Quantum manifestations of classical chaos in a Fermi accelerating disk

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Abstract

We study the classical and quantum mechanics of a two-dimensional version of a Fermi accelerator. The model consists of a free particle that collides elastically with the walls of a circular disk with the radius varying periodically in time. A complete quantum mechanical solution of the problem is possible for a specific choice of the time-periodic oscillating radius. The quasi-energy spectral properties of the model are obtained from direct evaluation of finite-dimensional approximations to the time evolution operator. As the scaled $\hbar$ is changed from large to small the statistics of the Quasienergy Eigenvalues (QEE) change from Poisson to circular orthogonal ensemble (COE). Different statistical tests are used to characterize this transition. The transition of the Quasienergy Eigenfunctions (QEF) is also studied using the $\chi^2$ test with $\nu$ degrees of freedom. The Porter-Thomas distribution is shown to apply in the COE regime, while the Poisson regime does not fit the $\chi^2$ test with $\nu = 0$. We find that the Poisson regime is associated with exponentially localized QEF whereas the eigenfunctions are extended in the COE regime.
To make a direct comparison between the classical and quantum solutions we change the representation of the model to one in which the boundary is fixed and the Hamiltonian acquires a quadratic term with a time-periodic frequency. We then carry out a successful comparison between specific classical phase space surface-of-section solutions and their corresponding quasi-energy eigenfunctions in the Husimi representation.

I. INTRODUCTION

In recent years a good deal of attention has been directed at trying to understand the quantum manifestations of classical chaos (QMCC). Although a complete understanding is not yet in sight, significant progress has been made in obtaining partial answers to this paradigm. This progress has primarily been achieved from studies of lower-dimensional models: 2-D for energy conserving models and 1-D for driven systems. In particular, very few studies have been carried out in two-dimensional time-dependent problems, for even in the classical limit the theoretical analysis is nontrivial. It is the purpose of this paper to consider a time-periodic two-dimensional problem which, because of the particular nature of the model, has a reduced degree of complexity and thus allows us to analyze its solutions in detail.

The model considered here is a two-dimensional version of the thoroughly studied one-dimensional Fermi accelerator [1]. The 1-D quantum problem was introduced in Ref. [2]. There it was shown that for a very specific form of the wall-oscillation function the Floquet evolution operator can be explicitly written and analyzed, without the complications of numerical time-ordering (which effectively precludes a thorough investigation). The model has been further analyzed by others [3–5]. The Fermi acceleration disk (FAD) studied here consists of a free particle bouncing elastically inside a circular boundary with a radius that oscillates periodically in time. Classically, the model exhibits a transition to chaos as the amplitude of the wall oscillation is increased. In the quantum case, the periodicity in the
wall oscillations allows us to use Floquet’s theory in terms of the one-period time evolution operator.

This model is of interest for several reasons: first, because the dynamics is not kicked, as in almost all other time-periodic models considered, e.g. the thoroughly investigated periodically kicked rigid rotator model (PKR) [6]. Second, the explicit form of the tight-binding-like model obtained for the dynamical equations decays algebraically instead of exponentially, as in the PKR. The FAD model then allows for a comparison of our results to those of the kicked models, for it is important to know if the corresponding results are generic. Third, the possibility exists here that the spectrum of the evolution operator contains a continuous component [5], implying nonrecurrent behavior of the wave functions. Evidence suggestive of such a continuous spectrum has been reported in [4].

One of the major advances in the understanding of the QMCC comes from the clear differences found between the eigenvalue spectral properties of models that classically exhibit chaos and those that do not [7]. The tools used to measure these differences are extensions of those developed in Random Matrix Theory (RMT) [8]. In this paper we present an analysis of the spectral properties of the time evolution operator using these tools.

While the FAD model provides considerable evidence for the spectral (or statistical) characterization of the transition from integrability to chaos, for reasons explained later, it turns out to be inconvenient in the study of the classical-quantum correspondence in phase-space. To overcome this difficulty, we map the FAD model onto a model of a particle subjected to a bounded, rotationally symmetric, inverted harmonic oscillator potential in the presence of time periodic kicks (the Fermi inverted parametric oscillator or FIPO model) [4]. In this new representation, we can construct a mapping in terms of classical canonical variables, which in turn permits us to make explicit connections between classical phase space solutions and their corresponding quantum counterparts in terms of the Husimi distributions. This change in representation also allows us to make connections with experiments in mesoscopic quantum dots [9].

The outline of the paper is as follows: In section II we define both the classical and
quantum FAD and FIPO models. The quantum problem is shown to be almost completely integrable for the specific functional form of the oscillating radius. In Section III we present the bulk of our quantum results for the statistics of the $QEE$ in the FAD representation, both for eigenvalues and eigenfunctions. Specifically, we discuss the eigenvalue statistical properties in terms of different measures that include the nearest-neighbor spacing probability distribution and its integrated form, the $\Delta_3$ statistic and the two-point function $\Sigma_2$. In fitting the data from the Poisson to the COE regime we use the Brody function as well as the Izrailev tests to try and quantitatively parametrize the transition region. We further discuss the eigenfunctions properties in different regimes using the $\chi^2$ distribution of $\nu$ freedoms as a convenient parameterization of the results [14], as well as their localization properties in terms of the time-averaged transition probabilities. In Section IV we discuss the explicit classical-quantum correspondences in phase space, carried out in the FIPO representation. We consider a series of classical orbits in the surface of section and then identify the Husimi contour plots of the eigenfunctions of the Floquet evolution operator that correspond to the classical solutions. Finally, in section V we present both our conclusions and also some questions left for future studies.

II. THE CLASSICAL AND QUANTUM MODELS

As we mentioned in the introduction we will use two representations of the model studied here. Both representations have their own strengths and thus their analysis leads to complementary results. In this section we explicitly define the FAD and FIPO models in the classical and quantum limits. The classical mechanics of the FAD is discussed briefly while the classical solutions of the FIPO are discussed in more detail since it will be in the FIPO representation that the quantum-classical correspondence will be carried out.
A. Classical Fermi accelerating disk (FAD)

The FAD model consists of a free particle confined to move inside a two-dimensional disk whose radius oscillates periodically in time; $R(\tau) = R_0 \theta(\tau)$ with $\theta(\tau + T) = \theta(\tau)$ and $T$ is the period. The circular wall is taken to represent an infinite potential barrier for the motion of the particle, that is, classically the particle undergoes perfectly elastic collisions, while quantum mechanically the particle’s wavefunction vanishes identically at the boundary.

The classical Hamiltonian written in polar coordinates is

$$H = \frac{1}{2m}(p_r^2 + p_\phi^2),$$

where $r$ is the radius with $0 \leq r \leq R(\tau)$, $\phi$ the azimuthal angle, and $m$ the mass of the particle. The resulting equations of motion are,

$$\frac{d^2r}{d\tau^2} = \frac{J^2}{m^2r^3}, \quad 0 \leq r(\tau) \leq R_0 \theta(\tau),$$

$$\frac{d\phi}{d\tau} = \frac{J}{mr^2}, \quad 0 \leq \phi(\tau) < 2\pi,$$

where $J$ is the magnitude of the conserved orbital angular momentum. In principle, we can study the dynamics of the problem for any form of the wall oscillation function $\theta(\tau)$. However, as we shall show in the quantum case, the problem dictates the appropriate $\theta(\tau)$ that allows an explicit evaluation of the time evolution operator. The specific form of $\theta(\tau)$ is found to be

$$\theta(\tau) = \sqrt{1 + 2\epsilon \left| \text{mod}(\tau, \frac{T}{2}) \right|},$$

where $\epsilon$ is the amplitude of the wall oscillation. We will henceforth measure $\epsilon$ in units of $R_0$ and time in units of the period $T$, while the mass will be taken as one henceforth. Note that $\theta(\tau)$ is already discontinuous in its first derivative, which is important in order to have chaotic solutions.

As mentioned above, the FAD representation is not the most appropriate one to discuss the classical-quantum correspondence of the solutions, of central interest in this paper.
However, to show that the FAD model does indeed show a transition to chaos we have solved the classical equations of motion at a period for different parameter values. Examples of $p_r$ vs $r$ phase space plots are shown in Figs. 1(a-c) for three values of $J$ and $\epsilon$. Note that since $\phi$ is related to $r(\tau)$ by Eq. (21), if $r(\tau)$ shows irregular behavior, so will $\phi(\tau)$.

B. Classical finite inverted parametric oscillator (FIPO)

To obtain the FIPO representation of the model studied here we begin by carrying out the Liouville transformation [16],

$$\tau = \int_{t_0}^{t} \theta^2(t') dt', \quad (4)$$

$$r(\tau) = \theta(t) \rho(t). \quad (5)$$

resulting in the new equation of motion for the $\rho$ coordinate,

$$\frac{d^2 \rho}{dt^2} + \omega^2(t) \rho - \frac{J^2}{m^2 \rho^3} = 0, \quad 0 \leq \rho(t) \leq R_0, \quad (6)$$

where

$$\omega^2(t) = \frac{d}{dt} \left( \frac{1}{\theta} \frac{d\theta}{dt} \right) - \left( \frac{1}{\theta} \frac{d\theta}{dt} \right)^2. \quad (7)$$

The equation of motion can be derived from the transformed Hamiltonian

$$H_{FIPO} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2(t) \rho^2 + \frac{J^2}{2m \rho^2}, \quad \text{with } 0 \leq \rho \leq R_0. \quad (8)$$

This hamiltonian describes the one-dimensional motion of a particle in an oscillator potential whose frequency changes with time, as given by equation Eq.(7), with a centrifugal barrier and confined to a stationary infinite well of width $R_0$. If, in addition, we choose the particular form of the wall oscillation function $\theta(t)$ as given in equation (3), the explicit time-dependent frequency of the oscillator is given by

$$\omega^2(t) = -\epsilon^2 + 2 \epsilon \sum_{n=-\infty}^{\infty} \{ \delta(t - (n + \frac{1}{2}) T_0) - \delta(t - nT_0) \}, \quad (9)$$

where
\[ T_0 = \int_0^T \frac{d\tau}{\theta^2(\tau)} = \frac{1}{\epsilon} \ln(1 + \epsilon T). \]  

(10)

Thus, the oscillator frequency consists of a constant, negative part, and two sets of periodic Dirac \( \delta \)-functions, one at integer, and the other at half-integer periods. This is the same form for \( \omega(t) \) obtained by Chu and José in their study the one-dimensional Fermi model \([4]\).

In normalized units, the full classical Hamiltonian \( H_{FIPO} \) reads

\[ H_{FIPO} = \frac{p^2}{2} - \frac{1}{2} \epsilon^2 \rho^2 + \frac{J^2}{2\rho^2} + 2 \epsilon \rho^2 \sum_{n=-\infty}^{\infty} \left\{ \delta(t - (n + \frac{1}{2})) - \delta(t - n) \right\}. \]  

(11)

The free parameters in the problem are \( \epsilon \) (associated with the amplitude of wall motion) and \( J \) (the angular momentum of the particle). Note that the harmonic oscillator kick strength alternates in sign every half-period.

When the particle is free, that is when it does not hit the wall nor does it get kicked, the equation of motion is

\[ \ddot{\rho} = \epsilon^2 \rho + \frac{J^2}{\rho^3}. \]  

(12)

The solution to Eq.(12) is easily verified to be

\[ \rho(t) = \sqrt{\rho_0 \cosh{\epsilon(t-t_0)} + \frac{p_0}{\epsilon} \sinh{\epsilon(t-t_0)}} \left[ \cosh{2\epsilon(t-t_0)} + \cosh^{-1}\left(\frac{\rho_0^2 + E_0/\epsilon^2}{a}\right) \right]^{\frac{1}{2}} + \frac{J^2}{\rho_0^2 \epsilon} \sinh^2{\epsilon(t-t_0)}}, \]  

(13)

\[ p(t) = \dot{\rho}(t), \]  

(14)

or, in terms of the initial energy

\[ E_0 = \frac{1}{2} \left( \frac{p_0^2}{\rho_0^2} - \epsilon^2 \rho_0^2 + \frac{J^2}{\rho_0^2} \right), \]  

(15)

the solutions are given by

\[ \rho(t) = \sqrt{a \cosh{2\epsilon(t-t_0)} + \cosh^{-1}\left(\frac{\rho_0^2 + E_0/\epsilon^2}{a}\right)}} - \frac{E_0}{\epsilon^2}, \]  

(16)

\[ p(t) = \pm \sqrt{2E_0 + \epsilon^2 \rho_0^2 - \frac{J^2}{\rho_0^2}}. \]  

(17)
The sign of $p$ is positive or negative depending on whether the particle’s last collision was with the centrifugal barrier or the wall. Here, $t_0$, $\rho_0$ and $E_0$ are determined by the initial conditions, and the constant $a$ is defined as

$$a^2 = \frac{j^2}{\epsilon^2} + \frac{E_0^2}{\epsilon^2}. \quad (18)$$

For a given energy $E$ the radial coordinate is constrained by the wall and the centrifugal barrier:

$$\rho_{\text{min}} \leq \rho \leq 1, \quad \rho_{\text{min}} = \sqrt{a - E_0/\epsilon^2}. \quad (19)$$

Note that these equations of motion are non linear, even in the wall’s absence. Formally, one can represent either of the set of equations (13)-(14) or (16)-(17) as

$$\begin{pmatrix} \rho(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} F(\rho_0, p_0, t - t_0) \\ G(\rho_0, p_0, t - t_0) \end{pmatrix}, \quad (20)$$

but we can investigate the stability of the mapping by linearizing about fixed points only numerically.

The effect of collisions with the wall (or potential barrier) is simply to reverse the direction of motion of the particle:

$$\begin{pmatrix} \rho(t_c^+) \\ p(t_c^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho(t_c^-) \\ p(t_c^-) \end{pmatrix}, \quad (21)$$

where $t_c$ is the time of collision of the particle with the wall (or potential barrier), and the minus and plus signs indicate times just before and after the collision.

The effect of the kicks at every half period are obtained by integrating the equations of motion over an infinitesimal kick time. The results are, for half-integer and integer period kicks,

$$\begin{pmatrix} \rho(t_{\frac{1}{2}}^+) \\ p(t_{\frac{1}{2}}^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2\epsilon & 1 \end{pmatrix} \begin{pmatrix} \rho(t_{\frac{1}{2}}^-) \\ p(t_{\frac{1}{2}}^-) \end{pmatrix}, \quad (22)$$
\[
\begin{pmatrix}
\rho(t^+_1) \\
p(t^+_1)
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
2\varepsilon & 1
\end{pmatrix} \begin{pmatrix}
\rho(t^-_1) \\
p(t^-_1)
\end{pmatrix},
\]
respectively. The non-integrability of the problem arises precisely because of these kicks.

In contrast to the one-dimensional (or \( J = 0 \)) case, because the particle here can never be exactly at the origin at finite energies, the kicks are always relevant (in the sense that they always couple the particle’s position and momentum).

While so far we have only looked at the \((\rho, p)\) plane, we can also take a coordinate-space cut. From the definition of angular momentum,

\[
\frac{d\phi}{dt} = \frac{J}{\rho^2(t)},
\]
we obtain that during a ‘free’ propagation, the angle advances according to

\[
\phi(t) - \phi_0 = \frac{1}{2} \left\{ \sin^{-1} \left( \frac{1}{a} \left( \frac{1}{\rho^2_0} - \frac{E_0}{J^2} \right) \right) - \sin^{-1} \left( \frac{1}{a} \left( \frac{1}{\rho(t)^2} - \frac{E_0}{J^2} \right) \right) \right\} \pmod{2\pi}
\]
where \( a \) was previously defined. In this paper we concentrate on the \((\rho, p)\) phase space plane.

We are now in a position to construct a map in phase space which takes an initial set of coordinates to a final one. The algorithm actually used was to propagate the initial conditions via Eqs. (16)-(17) until it either hit a barrier, in which case Eq.(21) was used to change the phase space coordinates, or until the time elapsed was a half-integer or integer multiple of the period, in which case Eqs.(22) or (23) are used to transform the coordinates.

We emphasize that in between kicks there may be many collisions between the particle and the walls, which effectively means new initial conditions for propagation under Eq. (16)-(17). In general, the map is quite complicated, and very sensitive to initial conditions. By recording the values at each successive period, we obtain a surface-of-section of the trajectory of the particle in phase space.

We shall discuss specific results under different conditions in Section IV.
C. Quantum FAD model

The Schrödinger equation of the quantum mechanical version of the FAD (QFAD) model introduced in Section (II A) in polar coordinates is

\[ i\hbar \frac{\partial \Lambda}{\partial \tau} = -\frac{\hbar^2}{2m} \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right\} \Lambda(r, \phi, \tau), \]  

(26)

with the boundary condition \( \Lambda(r(\tau) = R_0 \theta(\tau), \phi, \tau) = 0 \), and normalization

\[ \int_0^{R_0 \theta(\tau)} r \, dr \int_0^{2\pi} d\phi \, |\Lambda(r, \phi, \tau)|^2 = 1. \]  

(27)

These equations completely specify the QFAD model considered in this paper. To find the general solution to the QFAD model we expand the wave function \( \Psi(r, \phi, \tau) \) in terms of the natural complete set of basis functions \( \psi_{nl} \),

\[ \Lambda(r, \phi, \tau) = \sum_{l=-\infty}^{+\infty} \sum_{n=1}^{\infty} C_{nl}(\tau) \psi_{nl}(r, \phi, \tau), \]  

(28)

where the \( \psi_{nl} \)'s are the instantaneous orthonormalized eigenfunctions for a given \( R(t) \), i.e. the solutions to

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi_{nl}(r, \phi, \tau) = E_{nl}(\tau) \psi_{nl}(r, \phi, \tau). \]  

(29)

Here \( \tau \) is a parameter and \( \psi_{nl} \) satisfies the same boundary and normalization conditions as \( \Lambda \). The instantaneous solutions are given by

\[ \psi_{nl}(r, \phi, \tau) = \frac{1}{\sqrt{\pi R(\tau)} J_{l+1}(\beta_{nl})} J_l(\beta_{nl} \frac{r}{R(\tau)}) \exp(i l \phi), \]  

(30)

where \( J_l \) is the Bessel function of order \( l \) and \( \beta_{nl} \) is its \( n \)-th zero. The corresponding instantaneous energies are

\[ E_{nl}(\tau) = \frac{\hbar^2}{2m} \left( \frac{\beta_{nl}}{R_0 \theta(\tau)} \right)^2. \]  

(31)

We now substitute the expansion (28) into the Schrödinger equation (26). After using orthogonality and known identities involving Bessel functions, one can evaluate the integral
\[ K_l(m, n) = \int_0^\infty x^2 J_l(\beta_{ml} x) \frac{d}{dx} J_l(\beta_{nl} x) \, dx \]

\[ = \frac{\beta_{ml} \beta_{nl}}{\beta_{nl}^2 - \beta_{ml}^2} J_{l-1}(\beta_{ml}) J_{l-1}(\beta_{nl}) \quad (m \neq n), \quad (32) \]

so that the final result can be cast in the form

\[ i \hbar \dot{C}_{nl} = \sum_{m,k} \left[ E_{nl} \delta_{k,l} \delta_{n,m} - i \hbar \frac{\dot{\theta}}{\theta} \frac{2 \beta_{mk} \beta_{nk}}{\beta_{mk}^2 - \beta_{nk}^2} \delta_{k,l} (1 - \delta_{m,n}) \right] C_{mk} \quad (33) \]

\[ \equiv \sum_{m,k} H_{nm;kl} C_{mk}, \]

where the overdot denotes differentiation with respect to \( \tau \). Equations (33) are completely equivalent to the Schrödinger equation (26). The time development of the system is obtained from

\[ C_{nl}(\tau) = \sum_{m,k} U_{nm;kl}(\tau, \tau') C_{mk}(\tau'), \quad (34) \]

with the time evolution operator defined by

\[ U_{nm;kl}(\tau, \tau') = T \exp \left( -\frac{i}{\hbar} \int_{\tau'}^{\tau} H_{nm;kl}(s) \, ds \right), \quad (35) \]

where \( T \) formally represents the time-ordering operation. We can, however, get rid of the time-ordering if the effective Hamiltonian \( H \) factors into an overall time-dependent factor multiplying a time-independent part. For this to happen, we see from equations (31) and (33) that we must have

\[ \frac{\dot{\theta}}{\theta} \sim \frac{1}{\theta^2}. \quad (36) \]

An appropriate form for \( \theta(\tau) \), satisfying the periodicity and time reversal invariance conditions is the one given in (3). Using this form for \( \theta(\tau) \) the matrix elements of the one-period time evolution operator are exactly given by

\[ U_{nl;mk}(\tau + T, \tau) = \exp \left( -\frac{i}{\hbar} H_{nl;mk} \int_{\tau}^{\tau + T} ds \frac{ds}{\theta^2(s)} \right). \quad (37) \]
Note that the time evolution matrix is nominally four-dimensional but, because of the conservation of angular momentum, it reduces to a two-dimensional matrix, for a given value of $l$.

The required zeros of the Bessel functions are calculated using the Newton-Raphson technique after bracketing the root using the fact that separation between successive zeros asymptotically approaches $\pi$. This method is very efficient, since finding the roots even for the high zeros and large $l$ values can be done to machine accuracy. The quasi-energy spectrum is determined once the zeros of the Bessel functions are found and the $U$-matrix is diagonalized numerically (with unitarity preserved to machine precision). Since $U$ is unitary, its eigenvalues are unimodular, and we write it’s eigenvalue equation as

$$U(\tau + T, \tau) |\lambda_n\rangle = e^{i\varepsilon_n T} |\lambda_n\rangle \quad 0 \leq \varepsilon_n \leq 2\pi.$$  \hspace{1cm} (38)

Complete knowledge of $U$ implies full knowledge of the evolution of the system at all integer multiples of the period because $U(\tau + NT, \tau) = U^N(\tau + T, \tau)$. This product of $U$’s is the quantum counterpart of the classical one-period maps. The quasi-energy eigenvalues $\varepsilon_n$, being invariant under unitary transformations, are representation independent and the quasi-energy spectrum obtained this way is of high quality from a numerical point of view. For our calculations, we used a truncated basis of 400 states for calculating properties of the $QEF$, while we used a basis of 500 states to calculate the $QEE$. We checked that the $QEE$ spectrum was stationary, \textit{i.e.} the eigenvalues remain basically unchanged to within 1% in this case as we increase the size of the $U$ matrix from $400 \times 400$ to $600 \times 600$.

D. Quantum FIPO model

Quantum mechanically, the required set of Liouville and unitary transformations are a little more complicated (because of the additional requirement of normalization), but are nevertheless doable. We first transform the Schrödinger equation (26) to an effectively one-dimensional problem by writing the wavefunction as
\[ \Lambda(r, \phi, \tau) = \frac{1}{\sqrt{2\pi r}} e^{i\phi} \Phi(r, \tau). \]  

(39)

the Schrödinger equation (26) now reads

\[ i\hbar \frac{\partial}{\partial \tau} \Phi(r, \tau) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \Phi(r, \tau) + \frac{(l^2 - \frac{1}{4})\hbar^2}{2mr^2} \Phi(r, \tau). \]  

(40)

We see from the above equation that the centrifugal potential barrier, \((l^2 - \frac{1}{4})\hbar^2/2mr^2\), diverges at the origin for \(l \neq 0\). We hereafter consider only the nontrivial case of nonzero angular momentum, so we require in addition that \(\Phi\) vanishes at the origin. The \(\Phi\) normalization and boundary conditions are now

\[ \int_0^{R_0} \theta(\tau) |\Phi(r, \tau)|^2 dr = 1, \]  

(41)

\[ \Phi(r = 0, \tau) = 0 = \Phi(r = R_0, \theta(\tau), \tau). \]  

(42)

Equations (40-42) define an effective one-dimensional problem for \(r\).

We now proceed to apply the Liouville transformations, Eq.(4). the transformed wavefunction \(\Psi(\rho, t)\) is obtained from the unitary transformation [4]

\[ \Phi(r, \tau) = \frac{1}{\sqrt{\theta(t)}} \exp \left( \frac{im}{2\hbar} \frac{\theta^2}{\theta(t)} \right) \Psi(\rho, t), \]  

(43)

where the overdot denotes now differentiation with respect to \(t\). Following the definitions and applying the rules of partial differentiation, we have

\[ \frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial r} \frac{\partial}{\partial t} = \frac{1}{\theta(t)} \frac{\partial}{\partial \rho}, \]  

(44)

\[ \frac{\partial^2}{\partial r^2} = \frac{1}{\theta^2(t)} \frac{\partial^2}{\partial \rho^2}, \]  

(45)

\[ \frac{\partial}{\partial \tau} = \frac{\partial \rho}{\partial \tau} \frac{\partial}{\partial \rho} + \frac{\partial t}{\partial \tau} \frac{\partial}{\partial t} = -\frac{\rho \dot{\theta}}{\theta^2(t)} \frac{\partial}{\partial \rho} + \frac{1}{\theta^2(t)} \frac{\partial}{\partial t}. \]  

(46)

Applying these results to the set of equations (40-43), after some algebra, we finally obtain Schrödinger equation satisfied by \(\Psi(\rho, t)\),

\[ i\hbar \frac{\partial}{\partial t} \Psi(\rho, t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \rho^2} + \frac{1}{2} m\omega^2(t)\rho^2 + \frac{(l^2 - \frac{1}{4})\hbar^2}{2mr^2} \right\} \Psi(\rho, t), \]  

(47)
along with the transformed normalization and boundary conditions

\[ \int_0^{R_0} |\Psi(\rho, t)|^2 d\rho = 1, \quad (48) \]

\[ \Psi(\rho = 0, t) = 0 = \Psi(\rho = R_0, t). \quad (49) \]

The corresponding quantum FIPO (QFIPO) Hamiltonian is

\[ H_{QFIPO} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2(t)\rho^2 + \frac{J^2}{2m\rho^2}, \quad (50) \]

where

\[ J \equiv \hbar \sqrt{l^2 - 1/4} \quad l > 0. \quad (51) \]

The hamiltonian \( H_{QFIPO} \) is simply the quantized version of the corresponding classical Hamiltonian. What is significant is that the frequency law for \( \omega^2(t) \) is the same in both classical and quantum mechanics. As pointed out earlier in [4], this is a special effect arising out of the classical action consisting only of linear and quadratic terms.

The one-period time evolution operator is then given (in units where the mass, time and length are scaled in units of \( m, T_0 \) and \( R_0 \) respectively) by

\[ U(1, 0) = \exp \left( i\frac{\epsilon}{\hbar} \rho^2 \right) \cdot \exp \left( -i \frac{H_0}{2\hbar} \right) \cdot \exp \left( -i \frac{\epsilon}{\hbar} \rho^2 \right) \cdot \exp \left( -i \frac{H_0}{2\hbar} \right), \quad (52) \]

where

\[ H_0 = \frac{p^2}{2} - \frac{1}{2} \epsilon^2 \rho^2 + \frac{J^2}{2\rho^2}. \quad (53) \]

The first and third terms in Eq.(52) correspond to the kicks at full and half periods, while the other two terms correspond to the evolution of the state under \( H_0 \). We note here that while the classical problem has two free parameters \( \epsilon \) and \( J \), in the quantum case we have the additional parameter \( \hbar \). By writing the Schrödinger equation (47) in terms of dimensionless quantities, it is easy to verify that the parameter dependence of \( U \) is via \( \epsilon, \epsilon/\hbar \) and \( J \).

For convenience we choose a set of Fourier sine functions as the basis on which to diagonalize the evolution operator to span the Hilbert space. While unitarity requires a large
number of Fourier components as the scaled $\hbar$ gets smaller, this choice of basis has several advantages-it not only ensures automatic satisfaction of the boundary conditions, but also considerably simplifies the numerical calculations involved. In a sense, the Fourier basis is also the natural choice to handle the very short-wavelength oscillations encountered in the QFAD model. The details of obtaining the matrix elements of the evolution operator in this basis are presented in the Appendix.

III. STATISTICS OF QEE SPECTRA IN THE QFAD MODEL

As mentioned in the introduction, one of the clear QMCC emerges when one compares the spectral properties of specific model systems as appropriate parameters are tuned to classically produce a transition from integrable to completely chaotic regimes. In this section we follow the general thinking developed in RMT to implement different tests to quantify the spectral properties of the QFAD model. These properties are obtained from a direct diagonalization of the one-period time evolution matrix. For the results presented here we vary the value of $J$ and $\epsilon/\hbar$, while we take the wall amplitude fixed at $\epsilon = 1$. This value corresponds to a non-perturbative value, where the kinetic energy, the centrifugal barrier and the energy imparted by the wall all have about equal magnitudes. Furthermore, we rescale all lengths by $\epsilon$ and all energies by $\hbar$. The semiclassical limit of interest here is then obtained when $l \uparrow \infty$, $\hbar \downarrow 0$, with $J = \hbar \sqrt{l^2 - \frac{1}{4}}$ kept constant.

Next we discuss the RMT tests and their application to the results obtained for the QEE of the QFAD model.

A. Nearest neighbor QEE distributions

A local measure often used in RMT is the distribution of nearest-neighbor energy level separations, $P(s)$, where $s = \epsilon_{n+1} - \epsilon_n$. In the extreme integrable and chaotic regimes it has been established that $P(s)$ takes the Poisson or Wigner distribution forms,
\[ P_P(s) = e^{-s}, \quad (54) \]
\[ P_W(s) = \frac{\pi}{2} s e^{-\frac{s^2}{4}}, \quad (55) \]

respectively. A convenient and often successful parameterization of the \( P(s) \) obtained in the transition between \( P_P \) to \( P_W \) is provided by the Brody interpolation formula [11]:

\[ P_\alpha(s) = \gamma (\alpha + 1) s^\alpha \exp(-\gamma s^{\alpha+1}), \quad (56) \]

where \( \gamma = \left[ \Gamma \left( \frac{\alpha+2}{\alpha+1} \right) \right]^{\alpha+1} \), and \( \Gamma(x) \) is the Gamma function. This distribution is normalized and, by construction, has mean spacing \( \langle s \rangle = 1 \). We recover the Poisson case taking \( \alpha = 0 \) and Wigner for \( \alpha = 1 \). A criticism to the Brody distribution is, however, that there is no first principles justification for its validity. The fact remains that it does fit the specific results found when considering explicit model systems.

Instead of first looking at the \( P(s) \) we found that the fits are better if we start by calculating the integrated distribution

\[ \Pi_\alpha(s) = \int_0^s ds' P_\alpha(s'), \quad (57) \]

where we use all the eigenvalue spacings calculated. The results for \( \Pi_\alpha(s) \) are shown in Fig. 2. Next we use the Brody form to find the exponent \( \alpha \) to fit the data to \( \Pi_\alpha = 1 - e^{-\gamma s^{\alpha+1}} \).

The \( \alpha \) exponent is determined from a linear fit to \( \ell n \{ \ell n (1 - \Pi_\alpha) \} = \ell n \{ \gamma \} + (1+\alpha) \ell n \{ s \} \).

The fits shown in Fig. 2 are obtained by using these values of \( \alpha \). These results clearly show the general trend. As the nonintegrability parameter \( \epsilon/\hbar \) is increased, the \( \Pi(s) \) goes from being Poisson-like to Wigner-like. For small values of \( s \) (\( \leq 0.1 \)), the Poisson limit is well fitted by the Brody function while for the COE the fit is not as good, as one would expect due to the level repulsion. The corresponding results for \( P(s) \) are shown in Fig. 3, with the Brody fits using the \( \alpha \) values obtained from \( \Pi_\alpha(s) \). We found that the fits are better by analyzing the data this way rather than first fitting \( P(s) \) since in the binning process we lose information.

An alternative phenomenological interpolation formula with some justification was proposed by Izrailev [12] who used the analogy between the partition function of a two-
dimensional Coulomb gas of charged particles on a circle (the Dyson gas) and the joint probability distribution of the quasi-energy eigenvalues of the COE to construct an approximate form for $P(s)$ given by

$$P_\beta(s) = A \left( \frac{\pi}{2} s \right)^\beta \exp \left\{ -\frac{\beta}{4} \left( \frac{\pi}{2} s \right)^2 - \left( \frac{2B}{\pi} - \frac{\beta}{2} \right) \frac{\pi}{2} s \right\},$$

where $A(\beta)$ and $B(\beta)$ are constants fixed from normalization and the condition $\langle s \rangle = 1$. This formula also reduces to the appropriate Poisson and Wigner limits for $\beta = 0$ and 1, respectively. As we discuss below, this interpolation formula is also good for the QFAD but the Brody distribution provides a slightly better fit to our results. The interesting aspect of Izrailev’s distribution is that in principle it provides a quantitative connection between the energy level repulsion parameter $\beta$ and the degree of localization of the QEF. The connection is established by considering the ensemble averaged “entropy localization length” $l_s$ for the $u_n$ components of QEF for a chosen basis and defined as $\langle l_s \rangle = N \exp \left( \langle S_N \rangle - S_{COE}^N \right)$, where the information entropy

$$S_N(u_1, \ldots, u_N) = - \sum_{n=1}^{N} w_n \ln w_n, \quad w_n = |u_n|^2.$$  

Note that $S_N$ is essentially the logarithm of the number of sites significantly populated by the corresponding eigenstate, and $S_{COE}^N$ is the entropy corresponding to the random (COE) limit, introduced so that for completely chaotic states the exact limiting value of $\langle l_s \rangle = N$. The conjecture is that the quantity $\langle l_s \rangle / N$, which varies between 0 for completely localized states to 1 for fully extended ones, is identically equal to the repulsion parameter $\beta$ in Eq.(58).

To check the validity of this conjecture, we fitted the level spacing distribution to $P_\beta(s)$ given in Eq.(58) and determined the value $\beta_{hist}$ which minimized the $\chi^2$. We also calculated the ensemble-averaged entropy localization length as defined above, and calculated the corresponding $\beta_S$, and compared the two. The results are presented in Figure 4. We see that the agreement is good where the classical chaos is strong, but gets worse as the classical motion is more regular. Otherwise, the transition from COE to Poisson statistics
is consistent with the general trends observed previously. We conclude therefore that while
the Izrailev model is intuitively more appealing, the evidence to support it is not compelling
enough (in our model at least) to prefer it over the Brody formula.

B. $\Delta_3$ and $\Sigma^2$ QEE statistics

We also calculated higher-order eigenvalue spectral correlations. The average number
of levels in an interval of length $L$ is $<n(L) >= \frac{1}{L} \sum_\alpha n(\alpha, L)$, where the $<$ stands for
spectral average, and $n(\alpha, L)$ is the number of levels in an interval of length $L$ starting at $\alpha$
and ending at $\alpha + L$. Also important are the various moments of the level distribution. The
one considered here is the second moment of the average number of levels in a given stretch
of length $L$ of the spectrum, the $\Sigma^2(L)$ statistic [13]

$$< \Sigma^2(L) >= \langle (n(\alpha, L) - <n(\alpha, L)>)^2 \rangle. \quad (60)$$

Another often calculated statistic is the the Dyson-Mehta $\Delta_3(L)$ which measures the stiffness
of the spectrum. This is defined by

$$\Delta_3(L, \alpha) = \frac{1}{L} \min_{A,B} \int_\alpha^{\alpha+L} [\tilde{N}(x) - Ax - B]^2 dx, \quad (61)$$

where $\tilde{N}(x)$ is the unfolded number density. In our case there is no need to unfold the
spectrum. $\Delta_3$ is just the least mean square deviation of $\tilde{N}(x)$ from the mean straight line
behavior. This statistic is directly proportional to the $< \Sigma^2 >$ by $\Delta_3(L) = \frac{2}{L^2} \int_0^L (L^3 - sL^2 x + x^3) \Sigma^2(x) dx$, and thus can be calculated for the circular ensembles as well. The specific
theoretical predictions for the averaged $< \Delta_3(L) > = \frac{1}{L} \sum_\alpha \Delta_3(L, \alpha)$, are $\Delta_3^{(COE)}(L) = \frac{1}{\pi^2} \ell n\{L\} - 0.007$, and $< \Delta_3^{(Poisson)}(L) > = \frac{L}{15}$. These results are correct in the asymptotic
limit valid for $15 \leq L$.

In Fig. 5 we present our results for $< \Delta_3 >$ and $< \Sigma^2 >$. In these figures one clearly sees
the transition from Poisson-like (dashes) to COE-like (solid line) behavior as $\epsilon/h$ is varied.
We note that the $\Delta_3$ statistic does not saturate in the $COE$ limit, even for the maximum
interval \( L \) that we looked at, as would be expected from semiclassical arguments originally proposed by Berry. Furthermore, for the largest \( L \) considered the Poisson limit does not present the knee seen in other completely integrable systems. All in all the results shown in Fig. 5 are consistent with what we have come to expect for the transition region.

C. \( \chi^2 \) eigenfunctions distribution

To study the behavior of the eigenfunctions we begin by discussing the statistics of the overlap of the eigenfunctions with the natural basis vectors. Several authors \[14\] have conjectured that as the classical motion changes from chaotic to regular, this distribution can be represented by a \( \chi^2 \)-distribution in \( \nu \) degrees of freedom, with \( \nu \) decreasing from 1 (the Porter-Thomas limit) to 0 (the Poisson limit):

\[
P_\nu(y) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} y^{\nu/2-1} \exp(-\nu y/2), \tag{62}
\]

where \( y \equiv |\langle \lambda | n l \rangle|^2 \), \( |\lambda\rangle \) label the QEF and \( |n l\rangle \) label a set of \( N \) orthogonal basis vectors. (The \( y \)'s have been rescaled so that \( \langle y \rangle = 1 \).) We have tested this hypothesis for the overlap strengths. The results, plotted on a logarithmic scale in Figure 6, show the general trend of decreasing \( \nu \) as the corresponding classical system becomes more regular. However, we note that as we move from the COE to the Poisson limit the fits to the \( \chi^2 \) get worse. Note especially the appearance of a sharp second peak well away from 0 as \( \nu \) decreases. This discrepancy is related to the fact that the results are basis dependent when not in the COE limit. Equivalently, we do not expect to have a good fit to the \( \chi^2 \) except perhaps if we take the special basis obtained from a semiclassical calculation. Even if we manage to get good agreement with the \( \chi^2 \) for a properly chosen basis the result will not be generic and therefore the statistical analysis of amplitudes would lose its universal meaning. Universality does apply, however, in the COE limit.
D. Transition between localized to extended QEF

The phenomenon of dynamic localization gives another QMCC \[15\]. In this section we discuss the localization properties of the quasi-energy eigenfunctions when projected onto the natural basis. A physically meaningful quantity to calculate is the time averaged transition probabilities \( P(n, m) \) where \( \{ |n\rangle, n = 1..N \} \) label the basis states. While for small \( \epsilon/\hbar \) the probabilities are peaked around the diagonal (i.e. the QEF are localized in energy space), for larger values the probabilities are spread out, indicating delocalized states. To quantify this transition we define the pair-correlation function \( C(\Delta) \) by

\[
C(\Delta) = \frac{1}{N-\Delta} \sum_{m=1}^{N-\Delta} P(m, m+\Delta) \quad \Delta = 0, 1...N-1
\]

where \( P(n_0, m) = \lim_{T \to \infty} \frac{1}{T} \int_0^T |< n_0 | m >|^2 \) is the transition probability from the initial unperturbed state \( |n_0\rangle \) to the unperturbed state \( |m\rangle \). We find good fits of our evaluations of this correlation by the form,

\[
C(\Delta) \sim e^{-\Delta/\xi},
\]

where \( \xi \) is a localization length. The results are displayed in Fig. 7. We note that close to the near-integrable regime \( \xi << N \), indicating localization of the overlaps, while for larger \( \epsilon/\hbar \), \( \xi \geq N \). These results are consistent with the statistical analyses and with the expectation of a smooth transition from localized to extended solutions as we change from the Poisson to the COE regimes. We note that this results depends directly on the size of the \( U \) matrix being considered.

IV. CLASSICAL TO QUANTUM PHASE SPACE CORRESPONDENCE

In this section we consider the question of how to relate the quantum to the classical results. The natural habitat of nonintegrability in hamiltonian systems is phase space. As we mentioned in the introduction the FAD representation does not lend itself to an easy connection between the classical and quantum results \[3\]. It is in the FIPO representation that
we can carry out this program. Here we start by discussing a set of explicit classical phase space FIPO solutions that we will then directly connect to the QFIPO model results. The connection is done by using the Husimi representation of the quasi-energy eigenfunctions.

A. Classical FIPO model results

In Figs. 8(a-c) we present results of the classical Poincaré sections for three different angular momentum values, \( l = 1, 5, 10 \), for the parameter values indicated in the captions. We observe from these figures that there is a symmetry line in the \((\rho, p)\) section of phase space. As pointed out earlier [4] for the one-dimensional case, the reason for the existence of this symmetry line is as follows. Consider a particle kicked at \( t = 0 \). The position is unchanged, but the momentum changes to \( p^{(+)} = p^{(-)} + 2\epsilon\rho_0 \). If we denote \( p^{(+) by p_0} \), then at time \( 0^{(-)} \) the particle had momentum \( p^{(-)} = -2\epsilon\rho_0 + p_0 \). Since the Hamiltonian is time reversal invariant, we see that propagating a particle forward in time from \((\rho_0, p_0)\) is the same as propagating it backward from \((\rho_0, 2\epsilon\rho_0 - p_0)\). Thus, all results are symmetric about the line \( p = \epsilon\rho \). This symmetry is also present in the quantum problem, as will be discussed below.

Although the map is complicated, there are a few cases which can be studied analytically. The first of these is when the energy of the particle arises solely from its angular motion. The particle then does not feel the effects of the walls. The points \( p_0 = \epsilon\rho_0 \) lying on the symmetry line are then fixed points of order one of the map. To prove this, we begin by noting that the energy corresponding to this initial condition is given by Eq. (13) is

\[
E_0 = \frac{j^2}{2\rho_0^2},
\]

while the momentum just after the kick at half period is related to that just before by

\[
p^{(+)}_\frac{1}{2} = p^{(-)}_\frac{1}{2} - 2\epsilon\rho_\frac{1}{2}.
\]

The energy just after the kick is then related to that just before by
The trajectory of the particle during the second half-period is simply the time reversed trajectory during the first half, so that the energy has to be the same before and after the kick. Thus, Eqs. (66) and (67) imply
\[
p_{\frac{1}{2}^-} = \epsilon \rho_{\frac{1}{2}^+}, \quad p_{\frac{1}{2}^+} = -\epsilon \rho_{\frac{1}{2}^-}.
\] (69)

From Eq. (69), and using the definition of momentum from Eq. (20),
\[
p_{\frac{1}{2}^-} = \sqrt{2E_0 + \epsilon^2 \rho_{\frac{1}{2}^+}^2 - J^2 \rho_{\frac{1}{2}^+}^2},
\] (70)
we can deduce that
\[
E_0 = \frac{J^2}{2\rho_{\frac{1}{2}^+}^2}.
\] (71)

Comparing Eqs. (63) and (71) we see that \( \rho_{\frac{1}{2}} = \rho_0 \), and because of time reversal symmetry then tells us that at \( t = 1 \) we must have \( \rho_1 = \rho_0 \). The kick at full period now changes the momentum according to
\[
p_{\frac{1}{2}^+} = p_{\frac{1}{2}^-} + 2\epsilon \rho_0.
\] (72)

Using the same arguments for a full period as those used at half period, we can see that
\[
p_{1}^{(+)} = \epsilon \rho_0 = p_0.
\] (73)

Thus, \((\rho_0, \epsilon \rho_0)\) is a fixed point of order one of the map, and numerically we can verify that it is of parabolic type, with a maximum allowable value of \( \rho_0 \) for a given \( \epsilon \) and \( J \). That is, this line of fixed points, which starts at \( (\rho_{\text{min}}, \epsilon \rho_{\text{min}}) \) ends at a point determined by the condition \( \rho_{\frac{1}{2}} \leq 1 \). From substituting the initial conditions into Eq. (13) at half-period, we find that this point is given by
\[
\rho_{0}^{\text{max}} = \left(1 - \left(\frac{J}{\epsilon} \sinh\left(\frac{\epsilon}{2}\right) \exp\left(\frac{\epsilon}{2}\right)\right)^2\right)^{1/2}.
\] (74)
The reality of this maximum imposes yet another restriction on the allowable values of the parameters, viz,

\[ \frac{J}{\epsilon} \sinh\left(\frac{\epsilon}{2}\right) \exp(\epsilon/2) \leq 1. \]  

(75)

We now proceed to calculate one of the simplest nontrivial period-2 orbits, shown in Fig. 9. The particle starts at \((\rho_0, p_0)\) (shown as point A in the figure), with energy \(E_0\). The initial conditions are such that the particle bounces off the wall at \(\rho = 1\), reverses, hits the barrier at \(\rho = \rho_{\text{min}}\) (point C) and reverses momentum again before reaching the point \((\rho_{1/2}^{-}, p_{1/2}^{(-)})\) just before the kick at half period (point D). Now, from Eq.(16) we can write the time elapsed in motion under the influence of the potential only as

\[ t - t_0 = \frac{1}{2\epsilon} \left\{ \cosh^{-1} \left( \frac{\rho_0^2 + E_0/\epsilon^2}{a} \right) - \cosh^{-1} \left( \frac{\rho_0^2 + E_0/\epsilon^2}{a} \right) \right\}. \]  

(76)

Using this result, and substituting the appropriate \(\rho\)’s, we can write down the times elapsed during the three sections of the trajectory, \(t_{AB}, |t_{BC}| = t_{CB}\) and \(t_{CD}\). Equating the sum of these three times to 1/2, we can determine the position at D in terms of \(\rho_0\) and \(E_0\). The result is,

\[ \rho_{1/2} = \sqrt{a \cosh\{\epsilon + b(E_0)\} - \frac{E_0}{\epsilon^2}}, \]  

(77)

where

\[ b(E_0) = 2 \left\{ \cosh^{-1} \left( \frac{\rho_{\text{min}}^2 + E_0/\epsilon^2}{a} \right) - \cosh^{-1} \left( \frac{1 + E_0/\epsilon^2}{a} \right) + \cosh^{-1} \left( \frac{\rho_0^2 + E_0/\epsilon^2}{a} \right) \right\}. \]  

(78)

Now, the relation between the momenta just before and after the kick, and the argument about conservation of energy (based on time reversal invariance) lead to the relation (71). Inserting this into Eq.(77), we finally get

\[ \frac{J^2}{2E_0} + \frac{E_0}{\epsilon^2} = a \cosh\{\epsilon + b(E_0)\}. \]  

(79)

Since \(E_0\) is expressed in terms of \((\rho_0, p_0)\) and \(a = a(E_0)\), the above relation provides, in principle, a relation between \(\rho_0\) and \(p_0\).
The particle, after the kick at $t = 1/2$, follows the time reversed trajectory to the point G (which is the same as A) at $t = 1^{-}$ as shown, at which point it will have position $\rho_0$ and momentum $-p_0$. Then it gets kicked, which now changes its energy to $\tilde{E}_0$. This time, the momentum after the kick is given by

$$p_1^{(+)} = -p_0 + 2\epsilon \rho_0.$$  \hfill (80)

Using conservation of energy under time reversal again, we find that

$$\tilde{E}_0 = E_0 + 2\epsilon \rho_0 (\epsilon \rho_0 - p_0),$$  \hfill (81)

which also changes the other functions of energy:

$$\tilde{a} = \sqrt{(J/\epsilon)^2 + (\tilde{E}_0/\epsilon^2)^2},$$  \hfill (82)

$$\tilde{\rho}_{\text{min}} = \sqrt{\tilde{a} - \tilde{E}_0/\epsilon^2}. $$  \hfill (83)

It is important to remember that both $\tilde{a}$ and $\tilde{\rho}_{\text{min}}$ are functions of $\rho_0$ and $p_0$ via Eq.(82).

After getting kicked at H, the particle hits the barrier at I, reverses and gets to point J before getting kicked again, and following the time reversed trajectory back to the initial point A, as shown. The logic of determining the second relation between the initial conditions is the same as that used during the first quarter of its trajectory. By setting the total time taken $|t_{HI}| = t_{IH} + t_{IJ} = 1/2$, we first obtain a relation similar to Eq.(77), and repeating the argument about energy conservation due to time reversal invariance, we find

$$\tilde{E}_0 = \frac{J^2}{2\rho_1^2},$$  \hfill (84)

with

$$\rho_1 = \sqrt{\tilde{a} \cosh \left\{ \epsilon + c(\tilde{E}_0) \right\} - \frac{\tilde{E}_0}{\epsilon^2}},$$  \hfill (85)

and

$$c(\tilde{E}_0) = 2 \cosh^{-1} \left( \frac{\tilde{\rho}_{\text{min}}^2 + \tilde{E}_0/\epsilon^2}{\tilde{a}} \right) - \cosh^{-1} \left( \frac{\rho_0^2 + \tilde{E}_0/\epsilon^2}{\tilde{a}} \right).$$  \hfill (86)
Putting all these equations together, the final result is

$$\frac{J^2}{2E_0} + \frac{\tilde{E}_0}{\epsilon^2} = \tilde{a} \cosh \{ \epsilon + c(\tilde{E}_0) \}. \quad (87)$$

In principle, a physical solution to Eqs. (79) and (87) may be found analytically for the fixed point \((\rho_0, p_0)\), in practice we find it numerically. The fixed point shown in the Fig. 8(b) (for \(l = 10, \hbar = 0.026\) and \(\epsilon = 1\)), labelled F, is given by \(\rho_0 = 0.657385 \ldots, p_0 = 2.659181 \ldots\). When substituted back, this fixed point satisfies the equations given above to machine accuracy.

This analysis to determine this fixed point can be generalized to a family of fixed points of period \(2 \cdot (m + n + 1)\), with \(n, m = 0, 1, \ldots\) by letting the particle bounce between the wall and the centrifugal barrier \((2m + 1)\) times during the first half period (between B and C), and \((2n + 1)\) times during the third (between H and I). The equations to be solved are now

$$\frac{J^2}{2E_0} + \frac{E_0}{\epsilon^2} = a \cosh \{ \epsilon + b_m(E_0) \}, \quad (88)$$

where

$$b_m(E_0) = (2m + 1) b(E_0) - 2m \cosh^{-1} \left( \frac{\rho_0^2 + E_0/\epsilon^2}{\tilde{a}} \right), \quad (89)$$

and,

$$\frac{J^2}{2E_0} + \frac{\tilde{E}_0}{\epsilon^2} = \tilde{a} \cosh \{ \epsilon + c_n(\tilde{E}_0) \}, \quad (90)$$

where

$$c_n(\tilde{E}_0) = (2n + 1) c(\tilde{E}_0) - 2n \cosh^{-1} \left( \frac{\tilde{\rho}_{\text{min}}^2 + \tilde{E}_0/\epsilon^2}{\tilde{a}} \right), \quad (91)$$

and \(\tilde{E}_0, \tilde{a}\) and \(\tilde{\rho}_{\text{min}}\) are obtained in analogy to the period-2 case. In principle, higher order fixed points (and their associated families) can be calculated analytically in the above manner by writing larger numbers of such coupled nonlinear equations. In practice, however, that is quite a formidable task. To determine the stable manifolds we exploit the existence of the
symmetry line around hyperbolic fixed points, which are simply the unstable manifolds under time reversal. In practice, numerical difficulties preclude determining the stable manifolds in any other manner.

B. Correspondence

We can now make a direct comparison between the classical and quantum FIPO results by employing a phase space approach. To do this, we use the Husimi representation of the QEF [17]. The Husimi distribution, interpreted as a probability density, is a coarse-grained version of the Wigner function which goes smoothly to the semiclassical limit. In practice, the most often used technique of coarse-graining is to take the overlap of the QEF with coherent oscillator states. For the radial coordinate the coherent state is

\[ \Psi^G_{\rho_0,p_0}(\rho) = \left( \frac{\sigma}{\pi \hbar} \right)^{\frac{1}{4}} \exp \left\{ -\frac{\sigma}{2\hbar}(\rho - \rho_0)^2 + \frac{i}{\hbar} p_0 (\rho - \rho_0) \right\}, \tag{92} \]

which is a minimum-uncertainty Gaussian wavepacket centered at \((\rho_0, p_0)\), with root mean-squared deviations given by \(\Delta \rho = \sqrt{\hbar/2\sigma}, \Delta p = \sqrt{\hbar \sigma/2}\), and \(\sigma\) is the ‘squeezing’ parameter. This parameter is adjusted when making comparisons to the classical phase-space plots. The Husimi distribution of a single QEF \(\psi_z(\rho)\), is then defined by

\[ \mathcal{F}_z(\rho_0,p_0) = \left| \int_0^1 \Psi^G_{\rho_0,p_0}(\rho) \psi_z(\rho) \, d\rho \right|^2. \tag{93} \]

The Husimi distribution is obtained by scanning through the values of \((\rho_0, p_0)\) in the region of interest in phase space, and the result is compared with the classical surface-of-section. We begin the comparison by noting the symmetry about the line \(p = \epsilon \rho\) in the Husimi contour plots in Figs. 10(a-d). As mentioned earlier, this feature carries over from the classical results for the same reasons as there, and it is in fact used to effectively halve the numerical effort.

All calculations reported here were carried out for the wall oscillation amplitude \(\epsilon = 1\). In this case, all terms in the Hamiltonian are comparable in magnitude, which means that
we are in a non-perturbative regime. A few calculations were done for different $\epsilon$’s, but no new qualitative features emerged. In choosing a value of $\hbar$, we were guided by the following considerations. The value of $\hbar$ has to be small enough so that the system is well into the semiclassical regime, yet large enough so that the dimension of the truncated Hilbert space $N$ (which grows with decreasing $\hbar$) is large enough to preserve unitarity. Moreover, $N$ has to be such that the largest eigenenergy of $H_{FIPO}$ has to be larger than the maximum energy of the classical particle in the region of interest in phase space. These restrictions eventually led us to choose $\hbar = 0.026$ for $N = 100$. All the interesting features seen in this model are manifested in this regime. Finally, the classical conserved angular momentum was kept identical to the quantum value, $\hbar \sqrt{l^2 - 1/4}$.

In general, during our numerical calculations, we found that about 80% of the $QEE$ were ‘reliable’, in the sense that their imaginary parts were $< 10^{-3}$. This is one of the limitations of our numerical determination of the $QEF$, since being interested in the semiclassical limit we needed to choose small values of $\hbar$ (indicated in the right hand corner of Figs. (10-12). This meant that a large number of Fourier components were needed to accurately represent the $QEF$. The answers for the $QEF$ were quite reliable when calculating the Husimi distributions. However, the result were not good enough to do a RMT statistical analysis of the QFIPO model spectrum. This is why we resorted to the QFAD model, already discussed in the first part of this paper. Since we had precisely the opposite difficulty with the QFAD model we see that these two representations lead to results which are complimentary.

The classical analysis was carried out for different values of the angular momentum $J$. First, we iterated a single (arbitrarily chosen) initial condition several thousand times, which typically leads to the chaotic background as shown in the figures. Imbedded in this background are KAM tori centered around elliptic fixed points, defined by choosing appropriate initial conditions. In Figure 2(a), we show several such tori, and in particular, a fixed point of period 2 (marked as F1 and F2) which was determined earlier analytically. Also shown in each of the figures is a hyperbolic fixed point of order 6, marked by its stable and unstable manifolds. The fixed points were determined by using a modified Powell method \cite{8} of
determining zeros of coupled nonlinear sets of equations. This method, like all multidimensional root-finding techniques, requires a good initial guess to converge to the fixed point, but once given that, determines the root and the Monodromy matrix (the Jacobian or the determinant of the linearized version of the map equations) reliably and accurately. The fixed point is elliptic, parabolic or hyperbolic if the discriminant obtained from the eigenvalues (i.e., \((\text{Trace})^2 - 4 \cdot (\text{Determinant})\)) is negative, zero or positive, respectively. In all cases, it was verified within numerical error that the map was area-preserving, i.e., the determinant was equal to one. The unstable manifold was obtained by iterating the map along the direction given by the eigenvector corresponding to the eigenvalue larger than one. The stable manifold is given by the time reversed version of the unstable one.

Comparison of the Husimi distributions \(F_\psi(\rho_0, p_0)\) with the classical phase space plots show some striking similarities. There are, for many QEF, many structures which unmistakably correspond to elliptic, parabolic and hyperbolic periodic orbits, as seen in Figs. 10(a-c), 11(a-c) and 12(a-c). In the cases of \(l = 1, 5, 10\), for example, the Husimi representation of one of the QEF given in Fig. 12(a), 12(b) and 12(c) all sit on top of the analytic period-two fixed point marked as F. Also, seen in the figures there are Husimis which peak exactly on top of the unstable hyperbolic period-6 fixed point, referred to in the literature as “scars”. This correspondence is so robust, in fact, that often when a good guess to the classical hyperbolic fixed points are unavailable, the Husimis are used as a guide to the location of the fixed point (being unstable, hyperbolic fixed points cannot be located without a very good initial guess). These enhanced probability densities are conjectured to play as important a role in quantum mechanics as the hyperbolic orbits play in classical chaos. Finally, a rare but persistent occurrence in all the cases considered is that of a single Husimi distributions peaked simultaneously over both elliptic and hyperbolic fixed points, reflecting a purely quantum-mechanical tunneling between the KAM tori. Here we have only shown representative results of the correspondence between Husimi distributions and classical solutions.
V. DISCUSSION AND OUTLOOK

In this paper we have presented results of a thorough analysis of the quantum manifestations of classical chaos in a Fermi accelerating disk (FAD). We started by presenting results of the statistical properties of the quasi-energy spectrum for both the eigenvalues and the eigenfunctions. Most of the established tests of the manifestations of chaos in the quantum limit were implemented successfully with a few exceptions. To wit, we found clear evidence for the transition from Poissonian to \textit{COE}-like behavior in the eigenvalue statistics when we changed $\hbar$ from large to small. We have also connected the Poisson to \textit{COE} transition in the eigenvalues to the corresponding localized to extended regime of the eigenfunctions. In the \textit{COE} regime the Porter-Thomas distribution gives a good account of the statistical properties of the probability amplitudes, while in the Poisson limit the $\chi^2$ test fails to give agreement with our results.

While the statistical tests have become the hallmarks of the QMCC it is in phase space where a direct application of the correspondence principle can be probed. We found that the standard description in the FAD representation is not adequate to exhibit the quantum-classical correspondence. By carrying out a classical Liouville transformation plus the corresponding quantum unitary transformation, leading to the Fermi inverted parametric oscillator (FIPO), the correspondence was implemented explicitly for a selected set of classical trajectories in terms of their Husimi quasienergy distributions in phase space. The classical dynamics was studied in terms of a stroboscopic phase space plot. Various periodic orbits were found either analytically or numerically for different values of angular momentum. The quantum quasi-energy eigenfunctions were represented in phase space using the Husimi distribution. A direct correspondence was found between the classical stroboscopic phase space plot and the Husimi distributions of the quantum quasi-energy eigenfunctions. Periodic orbits present in the classical phase space were identified with corresponding high probability distribution in the quantum quasi-energy eigenfunctions. We found the “scars” in the quantum quasi-energy eigenfunctions that correspond to the classical unstable periodic orbits.
We also found quantum quasi-energy eigenfunctions that corresponds to tunneling across KAM curves between stable and unstable periodic orbits. While classically the KAM curves block the chaotic diffusion, the quantum tunneling allows it.

Our study extends the analysis carried out in one dimensional driven systems to quasi-two dimensional ones. This study is then the first step in the direction of testing the results hereby obtained with experiments, in particular in mesoscopic quantum dots in the presence of time varying magnetic fields. We shall discuss this problem elsewhere [9].

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APPENDIX:

In this Appendix, we set up the matrix elements of the evolution operator $U$. The Fourier sine basis we choose is

$$
\langle \rho, \phi | l, m \rangle = \sqrt{\frac{2}{\pi}} \sin(m \pi \rho) \cos(l \phi)
$$

(A1)

The basis is orthonormal:

$$
\langle l', m | l, n \rangle = \delta_{l', l} \delta_{m, n}
$$

(A2)

Then, the matrix elements of $H_0$ are

$$
\langle l', m | H_0 | l, n \rangle = H_0^{mn} \delta_{l', l}
$$

(A3)

where
\[ H_{mn}^{0} = -\frac{\hbar^2}{2} \langle m | (p^2 + \epsilon^2 \rho^2) | n \rangle + \frac{(l^2 - \frac{1}{4}) \hbar^2}{2} \langle m | \frac{1}{\rho^2} | n \rangle \]  
(A4)

\[ = \frac{\hbar^2 \pi^2 m^2}{2} \delta_{m,n} - \frac{\epsilon^2}{\pi^2} \cdot \begin{cases} 
(-1)^{m-n} \frac{4mn}{(m^2 - n^2)^2} & m \neq n \\
\frac{\pi^2}{6} - \frac{1}{4m^2 \pi^2} & m = n
\end{cases} \]  
(A5)

\[ + \frac{(l^2 - \frac{1}{4}) \hbar^2}{2} \pi \{(m + n) \text{Si} \left[ \pi (m + n) \right] - (m - n) \text{Si} \left[ \pi (m - n) \right]\} \]  
(A6)

where \( \text{Si}(x) \) is the sine integral,

\[ \text{Si}(x) = \int_{0}^{x} \frac{\sin(u)}{u} du. \]  
(A7)

We diagonalized \( H_0 \) numerically to obtain its eigenvalues and eigenvectors:

\[ H_0 |\mu\rangle = \hbar \omega_\mu |\mu\rangle \]  
(A8)

\[ |\mu\rangle = \sum_{n=1}^{N} \langle n | \mu \rangle |n\rangle \]  
(A9)

\[ \equiv \sum_{n=1}^{N} c_{n,\mu} |n\rangle. \]  
(A10)

Here we dropped the angular momentum quantum number index \( l \), because angular momentum conservation forbids transitions between states of different \( l \). Using the completeness and orthonormality properties of the basis set, we obtain the matrix elements of \( U \) as

\[ \langle m | U | n \rangle = \sum_{\mu,\nu=1}^{N} e^{-i(\omega_\mu + \omega_\nu)/2} \times \]  
(A11)

\[ \sum_{p,q,r} c_{n,\mu}^* c_{q,\mu} c_{r,\nu} \langle m | e^{i\epsilon \hat{\rho}^2/\hbar} | r \rangle \langle q | e^{-i\epsilon \hat{\rho}^2/\hbar} | p \rangle. \]  
(A12)

The last two quantities are again evaluated numerically by using methods for calculating integrals with rapidly oscillatory integrands.
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FIGURES

FIG. 1. Poincaré surfaces of section in the $(r, p_r)$ plane. (a) Almost regular behavior for small wall amplitude ($\epsilon = 0.001, l = 1$). (b) Intermediate case where regular and chaotic motions coexist ($\epsilon = 1.0, l = 10$). (c) Fully chaotic case ($\epsilon = 10, l = 100$).

FIG. 2. (a)-(d) Transition from Poisson to Wigner for $\Pi_\alpha$ as the tuning parameter is changed. The solid line in (a-d) corresponds to a $\Pi_\alpha(s)$ fit with the $\alpha$ value obtained from the best straight line fit to $\ell n \ell n [1 - \Pi_\alpha(s)]^{-1}$. The Brody fit is essentially on top of the calculated data. The values of $(\bar{h}, l)$ were (a) (1.0, 10), (b) (0.5, 20), (c) (0.1, 100) and (d) (0.01, 1000) for $\epsilon = 1$. The repulsion parameter $\alpha$ for (a-d) are 0.10, 0.20, 0.60 and 0.86, respectively.

FIG. 3. (a)-(d) Nearest-neighbor spacing distribution $P(s)$ for the same parameters as in Fig. 2. The dashed line in (a-b) corresponds to the Poisson limit, and to the COE case in (c) and (d). The solid line in (a-d) correspond to a $P_\alpha$ fit with the $\alpha$ values obtained in the fits of fig. 2.

FIG. 4. (a)-(d) Fits of the data to the Izrailev formula $P_\beta(s)$ (solid line). The dots are the fits as given by $\beta_S$ (see text for details). The best-fit and predicted values for $(\beta_{hist}, \beta_S)$ are, (a) (0.20,0.05), (b) (0.22,0.12), (c) (0.48,0.50) and (d) (1.14,1.00).

FIG. 5. The $< \Delta_3(L) >$ (upper plot) and the $\Sigma^2(L)$ statistics (lower plot). The parameter values are the same as in Fig. 2 with (a) +, (b) $\bullet$, (c) $\diamond$, and (d) for $\times$. The dashed line is the exact Poisson result while the continuous line is the corresponding COE RMT results. We observe a continuous transition from Poisson to COE. In the Poisson case the calculated values are slightly above the predicted behavior but it does not show the bending characteristic of picket fence spectra. In the COE regime no knee is found either.
FIG. 6. (a)-(d) Distribution of amplitudes of $QEF$ with the natural basis states for the same parameter values as in Fig. 2. Close to the $COE$ limit (d) the amplitudes are nearly gaussian or Porter-Thomas randomly distributed. Away from this limit the distributions are not well fitted by the $\chi^2$ distribution with a significant difference seen close to the Poisson limit. This discrepancy is explained in the text. The values of $\nu$ from the fits are, (a) 0.10, (b) 0.15, (c) 0.52, and (d) 1.

FIG. 7. (a)-(d) Logarithm of the pair-correlation function defined in Eq.(64) as a function of $\Delta$ for the same parameter values as in Fig.2. The transition from localized to extended behavior, exemplified by the value of the localization length $\xi$ as compared to the matrix sizes (N=400 in this case).

FIG. 8. Classical Poincaré section for $l = 1$ (a), 5 (b) and 10 (c). The period-2 fixed points are denoted as F. The hyperbolic fixed points in (b) are obtained as explained in the text. Here, and in the following, $\hbar = 0.026, \epsilon = 1.0$

FIG. 9. Schematic representation of the trajectory of the particular analytically calculated period-2 orbit, as explained in the text. The vertical scale is in arbitrary energy units, and the small separations correspond to the same energy. The horizontal scale is in arbitrary length units.

FIG. 10. (a-c) Contour plots of the Husimi distribution of a $QEF$ which corresponds to the three classical solutions corresponding to Fig. 8(a) ($l = 1$). In (a) we have to Husimi distribution that corresponds to the period-2 solution calculated analytically in the classical limit. (b) Husimi distribution of another $QEF$ which corresponds to the period-6 hyperbolic orbit marked by its stable and unstable manifolds - a ‘scarred’ eigenfunction. (c) A mixed case, where a single $QEF$ corresponds to both a stable, period-2 elliptic orbit, and the unstable, hyperbolic period-6 orbit.

FIG. 11. Same as in the previous figure corresponding to the case $l = 5$ of Fig. 8(b).