RICCI CURVATURE AND QUANTUM GEOMETRY

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We describe a few elementary aspects of the circle of ideas associated with a quantum field theory (QFT) approach to Riemannian Geometry, a theme related to how Riemannian structures are generated out of the spectrum of (random or quantum) fluctuations around a background fiducial geometry. In such a scenario, Ricci curvature with its subtle connections to diffusion, optimal transport, Wasserstein geometry and renormalization group, features prominently.

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1. Introduction: Ricci curvature

In this short review paper, we discuss some unconventional aspects that Ricci curvature still holds in store and which stress its basic role in a quantum field theory approach to Riemannian geometry. Even if the emphasis on the unconventional somehow implies that we assume the reader familiar with the conventional properties of Ricci curvature, it is worthwhile to recall some basic definitions involving our sponsor. Hence, in what follows, if not otherwise stated, \((M, g)\) is a smooth compact \(n\)-dimensional manifold \((n \geq 3)\) without boundary whose tangent bundle \(TM\) is endowed with a Riemannian metric \(g\). We respectively denote by \(d\mu_g\), \(\nabla\) and \(\text{Rm}(g)\) the Riemannian measure, the Levi–Civita connection and Riemann curvature tensor associated with \(g\), where \(\text{Rm}(g)(X,Y)Z :=\)

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The geometric nature of the Ricci tensor $\text{Ric}(g)$ is made manifest by its equivariance properties under the action of $\text{Diff}(M)$, the diffeomorphisms group of the underlying manifold $M$. In particular, if we denote by $\tilde{\phi}_t^* g$ the pull–back of the metric $g$ under a one-parameter group of diffeomorphism $[0,1] \ni t \mapsto \phi_t \in \text{Diff}(M)$ with $\phi_{t=0} = \text{Id}_M$, then we get the equivariance relation $\text{Ric}(\tilde{\phi}_t^* g) = \phi_t^* \text{Ric}(g)$, which in the limit $t \to 0$ gives rise [1] to the contracted Bianchi identity $\nabla^t \text{Ric}_{ik} = \frac{1}{2} \nabla_k \text{Ric}$. The $\text{Diff}(M)$–equivariance is the rationale underlying the interpretation of Ricci curvature (and the other curvatures) as a second order partial differential equation acting on the metric tensor. In particular, according to a carefully crafted analysis [2], [3], the Bianchi identity represents the basic obstruction for a symmetric bilinear form $A \in C^\infty(M, \otimes^2 TM^*)$ to be realized as the Ricci curvature of some Riemannian metric $\tilde{g}$, i.e., the local obstruction to solve the system of (weakly)-elliptic partial differential equations $\text{Ric}(\tilde{g}) = A$. (weakly here refers to the fact that the Ricci tensor, thought of as a second order partial differential operator acting on the metric, is non–degenerate only in the direction transversal to the $D\text{iff}(M)$–orbit of $g$). A further and subtle set of geometrical properties of $\text{Ric}(g)$ is related to its scaling properties, a consequence of the fact that besides diffeomorphisms, the metric $g$ is naturally acted upon also by overall rescalings according to $g \mapsto \lambda g$ for all $\lambda \in R_{>0}$ (in local coordinates this takes the form $g_{ik} \mapsto \lambda g_{ik}$ and $g^{ik} \mapsto \lambda^{-1} g^{ik}$). Correspondingly, we have the following induced scalings in the Riemannian volume, in the Levi-Civita connection, and in the associated curvatures: $\text{Vol}(\lambda g)(\Sigma) = \lambda^{\frac{n}{n-2}} \text{Vol}(g)(\Sigma)$, $\nabla^{(\lambda g)} = \nabla^{(g)}$, $\text{Rm}(\lambda g) = \text{Rm}(g)$, $\text{Ric}(\lambda g) = \text{Ric}(g)$, and $R(\lambda g) = \lambda^{-1} R(g)$. Hence both the Riemann and the Ricci tensor are scale–invariant. Diffeomorphisms equivariance and scaling properties of the Ricci tensor, with their PDE implications lurking in the background, play a basic role in discussing the critical points of the (volume–normalized) Einstein–Hilbert functional $\mathcal{S}_{E-H}[g] = \int_M \text{Ric}(g) \, d\mu_g$, where (for $M$ compact) $\Vol_g(M) := \int_M d\mu_g$. These critical points are the Einstein manifolds, namely Riemannian manifolds whose Ricci tensor is such that $\text{Ric}(g) = \rho(g) g$ for some constant $\rho(g)$. Notice that the Einstein constant $\rho(g)$ scales non–trivially: Since $\text{Ric}(g)$ is scale–invariant, we must have $\rho(\lambda g) \mapsto \lambda^{-1} \rho(g)$. 

\[ (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} ) Z, \] for vector fields $X,Y,Z \in C^\infty(M,TM)$ and where $[X,Y]$ denotes the commutator of $X,Y$. The Ricci curvature in the direction of the vector field $u \in C^\infty(M,TM)$ is defined by tracing the Riemann curvature according to $\text{Ric}(g)(u,u) := \text{trace}_\xi (\xi \mapsto \text{Rm}(g)(\xi,u)u)$. In terms of an orthonormal frame $\{e_a\}_{a=1}^n$, $g(e_a,e_b) = \delta_{ab}$, we have the Ricci (curvature) tensor expressed as $\text{Ric}(g)(u,v) = \sum_{a=1}^n g(\text{Rm}(g)(e_a,u)v, e_a)$. In local coordinates, $R_{ik} = R^h_{hik} -$ (notice that the upper index on the Riemann tensor is lowered into the 4–th position 

\[ \text{R}(g) := \sum_{a=1}^n \text{Ric}(g)(e_a,e_a) = g^{ik} R_{ik}. \]
A well–known geometrical interpretation of the Ricci curvature is provided by its expression in normal geodesic coordinates obtained by pulling back the Riemannian measure $d\mu_g$ to the tangent space $T_p M$, via the exponential map $\exp_p$ based at the point $p \in M$, (see e.g. [4]). If we denote by $d\mu_E$ the Euclidean measure on $T_p M$, then to leading order in the geodesic distance, this pull-back is provided by the Bertrand-Puiseaux formula $\exp^* p (d\mu_g) = \left(1 - \frac{1}{6} R_{ik}(p) u^i(q) u^k(q) + \ldots \right) d\mu_E$. It describes the Ricci curvature at the generic point $p \in (M, g)$ as the distortion (with respect to the Euclidean measure) of the Riemannian solid angle subtended by a small pencil of geodesics issued from $p$ in the direction $u = \exp_p^{-1}(q)$. From the point of view of geometric analysis a deeper insight on the nature of Ricci curvature is provided by the expression of its components in local harmonic coordinates $(U, \{x^i\}; \Delta_g x^i = 0)$. These latter are defined by requiring that each coordinate function defined on the coordinate neighborhood $U \subset M$, $x^k : U \rightarrow \mathbb{R}$ is harmonic. In particular, local solvability for elliptic PDEs implies (compare with [5], Corollary 3.30) that for any given point $p \in (M, g)$ there always exists a neighborhood $U_p \subset M$ of $p$ such that $\Delta_g x^i = \frac{\partial}{\partial x^i} g^{ij} + g^{ij} \frac{\partial}{\partial x^i} (\ln \sqrt{\det g}) = 0$. When passing from normal geodesic coordinates to harmonic coordinates we gain control on the components of the metric tensor in terms of the Ricci curvature rather than of the full Riemann tensor (cf. [6] for a particularly clear comparison of these two coordinate systems from the point of view of geometric analysis). In particular, as first stressed by C. Lanczos [7], in harmonic coordinates the components of the Ricci tensor take the suggestive form, (see e.g. [8] Lemma 2.6 for an elegant computation),

$$R_{ik} = \frac{1}{2} \Delta_g (g_{ik}) + Q_{ik} (g^{-1}, \partial g),$$  \hspace{1cm} (1)

where for each fixed pair of indexes $(i, k)$, the expression $\Delta_g (g_{ik})$ denotes the Laplace–Beltrami operator applied componentwise to each $g_{ik}$ as if it were a scalar function, (in particular $\Delta_g g_{ik}$ is not the tensorial (rough) Laplacian $g^{ab} \nabla_a \nabla_b g_{ik}$ of the metric tensor $g$, a quantity that obviously vanishes identically for the Levi–Civita connection). In [11], $Q (g^{-1}, \partial g)$ denotes a sum of terms quadratic in the components of $g, g^{-1}$ and their first derivatives, and whose explicit expression does not concern us here (compare with [8]). Hence in harmonic coordinates the Ricci curvature acts as a quasi–linear elliptic operator on the components of the metric, an observation that can be exploited to prove [12] that the metric tensor $g$, if not smooth, has maximal regularity in harmonic coordinates. It is also worthwhile to observe that the factor $1/2$ multiplying the Laplacian in (1), is not as incidental as is typically assumed. This is related to the well–known and, in the words of Elton P. Hsu [10], somewhat baffling fact that $\frac{1}{2} \Delta_g$ and not $\Delta_g$ is the generator of Brownian motion on $(\Sigma, g)$. This latter remark directly points to one of the subtlest meanings of Ricci curvature, related to its ubiquitous role in discussing diffusion over Riemannian manifolds.
2. Ricci curvature and Riemannian manifolds with density

Both the harmonic coordinate relation \([11]\) as well as the normal geodesic coordinates expression \(R_{i}(u,u) \propto \exp_{p}^{-1}(q)\), are prescient signals that Ricci curvature may find a deeper interpretative framework in the more general setting of \(n\)-dimensional compact Riemannian manifolds with density \([11], [12]\). This is the set of smooth orientable manifolds \((M,g, d\omega)\) without boundary endowed with a Riemannian metric \(g\) and a positive Borel measure such that \(d\omega \ll d\mu_{g}\). The absolute continuity requirement is with respect to the Riemannian volume element \(d\mu_{g}\), i.e., \(d\omega = e^{-f} d\mu_{g}\), for some function \(f \in C_{\infty}(M, \mathbb{R})\). In such a framework, the relevant differential operators on \((M,g,d\omega)\) are the \(d\omega\)-weighted divergence \(\nabla_{(\omega)} k v_{k} := e^f \nabla_{k} (e^{-f} v_{k})\), where \(v_{k}\) are the components of a smooth vector field \(v \in C_{\infty}(M,TM)\), and the \(d\omega\)-weighted Laplacian \(\Delta_{(\omega)} \psi := (\Delta_{g} - \nabla f \cdot \nabla) \psi\), for \(\psi \in C_{\infty}(M, \mathbb{R})\) \([11]\), where \(\Delta_{g}\) denotes the Laplace–Beltrami Laplacian on \((M,g)\).

In a similar vein, the role of the Ricci tensor is taken over by the Bakry–Emery Ricci curvature
\[
Ric_{BE}(g,d\omega) := Ric(g) + Hess_{g} f = Ric(g) + \frac{1}{2} \mathcal{L}_{\nabla f} g,
\]
where \(Hess_{g} f := \nabla \nabla f\) denotes the Hessian of \(f\) and \(\mathcal{L}_{\nabla f} g\) is the Lie derivative of the metric \(g\) along the gradient vector field \(\nabla f\). We also have a natural generalization of the contracted Bianchi identity \(\nabla^{i} R_{ik} \equiv \frac{1}{2} \nabla_{k} R\) which, for a Riemannian manifold with density \((M,g, d\omega = e^{-f} d\mu_{g})\), takes the form
\[
\nabla^{i}_{(\omega)} R_{ik}^{BE} \equiv \frac{1}{2} \nabla_{k} R_{Per},
\]
where
\[
R_{Per}(g) := R(g) + 2 \Delta_{g} f - |\nabla f|^{2}_{g} = R(g) + 2 \Delta^{(\omega)}_{g} f + |\nabla f|^{2}_{g}
\]
is Perelman’s modified scalar curvature \([13]\). In such a framework, the role of the Einstein-Hilbert functional is played by Perelman’s \(F(g; f)\)-energy \([13]\)
\[
F(g; f) \equiv \int_{M} R_{Per}(g) d\omega = \int_{M} \left( R(g) + |\nabla f|^{2}_{g} \right) e^{-f} d\mu_{g}
\]
and by the associated geometric functional defined on the underlying Riemannian manifold \((M,g)\) by
\[
F[g] \equiv \inf_{\{f \in W^{1,2}(M), e^{-f} d\mu_{g} = 1\}} \int_{M} R_{Per}(g) d\omega
\]
where \(W^{1,2}(M)\) denotes the Sobolev space of functions on \((M,g)\) which, together with their first derivatives are square summable with respect to the Riemannian measure \(d\mu_{g}\). If we set \(\psi := e^{-f/2}\), then one can characterize the functional \(F[g]\) as the lowest eigenvalue \(\lambda_{1}[g]\) of the Schrödinger-like operator on \((M,g)\) defined by
\[
-4 \Delta g \psi + R(g) \psi = \lambda_{1}[g] \psi.
\]
The reader familiar with R. Hamilton’s Ricci flow theory \cite{14}, \cite{15}, \cite{16} will have certainly recognized in the above definitions some of the characters featuring in Perelman’s celebrated proof \cite{13}, \cite{17}, \cite{18} of Thurston’s geometrization program \cite{19}, \cite{20}, \cite{21}. This is a direct consequence of the scaling and \text{Diff}(M)–equivariance properties of the Ricci curvature which imply that not only Einstein, but also quasi–Einstein metrics do matter in Riemannian geometry. Quasi–Einstein metrics are characterized by a Ricci tensor which can be written as

\[ \text{Ric}(g) = \rho(g) g - \frac{1}{2} \mathcal{L}_{V(g)} g = \rho(g) g - \frac{1}{2} (\nabla_i V_k + \nabla_k V_i), \]

for some constant $\rho(g)$ and some complete vector field $V(g) \in C^\infty(M, TM)$. If $V$ is a gradient, $V_i = g^{ik} \partial_k f$ for some $f \in C^\infty(M, \mathbb{R})$, then the quasi–Einstein condition becomes

\[ \text{Ric} + \text{Hess}_g f = \frac{\rho(g)}{2} g, \]

i.e. the isotropy of the Bakry–Emery Ricci curvature of the Riemannian manifold with density $(M, g, d\omega := e^{-f} d\mu_g)$. More flavor to this remark is added if we note that quasi–Einstein metrics have a significant dynamical characterization. To this end, let us introduce a parameter $\beta$ with $0 \leq \beta < \epsilon < \frac{1}{2\rho(g)}$ and let us define $\lambda(\beta) := (1 - 2\rho(g) \beta)$. Consider \cite{22} the one–parameter family of diffeomorphisms $\phi_\beta : M \longrightarrow M$ solution of the non–autonomous ordinary differential equation

\[ \frac{\partial}{\partial \beta} \phi_\beta(p) = \frac{1}{\lambda(\beta)} V(g)(\phi_\beta(p)), \quad \phi_{\beta=0} = \text{id}_M. \]

If we define the one–parameter family of metrics $g(\beta) := \lambda(\beta) \phi_\beta^* g$ ($g(\beta = 0) = g$), obtained by pulling back the metric $g$ under the action of the family of diffeomorphisms $\phi_\beta$ and rescalings $\lambda(\beta)$, then we can easily verify that, for $0 \leq \beta < \epsilon < \frac{1}{2\rho(g)}$, the flow $\beta \mapsto g(\beta)$ satisfies the evolution

\[ \frac{\partial}{\partial \beta} g(\beta) = -2 \rho(g(\beta)) g(\beta) + \mathcal{L}_{V(g(\beta))} g(\beta) = -2 \text{Ric}(g(\beta)), \]

with the initial condition $g(\beta = 0) = g$. In other words, under the combined action of this family of diffeomorphisms and scalings, the quasi–Einstein metric $g$ generates a self–similar solution $g(\beta) := \lambda(\beta) \phi_\beta^* g$, $0 \leq \beta < \epsilon$, of the Ricci flow \cite{14}

\[ \frac{\partial}{\partial \beta} g_{ab}(\beta) = -2 R_{ab}(\beta), \]

\[ g_{ab}(\beta = 0) = g_{ab}, \quad 0 \leq \beta < \frac{1}{2\rho(g)}, \]

viz., quasi-Einstein metrics feature as those solutions of the Ricci flow which evolve only under the action of diffeomorphisms and scalings: the Ricci solitons \cite{23}.

3. Ricci curvature, Wasserstein distance and the heat kernel

The connection between Ricci curvature, diffeomorphisms, scalings, and Ricci flow comes further to the fore if we remove the requirement of absolute continuity of
the measure \( d\omega \) with respect to the reference Riemannian measure \( d\mu_g \). We refer to any such manifold \((M, g, d\omega)\) as a weighted Riemannian manifold and introduce the corresponding \( \infty \)-dimensional space \( \text{Met}(M) \times [\text{Prob}(M), d\omega]_g^W \) of all weighted Riemannian manifolds, where \( \text{Met}(M) \) is the space of all smooth Riemannian metrics over \( M \), and \( \text{Prob}(M) \) denotes the space of all Borel probability measures \( d\omega \) over \( M \) endowed with the quadratic Wasserstein (or more appropriately, Kantorovich-Rubinstein) distance \( d\omega_g^W \). Since this notion of distance plays a basic role in discussing Ricci curvature it is worthwhile to review its definition in some detail.

Let \( \text{Prob}(M \times M) \) denote the set of Borel probability measures on the product space \( M \times M \), and let us consider the set of measures \( d\sigma \in \text{Prob}(M \times M) \) which reduce to \( d\omega_1 \) when restricted to the first factor and to \( d\omega_2 \) when restricted to the second factor, \( i.e. \)

\[
\text{Prob}_{\omega_1, \omega_2}(M \times M) := \left\{ d\sigma \in \text{Prob}(M \times M) \mid \pi_1^*(d\sigma) = d\omega_1, \pi_2^*(d\sigma) = d\omega_2 \right\},
\]

where \( \pi_i^* \) and \( \pi_j^* \) refer to the push–forward of \( d\sigma \) under the projection maps \( \pi_i \) onto the factors of \( M \times M \). Measures \( d\sigma \in \text{Prob}_{\omega_1, \omega_2}(M \times M) \) are often referred to as couplings between \( d\omega_1 \) and \( d\omega_2 \). Given a (measurable and non–negative) cost function \( c : M \times M \to \mathbb{R} \), an optimal transport plan \( \sqrt{d\sigma_{opt}} \in \text{Prob}_{\omega_1, \omega_2}(M \times M) \) between the probability measures \( d\omega_1 \) and \( d\omega_2 \in \text{Prob}(M) \) is defined by the infimum, over all \( d\sigma(x, y) \in \text{Prob}_{\omega_1, \omega_2}(M \times M) \), of the total cost functional

\[
\int_{M \times M} c(x, y) \, d\sigma(x, y).
\]

On a Riemannian manifold \((M, g)\), the usual cost function is provided by the squared Riemannian distance function \( d_g^2(\cdot, \cdot) \), and a major result of the theory is that for any pair \( d\omega_1 \) and \( d\omega_2 \in \text{Prob}(M) \), there is an optimal transport plan \( \sqrt{d\sigma_{opt}} \), induced by a map \( \Upsilon_{opt} : M \to M \) coming from a gradient. The resulting expression for the total cost of the plan

\[
d_g^W(d\omega_1, d\omega_2) := \left( \inf_{\sigma \in \text{Prob}_{\omega_1, \omega_2}(M \times M)} \int_{M \times M} d_g^2(x, y) \, d\sigma(x, y) \right)^{1/2},
\]

characterizes the quadratic Wasserstein distance between the two probability measures \( d\omega_1 \) and \( d\omega_2 \). Note that there can be distinct optimal plans \( d\sigma_{opt} \) connecting general probability measures \( d\omega_1 \) and \( d\omega_2 \in \text{Prob}(M) \), whereas on the subset of \( d\mu_g \)–absolutely continuous measures the optimal transport plan is unique. The quadratic Wasserstein distance \( d_g^W \) defines a finite metric on \( \text{Prob}(M) \) and it can be shown that \((\text{Prob}(M), d_g^W)\) is a geodesic space, endowed with the weak–* topology (we refer to \([31],[32],[33]\) for the relevant properties of Wasserstein geometry and optimal transport we freely use in the following). Notice that if we denote by \((\text{Prob}_{\mu_g}(M), d_g^W)\) the set of all \( d\mu_g \)–absolutely continuous Borel measures on the Riemannian manifold \((M, g)\), then the space of Riemannian manifolds with density,
\[ \text{Met}(M) \times [\text{Prob}_w(M), d^W_g], \] is a dense subset of \[ \text{Met}(M) \times [\text{Prob}(M), d^W_g]. \]

A deep rationale on Ricci curvature, scaling, \( \text{Diff}(M) \)-equivariance, and the induced Ricci flow follows by observing that on the (dense) subset of Riemannian manifold with density, the weighted Laplacian \( \Delta^\omega_g := (\Delta_g - \nabla f \cdot \nabla) \) is a symmetric operator with respect to the defining measure \( d\omega \) and can be extended to a self–adjoint operator in the space of square \( \omega \)-summable functions \( L^2(M, d\omega) \) generating the heat semigroup \( e^{t\Delta^\omega_g}, t \in \mathbb{R}_{>0} \). The associated heat kernel \( p^\omega_t(\cdot, z) \) is defined as the minimal positive solution of

\[
\left( \frac{\partial}{\partial t} - \Delta^\omega \right) p^\omega_t(y, z) = 0 ,
\]

with \( \delta_z \) the Dirac measure at \( z \in (M, d\omega) \). The heat kernel \( p^\omega_t(y, z) \) is \( C^\infty \) on \( \mathbb{R}_{>0} \times M \times M \), is symmetric \( p^\omega_t(y, z) = p^\omega_t(z, y) \), satisfies the semigroup identity \( p^\omega_t(y, z) = \int_M p^\omega_s(y, x) p^\omega_{t-s}(x, z) \, d\omega(x), \) and \( \int_M p^\omega_t(y, z) \, d\omega(z) = 1 \). Moreover, Varadhan’s large deviation formula \cite{34} holds

\[
- \lim_{t \to 0^+} t \ln \left[ p^\omega_t(y, z) \right] = \frac{d^2_g(y, z)}{4} ,
\]

where \( d_g(y, z) \) is the Riemannian distance between the points \( y \) and \( z \) on \((M, g, d\omega)\), and the convergence is uniform over all \((M, g, d\omega)\). Since the map \((M, g) \to (\text{Prob}(M), d^W_g)\) defined by \( z \mapsto \delta_z \) is an isometry, (one directly computes \( d^W_g(\delta_u, \delta_z) = d_g(y, z) \), by using the trivial optimal plan \( d\sigma(u, v) = \delta_g(u) \otimes \delta_z(v) \) in \((\mathbb{M})\), Varadhan’s formula suggests that we can use the (weighted) heat kernel of \((M, g, d\omega)\), \( (t, \delta_z) \mapsto p^\omega_{t^2}(\cdot, x) \), with source at \( x \in M \), to generate an injective embedding of \((M, g, d\omega)\)

\[
\nu^W_{t^2} : (M, g) \to (\text{Prob}(M), d^W_g)
\]

\[
x \mapsto \nu^W_{t^2}(x) := p^\omega_{t^2}(\cdot, x).
\]

in the space \((\text{Prob}(M), d^W_{g,t^2})\) of all probability measures over \( M \) endowed with the quadratic Wasserstein distance, (see \cite{35} for the pure Riemannian case and \cite{36} for the general case of the embedding of \((M, g, d\omega)\)). If we exploit this immersion, by pulling–back via \( \nu^W_{t^2} \) the Wasserstein distance \( d^W_g(p^\omega_{t^2}(\cdot, x), p^\omega_{t^2}(\cdot, y)) \) to \( M \), then we get the \( t \)-dependent metric tensor on \( M \) defined by

\[
g_t(v(x), w(x)) := \int_M g(\nu^y\hat{\psi}_{t(x,v)}(y) \nabla_{k} \hat{\psi}_{t(x,v,y)}(y) \nabla_{k} \hat{\psi}_{t(x,w)}(y) p^\omega_{t^2}(y, x) \, d\mu_g(y) ,
\]

where \( \hat{\psi}_{(t,v)}, \hat{\psi}_{(t,w)} \) are the tangent vectors in \( \text{Prob}(M) \) associated with the manifold tangent vectors \( v \) and \( w \). This procedure generates a scale dependent metric on \( M \) such that \( \lim_{t \to 0} g_t(v, v) = g(v, v), \) \( v \in T_y M, y \in M, \) and

\[
g_t(v, v) \leq e^{-2K^0_y t} g(v, v) ,
\]
where $K^{B-E}_g$ denotes the lower bound of the Bakry–Emery Ricci curvature of $(M, g, d\omega)$. A rather sophisticated use of the (weak) Riemannian geometry of the Wasserstein space $(\text{Prob}(M, g), d^W_g)$, (related to the Riemannian geometry of the diffeomorphisms group $\text{Diff}(M)$ a’la Arnold [37]), allows to compute [36] a full–fledged flows for the metric $t \mapsto g(t)$ and for the measure field $t \mapsto f(t)$, viz.

$$\partial \frac{\partial}{\partial t} f = - \triangle^z f$$

$$\partial \frac{\partial}{\partial t} g_t(u, w) = - 2 \text{Ric}^{(t)}(u, w) - 2 \text{Hess} f(u, v)$$

$$- 2 \int_M \left( \text{Hess} \hat{\psi}_{(t, u)} \cdot \text{Hess} \hat{\psi}_{(t, w)} \right) p_t(\omega)(y, z) d\omega(y),$$

where $\text{Ric}^{(t)}$ denotes the Ricci curvature of the evolving metric $(M, g_t)$, and where $\hat{\psi}_{(t,u)}, \hat{\psi}_{(t,w)}$ are the tangent vectors in $\text{Prob}(M)$ representing the manifold tangent vectors $u$ and $w$, respectively. Hence we get an extended Ricci flow coupled with a (backward) parabolic evolution for the measure $d\omega = e^{-f} d\mu_g$.

4. Ricci curvature and quantum Riemannian geometry

The connection between Ricci curvature, weighted heat kernel, Ricci flow and the Wasserstein geometry of $\text{Met}(M) \times [\text{Prob}(M), d^W_g]$ can be extended to the discussion of the scale–dependent random fluctuations of maps between Riemann surfaces $(\Sigma, h)$ and weighted Riemannian manifolds $(M, g, d\omega)$. This analysis can be used to describe in a mathematically rigorous way a quantum field theory (QFT) renormalization group perspective on Ricci curvature. With this remark we are coming to full circle. Indeed, it is worth recalling that Quasi-Einstein metrics originated from theoretical physics [38], [39], [40], in the perturbative analysis of the Renormalization Group for (Dilatonic) Non-Linear Sigma Model (NLSM), governed by an action which is the quantum field theory avatar of a harmonic map functional. To wit, to a localizable\footnote{A suitable form of localizability is required since maps in $W^{1,2}(\Sigma, M)$ may fail to have continuous representative. This delicate issue is discussed at length in [36].} map $\phi : (\Sigma, h) \mapsto (M, g, d\omega = e^{-f} d\mu_g)$ varying in the space of Sobolev maps $W^{1,2}(\Sigma, M)$ we associate the action

$$S[\phi; a, f, g] := \frac{1}{2a} \int_{\Sigma} d^2x \sqrt{h} \left[ h^{\mu\nu} \partial_{\mu} \phi^i \partial_{\nu} \phi^k - g_{ik} + 2a K_h(x) f(\phi(x)) \right]$$

where $K_h$ is the Gauss curvature of the surface $(\Sigma, h)$, $a^{-1} g$ is the metric coupling associated with the (energy) scale parameter $a \in \mathbb{R}_{>0}$, and where the scalar function $f : M \mapsto \mathbb{R}$ characterizes the dilaton coupling of the model. Heuristically, quantum (actually, random) fluctuations of $\phi : \Sigma \mapsto M$ around a classical configuration $\phi_{cm}$, (typically, the center of mass of a large collection $(\rightarrow \infty)$ of constant maps $\{\phi_{(i)}\}$, identically and independently distributed with respect to a sampling
functional measure $D[\phi]$ on $W^{1,2}(\Sigma, M)$, can modify the Riemannian geometry of $(M, g, d\omega)$. The strategy is to exploit a scale dependent ($\beta = \alpha t, t \geq 0$) perturbative renormalization to get a renormalization group (RG) flow for the metric and the dilaton couplings, $a^{-1}g$ and $f$, controlled by a large deviation mechanism w.r.t. the Gaussian fluctuations around the (classical) background $\phi_{cm}$, (i.e. by the control of the exponential decline of large field fluctuations, around $\phi_{cm}$, as the energy scale $\beta$ varies). This procedure (re)constructs perturbatively the geometry in a ball around $\phi_{cm}$ as a function of the parameter $\beta$ according to [38], [39], [40]

$$\frac{\partial}{\partial \beta} g_{ik}(\beta) = -2 R_{ik}(\beta) - 2 \nabla_i \nabla_k f - \frac{a}{2} (R_{imn} R_m^{kn}) + \mathcal{O}(a^2)$$

$$\frac{\partial}{\partial \beta} f(\beta) = \Delta f(\beta) - |\nabla f(\beta)|^2 + \mathcal{O}(a^2).$$

As long as we are in a weak coupling regime, characterized by the condition $a |Rm(g(\beta))|^{1/2} \ll 1$, we have the connection with Ricci flow in the DeTurck version [41]

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = -2R_{ab}(\beta) - 2\nabla_a \nabla_b f, \quad g_{ab}(\beta = 0) = g_{ab}$$
coupled to the $d\omega$–weighted forward heat equation for the dilaton

$$\frac{\partial}{\partial \beta} f(\beta) = \Delta g(\beta) f.$$

All this can be made mathematically rigorous either from the point of view of geometric analysis by exploiting Wasserstein geometry along the lines described above [36] or, from the point of view of quantum theory, by framing the interplay between NLSM and Ricci flow into the algebraic quantum field theory approach [42]. Question of space do not allow us to provide a more detailed description. The interested reader may find full details in the quoted papers. It must be stressed that a full-fledged analysis of the (weakly) parabolic PDEs featuring in discussing Ricci flow and RG flow requires a change of perspective in the role of the dilaton coupling $f$. We need to impose Perelman’s coupling: $viz.$ we need to conjugate the dilaton $f$ rescaling to the $\beta$–evolving Riemannian measure $d\mu_g(\beta)$, by replacing $f \rightarrow \tilde{f}$, with

$$\frac{\partial}{\partial \beta} e^{-\tilde{f}(\beta)} d\mu_{g(\beta)} = 0,$$

couples the Hamilton-DeTurck Ricci flow above with the time–reversed, $\eta = \beta^* - \beta$, parabolic flow

$$\frac{\partial}{\partial \eta} \tilde{f}(\eta) = \Delta g(\eta) \tilde{f}(\eta) - R(\eta) \tilde{f}(\eta).$$

In this way one recovers the celebrated monotonicity of Perelman’s $F[g]$–energy along the Ricci flow and its gradient–like nature. This change of perspective is not confined to the mathematical analysis of the one-loop contribution (Ricci flow) to the perturbative renormalization group for NLSM, but also to the nature of higher order terms such as $a (R_{imn} R_k^{mn})$, as discussed in detail in [43].
References

[1] J. L. Kazdan, Another proof of Bianchi’s identity in Riemannian geometry, Proc. Amer. Math. Soc. 81 (1981), 341-342.
[2] D. DeTurck, Metrics with prescribed Ricci curvature, in Seminar on Differential Geometry, Ed. S.-T. Yau, Annals of Math. Studies, 102, Princeton Univ. Press (1982), 525A–537.
[3] D. DeTurck, Existence of metrics with prescribed Ricci curvature: local theory, Inventiones Math. 65 (1981), 179A–201.
[4] A. L. Besse, Einstein Manifolds, Ergeb. der Math. Vol. 10, Springer-Verlag Berlin Heidelberg, 1987.
[5] B. Chow, D. Knopf, The Ricci Flow: An Introduction, Math. Surveys and Monographs 110, (2004) Am. Math. Soc.
[6] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Institute Lecture Notes 5, Ann. Math. Soc. Providence (2000).
[7] C. Lanczos, Ein Vereinfachendes Koordinatensystem für die Einsteinschen Gravitationsgleichungen, Phys. Zeits. 23 (1922), 537-539.
[8] P. Petersen, Riemannian geometry Graduate Text in Math. 171 Springer-Verlag, New York, (1998).
[9] D. M. DeTurck, J. L. Kazdan, Some regularity theorems in Riemannian geometry, Ann. Sci. École Norm. Sup. 14 (1981) 249-260.
[10] E. P. Hsu, Stochastic Analysis on Manifolds AMS Graduate Series in Mathematics, 38, American Math. Soc. (2002).
[11] A. Grigor’yan, Heat kernels on weighted manifolds and applications. In The ubiquitous heat kernel, Contemp. Math., 398, Amer. Math. Soc., Providence, RI, (2006) 93A–191.
[12] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Progress in Math., 152, Birkhäuser, Boston (1999).
[13] G. Perelman, The entropy formula for the Ricci flow and its geometric applications math.DG/0211159v1, 11 Nov 2002
[14] R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982), 255A–306.
[15] R.S. Hamilton, The formation of singularities in the Ricci flow, in Surveys in differential geometry, Vol. II, Internat. Press, Cambridge, MA, 1995, 7A–136.
[16] R.S. Hamilton, Non-singular solutions of the Ricci flow on three-manifolds, Comm. Anal. Geom. 7 (1999), 695A–729.
[17] G. Perelman, Ricci Flow with Surgery on Three-Manifolds, arXiv:math.DG/0303109 10 Mar 2003.
[18] G. Perelman, Finite Extinction Time for the Solutions to the Ricci Flow on Certain Three-Manifolds, arXiv:math.DG/0307245 17 Jul 2003
[19] W.P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, in The Mathematical heritage of Henri Poincare, Proc. Symp. Pure Math.39(1983), Part 1.(Also in Bull. Amer. Math. Soc. 6 (1982), 357A–381.)
[20] W.P. Thurston, Hyperbolic structures on 3-manifolds, I, deformation of acyclic manifolds, Ann. Math. 124(1986), 203A–246
[21] W.P. Thurston, Three-Dimensional Geometry and Topology, Vol. 1, ed. by Silvio Levy,Princeton Mathematical Series 35, Princeton University Press, Princeton, 1997.
[22] A. Derdzinski, Compact Ricci solitons, preprint (2009).
[23] R. S. Hamilton, The Ricci flow on surfaces, Contemp. Math. 71, 237-261 (1988).
[24] L. V. Kantorovich, On the translocation of masses, Dokl. Akad. Nauk. (new ser.), 37 (1942), 199–201 (in English: J. Math. Sci., 133 (2006), 1381–1382.
[25] J. Lott, *Optimal transport and Ricci curvature for metric-measure spaces*, in Metric and comparison geometry (eds. J. Cheeger and K. Grove), Surv. Diff. Geom., 11, Int. Press, Somerville, MA (2007), 229–257.
[26] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. 169 (2009), 903–991.
[27] R. J. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. 11 (2001), 586–608.
[28] K-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math., 196 (2006), 65–131.
[29] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Commun. Pure Appl. Math., 44 (1991), 375–417.
[30] D. Cordero-Erausquin, R. J. McCann, M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math., 146 (2001) 219–257.
[31] C. Villani, *Topics in Optimal Transportation*, Grad. Studies in Math. 58, Amer. Math. Soc. Providence, Rhode Island (2003).
[32] L. Ambrosio, N. Gigli, G. Savaré, *Gradient Flows in metric spaces and in the space of probability measures*, ETH Lect. Notes in Math., Birkhäuser Verlag, Basel (2005).
[33] C. Villani, *Optimal Transport, Old and New*, Grundlehren der Matematischen Wissenschaften, 338 (2008), Springer-Verlag.
[34] S. R. Varadhan, *Diffusion processes in a small time interval*, Commun. Pure Appl. Math., 20 (1967) 659-685.
[35] N. Gigli, C. Mantegazza, *A flow tangent to the Ricci flow via heat kernels and mass transport*, arXiv:1208.5815v1 (2012).
[36] M. Carfora, *The Wasserstein geometry of nonlinear sigma models and the Hamilton–Perelman Ricci flow*, Reviews in Mathematical Physics Vol. 29, No. 01, 1750001-1750071 (2017)
[37] V. I. Arnol’d, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits*, Ann. Inst. Grenoble 16 (1966), 319-361.
[38] D. Friedan, *Nonlinear Models in 2, +ε Dimensions*, Ph. D. Thesis (Berkeley) LBL-11517, UMI-81-13038, Aug 1980. 212pp.
[39] D. Friedan, *Nonlinear Models in 2 + ε Dimensions*, Phys.Rev.Lett. 45 1057 (1980).
[40] D. Friedan, *Nonlinear Models in 2 + ε Dimensions*, Ann. of Physics 163, 318-419 (1985).
[41] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensor*, J. Differential Geom. 18 (1983), 157–162.
[42] M. Carfora, C. Dappiaggi, N. Drago, P. Rinaldi, *Ricci Flow from the Renormalization of Nonlinear Sigma Models in the Framework of Euclidean Algebraic Quantum Field Theory*, Commun. Math. Phys. (2019). https://doi.org/10.1007/s00220-019-03508-2
[43] M. Carfora, C. Guenther *Scaling and Entropy for the RG-2 Flow*, arXiv:1805.09773 math.DG