TRANSVERSALITY FOR TIME-PERIODIC
COMPETITIVE-COOPERATIVE TRIDIAGONAL SYSTEMS

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Abstract. Transversality of the stable and unstable manifolds of hyperbolic
periodic solutions is proved for tridiagonal competitive-cooperative time-periodic
systems. We further show that such systems admit the Morse-Smale property
provided that all the fixed points (of the corresponding Poincaré map) are
hyperbolic. The main tools used here are the integer-valued Lyapunov func-
tion, as well as the Floquet theory developed in [1] for general time-dependent
tridiagonal linear systems.

1. Introduction. In this paper, we consider the dynamical properties of the fol-
lowing system with tridiagonal structure:
\[ \begin{align*}
\dot{x}_1 &= f_1(t, x_1, x_2), \\
\dot{x}_i &= f_i(t, x_{i-1}, x_i, x_{i+1}), \quad 2 \leq i \leq n-1, \\
\dot{x}_n &= f_n(t, x_{n-1}, x_n),
\end{align*} \] (1.1)
where \( f = (f_1, f_2, \cdots, f_n) \) is a \( C^1 \)-function defined on \( \mathbb{R} \times \mathbb{R}^n \), it is assumed that
there exists \( T > 0 \) such that
\[ f(t+T, x) = f(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n. \] (1.2)
Equations of the form (1.1) usually arise in modelling ecosystems of \( n \) species \( x_1, x_2, \cdots, x_n \) with a certain hierarchical structure. In such hierarchy, \( x_1 \) interacts only with \( x_2 \), \( x_n \) only with \( x_{n-1} \), and for \( i = 2, \cdots, n-1 \), species \( x_i \) interacts with \( x_{i-1} \) and \( x_{i+1} \).

Our standard assumption on the tridiagonal system (1.1) is that the variable \( x_{i+1} \) affects \( \dot{x}_i \) and \( x_i \) affects \( \dot{x}_{i+1} \) monotonically in the same fashion. More precisely, there are \( \delta_i \in \{-1, +1\}, i = 1, \cdots, n-1 \), such that
\[ \delta_i \frac{\partial f_i}{\partial x_{i+1}}(t, x) > 0, \quad \delta_i \frac{\partial f_{i+1}}{\partial x_i}(t, x) > 0, \quad 1 \leq i \leq n-1, \]
for all \((t, x) \in \mathbb{R} \times \mathbb{R}^n\). This is one of the most commonly studied tridiagonal
systems, called competitive-cooperative systems, in which individuals either compete
or cooperate with their neighboring species. In particular, if \( \delta_i = -1 \) for all \( i \), then
(1.1) is called competitive. If \( \delta_i = 1 \) for all \( i \), then (1.1) is called cooperative. Of course, when \( n = 2 \), system (1.1) naturally reduces to a two-dimensional competitive or cooperative system.

In the case where \( f \) is independent of \( t \), Smillie [10] showed that all bounded trajectories of system (1.1) converge to equilibria. Under the assumption that \( f \) is time-periodic with periodic \( T > 0 \) (i.e., (1.2)), Hale and Somolinos [4] proved that if \( n = 2 \), then all bounded solutions are asymptotic to \( T \)-periodic solutions (see also [8] for Lotka-Volterra systems). Later, Smith [11] studied the \( T \)-periodic system (1.1) for general \( n \)-species and showed that every bounded solution is asymptotic to a \( T \)-periodic solution. When \( f \) is quasi-periodic or almost-periodic, Wang [12] investigated system (1.1) in the framework of skew-product flows and showed that the \( \omega \)-limit set of any bounded orbit contains at most two minimal sets, and each minimal set is an almost automorphic extension of the base flow, which is driven by the time-translation for \( f \). Very recently, by developing the theory of Floquet bundles for the associated time-dependent linear system of (1.1), Fang, Gyllenberg and Wang [1] proved that any hyperbolic \( \omega \)-limit set is a 1-1 extension of the base flow.

Following [11], we now let \( \hat{x}_i = \mu_i x_i, \mu_i \in \{+1, -1\}, 1 \leq i \leq n \), with \( \mu_1 = 1, \mu_i = \delta_{i-1} \mu_{i-1} \). Then (1.1) transforms into a new system of the same type with \( \hat{\delta}_i = \mu_i \delta_{i+1} \mu_i = \mu_i^2 \delta_i^2 = 1 \). Therefore, we can always assume, without loss of generality, that the tridiagonal system (1.1) is cooperative, which means that

\[
\frac{\partial f_i}{\partial x_{i+1}} (t, x) > 0, \quad \frac{\partial f_{i+1}}{\partial x_i} (t, x) > 0, \quad 1 \leq i \leq n-1, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.3}
\]

In the theory of dynamical systems, transversality of stable and unstable manifolds of critical elements plays a central role in connection with structural stability. For the autonomous case (i.e., \( f \) is independent of \( t \)), Fusco and Oliva [3] have established transversality of the stable and unstable manifolds of hyperbolic equilibria and the structural stability of system (1.1). One can also refer to Hirsch [5] for the lower-dimensional (\( n \leq 3 \)) related cases.

In this article, we will focus on transversality of the stable and unstable manifolds, as well as structural stability, for the time-periodic tridiagonal competitive-cooperative system (1.1). More precisely, we show that the stable and unstable manifolds of any hyperbolic fixed points for the Poincaré map associated with system (1.1)-(1.3) always intersect transversely (see Theorem 3.1). Based on this, we further prove that the time-periodic system (1.1) admits the Morse-Smale property provided that all the fixed points (of the corresponding Poincaré map) are hyperbolic (see Theorem 4.2). Our results here are a natural generalization of the results of Fusco and Oliva [3] to the time-periodic systems.

The main tool we used here is the integer-valued Lyapunov function \( \sigma \), first defined by Smillie [10] (or Fusco and Oliva [2, 3], see also similar forms by Mallet-Paret and Smith [7], and Mallet-Paret and Sell [6]). Besides, we further utilize the approach of the skew-product flows to generalize a technical lemma (by Fusco and Oliva [3, Lemma 9]) from the asymptotically autonomous cases to the asymptotically time-periodic cases (see Lemma 3.2). We then accomplished our approach by combining this lemma with the Floquet theory (which relates the values of \( \sigma \) to the Floquet solutions) developed in [1] for the general time-dependent linear tridiagonal systems.
The paper is organized as follows. In section 2, we introduce an integer-valued Lyapunov function and recall results from Floquet theory (established in [1]) for time-dependent linear systems that will play an important role in the proof of our main results. In section 3, we focus on transversality of the stable and unstable of hyperbolic periodic solutions of system (1.1). Finally, we discuss the Morse-Smale main results. In section 3, we focus on transversality of the stable and unstable of time-dependent linear systems that will play an important role in the proof of our Lyapunov function and recall results from Floquet theory (established in [1]).

We further assume that there is an \( \varepsilon \in \mathbb{R} \) with all the coefficient functions being bounded and uniformly continuous on \( A_t \) for all \( t \in \mathbb{R} \) and \( 1 \leq i \leq n - 1 \). Hereafter, we also write the corresponding matrix \( A(t) = (a_{ij}(t))_{n \times n} \).

We will introduce an integer-valued Lyapunov function \( \sigma \) associated with (2.1). Following [10], we define a continuous map \( \sigma : \Lambda \to \{0, 1, 2, \ldots, n - 1\} \) on \( \Lambda = \{v \in \mathbb{R}^n : v_1 \neq 0, v_n \neq 0 \text{ and if } v_i = 0 \text{ for some } i, 2 \leq i \leq n - 1, \text{ then } v_{i-1}v_{i+1} < 0 \} \) by

\[
\sigma(v) = \# \{i | v_i = 0 \text{ or } v_{i-1}v_{i+1} \leq 0\},
\]

where \# denotes the cardinality of the set. Note that \( \Lambda \) is open and dense in \( \mathbb{R}^n \) and it is the maximal domain on which \( \sigma \) is continuous. Besides, it is also useful to define two integer-valued functions (see, e.g. [3])

\[
\sigma_m : \mathbb{R}^n \to \{0, 1, \ldots, n - 1\}, \\
\sigma_M : \mathbb{R}^n \to \{0, 1, \ldots, n - 1\},
\]

by letting \( \sigma_m, \sigma_M \) be the minimum and maximum value of \( \sigma(x') \) for \( x' \in U \cap \Lambda \), where \( U \) is a small neighborhood of \( x \). Note that \( x \in \Lambda \) is equivalent to \( \sigma_m(x) = \sigma(x) = \sigma_M(x) \). We are now collecting certain properties of \( \sigma \), which are stated in the following lemma.

**Lemma 2.1.** If \( x(t) \) is a nontrivial solution of (2.1) on \( \mathbb{R} \) then

(i) \( x(t) \in \Lambda \) except possibly for isolated values of \( t \);

(ii) \( \sigma(x(t)) \) is nonincreasing as \( t \) increases with \( x(t) \in \Lambda \). Moreover, if \( x(s) \notin \Lambda \) for some \( s \in \mathbb{R} \), then for \( \varepsilon > 0 \) sufficiently small, we have \( \sigma(x(s + \varepsilon)) < \sigma(x(s - \varepsilon)) \);

(iii) there exists a \( t_0 > 0 \) such that \( x(t) \in \Lambda \) and \( \sigma(x(t)) \) is constant for \( t \in [t_0, +\infty) \) and for \( t \in (-\infty, -t_0] \), respectively.

**Proof.** See [11, Proposition 1.2] or [1, Lemma 2.1]. \( \square \)

When \( A(t) \) is periodic in time \( t \), we have the following lemma concerning the Floquet multipliers of (2.1) and the corresponding eigenspaces.

**Lemma 2.2.** If \( A(t) \) is periodic in time with period \( T > 0 \), then

(i) the system (2.1) has \( n \) distinct positive Floquet multipliers \( \alpha_1, \alpha_2, \ldots, \alpha_n \), satisfying \( \alpha_1 > \alpha_2 > \cdots > \alpha_n > 0 \);
(ii) if $E_{\alpha_m}$ is the one-dimensional eigenspace associated with $\alpha_m$, then $E_{\alpha_m} \setminus \{0\} \subset A$ and

$$\sigma(E_{\alpha_m} \setminus \{0\}) = m - 1, \quad 1 \leq m \leq n.$$ 

Proof. See [11, Theorem 1.3]. \qed

For the general time-dependent system (2.1) without periodicity assumption on $A(t)$, we now introduce the so called “Floquet space” of $A(t)$, which will be useful in the forthcoming sections.

For any fixed integers $m$ and $l$ satisfying $0 \leq m \leq l \leq n-1$, we define the set, called the Floquet space of $A$,

$$W_{m,l}(A) = \{ x_0 \in \mathbb{R}^n \setminus \{0\} : \text{the solution } x(t) \text{ of (2.1) with } x(0) = x_0 \text{ satisfies } m \leq \sigma(x(t)) \leq l, \text{ whenever } x(t) \in A \} \cup \{0\}. $$

When $m = l$, we write $W_{m,l}(A)$ as $W_m(A)$ for brevity. The following proposition was proved in [1].

Lemma 2.3. For $0 \leq m \leq l \leq n-1$, the Floquet space $W_{m,l}(A)$ is a linear subspace of $\mathbb{R}^n$ with

$$\dim(W_{m,l}(A)) = l - m + 1.$$ 

Moreover, $W_{m,l}(A)$ has the direct sum decomposition $W_{m,l}(A) = \bigoplus_{k=m}^l W_k(A)$. In particular, if $A(t)$ is periodic in $t$, then $W_{m,l}(A) = \bigoplus_{j=m}^l E_{\alpha_j}$, where $E_{\alpha_j}$ is the one-dimensional eigenspace associated with the Floquet multiplier $\alpha_j$.

Proof. See [1, Lemma 2.4 and Remark 2.7]. \qed

3. Transversality. In this section, we shall prove that the stable and unstable manifolds of hyperbolic $T$-periodic solutions of (1.1)-(1.3) intersect transversely.

Let $p(t)$ be a $T$-periodic solution of (1.1)-(1.3). Consider the linearized equation of (1.1) along $p(t)$:

$$\dot{z} = Df(t, p(t))z, \quad t \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad (3.1)$$

which is a $T$-periodic linear equation in the form of (2.1). $p(t)$ is called hyperbolic if none of its Floquet multipliers is on the unit-circle $S^1 \subset \mathbb{C}$.

Hereafter, for each $\tau \in \mathbb{R}$, we write $\phi(t, \tau, x)$ as the solution of (1.1)-(1.3) satisfying $\phi(\tau, \tau, x) = x$. Define the stable (resp. unstable) manifold $W^s(\tau(x))$ (resp. $W^u(\tau(x))$) of $p(\tau)$ as

$$W^s(\tau(x)) = \{ x \in \mathbb{R}^n | \lim_{n \to +\infty} \phi(nT + \tau, \tau, x) = p(\tau) \},$$

$$W^u(\tau(x)) = \{ x \in \mathbb{R}^n | \lim_{n \to +\infty} \phi(-nT + \tau, \tau, x) = p(\tau) \}. \quad (3.2)$$

Two smooth submanifolds $M$ and $N$ of $\mathbb{R}^n$ are said to intersect transversely (written as $M \cap N = \emptyset$) if either $M \cap N = \emptyset$ or at each point $x \in M \cap N$, the tangent spaces $T_xM, T_xN$ span $\mathbb{R}^n$.

Our main result in this section is the following

Theorem 3.1. Let $p^-(t), p^+(t)$ be two hyperbolic $T$-periodic solutions of (1.1)-(1.3), then for any $\tau \in \mathbb{R}$,

$$W^u(\tau(p^-)) \cap W^s(\tau(p^+)).$$

In order to prove Theorem 3.1, we first need the following crucial lemma, which generalizes the asymptotically autonomous cases (in [3, Lemma 9]) to the asymptotically time-periodic cases.
Lemma 3.2. Let $A(t)$ be the coefficient matrix of (2.1). Assume that there exist $A^{+}(t)$, $A^{-}(t)$ such that $\lim_{t \to \pm \infty} \|A(t) - A^{\pm}(t)\| = 0$, where $A^{\pm}(t)$ are $T$-periodic matrix-valued functions in the form of (2.1). Let $x(t)$ be a nontrivial solution of the system $\dot{x} = A(t)x$. Then one can find a Floquet multiplier $\alpha^{+}$ (resp. $\alpha^{-}$) of the equation $\dot{x} = A^{+}(t)x$ (resp. $\dot{x} = A^{-}(t)x$) such that
\[
\lim_{n \to \pm \infty} \frac{x(nT + \tau)}{\|x(nT + \tau)\|} = \frac{q^{\pm}(\tau)}{\|q^{\pm}(\tau)\|},
\]
uniformly for all $\tau \in [0, T]$, where $q^{+}(\tau)$ (resp. $q^{-}(\tau)$) is a $T$-periodic function with $q^{\pm}(0) \in E_{\alpha^{\pm}}$ satisfying $\|q^{\pm}(0)\| = 1$.

**Proof.** For any $\tau \in \mathbb{R}$, we define the $\tau$-shift of $A$ by $A_{\tau}(t) = A(t + \tau), t \in \mathbb{R}$. Let
\[
H(A) := d\{A_{\tau}| \tau \in \mathbb{R}, A_{\tau}(t) = A(t + \tau)\}
\]
be the *halfl of $A$, where the closure is taken in the compact open topology. Clearly, $A^{+}$, $A^{-} \in H(A)$. Moreover, it follows from the Ascoli-Arzela’s lemma that the shift $\tau$ introduces a compact flow, denoted by $\theta_{\tau}$, on $H(A)$.

Then system $\dot{x} = A(t)x$ generates a skew-product flow $\Pi'$ on $\mathbb{R}^{n} \times H(A)$, as
\[
\Pi'(x, B) = (\varphi(t, x, B), \theta_{\tau}B),
\]
for all $t \in \mathbb{R}$ and $(x, B) \in \mathbb{R}^{n} \times H(A)$, where $\varphi(t, x, B)$ is the solution of $\dot{x} = B(t)x$ for the initial value $x$. So, $x(t)$ and $\varphi(t, x, A)$ denote the same solution of $\dot{x}(t) = A(t)x$, and we will use them without any confusion in the following context. Define an equivalence relation on $\mathbb{R}^{n} \setminus \{0\}$ by declaring that $x_{1} \sim x_{2}$ if and only if $x_{1} = \alpha x_{2}$ for some $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. We denote by $[x]$ the equivalence class of $x$. Then the skew-product flow $\Pi'$ induces a natural way a projection compact flow $\tilde{\Pi}^{+} : \mathbb{R}^{n-1} \times H(A) \to \mathbb{R}^{n-1} \times H(A)$ by $\tilde{\Pi}^{+}([x], B) = ([\varphi(t, x, B)], B)$, where $\mathbb{R}^{n-1}$ is the real $(n-1)$-dimensional projective space (see, e.g. [9]). For the period $T > 0$, we consider the discrete-time skew-product flow associated with $\tilde{\Pi}^{+}$, which is defined as $\tilde{\Pi}^{\pm} = \tilde{\Pi}^{\pm T}([x], B)$ for any $(x, [z], B) \in \mathbb{Z} \times \mathbb{R}^{n-1} \times H(A)$.

We only consider the case that $n$ tends to positive infinity, the other case is similar. Let $\alpha^{+}_{m} > 0$, $m = 1, 2, \cdots, n$, be the $m$-th Floquet multiplier of the system
\[
\dot{x} = A^{+}(t)x,
\]
and $E_{\alpha^{+}_{m}}$ be the 1-dim eigenspace associated with $\alpha^{+}_{m}$. Let $q^{+}_{m}(t)$ be the $T$-periodic function with $q^{+}_{m}(0) \in E_{\alpha^{+}_{m}} \setminus \{0\}$ and $\|q^{+}_{m}(0)\| = 1$. Then, for any nontrivial solution $\varphi(t, x, A^{+})$ of (3.4), there exists a nonzero constant vector $(c_{1}, \cdots, c_{n})$ such that
\[
\varphi(t, x, A^{+}) = \sum_{m=1}^{n} c_{m}e^{\mu^{+}_{m}t}q^{+}_{m}(t),
\]
where $\mu^{+}_{m} = \log \alpha^{+}_{m}$. Clearly, the fibre $\mathbb{R}^{n-1} \times \{A^{+}\}$ is invariant with respect to $\tilde{\Pi}^{+}$. By virtue of (3.5), $(q^{+}_{m}(0), A^{+})$, $m = 1, 2, \cdots, n$, are $n$ fixed points of $\tilde{\Pi}^{+}$ in the sense that $\tilde{\Pi}^{+}([q^{+}_{m}(0)], A^{+}) = ([q^{+}_{m}(0)], A^{+})$ for all $z \in \mathbb{Z}$; and moreover, any nontrivial orbit under $\tilde{\Pi}^{+}$ in $\mathbb{R}^{n-1} \times \{A^{+}\}$ connects two of these fixed points. As a consequence, any compact invariant set $\Gamma$ of $\tilde{\Pi}^{+}$ in $\mathbb{R}^{n-1} \times \{A^{+}\}$ contains at least two distinct fixed points unless $\Gamma$ is a singleton.

Now, for any $[x_{0}] \in \mathbb{R}^{n-1}$, let $\omega_{\tilde{\Pi}^{+}}([x_{0}], A) = \{([z], A^{+}) : \tilde{\Pi}^{n_{k}}([x_{0}], A) \to ([z], A^{+}) \text{ for some } n_{k} \to +\infty\}$. Clearly, $\omega_{\tilde{\Pi}^{+}}([x_{0}], A)$ is an invariant compact set of $\tilde{\Pi}^{+}$ in $\mathbb{R}^{n-1} \times \{A^{+}\}$. Suppose that $\omega_{\tilde{\Pi}^{+}}([x_{0}], A)$ is not a singleton. Then, as we
mentioned above, \( \omega_{\Omega^l}(x_0, A) \) contains at least two fixed points, say \((q^+_m(0), A^+)\) and \((q^+_n(0), A^+)\) with \( m \neq j \). So, one can find a sequence \( n_i \rightarrow +\infty \) such that
\[
\Omega^{n_i}(x_0, A) \rightarrow (q^+_m(0), A^+) \quad \text{and} \quad \Omega^{n_i+1}(x_0, A) \rightarrow (q^+_n(0), A^+)
\]
which implies that \([\varphi(n_{2l}, x_0, A)] \rightarrow [q^+_m(0)]\) and \([\varphi(n_{2l+1}, x_0, A)] \rightarrow [q^+_n(0)]\) as \( l \rightarrow +\infty \). Due to the continuity of \( \sigma \), we deduce that \( \sigma(\varphi(n_{2l}, x_0, A)) = j-1 \) and \( \sigma(\varphi(n_{2l+1}, x_0, A)) = m-1 \) for \( l \) sufficiently large. This contradicts Lemma 2.1(iii). Thus, we have proved that \( \omega_{\Omega^l}(x_0, A) \) is a singleton, say \((q^+_m(0), A^+)\), on the fibre \( RP^{n-1} \times \{A^+\} \).

Again, due to Lemma 2.1(iii), it entails that the sign of the first coordinate of \( \varphi(nT, x_0, A) \) is preserving for all \( n \) sufficiently large. Together with \( \omega_{\Omega^l}(x_0, A) = (q^+_m(0), A^+) \), this implies that \( \varphi(nT, x_0, A) \rightarrow q^+(0) \), for some \( q^+(0) \in E_{\alpha^+} \setminus \{0\} \) with \( ||q^+(0)|| = 1 \), where \( \log_\alpha^+ = \mu_i^+ \) for some \( i \in \{1, \ldots, n\} \).

Finally, for any \( x \in \mathbb{R}^n \setminus \{0\} \), it is clear that
\[
\varphi(\tau, \varphi(nT, x, A), \theta_{nT}A) \rightarrow \varphi(\tau, q^+(0), A^+)
\]
as \( n \rightarrow +\infty \), uniformly for all \( \tau \in [0, T] \). So it yields that
\[
\frac{\varphi(nT + \tau, x, A)}{\|\varphi(nT + \tau, x, A)\|} = \frac{\varphi(\tau, \varphi(nT, x, A), \theta_{nT}A)}{\|\varphi(\tau, \varphi(nT, x, A), \theta_{nT}A)\|} \rightarrow \frac{\varphi(\tau, q^+(0), A^+)}{\|\varphi(\tau, q^+(0), A^+)\|} = \frac{q^+(\tau)}{\|q^+(\tau)\|}
\]
as \( n \rightarrow \infty \), which completes the proof.

Motivated by \cite{3}, for any given integer \(-1 \leq h \leq n - 1\), let \( K_h \) and \( K^h \) be the sets
\[
K_h = \{0\} \cup \{x \in \mathbb{R}^n | \sigma_M(x) \leq h\},
\]
\[
K^h = \{0\} \cup \{x \in \mathbb{R}^n | \sigma_M(x) > h\}.
\]
In particular, we set \( K_{-1} = \{0\} \) and \( K^{-1} = \mathbb{R}^n \). The sets \( K_h \) and \( K^h \) so defined are cones. Moreover, \( K_h \) and \( K^h \) are open sets, \( K_h \cap K^h = \{0\} \) and \( \text{cl}(K_h \cup K^h) = \mathbb{R}^n \).

**Lemma 3.3.** Let \( A(t) \) be the coefficient matrix of (2.1) and \( K^h \), \( K_h \) be the corresponding cones. Then:

(i) For integers \( 0 \leq m \leq l \leq n - 1 \), the Floquet space \( W_{m,l}(A) \subset K^{m-1} \cap K_l \).

(ii) If \( \Sigma_0 \subset K_h \) is a linear subspace and \( \Sigma_t \) is the image of \( \Sigma_0 \) at time \( t \) under (2.1), then \( \dim \Sigma_t = \dim \Sigma_0 \) and \( \Sigma_t \subset K_h \), for any \( t \geq 0 \).

(iii) If \( \Sigma_0 \subset K^h \) is a linear subspace and \( \Sigma_t \) is the image of \( \Sigma_0 \) under (2.1), then \( \dim \Sigma_t = \dim \Sigma_0 \) and \( \Sigma_t \subset K^h \), for any \( t \leq 0 \).

**Proof.** For (i), we only prove that \( W_{m,l}(A) \subset K^{m-1} \), since the other is similar. It is obvious that \( W_{0,l} \subset K^{-1} = \mathbb{R}^n \). For any \( m > 0 \), if \( x \in W_{m,l}(A) \cap \Lambda \), then it follows from the definition of \( W_{m,l}(A) \) that \( \sigma_M(x) = \sigma(x) > m - 1 \), so we only need to consider the case that a nonzero vector \( x \in W_{m,l}(A) \setminus \Lambda \). Choose a small \( t_0 > 0 \) so that \( x(t_0) \in \Lambda \). By the definition of \( W_{m,l}(A) \), one has \( \sigma(x(t_0)) > m - 1 \). Suppose now that \( \sigma_M(x) \leq m - 1 \), then it follows from the definition of \( \sigma_M \) that there exists a sequence \( y_n \in \Lambda \) with \( \sigma(y_n) \leq m - 1 \) such that \( y_n \rightarrow x \) as \( n \rightarrow +\infty \). Then the continuity of \( \sigma \) implies that \( \sigma(y_n(t_0)) = \sigma(x(t_0)) > m - 1 \), for \( n \) sufficiently large. On the other hand, clearly \( \sigma(y_n(t_0)) \leq \sigma(y_n) \leq m - 1 \), a contradiction. This
implies that $W_{m,t} \subset K^{m-1}$. Thus (i) has been proved. See [3, Lemma 8] for (ii) and (iii).

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Without loss of generality, we only prove $W^u(p^+(0)) \cap W^s(p^-(0))$, because the general “$p$” case can be deduced similarly. And also, for brevity, we just let the period $T = 1$.

Assume that $W^s(p^+(0)) \cap W^u(p^-(0)) \neq \emptyset$. Let $y(t) = p^+(t) - p^-(t)$, then $y(t)$ be the solution of type (2.1), where

$$a_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial x_j}(t, u_{i-1}(s, t), u_i(s, t), u_{i+1}(s, t))ds,$$

with $u_i(t, s) = sp_i(t) + (1-s)p_i'(t)$, $l = i-1, i, i+1$. Then there exists an $x_0 \in \mathbb{R}^n$ such that $\phi(n, 0, x_0) \rightarrow p^+(0)$ and $\phi(-n, 0, x_0) \rightarrow p^-(0)$ as $n \rightarrow \infty$. Since $y(t)$ is periodic in $t$, $y(t) \in \Lambda$ for all $t \in \mathbb{R}$. Hence, $\sigma(y(t)) \equiv \text{const}$ for any $t \in \mathbb{R}$. By Lemma 2.1(iii), it then follows that there exists $t_0 > 0$, such that

$$\sigma(\phi(t, 0, x_0) - p^-(t)) = \sigma(p^+(0) - p^-(0)) \quad \text{for all } t \geq t_0.$$  \hspace{1cm} (3.6)

For brevity, we hereafter write $\phi(t)$ as $\phi(t, 0, x_0)$.

Now, let $w_n^- = \frac{\phi(-n) - p^-(0)}{\|\phi(-n) - p^-(0)\|}$ and define the matrix $A(t) = (\bar{a}_{ij}(t))$, where $\bar{a}_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial x_j}(t, \bar{u}_{i-1}(s, t), \bar{u}_i(s, t), \bar{u}_{i+1}(s, t))ds$, with $\bar{u}_i(s, t) = s\phi_i(t) + (1-s)p_i'(t)$, $l = i-1, i, i+1$. Clearly, $A(t)$ is asymptotic to a matrix-valued periodic function $A^-(t) = (\frac{\partial f_i}{\partial x_j}(t, p^-(t)))$ as $t \rightarrow -\infty$. By virtue of Lemma 3.2, we have $w_n^- \rightarrow w^-$ as $n \rightarrow +\infty$. Here $w^- \in E_{\alpha^-_j} \setminus \{0\} \subset \Lambda$ with $E_{\alpha^-_j}$ being the eigenspace associated with the $j$-th Floquet multiplier $\alpha^-_j$ of the linear system $\dot{x} = A^-(t)x$. Note also that $\phi(-n) \rightarrow p^-(0)$, then $w^- \in T_{p^-(0)}W^u(p^-(0))$ and $\alpha^-_j > 1$. Replacing $A(t)$ by $A^-(t)$ in Lemma 2.2, we obtain that $\sigma(w^-) = j-1$. So $\sigma(w^-_n) = j-1$, for all $n$ sufficiently large. As a consequence, $\sigma(\phi(t) - p^-(t)) = j - 1$ for large negative $t$. Therefore, $\sigma(\phi(t) - p^-(t)) \leq j - 1$ for all $t \in \mathbb{R}$, and hence, by (3.6) one has $\sigma(\phi(t) - p^-(0)) \leq j - 1$.

Similarly, we write $w_n^+ = \frac{p^+(0) - \phi(n)}{\|p^+(0) - \phi(n)\|}$. Again, by Lemma 3.2 and the similar arguments above, one has $w_n^+ \rightarrow w^+$ as $n \rightarrow +\infty$, where $w^+ \in E_{\alpha^+_k} \setminus \{0\} \subset \Lambda$, with $E_{\alpha^+_k}$ being the eigenspace associated with the $k$-th Floquet multiplier $\alpha^+_k < 1$ of the linear system $\dot{x} = D_x f(t, p^+(0))x = A^+(t)x$, and moreover, $\sigma(w^+) = k - 1$. This implies that $\sigma(p^+(0) - p^-(0)) \geq k - 1$. Consequently, one has $k \leq j$. Together with the fact that $\alpha^+_k < 1 < \alpha^-_j$, we obtain that $\text{dim} W^u(p^-(0)) \geq j \geq k > \text{dim} W^u(p^+(0))$.

Let $m^\pm = \text{dim} W^u(p^\pm(0))$, then $m^+ \leq m^- - 1$. It then follows that the $n - m^-$ vectors $q^+_{m^-+1}, \ldots, q^+_n$ are in $T_{p^+(0)}W^s(p^+(0))$, where $q^+_j \in E_{\alpha^+_j} \setminus \{0\}$ with $\alpha^+_j$ being the eigenspace associated with the $j$-th Floquet multiplier of $\dot{x} = A^+(t)x$ for $j = m^- + 1, \ldots, n$. Let $\Sigma = \text{span}\{q^+_{m^-+1}, \ldots, q^+_n\} = \oplus_{j=m^-+1}^n E_{\alpha^+_j}$, then by Lemma 2.3, $\Sigma = W_{m^-+1, n}(A^+)$. Therefore, Lemma 3.3(i) implies that $\Sigma \subset K^{m^-}$.

Note that $K^{m^-} \setminus \{0\}$ is open and $W^s(p^+(0))$ is a smooth manifold. Then for any integer $z$, it follows from $\Sigma \subset K^{m^-} \cap T_{p^+(0)}W^s(p^+(0))$ that there exists some $z^0 > z$ sufficiently large such that $T_{\phi(z^0)}W^s(p^+(0))$ contains an $(n - m^-)$ dimensional linear space $\Sigma^0$, which is also contained in $K^{m^-}$. Let $\Sigma^z$ be the image of
Let \( A \) be Morse-Smale. Assume that condition (H), \( m \) be the non-negative real number that makes \( m \geq 0 \), and \( \chi \) be a solution of (1.1) with \( \phi(t,x) \). Indeed, by assumption (H), \( \phi(t,x) \) is decreasing on \([0, t_0 + \varepsilon]\) for some small \( \varepsilon > 0 \), and \( m(\phi(t,x^0)) = m(\phi(t,x^0)) < C \), which means that \( m(\phi(t,x^0)) \in A \) for all \( t \in [0, t_0 + \varepsilon] \). By a continuation argument, we conclude that \( \phi(t,x) \) remains in \( A \) for any \( t \geq 0 \).

For \( x^0 \notin A \), we claim that there exists a \( T_0 > 0 \) such that \( \phi(T_0,x^0) \in A \). Suppose not, then \( \phi(t,x^0) \notin A \) for any \( t \geq 0 \), which entails that \( m(\phi(t,x^0)) > C \) for any \( t > 0 \). By (H), we know that \( m(\phi(t,x^0)) \) is non-increasing with \( t \), so \( \lim_{t \to +\infty} m(\phi(t,x^0)) \) exists. Let \( s = \lim_{t \to +\infty} m(\phi(t,x^0)) \). Then \( m(\phi(t,x^0)) \geq s \geq C \), for all \( t \geq 0 \). Choose...
a subsequence \( t_k = n_k T \) such that \( \lim_{k \to +\infty} \phi(n_k T, x^0) = \hat{x} \) and \( m(\hat{x}) = s \) as \( k \to +\infty \).

Since \( m(\hat{x}) = s \geq C, \) by assumption (H), for \( \varepsilon > 0 \) small enough, \( m(\phi(\varepsilon, \hat{x})) \) must be strictly decreasing with respect to \( \varepsilon, \) that is to say \( m(\phi(\varepsilon, \hat{x})) < s. \) On the other hand, recall that \( \phi(\varepsilon, \hat{x}) = \lim_{k \to +\infty} \phi(k(T, x^0)) = \lim_{k \to +\infty} \phi(\varepsilon + n_k T, x^0), \) it then follows that \( m(\phi(\varepsilon + n_k T, x^0)) < s \) for \( k \) sufficiently large. This contradicts the fact that \( m(\phi(t, x^0)) \geq s \) for all \( t \geq 0. \) Thus, we have proved the claim. Then by the argument in the previous paragraph, we have obtained our conclusion.

**Proof of Theorem 4.2.** For convenience, we again let \( T = 1, \) and hence, the Poincaré map \( P(x_0) = \phi(1, x_0). \)

It follows from Lemma 4.3 that \( P \) is point dissipative. In order to prove (M1), it suffices to show that \( \mathcal{N}(P) \subset Fix(P). \) Here \( \mathcal{N}(P) \) is the set of the nonwandering points of \( P \) and \( Fix(P) \) the set of the fixed points of \( P. \)

Suppose that \( x_0 \in \mathcal{N}(P) \setminus Fix(P). \) Then, we consider the function

\[
\psi(t) = \phi(t + 1, x_0) - \phi(t, x_0).
\]

Clearly, \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \) is not identically zero because \( x_0 \notin Fix(P). \) Moreover, it satisfies the following equation

\[
\begin{align*}
\dot{\psi}_1 &= a_{11}(t)\psi_1 + a_{12}(t)\psi_2, \\
\dot{\psi}_i &= a_{i,i-1}(t)\psi_{i-1} + a_{i,i}(t)\psi_i + a_{i,i+1}(t)\psi_{i+1}, & 2 \leq i \leq n-1, \\
\dot{\psi}_n &= a_{n,n-1}(t)\psi_{n-1} + a_{n,n}(t)\psi_n,
\end{align*}
\]

where \( a_{i,j}(t) = \int_0^1 \frac{\partial f_i}{\partial x_j}(t, u_{i-1}(s, t), u_i(s, t), u_{i+1}(s, t))ds, \) with \( u_l = s\phi_l(t + 1, x_0) + (1 - s)\phi_l(t, x_0), l = i - 1, i, i + 1. \) By Lemma 2.1(iii), there exists an \( N \in \mathbb{N} \) such that \( \psi(t) \) belongs to \( \Lambda \) for all \( t \geq N \) and \( \sigma(\psi(t)) \) is constant for \( t \in [N, \infty). \) As a consequence, \( \psi(t) \neq 0 \) for any \( t \geq N. \) Without loss of generality, we assume that \( \psi_l(t) > 0 \) for \( t \geq N. \) Since \( x_0 \in \mathcal{N}(P), \) there exist two sequences \( x_l \to x_0 \) and \( k_l \to +\infty \) such that \( \phi(k_l, x_l) = P^{k_l}(x_l) \to x_0 \) as \( l \to +\infty. \) Due to the continuity of \( \phi(N, \cdot) \) with respect to \( x, \) one has

\[
\phi(N, x_l) \to \phi(N, x_0) \text{ and } \phi(k_l + N, x_l) \to \phi(N, x_0),
\]

as \( l \to \infty. \) Let \( \psi_l(t) = \phi(t + 1, x_l) - \phi(t, x_l) \) and recall that \( \psi(N) \in \Lambda. \) Then there is a positive integer \( K, \) such that

\[
\sigma(\psi_l(N)) = \sigma(\psi_l(k_l + N)) = \sigma(\psi(N)) \quad \text{for all } l \geq K.
\]

It then follows from Lemma 2.1 that \( \psi_l(t) \in \Lambda \) for any \( l \geq K \) and \( t \in [N, k_l + N]. \)

So, \( \psi_1(t) \neq 0 \) for any \( t \in [N, k_l + N] \) and \( l \geq K. \) Noticing that \( \psi_1(N) \to \psi_1(N) \) as \( l \to \infty, \) one has \( \psi_1(t) > 0 \) for any \( N \leq t \leq k_l + N, \) with \( l \geq K. \) As a consequence, \( \phi_1(k_l + N, x_l) > \phi_1(N + 1, x_l), \)

for any \( l \geq K. \) Letting \( l \to \infty, \) we conclude that \( \phi_1(N, x_0) \geq \phi_1(N + 1, x_0), \)

which contradicts the assumption that \( \phi_1(N) > 0. \) Thus, we have proved that \( \mathcal{N}(P) \subset Fix(P). \) Moreover, since \( P \) is satisfied for the condition (H) and \( Fix(P) \) is a closed set in \( \mathbb{R}^n, \) \( Fix(P) \subset A \) is compact. Due to the fact that all the fixed points \( P \) are hyperbolic, \( Fix(P) \) is a discrete set. So, \( \mathcal{N}(P) = Fix(P) \) is a finite set. Thus, (M1) is satisfied.

As for (M2), Theorem 3.1 already guarantees transversality of the stable and unstable manifolds between hyperbolic \( T \)-periodic solutions. Thus, we have completed the proof.
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