Whitehead products in symplectomorphism groups and Gromov-Witten invariants

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Abstract

Consider any symplectic ruled surface \((M^g_\lambda, \omega_\lambda) = (\Sigma^g \times S^2, \lambda\sigma_{\Sigma^g} \oplus \sigma_{S^2})\). We compute all natural equivariant Gromov-Witten invariants \(EGW^{g,0}_{g,k} (M^g_\lambda; H_k, A - kF)\) for all hamiltonian circle actions \(H_k\) on \(M^g_\lambda\), where \(A = [\Sigma^g \times pt]\) and \(F = [pt \times S^2]\). We use these invariants to show the nontriviality of certain higher order Whitehead products that live in the homotopy groups of the symplectomorphism groups \(G^g_\lambda\), \(g \geq 0\). Our results are sharper when \(g = 0, 1\) and enable us to answer a question posed by D.McDuff in [13] in the case \(g = 1\) and provide a new interpretation of the multiplicative structure in the ring \(H^*(BG_0^g; \mathbb{Q})\) studied by Abreu-McDuff in [2].

1 Introduction

We study some topological aspects of symplectomorphism groups \(G_\lambda\) of a continuous family of symplectic structures \((M, \omega_\lambda)\), \(\lambda \geq 0\) on a given compact manifold.

In section two we provide preliminary definitions and results on symplectic fibrations, topological aspects of symplectomorphism groups and Whitehead (and hence Samelson) products. Higher order Whitehead products of elements in \(\pi_* BG_\lambda\) are subsets of some \(\pi_N BG_\lambda\) and measure the obstructions of extending existing maps defined on the codimension 2 skeleton of an appropriate product of spheres to the product itself. These sets are nonempty only when the lower order products contain the nullhomotopic class. Samelson products in \(G_\lambda\) are desuspending Whitehead products.

We make use of these obstruction theoretic properties in section three where we introduce and prove proposition-construction 3.1. This proposition relates, in certain circumstances, higher order Whitehead products to towers of symplectic fibrations over projective spaces.

Section four provides background material on parametric Gromov-Witten invariants PGW and provides a relation between these invariants and Whitehead products in the symplectomorphism groups. Roughly speaking, one can relate continuous deformations with respect to the parameter \(\lambda\) of Whitehead products in \(BG_\lambda\) to a symplectic deformation problem on symplectic fibrations. Parametric Gromov-Witten invariants are precisely invariants of such fibrations.

Computing PGW invariants is, aside from their consequences on the symplectomorphism groups, of interest on its own. We provide such computation in section five where we study ruled surfaces \((M^g_\lambda, \omega_\lambda) = (\Sigma^g \times S^2, \lambda\sigma_{\Sigma^g} \oplus \sigma_{S^2})\), with symplectomorphism groups \(G^g_\lambda\). If \(k = \lfloor \lambda \rfloor\) then \(M^g_\lambda\) admits \(k\) different hamiltonian circle actions \(H_i\) each with two fixed point sets given by holomorphic curves in classes \(A \pm iF, 1 \leq i \leq k\). Equivariant Gromov-Witten invariants EGW count precisely \(H_i\) invariant curves. We show that the only natural EGW (those counting generically isolated curves with no marked points in certain associated fibrations) are given by:

**Theorem 1.1** For any arbitrary genus \(g\), and a hamiltonian circle action with Lie group \(H_k\) on \(M^g_\lambda, \lambda > k\) as in (21), and equivariant almost complex structure \(J^{(k)}(g)\) we have

\[
EGW^{g,k}_{g,0} (M^g_\lambda; H_k; s_{A - kF}) = \pm 1 \cdot u^{2k+g-1} \in H^*(BS^1, \mathbb{Q})
\]
In order to successfully tie our result on equivariant Gromov-Witten invariants to nontrivial Samelson products in symplectomorphism groups of ruled surfaces we need to make use of several previously known results. Building on results from M. Gromov [8], M. Abreu [1] and M. Abreu-D. McDuff [2] these results are provided by D. McDuff in [13]. Denote by $A^0_\lambda$ the space of almost complex structures $J$ that tame some symplectic form cohomologous to $\omega_\lambda$. The results in [13] relate the homotopy groups of $G^0_\lambda$ to a good understanding of strata $A^0_{\lambda k}$ of almost complex structures that admit curves in class $A - kF$. A complete description of this stratification, for $g > 0$, is prevented by difficult gluing problems. When seeking nontrivial Samelson products in $\pi_sG^0_\lambda$, we circumvent some of these issues by using the topological structures laid out in the previous sections, and obtain:

**Proposition 1.2**  
(i) There exists an element $\tilde{\gamma} \in \pi_1G^1_1$ such that $[\tilde{\gamma}, \tilde{\gamma}]_s \in \pi_2G^1_1$ is a nontrivial element that disappears in $\pi_2G^1_\lambda$, $\lambda > 1$.

(ii) For all $k \geq 1$ and $k < \lambda \leq k + 1$ there exist elements $\tilde{\gamma}^0_\lambda \in \pi_1G^0_\lambda \otimes \mathbb{Q}$ such that the Samelson product of order $2k + 1$, $S^{2k+1}(\tilde{\gamma}^0_\lambda) = \{0, \tilde{w}_k\} \subset \pi_{4k}(G^0_{k+1})$ where $\tilde{w}_k$ is a nontrivial homotopy class that disappears when $\lambda > k + 1$.

(iii) For all genus $g \geq 2$ and all $k > [g/2]$ there exist elements $\tilde{\gamma}^0_\lambda \in \pi_1G^0_\lambda \otimes \mathbb{Q}$ and nonvanishing Samelson product of order $p$ with $g \leq p \leq 2k + g - 1$, $0 \neq \tilde{w}_p^0 \in S^{p}(\tilde{\gamma}^0_\lambda) \subset \pi_{2p-2}(G^0_\lambda)$.

In the case when $g = 0$ we use this theorem and the additive structure on $\pi_sG^0_\lambda \otimes \mathbb{Q}$ from [2], to give a new proof for the following:

**Theorem 1.3** (Abreu-McDuff)/[2] Fix an integer $k \geq 0$. For $k < \lambda \leq k + 1$ we have

$$H^*(BG^0_\lambda, \mathbb{Q}) = S(A, X, Y)/\{A(X - Y)(X - 4Y) \cdots (X - k^2Y) = 0\}$$

with $\deg A = 2$ and $\deg X = \deg Y = 4$.

## 2 Preliminaries

### 2.1 Symplectic fibrations

Consider a triple $(M, \omega_0, J)$ where $J$ is an almost complex structure that tames $\omega_0$ and has a canonical class $c_1(M)$.

**Definition 2.1** A locally trivial fibration $\pi : Q \to B$ is a symplectic fibration if the fiber is the compact symplectic manifold $(M, \omega_0)$ and there exist a two form $\Lambda_0$ on $Q$ which is vertically closed i.e. $i(v_1, v_2)d\Lambda_0 = 0$ for all vertical vectors $v_i$ and whose restriction to each fiber is the symplectic form of the fiber.

As shown in [9], such forms correspond to symplectic connections on the fibration. Consider $(U_\alpha)$ an atlas covering the base $B$ and a trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \to M \times U_\alpha$, that yields a collection of transition maps $\phi_{\alpha\beta} : U_\alpha \cup U_\beta \to \text{Diff}(M)$. An equivalent definition of the symplectic fibration is that $\phi_{\alpha\beta} \subset \text{Symp}(M, \omega_0)$. Indeed, given such trivialization the form $\Lambda_0$ is obtained via a partition of unity from canonical forms on $\pi^{-1}(U_\alpha)$ such that it restricts on each fiber $M_b$ to $\omega_b = \phi_\alpha(b)^*\omega_0$.

Given a symplectic fibration, we consider the associated cohomological respectively homological bundles $H^*(M, \mathbb{R}) \to \mathbb{Q}^* \to B$ and $H_*(M, \mathbb{Z}) \to \mathbb{Q}_* \to B$. These are obtained by simply considering the same atlas for the base and automorphisms naturally induced by the maps $\phi_\alpha$ on homology respectively cohomology.

In a similar manner one constructs an associated bundle $J(B, M)$ whose fiber over each $b$ is the space of almost complex structures $J$ on $M$ that are compatible with $\omega$. As explained in Le-Ono [11], since the fibers are contractible, one can always pick a section $b \to J_b$ in this bundle.
The above alternative descriptions of a symplectic fibration imply that there exist constant sections \( s^{[\omega_0]} : B \to Q^* \) with the value \([\omega_0] \in H^*(M, \mathbb{R})\) and \( s^{[\gamma]} : B \to Q^* \) that takes the integer value \( c_1(M, \omega_0) \in H^2(M, \mathbb{Z}) \).

We say that a symplectic fibration is a \textbf{hamiltonian fibration} if the structure group further reduces to \( \text{Ham}(M, \omega_0) \subseteq \text{Symp}(M, \omega_0) \).

By a result of Guillemin and Sternberg \([9]\), a symplectic fibration with a simply connected base \( B \) is hamiltonian if and only if there exist a \textit{closed extension} \( \Lambda_0 \in \Omega^2(Q) \). Moreover, a result of Thurston \([14]\) (page 333) guarantees that if the base \( B \) carries a symplectic form \( \sigma_B \), then for \( t \) sufficiently large the form \( \Lambda_0 \) can be chosen symplectic representing the class \( [\omega_0] + t[\pi^* \sigma_B] \).

If \( \pi_1 B \) acts trivially on the associated fibration \( H_*(M, \mathbb{Z}) \to Q_\ast \to B \) (e.g. if \( B \) is simply connected), then for each \( D \in H_2(M, \mathbb{Z}) \) there also exists a constant section \( s_D : B \to Q_\ast \) that takes the value \( D \).

Let us consider a symplectic deformation \((M, \omega_\lambda)_{\lambda \geq 0}\) of the symplectic structure \((M, \omega_0)\).

\textbf{Definition 2.2} We say that a continuous one parameter family of vertically closed 2-forms \((\Lambda_\lambda)_{\lambda \geq 0}\) on \( Q \) that satisfy the conditions in definition (2.1) for symplectic fibers \((M, \omega_\lambda)\), represents a fiberwise symplectic deformation based on the family \((M, \omega_\lambda)_{\lambda \geq 0}\).

These fibrations carry \textit{vertical almost complex structures} \( \tilde{J}_\lambda \). That is, almost complex structures on \( Q \) taming \( \Lambda_\lambda \) and compatible with the fibration. We will refer to such triples \((Q, \Lambda_\lambda, \tilde{J}_\lambda)\) as compatible to the symplectic fibration with fiber \((M, \omega_\lambda)\).

\section{2.2 Some topological aspects of the symplectomorphism groups}

In the rest of the paper we will study Samelson products in the symplectomorphism groups \( G_\lambda = \text{Symp}(M, \omega_\lambda) \cap \text{Diff}(M) \).

We will use greek letters such as \( \gamma \) for \( S^k \)-cycles in the symplectomorphism groups \( G_\lambda \) and use the notation \( \overline{\gamma} \) for their homotopy classes in \( \pi_k G_\lambda \). \( E(\gamma) \) and \( \overline{E(\gamma)} \) will be the corresponding \( S^{k+1} \)-cycle in the classifying space and respectively its homotopy class in \( \pi_{k+1} BG_\lambda \).

There is no direct inclusion of elements from \( G_\lambda \) in \( G_{\lambda+} \). The following proposition provides an adequate substitute:

\textbf{Proposition 2.3} Buse \([7]\) Denote by \( G_{[0,\infty)} = \bigcup_{\lambda>0} G_\lambda \times \{\lambda\} \subseteq \text{Diff}(M) \times [0,\infty) \).

Consider \( K \) an arbitrary compact set in \( G_\lambda \). Then there is an \( \epsilon_K > 0 \) and a continuous map \( h : [-\epsilon_K, \epsilon_K] \times K \to G_{[0,\infty)} \) such that the following diagram commutes
\[
\begin{array}{ccc}
[-\epsilon_K, \epsilon_K] \times K & \xrightarrow{pr_1} & [-\epsilon_K, \epsilon_K] \\
\downarrow & & \downarrow \text{incl} \\
[-\epsilon_K, \epsilon_K] & \xrightarrow{pr_2} & (-\infty, \infty).
\end{array}
\]

Moreover, for any two maps \( h \) and \( h' \) satisfying this diagram and which coincide on \( 0 \times K \), there exists, for small enough \( \epsilon' > 0 \), a homotopy \( H : [0,1] \times [-\epsilon', \epsilon'] \times K \to G_{[0,\infty]} \) between \( h \) and \( h' \) which also satisfies \( H \circ pr_2 = pr_1 \circ \text{incl} \).

Let \( \gamma_0 : S^k \to G_0 \) be a cycle in \( G_0 \). An \textbf{extension} \( \gamma_\lambda, \lambda \geq 0 \) of \( \gamma_0 \) is a smooth family of cycles \( \gamma_\lambda : S^k \to G_\lambda \) defined for small \( \lambda \) and satisfying (2).

\textbf{Definition 2.4} We say that an element \( \overline{\gamma_0} \in \pi_1 G_0 \) is \textbf{fragile} if it admits a null homotopic extension to the right \( 0 = \overline{\gamma_\lambda} \in \pi_1 G_\lambda \) for \( \lambda > 0 \). The element is said to be \textbf{robust} if it admits an essential extension to the right \( 0 \neq \overline{\gamma_0} \in \pi_1 G_\lambda \).

A continuous family \( \gamma_\lambda : B \to G_\lambda, \lambda > 0 \) is \textbf{new} if it is not the extension of a map \( \gamma_0 : B \to G_0 \).
2.3 Whitehead and Samelson products

Consider a topological group \( G \) and its classifying space \( X = BG \) with \( \Omega X = G \). Any Whitehead products can be introduced for an arbitrary topological space \( X \).

Consider elements \( \eta_i : S^j \to G \) representing elements \( \bar{\eta}_i \) in \( \pi_*(G) \), and their suspensions \( E(\eta_i) : S^j \to BG \).

The Samelson products \( [\bar{\eta}_1, \bar{\eta}_2]_s \in \pi_{j_1+j_2}(G) \) are given by the quotient of the commutator

\[
[\eta_1, \eta_2] = \eta_1 \eta_2 \eta_1^{-1} \eta_2^{-1}
\]

to \( S^{j_1+j_2} = S^{j_1} \times S^{j_2} / S^{j_1} \vee S^{j_2} \).

The ordinary second order Whitehead product \( [\bar{E}(\eta_1), \bar{E}(\eta_2)]_w \in \pi_{j_1+j_2+1}(BG) \) is given by the obstruction to extending the wedge map \( E(\eta_1) \vee E(\eta_2) : S^{j_1+1} \vee S^{j_2+1} \to BG \) to a map with the domain \( S^{j_1+1} \times S^{j_2+1} \). A classical result states that \( [\bar{\eta}_1, \bar{\eta}_2]_s \) is, up to a sign, the desuspension of \( [\bar{E}(\eta_1), \bar{E}(\eta_2)]_w \).

Following [17], the \( k \)th order higher Whitehead products \( [\bar{E}(\eta_1), \ldots, \bar{E}(\eta_k)]_w \) is a (possibly empty) subset of homotopy classes in \( \pi_{-1}(BG) \), with \( r = j_1 + 1 + \ldots + j_k + 1 \), defined as follows:

Let \( P = \Pi_{k=1}^r (S^{j_i+1}) \). The fat wedge product \( T \) is the \( r-2 \) skeleton inside \( P \) and consists of all the \( k \)-tuples \( (x_1, \ldots, x_k) \) of points in \( P \) such that at least one of their coordinates coincides the coordinate of a base point \( x_0 \). Clearly \( P \) is obtained from \( T \) by attaching an \( r \) dimensional cell with an attaching map \( a : S^{r-1} \to T \) also called the universal Whitehead product.

Given the set of homotopy classes \( \bar{E}(\eta_i) \) we have the following wedge map, unique up to homotopy:

\[
g = E(\eta_1) \vee \ldots \vee E(\eta_k) : S^{j_1+1} \vee \ldots \vee S^{j_k+1} \to BG
\]

such that \( g \circ i = E(\eta_i) \) with \( i \) the obvious inclusions.

Consider now the canonical inclusions \( i : S^{j_1+1} \vee \ldots \vee S^{j_k+1} \to T \) and take the set of all possible extensions of \( g \)

\[
W := \{ \bar{g} \bar{g} : T \to BG, \bar{g} \circ i = g \}
\]

Then the \( k \)th order higher Whitehead product is defined as the set of elements in \( \pi_{r-1}(BG) \) given by the maps \( a \circ \bar{g} : S^{r-1} \to BG \) for all possible extensions \( \bar{g} \in W \) and canonical attaching maps \( a : S^{r-1} \to T \). \( W \) is nonempty if and only if all the lower Whitehead products contain the element 0. It is immediate that the set of elements in \( [\bar{E}(\eta_1), \ldots, \bar{E}(\eta_k)]_w \) represents the obstructions to extending all possible maps \( \bar{g} \) to the product \( P \).

If one is interested only in those homotopy elements in Whitehead products that have infinite order those can be obtained as Whitehead products in a space \( X_\emptyset \) called the rationalization of \( X \), or equivalently, localization at \( \emptyset \) (cf. [3] and references therein). There exist localization maps

\[
e : X \to X_\emptyset
\]

such that any for any \( x \in [e_1(\bar{E}(\eta_1)), \ldots, e_k(\bar{E}(\eta_k))]_w \subset \pi_N X_\emptyset \) there exist an integer numbers \( M_i \) such that \( M_x \) is \( e_\epsilon z \), with \( z \in [M_1(\bar{E}(\eta_1)), \ldots, M_k(\bar{E}(\eta_k))]_w \).

The Whitehead products between elements \( e_\epsilon(\bar{E}(\eta)) \) in the rationalization \( X_\emptyset \) are called rational Whitehead products. These products are multilinear. In light of the above correspondence and since we will be interested in nontrivial elements of infinite order only up to multiplication with a factor, we will often say that the rational Whitehead products considered are of elements in \( \pi_{*}BG \otimes \mathbb{Q} \). This correspondence can be well formalized if we consider other definitions of Whitehead products cf. Allday [4], who defines rational Whitehead products on the graded differential Lie algebra \( \pi_* BG \otimes \mathbb{Q} \) (see remark 5.9).

Definition 2.5 (Andrew-Arkowitz [3]) We say that the \( k \)th order (rational Whitehead product \( [\bar{E}(\eta_1), \ldots, \bar{E}(\eta_k)]_w \) vanishes if it only contains the trivial element.
Definition 2.6 We call $r \geq 2$ the rational minimal Whitehead order of a topological space $X$ if it is the minimal order in which there exists a nonvanishing rational Whitehead product.

We will need the following result:

Proposition 2.7 (Andrew-Arkowitz [3]) If each group homotopy group $\pi_\ast X_0$ of the rationalization of a space $X$ is finitely generated and if $r$ is the minimal rational Whitehead order then any rational Whitehead product of order $r$ contains exactly one element.

We use these products to introduce the higher order Samelson products. In the present paper we will only consider the case when all elements $E(\eta_i)$ are the same and are given as suspensions of a circle map $\gamma : S^1 \to G$. In this situation we will use, for brevity, the notation $W(\gamma) \subset \pi_{2k-1}BG$. The $k^{th}$ order Samelson product $S(k)(\gamma) := [\gamma, \ldots, \gamma]_k$ will be a set in $\pi_{2k-2}(G)$ consisting of all the desuspensions of the elements in the set $W(k)(\gamma)$.

3 Whitehead products as obstructions to the existence of symplectic fibrations

3.1 Towers of symplectic fibrations

The results in this section will hold for any continuous family of symplectic structures $(M, \omega_\lambda)$ on a compact manifold $M$ with symplectomorphism groups $G_\lambda = \text{Symp}(M, \omega_\lambda) \cap \text{Diff}_0(M)$.

To each circle map $\gamma_\lambda : S^1 \to G_\lambda$ one can associate a symplectic fibration $P_{\gamma_\lambda} \to S^2$, with fiber $(M, \omega_\lambda)$, obtained by clutching $M \times D^+$ and $M \times D^-$ via the identification $(x, z) = (\gamma_\lambda(z)(x), 1/z)$ whenever $z \in \partial D^+$. $P_{\gamma_\lambda}$ is determined up to symplectic isotopy by the homotopy class $\tilde{\gamma}_\lambda$.

The main proposition of this section shows that triviality of certain Whitehead products yields a construction of towers of symplectic fibrations built on the fibration $P_{\gamma_\lambda}$, and that these towers behave well under symplectic deformations.

Proposition 3.1 a) Fix $\lambda$ and assume that there exist a map $\gamma_\lambda : S^1 \to G_\lambda$, that yield a nontrivial $\tilde{\gamma}_\lambda \in \pi_1 G_\lambda \otimes \mathbb{Q}$ for which all rational Whitehead products of orders $k$ smaller or equal that a given $p$ vanish:

$$\{0\} = W(k)(\tilde{\gamma}_\lambda), k \leq p \quad (7)$$

Then we can build on $\tilde{\gamma}_\lambda$ a tower of symplectic fibrations of length $p$:

$$
\begin{array}{c}
\xymatrix{
\mathbb{C}P^1 \ar[r]^-{\pi_1} \ar[d]^-{\pi_1} & \mathbb{C}P^2 \ar[r]^-{\pi_2} \ar[d]^-{\pi_2} & \cdots \ar[r]^-{\pi_{p-1}} & \mathbb{C}P^p \ar[r]^-{\pi_p} & \mathbb{C}P^p \ar[d]^-{\pi_p} \\
Q(1) \Lambda(1) & Q(2) \Lambda(2) & \cdots & Q(p-1) \Lambda(p-1) & Q(p) \Lambda(p) \\
\end{array}
$$

where the forms $\Lambda_{\lambda}^{(i)}$ are vertically closed 2-forms on $Q^{(i)}$ as in 8, and $Q^{(1)}$ is a clutching fibration $P_{\gamma_\lambda}$, for some $\gamma_\lambda'$ homotopy equivalent to a power of $\gamma_\lambda$.

In this diagram the morphisms preserve the fibration structure.

b) Assume now that there exist some other tower of length $p$ as in 8 built on the element $\tilde{\gamma}'_\lambda \in \pi_1 G_\lambda \otimes \mathbb{Q}$. Then $0 \in W(k)(\tilde{\gamma}_\lambda)$, $k \leq p$ and hence the rational Whitehead product $W(p+1)(\tilde{\gamma}_\lambda)$ is defined.

Moreover, if this tower and any of its $N$ coverings, obtained by taking an $N$ covering of $Q^{(k)}_\lambda$ at each step, are obstructed to extend to towers of length $p+1$ then the rational Whitehead product $W(p+1)(\tilde{\gamma}_\lambda)$ must contain a nontrivial element.
c) (Extension with respect to the parameter) Consider now a continuous family of homotopically nontrivial circle maps \( \gamma_\lambda : S^1 \to G_\lambda, \lambda \geq 0 \) that yield a family of nontrivial robust homotopy elements \( \tilde{\gamma}_\lambda \in \pi_1 G_\lambda \otimes \mathbb{Q} \).

Then for any existing tower of length \( s \) as in \( \lambda = 0 \) must extend continuously to towers of length \( s \) as in \( \lambda > 0 \) as in the following diagram

\[
\begin{array}{ccc}
(Q^{(1)} \times [0, \epsilon_1], \Lambda^{(1)})^\wedge_i \overset{\pi(i)}{\longrightarrow} (Q^{(s-1)} \times [0, \epsilon_{s-1}], \Lambda^{(s-1)})^\wedge_i \overset{\pi(s)}{\longrightarrow} (Q^{(s)} \times [0, \epsilon_s], \Lambda^{(s)}) & \\
\mathbb{C}P^1 \times [0, \epsilon_1]^\wedge_i \overset{\pi(i)}{\longrightarrow} \mathbb{C}P^{s-1} \times [0, \epsilon_{s-1}]^\wedge_i \overset{\pi(s)}{\longrightarrow} \mathbb{C}P^s \times [0, \epsilon_s]
\end{array}
\]  

(9)

where \( \epsilon_k > 0 \), and at each level \( k \) the morphisms \( \pi(k) \) commute with the projections on the second factors \([0, \epsilon_k]\) and the two forms \( \Lambda^{(k)} \) restrict to the symplectic forms \( \Lambda^{(k)}_\lambda \) on \( Q^{(k)} \times \{\lambda\} \).

d) (Hamiltonian case) The tower of symplectic fibrations \( Q^{(p)} \) is a tower of hamiltonian fibrations if and only if \( \gamma_\lambda : S^1 \to \text{Ham}(M, \omega_\lambda) \). In this case the forms \( \Lambda^{(i)}_\lambda \) can be chosen symplectic.

**Proof of Proposition 3.1:** Let \( P^{(k)} = (S^2)^k \) and \( T^{(k)} \) its corresponding fat wedge. Recall that there is a covering map

\[
pr_{(k)} : P^{(k)} \to \mathbb{C}P^k = P^{(k)}/S_k
\]

(10)

where \( S_k \) is the \( k \)-th group of permutation.

We will denote by \( h_{(k)} : S^{2k+1} \to \mathbb{C}P^k \) the maps used to attach a \((2k + 2)\)-dimensional cell to \( \mathbb{C}P^k \) in order to obtain \( \mathbb{C}P^{k+1} \).

Consider the universal fibration \( EG_\lambda \to BG_\lambda \) and let \( M_{G_\lambda} = M \times_{G_\lambda} EG_\lambda \). Well known properties of classifying spaces imply that, up to symplectic isotopy, all symplectic fibrations with fiber \((M, \omega_\lambda)\) and base \( B \) are obtained as \( f^*(M_{G_\lambda}) \) for some homotopy class of maps \( f : B \to BG_\lambda \). In particular the clutching fibration \( P_{\gamma_\lambda} \) is just \((E(\gamma_\lambda))^*(M_{G_\lambda})\). Therefore the existence of a tower of fibrations with basis \( B^{(1)} \subset B^{(2)} \subset \cdots \subset B^{(p)} \) is equivalent with the existence of maps \( \phi_{\lambda, k} : B^{(k)} \to BG_\lambda \) that commute with the inclusions \( i : B_k \to B_{k+1} \).

In order to prove the direct implication in part (a) for \( p > 1 \), first assume that for an integer \( p > 1 \) and a given value \( \lambda, 0 \in W^{(p)}(E(\gamma_\lambda)) \). Then clearly \( 0 \in W^{(k)}(E(\gamma_\lambda)) \) for all \( k \leq p \). Therefore the wedge map \( E(\gamma_\lambda) \vee E(\gamma_\lambda) \) admits extensions

\[
g_{\lambda, (k)} : T^{(k)} \to BG_\lambda, 1 \leq k \leq p
\]

(11)

which commute with the inclusions \( i : T^{(k)} \to T^{(k+1)} \).

A map defined on a subset of \( P^{(k)} \) invariant under the \( S_k \) action is symmetric if it commutes with the action.

**Claim:**

(1) The maps \( g_{\lambda, (k)}^{sym} \) in (11) can be chosen symmetric and they extend to symmetric maps

\[
g_{\lambda, (k)}^{ext} : P^{(k)} \to BG_\lambda.
\]

(12)

(2) There exist maps \( f_{\lambda, (k)} : \mathbb{C}P^k \to BG_\lambda \) that commute with the inclusions \( i : \mathbb{C}P^k \to \mathbb{C}P^{k+1} \), such that \( f_{\lambda, (1)} = E(\gamma_\lambda) \) and \( g_{\lambda, (k)}^{ext} = f_{\lambda, (k)} \circ pr_{(k)} \).

We will use induction to prove the claim:

**Proof of the claim for \( k=2 \):** Take \( g_{\lambda, (2)} = E(\gamma_\lambda) \vee E(\gamma_\lambda) \), clearly symmetric.

The obstruction to extend the map \( f_{\lambda, (1)} = E(\gamma_\lambda) \) from \( S^2 = \mathbb{C}P^1 \) to \( \mathbb{C}P^2 \) is given by the homotopy class \([E(\gamma_\lambda) \circ h_{(1)}] \in \pi_3 BG_\lambda \) and it satisfies \( 2[E(\gamma_\lambda) \circ h_{(1)}] = [g_{\lambda, (2)} \circ a_{(2)}] \).
$W^2(\bar{E}(\gamma_\lambda)) = 0$, where $a(2) : S^3 \to P^2$ is the universal Whitehead product map, used to attach the top cell of dimension 4 on $T^2$ to obtain $P^{(2)}$

But since we work rationally, at the expense of replacing $\gamma_\lambda$ with a multiple we can kill torsion in the obstructions of the extending maps from $\mathbb{C}P^k$ to $\mathbb{C}P^{k+1}$ and hence we conclude that $[E(\gamma_\lambda) \circ h(1)]$ must also be zero. Therefore we can extend the map $f_{\lambda,(1)}$ to a map

$$f_{\lambda,(2)} : \mathbb{C}P^2 \to B\Gamma_\lambda$$

Then we take the map $g^{ext}_{\lambda,(2)} : P^{(2)} \to B\Gamma_\lambda$ to be $g^{ext}_{\lambda,(2)} = f_{\lambda,(2)} \circ pr(2)$, which is clearly symmetric and extends $g_{\lambda,(2)}$.

**Proof that the claim for $k$ implies the claim for $k+1$:**

We have that $T^{(k+1)} = \bigvee_{j=0}^{p(k)} P_j^{(k)}$ where $P_j^{(k)}$ is an identification of the product $P^{(k)}$ with the space of $(k+1)$-tuples that have the coordinate in position $j$ at the base point $x_j$.

By the induction step, we already have $k+1$ identical copies of the symmetric map $g^{ext}_{\lambda,(k)}$ and a map $f_{\lambda,(k)}$ with $g^{ext}_{\lambda,(k)} = f_{\lambda,(k)} \circ pr(k)$ which give (using the relation above) a symmetric map $g_{\lambda,(k+1)} : T^{(k+1)} \to B\Gamma_\lambda$.

Moreover, the obstruction to extend the latter map to the product is $[g_{\lambda,(k)} \circ a(k)] = N[f_{\lambda,(k)} \circ h(k)] = 0$ by hypothesis.

Again as before we may conclude that $[f_{\lambda,(k)} \circ h(k)] = 0$ and hence the map $f_{\lambda,(k)}$ can be extended to $f_{\lambda,(k+1)} : \mathbb{C}P^{k+1} \to B\Gamma_\lambda$. As before, we define $g^{ext}_{\lambda,(k+1)} = f_{\lambda,(k+1)} \circ pr(k+1)$, which is a symmetric extension of $g_{\lambda,(k+1)}$.

From point (2) of the claim we obtain the tower of fibrations (8). Note that the forms $\Lambda_{(k)}$ can be chosen to extend one another by defining them as pull-backs from the universal fibration.

To prove point (b) let us assume a tower of fibrations as in (8) of length $p$ extending $P_{\lambda}$. This gives a sequence of maps $f_{\lambda,(k)} : \mathbb{C}P^k \to B\Gamma_\lambda$, $1 \leq k \leq p$ commuting with the inclusions that extend $E(\gamma_\lambda)$. If we take $g^{ext}_{\lambda,(k)} = f_{\lambda,(k)} \circ pr(k)$ then it is immediate that these maps are just extensions to the products $P^{(k)}$ of the recurrently constructed maps $g_{\lambda,(k)} : T^{(k)} \to B\Gamma_\lambda$, $1 \leq k \leq p$, with $g_{\lambda,(2)} = E(\gamma_\lambda) \vee E(\gamma_\lambda)$. But by definition this implies that $0 \in W^{(k)}(\bar{E}(\gamma_\lambda))$ for all $k \leq p$.

To show the remaining part of point (b) let us assume that $f_{\lambda,(p)} : \mathbb{C}P^p \to B\Gamma_\lambda$ cannot be extended over $\mathbb{C}P^{p+1}$. This implies that $[f_{\lambda,(p)} \circ h(p)] \neq 0$. We know that the obstruction to extend the map $g_{\lambda,(p)}$ satisfies $[g_{\lambda,(p)} \circ a(p)] = M[f_{\lambda,(p)} \circ h(p)]$. Again, if we work rationally we can insure (by considering a covering of the given fibration as in the hypothesis) that the homotopy class $[f_{\lambda,(p)} \circ h(p)]$ is of infinite order and hence $[g_{\lambda,(p)} \circ a(p)] \neq 0$.

Point (c) is obtained by applying the Proposition 2.3 for all the maps $g_{\theta,(k)} : T^{(k)} \to B\Gamma_0$. Note that if maps from $G_0 \to G_\lambda$ exist for all $\lambda$ then the fibrations extend for all parameters $\lambda$.

For point (d) let us first observe that we can replace $G_\lambda$ with its subgroup $\text{Ham}(M, \omega_\lambda)$ and repeat point (a) to argue that we have a tower of hamiltonian fibrations. Moreover, the forms $\Lambda_{(k)}$ can be chosen symplectic by taking a choice of closed forms $\Lambda'_{(k)}$ as in point (a). Then we replace them with $\Lambda'_{(k)} \oplus s \cdot \omega_{\mathbb{C}P^k}$ for large enough $s$.

**Remark 3.2**

- We do not need the family $\gamma_\lambda$ to exist for all values $\lambda \geq 0$. One can use Proposition 2.3 and restate the results for $\lambda$ in small intervals $(0, \epsilon_K)$.

- The tower of fibrations we use in Theorem 3.1 does not allow us to keep track of the torsion elements in $\pi_\ast G_\lambda$. It would be interesting to see if a version of these results could be set up in order to keep track of the torsion elements and hence find all Samelson products in $\pi_\ast G_\lambda$.

- One can use different types of symplectic fibrations to decide whether Samelson products between distinct robust elements, not necessarily in $\pi_1 G_\lambda$, are nontrivial. G.-Y. Shi [20] has an approach in this direction.
4 Parametric Gromov-Witten invariants and Whitehead products

Parametric Gromov-Witten invariants count vertical almost holomorphic maps in a symplectic fibration \((M, \omega_f) \to Q \to B\) endowed with a compatible triple \((Q, \Lambda, \tilde{J})\), \(\Lambda\) and P. Seidel [18] have used them before to detect robust elements in the symplectomorphism groups. By contrast, we will use them to detect fragile elements.

We will carefully exploit their properties of being fiberwise symplectic deformation invariants, in our cases of interest, to show how, combined with Proposition 3.1, they yield nontrivial Whitehead products.

Equivariant Gromov-Witten invariants are a special case of the parametric Gromov-Witten invariants that are computed on manifolds \((M, \omega_f)\) that admit Hamiltonian actions by compact Lie group \(H\). We look at the case \(H \approx S^1\) and treat it in a separate section.

4.1 Definition and properties of parametric Gromov-Witten invariants

We will first make a summary of their defining properties. We will use results from Li-Tian [12] as well as results from Le-Ono [11].

Assume that the symplectic fibration \(\pi : Q \to B\) with fiber \((M, \omega_f)\), admits a closed extension \(\Lambda\) of \(\omega\). Then as explained in Subsection 2.1 we may consider a section \(\tilde{J} : B \to \mathcal{J}(B, M)\) that provides an almost complex structure on each fiber \(M_b\) compatible with the symplectic form \(\omega_b\).

For \(2g + m > 2\) let \(\mathcal{M}_{g,m}\) be the moduli space of genus \(g\) Riemann surfaces with \(k\) marked distinct points. As usual \(\mathcal{M}_{g,m}\) represents the \((3g - 3 + m)\)-dimensional orbifold consisting of Riemann surfaces of genus \(g\) with at most rational double points different from the marked points; that is the Deligne-Mumford compactification of \(\mathcal{M}_{g,m}\).

Fix a homology class \(D \in H_2(M, \mathbb{Z})\) and assume that \(D\) yields a constant section \(s_D : B \to H_2(Q, \mathbb{Z})\) in the corresponding homological bundle. This will be the case if, for instance, the base \(B\) is simply connected.

For any \((\Sigma_g, x_1 \cdots x_m) \in \mathcal{M}_{g,m}\) the map \(f : (\Sigma_g, x_1 \cdots x_m) \to Q\) is a vertical stable map if its image is contained in a fiber \(Q_b\) and the following two conditions are satisfied:

1. Any irreducible component \(\Sigma_{irred}\) of genus 0 on which \(f\) is constant must contain at least two marked points
2. Any irreducible component \(\Sigma_{irred}\) of genus 1 on which \(f\) is constant must contain at least one marked point

Let \(j \in \text{Teich}(\Sigma)\) be an arbitrary complex structure on \(\Sigma\).

Note that the condition \(2g + m > 2\) is not mandatory. A vertical map \(f\) with \(\text{im}(f) \subset Q_b\) is \(J_b\)-holomorphic if there is an arbitrary complex structure \(j \in \text{Teich}(\Sigma)\) on \(\Sigma\), such that \(\partial_j f = \frac{1}{2}(df + J_b \circ df \circ j) = 0\).

Consider \(f : (\Sigma, x_1 \cdots x_m) \to Q\) and \(f' : (\Sigma', x_1' \cdots x_m') \to Q\).

\((b, f, x_1 \cdots x_m)\) is equivalent to \((b', f', x_1' \cdots x_m')\) if \(b = b'\), both \(\text{im}(f)\) and \(\text{im}(f')\) are contained in the same fiber \(Q_b\), and there is a biholomorphism \(\phi : \Sigma \to \Sigma'\) that takes marked points to marked points, nodal points to nodal points (and hence irreducible components to irreducible components), and such that \(f \circ \phi = f'\).

Let \(\mathcal{F}_{g,m}(Q, \tilde{J}, s_D)\) be the moduli space of equivalence classes of triples \([b, f, x_1, \cdots, x_m]\) as above such that \(f\) is \(C^1\) smooth and \([\text{im}(f)] = s_D(b) \in H_2(Q_b, \mathbb{Z})\). We denote by \(\mathcal{M}_{g,m}(Q, \tilde{J}, s_D)\) the subset of \(\mathcal{F}_{g,m}(Q, \tilde{J}, s_D)\) consisting of \(J_b\)-holomorphic stable maps.

Let \(\text{Smooth}(\Sigma) \subset \Sigma\) be the set of all non-singular points of \(\Sigma\). We denote by \(\Omega^{(0,1)}(f^* T_{Q_b}^{vert})\) the set of all continuous sections \(\xi\) in \(\text{Hom}(T\text{Smooth}(\Sigma), f^* T_{Q_b}^{vert})\) that anticommute with \(j\) and \(J_b\). Any such section can be continuously extended over the nodal points of \(\Sigma\). We can construct a generalized bundle \(E\) over \(\mathcal{F}_{g,k}(Q, \tilde{J}, s_D)\) with fiber \(\Omega^{(0,1)}(f^* T_{Q_b}^{vert})\) and consider
a section in $E$ given by $\Phi = \frac{1}{2}(df + J_0 \circ df \circ j)$. Then $\Phi^{-1}(0)$ is exactly $\overline{M}_{g,m}(Q, \tilde{J}, s_D)$. The topology on the space $\text{Hom}(T\text{Smooth}(\Sigma), f^*Tq^\text{vert})$ will be defined as in Li-Tian [12].

**Proposition 4.1** For $l \geq 2$ and the section $\phi : \overline{\mathcal{M}}_{g,k}^I(Q, \tilde{J}, s_D) \to E$ as above, $\phi^{-1}(0) = \overline{M}^*_{g,m}(Q, \tilde{J}, s_D)$ is compact and $\phi$ gives rise to a generalized Fredholm orbifold bundle with a natural orientation and index $d = 2(\text{dim}_C M - 3)(1 - g) + 2c_1(D) + 2m + \text{dim} B$.

Following as in [12], the above result allows one to construct a virtual moduli class $[\overline{M}_{g,m}(Q, \tilde{J}, s_D)]^\text{virt} \in H_d(\overline{M}^*_{g,m}(Q, \tilde{J}, s_D), \mathbb{Q})$. Let us consider now the usual evaluation map $\text{ev} : \overline{M}_{g,m}(Q, \tilde{J}, s_D) \to Q^m$ given by $\text{ev}([b, f, x_1, \ldots, x_m]) = (f(x_1), \ldots, f(x_m))$ as well as the forgetful map $\text{forget} : \overline{M}_{g,m}(Q, \tilde{J}, s_D) \to \overline{M}_{g,m}$ whose value is the stabilized domain (collapsing unstable components) of $f$.

**Definition 4.2** The parametric Gromov-Witten invariants are maps

$$\text{PGW}^J_{g,m}(Q, s_D) : [H^*(Q; \mathbb{R})]^m \times H^*(\overline{M}_{g,m}, \mathbb{Q}) \to \mathbb{Q}$$

which, for $\alpha \in [H^*(Q; \mathbb{R})]^m$ and $\beta \in M_{g,m}$ are given as:

$$\text{PGW}^J_{g,m}(Q, s_D)(\beta, \alpha) = [\overline{M}_{g,m}(Q, \tilde{J}, s_D)]^\text{virt} \cap (\text{forget}^* \beta \cup \text{ev}^* \alpha)$$

These invariants are zero unless

$$2(\text{dim}_C M - 3)(1 - g) + 2c_1(D) + 2m + \text{dim} B = \text{dim} \alpha + \text{dim} \beta$$

Assume that the associated smooth bundle in homology is trivial. Let us focus on the case when $\beta = 0$ and $\alpha$ is the Poincaré dual of a product of $k$ cycles $a_i$ that can be represented in a fiber $Q_b$ for some arbitrary $b$. Then the invariants count all maps $[b, f, j, x_1, \ldots, x_m]$ (with no restrictions on the genus $g$ domain $(\Sigma, j)$), whose homology class is $[\text{im}(f)] = s_D \in H_2(Q_b, \mathbb{Z})$ and such that $f(x_i)$ lies in $a_i$.

We define the symplectic vertical taming cone $\mathcal{T}(\tilde{J})$ of a section $\tilde{J}$ to be the space of closed 2-forms $\Lambda$ on $Q$ that are compatible with the symplectic fibration $\pi : Q \to B$ with fiber $(M, \omega)$ and which satisfy the taming relation $\Lambda(v, \tilde{J}w) > 0$ for any vectors $v, w$ tangent to a fiber $Q_b$.

As in Li-Tian [12] and Le-Ono [11], the following properties of parametric Gromov-Witten invariants hold:

**Proposition 4.3** (Properties of parametric Gromov-Witten invariants). Consider a symplectic fibration $\pi : Q \to B$ with fiber $(M, \omega_0)$, with a closed extension $\Lambda_0$ of $\omega_0$ and an integral homology class $D \in H_2(M, \mathbb{Z})$.

(i) The parametric Gromov-Witten invariants $\text{PGW}^J_{g,m}(Q, s_D)$ are well defined and independent of the choice of the section of tamed vertical almost complex structure $\tilde{J}$ with $\Lambda_0 \in \mathcal{T}(\tilde{J})$.

(ii) The parametric Gromov-Witten invariants $\text{PGW}^J_{g,m}(Q, s_D)$ are independent of the choice of the taming closed extension $\Lambda$ and hence are fiberwise symplectic deformation invariants as long as the deformation is within some symplectic taming cone $\mathcal{T}(\tilde{J})$.

(iii) (Le-Ono [11]) Symplectic sum formula Let $Q = Q_1 \# Q_2$ be a fiber connected sum of two fibrations. Then

$$\text{PGW}^J_{g,0}(Q, s_D) = \text{PGW}^J_{g,0}(Q_1, s_D) + \text{PGW}^J_{g,0}(Q_2, s_D)$$

(iv) (Le-Ono [11]) If $f : B' \to B$ is a $N$ covering map then $\text{PGW}^J_{g,m}(Q, s_D) = N \cdot \text{PGW}^J_{g,m}(f^*Q, s_D)$.
4.2 Equivariant Gromov-Witten invariants

Equivariant Gromov-Witten invariants can be defined for any hamiltonian action of a compact Lie group $H$ on a symplectic manifold $(M, \omega)$. We will restrict ourselves to the case of hamiltonian circle actions.

Consider the universal symplectic fibration $M_{S^1} = M \times_{S^1} ES^1$ with fiber $(M, \omega)$. $M_{S^1}$ consists of an infinite tower of hamiltonian fibrations $\pi_{(k)} : M_{S^1}^{(k)} = M \times_{S^1} S^{2k+1} \to \mathbb{C}P^k$. Note that $M \times_{S^1} S^{2k+1}$ comes equipped with a natural $S^1$-invariant almost complex structure $J^{(k)}$ compatible with the fibration that makes the map $\pi_{(k)}$ almost holomorphic, as well as with the closed extensions $\Lambda^{(k)}$ consisting of symplectic $S^1$ invariant forms.

We say that $M_{S^1}$ admits the vertical almost complex structure $\tilde{J}$ if $\tilde{J}$ restricts to an usual vertical almost complex structure on each $M_{S^1}^{(k)}$. Similarly, we say that $\Lambda$ is a closed 2-form on $M_{S^1}$ if it restricts to a closed 2-form on each $M_{S^1}^{(k)}$. Both $\tilde{J}$ and $\Lambda$ can be chosen $S^1$-invariant and compatible.

We get an $S^1$-action on the spaces of maps $\overline{\mathcal{M}}_{g,m}(M_{S^1}, \tilde{J}, s_D)$ and $\overline{\mathcal{M}}_{g,m}(M_{S^1}, \tilde{J}, s_D)$ and we can construct an equivariant virtual class

$$[\overline{\mathcal{M}}_{g,m}(M, \tilde{J}, s_D)]_{\text{equiv}}^{\text{virt}} \in H_*(\overline{\mathcal{M}}_{g,m}(M_{S^1}, \tilde{J}, s_D), \mathbb{Q})$$

Then the equivariant Gromov-Witten invariants are maps

$$EGW^{J}_{g,m}(M, s_D) : [H^*(M_{S^1})]^m \times H^*(\overline{\mathcal{M}}_{g,m}, \mathbb{Q}) \to H^*(BS^1, \mathbb{Q})$$

which, for $\alpha \in [H^*(M_{S^1}, \mathbb{Q})]^k$ and $\beta \in \overline{\mathcal{M}}_{g,k}$ are given as:

$$EGW^{J}_{g,k}(M, s_D)(\beta, \alpha) = [\overline{\mathcal{M}}_{g,k}(Q, \tilde{J}, s_D)]_{\text{equiv}}^{\text{virt}} \cap (\text{forget} \ast \beta \cup \text{ev}^*\alpha),$$

where the “$\cup$” is obtained from equivariant integration. The following proposition gives properties of equivariant Gromov-Witten invariants:

**Proposition 4.4**

1. For any vertical $S^1$-invariant almost complex structure $\tilde{J}$ compatible with the fibration and for any $S^1$-invariant taming form $\Lambda$ on $M_{S^1}$, the invariants $EGW^{J}_{g,m}(M, s_D)$ are well defined and independent of the choice of the invariant taming vertical almost complex structure $\tilde{J}$.

2. If the equivariant class $\alpha = \bigoplus_{k=1}^{\infty} \alpha^{(k)} u^k \in H^*(BS^1, \mathbb{Q})$ then we immediately have

$$EGW^{J}_{g,k}(M, s_D)(\beta, \alpha) = \bigoplus_{k=1}^{\infty} EGW^{J}_{g,k}(Q^{(k)}, s_D)(\beta, \alpha^{(k)}) u^k$$

where $EGW^{J}_{g,k}(Q^{(k)}, s_D)(\beta, \alpha^{(k)})$ are the parametric Gromov-Witten invariants of the fibration $Q^{(k)}$ and are zero unless

$$2(\dim C M - 3)(1 - g) + 2c_1(D) + 2k + 2m = \dim \alpha^{(k)} + \dim \beta$$

4.3 Parametric Gromov-Witten invariants and Whitehead products

**Lemma 4.5** Consider a symplectic deformation $(M, \omega_\lambda)_{\lambda \geq 0}$ and a homology class $D \in H_2(M, \mathbb{Z})$ with $[\omega_0](D) = 0$. Assume that there exists a smooth symplectic fibration $\pi : Q \to B$ endowed with a continuous family of closed two extensions $(\Lambda_\lambda)_{\lambda \geq 0}$ of the symplectic fibers $(M, \omega_\lambda)$. Moreover, assume that the maps $PGW^{J}_{g,m}(Q_{\lambda}, s_D)$ are nontrivial.

Then the family $(\Lambda_{\lambda})_{\lambda \geq 0}$ cannot extend to a fiberwise symplectic deformation $(\Lambda_{\lambda})_{\lambda \geq 0}$ based on the given family $(M, \omega_\lambda)_{\lambda \geq 0}$.
The proof is immediate. Indeed, the existence of a $J$-holomorphic curve in a class $D \in H_2(M, \mathbb{Z})$ implies that any taming symplectic form $\omega_0$ must satisfy $[\omega_0](D) > 0$. The result is a consequence of point (ii) in Proposition 4.3.

We will effectively use the lemma above to show that extensions with respect to the parameter as in Proposition 3.1 cannot exist in the presence of certain nontrivial PGW invariants:

**Corollary 4.6** Assume that we are in the conditions of Proposition 3.1 point (c) and we have a tower at level $\lambda = 0$ of length $p \geq 1$. Then the resulting fibrations $Q^{(k)}_\lambda, k \leq p, (\lambda > 0)$ obtained by extending with respect to the parameter the fibrations $Q^k_0, k \leq p$ must satisfy $0 = PGW_{g,m}(Q^{(p+1)}_\lambda, s_D)$ whenever $[\omega_0](D) = 0$.

The crux of the argument that $\{0\} \not= W^{(p+1)}(\overline{E}(\gamma_0))$, will be to show that some fibration $Q^{(p+1)}_0$ obtained by extending $Q^{(p+1)}_0$ with respect to the parameter, must have a nontrivial parametric Gromov-Witten invariant as above which would contradict the above corollary.

## 5 Ruled surfaces

A ruled surface $M^{g}_{\lambda}$ is the total space of the topologically trivial symplectic fibration $(S^{2} \times \Sigma_{g}, \sigma_{S^{2}} \oplus \lambda \sigma_{\Sigma_{g}}) \rightarrow (\Sigma_{g}, \sigma_{\Sigma_{g}})$. Accordingly, we let the symplectomorphism groups $G^{g}_{\lambda}$ be $\text{Symp}(S^{2} \times \Sigma_{g}, \sigma_{S^{2}} \oplus \lambda \sigma_{\Sigma_{g}}) \cap \text{Diff}_{0}(M)$.

### 5.1 Prior results

We will present here results that are essentially contained in McDuff [13]. Let us denote by $S_\lambda$ the space of symplectic forms that are strongly isotopic with $\omega_\lambda$, and by $A_\lambda$ the space of almost complex structures that are tamed by some form in $S_\lambda$. Then there exists a fibration $G_3 \rightarrow \text{Diff}_{0}(M) \rightarrow S_{\lambda}$ and, since $S_{\lambda}$ is homotopy equivalent with $A_{\lambda}$, there is also a homotopy fibration

$$G_\lambda \rightarrow \text{Diff}_{0}(M) \rightarrow A_\lambda. \quad (19)$$

Let $D_k = A - kF \in H_2(M^{g}_{\lambda}, \mathbb{Z})$ where $A$ and $F$ are the homology classes of the base and the fiber respectively. The subsets $A^{\ell+ \varepsilon}_{\lambda,k}$ of $A^{\ell}_{\lambda}$ consisting of almost complex structures that admit $J$-holomorphic curves in the class $D_k$ provide a stratification of $A^{\ell}_{\lambda}$ as in the following:

**Proposition 5.1** (McDuff[13])

(i) $A^{\ell}_{\lambda} \subset A^{\ell+ \varepsilon}_{\lambda,k}$ and hence, via (19) one obtains maps $h_{\lambda,\lambda+\varepsilon}: G_{\lambda} \rightarrow G_{\lambda+\varepsilon}$.

(ii) $A^{\ell}_{\lambda,k}$ is a Frechet suborbifold of $A^{\ell}_{\lambda}$ of codimension $4k - 2 + 2g$.

(iii) $A^{\ell}_{\lambda}$ is constant on all the intervals $(\ell, \ell + 1]$ and $A^{0}_{\lambda} = A^{0}_{k+\varepsilon,k}$.

(iv) The homotopy type of $G^{g}_{\lambda}$ is constant for $k < \lambda \leq k + 1$, with $k$ an integer greater than zero. For this range of $\lambda$ there exists a nontrivial fragile element $w_k \in \pi_{4k}(G^{g}_{\lambda}) \otimes \mathbb{Q}$ that disappears when $\lambda$ passes the critical value $k + 1$, while a new fragile element $w_{k+1}$ appears.

(v) There exists a fragile element $\rho \in \pi_{2}G^{g}_{1}$ that disappears in $\pi_{2}G^{g}_{1+\varepsilon}$.

Moreover the inclusions $i : G^{g}_{\lambda} \rightarrow \text{Diff}_{0}(M^{g})$ lift to maps $\tilde{i} : G^{g}_{\lambda} \rightarrow D^{0}_{g}$ where $D^{0}_{g}$ is the subgroup of diffeomorphisms that preserve the $S_{\lambda}^2$ fibers. The following proposition shows that all essential elements in $\pi_{i}(D^{0}_{g})$ are retained in the homotopy groups of symplectomorphism groups:

**Proposition 5.2** McDuff[13]

(i) The vector space $\pi_{i}(D^{0}_{g}) \otimes \mathbb{Q}$ has dimension 1 when $i = 0, 1, 3$ except in the cases $i = g = 1$ when the dimension is 3, and $g = 0, i = 3$ when the dimension is 2. It has dimension $2g$ when $i = 2$ and is zero otherwise.
(ii) There exist maps $\gamma^a_k : S^1 \to D^0_\lambda$ that induce a surjection on all rational homotopy groups for all $g > 0$ and $\lambda \geq 0$. The map is actually an isomorphism on $\pi_i$, $i = 1, \ldots, 2g - 1$ when we restrict to the range $\lambda > k$ where $g = 2k$ or $g = 2k + 1$ depending on the parity.

(iii) The map $\gamma^a_k$ also gives an isomorphism on $\pi_i$ for $g = 1, i = 2, 3, 4, 5$ and $\lambda > 3/2$.

(iv) The homotopy limit $G^g_\infty = \lim_{\lambda \to \infty} G^g_\lambda \approx D^g_0$

5.2 Hamiltonian circle actions on ruled surfaces, robust elements and equivariant Gromov-Witten invariants

We will first describe all possible hamiltonian circle actions on the manifolds $M^g_\lambda$. This is provided for instance in M. Audin [6]. For these actions we will give a complete description of the equivariant Gromov-Witten invariants that count isolated curves of genus $g$. We also describe families of robust elements that satisfy hypothesis $H_1$ which, combined with the nontrivial count of EGW yields nontrivial Whitehead products.

The Lie groups $H_k \approx S^1$ act on the manifolds $M^g_\lambda$, $\lambda > k$ as follows:

We denote by $\mathcal{O}(-2k)_g$ to be a holomorphic line bundle of degree $-2k$ over the surface $\Sigma_g$, and consider the projectivized line bundles $\pi : P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g) \to \Sigma_g$. The Kähler manifolds $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$ are endowed with naturally integrable almost complex structures denoted by $J^{(k),a}$. Topologically, they are just $\Sigma_g \times S^2$ and it is easy to see that these bundles admit a holomorphic circle action that rotates the fibers while fixing the zero section and the section at infinity that represent the classes $A - kF$ and $A + kF$ respectively:

$$\gamma^a_k : S^1 \times P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g) \to P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$$

(20)

In coordinates, this action is given by $e^{it} : (h, [v_1: v_2]) = (h, [e^{it}v_1 : v_2])$.

We will view the $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$ as the symplectic manifolds $M^g_\lambda$ endowed with the $S^1$-invariant taming complex structures $J^{(k),a}$ whenever $\lambda > k$.

The circle actions (20) become hamiltonian with respect to the symplectic forms $\omega_{\lambda}$, whenever $\lambda > k$; this is for example explained in Audin [6]. The ruled surfaces $M^g_\lambda$, for $\lambda > 0$, can be constructed via symplectic reduction from disk bundles $D_a(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$ with appropriate radii $a$.

This construction fails to work if $g > 0$ and $\lambda = 1$. In fact, it follows from Karshon [10] that the symplectic manifolds $M^g_0$ do not admit any such hamiltonian $S^1$-action. Moreover, it is clear that the actions (20) cease to be symplectic whenever $\lambda \leq k$. To stress this distinction we will use the following notation for the hamiltonian actions

$$\gamma^a_{k,\lambda} : S^1 \times M^g_\lambda \to M^g_\lambda, \lambda > k$$

(21)

or, equivalently,

$$E(\gamma^a_{k,\lambda}) : S^2 \to BH_k \subset BG^g_\lambda, \lambda > k.$$  

(22)

From Proposition 5.2 we see that the cycles $\tilde{\gamma}^a_{k,\lambda}$ are essential in $D^g_0$ and represent an element $\tilde{\gamma}^a_k \in D^g_0 \otimes \mathbb{Q}$.

In fact, a smooth representative for $\tilde{\gamma}^a_k \in D^g_0 \otimes \mathbb{Q}$ can be given as $\gamma^a_k : S^1 \to D^g_0$

$$\gamma^a_k(\theta)(w, z) = (z, \rho(R^a_0(w))$$

(23)

where $R^a_0(z)$ rotates the fiber sphere in $S^2 \times S^2$ with an angle $\theta$ about a point $z$ in the base sphere, and $\rho : \Sigma_g \to S^3$ is a covering map of degree $k$.

In the case $g = 0$ the hamiltonian $S^1$-actions (20) are in fact induced from a $T^2$ toric action on $M^g_\lambda$, $\lambda \geq 1$ can be obtained through symplectic reduction in $[\lambda]$ different ways as $M^0_0 = \mathbb{C}^4/T^2$, for any $0 \leq k < \lambda$ where the two generators $\xi_1, \xi_2$ of $T^2$ act on $\mathbb{C}^4$ with weights $(1, 1, 0, 0)$ and $(2k, 0, 1, 1)$. The group of toric automorphisms is a subgroup of the symplectomorphism group and it contains a Lie subgroup $K_k = S^1 \times SO(3)$ for $k > 0$ and $K_0 = SO(3) \times SO(3)$ such that the map $\pi_1 K_k \to G^g_0$ induces an injection on the homotopy groups. In this case we have
$H_k \subset K_k$. It follows that $\pi_1 K_k$ contains the generators $\tilde{\gamma}_{\lambda,k}^0 \in \pi_1 G_\lambda^0$ and $\tilde{\alpha}_k \in \pi_3 G_\lambda^0$ whenever $\lambda > k > 1$, and $\tilde{\alpha}$ and $\tilde{\eta}$ in $\pi_3 G_\lambda^0$ whenever $\lambda > 1$.

**Lemma 5.3** Consider (1) $g > 0$ and $k \geq 1$ or (2) $g = 0$ and $k \geq 2$.

(i) \[
\tilde{\gamma}_k^g = k \tilde{\gamma}_k^0 \in \pi_1 D_0^g \otimes \mathbb{Q}
\]  
(ii) If, in addition, we assume $\lambda > k > \frac{g}{2}$ then the same relation takes place in $\pi_1 G_\lambda^g \otimes \mathbb{Q}$:
\[
\tilde{\gamma}_{\lambda,k}^g = k \tilde{\gamma}_{\lambda,1}^0 \in \pi_1 G_\lambda^g \otimes \mathbb{Q}.
\]
(iii) (for $g = 0$)
\[
\tilde{\alpha}_k = \tilde{\alpha} + k^2 \tilde{\eta} \in H_3(G_\lambda, \mathbb{Q})
\]
(iv) There exist a continuous family of robust elements of infinite order $\gamma_{\lambda,k}^g : S^1 \to G_\lambda^g$, for $\lambda \geq k$ which for $\lambda > k$ is homotopy equivalent with the circle maps $\tilde{\gamma}_{\lambda,k}^g$ given by the group action $H_k$.

Moreover, with the exception of the case $g = k = 1$, at the critical values $\lambda = k$ we can deduce that there are integers $M$ so that
\[
\tilde{\gamma}_k^g = M \tilde{\gamma}_1^0 \in \pi_1 G_k^g \otimes \mathbb{Q}
\]

**Proof:** The proof of (i) is an immediate adaptation of Lemma 2.10 proved in Abreu-McDuff [2] for the case $g = 0$. In fact they actually compute the difference between the two terms as a 2-torsion element. Similarly, when we restrict to the given range for $\lambda$ the morphisms $\tilde{i}$ give an isomorphism on $\pi_1$ and hence the relation in (i) continues to hold in $\pi_1 G_\lambda^g \otimes \mathbb{Q}$. Part (iii) is also contained in Lemma 2.10 proved in Abreu-McDuff [2].

The existence of the robust family in part (iv) is an immediate consequence of Proposition 5.2. Indeed, since the maps $\tilde{i}$ induce a surjection on the first rational homotopy groups for $\lambda$ in that range and the family can be obtained by pulling back the smooth representative $\tilde{\gamma}_k^g$ to the symplectomorphism groups. Finally, the relation (27) follows for instance from the fact that the vector space $\pi_1 G_k^g \otimes \mathbb{Q}$ is one dimensional (proposition 5.2 point (i)).

\[\Box\]

### 5.3 Equivariant Gromov-Witten invariants

**Proof of Theorem 1.1:** Denote by
\[
(Q^{(k),(p),g}_\lambda, J^{(k),(p),g}) = M_\lambda^g \times_{H_k} S^{2p+1}
\]
the associated symplectic fibration with fiber $(M_\lambda^g, \omega_\lambda)$ endowed with the $S^1$-invariant symplectic form $\Lambda^{(k),(p)}$ and compatible almost complex structure $J^{(k),(p),g}$. Then according with Proposition 4.4 we need to show that $EGW_{g,b}^{(p),J^{(k),(p),g}}((Q^{(k),g}_\lambda, s_{D_k})$ is $\pm 1$ if $p = 2k + g - 1$ and zero otherwise.

The dimension condition in 18 translates into saying that
\[
(dim_{C} M_\lambda^g - 3)(1 - g) + c_1(A - kF) + 2p = g - 1 + (A - kF)^2 + 2 - 2g + 2p =
\]
\[
= g - 1 - 2k + 2 - 2g = -2k - g + 1 + 2p
\]
must be 0. Therefore all such invariants are zero unless $p = 2k + g - 1$.

In this situation there exists exactly one embedded vertical $J^{(k),(p),g}$-holomorphic map representing $s_{D_k}$ in each fiber $Q^{(k),g}_b$ for each $b \in \mathbb{C}P^p$. More precisely, each fiber is biholomorphic to $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_b)$. The only possible bubbling for vertical almost holomorphic curves in $Q^{(k),g}_b$ must take place within a fiber. It immediately follows that the only $J^{(k),(p),g}$ maps
in each fiber representing the class \(D_k\) is the zero section of the bundle \(P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)\). Therefore the moduli space \(\mathcal{M}_{g,0}(Q^{(k),(p)g}, J^{(k),(p)g}, s_{D_k})\) is naturally diffeomorphic with \(\mathbb{CP}^p\).

Given such \(J^{(k),(p)g}\) holomorphic map \(f : (\Sigma_g, J_g) \to (M^g_k, J^{(k),(p)g})\) in the class \(D_k\) the linearized operator \(D\phi\) of index zero is

\[
D\phi([b, f, j_g]) : T_b\mathbb{CP}^p \times C^\infty(f^*TM^g_k) \times T_{j_g}\text{Teich}_g \to \Omega^{(0,1)}(f^*TM^g_k)
\]

where the component corresponding to the Teichmüller space appears when \(g > 0\).

The actual dimension of \(\mathcal{M}_{g,0}(Q^{(k),(p)g}, J^{(k),(p)g}, s_{D_k})\) is larger than its formal dimension 0. This is because the fiberwise almost complex structure \(J^{(k),(p)g}\) is not \(D_k\)-regular, or equivalently, the linearized operator (30) is not onto. The computation of the invariants then follows from the following:

**Lemma 5.4**

(i) \(EGW^{(p),(k),(g)}_{g,0} (Q^{(k),(p)g}, J^{(k),(p)g}, s_{D_k}) = e(\mathcal{O}^g)\) where \(e(\mathcal{O}^g)\) represents the Euler class of the obstruction bundle \(\mathcal{O}^g \to \mathcal{M}_{g,0}(Q^{(k),(p)g}, J^{(k),(p)g}, s_{D_k})\) induced by the section \(\phi\) whose fiber over a point \([b, f, j_g]\) is given by \(\text{coker}D\phi([b, f, j_g])\).

(ii) Whenever \(p = 2k + g - 1\) the obstruction bundle \(\mathcal{O}^g \to \mathcal{M}_{g,0}(Q^{(k),(p)g}, J^{(k),(p)g}, s_{D_k})\) is isomorphic to \(\mathcal{O}_{\mathbb{CP}^p}(-1)^p \to \mathbb{CP}^p\).

**Proof:**

(i) This follows immediately from the setup in the general theory as in Li-Tian [12], since in this particular case the moduli space \(\phi^{-1}(0)\) is smooth and hence the generalized Fredholm orbifold is in fact a smooth vector bundle over \(\mathbb{CP}^p\).

(ii) Since \(f\) represents the zero section in the fiber \(Q_b = P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)\), the vertical tangent bundle \(T_{p\text{vert}}(Q^{(k),(p)g})\big|_{\text{vert}} = T(M^g_k)\big|_{\text{vert}}\) splits holomorphically in the direct sum \(T\Sigma_g \oplus \nu_g\), where \(\nu_g\) is the normal bundle to the image \(\Sigma_g\) of the zero section \(f\). It is immediate that the normal bundle is in fact \(\mathcal{O}(-2k)_g \to \Sigma_g\). The operator (30) becomes:

\[
D\phi([b, f, j_g]) : T_b\mathbb{CP}^p \oplus C^\infty(\Sigma_g, \nu^g_k) \oplus C^\infty(\Sigma_g, T\Sigma_g) \oplus T_{j_g}\text{Teich}_g \to \Omega^{(0,1)}(\Sigma_g, \nu^g_k) \oplus \Omega^{(0,1)}(\Sigma_g, T\Sigma_g)
\]

and hence

\[
D\phi\big|([b, f, j_g]) : T_b\mathbb{CP}^p \oplus C^\infty(\Sigma_g, \mathcal{O}(-2k)_g) \oplus C^\infty(\Sigma_g, T\Sigma_g) \oplus T_{j_g}\text{Teich}_g \to \Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-2k)_g) \oplus \Omega^{(0,1)}(\Sigma_g, T\Sigma_g)
\]

We will study the cokernel in the case \(g = 0\) separately. If \(g > 0\), then the component of \(D\phi([b, f, j_g])\) that is not onto is

\[
D\phi\text{restr}([b, f, j_g]) : C^\infty(\Sigma_g, \mathcal{O}(-2k)_g) \to \Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-2k)_g)
\]

whose cokernel is \(H^{(0,1)}(\Sigma_g, \mathcal{O}(-2k)_g)\). If we denote by \(K_g\) the degree \(2g - 2\) canonical bundle over \(\Sigma_g\) then, by Serre duality, \(\text{coker}D\phi([b, f, j_g])\) will be precisely the space of holomorphic sections \((H^0(\Sigma_g, \mathcal{O}(-2k)_g^* \otimes K_g))^*\). By the Riemann-Roch theorem this space has complex dimension \(2k + 2g - 2 - g + 1 = 2k + g - 1\). To find out how these fibers fit together topologically in the obstruction bundle we need to understand what is the induced \(S^1\)-action on \((H^0(\Sigma_g, \mathcal{O}(-2k)_g^* \otimes K_g))^*\) such that

\[
\mathcal{O}^g = (H^0(\Sigma_g, \mathcal{O}(-2k)_g^* \otimes K_g))^* \times_{S^1} S^{2p+1}.
\]

Since \(S^1\) acts with weight 1 on the normal bundle \(\mu_g = \mathcal{O}(-2k)_g\), and correspondingly on its dual \(\mathcal{O}(-2k)_g^*\), the space of sections inherits a diagonal \(S^1\)-action with equal weights given by either 1 or \(-1\). Since it will be enough to determine the EGW up to a sign, we will assume for simplicity that the weights are equal to 1. Since \(\text{im}(f)\) is a fixed set of the canonical
bundle $S^1$-action, $T(Q^{(k),(p),g})_{\text{inf}}$ is also fixed by the induced $S^1$-action and therefore so is the $K_g$. Hence the action on $(H^0(\Sigma_g, \mathcal{O}(-2k)\otimes K_g))^*$ is induced by the $S^1$-action with weights $(1, \ldots, 1)$ on $\mathcal{O}(-2k)\otimes K_g$ and hence it is diagonal with weights $(1, \ldots, 1)$. It immediately follows that $(H^0(\Sigma_g, \mathcal{O}(-2k)\otimes K_g))^* \times_{S^1} S^{2p+1}$ is given by $\mathcal{O}_{CP^p}(-1)^p \to CP^p$.

In the case $g = 0$ the moduli spaces involved in the computation must be of unparameterized curves, which means we have to quotient out the 6-dimensional group $PGL(2, \mathbb{C})$ representing the reparametrizations of the domain. The linearized operator will be

$$D\phi([b, f, j_0]) : T_b CP^p \oplus C^\infty(\Sigma_0, \nu_k^0) \oplus C^\infty(S^2, TS^2) \to \Omega^{(0,1)}(S^2, TS^2)$$

with the cokernel given by:

$$D\phi_{\text{ restr}}([b, f, j]) : C^\infty(S^2, \mathcal{O}(-2k)) \to \Omega^{(0,1)}(S^2, \mathcal{O}(-2k))$$

A similar line of thought as above then applies. In this case the canonical bundle is of negative degree $\mathcal{O}(-2)$ and the fiber of the obstruction bundle is $(H^0(S^2, \mathcal{O}(-2k)\otimes \mathcal{O}(-2))^* = (H^0(S^2, \mathcal{O}(2k - 2))^*$ of complex dimension $2k - 1$.

Hence whenever $p = 2k + g - 1$ we have $EGW_{\eta,0}^{(k),(p),g}((Q^{(k),(p),g}, s_{D_h}) = e(\mathcal{O}^g) = c_n(\mathcal{O}_{CP^p}(-1)^p) = c_1(\mathcal{O}_{CP^p}(-1))^p = 1$.  

Remark 5.5 As in Proposition 3.1 point (b) we also need to consider towers of fibrations that are finite covers of the original ones. Note that any convering of $Q^{(k),(p),g}$ must also have nontrivial PGW cf. Proposition 4.3(iii).

5.4 A non-trivial Whitehead product in the symplectomorphism group of $T^2 \times S^2$

Proof of Proposition 1.2(i): The result will follow from:

Claim: The Whitehead product $[\tilde{E}(\gamma_{1,1}^1), \tilde{E}(\gamma_{1,1}^1)]_w \in \pi_3 BG^1_1$ is nontrivial and yields, by desuspension, a nontrivial fragile element $w = [\gamma_{1,1}^1, \gamma_{1,1}^1] \in \pi_2 G^1_1$.

Let us assume that the claim is false, therefore $[\tilde{E}(\gamma_{1,1}^1), \tilde{E}(\gamma_{1,1}^1)]_w = 0$.

Then, according with Proposition 3.1(iii) there exists a continuous family of fibrations $(Q^{(2),\lambda})_\lambda \to CP^2$, for $\lambda \geq 1$ sufficiently close to 1 which fits in the tower (9). From corollary 4.6 we must have

$$PGW_{1,0}^{J}(Q^{(2),\lambda}, s_{A-F}) = 0$$  (32)

Then under the triviality assumption we claim the following:

Lemma 5.6 Take $E^{(2)}_\lambda \to CP^2$ be a fibration obtained as a N-covering of the fibration $Q^{(1),(2),1}_\lambda$ defined in (28).

1. For $\lambda > 1$ sufficiently close to 1, there exists a continuous family of fibrations $R_\lambda \to S^4$

   given by a family of elements $\eta_\lambda : S^3 \to G_\lambda$, such that $E^{(2)}_\lambda \# R_\lambda$ is symplectically isotopic to $Q^{(2)}_\lambda$.

2. $PGW_{1,0}^{J}(R_\lambda, s_{A-F}) \neq 0$

3. The family $\eta_\lambda : S^3 \to G_\lambda$ is new.

Proof of the lemma: Let us remind the reader that in order to obtain the family of fibrations $Q^{(2)}_\lambda$ we take the symmetric wedge map $g^{ex}_{(1),(2)}$, its extensions to the product $g^{ex}_{1,2}$ and $f^{ex}_{1,2}$ to the product $S^2 \times S^2$ and to $CP^2$ respectively and extend them continuously with respect to the parameter. Thus we obtain maps $g^{ex}_{(1),(2)}$, $g^{ex}_{(1),(2)}$ and $f^{ex}_{(1),(2)}$ that give a choice of extensions involved in the definition of the trivial Whitehead product $[\tilde{E}(\gamma_{1,1}^1), \tilde{E}(\gamma_{1,1}^1)]_w$. 
From Proposition 5.3(iv) and the uniqueness up to homotopy of the wedge map that gives a Whitehead product of order 2, it follows that the map \( g_{\lambda}^{(2)} \) has to be homotopy equivalent to a \( N \) covering of the map \( E(\gamma_{\lambda,1}) \oplus E(\gamma_{\lambda,1}) : S^2 \vee S^2 \to BH_1 \subset BG_\lambda^1 \). The latter extends to \( CP^2 \) and gives \( E_{\lambda}^{(k)} \).

Therefore the restrictions of both \( E_{\lambda}^{(k)} \) and \( Q_{\lambda}^{(k)} \) to their 2-skeletons must be isotopic and part (i) of the lemma follows.

Part (ii) is an immediate consequence of Theorem 1.1, Proposition 4.3 and (32).

Part (iii) then follows from part (ii) and Corollary 4.6. \( \square \)

Consider now the long exact sequences in homotopy:

\[
\begin{array}{ccccccc}
\pi_4 \text{Diff}_0(T^2 \times S^2) & \to & \pi_4 A_1^\lambda & \to & \pi_3 G_1^\lambda & \to & \pi_3 \text{Diff}_0(T^2 \times S^2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_4 \text{Diff}_0(T^2 \times S^2) & \to & \pi_4 A_{1+t}^\lambda & \to & \pi_3 G_{1+t}^\lambda & \to & \pi_3 \text{Diff}_0(T^2 \times S^2) \\
\end{array}
\]

Take a value \( \lambda \) close to 1 and assume that \( \eta_\lambda \) maps to a nontrivial element in \( \pi_3 \text{Diff}_0(T^2 \times S^2) \). Then it follows from Proposition 5.2(iv) that \( \eta_\lambda \) has to lift to a nontrivial element in \( \pi_3 D_0 \). But from Proposition 5.2(ii) it follows that there should be an element \( \xi \) in \( G_1^1 \) with the same image in \( \pi_3 \text{Diff}_0(T^2 \times S^2) \) as \( \eta_\lambda \). Any extension \( \xi_{\lambda, \lambda} > 0 \) must have trivial PGW in class \( A - K \), from Corollary 4.6. From lemma 5.6 and the symplectic sum formula it follows that we have an element \( \eta_{\lambda} \) representing the class \( \xi_{\lambda} - \bar{\eta}_\lambda \) with nontrivial PGW in class \( A - F \) and whose image in \( \pi_3 \text{Diff}_0(T^2 \times S^2) \) is trivial.

Therefore \( \eta_{\lambda} \) must be the boundary of some cycle \( b_\lambda : S^4 \to A_1^\lambda \) and this extends continuously (since \( A_1 \subset A_{1+t} \)) to a family of maps \( b_{1+t} : S^4 \to A_{1+t} \).

The boundaries of the cycles \( b_{1+t} \) provide a continuous family of maps \( \eta_{1+t}^\lambda : S^3 \to G_{1+t}^1 \) for any \( t \geq 0 \), which have a nullhomotopic image in \( \pi_3 \text{Diff}_0(T^2 \times S^2) \) and hence it does not lift to \( \pi_3 D_0 \) for any \( t \geq 0 \). Since PGW are symplectic deformation invariants from Lemma 5.6(ii) it follows that all elements \( \eta_{\lambda} \) must give nontrivial elements in all \( \pi_3 G_{1+t} \); elements that become nullhomotopic in \( \pi_3 D_0 \).

But for \( t > 3/2 \) this contradicts Proposition 5.2(iv) which says that all nontrivial elements in \( \pi_3 G_{1+t} \) lift to nontrivial elements in \( \pi_3 D_0 \). Therefore the featured Whitehead product must be nontrivial. This concludes the proof of Proposition 5.8(i).

### 5.5 On the rational homotopy type of \( B\text{Symp}_0(S^2 \times S^2, \omega_\lambda) \)

The following theorem is proved in [2] and describes rational cohomology of \( G_\lambda^0 \) and implicitly the additive structure of \( \pi_* G_\lambda^0 \otimes \mathbb{Q} \):

**Theorem 5.7** (Abreu-McDuff) [2]

1. Let \( k < \lambda \leq k + 1 \) for some natural number \( k \geq 0 \). We have

\[
H^*(G_\lambda^0, \mathbb{Q}) = \Lambda(a, x, y) \otimes S(w_k)
\]

(33)

where \( \Lambda(a, x, y) \) is an exterior algebra with generators of degrees \( \deg a = 1, \deg x = \deg y = 3 \) and \( S(w_k) \) is a polynomial algebra with one generator of degree \( 4k \).

2. For \( k < \lambda \leq k + 1 \) a complete set of generators for \( \pi_* G_\lambda^0 \otimes \mathbb{Q} \), dual to \( a, x, y, w \) respectively, is given by \( \gamma_{\lambda,1} \in \pi_1 G_\lambda^0, \bar{\alpha} \) and \( \bar{\eta} \) in \( \pi_3 G_\lambda^0 \) and \( \bar{w}_k \) in \( \pi_{4k} G_\lambda^0 \).

Based on the additive structure provided in Theorem 5.7 we will give a new proof of 1.3. Basically, in order to find the multiplicative structure of the ring (1) one must understand all the rational Samelson products among elements in the homotopy groups \( \pi_* G_\lambda \otimes \mathbb{Q} \) that are dual to the given complete set of generators \( a, x, y, w \). Under suitable conditions, each nontrivial such
Samelson (hence Whitehead) product gives a relation in the ring $H^*(BG_\lambda, \mathbb{Q})$. Our contribution will be the following:

**Proposition 5.8** For all $k \geq 1$ and $k < \lambda \leq k + 1$ the Samelson product of order $2k + 1$, $S^{(2k+1)}(\gamma^0_{k+1}) = \{0, \bar{w}_k\} \subset \pi_{4k}(BG_{k+1})$ where $w_k$ is a nontrivial fragile element.

**Proof**: As stated in Proposition 5.1(ii) there exist maps $h_{k+1, k+1+\epsilon} : G_{k+1} \to G_{k+1+\epsilon}$ and hence one gets maps $h_{k+1, k+1+\epsilon}^t : BG_{k+1} \to BG_{k+1+\epsilon}$.

Part (iv) in the same proposition implies that this maps induce an isomorphism on $\pi_i BG_\lambda$ for $i \leq 4k$. In particular any map defined on a CW-complex $B$ of dimension less or equal to $4k$, $f : B \to BG_{k+1+\epsilon}$ must belong to a continuous family $f_\lambda : B \to BG_\lambda$.

Let us fix $\epsilon$ small and apply this conclusion to the map derived from (22):

$$f_{k+1+\epsilon, (2k)} : \mathbb{C}P^{2k} \to BH_{k+1} \subset BG_{k+1+\epsilon}$$  \hspace{1cm} (34)

and obtain a continuous family

$$f_{\lambda, (2k)} : \mathbb{C}P^{2k} \to BG_\lambda, k + 1 \leq \lambda \leq k + 1 + \epsilon$$  \hspace{1cm} (35)

which for $\lambda = k + 1 + \epsilon$ coincides with (34).

The maps $f_{\lambda, (i)} : S^2 \to BG_{k+1}, i = 1, \ldots, 2k$ give a family of towers of fibrations of length $2k$ as in (9), for some elements in $\pi_2 BG_\lambda$ that are homotopy equivalent to a multiple of $\tilde{E}((\gamma^0_{k+1})$. After multiplying with an appropriate high power $N$, we can assume that the family $\gamma^0_{k+1}$ satisfies the hypothesis of Proposition 3.1 when we set $p = 2k$.

Then we claim that the fibration $Q^{(p)}_{k+1} \to \mathbb{C}P^{2k}$ cannot extend to a $Q^{(p+1)}_{k+1} \to \mathbb{C}P^{p+1}$. Indeed if it were, then $Q^{(p+1)}_{k+1}$ admitted a deformation $Q^{(p+1)}_{\lambda+k+1}$ with respect to the parameter as in Proposition 3.1 and from (34) and (35) above, it would follow that there exist an appropriate symplectic fibration $R^{(p+1)}_{k+1+\epsilon} \to S^{4k+2}$ such that:

$$Q^{(p+1)}_{k+1+\epsilon} = E^{(p+1)}_{k+1+\epsilon} \# R^{(p+1)}_{k+1+\epsilon}$$  \hspace{1cm} (36)

for $E^{(p+1)}_{k+1+\epsilon}$ a $N$-covering of the associated fibration $Q^{(k+1),(p+1),0}_{k+1+\epsilon}$ defined in (28).

As argued in the $g = 1$ case, any invariants counting maps in class $A - (k+1)D$ on $Q^{(p+1)}_{k+1+\epsilon}$ must be zero and from Theorem 1.1 and the symplectic sum formula (proposition 4.3 (iii)) it follows that:

$$PGW^J_{1,0}(R^{(p+1)}_{\lambda+1}, s_{A-(k+1)D}) \neq 0, \lambda \geq k + 1 + \epsilon$$  \hspace{1cm} (37)

But this implies (again using Proposition 5.1(iv)), that there exist essential maps $a_\lambda : S^{4k+1} \to G_\lambda$, which is false for sufficiently large $\lambda$.

\[\square\]

**Remark 5.9** We have relied throughout the paper on the classical definition of the higher order Whitehead products because we made use of their obstruction theoretic properties. Allday’s [4] definition of rational Whitehead products in the graded differential Lie algebra $\pi_* BG_\lambda \otimes \mathbb{Q}$ is being used in showing some of the results in the following lemma, as they sometimes use homology rather than homotopy relations. These invariants are in one to one correspondence with the usual rational Whitehead products, and in this case with Samelson products in $\pi_* G_\lambda$.

The rational Whitehead products in $\pi_* BG_\lambda$ are multilinear. To ease the computations we will write $A = \tilde{E}(\gamma^0_{\lambda+1}) \in \pi_2 BG_\lambda \otimes \mathbb{Q}, \ Y = \tilde{E}(\alpha) \in \pi_4 BG_\lambda \otimes \mathbb{Q}, \ X = \tilde{E}(\eta) \in \pi_4 BG_\lambda \otimes \mathbb{Q}$ and $\tilde{W}_k = \tilde{E}(w_k) \in \pi_{4k+1} BG_\lambda \otimes \mathbb{Q}$.

**Lemma 5.10** For any $k \geq 1$ and $k < \lambda \leq k + 1$ we have
1. Any Whitehead product of order less than \( k + 1 \) is vanishing and also the following order \( k + 1 \) products vanish:

\[
[\bar{A}, \ldots, \bar{A}, X + \bar{Y}, \ldots, X + \bar{Y}] = [\bar{A}, \ldots, \bar{A}, X + 4\bar{Y}, \ldots, X + 4\bar{Y}] = \\
\ldots = [\bar{A}, \ldots, \bar{A}, X + k^2\bar{Y}, \ldots, X + k^2\bar{Y}] = 0
\]  

(38)

2. The following Whitehead product of order \( k + 1 \) is nontrivial and consists of only one element in \( \pi_{4k+1}BG_\Lambda \):

\[
0 \neq [\bar{A}, \bar{X}, \ldots, \bar{X}]
\]

(39)

**Proof:** Clearly \([\bar{A}, \bar{A}] = 0\). Considerations of the dimension of \( \pi_*BG_\Lambda \) imply that any other Whitehead products of order strictly less than \( k + 1 \) must also vanish. Therefore Proposition 2.7(a) implies that any Whitehead product of order \( k + 1 \) is defined and contains only one element. Since all the Lie subgroups \( K_i, i \leq k \) embed in \( G_\Lambda \), lemma 5.3(i) and (ii) yield part (i) of the present lemma. We use here the fact that the classifying space of a Lie group is an H-space and hence it has vanishing rational Whitehead products.

To prove the second part let us first notice that the indeterminacy in the Whitehead product \( W^{(2k+1)}(\bar{A}) \) obtained in Proposition 5.8 implies, according with Proposition 2.7(a), that nonvanishing lower order Whitehead products must exist. Again, from dimension considerations, it follows that they can only be of order \( p + s < 2k + 1, p > 0, s > 0 \) and \( 2p + 4s = 4k + 2 \):

\[
0 \neq [\bar{A}, \ldots, \bar{A}, a_1\bar{X} + b_1\bar{Y}, \ldots, a_s\bar{X} + b_s\bar{Y}]
\]

(40)

**Claim:** The minimum Whitehead order is \( k + 1 \).

**Proof of the claim:** Assume that the minimum Whitehead order is \( p + s > k + 1 \). Hence \( p > 1 \) and as above \( 2p + 4s = 4k + 2 \).

Consider the following equation in \( b \):

\[
0 = [\bar{A}, \ldots, \bar{A}, \bar{X} + b\bar{Y}, \ldots, \bar{X} + b\bar{Y}]
\]

(41)

This equation has degree \( s \) and coefficients in \( \pi_{4k+1}BG_\Lambda \otimes Q \) given by Whitehead products (containing only one element) of type \((p, s)\) that give a basis for all the possible Whitehead products of type \((p, s)\).

Moreover, Proposition 5.3 implies that the equation must have \( k \) solutions \( b = 1, 4, \ldots, k^2 \) provided by the \( k \) different Lie groups actions.

But \( k = \frac{2p + 4s - 2}{4} > s \) whenever \( p > 1 \) and hence all the coefficients must be zero in this case. Since these coefficients generate all Whitehead products of the given type \((p, s)\), it follows that \( p \) must be 1 and hence the minimum order of an existing nontrivial product of type \((p, s)\) must be \( k + 1 \).

\[\Box\]

Combined with part (ii) of our lemma this implies part (ii).

**Proof of Theorem 1.3:** The proof will now follow the same lines as the proof in [2]: One has to build the Sullivan minimal model for \( H^*(BG_\Lambda, Q) \) by giving a complete set of generators and relations.

As explained in Andrews-Arkovitz a complete set of generators for the Sullivan minimal model's differential algebra \( M \) of \( BG_\Lambda \) is given by elements in the dual homotopy groups \( \text{Hom}(\pi_*(BG_\Lambda^0 \otimes Q, Q)) \).

We therefore take a complete set of generators \( A \in M^2, X, Y \in M^4, \) and \( W_k \in M^{4k+1} \), the duals of the homotopy elements \( A, X, \bar{Y} \) and \( \bar{W}_k \). We need to understand the degree 1 differential \( d \) on \( M \).

Consider first the case \( 1 < \lambda \leq 2 \).

If we denote by \( M^0 \) the quotient of \( M \) by the elements of degree 0, then any complete set of generators on \( M \) induces a filtration \( M^0_s \) on \( M^0 \), with \( M^0_s \) being the subalgebra generated by products of \( s \) generators. In this case the Whitehead minimal order is \( r = 2 \).
According with [3, Proposition 6.4] for any $\mu \in \mathcal{M}$ we must have $d\mu \in \mathcal{M}_2^\ast$. This, and degree considerations immediately imply that $dA = dX = dY = 0$ (all elements in $\mathcal{M}^4$ are indecomposable) and these elements transgress to generators in $H^*(BG^\mathbb{Z}_2, \mathbb{Q})$.

Theorem 5.4 in [3] states that for any $\mu$ with $d\mu \in \mathcal{M}_2^\ast$, and $z \in [x_1, x_2, \ldots, x_s] \in \pi_\ast(BG^\mathbb{Z}_2) \otimes \mathbb{Q}$, the Sullivan pairing $(\mu, z)$ can be computed in terms of suitable coefficients coming from the corresponding universal Whitehead products. This ultimately allows one to write $d\mu$ as a relation between the generators that will give a relation in the cohomology ring.

All we need to find in this case is who will $dW$ be in this case. On one hand $dW \in \mathcal{M}_2^\ast$ and on the other hand it corresponds (via Sullivan’s pairing) to a (minimal order) Whitehead product of order 2. It follows that $dW$ must be equal to a homogeneous function $F_2$ of order two in the remaining variables $A, X, Y$. Exactly as explained in [2], one may think of it as a symmetric bilinear function on a vector space spanned over $\mathbb{Q}$ by the base dual to $A, X, Y$; namely, $\tilde{A}, \tilde{X}, \tilde{Y}$.

But from (38) and (39) if follows that $F_2(\tilde{A}, \tilde{X} + \tilde{Y}) = 0$ and $F_2(\tilde{A}, \tilde{X}) \neq 0$ and hence $F_2 = A(X - Y)$ (up to a multiple).

The situation is similar when $k < \lambda \leq k + 1$ for arbitrary $k$.

In this case the free graded differential algebra $\mathcal{M}$ has generators $A \in \mathcal{M}^2$, $X, Y \in \mathcal{M}^4$, and $W_k \in \mathcal{M}^{4k+1}$ and the minimal Whitehead order is $k + 1$. As before, if follows that any $d\mu \in \mathcal{M}_k^{\ast}$ and hence $DA = DX = DY = 0$.

In this situation $dW_k$ is in $\mathcal{M}_{k+1}^\ast$ and corresponds (via Sullivan’s pairing) to a (minimal order) Whitehead product of order $k + 1$ given by (39). Hence

$$dW = F_{k+1}(A, X, Y)$$

where $F_{k+1}$ is homogeneous of degree $k + 1$ and corresponds to a symmetric $(k + 1)$-linear function defined on a vector space spanned over $\mathbb{Q}$ by the basis $\tilde{A}, \tilde{X}, \tilde{Y}$. Moreover, from Lemma 5.10 we have that

$$F_{k+1}(\tilde{A}, \tilde{X} + i^2\tilde{Y}, \ldots, \tilde{X} + (i^2\tilde{Y}) = 0, \ i = 1, \ldots, k$$

and

$$F_{k+1}(\tilde{A}, \tilde{X}, \ldots, \tilde{X}) \neq 0$$

One can check that the multilinear function $F_{k+1} = A(X - Y)(X - 4Y) \ldots (X - k^2Y)$ satisfies this relations and is unique up to a constant.

5.6 Higher genus cases

Proof of Proposition 1.2(iii): The proof will be a slightly more elaborated version of the proof of Proposition 5.8.

Let us fix $\epsilon$ small and consider the obvious maps derived from (22):

$$f_{k+\epsilon,(p)} : \mathbb{C}P^p \to BH_k \subset BG^\mathbb{Z}_{k+\epsilon}$$

Claim: For $k > \lfloor g/2 \rfloor$ the map $f_{k+\epsilon,(g)}$ belongs to a continuous family

$$f_{l,(g)}^\mathbb{Z} : \mathbb{C}P^g \to BG^\mathbb{Z}_l, k \leq \lambda \leq k + \epsilon$$

The claim follows from Proposition 5.2(ii). Indeed since the maps $h_{l,(k+\epsilon)} : BG^\mathbb{Z}_l \to BG^\mathbb{Z}_{k+\epsilon}$ induce an isomorphism on $\pi_i, i = 1, \ldots, 2g$ for $\lfloor g/2 \rfloor < k \leq \lambda \leq k + \epsilon$, any map defined on a CW-complex of dimension less than $2g$ must extend to a continuous family as in (46). Moreover $\pi_1BG^\mathbb{Z}_l \otimes \mathbb{Q}$ is 1-dimensional and hence after passing to high powers (hence finite coverings of the induced fibration) we may assume that the family of towers of fibrations $Q^{(p)}$ of length $p$ given by $f_{l,(p)}^\mathbb{Z}$, and their restrictions to the lower skeletons, gives a choice of tower for the Whitehead products of $\tilde{E}(\gamma^q_{\lambda,k}), k \leq \lambda \leq k + \epsilon$. 

Assumption A1 Let us assume that all the Whitehead products $W^{(r)}(\tilde{E}(\gamma_{k,k})) \in \pi_{2r-1}(BG^g_k)$ of order $g \leq r \leq 2k + g - 1$ vanish.

Then the tower $Q^{(g)}_k$ of length $g$ obtained at level $\lambda = k$, must extend to a tower $Q^{(2k+g-1)}_k$ of length $2k + g - 1$ as in Proposition 3.1 point (b). Furthermore, the tower extends again with respect to the parameter and we obtain once more families of fibrations

$$Q^{(r)}_\lambda$$

for $1 \leq r \leq 2k + g - 1$ and $k \leq \lambda \leq k + \epsilon$ (47)

As before, Denote by $E^{(2k+g-1)}_{k+\epsilon}$ a $N$-covering of the fibration (28) arising from the circle action $H_k$.

The construction above implies that two fibration $Q^{(2k+g-1)}_{k+\epsilon} \to CP^{2k+g-1}$ and $E^{(2k+g-1)}_{k+\epsilon} \to CP^{2k+g-1}$ agree over the $2g$ skeleton. We should point out that the corresponding restriction to the $2g$ skeletons are just the fibrations $Q^{(g)}_k \to CP^g$ and $E^{(g)}_k$ in the towers. Under the vanishing assumption A1 we have the following:

Lemma 5.11 There exist fiberwise symplectic deformations $Q^{(2k+g-1)}_\lambda \to CP^{2k+g-1}$ and $E^{(2k+g-1)}_{k+\epsilon} \to CP^{2k+g-1}$ such that, for a value $\lambda = a$ sufficiently large, the two corresponding fibrations are symplectically isotopic.

Proof: Let us remind the reader that $\pi_1 BD_0^g$ is trivial if $i > 4$. That will definitely be the case when $i > 2g \geq 4$.

Firstly, let us notice that since they agree (isotopic would be enough) on the the $2g$ skeleton we have

$$Q^{(g+1)}_{k+\epsilon} = E^{(g+1)}_{k+\epsilon} \# R^{(g+1)}_{k+\epsilon}$$

where $R^{g+1}_{k+i} \to S^{2g+1}$ is a symplectic fibration that corresponds to an element $\eta_{k+i}$ in $\pi_2 g + 1 BG^g_{k+i}$. Since (since $2g + 1 > 4$) the image of $\eta_{k+i}$ in $\pi_1 BD_0^g$ vanishes and hence so does its image in $\pi_2 g + 1 BD_0^g$. Therefore $\eta_{k+i}$ hence it must be the image of an element $\beta_{k+i} \in \pi_2 g + 2 A^g_0 + \epsilon$, in the following diagram:

Consider now the long exact sequences in homotopy:

$$\pi_{2g+2} Diff_0(S^g \times S^2) \to \pi_{2g+2} A^g_0 \to \pi_{2g+1} G^g_0 \to \pi_{2g+1} Diff_0(S^g \times S^2)$$

$$\pi_{2g+2} Diff_0(S^g \times S^2) \to \pi_{2g+2} A^g_0 \to \pi_{2g+1} G^g_0 \to \pi_{2g+1} Diff_0(S^g \times S^2)$$

There clearly is a continuous family $\beta_\lambda : S^{2g+2} \to A^g_0$ whose boundaries give a continuous extension $\eta_\lambda : S^{2g+1} \to BG^g_0$, $\lambda \geq k + \epsilon$. Assume that the latter family is essential in the homotopy of $BG^g_0$ for all values of $\lambda$. Then $\beta_\lambda \in \pi_2 g + 2 A^g_0$ must be nontrivial for all $\lambda$, and it immediately follows that it must also give essential element $\beta_\infty$ in $\pi_2 g + 2 A^g_0$, where $A^g_0 = \cup_{\lambda > 0} A^g_\lambda$. We now look at the homotopy fibration:

$$G^g_\infty \to Diff_0(S^g \times S^2) \to A^g_\lambda$$

In the long exact sequence in homotopy for this fibration the element $\beta_\infty$ does not lift to $\pi_{2g+2} Diff_0(S^g \times S^2)$. This is because none of the elements $\beta^{2g+1}_\lambda$ lift to $\pi_{2g+2} Diff_0(S^g \times S^2)$. Therefore $\beta_\infty$ must have a nontrivial boundary and give a nontrivial element $\eta_\infty \in \pi_2 g + 1 G^g_\infty$ which is impossible from Proposition 5.2 part (i).

Therefore there exist a value $\lambda_0$ where $\beta_{\lambda_0}^{2g+1}$ becomes inessential in $BG^g_{\lambda_0}$. Using the continuous family $\beta_\lambda$ we obtain two fiberwise symplectic deformations $Q^{(2g+1)}_\lambda$ and $E^{(2g+1)}_\lambda$, $\lambda \geq k + \epsilon$, that are isotopic at $\lambda = \lambda_0$. On them we build two deformations $Q^{(2k+g-1)}_\lambda$ and $E^{(2k+g-1)}_\lambda$, $\lambda \geq k + \epsilon$, whose restrictions to the $2g + 1$ skeleton are isotopic at $\lambda = \lambda_0$. 
We repeat this process $2k$ more steps, in each of which we obtain fibrations that agree on a skeleton of dimension two bigger. In the end we get a large value $\lambda = \alpha$ and two fiberwise deformations $Q^\lambda_{(2k+g-1)}$ and $E^\lambda_{(2k+g-1)}$, $k + \epsilon \leq \lambda \leq \alpha$ such that the following isotopy holds

$$Q^\lambda_{(2k+g-1)} \approx E^\lambda_{(2k+g-1)} \quad (49)$$

From here the result is straightforward: on one hand $Q^\lambda_{(2k+g-1)}$ must have trivial PGW in class $A - kF$ due to corollary 4.6 and on the other hand $E^\lambda_{(2k+g-1)}$ must have a nontrivial PGW in class $A - kF$ from Theorem 1.1 and invariance under deformation.

But this is impossible, therefore the assumption A1 must be false.

**Remark 5.12**

1. It is very likely that the proposition 1.2 (iii) can be strengthened to a statement about finding the nontrivial elements in the Samelson products $S^{(2k+g-1)}(\gamma^g_{k,k}) \in \pi_{4k+2g}{G_k^g}$, as conjectured in [13]. For that, we need to extend the gluing argument from [13] page 20 to show that the maps

$$f_{k+\epsilon,(2k+g-2)}: \mathbb{C}P^{2k+g-2} \to BH_k \subset BG_k^g$$

extend to continuous families as in (46).

2. The lemma 5.11 proves in fact that any two fibrations that agree on a the 4 skeletons deform into isotopic fibrations for large $\lambda$.

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