Title: Different Approximation To Fuzzy Ring Homomorphisms

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Received: 2018-01-16 15:10:16
Accepted: 2019-08-11 14:12:43

Article Type: Research Article
Volume: 23
Issue: 6
Month: December
Year: 2019
Pages: 1163-1172

How to cite
Ümit Deniz; (2019), Different Approximation To Fuzzy Ring Homomorphisms. Sakarya University Journal of Science, 23(6), 1163-1172, DOI: 10.16984/saufenbilder.379634
Access link
http://www.saujs.sakarya.edu.tr/issue/44246/379634
Different Approximation to Fuzzy Ring Homomorphisms

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Abstract:

In this study we approach the definition of $\mathcal{T}_L$–ring homomorphism. In the literature, the definition of fuzzy ring homomorphism is given by Malik and Mordeson by using their fuzzy function definition. In this study, we give the definition of fuzzy ring homomorphism by using the definition of Mustafa Demirci’s fuzzy function. Some definition and theorems of ring homomorphism in classic algebra are adapted to fuzzy algebra and proved.

Keywords: Fuzzy sets, Fuzzy Relations, Fuzzy Functions, Fuzzy Ring Homomorphisms.

1. INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh [10]. Fuzzy sets gives opportunity to constitute the uncertain problems in real life to mathematical models. Most of the problems in engineering, economics, medical science etc, have various uncertainties. The fuzzy set theory helps to modelling and solving these problems. Many mathematician tried to transfer the classic set theory to use the definition of Zadeh’s fuzzy set. Rosenfeld [12] gave the definition of fuzzy groups and fuzzy grupoids. Liu [9,10] gave the definition of fuzzy subrings and fuzzy ideals of a ring. Fuzzy relations are playing an important role in fuzzy modelling, fuzzy control and significant applications in relational databases, approximate reasoning, medical diagnosis. Malik and Mordeson gave some conditions to fuzzy relations to define fuzzy function [11]. With this definition, they introduced fuzzy ring homomorphism. In these studies, they used fuzzy subsets $\mu:X \rightarrow [0,1]$ and they used infimum for operation on $[0,1]$. In literature there isn’t a certain fuzzy function definition and therefore there isn’t a certain fuzzy ring homomorphism definition. In this study, we gave a different definition of fuzzy ring homomorphism. To give this definition, we used the fuzzy function definition of Demirci [2,3] and we used L-subsets $\mu:X \rightarrow L$ which L is a complete lattice and T-norms as operation of L.

In this study, we used the definition of fuzzy subrings and fuzzy ideals of a ring from Wang [14,15]. Some definitions and theorems of ring homomorphism in the classic algebra are adapted to fuzzy algebra with this definition and proved.

2. PRELIMINARY

In this section, we have presented the basic definitions and results of fuzzy algebra which may be found in the earlier studies.

Definition 2.1. [1] Let $(L, \leq)$ be a complete lattice with top and bottom elements 1, 0, respectively. A triangular norm (briefly t-norm) is a binary operation $T$ on $L$ which is
commutative, associative, monotone and has 1 as a neutral element, i.e., it is a function.

\( T: L^2 \rightarrow L \) such that for all \( x, y, z \in L \)

\[(T1) \ T(x, y) = T(y, x). \]

\[(T2) \ T(x, T(y, z)) = T(T(x, y), z). \]

\[(T3) \ T(x, y) \leq T(x, z) \) whenever \( y \leq z. \)

\[(T4) \ T(x, 1) = x. \]

**Definition 2.2. [1]**

a) A t-norm \( T \) on a lattice \( L \) is called \( \vee \)-distributive if

\[ T(a, b_1 \vee b_2) = T(a, b_1) \vee T(a, b_2). \]

b) A t-norm \( T \) on a complete lattice \( L \) is called infinitely \( \vee \)-distributive if

\[ T \left( a, \bigvee_{\tau \in Q} b_\tau \right) = \bigvee_{\tau \in Q} T(a, b_\tau) \]

for any subset \( \{a, b_\tau \in L, \tau \in Q\} \) of \( L \).

**Theorem 2.3. [1]** Let \( L \) be a complete lattice. If \( T \) is an infinitely \( \vee \)-distributive t-norm then

\[ \bigvee_{i \in I} \bigvee_{j \in J} T(a_i, b_j) = T \left( \bigvee_{i \in I} a_i, \bigvee_{j \in J} b_j \right). \]

**Definition 2.4. [17]** Let \( L \) be a complete lattice. With a \( L \)-subset of \( X \) we mean a function from \( X \) into \( L \). We denote all \( L \)-subsets by \( F(X, L) \). In particular, when \( L = [0,1] \), the \( L \)-subsets of \( X \) are called \( fuzzy \) subsets.

**Definition 2.5. [3]** If \( X \) and \( Y \) are sets then the function \( f: X \times Y \rightarrow L \) is called a \( L \)-relation and the set of all \( L \)-relations is denoted by \( F(X \times Y, L) \).

**Definition 2.6. [2]** Let \( L \) be a complete lattice. \( E: X \times X \rightarrow L \) a \( L \)-relation \( E \) on a set \( X \) is a \( TL \)-equivalence relation if and only if for all \( a, b, c \in X \) the following properties are satisfied:

\[(E1) \ E(a, a) = 1. \]

\[(E2) \ E(a, b) = E(b, a). \]

\[(E3) \ T(E(a, b), E(b, c)) \leq E(a, c). \]

\( E \) is called a \( separable \) \( TL \)-equivalence relation or a \( TL \)-equality if in addition, \( (E4) \ E(a, b) = 1 \) implies \( a = b \).

If \( E \) is a \( TL \)-equivalence relation on \( X \) it is shown by \( (X, E) \).

**Example. [2]** Let \( X \) be a non-empty set and \( \alpha \in L \).

Then

i) \( EX_M(x, y) = 1 \)

ii) \( EX_L(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \)

are \( TL \)-equivalence relations of \( X \).

**Theorem 2.7. [2]** Let \((X, E)\) and \((Y, F)\) be two equivalence relations. Then \( TL \)-subset \( E \times F \) \( E \times F: (X \times Y) \times (X \times Y) \rightarrow L \) defined by

\( (E \times F)((x, y), (x', y')) = T(E(x, x'), F(y, y')) \)

is a \( TL \)-equivalence relation.

**Definition 2.8. [2]** Let \( E \) be a \( TL \)-equivalence relation on a set \( X \). A \( L \)-subset \( \mu \) of \( X \) is extensional or observable w.r.t. \( E \) if and only if

\( T(\mu(b), E(a, b)) \leq \mu(a) \quad \forall a, b \in X. \)

**Definition 2.9. [13]** Let \( X, Y \) and \( Z \) are sets and \( f: X \times Y \rightarrow L \) and \( g: Y \times Z \rightarrow L \) be \( L \)-relations. Then \( g \circ f: X \times Z \rightarrow L \) \( L \)-relation is called composition of \( f \) and \( g \) such that

\( g \circ f(x, z) = \bigvee_{y \in Y} T(f(x, y), g(y, z)) \)

**Definition 2.10. [3]** Let \( f: X \times Y \rightarrow L \) be a \( L \)-relation then we call the function

\( f^{-1}: Y \times X \rightarrow L \) defined by

\( f^{-1}(y, x) = f(x, y) \) the inverse of \( f \)-relation.

**Definition 2.11. [3]** Let \( f: X \times Y \rightarrow L \) be a \( L \)-relation and \( A \in F(X, L) \) and \( B \in F(Y, L) \). The \( L \)-subsets \( f(A), f^{-1}(B) \) defined by for all \( x \in X, y \in Y \)

\( f(A)(y) = \bigvee_{x \in X} T(A(x), f(x, y)) \)

and

\( f^{-1}(B)(x) = \bigvee_{y \in Y} T(B(y), f(x, y)) \)
are respectively the image of A and the inverse image of B.

**Definition 2.12.** [2] Let \((X, E), (Y, F)\) be two TL-equivalence relations and \(f \in F(X \times Y, L)\). Then;

a) \(f\) is called an \(E\)-extensional if the inequality \(T(f(x,y), E(x,x')) \leq f(x,y')\) is satisfied for all \(x, x' \in X\) and for all \(y \in Y\). We call all E-extensional L-relations set as \(F(X \times Y, E, L)\).

b) \(f\) is called a \(F\)-extensional if the inequality \(T(f(x,y), F(y,y')) \leq f(x,y')\) is satisfied for all \(x \in X\) and for all \(y, y' \in Y\). We denote all F-extensional L-relations set by \(F(X \times Y, F, L)\).

c) A L-relation such as \(f\) is called \(E\)-\(F\)-extensional if \(f\) is E-extensional and F-extensional and denote all E-F-extensional relations set by \(F(X \times Y, E, F, L)\).

**Definition 2.13.** [2] Let \((X, E)\) and \((Y, F)\) be two TL-equivalence relations and \(f \in F(X \times Y, E, F, L)\) then;

a) \(f\) is called partial TL-function if \(T(f(x,y), f(x,y')) \leq F(y,y')\) is satisfied for all \(x \in X\) and for all \(y, y' \in Y\).

b) \(f\) is called fully defined if \(f\) fulfills the condition \(\bigvee_{z \in Y} f(x,z) = 1\) for all \(x \in X\).

c) A fully defined partial TL-function is called a TL-function.

**Definition 2.14.** [3] Let \(f \in F(X \times Y, E, F, L)\) be a TL-function;

a) \(f\) is called surjective if and only if \(\bigvee_{x \in X} f(x,y) = 1\) for all \(y \in Y\).

b) \(f\) is called injective if and only if \(T(f(x,y), f(x',y)) \leq E(x,x')\) for all \(x, x' \in X\) and \(y \in Y\).

**Proposition 2.15.** Let \((X, E)\) and \((Y, F)\) be two TL-equivalence relations, \(f \in F(X \times Y, E, F, L)\) and \(A \in F(X, L)\). If \(T\) is a infinitely \(V\)-distributive t-norm then \(f^{-1}(A)\) is F-extensional L-subset.

**Proof.** For \(x, x' \in X \times Y, y, y' \in Y\) and

\[
T(f(A)(y), F(y,y')) = T \left( \bigvee_{x \in X} T(A(x), f(x,y), F(y,y')) \right)
\]

\[
= \bigvee_{x \in X} T \left( T(A(x), f(x,y), F(y,y')) \right)
\]

\[
\leq \bigvee_{x \in X} T(A(x), f(x,y), F(y,y')) = f^{-1}(B)(x').
\]

**Theorem 2.17.** Let \(f \in F(X \times Y, E, F, L)\), \(g \in F(Y \times Z, F, G, L)\) be TL-functions and \(T\) be a infinitely \(V\)-distributive t-norm. Then \(g \circ_T f \in F(X \times Z, E, G, L)\) is a TL-function.

**Proof.** For \(x, x' \in X \times Y, z, z' \in Z\)

\[
T \left( (g \circ_T f)(x, z), E(x,x') \right)
\]

\[
= T \left( \bigvee_{y \in Y} T(f(x,y), g(y,z), E(x,x')) \right)
\]

\[
= \bigvee_{y \in Y} T \left( T(f(x,y), E(x,x')), g(y,z) \right)
\]

\[
\leq \bigvee_{y \in Y} T(f(x',y), g(y,z)) = (g \circ_T f)(x',z)
\]
and so \( g \circ f \) is an E-extensional.
\[
T\left((g \circ f)(x, z), G(z, z')\right)
= T\left(\bigvee_{y \in Y} T\left(f(x, y), g(y, z)\right), G(z, z')\right)
\leq \bigvee_{y \in Y} T\left(T(f(x, y), T\left(G(z, z'), g(y, z)\right))\right)
\leq \bigvee_{y \in Y} T\left(f(x, y), g(y, z')\right) = (g \circ f)(x, z', z').
\]

Hence \( g \circ f \) is a G-extensional.
\[
T\left((g \circ f)(x, z), (g \circ f)(x, z')\right)
= T\left(\bigvee_{y \in Y} T\left(f(x, y), g(y, z), \bigvee_{y' \in Y} T\left(f(x, y'), g(y', z')\right)\right)\right)
\leq \bigvee_{y \in Y} T\left(f(x, y), g(y, z), g(y', z')\right)
\leq \bigvee_{y \in Y} G(z, z') = G(z, z').
\]

Hence \( g \circ f \) is a partial TL-function.
\[
\bigvee_{x \in X} (g \circ f)(x, z) = \bigvee_{x \in X} T\left(f(x, y), g(y, z)\right)
= \bigvee_{x \in X} T\left(1, g(y, z)\right) = \bigvee_{x \in X} g(y, z) = 1.
\]

Hence \( g \circ f \) is full defined.

Finally \( g \circ f \) is TL-function.

**Theorem 2.18.** Let \((R_1, E_1), (R_2, E_2), (S_1, F_1)\) and \((S_2, F_2)\) be TL-equivalence relations and \(f : R_1 \times S_1 \rightarrow L\) \(g : R_2 \times S_2 \rightarrow L\) be TL-functions. Then the TL-equivalence relation is defined by
\[
\begin{align*}
T\left(\left((g \times f)\right)(x_1, x_2), (y_1, y_2)\right) &= T\left(f(x_1, y_1), g(x_2, y_2)\right)
= T(f(x_1, y_1), g(x_2, y_2))
\end{align*}
\]
and is a TL-function.

**Proof.**

i) For \(x_1, x'_1 \in R_1, x_2, x'_2 \in R_2, y_1 \in S_1, y_2 \in S_2\)
\[
T\left(\left((g \times f)\right)(x_1, x_2), (y_1, y_2)\right) = T\left(f(x_1, y_1), g(x_2, y_2)\right)
\]

\(g \times f : (R_1 \times R_2) \times (S_1 \times S_2) \rightarrow L\) such that
\[
\begin{align*}
(g \times f)(x_1, x_2), (y_1, y_2)
= T\left(f(x_1, y_1), g(x_2, y_2)\right)
\end{align*}
\]

ii) For \(x_1 \in R_1, x_2 \in R_2, y_1, y'_1 \in S_1, y_2, y'_2 \in S_2\)
\[
T\left(\left((g \times f)\right)(x_1, x_2), (y_1, y_2)\right) = T\left(f(x_1, y_1), g(x_2, y_2)\right)
\]

\(g \times f : (R_1 \times R_2) \times (S_1 \times S_2) \rightarrow L\) such that
\[
\begin{align*}
(g \times f)(x_1, x_2), (y_1, y_2)
= T\left(f(x_1, y_1), g(x_2, y_2)\right)
\end{align*}
\]

iii) For \(x_1 \in R_1, x_2 \in R_2, y_1, y'_1 \in S_1, y_2, y'_2 \in S_2\)
\[
T\left(\left((g \times f)\right)(x_1, x_2), (y_1, y_2)\right) = T\left(f(x_1, y_1), g(x_2, y_2)\right)
\]

\(g \times f : (R_1 \times R_2) \times (S_1 \times S_2) \rightarrow L\) such that
\[
\begin{align*}
(g \times f)(x_1, x_2), (y_1, y_2)
= T\left(f(x_1, y_1), g(x_2, y_2)\right)
\end{align*}
\]

iv) For each \((x_1, x_2) \in (R_1 \times R_2)\)
\[
\bigvee_{(y_1,y_2) \in S_1 \times S_2} (g \times f)((x_1, x_2), (y_1, y_2))
\]
\[
= \bigvee_{(y_1,y_2) \in S_1 \times S_2} T(f(x_1, y_1), g(x_2, y_2))
\]
\[
= T\left( \bigvee_{y_1 \in S_1} f(x_1, y_1), \bigvee_{y_2 \in S_2} g(x_2, y_2) \right)
\]
\[
= T(1,1) = 1 \text{ (because f and g are fully defined)}
\]

Then, we prove \((g \times f)\) is fully defined and so \((g \times f)\) is TL-function.

**Definition 2.19.** [14] Let \(R\) be a ring and \(\mu: R \rightarrow L, \vartheta: R \rightarrow L\) are L-subsets. Define \(\mu + \vartheta\), \(\mu - \vartheta\), and \(\mu \vartheta\) as follows.

\[
(\mu + \vartheta)(x) = \bigvee \{T(\mu(y), \vartheta(z)) | y, z \in R, y + z = x\}
\]
\[
(-\mu)(x) = \mu(-x)
\]
\[
(\mu - \vartheta)(x) = \bigvee \{T(\mu(y), \vartheta(z)) | y, z \in R, y - z = x\}
\]
\[
(\mu \vartheta)(x) = \bigvee \{T(\mu(y), \vartheta(z)) | y, z \in R, y \cdot z = x\}
\]

Where \(x\) is any element of \(R\). \(\mu + \vartheta, \mu - \vartheta, \mu \vartheta\) are called the T-sum, T-difference, and T-product of \(\mu\) and \(\vartheta\), respectively, and \(-\mu\) is called the negative of \(\mu\).

**Definition 2.21.** [14] Let \(\mu: R \rightarrow L\) such that \(\mu\) satisfies conditions (R1), (R2) and (R3). Then \(\mu\) is called a TL-left ideal of \(R\) if it also satisfies the condition

\[
(R5)_L \quad \mu(xy) \geq \mu(y) \forall x, y \in R;
\]

a TL-right ideal of \(R\) if it also satisfies the condition

\[
(R5)_R \quad \mu(xy) \geq \mu(x) \forall x, y \in R;
\]

and a TL-two sided ideal or TL-ideal of \(R\) if it is also satisfies the condition

\[
(R5) \quad \mu(xy) \geq \mu(x) \forall x, y \in R.
\]

In particular, when \(T = \Lambda\), a TL-left ideal, TL-right ideal and TL-ideal of \(R\) are referred to as an L-left ideal, L-right ideal and L-ideal of \(R\), respectively we denote by TLI\((R)\), TLI\(_L\)(R) and TLI\(_R\)(R), respectively, the set of all TL-left ideals of \(R\), the set of all TL-right ideals of \(R\) and the set of all TL-ideals of \(R\).

### 3. TL-Ring Homomorphisms

In this section we gave the definition of TL-ring homomorphism and carry with this definition the most of the theorems and definitions about ring homomorphisms.

**Definition 3.1.** Let \(R\) and \(S\) be rings, \((R, E), (S, F)\) be TL-equivalence relations and \(f \in F(R \times S, E, F, L)\) be TL-function. \(f\) is called TL-ring homomorphism if it satisfies the following conditions

\(f(x + x', y + y') \geq T(f(x, y), f(x', y'))\)
\(f(x, y, y') \geq T(f(x, y), f(x', y'))\)

**Definition 3.2.** Let \(f \in F(R \times S, E, F, L)\) be a TL-ring homomorphism

a) If \(f(0,0) = 1\) then \(f\) is called a perfect TL-ring homomorphism.

b) If \(f(-x, -y) \geq f(x, y)\) for all \(x \in R\) and \(y \in S\), then \(f\) is called a strong TL-ring homomorphism.

c) If \(f\) is both perfect and strong TL-ring homomorphism, then it is called as a complete TL-ring homomorphism.

**Theorem 3.3.** Let \(f \in F(R \times S, E, F, L)\) and \(g \in F(S \times K, E, G, L)\) be TL-ring homomorphisms and \(T\) be a infinitely \(V\)-distributive \(t\)-norm. Then \(g \circ_T f \in F(R \times K, E, G, L)\) TL-function is a TL-ring homomorphism.

**Proof** From theorem 2.17 \(g \circ_T f\) is TL-function. Now we show that only \(g \circ_T f\) satisfies the TL-ring homomorphism conditions.

i) For \(x, x' \in R\) and \(y, y' \in S\) and \(z, z' \in K\)
\[
(g \circ_T f)(x + x', z + z')
\]
\[
= \bigvee_{y \in S} T(f(x + x', y), g(y, z + z'))
\]
\[
= \bigvee_{y, y' \in S} T(f(x + x', y + y'), g(y + y', z + z'))
\]
\[ \geq \bigvee_{y, y' \in S} T\left( T(f(x, y), f(x', y')), T(g(y, z), g(y', z')) \right) \]

\[ = T\left( \bigvee_{y \in S} T(f(x, y), g(y, z)), \bigvee_{y' \in S} T(f(x', y'), g(y', z')) \right) \]

\[ = T((g \circ_T f)(x, z), (g \circ_T f)(x', z')). \]

ii) For \( x, x' \in R \) and \( y, y' \in S \), let \( z, z' \in K \)

\[ \geq \bigvee_{y, y' \in S} T\left( f(x, x', y), g(y, z, z') \right) \]

\[ = \bigvee_{y, y' \in S} T\left( f(x, x', y'), g(y, y', z, z') \right) \]

\[ \geq \bigvee_{y, y' \in S} T\left( T(f(x, y), f(x', y')), T(g(y, z), g(y', z')) \right) \]

\[ = T\left( \bigvee_{y \in S} T(f(x, y), g(y, z)), \bigvee_{y' \in S} T(f(x', y'), g(y', z')) \right) \]

\[ = T((g \circ_T f)(x, z), (g \circ_T f)(x', z')). \]

**Theorem 3.4.** Let \( f \in F(R \times S, E, F, L) \) be a perfect TL-ring homomorphism and \( T \) be an infinitely \( V \)-distributive t-norm. If \( A: R \rightarrow L \) is a TL-subring of \( R \), then \( f(A): S \rightarrow L \) is a TL-subring of \( S \).

**Proof.** For \( x, x' \in R \) and \( y, y' \in S \)

\[ R1) \text{ Since for } x = 0 \quad T(A(0), f(0,0)) = 1 \text{ then } \]

\[ f(A)(0) = \bigvee_{x \in R} T(A(x), f(x, 0)) = 1 \]

\[ R2) f(A)(-y) = (-f(A))(y) \]

\[ = - \bigvee_{x \in R} T(A(x), f(x, y)) \]

\[ = \bigvee_{x \in R} T((-A)(x), f(x, y)) \]

\[ = \bigvee_{x \in R} T(A(-x), f(x, y)) \]

\[ \geq \bigvee_{x \in R} T(A(x), f(x, y)) = f(A)(y) \]

\[ R3) f(A)(y + y') = \bigvee_{x \in R} T(A(x), f(x, y + y')) \]

= \bigvee_{x, x' \in R} T\left( T(A(x), f(x, x')), T(f(x, y), f(x', y')) \right) \]

\[ \geq \bigvee_{x, x' \in R} T\left( T(A(x), A(x')), T(f(x, y), f(x', y')) \right) \]

\[ = T\left( \bigvee_{x \in R} T(A(x), f(x, y)), \bigvee_{x' \in R} T(A(x'), f(x', y')) \right) \]

\[ = T(f(A)(y), f(A)(y')) \]

**R4) f(A)(y, y') = \bigvee_{x \in R} T(A(x), f(x, y, y')) \]

\[ \geq \bigvee_{x, x' \in R} T\left( T(A(x), A(x')), T(f(x, y), f(x', y')) \right) \]

\[ = T\left( \bigvee_{x \in R} T(A(x), f(x, y)), \bigvee_{x' \in R} T(A(x'), f(x', y')) \right) \]

\[ = T(f(A)(y), f(A)(y')) \]

**Theorem 3.5.** Let \( f \in F(R \times S, E, F, L) \) be a perfect TL-ring homomorphism and \( T \) be an infinitely \( V \)-distributive t-norm. If \( B: S \rightarrow L \) is a TL-subring of \( S \), then \( f^{-1}(B): R \rightarrow L \) is a TL-subring of \( R \).

**Proof.** For \( x, x' \in R \) and \( y, y' \in S \)

\[ R1) f^{-1}(B)(0) = \bigvee_{y \in S} T(B(y), f(0,0)) \]

for \( y = 0 \) \( T(B(0), f(0,0)) = 1 \) then \( f^{-1}(B)(0) = 1 \).

\[ R2) f^{-1}(B)(x) = -f^{-1}(B)(x) \]

\[ = - \bigvee_{y \in S} T(B(y), f(x, y)) \]

\[ = \bigvee_{y \in S} T((-B)(y), f(x, y)) \]

\[ = \bigvee_{y \in S} T(B(-y), f(x, y)) \]

\[ \geq \bigvee_{y \in S} T(B(y), f(x, y)) = f^{-1}(B)(x). \]
R3) $f^{-1}(B)(x + x') = \bigvee_{y \in S} T(B(y), f(x + x', y))$

$= \bigvee_{y, y' \in S} T(B(y + y'), f(x + x', y + y'))$

$\geq \bigvee_{y, y' \in S} T \left( T(B(y), B(y')), T(f(x, y), f(x', y')) \right)$

$\geq T \left( \bigvee_{y \in S} T(B(y), f(x, y)), \bigvee_{y' \in S} T(B(y'), f(x', y')) \right)$

$= T(f^{-1}(B)(x), f^{-1}(B)(x'))$.

R4) $f^{-1}(B)(x, x') = \bigvee_{y \in S} T(B(y), f(x, x', y))$

$= \bigvee_{y, y' \in S} T(B(y, y'), f(x, x', y, y'))$

$\geq \bigvee_{y, y' \in S} T \left( T(B(y), B(y')), T(f(x, y), f(x', y')) \right)$

$\geq T \left( \bigvee_{y \in S} T(B(y), f(x, y)), \bigvee_{y' \in S} T(B(y'), f(x', y')) \right)$

$= T(f^{-1}(B)(x), f^{-1}(B)(x'))$.

Theorem 3.7. Let $f \in F(R \times S, E, F, L)$ be a perfect TL-ring homomorphism and $T$ be a infinitely $V$-distributive t-norm. If $B: S \rightarrow L$ is a TL-ideal of $S$, then $f^{-1}(B): R \rightarrow L$ is a TL-ideal of R.

Proof. The conditions R1, R2 and R3 are provided by Theorem 3.5.

For $x, x' \in R \ y, y' \in S$

R5) $f^{-1}(B)(x, x') = \bigvee_{y \in S} T(B(y), f(x, x', y))$

$= \bigvee_{y, y' \in S} T(B(y, y'), f(x, x', y, y'))$

$= T \left( \bigvee_{y \in S} T(B(y), f(x, y)), \bigvee_{y' \in S} f(x', y') \right)$

$\left( \bigvee_{y' \in S} f(x', y') = 1 \text{ for all } x' \in R \right)$

$\text{because } f \text{ is TL - function.}$

$= T \left( \bigvee_{y \in S} T(B(y), f(x, y)), 1 \right)$

$= \bigvee_{y \in S} T(B(y), f(x, y)) = f^{-1}(B)(x)$.

R5) $f^{-1}(B)(x, x') = \bigvee_{y \in S} T(B(y), f(x, x', y))$

$= \bigvee_{y, y' \in S} T(B(y, y'), f(x, x', y, y'))$

$= T \left( \bigvee_{y \in S} T(B(y), f(x, y)), \bigvee_{y' \in S} T(f(x, y), f(x', y')) \right)$

$\geq T \left( \bigvee_{y' \in S} f(x', y'), T(f(x, y), f(x', y')) \right)$

$= T(\bigvee_{y \in S} f(x, y), \bigvee_{y' \in S} T(B(y'), f(x', y'))$
\[
\left( \bigvee_{y \in S} f(x,y) = 1 \text{ for all } x \in R \right)
\]

because \( f \) is TL-function. Then

\[
= \bigvee_{y' \in S} T(B(y'), f(x', y')) = f^{-1}(B)(x').
\]

**Definition 3.8.** Let \( f \in F(R \times S, E, F, L) \) be a TL-ring homomorphism. Define \( \text{Ker} f \in F(R, L) \) and \( \text{Im} f \in F(S, L) \) as follows.

\[
(\text{Ker} f)(x) = f(x, 0) \text{ and } (\text{Im} f)(y) = \bigvee_{x \in R} f(x, y)
\]

where \( x \in R \) and \( y \in S \). \( \text{Ker} f \) is called the kernel of \( f \) and \( \text{Im} f \) is called the image of \( f \).

**Theorem 3.9.** Let \( f \in F(R \times S, E, F, L) \) be a complete TL-ring homomorphism then \( \text{Ker} f \in F(R, L) \) is a TL-subring of \( R \).

**Proof.** For \( x, y \in R \)

\[
\text{R1)} \quad (\text{Ker} f)(0) = f(0, 0) = 1 \quad \text{because } f \text{ is complete TL-ring homomorphism.}
\]

\[
\text{R2)} \quad (\text{Im} f)(y) = \bigvee_{x \in R} f(x, y) \geq f(x, 0)
\]

\[
= (\text{Im} f)(y) \quad \text{(because } f \text{ is complete).}
\]

\[
\text{R3)} \quad (\text{Im} f)(y + y') = \bigvee_{x \in R} f(x, y + y')
\]

\[
= (\text{Im} f)(y).
\]

Theorem 3.10. Let \( f \in F(R \times S, E, F, L) \) be a complete TL-ring homomorphism and \( T \) be a infinitely \( \bigvee \)-distributive \( t \)-norm. Then \( \text{Im} f \in F(S, L) \) is a TL-subring of \( S \).

**Proof.** For \( x, x' \in R \) \( y, y' \in S \)

\[
\text{R1)} \quad \text{It follows } (\text{Im} f)(0)
\]

\[
= \bigvee_{x \in R} f(x, 0) = 1 \quad \text{because for } x = 0
\]

\[
f(0,0) = 1.
\]

\[
\text{R2)} \quad (\text{Im} f)(-y) = \bigvee_{x \in R} f(x, -y) \geq \bigvee_{x \in R} f(x, y)
\]

\[
= (\text{Im} f)(y) \quad \text{(because } f \text{ is complete).}
\]

\[
\text{R3)} \quad (\text{Im} f)(y + y') = \bigvee_{x \in R} f(x, y + y')
\]

\[
= \bigvee_{x \in R} f(x + x', y + y')
\]

\[
\geq \bigvee_{x \in R} T(f(x, y), f(x', y'))
\]

\[
= T\left( \bigvee_{x \in R} f(x, y), \bigvee_{x' \in R} f(x', y') \right)
\]

\[
= T(\text{Im} f(y), \text{Im} f(y')).
\]

**Theorem 3.11.** Let \( R \) and \( S \) be rings, \((R, ER)\) and \((S, FS)\) be TL-equivalencies and \( f : R \rightarrow S \) function be a ring homomorphism. Then TL-function \( \tilde{f} \in F(R \times S, E, F, L) \) \( \tilde{f} : R \times S \rightarrow L \) defined by

\[
\tilde{f}(x, y) = \begin{cases} 
1 & f(x) = y \\
0 & f(x) \neq y
\end{cases}
\]

is a TL-ring homomorphism.

**Proof.** It is clear that \( \tilde{f} \) is a TL-function. Let obtain the TL-ring homomorphism conditions.

\[
a) \quad \tilde{f}(x + x', y + y')
\]

\[
= \begin{cases} 
1 & f(x + x') = y + y' \\
0 & f(x + x') \neq y + y'
\end{cases}
\]

\[
i) \quad \text{If } f(x + x') = y + y' \text{ then } \tilde{f}(x + x', y + y') = 1 \text{ and the inequality }
\]

\[
\tilde{f}(x + x', y + y') \geq T(\tilde{f}(x, y), \tilde{f}(x', y'))
\]
is satisfied.

**ii)** If \( f(x + x') \neq y + y' \) then
\[
f(x) = y \text{ and } f(x') = y' \text{ can’t both because if it can}
\]
\[
f(x) + f(x') = y + y' \Rightarrow f(x + x') = y + y'
\]
so one of them doesn’t exist then
\[
T(\bar{f}(x, y), \bar{f}(x', y')) = T(0, 1) = 0
\]
and the inequality
\[
f(x + x', y + y') \geq T(\bar{f}(x, y), \bar{f}(x', y'))
\]
is satisfied.

b) \( \bar{f}(x, y)yy' = \begin{cases}
1 & f(x') = yy' \\
0 & f(x') \neq yy'
\end{cases} \)

**i)** If \( f(x') = yy' \) then
\[
\bar{f}(x', yy') = 1 \text{ and } \bar{f}(x', yy')
\]
\[
\geq T(\bar{f}(x, y), \bar{f}(x', y')) \text{ is satisfied.}
\]

ii) If \( f(x') \neq yy' \) then
\[
f(x) = y \text{ and } f(x') = y' \text{ can’t both because if it can}
\]
\[
f(x)f(x') = yy' \Rightarrow f(x') = yy'
\]
so one of them doesn’t exist then
\[
T(\bar{f}(x, y), \bar{f}(x', y')) = T(0, 1) = 0 \text{ and the inequality}
\]
\[
\bar{f}(x', yy') \geq T(\bar{f}(x, y), \bar{f}(x', y')) \text{ is satisfied.}
\]

**Theorem 3.12.** Let \( f \in F(R \times S, E, F, L) \) and \( g \in F(S \times K, F, G, L) \) be TL-ring homomorphisms. Then

a) \( \text{Ker}(g \circ_T f) = f^{-1}(\text{Ker}g) \)

b) \( \text{Im}(g \circ_T f) = g(\text{Im}f) \).

**Proof.**
a) \( \text{Ker}(g \circ_T f) = (g \circ_T f)(x, 0) \)
\[
= \bigvee_{y \in S} T(f(x, y), g(y, 0))
\]
\[
= \bigvee_{y \in S} T(f(x, y), \text{Ker}g(y)) = f^{-1}(\text{Ker}g)(x).
\]

b) \( \text{Im}(g \circ_T f)(z) = \bigvee_{x \in R} (g \circ_T f)(x, z) \)
\[
= \bigvee_{x \in R} \left( \bigvee_{y \in S} T(f(x, y), g(y, z)) \right)
\]
\[
= \bigvee_{y \in S} \left( \bigvee_{x \in R} T(f(x, y), g(y, z)) \right)
\]
\[
= \bigvee_{y \in S} T\left( \bigvee_{x \in R} f(x, y), g(y, z) \right)
\]

\[
= \bigvee_{y \in S} T((\text{Im}f)(y), g(y, z)) = g(\text{Im}f)(z).
\]

**Theorem 3.13.** Let \( f \in F(R \times S, E, F, L) \) be a strong TL-ring homomorphism and \( T \) be a infinitely \( V \)-distributive t-norm. Then for \( A \in F(R, L) \),
\[
f^{-1}(f(A)) \leq A + \text{Ker}f.
\]

**Proof.**
\[
f^{-1}(f(A))(x) = \bigvee_{y \in S} T(f(A)(y), f(x, y))
\]
\[
= \bigvee_{y \in S} \left( \bigvee_{x' \in R} T(A(x'), f(x', y)) \right), f(x, y)
\]
\[
= \bigvee \bigvee_{x' \in R} T(A(x'), f(x', y), f(x, y))
\]
\[
\leq \bigvee \bigvee_{x' \in R} T(A(x'), f(x', y), f(-x, -y))
\]
\[
= \bigvee \bigvee_{x' \in R} T(A(x'), f(x', y), f(-x, -y))
\]
\[
\leq \bigvee \bigvee_{x' \in R} T(A(x'), f(x' - x, y - y))
\]
\[
\leq \bigvee \bigvee_{x' \in R} T(A(x'), f(x - x', 0))
\]
\[
= \bigvee T(A(x'), (\text{Ker}f)(x - x'))
\]
(by \( x' + (x - x') = x = (A + \text{Ker}f)(x) \).

**Theorem 3.14.** Let \( (R_1, E_1), (R_2, E_2), (S_1, F_1) \) and \( (S_2, F_2) \) be TL-equivalence relations and \( f: R_1 \times S_1 \rightarrow L \) \( g: R_2 \times S_2 \rightarrow L \) be TL-ring homomorphisms. Then the TL-function defined by
\[
g \times f: (R_1 \times R_2) \times (S_1 \times S_2) \rightarrow L
\]
\( (g \times f)((x_1, x_2), (y_1, y_2)) = T(f(x_1, y_1), g(x_2, y_2)) \)
is a TL-ring homomorphism.

**Proof.** We proved that \((g \times f)\) is a TL-function in Theorem 2.18. Now we will prove the TL-ring homomorphism conditions.

**H1.** \( x_1, x_1' \in R_1 \), \( x_2, x_2' \in R_2 \), \( y_1, y_1' \in S_1 \)
,\ y_2, y'_2 \in S_2
\begin{align*}
(g \times f)((x_1, x_2) + (x'_1, x'_2), (y_1, y_2) + (y'_1, y'_2)) \\
= (g \times f)((x_1 + x'_1, x_2 + x'_2), (y_1 + y'_1, y_2 + y'_2)) \\
= T(f(x_1 + x'_1, y_1 + y'_1), g(x_2 + x'_2, y_2 + y'_2)) \\
\geq T(T(f(x_1, y_1), f(x'_1, y'_1)), T(g(x_2, y_2), g(x'_2, y'_2))) \\
= T(T'(f(x_1, y_1), g(x_2, y_2), f(x'_1, y'_1), g(x'_2, y'_2))) \\
= T((g \times f)((x_1, x_2) + (x'_1, x'_2), (y_1, y_2) + (y'_1, y'_2))).
\end{align*}
\]

H2. $x_1, x'_1 \in R_1, x_2, x'_2 \in R_2, y_1, y'_1 \in S_1$
\[y_2, y'_2 \in S_2\]
\begin{align*}
(g \times f)((x_1, x_2), (x'_1, x'_2), (y_1, y_2), (y'_1, y'_2)) \\
= (g \times f)((x_1 \times x'_1, x_2 \times x'_2), (y_1 \times y'_1, y_2 \times y'_2)) \\
= T\left(T'(f(x_1, y_1), f(x'_1, y'_1)), T(g(x_2, y_2), g(x'_2, y'_2))\right) \\
\geq T\left(T'(f(x_1, y_1), g(x_2, y_2), f(x'_1, y'_1), g(x'_2, y'_2))\right) \\
= T\left(T'(f(x_1, y_1), g(x_2, y_2), f(x'_1, y'_1), g(x'_2, y'_2))\right) \\
= T\left((g \times f)((x_1, x_2), (y_1, y_2)), (g \times f)((x'_1, x'_2), (y'_1, y'_2))\right).
\end{align*}

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