Behavior of Time-varying Constants in Relativity

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In this paper, we consider Bianchi type III and Kantowski-Sachs space-times and discuss the behavior of time-varying constants $G$ and $\Lambda$ by using two symmetric techniques, namely, kinematic self-similarity and matter collineation. In the kinematic self-similarity technique, we investigate the behavior of the first and the second kinds. In the matter collineation technique, we consider usual, modified, and completely modified matter collineation equations while studying the behavior of these constants. Further, we reduce the results for dust, radiation, and stiff fluids. We find that $\Lambda$ is a decreasing time function while $G$ is an increasing time function. This corresponds to the earlier results available in the literature for other space-times. Further, we find that the deceleration parameter attains a negative value, which shows that the expansion of the universe is accelerating.

Keywords: Time-varying constants, Varying behavior.

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I. INTRODUCTION

The time-varying behavior of the gravitational and the cosmological constants, i.e., $G$ and $\Lambda$, have been among the most controversial issues in cosmology. Some authors reveal the idea that both $G$ and $\Lambda$ can be considered as non-constant, i.e., $G = G(t)$ and $\Lambda = \Lambda(t)$, coupling scalars, while solving the Einstein field equations (EFEs). The use of $G$ and $\Lambda$ in the EFEs plays a significant role in cosmological, astronomical, and quantum phenomena.

Improved astronomical techniques indicate that a non-zero value of $\Lambda$ is required to discuss distant supernova. Analysis of high red-shift supernova shows that the universe may be accelerating [1-3] due to the presence of some type of vacuum energy. Some people relate this vacuum energy density to a non-zero cosmological constant. This indicates that the cosmological constant is a cause of the expansion of the universe and plays an important role in the evolution of the universe. In gravitational collapse, the cosmological constant slows it down, which limits the size of a black hole [4]. However, there is a fundamental problem related with the value of $\Lambda$, which is assumed to be very small. This value is about 120 orders less than the magnitude of the vacuum energy-density calculated in quantum field theory. A phenomenological solution to this problem is suggested by considering $\Lambda$ as a function of time. In the early universe, $\Lambda$ was large, but decreased with the expansion of the universe and the creation of photons [5].

The gravitational attraction, explained as a result of the curvature of spacetime, is proportional to $G$. According to quantum theory, the non-homogeneity of the gravitational field causes $G$ to change rapidly in small intervals of time [6]. The EFEs with $G$ and $\Lambda$ are given by

$$R_{ab} - \frac{1}{2}g_{ab} R + \Lambda(t) g_{ab} = \frac{8\pi G(t)}{c^4} T_{ab}, \quad (a, b = 0, 1, 2, 3),$$

where $R_{ab}$, $g_{ab}$, and $T_{ab}$ represent the components of the Ricci, metric, and energy-momentum tensors respectively, and $R$ is the Ricci scalar.

The time variation of $G$ was originally raised by Dirac [7]. He proposed that the gravitational constant varied with age of the universe. Modern theories, like string theory and Brans-Dicke (BD) theory, do not necessarily require such a variation, but provide a natural and self-consistent framework for this variation by assuming the existence of additional dimensions. The time variation of $G$ in these multi-dimensional theories has recently been studied, and their consistency with variable observational data for distant supernova has been analyzed [8]. The variation of the gravitational constant is found to make distant supernova appear brighter. The recent
results of Shapiro et al. [9], based on an analysis of radar echo time delays, have set an experimental upper limit on the possible time variation of the gravitational constant as $|\dot{G}/G| < 3 \times 10^{-10} \text{yr}^{-1}$, where the dot represents a derivative with respect to time. Theoretical calculations by Dicke for zero-pressure Friedmann-type cosmologies, according to the BD theory, yield $|\dot{G}/G| \approx 10^{-11} \text{yr}^{-1}$ for a flat spacetime and $|\dot{G}/G| \approx 3 \times 10^{-11} \text{yr}^{-1}$ for closed spacetimes [10,11]. Although the time variation in $G$ is extremely small at the present epoch, Dicke has shown that there exist early epoch solutions for which the energy density associated with the time variation is much greater than the matter energy density. Recently, a constraint on the variation of $G$ has been obtained by using the Wilkinson Microwave Anisotropy Probe (WMAP) and the big bang nucleosynthesis observations, which comes out to be $-3 \times 10^{-13} \text{yr}^{-1} < (\dot{G}/G)_{\text{today}} < 4 \times 10^{-13} \text{yr}^{-1}$ [12].

Bekenstein [13] and Bertolami [14] introduced models in which both $G$ and $\Lambda$ are time dependent. Several authors [15-17] studied the variations of $G$ and $\Lambda$ in the framework of flat Friedmann-Robertson-Walker (FRW) symmetries. This work has been extended [18-21] to more complicated geometries like the Bianchi type I model, which is the simplest generalization of the FRW flat model, by using a perfect fluid. The same model was considered with viscous fluids [22-24] to discuss the time variations of $\Lambda$ and $G$. Kalligas et al. [25] discussed the behaviors of these varying constants by using Lie method. Darabi [26] found that the time variations of these constants lead the vacuum energy density to be time dependent as $\rho_v = \Lambda(t)/8\pi G(t)$. Therefore, for an early universe, where $\Lambda$ is so large and $G$ is so small compared with their current values, the vacuum energy is huge. At the present status of the universe, however, the vacuum energy is vanishing due to the time variations of both $\Lambda$ and $G$. Belinchen and Dvila [27] discussed time-varying constants in different spacetimes by using different symmetric techniques. In recent papers [28, 29], the same author analyzed the behaviors of time-varying $G$ and $\Lambda$ for a Bianchi type I model and made comparison of different techniques, including self-similarity, matter collineations, kinematic self-similar similarity, and the Lie method.

In this paper, we extend Belinchen’s work to Bianchi type III and Kantowski-Sachs spacetimes. We shall use two symmetric techniques, i.e., matter collineations (using energy-momentum tensor for a perfect fluid) and kinematic self-similarity (first and second kinds). The scheme of this paper is as follows: In Section II, we shall write the field equations with relevant quantities for Bianchi type III and Kantowski-Sachs spacetimes. Section III is devoted to a study of the behaviors of the time-varying constants by using kinematic self-similarity technique. In Section IV, we use the matter
collineations technique to investigate the behaviors of $G$ and $\Lambda$. The last section will provide a summary and a discussion of the results obtained.

II. BIANCHI TYPE III AND KANTOWSKI-SACHS SPACETIMES

Bianchi type III and Kantowski-Sachs spacetimes are spatially homogeneous spacetimes that admit an abelian group of isometries $G_3$ acting on a spacelike hypersurface. These are generated by spacelike Killing vectors $\xi_1 = \partial_r$, $\xi_2 = \partial_\theta$ and $\xi_3 = \partial_\phi$. In co-moving coordinates, the metric representing these spacetimes is written as

$$ds^2 = c^2 dt^2 - A^2(t)dr^2 - B^2(t)(d\theta^2 + f^2(\theta)d\phi^2),$$

where $A$ and $B$ are arbitrary functions of $t$ while $f(\theta)$ is defined as

$$f(\theta) = \sinh \theta \quad \text{corresponding to Bianchi type III spacetime},$$

$$f(\theta) = \sin \theta \quad \text{corresponding to Kantowski-Sachs spacetime}.$$ 

These metrics represent anisotropic generalizations of the open and closed FRW models, respectively.

The energy-momentum tensor for a perfect fluid is given by

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab},$$

where $u^a$ is the four-velocity and in co-moving coordinates, it is defined as follows:

$$u^a = \left( \frac{1}{c}, 0, 0, 0 \right) \quad \text{with} \quad u^a u_a = 1.$$ 

A perfect fluid can be characterized by a dimensionless number $k$ given by

$$k = \frac{p}{\rho} \quad \text{or} \quad p = k \rho.$$ 

This is called the equation of state and represents a dust fluid for $k = 0$, radiation for $k = \frac{1}{3}$, and stiff matter for $k = 1$. Using Eq. (5), the EFEs of Eq. (1) lead to the following three equations:

$$\frac{2 \dot{A} \dot{B}}{AB} - \frac{c^2 f''}{B^2 f} + \frac{\dot{B}^2}{B^2} = - \frac{8\pi G}{c^2} \rho + \Lambda c^2,$$

$$\frac{2 \dot{B}}{B} - \frac{c^2 f''}{B^2 f} + \frac{\dot{B}^2}{B^2} = - \frac{8\pi G}{c^2} k \rho + \Lambda c^2,$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A} \dot{B}}{AB} = - \frac{8\pi G}{c^2} k \rho + \Lambda c^2.$$
Here, the dot denotes the time derivative, and the prime denotes the derivative with respect to $\theta$.

The time derivatives of $G$ and $\Lambda$ can be related by the Bianchi identities as follows:

$$\left( R_{ab} - \frac{1}{2} R g_{ab} \right)^b = \left( \frac{8\pi G}{c^4} T_{ab} + \Lambda g_{ab} \right)^b.$$  \hspace{1cm} (9)

Simplification of this expression by fixing $a = 0$ and varying $b = 0, 1, 2, 3$ yields

$$\frac{8\pi}{c^4} \dot{G}_\rho + \dot{\Lambda} = \frac{8\pi G}{c^4} \left[ \dot{\rho} + \rho(1 + k) \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) \right].$$ \hspace{1cm} (10)

The conservation law of the energy-momentum tensor of matter field, $T_{ab}^{\text{mat}} = 0$, gives

$$\dot{\rho} + \rho(1 + k) \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) = 0.$$ \hspace{1cm} (11)

In view of this equation, Eq. (10) implies that

$$\dot{\Lambda} = -\frac{8\pi}{c^4} \dot{G}_\rho.$$ \hspace{1cm} (12)

It is mentioned here that for the possibilities $a = 1, 2, 3$ and $b = 0, 1, 2, 3$, Eq. (9) is satisfied identically.

We define Hubble’s parameter as an average expansion of the universe as follows:

$$H = \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} = H_1 + 2H_2,$$ \hspace{1cm} (13)

where

$$H_1 = \frac{\dot{A}}{A}, \quad H_2 = \frac{\dot{B}}{B}.$$  

Consequently, the deceleration parameter is defined as

$$q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1.$$ \hspace{1cm} (14)

Using Eq. (13) in Eq. (11), it follows that

$$\dot{\rho} + \rho(1 + k)H = 0.$$ \hspace{1cm} (15)

In the next two sections, we use a Bianchi type III spacetime to discuss the behaviors of $G$ and $\Lambda$. However, we include the discussion of the behaviors of $G$ and $\Lambda$ for Kantowski-Sachs spacetime in the last section.
III. TIME-VARYING BEHAVIOR OF $G$ AND $\Lambda$ BY USING A KINEMATIC SELF-SIMILARITY TECHNIQUE

In general relativity, self-similarity can be defined by the existence of a homothetic vector field. Cahill and Taub [31] were the pioneers to introduce the concept of self-similarity corresponding to homothety. Carter and Henriksen [32] introduced the concept of kinematic self-similarity as a natural generalization of the homothetic case. A kinematic self-similar vector field $\xi$ satisfies the following conditions [33]:

$$\mathcal{L}_{\xi} u_a = \alpha u_a, \quad (16)$$
$$\mathcal{L}_{\xi} h_{ab} = 2\delta h_{ab}, \quad (17)$$

where $\alpha$ and $\delta$ are constants and $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor. Kinematic self-similarity (KSS) can be classified into first and second kinds by using a scale independent ratio $\frac{\alpha}{\delta}$, referred to as the similarity index. The ratio $\frac{\alpha}{\delta} = 1$ leads to self-similarity of the first kind, also known as homothety, while the ratio $\frac{\alpha}{\delta} \neq 0, 1$ indicates self-similarity of the second kind. Here, we discuss the behaviors of $G$ and $\Lambda$ by using these kinds.

1. Kinematic Self-similarity of the First Kind

For $\alpha = \delta = 1$, the KSS vector field yields

$$g_{ab,c} \xi^c + g_{ac} \xi^c_b + g_{bc} \xi^c_a = 2g_{ab}.$$  \hspace{1cm} (18)

In this case, $\xi$ is known as a homothetic vector field. This gives the following system of ten equations:

$$\xi^0_{,0} = 1, \quad (19)$$
$$c^2 \xi^0_{,1} - A^2 \xi^1_{,0} = 0, \quad (20)$$
$$c^2 \xi^0_{,2} - B^2 \xi^2_{,0} = 0, \quad (21)$$
$$c^2 \xi^0_{,3} - B^2 \sinh^2 \theta \xi^3_{,0} = 0, \quad (22)$$
$$\frac{\dot{A}}{A} \xi^0 + \xi^1_{,1} = 1, \quad (23)$$
$$A^2 \xi^1_{,2} + B^2 \xi^2_{,1} = 0, \quad (24)$$
$$A^2 \xi^1_{,3} + B^2 \sinh^2 \theta \xi^3_{,1} = 0, \quad (25)$$
$$\frac{\dot{B}}{B} \xi^0 + \xi^2_{,2} = 1, \quad (26)$$

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\[ B^2 \xi_3^2 + B^2 \sinh^2 \theta \xi_2^3 = 0, \]  
\[ \left( \frac{B}{\xi_3} \right)^0 + \xi_3^3 = 1. \]  
Here, the derivatives with respect to 0, 1, 2, and 3 denote the partial derivatives with respect to \( t, r, \theta, \) and \( \phi, \) respectively. Solving Eqs. (19), (23), (26), and (28) simultaneously, we obtain

\[ \xi^0 = t + f_0(r, \theta, \phi), \]  
\[ \xi^1 = (1 - \frac{\dot{A}}{A}) \xi^0 r + f_1(t, \theta, \phi), \]  
\[ \xi^2 = (1 - \frac{\dot{B}}{B}) \xi^0 \theta + f_2(t, r, \phi), \]  
\[ \xi^3 = (1 - \frac{\dot{B}}{B}) \xi^0 \phi + f_3(t, r, \theta), \]  

where \( f_1(t, \theta, \phi) - f_3(t, r, \theta) \) are functions of integration. Substituting these values of \( \xi \) in Eqs. (23) and (26), respectively, we obtain

\[ 2A \dot{A} f_{0,1}(r, \theta, \phi) r = 0, \]  
\[ 2B \dot{B} f_{0,2}(r, \theta, \phi) \theta = 0. \]  

Since \( A, B, r, \) and \( \theta \neq 0, \) the above equations yield the following four cases:

(i) \( \dot{A} = 0 = \dot{B}, \)  
(ii) \( \dot{A} = 0, \dot{B} \neq 0, \)  
(iii) \( \dot{A} \neq 0, \dot{B} = 0, \)  
(iv) \( \dot{A}, \dot{B} \neq 0. \)  

The cases (i)-(iii) do not provide the behaviors of \( G \) and \( \Lambda \) as they vanish. In the following, we discuss the case when \( \dot{A}, \dot{B} \neq 0. \)  

In this case, Eqs. (33) and (34) yield

\[ f_0(r, \theta, \phi) \equiv f_0(\phi), \]  
which, in view of Eq. (28), finally gives

\[ f_0(\phi) \equiv c_0, \]  

where \( c_0 \) is an arbitrary constant. Thus, \( \xi^0 \) takes the form

\[ \xi^0 = t + c_0. \]  

When we make use of this value of \( \xi^0 \) in Eqs. (30)-(32), it follows that

\[ \xi^1 = (1 - \frac{\dot{A}}{A}(t + c_0)) r + f_1(t, \theta, \phi), \]
\[ \xi^2 = (1 - \frac{\dot{B}}{B}(t + c_0))\theta + f_2(t, r, \phi), \]  
(37)
\[ \xi^3 = (1 - \frac{\dot{B}}{B}(t + c_0))\phi + f_3(t, r, \theta). \]  
(38)

Now replacing these components of \( \xi \) in the above system of equations, the functions of integration \( f_1(t, \theta, \phi) - f_3(t, r, \theta) \) reduce to arbitrary constants \( c_1, c_2, \) and \( c_3 \) respectively along with the following ODEs as necessary and sufficient conditions:
\[ \dot{A}A + (t + c_0)(\ddot{A}A - \dot{A}^2) = 0, \]  
(39)
\[ \dot{B}B + (t + c_0)(\ddot{B}B - \dot{B}^2) = 0. \]  
(40)

Thus, the homothetic vector field becomes
\[ \xi = (t + c_0)\partial_t + [(1 - (t + c_0)\frac{\dot{A}}{A})r + c_1]\partial_r + [(1 - (t + c_0)\frac{\dot{B}}{B})\theta + c_2]\partial_\theta \]
\[ + [(1 - (t + c_0))\frac{\dot{B}}{B})\phi + c_3]\partial_\phi. \]  
(41)

Using Eq. (13), the solution of the ODEs in Eqs. (39) and (40), respectively, yield
\[ A = A_0(t + c_0)^{\alpha_1}, \quad B = B_0(t + c_0)^{\alpha_2}, \]  
(42)
where \( A_0, B_0, \alpha_1, \) and \( \alpha_2 \) are positive constants (for physical reasons) of integration. Making use of these values of \( A \) and \( B \) in Eq. (13), the Hubble and the deceleration parameters turn out to be
\[ H = (\alpha_1 + 2\alpha_2)(t + c_0)^{-1}, \]  
(43)
\[ q = \frac{1}{\alpha_1 + 2\alpha_2} - 1. \]  
(44)

Consequently, Eq. (15) yields
\[ \rho = \rho_0(t + c_0)^{-((1+k)(\alpha_1+2\alpha_2))}, \]  
(45)
where \( \rho_0 \) is a constant of integration.

Solving Eqs. (6) and (7) simultaneously and then using Eqs. (42) and (15), we have
\[ G = \frac{2c^2}{8\pi\rho_0(1+k)}(\alpha_1\alpha_2 - \alpha_2(\alpha_2 - 1))(t + c_0)^{-(\alpha_1+2\alpha_2)(1+k)}, \]  
(46)
\[ \Lambda = \frac{1}{c^2}2\alpha_1\alpha_2 + \alpha_2^2 - \frac{2}{1+k}(\alpha_1\alpha_2 - \alpha_2^2 + \alpha_2)(t + c_0)^{-2} \]
\[ - \frac{1}{B_0^2}(t + c_0)^{-2\alpha_2}. \]  
(47)
Using the values of $G$, and $\Lambda$ and Eq. (45) in Eq. (5), it follows that

$$\alpha_2(\alpha_2 - 1) + \alpha_1\alpha_2 + \alpha_1(\alpha_1 - 1) = 2\alpha_2(\alpha_2 - 1) + \alpha_2^2 - \frac{c^2}{B_0^2}(t + c_0)^{-2\alpha_2 + 2}.$$  

This shows that $\alpha_1$ will be constant only if we choose $\alpha_2 = 1$. Thus, we have

$$\alpha_1 = \sqrt{1 - \frac{c^2}{B_0^2}}.$$  

(48)

For $\alpha_1$ to be real, it is necessary that

$$B_0^2 \geq c^2 \quad \Rightarrow \quad B_0 \geq c \quad \text{or} \quad B_0 \leq -c.$$  

In other words, we can say that

$$B_0 \in (-\infty, -c] \cup [c, \infty).$$  

(49)

For $B_0$ to be positive, our interval of interest is only where $B_0 \in [c, \infty)$.

i. Behavior of $G$

Using $\alpha_2 = 1$ in Eq. (46), $G$ becomes

$$G = G_0(t + c_0)^{\alpha_1 + (\alpha_1 + 2)k},$$  

(50)

where

$$G_0 = \frac{2c^2\alpha_1}{8\pi\rho_0(1 + k)}.$$  

(51)

We note that Eqs. (45) and (50) yield

$$G \rho \approx (t + c_0)^{-2}.$$  

(52)

For $G_0 > 0$, i.e., $\alpha_1 = \sqrt{1 - \frac{c^2}{B_0^2}}$, the behavior of $G$ can be discussed as follows:

$$G \text{ is increasing} \quad \Leftrightarrow \quad \alpha_1 > -\frac{2k}{1 + k} \quad \Leftrightarrow \quad \sqrt{1 - \frac{c^2}{B_0^2}} > -\frac{2k}{1 + k}.$$  

(53)

When we take square of both sides, the above inequality yields the following two cases:
(i) Here, the inequality in Eq. (53) implies that
\[ 1 - \frac{c^2}{B_0^2} > \frac{4k^2}{(1 + k)^2} \quad \text{if} \quad \sqrt{1 - \frac{c^2}{B_0^2}} > \frac{2k}{(1 + k)}. \]
Thus,
\[ G \text{ is increasing } \iff B_0^2 > \left( \frac{c(1 + k)}{\sqrt{1 + 2k - 3k^2}} \right)^2 \]
\[ \iff B_0 \in (-\infty, -b) \cup (b, \infty), \quad (54) \]
where
\[ b = \frac{c(1 + k)}{\sqrt{1 + 2k - 3k^2}}, \quad \forall \quad k \in (\frac{-1}{3}, 1); \quad (55) \]
b becomes imaginary or infinite for all other values of \( k \). One can easily verify that \( b > c \quad \forall \quad k \in (-\frac{1}{3}, 1). \)

(ii) The inequality in Eq. (53) implies that
\[ 1 - \frac{c^2}{B_0^2} < \frac{4k^2}{(1 + k)^2} \quad \text{if} \quad \sqrt{1 - \frac{c^2}{B_0^2}} < \frac{2k}{(1 + k)}. \]
Thus,
\[ G \text{ is increasing } \iff B_0^2 < \left( \frac{c(1 + k)}{\sqrt{1 + 2k - 3k^2}} \right)^2 \]
\[ \iff B_0 \in (-b, -c) \cup (c, b), \quad (56) \]
where \( b \) is the same as defined in Eq. (55). Hence, from both the cases, we can conclude that \( G \) increases for all \( B_0 \in \mathbb{R}^+ \setminus (0, c) \) while \( G \) becomes constant at \( b \). It is mentioned here that \( G \) always vanishes at \( B_0 = c \).

Now, we discuss the behaviors of \( G \) in the dust, radiation, and stiff fluid cases. For dust, we take \( k = 0 \), and the behavior of \( G \) is the following:
\[ G \text{ is increasing } \iff B_0 \in \mathbb{R}^+ \setminus (0, c), \]
\[ G \text{ vanishes } \iff B_0 = c. \]
For the radiation case, we have \( k = 1/3 \); thus,
\[ G \text{ is increasing } \iff B_0 \in \mathbb{R}^+ \setminus (0, \frac{2c}{\sqrt{3}}), \]
\[ G \text{ is constant } \iff B_0 = \frac{2c}{\sqrt{3}}. \]
For the stiff fluid, \( G \) increases for all values of \( B_0 \in \mathbb{R}^+ \setminus (0, c). \)
ii. Behavior of $\Lambda$

To discuss the behavior of $\Lambda$, we substitute $\alpha_2 = 1$ in Eq. (47) so that $\Lambda$ becomes

$$\Lambda = \left[ \frac{(2\alpha_1 + 1)(1 + k)B_0^2 - c^2(1 + k) - 2\alpha_1 B_0^2}{c^2 B_0^2(1 + k)} \right] (t + c_0)^{-2} = \Lambda_0(t + c_0)^{-2},$$

(57)

where

$$\Lambda_0 = \frac{(2\alpha_1 + 1)(1 + k)B_0^2 - c^2(1 + k) - 2\alpha_1 B_0^2}{c^2 B_0^2(1 + k)}$$

(58)

is a constant. From this equation, we can discuss the behavior of $\Lambda$ as follows:

- $\Lambda$ is increasing if $\Lambda_0 < 0$ and $t > c_0$,
- or $\Lambda_0 > 0$ and $t < c_0$,
- $\Lambda$ is decreasing if $\Lambda_0 > 0$ and $t > c_0$,
- or $\Lambda_0 < 0$ and $t < c_0$,
- $\Lambda$ vanishes if $\Lambda_0 = 0$ or $t \to \infty$.

(59)

It is noticed that $\Lambda_0 > 0$ if

$$(1 + k)(2\alpha_1 B_0^2 + (B_0^2 - c^2)) > 2\alpha_1 B_0^2.$$  

Using Eq. (48), this inequality becomes

$$(1 + k)(2\alpha_1 + \alpha_1^2) > 2\alpha_1,$$

(60)

$$\Rightarrow \alpha_1 > \frac{-2k}{1 + k},$$

which is the same condition as given by the inequality in Eq. (53). Hence, we finally obtain

$$\Lambda_0 > 0 \quad \forall \quad B_0 \in \mathbb{R}^+ \setminus (0, c).$$

(61)

Similarly, $\Lambda_0 < 0$ if

$$(1 + k)(2\alpha_1 + \alpha_1^2) < 2\alpha_1,$$

which yields a contradiction. Further, $\Lambda_0$ vanishes at $B_0 = b$.

For different types of fluids, we can discuss the above conditions on $\Lambda_0$ as follows: In the dust and stiff fluid case, $\Lambda_0$ is always positive for any value of $B_0 \in \mathbb{R}^+ \setminus (0, c)$ while in the radiation case $\Lambda_0$ is positive for all $B_0 \in \mathbb{R}^+ \setminus (0, \frac{2c_0}{\sqrt{3}})$. 

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2. Kinematic Self-similarity of the Second Kind

For \( \alpha = \delta \neq 0,1 \), the definition of KSS vector field yields

\[
\begin{align*}
\xi_0^0 &= 2\alpha c^2, \\
c^2\xi_1^0 - A^2\xi_0^1 &= 0, \\
c^2\xi_2^0 - B^2\xi_0^2 &= 0, \\
c^2\xi_3^0 - B^2 \sinh^2 \theta \xi_0^3 &= 0,
\end{align*}
\]

\( (62) \) to \( (65) \)

\[
\begin{align*}
\frac{\dot{A}}{A}\xi_0^0 + \xi_1^1 &= 2\delta, \\
A^2\xi_2^1 + B^2\xi_1^2 &= 0, \\
A^2\xi_3^1 + B^2 \sin^2 \theta \xi_1^3 &= 0,
\end{align*}
\]

\( (66) \) to \( (69) \)

Solving this system of equations simultaneously by adopting the same procedure as in the first kind of KSS, it follows that

\[
\begin{align*}
\xi &= (\alpha t + \beta) \partial_t + [(\delta - \frac{\dot{A}}{A}(\alpha t + \beta)) r + c_5] \partial_r \\
&\quad + [(\delta - \frac{\dot{B}}{B}(\alpha t + \beta)) \theta + c_6] \partial_\theta + [(\delta - \frac{\dot{A}}{A}(\alpha t + \beta)) \phi + c_7] \partial_\phi.
\end{align*}
\]

\( (72) \)

\[
\begin{align*}
A &= A_0(t + \frac{\beta}{\alpha})^{\alpha_1}, \\
B &= B_0(t + \frac{\beta}{\alpha})^{\alpha_2}, \\
H &= (\alpha_1 + 2\alpha_2)(t + \frac{\beta}{\alpha})^{-1}, \\
\rho &= \rho_0(t + \frac{\beta}{\alpha})^{-1+(1+k)(\alpha_1+2\alpha_2)},
\end{align*}
\]

\( (73) \) to \( (75) \)

where \( A_0, B_0, \alpha_1, \alpha_2, \) and \( \rho_0 \) are constants of integration.

Similarly, \( G \) and \( \Lambda \) can be found to be

\[
\begin{align*}
G &= G_0(t + \frac{\beta}{\alpha})^{\alpha_1+(\alpha_1+2)k}, \\
\Lambda &= \Lambda_0(t + \frac{\beta}{\alpha})^{-2},
\end{align*}
\]

\( (76) \) and \( (77) \)

where \( G_0 \) and \( \Lambda_0 \) are the same as given in the first kind. Obviously, Eqs. \( (76) \) and \( (77) \) reduce exactly to Eqs. \( (50) \) and \( (57) \) for \( c_0 = \frac{\beta}{\alpha} \); hence, the
behavior of $G$ and $\Lambda$ will be the same as in case of KSS of the first kind just by replacing $c_0 = \frac{\beta}{\alpha}$ in the corresponding constraint equations.

**IV. TIME-VARYING BEHAVIOR OF $G$ AND $\Lambda$ BY USING THE MATTER COLLINEATION TECHNIQUE**

The energy-momentum tensor $T_{ab}$ represents the matter part of the field equations in Eq. (1). This enables us to understand the physical structure of spacetime. Symmetries of the energy-momentum tensor provide the conservation laws on matter fields. Matter collineation is defined as

$$\mathcal{L}_\xi T_{ab} = 0.$$ (78)

This does not give any information about the behaviors of $G$ and $\Lambda$. However, it can be modified to obtain the behavior of $G$ as follows [28]:

$$\mathcal{L}_\xi \left( \frac{8\pi G(t)}{c^4} T_{ab} \right) = 0.$$ (79)

If we introduce $\Lambda$, we get a complete modification of matter collineation, Eq. (78) [28]:

$$\mathcal{L}_\xi \left( \frac{8\pi G(t)}{c^4} T_{ab} + \Lambda(t) g_{ab} \right) = 0.$$ (80)

This equation helps us to discuss the behaviors of time-dependent $G$ and $\Lambda$.

1. **Matter Collineations**

Using the components of a perfect fluid energy-momentum tensor, the system of matter collineation (MC) equations yields

$$(pc^2)_{,0} \xi^0 + 2pc^2 \xi^0_{,0} = 0,$$ (81)

$$pc^2 \xi^0_{,1} + pA^2 \xi^1_{,0} = 0,$$ (82)

$$pc^2 \xi^0_{,2} + pB^2 \xi^2_{,0} = 0,$$ (83)

$$pc^2 \xi^0_{,3} + pB^2 \sinh^2 \theta \xi^3_{,0} = 0,$$ (84)

$$(pA^2)_{,0} \xi^0 + 2pA^2 \xi^1_{,1} = 0,$$ (85)

$$pA^2 \xi^1_{,1} + pB^2 \xi^2_{,1} = 0,$$ (86)

$$pA^2 \xi^1_{,3} + pB^2 \sinh^2 \theta \xi^3_{,1} = 0,$$ (87)

$$(pB^2)_{,0} \xi^0 + 2pB^2 \xi^2_{,2} = 0,$$ (88)
\[
pB^2 \xi_3^2 + p B^2 \sinh^2 \theta \xi_2^3 = 0, \tag{89}
\]
\[
(pB^2 \sinh^2 \theta)_{,0} \xi^0 + 2pB^2 \sinh^2 \theta \xi_3^3 = 0. \tag{90}
\]

It follows from Eq. (81) that
\[
\xi^0 = f_8(r, \theta, \phi) \rho^{-1/2}, \tag{91}
\]
where \(f_8(r, \theta, \phi)\) is a function of integration. Similarly, Eq. (85) yields
\[
\left(\frac{\dot{p}}{p} + 2\frac{\dot{A}}{A}\right) \xi^0 + 2\xi_1^0 = 0.
\]

Using Eqs. (5) and (91), this equation gives
\[
\xi^1 = \xi^0 \left(\frac{\dot{\xi}^0}{\xi^0} - \frac{\dot{A}}{A}\right) r + f_9(t, \theta, \phi), \tag{92}
\]
where \(f_9(t, \theta, \phi)\) is an integration function. Similarly, by solving Eqs. (88) and (90) and then integrating w.r.t. \(\theta\) and \(\phi\), respectively, we obtain
\[
\xi^2 = \xi^0 \left(\frac{\dot{\xi}^0}{\xi^0} - \frac{\dot{B}}{B}\right) \theta + f_{10}(t, r, \phi), \tag{93}
\]
\[
\xi^3 = \xi^0 \left(\frac{\dot{\xi}^0}{\xi^0} - \frac{\dot{B}}{B}\right) \phi + f_{11}(t, r, \theta), \tag{94}
\]
where \(f_{10}(t, r, \phi)\) and \(f_{11}(t, r, \theta)\) are functions of integration.

When we solve the above system of ten equations simultaneously by using these values of \(\xi\), as done in the KSS technique, the functions of integration, \(f_8(r, \theta, \phi) - f_{11}(t, r, \theta)\), reduce to arbitrary constants, which are termed as \(c_8, c_9, c_{10}\), and \(c_{11}\), respectively. Thus, the final form of the vector field, in terms of \(\xi^0 = c_8 \rho^{-1/2}\), can be written as
\[
\xi = \xi^0 \partial_t + [(\xi^0 - \frac{\dot{A}}{A} \xi^0) r + c_9] \partial_r + [(\xi^0 - \frac{\dot{B}}{B} \xi^0) \theta + c_{10}] \partial_\theta + [(\xi^0 - \frac{\dot{B}}{B}) \phi + c_{11}] \partial_\phi, \tag{95}
\]
along with the following constraint equations
\[
p \partial_t[\xi^0(\frac{\dot{p}}{2p} + \frac{\dot{A}}{A})] = 0, \quad p \partial_t[\xi^0(\frac{\dot{p}}{2p} + \frac{\dot{B}}{B})] = 0. \tag{96}
\]

These two equations reveal that either \(p = 0\), i.e., a dust case, or \(p \neq 0\), i.e., a perfect fluid case. The dust case is not interesting as the time-varying behaviors, of \(G\) and \(\Lambda\) cannot be discussed for this case by using MCs.
In the perfect fluid case, Eq. (96) implies that

\[ A = \frac{1}{\sqrt{\rho}} e^{a_1 \int \sqrt{\rho} dt + a_2}, \quad B = \frac{1}{\sqrt{\rho}} e^{b_1 \int \sqrt{\rho} dt + b_2}, \] (97)

where \(a_1, a_1, b_1,\) and \(b_2\) are arbitrary constants of integration. For these values of the metric functions, the EFEs give

\[ G = \frac{2c^2}{8\pi \rho(1 + k)} \left[ \frac{\ddot{\rho}}{\rho^2} - \frac{\dot{\rho}^2}{\rho^3} + c_{13} \frac{\ddot{\rho}}{\rho^{3/2}} - \frac{b_1}{\rho^{3/2}} + 2c_{14} \right], \] (98)

\[ \Lambda = \frac{1}{c^2} \left( \frac{b_1 \rho^{1/2} - \frac{\dot{\rho}}{2\rho}}{2} - \frac{c_{15} \rho^{1/2} e^{-b_1 \int \sqrt{\rho} dt + b_2}}{2} \right) + \frac{2k}{1 + k} \left( b_2 \rho - c_{13} \frac{\dot{\rho} \rho^{1/2}}{2} + \frac{\rho^2 \rho^{2/3}}{4} \right) + \frac{2}{1 + k} \left( c_{13} \rho + \frac{3\rho^2}{4\rho^2} - b_1 \left( \frac{\dot{\rho}}{\rho^{1/2}} - \frac{1}{2\rho^{1/2}} \right) - \frac{\ddot{\rho}}{\rho} \right). \] (99)

These equations show that the time-varying behaviors of \(G\) and \(\Lambda\) cannot be discussed unless \(\rho\) is given. We assume the following two cases to discuss the behaviors of \(G\) and \(\Lambda\):

(i) \(\rho(t) = \rho_0\) (constant);
(ii) \(\rho(t) = \frac{1}{(at + b)^2}\), where \(a = \frac{1}{c_8}\) and \(b = \frac{c_9}{c_8}\).

**Case (i)** When we substitute \(\rho(t) = \rho_0\) (constant) in Eqs. (98) and (99), it follows that

\[ G = \frac{2c^2}{8\pi \rho_0(1 + k)} \left( -\frac{b_1}{\rho_0^{3/2}} + 2c_{18} \right) \] (100)

and

\[ \Lambda = -\frac{1}{\rho_0} e^{-2b_1(t + b_2)} + \Lambda_0, \] (101)

where

\[ \Lambda_0 = \frac{1}{c^2} \left[ (b_1 \rho_0^{1/2})^2 + \frac{2k}{1 + k} b_2 \rho_0 + \frac{2}{1 + k} \left( c_{13} \rho_0 + \frac{b_1}{2\rho_0^{1/2}} \right) \right]. \]

We see from Eqs. (100) and (101) that \(G\) always remains constant while the behavior of \(\Lambda\) is as follows:

\[ \Lambda \text{ is increasing} \iff b_1 > 0, \]
\[ \Lambda \text{ is decreasing} \iff b_1 < 0, \]
\[ \Lambda \text{ is constant} \iff b_1 = 0. \] (102)
**Case (ii)** In this case, we get $\xi_0^0 = t + c_0$. Using this in Eq. (95), we obtain the same homothetic vector field as in case of KSS of the first kind given in Eq. (11). It is mentioned here that this homothetic vector field satisfies the relation

$$L_{HO}(T_{ab}) = 0$$

with the same ODEs as given by Eqs. (39) and (57). Thus, the behaviors of $G$ and $\Lambda$ will be the same as in the case of KSS of the first kind.

2. Modified Matter Collineations

The modified MC equations, Eq. (79), can be written as

$$\xi^0 \dot{G} T_{ab} + G(T_{ab,c} \xi_c^c + T_{ac} \xi_b^b + T_{bc} \xi_a^a) = 0.$$ (104)

The corresponding system of MC equations yields

$$\xi^0 = c_{17}(G\rho)^{-1/2},$$ (105)

$$\xi^1 = \xi^0 \left( \frac{\xi^0}{\xi^0} - \frac{\dot{A}}{A} \right) r + c_{18},$$ (106)

$$\xi^2 = \xi^0 \left( \frac{\xi^0}{\xi^0} - \frac{\dot{B}}{B} \right) \theta + c_{19},$$ (107)

$$\xi^3 = \xi^0 \left( \frac{\xi^0}{\xi^0} - \frac{\dot{B}}{B} \right) \phi + c_{20}. $$ (108)

Here, $c_{17}, c_{18}, c_{19},$ and $c_{20}$ are arbitrary constants of integration satisfying the following ODEs

$$p \partial_t [\xi^0 \left( \frac{1}{2} \left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) + \frac{\dot{A}}{A} \right)] = 0,$$ (109)

$$p \partial_t [\xi^0 \left( \frac{1}{2} \left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) + \frac{\dot{B}}{B} \right)] = 0.$$ (110)

For $p \neq 0$, Eqs. (109) and (110) provide

$$A = \frac{1}{\sqrt{G\rho}} e^{c_{21}} \int \sqrt{G\rho} \, dt + c_{22}, \quad B = \frac{1}{\sqrt{G\rho}} e^{c_{23}} \int \sqrt{G\rho} \, dt + c_{24}. $$ (111)

For these values of $A$ and $B$, the EFEs yield

$$G = \frac{2c^2}{8\pi(1+k)} \left( \left( c_{21}(G\rho)^{1/2} - \frac{1}{2} \left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) \right) \left( c_{25}(G\rho)^{1/2} - \frac{1}{2} \left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) \right) \right).$$
\[- \left\{ c_{25}(G\rho)^{1/2} - \frac{1}{2}\left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) \right\}^2 - \frac{1}{2} c_{25}(\dot{G}\rho + G\dot{\rho})(G\rho)^{-1/2} \]

\[- \frac{1}{2}(\frac{\ddot{G}}{G} - \frac{\dot{\rho}^2}{\rho^2} + \frac{\dot{G}}{G} - \frac{G^2}{G^2}) \],

\[
\Lambda = \frac{1}{c^2}\left\{ c_{25}(G\rho)^{1/2} - \frac{1}{2}\left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) \right\}^2 - \frac{1}{2}\frac{c}{(G\rho)^{1/2}} e^{-c_{24}\int \sqrt{G\rho} dd + c_{25}}^2 
\]

\[+ \frac{2k}{1+k}\left\{ c_{21}(G\rho)^{1/2} - \frac{1}{2}\left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) \right\} \left\{ c_{25}(G\rho)^{1/2} - \frac{1}{2}\left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right) \right\} 
+ \frac{2}{1+k}\left\{ (c_{25}(G\rho)^{1/2} - \frac{1}{2}\left( \frac{\dot{\rho}}{\rho} + \frac{\dot{G}}{G} \right))^2 + \frac{1}{2} c_{25}(\dot{G}\rho + G\dot{\rho})(G\rho)^{-1/2} 
- \frac{1}{2}(\frac{\ddot{\rho}}{\rho} - \frac{\dot{\rho}^2}{\rho^2} + \frac{\ddot{G}}{G} - \frac{G^2}{G^2}) \right\} \].

(112)

It is obvious from Eqs. (112) and (113) that the expressions of \( G \) and \( \Lambda \) are too complicated to discuss. Thus, we assume \( \xi_0^0 = t + c_0 \), i.e., \( G(t)\rho(t) = \frac{1}{(at+b)^2} \) (as found in homothetic case), such that the vector field becomes homothetic and satisfies the equation

\[
\mathcal{L}_{HO} \left( \frac{8\pi G(t)}{c^4} T_{ab} \right) = 0
\]

(114)

along with the same ODEs as given by Eqs. (39) and (40). This corresponds to the KSS of the first kind. It is worth mentioning here that this case leads to

\[
\frac{\dot{G}}{G} + \frac{\dot{\rho}}{\rho} = -\frac{2}{(t+c_0)^2} \iff G\rho \approx (t+c_0)^{-2},
\]

(115)

which is exactly the same relation for the product \( G\rho \) as found by using the KSS technique.

3. Completely Modified Matter Collineations

This is the case in which the variation of \( \Lambda \) is included, as well. Here, we consider Eq. (80) and apply the definition of the Lie derivative to obtain

\[
\xi^0 \mathcal{G} T_{ab} + G(T_{ab,c} \xi^c + T_{ac} \xi^c_b + T_{bc} \xi^c_a) 
+ [\Lambda \xi^0_0 g_{ab} + \Lambda (g_{ab,c} \xi^c + g_{ac} \xi^c_b + g_{bc} \xi^c_a)] = 0.
\]

(116)

The system of equations here leads to the following solution:

\[
\frac{\dot{\xi}^0}{\xi} = -\frac{1}{2} \left[ \frac{8\pi}{c^4}(\dot{G}\rho + \dot{\rho} G) + \dot{\Lambda} \right].
\]
\[ \xi^0 = c_{26}(r, \theta, \phi) \left( \frac{8\pi G\rho}{c^4} + \Lambda \right)^{-1/2}. \] (117)

The remaining components of \( \xi \) are
\[
\begin{align*}
\xi^1 &= -\frac{1}{2} \left[ \frac{\frac{8\pi}{c^4} (\dot{G}p + \dot{p}G) - \dot{\Lambda}}{A} + 2\frac{\dot{A}}{A} \right] \xi^0_r + c_{27}(t, \theta, \phi), \\
\xi^2 &= -\frac{1}{2} \left[ \frac{\frac{8\pi}{c^4} (\dot{G}p + \dot{p}G) - \dot{\Lambda}}{B} + 2\frac{\dot{B}}{B} \right] \xi^0_\theta + c_{28}(t, r, \phi), \\
\xi^3 &= -\frac{1}{2} \left[ \frac{\frac{8\pi}{c^4} (\dot{G}p + \dot{p}G) - \dot{\Lambda}}{B} + 2\frac{\dot{B}}{B} \right] \xi^0_\phi + c_{29}(t, r, \theta).
\end{align*}
\] (118, 119, 120)

The following ODEs become the necessary and sufficient conditions:
\[
\begin{align*}
\partial_t \left[ \frac{\frac{8\pi}{c^4} (\dot{G}p + \dot{p}G) - \dot{\Lambda}}{A} + 2\frac{\dot{A}}{A} \left( \frac{8\pi G\rho}{c^4} + \Lambda \right)^{-1/2} \right] &= 0, \\
\partial_t \left[ \frac{\frac{8\pi}{c^4} (\dot{G}p + \dot{p}G) - \dot{\Lambda}}{B} + 2\frac{\dot{B}}{B} \left( \frac{8\pi G\rho}{c^4} + \Lambda \right)^{-1/2} \right] &= 0.
\end{align*}
\] (121, 122)

With the equation of state and then replacing \( \frac{8\pi G\rho}{c^4} = \Phi \), the above ODEs provide
\[
\begin{align*}
A &= A_0(\Phi - \Lambda)^{-1/2} + e^{a_3} \int \sqrt{\Phi + \Lambda} dt, \\
B &= B_0(\Phi - \Lambda)^{-1/2} + e^{a_4} \int \sqrt{\Phi + \Lambda} dt,
\end{align*}
\] (123, 124)

where \( A_0, B_0, a_3, \) and \( a_4 \) are constants of integration. If we substitute these values of the metric functions in the EFEs, we get much more complicated expressions for \( G \) and \( \Lambda \). However, if we assume \( \xi^0 = t + c_0 \), then the following relation is satisfied:
\[
\mathcal{L}_{HO} \left( \frac{8\pi G(t)}{c^4} T_{ab} + \Lambda(t) g_{ab} \right) = 0,
\] (125)

yielding the metric functions in the form
\[
\begin{align*}
A &= A_0(t + c_0)^{a_1} (\Phi - \Lambda)^{-1/2}, \\
B &= B_0(t + c_0)^{a_2} (\Phi - \Lambda)^{-1/2}.
\end{align*}
\] (126, 127)

Here, \( a_1 \) and \( a_2 \) are constants of integration. When we use these values of the metric functions in the EFEs, we are again unable to discuss the
behaviors of the $G$ and $\Lambda$ due to their complicated expressions. We note that if we replace $\Lambda$ by $\Phi^{-1}$, i.e., $\Lambda = \frac{8\pi G \rho}{c^4} - 1$, then it also corresponds to the homothetic case of the KSS.

It is interesting to note that if we write Eq. (80) as
\[ \mathcal{L}_\xi \left( \frac{8\pi G(t)}{c^4} T_{ab} \right) = 0 = \mathcal{L}_\xi (\Lambda(t) g_{ab}), \] (128)
which is a special case, we may get some insight. In this way, we reach again homothetic cases and obtain the following results:
\[ G \rho \approx (t + c_0)^{-2} \quad \text{and} \quad \Lambda = \Lambda_0 (t + c_0)^{-2}, \] (129)
where $\Lambda_0$ is an arbitrary constant of integration. Further discussion on the behavior of $\Lambda$ is the same as given in Eq. (59).

V. SUMMARY AND DISCUSSION

We have studied the perfect fluid Bianchi type III and Kantowski-Sachs spacetimes with time-varying constants $G$ and $\Lambda$. Due to the time-varying nature of these constants, Bianchi identities, along with the energy-momentum conservation law, $T_{ab}^b = 0$, yield a time-dependent expression of the energy density. This expression helps us to define Hubble parameter and the deceleration parameter.

In the KSS, we studied the behaviors of $G$ and $\Lambda$ for the first and the second kinds. When we solve the system of ten self-similar equations simultaneously, there arise two ODEs as necessary and sufficient conditions. The solution of these ODEs yields the metric functions $A$ and $B$, which make the variational behaviors of $G$ and $\Lambda$ possible. We also discuss the dust, radiation, and stiff fluid cases. Further, we discuss the behaviors of $G$ and $\Lambda$ by using MCs only the case that corresponds to the KSS.

In the KSS of the first kind, the metric functions take the form
\[ A = A_0(t + c_0)^{\alpha_1} \quad \text{and} \quad B = B_0(t + c_0)^{\alpha_2}, \]
where $\alpha_1$, $\alpha_2$, $A_0$, and $B_0$ are arbitrary positive constants. It is worth mentioning here that the physical situation is only possible if we assume $\alpha_2 = 1$. Then, $G$ takes the form
\[ G = G_0(t + c_0)^{\alpha_1 + (\alpha_1 + 2)k}, \]
where $G_0$ is a constant given by Eq. (51).
For a Bianchi Type III metric, \( \alpha_1 = \sqrt{1 - \frac{c^2}{B_0^2}} \), and the behavior of \( G \) depends on \( b \), Eq. \([55]\). For the Kantowski-Sachs metric, \( \alpha_1 \) turns out to be \( \sqrt{1 + \frac{c^2}{B_0^2}} \) and the behavior of \( G \) depends on
\[
\frac{c(1+k)}{\sqrt{3k^2-1-2k}} \quad \forall \; k \in (-\infty, -\frac{1}{3}) \cup (1, \infty).
\]

**Table 1.** Behavior of \( G \) for both spacetimes.

| Spacetime       | \( G \) is increasing | Constant |
|-----------------|------------------------|----------|
| Bianchi type III| \( B_0 \in \mathbb{R}^+ \setminus (0, c) \) | \( B_0 = b \) |
| Kantowski-Sachs  | Any value of \( B_0 \in \mathbb{R}^+ \) | \( B_0 = d \) |

The dust, radiation and stiff fluid cases for Bianchi Type III metric are given in Table 2.

**Table 2.** Behavior of \( G \) for different cases of fluids.

| Behavior of \( G \) | Dust case | Radiation case | Stiff matter |
|----------------------|-----------|----------------|-------------|
| \( G \) is increasing| \( B_0 \in \mathbb{R}^+ \setminus (0, c) \) | \( B_0 \in \mathbb{R}^+ \setminus \left(0, \frac{c}{\sqrt{3}}\right)\) | \( B_0 \in \mathbb{R}^+ \setminus (0, c) \) |
| \( G \) is Constant  | \( B_0 = c \) | \( B_0 = \frac{c}{\sqrt{3}} \) | - |

In the case of the Kantowski-Sachs metric, the dust and radiation cases are not physical while for stiff matter, \( G \) is increasing \( \forall \; B_0 \in \mathbb{R}^+ \). In the KSS of the first kind, \( \Lambda \) is given by the following equation:

\[
\Lambda = \Lambda_0 (t + c_0)^{-2},
\]

which yields the behavior of \( \Lambda \) given below in Table 3. In Bianchi type III and Kantowski-Sachs metrics, the possible values of \( \Lambda_0 \) are discussed in Table 4. For various fluids, \( \Lambda_0 \) reduces for Bianchi type III, as given in Table 5. In Kantowski-Sachs metric, the solutions corresponding to the dust and the radiation cases are not physical while for stiff matter \( \Lambda_0 > 0 \) \( \forall \; B_0 \in \mathbb{R}^+ \).

**Table 3.** Behavior of \( \Lambda \).

| Behavior of \( \Lambda \) | Value of \( \Lambda_0 \) and \( t \) |
|---------------------------|----------------------------------|
| \( \Lambda \) is increasing| \( \Lambda_0 < 0 \) and \( t > c_0 \) or \( \Lambda_0 > 0 \) and \( t < c_0 \) |
| \( \Lambda \) is decreasing  | \( \Lambda_0 > 0 \) and \( t > c_0 \) or \( \Lambda_0 < 0 \) and \( t < c_0 \) |
| \( \Lambda \) vanishes       | \( \Lambda_0 = 0 \) or \( t \to \infty \) |

**Table 4.** Possible values of \( \Lambda_0 \).

| Spacetime             | \( \Lambda_0 > 0 \) | \( \Lambda_0 = 0 \) |
|-----------------------|---------------------|---------------------|
| Bianchi type III      | \( B_0 \in \mathbb{R}^+ \setminus (0, c) \) | \( B_0 = b \) |
| Kantowski-Sachs       | \( \forall \; B_0 \in \mathbb{R}^+ \) | \( B_0 = d \) |
Table 5. Possible values of $\Lambda_0$ for different types of fluids.

| Cases   | Dust Fluid | Radiation Case | Stiff Matter |
|---------|------------|----------------|--------------|
| $\Lambda_0 > 0$ | $B_0 \in \mathbb{R}^+ \setminus (0, c)$ | $B_0 \in \mathbb{R}^+ \setminus (0, \frac{c}{\sqrt{3}})$ | $B_0 \in \mathbb{R}^+ \setminus (0, c)$ |
| $\Lambda_0 = 0$ | $B_0 = c$ | $B_0 = \frac{2}{\sqrt{3}}$ | - |

We note that for the KSS of the second kind, the results coincide with those of the first kind as given in the above tables except that in the expressions for the metric functions, $c_0$ is replaced with a fraction of two constants $\frac{2}{3}$.

Using the MCs technique, the behaviors of $G$ and $\Lambda$ are not straightforward as in the case of the KSS due to the complicated metric functions. However, we have managed to discuss two particular cases depending upon $\rho$. Firstly, for $\rho = \rho_0$ (a constant), $G$ becomes constant while $\Lambda$ varies as given in Eq. (102). Secondly, the case for $\rho = \frac{1}{(at+b)^2}$, corresponds to the homothetic case. In modified MCs, the behaviors of $G$ and $\Lambda$ could not be discussed generally, but we obtained a homothetic case by assuming $G(t)p(t) = \frac{1}{(at+b)^2}$. Further, we obtained a relationship $G \rho \approx (t + c_0)^{-2}$, which was the same as obtained in the KSS technique. Similarly, in the completely modified MCs case, we again obtained a homothetic case by assuming $\xi_0 = (t + c_0)$ and $\Lambda = \frac{8\pi Gp}{c^2} - 1$. Further, when we re-interpreted the completely modified MC equations, given in Eq. (80), by Eq. (128), we directly obtained a homothetic case giving $\Lambda = \Lambda_0 (t + c_0)^{-2}$. Consequently, the behavior of $\Lambda$ turns out to be the same as given in Eq. (59) with the KSS. It is mentioned here that the metric functions and vector fields are the same for both spacetimes. However, we obtain different behaviors of $G$ and $\Lambda$ due to slight changes in the EFEs.

We found that the cosmological constant $\Lambda$ turned out to be a time decreasing function for $\Lambda_0 > 0$ while the gravitational constant $G$ was a time increasing function when $G_0 > 0$ for all values of $k > 0$. It is worth mentioning here that our results verify the results obtained by Belinchon [29]. For these behaviors of $G$ and $\Lambda$, the time-dependent vacuum energy density relation is also satisfied, according to which both these constants are changing in a reciprocal way [26]. Further, we found that for $\alpha_2 = 1$, the deceleration parameter attained a negative value, which showed that the expansion of the universe was accelerating. Thus, we can say that with the expansion of the universe, $\Lambda$ is going to reduce [5].

We would like to mention here that the above mentioned time-varying behaviors of $G$ and $\Lambda$ can only be discussed in the homothetic case, i.e, the KSS of the first kind for both Bianchi type III and Kantowski-Sachs spacetimes. Moreover, we found that the vector field satisfying equation
\[ \mathcal{L}_\xi g_{ab} = 2g_{ab} \] also satisfied equation \[ \mathcal{L}_\xi T_{ab} = 0. \] By modifying the MC equations in an appropriate way, we were able to find the same relationships as in the case of the KSS solution.

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