Exceptional Point of Degeneracy in Backward-Wave Oscillator with Distributed Power Extraction

Tarek Mealy, Ahmed F. Abdelshafy and Filippo Capolino

Department of Electrical Engineering and Computer Science, University of California, Irvine, CA 92697 USA
tmealy@uci.edu, abdelsha@uci.edu and f.capolino@uci.edu

Abstract—We show how an exceptional point of degeneracy (EPD) is formed in a system composed of an electron beam interacting with an electromagnetic mode guided in a slow wave structure (SWS) with distributed power extraction from the interaction zone. Based on this kind of EPD, a new regime of operation is devised for backward wave oscillators (BWOs) as a synchronous and degenerate regime between a backward electromagnetic mode and the charge wave modulating the electron beam. Degenerate synchronization under this EPD condition means that two complex modes of the interactive system do not share just the wavenumber, but they rather coalesce in both their wavenumbers and eigenvectors (polarization states). In principle this new condition guarantees full synchronization between the electromagnetic wave and the beam’s charge wave for any amount of output power extracted from the beam, setting the threshold of this EPD-BWO to any arbitrary, desired, value. Indeed, we show that the presence of distributed radiation in the SWS results in having high-threshold electron-beam current to start oscillations which implies higher power generation. This may lead to very high power efficient BWOs with very high output power and excellent spectral purity.

Index Terms—Exceptional point of degeneracy, Slow-wave structures, Backward-wave oscillators, High power microwave.

I. INTRODUCTION

Exceptional points of degeneracy (EPDs) are points in parameter space of a system at which two or more eigenmodes coalesce in their eigenvalues (wavenumbers) and eigenvectors (polarization states). EPDs are found in uniform waveguides at their cutoff frequencies and in periodic waveguides at the regular band edge (RBE) and at the degenerate band edge (DBE). However, these RBE and DBE conditions are EPD realized in lossless structures [1]–[6]. Here, we investigate an EPDs that requires both losses and gain being simultaneously present in the SWS. A particular case of simultaneous existence of symmetric gain and loss is based on Parity-time (PT)-symmetry, which is a well-known condition that leads to the occurrence of an EPD [7]–[11], that however requires a spatial symmetry in gain and loss. The EPD considered here is far from that condition, involving two completely different media that support waves, a plasma and a waveguide for electromagnetic waves, but still requires their interaction and the simultaneous presence of gain and loss. In this paper the term “loss” is not associated to damping energy, but it is rather referred to a waveguide perspective where energy exits the waveguide in a distributed fashion, a mechanism referred to as distributed power extraction (e.g., radiation) from the interaction zone, for example by realizing distributed long slot along the SWS or by periodically spaced holes.

In this paper the linear electron beam is modeled using the description presented in [12]. Then we use the well established Pierce method [13] to account for the interaction of the electromagnetic (EM) wave in the SWS and the electron beam, assuming small signal modulation of the beam, that consists of an equivalent transmission line (TL) telegrapher’s equations coupled to the equations that describe the beam dynamics. The distributed radiation coming out of the SWS is conventionally represented by a distributed equivalent radiation resistance in the TL [14], [15], [16]. Recently, two coupled transmission lines with balanced gain and loss (balanced refers to the combination that generates an EPD) were shown to support EPDs without the need of PT-symmetry [17], [18], however in this paper the physics is described by a TL supporting backward wave propagation that interacts with an electron beam which is a plasma medium that supports two non-reciprocal waves. We show how the new EPD condition found in this paper can be used in high power electron beam devices.

Backward-wave oscillators (BWOs) are widely used as high power sources in radars, satellite communications, and various other applications. A BWO is composed of a SWS that guides EM backward-waves, where their phase and group velocities have opposite directions. The interaction between the electron beam and a backward EM mode constitutes a distributed feedback mechanism that makes the whole system unstable at certain frequencies [19], [20]. The extracted power in a conventional BWO is taken from one of the SWS waveguide ends. The most challenging issue in BWOs is the limitation in power generation level, i.e., the extracted power from the electron beam relative to its total power. Indeed conventional BWOs exhibit small starting beam current (to induce sustained oscillations) and limited power efficiency without reaching very high output power level. In this paper we propose to use an EPD for a new regime of operation in BWOs by introducing two special features: the “degenerate synchronization” mechanism based on the EPD with the simultaneous presence of continuous distributed power extraction, rather than power extraction from the waveguide end. Therefore in this paper the term “loss” does not refer to material loss but rather to a distributed power extraction (Fig. 1) that in antennas terminology is referred to as “radiation loss”, while the gain is provided by the electron beam interacting with the SWS. This mechanism differs from a conventional BWO
setup, since power exists the interaction zone via a distributed set of radiating slots along the SWS (Fig. 1), i.e., by extracting power in a distributed fashion. Alternatively, the power exiting the slots could be collected by an adjacent waveguide. Recently, SWSs exhibiting a degenerate band edge (DBE), which is an EPD of order four in a lossless periodic waveguide, were proposed to enhance the performance of high power devices [21]-[24]. It is important to point out that the EPDs obtained in these previously mentioned works were obtained in the "cold" SWS based on periodicity. Here, instead, we proposed a new regime where the EPD is maintained in the "hot" SWS, i.e., in presence of the interacting electron beam, that is a very large source of distributed gain. The degenerate synchronization between the charge wave induced on the electron beam and the EM slow wave is occurring at the EPD, making this degenerate synchronization regime a very special condition not yet explored.

II. SYSTEM MODELING

Consider the SWS shown in Fig. 1(a) supporting a Floquet-Bloch mode whose slow wave spatial harmonic interacts with an electron beam, and radiates via a distributed set of slots (radiation occurs via a fast wave spatial harmonic, usually the so called "1" harmonic). The configuration in Fig. 1(a) is given as a possible realization, though another one may consist in collecting the power exiting the interaction zone in a distributed fashion by an adjacent waveguide. We assume that the interaction between the SWS mode and the electron beam is modeled via the Pierce small-signal theory of traveling-wave tubes [12], [13], [25]-[29] that describes the evolution of the EM fields and electron beam dynamics assuming small signal modulation in the beam’s electron velocity and charge density. The beam electrons have average velocity and linear charge density $u_0$ and $\rho_0$, respectively. The electron beam has an average (dc) current $I_0 = -\rho_0 u_0$ and a dc (time-average) equivalent kinetic dc voltage $V_0 = \frac{1}{2} u_0^2 / \eta$ [20], [30], and $\eta = e / m = 1.758820 \times 10^{11} \text{ C/Kg}$ is the charge-to-mass ratio of the electron with charge equal to $-e$ and $m$ is its rest mass. The small signal modulation in electron beam velocity and charge density $u_b$ and $\rho_b$, respectively, form the so called “charge wave”. The linearized basic equations governing charge motion and continuity are written in their simplest form as

$$\partial_z u_b + u_0 \partial_z u_b = \eta E_z$$

and

$$\partial_z \rho_b = -\rho_0 \partial_z u_b - u_0 \partial_z \rho_b,$$

where $E_z$ is the electric field component in the z-direction of the EM mode in the SWS interacting with the electron beam, and the symbol $\partial_z$ indicates differentiation with respect to a variable $z$.

For convenience, we define equivalent kinetic beam voltage and current as $V_b = u_b u_0 / \eta$ and $I_b = u_b \rho_0 + u_0 \rho_b$. Assuming an $e^{i\omega t}$ time dependence, the two previously mentioned equations may be written in term of the beam equivalent voltage and current as $\partial_z V_b = -i \beta_0 V_b + E_z$ and $\partial_z I_b = i g V_b - i \beta I_b$, where $\beta_0 = \omega / u_0$ is the equivalent propagation constant of the unmodulated beam and $g = \frac{1}{2} I_0 \beta_0 / V_0$.

The EM mode propagating in the SWS is described by the equivalent transmission line in Fig. 1(b), with distributed per-unit-length series impedance $Z$ and shunt admittance $Y$, and equivalent voltage $V(z)$ and current $I(z)$ that satisfy the telegrapher’s equations $\partial_z V = -Z I$ and $\partial_z I = -Y V + i s$. The last term $i s$ accounts for the electron stream flowing in the SWS that loads the TL as a shunt displacement current according to [13], [26], [29] and whose expression is given by $i s = \partial_z I_b$. The propagation constant of the modes in the non-interactive EM system (i.e., when $i s = 0$) is given by $\beta_{ph} = \sqrt{-Z Y}$.

Based on the Pierce’s model, the EM wave couples to the electron beam with its longitudinal electric field given by $E_z = -\partial_z V$. For convenience, we define here a state vector $\Psi(z) = [ V(z), I(z), V_0(z), I_0(z) ]^T$ that describes the system evolution with coordinate $z$. Thus, the interacting EM mode and the beam modulation are described as [29]

$$\partial_z \Psi(z) = -i M \Psi(z)$$

where $M$ is the $4 \times 4$ system matrix

$$M = \begin{bmatrix}
0 & -iZ & 0 & 0 \\
-iY & 0 & g & -\beta_0 \\
0 & iZ & \beta_0 & 0 \\
0 & 0 & -g & \beta_0
\end{bmatrix}.$$

III. SECOND ORDER EPD IN AN INTERACTING ELECTROMAGNETIC WAVE AND AN ELECTRON BEAM’S CHARGE WAVE

Assuming a state vector $z$-dependence of the form $\Psi(z) \propto e^{-ikz}$, where $k$ is the wavenumber of a mode in the interacting system, i.e., in the hot SWS, the eigenmodes are obtained by solving the eigenvalue problem $k \Psi(z) = M \Psi(z)$, and the modal dispersion relation is given by

$$\partial_z \Psi(z) = -i M \Psi(z)$$
The “degenerate synchronization” for the interacting system \( \Psi \) obtained in conventional regimes promises to achieve electron beam devices with features not previously mentioned. Two conditions, it is found that the EPD occurs when the transmission line series impedance and shunt admittance are \( Z = Z_e \) and \( Y = Y_e \), where

\[
Z_e = \frac{i\beta_0^2\delta_e^3}{g_e}, \quad Y_e = \frac{ig_e(\delta_e + 1)^3}{\delta_e^3},
\]

and \( \delta_e = (k_e - \beta_{0e})/\beta_{0e} \) represents the relative deviation of degenerate modal wavenumber \( k_e \) (of the interactive system) from the beam equivalent propagation constant \( \beta_{0e} = \omega_e/\delta_e \). A detailed formulation of the derivation of the EPD condition in (4) is presented in Appendix C.

We simplify the two equations given in (3) to get rid of \( \delta_e \), thus the EPD conditions in (4) is simplified and the EPD SWS propagation constant \( \beta_{phe} \) (that is \( \beta_{ph} \) at the EPD frequency) and characteristic impedance \( Z_{ce} = \sqrt{Z_e/Y_e} \) satisfy

\[
\beta_{0e}\beta_{phe}^2 = \left( \sqrt{-j\delta_e Z_{ce}} \beta_{0e}\beta_{phe} + \beta_{0e} \right)^3.
\]

The above equation constrains all the system parameters to have an EPD, hence it is also saying that the EPD happens at a specific choice of the electron beam current denoted by \( I_e \). The interaction between the charge wave and the EM wave happens when they are synchronized, by matching the EM wave phase velocity \( \omega/\beta_{phe} \) to the electron beam propagation constant \( \omega_e/\beta_{0e} \). Note that this is just an initial criterion, because the phase velocity of the modes in the interactive systems are different from \( \omega/\beta_{ph} \) and \( \omega_e/\beta_{0e} \). When the system parameters are such that equation (5) is satisfied, there are two modes (in the interactive system) that have exactly the same phase velocity \( \omega/\Re(\kappa_e) \). Since this synchronization condition corresponds to an EPD, the two modes in the interactive system are actually identical also in their eigenvectors \( \Psi \). Note that the spatial evolution of the interacting EM and charge wave is described by four modes that are solutions of (4), three of which have a positive real part of \( k \). In this paper we enforce a very special kind of synchronization based on the EPD. We are enforcing that two of the resulting eigenmodes full coalesce in both their wavenumber and system state variable \( \Psi(z) \). This is a new kind of EPD found in physical systems and may be very promising to achieve electron beam devices with features not obtained in conventional regimes.

The EPD condition obtained in (5) guarantees that the system has two repeated eigenvalues \( k_e = \beta_{0e}\beta_{phe}^{\frac{1}{3}} \) and two coalescing eigenvectors at \( \Psi_e = [1, \, ik_e/Z_e, \, -1/\delta_e, \, g_e/(\beta_{0e}\delta_e^2)]^T \) that form the “degenerate synchronization”. For the interacting system of EM and charge wave, the EPD represents a point in parameter space at which the system matrix \( M \) in (2) is not diagonalizable and it is indeed similar to a matrix that contains a \( 2 \times 2 \) Jordan block. This implies that the solution of Eq. (1) includes an algebraic linear growth factor resulting in unusual wave propagation characteristics. Solution of (3) leads to four eigenmode wavenumbers, and due to the highly non-reciprocal physical nature of the electron beam, three waves have positive real part (i.e., \( \Re(k) > 0 \)) and one has it negative. According to the traveling-wave tube theory in [32], [13], conventionally applied to BWOs [19], there are three wavenumbers with \( \Re(k) > 0 \) that are participating to the synchronization, and the electric field in the SWS is generally represented as \( V(z) = V_1 e^{-ik_1z} + V_2 e^{-ik_2z} + V_3 e^{-ik_3z} \).

Remarkably, at the EPD the interactive system has two modes with the same wavenumber \( k_e \), resulting in an electric field with linear growth factor as

\[
V(z) = zV_1 e^{-ik_e z} + V_2 e^{-i2k_e z} + V_3 e^{-i3k_e z}
\]

which is completely different than any other regime of operations. The charge wave has an analogous description.

Using (4), we constrain \( \delta_e \) to provide \( Z_e \) and \( Y_e \) with positive real part, that means we use passive SWS, described by passive TLs, with gain coming from the interaction with the electron beam. Figure 2 shows two sectoral regions of allowed \( \delta_e \) that produce EPDs for different transmission line types and at the same time these two regions satisfy the passivity condition of the TL, \( \Re(Z) \geq 0 \) and \( \Re(Y) \geq 0 \), where the real part of \( Z \) and \( Y \) represents mainly radiation losses (we assume the waveguide made of copper that is an excellent conductor at radio frequency). We observe two important regions where EPDs are realized by exploiting either forward or backward propagating EM modes in the SWS as shown in the inset of Fig. 2. In this paper we focus on the light blue region that represents complex values of \( \delta_e \) with the EPDs resulting from the interaction of a backward electromagnetic wave (radiation is represented in TL by series or/and parallel losses) and the charge wave modulating the electron beam.

In the following we show an example based on an electron beam with dc voltage of \( V_0 = 23 \, \text{kV} \) and dc current of \( I_0 = 0.1 \, \text{A} \) (hence it corresponds to the EPD electron beam current \( I_e \), since the choice of the TL parameters are chosen accordingly) and assume the operational frequency at which the EPD occurs to be 1 GHz. We require (as an example) a relative degenerate wavenumber deviation of \( \delta_e = 0.01 + i0.017 \) that is lying exactly on the white dashed line shown in Fig. 2, with \( \angle \delta_e = \pi/3 \) (the phase of \( \delta_e \)). The corresponding wavenumber of two coalescing modes in the interactive system is \( k_e = (1.03 + i0.05) \beta_{0e} = 71.95 + i3.63 \, \text{l/m} \). By imposing the beam parameters and the chosen \( \delta_e \) in the EPD conditions in (4) we obtain the distributed per-unit-length TL impedance and admittance of SWS to be \( Z_e = -257.1 \, \text{Ohm/m} \) (i.e., capacitive) and \( Y_e = 1-i19.54 \, \text{Siemens/m} \) (i.e., inductive with losses), representing a backward wave in the SWS. Losses here are not assumed modeling energy damping but rather energy...
Fig. 2. The two sectoral regions represent complex values of \( \delta_e \), associated to EPDs resulting from the interaction of backward or forward electromagnetic waves (radiation is represented in TL by series or parallel losses) and the charge wave of electron beam. The value \( \delta_e = (k_e - \beta_1)/\beta_1 \) represents the wavenumber deviation of the EPD complex wavenumber \( k_e \) (of the interactive system) from the equivalent electron beam wavenumber \( \beta_1 \), that satisfies [2], assuming SWS realizations based on passive equivalent TLs. There are two regions of possible realizations (red and light blue), associated to the two distributed per-unit-length TL circuits on the right panel, that represent SWSs supporting either forward or backward wave propagation. In this paper we focus on EPDs obtained from backward electromagnetic waves interacting with an electron beam, i.e., those leading to \( \text{Re}(\delta_e) > 0 \) and \( \text{Im}(\delta_e) > 0 \).

Figure 3(a) shows the dispersion relation of three complex modes in the “hot” SWS, i.e., in the interactive EM wave electron beam system, obtained using the Pierce model as explained in [29]. (The forth mode with \( \text{Re}(k_e) < 0 \) is not shown since it does not have a significant role in the synchronization.) We show the real and imaginary parts of wavenumbers of the hot SWS versus normalized frequency (red curves), together with the dispersion of the beam alone (i.e., the beam “line” in green that actually represents two curves since we are neglecting the effect of the beam plasma frequency here) and of the backward wave in the cold SWS (blue curve, with negative slope). It is obvious from the figure that both the backward wave mode and a charge wave mode of the hot SWS (red curves) coalesce at the EPD frequency \( f_c \), forming an EPD of order two.

We recall that the SWS is a periodic structure, and is the slow wave harmonic of a mode that interacts with the electron beam charge wave. Though slow waves do not radiate, a spatial Floquet-Bloch harmonic of the mode is fast and able to radiate through the slots [33], justifying the presence of the conductance \( G = 1 \) Siemens/m, part of \( Y_e = 1 - i19.54 \) Siemens/m, in the TL model.

IV. EPD-BWO THRESHOLD CURRENT

The EPD is here employed to conceive a new regime of operation for BWOs based on “degenerate synchronization”. To explain that we consider the setup in Fig. 4(a) for a BWO that has “balanced gain and radiation-loss” per unit length, where distributed loss is actually representing distributed radiation (per unit length) via a shunt conductance \( G \) (see Fig. 2). The description is following the steps in [13] and [19] outlined for BWOs, where the TL is represented by a distributed circuit model shown in Fig. 4(b) that supports backward waves. An important difference from a standard BWO is that here the TL has shunt distributed “losses” that actually model the distributed radiation and whose presence is necessary to satisfy the EPD condition (5). We assume to have an unmodulated space charge at the beginning of the electron beam, i.e., \( V_b(z = 0) = 0 \) and \( I_b(z = 0) = 0 \) and we assume that the SWS waveguide length is \( \ell = N\lambda_e \), where \( \lambda_e = 2\pi/\beta_{0e} \) is the guided wavelength calculated at the EPD frequency and \( N \) is the normalized SWS length parameter. We also assume that the SWS waveguide is terminated by a load at \( z = 0 \) matched to the characteristic impedance of the TL (without loss and gain) \( R_v = \sqrt{L/C} \) and a short circuit at \( z = \ell \). We follow the same procedure used in [19] to obtain the oscillation condition which is based on imposing infinite voltage gain \( A_v = V(0)/V(\ell) \to \infty \). After simplification, and using the three-wave traveling-wave theory [13], [19], the voltage gain is written in term of the three modes concurring to the synchronization (those with \( \text{Re}(k) > 0 \) ) as

\[
A_v^{-1} e^{i\beta_0 \ell} = \frac{e^{-i\beta_0 \delta_1 \ell}}{(\delta_1 - \delta_2) (\delta_1 - \delta_3)} + \frac{e^{-i\beta_0 \delta_2 \ell}}{(\delta_2 - \delta_3) (\delta_2 - \delta_1)} + \frac{e^{-i\beta_0 \delta_3 \ell}}{(\delta_3 - \delta_1) (\delta_3 - \delta_2)} = 0,
\]

(7)
where $\delta_n = (k_n - \beta_0)/\beta_0$ and $k_1, k_2$ and $k_3$ are the three wavenumbers of the interactive EM-beam system with positive real part that are solutions of Eq. (3). In close proximity of the EPD there are two modes coalescing, with $\delta_1 = \delta_0 + \Delta/2$ and $\delta_2 = \delta_0 - \Delta/2$ where $\delta_0$ is the average and $|\Delta| \ll |\delta_0|$ is a very small quantity that vanishes at the EPD, and by observing that $|e^{-i\delta_0\ell}| \ll 1$ for very large $\ell$ because $\text{Im}(k_3) < 0$, the gain expression in (7) reduces to

$$A_G^{-1}e^{i\delta_0\ell} \approx \frac{e^{-i\delta_0\ell} \delta_0^2 \sin (\Delta \beta_0 \ell/2)}{(\delta_0 - 3) \Delta/2} = 0.$$  

(8)

From the above condition, assuming very large $\ell$, the first oscillation frequency occurs when $\Delta \beta_0 \ell = 2\pi$. This happens when the constraint on the wavenumbers $k_1 - k_2 = 2\pi/\ell$ is satisfied. This shows a very important fact, that for infinitely long structure $\ell \to \infty$ the starting oscillation condition is $k_1 = k_2 = k_e$, which corresponds to the EPD condition. This implies that the EPD is the exact condition for synchronization between the charge wave and EM wave, accounting for the interaction in infinitely long SWSs, that guarantees the generation of oscillations at the EPD frequency $f_e = \omega_e/(2\pi)$.

For finite length SWSs, oscillations occurs when $k_1 - k_2 = 2\pi/\ell$ is satisfied, assuming very large $\ell$, and since $|\Delta| \ll |\delta_0|$, we have both $k_1$ and $k_2$ very close to $k_e$, which implies that the systems is close to the EPD and hence the threshold beam current $I_{th}$ that starts oscillations is slightly different from $I_e$. A beam current slightly away from the EPD one causes the wavenumbers to bifurcate from the degenerate one $k_e$, following the Puiseux series approximation [34], (also called fractional power expansion) as $k_n - k_e \approx (-1)^n \alpha \sqrt{\Delta - I_e}$, with $n = 1, 2, \ldots$ and where $\alpha$ is a constant. This implies that $k_1 - k_2 \approx -2\alpha \sqrt{\Delta - I_e}$, and by comparing this with the $k_1 - k_2$ difference associated to the threshold beam current in a finite length SWS we infer that the threshold beam current $I_{th}$ that makes the EPD-BWO of finite length $\ell$ oscillate, asymptotically scales as

$$I_{th} \sim I_e + \left(\frac{\pi/\alpha}{\ell}\right)^2.$$  

(9)

A conventional BWO, that has no distributed power extraction and the power is delivered only to the load $R_c$, the threshold current scaling decrease with the SWS length asymptotically as $I_{th} \sim \zeta/\ell^3$, where $\zeta$ is a constant. Instead the EPD-BWO has a threshold current always larger than the EPB beam current $I_e$, which represents the current that keeps the oscillation going and simultaneously balance the distributed radiation power extracted from the beam for infinitely long SWSs. Importantly, this EPB beam current $I_e$ can be engineered to any desired value depending on how much power one wants to extract from the electron beam.

The above derivation was based on assuming $\ell \to \infty$, however a rigorous derivation for any EPB-BWO length is shown in Appendix D that leads to the determination of the threshold current and oscillation frequency. This rigorous result based on the Pierce model for the beam current threshold is used to compute the results shown in Fig. 4 when varying the EPD-BWO length or the shunt conductance $G$ that represents the distributed radiation.

Figure 4(c) shows the scaling of the threshold current versus SWS length for a system with the same parameters used in the previous example. The scaling shows that for infinitely long SWSs the oscillation starting current is equal to the EPD beam current $I_e$ which is consistent with the asymptotic relation in (9). Figure 4(c) also shows a comparison between the threshold current of the conventional BWO (that does not have distributed power extraction, i.e., $G = 0$) and that of the EPD-BWO (with distributed power extraction, represented by $G = 1$ Siemens/m) by showing their current scaling with the SWS length. The threshold beam current $I_{th}$ of the conventional BWO vanishes when the SWS length increases, whereas the threshold beam current $I_{th}$ of the EPD-BWO tends to the value of $I_e = 0.1$ A. Figure 4(d) shows the threshold beam current $I_{th}$ of the EPD-BWO (with distributed power extraction, represented by $G = 1$ Siemens/m) by showing its current scaling as a function of the radiation “loss” per unit length $G$. We also show the required starting beam current (i.e. the threshold) for oscillations (red curve) for a finite length “hot” SWS working at the EPD: we assume the SWS length normalized to the wavelength at the EPD frequency of $N = 70$. It is important to point out that the radiated power per-unit-length of the SWS is determined using $P_{\text{rad}}(z) = \frac{1}{2}G|V(z)|^2$, thus it is linearly proportional to the parameter $G$, i.e., higher values of $G$ imply higher radiated power per-unit-length and therefore higher level of energy extraction from the SWS.

We have shown two very important facts here: first, the threshold beam current is very close to the EPD beam current for any desired value of power extracted. This indicates that the EPD condition for hot SWSs with finite length is the condition that basically guarantees full synchronization between the EM guided mode and the beam’s charge wave. Secondly, the threshold current increases monotonically when increasing the required radiated power per unit length which implies a tight synchronization regime guaranteed for any high power generation. Therefore in principle the nonlinearity is maintained for any desired power output, according to the Pierce model, and this trend is definitely not observed in standard backward wave oscillators where the load is at the beginning of the SWS.

V. Conclusion

We have demonstrated the occurrence of an EPD in an interactive system made of a linear electron beam and an electromagnetic wave guided in a SWS. This have led to a possible new regime of operation of BWOs where the EPD guarantees the phase synchronism of a backward wave and a beam’s charge wave through enforcing the coalescence of two modes in both their wavenumber and state vector, a regime we named “degenerate synchronization”. A remarkable aspect of this EPD-BWO regime is that the “gain and radiation-loss balance” condition maintains a perfect degenerate synchronization between the charge wave and EM wave for any amount of required distributed power extraction. This distributed power
scaling of the threshold beam current for a finite-length SWS with power extraction, that is indeed given by 
\[ p \] backward waves. The distributed shunt conductance (unit-length) series capacitance and shunt inductance for a SWS that supports loss. "balanced gain and radiation-loss". (b) Equivalent transmission line model of the SWS with distributed (per-unit-length). We also show the scaling of the oscillation-threshold beam current \( I_{th} \) versus SWS length normalized to the wavelength \( N = \ell / \lambda_e \), where \( \lambda_e = 2\pi / \beta_0 \) is the guided wavelength calculated at the EPD frequency. It is obvious that for infinite long SWS where \( N \rightarrow \infty \) we have \( I_{th} \rightarrow I_e \) which implies that the EPD synchronization condition is also the threshold for infinitely long SWSs. (d) Scaling of the EPD beam current \( I_{th} = I_e \) versus radiation losses \( G \). We also show the scaling of the threshold beam current for a finite-length SWS with \( N = 70 \). Note that the threshold beam current is very close to the EPD current for any amount of required distributed extracted power which indicates that in principle the synchronism is achieved for any output power level.

Note that the threshold beam current is very close to the EPD current for any desired value by increasing the amount of power extracted from the electron beam subject to constant axial electric field across the beam’s cross section, we assume purely longitudinal beam model in [12]. We assume a narrow cylindrical beam of electrons subject to constant axial electric field and therefore the EPD-BWO exhibits high starting current that in principle can be set to any arbitrary value, in contrast to what happens in conventional BWOs where the beam’s starting-oscillation (i.e., the threshold) current tends to vanish when the SWS lengths increases. Remarkable, in the EPD-BWO regime the starting oscillation current is always larger than the EPD’s beam current that, in principle, can be set to any desired value by increasing the amount of power extracted per unit length. Therefore we have shown the fundamental principle that the amount of power generated under the EPD-BWO regime has no upper limit, which is imposed only by practical realizations, contrarily to conventional knowledge of BWOs.

Note that the degenerate regime discussed in this paper is very different from the ones discussed in [21]-[24]. There, it was the “cold” SWS that exhibited degeneracy conditions like the degenerate band edge (DBE), which is an EPD of order four, or the stationary inflection point (SIP), which is an EPD of order three, that were proposed to enhance the performance of high power devices. Those EPD conditions were obtained in "cold" SWSs based on periodicity, and the interaction of the EM modes in the SWS with the electron beam would perturb those degeneracy conditions, and indeed even destroy them for extremely large values of electron beam currents. Here, instead, we have proposed a new regime based on the concept of “distributed radiation and gain balance” where the EPD is maintained in the “hot” structure, i.e., in presence of the interacting electron beam, for any amount of electron beam current, in principle, and hence of any amount of power.

VI. ACKNOWLEDGMENT

This material is based upon work supported by the AFOSR under award number FA9550-18-1-0355.

APPENDIX A

ELECTRON BEAM MODEL

We show the fundamental equations that describe the evolution of electron beam dynamics in space and time. We follow the linearized equations that describe the space-charge wave on the electron beam presented in Pierce [13] based on the electron beam model in [12]. We assume a narrow cylindrical beam of electrons subject to constant axial electric field and negligible repulsion forces between electrons (hence we neglect space charge’s induced forces) compared to the force induced by the longitudinal electric field associated to the radio frequency mode in the slow wave structure (SWS). The total beam linear-charge density and electron speed are represented as

\[
\rho = \rho_0 + \rho_b, \quad u = u_0 + u_b, \quad (A.1)
\]

where the subscripts ‘0’ and ‘b’ denote dc (average value) and ac (modulation), respectively. The basic equations governing the charges’ motion and continuity are written in their simplest form as [13]

\[
\frac{\partial u}{\partial t} = -\eta E_z \quad (A.2)
\]

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial z}
\]

where \( \eta = e/m = 1.758820 \times 10^{11} \) C/Kg is the charge-to-mass ratio of the electron such that the electron charge is equal to \( -e \) and \( m \) is its rest mass. Assuming small signal modulation [12], [26], \( |u_0| \ll u_0 \) and \( |\rho_b| \ll |\rho_0| \), (A.2) is linearized to
Thus, the dc and ac beam voltage and current are determined as

\[ V = \frac{i^2}{2\eta} = V_0 + V_b, \]

\[ I = i\rho = -I_0 + I_b, \]  

(4.4)

Again, assuming small signal modulation \[12], [26], \[ |u_b| \ll u_0 \] and \[ |\rho_b| \ll \rho_0 \], we neglect the terms \[ u_b^2 \] and \[ u_b\rho_b \] in (4.4). Thus, the dc and ac beam voltage and current are determined as a function of the dc and ac beam charge density and speed \[U13, U30, U13\] as

\[ V_0 = \frac{1}{2} u_b^2 \eta, \]

\[ I_0 = -\rho_0 u_0, \]

\[ V_b = u_b \rho_b \eta, \]

\[ I_b = u_b \rho_b + \rho_0 u_b. \]  

(5.5)

By substituting (4.4) into (4.3), the evolution equations of beam equivalent kinetic voltage and current are written as

\[ \frac{\partial V_b}{\partial z} = -\frac{1}{u_0} \frac{\partial V_b}{\partial t} + E_z, \]

\[ \frac{\partial I_b}{\partial z} = \frac{i\rho_0}{u_0^2} \frac{\partial V_b}{\partial t} - \frac{1}{u_0} \frac{\partial I_b}{\partial t}. \]  

(6.6)

Assuming time-harmonic signals with time convention \(e^{i\omega t}\), that is omitted in the following for simplicity, (6.6) is simplified to

\[ \frac{\partial V_b}{\partial z} = \frac{-i\omega}{u_0} V_b + E_z, \]

\[ \frac{\partial I_b}{\partial z} = \frac{i\omega \rho_0}{u_0^2} V_b - \frac{i\omega}{u_0} I_b. \]  

(7.7)

When the electron beam is unmodulated, i.e., not interacting with an electromagnetic wave, i.e., when \(E_z = 0\), the charge wave has propagation constant \(\beta_0 = \omega / u_0\).

APPENDIX B
MODEL FOR AN ELECTRON BEAM INTERACTING WITH AN ELECTROMAGNETIC MODE IN A SLOW WAVE STRUCTURE

Consider a periodic slow wave structure (SWS) supporting an electromagnetic (EM) mode (two actually, one in each direction) that has a slow wave harmonic interacting with an electron beam as shown in Fig. 1(a) in the main body of the paper. The slow wave harmonic of the EM mode propagating in the SWS is described by the equivalent transmission line (TL) in Fig. 1(b), with distributed per-unit-length series impedance \(Z\) and shunt admittance \(Y\). The TL equivalent voltage \(V(z)\) and current \(I(z)\), that describe the guided electric and magnetic fields, respectively, satisfy the telegrapher’s equations

\[ \frac{\partial V}{\partial z} = -ZI, \]

\[ \frac{\partial I}{\partial z} = -YV + i_s. \]  

(B.1)

The last term \(i_s\) accounts for the electron stream flowing in the SWS that loads the TL as a shunt displacement current according to \[13], \[26], \[29\] and whose expression is given by \(i_s = \partial I_b / \partial z\). By ignoring the “space charge induced forces”, for simplicity, the longitudinal electric field in the SWS is given by \(E_z = -\partial V / \partial z\) \[13\]. The wavenumber of the EM modes in the non-interactive system (i.e., when \(i_s = 0\)) is given by \(\beta_{ph} = \sqrt{-ZY}\).

To study the interactive system made of a modulated charge wave interacting with an EM wave in a SWS, for convenience we define a state vector composed of the TL and e-beam voltages and currents \(\Psi(z) = [V(z), I(z), V_b(z), I_b(z)]^T\). Thus, by combining Eq. (7.7) and (B.1) the interactive system equations are rewritten as a multidimensional first order differential equation \[29\]

\[ \frac{\partial \Psi(z)}{\partial z} = -j\mathbf{M}(z)\Psi(z), \]  

(B.2)

where the matrix \(\mathbf{M}\) is of size \(4 \times 4\)

\[ \mathbf{M} = \begin{bmatrix} 0 & -iZ & 0 & 0 \\ -iY & 0 & g & -\beta_0 \\ 0 & iZ & \beta_0 & 0 \\ 0 & 0 & -g & \beta_0 \end{bmatrix} \]  

(B.3)

where

\[ g = \frac{\beta_0 I_0}{2iY}, \]

\[ \beta_0 = \frac{\omega}{u_0}. \]  

(B.4)

The differential equation in (B.2) represents the wave equation that governs the spatial evolution along the \(z\)-direction of the electromagnetic field and electron beam dynamics in the interactive system.

APPENDIX C
SECOND ORDER EPD IN A SYSTEM MADE OF AN ELECTROMAGNETIC WAVE INTERACTING WITH AN ELECTRON BEAM CHARGE’S WAVE

Assuming a state vector \(z\)-dependence of the kind \(\Psi(z) \propto e^{-jkz}\), where \(k\) is a wavenumber, the four solutions of (B.2) are found by solving the eigenvalue problem \(k\Psi(z) = \mathbf{M}\Psi(z)\). The characteristic equation of the system that determines the dispersion relation is calculated as

\[ D(\omega, k) = |\mathbf{M} - kI| = k^4 - 2\beta_0 k^2 + (\beta_0^2 + ZY - iZ\eta) k^2 - 2\beta_0 ZY k + \beta_0^2 ZY = 0, \]  

(C.1)
The above equation has four roots which represent the eigenvalues of the system. A necessary condition to have second order EPD is to have two repeated eigenvalues, which means that the characteristic equation should have two repeated roots as

$$D(\omega_e, k) \propto (k - k_e)^2$$  \hspace{1cm} (C.2)

where $\omega_e$ and $k_e$ are the degenerate angular frequency and wavenumber, respectively. The EPD condition in (C.2) is satisfied when [31]

$$D(\omega_e, k_e) = 0,$$

$$\frac{\partial D(\omega_e, k)}{\partial k} \bigg|_{k=k_e} = 0. \hspace{1cm} (C.3)$$

Substituting (C.1) into (C.3), the EPD conditions is found as

$$k_e^4 - 2\beta_0 k_e^3 + \left(\beta_0^2 + Y_c k_e - i g_e Z_e\right) \frac{k^2}{Z_e} - 2\beta_0 Z_e k_e + \beta_0^2 Z_e Y_c = 0,$$

$$4k_e^3 - 6\beta_0 k_e^2 + 2 \left(\beta_0^2 + Y_c k_e - i g_e Z_e\right) k_e - 2\beta_0 Z_e Y_c = 0,$$

where the EPD is designated with the subscript $e$, i.e., the parameters with the superscript $e$ are calculated at the EPD frequency; for example, $\beta_{0e} = \omega_e/u_0$. Only at a specific frequency (the EPD frequency) it is possible to find two identical eigenvalues. Therefore the above equation provides both the EPD radial frequency $\omega_e$ and wavenumber $k_e$.

The combination of the TL distributed series impedance $Z = Z_e$ and shunt admittance $Y = Y_e$ that provide the EPD are determined after making some mathematical manipulations in the two conditions in (C.4) and (C.5), leading to

$$Z_e = \frac{i g_e \delta_e^3}{\beta_{0e}}, \hspace{1cm} (C.6)$$

$$Y_e = \frac{i g_e (\delta_e + 1)^3}{\beta_{0e}}, \hspace{1cm} (C.7)$$

where $\delta_e = (k_e - \beta_{0e})/\beta_{0e}$ represents the relative deviation of degenerate modal wavenumber $k_e$ (of the interactive system) from the beam equivalent propagation constant $\beta_{0e} = \omega_e/u_0$.

The eigenvectors $\Psi_n$ of the system are determined by solving

$$(M - k_n I) \Psi_n = 0, \hspace{1cm} (C.8)$$

where $k_n$ with $n = 1, 2, 3, 4$ are the modes’ wavenumbers, and they are determined from (C.1). By solving (C.8), the eigenvectors are written in the form (each element of the eigenvector carries an implicit unit of Volt besides the units of the explicit parameters)

$$\Psi_n = \begin{bmatrix} 1, & ik_n/Z, & -(1 + \delta_n)/\delta_n, & g(1 + \delta_n)/(\beta_0 \delta_n^2) \end{bmatrix}^T, \hspace{1cm} (C.9)$$

where $\delta_n = (k_n - \beta_0)/\beta_0$. As shown in [13], three of these four modes are strongly affected by the synchronization of the electron beam and the EM mode. Assuming that these three wavenumbers of the beam-EM mode interactive system are a slight perturbation of the unperturbed beam’s propagation constant, i.e, $k_n \approx \beta_0$ with, $n = 1, 2, 3$ the eigenvector expression in (C.9) is approximated

$$\Psi_n \approx \begin{bmatrix} 1, & ik_n/Z, & -1/\delta_n, & g/(\beta_0 \delta_n^2) \end{bmatrix}^T. \hspace{1cm} (C.10)$$

It is worth mentioning that the eigenvector expression in (C.9) is valid for any of the four modes of the interacting system. Whereas the expression in (C.10) is only valid for the three modes with positive $\text{Re}(k)$, namely, for the three synchronous modes resulting from the interaction of the SWS EM mode with the electron beam. Those three synchronous modes are such that $k_n \approx \beta_0$ based on synchronization. Two of these eigenvectors coalesce at the EPD, forming the degenerate synchronization between the two modes $\Psi_1 = \Psi_2 = \Psi_e$, where

$$\Psi_e \approx \begin{bmatrix} 1, & ik_e/Z, & -1/\delta_e, & g_e/(\beta_0 \delta_e^2) \end{bmatrix}^T. \hspace{1cm} (C.11)$$

In summary, the two conditions in (C.6) and (C.7) represent constraints on the TL parameters, calculated at EPD frequency, that provide the second order EPD, where two eigenmodes of the interacting system have identical eigenvalues $k_1 = k_2 = k_e$ and eigenvectors $\Psi_1 = \Psi_2 = \Psi_e$. These two eigenmodes form the degenerate synchronization.

## Appendix D

### BACKWARD WAVE OSCILLATOR (BWO) THRESHOLD CURRENT

For a finite SWS with length $\ell$, the equivalent voltage that describes the electric field at any coordinate $z$ in the SWS is expanded as a combination of the four modes supported by the interactive beam-EM mode system (modes are calculated in a SWS of infinite length) as

$$V(z) = V_1 e^{-jk_1 z} + V_2 e^{-jk_2 z} + V_3 e^{-jk_3 z} + V_4 e^{-jk_4 z} \hspace{1cm} (D.1)$$

where $k_n$ with $n = 1, 2, 3, 4$ are the modes’ wavenumber, and they are determined from (C.1), and $V_n$ are the complex modes amplitudes. The associated TL current and beam dynamics are found by using (D.1) in (B.2) or by using the eigenvector expression in (20).
\[ I(z) = \frac{j k_1 V_1 e^{-j k_1 z}}{Z} + \frac{j k_2 V_2 e^{-j k_2 z}}{Z} + \frac{j k_3 V_3 e^{-j k_3 z}}{Z} + \frac{j k_4 V_4 e^{-j k_4 z}}{Z}, \]

\[ V_0(z) = -\frac{(1 + \delta_1) V_1 e^{-j k_1 z} - (1 + \delta_2) V_2 e^{-j k_2 z} - (1 + \delta_3) V_3 e^{-j k_3 z} - (1 + \delta_4) V_4 e^{-j k_4 z}}{\delta_1}, \]

\[ I_b(z) = \frac{g(1 + \delta_1) V_1 e^{-j k_1 z} + g(1 + \delta_2) V_2 e^{-j k_2 z} + g(1 + \delta_3) V_3 e^{-j k_3 z} + g(1 + \delta_4) V_4 e^{-j k_4 z}}{\delta_1}, \]

\[ \text{where } \delta_n = (k_n - \beta_n)/\beta_0. \]

The setup we use for the BWO with distributed power radiation shown in Fig. 4(a) in the main body of the paper, is similar to that used in [19], where we assume unmodulated space charge at the beginning of the electron beam (i.e., \( V_0(z = 0) = 0 \) and \( I_b(z = 0) = 0 \)), and we also assume that the output power is extracted at \( z = 0 \), terminated with a resistance matched to the characteristic impedance of the TL (without loss and gain) \( R_o = \sqrt{\lambda C} \), and short circuit at \( z = \ell \), where \( \ell = N \lambda_e \) is the SWS length (i.e. the TL length), and \( \lambda_e = 2\pi/\beta_0 \) is the guided wavelength calculated at the EPD frequency. (Note that in the absence of loss and gain in the TL, the TL distributed series impedance and shunt admittance are \( Z = 1/(i \omega C) \) and \( Y = 1/(i \omega L) \).)

The boundary conditions that describe the mentioned setup are

\[ V(0) + R_c I(0) = 0, \]
\[ V(\ell) = 0, \]
\[ V_b(0) = 0, \]
\[ I_b(0) = 0. \]

By imposing (D.3) in (D.2), we obtain a homogeneous system of linear equation

\[ A(\omega, I_0) V = 0 \]

where \( V = [ ] \]
\[ \text{Free oscillation in the interactive system occurs when there is a solution of (D.4) despite the absence of the source term (the right hand side of (D.4) is equal to zero). Therefore, oscillation occurs for a combination of radian frequency and electron beam current that satisfy} \]

\[ \text{det(} A(\omega, I_0) \text{)} = 0. \]

Since the solution of the above equation defines the threshold beam current to start oscillations, such solution is denoted by \( I_0 = I_{th} \) and \( \omega = \omega_{res} \) is the frequency of the oscillation.

**APPENDIX E**

**ASYMPTOTIC SCALING OF THE EPD-BWO THRESHOLD BEAM CURRENT**

In this section we derive the oscillation condition and the asymptotic scaling of the threshold beam current with the SWS length for the proposed EPD-BWO assuming that the length of the structure tends to infinity. We follow the traveling-wave tube theory used in [32, 13] and [19], since the synchronization involves mainly three waves, those with \( Re(k_n) > 0 \). Assuming that these three wavenumbers are a slight variation of the unperturbed beam wavenumber, i.e., \( k_n \approx \beta_n \), we use the eigenvector expression in (C.10) to write the TL circuit voltage and beam dynamics distribution along the z-direction which are represented in terms of three modes as \[ V(z) = V_1 e^{-j k_1 z} + V_2 e^{-j k_2 z} + V_3 e^{-j k_3 z}, \]

\[ V_0(z) = \frac{-V_1 e^{-j k_1 z} - V_2 e^{-j k_2 z} - V_3 e^{-j k_3 z}}{\delta_1}, \]

\[ I_b(z) = \frac{gV_1 e^{-j k_1 z} - gV_2 e^{-j k_2 z} - gV_3 e^{-j k_3 z}}{\delta_1}, \]

where \( \delta_n = (k_n - \beta_n)/\beta_0 \). The wavenumbers \( k_n \) with \( n = 1, 2, 3 \) are with positive real part and two of them are with positive imaginary part, let’s say \( n = 1, 2 \) and the other one is with negative imaginary part, \( n = 3 \). A numerical example of the wavenumbers is shown in Fig. 3. Here, we follow the same procedure used in [19] to obtain the oscillation condition which is based on imposing infinite voltage gain \( A_v = V(0)/V(\ell) \rightarrow \infty \). By imposing the beam boundary condition \( V_b(0) = 0 \) and \( I_b(0) = 0 \) in (E.1) and after some mathematical manipulations the gain expression is written in its simplest form as

\[ A_v^{-1} e^{i \beta_0 \ell} = \frac{e^{-i \delta_0 \delta_1 \ell} \delta_1^2}{(\delta_1 - \delta_2) (\delta_1 - \delta_3)} + \frac{e^{-i \delta_0 \delta_2 \ell} \delta_2^2}{(\delta_2 - \delta_3) (\delta_2 - \delta_1)}, \]

\[ + \frac{e^{-i \delta_0 \delta_3 \ell} \delta_3^2}{(\delta_3 - \delta_1) (\delta_3 - \delta_2)} = 0. \]

We first neglect the term with \( e^{-i \delta_0 \delta_3 \ell} \) in (E.2) since we consider very large SWS length \( \ell \) and know that \( Im(\delta_3) < 0 \). Therefore the gain expression reduces to

\[ A_v^{-1} e^{i \beta_0 \ell} \approx \frac{e^{-i \delta_0 \delta_1 \ell} \delta_1^2}{(\delta_1 - \delta_2) (\delta_1 - \delta_3)} + \frac{e^{-i \delta_0 \delta_2 \ell} \delta_2^2}{(\delta_2 - \delta_3) (\delta_2 - \delta_1)}, \]

By defining \( \Delta = (\delta_1 - \delta_2) \), hence \( \delta_1 = \delta_a + \Delta/2 \) and \( \delta_2 = \delta_a - \Delta/2 \) where \( \delta_a = (\delta_1 + \delta_2)/2 \) is the average, the gain expression is then written as

\[ A_v^{-1} e^{i \beta_0 \ell} \approx e^{-i \beta_0 \delta_a \ell} \left( \frac{\left(\delta_a + \Delta/2\right)^2 e^{-i \beta_0 \Delta \ell/2}}{(\Delta/2) (\delta_a - \delta_3 + \Delta/2)} + \frac{\left(\delta_a - \Delta/2\right)^2 e^{i \beta_0 \Delta \ell/2}}{(\Delta/2) (\delta_a - \delta_3 - \Delta/2)} \right). \]

In close proximity of the EPD we know that two modes coalesce, i.e., \( \delta_1 \approx \delta_2 \), thus we can assume that \( |\Delta| \ll |\delta_a| \), i.e.,
\[\Delta \to 0\] as we approach the EPD. It is important to point out that although \(\Delta\) is a very small value we can not neglect its effect in the exponential function in (E.4) because we assume the SWS to be very long, however we can neglect \(\Delta\) in other places, i.e., we have \(\delta_0 \pm \Delta \approx \delta_0\) and \(\delta_0 - \delta_3 \pm \Delta \approx \delta_0 - \delta_3\). Thus the gain expression finally reduces to

\[
A_{\text{ve}}^{-1} e^{i\beta_0 \ell} \approx \frac{-2ie^{-i\delta_0/\delta_3^2} \sin (\Delta \beta_0 \ell/2)}{(\delta_0 - \delta_3^2/\Delta^2).} \quad \text{(E.5)}
\]

From the above gain expression, the first oscillation condition occurs when \(\Delta \beta_0 \ell = 2\pi\), this indeed provides \(A_{\text{ve}}^{-1} \approx 0\). This happens when the constraint on the wavenumbers \(k_1 - k_2 = 2\pi/\ell\) is satisfied, and it shows a very important fact about the EPD, that for infinitely long structure, i.e., when \(\ell \to \infty\), the oscillation condition is \(k_1 = k_2 = k_c\), namely the oscillation occurs exactly at the EPD point. This implies that the EPD corresponds also to the threshold condition resulting from the synchronization between the charge wave and the EM wave, accounting for their interaction, in infinitely long SWSs, that guarantees the generation of oscillations at the EPD frequency \(f_c = \omega_c/(2\pi)\).

For finite length SWSs, the threshold for oscillation occurs when \(k_1 - k_2 = 2\pi/\ell\) is satisfied, assuming very large \(\ell\), and since \(\Delta \ll \delta_0\), we have both \(k_1\) and \(k_2\) very close to \(k_c\), which implies that the systems is close to the EPD and hence the threshold beam current \(I_0\), that starts oscillations is slightly different from \(I_c\). A beam current slightly away from the EPD one causes the wavenumbers to bifurcate from the degenerate one \(k_c\), following the Puiseux series approximation (also called fractional power expansion) as \(k_n - k_c \approx (-1)^n \alpha \sqrt{\ell_0} - T_c\), with \(n = 1, 2\), where \(\alpha\) is a constant. This implies that \(k_1 - k_2 \approx -2\alpha \sqrt{\ell_0} - T_c\), and by comparing this with the \(k_1 - k_2\) difference in the case of a finite length SWS we infer that the threshold beam current \(I_{th}\) that makes the EPD-BWO of finite length \(\ell\) oscillates, asymptotically scales as

\[
I_{th} \sim I_c + \left(\frac{\pi/\alpha}{\ell}\right)^2. \quad \text{(E.6)}
\]

This asymptotic trend is confirmed in Fig. 4 of the main body of the paper, where the threshold current \(I_{th}\) is calculated numerically by solving \(\det(A(\omega, I_0)) = 0\) for the beam current \(I_0\), as discussed in the previous section. Note that this equation also provides the \(\omega\) of the oscillation frequency, associated to the beam current threshold.

REFERENCES

[1] A. Figotin and I. Vitebskiy, "Oblique frozen modes in periodic layered media," Physical Review E, vol. 68, no. 3, p. 036609, 2003.
[2] A. Figotin and I. Vitebskiy, "Gigantic transmission band-edge resonance in periodic stacks of anisotropic layers," Physical Review E, vol. 72, no. 3, p. 036619, 2005.
[3] A. Figotin and I. Vitebskiy, "Frozen light in photonic crystals with degenerate band edge," Physical Review E, vol. 74, no. 6, p. 066613, 2006.
[4] M. A. Othman, X. Pan, G. Atmatzakis, C. G. Christodoulou, and F. Capolino, "Experimental demonstration of degenerate band edge in metallic periodically loaded circular waveguide," IEEE Transactions on Microwave Theory and Techniques, vol. 65, no. 11, pp. 4037–4045, 2017.
[5] M. Y. Nada, M. A. Othman, and F. Capolino, "Theory of coupled resonator optical waveguides exhibiting high-order exceptional points of degeneracy," Physical Review B, vol. 96, no. 18, p. 184304, 2017.
[6] J. T. Sloan, M. A. Othman, and F. Capolino, "Theory of double ladder lumped circuits with degenerate band edge," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 65, no. 1, pp. 3–13, 2018.
[7] C. M. Bender and S. Boettcher, "Real spectra in non-Hermitian Hamiltonians having PT symmetry," Physical Review Letters, vol. 80, no. 24, p. 5243, 1998.
[8] W. Heiss, M. Müller, and I. Rotter, "Collectivity, phase transitions, and exceptional points in open quantum systems," Physical Review E, vol. 58, no. 3, p. 2894, 1998.
[9] R. El-Ganainy, K. Makris, D. Christodoulides, and Z. H. Musslimani, "Theory of coupled optical PT-symmetric structures," Optics Letters, vol. 32, no. 17, pp. 2632–2634, 2007.
[10] A. Guo, G. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. Siviloglou, and D. Christodoulides, "Observation of PT symmetry breaking in complex optical potentials," Physical Review Letters, vol. 103, no. 9, p. 093902, 2009.
[11] S. Bittner, B. Dietz, U. Güntner, H. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer, "PT-symmetry and spontaneous symmetry breaking in a microwave billiard," Physical Review Letters, vol. 108, no. 2, p. 024101, 2012.
[12] D. Bohm and E. P. Gross, "Theory of plasma oscillations. A. Origin of medium-like behavior," Physical Review, vol. 75, no. 12, p. 1851, 1949.
[13] J. Pierce, "Waves in electron streams and circuits," Bell System Technical Journal, vol. 30, no. 3, pp. 626–651, 1951.
[14] R. C. Hansen, Phased array antennas, vol. 213. John Wiley & Sons, NJ, USA, 2009.
[15] W. L. Stutzman and G. A. Thiele, Antenna theory and design. John Wiley & Sons, NY, USA, 2012.
[16] E. Wang, S.-S. Zhong, Y.-M. Zhang, and X.-L. Liang, "A broadband slotted ridge waveguide antenna array," IEEE Transactions on Antennas and Propagation, vol. 54, no. 8, pp. 2416–2420, 2006.
[17] M. A. Othman and F. Capolino, "Theory of exceptional points of degeneracy in uniform coupled waveguides and balance of gain and loss," IEEE Transactions on Antennas and Propagation, vol. 65, no. 10, pp. 5289–5302, 2017.
[18] A. F. Abdelshafy, M. A. Othman, D. Oshmarin, A. Al-Mutawa, and F. Capolino, "Exceptional points of degeneracy in periodically-coupled waveguides and the interplay of gain and radiation loss: Theoretical and experimental demonstration," arXiv preprint arXiv:1809.05256, 2018.
[19] H. R. Johnson, "Backward-wave oscillators," Proceedings of the IRE, vol. 43, no. 6, pp. 684–697, 1955.
[20] S. E. Tsimring, Electron beams and microwave vacuum electronics. John Wiley & Sons, NJ, USA, 2007.
[21] M. A. Othman, M. Veysi, A. Figotin, and F. Capolino, "Low starting electron beam current in degenerate band edge oscillators," IEEE Transactions on Plasma Science, vol. 44, no. 6, pp. 918–929, 2016.
[22] M. A. Othman, M. Veysi, A. Figotin, and F. Capolino, "Giant amplification in degenerate band edge slow-wave structures interacting with an electron beam," Physics of Plasmas, vol. 23, no. 3, p. 033112, 2016.
[23] A. F. Abdelshafy, M. A. Othman, F. Yazdi, M. Veysi, A. Figotin, and F. Capolino, "Electron-beam-driven devices with synchronous multiple degenerate eigenmodes," IEEE Transactions on Plasma Science, vol. 46, no. 8, pp. 3126–3138, 2018.
[24] M. A. Othman, V. A. Tamma, and F. Capolino, "Theory and new amplification regime in periodic multimodal slow wave structures with degeneracy interacting with an electron beam," IEEE Transactions on Plasma Science, vol. 44, no. 4, pp. 594–611, 2016.
[25] J. Pierce, "Theory of the beam-type traveling-wave tube," Proceedings of the IRE, vol. 35, no. 2, pp. 111–123, 1947.
[26] S. Ramo, "Space charge and field waves in an electron beam," Physical Review, vol. 56, no. 3, p. 276, 1939.
[27] R. Kompfner, "The traveling-wave tube as amplifier at microwaves," Proceedings of the IRE, vol. 35, no. 2, pp. 124–127, 1947.
[28] L. I. Chu and J. D. Jackson, "Field theory of traveling-wave tubes," Physical Review, vol. 56, no. 3, p. 276, 1939.
[29] T. Coffey and R. Kompfner, "The traveling-wave tube as amplifier at microwave frequencies," Physical Review, vol. 75, no. 6, p. 911, 1949.
[29] V. A. Tamma and F. Capolino, “Extension of the pierce model to multiple transmission lines interacting with an electron beam,” *IEEE Transactions on Plasma Science*, vol. 42, no. 4, pp. 899–910, 2014.
[30] A. Gilmour, *Klystrons, traveling wave tubes, magnetrons, crossed-field amplifiers, and gyrotrons*. Artech House, MA, USA, 2011.
[31] G. W. Hanson, A. B. Yakovlev, M. A. Othman, and F. Capolino, “Exceptional points of degeneracy and branch points for coupled transmission lines—Linear-algebra and bifurcation theory perspectives,” *IEEE Transactions on Antennas and Propagation*, vol. 67, no. 2, pp. 1025–1034, 2019.
[32] J. Pierce, “Theory of the beam-type traveling-wave tube,” *Proceedings of the IRE*, vol. 35, no. 2, pp. 111–123, 1947.
[33] D. R. Jackson and A. A. Oliner, “Leaky-wave antennas,” *Modern antenna handbook*, pp. 325–367, 2008.
[34] A. Welters, “On explicit recursive formulas in the spectral perturbation analysis of a jordan block,” *SIAM Journal on Matrix Analysis and Applications*, vol. 32, no. 1, pp. 1–22, 2011.