Holes with hats and Erdős-Hajnal

Maria Chudnovsky¹
Princeton University, Princeton, NJ 08544

Paul Seymour²
Princeton University, Princeton, NJ 08544

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Abstract

A “hole-with-hat” in a graph $G$ is an induced subgraph of $G$ that consists of a cycle of length at least four, together with one further vertex that has exactly two neighbours in the cycle, adjacent to each other, and the “house” is the smallest, on five vertices. It is not known whether there exists $\varepsilon > 0$ such that every graph $G$ containing no house has a clique or stable set of cardinality at least $|G|^\varepsilon$; this is one of the three smallest open cases of the Erdős-Hajnal conjecture and has been the subject of much study.

We prove that there exists $\varepsilon > 0$ such that every graph $G$ with no hole-with-hat has a clique or stable set of cardinality at least $|G|^\varepsilon$. 
1 Introduction

Graphs in this paper are finite and simple, and \(|G|\) denotes the number of vertices of a graph \(G\). A graph is \(H\)-free if it has no induced subgraph isomorphic to \(H\). The Erdős-Hajnal conjecture \([7, 8]\) asserts:

1.1 Conjecture: For every graph \(H\), there exists \(\varepsilon > 0\) such that every \(H\)-free graph \(G\) has a clique or stable set of cardinality at least \(|G|\varepsilon\).

This has not yet been proved when \(H\) is the five-vertex path \(P_5\), and that problem motivates the work of this paper. The complement of \(P_5\) is the house, the graph consisting of a cycle of length four, together with one extra vertex with two neighbours in the cycle, adjacent. By taking complements, we see that proving 1.1 when \(H\) is the house is the same problem as proving it when \(H = P_5\). The house is the smallest example of a “hole-with-hat”.

![Figure 1: A house.](image)

A hole in a graph \(G\) is an induced cycle of length at least four. If \(C\) is a hole in \(G\), a vertex \(v \in V(G) \setminus V(C)\) is said to be a hat for \(C\) if \(v\) has exactly two neighbours \(x, y \in V(C)\), and \(x, y\) are adjacent. The subgraph induced on \(V(C) \cup \{v\}\) is then said to be a hole-with-hat in \(G\); and we say \(G\) is hole-with-hat-free if there is no hole-with-hat in \(G\).

The main result of this paper is:

1.2 There exists \(\varepsilon > 0\) such that for every hole-with-hat-free graph \(G\), there is a clique or stable set in \(G\) of cardinality at least \(|G|\varepsilon\).

Here are some earlier theorems with a similar nature:

- If \(G\) contains no hole, there is a clique or stable set in \(G\) of cardinality at least \(|G|^{1/2}\). (This is immediate because such graphs are perfect.)

- If \(G\) contains no house and no hole of odd length, then again there is a clique or stable set in \(G\) of cardinality at least \(|G|^{1/2}\). (Again, because such graphs are perfect, a consequence of the “strong perfect graph theorem” \([5]\).)

- For each \(\ell > 0\), there exists \(\varepsilon > 0\) such that if \(G\) contains no house and no hole of length at least \(\ell\), then \(G\) has a clique or stable set of cardinality at least \(|G|\varepsilon\). (This is a combination of a theorem of Bousquet, Lagoutte, and Thomassé \([3]\) and a theorem of Bonamy, Bousquet and Thomassé \([2]\).)
A graph $P$ is perfect if chromatic number equals clique number for every induced subgraph of $P$. We denote the set of nonnegative real numbers by $\mathbb{R}^+$. Let $G$ be a graph and $f : V(G) \to \mathbb{R}^+$; if $X \subseteq V(G)$, we define $f(X) = \sum_{v \in X} f(v)$, and if $P$ is an induced subgraph of $G$ we define $f(P) = f(V(P))$. We say that $f$ is good on $G$ if $f(P) \leq 1$ for every perfect induced subgraph $P$ of $G$. Now let $\alpha \geq 1$. We denote by $f^\alpha$ the function $g$ on $G$ defined by $g(v) = (f(v))^\alpha$ for each $v \in V(G)$. Let us say that $G$ is $\alpha$-narrow if $f^\alpha(G) \leq 1$ for every good function $f$ on $G$. Here is a result of Chudnovsky and Zwols [6]:

1.3 If $G$ is $\alpha$-narrow then $G$ has a clique or stable set of cardinality at least $|G|^{\varepsilon}$, where $\varepsilon = 1/(2\alpha)$.

Proof. Let $P$ be a perfect induced subgraph of $G$ with as many vertices as possible, and let $p = |P|$. Let $f(v) = 1/p$ for all $v \in V(G)$. Then $f$ is good on $G$, and so $f^\alpha(G) \leq 1$, that is, $p^{-\alpha}|G| \leq 1$, and so $p \geq |G|^{1/\alpha} = |G|^{2\varepsilon}$. But $P$ is perfect, and so $P$, and hence $G$, has a clique or stable set of cardinality at least $p^{1/2} \geq |G|^\varepsilon$. This proves 1.3.

In view of 1.3 the following implies 1.2.

1.4 There exists $\alpha \geq 1$ such that every hole-with-hat-free graph is $\alpha$-narrow.

Let us say a pair $(G, f)$ is $\alpha$-critical if $G$ is a graph and $f : V(G) \to \mathbb{R}^+$ is a good function on $G$, such that

- every proper induced subgraph of $G$ is $\alpha$-narrow; and
- $f^\alpha(G) > 1$.

To prove 1.4 choose an appropriately large value of $\alpha$, and suppose (for a contradiction) that 1.4 is not satisfied, and look at a counterexample $G$ with as few vertices as possible. Hence there is a good function $f$ on $G$ with $f^\alpha(G) > 1$, and so $(G, f)$ is $\alpha$-critical. Consequently, 1.4 can be reformulated as:

1.5 There exists $\alpha \geq 1$ such that for every $\alpha$-critical pair $(G, f)$, there is a hole-with-hat in $G$.

We will prove this at the end of the final section. But the main part of the argument lies in proving 1.3 which concerns the structure of the remainder of a hole-with-hat-free graph, after we have removed all the parts that can conveniently be removed.

2 Complete pairs of sets

Hole-with-hat-free graphs have some convenient structural properties, that we will prove next. Let $A, B \subseteq V(G)$ be disjoint; we say they are complete to each other if every vertex in $A$ is adjacent to every vertex in $B$, and anticomplete if there are no edges between $A, B$. We are concerned in this section with how the remainder of a hole-with-hat-free graph can attach to a pair of sets of vertices that are complete to each other. A graph is anticonnected if its complement is connected; and its anticomponents are the complements of the components of its complement. If $X \subseteq V(G)$, we say that $X$ is connected if $G[X]$ is connected, and anticonnected if $G[X]$ is anticonnected. If $C \subseteq V(G)$, a vertex $v \in V(G) \setminus C$ is mixed on $C$ if $v$ is neither complete not anticomplete to $C$. We begin with:
2.1 Let $G$ be a hole-with-hat-free graph. Let $C, D$ be disjoint anticonnected subsets of $V(G)$, complete to each other. Then no vertex of $(V(G) \setminus (C \cup D))$ is both mixed on $C$ and mixed on $D$.

**Proof.** Suppose that $v \in (V(G) \setminus (C \cup D))$ is both mixed on $C$ and mixed on $D$. Since $C$ is anticonnected, there exist nonadjacent $c_1, c_2 \in C$ such that $v$ is adjacent to $c_1$ and not to $c_2$; and choose $d_1, d_2 \in D$ similarly. Then the subgraph induced on $\{c_1, c_2, d_1, d_2, v\}$ is a house, contradicting that $G$ is hole-with-hat-free. This proves 2.1.

2.2 Let $G$ be a hole-with-hat-free graph. Let $C, D$ be disjoint subsets of $V(G)$, complete to each other, such that $C$ is connected. Let $P$ be a connected subgraph of $G \setminus (C \cup D)$, such that some vertex of $P$ has a neighbour in $C$, and no vertex of $P$ is complete to $C$. For every $v \in V(P)$, there exists $u \in V(P)$, mixed on $C$, such that every vertex in $D$ adjacent to $v$ is also adjacent to $u$.

**Proof.** Suppose the claim is false, and choose a counterexample with $P$ minimal. Choose $v \in V(P)$ such that no vertex of $P$ has a neighbour in $C$ and is adjacent to all neighbours of $v$ in $D$. Choose $u \in V(P)$ mixed on $C$. By the minimality of $P$, $P$ is an induced path between $u,v$, and $u$ is the only vertex of $P$ with a neighbour in $C$. Choose $d \in D$ adjacent to $v$ and not to $u$. By the minimality of $D$, $v$ is the only vertex of $P$ adjacent to $d$. Since $C$ is connected, there exist adjacent $c_1, c_2 \in C$ such that $u$ is adjacent to $c_1$ and not to $c_2$. But then the subgraph induced on $V(P) \cup \{c_1, c_2, d\}$ is a hole-with-hat, a contradiction. This proves 2.2.

2.3 Let $G$ be a hole-with-hat-free graph. Let $C, D$ be disjoint subsets of $V(G)$, complete to each other, such that $C$ is connected and $D$ is anticonnected. Let $P$ be a connected subgraph of $G \setminus (C \cup D)$, such that some vertex of $P$ has a neighbour in $C$, and no vertex of $P$ is complete to $C$. If some vertex of $P$ has a neighbour in $D$, then some vertex of $P$ is mixed on $C$ and complete to $D$.

**Proof.** If some vertex of $P$ has a neighbour in $D$, then by 2.2 some vertex $v \in C$ is mixed on $C$ and has a neighbour in $D$, and therefore by 2.1 $v$ is complete to $D$. This proves 2.3.

2.4 Let $G$ be a hole-with-hat-free graph. Let $C, D$ be disjoint nonempty subsets of $V(G)$, complete to each other, such that $C$ is connected and anticonnected, and $D$ is anticonnected. Let $P$ be a connected subgraph of $G$ with $V(P) \cap (C \cup D) = \emptyset$, such that some vertex of $P$ has a neighbour in $C$, and no vertex of $P$ is complete to $C$. Then no vertex of $P$ is mixed on $D$.

**Proof.** Suppose not, and choose $C, D, P$ not satisfying the theorem, with $P$ minimal. Choose $u \in V(P)$ with a neighbour in $C$, and therefore mixed on $C$; and choose $v \in V(P)$ mixed on $D$. From the minimality of $P$, it follows that $P$ is an induced path with ends $u,v$, and $u$ is the only vertex of $P$ with a neighbour in $C$, and no vertex of $P$ different from $v$ is mixed on $D$. By 2.3 $u$ is complete to $D$, and in particular $u \neq v$. Let $u'$ be the neighbour of $u$ in $P$. It follows that $C \cup \{u\}$ is connected and anticonnected, and $u'$ is mixed on it; and this contradicts the minimality of $P$. This proves 2.4.
2.5 Let $G$ be a hole-with-hat-free graph. Let $C, D$ be disjoint nonempty subsets of $V(G)$, complete to each other, such that $C$ is connected and anticonnected, and $D$ is connected and anticonnected. Then there do not exist connected subgraphs $P, Q$ of $G \setminus (C \cup D)$, with $V(P \cap Q) \neq \emptyset$, such that

- some vertex of $P$ has a neighbour in $C$, and no vertex of $P$ is complete to $C$;
- some vertex of $Q$ has a neighbour in $D$, and no vertex of $Q$ is complete to $D$.

**Proof.** Suppose that such $P, Q$ exist, and choose $C, D, P, Q$ with $|V(P)| + |V(Q)|$ minimal. Choose $w \in V(P \cap Q)$, and choose $p \in V(P)$ with a neighbour in $C$, and $q \in V(Q)$ with a neighbour in $D$. From the minimality of $|V(P)| + |V(Q)|$, it follows that $P$ is an induced path with ends $p, w$, and no vertex of $P$ different from $p$ has a neighbour in $C$; and similarly for $Q$; and $V(P \cap Q) = \{w\}$. Suppose that $p$ is complete to $D$. Hence $p \notin V(Q)$, and so $p \neq w$. Then $C \cup \{p\}$ is connected and anticonnected, and complete to $D$, and the two paths $P \setminus p$ and $Q$ contradict the minimality of $|V(P)| + |V(Q)|$. Thus $p$ is not complete to $D$. Since no other vertex of $P$ has a neighbour in $C$, it follows from 2.2 that no vertex of $P$ has a neighbour in $D$. Similarly no vertex of $Q$ has a neighbour in $C$. In particular, no vertex of $P \cup Q$ is complete to $C$, contrary to 2.3. This proves 2.5. ■

3 Decomposing hole-with-hat-free graphs

If $v \in V(G)$, we denote by $N(v) = N_G(v)$ the set of all neighbours of $v$ in $G$. If $N \subseteq V(G)$, we denote by $G[N]$ the induced subgraph with vertex set $N$. A *weighted graph* is a pair $(G, w)$, where $G$ is a graph and $w : V(G) \to \mathbb{R}^+$ is a function, such that $w(G) = 1$. Let $\varepsilon > 0$. We say a weighted graph $(G, w)$ is $\varepsilon$-coherent if

- for every $v \in V(G)$, $w(v) < \varepsilon$;
- for every $v \in V(G)$, $w(N_G(v)) < \varepsilon$; and
- if $A, B \subseteq V(G)$ are disjoint and anticonnected then $\min(w(A), w(B)) < \varepsilon$.

First we need:

3.1 Let $(G, w)$ be an $\varepsilon$-coherent weighted graph. If $X \subseteq V(G)$ with $w(X) \geq 3\varepsilon$, there is a component $Y$ of $G[X]$ with $w(Y) > w(X) - \varepsilon$.

**Proof.** Let $Z$ be a union of components of $G[X]$, minimal such that $w(Z) \geq \varepsilon$. Since $X \setminus Z$ is anticonnected to $Z$, it follows that $w(X \setminus Z) < \varepsilon$, and so $w(Z) > w(X) - \varepsilon$. Choose a component $Y$ of $G \setminus X$ with $Y \subseteq Z$. From the minimality of $Z$, $w(Z \setminus Y) < \varepsilon$, and so $w(Y) \geq w(X) - \varepsilon - \varepsilon \geq \varepsilon$, and therefore $Z = Y$ from the minimality of $Z$. But then $w(Y) = w(Z) > w(X) - \varepsilon$. This proves 3.1. ■

The component $Y$ of 3.1 satisfies $w(Y) > w(X) - \varepsilon \geq 2\varepsilon$, and since the remainder of $G[X]$ is anticonnected to $Y$ and therefore has weight less than $\varepsilon$, it follows that $Y$ is unique. We call $Y$ the big component of $G[X]$.

If $A \subseteq V(G)$, each vertex in $V(G) \setminus A$ with a neighbour in $A$ is called an *attachment* of $A$. Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let $(G, w)$ be an $\varepsilon$-coherent weighted graph. Let $C, D$ be disjoint subsets of $V(G)$, such that:
\begin{itemize}
  \item $|C| \geq 2$, and $G[C]$ is connected and anticonnected;
  \item $D \neq \emptyset$, and $D$ is the set of all vertices in $V(G) \setminus C$ that are complete to $C$; and
  \item $C$ contains no attachment of the big component of $G \setminus (C \cup D)$.
\end{itemize}

(Note that since there is a vertex in $D$ complete to $C$, it follows that $w(C) \leq \varepsilon$, and similarly $w(D) \leq \varepsilon$. Since $5\varepsilon \leq 1$, there is a big component of $G \setminus (C \cup D)$.) In these circumstances we call $(C, D)$ a split of $(G, w)$. As we shall see, splits are a useful kind of decomposition in hole-with-hat-free graphs.

Let us say a forcer is a graph $F$ with eight vertices $v_1, \ldots, v_8$, where $v_1v_2v_3v_4$ and $v_5v_6v_7v_8$ are induced paths of $F$, and $\{v_1, \ldots, v_4\}$ is complete to $\{v_5, \ldots, v_8\}$. We call these two paths the constituent paths of the forcer. A forcer in $G$ means an induced subgraph of $G$ that is a forcer, and $G$ is forcer-free if there is no forcer in $G$. Now we prove the main result of this section:

3.2 Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let $(G, w)$ be a $\varepsilon$-coherent weighted graph, where $G$ is hole-with-hat-free. Let $F$ be a forcer in $G$. Then there is a split $(C, D)$ of $G$ such that $G[C], G[D]$ both contain a constituent path of $F$.

\textbf{Proof.} Let $F$ be a forcer, and let $P_1, P_2$ be the constituent paths of $F$. Consequently there are disjoint subsets $X_1, X_2$ of $V(G)$, such that

\begin{itemize}
  \item $V(P_1) \subseteq X_1$, and $X_1$ is connected and anticonnected;
  \item $V(P_2) \subseteq X_2$, and $X_2$ is connected and anticonnected; and
  \item $X_1, X_2$ are complete to one another.
\end{itemize}

Choose such $(X_1, X_2)$ maximal in the sense that there is no choice of $(X_1', X_2')$ satisfying the same conditions, with $X_i \subseteq X_i'$ for $i = 1, 2$ and $|X_1' \cup X_2'| > |X_1 \cup X_2|$. We call this property the maximality of $(X_1, X_2)$. Let $X_3$ be the set of all vertices in $V(G) \setminus (X_1 \cup X_2)$ that are complete to $X_1 \cup X_2$, and let $R = V(G) \setminus (X_1 \cup X_2 \cup X_3)$. For $i = 1, 2$, let $R_i$ be the set of vertices in $R$ that are complete to $X_i$.

(1) $R_1$ is anticomplete to $X_2$, and $R_2$ is anticomplete to $X_1$, and so $R_1 \cap R_2 = \emptyset$.

Suppose that $v \in R_1$ has a neighbour in $X_2$, say. Since $v \notin X_3$, $v$ is mixed on $X_2$. But then $X_2' = X_2 \cup \{v\}$ is connected and anticonnected, and the pair $(X_1, X_2')$ violates the maximality of $(X_1, X_2)$. This proves (1).

For $i = 1, 2$, let $S_i$ be the union of all components of $G[R \setminus (R_1 \cup R_2)]$ that have an attachment in $X_i$. Let $S_3 = R \setminus (R_1 \cup R_2 \cup S_1 \cup S_3)$.

(2) $S_1 \cap S_2 = \emptyset$. Moreover, $S_1$ is anticomplete to $X_2 \cup R_2 \cup S_2$, and $S_2$ is anticomplete to $X_1 \cup R_1 \cup S_1$.

By [2.3] $S_1 \cap S_2 = \emptyset$. By [2.3], $S_1$ is anticomplete to $R_2$, and $S_2$ is anticomplete to $R_1$. This proves (2).
Thus, in summary, the sets $X_1, X_2, X_3, R_1, R_2, S_1, S_2, S_3$ are pairwise disjoint and have union $V(G)$. The pairs

$$(X_1, X_2), (X_1, X_3), (X_2, X_3), (R_1, X_1), (R_2, X_2)$$

are complete to each other; the pairs

$$(R_1, X_2), (R_2, X_1), (S_1, X_2), (S_2, X_1), (S_3, X_1), (S_3, X_2), (S_1, S_2), (S_1, S_3), (S_2, S_3)$$

are anticomplete; and there may be edges between the pairs not listed. Every component of $G[S_i]$ has an attachment in $X_i$ for $i = 1, 2$.

Define $T = X_1 \cup X_2 \cup X_3 \cup R_1 \cup R_2$. Choose $x_1 \in X_1$ and $x_2 \in X_2$. Then every vertex in $T$ is adjacent to one of $x_1, x_2$, and so $w(T) \leq 2\varepsilon$. Hence $G \setminus T$ has a big component $Y$, and since the sets $S_1, S_2, S_3$ are pairwise anticomplete, we may assume that $Y$ is disjoint from $S_1$, by exchanging $X_1, X_2$ if necessary. But then $(X_1, X_2 \cup X_3 \cup R_1)$ is a split of $G$ satisfying the theorem. This proves 3.2.

Now we prove the main result of this section. Let us say a split $(C, D)$ of $G$ is *optimal* if there is no split $(C', D')$ with $C \subseteq C'$ and $C' \neq C$. Let $(C, D)$ be an optimal split. Let $A$ be the union of all components of $G \setminus (C \cup D)$ that have an attachment in $C$; and let $B$ be the union of all other components of $G \setminus (C \cup D)$ (including the big component). Let us call $(A, C, D, B)$ a *fracture* of $G$. (Note that there are no edges between $B$ and $A \cup C$ but there may well be edges between $A$ and $D$. Also, $B \neq \emptyset$, since it contains the big component of $G \setminus (C \cup D)$, but $A$ might be empty.) From 3.2 we have immediately:

3.3 Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let $(G, w)$ be an $\varepsilon$-coherent weighted graph, where $G$ is hole-with-hat-free. Let $F$ be a forcer in $G$. Then there is a fracture $(A, C, D, B)$ of $G$ such that $G[C]$ contains a constituent path of $F$.

We need some observations about fractures.

3.4 Let $\varepsilon > 0$, with $5\varepsilon \leq 1$, and let $(G, w)$ be an $\varepsilon$-coherent weighted graph, where $G$ is hole-with-hat-free. Let $(A, C, D, B)$ be a fracture of $G$. 
• For each \( a \in A \), there is an attachment of the big component of \( G \setminus (C \cup D) \) that is nonadjacent to \( a \).

• For each \( a \in A \), and every anticomponent \( X \) of \( G[D] \), \( a \) is not mixed on \( X \).

**Proof.** Let \( Y \) be the big component of \( G \setminus (C \cup D) \); thus \( Y \subseteq B \), and all its attachments belong to \( D \). Suppose that \( a \in A \), and \( a \) is adjacent to every vertex of \( D \) that has a neighbour in \( Y \). Let \( P \) be the component of \( G[A] \) that contains \( a \); then some attachment of \( P \) belongs to \( C \). By 2.2, we may choose \( v \in P \) mixed on \( C \), such that every vertex in \( D \) adjacent to \( a \) is also adjacent to \( v \). Let \( D' \) be the set of all neighbours of \( v \) in \( D \). Then \((C \cup \{v\}, D')\) is a split (because \( D' \) contains all attachments of \( Y \)), contradicting that \((C, D)\) is optimal. This proves the first assertion.

Now suppose that \( a \in A \) is mixed on an anticomponent \( X \) of \( G[D] \). Let \( P \) be the component of \( G[A] \) that contains \( a \). Choose \( v \in P \) mixed on \( C \); then \( P \) contradicts 2.4 applied to the connected anticonnected set \( C \) and the anticonnected set \( X \). This proves 3.4.

### 4 Multiple fractures

A fracture \((A, C, D, B)\) of \( G \) is a kind of separation of \( G \), because deleting \( C \cup D \) disconnects \( A \) from \( B \). (But \( A \) might be empty.) Also the order of this separation is small, since \( w(C \cup D) \leq 2\varepsilon \) in the usual notation. It would be nice if these separations did not “cross”, so that they give us a tree-decomposition of \( G \), but that is not true. Nevertheless, something like that is true, as we see in this section.

**4.1** Let \( \varepsilon > 0 \), with \( 6\varepsilon \leq 1 \), and let \((G, w)\) be an \( \varepsilon \)-coherent weighted graph, where \( G \) is hole-without-hat-free. Let \((A, C, D, B)\) and \((A', C', D', B')\) be fractures in \( G \). Then either

- every connected subgraph of \( G[A \cup A'] \) is contained in one of \( A, A' \); or

- the big component of \( G \setminus (C \cup D) \) equals the big component of \( G \setminus (C' \cup D') \).

**Proof.** We suggest that, to follow this argument, the reader imagine a \( 4 \times 4 \) matrix with rows labelled \( A, C, D, B \) and columns \( A', C', D', B' \). We remind the reader that \( C \) is complete to \( D \), and \( A \cup C \) is anticomplete to \( B \), and the same for \((A', C', D', B')\).

\[
\begin{array}{cccc}
A' & C' & D' & B' \\
| & | & | & | \\
A & & & \\
| & | & | & | \\
C & & & \\
| & | & | & | \\
D & & & \\
| & | & | & | \\
B & & & \\
\end{array}
\]

Figure 3: Two fractures.
Let $Y$ be the big component of $G \backslash (C \cup D)$, and define $Y'$ similarly. Since $w(Y), w(Y') > 1 - 3\epsilon \geq 1/2$, it follows that $Y \cap Y' \neq \emptyset$, and since $Y \subseteq B$ and $Y' \subseteq B'$, we deduce that $Y \cap Y' \cap B \cap B' \neq \emptyset$.

(1) If $C \cap B' \neq \emptyset$ then $D \cap (A' \cup C') = \emptyset$, and if $B \cap C' \neq \emptyset$ then $(A \cup C) \cap D' = \emptyset$.

Let $u \in C \cap B'$. If $v \in D \cap (A' \cup C')$, then $v$ is adjacent to $u$ (because $C$ is complete to $D$), and yet $v$ is nonadjacent to $u$ (because $A' \cup C', D'$ are anticomplete), a contradiction. This proves the first statement, and the second follows by symmetry.

(2) We may assume that $A \cap (C' \cup D') = \emptyset$, and $(C \cup D) \cap A' = \emptyset$, and at least one of $B \cap D', D \cap B'$ is nonempty.

If $A \cap (C' \cup D') = \emptyset$, then the first outcome of the theorem holds, and similarly if $(C \cup D) \cap A' = \emptyset$. If $B \cap D', D \cap B'$ are both empty, then $Y, Y' \subseteq B \cap B'$, and $Y = Y'$ and the second outcome of the theorem holds. This proves (2).

From the third assertion of (2) and symmetry, we may assume that $B \cap D' \neq \emptyset$. By (1), $(A \cup C) \cap C' = \emptyset$. From (2), $A \cap D' \neq \emptyset$; so by (1) $B \cap C' = \emptyset$. Hence $D \cap C' \neq \emptyset$, because $C' \neq \emptyset$. Every vertex in $C \cap A'$ is complete to $D \cap C'$ and hence to $C'$; but no vertex in $A'$ is complete to $C'$ from the definition of a fracture, and so $C \cap A' = \emptyset$. By (2), $D \cap A' \neq \emptyset$. By (1), $C \cap B' = \emptyset$, and so $C \cap D' \neq \emptyset$.

![Figure 4: A solid dot means a nonempty set, and ? means we don’t know.](image)

Since $C = C \cap D'$ is anticonnected, and every vertex in $A$ has a nonneighour in $C$, and every vertex in $B \cap D'$ has a nonneighbour in $A \cap D'$, it follows that $(A \cup B \cup C) \cap D'$ is anticonnected. But each vertex in $D \cap A'$ has a neighbour in $(A \cup B \cup C) \cap D'$ (namely, in $C \cap D'$), and by 3.4 it follows that $D \cap A'$ is complete to $(A \cup B \cup C) \cap D'$. Similarly, since $D \cap (A' \cup B' \cup C')$ is anticonnected, it follows that $A \cap D'$ is complete to $D \cap (A' \cup B' \cup C')$. (Thus we almost have symmetry between $(A, C, D, B)$ and $(A', C', D', B')$; but not quite, because we do not know that $D \cap B' \neq \emptyset$.)

Let $Q$ be the set of vertices in $D \cap D'$ that are not complete to $D \cap A'$, and let $Q'$ be the set of vertices in $D \cap D'$ that are not complete to $A \cap D'$. Let $R = (D \cap D') \setminus (Q \cup Q')$.

(3) $Q \cap Q' = \emptyset$, and $Q, Q', R$ are pairwise complete.

Since there is a vertex in $A \cap D'$ and it is complete to $D \cap A'$, 3.4 implies that $A \cap D'$ is complete to $D \cap A'$, and so $A \cap D'$ is anticonnected.
lete to $Q$; and so $Q \cap Q' = \emptyset$. If $u \in Q$ and $v \in Q' \cup R$, there is a vertex in $D \cap A'$ adjacent to $v$ and not to $u$; so $3.3$ implies that $u, v$ are adjacent. Thus $Q$ is complete to $Q' \cup R$, and similarly $Q'$ is complete to $R$. This proves (3).

(4) $A' \cap B$ is anticomplete to $Q' \cup (B \cap D')$, and $B' \cap A$ is anticomplete to $Q \cup (D \cap B')$.

Each vertex in $A' \cap B$ is anticomplete to $A \cap D'$, and so by $3.4$ also anticomplete to $Q' \cup (B \cap D')$. Similarly $B' \cap A$ is anticomplete to $Q \cup (D \cap B')$. This proves (4).

Choose $v \in D \cap A'$; then by $3.4$ there is an attachment $q$ of $Y'$ nonadjacent to $v$. Since $v$ is adjacent to all vertices of $D' \setminus Q$, it follows that $q \in Q$. Similarly there is an attachment $q'$ of $Y$ with $q' \in Q'$. Let $X = (A \cup C) \cap D' \cup Q'$, and $X' = (D \cap (A' \cup C')) \cup Q$. Then $X, X'$ are disjoint, and complete to each other, and each of them is both connected and anticonnected. Now some vertex of $Y$ is adjacent to $q'$, and so has a neighbour in $X'$, and no vertex of $Y$ is complete to $X$ (because $Y \subseteq B \cap (B' \cap D')$, since there are no edges between $B \cap A'$ and $B \cap D'$). Similarly some vertex of $Y'$ has a neighbour in $X'$, and no vertex of $Y'$ is complete to $X'$. Since $Y \cap Y'$ is non-null, this contradicts 2.5.

4.2 Let $\varepsilon > 0$, with $6\varepsilon \leq 1$, and let $(G, w)$ be an $\varepsilon$-coherent weighted graph, where $G$ is hole-with-hat-free. Let $F$ be the set of all fractures of $G$, and let $A$ be the union of all the sets $A$ for $(A, C, D, B) \in F$. Then $w(A) < 3\varepsilon$.

Proof. Let $Z$ be the vertex set of a component of $G[A]$. For each $(A, C, D, B) \in F$, we call each component of $G[A]$ a piece; let $H$ be the set of all maximal pieces (taken over all $(A, C, D, B) \in F$). Thus $Z$ can be expressed as the union of vertex sets of maximal pieces. For each maximal piece $X$, let $(A, C, D, B) \in F$ such that $X$ is a component of $G[A]$, and let $Y$ be the big component of $G \setminus (C \cup D)$; we call $Y$ the fulcrum of $X$. (There may be more than one choice of $(A, C, D, B) \in F$ for a given set $X$, and correspondingly more than one choice of fulcrum: choose one, arbitrarily).

We observe:

(1) If $X, X'$ are maximal pieces such that either $V(X \cap X') \neq \emptyset$, or $X$ is not anticomplete to $X'$, then $X, X'$ have the same fulcrum.

Suppose not. Let $X$ be a component of $G[A]$ where $(A, C, D, B) \in F$, and define $(A', C', D', B')$ similarly. By 4.1 it follows that every connected subgraph of $G[A \cup A']$ is a subgraph of one of $G[A], G[A']$, and in particular the connected subgraph induced on $V(X) \cup V(X')$ is a subgraph of one of $G[A], G[A']$, say $G[A]$. But $X$ is a component of $G[A]$, so $V(X) = V(X \cup X')$, contradicting that $X'$ is a maximal piece. This proves (1).

Choose a connected subgraph $H$ of $G[Z]$, maximal such that $V(H)$ is the union of maximal pieces all with the same fulcrum $Y$. Suppose that $V(H) \neq Z$. Since $G[Z]$ is connected, there is a vertex $v_1 \in Z \setminus V(H)$ with a neighbour $v_2 \in V(H)$. Choose a maximal piece $X_1$ containing $v_1$, and a maximal piece $X_2$ containing $v_2$ with fulcrum $Y$. By (1), $X_1$ has fulcrum $Y$, contrary to the maximality of $H$. Thus $V(H) = Z$, and so $Y$ is anticomplete to $Z$. Since $w(Y) \geq \varepsilon$, it follows that $w(Z) < \varepsilon$. Since this holds for each component of $G[A]$, $3.1$ implies that $w(A) < 3\varepsilon$. This proves

4.2
Let us say $X \subseteq V(G)$ is a homogeneous set of $G$ if for every vertex $v \in V(G) \setminus X$, either $v$ is complete or anticomplete to $X$. Let $G$ be a graph; we say that $G$ is guarded if for every forcer $F$ in $G$, there is a homogeneous set $X$ of $G$ with $X \neq V(G)$ such that $G[X]$ contains a constituent path of $F$.

4.3 Let $\varepsilon > 0$, with $6\varepsilon \leq 1$, and let $(G, w)$ be an $\varepsilon$-coherent weighted graph, where $G$ is hole-without-hat-free. Then there exists $Z \subseteq V(G)$ with $|Z| > 1$, such that $G[Z]$ is connected and guarded, and $w(Z) > 1 - 4\varepsilon$.

**Proof.** Define $\mathcal{F}, A$ as in 4.2 and let $W = V(G) \setminus A$. By 4.2, $w(W) > 1 - 3\varepsilon \geq 3\varepsilon$. By 3.1, $G[W]$ has a big component, with vertex set $Z$ say, where $w(Z) \geq 1 - 4\varepsilon$. Hence $|Z| > 1$, since $w(v) \leq \varepsilon < 1 - 4\varepsilon$ for each vertex $v$. Let $F$ be a forcer in $G[Z]$. Then by 3.3, there is a fracture $(A, C, D, B)$ of $G$ such that $|V(F) \cap C| \geq 4$. Let $X = C \cap Z$. Since $C$ is a homogeneous set of $G \setminus A$, it follows that $X$ is a homogeneous set of $G[Z]$, and it contains a constituent path of $F$. This proves 4.3. 

5  \(\alpha\)-critical pairs

In this section we explore the properties of $\alpha$-critical pairs, and combine these results with 4.3 to prove 1.5.

5.1 Let $\alpha \geq 2$, and let $(G, f)$ be $\alpha$-critical. Then $f(w) < 1 - 4^{-1/\alpha}$ for each $w \in V(G)$.

**Proof.** Let $w \in V(G)$, and let $c = f(w)$. Let $N = N_G(w)$, and let $M = V(G) \setminus (N \cup \{w\})$. Since $(G, f)$ is $\alpha$-critical, it follows that $G[N]$ is $\alpha$-narrow, and so is $G[M]$. Let $p$ be the maximum of $f(P)$ over all perfect induced subgraphs of $G[N]$, and let $q$ be the maximum of $f(Q)$ over all perfect induced subgraphs $Q$ of $G[M]$. We claim that $f^\alpha(M) \leq p^\alpha$. If $f(v) = 0$ for every $v \in N$ then the statement is true, so we may assume that $f(v) > 0$ for some $v \in N$, and hence $p > 0$. So the function $f(v)/p$ ($v \in N$) is a good function on $G[N]$, and since $G[N]$ is $\alpha$-narrow, we deduce that $f^\alpha(N) \leq p^\alpha$.

Similarly $f^\alpha(M) \leq q^\alpha$.

But if $P$ is a perfect induced subgraph of $G[N]$ then $G[V(P) \cup \{w\}]$ is perfect, and therefore $f(V(P) \cup \{w\}) \leq 1$; and so $p \leq 1 - c$, and similarly $q \leq 1 - c$. Thus

$$1 < f^\alpha(G) = f^\alpha(N) + f^\alpha(M) + f^\alpha(w) \leq p^\alpha + q^\alpha + f^\alpha(w) \leq 2(1 - c)^\alpha + c^\alpha.$$ 

Now for $0 \leq x \leq 1$, the function $g(x) = 2(1 - x)^\alpha + x^\alpha$ has the value 1 when $x = 1$, and its value increases with $x$ for $2/3 \leq x \leq 1$, since $\alpha \geq 2$ (as can be seen by taking the derivative). Thus $g(x) \leq 1$ for $2/3 \leq x \leq 1$. Since $g(c) > 1$, it follows that $c < 2/3$, and so $c^\alpha \leq 1/2$; and consequently $2(1 - c)^\alpha > 1/2$, that is, $c < 1 - 4^{-1/\alpha}$. This proves 5.1.

5.2 Let $\alpha \geq 1$, and let $(G, f)$ be $\alpha$-critical. Let $A, B \subseteq V(G)$ be disjoint and either complete or anticomplete. Then not both $f^\alpha(A), f^\alpha(B) > 2^{-\alpha}$.

**Proof.** Let $P$ be a perfect induced subgraph of $G[A]$ with $f(P)$ maximum, and choose $Q$ in $G[B]$ similarly. Since $P \cup Q$ is a perfect induced subgraph of $G$, it follows that $f(P) + f(Q) \leq 1$, and from the symmetry we may assume that $f(P) \leq 1/2$. We may also assume that $f(P) > 0$, $f(P) = p$ say, and so $f(v)/p$ ($v \in A$) is a good function on $G[A]$; and we may assume that $Y \neq \emptyset$, and so $G[A]$ is $\alpha$-narrow, and consequently $f^\alpha(A) \leq p^\alpha \leq 2^{-\alpha}$. This proves 5.2.
Next we need the following consequence of a theorem of Rödl [10]:

5.3 For all \( \varepsilon > 0 \) and every graph \( H \), there exists \( \delta > 0 \) such that for every \( H \)-free graph \( G \), there is a subset \( X \subseteq V(G) \) with \( |X| \geq \delta |V(G)| \) such that one of \( G[X], \overline{G}[X] \) has maximum degree less than \( \varepsilon |X| \).

We also need the following theorem of Bousquet, Lagoutte and Thomassé [3]:

5.4 For every path \( H \), there exists \( \varepsilon > 0 \) such that for every \( H \)-free graph \( G \) with \( |G| > 1 \), either some vertex of \( G \) has degree at least \( \varepsilon |G| \), or there are disjoint anticomplete subsets \( A, B \subseteq V(G) \) with \( |A|, |B| \geq \varepsilon |G| \).

We recall that the house is the complement of \( P_5 \). Let us say \( G \) is house-free if \( G \) contains no house.

5.5 For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, if \( G \) is house-free and \( |G| > 1 \), then either

- there are disjoint sets \( A, B \subseteq V(G) \), complete to each other, with \( |A|, |B| \geq \varepsilon \delta |G| \), or
- there exists \( X \subseteq V(G) \) with \( |X| \geq \delta |G| \) such that \( G[X] \) has maximum degree less than \( \varepsilon |X| \).

Proof. Choose \( \varepsilon' > 0 \) such that 5.4 holds with \( H, \varepsilon \) replaced by \( P_5, \varepsilon' \) respectively. Now let \( \varepsilon > 0 \); we must show that there exists \( \delta \) as in the theorem. Thus we may assume that \( \varepsilon \leq \varepsilon' \), by reducing \( \varepsilon \) if necessary. Choose \( \delta \) as in 5.3. Now let \( G \) be a house-free graph with \( |G| > 1 \). The complement \( \overline{G} \) of \( G \) is \( P_5 \)-free, and so by 5.3 there is a subset \( X \subseteq V(G) \) with \( |X| \geq \delta |V(G)| \) such that one of \( G[X], \overline{G}[X] \) has maximum degree less than \( \varepsilon |X| \).

If \( G[X] \) has maximum degree less than \( \varepsilon |X| \) then the theorem holds, so we assume that \( G[X] \) has maximum degree at least \( \varepsilon |X| \) (and so \( |X| > 1 \)), and therefore \( \overline{G}[X] \) has maximum degree less than \( \varepsilon |X| \). By 5.4 applied to \( \overline{G}[X] \), there are disjoint anticomplete (in \( \overline{G} \)) subsets \( A, B \subseteq X \) with \( |A|, |B| \geq \varepsilon |X| \). But then \( A \) is complete to \( B \) in \( G \), and \( |A|, |B| \geq \varepsilon |X| \geq \delta \varepsilon |G| \). This proves 5.5.

From 5.5 we deduce:

5.6 Let \( \varepsilon > 0 \), and choose \( \delta > 0 \) satisfying 5.5. Let \( \alpha \geq 1 \), such that \( \varepsilon \delta 2^\alpha > 1 \). Let \( (G, f) \) be \( \alpha \)-critical, where \( G \) is house-free. Then there is a subset \( X \subseteq V(G) \) and a good function \( g \) on \( G[X] \) with \( g(v) \leq f(v) \) for each \( v \in X \), such that \( g^\alpha(X) \geq \delta f^\alpha(G) \) and \( g^\alpha(N_G(v) \cap X) < \varepsilon g^\alpha(X) \) for every vertex \( v \in X \).

Proof. By rational approximation, we may assume that \( f^\alpha \) is rational. Choose an integer \( T > 0 \) such that \( T f^\alpha(v) \) is an integer for all \( v \in V(G) \). Let \( G' \) be obtained from \( G \) by replacing each vertex \( v \) by a clique \( W_v \) of cardinality \( T f^\alpha(v) \), where

- the sets \( W_v \) (\( v \in V(G) \)) are pairwise disjoint;
- for all distinct \( u, v \in V(G) \) adjacent in \( G \), \( W_u \) is complete to \( W_v \) in \( G' \), and
- for all distinct \( u, v \in V(G) \) nonadjacent in \( G \), \( W_u \) is anticomplete to \( W_v \) in \( G' \).

It follows that \( G' \) is also house-free, and \( |G'| = T f^\alpha(G) > T \geq 1 \). From 5.5 applied to \( G' \), we deduce that either
• there are disjoint sets $A', B' \subseteq V(G')$, complete to each other, with $|A'|, |B'| \geq \varepsilon \delta |G'|$, or

• there exists $X' \subseteq V(G')$ with $|X'| \geq \delta |G'|$ such that $G'[X']$ has maximum degree less than $\varepsilon |X'|$.

Suppose that the first bullet holds. Let $A$ be the set of vertices in $G$ such that $W_v \cap A' \neq \emptyset$, and define $B$ similarly. Then $A$ is complete to $B$ in $G$. Moreover

$$f^\alpha(A) \geq |A'|/T \geq \varepsilon \delta |G'|/T \geq \varepsilon \delta f^\alpha(G) > \varepsilon \delta,$$

and similarly $f^\alpha(B) \geq \varepsilon \delta f^\alpha(G)$. By $5.5$, $\varepsilon \delta \leq 2^{-\alpha}$, contrary to the hypothesis.

Thus the second bullet holds. Let $X$ be the set of all $v \in V(G)$ such that $W_v \cap X' \neq \emptyset$; and for each $v \in V(G)$ let $g(v)$ satisfy $T(g(v))^\alpha = |W_v \cap X'|$. Thus $g^\alpha(X) = |X'| \geq \delta |G'| = \delta f^\alpha(G)$, and $g(v) \leq f(v)$ for each $v \in V(G)$. Let $v \in X$. The union of the sets $W_u \cap X'$ over all $u \in N(v) \cap X$ has cardinality less than $\varepsilon |X'|$ (indeed, less than $\varepsilon |X'| - |W_v \cap X'| + 1$); and so $Tg^\alpha(N(v) \cap X) < \varepsilon |X'| = \varepsilon Tg^\alpha(X)$, that is, $g^\alpha(N(v)) < \varepsilon g^\alpha(X)$. This proves $5.6$.

Since $P_4$-free graphs are perfect, a theorem of Erdős and Hajnal [8] (see also Alon, Pach and Solymosi [11]) implies:

5.7 There exists $\varepsilon > 0$ such that if $G$ is forcer-free then $G$ has a clique or stable set of cardinality at least $|G|^\varepsilon$.

We also need a theorem of Jacob Fox (he did not publish his proof, but we gave a proof in [4]):

5.8 Let $H$ be a graph for which there exists a constant $\delta > 0$ such every $H$-free graph $G$ has a clique or stable set of cardinality at least $|G|^\delta$. Then every $H$-free graph is $\frac{3}{8}$-narrow.

By combining 5.7 and 5.8 we obtain:

5.9 There exists $\alpha \geq 1$ such that every forcer-free graph is $\alpha$-narrow.

We deduce:

5.10 Let $\alpha' \geq 1$ such that every forcer-free graph is $\alpha'$-narrow. Let $\alpha \geq \alpha'$, and let $G$ be a graph such that every proper induced subgraph is $\alpha$-narrow. Let $g$ be a good function on $G$. Let $Z \subseteq V(G)$ with $|Z| > 1$, such that $G[Z]$ is connected and guarded. Let $d$ be the maximum of $g^\alpha(N_G(v) \cap Z)$ over all $v \in Z$. Then $g^\alpha(Z) \leq \max(2d, d^{1-\alpha'/\alpha})$.

Proof. If $G[Z]$ is not anticonnected, there are two vertices $u, v \in Z$ such that $N(u) \cup N(v) = Z$, and so $g^\alpha(Z) \leq 2d$ as required. So we may assume that $G[Z]$ is anticonnected. Let us list all subsets $X$ of $Z$ with the properties that $X$ is a homogeneous set of $G[Z]$, and $X \neq Z$, and $X$ is maximal with these two properties; let these subsets be $W_1, \ldots, W_k$ say. Thus $W_1 \cup \cdots \cup W_k = Z$, because $|Z| \leq 2$ and so each singleton subset of $Z$ is a subset of one of $W_1, \ldots, W_k$.

We claim that $W_1, \ldots, W_k$ are pairwise disjoint. Suppose that $W_1 \cap W_2 \neq \emptyset$ say. Choose $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. If $w_1, w_2$ are nonadjacent, then since $W_2$ is a homogeneous set, $w_1$ has no neighbours in $W_2$, and so, since $W_1$ is homogeneous, each vertex of $W_2$ has no neighbour in $W_1$; and so $G[W_1 \cup W_2]$ is not connected. If $w_1, w_2$ are adjacent, then similarly $G[W_1 \cup W_2]$ is not anticonnected. Since $G[Z]$ is both connected and anticonnected, it follows that $W_1 \cup W_2 \neq Z$. 12
But $W_1 \cup W_2$ is a homogeneous set of $G[Z]$, contrary to the maximality of $W_1$. This proves that $W_1, \ldots, W_k$ form a partition of $Z$, and so $k > 1$.

Choose $w_i \in W_i$ for $1 \leq i \leq k$, and let $G'$ be the graph induced on $\{w_1, \ldots, w_k\}$. From the hypothesis, $G'$ is forcer-free, and so $\alpha'$-narrow. Let $t = \alpha/\alpha'$ and $r = 2^{-1/\alpha'}$; thus $r^{\alpha'} = 1/2$, and $r^\alpha = 2^{-t}$. For $1 \leq i \leq k$, let $P_i$ be a perfect subgraph of $G[w_i]$ with $g(P_i)$ maximum, and let $f(w_i) = g(P_i)$. By an application of Lovász’ substitution lemma [9], it follows that $f$ is a good function on $G'$, and so $f^\alpha(G') \leq 1$. For $1 \leq i \leq k$, $G[w_i]$ is $\alpha$-narrow (because $k > 1$). Since $g(v)/f(w_i)$ ($v \in W_i$) is good on $G[w_i]$, it follows that $g^\alpha(W_i) \leq f(w_i)^\alpha$. But also, since $G[Z]$ is connected, and $W_i \neq Z$, there is a vertex $v \in Z \setminus W_i$ complete to $W_i$; and so $g^\alpha(W_i) \leq d$ by hypothesis. Thus

$$g^\alpha(W_i) \leq \min(f(w_i)^\alpha, d) \leq f(w_i)^\alpha d^{1-1/t}.$$

Hence

$$g^\alpha(Z) = \sum_{1 \leq i \leq k} g^\alpha(W_i) \leq d^{1-1/t} \sum_{1 \leq i \leq k} f(w_i)^\alpha \leq d^{1-1/t} f^\alpha(G') \leq d^{1-1/t}.$$

This proves 5.10.

We deduce 1.5, which we restate:

**5.11** There exists $\alpha \geq 1$ such that for every $\alpha$-critical pair $(G, f)$, there is a hole-with-hat in $G$.

**Proof.** Let $\varepsilon = 1/6$, and choose $\delta > 0$ satisfying 5.5. From 5.7, there exists $\alpha' \geq 1$ such that every forcer-free graph is $\alpha'$-narrow. Let $\alpha \geq 2$, such that $\varepsilon \delta 2^\alpha/\alpha' > 1$. We claim that $\alpha$ satisfies the theorem. Suppose not; then there is an $\alpha$-critical pair $(G, f)$, such that $G$ is hole-with-hat-free. By 5.6, there is a subset $X \subseteq V(G)$ and a good function $g$ on $G[X]$ with $g(v) \leq f(v)$ for each $v \in X$, such that $g^\alpha(X) \geq \delta f^\alpha(G) > \delta$ and $g^\alpha(N_G(v) \cap X) < \varepsilon g^\alpha(X)$ for every vertex $v \in X$. Let $g^\alpha(X) = \lambda$; then $\delta \leq \lambda \leq f^\alpha(G)$. Let $H = G[X]$, and define $w(v) = g^\alpha(v)/\lambda$ for each $v \in X$. Then $(H, w)$ is a weighted graph.

1. $(H, w)$ is $\varepsilon$-coherent.

By 5.1, for each $v \in V(H)$, $g(v) \leq f(v) < 1 - 4^{-1/\alpha} \leq 1/2$, and so $w(v) < (1 - 4^{-1/\alpha})^\alpha/\lambda \leq 2^{-\alpha}/\lambda \leq \varepsilon$ (since $\lambda \geq \delta$). Also, for each vertex $v \in V(H)$,

$$\lambda w(N_H(v)) = g^\alpha(N_G(v) \cap X) < \varepsilon g^\alpha(X) = \varepsilon \lambda,$$

and so $w(N_H(v)) < \varepsilon$. Third, by 5.2 if $A, B \subseteq V(H)$ are disjoint and anticomplete, then not both $w(A), w(B) \geq \varepsilon$, since $\varepsilon > 2^{-\alpha}/\lambda$ (because $\lambda \geq \delta$). Consequently $(H, w)$ is $\varepsilon$-coherent. This proves (1).

By 4.3 and since $\varepsilon = 1/6$, there exists $Z \subseteq V(H)$ with $|Z| > 1$, such that $H[Z]$ is connected and guarded, and $w(Z) > 1 - 4\varepsilon$. Let $d$ be the maximum of $\lambda w(N_G(v) \cap Z)$ over all $v \in Z$. Hence

$$d \leq \varepsilon \lambda.$$

By 5.10,

$$\lambda w(Z) \leq \max(2d, d^{1-\alpha'/\alpha}) \leq \max(2\varepsilon \lambda, (\varepsilon \lambda)^{1-\alpha'/\alpha}).$$

But $2\varepsilon \lambda \geq (\varepsilon \lambda)^{1-\alpha'/\alpha}$, since $2(\varepsilon \lambda)^{\alpha'/\alpha} \geq 1$ (since $\lambda \geq \delta$). Thus $\lambda w(Z) \leq 2\varepsilon \lambda$, and so $w(Z) \leq 2\varepsilon$, contradicting that $w(Z) > 1 - 4\varepsilon$ and $\varepsilon = 1/6$. This proves 5.11.
References

[1] N. Alon, J. Pach and J. Solymosi, “Ramsey-type theorems with forbidden subgraphs”, *Combinatorica* 21 (2001), 155–170.

[2] M. Bonamy, N. Bousquet and S. Thomassé, “The Erdős-Hajnal conjecture for long holes and antiholes”, *SIAM J. Discrete Math.*, 30, 1159–1164.

[3] N. Bousquet, A. Lagoutte, and S. Thomassé, “The Erdős-Hajnal conjecture for paths and antipaths”, *J. Combinatorial Theory, Ser. B*, 113 (2015), 261–264.

[4] M. Chudnovsky and P. Seymour, “Excluding paths and antipaths”, *Combinatorica*, 35 (2015), 389–412.

[5] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, “The strong perfect graph theorem”, *Annals of Math.* 164 (2006), 51–229.

[6] M. Chudnovsky and Y. Zwols, “Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement” *J. Graph Theory*, 70 (2012), 449–472.

[7] P. Erdős and A. Hajnal, “On spanned subgraphs of graphs”, *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977), www.renyi.hu/~p_erdos/1977-19.pdf.

[8] P. Erdős and A. Hajnal, “Ramsey-type theorems”, *Discrete Applied Mathematics* 25 (1989), 37–52.

[9] L. Lovász, “Normal hypergraphs and the perfect graph conjecture”, *Discrete Math.* 2 (1972), 253–267.

[10] V. Rödl, “On universality of graphs with uniformly distributed edges”, *Discrete Math.* 59 (1986), 125–134.