CONJUGATE GENERATORS OF KNOT AND LINK GROUPS

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ABSTRACT. This note shows that if two elements of equal trace (e.g., conjugate elements) generate an arithmetic two-bridge knot or link group, then the elements are parabolic. This includes the figure-eight knot and Whitehead link groups. Similarly, if two conjugate elements generate the trefoil knot group, then the elements are peripheral.

1. INTRODUCTION

By a knot or link group, we will mean the fundamental group of the knot or link complement in $S^3$. It is well known that a two-bridge knot or link group is generated by two meridians of the knot or link (see [5]). The converse is also known; it is proved for link complements in $S^3$ in [4] (Corollary 3.3), for hyperbolic 3-manifolds of finite volume in [1] (Theorem 4.3), and for the most general class of 3-manifolds in [3] (Corollary 5):

**Theorem 1.1.** If $M$ is a compact, orientable, irreducible 3-manifold with incompressible boundary and $\pi_1M$ is generated by two peripheral elements, then $M$ is homeomorphic to the exterior of a two-bridge knot or link in $S^3$.

Arising from work on Simon’s Conjecture (see Section 6 for statement and Problem 1.12 of [11]), A. Reid and others proposed the following conjecture, which for convenience we will call:

**Reid’s Conjecture.** Let $K$ be a knot for which $\pi_1(S^3 \setminus K)$ is generated by two conjugate elements. Then the elements are peripheral (and hence the knot is two-bridge by above).

A knot in $S^3$ is a hyperbolic, satellite, or torus knot (Corollary 2.5 of [18]). By Proposition 17 of [8], the $(p, q)$-torus knot group can be generated by two conjugate elements only when $p = 2$, i.e., when the torus knot is two-bridge. In Section 5 we establish Reid’s Conjecture for the $(2, 3)$-torus knot group (i.e., the trefoil knot group), and we prove in Section 6 that Reid’s Conjecture implies Simon’s Conjecture for two-bridge knots. The latter conjecture has recently been proved in [2].

When a knot or link complement in $S^3$ is hyperbolic, peripheral elements such as meridians map to parabolic elements under the discrete faithful representation of the knot or link group into $\text{PSL}_2\mathbb{C}$. Conjugate elements have equal trace, and we prove in Section 3 a stronger version of Reid’s Conjecture for the figure-eight knot (whose complement in $S^3$ is well known to be hyperbolic by [10]):

**Theorem 1.2.** If two elements of equal trace generate the figure-eight knot group, then the elements are parabolic.

The figure-eight knot group can, however, be generated by three conjugate loxodromic elements, so this result is, in some sense, sharp.
The proof of Theorem 1.2 relies heavily on the arithmeticity of the figure-eight knot complement in $S^3$. Since the figure-eight knot is the only knot with arithmetic hyperbolic complement in $S^3$ (13), extending our result to all hyperbolic knot groups would require new techniques. By Section 5 of [6], however, there are exactly four arithmetic Kleinian groups generated by two parabolic elements; each is the fundamental group of a hyperbolic two-bridge knot or link complement in $S^3$: the figure-eight knot, the Whitehead link, and the links $6_2^2$ and $6_3^3$. In Section 4 we again exploit arithmeticity to extend our result to these:

**Theorem 1.3.** If two elements of equal trace generate an arithmetic two-bridge knot or link group, then the elements are parabolic.

### 2. Preliminaries

We collect several preliminary results that will be useful in what follows. The first is Lemma 7.1, together with the comments and definitions that precede it, in [7].

**Lemma 2.1.** Let $\Gamma$ be a finite-covolume Kleinian group whose traces lie in $R$, the ring of integers in $\mathbb{Q}(\text{tr}\Gamma)$. If $\langle X, Y \rangle$ is a non-elementary subgroup of $\Gamma$, then $O = R[1, X, Y, XY]$ is an order in the quaternion algebra

$$A\Gamma = \left\{ \sum a_i \gamma_i : a_i \in \mathbb{Q}(\text{tr}\Gamma), \gamma_i \in \Gamma \right\}$$

over $\mathbb{Q}(\text{tr}\Gamma)$. Its discriminant $d(O)$ is the ideal $\langle 2 - \text{tr}[X, Y] \rangle$ in $R$.

The second is Theorem 6.3.4 in [12].

**Theorem 2.2.** Let $O_1$ and $O_2$ be orders in a quaternion algebra over a number field. If $O_1 \subset O_2$, then $d(O_2) | d(O_1)$, and $O_1 = O_2$ if and only if $d(O_1) = d(O_2)$.

Our application is the following.

**Corollary 2.3.** Let $\Gamma$ be a finite-covolume Kleinian group whose traces lie in $R$, the ring of integers in $\mathbb{Q}(\text{tr}\Gamma)$. If $\langle A, B \rangle = \Gamma = \langle X, Y \rangle$, then $2 - \text{tr}[X, Y]$ is a unit multiple of $2 - \text{tr}[A, B]$ in $R$.

*Proof.* By Lemma 2.1, $O_1 = R[1, A, B, AB]$ and $O_2 = R[1, X, Y, XY]$ are orders in $A\Gamma$. Furthermore, $d(O_1) = \langle 2 - \text{tr}[A, B] \rangle$ and $d(O_2) = \langle 2 - \text{tr}[X, Y] \rangle$ are ideals in $R$. The Cayley-Hamilton Theorem yields the identity

$$X + X^{-1} = \text{tr}X \cdot 1$$

which implies $A^{-1}, B^{-1} \in O_1$ and $X^{-1}, Y^{-1} \in O_2$. Since $O_1$ and $O_2$ are ideals that are also rings with 1, we have

$$\Gamma = \langle A, B \rangle \subset O_1 \text{ and } \Gamma = \langle X, Y \rangle \subset O_2.$$

Clearly, $R \subset O_1, O_2$, so $R\Gamma \subset O_1, O_2$, and $O_1, O_2 \subset R\Gamma$ by definition. Therefore,

$$O_1 = R\Gamma = O_2.$$

Hence, $d(O_1) = d(O_2)$ by Theorem 2.2 and the result follows. □

We conclude with two technical lemmas.
Lemma 2.4. Let $\Gamma$ be a Kleinian group whose traces lie in $R$, the ring of integers in $\mathbb{Q}(\text{tr} \Gamma)$, and $X, Y \in \Gamma$ with $\text{tr} X = \text{tr} Y$. If 

$$x = \text{tr} X = \text{tr} Y, \ y = \text{tr} XY - 2, \text{ and } z = 2 - \text{tr} [X, Y],$$

then $y \mid z$ in $R$, and 

$$x^2 = \frac{z}{y} + y + 4.$$ 

Proof. Standard trace relations (e.g., relation 3.15 in Section 3.4 of [12]) yield 

$$z = 4 + x^2 \text{tr} XY - \text{tr}^2 XY - 2x^2$$ 

$$= (\text{tr} XY - 2)x^2 - (\text{tr} XY - 2)(\text{tr} XY + 2)$$ 

$$= y(x^2 - (y + 4))$$ 

The result now follows. 

For the remainder of this note, let $\omega = e^{\pi i/3}$.

Lemma 2.5. Let $x = a + b\omega \in \mathbb{Z}[\omega]$, and 

$$x^2 = a^2 - b^2 + (2ab + b^2)\omega = n + m\omega \in \mathbb{Z}[\omega].$$ 

If $-4 \leq m \leq 4$, then the following are the only possibilities for $x^2$.

- If $m = 0$, then $x^2 = a^2 \in \mathbb{Z}^2$ or $x^2 \leq 0$.
- If $m = \pm 1$, then $x^2 = -1 + \omega$ or $x^2 = -\omega$.
- If $m = \pm 2$, then $x^2 \not\in \mathbb{Z}[\omega]$, i.e., $m = \pm 2$ is not possible.
- If $m = \pm 3$, then $x^2 \in \{ 3\omega, -8 + 3\omega, 3 - 3\omega, -5 - 3\omega \}$.
- If $m = \pm 4$, then $x^2 = -4 + 4\omega$ or $x^2 = -4\omega$.

Proof. For each case, we have the following.

- If $m = 2ab + b^2 = 0$, then $b = 0$ or $a = -\frac{b}{2}$, so $x^2 = a^2 \in \mathbb{Z}^2$ or 

$$x^2 = -\frac{3b^2}{4} \leq 0.$$ 

- If $m = 2ab + b^2 = \pm 1$, then $a = \frac{1 - b^2}{2b}$, so $b \mid 1$. Thus, $b = \pm 1$, and 

$$(a, b) \in \{ (0, \pm 1), \pm (1, -1) \}.$$ 

Therefore, $x^2 = a^2 - b^2 + (2ab + b^2)\omega = -1 + \omega$ or $x^2 = -\omega$.

- If $m = 2ab + b^2 = \pm 2$, then $a = \frac{1 - b^2}{2b}$, so $b \mid 2$ and $2 \mid b$. Thus, $b = \pm 2$, and 

$$(a, b) \in \left\{ \pm \left( \frac{1}{2}, -2 \right), \pm \left( \frac{3}{2}, -2 \right) \right\}.$$ 

Therefore, $x^2 = a^2 - b^2 + (2ab + b^2)\omega \not\in \mathbb{Z}[\omega]$.

- If $m = 2ab + b^2 = \pm 3$, then $a = \frac{1 - b^2}{2b}$, so $b \mid 3$. Thus, $b \in \{ \pm 1, \pm 3 \}$, and 

$$(a, b) \in \{ \pm (1, 1), \pm (1, -3), \pm (2, -1), \pm (2, -3) \}.$$ 

Therefore, $x^2 = a^2 - b^2 + (2ab + b^2)\omega \in \{ 3\omega, -8 + 3\omega, 3 - 3\omega, -5 - 3\omega \}.$
• If \( m = 2ab + b^2 = \pm 4 \), then \( a = \frac{\pm 4 - b^2}{2b} \), so \( b \mid 4 \) and \( 2 \mid b \). Thus, \( b \in \{\pm 2, \pm 4\} \), and

\[
(a, b) \in \left\{ (0, \pm 2), (2, -2), \pm \left( \frac{3}{2}, -4 \right), \pm \left( \frac{5}{2}, -4 \right) \right\}.
\]

Of these, the only values of \( x^2 = a^2 - b^2 + (2ab + b^2)\omega = \omega \in \mathbb{Z}[\omega] \) are \( x^2 = -4 + 4\omega \) and \( x^2 = -4\omega \).

### 3. The Figure-Eight Knot

The discrete faithful representation of the figure-eight knot group

\[
\pi_1(S^3 \setminus K) = \langle a, b \mid a^{-1}bab^{-1}a = ba^{-1}bab^{-1} \rangle
\]

into \( \text{PSL}_2\mathbb{C} \) is generated by

\[
a \mapsto A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b \mapsto B = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}
\]

(see [10]). Note our mild abuse of notation which will be continued tacitly throughout: we blur the distinction between elements of \( \text{PSL}_2\mathbb{C} \) and their lifts to \( SL_2\mathbb{C} \).

Then \( \Gamma = \langle A, B \rangle \) is a finite-covolume Kleinian subgroup of \( \text{PSL}_2(\mathbb{Z}[\omega]) \), so \( \text{tr} \Gamma \subset \mathbb{Z}[\omega] \), the ring of integers in the trace field \( \mathbb{Q}(\omega) \). The invariant quaternion algebra is \( M_2(\mathbb{Q}(\omega)) \). Theorem \ref{thm:main} follows immediately from:

**Theorem 3.1.** If \( \Gamma = \langle X, Y \rangle \) with \( \text{tr}X = \text{tr}Y \), then \( X \) and \( Y \) are parabolic.

**Proof.** By Corollary \ref{cor:main} \( 2 - \text{tr}[X, Y] \) is a unit multiple of \( 2 - \text{tr}[A, B] = -\omega^2 \) in \( \mathbb{Z}[\omega] \). The complete group of units in \( \mathbb{Z}[\omega] \) is given by

\[
(\mathbb{Z}[\omega])^* = \{ 1, \omega, \omega^2 = \omega - 1, \omega^3 = -1, \omega^4 = -\omega, \omega^5 = 1 - \omega \}.
\]

Since \( \Gamma \) is torsion-free, \( \text{tr}[X, Y] \notin \{ -2, 2 \} \), so

\[
2 - \text{tr}[X, Y] = \omega^n \quad \text{for some} \quad n \in \{1, 2, 3, 4, 5\}.
\]

Let

\[
x = \text{tr}X = \text{tr}Y = a + b\omega \in \mathbb{Z}[\omega], \quad y = \text{tr}XY - 2, \quad \text{and} \quad z = 2 - \text{tr}[X, Y].
\]

Lemma \ref{lem:main} implies \( y \mid z \) in \( \mathbb{Z}[\omega] \), so \( y \) is also a unit in \( \mathbb{Z}[\omega] \). Since \( \text{tr}XY \notin \{ -2, 2 \} \),

\[
y = \omega^m \quad \text{for some} \quad m \in \{0, 1, 2, 4, 5\}.
\]

Varying \( m \) and \( n \) as above generates the following table of values for

\[
x^2 = \frac{z}{y} + y + 4 = \omega^{n-m} + \omega^m + 4.
\]

| \( n \) | \( m = 0 \) | \( m = 1 \) | \( m = 2 \) | \( m = 4 \) | \( m = 5 \) |
|---|---|---|---|---|
| 1 | 5 + \omega | 5 + \omega | 4 | 3 - \omega | 4 |
| 2 | 4 + \omega | 4 + 2\omega | 4 + \omega | 4 - 2\omega | 4 - \omega |
| 3 | 4 | 3 + 2\omega | 3 + 2\omega | 5 - 2\omega | 5 - 2\omega |
| 4 | 5 - \omega | 3 + \omega | 2 + 2\omega | 5 - \omega | 6 - 2\omega |
| 5 | 6 - \omega | 4 | 2 + \omega | 4 | 6 - \omega |

Of these, by Lemma \ref{lem:main} the only possible value for \( x^2 \) is 4, i.e., \( X \) and \( Y \) must be parabolic if they are to generate \( \Gamma \) and have equal trace. \qed
Varying

Let $Z$ Since (By Lemma 2.4, 2

Proof. $X$ and $Y$ are parabolic.

Remark 3.2. The figure-eight knot group can be generated by three conjugate loxodromic elements.

Proof. Let $\alpha = a^{-1}b^2$, $\beta = bab^{-1} = ba^{-1}b$, and $\gamma = b^{-1}ab = b^{-1}a^{-1}b^3$. Then

$$
\beta^{-1}\alpha\gamma^{-1}\alpha\beta^{-1}\alpha^2 = b^{-1}ab^{-1}a^{-1}b^2b^{-3}aba^{-1}b^2b^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2 = b^{-1}ab^{-1}a^{-1}b^{-1}a(ba^{-1}bab^{-1})a^{-1}b^2a^{-1}b^2 = b
$$

which implies $b \in \langle \alpha, \beta, \gamma \rangle$, so $b^2\alpha^{-1} = a \in \langle \alpha, \beta, \gamma \rangle$. Hence, $\langle \alpha, \beta, \gamma \rangle = \pi_1(S^3 \setminus K)$, and

$$
\alpha \mapsto \begin{pmatrix} 1 - 2\omega & -1 \\ 2\omega & 1 \end{pmatrix}, \beta \mapsto \begin{pmatrix} 1 - \omega & -1 \\ 1 + \omega & 1 - \omega \end{pmatrix}, \text{ and } \gamma \mapsto \begin{pmatrix} 1 - 3\omega & -1 \\ -3 + 5\omega & 1 + \omega \end{pmatrix}.
$$

Thus, the figure-eight knot group is generated three conjugate loxodromic elements.

4. The Arithmetic Two-Bridge Links

For the remainder of this note, let $\theta = \frac{1 + \sqrt{7}}{2}$. The discrete faithful representation of an arithmetic two-bridge link group into $\text{PSL}_2 \mathbb{C}$ is generated by the matrices

$$
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}
$$

where $\xi = 1 + i, 1 + \omega$, and $\theta$ for the Whitehead link and the links $62^2$ and $63^3$ respectively (see Section 5 of [8]). Then $\Gamma = \langle A, B \rangle$ is a finite-covolume Kleinian group whose traces lie in $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, and $\mathbb{Z}[\theta]$ respectively. These are the rings of integers in the respective trace fields $\mathbb{Q}(i)$, $\mathbb{Q}(\omega)$, and $\mathbb{Q}(\theta)$. The respective invariant quaternion algebras are $M_2(\mathbb{Q}(i))$, $M_2(\mathbb{Q}(\omega))$, and $M_2(\mathbb{Q}(\theta))$. We now establish Theorem 1.3 for each arithmetic two-bridge link group.

Theorem 4.1. Let $\Gamma$ be the discrete faithful representation in $\text{PSL}_2(\mathbb{Z}[i])$ of the Whitehead link group with notation as above. If $\Gamma = \langle X, Y \rangle$ with $\text{tr}X = \text{tr}Y$, then $X$ and $Y$ are parabolic.

Proof. By Corollary 2.3, $2 - \text{tr}[X, Y]$ is a unit multiple of $2 - \text{tr}[A, B] = -2i$ in $\mathbb{Z}[i]$. Since $(\mathbb{Z}[i])^* = \{ \pm 1, \pm i \}$ and $\text{tr}[X, Y] \not\in (-2, 2),

$$
2 - \text{tr}[X, Y] \in \{-2, \pm 2i\}.
$$

Let

$$
x = \text{tr}X = \text{tr}Y = a + bi \in \mathbb{Z}[i], \quad y = \text{tr}XY - 2, \text{ and } z = 2 - \text{tr}[X, Y].
$$

By Lemma 2.3, $y \mid z$ in $\mathbb{Z}[i]$, so, since $\text{tr}XY \not\in (-2, 2)$, we have

$$
y \in \{ 1, \pm i, \pm (1 + i), \pm (1 - i), 2, \pm 2i \}.
$$

Varying $y$ and $z$ as above generates the following table of possible values for

$$
x^2 = \frac{z}{y} + y + 4.
$$
We now check which of these values have the form $x^2 = a^2 - b^2 + 2abi \in \mathbb{Z}[i]$ based on $0$, $\pm 1$, $\pm 2$, and $\pm 3$ being the only imaginary parts that arise in the table.

**Case 1:** The imaginary part of $x^2$ is $0$; that is,

\[
2ab = 0 \implies a = 0 \text{ or } b = 0
\]

\[
x^2 = -b^2 \text{ or } x^2 = a^2 \in \mathbb{Z}^2
\]

The only value in the table with imaginary part $0$ that can be expressed in either of these forms is $x^2 = 4$, i.e., $X$ and $Y$ are parabolic.

**Case 2:** The imaginary part of $x^2$ is $\pm 1$; that is, $2ab = \pm 1$, which is impossible for $a, b \in \mathbb{Z}$. Hence, $x^2$ cannot have imaginary part $\pm 1$.

**Case 3:** The imaginary part of $x^2$ is $\pm 2$; that is,

\[
2ab = \pm 2 \implies a^2 = b^2 = 1
\]

\[
x^2 = \pm 2i
\]

But $\pm 2i$ does not appear in the table, so $x^2$ cannot have imaginary part $\pm 2$.

**Case 4:** The imaginary part of $x^2$ is $\pm 3$; that is, $2ab = \pm 3$, which is impossible for $a, b \in \mathbb{Z}$. Hence, $x^2$ cannot have imaginary part $\pm 3$.

This exhausts the table of possible values for $x^2$; therefore, $X$ and $Y$ must be parabolic if they are to generate $\Gamma$ and have equal trace.

**Theorem 4.2.** Let $\Gamma$ be the discrete faithful representation in $\text{PSL}_2(\mathbb{Z}[\omega])$ of the $6_2^2$ link group with notation as before. If $\Gamma = \langle X, Y \rangle$ with $\text{tr}X = \text{tr}Y$, then $X$ and $Y$ are parabolic.

**Proof.** By Corollary 2.3, $2 - \text{tr}[X, Y]$ is a unit multiple of $2 - \text{tr}[A, B] = -3\omega$ in $\mathbb{Z}[\omega]$. Since $\text{tr}[X, Y] \not\in (-2, 2)$, we have

\[
2 - \text{tr}[X, Y] \in \{\pm 3\omega, \pm 3(1 - \omega), -3\} \subset 3(\mathbb{Z}[\omega])^*.
\]

Let

\[
x = \text{tr}X = \text{tr}Y = a + b\omega \in \mathbb{Z}[\omega], \ y = \text{tr}XY - 2, \text{ and } z = 2 - \text{tr}[X, Y].
\]

Lemma 2.4 implies $y \mid z$, and hence $y \mid 3$, in $\mathbb{Z}[\omega]$. Therefore, since $\text{tr}XY \not\in (-2, 2)$, we have

\[
y \in \{\pm(3 - 3\omega), \pm(2 - \omega), \pm(1 + \omega), \pm(1 - \omega), \pm(1 - 2\omega), \pm\omega, \pm 3\omega, 1, 3\}.
\]
Varying $y$ and $z$ as above generates the following table of values for $x^2 = \frac{z}{y} + 4$.

| $y$       | $z$       | $y$       | $z$       | $y$       | $z$       |
|-----------|-----------|-----------|-----------|-----------|-----------|
| $3 - 3\omega$ | $6 - 2\omega$ | $8 - 4\omega$ | $8 - 3\omega$ | $6 - 3\omega$ | $7 - 4\omega$ |
| $-3 + 3\omega$ | $2 + 2\omega$ | $4\omega$ | $3\omega$ | $2 + 3\omega$ | $1 + 4\omega$ |
| $2 - \omega$ | $5 + \omega$ | $7 - 3\omega$ | $8 - 2\omega$ | $4$ | $5 - 2\omega$ |
| $-2 + \omega$ | $3 - \omega$ | $1 + 3\omega$ | $2\omega$ | $4$ | $3 + 2\omega$ |
| $1 + \omega$ | $6 + 2\omega$ | $4$ | $6 - \omega$ | $4 + 3\omega$ | $3 + 2\omega$ |
| $-1 - \omega$ | $2 - 2\omega$ | $4$ | $2 + \omega$ | $4 - 3\omega$ | $5 - 2\omega$ |
| $1 - \omega$ | $2 + 2\omega$ | $8 - 4\omega$ | $8 - \omega$ | $2 - \omega$ | $5 - 4\omega$ |
| $-1 + \omega$ | $6 - 2\omega$ | $4\omega$ | $6 + \omega$ | $3 + 4\omega$ |
| $1 - 2\omega$ | $3 - \omega$ | $7 - 3\omega$ | $6 - \omega$ | $4 - 3\omega$ | $6 - 4\omega$ |
| $-1 + 2\omega$ | $5 + \omega$ | $1 + 3\omega$ | $2 + \omega$ | $4 + 3\omega$ | $2 + 4\omega$ |
| $\omega$ | $7 + \omega$ | $1 + \omega$ | $4 - 2\omega$ | $4 + 4\omega$ | $1 + 4\omega$ |
| $-\omega$ | $1 - \omega$ | $7 - \omega$ | $4 + 2\omega$ | $4 - 4\omega$ | $7 - 4\omega$ |
| $3\omega$ | $5 + 3\omega$ | $3 + 3\omega$ | $4 + 2\omega$ | $4 + 4\omega$ | $3 + 4\omega$ |
| $-3\omega$ | $3 - 3\omega$ | $5 - 3\omega$ | $4 - 2\omega$ | $4 - 4\omega$ | $5 - 4\omega$ |
| $1$ | $5 + 3\omega$ | $5 - 3\omega$ | $8 - 3\omega$ | $2 + 3\omega$ | $2$ |
| $3$ | $7 + \omega$ | $7 - \omega$ | $8 - \omega$ | $6 + \omega$ | $6$ |

Of these, by Lemma 2.5, the only possible values for $x^2$ are $4, 3\omega,$ and $3 - 3\omega$. If $x^2 = \text{tr}^2X = \text{tr}^2Y = 3\omega$, then $\text{tr}X = \text{tr}Y = \pm(1 + \omega)$, so the axes of $X$ and $Y$ project closed geodesics in $\mathbb{H}^3/\Gamma = S^3 \setminus 6\ell_2^2$ of length

$$\text{Re} \left( 2 \cosh^{-1} \left( \pm \frac{1 + \omega}{2} \right) \right) \approx 1.087070145.$$

Similarly, if $x^2 = \text{tr}^2X = \text{tr}^2Y = 3 - 3\omega$, then $\text{tr}X = \text{tr}Y = \pm(2 - \omega)$, so the axes of $X$ and $Y$ project closed geodesics in $\mathbb{H}^3/\Gamma = S^3 \setminus 6\ell_2^2$ of length

$$\text{Re} \left( 2 \cosh^{-1} \left( \pm \frac{2 - \omega}{2} \right) \right) \approx 1.087070145.$$

But rigorous computation of the length spectrum in SnapPea ([19], [10]) shows that the shortest two closed geodesics in $S^3 \setminus 6\ell_2^2$ have length

0.86255462766206 and 1.66288589105862.

Thus, the only possible value for $x^2$ is 4, i.e., $X$ and $Y$ must be parabolic if they are to generate $\Gamma$ and have equal trace. ■

**Theorem 4.3.** Let $\Gamma$ be the discrete faithful representation in $\text{PSL}_2(\mathbb{Z}[\theta])$ of the $6\ell_2^3$ link group with notation as before. If $\Gamma = \langle X, Y \rangle$ with $\text{tr}X = \text{tr}Y$, then $X$ and $Y$ are parabolic.
Proof. By Corollary 2.3, $2 - \text{tr}[X, Y]$ is a unit multiple of $2 - \text{tr}[A, B] = -\theta^2 = 2 - \theta$ in $\mathbb{Z}[\theta]$, i.e., $2 - \text{tr}[X, Y] = \pm(2 - \theta)$. Let

$$x = \text{tr}X = \text{tr}Y = a + b\theta \in \mathbb{Z}[\theta], \quad y = \text{tr}XY - 2,$$

and $z = 2 - \text{tr}[X, Y]$. Lemma 2.4 implies $y | z$ in $\mathbb{Z}[\theta]$. Since $\text{tr}XY \notin (-2, 2)$, we have

$$y \in \{1, \pm(2 - \theta), \pm \theta\}.$$

Therefore,

$$x^2 = \frac{z}{y} + y + 4 \in \{4, 1 + \theta, 3 + \theta, 4 \pm 2\theta, 5 - \theta, 7 - \theta\}.$$

Of these, arguing as before, the only possible value for $x^2 = a^2 - 2b^2 + (2ab + b^2)\theta \in \mathbb{Z}[\theta]$ is 4, i.e., $X$ and $Y$ must be parabolic if they are to generate $\Gamma$ and have equal trace.

\[\square\]

5. Torus Knots

Throughout this section, we assume $\gcd(p, q) = 1$ and $2 \leq p < q$. As is well known, the $(p, q)$-torus knot group admits the presentation

$$\pi_1(S^3 \setminus K_{p,q}) = \langle c, d \mid c^p = d^q \rangle,$$

which clearly surjects $\mathbb{Z}_p \ast \mathbb{Z}_q = \langle s, t \mid s^p = t^q = 1 \rangle$ via $c \mapsto s$, $d \mapsto t$ (see [5]).

We begin our investigation of conjugate generators for torus knot groups by paraphrasing Proposition 17 of [8]:

**Proposition 5.1.** If $\mathbb{Z}_p \ast \mathbb{Z}_q = \langle s, t \mid s^p = t^q = 1 \rangle$ can be generated by two conjugate elements, then $p = 2$.

Thus, via the surjection above, the $(p, q)$-torus knot group can be generated by two conjugate elements only when $p = 2$, i.e., when the torus knot is two-bridge with normal form $(q/1)$. The $(2, q)$-torus knot group $\langle c, d \mid c^2 = d^q \rangle$ has a parabolic representation into $\text{PSL}_2\mathbb{C}$ generated by

$$c \mapsto C = \begin{pmatrix} 0 & (2\cos(\pi/q))^{-1} \\ -2\cos(\pi/q) & 1 \end{pmatrix}, \quad d \mapsto D = \begin{pmatrix} 1 - 4\cos^2(\pi/q) & 1 \\ -4\cos^2(\pi/q) & 1 \end{pmatrix}$$

(Theorem 6 of [15]). Then $\Gamma_q = \langle C, D \rangle$ is a finite-coarea subgroup of $\text{PSL}_2\mathbb{R}$ and has a presentation $\langle C, D \mid C^2 = D^q = 1 \rangle = \mathbb{Z}_2 * \mathbb{Z}_q$. Furthermore, the traces of $\Gamma_q$ are algebraic integers in $\mathbb{Q}(\text{tr}\Gamma) = \mathbb{Q}(\cos(\pi/q))$ (cf. Section 8.3 of [12]).

To analyze the case $q = 3$ (i.e., the trefoil knot), we recall the classification of generating pairs for the groups $\mathbb{Z}_p \ast \mathbb{Z}_q$ up to Nielsen equivalence (Corollary 4.14 of [20] and Theorem 13 of [8]):

**Theorem 5.2.** Every generating pair for the group $\mathbb{Z}_p \ast \mathbb{Z}_q = \langle s, t \mid s^p = t^q = 1 \rangle$ is Nielsen equivalent to exactly one generating pair of the form $(s^m, t^n)$ where

$$\gcd(m, p) = \gcd(n, q) = 1, \quad 0 < 2m \leq p, \quad \text{and} \quad 0 < 2n \leq q.$$

**Corollary 5.3.** If two conjugate elements generate the $(2, 3)$-torus knot group (i.e., the trefoil knot group), then the elements are peripheral.
Proof. Suppose \((\alpha, \beta) = \pi_1(S^3 \setminus K_{2,3})\) with \(\alpha\) conjugate to \(\beta\). Let \(\alpha \mapsto X\) and \(\beta \mapsto Y\) via the representation above. Then \((X,Y) = \Gamma_3 = \mathbb{Z}_2 \ast \mathbb{Z}_3\), and \(X\) is conjugate to \(Y\), so \(\text{tr}X = \text{tr}Y\). Thus, by Theorem 5.2 with \(p = 2\) and \(q = 3\), \((X,Y)\) is Nielsen equivalent to \((C,D)\). Since commutators of Nielsen equivalent pairs have equal trace, we have
\[2 - \text{tr}[X,Y] = 2 - \text{tr}[C,D] = -1.\]
Lemma 2.4 then implies \(\text{tr}XY = 2 = \pm 1\), and so \(\text{tr}^2X = \text{tr}^2Y = 4\). Hence, \(\alpha\) and \(\beta\) are peripheral.

As defined in this section, \(\Gamma_q\) is a discrete faithful representation into \(\text{PSL}_2\mathbb{R}\) of the \((2,q,\infty)\)-triangle group. Thus, the proof of Corollary 6.3 shows that if two elements of equal trace (e.g., conjugate elements) generate the \((2,3,\infty)\)-triangle group (i.e., the modular group), then the elements are parabolic. Following Section 13.5 of [12] (cf. [17]), the \((2,q,\infty)\)-triangle group is arithmetic only when \(q = 3, 4, 6, \) or \(\infty\). Similar methods can then be used to show that if two elements of equal trace generate an arithmetic \((2,q,\infty)\)-triangle group, then the elements are parabolic. The \((2,q,\infty)\)-triangle group can, however, be generated by two conjugate hyperbolic elements when \(q > 3\) is odd: with \(\Gamma_q = \langle C,D \rangle\) as above, let \(X = CD\) and \(Y = C^{-1}XC = DC\); then \(\langle YX \rangle = \langle C,D \rangle = D\), so \(\langle X,Y \rangle = \Gamma_q\), and \(\text{tr}X = \text{tr}Y = -4\cos(\pi/q) < -2\) since \(q > 3\), so \(X\) and \(Y\) are hyperbolic.

6. SIMON’S CONJECTURE

The following is attributed to J. Simon (cf. Problem 1.12 of [11]).

Simon’s Conjecture. A knot group can surject only finitely many other knot groups.

To show Reid’s Conjecture implies Simon’s Conjecture for two-bridge knots, we first note Theorem 5.2 of [14] (recall that a knot complement is called small if it does not contain a closed embedded essential surface):

Theorem 6.1. If \(M\) is a small hyperbolic knot complement in \(S^3\), then there exist only finitely many hyperbolic 3-manifolds \(N\) for which there is a peripheral preserving epimorphism \(\pi_1M \twoheadrightarrow \pi_1N\).

Theorem 6.2. Reid’s Conjecture implies Simon’s Conjecture for two-bridge knots.

Proof. Let \(K\) be a two-bridge knot and \(\varphi_i : \pi_1(S^3 \setminus K) \twoheadrightarrow \pi_1(S^3 \setminus K_i)\) a collection of epimorphisms \(\varphi_i\) and knots \(K_i\).

If \(K_i\) is a \((p,q)\)-torus knot, then its Alexander polynomial has degree \((p-1)(q-1)\) and divides the Alexander polynomial of \(K\) since \(\varphi_i\) is an epimorphism (see [5]).

This can occur for only finitely many integer pairs \((p,q)\); hence, only finitely many \(K_i\) are torus knots.

Now assume \(K_i\) is not a torus knot. Since \(K\) is two-bridge, \(\pi_1(S^3 \setminus K)\) is generated by two conjugate meridians, \(a\) and \(b\). Then \(\varphi_i(a)\) and \(\varphi_i(b)\) are conjugate elements that generate \(\pi_1(S^3 \setminus K_i)\), so Reid’s Conjecture implies that they are peripheral. Hence, by Theorem 1.3, \(K_i\) is two-bridge and therefore hyperbolic since it is not a torus knot.

Let \(\lambda \in \pi_1(S^3 \setminus K)\) such that \(\langle a, \lambda \rangle\) is a peripheral subgroup of \(\pi_1(S^3 \setminus K)\). Then \(\lambda\) commutes with \(a\) in \(\pi_1(S^3 \setminus K)\), so \(\varphi_i(\lambda)\) commutes with \(\varphi_i(a)\) in \(\pi_1(S^3 \setminus K_i)\) and hence is a peripheral element of \(\pi_1(S^3 \setminus K_i)\). Therefore, \(\varphi_i\) is peripheral.
preserving, so the result follows from Theorem 6.1 since hyperbolic two-bridge knot complements in $S^3$ are small \[9\].

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