FULLY DISCRETE FINITE ELEMENT APPROXIMATION FOR THE STABILIZED GAUGE-UZAWA METHOD TO SOLVE THE BOUSSINESQ EQUATIONS

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Abstract. The stabilized Gauge-Uzawa method [SGUM], which is a 2nd order projection type algorithm used to solve Navier-Stokes equations, has been newly constructed in [11]. In this paper, we apply the SGUM to the evolution Boussinesq equations, which model the thermal driven motion of incompressible fluids. We prove that SGUM is unconditionally stable and we perform error estimations on the fully discrete finite element space via variational approach for the velocity, pressure and temperature, the three physical unknowns. We conclude with numerical tests to check accuracy and physically relevant numerical simulations, the Benard convection problem and the thermal driven cavity flow.

1. Introduction

The stabilized Gauge-Uzawa method [SGUM] is a 2nd order projection type method to solve the evolution Navier-Stokes equations. In this paper, we extend SGUM to the evolution Boussinesq equations: given a bounded polygon $\Omega$ in $\mathbb{R}^d$ with $d = 2$ or $3$,

$$
\begin{align*}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} + \kappa \mu^2 \mathbf{g} \theta &= f, & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0, & \text{in } \Omega, \\
\theta_t + \mathbf{u} \cdot \nabla \theta - \lambda \mu \Delta \theta &= b, & \text{in } \Omega,
\end{align*}
$$

(1.1)

with initial conditions $\mathbf{u}(x,0) = \mathbf{u}^0, \theta(x,0) = \theta^0$ in $\Omega$, vanishing Dirichlet boundary conditions $\mathbf{u} = 0, \theta = 0$ on $\partial \Omega$, and pressure mean-value $\int_{\Omega} p = 0$. The forcing functions $f$ and $b$ are given and $\mathbf{g}$ is the vector of gravitational acceleration. The nondimensional numbers $\mu = Re^{-1}$ and $\lambda = Pr^{-1}$ are reciprocal of the Reynolds and Prandtl numbers, respectively, whereas $\kappa$ is the Grashof number. The Boussinesq system (1.1) describes fluid motion due to density differences which are in turn induced by temperature gradients: hot, and thus less dense, fluid tends to rise against gravity and cooler fluid falls in its place. The simplest governing equations are thus the Navier-Stokes equations for motion of an incompressible fluid, with forcing $\kappa \mu^2 \mathbf{g} \theta$ due to buoyancy, and the heat equation for diffusion and transport of heat. Density differences are thus ignored altogether except for buoyancy.

The projection type methods are representative solvers for the incompressible flows and the Gauge-Uzawa method is a typical projection method. The Gauge-Uzawa method was constructed in [8] to solve Navier-Stokes equations and extended...
to more complicated problems, the Boussinesq equations in [9] and the non-constant density Navier-Stokes equations in [13]. However, most of studies for the Gauge-Uzawa method have been limited only for the first order accuracy backward Euler time marching algorithm. The second order Gauge-Uzawa method using BDF2 scheme was introduced in [12] and proved superiority for accuracy on the normal mode space, but we couldn’t get any theoretical proof via energy estimate even stability and we suffer from weak stability performance on the numerical test. Recently, we construct SGUM in [11] which is unconditionally stable for semi-discrete level to solve the Navier-Stokes equations. The goal of this paper is to extend SGUM to the Boussinesq equations (1.1), which model the motion of an incompressible viscous fluid due to thermal effects [4, 10]. We will estimate errors and stability on the fully discrete finite element space. The main difficulties in the fully discrete estimation arise from losing the cancellation law due to the failing of the divergence free condition of the discrete velocity function. The strategy of projection type methods compute first an artificial velocity and then decompose it to divergence free velocity and curl free functions. However the divergence free condition cannot be preserved in discrete finite element space, and so the cancellation law (1.9) can’t be satisfied any more. In order to solve this difficulty, we impose the discontinuous velocity on across inter-element boundaries to make full discrete divergence free velocity (1.9) automatically. We will discuss this issue at Remark 1.1 below. This discontinuity makes it difficult to treat non-linear term and to apply the integration by parts, because the discontinuous solution is not included in $H^1(\Omega)$. So we need to hire technical skills in proof of this paper.

One more remarkable discovery is in the second numerical test at the last section which is the Benard convection problem with the same setting in [9]. In this performance, we newly find out that the number of circulations depends on the time step size $\tau$. We obtain similar simulation within [9] for a relatively larger $\tau$. We stress that the space $H^s(\Omega)$, where $s$ is the norm in $H^s(\Omega)$, and with $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\Omega)$. Let $\mathcal{K} = \{K\}$ be a shape-regular quasi-uniform partition of $\Omega$ of meshsize $h$ into closed elements $K$ [1, 2, 5]. The vector and scalar finite element spaces are:

$$
\mathcal{W}_h := \{ w_h \in L^2(\Omega) : w_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathcal{K} \}, \quad \mathcal{V}_h := \mathcal{W}_h \cap H^1_0(\Omega),
$$

$$
\mathcal{T}_h := \{ \theta_h \in H^1_0(\Omega) \cap C^0(\Omega) : \theta_h|_K \in \mathcal{Q}(K) \quad \forall K \in \mathcal{K} \},
$$

$$
\mathcal{P}_h := \{ q_h \in L^2(\Omega) \cap C^0(\Omega) : q_h|_K \in \mathcal{R}(K) \quad \forall K \in \mathcal{K} \},
$$

where $\mathcal{P}(K)$, $\mathcal{Q}(K)$, and $\mathcal{R}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathcal{K}$ [2, 5]. We stress that the space $\mathcal{P}_h$ is composed
of continuous functions to ensure the crucial equality
\[ \langle \nabla \cdot w_h, \phi_h \rangle = - \langle w_h, \nabla \phi_h \rangle, \quad \forall w_h \in \mathcal{V}_h, \forall \phi_h \in \mathbb{P}_h. \]

Using the following discrete counterpart of the form \( \mathcal{N}(u, v, w) := \langle (u \cdot \nabla)v, w \rangle \)

\begin{equation}
(1.2) \quad \mathcal{N}(u_h, v_h, w_h) := \frac{1}{2} \langle (u_h \cdot \nabla)v_h, w_h \rangle - \frac{1}{2} \langle (u_h \cdot \nabla)w_h, v_h \rangle,
\end{equation}

we now introduce the fully discrete SGUM to solve the evolution Boussinesq equations (1.1).

**Algorithm 1** (The fully discrete stabilized Gauge-Uzawa method). Compute \( \theta_h^1, u_h^1 \) and \( p_h^1 \) via any method satisfying Assumption 4 below and set \( q_h^1 = 0 \). Repeat for \( 1 \leq n \leq N = \lceil \frac{T}{T} \rceil - 1 \).

**Step 1:** Set \( u_h^0 = 2u_h^0 - u_h^{-1} \) and \( \theta_h^0 = 2\theta_h^0 - \theta_h^{-1} \) and then find \( \hat{u}_h^{n+1} \in \mathcal{V}_h \) as the solution of
\begin{equation}
(1.3) \quad \frac{1}{2\tau} \langle 3\hat{u}_h^{n+1} - 4u_h^n + u_h^{n-1}, w_h \rangle + \langle \nabla p_h^n, w_h \rangle + \mathcal{N}(u_h^n, \hat{u}_h^{n+1}, w_h) + \mu \langle \nabla \hat{u}_h^{n+1}, \nabla w_h \rangle + \kappa \mu \langle \theta_h^n, w_h \rangle = \langle f(t^{n+1}), w_h \rangle, \quad \forall w_h \in \mathcal{V}_h.
\end{equation}

**Step 2:** Find \( \psi_h^{n+1} \in \mathbb{P}_h \) as the solution of
\begin{equation}
(1.4) \quad \langle \nabla \psi_h^{n+1}, \nabla \phi_h \rangle = \langle \nabla \psi_h^n, \nabla \phi_h \rangle + \langle \nabla \cdot \hat{u}_h^{n+1}, \phi_h \rangle, \quad \forall \phi_h \in \mathbb{P}_h.
\end{equation}

**Step 3:** Update \( u_h^{n+1} \) and \( q_h^{n+1} \in \mathbb{P}_h \) according to
\begin{equation}
(1.5) \quad u_h^{n+1} = \hat{u}_h^{n+1} + \nabla \left( \psi_h^{n+1} - \psi_h^n \right),
\end{equation}

\begin{equation}
(1.6) \quad \langle q_h^{n+1}, \phi_h \rangle = \langle q_h^n, \phi_h \rangle - \langle \nabla \cdot \hat{u}_h^{n+1}, \phi_h \rangle, \quad \forall \phi_h \in \mathbb{P}_h.
\end{equation}

**Step 4:** Update pressure \( p_h^{n+1} \) by
\begin{equation}
(1.7) \quad p_h^{n+1} = -\frac{3\psi_h^{n+1}}{2\tau} + \mu q_h^{n+1}.
\end{equation}

**Step 5:** Find \( \theta_h^{n+1} \in \mathbb{T}_h \) as the solution of
\begin{equation}
(1.8) \quad \frac{1}{2\tau} \langle 3\theta_h^{n+1} - 4\theta_h^n + \theta_h^{n-1}, \phi_h \rangle + \mathcal{N}(u_h^{n+1}, \theta_h^{n+1}, \phi_h) + \lambda \mu \langle \nabla \theta_h^{n+1}, \nabla \phi_h \rangle = \langle b(t^{n+1}), \phi_h \rangle, \quad \forall \phi_h \in \mathbb{T}_h.
\end{equation}

**Remark 1.1** (Discontinuity of \( u_h^{n+1} \)). We note that \( u_h^{n+1} \) is a discontinuous function across inter-element boundaries and that, in light of (1.4) and (1.5), \( u_h^{n+1} \) is discrete divergence free in the sense that
\begin{equation}
(1.9) \quad \langle u_h^{n+1}, \nabla \phi_h \rangle = \langle \hat{u}_h^{n+1} + \delta \nabla \psi_h^{n+1}, \nabla \phi_h \rangle = 0, \quad \forall \phi_h \in \mathbb{P}_h.
\end{equation}

We now summarize the results of this paper along with organization. We introduce appropriate Assumptions 1-5 in §2 and introduce well known lemmas. In §3, we prove the stability result:
Theorem 1 (Stability). The SGUM is unconditionally stable in the sense that, for all $\tau > 0$, the following a priori bound holds:

$$
\|u^{N+1}_h\|^2 + \|\tilde{u}^{N+1}_h\|^2_0 + \|2u^{N+1}_h - \tilde{u}^{N+1}_h\|^2 + \|2\theta^{N+1}_h - \tilde{\theta}^{N+1}_h\|^2 + \|\theta^{N+1}_h\|^2_0 \\
+ \sum_{n=1}^{N} (\|\delta u^{n+1}_h\|^2 + \|\delta \theta^{n+1}_h\|^2 + \|\nabla \delta \psi^{n+1}_h\|^2) + 3\|\nabla \psi^{N+1}_h\|^2 + 2\mu \tau \|q^{N+1}_h\|^2 \\
+ \mu \tau \sum_{n=1}^{N} (\|\nabla \tilde{u}^{n+1}_h\|^2 + \lambda \|\nabla \tilde{\theta}^{n+1}_h\|) \leq \|u_1\|^2 + \|2u_1 - u^0\|^2 + \|2\theta^1 - \theta^0\|^2 \\
+ \|\theta_1\|^2 + 3\|\nabla \psi^1\|^2 + 2\mu \tau \|q^1\|^2 + C\tau \sum_{n=1}^{N} \left(\|f(t^{n+1})\|^2 - \|b(t^{n+1})\|^2\right).
$$

We then will carry out the following optimal error estimates through several lemmas in §4.

Theorem 2 (Error estimates). Suppose the exact solution of (1.1) is smooth enough and $\tau \leq Ch$. If Assumptions 1 and 3-5 below hold, then the errors of Algorithm 1 will be bound by

$$
\tau \sum_{n=1}^{N} \left(\|u(t^{n+1}) - u_h^{n+1}\|^2 + \|u(t^{n+1}) - \tilde{u}_h^{n+1}\|^2\right) \leq C (\tau^4 + h^4),
$$

$$
\tau \sum_{n=1}^{N} \left(\|u(t^{n+1}) - u_h^{n+1}\|^2_1 + h^2\|\theta(t^{n+1}) - \tilde{\theta}_h^{n+1}\|^2_1\right) \leq C (\tau^2 + h^2),
$$

$$
\tau \sum_{n=1}^{N} \|\theta(t^{n+1}) - \tilde{\theta}_h^{n+1}\|^2 \leq C (\tau^4 + h^4).
$$

Moreover, if Assumptions 2 also hold, then we have

$$
\tau \sum_{n=1}^{N} \|p(t^{n+1}) - p_h^{n+1}\|^2_0 \leq C (\tau^2 + h^2).
$$

We note that the condition $\tau \leq Ch$ in Theorem 2 can be omitted for the linearized Boussinesq equations (see Remark 4.4). Finally, we perform numerical tests in §5 to check accuracy and physically relevant numerical simulations, the Benard convection problem and the thermal driven cavity flow.

2. Preliminaries

In this section, we introduce 5 assumptions and well known lemmas to use in proof of main theorems. We resort to a duality argument for

$$
-\Delta v + \nabla r = z \quad \text{and} \quad \nabla \cdot v = 0, \quad \text{in } \Omega,
$$

which is the stationary Stokes system with vanishing boundary condition $v = 0$, as well as the Poisson equation

$$
-\Delta \omega = \xi, \quad \text{in } \Omega,
$$

with boundary condition $\omega = 0$.

We now state a basic assumption about $\Omega$. 

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Assumption 1 (Regularity of \((v, r)\) and \( \omega \)). The unique solutions \( \{v, r\} \) of (2.1) and \( \omega \) of (2.2) satisfy
\[
\|v\|_2 + \|r\|_1 \leq C\|z\|_0 \quad \text{and} \quad \|\omega\|_2 \leq C\|\xi\|_0.
\]
We remark that the validity of Assumption 1 is known if \( \partial \Omega \) is of class \( C^2 \) [3, 6], or if \( \partial \Omega \) is a two-dimensional convex polygon [7], and is generally believed for convex polyhedral [6].

We impose the following properties for relations between the spaces \( V_h \) and \( P_h \).

Assumption 2 (Discrete Inf-Sup). There exists a constant \( \beta > 0 \) such that
\[
\inf_{s_h \in \mathbb{P}_h} \sup_{w_h \in V_h} \frac{\langle \nabla \cdot w_h, s_h \rangle}{\|w_h\|_1 \|s_h\|_0} \geq \beta.
\]

Assumption 3 (Shape regularity and quasiuniformity [1, 2, 5]). There exists a constant \( C > 0 \) such that the ratio between the diameter \( h_K \) of an element \( K \in \mathcal{T} \) and the diameter of the largest ball contained in \( K \) is bounded uniformly by \( C \), and \( h_K \) is comparable with the meshsize \( h \) for all \( K \in \mathcal{T} \).

In order to launch Algorithm 1, we need to set \((u_h^1, p_h^1, \theta_h^1)\) via any first order methods which hold the following conditions.

Assumption 4 (The setting of the first step values). Let \((u(t^1), p(t^1), \theta(t^1))\) be the exact solution of (1.1) at \( t = t^1 \). The first step value \((u_h^1, p_h^1, \theta_h^1)\) satisfies
\[
\|u(t^1) - u_h^1\|_1 + \|\theta(t^1) - \theta_h^1\|_0 \leq C (\tau^2 + h^2),
\]
\[
\|u(t^1) - u_h^1\|_1 + \|p(t^1) - p_h^1\|_1 + \|\theta(t^1) - \theta_h^1\|_1 \leq C (\tau + h).
\]

Assumption 5 (Approximability [1, 2, 5]). For each \((w, \eta, s) \in H^2(\Omega) \times H^2(\Omega) \times H^1(\Omega)\), there exist approximations \((w_h, \eta_h, s_h) \in V_h \times P_h \times \mathbb{P}_h\) such that
\[
\|w - w_h\|_0 + h\|w - w_h\|_1 \leq C h^2\|w\|_2,
\]
\[
\|\eta - \eta_h\|_0 + h\|\eta - \eta_h\|_1 \leq C h^2\|\eta\|_2,
\]
\[
\|s - s_h\|_0 \leq C h\|s\|_1.
\]

The following elementary but crucial relations are derived in [14].

Lemma 2.1 (Inverse inequality). If \( I_h \) denotes the Clement interpolant, then
\[
\|I_h w\|_{L_2(\Omega)} \leq C h^{-d/6} \|w\|_{L_2(\Omega)}, \quad \text{and} \quad \|w - I_h w\|_{L_2(\Omega)} \leq C h^{2 - d/6} \|w\|_2.
\]

Lemma 2.2 (Div-grad relation). If \( w \in H_h^0(\Omega) \), then
\[
\|\nabla \cdot w\|_0 \leq \|\nabla w\|_0.
\]

Let now \((v_h, r_h) \in V_h \times P_h\) indicate the finite element solution of (2.1), namely,
\[
\langle \nabla v_h, \nabla w_h \rangle + \langle \nabla r_h, w_h \rangle = \langle z, w_h \rangle, \quad \forall w_h \in V_h,
\]
\[
\langle \nabla \cdot v_h, s_h \rangle = 0, \quad \forall s_h \in P_h.
\]

Then we can find the well known lemmas in [1, 2, 5]:

Lemma 2.3 (Error estimates for mixed FEM). Let \((v, r) \in H_h^0(\Omega) \times L_h^0(\Omega)\) be the solutions of (2.1) and \((v_h, r_h) = \Theta_h(v, r) \in V_h \times P_h\) be the Stokes projection defined by (2.3), respectively. If Assumptions 1-3 and 5 hold, then
\[
\|v - v_h\|_0 + h\|v - v_h\|_1 + h\|r - r_h\|_0 \leq C h^2 (\|v\|_2 + \|r\|_1),
\]
\[
\|v - v_h\| := \|v - v_h\|_{L_\infty(\Omega)} + \|\nabla (v - v_h)\|_{L_2(\Omega)} \leq C \|z\|_0.
\]
Proof. Inequality (2.4) is standard [1, 2, 5]. To establish (2.5), we just deal with the $L^\infty$-norm since the other can be treated similarly. If $I_h$ denotes the Clement interpolant, then $\|v - I_h v\|_{L^\infty(\Omega)} \leq C\|v\|_2$ and

$$\|I_h v - v_h\|_{L^\infty(\Omega)} \leq Ch^{-d/2}\|I_h v - v_h\|_{L^2(\Omega)} \leq C\|v\|_2$$

as a consequence of an inverse estimate. This completes the proof.

Remark 2.4 ($H^1$ stability of $r_h$). The bound $\|\nabla r_h\|_0 \leq C(\|v\|_2 + \|r\|_1)$ is a simple by-product of (2.1). To see this, we add and subtract $I_h q$, use the stability of $I_h$ in $H^1$, and observe that (2.4) implies $\|\nabla (r_h - I_h r)\|_0 \leq C h^{-1} \|r_h - I_h r\|_0 \leq C$.

Lemma 2.5 (Error Estimates for Poisson’s equation [1, 5]). Let $\omega \in H^2_0(\Omega)$ be the solution of the Poisson’s equation (2.2) and $\omega_h \in \mathcal{T}_h$ be its finite element approximation

$$(\nabla \omega_h, \nabla \phi_h) = (\xi, \phi_h), \ \forall \phi_h \in \mathcal{T}_h.$$ \hspace{1cm}(2.6)

If Assumptions 1, 3, and 5 hold, then there exists a positive constant $C$ satisfies

$$\|\omega - \omega_h\|_0 + h\|\omega - \omega_h\|_1 \leq C h^2 \|\omega\|_2 \leq C h^2 \|\xi\|_0, \ \|\omega - \omega_h\|_2 \leq C \|\xi\|_0.$$ \hspace{1cm}(2.7)

We finally state without proof several properties of the nonlinear form $N$. In view of (1.2), we have the following properties of $N$ for all $u_h, v_h, w_h \in V_h$:

$$N(u_h, v_h, w_h) = -N(u_h, v_h, v_h), \ N(u_h, v_h, v_h) = 0,$$

and

$$\nabla \cdot u = 0 \Rightarrow N(u, v_h, w_h) = (u \cdot \nabla) v_h, w_h = - (u \cdot \nabla) v_h, v_h).$$

Applying Sobolev imbedding Lemma yields the following useful results.

Lemma 2.6 (Bounds on nonlinear convection [6, 8]). Let $u, v \in H^2(\Omega)$ with $\nabla \cdot u = 0$, and let $u_h, v_h, w_h \in V_h$. Then

$$(2.8) \quad N(u, v, w_h) \leq C \left\{ \begin{array}{c} \|u\|_1 \|v_h\|_1 \|w_h\|_1 \\ \|u_h\|_0 \|v_h\|_0 \|w_h\|_1 \\ \|u\|_2 \|v_h\|_0 \|w_h\|_1, \end{array} \right.$$ \hspace{1cm}(2.9)

In addition

$$(2.10) \quad N(u_h, v, w_h) \leq C \left\{ \begin{array}{c} \|u_h\|_0 \|v\| \|w_h\|_1 \\ \|u_h\|_{L^\infty(\Omega)} \|v_h\|_1 \|w_h\|_1. \end{array} \right.$$ \hspace{1cm}(2.10)

Remark 2.7 (Treatment of convection term). As we mention in Remark 1.1, $u_h^{n+1}$ is discontinuous across inter-element boundaries and so $u_h^{n+1} \notin H^1(\Omega)$. Thus we can’t directly apply (2.8) anymore to treat the convection term. To solve this difficulty, we apply (2.10) together with (2.5) and inverse inequality Lemma 2.1.

We will use the following algebraic identities frequently to treat time derivative terms.

Lemma 2.8 (Inner product of time derivative terms). For any sequence $\{z^n\}_{n=0}^N$, we have

$$(2.11) \quad 2 \langle z^{n+1} - 4z^n + z^{n-1}, z^{n+1} \rangle = \delta \|z^{n+1}\|_0^2 + \delta \|z^{n-1}\|_0^2 + \|\delta z^{n+1}\|_0^2.$$
(2.12) \[ 2 \langle z^{n+1} - z^n, z^{n+1} \rangle = \|z^{n+1}\|_0^2 - \|z^n\|_0^2 + \|z^{n+1} - z^n\|_0^2, \]

and

(2.13) \[ 2 \langle z^{n+1} - z^n, z^n \rangle = \|z^{n+1}\|_0^2 - \|z^n\|_0^2 - \|z^{n+1} - z^n\|_0^2. \]

3. Proof of stability

We show that the SGUM is unconditionally stable via a standard energy method in this section. We start to prove stability with rewriting the momentum equation (1.3) by using (1.5) and (1.7) as follows:

\[
\frac{1}{2\tau} \langle 3u_h^{n+1} - 4u_h^n + u_h^{n-1}, w_h \rangle + \mathcal{N} (u_h^*, \hat{u}_h^{n+1}, w_h) + \kappa \mu^2 \langle g\theta_h^*, w_h \rangle \\
- \left\langle \nabla \left( \frac{3}{2\tau} \psi_h^{n+1} - \mu \psi_h^n \right), w_h \right\rangle + \mu \left\langle \nabla \hat{\psi}_h^{n+1}, \nabla w_h \right\rangle = \langle f(t^{n+1}), w_h \rangle.
\]

We now choose \( w_h = 4\tau \hat{u}_h^{n+1} \) and apply (2.11) to obtain

(3.1) \[ \delta \|u_h^{n+1}\|_0^2 + \delta \|2u_h^{n+1} - u_h^n\|_0^2 + \|\delta u_h^{n+1}\|_0^2 + 4\tau \mu \|\nabla \hat{u}_h^{n+1}\|_0^2 = \sum_{i=1}^A A_i, \]

where

\[
A_1 := 6 \left\langle \nabla \psi_h^{n+1}, \hat{u}_h^{n+1} \right\rangle, \quad A_2 := 4\tau \mu \left\langle g_h^n, \nabla \cdot \hat{u}_h^{n+1} \right\rangle, \\
A_3 := 4\tau \left\langle f(t^{n+1}), \hat{u}_h^{n+1} \right\rangle, \quad A_4 := -4\tau \kappa \mu^2 \left\langle g \left( \theta_h^n - \theta_h^{n-1} \right), \hat{u}_h^{n+1} \right\rangle.
\]

We give thanks to (2.7) for eliminating the convection term. In light of \( \hat{u}_h^{n+1} = u_h^{n+1} - \nabla \delta \psi_h^{n+1} \), (1.9) and (2.12) yield

\[
A_1 = -6 \left\langle \nabla \psi_h^{n+1}, \nabla \delta \psi_h^{n+1} \right\rangle = -3 \left( \|\nabla \psi_h^{n+1}\|_0^2 - \|\nabla \psi_h^n\|_0^2 + \|\nabla \delta \psi_h^{n+1}\|_0^2 \right).
\]

Before we estimate \( A_2 \), we evaluate an inequality via choosing \( \phi_h = \delta q_h^{n+1} \) in (1.6) to get

\[ \|\delta q_h^{n+1}\|_0^2 = - \langle \nabla \cdot \hat{u}_h^{n+1}, \delta q_h^{n+1}\rangle \leq \|\nabla \cdot \hat{u}_h^{n+1}\|_0 \|\delta q_h^{n+1}\|_0. \]

Lemma 2.2 derives \[ \|\delta q_h^{n+1}\|_0^2 \leq \|\nabla \cdot \hat{u}_h^{n+1}\|_0^2 \leq \|\nabla \hat{u}_h^{n+1}\|_0^2, \] and so (1.6) and (2.13) lead us

\[
A_2 = -4\mu \tau \left\langle q_h^n, \delta q_h^{n+1} \right\rangle = -2\mu \tau \left( \|q_h^{n+1}\|_0^2 - \|q_h^n\|_0^2 - \|\delta q_h^{n+1}\|_0^2 \right) \\
\leq -2\mu \tau \left( \|q_h^{n+1}\|_0^2 - \|q_h^n\|_0^2 + \|\delta \varphi_h^{n+1}\|_0^2 \right).
\]

Clearly, we have

\[
A_3 \leq C \tau \left\| f(t^{n+1}) \right\|_{-1} + \tau \mu \|\nabla \hat{u}_h^{n+1}\|_0^2.
\]

We now attack \( A_4 \) with \[ \|\hat{u}_h^{n+1}\|_0^2 = \|u_h^{n+1}\|_0^2 + \|\nabla \delta \psi_h^{n+1}\|_0^2 \] which comes form (1.5) and (1.9). Then we arrive at

\[
A_4 \leq C \kappa \mu^4 \tau \|2\theta_h^n - \theta_h^{n-1}\|_0^2 + C \tau \|u_h^{n+1}\|_0^2 + \|\nabla \delta \psi_h^{n+1}\|_0^2.
\]
We also define \( \Theta^{n+1} \) and \( \phi^{n+1} \) in (1.8) and use (2.11), then we obtain
\[
\delta \| \theta^{n+1} \|_0^2 + \delta \| \Delta \theta_h^{n+1} - \theta_h^{n+1} \|_0^2 + \| \Delta \theta_h^{n+1} \|_0^2 + 4\tau \lambda \mu \| \nabla \theta_h^{n+1} \|
\]
(3.2)
\[
= 4\tau \langle b(t^{n+1}) , \theta_h^{n+1} \rangle \leq C \frac{\tau}{\lambda \mu} \| b(t^{n+1}) \|_{L^2}^2 + \lambda \mu \| \nabla \theta_h^{n+1} \|_0^2.
\]
Inserting \( A_1-A_4 \) back into (3.1), adding with (3.2), and then summing over \( n \) from 1 to \( N \) lead (1.10) by help of discrete Gronwall inequality and the equality \( \| u_h^{n+1} \|_0^2 = \| u_h^n \|_0^2 + \| \nabla \phi_h^{n+1} \|_0^2 \). So we finish the proof of Theorem 1.

4. Error estimates

We prove Theorem 2 which is error estimates for SGUM of Algorithm 1. This proof is carried out through several lemmas. We start to prove with defining \( \Theta^{n+1} \)
and \( \phi^{n+1} \) in (4.3), and then by comparing (4.15) and (1.8). We derive strong estimates of order 1 and this result is instrumental in proving weak estimates of order 2 for the errors
\[
E_h^{n+1} := U_h^{n+1} - U_h^n, \quad \hat{E}_h^{n+1} := U_h^{n+1} - \hat{U}_h^{n+1},
\]
(4.2)
\[
E_h^{n+1} := P_h^{n+1} - P_h^n, \quad \hat{E}_h^{n+1} := \theta_h^{n+1} - \hat{\theta}_h^{n+1}.
\]
Then, in conjunction with (4.2), we can readily get of the same accuracy for the errors
\[
E^{n+1} := u(t^{n+1}) - u_h^{n+1}, \quad \hat{E}_h^{n+1} := u(t^{n+1}) - \hat{u}_h^{n+1},
\]
(4.3)
\[
e^{n+1} := p(t^{n+1}) - p_h^{n+1}, \quad \hat{\theta}^{n+1} := \theta(t^{n+1}) - \hat{\theta}_h^{n+1}.
\]
Additionally, we denote
\[ e_h^{n+1} := P_h^{n+1} + \frac{3\delta e_h^{n+1}}{2\tau}. \]
We readily obtain the following properties
\[ \mathbf{G}^{n+1} = \mathbf{E}^{n+1} - \mathbf{E}_h^{n+1} = \hat{\mathbf{E}}^{n+1} - \hat{\mathbf{E}}_h^{n+1} \quad \text{and} \quad \hat{\mathbf{E}}_h^{n+1} = \mathbf{E}_h^{n+1} + \nabla \delta \psi_h^{n+1}, \]
whence we deduce crucial orthogonality properties:
\[ \langle \mathbf{E}_h^{n+1}, \nabla \phi_h \rangle = \langle \mathbf{G}^{n+1}, \nabla \phi_h \rangle = 0, \quad \forall \phi_h \in \mathcal{P}_h, \]
as well as, from (1.9),
\[ \langle \mathbf{E}^{n+1}, \nabla \phi_h \rangle = \langle \mathbf{G}^{n+1}, \nabla \phi_h \rangle = 0, \quad \forall \phi_h \in \mathcal{P}_h, \]
whence we deduce crucial orthogonality properties:
\[ \| \hat{\mathbf{E}}_h^{n+1} \|_0^2 = \| \mathbf{E}_h^{n+1} \|_0^2 + \| \nabla \delta \psi_h^{n+1} \|_0^2. \]
We also point out that, owing to Lemma 2.2, \( q_h^{n+1} \in \mathcal{P}_h \) defined in (1.6) satisfies
\[ \| q_h^{n+1} - q_h \|_0 \leq \| \nabla \mathcal{E}_h^{n+1} \|_0. \]
In conjunction with \( \eta^{n+1} = \theta(t^{n+1}) - \mathcal{G}_h^{n+1} = \eta^{n+1} - \eta_h^{n+1}, \) Assumption 5 leads
\[ \| \eta^{n+1} \|_0^2 + h^2 \eta^{n+1} \|_1 \leq Ch^4 \| \theta(t^{n+1}) \|_2^2, \]
and
\[ \| \delta \eta^{n+1} \|_0^2 + h^2 \| \delta \eta^{n+1} \|_1^2 \leq C \tau h^4 \int_{t_n}^{t_{n+1}} \| \theta_t \|_2 dt. \]
We now estimate the first order accuracy for velocity and temperature in Lemma 4.1, and then the 2nd order accuracy for time-derivative of velocity and temperature in Lemma 4.3. The result of Lemma 4.1 is instrumental to treat the convection term in proof of Lemma 4.3. We will use Lemmas 4.1 and 4.3 to prove optimal error accuracy in Lemma 4.5. Finally, we will prove pressure error estimate in Lemma 4.6.

**Lemma 4.1 (Reduced rate of convergence for velocity and temperature).** Suppose the exact solution of (1.1) is smooth enough. If Assumptions 3-5 hold, then the velocity and temperature error functions satisfy
\[ \left\| \mathbf{E}_h^{N} \right\|_0^2 + \left\| \mathbf{E}_h^{N} \right\|_0^2 + \left\| 2\mathbf{E}_h^{N} - \mathbf{E}_h^{N} \right\|_0^2 + 2\mu \| q_h^{N} \|_0^2 + \frac{4\tau^2}{3} \left( \left\| \mathbf{E}_h^{N+1} \right\|_0^2 + \lambda \| \nabla \theta^{n+1} \|_0^2 \right) \right. \]
\[ \left. + \left( \left\| \delta \theta_h^{N+1} \right\|_0^2 + \left\| 2\delta \theta_h^{N+1} - \delta \theta_h^{N+1} \right\|_0^2 + \mu \sum_{n=1}^{N} \left( \left\| \nabla \mathbf{E}_h^{N+1} \right\|_0^2 + \lambda \left\| \nabla \theta_h^{n+1} \right\|_0^2 \right) \right) \right) \leq C \left( \tau^2 + h^2 \right). \]

**Proof.** We resort to the Taylor theorem to write (1.1) as follows:
\[ \frac{3\mathbf{u}(t^{n+1}) - 4\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})}{2\tau} + (\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{u}(t^{n+1}) + \nabla p(t^{n+1}) - \mu \nabla \mathbf{u}(t^{n+1}) + \kappa \mathbf{g}^\theta(t^{n+1}) = \mathbf{R}^{n+1} + \mathbf{f}(t^{n+1}) \]
and
\[ \frac{3\theta(t^{n+1}) - 4\theta(t^n) + \theta(t^{n-1})}{2\tau} + \mathbf{u}(t^{n+1}) \cdot \nabla \theta(t^{n+1}) - \lambda \mu \Delta \theta(t^{n+1}) = Q^{n+1} + b(t^{n+1}), \]
where \( R^{n+1} := \frac{1}{\tau} \int_{t^n}^{t^{n+1}} u_{tt}(s)(s-t^n)^2 ds - \frac{1}{\tau} \int_{t^{n-1}}^{t^n} u_{tt}(s)(t^n-s)^2 ds \) and \( Q^{n+1} := \frac{1}{\tau} \int_{t^n}^{t^{n+1}} \theta_{tt}(s)(s-t^n)^2 ds - \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \theta_{tt}(s)(t^n-s)^2 ds \) are the truncation errors. In conjunction with the definition of the Stokes projection \( \{ U_h^{n+1}, P_h^{n+1} \} \), we readily get a weak formulation for (4.8), \( \forall w_h \in V_h \),

\[
\frac{1}{2\tau} \left( 3\tilde{E}_{h}^{n+1} - 4E^n + E^{n-1} \right) + \mu \tau \left( \nabla \tilde{E}_{h}^{n+1}, \nabla w_h \right)
\]
(4.10)

\[
= -\left( \nabla (\delta P_h^{n+1} + \varepsilon_h^n), w_h \right) + \mu \left( \nabla q_h^n, w_h \right)
\]
\[
+ \mathcal{N} (2u^n - u_h^{n-1}, \tilde{u}_h^{n+1}, w_h) - \mathcal{N} (u(t^{n+1}), u(t^{n+1}), w_h)
\]
\[
- \kappa \mu^2 \left( \left( \theta(t^{n+1}) - 2\theta_h^n + \theta_h^{n-1} \right), w_h \right) + \left( R^{n+1}, w_h \right).
\]

Choosing \( w_h = 4\tau \tilde{E}_{h}^{n+1} = 4\tau \left( \tilde{E}_{h}^{n+1} - G^{n+1} \right) = 4\tau \left( E_{h}^{n+1} + \nabla \delta \psi_h^{n+1} \right) \) in (4.11) and using (4.5) and (2.11) yield

\[
\begin{align*}
\delta \left\| E_h^{n+1} \right\|_0^2 + \delta \left\| 2E_{h}^{n+1} - E_h^n \right\|_0^2 + \left\| \delta E_h^{n+1} \right\|_0^2
\end{align*}
\]
(4.12)

\[
+ 6\left\| \nabla \delta \psi_h^{n+1} \right\|_0^2 + 4\mu \tau \left\| \nabla \tilde{E}_{h}^{n+1} \right\|_0^2 = \sum_{n=1}^7 A_i,
\]

where

\[
A_1 := -4\tau \mathcal{N} \left( u(t^{n+1}), u(t^{n+1}), \tilde{E}_{h}^{n+1} \right) + 4\tau \mathcal{N} \left( 2u^n - u_h^{n-1}, \tilde{u}_h^{n+1}, \tilde{E}_{h}^{n+1} \right),
\]
\[
A_2 := -2 \left( 3G^{n+1} - 4G^n + G^{n-1}, \tilde{E}_{h}^{n+1} \right),
\]
\[
A_3 := -4\tau \left( \nabla \delta P_h^{n+1}, \tilde{E}_{h}^{n+1} \right),
\]
\[
A_4 := -4\tau \left( \nabla \varepsilon_h^n, \tilde{E}_{h}^{n+1} \right),
\]
\[
A_5 := 4\mu \tau \left( \nabla q_h^n, \tilde{E}_{h}^{n+1} \right),
\]
\[
A_6 := 4\tau \left( R^{n+1}, \tilde{E}_{h}^{n+1} \right),
\]
\[
A_7 := -4\tau \kappa \mu^2 \left( \left( \theta(t^{n+1}) - 2\theta_h^n + \theta_h^{n-1} \right), \tilde{E}_{h}^{n+1} \right).
\]

We now estimate all the terms from \( A_1 \) to \( A_7 \) respectively. To tackle \( A_1 \), we first add and subtract \( 2u(t^n) - u(t^{n-1}) \) to obtain

\[
A_1 = -4\tau \mathcal{N} \left( \delta \delta u(t^{n+1}), u(t^{n+1}), \tilde{E}_{h}^{n+1} \right) - 4\tau \mathcal{N} \left( 2u_h^n - u_h^{n-1}, \tilde{E}_{h}^{n+1}, \tilde{E}_{h}^{n+1} \right)
\]
\[
- 4\tau \mathcal{N} \left( 2E^n - E^{n-1}, u(t^{n+1}), \tilde{E}_{h}^{n+1} \right).
\]

Because of \( \mathcal{N} \left( 2u_h^n - u_h^{n-1}, \tilde{E}_{h}^{n+1}, \tilde{E}_{h}^{n+1} \right) = 0 \), which comes from (2.7), the second term of \( A_1 \) can be replaced by

\[
4\tau \mathcal{N} \left( 2E^n - E^{n-1} - 2u(t^n) + u(t^{n-1}), G^{n+1}, \tilde{E}_{h}^{n+1} \right).
\]
If we apply Lemma 2.6, then we can readily obtain
\[
A_1 \leq C\tau \left( \| \delta \mathbf{u}(t^{n+1}) \|_0 \| \mathbf{u}(t^{n+1}) \|_2 + 2\| \mathbf{E}^n - \mathbf{E}^{n-1} \|_0 \| \mathbf{u}(t^{n+1}) \|_2 \right) \| \tilde{\mathbf{E}}^{n+1} \|_1 \\
+ C\tau \left( \| 2\mathbf{E}^n - \mathbf{E}^{n-1} \|_0 \| \mathbf{G}^{n+1} \| + \| 2\mathbf{u}(t^n) - \mathbf{u}(t^{n-1}) \|_2 \| \mathbf{G}^{n+1} \|_0 \right) \| \tilde{\mathbf{E}}^{n+1} \|_1.
\]
Since we have \( \| \mathbf{u}(t^{n+1}) \|_2 + \| \mathbf{G}^{n+1} \| \leq M \) according to (4.4), we arrive at
\[
A_1 \leq C\tau \left( \| 2\mathbf{E}^n - \mathbf{E}^{n-1} \|_0^2 + 2\| \mathbf{G}^n - \mathbf{G}^{n-1} \|_0^2 + \| \mathbf{G}^{n+1} \|_0^2 \right) \\
+ \mu\tau \left\| \nabla \tilde{\mathbf{E}}^{n+1} \right\|_0^2 + \frac{C\tau^4}{\mu} \int_{t^{n-1}}^{t^{n+1}} \| \mathbf{u}_{ttt}(t) \|_0^2 dt.
\]
In light of (4.3) and (4.5), \( A_2 \) becomes
\[
A_2 \leq C\tau \left\| \mathbf{E}_h^{n+1} \right\|_0^2 + \frac{1}{4} \left\| \nabla \delta \psi_h^{n+1} \right\|_0^2 + C\tau^4 \int_{t^n}^{t^{n+1}} \left( \| \mathbf{u}_t(t) \|_2^2 + \| p_t(t) \|_1^2 \right) dt.
\]
In order to estimate \( A_3 \) and \( A_4 \), we note that the cancellation law (1.9) gives \( \langle \nabla \varepsilon_h^n, \mathbf{E}_h^{n+1} \rangle = 0 \). Then \( \tilde{\mathbf{E}}^{n+1}_h = \mathbf{E}^{n+1}_h + \nabla \delta \psi_h^{n+1} \) yields
\[
A_3 = -4\tau \langle \nabla \delta P_h^{n+1}, \nabla \delta \psi_h^{n+1} \rangle \\
\leq \frac{1}{4} \left\| \nabla \delta \psi_h^{n+1} \right\|_0^2 + C\tau^3 \int_{t^n}^{t^{n+1}} \left( \| \mathbf{u}_t(t) \|_2^2 + \| p_t(t) \|_1^2 \right) dt.
\]
In conjunction with the definition \( \varepsilon_h^{n+1} = P_h^{n+1} + \frac{3\delta \psi_h^{n+1}}{2\tau} \), \( A_4 \) can be evaluated by
\[
A_4 = -4\tau \langle \nabla \varepsilon_h^n, \nabla \delta \psi_h^{n+1} \rangle = -\frac{8\tau^2}{3} \langle \nabla \varepsilon_h^n, \nabla (\delta \varepsilon_h^{n+1} - \delta P_h^{n+1}) \rangle \\
\leq -\frac{4\tau^2}{3} \left( \| \nabla \varepsilon_h^n \|_0^2 - \| \nabla \varepsilon_h^n \|_0^2 - \| \nabla \delta \varepsilon_h^{n+1} \|_0^2 \right) + C\tau^3 \| \nabla \varepsilon_h^n \|_0^2 + C\tau \| \nabla \delta P_h^{n+1} \|_0^2.
\]
If we now apply inequality \( (a + b)^2 \leq 4a^2 + \frac{4}{3}b^2 \), then we can get
\[
\frac{4\tau^2}{3} \| \nabla \delta \varepsilon_h^{n+1} \|_0^2 = \frac{4\tau^2}{3} \left\| \nabla \delta P_h^{n+1} + \frac{3}{2\tau} \nabla \delta \psi_h^{n+1} \right\|_0^2 \\
\leq C\tau^2 \| \nabla \delta P_h^{n+1} \|_0^2 + 4 \| \nabla \delta \psi_h^{n+1} \|_0^2.
\]
So we arrive at
\[
A_4 \leq -\frac{4\tau^2}{3} \left( \| \nabla \varepsilon_h^n \|_0^2 - \| \nabla \varepsilon_h^n \|_0^2 \right) + C\tau^3 \| \nabla \varepsilon_h^n \|_0^2 \\
+ 4 \| \nabla \delta \psi_h^{n+1} \|_0^2 + C\tau^2 \int_{t^n}^{t^{n+1}} \left( \| \mathbf{u}_t(t) \|_2^2 + \| p_t(t) \|_1^2 \right) dt.
\]
In light of \( \nabla \cdot \mathbf{u}(t^{n+1}) = 0 \) and (2.13), (1.6) and (4.6) yield
\[
A_5 = 4\mu\tau \langle \dot{q}_h^n, \nabla \cdot \tilde{\mathbf{u}}_h^{n+1} \rangle = -4\mu\tau \langle \dot{q}_h^n, \delta q_h^{n+1} \rangle \\
= -2\mu\tau \left( \| q_h^{n+1} \|_0^2 - \| q_h^n \|_0^2 - \| \delta q_h^{n+1} \|_0^2 \right) \\
\leq -2\mu\tau \left( \| q_h^{n+1} \|_0^2 - \| q_h^n \|_0^2 \right) + 2\mu\tau \| \nabla \tilde{\mathbf{E}}_h^{n+1} \|_0^2.
\]
Also we readily get
\[ A_6 \leq C\tau \|E_h^{n+1}\|_0^2 + \frac{1}{4} \|
abla\delta\nu_h^{n+1}\|_0^2 + C\tau^4 \int_{t_{n-1}}^{t_{n+1}} \|u_{tt}(t)\|_0^2 dt. \]

The Hölder inequality and (4.7) yield
\[
\begin{align*}
& A_7 = -4\kappa\mu^2 \tau \left( g (\delta\theta(t^{n+1}) - 2\vartheta^n + \vartheta^{n-1}), \hat{E}_h^{n+1} \right) \\
& \leq C\kappa^2 \mu^4 \tau \left( \|2\vartheta_h^n - \vartheta_h^{n-1}\|_0^2 + \|2\eta^n - \eta^{n-1}\|_0^2 \right) + \tau \|E_h^{n+1}\|_0^2 \\
& + \frac{1}{4} \|
abla\delta\psi_h^{n+1}\|_0^2 + C\tau^4 \int_{t_{n-1}}^{t_{n+1}} \|\theta_{tt}(t)\|_0^2 dt.
\end{align*}
\]

Inserting the above estimates into (4.12) gives
\[
\begin{align*}
& \delta \|E_h^{n+1}\|_0^2 + \delta \|2E_h^{n+1} - E_h^{n-1}\|_0^2 + \|\delta\psi_h^{n+1}\|_0^2 + \|
abla\delta\psi_h^{n+1}\|_0^2 + \mu \tau \|
abla\hat{E}_h^{n+1}\|_0^2 \\
& + \frac{4\tau^2}{3} \|
abla\varphi_h^n\|_0^2 + 2\mu \tau \|\varphi_h^n\|_0^2 + \frac{4\tau^2}{3} \|
abla\varphi_h^n\|_0^2 + C\tau^3 \|
abla\varphi_h^n\|_0^2 + 2\mu \tau \|\varphi_h^n\|_0^2 \\
& + C\tau \left( \|E_h^{n+1}\|_0^2 + \|2E_h^{n+1} - E_h^{n-1}\|_0^2 + \|2G^n - G^{n-1}\|_0^2 + \|G^{n+1}\|_0^2 \right) \\
& + C\kappa^2 \mu^4 \tau \|2\vartheta_h^n - \vartheta_h^{n-1}\|_0^2 + C\tau^4 \int_{t_{n-1}}^{t_{n+1}} \left( \|\theta_{tt}(t)\|_0^2 + \|\theta_{t}(t)\|_0^2 \right) dt \\
& + C\kappa^2 \mu^4 \tau \|2\vartheta_h^n - \vartheta_h^{n-1}\|_0^2 + C(\tau^2 + \mu^4) \int_{t_{n-1}}^{t_{n+1}} \left( \|\theta_{t}(t)\|_2^2 + \|p_t(t)\|_1^2 \right) dt.
\end{align*}
\]

On the other hand, the definition (4.1) of $\Theta_h^{n+1} \in T_h$ leads a weak formula of (4.9) as, for all $\phi_h \in T_h$,
\[
\begin{align*}
& \frac{1}{2\tau} \left( 3\theta(t^{n+1}) - 4\theta(t^n) + \theta(t^{n-1}), \phi_h \right) + \mathcal{N} \left( u(t^{n+1}), \theta(t^{n+1}), \phi_h \right) \\
& + \lambda\mu \left( \nabla\Theta_h^{n+1}, \nabla\phi_h \right) = \left( Q^{n+1} + b(t^{n+1}), \phi_h \right).
\end{align*}
\]

We now subtract (1.8) from (4.15) to derive
\[
\begin{align*}
& \frac{1}{2\tau} \left( 3\theta^{n+1} - 4\vartheta^n + \vartheta^n - \vartheta^{n-1}, \phi_h \right) + \mathcal{N} \left( u(t^{n+1}), \theta(t^{n+1}), \phi_h \right) \\
& - \mathcal{N} \left( u^{n+1}, \theta_h^{n+1}, \phi_h \right) + \lambda\mu \left( \nabla\vartheta_h^{n+1}, \nabla\phi_h \right) = \left( Q^{n+1}, \phi_h \right).
\end{align*}
\]

Choosing $\phi_h = 4\tau\vartheta_h^{n+1} = 4\tau (\vartheta^{n+1} - \eta^{n+1})$ yields
\[
\begin{align*}
& \delta \|\vartheta_h^{n+1}\|_0^2 + \delta \|2\vartheta_h^{n+1} - \vartheta_h^{n-1}\|_0^2 + \|\delta\vartheta_h^{n+1}\|_0^2 + 4\lambda\mu \tau \|
abla\vartheta_h^{n+1}\|_0^2 = A_8 + A_9,
\end{align*}
\]
where
\[
\begin{align*}
A_8 := -4\tau\mathcal{N} \left( u(t^{n+1}), \theta(t^{n+1}), \vartheta_h^{n+1} \right) + 4\tau\mathcal{N} \left( u_h^{n+1}, \theta_h^{n+1}, \vartheta_h^{n+1} \right), \\
A_9 := -2 \left( 3\eta^{n+1} - 4\eta^n + \eta^{n-1}, \vartheta_h^{n+1} \right) + 4\tau \left( Q^{n+1}, \vartheta_h^{n+1} \right).
\end{align*}
\]
To estimate $A_8$, we note $\mathcal{N}(E^{n+1} - u(t^{n+1}), \vartheta_h^{n+1}, \vartheta_h^{n+1}) = 0$ which comes from (2.7), then $\|E_{N+1}\|_{\mathbf{L}^2(\Omega)} \leq C h^{-\frac{1}{2}} \|E_{N+1}\|_0$ and $\|\eta_{N+1}\|_1 \leq C h \|\theta(t^{n+1})\|_2$ yield

$$A_8 = -4\tau \mathcal{N}(E^{n+1}, \theta(t^{n+1}), \vartheta_h^{n+1}) + 4\tau \mathcal{N}(E^{n+1} - u(t^{n+1}), \eta^{n+1}, \vartheta_h^{n+1})$$

\leq C\tau \|E^{n+1}\|_0 \|\theta(t^{n+1})\|_2 \|\vartheta_h^{n+1}\|_1

+ C\tau \left( \|E^{n+1}\|_{\mathbf{L}^2(\Omega)} \|\eta^{n+1}\|_1 + \|u(t^{n+1})\|_2 \|\eta^{n+1}\|_0 \right) \|\vartheta_h^{n+1}\|_1

\leq C\tau \left( \|E^{n+1}\|_0^2 + \|\eta^{n+1}\|_0^2 \right) + \lambda \mu \|\nabla \vartheta_h^{n+1}\|_0^2.

We use $\|\delta \eta^{n+1}\|^2_0 \leq C \tau h^4 \int_{t^{n+1}}^{t^{n+1}} \|\theta(t)\|_2^2 dt$ which is (4.7) to attack $A_9$,

$$A_9 \leq C \left| 3\eta^{n+1} - 4\eta^n + \eta^{n-1} \right|_0 \|\vartheta_h^{n+1}\|_0 + C\tau \|Q^{n+1}\|_0 \|\vartheta_h^{n+1}\|_0$$

\leq C\tau \|\vartheta_h^{n+1}\|_0^2 + C\tau^4 \int_{t^{n-1}}^{t^{n+1}} \|\theta(t)\|_2^2 dt + C\tau^4 \int_{t^{n-1}}^{t^{n+1}} \|\theta(t)\|_2^2 dt.

Inserting the above estimates into (4.17) gives

$$\delta \left( \|\vartheta_h^{n+1}\|_0^2 + \|2\vartheta_h^{n+1} - \vartheta_h^2\|_0^2 + \|\delta \vartheta_h^{n+1}\|_0^2 + 3\lambda \mu \|\nabla \vartheta_h^{n+1}\|_0^2 \right) + \lambda \mu \|\nabla \vartheta_h^{n+1}\|_0^2$$

\leq C\tau \|\vartheta_h^{n+1}\|_0^2 + C\tau \left( \|E^{n+1}\|_0^2 + \|\eta^{n+1}\|_0^2 \right)

\leq C\tau \left( \|\vartheta_h^{n+1}\|_0^2 + \|2\vartheta_h^{n+1} - \vartheta_h^2\|_0^2 + \|\delta \vartheta_h^{n+1}\|_0^2 + 3\lambda \mu \|\nabla \vartheta_h^{n+1}\|_0^2 \right)

\leq C\tau \left( \|\vartheta_h^{n+1}\|_0^2 + \|\eta^{n+1}\|_0^2 \right) + \lambda \mu \|\nabla \vartheta_h^{n+1}\|_0^2

+ C(h^4 + \tau^4) \int_{t^{n-1}}^{t^{n+1}} \left( \|\theta(t)\|_2^2 + \|\theta(t)\|_2^2 \right) dt.

Adding 2 inequalities (4.14) and (4.18) and summing over $n$ from 1 to $N$ imply

$$\|E_h^{N+1}\|^2_0 + \|2E_h^{N+1} - E_h^N\|^2_0 + \|\vartheta_h^{N+1}\|^2_0 + \|2\vartheta_h^{N+1} - \vartheta_h^N\|^2_0 + \|\vartheta_h^{n+1} - \vartheta_h^n\|^2_0 + \|\vartheta_h^{n+1} - \vartheta_h^n\|^2_0$$

\begin{align*}
&+ \frac{4\tau^2}{3} \|\nabla \varepsilon_h^{n+1}\|_0^2 + \sum_{n=1}^N \left( \|\delta \vartheta_h^{n+1}\|^2_0 + \|\delta \delta \vartheta_h^{n+1}\|^2_0 + \|\nabla \delta \vartheta_h^{n+1}\|^2_0 \right)

&\leq \|E_h^1\|^2_0 + \|2E_h^1 - E_h^0\|^2_0 + \|\vartheta_h^1\|^2_0

&+ 2\|\vartheta_h^0 - \vartheta_h^0\|^2_0 + \|\vartheta_h^0 - \vartheta_h^0\|^2_0 + \|\vartheta_h^0 - \vartheta_h^0\|^2_0

&+ \tau \sum_{n=1}^N \left( \|\vartheta_h^{n+1}\|^2_0 + \|2\vartheta_h^n - \vartheta_h^{n-1}\|^2_0 + \|2\vartheta_h^n - \vartheta_h^{n-1}\|^2_0 + \|\vartheta_h^n - \vartheta_h^{n-1}\|^2_0 \right)

&+ C\tau \sum_{n=1}^N \left( \|\vartheta_h^{n+1}\|^2_0 + \|2\vartheta_h^n - \vartheta_h^{n-1}\|^2_0 + \|2\vartheta_h^n - \vartheta_h^{n-1}\|^2_0 \right)

&+ C (h^4 + \tau^4) \int_{t^{n-1}}^{t^{n+1}} \left( \|u(t)\|_2^2 + \|p(t)\|_1^2 \right) dt

&+ C (\tau^2 + h^4) \int_{t^{n-1}}^{t^{n+1}} \left( \|u(t)\|_2^2 + \|p(t)\|_1^2 \right) dt.
\end{align*}

Because of $q^1 = 0$, we have $e_h^1 = P_h^1 + \frac{\mu}{\tau} \psi_h^1 = e_h^1$, and thus Assumption 4 yields $\|\nabla e_h^1\|_0^2 \leq C$ and the first 4 terms in the right hand side can be bound by Assumption 4 and properties $E^0 = 0$ and $q^1 = 0$ which are directly deduced from the conditions in Algorithm 1. As well as the remaining terms can be treated by
the discrete Gronwall lemma. Finally, in conjunction with (4.5), we conclude the desired result and complete this proof.

**Remark 4.2 (Optimal estimation).** In order to get optimal accuracy, we must get rid of the terms of \( A_3 \) and \( A_4 \) by applying duality argument in Lemma 4.5. To do this, we first evaluate the errors for time-derivative of velocity and temperature in Lemma 4.3. Thus we need to evaluate optimal initial errors for the case \( n = 1 \), and so we have to recompute again (4.13). We start to rewrite \( A_4 \) as

\[
A_4 = -\frac{8\tau^2}{3} \left\langle \nabla \epsilon_h^1, \nabla (\delta \epsilon_h^2 - \delta P_h^2) \right\rangle \\
\leq -\frac{4\tau^2}{3} \left( ||\nabla \epsilon_h^2||_0^2 - ||\nabla \epsilon_h^1||_0^2 + C\tau^2 ||\nabla \epsilon_h^1||_0^2 + C\tau^2 ||\nabla \delta P_h^{n+1}||_0^2 \right)
\leq -\frac{4\tau^2}{3} ||\nabla \epsilon_h^2||_0^2 + C\tau^2 ||\nabla \epsilon_h^1||_0^2 + 4||\nabla \delta \psi_h^n||_0^2 + C\tau^2 \int_0^T \left( ||p(t)||_1^2 + ||u(t)||_2^2 \right) dt.
\]

In light of Assumption 4, we arrive at (4.19)

\[
\left|\nabla \bar{\psi}_h^n\right|^2 + \left|\nabla \bar{\psi}_h^{n+1}\right|^2 + \left|\nabla \bar{\psi}_h^2 - \bar{\psi}_h^1\right|^2 + \frac{1}{2} \left|\nabla \delta \psi_h^n\right|^2 + \left|\nabla \delta \psi_h^{n+1}\right|^2 + \left|\nabla \delta \vartheta_h^n\right|^2 + \left|\nabla \delta \vartheta_h^{n+1}\right|^2 + \left|\nabla \delta \psi_h^n\right|^2 + \frac{4\tau^2}{3} \left( ||\nabla \epsilon_h^2||_0^2 + \mu \lambda \tau \left|\nabla \delta \vartheta_h^n\right|^2 \right) \leq C (\tau^2 + h^2).
\]

We now start to estimate errors for time-derivative of velocity.

**Lemma 4.3 (Error estimate for time-derivative of velocity and temperature).** Suppose the exact solution of \((1.1)\) is smooth enough and \( \tau \leq Ch \). If Assumptions 3-5 hold, then the time derivative velocity and temperature error functions satisfy (4.20)

\[
\left|\nabla \phi_h^{N+1}\right|^2 + \left|\nabla \bar{\phi}_h^{N+1}\right|^2 + \left|\nabla \phi_h^N - \nabla \phi_h^{N+1}\right|^2 + \frac{1}{2} \left|\nabla \delta \phi_h^n\right|^2 + \left|\nabla \delta \phi_h^{n+1}\right|^2 + \left|\nabla \delta \vartheta_h^n\right|^2 + \left|\nabla \delta \vartheta_h^{n+1}\right|^2 + \left|\nabla \delta \phi_h^n\right|^2 + \frac{4\tau^2}{3} \left( ||\nabla \epsilon_h^2||_0^2 + \mu \lambda \tau \left|\nabla \delta \vartheta_h^n\right|^2 \right) \leq C (\tau^2 + h^2).
\]

**Remark 4.4 (The condition \( \tau \leq Ch \)).** The assumption \( \tau \leq Ch \) requires to control convection terms which are used at only (4.23) and (4.26), so we can omit this condition for the linearized Boussinesq equations. However, this condition can not be removed for non-linear equation case, because (4.22) below must be bounded by \( h \).

**Proof.** We subtract two consecutive formulas of (4.11) and impose \( w_h = 4\tau \delta \bar{\phi}_h^{n+1} \) to obtain

\[
\left|\nabla \phi_h^{N+1}\right|^2 + \left|\nabla \bar{\phi}_h^{N+1}\right|^2 + \left|\nabla \phi_h^N - \nabla \phi_h^{N+1}\right|^2 + \frac{1}{2} \left|\nabla \delta \phi_h^n\right|^2 + \left|\nabla \delta \phi_h^{n+1}\right|^2 + 6 \left|\nabla \delta \psi_h^{n+1}\right|^2
\]

(4.21)

\[
-\left|\nabla \phi_h^n\right|^2 - \left|\nabla \phi_h^{n-1}\right|^2 + 4\mu \tau \left|\nabla \delta \phi_h^n\right|^2 = \sum_{i=1}^7 A_i,
\]
where

\[
A_1 := 4\tau N \left( u(t^n), u(t^n), \delta \hat{E}_h^{n+1} \right) - 4\tau N \left( 2u_{h}^{n-1} - u_{h}^{n-2}, \hat{u}_{h}^{n}, \delta \hat{E}_h^{n+1} \right) - 4\tau N \left( u(t^{n+1}), u(t^{n+1}), \delta \hat{E}_h^{n+1} \right) + 4\tau N \left( 2u_{h}^{n} - u_{h}^{n-1}, \hat{u}_{h}^{n+1}, \delta \hat{E}_h^{n+1} \right),
\]

\[
A_2 := -2 \left\langle 3\delta G^{n+1} - 4\delta G^n + \delta G^{n-1}, \delta \hat{E}_h^{n+1} \right\rangle,
\]

\[
A_3 := -4\tau \left\langle \nabla \delta \hat{E}_h^{n+1}, \delta \hat{E}_h^{n+1} \right\rangle,
\]

\[
A_4 := -4\tau \left\langle \hat{u}_{h}^{n}, \delta \hat{E}_h^{n+1} \right\rangle,
\]

\[
A_5 := 4\mu \tau \left\langle \nabla \delta \hat{u}_{h}^{n}, \delta \hat{E}_h^{n+1} \right\rangle,
\]

\[
A_6 := 4\tau \left\langle \hat{R}_h^{n+1}, \delta \hat{E}_h^{n+1} \right\rangle,
\]

\[
A_7 := 4\kappa \mu^2 \tau \left\langle \delta \left( \theta(t^{n+1}) - 2\theta^n + \theta^{n-1} \right), \delta \hat{E}_h^{n+1} \right\rangle.
\]

We now express each term $A_1$ to $A_7$ separately. The convection term $A_1$ can be rewritten as follows:

\[
A_1 = 4\tau N \left( \delta \hat{u}(t^n), u(t^n), \delta \hat{E}_h^{n+1} \right) - 4\tau N \left( \delta \hat{u}(t^{n+1}), u(t^{n+1}), \delta \hat{E}_h^{n+1} \right) + 4\tau N \left( 2E_{h}^{n-1} - E_{h}^{n-2}, u(t^n), \delta \hat{E}_h^{n+1} \right) - 4\tau N \left( 2E_{h}^{n} - E_{h}^{n-1}, u(t^{n+1}), \delta \hat{E}_h^{n+1} \right) + 4\tau N \left( 2u_{h}^{n-1} - u_{h}^{n-2}, \hat{E}_{h}^{n}, \delta \hat{E}_h^{n+1} \right) - 4\tau N \left( 2u_{h}^{n} - u_{h}^{n-1}, \hat{E}_{h}^{n+1}, \delta \hat{E}_h^{n+1} \right),
\]

In estimating convection terms, we will use Lemma 2.6 frequently without notice. We recall $\|u(t)\|_2 \leq C$ to obtain

\[
A_{1,1} + A_{1,2} \leq \frac{\mu \tau}{6} \left\| \nabla \delta \hat{u}_{h}^{n+1} \right\|_0^2 + \frac{C\tau^4}{\mu} \int_{t_{n-2}}^{t_{n+1}} \|u_{h}(t)\|_0^2 dt.
\]

The result in Lemma 4.1, $\|2E_{h}^{n} - E_{h}^{n-1}\|_0 \leq C(\tau + h)$, is essential to treat the next 2 convection terms. Invoking (2.9), we have

\[
A_{1,3} + A_{1,4} \leq C\tau \|2\delta \hat{E}_{h}^{n} - \delta \hat{E}_{h}^{n-1}\|_0 \|u(t^{n+1})\|_2 \left\| \delta \hat{E}_h^{n+1} \right\|_1 + \frac{C\tau}{\mu} \|2\delta \hat{E}_{h}^{n} - \delta \hat{E}_{h}^{n-1}\|_0 \left\| \delta \hat{E}_h^{n+1} \right\|_1 \leq \frac{\mu \tau}{6} \left\| \nabla \delta \hat{E}_h^{n+1} \right\|_0^2 + \frac{C\tau^2 (\tau^2 + h^2)}{\mu} \int_{t_{n-1}}^{t_{n}} \|u_{h}(t)\|_0^2 dt.
\]

We note $N \left( 2u_{h}^{n} - u_{h}^{n-1}, \delta \hat{E}_{h}^{n+1}, \delta \hat{E}_h^{n+1} \right) = 0$ which comes from (2.7). Then we obtain

\[
A_{1,5} + A_{1,6} = -4\tau N \left( 2\delta u_{h}^{n} - \delta u_{h}^{n-1}, G^{n+1} + \hat{E}_{h}^{n+1}, \delta \hat{E}_h^{n+1} \right) - 4\tau N \left( 2u_{h}^{n-1} - u_{h}^{n-2}, \delta G^{n+1}, \delta \hat{E}_h^{n+1} \right) = 4\tau N \left( 2\delta E^{n} - \delta E^{n-1} - 2\delta u(t^n) + \delta u(t^{n-1}), G^{n+1} + \hat{E}_{h}^{n+1}, \delta \hat{E}_h^{n+1} \right) + 4\tau N \left( 2E^{n-1} - E^{n-2} - 2u(t^{n-1}) + u(t^{n-2}), \delta G^{n+1}, \delta \hat{E}_h^{n+1} \right) = B_1 + B_2.
\]
To attack $B_1$, we first note Lemma 2.1 which is, for any $w_h \in \mathbb{V}_h$, $\|w_h\|_{L^2(\Omega)} \leq Ch^{-d/6}\|w_h\|_0$. If we apply
\begin{equation}
\|\hat{\mathbf{E}}^{n+1} + \mathbf{G}^{n+1}\|_0 + \sqrt{\tau + h} \|\hat{\mathbf{E}}^{n+1} + \mathbf{G}^{n+1}\|_1 \leq C(\tau + h)
\end{equation}
which is the result of Lemma 4.1, then we can conclude, in light of (2.10),
\begin{equation}
B_1 \leq C\tau \|2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1}\|_{L^2(\Omega)} \|\hat{\mathbf{E}}^{n+1}_h + \mathbf{G}^{n+1}\|_1 \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0 \\
+ C\tau \|2\delta \mathbf{u}(t^n) - \delta \mathbf{u}(t^{n-1})\|_2 \|\hat{\mathbf{E}}^{n+1}_h + \mathbf{G}^{n+1}\|_0 \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0 \\
\leq \frac{\mu \tau}{6} \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0^2 + C\tau \left(1 + \frac{\tau}{h}\right) \|2\delta\mathbf{E}^n - \delta\mathbf{E}^{n-1}\|_0^2 \\
+ \frac{C\tau^2}{\mu} (\tau + h)^2 \int_{t^{n-2}}^{t^n} \|\mathbf{u}_t(t)\|_2^2 \, dt.
\end{equation}
We now estimate $B_2$ using $\|2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}\|_{L^2(\Omega)} \leq Ch^{-d/6}\|2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}\|_0 \leq M$,
\begin{equation}
B_2 \leq C\tau \|2\mathbf{E}^{n-1} - \mathbf{E}^{n-2}\|_{L^2(\Omega)} \|\delta\mathbf{G}^{n+1}\|_1 \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0 \\
+ C\tau \|2\mathbf{u}(t^{n-1}) - \mathbf{u}(t^{n-2})\|_1 \|\delta\mathbf{G}^{n+1}\|_1 \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0 \\
\leq C\tau \|\delta\mathbf{G}^{n+1}\|_0^2 + \frac{\mu \tau}{6} \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0^2 \\
\leq \frac{\mu \tau}{6} \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0^2 + C\tau^2 h^2 \int_{t^{n-1}}^{t^{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right) \, dt.
\end{equation}
In light of Hölder inequality, (4.3) yields
\begin{equation}
A_2 = -2 \left(3\delta\hat{\mathbf{E}}^{n+1}_h + 4\delta\mathbf{G}^n + \delta\mathbf{G}^{n+1}, \delta\hat{\mathbf{E}}^{n+1}_h\right) \\
\leq \frac{\mu \tau}{6} \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0^2 + Ch^4 \int_{t^{n-2}}^{t^{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right) \, dt.
\end{equation}
Integral by parts leads
\begin{equation}
A_3 = 4\tau \left(\delta\hat{\mathbf{E}}^{n+1}_h, \nabla \cdot \delta\hat{\mathbf{E}}^{n+1}_h\right) \\
\leq \frac{\mu \tau}{6} \|\nabla \delta\hat{\mathbf{E}}^{n+1}_h\|_0^2 + \frac{C\tau^4}{\mu} \int_{t^{n-1}}^{t^{n+1}} \left(\|\mathbf{u}_t(t)\|_2^2 + \|p_t(t)\|_1^2\right) \, dt.
\end{equation}
In order to tackle $A_4$, marking use of $\delta\hat{\mathbf{E}}^{n+1}_h = \delta\mathbf{E}^{n+1}_h + \nabla \delta\mathbf{p}^{n+1}_h$. We readily get
\begin{equation}
A_4 = -4\tau \left(\nabla \delta\hat{\mathbf{E}}^{n+1}_h, \nabla \delta\hat{\mathbf{E}}^{n+1}_h + \nabla \delta\mathbf{p}^{n+1}_h\right) \\
= -\frac{8\tau^2}{3} \left(\nabla \delta\mathbf{E}^n_h, \nabla \left(\delta\hat{\mathbf{E}}^{n+1}_h - \delta\mathbf{p}^{n+1}_h\right)\right) \\
\leq -\frac{4\tau^2}{3} \left(\|\nabla \delta\mathbf{E}^n_h\|_0^2 - \|\nabla \delta\mathbf{E}^n_h\|_0^2 - \|\nabla \delta\mathbf{E}^{n+1}_h\|_0^2\right) \\
+ C\tau^3 \|\nabla \delta\mathbf{E}^{n+1}_h\|_0^2 + C\tau \|\nabla \delta\mathbf{p}^{n+1}_h\|_0^2.
\end{equation}
If we now apply inequality \((a + b)^2 \leq 4a^2 + \frac{4}{3}b^2\), then we can have
\[
\frac{4\tau^2}{3} \left\| \nabla \delta \varepsilon_h^{n+1} \right\|_0^2 = \frac{4\tau^2}{3} \left\| \nabla \delta P_h^{n+1} + \frac{3}{2\tau} \nabla \delta \phi_h^{n+1} \right\|_0^2 \\
\leq C \tau^2 \left\| \nabla \delta P_h^{n+1} \right\|_0^2 + \frac{4}{3} \left\| \nabla \delta \phi_h^{n+1} \right\|_0^2.
\]

So we arrive at
\[
A_4 \leq -\frac{4\tau^2}{3} \left( \left\| \nabla \delta \varepsilon_h^{n+1} \right\|_0^2 - \left\| \nabla \delta \varepsilon_h^n \right\|_0^2 \right) + 4 \left\| \nabla \delta \phi_h^{n+1} \right\|_0^2 + C \tau^4 \int_{t_{n-1}}^{t_n} \left( \left\| \mathbf{u}_{tt}(t) \right\|_2^2 + \left\| p_{tt}(t) \right\|_1^2 \right) dt.
\]

Invoking (1.6), (2.13) and (4.7) lead
\[
A_5 = -4\mu \tau \left( \left\| \delta q_h^n \right\|_0^2 - \left\| \delta q_h^{n+1} \right\|_0^2 \right) \\
= -2\mu \tau \left( \left\| \delta q_h^{n+1} \right\|_0^2 - \left\| \delta q_h^n \right\|_0^2 \right) \\
\leq -2\mu \tau \left( \left\| \delta q_h^{n+1} \right\|_0^2 - \left\| \delta q_h^n \right\|_0^2 \right) + 2\mu \tau \left\| \nabla \delta \psi_h^{n+1} \right\|_0^2.
\]

Finally, the \(A_6\) term becomes
\[
A_6 \leq C \tau \left\| \delta \mathbf{R}^{n+1}_h \right\|_0 \left\| \nabla \psi_h^{n+1} \right\|_0 + C \tau^4 \int_{t_{n-1}}^{t_n} \left\| \mathbf{u}_{tt}(t) \right\|_2^2 dt.
\]

For the new term \(A_7\) we argue as in Lemma 4.1 to arrive at
\[
A_7 \leq C \kappa^2 \mu^4 \tau \left( \left\| 2\delta \theta_h^{n+1} \right\|_0^2 + \left\| 2\delta \eta_h^n - \delta \eta_h^{n-1} \right\|_0^2 \right) + \tau \left\| \delta \mathbf{E}_h^{n+1} \right\|_0^2 \\
+ \frac{1}{2} \left\| \nabla \delta \psi_h^{n+1} \right\|_0^2 + C \tau^4 \int_{t_{n-2}}^{t_{n-1}} \left\| \theta_{tt}(t) \right\|_0^2 dt.
\]

We now insert the above estimates into (4.21) to obtain
\[
(4.24) \quad \delta \left\| \delta \mathbf{E}_h^{n+1} \right\|_0^2 + \delta \left\| 2\delta \mathbf{E}_h^{n+1} - \delta \mathbf{E}_h^n \right\|_0^2 + \delta \left\| \delta \mathbf{E}_h^{n+1} \right\|_0^2 + \mu \tau \left\| \nabla \delta \psi_h^{n+1} \right\|_0^2 \\
\leq C \mu \left( \left\| 2\delta \mathbf{G}_h^n - \delta \mathbf{G}_h^{n-1} \right\|_0^2 + \left\| 2\delta \mathbf{G}_h^n - \delta \mathbf{G}_h^{n-1} \right\|_0^2 \right) + \frac{4\tau^2}{3} \left\| \nabla \delta \varepsilon_h^n \right\|_0^2 \\
+ C \mu^4 \tau \left( \left\| 2\delta \theta_h^n - \delta \theta_h^{n-1} \right\|_0^2 + \left\| 2\delta \eta_h^n - \delta \eta_h^{n-1} \right\|_0^2 \right) + \tau \left\| \delta \mathbf{E}_h^{n+1} \right\|_0^2 \\
+ 2\mu \tau \left\| \delta q_h^n \right\|_0^2 + C \tau^4 \int_{t_{n-1}}^{t_n} \left( \left\| \mathbf{u}_{tt}(t) \right\|_2^2 + \left\| \mathbf{u}_{tt}(t) \right\|_2^2 + \left\| p_{tt}(t) \right\|_1^2 + \left\| \theta_{tt}(t) \right\|_0^2 \right) dt \\
+ C \tau^3 \left\| \nabla \delta \varepsilon_h^n \right\|_0^2 + C \left( \tau^2 + h^2 \right)^2 \int_{t_{n-2}}^{t_{n-1}} \left( \left\| \mathbf{u}_{tt}(t) \right\|_2^2 + \left\| \mathbf{u}_{tt}(t) \right\|_2^2 + \left\| p_{tt}(t) \right\|_1^2 \right) dt.
\]

To evaluate errors of \(\delta \theta_h^{n+1}\), we subtract two consecutive formulas (4.16) and choose by \(\phi_h = 4\delta \delta \theta_h^{n+1} = 4\tau \left( \delta \theta_h^{n+1} - \delta \eta_h^{n+1} \right)\) as follows:
\[
(4.25) \quad \delta \left\| \delta \theta_h^{n+1} \right\|_0^2 + \delta \left\| 2\delta \theta_h^{n+1} \right\|_0^2 + \delta \left\| \delta \delta \theta_h^{n+1} \right\|_0^2 + 4\lambda \mu \tau \left\| \nabla \delta \theta_h^{n+1} \right\|_0^2 = A_8 + A_9,
\]
where
\[ A_8 := 4\tau N\left( u(t^n) , \theta(t^n) , \delta\vartheta_h^{n+1} \right) - 4\tau N\left( u_h^n , \theta_h^n , \delta\vartheta_h^{n+1} \right) \]
\[ A_9 := -2\left(3\delta\eta^{n+1} - 4\delta\eta^n + \delta\eta^{n-1} , \delta\vartheta_h^{n+1} \right) + 4\tau \left( \delta Q^n , \delta\vartheta_h^{n+1} \right). \]

We treat the \( A_8 \) term first by rewriting it as follows:
\[ A_8 = -4\tau N\left( E^{n+1} , \theta(t^n) , \delta\vartheta_h^{n+1} \right) - 4\tau N\left( u_h^{n+1} , \vartheta^{n+1} , \delta\vartheta_h^{n+1} \right) \]
\[ + 4\tau N\left( E^n , \theta(t^n) , \delta\vartheta_h^{n+1} \right) + 4\tau N\left( u_h^n , \vartheta^n , \delta\vartheta_h^{n+1} \right) := \sum_{i=1}^{4} A_{8,i}. \]

In estimating convection terms, we will use Lemma 2.6 frequently without notice. We recall \( \|\theta(t)\|_2 \leq C \) and \( \|E^n\|_0 \leq C(\tau + h) \) which is the result in Lemma 4.1 to obtain
\[ A_{8,1} + A_{8,3} = -4\tau N(\delta E^{n+1} , \theta(t^n) , \delta\vartheta_h^{n+1} ) - 4\tau N(E^n , \delta\vartheta(t^{n+1}) , \delta\vartheta_h^{n+1} ) \]
\[ \leq C\tau \left( \|\delta E^{n+1}\|_0 \|\theta(t^n)\|_2 + \|E^n\|_0 \|\delta\vartheta(t^{n+1})\|_2 \|\delta\vartheta_h^{n+1}\|_1 \right) \]
\[ \leq \lambda\mu\tau \|\nabla\delta\vartheta_h^{n+1}\|_0^2 + \frac{C\tau^2}{\lambda\mu} \int_0^{t^n} \|\theta(t)\|_2^2 dt. \]

In conjunction with \( N\left( u_h^{n+1} , \delta\vartheta_h^{n+1} , \delta\vartheta_h^{n+1} \right) = 0 \) which comes from (2.7), we rewrite \( A_{8,2} + A_{8,4} \) as
\[ A_{8,2} + A_{8,4} = 4\tau N(E_h^{n+1} - U_h^{n+1} , \delta\eta^{n+1} , \delta\vartheta_h^{n+1} ) \]
\[ + 4\tau N(\delta E_h^{n+1} - \delta U_h^{n+1} , \vartheta^n , \delta\vartheta_h^{n+1} ) := B_3 + B_4. \]

Because we can readily get \( \|E_h^{n+1} - U_h^{n+1}\|_{L^2(\Omega)} \leq M \) via Lemmas 2.1 and 4.1, we conclude, in conjunction with (4.7),
\[ B_3 \leq C\tau \left( \|E_h^{n+1} - U_h^{n+1}\|_{L^2(\Omega)} \|\delta\eta^{n+1}\|_1 \|\delta\vartheta_h^{n+1}\|_1 \right) \]
\[ \leq \lambda\mu\tau \|\nabla\delta\vartheta_h^{n+1}\|_0^2 + \frac{C\tau^2}{\lambda\mu} \int_0^{t^n} \|\theta(t)\|_2^2 dt. \]

Before we attack \( B_4 \), we note that the assumption \( \tau \leq Ch \) is required to apply \( \|\delta E_h^{n}\|_{L^2(\Omega)} \leq C\tau^{-\frac{1}{2}} \|\delta E_h^{n}\|_0 \). We now apply \( \|\vartheta^n\|_0^2 + \tau \|\vartheta^n\|_1^2 \leq C(\tau^2 + h^2) \) and (4.3) to get
\[ B_4 = 4\tau N(\delta G^{n+1} - \delta u(t^{n+1}) + \delta E_h^{n+1} , \vartheta^n , \delta\vartheta_h^{n+1} ) \]
\[ \leq C\tau \left( \left( \|\delta G^{n+1}\|_1 + \|\delta E_h^{n+1}\|_{L^2(\Omega)} \right) \|\vartheta^n\|_1 + \|\delta u(t^n)\|_2 \|\vartheta^n\|_0 \right) \|\delta\vartheta_h^{n+1}\|_1 \]
\[ \leq \frac{C\tau}{\lambda\mu} \|\delta E_h^{n+1}\|_0^2 + \lambda\mu\tau \|\nabla\delta\vartheta_h^{n+1}\|_0^2 \]
\[ + \frac{C\tau^2}{\lambda\mu} \int_0^{t^n} \left( \|u(t)\|_2^2 + \|p(t)\|_2^2 \right) dt. \]

In light of Hölder inequality, \( A_9 \) becomes
\[ A_9 \leq C\tau \|\delta\vartheta_h^{n+1}\|_0^2 + C(\tau^4 + h^4) \int_0^{t^n} \left( \|\theta(t)\|_2^2 + \|\theta(t)\|_0^2 \right) dt. \]
Inserting the above estimates into (4.25) yields
\begin{equation}
\delta \| \nabla \vartheta_{n+1} \|^2 _0 + \delta \| \Delta \vartheta_{n+1} - \vartheta_{n} \|^2 _0 + \| \vartheta_{n+1} - \vartheta_{n} \|^2 _0 \leq C \tau \| \nabla \vartheta_{n+1} \|^2 _0 + \frac{C \tau}{\lambda \mu} \left( \| \delta E_{n+1} \|^2 _0 + \| \delta E_{n} \|^2 _0 + \| \delta F_{n+1} \|^2 _0 \right) + C (\tau^2 + h^2)^2 \int_{t^n}^{t^{n+1}} \left( \| u(t) \|^2 _{L^2} + \| p(t) \|^2 _{L^2} + \| \theta(t) \|^2 _{L^2} + \| \theta_{tt} \|^2 _{L^2} \right) dt.
\end{equation}

Adding 2 equations (4.24) and (4.27) and then summing up n from 2 to N lead up to (4.20) and complete the proof.

We now estimate optimal accuracy for velocity and temperature.

**Lemma 4.5** (Full rate of convergence for velocity and temperature). We denote that \((v^{n+1}, r^{n+1})\) and \((v_h^{n+1}, r_h^{n+1})\) are the solutions of (2.1) and (2.3) with \(z = E_{n+1}^h\), respectively. And let \(\omega^{n+1}\) and \(\omega_{n+1}^h\) be solutions of (2.2) and (2.6) with \(\xi = \vartheta_{n+1}^h\). Let the exact solution of (1.1) be smooth enough and \(\tau \leq C h\). If Assumptions 1 and 3-5 hold, then we have
\begin{equation}
\| \nabla v_{n+1}^h \|^2 _0 + \| \nabla (2v_{n+1}^h - v_{n}^h) \|^2 _0 + \| \nabla \omega_{n+1}^h \|^2 _0 + \| \nabla (2\omega_{n+1}^h - \omega_{n}^h) \|^2 _0 + \sum_{n=1}^{N} \left( \| \nabla \delta v_{n+1}^h \|^2 _0 + \| \nabla \delta \omega_{n+1}^h \|^2 _0 \right) + 2\mu \tau \sum_{n=1}^{N} \left( \| E_{n+1}^h \|^2 _0 + \lambda \| \vartheta_{n+1}^h \|^2 _0 \right)
\leq C (\tau^4 + h^4).
\end{equation}

**Proof.** We choose \(w_h = 4\tau v_{n+1}^h \in V_h\) in (4.11) to obtain
\begin{equation}
\delta \| \nabla v_{n+1}^h \|^2 _0 + \| \nabla (2v_{n+1}^h - v_{n}^h) \|^2 _0 + \| \nabla \delta v_{n+1}^h \|^2 _0 + 4\mu \tau \| E_{n+1}^h \|^2 _0 = \sum_{i=1}^{5} A_i,
\end{equation}
where
\begin{align*}
A_1 := & \ 4\tau N \left( 2u_h^0 - u_h^1, \vartheta_{n+1}^h, v_{n+1}^h \right) - 4\tau N \left( u(t_{n+1}), u(t_n), v_{n+1}^h \right), \\
A_2 := & \ \left( 3G_{n+1}^h - 4G_n, G_{n+1}^h, v_{n+1}^h \right), \\
A_3 := & \ 4\mu \tau \left( E_{n+1}^h, r_{n+1}^h \right), \\
A_4 := & \ 4\tau \left( R_{n+1}^h, v_{n+1}^h \right), \\
A_5 := & \ 4\mu \tau \left( g \left( \theta(t_{n+1}) - 2\theta_h^0 + \vartheta_{n+1}^h \right), v_{n+1}^h \right).
\end{align*}

We now estimate all the terms from \(A_1\) to \(A_5\) respectively. The convection term \(A_1\) can be rewritten as follows:
\begin{align*}
A_1 = & \ -4\tau N \left( \delta \nabla u(t_{n+1}), u(t_{n+1}), v_{n+1}^h \right) - 4\tau N \left( 2u(t_n) - u(t_{n+1}), \tilde{E}_{n+1}^h, v_{n+1}^h \right) \\
& \ + 4\tau N \left( 2E_n - E_{n+1}^h, \tilde{E}_{n+1}^h, v_{n+1}^h \right) - 4\tau N \left( 2E_n - E_{n+1}^h, u(t_{n+1}), v_{n+1}^h \right),
\end{align*}
and we denote by \(A_{1,i}\), for \(i = 1, 2, \cdots, 4\) the four terms in the right hand side. To estimate convection terms, we will use frequently Lemma 2.6 without notice. Using \(\| u(t_{n+1}) \| \leq M\), we can readily get
\begin{align*}
A_{1,1} \leq & \ C \tau \| \delta \nabla u(t_{n+1}) \|_0 \| u(t_{n+1}) \|_1 \| v_{n+1}^h \|_1 \\
\leq & \ C \tau \| \nabla v_{n+1}^h \|_0^2 + C \mu \tau \int_{t_{n-1}}^{t_{n+1}} \| u_{tt}(t) \|_0^2 dt.
\end{align*}
and
\[ A_{1,4} \leq C \tau \left( \|E^n\|_0 + \|\delta E^n\|_0 + \|u(t^{n+1})\|_2 \right) \|v_h^{n+1}\|_1 \\
\leq \frac{\mu T}{2} \left( \|E_h^{n+1}\|_0^2 + \|G^n\|_0^2 + \|\delta E^n\|_0^2 + \|\delta G^n\|_0^2 \right) + \frac{C \tau}{\mu} \|\nabla v_h^{n+1}\|_0^2. \]

Because \( \nabla \cdot (2u(t^n) - u(t^{n-1})) = 0 \) and \( 2u(t^n) - u(t^{n-1}) = 0 \) on boundary, we can use (2.8) and so we get
\[ A_{1,2} \leq C \tau \|2u(t^n) - u(t^{n-1})\|_2 \left( \|\nabla E^n\|_0 \right) \|v_h^{n+1}\|_1 \\
\leq \frac{\mu T}{2} \left( \|E_h^{n+1}\|_0^2 + \|G^n\|_0^2 + \|\nabla \psi_h^{n+1}\|_0^2 \right) + \frac{C \tau}{\mu} \|\nabla v_h^{n+1}\|_0^2. \]

In light of \( \varepsilon_h^{n+1} = P_h^{n+1} + \frac{3\phi_h^{n+1}}{2} \), we can obtain
\[ \|\nabla \psi_h^{n+1}\|_0^2 = \frac{4\tau^2}{9} \|\nabla (\delta \varepsilon_h^{n+1} - \delta P_h^{n+1})\|_0^2 \\
\leq C \tau^2 \|\nabla \varepsilon_h^{n+1}\|_0^2 + C \tau^3 \int_{t^n}^{t^{n+1}} \left( \|u(t)\|_2 + \|p(t)\|_1 \right) dt, \]

and so we can conclude
\[ A_{1,2} \leq \frac{\mu T}{2} \left( \|E_h^{n+1}\|_0^2 + \|G^n\|_0^2 \right) + C \tau^2 \|\nabla \varepsilon_h^{n+1}\|_0^2 \\
+ C \tau^4 \int_{t^n}^{t^{n+1}} \left( \|u(t)\|_2 + \|p(t)\|_1 \right) dt + \frac{C \tau}{\mu} \|\nabla v_h^{n+1}\|_0^2. \]

If we apply \( \|2E^n - E^{n-1}\|_{L^3(\Omega)} \leq C h^{-d/6} \|2E^n - E^{n-1}\|_0 \leq C h^{-d/6} (\tau + h) \) which derives from Lemmas 2.1 and 4.1, then we can get, by the help of Lemma 2.6 and Assumption 1,
\[ A_{1,3} \leq C \tau \|2E^n - E^{n-1}\|_0 \|\nabla E^n\|_1 \|v_h^{n+1}\|_2 \\
+ C \tau \|2E^n - E^{n-1}\|_{L^3(\Omega)} \|\nabla E^n\|_1 \|v_h^{n+1} - v_h^{n}\|_1 \\
\leq C \tau (\tau + h) \|\nabla E^n\|_1 \|v_h^{n+1}\|_2 + C \tau h \|E_h^{n+1}\|_1 \|v_h^{n+1}\|_2 \\
\leq \frac{C}{\mu} \tau (\tau + h)^2 \left( \|\nabla E_h^{n+1}\|_0^2 + \|\nabla G^n\|_0^2 \right) + \frac{\mu T}{2} \|E_h^{n+1}\|_0^2. \]

In conjunction with (4.3), we can have
\[ A_2 \leq \frac{C}{\tau} \left( \|\delta G^n\|_0^2 + \|\delta G^n\|_0^2 \right) + C \tau \|v_h^{n+1}\|_0^2 \\
\leq C \tau \|\nabla v_h^{n+1}\|_0^2 + C h^4 \int_{t^n}^{t^{n+1}} \left( \|u(t)\|_2 + \|p(t)\|_1 \right) dt. \]

The definition \( \tilde{E}_h^{n+1} = E_h^{n+1} + \nabla \psi_h^{n+1} \) and Assumption 1 give us
\[ A_3 = 4 \mu \tau \left( \nabla \psi_h^{n+1}, \nabla \psi_h^{n+1} \right) \leq \mu \tau \|E_h^{n+1}\|_0^2 + \frac{C \tau}{\mu} \|\nabla \psi_h^{n+1}\|_0^2. \]

If we apply (4.30) again, then we arrive at
\[ A_3 \leq \mu \tau \|E_h^{n+1}\|_0^2 + \frac{C \tau^3}{\mu} \|\nabla \varepsilon_h^{n+1}\|_0^2 + \frac{C \tau^4}{\mu} \int_{t^n}^{t^{n+1}} \|p(t)\|_0^2 dt. \]
On the other hand, the truncation error term becomes

$$ A_4 = 4\tau \left< \mathbf{R}^{n+1}, \mathbf{v}_h^{n+1} \right> \leq C_\tau \| \nabla \mathbf{v}_h^{n+1} \|^2_0 + C_\tau \int_{t_{n-1}}^{t_n} \| \mathbf{u}_tt(t) \|^2_0 dt. $$

For the new term $A_5$, we observe $\| \delta \delta \theta (t^{n+1}) \|^2_0 \leq C_\tau \int_{t_{n-1}}^{t_n} \| \theta_t(t) \|^2_0 dt$, whence

$$ A_5 = 4\kappa \mu^2 \tau \left( g \left( \delta \delta \theta (t^{n+1}) + 2\theta^n - \vartheta^{n-1} \right), \mathbf{v}_h^{n+1} \right) $$

$$ \leq C_\tau \| \nabla \mathbf{v}_h^{n+1} \|^2_0 + C_\kappa \mu^4 \tau \| \theta_t(t) \|^2_0 dt. $$

Invoking $\mathbf{v}^0 = 0$, inserting the above estimates from $A_1$ and $A_5$ into (4.29) lead us

$$ \delta \| \nabla \mathbf{v}_h^{n+1} \|^2_0 + \delta \| \nabla \left( 2\mathbf{v}_h^{n+1} - \mathbf{v}_h^n \right) \|^2_0 + \| \nabla \delta \mathbf{v}_h^{n+1} \|^2_0 + 2\mu \| \mathbf{E}_h^{n+1} \|^2_0 $$

$$ \leq \frac{\mu^2}{2} \left( \| \mathbf{E}_h^n \|^2_0 + \| \mathbf{G}^{n+1} \|^2_0 + \| \mathbf{G}^{n+1} \|^2_0 + \| \delta \mathbf{E}_h^n \|^2_0 + \| \delta \mathbf{G}^{n+1} \|^2_0 \right) $$

$$ + C_\kappa \mu \tau \left( \| \nabla \mathbf{v}_h^{n+1} \|^2_0 + \| \nabla \mathbf{G}^{n+1} \|^2_0 \right) $$

$$ + C_\mu \| \nabla \mathbf{E}_h^{n+1} \|^2_0 + C_\mu \| \nabla \mathbf{E}_h^{n+1} \|^2_0 + \| \theta_t(t) \|^2_0 \right) dt $$

$$ + C_\tau \| \nabla \mathbf{v}_h^{n+1} \|^2_0 + C_\tau \int_{t_{n-1}}^{t_n} \left( \| \mathbf{u}_tt(t) \|^2_0 + \| \mathbf{u}_{tt}(t) \|^2_0 + \| \theta_t(t) \|^2_0 \right) dt. $$

In the other hand, we choose $\phi_h = 4\tau \omega_h^{n+1}$ in (4.16) to obtain

$$ \delta \| \nabla \omega_h^{n+1} \|^2_0 + \delta \| \nabla \left( 2\omega_h^{n+1} - \omega_h^n \right) \|^2_0 + \| \nabla \delta \omega_h^{n+1} \|^2_0 + 4\lambda \mu \tau \| \vartheta_h^{n+1} \|^2_0 = A_6 + A_7, $$

where

$$ A_6 = 4\tau \mathcal{N} \left( \mathbf{u}_h^{n+1}, \theta_h^{n+1}, \omega_h^{n+1} \right) - 4\tau \mathcal{N} \left( \mathbf{u}(t^{n+1}), \theta(t^{n+1}), \omega_h^{n+1} \right), $$

$$ A_7 = -2 \left< 3\eta^{n+1} - 4\eta^n + \| \eta^{n+1} \|^2_0 \right) dt. $$

In order to estimate $A_6$, we note first $\| \mathbf{E}^{n+1} \|_{L^2(\gamma)} \leq C h^{-\frac{1}{2}} \| \mathbf{E}^{n+1} \|_0$ and $\| \vartheta^{n+1} \|^2_1 \leq C \frac{\tau^2 + h^2}{2}$ which come from Lemma 2.1 and Lemma 4.1, respectively. Then Lemma 2.6 and the assumption $\tau \leq C h$ yield

$$ A_6 = -4\tau \mathcal{N} \left( \mathbf{E}^{n+1} + \mathbf{E}^{n+1}, \vartheta^{n+1}, \omega_h^{n+1} \right) + 4\tau \mathcal{N} \left( \mathbf{E}^{n+1} - \mathbf{u}(t^{n+1}), \vartheta^{n+1}, \omega_h^{n+1} \right) $$

$$ \leq C \tau \| \mathbf{E}^{n+1} \|_0 \| \vartheta(t^{n+1}) \|_2 \| \omega_h^{n+1} \|_1 + C \tau \| \mathbf{E}^{n+1} \|_{L^2(\gamma)} \| \vartheta^{n+1} \|_1 \| \omega_h^{n+1} \|_1 $$

$$ + C \tau \| \mathbf{u}(t^{n+1}) \|_2 \| \omega_h^{n+1} \|_1 + \mu \tau \| \mathbf{E}^{n+1} \|^2_0 + \lambda \mu \tau \left( \| \theta_h^{n+1} \|^2_0 + \| \eta^{n+1} \|^2_0 \right). $$

Since $\| \delta \eta^{n+1} \|^2_0 \leq C h^4 \tau \int_{t_n}^{t_{n+1}} \| \theta_t(t) \|^2 dt$, we can readily get

$$ A_7 \leq C \tau \| \nabla \omega_h^{n+1} \|^2_0 + C (\tau^4 + h^4) \int_{t_{n-1}}^{t_n} \left( \| \theta_t(t) \|^2_0 + \| \theta_{ttt}(t) \|^2_0 \right) dt. $$
Inserting above estimates into (4.32) yields
\[
\delta \| \nabla w_h^{n+1} \|^2_0 + \delta \left\| \nabla \left( 2 \omega_h^{n+1} - \omega_h^n \right) \right\|^2_0 + \| \nabla \delta \omega_h^{n+1} \|^2_0 + 3 \lambda \mu \tau \| \theta_h^{n+1} \|^2_0 \\
\leq C \tau \| \nabla w_h^{n+1} \|^2_0 + \lambda \mu \tau \| \eta^{n+1} \|^2_0 + \mu \tau \| E^{n+1} \|^2_0 + C (\tau^4 + h^4) \int_{n-1}^{n} \left( \| \theta(t) \|_2^2 + \| \theta_{tt}(t) \|_2^2 \right) dt.
\]
(4.33)

In conjunction with the discrete Gronwall inequality, adding (4.31) and (4.33) and summing over \( n \) from 1 to \( N \) leads (4.28).

We now estimate the pressure error in \( L^2(0, T; L^2(\Omega)) \). This hinges on the error estimates for the time derivative of velocity and temperature of Lemma 4.3.

**Lemma 4.6 (Pressure error estimate).** Let the exact solution of (1.1) is smooth enough and \( \tau \leq Ch \). If Assumptions 1-5 hold, then we have
\[
\tau \sum_{n=1}^{N} \| e_h^{n+1} \|^2_0 \leq C (\tau^2 + h^2).
\]
(4.34)

**Proof.** We first recall again inf-sup condition in Assumption 2. Consequently, it suffices to estimate \( \left\langle e_h^{n+1}, \nabla \cdot w_h \right\rangle \) in terms of \( \| \nabla w_h \|_0 \). In conjunction with (1.7), we can rewrite (4.11) as
\[
\left\langle e_h^{n+1}, \nabla \cdot w_h \right\rangle = \frac{1}{2 \tau} \left\langle 3 E^{n+1} - 4 E^n + E^{n-1}, w_h \right\rangle + \mu \left\langle \nabla \tilde{E}_h^{n+1}, \nabla w_h \right\rangle \\
+ N \left( \delta \tilde{u}(t^{n+1}), u(t^{n+1}), w_h \right) + N \left( 2 u(t^{n+1}) - u(t^n), \tilde{E}_h^{n+1}, w_h \right) \\
+ N \left( 2 E^n - E^{n-1}, \tilde{u}_h^{n+1}, w_h \right) - \mu \left\langle \nabla \delta \theta_h^{n+1}, w_h \right\rangle \\
- \left\langle R^{n+1}, w_h \right\rangle + \kappa \mu^2 \left\langle g \left( \theta(t^{n+1}) - 2 \theta_h^n + \theta_h^{n-1} \right), w_h \right\rangle := \sum_{i=1}^{8} A_i.
\]
(4.35)

We now proceed to estimate each term \( A_1 \) to \( A_7 \) separately. We readily obtain

\[ A_1 \leq C \left( \frac{\| \delta E^{n+1} \|^2_0 + \| \delta E^n \|_0 \| w_h \|_0 \leq C \left( \| \delta E^{n+1} \|^2_0 + \| \delta E^n \|_0 \| \nabla w_h \|_0 \right) \right. \]

and
\[ A_2 \leq C \left\| \nabla \tilde{E}_h^{n+1} \right\|_0 \| \nabla w_h \|_0. \]

Term \( A_3 \) and \( A_4 \) can be dealt with thanks to the aid of Lemma 2.6 and \( \| u(t^{n+1}) \|_2 \leq M \) as follows:
\[ A_3 \leq C \left( \| \delta \tilde{u}(t^{n+1}) \|_0 \| u(t^{n+1}) \|_2 \| w_h \|_1 \leq C \left( \| \delta u(t^{n+1}) \|_1 \| \nabla w_h \|_1 \right. \right. \]

and
\[ A_4 \leq C \left( \| 2 u(t^{n+1}) - u(t^n) \|_2 \| \tilde{E}_h^{n+1} \|_0 \| w_h \|_1 \leq C \left( \| \tilde{E}_h^{n+1} \|_0 \| \nabla w_h \|_0. \right) \right. \]

In light of \( \| \tilde{u}_h^{n+1} \|^2_1 = \| \tilde{E}_h^{n+1} - u(t^{n+1}) \|^2_1 \leq C \) from Lemma 4.1, we can have
\[ A_5 \leq C \left( \| 2 E^n - E^{n-1} \|_{L^4(\Omega)} \| \tilde{u}_h^{n+1} \|_1 \| w_h \|_1 \leq C \left( \| E^{n+1} \|^2_0 + \| E^n \|_0 \| \nabla w_h \|_0. \right. \right. \]

Integrate by parts and Hölder inequality yield
\[ A_6 \leq C \left( \| \delta \theta_h^{n+1} \|^2_0 \| \nabla w_h \|_0. \right. \]
On the other hand, we have
\[ A_T \leq \| R^{n+1} \|_{-1} \| \nabla w \|_0. \]

The new term \( A_8 \) can be bound by the Hölder inequality as
\[
A_8 = \kappa \mu^2 \langle g (\delta \tilde{\theta}(t^{n+1}) - 2 \tilde{\theta}^n + \tilde{\theta}^{n-1}) \cdot w_h^{n+1} \rangle \\
\leq C \kappa \mu^2 \left( \| \delta \tilde{\theta}(t^{n+1}) \|_0 + \| 2 \tilde{\theta}^n - \tilde{\theta}^{n-1} \|_0 + \| 2 \tilde{\theta}^n - \tilde{\theta}^{n-1} \|_0 \| \nabla w \|_0. \right)
\]

Inserting the estimates for \( A_1 \) to \( A_7 \) back into (4.35), and employing the discrete inf-sup condition in Assumption 2, we obtain
\[
C e^n_{n+1} \leq \frac{1}{\tau} \left( \| \delta E_{n+1} \|_0 + \| \delta E^n \|_0 \right) + \frac{1}{\sqrt{h}} \left( \| E_{n+1} \|_0 + \| E^n \|_0 \right) + \| \delta \theta_{n+1} \|_0 \\
+ \| \nabla E_{h_{n+1}} \|_0 + \| \delta u(t^{n+1}) \|_1 + \| \tilde{E}_{n+1} \|_0 + \| R^{n+1} \|_{-1} \\
+ \| \delta \tilde{\theta}(t^{n+1}) \|_0 + \| 2 \tilde{\theta}^n - \tilde{\theta}^{n-1} \|_0 + \| 2 \tilde{\theta}^n - \tilde{\theta}^{n-1} \|_0. \right)
\]

If we now square it, multiply it by \( \tau \), and sum over \( n \) from 1 to \( N \), then Lemmas 4.1, 4.3 and 4.5 derive (4.34).

5. Numerical Experiments

We finally document 3 computational performance of SGUM. The first is to check accuracy and then the next 2 examples are physically relevant numerical simulations, the Benard convection problem and thermal driven cavity flow. We perform the last 2 examples under the same set within [9], but we conclude with different numerical simulation for the second test, the Benard convection problem, to that of [9]. We impose Taylor-Hood (\( P_2 - P_1 \)) in all 3 experiments.

5.1. Example 1: Mesh analysis. In this first experiment, we choose square domain \([0, 1] \times [0, 1]\) and impose forcing term the exact solution to become
\[
u = \pi (t^2 - t + 1)x^2(1 - x)^2 \sin(2 \pi y), \\
v = -2(t^2 - t + 1)x(2x - 1)(x - 1) \sin^2(\pi y), \\
p = -(t^2 - t - 1) \cos(\pi x)(y^2 + 1), \\
\theta = \cos(t) \sin(\pi x)y(1 - y).
\]

Table 1 is error decay with \( \mu = 1 \) and \( \tau = h \). We conclude that the numerical accuracy of SGUM is optimal and consists with the result of Theorem 2.

5.2. Example 2: Benard convection. In order to explore the applicability of the SGUM, we consider the Benard convection on the domain \( \Omega = [0, 5] \times [0, 1] \) with forcing \( f = 0 \) and \( b = 0 \). Figure 1 displays the initial and boundary conditions for
Table 1. Error decay for Algorithm 1 with $\tau = h$ and $\mu = 1$

| $\tau = h$ | $1/16$ | $1/32$ | $1/64$ | $1/128$ | $1/256$ |
|------------|--------|--------|--------|--------|--------|
| $\|E\|_0$  | 0.00198001 | 0.000651734 | 0.000188113 | 5.0822e-05 | 1.32686e-05 |
| Order      | 1.603153 | 1.792684 | 1.888075 | 1.937437 |
| $\|E\|_{L^\infty}$ | 0.00837795 | 0.00282504 | 0.000825997 | 0.000225008 |
| Order      | 1.568326 | 1.774063 | 1.876160 | 1.930060 |
| $\|e\|_0$  | 0.0279984 | 0.00950852 | 0.00280233 | 0.00077052 |
| Order      | 1.558052 | 1.762594 | 1.862723 | 1.909660 |
| $\|e\|_{L^\infty}$ | 0.197857 | 0.107171 | 0.0453294 | 0.0170497 |
| Order      | 1.470303 | 1.711876 | 1.832716 | 1.896142 |
| $\|\vartheta\|_0$ | 0.000025482 | 5.3851e-05 | 1.34612e-05 | 3.36511e-06 |
| Order      | 0.884544 | 1.241396 | 1.410701 | 1.481586 |
| $\|\vartheta\|_{L^\infty}$ | 0.000197687 | 4.98479e-05 | 1.25012e-05 | 3.1314e-06 |
| Order      | 1.987613 | 1.995466 | 1.997187 | 1.999059 |
| $\|\vartheta\|_1$  | 0.0167834 | 0.00893145 | 0.0041957 | 0.00104892 |
| Order      | 1.000043 | 1.000009 | 1.000007 | 1.000000 |

velocity $u$ and temperature $\theta$, as already studied in [9]. Figures 2-4 are simulations at $t = 1$ with the nondimensional parameters $\kappa = 10^4, \lambda = 1, \mu = 1$. Figure 2 is the result for the case $\tau = 10^{-2}, h = 2^{-4}$ which is the same condition in [9] and so it displays similar behavior within [9] including 6 circulations in the velocity stream line. However, Figures 3-4, the higher resolution simulations with $\tau = 10^{-3}, h = 2^{-4}$ and $\tau = 10^{-4}, h = 2^{-6}$, display 8 circulations in the stream line. So we conclude that the high resolution result is correct simulation and thus Figure 2 and the result in [9] are not eventual simulation.

5.3. Example 3: Thermal Driven Cavity Flow. We consider the thermal driven cavity flow in an enclosed square $\Omega = [0, 1]^2$, as already studied in several papers [4, 10, 9]. The experiment is carried out with the same setting as in Gresho-Lee-Chan[4], which is shown in Figure 5. Figure 6 displays the evolution from rest ($t=0$) to steady state ($t=0.2$).

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Figure 2. Example 2: Streamlines of velocity and isolines of temperature and pressure, at time $t = 1.0$. The nondimensional parameters are $\kappa = 10^4, \lambda = 1, \mu = 1$, and the discretization parameters are $\tau = 10^{-2}, h = 2^{-4}$.

Figure 3. Example 2: Streamlines of velocity and isolines of temperature and pressure, at time $t = 1.0$. The nondimensional parameters are $\kappa = 10^4, \lambda = 1, \mu = 1$, and the discretization parameters are $\tau = 10^{-3}, h = 2^{-4}$.

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Figure 4. Example 2: Streamlines of velocity and isolines of temperature and pressure, at time $t = 1.0$. The nondimensional parameters are $\kappa = 10^4$, $\lambda = 1$, $\mu = 1$, and the discretization parameters are $\tau = 10^{-4}$, $h = 2^{-6}$.

Figure 5. Example 3: Initial and boundary values for thermal driven cavity flow.

\[ \begin{align*}
\theta &= \frac{1}{2} \\
\theta^0 &= 0 \\
\theta &= -\frac{1}{2} \\
\partial_t \theta &= 0 \\
\partial_n \theta &= 0
\end{align*} \]

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Figure 6. Example 3: Time sequence $t = 0.003, 0.01, 0.02, 0.04$, and 0.1 for the driven cavity. The first two columns are the streamlines and vector fields for velocity, and the third and fourth are the contour lines for pressure and temperature, respectively. The nondimensional parameters are $\kappa = 10^5, \lambda = 1, \mu = 1$, and the discretization parameters are $\tau = 10^{-4}, h = 2^{-5}$. Note that $u_{\text{max}}$ stands for $\|u\|_{\infty}$.

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