Disjoint paths in tournaments

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\textsuperscript{1}Supported by NSF grants DMS-0758364 and DMS-1001091.
\textsuperscript{2}Supported by NSF grant DMS-0901075 and ONR grant N00014-10-1-0680.
Abstract

Given $k$ pairs of vertices $(s_i, t_i)$ ($1 \leq i \leq k$) of a digraph $G$, how can we test whether there exist $k$ vertex-disjoint directed paths from $s_i$ to $t_i$ for $1 \leq i \leq k$? This is NP-complete in general digraphs, even for $k = 2$ [2], but for $k = 2$ there is a polynomial-time algorithm when $G$ is a tournament (or more generally, a semicomplete digraph), due to Bang-Jensen and Thomassen [1]. Here we prove that for all fixed $k$ there is a polynomial-time algorithm to solve the problem when $G$ is semicomplete.
1 Introduction

Let $s_1, t_1, \ldots, s_k, t_k$ be vertices of a graph or digraph $G$. The $k$ vertex-disjoint paths problem is to determine whether there exist vertex-disjoint paths $P_1, \ldots, P_k$ (directed paths, in the case of a digraph) such that $P_i$ is from $s_i$ to $t_i$ for $1 \leq i \leq k$. For undirected graphs, this problem is solvable in polynomial time for all fixed $k$; this was one of the highlights of the Graph Minors project of Robertson and the third author [4]. The directed version is a natural and important question, but it was shown by Fortune, Hopcroft and Wyllie [2] that, without further restrictions on the input $G$, this problem is NP-complete for digraphs, even for $k = 2$. This motivates the study of subclasses of digraphs for which the problem is polynomial-time solvable.

In this paper, all graphs and digraphs are finite, and without loops or parallel edges; thus if $u, v$ are distinct vertices of a digraph then there do not exist two edges both from $u$ to $v$, although there may be edges $uv$ and $vu$. Also, by a “path” in a digraph we always mean a directed path. A digraph is a tournament if for every pair of distinct vertices $u, v$, exactly one of $uv, vu$ is an edge; and a digraph is semicomplete if for all distinct $u, v$, at least one of $uv, vu$ is an edge. It was shown by Bang-Jensen and Thomassen [1] that

- the $k$ vertex-disjoint paths problem (for digraphs) is NP-complete if $k$ is not fixed, even when $G$ is a tournament;
- the two vertex-disjoint paths problem is solvable in polynomial time if $G$ is semicomplete.

We shall show:

1.1 For all fixed $k \geq 0$, the $k$ vertex-disjoint paths problem is solvable in polynomial time if $G$ is semicomplete.

In fact we will prove a result for a wider class of digraphs, that we define next. Let $P$ be a path of a digraph $G$, with vertices $v_1, \ldots, v_n$ in order. We say $P$ is minimal if $j \leq i + 1$ for every edge $v_iv_j$ of $G$ with $1 \leq i, j \leq n$. Let $d \geq 1$; we say that a digraph $G$ is $d$-path-dominant if for every minimal path $P$ of $G$ with $d$ vertices, every vertex of $G$ either belongs to $V(P)$ or has an out-neighbour in $V(P)$ or has an in-neighbour in $V(P)$. Thus a digraph is 1-path-dominant if and only if it is semicomplete; and 2-path-dominant if and only if its underlying simple graph is complete multipartite. We will show:

1.2 For all fixed $d, k \geq 1$, the $k$ vertex-disjoint paths problem is solvable in polynomial time if $G$ is $d$-path-dominant.

We stress here that we are looking for vertex-disjoint paths. One can ask the same for edge-disjoint paths, and that question has also been recently solved for tournaments, and indeed for digraphs with bounded independence number [3], but the solution is completely different. We do not know a polynomial-time algorithm for the two vertex-disjoint paths problem for digraphs with independence number two.

But we can extend 1.2 in a different way:

1.3 For all $d, k \geq 1$, there is a polynomial-time algorithm as follows:
• **Input:** Vertices \( s_1, t_1, \ldots, s_k, t_k \) of a \( d \)-path-dominant digraph \( G \), and integers \( x_1, \ldots, x_k \geq 1 \).

• **Output:** Decides whether there exist pairwise vertex-disjoint directed paths \( P_1, \ldots, P_k \) of \( G \) such that for \( 1 \leq i \leq k \), \( P_i \) is from \( s_i \) to \( t_i \) and has at most \( x_i \) vertices.

Let \( s_1, t_1, \ldots, s_k, t_k \) be vertices of a digraph \( G \). We call \((G, s_1, t_1, \ldots, s_k, t_k)\) a problem instance. A linkage in a digraph \( G \) is a sequence \( L = (P_i : 1 \leq i \leq k) \) of vertex-disjoint paths, and \( L \) is a linkage for a problem instance \((G, s_1, t_1, \ldots, s_k, t_k)\) if \( P_i \) is from \( s_i \) to \( t_i \) for each \( i \). (With a slight abuse of notation, we shall call \( k \) the “cardinality” of \( L \), and \( P_1, \ldots, P_k \) its “members”. Also, every subsequence of \((P_i : 1 \leq i \leq k)\) is a linkage \( L' \), and we say \( L \) “includes” \( L' \).) If \( x = (x_1, \ldots, x_k) \) is a \( k \)-tuple of integers, we say a linkage \((P_i : 1 \leq i \leq k)\) is an \( x \)-linkage if each \( P_i \) has \( x_i \) vertices. We say a \( k \)-tuple of integers \( x = (x_1, \ldots, x_k) \) is a quality of \((G, s_1, t_1, \ldots, s_k, t_k)\) if there is an \( x \)-linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\). If \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_k) \), we say \( x \leq y \) if \( x_i \leq y_i \) for \( 1 \leq i \leq k \); and \( x < y \) if \( x \leq y \) and \( x \neq y \). We say a quality \( x \) of \((G, s_1, t_1, \ldots, s_k, t_k)\) is key if there is no quality \( y \) with \( y \leq x \) and \( y < x \). Our main result is the following:

1.4 For all \( d, k \), there is an algorithm as follows:

• **Input:** A problem instance \((G, s_1, t_1, \ldots, s_k, t_k)\) where \( G \) is \( d \)-path-dominant.

• **Output:** The set of all key qualities of \((G, s_1, t_1, \ldots, s_k, t_k)\).

• **Running time:** \( O(n^t) \) where \( t = 6k^2d(k + d) + 13k \).

The idea of the algorithm for 1.2 is easy described. We define an auxiliary digraph \( H \) with two special vertices \( s_0, t_0 \), and prove that there is a path in \( H \) from \( s_0 \) to \( t_0 \) if and only if there is a linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\). Thus to solve the problem of 1.2 it suffices to construct \( H \) in polynomial time. The more general question of 1.4 is solved similarly, by assigning appropriate weights to the edges of \( H \).

Recently we have been able to extend 1.1 to a more general class of digraphs, namely the digraphs whose vertex set can be partitioned into a bounded number of subsets such that each subset induces a semicomplete digraph. The proof is by a modification of the method of this paper, but it is considerably more difficult and not included here.

### 2 A useful enumeration

If \( P \) is a path of a digraph \( G \), its **length** is \(|E(P)|\) (every path has at least one vertex); and \( s(P), t(P) \) denote the first and last vertices of \( P \), respectively. If \( F \) is a subdigraph of \( G \), a vertex \( v \) of \( G \setminus V(F) \) is **\( F \)-outward** if no vertex of \( F \) is adjacent from \( v \) in \( G \); and **\( F \)-inward** if no vertex of \( F \) is adjacent to \( v \) in \( G \). If \( F \) is a digraph and \( v \in V(F) \), \( F \setminus v \) denotes the digraph obtained from \( F \) by deleting \( v \); if \( X \subseteq V(F) \), \( F \setminus X \) denotes the subdigraph of \( F \) induced on \( X \); and \( F \setminus X \) denotes the subdigraph obtained by deleting all vertices in \( X \).

Now let \( L = (P_i : 1 \leq i \leq k) \) be a linkage in \( G \). We define \( V(L) \) to be \( V(P_1) \cup \cdots \cup V(P_k) \). A vertex \( v \) is an **internal vertex** of \( L \) if \( v \in V(L) \), and \( v \) is not an end of any member of \( L \). A linkage \( L \) is **internally disjoint** from a linkage \( L' \) if no internal vertex of \( L \) belongs to \( V(L') \) (note that this does not imply that \( L' \) is internally disjoint from \( L \)); and we say that \( L, L' \) are **internally disjoint**
if each of them is internally disjoint from the other (and thus all vertices in $V(L) \cap V(L')$ must be ends of paths in both $L$ and $L'$).

Let $Q, R$ be vertex-disjoint paths of a digraph $G$. A planar $(Q, R)$-matching is a linkage $(M_j : 1 \leq j \leq n)$ for some $n \geq 0$, such that

- $M_1, \ldots, M_n$ each have either two or three vertices;
- $s(M_1), \ldots, s(M_n)$ are vertices of $Q$, in order in $Q$; and
- $t(M_1), \ldots, t(M_n)$ are vertices of $R$, in order in $R$.

Fix $d, k \geq 1$, and let $L = (P_1, \ldots, P_k)$ be a linkage in a $d$-path-dominant digraph $G$. A subset $B \subseteq V(L)$ is said to be acceptable (for $L$) if

- for $1 \leq j \leq k$, if $uv$ is an edge of $P_j$ and $v \in B$ then $u \in B$ (and so $Q_j = P_j | B$ and $R_j = P_j | (V(G) \setminus B)$ are paths if they are non-null);
- for $1 \leq i, j \leq k$, there is no planar $(Q_i, R_j)$-matching of cardinality $(k - 1)d + k^2 + 2$ internally disjoint from $L$.

Thus $\emptyset$ and $V(L)$ are acceptable.

2.1 Let $d \geq 1$, let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance, where $G$ is $d$-path-dominant, let $x$ be a key quality, and let $L = (P_1, \ldots, P_k)$ be an $x$-linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$. Suppose that $B \subseteq V(L)$ is acceptable for $L$ and $B \neq V(L)$. Then there exists $v \in V(L) \setminus B$ such that $B \cup \{v\}$ is acceptable for $L$.

**Proof.** Let $A = V(G) \setminus B$. For $1 \leq j \leq k$, let $Q_j = P_j | B$ and $R_j = P_j | A$. Let $q_j, r_j$ be the last vertex of $Q_j$ and the first vertex of $R_j$, respectively (if they exist).

(1) For $1 \leq j \leq k$, $P_j$ is a minimal path of $G$. In particular, the only edge of $G$ from $V(Q_j)$ to $V(R_j)$ (if there is one) is $q_j r_j$. Moreover, every three-vertex path from $V(Q_j)$ to $V(R_j)$ with internal vertex in $V(G) \setminus V(L)$ uses at least one of $q_j, r_j$. Consequently, there is no planar $(Q_j, R_j)$-matching of cardinality three internally disjoint from $L$.

For suppose there is an edge $uv$ of $G$ such that $u, v \in V(P_j)$ and $u$ is before $v$ in $P_j$, and there is at least one vertex of $P_j$ between $u$ and $v$. If we delete from $P_j$ the vertices of $P_j$ strictly between $u$ and $v$, and add the edge $uv$, we obtain a path from $s_j$ to $t_j$ disjoint from every member of $L$ except $P_j$, and with strictly fewer vertices than $P_j$, contradicting that $x$ is key. Thus $P_j$ is induced. Similarly there is no three-vertex path from $V(Q_j)$ to $V(R_j)$ with internal vertex in $V(G) \setminus V(L)$ containing neither of $q_j, r_j$. The final assertion follows. This proves (1).

From (1), the theorem holds if $k = 1$, so we may assume that $k \geq 2$.

(2) We may assume that for all $i \in \{1, \ldots, k\}$, if $R_i$ is non-null then for some $j \in \{1, \ldots, k\}$ with $j \neq i$, there is a planar $(Q_i, R_j \setminus r_j)$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$. 

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For suppose that some $i$ does not satisfy the statement of (2). Thus $R_i$ is non-null, and there is no $j$ as in (2). Since $R_i$ is non-null, it follows that $r_i$ exists. We may assume that $B \cup \{r_i\}$ is not acceptable. Consequently, one of the two conditions in the definition of “acceptable” is not satisfied by $B \cup \{r_i\}$. The first is satisfied since $r_i$ is the first vertex of $R_i$. Thus the second is false, and so for some $i', j \in \{1, \ldots, k\}$, there is a planar $(P_i|(B \cup \{r_i\}), P_j|(A \setminus \{r_i\}))$-matching of cardinality $(k - 1)d + k^2 + 2$ internally disjoint from $L$. Since there is no planar $(Q_i', R_j')$-matching of cardinality $(k - 1)d + k^2 + 2$ internally disjoint from $L$, and $P_j|(A \setminus \{r_i\})$ is a subpath of $R_j$, it follows that $P_i|(B \cup \{r_i\}) \neq Q_j$, and so $i' = i$. Since only one vertex of $P_i|(B \cup \{r_i\})$ does not belong to $Q_i$, it follows that there is a planar $(Q_i, R_j \setminus r_i)$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$. Since $(k - 1)d + k^2 \geq 4$ (because $k \geq 2$), (1) implies that $j \neq i$. This proves (2).

(3) We may assume that for some $p \geq 2$, and for all $i$ with $1 \leq i < p$, there is a planar $(Q_i, R_{i+1} \setminus r_{i+1})$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$, and there is a planar $(Q_p, R_1 \setminus r_1)$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$.

For by hypothesis, there exists $i \in \{1, \ldots, k\}$ such that $R_i$ is non-null. By repeated application of (2), there exist distinct $h_1, \ldots, h_p \in \{1, \ldots, k\}$ such that for $1 \leq i \leq p$ there is a planar $(Q_{h_i}, R_{h_i+1} \setminus r_{h_i+1})$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$, where $h_{p+1} = h_1$; and $p \geq 2$ by (1). Without loss of generality, we may assume that $h_i = i$ for $1 \leq i \leq p$. This proves (3).

Let us say a planar $(Q, R)$-matching is $s$-spaced if no subpath of $Q$ with at most $s$ vertices meets more than one member of the matching, and no subpath of $R$ with at most $s$ vertices meets more than one member of the matching.

(4) We may assume that for some $p \geq 2$, and for all $i$ with $1 \leq i < p$, there is a planar $(Q_i, R_{i+1} \setminus r_{i+1})$-matching $L_i$, and there is a planar $(Q_p, R_1 \setminus r_1)$-matching $L_p$, such that

- $L_1, \ldots, L_p$ all have cardinality $k$;
- they are pairwise internally disjoint;
- each of $L_1, \ldots, L_p$ is internally disjoint from $L$; and
- each of $L_1, \ldots, L_p$ is $(d + 1)$-spaced.

For let $L'_i$ be a planar $(Q_i, R_{i+1} \setminus r_{i+1})$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$, for $1 \leq i < p$, and let $L'_p$ be a planar $(Q_p, R_1 \setminus r_1)$-matching of cardinality $(k - 1)d + k^2$ internally disjoint from $L$. We choose $L_i \subseteq L'_i$ inductively. Suppose that for some $h < p$, we have chosen $L_1, \ldots, L_h$, such that

- $L_1, \ldots, L_h$ all have cardinality $k$;
- they are pairwise internally disjoint;
- each of $L_1, \ldots, L_h$ is internally disjoint from $L$; and
- each of $L_1, \ldots, L_h$ is $(d + 1)$-spaced.


We define \( L_{h+1} \) as follows. The union of the sets of internal vertices of \( L_1, \ldots, L_h \) has cardinality at most \( h k \leq k(k - 1) \), and so \( L_{h+1}^\prime \) includes a planar \((Q_{h+1}, R_{h+2} \setminus r_{h+2})\)-matching (or \((Q_p, R_1 \setminus r_1)\)-matching, if \( h = p - 1 \)) of cardinality \((k - 1)d + k^2 - k(k - 1) = 1 + (k - 1)(d + 1)\), internally disjoint from each of \( L_1, \ldots, L_h \). By ordering the members of this matching in their natural order, and taking only the \( i \)th terms, where \( i = 1, 1 + (d + 1), 1 + 2(d + 1) \ldots \), we obtain a \((d + 1)\)-spaced matching of cardinality \( k \). Let this be \( L_{h+1} \). This completes the inductive definition of \( L_1, \ldots, L_p \), and so proves (4).

For \( 1 \leq i \leq p \), let \( L_i = \{ M^1_i, \ldots, M^k_i \} \), numbered in order; thus, if \( q^h_i \) and \( r^h_i \) denote the first and last vertices of \( M^h_i \), then \( q^h_1, \ldots, q^h_k \) are distinct and in order in \( Q_i \), and \( r^h_1, r^h_1, \ldots, r^h_k \) are distinct and in order in \( R_{i+1} \) (or in \( R_1 \) if \( i = p \)). For \( 1 \leq i \leq p \) and \( 2 \leq h \leq k \), let \( Q^h_i \) be the subpath of \( P_i \) with \( d \) vertices and with last vertex \( q^h_i \). (Thus \( Q^h_i \) does not belong to \( Q^h_p \) since \( L_i \) is \( d \)-spaced, and indeed \((d + 1)\)-spaced.) Since \( P_i \) and hence \( Q^h_i \) is a minimal path of \( G \), and \( G \) is \( d \)-path-dominant, it follows that for \( 1 \leq i \leq p \) and \( 2 \leq h \leq k \), \( r^h_i \) is adjacent to or from some vertex \( v \) of \( Q^h_i \). Since \( r^h_{i-1} \neq r_i \), (1) implies that \( r^h_{i-1} \) is not adjacent from any vertex of \( Q^h_i \); and so there is a path \( T_i \) of length at most \( d \), such that all its internal vertices belong to \( Q^h_i \). For \( 1 \leq i \leq p \), and \( 1 \leq h < k \), let \( S^h_i \) be the path

\[
q^h_i \rightarrow M^1_i \rightarrow r^h_i \rightarrow R^h_{i+1} \rightarrow q^h_{i+1},
\]

or

\[
q^h_{i-1} \rightarrow M^1_i \rightarrow r^h_i \rightarrow R^h_{i+1} \rightarrow q^h_{i+1}
\]

if \( i = p \); then \( S^h_i \) is a path from \( q^h_i \) to \( q^h_{i+1} \) (or to \( q^h_{i+1} \) if \( i = p \)), of length at most \( d + 2 \). Thus (reading subscripts modulo \( p \)) concatenating \( S^h_1, S^h_2, \ldots, S^h_{i-1} \) and \( M^p_i \) gives a path \( T_i \) from \( q^h_1 \) to \( r^h_p \) of length at most \( (p - 1)(d + 2) + 2 \). The subpath \( T_i \) of \( P_i \) from \( q^h_1 \) to \( r^h_p \) has length at least \( (p + k - 2)(d + 1) + 2 \), since \( L_{i-1}, L_i \) are \((d + 1)\)-spaced and \( r_i \) is different from \( r^h_1 \); and since \( p + k - 1 \geq 2(p - 1) \) and \( d + 1 > (d + 2)/2 \), it follows that \( T_i \) has length strictly greater than that of \( T^h_i \). Let \( P^h_i \) be obtained from \( P_i \) by replacing the subpath \( T_i \) by \( T^h_i \), for \( 1 \leq i \leq p \), and let \( P^h_{i-1} \) for \( p + 1 \leq i \leq k \). Then \( \{ P^h_1, \ldots, P^h_k \} \) is a linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\), contradicting that \( x \) is key. This proves 2.1.

We deduce:

2.2 Let \( d \geq 1 \), let \((G, s_1, t_1, \ldots, s_k, t_k)\) be a problem instance where \( G \) is \( d \)-path-dominant, let \( x \) be a key quality, and let \( L = (P_1, \ldots, P_k) \) be an \( x \)-linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\). Let \( c = (k - 1)d + k^2 + 2 \). Then there is an enumeration \((v_1, \ldots, v_n)\) of \( V(L) \), such that

- for \( 1 \leq h \leq k \) and \( 1 \leq p, q \leq n \), if \( v_pv_q \) is an edge of \( P_h \) then \( p < q \);

- for \( 1 \leq h, i \leq k \) and \( 1 \leq p \leq n - 1 \), and every \( cd \)-vertex subpath \( Q \) of \( P_h \)\{\(v_1, \ldots, v_p\}\}, and every \( cd \)-vertex subpath \( R \) of \( P_i \)\{\(v_{p+1}, \ldots, v_n\}\}, there are at most \( c(2k + 1) \) vertices of \( G \) that are both \( Q \)-outward and \( R \)-inward.

Proof. Since \( \emptyset \) is acceptable for \( L \), by repeated application of 2.1 implies that there is an enumeration \((v_1, \ldots, v_n)\) of \( V(L) \), such that \( \{v_1, \ldots, v_p\} \) is acceptable for \( 0 \leq p \leq n \). We claim that this enumeration satisfies the theorem. For certainly the first bullet holds; we must check the second.
Thus, let \(1 \leq p \leq n\), and let \(B = \{v_1, \ldots, v_p\}\) and \(A = \{v_{p+1}, \ldots, v_n\}\). For \(1 \leq h \leq k\), let \(Q_h = P_h|B\) and \(R_h = P_h|A\). Now let \(1 \leq h, i \leq k\), and let \(Q, R\) be \(cd\)-vertex subpaths of \(Q_h, R_i\) respectively. Let \(X\) be the set of all vertices of \(G\) that are both \(Q\)-outward and \(R\)-inward. We must show that \(|X| \leq c(2k + 1)\).

1. If \(x_1, \ldots, x_c \in X\) are distinct, then there exist \(y_1, \ldots, y_c \in V(Q)\), distinct and in order in \(Q\), such that \(y_jx_j\) is an edge for \(1 \leq j \leq c\).

   For \(Q\) has \(cd\) vertices; let its vertices be \(q_1, \ldots, q_{cd}\) in order. Let \(1 \leq j \leq c\). The subpath of \(Q\) induced on \(\{q_s : (j - 1)d < s \leq jd\}\) has \(d\) vertices, and since \(Q\) is a minimal path of \(G\) and \(G\) is \(d\)-path-dominant, and \(X \cap V(Q) = \emptyset\), it follows that \(x_j\) is in- or out-adjacent to a vertex of this subpath, say \(y_j\). Since \(x_j \in X\) and hence is \(Q\)-outwards, it follows that \(x_jy_j\) is not an edge, and so \(y_jx_j\) is an edge. But then \(y_1, \ldots, y_c\) satisfy (1). This proves (1).

2. The sets \(X \setminus V(L)\), \(X \cap V(Q_g) \ (1 \leq g \leq k)\) and \(X \cap V(R_g) \ (1 \leq g \leq k)\) all have cardinality at most \(c - 1\), and hence \(|X| \leq (2k + 1)(c - 1)\).

For suppose that there exist distinct \(x_1, \ldots, x_c \in X \setminus V(L)\). By (1) there exist distinct \(y_1, \ldots, y_c \in V(Q)\), in order in \(Q\), such that \(y_jx_j\) is an edge for \(1 \leq j \leq c\); and similarly there exist \(z_1, \ldots, z_c \in V(R)\), in order in \(R\), such that \(x_jz_j\) is an edge for \(1 \leq j \leq c\). But then the \(c\) paths \(y_j-x_j-z_j \ (1 \leq j \leq c)\) form a planar \((Q_h, R_i)\)-matching of cardinality \(c\), internally disjoint from \(L\), contradicting that \(\{v_1, \ldots, v_p\}\) is acceptable. Thus \(|X \setminus V(L)| \leq c - 1\). Now suppose that for some \(g \in \{1, \ldots, k\}\), there exist distinct \(x_1, \ldots, x_c \in X \cap V(R_g)\), numbered in order in \(R_g\). Choose \(y_1, \ldots, y_c\) as in (1); then the paths \(y_jx_j \ (1 \leq j \leq c)\) form a planar \((Q_h, R_g)\)-matching of cardinality \(c\), internally disjoint from \(L\), contradicting that \(\{v_1, \ldots, v_p\}\) is acceptable. Thus \(|X \cap V(R_g)| \leq c - 1\), and similarly \(|X \cap V(Q_g)| \leq c - 1\), for \(1 \leq g \leq k\). This proves (2).

From (2), the theorem follows.

3 Confusion and the auxiliary digraph

Let \((G, s_1, t_1, \ldots, s_k, t_k)\) be a problem instance, and let \(L = (M_1, \ldots, M_k)\) be a linkage in \(G\) (not necessarily a linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\)). Let \(A(L)\) be the set of all vertices in \(V(G) \setminus V(L)\) that are \(M_j \setminus t(M_j)\)-inward for some \(j \in \{1, \ldots, k\}\) such that \(t(M_j) \neq t_j\) and let \(B(L)\) be the set of all vertices in \(V(G) \setminus V(L)\) that are \(M_j \setminus s(M_j)\)-outward for some \(j \in \{1, \ldots, k\}\) such that \(s(M_j) \neq s_j\). We call \(|A(L) \cap B(L)|\) the confusion of \(L\); and it is helpful to keep the confusion small, as we shall see.

A \((k, m, c)\)-rail in a problem instance \((G, s_1, t_1, \ldots, s_k, t_k)\) is a triple \((L, X, Y)\), where

- \(L\) is a linkage in \(G\) consisting of \(k\) paths \((M_1, \ldots, M_k)\) (but not necessarily a linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\));
- for \(1 \leq j \leq k\), \(M_j\) has at most \(2m\) vertices, and if it has fewer than \(2m\) vertices then \(M_j\) either has first vertex \(s_j\) or last vertex \(t_j\);
• $L$ has confusion at most $c$;

• $X, Y$ are disjoint subsets of $V(G) \setminus V(L)$; and

• $X \subseteq A(L), Y \subseteq B(L)$, and $X \cup Y = A(L) \cup B(L)$.

3.1 For all $k, m, c \geq 0$, if $(G, s_1, t_1, \ldots, s_k, t_k)$ is a problem instance and $G$ has $n$ vertices then there are at most $2^n n^{2km(2km)}$ $(k, m, c)$-rails in $(G, s_1, t_1, \ldots, s_k, t_k)$. Moreover, for all fixed $k, m, c \geq 0$, there is an algorithm which, with input a problem instance $(G, s_1, t_1, \ldots, s_k, t_k)$, finds all its $(k, m, c)$-rails in time $O(n^{2km+1})$, where $n = |V(G)|$.

Proof. First, if $L$ is a linkage with $k$ paths each with at most $2m$ vertices, then $|V(L)| \leq 2km$, and so the number of such linkages is at most $n^{2km(2km)^k}$, as is easily seen. Now fix a linkage $L$ satisfying the first two bullets in the definition of $(k, m, c)$-rail; let us count the number of pairs $(X, Y)$ such that $(L, X, Y)$ is a $(k, m, c)$-rail. There are none unless $|A(L) \cap B(L)| \leq c$; and in that case, there are at most $2^c$ possibilities for the pair $(X, Y)$, since $X$ consists of $A(L) \setminus B(L)$ together with some subset of $A(L) \cap B(L)$, and $Y = (A(L) \cup B(L)) \setminus X$.

For the algorithm, we first find all linkages $L$ with $k$ paths each with at most $2m$ vertices, by examining all ordered $2km$-tuples of distinct vertices of $G$. For each such $L$, we check whether it satisfies the first three bullets in the definition of $(k, m, c)$-rail (this takes time $O(n)$); if not we discard it and otherwise we partition $A(L) \cap B(L)$ into two subsets in all possible ways, and output the corresponding $(k, m, c)$-rails. The result follows.

Let $(L, X, Y)$ and $(L', X', Y')$ be distinct $(k, m, c)$-rails in $G$, and let $L = (P_1, \ldots, P_k)$ and $L' = (P'_1, \ldots, P'_k)$. We write $(L, X, Y) \rightarrow (L', X', Y')$ if the following hold:

• for $1 \leq i \leq k$, $P_i \cup P'_i$ is a path from the first vertex of $P_i$ to the last vertex of $P'_i$;

• for $1 \leq i \leq k$, $V(P'_i) \subseteq V(P_i) \cup X$, and $V(P_i) \subseteq V(P'_i) \cup Y'$; and

• $X' \subseteq X$, and $Y \subseteq Y'$.

Let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance, and let $T$ be the set of all $(k, m, c)$-rails in $(G, s_1, t_1, \ldots, s_k, t_k)$. Take two new vertices $s_0, t_0$, and let us define a digraph $H$ with vertex set $T \cup \{s_0, t_0\}$ as follows. Let $u, v \in V(H)$. If $u, v \in T$ are distinct, then $uv \in E(H)$ if and only if $u \rightarrow v$. If $u = s_0$ and $v \in T$, let $v = (L, X, Y)$ where $L = (M_1, \ldots, M_k)$; then $u v \in E(H)$ if and only if $M_j$ has first vertex $s_j$ for all $j \in \{1, \ldots, k\}$. Similarly, if $u \in T$ and $v = t_0$, let $u = (L, X, Y)$ where $L = (M_1, \ldots, M_k)$; then $uv \in E(H)$ if and only if $M_j$ has last vertex $t_j$ for all $j \in \{1, \ldots, k\}$. This defines $H$. We call $H$ the $(k, m, c)$-tracker of $(G, s_1, t_1, \ldots, s_k, t_k)$.

We shall show that with an appropriate choice of $m, c$, when $G$ is $d$-path-dominant we can reduce our problems about linkages for $(G, s_1, t_1, \ldots, s_k, t_k)$ to problems about paths from $s_0$ to $t_0$ in the $(k, m, c)$-tracker. Let $(G, s_1, t_1, \ldots, s_k, t_k)$ be a problem instance, let $(P_1, \ldots, P_k)$ be a linkage for $(G, s_1, t_1, \ldots, s_k, t_k)$, and let $P$ be a path from $s_0$ to $t_0$ in the $(k, m, c)$-tracker. Let $P$ have vertices

$s_0, (L_1, X_1, Y_1), \ldots, (L_n, X_n, Y_n), t_0$

in order, and let $L_p = (M_{p,1}, \ldots, M_{p,k})$ for $1 \leq p \leq n$. We say that $P$ traces $(P_1, \ldots, P_k)$ if $P_j$ is the union of $M_{1,j}, \ldots, M_{n,j}$ for all $j \in \{1, \ldots, k\}$.
3.2 Let \( k, m, c \geq 0 \) be integers, and let \((G, s_1, t_1, \ldots, s_k, t_k)\) be a problem instance, with \((k,m,c)\)-tracker \( H \). Every path in \( H \) from \( s_0 \) to \( t_0 \) traces some linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\).

**Proof.** Let \( P \) be a path of \( H \), with vertices

\[ s_0, (L_1, X_1, Y_1), \ldots, (L_n, X_n, Y_n), t_0 \]

in order, and let \( L_p = (M_{p,1}, \ldots, M_{p,k}) \) for \( 1 \leq p \leq n \). For \( 1 \leq p \leq n \) and \( 1 \leq j \leq k \), let \( P_{p,j} \) be the union of \( M_{1,j}, \ldots, M_{p,j} \).

(1) For \( 1 \leq p \leq n \) and \( 1 \leq j \leq k \), every vertex of \( P_{p,j} \) belongs to \( Y_p \cup V(M_{p,j}) \).

We prove this by induction on \( p \). If \( p = 1 \) the claim is true, since then \( P_{1,j} = M_{1,j} \). We assume then that \( p > 1 \) and the result holds for \( p-1 \). Let \( v \in V(P_{p,j}) \). If \( v \in V(M_{p,j}) \) then the claim is true, so we assume not. Since \( v \in V(P_{p,j}) \), and \( P_{p,j} = P_{p-1,j} \cup M_{p,j} \), it follows that \( v \in V(P_{p-1,j}) \), and so from the inductive hypothesis, \( v \in Y_{p-1} \cup V(M_{p-1,j}) \). But since \( (L_{p-1}, X_{p-1}, Y_{p-1}) \rightarrow (L_p, X_p, Y_p) \), we deduce that \( Y_{p-1} \subseteq Y_p \), and \( V(M_{p-1,j}) \subseteq V(M_{p,j}) \cup Y_p \), and so \( v \in V(M_{p,j}) \cup Y_p \). This proves (1).

(2) For \( 1 \leq p \leq n \) and \( 1 \leq j \leq k \), \( P_{p,j} \) is a path from \( s_j \) to the last vertex of \( M_{p,j} \).

The claim holds if \( p = 1 \); so we assume that \( p > 1 \) and the claim holds for \( p-1 \). Thus \( P_{p-1,j} \) is a path from \( s_j \) to the last vertex of \( M_{p-1,j} \); and also, \( M_{p-1,j} \cup M_{p,j} \) is a path, from the first vertex of \( M_{p-1,j} \) to the last vertex of \( M_{p,j} \), since \((L_{p-1}, X_{p-1}, Y_{p-1}) \rightarrow (L_p, X_p, Y_p)\). We claim that every vertex \( v \) that belongs to both of \( P_{p-1,j}, M_{p,j} \) also belongs to \( M_{p-1,j} \). For suppose not; then by (1), \( v \in Y_{p-1} \) since \( v \in V(P_{p-1,j}) \setminus V(M_{p-1,j}) \), and \( v \in X_{p-1} \), since \( v \in V(M_{p,j}) \setminus V(M_{p-1,j}) \). This is impossible since \( X_{p-1} \cap Y_{p-1} = \emptyset \). This proves that every vertex that belongs to both of \( P_{p-1,j}, M_{p,j} \) also belongs to \( M_{p-1,j} \). Since \( M_{p-1,j} \) is non-null, we deduce that \( P_{p-1,j} \cup M_{p,j} \) is a path from \( s_j \) to the last vertex of \( M_{p,j} \). This proves (2).

(3) For \( 1 \leq p \leq n \), the paths \( P_{p,1}, \ldots, P_{p,k} \) are pairwise vertex-disjoint.

For again we proceed by induction on \( p \), and may assume that \( p > 1 \) and the result holds for \( p-1 \). Suppose that \( v \) belongs to two of the paths \( P_{p,1}, \ldots, P_{p,k} \), say to \( P_{p,1} \) and \( P_{p,2} \). From the inductive hypothesis, \( v \) does not belong to both of \( P_{p-1,1} \) and \( P_{p-1,2} \), so we may assume that \( v \in V(M_{p,1}) \). Now \( v \notin V(M_{p,2}) \), because \( L_p \) is a linkage, and so \( v \in V(P_{p-1,2}) \). From (1) we deduce that \( v \in Y_{p-1} \cup V(M_{p-1,2}) \). But \( Y_{p-1} \subseteq Y_p \), and \( V(M_{p-1,2}) \setminus V(M_{p,2}) \subseteq Y_p \), and so \( v \in Y_p \); but \( Y_p \cap V(L_p) = \emptyset \) since \((L_p, X_p, Y_p)\) is a \((k,m,c)\)-rail, a contradiction. This proves (3).

From (2) and (3) we deduce that \( P_{n,1}, \ldots, P_{n,k} \) is a linkage \( L \) for \((G, s_1, t_1, \ldots, s_k, t_k)\). Thus \( P \) traces \( L \). This proves 3.2.

The next result is a kind of partial converse; but we have to choose \( m, c \) carefully, and we need \( G \) to be \( d \)-path-dominant, and the proof only works for linkages that realize a key quality.
3.3 Let \( d, k \geq 1 \) be integers, and let
\[
c = ((k - 1)d + k^2 + 2)(2k + 1)k^2
\]
\[
m = ((k - 1)d + k^2 + 2) + 1.
\]
Let \((G, s_1, t_1, \ldots, s_k, t_k)\) be a problem instance where \( G \) is \( d \)-path-dominant, let \( x \) be a key quality, and let \((P_1, \ldots, P_k)\) be an \( x \)-linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\). Let \( H \) be the \((k, m, c)\)-tracker of \((G, s_1, t_1, \ldots, s_k, t_k)\). Then there is a path in \( H \) from \( s_0 \) to \( t_0 \) tracing \((P_1, \ldots, P_k)\).

**Proof.** Let \( L = (P_1, \ldots, P_k) \). By 2.2, there is an enumeration \((v_1, \ldots, v_n)\) of \( V(L) \), such that

- for \( 1 \leq j \leq k \) and \( 1 \leq p, q \leq n \), if \( v_p v_q \) is an edge of \( P_j \) then \( p < q \);
- for \( 1 \leq i, j \leq k \) and \( 1 \leq p \leq n - 1 \), and every \((m-1)\)-vertex subpath \( Q \) of \( P_i \{v_1, \ldots, v_p\} \), and every \((m-1)\)-vertex subpath \( R \) of \( P_j \{v_{p+1}, \ldots, v_n\} \), there are at most \(((k - 1)d + k^2 + 2)(2k + 1)\) vertices of \( G \) that are both \( Q \)-outward and \( R \)-inward.

For each \( v \in V(L) \), let \( \phi(v) = i \) where \( v = v_i \); thus \( \phi \) is a bijection from \( V(L) \) onto \( \{1, \ldots, n\} \).

For all \( p \in \{0, \ldots, n\} \) and all \( j \in \{1, \ldots, k\} \), if \( \phi(s_j) \leq p \), let \( Q_{p,j} \) be the maximal subpath of \( P_j \) with at most \( m \) vertices and with last vertex \( v_q \), where \( q \leq p \) is maximum such that \( v_q \in V(P_j) \). If \( \phi(s_j) > p \), let \( Q_{p,j} \) be the null digraph. Similarly, if \( \phi(t_j) > p \), let \( R_{p,j} \) be the maximal subpath of \( P_j \) with at most \( m \) vertices and with first vertex \( v_r \), where \( r > p \) is minimum such that \( v_r \in V(P_j) \). If \( \phi(t_j) \leq p \), let \( R_{p,j} \) be the null digraph. Thus, if \( Q_{p,j}, R_{p,j} \) are both non-null, then \( t(Q_{p,j}) \) and \( s(R_{p,j}) \) are consecutive in \( P_j \).

For all \( p \in \{0, \ldots, n\} \) and all \( j \in \{1, \ldots, k\} \), let \( M_{p,j} \) be the subpath of \( P_j \) defined as follows: if both \( Q_{p,j}, R_{p,j} \) are non-null, \( M_{p,j} \) consists of \( Q_{p,j} \cup R_{p,j} \) together with the edge of \( P_j \) from \( t(Q_{p,j}) \) to \( s(R_{p,j}) \), while if one of \( Q_{p,j}, R_{p,j} \) is null, \( M_{p,j} \) equals the other (not both can be null). We see that, for all \( p, j \), \( M_{p,j} \) has at most \( 2m \) vertices; and either it has exactly \( 2m \), or its first vertex is \( s_j \), or its last vertex is \( t_j \). For all \( p \in \{0, \ldots, n\} \), let \( L_p \) be the linkage \((M_{p,1}, \ldots, M_{p,k})\).

(1) For all \( p \in \{0, \ldots, n\} \), \( L_p \) has confusion at most \( c \).

Let \( v \in A(L_p) \cap B(L_p) \), where \( A(L_p), B(L_p) \) are as in the definition of confusion. Thus there exists \( j \in \{1, \ldots, k\} \) such that \( v = M_{p,j} \setminus t(M_{p,j}) \)-inward and \( t(M_{p,j}) \neq t_j \). Since \( t(M_{p,j}) \neq t_j \), it follows from the choice of \( R_{p,j} \) that \( R_{p,j} \) has exactly \( m \) vertices. Moreover, \( v \in R_{p,j} \setminus t(R_{p,j}) \)-inward, since \( v \) is \( M_{p,j} \setminus t(M_{p,j}) \)-inward. Similarly, there exists \( i \in \{1, \ldots, k\} \) such that \( v = Q_{p,i} \setminus s(Q_{p,i}) \)-outward and \( Q_{p,i} \) has \( m \) vertices. For each choice of \( i, j \in \{1, \ldots, k\} \), there are at most \(((k - 1)d + k^2 + 2)(2k + 1)\) vertices that are both \( Q_{p,i} \setminus s(Q_{p,i}) \)-outward and \( R_{p,j} \setminus t(R_{p,j}) \)-inward, from the choice of the enumeration \((v_1, \ldots, v_n)\). Consequently in total there are only \( c \) possibilities for \( v \), and so \(|A(L_p) \cap B(L_p)| \leq c\). This proves (1).

(2) For \( 0 \leq p \leq n \) and each \( v \in V(L) \setminus V(L_p) \), if \( \phi(v) > p \) then \( v \in A(L_p) \), and if \( \phi(v) \leq p \) then \( v \in B(L_p) \).

For let \( v \in V(P_j) \) say. Assume first that \( \phi(v) > p \). Since \( v \notin V(L_p) \), it follows that \( M_{p,j} \) does not have last vertex \( t_j \); and since \( x \) is key, \( v \) is not adjacent from any vertex in \( M_{p,j} \) except possibly
From (1), it suffices to check that
\[ (3) \quad \forall Y \subseteq A, \quad B \subseteq M \text{ and } v \neq t \implies \forall j \in \mathbb{Z}, \quad v / j, \text{ and so if } v \neq t. \]

Consequently, we deduce that \( v \neq t \) since \( v \neq t \). Thus \( v \neq t \) and we may assume they are distinct, and so \( v \neq t \). Consequently \( v \neq t \) as required. This proves (3).

(4) For all \( p \in \{0, \ldots, n - 1\} \), and all \( j \in \{1, \ldots, k\} \), \( M_{p,j} \cup M_{p+1,j} \) is a path from the first vertex of \( M_{p,j} \) to the last vertex of \( M_{p+1,j} \).

For \( M_{p,j}, M_{p+1,j} \) are both subpaths of \( P_j \), and we may assume they are distinct, and so \( v_{p+1} \in V(P_j) \). Hence, since \( m > 0 \), \( v_{p+1} \) is the first vertex of \( R_{p,j} \), and the last vertex of \( Q_{p+1,j} \); and so \( M_{p,j} \cup M_{p+1,j} \) is a path. Moreover, it follows from the definition of the paths \( M_{p,j} \) that \( M_{p,j} \cup M_{p+1,j} \) is a path from the first vertex of \( M_{p,j} \) to the last vertex of \( M_{p+1,j} \). This proves (4).

(5) For all \( p \in \{0, \ldots, n - 1\} \), and all \( j \in \{1, \ldots, k\} \), \( A(L_{p+1}) \subseteq A(L_p) \cup V(L) \) and \( B(L_p) \subseteq B(L_{p+1}) \cup V(L) \).

For let \( v \in A(L_{p+1}) \). We need to prove that \( v \in A(L_p) \cup V(L) \), and so we may assume that \( v \notin V(L) \). Choose \( j \) with \( 1 \leq j \leq k \) such that \( v \) is \( M_{p+1,j} \setminus t(M_{p+1,j}) \)-inward and \( t(M_{p+1,j}) \neq t_j \). Consequently \( t(M_{p,j}) \neq t_j \), and so \( v \) is \( M_{p,j} \setminus t(M_{p,j}) \)-inward then \( v \in A(L_p) \) as required, so we may assume that \( v \) is adjacent from some vertex of \( M_{p,j} \). In particular, \( M_{p,j} \neq M_{p+1,j} \) and so \( v_{p+1} \in V(P_j) \), and \( v_{p+1} = s(R_{p,j}) = t(Q_{p+1,j}) \). Moreover, since \( s(M_{p,j}) \) is the only vertex of \( M_{p,j} \) that may not belong to \( M_{p+1,j} \), we deduce that \( s(M_{p,j}) \) is adjacent to \( v \), and \( s(M_{p,j}) \) does not belong to \( M_{p+1,j} \). Consequently \( s(M_{p+1,j}) \neq s_j \), and so \( Q_{p+1,j} \) has \( m \) vertices. Since \( v \) is \( M_{p+1,j} \setminus t(M_{p+1,j}) \)-inward, and \( G \) is \( d \)-path-dominant, and \( M_{p+1,j} \setminus t(M_{p+1,j}) \) is a minimal path of \( G \), and it has \( m - 1 > d + 2 \) vertices, there is a subpath of \( M_{p+1,j} \setminus t(M_{p+1,j}) \) with \( d \) vertices, not containing the first or second vertex of \( M_{p+1,j} \setminus t(M_{p+1,j}) \); and so \( v \) is adjacent to some vertex \( w \) of \( M_{p+1,j} \setminus t(M_{p+1,j}) \) different from its first and second vertices. But \( v \) is adjacent from \( u \), so by replacing the subpath of \( P_j \) between \( u \) and \( w \) by the path \( u \rightarrow w \), we contradict that \( x \) is key. This proves that \( v \in A(L_p) \), and so \( A(L_{p+1}) \subseteq A(L_p) \cup V(L) \). Similarly \( B(L_p) \subseteq B(L_{p+1}) \cup V(L) \). This proves (5).
(6) For all \( p \in \{0, \ldots, n-1\} \), \( X_{p+1} \subseteq X_p \) and \( Y_p \subseteq Y_{p+1} \).

Let \( v \in X_{p+1} \). Suppose first that \( v \notin V(L) \). Then \( v \in A(L_{p+1}) \setminus B(L_{p+1}) \). By (5), \( v \in A(L_p) \setminus B(L_p) \), and so \( v \in X_p \) as required. Thus we may assume that \( v \in V(L) \). Since \( v \in X_{p+1} \), it follows that either \( \phi(v) > p + 1 \), or \( v \in B(L_{p+1}) \). If \( \phi(v) > p + 1 \), then since \( v \notin V(L_{p+1}) \), it follows that \( v \notin V(L_p) \), and hence \( v \in X_p \) from the definition of \( X_p \). Thus we may assume that \( \phi(v) \leq p + 1 \) and \( v \notin B(L_{p+1}) \), contrary to (2). This proves that \( X_{p+1} \subseteq X_p \).

For the second inclusion, let \( v \in Y_p \). Suppose first that \( v \notin V(L) \). Then \( v \in B(L_p) \); and so \( v \in B(L_{p+1}) \) by (5), and hence \( v \in Y_{p+1} \) as required. Thus we may assume that \( v \in V(L) \). Since \( v \in Y_p \), it follows that \( \phi(v) \leq p \). Now \( v \notin V(L_p) \), and therefore \( v \notin V(L_{p+1}) \). But \( \phi(v) \leq p + 1 \), and so by (2), \( v \in B(L_{p+1}) \), and consequently \( v \notin X_{p+1} \). Thus \( v \in Y_{p+1} \), as required. This proves that \( Y_p \subseteq Y_{p+1} \), and so proves (6).

(7) For all \( p \in \{0, \ldots, n-1\} \), and all \( j \in \{1, \ldots, k\} \), \( V(P_{p+1,j}) \subseteq V(P_{p,j}) \cup X_p \) and \( V(P_{p,j}) \subseteq V(P_{p+1,j}) \cup Y_{p+1} \).

To prove the first assertion, let \( v \in V(P_{p+1,j}) \setminus V(P_{p,j}) \). It follows that \( \phi(v) > p \); but then \( v \in X_p \) from the definition of \( X_p \). For the second assertion, let \( v \in V(P_{p,j}) \setminus V(P_{p+1,j}) \); then \( \phi(v) \leq p + 1 \), and so \( v \in B(L_{p+1}) \) by (2). Consequently \( v \notin X_{p+1} \), and so \( v \in Y_{p+1} \) as required. This proves (7).

(8) For all \( p \in \{0, \ldots, n-1\} \), \( (L_p, X_p, Y_p) \rightarrow (L_{p+1}, X_{p+1}, Y_{p+1}) \).

This is immediate from (4), (6) and (7).

Now \((L_1, X_1, Y_1), \ldots, (L_n, X_n, Y_n)\) are not necessarily all distinct. But we have:

(9) For all \( p, r \) with \( 0 \leq p \leq r \leq n \), if \((L_p, X_p, Y_p) = (L_r, X_r, Y_r)\), then \((L_p, X_p, Y_p) = (L_q, X_q, Y_q)\) for all \( q \) with \( p \leq q \leq r \).

For (6) implies that \( X_q \subseteq X_p \), and \( X_r \subseteq X_q \), and so \( X_p = X_q \), and similarly \( Y_p = Y_q \). If some vertex \( v \) belongs to \( V(L_q) \setminus V(L_p) \), then by (7) and (6), \( v \in X_p = X_q \), a contradiction. Similarly, if \( v \in V(L_p) \setminus V(L_q) \) then \( v \in Y_q = Y_p \), a contradiction. This proves (9).

(10) For all \( j \in \{1, \ldots, k\} \), \( M_{0,j} \) has first vertex \( s_j \), and \( M_{n,j} \) has last vertex \( t_j \).

This follows from the definitions of \( M_{0,j} \) and \( M_{n,j} \).

We recall that \( H \) is the \((k, m, c)\)-tracker, with two special vertices \( s_0, t_0 \). Now (10) implies that \( s_0 \) is adjacent to \((L_1, X_1, Y_1)\) in \( H \), and \((L_n, X_n, Y_n)\) is adjacent to \( t_0 \). From (8) and (9), there is a subsequence of the sequence

\( s_0, (L_1, X_1, Y_1), \ldots, (L_n, X_n, Y_n), t_0, \)

which lists the vertex set in order of a path of \( H \) from \( s_0 \) to \( t_0 \). By 3.2, this path traces some linkage \( L' \) for \((G, s_1, t_1, \ldots, s_k, t_k) \). But for all \( j \in \{1, \ldots, k\} \), \( M_{0,j}, M_{1,j}, \ldots, M_{n,j} \) are all subpaths of \( P_j \);
and since their union is a path from $s_j$ to $t_j$, it follows that their union is $P_j$. Hence $L' = L$. This proves 3.3.

4 The algorithm

Next, we need a polynomial algorithm to solve a kind of vector-valued shortest path problem. If $n \geq 0$ is an integer, $K_n$ denotes the set of all $k$-tuples $(x_1, \ldots, x_k)$ of nonnegative integers such that $x_1 + \cdots + x_k \leq n$.

4.1 There is an algorithm as follows:

- **Input:** A digraph $H$, and distinct vertices $s_0, t_0 \in V(H)$; an integer $n \geq 0$; and for each edge $e$ of $H$, a member $l(e)$ of $K_n$.
- **Output:** The set of all minimal (under component-wise domination) vectors $l(P)$, over all paths $P$ of $H$ from $s_0$ to $t_0$; where for a path $P$ with edge set $\{e_1, \ldots, e_p\}$, $l(P) = l(e_1) + \cdots + l(e_p)$.
- **Running time:** $O(n^k|V(H)||E(H)|)$.

**Proof.** Let $Q_0(s_0) = \{(0, \ldots, 0)\}$, and let $Q_0(v) = \emptyset$ for every other vertex $v$ of $D$. Inductively, for $1 \leq i \leq |V(H)|$, let $Q_i(v)$ be the set of minimal vectors in $K_n$ that either belong to $Q_{i-1}(v)$ or are expressible in the form $l(e) + x$ for some edge $e = uv$ of $H$ and some $x \in Q_{i-1}(u)$.

Now here is an algorithm for the problem:

- For $i = 1, \ldots, |V(H)|$ in turn, compute $Q_i(v)$ for every $v \in V(H)$.
- Output $Q_{|V(H)|}(t_0)$.

It is easy to check that this output is correct, and we leave it to the reader. To compute $Q_i(v)$ at the $i$th step takes time $O(n^k d^-(v))$, where $d^-(v)$ is the in-degree of $v$ in $H$ (since $K_n$ has at most $(n + 1)^k$ members), and so the $i$th step in total takes time $O(n^k|E(H)|)$. Thus the running time is $O(n^k|V(H)||E(H)|)$. Finally, we can give the main algorithm, 1.4, which we restate.

4.2 For all $d, k \geq 1$, there is an algorithm as follows:

- **Input:** A problem instance $(G, s_1, t_1, \ldots, s_k, t_k)$ where $G$ is $d$-path-dominant.
- **Output:** The set of all key qualities of $(G, s_1, t_1, \ldots, s_k, t_k)$.
- **Running time:** $O(n^t)$ where $t = 6k^2d(k + d) + 13k$.

**Proof.** Here is the algorithm.
• Compute the \((k, m, c)\)-tracker \(H\), where

\[
c = ((k - 1)d + k^2 + 2)(2k + 1)k^2
\]
\[
m = ((k - 1)d + k^2 + 2)d + 1.
\]

• For each edge \(e = uv\) of \(H\), define \(l(e)\) as follows:
  - if \(u = s_0\) and \(v = (L, X, Y)\) where \(L = (M_1, \ldots, M_k)\), let \(l(e) = (|V(M_1)|, \ldots, |V(M_k)|)\);
  - if \(u = (L, X, Y)\) where \(L = (M_1, \ldots, M_k)\), and \(v = (L', X', Y')\) where \(L' = (M'_1, \ldots, M'_k)\), let \(l(e) = (|V(M'_1) \setminus V(M_1)|, \ldots, |V(M'_k) \setminus V(M_k)|)\);
  - if \(v = t_0\) let \(l(e) = (0, \ldots, 0)\).

• Run the algorithm of 4.1 with input \(H, s_0, t_0, l\).

• Output its output.

To see its correctness, we must check that every key quality is in the output, and everything in the output is a key quality. We show first that every vector in the output is a quality. For let \(x\) be in the output, and let \(P\) be a path in \(H\) from \(s_0\) to \(t_0\) with \(l(P) = x\). By 3.2, \(P\) traces some linkage \(L = (P_1, \ldots, P_k)\) for \((G, s_1, t_1, \ldots, s_k, t_k)\); and so \((|V(P_1)|, |V(P_2)|, \ldots, |V(P_k)|) = l(P) = x\). Hence \(x\) is a quality.

Next, we show that every key quality is in the output. For let \(x\) be a key quality. Let \(L\) be an \(x\)-linkage for \((G, s_1, t_1, \ldots, s_k, t_k)\). By 3.3, there is a path \(P\) of \(H\) from \(s_0\) to \(t_0\) tracing \(L\); and hence \(l(P) = x\) (where \(l(P)\) is defined as in the statement of 4.1). Thus the output of 4.1 contains a vector dominated by \(x\). But \(x\) does not dominate any other quality, since it is key; and since every member of the output is a quality, it follows that \(x\) belongs to the output.

Third, we show that every member of the output is key. For let \(x\) be in the output, and suppose it is not key. Hence \(x\) dominates some other quality, and hence dominates some other key quality \(y\) say. Consequently \(y\) is in the output. But no two members of the output dominate one another, a contradiction. This proves that every member of the output is key, and so completes the proof that the output of the algorithm is as claimed.

Finally, for the running time: by 3.1, we can find all \((k, m, c)\)-rails in time \(O(n^{2km+1})\); and since there are at most \(O(n^{2km})\) of them (by 3.1), we can compute \(H\) and the function \(l\) in time \(O(n^{4km})\). Then running 4.1 takes time \(O(n^k|V(H)|^3)\), and hence time at most \(O(n^{6km+k})\). Thus the total running time is \(O(n^{6km+k})\). Since \(m = ((k - 1)d + k^2 + 2)d + 1\), the running time is \(O(n^t)\) where

\[
t = 6k(k - 1)d^2 + 6k(k^2 + 2)d + 7k = 6k^2d^2 + 6k^3d + 12kd + 7k - 6kd^2 \leq 6k^2d(k + d) + 13k
\]
as claimed. This proves 4.2.

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