WEIGHTED $\theta$-INCOMPLETE PLURIPOTENTIAL THEORY

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ABSTRACT. Weighted pluripotential theory is a rapidly developing area; and Callaghan \cite{Cal07} recently introduced $\theta$-incomplete polynomials in $\mathbb{C}^d$ for $d > 1$. In this paper we combine these two theories by defining weighted $\theta$-incomplete pluripotential theory. We define weighted $\theta$-incomplete extremal functions and obtain a Siciak-Zahariuta type equality in terms of $\theta$-incomplete polynomials. Finally we prove that the extremal functions can be recovered using orthonormal polynomials and we demonstrate a result on strong asymptotics of Bergman functions in the spirit of \cite{Ber}.

1. Introduction

The theory of $\theta$-incomplete polynomials in $\mathbb{C}^d$ for $d > 1$ was recently developed by Callaghan \cite{Cal07}. It has many applications in approximation theory. He also defined interesting extremal functions in terms of $\theta$-incomplete polynomials and related plurisubharmonic functions.

This paper has three goals. The first one is to further develop the $\theta$-incomplete pluripotential theory of Callaghan. The second goal is to combine this theory with weighted pluripotential theory and get a unified theory by defining weighted $\theta$-incomplete pluripotential theory in $\mathbb{C}^d$. If $\theta = 0$, we get weighted pluripotential theory, and for the weight $w = 1$, we get $\theta$-incomplete pluripotential theory. Finally we show that extremal functions in these settings can be recovered asymptotically using orthonormal polynomials.

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In this section we recall some definitions and major results of weighted pluripotential theory and we recall Berman’s paper [Ber] which is a special case of weighted pluripotential theory. Our initial goal was to study Berman’s recent work on globally defined weights within the framework of $\theta$-incomplete pluripotential theory. We were able to prove many results for admissible weights defined on closed subsets of $\mathbb{C}^d$.

In the second section we recall some important results of $\theta$-incomplete pluripotential theory. We improve a result of Callaghan and we extend a result of Bloom and Shiffman [BS07] to the $\theta$-incomplete extremal function $V_{K,\theta}$ associated to a compact set $K$ for $0 \leq \theta < 1$.

In the third section we work on closed subsets of $\mathbb{C}^d$. We define the weighted $\theta$-incomplete extremal function $V_{K,Q,\theta}$ for a closed set $K$ and an admissible weight function $w$ and we give various properties of this extremal function. We also show that $V_{K,Q,\theta}$ can be obtained via taking the supremum of $\theta$-incomplete polynomials whose weighted norm is less then or equal to 1 on $K$, generalizing the analogous result for $V_{K,\theta}$ (unweighted case) from the previous section. In particular we state analogous results in the case of global weights.

In the last section we recall the Bernstein-Markov property relating the sup norms and $L^2(\mu)$ norms of polynomials on a compact set $K$ with measure $\mu$. We define a version of the Bernstein-Markov property for $\theta$-incomplete polynomials in the weighted setting. Then we prove results on asymptotics of orthonormal polynomials to extremal functions in the $\theta$-incomplete and weighted setting. Finally in Theorem [4.7] we prove a result on strong asymptotics of Bergman functions analogous to the main theorem in [Ber].

1.1. Weighted Pluripotential Theory. We give some basic definitions from weighted pluripotential theory. A good reference is Saff and Totik’s book [ST97] for $d = 1$ and Thomas Bloom’s Appendix B of [ST97] for $d > 1$.

Let $K$ be a non-pluripolar closed subset of $\mathbb{C}^d$. An upper semicontinuous function $w: K \to [0, \infty)$ is called an admissible weight function on $K$ if

i) the set $\{z \in K \mid w(z) > 0\}$ is not pluripolar and
ii) If $K$ is unbounded, $|z| w(z) \to 0$ as $|z| \to \infty$, $z \in K$.

We define $Q = Q_w = -\log w$, and we will use $Q$ and $w$ interchangeably.

The weighted pluricomplex extremal function of $K$ with respect to $Q$ is defined as

\begin{equation}
V_{K,Q}(z) := \sup \{ u(z) \mid u \in L, u \leq Q \text{ on } K \},
\end{equation}

where the Lelong class $L$ is defined as

\begin{equation}
L := \{ u \mid u \text{ is plurisubharmonic on } \mathbb{C}^d, u(z) \leq \log^+ |z| + C \}.
\end{equation}

We recall that the upper semicontinuous regularization of a function $v$ is defined by $v^*(z) := \limsup_{w \to z} v(w)$ and it is well known that the upper semicontinuous regularization of $V_{K,Q}$ is plurisubharmonic and in $L^+$ where

\[ L^+ := \{ u \in L \mid \log^+ |z| + C \leq u(z) \}. \]

By Lemma 2.3 of Bloom’s Appendix B of [S197], the support, $S_w$, of $(dd^cV_{K,Q}^*)^d$ is a subset of $S^*_w := \{ z \in K \mid V_{K,Q}^*(z) \geq Q(z) \}$.

Here $dd^c v = 2i \partial \bar{\partial} v$ and $(dd^c v)^d$ is the complex Monge-Ampère operator defined by $(dd^c v)^d = dd^c v \wedge \cdots \wedge dd^c v$ for plurisubharmonic functions which are $C^2$. For the cases considered in this paper see [Kli91, Dem87] for the details of the definition.

A set $E$ is called pluripolar if $E \subset \{ z \in \mathbb{C}^d \mid u(z) = -\infty \}$ for some plurisubharmonic function $u$. If a property holds everywhere except on a pluripolar set we will say the property holds quasi everywhere.

1.2. A Special Case of Weighted Pluripotential Theory. We recall some definitions from Berman’s paper [Ber], where the weight is defined globally in $\mathbb{C}^d$. Let $\phi$ be a lower semicontinuous function, and $\phi(z) \geq (1 + \varepsilon) \log |z|$ for $z \gg 1$ for some fixed $\varepsilon > 0$. The weighted extremal function is defined as

\begin{equation}
V_{\phi}(z) := \sup \{ u(z) \mid u \in L \text{ and } u \leq \phi \text{ on } \mathbb{C}^d \}.
\end{equation}
We define

\begin{align}
S_\phi^* &:= \{ z \in \mathbb{C}^d \mid V_\phi^*(z) \geq \phi(z) \} \text{ and} \\
S_\phi &:= \text{supp}((dd^c V_\phi)^d). \tag{1.4} \\
S_\phi &:= \text{supp}((dd^c V_\phi)^d). \tag{1.5}
\end{align}

This is a special case of weighted pluripotential theory with \( K = \mathbb{C}^d \) and \( Q = \phi \). Hence \( S_\phi \subset S_\phi^* \).

Berman [Ber] studied the case where the global weight \( \phi \in C^{1,1}(\mathbb{C}^d) \). In this case we define

\begin{align}
D_\phi &= \{ z \in \mathbb{C}^d \mid V_\phi(z) = \phi(z) \}, \\
P &= \{ z \in \mathbb{C}^d \mid dd^c \phi(z) \text{ exist and is positive} \}. \tag{1.6} \tag{1.7}
\end{align}

We remark that \( D_\phi \) is a compact set and \( S_\phi \subset D_\phi \). By Proposition 2.1 of [Ber], if \( \phi \in C^{1,1}(\mathbb{C}^d) \), then we have \( V_\phi \in C^{1,1}(\mathbb{C}^d) \) and \( (dd^c V_\phi)^d = (dd^c \phi)^d \) on \( D_\phi \cap P \) almost everywhere as \( (d,d) \) forms with \( L^\infty \) coefficients.

**Example 1.1.** Let \( \phi(z) = |z|^2 \). Then we have

\begin{equation}
V_\phi(z) = \begin{cases} 
|z|^2 & \text{if } |z| \leq \frac{1}{\sqrt{2}}, \\
\log |z| + \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} & \text{if } |z| \geq \frac{1}{\sqrt{2}}.
\end{cases} \tag{1.8}
\end{equation}

Clearly the plurisubharmonic function, \( V \), on the right hand side is less than or equal to \( \phi \), hence \( V \leq V_\phi \). On the other hand the support of the Monge-Ampère measure of \( V \) is the closed ball of radius \( 1/\sqrt{2} \) centered at the origin. Since any competitor, \( u \), for the extremal function is less than or equal to \( |z|^2 \) on this closed ball, by the domination principle, (see Appendix B of [ST97] or Theorem 2.1 below) \( u \) is less than or equal to \( V \) on \( \mathbb{C}^d \). Therefore \( V_\phi \leq V \) and hence equality holds.
2. \(\theta\)-Incomplete Pluripotential Theory

We recall the basic notions of \(\theta\)-incomplete pluripotential theory from \cite{Cal07}. We fix \(0 \leq \theta \leq 1\). A \(\theta\)-incomplete polynomial in \(\mathbb{C}^d\) is a polynomial of the form

\[
P(z) = \sum_{|\alpha| = \lfloor N \theta \rfloor} c_\alpha z^\alpha,
\]

where \([x]\) is the least integer greater than or equal to \(x\). Here we use the following multi-index notations. Let \(z = (z_1, \ldots, z_d) \in \mathbb{C}^d\) and \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d\), then

\[
z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d} \quad \text{and} \quad |\alpha| = \alpha_1 + \cdots + \alpha_d
\]

The set of all \(\theta\)-incomplete polynomials of the form (2.1) will be denoted by \(\pi_{N,\theta}\). We remark that when \(\theta = 0\), \(\pi_{N,\theta}\) is the set of all polynomials of degree at most \(N\); and when \(\theta = 1\), \(\pi_{N,\theta}\) is the set of homogenous polynomials of degree \(N\).

Related classes of plurisubharmonic functions are defined as follows (See \cite{Cal07} for details).

\[
L_\theta = \{ u \in L | u(z) \leq \theta \log |z| + C \text{ for } |z| < 1 \},
\]

\[
L_\theta^+ = \{ u \in L_\theta | \max(\theta \log |z|, \log |z|) + C \leq u(z) \text{ for all } z \in \mathbb{C}^d \}.
\]

We remark that if \(P \in \pi_{N,\theta}\) then \(\frac{1}{N} \log |P| \in L_\theta\). Another observation is if \(\theta_1 \geq \theta_2\), then \(L_{\theta_2} \subset L_{\theta_1}\).

The next theorem gives a domination principle for \(L_\theta\) classes.

**Theorem 2.1.** \cite{Cal07} Theorem 3.15] Let \(0 \leq \theta < 1\). If \(u \in L_\theta\) and \(v \in L_\theta^+\) and if \(u \leq v\) holds almost everywhere with respect to \((dd^c v)^d\), then \(u \leq v\) on \(\mathbb{C}^d\).

We remark that for \(0 < \theta < 1\), we have \(u(0) = v(0) = -\infty\) and the origin is a distinguished point as it is charged by \((dd^c v)^d\).

Callagham \cite{Cal07} defined the following extremal function for a set \(E \subset \mathbb{C}^d\):

\[
V_{E,\theta}(z) := \sup \{ u(z) : u \in L_\theta \text{ and } u \leq 0 \text{ on } E \}.
\]
We will call it the $\theta$-incomplete extremal function of $E$. The upper semicontinuous regularization $V_{E,\theta}^*$, is in $L^+_\theta$ if $E$ is not pluripolar by Lemma 3.7 of [Cal07]. Also if $K$ is a regular compact set in $\mathbb{C}^d$, then $V_{K,\theta}^* = V_{K,\theta}$. Hence it is continuous except at $z = 0$. Here regular means the extremal function of $K$, $V_K := V_{K,0}$ is continuous. We remark that $(dd^c V_{E,\theta}^*)^d$ is supported in $\bar{E} \cup \{0\}$.

According to [Cal07] we have the following result for compact sets $K$,}

\begin{equation}
V_{K,\theta} = \log \Phi'_{K,\theta},
\end{equation}

where

$$\Phi'_{K,\theta}(z) = \sup\{|f(z)|^{1/N} : f \in \pi_{N,\theta} \text{ for some } N \geq 1, \|f\|_K \leq 1\}.$$\n
We define the following functions for a compact set $K$. For $N \geq 1$ we let

\begin{equation}
\Phi_{K,\theta,N}(z) = \sup\{|f(z)| : f \in \pi_{N,\theta}, \|f\|_K \leq 1\}
\end{equation}

and

\begin{equation}
\Phi_{K,\theta} = \sup_N (\Phi_{K,\theta,N})^{1/N}.
\end{equation}

The next proposition shows that the supremum in (2.7) is actually a limit.

**Proposition 2.2.** With the above notation we have

$$\sup_N \frac{1}{N} \log \Phi_{K,\theta,N} = \lim_{N \to \infty} \frac{1}{N} \log \Phi_{K,\theta,N}$$

and $\Phi'_{K,\theta} = \Phi_{K,\theta}$.

Hence we have $\lim_{N \to \infty} \frac{1}{N} \log \Phi_{K,\theta,N} = V_{K,\theta}$.

**Proof.** First of all we have $\Phi_{K,\theta,J} \Phi_{K,\theta,I} \leq \Phi_{K,\theta,J+I}$ for all integers $I, J \geq 0$. For if $P(z) = \sum_{|\alpha| = [\theta J]} a_\alpha z^\alpha$ and $Q(z) = \sum_{|\alpha| = [\theta I]} b_\alpha z^\alpha$, then $PQ(z) = \sum_{|\alpha| = [\theta (J+I)]} c_\alpha z^\alpha$ is in $\pi_{J+I,\theta}$, since $[\theta J] + [\theta I] \geq [\theta (J+I)]$.

By taking logarithms we get

\begin{equation}
\log \Phi_{K,\theta,J} + \log \Phi_{K,\theta,I} \leq \log \Phi_{K,\theta,J+I},
\end{equation}

so by Theorem 4.9.19 of [BG91], $\lim_{N \to \infty} \frac{1}{N} \log \Phi_{K,\theta,N}$ exists and equals $\sup_N \frac{1}{N} \log \Phi_{K,\theta,N}$. Now by Callaghan’s result (2.5) we get the last equality $\Phi'_{K,\theta} = \Phi_{K,\theta}$.

□
In the next section we will extend this result to the weighted case. This proposition also fixes a gap in the proof of Theorem 8.2 in [Cal06] and we will use it in the proof of Theorem 4.3.

The following theorem extends a result of Bloom and Shiffman [BS07] to the $\theta$-incomplete case.

**Theorem 2.3.** Let $K$ be a regular compact set in $\mathbb{C}^d$. Then

$$\frac{1}{N} \log \Phi_{K,\theta,N} \rightarrow V_{K,\theta}$$

uniformly on compact subsets of $\mathbb{C}^d \setminus \hat{K}_\theta$.

Here $\hat{K}_\theta$ is the $\theta$-incomplete hull of $K$ defined for a compact set $K$ as

$$\hat{K}_\theta = \{ z \in \mathbb{C}^d \mid |p(z)| \leq \|p\|_K \text{ for all } p \in \pi_{N,\theta} \text{ for } N = 0, 1, \ldots \}. $$

It is clear that for $\theta > 0$, the origin always belongs to $\hat{K}_\theta$ for any set $K$, so $\hat{K}_\theta$ is often larger then the usual polynomially convex hull $\hat{K} := \hat{K}_0$. It is also easy to see that $\hat{K}_\theta = \{ z \in \mathbb{C}^d \mid V_{K,\theta} \leq 0 \}$.

**Proof.** Let $E$ be a compact set in $\mathbb{C}^d \setminus \hat{K}_\theta$. First we want to show that there exists $N_0$ such that $\Phi_{K,\theta,N}(z) > 1$ for all $N > N_0$ for all $z \in E$.

We fix $z_0 \in E$ and $\delta > 0$ such that $V_{K,\theta}(z_0) = 2\delta$. By the above proposition we have

$$\lim_{N \to \infty} \frac{1}{N} \log \Phi_{K,\theta,N}(z_0) = 2\delta,$$

so there exists an integer $N_{z_0}$ such that for all $N \geq N_{z_0}$ we have $\frac{1}{N} \log \Phi_{K,\theta,N}(z_0) > \delta$. In particular $\Phi_{K,\theta,N}(z_0) > 1$ for all $N > N_{z_0}$.

Since $\Phi_{K,\theta,N_{z_0}}$ is the supremum of continuous plurisubharmonic functions, it is lower semicontinuous. Hence $U_{z_0} := \{ z \in \mathbb{C}^d \mid \Phi_{K,\theta,N_{z_0}}(z) > 1 \}$ is open. Now we can cover $E$ by the sets $U_{z_0}, \ldots, U_{z_m}$, of $E$. Hence taking $N_0$ to be the largest of $N_{z_1}, \ldots, N_{z_m}$, we can conclude that $\Phi_{K,\theta,N}(z) > 1$ for all $z \in E$ and for all $N \geq N_0$. Thus we have $1 \leq \Phi_{K,\theta,I} \leq \Phi_{K,\theta,I} \leq \Phi_{K,\theta,I+1}$ for all $I, J \geq N_0$ on $E$.

We follow [BS07] to prove that the sequence converges uniformly on $E$. We will write $\psi_N = \frac{1}{N} \log \Phi_{K,\theta,N}$. We note that $\psi_{N_k} \geq \psi_N$ for all $N \geq N_0$. We see this by
\[ \psi_{Nk} = \frac{1}{Nk} \log \Phi_{K,\theta,Nk} \geq \frac{1}{Nk} \log(\Phi_{K,\theta,N})^k = \frac{k}{Nk} \log \Phi_{K,\theta,N} = \psi_N \text{ for } N \geq N_0. \]

From (2.8), we have \( Nk\psi_{Nk} + j\psi_j \leq (Nk + j)\psi_{Nk+j} \) for \( N, k \geq 1, j \geq 0 \). Since \( \psi_j > 0 \) on \( E \) for \( j > N_0 \), using \( \psi_{Nk} \geq \psi_N \) for such \( j \) we get

\[ (2.10) \quad \psi_{Nk+j} \geq \frac{Nk}{Nk + j} \psi_N + \frac{j}{Nk + j} \psi_j \geq \frac{Nk}{Nk + j} \psi_N. \]

Let \( \varepsilon > 0 \). For each \( a \in E \) we can choose \( N_a > N_0 \) large so that \( V_{K,\theta}(a) - \psi_{N_a}(a) < \varepsilon \) and \( \frac{V_{K,\theta}(a)}{N_a} < \varepsilon \), and then we can find an open neighborhood \( U_a \) of \( a \) such that \( |V_{K,\theta}(z) - V_{K,\theta}(a)| < \varepsilon, \psi_{N_a}(z) > \psi_{N_a}(a) - \varepsilon \), and \( \frac{V_{K,\theta}(z)}{N_a} < \varepsilon \) for \( z \in U_a \). This is possible by the facts that regularity of \( K \) implies the continuity of \( V_{K,\theta} \) and that \( \psi_{N_a} \) is lower semicontinuous.

Now we find a finite number of points \( a_1, \ldots, a_M \) in \( E \) such that the open sets \( U_{a_1}, \ldots, U_{a_M} \) cover \( E \). We choose \( N_1 = \max_{a_1, \ldots, a_M} (N_{a_1}^2 + N_{a_i}) \). Now for each \( a_i \) if \( N \geq (N_{a_1}^2 + N_{a_i}) \), we write \( N = N_a(k - 1) + j \), where \( k \geq N_a \), and \( N_a \leq j \leq 2N_a \). By Proposition 2.2 and (2.10) we get

\[ 0 \leq V_{K,\theta} - \psi_N \leq V_{K,\theta} - \frac{N_a(k - 1)}{N_a(k - 1) + j} \psi_{N_a} \leq V_{K,\theta} - \frac{N_a}{N_a} \psi_{N_a} \leq V_{K,\theta} - \psi_{N_a} + \frac{2}{N_a} V_{K,\theta}. \]

Let \( z \in E \), then \( z \in U_{a_i} \) for some \( a_i \), hence for all \( N \geq N_1 \) we have

\[ 0 \leq V_{K,\theta}(z) - \psi_N(z) < V_{K,\theta}(z) - \psi_{N_a}(z) + 2\varepsilon \]

\[ = [V_{K,\theta}(z) - V_{K,\theta}(a_i)] + [V_{K,\theta}(a_i) - \psi_{N_a}(a_i)] + [\psi_{N_a}(a_i) - \psi_{N_a}(z)] + 2\varepsilon \]

\[ \leq 5\varepsilon. \]

Thus we have the desired uniform convergence on \( E \).

3. Weighted \( \theta \)-Incomplete Pluripotential Theory

In this section we define and develop two weighted versions of \( \theta \)-incomplete pluripotential theory. The first one is the \( \theta \)-incomplete version of the weighted pluripotential theory in closed subsets of \( \mathbb{C}^d \) and the second one is the \( \theta \)-incomplete version of the special case of weighted pluripotential theory studied in [Ber]. As in the \( \theta = 0 \) case the second version is a special case of the first.
3.1. Weighted $\theta$-Incomplete Pluripotential Theory with Weight Defined on Closed Sets. Let $K$ be a closed set in $\mathbb{C}^d$ and $w$ be an admissible weight on $K$ as defined in Subsection 1.1. Then we define

$$V_{K,Q,\theta}(z) := \sup \{u(z) \mid u \in L_\theta, u \leq Q \text{ on } K\}. \tag{3.1}$$

We remark that $V_{K,Q,\theta_1} \leq V_{K,Q,\theta_2}$ if $\theta_1 > \theta_2$. The $\theta = 0$ case gives the classical weighted pluripotential theory. Following Siciak [Sic81], it can be shown that $V_{K,Q,\theta} = V_{K,Q,\theta}^*$, so that $V_{K,Q,\theta}$ is continuous on $\mathbb{C}^d \setminus \{0\}$, for $K$ locally regular and $Q$ continuous.

Here $K$ locally regular means for all $a \in K$, we have $K \cap B(a,r)$ is regular for all $r > 0$, where $B(a,r) := \{z \in \mathbb{C}^d \mid |z - a| < r\}$.

Comparing the defining families we get the following obvious inequalities.

**Proposition 3.1.** Let $K_1 \subset K_2$ and let $w$ be a function defined on $K_2$ which is an admissible weight on both $K_1$ and $K_2$. Then $V_{K_1,Q,\theta} \leq V_{K_2,Q,\theta}$.

Using (ii) in the definition of admissibility from section 1.1, we show that $V_{K,Q,\theta}$ coincides with the weighted $\theta$-incomplete extremal function of a compact subset of $K$.

**Lemma 3.2.** If $K$ is unbounded then $V_{K,\rho,Q,\theta}^* = V_{K,Q,\theta}^*$, for some $\rho > 0$ where $K_\rho = \{z \in K \mid |z| \leq \rho\}$.

**Proof.** Since $V_{K,\rho,Q,\theta}^* \in L_\theta$, there exists $C$ and $\rho$ such that

$$V_{K,\rho,Q,\theta}^*(z) \leq \log |z| + C \text{ for } |z| > \rho.$$ 

Now by the second condition of admissibility we may choose $\rho$ large enough that

$$Q(z) - \log |z| \geq C + 1 \text{ for } z \in K \setminus K_\rho.$$ 

If $u \in L_\theta$ and $u \leq Q$ on $K_\rho$, so that $u \leq V_{K_\rho,Q,\theta}^*$, by the above inequalities we get $u \leq Q$ on $K$. Hence we get $V_{K,\rho,Q,\theta}^* \leq V_{K,Q,\theta}^*$. The other inequality is given by Proposition 3.1, which gives the equality. \hfill \square

**Proposition 3.3.** Let $K$ be a closed subset of $\mathbb{C}^d$ and let $w$ be an admissible weight function on $K$ then $V_{K,Q,\theta}^* \in L_\theta^+$.  

Proof. The case $\theta = 0$ is the classical case and is well known. For $0 < \theta \leq 1$ we will follow the proof of Lemma 3.7 of [Cal07].

Since $V_{K,Q,\theta}^* \leq V_{K,Q}^*$ and $V_{K,Q}^* \in L^+$, we have $V_{K,Q,\theta}^* \in L$.

Next we show that $V_{K,Q,\theta}^* \in L_\theta$. Let $M := \sup_{z \in B(0,1)} V_{K,Q,\theta}^*(z)$ and $u$ be in the defining class for $V_{K,Q,\theta}$. Then $\frac{1}{\theta}(u - M) \leq 0$ on $B(0,1)$. Hence it is a competitor for the pluricomplex Green function of the unit ball $B(0,1)$ with logarithmic pole at the origin. The pluricomplex Green function of a bounded domain $\Omega$ with logarithmic pole at $a \in \Omega$ is defined by

$$g_{\Omega}(z,a) := \sup\{u(z) \mid u \text{ plurisubharmonic on } \Omega, u \leq 0 \text{ and } u(z) - \log |z - a| \leq C \text{ as } z \to a\},$$

and $g_{B(0,1)}(z,0) = \log |z|$. Hence $\frac{1}{\theta}(u - M) \leq \log |z|$ on the unit ball. Since $u$ is arbitrary we get $V_{K,Q,\theta}^*(z) \leq \theta \log |z| + M$ on $B(0,1)$. Thus $V_{K,Q,\theta}^* \in L_\theta$.

By Lemma 3.2 we may assume $K \subset B(0,R)$ for some $R$. Let $A := \sup_{z \in B(0,R)} (\theta \log |z| - Q(z))$, then $u(z) = \max(\theta \log |z|, \log |z|) - A$ is a competitor for the extremal function $V_{K,Q,\theta}$ and $u \in L_\theta^+$, hence $V_{K,Q,\theta}^* \in L_\theta^+$. \hfill \Box

We define the following sets:

$$S_{K,Q,\theta}^* := \{z \in K \mid V_{K,Q,\theta}^*(z) \geq Q(z)\} \quad \text{and}$$

$$(3.2) \quad (3.3) \quad S_{K,Q,\theta} := \text{supp}((dd^cV_{K,Q,\theta}^*)^d).$$

Lemma 3.4. Let $K$ be closed in $\mathbb{C}^d$ and let $w$ be an admissible weight on $K$. Then $S_{K,Q,\theta} \subset S_{K,Q,\theta}^* \cup \{0\}$ if $0 < \theta \leq 1$ and $S_{K,Q,\theta} \subset S_{K,Q,\theta}^*$ if $\theta = 0$.

Proof. The classical case, i.e. when $\theta = 0$, is Lemma 2.3 of Appendix B of [ST97]. Therefore we assume $0 < \theta \leq 1$. Let $z_0$ be a point in $K \setminus \{0\}$ such that $V_{K,Q,\theta}^*(z_0) < Q(z_0) - \varepsilon$ for some positive $\varepsilon$. We will show that $V_{K,Q,\theta}^*$ is maximal in a neighborhood of $z_0$, i.e. $(dd^cV_{K,Q,\theta}^*)^d = 0$ there.

Since $Q$ is lower semicontinuous we have $\{z \in K \mid Q(z) > Q(z_0) - \varepsilon/2\}$ is open relative to $K$. Similarly we have $\{z \in \mathbb{C}^d \mid V_{K,Q,\theta}^*(z) < V_{K,Q,\theta}^*(z_0) + \varepsilon/2\}$ is open. Thus
we may find a ball of radius \( r \) around \( z_0 \) such that \( \sup_{z \in B(z_0, r)} V_{K,Q,\theta}^*(z) < \inf_{z \in B(z_0, r) \cap K} Q(z) \) and \( 0 \notin B(z_0, r) \).

By Theorem 1.3 of Appendix B in [ST97], we can find a plurisubharmonic function \( u \) with \( u \geq V_{K,Q,\theta}^* \) on \( B(z_0, r) \), \( u = V_{K,Q,\theta}^* \) on \( \mathbb{C}^d \setminus B(z_0, r) \), and \( u \) maximal on \( B(z_0, r) \). Then \( u \leq V_{K,Q,\theta}^* \) because \( u(z) \leq \sup_{z \in B(z_0, r)} V_{K,Q,\theta}^*(z) < \inf_{z \in B(z_0, r) \cap K} Q(z) \) for all \( z \in B(z_0, r) \). Since \( B(z_0, r) \cap \{0\} \) we have \( u \in L_\theta \). Hence \( u \equiv V_{K,Q,\theta}^* \). Therefore we get \( V_{K,Q,\theta}^* \) is maximal in a neighborhood of \( z_0 \). Hence \( z_0 \) is not in \( S_{K,Q,\theta} \). □

A special case of this is when the admissible weights are globally defined. Let \( \phi : \mathbb{C}^d \rightarrow \mathbb{R} \) be an admissible weight function. Generalizing the case of [Ber] we define weighted \( \theta \)-incomplete extremal functions by

\[
V_{\phi,\theta}(z) = \sup \{ u(z) \mid u \in L_\theta \text{ and } u \leq \phi \} \text{ for } 0 \leq \theta \leq 1.
\]

Observe that \( V_{\phi,\theta}^* = V_{\phi,\theta} \) if \( \phi \) is continuous, for in this case \( V_{\phi,\theta}^* \leq \phi \) on \( \mathbb{C}^d \) so that \( V_{\phi,\theta}^* \leq V_{\phi,\theta} \). We also remark that \( \theta = 0 \) gives \( V_{\phi,0} = V_{\phi} \) and \( V_{\phi,\theta_1} \leq V_{\phi,\theta_2} \) if \( \theta_1 > \theta_2 \) since \( L_{\theta_1} \subset L_{\theta_2} \).

We define the following sets:

\[
D_{\phi,\theta} := \{ z \in \mathbb{C}^d \mid V_{\phi,\theta}^*(z) \geq \phi(z) \} \quad \text{and} \quad
S_{\phi,\theta} := \text{supp}((dd^c V_{\theta,\theta}^*)^d).
\]

If \( \theta = 0 \), we will write \( D_{\phi,0} = D_{\phi} \) and \( S_{\phi,0} = S_{\phi} \). If \( \phi \) is continuous then \( V_{\phi,\theta} \) is continuous and we have

\[
D_{\phi,\theta} = \{ z \in \mathbb{C}^d \mid V_{\phi,\theta}(z) = \phi(z) \}.
\]

If \( \phi \) is a globally defined admissible weight function then we define \( K := D_{\phi,\theta} \) and \( Q := \phi|_K \). Clearly \( V_{\phi,\theta}^* \leq Q \) quasi everywhere in \( K \) so \( V_{\phi,\theta}^* \leq V_{K,Q,\theta}^* \).

Conversely, on \( K \), \( V_{K,Q,\theta} \leq Q = \phi = V_{\phi,\theta} \) quasi everywhere. Since \( (dd^c V_{\phi,\theta}^*)^d \) is supported on \( K \cup \{0\} \), by Theorem 2.1 we have \( V_{K,Q,\theta}^* \leq V_{\phi,\theta}^* \). Hence \( V_{K,Q,\theta}^* = V_{\phi,\theta}^* \).

This shows that we may reduce the global weighted situation to the compact case by considering the sets \( D_{\phi,\theta} \).
As a consequence of the above definitions, Lemma 3.3 and earlier results of this section we have the following corollary.

**Corollary 3.5.** Let $\phi$ be a globally defined admissible weight, then we have

1) $S_{\phi, \theta} = \text{supp}((dd^c V_{\phi, \theta}^*)^d) \subset D_{\phi, \theta} \cup \{0\}$ if $\theta > 0$, and for $\theta = 0$,

\[
\text{supp}((dd^c V_{\phi}^*)^d) \subset D_{\phi},
\]

2) $D_{\phi, \theta_1} \subset D_{\phi, \theta_2} \subset D_{\phi, 0} = D_{\phi}$ where $\theta_1 > \theta_2$,

3) $V_{\phi, \theta}$ is in $L^+_{\theta}$ for $0 \leq \theta \leq 1$,

4) if $u \in L_{\theta}$ and $u \leq \phi$ on $D_{\phi, \theta}$ then $u \leq V_{\phi, \theta}$.

The next lemma shows the monotonicity of the extremal functions under increasing and decreasing $\theta$.

**Lemma 3.6.** Let $K \subset \mathbb{C}^d$ be a closed set and let $\omega$ be an admissible weight on $K$. For $0 \leq \theta_0 < 1$, as $\theta \searrow \theta_0$ we have $V_{K, Q, \theta}^*$ increases to $V_{K, Q, \theta_0}^*$ quasi everywhere. If $\theta \nearrow \theta_0$ we have $V_{K, Q, \theta}^*$ decreases to $V_{K, Q, \theta_0}^*$.

**Proof.** The last statement is clear, thus we consider $\theta \searrow \theta_0$. Clearly we have monotonicity of the $V_{K, Q, \theta}^*$. Since $V_{K, Q, \theta}^*$ are bounded above by $V_{K, Q, \theta_0}^*$, we have $V_{K, Q, \theta}^*$ increases to a function, $v$, whose upper semicontinuous regularization $v^*$ is plurisubharmonic and again bounded above by $V_{K, Q, \theta_0}^*$.

Since $V_{K, Q, \theta}^* \in L^+_{\theta}$ we have $V_{K, Q, \theta}^*(z) \geq \max(\theta \log |z|, \log |z|) + M_\theta$ where $M_\theta$ is a constant depending on $\theta$. As $\theta \searrow \theta_0$ we get $v^* \in L^+_{\theta_0}$ since $v^* \leq V_{K, Q, \theta_0}^*$. Also by monotonicity we get $(dd^c V_{K, Q, \theta}^*)^d \to (dd^c v^*)^d$ weak-*.

We will write $S := \text{supp}(dd^c v^*)^d \setminus \{0\}$ and $S' := \{z \in K \mid v^*(z) \geq Q(z)\}$. By the lower semicontinuity of $Q$, and upper semicontinuity of $v^*$, we have $S'$ is closed. Next we will show that $v^* \geq Q$ on $S$ by showing that $S \subset S'$.

Since $(dd^c V_{K, Q, \theta}^*)^d \to (dd^c v^*)^d$ we have $S \subset \bigcup_{\theta > \theta_0} S_{K, Q, \theta} \setminus \{0\}$. By Proposition 3.4, we have $\bigcup_{\theta > \theta_0} S_{K, Q, \theta} \setminus \{0\} \subset \bigcup_{\theta > \theta_0} S_{K, Q, \theta}^* \setminus \{0\} \subset \{z \in K \mid v(z) \geq Q(z)\} \subset S'$. Since $S'$ is closed, we get $\bigcup_{\theta > \theta_0} S_{K, Q, \theta_0} \setminus \{0\} \subset S'$. Therefore $S \subset S'$. Since $V_{K, Q, \theta_0}^* \leq Q$ quasi everywhere on $K$ and $(dd^c v^*)^d$ does not charge pluripolar sets except the origin,
we have $V^*_{K,Q,\theta_0} \leq v^*$ almost everywhere with respect to $(dd^c v^*)^d$ on the support of $(dd^c v^*)^d$. Here we recall that if $\theta > 0$ then $V^*_{K,Q,\theta_0}(0) = v^*(0) = -\infty$. Therefore by the domination principle (Theorem 2.1) we get $V^*_{K,Q,\theta_0} \leq v^*$ on $\mathbb{C}^d$, so that $V^*_{K,Q,\theta_0} = v^*$. □

**Corollary 3.7.** Let $\phi$ be a globally defined admissible weight. Let $0 \leq \theta_0 < 1$, as $\theta \searrow \theta_0$ we have $V^*_{\phi,\theta}$ increases to $V^*_{\phi,\theta_0}$ quasi everywhere, and if $\theta \nearrow \theta_0$ we have $V^*_{\phi,\theta}$ decreases to $V^*_{\phi,\theta_0}$.

The following example illustrates the above corollary.

**Example 3.8.** Let $\phi(z) = |z|^2$. Then we have for $0 < \theta < 1$

$$V_{\phi,\theta}(z) = \begin{cases} 
\theta \log |z| + \frac{\theta}{2} - \frac{\theta}{2} \log \frac{\theta}{2} & \text{if } |z| < \sqrt{\frac{\theta}{2}}, \\
|z|^2 & \text{if } \sqrt{\frac{\theta}{2}} \leq |z| \leq \sqrt{\frac{1}{2}}, \\
\log |z| + \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} & \text{if } |z| \geq \sqrt{\frac{1}{2}}.
\end{cases}$$

If $\theta = 1$ we get

$$V_{\phi,\theta}(z) = V_{\phi,1}(z) = \log |z| + \frac{1}{2} - \frac{1}{2} \log \frac{1}{2}.$$  

We had given $V_{\phi,\theta}$ earlier in (1.8).

Note that $D_{\phi,\theta} = \overline{B(0, \sqrt{\frac{\theta}{2}})} \setminus B(0, \sqrt{\frac{\theta}{2}})$ which increases to $\overline{B(0, \frac{1}{\sqrt{2}})} \setminus \{0\}$ as $\theta$ decreases to 0.

We define the following notions. Let $K \subset \mathbb{C}^d$ be compact and $w$ be an admissible weight on $K$. We define

$$\Phi_{K,Q,\theta}^N(z) := \sup\{|P(z)|^{1/N} \mid \|w^N P_N\|_K \leq 1 \text{ where } P_N \in \pi_{N,\theta}\}$$

and

$$\Phi_{K,Q,\theta} := \sup_N \Phi_{K,Q,\theta}^N = \lim_{N \to \infty} \Phi_{K,Q,\theta}^N.$$

We can see that the supremum is actually a limit by following the proof of Proposition 2.2.

**Theorem 3.9.** Let $0 \leq \theta < 1$. Let $K \subset \mathbb{C}^d$ be a compact set and $w$ be a continuous admissible weight on $K$. Then $V_{K,Q,\theta} = \log \Phi_{K,Q,\theta}$.
Proof. Let $P_N \in \pi_{N,\theta}$ satisfying $\|w^N P_N\|_K \leq 1$. Then we have
\[
\frac{1}{N} \log |P_N(z)| \leq Q(z) \text{ on } K.
\]
Hence we get
\[
(3.9) \quad \log \Phi_{K,Q,\theta} \leq V_{K,Q,\theta}.
\]

The rest of the proof essentially follows the proof of Callaghan [Cal07]. We will modify the last step using a result of Brelot-Cartan instead of Hartog’s lemma.

We fix $\varepsilon > 0$ such that $\theta + \varepsilon < 1$. Let $u \in L_{\theta + \varepsilon}$ and $u \leq Q$ on $K$. By Theorem 2.9 of Appendix B of [ST97], we have
\[
u(z) = \lim_{j \to \infty} \frac{1}{N_j} \max_{1 \leq k \leq t_j} \log |P_{k,j}(z)|,
\]
where the sequence is decreasing and each $P_{k,j}$ is a polynomial of degree at most $N_j$. Here $t_j$ is a finite number depending on $j$.

As in [Cal07] we write
\[
P_{k,j}(z) := \sum_{|\alpha| = 0}^{N_j} c_{\alpha,k,j} z^\alpha
\]
and
\[
P'_{k,j}(z) := \sum_{|\alpha| = 0}^{\lfloor N_j \theta \rfloor} c_{\alpha,k,j} z^\alpha,
\]
where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$.

We remark that $P_{k,j} - P'_{k,j}$ is a $\theta$-incomplete polynomial. By Callaghan’s asymptotic estimates we get
\[
u(z) = \lim_{j \to \infty} \frac{1}{N_j} \max_{1 \leq k \leq t_j} \log |P_{k,j}(z) - P'_{k,j}(z)|
\]
pointwise on $\mathbb{C}^d$.

By Theorem 3.4.3 c) of [Ran95], for $\varepsilon_1 > 0$, there exists $j_1$ such that for $j \geq j_1$ we have
\[
\frac{1}{N_j} \max_{1 \leq k \leq t_j} \log |P_{k,j}(z) - P'_{k,j}(z)| \leq Q + \varepsilon_1 \text{ on } K,
\]
since $Q$ is continuous. Now we have
\[ u(z) = \lim_{j \to \infty} \frac{1}{N_j} \max_{1 \leq k \leq t_j} \log |P_{k,j}(z) - P'_{k,j}(z)| \leq \log \Phi_{K,Q,0}(z) + \varepsilon_1 \]
for any $\varepsilon_1$ and therefore $u(z) \leq \log \Phi_{K,Q,0}(z)$. Hence we get
\[ V_{K,Q,\varepsilon}(z) \leq \log \Phi_{K,Q,0}(z). \]

By Lemma 3.6 as $\varepsilon \to 0$ we get
\[ V_{K,Q,0}(z) \leq \log \Phi_{K,Q,0}(z). \]

Combining (3.10) with (3.9) we get the desired result. □

Note that if $\theta = 0$, we recover
\[ (3.11) \quad V_{K,Q} = \log \Phi_{K,Q} \text{ where } \Phi_{K,Q} := \Phi_{K,Q,0} \]

**Corollary 3.10.** Let $0 \leq \theta < 1$. Let $\phi$ be a globally defined continuous admissible weight, then we have $V_{\phi,\theta} = \log \Phi_{\phi,\theta}$, where
\[ \Phi_{\phi,\theta}^N(z) := \sup \{ |P(z)|^{1/N} | \| e^{-N\phi} P_N \|_{D_{\phi,\theta}} \leq 1 \text{ where } P_N \in \pi_{N,\theta} \} \]
and
\[ \Phi_{\phi,\theta} := \sup_N \{ \Phi_{\phi,\theta}^N \}. \]

**Corollary 3.11.** Let $0 \leq \theta < 1$. Let $\phi$ be a globally defined continuous admissible weight, then we have $V_{\phi,\theta} = \log \tilde{\Phi}_{\phi,\theta}$, where
\[ \tilde{\Phi}_{\phi,\theta}^N(z) := \sup \{ |P(z)|^{1/N} | \| e^{-N\phi} P_N \|_{C^d} \leq 1 \text{ where } P_N \in \pi_{N,\theta} \} \]
and
\[ \tilde{\Phi}_{\phi,\theta} := \sup_N \{ \tilde{\Phi}_{\phi,\theta}^N \}. \]

**Proof.** It is sufficient to show that for any $P_N \in \pi_{N,\theta}$, $\| e^{-N\phi} P_N \|_{C^d} \leq 1$ if and only if $\| e^{-N\phi} P_N \|_{D_{\phi,\theta}} \leq 1$. The "only if" direction is trivial. For the other direction let $P_N \in \pi_{N,\theta}$ and $\| e^{-N\phi} P_N \|_{D_{\phi,\theta}} \leq 1$. We will show that $\| e^{-N\phi} P_N \|_{C^d} \leq 1$. We have $e^{-N\phi(z)} P_N(z) \leq 1$ for $z \in D_{\phi,\theta}$ so we get $\frac{1}{N} \log |P_N(z)| \leq \phi(z)$ on $D_{\phi,\theta}$. 
Hence it is a competitor for the extremal function $V_{\phi,\theta}$, and we have $\frac{1}{N} \log |P_N(z)| \leq V_{\phi,\theta}(z) \leq \phi(z)$ for all $z \in \mathbb{C}^d$. Therefore we get $e^{-N\phi(z)}P_N(z) \leq 1$ for all $z \in \mathbb{C}^d$. □

4. Asymptotics

Let $K$ be a compact set in $\mathbb{C}^d$ and $\mu$ be a Borel probability measure whose support is in $K$. We say that the pair $(K, \mu)$ satisfies a Bernstein-Markov property if for any $\varepsilon > 0$ there exists $C > 0$ such that

(4.1) \[ \|P\|_K \leq C e^{\varepsilon N} \|P\|_{L^2(\mu)} \]

holds for all polynomials of degree at most $N$. Equivalently, there exists $M_N$ with $(M_N)^{\frac{1}{N}} \to 1$ as $N \to \infty$ such that the following inequality holds for all polynomials of degree at most $N$:

(4.2) \[ \|P\|_K \leq M_N \|P\|_{L^2(\mu)}. \]

We remark that if $K$ is a regular compact set then $(K, (dd^c V_K)^d)$ satisfies the Bernstein-Markov property. See [Zé85] for details.

We fix $0 \leq \theta \leq 1$. If these inequalities are satisfied for all $P \in \pi_{N,\theta}$ for all $N \geq 0$, then we say the pair $(K, \mu)$ satisfies a Bernstein-Markov property for $\theta$-incomplete polynomials.

Let $\mu$ be a measure such that $(K, \mu)$ satisfies the Bernstein-Markov property for $\theta$-incomplete polynomials. Let $\{P_j\}$ be an orthonormal basis of $\pi_{N,\theta}$ with respect to the inner product $\langle f, g \rangle := \int f \overline{g} \, d\mu$. We define the Bergman function $K_{N,\theta}(z, w) := \sum_{j=1}^{d(N,\theta)} P_j(z)\overline{P_j(w)}$, where $d(N, \theta)$ is the dimension of $\pi_{N,\theta}$.

The following two lemmas are generalizations of results of Bloom and Shiffman [BS07].

Lemma 4.1. If $(K, \mu)$ satisfies the Bernstein-Markov property for $\theta$-incomplete polynomials, then for all $\epsilon > 0$, there exists $C > 0$ such that

(4.3) \[ \frac{(\Phi_{K,\theta,N}(z))^2}{d(N, \theta)} \leq K_{N,\theta}(z, z) \leq C e^{\epsilon N} (\Phi_{K,\theta,N}(z))^2 d(N, \theta) \]

for all $z \in \mathbb{C}^d$. 
Proof. To show the first inequality we take $P \in \pi_{N,\theta}$ and $\|P\|_K \leq 1$. Then we have

$$|P(z)| = \left| \int_K K_{N,\theta}(z,w)P(w)d\mu(w) \right| \leq \int_K |K_{N,\theta}(z,w)|d\mu(w) \leq \int_K (K_{N,\theta}(z,z))^\frac{1}{2}(K_{N,\theta}(w,w))^\frac{1}{2}d\mu(w) = (K_{N,\theta}(z,z))^\frac{1}{2}\|K_{N,\theta}(w,w)\|_{L^1(\mu)} \leq (K_{N,\theta}(z,z))^\frac{1}{2}\|1\|_{L^2(\mu)}\|K_{N,\theta}(w,w)\|_{L^2(\mu)} = (K_{N,\theta}(z,z))^\frac{1}{2}d(N,\theta)^\frac{1}{2}.$$

Taking the supremum of all $P$ as above we have $\Phi_{K,\theta,N}(z) \leq (K_{N,\theta}(z,z))^\frac{1}{2}d(N,\theta)^\frac{1}{2}$, which gives the first inequality.

For the second inequality, let $\{P_j\}$ be an orthonormal basis of $\pi_{N,\theta}$. Then by the Bernstein-Markov property we have $\|P_j\|_K \leq C\varepsilon N$, hence $|P_j(z)| \leq \|P_j\|_K \Phi_{K,\theta,N}(z)$ for all $P_j$. Thus we have

$$K_{N,\theta}(z,z) = \sum_{j=1}^{d(N,\theta)} |P_j(z)|^2 \leq d(N,\theta)C^2\varepsilon N(\Phi_{K,\theta,N}(z))^2.$$ 

Hence we get the second inequality. \qed

Lemma 4.2. Let $0 \leq \theta < 1$. Let $K$ be a regular compact set in $\mathbb{C}^d$. If $(K,\mu)$ satisfies the Bernstein-Markov property for $\theta$-incomplete polynomials, then we have

$$\frac{1}{2N}\log K_{N,\theta}(z,z) \to V_{K,\theta}(z)$$

uniformly on compact subsets of $\mathbb{C}^d \setminus \hat{K}_\theta$.

Proof. We remark that $d(N,\theta) \leq d(N) := d(N,0)$ and $d(N) = (N+d)^d$. Taking logarithms in (4.3), we obtain

$$-\frac{d(N,\theta)}{N} \leq \frac{\log(K_{N,\theta}(z,z))}{\log(\Phi_{K,\theta,N}(z))^2} \leq \frac{\log(C\varepsilon N d(N,\theta))}{N}.$$ 

By the above observation we get

$$-\frac{d}{N} \log(N+d) \leq \frac{1}{N} \log\left(\frac{K_{N,\theta}(z,z)}{(\Phi_{K,\theta,N}(z))^2}\right) \leq \frac{\log C}{N} + \varepsilon + \frac{d}{N} \log(N+d).$$

Since $\varepsilon$ is arbitrary we have $\frac{1}{N} \log\left(\frac{K_{N,\theta}(z,z)}{(\Phi_{K,\theta,N}(z))^2}\right) \to 0$, which gives the desired result by Theorem 2.3. \qed
Let $K$ be a compact set with admissible weight $w$ on $K$. Let $\mu$ be a Borel probability measure on $K$. We say the triple $(K, \mu, w)$ satisfies a **weighted Bernstein-Markov property** if there exists $M_N > 0$ with $(M_N)^{1/N} \to 1$ such that for any polynomial $P_N$ of degree $N$,

\[(4.4) \quad \|w^N P_N\|_K \leq M_N \|w^N P_N\|_{L^2(\mu)}.\]

We remark that if $K$ is locally regular and $Q$ is continuous then $(K, (dd^c V_{K,Q})^d, w)$ satisfies a weighted Bernstein-Markov property by Corollary 3.1 of [Blo06]. Also $(D\phi, (dd^c V_{\phi})^d, e^{-\phi})$ satisfies the weighted Bernstein-Markov property if $\phi$ is continuous by Theorem 4.5 of [BB].

**Theorem 4.3.** Let $K$ be a compact set with a continuous admissible weight $w$ on $K$. Let $\mu$ be a probability measure on $K$ such that $(K, \mu, w)$ satisfies a weighted Bernstein-Markov property. Then we have

\[(4.5) \quad \lim_{N \to \infty} \sup_{k=1, \ldots, d(N)} (|B_{k,N}(z)|)^{1/N} = e^{V_{K,Q}(z)},\]

where $\{B_{k,N}\}_{k=1}^{d(N)}$ is an orthonormal basis for the polynomials with degree at most $N$ with respect to the measure $w^{2N} \mu$.

We remark that unlike the unweighted case, where $w = 1$, each time $N$ changes the basis and the $L^2$ norms change.

**Proof.** By the weighted Bernstein-Markov property we have

\[\|w^N B_{k,N}\|_K \leq M_N \|w^N B_{k,N}\|_{L^2(\mu)},\]

so we get

\[\frac{1}{N} \log \frac{|B_{k,N}(z)|}{M_N} \leq Q(z) \text{ on } K.\]

Hence

\[\frac{1}{N} \log \frac{|B_{k,N}(z)|}{M_N} \leq V_{K,Q}(z) \text{ on } \mathbb{C}^d.\]

Since $(M_N)^{1/N} \to 1$, we have

\[\limsup_{N \to \infty} \sup_{k=1, \ldots, d(N)} (|B_{k,n}(z)|)^{1/N} \leq \limsup_{N \to \infty} (e^{V_{K,Q}(z)} M_N^{1/N}) \leq e^{V_{K,Q}(z)}.\]
Now we want to show that \( \liminf_{N \to \infty} (\sup_{P_k=1} d(N) |B_k,N(z)|)^{1/N} \geq e^{V_{K,Q}(z)} \), for \( V_{K,Q}(z) > 0 \).

Let \( P \) be a polynomial of degree at most \( N \) such that \( \|w^N P\|_K \leq 1 \). We will write \( w = e^{-Q} \). Since \( \{B_{k,N}\}_{k=1}^{d(N)} \) is an orthonormal basis we have

\[
P(z) = \sum_{j=1}^{d(N)} \left( \int_K P \bar{B}_{j,N} e^{-2NQ} d\mu \right) B_{j,N}(z).
\]

By the triangle inequality we have

\[
|P(z)| \leq \sum_{j=1}^{d(N)} \left| \int_K P \bar{B}_{j,N} e^{-2NQ} d\mu \right| |B_{j,N}(z)|.
\]

By the Cauchy-Schwarz inequality we have

\[
|P(z)| \leq \sum_{j=1}^{d(N)} \left( \int_K |P|^2 e^{-2NQ} d\mu \right)^{1/2} \left( \int_K |B_{j,N}|^2 e^{-2NQ} d\mu \right)^{1/2} |B_{j,N}(z)|.
\]

Now since \( \|w^N P\|_K \leq 1 \) and \( \{B_{k,N}\}_{k=1}^{d(N)} \) is an orthonormal basis we get

\[
|P(z)| \leq \sum_{j=1}^{d(N)} |B_{j,N}(z)|.
\]

This implies that

\[
|P(z)| \leq (d(N)) \sup_{j=1}^{d(N)} |B_{j,N}(z)| \quad \text{for any } z \in \mathbb{C}^d.
\]

We fix \( z \in \mathbb{C}^d \). Then we have

\[
e^{V_{K,Q}(z)} \leq \liminf_{N \to \infty} \left( \sup_{P \in \pi_{N,0}, \|w^N P\|_K \leq 1} |P(z)|^{1/N} \right) \leq \liminf_{N \to \infty} (d(N))^{1/N} \left( \sup_{j=1}^{d(N)} |B_{j,N}(z)| \right)^{1/N}.
\]

Here \( e^{V_{K,Q}} \leq \liminf_{N \to \infty} \left( \sup_{P \in \pi_{N,0}, \|w^N P\|_K \leq 1} |P(z)|^{1/N} \right) \) follows from (3.11). Now since \( (d(N))^{1/N} \to 1 \) we get the result. \( \square \)

**Corollary 4.4.** Let \( \phi \) be a globally defined continuous admissible weight and \( \mu \) be a Borel probability measure on \( D_\phi \) such that \( (D_\phi, \mu, e^{-\phi}) \) satisfies a weighted Bernstein-Markov property. Then we have

\[
\lim_{N \to \infty} \sup_{k=1, \ldots, d(N)} |B_{k,N}(z)|^{1/N} = e^{V_\phi(z)}.
\]
Here \( \{B_{k,N}\}_{k=1}^{d(N)} \) is an orthonormal basis for the polynomials with degree at most \( N \) with respect to the measure \( e^{-2N\phi}\mu \).

If \([4.4]\) holds for any \( P_N \in \pi_{N,\theta} \) then we say \((K,\mu,w)\) satisfies a \textbf{weighted Bernstein-Markov property for \( \theta \)-incomplete polynomials}.

We remark that if a triple \((K,\mu,w)\) satisfies a weighted Bernstein-Markov property, then it satisfies the weighted Bernstein-Markov property for \( \theta \)-incomplete polynomials.

Using only the orthonormal basis for \( \pi_{N,\theta} \) and using Theorem \( 3.9 \) instead of \( (3.11) \) we get the following theorem by the same proof as for Theorem \( 4.3 \).

\textbf{Theorem 4.5.} Let \( 0 \leq \theta < 1 \). Let \( K \) be a compact set with a continuous admissible weight \( w \) on \( K \). Let \( \mu \) be a measure on \( K \) such that \((K,\mu,w)\) satisfies the weighted Bernstein-Markov property for \( \theta \)-incomplete polynomials. Then we have

\[
\lim_{N \to \infty} \sup_{k=1,\ldots,d(N,\theta)} \left( |B_{k,N}(z)| \right)^{1/N} = e^{V_{K,Q,\theta}(z)},
\]

where \( \{B_{k,N}\}_{k=1}^{d(N,\theta)} \) is an orthonormal basis for \( \pi_{N,\theta} \) with respect to the measure \( w^{2N}\mu \).

\textbf{Corollary 4.6.} Let \( 0 \leq \theta < 1 \). Let \( \phi \) be a globally defined continuous admissible weight.

If \((D_{\phi},\mu,e^{-\phi})\) satisfies a weighted Bernstein-Markov property then we have

\[
\lim_{N \to \infty} \sup_{k=1,\ldots,d(N,\theta)} \left( |B_{k,N}(z)| \right)^{1/N} = e^{V_{\phi,\theta}(z)},
\]

where \( \{B_{k,N}\}_{k=1}^{d(N,\theta)} \) is an orthonormal basis for \( \pi_{N,\theta} \) with respect to the measure \( e^{-2N\phi}\mu \).

Finally, we prove the strong Bergman asymptotics in the weighted \( \theta \)-incomplete setting following \cite{Ber} closely. We fix \( 0 \leq \theta < 1 \). Let \( \phi \) be a globally defined admissible weight and \( \phi(z) \geq (1 + \varepsilon) \log|z| \) if \( |z| \gg 1 \). Let \( \{p_1,\ldots,p_{d(N,\theta)}\} \) be an orthonormal basis for \( \pi_{N,\theta} \) with respect to the inner product \( \langle f,g \rangle := \int_{\mathbb{C}^d} f g e^{-2N\phi}\omega_d \) where \( \omega_d(z) = (dd^c|z|^2)^d/4^dd! \) on \( \mathbb{C}^d \). We denote the \( L^2 \)-norm by \( ||p_N||_{L^2_{\phi}} := ||p_N||_{w_{d,N,\phi}}^2 = \int_{\mathbb{C}^d} |p_N(z)|^2 e^{-2N\phi(z)}\omega_d(z) \). We define the \( N \)-th \( \theta \)-incomplete Bergman function by

\[
K_N(z) := K_{N,\theta}(z,z) = \sum_{j=1}^{d(N,\theta)} |p_j(z)|^2 e^{-2N\phi(z)}.
\]
By the reproducing property of the Bergman functions we have

\[(4.12) \quad K_N(z) = \sup_{p_N \in \pi_{N,\theta} \setminus \{0\}} |p_N(z)|^2 e^{-2N\phi(z)}/\|p_N\|_{N,\theta}^2.\]

**Theorem 4.7.** Let \( \phi \in C^2(\mathbb{C}^d) \) with \( \phi(z) \geq (1 + \epsilon) \log |z| \) for \( |z| \gg 1 \). If \( V_{\phi,\theta} \in C^{1,1}(\mathbb{C}^d \setminus \{0\}) \) then \( (dd^cV_{\phi,\theta})^d \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{C}^d \setminus \{0\} \) and \( \det(dd^c\phi)\omega_d = (dd^cV_{\phi,\theta})^d \) on \( \mathbb{C}^d \setminus \{0\} \) as \((d,d)\) forms with \( L^\infty_{\text{loc}}(\mathbb{C}^d) \) coefficients. For a compact set \( K \) we have a local bound

\[(4.13) \quad \frac{1}{d(N,\theta)} K_N(z) \leq C = C(K) \text{ for } z \in K.\]

Moreover we have

\[(4.14) \quad \frac{1}{d(N,\theta)} K_N \to \frac{1}{(1 - \theta^d)} \chi_{D_{\phi,\theta} \cap P} \frac{\det(dd^c\phi)}{(2\pi)^d} \text{ in } L^1(\mathbb{C}^d)\]

and

\[(4.15) \quad \frac{1}{d(N,\theta)} K_N \omega_d \to \frac{1}{(1 - \theta^d)} \frac{(dd^cV_{\phi,\theta})^d}{(2\pi)^d} \text{ weak-* on } D_{\phi,\theta} \cap P.\]

Here \( \det(dd^c u) := \frac{(dd^c u)^d}{\omega_d} \) and for a twice continuously differentiable function \( u \) we have \( \det(dd^c u) = 2i \frac{\partial^2 u}{\partial z \partial \bar{z}} \), \( j,k=1,\ldots,d \). The characteristic function of a set \( A \) is denoted by \( \chi_A \). We remark that we assume \( V_{\phi,\theta} \in C^{1,1}(\mathbb{C}^d \setminus \{0\}) \).

We will use the following lemma from measure theory in the proof of the theorem.

**Lemma 4.8.** [Ber06, Lemma 2.2] Let \((X,\mu)\) be a measure space and let \( \{f_N\} \) be a sequence of uniformly bounded, integrable functions on \( X \). If \( f \) is a bounded, integrable function on \( X \) with

1. \( \lim_{N \to \infty} \int_X f_N d\mu = \int_X f d\mu \) and
2. \( \limsup_{N \to \infty} f_N \leq f \) a.e. with respect to \( \mu \)

then \( f_N \) converges to \( f \) in \( L^1(X,\mu) \).

**Proof of Theorem 4.7.** The \( \theta = 0 \) case is proven by Berman in [Ber], so we assume \( 0 < \theta < 1 \).
By assumption $V_{\phi, \theta} = \phi$ on $D_{\phi, \theta} \cap P$ and both are $C^{1,1}$ on $D_{\phi, \theta} \cap P$. Therefore $\det(dd^c\phi)\omega_d = (dd^cV_{\phi, \theta})^d$ on $D_{\phi, \theta} \cap P$ almost everywhere as $(d,d)$ forms with $L^\infty$ coefficients by the argument in Section 12 of [Dem92].

First of all using (4.12) to prove an asymptotic upper bound on $\frac{1}{d(N, \theta)}K_N(z)$ at a point $z_0 = (z_0^1, \ldots, z_0^d)$, we can assume that near $z_0$, $\phi$ is of the form

\begin{equation}
\phi(z) = \sum_{j=1}^d \lambda_j|z_j - z_0^j|^2 + 0(|z - z_0|^3)
\end{equation}

as in [Ber]. Namely we assume that $\phi(z_0) = 0$ and the first order partial derivatives of $\phi$ vanish at $z_0$.

Following [Ber], we have for each $z_0 \in \mathbb{C}^d$ there exist $R > 0$ and a constant $C$ such that

\begin{equation}
|\phi(z)| \leq C|z - z_0|^2 \text{ on } B(z_0, R),
\end{equation}

and for any $R > 0$ we have

\begin{equation}
\lim_{N \to \infty} \left[ \sup_{z \in B(0,R)} \left| N\phi(z/\sqrt{N} + z_0) - \sum_{j=1}^d \lambda_j|z_j|^2 \right| \right] = 0.
\end{equation}

We fix $z_0$ be a point in $\mathbb{C}^d$. We take a polynomial $p_N \in \pi_{N, \theta}$ satisfying the extremal property (4.12) at $z_0$. Then we have

\begin{equation}
\frac{1}{d(N, \theta)}K_N(z_0) = \frac{|p_N(z_0)|^2 e^{-2N\phi(z_0)}}{d(N, \theta) \int_{\mathbb{C}^d} |p_N(z)|^2 e^{-2N\phi(z)} \omega_d(z)} = \frac{|p_N(z_0)|^2}{d(N, \theta) \int_{|z - z_0| \leq R/\sqrt{N}} |p_N(z)|^2 e^{-2N\phi(z)} \omega_d(z)}.
\end{equation}

By positivity of the integrand we have

\begin{equation}
\frac{1}{d(N, \theta)}K_N(z_0) \leq \frac{|p_N(z_0)|^2}{d(N, \theta) \int_{|z - z_0| \leq R/\sqrt{N}} |p_N(z)|^2 e^{-2N\phi(z)} \omega_d(z)}.
\end{equation}

We choose $R$ as in (4.17) so that we can replace $\phi(z)$ by $C|z - z_0|^2$ in the integrand and thus we have

\begin{equation}
\frac{1}{d(N, \theta)}K_N(z_0) \leq \frac{|p_N(z_0)|^2}{d(N, \theta) \int_{|z - z_0| \leq R/\sqrt{N}} |p_N(z)|^2 e^{-2NC|z - z_0|^2} \omega_d(z)}.
\end{equation}
We apply the subaveraging property to the subharmonic function \(|p_N|^2\) on the ball \(\{|z - z_0| \leq R/\sqrt{N}\}\) with respect to the radial probability measure with center \(z_0\)

\[
\int_{|z-z_0| \leq R/\sqrt{N}} e^{-2NC|z-z_0|^2} \omega_d(z)
\]

to obtain

\[
\frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{1}{d(N, \theta)} \int_{|z-z_0| \leq R/\sqrt{N}} e^{-2NC|z-z_0|^2} \omega_d(z)
\]

\[
\leq \frac{N^d}{d(N, \theta) \int_{|z'| \leq R} e^{-2C|z'|^2} \omega_d(z')}
\]

For the last inequality we used a change of variable \(z \to z' := (z - z_0)\sqrt{N}\), where \(\omega_d(z') = N^d \omega_d(z)\). Since \(d(N, \theta) \geq (1 - \theta^d)d(N, 0)\), we have \(d(N, \theta) \geq (1 - \bar{\theta}^d)d(N, 0)\) for all \(N \geq N_0\) for some \(\bar{\theta} \geq \theta\). Now using the estimate \(d(N, \theta) \geq (1 - \bar{\theta}^d)d(N, 0) = (1 - \bar{\theta}^d)\left(\frac{d^N}{d!}\right) \geq (1 - \bar{\theta}^d)N^d/d!\) for all \(N \geq N_0\), we get

\[
\frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{d!}{(1 - \bar{\theta}^d) \int_{|z'| \leq R} e^{-2C|z'|^2} \omega_d(z')}
\]

for all \(N \geq N_0\).

The right hand side of the inequality is uniformly bounded. As \(z_0\) varies on the compact set \(K\), we get a constant \(C(K)\) giving a local bound for all \(N \geq N_0\). By continuity of \(\frac{1}{d(N, \theta)} K_N(z)\), and considering the \(\max_{N=1, \ldots, N_0} \sup_{z \in K} \frac{1}{d(N, \theta)} K_N(z)\) we get the local bound \((4.13)\) holds at each point of \(K\).

For the rest of the proof, we fix \(z_0\) and start with the inequality

\[
\frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{|p_N(z_0)|^2}{d(N, \theta) \int_{|z-z_0| \leq R/\sqrt{N}} |p_N(z)|^2 e^{-2N\phi(z)} \omega_d(z)}
\]

which holds for any \(R > 0\). By using the same change of variable and estimates as above we get

\[
\frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{d!|p_N(z_0)|^2}{(1 - \bar{\theta}^d) \int_{|z'| \leq R} |p_N(z'/\sqrt{N} + z_0)|^2 e^{-2N\phi(z'/\sqrt{N} + z_0)} \omega_d(z')}\]

for all \(N \geq N_0\) where \(\bar{\theta} \geq \theta\). Multiplying the integrand by \(e^{-2\sum_{j=1}^d \lambda_j |z'_j|^2} e^{2\sum_{j=1}^d \lambda_j |z'_j|^2}\) and taking the infimum of \(\exp \left[-2 \left( N\phi(z'/\sqrt{N}) - \sum_{j=1}^d \lambda_j |z'_j|^2 \right) \right]\) on \(B(0, R)\) out of the
integral, we get
\[
\frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{d!|p_N(z_0)|^2 \exp \left[ 2 \sup_{|z'| \leq R} \left| N \phi(z'/\sqrt{N}) - \sum_{j=1}^d \lambda_j |z'_j|^2 \right| \right]}{(1 - \tilde{\theta}^d) \int_{|z'| \leq R} |p_N(z'/\sqrt{N} + z_0)|^2 e^{-2\sum_{j=1}^d \lambda_j |z'_j|^2} \omega_d(z')},
\]
for all $N \geq N_0$. We apply the subaveraging property to the subharmonic function $|p_N(z'/\sqrt{N} + z_0)|^2$ with respect to radial probability measure $\frac{e^{-2\sum_{j=1}^d \lambda_j |z'_j|^2} \omega_d(z')}{\int_{|z'| \leq R} e^{-2\sum_{j=1}^d \lambda_j |z'_j|^2} \omega_d(z')}$ and we get
\[
\frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{d! \exp \left[ 2 \sup_{|z'| \leq R} \left| N \phi(z'/\sqrt{N}) - \sum_{j=1}^d \lambda_j |z'_j|^2 \right| \right]}{(1 - \tilde{\theta}^d) \int_{|z'| \leq R} e^{-2\sum_{j=1}^d \lambda_j |z'_j|^2} \omega_d(z')}.
\]
for all $N \geq N_0$. By (4.18), $\exp \left[ 2 \sup_{|z'| \leq R} \left| N \phi(z'/\sqrt{N}) - \sum_{j=1}^d \lambda_j |z'_j|^2 \right| \right] \to 1$ as $N \to \infty$. Therefore we have
\[
\limsup_{N \to \infty} \frac{1}{d(N, \theta)} K_N(z_0) \leq \frac{d!}{(1 - \tilde{\theta}^d) \int_{|z'| \leq R} e^{-2\sum_{j=1}^d \lambda_j |z'_j|^2} \omega_d(z')}.
\]
As $R \to \infty$ the Gaussian integral on the right hand side goes to $\frac{\pi^d}{2^d \lambda_1 \ldots \lambda_d}$ if all $\lambda_j > 0$ and to $+\infty$ otherwise. Since $\det(dd^c \phi(z_0)) = 4^d d! \lambda_1 \ldots \lambda_d$ we have
\[
(4.19) \quad \limsup_{N \to \infty} \frac{1}{d(N, \theta)} K_N(z) \leq \frac{1}{(1 - \tilde{\theta}^d) \chi_{D_\phi, \theta \cap P}} \frac{\det(dd^c \phi)}{(2\pi)^d} \quad \text{a.e on } \mathbb{C}^d.
\]
Letting $\tilde{\theta} \to \theta$ we obtain
\[
(4.20) \quad \limsup_{N \to \infty} \frac{1}{d(N, \theta)} K_N(z) \leq \frac{1}{(1 - \theta^d) \chi_{D_\phi, \theta \cap P}} \frac{\det(dd^c \phi)}{(2\pi)^d} \quad \text{a.e on } \mathbb{C}^d.
\]
By the definition of $\limsup$ and using the extremal property (4.12), we get
\[
(4.21) \quad \frac{1}{N^d} |p_N(z)|^2 e^{-2N\phi(z)} / ||p_N||_{N\phi}^2 \leq C_N \text{ on } D_\phi, \theta \quad \text{for any } p_N \in \pi_{N, \theta},
\]
where $C_N = \frac{1}{(1 - \theta^d) \sup_{z \in D_\phi, \theta \cap P} \frac{\det(dd^c \phi(z))}{(2\pi)^d}}$. Next we will show that
\[
(4.22) \quad \frac{1}{N^d} K_N(z) \leq C_N e^{-2N(\phi(z) - V_{\phi, \theta}(z))} \text{ on } \mathbb{C}^d.
\]
Let $p_N \in \pi_{N, \theta}$ such that $||p_N||_{N\phi}^2 = N^{-d}$, then by (4.21) we have
\[
|p_N(z)|^2 e^{-2N\phi(z)} \leq C_N \text{ on } D_\phi, \theta.
\]
By taking logarithms we get
\[
\frac{1}{2N} \log |p_N(z)|^2 \leq \phi(z) + \frac{1}{2N} \log C_N \text{ on } D_{\phi,\theta}
\]
and thus we have
\[
\frac{1}{2N} \log |p_N(z)|^2 \leq V_{\phi,\theta}(z) + \frac{1}{2N} \log C_N \text{ on } \mathbb{C}^d.
\]
So from the extremal property of Bergman functions (4.12) we obtain
\[
\frac{1}{N^d} K_N(z) = \sup_{|p_N|_{K_{\phi,\theta}=N^{-d}}} |p_N(z)|^2 e^{-2N\phi(z)} \leq C_N e^{-2N(\phi(z)-V_{\phi,\theta}(z))} \text{ on } \mathbb{C}^d.
\]
Since \(\phi(z) > V_{\phi,\theta}(z)\) on \(\mathbb{C}^d \setminus D_{\phi,\theta}\), we obtain
\[
\lim_{N \to \infty} \frac{1}{N^d} K_N(z) = 0 \text{ on } \mathbb{C}^d \setminus D_{\phi,\theta}.
\]
Using \(d(N, \theta) \asymp (1 - \theta^d)d(N, 0)\), we obtain
\[
\lim_{N \to \infty} \frac{1}{d(N, \theta)} K_N(z) = 0 \text{ on } \mathbb{C}^d \setminus D_{\phi,\theta}.
\]
From (4.22) and the growth assumption on \(\phi\), for a sufficiently large \(R\), there is a \(C\) with
\[
(4.23) \quad \frac{1}{N^d} K_N(z) \leq C|z|^{-2N} \text{ for } |z| > R.
\]
By combining the local bound (4.13) and above estimate (4.23) we get a global bound for \(\frac{1}{d(N, \theta)} K_N\). Therefore Lebesgue’s dominated convergence theorem gives that
\[
(4.24) \quad \lim_{N \to \infty} \int_{\mathbb{C}^d \setminus D_{\phi,\theta}} \frac{1}{d(N, \theta)} K_N \omega_d = 0.
\]
Next we show that
\[
(4.25) \quad \lim_{N \to \infty} \int_{D_{\phi,\theta} \cap P} \frac{1}{d(N, \theta)} K_N \omega_d = \frac{1}{(1 - \theta^d)} \int_{D_{\phi,\theta} \cap P} \frac{\det(dd^c \phi)}{(2\pi)^d} \omega_d.
\]
To prove (4.25), we know that
\[
\int_{\mathbb{C}^d} K_N \omega_d = d(N, \theta)
\]
and using \((4.24)\) we have

\[
1 = \lim_{N \to \infty} \int_{\mathbb{C}^d} \frac{1}{d(N, \theta)} K_N \omega_d = \lim_{N \to \infty} \int_{D_{\phi, \theta} \cap P} \frac{1}{d(N, \theta)} K_N \omega_d.
\]

On the other hand, using the positivity of the integrand and applying \((4.20)\) on \(D_{\phi, \theta} \cap P\), we have

\[
1 = \lim_{N \to \infty} \int_{D_{\phi, \theta} \cap P} \frac{1}{d(N, \theta)} K_N \omega_d \leq \frac{1}{(1 - \theta^d)} \int_{D_{\phi, \theta} \cap P} \left( \frac{\det(\ddc^{\phi})}{(2\pi)^d} \right) \omega_d.
\]

By the first part of this theorem, we can replace \(\det(\ddc^{\phi}) \omega_d\) by \(\left( \frac{\det(\ddc^V_{\phi, \theta})}{(2\pi)^d} \right)\) which has total mass \((2\pi)^d(1 - \theta^d)\) on \(D_{\phi, \theta} \cap P\), hence we have

\[
1 = \lim_{N \to \infty} \int_{D_{\phi, \theta} \cap P} \frac{1}{d(N, \theta)} K_N \omega_d \leq \frac{1}{(1 - \theta^d)} \int_{D_{\phi, \theta} \cap P} \left( \frac{\det(\ddc^V_{\phi, \theta})}{(2\pi)^d} \right) = \frac{(2\pi)^d(1 - \theta^d)}{(2\pi)^d(1 - \theta^d)} = 1.
\]

This gives \((4.25)\). We will use this relation, together with \((4.14)\), to show that

\[
\frac{1}{d(N, \theta)} K_N \to \frac{1}{(1 - \theta^d)} \chi_{D_{\phi, \theta} \cap P} \frac{\det(\ddc^V_{\phi, \theta})}{(2\pi)^d} \text{ in } L^1(\mathbb{C}^d).
\]

We set \(f_N := \frac{1}{d(N, \theta)} K_N\) and \(f := \frac{1}{(1 - \theta^d)} \chi_{D_{\phi, \theta} \cap P} \frac{\det(\ddc^V_{\phi, \theta})}{(2\pi)^d}\). By the upper bound \((4.20)\) we have \(\limsup_{N \to \infty} f_N \leq f\) almost everywhere and by \((4.24)\) and \((4.25)\) we have \(\lim_{N \to \infty} \int_{\mathbb{C}^d} f_N \omega_d = \int_{\mathbb{C}^d} f \omega_d\). Thus by Lemma \([4.8]\) we get the convergence of \(\frac{1}{d(N, \theta)} K_N\) to \(\frac{1}{(1 - \theta^d)} \chi_{D_{\phi, \theta} \cap P} \frac{\det(\ddc^V_{\phi, \theta})}{(2\pi)^d}\omega_d\) in \(L^1(\mathbb{C}^d)\). This implies the weak-* convergence of \(\frac{1}{d(N, \theta)} K_N \omega_d\) to \(\frac{1}{(1 - \theta^d)} \chi_{D_{\phi, \theta} \cap P} \frac{\det(\ddc^V_{\phi, \theta})}{(2\pi)^d} \omega_d\) and completes the proof of the theorem.

\begin{flushright}
\(\square\)
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