On abstract generalized topological spaces generated by the density type operators

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Abstract

In the paper we concentrate on a generalized topological space generated by a density type operator on a measurable space. The properties of such generalized topological space are investigated. Moreover, the properties of nowhere dense sets, meager sets and compact sets in this generalized topological space are studied.

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1. Introduction

Let $X$ be a non-empty set, $S$ be an algebra of subsets of $X$ (i.e. the empty set belongs to $S$ and $S$ is closed under the finite unions of sets and the complements of sets) and $J \subseteq S$ be a proper ideal of subsets of $X$ (i.e. if $A \in J$ and $B \subseteq A$ then $B \in J$ and $J$ is closed under the finite unions of sets). We will focus on a measurable space i.e. a triple $(X, S, J)$, where $J \subseteq S$ is a proper ideal of sets such that all singletons belong to $J$. Moreover, if it is necessary, we will assume that $J$ is the $\sigma$-ideal of sets it means $J$ is additionally closed under the countable unions of sets. The density type operators defined on some families of subsets of this space will also play a special role in our considerations.

The family of all subsets of a non-empty set $X$ will be denoted by $2^X$. For any $A, B \in 2^X$ the symbol $A \triangle B$ will stand for the set $(A \setminus B) \cup (B \setminus A)$. Moreover, for any measurable space $(X, S, J)$ and $A, B \subseteq X$ we will write $A \sim B$ iff $A \triangle B \in J$.

Let $(X, S, J)$ be a measurable space. A measurable hull of a set $A \subseteq X$ is any set $B \in S$ such that $A \subseteq B$ and for any $C \subseteq B \setminus A$ if $C \in S$ then $C \in J$. The set $B$ described above is called an $S$-measurable hull of a set $A$. We will write it simply “a measurable hull of $A$” when no confusion can arise. The family of all measurable hulls of a set $A \subseteq X$ will be denoted by $\mathcal{H}(A)$. We shall say that $(X, S, J)$ has the hull property if $\mathcal{H}(A) \neq \emptyset$ for any set $A \subseteq X$.

In the next part of the paper a notion of a generalized topological space, introduced in [1] by Á. Császár, will be used. We shall say that a family $\gamma \subseteq 2^X$ is a generalized topology on $X$ if $\emptyset \in \gamma$ and $\bigcup_{t \in T} G_t \in \gamma$ whenever $\{G_t : t \in T\} \subseteq \gamma$. The pair $(X, \gamma)$ is...
called a generalized topological space. If $X \in \gamma$ then we shall say that $(X, \gamma)$ is a strong generalized topological space.

In the theory of generalized topological spaces almost all notions (e.g. an interior of a set, a closure of a set, a boundary of a set, a compact set) are defined as in standard topological spaces (see [1,2]). The interior, the closure and the boundary of $A \subset X$ will be denoted by $\text{int}_\gamma(A)$, $\text{cl}_\gamma(A)$ and $\text{Fr}_\gamma(A)$, respectively. Moreover, we will write $\gamma$-open, $\gamma$-closed, etc. if we want to emphasize that the considerations concern the space $(X, \gamma)$. Similarly to the classical topological space we define a base of a generalized topological space or a connected set in such space (see [3,8]). Separation axioms for a generalized topological space are defined as in the case of the classical topological space [4]. Moreover, the definitions of a separable space, Lindelöf space, first countable and second countable space can be adopt from the classical topological space.

In the case of a topological space, the notion of a nowhere dense set may be introduced by different equivalent definitions. One can say that $A$ is a nowhere dense set if the interior of the closure of $A$ is an empty set. On the other hand one can say that $A$ is a nowhere dense set if any nonempty open set contains a nonempty open subset which is disjoint with $A$. In the case of a generalized topological space, these two conditions can lead to different notions. In [9] one can find two notions connected with nowhere density in generalized topological space. We say that a set $A \subset X$ is $\gamma$-nowhere dense if $\text{int}_\gamma(\text{cl}_\gamma(A)) = \emptyset$. A set $A \subset X$ is $\gamma$-strongly nowhere dense if for $V \in \gamma \setminus \{\emptyset\}$ there exists $U \in \gamma \setminus \{\emptyset\}$ such that $U \subset V$ and $A \cap U = \emptyset$. It is easy to see that if $A$ is $\gamma$-strongly nowhere dense then it is $\gamma$-nowhere dense. The converse theorem is not true in general (see [9]).

At the beginning of this section we mentioned that the particular operators will play a special role in our consideration, so we start with their definitions. First we shortly recall the definition of the lower density operator which is investigated by many mathematicians (e.g. [5,12]).

**Definition 1.1.** We shall say that an operator $\Phi : S \to S$ is the lower density operator on $(X, S, J)$ if

1. $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$;
2. $\forall A \in S \forall B \in S \subseteq \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
3. $\forall A \in S \forall B \in S \subseteq A \Delta B \in J \Rightarrow \Phi(A) = \Phi(B)$;
4. $\forall A \in S \subseteq \Phi(A) \Delta A \in J$.

Obviously, the classical density operator defined in [12] is an example of the lower density operator. If $(X, S, J)$ is a measurable space with the hull property and $\Phi$ is the lower density operator on $(X, S, J)$, then the following theorem is true (see [11], p. 213).

**Theorem 1.2.** The family $T_\Phi = \{A \in S : A \subset \Phi(A)\}$ is a topology on $X$, which is called a topology generated by $\Phi$.

**Proof.** One can find the proof of this theorem in [11], but for the convenience of the reader we will present that any union of elements of $T_\Phi$ belongs to $T_\Phi$. Let $\{A_w\}_{w \in W} \subset T_\Phi$. Since $(X, S, J)$ is a measurable space with the hull property, we obtain that there exists a set $B$ being a measurable kernel of the set $\bigcup_{w \in W} A_w$ (i.e. $B \in S$, $B \subset \bigcup_{w \in W} A_w$ and for any measurable set $C \subset \bigcup_{w \in W} A_w \setminus B$ we have that $C \in J$). Obviously $(A_w \cap B) \Delta A_w \in J$ for any $w \in W$ and

$$B \subset \bigcup_{w \in W} A_w \subset \bigcup_{w \in W} \Phi(A_w) = \bigcup_{w \in W} \Phi(A_w \cap B) \subseteq \Phi(B).$$
Condition 4° from Definition 1.1 implies that \( \Phi(B) \setminus B \in \mathcal{J} \) and, in consequence, \( \bigcup_{w \in \mathcal{W}} A_w \in \mathcal{S} \). Now, it is easy to see that

\[
\bigcup_{w \in \mathcal{W}} \Phi(A_w) \subset \Phi\left( \bigcup_{w \in \mathcal{W}} \Phi(A_w) \right),
\]

so \( \bigcup_{w \in \mathcal{W}} A_w \in \mathcal{T}_{\Phi} \).  

Obviously topology described in the above theorem is an example of an abstract density topology. The papers [6,7,10,13] contain many results and properties relevant to abstract density topologies and lower density operators. Now, we are following the lower density operator \( \Phi \) on \( \langle X, \mathcal{S}, \mathcal{J} \rangle \). From now on, we will assume that \( \langle X, \mathcal{S}, \mathcal{J} \rangle \) has the hull property.

Let \( \Phi \) be the lower density operator on \( \langle X, \mathcal{S}, \mathcal{J} \rangle \). Let us consider an operator \( \Phi^* : 2^X \to \mathcal{S} \) defined in the following way:

\[
\forall_{A \subset X} \Phi^*(A) = \Phi(B),
\]

where \( B \) is an \( \mathcal{S} \)-measurable hull of a set \( A \). By condition 3° of Definition 1.1 we have that \( \Phi^* \) is defined correctly. Clearly, if \( A \in \mathcal{S} \) then \( \Phi^*(A) = \Phi(A) \). Moreover, we have the following propositions.

**Proposition 1.3.** 1° \( \Phi^*(\emptyset) = \emptyset \) and \( \Phi^*(X) = X \);

2° \( \forall_{A \subset X} \forall_{B \in \mathcal{S}} \Phi^*(A \cap B) = \Phi^*(A) \cap \Phi(B) \);

3° \( \forall_{A \subset X} \forall_{B \subset \mathcal{S}} A \Delta B \in \mathcal{J} \Rightarrow \Phi^*(A) = \Phi^*(B) \);

4° \( \forall_{A \subset X} A \setminus \Phi^*(A) \in \mathcal{J} \).

**Proof.** Condition 1° is obvious. Let \( A \subset X \) and \( B \in \mathcal{S} \). If \( C \in \mathcal{S} \) is a measurable hull of \( A \) then \( C \cap B \) is a measurable hull of \( A \cap B \). Hence \( \Phi^*(A \cap B) = \Phi(C \cap B) = \Phi(C) \cap \Phi(B) = \Phi^*(A) \cap \Phi(B) \), so condition 2° is satisfied. To prove 3° let us observe that if \( A \Delta B \in \mathcal{J} \) and \( C_1, C_2 \) are measurable hulls of \( A \) and \( B \), respectively, then \( C_1 \cap C_2 \subseteq \mathcal{J} \). It implies that \( \Phi(C_1) = \Phi(C_2) \) and finally, \( \Phi^*(A) = \Phi^*(B) \). In the case of 4° if \( C \) is an \( \mathcal{S} \)-measurable hull of \( A \) then \( A \setminus \Phi^*(A) \subseteq C \setminus \Phi(C) \in \mathcal{J} \). □

**Proposition 1.4.** For every \( A \subset X \) the following properties hold:

i) \( \Phi(\Phi^*(A)) = \Phi^*(A) \);

ii) \( \Phi^*(A \cap \Phi^*(A)) = \Phi^*(A) \).

**Proof.** Let \( B \) be a measurable hull of \( A \). Then \( \Phi(\Phi^*(A)) = \Phi(\Phi(B)) = \Phi(B) = \Phi^*(A) \). It means that i) is satisfied. In the case of ii) we have \( \Phi^*(A \cap \Phi^*(A)) = \Phi^*(A \cap \Phi(B)) = \Phi^*(A) \cap \Phi(B) = \Phi^*(A) \cap \Phi^*(A) = \Phi^*(A) \). □

**Proposition 1.5.** For every \( A \subset X \) we have

(i) \( A \cap \Phi^*(A) = \emptyset \) if \( A \notin \mathcal{J} \);

(ii) \( A \cap \Phi^*(A) \in \mathcal{S} \) if \( A \in \mathcal{S} \).

**Proof.** If \( A \notin \mathcal{J} \) then by Definition 1.1 we have \( \Phi(A) = \emptyset \), so that \( A \cap \Phi(A) = \emptyset \). Let \( A \notin \mathcal{J} \). Then \( A = (A \cap \Phi^*(A)) \cup (A \setminus \Phi^*(A)) \). Since, by Proposition 1.3, \( A \setminus \Phi^*(A) \in \mathcal{J} \) we get that \( A \cap \Phi^*(A) \notin \mathcal{J} \). It implies that \( A \cap \Phi^*(A) \neq \emptyset \) and condition (i) is satisfied.

Now, we prove condition (ii). If \( A \in \mathcal{S} \) then \( \Phi^*(A) = \Phi(A) \) and, by Definition 1.1, we get that \( \Phi(A) \in \mathcal{S} \). It implies that \( A \cap \Phi^*(A) \in \mathcal{S} \). If \( A \cap \Phi^*(A) \in \mathcal{S} \) then, by Proposition 1.3, \( A \setminus \Phi^*(A) \notin \mathcal{J} \) and we get that \( A \notin \mathcal{S} \). □
2. A generalized topological space connected with $\Phi^*$

In this section, we will study the family

$$\mathcal{J}_{\Phi^*} = \{ A \subset X : A \subset \Phi^*(A) \}$$

generated by the operator $\Phi^*$.

**Remark 2.1.** The family $\mathcal{J}_{\Phi^*}$ has not to be closed with respect to the finite intersection.

Indeed, let $\mathbb{R}$ be the set of all real numbers, $\mathcal{L}$ be the $\sigma$-algebra of Lebesgue measurable sets and $\mathcal{L}$ be the $\sigma$-ideal of Lebesgue measure zero sets in $\mathbb{R}$. If $\Phi$ is the density operator on $(\mathbb{R}, \mathcal{L}, \mathbb{L})$ and $B$ is a Bernstein set then $\Phi^*(B) = \Phi(\mathbb{R}) = \mathbb{R}$ and $\Phi^*(\mathbb{R} \setminus B) = \Phi(\mathbb{R}) = \mathbb{R}$. Additionally, for every $x \in \mathbb{R}$ we get that $B \cup \{ x \} \in \mathcal{J}_{\Phi^*}$ and $(\mathbb{R} \setminus B) \cup \{ x \} \in \mathcal{J}_{\Phi^*}$, but $(B \cup \{ x \}) \cap ((\mathbb{R} \setminus B) \cup \{ x \}) = \{ x \} \notin \mathcal{J}_{\Phi^*}$.

However, it is easy to prove the following theorem:

**Theorem 2.2.** The family $\mathcal{J}_{\Phi^*}$ is a strong generalized topology on $X$ and $\mathcal{J}_\Phi \subset \mathcal{J}_{\Phi^*}$.

If we consider the classical density operator $\Phi$ on $(\mathbb{R}, \mathcal{L}, \mathbb{L})$, then it is easy to see that a Bernstein set belongs to $\mathcal{J}_\Phi \setminus \mathcal{J}_{\Phi^*}$. Proposition 1.3 implies that

**Remark 2.3.** If $W \in \mathcal{J}_\Phi$ and $A \in \mathcal{J}_\Phi$ then $W \setminus A \in \mathcal{J}_{\Phi^*}$. Moreover, if $W \in \mathcal{J}_\Phi \setminus \{ \emptyset \}$ then $W \notin \mathcal{J}$.

**Proof.** Let $W \in \mathcal{J}_\Phi$, $A \in \mathcal{J}$ and $V = W \setminus A$. Clearly, $V \Delta W \in \mathcal{J}$, so Condition 3° in Proposition 1.3 gives that $\Phi^*(V) = \Phi^*(W)$. Obviously, we have that $W \subset \Phi^*(W)$, because $W \in \mathcal{J}_\Phi$. Thus $V \subset W \subset \Phi^*(W) = \Phi^*(V)$, which gives that $V \in \mathcal{J}_{\Phi^*}$. Let now $W \in \mathcal{J}_\Phi \setminus \{ \emptyset \}$. Suppose, contrary to our claim that $W \in \mathcal{J}$, Proposition 1.3 Conditions 1° and 3° imply that $\Phi^*(W) = \Phi^*(\emptyset) = \emptyset$. Since $W \subset \Phi^*(W)$ we obtain that $W = \emptyset$, which is impossible. \qed

By Proposition 1.4 we have that

**Remark 2.4.** For every $A \subset X$ the sets $\Phi^*(A)$ and $A \cap \Phi^*(A)$ are the members of $\mathcal{J}_{\Phi^*}$.

Moreover, we have the following properties:

**Proposition 2.5.** For every $A \subset X$ we have

$$\text{int}_{\mathcal{J}_{\Phi^*}}(A) = A \cap \Phi^*(A).$$

**Proof.** By Remark 2.4 we have $A \cap \Phi^*(A) \in \mathcal{J}_{\Phi^*}$, so that $A \cap \Phi^*(A) \subset \text{int}_{\mathcal{J}_{\Phi^*}}(A)$. Let $V \in \mathcal{J}_{\Phi^*}$ and $V \subset A$. Then $\Phi^*(V) \subset \Phi^*(A)$. So that $V \subset A \cap \Phi^*(A)$. Finally, $\text{int}_{\mathcal{J}_{\Phi^*}}(A) \subset A \cap \Phi^*(A)$. \qed

**Proposition 2.6.** $\text{Fr}_{\mathcal{J}_{\Phi^*}}(A) \in \mathcal{J}$ for every set $A \subset X$.

**Proof.** Obviously, $\text{Fr}_{\mathcal{J}_{\Phi^*}}(A) = \text{cl}_{\mathcal{J}_{\Phi^*}}(A) \setminus \text{int}_{\mathcal{J}_{\Phi^*}}(A) = \{ A \cup (X \setminus \Phi^*(X \setminus A)) \setminus (A \cap \Phi^*(A)) = (A \setminus \Phi^*(A)) \cup ((X \setminus \Phi^*(X \setminus A)) \cap ((X \setminus A) \cup (X \setminus \Phi^*(A)))) \subset (A \setminus \Phi^*(A)) \cup ((X \setminus A) \setminus \Phi^*(X \setminus A)) \in \mathcal{J}. \quad \Box$

The next property is the characterization of nowhere dense sets in the generalized topological space $(X, \mathcal{J}_{\Phi^*})$.

**Theorem 2.7.** Let $A \subset X$. Then the following conditions are equivalent:

i) $\forall W \in \mathcal{J}_{\Phi^*} \setminus \{ \emptyset \} \exists V \in \mathcal{J}_{\Phi^*} \setminus \{ \emptyset \}$ ($V \subset W \land V \cap A = \emptyset$);

ii) $\text{int}_{\mathcal{J}_{\Phi^*}}(\text{cl}_{\mathcal{J}_{\Phi^*}}(A)) = \emptyset$;

iii) $A \in \mathcal{J}$. 

Proposition 2.10. Let $A \subset X$. Then $A \in J$ if and only if $A$ is $J_{\Phi^*}$-closed and $J_{\Phi^*}$-nowhere dense.

Proof. Let $A \in J$. Obviously $X \setminus A \in J_{\Phi^*}$. Thus $A$ is $J_{\Phi^*}$-closed and evidently, by Corollary 2.8, $A$ is $J_{\Phi^*}$-nowhere dense. Sufficiency is the consequence of Corollary 2.8.

As the consequence of this property we have

Proposition 2.11. If $A \in J$ then $A$ is $J_{\Phi^*}$-closed and $J_{\Phi^*}$-discrete.

Also the following property is obvious.

Property 2.12. If $J$ is a $\sigma$-ideal and $A \in J_{\Phi^*} \setminus \{\emptyset\}$ then $A$ is $J_{\Phi^*}$-second category, i.e. $A \not\in K(J_{\Phi^*})$.

Moreover, we have

Proposition 2.13. If $J$ is a $\sigma$-ideal then a set $A \subset X$ is $J_{\Phi^*}$-compact if and only if $A$ is finite.

Proof. Sufficiency is obvious. Let us assume that $A \subset X$ is $J_{\Phi^*}$-compact and infinite. Let $B \subset A$ be infinite and countable. Then $(X \setminus B) \cup \{x\} \in J_{\Phi^*}$ for every $x \in B$. Indeed, clearly $B \in J$ and $\{x\} \in J$ for any $x \in B$, so $X \Delta ((X \setminus B) \cup \{x\}) \in J$ and, in consequence, Proposition 1.3 Conditions 1° and 3° give that $(X \setminus B) \cup \{x\} \subset X = \Phi^*((X \setminus B) \cup \{x\})$. Thus $(X \setminus B) \cup \{x\} \in J_{\Phi^*}$. Obviously the family $(X \setminus B) \cup \{x\}_{x \in B}$ is an open cover of $A$ which does not contain a finite subcover of $A$. It contradicts the fact that $A$ is $J_{\Phi^*}$-compact.

Proposition 2.14. If $J$ is a $\sigma$-ideal then the space $(X, J_{\Phi^*})$ neither fulfills the first nor the second axiom of countability and is not separable.

Proof. Let us suppose that $(X, J_{\Phi^*})$ fulfills the first axiom of countability. Let $x \in X$ and $\{V_n\}_{n \in \mathbb{N}}$ be a sequence of all $J_{\Phi^*}$-open sets from a countable base of $J_{\Phi^*}$ at $x$. Let $x_n \in V_n \setminus \{x\}$ for $n \in \mathbb{N}$. Putting $V = V_1 \setminus \{x_n : n \in \mathbb{N}\}$ we get that $V \in J_{\Phi^*}$, $x \in V$ and $V$
does not contain any set $V_n$ for $n \in \mathbb{N}$. Hence $(X, \mathcal{T}_f)$ does not fulfill the first countability axiom and therefore does not fulfill the second countability axiom. Since every countable set belongs to $\mathcal{J}$ so thus we infer that $(X, \mathcal{T}_f)$ is not separable. □

**Proposition 2.15.** If $\mathcal{J}$ contains an uncountable set then $(X, \mathcal{T}_f)$ is not a Lindelöf space.

**Proof.** Let $D \in \mathcal{J}$ be an uncountable set then $(X \setminus D) \cup \{x\} \in \mathcal{T}_f$, for every $x \in D$ and $\{(X \setminus D) \cup \{x\}\}_{x \in D} \in \mathcal{T}_f$ is an open cover of $X$ which does not contain a countable subcover. □

Since $V = X \setminus \{x\} \in \mathcal{T}_f$ for any $x \in X$, we see at once

**Proposition 2.16.** The space $(X, \mathcal{T}_f)$ is a $T_1$-space.

We end this section with the interesting property of the functions continuous with respect to the generalized topology $\mathcal{T}_f$.

**Theorem 2.17.** If $\mathcal{J}$ is a $\sigma$-ideal then for an arbitrary function $f : X \to Y$, where $(Y, \mathcal{T})$ satisfies the second countability axiom, there exists a set $A \in \mathcal{J}$ such that for every $x \in X \setminus A$ the function $f$ is $\mathcal{T}_f$-continuous at $x$.

**Proof.** Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable base of $(Y, \mathcal{T})$. For every $n \in \mathbb{N}$ we have that $f^{-1}(B_n) = C_n \cup D_n$, where $C_n = \Phi^n(f^{-1}(B_n)) \cap f^{-1}(B_n)$ and $D_n = f^{-1}(B_n) \setminus \Phi^n(f^{-1}(B_n))$. By Proposition 2.5, $C_n \in \mathcal{T}_f$ for any $n \in \mathbb{N}$. Moreover, by Proposition 1.3, $A = \bigcup_{n=1}^{\infty} D_n \in \mathcal{J}$. Let $x_0 \in X \setminus A$ and $W \in \mathcal{T}_f$ be such that $f(x_0) \in W$. Thus there exists $n_0 \in \mathbb{N}$ such that $B_{n_0} \subset W$ and $x_0 \in f^{-1}(B_{n_0})$. Hence $x_0 \in C_{n_0} \in \mathcal{T}_f$, and $f(C_{n_0}) \subset W$. It means that $f$ is $\mathcal{T}_f$-continuous for every $x \in X \setminus A$. □

3. The $(\ast)$ property and the $(\ast\ast)$ property

In this section we concentrate on the family $\mathcal{T}_f$, in a space $(X, \mathcal{S}, \mathcal{J})$ having two special properties: the $(\ast)$ property and the $(\ast\ast)$ property. We start with the definition of these properties.

**Definition 3.1.** We shall say that $(X, \mathcal{S}, \mathcal{J})$ has

- the $(\ast)$ property if there exist $B \subset X$ such that $X \in \mathcal{H}(B) \cap \mathcal{H}(X \setminus B)$;
- the $(\ast\ast)$ property if for every $A \subset X$ there exist $B \subset A$ and $C \in \mathcal{H}(B)$ such that $C \in \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$.

It is easy to see that $(X, \mathcal{S}, \mathcal{J})$ has the $(\ast\ast)$ property if for every $A \subset X$ there exists $B \subset A$ such that $\mathcal{H}(B) \cap \mathcal{H}(A \setminus B) \cap \mathcal{H}(A) \neq \emptyset$. Moreover, we see at once that if $(X, \mathcal{S}, \mathcal{J})$ has the $(\ast)$ property and $B \subset X$ is such that $\mathcal{H}(B) \cap \mathcal{H}(X \setminus B) \neq \emptyset$ then $\mathcal{H}(B) \cap \mathcal{H}(X \setminus B) = \{X\}$.

Let $\mathcal{B}_a$, $\mathcal{B}$ and $\mathbb{K}$ be the family of all sets having the Baire property, the family of Borel sets and the family of all meager sets with respect to the natural topology $\tau_0$, respectively. Note that the measurable space $(\mathbb{R}, \mathcal{B}_a, \mathcal{K})$ has the $(\ast)$ property. Indeed, if $\mathcal{C} \subset \mathbb{R}$ is a Bernstein set then $\mathbb{R} \in \mathcal{H}(\mathcal{C}) \cap \mathcal{H}(\mathbb{R} \setminus \mathcal{C})$.

Moreover if additivity of $\sigma$-ideal $\mathbb{K}$ is equal to $\epsilon$ then the measurable space $(\mathbb{R}, \mathcal{B}_a, \mathcal{K})$ has the $(\ast\ast)$ property. Indeed, if $A \subset \mathbb{R}$ and $A \in \mathbb{K}$, then for any $B \subset A$ we have that $A \in \mathcal{H}(B) \cap \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$. If $A \subset \mathbb{R}$ and $A \notin \mathbb{K}$, then the cardinality of the family $\mathcal{F} = \{F \in \mathcal{B} : A \cap F \notin \mathbb{K}\}$ equals $\epsilon$. Therefore, one can find sets $P_1 = \{x_\alpha : \alpha < \epsilon\}$ and $P_2 = \{y_\alpha : \alpha < \epsilon\}$ such that $P_1 \cup P_2 \subset A$, $P_1 \cap P_2 = \emptyset$, the cardinality of $P_1$ and $P_2$ is equal to $\epsilon$ and $P_1 \cap F \neq \emptyset \neq P_2 \cap F$ for any $F \in \mathcal{F}$. Putting $B = P_1$, we obtain that $\mathcal{H}(B) \cap \mathcal{H}(A \setminus B) \cap \mathcal{H}(A) \neq \emptyset$. Indeed, let $V \in \mathcal{H}(A)$. Let $W \subset V \setminus B$ have the Baire property. Suppose that $W \cap A \notin \mathbb{K}$. Obviously, one can find a set $Z \subset W$ and $Z \cap A \notin \mathbb{K}$. Thus $\emptyset \neq Z \cap P_1 \subset W \cap P_1$, which is impossible. Therefore, we have that $W \cap A \in \mathbb{K}$ and, in consequence, $W \setminus A$ has the Baire property. Since $\mathbb{R} \in \mathcal{H}(A)$, we
obtain that $W \setminus A \in \mathcal{K}$. Hence $W \in \mathcal{K}$. Finally, we have that $V \in \mathcal{H}(B)$. By a similar argument, $V \in \mathcal{H}(A \setminus B)$.

**Theorem 3.2.** If $(X, S, J)$ has the $(\ast)$ property then the smallest topology $\sigma(J_{\Phi^*})$ containing $T_{\Phi^*}$ is equal to $2^X$.

**Proof.** Let $x \in X$ and $B \subset X$ be such that $\mathcal{H}(B) \cap \mathcal{H}(X \setminus B) \neq \emptyset$. Thus $B \cup \{x\} \in T_{\Phi^*}$ and $(X \setminus B) \cup \{x\} \in T_{\Phi^*}$. It implies that $(B \cup \{x\}) \cap ((X \setminus B) \cup \{x\}) \in \sigma(J_{\Phi^*})$, so that $\sigma(J_{\Phi^*}) = 2^X$. \hfill $\square$

**Theorem 3.3.** If $(X, S, J)$ has the $(\ast)$ property then $T_{\Phi^*}$ does not include the supremum of the topologies included in $T_{\Phi^*}$.

**Proof.** Let us suppose that $\mathcal{T}$ is the supremum of the topologies included in $T_{\Phi^*}$. Let $B \subset X$ be such that $X \in \mathcal{H}(B) \cap \mathcal{H}(X \setminus B)$. Let $x_0 \in X$. Put $\mathcal{T}_1 = \{\emptyset, B \cup \{x_0\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, (X \setminus B) \cup \{x_0\}, X\}$. It is easy to see that $\mathcal{T}_1, \mathcal{T}_2$ are topologies contained in $T_{\Phi^*}$, so $\mathcal{T}_1 \cup \mathcal{T}_2 \subset \mathcal{T}$. Moreover, $(B \cup \{x_0\}) \cap ((X \setminus B) \cup \{x_0\}) = \{x_0\} \in \mathcal{T} \subset T_{\Phi^*}$. It is a contradiction with the fact that $\{x_0\} \notin T_{\Phi^*}$.

However, we have the following property.

**Theorem 3.4.** There exists a maximal topology in the family $A$ of all topologies contained in $T_{\Phi^*}$ and ordered by the inclusion.

**Proof.** Let $\{\mathcal{T}_\lambda\}_{\lambda \in \Delta}$ be an arbitrary chain in $A$. Put $\mathcal{T} = \{\bigcup_{\omega \in \mathcal{W}} A_\omega : A_\omega \mid_{\omega \in \mathcal{W}} \subset \bigcup_{\lambda \in \Delta} \mathcal{T}_\lambda\}$. We see at once that $\emptyset, X \in \mathcal{T}$, $\mathcal{T}$ is closed under arbitrary unions and $\mathcal{T} \subset T_{\Phi^*}$. Since $\{\mathcal{T}_\lambda\}_{\lambda \in \Delta}$ is a chain we obtain that $\mathcal{T}$ is closed under finite intersections. Therefore $\mathcal{T}$ is a topology contained in $T_{\Phi^*}$ and simultaneously, it is the upper bound of $\{\mathcal{T}_\lambda\}_{\lambda \in \Delta}$. By Kuratowski-Zorn Lemma we get the existence of a maximal topology in $A$. \hfill $\square$

**Proposition 3.5.** If $(X, S, J)$ has the $(\ast)$ property then $(X, \mathcal{T}_{\Phi^*})$ is a Hausdorff space.

**Proof.** Let $x, y \in X$ and $x \neq y$. Let $B \subset X$ be such that $X \in \mathcal{H}(B) \cap \mathcal{H}(X \setminus B)$. If $x \in B$ and $y \in X \setminus B$ then putting $V_1 = B$ and $V_2 = X \setminus B$ we get that $V_1, V_2 \in T_{\Phi^*}, V_1 \cap V_2 = \emptyset, x \in V_1$ and $y \in V_2$. If $x, y \in B$ then it is enough to consider the sets $V_1 = B \setminus \{y\} \in T_{\Phi^*}$ and $V_2 = (X \setminus B) \cup \{y\} \in T_{\Phi^*}$. If $x, y \in X \setminus B$ the proof runs in the similar way. \hfill $\square$

**Proposition 3.6.** If $(X, S, J)$ has the $(\ast\ast)$ property then $(X, \mathcal{T}_{\Phi^*})$ is a normal space.

**Proof.** Let $F_1, F_2$ be nonempty and disjoint $T_{\Phi^*}$-closed subsets of $X$. If $A = X \setminus (F_1 \cup F_2) \in J$ then putting $V_1 = (X \setminus F_2) \setminus A$ and $V_2 = (X \setminus F_1) \setminus A$ we get that $F_1 \subset V_1, F_2 \subset V_2, V_1 \cap V_2 = \emptyset$. We prove first that $V_1 \in T_{\Phi^*}$. In this purpose we show that $F_1 \cup B \subset \Phi^*(F_1 \cup B)$. Since $\Phi^*(F_1 \cup B) = \Phi^*(F_1 \cup C) = \Phi^*(F_1 \cup A) = \Phi^*(X \setminus F_2)$. Suppose that $x \in (F_1 \cup B) \setminus \Phi^*(F_1 \cup B) \subset X \setminus \Phi^*(X \setminus F_2)$. Because $F_2$ is $T_{\Phi^*}$-closed it means that $X \setminus F_2 \subset \Phi^*(X \setminus F_2)$ and finally $x \in (F_1 \cup B) \setminus (X \setminus F_2) = (F_1 \cup B) \setminus (F_1 \cup A) = \emptyset$. This contradiction infer that $V_1 \in T_{\Phi^*}$. Similarly, we can prove that $V_2 \in T_{\Phi^*}$. It ends the proof. \hfill $\square$

**Theorem 3.7.** If $(X, S, J)$ has the $(\ast\ast)$ property then every $T_{\Phi^*}$-closed subset of $X$ is $G_\delta$-set in the space $(X, \mathcal{T}_{\Phi^*})$.

**Proof.** Let $F \subset X$ be $T_{\Phi^*}$-closed subset of $X$. Let $A = X \setminus F$. If $A \notin J$ then $F = X \setminus A \notin T_{\Phi^*}$.

Let us assume that $A \notin J$. By the $(\ast\ast)$ property there exist $B \subset A$ and $C \in \mathcal{H}(B)$ such that $C \in \mathcal{H}(A \setminus B) \cap \mathcal{H}(A)$. Let $V_1 = F \cup B$ and $V_2 = F \cup (A \setminus B)$. Simultaneously as in the proof of the previous theorem we get that $V_1, V_2 \in T_{\Phi^*}$. Since $F = V_1 \cap V_2$, we get that $F$ is $G_\delta$-set in the space $(X, \mathcal{T}_{\Phi^*})$. \hfill $\square$
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