GLOBAL ATTRACTIVITY IN ALMOST PERIODIC SINGLE SPECIES MODELS

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Abstract. Using properties of asymptotically almost periodic solutions we prove existence theorem for piece-wise continuous almost periodic solutions of differential equations with delay and impulses. We apply these results to study almost periodic single species model with stage structure and impulses.

Key Words. Almost periodic, delay, impulsive action, single species models, stage structure

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1. Introduction. We investigate the existence and attractivity properties of piece-wise continuous almost periodic solutions for systems of differential equations with delay and impulsive action. Since solutions of impulsive system have discontinuities, almost periodicity of impulsive system can be understood in different way. In our paper we use conception of discontinuous almost periodic function proposed in [6] and then investigated in [11, 12, 13, 14, 15] and other works. Following ideas of [18], we first prove the existence theorem for an asymptotically almost periodic solution of impulsive system. This implies the existence of an discontinuous almost periodic solution.

Single species model with time-delay stage structure was proposed in [1] and then many authors studied different kinds of stage structure biological models (see, for example, [2, 3, 9, 10, 17]). Using results of first part of our paper we obtain sufficient conditions for the existence and global attractivity of discontinuous almost periodic solutions of almost periodic single species models with stage structure and impulses.

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2. Discontinuous almost periodic solutions. We will consider the space $\mathcal{PC}^k(J, \mathbb{R}^n)$, $J \subset \mathbb{R}$, of all piece-wise continuous functions $x : J \rightarrow \mathbb{R}^n$ such that

i) the set $T = \{t_j \in J, t_{j+1} > t_j, j \in \mathbb{Z}\}$ is the set of discontinuities of $x$;

ii) $x(t_j - 0) = x(t_j)$ and there exists $\lim_{t \rightarrow t_j^+} x(t) = x(t_j + 0) < \infty$;

iii) the function $x(t)$ is $C^k$ smooth in $J \setminus T$.

**Definition 1.** [12] The sequence $\{t_k\}$ has uniformly almost periodic differences if for any $\varepsilon > 0$ there exists a relatively dense set of $\varepsilon$-almost periods common for all sequences $\{t_k\}$, where $t_k = t_{k+j} - t_k, j \in \mathbb{Z}$.

**Definition 2.** [12] The function $\varphi(t) \in \mathcal{PC}^k(\mathbb{R}, \mathbb{R}^n)$ is said to be W-almost periodic (W.a.p.) if

i) the sequence $\{t_k\}$ of discontinuities of $\varphi(t)$ has uniformly almost periodic differences;

ii) for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points $t'$ and $t''$ belong to the same interval of continuity and $|t' - t''| < \delta$ then $\|\varphi(t') - \varphi(t'')\| < \varepsilon$ ($\|\cdot\|$ is usual norm in $\mathbb{R}^n$);

iii) for any $\varepsilon > 0$ there exists a relatively dense set $\Gamma$ of $\varepsilon$-almost periods such that if $\tau \in \Gamma$, then $\|\varphi(t + \tau) - \varphi(t)\| < \varepsilon$ for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_k| \geq \varepsilon, k \in \mathbb{Z}$.

**Definition 3.** [11] Piece-wise continuous function $\varphi_1(t) \in \mathcal{PC}(J, \mathbb{R}^n)$ is situated in the $\varepsilon$-neighborhood of function $\varphi_2(t) \in \mathcal{PC}(J, \mathbb{R}^n)$ if $\|\varphi_1(t) - \varphi_2(t)\| < \varepsilon$ for all $t \in J$ such that $|t - \tau_i^1| > \varepsilon, |t - \tau_i^2| > \varepsilon,$ and $|\tau_i^1 - \tau_i^2| < \varepsilon,$ $i \in \mathbb{Z},$ where $\{\tau_i^1\}$ and $\{\tau_i^2\}$ are sequences of discontinuities of $\varphi_1(t)$ and $\varphi_2(t)$ respectively. In this case we will write $\rho(\varphi_1, \varphi_2) < \varepsilon$.

The sequence $\{f_k(t)\}$ of functions $f_k \in \mathcal{PC}(J, \mathbb{R}^n), J \subset \mathbb{R}$, converges in W-topology to function $f \in \mathcal{PC}(J, \mathbb{R}^n)$ if for any $\varepsilon > 0$ there exists positive integer $N = N(\varepsilon)$ such that $\|f_k(t) - f(t)\| < \varepsilon$ for all $k \geq N$ and $|t - \tau_i| > \varepsilon$ ($\tau_i$ are points of discontinuities of the function $f$ at the set $J$) and points of discontinuities of functions $f_k(t)$ which are contained in $J$ converges to points $\tau_i$ uniformly with respect to $i$.

We consider the system with delay and impulsive action

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(t, x(t), x(t - h)), \quad t \neq t_k, \\
x(t_k + 0) &= x(t_k) + I_k(x(t_k)), \quad k \in \mathbb{Z},
\end{align*}
\]

where $x \in \mathbb{R}^n, h = \text{const} > 0$. We assume that

1) sequence of real numbers $t_k$ has uniformly almost periodic differences;

2) function $f(t, x, y)$ is W-almost periodic in $t$ and Lipschitz in $x$ and $y$ uniformly for $x, y$ from compact sets, the sequence of discontinuities of $f$ is the sequence $\{t_k\}$;
3) the sequence \( \{I_k(x)\} \) is almost periodic uniformly with respect to \( x \) from compact sets. Functions \( I_k(x) \) are Lipschitz in \( x \).

We denote by \( x \) the function \( x(t+\theta), \theta \in [-h,0] \), where \( x(t) \in PC(\mathbb{R}, \mathbb{R}^n) \).

**Definition.** The piece-wise continuous function \( a(t) \) is \( W \)-asymptotically almost periodic (W.a.a.p.) if it is a sum of \( W \)-almost periodic function \( p(t) \) and function \( q(t) \in PC \) such that \( q(t) \to 0 \) as \( t \to \infty \).

**Proposition.** A solution \( \xi(t) \) of system (1), (2) is W.a.a.p. if and only if for any sequence of real numbers \( \{\tau_k\} \) such that \( \tau_k \to \infty \) as \( k \to \infty \) there exists a subsequence \( \{\tau_{k_j}\} \) for which \( \xi(t+\tau_{k_j}) \) converges on \( 0 \leq t < \infty \) in \( W \)-topology.

**Proof.** Necessity. Let \( \xi(t) \) be W.a.a.p., then \( \xi(t) = p(t) + q(t) \) where \( p(t) \) is W.a.p. and \( q(t) \to 0 \) as \( t \to \infty \). By [11], a piece-wise continuous function \( p(t) \) is W.a.p. if and only if every infinite set of shifts \( \{\varphi(t+\tau_n)\} \) is compact relative to \( W \)-topology. Since \( q(t+\tau_k) \to 0 \) as \( \tau_k \to \infty \) for all \( t \geq 0 \), then there exists a subsequence \( \{\tau_{k_j}\} \) for which \( \xi(t+\tau_{k_j}) \) converges on \( 0 \leq t < \infty \) in \( W \)-topology.

**Sufficiently.** Since the sequence \( \{t_k\} \) of points of impulses has uniformly almost periodic differences then by [14, 15] for any sequence \( \{\tau_k\} \) there exist subsequence (which we denote by \( \{\tau_k\} \) again), sequence \( \{p_k\} \) with uniformly almost periodic differences, and sequence \( \{\alpha(k)\} \) such that

\[
\lim_{k \to \infty} (t_n+\alpha(k) - \tau_k) = p_n
\]

uniformly in \( n \in \mathbb{Z} \). Taking into account (3) and the W-convergence of \( \xi(t+\tau_k) \), we have that for every \( \varepsilon > 0 \) there exists positive integer \( N = N(\varepsilon) \) such that \( |t_n+\alpha(k) - \tau_k - p_n| < \varepsilon \) for all \( k \geq N \) and \( n \in \mathbb{Z} \) and \( \|\xi(t+\tau_k) - \xi(t+\tau_m)\| < \varepsilon \) for all \( k, m \geq N \) and \( t \in [0, \infty), |t+\tau_k - t_n| > \varepsilon, |t+\tau_m - t_n| > \varepsilon \).

First, we prove that for every \( \varepsilon > 0 \) there is \( T = T(\varepsilon) \) such that the set

\[
\{\tau : \sup_{t \geq T(\varepsilon)} \|\xi(t+\tau) - \xi(t)\| < \varepsilon, |t-t_k| > \varepsilon\}
\]

is relatively dense in \( \mathbb{R} \). Conversely, suppose that there exists \( \varepsilon_0 > 0 \) such that set (4) is not relatively dense. Then for this \( \varepsilon_0 \) and any \( T(\varepsilon_0) \) there exists a sequence of intervals \( [h_n - l_n, h_n + l_n] \) such that \( \sup_{t \geq T(\varepsilon_0), |t-t_k| \geq \varepsilon_0} \|\xi(t+\tau) - \xi(t)\| \geq \varepsilon_0 \) for all points \( \tau \in [h_n - l_n, h_n + l_n] \). Let \( l_1 \) be arbitrary and \( l_n > \max m < n l_m \), then \( h_n - h_m \in [h_n - l_n, h_n + l_n] \) if \( m < n \). Therefore

\[
\sup_{t \geq h_n, |t+h_n-h_k| \geq \varepsilon_0} \|\xi(t+h_n) - \xi(t+h_m)\| = \sup_{t \geq 0, |t-t_k| \geq \varepsilon_0} \|\xi(t) - \xi(t+h_n-h_m)\| \geq \varepsilon_0.
\]

This contradicts the W-convergence of \( \{\xi(t+h_n)\} \).
Assume that the sequence \( \{\xi(t + \tau_k)\} \) converges on \([0, \infty)\) in W-topology to function \( p(t) \) with the sequence of discontinuities \( \{p_n\} \). As a limit of shifts of sequence \( \{t_n\} \) the sequence \( \{p_n\} \) has uniformly almost periodic differences. Choosing a subsequence of \( \{\tau_k\} \) if necessary we can construct function \( p(t) \) on the hole axis such that \( \xi(t + \tau_k) \) converges to \( p(t) \) in W-topology on compact subsets of \( \mathbb{R} \).

Now we prove that \( p(t) \) is W.a.p. We have proved that if \( \varepsilon > 0 \) is given then there exist \( T(\varepsilon) \) and relatively dense set of numbers \( \tau \) such that \( \|\xi(t + \tau) - \xi(t)\| < \varepsilon \) if \( t \geq T(\varepsilon), t + \tau \geq T(\varepsilon) \) and \( |t - t_k| > \varepsilon \). Introducing \( \tau_n \) with sufficiently large \( n \) we have \( \|\xi(t + \tau_n + \tau) - \xi(t + \tau_n)\| < \varepsilon \) if \( t \geq T(\varepsilon) - \tau_n, t + \tau \geq T(\varepsilon) - \tau_n \) and \( |t + \tau_n - t_k| > \varepsilon \). Fix \( t \) and \( \tau \) and take \( n \) large enough so the last inequalities are correct. Then taking limits \( n \to \infty \) we have \( \|p(t + \tau) - p(t)\| < \varepsilon \). This holds for \( t \in \mathbb{R}, |t - p_k| > \varepsilon \) and relatively dense set of \( \varepsilon \)-almost periods \( \tau \). Thus \( p(t) \) is W.a.p. Analogously with \([5]\), p. 158, we can show that \( \xi(t) - p^*(t) \to 0 \) as \( t \to \infty \), where W.a.p. function \( p^*(t) \) is some translation of W.a.p. function \( p(t) \).

**Theorem 1.** Suppose that system (1), (2) has a solution \( \xi(t) \) defined on \( I = [0, \infty) \) such that \( \|\xi(t)\| \leq B \) for all \( t \geq 0 \). If \( \xi(t) \) is W.a.a.p. then the system (1), (2) has an W.a.p. solution \( p(t) \).

**Proof.** W.a.a.p. solution \( \xi(t) \) has form \( \xi(t) = p(t) + q(t) \) where \( p(t) \) is W.a.p. and \( q(t) \to 0 \) as \( t \to \infty \). Let \( \{\tau_k\} \) be a sequence such that there exist a sequence \( \{\alpha(k)\} \) and \( N \) such that \( |t_{n+\alpha(k)} - \tau_k - t_n| < \varepsilon \) and \( \rho(p(t + \tau_k), p(t)) < \varepsilon \) for all \( n \in \mathbb{Z} \) and \( k \geq N \). Obviously, \( q(t + \tau_k) \to 0 \) as \( k \to \infty \).

The function \( \xi(t + \tau_m) \) satisfies equation

\[
\begin{align*}
\dot{x}(t) &= f(t + \tau_m, x(t), x(t-h)), \\
x(t_n - \tau_m + 0) &= x(t_n - \tau_m) + I_n(x(t_n - \tau_m)), \quad n = 0, 1, ...
\end{align*}
\]  
(5)  
(6)

Denote \( t_n - \tau_m = t'_n - \alpha(m) \). Let \( n - \alpha(m) = j \), then \( n = j + \alpha(m) \) and equation (5), (6) is written in the form

\[
\begin{align*}
\dot{x}(t) &= f(t + \tau_m, x(t), x(t-h)), \\
x(t'_j + 0) &= x(t'_j) + I_{j+\alpha(m)}(x(t'_j)), \quad j = 0, 1, ...
\end{align*}
\]  
(7)  
(8)

For \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that

\[
\begin{align*}
\|f(t + \tau_m, \xi(t + \tau_m), \xi(t + \tau_m - h)) - & \ f(t + \tau_m, p(t + \tau_m), p(t + \tau_m - h))\| < \varepsilon, \\
||I_{j+\alpha(m)}(\xi(t'_j)) - I_{j+\alpha(m)}(p(t'_j))|| < \varepsilon
\end{align*}
\]

if \( \|\xi(t + \tau_m) - p(t + \tau_m)\| = \|q(t + \tau_m)\| < \delta \) and \( \|\xi(t'_j) - p(t'_j)\| = \|q(t'_j)\| < \delta \).
Let \([\bar{t}_1, \bar{t}_2]\) be some subinterval of \(\mathbb{R}\). For \(\delta(\varepsilon)\) there exists positive integer \(N\) such that \(\rho(p(t), p(t + \tau_m)) < \delta\) for all \([\bar{t}_1, \bar{t}_2]\) and \(m \geq N\).

We write system (7), (8) in the integral form

\[
\xi(t + \tau_m) = \xi(\bar{t}_1 + \tau_m) + \int_{\bar{t}_1}^{t} f(s + \tau_m, \xi(s + \tau_m), \xi(s + \tau_m - h))ds + \\
+ \sum_{\bar{t}_1 \leq \tau_m < t} I_{j+\alpha(m)}(\xi(t_j' + \alpha_m)).
\]

Making \(\tau_m \to \infty\) we have

\[p(t) = p(\bar{t}_1) + \int_{\bar{t}_1}^{t} f(s, p(s), p(s - h))ds + \sum_{\bar{t}_1 \leq t_j < t} I_{j}(p(t_j)).\]

Hence, W.a.p. function \(p(t)\) is a solution of system (1), (2).

**Theorem 2.** Let bounded solution \(\xi(t)\) of the system (1), (2) be uniformly asymptotically stable for \(t \geq 0\). Then \(\xi(t)\) is W.a.p. and system (1), (2) has W.a.p. solution which is uniformly asymptotically stable for \(t \geq 0\).

**Proof.** Since \(\xi(t)\) is uniformly asymptotically stable, then for any \(\varepsilon > 0\) there exist \(\delta = \delta(\varepsilon) > 0\) and \(T(\varepsilon) > 0\) such that if \(x(t)\) is a solution of (1), (2) such that \(\rho(x_0, x_0) < \delta\) then \(\|\xi(t) - x(t)\| < \varepsilon/2\) for \(t \geq 0\) and \(\|\xi(t) - x(t)\| < \delta_1/2\) for all \(t \geq T(\varepsilon), \delta_1 = \min(\varepsilon, \delta)\).

Let \(\{\tau_m\}\) be a sequence such that \(\tau_{m+1} > \tau_m, \tau_m \to \infty\) as \(m \to \infty\).

Denote \(\xi^m(t) = \xi(t + \tau_m)\). Then \(\xi^m(t)\) is a solution of system (7), (8) and \(\xi^m(t)\) is uniformly asymptotically stable with same \(\delta(\varepsilon)\) and \(T(\varepsilon)\) as for \(\xi(t)\).

There exists a subsequence of \(\{\tau_m\}\) (which we denote by \(\{\tau_m\}\) again) such that there exist sequence \(\{p_n\}\) with uniformly almost periodic differences and sequence \(\alpha(m)\) such that:

\[
\lim_{m \to \infty}(t_{i+\alpha(m)} - \tau_m) = p_i, \quad \lim_{m \to \infty}(I_{i+\alpha(m)}(x)) = J_i(x) \text{ uniformly with respect to } i \in \mathbb{Z} \text{ and } \|x\| \leq K,
\]

\[
f(t + \tau_m, x, y) \text{ tends to } g(t, x, y) \text{ in W-topology uniformly with respect to } x, y, \|x\| \leq K, \|y\| \leq K,
\]

\[
\xi^m_0 = \{\xi^m(\theta), \theta \in [-h, 0]\} \text{ converges in W-topology to } \xi_0 = \{\xi(\theta), \theta \in [-h, 0]\}.
\]

Therefore, for some \(\delta_2 > 0\) there exists a positive integer \(k_0(\varepsilon)\) such that if \(k \geq m \geq k_0(\varepsilon)\) then

\[
\|f(t + \tau_k, x, y) - f(t + \tau_m, x, y)\| < \delta_2(\varepsilon)
\]

for all \(\|x\| \leq K, \|x\| \leq K\) and all \(t \in \mathbb{R}, |t + \tau_k - t_j| > \delta_2, |t + \tau_m - t_j| > \delta_2, j \in \mathbb{Z}, \text{ and}
\]

\[
\|\xi(\theta + \tau_k) - \xi(\theta + \tau_m)\| < \delta_1(\varepsilon)
\]
for all \( \theta \in [-h, 0], |\theta + \tau_k - t_j| > \delta, |\theta + \tau_m - t_j| > \delta, j \in \mathbb{Z}. \)

Let \( \eta(t) \) be the solution of system (7), (8) with initial function \( \eta_0 = \xi_0^k \).

Since system (7), (8) is uniformly asymptotically stable then \( \|\xi^m(t) - \eta(t)\| < \varepsilon/2 \) for all \( t \geq 0 \) and \( \|\xi^m(t) - \eta(t)\| < \delta_1/2 \) for all \( t \geq T(\varepsilon) \).

Let us estimate the difference \( \eta(t) - \xi^k(t) \) if \( t \in [0, T(\varepsilon) + h] \). The function \( \xi^k(t) \) satisfies equation

\[
\begin{align*}
\dot{y}(t) &= f(t + \tau_k, y(t), y(t - h)), \\
y(t'' + 0) &= y(t''_j) + I_{j+\alpha(k)}(y(t''_j)), \quad j = 0, 1, \ldots,
\end{align*}
\]

where \( t''_j = t_{j+\alpha(k)} - \tau_k \).

Difference \( \eta(t) - \xi^k(t) \) satisfies following integral equation

\[
\begin{align*}
\eta(t) - \xi^k(t) &= \eta(0) - \xi^k(0) + \\
+ \int_0^t &\left( f(s + \tau_m, \eta(s), \eta(s - h)) - f(s + \tau_k, \xi^k(s), \xi^k(s - h)) \right) ds + \\
+ \sum_{0 \leq t_j < t} &I_{j+\alpha(m)}(\eta(t_j)) - \sum_{0 \leq t''_j < t} I_{j+\alpha(k)}(\xi(t''_j)).
\end{align*}
\]

By [12], p. 191, there exist a number \( l > 0 \) and a positive integer \( q \) such that any interval of the time axis of length \( l \) contains no more then \( q \) terms of the sequence \( \{p_n\} \). Since \( t'_n = t_{n+\alpha(m)} - \tau_m, t''_n = t_{n+\alpha(k)} - \tau_k \), and \( t'_n \to p_n, t''_n \to p_n \), as \( m \to \infty, k \to \infty \), then interval \( [0, T(\varepsilon) + h] \) contains finite number of points from the sequence \( \{p_n\} \) and the same number of points \( t'_n \) and \( t''_n \) (if \( k \) and \( m \) are sufficiently large.)

Analogously to [4], Theorem 5, we estimate difference \( \eta(t) - \xi^k(t) \) in succession on interval \( [0, \min(t'_1, t''_1)] \), at point \( \max(t'_1, t''_1) \), on the interval \( [\max(t'_1, t''_1), \min(t'_2, t''_2)] \) and so on.

As result, for \( \delta_1/2 \) there exists \( \delta_2(\varepsilon) \) such that if \( |t'_j - t''_j| < \delta_2, \|I_{j+\alpha(k)} - I_{j+\alpha(m)}\| < \delta_2, \|f(t + \tau_k, x, y) - f(t + \tau_m, x, y)\| < \delta_2 \) then \( |\xi^k(t) - \eta(t)| < \delta_1/2 \) for all \( t \in [0, T(\varepsilon) + h], |t + \tau_k - t_j| > \delta/2, |t + \tau_m - t_j| > \delta/2, j \in \mathbb{Z}. \)

Therefore, we have inequality \( \rho(\xi^k, \xi^m) < \varepsilon \) on the interval \( t \in [0, T(\varepsilon) + h] \) and \( \rho(\xi^k, \xi^m) < \delta_1 \leq \delta \) on the interval \( t \in [T(\varepsilon), T(\varepsilon) + h] \).

By the same argument as the above, we can see that if \( m \geq k \geq k_0(\varepsilon) \), then

\[
\rho(\xi^k_t, \xi^m_t) < \varepsilon \quad \text{for} \quad t \in [T(\varepsilon), 2T(\varepsilon)]
\]

and, in general,

\[
\rho(\xi^k_t, \xi^m_t) < \varepsilon \quad \text{for} \quad t \in [qT(\varepsilon), (q + 1)T(\varepsilon)] \text{,} \quad q = 1, 2, \ldots.
\]

Hence, \( \xi \) is W.a.a.p. and equation has W.a.p. solution. \( \square \)
3. Stage structure model. We consider the system of differential equations with impulsive action, which describe the behavior of biological species with two stages, immature and mature

\[
\begin{align*}
\dot{x}_i(t) &= \alpha(t)x_m(t) - \gamma(t)x_i(t) - \alpha(t-h)e^{-\int_{t-h}^t \gamma(s)ds}x_i(t-h), \\
\dot{x}_m(t) &= \alpha(t-h)e^{-\int_{t-h}^t \gamma(s)ds}x_m(t-h) - \beta(t)x_m^2(t),
\end{align*}
\]

for \( t \neq t_k \) and impulsive action

\[
x_m(t_k + 0) = (1 + d_k)x_m(t_k),
\]

at moments \( t_k, k \in \mathbb{Z} \). We assume that the sequence \( \{t_k\} \) of moments of impulsive action has uniformly almost periodic differences, the sequence \( \{d_k\} \) is almost periodic, \( d_k \in (-1, d], d > 0 \), functions \( \alpha(t), \beta(t) \) and \( \gamma(t) \) are piecewise continuous positive and W-almost periodic. For the sake of simplicity, we assume that points of discontinuities of \( f \) are \( t_k, k \in \mathbb{Z} \).

Here, \( x_i(t) \) and \( x_m(t) \) denote the density of immature and mature populations respectively. The birth of immature population at time \( t > 0 \) is proportional to the existing mature population with birth rate \( \alpha(t) \), \( \gamma(t) \) is the immature death rate, \( \beta(t) \) is the mature death and overcrowding rate, \( h \) is the time to maturity. The term \( \alpha(t-h)e^{-\int_{t-h}^t \gamma(s)ds}x_m(t-h) \) represents the transformation of immatures to matures.

According to biological interpretation we consider nonnegative solutions of (11) - (13) with initial conditions

\[
\begin{align*}
x_i(0) &= \varphi_i > 0, \\
x_m(\theta) &= \psi_m(\theta) \geq 0, \ \theta \in [-h, 0], \ \psi_m(0) > 0.
\end{align*}
\]

For a function \( g(t) \) bounded on the real axis, we denote \( g^L = \inf_t g(t), g^M = \sup_t g(t) \).

For almost periodic sequence \( \{d_n\} \), there exists

\[
\sigma = \lim_{T \to \infty} \frac{1}{T} \sum_{0 \leq t_k < T} \ln(1 + d_k).
\]

We assume that the function

\[
\omega(t) = \prod_{0 \leq t_k < t} (1 + d_k)e^{-\sigma t}
\]

is W-almost periodic. Then functions

\[
A(t) = \prod_{t-h \leq t_k < t} (1 + d_k)^{-1} \alpha(t-h) \exp \left( \sigma h - \int_{t-h}^t \gamma(s)ds \right)
\]
and

\[ C(t) = \prod_{0 \leq t_k < t} (1 + d_k)e^{-\sigma t} \beta(t) \]

are also W-almost periodic.

**Theorem 3.** Assume that the inequality

\[ \sigma + \sup_t \left( \alpha(t-h)e^{\sigma - \int_t^{t-h} \gamma(s)ds} \prod_{t-h \leq t_k < t} (1 + d_k)^{-1} \right) > 0 \]

is fulfilled. Then system (11) - (13) is permanent, i.e., there exist positive constants \( m_0 \) and \( M_0 \) such that all its solutions with initial values (14), (15) satisfy inequalities

\[ \lim inf_{t \to \infty} x_i(t) \geq m_0, \quad \lim sup_{t \to \infty} x_i(t) \leq M_0, \quad \lim inf_{t \to \infty} x_m(t) \geq m_0, \quad \lim sup_{t \to \infty} x_m(t) \leq M_0. \]

If, in addition, the inequality

\[ (A^M + \sigma)C^M < 2C^L(A^L + \sigma) \]

is satisfied, then system (11) - (13) has unique positive W-almost periodic solution which is globally attractive.

**Proof.** First, we prove that \( x_m(t, \varphi) > 0 \), \( t > 0 \) if \( \psi_m(\theta) \geq 0 \), \( \theta \in [-h, 0] \), \( \psi_m(0) > 0 \). Really, if \( t \in [0, h] \) then equation (12), (13) has form

\[
\begin{align*}
\dot{x}_m(t) &= \alpha(t-h)e^{-\int_t^{t-h} \gamma(s)ds} \psi_m(t-h) - \beta(t)x^2_m(t), \\
M(t_k + 0) &= (1 + d_k)x_m(t_k).
\end{align*}
\]

The solution of equation (18) - (19) with initial value \( x_m(0) = \psi_m(0) > 0 \) is estimated from below by corresponding solution of the equation

\[
\dot{u}(t) = -\beta(t)u^2(t), \quad t \neq t_k, \\
u(t_k + 0) = (1 + d_k)u(t_k).
\]

The solution of last equation is strictly positive for \( t \in (0, h] \) since \( u(0) = \varphi(0) > 0 \) and \( (1 + d_k) > 0 \). Analogously, considering equation on intervals \([h, 2h], [2h, 3h], \ldots \), we obtain positiveness for all \( t > 0 \).

To prove \( x_i(t) > 0, t > 0 \), we use the following argument. The number of immatures which was born at time \( s \) and survived to time \( t \) is given by \( \alpha(s)x_m(s)e^{-\int_s^t \gamma(\xi)d\xi} \). Since \( t - s \leq h \), then

\[ x_i(t) = \int_{t-h}^t \alpha(s)x_m(s)e^{-\int_s^t \gamma(\xi)d\xi}ds. \]
By (20) we have \( x_i(t) > 0 \) for all \( t \geq 0 \) since \( x_m(s) > 0 \).

We make change of variables

\[ x_m(t) = \omega(t) v(t) = \prod_{0 \leq t_k < t} (1 + d_k)e^{-\sigma t}v(t) \]

at the equation (12) - (13) and obtain the following equation without impulses

\[ \dot{v}(t) = A(t)v(t-h) + \sigma v(t) - C(t)v^2(t). \]

Solutions of equation (22) are continuous with left continuous derivatives.

Parallel with (22) we consider two equations

\[ \dot{v}_L(t) = A_L v_L(t-h) + \sigma v_L(t) - C^L v^2_L(t), \]
\[ \dot{v}_M(t) = A_M v_M(t-h) + \sigma v_M(t) - C^M v^2_M(t). \]

Let \( v(t, \varphi), v_L(t, \varphi) \) and \( v_M(t, \varphi) \) be solutions of equations (22), (23) and (24) respectively with the same initial function. By [16], p. 79, they satisfy inequalities

\[ v_L(t, \varphi) \leq v(t, \varphi) \leq v_M(t, \varphi), \quad t \geq 0. \]

By [8], equation (23) has unique positive asymptotically stable equilibrium \( \bar{m}_0 = (A_L + \sigma)/C^M \) (if \( A_L > -\sigma \)). Analogously, equation (24) has unique positive asymptotically stable equilibrium \( \bar{M}_0 = (A_M + \sigma)/C^L \). Therefore,

\[ \liminf_{t \to \infty} v(t, \varphi) \geq \frac{A_L + \sigma}{C^M}, \quad \limsup_{t \to \infty} v(t, \varphi) \leq \frac{A_M + \sigma}{C^L}. \]

The permanence of equation (11) follows from (20) directly.

Now we prove uniform asymptotic stability of solutions of (22). We consider two solutions \( x(t) \) and \( y(t) \) such that \( x(t) \geq m_0 - \delta \) and \( y(t) \geq \bar{m}_0 - \delta \) for all \( t \geq 0 \), where \( \delta \) is some small positive constant. The difference \( w(t) = x(t) - y(t) \) satisfies linear equation

\[ \frac{d}{dt}w(t) = A(t)w(t-h) - w(t) (C(t)(x(t) + y(t)) - \sigma). \]

By [7], p. 111, equation (27) is uniformly asymptotically stable if

\[ A^M < \inf_{t} (C(t)(x(t) + y(t)) - \sigma). \]

Using (26), we obtain (17). By Theorem 2, equation (12), (13) has unique positive piece-wise continuous almost periodic solution which is globally attractive. By (20), we verify W-almost periodicity of \( x_i(t) \) if \( x_m(t) \) is W-almost periodic.
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