpha_g(a)) \lambda_{gs^{-1}} \otimes M_F l_{gs^{-1}} \, ds \]
\[ = \int_G f(g^{-1} sg) \tilde{\pi}(\alpha_g(a)) \lambda_s \otimes M_F l_s \, ds. \]  
(4.23)

Therefore

\[ T(Ad_{\lambda_g \otimes l_g r_g}(g)) = \int_G f(g^{-1} sg) \omega_\eta(M_F l_s) \tilde{\pi}(\alpha_g(a)) \lambda_s \, ds. \]  
(4.24)

It remains to observe that, for any \( s \in G \),

\[ \omega_\eta(M_F l_s) = \omega_\eta(M_F l_{g^{-1}sg}). \]  
(4.25)

Indeed,

\[ \omega_\eta(M_F l_{g^{-1}sg}) = \int_G (M_F l_{g^{-1}sg}) \eta(r) dr \]
\[ = \int_G F(r) \eta(g^{-1}s^{-1}gr) \eta(r) \, dr \]
\[ = \int_G F(r) \eta(s^{-1}gr^{-1}) \eta(grg^{-1}) \, dr \]
\[ = \int_G F(g^{-1}rg) \eta(s^{-1}r) \eta(r) \, dr \]
\[ = \omega_\eta(M_F l_s), \]  
(4.26)

completing the proof. \( \square \)
5 Main results

In this section we will use our preceding results to compare the entropies of various automorphisms. Given a subgroup $H$ of continuous automorphisms of $G$, a function $f$ on $G$ is $H$-invariant if, for all $\phi \in H$, $f(\phi(x)) = f(x)$ everywhere or almost everywhere, as appropriate. If $H$ is generated by $Ad_g$ for some fixed $g \in G$, then we may also use the terminology $g$-invariant. In the case when $H$ is the closure of the inner automorphisms, we will just say that $f$ is invariant. A set $N$ is called $g$-invariant if $gNg^{-1} = N$.

Lemma 5.1. Let $G$ be a locally compact group and let $g$ be a fixed element of $G$.

(i) If the group has a basis of compact $g$-invariant neighborhoods of $e$, then $\Delta(g) = 1$;

(ii) If the closed subgroup of Aut($G$) generated by the inner automorphism $Ad_g$ is compact, then there is a basis of compact $g$-invariant neighborhoods of $e$.

Proof. Any compact invariant neighborhood $N$ satisfies $gNg^{-1} = N$, from which the equality $\Delta(g) = 1$ follows. To prove the second part, consider $g \in G$ and let $H$ be the closed subgroup of Aut($G$) generated by $Ad_g$, which is compact by hypothesis. It follows from [3, Theorem 4.1] that $G$ has small $H$-invariant neighborhoods of the identity, as required.

Recall that a group is said to be [SIN] if there is a basis of compact invariant neighborhoods of $e$. Such groups are unimodular, [7]. If, for each $g \in G$, a basis of $g$-invariant neighborhoods can be found, then we say that $G$ is locally [SIN]. It is then clear that the following theorem applies to both classes of groups, while being more general.

Theorem 5.2. Let $g \in G$ be a fixed element for which the identity has a basis of compact $g$-invariant neighborhoods. Then

$$ht_A(\alpha_g) \leq ht_{A \times \alpha G}(Ad_{\lambda g}).$$

(5.1)

Proof. We prove that the pair $(A, \alpha_g)$, $(A \times \alpha G, Ad_{\lambda g})$ satisfies the CCCFP of Section 2. Given $\varepsilon > 0$ and $\omega = \{a_1, a_2, \ldots, a_n\} \subseteq A_1$, choose $N$ to be a compact symmetric $g$-invariant neighborhood of $e$ on which

$$\|\alpha_{t^{-1}}(a_i) - a_i\| \leq \varepsilon/2, \quad 1 \leq i \leq n.$$  

(5.2)

Let $\eta$ be the characteristic function of $N$, normalized in the $L^2$-norm. Clearly $\eta$
is $g$-invariant. Choose $N_1 \subseteq N$ a compact symmetric $g$-invariant neighborhood of $e$ such that
\[ \int_G \eta(t)\eta(s^{-1}t)\,dt > 1 - \varepsilon/2, \quad s \in N_1. \]
Pick $f \in C_c(G)$ to be a positive, $g$-invariant function supported on $N_1$ (as in [7], this can be achieved by choosing $N_2$ to be a compact neighborhood of $e$ such that $N_2 \cdot N_2 \subseteq N_1$ and then letting $f = \chi_{N_2} * \check{\chi}_{N_2}$). Upon normalizing $f$ in the $L^1$ norm, we may assume that $\|f\|_1 = 1$. Let
\[ h(t) = \eta(t) \int_G f(s)\eta(s^{-1}t)\,ds, \quad t \in G. \] (5.3)
Clearly,
\[ 1 \geq \int_G h(t)\,dt = \int_{N_1} f(s) \int_G \eta(t)\eta(s^{-1}t)\,dt\,ds \geq (1 - \varepsilon/2) \int_G f(s)\,ds = 1 - \varepsilon/2. \] (5.4)
Consider now the complete contractions $S_f$ and $T_\eta$, introduced in Propositions 4.2 and 4.3 (see also [10]). A simple calculation gives
\[ T_\eta S_f(a) - a = \int_N h(t)(\alpha_t^{-1}(a) - a)\,dt + \left( \int_G h(t)\,dt - 1 \right)a. \] (5.5)
But then, for $i = 1, 2, \ldots, n$,
\[ \|T_\eta S_f(a_i) - a_i\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \] (5.6)
using (5.2), (5.4), and the assumption $\|a_i\| \leq 1$. By Lemma 5.1, $\Delta(g) = 1$, and so the theorem now follows by combining Propositions 4.2 and 4.3. \hfill \Box

We are now able to state and prove the main result of this paper.

Recall from Section 1 that the class of locally $[FIA]^{-}$ groups in the next result is defined by the requirement that the group of automorphisms generated by a fixed but arbitrary inner automorphism should have compact closure in $\text{Aut}(G)$.

**Theorem 5.3.** Let $G$ be an amenable group and let $g \in G$ be such that the closure of the group generated by $\text{Ad}_g$ in $\text{Aut}(G)$ is compact. Then
\[ \text{ht}_{A \times \alpha G}(\text{Ad}_g) = \text{ht}_A(\alpha_g). \] (5.7)
In particular, equality holds for all elements of an amenable locally $[FIA]^{-}$ group $G$.

**Proof.** The inequality $\text{ht}_{A \times \alpha G}(\text{Ad}_g) \geq \text{ht}_A(\alpha_g)$ is an immediate consequence of the preceding three results, and so we consider only the reverse. We will
prove that the pair \((A \times_{\alpha} G, Ad_{\lambda_g}), ((A \times_{\alpha} G) \times_{\hat{\alpha}} G, Ad_{\lambda_g \otimes \ell_g \gamma_g})\) has the CCCFP, which will imply that

\[
ht_{A \times_{\alpha} G}(Ad_{\lambda_g}) \leq ht_{(A \times_{\alpha} G) \times_{\hat{\alpha}} G}(Ad_{\lambda_g \otimes \ell_g \gamma_g}).
\] (5.8)

Using the version of the Imai-Takai duality theorem stated in Theorem 4.1 and basic properties of the topological entropy from [1], we will obtain

\[
ht_{(A \times_{\alpha} G) \times_{\hat{\alpha}} G}(Ad_{\lambda_g \otimes \ell_g \gamma_g}) = ht_{A \otimes K(L^2(G))}(\alpha_{\gamma} \otimes Ad_{\ell_g \gamma_g}) \\
\leq ht_{A}(\alpha_{\gamma}) + ht_{K(L^2(G))}(Ad_{\ell_g \gamma_g}).
\] (5.9)

The result will then follow from the vanishing of the topological entropy of any automorphism of the algebra of compact operators. This is folklore for which we have no reference. However, it is simple to prove by applying the definition to finite sets of rank one projections, whose span is norm dense in \(K(H)\).

To prove that the pair \((A \times_{\alpha} G, Ad_{\lambda_g}), ((A \times_{\alpha} G) \times_{\hat{\alpha}} G, Ad_{\lambda_g \otimes \ell_g \gamma_g})\) satisfies the CCCFP it is enough to consider finite sets of the form

\[
\omega = \{f_1 \cdot a_1, f_2 \cdot a_2, \ldots, f_n \cdot a_n\} \subseteq A \times_{\alpha} G,
\] (5.10)

where \(a_i \in A\) and \(f_i \in C_c(G)\) for \(1 \leq i \leq n\). Let \(K\) be a compact set containing the supports of all functions \(f_i\), \(1 \leq i \leq n\). Fix \(\varepsilon > 0\), let

\[
M = \max_{1 \leq i \leq n} \|f_i\|_1 \|a_i\|,
\] (5.11)

and define \(\delta\) to be \(\varepsilon^2/4M^2\).

Denote by \(H\) the closed subgroup of \(\text{Aut}(G)\) generated by the inner automorphism \(Ad_g\). Since the set

\[
\{\phi(t) : t \in K \cup K^{-1}, \phi \in H\}
\] (5.12)

is compact, we may assume that \(K\) is \(H\)-invariant and \(K = K^{-1}\). By amenability of \(G\) and using [14, Proposition 7.3.8], there exists \(f \in C_c(G)\), \(f \geq 0\), \(\|f\|_1 = 1\), such that

\[
\|s f - f\|_1 < \delta, \quad s \in K,
\] (5.13)

where \(s f(\cdot)\) is defined by \(s f(t) = f(s^{-1}t)\). As in [8], define the function \(f^\# \in L^1(G)\) by

\[
f^\#(t) = \int_H f(\phi(t)) d\phi, \quad t \in G,
\] (5.14)

where \(d\phi\) is normalized Haar measure on the compact abelian group \(H\). Then \(f^\# \in C_0(G)\), being a vector integral of elements of \(C_c(G)\) over a compact
group. Since $f \geq 0$,

$$
\|f^#\|_1 = \int_G \int_H f(\phi(t)) \, d\phi \, dt = \int_H \int_G f(\phi(t)) \, dt \, d\phi = \int_H \|f\|_1 \, d\phi = 1. \tag{5.15}
$$

The last equality holds because any continuous automorphism $\phi$ in the closure of the inner automorphisms of a unimodular group preserves the Haar measure. Moreover, by invariance of $d\phi$,

$$
f^#(gtg^{-1}) = f^#(t), \quad t \in G. \tag{5.16}
$$

Also, for $s \in K$,

$$
\|sf^# - f^#\|_1 = \int_G \left| \int_H \left[ f(\phi(s^{-1})\phi(t)) - f(\phi(t)) \right] \, d\phi \right| \, dt \\
\leq \int_H \int_G \left| f(\phi(s))f(\phi(t)) - f(\phi(t)) \right| \, dt \, d\phi \\
= \int_H \|f(\phi(s))f - f\|_1 \, d\phi < \delta, \tag{5.17}
$$

since $\phi(s) \in K$. Let $\eta = \sqrt{f^#}$. Then $\|\eta\|_2 = 1$ and by [12, p.126], for any $s \in K$

$$
\|s\eta - \eta\|_2 \leq \|sf^# - f^#\|_1^{1/2} < \sqrt{\delta}. \tag{5.18}
$$

It follows that, for any $s \in K$,

$$
1 - \sqrt{\delta} < \int_G \eta(s^{-1}t)\eta(t) \, dt \leq 1, \tag{5.19}
$$

using the Cauchy–Schwarz inequality and the preceding estimate. Now choose $L \subseteq G$ to be a large $H$-invariant compact set such that

$$
\int_L (\eta(t))^2 \, dt > 1 - \sqrt{\delta}. \tag{5.20}
$$

Then there exists an $H$-invariant function $F \in C_0(G)$ such that $F|_L = 1$ and $0 \leq F \leq 1$ (as before, choose $F_1 \in C_0(G)$, $F_1|_L = 1$, $0 \leq F_1 \leq 1$ and let $F = F_1^#$).

To summarize, we have found a positive $g$-invariant function $F \in C_0(G)$ and a $g$-invariant unit vector $\eta \in L^2(G)$ such that

$$
1 \geq \int_G F(t)\eta(s^{-1}t)\eta(t) \, dt > 1 - 2\sqrt{\delta}, \quad s \in K. \tag{5.21}
$$

Define the maps $S_F$ and $T_\eta$ as in Propositions 4.4 and 4.5 respectively. Then,
for any \( a \in \mathcal{A} \) and \( f \in C_c(G) \),

\[
T\eta S_F(f \cdot a) - f \cdot a = \int_G (h(s) - 1)f(s)\tilde{\pi}(a)\lambda_s \, ds,
\]

(5.22)

where \( h \) is defined by

\[
h(s) = \int_G F(t)\eta(s^{-1}t)\eta(t) \, dt.
\]

(5.23)

In particular, for any \( 1 \leq i \leq n \),

\[
\|T\eta S_F(f_i \cdot a_i) - f_i \cdot a_i\| \leq 2\sqrt{\delta M} = \varepsilon,
\]

(5.24)

using the estimate (5.21), which is valid since each \( f_i \) is supported in \( K \). Since \( S_F \) and \( T\eta \) satisfy the conditions of Propositions 4.4 and 4.5 respectively, the theorem is proved. \( \square \)
References

[1] N. Brown, *Topological entropy in exact C*-algebras*, Math. Ann. 314 (1999), 347–367.

[2] M.–D. Choi and E. G. Effros, *Nuclear C*-algebras and the approximation property*, Amer. J. Math. 100 (1978), 61–79.

[3] S. Grosser and M. Moskowitz, *On central topological groups*, Trans. A.M.S., 127 (1967), 317–340.

[4] S. Imai and H. Takai, *On a duality for C*-crossed products by a locally compact group*, J. Math. Soc. Japan 30 (1978), 495–504.

[5] E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. Reine Angew. Math. 452 (1994), 39–77.

[6] E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. 129 (1995), 35–63.

[7] R. D. Mosak, *Central functions in group algebras*, Proc. Amer. Math. Soc. 29 (1971), 613–616.

[8] R. D. Mosak, *The $L^1$- and C*-algebras of $[FIA]_B$ groups, and their representations*, Trans. Amer. Math. Soc. 163 (1972), 277–310.

[9] G. J. Murphy, *C*-algebras and operator theory*, Academic Press, New York, 1990.

[10] M. M. Nilsen and R. R. Smith, *Approximation properties for crossed products by actions and coactions*, Internat. J. Math. 12 (2001), 595–608.

[11] T. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. 8 (1978), 683–741.

[12] A. L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs 29 American Mathematical Society, Providence, RI, 1988.

[13] V. I. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series 146 Longman, Harlow, 1986.

[14] G. K. Pedersen, *C*-algebras and their automorphism groups*, Academic Press, New York, 1979.

[15] J. Peters and T. Sund, *Automorphisms of locally compact groups*, Pacific J. Math. 76 (1978), 143–156.

[16] J. C. Quigg, *Duality for reduced twisted crossed products of C*-algebras*, Indiana Univ. Math. J. 35 (1986), 549–571.

[17] A. M. Sinclair and R. R. Smith, *The completely bounded approximation property for discrete crossed products*, Indiana Univ. Math. J. 46 (1997), 1311–1322.
[18] R. R. Smith, *Completely contractive factorizations of $C^*$-algebras*, J. Funct. Anal. **64** (1985), 330–337.

[19] W. F. Stinespring, *Positive functions on $C^*$-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.

[20] E. Størmer, *Entropy in operator algebras*, Astérisque **232** (1995), 211–230.

[21] H. Takai, *On a duality for crossed products of $C^*$-algebras*, J. Funct. Anal. **19** (1975), 25–39.

[22] D. Voiculescu, *Dynamical approximation entropies and topological entropy in operator algebras*, Comm. Math. Phys. **170** (1995), 249–281.

[23] S. Wassermann, *Tensor products of free-group $C^*$-algebras*, Bull. London Math. Soc. **22** (1990), 375–380.