Generic vanishing in characteristic $p > 0$ and the geometry of theta divisors

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Abstract
In this paper, we prove a strengthening of the generic vanishing result in characteristic $p > 0$ given in Hacon and Patakfalvi (Am J Math 138(4):963–998, 2016). As a consequence of this result, we show that irreducible $\Theta$ divisors are strongly F-regular and we prove a related result for pluri-theta divisors.

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1 Introduction

Since the work of Green and Lazarsfeld [3,4] it has been clear that generic vanishing plays a fundamental role in the geometry of complex projective irregular varieties, i.e. varieties such that the Albanese map is non-trivial. Combining these results with the Fourier–Mukai transform [15] and Kollár’s results on (higher) direct images of dualizing sheaves [13,14], many surprisingly far reaching and precise results on the birational geometry of complex projective varieties have been proven over the last couple of decades. In [7], a new more functorial perspective was introduced and in particular it was shown that generic vanishing is a consequence of traditional vanishing theorems and Kollár’s results above on higher direct images. Unluckily, it is well known that (generic) vanishing theorems and the results of [13,14] fail in positive characteristic [9]. Nevertheless recently, using the theory of Cartier modules and Serre vanishing, a technical generalization of generic vanishing was proven in [10] for projective varieties over a field of characteristic $p > 0$. Despite its technical nature, this result leads the way to some remarkable geometric applications such as a characterization of (ordinary) abelian varieties [11]. In this paper we further refine the results of [10] proving a more precise generic vanishing result in characteristic $p > 0$ and we use this result to investigate the geometry of pluri-theta divisors.

1.1 Generic vanishing theorems

1.1.1 Characteristic zero background

Let $X$ be a complex projective manifold and $a : X \to A$ its Albanese morphism. Green and Lazarsfeld study the cohomological support loci

$$V^i(\omega_X) := \{ P \in \text{Pic}^0(A) \mid h^i(\omega_X \otimes P) \neq 0 \}.$$  

In [3,4], they show that these loci are translates of abelian subvarieties of $\text{Pic}^0(A)$ of codimension $\geq i - (\dim X - \dim a(X))$. In particular if $X$ is of maximal Albanese dimension so that $\dim X = \dim a(X)$, then $\text{codim} V^i(\omega_X) \geq i$. They also conjecture the following more functorial version of this result (see Sect. 2 for the definition of the Grothendieck dual $D_A$ and the Fourier Mukai transform $R\hat{S}$):

$$\text{if } X \text{ is of maximal Albanese dimension, then } R\hat{S}(D_A(Ra_\omega X)) \text{ is a sheaf, or equivalently } H^i\left(R\hat{S}(D_A(Ra_\omega X))\right) = 0 \text{ for } i \neq 0. \quad (1)$$

In [7, Theorem 1.2.3] a stronger version of conjecture (1) was proven. Namely it is shown that if $F$ is a coherent sheaf on $A$ and $L$ is an ample line bundle on $A$ such that $H^i(A, F \otimes \hat{L}^\vee) = 0$ for any $i > 0$ and any sufficiently ample line bundle $L$, then $R\hat{S}(D_A(F))$ is a sheaf (see Sect. 2 for the definition of $\hat{L}^\vee$). Note that by [14],

$$Ra_\omega X = \sum_{j=0}^{\dim X - \dim a(X)} R^i a_\omega X[-j]$$

and by [13],

$$H^i(A, R^i a_\omega X \otimes \hat{L}^\vee) = 0, \quad \text{for } i > 0.$$
In particular this implies (1), which in turn implies \( \dim X = \dim a(X) \). The original generic vanishing results then follow easily.

### 1.1.2 Positive characteristic background

In characteristic \( p > 0 \), it is known that the results of [13,14] fail frequently. For example, there exist smooth surfaces such that \( H^1(\omega_X \otimes L) \neq 0 \) for an ample line bundle \( L \) and there are examples where the sheaves \( R^i a_* \omega_X \) are torsion (even when \( a \) is surjective). Thus, traditional generic vanishing results must fail as shown in [9]. On the other hand it is well known that one can recover some weaker vanishing results by combining Serre Vanishing and the Frobenius morphism. For example, note that if \( f : X \to Y \) is a projective morphism of smooth projective varieties then \( R^i f_* \omega_X \) is a Cartier module. By definition this means that we have an \( O_X \)-module homomorphism \( F_* R^i f_* \omega_X \to R^i f_* \omega_X \), which in this case is given by applying \( R^i f_* (_) \) to the Grothendieck trace of the Frobenius morphism of \( X \). Let \( \Omega := F_* R^1 f_* \omega_X \) and let \( H \) be an ample line bundle on \( A \). Then, by the projection formula and Serre vanishing we have that for every \( e > 0 \),

\[
H^j(A, \Omega \otimes H) \cong H^j(A, F^e(R^1 f_* \omega_X \otimes H^p)) = 0 \quad j > 0.
\]

Since the inverse system \( \Omega_e \) satisfies the Mittag–Leffler property, we have \( H^i(\Omega \otimes H) = 0 \) for \( j > 0 \) where \( \Omega := \varprojlim \Omega_e \). Note that in fact this works for any Cartier module \( \Omega : F_* \Omega_0 \to \Omega_0 \), and hence one can deduce that if \( \Omega_0 \) is a Cartier module on an abelian variety, then \( H^i(\Omega \otimes \hat{L}^v) = 0 \) for \( i > 0 \) [10, Lem 3.1.2]. However, since \( \Omega \) is not a coherent sheaf it is unclear if it satisfies a generic vanishing statement. The situation is clarified in [10, Thm 3.1.1] where we show that if \( \Lambda_e = R\hat{S}(D_A(\Omega_e)) \) then \( \Lambda := \varinjlim \Lambda_e \) is a sheaf and \( \Omega = (-1)^r D_A(RS(\Lambda))[-g] \). Despite its technical nature, this statement proves to be very useful in several geometric contexts. Its applications are however somewhat limited in scope since it is not immediately clear how to recover more traditional generic vanishing statements such as statements on the codimension of the loci \( V^i(\omega_X) \) or more functorially on the codimension of the supports of the sheaves \( \mathcal{H}^i(R\hat{S} Ra_* \omega_X) \). In this paper we take a significant step in this direction. In particular we prove the following.

### 1.2 Generic vanishing type results

From now for the entire article, the base field \( k \) is algebraically closed and of characteristic \( p > 0 \). The Poincaré bundle on \( A \times \hat{A} \) is denoted by \( \mathcal{P} \). In particular, \( \mathcal{P}|_{0 \times \hat{A}} = \mathcal{O}_{\hat{A}} \) and if \( y \in \hat{A} \) is a closed point then \( \mathcal{P}_y \cong \mathcal{P}|_{A \times y} \) is the numerically trivial line bundle on \( A \) corresponding to \( y \in \hat{A} \).

**Theorem 1.1** Let \( A \) be an abelian variety over \( k \) and let \( F_* \Omega_0 \to \Omega_0 \) be a Cartier module on \( A \). Then, for each integer \( 0 \leq i \leq \dim A \), there are closed subsets \( W^i \subseteq \hat{A} \) such that

- (a) **CODIMENSION**: \( \dim W^i \geq i \),
- (b) **VANISHING**: there is an integer \( t > 0 \) such that for every \( i \geq 0 \) and for every closed point \( y \in \hat{A} \setminus W^i \), the following homomorphism is zero:

\[
H^i(A, \mathcal{P}_y \otimes F_*^{\ell t} \Omega_0) \to H^i(A, \mathcal{P}_y \otimes \Omega_0),
\]
(c) **Non-vanishing**: if $Z$ is an irreducible component of $W^i$ of codimension $i$, then for very general closed points $y \in Z$,

$$0 \neq \lim_{e \to \infty} H^i(A, \mathcal{P}_y \otimes F^e_{*} \Omega_0).$$

(d) **Torsion Property**: if $A$ does not have supersingular factors, then every irreducible component $Z$ of $W^i$ of codimension $i$ is a torsion translate of an abelian subvariety.

(e) **Full Limit Versions** (direct consequences of points (b) and (c)):

$$y \notin \bigcup_{j \in \mathbb{N}} [p^j]^{-1} W^i \text{ is a closed point} \implies 0 = \lim_{e \to \infty} H^i(A, \mathcal{P}_y \otimes F^e_{*} \Omega_0),$$

and if $Z$ is an irreducible component of $[p^j]^{-1} W^i$ of codimension $i$, then

$$y \in Z \text{ a very general closed point} \implies 0 \neq \lim_{e \to \infty} H^i(A, \mathcal{P}_y \otimes F^e_{*} \Omega_0).$$

(f) **Connection to Fourier-Mukai Transforms**: independently of $A$ having a supersingular factor or not, we have:

$$y \in W^i \setminus \left( \bigcup_{r > i} W^r \right) \implies i = \text{codim } y + \text{depth}_{\hat{\mathcal{O}}_{\hat{A}, y}} \tilde{\Lambda}_{0, y},$$

where for every $e \gg 0$:

$$\tilde{\Lambda}_0 := \text{im} \left( \mathcal{H}^0 \left( R^\mathbb{S} \left( D_A(\Omega_0) \right) \right) \to \mathcal{H}^0 \left( R^\mathbb{S} \left( D_A(F^e_{*} \Omega_0) \right) \right) \right)$$

and additionally

$$\text{Supp } \tilde{\Lambda}_0 \subseteq \left\{ y \in \hat{A} \left| H^0(A, \mathcal{P}_y \otimes \Omega_0) \neq 0 \right. \right\}.$$

In particular, the abelian subvarieties of point (d) are exactly the closures of the associated primes of $\tilde{\Lambda}_0$.

Surprisingly, one cannot remove the non-existence of supersingular factors from point (d) of Theorem 1.1. This is stated in the next proposition, by choosing $y$ of the proposition to be a non-torsion point.

**Proposition 1.2** (= Example 5.5) **Point (d) of Theorem 1.1 does not hold if $A = (E \times E) \otimes \mathbb{F}_p$**, $k$, where $E$ is a supersingular elliptic curve over $\mathbb{F}_p$. That is, for each curve $C \subseteq \hat{A}$ defined over $\mathbb{F}_p$, there is a Cartier module $F^e_{*} \Omega_0 \to \Omega_0$ such that $\text{codim } W^1 = 1$ and $W^1 = \hat{C}$.

**Remark 1.3** An admitted aesthetic defect of Theorem 1.1 is that the non-vanishing of $\lim_{e \to \infty} H^i(A, \mathcal{P}_y \otimes F^e_{*} \Omega_0)$ does not happen for all points $y \in W^i$, just for very general points of the components of $W^i$ of codimension $i$. Because of the presence of (infinitely many) homomorphisms in these inverse limits, there is not much hope to remove completely the genericity condition from the statement, although it would be interesting to see whether a statement with very general replaced by general can be found. This would go beyond the scope of the present article, so we leave it as an open question.

It is also unclear at the moment whether our statement could be improved to a non-vanishing (at (very) general points of) all components of $W^i$. Let us explain below the reason why this non-vanishing happens only at generic points of components of codimension $i$ in Theorem 1.1. Intuitively, $\lim_{e \to \infty} H^i(A, \mathcal{P}_y \otimes F^e_{*} \Omega_0)$ could be zero for two reasons:
(a) the cohomology group corresponding to \( H^i(A, \mathcal{P}_y \otimes \Omega_0) \) is vanishing in characteristic zero,
(b) the corresponding cohomology group is not vanishing in characteristic zero, but the action on cohomology induced by \( F_*^\mathcal{S} \Omega_0 \to \Omega_0 \) is still nilpotent because of some particular arithmetic behavior.

Our approach is based on an inverse system of complexes, which we denote by \( (R\hat{S}(\Omega_e))_{e \geq 0} \) and in fact, \( W^i \) of Theorem 1.1 is defined as \( \bigcup_{j \geq i} \text{Supp} R^j\hat{S}(\Omega_e) \). We refer to Sect. 4 for the details, here we focus on the important point that these complexes are able to distinguish points \( y \in \hat{A} \) of type (a) from the rest, but not points of type (b). To repair the above defect, one should find an inverse system \( (\mathcal{C}_e)_{e \geq 0} \) of bounded complexes of coherent sheaves, which is equivalent to \( (R\hat{S}(\Omega_e))_{e \geq 0} \) in the sense of Lemma 4.2, and for which the complexes \( \mathcal{C}_e \) can somehow also tell apart points of type (b) from the rest of the points of \( \hat{A} \). Unfortunately, we were not able to find such an inverse system, and hence we leave it also as open question to decide whether one could strengthen our non-vanishing result beyond the components of codimension \( i \).

Another byproduct of our methods is the following non-vanishing statement:

**Theorem 1.4** Let \( F_*^\mathcal{S} \Omega_0 \to \Omega_0 \) be a Cartier module on an abelian variety \( A \) without supersingular factors, and let \( \hat{W} \subseteq \hat{A} \) be an abelian subvariety such that the generic point of a translate of \( \hat{W} \) is an associated prime of \( \text{im} \left( R\hat{S}(D_A(\Omega_0)) \to R\hat{S}(D_A(F_*^\mathcal{S} \Omega_0)) \right) \) for every \( e \gg 0 \) (such \( \hat{W} \) exists for all associated points according to Corollary 4.4). Let \( \pi : A \to W \) be the morphism of abelian varieties dual to the inclusion \( \hat{W} \hookrightarrow \hat{A} \), and let \( j \) be the relative dimension of \( \pi \). Then, \( R^j\pi_* (\Omega_0 \otimes Q) \neq 0 \) for some \( Q \in \tilde{A} \).

**Remark 1.5** It is interesting to note that in characteristic 0, if \( \Omega_0 \) is a direct summand of the pushforward of the canonical bundle of a smooth variety (or more generally the push forward of a sheaf of the form \( \mathcal{O}_X(D) \) where \( D \) is a Weil divisor numerically equivalent to \( K_X + B \) for some klt pair \((X, B))\), and if \( Z \) is the support of \( \Omega_0 \), then \( R^j\pi_* (\Omega_0 \otimes Q) \) is a torsion free sheaf supported on \( \pi(Z) \). It then follows that \( \dim Z_w \geq j \) for every closed point \( w \in \pi(Z) \) and hence \( Z = \pi^{-1}\pi(Z) \), i.e., \( Z \) is fibred by tori (cf. [3, Theorem 1] and [2, Theorem 3]). In order to make further progress in understanding the geometry of varieties in positive characteristics it would be useful to answer the question below.

**Question 1.6** Let \( f : X \to Y \) be a morphism of smooth projective varieties over an algebraically closed field of characteristic \( p > 0 \). Is it the case in some sense that:
(a) \( \lim F_*^\mathcal{S} R^i f_* \omega_X \) is torsion free on \( f(Y) \) and in particular \( \lim F_*^\mathcal{S} R^i f_* \omega_X = 0 \) for \( i > \dim X - \dim f(X) \), and
(b) \( \dim \lim R f_* F_*^\mathcal{S} \omega_X = \sum \lim F_*^\mathcal{S} R^i f_* \omega_X \{-i\} \)?

The words “some sense” are added because the ideal framework for the above questions is not entirely clear at this point. For example in case of (b), one could also ask if the equality holds in some category of derived Cartier modules localized by nilpotence.

### 1.3 Geometry of pluri-theta divisors

By [2], it is known that if \( (A, \Theta) \) is a principally polarized abelian variety (PPAV) over \( \mathbb{C} \) the field of complex numbers such that \( \Theta \) is irreducible, then \( \Theta \) is normal and has rational singularities. In fact \( (A, \Theta) \) is plt or equivalently \( \Theta \) has canonical singularities (since
it is Gorenstein). We prove an analog of this result in positive characteristic. We recall:

for the entire article, the base field \( k \) is algebraically closed and of characteristic \( p > 0 \).

**Theorem 1.7** (Watson) Let \((A, X)\) be a PPAV over \( k \) such that \( X \) is irreducible and \( A \) has no supersingular factors, then \( X \) is strongly F-regular.

**Remark 1.8** By [8] we already know that \((A, X)\) is F-pure, however the strongly F-regular statement is substantially more involved. In particular it is natural to ask if it implies Witt rationality.

More generally one expects a positive answer to the following.

**Question 1.9** Let \( X \) be a strongly F-regular variety, then does \( X \) have Witt rational singularities?

**Remark 1.10** We emphasize that several of our techniques are based on and sometimes closely follow the arguments of [21] and as indicated above Theorem 1.7 should be credited to A. Watson. However [21] is yet unpublished and appears to contain several imprecise statements. We hope that the simplifications that we introduce will help to clarify and disseminate Watson’s result.

**Theorem 1.11** Let \((A, X)\) be a PPAV over \( k \) with no supersingular factors and with \( X \) irreducible. Furthermore, let \( D \in |mX| \) be a divisor such that \((m, p) = 1\) and \([D/m] = 0\). Then, \((X, D/m)\) is purely F-regular. In particular \((X, D/m)\) is klt and \( \text{mult}_P(D) \leq mj \) for any codimension \( j \) point.

**Remark 1.12** We refer the reader to [8], for a related result in characteristic 0.

### 1.4 \( V \)-modules

A biproduct of our arguments is a new method of constructing examples of Cartier modules \( F^*_s \Omega_0 \to \Omega_0 \) on an abelian variety \( A \) with a good control on the loci

\[
W^i_F = \left\{ Q \in \hat{A} \mid \lim_{\varepsilon} H^i(A, \Omega_0 \otimes Q^{p^\varepsilon}) \neq 0 \right\}.
\]  

We hope that this will lead in the long run to a much better understanding of generic vanishing in positive characteristic in general. The central notion of this approach is the notion of a \( V \)-module on \( \hat{A} \) which is the abstraction of the sheaf \( \tilde{\Lambda}_0 \) of point (f) of Theorem 1.1. According to Definition 5.1, it is a coherent sheaf \( M \) on \( \hat{A} \) together with an \( O_{\hat{A}} \)-linear morphism \( M \to V^{s,*}M \), where \( s > 0 \) is an integer and \( V \) is the Verschiebung isogeny of \( \hat{A} \). The main statement that allows one to run the above mentioned general process of constructing examples of Cartier modules is the following:

**Theorem 1.13** (Equivalent of Theorem 5.2) Let \( A \) be an abelian variety over \( k \). If \( M \to V^{s,*}M \) is a \( V \)-module on \( \hat{A} \), then \( \lim_{\varepsilon} \mathcal{H}^i\left(\text{RS}(D_{\hat{A}}(V^{es,*}M))\right) = 0 \) for every \( i \neq 0 \).

Using Theorem 1.13, one can see that the Cartier module \( \mathcal{H}^0\left(\text{RS}(D_{\hat{A}}(M))\right) \) on \( A \) is a Cartier module for which many properties of the associated loci \( W^i_F \) can be read off directly from the \( V \)-module \( M \). We refer the reader to Corollary 5.3 for the precise statement of some of these properties. We believe for example that using Corollary 5.3, the possibilities for \( W^0_F \) can be completely classified, which question we leave for future articles. Here, we only state one sample consequence of Corollary 5.3:
Corollary 1.14 Let $L$ be a line bundle on the dual $\hat{\mathbb{A}}$ of an abelian variety $A$ over $k$ and let $W^0_F$ be the 0-th Frobenius stable cohomology support locus defined in (2) for a Cartier module $\Omega_0$ specified below.

(a) Let $E$ be a supersingular elliptic curve over $\mathbb{F}_p$, and set $A_{\mathbb{F}_p} = \times_{i=1}^g E$. Assume that $A = A_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} k$ and that $L = L_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} k$ for some line bundle $L_{\mathbb{F}_p}$ on $\hat{\mathbb{A}}_{\mathbb{F}_p}$. If $0 \neq s_{\mathbb{F}_p} \in H^0(\hat{\mathbb{A}}_{\mathbb{F}_p}, L^{-1} \otimes F^*_p L_{\mathbb{F}_p})$, then there is a Cartier module $\Omega_0$ on $A$ such that $W^0_F = \hat{\mathbb{A}} \setminus \{y \in \hat{\mathbb{A}} \mid \text{there are infinitely many } n \in \mathbb{N} \text{ such that } p^n y \in V(s)\}$.

(b) If $A$ has no supersingular factor, and $0 \neq s \in H^0(\hat{\mathbb{A}}, L^{-1} \otimes V^* L)$, then there is a Cartier module $\Omega_0$ on $A$ such that $W^0_F = \hat{\mathbb{A}} \setminus \{y \in \hat{\mathbb{A}} \mid \text{there are infinitely many } n \in \mathbb{N} \text{ such that } p^n y \in V(s)\}$.

In particular, if $A$ is a surface and $V(s)$ has no component that is a torsion translate of an abelian subvariety, then $\hat{\mathbb{A}} \setminus W^0_F$ is the union of countably many closed points.

We note that as the degree of $V^* L$ is bigger than that of $L$, an $s$ as in point (b) of Corollary 1.14 can be found for any ample enough line bundle $L$ on $\hat{\mathbb{A}}$.

We also note that in the surface case of point (b) of Corollary 1.14 whether $\hat{\mathbb{A}} \setminus W^0_F$ is infinite seems to be connected to the arithmetic behavior of $V(s)$. That is, the only obvious points that are contained in $\hat{\mathbb{A}} \setminus W^0_F$ are the prime-to-$p$ points of $V(s)$, which can be infinite only if either $V(s)$ contains a positive dimensional abelian subvariety or an irreducible component defined over a finite subfield of $k$, by the Manin-Mumford conjecture [18, Thm 3.6].

Sample questions that our above discussion raises are the following:

Question 1.15 For an abelian variety $A$, for which $\overline{W^0_F} = \hat{\mathbb{A}}$:

(a) Is $\hat{\mathbb{A}} \setminus W^0_F$ the countable union of constructible subsets?
(b) Is the codimension 1 part of $\hat{\mathbb{A}} \setminus W^0_F$ a closed, or at least constructible subset?
(c) If $A$ has no supersingular factor, then is every codimension 1 component of $\hat{\mathbb{A}} \setminus W^0_F$ a torsion translate of an abelian subvariety?
(d) If $A$ has no supersingular factor, is then the codimension 2 part of $\hat{\mathbb{A}} \setminus W^0_F$ not constructible if and only if $V(s)$ contains a component of arithmetic flavor as in the Manin-Mumford conjecture [18, Thm 3.6]?

1.5 Structure of the article

The article is structured based on the theorems mentioned in the introduction.

- In Sect. 2, we collect some definitions and background statements.
- In Sect. 3, we collect a few lemmas about the nilpotence behavior of direct and inverse systems in $D^b_c(X)$.
- In Sect. 4, we show Theorem 1.1.
- In Sect. 5, we show Theorem 1.13, Corollary 1.14 and Proposition 1.2.
- In Sect. 6, we show Theorem 1.4.
- In Sect. 7, we show Theorem 1.7 and Theorem 1.11.
2 Preliminaries and notation

We work over a fixed algebraically closed field $k$ of characteristic $p > 0$. We fix the following notation:

- $A$ is a $g$-dimensional abelian variety over $k$.
- $\hat{A}$ is the dual abelian variety of $A$.
- $\mathcal{P}$ is the Poincaré line bundle of $A$, which is a line bundle on $A \times \hat{A}$.
- If we write $Q \in \hat{A}$ for a sheaf $Q$ on $A$, then we mean that $Q = \mathcal{P}_y$ for some closed point $y \in \hat{A}$.
- The Fourier-Mukai transforms $R\hat{S} : D(A) \to D(\hat{A})$ and $RS : D(\hat{A}) \to D(A)$ are defined by
  $$R\hat{S}(?) = Rp_{\hat{A},*}(Lp_A^*? \otimes \mathcal{P}), \quad RS(?) = Rp_{A,*}(Lp_{\hat{A}}^*? \otimes \mathcal{P}).$$
- If $L$ is an ample line bundle on $A$, then let $\hat{L} = R^0\hat{S}(L)$. In this case, $\phi_L^*(\hat{L}^\vee) \cong \oplus \mathcal{H}^0(L)_L$, where $\phi_L : A \to \hat{A}$ is the isogeny defined by $\phi_L(a) = t^a_L L^\vee \otimes L$ for any $a \in A$. In particular, if $L$ is the line bundle of a principal polarization, then $\deg \phi_L = 1$ and $\mathcal{H}^0(L) = 1$.
- A variety is a scheme of finite type over $k$.
- As in [15], if $X$ is a variety over $k$, then $D(X)$ is the derived category of sheaves of $\mathcal{O}_X$-modules, and we denote certain full subcategories by putting adequate subscripts and superscripts on $D(X)$. Subscript $c$ and $qc$ means that the cohomology sheaves are required to be coherent and quasi-coherent, respectively. Superscript $b$, $-$ and $+$ means that the cohomology sheaves are zero outside of a finite range, above a finite cohomological degree and below a finite cohomological degree.
- We use homotopy-colimits, which are derived category versions of colimits. The only important fact for the article about them is that $\mathcal{H}^i(\text{hocolim}_n(\_)) = \lim_{\longrightarrow} (\mathcal{H}^i(\_))$.
- When working with elements of $D(X)$ and its variants, we also use the abbreviation $\mathcal{H}^i(X,\_\_\_) = R^i\Gamma(X,\_\_\_)$.
- $F$, $V$ and $[p]$ are the usual isogenies of $A$ (Frobenius, Verschiebung and multiplication by $p$). Note that $k = \bar{k}$ and therefore $k$ is perfect. Hence, the isogeny $F$ identifies with the absolute Frobenius morphism of $A$. That is, $F$ is the upper horizontal arrow in (3) either if considered as an isogeny or as the absolute Frobenius morphism, with the only difference being the $k$-structure on the source.
  - If $F$ is regarded as an isogeny then for the source $X$ is considered with the alternative $k$-structure denoted by $X^{(-1)}$.
  - On the other hand, if $F$ is regarded as the absolute Frobenius, then the source of $F$ is $X$ with its original $k$-structure. Note that in particular, in this situation $F$ is not $k$-linear, but only twisted $k$-linear, also called $1/p$-linear or $\sigma^{-1}$-linear.

Similarly, $V$ a priori is an isogeny $X \to X^{(-1)}$, but as shown in (3), it can be also regarded as a (twisted $k$-linear) morphism $X \to X$. 

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In the present article we regard $F$ and $V$ as twisted linear unless otherwise stated. This way, as shown in (3), we retain the equalities $[p] = F \circ V = V \circ F$ known from the isogeny point of view.

Additionally, if $X = Y \otimes \mathbb{F}_p k$, for a variety $Y$ over $\mathbb{F}_p$, then there is an additional isomorphism that one can apply to $X$ in the middle of the top row of the following diagram, which identifies $F$ with the base-extension $F_Y \otimes \text{id}_k$ of $F_Y$ to $k$:

If furthermore $Y$, and hence also $X$, is a supersingular elliptic curve, then composing with $\text{id}_Y \otimes \mathbb{F}_p F$ identifies $V$ with $F_Y \otimes \mathbb{F}_p \text{id}_k$. To see this, note that one of the defining properties of being supersingular for an elliptic curve $X$ over $k$ is that $[p]^{-1}(0)$ has a single point. In particular, $[p]$ is purely inseparable. As $[p] = V \circ F$, it follows that $V$ is also purely inseparable. As discussed above, we typically view $V$ as a $1/p$-linear morphism. However, then if we compose it with $\text{id}_Y \otimes F$, it becomes a $k$-morphism. So, $(\text{id}_Y \otimes \mathbb{F}_p F) \circ V$ is a purely inseparable $k$-morphism of degree $p$. The only such morphism is $F_Y \otimes \mathbb{F}_p \text{id}_k$, as $X$ is a curve and hence its Frobenius has degree $p$ (so there is no room for multiple degree $p$ purely inseparable morphisms).

Note also that $F = \hat{V}$ if either both $F$ and $V$ are considered to be $k$-linear or both of them are considered to be twisted linear. See [10, Sec 2.3] for further details.

Recall the following.

Theorem 2.1 (Mukai [15]) The following equalities hold in $D_{\text{qc}}(A)$ and $D_{\text{qc}}(\hat{A})$:

$$RS\hat{S} \circ RS = (-1_A)^*[-g], \quad RS \circ \hat{R}S = (-1_A)^*[-g],$$

$$DA \circ RS \cong ((-1_A)^* \circ RS \circ D_A)[g],$$

$$RS \circ T_x \cong (\otimes P_{-x}) \circ RS,$$

$$\phi^* \circ RS = RS \circ \hat{\phi}_x^*, \quad \psi^* \circ RS = RS \circ \hat{\psi}^*$$

where $x \in \hat{A}$, $P_x = P|_{A \times x}$, $T_x$ denotes translation by $x$ on $\hat{A}$, $[g]$ denotes shift by $g$ spaces to the left, and $\phi$ and $\psi$ are any isogenies of $A$. 

\text{Springer}
2.1 Truncation functors

We recall some results from [16, Sect. 3.3]. Let $\mathcal{A}$ be an abelian category and $A^\bullet$ be a complex of objects of $\mathcal{A}$, then there are two natural truncations given by

$$
\tau_{\leq n}(A^\bullet)^p = \begin{cases} 
A^p & \text{for } p < n \\
\ker d^n & \text{for } p = n \\
0 & \text{for } p > n
\end{cases}
$$

$$
\tau_{\geq n}(A^\bullet)^p = \begin{cases} 
0 & \text{for } p < n \\
coker d^{n-1} & \text{for } p = n \\
A^p & \text{for } p > n.
\end{cases}
$$

These truncations induce functors on the derived category

$$
\tau_{\leq n} : D(\mathcal{A}) \to D^- (\mathcal{A}), \quad \text{and} \quad \tau_{\leq n} : D(\mathcal{A}) \to D^+ (\mathcal{A}).
$$

For every $X \in D(\mathcal{A})$, we have morphisms in $D(\mathcal{A})$

$$
i : \tau_{\leq n}(X) \to X, \quad \text{and} \quad q : X \to \tau_{\geq n}(X)
$$

such that the induced map $H^p(X) \to H^p(\tau_{\leq n}(X))$ is an isomorphism if $p \geq n$ and zero otherwise and $H^p(\tau_{\leq n}(X)) \to H^p(X)$ is an isomorphism if $p \leq n$ and zero otherwise. For $A^\bullet$ as above, there is a short exact sequence of complexes

$$
0 \to \tau_{\leq n}(A^\bullet) \to A^\bullet \to Q^\bullet \to 0,
$$

where $Q^i = 0$, $\coim(d^n)$, $A^p$, if $p < n$, $p = n$, $p > n$. There is an induced projection $Q^\bullet \to \tau_{\geq n+1}(Q^\bullet) = \tau_{\geq n+1}(A^\bullet)$. We have [16, Proposition 3.6.1]

**Proposition 2.2** There exists a unique morphism $h : \tau_{\geq n+1}(A^\bullet) \to \tau_{\leq n}(A^\bullet)[1]$ inducing an exact triangle in $D(\mathcal{A})$

$$
\tau_{\leq n}(A^\bullet) \longrightarrow A^\bullet \longrightarrow \tau_{\geq n+1}(A^\bullet) \longrightarrow h
$$

**Lemma 2.3** Let $X$ be an object of $D(\mathcal{A})$. There is an exact triangle

$$
\tau_{\leq m}(X) \longrightarrow \tau_{\leq m+1}(X) \longrightarrow \mathcal{H}^{m+1}(X)[-m-1] \longrightarrow h
$$

**Proof** Applying $\tau_{\leq m+1}$ to the morphism $\tau_{\leq m}(X) \to X$ we obtain a morphism $\tau_{\leq m+1}(\tau_{\leq m}(X)) \to \tau_{\leq m+1}(X)$. Since $\tau_{\leq m+1}(\tau_{\leq m}(X)) = \tau_{\leq m}(X)$, we have a triangle $\tau_{\leq m}(X) \to \tau_{\leq m+1}(X) \to Y$. Applying $\mathcal{H}^\bullet$, since $\mathcal{H}^p(\tau_{\leq m}(X)) \to \mathcal{H}^p(\tau_{\leq m+1}(X))$ is an isomorphism for $p \neq m + 1$, we obtain $\mathcal{H}^p(Y) = 0$ for $p \neq m + 1$ and $\mathcal{H}^{m+1}(Y) = \mathcal{H}^{m+1}(X)$ so that $Y$ is quasi isomorphic to $\mathcal{H}^{m+1}(X)[-m-1]$. \qed

3 Nilpotent direct systems in the derived category

This section is a collection of lemmas about nilpotence behavior of direct and inverse systems in the bounded derived category of coherent sheaves. These statements will be used extensively in Sects. 4 and 5. We will consider arbitrary direct systems of the above type, but for inverse systems we make the assumption that they are associated to (derived) Cartier modules. We recall that a Cartier module on a scheme $X$ over $k$ is a pair $(\mathcal{M}, \phi)$, where $\phi$ is an $O_X$-homomorphism $F_s^! \mathcal{M} \to \mathcal{M}$ for some integer $s > 0$. 
Definition 3.1  Let $X$ be a variety over $k$. An inverse system $(D_e, \alpha_e : D_{e+1} \to D_e)_{e \geq 0}$ in $D^b_c(X)$ is associated to a derived Cartier module, if there is an integer $s > 0$ such that $D_e \cong F^s \cdot D_0$ and via these isomorphisms $\alpha_e$ identifies with $F^s(\alpha_0)$.

Remark 3.2  In the situation of Definition 3.1, as $F$ is affine, for each integer $i > 0$, the inverse system $(\mathcal{H}^i(D_e), \mathcal{H}^i(\alpha_e))_{e \geq 0}$ comes from the iterations of a Cartier module. Hence, for every integer $e \geq 0$ the images im$(\mathcal{H}^i(D_e) \to \mathcal{H}^i(D_e))$ stabilize for $e' \gg 0$ [1, Proposition 8.1.4].

Definition 3.3  Let $X$ be a variety over $k$. A direct system $(C_e; \alpha_e : C_e \to C_{e+1})_{e \geq 0}$ in $D^b_c(X)$ is nilpotent in cohomological degree $i$, if for every integer $e > 0$ the homomorphism $\mathcal{H}^i(C_e) \to \mathcal{H}^i(C'_e)$ is zero for $e' \gg e$. Note that $\mathcal{H}^i(C_e)$ is Noetherian, the images $\mathcal{H}^i(C_e) \to \mathcal{H}^i(C'_e)$ stabilize for $e' \gg 0$, and hence being nilpotent in cohomological degree $i$ is equivalent to $\lim_{e} \mathcal{H}^i(C_e) = 0$.

The same direct system is nilpotent outside of cohomological degree $i$, if it is nilpotent in each cohomological degree $j \neq i$.

The same direct system is nilpotent if for every integer $e > 0$ there is an integer $e' \geq e$ such that $C_e \to C_{e'}$ is zero (where the bigness of $e'$ can depend on $e$; note that this is not the same as requiring $\mathcal{H}^i(C_e) \to \mathcal{H}^i(C_e')$ to be zero for all $i$).

Similarly, an inverse system $(\mathcal{D}_e : \alpha_e : D_{e+1} \to D_e)$ associated to a derived Cartier module is nilpotent in cohomological degree $i$ if for every integer $e > 0$, the homomorphism $\mathcal{H}^i(D_{e'}) \to \mathcal{H}^i(D_e)$ is zero for every $e' \gg e$. According to Remark 3.2, this is equivalent to $\lim_{e} \mathcal{H}^i(D_{e'}) = 0$. Being nilpotent outside of cohomological degree $i$ and being nilpotent is then defined analogously as above for direct systems.

Lemma 3.4  Let $X$ be a variety over $k$, and let $(\mathcal{F}_e)_{e \geq 0}$ be a direct system in $D^b_c(X)$, which is nilpotent in all cohomological degrees, and for which there are integers $c \leq d$ such that $\mathcal{H}^c(\mathcal{F}_e) = 0$ for all $e \in \mathbb{N}$ whenever $i < c$ or $i > d$. Then, $(\mathcal{F}_e)_{e \geq 0}$ is nilpotent.

Proof Let $\phi_{e,s} : \mathcal{F}_e \to \mathcal{F}_s$ be the structural homomorphism. The proof is by induction on $j := d - c$. For $d = c$, we have $\mathcal{H}^c(\mathcal{F}_e)[-c] \cong \mathcal{F}_e$. Hence, in this case, for all $s \gg e$ we have that $\phi_{e,s}$ agrees as an arrow of $D(X)$ with the shift by $-c$ of the zero homomorphism $\mathcal{H}^c(\mathcal{F}_e) \to \mathcal{H}^c(\mathcal{F}_s)$, which is zero in $D(X)$.

So, let us assume that $j > 0$ and that we know the statement for $j$ replaced by $j - 1$. In the rest of the proof we prove the statement for $j$ under these assumptions. Note first that the composition $\mathcal{F}_e \to \mathcal{F}_s \to \mathcal{H}^d(\mathcal{F}_e)[-d]$ agrees with the composition $\mathcal{F}_e \to \mathcal{H}^d(\mathcal{F}_e)[-d] \to \mathcal{H}^d(\mathcal{F}_s)[-d]$. As the second map is zero in the latter composition for $s \gg e$, we obtain that both compositions are zero in $D(X)$ for $s \gg e$. Fix such an integer $s > e$. Consider now the distinguished triangle

$$
\tau_{d-1} \mathcal{F}_s \to \mathcal{F}_s \to \mathcal{H}^d(\mathcal{F}_s)[-d] \to ^{+1}.
$$

Applying $\text{Hom}_X(\mathcal{F}_e, -)$ we obtain an exact sequence, where the image of $\phi_{e,s}$ on the right is zero by the above discussion.

$$
\text{Hom}_X(\mathcal{F}_e, \tau_{d-1} \mathcal{F}_s) \to \text{Hom}_X(\mathcal{F}_e, \mathcal{F}_s) \to \text{Hom}_X(\mathcal{F}_e, \mathcal{H}^d(\mathcal{F}_s)[-d]) \to 0.
$$

By the exact sequence (5), we obtain that $\phi_{e,s}$ descends to an arrow $\widetilde{\phi}_{e,s} : \mathcal{F}_e \to \tau_{d-1} \mathcal{F}_s$ in $D(X)$. By our induction hypothesis, $\tau_{d-1} \phi_{s,e'} : \tau_{d-1} \mathcal{F}_s \to \tau_{d-1} \mathcal{F}_e'$ is zero for $e' \gg s$. 

$\circ$ Springer
Hence, the following diagram then concludes our proof for $e' \gg s$:

$$\begin{array}{cccc}
\mathcal{F}_e & \xrightarrow{\phi_{e,s}} & \tau_{\leq d-1}\mathcal{F}_s & \xrightarrow{\tau_{\leq d-1}\phi_{s,e}=0} & \tau_{\leq d-1}\mathcal{F}_e' \\
\downarrow \phi_{e,e'} & & \downarrow \phi_{s,e'} & & \downarrow \phi_{s,e'} \\
\mathcal{F}_s & & \mathcal{F}_s & & \mathcal{F}_e' \\
\end{array}$$

Lemma 3.5 Let $(C_e)_{e \geq 0}$, $(D_e)_{e \geq 0}$, $(E_e)_{e \geq 0}$ be direct systems in $D^b_c(X)$ that form an exact triangle

$$C_\bullet \longrightarrow D_\bullet \longrightarrow E_\bullet \longrightarrow +1$$

(That is, we have an exact triangle as above for each $\bullet = e \geq 0$, which commute with the structure maps of the systems.)

Then,

(a) if both $(C_e)_{e \geq 0}$ and $(E_e)_{e \geq 0}$ are nilpotent in cohomological degree $i$, then so is $(D_e)_{e \geq 0}$, and

(b) if $(E_e)_{e \geq 0}$ is not nilpotent in cohomological degree $i$, and $(C_e)_{e \geq 0}$ is nilpotent in cohomological degree $i+1$, then $(D_e)_{e \geq 0}$ is not nilpotent in cohomological degree $i$.

Proof Passing to cohomology sheaves we obtain a long exact sequence of direct systems

$$\ldots \longrightarrow (\mathcal{H}^{i-1}(E_e))_{e \geq 0} \longrightarrow (\mathcal{H}^i(C_e))_{e \geq 0} \longrightarrow (\mathcal{H}^i(D_e))_{e \geq 0} \longrightarrow \ldots$$

As taking direct limit is exact [19, Tag 04B0], we obtain a long exact sequence

$$\ldots \longrightarrow \lim_{\longrightarrow} \mathcal{H}^{i-1}(E_e) \longrightarrow \lim_{\longrightarrow} \mathcal{H}^i(C_e) \longrightarrow \lim_{\longrightarrow} \mathcal{H}^i(D_e) \longrightarrow \ldots$$

As a direct system being nilpotent is equivalent to the corresponding direct limit being zero, we obtain both statements of the lemma directly from (7).

Proposition 3.6 For a variety $X$ over $k$, let $(\mathcal{F}_e)_{e \geq 0}$ be a direct system of complexes of coherent sheaves, such that:

(a) $\mathcal{H}^i(\mathcal{F}_e) = 0$ for $i > 0$,

(b) there is $0 \geq c \in \mathbb{Z}$ such that $\mathcal{H}^i(\mathcal{F}_e) = 0$ whenever $i < c$, and

(c) $(\mathcal{F}_e)_{e \geq 0}$ is nilpotent in cohomological degrees $i < 0$.

Set $G_e := \text{im} (\mathcal{H}^0(\mathcal{F}_e) \to \mathcal{H}^0(\mathcal{F}_s))$ for $s \gg e$. Then, for every integer $e > 0$, and for every integer $e' \gg e$ (where the bigness depends on $e$), there is a homomorphism as shown by the dashed arrow of the following commutative diagram:

$$\begin{array}{cccc}
\mathcal{F}_e & \xrightarrow{\phi_{e,e'}} & \mathcal{F}_e' \\
\downarrow \alpha_e & & \downarrow \alpha_{e'} \\
G_e & \xrightarrow{\eta_{e,e'}} & G_{e'} \\
\end{array}$$

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**Proof**  

**STEP 1.** *It is enough to find a dashed arrow making the following diagram commute:*

\[
\begin{array}{ccc}
\mathcal{F}_e & \xrightarrow{\phi_{e,e'}} & \mathcal{F}_{e'} \\
\alpha_e & \searrow & \xi_{e,e'} \\
\mathcal{G}_e & \nearrow & \mathcal{G}_{e'}
\end{array}
\]

Indeed, then we have the commutativity of the following two diagrams:

\[
\begin{array}{cccc}
\mathcal{H}^0(\mathcal{F}_e) & \xrightarrow{\mathcal{H}^0(\phi_{e,e'})} & \mathcal{H}^0(\mathcal{F}_{e'}) & \xrightarrow{\mathcal{H}^0(\phi_{e,e'}')} \\
\mathcal{G}_e & \searrow & \mathcal{G}_{e'} & \nearrow \\
\mathcal{G}_e & \nearrow & \mathcal{G}_{e'}
\end{array}
\]

and

\[
\begin{array}{cccc}
\mathcal{H}^0(\mathcal{F}_e) & \xrightarrow{\mathcal{H}^0(\alpha_e)} & \mathcal{H}^0(\mathcal{F}_{e'}) & \xrightarrow{\mathcal{H}^0(\alpha_{e'})} \\
\mathcal{G}_e & \searrow & \mathcal{G}_{e'} & \nearrow \\
\mathcal{G}_e & \nearrow & \mathcal{G}_{e'}
\end{array}
\]

In particular, the surjectivity of the left arrows of the diagrams of (9) shows that the bottom triangle of diagram (10) commutes. Combining this with the commutativity of the upper triangle of diagram (10), we obtain the commutativity of (8):

\[
\begin{array}{cccc}
\mathcal{H}^0(\mathcal{F}_e) & \xrightarrow{\mathcal{H}^0(\phi_{e,e'})} & \mathcal{H}^0(\mathcal{F}_{e'}) & \xrightarrow{\mathcal{H}^0(\phi_{e,e'}')} \\
\mathcal{G}_e & \searrow & \mathcal{G}_{e'} & \nearrow \\
\mathcal{G}_e & \nearrow & \mathcal{G}_{e'}
\end{array}
\]

**STEP 2:** *The cone of $\mathcal{F}_e \rightarrow \mathcal{G}_e$ is nilpotent in all cohomological degrees:* for each $e \geq 0$, fix complexes forming the following exact triangles

\[
\begin{array}{cccc}
\mathcal{K}_e & \xrightarrow{\iota_e} & \mathcal{F}_e & \xrightarrow{\alpha_e} & \mathcal{G}_e & \xrightarrow{+1}\\
\psi_{e,e+1} & \downarrow & \phi_{e+1} & \downarrow & \eta_{e+1} & \downarrow \\
\mathcal{K}_{e+1} & \xrightarrow{\iota_{e+1}} & \mathcal{F}_{e+1} & \xrightarrow{\alpha_{e+1}} & \mathcal{G}_{e+1} & \xrightarrow{+1}
\end{array}
\]  

(11)

and fix also arrows $\psi_{e,e+1}$ making the following diagram commute:

\[
\begin{array}{cccc}
\mathcal{K}_e & \xrightarrow{\iota_e} & \mathcal{F}_e & \xrightarrow{\alpha_e} & \mathcal{G}_e & \xrightarrow{+1}\\
\psi_{e,e+1} & \downarrow & \phi_{e+1} & \downarrow & \eta_{e+1} & \downarrow \\
\mathcal{K}_{e+1} & \xrightarrow{\iota_{e+1}} & \mathcal{F}_{e+1} & \xrightarrow{\alpha_{e+1}} & \mathcal{G}_{e+1} & \xrightarrow{+1}
\end{array}
\]  

(12)

This defines an inverse system $(\mathcal{K}_e)_{e \geq 0}$, and a morphism of inverse systems $(\iota_e)_{e \geq 0} : (\mathcal{K}_e)_{e \geq 0} \rightarrow (\mathcal{F}_e)_{e > 0}$. Define $\psi_{e,e'} := \psi_{e+1,e'} \circ \cdots \circ \psi_{e,e+1} : \mathcal{K}_e \rightarrow \mathcal{K}_{e'}$, and consider the corresponding commutative diagram:

\[
\begin{array}{cccc}
\mathcal{K}_e & \xrightarrow{\iota_e} & \mathcal{F}_e & \xrightarrow{\alpha_e} & \mathcal{G}_e & \xrightarrow{+1}\\
\psi_{e,e'} & \downarrow & \phi_{e,e'} & \downarrow & \eta_{e,e'} & \downarrow \\
\mathcal{K}_{e'} & \xrightarrow{\iota_{e'}} & \mathcal{F}_{e'} & \xrightarrow{\alpha_{e'}} & \mathcal{G}_{e'} & \xrightarrow{+1}
\end{array}
\]  

(12)

By point (a) of Lemma 3.5, we obtain that $(\mathcal{K}_e)_{e \geq 0}$ is nilpotent in all cohomology degrees $i$, except possibly $i = 0$ and $i = 1$. For $i = 1$, we have even $\mathcal{H}^1(\mathcal{K}_e) = 0$ by assumption (a) and by the surjectivity of $\mathcal{H}^0(\alpha_e)$. 

\[\text{Springer}\]
So, to finish we are left to show that $(K_e)_{e \geq 0}$ is also nilpotent in cohomological degree $i = 0$. This is a diagram chase which we explain in the rest of Step 2. For that, denote by upper index $i$ the homomorphisms obtained by applying $H^i(\_)\cup$ to the maps of diagram (12). Additionally, denote by $\gamma^j_e : H^j(G_e) \to H^{j+1}(K_e)$ the edge homomorphisms of the rows of diagram (12). Fix then, $x \in H^0(K_e)$. We show that $\psi_{e,e'}(x) = 0$ for $e' \gg e$. Indeed, $\alpha^0_e(\gamma^0_e(x)) = 0$ by the exactness of the rows of (12). Hence, by the definition of $G_e$, this means that $\phi^0_{e,e'}(\gamma^0_e(x)) = 0$ for $e' \gg 0$. However, then by the commutativity of (12), we obtain that $i^0_e(\psi^0_{e,e'}(x)) = 0$. By the exactness of the rows of (12) and by the fact that $H^{-1}(G_e) = 0$, this means that $\psi^0_{e,e'}(x) = 0$ holds, which concludes the proof of Step 2.

**Step 3:** Concluding the argument.

According to Step 1, it is enough to lift $\phi_{e,e'}$ to an element of $\text{Hom}_X(G_e, F_e)$ for $e' \gg e$. This is shown to be true by the following exact sequence:

\[ \begin{array}{cccccc}
\psi_{e,e'} = 0 \\
\text{by Step 2} & \text{and Lemma} & 3.4
\end{array} \]

\[
\begin{array}{ccc}
\phi_{e,e'} & \longrightarrow & \phi_{e,e'} \circ \iota_e = (e' \circ \psi_{e,e'}) = 0 \\
\downarrow & & \downarrow \\
\text{Hom}_X(G_e, F_e) & \longrightarrow & \text{Hom}_X(F_e, F_e') & \longrightarrow & \text{Hom}_X(K_e, F_e') \\
\end{array}
\]

Lemma 3.7 Let $X$ be a variety over $k$. If $(D_e : \alpha_e : D_{e+1} \to D_e)_{e \geq 0}$ is an inverse system associated to a derived Cartier module, which is nilpotent in all cohomological degrees, then $(D_e)_{e \geq 0}$ is nilpotent.

**Proof** By Definition 3.1, $D_e$ are supported in the same cohomological degrees, say in the interval $[c, d]$, for some integers $c < d$. It is enough to show that the direct system $(D_X(D_e))_{e \geq 0}$ is nilpotent. As $(H^j(D_e))_{e \geq 0}$ are nilpotent, we know that so are the direct systems $(D_X(H^j(D_e)))_{e \geq 0}$. Hence, $(D_X(H^j(D_e)))_{e \geq 0}$ are also nilpotent in every cohomological degree. Applying then point (a) of Lemma 3.5 to the following exact triangles inductively for $c \leq j \leq d$ implies that the direct system $(D_X(D_e))_{e \geq 0}$ is nilpotent in all cohomological degrees.

\[
\begin{array}{cccc}
D_X(H^j(D_e))[j] & \longrightarrow & D_X(\tau_\leq j D_e) & \longrightarrow & D_X(\tau_\leq j-1 D_e) \oplus 1 \\
\end{array}
\]

Lemma 3.4 then concludes our proof. \qed

4 Frobenius stable cohomology support loci

In this section we prove Theorem 1.1.

**Notation 4.1** Let $F^e_* \Omega_0 \to \Omega_0$ be a Cartier module on $A$. We set for every integer $e \geq 0$:

- $\Omega_e := F^e_* \Omega_0$.
- $\Lambda_e := R\mathcal{S}(D_A(\Omega_e)) \cong \mathcal{V}^{e\cdot \ast} \Lambda_0$, and
\[
\Lambda := \hocolim_{\varepsilon} \Lambda_{\varepsilon}.
\]

According to [10, Cor 3.1.4], \( \Lambda \) is supported in cohomological degree 0, that is, it is quasi-isomorphic to a sheaf, or equivalently \( \Lambda \cong \lim_{\to} \mathcal{H}^0(\Lambda_{\varepsilon}) \). In particular, we may set the following notation:

- \( \tilde{\Lambda}_{\varepsilon} := \text{Im}(\mathcal{H}^0(\Lambda_{\varepsilon}) \to \Lambda) \),
- \( \tilde{\Omega}_{\varepsilon} := (-1)^{\varepsilon} D_{\Lambda} RS(\tilde{\Lambda}_{\varepsilon})[-1] = RS(D_{\Lambda}(\tilde{\Lambda}_{\varepsilon})) \),
- \( \text{Ass}_{\varepsilon} := \{ \xi \in \hat{A} \mid \xi \text{ is an embedded point of } \tilde{\Lambda}_{\varepsilon} \} \), and
- \( \text{Ass} := \bigcup_{\varepsilon} \text{Ass}_{\varepsilon} \).

Lemma 4.2 implies that the inverse systems \( \left( R^j \hat{S}(\Omega_{\varepsilon}) \right)_{\varepsilon \geq 0} \) and \( \left( R^j \hat{S}(\tilde{\Omega}_{\varepsilon}) \right)_{\varepsilon \geq 0} \) become naturally isomorphic when passing to the inverse limits.

**Lemma 4.2** In the situation of Notation 4.1, there is a monotone increasing sequence \((e_i)_{i \geq 0}\) such that there are morphisms of direct systems given by \( \alpha_i : \Lambda_{e_i} \to \tilde{\Lambda}_{e_i} \) and \( \beta_i : \Lambda_{e_i} \to \Lambda_{e_{i+1}} \), such that both \((\alpha_{i+1})_{i \geq 0} \circ (\beta_{i})_{i \geq 0}\) and \((\beta_{i})_{i \geq 0} \circ (\alpha_{i})_{i \geq 0}\) are shifts by 1 in the positive direction of the index \( i \). We may also choose \( i_0 = 0 \).

**Proof** [10, Cor 3.1.4] tells us that we may apply Proposition 3.6 to the direct system \((\Lambda_{e_i})_{e \geq 0}\). That is, we can define \( e_i \) inductively so that the following diagram commutes:

\[
\begin{array}{c}
\Lambda_{e_0} & \longrightarrow & \Lambda_{e_1} & \longrightarrow & \Lambda_{e_2} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\tilde{\Lambda}_{e_0} & \longrightarrow & \tilde{\Lambda}_{e_1} & \longrightarrow & \tilde{\Lambda}_{e_2} & \longrightarrow & \\
\end{array}
\]

This concludes our proof. \( \square \)

**Lemma 4.3** In the situation of Notation 4.1, let \( \phi : \hat{A} \to \hat{A} \) be the multiplication by \( p^s \) isogeny. Then:

(a) \( \forall \xi \in \text{Ass}_{\varepsilon}, \forall \eta \in \phi^{-1}(\xi) : \eta \in \text{Ass}_{\varepsilon+1} \).

(b) \( \forall \xi \in \text{Ass}_{\varepsilon+1} : \phi(\xi) \in \text{Ass}_{\varepsilon} \), and

(c) \( \text{If } \xi \in \text{Ass}_{\varepsilon}, \text{ then } \xi \in \text{Ass}_{\varepsilon+1} \).

**Proof** Topologically \( V^s \) and \([p]^s\) agree, hence we may redefine \( \phi \) to be \( V^s \). Then, we have \( V^s, \ast \tilde{\Lambda}_{\varepsilon} \cong \tilde{\Lambda}_{e_{i+1}} \) [10, Lem 2.3.1, Lem 2.4.5]. Additionally, as \( V^s \) is faithfully flat, it follows that the embedded points of \( \tilde{\Lambda}_{e_{i+1}} \) are exactly the points lying over embedded points of \( \tilde{\Lambda}_{\varepsilon} \), via \( V^s \) [5, Prop 6.3.1]. This yields points (a) and (b). Point (c) follows from the fact that we have natural injection \( \tilde{\Lambda}_{\varepsilon} \hookrightarrow \tilde{\Lambda}_{e_{i+1}} \). \( \square \)

**Corollary 4.4** In the situation of Notation 4.1, if \( \Lambda \) has no supersingular factors, then for any integer \( e > 0 \) there exist integers \( k_0, k_1 > 0 \) such that for every \( \xi \in \text{Ass}_{\varepsilon} \), there is an abelian subvariety \( \hat{W} \subseteq \hat{A} \) and a \( p^{k_0s}(p^{1+5s} - 1) \)-torsion point \( Q \in \hat{A} \), such that \( \xi = \hat{W} + Q \).

**Proof** Fix \( \xi \in \text{Ass}_{\varepsilon} \). Points (b) and (c) of Lemma 4.3, yield that \( p^s(\text{Ass}_{\varepsilon}) \subseteq \text{Ass}_{\varepsilon} \). Since there are only finitely many points in \( \text{Ass}_{\varepsilon} \), and \( \text{Ass}_{\varepsilon} \supseteq p^s(\text{Ass}_{\varepsilon}) \supseteq p^{2s}(\text{Ass}_{\varepsilon}) \ldots \), we may fix \( k_0 > 0 \) such that

\[
p^{k_0s}\text{Ass}_{\varepsilon} = \cap_{i \geq 1} p^i s\text{Ass}_{\varepsilon}
\]
and in particular \( p^g(p^{k_0\mathcal{S}} \mathcal{Ass}_e) = p^{k_0\mathcal{S}} \mathcal{Ass}_e \). We may then pick \( k_1 > 0 \) such that \( p^{k_1\mathcal{S}}(\xi') = \xi' \) for any \( \xi' \in p^{k_0\mathcal{S}} \mathcal{Ass}_e \) and hence \( p^{(k_1+k_0)\mathcal{S}}(\xi) = p^{k_0\mathcal{S}} \xi \) for any \( \xi \in \mathcal{Ass}_e \). \cite[Thm III.3.4.b]{10} yields that if \( \xi \in p^{k_0\mathcal{S}} \mathcal{Ass}_e \), then \( p^{k_0\mathcal{S}} \xi = \hat{W} + Q \), except that \( Q \) might be arbitrary torsion point. However, then

\[
p^{k_1\mathcal{S}}(\hat{W} + Q) = p^{k_1\mathcal{S}} \left( p^{k_0\mathcal{S}} \xi \right) = p^{k_0\mathcal{S}} \xi = \hat{W} + Q \implies (p^{k_1\mathcal{S}} - 1)Q \in \hat{W}.
\]

Hence, there is \( Q' \in \hat{W} \), such that \( (p^{k_1\mathcal{S}} - 1)Q' = -(p^{k_1\mathcal{S}} - 1)Q \). For this \( Q' \) we have \( \hat{W} + Q = \hat{W} + (Q + Q') \), and also \( (p^{k_1\mathcal{S}} - 1)(Q + Q') = 0 \). Therefore, replacing \( Q \) by \( Q + Q' \) we may assume that \( (p^{k_1\mathcal{S}} - 1)Q = 0 \).

Hence, we have \( p^{k_0\mathcal{S}} \xi = p^{k_0\mathcal{S}} \xi = \hat{W} + Q \), where \( (p^{k_1\mathcal{S}} - 1)Q = 0 \). Thus \( \xi = \hat{W} + R \) where \( (p^{k_1\mathcal{S}} - 1)p^{k_0\mathcal{S}}R = 0 \).

**Notation 4.5** In the situation of Notation 4.1, fix an integer \( e \geq 0 \) and an arbitrary (scheme theoretic) point \( \eta \in \bar{A} \). Set:

- \( -\eta := (-1)\hat{\eta} \),
- \( j := \text{codim} \eta \),
- \( d := \begin{cases} j + 1, & \text{if } \bar{A}_{e,-\eta} = 0 \\ \text{depth}_{\bar{A}_{e,-\eta}} \bar{A}_{e,-\eta}, & \text{otherwise} \end{cases} \)
- \( \mathcal{P}_\eta := \mathcal{P}|_{A \times \eta} \),
- \( \text{pr}_\eta : A \times \eta \rightarrow A \)

**Lemma 4.6** In the situation of Notation 4.5, the following holds:

(a) \( R^i\hat{S}(\tilde{\mathcal{O}}_e) \otimes k(\eta) = 0 \) for every integer \( i > j - d \),

(b) if \( \bar{A}_{e,-\eta} \neq 0 \), then \( R^{j-d}\hat{S}(\tilde{\mathcal{O}}_e) \otimes k(\eta) \neq 0 \), and

(c) if \( d = 0 \), then \( \lim_{r \to 0} \left( R^j\hat{S}(\tilde{\mathcal{O}}_e) \otimes k(\eta) \right) \) is a non-zero \( k(\eta) \)-vector space.

**Proof** \textbf{Step 1.} \textit{The proof of points (a) and (b):} First, note that

\[
R\hat{S}(\tilde{\mathcal{O}}_e) \xrightarrow{\text{definition of } \tilde{\mathcal{O}}_e} R\hat{S}(R\mathcal{S}(D_{\bar{A}}(\hat{\mathcal{A}}))) \xrightarrow{\text{Theorem 2.1}} (-1)^* D_{\bar{A}}(\hat{\mathcal{A}})[-g] = (-1)^* R\mathcal{H}\text{om}(\hat{\mathcal{A}}, \mathcal{O}_{\bar{A}}).
\]

(13)

This implies the following, where it is useful to recall that the Matlis duality functor is faithful and exact, and hence the Matlis-dual of a non-zero sheaf is non-zero:

\[
(-1)^*_{\eta} R^i\hat{S}(\tilde{\mathcal{O}}_e)_{\eta} \cong \text{Ext}^i_{\bar{A}_{e,-\eta}}(\hat{\mathcal{A}}_{e,-\eta}, \mathcal{O}_{\bar{A}_{e,-\eta}}) \cong \left( H^{j-i}_{\bar{A}_{e,-\eta}}(\hat{\mathcal{A}}_{e,-\eta}) \right)_{\text{Matlis dual}}
\]

(14)

Note that \( (-1)^*_{\eta} \) is also fully faithful, as \( -1 \) is an isomorphism. Hence, Eq. (14) yields directly points (a) and (b) by the characterization of depth via the vanishing/non-vanishing of local cohomology \cite[Exc III.3.4.b]{12}.

\( \square \) Springer
**STEP 2.** If \( d = 0 \), then \( 0 \neq \lim_{r} (R^j \hat{S}(\tilde{\Omega}_r)_\eta) \). By (14), it is enough to show that

\[
0 \neq \lim_{r} H^0_{m_{\tilde{\lambda}, -\eta}}(\tilde{\Lambda}_r, -\eta) = H^0_{m_{\tilde{\lambda}, -\eta}}(\lim_{r} \tilde{\Lambda}_r, -\eta) = H^0_{m_{\tilde{\lambda}, -\eta}}(\Lambda, -\eta)
\]  

(15)

direct limits commute with local cohomology

direct limits commute with localization

However, this is true, as \( \tilde{\Lambda}_{e, -\eta} \to \Lambda, -\eta \) is an injection, and \( H^0_{m_{\tilde{\lambda}, -\eta}}(\tilde{\Lambda}_{e, -\eta}) \neq 0 \) by the \( d = 0 \) assumption.

**STEP 3.** If \( d = 0 \), then \( R^j \hat{S}(\tilde{\Omega}_r)_\eta \) is an Artinian \( \mathcal{O}_{\tilde{\lambda}, \eta} \)-module for every integer \( \mathbb{R} \geq 0 \); for this it is enough to show that \( R^j \hat{S}(\tilde{\Omega}_r)_\eta \) is supported on \( m_{\tilde{\lambda}, \eta} \) or equivalently that if \( \xi \in \tilde{\lambda} \) is a generalization of \( \eta \), then \( R^j \hat{S}(\tilde{\Omega}_r)_\xi = 0 \). However, in this case \( \text{codim} \xi < j \), and hence by applying Step 1 to \( \xi \) instead of \( \eta \) we obtain this vanishing.

**STEP 4.** Concluding the argument: By Step 3, for every integer \( r \geq 0 \), there is a unique \( \mathcal{O}_{\tilde{\lambda}, \eta} \)-submodule \( M_r \subseteq R^j \hat{S}(\tilde{\Omega}_r)_\eta \) such that for every \( s \gg r \):

\[
M_r = \text{im} \left( R^j \hat{S}(\tilde{\Omega}_s)_\eta \to R^j \hat{S}(\tilde{\Omega}_r)_\eta \right).
\]

Hence, we may find a strictly monotone sequence \( 0 = r_0 < r_1 < \ldots \), such that we have a commutative diagram as follows

\[
\begin{array}{cccccc}
M_{r_0} & \xrightarrow{f} & M_{r_1} & \xrightarrow{f} & M_{r_2} & \rightarrow \ldots \\
\downarrow & & \downarrow & & \downarrow \\
R^j \hat{S}(\tilde{\Omega}_{r_0})_\eta & \xrightarrow{f} & R^j \hat{S}(\tilde{\Omega}_{r_1})_\eta & \xrightarrow{f} & R^j \hat{S}(\tilde{\Omega}_{r_2})_\eta & \rightarrow \ldots
\end{array}
\]

(16)

Applying \( \_ \otimes k(\eta) \) yields the following other commutative diagram, using that tensoring is right-exact:

\[
\begin{array}{cccccc}
M_{r_0} \otimes k(\eta) & \xrightarrow{f} & M_{r_1} \otimes k(\eta) & \xrightarrow{f} & M_{r_2} \otimes k(\eta) & \rightarrow \ldots \\
\downarrow & & \downarrow & & \downarrow \\
R^j \hat{S}(\tilde{\Omega}_{r_0}) \otimes k(\eta) & \xrightarrow{f} & R^j \hat{S}(\tilde{\Omega}_{r_1}) \otimes k(\eta) & \xrightarrow{f} & R^j \hat{S}(\tilde{\Omega}_{r_2}) \otimes k(\eta) & \rightarrow \ldots
\end{array}
\]

(17)

As Eq. (16) yields a relation as in Lemma 4.2 between the inverse systems \( \left( R^j \hat{S}(\tilde{\Omega}_r)_\eta \right)_{r \geq 0} \) and \( (M_r)_{r \geq 0} \). Hence, we have \( \lim_{r} (R^j \hat{S}(\tilde{\Omega}_r)_\eta) = \lim_{r} M_r \). Combining this with Step 2 we obtain \( 0 \neq \lim_{r} M_r \). In particular, \( M_r \neq 0 \) for all \( r \gg 0 \). Hence, using that \( M_{r+1} \otimes k(\eta) \to M_r \otimes k(\eta) \) is surjective, \( 0 \neq \lim M_r \otimes k(\eta) \). Then, as diagram (17) yields also a relation as in Lemma 4.2, we obtain point (c).

For the next statement recall that when working with elements of \( D(X) \) and its variants, we also use the abbreviation \( H^i(\_ , \_ ) = R^i \Gamma(\_ , \_ ) \).

**Corollary 4.7** In the situation of Notation 4.5, we have:

\[
H^i \left( A \times \eta, \mathcal{P}_\eta \otimes \text{pr}_\eta^* \tilde{\Omega}_e \right) \cong R^j \hat{S}(\tilde{\Omega}_e) \otimes k(\eta) \cong \begin{cases} 0 & \text{if } i > j - d \\ \neq 0 & \text{if } i = j - d \text{ and } \tilde{\Lambda}_{e, -\eta} \neq 0 \end{cases}
\]

(18)
Proof Lemma 4.6 shows the second isomorphism of (18). So, we are left to show that for \(i > j - d\) the left hand side of (18) is 0, and for \(i = j - d\) it is isomorphic to \(R^j \hat{\Sigma}(\Omega_e) \otimes k(\eta)\).

Let \(\iota : \eta \to \hat{\Lambda}, j : A \times \eta \to A \times \hat{\Lambda}\) and \(q : A \times \hat{\Lambda} \to A\) be the natural morphisms. Then:

\[
L^i R^j \hat{\Sigma}(\Omega_e) \cong RT\left(A \times \eta, L^j (\mathcal{P} \otimes q^* \hat{\Omega}_e)\right) \cong RT\left(A \times \eta, \mathcal{P}_\eta \otimes pr^*_\eta \hat{\Omega}_e\right)
\]  

(19)

According to Lemma 4.6, for every integer \(i > j - d\), we have \(R^i \hat{\Sigma}(\Omega_e)_\eta = 0\). Hence, there is an open neighborhood \(\eta \in U \subseteq \hat{\Lambda}\) and a distinguished triangle

\[
\tau_{<j-d} R^j \hat{\Sigma}(\Omega_e)|_U \to R^j \hat{\Sigma}(\Omega_e)|_U \to R^{i-d} \hat{\Sigma}(\Omega_e)|_{d-j}|_U \xrightarrow{+1}
\]  

(20)

Note that as \(\tau_{<j-d} R^j \hat{\Sigma}(\Omega_e)\) is supported in cohomological degrees smaller than \(j - d\), equations (19) and (20) imply that for every integer \(i \geq j - d\)

\[
H^i\left(A \times \eta, \mathcal{P}_\eta \otimes pr^*_\eta \hat{\Omega}_e\right) \cong L^i \iota^*\left(R^{j-d} \hat{\Sigma}(\Omega_e)|_{d-j}\right) = L^{i-(j-d)} \iota^*\left(R^{i-d} \hat{\Sigma}(\Omega_e)\right).
\]  

(21)

Using that \(L^s \iota^*(\_\_)\) applied to a sheaf is zero for \(s > 0\) and it is the same as the application of \((\_\_) \otimes k(\eta)\) for \(s = 0\), we obtain from (21) exactly the description of the left hand side of (18) we were looking for.

\(\square\)

Corollary 4.8 In the situation of Notation 4.5, define \(W^i := \bigcup_{r \geq i} \text{Supp } R^r \hat{\Sigma}(\Omega_e)\). Then, for every integer \(0 \leq i \leq g\) there are the following two options:

(a) if \(\eta \notin W^i\), then \(H^i\left(A \times \eta, \mathcal{P}_\eta \otimes pr^*_\eta \hat{\Omega}_e\right) = 0\) and \(j - d < i\), and

(b) if \(\eta \in W^i \setminus W^{i+1}\), then \(H^i\left(A \times \eta, \mathcal{P}_\eta \otimes pr^*_\eta \hat{\Omega}_e\right) \cong R^i \hat{\Sigma}(\Omega_e) \otimes k(\eta) \neq 0\), and \(j - d = i\).

Proof One way to read Corollary 4.7 is that as we descend with the index \(i\), starting from \(i = g\), if ever the cohomology group \(H^i\left(A \times \eta, \mathcal{P}_\eta \otimes pr^*_\eta \hat{\Omega}_e\right)\) becomes non-zero, then the first time this happens at \(i = j - d\). Additionally, also by Corollary 4.7, for this value of \(i\) we have \(\eta \in W^i \setminus W^{i+1}\). This is worded precisely in the present corollary, taken into account that we have \(W^g \subseteq W^{g-1} \subseteq \cdots\), and with the added expression for \(H^{j-d}(A \times \eta, \mathcal{P}_\eta \otimes pr^*_\eta \hat{\Omega}_e)\) coming from Corollary 4.7.

\(\square\)

Proof of Theorem 1.1 In all applications of the statements of the present section we set \(e = 0\), except for the proof of point (e), where this will be stated carefully. Define \(W^i\) as in Corollary 4.8. Set \(t\) to be the \(e_1\) of Lemma 4.2. That is, we have a factorization

\[
\Omega_t = F^{*s}_t \Omega_0 \xrightarrow{\Omega_0} \Omega_0
\]

Taking cohomology we obtain homomorphisms

\[
H^i(A, \mathcal{P}_y \otimes F^{*s}_t \Omega_0) \to H^i(A, \mathcal{P}_y \otimes \hat{\Omega}_0) \to H^i(A, \mathcal{P}_y \otimes \Omega_0).
\]
By point (a) of Corollary 4.8, the middle term vanishes whenever \( \eta \in \hat{\Lambda} \setminus W^t \). This implies point (b) of the present theorem.

Point (b) of Corollary 4.8 implies that for every \( \eta \in W^i \setminus W^{i+1} \) we have \( j = \text{codim} \eta \geq i \). Hence, \( i \leq \text{codim} (W^i \setminus W^{i+1}) \). Then, by downwards induction on \( i \) it follows that \( i \leq \text{codim} W^i \). This is the statement of point (a) of the present theorem.

Let us assume for the present paragraph that \( \eta \) is the generic point of an irreducible component of \( W^i \) of codimension \( i \). This is equivalent to assuming that \( \text{codim} \eta = i \). Then, as we have already showed point (a), we have \( \eta \notin W^r \) for \( r > i \). Hence, by point (b) of Corollary 4.8, we obtain that \( 0 = \text{depth}_{\mathcal{O}_{\hat{\Lambda} \setminus \eta}} \hat{\Lambda}_{0, \eta} = 0 \). That is, \(-\eta\) is an associated prime of \( \hat{\Lambda}_0 \). Note now that by Corollary 4.7, cohomology and base-change holds at \( \eta \) for \( \mathcal{F}_e := R^j \text{pr}_\ast (\mathcal{P} \otimes \mathcal{Q}_e^\ast \hat{\Omega}_e) \), where \( \text{pr} : A \times \hat{\Lambda} \to \hat{\Lambda} \) and \( q : A \times \hat{\Lambda} \to A \) are the projections. Then, it also holds for very general closed points of \( \bar{\eta} \), for every integer \( e \geq 0 \). Point (c) of Lemma 4.6 also states that \( \lim_{\leftarrow e} \big( \mathcal{F}_e \otimes \mathcal{k}(\eta) \big) = 0 \), which then holds for \( \eta \) replaced by very general closed points of \( \bar{\eta} \).

That is, we obtain that for very general closed point \( y \in \bar{\eta} \) we have \( 0 = \lim_{\leftarrow e} H^i \big( A, \mathcal{P}_y \otimes \hat{\Omega}_e \big) \).

Then, by Lemma 4.2 we obtain that \( 0 \neq \lim_{\leftarrow e} H^i \big( A, \mathcal{P}_y \otimes \hat{\Omega}_e \big) \) holds too. This concludes the proof of point (c).

To obtain point (d), take \( \eta \) as in the situation of the previous paragraph. As we have seen there, \(-\eta\) is an associated point of \( \hat{\Lambda}_0 \). Hence, by Corollary 4.4, we obtain that \( \bar{\eta} \) is the torsion translate of an abelian subvariety.

To prove point (e) of the theorem let us introduce the notation \( W^i_e \), which is defined as \( W^i \) but for arbitrary \( e \) instead of just \( e = 0 \). As \( R^j \hat{S}(\hat{\Omega}_e) \cong R^j \hat{S}(F^es_\ast \hat{\Omega}_0) \cong V^{es, \ast} R^j \hat{S}(\hat{\Omega}_0) \), we have \( W^i_e = V^{es, \ast} W^i = \left[ p^{es} \right]^{-1} W^i \). Hence, points (b) and (c), which we already proved, hold for \( \hat{\Omega}_0 \) replaced by \( \Omega_e \) if we also replace \( W^i \) by \( \left[ p^{es} \right]^{-1} W^i \). This \( \Omega_e \) version of point (c) yields the non-vanishing part of point (e) directly. Additionally, by taking inverse limits of the \( \Omega_e \) versions of point (b) we obtain the vanishing part of point (e).

Point (f) follows directly from Corollary 4.8, using that being an embedded point of \( \Omega_0 \) is the same as \( \text{depth}_{\mathcal{O}_{\hat{\Lambda} \setminus \eta}} \hat{\Lambda}_{0, \eta} = 0 \) being zero.

\[ \square \]

5 \( V \)-modules and examples

The main purpose of this section is to show that many properties of cohomology support loci of Cartier modules on \( \hat{\Lambda} \) can be understood by understanding \( V \)-modules on \( \hat{\Lambda} \), which are defined in the next definition.

**Definition 5.1** A \( V \)-module is a pair \( (\mathcal{M}, \phi) \), where \( \mathcal{M} \) is a coherent sheaf on \( \hat{\Lambda} \), and \( \phi : \mathcal{M} \to V^{s, \ast} \mathcal{M} \) is an \( \mathcal{O}_{\hat{\Lambda}} \)-homomorphism for some integer \( s > 0 \). A \( V \)-module is injective if the structure homomorphism \( \phi \) is injective.

The main technical statement is the following, which can be thought of as the dual of [10, Thm 3.1.1].

**Theorem 5.2** If \( \mathcal{M} \to V^{s, \ast} \mathcal{M} \) is a \( V \)-module, then \( \left( RS(D^*_\hat{\Lambda}(V^{es, \ast} \mathcal{M})) \cong F^es RS(D^*_\hat{\Lambda}(\mathcal{M})) \right)_{e \geq 0} \) is an inverse system associated to a derived Cartier module, see Definition 3.1, and it is nilpotent in all cohomological degrees outside 0.
First we note that the isomorphism \( RS(\hat{D}_A(V^{es.\ast}M)) \cong F^e_s RS(\hat{D}_A(M)) \) is obtained by applying Theorem 2.1 multiple times:

\[
RS(\hat{D}_A(V^{es.\ast}M)) \cong (-1)^* D_A(RS(V^{es.\ast}M))[g] \cong (-1)^* D_A(F^e_s RS(M))[g] \\
\cong F^e_s (-1)^* D_A(RS(M))[g] \cong F^e_s RS(\hat{D}_A(M)). \tag{22}
\]

Set then:

\[
\mathcal{M}_e := V^{es.\ast}M, \quad \text{and} \quad \mathcal{K}_e := RS(\hat{D}_A(\mathcal{M}_e)) \cong F^e_s \mathcal{K}_0.
\]

As \((\mathcal{M}_e)_{e \geq 0}\) is associated to a \(V\)-module, and as the isomorphisms of (22) are functorial, \((\mathcal{K}_e)_{e \geq 0}\) is associated to the derived Cartier module \(\mathcal{K}_1 \cong F^s_e \mathcal{K}_0 \to \mathcal{K}_0\). Note that \(\mathcal{K}_e\) are supported in the same cohomological degrees, which lie in the interval \([-g, 0]\). So, assume that \((\mathcal{K}_e)_{e \geq 0}\) is not nilpotent in all cohomological degrees outside of 0. Let \(j\) be the lowest cohomological degree in which \((\mathcal{K}_e)_{e \geq 0}\) is not nilpotent. By our assumption, to which we would like to find a contradiction, we have \(j < 0\). Consider then the following exact triangles for every \(i\) and \(e\):

\[
\begin{array}{ccc}
R\hat{S}(D_A(\tau_{\geq i+1} \mathcal{K}_e)) & \rightarrow & R\hat{S}(D_A(\tau_{\geq i} \mathcal{K}_e)) \\
& \rightarrow & R\hat{S}(D_A(\mathcal{H}^i(\mathcal{K}_e)))[i] \rightarrow 1
\end{array}
\]

We claim that the direct system \(\left( R\hat{S}(D_A(\mathcal{H}^i(\mathcal{K}_e)))[i] \right)_{e \geq 0}\) is nilpotent outside of cohomological degrees (strictly) higher than \(-j\). Indeed, as \((\mathcal{H}^i(\mathcal{K}_e))_{e \geq 0}\) is an inverse system associated to a Cartier module, by [10, Thm 3.1.1] we obtain that \(\left( R\hat{S}(D_A(\mathcal{H}^i(\mathcal{K}_e))) \right)_{e \geq 0}\) is nilpotent outside of cohomological degree zero. This solves the claim for \(i \geq j\). For \(i < j\) on the other hand \((\mathcal{H}^i(\mathcal{K}_e))_{e \geq 0}\) itself is nilpotent by the choice of \(j\), and hence so is \(\left( R\hat{S}(D_A(\mathcal{H}^i(\mathcal{K}_e))) \right)_{e \geq 0}\) in all cohomological degrees. This concludes our claim.

By induction on \(i\), using our above claim, point (a) of Lemma 3.5 and the exact triangles (23), we obtain that \(\left( R\hat{S}(D_A(\tau_{\geq j} \mathcal{K}_e)) \right)_{e \geq 0}\) is nilpotent in cohomological degree \(-j + 1\) for all \(i\). Additionally, by the arguments with which we proved the above claim, using [10, Thm 3.1.1] again, we see that \(\left( R\hat{S}(D_A(\mathcal{H}^j(\mathcal{K}_e))) \right)_{e \geq 0}\) is not nilpotent in cohomological degree \(-j\). Applying (b) of Lemma 3.5 to the triangle (23) for \(i \leq j\) we obtain inductively that \(\left( R\hat{S}(D_A(\tau_{\geq j} \mathcal{K}_e)) \right)_{e \geq 0}\) is not nilpotent in cohomological degree \(-j\) for every integer \(i \leq j\).

For \(i = -g\) this yields that \(\left( R\hat{S}(D_A(\mathcal{K}_e)) \right)_{e \geq 0}\cong (\mathcal{M}_e)_{e \geq 0}\) is not nilpotent in cohomological degree \(-j > 0\). This is a contradiction as \(\mathcal{M}\) is a sheaf. \(\square\)

The next corollary states that many features of cohomology support loci of Cartier modules on \(A\) can be understood by just understanding \(V\)-modules on \(\hat{A}\), from which the corresponding properties can be read off directly. An example precise statement in Corollary 5.3 is that this works for understanding the behavior of the loci

\[
W^0_P = \left\{ Q \in \hat{A} \mid \lim_{e} H^0(A, \Omega_{0} \otimes Q^P) \neq 0 \right\}
\]

for any Cartier module \(\Omega_{0}\) on \(A\). In fact, Corollary 5.3 collects only some of the properties that can be read off directly from \(V\)-modules. We hope this list will be further extended in later articles.
Corollary 5.3 If $\mathcal{M} \to V^{s,*}\mathcal{M}$ is an injective $V$-module, then we have an embedding of $V$-modules,

$$\tilde{\Lambda}_e \hookrightarrow V^{s,*} \tilde{\Lambda}_e = \tilde{\Lambda}_1$$

$$\mathcal{M} \hookrightarrow V^{s,*} \mathcal{M},$$

where $\tilde{\Lambda}_e$ is as in Notation 4.1 by setting $\Omega_0 := \mathcal{H}^0(RS(D_A(\mathcal{M})))$, and additionally we have $\lim_{e \to -\infty} V^{e,0} \mathcal{M} = \lim_{e \to -\infty} \tilde{\Lambda}_e$.

In particular,

(a) for every closed point $y \in \hat{A}$ the direct systems $\left(\mathcal{H}^0(\mathcal{M})^\vee\right)_{e \geq 0}$ and $\left(k(y) \otimes V^{e,0} \mathcal{M}\right)_{e \geq 0}$ are equivalent in the sense of Lemma 4.2. Therefore, we have

$$\lim_{e \to -\infty} \mathcal{H}^0(\mathcal{M})^\vee \cong \left(\lim_{e \to -\infty} k(y) \otimes V^{e,0} \mathcal{M}\right)^\vee$$

(b) if $A$ has no supersingular factors, the abelian subvarieties appearing in Theorem 1.1 for the above choice of $\Omega_0$ can be read of from $\mathcal{M}$: the embedded points of $\mathcal{M}$ are torsion translates of exactly these abelian subvarieties.

Proof The main task is to prove the part of the statement before “In particular”. Indeed, assuming everything is proved before “In particular”, point (a) follows from [10, Cor 3.2.1], and point (b) follows from points (d) and (f) of Theorem 1.1. So, we are left to prove the part of the statement before “In particular”.

Set $\mathcal{M}_e := V^{e,0} \mathcal{M}$. For each integer $e \geq 0$, we have an exact triangle:

$$E_e := \tau_{\leq 0} RS(D_A(\mathcal{M}_e)) \to RS(D_A(\mathcal{M}_e)) \to \Omega_e = \mathcal{H}^0(RS(D_A(\mathcal{M}_e))) +1$$

where $E_e := F_\ast^e E_0$ because of the isomorphism $RS(D_A(\mathcal{M}_e)) \cong F_\ast^e (RS(D_A(\mathcal{M}_e)))$ stated in Theorem 5.2. In particular, $E_e$ is nilpotent in all cohomological degrees by Theorem 5.2, and hence according to Lemma 3.7, $(\mathcal{M}_e)_{e \geq 0}$ is nilpotent.

Applying $R\hat{S}(D_A(\_))$ we obtain the exact triangle

$$\Lambda_e := R\hat{S}(D_A(\Omega_e)) \to \mathcal{M}_e \to C_e := R\hat{S}(D_A(E_e)) +1$$

(24)

where $(C_e)_{e \geq 0}$ is nilpotent. Additionally the triangles (24) form a direct system themselves too as we vary the integer $e \geq 0$. Hence, we have a homomorphism $(\mathcal{H}^0(\Lambda_e))_{e \geq 0} \to (\mathcal{M}_e = \mathcal{H}^0(\mathcal{M}_e))_{e \geq 0}$ of direct systems with nilpotent kernel and cokernel. We claim that this induces an injective homomorphism of direct systems $(\beta_e)_{e \geq 0} : (\tilde{\Lambda}_e)_{e \geq 0} \to (\mathcal{M}_e)_{e \geq 0}$ with nilpotent cokernel. Denote by

$$\psi_{e,\infty} : \mathcal{H}^0(\Lambda_e) \to \lim_{-\to} \mathcal{H}^0(\Lambda_{e'})$$

and $\alpha_e : \mathcal{H}^0(\Lambda_e) \to \mathcal{M}_e$ the induced homomorphisms. As the maps $\alpha_e$ induce a homomorphism of direct systems, and as the maps $\mathcal{M}_e \to \mathcal{M}_{e'}$ are injective, we obtain that $\ker \psi_{e,\infty} \subseteq \ker \alpha_e$ for every integer $e \geq 0$. However, as $(\ker \alpha_e)_{e \geq 0}$ is nilpotent, we have in fact $\ker \psi_{e,\infty} = \ker \alpha_e$. It follows then that there is an induced injection

$$\beta_e : \tilde{\Lambda}_e = \mathcal{H}^0(\Lambda_e) / \ker \psi_{e,\infty} = \mathcal{H}^0(\Lambda_e) / \ker \alpha_e \to \mathcal{M}_e$$
that forms the injective homomorphism of direct systems as in the claim. Additionally, by construction it follows that im \( \beta_e = \im \alpha_e \). Hence, the cokernels of \( \beta_e \) form a nilpotent direct system just as for \( \alpha_e \).

Having showed our claim, the direct limit statement of the corollary then follows from the exactness of taking direct limits \([19, \text{Tag 04B0}]\).

**Remark 5.4** Note that the induced \( V \)-module is also injective and hence we may iterate the procedure. This gives a decreasing sequence of injective \( V \)-modules \( \Lambda_0 \supseteq \Lambda_1 \supseteq \Lambda_2 \supseteq \ldots \) with \( \lim_{\mathcal{M}} V^{e,*} \mathcal{M} = \lim_{\mathcal{M}} \Lambda_i \) for any \( i > 0 \). We do not know if this sequence stabilizes i.e. \( \cap_{i \geq 0} \Lambda_i^j = \Lambda_0^j \) for any \( j \gg 0 \).

**Proof of Corollary 1.14** In the case of point (b), we construct the Cartier module \( \Omega_0 \) by setting \( \mathcal{M} = L \), we let \( \phi : L \rightarrow V'^*L \) be given by \( s \), and finally we set \( \Omega_0 = \mathcal{H}^0(\mathcal{R}S(D_{\hat{A}}(\mathcal{M}))) \), as in Corollary 5.3.

In the case of (a), we can also define \( \Omega_0 \) as above, as soon as we are able to exhibit \( \phi : L \rightarrow V'^*L \), such that \( V(\phi) = V(s_{\hat{F}_p}) \otimes_{\widehat{\mathbb{F}}_p} k \). To this end consider \( \phi' : L_{\hat{F}_p} \rightarrow F_{\hat{F}_p}^* L_{\hat{F}_p} \) induced by \( s_{\hat{F}_p} \). Base-extending \( \phi' \) to \( k \) yields \( \phi'' : L \rightarrow (F_{\hat{F}_p} \otimes \id_k)^* L \). Note now that as \( L = L_{\hat{F}_p} \otimes_{\hat{F}_p} k \), pullback by any power of the non-\( k \)-linear automorphism \( \id_{\hat{A}_p} \otimes F_k \) of \( \hat{A} \) fixes \( L \). That is, there is an isomorphism \( \phi''' : L \rightarrow (\id_{\hat{A}_p} \otimes F_k)^* L \). We obtain the following composition, see (4) for explanation on the last equality:

\[
\phi : L \xrightarrow{\phi'''} (\hat{F}_p \otimes \id_k)^* L \xrightarrow{\phi''} (\hat{F}_p \otimes \id_k)^* (\id_{\hat{A}_p} \otimes F_k)^* L = V'^* L.
\]

Note that as \( \phi''' \) is an isomorphism, the vanishing locus of \( \phi \), agrees with the vanishing locus of \( \phi'' \), that is, with \( V(s_{\hat{F}_p}) \otimes_{\hat{F}_p} k \). This concludes the construction of a \( V \)-module \( \phi : L \rightarrow V'^*L \), such that \( V(\phi) = V(s_{\hat{F}_p}) \otimes \hat{F}_p \), in the case of point (a).

So, in the case of either points, we obtained the Cartier module \( \Omega_0 \) associated to \( \phi : L \rightarrow V'^*L \) corresponding to \( s \in \mathcal{H}^0(\hat{A}, L \otimes V'^*L^{-1}) \). Then, we have

\[
W_F^0 = \{ y \in \hat{A} \mid k(y) \otimes \lim_{\mathcal{M}} V^{e,*} \mathcal{M} = 0 \} = \hat{A} \setminus \{ y \in \hat{A} \mid p^e(y) \in V(s) \text{ for infinitely many } e \} \quad (25)
\]

**[point (a) of Corollary 5.3]** \( V = [p] \) topologically

Point (a) follows then directly from (25), using that in this case topologically \( [p] = (F_{\hat{F}_p} \otimes \id_k)^2 \), and hence it leaves \( V(s) \) fixed.

For point (b), we only have to show the addendum about the case when \( A \) is a surface. So, assume that \( V(s) \) does not contain any torsion translate of an abelian subvariety. *First we claim that \( V(s) \) and \( p^e(V(s)) \) cannot have common components for all integers \( e \gg 0 \). Indeed, otherwise \( p \) permutes such components, and hence for every integer \( j \) divisible enough \( p^j \) leaves such components fixed. However, then by [10, Thm 2.3.6] such components would be torsion translates of abelian subvarieties. This concludes our claim. Let \( e_0 \) be the threshold such that for \( e \geq e_0 \) the claim holds. Then the following computation concludes that when \( A \) is a surface without supersingular factors, then \( \hat{A} \setminus W_F^0 \) is countable:

\[
\hat{A} \setminus W_F^0 \subseteq \bigcup_{j,l \in \mathbb{N}, j \geq j + e_0} p_j^{-1} V(s) \cap p_l^{-1} V(s) = \bigcup_{j,l \in \mathbb{N}, j \geq j + e_0} p_j^{-1} (V(s) \cap p_l^{-1}) V(s).
\]

**[finite by the above claim]**
The following example shows, using $V$-modules, that the torsion property of Theorem 1.1 is sharp. That is, it does not hold if $\hat{A}$ is supersingular. This is the statement of Proposition 1.2.

Example 5.5 Set $A_{F_p} = E \times E$. First note that topologically on $\hat{A}$ we have $V = [p] = F_{A_{F_p}} \otimes \mathbb{F}_p$, see (4). As $C$ is defined over $\mathbb{F}_p$, the automorphism $V$ fixes $C$.

We use the notation of Corollary 5.3. Let $L$ be an ample enough line bundle, and consider $\mathcal{M} = L|_C$. We endowed $\mathcal{M}$ with a $V$-module structure. As $V^*$ is equal to the Frobenius up-to twisting the $k$-structure (see (4)), $V^*\mathcal{M} \cong L^p/L^p(-pC)$. In particular, it has a subsheaf $\mathcal{K} = L^p(-(p-1)C)/L^p(-(p-1)C)|_C$. As we took $L$ to be sufficiently ample, we see that there is a non-zero homomorphism $\phi : \mathcal{M} \hookrightarrow \mathcal{K} \subseteq V^*\mathcal{M}$. Let us take this as the $V$-module structure on $\mathcal{M}$.

As $L$, or rather $L|_C$ is sufficiently ample, $R\hat{S}(\mathcal{M}) = R^0\hat{S}(\mathcal{M})$ is a vector bundle. In particular, the same holds for $\Omega_0 = RS(D_\hat{A}(\mathcal{M}))$. As $R\hat{S}(D_\hat{A}(-))$ is the inverse of $RS(D_\hat{A}(-))$ up to a reflection, we see that $\Lambda_0 = \hat{\Lambda}_0 = R\hat{S}(D_A(\Omega_0)) = (-1_A)^*L|_C$. We also obtain then that $\Omega_0 = \hat{\Omega}_0$.

Additionally, as explained in the proof of [10, Thm 3.1.1], $H^i(A, \Omega_0 \otimes \hat{N}) \cong R^{-i}\Gamma(\hat{\Lambda}, \Lambda_0 \otimes N)$ for any line bundle $N$ on $\hat{A}$. However, as $\Lambda_0$ is supported in cohomological degree 0, this means that $H^i(A, \Omega_0 \otimes \hat{N}) = 0$ for all $i > 0$. In particular, $\Omega_0$ is a GV sheaf by [17, Thm A]. Hence $R^2\hat{S}(\Omega_0)$ is supported at finitely many points. Set $C^0 := C\setminus \text{Supp} R^2\hat{S}(\Omega_0)$, which is then a non-empty irreducible locally closed subset of $\hat{A}$ of dimension 1. Note now:

- If $\mathcal{P} \in \hat{A}\setminus C$, then $H^i(A, \Omega_0 \otimes \mathcal{P}) = 0$ for all $i$. Indeed, we have $R\hat{S}(\Omega_0) = (-1_A)^*\Lambda_0[-g] = \mathcal{M}[-g]$. So, $C \supseteq \text{Supp} R^1\hat{S}(\Omega_0)$ for all $i$. Then, cohomology and base-change yields the statement of the present point.
- A consequence of the previous point is that for all $\mathcal{P} \in \hat{A}$ we have $\chi(A, \Omega_0 \otimes \mathcal{P}) = 0$.
- If $\mathcal{P} \in C^0$, then $H^2(\Omega_0 \otimes \mathcal{P}) = 0$, but $H^0(\Omega_0 \otimes \mathcal{P}) \neq 0$. Using that the Euler-characteristic is zero, we obtain that consequently, $H^1(\Omega_0 \otimes \mathcal{P}) \neq 0$.

This in particular, by cohomology and base-change also implies that $\text{Supp} R^1\hat{S}(\Omega_0) = C$.

Recall that in the proof of Theorem 1.1 we defined $W^i$ as in Corollary 4.8. Hence, by the above facts about $\text{Supp} R^1\hat{S}(\Omega_0)$ we obtain that $\text{Supp} W^2$ is finitely many points, and $W^1 = W^0 = C$.

6 Maximal dimensional components of Frobenius stable cohomology support loci

In this section, we prove Theorem 1.4.

Notation 6.1 We use Notation 4.1, for a Cartier module $F_{\ast}\Omega_0 \to \Omega_0$ on an abelian variety $A$ without supersingular factors. Let $\eta \in \hat{A}$ be an associated prime of $\hat{\Lambda}_0$ of codimension $j$. According to Corollary 4.4, there exist integers $k_0$ and $k_1$ such that $\bar{\eta} = \hat{W} + y$, where $y \in \hat{A}$ is a closed $p^{k_0}(p^{k_1} - 1)$-torsion point, and $\hat{W} \subseteq \hat{A}$ is an abelian subvariety. By Lemma 4.3, $\eta' := p^{k_0}\eta$ is also an associated prime of $\Lambda_0$ of codimension $j$ and its closure is $\hat{W} + y'$ where $y' = p^{k_0}y$ is a $(p^{k_1} - 1)$-torsion point. Replacing $\eta$ by $\eta'$, we may assume that $\bar{\eta} = \hat{W} + y$, where $y \in \hat{A}$ is a closed $(p^{k_1} - 1)$-torsion point. Using the equation $p^{k_1}\bar{\eta} = p^{k_1}(\hat{W} + y) = \hat{W} + p^{k_1}y = \hat{W} + y = \bar{\eta} \implies p^{k_1}\eta = \eta$. Springer
in conjunction with Lemma 4.3, we define:

\[
\Omega'_e := P_y \otimes \Omega_{k1e} = P_y \otimes F_{*e}^{k1s} \Omega_0 \cong F_{*e}^{k1s} \left( P_{p^{k1s}y} \otimes \Omega_0 \right) \cong F_{*e}^{k1s} (P_y \otimes \Omega_0) = F_{*e}^{k1s} \Omega'_0.
\]

That is \( F_{*e}^{k1s} \Omega_0 \rightarrow \Omega'_0 \) is naturally a Cartier module, such that \( F_{*e}^{k1s} \Omega'_0 \cong \Omega'_e \). Similarly to Notation 6.1 let us introduce also

- \( \Lambda'_e := R\hat{S}(D_A(\Omega'_e)) \), and
- \( \Lambda' := \hocolim_{\tau} \Lambda'_e. \)
- \( \tilde{\Lambda}'_e := \text{Im}(\text{h}^0(\Lambda'_e) \rightarrow \Lambda), \)
- \( \tilde{\Omega}'_e := (-1) \text{d}^* D_A RS(\tilde{\Lambda}'_e)[-g] = RS(D_A(\tilde{\Lambda}'_e)). \)

In particular, then we have:

\[
\tilde{\Lambda}'_e \cong T^n_{-y} \tilde{A}_{ek_1}.
\]

Hence, the generic point \( \eta' \) of \( \hat{W} \) is an associated prime of \( \tilde{\Lambda}'_e \) for all integers \( e \geq 0 \).

Let \( W \) be the dual abelian variety of \( \hat{W} \), and let \( \pi : A \rightarrow W \) be the dual morphism of \( \hat{W} \rightarrow \hat{A} \). Note that \( \pi \) is of relative dimension \( j \).

**Assumption 6.2** In the situation of Notation 6.1, assume that \( R^j \pi_* \Omega'_e = 0 \).

**Claim 6.3** Under Assumption 6.2, we have that

\[
\lim_{e \rightarrow} \text{h}^i \left( RS_{W, \hat{W}} D_W (R \pi_* \Omega'_e) \right) = 0 \quad \text{for} \quad l \notin [-j + 1, \ldots, 0],
\]

or equivalently the direct system \(( RS_{W, \hat{W}} D_W (R \pi_* \Omega'_e) )_{e \geq 0} \) is nilpotent outside of cohomological degrees \([-j + 1, \ldots, 0]\).

**Proof** First, note that

\[
R^j \pi_* \Omega'_e = R^j \pi_* \left( F_{*e}^{k1s} \Omega'_0 \right) \cong F_{*e}^{k1s} \left( R^j \pi_* \Omega'_0 \right) = 0.
\]

Then, we claim that:

\[
\lim_{e \rightarrow} \text{h}^i \left( RS_{W, \hat{W}} D_W \left( \tau^{\leq m} R \pi_* \Omega'_e \right) \right) = 0 \quad \text{for} \quad i \notin [-m, \ldots, 0].
\]

Since \( \tau^{\leq 0} R \pi_* \Omega'_e = R^0 \pi_* \Omega'_e \), the base of the induction holds by [10, Thm 3.1.1]. Consider now the triangle

\[
\tau^{\leq m} R \pi_* \Omega'_e \longrightarrow \tau^{\leq m+1} R \pi_* \Omega'_e \longrightarrow R^{m+1} \pi_* \Omega'_e [-m - 1] \longrightarrow 1.
\]

By induction we obtain the following two vanishings:

\[
\forall i \notin [-m, \ldots, 0] : \lim_{e \rightarrow} \text{h}^i \left( RS_{W, \hat{W}} D_W \left( \tau^{\leq m} R \pi_* \Omega'_e \right) \right) = 0
\]

\[
\forall i \neq -m - 1 : \lim_{e \rightarrow} \text{h}^i \left( RS_{W, \hat{W}} D_W \left( R^{m+1} \pi_* \Omega'_e [-m - 1] \right) \right) \cong \lim_{e \rightarrow} \text{h}^i \left( RS_{W, \hat{W}} D_W \left( R^{m+1} \pi_* \Omega'_e \right) \right) \left[ m + 1 \right] = 0.
\]
Applying $\mathcal{H}^\bullet$ and $\lim$ to the triangle in (27) concludes our claim.

To conclude the proof, we just observe that by (26) we have that $R\pi_*\Omega'_e \cong R\tau_{\leq -1}\pi_*\Omega'_e$. \hfill \qed

**Proof of Theorem 1.4** We may use Notation 6.1 throughout the proof. We have to arrive to contradiction under Assumption 6.2.

We start by introducing some additional notation. Consider the morphisms given in the following diagram:

\[
\begin{array}{ccc}
A \times \eta & \xrightarrow{\tau_\eta} & \eta \\
\downarrow j_\hat{\omega} & & \downarrow \hat{\tau}_\hat{\omega} \\
A \times \hat{W} & \xrightarrow{\rho = \pi \times \id_W} & \hat{W}
\end{array}
\]

Set $\mathcal{P}^{A \times \hat{W}} := \mathcal{P}|_{A \times \hat{W}}$ and $\mathcal{P}^{A \times \eta} := \mathcal{P}|_{A \times \eta}$, and notice that

\[
\forall \hat{\omega} \in \hat{W} \subset \hat{A} : \mathcal{P}^{A \times \hat{W}}|_{A \times \hat{\omega}} \cong \rho^* \mathcal{P}^{W \times \hat{W}}|_{A \times \hat{\omega}}, \quad \text{and} \quad \mathcal{P}^{A \times \hat{W}}|_{O_A \times \hat{W}} = \rho^* \mathcal{P}^{W \times \hat{W}}|_{O_A \times \hat{W}}.
\]

(28)

By Claim 6.3, we know that the direct system in $D(\hat{A})$ formed out of the following sheaves is nilpotent in cohomological degree $-j$:

\[
(-1)^j R\tau_{W \times \hat{W},*} D_W (R\pi_*\Omega'_e) \cong D_{\hat{W}} R\tau_{W \times \hat{W},*} (R\pi_*\Omega'_e)[-g + j]
\]

Identity for exchanging $RS$ and $D$ for $W$, see Theorem 2.1

\[
\cong D_{\hat{W}} R\tau_{W \times \hat{W},*} (\mathcal{P}^{W \times \hat{W}} \otimes \rho^* \mathcal{P}^{W \times \hat{W}} R\pi_*\Omega'_e)[-g + j]
\]

Definition of $RS_{W \times \hat{W}}$

\[
\cong D_{\hat{W}} R\tau_{W \times \hat{W},*} (\mathcal{P}^{W \times \hat{W}} \otimes R\rho_* \rho^* \mathcal{P}^{W \times \hat{W}} R\pi_*\Omega'_e)[-g + j] \cong D_{\hat{W}} R\tau_{W \times \hat{W},*} (\mathcal{P}^{A \times \hat{W}} \otimes \rho^* \mathcal{P}^{W \times \hat{W}} R\pi_*\Omega'_e)[-g + j]
\]

Flat base-change

\[
\cong R\tau_{W \times \hat{W},*} D_{A \times \hat{W}} (\mathcal{P}^{A \times \hat{W}} \otimes \rho^* \mathcal{P}^{W \times \hat{W}} R\pi_*\Omega'_e)[-g + j] \cong R\tau_{W \times \hat{W},*} D_{A \times \hat{W}} / \hat{W} (\mathcal{P}^{A \times \hat{W}} \otimes \rho^* \mathcal{P}^{W \times \hat{W}} R\pi_*\Omega'_e).
\]

(29)

Grothendieck duality

\[
\omega^*_{A \times \hat{W}}[-(g - j)] \cong \omega^*_{A \times \hat{W} / \hat{W}}
\]

Then, as $\eta \to \hat{W}$ is flat, the direct system in $D(\hat{A})$ formed out of the following sheaves is also nilpotent in cohomological degree $-j$:
\[
\left(-1 \cdot \hat{W}^* R S_{\hat{W}, \hat{W}} D_{W} (R \pi_* \Omega'_e) \right) \otimes k(\eta) \cong \left( R \tau_{\hat{W}, e} D_{A_{\times \hat{W}}} (\mathcal{P}^A_{\times \hat{W}} \otimes \text{pr}_e^* \Omega'_e) \right) \otimes k(\eta)
\]
\[
\cong \left( R \tau_{\eta, e}^* j_{\hat{W}}^* D_{A_{\times \hat{W}}} (\mathcal{P}^A_{\times \hat{W}} \otimes \text{pr}_e^* \Omega'_e) \right) \cong \left( D_{k(\eta)} R \Gamma (A_{\times \eta}, \mathcal{P}^A_{\times \eta} \otimes \text{pr}_e^* \Omega'_e) \right).
\]

So, we obtain that
\[
0 = D_{k(\eta)} \left( \lim_{\rightarrow} \mathcal{H}^{-j} (D_{k(\eta)} R \Gamma (A_{\times \eta}, \mathcal{P}^A_{\times \eta} \otimes \text{pr}_e^* \Omega'_e)) \right)
\]
\[
= \lim_{\leftarrow} R^j \Gamma (A_{\times \eta}, \mathcal{P}^A_{\times \eta} \otimes \text{pr}_e^* \Omega'_e)
\]

However, by point (c) of Lemma 4.6, Corollary 4.7 we have
\[
0 \neq \lim_{\leftarrow} R^j \Gamma (A_{\times \eta}, \mathcal{P}^A_{\times \eta} \otimes \text{pr}_e^* \tilde{\Omega}'_e).
\]

This contradicts the relation given by Lemma 4.2 between the inverse systems \((\tilde{\Omega}'_e)_{e \geq 0}\) and \((\Omega'_e)_{e \geq 0}\).

\[\Box\]

7 Proof of Theorem 1.7

Proof of Theorem 1.7 Let \(X \subset A\) be a theta divisor, and set \(\tau := \tau(X)\) to be the test ideal of \(X\). Since \(X\) is a theta divisor, it is reduced. In particular, this also means that we may assume that \(\text{dim} A \geq 2\).

Consider then the following diagram of Cartier modules on \(A\):

\[
\begin{array}{ccccccccc}
F_* \omega_A & \xrightarrow{F_* (a)} & F_* \omega_A (X) & \xrightarrow{F_* (b)} & F_* \omega_X & \leftarrow F_* (\omega_X \otimes \tau) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\omega_A & \xrightarrow{a} & \omega_A (X) & \xrightarrow{b} & \omega_X & \xleftarrow{\omega_X \otimes \tau} \\
\end{array}
\]

Diagram (30), yields a natural Cartier module structure on \(\omega_A (X) \otimes \tau' := b^{-1} (\omega_X \otimes \tau)\), together with the exact sequence

\[
0 \rightarrow \omega_A \xrightarrow{a} \omega_A (X) \otimes \tau' \xrightarrow{b} \omega_X \otimes \tau \rightarrow 0.
\]

Notice that \(X\) is strongly F-regular if and only if \(\tau = \mathcal{O}_X\) or equivalently if \(\tau' = \mathcal{O}_A\). Let \(\Omega_0 := \omega_X \otimes \tau\). Our goal is to show that \(\Omega_0 = \omega_X\). Proceeding by contradiction we assume that \(\Omega_0 \neq \omega_X\). Note that as \(X\) is reduced, the cosupport of \(\tau\) is a proper closed subset of \(X\).
From now, we use the additional notation of Notation 4.1 given by $\Omega_0$ and $\Lambda$. We show several statements next about

$$\text{Supp } \Lambda_0 \subseteq \left\{ Q \in \hat{A} \mid H^0(A, \Omega_0 \otimes Q^{-1}) \neq 0 \right\}. \quad (32)$$

[10, Cor 3.2.1], and the fact that $\text{Supp } \Lambda_0 \supseteq \text{Supp } \Lambda_0$

Supp $\Lambda_0 \neq \emptyset$: as $\Lambda_v = V^{e_v} \Lambda_0$, this is equivalent to showing that $\Lambda \neq 0$. This follows from [10, Theorem 3.1.1] since $\Omega = \varprojlim \Omega_0 \neq 0$.

Supp $\Lambda_0 \neq A$: note that $H^0(A, \mathcal{O}_A(X) \otimes \tau \otimes Q) = 0$ for general $Q \in \hat{A}$, otherwise the support of every translate of $X$ would contain $Z(\tau')$, which would imply that $Z(\tau') = \emptyset$. As, $H^1(A, \tau) = 0$ for $Q \in \hat{A} \setminus \{\mathcal{O}_A\}$, using (31), we obtain that $H^0(A, \omega_X \otimes \tau \otimes Q) = 0$ for $Q \in \hat{A}$ general. Then, (32) concludes that Supp $\Lambda_0 \neq \hat{A}$.

This concludes our statements about Supp $\Lambda_0$. As a consequence we may assume that Supp $\Lambda_0$ has an irreducible component $U$, such that $0 < \dim U < g$. According to Corollary 4.4, $U = P + \hat{W}$ where $P$ is a $p^{k_0}(p^{k_1} - 1)$-torsion element of $\hat{A}$ and $\hat{W} \subset \hat{A}$ is an abelian subvariety of codimension $0 < j < g$.

Consider the projection $\pi : A \to W$ dual to the inclusion $\hat{W} \to \hat{A}$. For any $a \in A$ we let $T_a : A \to A$ be the translation by $a$ and $\lambda(a) = \mathcal{O}_A(X)^{V} \otimes \mathcal{O}_A(T_a^*X)$ which gives a morphism $\lambda : A \to \hat{A}$. Since $(A, X)$ is a PAV, then $\lambda$ is an isomorphism. Let $\bar{W} = \lambda^{-1}(\hat{W} + P) \subset A$ be the induced abelian subvariety of $A$ and $\bar{\pi} = \pi|_{\bar{W}} : \bar{W} \to W$ be the induced isogeny.

By Theorem 1.4 we know that $R^j\pi_* (Q \otimes \Omega_0) \neq 0$ for some $Q \in \hat{A}$. Hence, it is enough to show the following:

**Claim 7.1** For every $Q \in \hat{A}$, we have $R^j\pi_* (Q \otimes \Omega_0) = 0$.

**Proof** Assume the contrary, that is, for some fixed $Q \in \hat{A}$ we have

$$R^j\pi_* (Q \otimes \omega_X \otimes \tau) = R^j\pi_* (Q \otimes \Omega_0) \neq 0. \quad (32)$$

Consider the short exact sequence

$$0 \to Q \to Q \otimes \mathcal{O}_A(X) \to \omega_X \otimes Q \to 0.$$

Pushing forward to $W$, we have $R^{j+1}\pi_* Q = 0$, since $\dim (A/W) = j$; as well as $R^i\pi_* Q(X) = 0$ for $i > 0$, since $Q(X)$ is ample and so $H^i(A_y, Q(X)|_{A_y}) = 0$ for any $y \in W$. Thus $R^j\pi_* (\omega_X \otimes Q) = 0$. Now consider the short exact sequence

$$0 \to \omega_X \otimes \tau \otimes Q \to \omega_X \otimes Q \to C \to 0. \quad (33)$$

Pushing forward to $W$, we have a surjection

$$R^{j-1}\pi_* C \to R^j\pi_* (\omega_X \otimes \tau \otimes Q) \neq 0. \quad (35)$$

It follows that $Z_{w_0} = Z(\tau)|_{A_{w_0}}$ has codimension $\leq 1$ for some fiber $A_{w_0}$ of $\pi$. 
Note that for \( R \in (P + \hat{W}) \setminus \{O_A\} \), we have
\[
H^0(A, O_A(X) \otimes \tau' \otimes R^{-1}) \rightarrow H^0(A, \omega_X \otimes \tau \otimes R^{-1}) \neq 0
\]
(31) and \( H^1(A, R) = 0 \) for \( R \in \hat{A} \setminus \{O_A\} \) (32)

Then, by semicontinuity, using that \( \dim \) is a finite isogeny, it follows that
\[
\forall R \in P + \hat{W} : H^0(A, O_A(X) \otimes \tau' \otimes R^{-1}) \neq 0
\]
\[
\implies \forall w \in \hat{W} : H^0(A, O_A(T^*_w X) \otimes \tau') \neq 0
\]
This means that \( T^*_w X \) is contained in the support of \( X \) for every \( w \in \hat{W} \). Since \( \hat{W} \rightarrow W \) is a finite isogeny, it follows that \( \hat{W} + Z_{w_0} \) is a divisor contained in \( X \). Since \( X \) is irreducible, \( X = \hat{W} + Z_{w_0} \) which is not ample (for example because it is disjoint from \( \hat{W} + z \) for any \( z \in A(\hat{A}) \)). This is impossible.

As mentioned before Claim 7.1, by Theorem 1.4 we know that \( R^j \pi_*(Q \otimes O_\hat{A}) \neq 0 \) for some \( Q \in \hat{A} \). Hence, Claim 7.1 yields a contradiction. That is, our assumption that \( \tau \neq O_X \) is false. Equivalently, \( X \) is strongly \( F \)-regular.

**Proof of Theorem 1.11** Choose some integer \( e > 0 \) such that \( m|p^e - 1 \). Consider then the Frobenius trace maps
\[
\Phi^e : F^e_* O_A(X) \cong F^e_* O_A((1 - p^e)(K_A + D/m) + p^e X) \rightarrow O_A(X)
\]
and let \( \tau \otimes O_A(X) \) be the image of \( \Phi^e \) for \( e \gg 0 \). We aim to show that \( \tau = O_A \). Proceeding by contradiction, assume that \( \tau \neq O_A \). Since \( [D/m] = 0 \), then \( \tau = T_Z \) where \( Z \) is a subscheme of \( A \) of codimension \( \geq 2 \). We set \( \Omega_0 := \tau \otimes O_A(X) \) and we use the associated notation defined in Notation 4.1. As above, we have \( \Lambda := \hocolim(\Lambda_e) = H^0(\Lambda) \neq 0 \) and hence \( \tilde{\Lambda}_0 \neq 0 \). Note that as \( X \) does not contain every translate of \( Z \), we have \( H^0(\tau \otimes O_A(X) \otimes P) = 0 \) for some (and hence general) \( P \in \hat{A} \). Thus \( 0 \leq \dim \text{Supp}(\tilde{\Lambda}_0) < \dim A \).

We will now show that \( \dim \text{Supp}(\tilde{\Lambda}_0) \geq 1 \). Suppose instead that \( \dim \text{Supp}(\tilde{\Lambda}_0) = 0 \), then \( \tilde{\Lambda}_0 \) is Artinian and hence each \( \tilde{\Omega}_e \) is a homogeneous vector bundle (i.e. a successive extension of topologically trivial line bundles). But then since the composition
\[
\Omega_t = F^t_* \Omega_0 \rightarrow \tilde{\Omega}_0 \rightarrow \Omega_0
\]
is surjective, then \( F^t_*(O_A(X) \otimes \tau) \rightarrow \tilde{\Omega}_0 \) is non-trivial. But as \( \tilde{\Omega}_0 \) is a successive extension of topologically trivial line bundles, we must have
\[
0 \neq \text{Hom}(F^t_*(O_A(X) \otimes \tau), P) = \text{Hom}(F^t_*(O_A(X') \otimes \tau), O_A)
\]
for some element \( P \in \text{Pic}^0(A) \) and some translate \( X' \) of \( X \) such that \( O_A(X') \otimes P^{p^t} = O_A(X) \).

By Grothendieck duality, we have
\[
\text{Hom}(F^t_*(O_A(X') \otimes \tau), O_A) = F^t_* \text{Hom}((O_A(X') \otimes \tau), O_A).
\]
But, taking reflexive hulls, any non-trivial homomorphism \( O_A(X') \otimes \tau \rightarrow O_A \) gives rise to a non-trivial homomorphism \( O_A(X') \rightarrow O_A \), which is impossible as \( H^0(O_A(-X')) = 0 \).

If \( \eta \) is a maximal dimensional associated prime of \( \tilde{\Lambda}_0 \), then \( \tilde{\eta} = P + \hat{W} \) where \( P \) is torsion and \( \hat{W} \subset \hat{A} \) is an abelian subvariety of codimension \( j \geq 1 \). Let \( \pi : A \rightarrow W \) be the dual projection, \( \lambda : A \rightarrow \hat{A} \) be the morphism defined by \( a \rightarrow O_A(T^*_a X - X) \) and \( \hat{W} = \lambda^{-1}(\hat{W} + P) \subset A \), as in the proof of Theorem 1.7.
Claim 7.2 $R^j\pi_*(\Omega_0 \otimes Q) = 0$ for every $Q \in \text{Pic}^0(A)$.

Proof Suppose the opposite, $R^j\pi_*(\Omega_0 \otimes R) \neq 0$ for some $R \in \hat{A}$. Consider the short exact sequence

$$0 \to \mathcal{O}_A(X) \otimes \tau \otimes R \to \mathcal{O}_A(X) \otimes R \to Q \to 0$$

Since $R^j\pi_*(\mathcal{O}_A(X) \otimes R) = 0$, we have $R^{j-1}\pi_*(Q) \neq 0$. Hence, if $\mathcal{I}_Z = \tau$, then there exists a fiber $Z_{w_0}$ of dimension $\geq j - 1$ for some $w_0 \in W$. But then $H^0(A, \mathcal{O}_A(T^*_wX) \otimes \tau) \neq 0$ for any $w \in \hat{W}$ which implies that $X$ contains the non-ample divisor $Z_{w_0} + \hat{W}$. This is impossible as $X$ is irreducible.

By Theorem 1.4 we know that $R^j\pi_*(Q \otimes \Omega_0) \neq 0$ for some $Q \in \hat{A}$. Hence, Claim 7.2 yields a contradiction. That is, our assumption that $\tau \neq \mathcal{O}_A$ is false. Equivalently, $(X, D/m)$ is purely $F$-regular. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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