Non-Hermitian $\mathcal{PT}$-Symmetric Dirac-Pauli Hamiltonians with Real Energy Eigenvalues in the Magnetic Field

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Abstract

The modified Dirac-Pauli equations, which is entered by means of $\gamma_5$-mass extension of Hamiltonian operators, are considered. We also take into account the interaction of fermions with the intensive homogenous magnetic field focusing attention on $(g-2)$ gyromagnetic factor of particles with spin $1/2$. Without the use of perturbation theory in the external field the exact energy spectra are deduced with regard to spin effects of fermions. We discuss the possible proposals of experimental measuring of properties of new particles which arising in this model.

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1 Introduction

Now it is well-known fact, that the reality of the spectrum in models with a non-Hermitian Hamiltonian is a consequence of $\mathcal{PT}$-invariance of the theory, i.e. a combination of spatial and temporary parity of the total Hamiltonian: $[H, \mathcal{PT}]\psi = 0$. When the $\mathcal{PT}$ symmetry is unbroken, the spectrum of the quantum theory is real. This surprising results explain the growing interest in this problem which was initiated by Bender and Boettcher’s observation [1]. For the past a few years has been studied a lot of new non-Hermitian $\mathcal{PT}$-invariant systems (see, for example [2] - [19]).

The non-Hermitian $\mathcal{PT}$-symmetric $\gamma_5$-extension of the Dirac equation is first studied in [2] and further developed in [3]-[8]. The purpose of this paper
is the continuation of the studying examples of pseudo-Hermitian relativistic Hamiltonians, investigations of which was started by us earlier.

The quantum field theoretic problem of the motion of charged fermions in a uniform magnetic field has been solved in the paper [20]. In this paper the energy spectra of the charged fermions have been obtained as solutions of the Dirac equation at the neglect of the anomalous magnetic moment (AMM) of the fermion with a uniform magnetic field.

New stage of the research of exact solutions in the external field was opened by the paper Ternov, Bagrov and Zhukovskii [21]. In this paper, for the first time was found energy spectrum of fermions moving in the homogeneous magnetic field and was obtained wave functions for the case of fermions taking into account the vacuum supplements to the Bohr magneton. After this exact solutions of the Dirac equation with the Pauli term for charged and neutral fermions has been confirmed in [22]-[24].

The exact results, obtained in [21]-[24], in particular have been applied to the analysis of the synchrotron radiation and the neutron decay rates in very strong magnetic fields (see, for example [25]). Note also that intensive magnetic fields intensity of the order of $10^{12} \div 10^{13}$ Gauss observed near pulsars. Here also may be included the recent opening of such objects as sources soft repeated gamma-ray burst and anomalous x-ray pulsars. For them magnetorotational models are proposed, and they were named as magnetars. It was showed that for such objects achievable magnetic fields with intensity up to $10^{15}$ Gauss. It is very important that the share of magnetars in the General population of neutron stars reaches 10% [26]. In this regard, we note that the processes with the participation of neutrinos in the presence of such strong magnetic fields can have a significant influence on the processes which may determine the evolution of astrophysical objects.

In 1965, a hypothesis was proposed by M.A. Markov [27], according to which the mass spectrum of particles should be limited by the Planck mass $m_{\text{Planck}} = 10^{19}$ GeV. The particles with the limiting mass

$$m \leq m_{\text{Planck}}$$

were named maximons by the author. However, condition (1) initially was purely phenomenological and it has seemed until recently that the SM can be applied adequately up to Plank masses. In the current situation, however, more and more data have accumulated that are in favor of the necessity of
revising some physical principles. In particular, this is confirmed by abundant
evidence that dark matter, apparently exists and absorbs a substantial part
of the energy density in the Universe.

In the late 1970s, a new radical approach was offered by V.G.Kadyshevsky
[28] (see also [29], [30]), in which the Markov idea of the existence of a maximal
mass of particles was accepted as a new fundamental principle of the quantum
field theory (QFT). As it is known, the particle mass \( m \) in the SM can possess
a value in the interval \( 0 \leq m < \infty \). In the new geometrical theory, the
condition of the mass spectrum finiteness is postulated

\[
m \leq \mathcal{M},
\]

where the maximal mass parameter \( \mathcal{M} \), which is called by the \textit{fundamental mass}, is a \textit{new physical constant}. The quantity \( \mathcal{M} \) is considered as a curvature
radius of a five dimensional hyperboloid whose surface is a realization of the
curved momentum 4-space, or the anti de Sitter space. Objects with a mass
larger than \( \mathcal{M} \) cannot be regarded as elementary particles because no local
fields correspond to them. For a free particle, condition (2) automatically
holds on surface of a five dimensional hyperboloid. In the approximation
\( \mathcal{M} \gg m \) the anti de Sitter geometry goes over into the Minkowski geometry
in the four dimensional pseudo Euclidean space (“flat limit”).

We consider \( \mathcal{PT} \)-symmetric Hamiltonians from the standpoint of the al-
gebraic approach developed in works devoted to studying of non-Hermitian
quantum theory [3]-[8]. Hamiltonians under \( \gamma_5 \)-extension are non-Hermitian
but \( \mathcal{PT} \)-symmetric. As has been already noted, \( H \) is non-Hermitian due to
the summand with \( m_2 \) changing the sign at the Hermitian conjugation of
the Hamiltonian \( (H^+ \neq H) \). It is easy to verify that \( H \) is also non-invariant
at individual transforms \( \mathcal{P} \) or \( \mathcal{T} \) because the summand with \( m_2 \) changes the
sign under impact of any of these transforms. However, \( H \) is invariant with
respect to the joint transform. A similar model was considered in [2], where
the \( \mathcal{PT} \)-symmetric massive Thirring model was investigated in the \((1 + 1)\)
dimension.

The inequality \( m_1 \geq m_2 \) in this theory which is following from the con-
dition \( m^2 = m_1^2 - m_2^2 \), is the basic requirement that defines a domain of
the unbroken symmetry of the Hamiltonian under study [2]. However, this
inequality between \( m_1, m_2 \) and determination of physical mass of particle \( m \)
are not a single mass condition. Therefore we can write the new condition
for the physical mass \( m \), which, may be more substantial. Indeed, using the
simple mathematical theorem, we can obtain \[7\]

\[m \leq m_1^2/2m_2 = M.\] (3)

We recall that we are investigating the issue of the existence of constraints on mass parameters in the given theory. We suggest that there is a constraint on the parameter \(m\). In this case, there are reasons to believe that a relationship exists between \(M\) obtained by algebraic transformation and \(M\) from the geometric theory with limited mass \([3]-[7]\).

If one use the notation of fundamental length then we can write relationship \(l_f = 1/\mathcal{M}\), where \(h = c = 1\), which is mentioned in the works \([28]-[30]\). The non-Hermitian \(\mathcal{PT}\)-symmetric Hamiltonians may be considered as a kind of a very fruitful environment for the creation of new physics beyond the SM. The pioneering papers in this field were accomplished by Miloslav Znojil \([31],[32]\) where contains the development theories with \(\mathcal{PT}\)-symmetric Hamiltonians for considering the possible existence of the fundamental length in quantum physics. In this connection, investigation of the results of \([31],[32]\), together with studying of consequences of the limitations of the spectrum of masses of elementary particles could shed light on the possibility of comparing the expressions for the fundamental lengths obtained in different approaches.

Here we are producing our investigation of non-Hermitian systems with \(\gamma_5\)-mass term extension taking into account AMM of fermions in external magnetic field. We are studying the spectral and polarization properties of such systems (Section 2.). The novelty of developed by us approach is associated with predictions of new phenomena caused by a number of additional terms of the non-Hermitian Hamiltonians, which radically changes the picture of interactions (Section 3.). Most intriguing predictions developed in our paper is devoted to non-Hermitian mass extension \(m \to m_1 + m_2\gamma_5\) associated with the appearance in this the algebraic approach of some new particles (Section 4.). It is important that previously such particles (“exotic particles”) was observed only in the framework of the geometric approach to the construction of QFT.
2 Modified model for the study of non-Hermitian mass parameters

Let us now consider the solutions of modified Dirac equations for free massive particles using the $\gamma_5$-factorization of the ordinary Klein-Gordon operator. In this case similar to the Dirac procedure one can represent the Klein-Gordon operator in the form of a product of two commuting matrix operators:

\[
\left( \partial_\mu^2 + m^2 \right) = \left( i \partial_\mu \gamma^\mu - m_1 - \gamma_5 m_2 \right) \left( - i \partial_\mu \gamma^\mu - m_1 + \gamma_5 m_2 \right), \tag{4}
\]

where the physical mass of particles $m$ is expressed through the parameters $m_1$ and $m_2$

\[
m^2 = m_1^2 - m_2^2. \tag{5}
\]

For so the function would obeyed to the equations of Klein-Gordon

\[
\left( \partial_\mu^2 + m^2 \right) \tilde{\psi}(x,t) = 0 \tag{6}
\]

one can demand that it also satisfies one of equations of the first order

\[
\left( i \partial_\mu \gamma^\mu - m_1 - \gamma_5 m_2 \right) \tilde{\psi}(x,t) = 0; \quad \left( - i \partial_\mu \gamma^\mu - m_1 + \gamma_5 m_2 \right) \tilde{\psi}(x,t) = 0 \tag{7}
\]

Equations (7) of course, are less common than (6), and although every solution of one of the equations (7) satisfies to (6), reverse approval has not designated. It is also obvious that the Hamiltonians, associated with the equations (7), are non-Hermitian, because in it the $\gamma_5$-dependent mass components appear ($H \neq H^+$):

\[
H = \bar{\psi} \sigma p + \beta(m_1 + \gamma_5 m_2) \tag{8}
\]

and

\[
H^+ = \bar{\psi} \sigma p + \beta(m_1 - \gamma_5 m_2). \tag{9}
\]

Here matrices $\alpha_i = \gamma_0 \cdot \gamma_i$, $\beta = \gamma_0$, $\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3$. It is easy to see from (5) that the mass $m$, appearing in the equation (6) is real, when the inequality

\[
m_1^2 \geq m_2^2. \tag{10}
\]

is accomplished.
In this section, we will touch upon also question of describing the motion of Dirac particles, if their own magnetic moment is different from the Bohr magneton. As it was shown by Schwinger [33], that the Dirac equation of particles in the external electromagnetic field $A^{ext}$ taking into account the radiative corrections may be represented in the form

$$(\mathcal{P}_\gamma - m) \Psi(x) - \int \mathcal{M}(x, y|A^{ext}) \Psi(y) dy = 0,$$

(11)

where $\mathcal{M}(x, y|A^{ext})$ is the mass operator of fermion in external field and $\mathcal{P}_\mu = p_\mu - eA^{ext}_\mu$. From equation (11) by means of expansion of the mass operator in series according to $eA^{ext}$ with precision not over then linear field terms one can obtain the modified equation. This equation preserves the relativistic covariance and consistent with the phenomenological equation of Pauli obtained in his early papers [25].

Now let us consider the model of massive fermions with $\gamma_5$-extension of mass $m \rightarrow m_1 + \gamma_5m_2$ taking into account the interaction of their charges and AMM with the electromagnetic field $F_{\mu\nu}$:

$$\left(\gamma^\mu \mathcal{P}_\mu - m_1 - \gamma_5m_2 - \frac{\Delta \mu}{2} \sigma^{\mu\nu} F_{\mu\nu}\right) \tilde{\Psi}(x) = 0,$$

(12)

where $\Delta \mu = (\mu - \mu_0) = \mu_0(g - 2)/2$. Here $\mu$ - magnetic moment of a fermion, $g$ - fermion gyromagnetic factor, $\mu_0 = |e|/2m$ - the Bohr magneton, $\sigma^{\mu\nu} = i/2(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$. Thus phenomenological constant $\Delta \mu$, which was introduced by Pauli, is part of the equation and gets the interpretation with the point of view QFT.

The Hamiltonian form of (12) in the homogeneous magnetic field is the following

$$i \frac{\partial}{\partial t} \tilde{\Psi}(r, t) = H_{\Delta \mu} \tilde{\Psi}(r, t),$$

(13)

where

$$H_{\Delta \mu} = \vec{\alpha} \vec{P} + \beta(m_1 + \gamma_5m_2) + \Delta \mu \beta(\vec{\sigma} \vec{H}).$$

(14)

Given the quantum electrodynamic contribution in AMM of an electron with accuracy up to $e^2$ order we have $\Delta \mu = \frac{\alpha}{2\pi} \mu_0$, where $\alpha = e^2 = 1/137$ - the fine-structure constant and we still believe that the potential of an external field satisfies to the free Maxwell equations.

It should be noted that now the operator projection of the fermion spin at the direction of its movement - $\vec{\sigma} \vec{P}$ is not commute with the Hamiltonian.
and hence it is not the integral of motion. The operator $\mu_3$, which is commuting with this Hamiltonian is operator of polarization in the form of the third component of the polarization tensor $[25]$ in the direction of the magnetic field, and

$$\mu_3 = m_1 \sigma_3 + \rho_2 [\tilde{\sigma} \tilde{P}]_3$$

(15)

where matrices

$$\sigma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}.$$  

Subjecting the wave function $\tilde{\psi}$ to requirement to be eigenfunction of the operator polarization $\mu_3$ and Hamilton operator (14) we can obtain:

$$\mu_3 \tilde{\psi} = \zeta k \tilde{\psi}, \quad \mu_3 = \begin{pmatrix} m_1 & 0 & 0 & \mathcal{P}_1 - i\mathcal{P}_2 \\ 0 & -m_1 & -\mathcal{P}_1 - i\mathcal{P}_2 & 0 \\ 0 & -\mathcal{P}_1 + i\mathcal{P}_2 & m_1 & 0 \\ \mathcal{P}_1 + i\mathcal{P}_2 & 0 & 0 & -m_1 \end{pmatrix},$$

(16)

where $\zeta = \pm 1$ are characterized the projection of fermion spin at the direction of the magnetic field, and

$$H_{\Delta \mu} \tilde{\psi} = E \tilde{\psi}, \quad H_{\Delta \mu} = \begin{pmatrix} m_1 + H \Delta \mu & 0 & \mathcal{P}_3 - m_2 & \mathcal{P}_1 - i\mathcal{P}_2 \\ 0 & m_1 - H \Delta \mu & \mathcal{P}_1 + i\mathcal{P}_2 & -m_2 - \mathcal{P}_3 \\ m_2 + \mathcal{P}_3 & \mathcal{P}_1 - i\mathcal{P}_2 & -m_1 - H \Delta \mu & 0 \\ \mathcal{P}_1 + i\mathcal{P}_2 & m_2 - \mathcal{P}_3 & 0 & H \Delta \mu - m_1 \end{pmatrix}.$$  

(17)

A feature of the model with $\gamma_5$-mass contribution is that it may contain another any restrictions of mass parameters in addition to (10). Indeed while that for the physical mass $m$ one may be constructed by infinite number combinations of $m_1$ and $m_2$, satisfying to (5), however besides it need take into account the rules of conformity of this parameters in the Hermitian limit $m_2 \to 0$. Without this assumption the developing of Non-Hermitian models may not be adequate. With this purpose one can determine an additional mass scale which will depend on $m_1$, $m_2$ and which would put an upper bound on the mass spectrum of particles. These considerations make search in the frame of the stringent restriction of $m \leq m_1$, the existence of more complicated non-linear dependence on limiting mass value

$$m \leq M(m_1, m_2).$$

(18)
This expression meets the requirements of the principle of conformity: for all ordinary fermions, when $M \to \infty$ we should obtain ordinary Hermitian theory. In this sense the principle of conformity identical to the transition to the "flat limit" in the geometrical model (see page 3.).

Possibly explicit expression for $M(m_1, m_2)$, may be obtained from the simple mathematical theorem: the arithmetical average of two non-negative real numbers always is not less than the geometrical mean of the same numbers. Really, we have

$$\frac{m^2 + m_2^2}{2} \geq \sqrt{m^2 \cdot m_2^2}$$

and substitution (15), we can get the inequality

$$m \leq m_1^2/2m_2 = M(m_1, m_2). \quad (19)$$

Values of $M$ is now defined by two parameters $m_1, m_2$ and in the Hermitian limit, when $m_2 \to 0$ the value of the maximal mass $M$ tends to infinity. It is very important that in this limit the restriction of mass values completely disappear. In such a way one can demonstrate a natural transition from the Modified Model to the SM, which contains any values of the physical mass $m$.

Using (5) and expression (19) we can also obtain the system of two equations

$$\begin{cases} m = m_1^2 - m_2^2 \\ M = m_1^2/2m_2 \end{cases} \quad (20)$$

Thus, the solution of this system relative to the parameters $m_1$ and $m_2$ may be represented in the form

$$m_1^{\mp} = \sqrt{2M} \sqrt{1 \mp \sqrt{1 - m^2/M^2}}; \quad (21)$$

$$m_2^{\mp} = M \left(1 \mp \sqrt{1 - m^2/M^2}\right). \quad (22)$$

It is easy to verify that the obtained values of the mass parameters satisfy the conditions (5) and (10) regardless of which the sign will be chosen. Besides formulas (21), (22) in the case of the upper sign are agreed with conditions
\[ m_2 \to 0 \]

and

\[ m_1 \to m \]

when

\[ M \to \infty, \]

i.e. when the Hermitian limit is exist.

However, if one choose a lower sign (i.e. for the \( m_1^+ \) and \( m_2^+ \)) such limit is absent. Thus we can see that the nonlinear scheme of mass restrictions (see (19)) additionally contains the solutions satisfying to the some new particles. However this solutions should be considered only as an indication of the principal possibility of the existence of such particles. In this case, as follows from (21), (22), for each of ordinary particle may be exist the new partner, possessing the same mass and possibly having a number of another similar properties.

Let’s consider the “normalized” parameter of the modified model with the maximal mass \( M \):

\[ \xi = \frac{m_1}{M} = \frac{2m_2}{m_1} \quad \text{(23)} \]

and using (5), we can obtain

\[ \frac{m_2}{M} = \frac{\xi^2}{2} \quad \text{(24)} \]

\[ \frac{m}{M} = \xi \sqrt{1 - \xi^2 / 4} \quad \text{(25)} \]

At Fig.1 one can see the dependence of the normalized parameters \( m/m_1 \), \( M/m_1 \) and \( m/M \) on the relative parameter \( \xi = m_1/M = 2m_2/m_1 \). In particular, the maximum value of the particle mass \( m = M \) is achieved at the ratio of the subsidiary masses is equal to \( m_2 = m_1 / \sqrt{2} \). Till to this value for each mass of ordinary particles, one can find the parameters \( m_1 \) and \( m_2 \),

\(^1\)As the exotic particles do not agree in the “flat limit” with the ordinary Dirac expressions then one can assume that in this case we deal with a description of some new particles, properties of which have not yet been studied. This fact for the first time has been fixed by V.G.Kadyshevsky in his early works in the geometric approach to the development of the quantum field theory with a fundamental mass” Ref.[28] in curved de-Sitter momentum space. Besides in Ref.[29], [30] it was noted that the most intriguing prediction of the new approach is the possible existence of exotic fermions with no analogues in the SM, which may be candidates for constituents of dark matter.
for which a limit transition to regular Dirac theory exist. Further increasing of \( m_2 \), leads to the descending branch of the \( m/M \), where the Dirac limit is not exist and at the point \( m_2 = m_1 \) the value of \( m \) is equal to zero. Thus, it is the region \( m_1 > \sqrt{2}m_2 \) \( (m_2 > M) \) corresponds to the description of the "exotic particles", for which there is not transition to Hermitian limit. Note, that the equality \( m_2 = m_1 \) corresponds to the case of massless exotic fermions.

3 Exact solutions of Dirac-Pauli equations in the intensive uniform magnetic field

Performing calculations in many ways reminiscent of similar calculations carried out in the ordinary model in the magnetic field [25], in a result, for modified Dirac-Pauli equation one can find the exact solution for energy spectrum
E(ζ, p_3, 2γn, H) = \sqrt{p_3^2 - m_2^2 + \left[ \sqrt{m_1^2 + 2\gamma n + \zeta \Delta \mu H} \right]^2} \tag{26}

and for eigenvalues of the operator polarization μ_3 we can write in the form

\[ k = \sqrt{m_1^2 + 2\gamma n}. \tag{27} \]

From (26) it follows that in the field where \(PT\) symmetry is unbroken \(m \leq M\), all energy levels are real for the case of spin orientation along the magnetic field direction \(\zeta = +1\). We can clearly see the dependence of the set of energy values from the parameter \(x = m/M\) on Fig.2.

![Figure 2: Dependence of \(E(+1, 0, 0.4n, 0.1)\) on the parameter \(x = m/M\) for the cases \(n = 0, 1, 2, 3, 4\) and \(\Delta \mu H = 0.1\).]

However, in the opposite case \(\zeta = -1\) we have the imaginary part from the ground state of fermion \(n = 0\) and other low energy levels, see on Fig.3. For the cases of increasing parameter \(\Delta \mu H = 0.2\) we can watch overlapping of different levels (see Fig.4 and Fig.5).
Figure 3: Dependence of $E(-1, 0, 0.4n, 0.1)$ on the parameter $x = m/M$ for the cases $n = 0, 1, 2, 3, 4$ and $\Delta \mu H = 0.1$.

It is easy to see that in the case $\Delta \mu = 0$ from (26) one can obtain the ordinary expression for energy of charged particle in the magnetic field (Landau levels). Besides it should be emphasized that from the expression (26), in the Hermitian limit putting $m_1 = 0$ and $m_2 = m$ one can obtain:

$$E(\zeta, p_3, 2\gamma n, H) = \sqrt{p_3^2 + \left[ \sqrt{m^2 + 2\gamma n + \zeta \Delta \mu H} \right]^2}. \quad (28)$$

Note that in the paper [21] was previously obtained result analogical to (28) by means of using of the Hermitian Dirac-Pauli approach. Direct comparison of formula (28) with the result [21] shows their coincidence. It is easy to see that the expression (26) contains dependence on parameters $m_1$ and $m_2$ separately, which are not combined into a mass of particles, that essentially differs from the examples which were considered early [2]-[7].

Thus, here the calculation of interaction AMM of fermions with the magnetic field allow to put the question about the possibility of experimental
Figure 4: Dependence of $E(+1, 0, 0.4n, 0.2)$ on the parameter $x = m/M$ for the cases $n = 0, 1, 2, 3, 4$ and $\Delta \mu H = 0.2$.

studies of the non-Hermitian effects of $\gamma_5$-extensions of a fermion mass. Thus, taking into account the expressions (21) and (22) we obtain that the energetic spectrum (26) is expressed through the fermion mass $m$ and the value of the maximal mass $M$. Thus, taking into account that the interaction $\Delta M M$ with magnetic field removes the degeneracy on spin variable, we can obtain the energy of the ground state ($\zeta = -1$) in the form

$$E(-1, 0, 0, H, x) = m \sqrt{- \left( \frac{1 \mp \sqrt{1 - x^2}}{x} \right)^2 + \left( \frac{\sqrt{2} \sqrt{1 \mp \sqrt{1 - x^2}}}{x} - \frac{\Delta \mu H}{m} \right)^2},$$

where $x = m/M$ and the upper sign corresponds to the ordinary particle and the lower sign defines their "exotic" partners.

(29)
Figure 5: Dependence of $E(-1, 0, 0.4n, 0.2)$ on the parameter $x = m/M$ for the cases $n = 0, 1, 2, 3, 4$ and $\Delta \mu H = 0.2$.

Through decomposition of functions $-m_1$ and $-m_2$ we can obtain

$$-m_1/m = \begin{cases} 1 + \frac{x^2}{8} + \frac{7x^4}{128}, & x \ll 1 \\ \frac{\sqrt{x}}{x}, & x \to 1 \end{cases}$$

$$-m_2/m = \begin{cases} \frac{x}{2} + \frac{x^3}{8} + \frac{x^5}{16}, & x \ll 1 \\ \frac{1}{x}, & x \to 1 \end{cases}$$

Similarly, for $+m_1$ and $+m_2$ we have

$$+m_1/m = \begin{cases} \frac{2}{x} - \frac{x}{4} - \frac{5x^3}{64}, & x \ll 1 \\ \frac{\sqrt{x}}{x}, & x \to 1 \end{cases}$$

$$+m_2/m = \begin{cases} \frac{2}{x} - \frac{x}{2} - \frac{x^3}{8}, & x \ll 1 \\ \frac{1}{x}, & x \to 1 \end{cases}$$

Thus, it is shown that the main progress, is obtained by us in the algebraic way of the construction of the fermion model with $\gamma_5$-mass term is consists of describing of the new energetic scale, which is defined by the parameter $M = \ldots$
This value on the scale of the masses is a point of transition from the ordinary particles \( m_2 < M \) to exotic \( m_2 > M \). Furthermore, description of the exotic fermions in the algebraic approach are turned out essentially the same as in the model with a maximal mass, which was investigated by V.G.Kadyshevsky with colleagues on the basis of geometrical approach [28]-[30].

It should be noted that the formula (26) is valid not only for charged fermions, but and for the neutral particles possessing AMM. In this case one must simply replace the value of quantized transverse momentum of a charged particle in a magnetic field on the ordinary value \( 2\gamma n \rightarrow p_1^2 + p_2^2 \).

Thus, for the case of ultra cold polarized ordinary electronic neutrino with precision not over then linear field terms we can write

\[
E(-1, 0, 0, H, M \rightarrow \infty) = m_{\nu_e} \sqrt{1 - \frac{\mu_{\nu_e} H}{\mu_0 H_c}}. \tag{32}
\]

However, in the case of exotic electronic neutrino we have

\[
E(-1, 0, 0, H, m_{\nu_e}/M) = m_{\nu_e} \sqrt{1 - \frac{\mu_{\nu_e} 2MH}{\mu_0 m_{\nu_e} H_c}}. \tag{33}
\]

It is well known [34],[35] that in the minimally extended SM the one-loop radiative correction generates neutrino magnetic moment which is proportional to the neutrino mass

\[
\mu_{\nu_e} = \frac{3}{8\sqrt{2\pi^2}} |e| G_F m_{\nu_e} = \left(3 \cdot 10^{-19}\right) \mu_0 \left(\frac{m_{\nu_e}}{1\text{eV}}\right), \tag{34}
\]

where \( G_F \)-Fermi coupling constant and \( \mu_0 \) is Bohr magneton. However, so far, the most stringent laboratory constraints on the neutrino magnetic moment come from elastic neutrino-electron scattering experiments: \( \mu_{\nu_e} < (1.5 \cdot 10^{-10})\mu_0 \)[36]. Besides the discussion of problem of measuring the mass of neutrinos (either active or sterile) show that for an active neutrino model we have \( \sum m_\nu = 0.320\text{eV} \), whereas for a sterile neutrino \( \sum m_\nu = 0.06\text{eV} \)[37].

### 4 Conclusions

One can also estimate the change of the border of region of unbroken \( \mathcal{PT} \)-symmetry due to the shift of the lowest-energy state in the magnetic field.
Using formulas (32) and (33) we obtain correspondingly regions of undisturbed $\mathcal{PT}$-symmetry in the form

$$H_{\nu_e(\text{ordinary})} \leq \frac{\mu_0}{\mu_{\nu_e}} H_c; \quad (35)$$

$$H_{\nu_e(\text{exotic})} \leq \frac{m_{\nu_e} \mu_0}{2M \mu_{\nu_e}} H_c. \quad (36)$$

Indeed let us take the following parameters of neutrino: the mass of the electronic neutrino is equal to $m_{\nu_e} = 1eV$ and magnetic moment equal to $\mu_0$. If we assume that the values of mass and magnetic moment of exotic neutrino identical to parameters of ordinary neutrinos, we can obtain the estimates of the border area undisturbed $\mathcal{PT}$ symmetry for (35) in the form

$$H_{cr\nu_e(\text{ordinary})} = \frac{\mu_0}{\mu_{\nu_e}} H_c \sim 10^{32} \text{Gauss.} \quad (37)$$

However in the case (36) the situation may change radically

$$H_{cr\nu_e(\text{exotic})} = \frac{\mu_0 m_{\nu_e}}{\mu_{\nu_e} 2M} H_c \sim 10^4 \text{Gauss.} \quad (38)$$

In (37) and (38) we used the values of quantum-electrodynamic constant $H_c = 4.41 \cdot 10^{13} \text{Gauss}$ and the Planck mass $M = m_{\text{Planck}} \approx 10^{19} \text{GeV}$.

In the case of (37) one can see that the experimentally possible field corrections are extremely small, because the critical values of the magnetic field are fantastic large. On the other hand it is obvious that the critical value of magnetic field (38) is attainable in the sense of ordinary terrestrial experiments. We do not know if there is an upper limit of spectrum masses of elementary particles consistent with the Markov’s conjecture [27]. However contemporary precision of alternative laboratory measurements at low energy in the magnetic field may in principle allow to achieve the required values of exotic particles in the near future. Thus, consequences from the obtained formulas (33) - (38) allow to be convinced not only in the existence of the Maximal Mass but and in reality of the so-called exotic particles, because this phenomena are inextricably related.

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