Relativistic Wave Equations in the Helicity Basis

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Abstract

The principal series of unitary representations of the Lorentz group has been considered in the helicity basis. Decompositions of tensor products of the spinspace are studied in the framework of projective representations of the symmetric group. Higher-spin Gel’fand-Yaglom equations are defined in the helicity basis over an arbitrary representation space. Applications of decomposable and indecomposable Gel’fand-Yaglom equations to particle physics are discussed.

1 Introduction

As known [17], a root subgroup of a semisimple Lie group $O_4$ (a rotation group of the 4-dimensional space) is a normal divisor of $O_4$. For that reason the 6-parameter group $O_4$ is semisimple, and is represented by a direct product of the two 3-parameter unimodular groups. By analogy with the group $O_4$, a double covering $SL(2, \mathbb{C})$ of the proper orthochronous Lorentz group $\mathfrak{G}_+$ (a rotation group of the 4-dimensional Minkowski spacetime) is semisimple, and is represented by a direct product of the two 3-parameter special unimodular groups, $SL(2, \mathbb{C}) \simeq SU(2) \otimes SU(2)$. An explicit form of this isomorphism can be obtained by means of a complexification of the group $SU(2)$, that is, $SL(2, \mathbb{C}) \simeq \text{complex}(SU(2)) \simeq SU(2) \otimes SU(2)$ [14].

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Moreover, in the works [3, 12] the Lorentz group is represented by a product $SU_R(2) \otimes SU_L(2)$, and spinors

$$\psi(p^\mu) = \begin{pmatrix} \phi_R(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix}$$

are transformed within $(j, 0) \oplus (0, j)$ representation space. The components $\phi_R(p^\mu)$ and $\phi_L(p^\mu)$ correspond to different helicity states (right- and left-handed spinors).

## 2 Helicity basis

Let $g \to T_g$ be an arbitrary linear representation of group $\mathfrak{g}_+$ and let $A_i(t) = T_{a_i(t)}$ be an infinitesimal operator corresponded the rotation $a_i(t) \in \mathfrak{g}_+$. Analogously, we have $B_i(t) = T_{b_i(t)}$, where $b_i(t) \in \mathfrak{g}_+$ is a hyperbolic rotation. The operators $A_i$ and $B_i$ satisfy the following commutation relations:

\[
\begin{align*}
[A_1, A_2] &= A_3, & [A_2, A_3] &= A_1, & [A_3, A_1] &= A_2, \\
[B_1, B_2] &= -A_3, & [B_2, B_3] &= -A_1, & [B_3, B_1] &= -A_2, \\
A_1, B_1 &= 0, & [A_2, B_2] &= 0, & [A_3, B_3] &= 0, \\
A_1, B_2 &= B_3, & [A_1, B_3] &= -B_2, \\
A_2, B_3 &= B_1, & [A_2, B_1] &= -B_3, \\
A_3, B_1 &= B_2, & [A_3, B_2] &= -B_1.
\end{align*}
\]

(1)

Denoting $l^{12} = A_1$, $l^{11} = A_2$, $l^{12} = A_3$, and $l^{01} = B_1$, $l^{02} = B_2$, $l^{03} = B_3$ we can write the relations (1) in a more compact form:

$$[l^{\mu\nu}, l^{\rho\lambda}] = \delta_{\mu\rho}l^{\lambda\nu} + \delta_{\nu\lambda}l^{\mu\rho} - \delta_{\nu\rho}l^{\mu\lambda} - \delta_{\mu\lambda}l^{\nu\rho}.$$ 

Let us consider the operators

$$X_k = \frac{1}{2}(A_k + iB_k), \quad Y_k = \frac{1}{2}(A_k - iB_k), \quad (k = 1, 2, 3).$$

(2)

Using the relations (1) we find that

\[
\begin{align*}
[X_1, X_2] &= X_3, & [X_2, X_3] &= X_1, & [X_3, X_1] &= X_2, \\
[Y_1, Y_2] &= Y_3, & [Y_2, Y_3] &= Y_1, & [Y_3, Y_1] &= Y_2, \\
X_k, Y_l &= 0, \quad (k, l = 1, 2, 3).
\end{align*}
\]

(3)
Further, taking
\[ X_+ = X_1 + iX_2, \quad X_- = X_1 - iX_2, \]
\[ Y_+ = Y_1 + iY_2, \quad Y_- = Y_1 - iY_2, \]
we see that in virtue of commutativity of the relations (3) a space of an irreducible finite-dimensional representation of the group \( G_+ \) can be stretched on the totality of \((2l + 1)(2\ell + 1)\) basis vectors \( | l, m; \ell, \ell \rangle \), where \( l, m, \ell, \ell \) are integer or half-integer numbers, \(-l \leq m \leq l, \ -\ell \leq \ell \leq \ell \). Therefore,
\[ X_- | l, m; \ell, \ell \rangle = \sqrt{(\ell + m)(\ell - m + 1)} | l, m; \ell, \ell - 1 \rangle \quad (m > -\ell), \]
\[ X_+ | l, m; \ell, \ell \rangle = \sqrt{(\ell - m)(\ell + m + 1)} | l, m; \ell, \ell + 1 \rangle \quad (m < \ell), \]
\[ X_3 | l, m; \ell, \ell \rangle = \ell | l, m; \ell, \ell \rangle, \]
\[ Y_- | l, m; \ell, \ell \rangle = \sqrt{(l + m)(l - m + 1)} | l, m - 1, \ell, \ell \rangle \quad (m > -l), \]
\[ Y_+ | l, m; \ell, \ell \rangle = \sqrt{(l - m)(l + m + 1)} | l, m + 1, \ell, \ell \rangle \quad (m < l), \]
\[ Y_3 | l, m; \ell, \ell \rangle = m | l, m; \ell, \ell \rangle. \] (5)

From the relations (3) it follows that each of the sets of infinitesimal operators \( X \) and \( Y \) generates the group \( SU(2) \) and these two groups commute with each other. Thus, from the relations (3) and (5) it follows that the group \( G_+ \), in essence, is equivalent to the group \( SU(2) \otimes SU(2) \). In contrast to the Gel’fand–Naimark representation for the Lorentz group \([15, 26]\) which does not find a broad application in physics, a representation (5) is a most useful in theoretical physics (see, for example, \([1, 36, 33, 34]\)). This representation for the Lorentz group was firstly given by Van der Waerden in his brilliant book \([51]\).

As showed in \([43]\) the spinspace
\[ S_{2^{k+r}} \simeq S_2 \otimes S_2 \otimes \cdots \otimes S_2 \otimes \hat{S}_2 \otimes \hat{S}_2 \otimes \cdots \hat{S}_2 \] (6)
is the minimal left ideal of the complex Clifford algebra
\[ C_{2n} \simeq C_2 \otimes C_2 \otimes \cdots \otimes C_2 \otimes \hat{C}_2 \otimes \hat{C}_2 \otimes \cdots \hat{C}_2. \] (7)

The algebras \( C_2 (\hat{C}_2) \) and the spinspaces \( S_2 (\hat{S}_2) \) correspond to fundamental representations \( \tau_{\frac{1}{2}, 0} (\hat{\tau}_{0, \frac{1}{2}}) \) of the Lorentz group \( G_+ \).
In general case the spin space (5) is reducible, that is, there exists a decomposition of the original spin space $S_{2k+r}$ into a direct sum of irreducible subspaces with respect to a representation

$$
\tau_{\frac{1}{2},0} \otimes \tau_{\frac{1}{2},0} \otimes \cdots \otimes \tau_{\frac{1}{2},0} \otimes \tau_{0,\frac{1}{2}} \otimes \cdots \otimes \tau_{0,\frac{1}{2}}. 
$$

(8)

First of all, let us consider a decomposition of the spin space

$$S_{2m} \simeq S_2 \otimes S_2 \otimes \cdots \otimes S_2
$$

(9)

into irreducible subspaces with respect to an action of the representation

$$\tau_{\frac{1}{2},0} \otimes \tau_{\frac{1}{2},0} \otimes \cdots \otimes \tau_{\frac{1}{2},0}. 
$$

(10)

### 2.1 Projective representations of the symmetric group

A group ring $\mathbb{Z}(S_m)$ and a group algebra $\mathbb{C}(S_m)$ of the symmetric group $S_m$ act in the spin space (5). In 1911 [35] (see also [25, 39]), Schur showed that over the field $\mathbb{F} = \mathbb{C}$ there exists a nontrivial $\mathbb{Z}_2$-extension $T_n$ of the symmetric group $S_n$ defined by the sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow T_n \rightarrow S_n \rightarrow 1,$$

where

$$T_n = \{z, t_1, \ldots, t_{n-1} : z^2 = 1, zt_i = t_iz, t_i^2 = z, (t_jt_{j+1})^3 = z, t_k t_l = zt_l t_k \}$$

$1 \leq i \leq n-1, 1 \leq j \leq n-2, k \leq l-2.$

The group $T_n$ has order $2(n!)$. The subgroup $\mathbb{Z}_2 = \{1, z\}$ is central and, is contained in the commutator subgroup of $T_n$, $T_n/\mathbb{Z}_2 \simeq S_n$ ($n \geq 4$). If $n < 4$, then every projective representation of $S_n$ is projectively equivalent to a linear representation.

Spinor representations of the transpositions $t_k$ are defined by elements of the group algebra $\mathbb{C}(S_n)$:

$$t_k = \sqrt{\frac{k-1}{2k}} e_{k-1} - \sqrt{\frac{k+1}{2k}} e_k, \quad k = 1, \ldots, m,$$
where

\[ \mathcal{E}_1 = \sigma_1 \otimes 1_2 \otimes \cdots \otimes 1_2 \otimes 1_2, \]
\[ \mathcal{E}_2 = \sigma_3 \otimes \sigma_1 \otimes 1_2 \otimes \cdots \otimes 1_2 \otimes 1_2, \]
\[ \mathcal{E}_3 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1_2 \otimes \cdots \otimes 1_2, \]
\[ \cdots \cdots \cdots \cdots \cdots \cdots \]
\[ \mathcal{E}_m = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes 1_1, \]
\[ \mathcal{E}_{m+1} = \sigma_2 \otimes 1_2 \otimes \cdots \otimes 1_2, \]
\[ \mathcal{E}_{m+2} = \sigma_3 \otimes \sigma_2 \otimes 1_2 \otimes \cdots \otimes 1_2, \]
\[ \cdots \cdots \cdots \cdots \cdots \cdots \]
\[ \mathcal{E}_{2m} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2. \]

are the tensor products of \( m \) Pauli matrices. These matrices form a spinor basis of the complex Clifford algebra \( \mathbb{C}_n \) \((n = 2m)\) in the Brauer-Weyl representation [7]. At this point there is an isomorphism \( \mathbb{C}_{2m} \cong M_{2^n}(\mathbb{C}) \), where \( M_{2^n}(\mathbb{C}) \) is a full matrix algebra over the field \( \mathbb{F} = \mathbb{C} \). If \( n = 2m + 1 \) we add one more matrix

\[ \mathcal{E}_{2m+1} = \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{m \text{ times}}. \]

In virtue of an isomorphism \( \mathbb{C}_{2m+1} \cong \mathbb{C}_{2m} \oplus \mathbb{C}_{2m} \) the units of \( \mathbb{C}_{2m+1} \) are represented by direct sums \( \mathcal{E}_1 \oplus \mathcal{E}_1, \mathcal{E}_2 \oplus \mathcal{E}_2, \ldots, \mathcal{E}_{2m} \oplus \mathcal{E}_{2m} \). Besides, there exists a mapping \( \epsilon : \mathbb{C}_{2m+1} \to \mathbb{C}_{2m} \). In the result of the homomorphism \( \epsilon \) we have a quotient algebra \( \mathbb{C}_{2m+1} \cong \mathbb{C}_{2m} / \text{Ker} \epsilon \), where \( \text{Ker} \epsilon = \{ A^1 - \epsilon \omega A^1 \} \) is a kernel of \( \epsilon \), \( A^1 \) is an arbitrary element of the subalgebra \( \mathbb{C}_{2m} \), \( \omega \) is a volume element of \( \mathbb{C}_{2m+1} \) [43].

Recently, projective representations of the symmetric group have been used at the study of fractional quantum Hall effect [29, 52, 53] and non-Abelian statistics [13].

In accordance with a general Weyl scheme [54] a decomposition of tensor products of representations of the groups \( U(n) \) \((SU(n))\) is realized via a decomposition of a module of the group algebra \( \mathbb{C}(S_m) \), \( 1 = f(\alpha_1) + f(\alpha_2) + \ldots + f(\alpha_s) \), where \( f(\alpha_i) \) are the primitive idempotents of \( \mathbb{C}(S_m) \) (Young symmetrizers), \( \alpha_i \) are the Young diagrams, \((i = 1, \ldots, s)\). The Weyl scheme fully admits the Schur’s \( \mathbb{Z}_2 \)-extension \( T_n \), but an explicit form of the Young’s orthogonal representation has been unanswered so far [27, 28].
2.2 Decomposition of the spinspace

The decomposition of the spinspace \( \mathbb{H} \) with respect to \( SL(2, \mathbb{C}) \) is a simplest case of the Weyl scheme. Every irreducible representation of the group \( SL(2, \mathbb{C}) \) is defined by the Young diagram consisting of only one row. For that reason the representation \( \mathbb{H} \) is realized in the space \( \text{Sym}_{(m,0)} \) of all symmetric spintensors of the rank \( m \). Dimension of \( \text{Sym}_{(m,0)} \) is equal to \( m + 1 \).

In its turn, every element of the spinspace \( \mathbb{H} \), related with the representation \( \mathbb{H} \), corresponds to an element of \( S_{2k} \otimes S_{2r} \) (representations \( \tau_{k/2,0} \otimes \tau_{r/2,0} \) and \( \tau_{k/2,0} \otimes \tau_{0,r/2} \) are equivalent). This equivalence can be described as follows

\[
\varphi \otimes \psi \longrightarrow \varphi \otimes \psi I,
\]

(12)

where \( \varphi, \psi \in S_{2k}, \psi I \in S_{2r} \) and

\[
I = \lambda \left( \begin{array}{ccc}
0 & & -1 \\
-1 & \ddots & \\
& \ddots & 0
\end{array} \right)
\]

is the matrix of a bilinear form (this matrix is symmetric if \( r+k \equiv 0 \) (mod 2) and skewsymmetric if \( r+k \equiv 1 \) (mod 2). In such a way, the representation \( \mathbb{H} \) is realized in a symmetric space \( \text{Sym}_{(k,r)} \) of dimension \( (k+1)(r+1) \) (or \( (2l_1+1)(2l_2+1) \) if suppose \( l_1 = k/2 \) and \( l_2 = r/2 \)). The decomposition of \( \mathbb{H} \) is given by a Clebsh-Gordan formula

\[
\tau_{l_1l_2'} \otimes \tau_{l_2l_2'} = \sum_{|l_1-l_2| \leq k \leq l_1+l_2, |l_2'-l_2| \leq k' \leq l_2'+l_2} \tau_{kk'},
\]

where the each \( \tau_{kk'} \) acts in the space \( \text{Sym}_{(k,k')} \). In its turn, every space \( \text{Sym}_{(k,r)} \) can be represented by a space of polynomials

\[
p(z_0, z_1, \bar{z}_0, \bar{z}_1) = \sum_{(\alpha_1, \ldots, \alpha_k, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r)} \frac{1}{k! r!} a_{\alpha_1, \ldots, \alpha_k, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r} z_{\alpha_1} \cdots z_{\alpha_k} \bar{z}_{\tilde{\alpha}_1} \cdots \bar{z}_{\tilde{\alpha}_r}
\]

(13)

\((\alpha_i, \tilde{\alpha}_i = 0, 1)\)
with the basis (the basis of a representation \((l_1 l_2) \oplus (l_2 l_1)\))

\[
\frac{\mathcal{C}_{l_1 l_2}^{l_1 l_2}}{\mathcal{C}_{m_1 m_2}^{l_1 l_2}} = \frac{\zeta_{l_1}^{l_1 + m_1} \zeta_{l_2}^{l_1 - m_1} \mathcal{C}_{l_1}^{l_1 + m_2} \mathcal{C}_{l_2}^{l_2 - m_2} (\zeta_{l_1} \zeta_{l_2} - \zeta_{l_1} \zeta_{l_2})^{2l_1}}{(l_1 + m_1)!(l_1 - m_1)!(l_2 + m_2)!(l_2 - m_2)!},
\]

\[
\frac{\mathcal{C}_{l_1 l_2}^{l_2 l_1}}{\mathcal{C}_{m_1 m_2}^{l_2 l_1}} = \frac{\zeta_{l_1}^{l_1 + m_1} \zeta_{l_2}^{l_1 - m_1} \mathcal{C}_{l_1}^{l_1 + m_2} \mathcal{C}_{l_2}^{l_2 - m_2} (\zeta_{l_1} \zeta_{l_2} - \zeta_{l_1} \zeta_{l_2})^{2l_1}}{(l_1 + m_1)!(l_1 - m_1)!(l_2 + m_2)!(l_2 - m_2)!}.
\]

The vectors \(z_{mm'}^{ll'}\) of the canonical helicity basis have the form

\[
z_{mm'}^{ll'} = \sum_{j+k=m,j'+k'=k'} C(l_1, l_2, l; j, k, m) C(l_1', l_2', l'; j', k', m') \zeta_{jj'} \otimes \zeta_{kk'},
\]

where

\[
C(l_1, l_2, l; j, k, m) C(l_1', l_2', l'; j', k', m') = B_{i_1, i_2, i_3, i_4}^{k, k_1, j, j_1, m, m_1}
\]

are the Clebsch–Gordan coefficients of the group \(SL(2, \mathbb{C})\). Expressing the Clebsch–Gordan coefficients \(C(l_1, l_2, l; j, k, j + k)\) of the group \(SU(2)\) via a generalized hypergeometric function \(\mathfrak{g}_F\) (see, for example, [32, 24, 17, 50]) we see that CG–coefficients of \(SL(2, \mathbb{C})\) have the form

\[
B_{i_1, i_2, i_3, i_4}^{k, k_1, j, j_1, m, m_1} = (-1)^{t_1 + t_2 - j - j'}  
\]

\[
\times \frac{\Gamma(l_1 + l_2 - m + 1) \Gamma(l_1' + l_2' - m' + 1)}{\Gamma(l_2 - l_1 + m + 1) \Gamma(l_2' - l_1' + m' + 1)} 
\]

\[
\times \frac{(l - m)!(l + l_2 - l_1)!(l_1 - j)!(l_2 + k)! (l + m)!(2l + 1)}{(l_1 - l_2 + l)!(l_1 + l_2 - l)!(l_1 + l_2 + l)!(l_1 - j)!(l_2 - k)!}^{\frac{1}{2}} 
\]

\[
\times \frac{(l' - m')!(l' + l_2' - l_1')!(l_1' - j')!(l_2' + k')!(2l' + 1)}{(l_1' - l_2' + l')!(l_1' + l_2' - l')!(l_1' + l_2' + l')!(l_1' - j')!(l_2' - k')!}^{\frac{1}{2}} 
\]

\[
\times \frac{3F_2\left(l + m + 1, -l + m, -l_1 + j 
\right.}{l_1 - l_2 + m, l_2 - l_1 + m + 1} 
\]

\[
\left. \times \frac{3F_2\left(l' + m' + 1, -l' + m', -l_1' + j' 
\right.}{l_1' - l_2' + m', l_2' - l_1' + m' + 1} \right),
\]

where \(m = j + k, m' = j' + k'\).
Infinitesimal operators of $\Phi_+$ in the helicity basis have a very simple form

\begin{align}
A_1 &= -\frac{i}{2} \alpha^l_m \xi_{m-1} - \frac{i}{2} \alpha^l_{m+1} \xi_{m+1}, \\
A_2 &= \frac{1}{2} \alpha^l_m \xi_{m-1} - \frac{1}{2} \alpha^l_{m+1} \xi_{m+1}, \\
A_3 &= -im \xi_m, \\
B_1 &= \frac{1}{2} \alpha^l_m \xi_{m-1} + \frac{1}{2} \alpha^l_{m+1} \xi_{m+1}, \\
B_2 &= \frac{i}{2} \alpha^l_m \xi_{m-1} - \frac{i}{2} \alpha^l_{m+1} \xi_{m+1}, \\
B_3 &= m \xi_m, \\
\tilde{A}_1 &= \frac{i}{2} \alpha^l_m \xi_{m-1} + \frac{i}{2} \alpha^l_{m+1} \xi_{m+1}, \\
\tilde{A}_2 &= -\frac{1}{2} \alpha^l_m \xi_{m-1} + \frac{1}{2} \alpha^l_{m+1} \xi_{m+1}, \\
\tilde{A}_3 &= im \xi_m, \\
\tilde{B}_1 &= -\frac{1}{2} \alpha^l_m \xi_{m-1} - \frac{1}{2} \alpha^l_{m+1} \xi_{m+1}, \\
\tilde{B}_2 &= -\frac{i}{2} \alpha^l_m \xi_{m-1} + \frac{i}{2} \alpha^l_{m+1} \xi_{m+1}, \\
\tilde{B}_3 &= -m \xi_m,
\end{align}

where

$$\alpha^l_m = \sqrt{(l + m)(l - m + 1)}.$$
2.3 Gel’fand-Naimark basis

There exists another representation basis for the Lorentz group $H_3\xi_{lm} = m\xi_{lm}$, $H_+\xi_{lm} = \sqrt{(l+m+1)(l-m)}\xi_{l,m+1}$, $H_-\xi_{lm} = \sqrt{(l+m)(l-m+1)}\xi_{l,m-1}$, $F_3\xi_{lm} = C_l\sqrt{l^2-m^2}\xi_{l-1,m} - A_l m\xi_{l,m} - C_{l+1}\sqrt{(l+1)^2-m^2}\xi_{l+1,m}$, $F_+\xi_{lm} = C_l\sqrt{(l-m)(l+m+1)}\xi_{l-1,m+1} - A_l\sqrt{(l-m)(l+m+1)}\xi_{l,m+1} + C_{l+1}\sqrt{(l+m+1)(l+m+2)}\xi_{l+1,m+1}$, $F_-\xi_{lm} = -C_l\sqrt{(l+m)(l-m+1)}\xi_{l-1,m-1} - A_l\sqrt{(l+m)(l-m+1)}\xi_{l,m-1} - C_{l+1}\sqrt{(l-m+1)(l-m+2)}\xi_{l+1,m-1}$, $A_l = \frac{il_1}{l(l+1)}$, $C_l = \frac{i}{7}\sqrt{\frac{(l^2-l_0^2)(l^2-l_1^2)}{4l^2-1}}$, $m = -l, -l+1, \ldots, l-1, l$, $l = l_0, l_0+1, \ldots$, where $l_0$ is a positive integer or half-integer number, $l_1$ is an arbitrary complex number. These formulas define a finite-dimensional representation of the group $\mathfrak{g}_+$ when $l_1^2 = (l_0 + p)^2$, $p$ is some natural number. In the case $l_1^2 \neq (l_0 + p)^2$ we have an infinite-dimensional representation of $\mathfrak{g}_+$. The operators $H_3, H_+, H_-, F_3, F_+, F_-$ are $H_+ = iA_1 - A_2$, $H_- = iA_1 + A_2$, $H_3 = iA_3$, $F_+ = iB_1 - B_2$, $F_- = iB_1 + B_2$, $F_3 = iB_3$.

This basis was firstly given by Gel’fand in 1944 (see also [18, 14, 26]). The following relations between generators $Y_\pm, X_\pm, Y_3, X_3$ and $H_\pm, F_\pm, H_3, F_3$...
define a relationship between the helicity (Van der Waerden) and Gel’fand-
Naimark basises

\[ Y_+ = -\frac{1}{2}(F_+ + iH_+), \quad X_+ = \frac{1}{2}(F_+ - iH_+), \]
\[ Y_- = -\frac{1}{2}(F_- + iH_-), \quad X_- = \frac{1}{2}(F_- - iH_-), \]
\[ Y_3 = -\frac{1}{2}(F_3 + iH_3), \quad X_3 = \frac{1}{2}(F_3 - iH_3). \]

2.4 Complexification of SU(2)

As noted previously the explicit form of the isomorphism \( SL(2, \mathbb{C}) \cong SU(2) \otimes SU(2) \) can be obtained via the complexification of \( SU(2) \). Indeed, the group \( SL(2, \mathbb{C}) \) is a group of all complex matrices

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

of 2-nd order with the determinant \( \alpha\delta - \gamma\beta = 1 \). The group \( SU(2) \) is one of the real forms of \( SL(2, \mathbb{C}) \). The transition from \( SU(2) \) to \( SL(2, \mathbb{C}) \) is realized via the complexification of three real parameters \( \varphi, \theta, \psi \) (Euler angles). Let \( \theta^c = \theta - i\tau, \varphi^c = \varphi - i\epsilon, \psi^c = \psi - i\varepsilon \) be complex Euler angles, where

\[
0 \leq \text{Re} \theta^c = \theta \leq \pi, \quad -\infty < \text{Im} \theta^c = \tau < +\infty, \]
\[
0 \leq \text{Re} \varphi^c = \varphi < 2\pi, \quad -\infty < \text{Im} \varphi^c = \epsilon < +\infty, \]
\[
-2\pi \leq \text{Re} \psi^c = \psi < 2\pi, \quad -\infty < \text{Im} \psi^c = \varepsilon < +\infty. \]

Replacing in \( SU(2) \) the angles \( \varphi, \theta, \psi \) by the complex angles \( \varphi^c, \theta^c, \psi^c \) we come to the following matrix

\[
g = \begin{pmatrix}
\cos \frac{\theta^c}{2} e^{i\varphi^c - \psi^c} & i\sin \frac{\theta^c}{2} e^{i(\varphi^c + \psi^c)} \\
i\sin \frac{\theta^c}{2} e^{i(\varphi^c - \psi^c)} & \cos \frac{\theta^c}{2} e^{-i(\varphi^c + \psi^c)}
\end{pmatrix} =
\begin{pmatrix}
\cos \frac{\theta}{2} \cosh \frac{\tau}{2} + i \frac{\sin \theta}{2} \sinh \frac{\tau}{2} & e^{-\frac{i}{2}(\psi^c + \varphi^c)} \\
\cos \frac{\theta}{2} \sinh \frac{\tau}{2} + i \frac{\sin \theta}{2} \cosh \frac{\tau}{2} & e^{\frac{i}{2}(\psi^c + \varphi^c)}
\end{pmatrix}
\begin{pmatrix}
\cos \frac{\theta}{2} \sinh \frac{\tau}{2} + i \frac{\sin \theta}{2} \cosh \frac{\tau}{2} & e^{-\frac{i}{2}(\psi^c - \varphi^c)} \\
\cos \frac{\theta}{2} \cosh \frac{\tau}{2} + i \frac{\sin \theta}{2} \sinh \frac{\tau}{2} & e^{\frac{i}{2}(\psi^c - \varphi^c)}
\end{pmatrix},
\]

(20)
since \[ \cos \frac{1}{2}(\theta - i\tau) = \cos \frac{\theta}{2} \cosh \frac{\tau}{2} + i \sin \frac{\theta}{2} \sinh \frac{\tau}{2}, \] and \[ \sin \frac{1}{2}(\theta - i\tau) = \sin \frac{\theta}{2} \cosh \frac{\tau}{2} - i \cos \frac{\theta}{2} \sinh \frac{\tau}{2}. \] It is easy to verify that the matrix (20) coincides with a matrix of the fundamental representation of the group \( SL(2, \mathbb{C}) \) (in Euler parametrization):

\[
g(\varphi, \epsilon, \theta, \tau, \psi, \varepsilon) =
\begin{pmatrix}
e^{\frac{i\varphi}{2}} & 0 & e^{\frac{i\psi}{2}} & 0 \\
0 & e^{-\frac{i\varphi}{2}} & 0 & e^{-\frac{i\psi}{2}} \\
e^{\frac{\theta}{2}} & i \sin \frac{\theta}{2} & \cosh \frac{\tau}{2} & \sinh \frac{\tau}{2} \\
i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & \sinh \frac{\tau}{2} & \cosh \frac{\tau}{2}
\end{pmatrix}
\begin{pmatrix}
e^{\frac{i\varphi}{2}} & 0 & e^{\frac{i\psi}{2}} & 0 \\
0 & e^{-\frac{i\varphi}{2}} & 0 & e^{-\frac{i\psi}{2}} \\
e^{\frac{\theta}{2}} & i \sin \frac{\theta}{2} & \cosh \frac{\tau}{2} & \sinh \frac{\tau}{2} \\
i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & \sinh \frac{\tau}{2} & \cosh \frac{\tau}{2}
\end{pmatrix}
\begin{pmatrix}
e^{\frac{i\varphi}{2}} & 0 & e^{\frac{i\psi}{2}} & 0 \\
0 & e^{-\frac{i\varphi}{2}} & 0 & e^{-\frac{i\psi}{2}} \\
e^{\frac{\theta}{2}} & i \sin \frac{\theta}{2} & \cosh \frac{\tau}{2} & \sinh \frac{\tau}{2} \\
i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & \sinh \frac{\tau}{2} & \cosh \frac{\tau}{2}
\end{pmatrix}.
\]

Moreover, the complexification of \( SU(2) \) gives us the most simple and direct way for calculation of matrix elements of the Lorentz group. It is known that these elements have a great importance in quantum field theory and widely used at the study of relativistic amplitudes. The most degenerate representation of such the elements was firstly obtained by Dolginov, Toptygin and Moskalev \[4, 10, 11\] via an analytic continuation of representations of the group \( O_4 \). Later on, matrix elements are studied on the hyperboloid \[48, 46\], on the direct product of the hyperboloid and sphere \[22, 21\]. The matrix elements of the principal series are studied by Ström \[40, 41\] in the Gel’fand-Naimark basis. However, all the matrix elements, calculated in the GN-basis, have a very complicated form.
Matrix elements of $SL(2, \mathbb{C})$ in the helicity basis (see [44]) are

$$M_{mn}^{l}(g) = e^{-m(\epsilon + i\varphi) - n(\epsilon + i\psi)} Z_{mn}^{l} = e^{-m(\epsilon + i\varphi) - n(\epsilon + i\psi)} \times$$

$$\sum_{k=-l}^{l} i^{m-k} \sqrt{\Gamma(l - m + 1) \Gamma(l + m + 1) \Gamma(l - k + 1) \Gamma(l + k + 1)} \times$$

$$\cos^{2l} \frac{\theta}{2} \tan^{m-k} \frac{\theta}{2} \times$$

$$\min(l-m,l+k) \sum_{j=\max(0,k-m)}^{\min(l+m,l+k)} i^{2j} \tan^{2j} \frac{\theta}{2} \times$$

$$\Gamma(j + 1) \Gamma(l - m - j + 1) \Gamma(l + k - j + 1) \Gamma(m - k + j + 1) \times$$

$$\sqrt{\Gamma(l - n + 1) \Gamma(l + n + 1) \Gamma(l - k + 1) \Gamma(l + k + 1)} \cosh^{2l} \frac{\tau}{2} \tanh^{n-k} \frac{\tau}{2} \times$$

$$\min(l-n,l+k) \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \tanh^{2s} \frac{\tau}{2} \times$$

$$\Gamma(s + 1) \Gamma(l - n - s + 1) \Gamma(l + k - s + 1) \Gamma(n - k + s + 1).$$

(21)

We will call the functions $Z_{mn}^{l}$ in (21) as ***hyperspherical functions***. The hyperspherical functions $Z_{mn}^{l}$ can be written via the hypergeometric series:

$$Z_{mn}^{l} = \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \sum_{k=-l}^{l} i^{m-k} \tan^{m-k} \frac{\theta}{2} \tanh^{n-k} \frac{\tau}{2} \times$$

$$2F_{1} \left( \begin{array}{c} m - l + 1, l - k \\ m - k + 1 \end{array} \right) i^{2j} \tan^{2j} \frac{\theta}{2} \times$$

$$2F_{1} \left( \begin{array}{c} n - l + 1, l - k \\ n - k + 1 \end{array} \right) \tanh^{2} \frac{\tau}{2}.\right)$$

(22)

Therefore, relativistic spherical functions can be expressed by means of the function (a generalized hyperspherical function)

$$M_{mn}^{l}(g) = e^{-m(\epsilon + i\varphi)} Z_{mn}^{l} e^{-n(\epsilon + i\psi)},$$

(23)

where

$$Z_{mn}^{l} = \sum_{k=-l}^{l} \tilde{P}_{mk}^{l}(\cos \theta) \tilde{M}_{kn}^{l}(\cosh \tau),$$

(24)
here $P_{mn}^l(\cos \theta)$ is a generalized spherical function on the group SU(2) (see [13]), and $\mathcal{P}_{mn}^l$ is an analog of the generalized spherical function for the group QU(2) (so-called Jacobi function [49]). QU(2) is a group of quasiunitary unimodular matrices of second order. As well as the group SU(2), the group QU(2) is one of the real forms of SL(2, C) (QU(2) is noncompact). Further, from (22) we see that the function $Z_{mn}^l$ depends on two variables $\theta$ and $\tau$. Therefore, using Bateman factorization we can express the hyperspherical functions $Z_{mn}^l$ via Appell functions $F_1–F_4$ (hypergeometric series of two variables [4, 6]).

3 Gel’fand-Yaglom equations in the helicity basis

As a direct consequence of the isomorphism $SL(2, \mathbb{C}) \simeq SU(2) \otimes SU(2)$ we have equations

$$\sum_{i=1}^{3} \Lambda_i \frac{\partial \psi}{\partial x_i} - i \sum_{i=1}^{3} \Lambda_i \frac{\partial \psi}{\partial x_i^*} + \kappa c \psi = 0,$$

$$\sum_{i=1}^{3} \Lambda_i^* \frac{\partial \dot{\psi}}{\partial \tilde{x}_i} + i \sum_{i=1}^{3} \Lambda_i^* \frac{\partial \dot{\psi}}{\partial \tilde{x}_i^*} + \kappa c \dot{\psi} = 0,$$

with invariance conditions

$$\sum_k g_{ik} T_{\theta} \Lambda_k T_{\theta}^{-1} = \Lambda_i,$$

$$\sum_k g_{ik}^* \Lambda_k^* T_{\theta}^{-1} = \Lambda_i^*.$$  (26)

The equations (25) act in a 3-dimensional complex space $\mathbb{C}^3$. In its turn, the space $\mathbb{C}^3$ is isometric to a 6-dimensional bivector space $\mathbb{R}^6$ (parameter space or group manifold of the Lorentz group [19, 30]).

Let us find commutation relations between the matrices $\Lambda_i, \Lambda_i^*$ and infinitesimal operators (13), (17), (18), (19). First of all, let us present transformations $T_{\theta}$ ($\tilde{T}_{\theta}$) in the infinitesimal form, $1 + A_i \xi + \ldots, 1 + B_i \xi + \ldots, 1 + \tilde{A}_i \xi + \ldots, 1 + \tilde{B}_i + \ldots$. Substituting these transformations into invariance
conditions (26) we obtain with an accuracy of the terms of second order the following commutation relations

\[
\begin{align*}
[A_1, \Lambda_1] &= 0, & [A_1, \Lambda_2] &= \Lambda_3, & [A_1, \Lambda_3] &= -\Lambda_2, \\
[A_2, \Lambda_1] &= -\Lambda_3, & [A_2, \Lambda_2] &= 0, & [A_2, \Lambda_3] &= \Lambda_1,
\end{align*}
\]

(27)

\[
\begin{align*}
[B_1, \Lambda_1] &= 0, & [B_1, \Lambda_2] &= i\Lambda_3, & [B_1, \Lambda_3] &= -i\Lambda_2, \\
[B_2, \Lambda_1] &= -i\Lambda_3, & [B_2, \Lambda_2] &= 0, & [B_2, \Lambda_3] &= i\Lambda_1,
\end{align*}
\]

(28)

\[
\begin{align*}
[\tilde{A}_1, \Lambda_1^*] &= 0, & [\tilde{A}_1, \Lambda_2^*] &= -\Lambda_3^*, & [\tilde{A}_1, \Lambda_3^*] &= \Lambda_2^*, \\
[\tilde{A}_2, \Lambda_1^*] &= \Lambda_3^*, & [\tilde{A}_2, \Lambda_2^*] &= 0, & [\tilde{A}_2, \Lambda_3^*] &= -\Lambda_1^*, \\
[\tilde{A}_3, \Lambda_1^*] &= -\Lambda_2^*, & [\tilde{A}_3, \Lambda_2^*] &= \Lambda_1^*, & [\tilde{A}_3, \Lambda_3^*] &= 0.
\end{align*}
\]

(29)

\[
\begin{align*}
[\tilde{B}_1, \Lambda_1^*] &= 0, & [\tilde{B}_1, \Lambda_2^*] &= +i\Lambda_3^*, & [\tilde{B}_1, \Lambda_3^*] &= -i\Lambda_2^*, \\
[\tilde{B}_2, \Lambda_1^*] &= -i\Lambda_3^*, & [\tilde{B}_2, \Lambda_2^*] &= 0, & [\tilde{B}_2, \Lambda_3^*] &= +i\Lambda_1^*, \\
[\tilde{B}_3, \Lambda_1^*] &= +i\Lambda_2^*, & [\tilde{B}_3, \Lambda_2^*] &= -i\Lambda_1^*, & [\tilde{B}_3, \Lambda_3^*] &= 0.
\end{align*}
\]

(30)

Further, using the latter relations and taking into account (2) it is easy to establish commutation relations between \(\Lambda_3\) and generators \(Y_\pm, Y_3, X_\pm, X_3\):

\[
\begin{align*}
[Y_+, [\Lambda_3, Y_-]] &= 2\Lambda_3, \\
[\Lambda_3, Y_3] &= 0, \\
[\Lambda_3, X_-] &= 0, \\
[\Lambda_3, X_+] &= 0, \\
[\Lambda_3, X_3] &= 0,
\end{align*}
\]

(31)
\[
\begin{align*}
[X_+, [\Lambda_3^*, X_-]] &= 2\Lambda_3^*, \\
[\Lambda_3^*, X_3] &= 0, \\
[\Lambda_3^*, Y_-] &= 0, \\
[\Lambda_3^*, Y_+] &= 0, \\
[\Lambda_3^*, Y_3] &= 0.
\end{align*}
\]

(32)

From (31) and (32) it immediately follows that elements of the matrices \(\Lambda_3^*\) and \(\Lambda_3^*\) are

\[
\begin{align*}
\Lambda_3^* : \\
&\begin{cases}
  c_{l-1,l,m}^{k'k} = c_{l-1,l}^{k'k}\sqrt{l^2 - m^2}, \\
c_{l,l,m}^{k'k} = c_{l}^{k'k}m, \\
c_{l+1,l,m}^{k'k} = c_{l+1,l}^{k'k}\sqrt{(l+1)^2 - m^2}.
\end{cases}
\end{align*}
\]

(33)

All other elements are equal to zero. If we know the elements of \(\Lambda_3\) and \(\Lambda_3^*\), then the elements of \(\Lambda_1, \Lambda_2\) and \(\Lambda_1^*, \Lambda_2^*\) can be obtained via the relations (27)–(30).

With a view to separate the variables in (25) let us assume

\[
\psi_{lm}^k = f_{lmk}(r)M_{mn}^l(\varphi, \epsilon, \theta, \tau, 0, 0),
\]

(35)

where \(l_0 \geq l\), and \(-l_0 \leq m, n \leq l_0\). The full separation of variables in (25) has been given in the recent paper [45], where the hyperspherical functions \(M_{mn}^l\) are defined on a 2-dimensional complex sphere (about the 2-dimensional complex sphere see [37]). In the result of the separation we come to a system of ordinary differential equations which depends on the
radial functions,

\[ \sum_{k'} c_{i,l-1}^{k,k'} \left[ 2\sqrt{l^2 - m^2} \frac{d f_{l-1,m,k'}^0(r)}{dr} + \right. 
\]
\[ - \frac{1}{r} (l + 1) \sqrt{l^2 - m^2} f_{l-1,m,k'}^0(r) + 
\]
\[ + \frac{i}{r} \sqrt{(l + m)(l + m - 1)} \sqrt{(l_0 + \hat{m})(l_0 - \hat{m} + 1)} f_{l-1,m-1,k'}^0(r) + 
\]
\[ + \frac{i}{r} \sqrt{(l - m)(l - m - 1)} \sqrt{(l_0 + \hat{m} + 1)(l_0 - \hat{m})} f_{l-1,m+1,k'}^0(r) \] 

\[ \sum_{k'} c_{i,l}^{k,k'} \left[ 2m \frac{d f_{l,m,k'}^0(r)}{dr} - \frac{1}{r} m f_{l,m,k'}^0(r) - 
\]
\[ - \frac{i}{r} \sqrt{(l + m)(l - m + 1)} \sqrt{(l_0 + \hat{m})(l_0 - \hat{m} + 1)} f_{l,m-1,k'}^0(r) + 
\]
\[ + \frac{i}{r} \sqrt{(l + m + 1)(l - m)} \sqrt{(l_0 + \hat{m} + 1)(l_0 - \hat{m})} f_{l,m+1,k'}^0(r) \] 

\[ \sum_{k'} c_{i,l+1}^{k,k'} \left[ 2\sqrt{(l + 1)^2 - m^2} \frac{d f_{l+1,m,k'}^0(r)}{dr} - 
\]
\[ + \frac{1}{r} l \sqrt{(l + 1)^2 - m^2} f_{l+1,m,k'}^0(r) - 
\]
\[ - \frac{i}{r} \sqrt{(l - m + 1)(l - m + 2)} \sqrt{(l_0 + \hat{m})(l_0 - \hat{m} + 1)} f_{l+1,m-1,k'}^0(r) - 
\]
\[ - \frac{i}{r} \sqrt{(l + m + 1)(l + m + 2)} \sqrt{(l_0 + \hat{m} + 1)(l_0 - \hat{m})} f_{l+1,m+1,k'}^0(r) + 
\]
\[ + \kappa^c f_{lmk}^0(r) = 0, \]
\[
\sum_{k'} c_{i,l-1}^{kk'} \left[ 2\sqrt{l^2 - \hat{m}^2} \frac{df^{i0}_{l-1,m,k'}(r)}{dr^*} - \frac{1}{r^*}(\hat{l} + 1)\sqrt{l^2 - \hat{m}^2} f^{i0}_{l-1,m,k'}(r) + \frac{i}{r^*} \sqrt{(l + \hat{m})(l + \hat{m} - 1)}(l_0 + m)(l_0 - m + 1) f^{i0}_{l-1,m-1,k'}(r) + \frac{i}{r^*} \sqrt{(l - \hat{m})(l - \hat{m} - 1)}(l_0 + m + 1)(l_0 - m) f^{i0}_{l-1,m+1,k'}(r) \right] + \\
\sum_{k'} c_{i,l}^{kk'} \left[ 2\hat{m} \frac{df^{i0}_{l,m,k'}(r)}{dr^*} - \frac{1}{r^*} \hat{m} f^{i0}_{l,m,k'}(r) - \frac{i}{r^*} \sqrt{(l + \hat{m})(l - \hat{m} - 1)}(l_0 + m)(l_0 - m + 1) f^{i0}_{l,m-1,k'}(r) + \frac{i}{r^*} \sqrt{(l + \hat{m} + 1)(l - \hat{m})}(l_0 + m + 1)(l_0 - m) f^{i0}_{l,m+1,k'}(r) \right] + \\
\sum_{k'} c_{i,l+1}^{kk'} \left[ 2\sqrt{(\hat{l} + 1)^2 - \hat{m}^2} \frac{df^{i0}_{l+1,m,k'}(r)}{dr^*} + \frac{1}{r^*} i \sqrt{(\hat{l} + 1)^2 - \hat{m}^2} f^{i0}_{l+1,m,k'}(r) - \frac{i}{r^*} \sqrt{(l - \hat{m} + 1)(l - \hat{m} + 2)}(l_0 + m)(l_0 - m + 1) f^{i0}_{l+1,m-1,k'}(r) - \frac{i}{r^*} \sqrt{(l + \hat{m} + 1)(l + \hat{m} + 2)}(l_0 + m + 1)(l_0 - m) f^{i0}_{l+1,m+1,k'}(r) \right] + \kappa^c f^{i0}_{l,m,k'}(r) = 0. \tag{36}
\]

This system is solvable for any spin in a class of the Bessel functions. Therefore, according to (35) relativistic wave functions are expressed via the products of cylindrical and hyperspherical functions. For that reason the wave function hardly depends on the parameters of the Lorentz group. Moreover, it allows to consider solutions of RWE as the functions on the Lorentz group. On the other hand, a group theoretical description of RWE allows
to present all the physical fields on an equal footing. Namely, all these fields are the functions on the Lorentz (Poincaré) group. Such a description corresponds to a quantum field theory on the Poincaré group introduced by Lurçat [23] (see also [3, 21, 12, 16] and references therein).

3.1 Decomposable and indecomposable RWE

In accordance with equivalence (12) and decompositions of (13) and (8) the matrix $\Lambda_3$ can be written in the form

$$\Lambda_3 = \begin{pmatrix} C^1 & 0 & \cdots & 0 \\ C^2 & & \cdots & \\ \vdots & & & \cdots \\ 0 & \cdots & & C^s \end{pmatrix}.$$ (37)

The matrix $\Lambda_3^*$ has the same form. $C^s$ is a spin block. The each spin block $C^s$ is realized in the space $\text{Sym}_s$. The matrix elements of the corresponded representations are expressed via the hyperspherical functions $s$. If the spin block $C^s$ has non–null roots, then the particle possesses the spin $s$ [14, 3, 31]. The spin block $C^s$ in (37) consists of the elements $c^{s}_{\tau_1,\tau_2}$, where $\tau_{l_1,l_2}$ and $\tau'_{l'_1,l'_2}$ are interlocking irreducible representations of the Lorentz group, that is, such representations, for which $l'_1 = l_1 \pm \frac{1}{2}$, $l'_2 = l_2 \pm \frac{1}{2}$. At this point the block $C^s$ contains only the elements $c^{s}_{\tau_1,\tau_2}$ corresponding to such interlocking representations $\tau_{l_1,l_2}$, $\tau'_{l'_1,l'_2}$ which satisfy the conditions

$$|l_1 - l_2| \leq s \leq l_1 + l_2, \quad |l'_1 - l'_2| \leq s \leq l'_1 + l'_2.$$ 

According to a de Broglie theory of fusion [8, 38] interlocking representations give rise to indecomposable RWE. Otherwise, we have decomposable equations. As known, the indecomposable RWE correspond to composite particles. A wide variety of such representations and RWE it seems to be sufficient for description of all particle world.

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