Low-energy behavior of the spin-tube and spin-orbital models

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(March 24, 2022)

The low-energy effective Hamiltonian of three coupled spin chains with periodic boundary conditions (spin tube) is expressed, in the limit of strong interchain coupling, in terms of XXZ chains coupled by biquadratic exchange interaction. A similar effective model was also proposed to describe the coupling of spins to orbital degrees of freedom in materials such as NaV$_2$O$_5$. We investigate the effective model by means of bosonization and renormalization group techniques, and find that the generic phase diagram comprises a gapless region and gapped regions consisting of a spin liquid phase and various antiferromagnetic phases. We discuss the properties of the spin liquid phase, in particular the nature of the ground state and of the elementary excitations above it. We then study the effect of a magnetic field, and conclude that a strong enough magnetic field can suppress the dimerized phase leading to a two component Luttinger liquid. The critical exponents at the transition gapful-gapless are calculated and shown to be non-universal in the spin tube case or the generic spin orbital problem.

PACS numbers:75.10.Jm, 75.30.Kz, 75.40.Gb

Keywords: spin-tube, bosonization and renormalization group, spin-liquid

I. INTRODUCTION

Coupled spin 1/2 chains have attracted much attention lately due to the large number of their experimental realizations, as well as to the variety of theoretical techniques, both analytical and numerical, available to study the relevant models. Spin 1/2 chains, owing to the Jordan-Wigner transformation, show properties remarkably similar to those of interacting one dimensional fermions with their low energy properties described by an effective Luttinger liquid theory. Recently it has been realized that these properties are drastically modified when the spin 1/2 chains are coupled together forming ladder systems. In this case, in a way very similar to Haldane's spin-S problem, a gap is found to open for an even number of chains while the system remains gapless if the number of chains is odd. This phenomenon has been thoroughly investigated both analytically and numerically, and corresponding experimental systems were identified. Typical examples of two-chain spin 1/2 ladders exhibiting a gap are SrCu$_2$O$_3$ and Cu$_2$(C$_5$H$_{12}$N$_2$)$_2$Cl. On the other hand, an example of a gapless three-chain ladder is Sr$_2$Cu$_3$O$_6$. From the theoretical point of view, the difference between odd and even number of legs is odd, and even spin has been understood qualitatively through a generalization of the Lieb-Schultz-Mattis Theorem. The theorem states that if the ground state is unique the system will necessarily be gapless. This is indeed the case for an odd number of coupled spin 1/2 chains.

More recent theoretical work has emphasized the role of boundary conditions in the transverse direction in the formation of spin gaps. The results quoted above - the opening of a gap for an even number of coupled chains, gaplessness if the number is odd - are valid for open boundary conditions (OBC). It turns out that when periodic boundary conditions (PBC) are imposed on the transverse direction a gap opens for both cases of even and odd number of chains although the underlying reasons and the natures of the gaps are different. In the case of an even number of coupled chains, the reason for the gap is the formation of spin singlets along the transverse direction, similarly to the case of chains with open boundary conditions. When the number of coupled chains is odd a two fold degenerate dimerized ground state is obtained in the case of PBC - in contrast to its uniqueness in the OBC - allowing for a gap in the spin excitations. The degeneracy of the ground state in the case of PBC can be understood as a consequence of the fact that PBC are frustrating for an odd number of legs. To date, no experimental system described by coupled antiferromagnetic spin 1/2 chains with periodic boundary conditions and an odd number of legs has been reported.

When, further, leg-leg biquadratic interactions are included new states emerge. A system of two spin-1/2 spin chains enters a spontaneously dimerized phase with a gapped spectrum exhibiting non-Haldane spin-liquid properties. The elementary excitations are neither spinons nor magnons, but pairs of propagating triplet or singlet solitons connecting...
two spontaneously dimerized ground states. Subsequently more non-Haldane spin-liquid models have been proposed. Recently, these models of spin ladders with biquadratic exchange have been advocated as possible models for the formation of a spin gap in NaV$_2$O$_5$ and Na$_2$Ti$_2$Sb$_2$O.

In this paper we study the low energy physics of the three-leg ladder with periodic boundary conditions (see Fig. 1). This model is called the spin-tube model in the following. By analyzing in detail its effective low energy Hamiltonian (LEH), which consists of two non-equivalent coupled XXZ chains of spins and chiral degrees of freedom, in the presence of a biquadratic exchange, we will show that the dimerized ground state of this model falls in the universality class of the non-Haldane spin liquids. In the case of XXZ chains with a biquadratic coupling and for $J_z/J > 1$, we find that the Ising antiferromagnetic order can compete with the dimer order, and we will describe the resulting phase diagram.

The organization of the paper is as follows. In Sec. 1 we introduce the spin tube model. We sketch the derivation of the strong interchain coupling effective Hamiltonian and discuss its relation with two spin 1/2 chains coupled by a biquadratic interaction. We then discuss the symmetries and recall the results already known on the two chains with biquadratic interactions, in particular the prediction of a spin gap.

In Sec. 11 we discuss the bosonization treatment of two non-equivalent XXZ chains coupled by a biquadratic interaction. We show that this Hamiltonian contains a term $\sin 2\phi_1 \sin 2\phi_2$ that is responsible for the formation of a spin-gap and singlet order, as well as terms $\cos 4\phi_{1,2}$ that cause antiferromagnetic order. To analyze the competitions of these terms, we derive renormalization group equations. Using these equations, we estimate in which part of the phase diagram we should expect a competition of antiferromagnetism and dimer order. We also estimate the equation of the phase boundary between the singlet and the Antiferromagnetic order.

In Sec. 1V we analyze in details the dimerized phase. The dimerized ground state is two-fold degenerate and is formed of alternating singlets of spins and pseudospins. The elementary excitations above the ground state carry a two component Luttinger liquid. In the case of spin-orbital models, spin and orbital degrees of freedom decouple completely but the presence of orbital modes should affect the specific heat. In the case of the spin-tube, we show that in contrast to spin-orbital models, the spin-correlation functions are affected by the presence of auxiliary gapless modes. We also show that for non-equivalent chains (a case that is realized in spin-orbital models) the exponent of the spin correlations is non-universal at the transition. Finally, in Sec. 1VI we will give some concluding remarks.

II. THE SPIN-TUBE MODEL AND COUPLED XXZ CHAINS

A. The three chain ladder with periodic boundary conditions

We wish to study the three-chain ladder Hamiltonian,

$$H = J \sum_{i=1}^{N} \sum_{p=1}^{3} \vec{S}_{i,p} \vec{S}_{i+1,p} + J_\perp \sum_{i=1}^{N} \sum_{p=1}^{3} \vec{S}_{i,p} \vec{S}_{i+1,p},$$

(2.1)

where $p$ (resp. $i$) is a chain (resp. site) index, $J$ is the coupling along the chain and $J_\perp$ the transverse coupling. We impose periodic boundary conditions along the rungs by identifying $\vec{S}_{i,4} = \vec{S}_{i,1}$. We call the resulting model the spin-tube (see Fig. 1) since it can be realized by placing 3 spin 1/2 chains forming an equilateral triangle. In order to investigate the low-energy physics of the spin-tube, we consider the limit of strong interchain coupling ($J_\perp \gg J$). This is the appropriate starting point that yields a good effective description of the properties of the spin tube in the whole range of $J_\perp/J$. (Starting from the opposite limit $J_\perp \ll J$, and treating $J_\perp$ as a perturbation, one finds that it gives rise to relevant terms in the Hamiltonian. As a result, the initial $J_\perp$ grows until it is of order of $J$ at which point the weak coupling bosonization scheme is no more valid.)

To derive the effective low-energy Hamiltonian, let us first consider the case $J = 0$. The system is a collection of independent rungs, each described by the following Hamiltonian

$$H = J_\perp (\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1) = J_\perp ((\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 - 9/4)/2.$$

(2.2)
The ground state of each rung is four-fold degenerate, composed of two doublets of spin 1/2 excitations, corresponding to the left and right chirality (−/+), with energy \((-3/4J_{\perp})\) \(^{32}\), and the excited states form a spin 3/2 quadruplet with energy \(3/4J_{\perp}\).

Turning on \(J > 0\) allows the rungs to exchange spins. In the limit \(J_{\perp} \gg J\), the quadruplet of \(S=3/2\) excitations can be neglected, and only the degenerate low energy subspace of spin 1/2 states needs to be taken into account. In this subspace the Hamiltonian transforms into an effective Hamiltonian with a biquadratic coupling between the spin and chirality degrees of freedom \(^{23}\),

\[
H_{\text{eff}} = \frac{J}{3} \sum_i \hat{S}_i \hat{S}_{i+1} [1 + 2(\tau_i^z \tau_{i+1}^z + \tau_i^y \tau_{i+1}^y)],
\]

(2.3)

where \(S\) is the total spin, and \(\tau_i\) are operators exchanging left and right chiralities. The original spin operators can be expressed in terms of the effective spin \(S\) and the chirality \(\tau\) in the following way:

\[
\begin{align*}
S^+_{i,p} &= -\frac{1}{3} S_1^+ + \frac{2}{3} j_2^p S^+_{i} \tau_i^- + \frac{2}{3} j_3^p S^+_{i} \tau_i^+ \\
S^z_{i,p} &= \frac{1}{3} S_1^z + \frac{2}{3} j_2^p S^z_{i} \tau_i^- + \frac{2}{3} j_3^p S^z_{i} \tau_i^+ 
\end{align*}
\]

(2.4)

The spin tube model has already been investigated numerically \(^{15}\) using DMRG in the case of coupled XXX chains. The system was shown to exhibit a spin gap \(\Delta = 0.28J\), with exponentially decaying correlation functions \((-)^n \langle S_0^z S_n^z \rangle\) and \((-)^n \langle \tau_0^z \tau_n^z \rangle\). This behavior was shown to be related to the formation of a dimer order. A qualitative discussion of the origin of this dimer order can be found \(^{25}\). In Ref. \(^{15}\) the dispersion relation of excitations having \(\tau^z = 0\) was obtained numerically for a system of 12 sites showing only gapped excitations.

In Ref. \(^{20}\) a generalization of the effective Hamiltonian Eq. (2.3) to coupled anisotropic spin chains has been derived \(^{20}\), \(^{21}\). Numerical diagonalizations were performed. A gap to \(S^z = 1\) excitations was obtained for \(0 < J_z/J < 1.2\) with results in good agreement with those of Ref. \(^{15}\) for \(J_z = J\). It was also shown that the ground state was degenerate in agreement with the dimer order picture of Ref. \(^{15}\). The dispersion relation of excitations having \(\tau^z = 0\) and total spin \(S = 0\) or \(S = 1\) was obtained \(^{22}\). It was shown to be the bottom of a two particle continuum. The fundamental particle, the spinon, was conjectured to have \(\tau^z = \pm 1/2\), \(S = 1/2\). The spinon dispersion relation was obtained numerically by considering a system with an odd number of sites \(^{22}\), showing that the spinons were massive for all momentum.

**B. Generalized spin-tube model and XXZ chains coupled by a biquadratic exchange**

The Hamiltonian (2.3) is part of the class of the Hamiltonians consisting of two non-equivalent coupled XXZ chains in the presence of a biquadratic exchange,

\[
H = H_0 + H_B,
\]

(2.5)

where

\[
H_0 = \sum_i \sum_{\alpha = 1,2} J_{\alpha}(S_{\alpha,i}^x S_{\alpha,i+1}^x + S_{\alpha,i}^y S_{\alpha,i+1}^y) + J_{\alpha}^z S_{\alpha,i}^z S_{\alpha,i+1}^z
\]

(2.6)

\[
H_B = \gamma \sum_i (S_{1,i}^x S_{1,i+1}^x + S_{1,i}^y S_{1,i+1}^y + \Delta_1 S_{1,i}^z S_{1,i+1}^z) (S_{2,i}^x S_{2,i+1}^x + S_{2,i}^y S_{2,i+1}^y + \Delta_2 S_{2,i}^z S_{2,i+1}^z).
\]

(2.7)

In the case of the spin-tube, \(S_{1,i}\) corresponds to spin \(S_i\), and \(S_{2,i}\) is associated with the chiral degrees of freedom \(\tau_i\). Another way of writing a class of Hamiltonian generalizing the spin tube model (2.3) is:

\[
H = \sum_{i=1}^N [u + \gamma(S_{i-1}^+ S_{i-1}^- + S_i^+ S_i^-) + J_z S_i^x S_{i+1}^x][v + \alpha(\tau_i^+ \tau_{i+1}^- + \tau_i^- \tau_{i+1}^+) + J'_z \tau_i^+ \tau_{i+1}^-].
\]

(2.8)

The effective Hamiltonian for the spin tube is obtained for,

\[
u = 1, \alpha = 1, J'_z = 0.
\]

(2.9)
For a tube made of XXZ chains, one has, instead:\[\]
\[
\begin{align*}
    u &= 0, \quad \gamma = J/6, \quad J_z = J/3 \Delta \\
    v &= 1, \quad \alpha = 1, \quad J'_z = 0.
\end{align*}
\]
(2.10)
The parameters \(J_z/\gamma\) and \(J'_z/\alpha\) in Eq. (2.8) measure the XXZ anisotropy for spin and chirality, respectively. When both of them are equal to 1, the Hamiltonian Eq. (2.8) is \(SU(2) \times SU(2)\) symmetric. For \(u = v, \quad \alpha = \gamma, \quad J_z = J'_z\) the two chains are equivalent, and can be parametrized as:
\[
    u = v = \frac{p}{2}, \quad \alpha = \gamma = 1, \quad J_z = J'_z = 2q.
\]
(2.11)
Hamiltonians of the type Eq. (2.8) can be mapped onto Hamiltonians of the type Eq. (2.3). The correspondence is given by:
\[
    \begin{align*}
    J_1 &= 2u\alpha, \quad J_2 = 2v\gamma \\
    J'_1 &= vJ_z, \quad J'_2 = uJ'_z \\
    \Delta_1 &= \frac{J_z}{2\gamma}, \quad \Delta_2 = \frac{J'_z}{2\alpha} \\
    \lambda &= 4\alpha\gamma
    \end{align*}
\]
(2.12)
with the identification \(\vec{S}_1 \equiv \vec{S}, \quad \vec{S}_2 \equiv \vec{\tau}\). Since writing the Hamiltonian in the form (2.3) is less restrictive than in the form (2.8) (i.e. the former includes the case \(\Delta_\alpha \neq \frac{J'_z}{\alpha}\)), we focus on the Hamiltonian (2.3). Hamiltonians of the type Eq. (2.3) are also encountered in a different context than the spin-tube model. In particular, a Hamiltonian of the type (2.3) has been proposed by Mostovoy and Khomskii as a model for the spin gap formation in the \(\text{NaV}_2\text{O}_5\) and \(\text{Na}_2\text{Ti}_2\text{Sb}_2\text{O}_6\) compounds. In that case, the \(S_1\) spins correspond to the real spin of the system whereas the \(S_2\) spins are pseudospins associated with orbital degrees of freedom. These spin-orbital models can be derived from a multiband Hubbard model. The derivation is reviewed for instance in Ref. 23.

Let us discuss first the case \(J_1 = J_2 = J'_1 = J'_2 = J, \quad \Delta_1 = \Delta_2 = 1\). The Hamiltonian describes two coupled Heisenberg chains with a biquadratic coupling preserving the \(SU(2)\) symmetry. Actually, the full symmetry group is larger than \(SU(2)\). One has:
\[
[H, \vec{S}_{1,\text{tot}}] = 0 \quad [H, \vec{S}_{2,\text{tot}}] = 0
\]
(2.13)
and the full symmetry group is therefore \(SU(2) \times SU(2)\) rather than the \(SU(2)\) symmetry that follows from \([H, \vec{S}_{1,\text{tot}} + \vec{S}_{2,\text{tot}}] = 0\). As a result the spectrum consists of \(SU(2) \times SU(2) \sim SO(4)\) multiplets. For \(J = \frac{1}{4}\), the Hamiltonian (2.3) has been shown to have an even larger \(SU(4)\) symmetry and to reduce to an integrable \(SU(4)\) spin chain.\[\]
The spectrum of the \(SU(4)\) spin chain has been obtained by the Bethe Ansatz and the correlations functions have been obtained by non-abelian bosonization techniques, identifying the low energy effective theory describing the spin chain as the \(SU(4)\) WZNW model. The integrable \(SU(4)\) spin-chain has also been intensively studied numerically using Quantum Monte Carlo (QMC) or Density Matrix Renormalization Group (DMRG) and Lanzcos Exact Diagonalization (ED) in the context of the spin-orbital models. The ground state energy and excited state energy were obtained in good agreement with analytical calculation using the Bethe Ansatz. The numerical calculation of the correlation functions reproduces results of the the continuum field theoretical treatment. At \(J \neq \lambda/4\) perturbations are generated which lower the \(SU(4)\) symmetry to \(SU(2) \times SU(2)\) and render the Hamiltonian non-integrable. These perturbations have been recently studied by describing the chain away from the integrable point as a perturbed \(SU(4)\) WZW model. It was found that for \(J < \lambda/4\), a gapless phase is obtained while for \(J > \lambda/4\), a gap is formed. A different field theoretical treatment in the limit \(\lambda \ll J\) also leads to the appearance of a gapped phase. For \(J = 3\lambda/4\), the ground state wavefunction could be obtained exactly in matrix product form of singlet states along the legs of the ladder. This picture is in good agreement with the predictions of the field theoretical treatment. The dependence of the gap on the coupling for the range \(0 < \lambda/J < 4\) was obtained by DMRG calculations. It was found that for \(\lambda/J \ll 1\), the gap increases proportionally to \(\lambda\) in agreement with the weak coupling bosonization treatment, and vanishes for \(\lambda/J = 1/4\), which is the \(SU(4)\) symmetric point as predicted. The DMRG calculations of Ref. 26 showed however a power law gap opening. Such power law gap opening can only be explained by the presence of a relevant operator in the continuum description. However, no such operator was obtained in the bosonization treatment. Moreover, if a relevant operator was present in the continuum theory, the absence of a gap at \(\lambda/J = 4\) would be the result of the coefficient of this operator vanishing precisely at \(\lambda/J = 4\).
then, in contrast to what is observed in numerical calculations, a gap would also obtain for $\lambda/J > 1/4$. A solution to this puzzle has been suggested recently\cite{22}. In Ref.\cite{22} the gap has been calculated by assuming that the first excited state was in the subspace $(S_1, S_2) = (1, 1)$. This assumption was shown to be incorrect in the gapped phase in which the first excited state lies in the subspace $(S_1, S_2) = (1, 0)$. When corrected\cite{23} a slow increase with $4 - \lambda/J$ is found, compatible with an exponential gap opening. The correlation functions\cite{24} do not show incommensurability, in agreement with the field theoretic approach\cite{25}.

Taking $J_1 = J_2 \neq J_3 = J_4$ and $\Delta_1 = \Delta_2 = 1$ in (2.3) preserves the $SU(2) \times SU(2)$ symmetry. This case has been investigated numerically\cite{26} with $J_1 = 0.7 J_2 - 0.3 \lambda / 4$. It was shown that a gapless–gapped phase transition obtained as $J_2 / \lambda$ was increased. This work was followed by analytical investigation based on the perturbed $SU(4)_1$ WZW continuum theory\cite{27}. The analytical investigations established the existence for a given $\lambda$ of an extended gapless region in the plane $J_1 - J_2$ that contains the line $J_1 = J_2 < \lambda / 4$ previously discussed.

In the present work, we consider the general case of $J_3^z \neq J_4$ and $\Delta_\alpha \neq 1$. This case includes in particular the spin tube model. We will focus on the regime $\lambda \ll J_{1,2}$ and apply methods similar to those of Ref.\cite{16}.

III. PHASE DIAGRAM

In this section, we derive the phase diagram of two XXZ spin chains weakly coupled by a biquadratic exchange. We first recall the derivation of the bosonized Hamiltonian and spin operators that describe an isolated XXZ chain. We follow the well known abelian bosonization procedure for spins\cite{18,22,23}. The XXZ spin chain is described by the Hamiltonian:

$$H_{XXZ} = J \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z \sum_i S_i^z S_{i+1}^z$$  \hspace{1cm} (3.1)

The XXZ spin chain Hamiltonian is first transformed into an interacting fermionic system on the lattice by expressing the spin operators $S^+, S^-, S^z$ in terms of fermion operators $a^\dagger, a$, using the Jordan-Wigner transformation\cite{22}:

$$S_i^+ = (-1)^i a_i^\dagger \cos \left( \pi \sum_{j=0}^{i-1} a_j^\dagger a_j \right)$$ \hspace{1cm} (3.2)

$$S_i^- = (-1)^i \cos \left( \pi \sum_{j=0}^{i-1} a_j^\dagger a_j \right) a_i$$ \hspace{1cm} (3.3)

$$S_i^z = a_i^\dagger a_i - \frac{1}{2}$$ \hspace{1cm} (3.4)

This transformation turns the XXZ Hamiltonian into a model of spinless fermions with nearest neighbor interaction described by the Hamiltonian,

$$H_{XXZ} = -\frac{J}{2} \sum_{i} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + J_z \sum_i (a_i^\dagger a_i - \frac{1}{2})(a_{i+1}^\dagger a_{i+1} - \frac{1}{2})$$ \hspace{1cm} (3.5)

For $J_z = 0$, the Hamiltonian (3.5) describes non-interacting fermions and is easily diagonalized. To proceed, we restrict our attention to the low-energy sector of the theory, captured by the continuum theory. Introduce the left (right) chiral fermion fields $\psi_L(x) (\psi_R(x))$ containing momenta close to the Fermi points $k_F = \pm \frac{\pi}{2a}$, ($x = na$ with $a$ the lattice spacing),

$$\frac{a_n}{\sqrt{a}} = e^{i \frac{\pi}{2a}} \psi_R(0a) + e^{-i \frac{\pi}{2a}} \psi_L(0a).$$ \hspace{1cm} (3.6)
The continuum Fermi fields are then reexpressed in terms of bosonic fields as follows:

$$\psi_R(x) = \frac{e^{i(\theta - \phi)(x)}}{\sqrt{2\pi a}},$$

$$\psi_L(x) = \frac{e^{i(\theta + \phi)(x)}}{\sqrt{2\pi a}},$$

where the pair of conjugate fields, $\Pi, \phi$ satisfy the following commutation relation:

$$[\Phi(x), \Pi(x')] = i\delta(x - x'),$$

and the field $\theta$, dual to $\phi$, is defined as:

$$\theta(x) = \pi \int^x \Pi(x') dx'.$$

The spin operators Eqs. (3.3)-(3.4), can be expressed in the continuum limit as:

$$S^+ (x) = \frac{S^+_n}{\sqrt{a}} = \frac{e^{i\theta(x)}}{\sqrt{2\pi a}} \left[ e^{\pi x/a} + \cos 2\phi \right],$$

$$S^z(x) = \frac{S^z_n}{a} = -\frac{\partial_x \phi}{\pi a} + \frac{e^{\pi x/a}}{\pi a} \cos 2\phi,$$

Introducing normal ordering with respect to the fermion vacuum, one has:

$$S^+_i S^-_{i+1} + S^z_i S^z_{i+1} + \Delta S^z_i S^z_{i+1} = \langle S^+_i S^z_{i+1} + S^z_i S^z_{i+1} + \Delta S^z_i S^z_{i+1} \rangle +$$

$$\quad + : S^+_i S^-_{i+1} + S^z_i S^z_{i+1} + \Delta S^z_i S^z_{i+1} :,$$

where $\ldots :$ indicates normal ordering. The average $A = \langle S^+_i S^-_{i+1} + S^z_i S^z_{i+1} + \Delta S^z_i S^z_{i+1} \rangle$ is the ground state energy of the chain and can be found by the Bethe Ansatz. However, as long as one is only interested in the correlation functions of the single chain, one can simply drop this contribution and focus on the normal ordered terms that describe the excitations above the ground state. Our task is thus to derive a bosonized expression of the normal ordered product in Eq. (3.11). Some care is needed in order to obtain correct results, and one finds:

$$: S^+_i S^-_{i+1} + S^z_i S^z_{i+1} : = \frac{1}{\pi} \left[ (\pi \Pi)^2 + (\partial_x \phi)^2 \right] + \frac{e^{\pi x/a}}{\pi} \sin 2\phi$$

$$: S^z_i S^z_{i+1} + \Delta S^z_i S^z_{i+1} : = \frac{2}{\pi^2} (\partial_x \phi)^2 + \frac{2}{(\pi a^2)} e^{\pi x/a} \sin 2\phi + \frac{2\cos 4\phi}{(2\pi a)^2} + \text{irrelevant terms} \ldots$$

The oscillating terms in Eq. (3.12) are dropped from the Hamiltonian after integration over $x$. The Hamiltonian $H_{XXZ}$, Eq. (3.3) then becomes:

$$H_{XXZ} = \int \frac{dx}{2\pi} \left[ uK(\pi \Pi)^2 + \frac{u}{K}(\partial_x \phi)^2 \right] - \frac{2\delta}{(2\pi a)^2} \int dx \cos(4\phi),$$

where for $J_z \ll J$,

$$u = aJ \left( 1 + \frac{4J_z}{\pi J} \right)^{1/2}$$

$$K = \left( 1 + \frac{4J_z}{\pi J} \right)^{-1/2}$$

$$\delta = J_z a.$$

Thus, the bosonized form of $H_{XXZ}$ reduces to a sine-Gordon Hamiltonian, where the cosine terms come from the intrachain Umklapp processes. The renormalization group treatment shows that in the vicinity of the XY point, the cosine terms are irrelevant, so that asymptotic properties are described by a free scalar field with renormalized $u^*, K^*$. Since the XXZ chain is integrable, it can be shown that the gapless spectrum extends to the whole region $|J_z| < J$. Moreover, it is possible to obtain an analytic expression for the renormalized $u^*, K^*$ from the exact solution. One finds:
The isotropic point $J_z = J$ (Antiferromagnetic Heisenberg model) corresponds, in the bosonization description, to $K^* = 1/2$ and $\delta^* = 0$. At this point $\cos(4\theta)$ is marginally irrelevant. For $J_z > J$, the exact solution shows that a gap opens in the excitations of the system (3.18), and that an Ising order of the spins along the z axis is obtained. This result can also be found from a bosonization procedure valid in the vicinity of the isotropic point.

**B. Two coupled XXZ chains with a biquadratic exchange**

We now proceed to derive, using the results reviewed in the previous subsection, the bosonized Hamiltonian of two non-equivalent coupled XXZ chains. To bosonize the Hamiltonian $H_0$, Eq. (2.10) describing two decoupled XXZ chains, we introduce two pairs of dual fields (one pair for each chain) $\theta_\alpha, \phi_\alpha \ (\alpha = 1, 2)$ as defined in Sec. III A. The bosonized form of the Hamiltonian $H_0$ is then:

$$H_0 = \sum_{\alpha=1,2} \left\{ \int \frac{dx}{2\pi} \left[ u_\alpha K_\alpha (\pi \Pi_\alpha)^2 + \frac{u_\alpha}{K_\alpha} (\partial_x \phi_\alpha)^2 \right] - \frac{2\delta_\alpha}{(2\pi a)^2} \int dx \cos 4\phi_\alpha \right\},$$

with the fields $\phi_1$ and $\phi_2$ having a priori different velocities and Luttinger couplings,

$$u_\alpha^* = \frac{\pi \sqrt{J_\alpha^2 - (J_z \phi_\alpha)^2}}{2 \arccos \frac{J_z \phi_\alpha}{J_\alpha}},$$

$$K_\alpha^* = \frac{1}{2 - \frac{2}{\pi} \arccos \frac{J_z \phi_\alpha}{J_\alpha}},$$

$$\delta_\alpha = J_\alpha^* a.$$

The bosonization formulas for the spins (3.10) are unchanged except for the obvious introduction of a chain index.

In order to have the full bosonized Hamiltonian, we now have to derive the bosonized form of the biquadratic exchange (2.4). The first step is to normal order using Eq. (3.12). This step is important since it leads to non trivial excitations. Such a case is realized in the spin-tube problem where the exchange constant of the pseudospins $\tau$ is zero. We see that the mean-field like contribution of the spin fluctuations contributes in that system to provide an exchange constant and thus a finite velocity to the pseudospin excitations. Of course, in such case, the bosonization
procedure is not really justified since interactions are of the order of magnitude of the bandwidth of the spin or pseudospin excitations. However, it is usual in quasi-one dimensional systems to have a continuity between the weak and the strong coupling regime. Moreover, a mean field theoretical treatment in the XY limit \( J_1^z = J_2^z = \Delta_1^z = \Delta_2^z \) leads to similar results to the bosonization treatment. Details can be found in App. A. We will therefore assume that although not fully justified in the spin tube case, bosonization nevertheless leads to qualitatively correct results concerning the phases of the system and the overall behavior of correlations in these various phases. A quantitative treatment (in particular of the phase boundaries) requires numerical simulations that are beyond the scope of this paper.

The present treatment shows us that in weak coupling the two chains are described by a bosonized Hamiltonian:

\[
H = \int \frac{dx}{2\pi} \sum_{\alpha=1,2} \left[ u_\alpha K_\alpha (\pi \Pi_\alpha)^2 + \frac{u_\alpha}{K_\alpha} (\partial_x \phi_\alpha)^2 - \frac{4\pi \delta_\alpha}{(2\pi)^2} \cos 4\phi_\alpha \right] + \frac{2g}{(2\pi)^2} \int dx \sin 2\phi_1 \sin 2\phi_2
\]  

(3.24)

We shall analyze the model perturbatively, with results valid up to a given value of \( \lambda \); it is known that in the strong coupling regime of the two chains with biquadratic exchange there is a special value of the interchain coupling at which the model has \( SU(4) \) symmetry. At this special point, the model develops a gapless phase described by a \( SU(4)_1 \) WZNW model. Such an effect is non-perturbative: the resulting critical point has a conformal anomaly \( c = 3 \) whereas the original unperturbed model has \( c = 2 \). This implies by Zamolodchikov’s c–theorem that the transition to the \( SU(4)_1 \) cannot be predicted by a RG calculation that always leads to a decrease of \( c \). Beyond this special value of \( \lambda \), the weak coupling theory would lead to incorrect predictions and an alternative approach such as the one of Lecheminant and Azaria is needed. On the other hand, it is expected to give qualitatively correct predictions when the coupling is smaller than the critical value. In the remainder of the paper we shall thus work in the weak coupling regime where the weak coupling theory is valid. In the following section, we will discuss the RG treatment of the weak coupling model.

C. Renormalization group equations

In the following analysis, we will neglect the velocity difference between chains 1 and 2 since usually velocity differences do not play an important role in the derivation of the phase diagram by RG techniques. However, for the sake of completeness, we have given in App. B the RG equations for non-equal velocities derived using a momentum shell integration technique. When velocity differences are neglected, the renormalization group equations for \( K_\alpha, \delta_\alpha, g \) can be easily derived from the Hamiltonian (3.19)–(3.23) using Operator Product Expansions (OPE)\[14\]. The renormalization group equations for \( \delta_\alpha, g \) neglecting velocity differences are:

\[
\frac{d}{dl} \left( \frac{\delta_1}{\pi u} \right) = (2 - 4K_1) \frac{\delta_1}{\pi u} - \frac{g^2}{8\pi^2 u^2}
\]

\[
\frac{d}{dl} \left( \frac{\delta_2}{\pi u} \right) = (2 - 4K_2) \frac{\delta_2}{\pi u} - \frac{g^2}{8\pi^2 u^2}
\]

\[
\frac{d}{dl} \left( \frac{g}{\pi u} \right) = (2 - K_1 - K_2) \frac{g}{\pi u} - \frac{g(\delta_1 + \delta_2)}{2\pi^2 u^2}
\]

(3.25)

while the renormalization group equations for \( K_1, K_2 \) are:

\[
\frac{d}{dl} \left( \frac{1}{K_1} \right) = \left( \frac{\delta_1}{\pi u} \right)^2 + \frac{1}{8} \left( \frac{g}{\pi u} \right)^2
\]

\[
\frac{d}{dl} \left( \frac{1}{K_2} \right) = \left( \frac{\delta_2}{\pi u} \right)^2 + \frac{1}{8} \left( \frac{g}{\pi u} \right)^2
\]

(3.26)

We now proceed to deduce the weak coupling phase diagram of the model. Eq. (3.24) indicates that \( g \) is a relevant variable when \( (K_1 + K_2) < 2 \), while the variables \( \delta_1, \delta_2 \) become relevant when respectively, \( K_1 < 1/2, K_2 < 1/2 \). There are therefore four cases to distinguish. In the first case: \( K_1 + K_2 > 2 \), there are no relevant operators and the system is in a gapless state. In the second case: \( K_1 + K_2 < 2, K_1, K_2 > 1/2 \), there is a single relevant operator; for \( K_1 > 1/2, K_2 < 1/2 \) (or equivalently \( K_1 < 1/2, K_2 > 1/2 \)), there are two, while for \( K_1, K_2 < 1/2 \) there are three relevant operators. These four different regions are shown in Fig. 3.
In the presence of relevant operators, RG equations cease to be valid as soon as the dimensionless coupling become $O(1)$, and we have to determine the nature of the strong coupling fixed points in order to predict the phase diagram. We see that $K_1, K_2$ are driven to zero by the flow of the RG when there is a relevant operator so that the fields become classical. As a result, $\phi_1, \phi_2$ are locked at average values $\langle \phi_1 \rangle, \langle \phi_2 \rangle$ that minimize the ground state energy and a gap opens in the excitations of these fields. When these fields are locked, it is also known that the exponentials of the dual field have exponential decay.

Let us consider first the case with $g$ the only relevant operator. Then, the minimization of the ground state energy requires $\sin 2 \langle \phi_1 \rangle \sin 2 \langle \phi_2 \rangle = -1$, i.e. $\langle \phi_1 \rangle = -\langle \phi_2 \rangle = \pm \frac{\pi}{4}$. It is then clear that $\cos 2 \phi_1, 2 \phi_2$ as well as $e^{\phi_1}$, which are the staggered component of $S^z$ and $S^+$, will display an exponential decay. It can also be shown that the uniform component of the spins also present an exponential decay. As a result we have a spin-liquid phase that presents only short range order in all its spin correlations. As we shall see, this spin-liquid phase is a dimerized phase whose properties are discussed extensively in Sec. IV. In the case $g = 0$ (decoupled chains) the analysis is even simpler. Then the two chains remain decoupled, and we are left with the analysis of the usual sine-Gordon model. It is then known that when $\delta$ is relevant, the field $\phi_\alpha$ is locked. The analysis of the resulting strong coupling fixed point can be found for instance in Ref. 16. It is found that the strong coupling fixed point is associated with the Ising Antiferromagnetic phase of the XXZ chain at $J_z > J$ in which the staggered component of $S^z$ has a non zero expectation value in the ground state.

When $g$ and at least one of the $\delta$ are relevant, there are two possible candidates for the ground state. Considering Eq. (3.25), we see that the effect of the biquadratic interchain interaction is to reduce the effective $\delta_1, \delta_2$. Physically, this means that the tendency to form singlets competes with the tendency to form an Ising antiferromagnet. Two scenarios are possible. One is that there is a well defined phase boundary between a pure spin-liquid state and a pure Ising antiferromagnet state. The second scenario is that there is a crossover between the two pure states as $\lambda/J_z^1, 2$ is varied. In such case, increasing $\lambda$ would lead to a gradual disappearance of antiferromagnetic order leaving a purely singlet ground state. Since in both phases it is the same field that orders, there is a priori no reason to exclude a mixed spin-liquid antiferromagnet order. Thus, the first scenario appears extremely unlikely. A numerical investigation of the crossover could be very interesting as a toy model of a crossover from spin-liquid to antiferromagnetism. It is interesting to remark that if $K_1 = K_2 = 0$, and $\delta_1 = \delta_2 = \delta$, the equations (3.25) can be integrated analytically. Two phases are obtained, separated by a line $g = 2\sqrt{2}\delta$. In the first one, $g \rightarrow +\infty$ and $\delta \rightarrow -\infty$ which corresponds to a ground state with singlet order. In the second one, $g \rightarrow 0$ and $\delta \rightarrow +\infty$, which corresponds to antiferromagnetic order. The RG equations cease to be valid for $\delta_1/(\pi u) \sim 1$ or $g/(\pi u) \sim 1$. If when this scale is reached $g$ and $\delta$ are of the same order of magnitude, there is a possibility of obtaining a mixing of antiferromagnetism and dimer order. Note that even on the line $g = 2\sqrt{2}\delta$, there is a finite correlation length. This is a further evidence for a progressive crossover from dominant Antiferromagnetic order to dominant dimer order.

It is also possible to give a purely classical treatment for $K_1 = K_2 = 0$ by simply minimizing the ground state energy with respect to $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$. In the case $\delta_2 > \delta_1$, one finds that there are three different regimes. For $g < 4\delta_2$ one obtains $\langle \phi_1 \rangle = -\langle \phi_2 \rangle = \pm \frac{\pi}{4}$ corresponding to a purely dimerized phase. For $4\sqrt{\delta_1 \delta_2} < g < 4\delta_2$, one obtains $\langle \phi_1 \rangle = \pm \frac{\pi}{4}$ and $\sin 2 \langle \phi_2 \rangle = \pm g/(4\delta_2)$. This corresponds to persistence of dimerization in the chain with the smallest tendency to antiferromagnetic order, whereas the chain with the strongest tendency to antiferromagnetism is found in state with mixed dimer and antiferromagnetic order. The antiferromagnetic order parameter in that chain, $\cos 2 \langle \phi_2 \rangle$, then assumes the value $\pm \sqrt{1 - (g/(4\delta_2))^2}$. Finally in the region $g < 4\sqrt{\delta_1 \delta_2}$, both chains display antiferromagnetism with $\langle \phi_1,2 \rangle = \pm \pi/2$. These results are summarized on figure 3. For $\delta_1 = \delta_2$, the region with mixed antiferromagnetic and dimer order shrinks to a single point. It would be interesting to see how quantum fluctuations affect the present picture and in particular determine whether sharp transitions are preserved or if they evolve into crossovers.

D. Phase diagram in zero external magnetic field

In this section, we try to estimate the position of the crossover between the spin-liquid and the antiferromagnet. This can be done roughly by comparing the correlation lengths in the antiferromagnet and in the spin-liquid phase. Using the RG equations, for small $g, \delta_1, \delta_2$, we can neglect the renormalization of $K_1, K_2$. This leads to:

\begin{align}
g(l) &= g(0)e^{(2-K_1-K_2)l} \\
\delta_1(l) &= \delta_1(0)e^{(2-4\delta_1)l} \\
\delta_2(l) &= \delta_2(0)e^{(2-4\delta_2)l}
\end{align}

(3.27)

when any of these quantities become of the order of the energy cutoff $\pi v_F/a$, the RG equations cease to be valid and the phase that is obtained is determined by minimizing the ground state energy.
For $g$, the strong coupling is obtained at a length scale:

$$L_{\text{dim.}} = a \left( \frac{\pi v_F}{ag} \right)^{\frac{1}{2 - 4\lambda_1 - 4\lambda_2}}$$  \hspace{1cm} (3.28)$$

This is the correlation length of spin fluctuations in the spin-liquid phase. For $\delta_1, \delta_2$ the strong coupling is obtained respectively at length scale:

$$L_{\text{AF,1}} = a \left( \frac{\pi v_F}{a|\delta_1|} \right)^{\frac{1}{2 - 4\lambda_1}}$$  \hspace{1cm} (3.29)$$

$$L_{\text{AF,2}} = a \left( \frac{\pi v_F}{a|\delta_2|} \right)^{\frac{1}{2 - 4\lambda_2}}$$  \hspace{1cm} (3.30)$$

It is clear that the shortest length corresponds to the first operator to attain strong coupling. Therefore, the phase that is obtained is the one with the shortest correlation length. For $K_1 = K_2 = 0$, this is in agreement up to a constant with the criterion derived from the RG equations (3.25)–(3.26). The comparison of correlation length allows to draw a rough phase boundary between the antiferromagnet and the dimerized phase that could also be obtained by numerically integrating the RG equations (3.25)–(3.26) starting from weak coupling and any $K_1, K_2$. The equation of the phase boundary is in the case of equivalent chains $\delta_1 = \delta_2 = \delta$, $K_1 = K_2 = K$:

$$g = \frac{\pi v_F}{a} \left( \frac{|\delta| a}{\pi v_F} \right)^{\frac{2 - 2K}{4\lambda}}$$  \hspace{1cm} (3.31)$$

Let us note that in the isotropic spin-tube case, the operators causing antiferromagnetic order are (marginally) irrelevant so that there is only singlet order. However, in the case of an anisotropic spin tube (3 coupled XXZ spin chain with $J_z < J$), such competition becomes possible. Completely decoupled chains exhibit for $J_z < J$ an Ising antiferromagnetic phase. Introducing a strong enough biquadratic interchain coupling favors on the other hand a spin liquid phase. The competition of the two should produce a crossover from the Ising Antiferromagnet to the spin liquid of the type discussed in the preceding section.

IV. SPIN LIQUID PHASE

In this section, we discuss the properties of the spin liquid phase. In the case of equivalent chains, further progress can be made by using symmetric and antisymmetric modes allowing in particular a refermionization of the problem and the calculation of some correlation functions. This will be discussed in Sec. IV A. In the general case with inequivalent chains, such decoupling is no longer possible. However, it is still possible to present a simple semiclassical picture of the nature of excitations above the ground state. This will be the subject of section IV B.

We will focus on region $\frac{1}{5} < K_1 = K_2 < 2$, $K_1 + K_2 < 2$, in which the only relevant operator is the biquadratic exchange, $\sin 2\phi_1 \sin 2\phi_2$. The ground state shows long-range order of the fields $\phi_1$ and $\phi_2$. The expectation values of the ordered fields are:

$$\langle \phi_1 \rangle = \pm \frac{\pi}{4}, \quad \langle \phi_2 \rangle = \mp \frac{\pi}{4},$$  \hspace{1cm} (4.1)$$

and as a result, $\langle \sin 2\phi_{1,2} \rangle \neq 0$. Using Eq. (3.12), this implies that a dimerized order develops both in spin variables and pseudospin variables, i.e. $(-1)^{i} S_{\alpha i} S_{\alpha i+1} \neq 0 \ (\alpha = 1, 2)$. In parallel with that, we have $\langle \cos 2\phi_{1,2} \rangle = 0$, so that by Eq. (3.10) $(-1)^{i} S_{\alpha i}^z = 0$, and the correlation functions $(-)^{|i-j|} S_{\alpha i}^z S_{\alpha j}^z \rightarrow 0$ as $|i-j| \to \infty$. It is also well known that when the fields $\phi_0$ are ordered, the correlation functions of the disorder operators $e^{i\theta_0}$ decay exponentially at large distances. Using again the bosonization formulas for the spins, Eq. (3.10), this implies an exponential decay of all correlation functions : $(-)^{|i-j|} \langle S_{\alpha i}^z S_{\alpha j}^z \rangle$ and $\langle S_{\alpha i}^z S_{\alpha j}^y \rangle$ where $a = x, y, z$. Therefore, the dimerized phase appears as a spin liquid state formed of singlets of spins $S_{\alpha i}$ on both chain 1 and chain 2. Such a conclusion was reached previously in a numerical investigation of the spin tube\cite{4} and by considering the equivalent isotropic chains at the solvable point $\lambda = 3J/4$ where the ground state wavefunction can be obtained exactly in Matrix Product Form. Our results show that such mechanism of spin liquid ground state formation does not require SU(2) symmetry.

This mechanism is somewhat reminiscent of the spin-Peierls transition\cite{3}, the pseudospins playing here the role of the phonons. This is the essence of the Mostovoy-Khomskii model\cite{4} for the “spin-Peierls” transition at $T_c = 34K$ in NaV$_2$O$_5$. 

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In the case of the spin-tube the same picture of the ground state obtains, with the ground state of the spin-tube formed by singlet of spins on even bonds and singlet of chiralities on odd bonds or by singlet of spins on odd bonds and singlet of chiralities on even bonds (see Fig. 3).

For the moment, we have only been able to discuss the nature of the ground state. However, it is also important to discuss the nature of the excitations as well as the various correlation functions. In order to do that, it is worth to restrict first to the simple case of two equivalent chains, in which the physical picture is the clearest.

A. Equivalent chains

When the two chains are equivalent, we introduce the fields:

\[
\phi_a = (\phi_1 + \phi_2)/\sqrt{2}, \quad \phi_s = (\phi_1 - \phi_2)/\sqrt{2}
\]

and their conjugate fields:

\[
\Pi_a = (\Pi_1 + \Pi_2)/\sqrt{2}, \quad \Pi_s = (\Pi_1 - \Pi_2)/\sqrt{2},
\]

so that the total Hamiltonian can be completely decoupled into the symmetric and antisymmetric parts,

\[
H = H_s + H_a,
\]

\[
H_s = \int \frac{dx}{2\pi} \left[ uK(\pi \Pi_s)^2 + \frac{u}{K}(\partial_x \phi_s)^2 \right] - \frac{g}{(2\pi a)^2} \int dx \cos \sqrt{8}\phi_s
\]

\[
H_a = \int \frac{dx}{2\pi} \left[ uK(\pi \Pi_a)^2 + \frac{u}{K}(\partial_x \phi_a)^2 \right] + \frac{g}{(2\pi a)^2} \int dx \cos \sqrt{8}\phi_a,
\]

where the magnetic field coupling only to \( \phi_s \), and only the most relevant operators have been taken into account. The elementary excitations can be discussed in terms of solitons of two decoupled sine-Gordon models. The solitons of the \( \phi \) are represented in Fig. 5. It is also convenient to use the canonical transformation

\[
\rho = \frac{e^{i(\theta_x - \phi_s)}}{\sqrt{2\pi a}},
\]

\[
\tilde{\psi}_{R,a} = \frac{e^{i(\theta_x + \phi_s)}}{\sqrt{2\pi a}}
\]

where \( \nu = a, s \), one can finally rewrite \( H_{a,s} \) in the form:

\[
H_{\nu} = -v \int dx(\tilde{\psi}_{R,\nu}^\dagger \partial_x \tilde{\psi}_{R,\nu} - \tilde{\psi}_{L,\nu}^\dagger \partial_x \tilde{\psi}_{L,\nu}) - \mu_{\nu} \int dx(\tilde{\psi}_{R,\nu}^\dagger \tilde{\psi}_{L,\nu} + \tilde{\psi}_{L,\nu}^\dagger \tilde{\psi}_{R,\nu}) + \lambda \int dx \rho_{\nu}(x)^2
\]

where \( \rho_{\nu}(x) = \tilde{\psi}_{L,\nu}^\dagger \tilde{\psi}_{L,\nu} + \tilde{\psi}_{R,\nu}^\dagger \tilde{\psi}_{R,\nu} \) and \( \nu = a, s \). The couplings are given by:

\[
v = 2uK
\]

\[
\lambda = \pi u \left( \frac{1}{4K} - K \right)
\]

\[
\frac{g}{4\pi a} = \mu_a = -\mu_s
\]

At the isotropic point \( (K = 1/2) \), one has \( \lambda = 0 \), so that \( H_{\nu} \) is a free fermion Hamiltonian. Similarly to the spin ladder, where \( [\tilde{S}_{tot}, H] = 0 \), the excitation spectrum can be split into a singlet and a triplet with spin \(-1,0,1\). However, in contrast to the spin ladder case, the triplet and the singlet here have the same mass. This is the signature of a larger symmetry group, \( SU(2) \times SU(2) \sim SO(4) \). The correlation functions can be obtained from mapping the free fermion Hamiltonian onto two non-critical Ising model exhibiting the \( SU(2) \times Z_2 \) symmetry. Remarkably,
although the system has a spin gap, these correlation functions are very different from those of a spin-1 chain or a spin ladder. In the chain with biquadratic exchange, the response functions do not show any particle-like delta function peak in their imaginary part, but only a two particle continuum even in the vicinity of \( q = \pi \). This is to be contrasted with the spin ladder which shows a delta function peak associated with a single particle excitation at \( q = \pi \). The two chains with biquadratic interactions thus form a “non-Haldane” spin liquid.

Away from the isotropic point, the Hamiltonian can still be refermionized but the fermions (solitons) have in-

\[ H = \int dx \sum_{r=s,a} \left[ -i \nu (\psi_{R,r}^{\dagger} \partial_x \psi_{R,r} - \psi_{L,r}^{\dagger} \partial_x \psi_{L,r}) + m (\psi_{R,r}^{\dagger} \psi_{L,r} - + \psi_{L,r}^{\dagger} \psi_{R,r}) \right] + g (\rho_{R,s}(x) \rho_{R,a}(x) + \rho_{L,s}(x) \rho_{L,a}(x)) + \tilde{g}_1 \sum_{r \neq r'} \rho_{R,r}(x) \rho_{L,r'}(x) + \tilde{g}_2 \sum_r \rho_{R,r}(x) \rho_{L,r}(x) \] 

(4.11)

As a result, there is now an interaction between the excitations of spin \( S^z = 0 \) (the \( a \) fermions) and the excitations of spin \( S^z = \pm 1 \) (the \( s \) fermions). We have:

\[ v = \frac{1}{2} \left[ u_1 \left( K_1 + \frac{1}{4K_1} \right) + u_2 \left( K_2 + \frac{1}{4K_2} \right) \right] \] 

(4.12)

\[ g = \pi \left[ u_1 \left( K_1 + \frac{1}{4K_1} \right) - u_2 \left( K_2 + \frac{1}{4K_2} \right) \right] \] 

(4.13)

\[ \tilde{g}_1 = \pi \left[ u_1 \left( K_1 - \frac{1}{4K_1} \right) + u_2 \left( K_2 - \frac{1}{4K_2} \right) \right] \] 

(4.14)

\[ \tilde{g}_2 = \pi \left[ u_1 \left( \frac{1}{4K_1} - K_1 \right) - u_2 \left( K_2 - \frac{1}{4K_2} \right) \right] \] 

(4.15)

\[ m = \frac{g}{4\pi a} \] 

(4.16)

The fermionic version of the model is a generalization of the massive Thirring model. The fields carry spin and the interactions break spin rotation symmetry. It can be checked that the interaction of \( a \) with \( s \) fermions disappears only for \( u_1 = u_2, K_1 = K_2 \). Not much can be said of the elementary excitations of Hamiltonian \( (4.11) \) due to the absence of an exact solution. In order to gain some insight into the elementary excitations of the dimerized state in the case of non-equivalent chains, we resort to semi-classical approximations. Clearly, we can search for a semi-classical minimum of \( (4.10) \) with either \( \phi_a = 0 \) or \( \phi_a = \pi / 8 \). This corresponds to a single soliton in the system. In the general case, such excitations are associated with a single fermion, either \( a \) or \( s \). In order to obtain a physical picture for this type of elementary excitations, let us calculate the average “magnetization” for the fields \( \phi_1 \) and \( \phi_2 \) when there is a soliton.
connecting the two degenerate dimerized ground-states. In the case \( \phi_a = 0 \), \( \phi_1 \) decreases from \( \pi/4 \) to \( -\pi/4 \) while \( \phi_2 \) increases from \( -\pi/4 \) to \( \pi/4 \). We immediately get:

\[
m_1 = -\frac{1}{\pi} \int_{-\infty}^{\infty} \partial_x \phi_1 = -\frac{1}{\pi} [\phi_1(\infty) - \phi_1(-\infty)] = -\frac{1}{2}
\]

\[
m_2 = -\frac{1}{\pi} \int_{-\infty}^{\infty} \partial_x \phi_2 = -\frac{1}{\pi} [\phi_2(\infty) - \phi_2(-\infty)] = m_1 = \frac{1}{2}
\]

Another solution is obtained by reversing the sign of \( \phi_a \) leading to \( m_1 = -m_2 = 1/2 \). The case \( \phi_a = \pi/\sqrt{8} \) corresponds to \( m_1 = m_2 = \pm 1/2 \). In each case, the elementary excitations are formed by breaking a singlet on neighboring sites on each chain. Such objects then propagate coherently. In the case of the spin tube, this corresponds to having one unpaired spin associated with one unpaired chirality pseudospin. Such an excitation has \( \tau = \pm 1/2 \), \( S^z = \pm 1/2 \). It is the spinon of Ref. 20. Thus the elementary excitations of the model can be easily visualized as an unpaired spin and an unpaired chirality forming a triplet or a singlet diagonal bond (see Fig. 7). This is the soliton in the dimer order. This picture generalizes from isotropic equivalent spin chains to the case of inequivalent and anisotropic chains. An open question is whether in the case of inequivalent chains bound states of these elementary excitations can be obtained in contrast to the case of equivalent chains. A necessary condition is that there exists attractive interactions between \( a \) and \( s \) fermions. The study of such bound states could be of interest in relation with light scattering experiments on NaV\(_2\)O\(_5\). Besides semi-classical approximation, another approximate treatment is possible. In the case where \( u_1 = u_2 \), \( \delta_1 = \delta_2 = 0 \) and \( K_1 + K_2 = 1 \), the Hamiltonian (3.24) is the double sine-Gordon model or the Bukhvonst-Lipatov model. The model is known to be Bethe Ansatz solvable but its elementary excitations and \( S \) matrix have only been obtained recently and shown to be identical to those of the double sine-Gordon model on its integrable line. The spectrum of the model is consists of four massive particles carrying two quantum numbers \( Q_1, Q_2 = \pm 1 \), and the \( S \) matrix is:

\[
S = S^{K_1}_{SG} \otimes S^{K_2}_{SG},
\]

where \( S^{K}_{SG} \) is the \( S \) matrix of a sine Gordon model having the parameter \( K \) and \( \tilde{K}_a = 4K_a/(1 + 2K_a) \). As a result, one of the sine Gordon models is in the attractive regime and has a spectrum made of fermions and bound states of fermions whereas the second sine-Gordon model is in the repulsive regime and has a spectrum containing only fermions. The mass of the fundamental fermion is \( m = g/4\pi a \sin(\pi K_1) \). For \( K_1 = 1/2 \), one recovers the equivalent chain and the mass of the bound states is then:

\[
m_n = g/4\pi a \sin(n \pi K_1), \quad n \in \mathbb{N}, nK_1 < 1/2
\]

Since for \( 1/4 < K_1 < 3/4 \), the operator \( \sin(2\phi_1) \sin(2\phi_2) \) is the most relevant, it is reasonable to expect that neither the marginal perturbations due to \( u_1 \neq u_2 \) nor the less relevant perturbations \( \delta_{1,2} \neq 0 \) change the gapped nature of the spectrum. In this regime, one should not observe any bound state, since the \( n = 2 \) bound state only exists for \( K < 1/4 \). In terms of the original spin chain, the double sine-Gordon regime should be accessible if one chooses to have one chain with \( J_z < J \) and the other chain with \( J_z > J \) in such way that \( K_1 + K_2 = 1 \) and \( J_z \ll J, J_z \). However, the double sine-Gordon regime would not be observed in the spin tube case, in which we should have \( K_1 + K_2 = 3/2 \). We expect that in the spin tube case, the system has more quantum fluctuations than in the double sine-Gordon regime. Therefore, no bound states of spinons will form. This heuristic argument agrees with the numerical calculations on the spin tube that show no bound state of spinons.

**V. EFFECT OF A MAGNETIC FIELD ON THE DIMERIZED PHASE**

In this section, we discuss the effect of the application of a magnetic field on the dimerized phase. In general, the application of a magnetic field to a gapped one dimensional spin liquid system results in the closure of the gap for a magnetic field of the order of magnitude of the gap. Below the critical magnetic field the magnetization is zero. Above the critical magnetic field the magnetization increases as \( (h - h_c)^{1/2} \), the system is in a single component Luttinger liquid state and has incommensurate spin correlations. Here, we have to consider two \textit{a priori} different cases. In the first one and most academic, the two coupled chains carry real spins that couple to the magnetic field. In the second case, only one chain carry a spin that couple to the magnetic field, the other one carrying only a pseudospin degree of freedom that does not couple to a magnetic field. This last case corresponds to the spin-orbital models and to the spin tube model. The behavior of the spin-orbital model in a magnetic field has been discussed in Refs.
Both references study models with $SU(2) \times SU(2)$ symmetry in the absence of the magnetic field. It is shown that in the vicinity of the SU(4) point, the spin-orbital model becomes equivalent to the $O(4)$ Gross Neveu model describing the gapped modes plus a $c = 1$ CFT that describes the gapless magnetic modes. Also, in Ref. [59], the weak coupling case has been discussed. The case where both chains carry spin will be dealt with in Sec. V A and the case of a single chain carrying spin in Sec. V B.

A. Both chains carry spin

If both chains carry spin, the coupling to the magnetic field (taken parallel to the z axis) is:

$$- \frac{h}{\pi} \int dx \partial_x (\phi_1 + \phi_2) \tag{5.1}$$

In that case, we can use the decoupling of Sec. IV A, Eqs. (4.4) and (4.6). The coupling to the magnetic field is:

$$- \sqrt{2} \frac{h}{\pi} \int dx \partial_x \phi_s \tag{5.2}$$

1. The case of equivalent chains

We consider the case where the two chains are equivalent. This problem has been discussed in the context of spin ladder. The gap in the antisymmetric modes $\phi_s$ is not affected by the presence of the magnetic field. On the other hand, for a sufficiently large magnetic field $h > h_{c1}$, the gap in the symmetric modes closes and the magnetization $m = - \frac{\sqrt{2}}{x} \partial_x \phi_s$ behaves as $m \sim (h - h_{c1})^{1/2}$ for $h > h_{c1}$. Moreover, it can be shown that the low energy modes of $\phi_s$ are described by the following Hamiltonian:

$$H_s = \int \frac{dx}{2\pi} \left[ u^*_s K_s^*(\pi \Pi_s)^2 + \frac{u^*_s}{K_s^*} (\partial_x \phi_s)^2 \right] \tag{5.3}$$

And the exponent $K_s^*$ takes the universal value 1/2 at $h = h_{c1}$. Correlation functions in the incommensurate phase can be obtained in the isotropic case by a calculation similar to the spin-ladder case. This time, $\langle \phi_s^2 \rangle$ is finite so that $e^{i\theta_s/\sqrt{2}}$ has exponentially decaying correlations. As a result, one has:

$$\langle S^+(x)S^-(0) \rangle \sim e^{-x/\xi}, x \ll \xi \tag{5.4}$$

A simplified expression for the operator $S^z = S^+_z + S^-_z$ is:

$$S^z = - \frac{\sqrt{2}}{\pi} \partial_x \phi_s + \cos(\sqrt{8} \phi_s - 2\pi mx) \tag{5.5}$$

and the correlations of $S^z$ are:

$$\langle T_x S^z(x, \tau) S^z(0, 0) \rangle = \frac{x^2 - (ur)^2}{(x^2 + (ur)^2)^2} + \text{constant} \left( \frac{a^2}{x^2 + (ur)^2} \right)^2 \cos(2\pi mx) \tag{5.6}$$

There are subleading power law corrections at $Q = 2\pi mx$. As shown by Furusaki and Zhang, these corrections are missed if one naively neglects the band curvature after refermionization. These corrections can also be obtained by using the Haldane expansion of the spin operators and retaining the terms up to $4k_F$ as we did here.

2. The case of non-equivalent chains

In the case of non-equivalent chains, the problem gets more complicated. The magnetic field still couples only to $\phi_s$. However, when the magnetic field $h$ exceeds the field $h_{c1}$ needed to close the gap in the $a$ modes, the appearance of a non-zero $\langle \partial_x \phi_s \rangle$ creates an effective magnetic field that couples to $\partial_x \phi_a$. If this magnetic field is not strong enough to close the gap in the $\phi_a$ modes, the system remains in a one-component Luttinger liquid for fields $h$ not much stronger than the critical field $h_{c1}$. For $h$ sufficiently large, the generated effective field can close the gap in the $a$ modes leading to a two component Luttinger Liquid. Since no experimental ladder system with a biquadratic exchange much larger than the quadratic exchange much larger than the quadratic exchange and made of two non-equivalent chains is presently available, this two-step transition to a two component Luttinger liquid is unlikely to be observed experimentally.
B. Only one chain carries spin

This case includes the spin tube and the Mostovoy-Khomskii model and is therefore the most relevant physically. This case has been discussed for two coupled $SU(2) \times SU(2)$ symmetric chains in Ref. 59. In the case where only one chain, say chain 1, to fix notations, carries spin, the interaction with the magnetic field is given by a term:

$$-\frac{\hbar}{\pi} \int dx \partial_x \phi_1,$$

(5.7)

In the case of equivalent chains, we can use the same decoupling of Sec. IV A, Eqs. (4.3) and (4.4) as in the preceding section. However, there is an important difference. Now, we have the coupling to the magnetic field in the form:

$$-\frac{\hbar}{\pi \sqrt{2}} \int dx \partial_x (\phi_s + \phi_a)$$

(5.8)

As a result, now the magnetic field couples to both $\phi_a$ and $\phi_s$. Moreover, the strength of the couplings is exactly the same. We start with the discussion of equivalent chains. Then, we discuss non equivalent chain. We will show that in both case, the closure of the gap leads to a two-component Luttinger liquid behavior in contrast with the usual spin-liquid systems that lead to a single component Luttinger liquid behavior. We will also give the expression of correlation functions in the Luttinger liquid phase.

1. Equivalent chains

As a result of the symmetry between $\phi_a$ and $\phi_s$, the gap in the symmetric and the antisymmetric mode close simultaneously, leading to a two component Luttinger liquid ground state under strong enough magnetic field. Contrarily to the case where the magnetic field couples to both 1 and 2 spins, there is no intermediate single component Luttinger liquid phase. We expect that the magnetization $m = -\partial_x \phi_1/\pi$ behaves as $(\hbar - h_c)^{1/2}$ close to the threshold. It is also important to note that, the two sine-Gordon model being equivalent, one has for the fixed point Hamiltonian $u_a^* = u_a^*$ and $K_a^* = K_a^*$. As a result, the fixed point Hamiltonian is invariant under any rotation in the $(\phi_a, \phi_s)$ plane. We can thus write the fixed point Hamiltonian as:

$$H = \int \frac{dx}{2\pi} \left[ u_s^* K_s^*(\pi \Pi)^2 + \frac{u_s^*}{K_s^*}(\partial_x \phi)^2 \right]$$

(5.9)

Where $\bar{\phi} = (\phi_1, \phi_2)$ and $\bar{\Pi} = (\Pi_1, \Pi_2)$. Moreover, in the case of $SU(2) \times SU(2)$ symmetry, we have $K_1 = K_2 = 1/2$. It can then be shown easily, using the refermionization procedure that $K^* = 1/2$ for any magnetic field. Equation (5.9) has important consequences for the correlation functions, which are of the form (for equivalent chains):

$$\langle S_\alpha^+(x,t)S_\alpha^-(0,0) \rangle = (-1)^{\alpha}(x^2 - t^2)^{-\pi i \alpha} \begin{bmatrix} e^{2i\pi m_\alpha} \left( x - \frac{\pi c}{2} \right)^2 + e^{2i\pi m_\alpha} \left( x + \frac{\pi c}{2} \right)^2 \end{bmatrix}$$

$$\langle (S_\alpha^-(x,t) - m_\alpha)(S_\alpha^+(0,0) - m_\alpha) \rangle = \cos(\pi x(1 - 2m_\alpha))(x^2 - t^2)^{-\pi i} \begin{bmatrix} 1 \left( x^2 + \frac{\pi c}{2} \right)^2 + \left( x^2 - \frac{\pi c}{2} \right)^2 \end{bmatrix},$$

where the index $\alpha$ indicates the spin in chain 1 or 2, $m_\alpha$ is the magnetization, with $m_1 = m$ (total magnetization) and $m_2 = 0$. From the above expressions we deduce the following: the correlation function parallel to the field, $\langle S_1^+ S_1^- \rangle$, has a staggered part shifted from the wave vector $q = \pi$ to $q = \pi(1 - 2m)$, while the correlation function perpendicular to the field, $\langle S_1^+ S_2^- \rangle$, has an unshifted staggered mode and the uniform magnetization mode shifted to $q = 2\pi m$. The correlation functions for the spin of type 2, instead, are completely unaffected by the presence of an external magnetic field.

2. General case

In this section, we consider the case where the two chains are not necessarily equivalent. In particular, this is the case that is realized in the spin-tube under a magnetic field.

In this case, it is convenient to use the rotation,
\[ \phi_s = (\phi_1 + \phi_2)/\sqrt{2}, \quad \phi_a = (\phi_1 - \phi_2)/\sqrt{2} \]
\[ \Pi_s = (\Pi_1 + \Pi_2)/\sqrt{2}, \quad \Pi_a = (\Pi_1 - \Pi_2)/\sqrt{2} \]  
(5.10)

to bring the Hamiltonian to the form:
\[ H = \int \frac{dx}{2\pi} \left[ \frac{(u_1 K_1 + u_2 K_2)}{2}, (\pi \Pi_a)^2 + (\pi \Pi_a)^2 \right] + \frac{(u_1 K_1 + u_2 K_2)}{2}\left[(\partial_x \phi_s)^2 + (\partial_x \phi_a)^2\right] 
+ \frac{u_1 K_1 - u_2 K_2}{2} \pi^2 \Pi_a + \frac{u_1 K_1 - u_2 K_2}{2} \phi_s \phi_s \partial_x \phi_a \right] + 
\]
\[ + \frac{2g}{(2\pi a)^2} \int dx [\cos \sqrt{8} \phi_s - \cos \sqrt{8} \phi_a] - \frac{\hbar}{\sqrt{2}} \int dx \partial_x (\phi_s + \phi_a) \]  
(5.11)

Shifting \( \phi_a \rightarrow \phi_a + \pi/\sqrt{8} \), renders Hamiltonian invariant under the interchange of \( \phi_s \) and \( \phi_a \). As a consequence, the gaps in \( \phi_s \) and \( \phi_a \) are identical and have to close simultaneously. Therefore, as in the case of identical chain, there is a transition from a gapped phase into a gapless two-component Luttinger Liquid phase. The most general Hamiltonian for a two-component Luttinger liquid is:
\[ H = \int \frac{dx}{2\pi} \left[ M_{aa}(\pi \Pi_a)^2 + 2M_{as} \pi \Pi_a \pi \Pi_a + M_{ss}(\pi \Pi_a)^2 + N_{aa}(\partial_x \phi_a)^2 + 2N_{as} \partial_x \phi_a \partial_x \phi_s + N_{ss}(\partial_x \phi_s)^2 \right]. \]  
(5.12)

In the case of coupled XXZ chain, the effective Hamiltonian (5.12) can be simplified by making use of symmetry. We know that the original Hamiltonian, Eq. (5.11) is invariant under the transformation:
\[ \phi_a \leftrightarrow \phi_s \]
\[ \Pi_a \leftrightarrow \Pi_s \]  
(5.13)

This symmetry must be preserved by the renormalized Hamiltonian. Therefore, one must have \( M_{aa} = M_{ss} \) and \( N_{aa} = N_{ss} \). As a result, the effective Hamiltonian (5.12) is diagonalized by returning to the original variables \( \phi_1, \phi_2 \). The effective Hamiltonian is therefore the following:
\[ H = \int \frac{dx}{2\pi} \left[ u_1^* K_1^*(\pi \Pi_a)^2 + \frac{u_1^*}{K_1^*} (\partial_x \phi_1)^2 + u_2^* K_2^*(\pi \Pi_2)^2 + \frac{u_2^*}{K_2^*} (\partial_x \phi_2)^2 \right] \]  
(5.14)

The absence of coupling between \( \phi_1 \) and \( \phi_2 \) in this Hamiltonian is somewhat surprising. This can however be understood by the fact that the Hamiltonian even in the presence of a magnetic field is invariant by a rotation by \( \pi S_2^{\text{z}} \rightarrow -S_2^{\text{z}} \). Thus, \( (S_2^z) = 0 \) for any \( h \), implying that \( \partial_x \phi_1 \partial_x \phi_2 \) terms cannot appear in the two component Luttinger liquid Hamiltonian. In contrast to the spin ladder case\(^3\), the exponents \( K_{1,2}^* \) in the Hamiltonian Eq. (5.14) are non-universal. This can be shown in the following way. Let us consider the case where \( u_1 K_1 - u_2 K_2 \) and \( u_1 K_1 - u_2 K_2 \) are small compared to \( u_1 \) and \( u_2 \), namely where the couplings in the two chains are nearly identical. In that case, we can neglect in a first approximation the terms \( \Pi_a \Pi_a \) and \( \partial_x \phi_a \partial_x \phi_a \) in the Hamiltonian (5.11). We are thus left with two decoupled sine-Gordon models under a magnetic field. These sine-Gordon models undergo a commensurate-incommensurate transition at a critical magnetic field. It is well known\(^4\) that the exponent at the transition assumes a universal value that renders scaling dimension of the cosine term equal to one. Therefore, in our case, the universal value of the exponent at the transition is: \( K^* = 1/2 \). As a result, close to the transition the effective Hamiltonian is:
\[ H = \int \frac{dx}{2\pi} \sum_{\nu=a,s} \left[ u^* K^*(\pi \Pi_\nu)^2 + \frac{u^*}{K^*} (\partial_x \phi_\nu)^2 \right] + \int \frac{dx}{2\pi} \left[ \frac{u_1 K_1 - u_2 K_2}{2} \pi^2 \Pi_a + \frac{u_1 K_1 - u_2 K_2}{2} \partial_x \phi_a \partial_x \phi_a \right]. \]  
(5.15)

Returning to \( \phi_1 \) and \( \phi_2 \), one obtains an Hamiltonian of the form Eq. (5.14) with:
\[ K_1^* = \sqrt{\frac{1 + (u_1 K_1 - u_2 K_2)}{1 + (u_1 K_1 - u_2 K_2)}} \]
\[ u_1^* = u^* \sqrt{\left( 1 + \frac{u_1 K_1 - u_2 K_2}{u^*} \right) \left( 1 + \frac{u_1 K_1 - u_2 K_2}{u^*} \right)} \]
indicating that except in the case of equivalent chains, one should not expect universal exponents at the transition. This should be the case in particular for the spin-tube. It may provide an experimental test for the spin-orbital model of NaV$_2$O$_5$ since the exponent $K_1^2$ controls the temperature dependence of the NMR relaxation rate. However, since the transition temperature between the gapped and the gapless phase in NaV$_2$O$_5$ is $T_c = 35 K$, the magnetic field needed to close the gap should be of order $52.5 T$ which could make the experiment impossible. In Na$_2$Ti$_2$Sb$_3$O, with $T_c = 110 K$, the situation is even worse.

The spin and pseudospin operators are,

$$S^z(x) = m - \frac{\partial_x \phi_1}{\pi} + \frac{e^{i\frac{\pi x}{a}}}{\pi a} \sin(2\phi_1 - 2\pi mx), \quad S^+(x) = \frac{e^{i\theta_1}}{\sqrt{\pi a}} \left[u + \sin(2\phi_1 - 2\pi mx)\right]$$

$$\tau^z(x) = -\frac{\partial_x \phi_2}{\pi} + \frac{e^{i\frac{\pi x}{a}}}{\pi a} \sin 2\phi_2, \quad \tau^+(x) = \frac{e^{i\theta_2}}{\sqrt{\pi a}} \left[u + \sin 2\phi_2\right],$$

where $m$ is the total magnetization. In the case of the spin-orbital model, the spin-spin correlation functions are therefore given by the usual formulas. The situation is however more interesting in the case of the spin tube. Using the formula (2.4), one has:

$$\langle TS_p^\beta(x,t)S_p^\beta(0,0)\rangle = \langle TS^\beta(x,t)S^\beta(0,0)\rangle \times \left[\frac{1}{9} + \frac{8}{9} \langle T\tau^+(x,t)\tau^-(0,0)\rangle\right],$$

where $\beta = (+, -, \bar{z})$, and $p$ is the chain index. Explicitly,

$$\langle TS^+(x,t)S^-(0,0)\rangle = (-1)^{x/a}(x^2 - (u_1 t)^2)^{-\frac{1}{1+K_1}} + \text{const} \times \left(x^2 - (u_1 t)^2\right)^{-2\frac{1}{1+K_1}} \times \frac{e^{2i\pi mx} + e^{-2i\pi mx}}{(x - u_1 t)^2 + (x + u_1 t)^2}$$

$$\langle (S^2(x,t) - m)(S^2(0,0) - m)\rangle = \cos(\pi x(1 - 2m))(x^2 - (u_1 t)^2)^{-K_1} \times \frac{1}{4\pi^2} \left(\frac{1}{(x - u_1 t)^2} + \frac{1}{(x + u_1 t)^2}\right)$$

$$\langle T\tau^+(x,t)\tau^-(0,0)\rangle = (-1)^{x}(x^2 - (u_2 t)^2)^{-\frac{1}{1+K_2}} + \text{const} \times \left(x^2 - t^2\right)^{-2\frac{1}{1+K_2}} \times \left(\frac{2(x^2 + (u_2 t)^2)}{(x^2 - (u_2 t)^2)^2}\right).$$

This correlation function, which enters in particular in the calculation of the NMR relaxation rate, contains power-law divergences at wave vector $q \sim 0, \pi(1 \pm 2m)$ but also $\pm 2m$ due to the fluctuations of chiralities.

VI. CONCLUSIONS

We have presented a field-theoretical analysis of the low-energy physics of the anisotropic spin-orbital model and the three-leg ladder with periodic boundary conditions (the spin tube) in the strong interchain coupling limit. We gave a derivation of the field theoretical model from the lattice Hamiltonian and then analyzed the phase-diagram using renormalization group equations. The system is found to exhibit a gapless phase, a spin-liquid phase or an Ising Antiferromagnetic phase depending on the microscopic couplings. The spin liquid ground state is two-fold degenerate, formed either by singlets of spins on even bonds and singlets of orbital pseudospins (chirality in the spin-tube case) on odd bonds or the other way round. The antiferromagnetic phase competes with the spin-liquid and we have discussed this competition briefly. The spin liquid phase obtains in the case of the spin tube with $SU(2)$ symmetry, and we discussed the nature of excitations above this spin liquid ground state. These excitations have spin $S^z = \pm 12$ and pseudospin $\tau^z = \pm 12$. They are formed by introducing a free spin as a defect in the spin singlet pattern as well as a pseudospin in the pseudospin singlet pattern. These excitations lead to a kind of non Haldane spin liquid analogous to the one discussed by Nersesyan and Tsvelik. An interesting consequence is the absence of a magnon peak at $q = \frac{\pi}{3}$ in the spin-spin correlation functions of the spin tube. This behavior could be tested in numerical simulations. We have investigated the effect of an applied magnetic field $h$ on the dimerized phase. A strong enough magnetic field $h > h_c$ causes the closure of the gap and the disappearance of dimer order. The resulting gapless phase is a two component Luttinger liquid in contrast to the one component Luttinger Liquid that is observed in spin ladders.
The exponents appeared to be *non-universal* at the transition point, in contrast with the spin ladder case. It would be interesting to obtain numerically the Luttinger liquid exponents for the spin tube or the anisotropic spin orbital model. For the spin tube, we have also shown that new soft modes appeared in the spin-spin correlation functions above $h_c$, by comparison with the soft modes of the single chain. This is the result of the presence of soft chirality modes. This may be tested numerically.

ACKNOWLEDGMENTS

E. O. acknowledges support from NSF under grant NSF-DMR 96-14999. We thank P. Lecheminant and H. Saleur for discussions and useful comments on the manuscript.

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APPENDIX A: MEAN FIELD TREATMENT OF THE TWO XY CHAINS COUPLED WITH A BICOQUADRATIC EXCHANGE RESPECTING THE XY SYMMETRY

1. Equivalent chains

We start from the Hamiltonian:

\[
H = \sum_{n=1}^{N} \left( S_{n,a}^+ S_{n+1,a}^- + S_{n,a}^- S_{n+1,a}^+ + \lambda \sum_{n} (S_{n,2}^+ S_{n+1,2}^- + S_{n,2}^- S_{n+1,2}^+)(S_{n,1}^+ S_{n+1,1}^- + S_{n,1}^- S_{n+1,1}^+) \right) \tag{A1}
\]

After the usual Jordan Wigner Fermionization, this gives:

\[
H = \sum_{n=1}^{N} \left( a_{n,a}^+ a_{n+1,a}^- + a_{n,a}^- a_{n+1,a}^+ + \lambda \sum_{n} (a_{n,2,a}^+ a_{n+1,2}^- + a_{n,2}^- a_{n+1,2}^+)(a_{n,1,a}^+ a_{n+1,1}^- + a_{n,1}^- a_{n+1,1}^+) \right) \tag{A2}
\]

The mean field approximation is obtained by taking: (recall, \( a = 1, 2 \) is a chain index).

\[
\langle a_{n,a}^+ a_{n+1,a}^- + a_{n,a}^- a_{n+1,a}^+ \rangle = t + (-)^{n+a} \delta \tag{A3}
\]

The mean field equation are then:

\[
1 = \frac{2\lambda}{\pi} \sqrt{1 + \left( \frac{\delta}{1 + \lambda t} \right)^2} \left[ 2K \left( 1 - \left( \frac{\delta}{1 + \lambda t} \right)^2 \right) - \frac{E \left( 1 - \left( \frac{\delta}{1 + \lambda t} \right)^2 \right)}{1 - \left( \frac{\delta}{1 + \lambda t} \right)^2} \right]
\]

\[
t = -\frac{2}{\pi} (1 + \lambda t) \sqrt{1 + \left( \frac{\delta}{1 + \lambda t} \right)^2} \frac{E \left( 1 - \left( \frac{\delta}{1 + \lambda t} \right)^2 \right)}{1 - \left( \frac{\delta}{1 + \lambda t} \right)^2} \tag{A4}
\]

Where \( K, E \) are the complete elliptic integrals of the first and second kind respectively. For small \( \delta \), the mean field equations (A4) reduce to:

\[
\frac{t}{1 + \lambda t} = -\frac{2}{\pi} \frac{4\lambda}{\pi} \ln \left( \frac{\sqrt{8} \left| 1 + \lambda t \right|}{\left| \delta \right|} \right) \tag{A5}
\]

And one obtains:

\[
t = -\frac{2}{\pi + 2\lambda} \delta = \sqrt{8\pi - 2\lambda} \exp \left( -\frac{\pi}{4\lambda} \right) \tag{A6}
\]

This result from the mean field theory agrees with the prediction of bosonization since bosonization would predict the same essential singularity in \( \delta \) at small \( \lambda \), the interchain coupling being marginal.

2. Non equivalent XY chains

In this section, we consider a problem with a Hamiltonian of the form:

\[
H = \sum_{n} (S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+)(1 + \alpha (\tau_n^+ \tau_{n+1}^- + \tau_n^- \tau_{n+1}^+)) \tag{A7}
\]

After a Jordan Wigner transformation, the Hamiltonian becomes:
\[ H = \sum_n (a_n^+ a_{n+1}^- + a_n^- a_{n+1}^+)(1 + \alpha (b_n^+ b_{n+1}^- + b_n^- b_{n+1}^+)) \]  \hspace{1cm} (A8)

The mean field approximation is now:

\[ \langle a_n^+ a_{n+1}^- + a_n^- a_{n+1}^+ \rangle = t_1 + (-)^n \delta_1 \]  \hspace{1cm} (A9)

\[ \langle b_n^+ b_{n+1}^- + b_n^- b_{n+1}^+ \rangle = t_2 + (-)^n \delta_2 \]  \hspace{1cm} (A10)

The mean field equations are now:

\[ t_1 = -\frac{2}{\pi} (1 + \alpha t_2) \sqrt{1 + \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2} \left[ 2K \left( 1 - \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2 \right) - \frac{E \left( 1 - \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2 \right)}{1 - \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2} \right] \]  \hspace{1cm} (A11)

\[ \delta_1 = -\frac{2 \alpha \delta_2}{\pi} \sqrt{1 + \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2} \left[ 2K \left( 1 - \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2 \right) - \frac{E \left( 1 - \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2 \right)}{1 - \left( \frac{\delta_2}{1 + \alpha t_2} \right)^2} \right] \]  \hspace{1cm} (A12)

\[ t_2 = -\frac{2}{\pi} (1 + \alpha t_2) \sqrt{1 + \left( \frac{\delta_1}{t_1} \right)^2} \left[ 2K \left( 1 - \left( \frac{\delta_1}{t_1} \right)^2 \right) - \frac{E \left( 1 - \left( \frac{\delta_1}{t_1} \right)^2 \right)}{1 - \left( \frac{\delta_1}{t_1} \right)^2} \right] \]  \hspace{1cm} (A13)

\[ \delta_2 = -\frac{2 \alpha \delta_1}{\pi} \sqrt{1 + \left( \frac{\delta_1}{t_1} \right)^2} \left[ 2K \left( 1 - \left( \frac{\delta_1}{t_1} \right)^2 \right) - \frac{E \left( 1 - \left( \frac{\delta_1}{t_1} \right)^2 \right)}{1 - \left( \frac{\delta_1}{t_1} \right)^2} \right] \]  \hspace{1cm} (A14)

for small \( \alpha \), the mean field equations for \( t_1 \) and \( \delta_1 \) reduce to:

\[ t_1 = -\frac{2}{\pi} \]  \hspace{1cm} (A15)

\[ \delta_1 = -\frac{4 \alpha \delta_2}{\pi} \ln \left( \frac{\sqrt{\delta_2}}{1 + \alpha t_2} \right) \]  \hspace{1cm} (A16)

One sees that a finite bandwidth \( 2\alpha/\pi \) is produced for the \( \tau \) spin waves at least for small \( \alpha \). The gap equations admit solutions at small \( \delta \). One expects at weak coupling an essential singularity in \( \delta_1 \sim \exp(-\text{Cte}/\alpha) \). Therefore, \( \delta_1/t_1 \) should go to zero for \( \alpha \to 0 \). This implies that one can write:

\[ t_2 = -\frac{2 \alpha t_1}{\pi} \]  \hspace{1cm} (A17)

\[ \delta_2 = -\frac{2 \alpha \delta_1}{\pi} \ln \left( \frac{\sqrt{\delta_1}}{t_1} \right) \]  \hspace{1cm} (A18)

One can solve the mean field equations for \( \delta_1 \) and \( \delta_2 \). One obtains:

\[ \delta_1 = \frac{4}{\sqrt{\pi}} (1 + o(1)) \exp \left( -\frac{\pi}{4\alpha} \right) \]  \hspace{1cm} (A19)

\[ \delta_2 = \frac{4}{\sqrt{\pi}} (1 + o(1)) \exp \left( -\frac{\pi}{4\alpha} \right) \]  \hspace{1cm} (A20)

We obtain therefore self-consistently a gap much smaller than the smallest bandwidth. As a consequence, the correlation length is much larger than the lattice spacing, which justifies a continuum approximation. The bosonization treatment is therefore valid for \( \alpha \to 0 \) and close to the XY limit.
APPENDIX B: RENORMALIZATION GROUP EQUATIONS DERIVED BY MOMENTUM SHELL INTEGRATION

1. The Quantum sine Gordon Model

In this section, we derive RGE for the quantum sine Gordon model using the method of Knops and Den Ouden\textsuperscript{47}. The quantum sine Gordon model is defined by the following lattice Hamiltonian:

\[ H = \sum_i \left\{ \frac{v_F a}{2\pi} \left[ (\pi \Pi_i)^2 + (\phi_{i+1} - \phi_i)^2 \right] - \frac{2g}{(2\pi a)^2} \cos 4\phi_i \right\} \]  

(B1)

The cutoff in this model is only on space and not on time.

The Euclidean action of the Quantum Sine Gordon model is obtained as:

\[ S_E[\phi] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\pi a}^{\pi a} \frac{dk}{\pi} \pi u^2 \left( k^2 + \frac{\omega^2}{u^2} \right)^2 |\phi(k, \omega)|^2 - \frac{2g}{(2\pi a)^2} \int dx d\tau \cos 4\phi \]  

(B2)

Where \( \int dx \rightarrow a \sum_i \). One can integrate from \( -\infty \) to \( \infty \) using a cutoff function \( \varphi(ak) = \Theta(\frac{\pi}{a} - |k|) \). This action breaks rotation invariance in the \( x, \tau \) space in contrast with the action of the classical sine-Gordon model\textsuperscript{47}. Instead of the sharp cutoff, one can use any cutoff function \( \varphi(ak) = e^{-ak|k|} \). The technique developed by Knops and Den Ouden\textsuperscript{47} for the classical sine-Gordon model is straightforwardly adapted to the Quantum Sine-Gordon model. One obtains:

\[ \frac{dg}{dl} = (2 - 4K)g \]  

(B3)

\[ \frac{d}{dl} \left( \frac{u}{K} \right) = u \left( \frac{g}{\pi u} \right)^2 \int_0^\infty d\rho \rho^{(1-2K)} \frac{\partial F_2}{\partial \rho}(\rho) \]  

(B4)

\[ \frac{d}{dl} (uK) = \frac{u}{a} \left( \frac{g}{\pi u} \right)^2 \int_0^\infty d\rho \rho^{(1-2K)} \frac{\partial F_1}{\partial \rho}(\rho) \]  

(B5)

Where:

\[ F_1(\rho) = \int_0^{2\pi} \sin^2 \theta e^{16K[\psi(\rho, \theta) + \frac{1}{2} \ln \rho]} \frac{d\theta}{\pi} \]  

\[ F_2(\rho) = \int_0^{2\pi} \cos^2 \theta e^{16K[\psi(\rho, \theta) + \frac{1}{2} \ln \rho]} \frac{d\theta}{\pi} \]  

(B6)

And:

\[ V(\rho, \theta) = \frac{1}{2} \int_0^\infty \frac{dK}{\kappa} \varphi(\kappa)(\cos(\kappa \rho \cos \theta)e^{-\kappa|\sin \theta|} - 1) \]  

(B7)

For instance, if \( \varphi(\kappa) = e^{-\kappa} \), one has:

\[ V(\rho, \theta) = -\frac{1}{2} \ln \left( \frac{\sqrt{\rho^2 \cos^2 \theta + (1 + \rho |\sin \theta|)^2}}{a} \right) \]  

(B8)

The Eqs. (B3) have to be contrasted with the usual RG equations for the sine Gordon model with a cutoff isotropic in the \( x, u \tau \) space. In the latter case, the equations are such that \( du/dl = 0 \), and only \( K \) is flowing under renormalization. In the case we have considered, both \( u \) and \( K \) are flowing under RG transformation. However, for \( K = 1/2 \), the RG equations (B3) are considerably simplified. Since \( F_1(0) = F_2(0) = 0 \) and \( F_1(\infty) = F_2(\infty) = 1 \), one has the following simplified RG equations:
the vicinity of the BKT transition point, the anisotropy between space and time does not matter.

These universal coefficients are respectively:

\[
\frac{du}{dl} = 0
\]

\[
\frac{dK}{dl} = \left(\frac{g}{\pi u}\right)^2
\]

\[
\frac{dg}{dl} = (2 - 4K)g
\]

(B9)

These RG equations are identical to the ones obtained with a cutoff isotropic in \(x,u,\tau\) space. We conclude that in the vicinity of the BKT transition point, the anisotropy between space and time does not matter.

2. Renormalization of the dimerization term

We consider the case where \(\delta_1 = \delta_2 = 0\), with Euclidean action:

\[
S = \sum_{\alpha=1,2} \int dx dr \left[\frac{u_\alpha(\partial_x \phi_\alpha)^2}{2\pi K_\alpha} + \frac{(\partial_r \phi_\alpha)^2}{2\pi u_\alpha K_\alpha} - \frac{2g}{(2\pi)^2} \sin 2\phi_1 \sin 2\phi_2\right]
\]

(B10)

Using the method of Knops and Den Ouden, we obtain the following RG equations:

\[
\frac{dg}{dl} = (2 - K_1 - K_2)g
\]

\[
\frac{d}{dl} \left(\frac{u_1}{K_1}\right) = \frac{u_1}{8} \left(\frac{g}{\pi u_1}\right)^2 \int_0^\infty d\rho \rho^{2(2-K_1-K_2)} \partial_\rho F_3(\rho)
\]

\[
\frac{d}{dl} \left(\frac{1}{u_1 K_1}\right) = \frac{1}{8u_1} \left(\frac{g}{\pi u_1}\right)^2 \int_0^\infty d\rho \rho^{2(2-K_1-K_2)} \partial_\rho F_4(\rho)
\]

\[
\frac{d}{dl} \left(\frac{u_2}{K_2}\right) = \frac{u_2}{8} \left(\frac{g}{\pi u_2}\right)^2 \int_0^\infty d\rho \rho^{2(2-K_1-K_2)} \partial_\rho F_5(\rho)
\]

\[
\frac{d}{dl} \left(\frac{1}{u_2 K_2}\right) = \frac{1}{8u_2} \left(\frac{g}{\pi u_2}\right)^2 \int_0^\infty d\rho \rho^{2(2-K_1-K_2)} \partial_\rho F_6(\rho)
\]

(B11)

Where:

\[
F_3(\rho) = \int_0^{2\pi} \frac{d\theta}{\pi} \cos^2 \theta \exp \left(4K_1 V(\rho, \theta, 1) + 4K_2 V(\rho, \theta, \frac{u_2}{u_1}) + 2(K_1 + K_2) \ln \rho\right)
\]

\[
F_4(\rho) = \int_0^{2\pi} \frac{d\theta}{\pi} \sin^2 \theta \exp \left(4K_1 V(\rho, \theta, 1) + 4K_2 V(\rho, \theta, \frac{u_2}{u_1}) + 2(K_1 + K_2) \ln \rho\right)
\]

\[
F_5(\rho) = \int_0^{2\pi} \frac{d\theta}{\pi} \cos^2 \theta \exp \left(4K_1 V(\rho, \theta, \frac{u_1}{u_2}) + 4K_2 V(\rho, \theta, 1) + 2(K_1 + K_2) \ln \rho\right)
\]

\[
F_4(\rho) = \int_0^{2\pi} \frac{d\theta}{\pi} \sin^2 \theta \exp \left(4K_1 V(\rho, \theta, \frac{u_1}{u_2}) + 4K_2 V(\rho, \theta, 1) + 2(K_1 + K_2) \ln \rho\right)
\]

(B12)

And:

\[
V(\rho, \theta, \alpha) = \frac{1}{2} \int_0^\infty \frac{d\kappa}{\kappa} \varphi(\kappa) \left(\cos(\kappa \rho \cos \theta) e^{-\kappa \alpha |\sin \theta|} - 1\right)
\]

(B13)

It is easily seen that similarly to the case in which \(K_1 + K_2 = 2\), universal coefficients appear in the RG equations. These universal coefficients are respectively:

\[
F_3(\infty) = \int_0^{2\pi} \frac{d\theta}{\pi} \frac{\cos^2 \theta}{\left(\cos^2 \theta + \left(\frac{u_2}{u_1}\right)^2 \sin^2 \theta\right)^{K_1}}
\]

\[
F_4(\infty) = \int_0^{2\pi} \frac{d\theta}{\pi} \frac{\sin^2 \theta}{\left(\cos^2 \theta + \left(\frac{u_2}{u_1}\right)^2 \sin^2 \theta\right)^{K_2}}
\]
\[ F_5(\infty) = \int_0^{2\pi} \frac{\cos^2 \theta}{\left( \cos^2 \theta + \left( \frac{u_1}{u_2} \right)^2 \sin^2 \theta \right)^{K_1} / \pi} d\theta \]
\[ F_6(\infty) = \int_0^{2\pi} \frac{\sin^2 \theta}{\left( \cos^2 \theta + \left( \frac{u_1}{u_2} \right)^2 \sin^2 \theta \right)^{K_1} / \pi} d\theta \] (B14)

One can check easily that if \( u_1 = u_2 \), the RG equations (B11) reduce to
\[ \frac{d}{dl} \left( \frac{1}{K_1} \right) = \frac{g^2}{8\pi^2 u_2^2} \]
\[ \frac{d}{dl} \left( \frac{1}{K_2} \right) = \frac{g^2}{8\pi^2 u_2^2} \] (B15, B16)

which is equation (3.24) in the spintube paper for \( \delta_1 = \delta_2 = 0 \).

3. The full problem

For the full problem, the Lagrangian is:
\[ L = \int dxd\tau \left[ \frac{u_1(\partial_x \phi_1)^2}{2\pi K_1} + \frac{(\partial_t \phi_1)^2}{2\pi u_1 K_1} - \frac{2\delta_1}{(2\pi a)^2} \cos 4\phi_1 \right. \]
\[ \left. + \frac{u_2(\partial_x \phi_2)^2}{2\pi K_2} + \frac{(\partial_t \phi_2)^2}{2\pi u_2 K_2} - \frac{2\delta_2}{(2\pi a)^2} \cos 4\phi_2 \right. \]
\[ \left. + \frac{2g}{(2\pi a)^2} 2\sin 2\phi_1 \sin 2\phi_2 \right] \] (B18)

Applying the Knops Den Ouden Method, we obtain the following Renormalization Group equations:
\[ \frac{d}{dl} \left( \frac{u_1}{K_1} \right) = \frac{u_1}{8} \left( \frac{g}{\pi u_1} \right)^2 \int_0^\infty d\rho \rho^4(2K_1 - K_2) \frac{\delta_1}{\pi u_1} \int_0^\infty d\rho \rho^2(1-2K_1) \partial_\rho F_3 \]
\[ + u_1 \left( \frac{\delta_1}{\pi u_1} \right)^2 \int_0^\infty d\rho \rho^2(1-2K_1) \partial_\rho F_4 + \frac{1}{u_1} \left( \frac{\delta_1}{\pi u_1} \right)^2 \int_0^\infty d\rho \rho^4(1-2K_1) \partial_\rho F_1 \]
\[ \frac{d}{dl} \left( \frac{u_2}{K_2} \right) = \frac{u_2}{8} \left( \frac{g}{\pi u_2} \right)^2 \int_0^\infty d\rho \rho^4(2K_1 - K_2) \frac{\delta_2}{\pi u_2} \int_0^\infty d\rho \rho^2(1-2K_1) \partial_\rho F_5 + \frac{1}{u_2} \left( \frac{\delta_2}{\pi u_2} \right)^2 \int_0^\infty d\rho \rho^4(1-2K_1) \partial_\rho F_2 \]
\[ \frac{d}{dl} \left( \frac{\delta_1}{\pi u_1} \right) = (2-4K_1) \frac{\delta_1}{\pi u_1} - \frac{1}{8} \left( \frac{g}{\pi u_1} \right)^2 \int_0^\infty d\rho \rho^2(1+K_2 - K_1) \partial_\rho F_6 \]
\[ + \frac{1}{u_1} \left( \frac{\delta_1}{\pi u_1} \right)^2 \int_0^\infty d\rho \rho^2(1+K_2 - K_1) \partial_\rho F_7 \] (B19)
\[ \frac{d}{dl} \left( \frac{\delta_2}{\pi u_2} \right) = (2-4K_2) \frac{\delta_2}{\pi u_2} - \frac{1}{8} \left( \frac{g}{\pi u_2} \right)^2 \int_0^\infty d\rho \rho^2(1+K_2 - K_1) \partial_\rho F_8 \]
\[ + \frac{1}{u_2} \left( \frac{\delta_2}{\pi u_2} \right)^2 \int_0^\infty d\rho \rho^2(1+K_2 - K_1) \partial_\rho F_9 \]
\[ \frac{d}{dl} \left( \frac{g}{\pi u_1} \right) = (2-K_1 - K_2) \frac{g}{\pi u_1} - \frac{1}{2} \left( \frac{\delta_1}{\pi u_1} \right) \int_0^\infty d\rho \rho^2 - 4K_1 \partial_\rho F_0 + \frac{\delta_2}{u_2} \int_0^\infty d\rho \rho^2 - 4K_2 \partial_\rho F_9 \]

Where:
\[ F_7(\rho) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{4 \left[ K_1 V(\rho, \theta, \frac{\rho}{\pi u_1}) - K_2 V(\rho, \theta) \right] - 2(K_1 - K_2) \ln \rho} \]
\[ F_8(\rho) = \int_0^{2\pi} \frac{d\theta}{2\pi} \rho^4 \left[ K_1 V(\rho, \theta) - K_2 V(\rho, \theta, \frac{u_2}{u_1}) \right] - 2(K_2 - K_1) \ln \rho \]

\[ F_9(\rho) = \int_0^\infty e^{8K_1 V(\rho, \theta)} + \frac{1}{2} \ln \rho \]

(B20)

One can check that if \( u_1 = u_2 = u \), these equations reduce to those derived using OPE techniques. Clearly, the presence of a finite velocity difference results in different coefficients in the RG equations. However, this should not affect the topology of the phase diagram.

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FIG. 1. Cylindrical three-leg ladder (spin-tube). The choice of the topology affects the strong-coupling limit.

FIG. 2. Phase diagram of the spin-tube model. For $K_1 + K_2 > 2$ the system is in a gapless phase, while it is gapped for $K_1 + K_2 < 2$. The dashed line corresponds to the isotropic line with $K_1 = K_2$. The isotropic point ($K_1 = 1, K_2 = 1$) corresponds to two $XY$ chains, while the isotropic point ($K_1 = 1/2, K_2 = 1/2$) corresponds to two-equivalent spin-chains considered by Nersesyan and Tsvelik.
FIG. 3. Representation of the two-fold degenerate ground-state, formed by singlet of spins on even bonds and singlet of chiralities on odd bonds or singlet of spins on odd bonds and singlet of chiralities on even bonds.

FIG. 4. Phase diagram in the limit $K_1 = K_2 = 0$ as a function of $g$ for fixed $\delta_1 < \delta_2$. An intermediate phase with dimer order in chain 1 and mixed dimer and antiferromagnetic order in chain 2 is obtained when $4\sqrt{\delta_1\delta_2} < g < 4\delta_2$. 
FIG. 5. (a) magnetic soliton. These solitons are associated with the triplet excitations having $m = \pm 1$. On the figure, $m = -1$; (b) non-magnetic soliton. These solitons are associated with the triplet excitation having $m = 0$ or the singlet excitation.
FIG. 6. (a) triplet excitation of the spin ladder with biquadratic exchange; (b) singlet excitation of the spin ladder with biquadratic exchange.

FIG. 7. Physical picture of the spin excitations above the ground state. The elementary excitations confine to form singlets or triplets between one unpaired spin and one unpaired chirality.