SATellite OPERATORS AS GROUP ACTIONS ON KNOT CONCORDANCE

CHRISTOPHER DAVIS AND ARUNIMA RAY

ABSTRACT. Any knot in a solid torus (called a satellite operator) acts on knots in $S^3$. We introduce a generalization of satellite operators which act on knots in homology 3–spheres. Unlike traditional satellite operators, these generalized operators form a group, modulo an appropriate generalization of concordance. By studying the action of this group on knots in homology 3–spheres we recover the very recent result of Cochran–Davis–Ray that satellite operators with strong winding number one give injective functions on smooth knot concordance in $S^3 \times [0,1]$, modulo the smooth 4–dimensional Poincaré Conjecture. We also describe how the notion of generalized satellite operators provides a new framework within which to consider the question of surjectivity of satellite operators and make some progress towards answering this question. We also construct a new example of a bijective satellite operator.

1. Introduction

The satellite construction is a classical and well-studied family of functions on the set of knots in $S^3$ and is described in Figure 1. The arguments of the satellite construction are a satellite operator, $P$, i.e. a knot in a solid torus $V = S^1 \times D^2$, and a knot $K$ in $S^3$. Loosely speaking, in order to form the satellite knot $P(K)$, we tie the solid torus $V$ into the knot $K$ and see where $P \subseteq V$ goes. More precise definitions can be found in Section 2.

![Figure 1. The satellite operation on knots in $S^3$.](image-url)
The satellite construction descends to a well-defined action on the smooth knot concordance group, $\mathcal{C}$. This action has been used to construct knots which are distinct in knot concordance but which many classical invariants fail to distinguish. Examples of this philosophy can be found in [CHL11] and [COT04]. In [CFHH13], winding number one satellite operators are used to construct non-concordant knots which have homology cobordant zero surgery manifolds. Satellite operations also find applications in the more general context of 3– and 4–manifold topology. For example, they were used in [Har08] to modify a 3–manifold while fixing its homology type. Winding number one satellite operators, which are of particular interest in this paper, are related to Mazur 4–manifolds [AK79] and Akbulut corks [Akb91]. In [AY08] it was shown that changing the attaching curve of a 2–handle in a handlebody description of a 4–manifold by a winding number one satellite operation can change the diffeomorphism type while fixing the homeomorphism type!

As a result, there has been considerable interest in understanding how satellite operators act on $\mathcal{C}$. For example, it is a famous conjecture that the Whitehead double of a knot $K$ is smoothly slice if and only if $K$ is smoothly slice [Kir97, Problem 1.38]. This question might be generalized to ask if operators are injective, that is, given an operator $P$, does $P(K) = P(J)$ imply $K = J$ in smooth concordance? A survey of such work on the Whitehead doubling operator may be found in [HK12]. In [CHL11], several robust doubling operators were introduced and some evidence was provided for their injectivity. This is the current state of knowledge in the winding number zero case. For operators with nonzero winding numbers, there has been more success. In [CDR12] Cochran and the authors showed that a certain large class of winding number one operators are injective on $\mathcal{C}$, modulo the 4–dimensional smooth Poincaré Conjecture. These winding number one operators were also shown to be injective on the topological knot concordance group. Similar results are true for operators with greater winding numbers with respect to appropriate generalizations of concordance.

In this paper we will recast the satellite operation on knot concordance classes, a monoid action, in terms of a group action by a larger set. Specifically in the main result of this paper, we show that satellite operators form a subgroup of the group of homology cobordism classes of homology cylinders. This group was introduced by Levine in [Lev01] and studied by Cha–Friedl–Kim in [CFK11]. We additionally show that this subgroup acts on concordance classes of knots in homology spheres in a way that is compatible with the classical satellite construction. That is, we prove a theorem of the following type. The precise statement is given in Section 2.

**Main Theorem** (See Section 2). For $\mathcal{C}_*$ an appropriate generalization of knot concordance and $\mathcal{S}_*$ a submonoid of the set of all satellite operators which acts on $\mathcal{C}_*$ there is an enlargement of $\mathcal{C}_*$, $\Psi : \mathcal{C}_* \hookrightarrow \hat{\mathcal{C}}_*$, and a monoid morphism, $E : \mathcal{S}_* \to \hat{\mathcal{S}}_*$, where $\hat{\mathcal{S}}_*$ is a group which acts on $\hat{\mathcal{C}}_*$ making the following
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Diagram commute for all \( P \in S_* \):

\[
\begin{array}{ccc}
\mathcal{C}_* & \xrightarrow{P} & \mathcal{C}_* \\
\downarrow \Psi & & \downarrow \Psi \\
\widehat{\mathcal{C}}_* & \xrightarrow{E(P)} & \widehat{\mathcal{C}}_*
\end{array}
\]

The generalization \( \mathcal{C}_* \) might denote the topological concordance group or (if the smooth 4-dimensional Poincaré conjecture is true) the smooth concordance group. \( \widehat{\mathcal{C}}_* \) is roughly the set of all knots in homology spheres modulo a corresponding sense of concordance.

Since \( E(P) \) is an element of a group acting on \( \widehat{\mathcal{C}}_* \), \( E(P) : \widehat{\mathcal{C}}_* \to \widehat{\mathcal{C}}_* \) is a bijection. The results mentioned above of [CDR12] that \( P : \mathcal{C}_* \to \mathcal{C}_* \) is an injection follow now from an elementary diagram chase (see Corollary 2.5).

1.1. **Applications and questions:** Considering the satellite construction as a group action provides a novel approach to the problem of finding nontrivial surjective operators on \( \mathcal{C}_* \), which we describe in Section 4. While it is elementary to show that satellite operators with winding number other than ±1 cannot give surjections on knot concordance (see Proposition 4.5), very little is known in the case of satellite operators of winding number ±1. For instance, a conjecture of Akbulut [Kir97, Problem 1.45] claiming that there exists a winding number one satellite operator \( P \) such that \( P(K) \) is not slice for any knot \( K \) is wide open. This claim is equivalent to the unknot not being in the image of \( P : \mathcal{C} \to \mathcal{C} \).

As an element of a group acting on \( \widehat{\mathcal{C}}_* \), each satellite operator \( P \in S_* \) gives a bijection \( E(P) : \widehat{\mathcal{C}}_* \to \widehat{\mathcal{C}}_* \), and \( E(P)^{-1} \) makes sense. Therefore, instead of asking whether a knot \( K \) is in the image of \( P \) we may ask if \( E(P)^{-1}(K) \in \widehat{\mathcal{C}}_* \) is in the image of \( \Psi : \mathcal{C}_* \hookrightarrow \widehat{\mathcal{C}}_* \). This turns out to be an easier question to address and allows us to find a class of operators on \( \mathcal{C}_* \) in Section 5 which are surjective (as well as injective). An example of such an operator is shown in Figure 2.

**Figure 2.** This satellite operator gives a bijective map on \( \mathcal{C}_* \).

See Proposition 5.2

As another application of our Main Theorem we draw a connection between the surjectivity of satellite operators and the question of existence of knots...
in homology 3–spheres which are not concordant to any knot in \( S^3 \). That is, we connect Akbulut’s conjecture mentioned above to a question of Matsumoto [Kir97, Problem 1.30] asking if every knot in a 3–manifold homology cobordant to \( S^3 \) is concordant in a homology cobordism to \( S^3 \) to a knot in \( S^3 \). (This is slightly dishonest, as Matsumoto’s question is stated in the piecewise linear category which we do not study in this paper.) In Section 6 we show that if Akbulut’s conjecture is true then the answer to Matsumoto’s question is no.

2. Background and statement of the main theorem

In this section we recall the precise meaning of the satellite construction and state the categories in which we work. We will close this section with an explicit statement of the Main Theorem and recover the injectivity result of [CDR12].

2.1. Satellite operators. A pattern, or a satellite operator, is a knot in the standard solid torus \( V = S^1 \times D^2 \). Let \( S \) denote the set of all satellite operators.

For a pattern \( P \subseteq V \), let \( E(P) \) denote the complement of a regular neighborhood of \( P \). There are four important curves on the boundary of \( E(P) \):

1. \( m(P) \), the meridian of the pattern,
2. \( \ell(P) \), the longitude of the pattern,
3. \( m(V) = m(V(P)) = \{p\} \times \partial D^2 \), the meridian of the solid torus and
4. \( \ell(V) = \ell(V(P)) = S^1 \times \{p\} \), the longitude of the solid torus.

Since \( V = S^1 \times D^2 \) (and not merely homeomorphic to \( S^1 \times D^2 \)) we can say that \( \ell(V) = S^1 \times \{p\} \) meaningfully. There is a unique isotopy class of pushoffs of \( P \) in \( V \) which are homologous in \( E(P) \) to a multiple of \( \ell(V) \). This defines \( \ell(P) \). We say that \( P \) has winding number \( w = w(P) \in \mathbb{Z} \) if \( \ell(P) \) is homologous to \( w \cdot \ell(V) \).

\( S \) is a monoid in a natural way (see Figure 3). Given two satellite operators \( S \) and \( P \), we can define an operator \( S \ast P \) as follows. Let \( V(S) \) and \( V(P) \) be the standard solid tori containing \( S \) and \( P \) respectively. Glue \( E(P) \) and \( V(S) \) together by identifying \( \partial E(P) \) with \( \partial V(S) \) via \( m(P) \sim m(V(S)) \) and \( \ell(P) \sim \ell(V(S)) \). \( S \ast P \) is the image of \( P \) after this identification and the resulting manifold is still a solid torus \( V(S \ast P) \).

![Figure 3. The monoid operation on patterns.](image-url)
It is a good exercise to see that $\ast$ is associative, i.e. $P \ast (Q \ast S) = (P \ast Q) \ast S$, and the monoid identity is given by the core of the solid torus, i.e. the trivial satellite operator.

Satellite operators act on knots in $S^3$ as we described in Figure 1. To obtain $P(K)$ from a pattern $P \subseteq V$ and a knot $K \subseteq S^3$, start with the knot complement $E(K)$. The toral boundary contains $\ell(K)$, the longitude of $K$, and $m(K)$, the meridian of $K$. Glue in $V(P)$ by identifying $\ell(V) \sim \ell(K)$ and $m(V) \sim m(K)$. The resulting 3–manifold is $S^3$ and the image of $P$ is the satellite knot $P(K)$. For more details about the satellite construction see [Lic97, p. 10] or [Rol90, p. 111].

Let $K$ denote the set of knots in $S^3$ modulo isotopy. For patterns $P$ and $Q$ and a knot $K$, $(P \ast Q)(K) = P(Q(K))$. Therefore, we have a monoid homomorphism $S \rightarrow \text{Maps}(K, K)$, that is, a monoid action on the set of isotopy classes of knots in $S^3$. It is worth noting that $S$ is far from being a group under the $\ast$ operation.

**Proposition 2.1.** The only element of $S$ which has an inverse under the $\ast$ operation is the trivial operator, namely the operator given by the core of the solid torus.

**Proof.** We can see this using the notion of bridge index of a knot [Rol90, p. 114]. Suppose the inverse of an operator $P \subseteq V$ is denoted $P^{-1}$. For any satellite knot $P(K)$, we know from [Sch54] that

$$b(P(K)) \geq n \cdot b(K)$$

where $b(\cdot)$ denotes the bridge index and $n$ is the geometric winding number of $P$, namely $n$ is the number of times $\ell(P)$ intersects a generic meridional disk of $V$. Specifically, this means that for any operator $P$ and knot $K$, $b(P(K)) \geq b(K)$. Therefore,

$$b(K) = b((P^{-1} \ast P)(K)) = b(P^{-1}(P(K))) \geq b(P(K)) \geq b(K)$$

This implies that $b(P(K)) = b(K)$. Since $b(P(K)) \geq n \cdot b(K)$, we must have that $n = 1$, which implies that $P$ is a so-called connected-sum operator, i.e. $P = Q_J$ where $Q_J(K) = J \# K$ for all $K \in \mathcal{K}$ and some specific knot $J$. See Figure 4.

![Diagram](image)

**Figure 4.** For a knot $J$, the pattern $Q_J$ depicted above is the connected-sum operator. For any knot $K$, $Q_J(K) = J \# K$.

However, we know that a non-trivial connected-sum as a function on $\mathcal{K}$ does not have an inverse, due to the additivity of genus under connected-sum.
Therefore, the only pattern which has an inverse in $S$ is the trivial operator $Q_U$, i.e. connected-sum with the unknot.

The following submonoids of $S$ will be of particular interest in this paper.

**Definition 2.2.** Let $P$ be a pattern and $R \subseteq \mathbb{Q}$ be a localization of $\mathbb{Z}$ (possibly $R = \mathbb{Z}$).

1. $P$ is said to lie in $S_R$ if $w(P)$ is invertible in $R$, that is, $\frac{1}{w(P)} \in R$.
2. $P$ is said to lie in $S_{\text{str}}$ (and have **strong winding number one**) if $w(P) = \pm 1$ and the sets $\{m(V), \ell(V)\}$ and $\{m(P), \ell(P)\}$ each normally generates $\pi_1(E(P))$.

For a pattern $P \subseteq V = S^1 \times D^2$, one obtains a knot in $S^3$, $\tilde{P} = P(\text{unknot})$, by adding a 3–dimensional 2–handle along $\ell(V)$ and then a 3–dimensional 3–handle along the resulting 2–sphere boundary component. The following is a definition from [CDR12].

(2') [CDR12] $P$ is said to have strong winding number one if $w(P) = \pm 1$ and $m(V)$ normally generates $\pi_1(S^3 - \tilde{P})$.

At first glance the definition of strong winding number one operators in [CDR12] appears different from ours given in Definition 2.2 but we prove their equivalence below.

**Proposition 2.3.** The two notions of strong winding number one satellite operators given above are equivalent.

**Proof.** Suppose $P$ satisfies (2). $S^3 - \tilde{P}$ is obtained by adding a 2–handle to $E(P)$ along $\ell(V)$ (followed by a 3–handle). As a result, $\pi_1(S^3 - \tilde{P}) = \pi_1(E(P))/\langle\langle\ell(V)\rangle\rangle$, where $\langle\langle\cdot\rangle\rangle$ denotes normal closure. Since $\{m(V), \ell(V)\}$ normally generates $\pi_1(E(P))$, $\pi_1(S^3 - \tilde{P})$ is normally generated by $m(V)$ and Condition (2') holds.

Conversely, suppose that (2') holds. Then $m(V)$ normally generates $\pi_1(S^3 - \tilde{P}) = \pi_1(E(P))/\langle\langle\ell(V)\rangle\rangle$. Thus, $\{m(V), \ell(V)\}$ normally generates $\pi_1(E(P))$. In order to see that $\{m(P), \ell(P)\}$ normally generates, notice that $V$ is obtained by adding a 2–handle along $m(P)$ (followed by a 3–handle) so that $\mathbb{Z} \cong \pi_1(V) = \pi_1(E(P))/\langle\langle m(P)\rangle\rangle$. Since $P$ is winding number one, $\pi_1(\tilde{V})$ is generated by $\ell(P)$ which is homotopic to $\ell(V)$. Thus $\{m(P), \ell(P)\}$ normally generates and Condition (2) holds.

2.2. **Notions of knot concordance.** Let $K$ denote the set of all knots in $S^3$ modulo isotopy. $C$ is the set of all knots modulo **smooth concordance**. We will work in several variants of knot concordance.

1. $C_{\text{ex}}$ (short for $C_{\text{exotic}}$) is the set of knots modulo smooth concordance with respect to a **possibly exotic smooth structure** on $S^3 \times [0,1]$.
2. $C_{\text{top}}$ is the set of all knots modulo **topological concordance**.
(3) For $R$ a localization of $\mathbb{Z}$, $\mathcal{C}_R$ is the set of all knots modulo concordance in **smooth $R$–homology cobordisms** from $S^3$ to $S^3$.

(4) For $R$ a localization of $\mathbb{Z}$, $\mathcal{C}_{R,\text{top}}$ is the set of all knots modulo concordance in **topological $R$–homology cobordisms** from $S^3$ to $S^3$.

It is enlightening to see how these variants of concordance relate to each other. Suppose that $R \subseteq S \subseteq \mathbb{Q}$ are localizations of $\mathbb{Z}$. The following diagram commutes, where each map sends the class of a knot to the corresponding class of the same knot.

$$
\begin{array}{cccc}
\mathcal{C} & \rightarrow & \mathcal{C}_{\text{ex}} & \rightarrow & \mathcal{C}_R & \rightarrow & \mathcal{C}_S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}_\text{top} & \rightarrow & \mathcal{C}_{R,\text{top}} & \rightarrow & \mathcal{C}_{S,\text{top}}
\end{array}
$$

If the smooth 4–dimensional Poincaré Conjecture is true, $\mathcal{C}_{\text{ex}} = \mathcal{C}$.

We can easily see that if two knots $K$ and $J$ are concordant (in any of the senses above), then $P(K)$ is concordant to $P(J)$, for any satellite operator $P$. Thus, $\mathcal{S}$ (and so its submonoids $\mathcal{S}_R$ and $\mathcal{S}_{\text{str}}$) act on all of the above notions of knot concordance.

We can now state the main result of this paper, which shows that these actions come from group actions on an enlargement of knot concordance.

**Main Theorem.** Let $R$ be a localization of $\mathbb{Z}$. Let $* \in \{\text{ex, top, } R, (R, \text{top})\}$. There exist an enlargement of $\mathcal{C}_*$, $\Psi : \mathcal{C}_* \hookrightarrow \mathcal{\hat{C}}_*$. There exist homomorphisms, each of which we call $E$; $\mathcal{S}_{\text{str}} \rightarrow \mathcal{\hat{S}}_{\text{ex}}, \mathcal{S}_{\text{str}} \rightarrow \mathcal{\hat{S}}_{\text{top}}, \mathcal{S}_R \rightarrow \mathcal{\hat{S}}_R, \text{ and } \mathcal{S}_R \rightarrow \mathcal{\hat{S}}_{R,\text{top}}$ where each $\mathcal{\hat{S}}_*$ is a group which acts on $\mathcal{\hat{C}}_*$.

The following diagrams commute for $P \in \mathcal{S}_{\text{str}}$.

$$
\begin{array}{cccc}
\mathcal{C}_{\text{ex}} & \rightarrow & \mathcal{C}_{\text{ex}} & \rightarrow & \mathcal{C}_{\text{top}} & \rightarrow & \mathcal{C}_{\text{top}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{\hat{C}}_{\text{ex}} & \rightarrow & \mathcal{\hat{C}}_{\text{ex}} & \rightarrow & \mathcal{\hat{C}}_{\text{top}} & \rightarrow & \mathcal{\hat{C}}_{\text{top}}
\end{array}
$$

(1)

The following diagrams commute for $P \in \mathcal{S}_R$.

$$
\begin{array}{cccc}
\mathcal{C}_R & \rightarrow & \mathcal{C}_R & \rightarrow & \mathcal{C}_{R,\text{top}} & \rightarrow & \mathcal{C}_{R,\text{top}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{\hat{C}}_R & \rightarrow & \mathcal{\hat{C}}_R & \rightarrow & \mathcal{\hat{C}}_{R,\text{top}} & \rightarrow & \mathcal{\hat{C}}_{R,\text{top}}
\end{array}
$$

(2)
In the interest of clarity, we restate the theorem in the exotic case.

**Theorem 2.4.** There is an enlargement of $C_{ex}$, $\Psi : C_{ex} \hookrightarrow \widehat{C}_{ex}$. There is a homomorphism $E : S_{ex} \to \widehat{S}_{ex}$, where $\widehat{S}_{ex}$ is a group acting on $\widehat{C}_{ex}$. The following diagram commutes.

\[
\begin{array}{ccc}
C_{ex} & \xrightarrow{P} & C_{ex} \\
\Psi \downarrow & & \downarrow \Psi \\
\widehat{C}_{ex} & \xrightarrow{E(P)} & \widehat{C}_{ex}
\end{array}
\]

The proofs in the four cases are basically identical. Much of Section 3 consists of defining $\widehat{C}_{\ast}$, $\widehat{S}_{\ast}$, $E$ and $\Psi$. In brief, $\widehat{C}_{\ast}$ is the set of appropriately defined concordance classes of knots in homology 3–spheres, $\widehat{S}_{\ast}$ is a subgroup of the group of homology cobordism classes of homology cylinders as defined by Levine in [Lev01], and $E(P)$ is the exterior of the pattern $P$ regarded as a homology cylinder.

As an immediate corollary of the Main Theorem we recover the following result from [CDR12].

**Corollary 2.5** (Theorem 5.1 of [CDR12]). Let $P$ be a pattern.

- If $P$ has winding number $n \neq 0$ then $P : C_{\mathbb{Z}[1/n]} \to C_{\mathbb{Z}[1/n]}$ and $P : \widehat{C}_{\mathbb{Z}[1/n],\text{top}} \to \widehat{C}_{\mathbb{Z}[1/n],\text{top}}$ are injective.
- If $P$ has strong winding number one then $P : C_{ex} \to C_{ex}$ and $P : C_{\text{top}} \to \widehat{C}_{\text{top}}$ are injective.

**Proof.** The proof is a straightforward diagram chase.

Suppose that $P(K) = P(J)$ in $\widehat{C}_{\ast}$. Then $\Psi(P(K)) = \Psi(P(J))$. According to the commutativity of Diagrams 1 and 2 in the Main Theorem we have that $(E(P))(\Psi(K)) = (E(P))(\Psi(J))$. Since $E(P) \in \widehat{S}_{\ast}$ is an element of a group which acts on $\widehat{C}_{\ast}$, it has an inverse. Therefore, the map $E(P)$ is bijective and in particular injective. Thus, $\Psi(K) = \Psi(J)$. But $\Psi$ is also injective and therefore we can conclude that $K = J$ in $\widehat{C}_{\ast}$.

3. **Generalized satellite operators and the proof of the Main Theorem**

In this section we provide the definitions needed in the Main Theorem. The enlargement $\Psi : C_{\ast} \hookrightarrow \widehat{C}_{\ast}$ is described in Section 3.1. In Section 3.2 we recall the group of homology cylinders, $H_{\ast}$ of Levine [Lev01]. In Section 3.3 we provide a homomorphism $E$ from $S_{\ast}$ to a subgroup $\widehat{S}_{\ast} \leq H_{\ast}$. In Section 3.4 we show that $\widehat{S}_{\ast}$ acts on $\widehat{C}_{\ast}$ in a manner compatible with the action of $S_{\ast}$ on $C_{\ast}$. Together, these pieces will comprise the proof of the Main Theorem.
3.1. Generalizations of knot concordance.

Definition 3.1. Let $K$ and $J$ be knots in the $\mathbb{Z}$–homology spheres $X$ and $Y$ respectively.

- $(K, X)$ and $(J, Y)$ are called **exotically concordant** if there is a smooth $\mathbb{Z}$–homology cobordism $W$ from $X$ to $Y$ with $\pi_1(W)$ normally generated by the images of each of $\pi_1(X)$ and $\pi_1(Y)$ and in which $K$ and $J$ cobound a smooth annulus. $\hat{C}_{\text{ex}}$ is the set of all knots in $\mathbb{Z}$–homology spheres modulo exotic concordance.

- $(K, X)$ and $(J, Y)$ are called **topologically concordant** if there is a $\mathbb{Z}$–homology cobordism $W$ from $X$ to $Y$ with $\pi_1(W)$ normally generated by the images of each of $\pi_1(X)$ and $\pi_1(Y)$ and in which $K$ and $J$ cobound a locally flat annulus. $\hat{C}_{\text{top}}$ is the set of all knots in $\mathbb{Z}$–homology spheres modulo topological concordance.

Now suppose that $R$ is a localization of $\mathbb{Z}$ and $X$ and $Y$ are only $R$–homology spheres.

- $(K, X)$ and $(J, Y)$ are called **$R$–concordant** if there is a smooth $R$–homology cobordism $W$ from $X$ to $Y$ in which $K$ and $J$ cobound a smooth annulus. $\hat{C}_R$ is the set of all knots in $R$–homology spheres modulo $R$–concordance.

- $(K, X)$ and $(J, Y)$ are called **topologically $R$–concordant** if there is an $R$–homology cobordism $W$ from $X$ to $Y$ in which $K$ and $J$ cobound a locally flat annulus. $\hat{C}_{R, \text{top}}$ is the set of all knots in $R$–homology spheres modulo topological $R$–concordance.

For every choice of $*$, there is an obvious injection $C_* \hookrightarrow \hat{C}_*$.

Proposition 3.2. For each $* \in \{\text{ex, top, } R, (R, \text{ top})\}$, the map $\Psi : C_* \to \hat{C}_*$ which sends each knot $K$ in $S^3$ to the pair $(K, S^3)$ is well-defined and injective.

Proof. We start with the proof in the case $* = \text{ex}$. If $K = J$ in $C_{\text{ex}}$ then $K$ and $J$ cobound a smooth annulus in $W := S^3 \times [0, 1]$ with a possibly exotic smooth structure. Here $W$ is a homology cobordism from $S^3$ to $S^3$ and $\pi_1(W) = 0$ is normally generated by the components of $\partial W$. Thus $(K, S^3) = (J, S^3)$ in $\hat{C}_{\text{ex}}$ and the map $\Psi$ is well-defined.

Conversely, if $(K, S^3) = (J, S^3)$ in $\hat{C}_{\text{ex}}$ then $K$ and $J$ are concordant in a smooth homology cobordism $W$ from $S^3$ to $S^3$ with $\pi_1(W)$ normally generated by $\pi_1(S^3) = 0$. Thus, $W$ is simply connected, and by Freedman’s proof of the topological 4–dimensional Poincaré Conjecture [Fre82], $W$ is homeomorphic to $S^3 \times [0, 1]$ (but not necessarily diffeomorphic). We conclude that $K = J$ in $C_{\text{ex}}$.

If we remove the word smooth in the above discussion we obtain the proof in the case $* = \text{top}$. If we replace $S^3 \times [0, 1]$ with a (smooth or topological) $R$–homology cobordism then we complete the proof in the cases $* = R, (R, \text{ top})$. □
3.2. Homology cobordism classes of homology cylinders. In [Lev01] Levine defined the group of integral homology cylinders over a surface, with the goal of producing an enlargement of the mapping class group. For completeness, and since we require slight variants and generalizations, we recall the definitions below.

**Definition 3.3** (See [Lev01]). Let \( T = S^1 \times S^1 \) be the torus and \( R \) a localization of \( \mathbb{Z} \). An \( R \)-**homology cylinder on** \( T \), or an \( R \)-**cylinder**, is a triple \((V, i_+, i_-)\) where

- \( V \) is a 3–manifold
- For \( \epsilon = \pm 1 \), \( i_\epsilon : T \to \partial V \) is an embedding
- \( i_+ \) is orientation-preserving and \( i_- \) is orientation-reversing
- \( \partial V = i_+(T) \sqcup i_-(T) \)
- \( (i_\epsilon)_* : H_*(T; R) \to H_*(V; R) \) is an isomorphism

A \( \mathbb{Z} \)-cylinder \((V, i_+, i_-)\) is called a **strong cylinder** if \( \pi_1(V) \) is normally generated by each of \( \text{Im}(i_+)_* \) and \( \text{Im}(i_-)_* \).

Let \( H_R \) denote the set of all \( R \)-cylinders and \( H_{\text{str}} \) denote the set of all strong cylinders.

For \(* \in \{\text{str}, R\}\), there is a monoid operation on \( H_* \) given by stacking:

\[
(V, i_+, i_-) \ast (U, j_+, j_-) = ((V \cup U)/(i_+ = j_-), j_+, i_-)
\]

where \((T \times [0, 1], \text{Id} \times \{1\}, \text{Id} \times \{0\})\) is the identity element with respect to this operation.

**Definition 3.4** (See [Lev01]). Two \( R \)-cylinders \((V, i_+, i_-)\) and \((U, j_+, j_-)\) are said to be **\( R \)-cobordant** if there is smooth 4–manifold \( W \) with \( \partial W = (V \cup -U)/(i_+ = j_+, i_- = j_-) \) such that \( H_*(V; R) \to H_*(W; R) \) and \( H_*(U; R) \to H_*(W; R) \) are isomorphisms. Such a \( W \) is called an **\( R \)-cobordism**. Equivalently, one could require that the composition \( H_*(T; R) \xrightarrow{(i_\epsilon)_*} H_*(V; R) \to H_*(W; R) \) is an isomorphism. \( \mathcal{H}_R \) is the set of all \( R \)-cobordism classes of \( R \)-cylinders.

If the 4–manifold \( W \) in the above paragraph is only a topological 4–manifold, then \((V, i_+, i_-)\) and \((U, j_+, j_-)\) are called **topologically \( R \)-cobordant**. \( \mathcal{H}_{R, \text{top}} \) is the set of all topological \( R \)-cobordism classes of \( R \)-cylinders.

Two strong cylinders \((V, i_+, i_-)\) and \((U, j_+, j_-)\) are said to be **strongly cobordant** if there exists a smooth \( \mathbb{Z} \)-cobordism \( W \) between \( V \) and \( U \) such that \( \pi_1(W) \) is normally generated by each of \( \pi_1(V) \) and \( \pi_1(U) \). Such a \( W \) is called a **strong cobordism**. This is equivalent to the image of \( \pi_1(T) \xrightarrow{(i_\epsilon)_*} \pi_1(V) \to \pi_1(W) \) normally generating. \( \mathcal{H}_{\text{str}} \) is the set of all strong cobordism classes of strong cylinders.

Similarly, if we remove the smooth condition then \((V, i_+, i_-)\) and \((U, j_+, j_-)\) are called **strongly topologically cobordant**. \( \mathcal{H}_{\text{top}} \) is the set of strong topological cobordism classes of strong cylinders.
In [Lev01], Levine proves that the binary operation $*$ on $H_Z$ is well-defined on $H_Z$ and that $H_Z$ is a group. Indeed, $V \times [0, 1]$ can be seen to be a cobordism between $(V, i_+, i_-) \ast (-V, i_-, i_+)$ and the identity element of the monoid operation $(T \times [0, 1], \Id \times \{0\}, \Id \times \{1\})$. Thus, the group inverse is given by $(V, i_+, i_-)^{-1} = (-V, i_-, i_+)$, where $-V$ denotes the orientation-reverse of $V$. Clearly $V \times [0, 1]$ is also a (smooth and topological) $R$–cobordism so that $\mathcal{H}_R$ and $\mathcal{H}_{R, \text{top}}$ are groups for all $R$. Similarly, in the case that $V$ is a strong cylinder it is easy to see that $V \times [0, 1]$ is a (smooth and topological) strong cobordism. Thus, $\mathcal{H}_{\text{str}}$ and $\mathcal{H}_{\text{top}}$ are also groups.

3.3. Patterns as homology cylinders. For any pattern $P \in \mathcal{S}_R$, the exterior of $P$, $E(P)$, can be seen to be an $R$–homology cylinder in a natural way. Let $i_+$ be the identification $S^1 \times S^1 \rightarrow \partial V$ sending $m \mapsto m(V)$ and $\ell \mapsto \ell(V)$. Similarly let $i_-$ be the identification of the boundary of a tubular neighborhood of $P$ with $S^1 \times S^1$ which sends $\ell \mapsto \ell(P)$ and $m \mapsto m(P)$. A Mayer–Vietoris argument easily reveals that $(E(P), i_+, i_-) \in H_R$ is an $R$–cylinder. It follows immediately from our definitions that if $P \in \mathcal{S}_{\text{str}}$ is strong winding number one then $(E(P), i_+, i_-) \in H_{\text{str}}$. Henceforth we will often abuse notation by letting $E(P)$ denote the $*$–cylinder $(E(P), i_+, i_-)$, where $* \in \{\text{str, top, } R, (R, \text{top})\}$.

For each $*$, we have a map $E : \mathcal{S}_* \rightarrow H_*$ sending a satellite operator to its exterior. It is easy to check that $E$ is a monoid homomorphism.

Notice the following fact. Let $P$ be a pattern of winding number $w \neq 0$, and $E(P) = (E(P), i_+, i_-)$ be the corresponding cylinder. $\ell(P)$ is homologous in $E(P)$ to $w \cdot \ell(V)$ and $m(P)$ is homologous to $(1/w) \cdot m(V)$. Thus, with respect to the basis $\{\ell, m\}$ for $H_1(S^1 \times S^1)$, the composition $(i_+^{-1} \circ i_-)$ is given by the matrix $\begin{bmatrix} w & 0 \\ 0 & 1/w \end{bmatrix}$.

Motivated by this we make the following definition.

**Definition 3.5.** Let $R$ be a localization of $\mathbb{Z}$ and $\hat{\mathcal{S}}_R^0 \subseteq H_R$ be the submonoid of all $R$–cylinders $(Y, i_+, i_-)$ for which the map $(i_+^{-1} \circ (i_-)_*) : H_1(T; R) \rightarrow H_1(T; R)$ has determinant $1$ and is diagonal with respect to the basis $\{\ell, m\}$.

$\hat{\mathcal{S}}_{\text{str}}^0 \subseteq H_{\text{str}}$ is the submonoid of all strong cylinders $(Y, i_+, i_-)$ for which the map $(i_+^{-1} \circ (i_-)_*) : H_1(T; \mathbb{Z}) \rightarrow H_1(T; \mathbb{Z})$ is $\pm \Id$.

Elements of $\hat{\mathcal{S}}_*^0$ will be called **generalized $*$–satellite operators**.

As a result of the preceding analysis we see the following result.

**Proposition 3.6.** For $* \in \{R, \text{str}\}$ there is a monoid homomorphism $E_* : \mathcal{S}_* \rightarrow \hat{\mathcal{S}}_*^0$. 

The submonoids \( \hat{\mathcal{S}}_*^0 \) of \( H_* \) are closed under the action of taking inverses, so that each of
\[
\hat{\mathcal{S}}_{\text{ex}} := \frac{\mathcal{S}_{\text{str}}^0}{\text{strong cobordism}} \quad \hat{\mathcal{S}}_{\text{top}} := \frac{\mathcal{S}_{\text{top}}^0}{\text{strong topological cobordism}}
\]
\[
\hat{\mathcal{S}}_R := \frac{\mathcal{S}_{R}^0}{R\text{--cobordism}} \quad \hat{\mathcal{S}}_{R,\text{top}} := \frac{\mathcal{S}_{R,\text{top}}^0}{\text{topological } R\text{--cobordism}}
\]
is a a subgroup of the appropriate \( H_* \).

By considering the compositions \( \hat{\mathcal{S}}_* \xrightarrow{E} \hat{\mathcal{S}}_*^0 \rightarrow \hat{\mathcal{S}}_* \) we see the following result.

**Proposition 3.7.** There are monoid homomorphisms each of which we call \( E : \mathcal{S}_R \rightarrow \hat{\mathcal{S}}_{R}^0, \mathcal{S}_R \rightarrow \hat{\mathcal{S}}_{R,\text{top}}, \mathcal{S}_{\text{str}} \rightarrow \hat{\mathcal{S}}_{\text{ex}}, \mathcal{S}_{\text{str}} \rightarrow \hat{\mathcal{S}}_{\text{top}}. \)

### 3.4 Generalized satellite operators act on knots in homology spheres.

In this section, we will show that for \( * \in \{ \text{ex, top, } R, (R, \text{ top}) \} \), \( \hat{\mathcal{S}}_* \) has a well-defined action on \( \hat{\mathcal{C}}_* \).

**Proposition 3.8.** The monoid \( \hat{\mathcal{S}}_{R}^0 \) acts on the set of knots in \( R\text{--homology spheres}. \) For the map \( E : \mathcal{S}_R \rightarrow \hat{\mathcal{S}}_{R}^0, \) if \( P \in \mathcal{S}_R \) and \( K \subseteq S^3 \) is a knot, \((P(K), S^3)\) is isotopic to \((E(P))(K, S^3)\).

Since \( \mathcal{S}_{\text{str}}^0 \subseteq \mathcal{S}_*, \hat{\mathcal{S}}_{\text{str}}^0 \) also acts on the set of knots in \( \mathbb{Z}\text{-homology spheres}. \)

**Proof.** Let \((K, Y)\) be a knot in an \( R\text{--homology sphere}. \) Let \((X, i_+, i_-)\) be an \( R\text{--homology cylinder}. \) Construct a 3–manifold \( Y' \) as follows—start with \( X \), glue on a solid torus along \( i_-(T) \) such that \( i_-(m) \) bounds a disk, and glue in \( Y - K \) such that \( i_+(t) \sim t(K) \) and \( i_+(m) \sim m(K) \). Therefore:
\[
Y' = S^1 \times D^2 \cup_{\partial D^2 \sim i_-(m)} X \cup_{i_+(m) \sim m(K)} Y - K.
\]

It is easy to check that \( Y' \) is an \( R\text{--homology sphere} \) when \( (X, i_+, i_-) \in \hat{\mathcal{S}}_R \).

Let \( K' \) be the core of the solid torus \( S^1 \times D^2 \) in this decomposition. The above construction gives the desired action on knots in \( R\text{--homology spheres}, \) that is,
\[
(X, i_+, i_-) \cdot (K, Y) := (K', Y')
\]

It is straightforward to see that for all homology cylinders \( X \) and \( W \) and any knot in an \( R\text{--homology sphere} \) \((K, Y)\), \((X \ast W) \cdot (K, Y) = X \cdot (W \cdot (K, Y)) \).

Therefore we have a monoid action. \((P(K), S^3)\) is isotopic to \((E(P))(K, S^3)\) since the gluing instructions given above are identical to those in the classical satellite construction. \( \square \)

It remains only to show that this action of generalized satellite operators on knots in homology spheres induces an action of \( \hat{\mathcal{S}}_* \) on \( \hat{\mathcal{C}}_* \).

**Proposition 3.9.** Let \( R \) be a localization of \( \mathbb{Z} \) and \( * \in \{ \text{ex, top, } R, (R, \text{ top}) \} \). Suppose \((V, i_+, i_-), (U, j_+, j_-) \in \hat{\mathcal{S}}_*^0 \) are \(*\text{-cobordant} \) and \((K, Y), (J, X) \in \hat{\mathcal{C}}_* \) are \(*\text{-concordant} \). Then \((V, i_+, i_-) \cdot (K, Y) \) is \(*\text{-concordant} \) to \((U, j_+, j_-) \cdot (J, X) \).
Therefore, for each choice of $\ast \in \{\text{ex}, \text{top}, R, (R, \text{top})\}$, $\hat{S}_\ast$ acts on $\hat{C}_\ast$.

Proof. The proof is the same in spirit as the proof that classical satellite operators are well-defined on concordance, and is guided by the schematic picture in Figure 5.

Recall that $(V, i_+, i_-) \cdot (K, Y)$ and $(U, j_+, j_-) \cdot (J, X)$ are knots in the 3-manifolds $Y' = S^1 \times D^2 \cup V \cup E(K)$ and $X' = S^1 \times D^2 \cup U \cup E(J)$. To be precise, the knots are given by the cores of the $S^1 \times D^2$-pieces.

Since $(K, Y)$ and $(J, X)$ are $\ast$-concordant, we have a concordance $C$ between them in some 4-manifold. Let $E(C)$ be the complement of $C$. Since $(V, i_+, i_-)$ and $(U, j_+, j_-)$ are $\ast$-cobordant there is a $\ast$-cobordism $W_0$ between $V$ and $U$.

The gluing instructions used to build $X'$ and $Y'$ extend to gluing instructions for a 4-manifold

$$W = S^1 \times D^2 \times [0, 1] \cup W_0 \cup E(C)$$

with $\partial W = Y' \sqcup -X'$ and in which $(V, i_+, i_-) \cdot (K, Y)$ and $(U, j_+, j_-) \cdot (J, X)$ cobound the annulus given by the core of $S^1 \times D^2 \times [0, 1]$.

From here the proof is most obvious in the case $\ast = (R, \text{top})$. Since each of the building blocks of $W = S^1 \times D^2 \times [0, 1] \cup W_0 \cup E(C)$ is an $R$-homology cobordism, it follows from a Mayer–Vietoris argument that $W$ is an $R$–homology cobordism in which there is a locally flat concordance (the core of $S^1 \times D^2 \times [0, 1]$) between the knots.

Suppose $\ast = R$. We saw in the $\ast = (R, \text{top})$ case that $W$ is an $R$–homology cobordism in which the knots cobound an annulus. Since each of the pieces of...
W is smooth, W is smooth as well. Furthermore the concordance is smoothly embedded.

Suppose \(* = \text{top}*. Since each of the pieces of W has fundamental group normally generated by the boundary pieces, W has fundamental group normally generated by each of its boundary components by a Seifert–Van Kampen argument. As in the \(* = (R, \text{top})* case the concordance is locally flat.

Suppose \(* = \text{ex}*. As in the \(* = \text{top})* case, we see that the fundamental group of W is normally generated by its boundary components. As in the \(* = R)* case W is smooth and the concordance is smoothly embedded. □

Finally we combine the results of this section together to prove the Main Theorem.

**Proof of Main Theorem.** By Proposition 3.2, we have an injection \(\Psi : \mathcal{C}_* \rightarrow \hat{\mathcal{C}}_*\).

By Proposition 3.7 we have a group homomorphism \(E\). By Proposition 3.9, \(\hat{S}_*\) acts on \(\hat{\mathcal{C}}_*\). By Proposition 3.8, \(\Psi(P(K)) = (E(P)) (\Psi(K))\) for all \(K \in \mathcal{C}_*\). This completes the proof. □

4. Surjectivity of satellite operators

Since satellite operators have now been shown to be injective in several categories (in Section 2 and [CDR12]), it is natural to ask whether there exists a non-trivial satellite operator that is *surjective* on knot concordance. In particular, we ask the following question:

**Question 4.1.** For \(P \in \mathcal{S}_*\) is the map \(P : \mathcal{C}_* \rightarrow \mathcal{C}_*\) surjective?

We seek satellite operators that are not the same as a connected-sum operator (Figure 4), since such operators are easily seen to be bijective.

Our notion of generalized satellite operators allows us to approach the surjectivity question in a novel way. For any honest satellite operator \(P \in \mathcal{S}_*\), \(E(P) \in \hat{S}_*\) has an inverse when considered as an element of the group \(\hat{S}_*\). If \(E(P)^{-1}\) is the image of an honest satellite operator (under \(\mathcal{S}_* \rightarrow \hat{S}_*\)), \(P\) must have been surjective on \(\mathcal{C}_*\). As a result, we are led to ask:

**Question 4.2.** Given a satellite operator \(P \in \mathcal{S}_*\), when is \(E(P)^{-1} \in \hat{S}_*\) in the image of \(E : \mathcal{S}_* \rightarrow \hat{S}_*\)?

For instance, if \(P = Q_K\) is the satellite operator corresponding to connected-sum with some knot \(K\), \(E(P)^{-1} = Q_{-K}\) (in \(\hat{S}_*\)) is the operator corresponding to connected-sum with \(-K\), the concordance inverse of \(K\). Are there any examples which are not connected-sum operators?

In fact, we actually require less of \(E(P)^{-1}\) in order to infer that \(P\) is surjective on \(\mathcal{C}_*\). Indeed, as long as \(E(P)^{-1}(\mathcal{C}_*) \subseteq \mathcal{C}_*\), we may infer that \(P\) is surjective on \(\mathcal{C}_*\). This leads us to the following two questions:

**Question 4.3.** Given a satellite operator \(P \in \mathcal{S}_*\), is \(E(P)^{-1}(\mathcal{C}_*) \subseteq \mathcal{C}_*\)?
Question 4.4. Given a generalized satellite operator \( P \in \hat{S}_* \) with \( P(C_*) \subseteq C_* \), is \( P \) in the image \( E : S_* \to \hat{S}_* \)?

In the remainder of this paper we make some progress towards answering the above questions. The following proposition shows that if \( P \) is a pattern with winding number not equal to \( \pm 1 \) then the answer to Question 4.1 is no. It remains only to consider operators with winding number \( \pm 1 \):

Proposition 4.5. Satellite operators with winding number \( n \), with \( |n| > 1 \), are not surjective, even on \( C_Q \).

Proof. Consider a pattern \( P \) of winding number \( n \) acting on a knot \( K \). If we let \( \sigma(\cdot, \omega) \) denote the Levine-Tristram signatures of a knot at \( \omega \in \mathbb{C}, |\omega| = 1 \), we know from [Lit79, LM85] that

\[
\sigma(P(K), \omega) = \sigma(P, \omega) + \sigma(K, \omega^n)
\]

Since \( P \) is fixed, this imposes restrictions on the signature function of \( P(K) \); for example, if \( P \) were slice when considered as a knot in \( S^3 \), this would imply that \( P(K) \) has a periodic signature function. Since the jump function of the signature function is an invariant of rational concordance, [CK02, Theorem 1.1] a knot with non-periodic signature function (for example the trefoil knot) is not in the image of \( P : C_Q \to C_Q \), which therefore is not surjective. \( \square \)

Proposition 4.6. All satellite operators in \( S_* \) are surjective on \( C_* \) if and only if all satellite operators in \( S_* \) have the unknot in their images as maps from \( C_* \) to \( C_* \).

Proof. The \( \Rightarrow \) direction is trivial. For the \( \Leftarrow \) direction, start with any \( P \in S_* \) and any knot \( K \). Construct the operator \( \overline{P} = Q_{-K} \ast P \in S_* \) (see Figure 4). It is easy to check that \( Q_K \) is in \( S_* \), so that \( \overline{P} \) is in \( S_* \). Thus, by assumption there is some knot \( J \) such that \( \overline{P}(J) = O \) is the \( \ast \)-concordance class of the unknot. But note that \( \overline{P}(J) = Q_{-K}(P(J)) = -K \# P(J) \). Therefore, \( -K \# P(J) = O \), i.e. \( P(J) = K \). \( \square \)

Corollary 4.7. For any integer \( n, n \neq \pm 1 \) and \( R = \mathbb{Z}[1/n] \), there exist satellite operators in \( S_R \) which do not have the unknot in their image, even as maps on \( C_Q \).

Proof. The proof is straightforward using the above two propositions. \( \square \)

5. Examples of bijective operators

In this section we prove that the satellite operator of Figure 2 is bijective on \( C_* \) for each choice of \( \ast \in \{ \text{ex}, \text{top} \} \). In order to do so we will need the following proposition.

Proposition 5.1. Let \( P \in S_Z \) be a pattern of winding number one. If \( m(P) \) is in the normal subgroup of \( \pi_1(E(P)) \) generated by \( m(V) \) then \( P : C_* \to C_* \) is bijective for \( \ast = \text{ex} \) or \( \text{top} \).
Proof. Motivated by the discussion following Question 3 we see that it is relevant to check whether $E(P)^{-1} \in \hat{S}_*\text{ sends } \hat{C}_* \to \hat{C}_*$. Let $K$ be any knot in $S^3$. Then $E(P)^{-1}(K)$ is a knot in the following 3–manifold:

$$Y = S^1 \times D^2 \cup_{\partial D^2 \sim m(V)} E(P) \cup_{m(P) \sim m(K)} E(K) \cup_{\ell(P) \sim \ell(K)} E(K)$$

Notice that for any pattern $P$ with winding number 1, $\{m(P), \ell(P)\}$ normally generates $\pi_1(E(P))$. In the group

$$\pi_1 \left( S^1 \times D^2 \cup_{\partial D^2 \sim m(V)} E(P) \right),$$

$m(V)$ (and therefore $m(P)$) is trivial. In

$$\pi_1 \left( E(P) \cup_{m(P) \sim m(K)} E(K) \right),$$

$\ell(P) \sim \ell(K)$ is in the subgroup normally generated by $m(K) \sim m(P)$. These facts together with a Seifert–Van Kampen argument reveals that $\pi_1(Y) = 0$. Then, by the 3–dimensional Poincaré Conjecture, $Y$ is diffeomorphic to $S^3$ and $E(P)^{-1}(K, S^3)$ is a knot in $S^3$.

Thus, $E(P)^{-1} : \hat{C}_* \to \hat{C}_*$ sends $\hat{C}_*$ to $\hat{C}_*$. A straightforward chase of diagrams 1 and 2 from the Main Theorem now shows that $P : \hat{C}_* \to \hat{C}_*$ is surjective. If $m(P)$ is normally generated by $m(V)$, then the sets $\{m(V), \ell(V)\}$ and $\{m(P), \ell(P)\}$ each normally generate $\pi_1(E(P))$ and therefore, $P$ is strong winding number one. We know that this implies that $P$ is injective, from Corollary 2.5 and Theorem 5.1 of [CDR12].

We are now ready to prove that the satellite operator of Figure 2 is surjective.

**Proposition 5.2.** The pattern $P$ of Figure 2 satisfies the conditions of Proposition 5.1 and therefore, $P : \hat{C}_* \to \hat{C}_*$ is surjective for $* = \text{ex or top}$.

Proof. Notice that $m(P)$ being contained in the normal subgroup generated by $m(V)$ is equivalent to $m(P)$ being nullhomotopic in the 3–manifold $N$ obtained from $E(P)$ by adding a 2–handle to $m(V)$. The result of sliding $P$ over this 2–handle (an isotopy in $V \cup 2$–handle) is depicted in Figure 6. In the result of the isotopy, the meridian of $P$ cobounds an annulus with the meridian of $V$ and so bounds a disk in $V$. This completes the proof.

It is very relevant to ask whether the operator shown in Figure 2 and any others with satisfy Proposition 5.1 are equivalent to connected-sum operators. That is, does there exist some knot $J$ such that $P(K) = Q_J(K) = J \# K$ for every knot $K$, where $P$ is the operator in Figure 2? (Such a $J$ would necessarily have to be $P(\text{unknot})$.) We have been unable to address this question at present.
Akbulut conjectured that there exists a winding number one satellite operator \( P \) which does not have the unknot in its image under \( P : \mathcal{C}_{\text{ex}} \to \mathcal{C}_{\text{ex}} \).

By Proposition 4.6 this conjecture is equivalent to the conjecture that not all winding number one operators are surjective. We restate Akbulut’s conjecture in these terms for \( * \in \{ \text{ex}, \text{top}, R, (R, \text{top}) \} \).

**Conjecture 6.1** (Problem 1.45 of [Kir97]). There is a satellite operator of winding number one, \( P \in \mathcal{S}_{*} \), such that \( P : \mathcal{C}_{*} \to \mathcal{C}_{*} \) is not surjective.

Matsumoto asked whether the only obstruction to a knot in a 3–manifold \( M \) being concordant (in the sense of \( \hat{\mathcal{C}}_{*} \), see Definition 3.1) to a knot in \( S^3 \) is the *–cobordism class of \( M \). (This is slightly dishonest, as Matsumoto’s question is in the piecewise linear category, which is not one of the categories discussed in this paper.) We restate Matsumoto’s question as a conjecture.

**Conjecture 6.2** (Problem 1.31 of [Kir97]). The image of \( \Psi : \mathcal{C}_{*} \to \hat{\mathcal{C}}_{*} \) is the set of all concordance classes \( (K, M) \) of knots \( K \) in 3–manifolds \( M \) where \( M \) is *–cobordant to \( S^3 \).

We use the group action given in the Main Theorem to prove that these two conjectures cannot simultaneously be true.

**Proposition 6.3.** For any \( P \in \mathcal{S}_{*} \) and any \( K \in \mathcal{C}_{*} \), if \( K \notin \text{Im}(P : \mathcal{C}_{*} \to \mathcal{C}_{*}) \) then \( E(P)^{-1}(\Psi(K)) \) is:

1. Not in the image of \( \Psi : \mathcal{C}_{*} \to \hat{\mathcal{C}}_{*} \) and
2. A knot in a 3–manifold which is *–cobordant to \( S^3 \).

**Proof.** To see the first claim, suppose that \( E(P)^{-1}(\Psi(K)) \) were equal in \( \hat{\mathcal{C}}_{*} \) to \( \Psi(J) \) for some \( J \in \mathcal{C}_{*} \) then

\[
\Psi(K) = (E(P)(E(P)^{-1}(\Psi(K)))) = E(P)(\Psi(J))
\]

so that by the diagrams (1) and (2) of the Main Theorem \( K = P(J) \) is in the image of \( P : \mathcal{C}_{*} \to \mathcal{C}_{*} \).

In order to see the second result we make use of the following lemma:
Lemma 6.4. If $P \in S_*$ and $(K,Y) \in \widehat{C}_*$, then $E(P)^{-1}(K,Y)$ is a knot in a 3–manifold which is $\ast$–cobordant to $Y$.

Proof. First we consider $P \in S_*$, then $E(P)(K,Y)$ is a knot in the 3–manifold

$$Y' = S^1 \times D^2 \cup_{\partial D^2 \sim m(P)} E(P) \cup_{m(V) \sim m(K)} Y - K$$

but $S^1 \times D^2 \cup E(P)$ is a solid torus with meridian $m(V)$. These gluing instructions cut a solid torus out of $Y$ and then glue it back in the same way. Therefore, $Y'$ is the same as $Y$ and $E(P)(K,Y)$ is a knot in $Y$.

Let $E(P)^{-1}(K,Y) = (K',Y')$ and $E(P)(K',Y') = (K'',Y'')$. By the preceding analysis, $Y'' = Y'$. Since $E(P) \circ E(P)^{-1}$ is the identity map on $\widehat{C}$, $K$ is concordant to $K''$ in a $\ast$–cobordism between $Y$ and $Y''$. In particular $Y$ is $\ast$–cobordant to $Y'' = Y'$ and $E(P)^{-1}(K,Y) = (K',Y')$ is a knot in a 3–manifold which is $\ast$–cobordant to $Y'$, as claimed. \qed

The second claim now follows immediately. Indeed, since $\Psi(K) = (K,S^3)$ is a knot in $S^3$, $E(P)^{-1}(\Psi(K)) = E(P)^{-1}(K,S^3)$ is a knot in a 3–manifold which is $\ast$–cobordant to $S^3$. \qed

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Department of Mathematics, The University of Wisconsin at Eau Claire

E-mail address: daviscw@uwec.edu

URL: www.math.osu.edu/~Davis.3929

Department of Mathematics, Rice University

E-mail address: arunima.ray@rice.edu

URL: www.math.rice.edu/~ar25