A COMBINATORIAL IDENTITY FOR THE JACOBIAN OF $t$-SHIFTED INVARIANTS

OKSANA YAKIMOVA

ABSTRACT. Let $g$ be a simple Lie algebra. There are classical formulas for the Jacobians of the generating invariants of the Weyl group of $g$ and of the images under the Harish-Chandra projection of the generators of $ZU(g)$. We present a modification of these formulas related to Takiff Lie algebras.

INTRODUCTION

Let $g$ be a simple complex Lie algebra, $h \subset g$ be a Cartan subalgebra. Fix a triangular decomposition $g = n^- \oplus h \oplus n^+$. Let $\Delta \subset h^*$ be the corresponding root system with $\Delta^+ \subset \Delta$ being the subset of positive roots. Define $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let $W = W(g,h)$ be the Weyl group of $g$. Set $n = \text{rk } g$ and let $d_i - 1$ with $1 \leq i \leq n$ be the exponents of $g$. For $\alpha \in \Delta^+$, let $\{f_\alpha, h_\alpha, e_\alpha\} \subset g$ be an $sl_2$-triple with $e_\alpha \in g_\alpha$. Finally choose a basis $\{h_1, \ldots, h_n\}$ of $h$.

For polynomials $P_1, \ldots, P_n \in S(h) \cong \mathbb{C}[h^*]$, the Jacobian $J(\{P_i\})$ is defined by the property

$$dP_1 \wedge dP_2 \wedge \ldots \wedge dP_n = J(\{P_i\})dh_1 \wedge \ldots \wedge dh_n.$$ 

If $\hat{P}_1, \ldots, \hat{P}_n \in S(h)^W$ are generating invariants (with $\text{deg } \hat{P}_i = d_i$), then

$$J(\{\hat{P}_i\}) = C \prod_{\alpha \in \Delta^+} h_\alpha \quad \text{with } C \in \mathbb{C}, C \neq 0$$

by a classical argument, which is presented, for example, in [H90, Sec. 3.13].

Let $ZU(g)$ denote the centre of the enveloping algebra $U(g)$. Then $ZU(g)$ has a set $\{P_i \mid 1 \leq i \leq n\}$ of algebraically independent generators such that $P_i \in U_{d_i}(g)$. Let $P_i \in S(h)$ be the image of $P_i$ under the Harish-Chandra projection. Then $\hat{P}_i \in S(h)^W$ for $\hat{P}_i(x) = P_i(x - \rho)$, see e.g. [Di74, Sec. 7.4], and

$$J(\{P_i\}) = C \prod_{\alpha \in \Delta^+} (h_\alpha + \rho(h_\alpha)).$$

For any complex Lie algebra $l$, let $\varpi : S(l) \to U(l)$ be the canonical symmetrisation map. Let $S(l)^l$ denote the ring of symmetric $l$-invariants. Since $\varpi$ is an isomorphism of $l$-modules, it provides an isomorphism of vector spaces $S(l)^l \cong ZU(l)$.

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Suppose next that \( \mathcal{P}_i = \varpi(H_i) \) is the symmetrisation of \( H_i \) and that \( H_i \in S(\mathfrak{g})^g \) is a homogeneous generator of degree \( d_i \). Let \( T : \mathfrak{g} \to \mathfrak{g}[t] \) be the \( \mathbb{C} \)-linear map sending each \( x \in \mathfrak{g} \) to \( xt \). Then \( T \) extends uniquely to the commutative algebras homomorphism
\[
T : S(\mathfrak{g}) \to S(\mathfrak{g}[t]).
\]
Set \( H_i^{[1]} = T(H_i) \) and \( \mathcal{P}_i^{[1]} = \varpi(H_i^{[1]}) \). Here \( \mathcal{P}_i^{[1]} \in \mathcal{U} (t \mathfrak{g}[t]) \).

The triangular decomposition of \( \mathfrak{g} \) extends to \( \mathfrak{g}[t] \) as \( \mathfrak{g}[t] = \mathfrak{n}^- [t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t] \). Let \( P_i^{[1]} \in S(\mathfrak{h}[t]) \) be the image of \( \mathcal{P}_i^{[1]} \) under the Harish-Chandra projection. In order to define the Jacobian \( J(\{P_i^{[1]}\}) \) as an element of \( S(\mathfrak{h}) \), set at first \( \partial_{x_j} (xt^k) = kt^{k-1} \) for every \( x \in \mathfrak{h} \) and every \( k \geq 1 \), \( \partial_{h_i} (h_j t^k) = 0 \) for \( i \neq j \). Then the desired formula reads
\[
J(\{P_i^{[1]}\}) = \det (\partial_{h_j} P_i^{[1]})_{t=1}.
\]

**Theorem 1.** We have the following identity
\[
J(\{P_i^{[1]}\}) = C \prod_{\alpha \in \Delta^+} (h_\alpha + \rho(h_\alpha) + 1).
\]

Our proof of Theorem 1 interprets the zero set of \( J(\{P_i^{[1]}\}) \) in terms of the Takiff Lie algebra \( \mathfrak{q} = \mathfrak{g}[u]/(u^2) \) and then uses the extremal projector associated with \( \mathfrak{g} \), see Section 2.1 for the definition.

In 1971, Takiff proved that \( S(\mathfrak{q})^g \) is a polynomial ring whose Krull dimension equals \( 2 \rk \mathfrak{g} \) [Ta71]. This has started a serious investigation of these Lie algebras and their generalisations, see e.g. [PY] and reference therein. Verma modules and an analogue of the Harish-Chandra homomorphism for \( \mathfrak{q} \) were defined and studied in [G95, W11]. We remark that \( \mathfrak{q} \)-modules appearing in this paper are essentially different.

1. **Several combinatorial formulas**

Keep the notation of the introduction. In particular, \( H_i \in S(\mathfrak{g})^g \) stands for a homogeneous generator of degree \( d_i \), \( \mathcal{P}_i \) is the image of \( \mathcal{P}_i = \varpi(H_i) \) under the Harish-Chandra projection, \( \hat{P}_i \in S(\mathfrak{h})^W \) is the \((-\rho)\)-shift of \( P_i \), i.e., \( \hat{P}_i(x) = P_i(x - \rho) \), and \( P_i^{[1]} \) is the image of \( \varpi(T(H_i)) \) under the Harish-Chandra projection related to \( \mathfrak{g}[t] \). Let also \( P_i^o \) be the highest degree component of \( P_i \). Then \( P_i^o = H_i \mid \mathfrak{h} \). By the Chevalley restriction theorem, the polynomials \( P_i^o \) with \( 1 \leq i \leq n \) generate \( S(\mathfrak{h})^W \). The constant \( C \) is fixed by the equality
\[
J(\{P_i^o\}) = C \prod_{\alpha \in \Delta^+} h_\alpha. \quad \text{It is clear that} \quad J(\{P_i^o\}) = J(\{\hat{P}_i\}) .
\]

**Lemma 1.1.** The highest degree component of \( J(\{P_i^{[1]}\}) \) is equal to \( C \prod_{\alpha \in \Delta^+} h_\alpha \).

**Proof.** The highest degree component of \( P_i^{[1]} \) is \( T(P_i^o) \in S(\mathfrak{h} t) \). Each monomial of \( P_i^{[1]} \) is of the form \( (x_1 t) \ldots (x_d t) \) with \( x_j \in \mathfrak{h} \) for each \( j \). By the construction, \( \partial_{h_j} T(P_i^o)_{t=1} = \partial_{h_j} P_i^o \). The result follows. \( \square \)
In order to prove the next lemma, we need a well-known equality, namely \( \prod_{i=1}^{n} d_i = |W| \).

**Lemma 1.2.** We have \( J(\{P_i^{[1]}\})(0) = C \prod_{\alpha \in \Delta^+} (\rho(h_\alpha) + 1) \).

**Proof.** Clearly \( P_i^{[1]} |_{t=1} = P_i \). Since \( H_i \) is a homogeneous polynomial of degree \( d_i \), the linear in \( h \) part of \( P_i^{[1]} \) has degree \( d_i \) in \( t \). It follows that

\[
J(\{P_i^{[1]}\})(0) = (d_1 \ldots d_n) J(\{P_i\})(0) = |W| C \prod_{\alpha \in \Delta^+} \rho(h_\alpha).
\]

According to a formula of Kostant:

\[
\prod_{\alpha \in \Delta^+} \frac{\rho(h_\alpha) + 1}{\rho(h_\alpha)} = |W^\vee| = |W|.
\]

This completes the proof. \( \square \)

**Remark 1.3.** The Kostant formula (1.1) is a particular case of another combinatorial statement. Let \( W(t) = \sum_{w \in W} t^{l(w)} \) be the Poincaré polynomial of \( W \). Then

\[
W^\vee(t) = \prod_{\alpha \in \Delta^+} t^{(\rho, \alpha^\vee) + 1} - 1 \; t^{(\rho, \alpha^\vee) - 1},
\]

see Equation (34) in [H90, Sec. 3.20]. Since \( \rho(h_\alpha) = (\rho, \alpha^\vee) \), evaluating at \( t = 1 \) one gets exactly Eq. (1.1).

**Example 1.4.** Take \( g = \mathfrak{sl}_2 \) with the usual basis \( \{e, h, f\} \). Then \( H = H_1 = 4ef + h^2 \), \( H^{[1]} = 4ef + (ht)^2 \), and

\[
P^{[1]} = \pi(H^{[1]}) = 2ef t + 2ft e + (ht)^2 = (ht)^2 + 2ht^2 + 4ft e.
\]

Therefore \( P^{[1]} = P^{[1]}_1 = (ht)^2 + 2ht^2 \). Computing the partial derivative and evaluating at \( t = 1 \), we obtain \( J(\{P^{[1]}\}) = 2h + 4 = 2(h + 2) \). Observe that \( \rho(h) = 1 \).

## 2. TAKIFF LIE ALGEBRAS AND BRANCHING

Theorem 1 can be interpreted as a statement in representation theory of Takiff Lie algebras

\[
q = g \times g^{ab} \cong g[u]/(u^2).
\]

The first factor, the non-Abelian copy of \( g \), acts on \( g^{ab} = gu \) as a subalgebra of \( \mathfrak{gl}(g) \). Therefore there is the canonical embedding \( q \subset \mathfrak{gl}(g) \times g^{ab} \). Set \( \ell = \dim g + 1 \). In its turn, \( \mathfrak{gl}(g) \times g^{ab} \) can be realised as a subalgebra of \( \mathfrak{gl}(g \oplus \mathbb{C}) \cong \mathfrak{gl}_\ell(\mathbb{C}) \). The Lie algebra \( \mathfrak{gl}_\ell(\mathbb{C}) \) is equipped with the standard triangular decomposition. Let \( b_\ell \subset \mathfrak{gl}_\ell(\mathbb{C}) \) be the corresponding positive Borel. Recall that we have chosen a triangular decomposition
Lemma 2.1. The map \( T \) takes \( \sum d \) with \( \xi_i \in g \) to
\[
\sum_{i=1}^{d} \xi_1 \ldots \xi_{i-1}(\xi_i u)\xi_{i+1} \ldots \xi_d .
\]
Set \( R_i = \varpi(\psi(H_i)) \). The elements \( R_1, \ldots, R_n \in U(q) \) are not necessary central. If we assume that \( d_1 = 2 \), then \( R_1 \in \mathcal{Z}U(q) \). However, since both maps, \( \psi \) and \( \varpi \), are homomorphisms of \( g \)-modules, each \( R_i \) commutes with \( g \). Note that the elements \( R_i \) have degree 1 in \( gu \). They are crucial for further considerations and our next goal is to relate them to \( P_i^{[1]} \in U(tg[t]) \).

The map \( \psi \) is also well-defined for the tensor algebra of \( g \), but not for \( U(g) \), because of the following obstacle
\[
\psi((\xi_1, \xi_2) - (\xi_2, \xi_1)) = (\xi_1 u)\xi_2 + \xi_1(\xi_2 u) - (\xi_2 u)\xi_1 - \xi_2(\xi_1 u) = [\xi_1 u, \xi_2] + [\xi_1, \xi_2 u] = 2[\xi_1, \xi_2]u \neq [\xi_1, \xi_2]u.
\]
The remedy is to pass to the current algebras \( g[t] \) and \( q[t] \). Let \( T: U(tg[t]) \to U(q[t]) \) be a \( C \)-linear map such that
\[
T(\xi^k) = k(\xi u)t^{k-1} \text{ for each } \xi \in g,
\]
\[
T(ab) = T(a)b + aT(b) \text{ for all } a, b \in U(tg[t]), T \text{ is a derivation}.
\]
Of course, one has to check that \( T \) exists.

**Lemma 2.1.** The map \( T \) is well-defined.

**Proof.** Take \( \xi, \eta \in g \). Then
\[
T(\xi^k) = k(\xi u)t^{k-1}\eta^m + m\xi^k(\eta u)t^{m-1} - m(\eta u)t^{m-1}\xi^k - k\eta^m(\xi u)t^{k-1} = k[\xi u, \eta]t^{k-1-m} + m[\xi, \eta u]t^{k+m} = (k+m)[\xi, \eta]u)t^{k+m} = T(\xi^k).
\]

Now, having the map \( T \), we can state that \( R_i = T(P_i^{[1]})|_{t=1} \).

A word of caution, in \( U(b^-)(n^+ u) \) and similar expressions, \( (n^+ u) \) stands for the subspace \( n^+ u \subset g^{ab} \) and \textbf{not} for an ideal generated by \( n^+ u \). The same applies to \( (bu), (gu) \), etc.

**Lemma 2.2.** Let \( M_\lambda = U(b^-)v_\lambda \) with \( \lambda \in \mathbb{C}^t \) be a Verma module of \( gl(g^{ab} \mathbb{C}) \). Set \( \mu = \lambda|_b \). There exists a non-trivial linear combination \( R = \sum c_i R_i \) such that \( Rv_\lambda \in n^-U(b^-)(n^+ u)v_\lambda \) if and only if \( J(\{ P_i^{[1]} \})(\mu) = 0 \).

**Proof.** We have
\[
P_i^{[1]} \in P_i^{[1]} + n^-[t]U(g[t])n^+[t].
\]
Accordingly \( R_i = T(P_i^{[1]})|_{t=1} + \mathcal{X} \), where \( \mathcal{X} \) is the image of the second summand of \( P_i^{[1]} \). Let \( X = x_1 \ldots x_r \) be a monomial appearing in \( \mathcal{X} \). If \( x_r \in n^+ \), then \( Xv_\lambda = 0 \). Assume that
Then necessary \( x_r \in \mathbb{n}^+ u \) and \( x_1, \ldots, x_{r-1} \in \mathfrak{g} \). If \( x_i \in \mathbb{n}^+ \) for some \( i \leq (r-1) \), then we replace \( X \) by \( x_1 \ldots x_{i-1} [x_i, x_{i+1} \ldots x_r] \). Note that here \([x_i, x_r] \in \mathbb{n}^+ u \). Applying this procedure as long as possible one replaces \( X \) by an element of \( \mathfrak{u}(b^-)(\mathbb{n}^+ u) \) without altering \( Xv_\lambda \). Since \( X \) is an invariant of \( \mathfrak{h} \), the new element lies in \( \mathbb{n}^- \mathfrak{u}(b^-)(\mathbb{n}^+ u) \). Summing up,

\[ (2.1) \quad \mathcal{R}(P_i^{(1)})|_{t=1} = \sum_{j=1}^{n} (\partial_{h_j} P_i^{(1)})|_{t=1} h_j u, \]

where the partial derivatives are understood in the sense of the introduction. Exactly these derivatives have been used in order to define \( J(\{P_i^{(1)}\}) \). Hence \( J(\{P_i^{(1)}\})(\mu) = 0 \) if and only if there is a non-zero vector \( \bar{c} = (c_1, \ldots, c_n) \) such that \( \sum c_i \mathcal{T}(P_i^{(1)})|_{t=1,\mu} = 0 \). This shows that if \( J(\{P_i^{(1)}\})(\mu) = 0 \), then \( \mathcal{R}v_\lambda \in \mathbb{n}^- \mathfrak{u}(b^-)(\mathbb{n}^+ u) v_\lambda \).

Suppose now that \( \mathcal{R}v_\lambda \in \mathbb{n}^- \mathfrak{u}(b^-)(\mathbb{n}^+ u) v_\lambda \subseteq \mathfrak{u}(\mathbb{n}^-)(\mathbb{n}^+ u) v_\lambda \). Then

\[ \sum c_i \mathcal{T}(P_i^{(1)})|_{t=1,\mu} v_\lambda \in \mathfrak{u}(\mathbb{n}^-)(\mathbb{n}^+ u) v_\lambda. \]

Since we are working with a Verma module of \( \mathfrak{gl}_\ell(\mathbb{C}) \) and since \( \mathcal{T}(P_i^{(1)})|_{t=1,\mu} \in \mathfrak{h}u \subseteq \mathfrak{n}_\ell^-, \mathbb{n}^+ u \subseteq \mathfrak{n}_\ell^- \), we have

\[ \sum c_i \mathcal{T}(P_i^{(1)})|_{t=1,\mu} \in \mathfrak{u}(\mathbb{n}^-)(\mathbb{n}^+ u). \]

At the same time \( \mathfrak{h}u \cap \mathbb{n}^+ u = 0 \). Therefore \( \sum_{i=1}^{n} c_i (\partial_{h_j} P_i^{(1)})|_{t=1,\mu} = 0 \) for each \( j \) and thus \( J(\{P_i^{(1)}\})(\mu) = 0 \). \( \square \)

For \( \gamma \in \mathfrak{h}^* \), let \( M_{\lambda,\gamma} \) be the corresponding weight subspace of \( \mathfrak{u}(q)v_\lambda \subseteq M_\lambda \). Since \( \mathfrak{h}u \subseteq \mathfrak{n}^-_\ell \), either \( M_{\lambda,\gamma} = 0 \) or \( \dim M_{\lambda,\gamma} = \infty \). We have also \( (\mathfrak{h}u)v_\lambda \neq 0 \). Because of these facts, the q-modules \( M_\lambda \) and \( \mathfrak{u}(q)v_\lambda \) do not fit in the framework of the highest weight theory developed in [G95, W11]. Nevertheless, they may have some nice features.

Lemma 2.2 relates \( J(\{P_i^{(1)}\}) \) to a property of the branching \( q \downarrow \mathfrak{g} \) in a particular case of the q-module \( \mathfrak{u}(q)v_\lambda \). In order to get a better understanding of this branching problem, we employ a certain projector introduced by Asherova, Smirnov, and Tolstoy in [AST].

### 2.1. The extremal projector

Recall that \( \{f_\alpha, h_\alpha, e_\alpha\} \subset \mathfrak{g} \) is the \( \mathfrak{sl}_2 \)-triple corresponding to \( \alpha \in \Delta^+ \). Set

\[ p_\alpha = 1 + \sum_{k=1}^{\infty} f_\alpha^k e_\alpha^k \frac{(-1)^k}{k!(h_\alpha + \rho(h_\alpha) + 1) \ldots (h_\alpha + \rho(h_\alpha) + k)}. \]

Set \( N = |\Delta^+| \). Choose a numbering of positive roots, \( \alpha_1, \ldots, \alpha_N \). Each \( p_\alpha \), as well as any product of finitely many of them, is a formal series with coefficients in \( \mathbb{C}(\mathfrak{h}^*) \) in monomials

\[ f_{\alpha_1}^{r_1} \ldots f_{\alpha_N}^{r_N} e_{\alpha_N}^{k_N} \ldots e_{\alpha_1}^{k_1} \]

such that \( (k_1 - r_1)\alpha_1 + \ldots + (k_N - r_N)\alpha_N = 0 \).
A total order on $\Delta^+$ is said to be normal if either $\alpha < \alpha + \beta < \beta$ or $\beta < \alpha + \beta < \alpha$ for each pair of positive roots $\alpha, \beta$ such that $\alpha + \beta \in \Delta$. There is a bijection between the normal orders and the reduced decompositions of the longest element of $W$.

Choose a normal order $\alpha_1 < \ldots < \alpha_N$, and define

$$p = p_{\alpha_1} \cdots p_{\alpha_N}$$

accordingly. The element $p$ is called the extremal projector. It is independent of the choice of a normal order. For proofs and more details on this operator see [M07, §9.1]. Most importantly, it has the property that

$$(2.2)\quad e_\alpha p = pf_\alpha = 0$$

for each $\alpha$.

The nilpotent radical $n_\ell \subset b_\ell$ acts on $M_\lambda$ locally nilpotently. Recall that $n^+ \subset n_\ell$. Let $v \in M_\lambda$ be an eigenvector of $h_i$ of weight $\gamma \in h^*_i$. First of all, $pv$ is a finite sum of vectors of $M_\lambda$ with coefficients in $\mathbb{C}(h^*_i)$. Second, if all the appearing denominators are non-zero at $\gamma$, then $pv$ is a well-defined vector of $M_\lambda$ of the same weight $\gamma$.

3. PROOF OF THEOREM 1

Let $\lambda$, $\mu$, and $M_\lambda$ be as in Lemma 2.2. Keep in mind that $\lambda$ and $\mu$ are arbitrary elements of $\mathbb{C}^\ell$ and $\mathbb{C}^n$. Since each $R_i$ commutes with $g$, each $R_i v_\lambda$ is a highest weight vector of $g$.

We use the extremal projector $p$ associated with $g$. If $p$ can be applied to a highest weight vector $v$, then $pv = v$. Suppose that $p$ is defined at $\mu$. Then, in view of (2.1) and (2.2),

$$R_i v_\lambda = p R_i v_\lambda = p \mathcal{T}(P_i^{[1]}|_{t=1}) v_\lambda.$$ 

Assume that $J(\{P_i^{[1]}\})(\mu) = 0$. Then there is a non-trivial linear combination $\mathcal{R} = \sum c_i R_i$ such that

$$\mathcal{R} v_\lambda \in n^- \mathcal{U}(b^-) (n^+ u) v_\lambda,$$

see Lemma 2.2. Here $p\mathcal{R} v_\lambda = 0$ and hence $\mathcal{R} v_\lambda = 0$ as well.

Since we are considering a Verma module of $gl_\ell(\mathbb{C})$, this implies that

$$\mathcal{R} \in \mathcal{U}(gl_\ell(\mathbb{C})) b_\ell \cap \mathcal{U}(q) = \mathcal{U}(q)b.$$ 

Hence the symbol $gr(\mathcal{R})$ of $\mathcal{R}$ lies in the ideal of $S(q)$ generated by $b$.

The decomposition $g = n^- \oplus b$ defines a bi-grading on $S(g)$. Let $H_i^*$ be the bi-homogeneous component of $H_i$ having the highest degree w.r.t. $n^-$. According to [PY12, Sec. 3], $H_i^* \in b S^{d_i-1}(n^-)$ and the polynomials $H_1^*, \ldots, H_n^*$ are algebraically independent. We have

$$\psi(H_i^*) \in (bu) S^{d_i-1}(n^-) \oplus b(n^- u) S^{d_i-2}(n^-).$$

Write this as $\psi(H_i^*) \in H_{i,1} + b(n^- u) S^{d_i-2}(n^-)$. Then the polynomials $H_{i,1}$ with $1 \leq i \leq n$ are still algebraically independent. As can be easily seen, $\psi(H_i) \in H_{i,1} + b S(q)$. 

Set \( d = \max_{i, c_i \neq 0} d_i \). Then

\[
\text{gr}(\mathcal{R}) = \sum_{i, d_i = d} c_i \psi(H_i)
\]

and it lies in \((h) \triangleleft S(q)\) if and only if \( \sum_{i, d_i = d} c_i H_{i,1} = 0 \). Since at least one \( c_i \) in this linear combination is non-zero, we get a contradiction. The following is settled: if \( p \) is defined at \( \mu \), then \( J(\{P_i^{[1]}\})(\mu) \neq 0 \).

Now we know that the zero set of \( J(\{P_i^{[1]}\}) \) lies in the union of hyperplanes \( h_\alpha + \rho(h_\alpha) = -k \) with \( k \geq 1 \). At the same time this zero set is an affine subvariety of \( \mathbb{C}^n \) of codimension one. Therefore it is the union of \( N \) hyperplanes and \( J(\{P_i^{[1]}\}) \) is the product of \( N \) linear factors of the form \( (h_\alpha + \rho(h_\alpha) + k_\alpha) \). A priori, a root \( \alpha \) may appear in several factors with different constants \( k_\alpha \).

By Lemma 1.1, the highest degree component of \( J(\{P_i^{[1]}\}) \) is equal to \( C \prod_{\alpha \in \Delta^+} h_\alpha \). Therefore each \( \alpha \in \Delta^+ \) must appear in exactly one linear factor of \( J(\{P_i^{[1]}\}) \). Observe that \( \rho(h_\alpha) \geq 1 \) and that \( \rho(h_\alpha) + k_\alpha \geq \rho(h_\alpha) + 1 \). If for some \( \alpha \), we have \( k_\alpha > 1 \), then

\[
|J(\{P_i^{[1]}\})(0)| > |C| \prod_{\alpha \in \Delta^+} (\rho(h_\alpha) + 1).
\]

But this cannot be the case in view of Lemma 1.2.

\[ \square \]

4. CONCLUSION

The elements \( \mathcal{R}_i \) are rather natural \( g \)-invariants in \( \mathcal{U}(q) \) of degree one in \( gu \). Note that because \( gu \) is an Abelian ideal of \( q \), where is no ambiguity in defining the degree in \( gu \). The involvement of these elements in the branching rules \( q \downarrow g \) remains unclear. However, combining Lemma 2.2 with Theorem 1, we obtain the following statement.

**Corollary 4.1.** In the notation of Lemma 2.2, there is a non-trivial linear combination \( \mathcal{R} = \sum c_i \mathcal{R}_i \) such that \( \mathcal{R}v_\lambda \in n^- \mathcal{U}(q)v_\lambda \) if and only if \( \mu(h_\alpha) = -\rho(h_\alpha) - 1 \) for some \( \alpha \in \Delta^+ \). \( \square \)

As the theory of finite-dimensional representations suggests, it is unusual for a highest weight vector of \( g \) to belong to the image of \( n^- \). The proof of Theorem 1 shows that \( \mathcal{R}v_\lambda \neq 0 \) for the linear combination of Corollary 4.1.

**Remark 4.2.** The subspace \( \mathcal{V}[1] = (\mathcal{U}(g)(gu))^0 \subset \mathcal{U}(q) \) is a \( \mathbb{Z}\mathcal{U}(g) \)-module. From a well-known description of \( (g \otimes S(q))^0 \), one can deduce that \( \mathcal{V}[1] \) is freely generated by \( \mathcal{R}_1, \ldots, \mathcal{R}_n \) as a \( \mathbb{Z}\mathcal{U}(g) \)-module. There are other choices of generators in \( \mathcal{V}[1] \) and it is not clear, whether one can get nice formulas for the corresponding Jacobians.
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(O. Yakimova) Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Deutschland

E-mail address: yakimova.oksana@uni-koeln.de