Spectral types of linear $q$-difference equations and $q$-analog of middle convolution

Hidetaka Sakai and Masashi Yamaguchi
Graduate School of Mathematical Sciences, The university of Tokyo,
Komaba, Tokyo 153-8914, Japan.

Abstract

We give a $q$-analog of middle convolution for linear $q$-difference equations with rational coefficients. In the differential case, middle convolution is defined by Katz, and he examined properties of middle convolution in detail. In this paper, we define a $q$-analog of middle convolution. Moreover, we show that it also can be expressed as a $q$-analog of Euler transformation. The $q$-middle convolution transforms Fuchsian type equation to Fuchsian type equation and preserves rigidity index of $q$-difference equations.

1 Introduction

In this paper, we give a $q$-analog of middle convolution for linear $q$-difference equations with rational coefficients, and we show properties of the $q$-middle convolution. Before that, we briefly look over the theory of middle convolution for differential equations.

At first, we look over a theory of Katz in [1]. He defined addition and middle convolution for solutions of differential equations of Schlesinger normal form

$$\frac{dY}{dx}(x) = A(x)Y(x), \quad A(x) = \sum_{k=1}^{N} \frac{A_k}{x - t_k} \quad (t_k \in \mathbb{C}, A_k \in M_m(\mathbb{C})).$$

(1)

These operations transform Fuchsian equation to Fuchsian equation and preserve rigidity index of the equation. Rigidity index is the integer related to the number of accessory parameters. Accessory parameters are parameters which are independent of eigenvalues of $A_k$, $A_\infty = -(A_1 + \cdots + A_N)$. If the equation (1) has no accessory parameters, it is called “rigid”. Katz showed that any irreducible rigid Fuchsian equations can be obtained from a certain 1st order equation by finite iterations of additions and middle convolutions. Katz’s theorem tells that there exists integral representation of solutions of any irreducible rigid Fuchsian equations, because an addition transforms solution $Y(x)$ of the equation (1) to

$$\prod_{k=1}^{r} (x - a_k)^{b_k} \cdot Y(x) \quad (a_k, b_k \in \mathbb{C})$$

and a middle convolution is integral transformation for solution $Y(x)$ of the equation (1).

Remark 1.1 There are two types, “additive version” and “multiplicative version” of middle convolution defined by Katz. Additive version is transformation for equations. Multiplicative version
is transformation for solutions. Multiplicative middle convolution induces a transformation of monodromy representation. In this paper, we treat the similar version to the former, which should be called “additive version” $q$-middle convolution. In the $q$-difference case, we think that connection matrix between two local solutions at singularities $x = 0, \infty$ correspond to monodromy of differential equation. Birkhoff studied the connection matrix $P(x)$ for local solutions $Y_0(x), Y_\infty(x)$ at singularities $x = 0, \infty$ of linear $q$-difference system with polynomial coefficient $Y(qx) = A(x)Y(x)$.

Furthermore, Sauloy considered a category of linear $q$-difference systems with polynomial, a category of solutions and a category of connection data in [6]. He gave Riemann-Hilbert correspondence for these categories. Based on the Sauloy’s result, Roques studied rigidity of connection of linear $q$-difference systems with rational coefficients in [7]. □

We referred to an easier construction of Dettweiler and Reiter in order to define the $q$-analog of middle convolution. Let us look over a result of Dettweiler and Reiter in [2, 3]. They express Katz’s middle convolution in terms of matrices. The next transformation is called “convolution” with parameter $\lambda \in \mathbb{C}$:

$$\frac{dZ}{dx}(x) = G(x)Z(x), \quad G(x) = \sum_{k=1}^{N} \frac{G_k}{x - t_k} \quad (G_k \in M_{mN}(\mathbb{C})), \quad (2)$$

$$G_k = \begin{pmatrix} O & \cdots & A_k + \lambda_1 \cdot \cdots \cdot A_N \\ A_1 & \cdots & A_k + \lambda_1 \cdot \cdots \cdot A_N \\ \vdots \\ O \end{pmatrix} \quad (k \text{th entry}) \quad (1 \leq k \leq N, \quad 1_m = \{\delta_{i,j}\}_{1 \leq i,j \leq m} \in M_{m}(\mathbb{C})). \quad (3)$$

Moreover, we define two linear spaces

$$\mathcal{K} = \begin{pmatrix} \ker A_1 \\ \vdots \\ \ker A_N \end{pmatrix}, \quad \mathcal{L} = \ker(G_1 + \cdots + G_N). \quad (4)$$

Let $G_k$ be a matrix induced by the action of $G_k$ on $\mathbb{C}^{mN}/(\mathcal{K} + \mathcal{L})$. We define middle convolution

$$mc_\lambda : (A_1, \ldots, A_n) \mapsto (\overline{G}_1, \ldots, \overline{G}_n).$$

We obtain a similar transformation by considering the Dettweiler and Reiter’s setting in the $q$-difference case.

Let

$$B = t^t(B_1, \ldots, B_N, B_\infty) \in (M_m(\mathbb{C}))^{N+1},$$

$$b = t^t(b_1, \ldots, b_N) \in (\mathbb{C}\{0\})^N \quad (b_i = b_j \Rightarrow i = j).$$

We set an equation

$$E_{B,b} : \sigma_x Y(x) = B(x)Y(x), \quad B(x) = B_\infty + \sum_{i=1}^{N} \frac{B_i}{1 - \frac{x}{b_i}}. \quad (5)$$

For an equation $E_{B,b}$, we define the $q$-convolution.
Definition 1.2 (q-convolution) Let $\mathcal{E}$ be the set of $E_{B,b}$’s. For $E_{B,b} \in \mathcal{E}, \lambda \in \mathbb{C}$, we define $q$-convolution $c_\lambda : \mathcal{E} \to \mathcal{E}$ ($E_{B,b} \mapsto E_{F,b}$) as

\[ F = (F_1, \ldots, F_N, F_\infty) \in (M_{(N+1)m}(\mathbb{C}))^{N+1}, \]

\[ F_i = \begin{pmatrix} O & \cdots & B_i - (1 - q^\lambda)1_m & \cdots & B_N \end{pmatrix} \quad \text{(i+1 th entry), } 1 \leq i \leq N, \]

\[ F_\infty = 1_{(N+1)m} - \hat{F}, \]

\[ \hat{F} = \begin{pmatrix} \hat{F}_{11} & \cdots & \hat{F}_{1N} \\ \vdots & \ddots & \vdots \\ \hat{F}_{N1} & \cdots & \hat{F}_{NN} \end{pmatrix}, \quad \hat{F}_{ij} = 1_m - B_\infty - \sum_{j=1}^N B_j. \]

Furthermore, we define the $q$-middle convolution.

Definition 1.3 (q-middle convolution) Let $\mathcal{V} = \mathbb{C}^m$ and $F$-invariant subspaces of $\mathcal{V}^{N+1}$ as

\[ \mathcal{K} = \mathcal{K}_\mathcal{V} = \bigoplus_{i=0}^N \ker B_i, \quad \mathcal{L} = \mathcal{L}_\mathcal{V}(\lambda) = \ker(\hat{F} - (1 - q^\lambda)1_{(N+1)m}). \] (7)

Let $F_k$ be a matrix induced by the action of $F_k$ on $\mathcal{V}^{N+1}/(\mathcal{K} + \mathcal{L})$, and we define the q-middle convolution $mc_\lambda$ as $\mathcal{E} \to \mathcal{E}$ ($E_{B,b} \mapsto E_{F,b}$).

We abbreviated that modules $(B, \mathcal{V}), (F, \mathcal{V}^{N+1}), (\hat{F}, \mathcal{V}^{N+1}/(\mathcal{K} + \mathcal{L}))$ are $\mathcal{V}$, $\mathcal{V}^{N+1}$, $\mathcal{V}^{N+1}/(\mathcal{K} + \mathcal{L})$ respectively. Moreover, we set

\[ c_\lambda(B) = F, \quad c_\lambda(\mathcal{V}) = \mathcal{V}^{N+1}, \quad mc_\lambda(B) = \hat{F}, \quad mc_\lambda(\mathcal{V}) = \mathcal{V}^{N+1}/(\mathcal{K} + \mathcal{L}). \]

Here a $q$-analog of middle convolution was defined. We can also give an integral representation of $q$-convolution by $q$-analog of Euler transformation. We will describe it in detail in Section 2.

By the way, we would like to understand q-middle convolution as a transformation for the analog of Fuchsian equation. From now on, we set $q \in \mathbb{C}, 0 < |q| < 1$, $\sigma_x : x \mapsto qx$. We set a linear $q$-difference equation with polynomial coefficient

\[ E_A : \sigma_x Y(x) = A(x)Y(x), \quad A(x) = \sum_{k=0}^N A_k x^k \quad (A_k \in M_m(\mathbb{C})). \] (8)

Moreover, we let $A_\infty = A_N$. We define “Fuchsian” $q$-difference equations.

Definition 1.4 (Fuchsian type equation) For an equation $E_A$, if $A_0, A_\infty \in \text{GL}_m(\mathbb{C})$, then we call $E_A$ Fuchsian type $q$-difference equation.

Although we cannot apply the $q$-middle convolution to this Fuchsian equation directly, we see that the equation $E_A$ is connected with $E_{B,b}$ by simple transformations. We consider $m \times m$ matrix system $E_R$ with rational coefficients

\[ E_R : \sigma_x Y(x) = R(x)Y(x). \] (9)
As gauge transformations for the solution $Y(x)$ of the equation $E_R$, we consider only two types in this paper. The first one is the transformation

$$
\varphi_P : Y(x) \mapsto \tilde{Y}(x) = PY(x) \quad (P \in \text{GL}_m(\mathbb{C})).
$$

(10)

The second one is the transformation

$$
\varphi_f : Y(x) \mapsto \tilde{Y}(x) = f(x)Y(x),
$$

(11)

where $f(x)$ is solution of $\sigma_x f(x) = Q(x)f(x)$ ($Q(x)$ is a scalar rational function). This function $f(x)$ can be expressed by using the functions

$$(ax; q)_\infty, \quad \vartheta_q(x).$$

Here we set

$$(a_1, \ldots, a_n; q)_0 = 1,$$

$$(a_1, \ldots, a_n; q)_m = \prod_{i=1}^n \prod_{j=0}^{m-1} (1 - a_i q^j) \quad (m \in \mathbb{Z}_{>0}),$$

$$(a_1, \ldots, a_n; q)_\infty = \lim_{m \to \infty} (a_1, \ldots, a_n; q)_m,$$

$$\vartheta_q(x) = \prod_{n=0}^{\infty} (1 - q^{n+1})(1 + xq^n)(1 + x^{-1}q^{n+1}).$$

To be specific, for the solution $Y(x)$ of the equation $E_R$,

if we put $\tilde{Y}(x) = (ax; q)_\infty Y(x)$, then $\sigma_x \tilde{Y}(x) = (1 - ax)R(x)\tilde{Y}(x);$  
if we put $\tilde{Y}(x) = \frac{1}{\vartheta_q(x)} Y(x)$, then $\sigma_x \tilde{Y}(x) = xR(x)\tilde{Y}(x);$  
if we put $\tilde{Y}(x) = \frac{\vartheta_q(x)}{\vartheta_q(ax)} Y(x) \quad (a \in \mathbb{C}\{0\})$, then $\sigma_x \tilde{Y}(x) = aR(x)\tilde{Y}(x).$

We define a family of equations by modulo $\varphi_P$ and $\varphi_f$. We interpret the $q$-middle convolution as the transformation of the family of equations. From arbitrary equation $E_R$, we obtain $\tilde{E}_R$:

$$
\tilde{E}_R : \sigma_x \tilde{Y}(x) = A(x)\tilde{Y}(x),
$$

(12)

$$
A(x) = \sum_{i=0}^N A_i x^i \quad (A_k \in \text{M}(m, \mathbb{C}), \ A_0, A_N \neq 0, \ \forall a \in \mathbb{C}; \ A(a) \neq 0),
$$

(13)

which is determined up to multiplication of constant and similarity transformations by $\varphi_P$.

We call $\tilde{E}_R$ the canonical form of the equation $E_R$. In general case, for canonical form $\sigma_x \tilde{Y}(x) = A(x)\tilde{Y}(x)$ of $E_{B,b}$, we obtain
\[ A(x) = T(x)B(x), \quad T(x) = \prod_{i=1}^{N} \left(1 - \frac{x}{b_i}\right), \quad (14) \]

\[ A_0 = 1_m - B_0, \quad A_\infty = b_\infty B_\infty, \quad B_\infty = 1_m - \sum_{i=1}^{N} B_i - B_\infty, \quad b_\infty = \prod_{i=1}^{N} (-b_i^{-1}), \quad (15) \]

\[ \text{rank} B_i = \begin{cases} m - n^k_i & (b_i = a_k \in \mathbb{Z}_R) \\ m & (b_i \notin \mathbb{Z}_R) \end{cases}, \quad (1 \leq i \leq N, \ n^k_i = \dim \ker A(a_k)). \quad (16) \]

**Remark 1.5** The definition of the Fuchsian type equation may not be appropriate. We look at Heine’s \(q\)-hypergeometric function

\[ _2\varphi_1(\alpha, \beta, \gamma; q; x) = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (\beta; q)_n}{(q; q)_n (\gamma; q)_n} x^n. \quad (17) \]

Here \( u(x) = _2\varphi_1(\alpha, \beta, \gamma; q; x) \) satisfies the equation

\[ \{(1 - \sigma_x)(1 - q^{-1}\gamma\sigma_x) - x(1 - \alpha\sigma_x)(1 - \beta\sigma_x)\} u(x) = 0. \quad (18) \]

If we set \( v(x) = \frac{1}{x} \sigma_x u(x) \) and \( Y(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \), then we obtain

\[ \sigma_x Y(x) = \frac{1}{x(q\alpha\beta x - \gamma)} \begin{pmatrix} 0 & x^2(q\alpha\beta x - \gamma) \\ -x + 1 & x\{(\alpha + \beta)x - q^{-1}\gamma - 1\} \end{pmatrix} Y(x). \quad (19) \]

Although this is not Fuchsian \(q\)-difference equation in our sense, this equation transforms to Fuchsian type equation by a simple transformation:

\[ Y(x) \leftrightarrow \tilde{Y}(x) = \begin{pmatrix} 1 & 0 \\ 1 & -x \end{pmatrix} Y(x) = \begin{pmatrix} u(x) \\ (1 - \sigma_x) u(x) \end{pmatrix}. \quad (20) \]

\( \tilde{Y}(x) \) satisfies Fuchsian \(q\)-difference equation

\[ \sigma_x \tilde{Y}(x) = \frac{1}{\alpha\beta x - q^{-1}\gamma} \begin{pmatrix} \alpha\beta x - q^{-1}\gamma & -\alpha\beta x + q^{-1}\gamma \\ (1 - \alpha)(1 - \beta) & (\alpha + \beta - \alpha\beta)x - 1 \end{pmatrix} \tilde{Y}(x). \quad (21) \]

Although we do not introduce such transformations, Saloy used a transformation by rational component matrix as a gauge transformation in \[9\]. We think that our Fuchsian \(q\)-difference equation corresponds to the Schlesinger normal form in the differential case. Although we do not call the equation \[10\] Fuchsian type, we might have to do. On the other hand, in the differential case, there exists Fuchsian differential equations which cannot be written in the Schlesinger normal form. We set \( y_i(x) (i = 1, 2) \) the components of a solution \( Y(x) \) of an equation

\[ \frac{dY}{dx}(x) = R(x)Y(x) \quad (R(x) \text{ is rational function}). \quad (22) \]
If singularities of \( y_i(x) \) are at most regular singularities, we call the equation (22) Fuchsian differential equation. Regular singularity is defined from local properties of solution. In more detail, if function \( y(x) \) is not holomorphic at \( x = x_0 \) and for any \( \theta, \theta' < \theta \), there exists \( n_0 \in \mathbb{Z}_{>0} \) such that

\[
\lim_{\theta < \arg(x-x_0) < \theta', x \to x_0} |x-x_0|^{n_0} |y(x)| = 0,
\]

we call \( x = x_0 \) the regular singularity of \( y(x) \). Here we consider the equation of Schlesinger normal form

\[
dY(x) = \left( \sum_{i=1}^N \frac{A_i}{x - a_i} \right) Y(x) \quad (a_i \in \mathbb{C}, A_i \in M_m(\mathbb{C})),
\]

that is, a special case of the Fuchsian differential equation. □

We can think that our Fuchsian type equation actually Fuchsian because Carmichael’s theorem has been establish in [4].

**Theorem 1.6** (Carmichael) Let \( \theta_j, \kappa_j \ (1 \leq j \leq m) \) the eigenvalues of \( A_0, A_\infty \in \text{GL}_n(\mathbb{C}) \) respectively, we assume further that \( A_0, A_\infty \) are semi-simple and

\[
\frac{\theta_j}{\theta_k}, \frac{\kappa_j}{\kappa_k} \notin q^{\mathbb{Z}_{>0}} \quad (\forall j, \forall k).
\]

Then, there exist unique solutions \( Y_0(x), Y_\infty(x) \) of the equation (8) with the following properties,

\[
Y_0(x) = \hat{Y}_0(x)x^{D_0}, \quad Y_\infty(x) = q^{\frac{x}{2}u(u-1)}\hat{Y}_\infty(x)x^{D_\infty}, \quad u = \frac{\log x}{\log q}.
\]

Here \( \hat{Y}_0(x) \) (resp. \( \hat{Y}_\infty(x) \)) is a holomorphic and invertible matrix at \( x = 0 \) (resp. \( x = \infty \)) such that \( \hat{Y}(0) = C_0 \) (resp. \( \hat{Y}(\infty) = C_\infty \)).

**Remark 1.7** The functions used in the above theorem

\[
x^{\log \theta/\log q}, \quad q^{u(u-1)/2} \quad (u = \log x/\log q)
\]

are solutions of the following equations, respectively,

\[
\sigma_x f(x) = \theta f(x), \quad \sigma_x f(x) = xf(x).
\]

Hence instead of these functions, we can use the following single-valued functions as solutions of the above equations,

\[
\frac{\theta_q(x)}{\theta_q(\theta x)}, \quad \frac{1}{\theta_q(x)}.
\]

These functions are widely used in recent years, we use these in this paper. □
The purpose of this study is to examine properties of the \( q \)-middle convolution. Let us describe the contents of this paper. In the 2nd section, we show that \( q \)-convolution can be expressed by a \( q \)-analog of Euler transformation. In the 3rd section, we define the spectral type and the rigidity index for the equation \( E_R \). Spectral types are defined by the size of Jordan cells of \( A_0 \), \( A_\infty \) and types of elementary divisors of \( A(x) \). Notice that the rigidity index is not only determined by data of \( B_k \) of \( B(x) \), but also by data of elementary divisors of coefficient polynomial \( A(x) \) of canonical form \( \tilde{E}_R \). In the 4th section, we prove the three main theorems.

**Theorem 1.8** (Fuchsian type equation) If equation \( E_R \) is Fuchsian type equation, then \( mc_\lambda(E_R) \) is also Fuchsian type equation.

Here we assume that next conditions (\( * \)), (\( ** \)) after the manner of Dettweiler and Reiter in [2]. (These conditions are generally satisfied if \( \dim \mathcal{V} = 1 \) or \( \dim \mathcal{V} > 1 \) and \( B \) is irreducible)

**Definition 1.9** We define the conditions (\( * \)), (\( ** \)):

\[
(\ast) \colon \forall i \in \{0, \ldots, N\}, \forall \tau \in \mathbb{C} ; \quad \bigcap
_{j \neq i} \ker B_j \cap \ker (B_i + \tau 1_m) = 0,
\]

\[
(\ast\ast) \colon \forall i \in \{0, \ldots, N\}, \forall \tau \in \mathbb{C} ; \quad \sum
_{j \neq i} \text{im} B_j + \text{im} (B_i + \tau 1_m) = \mathcal{V}.
\]

**Theorem 1.10** (irreducibility) If (\( \ast \)), (\( \ast\ast \)) are satisfied, then \( \mathcal{V} \) is irreducible if and only if \( mc_\lambda(\mathcal{V}) \) is irreducible.

**Theorem 1.11** (rigidity index) If (\( \ast \)), (\( \ast\ast \)) are satisfied, then \( mc_\lambda \) preserves rigidity index of Fuchsian equation \( E_R \).

To prove these theorems, we do not need for the following conditions in the Theorem 1.6:

\[ A_0, A_\infty : \text{semi-simple, } \frac{\theta_i}{\theta_k}, \frac{\kappa_i}{\kappa_k} \not\in q \mathbb{Z}^> 0 \quad (\theta_i, \kappa_i : \text{eigenvalues of } A_0, A_\infty \text{ respectively}). \]

We will explain “rigidity index” in the section 3. This is defined by “spectral type” of the Fuchsian equation \( E_R \).

### 2 Integral representation of \( q \)-convolution

We gave a \( q \)-analog of convolution as a transformation of the \( q \)-difference equations. We can also give an integral representation of “\( q \)-convolution” by a \( q \)-analog of Euler transformation. In this section, we show

**Theorem 2.1** For the solution \( Y(x) \) of the equation \( E_{B,b} \), let \( \tilde{Y}(x) = t(\tilde{Y}_0(x), \ldots, \tilde{Y}_N(x)) \) by

\[
\tilde{Y}_i(x) = \int_0^\infty \frac{P_\lambda(x,s)}{s - b_i} Y(s) \, dq_s, \quad b_0 = 0, \quad P_\lambda(x,s) = \frac{(q^{\lambda+1}sx^{-1};q)_\infty}{(qsx^{-1};q)_\infty}.
\] (28)
Then, \( \tilde{Y}(x) \) is the solution of the equation \( E_{F,b} \) (see Definition 1.2). Here Jackson integral is defined by

\[
\int_0^\infty f(x) \, dq \, x = (1 - q) \sum_{n=-\infty}^\infty q^n f(q^n).
\]

**Proof.** \( P_\lambda(x,s) \) is a solution of partial difference equations

\[
(\sigma_x - \sigma_s^{-1})y(x,s) = 0, \quad \sigma_x y(x,s) = \frac{1 - q^s x^{-1}}{1 - s x^{-1}} y(x,s).
\]

Hence \( P_\lambda(x,s) \) is a solution of

\[
\frac{\sigma_x P_\lambda(x,s)}{s - b_i} = \frac{x - q^s b_i P_\lambda(x,s)}{x - b_i} + \frac{x}{s - b_i} \frac{\sigma_s^{-1} - 1}{s} P_\lambda(x,s).
\]

Moreover, this function is independent to \( b_i \in \mathbb{C} \). By multiplying \( Y(s) \), and by Jackson integral calculation, we obtain

\[
\sigma_x \tilde{Y}_i(x) = \left\{ 1 + \left(1 - q^\lambda \right)b_i \right\} \tilde{Y}_i(x) + \frac{x}{x - b_i} \int_0^\infty \frac{\sigma_s^{-1} - 1}{s} P_\lambda(x,s) Y(s) \, dq \, s.
\]

Meanwhile, we obtain

\[
\int_0^\infty \frac{\sigma_s^{-1} - 1}{s} P_\lambda(x,s) \cdot Y(s) \, dq \, s
\]

\[
= \int_0^\infty P_\lambda(x,s) \frac{1}{s} \left\{ \sigma_s Y(s) - Y(s) \right\} \, dq \, s
\]

\[
= \int_0^\infty P_\lambda(x,s) \frac{1}{s} \left( B_\infty + \sum_{j=1}^N \frac{B_j}{1 - \frac{1}{b_j}} - 1_m \right) Y(s) \, dq \, s
\]

\[
= \int_0^\infty P_\lambda(x,s) \left\{ \frac{1}{s} \left( B_\infty + \sum_{j=1}^N B_j - 1_m \right) - \sum_{j=1}^N \frac{1}{s - b_j} B_j \right\} Y(s) \, dq \, s
\]

\[
= - \int_0^\infty P_\lambda(x,s) \sum_{j=0}^N \frac{1}{s - b_j} B_j \cdot Y(s) \, dq \, s \left( b_0 = 0, \quad B_0 = 1_m - \sum_{i=1}^N B_i - B_\infty \right)
\]

\[
= - \sum_{j=0}^N B_j \int_0^\infty \frac{P_\lambda(x,s)}{s - b_j} Y(s) \, dq \, s
\]

\[
= - \sum_{j=0}^N B_j \tilde{Y}_j(x).
\]

Here \( \tilde{Y}_i(x) \) satisfies

\[
\sigma_x \tilde{Y}_i(x) = \left\{ 1 + \left(1 - q^\lambda \right)b_i \right\} \tilde{Y}_i(x) - \frac{x}{x - b_i} \sum_{j=0}^N B_j \tilde{Y}_j(x)
\]

\[
= \tilde{Y}_i(x) - \sum_{j=0}^N B_j \tilde{Y}_j(x) + \frac{1}{1 - q^\lambda} \left\{ - (1 - q^\lambda) \tilde{Y}_i(x) + \sum_{j=0}^N B_j \tilde{Y}_j(x) \right\}.
\]

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Therefore, $\tilde{Y}(x)$ is a solution of the equation $E_{F,b}$. $\square$

From the above, we proved that $q$-convolution can be expressed by a $q$-analog of Euler transformation.

3 Rigidity index of $q$-difference equations

In this section, we define the spectral type and the rigidity index of the equation $E_R$. We set the coefficient $A(x) = \sum_{k=0}^{N} A_k x^k$ of the canonical form of a Fuchsian equation $E_R$.

**Definition 3.1** Let $A_\xi \sim \bigoplus_{i=1}^{l_\xi} \bigoplus_{j=1}^{s_{i,j}^\xi} J(\alpha_i^\xi, t_{i,j}^\xi)$ ($J(\alpha, t)$: Jordan cell, $t_{i,j}^\xi \leq t_{i,j+1}^\xi$). Moreover, let $\{m_{i,k}^\xi\}_{k}$ denote the conjugate of $\{t_{i,j}^\xi\}_{j}$ in Young diagram. We call

$$S_\xi : m_{1,1}^\xi \ldots m_{1,t_{1,1}^\xi} \ldots \cdot m_{k,1}^\xi \ldots m_{1,t_{1,1}^\xi}$$

spectral type of $A_\xi$.

**Definition 3.2** Let $Z_A = \{a \in \mathbb{C}; \det A(a) = 0\}$ and denote by $d_i$ ($1 \leq i \leq m$) the elementary divisors of $\det A(x) (d_{i+1} | d_i)$. For any $a_i \in Z_A$, we denote by $\{\tilde{n}_k^i\}_k$ the orders of zeros $a_i$ of $\{d_k\}_k$. We set $\{n_k^i\}_j$ the conjugate of $\{\tilde{n}_k^i\}_k$. We call

$$S_{\text{div}} : n^1_1 \ldots n^1_{k_1}, \ldots, n^l_1 \ldots n^l_{k_l}$$

spectral type of $A(x)$.

**Definition 3.3** We call $S(E_R) = (S_0; S_\infty; S_{\text{div}})$ spectral type of $E_R$.

From the above, we define the rigidity index.

**Definition 3.4** We define the rigidity index $\text{idx}(E_R)$ of the equation $E_R$ as

$$\text{idx}(E_R) = \sum_{\xi=0, \infty} \sum_{i=1}^{l_\xi} \sum_{j=1}^{s_{i,j}^\xi} (m_{i,j}^\xi)^2 + \sum_{i=1}^{l} \sum_{j=1}^{s_{i,j}^i} (n_{i,j}^i)^2 - m^2 N. \quad (29)$$

For example, we consider

$E_1 : A(x) = A(x)Y(x)$, $A(x) = A_0 + A_1 x + A_\infty x^2$,

$A_0 \sim J(\alpha^0, 1) \oplus J(\alpha^0, 1) \oplus J(\alpha^0, 1) \oplus J(\alpha^0, 1) \oplus J(\alpha^0, 1)$, $A_\infty \sim J(\alpha^\infty, 1) \oplus J(\alpha^\infty, 1) \oplus J(\alpha^\infty, 1)$,

$A(x) \sim \text{diag}((x - a_1)(x - a_2)(x - a_3)(x - a_4), (x - a_1)(x - a_2), (x - a_1)(x - a_2), x - a_1, 1)$

$(a_i^0 \neq a_j^0, a_i^\infty \neq a_j^\infty, a_i \neq a_j (i \neq j))$.

Spectral type and rigidity index of the equation $E_1$ are

$$S(E_1) : 31, 1; 3, 1, 1; 4, 31, 1, 1, \quad \text{idx}(E_1) = 0.$$
Remark 3.5 We can also express the rigidity index \( \text{idx}(E_R) \) of the equation \( E_R \) as

\[
\text{idx}(E_R) = \dim Z(A_0) + \dim Z(A_\infty) + \sum_{i=1}^{l} \sum_{j=1}^{k_i} (n_i^j)^2 - m^2 N. \tag{30}
\]

Here, we let \( Z(A) = \{ X \in \text{GL}_m(\mathbb{C}) ; AX = XA \} (A \in \text{M}_m(\mathbb{C})) \). □

We can easily check the next facts.

Proposition 3.6

(i) \( \sum_{i=1}^{l} \sum_{j=1}^{k_i} m_{i,j}^\xi = m, \quad \sum_{i=1}^{l} \sum_{j=1}^{k_i} n_j^i = Nm. \)
(ii) \( n_i = \sum_{j=1}^{k_i} n_j^i \) is a multiplicity of \( \det A(x) \) of zeros \( a_i \in Z_A. \)
(iii) \( \text{idx}(E_R) \) is even number.

After the definition of \( q \)-analog of spectral type and rigidity index, let’s look at some examples. At first, we consider the Heine’s \( q \)-hypergeometric equation \( E_2 : (21) \). It is easy to confirm that the equation \( E_2 \) has generally the following data:

\[
S(E_2) : 1,1; 1,1, \quad \text{idx}(E_2) = 2. \tag{31}
\]

Moreover, we consider generalized \( q \)-hypergeometric equation

\[
E_3 : \sigma_x Y(x) = A(x)Y(x), \quad A(x) = \begin{pmatrix} 0 & f_0 & & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & f_0 & \\ & & -f_m & \cdots & -f_2 & -f_1 \end{pmatrix}, \tag{32}
\]

\[
f_0 \sigma_x^m + f_1 \sigma_x^{m-1} + \cdots + f_m = \prod_{k=1}^{m} \left( \frac{b_k}{q} \sigma_x - 1 \right) - \lambda x \prod_{k=1}^{m} (a_k \sigma_x - 1) \quad (a_k, b_k, \lambda \in \mathbb{C}^*). \tag{33}
\]

We set \( A(x) = A_0 + A_\infty x \ (A_k \in \text{M}_m(\mathbb{C})) \). We obtain the data of the equation \( E_3 \) as

\[
\text{Ev}(A_0) = \left\{ \frac{q}{b_1}, \ldots, \frac{q}{b_m} \right\}, \quad \text{Ev}(A_\infty) = \left\{ \frac{1}{a_1}, \ldots, \frac{1}{a_m} \right\}, \tag{34}
\]

\[
\text{zeros of } \det A(x) \text{ are } \frac{1}{\lambda} \text{ and } \frac{1}{\lambda} \prod_{k=1}^{m} \frac{b_k}{aq_k} \text{ (multiplicity : } m-1). \tag{35}
\]

Here we denote by \( \text{Ev}(A_\xi) (\xi = 0, \infty) \) the set of eigenvalues of \( A_\xi \). Therefore, we generally obtain rigidity index of the equation \( E_3 \) as

\[
\text{idx}(E_3) = 1^2 \times m + 1^2 \times m + 1^2 + (m-1)^2 - 1 \times m^2 = 2.
\]

In the next section, we study how these data are changed by \( q \)-middle convolution in detail.
4 Properties of $q$-middle convolution

In this section, we prove the three theorems.

**Theorem 1.8** (Fuchsian type equation) If equation $E_R$ is Fuchsian type equation, then $mc_\lambda(E_R)$ is also Fuchsian type equation.

**Theorem 1.10** (irreducibility) If $(\ast), (\ast\ast)$ are satisfied, then $\mathcal{V}$ is irreducible if and only if $mc_\lambda(\mathcal{V})$ is irreducible.

**Theorem 1.11** (rigidity index) If $(\ast), (\ast\ast)$ are satisfied, then $mc_\lambda$ preserves rigidity index of Fuchsian equation $E_R$.

About $(\ast), (\ast\ast)$, see Definition 1.9. Theorem 1.8 is proved easily by examining coefficient polynomial of canonical form of $c_\lambda(\tilde{E}_R)$. Although many preparations are necessary for us to prove Theorem 1.10, the outline is the same as method of Dettweiler and Reiter in [2]. Finally, Theorem 1.11 is proved by investigating in detail the change of spectral type of the equation $E_R$.

4.1 Proof of Theorem 1.8.

Here we prove the next theorem.

**Theorem 1.8** (Fuchsian type equation) If equation $E_R$ is Fuchsian type equation, then $mc_\lambda(E_R)$ is also Fuchsian type equation.

**Proof.** We put coefficients $A(x) = \sum_{k=0}^{N} A_k x^k (A_0, A_\infty \in \text{GL}_m(\mathbb{C}))$, $G(x) = \sum_{k=0}^{N} G_k x^k$ of canonical form of $E_{B,b}$, $E_{F,b}(F = c_\lambda(B))$. From the relations [15]:

$$A_0 = 1 - m - B_0, \quad A_\infty = b_\infty B_\infty, \quad B_0 = 1 - \sum_{i=1}^{N} B_i - B_\infty, \quad b_\infty = \prod_{i=1}^{N} (-b_i^{-1}) \neq 0,$$

we obtain $B_0 - 1, B_\infty \in \text{GL}_m(\mathbb{C})$. For any $v = t(v_0, \ldots, v_N) \in \ker F_\infty (v_k \in \mathcal{V})$, we get $G_\infty \in \text{GL}_{(N+1)m}(\mathbb{C})$ because

$$0 = G_\infty v = b_\infty F_\infty v = b_\infty t(B_\infty s), \ldots, t(B_\infty s)) (s = \sum_{i=0}^{N} B_i v_i).$$

Meanwhile, for any $v = t(v_0, \ldots, v_N) \in \ker G_0$, since

$$0 = G_0 v = (1_{(N+1)m} - F_0)v \neq (\sum_{i=1}^{N} F_i + F_\infty)v \neq (\sum_{i=1}^{N} F_i + 1_{(N+1)m} - \tilde{F})v,$$

we obtain $v = 0$. Hence $G_0 \in \text{GL}_{(N+1)m}(\mathbb{C})$. Therefore, $mc_\lambda(E_R)$ is a Fuchsian type equation. □
4.2 Proof of Theorem 1.10.

Here we derive a dimension formula of $q$-middle convolution. Moreover, we prove that $q$-middle convolution preserves irreducibility of the equation. The outline is the same as calculations of Dettweiler and Reiter in [2].

At first, linear spaces $K, L$ satisfy the next proposition.

**Proposition 4.1** $K, L$ are $F$-invariant subspaces of $V^{N+1}$.

**Proof.** (i) Let $J = \{1, \ldots, N\}$. For any \( v = (t v_0, \ldots, t v_N) \in K \) \((v_k \in \text{ker}B_k)\), we get

\[
F_j v = \sum_{i=0}^{j+1} (q^\lambda - 1)^j v_i, 0, \ldots, 0) \in K \quad (j \in J).
\]

Hence $F_j K$ is subspace of $K$. In the meantime, $F_{\infty} K$ is subspace of $K$ because for any $v \in K$, we obtain $F_{\infty} v = (1_{(N+1)m - \hat{F}}) v = v \in K$. Therefore, $K$ is $F$-invariant subspace of $V^{N+1}$.

(ii) Let

\[
1_{m,k} = \{\delta_{i,k+1}\delta_{j,k+1}1_m\}_{0 \leq i,j \leq N+1} = \text{diag}(0, \ldots, 1_m, \ldots, 0).
\]

For any $v \in L$, we get

\[
(\tilde{F} - (1 - q^\lambda)1_{(N+1)m}) F_j v = (\tilde{F} - (1 - q^\lambda)1_{(N+1)m}) 1_{m,j}(\tilde{F} - (1 - q^\lambda)1_{(N+1)m}) v = 0 \quad (j \in J).
\]

Hence $F_j L$ is subspace of $L$. Moreover, $F_{\infty} L$ is subspace of $L$ because for any $v \in L$, we obtain

\[
(\tilde{F} - (1 - q^\lambda)1_{(N+1)m}) F_{\infty} v = (\tilde{F} - (1 - q^\lambda)1_{(N+1)m})(1_{(N+1)m} - \tilde{F}) v = 0.
\]

Therefore, $L$ is $F$-invariant subspace of $V^{N+1}$. □

The next facts are important as “dimension formula”.

**Proposition 4.2**

(i) If $\lambda = 0$, then $K$ is subspace of $L$ and satisfies

\[
L = \{ t(v_0, \ldots, t v_N); \sum_{j=0}^{N} B_j v_j = 0 \}.
\]

(ii) If $\lambda \neq 0$, then $K \cap L = 0$, $L = \{ t(h, \ldots, t h); h \in \text{ker}(A_{\infty} - q^\lambda b_{\infty} 1_m) \}$ and

\[
\dim (mc_\lambda(V)) = (N + 1)m - \sum_{i=1}^{N} \dim \ker B_i - \dim \ker (A_0 - 1_m) - \dim \ker (A_{\infty} - q^\lambda b_{\infty} 1_m).
\]

**Proof.** (i) If $\lambda = 0$, then $L = \ker \tilde{F}$. Here for any $v \in K$, we obtain $\tilde{F} v = 0$. Hence $v \in L$. Moreover, we obtain $L = \{ t(v_0, \ldots, t v_N); \sum_{j=0}^{N} B_j v_j = 0 \}$.

(ii) If $\lambda \neq 0$, for any $v \in K \cap L$, we obtain

\[
0 = (\tilde{F} - (1 - q^\lambda)1_{(N+1)m}) v = \tilde{F} v - (1 - q^\lambda) v = (q^\lambda - 1) v.
\]
Hence we get $v = 0$. For any $v = t(tv_0, \ldots , tv_N) \in \mathcal{L}$, we obtain $\hat{F}v = (1 - q^\lambda)v$. Consequently, we see $\sum_{j=0}^N B_jv_j = (1 - q^\lambda)v_i (i \in I = \{0, \ldots , N\})$. Here $v_0 = \cdots = v_N$ and
\[
\mathcal{L} = \{t(t^ih, \ldots , t^ih) ; h \in \text{ker}(A_\infty - q^\lambda b_\infty 1_m)\}.
\]
Therefore, we can compute $\dim(\text{mc}_\lambda(V))$:
\[
\dim(\text{mc}_\lambda(V)) = \dim(V^{N+1}/(K + \mathcal{L})) = \dim(V^{N+1}) - \dim(K + \mathcal{L}) = \dim(V^{N+1}) - \dim K - \dim \mathcal{L} \quad (: \; K \cap \mathcal{L} = 0)
= (N + 1)m - \sum_{i=0}^N \dim \text{ker } B_i = \dim \text{ker}(B_\infty - q^\lambda 1_m)
= (N + 1)m - \sum_{i=1}^N \dim \text{ker } B_i - \dim \text{ker}(A_0 - 1_m) - \dim \text{ker}(A_\infty - q^\lambda b_\infty 1_m). \quad \Box
\]

**Proposition 4.3** If $W$ is $B$-invariant subspace of $V$, then $W^{N+1}$ is $F$-invariant subspace. Moreover, $\text{mc}_\lambda(W)$ is submodule of $\text{mc}_\lambda(V)$.

**Proof.** For any $w = t(tw_0, \ldots , tw_N) \in W^{N+1}$ and $j \in J = \{1, \ldots , N\}$, it is clear that
\[
F_jw = t(0, \ldots , \sum_{i=0}^N t(B_iw_i) - (1 - q^\lambda)t^i w_j, \ldots , 0) \in W^{N+1}.
\]
Since $F_\infty w = (1_{(N+1)m} - \hat{F})w = w - \hat{F}w \in W^{N+1}$, $W^{N+1}$ is $F$-invariant subspace of $V^{N+1}$. The second claim follows from
\[
W^{N+1} \cap (K_V + \mathcal{L}_V) = K_W + \mathcal{L}_W. \quad (37)
\]
Hence we prove (37). If $\lambda = 0$, $K$ is subspace of $\mathcal{L}$. If $\lambda \neq 0$, then
\[
K_W + \mathcal{L}_W = K_W \cap W + \mathcal{L}_W W
\]
is subspace of $W^{N+1} \cap (K_V + \mathcal{L}_V)$. Moreover, for any $w = t(tw_0, \ldots , tw_N) \in W^{N+1} \cap (K_V + \mathcal{L}_V)$ and $i \in I = \{0, \ldots , N\}$, we can let
\[
w_i = k_i + h (k_i \in \text{ker } B_i, h \in \text{ker}(A_\infty - q^\lambda b_\infty 1_m)).
\]
Here we obtain $W \ni \sum_{i=0}^N B_iw_i = \sum_{i=0}^N B_i(k_i + h) = (1 - q^\lambda)h$. Consequently, $h \in W$. Moreover, we find $w \in K_W + \mathcal{L}_W$ from $k_i = w_i - h \in W$. Therefore, $W^{N+1} \cap (K_V + \mathcal{L}_V)$ is subspace of $K_W + \mathcal{L}_W$. \Box

From now on, we assume the conditions $(*), (**).$ Here we can prove

**Proposition 4.4** If $(**)$ is satisfied, then $\text{mc}_0(V) \simeq V$. 

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Proof. If $\lambda = 0$, then we get $\mathcal{K} + \mathcal{L} = \mathcal{L} = \{t(t^0, \ldots, t^N) ; \sum_{j=0}^{N} B_j v_j = 0\}$. Let
\[
\phi : t(t^0, \ldots, t^N) \mapsto \sum_{j=0}^{N} B_j v_j.
\]
Then $\phi : \psi^m \rightarrow \psi$ is surjection from a condition (**) For any $v = t(t^0, \ldots, t^N) \in \psi^{N+1}$, we get
\[
\phi \circ F_j (v) = \phi(t(0, \ldots, t^s, \ldots, 0)) = B_j s = (B_j \circ \phi)(v),\ s = \sum_{i=0}^{N} B_i v_i (j \in J = \{1, \ldots, N\}),
\]
\[
(\phi \circ F_{\infty})(v) = (\phi \circ (1_{(N+1)m} - \tilde{F}))(v) = \phi(t(t^0 - t^s, \ldots, t^N - t^s)) = B_{\infty} s = (B_{\infty} \circ \phi)(v).
\]
Therefore, we obtain
\[
\psi = \text{im}(\phi) \simeq \psi^{N+1} / \ker(\phi) = \psi^{N+1} / (\mathcal{K} + \mathcal{L}) = mc_0(\psi). \Box
\]

Here we introduce a transformation $\psi_{m}$ in expedient.

**Definition 4.5** For $T = (T_1, \ldots, T_N, T_{\infty}) \in (M_{(N+1)m}(\mathbb{C}))^{N+1}$, we define
\[
\psi_{m} : (M_{(N+1)m}(\mathbb{C}))^{N+1} \longrightarrow (M_{(N+1)m}(\mathbb{C}))^{N+1}, \quad (T_1, \ldots, T_N, T_{\infty}) \mapsto (T_1, \ldots, T_N, T_{\infty} + \mu_{1(N+1)m}).
\]

We set the module $\psi_{m}(\psi) = (\psi_{m}(T), \psi)$.

Here $\psi_{m}$ preserves irreducibility of equations clearly. Moreover, we introduce a transformation $\psi_{\lambda}$.

**Definition 4.6** We define $\psi_{\lambda} : \psi \longrightarrow \psi$,
\[
\psi_{\lambda} = \psi_{1-q_{\lambda}} \circ c_{\lambda}.
\]

Let $\tilde{F} = \psi_{\lambda}(B)$, $\psi_{\lambda}(\psi) = (\tilde{F}, \psi^{N+1})$. We let $\tilde{F}_k$ be a matrix induced by the action of $F_k$ on $\psi^{N+1}/(\mathcal{K} + \mathcal{L})$. Moreover, we define $\psi_{\lambda} : \psi \longrightarrow \psi$,
\[
\psi_{\lambda}(B) = \tilde{F}, \quad \psi_{\lambda}(\psi) = \psi^{N+1}/(\mathcal{K} + \mathcal{L}) = (\tilde{F}, \psi^{N+1}/(\mathcal{K} + \mathcal{L})).
\]

Here the following facts are proved in the same way as above.

**Proposition 4.7** $\mathcal{K}, \mathcal{L}$ are $\tilde{F}$-invariant.

**Proposition 4.8** If $\mathcal{W}$ is $B$-invariant subspace of $\psi$, then $\mathcal{W}^{N+1}$ is $\tilde{F}$-invariant subspace. Moreover, $\psi_{\lambda}(\mathcal{W})$ is submodule of $\psi_{\lambda}(\psi)$.

From $\psi_0 = \text{id}_{\psi^{N+1}}, mc_0 = \psi_0$, the next proposition is obvious.

**Proposition 4.9** If (**) is satisfied, then $\psi_{\lambda}(\psi) \simeq \psi$.

Proof of the Proposition 4.10, 4.11 are similar to Dettweiler and Reiter’s paper [2].
Proposition 4.10  If (*), (**) are satisfied, then for any $\lambda, \mu \in \mathbb{C}$, \( \overline{\Psi}_\mu \circ \overline{\Psi}_\lambda(\mathcal{V}) \simeq \overline{\Psi}_{\mu}(\mathcal{V}^{N+1})/\overline{\Psi}_\mu(\mathcal{K}_V + \mathcal{L}_V(\lambda)) \).

Proof. If $\mu = 0$, it is easily seen that
\[
\overline{\Psi}_0 \circ \overline{\Psi}_\lambda(\mathcal{V}) \simeq \overline{\Psi}_\lambda(\mathcal{V}) = \mathcal{V}^{N+1}/(\mathcal{K}_V + \mathcal{L}_V(\lambda)) \simeq \overline{\Psi}_0(\mathcal{V}^{N+1})/\overline{\Psi}_0(\mathcal{K}_V + \mathcal{L}_V(\lambda)).
\]
Here we assume $\mu \neq 0$. We set
\[
\lambda' = q^\lambda - 1, \quad \mu' = q^\mu - 1, \quad K_1 = \mathcal{K}_V, \quad \mathcal{L}_1 = \mathcal{L}_V(\lambda), \quad K_2 = \mathcal{K}_V^{N+1}, \quad \mathcal{L}_2 = \mathcal{L}_V^{N+1}(\mu),
\]
\[
\tilde{F} = \overline{\Psi}_\lambda(B), \quad \tilde{F} = \overline{\Psi}_\lambda(B), \quad \mathcal{M} = \overline{\Psi}_\lambda(\mathcal{V}), \quad \mathcal{H} = K_1 + \mathcal{L}_1. \tag{41}
\]
Let us first prove
\[
(i) \quad \mathcal{K}_\mathcal{M} = (K_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1}, \quad (ii) \quad \mathcal{L}_\mathcal{M} = (\mathcal{L}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1}. \tag{43}
\]
(i) We set $\tilde{F}_0 = 1_m - \sum_{i=1}^N \tilde{F}_i - \tilde{F}_\infty$. For any $k + \mathcal{H}^{N+1} = t^i(k_0, \ldots, k_N) + \mathcal{H}^{N+1} \in (K_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1}$, we obtain $k + \mathcal{H}^{N+1} \in \mathcal{K}_\mathcal{M}$ from $\tilde{F}_i(k_i + \mathcal{H}) = \mathcal{H}(i \in I = \{0, \ldots, N\})$. Therefore, $(K_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1}$ is subspace of $\mathcal{K}_\mathcal{M}$. On the other hand, for any $v + \mathcal{H}^{N+1} = t^i(v_0, \ldots, v_N) + \mathcal{H}^{N+1} \in \mathcal{K}_\mathcal{M}$, $v_i = t^i(v_{i0}, \ldots, v_{iN}) (v_{ij} \in \mathcal{V})$, we compute $\tilde{F}_0 v_0$:
\[
\tilde{F}_0 v_0 = (1_m - \sum_{i=1}^N \tilde{F}_i - \tilde{F}_\infty) v_0 = (\tilde{F} - \sum_{i=1}^N \tilde{F}_i) v_0 = t^i(\sum_{j=0}^N t(B_j v_{ij}), -\lambda t v_{i0}, \ldots, -\lambda t v_{iN})
\]
and we find
\[
\tilde{F}_j v_j = t^j(0, \ldots, \sum_{i=0}^N t(B_i v_{ji}) + \lambda t v_{jj}, \ldots, 0) (j \in J = \{1, \ldots, N\}).
\]
(i-1) If $\lambda = 0$, then it is clear that $\tilde{F}_i v_i = t^i(0, \ldots, \sum_{j=0}^N t(B_j v_{ij}), \ldots, 0) (i \in I)$. Moreover, $\tilde{F}_i v_i \in \mathcal{H} = \mathcal{K} + \mathcal{L} = \mathcal{L} = \{t^i(w_0, \ldots, w_N); \sum_{j=0}^N B_j w_j = 0\}$ and $B_i \sum_{j=0}^N B_j v_{ij} = 0$. Hence we get
\[
\tilde{F}_i v_i \in t^i(0, \ldots, \ker B_i, \ldots, 0).
\]
Therefore, we obtain $v_i \in \ker \tilde{F}_i + \mathcal{K}_1$.

(i-2) If $\lambda \neq 0$, then
\[
\tilde{F}_i v_i = (t^i k_0 + t^i h_i, \ldots, t^i k_i + t^i h_i) (k_{ij} \in \ker B_j, h_i \in \ker (A_\infty - b_\infty q^\lambda 1_m), i \in I).
\]
If $i \neq 0$, we get $h_i = -k_{ij} \in \ker B_j (j \in I \setminus \{i\})$. Hence we see $h_i \in \ker (B_i + \lambda 1_m)$ from $h_i \in \ker (A_\infty - b_\infty q^\lambda 1_m) = \ker (\sum_{r=0}^N B_r + \lambda 1_m)$. Since (**) is satisfied, we get $h_i = 0$. Here
\[
\tilde{F}_i v_i \in t^i(0, \ldots, \ker B_i, \ldots, 0).
\]
If $i = 0$, then it results in the case $i \neq 0$ because
\[
F_0 = 1_{(N+1)m} - \sum_{r=1}^N F_r + \lambda 1_{(N+1)m} - F_\infty = \left(\begin{array}{ccc}
B_0 + \lambda 1_m & \cdots & B_N \\
0 & \cdots & 0
\end{array}\right). \tag{44}
\]

Hence we find \( v + \mathcal{H}^{N+1} \in (\mathcal{K}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1} \). Moreover, \( \mathcal{K}_M \) is a subspace of \( (\mathcal{K}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1} \). Therefore, we obtain \( \mathcal{K}_M = (\mathcal{K}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1} \).

(ii) For any

\[
v + \mathcal{H}^{N+1} = t(t^h, \ldots, t^h) + \mathcal{H}^{N+1} \in (\mathcal{L}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1} (h \in \ker(\tilde{F}_\infty - \mu 1_{(N+1)m})),
\]

we let \( \tilde{H} = (F_{t^{-1}})_{1 \leq s, t \leq N+1}, \tilde{H} = (F_{t^{-1}})_{1 \leq s, t \leq N+1} \). Then we obtain

\[
(\tilde{H} + \mu 1_{(N+1)^2})(v + \mathcal{H}^{N+1}) = (\tilde{H} + \mu 1_{(N+1)^2})v + \mathcal{H}^{N+1} = \mathcal{H}^{N+1}.
\]

Consequently, we find \( v + \mathcal{H}^{N+1} \in \mathcal{L}_M \). Meanwhile, for any

\[
v + \mathcal{H}^{N+1} = t(t^h, \ldots, t^h) + \mathcal{H}^{N+1} \in \mathcal{L}_M (h \in \ker(\tilde{F}_\infty - \mu 1_{(N+1)m})),
\]

we see \( v + \mathcal{H}^{N+1} \in (\mathcal{L}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1} \). Therefore, we obtain \( \mathcal{L}_M = (\mathcal{L}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1} \).

Let us remember the isomorphism theorems. For a linear space \( V \) and subspaces \( W, W' \) of \( V \),

(iii) if \( W' \subset W \), then \( (V/W')/(W/W') \simeq V/W \);

(iv) \( W'/(W \cap W') \simeq (W + W')/W \).

From the above, we can compute \( mc_\mu \circ mc_\lambda (V) \):

\[
mc_\mu \circ mc_\lambda (V) = mc_\mu (V^{N+1}/\mathcal{H})
= (V^{N+1}/\mathcal{H})^{N+1}/(\mathcal{K}_M + \mathcal{L}_M)
= (V^{(N+1)^2}/\mathcal{H}^{N+1})/((\mathcal{K}_2 + \mathcal{L}_2 + \mathcal{H}^{N+1})/\mathcal{H}^{N+1}) \quad (\because (i), (ii))
\simeq (V^{(N+1)^2}/(\mathcal{K}_2 + \mathcal{L}_2))/((\mathcal{K}_2 + \mathcal{L}_2 + \mathcal{H}^{N+1})/(\mathcal{K}_2 + \mathcal{L}_2)) \quad (\because (iii))
\simeq mc_\mu (V^{N+1})/(\mathcal{H}^{N+1})/((\mathcal{K}_2 + \mathcal{L}_2) \cap \mathcal{H}^{N+1}) \quad (\because (iv))
= mc_\mu (V^{N+1})/mc_\mu (\mathcal{K}_V + \mathcal{L}_V(\lambda)).
\]

**Proposition 4.11** \( mc_\lambda \) preserves conditions \((*)\), \((***)\).

**Proof.** It is sufficient to prove that \( \overline{\mathcal{V}}_\lambda \) preserves conditions \((*)\), \((***)\). In the case \( \lambda = 0 \) is obvious because of Proposition 4.9. Hence we assume \( \lambda \neq 0 \) and \( \mathcal{V} \) satisfy \((*)\), \((***)\). Here we use notations in proof of previous proposition. If \( \tau = 0 \), for any \( v + \mathcal{H} = t(t^v_0, \ldots, t^v_N) + \mathcal{H} \in \bigcap_{i=0}^N \ker \widetilde{F}_i \), it is clear that \( \widetilde{F}_0 v \in \mathcal{H} \). Here we get \( v \in \mathcal{H} \) from Proposition 4.10 (i-2). Consequently, we obtain \( \bigcap_{i=0}^N \ker \widetilde{F}_i = \{ \mathcal{H} \} \).

If \( \tau \neq 0 \), for any \( v + \mathcal{H} \in \bigcap_{j \neq i} \ker \widetilde{F}_j \cap (\widetilde{F}_i + \tau 1_{(N+1)m}) \) \( (i \in J = \{1, \ldots, N\}) \), we get \( v \in \mathcal{H} \) from \( \widetilde{F}_0 v \in \mathcal{H} \). Hence we obtain \( \bigcap_{j \neq i} \ker \widetilde{F}_j \cap (\widetilde{F}_i + \tau 1_{(N+1)m}) = \{ \mathcal{H} \} \). The case \( i = 0 \) is reduced to the case \( i \in J \). Therefore, \( \overline{\mathcal{V}}_\lambda (\mathcal{V}) \) satisfies \((*)\).

In the meantime, we put any \( \tau \in \mathcal{C} \) and \( v = t(t^v_0, \ldots, t^v_N) \in \mathcal{V}^{N+1} \). If \( i \in J \), then

\[
\widetilde{F}_i v = t(0, \ldots, \sum_{j=0}^{i-1} t(B_j v_j) + \lambda^i v_i, \ldots, 0).
\]
Hence $\tilde{F}_i v$ spans the linear space $t(0, \ldots, V, \ldots, 0)$. Moreover, it is clear that
\[(\tilde{F}_0 + \tau 1_{(N+1)m})v = t\left(\sum_{j=0}^{N} t(B_j v_j) + (\lambda + \tau)^t v_0, \tau^t v_1, \ldots, \tau^t v_N\right)\).
Consequently, $\sum_{j=0}^{N} B_j v_j + (\lambda + \tau) v_0$ spans $\mathcal{V}$. Here the case $i = 0$ is reduced to the case $i \in J$. Therefore, we obtain $\sum_{j\neq i} \text{im} F_j + \text{im}(\tilde{F}_i + \tau 1_{(N+1)m}) = \mathcal{V}^{N+1} + \mathcal{H}(i \in J)$. From the above, $\Psi_{\lambda}(\mathcal{V})$ satisfies (**). □

Here the $\Psi_{\lambda}$ satisfies the next proposition.

**Proposition 4.12** If (**), (***) are satisfied, then for any $\lambda, \mu \in \mathbb{C}$, $\Psi_{\mu} \circ \Psi_{\lambda}(\mathcal{V}) \simeq \Psi_{\log_{q}(q^\lambda + q^\mu - 1)}(\mathcal{V})$.

**Proof.** If $\lambda \mu = 0$, it is obvious. We assume $\lambda \mu \neq 0$ and set
\[
\tilde{F} = \Psi_{\lambda}(B), \quad F' = \Psi_{\log_{q}(q^\lambda + q^\mu - 1)}(B), \quad H = \Psi_{\mu}(\tilde{F}), \quad K_1 = K_{\mathcal{V}}, \quad L_1 = L_{\mathcal{V}}(\lambda),
\]
\[
K_2 = (K_{\mathcal{V}}, \tilde{F}), \quad L_2 = (L_{\mathcal{V}}(N+1,m), \tilde{F}'), \quad L' = L_{\mathcal{V}}(1) = V, \quad \mathcal{H} = K_1 + L_2.
\]
Here we prove that induced mapping $\phi : \Psi_{\mu} \circ \Psi_{\lambda} \longrightarrow \Psi_{\log_{q}(q^\lambda + q^\mu - 1)}(\mathcal{V})$ is isomorphism from
\[
\phi = \Psi_{\mu} \circ \Psi_{\lambda}(V) \longrightarrow \Psi_{\log_{q}(q^\lambda + q^\mu - 1)}(\mathcal{V}) \left(t(t v_0, \ldots, t v_N) \longrightarrow \sum_{i=0}^{N} \tilde{F}_i v_i\right).
\]
We first find
\[
\Psi_{\mu} \circ \Psi_{\lambda}(\mathcal{V}) \simeq \Psi_{\mu}(\mathcal{V}^{N+1})/\Psi_{\mu}(K_{\mathcal{V}} + L_{\mathcal{V}}(\lambda)) \simeq \mathcal{V}^{(N+1)^2}/(K_2 + L_2 + H)^{N+1}.
\]
It is easy to check that $(L_1)^{N+1}$ is subspace of $K_2 = \ker(\phi)$. Moreover, we get $\phi((K_1)^{N+1}) = \sum_{i=0}^{N} \tilde{F}_i K_1 = K_1$ and $L_2 = \{t(t h, \ldots, t h); h \in \ker(\tilde{F}_{\infty} - q^\mu 1_{(N+1)m})\}$. Hence we obtain
\[
\phi(L_2) = \sum_{i=0}^{N} \tilde{F}_i \ker F'_{\infty} = \left(\sum_{i=0}^{N} \tilde{F}_i\right) \ker F'_{\infty} = \ker F'_{\infty} = L' = F'_{\infty} = F_{\infty} - q^\mu 1_{(N+1)m}.
\]
Here we compute $\dim(L_2)$:
\[
\dim(K_2) = \sum_{i=0}^{N} \dim \ker \tilde{F}_i = \sum_{i=0}^{N} \dim(V^{N+1}) - \dim(\ker F_i) = \sum_{i=0}^{N} ((N+1)m - m) = N(N+1)m.
\]
Consequently, we can calculate $\dim(\Psi_{\mu} \circ \Psi_{\lambda}(\mathcal{V}))$:
\[
\dim(\Psi_{\mu} \circ \Psi_{\lambda}(\mathcal{V})) = \dim(V^{(N+1)^2}/(K_2 + L_2 + H)^{N+1})
\]
\[
= \dim(V^{(N+1)^2}) - \dim(K_2 + L_2 + (K_1)^{N+1} + (L_1)^{N+1})
\]
\[
= (N+1)^2 m - \dim(K_2 + L_2 + (K_1)^{N+1})
\]
\[
= (N+1)^2 m - \dim(K_2) - \dim(L_2 + (K_1)^{N+1})
\]
\[
= (N+1)^2 m - N(N+1)m + \dim(K_1 + L')
\]
\[
= (N+1)m - \dim(K_1 + L')
\]
\[
= \dim(V^{N+1}) - \dim(K_1 + L')
\]
\[
= \dim(V^{N+1}/(K_1 + L')).
\]
Here we set \( \lambda' = q^\lambda - 1, \mu' = q^\mu - 1 \). For any
\[
v = t^i(v_0, \ldots, t^jv_N) \in V^{(N+1)^2}, \quad (v_j = t^i(v_{j0}, \ldots, t^jv_{jN}), v_{ij} \in V),
\]
we get the following relations.
\[
(F_i' \circ \phi)(v) = t^i(0, \ldots, t^jw_i, \ldots, 0) = (\phi \circ H_i)(v) (i \in \{0, \ldots, N\}),
\]
\[
w_i = \sum_{j=0}^N B_j \left\{ \sum_{k=0}^N B_kv_{jk} + \lambda'B_jv_{jj} + (\lambda' + \mu')v_{ij} \right\} + \lambda'(\lambda' + \mu')v_{ii},
\]
\[
F'_\infty \circ \phi = \phi - \sum_{i=0}^N (F_i' \circ \phi) = \phi - \sum_{i=0}^N (\phi \circ H_i) = \phi \circ H_\infty.
\]
Therefore, we obtain \( \overline{\Psi}_\mu \circ \overline{\Psi}_\lambda(V) \simeq \overline{\Psi}_{\log(q^\lambda - q^\mu - 1)}(V) \).

From the above, Theorem 1.10 is shown.

**Theorem 1.10 (irreducibility)** If (*) (** are satisfied, then \( V \) is irreducible if and only if \( mc_\lambda(V) \) is irreducible.

**Proof.** For any non-zero irreducible module \( V \) and \( \lambda \in \mathbb{C} \), we put \( M = \overline{\Psi}_\lambda(V) \) and non-zero submodule \( M' \) of \( M \). Here \( W = \overline{\Psi}_{\log(1-q^\lambda)}(M') \) is submodule of
\[
\overline{\Psi}_{\log(1-q^\lambda)}(M) = (\overline{\Psi}_{\log(1-q^\lambda)} \circ \overline{\Psi}_\lambda)(V) \simeq \overline{\Psi}_0(V) = mc_\lambda(V) \simeq V.
\]
Hence we obtain \( W = 0 \) or \( V \). If \( W = 0 \), then we get \( M' \simeq \overline{\Psi}_\lambda(W) = \overline{\Psi}_\lambda(0) = 0 \). This is a contradiction. Consequently, we find \( W = V \). Moreover, we get
\[
M' = \overline{\Psi}_\lambda(W) = \overline{\Psi}_\lambda(V) = M.
\]
Hence \( W \) is irreducible module. Here \( \overline{\Psi}_\lambda(V) \) is irreducible if and only if \( mc_\lambda(V) \) is irreducible. Therefore, \( V \) is irreducible if and only if \( mc_\lambda(V) \) is irreducible. The proof of the theorem has been completed.

**4.3 Proof of Theorem 1.11.**

In this section, we prove that \( mc_\lambda \) preserves rigidity index of equation \( E_R \). At first, we examine the change of spectral types \( S_0, S_\infty \).

**Lemma 4.13** We set coefficient polynomial \( A(x) = \sum_{k=0}^N A_kx^k \) (resp. \( G(x) = \sum_{k=0}^N G_kx^k \)) of canonical form of \( E_{B,b} \) (resp. \( E_{F,b} \)), we let \( \text{Ev}(M) \) be the set of eigenvalues of \( M \in M_n(\mathbb{C}) \). If
\[
A_0 \sim \bigoplus_{\theta \in \text{Ev}(A_0)} J(\theta, t_{0,j}^0), \quad A_\infty \sim \bigoplus_{\kappa \in \text{Ev}(A_\infty)} J(\kappa, t_{j,\kappa}^\infty),
\]
and (**) is satisfied, then we obtain

\[
G_0 \sim \bigoplus_{\theta \in \text{Ev}(A_0) \setminus \{q^\lambda\}} s_0^0 \bigoplus_{j=1} J(\theta, t_{\theta, j}^0) \bigoplus_{j=1} J(\theta^\lambda, t_{\theta^\lambda, j}^0 + 1) \bigoplus J(q^\lambda, 1)^{(N \cdot s_0^0)} \\
G_\infty \sim \bigoplus_{\kappa \in \text{Ev}(A_\infty) \setminus \{b_\infty\}} s_\infty^\kappa \bigoplus_{j=1} J(\kappa, t_{\kappa, j}^\infty) \bigoplus_{j=1} J(\kappa^\infty, t_{\kappa^\infty, j}^\infty + 1) \bigoplus J(\kappa^\infty, 1)^{(N \cdot s_\infty^\kappa)}.
\]

**Proof.** It is easily seen that \( G_0 = 1_m - F_0, \ G_\infty = b_\infty F_\infty, \ F_0 = 1_m - \sum_{i=1}^N F_i - F_\infty \) and

\[
\theta 1_{(N+1)m} - G_0 = \begin{pmatrix}
(\theta 1_m - A_0 & B_1 & \cdots & B_N \\
(\theta - q^\lambda)1_m & \ddots & \sin & \sin \\
& & (\theta - q^\lambda)1_m
\end{pmatrix} \quad (\theta \in \mathbb{C}). \quad (49)
\]

(i) If \( \theta \neq q^\lambda \), then \( \dim \ker((\theta 1_{(N+1)m} - G_0)^n) = \dim \ker((\theta 1_m - A_0)^n) \) \( (n \in \mathbb{Z}_{>0}) \).

(ii) If \( \theta = q^\lambda \), then for any \( v = t(v_0, \ldots, v_N) \in V^{N+1} \) \( (v_i \in V) \), we get

\[
(\theta 1_{(N+1)m} - G_0)v = t(v', 0, \ldots, 0), \quad v' = \sum_{k=0}^N B_k v_k + (\theta - 1)1_m.
\]

Here \( v' \) spans \( V \) because condition (**) Hence we obtain

\[
\dim \ker(\theta 1_{(N+1)m} - G_0) = Nm, \quad \dim \ker((\theta 1_{(N+1)m} - G_0)^{n+1}) = \dim \ker((\theta 1_m - A_0)^n) \) \( (n \in \mathbb{Z}_{>0}) \).

(iii) If \( \kappa \neq b_\infty \), then for any \( v = t(v_0, \ldots, v_N) \in \ker((\kappa 1_{(N+1)m} - G_\infty)^n) \) \( (v_i \in V, n \in \mathbb{Z}_{>0}) \), we get

\[
0 = (\kappa 1_{(N+1)m} - G_\infty)^n v = \{ (\kappa - b_\infty)1_{(N+1)m} + b_\infty \hat{F} \}^n v = (\kappa - b_\infty)^n v + P \hat{F} v \quad (P \in \mathbb{M}_m(\mathbb{C})).
\]

Hence we find \( v_0 = \cdots = v_N \). Moreover, it is clear that

\[
(\kappa 1_{(N+1)m} - G_\infty)^n v = \{ (\kappa - b_\infty)1_{(N+1)m} + b_\infty \hat{F} \}^n v = t(v', \ldots, v'), \quad v' = (\kappa 1_m - A_\infty)^n v_0.
\]

Therefore, we obtain \( \dim \ker((\kappa 1_{(N+1)m} - G_\infty)^n) = \dim \ker((\kappa 1_m - A_\infty)^n) \).

(iv) If \( \kappa = b_\infty \), then we obtain

\[
\dim \ker(\kappa 1_{(N+1)m} - G_\infty) = \dim \ker \hat{F} = (N + 1)m - \dim \text{im} \hat{F} = (N + 1)m - m = Nm
\]

from \( \kappa 1_{(N+1)m} - G_\infty = b_\infty \hat{F} \) and (**) Here for any

\[
v = t(v_0, \ldots, v_N) \in \ker((\kappa 1_{(N+1)m} - G_\infty)^{n+1}) \quad (v_i \in V, n \in \mathbb{Z}_{>0}),
\]

it is easily seen that

\[
(\kappa 1_{(N+1)m} - G_\infty)v = b_\infty \hat{F} v = t(v', \ldots, v') \quad v' = b_\infty \sum_{k=0}^N B_k v_k
\]

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and

$$(\kappa_1(N+1)m - G_\infty)^{N+1}v = t(tw, \ldots, tw), \ w = (\kappa_1m - A_\infty)^n v'.$$

Therefore, we obtain $\dim \ker((\kappa_1(N+1)m - G_\infty)^{N+1}) = \dim \ker((\kappa_1m - A_\infty)^n)$. □

We prepare for examining changes of spectral type $S_{\text{div}}$.

**Lemma 4.14** We can reduce $G(x)$ to $\tilde{G}(x)$:

$$\tilde{G}(x) = \begin{pmatrix} T(x)1_m & \cdots & T(x)1_m \\ V_1(x) & \cdots & V_N(x) & A(q^{-\lambda}x) \end{pmatrix}$$  \hspace{1cm} (50)

by elementary matrices. Here $V_i(x) (i = 1, \ldots, N)$ are polynomials and $T(x) = \prod_{k=1}^N (1 - \frac{x}{b_k})$. 

**Proof.** For any $\lambda \in \mathbb{C}, k \in J = \{1, \ldots, N\}, b_k \in \mathbb{C} \setminus \{0\}$, let $s_k = 1 - \frac{x}{b_k}, \ s'_k = 1 - \frac{x}{q^{-\lambda}b_k}, \ T_k = \frac{T(x)}{s_k}, \ b_{i,j} = 1 - \frac{b_i}{b_j}$. It is clear that

$$G(x) = T(x)F(x)$$ \hspace{1cm} (51)

$$= \left( \prod_{k=1}^N s_k \right) \cdot \left( F_\infty + \sum_{l=1}^N \frac{F_l}{s_l} \right)$$ \hspace{1cm} (52)

$$= T(x)1_m \oplus \bigoplus_{k=1}^N q^{\lambda}s_k T_k(x)1_m + \left( -T(x)1_m \oplus \bigoplus_{k=1}^N \frac{xT_k(x)}{b_k} \right) \begin{pmatrix} 1_m \\ \vdots \\ 1_m \end{pmatrix} \begin{pmatrix} B_0 & \cdots & B_n \end{pmatrix}. \hspace{1cm} (53)$$

Here we row reduce $G(x)$ by the elementary matrix

$$\begin{pmatrix} (1 - s_1)1_m & s_11_m \\ -1_m & 1_m \\ \vdots & \vdots \\ -1_m & 1_m \end{pmatrix}.$$ \hspace{1cm} (54)

Next, we column reduce by the elementary matrix

$$\begin{pmatrix} 1_m \\ 1_m \\ \vdots \\ 1_m \end{pmatrix}.$$ \hspace{1cm} (55)

Then we obtain

$$q^\lambda \begin{pmatrix} T_1m & s'_1T_1m \\ s'_1T_1m & \cdots \end{pmatrix} + \begin{pmatrix} O_m \\ T_11_m \\ \vdots \\ T_N1_m \end{pmatrix} (q^\lambda1_m - B_\infty B_1 \cdots B_n).$$ \hspace{1cm} (56)
We set
\[ f_{i,j} = (-1)^{i+j} b_i b_j^{-1} b_{i+1,j}^{-1} \prod_{k=1}^{j-1} (b_{j,k}^{-1} b_{i,k}) \cdot \prod_{k=j+1}^{i-1} (b_k b_j^{-1} b_{k,j}^{-1} b_{i,k}) \quad (b_{N+1,j} = 1), \] (57)

\[ g_i = - \prod_{k=1}^{i} b_i^{-1} b_{i+1,k} \cdot \prod_{k=1}^{i-1} b_{i,k} (\neq 0) \] (58)

and \( C_0 = (C_{i,j}^0)_{1 \leq i,j \leq N+1} \in M_{(N+1)m}(\mathbb{C}) \) \((C_{i,j}^1 \in M_m(\mathbb{C}))\) as
\[
C_{i,j}^0 = \begin{cases} 
1_m & (i = j = 1) \\
F_{i-1,j-1} s_i^{-1} 1_m & (2 \leq i,j \leq N, i \geq j) \\
g_{i-1} s_i 1_m & (2 \leq i = j - 1 \leq N) \cdot \\
F_{N,j-1} 1_m & (i = N + 1, j \neq 1) \\
O_m & \text{(otherwise)}
\end{cases} \] (59)

Here \( C_0 \) is an elementary matrix. Let \( j \in \{1, \ldots, N\}, I_j = \{1, \ldots, j\} \). We prove

(i) \( \sum_{k=1}^{l} f_{l,k} = -g_l \quad (l \in I_{N-1}) \),

(ii) \( \sum_{k=1}^{N} f_{N,k} T_k(x) = t_0 \quad \left( t_0 = \prod_{k=1}^{N-1} b_{N,k} \neq 0 \right) \).

(i) It is clear that \( \sum_{k=1}^{l} f_{l,k} (b_{l+1}) \equiv 0 \pmod{g_l(b_{l+1})} \). Here we set \( f(b_{l+1}) = - (g_l)^{-1} \sum_{k=1}^{l} f_{l,k} \), then we find \( \deg f(b_{l+1}) \leq l - 1 \) and \( f(b_s) = 1 \) \((s \in I_l)\). Therefore, for any \( b_{l+1} \in \mathbb{C} \), we obtain \( f(b_{l+1}) = 1 \).

(ii) Let \( g(x) = \sum_{k=1}^{N} f_{N,k} T_k(x) \), then we find \( \deg g(x) \leq N - 1 \) and \( g(b_s) = t_0 \) \((s \in I_N)\). Therefore, \( g(x) = t_0 \). Hence we get
\[
C_0 \begin{pmatrix} O_m \\ T_1 1_m \\ \vdots \\ T_N 1_m \end{pmatrix} = \begin{pmatrix} O_m \\ \vdots \\ O_m \end{pmatrix}. \] (60)

Here let us reduce
\[
q^\lambda C_0 \begin{pmatrix} T_1 1_m & s'_1 T_1 1_m \\ \vdots \\ s'_N T_N 1_m \end{pmatrix}. \] (61)
We set $U_{i,j}(p) := (p \delta_{i,j} t_{1,m})_{1 \leq s, t \leq N+1} \in M_{(N+1)m}(\mathbb{C}) (p \in \mathbb{C})$ and

$$h_{i,j} = g_j^{-1} \sum_{k=1}^j f_{i,k},$$

$$C_l = 1_{(N+1)m} + \sum_{k=l+2}^N U_{k,l+1}(h_{k-1,l}) \quad (1 \leq l \leq N - 2),$$

$$C_{N-1} = 1_m \oplus (-g_1^{-1})_m \oplus \cdots \oplus (-g_{N-1}^{-1})_m \oplus 1_m,$$

$$C = C_{N-1} C_{N-2} \cdots C_1 C_0.$$  

Then we obtain

$$\begin{pmatrix} T_{1m} & s'_1 T_{1m} \\ s'_1 T_{1m} & \ddots & s'_N T_{N1m} \end{pmatrix} = \begin{pmatrix} T_{1m} & s'_1 T_{1m} & -s'_2 T_{1m} & \cdots & -s'_N T_{1m} \\ s'_1 T_{1m} & s'_2 T_{1m} & \ddots & \ddots \iddots & \iddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ f_{N,1} s'_1 T_{11m} & f_{N,2} s'_2 T_{21m} & \cdots & f_{N,N-1} s'_{N-1} T_{N-11m} & f_{N,N} s'_N T_{N1m} \end{pmatrix},$$

For any $i \in I_{N-1}$, we set

$$u_i = \prod_{k=1}^i s'_k, \quad u'_i = \prod_{k=1}^i b_{i+1,k}, \quad \tilde{u}_i = s_{i+1}^{-1} (u_i - u'_i),$$

$$D_0 = 1_{(N+1)m} - U_{1,2}(s'_1),$$

$$D_{1,i} = (1_{(N+1)m} + U_{i+2,i+1}(\tilde{u}_i))(1_{(N+1)m} + U_{i+2,i+2}(u'_i - 1))(1_{(N+1)m} + U_{i+1,i+2}(s'_{i+1})), \quad D_{2,i} = (1_{(N+1)m} + U_{i+2,i+1}(-\tilde{u}_i s'_{i+1}))(1_{(N+1)m} + U_{i+1,i+1}(u'_i - 1))(1_{(N+1)m} + U_{i+1,i+2}(s'_{i+1})), \quad D_1 = D_0 D_{1,1} \cdots D_{1,N-1}, \quad D_2 = D_2 D_{N-1} \cdots D_{2,1}.$$  

Here we remember $A(q^{-\lambda} x) = T(q^{-\lambda} x) B(q^{-\lambda} x) = (\prod_{k=1}^N s'_k)(B_{\infty} + \sum_{l=1}^N s'_{l-1} B_l)$, and we compute

$$D_2 C \left\{ q^\lambda \begin{pmatrix} T_{1m} & s'_1 T_{1m} \\ s'_1 T_{1m} & \ddots & s'_N T_{N1m} \end{pmatrix} + \begin{pmatrix} O_m \\ T_{1m} \\ \vdots \\ T_{N1m} \end{pmatrix} \right\} D_1.$$  

This is the $\tilde{G}(x)$. □

We prove the next lemma for examining type of elementary divisors of $G(x)$. 

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Lemma 4.15  For coefficient polynomial $A(x) = \sum_{k=0}^{N} A_k x^k$ of canonical form of Fuchsian equation $E_R$, we define $P_A \in M_{Nm}(C)$ as

$$
\begin{pmatrix}
1_m & & \\
& \ddots & \\
& & 1_m \\
-A_{-1}^{-1}A_0 & -A_{-1}^{-1}A_1 & \cdots & -A_{-1}^{-1}A_{N-1}
\end{pmatrix}.
$$

(69)

Then, for any $a_i \in Z_R = \{a \in C; \det(A(a)) = 0\}$, we obtain

$$n_j = \dim \ker((a_i1_{Nm} - P_A)^j) - \dim \ker((a_i1_{Nm} - P_A)^{j-1}) \quad (j \in Z_{>0}).$$

(70)

Proof.  $x1_{Nm} - P_A$ can be transformed to $1_{(N-1)m} \oplus A(x)$ by elementary matrices. Therefore, type of elementary divisors of $x1_{Nm} - P_A$ and type of elementary divisors of $A(x)$ are equal except for $1_{(N-1)m}$. □

We obtain the following lemma by calculating the dimensions of the generalized eigenspaces of $P_A$.

Lemma 4.16  Let $I_j = \{1, \ldots, j\}, j_1 = \min\{N + 1, j\}, j_2 = \max\{N + 1, j\} \quad (j \in Z_{>0}), I'_j = \{1, \ldots, j_2\}$. For any $a \in C \setminus \{0\}$, the following conditions are equivalent:

(i) $t(t^{v_1}, \ldots, t^{v_N}) \in \ker(a1_{Nm} - P_A) \quad (v_k \in \mathcal{V})$,

(ii) There exist $v_{j_1}, \ldots, v_{j_2} \in \mathcal{V}$ such that for $w_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} a_k^{l-1} v_l \quad (k \in I_j)$,

$$v_k = \sum_{l=1}^{j} (-1)^{l-1} \binom{k-1}{l-1} a_k^{l-1} w_l \quad (k \in I'_j) \quad \text{and} \quad \sum_{i=0}^{k-1} (-1)^i \frac{d^i A}{dx^i} (a) w_{i+j-k+1} = 0 \quad (k \in I_j).$$

Proof.  If $j = 1$, then for any $v = t(t^{v_1}, \ldots, t^{v_N}) \in \ker(a1_{Nm} - P_A) \quad (v_k \in \mathcal{V})$, we put $w_1 = v_1, v_{N+1} = a^N v_1$. Here we get $w_k = a_k^{k-1} v_1 = a_k^{k-1} w_1 \quad (k \in I'_1)$ and $A(a) w_1 = \sum_{k=0}^{N} A_k a_k^{k} v_1 = \sum_{k=0}^{N} A_k v_{k+1} = 0$ from $(P_A - a1_{Nm}) v = 0$. We assume that the equivalence is satisfied in the case $j = j' \in Z_{>0}$. For any $v = t(t^{v_1}, \ldots, t^{v_N}) \in \ker((a1_{Nm} - P_A)^{j'+1}) \quad (v_k \in \mathcal{V})$, we let

$$u = t(t^{u_1}, \ldots, t^{u_N}) = (a1_{Nm} - P_A)v \quad (u_k \in \mathcal{V}), \quad v_{N+1} = -A_{-1}^{-1} \sum_{k=0}^{N-1} A_k v_{k+1}.$$  

Then we find $u_k = av_k - v_{k+1}$ and $\sum_{k=0}^{N} A_k v_{k+1} = 0 \quad (k \in I_0 = \{1, \ldots, N\})$. Here we set

$$\tilde{w}_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} a_k^{l-1} u_l \quad (k \in I'_j).$$

There exist $u_{j'_1}, \ldots, u_{j'_2} \in \mathcal{V} \quad (j'_1 = \min\{N + 1, j'_1\}, j'_2 = \max\{N + 1, j'_2\})$ such that

$$\tilde{w}_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} a_k^{l-1} u_l = \sum_{l=1}^{k+1} (-1)^{l-1} \binom{k}{l-1} a^{k+1-l} v_l \quad (k \in I'_j).$$  

(71)
Let \( w_1 = v_1, w_k = \tilde{w}_{k-1} \) (\( k \in I_{j'_1+1} \setminus \{1\} \)), we obtain

\[
w_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{k-1}{l-1} a^{k-l} v_l \quad (k \in I_{j'_1+1}).
\]

Here we find \( u_k = \sum_{l=1}^{j'_1} (-1)^{l-1} \binom{k-1}{l-1} a^{k-l} w_{l+1} \) (\( k \in I_{j'} \)). We put \( v_k \in \mathcal{V} \) such that \( a v_k - v_{k+1} = u_k \) (\( k \in \{j'_1, \ldots, j'_2\} \)). For any \( k \in I_{j'} \), we get

\[
a v_k - v_{k+1} = \sum_{l=1}^{j'_1} (-1)^{l-1} \binom{k-1}{l-1} a^{k-l} w_{l+1} = \sum_{l=2}^{j'_1+1} (-1)^{l-2} \binom{k-1}{l-2} a^{k+1-l} w_l \quad (72)
\]

and

\[
v_k = \sum_{l=1}^{j'_1+1} (-1)^{l-1} \binom{k-1}{l-1} a^{k-l} w_l. \quad (73)
\]

Moreover, we obtain

\[
\sum_{l=1}^{k-1} (-1)^i \frac{d^i A}{dx^i}(a) w_{i+(j'_1+1)-k+1} = 0
\]

from \( \sum_{i=0}^{k-1} (-1)^i \frac{d^i A}{dx^i}(a) \tilde{w}_{i+j'_1-k+1} = 0 \) (\( k \in I_{j'} \)). On the other hand, by the computation:

\[
0 = \sum_{k=0}^{N} A_k v_{k+1}
= \sum_{k=0}^{N-1} A_k \sum_{l=1}^{j'_1+1} (-1)^{l-1} \binom{k}{l-1} a^{k-l} w_l
+ A_N \left\{ \sum_{l=1}^{j'_1+1} (-1)^{l-1} \binom{N-1}{l-1} a^{N-l} w_l - \sum_{l=1}^{j'_1} (-1)^{l-1} \binom{N-1}{l-1} a^{N-l} w_{l+1} \right\}
= \sum_{l=0}^{(j'_1+1)-1} \left( \frac{(-1)^l}{l!} \sum_{k=0}^{N} \frac{k!}{(k-l)!} A_k a^{k-l} w_{l+1} \right)
= \sum_{l=0}^{(j'_1+1)-1} \left( \frac{(-1)^l}{l!} \frac{d^l A}{dx^l}(a) w_{l+(j'_1+1)-(j'_1+1)+1} \right),
\]

(ii) is satisfied in the case \( j = j'_1 + 1 \in \mathbb{Z}_{>0} \). The proof of the lemma has been completed. \( \square \)

From the above, we can calculate the type of elementary divisors of \( G(x) = c_\lambda(A)(x) \). We obtain the next lemma by calculating the dimension of the generalized eigenspaces of \( P_{\overline{G}} \in M_{N(N+1)n} \).

**Lemma 4.17** If \((*)\), \((**)) are satisfied, then for any \( a \in \mathbb{Z}_0 = \{ a \in \mathbb{C}; \det A(a) = 0 \} \) and \( j \in \mathbb{Z}_{>0} \), we obtain (i),(ii):
(i) If \( q^\lambda a \in q^\lambda Z_R \setminus \{b_k; k \in \{1, \ldots, N\}\}, \) then
\[
\dim \ker((q^\lambda a_1 N(N+1)m - P_G)^j) = \dim \ker((a_1 N_m - P_A)^j).
\]
(ii) If \( q^\lambda a \in q^\lambda Z_R \cap \{b_k; k \in \{1, \ldots, N\}\}, \) then \( \dim \ker(q^\lambda a_1 N(N+1)m - P_G) = Nm \) and
\[
\dim \ker((q^\lambda a_1 N(N+1)m - P_G)^j+1) = \dim \ker((a_1 N_m - P_A)^j).
\]

\textbf{Proof.} (i) For any
\[
v = t(t_1 v_1, \ldots, t v_N) \in \ker(a_1 (N+1)m - P_G), \quad (v_k = t(t_k v_0, \ldots, t v_k N), v_{k,l} \in \mathcal{V}),
\]
we find \( v_k = a^{k-1} v_1, \tilde{G} q^\lambda a v_1 = 0. \) Moreover, we obtain \( A(a) v_{1,N} = 0, \dim \ker(q^\lambda a_1 N(N+1)m - P_G) = \dim \ker A(a) = \dim \ker(a_1 N_m - P_A) \) from \( \tilde{G} q^\lambda a v_1 = 0 \iff v_{1,l} = 0 (j \neq N). \) Meanwhile, we assume \( \dim \ker((q^\lambda a_1 N(N+1)m - P_G)^j) = \dim \ker((a_1 N_m - P_A)^j) (j = j' \in \mathbb{Z}_{>0}). \) In another expression, \( w_k = t(t_k w_0, \ldots, t w_{k,l}) \in \mathcal{V}^{N+1} (w_{k,N} \in \mathcal{V}, k \in J = \{1, \ldots, j\}'), \)
\[
\sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \frac{d^i \tilde{G}}{d x^i} (q^\lambda a) w_{i+j' - k+1} = 0
\]
\[
\iff \sum_{i=0}^{k-1} q(i-j-1)\lambda (\frac{-1}{i!} \frac{d^i A}{d x^i} (a) w_{i+j' - k+1,N} = 0, \quad w_{k,l} = 0 (l \neq N).
\]

Here if there exist
\[
w_k = t(t_k w_0, \ldots, t w_{k,N}) \in \mathcal{V}^{N+1} (w_{k,l} \in \mathcal{V}, k \in J' = \{1, \ldots, j' + 1\})
\]
such that \( \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \frac{d^i \tilde{G}}{d x^i} (q^\lambda a) w_{i+j' - k+2} = 0. \) Then we get \( w_{k,l} = 0 (k \neq 1, l \neq N). \) Moreover, we find
\[
w_{1,l} = 0, \quad \sum_{i=0}^{j'-1} q(i-j'i)\lambda (\frac{-1}{i!} \frac{d^i A}{d x^i} (a) w_{i+j' - k+2,N} = 0 (k \in J', l \neq N),
\]
because \( \sum_{i=0}^{j'} \frac{(-1)^i}{i!} \frac{d^i \tilde{G}}{d x^i} (q^\lambda a) w_{i+1} = 0. \) Therefore, we obtain
\[
\dim \ker((q^\lambda a_1 N(N+1)m - P_G)^j+1) = \dim \ker((a_1 N_m - P_A)^j+1).
\]

(ii) If \( q^\lambda a = b_{k_0} \in q^\lambda Z_R \cap \{b_k; k \in \{1, \ldots, N\}\} (k_0 \in \{1, \ldots, N\}), \) then we obtain
\[
\dim \ker(q^\lambda a_1 N(N+1)m - P_G) = \dim \ker \tilde{G}(k_0) = \dim \ker G(k_0) = (N+1)m - \dim \im G(k_0) = Nm.
\]
We assume that there exist \( w_k = t(t_k w_0, \ldots, t w_{k,N}) \in \mathcal{V}^{N+1} (w_{k,l} \in \mathcal{V}, k = 1, 2) \) such that
\[
\tilde{G}(q^\lambda a) w_2 = 0, \quad \frac{d \tilde{G}}{d x}(q^\lambda a) w_2 = \tilde{G}(q^\lambda a) w_1.
\]
Then it is clear that \( \frac{d \tilde{G}}{d x}(b_{k_0}) \neq 0. \) Hence we get
\[
w_{2,l} = 0 (l \neq N), \quad A(a) w_{2,N} = 0, \quad \frac{d A}{d x} (a) w_{2,N} = q^\lambda \sum_{l=0}^{N} U_l(q^\lambda a) w_{1,l}.
\]
Here \( q^\lambda \sum_{t=0}^{N} U_t(q^\lambda a)w_{1,t} \) spans \( \mathcal{V} \) from condition (**). Moreover, we find
\[
\dim \ker((q^\lambda a^1_{N(N+1)m} - P_G)^2) = \dim \ker A(a) = \dim \ker(a_{1N_m} - P_A).
\]
Therefore, we obtain
\[
\dim \ker((q^\lambda a^1_{N(N+1)m} - P_G)^{j+2}) = \dim \ker((a_{1N_m} - P_A)^{j+1}). \quad \square
\]

From the above, the next proposition is obvious.

**Proposition 4.18** If \((*)\), \((**)*\) are satisfied and the spectral type \( S(E_R) = (S_0; S_\infty; S_{\text{div}}) \) of Fuchsian equation \( E_R \) is given as
\[
S_\xi = (m^\xi_{1,1} \ldots m^\xi_{1,N_1} \ldots , m^\xi_{1,1} \ldots m^\xi_{1,N_1}) (\xi = 0, \infty),
\]
\[
S_{\text{div}} = (n^k_1 \ldots n^k_1, \ldots , n^k_1 \ldots n^k_1),
\]
then spectral type \( S(e\lambda (E_R)) = (S'_0; S'_\infty; S'_{\text{div}}) \) satisfies
\[
S'_0 = \begin{cases}
(Nm m^0_{1,1} \ldots m^0_{1,N_1} \ldots , m^0_{1,1} \ldots m^0_{1,N_1}) (q^\lambda = \alpha^0_1) \\
(Nm, m^0_{1,1} \ldots m^0_{1,N_1} \ldots , m^0_{1,1} \ldots m^0_{1,N_1}) (q^\lambda \notin \text{Ev}(A_0))
\end{cases},
\]
\[
S'_\infty = \begin{cases}
(Nm m^\infty_{1,1} \ldots m^\infty_{1,N_1} \ldots , m^\infty_{1,1} \ldots m^\infty_{1,N_1}) (b^\infty = \alpha^\infty_1) \\
(Nm, m^\infty_{1,1} \ldots m^\infty_{1,N_1} \ldots , m^\infty_{1,1} \ldots m^\infty_{1,N_1}) (b^\infty \notin \text{Ev}(A_\infty))
\end{cases},
\]
\[
S'_{\text{div}} = (Nm, Nm, Nm n^k_1 \ldots n^k_1, \ldots , Nm n^k_1 \ldots n^k_1) \quad (b_1, \ldots , b_{r_1} \in \{b_k ; k \in \{1, \ldots , N\}\} \setminus q^\lambda Z_A, \ q^\lambda a_1, \ldots , q^\lambda a_{r_2} \in \{b_k ; k \in \{1, \ldots , N\}\}).
\]

We show the next lemma in order to examine how \( q \)-middle convolution changes the spectral type.

**Lemma 4.19** If \( \lambda \neq 0 \), for \( \theta, \kappa, a \in \mathbb{C} \setminus \{0\}, I = \{1, \ldots , N\}, \) we obtain
\[
\dim (\ker(\theta_1_{(N+1)m} - G_0) \cap \mathcal{K}) = \begin{cases}
\dim \ker(A_0 - 1_m) & (\theta = 1) \\
\sum_{k=1}^{N} \dim \ker B_k & (\theta = q^\lambda) \\
0 & (\theta \neq 1, q^\lambda)
\end{cases},
\]
\[
\dim (\ker(\kappa_1_{(N+1)m} - G_\infty) \cap \mathcal{K}) = \begin{cases}
\dim \ker(A_0 - 1_m) + \sum_{k=1}^{N} \dim \ker B_k & (\kappa = b^\infty) \\
0 & (\kappa \neq b^\infty)
\end{cases},
\]
\[
\dim(\ker G(a) \cap \mathcal{K}) = \begin{cases} 
\dim \ker (A_0 - 1_m) + \sum_{k \neq j} \dim \ker B_k & (a = b_j) \\
\dim \ker B_j & (a = q^\lambda b_j \in q^\lambda Z_A \setminus \{b_k; k \in I\}) \\
0 & \text{(otherwise)}
\end{cases}
\]
(80)

\[
\dim \left( \frac{dG}{dx}(a) \right)^{-1}(\text{im} G(a)) \cap \ker G(a) \cap \mathcal{K} = \begin{cases} 
\dim \ker B_j & (a = q^\lambda b_j \in q^\lambda Z_A) \\
0 & \text{(otherwise)}
\end{cases}
\]
(82)

\[
\dim(\ker (\theta 1_{(N+1)m} - G_0) \cap \mathcal{L}) = \begin{cases} 
\dim \ker (A_{\infty} - q^\lambda b_\infty 1_m) & (\theta = q^\lambda) \\
0 & (\theta \neq q^\lambda)
\end{cases}
\]
(83)

\[
\dim(\ker (\kappa 1_{(N+1)m} - G_\infty) \cap \mathcal{L}) = \begin{cases} 
\dim \ker (A_{\infty} - q^\lambda b_\infty 1_m) & (\kappa = q^\lambda b_\infty) \\
0 & (\kappa \neq q^\lambda b_\infty)
\end{cases}
\]
(84)

\[
\dim(\ker G(a) \cap \mathcal{L}) = \begin{cases} 
\dim \ker (A_{\infty} - q^\lambda b_\infty 1_m) & (a \in \{b_k; k \in I\}) \\
0 & (a \notin \{b_k; k \in I\})
\end{cases}
\]
(85)

**Proof.** (i) (Change of \( S_0 \) due to the \( \mathcal{K} \)) For \( \theta \in \mathbb{C} \) and any \( v = t(v_0, \ldots, v_N) \in \ker (\theta 1_{(N+1)m} - G_0) \cap \mathcal{K} (v_k \in \mathcal{V}) \), it is easily seen that
\[
0 = (\theta 1_{(N+1)m} - G_0)v = t(\sum_{k=0}^{N} t(B_k v_k) + (\theta - 1) t v_0, (\theta - q^\lambda)^t v_1, \ldots, (\theta - q^\lambda)^t v_N).
\]

If \( \theta = 1 \), then it is clear that \( \theta \neq q^\lambda \) and \( v_k = 0 \) (\( k \in I = \{1, \ldots, N\} \)), \( v_0 \in \ker (A_0 - 1_m) \). Here we get \( \dim(\ker (\theta 1_{(N+1)m} - G_0) \cap \mathcal{K}) = \dim \ker (A_0 - 1_m) \).

If \( \theta = q^\lambda \), then we find \( v_k \in \ker B_k (k \in I) \). Therefore, we obtain \( \dim(\ker (\theta 1_{(N+1)m} - G_0) \cap \mathcal{K}) = \sum_{k=1}^{N} \dim \ker B_k \).

(ii) (Change of \( S_\infty \) due to the \( \mathcal{K} \)) For \( \kappa \in \mathbb{C} \) and any \( v \in \ker (\kappa 1_{(N+1)m} - G_\infty) \cap \mathcal{K} \), we get
\[
0 = (\kappa 1_{(N+1)m} - G_\infty)v = (\kappa 1_{(N+1)m} - b_\infty F_\infty)v = \{\kappa 1_{(N+1)m} - b_\infty (1_{(N+1)m} - \widehat{F})\}v = (\kappa - b_\infty)v.
\]

If \( \kappa = b_\infty \), then we obtain \( \dim(\ker (\kappa 1_{(N+1)m} - G_\infty) \cap \mathcal{K}) = \dim \mathcal{K} = \dim \ker (A_0 - 1_m) + \sum_{k=1}^{N} \dim \ker B_k \).

(iii) (Change of \( S_{\text{div}} \) due to the \( \mathcal{K} \))

(iii-a) For any \( v = t(v_0, \ldots, v_N) \in \ker G(b_k) \cap \mathcal{K} (v_k \in \mathcal{V}, k \in I) \), it is clear that \( v_k = 0 \). Hence we get \( \dim(\ker G(b_k) \cap \mathcal{K}) = \dim \ker (A_0 - 1_m) + \sum_{l \neq k} \dim \ker B_l \).

(iii-b) If \( q^\lambda a_i \in q^\lambda Z_A \setminus \{b_k; k \in I\} \), then \( T(q^\lambda a_i) \neq 0 \). Hence we obtain
\[
\ker G(q^\lambda a_i) = \ker F(q^\lambda a_i) = \ker \left( 1_{(N+1)m} - \widehat{F} + \sum_{k=1}^{N} \frac{F_k}{1 - q^\lambda a_i b_k^{-1}} \right) .
\]
(86)

For any \( v = t(v_0, \ldots, v_N) \in \ker G(q^\lambda a_i) \cap \mathcal{K} (v_k \in \mathcal{V}) \), we get
\[
0 = \{1 - \widehat{F} + \sum_{k=1}^{N} (1 - q^\lambda a_i b_k^{-1})^{-1} F_k\}v = \{1_m \oplus_{k=1}^{N} q^\lambda (1 - a_i b_k^{-1})(1 - q^\lambda a_i b_k^{-1})^{-1} 1_m \}v.
\]
Here if \( a_i \notin \{ b_k \mid k \in I \} \), then \( v = 0 \). In the meantime, if \( a_i = b_j (j \in I) \) and \( k \neq j \), then \( v_k = 0 \). Therefore, we find \( v_j \in \ker B_j \). From the above, we obtain

\[
\dim \left( \frac{dG}{dx}(q^\lambda a_i)^{-1} (\text{im } G(q^\lambda a_i)) \cap \ker G(q^\lambda a_i) \cap K \right) = \begin{cases} 
0 & (a_i \notin \{ b_k \mid k \in I \}) \\
\dim \ker B_j = n^j_i & (a_i = b_j) 
\end{cases}
\] (87)

(iii-c) If \( q^\lambda a_i = b_{j'} \in q^\lambda Z \cap \{ b_k \mid k \in I \} \) \((j' \in I)\), then we put \( w_k = (\ldots, w_{k,0}, \ldots, w_{k,N}) \in \nu^{N+1} (w_k \in \mathcal{V}, k = 1, 2) \) such that

\[
w_2 \in \ker G(q^\lambda a_i) \cap K, \quad \frac{dG}{dx}(q^\lambda a_i)w_2 = G(q^\lambda a_i)w_1.
\]

Hence we find \( \ker G(q^\lambda a_i) \cap K = \ker F_{j'} \cap K \) and \( q^\lambda \neq 1 \). Therefore, we get \( w_{2,j'} = 0 \). Moreover, \( G(q^\lambda a_i)w_1 \) spans \( \langle 0, \ldots, 0, \mathcal{V}, 0, \ldots, 0 \rangle \) from (**).

If \( a_i \notin \{ b_k \mid k \in I \} \), then we get \( w_{2,k} = 0 (k \neq j') \) from \( \frac{dG}{dx}(q^\lambda a_i)w_2 = G(q^\lambda a_i)w_1 \). Therefore, \( w_2 = 0 \). Meanwhile, if \( a_i = b_j (j \in I) \) and \( k \neq j \), then we find \( w_{2,k} = 0 \) and \( w_{2,j} \in \ker B_j \). From the above, we obtain

\[
\dim \left( \frac{dG}{dx}(q^\lambda a_i)^{-1} (\text{im } G(q^\lambda a_i)) \cap \ker G(q^\lambda a_i) \cap K \right) = \begin{cases} 
0 & (a_i \notin \{ b_k \mid k \in I \}) \\
\dim \ker B_j & (a_i = b_j) 
\end{cases}
\] (88)

(iv) (Change of \( S_0 \) due to the \( \mathcal{L} \)) For \( \theta \in \mathbb{C} \) and any \( v = (\ldots, t^h, \ldots) \in \ker (\theta \lambda^{(N+1)m} - G_0) \cap \mathcal{L} (h \in \ker (A^{\infty} - q^\lambda b^{\infty}1_m)) \), we find

\[
0 = (\theta \lambda^{(N+1)m} - G_0)v = (\ldots, (\theta - q^\lambda)^t h, \ldots, (\theta - q^\lambda)^t h).
\]

If \( \theta = q^\lambda \), then \( h \in \ker (A^{\infty} - q^\lambda b^{\infty}1_m) \). Therefore, we obtain \( \dim (\ker (\theta \lambda^{(N+1)m} - G_0) \cap \mathcal{L}) = \dim \mathcal{L} = \dim \ker (A^{\infty} - q^\lambda b^{\infty}1_m) \).

(v) (Change of \( S_{\infty} \) due to the \( \mathcal{L} \)) For \( \kappa \in \mathbb{C} \) and \( v = (\ldots, t^h, \ldots) \in \ker (\kappa \lambda^{(N+1)m} - G_\infty) \cap \mathcal{L} (h \in \ker (A^{\infty} - q^\lambda b^{\infty}1_m)) \), we get

\[
0 = (\kappa \lambda^{(N+1)m} - G_\infty)v = (\kappa - q^\lambda b_\infty)v.
\]

If \( \kappa = q^\lambda b_\infty \), then we obtain \( \dim (\ker (\kappa \lambda^{(N+1)m} - G_\infty) \cap \mathcal{L}) = \dim \mathcal{L} = \dim \ker (A^{\infty} - q^\lambda b^{\infty}1_m) \).

(vi) (Change of \( S_{\text{div}} \) due to the \( \mathcal{K} \)) For any \( k \in I \), \( \mathcal{L} \) is subspace of \( \ker G(b_k) = \ker F_k \). Therefore, we obtain

\[
\dim (\ker G(b_k) \cap \mathcal{L}) = \dim \mathcal{L} = \dim \ker (A^{\infty} - q^\lambda b^{\infty}1_m).
\]

From the above, Theorem 1.11 is shown.

**Theorem 1.11** (rigidity index) If (\ref{eq:1.1}), (\ref{eq:1.2}) are satisfied, then \( mc_\lambda \) preserves rigidity index of Fuchsian equation \( E_R \).
Proof. In the case $\lambda = 0$, it is obvious from Proposition 4.4. We assume $\lambda \neq 0$. Let coefficient
\[
\overline{G}(x) = \sum_{k=0}^{N} \overline{G}_k x^k \quad (\overline{G}_{\infty} = \overline{G}_N)
\]
of canonical form of $E_{\overline{F},b}(\overline{F} = mc_{\lambda}(B))$. It is clear that $q^\lambda \neq 1, q^\lambda b_{\infty} \neq b_{\infty}$. Here let $\alpha_{i_0}^0 = 1, \alpha_{i_\infty}^\infty = q^\lambda b_{\infty}$. we get
\[
\dim \ker(A_0 - 1_m) = m_{i_0,1}^0, \quad \dim \ker(A_\infty - q^\lambda b_{\infty} 1_m) = m_{i_\infty,1}^\infty.
\]
Moreover, we set
\[
b_k = \begin{cases} \quad a_k & (k \in \{1, \ldots, r\}) \\ \quad c_k & (k \in \{r+1, \ldots, N\}, c_k \notin Z_A) \end{cases}, \quad d_k = \dim \ker B_k, \quad d = \sum_{k=1}^{N} d_k.
\]
Then we find
\[
\dim(mc_\lambda(V)) = (N + 1)m - m_{i_0,1}^0 - m_{i_\infty,1}^\infty - d, \quad d = \sum_{k=1}^{r} n_k^i.
\]
Since these relations, we obtain
\[
p_0 = \dim \ker(\overset{\infty}{G} - q^\lambda 1_{\dim(mc_\lambda(V))}) =Nm - m_{i_\infty,1}^\infty - d,
\]
\[
p_\infty = \dim \ker(\overset{\infty}{G} - b_{\infty} 1_{\dim(mc_\lambda(V))}) =Nm - m_{i_0,1}^0 - d,
\]
\[
p_k = \dim \ker(\overset{\infty}{G}(b_k)) = Nm - m_{i_0,1}^0 - m_{i_\infty,1}^\infty - d + d_k (k \in \{1, \ldots, N\}).
\]
From the above, rigidity index, $\text{idx}(mc_\lambda(E_R))$, of equation $E_R$ is calculated:
\[
\text{idx}(mc_\lambda(E_R)) = \sum_{i=1}^{l_0} \sum_{j=1}^{l_0} (m_{i_0,1}^0)^2 + \sum_{j=2}^{l_0} (m_{i_0,1}^0)^2 + (p_0)^2 + \sum_{i=1}^{l_\infty} \sum_{j=1}^{l_\infty} (m_{i_\infty,1}^\infty)^2 + \sum_{j=2}^{l_\infty} (m_{i_\infty,1}^\infty)^2 + (p_\infty)^2
\]
\[
+ \sum_{i=1}^{r} \sum_{j=2}^{l_0} (n_j^i)^2 + \sum_{i=r+1}^{l_0} \sum_{j=1}^{l_0} (n_j^i)^2 + \sum_{k=1}^{N} (p_k)^2 - N\{\dim(mc_\lambda(V))\}^2
\]
\[
= \sum_{i=1}^{l_0} \sum_{j=1}^{l_0} (m_{i_0,1}^0)^2 - (m_{i_0,1}^0)^2 + (p_0)^2 + \sum_{i=1}^{l_\infty} \sum_{j=1}^{l_\infty} (m_{i_\infty,1}^\infty)^2 - (m_{i_\infty,1}^\infty)^2 + (p_\infty)^2
\]
\[
+ \sum_{i=1}^{r} \sum_{j=1}^{l_0} (n_j^i)^2 - \sum_{i=1}^{r} (n_j^1)^2 + \sum_{k=1}^{N} (p_k)^2 - N\{\dim(mc_\lambda(V))\}^2
\]
\[
= \text{idx}(E_R) - (m_{i_0,1}^0)^2 + (Nm - m_{i_\infty,1}^\infty - d)^2 - (m_{i_\infty,1}^\infty)^2 + (Nm - m_{i_0,1}^0 - d)^2 - \sum_{i=1}^{r} (n_j^i)^2
\]
\[
+ \sum_{k=1}^{N} (Nm - m_{i_0,1}^0 - m_{i_\infty,1}^\infty - d + d_k)^2 - N\{(N + 1)m - m_{i_0,1}^0 - m_{i_\infty,1}^\infty - d\}^2 + Nm^2
\]
\[
= \text{idx}(E_R).
\]

The proof of the theorem has been completed. □
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