Integrable Lax Hierarchies, their Symmetry Reductions and Multi-Matrix Models

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Abstract

Some new developments in constrained Lax integrable systems and their applications to physics are reviewed. After summarizing the tau function construction of the KP hierarchy and the basic concepts of the symmetry of nonlinear equations, more recent ideas dealing with constrained KP models are described. A unifying approach to constrained KP hierarchy based on graded $SL(r+n, n)$ algebra is presented and equivalence formulas are obtained for various pseudo-differential Lax operators appearing in this context.

It is then shown how the Toda lattice structure emerges from constrained KP models via canonical Darboux-Bäcklund transformations. These transformations enable us to find simple Wronskian solutions for the underlying tau-functions.

We also establish a relation between two-matrix models and constrained Toda lattice systems and derive from this relation expressions for the corresponding partition function.

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1 Introduction

1.1 Abbreviations; Diagram of Lectures

Abbreviations used in text:

KP hierarchy (equation) = Kadomtsev-Petviashvili hierarchy (equation)
cKP hierarchy = constrained KP hierarchy
DB transformation = Darboux-Bäcklund transformation
KdV hierarchy (equation) = Korteveg-de Vries hierarchy (equation)
BA function = Baker-Akhiezer function
AKNS hierarchy = Ablowitz-Kaup-Newell-Segur hierarchy
AKS approach = Adler-Kostant-Symes approach
GNLS equation = generalized non-linear Schrödinger equation

Fig. 1. Structure of exposition

![Diagram of Lectures]
1.2 Content of Lectures

In these lectures we study constrained KP hierarchy, its connection to the discrete Toda models and application via Toda models to the multi-matrix approach to the two-dimensional gravity.

A presentation follows the diagram shown on the previous page in the counter-clockwise direction. In Section 2 we collect some standard facts on KP hierarchy which are used in several places in these lectures. A link between the BA function and $\tau$ function is established and it is shown how to obtain the conserved current densities of the KP hierarchy from these concepts. A complementary presentation of KP hierarchy centered around multi-Hamiltonian/Poisson structures was given at the previous Swieca school [3].

The notion of constrained KP hierarchies is introduced in Section 3. Since this notion is significant for our lectures we introduce a variety of reduction schemes and elaborate on their mutual equivalency.

The picture which emerges from this discussion is most simply presented in the language of pseudo-differential operators. Two basic equivalent approaches look especially attractive and deserve to be mentioned here in the introduction.

One construction introduces the Lax operator of the constrained KP ($cKP$) hierarchy as a ratio

$$L_{m,n} \equiv \frac{L^{(m)}}{L^{(n)}} \quad n \leq m - 1 \quad (1.1)$$

of two purely differential operators

$$L^{(m)} = (D + v_m)(D + v_{m-1})\cdots(D + v_1) \quad ; \quad L^{(n)} = (D + \tilde{v}_n)(D + \tilde{v}_{n-1})\cdots(D + \tilde{v}_1) \quad (1.2)$$

with coefficients $v_i, \tilde{v}_i$ subject to a constraint:

$$\sum_{j=1}^{m} v_j - \sum_{l=1}^{n} \tilde{v}_l = 0 \quad (1.3)$$

Imposing condition (1.3) is equivalent to requiring tracelessness of the underlying graded $SL(m, n)$ algebra. For this reason we refer to this class of constrained KP ($cKP$) models as $SL(m, n)$ $cKP$ hierarchy. This construction is useful to describe a bi-Poisson structure of the $cKP$ hierarchy and in fact involves variables which almost abelianize the second Poisson structure. There is another formulation of the same class of models which is based on an alternative expression for the Lax operator given in (1.1):

$$L_{m-r+n,n} = D^r + \sum_{l=0}^{r-2} u_l D^l + \sum_{i=1}^{n} \Phi_i D^{-1} \Psi_i \quad (1.4)$$

A connection between variables from (1.1) and (1.4) takes a form of complicated Miura-like transformations (see Section 3 and [12]), but for $\Psi_i$'s the link is very simple since $\Psi_i$'s turn out to be elements of the $n$-dimensional kernel of $L^{(n)}$. Furthermore $\Phi_i, \Psi_i$ abelianize the first bracket structure. Their property of being eigenfunctions of $L_{r+n,n}$ makes the expression in (1.4) a proper framework for discussing the Lax equation in the setting of the $SL(m, n)$ $cKP$ hierarchy.
To go further and find $\tau$-functions corresponding to the $SL(m, n)$ cKP hierarchy we need a notion of the Darboux-Bäcklund (DB) transformations which we introduce in Section 4. With this notion we are able to derive simple Wronskian expressions for the $\tau$-functions. We also reveal a connection between on one side (constrained) Toda discrete models and on another cKP models endowed with DB symmetry structures. The picture which emerges is of the continuous integrable model with canonical symmetry mimicking the lattice shift of the corresponding Toda lattice. This scenario has been appearing in various forms in literature [22, 56, 43, 15, 14, 30, 49, 7, 8].

The observations of Section 4 provide a right scene for establishing the link to two-matrix model. The two-matrix model is rewritten in Section 5 as a linear discrete system with additional constraint; a string equation. Using the string equation we are able to cast the lattice equations into the form of a constrained Toda equation which can be represented via construction of Section 4 to the class of cKP models.

We have included two appendices on Schur polynomials and Wronskians, which may clarify few technical points.

1.3 Acknowledgements

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2 Pseudo-differential Operators, BA Functions and Hirota Equations

2.1 Lax Equation and Eigenfunction for the Lax Operator

We begin with the Sato theory of the KP hierarchy. Let \( \{t_j\} \) denote a set of independent variables with \( t_1 \equiv x \). The formulation of the KP hierarchy is based on the Lax equations

\[
\frac{\partial L}{\partial t_n} = [B_n, L] \quad n = 1, 2, \ldots
\]  

(2.1)

describing isospectral deformations of the pseudo-differential operator:

\[
L = \partial + \sum_{i=0}^{\infty} u_i \partial^{-i-1}
\]

(2.2)

Here \( u_n \) are the functions of \( \{t_j\} \), and \( B_n = (L^n)_+ \) is the truncation to the differential part of \( L^n \). A negative power part of the differential operator \( L^n \) will be written as \( (L^n)_- \). The Lax equations (2.1) have the zero-curvature representation taking the Zakharov-Shabat form:

\[
\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m] = 0, \quad n, m = 1, 2, \ldots
\]

(2.3)

It is often convenient to view (2.2) and (2.3) as integrability conditions of the linear system:

\[
L\psi(t, \lambda) = \lambda \psi(t, \lambda)
\]

(2.4)

\[
\frac{\partial \psi(t, \lambda)}{\partial t_n} = B_n \psi(t, \lambda)
\]

(2.5)

In connection with the above linear eigenvalue problem one introduces two class of functions.

**Definition.** A function \( \Phi \) is called eigenfunction for the Lax operator \( L \) satisfying Sato’s flow equation (2.1) if its flows are given by expression:

\[
\frac{\partial \Phi}{\partial t_m} = (L^m)_+ \Phi
\]

(2.6)

for the infinite many times \( t_m \).

A special role is played by an eigenfunction function, which also satisfies the spectral equation (2.4).

**Definition.** A function \( \psi(t, \lambda) \) satisfying relations (2.4) and (2.5) is called a Baker-Akhiezer function.

2.2 Conservation Laws of KP Hierarchy, and Tau-Function Representation of the Baker-Akhiezer Function

Note first that (2.2) can be inverted providing an expansion of \( D \) in powers of Lax operator:

\[
D = L + \sum_{i=1}^{\infty} \sigma^{(1)}_i L^{-i}
\]

(2.7)
Similar inversion relation can also be written for the differential operator $B_m$

$$B_m = L^m + \sum_{i=1}^{\infty} \sigma_i^{(m)} L^{-i}$$

(2.8)

Expand now $\ln \psi(t, \lambda)$ as

$$\ln \psi(t, \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n + \sum_{i=1}^{\infty} \psi_i \lambda^{-i},$$

(2.9)

Because of the linear equation (2.4) for $L$, one finds

$$\chi(\lambda) \equiv \partial \psi(t, \lambda)/\psi(t, \lambda) = \lambda + \sum_{i=1}^{\infty} \sigma_i^{(1)} \lambda^{-i}$$

(2.10)

where we have defined for convenience an auxiliary quantity $\chi = \partial \ln \psi$. Hence by comparing (2.7) and (2.9) we get

$$\partial \psi_i = \sigma_i^{(1)}.$$

The conservation laws for the KP hierarchy in form we will discuss here follow from the recurrence relation for the differential operators $B_m$. From (2.8) it follows namely

$$B_{m+1} = \left( D - \sum_{i=1}^{\infty} \sigma_i^{(1)} L^{-i} \right) L^m + \sum_{i=1}^{\infty} \sigma_i^{(m+1)} L^{-i}$$

$$= \left( D - \sum_{i=1}^{\infty} \sigma_i^{(1)} L^{-i} \right) \left( B_m - \sum_{j=1}^{\infty} \sigma_j^{(m)} L^{-j} \right) + \sum_{i=1}^{\infty} \sigma_i^{(m+1)} L^{-i}$$

(2.11)

$$= DB_m - \sum_{j=1}^{m} \sigma_j^{(1)} B_{m-j} - \sigma_1^{(m)} + O(D^{-1})$$

Since the left hand side is a pure differential operator it is obvious that terms with $D^{-k}$ with $k > 0$ cancel and consequently the only non-zero contribution comes from the first three terms on the right hand side. The following identity:

$$[D - \chi(\lambda)] B(\lambda) = \epsilon(\lambda)$$

(2.12)

with

$$\epsilon(\lambda) \equiv -1 + \sum_{m=1}^{\infty} \lambda^{-m-1} \sigma_1^{(m)} ; \quad B(\lambda) \equiv \sum_{m=0}^{\infty} \lambda^{-m-1} B_m$$

(2.13)

presents a compact way of expressing the recurrence relation of equation (2.11). We now follow reference [31] (see also [79, 21]) and apply (2.12) on the BA function $\psi(z)$ with expansion parameter being $z$ instead of $\lambda$:

$$\sum_{m=0}^{\infty} \lambda^{-m-1} \partial B_m \psi(z) - \chi(\lambda) \sum_{m=0}^{\infty} \lambda^{-m-1} B_m \psi(z) = \epsilon(\lambda) \psi(z)$$

(2.14)

or

$$\sum_{m=0}^{\infty} \lambda^{-m-1} (\partial B_m \psi(z)) \cdot \psi^{-1}(z) - \chi(\lambda) \sum_{m=0}^{\infty} \lambda^{-m-1} B_m \psi(z) \cdot \psi^{-1}(z) = \epsilon(\lambda)$$

(2.15)
Using that $B_m \psi(z) \cdot \psi^{-1}(z) = z^m + \sigma_1^{(m)} z^{-1} + \ldots$ we obtain

$$
\epsilon(\lambda) = \sum_{m=0}^{\infty} \lambda^{-m-1} \partial \left( B_m \psi(z) \cdot \psi^{-1}(z) \right) - \frac{\chi(z) - \chi(\lambda)}{z - \lambda} \tag{2.16}
$$

$$
\chi(\lambda) = \sum_{m=0}^{\infty} \lambda^{-m-1} \left( \sigma_1^{(m)} z^{-1} + \ldots \right) + \chi(z) \sum_{m=0}^{\infty} \lambda^{-m-1} \left( \sigma_1^{(m)} z^{-1} + \ldots \right)
$$

which upon taking the limit $z \to \lambda$ yields

$$
\frac{\partial \chi(\lambda)}{\partial \lambda} + \epsilon(\lambda) = \partial \sum_{m=0}^{\infty} \lambda^{-m-1} \left( B_m \psi(\lambda) \cdot \psi^{-1}(\lambda) \right) \tag{2.17}
$$

Hence modes of the left hand side of (2.17): $\sigma_1^{(l)} - l \sigma_1^{(1)}$, $l = 1, \ldots$ are conserved densities of the KP hierarchy. From

$$
\partial \left( B_m \psi \cdot \psi^{-1} \right) = \partial \left( \frac{\partial \psi}{\partial t_m} \cdot \psi^{-1} \right) = \partial \frac{\partial \psi}{\partial t_m} \left( \partial \psi \cdot \psi^{-1} \right) = \frac{\partial \chi}{\partial t_m} \tag{2.18}
$$

we find that also $\partial \sigma_1^{(l)} / \partial t_m$ define conserved densities of the KP hierarchy and moreover comparing (2.17) and (2.18) we get a relation between these conserved quantities:

$$
\sum_{j=1}^{m-1} \frac{\partial \sigma_1^{(m-j)}}{\partial t_j} = \sigma_1^{(m)} - m \sigma_m^{(1)} \tag{2.19}
$$

From an obvious identity

$$
0 = \left( \frac{\partial^2 \psi}{\partial t_k \partial t_j} - \frac{\partial^2 \psi}{\partial t_j \partial t_k} \right) \psi^{-1} = \left( \frac{\partial \sigma_1^{(j)}}{\partial t_k} - \frac{\partial \sigma_1^{(k)}}{\partial t_j} \right) \lambda^{-1} + \ldots \tag{2.20}
$$

we see that we can write $\sigma_1^{(m)} = \partial f / \partial t_m$ with some arbitrary function $f$, which we will choose to write as $f = -\partial_x \ln \tau$. Hence in this new notation $\sigma_1^{(m)} = -\partial_x \partial_m \ln \tau$ and clearly $\partial_n \sigma_1^{(m)} = \partial_m \sigma_1^{(n)}$. Furthermore we can now rewrite (2.19) as

$$
\frac{\partial}{\partial t_m} \partial_x \ln \tau = -m \sigma_1^{(1)} - \sum_{k=1}^{m-1} \frac{\partial \sigma_1^{(m-k)}}{\partial t_k} \tag{2.21}
$$

**Remark.** From (2.7) $\sigma_1^{(1)} = -\text{Res}_\partial(L)$ and therefore $u_0 = \partial_2^2 \ln \tau$. Recall that for the pseudo-differential operator $A = a_1 D^{-1} + \ldots$ we have $\text{Res}_\partial A = a_1$.

On basis of identity (A.3) (or (A.6)) for the Schur polynomials $p_n$ shown in Appendix A one verifies that (2.21) has a solution given by

$$
\sigma_n^{(1)} = \partial_x p_n(-\tilde{\partial}) \ln \tau \quad ; \quad n \geq 1 \tag{2.22}
$$

where

$$
\tilde{\partial} \equiv \left( \frac{\partial}{\partial t_1}, \frac{\partial}{2 \partial t_2}, \frac{\partial}{3 \partial t_3}, \ldots \right) \tag{2.23}
$$
Comparing with (2.9) and recalling that \(\partial \psi_i = \sigma_i^{(1)}\) we see that (up to an integration constant)

\[
\ln \psi(t, \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n + \sum_{i=1}^{\infty} \lambda^{-i} p_i (-\bar{\partial}) \ln \tau
\]

which yields the main result of this subsection:

\[
\psi(t, \lambda) = e^{\xi(t, \lambda)} \exp \left( -\sum_{i=1}^{\infty} \lambda^{-i} \partial \right) \tau(t) = e^{\xi(t, \lambda)} \sum_{n=0}^{\infty} \frac{p_n (-\bar{\partial}) \tau(t)}{\tau(t)} \lambda^{-n}
\]

where

\[
\xi(t, \lambda) \equiv \sum_{n=1}^{\infty} t_n \lambda^n
\]

### 2.3 Dressing Operator

It is possible to reproduce the Lax operator (2.2) through the dressing formula

\[
L = WDW^{-1}
\]

where the dressing operator \(W\) is the pseudo-differential operator:

\[
W = 1 + \sum_{i=1}^{\infty} w_i(t) D^{-i}
\]

satisfying the Sato equations:

\[
\frac{\partial W}{\partial t_n} = B_n W - WD^n = -(L^n)_- W
\]

\[
LW = WD.
\]

From the dressing formula it follows immediately that the BA function can be rewritten as

\[
\psi(t, \lambda) = W \exp \left( \sum_{n=1}^{\infty} t_n \lambda^n \right) = \hat{w}(t, \lambda) e^{\xi(t, \lambda)}
\]

where

\[
\hat{w}(t, \lambda) = \frac{\tau(t_i - 1/i\lambda_i)}{\tau(t_i)}
\]

as can be found from (2.25).

In a similar way an adjoint BA function \(\psi^*(t, \lambda)\) can be constructed as follows:

\[
\psi^*(t, \lambda) = (W^{-1})^* e^{-\xi(t, \lambda)} = \hat{w}^*(t, \lambda) e^{-\xi(t, \lambda)}
\]

\[
\hat{w}^*(t, \lambda) = \frac{\tau(t_i + 1/i\lambda_i)}{\tau(t_i)}
\]
and defines a linear system:

\[ L^* \psi^*(t, \lambda) = \lambda \psi^*(t, \lambda) \]  \hspace{1cm} (2.35)
\[ \frac{\partial \psi^*(t, \lambda)}{\partial t_n} = -B_n^* \psi^*(t, \lambda) \]  \hspace{1cm} (2.36)

where \( L^* \) is adjoint of \( L \) and \( B_n^* \) is a differential part of \((L^*)^n\). Recall that the adjoint operator \( A^* \) of \( A = a D^k + \ldots \) is given by \( A^* = (-1)^k D^k a + \ldots \).

### 2.4 Hirota Equations

As seen from (2.31) and (2.25) we can rewrite the dressing operator as

\[ W = \sum_{i=0}^{\infty} p_i (-\tilde{\partial}) \tau(t) D^{-i} \]  \hspace{1cm} (2.37)

leading to

\[ L^n = W D^n W^{-1} = \sum_{i,j}^{i+j=n+1} p_i (-\tilde{\partial}) \tau(t) \cdot p_j (\tilde{\partial}) \tau(t) \frac{D^{-i}}{\tau^2(t)} \]  \hspace{1cm} (2.38)

Let us insert the above expressions into the Sato equation (2.29). Noticing that

\[ -(L^n)_- = - \sum_{i,j \geq 0}^{i+j=n+1} p_i (-\tilde{\partial}) \tau(t) \cdot p_j (\tilde{\partial}) \tau(t) D^{-i} + \ldots \]  \hspace{1cm} (2.39)

and taking the residue on both sides of (2.29) we get:

\[ \tau \frac{\partial^2 \tau}{\partial t_n \partial t_1} - \frac{\partial \tau}{\partial t_n} \frac{\partial \tau}{\partial t_1} - \sum_{i,j \geq 0}^{i+j=n+1} p_i (-\tilde{\partial}) \tau(t) \cdot p_j (\tilde{\partial}) \tau(t) = 0 \]  \hspace{1cm} (2.40)

or

\[ \left( \frac{1}{2} D_1 D_n - p_{n+1}(\tilde{D}) \right) \tau \cdot \tau = 0 \]  \hspace{1cm} (2.41)

where we used Hirota’s operators defined by

\[ D_j^m a \cdot b = \frac{\partial^m}{\partial s_j^m} a(t_j + s_j) b(t_j - s_j) \bigg|_{s_j=0} \]  \hspace{1cm} (2.42)

The first non-trivial equation obtained from (2.41) for \( n = 3 \) (by dropping odd polynomials in \( D \), which do not make any non-zero contribution) is the KP equation:

\[ (D_1^4 + 3 D_2^2 - 4 D_1 D_3) \tau \cdot \tau = 0 \]  \hspace{1cm} (2.43)

which in components takes the usual form:

\[ \frac{\partial}{\partial x} \left( \frac{\partial u_0}{\partial t_3} - \frac{1}{4} \frac{\partial^2 u_0}{\partial x^3} - 3 u_0 \frac{\partial u_0}{\partial x} \right) - \frac{3}{4} \frac{\partial^2 u_0}{\partial t_2^2} = 0 \]  \hspace{1cm} (2.44)
with \( u_0 = \text{Res}_\theta L = \partial^2 \ln \tau \).

**Remark.** Eq. (2.41) is contained in the Hirota differential equations for the KP hierarchy taking the following expression:

\[
\sum_{j=0}^{\infty} p_j (2y) p_{j+1} (\tilde{D}_t) \exp \left\{ \sum_{l=1}^{\infty} y_l D_{t_1} \right\} \tau (t) \cdot \tau (t) = 0
\]

with \( y = (y_1, y_2, \ldots) \) being an extra multi-variable.

Indeed, it is easy to see that (2.41) can be obtained from (2.45) as a coefficient of the term linear in \( y_n \). It can be shown that the Hirota differential equations (2.45) can be obtained from the bilinear identity for the \( \tau \) functions:

\[
\int \tau \left( t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, \ldots \right) \tau \left( t'_1 + \frac{1}{\lambda}, t'_2 + \frac{1}{2\lambda^2}, \ldots \right) e^{\xi(t-t',\lambda)} d\lambda = 0
\]

by expanding arguments of both \( \tau \)-functions in (2.46) \( [24, 39] \). Hence (2.46) carries a complete information about the KP hierarchy. The KP formalism based on (2.46) instead on the pseudo-differential Lax operators can be analyzed in terms of the free fermions Fock space. These beautiful results are beyond the current discussion (see for details \([24, 39]\)).

### 2.5 Hamiltonian Densities and the \( \tau \)-Function

Let us now consider the product \( s \equiv \psi \psi^* \) of the BA function and its adjoint. We will now show that \( s \) generates infinitely many commuting KP flows. First as in \([45]\) note from (2.32) and (2.34) that

\[
\psi \psi^* = \frac{\tau (t_1 - \frac{1}{i\lambda}) \tau (t_1 + \frac{1}{i\lambda})}{\tau^2 (t_1)} = \frac{1}{\tau^2 (t_1)} \exp \left( \sum_{k=1}^{\infty} \frac{\partial}{k \lambda^k \partial \epsilon_k} \right) \tau (t + \epsilon) \tau (t - \epsilon) \bigg|_{\epsilon=0}
\]

\[
= \frac{1}{\tau^2 (t_1)} \sum_{k=0}^{\infty} p_k (\tilde{D}) \frac{\tau \cdot \tau}{\lambda^k} = \frac{1}{2\tau^2 (t)} \sum_{k=0}^{\infty} \frac{D_1 D_{k-1} \tau \cdot \tau}{\lambda^k}
\]

where in the last equation we have used the Hirota equation (2.41). This equation suggests to consider the Laurent coefficients of \( s \) defined through

\[
s = \psi \psi^* = \sum_{n=0}^{\infty} s_n \lambda^{-n} \quad \rightarrow \quad s_n = \frac{D_1 D_{n-1} \tau \cdot \tau}{2\tau^2}
\]

We now recall the following technical lemma \([26]\):

**Lemma.** Let \( P, Q \) be pseudo-differential operators. We have

\[
\text{Res}_\lambda \left[ \left( P e^{x\lambda} \right) \left( Q e^{-x\lambda} \right) \right] = \text{Res}_\theta \left( PQ \right)
\]

as follows from a direct verification.

Taking \( P = WD^n \) and \( Q = W^{-1} \) we obtain from the Lemma that

\[
s_{n+1} = \text{Res}_\lambda \left[ \lambda^n \psi \psi^* \right] = \text{Res}_\theta \left( L^n \right)
\]
From (2.47) we find

\[ s_n = \text{Res}_\partial \left( L^{n-1} \right) = \frac{1}{2\tau^2(t)} D_1 D_{n-1} \tau \cdot \tau = \partial_x \partial_{n-1} \ln \tau \]  

(2.51)

Since \( u_0 = \partial^2 \ln \tau \) we can rewrite \( \partial_x s_n \) as

\[ \partial_x s_n = \frac{\partial u_0}{\partial t_{n-1}} \]  

(2.52)

Note that in notation of the previous subsections the above equation shows that the conserved densities of the KP hierarchy \( \sigma_1^{(m)} \) are equal to Hamiltonian densities \( \mathcal{H}_m = \text{Res}_\partial \left( L^m \right) \) of the KP hierarchy, which generate commuting flows through relation \( \partial_x \mathcal{H}_n = \left( \partial / \partial t_n \right) u_0 \). Accordingly the KP hierarchy has an infinite set of commuting symmetries associated with infinite set of the conserved Hamiltonians. This translates into commutativity of Hamiltonians on the level of the Poisson brackets indicating integrability of the system. These symmetries are called *isospectral* symmetries as they preserve the spectrum of the underlying linear problem.
3 Symmetry Reduction of the Integrable Systems. Five Constructions of the cKP models.

3.1 Symmetry of Nonlinear Evolution Equation

We will discuss notion of symmetry of partial differential equation on the basis of an evolution equation:

\[ \frac{\partial u}{\partial t} = K[u(t)] \]  

(3.1)

with some nonlinear operator \( K \). An evolution equation \( \frac{\partial u}{\partial \sigma} = G[u(t)] \) is called a symmetry of (3.1) if the Fréchet derivative of \( K \) at the point \( u \) in the direction of \( G \)

\[ K'[u(t)](G) \equiv \left( \frac{\partial}{\partial \epsilon} K[x, u(t) + \epsilon G, u' + \epsilon G_x, \ldots] \right|_{\epsilon=0} \]  

(3.2)

\[ G_x \equiv \frac{\partial G}{\partial x} + \sum_{k \leq 1} \frac{\partial G}{\partial u^{(k)}} u^{(k+1)} \]  

(3.3)

satisfies

\[ K'[u(t)](G) \equiv \frac{\partial G}{\partial t} + G'[u(t)](K) \]  

(3.4)

Hence a condition that \( G \) is a symmetry of (3.1) is equivalent to having \( v \equiv G[u(t)] \) satisfy the linearized version of (3.1) with a background \( u(t) \) i.e. \( dv/dt = K'[u(t)](v) \). In other words \( u + \epsilon v \) satisfies (3.1) for all solutions \( u \) of (3.1) for an arbitrarily small \( \epsilon \). Therefore the linearized version of the nonlinear evolution equation contains all the pertinent information about its symmetries.

We can also rephrase condition (3.4) as commutativity \( [D_t, D_\sigma] = 0 \) of two differentiations associated with \( t \) and \( \sigma \)

\[ D_t \equiv \frac{\partial}{\partial t} + K \frac{\partial}{\partial u} + K_x \frac{\partial}{\partial u_x} + \ldots \]  

(3.5)

\[ D_\sigma \equiv \frac{\partial}{\partial \sigma} + G \frac{\partial}{\partial u} + G_x \frac{\partial}{\partial u_x} + \ldots \]  

(3.6)

Examples:

Consider the KdV equation: \( u_t = 6uu' - u'' \) in terms of \( \omega' = u \):

\[ \omega_t = K[\omega] = 3\omega'^2 - \omega'' \]  

(3.7)

Condition (3.4) gives

\[ dv/dt = K'[\omega(t)](v) = 6\omega'v' - v'' \]  

(3.8)

Consider Galilean symmetry of KdV:

\[ \omega \rightarrow \tilde{\omega} = \omega(x + 3t\epsilon) + \frac{1}{2} x \epsilon \]  

(3.9)

which can be cast in the form of “evolution” equation:

\[ v = G = \frac{d\tilde{\omega}}{d\epsilon} \bigg|_{\epsilon=0} = 3t\omega' + \frac{1}{2} x \]  

(3.10)
Taking into account that \( v_x = 3t\omega'' + \frac{1}{2}, \) \( v_{xxx} = 3t\omega^{IV} \) and also \( dv/dt = 3\omega' + 3t(3\omega'' - \omega^{IV})' \) we can verify that the Galilean transformation is a “non-isospectral” symmetry of the KdV equation.

For the KP eq. (2.44) its linearized version takes the form:

\[
dv/dt = \frac{1}{4} \frac{\partial^3 v}{\partial x^3} + 3 \frac{\partial (u_0 v)}{\partial x} + \frac{3}{4} \partial^{-1} \frac{\partial^2 v}{\partial t^2}
\] (3.11)

### 3.2 Symmetry Reduction and the cKP Hierarchy

We will first show that the quantity \( \partial_x (\Phi \Psi) \) is a symmetry of the KP equation with \( \Phi \) and \( \Psi \) are (adjoint) eigenfunctions of \( L \) i.e. they satisfy (2.3) and (2.36):

\[
\frac{\partial}{\partial t_m} \Phi = L^m \Phi ; \quad \frac{\partial}{\partial t_m} \Psi = -L^m* \Psi
\] (3.12)

where \( L^m* \) is the adjoint operator of \( L^m \).

Let us particularly stress that the above eigenfunctions do not need to be Baker-Akhiezer eigenfunctions of \( L \) and \( \partial_x (\Phi \Psi) \) in general differs from quantity \( s_x \equiv \partial_x \psi \psi^* \), which, as we have seen in the previous section, generates an infinite set of commuting symmetries associated with infinite set of the conserved Hamiltonians of the KP hierarchy.

For \( n = 2 \) equation (3.12) gives:

\[
\frac{\partial \Phi}{\partial t_2} = \frac{\partial^3 \Phi}{\partial x^3} + 2u_0 \Phi ; \quad \frac{\partial \Psi}{\partial t_2} = -\frac{\partial^2 \Psi}{\partial x^2} - 2u_0 \Psi
\] (3.13)

while for \( n = 3 \) we obtain from (3.12):

\[
\frac{\partial \Phi}{\partial t_3} = \frac{\partial^3 \Phi}{\partial x^3} + 3u_0 \frac{\partial \Phi}{\partial x} + \frac{3}{2} u'_0 \Phi + \frac{3}{2} \left( \partial^{-1}_x \frac{\partial u_0}{\partial t_2} \right) \Phi
\]

\[
\frac{\partial \Psi}{\partial t_3} = \frac{\partial^3 \Psi}{\partial x^3} + 3u_0 \frac{\partial \Psi}{\partial x} + \frac{3}{2} u'_0 \Psi - \frac{3}{2} \left( \partial^{-1}_x \frac{\partial u_0}{\partial t_2} \right) \Psi
\] (3.14)

Using equations (3.13) and (3.14) one can show as in [13, 18] that \( S \equiv \Phi \Psi \) satisfies the evolution equation

\[
\frac{\partial S}{\partial t_3} = \frac{1}{4} \frac{\partial^3 S}{\partial x^3} + 3u_0 \frac{\partial S}{\partial x} + \frac{3}{4} \partial^{-1} \frac{\partial^2 S_x}{\partial t^2}
\] (3.15)

and therefore the quantity \( S_x \equiv \partial_x S \) satisfies a linearized version of the KP equation (3.11). Hence according to the above definition \( S_x \) is a symmetry of the KP equation.

Recall now that in components the Lax equation (2.1) takes the following form of partial differential equations

\[
\partial_ r u_n = F_{n,r} (u_0, u_1, \ldots, u_{r+n-1}) \quad n = 0, \ldots
\] (3.16)

with differential polynomials \( F_{n,r} \) in \( u_0, u_1, \ldots, u_{r+n-1} \). In particular we have a hierarchy of flow equations:

\[
\partial_ r u_0 \equiv F_ r (u_0, u_1, \ldots, u_{r-1})
\] (3.17)
Equations (3.17) can be viewed as symmetries of the KP equation according to our above definition.

We will first introduce a constrained KP (cKP) hierarchy by imposing a constraint on symmetry represented by the evolution equations in (3.17).

We will discuss three (very related) approaches to define the cKP hierarchy and show their equivalence.

**Symmetry constraint, cKP hierarchy, Definition I**

This approach defines the constrained KP hierarchy by imposing the symmetry constraint on flows in (3.17) in terms of the eigenfunction and its adjoint entering the linear problems (2.5) and (2.36):

\[ F_r (u_0, u_1, \ldots, u_{r-1}) = \partial_x (\Phi \Psi) \]  

(3.18)

Hence for each \( r = 1, 2, \ldots \) we define the different cKP hierarchies by imposing equality of the flows/symmetries \( F_r \) to the symmetry \( \partial_x (\Phi \Psi) \) of the KP equation. The constraint allows us to eliminate the Lax coefficients \( u_{k \geq r-1} \) in terms of eigenfunctions \( \Phi \) and \( \Psi \) and \( u_{k < r-1} \).

The next two approaches will allow us to write down an explicit Lax representation of the cKP hierarchy.

**Symmetry constraint, cKP hierarchy, Definition II**

Consider the Lax operator \( Q = D + \sum_{i=0}^{\infty} u_i D^{-i-1} \) satisfying the flow equations as in (2.1) i.e.

\[ \frac{\partial Q}{\partial t_k} = [B_k, Q] \quad ; \quad B_k \equiv (Q^k)_+ \quad k = 1, 2, \ldots \]  

(3.19)

Let furthermore (like in (3.12)) \( \Phi \) and \( \Psi \) be, respectively, eigenfunction and adjoint eigenfunction of \( Q \), meaning that for \( k = 1, 2, \ldots \) it holds

\[ \frac{\partial \Phi}{\partial t_k} = B_k \Phi \quad ; \quad \frac{\partial \Psi}{\partial t_k} = -B_k^* \Psi \]  

(3.20)

where \( B_k^* \) is an adjoint operator of \( B_k \). We now impose the constraint on the KP hierarchy by requiring that the Lax operator \( Q \) satisfies a condition [20]:

\[ Q^r = B_r + \Phi D^{-1} \Psi \quad ; \quad B_r \equiv (Q^r)_+ \]  

(3.21)

for \( r \) being a fixed positive integer.

Let us first verify that **Definition II** implies the symmetry constraint of **Definition I**. Let us namely impose constraint (3.21) and notice that from (3.19) we get

\[ \frac{\partial u_0}{\partial t_r} = \text{Res}_0 (\partial Q/\partial t_r) = \text{Res}_0 [B_r, Q] = \text{Res}_0 [Q^r - \Phi D^{-1} \Psi, Q] \]

\[ = \text{Res}_0 [-\Phi D^{-1} \Psi, Q] = \partial_x (\Phi \Psi) \]

(3.22)

This clearly shows that the **Definition II** realizes directly in the Lax setting the symmetry constraint of **Definition I**.

**Lemma.** The time evolution of the pseudo-differential operator \( \Phi D^{-1} \Psi \) is given by:

\[ \frac{\partial}{\partial t_k} \Phi D^{-1} \Psi = [B_k, \Phi D^{-1} \Psi]_- \]  

(3.23)
The proof is a consequence of following technical observations based on (3.20):

\[
\left( B_k \Phi D^{-1} \Psi \right)_- = (B_k \Phi)_0 D^{-1} \Psi = \frac{\partial \Phi}{\partial t_k} D^{-1} \Psi
\]

(3.24)

and

\[- \left( \Phi D^{-1} \Psi B_k \right)_- = - \Phi D^{-1} (B_k^* \Psi)_0 = \Phi D^{-1} \frac{\partial \Psi}{\partial t_k} \]

(3.25)

What we learn from the above lemma is that if we define the Lax operator \( L \) such that its purely pseudo-differential part is \( L_- = \Phi D^{-1} \Psi \) then \( L_- \) satisfies automatically the KP flow equations: \( \partial L_- / \partial t_k = [B_k, L_-] \). Note that in the last equation we used that \([B_k, L]_- = [B_k, L_-]\). These observations lead us to the next definition of cKP hierarchy.

**Symmetry constraint, cKP hierarchy, Definition III**

Here we work with Lax operator of the cKP hierarchy defined as

\[ L = D^r + \sum_{l=0}^{r-2} U_l D^l + \Phi D^{-1} \Psi \]

(3.26)

\[ \frac{\partial L}{\partial t_k} = \left[ \left( L^{k/r} \right)_+, L \right] \]

(3.27)

We first address an issue of equivalence between Definition II and Definition III, which is easy to establish when making a connection \( Q^r \equiv L \). It is then easy to see that the two flows given below are equivalent (\( B_k = \left( Q^k \right)_+ = \left( L^{k/r} \right)_+ \)):

\[ \partial Q / \partial t_k = \left[ \left( Q^k \right)_+, Q \right] \leftrightarrow \partial L / \partial t_k = \left[ \left( Q^k \right)_+, L \right] = \left[ \left( L^{k/r} \right)_+, L \right] \]

(3.28)

We have already seen in (3.23) that the pseudo-differential part of \( L \) from (3.26) evolves according to the Sato’s formalism. Let us now investigate evolution of the differential part \( B_r = (Q^r)_+ = (L)_+ \) of \( L \):

\[ \partial B_r / \partial t_k = [B_k, \ L]_+ = [B_k, \ B_r] + [B_k, \ \Phi D^{-1} \Psi]_+ \]

(3.29)

which after using Zakharov-Shabat equation (2.3) becomes:

\[ \partial B_k / \partial t_r = [B_k, \ \Phi D^{-1} \Psi]_+ \]

(3.30)

valid for all \( k = 1, 2, \ldots \). Equation (3.30) describes how the cKP constraint condition is being imposed on the flows of KP coefficients in \( B_k \). Let us study consequences of (3.30) in some simple cases. The first nontrivial case occurs for \( k = 2 \) with \( B_2 = (Q^2)_+ = D^2 + 2u_0 = D^2 + U_0 \). From (3.30) it follows:

\[ 2 \partial u_0 / \partial t_r = [D^2, \ \Phi D^{-1} \Psi]_+ = 2 \partial_x(\Phi \Psi) \]

(3.31)

which reproduces the flow equation being basis for the Definition I of the cKP hierarchy, establishing henceforth equivalence between Definition I and Definition III.
For \( k = 3 \) we have \( B_3 = (Q^3)_+ = D^3 + 3u_0D + 3u_1 + 3\partial_xu_0 = D^3 + U_0D + U_1 \). From (3.30) it follows this time that:

\[
\partial(3u_0D + 3u_1 + 3u'_0)/\partial t_r = [D^3, \Phi D^{-1}\Psi]_+ = 3\Phi''\Psi + 3\Phi'\Psi' + 3(\Phi\Psi)'D
\]  

(3.32)

or

\[
\partial u_1/\partial t_r = -(\Phi\Psi)'
\]  

(3.33)

Consider the Lax operator \( L \) having the form as in (3.26) and satisfying (3.27). One can ask whether \( \Phi \) (\( \Psi \)) are automatically the (adjoint) eigenfunctions. The following Lemma answers this question affirmatively.

**Lemma.** Let \( L = L_+ + \Phi D^{-1}\Psi \), with \( r \) being the order of \( L_+ \). Then the Lax equations of motion

\[
\frac{\partial}{\partial t_k} L = [B_k, L] 
\]

imply:

\[
\left[ \left( \frac{\partial}{\partial t_k} \Phi - (B_k\Phi)_0 \right) D^{-1}\Psi + \Phi D^{-1} \left( \frac{\partial}{\partial t_k} \Psi + (B_k^*\Psi)_0 \right) \right] = 0 
\]  

(3.34)

where \( B_k = (L^{k/r})_+ \), \( B_k^* = (L^{k/r})_+^* \)

**Remark.** Eq. (3.34) is equivalent to the infinite set of equations:

\[
\left[ \left( \frac{\partial}{\partial t_k} \Phi - (B_k\Phi)_0 \right) \partial^l_x\Phi + \Phi \partial^l_x \left( \frac{\partial}{\partial t_k} \Psi + (B_k^*\Psi)_0 \right) \right] = 0, \quad l = 0, 1, 2, \ldots 
\]

(3.35)

An obvious solution of (3.35) seems to be:

\[
\frac{\partial}{\partial t_k} \Phi - (B_k\Phi)_0 = \frac{\partial c_r}{\partial t_k} \Phi \\
\frac{\partial}{\partial t_k} \Psi + (B_k^*\Psi)_0 = -\frac{\partial c_r}{\partial t_k} \Psi
\]

where \( c_r \) is \( x \)-independent. Thus, up to a \( x \)-independent phase transformation \( \Phi \rightarrow e^{c_r}\Phi \) and \( \Psi \rightarrow e^{-c_r}\Psi \), which does not change the Lax operator, \( \Phi \) (\( \Psi \)) are (adjoint) eigenfunctions.

**Examples:**

\( r = 1 \). The simplest case is when \( r = 1 \) for which we have:

\[
F_1 = \partial_xu_0 = \partial_x\Phi\Psi \rightarrow u_0 = \Phi\Psi + \text{const}
\]

(3.37)

Inserting this into (3.13) (with zero integration constant) we get

\[
\frac{\partial \Phi}{\partial t_2} = \frac{\partial^2 \Phi}{\partial x^2} + 2\Phi^2\Psi; \quad \frac{\partial \Psi}{\partial t_2} = -\frac{\partial^2 \Psi}{\partial x^2} - 2\Phi^2
\]

(3.38)

in which we recognize the Nonlinear Schrödinger (NLS) equations being the first (non-trivial) flow of the AKNS hierarchy. Eqs. (3.38) are also results of the Sato equation (2.1) for two-boson hierarchy [1, 3] defined by the Lax operator \( Q = L = D + \Phi D^{-1}\Psi \). Expressions for
the coefficients \( u_{k \geq 0} \) of the standard expansion for \( Q \) are easy to find in terms of the Faà di Bruno polynomials [5, 6].

**r=2.** Here \( F_2 = \partial_x^2 u_0 + 2 \partial_x u_1 = \partial_x (\Phi \Psi) \) which allows to express \( u_{n \geq 1} \) by \( u_0, \Phi, \Psi \). Alternatively writing in the spirit of the Definition III the Lax operator as \( L = Q^2 = D^2 + 2 u_0 + \Phi D^{-1} \Psi \) we obtain from the Sato eq. (3.28) or (3.12):

\[
\begin{align*}
\frac{\partial \Phi}{\partial t_2} & = \frac{\partial^2 \Phi}{\partial x^2} + 2 u_0 \Phi \\
\frac{\partial \Psi}{\partial t_2} & = -\frac{\partial^2 \Psi}{\partial x^2} - 2 u_0 \Psi \\
\frac{\partial u_0}{\partial t_2} & = \partial_x (\Phi \Psi)
\end{align*}
\]

(3.39)

which agrees with first flow equation of the so-called Yajima-Oikawa hierarchy [61, 40].

**r=3.** Here \( F_3 = \partial_x^3 u_0 + 3 \partial_x^2 u_1 + 3 \partial_x u_2 + 6 u_0 \partial_x u_0 = \partial_x (\Phi \Psi) \) which allows to express \( u_{n \geq 2} \) by \( u_0, u_1, \Phi, \Psi \). Alternatively, writing in the spirit of the Definition III the Lax operator as \( L = Q^3 = D^3 + 3 u_0 D + 3 u_1 + 3 \partial_x u_0 + \Phi D^{-1} \Psi \) we obtain from the Sato eq. (3.28):

\[
\begin{align*}
\frac{\partial \Phi}{\partial t_3} & = \frac{\partial^3 \Phi}{\partial x^3} + 3 u_0 \frac{\partial \Phi}{\partial x} + 3 \left( u_1 + \frac{\partial u_0}{\partial x} \right) \Phi \\
\frac{\partial \Psi}{\partial t_3} & = \frac{\partial^3 \Psi}{\partial x^3} + 3 u_0 \frac{\partial \Psi}{\partial x} - 3 u_1 \Psi \\
\frac{\partial u_0}{\partial t_3} & = \partial_x (\Phi \Psi) \\
\frac{\partial u_1}{\partial t_3} & = -(\Phi \Psi')' \nonumber
\end{align*}
\]

(3.41)

in which one recognize the \( t_2 \) flow of the so-called Melnikov system [18]. The flow eqs. for \( t_3 = t_r \) follow from (3.31), (3.32) and (3.12) (or (3.14)):

In the next subsection we will learn how to view the above examples as special cases of \( SL(r+1,1) \) cKP hierarchy with \( r = 1, 2, 3 \) respectively.

**Remark.** In the above we have kept for simplicity discussion restricted to a single pair \( \Phi, \Psi \) of (adjoint) eigenfunctions. It is very simple to generalize all formulas to the case of \( n \) \( \Phi_n, \Psi_n \) (adjoint) eigenfunctions. Clearly in expressions (3.24) and (3.28) we need to make a substitution: \( \Phi D^{-1} \Psi \rightarrow \sum_{i=1}^{n} \Phi_i D^{-1} \Psi_i \). This is the structure, which will arise naturally in the next subsection.
3.3 cKP Hierarchy as SL(m,n)-type Lax Hierarchy and its Bi-Poisson Structure

Let us introduce a ratio of two differential operators (with \( m = r + n \) in notation of the previous subsection).

\[
\mathcal{L}_{m,n} \equiv \frac{L^{(m)}}{L^{(n)}} \quad n \leq m - 1
\]

where

\[
L^{(m)} = D^m + u_{m-1}D^{m-1} + \ldots + u_1D + u_0 \;
\quad L^{(n)} = D^n + \tilde{u}_{n-1}D^{n-1} + \ldots + \tilde{u}_1D + \tilde{u}_0 \quad (3.43)
\]

Let \( \{ \psi^{(m)}_i \}, \ i = 1, \ldots, m \) and \( \{ \psi^{(n)}_i \}, \ i = 1, \ldots, n \) be a basis for the kernels of \( L^{(m)} \) and \( L^{(n)} \), respectively, i.e. we have \( L^{(\alpha)}\psi^{(\alpha)}_i = 0 \), \( i = 1, \ldots, \alpha = m \) or \( n \). Alternatively, we can rewrite (3.43) as

\[
L^{(m)} = (D + v_m)(D + v_{m-1}) \ldots (D + v_1) \;
\quad L^{(n)} = (D + \tilde{v}_n)(D + \tilde{v}_{n-1}) \ldots (D + \tilde{v}_1) \quad (3.44)
\]

with

\[
v_i = \partial \left( \log \frac{W_{i-1}[\psi^{(m)}_1, \ldots, \psi^{(m)}_{i-1}]}{W_i[\psi^{(m)}_1, \ldots, \psi^{(m)}_i]} \right) \quad , \quad W_0 = 1 \quad (3.45)
\]

and similar expressions for \( \tilde{v}_i \).

Taking the above into consideration we can rewrite (3.42) as \[12, 62\]

\[
\mathcal{L}_{m,n} = \prod_{j=m}^{1} (D + v_j) \prod_{l=1}^{n} (D + \tilde{v}_l)^{-1} \quad (3.46)
\]

(see also \[16, 25, 53\] for discussion of the pseudo-differential operators of the constrained KP hierarchy). We will now study the Poisson structures of the hierarchy defined by the Lax operators of the form given in (3.46). It is easy to prove the following Proposition \[62, 12\].

**Proposition.** The Poisson bracket relation:

\[
\{ v_i, v_j \}_{PB} = \delta_{ij} \delta'(x - y) \quad i, j = 1, \ldots, m
\]

\[
\{ \tilde{v}_r, \tilde{v}_s \}_{PB} = -\delta_{rs} \delta'(x - y) \quad r, s = 1, \ldots, n \quad (3.47)
\]

is equivalent to

\[
\{ \langle \mathcal{L}_{m,n} | X \rangle , \langle \mathcal{L}_{m,n} | Y \rangle \} _{PB} = \text{Tr}_A \left( (\mathcal{L}_{m,n}X)_+ \mathcal{L}_{m,n}Y - (X \mathcal{L}_{m,n})_+ Y \mathcal{L}_{m,n} \right) \quad (3.48)
\]

where the subscript \( PB \) stands for the Poisson bracket as defined by (3.47). Moreover the Adler trace \( \langle \mathcal{L}_{m,n} | X \rangle \) defines an invariant, non-degenerate bilinear form: \( \int dx \text{Res} (\mathcal{L}_{m,n}X) \) for the pseudo-differential operator \( \mathcal{L}_{m,n} \) and its dual \( X \) (see for instance \[3, 12\]).

We now introduce a Dirac constraint:

\[
\Psi_{m,n} = \sum_{j=1}^{m} v_j - \sum_{l=1}^{n} \tilde{v}_l = 0 \quad (3.49)
\]
which is second-class due to:

$$\{\Psi_{m,n},\Psi_{m,n}\}_{PB} = (m-n) \delta'(x-y) \tag{3.50}$$

Condition (3.49) expresses tracelessness of the underlying graded $SL(m,n)$ algebra. Corresponding Poisson algebra of diagonal part of the graded $SL(m,n)$ Kac-Moody algebra is obtained by the usual Dirac bracket calculation:

$$\{v_i, v_j\}_{DB} = \left(\delta_{ij} - \frac{1}{m-n}\right) \delta'(x-y) \quad i, j = 1, \ldots, m$$

$$\{\tilde{v}_r, \tilde{v}_s\}_{DB} = -\left(\delta_{rs} + \frac{1}{m-n}\right) \delta'(x-y) \quad r, s = 1, \ldots, n \tag{3.51}$$

Proposition. The Dirac bracket (3.51) takes the following form in the Lax operator representation:

$$\{\langle L_{m,n} \mid X \rangle, \langle L_{m,n} \mid Y \rangle\}_{DB} = \{\langle L_{m,n} \mid X \rangle, \langle L_{m,n} \mid Y \rangle\}_{PB} + \frac{1}{m-n} \int dx \text{Res} ([L_{m,n}, X]) \partial^{-1} \text{Res} ([L_{m,n}, Y])$$

The proof follows from evaluation of the extra term of the relevant Dirac bracket:

$$-\int \{\langle L_{m,n} \mid X \rangle, \Psi_{m,n}(z)\}_{PB} \{\Psi_{m,n}, \Psi_{m,n}\}_{DB}^{-1} \{\Psi_{m,n}, \langle L_{m,n} \mid Y \rangle\}_{DB} \tag{3.52}$$

One easily verifies (3.52) using (3.50):

$$\{\langle L_{m,n} \mid X \rangle, \Psi_{m,n}(z)\}_{PB} = -\{[\delta(x-z), L_{m,n}] \mid X\} = -\text{Res} ([X, L_{m,n}]) (z) \tag{3.53}$$

Formula (3.52) contains as special cases $n = m - 1$ corresponding to the constrained KP hierarchy with $L_+ = D$ and $n = 0$ corresponding to the $m$-KdV hierarchy (see e.g. [27]). For the intermediary cases $0 < n < m - 1$ this formula presents a compact expression for the bracket structure of the $SL(m,n)$ cKP hierarchy.

The Lax operator from (3.42) restricted to the constrained manifold defined by (3.49) can be parametrized as

$$L_{m,n} \equiv L_{m,n} \mid_{\Psi_{m,n}=0} = \prod_{l=m-1}^{n+1} (D + \tilde{c}_l) \prod_{l=n}^{1} (D + \tilde{c}_l + \tilde{v}_l) \left(D - \sum_{l=1}^{m-1} \tilde{c}_l\right) \prod_{l=1}^{n} (D + \tilde{v}_l)^{-1} \tag{3.54}$$

We can alternatively rewrite the expression (3.54) as

$$L_{m,n} = \sum_{l=1}^{n} \tilde{A}_l \prod_{i=l}^{n} (D + \tilde{v}_i)^{-1} + \sum_{l=0}^{m-n-2} \tilde{A}_{l+n+1} D^l + D^{m-n} \tag{3.55}$$

with the second bracket structure automatically given by the formula (3.52).
For the special case \(m - n = r = 1\) we have a canonical representation for variables \((v_1, \tilde{v}_1)\) with \(1 \leq i \leq m\), \(1 \leq l \leq m - 1\) in \(L_{m,m-1} = \prod_{j=m}^1 (D + v_j) \prod_{l=1}^{m-1} (D + \tilde{v}_l)^{-1}\) in terms of the pairs \((c_r, e_r)_{r=1}^{m-1}\), which are the “Darboux” canonical pairs for the second KP bracket satisfying \(\{c_i(x), e_i(y)\} = -\delta_{i l} \partial_x \delta(x - y)\) for \(i, l = 1, 2, \ldots, m - 1\). The representation is:

\[
\begin{align*}
\tilde{v}_l &= -e_l - \sum_{p=l}^{m-1} c_p \quad l = 1, 2, \ldots, m - 1 \quad (3.56) \\
v_i &= -e_{i-1} - \sum_{p=i}^{m-1} c_p = -c_i + \tilde{v}_i \quad i = 1, \ldots, m \quad (3.57)
\end{align*}
\]

Eq. (3.57) includes the special cases of \(v_1 = -\sum_{p=1}^{m-1} c_p\) and \(v_m = -e_1\). One checks easily that in this representation the Poisson algebra of \(\tilde{v}_1, v_i\) takes indeed the form of the bracket algebra of graded \(SL(m, m - 1)\) Kac-Moody algebra in a diagonal gauge:

\[
\begin{align*}
\{v_i, v_j\} &= (\delta_{ij} - 1) \delta'(x - y) \quad i, j = 1, \ldots, m \\
\{\tilde{v}_p, \tilde{v}_l\} &= - (\delta_{pl} + 1) \delta'(x - y) \quad p, l = 1, \ldots, m - 1 \\
\{v_i, \tilde{v}_l\} &= -\delta'(x - y)
\end{align*}
\]

We can interpret (3.54) as a superdeterminant of the graded \(SL(m, n)\) matrix in a diagonal gauge, which for KdV case \(n = 0\) becomes an ordinary determinant as in Fateev-Lukyanov [9] expression:

\[
L_{m,0} = \prod_{i=1}^m (D + v_i) = D^m + \bar{A}_{m-1} D^{m-2} + \ldots + \bar{A}_1 \quad ; \quad \sum_{i=1}^m v_i = 0 \quad (3.59)
\]

Example. As an example let us take \(m = 2, n = 1\) in (3.54). Then:

\[
L_{2,1} = (D + c_1 + \tilde{v}_1) (D - c_1) (D + \tilde{v}_1)^{-1} = -[(\tilde{v}_1 + c_1)' + c_1(\tilde{v}_1 + c_1)] (D + \tilde{v}_1)^{-1} + D
\]

as follows by direct calculation making use of an obvious identity \((D + c) (D + v)^{-1} = (c - v) (D + v)^{-1} + 1\).

One notices the absence of the term proportional to \(D^{m-n-1}\) in (3.55) due to the \(SL(m, n)\) trace zero condition. Another feature is the presence of the constant term \(A_{m+1}\) in the Lax operator \(L_{m,n}\) (for \(m - n > 1\)). This fact enables us to prove that \(SL(m, n)\)-cKP hierarchy is a bi-Poisson hierarchy. Consider namely \(L'_{m,n} = L_{m,n} - \lambda\) obtained by redefining the \(D^0 = 1\) term in the Lax operator by addition of the constant \(\lambda\). Clearly the value of the of bracket (3.52) for the new Lax is

\[
\begin{align*}
\left\{ \langle L'_{m,n} | X \rangle, \langle L'_{m,n} | Y \rangle \right\}_{DB} &= \text{Tr}_A \left( (L'_{m,n} X)_{+} L'_{m,n} Y - (X L'_{m,n})_{+} Y L'_{m,n} \right) \\
+ \frac{1}{m - n} \int dx \text{Res} \left( \left[ L'_{m,n}, X \right] \right) \partial^{-1} \text{Res} \left( \left[ L'_{m,n}, Y \right] \right) - \lambda \langle L'_{m,n} | [X, Y]_R \rangle
\end{align*}
\]

where we introduced an \(R\)-commutator \([X, Y]_R \equiv [X_+, Y_+] - [X_-, Y_-]\) with subscripts \(\pm\) denoting projection on pure differential and pseudo-differential parts of the pseudo-differential
operators $X, Y$. Define next an $R$-bracket $\{\cdot, \cdot\}_R$ as a bracket obtained by substituting $R$-commutator $[X, Y]_R$ for the ordinary commutator $[54, 58, 10]$:

$$\{ (L \mid X), (L \mid Y) \}_1^R \equiv - \langle L \mid [X, Y]_R \rangle$$

Relation (3.61) shows that $\{\cdot, \cdot\}_R$ satisfies the Jacobi identity. We can state this result as:

**Proposition.** $SL(m,n)$-cKP hierarchy is bi-Poisson with brackets $\{\cdot, \cdot\}_DB$ and $\{\cdot, \cdot\}_R$ defining a compatible pair of brackets.

This Proposition establishes the fundamental criterion for integrability of the $SL(m,n)$-cKP hierarchy. Clearly the argument holds also for the case $m - n = 1$, where one adds the constant $\lambda$ to zero representing the missing constant term.

Let us now concentrate on the pseudo-differential part of $L_{m,n}$ from (3.55). We can rewrite it as

$$(L_{m,n})_\pm = \sum_{i=1}^{n} r_i \prod_{i=1}^{n} D^{-1} q_i$$

where

$$r_i = A_i e^{-\int \tilde{v}_i} ; \quad q_n = e^{\int \tilde{v}_n} , \quad q_i = e^{\int (\tilde{v}_i - \tilde{v}_{i+1})} , \quad i = 1, \ldots, n - 1$$

Let us define the quantity

$$Q_{l,i} \equiv (-1)^{i-n} \int q_i \int q_{i-1} \int \ldots \int q_1 (dx^i)^{i-l+1} 1 \leq l \leq i \leq n$$

Then using that $D^{-1} Q_{1,i-1} q_i = D^{-1} Q_{1,i} D - Q_{1,i}$ we obtain for quantity in eq.(3.63)

$$(L_{m,n})_\pm = \sum_{i=2}^{n} r_i^{(1)} \prod_{i=1}^{n} D^{-1} q_i + r_1 D^{-1} (-Q_{1,n-1} q_n)$$

where

$$r_i^{(1)} \equiv r_i + r_1 Q_{1,i-1} \quad i = 2, \ldots, n$$

The above process can be continued to yield an expression

$$(L_{m,n})_\pm = \sum_{i=1}^{n} \Phi_i D^{-1} \Psi_i$$

with

$$\Phi_i = r_i + \sum_{n=1}^{i-1} r_n \sum_{s_i - n - 1 = s_i - n - 2 + 1}^{i-n} \ldots \sum_{s_2 = 1 + 1}^{i-n} \sum_{s_1 = 1}^{i-n} Q_{n,i-s_i-n-1-i} Q_{i-s_i-n-1,i-s_i-n-2-1} \ldots \cdot Q_{1-s_2,i-s_1-1} Q_{i-s_1,i-1}$$

$$1 \leq i \leq n$$

$$\Psi_n = q_n , \quad \Psi_i = (-1)^{n-i} q_n \int q_{n-1} \int \ldots \int q_i (dx^i)^{n-i} 1 \leq i \leq n - 1$$

Note that the new variables $\Psi_i$ coincide with elements $\psi_i^{(n)}$ in the kernel of $L^{(n)}$. It follows from the construction opposite to the one shown above and involving the following relation
\[(L_{m,n})_0 = \sum_{i=1}^{n} \Phi_i D^{-1} \Psi_i \quad (3.71)\]
\[
= \sum_{i=1}^{n} A_i^{(n)} (D + B_i^{(n)})^{-1} (D + B_{i+1}^{(n)})^{-1} \cdots (D + B_n^{(n)})^{-1} \quad (3.72)
\]

where the new variables are:

\[A_i^{(n)} = (-1)^{n-i} \sum_{s=1}^{i} \Phi_s W[\Psi_s, \ldots, \Psi_{i+1}, \Psi_i] \quad (3.73)\]
\[B_i^{(n)} = \partial_x \ln \frac{W[\Psi_n, \ldots, \Psi_{i+1}, \Psi_i]}{W[\Psi_n, \ldots, \Psi_{i+1}]} \quad (3.74)\]

From (3.72) we see that \(\tilde{v}_i = B_i^{(n)}\) and from (3.74) it follows that \(v_i^{(n)} = \Psi_i\).

**Example.** For \(n = 2\) relations (3.73)-(3.74) become

\[
A_2^{(2)} = \Phi_1 \Psi_1 + \Phi_2 \Psi_2, \quad A_1^{(2)} = -\Phi_1 \Psi_1 (\partial_x \ln \Psi_1^{-1}) \quad (3.75)
\]
\[
B_2^{(2)} = \partial_x \ln \Psi_2, \quad B_1^{(2)} = \partial_x \ln \left[\Psi_1 \left(\partial_x \ln \Psi_1^{-1}\right)\right] \quad (3.76)
\]

Equivalence between (3.72) and (3.71) follows then by inspection.

**Proposition.** \(\Phi_i, \Psi_i\) are canonical-Darboux fields for the first bracket of the \(SL(n+1, n)\) cKP hierarchy:

\[
\{ \Phi_i, \Psi_j \}_1 = -\delta_{ij} \delta(x - y) \quad , \quad i, j = 1, \ldots, n \quad (3.77)
\]

### 3.4 Affine \(sl(n + 1)\) Origin of \(SL(n + 1, n)\) cKP Hierarchy.

In this subsection we will establish a connection between Generalized Non-linear Schroedinger GNLS matrix hierarchy for the hermitian symmetric space \(sl(n+1)\) \[33\] and the constrained KP hierarchy.

We first introduce ZS-AKNS scheme arising from the linear matrix problem for \(A \in G\) with \(G\) being a Lie algebra \[32, 43\]:

\[
\partial \Psi = A \Psi \quad , \quad \partial = \frac{\partial}{\partial t_1} = \partial_x \quad (3.78)
\]
\[
\partial_{t_m} \Psi = B_m \Psi \quad m = 2, 3, \ldots \quad (3.79)
\]

The compatibility condition for the linear problem (3.78) and (3.79) leads to the Zakharov-Shabat (Z-S) integrability equations

\[
\partial_m A - \partial B_m + [A, B_m] = 0 \quad ; \quad \partial_m \equiv \partial_{t_m} \quad (3.80)
\]

Let us use the following decomposition in (3.78):

\[
A = \lambda E + A^0 \quad \text{with} \quad E = \frac{2\mu_a \cdot H}{\alpha_a^2} \quad (3.81)
\]
where $\mu_a$ is a fundamental weight and $\alpha_a$ are simple roots of $G$. The element $E$ is used to decompose the Lie algebra $G$ as follows:

$$G = \text{Ker}(\text{ad}E) \oplus \text{Im}(\text{ad}E) \quad (3.82)$$

where the Ker(ad$E$) has the form $K \times u(1)$ and is spanned by the Cartan subalgebra of $G$ and step operators associated to roots not containing $\alpha_a$. Moreover Im(ad$E$) is the orthogonal complement of Ker(ad$E$).

From now on we consider $G = \mathfrak{sl}(n+1)$ with roots $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ for some $i, j = 1, \ldots, n$, $E = \frac{2\mu_nH_a}{\alpha_a}$, $\mu_n$ is the $n$th fundamental weight and $H_a, a = 1, \ldots, n$ are the generators of the Cartan subalgebra. This decomposition generates the symmetric space $\mathfrak{sl}(n+1)/\mathfrak{sl}(n) \times u(1)$ (see [9] for details).

ZS-AKNS scheme becomes in this case the $\text{GNLS}_n (= \mathfrak{sl}(n+1) \text{GNLS})$ hierarchy. In matrix notation we have:

$$E = \frac{1}{n+1} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ r_1 & r_2 & \cdots & r_n \end{pmatrix} \quad (3.83)$$

The model is defined by $A^0 \in \text{Im}(\text{ad}E)$ and is parametrized by fields $q_a$ and $r_a, a = 1, \ldots, n$ according to:

$$A^0 = \sum_{a=1}^{n} \left( q_a E(\alpha_a + \ldots + \alpha_n) + r_a E(-\alpha_a + \ldots + \alpha_n) \right) \quad (3.84)$$

which in the matrix form can be rewritten as

$$A^0 = \begin{pmatrix} 0 & \cdots & 0 & \cdots & q_1 \\ 0 & 0 & \cdots & 0 & q_2 \\ \vdots & \ddots & \vdots \\ r_1 & r_2 & \cdots & r_n & 0 \end{pmatrix} \quad (3.85)$$

It can be shown that the flows as defined in (3.78) and (3.81) satisfy the recurrence relation:

$$\partial_m A^0 = \mathcal{R} \partial_{m-1} A^0 \quad ; \quad \mathcal{R} \equiv \left( \partial - \text{ad} A^0 \partial^{-1} \text{ad} A^0 \right) \text{ad} E \quad (3.86)$$

where we have defined a recursion operator $\mathcal{R}$ [3, 4].

The connection to the cKP hierarchy is first established between linear systems defining both hierarchies. With (3.83) and (3.85) the linear problem from (3.78) is explicitly given by:

$$\begin{pmatrix} \partial - \lambda/(n+1) & 0 & \cdots & 0 & q_1 \\ 0 & \partial - \lambda/(n+1) & 0 & \cdots & q_2 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & \partial - \lambda/(n+1) & q_n \\ r_1 & r_2 & \cdots & r_n & \partial + n\lambda/(n+1) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \\ \psi_{n+1} \end{pmatrix} = 0 \quad (3.87)$$

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Perform now the phase transformation:

$$
\psi_k \rightarrow \bar{\psi}_k = \exp \left( -\frac{1}{n+1} \int \lambda \, dx \right) \psi_k \quad k = 1, \ldots, n+1
$$

(3.88)

We now see that thanks to the special form of $E$ in $A = \lambda E + A^0$, (3.87) takes a simple and equivalent form:

$$
\begin{pmatrix}
\partial & 0 & \cdots & 0 & q_1 \\
0 & \partial & \cdots & q_2 \\
\vdots & \ddots & \ddots & \vdots \\
r_1 & r_2 & \cdots & r_n & \partial + \lambda
\end{pmatrix}
\begin{pmatrix}
\bar{\psi}_1 \\
\bar{\psi}_2 \\
\vdots \\
\bar{\psi}_n \\
\bar{\psi}_{n+1}
\end{pmatrix}
= 0
$$

(3.89)

The linear problem (3.89) after elimination of $\bar{\psi}_k$, $k = 1, \ldots, n$ takes a form of the scalar eigenvalue problem:

$$
- [\partial - \sum_{k=1}^{n} r_k \partial^{-1} q_k] \bar{\psi}_{n+1} = \lambda \bar{\psi}_{n+1}
$$

(3.90)

in terms of a single eigenfunction $\bar{\psi}_{n+1}$ and the pseudo-differential operator

$$
L_n = \partial - \sum_{k=1}^{n} r_k \partial^{-1} q_k
$$

(3.91)

This formally relates GNLS$_n$ hierarchy to the SL($n+1, n$) cKP hierarchy on basis of correspondence of the linear problems characterizing them. To fully establish their complete equivalence one can show that both models possess the same recurrence operators and therefore all the flows are identical see [9] (20] describes the the case $n = 1$). We first note that the successive flows (3.88) related by the recursion operator (3.88) are given by

$$
\partial_n \begin{pmatrix} r_i \\ q_l \end{pmatrix} = \mathcal{R}(i,i),(j,m) \begin{pmatrix} r_j \\ q_m \end{pmatrix} =
$$

(3.92)

$$
\begin{pmatrix}
-\partial + r_k \partial^{-1} q_k \\
- q_l \partial^{-1} q_j - q_j \partial^{-1} q_l
\end{pmatrix}
\delta_{ij} + r_i \partial^{-1} q_j
\begin{pmatrix}
r_i \partial^{-1} r_m + r_m \partial^{-1} r_i \\
\partial - q_k \partial^{-1} r_k \delta_{lm} - q_l \partial^{-1} q_m
\end{pmatrix}
\partial_{n-1} \begin{pmatrix} r_j \\ q_m \end{pmatrix}
$$

From relation between the recursion matrix and two first bracket structure $\mathcal{R} = P_2 P_1^{-1}$ and (3.92) one finds an explicit expression for the second bracket [9] to be:

$$
P_2 = \begin{pmatrix}
r_i \partial^{-1} r_j + r_j \partial^{-1} r_i \\
\partial - \sum_k r_k \partial^{-1} r_k \delta_{lj} - q_l \partial^{-1} q_j
\end{pmatrix}
\begin{pmatrix}
\partial - \sum_k r_k \partial^{-1} q_k \\
q_l \partial^{-1} q_m + q_m \partial^{-1} q_l
\end{pmatrix}
$$

(3.93)

in the same basis as in (3.92).

The following Proposition proves the equivalence between sl$(n+1)$ GNLS hierarchy defined and the SL$(n+1, n)$ cKP hierarchy introduced in the previous subsection.

**Proposition.** Flows

$$
\partial_{lm} L_n = [\{L_n^m \}, \ L_n]
$$

(3.94)

$$
\partial_{lm} r_i = \left( \left( -\partial + \sum_{k=1}^{n} q_k \partial^{-1} r_k \right) \right)_+ r_i
$$

(3.95)

$$
\partial_{lm} q_i = - \left( \left( -\partial + \sum_{k=1}^{n} q_k \partial^{-1} r_k \right) \right)_+ q_i
$$

(3.96)
of $\text{SL}(n+1, n)$ cKP hierarchy containing the Lax operator from (3.91) coincide with the flows produced by the recursion operator $R$ (3.92) of the $\text{sl}(n+1)$ GNLS hierarchy.

Proof is given in [9] and is accomplished by showing that both hierarchies have identical recursion operators. Especially (3.96) yields for $m = 2$:

$$\frac{\partial q_i}{\partial t_2} = -\partial_x^2 q_i + 2q_i \sum_{b=1}^{n} q_b r_b$$

$$\frac{\partial r_i}{\partial t_2} = \partial_x^2 r_i - 2r_i \sum_{b=1}^{n} q_b r_b$$

(3.97)

for $a = 1, \ldots, n$. These are the GNLS equations [33], which have been derived in [9] entirely from the AKS formalism [1] (see also [36]) associated with an affine Lie algebraic structure (loop algebra $\hat{\mathcal{G}} \equiv \mathcal{G} \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$ with $\mathcal{G} = \text{sl}(n+1)$). This construction reveals an affine Lie algebraic structure underlying the integrability of the $\text{SL}(n+1, n)$ cKP hierarchy.
4 Darboux-Bäcklund Techniques of SL\((p, q)\) cKP Hierarchies and Constrained Generalized Toda Lattices

4.1 Emergence of the Toda structure from Two-Bose KP System

The two-boson KP system defined by the Lax operator \(L = D + \Phi D^{-1} \Psi \equiv D + a (D - b)^{-1}\) is the most basic constrained KP structure. It belongs to the \(SL(2, 1)\) cKP system.

We start with the initial “free” Lax operator \(L^{(0)} = D\) and perform a following transformation:

\[
L^{(1)} = \left( \Phi^{(0)} D \Phi^{(0)} \right)^{-1} D \left( \Phi^{(0)} D^{-1} \Phi^{(0)} \right) = D + \left[ \Phi^{(0)} \left( \ln \Phi^{(0)} \right)^\prime \right] D^{-1} \left( \Phi^{(0)} \right)^{-1}
\] (4.1)

which we call a DB transformation. The construction below is a special application of properties listed in Appendix B including eq. (B.9).

Successive application of DB transformations leads to the following recursive expressions:

\[
L^{(k+1)} = \left( \Phi^{(k)} D \Phi^{(k)} \right)^{-1} L^{(k)} \left( \Phi^{(k)} D^{-1} \Phi^{(k)} \right) = D + \Phi^{(k+1)} D^{-1} \Psi^{(k+1)}
\] (4.2)

\[
\Phi^{(k+1)} = \Phi^{(k)} \left( \ln \Phi^{(k)} \right)^\prime + \left( \Phi^{(k)} \right)^2 \Psi^{(k)}, \quad \Psi^{(k+1)} = \left( \Phi^{(k)} \right)^{-1}
\] (4.3)

Introduce now

\[
\phi_k = \ln \Phi^{(k)} \quad \rightarrow \quad \Phi^{(k)} = e^{\phi_k} \quad k = 0, \ldots
\] (4.4)

which allows us to rewrite (4.3) as a (ordinary one-dimensional) Toda lattice equation:

\[
\partial^2 \phi_k = e^{\phi_{k+1} - \phi_k} - e^{\phi_k - \phi_{k-1}}
\] (4.5)

Related objects \(\psi_n\) are:

\[
\phi_n = \psi_{n+1} - \psi_n
\] (4.6)

which satisfy due to eq. (4.5) the following form of the Toda lattice equation:

\[
\partial^2 \psi_n = e^{\psi_{n+1} + \psi_{n-1}} - 2 \psi_n
\] (4.7)

with \(\psi_n = 0\) for \(n \leq 0\). Comparing (4.7) with Jacobi’s theorem (B.7) we find the Wronskian representation for \(\psi_n\):

\[
\psi_n = \ln W_n [\Phi, \partial \Phi, \ldots, \partial^{n-1} \Phi] \quad \text{with} \quad \Phi = \Phi^{(0)}
\] (4.8)

Correspondingly \(\Phi^{(k)}\) acquires the form:

\[
\Phi^{(k)} = \frac{W_{k+1} [\Phi, \partial \Phi, \ldots, \partial^k \Phi]}{W_k [\Phi, \partial \Phi, \ldots, \partial^{k-1} \Phi]}
\] (4.9)

We recognize at the right hand side of (4.7) a structure of the Cartan matrix for \(A_n\). Leznov considered such an equation with Wronskian solution (in two dimensions) in [12].

Hence, the solutions of the (ordinary one-dimensional) Toda lattice equations, with boundary conditions \(\psi_n = 0\) for \(n \leq 0\), reproduce the DB solutions of the ordinary two-boson KP hierarchy (4.9) upon taking into account that \(\Phi = \Phi^{(0)} = \exp (\phi_0) = \exp (\psi_1)\).
Note that we can write \( W_n = \tau_n \) with the \( \tau \)-function \( \tau_n \) satisfying Hirota’s bilinear equation for the Toda lattice:

\[
\tau_n \partial^2 \tau_n - (\partial \tau_n)^2 = \tau_{n+1} \tau_{n-1}
\]  

(4.10)
on basis of (3.8).

We will now find a linear discrete system (Toda spectral problem), which leads to the above structure. First define:

\[
R_n = \frac{\Phi^{(n+1)}}{\Phi^{(n)}} = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}
\]  

(4.11)

\[
S_n = \partial \left( \ln \Phi^{(n+1)} \right) = \partial \left( \ln \frac{\tau_{n+1}}{\tau_n} \right)
\]  

(4.12)

As result of (4.10) and their definition \( S_n, R_n \) satisfy the Toda equations of motion:

\[
\partial S_n = R_{n+1} - R_n
\]  

(4.13)

\[
\partial R_n = R_n (S_n - S_{n-1})
\]  

(4.14)

These equations can also be obtained as a consistency of the spectral system

\[
\partial \Psi_n = \Psi_{n+1} + S_n \Psi_n
\]  

(4.15)

\[
\lambda \Psi_n = \Psi_{n+1} + S_n \Psi_n + R_n \Psi_{n-1}
\]  

(4.16)

which defines a so-called Toda chain system.

As we will show this purely discrete system contains information about the underlying continuous structure. This structure is being revealed when we realize that the lattice jump \( n \to n + 1 \) can be given a meaning of the DB transformation. We start by rewriting (4.15) as follows:

\[
\partial \Psi_n = \Psi_{n+1} + S_n \Psi_n \sim \Psi_{n+1} = e^{\int S_n \partial} e^{-\int S_n \Psi_n}
\]  

(4.17)

or taking into account (4.14) as

\[
\Psi_{n+1} = R_n e^{\int S_{n-1} \partial} \left( R_n e^{\int S_{n-1} \partial} \right)^{-1} \Psi_n = \Phi(n) \partial \Phi^{-1}(n) \Psi_n = T(n) \Psi_n
\]  

(4.18)

where \( \Phi(n) = R_n e^{\int S_{n-1}} \) and \( T(n) = \Phi(n) \partial \Phi^{-1}(n) \) plays a role of the DB transformation operator generating the lattice translation \( n \to n + 1 \).

The remaining of the Toda spectral equation (4.16) can be given a form (with \( \Psi(n) \equiv e^{-\int S_{n-1}} \)):

\[
\lambda \Psi_n = \left( \partial + R_n (\partial - S_{n-1})^{-1} \right) \Psi_n = \left[ \partial + R_n e^{\int S_{n-1} \partial} e^{-\int S_n} \right] \Psi_n = \left[ \partial + \Phi(n) \partial^{-1} \Psi(n) \right] \Psi_n = L(n) \Psi_n
\]  

(4.19)

of the Lax eigenvalue problem of the two-bose KP Lax system with generic two-bose KP Lax operator \( L = D + \Phi D^{-1} \Psi \). Hence the Toda lattice spectral problem has been shown equivalent to the continuous cKP system possessing the symmetry with respect to the DB transformations.
4.2 On the DB Transformations of the SL(r + n, n) cKP Lax Operators

We shall here consider behavior of the general class of constrained KP Lax operators from $SL(r + n, n)$ cKP hierarchy under an arbitrary DB transformation $\tilde{L} = \chi D\chi^{-1}L\chi^{-1}D^{-1}$ where $\chi$ is an eigenfunction of the Lax operator $L$ as given in (3.26). The transformed Lax operator reads:

$$\tilde{L} = \chi D\chi^{-1}\left(L_+ + \sum_{i=1}^{n} \phi_i D^{-1} \psi_i \right) \chi D^{-1} \chi^{-1} \equiv \tilde{L}_+ + \tilde{L}_- \quad (4.20)$$

$$\tilde{L}_+ = L_+ + \chi \left( \partial_x \left( \chi^{-1} L_+ \chi \right) \right) \chi^{-1} \quad (4.21)$$

$$\tilde{L}_- = \tilde{\phi}_0 D^{-1} \tilde{\psi}_0 + \sum_{i=1}^{n} \tilde{\phi}_i D^{-1} \tilde{\psi}_i \quad (4.22)$$

$$\tilde{\phi}_0 = \chi \left[ \partial_x \left( \chi^{-1} L_+ \chi \right) + \sum_{i=1}^{n} \left( \partial_x \left( \chi^{-1} \phi_i \right) \right) \partial_x^{-1} \left( \psi_i \chi + \phi_i \psi_i \right) \right] \equiv \left( \chi D \chi^{-1} L \right) \chi \quad (4.23)$$

$$\tilde{\psi}_0 = \chi^{-1} \quad , \quad \tilde{\phi}_i = \chi \partial_x \left( \chi^{-1} \phi_i \right) \quad , \quad \tilde{\psi}_i = -\chi^{-1} \partial_x^{-1} \left( \psi_i \chi \right) \quad (4.24)$$

From the above discussion we know that all involved functions are (adjoint) eigenfunctions of $L$ (3.26), i.e., they satisfy:

$$\frac{\partial}{\partial t_k} f = L_k^* f \quad f = \phi, \psi \quad ; \quad \frac{\partial}{\partial t_k} \psi_i = -L_k^* \phi_i \quad (4.25)$$

We are interested in the special case when $\chi$ coincides with one of the original eigenfunctions of $L$, e.g. $\chi = \phi_1$. Then $\tilde{\phi}_1 = 0$ and the DB transformation (4.20) preserves the form (3.26) of the Lax operators involved, i.e., it becomes an auto-Bäcklund transformation. Applying the successive DB transformations in this case yields:

$$L^{(k)} = T^{(k-1)} L^{(k-1)} \left(T^{(k-1)} \right)^{-1} = \left( L^{(k)} \right)_+ + \sum_{i=1}^{n} \phi_i^{(k)} D^{-1} \psi_i^{(k)} \quad , \quad T^{(k)} \equiv \phi_1^{(k)} D \left( \phi_1^{(k)} \right)^{-1} \quad (4.26)$$

$$\phi_i^{(k+1)} = T^{(k)} \phi_i^{(k)} \quad , \quad \psi_i^{(k+1)} = \phi_i^{(k)} \quad (4.27)$$

$$\psi_i^{(k+1)} = -\left( \psi_i^{(k)} \right)^{-1} \partial_x^{-1} \left( \phi_i^{(k)} \Phi_1^{(k)} \right) \quad , \quad i = 2, \ldots, n \quad (4.28)$$

Using the first identity from (4.20), i.e., $L^{(k+1)} T^{(k)} = T^{(k)} L^{(k)}$, one can rewrite (4.27) in the form:

$$\phi_1^{(k)} = T^{(k-1)} T^{(k-2)} \ldots T^{(0)} \left( \left( L^{(0)} \right)^k \phi_1^{(0)} \right) \quad (4.30)$$

whereas:

$$\phi_i^{(k)} = T^{(k-1)} T^{(k-2)} \ldots T^{(0)} \phi_i^{(0)} \quad , \quad i = 2, \ldots, n \quad (4.31)$$
Finally, for the coefficient of the next-to-leading differential term in (3.26) \( u_{r-2} = r \text{Res} L^\perp = r \partial_2^2 \ln \tau \), we easily obtain from (1.21) (with \( \chi = \Phi_1 \)) its \( k \)-step DB-transformed expression:

\[
\frac{1}{r} (u_{r-2}^{(k)} - u_{r-2}^{(0)}) = \partial_x^2 \ln \tau^{(k)}(0) = \partial_x^2 \ln \left( \Phi_1^{(k-1)} \cdots \Phi_1^{(0)} \right)
\]  

(4.32)

Remark. Let us particularly stress that the eigenfunctions we are working with are not Baker-Akhiezer eigenfunctions of \( L \) from (3.20). Imagine, namely that we started with \( L = L_+ + \psi D^{-1} \psi^* \), where \( \psi, \psi^* \) are BA functions. Choosing \( \chi = \psi \) would result according to (4.23) and (4.24) in \( \tilde{L} = D \). Hence in such case we would be able to transform away the pseudo-differential part of the cKP Lax operator by a finite number of DB transformations.

4.3 DB Transformation and Eigenfunctions of the Lax Operator

After seeing DB transformation in action in the simple cases shown in the first two subsections of this chapter we are now ready to review the basic properties of DB transformation from the point of view of preserving the form of the Lax evolution equation. Related material can be found in e.g. [19, 51].

Lemma. For arbitrary pseudo-differential operator \( A \) we have the following identity [52]:

\[
\left( \chi D^{-1} A \chi D^{-1} \chi^{-1} \right)_+ = \chi D^{-1} (A)_+ \chi D^{-1} \chi^{-1} - \chi \partial_x \left( \chi^{-1}((A)_+ \chi) \right) D^{-1} \chi^{-1}
\]  

(4.33)

For \( L \) satisfying a general KP-KdV Lax equation \( \partial_k L = \left[ L^\perp, L \right] \) the transformed Lax operator \( \tilde{L} \equiv TLT^{-1} \) will satisfy:

\[
\partial_k \tilde{L} = \left[ T^\perp L^\perp T^{-1} + (\partial_k T)T^{-1}, \tilde{L} \right]
\]  

(4.34)

Let \( \Phi \) be an eigenfunction of \( L \) i.e. \( L^\perp \Phi = \partial_k \Phi \), which enters the DB transformation through \( T = \Phi D \Phi^{-1} \). One verifies easily that for \( A = L^\perp \) and \( \chi = \Phi \) equation (4.33) becomes

\[
\left( TL^\perp T^{-1} \right)_+ = TL^\perp T^{-1} + (\partial_k T)T^{-1}
\]  

(4.35)

Correspondingly (4.34) takes the form:

\[
\partial_k \tilde{L} = \left[ \tilde{L}^\perp, \tilde{L} \right]
\]  

(4.36)

and we have therefore established:

Proposition. The DB transformation with an eigenfunction \( \Phi \) preserves the form of the Lax equation (2.1) i.e. the DB transformed Lax operator satisfies the same evolution equation as the original Lax operator.

For the pseudo-differential operator \( A = D + a_1 D^{-1} + \ldots \) we have

\[
\text{Res} \left( \chi D^{-1} A \chi D^{-1} \chi^{-1} \right) = (\ln \chi)'' + \text{Res} (A)
\]  

(4.37)

For the case of the Lax operator \( L \) with the \( \tau \)-function satisfying \( \text{Res} L = \partial_2^2 \ln \tau \) we get therefore:

Proposition. Under the DB transformation with an eigenfunction \( \Phi \) the \( \tau \)-function associated to the Lax operator \( L \) transforms according to \( \tau \rightarrow \tilde{\tau} = \Phi \tau \).
4.4  Exact Solutions of SL(p, q) cKP hierarchy via DB Transformations

Armed with the Wronskian identities from Appendix B, we can represent the $k$-step DB transformation (4.30)—(4.32) in terms of Wronskian determinants involving the coefficient functions of the “initial” Lax operator

$$L^{(0)} = D^r + \sum_{i=1}^{n} \Phi_i^{(0)} D^i + \sum_{i=1}^{r-2} u_i^{(0)} D^i$$

only. Indeed, using identity (B.9) and defining:

$$\left( L^{(0)} \right)^k \Phi_1^{(0)} \equiv \chi^{(k)} \quad k = 1, 2, \ldots$$

we arrive at the following general result:

**Proposition.** The $k$-step DB-transformed eigenfunctions and the tau-function (4.30)—(4.32) of the SL($r + n, n$) cKP system (3.71) for arbitrary initial $L^{(0)}$ (4.38) are given by:

$$\Phi_1^{(k)} = \frac{W_{k+1}[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k)}]}{W_k[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}]}$$

$$\Phi_j^{(k)} = \frac{W_{k+1}[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}, \Phi_j^{(0)}]}{W_k[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}]}, \quad j = 2, \ldots, n$$

$$\tau^{(k)} = \frac{W_k[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}]}{\Psi^{(0)}(0)}$$

where $\tau^{(0)}$, $\tau^{(k)}$ are the $\tau$-functions of $L^{(0)}$, $L^{(k)}$, respectively, and $\chi^{(i)}$ is given by (4.39).

**Example** Consider $SL(2, 1)$ cKP hierarchy with the Lax operator: $L^{(0)} = D + \Phi^{(0)} D^{-1} \Psi^{(0)}$, which serves as a starting point of the successive DB transformations:

$$L^{(k)} = D + \Phi^{(k)} D^{-1} \Psi^{(k)} = T^{(k-1)} L^{(k-1)} \left( T^{(k-1)} \right)^{-1}$$

$$\Phi^{(k)} = T^{(k-1)} \ldots T^{(1)} T^{(0)} \left( (L^{(0)})^k \Phi^{(0)} \right) = \frac{W_{k+1}[\Phi^{(0)}, \chi^{(1)}, \ldots, \chi^{(k)}]}{W_k[\Phi^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}]}$$

$$\Psi^{(k)}(0) = (\Phi^{(k-1)})^{-1}$$

Comparing two alternative expressions involving the tau-function $\tau^{(k)}$:

$$\tau^{(k)} = \Phi^{(k)} \ldots \Phi^{(1)} \Phi^{(0)} \tau^{(0)} = W_{k+1}[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}] \tau^{(0)}$$

and $\text{Res} L^{(k)} = \Phi^{(k)} \Psi^{(k)} = \partial^2 \ln \tau^{(k)}$ we arrive at:

$$\Phi^{(0)} \Psi^{(0)} = -\partial^2 \ln W_k[\Phi^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}] + \frac{W_{k+1}[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}] W_{k-1}[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-2)}]}{W_k[\Phi_1^{(0)}, \chi^{(1)}, \ldots, \chi^{(k-1)}]}$$
which generalizes the structure of the Hirota equation (4.10) found for the case $\Phi^{(0)} = \Psi^{(0)} = 0$ in subsection 3.1.

Example: Construction of the $SL(3, 1)$ cKP Lax operator, i.e., $r = 2, n = 1$ in (3.20) i.e.

$$L = D^2 + u + A(D - B)^{-1} = D^2 + u + \Phi D^{-1} \Psi$$ (4.49)

starting from the “free” $L^{(0)} = D^2$. This example is pertinent to the simplest nontrivial string two-matrix model [11].

From the basic formulas for successive DB transformations (4.27) – (4.26), applied to (4.27) – (4.26), we obtain the following explicit solutions for $u^{(n)}$, $B^{(n)}$, and $A^{(n)}$.

$$u^{(n)} = 2 \partial_x^2 \ln W \left[ \Phi^{(0)}, \partial_x^2 \Phi^{(0)}, \ldots, \partial_x^{2(n-1)} \Phi^{(0)} \right]$$ (4.56)

$$B^{(n)} = \partial_x \ln \left( \frac{W \left[ \Phi^{(0)}, \partial_x^2 \Phi^{(0)}, \ldots, \partial_x^{2(n-1)} \Phi^{(0)} \right]}{W \left[ \Phi^{(0)}, \partial_x^2 \Phi^{(0)}, \ldots, \partial_x^{2(n-2)} \Phi^{(0)} \right]} \right)$$ (4.57)

$$A^{(n)} = \frac{W \left[ \Phi^{(0)}, \partial_x^2 \Phi^{(0)}, \ldots, \partial_x^{2n} \Phi^{(0)} \right]}{W \left[ \Phi^{(0)}, \partial_x^2 \Phi^{(0)}, \ldots, \partial_x^{2(n-2)} \Phi^{(0)} \right]}$$ (4.58)

In the more general case of $SL(r + 1, 1)$ cKP Lax operator for arbitrary finite $r$:

$$L = D^r + \sum_{l=0}^{r-2} u_l D^l + \Phi D^{-1} \Psi$$ (4.61)
which defines the integrable hierarchy corresponding to the general string two-matrix model (cf. [11, 12]), the generalizations of (4.56) and (4.57) read:

$$\Phi^{(k)} = T^{(k-1)} \ldots T^{(0)} \left( \partial_x^{r} \Phi^{(0)} \right) = \frac{W \left[ \Phi^{(0)}, \partial_x^{r} \Phi^{(0)}, \ldots, \partial_x^{k-1} \partial_x^{r} \Phi^{(0)} \right]}{W \left[ \Phi^{(0)}, \partial_x^{r} \Phi^{(0)}, \ldots, \partial_x^{k-1} \partial_x^{r} \Phi^{(0)} \right]}$$ (4.62)

$$\frac{1}{r} u_{r-2}^{(k)} = \text{Res} L^{\frac{1}{r}} = \partial_x^2 \ln \tau^{(k)} , \quad \tau^{(k)} = W \left[ \Phi^{(0)}, \partial_x \Phi^{(0)}, \ldots, \partial_x^{k-1} \partial_x \Phi^{(0)} \right]$$ (4.63)

where $\Phi^{(0)}$ is again given explicitly by (4.55).

### 4.5 Relation to the Constrained Generalized Toda Lattices

Here we shall establish the equivalence between the set of successive DB transformations of the $SL(r+1, 1)$ cKP system (4.61):

$$L^{(k+1)} = T^{(k)} L^{(k)} \left( T^{(k)} \right)^{-1} , \quad T^{(k)} = \Phi^{(k)} D \Phi^{(k)^{-1}}$$ (4.64)

$$L^{(0)} = D^r + \sum_{l=0}^{r-2} u_{l}^{(0)} D^l + \Phi^{(0)} D^{-1} \Psi^{(0)}$$ (4.65)

and the equations of motion of a constrained generalized Toda lattice system, underlying the two-matrix string model, which contains, in particular, the two-dimensional Toda lattice equations.

For simplicity we shall illustrate the above property on the simplest nontrivial case of $SL(3, 1)$ cKP hierarchy (4.49). We note that eqs.(4.52)–(4.54) (or (4.58)–(4.60)) can be cast in the following recurrence form:

$$\partial_x \ln A^{(n-1)} = B^{(n)} - B^{(n-1)}$$ (4.66)

$$u^{(n)} - u^{(n-1)} = 2 \partial_x B^{(n)}$$ (4.67)

$$A^{(n)} - A^{(n-1)} = \partial_x \left( \left( B^{(n)} \right)^2 + \frac{1}{2} \left( u^{(n)} + u^{(n-1)} \right) \right)$$ (4.68)

with “initial” conditions (cf. (4.55)):

$$A^{(0)} = B^{(0)} = u^{(0)} = 0 , \quad B^{(1)} = \partial_x \ln \Phi$$ (4.69)

where $\Phi$ is so far an arbitrary function. Now, we can view (4.66)–(4.68) as a system of lattice equations for the dynamical variables $A^{(n)}, B^{(n)}, u^{(n)}$ associated with each lattice site $n$ and subject to the boundary conditions:

$$A^{(n)} = B^{(n)} = u^{(n)} = 0 \quad , \quad n \leq 0$$ (4.70)

Taking (4.69) as initial data, one can solve the lattice system (4.66)–(4.68) step by step (for $n = 1, 2, \ldots$) and the solution has precisely the form of (4.58)–(4.60).
4.6 Darboux-Bäcklund Transformation and the Dressing Chain.

As a small digression we will here apply Darboux-Bäcklund (DB) transformation to the KdV hierarchy emphasizing similarity with the approach developed in the KP setting. Recall from (4.50)-(4.55) that for \( T = \Phi D \Phi^{-1} \) we have

\[
TD^2 T^{-1} = D^2 + 2 \partial_x^2 \ln \Phi + \Phi \left( \Phi^{-1} \Phi'' \right)' D^{-1} \Phi^{-1}
\]  

(4.71)

The inverse scattering problem corresponding to the KdV equation \( \partial_t u + 3u \partial u + \frac{1}{2} \partial^3 u = 0 \) is defined in terms of the differential operator \( L = D^2 + u \), which transforms as:

\[
L_1 \equiv T \left( D^2 + u \right) T^{-1} = D^2 + 2 \partial_x^2 \ln \Phi + \Phi \left( \Phi^{-1} \Phi'' \right)' D^{-1} \Phi^{-1} + u + \Phi u' D^{-1} \Phi^{-1}
\]  

(4.72)

In order for \( L_1 \) to be a differential operator we have to demand that the pseudo-differential part in (4.72) vanishes:

\[
\Phi \left( \Phi^{-1} \Phi'' \right)' + \Phi u' = 0
\]  

(4.73)

which translates into relation between \( u \) and \( \Phi \):

\[
u = -\frac{\Phi''}{\Phi} - \lambda_0 = -f_0^2 - f_0' - \lambda_0
\]  

(4.74)

where \( \lambda_0 \) is an integration constant and \( f_0 \equiv (\ln \Phi)' \). In this notation:

\[
L = D^2 + u = D^2 - f_0^2 - f_0' - \lambda_0 = (D + f_0)(D - f_0) - \lambda_0
\]  

(4.75)

Since \( T = D - f_0 \) and \( T^\dagger = -D - f_0 \) we can factorize the Lax operator \( L \) according to

\[
L = -T^\dagger T + \lambda_0
\]  

(4.76)

In this notation the DB transformation on \( L \):

\[
L_1 = TL T^{-1} = -TT^\dagger - \lambda_0 = D^2 - f_0^2 + f_0' - \lambda_0 = D^2 + u_1
\]  

(4.77)

amounts to reversing the \( T, T^\dagger \) operators in expression for \( L \). Moreover we find that

\[
u_1 = -f_0^2 + f_0' - \lambda_0 = -f_0^2 - f_1' - \lambda_1 = u + 2(\ln \Phi)''
\]  

(4.78)

where we introduced new variables \( f_1 = (\ln \Phi_1)' \) and \( \lambda_1 \), which allow us to rewrite \( L_1 \) as \( L_1 = -T_1^\dagger T_1 + \lambda_1 \) with \( T_1 = D - f_1 \). Clearly we have \( L_1 \Phi_1 = -\lambda_1 \Phi_1 \).

Successive use of the DB transformations [53] leads to the Lax operators \( L_n = D^2 + u_n \) with

\[
u_n = -f_n^2 + f_n' - \lambda_{n-1} = -f_n^2 - f_n' - \lambda_n = u + 2(\ln \Phi \cdots \Phi_{n-1})''
\]  

(4.79)

with a string of eigenvalue problems:

\[
L_n \Phi_n = -\lambda_n \Phi_n \quad \text{or} \quad \left( \partial^2 + u_i + \lambda_i \right) \Phi_i = 0 \quad \Phi \equiv \Phi_0 \quad , \quad u_0 \equiv u
\]  

(4.80)
for $i = 0, 1, \ldots$. Let $\Psi_1, \ldots, \Psi_n$ be solutions of the related eigenvalue problem:

$$L_0 \Psi_i = -\lambda_i \Psi_i \quad \text{or} \quad \left( \partial^2 + u + \lambda_i \right) \Psi_i = 0 \quad \Psi_0 \equiv \Phi, \quad i = 0, 1, \ldots \quad (4.81)$$

We have a following theorem shown by Crum [23], which establishes relation between $\Phi_i$ and $\Psi_i$ eigenfunctions.

**Proposition.** The function

$$\Phi_n \equiv \frac{W_{n+1}[\Psi_0, \Psi_1, \ldots, \Psi_n]}{W_n[\Psi_0, \Psi_1, \ldots, \Psi_{n-1}]} \quad (4.82)$$

satisfies the differential equation:

$$\left( \partial^2 + u_n + \lambda_n \right) \Phi_n = 0 \quad (4.83)$$

$$u_n = u + 2\partial^2 \ln W_n[\Psi_0, \Psi_1, \ldots, \Psi_{n-1}] \quad (4.84)$$

**Proof.** To prove it let us notice that $L_n = (T_{n-1} \cdots T_0) L_0 (T_{n-1} \cdots T_0)^{-1}$, with $T_i = \Psi_i \partial \Psi_i^{-1}$. From $L_n \Phi_n = -\lambda_n \Phi_n$ we find that $\Phi_n = T_{n-1} \cdots T_0 \Psi_n$. By use of induction and (B.9) one completes the proof. Equation (4.84) follows automatically from (4.82) and (4.79).

Note a clear analogy of the dressing chain construction with the Susy QM [47]. Define namely, $T_n = D - f_n$ with $T_n^\dagger = -(D + f_n)$ and introduce $H_\pm$ through:

$$T_n T_n^\dagger = -(D^2 - f_n^2 + f_n') = H_- - \lambda_n \quad ; \quad T_n^\dagger T_n = -(D^2 - f_n^2 - f_n') = H_+ - \lambda_n \quad (4.85)$$

These relations can be cast into the QM Susy algebra:

$$\{ Q_-, Q_+ \} = H - \lambda_n \quad ; \quad [ Q_\pm, H ] = 0 \quad (4.86)$$

in terms of the $2 \times 2$ matrices:

$$Q_+ = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \quad ; \quad Q_- = \begin{pmatrix} 0 & T^\dagger \\ 0 & 0 \end{pmatrix} \quad ; \quad H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \quad (4.87)$$

The eigenvalue problem $HV_n = \lambda_n V_n$ with $V_n = (V_-, V_+)^T$ takes a familiar form $(D^2 - f_n^2 I + f_n' \sigma_3) V = 0$ and solutions are given by

$$V_\pm = \exp \left( \pm \int f_n \right) = (\Phi_n)^{\pm 1} \quad (4.88)$$

as verified by rewriting the eigenvalue problem as $T^\dagger T V_+ = 0$ and $T T^\dagger V_- = 0$. Hence the Darboux techniques prove useful in constructing exact solution of the Schrödinger problems of the supersymmetric quantum mechanics.
4.7 Connection to Grassmannian Manifolds and n-Soliton Solution for the KP Hierarchy

Let \( \{\psi_1, \ldots, \psi_n\} \) be a basis of solutions of the \( n \)-th order equation \( L\psi = 0 \), where \( L = (D + v_n)(D + v_{n-1}) \cdots (D + v_1) \). If \( W_k \) denotes the Wronskian determinant of \( \{\psi_1, \ldots, \psi_k\} \) one can then show that \( v_i = \partial \left( \ln \frac{W_{i-1}}{W_i} \right) \) \( W_0 = 1 \) (4.89).

This allows to establish that the space of differential operators is parametrized by the Grassmannian manifold (see e.g. [60, 46]). Start namely with the given differential operator \( L_n = D^n + u_1 D^{n-1} + \cdots + u_n \) and determine the kernel of \( L_n \) given by \( n \)-dimensional subspace of some Hilbert space of functions \( \mathcal{H} \), spanned, let say, by \( \{\psi_1, \ldots, \psi_n\} \). This establishes the connection one way. On the other hand let \( \{\psi_1, \ldots, \psi_n\} \) be a basis of one point \( \Omega \) of a Grassmannian manifold \( \text{Gr}(n) \). Define the differential equation as \( L_n(\Omega)f = \frac{W_k(f)}{W_k} \).

From (B.11) this associates the differential operator \( L_n = (D^m + v_n D^{m-1} + \cdots + v_1) \) (4.90) given by a Miura correspondence to a given point of the Grassmannian.

Let us now comment on connection to n-soliton solution for the KP hierarchy. Assume that the above functions \( \psi_i \) \( i = 1, \ldots, n \) have the property \( \partial_m \psi_i = \partial^m \psi_i \) for arbitrary \( m \geq 1 \) \( (\partial_m \equiv \partial^{\text{t}} \partial_m) \), in other words \( \psi_i \) are eigenfunctions of \( L^{(0)} = D \). We introduce \( L = L_n DL_n^{-1} \), where \( L_n \) is defined in terms of \( \{\psi_1, \ldots, \psi_n\} \) as in (4.89) and (4.90). It is known [44, 26, 50] that such a Lax operator satisfies a generalized Lax equation \( \partial_m L = [L^{(m)}, L] \).

Using (B.9) we can rewrite the above Lax operator as a result of successive DB transformations applied on \( D \):

\[
L = L_n DL_n^{-1} = T_n T_{n-1} \cdots T_1 DT_1^{-1} \cdots T_{n-1}^{-1} T_n^{-1}
\]

(4.91)

where \( T_i \) are given in terms of Wronskians as in (B.10). It follows that \( L \) can be cast in a form of the Lax operator belonging to subclass of the \( \text{SL}(1 + n, n) \) cKP hierarchy and having the form as in (3.26) with \( r = 1 \). Using the formalism developed in this paper one can prove by induction that the corresponding \( \tau \)-function of \( L \) takes a Wronskian form \( \tau_n = W_n[\psi_1, \ldots, \psi_n] \) reproducing n-soliton solution to the KP equation derived in [34]. In fact, choosing \( \psi_i = \exp \left( \sum t_k \alpha_k^i \right) + \exp \left( \sum t_k \beta_k^i \right) \) allows to rewrite \( \tau_n \) in the conventional form of the \( n \)-soliton solution to the KP equation [37, 33].
5 Two-Matrix Model as a SL\((r + 1, 1)\)-cKP Hierarchy

5.1 Two-Matrix model, Orthogonal Polynomials Technique

We shall consider the two-matrix model with partition function:

\[
Z_N[t, \tilde{t}, g] = \int dM_1 dM_2 \exp \left\{ \sum_{r=1}^{p_1} t_r \text{Tr} M_1^r + \sum_{s=1}^{p_2} \tilde{t}_s \text{Tr} M_2^s + g \text{Tr} M_1 M_2 \right\}
\] (5.1)

where \(M_{1,2}\) are Hermitian \(N \times N\) matrices, and the orders of the matrix “potentials” \(p_{1,2}\) may be finite or infinite. As in ref.\[15\] and \[57, 4\] we will use the method of generalized orthogonal polynomials \[17\] to evaluate partition function (5.1). After angular integration in (5.1) we obtain

\[
Z_N[t, \tilde{t}, g] = \frac{1}{N!} \prod_{i=1}^{N} d\lambda_i d\tilde{\lambda}_i \exp \left\{ \sum_{i=1}^{N} \left( V(\lambda_i) + \tilde{V}(\tilde{\lambda}_i) + g \lambda_i \tilde{\lambda}_i \right) \right\} \Delta(\lambda_i) \Delta(\tilde{\lambda}_i)
\] (5.2)

where \(V(\lambda) = \sum t_k \lambda^k, \tilde{V}(\tilde{\lambda}) = \sum \tilde{t}_s \tilde{\lambda}^s\). \(\Delta(\lambda_i), \Delta(\tilde{\lambda}_i)\) are standard Van der Monde determinants. As in one-matrix model one can deal with them using orthogonal polynomials. Here we will work with two families of the orthogonal polynomials:

\[
P_n(\lambda_1) = \lambda_1^n + O(\lambda_1^{n-1}) ; \quad \tilde{P}_m(\lambda_2) = \lambda_2^m + O(\lambda_2^{m-1}) , \quad n, m = 0, 1, \ldots
\] (5.3)

which enter the orthogonal relation:

\[
\int d\lambda_1 d\lambda_2 \exp \left( V(\lambda_1) + \tilde{V}(\lambda_2) + g \lambda_1 \lambda_2 \right) P_n(\lambda_1) \tilde{P}_m(\lambda_2) = h_m \delta_{nm}
\] (5.4)

Following approach to one-matrix model we write down the recursion relations for the polynomials in (5.3):

\[
\lambda_1 P_n(\lambda_1) = \sum_{l=0}^{n+1} Q_{nl} P_l(\lambda_1) ; \quad \lambda_2 \tilde{P}_m(\lambda_2) = \sum_{l=0}^{n+1} \tilde{Q}_{l,n} \tilde{P}_l(\lambda_2)
\] (5.5)

From the definition of orthogonal polynomials it follows that

\[
Q_{n,n+1} = 1 ; \quad Q_{n,l} = 0 \quad l \geq n + 2 \quad \text{and} \quad \tilde{Q}_{n+1,n} = 1 ; \quad \tilde{Q}_{l,n} = 0 \quad l \geq n + 2
\] (5.6)

The orthogonal polynomials approach leads to expression for the partition function in terms of \(h_n\):

\[
Z_N = h_0 h_1 \cdots h_{N-2} h_{N-1}
\] (5.7)

From the orthogonal relation (5.4) and definitions (5.6) we obtain:

\[
\frac{\partial \ln h_n}{\partial t_r} = (Q^r)_{nn} ; \quad \frac{\partial \ln h_n}{\partial \tilde{t}_s} = (\tilde{Q}^s)_{nn}
\] (5.8)
as well as
\[
\frac{\partial P_n}{\partial t_r} = -\sum_{l=0}^{n-1} Q'_{nl} P_l(\lambda_1) ; \quad \frac{\partial \tilde{P}_n}{\partial t_r} = -\sum_{l=0}^{n-1} \tilde{P}_l(\lambda_2)(H^{-1}Q^r H)_{ln} \tag{5.9}
\]
\[
\frac{\partial P_n}{\partial t_s} = -\sum_{l=0}^{n-1} \tilde{Q}'_{nl} P_l(\lambda_1) ; \quad \frac{\partial \tilde{P}_n}{\partial t_s} = -\sum_{l=0}^{n-1} \tilde{P}_l(\lambda_2)\tilde{Q}'_{ln} \tag{5.10}
\]
where we introduced
\[
\tilde{Q}_{nm} \equiv \left(H\tilde{Q}H^{-1}\right)_{nm} \quad \text{with} \quad H_{nm} = h_n\delta_{nm} \tag{5.11}
\]
Define now a wave function
\[
\Psi_n(t, \tilde{t}, \lambda) = P_n(\lambda_1)e^{V(t, \lambda_1)} \tag{5.12}
\]
From definition of \(\Psi_n\) and (5.5), (5.9) and (5.10) we obtain the following eigenvalue problem:
\[
\lambda_1 \Psi = Q \Psi \quad ; \quad \frac{\partial \Psi}{\partial t_r} = Q'_{(+)} \Psi \tag{5.13}
\]
\[
\frac{\partial \Psi}{\partial t_s} = -\tilde{Q}'_{-} \Psi \tag{5.14}
\]
where \(\Psi\) is a semi-infinite column \((\ldots, \Psi_n, \Psi_{n+1}, \ldots)^T\) and \(Q\) and \(\tilde{Q}\) are semi-infinite matrices, \(i.e.,\) with indices running from 0 to \(\infty\), Furthermore we adhere to the following notation: the subscripts \(-/+\) denote lower/upper triangular parts of the matrix, whereas \((+)/(−)\) denote upper/lower triangular plus diagonal parts.

In addition we also have a relation:
\[
\frac{\partial \Psi}{\partial \lambda_1} = -g\tilde{Q}\Psi \tag{5.15}
\]
which can be derived using identity:
\[
0 = \int d\lambda_1 d\lambda_2 \frac{\partial}{\partial \lambda_i} \exp\left(V(\lambda_1) + \tilde{V}(\lambda_2) + g\lambda_1\lambda_2\right) P_n(\lambda_1)\tilde{P}_m(\lambda_2) \quad i = 1, 2 \tag{5.16}
\]

### 5.2 Two-Matrix Model as a Discrete Linear System

The compatibility of the eigenvalue problem (5.13)-(5.16)\(^\text{[15, 11]}\) gives rise to a discrete linear system which we shall call the \textit{constrained} generalized Toda lattice hierarchy:
\[
\frac{\partial Q}{\partial t_r} = \left[Q_{(+)}', Q\right] \quad , \quad \frac{\partial \tilde{Q}}{\partial t_r} = \left[Q_{(+)}', \tilde{Q}\right] \quad , \quad r = 1, \ldots, p_1 \tag{5.17}
\]
\[
\frac{\partial Q}{\partial t_s} = \left[Q, \tilde{Q}'\right] \quad , \quad \frac{\partial \tilde{Q}}{\partial t_s} = \left[Q, \tilde{Q}'\right] \quad , \quad s = 1, \ldots, p_2 \tag{5.18}
\]
\[
-\tilde{g} \left[Q, \tilde{Q}\right] = \mathbb{1} \tag{5.19}
\]
As will be shown later the lattice system (4.66)–(4.68) can be identified with the \( \tilde{t}_1 \equiv x \) evolution equations of the above discrete linear system and therefore provides a direct link between matrix models and the integrable hierarchies [28].

In what follows it will be convenient to define an explicit parametrization of matrices \( Q \) and \( \bar{Q} \). We choose the following parametrization which is consistent with (5.6):

\[
Q_{nn} = a_0(n) , \quad Q_{n,n+1} = 1 , \quad Q_{n,n-k} = a_k(n) \quad k = 1, \ldots, p_2 - 1
\]

\[
\bar{Q}_{nn} = b_0(n) , \quad \bar{Q}_{n,n-1} = R_n , \quad \bar{Q}_{n,n+k} = b_k(n)R_{n+1}^{-1} \cdots R_{n+k}^{-1} \quad k = 1, \ldots, p_1 - 1
\]

\[
\bar{Q}_{nm} = 0 \quad \text{for} \quad n - m \geq 2 , \quad m - n \geq p_2
\]

\[
Q_{nm} = 0 \quad \text{for} \quad n - m \geq 2 , \quad m - n \geq p_1
\]

In most examples used in this section we work with the number \( p_2 = 3 \) in (5.21), whereas the number \( p_1 \) in (5.22) remains finite or \( \infty \).

Let us consider for the moment the first evolution parameters \( t_1, \tilde{t}_1 \) as coordinates of a two-dimensional space, i.e., \( \tilde{t}_1 \equiv \tilde{x} \) and \( t_1 \equiv y \), so all modes \( a_k(n), b_k(n) \) and \( R_n \) depend on \( (\tilde{x}, y; t_2, \ldots, t_{p_1}; \tilde{t}_2, \ldots, \tilde{t}_{p_2}) \).

The second lattice equation of motion (5.19) for \( s = 1 \), using parametrization (5.22), gives:

\[
\partial_{\tilde{x}} \left( \frac{b_k(n)}{R_{n+1} \cdots R_{n+k}} \right) = \frac{b_{k+1}(n) - b_{k+1}(n-1)}{R_{n+1} \cdots R_{n+k}} , \quad k \geq 2
\]

Similarly, the second lattice equation of motion (5.18) for \( r = 1 \) gives:

\[
\partial_y b_0(n) = R_{n+1} - R_n , \quad \partial_y b_k(n) = R_{n+1}b_{k-1}(n+1) - R_{n+k}b_{k-1}(n)
\]

for \( k \geq 1 \). From the above equations one can express all \( b_k(n \pm \ell) \), \( k \geq 2 \) and \( R_{n \pm \ell} \) (\( \ell \) – arbitrary integer) as functionals of \( b_0(n), b_1(n) \) at a fixed lattice site \( n \).

Furthermore, the lattice equations of motion (5.18)-(5.19) for \( r = 1, s = 1 \) read explicitly:

\[
\partial_{\tilde{x}} a_0(n) = R_{n+1} - R_n , \quad \partial_{\tilde{x}} a_k(n) = R_{n-k}a_{k-1}(n) - R_n a_{k-1}(n-1)
\]

(with \( k \geq 1 \)) and

\[
\partial_y a_0(n) = a_1(n+1) - a_1(n) , \quad \partial_y \left( \frac{a_k(n)}{R_{n} \cdots R_{n-k+1}} \right) = \frac{a_{k+1}(n+1) - a_{k+1}(n)}{R_{n} \cdots R_{n-k+1}}
\]

with \( k \geq 1 \). Following (5.23), (5.25), (5.26) we obtain the “duality” relations:

\[
\partial_y b_1(n) = \partial_{\tilde{x}} R_{n+1} , \quad \partial_{\tilde{x}} a_0(n) = \partial_y b_0(n) , \quad \partial_{\tilde{x}} a_1(n) = \partial_y R_n
\]

From the above one gets the two-dimensional Toda lattice equation:

\[
\partial_y \ln R_n = a_0(n) - a_0(n-1) \quad \rightarrow \quad \partial_{\tilde{x}} \partial_y \ln R_n = R_{n+1} - 2R_n + R_{n-1}
\]

\[\footnote{Both numbers \( p_{1,2} \) indicating the number of non-zero diagonals, outside the main one, of the matrices \( Q \) and \( \bar{Q} \) are related with the polynomial orders of the corresponding string two-matrix model potentials, whereas the constant \( g \) in (5.20) denotes the coupling parameter between the two random matrices.} \]

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Eqs. (5.26)–(5.29) allow to express all \(a_k(n \pm \ell)\), \(k \geq 1\) and \(R_{n\pm\ell}\) as functionals of \(a_0(n)\) and \(R_n\) (or \(a_1(n)\) instead) at a fixed lattice site \(n\). Furthermore, due to eqs. (5.26) and (5.25) for \(k = 1\), all matrix elements of \(Q\) and \(\bar{Q}\) are functionals of \(b_0(n), b_1(n)\) at a fixed lattice site \(n\). Alternatively, due to (5.28) we can consider \(a_0(n)\) and \(R_{n+1}\) as independent functions instead of \(b_0(n), b_1(n)\).

Let us also add the explicit expressions for the flow eqs. for \(b_0(n), b_1(n), R_{n+1}\) resulting from (5.18) and (5.19):

\[
\frac{\partial}{\partial t_s} b_0(n) = \partial_x \left( Q_s \right)_{nn}, \quad \frac{\partial}{\partial t_s} b_1(n) = \partial_x \left[ R_{n+1} \left( Q^s \right)_{n,n+1} \right], \quad \frac{\partial}{\partial t_s} R_{n+1} = \partial_x \left( \bar{Q}^s \right)_{n+1,n} \quad (5.30)
\]

\[
\frac{\partial}{\partial t_r} b_0(n) = \partial_x \left( Q^r \right)_{nn}, \quad \frac{\partial}{\partial t_r} b_1(n) = \partial_x \left[ R_{n+1} \left( Q^r \right)_{n,n+1} \right], \quad \frac{\partial}{\partial t_r} R_{n+1} = \partial_x \left( \bar{Q}^r \right)_{n+1,n} \quad (5.31)
\]

### 5.3 String Equation

Note the presence of the non-evolution constraint equation (5.20), which is called a string equation. The lattice equations for the matrix elements \(a_k(n)\) of \(Q\) (the first eqs. (5.18) and (5.19)) can be solved explicitly as functionals of the matrix elements of \(\bar{Q}\):

\[
Q_{(-)} = \sum_{s=0}^{p_2-1} \alpha_s \bar{Q}^s_{(-)}; \quad \alpha_s = -(s+1) \frac{\tilde{t}_{s+1}}{g} \quad (5.32)
\]

Equation (5.32) follows simply from (5.17) with \(i = 2\).

Note also that there is a complete duality under \(p_1 \leftrightarrow p_2\) when the order \(p_1\) of the first matrix potential in (5.1) is also finite. For instance due to this duality we can obtain the analog of eq. (5.32):

\[
\bar{Q}_{(+)} = \sum_{r=0}^{p_1-1} \beta_r Q_{(+)}; \quad \beta_r = -(r+1) \frac{t_{r+1}}{g} \quad (5.33)
\]

by interchanging \(p_1 \leftrightarrow p_2, \tilde{t}_s \leftrightarrow t_r, Q_{(-)} \leftrightarrow \bar{Q}_{(+)}\). Equation (5.33) follows simply from (5.17) with \(i = 1\).

The first equations of (5.20) and (5.23), imply the following two additional constraints on the independent functions \(b_0(n)\) and \(R_{n+1}\) (or \(b_1(n)\))

\[
\partial_y b_0(n) = \partial_x \left( \sum_{s=0}^{p_2-1} \alpha_s \bar{Q}^s_{nn} \right); \quad \partial_y R_{n+1} = \partial_x \left( \sum_{s=1}^{p_2-1} \alpha_s \bar{Q}^s_{n+1,n} \right) \quad (5.34)
\]

In fact, eqs. (5.34) are nothing but the component form of the string equation (5.20). Indeed upon using (5.32) and (5.18), (5.19), eq. (5.20) can be rewritten in the following form:

\[
\left( \partial_y - \sum_{s=1}^{p_2-1} \alpha_s \partial/\partial \tilde{t}_s \right) Q = -\frac{1}{g} \mathbb{1} \quad (5.35)
\]
or equivalently
\[
\left( \partial_y - \sum_{s=1}^{p_2-1} \alpha_s \partial/\partial t_s \right) b_0(n) = -\frac{1}{g} \quad \text{and} \quad \left( \partial_y - \sum_{s=1}^{p_2-1} \alpha_s \partial/\partial t_s \right) R_{n+1} = 0, \quad \left( \partial_y - \sum_{s=1}^{p_2-1} \alpha_s \partial/\partial t_s \right) b_k(n) = 0 \quad (5.36)
\]
for \( k \geq 1 \). Now, inserting (5.30) into (5.34) we find that the latter two equations precisely coincide with the \( nn \) and \( n+1, n \) component of the string eq. (5.36). Hence the string equation would amount to identifying the flow \( \partial/\partial y = \partial/\partial t_1 \) with the flow
\[
\frac{\partial}{\partial t_{p_2-1}} \equiv \sum_{s=1}^{p_2-1} \alpha_s \frac{\partial}{\partial t_s} \quad (5.37)
\]
if not for the constant \(-1/g\) on the right hand side of (5.36). For this reason it is more convenient to make a change of variables and correspondingly introduce another matrix \( \hat{Q} \):
\[
\hat{Q}_{p_2-1}^s(-) \equiv \sum_{s=0}^{p_2-1} \alpha_s \hat{Q}_{(-)}^s \quad (5.38)
\]
We now generalize (5.38) to:
\[
\hat{Q}_{(-)}^s = \sum_{\sigma=0}^{s} \gamma_{s\sigma} \hat{Q}_{(-)}^\sigma \quad s = 1, \ldots, p_2 \quad (5.39)
\]
and parametrize \( \hat{Q} \) as in (5.22)) with matrix elements \( \hat{R}_n, \hat{b}_k(n) \) in place of \( R_n, b_k(n) \). Coefficients \( \gamma_{s\sigma} \) are simply expressed through \( \alpha_s \equiv \gamma_{p_2-1, s} \quad (5.40) \) as:
\[
\gamma_{s\sigma} = (\gamma_{11})^s, \quad \gamma_{s, s-1} = \frac{s}{2} (\gamma_{11})^{s-2} \gamma_{21}, \quad \gamma_{s, s-2} = (\gamma_{11})^{s-4} \left[ \frac{s(s-3)}{8} (\gamma_{21})^2 + \frac{s}{3} \gamma_{11} \gamma_{31} \right] \quad (5.40)
\]
\[
\gamma_{11} = (\alpha_{p_2-1})^{\frac{1}{2p_2-1}}, \quad \gamma_{21} = \frac{2}{p_2-1} \left( \frac{\alpha_{p_2-2}}{\alpha_{p_2-1}} \right)^{\frac{2p_2-3}{2p_2-1}} \quad (5.41)
\]
and for the \( \hat{Q} \)-matrix elements we obtain:
\[
\hat{R}_n = \gamma_{11} R_n \quad \hat{b}_0(n) = \gamma_{11} b_0(n) + \frac{\gamma_{21}}{2\gamma_{11}}, \quad \hat{b}_1(n) = \gamma_{11}^2 b_1(n) + \frac{\gamma_{31}}{3\gamma_{11}} - \left( \frac{\gamma_{21}}{2\gamma_{11}} \right)^2 \quad (5.42)
\]
etc.

We can now rewrite the string equation in the “hatted” variables from eq. (5.42). First we note that the explicit calculation based on (5.41) gives
\[
\frac{\partial}{\partial t_{p_2-1}} \gamma_{s, s-1} = -\frac{s}{g} \gamma_{s, s} \quad \text{and} \quad \frac{\partial}{\partial t_{p_2-1}} \gamma_{s, s-2} = -\frac{s+1}{2g} \gamma_{s, s-1} \quad (5.43)
\]
Especially, we find
\[ \frac{\partial}{\partial t_{p_2-1}} \gamma_{21} = -\frac{2}{g} \gamma_{11}^2 \]  
(5.44)
and
\[ \frac{\partial}{\partial t_{p_2-1}} \left[ \frac{\gamma_{31}}{3\gamma_{11}} - \left( \frac{\gamma_{21}}{2\gamma_{11}} \right)^2 \right] = 0 \]  
(5.45)
Hence we easily get
\[ \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_{p_2-1}} \right) \hat{b}_i(n) = 0 \quad i = 1, 2 \]  
(5.46)
and the same result for \( \hat{R}_n \). This allows us to postulate a general statement:
\[ \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_{p_2-1}} \right) \hat{Q} = 0 \]  
(5.47)
Accordingly in the new “hatted” framework the string equation boils down to identification of the flow \( \partial/\partial t_1 \) with \( \partial/\partial t_{p_2-1} \) flow.
Consider now an arbitrary matrix \( X(Q, \bar{Q}) \). From equations (5.19), (5.18) and (5.32) we find
\[ \frac{\partial X(Q, \bar{Q})}{\partial t_{p_2-1}} = [X, \bar{Q}] = \left[ Q_1, X \right] + \left[ X, Q \right] = \frac{\partial X}{\partial t_1} + \left[ X, Q \right] \]  
(5.48)
Comparing with (5.47) we find that \( [\hat{Q}, Q] = 0 \). It is therefore plausible to supplement (5.38) by condition
\[ \hat{Q}^{p_2-1}_{(+)} = I_+ \longrightarrow \hat{Q}^{p_2-1} = Q \]  
(5.49)
with \( (I_+)^{nm} = \delta_{n+1,m} \). The last equality follows from (5.32). Let us also introduce the new subset of evolution parameters \( \{ \hat{t}_s \} \) through the equations:
\[ \frac{\partial}{\partial \hat{t}_s} \equiv \sum_{\sigma=1}^{s} \gamma_{s\sigma} \frac{\partial}{\partial t_{\sigma}} \quad , \quad s = 1, \ldots, p_2 \]  
(5.50)
which for \( s = p_2 - 1 \) reproduces (5.37). Now, taking into account (5.38), (5.50) and condition (5.49), all constrained Toda lattice eqs. (5.18)–(5.20) can be re-expressed as a single set of the flow equations for the matrix \( \hat{Q} \):
\[ \frac{\partial}{\partial \hat{t}_s} \hat{Q} = \left[ \hat{Q}, \hat{Q}^s \right] \quad , \quad s = 1, \ldots, p_2, 2(p_2 - 1), 3(p_2 - 1), \ldots, p_1(p_2 - 1) \]  
(5.51)
with the identification \( t_r \equiv \hat{t}_{r(p_2-1)} \). The associated linear problem reads:
\[ \hat{Q}^{nm} \hat{\psi}_m = \lambda \hat{\psi}_n \quad , \quad \frac{\partial}{\partial \hat{t}_s} \hat{\psi}_n = - \left( \hat{Q}^s \right)_{nm} \hat{\psi}_m \]  
(5.52)
From the fact that \( \hat{Q} \) has the same form as \( \bar{Q} \) up to the “hatting” we can deduce that \( \hat{Q} \) satisfies the Zakharov-Shabat equation:
\[ \frac{\partial}{\partial t_r} \hat{Q}^s - \frac{\partial}{\partial \hat{t}_s} \hat{Q}^r + \left[ \hat{Q}^r, \hat{Q}^s \right] = 0 \]  
(5.53)
Recalling associated linear problem (5.52) we see that (5.53) translates into:

$$\left[ \frac{\partial}{\partial t_r}, \frac{\partial}{\partial t_s} \right] \hat{\psi} = 0 \quad (5.54)$$

establishing the commutativity of flows. In fact an explicit calculation for the simplest nontrivial case of $p_2 = 3$ gives indeed (5.54) for $r, s = 1, 2, 3$.

Recall now the equation (5.33). Together with the string equation (5.20) it imposes a following condition on $Q$:

$$\left( \sum_{r=1}^{p_1-1} \beta_r \frac{\partial}{\partial t_r} - \frac{\partial}{\partial t_1} \right) Q = \left( \sum_{r=1}^{p_1-1} \beta_r \frac{\partial}{\partial t_r} - \gamma_{11}^{-1} \frac{\partial}{\partial t_1} \right) Q = \frac{1}{g} \quad (5.55)$$

Due to identification between $\hat{Q}$ and $Q$ modes in (5.49) this extra condition translates into a condition for $\hat{Q}^{p_2-1}$.

We obtain from (5.51),(5.52) the following continuum Lax problem at a fixed lattice site $n$:

$$x^{p_2-1} \hat{\psi}_n = \hat{L}(n) \hat{\psi}_n, \quad \frac{\partial}{\partial t_s} \hat{\psi}_n = -\hat{L}_s(n) \hat{\psi}_n \quad (5.56)$$

$$\frac{\partial}{\partial t_s} \hat{L}(n) = [\hat{L}(n), \hat{L}_s(n)], \quad s = 1, \ldots, p_2, 2(p_2 - 1), 3(p_2 - 1), \ldots p_1(p_2 - 1) \quad (5.57)$$

$$\hat{L}(n) \equiv -D_x^{-1} \hat{R}_{n+1} + \frac{\hat{Q}^{p_2-1}}{\hat{Q}_{n-n-1}} \left( D_x - \partial_x \ln \left( \hat{R}_n \ldots \hat{R}_{n-k+2} \right) \right) \times \ldots \times \left( D_x - \partial_x \ln \hat{R}_n \right) D_x \quad (5.58)$$

$$\hat{L}_s(n) \equiv \sum_{k=1}^{s} \left( \frac{(-1)^k \hat{Q}^{s_{n-k}}_{n-k}}{\hat{R}_n \ldots \hat{R}_{n-k+1}} \right) \left( D_x - \partial_x \ln \left( \hat{R}_n \ldots \hat{R}_{n-k+2} \right) \right) \ldots D_x \quad (5.59)$$

where all coefficients are expressed in terms of $\hat{R}_{n+1}, \hat{b}_0(n), \ldots, \hat{b}_{p_1-2}(n)$ at a fixed site $n$ through the $t_1 \equiv x$ lattice equations of motion (5.51). Up to a gauge transformation and conjugation, the explicit form of $\hat{L}(n)$ reads:

$$e^{\int b_0(n)} \left( \hat{L}(n) \right)^* e^{-\int b_0(n)} = D_x^{p_2-1} + (p_2 - 1) \hat{b}_1(n) D_x^{p_2-3} + \cdots + \hat{R}_{n+1} \left( D_x - \hat{b}_0(n) \right)^{-1} \quad (5.60)$$

Eqs.(5.56)–(5.59) are the continuum analogs of the constrained Toda lattice Lax equations (5.13)–(5.16) and (5.18)–(5.20) without taking any continuum (double-scaling) limit. Let us particularly stress that they explicitly incorporate the whole information from the matrix-model string equation (5.20) through (5.32) and (5.38)–(5.42) which were used in their derivation.

Consider a case $p_2 = 3$. The Lax operator $\hat{L}(n)$ from (5.58) is given by:

$$\hat{L}(n) = -D_x^{-1} \hat{R}_{n+1} + 2 \hat{b}_1(n) + \hat{b}_n^2(n) - \partial_x \hat{b}_0(n) - 2 \hat{b}_0(n) D + D^2 \quad (5.61)$$
The corresponding lattice equations then read (we write down explicitly only the \( \hat{t}_1 \equiv x \) and \( \hat{t}_2 \) evolution equations):

\[
\begin{align*}
\partial_x \ln \hat{R}_{n+1} &= \hat{b}_0(n+1) - \hat{b}_0(n) , \quad \hat{b}_1(n) - \hat{b}_1(n-1) = \partial_x \hat{b}_0(n) \quad (5.62) \\
\hat{R}_{n+1} - \hat{R}_n &= \partial_x \left( \hat{b}_0^2(n) + \hat{b}_1(n) + \hat{b}_1(n-1) \right) \\
\frac{\partial}{\partial \hat{t}_2} \hat{R}_{n+1} &= \partial_x \left[ \partial_x \hat{R}_{n+1} + 2 \hat{b}_0(n) \hat{R}_{n+1} \right] \\
\frac{\partial}{\partial \hat{t}_2} \hat{b}_0(n) &= \partial_x \left[ 2 \hat{b}_1(n) + \hat{b}_0^2(n) - \partial_x \hat{b}_0(n) \right] , \quad \frac{\partial}{\partial \hat{t}_2} \hat{b}_1(n) = \partial_x \hat{R}_{n+1} \quad (5.65)
\end{align*}
\]

Now, we observe that the system of Darboux-Bäcklund equations for \( SL(3,1) \) cKP hierarchy (4.66)–(4.69) exactly coincides upon identification:

\[
B^{(n)} = \hat{b}_0(n-1) \quad , \quad u^{(n)} = 2 \hat{b}_1(n-1) \quad , \quad A^{(n)} = \hat{R}_n \quad (5.66)
\]

with the \( x \equiv \hat{t}_1 \) constrained Toda lattice evolution equations (5.62)–(5.63). Also, the higher Toda lattice evolution parameters can be identified with the following subset of evolution parameters of the \( SL(p_2,1) \) KP-KdV hierarchy (4.61) [11, 12] :

\[
\hat{t}_s \simeq t_s^{KP-KdV} \quad s = 2, \ldots, p_2 \quad ; \quad t_r \simeq t_r^{KP-KdV} \quad r = 1, \ldots, p_1 \quad (5.67)
\]

the second identification resulting from (5.38).

In particular, excluding \( \hat{b}_0(n) \equiv B^{(n+1)} \) and \( \hat{b}_1(n) \equiv \frac{1}{2} u^{(n+1)} \) in (4.68) using (5.62)–(5.63), we obtain the two-dimensional Toda lattice equation for \( A^{(n)} \equiv \hat{R}_n \):

\[
\partial_x \partial_{\hat{t}_2} \ln A^{(n)} = A^{(n+1)} - 2A^{(n)} + A^{(n-1)} \quad (5.68)
\]

### 5.4 Partition Function of the Two-Matrix String Model

The partition function \( Z_N \) of the two-matrix string model is simply expressed in terms of the \( \hat{Q} \) matrix element \( b_1(N-1) \) at the Toda lattice site \( N-1 \), where \( N \) indicates the size \( (N \times N) \) of the pertinent random matrices: \( \partial_x^2 \ln Z_N = b_1(N-1) \) (cf. [53, 11]). Thus, using (4.42) and (4.53) together with (5.67), and accounting for the relations (5.42)–(5.41), we obtain the following exact solution at finite \( N \) for the two-matrix model partition function:

\[
Z_N = W_N \left[ \hat{\Phi}(0), \partial_{\hat{t}_{p_2-1}} \hat{\Phi}(0), \ldots, \partial_{\hat{t}_{p_2-1}}^{N-1} \hat{\Phi}(0) \right] \exp -N \int_{\hat{t}_1}^{\hat{t}_2} \left( \frac{\gamma_{21}}{2\gamma_{11}} \right) \quad (5.69)
\]

\[
\hat{\Phi}(0) = \int \frac{d\lambda}{2\pi} c(\lambda) e^{\xi(\lambda, \{t, t\})} \ , \ \xi(\lambda, \{t, t\}) \equiv \sum_{s=1}^{p_2} \lambda^s t_s + \sum_{r=2}^{p_1} \lambda^{r(p_2-1)} t_r \quad (5.70)
\]

where \( W_N \ldots \equiv \det \left( \partial_x^{i-1} \partial_{\hat{t}_{p_2-1}}^{j-1} \hat{\Phi}(0) \right)_{1 \leq i,j \leq N} \) with \( x \equiv \hat{t}_1 \). The \( \gamma \)-coefficients are defined in (5.41). The “density” function \( c(\lambda) \) in (5.70) is determined from matching the expression for
\( \Phi^{(0)} : \partial_x \ln \Phi^{(0)} = \hat{b}_0(0) = \gamma_{11}b_0(0) + \gamma_{21}/2\gamma_{11} \) (cf. (4.69), (5.66), (5.42)), with the expression for \( b_0(0) \) in the orthogonal-polynomial formalism [15]:

\[
\int \frac{d\lambda}{2\pi} c(\lambda) \exp \left( \sum_{s=1}^{p_2} \lambda^s \hat{t}_s + \sum_{r=2}^{p_1} \lambda^r(p_2-1)t_r \right)
= \exp \left( \int \frac{\gamma_{21}}{2\gamma_{11}} \right) \int \int d\lambda_1 d\lambda_2 \exp \left( \sum_{r=1}^{p_1} \lambda_1^r t_r + \sum_{s=1}^{p_2} \lambda_2^s \hat{t}_s + g\lambda_1\lambda_2 \right) \bigg|_{t_1=t_{p_2-1}(\hat{t})} \tag{5.71}
\]

One can easily check that \( W_N \) from (5.69) satisfies the two-dimensional Toda lattice equation \( \partial^2 \ln W_N / \partial t_1 \partial t_{p_2-1} = W_{N+1}W_{N-1}/W_N^2 \). Furthermore \( Z_N^{\gamma_{11}^{-N(N-1)/2}} \) provides a solution to the two-dimensional Toda lattice equation \( \partial^2 \ln Z_N / \partial t_1 \partial \hat{t}_1 = Z_{N+1}Z_{N-1}/Z_N^2 \) underlying the two-matrix model. This shows relevance of methods based on the Darboux-Bäcklund transformations for obtaining partition functions of the multi-matrix models.
A Schur Polynomials

Definition. The Schur polynomials $p(x_1, x_2, \ldots)$ are polynomials of the multi-variable $x \equiv (x_1, x_2, \ldots)$ defined by the generating function:

$$\exp \left( \sum_{k=1}^{\infty} x_k z^k \right) = \sum_{k=0}^{\infty} p_k(x) z^k$$ (A.1)

and $p_k(x) = 0$ for $k < 1$.

In particular

$$p_0(x) = 1, \ p_1(x) = x_1, \ p_2(x) = x_1^2/2 + x_2, \ p_3(x) = x_1^3/6 + x_1 x_2 + x_3$$ (A.2)

and generally $p_n(x) = x_1^n/n! + \ldots + x_n$. Also $(\partial/\partial x_m) p_n(x) = p_{n-m}(x)$.

We need the following recurrence relation.

Lemma.\[
mp_m(\tilde{x}) = \sum_{k=1}^{m} x_k p_{m-k}(\tilde{x})
\] (A.3)

where $\tilde{x} \equiv (x_1, x_2, x_3, \ldots)$

Proof.

From definition (A.1) we get

$$\exp \left( \sum_{k=1}^{\infty} \frac{x_k}{k} z^k \right) = \sum_{k=0}^{\infty} p_k(\tilde{x}) z^k$$ (A.4)

Multiplying on both sides by $\sum_{l=0}^{\infty} x_l z_l$ we obtain

$$\sum_{m=0}^{\infty} \left( \sum_{k=1}^{m} x_k p_{m-k}(\tilde{x}) \right) z^m = \sum_{l=0}^{\infty} x_l z_l \exp \left( \sum_{k=1}^{\infty} \frac{x_k}{k} z^k \right) = z \frac{dz}{d\tilde{z}} \exp \left( \sum_{k=1}^{\infty} \frac{x_k}{k} z^k \right)$$

$$= z \frac{dz}{d\tilde{z}} \sum_{m=0}^{\infty} p_m(\tilde{x}) z^m = \sum_{m=0}^{\infty} m p_m(\tilde{x}) z^m$$ (A.5)

the final result follows now by comparing coefficients of $z^m$.

Especially taking $x_k = -\partial/\partial t_k$ in (A.3) we get

$$mp_m(-\tilde{\partial}) = -\sum_{k=1}^{m} \frac{\partial p_{m-k}(-\tilde{\partial})}{\partial t_k} = -\sum_{k=1}^{m-1} \frac{\partial p_{m-k}(-\tilde{\partial})}{\partial t_k} - \frac{\partial}{\partial t_m}$$ (A.6)

which reproduces (2.21) in the text when applied on $\ln \tau$ after an extra $x$-derivative is taken on both sides.

B Wronskian Preliminaries

We list here three basic properties of the Wronskian determinants.
1) The derivative $D'$ of a determinant $D$ of order $n$, whose entries are differentiable functions, can be written as:

$$D' = D_{(1)} + D_{(2)} + \ldots + D_{(n)}$$  \hspace{1cm} (B.1)

where $D_{(i)}$ is obtained from $D$ by differentiating the entries in the $i$-th row.

2) Jacobi expansion theorem:

$$W_k (f) W_{k-1} = W_k W'_{k-1} (f) - W'_k W_{k-1} (f)$$  \hspace{1cm} (B.2)

or

$$\partial \left( \frac{W_{k-1}(f)}{W_k} \right) = \frac{W_k (f) W_{k-1}}{W_k^2}$$  \hspace{1cm} (B.3)

where the Wronskians are

$$W_k \equiv W_k[\psi_1, \ldots, \psi_k] = \det \left( \frac{\partial^{i-1}\psi_j}{\partial x^{i-1}} \right)_{1 \leq i,j \leq k}$$  \hspace{1cm} (B.4)

and

$$W_{k-1} (f) \equiv W_k[\psi_1, \ldots, \psi_{k-1}, f].$$  \hspace{1cm} (B.5)

The proof (see also [2, 3]) of (B.2) uses the fact that the left hand side of (B.3) can be written as a linear combination $\sum_{p=1}^{k} a_p f^{(p)}$. Expression for the coefficients $a_p$ (up to multiplication with a common function) can be obtained by e.g. the Cramer’s rule due to the fact that $\sum_{p=1}^{k} a_p \psi_i^{(p)} = 0$ for $i = 1, \ldots, k$ as can be verified directly from (B.3). It is now easy to see that the left hand side must be proportional to $W_k (f)$. The proof follows now by establishing that the terms with $f^{(k-1)}$ on both sides agree.

Take now a special class of Wronskians

$$W_n \equiv W_n[\psi, \psi', \ldots, \partial^{n-1}\psi] = \det \left( \frac{\partial^{i+j-2}\psi}{\partial x^{i+j-2}} \right)_{1 \leq i,j \leq n}.$$  \hspace{1cm} (B.6)

Hence, from (B.2) we get:

$$W_n W''_n - W'_n W'_n = W_n W'_{n-1} (\partial^n \psi) - W'_n W_{n-1} (\partial^n \psi) = W_{n-1} W_{n+1}$$  \hspace{1cm} (B.7)

which can be rewritten as

$$\partial^2 \ln W_n = \frac{W_{n+1} W_{n-1}}{W_n^2}$$  \hspace{1cm} (B.8)

3) Iterative composition of Wronskians:

$$T_k T_{k-1} \ldots T_1 (f) = \frac{W_k (f)}{W_k}$$  \hspace{1cm} (B.9)

where

$$T_j = \frac{W_j}{W_{j-1}} \partial \frac{W_{j-1}}{W_j} = \left( \partial + \left( \ln \frac{W_{j-1}}{W_j} \right)' \right); \quad W_0 = 1$$  \hspace{1cm} (B.10)
The proof of (B.9) follows by simple iteration of (B.3) (see also the standard references on this subject [23, 38, 2]). For future use let us rewrite (B.9) as:

\[
(\partial + v_k)(\partial + v_{k-1}) \cdots (\partial + v_1) f = \frac{W_k(f)}{W_k}; \quad v_j \equiv \partial_x \ln \frac{W_{j-1}}{W_j}
\]  \hspace{1cm} (B.11)

or

\[
(D + v_k)(D + v_{k-1}) \cdots (D + v_1) = \frac{1}{W_k} \begin{vmatrix}
\psi_1 & \psi_2 & \cdots & 1 \\
\psi'_1 & \psi'_2 & \cdots & D \\
\vdots & \vdots & \cdots & \vdots \\
\psi^{(k)}_1 & \psi^{(k)}_2 & \cdots & D^k
\end{vmatrix}
\]  \hspace{1cm} (B.12)
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