SEMISIMPLICITY OF CELLULAR ALGEBRAS OVER
POSITIVE CHARACTERISTIC FIELDS

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Abstract. In this paper, we investigate semisimplicity of cellular algebras over positive characteristic fields. Our main result shows that the Frame number of cellular algebras characterizes semisimplicity of it. In a sense, this is a generalization of Maschke’s theorem.

1. INTRODUCTION

Cellular algebras are an object in algebraic combinatorics which were introduced by B. Yu. Weisfeiler and A. A. Lehman as cellular algebras and independently by D. G. Higman as coherent algebras (see [8] and [12]). They are by definition matrix algebras over a ring which is closed under the Hadamard multiplication and the transpose and containing the identity matrix and the all one matrix. Note that according to E. Bannai and T. Ito [2], a homogeneous coherent configuration is also called an association scheme (not necessarily commutative). Clearly, the adjacency algebra of a coherent configuration (or scheme) is a cellular algebra. Conversely, for each cellular algebra \( W \) there exists a coherent configuration whose adjacency algebra coincides with \( W \). So we prefer to deal with the adjacency algebra of a coherent configuration. In a sense, cellular algebras are generalization of group algebras, so it is natural to extend Maschke’s theorem (see [10] and [11, Theorem III.1.22]) to them. Also E. Bannai and T. Ito in [3, page 303], asked about determination by the parameters, association schemes and fields for which the adjacency algebras are semisimple, symmetric, Frobenius and quasi-Frobenius. We will answer this question about semisimplicity, for general case, cellular algebras. In order to do this, we use the Frame number of cellular algebras. This number which was introduced by J. S. Frame in 1941, is in relation with the double cosets of finite groups. In 1976 D. G. Higman extended this number to cellular algebras. Z. Arad in 1999 with the help of Frame number characterized semisimplicity of commutative cellular algebras (or commutative association schemes) over fields of prime order (see [1]). Finally, A. Hanaki

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in 2002 generalized the result by Z. Arad for homogeneous cellular algebras (or association schemes) over positive characteristic fields (see [6]). In this paper, we consider cellular algebras, not necessarily homogeneous, over positive characteristic fields. Actually we prove that a cellular algebra over a field $k$ is semisimple if and only if its Frame number is not divided by characteristic of $k$.

2. Definition and Notation

To make this paper self-contained we put in this section the notations and definitions concerning cellular algebras. For more details, we refer to [9].

**Definition 2.1.** Let $V$ be a finite set and $\mathcal{R}$ a set of nonempty binary relations on $V$. A pair $\mathcal{C} = (V, \mathcal{R})$ is called a **coherent configuration** or a **scheme** on $V$ if the following conditions are satisfied:

1. $\mathcal{R}$ forms a partition of the set $V^2$.
2. The diagonal $\Delta(V)$ of $V^2$ is a union of elements of $\mathcal{R}$.
3. For every $R \in \mathcal{R}$, $R^t := \{(u, v) : (u, v) \in R\} \in \mathcal{R}$.
4. For every $R, S, T \in \mathcal{R}$, the number $|\{v \in V : (u, v) \in R, (v, w) \in S\}|$ does not depend on the choice of $(u, w) \in T$ and is denoted by $c_{R, S}^T$.

The elements of $V$, the relations of $\mathcal{R} = \mathcal{R}(\mathcal{C})$ and the numbers from condition (C4) are called the **points**, the **basis relations** and the **intersection numbers** of $\mathcal{C}$, resp. The numbers $\deg(\mathcal{C}) = |V|$ and $\rk(\mathcal{C}) = |\mathcal{R}|$ are called the **degree** of $\mathcal{C}$ and the **rank** of $\mathcal{C}$, resp. Also $\mathcal{R}^*(\mathcal{C})$ is defined as the set of all relations of $\mathcal{C}$ each of which is a union of elements of $\mathcal{R}(\mathcal{C})$.

**Example 2.1.** Let $G \leq \text{Sym}(V)$ be a permutation group and $\mathcal{R} = \text{Orb}_2(G)$ be the set of orbitals of $G$. Then $\mathcal{R}$ forms a partition of the set $V^2$ such that $R^t$ belongs to $\mathcal{R}$ for all $R \in \mathcal{R}$. Moreover, the reflexive relation $\Delta(V)$ is a union of elements of $\mathcal{R}$. Finally, given $(u, v) \in V^2$ and $R, S \in \mathcal{R}$, if we set

$$p_{u, v}(R, S) = \{v \in V : (u, v) \in R, (v, w) \in S\},$$

then obviously $p_{u^g, v^g}(R^g, S^g) = p_{u^g, v^g}(R, S)$ for all $g \in G$. So the number $|p_{u, v}(R, S)|$ does not depend on the choice of the pair $(u, v) \in T$ for all $T \in \mathcal{R}$. Thus $\text{Inv}(G) := (V, \mathcal{R})$ is a scheme which is called **Schurian**. Also $\text{Inv}(\text{id}_V)$ is called the **trivial scheme** (see [9]).

**Adjacency algebra.** Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme and $\mathbb{Z}$ be the ring of integers. Given a relation $R \in \mathcal{R}$. Denote by $A(R)$ the **adjacency**
matrix of $R$: $A(R)$ is a $\{0,1\}$-matrix of the full matrix algebra $\text{Mat}_V(\mathbb{Z})$ such that $A(R)_{u,v} = 1$ iff $(u,v) \in R$. Then from the definition of $C$ it follows that the $\mathbb{Z}$-linear span $W = W(C)$ of the set $\{A(R) : R \in \mathcal{R}\}$ in $\text{Mat}_V(\mathbb{Z})$ satisfies the following conditions:

\begin{enumerate}
  \item[(C’1)] for every $R, S \in \mathcal{R}$, \quad $A(R)A(S) = \sum_{T \in \mathcal{R}} c_{R,S}^{T} A(T)$,
  \item[(C’2)] $I_V, J_V \in W$,
  \item[(C’3)] $W$ is closed under the Hadamard (componentwise) multiplication,
  \item[(C’4)] $W$ is closed under the transpose map,
\end{enumerate}

where $I_V$ is the identity matrix and $J_V$ is the matrix whose all entries are ones. In particular, $W$ is a ring with respect to the both multiplications with identities $I_V$ and $J_V$, resp. It is called the adjacency ring of the scheme $C$. If $\mathcal{R}$ is a ring, then we define $W^\mathcal{R} := \mathcal{R} \otimes_{\mathbb{Z}} W$, the adjacency algebra of $W$ over $\mathcal{R}$. If $\mathbb{Z}$ is a subring of $\mathcal{R}$, then $W$ is a subalgebra of $W^\mathcal{R}$. We call $\{A(R) : R \in \mathcal{R}\}$ the standard basis of $W(C)$.

**Definition 2.2.** An $\mathcal{R}$–subalgebra $W$ of the algebra $\text{Mat}_V(\mathcal{R})$ is called a cellular algebra on $V$ if it satisfies conditions (C’2)-(C’4).

**Example 2.2.** The adjacency ring of the trivial scheme on $V$ coincides with $\text{Mat}_V(\mathbb{Z})$ and the adjacency ring of a scheme of rank 2 has a standard basis $\{I_V, J_V - I_V\}$. Given a group $G$ and $g \in G$ the mapping $x \mapsto xg$ is a permutation of $G$ and denoted by $g_{\text{right}}$. The set $G_{\text{right}} = \{g_{\text{right}}|g \in G\}$ is a permutation group on $G$. Analogously, the group $G_{\text{left}}$ consists of permutations $x \mapsto g^{-1}x, g \in G$. Let $\mathcal{C} = \text{Inv}(G_{\text{left}})$ for a group $G$. Each basis relation of $\mathcal{C}$ is of the form $R_g = \{(x, xg) : x \in G\}$ for some $g \in G$. We observe that $A(R_g) = P_g$ where $P_g$ is the permutation matrix corresponding to $g_{\text{right}}$. Since obviously $P_gP_h = P_{gh}$ for all $g, h \in G$, the mapping

\[ \mathbb{Z}[G] \longrightarrow W(\mathcal{C}) \]

\[ g \mapsto P_g \]

induces an algebra isomorphism from the group ring $\mathbb{Z}[G]$ of the group $G$ to the cellular ring $W(\mathcal{C})$ of the scheme $\mathcal{C}$.

**Remark 2.3.** It is well known that there exists a one-to-one correspondence between the set of all cellular algebras on $V$ and the set of all schemes on $V$. Due to this correspondence, we can use freely both the language of matrices and the language of relations. In particular, a scheme $\mathcal{C}$ is commutative, if so is the adjacency algebra of $\mathcal{C}$. This is equivalent to the equalities $c_{R,S}^{T} = c_{S,R}^{T}$ for all $R, S, T \in \mathcal{R}(\mathcal{C})$. It is easy to see that any symmetric scheme is commutative (the scheme $\mathcal{C}$ is called symmetric if each basis relation of it is symmetric, i.e., $R = R^t$.
for all $R \in \mathcal{R}(C)$. 

**Cells and basis relations.** Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. Set 
$$\text{Cel}(\mathcal{C}) = \{ X \subseteq V : \Delta(X) \in \mathcal{R} \}, \quad \text{Cel}^*(\mathcal{C}) = \{ X \subseteq V : \Delta(X) \in \mathcal{R}^* \}.$$ 
Each element of the set cel$(\mathcal{C})$ is called a *cell* (resp. *cellular set*) of the scheme $\mathcal{C}$. For a permutation group $G$ we have 
$$\text{Cel}(\text{Inv}(G)) = \text{Orb}(G), \quad \text{Cel}^*(\text{Inv}(G)) = \text{Orb}^*(G),$$ 
in which $\text{Orb}(G)$ is the set of all orbits of $G$ on $X$ and $\text{Orb}^*(G)$ is the set of all invariant sets of $G$ on $X$. For instance, the Cells of the trivial scheme on $V$ are exactly the singletons of $V$, whereas a scheme of rank 2 on $V$ has the unique cell, namely, $V$. Let $X, Y \in \text{Cel}^*(\mathcal{C})$. Then $\Delta(X)$ and $\Delta(Y)$ are relations of the scheme $\mathcal{C}$ and so the adjacency matrices $I_X$ and $I_Y$ of them belong to the algebra $W = W(\mathcal{C})$. This implies that $I_X J_Y I_Y = A(X \times Y) \in W$ and hence $X \times Y \in \mathcal{R}^*(\mathcal{C})$. Moreover From (C2) it follows that $\Delta(V) = \bigcup_{X \in \text{Cel}(\mathcal{C})} \Delta(X)$. So $V$ is the disjoint union of cells and we have,

$$\mathcal{R}(\mathcal{C}) = \bigcup_{X,Y \in \text{Cel}(\mathcal{C})} \mathcal{R}_{X,Y} \quad \text{(disjoint union)},$$

where for $X, Y \in \text{Cel}(\mathcal{C})$ we set 
$$\mathcal{R}_{X,Y} = \mathcal{R}_{X,Y}(\mathcal{C}) = \{ R \in \mathcal{R} : R \subseteq X \times Y \}.$$ 
For $R \in \mathcal{R}_{X,Y}$ with $X, Y \in \text{Cel}(\mathcal{C})$, set 
$$d_{\text{out}}(R) = c_{R,R'}^{\Delta(X)}, \quad d_{\text{in}}(R) = c_{R',R}^{\Delta(Y)}.$$ 
If $A = A(R)$, then $d_{\text{out}}(\mathcal{R})$ (resp. $d_{\text{in}}(\mathcal{R})$) is the number of ones in each row $u$ (resp. each column $v$) of the matrix $A$ where $u \in X$ (resp. $v \in Y$). From the definition of intersection numbers it follows that given $(u, v) \in X \times Y$ we have 
$$d_{\text{out}}(R) = |R_{\text{out}}(u)|, \quad d_{\text{in}}(R) = |R_{\text{in}}(v)|,$$
where $R_{\text{out}}(u) = \{ w \in V : (u, w) \in R \}$ and $R_{\text{in}}(v) = \{ w \in V : (w, v) \in R \}$. Thus

$$\sum_{R \in \mathcal{R}_{X,Y}} d_{\text{out}}(R) = |Y|, \quad \sum_{R \in \mathcal{R}_{X,Y}} d_{\text{in}}(R) = |X|,$$

(5)

$$|X|d_{\text{out}}(R) = |R| = |Y|d_{\text{in}}(R).$$

A scheme $\mathcal{C}$ is called *homogeneous* or (*an association scheme*) if $|\text{Cel}(\mathcal{C})| = 1$ or equivalently, if $\Delta(V) \in \mathcal{R}$. (From (1) it follows that for a permutation group $G$ the scheme Inv$(G)$ is homogeneous iff the group $G$ is transitive.) In this case for given $R \in \mathcal{R}$ we have 
$$|V|d_{\text{out}}(R) = |R| = |V|d_{\text{in}}(R), \quad d_{\text{out}}(R) = d_{\text{in}}(R).$$
The latter number is denoted by \( d(R) \) and is called the degree of the relation \( R \). Thus each basis relation of a homogeneous scheme can be treated as the set of arcs of a regular digraph with the vertex set \( V \). From (4) it follows that

\[
\sum_R d(R) = |V|.
\]

Suppose that \( X \in \text{Cel}(\mathcal{C}) \) and denote by \( I_X \) the adjacency matrix of \( \Delta(X) \), then \( I_V = \sum_{X \in \text{Cel}(\mathcal{C})} I_X \) is an idempotent decomposition of \( I_V \). We observe that every commutative scheme is homogeneous. (Indeed, the commutativity of \( \mathcal{C} \) means the commutativity of the adjacency algebra \( W(\mathcal{C}) \). If \( X, Y \in \text{Cel}(\mathcal{C}) \) and \( X \neq Y \), then \( I_X A(R) = A(R) \) and \( I_X A(R^t) = 0 \) for all \( R \in \mathcal{R}_{X,Y} \).)

**Definition 2.4.** Let \( \mathcal{C} = (V, R) \) be a scheme with its adjacency ring \( W \). If \( ZV \) is a free \( Z \)-module of rank \( |V| \) indexed by \( V \), then \( W \) acts naturally on the basis set \( V \), namely \( ZV \) has the structure of a module over \( \text{Mat}_V(Z) \) according to

\[
u A := \sum_{v \in V} A_{u,v} v \quad (A \in \text{Mat}_V(Z), u \in V).\]

Assume that \( F \) is a field and define \( FV := F \otimes_Z ZV \). Then \( FV \) can be regarded as \( W^F \)-module. We call this the standard module of \( W \) (resp. \( \mathcal{C} \)) over \( F \). The character of \( W^F \) afforded by the standard module is called the standard character of \( W \). We shall denote the standard character of \( W \) by \( \rho \) which is calculated in the following lemma (By \( \delta \) we mean the Kronecker delta).

**Lemma 2.5.** For every \( R \in \mathcal{R} \) we have \( \rho(A(R)) = \sum_{X \in \text{Cel}(\mathcal{C})} \delta_{R, \Delta(X)} |X| \).

We state here some facts about finite dimensional algebras. Let \( A \) be a finite dimensional algebra over \( F \) (concerning finite dimensional algebras we refer to [III]). The Jacobson radical \( \text{Rad}(A) \) of \( A \) is the intersection of all maximal right ideals of \( A \). Also \( A \) is said to be semisimple if \( \text{Rad}(A) = 0 \). In section III we introduce another criterion for semisimplicity of finite dimensional algebras.

Let \( K \) be an extension field of \( F \). Then \( \text{Rad}(A) \otimes_F K \subseteq \text{Rad}(A \otimes_F K) \), since \( \text{Rad}(A) \otimes_F K \) is a nilpotent ideal of \( A \otimes_F K \). However, they do not necessarily coincide. But if \( K \) is a separable extension of \( F \), then the equality holds. Also \( A \) is called separable over \( F \) if \( A \) is semisimple and \( A \otimes_F K \) remains semisimple for any extension \( K \) of \( F \).

**Theorem 2.6.** [III Theorem II.5.4] If \( F \) is a perfect field (e.g., \( \text{char}(F) = 0 \) or \( F \) is finite), then every semisimple \( F \)-algebra is separable over \( F \).

We denote the complete set of representatives of isomorphism classes of irreducible \( A \)-modules by \( \text{IRR}(A) \).
It is well known that $A/\text{Rad}(A)$ is semisimple. If $A$ is a split $F$-algebra, namely $F$ is a splitting field for $A$, we have

$$A/\text{Rad}(A) = \bigoplus_{i=1}^{r} M_{f_i}(F),$$

where $f_i$’s are the degrees of irreducible representations of $A$. So we have the following (see [11] for details).

**Proposition 2.7.** Let $A$ be a split $F$-algebra with $\text{IRR}(A) = \{M_1, ..., M_r\}$. Then

$$\dim_F(A) = \sum_{i=1}^{r} (\deg M_i)^2 + \dim_F(\text{Rad}(A)).$$

Let $W$ be a cellular algebra and $k \leq K$ be fields. Then there is a natural isomorphism $W^k \otimes_k K \cong W^K$ of $K$-algebras such that $\alpha \otimes x \mapsto \alpha x$. Thus $W^K$ is just the scalar extension of $W^k$ for every subfield $k$ of $K$. This is quite useful in the study of cellular algebras. By using this we can prove the following result.

**Lemma 2.8.** [11, Lemma III.1.28] If $W^K$ is a cellular algebra over field $K$, then $W^K/\text{Rad}(W^K)$ is separable over $K$.

### 3. Discriminant of algebras

Let $\mathfrak{B}$ be a principal ideal domain, and $A$ a free $\mathfrak{B}$-algebra of finite rank $n$. Suppose that $M$ is a finite-dimensional $A$-module with a matrix representation $\mathbf{X}$, we define the discriminant of the representation module $M$ as follows. The map $\Phi_M : A \times A \rightarrow \mathfrak{B}$ defined by

$$\Phi_M(a, b) = \text{Tr}(\mathbf{X}(ab)),$$

is a symmetric bilinear form, where Tr is the usual trace of matrices. Let $a_1, \ldots, a_n$ be an $\mathfrak{B}$-basis of $A$. We put

$$D_{M,\{a_i\}}(A) = \det(\text{Tr}(\mathbf{X}(a_ia_j))).$$

Especially, when the representation $\mathbf{X}$ is the regular representation, we call $D_{M,\{a_i\}}(A)$ the discriminant of $A$, and denote it by $D(A)$. Note that $D_{M,\{a_i\}}(A) \neq 0$ iff $\Phi_M$ is nondegenerate. We note that $D_{M,\{a_i\}}(A)$ depends on the choice of the basis $\{a_i\}$ of $A$, but being nondegenerate is independent on it. i.e., if we take another basis $a'_1, \ldots, a'_r$, then $\det(\text{Tr}(\mathbf{X}(a'_ia'_j))) = \varepsilon^2 \det(\text{Tr}(\mathbf{X}(a_ia_j)))$ for some unit $\varepsilon$ in $\mathfrak{B}$. Hence, if $\mathfrak{B} = \mathbb{Z}$, then the discriminant is uniquely determined.

Suppose $A$ is not semisimple. Then $\text{Rad}(A) \neq 0$. If $0 \neq a \in \text{Rad}(A)$, then $\Phi_M(a, b) = 0$ for any $b \in A$, since each element of $\text{Rad}(A)$ is nilpotent. So $\Phi_M$ is degenerate.

Conversely, assume that $A$ is a semisimple split algebra and $\text{IRR}(A) =$
\{M_1, \ldots, M_r\}. We have \(W^K \simeq \bigoplus_{i=1}^r M_{f_i}(K)\), since \(W^K\) is a split \(K\)-algebra. We consider another basis \(B = \{e^{(i)}_{st} \mid 1 \leq i \leq r, \ 1 \leq s, t \leq f_i\}\) of \(W^K\), where \(e^{(i)}_{st}\) is matrix unit in \(M_{f_i}(K)\). If we put \(M = \bigoplus_{i=1}^r M_i\), then

\[
\Phi_M(A) = \bigoplus_{i=1}^r \Phi_{M_i}(A), \quad D_{M,B}(A) = \prod_{i=1}^r D_{M_i,B_i}(A),
\]

where \(B_i = \{e^{(i)}_{st} \mid 1 \leq s, t \leq f_i\}\) and \(\Phi_M(A)\) is the direct sum of \(\Phi_{M_i}(A)\) for \(1 \leq i \leq r\). We may assume that \(A = M_{f_i}(K)\) where \(f_i\) is the degree of \(M_i\). Given \(0 \neq a \in A\) with nonzero entry \(a_{ij}\), then

\[
\Phi_M(a, e_{ji}) = \text{Tr}(ae_{ji}) = \sum_{t=1}^{f_i} (ae_{ji})_{tt}
\]

\[
= \sum_{t=1}^{f_i} \sum_{k=1}^{f_i} a_{tk}(e_{ji})_{kt}
\]

\[
= \sum_{k=1}^{f_i} a_{tk}(e_{ji})_{ki} = a_{ij} \neq 0,
\]

where \(e_{ji}\) is matrix unit. Hence \(A\) is nondegenerate.

**Theorem 3.1.** Let \(C = (V, R)\) be a scheme with standard basis \(\{A_i\}\). If \(K\) is a field and \(N = KV\) is the standard module of \(C\), then

\[
D_{N,\{A_i\}}(W^K) = \varepsilon \prod_{R \in R} |R|; \quad \varepsilon \in \{1, -1\}.
\]

**Proof.** Assume that \(\rho\) is the standard character of \(C\). Then by lemma 2.3 and (3) we have

\[
\Phi_N(A(R), A(S)) = \rho(A(R)A(S))
\]

\[
= \sum_{T \in R} c_{R,S}^T \rho(A(T))
\]

\[
= \sum_{T \in R} \sum_{X \in \text{Cel}(C)} c_{R,S}^T \delta_{T,\Delta(X)} |X|
\]

\[
= \delta_{S,R'} c_{R,S}^{\Delta(X)} |X| \quad (R \subseteq X \times Y)
\]

\[
= \delta_{S,R'} d_{out}(R) |X| = \delta_{S,R'} |R|.
\]
So
\[
D_{N\{A_i\}}(W^K) = \sum_{\sigma \in \text{Sym}(R)} \text{sgn}(\sigma) \prod_{R \in R} \Phi_N(A(R), A(R^\sigma))
\]
\[
= \sum_{\sigma \in \text{Sym}(R)} \text{sgn}(\sigma) \prod_{R \in R} \delta_{R^\sigma, R} |R|
\]
\[
= \varepsilon \prod_{R \in R} |R|; \quad \varepsilon \in \{1, -1\}.
\]

4. Frame number

In this section, we define the Frame number of a scheme. This number was introduced by Frame [5] and was extended to cellular algebras by D.G. Higman [7].

Let \( C = (V, R) \) be a scheme and \( W \leq \text{Mat}_V(\mathbb{Z}) \) the adjacency ring of \( C \). Suppose that \( \text{IRR}(W^C) = \{M_1, \ldots, M_r\} \), \( f_i = \text{dim}(M_i) \) and \( CV \) is the standard module of \( C \) over complex field \( \mathbb{C} \). As \( W^C \) is semisimple (see Proposition [5.1]), then we have
\[
CV = \bigoplus_{i=1}^r m_iM_i.
\]

We call \( m_i \) the multiplicity of \( M_i \).

If we consider matrix form of Schur relations of \( C \)(see [7]), then we obtain the following
\[
\mathfrak{M}(C) = \frac{\prod_{X \in \text{Cel}(C)} |X|^{-2} \prod_{R \in R} |R|}{\prod_{i=1}^r m_i^{f_i}},
\]
where the number \( \mathfrak{M}(C) \) is called the Frame Quotient of \( C \). It is well known that \( \mathfrak{M}(C) \) is a rational integer (see [7]). We define the Frame number \( F(C) \) by
\[
F(C) = \frac{\prod_{R \in R} |R|}{\prod_{i=1}^r m_i^{f_i}}.
\]
It is clear that \( F(C) \) is a rational integer. This number is a criterion for the semisimplicity of cellular algebras, which will appear in our main result.

**Theorem 4.1.** Let \( K \) be a field of characteristic zero and \( W^K \) be a split cellular algebra over \( K \). If \( \text{IRR}(W^K) = \{M_1, \ldots, M_r\} \) and \( M = \bigoplus_{i=1}^r M_i \), then
\[
D_{M\{A_i\}}(W^K) = \varepsilon F(C); \quad \varepsilon \in \{1, -1\}
\]
where \( \{A_i\} \) is the standard basis basis of \( W \).
Proof. We have $W^K \simeq \bigoplus_{i=1}^r M_{f_i}(K)$, since $W^K$ is a split $K$-algebra. We consider another basis $B = \{c_{sl}^{(j)} | 1 \leq i \leq r, \ 1 \leq s, t \leq f_i \}$ of $W^K$. Let $P$ be the transformation matrix of the bases $\{A_i\}$ and $B$. Then

$$D_{M,\{A_i\}}(W^K) = D_{M,B}(W^K)(\det P)^2,$$

and we have $D_{M,B}(W^K) = \pm 1$ by $\Phi_M(c_{sl}^{(j)}, c_{uv}^{(j)}) = \delta_{ij}\delta_{sv}\delta_{tu}$.

Thus $D_{M,\{A_i\}} = \pm (\det P)^2$. Next we put $N = KV$, the standard module. Then $N = \bigoplus_{i=1}^r m_i M_i$ and $\Phi_N(c_{sl}^{(j)}, c_{uv}^{(j)}) = \delta_{ij}\delta_{sv}\delta_{tu} m_i$. Thus

$$D_{N,\{A_i\}}(W^K) = D_{N,B}(W^K)(\det P)^2 = \pm \prod_{i=1}^r m_i^{f_i} (\det P)^2.$$

By the Theorem 3.1 $D_{N,\{A_i\}}(W^K) = \varepsilon \prod_{R \in R} |R|$ where $\varepsilon \in \{1, -1\}$. Now we have

$$D_{M,\{A_i\}}(W^K) = \pm (\det P)^2 = \pm \frac{\prod_{R \in R} |R|}{\prod_{i=1}^r m_i^{f_i}} = \pm \mathcal{F}(C).$$

□

5. Semisimlicity

Let $A \leq \text{Mat}_V(C)$ and suppose that $A$ is closed under the complex conjugate transpose map. Then $A$ is semisimple. Let $C = (V, R)$ be a scheme with cellular ring $W$. Also suppose that $\{A_i\}$ is the standard basis of $W$. By definition of schemes $W^C$ is closed under the complex conjugate transpose map and thus it is semisimple.

Proposition 5.1. Let $k$ be a field of characteristic $p$. Then in the above notation, the following hold:

1. If $p \nmid \prod_{R \in R} |R|$, then $W^k$ is semisimple.
2. If $p \mid \prod_{X \in \text{Cel}(C)} |X|$, then $W^k$ is not semisimple.

Proof. For (1) see [13, Theorem 4.1.3]. If $p \nmid |X|$ for some $X$ in $\text{Cel}(C)$, then $P = \sum_{X \in \text{Cel}(C)} \prod_{Y \in \text{Cel}(C), Y \neq X} |Y| J_X$ is a nonzero central nilpotent element of $W^k$. Otherwise $|X| = p^{\alpha_X} m$, $(p, m) = 1$. If we put $\lambda := \text{Max} \{\alpha_X | X \in \text{Cel}(C)\}$, then $p^\lambda \sum_{X \in \text{Cel}(C)} J_X / |X|$ is a nonzero central nilpotent element of $W^k$ and thus $W^k$ is not semisimple.

□

Theorem 5.2. [6, Hanaki] Let $k$ be a field of characteristic $p$. Suppose $C = (V, R)$ is an association scheme. Then $W^k$ is semisimple if and only if the Frame number $\mathcal{F}(C)$ is not divided by $p$.

The following is our main result and it is a generalization of Theorem 5.2 in which we do not need that $W$ is homogeneous.
Theorem 5.3. Let $k$ be a field of characteristic $p$. Then $W^k$ is semisimple if and only if the Frame number $\mathcal{F}(C)$ is not divided by $p$.

In order to prove this result, we use the following tools which is quite useful in the study of modular representation theory.

Let $p$ be a prime, and let $(K, R, F)$ be a $p$-modular system for $W$. Namely, $R$ is a complete discrete valuation ring with the maximal ideal $(\pi)$, $K$ is the quotient field of $R$ and its characteristic is 0, and $F$ is the residue field $R/(\pi)$ and its characteristic is $p$. For more details about $p$-modular systems, see [11]. The simplest example of $p$-modular systems is $(\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_p)$, in which $\mathbb{Z}_p$ is the localization of $\mathbb{Z}$ at prime ideal $p\mathbb{Z}$. To simplify our argument, we suppose that $W$ and $W^F$ are split algebras. In this case, we say $(K, R, F)$ is a splitting $p$-modular system of $W$. For $x \in R$, we denote the image of the natural homomorphism $R \to F$ by $x^*$.

Each idempotent of $W^F$ is the image of an idempotent of $W^R$ by the natural epimorphism from $W^R$ to $W^F \simeq W^R/\pi W^R$. The primitivity of idempotents is preserved by this correspondence (see [11, Theorem I.14.2]). Moreover, there exists natural correspondence between the set of central primitive idempotents of $W^R$ and that of $W^F$ (see [4, Proposition 1.12]). Namely, if

$$1 = e_0 + e_1 + \cdots + e_r$$

is the central idempotent decomposition of 1 in $W^R$, then so is

$$1^* = e_0^* + e_1^* + \cdots + e_r^*$$

and we have the following proposition.

Proposition 5.4. [6, Hanaki] Suppose $W^F$ is semisimple. Then there exists a natural correspondence between $\text{IRR}(W^K)$ and $\text{IRR}(W^F)$ which preserves dimensions. Namely, If $\{e_0, e_1, \ldots, e_r\}$ is the set of central primitive idempotents of $W^K$, then so is $\{e_0^*, e_1^*, \ldots, e_r^*\}$ of that in $W^F$.

Let $\text{IRR}(W^K) = \{M_1, \ldots, M_r\}$ and let $X_i$ be a matrix representation of $W^K$ corresponding to $M_i$. Put $f_i = \dim M_i$. By [11, Theorem II.1.6], $M_i$ has an $R$-form, namely, we may assume that $X_i(A_j) \in M_{f_i}(R)$. Put $M = \bigoplus_{i=1}^r M_i$, and we can define an $W^R$-module $\tilde{M}$ such that $K \otimes \tilde{M} \simeq M$. Then we can define an $W^F$-module $M^* = \tilde{M}/\pi M$. Obviously, \((D_{M,\{A_i\}}(W^K))^* = D_{M^*,\{A_i^*\}}(W^F)\). By Proposition 5.4, if $W^F$ is semisimple, then \((D_{M,\{A_i\}}(W^K))^* \neq 0\). Also if $W^F$ is not semisimple, then \((D_{M,\{A_i\}}(W^K))^* = 0\).

Therefor we can say that $D_{M,\{A_i\}}(W^K)$ characterizes the semisimplicity of $W^F$. Now by Theorem 4.1, $D_{M,\{A_i\}}(W^K) = \pm \mathcal{F}(C)$ and thus Theorem 5.3 holds for $F$. As the prime field of order $p$ is a perfect field, by
Theorem 2.6. Theorem 5.3 holds for arbitrary field $k$ of characteristic $p$.

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