The distribution of the number of points modulo an integer on elliptic curves over finite fields

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Abstract

Let \( \mathbb{F}_q \) be a finite field and let \( b \) and \( N \) be integers. We study the probability that the number of points on a randomly chosen elliptic curve \( E \) over \( \mathbb{F}_q \) equals \( b \) modulo \( N \). We prove explicit formulas for the cases \( \gcd(N, q) = 1 \) and \( N = \text{char}(\mathbb{F}_q) \). In the former case, these formulas follow from a random matrix theorem for Frobenius acting on the \( N \)-torsion part of \( E \), obtained by applying density results due to Chebotarev to the modular covering \( X(N) \to X(1) \). As an additional application to this theorem, we estimate the probability that a randomly chosen elliptic curve has a point of order precisely \( N \).

1 Introduction

If one writes the number of rational points on an elliptic curve \( E \) over a finite field \( \mathbb{F}_q \) as \( q + 1 - T \), then the integer \( T \) is called the trace of Frobenius of \( E \). Hasse proved that \( T \in [-2\sqrt{q}, 2\sqrt{q}] \), but within this interval the trace of Frobenius is an unpredictable number, seemingly picked at random. Since the 1960’s, its statistical behaviour has become subject to extensive study.

To make the problem well-defined, the best-known approach is to fix an elliptic curve \( E \) over a number field \( K \), and to consider it modulo various prime ideals \( p \subset \mathcal{O}_K \) of good reduction. Based on experimental evidence, Sato and Tate conjecturally described how the traces of Frobenius of \( E \mod p \) are — after being normalized by \( 2\sqrt{N(p)} \) — distributed along \([-1, 1]\). See [5] for the details and an introduction to the recent progress on this subject.

Another approach is to fix the finite field \( \mathbb{F}_q \), and to consider all \( \mathbb{F}_q \)-isomorphism classes of elliptic curves \( E \) over it. Their traces of Frobenius \( T_E \) define a discrete probability measure \( \mu_q \) on \((-|2\sqrt{q}|, \ldots, |2\sqrt{q}|) \). As above, one can normalize to obtain a distribution \( \tilde{\mu}_q \) on \([-1, 1]\). Birch [3] and Deligne [7, 3.5.7] proved results on the limit behaviour of \( \tilde{\mu}_q \) as \( q \) tends to infinity, thereby lending support for the Sato-Tate conjecture. However, not all is said with this: some remarkable properties, related to the discrete nature of \( \mu_q \), become dissolved in the limit procedure. As an introductory exercise, the reader is invited to show...
that when \( q \) is odd, \( T_E \) favours even numbers. This is related to the fact that a randomly chosen cubic polynomial \( f(x) \in \mathbb{F}_q[x] \) has a rational root with a probability that tends to \( \frac{2}{3} \) as \( q \) gets big. This phenomenon illustrates the more general fact that for any positive integer \( N \), the probability that \( \#E(\mathbb{F}_q) = q + 1 - T_E \) is divisible by \( N \) tends to be strictly bigger than \( \frac{1}{N} \). Lenstra was the first to observe this, and proved in \([19]\) explicit estimates in the situation where \( N \) is a prime number different from \( p = \text{char}(\mathbb{F}_q) \), by using modular curves. His work was generalized to arbitrary \( N \) by Howe \([14]\), and has implications for integer factorization \([19]\) and cryptography \([10]\).

In this paper, we further generalize Lenstra’s work. For an arbitrary integer \( N \geq 2 \) and \( t \in \{0, 1, \ldots, N-1\} \), write \( P_{q,N}(t) \) for the probability that \( T_E \mod N \) equals \( t \). We prove

**Theorem 1** Write \( N = p^m \ell_1^{n_1} \ell_2^{n_2} \cdots \ell_r^{n_r} \) where the \( \ell_i \) are pairwise distinct primes different from \( p \).

(i) If \( \gcd(N,p) = 1 \), then \( P_{q,N} \) converges to a multiplicative arithmetic function in \( N \), i.e.

\[
\lim_{q \to \infty} \left( P_{q,N}(t) - \prod_{i=1}^r P_{q,\ell_i^{n_i}}(t \mod \ell_i^{n_i}) \right) = 0. \tag{1}
\]

If \( N = \ell^n \) for a prime \( \ell \neq p \), then there is an explicitly described function \( \varphi : \mathbb{Z} \to \mathbb{Z} \) for which

\[
\lim_{q \to \infty} \left( P_{q,N}(t) - \frac{\varphi(t^2 - 4q)}{\ell^{3n} - \ell^{3n-2}} \right) = 0.
\]

In case \( \ell \geq 3 \) and \( n = 1 \) we have \( \varphi : x \mapsto x^2 + (\frac{q}{\ell})x \), where \((\cdot)\) is the Legendre symbol. See Section 4 for the definition of \( \varphi \) in the general case.

(ii) If \( N = p \), then

\[
\lim_{k \to \infty} P_{p^k,N}(0) = 0 \quad \text{and} \quad \lim_{k \to \infty} P_{p^k,N}(t) = \frac{1}{p-1} \quad \text{if} \ t \neq 0.
\]

Explicit error terms are given in Section 4 and Section 5.

Note that if \( N \) is an arbitrary \( p \)-th power \( p^n \) \((n \geq 1)\), then (ii) trivially implies \( \lim_{k \to \infty} P_{p^k,N}(t) = 0 \) whenever \( t \equiv 0 \mod p \). Numerical experiments suggest that the other traces are again evenly distributed:

\[
\lim_{k \to \infty} P_{p^k,N}(t) = \frac{1}{p^n - p^{n-1}} \quad \text{if} \ t \neq 0 \mod p.
\]

This can be made rigorous for \( t = \pm 1 \), following Howe \([14]\) Theorem 1.1 and using quadratic twisting. Our numerical experiments also suggest that the independence expressed in \([11]\) extends to arbitrary \( N \), i.e. including \( p \mid N \). Together,
this would give a complete description of the distribution of \( T_E \mod N \) (as \( q \) tends to infinity).

The case \( \gcd(N,p) = 1 \) is obtained from an equidistribution theorem on matrices of Frobenius acting on the \( N \)-torsion group \( E[N] \) of \( E \). Recall that \( E[N] \cong \mathbb{Z}_N \oplus \mathbb{Z}_N \), where \( \mathbb{Z}_N \) abbreviates \( \mathbb{Z}/(N\mathbb{Z}) \). Then the \( q \)th-power Frobenius action on \( E[N] \) determines a unique \( \text{GL}_2(\mathbb{Z}_N) \)-conjugacy class \( \mathcal{F}_E \) of matrices having determinant \( q \). Denote the subset of \( \text{GL}_2(\mathbb{Z}_N) \) consisting of all matrices of determinant \( q \) by \( \mathcal{M}_q \). Then the theorem reads:

**Theorem 2** Fix a conjugacy class \( \mathcal{F} \subset \text{GL}_2(\mathbb{Z}_N) \) of matrices of determinant \( q \). Let \( E \) be a uniformly randomly chosen \( \mathbb{F}_q \)-isomorphism class of elliptic curves over \( \mathbb{F}_q \). Let \( P_\mathcal{F} \) be the probability that \( \mathcal{F}_E = \mathcal{F} \). Then

\[
\left| P_\mathcal{F} - \frac{\#\mathcal{F}}{\#\mathcal{M}_q} \right| \leq C \cdot \frac{N^2}{\sqrt{q}}
\]

where \( C \in \mathbb{R}_{>0} \) is an absolute and explicitly computable constant.

In other words, if \( q \) gets big, a Frobenius conjugacy class becomes as likely as its own relative size. See Section 3 for more details on the constant \( C \).

In its above form, Theorem 2 seems new and fits in the random matrix philosophy that dominates nowadays research on the statistical behaviour of Frobenius, both in the Sato-Tate setting (fixed curve, varying field) as in the Birch-Deligne setting (fixed field, varying curve). This was initialized by Deligne, who obtained his above-mentioned result as a consequence to an equidistribution theorem in étale cohomology. The random matrix idea has proven to provide well-working models for higher genus analogues of the Frobenius distribution problem \([17, 18]\), although many statements remain conjectural. We refer to the book by Katz and Sarnak \([17]\) for more details. This book also contains a refinement of Deligne’s equidistribution theorem \([17, 9.7]\) which was used by Achter to prove a variant of Theorem 2 that works in arbitrary genus \([2, \text{Theorem 3.1}]\). However, Achter’s result has a worse error bound and imposes certain weak restrictions on \( q \) and \( N \). Our attention will be devoted to a more elementary approach, based on the modular covering \( X(N) \to X(1) \) and (parts of the proof of) Chebotarev’s density theorem for function fields.

As an additional application to Theorem 2, we investigate the probability of a point of prescribed order coprime to \( q \).

**Theorem 3** Let \( N \geq 2 \) be an integer coprime to \( q \), and write \( N = \ell_1^{n_1} \ell_2^{n_2} \cdots \ell_r^{n_r} \), where the \( \ell_i \) are pairwise distinct primes. Let \( E \) be a uniformly randomly chosen \( \mathbb{F}_q \)-isomorphism class of elliptic curves over \( \mathbb{F}_q \). Write \( P_q'(N) \) for the probability that \( E \) has a point of order \( N \). Then

(i) \( P_q' \) converges to a multiplicative arithmetic function, i.e.

\[
\lim_{\gcd(q, N) \to 1} \left( P_q'(N) - \prod_{i=1}^{r} P_q'(\ell_i^{n_i}) \right) = 0.
\]
(ii) If \( \ell \neq p \) is a prime number, \( q_0 \) and \( n \geq 1 \) are integers with \( q_0 \not\equiv 0 \mod \ell \) and \( \nu \) is the \( \ell \)-adic valuation of \( q_0 - 1 \), then

\[
\lim_{q \to \infty} \left( P'_q(\ell^n) - \theta_{\ell^n} \right) = 0,
\]

where \( \theta_{\ell^n} \) equals \( 1/(\ell^n - \ell^{n-2}) \) if \( \nu \geq n \) and \( (\ell^{2\nu+1} + 1)/(\ell^{n+2\nu+1} - \ell^{n+2\nu-1}) \) in the other cases.

An explicit error term is given in Section 6.

It is worth remarking that several questions related to Theorem 1 and Theorem 3 were already posed by Gekeler in the weaker set-up where \( \mathbb{F}_q \) is a large prime field that has to be chosen at random; he studied the distribution of Frobenius traces [11] and various probabilities such as \( E[\ell^{\infty}](\mathbb{F}_q) \) having a given structure or \( E(\mathbb{F}_q) \) being cyclic [12, 13]. The latter probability has also been studied by Vlăduț in case \( \mathbb{F}_q \) is fixed [21], using Howe’s work. Still for \( \mathbb{F}_q \) fixed, Galbraith and McKee conjecturally estimated the probability that \( E(\mathbb{F}_q) \) is a prime number [10]. Achter and Sadornil studied the chance that \( E \) has a given number of rational isogenies of given prime degree emanating from it [3]. For higher genus curves \( C/\mathbb{F}_q \), Achter gave explicit estimates for the chance that \( \text{Jac}(C)[N](\mathbb{F}_q) \) has a given structure [11, 2], and Chavdarov proved that the numerator of the zeta function \( Z_C(T) \) is generically irreducible [6].

The article is organized as follows. Section 2 recalls the necessary background on modular curves, Section 3 contains the proof of Theorem 2 and we use this in Section 4 to deduce Theorem 1 for the case \( \gcd(N, p) = 1 \). Section 5 contains the proof for the case \( N = p \). Finally, Section 6 contains the proof of Theorem 3.

We also include an Appendix, which recalls certain facts about twisting, and which discusses some disambiguations on what is meant by a randomly chosen elliptic curve.

The authors are very grateful to Hendrik W. Lenstra for his suggestion to consider Chebotarev’s density theorem for the proof of Theorem 2.

2 Background on modular curves

An implicit reference for this section are the lecture notes by Deligne and Rapoport [8] and the earlier work by Igusa [15, 16] on which these build.

Let \( \mathbb{F}_p \) be the finite prime field with \( p \) elements, and let \( N \) be a positive integer, coprime to \( p \). Fix a primitive \( N \)-th root of unity \( \zeta_N \in \mathbb{F}_p \). Consider all triplets \((E, P, Q)\), where \( E \) denotes an elliptic curve over \( \mathbb{F}_p \), and \( P, Q \in E[N] \) satisfy \( e_N(P, Q) = \zeta_N \).

\[
e_N : E[N] \times E[N] \to \{N\text{-th roots of unity}\}
\]

is the Weil pairing, see [20, III.§8]. Two triplets \((E, P, Q)\) and \((E', P', Q')\) are called \emph{equivalent} if there exists an \( \mathbb{F}_p \)-isomorphism \( E \to E' \) mapping \( P \) to \( P' \) and \( Q \) to \( Q' \). As a special instance, using multiplication by \(-1\), we have that \((E, P, Q)\) is equivalent to \((E, -P, -Q)\).
The set of equivalence classes of such triplets can be given the structure of a nonsingular affine curve \( Y(N) \). Note that \( Y(1) \) merely parameterizes elliptic curves by their \( j \)-invariant; it has the structure of the affine line \( \mathbb{A}^1 \). The nonsingular completion of \( Y(N) \) is called the modular curve of level \( N \) and is denoted by \( X(N) \). In particular, \( X(1) \) can be identified with \( \mathbb{P}^1 \). The natural covering

\[
Y(N) \to \mathbb{A}^1 : (E, P, Q) \mapsto j(E)
\]

extends to an algebraic morphism \( \psi : X(N) \to \mathbb{P}^1 \), which is Galois, with Galois group \( \text{PSL}_2(\mathbb{Z}_N) \). On \( Y(N) \) this group acts through

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (E, P, Q) = (E, \alpha P + \beta Q, \gamma P + \delta Q).
\]

The morphism \( \psi \) is ramified at (and only at) \( j = 0, 1728, \infty \). The genus of \( X(N) \) equals \( 1 + \#\text{PSL}_2(\mathbb{Z}_N) \cdot (N - 6)/12N \).

The construction of \( Y(N) \) primarily provides a model that is defined over \( \mathbb{F}_p(\zeta_N) \). To remedy this, one repeats the above construction for all primitive \( N \)th-roots of unity. The union again parameterizes triplets \((E, P, Q)\) modulo equivalence, but now one only imposes that \((P, Q)\) is a basis of \( E[\mathbb{N}] \). Up to tensoring with \( \mathbb{F}_p(\zeta_N) \), this union is what Deligne and Rapoport denote by \( \mathcal{M}_0^N \). It is a reducible scheme decomposing into \( \varphi(N) \) copies of \( Y(N) \). Similar to (2), we have an action of

\[
H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \bigg| \alpha \in \mathbb{Z}_N^\times \right\} \subset \text{GL}_2(\mathbb{Z}_N)
\]
on \( \mathcal{M}_0^N \otimes \mathbb{F}_p(\zeta_N) \) which connects these components horizontally: every orbit \( \{ (E, P, Q) \} \) contains a unique point of each component. The quotient under this action can thus be identified with \( Y(N) \), and realizes it as a curve over the fixed field \( \mathbb{F}_p(\zeta_N)^{\text{det} H} \), where \( \alpha \in \text{det} H \) acts on \( \mathbb{F}_p(\zeta_N) \) as \( \zeta_N \mapsto \zeta_N^\alpha \). Hence it realizes \( Y(N) \) as a curve over \( \mathbb{F}_p \). As a consequence, \( X(N) \) is defined over \( \mathbb{F}_p \), and this also accounts for the morphism \( X(N) \to X(1) \).

From now on, let \( \mathbb{F}_q \supset \mathbb{F}_p \) be the finite field with \( q \) elements, and consider \( X(N) \) as a curve over \( \mathbb{F}_q \). Then it is endowed with a \( q \)th-power Frobenius action \( \Sigma \), where some caution is needed in describing it explicitly. Let \( \sigma \in \text{Gal}(\overline{\mathbb{F}_q}, \mathbb{F}_q) \) be the usual \( q \)th-power Frobenius automorphism. Then the map \((E, P, Q) \mapsto (E^\sigma, P^\sigma, Q^\sigma)\) is not well-defined on \( Y(N) \), as it does not preserve the Weil pairing. However, the \( H \)-orbit of \((E^\sigma, P^\sigma, Q^\sigma)\) contains a unique representant on which the Weil pairing acts properly, and this is

\[
\Sigma(E, P, Q) = (E^\sigma, q^{-1}P^\sigma, Q^\sigma).
\]

We end by commenting on the algebraic side of the above story, whilst fixing notation. The coordinate ring \( R \) of \( Y(1) \) (over \( \mathbb{F}_q \)) equals \( \mathbb{F}_q[j] \), in which the formal variable \( j \) can be seen as a universal \( j \)-invariant. Its field of fractions will be denoted by \( K \), while the function field of \( Y(N) \) (over \( \mathbb{F}_q \)) will be denoted
by $L$. The morphism $X(N) \to X(1)$ corresponds to a field extension $K \subset L$, which is normal and separable with Galois group $\text{PSL}_2(Z_N)$. We will write $g_L$ for the genus of $L$, which is $1 + [L : K] \cdot (N - 6)/12N$. The integral closure of $R$ in $L$ can be identified with the coordinate ring of $Y(N)$, and will be denoted by $S$. Here is a summarizing diagram:

$$
\begin{align*}
\mathbb{F}_q[j] &= R \subset K = \mathbb{F}_q(j) \\
\cap & \quad \cap
\mathbb{F}_q[Y(N)] = S \subset L = \mathbb{F}_q(Y(N)).
\end{align*}
$$

From now on, an elliptic curve with $j$-invariant $j_0 \in \overline{\mathbb{F}}_q$ will always be denoted by $E_{j_0}$.

### 3 The distribution of Frobenius matrices

We will now prove Theorem 2 by applying density results due to Chebotarev to the modular covering $X(N) \to X(1)$. Our main reference for the proof of the Chebotarev density theorem is [9, Section 5.4].

Let $E_{j_0} \in \mathbb{F}_q$. A triplet $E = (E_{j_0}, P, Q)$ on the modular curve $Y(N)$ corresponds to a maximal ideal $m_E$ in $S \otimes \overline{\mathbb{F}}_q$. Define $P_E := m_E \cap S$, which can be viewed as a closed point of $Y(N)$ as an $\mathbb{F}_q$-scheme. Suppose that $P_E$ is unramified over $K$, which is equivalent to the condition $j_0 \neq 0, 1728$. As explained in [9, Section 5.2], we can associate to $P_E$ its Frobenius automorphism $\left[\frac{L/K}{P_E}\right] \in \text{Gal}(L/K)$. With $P_E := P_E \cap R$ this automorphism is uniquely determined by the condition

$$
\left[\frac{L/K}{P_E}\right] x \equiv x^{N(P_E)} \mod P_E, \quad \text{for all } x \in S.
$$

We note that $j_0 \in \mathbb{F}_q$ implies that $P_E = (j - j_0)$ and hence $N(P_E) = q$. Geometrically, the above condition means that if

$$
\{(E_{j_0}, P_1, Q_1), (E_{j_0}, P_2, Q_2), \ldots, (E_{j_0}, P_{\deg P_E}, Q_{\deg P_E})\}
$$

is the set of points of $Y(N)$ (maximal ideals of $S \otimes \overline{\mathbb{F}}_q$) above $P_E$, then $\left[\frac{L/K}{P_E}\right] \in \text{PSL}_2(Z_N)$ permutes this set, in the same manner as $\Sigma$ does. If $P'$ is another prime ideal of $S$ above $P_E$, we have that the Frobenius automorphism $\left[\frac{L/K}{P'}\right]$ is conjugated to $\left[\frac{L/K}{P_E}\right]$. The Artin symbol

$$
\left(\frac{L/K}{P_E}\right)
$$

of $P_E$ is then defined as the conjugacy class of $\left[\frac{L/K}{P_E}\right]$ in $\text{Gal}(L/K)$. We can now formulate our main tool.
Lemma 1 Choose \( \tau \in \text{Gal}(L/K) \cong \text{PSL}_2(Z_N) \). Let \( A \) denote the set of points \( E = (E_{j_0}, P, Q) \in Y(N) \) for which \( j_0 \in \mathbb{F}_q \setminus \{0, 1728\} \) and \( \left[ \frac{L/K}{F_E} \right] = \tau \). Then we have
\[
|\#A - q| \leq (4[L : K] + 4g_L + 2) \cdot \sqrt{q}.
\]
We postpone the proof to the end of this section. Now recall that \( Y(N) \) parameterizes triplets \( (E_{j_0}, P, Q) \) up to \( \mathbb{F}_q \)-isomorphism, whereas we are interested in triplets up to \( \mathbb{F}_q \)-isomorphism. Using that all \( j_0 \in \mathbb{F}_q \setminus \{0, 1728\} \) correspond to two elliptic curves over \( \mathbb{F}_q \) (related to each other by quadratic twisting, see Corollary 3 in the Appendix below), we get the following result.

Corollary 1 Suppose \( N > 2 \). Choose \( F \in \text{GL}_2(Z_N) \) such that \( \det F = q \). Let \( B \) denote the set of triplets \( (E_{j_0}, P, Q) \) up to \( \mathbb{F}_q \)-isomorphism for which
\begin{itemize}
  \item[(i)] \( E_{j_0} \) is an elliptic curve over \( \mathbb{F}_q \) with \( j \)-invariant \( j_0 \neq 0, 1728 \),
  \item[(ii)] the points \( P, Q \in E_{j_0}[N] \) satisfy \( e_N(P, Q) = \zeta_N \), and
  \item[(iii)] the matrix of \( q \)-th-power Frobenius on \( E_{j_0}[N] \) with respect to the basis \( (P, Q) \) equals \( F \).
\end{itemize}
Then we have
\[
|\#B - q| \leq (4[L : K] + 4g_L + 2) \cdot \sqrt{q}.
\]

Proof. Let \( \tau \in \text{PSL}_2(Z_N) \) and suppose that \( T \in \text{SL}_2(Z_N) \) reduces to \( \tau \) mod \( \{\pm \text{Id}\} \). Every point \( E = (E_{j_0}, P, Q) \in Y(N) \) for which \( j_0 \in \mathbb{F}_q \setminus \{0, 1728\} \) and \( \left[ \frac{L/K}{P_E} \right] = \tau \), corresponds up to \( \mathbb{F}_q \)-isomorphism to precisely two triplets, namely \( (E_{j_0}, P, Q) \) and its quadratic twist. Their \( q \)-th-power Frobenius matrices differ by sign and are equal to
\[
\pm \begin{pmatrix}
q & 0 \\
0 & 1
\end{pmatrix} \cdot T \in \text{GL}_2(Z_N)
\]
(see 3) and the discussion preceding Lemma 1. Conversely, if we start with a triplet \( (E_{j_0}, P, Q) \in B \), we find
\[
\pm \begin{pmatrix}
q & 0 \\
0 & 1
\end{pmatrix}^{-1} \cdot F \in \text{PSL}_2(Z_N)
\]
as the Frobenius automorphism \( \left[ \frac{L/K}{P_E} \right] \in \text{Gal}(L/K) \) associated to the point \( E = (E_{j_0}, P, Q) \in Y(N) \). This induces a bijection between \( B \) and the set \( A \) of the previous lemma (for an appropriate choice of \( \tau \)). \( \square \)

Note 1 If \( N = 2 \), then \( \text{Id} = -\text{Id} \) in \( \text{GL}_2(Z_N) \). Therefore \( (E_{j_0}, P, Q) \) and its quadratic twist correspond to the same Frobenius matrix, so we have \( \#B = 2\#A \). In the proof of Theorem 4 below, this is compensated by the fact that \( \#\text{SL}_2(Z_N) = 2\#\text{PSL}_2(Z_N) \) if \( N > 2 \), whereas \( \#\text{SL}_2(Z_N) = \#\text{PSL}_2(Z_N) \) if \( N = 2 \).
We can now state and prove our main theorem.

**Theorem 4** Denote with \( M_q \) the subset of \( \text{GL}_2(\mathbb{Z}_N) \) of matrices with determinant \( q \), and let \( \mathcal{F} \) be a \( \text{GL}_2(\mathbb{Z}_N) \)-conjugacy class in this set. Let \( E_{j_0} \) represent a uniformly randomly chosen \( \mathbb{F}_q \)-isomorphism class of elliptic curves over \( \mathbb{F}_q \), and let \( \mathcal{F}_{E_{j_0}} \subset \text{GL}_2(\mathbb{Z}_N) \) be the conjugacy class determined by the action of \( q \)-th-power Frobenius on \( E_{j_0}[N] \). The probability \( P_{\mathcal{F}} \) that \( \mathcal{F}_{E_{j_0}} = \mathcal{F} \) satisfies

\[
\left| P_{\mathcal{F}} - \frac{\# \mathcal{F}}{\# M_q} \right| \leq \frac{\# \mathcal{F}}{\# M_q} \left( 4[L : K] + 4g_L + 2 \right) \frac{1}{\sqrt{q}} + \frac{23}{q}.
\]

The estimate in the theorem is easily seen to be \( O(N^2 q^{-1/2}) \), which gives an idea about how large \( q \) has to be with respect to \( N \) in order to find a meaningful result.

**Proof.** We suppose \( N > 2 \) (see Note 1 for the case \( N = 2 \)). The set \( \mathcal{W} \) of (\( \mathbb{F}_q \)-isomorphism classes of) elliptic curves over \( \mathbb{F}_q \) has \( 2^q + \delta \) elements, with \( 0 \leq \delta \leq 22 \) depending on the finite field \( \mathbb{F}_q \); see Corollary 3 in the Appendix below. Denote with \( \mathcal{V} \subset \mathcal{W} \) the set of elliptic curves \( E_{j_0} \) for which \( \mathcal{F}_{E_{j_0}} = \mathcal{F} \). By Corollary 3, \( \mathcal{V} \) contains at most \( \epsilon \leq 24 \) elliptic curves with \( j \)-invariant 0 or 1728, and all other curves in \( \mathcal{V} \) correspond to \( \# \text{PSL}_2(\mathbb{Z}_N) \) tuples \((E_{j_0}, P, Q)\) (with \( e_N(P, Q) = \zeta_N \)) up to \( \mathbb{F}_q \)-isomorphism. Combined with the definition of \( B \) from Corollary 1, this gives the equality

\[
(\# \mathcal{V} - \epsilon) \cdot \# \text{PSL}_2(\mathbb{Z}_N) = \# B \cdot \# \mathcal{F}.
\]

Now we can compute \( P_{\mathcal{F}} \) as follows:

\[
P_{\mathcal{F}} = \frac{\# \mathcal{V}}{\# \mathcal{W}} = \frac{\# B \cdot \# \mathcal{F}}{(2q + \delta) \# \text{PSL}_2(\mathbb{Z}_N)} + \frac{\epsilon}{2q + \delta}.
\]

A first estimate of this probability is then

\[
\left| P_{\mathcal{F}} - \frac{\# \mathcal{F}}{\# \text{PSL}_2(\mathbb{Z}_N)} \left( \frac{\# B}{q + \delta/2} \right) \right| \leq \frac{12}{q}.
\]

Using Corollary 1 and \( \# M_q = \# \text{PSL}_2(\mathbb{Z}_N) \) this implies

\[
\left| P_{\mathcal{F}} - \frac{\# \mathcal{F}}{\# M_q} \left( \frac{q}{q + \delta/2} \right) \right| \leq \frac{12}{q} + \frac{\# \mathcal{F}}{\# M_q} \left( (4[L : K] + 4g_L + 2) \frac{1}{\sqrt{q}} \right).
\]

Noting that \( \# \mathcal{F} \leq \# M_q \) and

\[
\left| 1 - \frac{q}{q + \delta/2} \right| \leq \frac{\delta}{2q},
\]

we finally arrive at

\[
\left| P_{\mathcal{F}} - \frac{\# \mathcal{F}}{\# M_q} \right| \leq \frac{12 + \delta/2}{q} + \frac{\# \mathcal{F}}{\# M_q} \left( (4[L : K] + 4g_L + 2) \frac{1}{\sqrt{q}} \right),
\]

8
which concludes the proof. □

We will now prove Lemma 1. Our proof essentially uses the proof of the Chebotarev density theorem for function fields as given in Section 5.4 of [9]. We remark that Theorem 4 seems not to follow from the density theorem itself; we really need parts of its proof. The reason is that the Frobenius matrix (up to sign) corresponding to a point \( E = (E_{j_0}, P, Q) \) \( \in Y(N) \) and the Frobenius automorphism in \( \text{PSL}_2(\mathbb{Z}_N) \) associated to the prime ideal \( P_E \subset S \) are only related through multiplication by

\[
\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \in H,
\]

which tears the conjugacy classes apart when \( q \not\equiv 1 \mod N \). In general, there is no bijection between the conjugacy classes of Frobenius automorphisms and the conjugacy classes of Frobenius matrices. Note that if \( q \equiv 1 \mod N \) then the above matrix becomes the identity, and it is indeed possible to use the Chebotarev density theorem rather directly.

**Proof of Lemma 1.** We denote with \( P(L) \) the set of prime ideals of \( S \) which are unramified over \( K \), and let \( P(K) \) be the set of prime ideals of \( R \). For \( P \in P(L) \) we write \( p_P := P \cap R \), the \( R \)-ideal below \( P \). The conjugacy class of \( \tau \in \text{PSL}_2(\mathbb{Z}_N) \) will be denoted by \( M_\tau \). Define

\[
C_1(L/K, M_\tau) := \left\{ \frac{L/K}{p} : \deg(p) = 1 \right\}.
\]

Note that the condition \( \deg(p) = 1 \) is equivalent to the associated \( j \)-invariant living in \( \mathbb{F}_q \). Let

\[
D_1(L/K, \tau) := \left\{ P \in P(L) : \left[ \frac{L/K}{P} \right] = \tau; \ p_P \in C_1(L/K, M_\tau) \right\}.
\]

If we look at [9, Proposition 5.16] and particularly the formulas (15), (16) and (17) appearing in its proof, we find with \( d = [K : \mathbb{F}_q(j)] = 1, n = k = 1, g_K = 0 \) and \( m = [L : K] \) that

\[
\left| \#C_1(L/K, M_\tau) - \frac{\#M_\tau}{[L : K]} \cdot q \right| \leq \#M_\tau \cdot \left( 4 + 2 \frac{2g_L + 1}{[L : K]} \right) \cdot \sqrt{q}. \tag{4}
\]

From [9, Lemma 5.9(b)] with \( C'_1 = C_1(L/K, M_\tau) \) and hence \( D'_1(\tau) = D_1(L/K, \tau) \) we see that

\[
\#C_1(L/K, M_\tau) = \#M_\tau \cdot \frac{\text{ord}(\tau)}{[L : K]} \cdot \#D_1(L/K, \tau).
\]

We insert this in equation (4) and divide by \( \#M_\tau \):

\[
\left| \frac{\text{ord}(\tau)}{[L : K]} \cdot \#D_1(L/K, \tau) - \frac{q}{[L : K]} \right| \leq \left( 4 + 2 \frac{2g_L + 1}{[L : K]} \right) \cdot \sqrt{q}. \tag{5}
\]
From [9] Lemma 5.9(a) it follows that the number of points \( \mathcal{E} = (E_{2t}, P, Q) \in Y(N) \) with \( m_2 \) lying above some fixed \( P \in D_1(L/K, \tau) \) equals \( \text{ord}(\tau) \), so that our lemma follows from [5], after multiplying both sides with \([L : K] \).

4 The distribution of Frobenius traces mod \( N \)

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements, take \( t \in \mathbb{Z} \) and let \( N \geq 2 \) be an integer coprime to \( q \). Using Theorem [3] we will estimate the probability that a randomly chosen elliptic curve over \( \mathbb{F}_q \) has trace of Frobenius congruent to \( t \) modulo \( N \). A first observation is that this probability converges to a multiplicative arithmetic function. Indeed, if \( N = A \cdot B \) with \( A \) and \( B \) coprime, then we have an obvious isomorphism \( \text{GL}_2(Z_N) \cong \text{GL}_2(Z_A) \oplus \text{GL}_2(Z_B) \), and this bijection respects the sets of matrices with determinant \( q \) and trace \( t \) (modulo \( N \) resp. \( A \) and \( B \)). Therefore, in order to make the formulas not too complicated, we will confine ourselves to \( N = \ell^n \), where \( \ell \) is a prime that does not divide \( q \).

It is easy to verify that \( \# \text{SL}_2(Z_{t^n}) = \ell^{3n-2}(\ell^2 - 1) \). With \( \alpha \in Z_\mathbb{Z}\setminus\{0\} \), we define the valuation \( \text{ord}(<\alpha>) \) as the \( \ell \)-adic valuation of \( \alpha \) embedded in \( \mathbb{Z} \), whereas we will put \( \text{ord}(0) = +\infty \). Let for \( \ell \geq 3 \) the map \( \varphi : \mathbb{Z} \to \mathbb{Z} \) be defined as \( \varphi = \psi \circ \chi \), where \( \chi : \mathbb{Z} \to \mathbb{Z}_{\ell^n} \) is the natural projection and \( \psi : \mathbb{Z}_{\ell^n} \to \mathbb{Z} \) is given by

\[
\begin{align*}
\Delta &\mapsto \begin{cases} 
\ell^{2n} + \ell^{2n-1} & \text{if } \Delta \text{ is a nonzero square}, \\
\ell^{2n} + \ell^{2n-1} - 2\ell^{2n-\frac{1}{2}} - 1 & \text{if } \Delta \text{ is no square, } k := \text{ord}(\Delta) \text{ is even}, \\
\ell^{2n} + \ell^{2n-1} - (\ell + 1)\ell^{2n-\frac{k+3}{2}} & \text{if } k := \text{ord}(\Delta) \text{ is odd}, \\
\ell^{2n} + \ell^{2n-1} - \ell^{\frac{2n-1}{2}} & \text{if } \Delta = 0 \text{ and } n \text{ is even}, \\
\ell^{2n} + \ell^{2n-1} - \ell^{\frac{2n-3}{2}} & \text{if } \Delta = 0 \text{ and } n \text{ is odd}.
\end{cases}
\end{align*}
\]

We refer to the end of this section for the definition of \( \varphi \) in case \( \ell = 2 \).

**Theorem 5** Let \( \mathbb{F}_q, t \) and \( \ell^n \) be as above and define \( \Delta_t := t^2 - 4q \). Let \( E \) be a uniformly randomly chosen \( \mathbb{F}_q \)-isomorphism class of elliptic curves over \( \mathbb{F}_q \), and let \( T \) be its trace of Frobenius. The probability \( P(t) \) that \( T \equiv t \) mod \( \ell^n \) satisfies

\[
\left| P(t) - \frac{\varphi(\Delta_t)}{\ell^{3n} - \ell^{3n-2}} \right| \leq \frac{4[L : K] + 4qL + 2}{\ell^n - \ell^{n-1}} \cdot \sqrt{72} + 23 \leq O(\ell^{2nq^{-\frac{1}{2}}}).
\]

Here \([L : K] = \# \text{PSL}_2(Z_{\ell^n})\) and \( g_L = 1 + [L : K](\ell^n - 6)/(12\ell^n) \) as in Section 2. Note that this theorem implies that \( \frac{P(t) - \varphi(\Delta_t)}{(\ell^{3n} - \ell^{3n-2})} \) for \( q \to \infty \) under the restriction that \( q \) stays in a single congruence class modulo \( \ell^n \).

Before proving Theorem 5 we discuss some corollaries. The number of rational points on an elliptic curve \( E \) over \( \mathbb{F}_q \) with trace of Frobenius \( T \) equals \( q + 1 - T \). Hence we can estimate the probability that \( \ell^n \# E(\mathbb{F}_q) \) by applying Theorem 5 with \( t = q + 1 \). Note that then \( t^2 - 4q \equiv (q - 1)^2 \mod \ell^n \). Using this, we partly recover the results of Howe [14].
If we suppose $\ell \geq 3$ and $n = 1$, then the above theorem becomes quite pretty, namely

$$P(t) \sim \begin{cases} \frac{\ell}{\ell^2 - 1} & \text{if } t^2 - 4q = 0 \text{ in } \mathbb{F}_\ell, \\ \frac{1}{\ell - 1} & \text{if } t^2 - 4q \in \mathbb{F}_\ell^\times \text{ is a square}, \\ \frac{1}{\ell + 1} & \text{if } t^2 - 4q \in \mathbb{F}_\ell 	ext{ is a nonsquare}. \end{cases}$$

This generalizes Lenstra’s result \[19\] which states that the probability of $\ell$-torsion approaches $\ell/(\ell^2 - 1)$ if $q \equiv 1 \bmod \ell$ and $1/(\ell - 1)$ otherwise.

The remainder of this section will be devoted to the proof of Theorem $5$. It suffices to show that the number of matrices in $\text{GL}_2(\mathbb{Z}_\ell n)$ with determinant $q$ and trace $t$ equals $\varphi(\Delta_t)$, with $\Delta_t = t^2 - 4q$. Indeed, then Theorem $4$ implies that $P(t)$ satisfies

$$\left| P(t) - \frac{\varphi(\Delta_t)}{\#M_q} \right| \leq \frac{\varphi(\Delta_t)}{\#M_q} \cdot (4[L : K] + 4gL + 2) \frac{1}{\sqrt{q}} + m \cdot 23 \cdot q,$$

where $m$ is the number of $\text{GL}_2(\mathbb{Z}_\ell n)$-conjugacy classes of such matrices. Since $m \leq \varphi(\Delta_t) \leq \ell^{2n} + \ell^{2n-1}$ and $\#M_q = \ell^{3n} - \ell^{3n-2}$, the theorem follows. Note that the counting of matrices described below was already done by Gekeler \[11\] Theorem $4.4$ for the case $n \geq 2 \cdot \text{ord}(\Delta) + 2$, using different techniques.

Let $(u \ x \ y \ z) \in \text{GL}_2(\mathbb{Z}_\ell n)$ have determinant $q$ and trace $t$. A trivial computation yields that these conditions are equivalent to the system of equations

$$u = t - z, \quad xy = z^2 - tz + q. \quad \text{(6)}$$

By completing the square, the above system has as many solutions as

$$u = t - z, \quad xy = z^2 - \Delta_t/4, \quad \text{(7)}$$

provided that $t/2$ exists modulo $\ell^n$. Suppose for the rest of the proof that $\ell \geq 3$ and $\Delta_t \in \mathbb{Z}_{\ell^n}$, we refer to the end of this section for the situation $\ell = 2$. Clearly all relevant properties (valuation, being a square or not) of $\Delta_t$ and $\Delta_t/4$ are the same, hence if we can show that the number of solutions to $xy = z^2 - \Delta_t$ equals $\varphi(\Delta_t)$, we are done. For each value of $z$, we will determine the valuation of $z^2 - \Delta_t$. Then the number of corresponding solutions $(x, y)$ can be computed using the following lemma.

**Lemma 2** Let $\ell$ be any prime number, let $n \in \mathbb{Z}_{\geq 1}$ and $\alpha \in \mathbb{Z}_{\ell^n}$. Write $k := \text{ord}(\alpha)$. Then the equation $xy = \alpha$ has the following number of solutions $(x, y)$ in $(\mathbb{Z}_{\ell^n})^2$:

$$\begin{cases} (k + 1)(\ell^n - \ell^{n-1}) & \text{if } \alpha \neq 0, \\ (n + 1)(\ell^n - \ell^{n-1}) + \ell^{n-1} & \text{if } \alpha = 0. \end{cases}$$

**Proof.** Suppose $\alpha \neq 0$, the other case works similarly. We can take $x$ to be any number with valuation $i \in \{0, 1, \ldots, k\}$. For each $i$, the number of such $x$ is
$\ell^{n-i} - \ell^{n-i-1}$. Every choice of $x$ fixes all but the last $i$ $\ell$-adic digits of $y$, hence we have $\ell^i$ possibilities for $y$. In total this amounts to

$$\sum_{i=0}^{k} (\ell^{n-i} - \ell^{n-i-1})\ell^i = \sum_{i=0}^{k} (\ell^n - \ell^{n-1}) = (k+1)(\ell^n - \ell^{n-1})$$

solutions $(x, y)$. □

Another tool will be the following formula, which is easily proven by induction:

**Lemma 3** Let $\ell$ be any prime number, let $n \geq 1$ be an integer and $k \in \{0, 1, \ldots, n\}$. Then

$$\sum_{i=0}^{k} (\ell^{n-i} - \ell^{n-i-1})(2i+1)(\ell^n - \ell^{n-1}) = \ell^{2n} + \ell^{2n-1} - (2k + 3)\ell^{2n-k-1} + (2k + 1)\ell^{2n-k-2}.$$  

Suppose first that $\Delta_i = 0$ and $n$ even. Then $\text{ord}(z^2 - \Delta_i) = \text{ord}(z^2)$ for all $z$, and the number of solutions to $xy = z^2 - \Delta_i$ with $\text{ord}(z) < n/2$ equals

$$\sum_{i=0}^{n/2-1} (\ell^{n-i} - \ell^{n-i-1})(2i+1)(\ell^n - \ell^{n-1}),$$

by Lemma 2. For $\text{ord}(z) \geq n/2$, we find

$$\ell^{n/2} ((n+1)(\ell^n - \ell^{n-1}) + \ell^{n-1})$$

additional solutions. Using Lemma 3 one verifies that the sum of these expressions equals $\varphi(0)$. If $n$ is odd, then the reasoning is similar.

Let us now assume that $\Delta_i$ is a nonzero square, i.e. $\Delta_i = \ell^{2k}\Delta^2$, where $2k < n$ and $\Delta$ is a unit. Under the change of variables $(x, y, z) \leftarrow (\Delta x, \Delta y, \Delta z)$ our equation becomes

$$xy = z^2 - \ell^{2k}. \quad (8)$$

We will use induction on $k$ to show that (8) has $\varphi(\Delta_i) = \ell^{2n} + \ell^{2n-1}$ solutions. For $k = 0$ we have $xy = z^2 - 1$. If $x$ is any units, we have $y = x^{-1}(z^2 - 1)$ and $z$ can be chosen arbitrarily. If $x$ is a nonunit and $y$ is arbitrary, we have 2 different solutions $z \equiv \pm 1$ modulo $\ell$, which can both be lifted to $\mathbb{Z}_{2\ell^n}$. In total this gives

$$(\ell^n - \ell^{n-1})\ell^n + 2\ell^n - 1\ell^n = \ell^{2n} + \ell^{2n-1}.$$ 

Suppose now that $k \geq 1$. There are $\ell^{2n} - \ell^{2n-1}$ solutions for which $x$ is a unit. There are $(\ell^n - \ell^{n-1})\ell^n$ solutions for which $y$ is a unit and $z$ — and hence $x$ — are nonunits. The solutions for which $x$ and $y$ are both nonunits
can be determined using the induction hypothesis. Indeed, a triplet \((x, y, z) = (\ell x', \ell y', \ell z')\) satisfies \(\phi\) if and only if \((x', y', z')\) satisfies

\[ x' y' = z'^2 - \ell^{2k-2} \text{ over } \mathbb{Z}_{\ell^n-2}, \]

which has \(\ell^{2n-4} + \ell^{2n-5}\) solutions. For each \(x' \in \mathbb{Z}_{\ell^n-2}\) there are \(\ell\) corresponding values for \(x = \ell x' \mod \ell^n\), and similar for \(y\) and \(z\). In total we find then

\[ \ell^{2n} - \ell^{2n-1} + (\ell^n - \ell^{n-1})\ell^{n-1} + \ell^3(\ell^{2n-4} + \ell^{2n-5}) = \ell^{2n} + \ell^{2n-1}. \]

Next, if \(k = \text{ord}(\Delta_t) < +\infty\) is odd, we find the following sum for the number of solutions

\[ \sum_{i=0}^{(k-1)/2} (\ell^{n-i} - \ell^{n-i-1})(2i + 1)(\ell - \ell^{n-1}) + \ell^{n-(k+1)/2}(k + 1)(\ell^n - \ell^{n-1}), \]

which by Lemma 3 equals \(\varphi(\Delta_t)\).

Finally, with \(k\) even but \(\Delta_t\) nonsquare we get

\[ \sum_{i=0}^{k/2-1} (\ell^{n-i} - \ell^{n-i-1})(2i + 1)(\ell - \ell^{n-1}) + \ell^{n-k/2}(k + 1)(\ell^n - \ell^{n-1}), \]

and again the result follows from Lemma 3. This completes the proof for \(\ell \geq 3\).

We end this section by considering the case \(\ell = 2\). The appropriate description of \(\varphi\) depends now on its argument mod \(2^{n+2}\) rather than mod \(2^n\). More precisely, \(\varphi = \psi \circ \chi\) where \(\chi : \mathbb{Z} \to \mathbb{Z}_{2^{n+2}}\) is the natural projection and \(\psi : \mathbb{Z}_{2^{n+2}} \to \mathbb{Z}\) is partially given by

\[ \Delta \mapsto \begin{cases} 2^{2n-1} & \text{if } \Delta \text{ is odd}, \\ 2^{2n} + 2^{2n-1} - 3 \cdot 2^{n-\frac{k+1}{2}} & \text{if } \Delta \neq 0 \text{ is even and } k := \text{ord}(\Delta) \text{ is odd}, \\ 2^{2n} + 2^{2n-1} - 2^{\frac{k+1}{2}} - 1 & \text{if } \Delta \equiv 0 \text{ mod } 2^{n+2} \text{ and } n \text{ is even}, \\ 2^{2n} + 2^{2n-1} - 2^{\frac{3n-1}{2}} & \text{if } \Delta \equiv 0 \text{ mod } 2^{n+2} \text{ and } n \text{ is odd}. \end{cases} \]

In case \(\Delta \neq 0\) is even and \(\text{ord}(\Delta) = 2k > 0\) is even as well, the definition of \(\psi\) is more complicated. Let \(D\) be such that \(\Delta = 2^{2k}D\). Then:

\[
\begin{align*}
&\text{if } n = 2k - 1: & \psi(\Delta) := 2^{2n} + 2^{2n-1} - 2^{\frac{4n-k}{2}}, \\
&\text{if } n = 2k, \quad D \equiv 1 \text{ mod } 4: & \psi(\Delta) := 2^{2n} + 2^{2n-1} - 2^{\frac{k+1}{2}}, \\
&\text{if } n = 2k, \quad D \equiv 3 \text{ mod } 4: & \psi(\Delta) := 2^{2n} + 2^{2n-1} - 3 \cdot 2^{\frac{k}{2}} - 1, \\
&\text{if } n \geq 2k + 1, \quad D \equiv 3 \text{ mod } 8: & \psi(\Delta) := 2^{2n} + 2^{2n-1} - 3 \cdot 2^{2n-k-1}, \\
&\text{if } n \geq 2k + 1, \quad D \equiv 5 \text{ mod } 8: & \psi(\Delta) := 2^{2n} + 2^{2n-1} - 2^{2n-k}, \\
&\text{if } n \geq 2k + 1, \quad D \equiv 1 \text{ mod } 8: & \psi(\Delta) := 2^{2n} + 2^{2n-1}.
\end{align*}
\]

We will now prove that for any \(t \in \mathbb{Z}\), the number of solutions (over \(\mathbb{Z}_{2^n}\)) to the system \((8)\) is precisely \(\varphi(\Delta_t)\), where \(\Delta_t = t^2 - 4q\). Note first that if \(t\)
(or equivalently $\Delta_t$) is odd, we have that $\text{ord}(z^2 - tz + q) = 0$ for all $z$. Then Lemma 2 gives a total of

$$2^n(2^n - 2^{n-1}) = 2^{2n-1} = \varphi(\Delta_t)$$

solutions.

Therefore suppose that $t$ is even. Then $\Delta_t \equiv 0 \mod 4$, and it makes sense to complete the square in (6) and analyze the system (7) instead. As we are interested in solutions modulo $2^n$, from now on we will consider $\Delta_t/4$ as an element of $\mathbb{Z}_{2^n}$. Note that this depends on $\Delta_t \mod 2^{n+2}$. Copying the proofs of the corresponding cases above, the system (7) has $\varphi(\Delta_t)$ solutions if $\Delta_t/4 = 0$ (in $\mathbb{Z}_{2^n}$) or if $\text{ord}(\Delta_t/4) < n$ is odd. Hence we assume that $\text{ord}(\Delta_t/4) = 2\kappa < n$ is even. Let $D \in \mathbb{Z}_{2^n}$ be such that $2^{2\kappa}D = \Delta_t/4$. If $i = \text{ord}(z) < \kappa$ we have $\text{ord}(z^2 - \Delta_t/4) = 2i$, so by Lemma 2 and Lemma 3 all such $z$ together account for

$$S := \sum_{i=0}^{\kappa-1} (2^{n-i} - 2^{n-i-1})(2i+1)(2^n - 2^{n-1}) = 2^{2n} + 2^{2n-1} - (2\kappa + 3)2^{2n-\kappa-1}$$

solutions $(x, y, z)$. From now on we assume $\text{ord}(z) \geq \kappa$ and put $z = 2^\kappa z'$, so that our equation becomes

$$xy = 2^{2\kappa}(z'^2 - D).$$

Note that $z'$ is only well-determined modulo $2^{n-\kappa}$, and that we are interested in $z'^2 - D \mod 2^{n-2\kappa}$.

If $n = 2\kappa + 1$ we have two possibilities: either $z' \equiv 0 \mod 2$, which gives $2^{n-\kappa-1}(2\kappa + 1)2^{n-1}$ solutions $(x, y, z' \mod 2^{n-\kappa})$, or $z' \equiv 1 \mod 2$, which gives $2^{n-\kappa-1}(n + 1)2^{n-1}$ solutions. If we add $S$ to these two numbers, we find the requested result.

Let $n = 2\kappa + 2$, then we have to distinguish between $D \equiv 1 \mod 4$ and $D \equiv 3 \mod 4$. For example, if $D \equiv 3 \mod 4$ and $z'$ is odd, the valuation of $2^{2\kappa}(z'^2 - D)$ equals $2\kappa + 1$, since 3 is not a quadratic residue modulo 4. We leave further details to the reader.

Finally we assume that $n \geq 2\kappa + 3$. The cases $D \equiv 3 \mod 4$ and $D \equiv 5 \mod 8$ are similar to the situation $n = 2\kappa + 2$ above, so we only go into more details for $D \equiv 1 \mod 8$. Then we know that $D$ is a square modulo $2^{n-2\kappa}$ and we can proceed as in the case $\ell \geq 3$ and $\Delta_t$ a nonzero square. However, things work differently for the induction step $\kappa = 0$, i.e. $xy = z^2 - 1 \mod 2^n$, $n \geq 3$. As the valuation of $z^2 - 1$ cannot be 1 or 2, we have to consider four situations. Firstly, $\text{ord}(x) = 0$, then $z$ can be chosen arbitrarily and we find $2^{n-1} \cdot 2^n$ solutions. Secondly, $\text{ord}(x) = 1$, then $\text{ord}(y) \geq 2$ and we can lift the four solutions $z \equiv 1, 3, 5, 7 \mod 8$ to $\mathbb{Z}_{2^n}$, which gives a total of $4 \cdot 2^{n-2}2^{n-2}$ solutions. Third, $\text{ord}(x) = 2$ and $\text{ord}(y) \geq 1$ which gives again $2^{2n-2}$ solutions. Finally, $\text{ord}(x) \geq 3$ and $y$ is arbitrary, which gives $4 \cdot 2^{n-3}2^n$ solutions. Adding all these terms together gives $2^{2n} + 2^{2n-1}$ solutions.
The distribution of Frobenius traces mod $p$

**Theorem 6** Let $p \geq 5$ be a prime number, let $k \geq 1$ be an integer, and let $t \in \{1, \ldots, p-1\}$. Let $S_t$ be the set of couples in

$$S = \{(A, B) \in (\mathbb{F}_{p^k})^2 \mid 4A^3 + 27B^2 \neq 0\}$$

for which the trace $T$ of $p$th-power Frobenius of the elliptic curve defined by $y^2 = x^3 + Ax + B$ satisfies $T \equiv t \mod p$.

Then $\#S = p^{2k} - p^k$ and

$$\left|\#S_t - \frac{\#S}{p-1}\right| \leq 3p^{\frac{k}{2}+1}.$$

**Proof.** We leave it as an exercise to show that $\#S = p^{2k} - p^k$.

For each $(A, B) \in S$, one has that $T \mod p$ equals the norm (with respect to $\mathbb{F}_{p^k}/\mathbb{F}_p$) of the coefficient $c_{A,B}$ of $x^{p-1}$ in

$$(x^3 + Ax + B)^{\frac{p-1}{2}}$$

(see the proof of [20, Theorem V.4.1(a)]). Lemma 4 below shows that for every $\gamma \in \mathbb{F}_{p^k} \setminus \{0\}$, the polynomial $c_{A,B} - \gamma$ is absolutely irreducible and nonzero.

Now write $S'_t$ for the set of couples $(A, B) \in (\mathbb{F}_{p^k})^2$ in which $c_{A,B}$ evaluates to an element $\gamma \in \mathbb{F}_{p^k} \setminus \{0\}$ with norm $t$ (regardless of the condition $4A^3 + 27B^2 \neq 0$). Note that there are $\frac{p^k - 1}{p-1}$ such $\gamma$'s. For each of these the polynomial $c_{A,B} - \gamma$ defines a plane affine curve, by the irreducibility proven above. Its degree is bounded by $d = 3(p-1)/2$, hence its (geometric) genus is at most $(d-1)(d-2)/2$, and the number of points at infinity is at most $d$. Therefore the set $S'_t \subset S'_t'$ of couples satisfying $c_{A,B} = \gamma$ is subject to

$$\left|\#S'_t - (p^k + 1)\right| \leq (d-1)(d-2)\sqrt{p^k} + d \leq \frac{9}{4}p^{\frac{k}{2}+2}$$

by the Hasse-Weil bound. Remark that this includes the singular case, where the number of points may become smaller, but the Hasse-Weil bound tightens at bigger speed.

Summing up, and using $(p^k - 1)/(p-1) \leq \frac{5}{4}p^{k-1}$ (since $p \geq 5$),

$$\left|\#S'_t - \frac{p^{2k} - 1}{p-1}\right| \leq \frac{45}{16}p^{\frac{2k}{2}+1}.$$ 

Therefore, because $\#(S'_t \setminus S_t) \leq p^k$ and $5p^{k-1} \leq p^k \leq \frac{16}{11}p^{\frac{2k}{2}+1}$, we obtain

$$\left|\#S_t - \frac{p^{2k} - p^k}{p-1}\right| \leq \left|\#S_t - \frac{p^{2k} - 1}{p-1}\right| + \frac{p^k - 1}{p-1} \leq \left(\frac{45}{16} + \frac{1}{11} + \frac{5}{4} \cdot \frac{1}{55}\right)p^{\frac{2k}{2}+1},$$

which ends the proof. □
**Corollary 2** Let \( p \) be any prime number. Let \( t \in \{0, \ldots, p-1\} \), and for each \( k \geq 1 \) we denote by \( P_k(t) \) the proportion of elliptic curves over \( \mathbb{F}_{p^k} \) (modulo \( \mathbb{F}_{p^k} \)-isomorphism) for which the trace of Frobenius is congruent to \( t \) mod \( p \). If \( t \neq 0 \), then

\[
\lim_{k \to \infty} P_k(t) = \frac{1}{p-1}
\]

whereas

\[
\lim_{k \to \infty} P_k(0) = 0.
\]

**Proof.** If \( t \neq 0 \) and \( p \geq 5 \), then the result easily follows from Theorem 6; see also the Appendix below.

If \( t = 0 \), then the curves of consideration are supersingular, and by [20, Theorem V.3.1] their \( j \)-invariants must be contained in \( \mathbb{F}_{p^2} \). Using Corollary 3, this implies

\[
\lim_{k \to \infty} P_k(0) \leq \lim_{k \to \infty} \frac{24p^2}{2p^k} = 0.
\]

If \( p = 2 \), the result then trivially follows from \( P_k(0) + P_k(1) = 1 \).

If \( p = 3 \), this works similarly, since quadratic twisting provides a bijection between the set of elliptic curves having trace 1 mod 3, and the set of elliptic curves with trace 2 mod 3. \(\square\)

**Lemma 4** Let \( p \geq 5 \) be a prime number and let \( c_{A,B} \in \mathbb{F}_p[A,B] \) be the coefficient of \( x^{p-1} \) in

\[
(x^3 + Ax + B) \frac{x^{p-1}}{x^{p-1}} \in \mathbb{F}_p[A,B][x].
\]

Then \( c_{A,B} \) is homogeneous of \((2,3)\)-weighted degree \((p-1)/2\), nonzero, and absolutely squarefree. As a consequence, for any \( \gamma \in \mathbb{F}_p \setminus \{0\} \), the polynomial

\[
c_{A,B} - \gamma \in \mathbb{F}_p[A,B]
\]

is irreducible.

**Proof.** One verifies that

\[
c_{A,B} = \sum_{i=\left[\frac{p-1}{3} \right]}^{\left[\frac{p-1}{3} \right] + \left[\frac{p-1}{2} \right]} \binom{\frac{p-1}{3}}{i} \binom{\frac{p-1}{2}}{i} A^{3i} B^{\frac{p-1}{3} - 2i} \tag{9}
\]

from which it immediately follows that \( c_{A,B} \) is nonzero and homogeneous of degree \((p-1)/2\) if we equip \( A \) and \( B \) with weights 2 and 3 respectively. Now

\[
\left\lfloor \frac{p-1}{6} \right\rfloor = \frac{p-1}{6} + \delta
\]

where \( \delta \) equals 0 or 1/3. From this we see that

\[
3 \left\lfloor \frac{p-1}{6} \right\rfloor - \frac{p-1}{2}
\]
equals 0 or 1. In particular, since all coefficients in \([9]\) are nonzero, we see that \(A\) appears as a factor at most once. Similarly, one checks that \(B\) appears as a factor at most once.

Let \(c'_{A,B}\) be obtained from \(c_{A,B}\) by deleting the factors \(A\) and \(B\) when possible. Define \(\epsilon_A\) (resp. \(\epsilon_B\)) to be 1 if a factor \(A\) (resp. \(B\)) was deleted, and 0 otherwise. Then \(c'_{A,B}\) is still homogeneous, of degree \((p - 1)/2 - 2\epsilon_A - 3\epsilon_B\).

After dividing by a suitable power of \(A\) and considering the resulting polynomial in the single variable \(B^2/A^3\), one verifies that \(c'_{A,B}\) splits (over \(\mathbb{F}_p\)) as

\[
c(B^2 - a_1A^3)(B^2 - a_2A^3) \cdots (B^2 - a_rA^3)
\]

with \(r = \frac{1}{2}((p - 1)/2 - 2\epsilon_A - 3\epsilon_B)\) and all \(c, a_i \neq 0\). Each of these factors corresponds to a \(j_i \neq 0, 1728\) for which the elliptic curve over \(\mathbb{F}_p\) with \(j\)-invariant \(j_i\) is supersingular, and conversely all supersingular \(j\)-invariants different from 0,1728 must be represented this way. Now one has that the number of supersingular \(j\)-invariants different from 0,1728 is precisely given by \(r\) (see the proof of \([20]\) Theorem V.4.1(c))]. Therefore, all factors in \([10]\) must be different, and in particular \(c_{A,B}\) must be squarefree.

Now let \(\gamma \in \mathbb{F}_p \setminus \{0\}\) and suppose we had a nontrivial factorization

\[
c_{A,B} - \gamma = (F_1 + X_1)(F_2 + X_2),
\]

where \(F_1\) and \(F_2\) are the components of highest degree of the respective factors. Then it follows that \(F_1F_2 = c_{A,B}\), so \(F_1\) and \(F_2\) cannot have a common factor. It also follows that

\[
X_1F_2 + X_2F_1 + X_1X_2 + \gamma = 0. \tag{11}
\]

Let \(X'_1\) and \(X'_2\) be the components of highest degree of \(X_1\) and \(X_2\) respectively. Suppose \(\deg X_1F_2 > \deg X_2F_1\). Then \(X'_1F_2\) is zero, because it cannot be cancelled in \([11]\). But then \(X'_1 = X_1 = 0\) and we run into a contradiction. By symmetry, we conclude that \(\deg X_1F_2 = \deg X_2F_1\). But then \(X'_1F_2 + X'_2F_1 = 0\).

So all factors of \(F_1\) must divide \(X'_1F_2\), which is impossible unless \(X'_1 = 0\), and we again run into a contradiction. \(\square\)

### 6 The probability of a point of order \(N\)

Let \(q\) be a prime power and let \(N \geq 2\) be any integer coprime to \(q\). In this section we ask for the probability \(P'(N)\) that a random \(\mathbb{F}_q\)-isomorphism class \(E\) of elliptic curves over \(\mathbb{F}_q\) has an \(\mathbb{F}_q\)-rational point of order precisely \(N\). It is well-known (see e.g. \([20]\) Exercise 5.6]) that

\[
E(\mathbb{F}_q), + \cong \mathbb{Z}_k \oplus \mathbb{Z}_m
\]

for integers \(k, m\) such that \(k|m\) and \(k|q - 1\). Hence if \(\gcd(N, q - 1) = 1\), then \(P'(N)\) equals the probability \(P(q + 1)\) that \(N|\#E(\mathbb{F}_q)\) (see Theorem \([5]\)).

As in the previous section we will use Theorem \([4]\) which implies that \(P'\) behaves as a multiplicative arithmetic function of \(N\) as \(q \to \infty\). So we can
assume that $N = \ell^n$ with $\ell$ prime and $\ell \nmid q$. As we just explained, this is only interesting when $\ell q - 1$.

**Theorem 7** Let $F$ be an elliptic curve and $\ell^n$ as above. Let $\nu \geq 1$ be the $\ell$-adic valuation of $q - 1$ and define $\theta_{\ell^n}$ by

$$
\theta_{\ell^n} := \begin{cases} 
\frac{1}{\ell^n - \ell^{n-2}} & \text{if } q \equiv 1 \mod \ell^n, \text{ i.e. } \nu \geq n, \\
\frac{\ell^{2\nu+1} + 1}{\ell^n + 2\nu - (\ell^2 - 1)} & \text{elsewhere}.
\end{cases}
$$

We have

$$
|P'(\ell^n) - \theta_{\ell^n}| \leq \frac{4[L : K] + 4gL + 2}{\ell^n - \ell^{n-1}} \cdot \frac{\sqrt{q} + 23}{q} = O(\ell^{2\nu}q^{-\frac{1}{2}}).
$$

We refer to Section 2 for the definition of $[L : K]$ and $g_L$. The following small example might shed some light on the difference between Theorems 5 and 7. Let $\ell^n = 9$, $q \equiv 1 \mod 9$ and $E$ a random elliptic curve over $\mathbb{F}_q$. The probability $P(q+1)$ that $\#E(\mathbb{F}_q) \equiv 0 \mod 9$ approaches (for $q \to \infty$) $11/72$. However, the approximate probability that $E$ has a point of order 9 is smaller, namely $9/72$. A corollary is that the probability that $E(\mathbb{F}_q)[9] \equiv Z_3 \oplus Z_3$ tends to $2/72$.

**Proof of Theorem 7** Let $E/\mathbb{F}_q$ be an elliptic curve and $F_E \in \text{GL}_2(\mathbb{Z}_{\ell^n})$ the matrix of $q$th-power Frobenius with respect to any basis of $E[\ell^n]$. If $E$ has an $\mathbb{F}_q$-rational point $P$ of order $\ell^n$, then we can take any $Q$ such that $(P,Q)$ is a basis of $E[\ell^n]$, and the matrix of Frobenius with respect to this basis equals $(1 \quad w \nmid q)$ for a certain $w \in \mathbb{Z}_{\ell^n}$. Moreover, $F_E$ is $\text{GL}_2(\mathbb{Z}_{\ell^n})$-conjugated to this matrix, and the converse implication holds as well: if $F_E$ is in the conjugacy class of a matrix $(1 \quad w \nmid q)$, then $E$ has an $\mathbb{F}_q$-rational point of order $\ell^n$. Note that this condition is equivalent to $F_E$ having an eigenvector with eigenvalue 1 which is not the zero vector modulo $\ell$. We will show that the number of such matrices equals $\theta_{\ell^n} \cdot \#\text{SL}_2(\mathbb{Z}_{\ell^n})$. Then the theorem follows using precisely the same argument we explained in the beginning of the proof of Theorem 5.

The conjugacy classes of matrices of the form $(1 \quad \frac{w}{q})$ are determined by their representants $M_a$ in Lemma 5 below. The size of the conjugacy class $\text{Cl}_a$ of $M_a$ can be computed as follows. Let $\text{St}_a$ be the stabilizer subgroup of $M_a$, then the classical orbit-stabilizer theorem states that $\#\text{St}_a \cdot \#\text{Cl}_a = \#\text{GL}_2(\mathbb{Z}_{\ell^n})$. Hence it suffices to compute the size of $\text{St}_a$. We know that $(x \quad y \nmid s \quad t) \in \text{St}_a$ if and only if $(\frac{y}{x} \quad \frac{t}{s})$ is invertible and

$$
\begin{pmatrix}
1 & \ell^n \\
0 & q
\end{pmatrix} \cdot \begin{pmatrix} x & y \\
s & t \end{pmatrix} = \begin{pmatrix} x & y \\
s & t \end{pmatrix} \cdot \begin{pmatrix} 1 & \ell^n \\
0 & q
\end{pmatrix}.
$$

This condition is equivalent to the system (using $a \leq \nu$)

$$
\begin{align*}
\ell^n s &\equiv 0 \mod \ell^n \\
\ell^n(t-x) &\equiv y(q-1) \mod \ell^n.
\end{align*}
$$

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We can choose \( x \) and \( y \) at random, so that \( t = x(y - 1)\ell^{-a} + x \mod \ell^{n-a} \) and \( s \equiv 0 \mod \ell^{n-a} \); we find a total of \( \ell^{2n+2a-1} \) matrices satisfying (12). From these we have to remove the singular matrices, which adds the condition \( xt \equiv sy \mod \ell \). If \( a < \nu \) we have by (13) that \( s \equiv 0 \mod \ell \) and \( t \equiv x \mod \ell \), hence the only additional restriction is that \( x \equiv 0 \mod \ell \). This gives \( \ell^{2n+2a-1} \) singular matrices and hence \( \#\text{St}_a = \ell^{2n+2a} - \ell^{2n+2a-1} \) for \( a < \nu \). If \( \nu = n \) it is obvious that \#\text{Cl} = 1, so we are left with considering \( \text{St}_\nu \) for \( \nu < n \). As shown in the proof of Lemma 5 the matrix \((\begin{smallmatrix} \ell^a \\ 0 \\ q \end{smallmatrix})\) is conjugated to \((\begin{smallmatrix} 1 \\ 0 \\ q \end{smallmatrix})\), and now it is an easy exercise to compute the number \( \#\text{St}_\nu = \ell^{2n+2\nu} - (2\ell^{2n-1} - \ell^{2n-2})\ell^{2\nu} \). Combined this gives that the number of matrices conjugated to some \((\begin{smallmatrix} 1 \\ 0 \\ w \end{smallmatrix})\) where \( \nu < n \) equals (note that \( \#\text{GL}_2(\mathbb{Z}_\ell) = \ell^{4n-4}(2^2 - \ell)(\ell^2 - 1) \)):

\[
\sum_{a=0}^{\nu-1} \frac{\ell^{4n-4}(\ell^2 - \ell)(\ell^2 - 1)}{\ell^{2n+2a} - \ell^{2n+2a-1}} + \frac{\ell^{4n-4}(\ell^2 - \ell)(\ell^2 - 1)}{\ell^{2n+2\nu} - 2\ell^{2n+2\nu-1} + \ell^{2n+2\nu-2}} = \ell^{2n} + \ell^{2n-2\nu-1}.
\]

Dividing this number by \( \#\text{SL}_2(\mathbb{Z}_\ell) \) gives the theorem for \( \nu < n \). If \( q \equiv 1 \mod \ell^n \) we similarly find

\[
\sum_{a=0}^{n-1} \frac{\ell^{4n-4}(\ell^2 - \ell)(\ell^2 - 1)}{\ell^{2n+2a} - \ell^{2n+2a-1}} + 1 = \ell^{2n}.
\]

This concludes the proof. \( \square \)

**Lemma 5** Let \( \nu = \text{ord}_\ell(q - 1) \). Each matrix over \( \mathbb{Z}_\ell \) of the form \((\begin{smallmatrix} 1 \\ 0 \\ w \end{smallmatrix})\) is conjugated to precisely one matrix of

\[
\left\{ M_a := \begin{pmatrix} 1 & \ell^a \\ 0 & q \end{pmatrix} \ igg| \ 0 \leq a \leq \nu \right\}.
\]

**Proof.** First we show that \((\begin{smallmatrix} 1 \\ 0 \\ q^{a-\nu} \end{smallmatrix})\) with \( a \geq \nu \) is conjugated to \((\begin{smallmatrix} 1 \\ 0 \\ q \end{smallmatrix})\). Write \( q = 1 + \ell^\nu q' \), then

\[
\begin{pmatrix} 1 & q^{a-\nu - 1} \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & \ell^a \\ 0 & q \end{pmatrix} \cdot \begin{pmatrix} 1 & q^{-1}(q^{a-\nu} - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ell^\nu \\ 0 & q \end{pmatrix}.
\]

Let \( w = \ell^a w' \) with \( w' \) a unit in \( \mathbb{Z}_\ell \), then

\[
\begin{pmatrix} w' & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & \ell^a w' \\ 0 & q \end{pmatrix} \cdot \begin{pmatrix} w' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ell^a \\ 0 & q \end{pmatrix},
\]

which implies that at least one matrix of the above set is conjugated to \((\begin{smallmatrix} 1 \\ 0 \\ w \end{smallmatrix})\). The fact that all matrices \( M_a \) define different conjugacy classes follows either from a direct reasoning (assuming that two of them are conjugated, the transformation matrix will have determinant 0 modulo \( \ell \)) or from the computations above which show that the conjugacy classes have different size. \( \square \)
In this section we excluded the case $p|N$. Note that $E[p^\infty](\F_q)$ is always cyclic (and even trivial when $E$ is supersingular). Therefore, asking for a point of order $p^n$ is the same as asking for $p^n$-torsion, the probability of which was described by Howe (see the discussion following Theorem 1 in the introduction). We then see that the only gap towards a complete description of the probability that an elliptic curve has a point of order $N$, is a proof of the presumed independence between $N = p^n$ and $N$ coprime to $q$.

**Note 2** It is possible to determine the probability of all kinds of group structures in a similar way. For example, let $0 \leq a \leq b$ be integers, $\ell$ a prime coprime to $q$ and suppose we want to know the probability that $E[\ell^\infty](\F_q) \cong \Z_{\ell^a} \oplus \Z_{\ell^b}$. This can be done as follows. Let $S$ be the set of matrices $M$ in $\text{GL}_2(\Z_{\ell^a+b+1})$ with determinant $q$ for which the following conditions hold:

(i) $\text{Tr}(M) \not\equiv q+1 \mod \ell^{a+b+1}$,

(ii) $\text{Tr}(M) \equiv q+1 \mod \ell^{a+b}$,

(iii) $M$ is conjugated to some $\left( \begin{array}{cc} 1 & \ast \\ 0 & q \end{array} \right)$ mod $\ell^b$, and

(iv) $M \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ mod $\ell^a$.

Then the requested probability tends to $\#S/\#\text{SL}_2(\Z_{\ell^a+b+1})$. Note that this question was also considered by Gekeler in [12] in the alternative setting mentioned in the introduction.

**Appendix: Twists, randomness, and disambiguations**

**Quadratic twisting.** The existing literature seems to contain varying definitions for the notion of quadratic twisting. We followed [20] X.2.4, Exercise A.2], which we recall here. Let $F_q$ be a finite field and let $E$ be an elliptic curve over $F_q$. If $\text{char}(F_q) \neq 2$, one takes a short Weierstrass model $y^2 = f(x)$ and a nonsquare $d \in F_q$. Then the quadratic twist of $E$ is the curve $E_t$ defined by $dy^2 = f(x)$. Its $F_q$-isomorphism class does not depend on the choice of the Weierstrass model, nor on the choice of $d$. We have an $F_q$-isomorphism $\iota : E_t \to E : (x, y) \mapsto (x, \sqrt{dy})$. If $\text{char}(F_q) = 2$ and $j(E) \neq 0$ then $E$ allows a model

$$y^2 + xy = x^3 + a_2 x^2 + a_6$$

(see [20] Appendix A). Let $d \in F_q$ have trace 1, then it is of the form $\beta^2 + \beta$ for some $\beta \in F_{q^2} \setminus F_q$. The quadratic twist $E^t$ is then given by

$$y^2 + xy = x^3 + (a_2 + d)x^2 + a_6.$$ 

This is again well-defined and we have an $F_q$-isomorphism $\iota : E^t \to E : (x, y) \mapsto (x, y + \beta x)$. Note that $E$ can be $F_q$-isomorphic to its quadratic twist, take for instance $q \equiv 3 \mod 4$, $E : y^2 = x^3 + x$ and $d = -1$.

Let $N$ be a positive integer, coprime to $q$, and let $(P, Q)$ be a basis of $E[N]$. Let $F$ be the matrix of $q$th-power Frobenius acting on $E[N]$ with respect to this basis.
Then it is an easy exercise to verify that \(-F\) is the matrix of \(q\)th-power Frobenius acting on \(E^\prime[N]\) with respect to the basis \((\iota^{-1}(P), \iota^{-1}(Q))\). As a consequence, if \(\mathcal{F}_E\) is the \(\text{GL}_2(\mathbb{Z}_N)\)-conjugacy class associated to \(q\)th-power Frobenius acting on \(E[N]\), then \(-\mathcal{F}_E = \{-M \mid M \in \mathcal{F}_E\}\) is the \(\text{GL}_2(\mathbb{Z}_N)\)-conjugacy class associated to \(q\)th-power Frobenius acting on \(E^\prime[N]\). Note that the example above shows that one might have \(\mathcal{F}_E = -\mathcal{F}_E\), even if \(N > 2\).

The number of twists of an elliptic curve. The following formula, due to Howe, summarizes what we need.

**Theorem 8** Let \(\mathbb{F}_q\) be a finite field and \(E/\mathbb{F}_q\) an elliptic curve. Let \([E]_{\mathbb{F}_q}\) be the set of \(\mathbb{F}_q\)-isomorphism classes of elliptic curves that are \(\mathbb{F}_q\)-isomorphic to \(E\). Then

\[
\sum_{E' \in [E]_{\mathbb{F}_q}} \frac{1}{\text{Aut}_{\mathbb{F}_q}(E')} = 1.
\]

**Proof.** See [14] Proposition 2.1. \(\square\)

**Corollary 3** One has \(#[E]_{\mathbb{F}_q} \geq 2\). If \(j(E) \neq 0, 1728\), then this becomes an equality, and \([E]_{\mathbb{F}_q}\) consists of \(E\) and its quadratic twist. Otherwise, we have the following upper bounds. If \(j(E) = 1728\) and \(\text{char}(\mathbb{F}_q) \neq 2, 3\) then \(#[E]_{\mathbb{F}_q} \leq 4\). If \(j(E) = 0\) and \(\text{char}(\mathbb{F}_q) \neq 2, 3\) then \(#[E]_{\mathbb{F}_q} \leq 6\). If \(j(E) = 0 = 1728\) and \(\text{char}(\mathbb{F}_q) = 3\) then \(#[E]_{\mathbb{F}_q} \leq 12\). Finally, if \(j(E) = 0 = 1728\) and \(\text{char}(\mathbb{F}_q) = 2\) then \(#[E]_{\mathbb{F}_q} \leq 24\).

**Proof.** Since \(\{\pm 1\} \subset \text{Aut}_{\mathbb{F}_q}(E')\), one must have that \(#[E]_{\mathbb{F}_q} \geq 2\). The upper bounds follow from \(\text{Aut}_{\mathbb{F}_q}(E') \subset \text{Aut}_{\mathbb{F}_q}(E)\) and [20] Theorem III.10.1. It remains to show that if \(j(E) \neq 0, 1728\), then \(E\) cannot be \(\mathbb{F}_q\)-isomorphic to its quadratic twist: indeed, this would give a non-rational automorphism of \(E\), which cannot exist since \(\text{Aut}_{\mathbb{F}_q}(E) = \{\pm 1\}\). \(\square\)

Randomly chosen elliptic curves. Throughout this article, by a randomly chosen elliptic curve over \(\mathbb{F}_q\) we always meant that the \(\mathbb{F}_q\)-isomorphism class of \(E\) was uniformly randomly chosen among the \(\mathbb{F}_q\)-isomorphism classes of elliptic curves over \(\mathbb{F}_q\). Note that from Corollary 3 above it follows that the number of such \(\mathbb{F}_q\)-isomorphism classes lies in \([2q, 2q + 22]\).

We will now briefly comment on two common disambiguations. Suppose first that \(\text{char}(\mathbb{F}_q) > 3\). Then it is natural to state that a random elliptic curve is given by

\[
y^2 = x^3 + Ax + B
\]

where \((A, B)\) was uniformly randomly chosen in the set

\[
S = \{(A, B) \in (\mathbb{F}_q)^2 \mid 4A^3 + 27B^2 \neq 0\}.
\]

Since not all \(\mathbb{F}_q\)-isomorphism classes are represented by an equal number of pairs \((A, B)\), this notion is nonequivalent to ours. However, as \(q\) gets big, the slight difference becomes negligible. Indeed, by Corollary 3 there are at most ten elliptic curves over \(\mathbb{F}_q\) having \(j\)-invariant 0 or 1728. These precisely correspond to the \(2q - 2\) Weierstrass models \(y^2 = x^3 + Ax + B\) for which \(AB = 0\). All other \(\mathbb{F}_q\)-isomorphism classes are represented by exactly \((q - 1)/2\) couples \((A, B) \in S\). As a consequence, under the
another disambiguation is to consider an elliptic curve defined by
\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \]
for uniformly randomly chosen \((a_1, a_2, a_3, a_4, a_6) \in \mathbb{F}_q^5\) subject to the appropriate smoothness condition. Again one can verify that Theorem I is still valid under this notion of randomness.

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