Recognizing The Semiprimitivity of N-graded Algebras via Gröbner Bases*

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Abstract. Let $K\langle X \rangle = K\langle X_1, \ldots, X_n \rangle$ be the free $K$-algebra on $X = \{X_1, \ldots, X_n\}$ over a field $K$, which is equipped with a weight $\mathbb{N}$-gradation (i.e., each $X_i$ is assigned a positive degree), and let $\mathcal{G}$ be a finite homogeneous Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$ of $K\langle X \rangle$ with respect to some monomial ordering $\prec$ on $K\langle X \rangle$. It is proved that if the monomial algebra $K\langle X \rangle/\langle \text{LM}(\mathcal{G}) \rangle$ is semi-prime, where $\text{LM}(\mathcal{G})$ is the set of leading monomials of $\mathcal{G}$ with respect to $\prec$, then the $\mathbb{N}$-graded algebra $A = K\langle X \rangle/I$ is semiprimitive (in the sense of Jacobson). In the case that $\mathcal{G}$ is a finite non-homogeneous Gröbner basis with respect to a graded monomial ordering $\prec_{gr}$, and the $\mathbb{N}$-filtration $FA$ of the algebra $A = K\langle X \rangle/I$ induced by the $\mathbb{N}$-grading filtration $FK\langle X \rangle$ of $K\langle X \rangle$ is considered, if the monomial algebra $K\langle X \rangle/\langle \text{LM}(\mathcal{G}) \rangle$ is semi-prime, then it is proved that the associated $\mathbb{N}$-graded algebra $G(A)$ and the Rees algebra $\tilde{A}$ of $A$ determined by $FA$ are all semiprimitive.

Key words: Semiprimitive algebra, graded algebra, monomial algebra, Gröbner basis, Ufnarovski graph.

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1. Introduction

Let \( K \langle X \rangle = K \langle X_1, \ldots, X_n \rangle \) be the free \( K \)-algebra on \( X = \{ X_1, \ldots, X_n \} \) over a field \( K \), which is equipped with a weight \( \mathbb{N} \)-gradation (i.e., each \( X_i \) is assigned a positive degree), and let \( G \) be a finite Gröbner basis for the ideal \( I = \langle G \rangle \) of \( K \langle X \rangle \) with respect to some monomial ordering \( \prec \) on \( K \langle X \rangle \). Consider the algebra \( A = K \langle X \rangle / I \), the associated monomial algebra \( A = K \langle X \rangle / \langle \text{LM}(G) \rangle \) of \( A \) (where \( \text{LM}(G) \) is the set of leading monomials of \( G \) with respect to \( \prec \)), the \( \mathbb{N} \)-filtration \( FA \) of \( A \) induced by the \( \mathbb{N} \)-grading filtration of \( K \langle X \rangle \), the associated \( \mathbb{N} \)-graded algebra \( G(A) \) and the Rees algebra \( \tilde{A} \) of \( A \) determined by \( FA \) (see Section 3 for the definitions). In [Li2] and [Li3], it has been proved that many structural properties of \( A \) can be transferred to \( A \), \( G(A) \) and \( \tilde{A} \) (see [Li4] for more details). In this paper, we first show that if \( G \) is a finite homogeneous Gröbner basis and if the monomial algebra \( A \) is semi-prime, then the \( \mathbb{N} \)-graded algebra \( A = K \langle X \rangle / I \) is semiprimitive (in the sense of Jacobson). In the case that \( G \) is a finite non-homogeneous Gröbner basis with respect to a graded monomial ordering \( \prec_{gr} \), if \( A \) is semi-prime, then we show that the \( \mathbb{N} \)-graded algebras \( G(A) \) and \( \tilde{A} \) are all semiprimitive. Since the semiprimeness of the monomial algebra \( A \) can be determined in an algorithmic way ([G-IL], [G-I]), our results are algorithmically realizable in case the algorithms given in loc. cit. are implemented on computer.

Throughout this paper, \( K \) denotes a field, algebras considered are associative \( K \)-algebras with multiplicative identity \( 1 \), and ideals considered in an algebra are meant two-sided ideals. For a subset \( U \) of an algebra \( A \), we write \( \langle U \rangle \) for the ideal generated by \( U \) in \( A \). Moreover, we use \( \mathbb{N} \), respectively \( \mathbb{Z} \), to denote the set of nonnegative integers, respectively the set of integers.

2. Some Known Results on Monomial Algebras

Let \( K \langle X \rangle = K \langle X_1, \ldots, X_n \rangle \) be the free \( K \)-algebra on \( X = \{ X_1, \ldots, X_n \} \), and \( B = \{ X_{i_1}^{\alpha_1}X_{i_2}^{\alpha_2} \cdots X_{i_s}^{\alpha_s} \ | \ X_{i_j} \in X, \ \alpha_j \in \mathbb{N} \} \) the standard \( K \)-basis of \( K \langle X \rangle \) consisting of all monomials (words) in \( X_{i_j} \)'s. For convenience, we use lowercase letters \( w, u, v, s, \ldots \) to denote monomials in \( B \). In this section we recall from [G-IL] and [G-I] how to recognize the Jacobson semiprimitivity of a finitely presented monomial algebra \( R = K \langle X \rangle / \langle \Omega \rangle \) in a computational way, where \( \Omega = \{ u_1, \ldots, u_s \} \subset B \) is a reduced finite subset of monomials (see the definition below) such that \( \Omega \cap X = \emptyset \).

For \( u, v \in B \), we say that \( v \) divides \( u \), denoted by \( v | u \), if \( u = wvs \) for some \( w, s \in B \).
We say that a subset \( \Omega \subset B \) is reduced if \( v, u \in \Omega \) and \( v \neq u \) implies \( v \not| u \).

2.1. Theorem Let \( \Omega = \{u_1, \ldots, u_s\} \) be a reduced finite subset of \( B - X \) and \( R = K\langle X \rangle / \langle \Omega \rangle \).
(i) ([G-IL], Theorem 16) The Jacobson radical \( J(R) \) of \( R \) coincides with the upper nil-radical \( \text{Nil}(R) \) of \( R \).
(ii) ([G-I], Theorem 2.27) \( R \) is semiprimitive (in the sense of Jacobson), i.e., \( J(R) = \{0\} \), if and only if \( R \) is semi-prime, i.e., \( a \in R \) and \( aRa = \{0\} \) implies \( a = 0 \).

Let \( \Omega = \{u_1, \ldots, u_s\} \) be a reduced finite subset of \( B \), and \( \langle \Omega \rangle \) the monomial ideal generated by \( \Omega \). Then the set of normal monomials (mod \( \langle \Omega \rangle \)) in \( B \) is defined as
\[
N(\Omega) = \{w \in B \mid u \not| w, u \in \Omega\}.
\]
For each \( u_i \in \Omega \), say \( u_i = X_{i_1}^{\alpha_1} \cdots X_{i_r}^{\alpha_r} \) with \( X_{i_j} \in X \) and \( \alpha_j \in \mathbb{N} \), we write \( l(u_i) = \alpha_1 + \cdots + \alpha_r \) for the length of \( u_i \). Put
\[
\ell = \max \{l(u_i) \mid u_i \in \Omega\}.
\]
Then the Ufnarovski graph of \( \Omega \) (in the sense of [Uf1]), denoted by \( \Gamma(\Omega) \), is defined as a directed graph, in which the set of vertices \( V \) is given by
\[
V = \{v_i \in N(\Omega) \mid l(v_i) = \ell - 1\},
\]
and the set of edges \( E \) contains the edge \( v_i \rightarrow v_j \) if and only if there exist \( X_k, X_t \in X \) such that \( v_iX_k = X_tv_j \in N(\Omega) \). Since \( \Omega \) is finite, the directed graph \( \Gamma(\Omega) \) is thereby practically constructible.

Remark (i) Note that we have defined the number \( \ell \) above as in [G-IL]. While in [G-I] this number was defined as \( m + 1 = \max \{l(u_i) \mid u_i \in \Omega\} \). So, in the subsequent results we shall use \( \ell \) and \( \ell - 1 \) instead of \( m + 1 \) and \( m \).
(ii) To better understand the practical application of \( \Gamma(\Omega) \), it is essential to notice that a Ufnarovski graph is defined by using the length \( l(u) \) of the monomial (word) \( u \in B \) instead of using the degree of \( u \) as a homogeneous element in \( K\langle X \rangle \) whenever a weight \( \mathbb{N} \)-gradation of \( K\langle X \rangle \) is used (see Section 3), though both notions coincide when each \( X_i \) is assigned the degree 1.

Basic notions from classical graph theory fully suit an Ufnarovski graph \( \Gamma(\Omega) \). For instance, a route of length \( m \) in \( \Gamma(\Omega) \) with \( m \geq 1 \) is a sequence of edges
\[
v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{m-1} \rightarrow v_m.
\]
If in a route no edge appears repeatedly then it is called a simple route. A simple route with \( m \geq 1 \) and \( v_0 = v_m \) is called a cycle. If, as an undirected graph, there is a route between any two distinct vertices of \( \Gamma(\Omega) \), then \( \Gamma(\Omega) \) is called connected. A connected component of \( \Gamma(\Omega) \) (as an undirected graph) is a connected subgraph which is connected to no additional vertices.

Let \( \Gamma(\Omega) \) be as above. It follows from [Uf1] that there is a one-to-one correspondence between the monomials (words) of length \( \geq \ell - 1 \) in \( N(\Omega) \) and the routes in \( \Gamma(\Omega) \), that is, if \( u \in N(\Omega) \) and \( u = X_{i_1} \cdots X_{i_t} \) with \( t \geq \ell - 1 \), then \( u \) is mapped to the route
\[
R(u) : v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m,
\]
where
\[
m = t - \ell + 1, \quad \text{and} \quad v_j = X_{i_{j+1}}X_{i_{j+2}}\cdots X_{i_{j+\ell-1}}, \quad 0 \leq j \leq m.
\]

In [G-I] the following notions are introduced. A vertex of \( \Gamma(\Omega) \) is called a cyclic vertex if it belongs to a cycle. A normal monomial \( u \neq 1 \) (i.e., \( u \in N(\Omega) - \{1\} \)) is called a cyclic normal monomial if \( l(u) \leq \ell - 1 \) and \( u \) is a suffix of some cyclic vertex of \( \Gamma(\Omega) \), or, if \( l(u) > \ell - 1 \) and the associated route \( R(u) \) of \( u \) is a subroute of some cyclic route.

2.2. Proposition Let \( \Omega = \{u_1, \ldots, u_s\} \subset B - X \) be reduced, \( R = K\langle X \rangle /\langle \Omega \rangle \), and let \( J(R) \) be the Jacobson radical of \( R \). If \( u \in N(\Omega) - \{1\} \), then the following statements hold.
(i) ([G-I], Lemma 2.13) \( u \) is cyclic if and only if there exists a monomial \( v \in B \) such that \( (uv)^q \notin \langle \Omega \rangle \) for all integer \( q \geq 0 \).
(ii) ([G-I], Corollary 2.17) \( \overline{u} \in J(R) \) if and only if \( u \) is noncyclic, where \( \overline{u} \) is the residue class represented by \( u \) in \( R \).

2.3. Theorem ([G-I], Theorem 2.21) Let the monomial algebra \( R = K\langle X \rangle /\langle \Omega \rangle \) be as in Proposition 2.2. Then \( R \) is semiprimitive if and only if any monomial \( u \in N(\Omega) \) with \( 1 \leq l(u) \leq \ell \) is cyclic.

\[ \square \]

Remark (i) By Theorem 2.1(ii), it is clear that if \( R = K\langle X \rangle /\langle \Omega \rangle \) is a prime ring, then \( R \) is semiprimitive.
(ii) The reader is referred to [G-IL] and [G-I] for the algorithms written for determining the semi-primeness and the primeness (and hence the semiprimitivity) of a finitely presented monomial algebra \( R = K\langle X \rangle /\langle \Omega \rangle \).
3. The Main Results

In this section, we prove the main results of this paper (Theorem 3.2; Theorem 3.3, Theorem 3.5). The Gröbner basis theory for ideals in a free $K$-algebra is referred to ([Ber], [Mor], [Gr], [Uf2]).

To begin with, let $K$ be a field and $K\langle X \rangle = K\langle X_1, \ldots, X_n \rangle$ the free $K$-algebra on $X = \{X_1, \ldots, X_n\}$. As before the standard $K$-basis of $K\langle X \rangle$ is denoted by $\mathcal{B}$. We fix a weight $\mathbb{N}$-gradation for $K\langle X \rangle$, that is, $K\langle X \rangle = \bigoplus_{p \in \mathbb{N}} K\langle X \rangle_p$ in which, each $X_i$ has an assigned positive degree $m_i, 1 \leq i \leq n$, and the degree-$p$ homogeneous part $K\langle X \rangle_p$ is the $K$-vector space spanned by all monomials of degree $p$. For a nonzero homogeneous element $H \in K\langle X \rangle_p$, we write $d(H)$ for the degree of $H$, i.e., $d(H) = p$. Note that every monomial $w \in \mathcal{B}$ is a homogeneous element. If $I$ is a graded ideal of $K\langle X \rangle$ (i.e., $I$ is generated by homogeneous elements), then $A = K\langle X \rangle / I$ is an $\mathbb{N}$-graded algebra, that is, $A = \bigoplus_{p \in \mathbb{N}} A_p$ with the degree-$p$ homogeneous part $A_p = (K\langle X \rangle_p + I) / I$.

Moreover, let $\prec$ be a monomial ordering on $\mathcal{B}$, i.e., $\prec$ is a well-ordering on $\mathcal{B}$ such that $u \prec v$ implies $wus \prec wvs$, and $v = wus$ with $w \neq 1$ or $s \neq 1$ implies $u \prec v$, for all $w, u, v, s \in \mathcal{B}$. With the monomial ordering $\prec$ fixed on $\mathcal{B}$, each subset $S$ of $K\langle X \rangle$ is associated to a subset of monomials $\text{LM}(S) = \{\text{LM}(f) \mid f \in S\} \subseteq \mathcal{B}$, where if $f \in S$ and $f = \sum_{i=1}^{s} \lambda_i w_i$ with $\lambda_i \in K$ and $w_i \in \mathcal{B}$ such that $w_1 \prec w_2 \prec \cdots \prec w_s$, then $\text{LM}(f) = w_s$. $\text{LM}(S)$ is usually referred to as the set of leading monomials of $S$. By the classical Gröbner basis theory of $K\langle X \rangle$, in principle every nonzero ideal $I$ of $K\langle X \rangle$ has a nontrivial (finite or infinite) Gröbner basis $\mathcal{G}$ in the sense that $\mathcal{G}$ is a proper subset of $I - \{0\}$ and $\langle \text{LM}(I) \rangle = \langle \text{LM}(\mathcal{G}) \rangle$. A Gröbner basis $\mathcal{G}$ consisting of homogeneous elements of $K\langle X \rangle$ is called a homogeneous Gröbner basis.

In proving our main theorems, we need a fundamental result concerning the Jacobson radical of a $\mathbb{Z}$-graded ring, which is due to G. Bergman (cf. [Row]).

3.1. Theorem Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a $\mathbb{Z}$-graded ring and $J(R)$ the Jacobson radical of $R$. Then
(i) $J(R)$ is a graded ideal of $R$; and
(ii) if $n \neq 0$ and $a \in R_n$, then $1 + a$ is invertible if and only if $a$ is nilpotent.

\[ \square \]

3.2. Theorem Let $K\langle X \rangle$ and the monomial ordering $\prec$ on $\mathcal{B}$ be as fixed above, and let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a finite homogeneous Gröbner basis for the ideal $I = \langle \mathcal{G} \rangle$, such that $\text{LM}(\mathcal{G}) \cap X = \emptyset$ and $\text{LM}(\mathcal{G})$ is reduced (in the sense of Section 2). If the monomial
algebra $\overline{A} = K\langle X \rangle/\langle LM(G) \rangle$ is semi-prime, then the $\mathbb{N}$-graded algebra $A = K\langle X \rangle/I$ is semiprimitive.

**Proof** Let $N(I)$ be the set of normal monomials (mod $I$) in $B$, i.e., $N(I) = \{w \in B \mid w \not\in \langle LM(I) \rangle \}$. Then, since $G$ is a Gröbner basis of $I$, $N(I) = \{u \in B \mid LM(g_i)u, g_i \in G \}$. If, with respect to the $\mathbb{N}$-gradation of $K\langle X \rangle$, $H \in K\langle X \rangle_+$ is a homogeneous element of degree $p$, and if $H \not\in I$, then by the division by the homogeneous Gröbner basis $G$, $H$ has a representation $H = \sum_{i,j} \lambda_{ij}w_{ij}v_{ij} + \sum_t \mu_tu_t$, where $\lambda_{ij}, \mu_t \in K - \{0\}$, $w_{ij}, v_{ij} \in B$, $g_j \in G$, and $u_t \in N(I)$ such that $d(H) = p = d(u_t)$ for all $t$. Putting $H' = \sum_t \mu_tu_t$ and considering the nonzero homogeneous element $\overline{H}$ of degree $p$ represented by $H$ in $A_p = (K\langle X \rangle_p + I)/I$, we have

$$\overline{H} = \overline{H'} = \sum_t \mu_t\overline{u_t}, \quad \mu_t \in K - \{0\}, \quad u_t \in N(I) \text{ with } d(u_t) = p = d(H). \quad (1)$$

Let $J(A)$ be the Jacobson radical of the $\mathbb{N}$-graded algebra $A = K\langle X \rangle/I$. If $J(A) \ne \{0\}$, then it follows from Theorem 3.1 that $J(A)$ is a graded ideal of $A$. Taking a nonzero homogeneous element of $J(A)$, say $\overline{H} \in J(A) \cap A_p$, where $A_p = (K\langle X \rangle_p + I)/I$ and $H \in K\langle X \rangle_p$, we may replace $\overline{H}$ by $\overline{H'}$ as in (1) above. Without loss of generality we assume that $LM(H') = u_1$ with respect to $\prec$. Our aim below is to show that

(•) the normal monomial $u_1$ is noncyclic, thereby $\overline{u_1} \in J(\overline{A})$ by Proposition 2.2, where $\overline{u_1}$ is the residue class represented by $u_1$ in $\overline{A}$.

Assume the contrary that $u_1$ is cyclic (see Section 2 for the definition). Then, by Proposition 2.2, there is a monomial $v \in B$ such that $(u_1v)^q \not\in \langle LM(I) \rangle = \langle LM(G) \rangle$, or equivalently, $(u_1v)^q \in N(I)$ for all $q \in \mathbb{N}$. Since $LM(H'v) = u_1v$, it turns out that

$$LM((H'v)^q) = (LM(H'v))^q = (u_1v)^q \in N(I), \quad q \in \mathbb{N}. \quad (2)$$

On the other hand, writing $\overline{v}$ for the residue class represented by $v$ in $A$, we have $\overline{H'v} = \overline{H'v} \in J(A)$. As $\overline{H'v}$ is again a homogeneous element of $J(A)$, it follows from Theorem 3.1 that $\overline{H'v}^m = 0$ for some integer $m > 0$. Hence, $(H'v)^m \in I$ and this gives rise to

$$(u_1v)^m = (LM(H'v))^m = LM((H'v)^m) \in \langle LM(I) \rangle = \langle LM(G) \rangle. \quad (3)$$

Clearly, (3) contradicts (2). Therefore, $u_1$ is noncyclic and consequently $\overline{u_1} \in J(\overline{A})$, proving the claim (•).

Finally, suppose that the monomial algebra $\overline{A} = K\langle X \rangle/\langle LM(G) \rangle$ is semi-prime. Then it follows from Theorem 2.1(ii) that $\overline{A}$ is semiprimitive and hence $J(\overline{A}) = \{0\}$. By the above argument, we conclude that $J(A) = \{0\}$; otherwise, by the claim (•), there would
be a normal monomial \( u_1 \in N(I) \) such that \( \overline{u_1} \in J(\overline{A}) = \{0\} \) and hence, \( u_1 \in \langle \text{LM}(\mathcal{G}) \rangle \), which is a contradiction. This shows that \( A \) is semiprimitive.

Let \( I \) be an arbitrary proper ideal of \( K\langle X \rangle \), \( A = K\langle X \rangle / I \). Consider the natural \( \mathbb{N} \)-grading filtration \( FK(X) = \{ F_pK(X) \}_{p \in \mathbb{N}} \) of \( K\langle X \rangle \) determined by a fixed weight \( \mathbb{N} \)-gradation for \( K\langle X \rangle \), that is, \( F_pK(X) = \oplus_{q \leq p} K(X)_q \) for \( p \in \mathbb{N} \). Then \( A \) has the natural \( \mathbb{N} \)-filtration \( FA = \{ F_pA \}_{p \in \mathbb{N}} \) induced by the \( \mathbb{N} \)-grading filtration \( FK(X) \), where \( F_pA = (F_pK(X) + I)/I \) for \( p \in \mathbb{N} \), and \( A \) has the associated \( \mathbb{N} \)-graded algebra \( G(A) = \bigoplus_{p \in \mathbb{N}} G(A)_p \) with \( G(A)_p = F_pA/F_{<p}A \), where \( F_{<p}A = \bigcup_{q < p} F_qA \) (conventionally we put \( F_{<0}A = \{0\} \)).

In case \( I \) is a graded ideal of \( K\langle X \rangle \), it is clear that \( G(A) \cong A \) as \( \mathbb{N} \)-graded algebras.

We also recall that any total ordering \( \prec \) on \( B \) induces an ordering \( \prec_{gr} \) on \( B \) subject to the rule: For \( u, v \in B \),

\[ u \prec_{gr} v \iff d(u) < d(v) \text{ or } d(u) = d(v) \text{ and } u \prec v, \]

where \( d(\ ) \) is the degree function on the homogeneous elements of \( K\langle X \rangle \). If \( \prec_{gr} \) is a monomial ordering on \( B \), then \( \prec_{gr} \) is called an \( \mathbb{N} \)-graded monomial ordering, for instance, the commonly used \( \mathbb{N} \)-graded lexicographic ordering on \( B \).

### 3.3. Theorem

With notation as before, let \( \prec_{gr} \) be an \( \mathbb{N} \)-graded monomial ordering on \( B \) with respect to a fixed weight \( \mathbb{N} \)-gradation of \( K\langle X \rangle \), and let \( \mathcal{G} = \{ g_1, \ldots, g_s \} \) be a finite (but not necessarily homogeneous) Gröbner basis for the ideal \( I = \langle \mathcal{G} \rangle \), such that \( \text{LM}(\mathcal{G}) \cap X = \emptyset \) and \( \text{LM}(\mathcal{G}) \) is reduced (in the sense of Section 2). Consider the algebra \( A = K\langle X \rangle / I \). If the monomial algebra \( \overline{A} = K\langle X \rangle / \langle \text{LM}(\mathcal{G}) \rangle \) is semi-prime, then \( G(A) \) is semiprimitive.

**Proof** Let \( \text{LH}(\mathcal{G}) = \{ \text{LH}(g_i) \mid g_i \in \mathcal{G} \} \) be the set of \( \mathbb{N} \)-leading homogeneous elements of \( \mathcal{G} \) with respect to the fixed weight \( \mathbb{N} \)-gradation of \( K\langle X \rangle \), that is, \( \text{LH}(g_i) = H_p \) if \( g_i = H_0 + H_1 + \cdots + H_p \) with \( H_j \in K\langle X \rangle_j \) and \( H_p \neq 0 \). Since we are using an \( \mathbb{N} \)-graded monomial ordering \( \prec_{gr} \), it follows from ([LWZ], Theorem 2.3.2) or ([Li2], Proposition 3.2) that \( \text{LH}(\mathcal{G}) \) is a Gröbner basis for the graded ideal \( \langle \text{LH}(\mathcal{G}) \rangle \) of \( K\langle X \rangle \), and that

\[ G(A) \cong K\langle X \rangle / \langle \text{LH}(\mathcal{G}) \rangle \]

as \( \mathbb{N} \)-graded algebras. Furthermore, under the \( \mathbb{N} \)-graded monomial ordering \( \prec_{gr} \) we have \( \text{LM}(\mathcal{G}) = \text{LM}(\text{LH}(\mathcal{G})) \). Hence the \( \mathbb{N} \)-graded algebra \( K\langle X \rangle / \langle \text{LH}(\mathcal{G}) \rangle \) has the associated monomial algebra \( \overline{A} = K\langle X \rangle / \langle \text{LM}(\mathcal{G}) \rangle \). Consequently, our assertion follows from Theorem 3.2 and the isomorphism given above. \( \square \)

Now, we turn to the Rees algebra \( \overline{A} \) of the \( \mathbb{N} \)-filtered algebra \( A = K\langle X \rangle / I \), where \( I \) is an arbitrary proper ideal of \( K\langle X \rangle \), the \( \mathbb{N} \)-filtration \( FA = \{ F_pA \}_{p \in \mathbb{N}} \) for \( A \) is as
constructed before Corollary 3.3, and \( \tilde{A} \) is defined as the \( \mathbb{N} \)-graded algebra \( \tilde{A} = \bigoplus_{p \in \mathbb{N}} F_p A \) with the multiplication induced by \( F_p A F_q A \subseteq F_{p+q} A \) for all \( p, q \in \mathbb{N} \). The relations between \( A \), \( G(A) \) and \( \tilde{A} \) are given by the algebra isomorphisms \( A \cong \tilde{A}/\langle 1 - Z \rangle \) and \( G(A) \cong \tilde{A}/\langle Z \rangle \), where \( Z \) is the homogeneous element of degree 1 in \( \tilde{A}_1 = F_1 A \) represented by the multiplicative identity element 1 of \( A \). Because of these relations, the structure of \( \tilde{A} \) is closely related to the study of the homogenized algebra of an algebra defined by relations, the regular central extension and the PBW-deformation of an \( \mathbb{N} \)-graded algebra defined by relations (cf. [LWZ], [Li1], [Li2]).

Consider the free \( K \)-algebra \( K\langle X, T \rangle = K\langle X_1, \ldots, X_n, T \rangle \) in which each \( X_i \) has the same positive degree as fixed in \( K\langle X \rangle \), and we assign \( d(T) = 1 \). Write \( \tilde{B} \) for the standard \( K \)-basis for \( K\langle X, T \rangle \). If \( \vartriangleleft_{gr} \) is some \( \mathbb{N} \)-graded lexicographic ordering on the standard \( K \)-basis \( B \) of \( K\langle X \rangle \), then \( \vartriangleleft_{gr} \) extends to a \( \mathbb{N} \)-graded lexicographic ordering \( \vartriangleleft_{T-gr} \) on \( \tilde{B} \) subject to \( T \vartriangleleft_{T-gr} X_i, 1 \leq i \leq n \). If \( f \in K\langle X \rangle \) has the linear representation \( f = \lambda LM(f) + \sum f_i w_i \) with \( \lambda, f_i \in K - \{0\} \), \( LM(f) \in B \cap K\langle X \rangle_{p}, w_i \in B \cap K\langle X \rangle_{q_i}, \) then the non-central homogenization of \( f \) with respect to \( T \) is the homogeneous element

\[
\tilde{f} = LC(f) LM(f) + \sum \lambda_i T^{p - q_i} w_i \in K\langle X, T \rangle_{p}.
\]

Clearly, \( LM(f) = LM(\tilde{f}) \), where \( LM(f) \) is taken with respect to \( \vartriangleleft_{gr} \) on \( B \) and \( LM(\tilde{f}) \) is taken with respect to \( \vartriangleleft_{T-gr} \) on \( \tilde{B} \). If \( I \) is an ideal of \( K\langle X \rangle \), then we put

\[
\tilde{I} = \{ \tilde{f} \mid f \in I \} \cup \{ X_i T - T X_i \mid 1 \leq i \leq n \},
\]

and call \( \langle \tilde{I} \rangle \), the graded ideal of \( K\langle X, T \rangle \) generated by \( \tilde{I} \), the non-central homogenization ideal of \( I \) in \( K\langle X, T \rangle \) with respect to \( T \).

3.4. Proposition With notation as fixed above, let \( I \) be an ideal of \( K\langle X \rangle \) and \( G \subset I \).

The following statements are equivalent.

(i) \( G \) is a Gröbner basis for \( I \) with respect to \( \vartriangleleft_{gr} \) on \( B \).

(ii) \( \tilde{G} = \{ \tilde{g} \mid g \in G \} \cup \{ X_i T - T X_i \mid 1 \leq i \leq n \} \) is a homogeneous Gröbner basis for \( \langle \tilde{I} \rangle \) with respect to \( \vartriangleleft_{T-gr} \) on \( \tilde{B} \).

(iii) The set of normal monomials \( \text{mod}(\tilde{I}) \) in \( \tilde{B} \), with respect to \( \vartriangleleft_{T-gr} \), is given by

\[
N(\langle \tilde{I} \rangle) = \{ T^r u \mid u \in N(I), r \in \mathbb{N} \},
\]

where \( N(I) \) is the set of normal monomial \( \text{mod} I \) in \( B \) with respect to \( \vartriangleleft_{gr} \).

Proof The equivalence (i) \( \Leftrightarrow \) (ii) is a strengthened version of ([LWZ], Theorem 2.3.2 (i) \( \Leftrightarrow \) (ii)), of which a detailed proof was given in [LS].
Noticing that \( N(I) = \{ u \in B \mid \text{LM}(g) \} \cup \{ u \mid g \in \mathcal{G} \} \), \( \text{LM}(\mathcal{G}) = \text{LM}(\tilde{G}) \) where \( \text{LM}(\mathcal{G}) \) is taken with respect to \( \prec_{\text{gr}} \) on \( B \) and \( \text{LM}(\tilde{G}) \) is taken with respect to \( \prec_{\text{gr}} \) on \( \tilde{B} \), and that \( \text{LM}(X_i T - TX_i) = X_i T \) with respect to \( \prec_{\text{gr}} \) on \( \tilde{B} \), the verification of the equivalence (ii) \( \Leftrightarrow \) (iii) is straightforward by referring to the well-known characterization of a Gröbner basis in terms of the remainder on division by \( \tilde{G} \).

With the preparation made above, we are ready to mention and prove the next

3.5. Theorem With notation as fixed above, let \( \mathcal{G} = \{ g_1, \ldots, g_s \} \) be a finite (but not necessarily homogeneous) Gröbner basis for the ideal \( I = \langle \mathcal{G} \rangle \) with respect to \( \prec_{\text{gr}} \) on \( B \), such that \( \text{LM}(\mathcal{G}) \cap X = \emptyset \) and \( \text{LM}(\mathcal{G}) \) is reduced (in the sense of Section 2). Consider the algebra \( A = K\langle X \rangle/I \) which has the \( \mathbb{N} \)-filtration \( FA \) as constructed before. If the monomial algebra \( \tilde{A} = K\langle X \rangle/(\text{LM}(\mathcal{G})) \) is semi-prime, then the Rees algebra \( \tilde{A} \) of \( A \) is semiprimitive.

Proof First note that if the \( \mathbb{N} \)-graded lexicographic ordering \( \prec_{\text{gr}} \) on \( B \) is defined subject to
\[
X_{i_1} \prec_{\text{gr}} X_{i_2} \prec_{\text{gr}} \cdots \prec_{\text{gr}} X_{i_n},
\]
then the \( \mathbb{N} \)-graded lexicographic ordering \( \prec_{\text{gr}} \) on \( \tilde{B} \) is defined subject to
\[
T \prec_{\text{gr}} X_{i_1} \prec_{\text{gr}} X_{i_2} \prec_{\text{gr}} \cdots \prec_{\text{gr}} X_{i_n}.
\]
Moreover, we also bear in mind that \( d(T) = 1 \), and that each \( X_i \) has the same positive degree as fixed in \( K\langle X \rangle \).

Let \( \langle \tilde{I} \rangle \) be the non-central homogenization ideal of \( I \) in \( K\langle X, T \rangle \) with respect to \( T \). Then, by Proposition 3.4(iii), the set of normal monomials (mod \( \langle \tilde{I} \rangle \)) in \( \tilde{B} \) with respect to \( \prec_{\text{gr}} \) is given by \( N(\langle \tilde{I} \rangle) = \{ T^r u \mid u \in N(I), \ r \in \mathbb{N} \} \), where \( N(I) \) is the set of normal monomial (mod \( I \)) in \( B \) with respect to \( \prec_{\text{gr}} \). If \( T^r u_1, T^s u_2 \in N(\langle \tilde{I} \rangle) \) with \( d(T^r u_1) = d(T^s u_2) \) such that \( T^r u_1 \prec_{\text{gr}} T^s u_2 \), then it follows from the definition of \( \prec_{\text{gr}} \) that
\[
r \geq s \text{ and } u_1 \prec_{\text{gr}} u_2.
\]

Consider the \( \mathbb{N} \)-gradation of \( K\langle X, T \rangle \) determined by the assigned degrees for \( T \) and \( X_i \)'s. If \( H \in K\langle X, T \rangle \) is a homogeneous element of degree \( p \), and if \( H \notin \langle \tilde{I} \rangle \), then since \( \tilde{G} = \{ g \mid g \in \mathcal{G} \} \cup \{ X_i T - TX_i \mid 1 \leq i \leq n \} \) is a homogeneous Gröbner basis for \( \langle \tilde{I} \rangle \) with respect to \( \prec_{\text{gr}} \) (Proposition 3.4(ii)), the division by \( \tilde{G} \) yields a representation \( H = D + \sum_i \lambda_i T^{r_i} u_i \), where \( D \in \langle \tilde{I} \rangle \), \( \lambda_i \in K \setminus \{ 0 \} \), and \( T^{r_i} u_i \in N(\tilde{I}) \) such that \( d(T^{r_i} u_i) = p = d(H) \) for all \( i \). Put \( H' = \sum_i \lambda_i T^{r_i} u_i \) and consider the \( \mathbb{N} \)-graded algebra
\[ K(X, T)/⟨\tilde{I}⟩ = \oplus_{p \in \mathbb{N}}(K(X, T)_p + ⟨\tilde{I}⟩)/⟨\tilde{I}⟩. \]

Then the nonzero homogeneous element \( \overline{π} \) of degree \( p \) represented by \( H \) in \((K(X, T)_p + ⟨\tilde{I}⟩)/⟨\tilde{I}⟩ \) has the representation

\[ \overline{H} = \overline{H}' = \sum \lambda_i T^{r_i} u_i, \quad \lambda_i \in K \setminus \{0\}, \quad u_i \in N(I) \text{ with } d(T^{r_i} u_i) = p = d(H). \quad (2) \]

Moreover, By the above (1), if \( \text{LM}(H') = T^{r_1} u_1 \) with respect to \( \triangleleft_{\text{gr}} \), then

\[ \text{LM}\left(\sum \lambda_i u_i\right) = u_1 \text{ with respect to } \triangleleft_{\text{gr}}. \quad (3) \]

By the definition of \( \tilde{I} \), it is not difficult to verify that the map

\[
\psi : \quad K(X, T)/⟨\tilde{I}⟩ \longrightarrow K(X)/I = A \quad \text{such that } (4) \\
\overline{T} \quad \mapsto \quad 1 \quad \text{for } 1 \leq i \leq n
\]

is well defined, and that \( \psi \) is a \( K \)-algebra epimorphism. If, as described in (2) above, \( \overline{H} = \overline{H}' = \sum \lambda_i T^{r_i} u_i \) is a nonzero homogeneous element of degree \( p \) in \( K(X, T)/⟨\tilde{I}⟩ \), then since \( u_i \in N(I) \) and conventionally the ideal \( I \) considered is a proper ideal, we have

\[
\psi(\overline{H}) = \psi(\overline{H}') = \sum \lambda_i \overline{u_i} = \sum \lambda_i u_i \neq 0. \quad (4)
\]

By ([LWZ], [Li1], [Li2]), \( \tilde{A} \cong K(X, T)/⟨\tilde{I}⟩ \) as \( \mathbb{N} \)-graded algebras, that is, the algebra isomorphism gives rise to isomorphisms of \( K \)-vector spaces

\[
\tilde{A}_p \cong (K(X, T)_p + ⟨\tilde{I}⟩)/⟨\tilde{I}⟩, \quad p \in \mathbb{N}.
\]

Identifying \( \tilde{A} \) with \( K(X, T)/⟨\tilde{I}⟩ \), we now proceed to deal with the Jacobson radical \( J(\tilde{A}) \) of \( \tilde{A} \). If \( J(\tilde{A}) \neq \{0\} \), then \( J(\tilde{A}) \) is a graded ideal of \( \tilde{A} \) by Theorem 3.1. Taking a nonzero homogeneous element of \( J(\tilde{A}) \), say \( \overline{π} \in J(\tilde{A}) \cap \tilde{A}_p \), where \( \tilde{A}_p = (K(X, T)_p + ⟨\tilde{I}⟩)/⟨\tilde{I}⟩ \) and \( H \in K(X, T)_p \), we may replace \( \overline{H} \) by \( \overline{H}' \) as in (2) above. Without loss of generality we assume that \( \text{LM}(H') = T^{r_1} u_1 \). Our aim below is to show that

\[(*) \text{ the normal monomial } u_1 \text{ is noncyclic, thereby } \overline{u_1} \in J(\tilde{A}) \text{ by Proposition 2.2, where } \overline{u_1} \text{ is the residue class represented by } u_1 \text{ in } \tilde{A}. \]

Assume the contrary that \( u_1 \) is cyclic (see Section 2 for the definition). Then, by Proposition 2.2, there is a monomial \( v \in B \) such that \((u_1 v)^q \not\in \langle \text{LM}(G) \rangle \) for all \( q \in \mathbb{N} \), or equivalently,

\[
(u_1 v)^q \in N(I) \text{ for all } q \in \mathbb{N}, \text{ in particular } u_1 v \in N(I). \quad (5)
\]
On the other hand, $H'v = H'v \in J(\tilde{A})$. As $H'v$ is again a homogeneous element of $J(A)$, it follows from Theorem 3.1 that

$$H'v^m = 0 \text{ for some integer } m > 0, \text{ i.e., } \langle H'v \rangle^m \subseteq \langle \tilde{I} \rangle. \quad (6)$$

Furthermore, put $H'' = \sum \lambda_i u_i v$. By the foregoing (3), $\text{LM}(H'') = u_1 v$ with respect to $\prec_{gr}$. Since $\psi(H') \neq 0$ and $u_1 v \in N(I)$ by (4) and (5), we have $\psi(H'v) = H'' \neq 0$ in $A$. But it follows from (6) that $H''$ is a nilpotent element in $A$, i.e.,

$$H''^m = \left( \sum \lambda_i u_i v \right)^m = \left( \psi(H'v) \right)^m = \psi(H''^m) = 0. \quad (7)$$

Hence, $(H'')^m \in I$ and this gives rise to

$$(u_1 v)^m = (\text{LM}(H''))^m = \text{LM}((H'')^m) \in \langle \text{LM}(I) \rangle = \langle \text{LM}(G) \rangle. \quad (8)$$

Clearly, (8) contradicts (5). Therefore, $u_1$ is noncyclic and consequently $u_1 \in J(\tilde{A})$, proving the claim ($\ast$).

Finally, suppose that the monomial algebra $\tilde{A} = K\langle X \rangle/\langle \text{LM}(G) \rangle$ is semi-prime. Then it follows from Theorem 2.1(ii) that $\tilde{A}$ is semiprimitive and hence $J(\tilde{A}) = \{0\}$. By the above argument, we conclude that $J(\tilde{A}) = \{0\}$; otherwise, by the claim ($\ast$), there would be a normal monomial $u_i \in N(I)$ such that $\tilde{u}_i \in J(\tilde{A}) = \{0\}$ and hence, $u_i \in \langle \text{LM}(G) \rangle$, which is a contradiction. This shows that $\tilde{A}$ is semiprimitive. \qed

We end this paper by the following

**Open question** Let the $K$-algebra $A = K\langle X \rangle/\langle G \rangle$ be as in Theorem 3.5. If the monomial algebra $\tilde{A} = K\langle X \rangle/\langle \text{LM}(G) \rangle$ is semi-prime, is $A$ semiprimitive?

References

[Ber3] G. Bergman, The diamond lemma for ring theory, *Adv. Math.*, 29(1978), 178–218.

[G-I] T. Gateva-Ivanova, Algorithmic determination of the Jacobson radical of monomial algebras, in: *Proc. EUROCAL’85*, LNCS Vol. 378, Springer-Verlag, 1989, 355–364.

[G-IL] T. Gateva-Ivanova and V. Latyshev, On recognizable properties of associative algebras, *J. Symbolic Computation*, 6(1988), 371–388.

[Gr] E. Green, An introduction to noncommutative Gröbner bases, in: *Computational Algebra*, Proceedings of the fifth meeting of the Mid-Atlantic Algebra Conference, 1993, (K. G. Fischer, P. Loustaunau, J. Shapiro, E. L. Green, and D. Farkas eds.),
Lecture Notes in Pure and Applied Mathematics, Vol. 151, Marcel Dekker, 1994, 167–190.

[Li1] H. Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*, Lecture Notes in Mathematics, Vol. 1795, Springer, 2002.

[Li2] H. Li, Γ-leading homogeneous algebras and Gröbner bases, in: *Recent Developments in Algebra and Related Areas* (F. Li and C. Dong eds.), Advanced Lectures in Mathematics, Vol. 8, International Press & Higher Education Press, Boston-Beijing, 2009, 155 – 200. arXiv:math.RA/0609583, http://arXiv.org

[Li3] H. Li, On the calculation of ${\text{gl.dim}}_{G}(A)$ and ${\text{gl.dim}}_{\tilde{A}}$ by using Gröbner bases, *Algebra Colloquium*, 16(2)(2009), 181–194. arXiv:math.RA/0805.0686, http://arXiv.org

[Li4] H. Li, *Gröbner Bases in Ring Theory*, Monograph, World Scientific Publishing Co., Oct. 2011.

[LS] H. Li and C. Su, On (de)homogenized Gröbner bases, *Journal of Algebra, Number Theory: Advances and Applications*, 3(1)(2010), 35–70. arXiv:0907.0526, http://arXiv.org

[LWZ] H. Li, Y. Wu and J. Zhang, Two applications of noncommutative Gröbner bases, *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.*, XLV(1999), 1–24.

[Mor] T. Mora, An introduction to commutative and noncommutative Gröbner Bases, *Theoretic Computer Science*, 134(1994), 131–173.

[Row] L.H. Rowen, *Ring Theory*, Vol. I, Pure and Applied Mathematics vol. 127, Academic Press, 1988.

[Uf1] V. Ufnarovski, On the use of graphs for computing a basis, growth and Hilbert series of associative algebras, (in Russian 1989), *Math. USSR Sbornik*, 180(11)(1989), 417-428.

[Uf2] V. Ufnarovski, Introduction to noncommutative Gröbner basis theory, in: *Gröbner Bases and Applications* (Linz, 1998), London Math. Soc. Lecture Notes Ser., 251, Cambridge Univ. Press, Cambridge, 1998, 259–280.