Scattering above energy norm of solutions of a loglog energy-supercritical Schrödinger equation with radial data

Tristan Roy

Institute For Advanced Study
School of Mathematics
Einstein Drive
Princeton, NJ 08540, USA

Abstract

We prove scattering of \( \tilde{H}^k := \dot{H}^k(\mathbb{R}^n) \cap \dot{H}^1(\mathbb{R}^n) \)- solutions of the loglog energy-supercritical Schrödinger equation

\[
i\partial_t u + \Delta u = |u|^{\frac{4}{n-2}} u \log^c (\log (10 + |u|^2)),
\]

where \( c < c_n, n = \{3, 4\} \), with radial data \( u(0) = u_0 \in \tilde{H}^k := \dot{H}^k(\mathbb{R}^n) \cap \dot{H}^1(\mathbb{R}^n), k > \frac{n}{2}. \)

This is achieved, roughly speaking, by extending Bourgain’s argument [1] (see also Grillakis [5]) and Tao’s argument [10] in high dimensions.

Key words: Global existence, Scattering, Concentration, Morawetz-type estimate

1 Introduction

We shall study the solutions of the following Schrödinger equation in dimension \( n, n \in \{3, 4\} \):

\[
i\partial_t u + \Delta u = |u|^{\frac{4}{n-2}} u g(|u|)
\]

with \( g(|u|) := \log^c (\log (10 + |u|^2)), 0 < c < c_n \) and

\[
c_n := \begin{cases} 
\frac{(n-2)^2(6-n)}{2n(4n^2-15n+22)(16n^2-70n+20)}, & n = 3 \\
\frac{(n+2)(6-n)}{(n^2+12n+4)(44n^2-62n+12)}, & n = 4 
\end{cases}
\]

Email address: triroy@math.ias.edu (Tristan Roy).

Preprint submitted to Elsevier

15 January 2010
This equation has many connections with the following power-type Schrödinger equation, $p > 1$

$$i\partial_t v + \Delta v = |v|^{p-1}v \quad (3)$$

(3) has a natural scaling: if $v$ is a solution of (3) with data $v(0) := v_0$ and if $\lambda \in \mathbb{R}$ is a parameter then $v_\lambda(t, x) := \frac{1}{\lambda^{\frac{n}{p-1}}} u \left( \frac{t}{\lambda}, \frac{x}{\lambda} \right)$ is also a solution of (3) but with data $v_\lambda(0, x) := \frac{1}{\lambda^{\frac{n}{p-1}}} u_0 \left( \frac{x}{\lambda} \right)$. If $s_p := \frac{n}{2} - \frac{2}{p-1}$ then the $\dot{H}^s$ norm of the initial data is invariant under the scaling: this is why (3) is said to be $\dot{H}^s$-critical. If $p = 1 + \frac{4}{n-2}$ then (3) is $\dot{H}^1$ (or energy) critical. The energy-critical Schrödinger equation

$$i\partial_t u + \Delta u = |u|^{4-\frac{n}{n-2}}u \quad (4)$$

has received a great deal of attention. Cazenave and Weissler [2] proved the local well-posedness of (4): given any $u(0)$ such that $\|u(0)\|_{\dot{H}^1} < \infty$ there exists, for some $t_0$ close to zero, a unique $u \in C_t([0, t_0], \dot{H}^1) \cap L_{t, x}^{2(n+2), 2(n+2)}([0, t_0])$ satisfying (4) in the sense of distributions

$$u(t) = e^{it\Delta}u(0) - i \int_0^t e^{i(t-t')\Delta} \left[ |u(t')|^{4-\frac{n}{n-2}} u(t') \right] dt' \quad (5)$$

Bourgain [1] proved global existence and scattering of radial solutions in the class $C_t \dot{H}^1 \cap L_{t, x}^{2(n+2), 2(n+2)}$ in dimension $n = 3, 4$. He also proved this fact that for smoother solutions. Another proof was given by Grillakis [5] in dimension $n = 3$. The radial assumption for $n = 3$ was removed by Colliander-Keel-Staffilani-Takaoka-Tao [4]. This result was extended to $n = 4$ by Rickman-Visan [7] and to $n \geq 5$ by Visan [11]. If $p > 1 + \frac{4}{n-2}$ then $s_p > 1$ and we are in the energy supercritical regime. The global existence of $\dot{H}^k$-solutions in this regime is an open problem. Since for all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that $\|u|^{\frac{4}{n-2}}u\| \lesssim \|u|^{\frac{4}{n-2}}ug(|u|)\| \leq c_\epsilon \max (1, ||u|^{\frac{4}{n-2}}+ u|)$ then the nonlinearity of (1) is said to be barely supercritical.

In this paper we are interested in establishing global well-posedness and scattering of $\dot{H}^k := H^k(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) -$ solutions of (1) for $n \in \{3, 4\}$. First we prove a local-wellposed result. The local well-posedness theory for (1) and for $\dot{H}^k$-solutions can be formulated as follows

**Proposition 1** “Local well-posedness ” Let $M$ be such that $\|u_0\|_{\dot{H}^k} \leq M$. Let $n \in \{3, 4\}$. Then there exists $\delta := \delta(M) > 0$ small such that if $T_1 > 0$
(T_l=\text{time of local existence}) satisfies
\[ \|e^{it\Delta}u_0\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2(n+2)}{n-2}}([0,T_l] \times \mathbb{R}^n)} \leq \delta \]  
(6)

then there exists a unique
\[ u \in C([0,T_l], \tilde{H}^k) \cap L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2(n+2)}{n-2}}([0,T_l]) \cap D^{-1}L_t^{\frac{2(n+2)}{n}}L_x^{\frac{2(n+2)}{n}}([0,T_l]) \cap D^{-k}L_t^{\frac{2(n+2)}{n}}L_x^{\frac{2(n+2)}{n}}([0,T_l]) \]  
(7)
such that
\[ u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-t')\Delta} \left( |u(t')|^{\frac{4}{n-2}}u(t')g(|u(t')|) \right) dt' \]  
(8)
is satisfied in the sense of distributions.

This allows to define the notion of maximal time interval of existence \( I_{\text{max}} \), that is the union of all the intervals I containing 0 such that (8) holds in the class \( C(I, \tilde{H}^k) \cap L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2(n+2)}{n-2}}(I) \cap D^{-1}L_t^{\frac{2(n+2)}{n}}L_x^{\frac{2(n+2)}{n}}(I) \cap D^{-k}L_t^{\frac{2(n+2)}{n}}L_x^{\frac{2(n+2)}{n}}(I) \).

Next we prove a criterion for global well-posedness:

**Proposition 2** “Global well-posedness: criterion” If \( |I_{\text{max}}| < \infty \) then
\[ \|u\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2(n+2)}{n-2}}(I_{\text{max}})} = \infty \]  
(9)

These propositions are proved in Section 2. With this in mind, global well-posedness follows from an \textit{a priori} bound of the form
\[ \|u\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2(n+2)}{n-2}}([-T,T])} \leq f(T, \|u_0\|_{\tilde{H}^k}) \]  
(10)

for arbitrary large time \( T > 0 \). In fact we shall prove that the bound does not depend on time \( T \); this is the preliminary step to prove scattering.

The main result of this paper is:

**Theorem 3** The solution of (1) with data \( u(0) := u_0 \in \tilde{H}^k, n = \{3, 4\}, k > \frac{n}{2} \) and \( 0 < c < c_n \) exists for all time \( T \). Moreover there exists a scattering state \( u_{0,+} \in \tilde{H}^k \) such that
\[ \lim_{t \to \infty} \|u(t) - e^{it\Delta}u_{0,+}\|_{\tilde{H}^k} = 0 \]  
(11)

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and there exists $C$ depending only on $\|u_0\|_{\tilde{H}^k}$ such that
\begin{equation}
\|u\|_{L_t^{2(n+2)} L_x^{2(n+2)}(\mathbb{R} \times \mathbb{R}^n)} \leq C(\|u_0\|_{\tilde{H}^k}) \tag{12}
\end{equation}

**Remark:** This implies global regularity by the Sobolev embedding $\|u\|_{L_t^\infty L_x^\infty} \lesssim \|u\|_{L_t^\infty \tilde{H}^k}$ for $k > \frac{n}{2}$.

We recall some estimates. The pointwise dispersive estimate is $\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{1}{|t|^n} \|f\|_{L^1(\mathbb{R}^n)}$. Interpolating with $\|e^{it\Delta} f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$ we have the well-known generalized pointwise dispersive estimate:
\begin{equation}
\|e^{it\Delta} f\|_{L^p(\mathbb{R}^n)} \lesssim \frac{1}{|t|^n (\frac{2}{p} - \frac{1}{2})} \|f\|_{L^{p'}(\mathbb{R}^n)} \tag{13}
\end{equation}

Here $2 \leq p \leq \infty$ and $p'$ is the conjugate of $p$. We recall some useful Sobolev inequalities:
\begin{equation}
\|u\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J)} \lesssim \|Du\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J)} \tag{14}
\end{equation}

and
\begin{equation}
\|u\|_{L_t^\infty L_x^\infty(J)} \lesssim \|u\|_{L_t^\infty \tilde{H}^k(J)} \tag{15}
\end{equation}

If $u$ is an $\tilde{H}^k$ solution of $i\partial_t u + \Delta u = F(u)$, $u(t = 0) := u_0$ and if $t_0 \in \mathbb{R}$ then, combining (14) with the Strichartz estimates (see for example [6]), we get
\begin{equation}
\begin{align*}
\|u\|_{L_t^\infty H_j(J)} + \|D^j u\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J)} + \|D^j u\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J)} + \|u_0\|_{\tilde{H}^j(J)} \lesssim \|D^j F\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J)} + \|u_0\|_{\tilde{H}^j(J)} \tag{16}
\end{align*}
\end{equation}

if $j \in \{1, k\}$ and we write
\begin{equation}
u(t) = u_{t,t_0}(t) + u_{nl,t_0}(t) \tag{17}\end{equation}

with $u_{t,t_0}$ denoting the linear part starting from $t_0$, i.e
\begin{equation}
u_{t,t_0} := e^{i(t-t_0)\Delta} u(t_0) \tag{18}\end{equation}
and \( u_{nl,t_0} \) denoting the nonlinear part from \( t_0 \), i.e

\[
\mathfrak{u}_{nl,t_0} := -i \int_{t_0}^t e^{i(t-s)\Delta} F(u(s)) \, ds
\]  

(19)

Moreover \( u \) has a finite energy

\[
E := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(t, x)|^2 + \int_{\mathbb{R}^n} F(u, \bar{u})(t, x) \, dx
\]

(20)

with

\[
F(z, \bar{z}) := \int_0^{|z|} t^{\frac{n+2}{2}} g(t) \, dt
\]

(21)

Indeed,

\[
|\int_{\mathbb{R}^n} F(u, \bar{u})(t, x) \, dx| \lesssim \|u(t)\|_{L_{L^\infty}^{n+2}} g(\|u(t)\|_{L^\infty})
\]

\[
\lesssim \|u(t)\|_{\dot{H}^k} g(\|u(t)\|_{\dot{H}^k})
\]

(22)

: this follows from a simple integration by part

\[
F(z, \bar{z}) \sim |z|^\frac{2n}{n-2} g(|z|)
\]

(23)

combined with (15). A simple computation shows that the energy is conserved, or, in other words, that \( E(u(t)) = E(u_0) \). Let \( \chi \) be a smooth, radial function supported on \( |x| \leq 2 \) such that \( \chi(x) = 1 \) if \( |x| \leq 1 \). If \( x_0 \in \mathbb{R}^n, \, R > 0 \) and \( u \) is an \( \dot{H}^k \) solution of (1) then we define the mass within the ball \( B(x_0, R) \)

\[
\text{Mass}(B(x_0, R), u(t)) := \left( \int_{B(x_0, R)} |u(t, x)|^2 \, dx \right)^{\frac{1}{2}}
\]

(24)

Recall (see [5]) that

\[
\text{Mass}(B(x_0, R), u(t)) \lesssim R \sup_{t' \in [0,t]} \|\nabla u(t')\|_{L^2}
\]

(25)

and that its derivative satisfies

\[
\partial_t \text{Mass}(u(t), B(x_0, R)) \lesssim \frac{\sup_{t' \in [0,t]} \|\nabla u(t')\|_{L^2}}{R}
\]

(26)

Now we set up some notation. We write \( a \ll b \) if \( a \leq \frac{1}{100} b \), \( a \gg b \) is \( a \geq 100b \) and \( a \sim b \) if \( \frac{1}{100} b \leq a \leq 100 b \), \( a \ll_E b \) if \( a \leq \frac{1}{100 \max(1,E)^{100n}} b \)
If $u \in \mathcal{X}(J)$ of (1) on an interval $J$, induction, that in fact this norm and other norms (such as $\|u\|_{L^\infty_t\tilde{H}^k(J)}$) can be bounded only by a constant only depending on the norm of the initial data. This already shows (by Proposition 2) global well-posedness of the $\tilde{H}^k$-solutions of (1). In fact we show that that these solutions of (1). In fact we show that that these
Proposition 4 “Bound of $L_t^{2(n+2)/n} L_x^{n/2-2}$ norm” Let $u$ be an $\tilde{H}^k$ solution of (1) on an interval $J$. There exist three constants $C_1 \gg_E 1$, $C_2 \gg_E 1$, $a_n > 0$ and $b_n > 0$ such that if $\|u\|_{L^\infty_t\tilde{H}^k} \leq M$ for some $M \gg 1$, then
\begin{equation}
\|u\|_{L_t^{2(n+2)/n} L_x^{n/2-2}} \leq (C_1 g^{a_n}(M))^{C_2 g^{b_n}(M)}
\end{equation}
with $b_n$ such that
\begin{equation}
b_n := \begin{cases} 
\frac{2n(4n^2-15n+22)(46n^2-70n+20)}{(n-2)^2(6-n)}, & n = 3 \\
\frac{(n^2+12n+4)(4n^2-62n+12)}{(n+2)(6-n)}, & n = 4
\end{cases}
\end{equation}
By combining this bound with the Strichartz estimates, we can prove, by induction, that in fact this norm and other norms (such as $\|u\|_{L^\infty_t\tilde{H}^k(J)}$, $\|Du\|_{L_t^{2(n+2)/n} L_x^{n/2-2}}$, etc.) can be bounded only by a constant only depending on the norm of the initial data. This already shows (by Proposition 2) global well-posedness of the $\tilde{H}^k$-solutions of (1). In fact we show that that these bounds imply a linear asymptotic behaviour of the solutions, or, in other words, scattering. The rest of the paper is devoted to prove Proposition 4. First we prove a weighted Morawetz-type estimate: it shows, roughly speaking, that the $L_t^{2n/2(n-2)} L_x^{n/2}$ norm of the solution cannot concentrate around the origin on long time intervals. Then we modify arguments from Bourgain [1], Grillakis [5] and mostly Tao [10]. We divide $J$ into subintervals $(J_l)_{1 \leq l \leq L}$ such that the $L_t^{2n/2(n-2)} L_x^{n/2}$ norm of $u$ is small but substantial. We prove that, on most
of these intervals, the mass on at least one ball concentrates. By using the radial assumption, we prove that in fact the mass on a ball centered at the origin concentrates. This implies, by using the Morawetz-type estimate that there exists a significant number of intervals (in comparison with $L$) that concentrate around a point $\bar{t}$ and such that the mass concentrates around the origin. But, by Hölder, this implies that $L$ is finite: if not it would violate the fact that the $L^\infty_t L^{2n/(n-2)}_x$ norm of the solution is bounded by some power of the energy. The process involves several tuning parameters. The fact that these parameters depend on the energy is not important; however, it is crucial to understand how they depend on $g(M)$ since this will play a prominent role in the choice of $c_n$ for which we have global well-posedness and scattering of $\tilde{H}^k$-solutions of (1) (with $g(|u|) := \log^c (\log (10 + |u|^2))$ and $c < c_n)$; see the proof of Theorem 3, Section 3.

2 Local well-posedness and criterion for global well-posedness

In this section we prove Proposition 1 and Proposition 2.

2.1 Proof of Proposition 1

This is done by a modification of standard arguments to establish a local well-posedness theory for (4).

We define

$$X := C([0, T_i], \tilde{H}^k) \cap D^{-1} L^{2(n+2)/n}_t L^{2(n+2)/n}_x ([0, T_i]) \cap D^{-k} L^{2(n+2)/n}_t L^{2(n+2)/n}_x ([0, T_i]) \cap L^{2n/(n-2)}_t L^{2n/(n-2)}_x ([0, T_i])$$

(30)

and, for some $C > 0$ to be chosen later,

$$X_1 := B \left( C([0, T_i], \tilde{H}^k) \cap D^{-1} L^{2(n+2)/n}_t L^{2(n+2)/n}_x ([0, T_i]), 2CM \right)$$

(31)

and

$$X_2 := B \left( L^{2(n+2)/n}_t L^{2(n+2)/n}_x ([0, T_i]), 2\delta \right)$$

(32)

$X_1 \cap X_2$ is a closed space of the Banach space $X$: therefore it is also a Banach
\[ \Psi := X_1 \cap X_2 \rightarrow X_1 \cap X_2 \]
\[ u \rightarrow e^{it\Delta}u(0) - i \int_0^t e^{i(t-t')\Delta} \left( |u|^{\frac{4}{n-2}}(t')u(t') \right) dt' \] (33)

- \( \Psi \) maps \( X_1 \cap X_2 \) to \( X_1 \cap X_2 \)

By the fractional Leibnitz rule (see Appendix with \( F(x) := g(x), G(x, \bar{x}) := |x|^{\frac{4}{n-2}}x \) and \( \beta := \frac{4}{n-2} \)) and (15) we have

\[
\left\| D^j \left( |u|^{\frac{4}{n-2}}ug(|u|) \right) \right\|_{L_t^{2(\frac{n+2}{n-2})} L_x^{2(\frac{n+2}{n-2})}([0,T])} \lesssim \left\| D^j u \right\|_{L_t^{2(\frac{n+2}{n})} L_x^{2(\frac{n+2}{n})}([0,T])} \left( \frac{4}{n-2} \right)^{\frac{1}{2}} \left\| u \right\|_{L_t^{\infty} H^k([0,T])} \]
\[
\left\| u \right\|_{L_t^{\infty} H^k([0,T])} + \left\| Du \right\|_{L_t^{2(\frac{n+2}{n})} L_x^{2(\frac{n+2}{n})}([0,T])} + \left\| D^k u \right\|_{L_t^{2(\frac{n+2}{n})} L_x^{2(\frac{n+2}{n})}([0,T])} \lesssim M + \delta^{\frac{4}{n-2}} M g(M) \] (34)

Moreover

\[
\left\| u \right\|_{L_t^{2(\frac{n+2}{n-2})} L_x^{2(\frac{n+2}{n-2})}([0,T])} - \left\| e^{it\Delta}u_0 \right\|_{L_t^{2(\frac{n+2}{n-2})} L_x^{2(\frac{n+2}{n-2})}([0,T])} \lesssim \left\| D \left( |u|^{\frac{4}{n-2}}ug(|u|) \right) \right\|_{L_t^{2(\frac{n+2}{n+4})} L_x^{2(\frac{n+2}{n+4})}([0,T])} \]
\[
\lesssim \delta^{\frac{4}{n-2}} M g(M) \] (36)

so that

\[
\left\| u \right\|_{L_t^{2(\frac{n+2}{n-2})} L_x^{2(\frac{n+2}{n-2})}([0,T])} - \delta \lesssim \delta^{\frac{4}{n-2}} M g(M) \] (37)

Therefore if let \( C \) be equal to the maximum of the constants determined by (35) and (37), then we see that \( \Psi(X_1 \cap X_2) \subset X_1 \cap X_2 \), provided that \( \delta = \delta(M) > 0 \) is small enough.
• Ψ is a contraction. Indeed, by the fundamental theorem of calculus

\[ \|\Psi(u) - \Psi(v)\|_X \lesssim \]

\[
\left( \|u\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n}}} \|v\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n}}} + \|D(u - v)\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}}} + \|D^k(u - v)\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}}} \right) \]

\[
\left( \|u\|_{L_t^{\frac{6-n}{2(n+2)}} L_x^{\frac{2(n+2)}{n}}} + \|v\|_{L_t^{\frac{6-n}{2(n+2)}} L_x^{\frac{2(n+2)}{n}}} + \|D(u - v)\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}}} + \|D^k(u - v)\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}}} \right) \]

\[
\left( g(\|u\|_{L_t^{\frac{6-n}{2(n+2)}}(0,T]})) + g(\|v\|_{L_t^{\frac{6-n}{2(n+2)}}(0,T]}) \right)
\]

\[ \lesssim \delta M \|u - v\|_X \]

and if \( \delta = \delta(M) > 0 \) is small enough then \( \Psi \) is a contraction.

2.2 Proof of Proposition 2

Again, this is done by a modification of standard arguments used to prove a criterion of global well-posedness of (3) (See [9] for similar arguments). Assume that \( \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}} (I_{\max})} < \infty \). Then

• First step: \( Q(I_{\max}, u) < \infty \). Indeed, let \( \epsilon << 1 \). Let \( C \) be the constant determined by \( \lesssim \) in (16). We divide \( I_{\max} \) into subintervals \( (I_j = [t_j, t_{j+1}])_{1 \leq j \leq J} \) such that

\[
\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}} (I_j)} = \frac{\epsilon}{g^{\frac{n-2}{n}}(M)} \]

if \( 1 \leq j < J \) and

\[
\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n}} (I_j)} \leq \frac{\epsilon}{g^{\frac{n-2}{n}}(M)} \]
Notice that such a partition always exists since, for \( J \) large enough,
\[
\sum_{j=1}^{J} \frac{1}{g(2^{(n+2)}(2C)^{j} \| u_{\bar{k}} \|_{\bar{H}^{k}})} \geq \sum_{j=1}^{J} \frac{1}{(2C)^{j} \| u_{\bar{k}} \|_{\bar{H}^{k}}} = \sum_{j=1}^{J} \frac{1}{(2C)^{j} + \log(\| u_{\bar{k}} \|_{\bar{H}^{k}})} \geq \| u \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} (I_{\max})
\]

By the fractional Leibnitz rule (see Appendix) and (16) we have
\[
Q(I_{1}, u) \leq C \| u_{0} \|_{\bar{H}^{k}} + C \| D^{\frac{1}{2}} u g(|u|) \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} (I_{1})
\]
\[
+ C \| D^{k} (|u|^{\frac{n-2}{2}} u g(|u|)) \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} (I_{1})
\]
\[
\leq C \| u_{0} \|_{\bar{H}^{k}} + C \left( \| Du \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} + \| D^{k} u \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} \right) (I_{1})
\]
\[
\leq C \| u_{0} \|_{\bar{H}^{k}} + 2CQ(I_{1}, u) \| u \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} (Q(I_{1}, u))
\]
and by a continuity argument, \( Q(I_{1}) \leq 2C \| u_{0} \|_{\bar{H}^{k}} \). By iteration \( Q(I_{j}) \leq (2C)^{j} \| u_{0} \|_{\bar{H}^{k}} \). Therefore \( Q(I_{\max}) < \infty \).

* Second step. We write \( I_{\max} = [a_{\max}, b_{\max}] \). We have
\[
\| e^{it\Delta} u_{0} \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} \geq \| u \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} (I_{\max})
\]
\[
+ \| Du \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} \| u \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} (I_{\max})
\]
\[
g(\| u \|_{L_{t}^{\infty} \bar{H}^{k}(I_{\max})}) < \infty
\]

and, by the dominated convergence theorem, there exists \( \tilde{t} \) such that
\[
\| e^{it\Delta} u_{0} \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} \leq \frac{\delta}{2} \text{ (with } \delta \text{ defined in Proposition 1). Hence}
\]
\[
\text{contradiction.}
\]

3 Proof of Theorem 3

The proof is made of two steps:

* finite bound of \( \| u \|_{L_{t}^{\infty} \bar{H}^{k}(\mathbb{R})} \), \( \| u \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} \), \( \| Du \|_{L_{t}^{2(n+2)} L_{x}^{2(n+2)}} \) and
\[ D^k u \|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}} \] By time reversal symmetry ¹ and by monotone convergence it is enough to find, for all \( T \geq 0 \), a finite bound of all these norms restricted to \([0, T]\) and the bound should not depend on \( T \). We define

\[ \mathcal{F} := \{ T \in [0, \infty) : \sup_{t \in [0, T]} Q([0, t], u) \leq M_0 \} \] (44)

We claim that \( \mathcal{F} = [0, \infty) \) for \( M_0 \), a large constant (to be chosen later) depending only on \( \|u_0\|_{\dot{H}^k} \). Indeed

- \( 0 \in \mathcal{F} \).
- \( \mathcal{F} \) is closed by continuity
- \( \mathcal{F} \) is open. Indeed let \( T \in \mathcal{F} \). Then, by continuity exists \( \delta > 0 \) such that for \( T' \in [0, T + \delta] \) we have \( Q([0, T']) \leq 2M_0 \). In view of (28), this implies, in particular, that

\[ \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0, T])} \leq C_1 (g^{\eta n}(2M_0))^{C_2 g^{\eta n}+(2M_0)} \] (45)

Let \( \epsilon << 1 \). We get from (16) and the Sobolev inequality \( \|u\|_{L_t^\infty L_x^\infty([0,a])} \lesssim \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0,a])} \)

\[ Q([0, a], u) \lesssim \|u_0\|_{\dot{H}^k} + \left( \|Du\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0,a])} + \|D^k u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0,a])} \right) \]

\[ \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0,a])} \gtrsim \|u_0\|_{\dot{H}^k} + Q([0, a], u) \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0,a])} \]

Let \( C \) be the constant determined by \( \lesssim \) in (46). We may assume without loss of generality that \( C >> \max \left( \frac{100}{\|u_0\|_{\dot{H}^k}}, \frac{1}{\|u_0\|_{\dot{H}^k}} \right) \). Let \( \epsilon << 1 \). Notice that if \( J \) satisfies \( \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}([0,a])} = \frac{1}{g^{\frac{\epsilon}{4}} (2C\|u_0\|_{\dot{H}^k})} \), then a simple continuity argument shows that

\[ Q([0, a], u) \leq 2C\|u_0\|_{\dot{H}^k} \] (47)

We divide \([0, T']\) into subintervals \((J_i)_{1 \leq i \leq I}\) such that \( \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}(J_i)} \leq \frac{1}{g^{\frac{\epsilon}{4}} (2C\|u_0\|_{\dot{H}^k})} \)

\[ \frac{\epsilon}{g^{\frac{\epsilon}{4}} (2C\|u_0\|_{\dot{H}^k})}, \quad 1 \leq i < I \quad \text{and} \quad \|u\|_{L_t^{\frac{n}{n-2}} L_x^{\frac{2(n+2)}{n}}(J_i)} \leq \frac{1}{g^{\frac{\epsilon}{4}} (2C\|u_0\|_{\dot{H}^k})}. \]

¹ i.e if \( t \to u(t, x) \) is a solution of (1) then \( t \to \bar{u}(-t, x) \) is also a solution of (1)
Notice that such a partition exists by (45) and the following inequality

\[(C_1 g^{a_n}(2M_0))^{C_2 g^{b_n}(2M_0)} \geq \sum_{i=1}^{I} \frac{1}{g \frac{1}{2} \left( (2C)^i \| u_0 \|_{\tilde{H}^k} \right)} \geq \sum_{i=1}^{I} \frac{1}{\log \left( \frac{(a+2)C}{2i} \right) \left( \log \left( 10 + (2C)^2 \| u_0 \|_{\tilde{H}^k} \right) \right)} \geq \sum_{i=1}^{I} \frac{1}{\log \left( \frac{(a+2)C}{2i} \right) \left( 2i \log (2C) + 2 \log (\| u_0 \|_{\tilde{H}^k}) \right)} \geq \| u_0 \|_{\tilde{H}^k} \sum_{i=1}^{I} \frac{1}{t^2} \geq \| u_0 \|_{\tilde{H}^k} I \frac{1}{2} \]

Moreover, by iterating the procedure in (46) and (47) we get

\[Q([0, T'], u) \leq (2C)^I \| u_0 \|_{\tilde{H}^k} \]

Therefore by (48) there exists \( C' = C' (\| u_0 \|_{\tilde{H}^k}) \)

\[
\log I \lesssim \log \left( C' + C_2 \log^{(b_n + \epsilon)} \log \left( 10 + 4M_0^2 \right) \right) \\
\log \left( C_1 \log^{a_n \epsilon} \log \left( 10 + 4M_0^2 \right) \right) \leq \log \left( \frac{M_0}{\| u_0 \|_{\tilde{H}^k}} \right) \quad \text{log} (C') + C_2 \log^{(b_n + \epsilon)} \log \left( 10 + 4M_0^2 \right) \log \left( C_1 \log^{a_n \epsilon} \log \left( 10 + 4M_0^2 \right) \right) \\
\log \left( \frac{M_0}{\| u_0 \|_{\tilde{H}^k}} \right) \quad \text{log} \left( C' \right) + C_2 \log^{(b_n + \epsilon)} \log \left( 10 +4M_0^2 \right) \log \left( C_1 \log^{a_n \epsilon} \log \left( 10 +4M_0^2 \right) \right) \left( \frac{M_0}{\log (2C)} \right) \rightarrow M_0 \rightarrow \infty 0 \]

\[\log (C') + C_2 \log^{(b_n + \epsilon)} \log \left( 10 +4M_0^2 \right) \log \left( C_1 \log^{a_n \epsilon} \log \left( 10 +4M_0^2 \right) \right) \left( \frac{M_0}{\log (2C)} \right) \rightarrow M_0 \rightarrow \infty 0 \]

- Scattering: it is enough to prove that \( e^{-it\Delta} u(t) \) has a limit as \( t \rightarrow \infty \) in \( \tilde{H}^k \).

If \( t_1 < t_2 \) then we have

\[
\| e^{-it_1 \Delta} u(t_1) - e^{-it_2 \Delta} u(t_2) \|_{\tilde{H}^k} \lesssim \| u(t_1) - u(t_2) \|_{\tilde{H}^k} \\
\lesssim \| D^k \left( \| u \|_{L_t^{4/3} L_x^{4/3}} \right) \|_{L_t^{4/3} L_x^{4/3}} \left( \left( 1 + \frac{1}{n-2} \right) \log \left( \frac{M_0}{\| u_0 \|_{\tilde{H}^k}} \right) \right) \left( \frac{M_0}{\log (2C)} \right) \rightarrow M_0 \rightarrow \infty 0 \]
and we conclude from the previous step that given $\epsilon > 0$ there exists $A(\epsilon)$ such that $t_2 \geq t_1 \geq A(\epsilon)$ such that $\|e^{-it_1\triangle}u(t_1) - e^{-it_2\triangle}u(t_2)\|_{\tilde{H}^k} \leq \epsilon$. The Cauchy criterion is satisfied. Hence scattering.

4 Proof of Proposition 4

The proof relies upon a Morawetz type estimate that we prove in the next subsection:

**Lemma 5** “Morawetz type estimate” Let $u$ be a smooth solution of (1) on an interval $I$. Let $A > 1$. Then

$$
\int_I \int_{|x| \leq A|I|^\frac{1}{2}} \frac{\tilde{F}(u, \bar{u})(t, x)}{|x|} dx \, dt \lesssim EA|I|^\frac{1}{4}
$$

with

$$
\tilde{F}(u, \bar{u})(t, x) := \int_0^{|u(t, x)|} s^{\frac{n+2}{n-2}} \left( \frac{4}{n-2} g(s) + sg'(s) \right) ds
$$

We prove now Proposition 4.

**Step 1**

We divide the interval $J = [t_1, t_2]$ into subintervals $(J_l := [t_l, t_{l+1}])_{1 \leq l \leq L}$ such that

$$
\|u\|_{L_t^{\frac{2(n+2)}{2(n-2)}} L_x^{\frac{2(n+2)}{2(n-2)}} (J_l)} = \eta_1
$$

$$
\|u\|_{L_t^{\frac{2(n+2)}{2(n-2)}} L_x^{\frac{2(n+2)}{2(n-2)}} (J_{L})} \leq \eta_1
$$

with $c_1 \ll_\|E\|$ and $\eta_1 = \frac{c_1(E)}{g^{\frac{2(n+2)}{6-n}} (M)}$. It is enough to find an upper bound of $L$ that would depend on the energy $E$ and $M$.

Notice that the value of this parameter, along with the values of the other parameters $\eta_2$, $\eta_3$ and $\eta$ are not chosen randomly: they are the largest ones (modulo the energy) such that all the constraints appearing throughout the proof are satisfied. Indeed, if we consider for example $\eta_1$, we basically want to minimize $L \eta_1$. If we go throughout the proof without assigning any value to $\eta_1$
we realize that basically $L \lesssim \left( \frac{1}{\eta_1} \right)^{\frac{1}{n}}$ and therefore $L \eta_1$ gets smaller as $\eta_1$ grows.

**Step 2**

We first prove that some norms on these intervals $J_t$ are bounded by a constant that depends on the energy.

**Result 1**

We have

$$\|Du\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J_t)} \lesssim E \quad (58)$$

**Proof:**

$$\|Du\|_{L_t^{2(n+2)} L_x^{2(n+2)}(J_t)} \lesssim \|Du(t)\|_{L^2} + \|D(|u|^{\frac{4}{n-2}} u g(|u|))\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_t)} \lesssim E^{\frac{1}{2}} + \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_t)} \|Du\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_t)} \quad (59)$$

Therefore, by a continuity argument, we conclude that $\|Du\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_t)} \lesssim E \quad 1.$

**Result 2**

Let $\tilde{J} \subset J$ be such that

$$\frac{M}{\eta_1} \leq \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(\tilde{J})} \leq \eta_1 \quad (60)$$

Then

$$\|u_{t,t_j}\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(\tilde{J})} \geq \eta_1 \quad (61)$$

for $j \in \{1, 2\}.$
Proof By Result 1 we have

\[
\| u - u_{t,j} \|_{L_\frac{n}{n-2} L_x \frac{n}{n-2}} \lesssim \| D(|u|^{\frac{4}{n-2}} u g(|u|)) \|_{L_t \frac{n}{n-2} L_x \frac{n}{n-2}} (\tilde{J}) \\
\lesssim \| Du \|_{L_t \frac{n}{n-2} L_x \frac{n}{n-2}} \| u \|_{L_t \frac{n}{n-2} L_x \frac{n}{n-2}} (\tilde{J}) g(M) \\
\lesssim E \| u \|_{L_t \frac{n}{n-2} L_x \frac{n}{n-2}} (J) g(M) \\
< \eta_1 \tag{62}
\]

Therefore (61) holds.

**Step 3**

We recall the notion of exceptional intervals and the notion of unexceptional intervals (such a notion appeared first in Tao [10]). Let

\[
\eta_2 := \begin{cases} 
\frac{c_2}{\gamma (6-n)(n-2)} (M) & n = 3 \\
\frac{c_2}{\gamma (3n^2+30n^2+20n+8) (M)} & n = 4
\end{cases}
\tag{63}
\]

with \( c_2 << c_1 \). An interval \( J_{l_0} = [t_{l_0}, t_{l_0+1}] \) of the partition \((J_l)_{1 \leq l \leq L}\) is exceptional if

\[
\| u_{t,j_1} \|_{L_t \frac{n}{n-2} L_x \frac{n}{n-2}} (J_{l_0}) + \| u_{t,j_2} \|_{L_t \frac{n}{n-2} L_x \frac{n}{n-2}} (J_{l_0}) \geq \eta_2 \tag{64}
\]

Notice that, in view of the Strichartz estimates (16), it is easy to find an upper bound of the cardinal of the exceptional intervals:

\[
\text{card } \{ J_l : J_l \text{ exceptional} \} \lesssim E \eta_2^{-1} \tag{65}
\]

**Step 4**

Now we prove that on each unexceptional subintervals \( J_l \) there is a ball for which we have a mass concentration.
Result 4 ("Mass Concentration")
There exists an \( x_i \in \mathbb{R}^n \), two constants \( c \ll E \) and \( C >> E \) 1 such that for each unexceptional interval \( J_i \) and for \( t \in J_i \)

- if \( n = 3 \)

\[
\text{Mass} \left( u(t, B(x_i, Cg^{\frac{2n^2 + 15n + 22}{6-n}}(M)|J_i|^{\frac{3}{2}}) \right) \geq cg^{\frac{(4n^2 + 15n + 22)}{2(6-n)}}(M)|J_i|^{\frac{3}{2}} (66)
\]

- if \( n = 4 \)

\[
\text{Mass} \left( u(t, B(x_i, Cg^{\frac{2n^2 + 12n + 4}{2n+2}(M)|J_i|^{\frac{1}{2}}) \right) \geq cg^{\frac{(2n^2 + 12n + 4)}{2(n+2)(6-n)}}(M)|J_i|^{\frac{1}{2}} (67)
\]

Proof
By time translation invariance \(^2\) we may assume that \( t_i = 0 \). By using the pigeonhole principle and the reflection symmetry (if necessary) \(^3\) we may assume that

\[
\int_{\frac{|J_i|}{2}}^{[J_i]} \int_{\mathbb{R}^n} |u(t, x)|^{\frac{2(n+2)}{n-2}} dx dt \geq \frac{\eta}{4} (68)
\]

By the pigeonhole principle there exists \( t_* \) such that \([t_3 - \eta_3)|J_i|, t_* |J_i| \subset [0, \frac{|J_i|}{2}]\) (with \( \eta_3 << 1 \)) and

\[
\int_{[t_* - \eta_3)|J_i|}^{[t_*]|J_i|} \int_{\mathbb{R}^n} |u(t, x)|^{\frac{2(n+2)}{n-2}} dx dt \leq \eta_1 \eta_3 \quad (69)
\]

\[
\int_{\mathbb{R}^n} |u, t_i ((t_* - \eta_3)|J_i|, x)|^{\frac{2(n+2)}{n-2}} dx \leq \frac{\eta_2}{|J_i|} \quad (70)
\]

Applying Result 2 to (68) we have

\[
\int_{[t_*]|J_i|}^{[t_*]|J_i|} \int_{\mathbb{R}^n} |e^{i(t_* - t_*|J_i|) \Delta} u(t_* |J_i|, x)|^{\frac{2(n+2)}{n-2}} dx dt \gtrless_E \eta_1 \quad (71)
\]

By Duhamel formula we have

\[
u(t_*|J_i|) = e^{i(t_*|J_i| - t_1) \Delta} u(t_1) - i \int_{[t_* - \eta_3)]|J_i|}^{[t_*]|J_i|} e^{i(t_*|J_i| - s) \Delta}(|u(s)|^{\frac{n}{n-2}} u(s) g(|u(s)|)) ds \quad (72)
\]

\[^2\] i.e if \( u \) is a solution of (1) and \( t_0 \in \mathbb{R} \) then \( t, x \to u(t - t_0, x) \) is also a solution of (1)

\[^3\] if \( u \) is a solution of (1) then \( t, x \to \bar{u}(-t, x) \) is also a solution of (1)
and, composing this equality with \( e^{i(t-t_\ast |J_i|)\Delta} \) we get

\[
e^{i(t-t_\ast |J_i|)\Delta} u(t_\ast |J_i|) = u_{t_\ast t_1}(t) - i \int_{t_1}^{(t_\ast - \eta_3)|J_i|} e^{i(t-s)\Delta} \left( |u(s)|^{\frac{4}{n-2}} u g(|u(s)|) \right) ds \]

\[-i \int_{(t_\ast - \eta_3)|J_i|}^{(t_\ast)|J_i|} e^{i(t-s)\Delta} \left( |u(s)|^{\frac{4}{n-2}} u g(|u(s)|) \right) ds \]

\[= u_{t_\ast t_1}(t) + v_1(t) + v_2(t) \tag{73}\]

We get from the Strichartz estimates (16) and the Sobolev inequality (14)

\[
\|v_2\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast |J_i|, |J_i|])} \lesssim \|D(|u|^{\frac{4}{n-2}} u g(|u|))\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast - \eta_3]|J_i|, |J_i|)} \]

\[
\lesssim \|Du\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}} ([t_\ast - \eta_3]|J_i|, |J_i|)} \]

\[
\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast - \eta_3]|J_i|, |J_i|)} \]

\[
2 \left( \eta_1 \eta_3 \right)^{\frac{1}{n+2}} g(M) \]

<< \eta_1^{\frac{n}{n+2}} \tag{74}\]

Notice also that \( \eta_2 << \eta_1 \) and that \( J_i \) is non-exceptional. Therefore

\[\|u_{t_\ast t_1}\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast |J_i|, |J_i|])} << \eta_1 \] and combining this inequality with (74) and (71) we conclude that the \( L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} \) norm of \( v_1 \) on \([t_\ast |J_i|, |J_i|]\) is bounded from below:

\[\|v_1\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast |J_i|, |J_i|])} \gtrsim \eta_1 \tag{75}\]

By (16), (73) and (74) we also have an upper bound of the \( L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} \) norm of \( v_1 \) on \([t_\ast |J_i|, |J_i|]\)

\[\|v_1\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast |J_i|, |J_i|])} \lesssim E 1 \tag{76}\]

Now we use a lemma that is proved in Subsection 4.1.

**Lemma 6** “*Regularity of \( v_1 \)”* We have

\[\|v_{1,h} - v_1\|_{L_t^{\infty} L_x^{\frac{2(n+2)}{n-2}} ([t_\ast |J_i|, |J_i|])} \lesssim E |\hat{h}|^\gamma |J_i|^{\beta_\gamma} \tag{77}\]

with
\[ \alpha = \frac{1}{3} \text{ if } n = 3; \quad \alpha = 1 \text{ if } n = 4 \]
\[ \beta = \frac{n+2}{n-2} \text{ if } n = 3; \quad \gamma = \frac{n+2}{n-2} (M) \text{ if } n = 4; \]

\[
\| v_{1,h} - v_1 \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim |J_1|^{\frac{n-2}{2(n+2)}} \| v_{1,h} - v_1 \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim_E |J_1|^\alpha |J_1|^{\beta + \frac{n-2}{2(n+2)} \gamma} (78)
\]

Therefore if \( h \) satisfies \( |h| = c_3 |J_1|^{\frac{\beta + \frac{n-2}{2(n+2)}}{\alpha}} \gamma \frac{1}{\alpha} \eta_1^{\frac{n-2}{2(n+2)}} \) then

\[
\| v_{1,h} \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \gtrsim \eta_1^{\frac{n-2}{2(n+2)}} (79)
\]

Now notice that by the Duhamel formula \( v_1(t) = u_{t, (t_s - \eta_3)} |J_1|(t) - u_{t, t^-} \) and therefore, by the Strichartz estimates (16) and the conservation of energy,

\[
\| v_1 \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim_E 1. \text{ From that we get } \| v_{1,h} \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim_E |J_1|^{\frac{n-2}{2(n+2)}} \text{ and, by interpolation,}
\]

\[
\| v_{1,h} \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim \| v_{1,h} \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \| v_{1,h} \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim |J_1|^{\frac{n-2}{2(n+2)}} (80)
\]

and, in view of (79)

\[
\| v_{1,h} \|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \gtrsim |J_1|^{-\frac{n-2}{4}} \eta_1^{\frac{n+2}{4}} (81)
\]

Writing \( \text{Mass}(v(t), B(x, r)) = r^{\frac{n+2}{4}} (\int_{|y| \leq 1} |v(t, x + ry)|^2 \ dy)^{\frac{1}{2}} \) we deduce from Cauchy Schwarz and (81) that there exists \( t_l \in [t_s |J_1|, |J_l|] \) and \( x_l \in \mathbb{R}^n \) such that

\[
\text{Mass} (v_1(t_l), B(x_l, |h|)) \gtrsim |J_1|^{-\frac{n-2}{4}} \eta_1^{\frac{n+2}{4}} |h|^{\frac{n+2}{4}} (82)
\]

Therefore, by (26) we see that if \( R = C_4(E) \eta_1^{\frac{2}{4}} |J_1|^{\frac{2}{4}} |h|^{\frac{n+2}{4}} \) then

\[
\text{Mass} (v_1((t_s - \eta_3) |J_1|), B(x_l, R)) \gtrsim |J_1|^{-\frac{n-2}{4}} \eta_1^{\frac{n+2}{4}} |h|^{\frac{n+2}{4}} (83)
\]
Notice that \( u((t^* - \eta_3)|J_i|) = u_{t_1}(t^* - \eta_3)|J_i| - iv_1((t^* - \eta_3)|J_i|) \). By Hölder inequality and by \((70)\)

\[
\text{Mass}(u_{t_1}, ((t^* - \eta_3)|J_i|), B(x_1, R)) \leq R^{\frac{2n}{n+2}} \frac{\eta_i^{n-2}}{|J_i|^{\frac{2(n+2)}{n}}}
\]

\[
<< |J_i|^{-\frac{2n}{n+2}} \eta_i^{n-2} |h|^{\frac{2}{n}}
\]

Therefore \( \text{Mass}(u((t^* - \eta_3)|J_i|), B(x_1, R)) \sim \text{Mass}(v_1((t^* - \eta_3)|J_i|), B(x_1, R)) \).

Applying again \((26)\) we get

\[
\text{Mass}(u(t), B(x_1, R)) \gtrsim |J_i|^{-\frac{n-2}{4}} \eta_i^{n-2} |h|^{\frac{n}{2}}
\]

for \( t \in J_i \). Putting everything together we get \((66)\) and \((67)\).

Next we use the radial symmetry to prove that, in fact, there is a mass concentration around the origin.

**Step 5**

**Result 4** (“Mass concentration around the origin”)

There exists a constant \( \tilde{C} >> E \) 1 such that on each exceptional interval \( J_i \) we have

- if \( n = 3 \)
  \[
  \text{Mass}\left(u(t), B(0, \tilde{C} g \frac{(4n^2-15n+2)}{(n-2)}(M)|J_i|^{\frac{1}{2}})\right) \geq cg \frac{(4n^2-15n+2)}{6-n} (M)|J_i|^{\frac{1}{2}} \quad (86)
  \]

- if \( n = 4 \)
  \[
  \text{Mass}\left(u(t), B(0, \tilde{C} g \frac{(n^2+12n+4)}{2(n+2)}(M)|J_i|^{\frac{1}{2}})\right) \geq cg \frac{(2-n)(n^2+12n+4)}{2(n+2)(6-n)} (M)|J_i|^{\frac{1}{2}} \quad (87)
  \]

**Proof**

We deal with the case \( n = 4 \). The case \( n = 3 \) is treated similarly and the proof is left to the reader.

Let \( A := \tilde{C} g \frac{(n^2+12n+4)}{2(n+2)}(M) \) for some \( \tilde{C} >> E \) \( C \) (Recall that \( C \) is defined in \((67)\)). There are (a priori) two options:
• $|x_t| \geq \frac{A}{2} |J_l|^{\frac{1}{2}}$. Then there are at least \( \frac{A}{100 C g} \frac{(n-2)(n^2+12n+4)}{2(n+2)(6-n)} (M) |J_l|^{\frac{1}{2}} \) rotations of the ball \( B(x_t, C g^{-\frac{(n-2)(n^2+12n+4)}{2(n+2)(6-n)}} (M) |J_l|^{\frac{1}{2}}) \) that are disjoint. Now, since the solution is radial, the mass on each of these balls \( B_j \) is equal to that of the ball \( B(x_t, C g^{-\frac{(n-2)(n^2+12n+4)}{2(n+2)(6-n)}} (M) |J_l|^{\frac{1}{2}}) \). But then by Hölder we have

\[
\|u(t)\|_{L^2(B_j)}^{\frac{2n}{n-2}} \leq \|u(t)\|_{L^{\frac{n}{n-2}}(B_j)}^{\frac{2n}{n-2}} \left( C g^{-\frac{(n-2)(n^2+12n+4)}{2(n+2)(6-n)}} (M) |J_l|^{\frac{1}{2}} \right)^{\frac{2n}{n-2}} \tag{88}
\]

and summing over \( j \) we see from the equality \( \|u(t)\|_{L^2(B_j)}^{\frac{2n}{n-2}} \lesssim E \) that

\[
\frac{A}{100 C g} \frac{(n-2)(n^2+12n+4)}{2(n+2)(6-n)} (M) \left( C g^{-\frac{(2-n)(n^2+12n+4)}{2(n+2)(6-n)}} (M) |J_l|^{\frac{1}{2}} \right)^{\frac{2n}{n-2}} \leq E \left( C g^{-\frac{(n-2)(n^2+12n+4)}{2(n+2)(6-n)}} (M) |J_l|^{\frac{1}{2}} \right)^{\frac{2n}{n-2}} \tag{89}
\]

must be true. But with the value of \( A \) chosen above we see that this inequality cannot be satisfied if \( \tilde{C} \) is large enough. Therefore this scenario is impossible.

• $|x_t| \leq \frac{A}{2} |J_l|^{\frac{1}{2}}$. Then by (67) and the triangle inequality, we see that (87) holds.

**Remark** In order to avoid too much notation we will still write in the sequel \( C \) for \( \tilde{C} \) in (87).

### Step 6

Combining the inequality (87) to the Morawetz type inequality found in Lemma 5 we can prove that at least one of the intervals \( J_l \) is large. More precisely

**Result 5** “One of the intervals \( J_l \) is large”

There exists a constant \( c \ll E \) (that we still denote by \( c \) to avoid too much notation) and \( \tilde{l} \in [1, \ldots, L] \) such that

- if \( n = 3 \)
  \[
  |J_{\tilde{l}}| \geq c g \frac{4(4n^2 - 15n + 22)(11n^2 - 16n + 4)}{(n-2)^2(6-n)} (M) |J| \tag{90}
  \]

- if \( n = 4 \)
  \[
  |J_{\tilde{l}}| \geq c g \frac{2(n^2 + 12n + 4)(11n^2 - 16n + 4)}{(n+2)(6-n)} (M) |J| \tag{91}
  \]

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Proof

Again we shall treat the case $n = 4$. The case $n = 3$ is left to the reader.

There are two options:

- $J_l$ is unexceptional. Let $R := C g \frac{(n^2 + 12n + 4)(5n - 2)}{2(n+2)(6-n)} (M) |J_l|^\frac{1}{2}$. By Hölder inequality (in space), by integration in time we have

\[ \int_{J_l} \int_{B(0,R)} \frac{|u(t,x)|^{2n}}{|x|} \, dx \, dt \geq |J_l| \text{Mass} \frac{2n}{n-2} (u(t), B(0, R)) R^{\frac{2-3n}{n-2}} \quad (92) \]

After summation over $l$ we see, by (87) and (54) that

\[ \sum_{l=1}^{L} |J_l| \left( g \frac{(2-n)(n^2 + 12n + 4)}{2(n+2)(6-n)} (M)|J_l|^\frac{1}{2} \right)^{\frac{2n}{n-2}} \left( C g \frac{(5n-2)(n^2 + 12n + 4)}{2(n+2)(6-n)} (M)|J_l|^\frac{1}{2} \right)^{\frac{2-3n}{n-2}} \quad (93) \]

and after rearranging, we see that

\[ \sum_{l=1}^{L} |J_l|^\frac{1}{2} g \frac{(5n-2)(n^2 + 12n + 4)}{2(n+2)(6-n)} (M) \lesssim CE|J|^\frac{1}{2} \quad (94) \]

and therefore, writing $\sum_{l=1}^{L} |J_l|^\frac{1}{2} \geq \frac{|J|}{\sup_{1 \leq l \leq L} |J_l|^\frac{1}{2}}$, we conclude that there exists a constant $\ll_E 1$ (still denoted by $c$) and $\tilde{l} \in [1, \ldots, L]$ such that

\[ |J_{\tilde{l}}| \geq c g \frac{2(n^2 + 12n + 4)(11n^2 - 16n + 4)}{(n+2)(6-n)} (M)|J| \quad (95) \]

Therefore (91) holds.

- $J_l$ is exceptional. In this case by (65) and

\[ \sum_{l=1}^{L} |J_l|^\frac{1}{2} \lesssim E \eta_1^{-1} \sup_{1 \leq l \leq L} |J_l|^\frac{1}{2} \lesssim E \eta_1^{-1} |J|^\frac{1}{2} \quad (96) \]

and therefore (see the end of the proof of the previous case) we see, after plugging the value of $\eta_2$ that there exists a constant $\ll_E 1$ such that (91) holds.

Step 7

We use a crucial algorithm due to Bourgain [1] to prove that there are many of those intervals that concentrate.
Result 6 ("Concentration of intervals") Let

\[
\eta := \begin{cases} 
  cg \frac{4(4n^2+15n+22)(11n^2-16n+4)}{(n-2)^2(6-n)} (M), & n = 3 \\
  cg \frac{2(2n^2+12n+4)(11n^2-16n+4)}{(n+2)(6-n)} (M), & n = 4 
\end{cases}
\] (97)

Assume that \(L > 1\). Then there exist a time \(\bar{t} > 0\) and intervals \(J_{l1}, \ldots, J_{lK}\) such that

\[
|J_{l1}| \geq 2|J_{l2}| \ldots \geq 2^{k-1}|J_{l_k}| \ldots \geq 2^{K-1}|J_{l_K}|
\] (98)

such that

\[
dist(\bar{t}, J_{lk}) \leq \eta^{-1}
\] (99)

and

\[
K \geq -\frac{\log(L)}{2 \log \left(\frac{\eta}{8}\right)}
\] (100)

Proof

There are several steps

(1) By Result 5 there exists an interval \(J_{l1}\) such that \(|J_{l1}| \geq \eta|J|\). We have \(dist(t, J_{l1}) \leq |J| \leq \eta^{-1}|J_{l1}|, t \in J\).

(2) Remove all the intervals \(J_l\) such that \(|J_l| \geq \frac{|J_{l1}|}{2}\). By the property of \(J_{l1}\), there are at most \(2\eta^{-1}\) intervals satisfying this property and consequently there are at most \(4\eta^{-1}\) remaining connected components resulting from this removal.

(3) If \(L \leq 100\eta^{-1}\) then we let \(K = 1\) and we can check that (100) is satisfied. If not: one of these connected components (denoted by \(K_1\)) contains at least \(\frac{\eta}{8}L\) intervals. Let \(L_1\) be the number of intervals making \(K_1\).

(4) Apply (1) again: there exists an interval \(J_{l2}\) such that \(|J_{l2}| \geq \eta|K_1|\) and \(dist(t, J_{l2}) \leq |K_1| \leq \eta^{-1}|J_{l1}|\). Apply (2) again: remove all the intervals \(J_l\) such that \(|J_l| \geq \frac{|J_{l2}|}{2}\). By the property of \(J_{l2}\), there are at most \(2\eta^{-1}\) intervals to be removed and there are at most \(4\eta^{-1}\) remaining connected components. Apply (3) again: if \(L_1 \leq 100\eta^{-1}\) then we let \(K = 2\) and we can check that (100) is satisfied, since \(K_1\) contains at least \(\frac{\eta}{8}L\) intervals; if \(L_1 \geq 100\eta^{-1}\) then one of the connected components (denoted by \(K_2\)) contains at least \(\frac{\eta}{8}L_1\) intervals. Let \(L_2\) be the number of intervals making \(K_2\). Then \(L_2 \geq \left(\frac{\eta}{8}\right)^2 L\).
(5) We can iterate this procedure $K$ times as long as $L_K \geq 1$. It is not difficult to see that there exists a $K$ satisfying (100) and $L_K \geq 1$, since $L_K \geq \left(\frac{\eta}{8}\right)^K L$.

**Step 8**

We prove that $L < \infty$, by using Step 7 and the conservation the energy. More precisely

**Result 7** “finite bound of $L$”

There exist two constants $C_1 >> E$ and $C_2 >> E$ 1 such that

- if $n = 3$

\[
L \leq \left( C_1 g \frac{4(4n^2 - 15n + 22)(11n^2 - 16n + 4)}{(n-2)^2(6-n)} (M) \right)^{C_2 g \frac{2n(4n^2 - 15n + 22)(46n^2 - 70n + 20)}{(n-2)^2(6-n)} (M)}
\]

- if $n = 4$

\[
L \leq \left( C_1 g \frac{4(n^2 + 12n + 4)(11n^2 - 16n + 4)}{(n+2)(6-n)} (M) \right)^{C_2 g \frac{(n^2 + 12n + 4)(44n^2 - 62n + 12)}{(n+2)(6-n)} (M)}
\]

**Proof**

Again we shall prove this result for $n = 4$. The case $n = 3$ is left to the reader.

Let $R := Cg \frac{(n^2 + 12n + 4)(44n^2 - 62n + 12)}{(n+2)(6-n)} (M) |J_l|^\frac{1}{2}$. By Result 3 we have

\[
\text{Mass} \left( u(t), B(x_{l_k}, R) \right) \geq cg \frac{(2-n)(n^2 + 12n + 4)}{2(n+2)(6-n)} (M) |J_l|^\frac{1}{2}
\]

for all $t \in J_{l_k}$. Even if it means redefining $C$ \footnote{i.e making it larger than its original value modulo a multiplication by some power of max (1, E)} then we see, by (26) and (99) that (103) holds of $t = \bar{t}$ with $c$ substituted for $\frac{\xi}{2}$. On the other hand we see that by (25) that \footnote{Notation: $\sum_{k'=k+N}^{K} a_{k'} = 0$, if $k' > K$}

\[
\sum_{k'=k+N}^{K} \int_{B(x_{l_k'}, R)} |u(\bar{t}, x)|^2 dx \leq \left( \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+1}} \right) ER^2
\]

\[
\leq \frac{1}{2^{n+1}} ER^2
\]

(104)
Now we let \( N = C' \log (g(M)) \) with \( C' \gg E - \log c \) so that
\[
\frac{ER^2}{2^n} \leq \frac{1}{2} c^2 g \frac{(2-n)(n^2+12n+4)}{(n+2)(6-n)} (M)|J_l|.
\]
By (103) we have
\[
\int_{B(x_{l_k}, R)} |u(\bar{t}, x)|^2 \, dx \leq \frac{1}{2} \int_{B(x_{l_k}, R)} |u(\bar{t}, x)|^2 \, dx
\]
(105)
Therefore
\[
\int_{B(x_{l_k}, R)/\cup K_{k'=k+N} B(x_{l_{k'}}, R)} |u(\bar{t}, x)|^2 \, dx \geq \frac{1}{2} \int_{B(x_{l_k}, R)} |u(\bar{t}, x)|^2 \, dx
\]
(106)
and by Hölder inequality, there exists a constant \(<E 1\) (that we still denote by \( c \)) such that
\[
\int_{B(x_{l_k}, R)/\cup K_{k'=k+N} B(x_{l_{k'}}, R)} |u(\bar{t}, x)|^{\frac{2n}{n-2}} \, dx \geq c g \frac{-(n^2+12n+4)(44n^2-62n+12)}{(n+2)(6-n)} (M)
\]
(107)
and after summation over \( k \), we get
\[
\frac{K}{N} c g \frac{-(n^2+12n+4)(44n^2-62n+12)}{(n+2)(6-n)} (M) \leq E
\]
(108)
since \( \sum_{k=1}^{K} \chi_{B(x_{l_k}, R)/\cup K_{k'=k+N} B(x_{l_{k'}}, R)} \leq N \) and \( \|u(t)\|_{L^{\frac{2n}{n-2}}} \leq E \). Rearranging we see that there exists a constant \( \gg E 1 \) (that we still denote by \( C \)) such that there are two constants \( C_1 \gg E 1 \) and \( C_2 \gg E 1 \) such that
\[
L \leq \left( C_1 g \frac{4(n^2+12n+4)(11n^2-16n+4)}{(n+2)(6-n)} (M) \right)^{C_2 g \frac{(n^2+12n+4)(44n^2-62n+12)}{(n+2)(6-n)} (M)}
\]
(109)
We see that (102) holds.

**Step 9**

This is the final step. Recall that there are \( L \) intervals \( J_i \) and that on each of these intervals except maybe the last one we have \( \|u\|_{L^{\frac{2(n+2)}{n-2}}} \leq \eta_1 \). Therefore, there are two constants \( \gg E 1 \) (that we denote by \( C_1 \) and \( C_2 \)) such that (102) holds.
4.1 Proof of Lemma 6

In this subsection we prove Lemma 6. There are two cases

- $n = 3$
  By the fundamental theorem of calculus (and the inequality $\|Dv_1\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \lesssim E_2^{1/2}$) we have
  \[
  \|u_h - u\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \leq E_2^{1/2}|h| \tag{110}
  \]
  Moreover, by Sobolev (and the inequality $\|u\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \lesssim E_2^{1/2}$) we have
  \[
  \|u_h - u\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \leq E_2^{1/2} \tag{111}
  \]
  Therefore, by interpolation of (110) and (111), we get
  \[
  \|u_h - u\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \leq E_2^{1/2}|h|^{1/2} \tag{112}
  \]
  Now, by the fundamental theorem of calculus, the inequality $|x|g'(|x|) \lesssim g(|x|)$, (23) and (20) we have
  \[
  \|\{(s)^{1/2} - (u(s))^{1/2} u(s) g(|u(s)|) - (u_h(s))^n - u_h(s) g(|u_h(s)|)\}_{L^1} \lesssim \|u_h(s) - u(s)\|_{L^3} \|u(s) g^{n/2} (u(s))\|_{L^2} \|g^{1/2} (u(s))\|_{L^\infty} \tag{113}
  \]
  and, by the dispersive inequality (13) we conclude that
  \[
  \|v_{1,h} - v_1\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \lesssim E_3^{1/2} |J|^{1/2} g^{1/2} \lesssim (M)|h|^{1/2} \tag{114}
  \]
  Interpolating this inequality with
  \[
  \|v_{1,h} - v_1\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} = \|u_{t,(t_*-\eta_3)}|J|,h - u_{t,t_1},h - (u_{t,(t_*-\eta_3)}|J|, - u_{t,t_1})\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \tag{115}
  \]
  we get (77).
  - $n = 4$ By the fundamental theorem of calculus we have
  \[
  \|v_{1,h} - v_1\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} \lesssim \|Dv_1\|_{L_t^\infty L_x^{n/2}([t_*|J||J|])} |h| \tag{116}
  \]
But, by interpolation

\[ \|Dv_1\|_{L_t^\infty L_x^{\frac{n}{n-2}}([t_*,|J_i|,|J_i|])} \lesssim \|Dv_1\|_{L_t^\infty L_x^{\frac{n+2}{n-2}}([t_*,|J_i|,|J_i|])} \lesssim E \|Dv_1\|_{L_t^\infty L_x^{\frac{2n}{2n-4}}([t_*,|J_i|,|J_i|])} \]  

(117)

So it suffices to estimate \( \|Dv_1\|_{L_t^\infty L_x^{\frac{2n}{2n-4}}([t_*|J_i|,|J_i|])} \). By (20), (23) and Result 1 we have

\[ \|D(|u|^{\frac{4}{n}} u g(|u|))\|_{L_t^\infty L_x^{\frac{2n}{2n+4}}([t_1, (t^* - \eta_3)|J_i|])} \lesssim \|Du\|_{L_t^\infty L_x^{\frac{2n}{2n+4}}([t_1, (t^* - \eta_3)|J_i|])} \|u g(|u|)\|_{L_t^\infty L_x^{\frac{2n}{2n+4}}([t_1, (t^* - \eta_3)|J_i|])} \]

(118)

\[ g^{\frac{n-2}{n}} (\|u\|_{L_t^\infty \mathcal{H}^{k}(|t_1, (t^* - \eta_3)|J_i|)}) \lesssim E \ g^{\frac{n-2}{n}} (M) \]

and by combining (118) with the dispersive inequality (13) we have

\[ \|Dv_1\|_{L_t^\infty L_x^{\frac{2n}{2n-4}}([t_*|J_i|,|J_i|])} \lesssim \left\| \int_{t_1}^{(t^* - \eta_3)|J_i|} \|De^{i(t-s)\Delta}(|u(s)|^{\frac{4}{n-2}} u(s)g(|u(s)|))\|_{L_x^\infty} \frac{2n}{2n-4} ds \right\|_{L_t^\infty([t_*|J_i|,|J_i|])} \]

(119)

\[ \lesssim g^{\frac{n-2}{n}} (M) \eta_3^{-1} |J_i|^{-1} \]

We conclude from (117) and (119) that (77) holds.

\[ \bullet \ n = 5 \]

### 4.2 Proof of Lemma 5

By (1) we have

\[ \partial_t \Im(\partial_k u \bar{u}) = \Re \left[ |u|^{\frac{4}{n-2}} \bar{u} g(|u|) \partial_k u - \partial_k (|u|^{\frac{4}{n-2}} \bar{u} g(|u|)) \right] + \Re \left( \Delta(\partial_k u \bar{u}) - \bar{u} \Delta \partial_k u \right) \]

(120)

Moreover

\[ \frac{1}{2} \partial_k \Delta(|u|^2) = 2 \partial_j \Re(\partial_k u \partial_j \bar{u}) - \Re(\partial_k u \Delta \bar{u}) + \Re(u \Delta \partial_k u) \]

(121)

Therefore, adding (120) and (121) leads to

\[ \partial_t \Im(\partial_k u \bar{u}) = -2 \partial_j \Re(\partial_k u \partial_j \bar{u}) + \frac{1}{2} \partial_k \Delta(|u|^2) + \Re \left[ |u|^{\frac{4}{n-2}} \bar{u} g(|u|) \partial_k u - \partial_k (|u|^{\frac{4}{n-2}} \bar{u} g(|u|)) \right] \]

(122)
It remains to understand \( \mathcal{R} \left[ |u|^{\frac{1}{n-2}} \bar{u} g(|u|) \partial_k u - \partial_k (|u|^{\frac{1}{n-2}} u g(|u|)) \bar{u} \right] \). We write

\[
\mathcal{R} \left[ |u|^{\frac{1}{n-2}} \bar{u} g(|u|) \partial_k u - \partial_k (|u|^{\frac{1}{n-2}} u g(|u|)) \bar{u} \right] = A_1 + A_2
\]  

with

\[
A_1 := \mathcal{R} \left[ |u|^{\frac{1}{n-2}} \bar{u} g(|u|) \partial_k u \right]
\]  

and

\[
A_2 := -\mathcal{R} \left( \partial_k (|u|^{\frac{1}{n-2}} u g(|u|)) \bar{u} \right)
\]

We are interested in finding a function \( F_1 : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \), continuously differentiable such that \( F_1(z, \bar{z}) = \overline{F_1(z, \bar{z})}, F_1(0, 0) = 0 \) and \( A_1 = \partial_k F_1(u, \bar{u}) \). Notice that the first condition implies in particular that \( \partial_z F_1(z, \bar{z}) = \overline{\partial_{\bar{z}} F_1(z, \bar{z})} \). Therefore we get, after computation

\[
\partial_z F_1(z, \bar{z}) = \frac{|z|^{\frac{1}{n-2}} z g(|z|)}{2} \quad \partial_{\bar{z}} F_1(z, \bar{z}) = \frac{|z|^{\frac{1}{n-2}} z g(|z|)}{2}
\]

and by the fundamental calculus, if such a function exists, then

\[
F_1(z, \bar{z}) = \int_0^1 F'_1(tz, t\bar{z}) \cdot (z, \bar{z}) \, dt
\]

\[
= 2\mathcal{R} \int_0^1 \partial_z F_1(tz, t\bar{z}) z \, dt
\]

\[
= \int_0^1 t|z|^{\frac{1}{n-2}} t|z|^2 g(t|z|) \, dt
\]

and, after a change of variable, we get

\[
F_1(z, \bar{z}) = \int_0^{|z|} t^{\frac{n+2}{n-2}} g(t) \, dt
\]

Conversely it is not difficult to see that \( F_1 \) satisfies all the required conditions.

We turn now to \( A_2 \). We can write

\[
A_2 = A_{2,1} + A_{2,2}
\]

with

\[
A_{2,1} := -\mathcal{R} \left( \partial_u (|u|^{\frac{1}{n-2}} u g(|u|)) \bar{u} \partial_k u \right)
\]
\[ A_{2,2} := -\Re \left( \partial_u (|u|^{\frac{4}{n-2}} u g(|u|)) \bar{u} \partial_k u \right) \]  

(131)

Again we search for a function \( F_{2,1} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \) and continuously differentiable such that \( F_{2,1}(z, \bar{z}) = F_{2,1}(z, \bar{z}) \) and \( A_{2,1} = \partial_k F_{2,1}(u, \bar{u}) \). By identification we have

\[ \partial_z F_{2,1}(z, \bar{z}) = -\frac{|z|^{\frac{4}{n-2}} \bar{z}}{2} \left( \frac{2}{n-2} g(|z|) + g'(|z|) |z| \right) \]  

\[ \partial_{\bar{z}} F_{2,1}(z, \bar{z}) = -\frac{|z|^{\frac{4}{n-2}} z}{2} \left( \frac{2}{n-2} g(|z|) + g'(|z|) |z| \right) \]  

(132)

and by the fundamental theorem of calculus

\[ F_{2,1}(z, \bar{z}) = \int_0^1 F'_{2,1}(tz, t\bar{z}) \cdot (z, \bar{z}) \, dt \]

\[ = \int_0^1 2\Re (\partial_z F_{2,1}(tz, t\bar{z}) z) \, dt \]

\[ = -\int_0^1 |tz|^{\frac{4}{n-2}} \left( \left( \frac{2}{n-2} + 1 \right) g(|tz|) + g'(|tz|) |tz| \right) t |z|^2 \, dt \]  

(133)

and, after a change of variable, we get

\[ F_{2,1}(z, \bar{z}) = -\int_0^{|z|} t^{\frac{n+2}{n-2}} \left( \frac{2}{n-2} + 1 \right) g(t) + g'(t) \, dt \]  

(134)

Again, we can easily check that \( F_{2,1} \) satisfies all the required conditions. By using a similar process we can prove that

\[ A_{2,2} = \partial_k F_{2,2}(u, \bar{u}) \]  

(135)

with

\[ F_{2,2}(z, \bar{z}) = -\int_0^{|z|} t^{\frac{n+2}{n-2}} \left( \frac{2}{n-2} g(t) + t g'(t) \right) \, dt \]  

(136)

Therefore we get the local momentum conservation identity

\[ \partial_t \Im (\partial_k u \bar{u}) = -2\partial_j \Re (\partial_k u \bar{u} \partial_j u) + \frac{1}{2} \partial_k \Delta (|u|^2) - \partial_k \left( \tilde{F}(u, \bar{u}) \right) \]  

(137)

with \( \tilde{F}(u, \bar{u}) \) defined in (55). This identity has a similar structure to the local momentum conservation that for a solution \( v \) of the energy-critical Schrödinger
\[
\partial_t \Im \partial_k v \bar{v} = -2\partial_j \Re (\partial_k v \partial_j \bar{v}) + \frac{1}{2} \partial_k \Delta (|v|^2) + \partial_k \left( -\frac{2}{n} |u|^{\frac{2n}{n-2}} \right)
\]  

(138)

With this in mind, we multiply (137) by an appropriate spatial cutoff, in the same spirit as Bourgain [1] and Grillakis [5], to prove a Morawetz-type estimate. We follow closely an argument of Tao [10]: we introduce the weight

\[ a(x) := \left( \epsilon^2 + \left( \frac{|x|}{|A|^{\frac{1}{2}}} \right)^2 \right)^{\frac{1}{2}} \chi \left( \frac{x}{|A|^{\frac{1}{2}}} \right) \]

where \( \chi \) is a smooth function, radial such that \( \chi(|x|) = 1 \) for \( |x| \leq 1 \) and \( \chi(|x|) = 0 \) for \( |x| \geq 2 \). We give here the details since this equation, unlike the energy-critical Schrödinger equation, has no scaling property. Notice that \( a \) is convex on \( |x| \leq |A|^{\frac{1}{2}} \) since it is a composition of two convex functions. We multiply (137) by \( \partial_k a \) and we integrate by parts

\[
\partial_t \int_{\mathbb{R}^n} \partial_k a \Im (\partial_k u \bar{u}) = 2 \int_{\mathbb{R}^n} \partial_j \partial_k a \Re (\partial_k u \partial_j \bar{u}) - \frac{1}{2} \int_{\mathbb{R}^n} \Delta (\triangle a) |u|^2 \, dx + \int_{\mathbb{R}^n} \triangle a \bar{F}(u, \bar{u})(t, x) \, dx
\]  

(139)

A computation shows that for \( 0 \leq |x| \leq |A|^{\frac{1}{2}} \)

\[
\triangle a = \frac{n-1}{(|A|^{\frac{1}{2}})^2} \left( \epsilon^2 + \frac{|x|^2}{|A|^{\frac{1}{2}}} \right)^{-\frac{1}{2}} + \frac{\epsilon^2}{(|A|^{\frac{1}{2}})^2} \left( \epsilon^2 + \frac{|x|^2}{|A|^{\frac{1}{2}}} \right)^{-\frac{3}{2}}
\]  

(140)

and

\[
-\Delta \triangle a = \frac{(n-1)(n-3)}{(|A|^{\frac{1}{2}})^4} \left( \epsilon^2 + \frac{|x|^2}{|A|^{\frac{1}{2}}} \right)^{-\frac{3}{2}} + \frac{6(n-3)\epsilon^2}{|A|^{\frac{1}{2}}} \left( \epsilon^2 + \frac{|x|^2}{|A|^{\frac{1}{2}}} \right)^{-\frac{5}{2}} + \frac{15\epsilon^4}{|A|^{\frac{1}{2}}} \]

(141)

Moreover we have \( -\triangle (\triangle a) \lesssim \frac{1}{(|A|^{\frac{1}{2}})^4} \), \( |\triangle a| \lesssim \frac{1}{(|A|^{\frac{1}{2}})^2} \) and \( |\partial_j \partial_k a| \lesssim \frac{1}{|A|^{\frac{1}{2}}} \) for \( |A| \leq |x| \leq 2|A|^{\frac{1}{2}} \) and \( |\partial_k a| \lesssim \frac{1}{|A|^{\frac{1}{2}}} \) for \( |x| \leq 2|A|^{\frac{1}{2}} \). Therefore by the previous estimates, (20), (23) and the inequality \( |x| g'(|x|) \lesssim g(|x|) \) we get, after integrating on \( I \times \mathbb{R}^n \) and letting \( \epsilon \) go to zero

\[
\frac{1}{|A|^{\frac{1}{2}}} \int_I \int_{|x| \leq |A|^{\frac{1}{2}}} \frac{\bar{F}(u, \bar{u})(t, x)}{|x|} \, dx \, dt - \left( \frac{C |A|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} \right)^{-2} E|I| + \left( \frac{C |A|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} \right)^{-4} E|A|^{\frac{1}{2}}^{2}|I| \right) \lesssim E
\]

(142)

for some constant \( C \geq 1 \). After rearranging we get (54).
We shall prove the following Leibnitz rule:

**Proposition 7** “A fractional Leibnitz rule” Let $0 \leq \alpha \leq 1$, $k \geq 2$, $(r, r_1, r_2) \in (1, \infty)^2$, $r_3 \in (1, \infty]$ be such that $\frac{1}{r} = \frac{\beta}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a $C^{k+1}$ function and let $G := \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be such that

$$F^{[i]}(x) = O\left(\frac{F(x)}{x^i}\right)$$

and

$$|G^{[i]}(x, \bar{x})| = \begin{cases} O(|x|^\beta + 1-i), & i \leq \beta + 1 \\ 0, & i > \beta + 1 \end{cases}$$

for $0 \leq i \leq k$. Then

$$\left\| D^{k-1+\alpha}(G(f, \bar{f})F(|f|)) \right\|_{L^r} \lesssim \left\| f \right\|_{L^{r_1}}^{\beta} \left\| D^{k-1+\alpha}f \right\|_{L^{r_2}} \left\| F(|f|) \right\|_{L^{r_3}}$$

Here $F^{[i]}$ and $G^{[i]}$ denote the $i$th derivatives of $F$ and $G$ respectively.

**Proof**

The proof relies upon an induction process, the usual product rule for fractional derivatives

$$\left\| D^{\alpha_1}(f g) \right\|_{L^q} \lesssim \left\| D^{\alpha_1}f \right\|_{L^{q_1}} \left\| g \right\|_{L^{q_2}} + \left\| f \right\|_{L^{q_3}} \left\| D^{\alpha_1}g \right\|_{L^{q_4}}$$

and the usual Leibnitz rule for fractional derivatives:

$$\left\| D^{\alpha_2}H(f) \right\|_{L^q} \lesssim \left\| H'(f) \right\|_{L^{q_1}} \left\| D^{\alpha_2}f \right\|_{L^{q_2}}$$

if $H$ is $C^1$, $0 < \alpha_1 < \infty$, $0 \leq \alpha_2 \leq 1$, $(q, q_1, q_2, q_3, q_4) \in (1, \infty)^5$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\frac{1}{q} = \frac{1}{q_3} + \frac{1}{q_4}$: (see Christ-Weinstein [3])

Moreover we shall use interpolation and the properties of $F$ to control the intermediate terms.

\(^6\) notice that in [3], they add the restriction $0 < \alpha_1 < 1$. It is not difficult to see that this restriction is not necessary: see Taylor [8] for example
Let $k = 2$. Then

\[
\left\| D^{2-\alpha} G(f, \tilde{f}) F(|f|) \right\|_{L^r} \sim \left\| D^\alpha \nabla (G(f, \tilde{f}) F(|f|)) \right\|_{L^r} \\
\lesssim \left\| D^\alpha (\partial_2 G(f, \tilde{f}) \nabla f F(|f|)) \right\|_{L^r} \\
+ \left\| D^\alpha (\partial_2 G(f, \tilde{f}) \nabla \bar{f} F(|f|)) \right\|_{L^r} \\
+ \left\| D^\alpha \left( F'(|f|) \left( 2 \Re \left( \frac{\tilde{f}}{f} \right) \right) \right) \nabla f G(f, \tilde{f}) \right\|_{L^r} \\
\lesssim A_1 + A_2 + A_3
\] (148)

We estimate $A_1, A_2$ is estimated in a similar fashion. By (146), (147) and the assumption $F'(x) = O \left( \frac{F(x)}{x} \right)$

\[
A_1 \lesssim \left\| D^\alpha (\partial_2 G(f, \tilde{f}) F(|f|)) \right\|_{L^r} \left\| Df \right\|_{L^{r_5}} + \left\| \partial_2 G(f, \tilde{f}) F(|f|) \right\|_{L^{r_6}} \left\| D^{(2-1)+\alpha} f \right\|_{L^{r_2}} \quad (149)
\]

with $\frac{1}{r} = \frac{1}{r_4} + \frac{1}{r_5}, \frac{1}{r} = \frac{1}{r_6} + \frac{1}{r_7}, \frac{1}{r_4} = \frac{\theta_1 - 1}{r_1} + \frac{1}{r_3} + \frac{1}{r_8}, \frac{1}{r_7} = \frac{1}{r_1} \frac{\theta_1}{r_2} + \frac{\theta_2}{r_2}$ and $\theta_1 = \frac{1}{1+\alpha}$. Notice that these relations imply that $\frac{1}{r_8} = \frac{\theta_1}{r_1} + \frac{1}{r_2}$. Now, by complex interpolation, we have

\[
\left\| D^\alpha f \right\|_{L^{r_9}} \lesssim \left\| f \right\|_{L^{r_1}}^{\theta_1} \left\| D^{(2-1)+\alpha} f \right\|_{L^{r_2}}^{1-\theta_1} \quad (150)
\]

and

\[
\left\| Df \right\|_{L^{r_5}} \lesssim \left\| f \right\|_{L^{r_1}}^{1-\theta_1} \left\| D^{(2-1)+\alpha} f \right\|_{L^{r_2}}^{\theta_1} \quad (151)
\]

Plugging (150) and (151) into (149) we get (145).

We estimate $A_3$.

\[
A_3 \lesssim \left\| D^\alpha \left( F'(|f|) \left( \frac{\tilde{f}}{f} \right) \frac{\bar{f}}{|\bar{f}|} G(f, \tilde{f}) \right) \right\|_{L^r} \left\| Df \right\|_{L^{r_5}} + \left\| D^{\alpha+1} f \right\|_{L^{r_2}} \left\| F'(|f|) \frac{\tilde{f}}{|\bar{f}|} G(f, \tilde{f}) \right\|_{L^{r_6}} \quad (152)
\]

Using the assumption $F'(x) = O \left( \frac{F(x)}{x} \right)$ we get $A_{3,2} \lesssim \left\| f \right\|_{L^{r_1}}^{\beta} \left\| D^{1+\alpha} f \right\|_{L^{r_2}} \left\| F(|f|) \right\|_{L^{r_3}}$.

Moreover, by (147), the assumptions on $F$ and $G$, (150) and (151) we get

\[
A_{3,1} \lesssim \left\| F(|f|) \right\|_{L^{r_1}}^{\beta-1} \left\| D^\alpha f \right\|_{L^{r_8}} \left\| Df \right\|_{L^{r_5}} \\
\lesssim \left\| f \right\|_{L^{r_1}}^{\beta} \left\| D^{1+\alpha} f \right\|_{L^{r_2}} \left\| F(|f|) \right\|_{L^{r_3}} \quad (153)
\]
with $\frac{1}{r_7} + \frac{1}{r_8} = \frac{1}{r_4}$. Now let us assume that the result is true for $k$. Let us prove that it is also true for $k + 1$. By (146) we have

\[
\|D^{k+\alpha}(G(f, \tilde{f})F(|f|))\|_{L^r} \sim \|D^{k-1+\alpha}\nabla(G(f, \tilde{f})F(|f|))\|_{L^r} \\
\lesssim \|D^{k-1+\alpha}\partial_z G(f, \tilde{f})\nabla f F(|f|)\|_{L^r} \\
+ \|D^{k-1+\alpha}\partial_z G(f, \tilde{f})\nabla \tilde{f} F(|f|)\|_{L^r} \\
+ \left\|D^{k-1+\alpha} \left[ G(f, \tilde{f})F'(|f|) \left(2\Re \left(\frac{f}{|f|}\right) \nabla f\right)\right]\right\|_{L^r} \\
\lesssim A'_1 + A'_2 + A'_3
\]

We estimate $A'_1$ and $A'_3$. $A'_2$ is estimated in a similar fashion as $A'_1$. By (146), (147) and the assumption $|\partial_z G(f, \tilde{f})| \lesssim |f|^\beta$ we have

\[
A'_1 \lesssim \|D^{k+\alpha}f\|_{L^2} \|\partial_z G(f, \tilde{f})F(|f|)\|_{L^{r_6}} + \|D^{k-1+\alpha}(\partial_z G(f, \tilde{f})F(|f|))\|_{L^{'r_4}} \|Df\|_{L^{'r_5}} \\
\lesssim \|f\|_{L^{r_1}} \|D^{(k+1)-1+\alpha}f\|_{L^{'r_4}} \|F(|f|)\|_{L^{r_3}} + A'_{1,1}
\]

with $r_4'$, $r_5'$ such that $\frac{1}{r_4} + \frac{1}{r_5} = \frac{1}{r_1}$, $\frac{1}{r_5} = \frac{1-\theta_4}{r_2} + \frac{\theta_5}{r_2}$ and $\theta_1' = \frac{1}{k+\alpha}$. Notice that, since we assumed that the result is true for $k$, we get, after checking that $\partial_z G$ satisfies the right assumptions

\[
\|D^{k-1+\alpha}(\partial_z G(f, \tilde{f})F(|f|))\|_{L^{'r_4}} \lesssim \|f\|_{L^{r_1}} \|D^{k-1+\alpha}f\|_{L^{'r_4}} \|F(|f|)\|_{L^{r_3}}
\]

with $r_8'$ such that $\frac{1}{r_4} = \frac{\beta-1}{r_1} + \frac{1-\theta_4}{r_2} + \frac{1}{r_3}$. Notice also that, by complex interpolation

\[
\|Df\|_{L^{'r_5}} \lesssim \|f\|_{L^{r_1}} \|D^{(k+1)-1+\alpha}f\|_{L^{'r_4}} \lesssim \|f\|_{L^{r_1}} \|D^{k+\alpha}f\|_{L^{'r_4}} \|Df\|_{L^{'r_5}}
\]

and

\[
\|D^{k-1+\alpha}f\|_{L^{'r_8}} \lesssim \|f\|_{L^{r_1}} \|D^{(k+1)-1+\alpha}f\|_{L^{'r_4}} \|Df\|_{L^{'r_5}} \lesssim \|f\|_{L^{r_1}} \|D^{k+\alpha}f\|_{L^{'r_4}} \|Df\|_{L^{'r_5}}
\]

Combining (156), (157) and (158) we have

\[
A'_{1,1} \lesssim \|f\|_{L^{r_1}} \|D^{k+\alpha}f\|_{L^{'r_4}} \|F(|f|)\|_{L^{r_3}}
\]

Plugging this bound into (155) we get the required bound for $A'_{1,1}$.

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We turn to $A_3$. We have

$$A_3' \lesssim \left\| D^{k-1+\alpha} \left[ G(f, f) F'(|f|) \frac{f}{|f|} \right] \right\|_{L_t' \infty} \left\| Df \right\|_{L_x' \infty}$$

$$+ \left\| D^{k+\alpha} f \right\|_{L_t^2} \left\| G(f, f) F'(|f|) \right\|_{L_t^6}$$

$$\lesssim \left\| f \right\|_{L_t^1} \left\| D^{k-1+\alpha} f \right\|_{L_x' \infty} \left\| Df \right\|_{L_x' \infty}$$

$$+ \left\| D^{k-\alpha} f \right\|_{L_t^2} \left\| F(|f|) \right\|_{L_t^3}$$

(160)

$$\lesssim \left\| f \right\|_{L_t^1} \left\| D^{k-\alpha} f \right\|_{L_t^2} \left\| F(|f|) \right\|_{L_t^3}$$

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