We report a global effect induced by the local complex field, associated with the spin-exchange interaction. High-order exceptional point up to \((N + 1)\)-level coalescence is created at the critical local complex field applied to the \(N\)-size quantum spin chain. The \((N + 1)\)-order coalescent level is a saturated ferromagnetic ground state in the isotropic spin system. Remarkably, the final state always approaches the ground state for an arbitrary initial state with any number of spin flips; even if the initial state is orthogonal to the ground state. Furthermore, the switch of macroscopic magnetization is solely driven by the time and forms a hysteresis loop in the time domain. The retentivity and coercivity of the hysteresis loop mainly rely on the non-Hermiticity. Our findings highlight the cooperation of non-Hermiticity and the interaction in quantum spin system, suggest a dynamical framework to realize magnetization, and thus pave the way for the non-Hermitian quantum spin system.

I. INTRODUCTION

Non-Hermitian quantum mechanics is an extension of standard quantum mechanics and describes dissipative systems in a minimalistic fashion. The research field of non-Hermitian physics has been greatly developed in the optical platforms, in particular, the peculiar features of the exceptional point (EP), which is the non-Hermitian phase transition point that solely presents in non-Hermitian system. The EP plays the pivotal role in the intriguing dynamics and application including asymmetric mode switching, unidirectional lasing, and enhanced optical sensing. Notably, the properties of EP highly depend on the level coalescence and its topology. Recently, the inspiring insights of non-Hermitian physics emerge rapidly in the condensed-matter systems. The non-Hermitian quantum spin models and the exotic quantum many-body effect ranging from non-Hermitian extensions of Kondo effect, Fermi surface in coordinate space, Kibble Zurek mechanism, many-body localization, to fermionic superfluidity are reported. These findings unveil the interesting and important impacts of the non-Hermiticity in the interacting systems.

In this paper, we uncover the influence of complex magnetic field in the quantum spin system. Remarkably, we find that a local critical complex field can induce the coalescence of substantial energy levels: the degenerate states with different symmetry of the Kibble quantum spin system coalesce at the critical complex field and form a high-order EP; the order of coalescence is solely determined by the degeneracy. The ground state associated with a saturated ferromagnetic ground state has the highest order of coalescence and thus enables the dynamic magnetization. For an initial state with any number of spin flips excited on the ground state, the final state always approaches the ground state. Furthermore, a hysteresis loop is formed in the time domain and it is driven by the time rather than the magnetic field in contrast to the traditional magnetism. The properties of the non-Hermitian quantum spin system are capable of been examined from the retentivity and coercivity.

The rest of this paper is organized as follows: In Sec. II, we investigate the non-Hermitian quantum spin model, the non-Hermiticity of which stems from the complex magnetic field and propose a general method to connect the non-Hermitian model to the Hermitian spin model. With these preparations, in Sec. III, we demonstrate that a local complex field can induce a global effect with the aid of the spin-exchange interaction. The formation of a high-order EP is therefore observed. Based on the performance of the dynamics of high-order EP, the dynamical generation of saturated ferromagnetic state and a hysteresis loop in the time domain are proposed in Sec. IV and Sec. V. Sec. VI concludes this paper. Some details of our calculation are placed in Appendix.

II. NON-HERMITIAN QUANTUM SPIN SYSTEM

In the quantum spin system, either a real or a complex field results in the splitting of the degenerate ground states, where the spins are aligned along the direction of the external field. However, the spectrum and the eigenstate of the system with a real spectrum do not experience dramatic change in the present of the external field; and the initial state exhibits a periodic oscillating behavior among all the possible spin orientations. However, the situation changes when a critical complex field is applied. The eigenstates coalesce and the dynamics encounter dramatic changes in the sense that all the initial state evolve to the coalescent state regardless of the initial spin orientation. It is interesting to find out the intriguing features of the quantum spin system in the presence of the complex field.

We consider a non-Hermitian spin system \(H = H_0 + H_1\) and show the unique properties determined by the competition between the non-Hermiticity and the interaction.
The quantum spin system
\[
H_0 = -\sum_{i,j \neq i} (J_{ij}/2) (s_i^+ s_j^- + s_i^- s_j^+) + \sum_{i,j \neq i} \Delta_{ij} s_i^x s_j^x, \quad (1)
\]
is subjected to an external complex field
\[
H_I = \sum_i g_i \mathbf{h} \cdot \mathbf{s}_i. \quad (2)
\]
The operators \(s_i^\pm = s_i^x \pm is_i^y\) and \(s_i^z\) are for the spin-1/2 at the \(i\)-th site, obeying Lie algebra \([s_i^x, s_j^z] = \pm s_i^z \delta_{ij}\) and \([s_i^z, s_j^z] = 2s_i^z \delta_{ij}\), where \(\delta_{ij}\) is the Dirac delta function. \(\sum_{i,j \neq i}\) means the summation over all the possible pair interactions at an arbitrary range. \(J_{ij}\) represents the inhomogeneous spin-spin interaction and \(\Delta_{ij}\) characterizes the anisotropy of the spin system \(H_0\). The non-Hermiticity of \(H_I\) originates from the complex magnetic field \(\mathbf{h}\) = \((1, -i\gamma, 0)\), which can be understood as the spin-dependent losses and are within the reach of ultracold atom experiments. The strength felt by each spin is \(g_i \mathbf{h}\) in the inhomogeneous complex magnetic field. The system \(H_0\) respects the time-reversal symmetry \(T\) ((i) \(Ts_i^z T^{-1} = -s_i^z\)), which leads to the Kramers degeneracy when the system possesses a half-integer total spin: the degeneracy breaks down when the external complex field presents (\(7T \mathbf{s} \cdot \mathbf{T}^{-1} = -\mathbf{s}^* \cdot \mathbf{s}\)). The external field also spoils the commutation relation \([\sum_j s_j^z, H] = 0\).

We first show that a local complex field dramatically changes the ground state property of a quantum spin system. The Hilbert space of the non-Hermitian system \(H\) cannot be decomposed into subspaces in which the spin number is specified even if \(H_0\) has homogeneous spin-spin interaction \(J_{ij} = J\), \(\Delta_{ij} = \Delta\), and \(\Delta \neq J\) (the XXZ model). Considering a local complex field \(H_I = g_N \mathbf{h} \cdot \mathbf{s}_N[\text{Fig. 1(a)}]\), \(H_0\) and \(H_I\) share two eigenstates even though \([H_0, H_I] \neq 0\). The pair of eigenstates
\[
|\psi\rangle_{xxz, \pm} = \pm \sqrt{1 - |\gamma|^2} |\psi\rangle + \sqrt{1 + |\gamma|^2} |\bar{\psi}\rangle, \quad (3)
\]
satisfy \(H|\psi\rangle_{xxz, \pm} = (-N\Delta/4 \pm \sqrt{1 - |\gamma|^2})|\psi\rangle_{xxz, \pm}\). Notably, \(|\psi\rangle_{xxz, \pm}\) coalesce at \(|\gamma| = 1\) (see Supplemental Material A). This indicates that the ground state is dramatically changed from degeneracy to coalescence by the local complex field.

Under a homogeneous global complex field [Fig. 1(b)], the Hamiltonian of the free spins in the absence of the interaction describes a \(\mathcal{PT}\)-symmetric hypercube graph of \(N\) dimension and the system can be projected onto several invariant subspaces denoted by \(s = (s = N/2, N/2 - 1, \ldots)\). \(\gamma = 1\) is the \(\mathcal{PT}\) symmetry limit of \(n = 2s + 1\) of \(n\)-eigenstate coalescence in each subspace. If the complex field is inhomogeneous, for example, the critical complex field is locally applied to only a single spin, \(\gamma = 1\) reduces to an EP2 of two-state coalescence. To gain more insights for the interacting spins under the homogeneous global complex field, we invite the exactly solvable non-Hermitian Ising model to show that all the energy levels and the eigenstates can be significantly affected by the complex field. To proceed, we introduce a similarity transformation \(S = \prod_j S_j\), where \(S_j = e^{-i\theta j}s_j^z\) represents a counter-clockwise spin rotation in the \(s_z\)-axis by an angle \(\theta\). Here \(\theta = \tan^{-1}(i\gamma)\) is a complex number depending on the strength of the complex field. Notably, the spin-rotation \(S_j\) is valid at arbitrary \(\gamma\) unless at the EP of \(H_I\), where \(\mathbf{h} \cdot \mathbf{s}_i\) is in a nondiagonalizable Jordan block form. Under the spin-rotation, \(H\) is transformed to \(\tilde{H} = H_0 (s \rightarrow \tau) + \sqrt{1 - \gamma^2} \sum_i g_i \tau_i^z\), where the new set of operators \(\tau_i^j = S_j s_i^j S_j^{-1}\) and \(\tau_i^z = S_j s_i^z S_j^{-1}\) also satisfies the Lie algebra, that is, \([\tau_i^+, \tau_i^-] = \pm \tau_i^z \delta_{ij}\) and \([\tau_i^+, \tau_i^-] = 2\tau_i^z \delta_{ij}\). Notice that \(\tau_i^\pm \neq (\tau_i^\mp)^\dagger\) due to the complex rotation angle \(\theta\). We set \(|\psi_n\rangle\) as the eigenstates of the operator \(\sum_i s_i^z\) that represents the complex rotation angle \(\theta\). The matrix form of \(\tilde{H}\) is Hermitian for \(|\gamma| < 1\). This directly leads to an entire real spectrum of \(\tilde{H}\). It is worth pointing out that the transformation depends on \(\gamma\) only and hence the spectrum is entirely real even though a non-zero \(g_i\) presents. This indicates that the presence of the local complex field breaks the SU(2) symmetry of the system but remains the entire real spectrum without symmetry protection. In general case, the EP of \(H\) or \(\tilde{H}\) may not be the EP of \(H_I\) at \(|\gamma| = 1\); however, we prove that \(|\gamma| = 1\) is the EP of the non-Hermitian Ising model \(H\) by taking \(J_{ij} = 0, \Delta_{ij} = 1\) and \(g_i = g\) (see Supplemental Material B) and all the eigenstates coalesce. In particular, there are two types of phase transitions in the non-Hermitian Ising model, the \(\mathcal{PT}\) symmetry breaking at \(\gamma = 1\) and the spontaneous symmetry breaking at \(g\sqrt{1 - \gamma^2} = 1\), are both modulated by the transverse complex field \(g\). Therefore, the homogeneous global complex field can induce the phase transitions of the non-Hermitian spin systems.

The interplay between the complex field and the spin-
spin interaction brings intriguing change to the system properties. In the aforementioned cases, the influences of the local and global complex fields are discussed, respectively. It is counter-intuitive that the local field associated with the inhomogeneous interaction can generate the effect induced by the homogeneous global complex field; for example, the strongly coalesced high-order EP\((N + 1)\) for an \(N\)-spin system can present in the quantum spin systems under a local complex field.

### III. HIGH-ORDER EP UNDER LOCAL COMPLEX FIELD

In the XXZ model \((\Delta > J)\), the ground states are two degenerate ferromagnetic states with all the spins aligned in the \(+z\) and \(−z\) directions; and two degenerate ground states coalesce at the critical complex field as an EP2. The underlying mechanism of cooperation between the local complex field and spin-spin interaction is elaborated as follows. For the ground state of the quantum spin system \(\left\{ H_0 \right\}\), the spin-spin interaction drives all the spins to behave like one spin. If a spin is subjected to a complex field, all the spins feel the same complex field because of the spin-correlation. Thus, all the degenerate ground states of the quantum spin system coalesce at the critical complex field. This enlightens us to propose the high-order EP. For a ferromagnetic Heisenberg chain of \(N\) spins, the ground states are \((N + 1)\)-fold degenerate with the angular momentum \(N/2\) such that the projection of the spin has \(N + 1\) values in an arbitrary direction. The response of the Hermitian system to the external field can be ascribed to the performance of such an angular momentum formed by \(N\) non-interacting spins under the global complex field. Therefore, the critical complex field turns all the possible spin orientations to the \(+z\) direction and an EP\((N + 1)\) is formed [Fig. 3(c)].

We consider the local complex field \(g_j h\) applied to the spin \(s_j\) in an isotropic ferromagnetic Heisenberg model with \(\Delta_{jj} = J_{jj}\), which is also referred to as the XXX model. The spin-1/2 Heisenberg antiferromagnet often serves as an effective low-energy description of the half-filled Hubbard model with interaction. \(H_0\) is rotationally invariant since it commutes with all the three components of the total spin \(s = \sum_{j=1}^{N} s_j\). Thus, the eigenstates of \(H_0\) can be classified in terms of the total spin number \(s\).

For the ferromagnetic Heisenberg model \(H_0\), a saturated ferromagnetic state, denoted as \(|\uparrow\rangle\), is the members of the ground state multiple. All the other degenerate ground states can be obtained by acting \(s^+_j\) on \(|\uparrow\rangle\) step by step; the ground state is \((N + 1)\)-folder degenerate.

The Hermitian \(H_0\) commutes with the non-Hermitian \(H_1\) for the homogeneous global critical complex field; however, the local complex field is a nontrivial case since \(H_1\) does not commute with \(H_0\). The local complex field breaks not only the spin-rotation symmetry, but also the time-reversal symmetry; thus, all the eigenstates including the ground state become non-degenerate. In principle, \(H_1\) and \(H_0\) do not share common eigenstates and we cannot infer the property of \(H\) from \(H_1\). However, an EP\((N + 1)\) emerges when \(|\gamma\rangle \rightarrow 1\). This is observed from the subspace spanned by \(\{|G_n\}\) that belongs to the subspace \(s = N/2\), where \(\{|G_n\}\) is given by \(G_n = (\sum_{j} s^-_j)^{n-1} |\uparrow\rangle\), \((n = 1, 2, \ldots N + 1)\). \(\{|G_n\}\) are the degenerate groundstates of \(H_0\) of \((N + 1)\)-folder degeneracy with all the spins aligned in the same direction. The condition of \(|\gamma\rangle \rightarrow 1\) guarantees the validity of the perturbation theory in the representation of \(H\) and yields the matrix form of \(H_1\) in the form of \(W_{m,n} = g_j \sqrt{(N + 1 - m)/(1 + \gamma)} \delta_{m,1,n} + (1 - \gamma) \delta_{m,n+1}/2N\) (see Supplemental Material C). It is a non-Hermitian hypercube with an EP\((N + 1)\) at \(|\gamma| = 1\). Similar as the exactly solvable non-Hermitian Ising model, \(|\gamma| = 1\) is also the EP of the non-Hermitian ferromagnetic Heisenberg model \(H\).

We plot the eigenenergies of \(H\) as functions of \(|\gamma|\) in Fig. 2 to show the high-order EP. \(O_n\) is introduced to quantify the similarity between the excited state and the ground state; \(O_n\) is defined as

\[
O_n = \langle \phi_1 (\gamma) | \phi_n (\gamma) \rangle / \langle \phi_1 (\gamma) | \phi_1 (\gamma) \rangle,
\]

where \(|\phi_n (\gamma)\rangle\) \((n = 1, \ldots N + 1)\) are the ground state and \(N\) excited states of \(H\). The overlaps \(O_n (|\gamma|) \rightarrow 1\) in the plots. A critical local magnetic field not only drives all the ground state coalescence, but also the excited states coalesce at different energies with multiple types of level coalescences and degeneracies, the order of level coalescence is determined by the degenerate levels of \(H_0\). Thus, the ground state has the highest order of \((N + 1)\)-level coalescence. Furthermore, Figs. 2(b) and 2(c) clearly show that the complicated spin structures induced by the interaction can harbour the high-order EPs regardless of the position of applied external complex field. This is a consequence that the ground states degeneracy is independent of the spin-configuration. This feature indicates our findings universally present.

### IV. DYNAMICAL GENERATION OF SATURATED FERROMAGNETIC STATE

The high-order EPs generated by the cooperation of the local complex field and the spin-spin interaction brings intriguing dynamics. The spectrum of the ferromagnetic Heisenberg model at the critical complex field is constituted by many coalesced levels; instead of being diagonalized, the system can be decomposed only into multiple Jordan blocks of various orders. In each subspace, an arbitrary initial state will evolve towards the coalescent state and its probability increases over time in power law according to the order of coalescence. The more levels coalesced to one, the higher order of the coalescence, and the faster probability increased in the dynamics. For an arbitrary initial state, the highest order of the coalescent state determines the final state for a long
We consider a homogeneous spin-spin interaction $J_{ij} = J$ in the ferromagnetic Heisenberg model. The coalesced ground state is $|\uparrow\rangle$ with all the spins aligned in the $+z$ direction. The initial state finally approaches the saturated ferromagnetic state because that the ground state has the highest order of coalescence at the critical complex field and its probability increases dominantly in the time-evolution process. For an arbitrary initial state $\sum_n c_n (0) |G_n \rangle$ within the subspace $s = N/2$, the coefficient $c_m (t)$ is

$$c_m (t) = c_m (0) + \sum_{n \neq m} (-it/N)^{n-m} (n-m)! h (n-m) \prod_{p=m}^{m-1} p(N+1-p)^{1/2} c_n (0). \quad (5)$$

where $h (n-m)$ is the Heaviside step function (see Supplemental Material D). It indicates that the coefficient $c_1 (t)$ of the evolved state always includes the highest power of time $t$. Thus, the component $c_1 (t)$ of the evolved state overwhelms the other components and the final state is the coalescent state $|\uparrow\rangle$. The fidelity $F (t) = |\langle \uparrow | e^{-iHt} |\downarrow\rangle / |\langle \uparrow | e^{-2\pi f t} |\downarrow\rangle|^2 = [1 + 1/\eta^2 (t)]^{-N}$ captures the full dynamics, where $\eta (t) = t/t_o$ and $t_o = N/g_1$. Obviously, $F (t \to \infty) = 1$ and any initial state with arbitrary spin flip evolves to $|\uparrow\rangle$ (see Supplemental Material D). This feature is important for the hysteresis loop in the time domain.

V. HYSTERESIS LOOP IN THE TIME DOMAIN

We consider a time-reversal process of two types of the system $s^z = 0$ initial states and observe their dynamics. The first type of initial state is a Neel state $|\Psi_I (0)\rangle = |\uparrow\downarrow\downarrow \cdots \uparrow\rangle$ and the second type of initial state $|\Psi_{II} (0)\rangle$ is the ground state of isotropic Heisenberg model with $s = N/2$. We inspect the time dependent average magnetization

$$M_{I(II)} (t) = N^{-1} \langle \Psi_{I(II)} (t) | \sigma^x | \Psi_{I(II)} (t) \rangle / |\Psi_{I(II)} (t)\rangle,$$  

where $|\Psi_{I(II)} (t)\rangle$ is the time-evolving state driven by the non-Hermitian Heisenberg Hamiltonian. The local complex field applied is in the form of $h = (1,i,0)$. We notice that any initial state finally evolves to the saturated ferromagnetic state after a specified relaxation time $t_f \gg t_o$. In Fig. 3, the dynamic magnetization is presented. The trajectory of the initial state that has never been previously magnetized follows the blue curve. After a relaxation time $t_f$, all the spins are aligned along the $+z$ direction; keep increasing $t$ will produce slight increase in $M(t)$. Then, we take the time-reversal action on $H$ and observe the inverse magnetization. The trajectories are obtained as (see Supplemental Material D)

$$M_\pm (t) = \pm [1 - \eta^2 (t \mp t_f)]/[1 + \eta^2 (t \mp t_f)], \quad (7)$$

for the magnetization $M_- (t)$ (in yellow) and the inverse magnetization $M_+ (t)$ (in red). The red curve corresponds to the inverse of the yellow curve. They form the hysteresis loop in the time domain and is independent of the initial state. The stark difference from the traditional hysteresis loop is that the switch of macroscopic magnetization is driven in the time domain rather than the external field. When $t$ is reduced to zero, some magnetic
flux remains in the evolved state even though the time is back to the origin. Similar to the hysteresis loop driven by the external field, a non-zero magnetization point can be dubbed as retentivity $M_1 = 1 - 2/(1 + t_o^2/t_f^2)$ (red solid circle) in Fig. 3(b) and indicates the remanence of residual magnetism in the evolved state. As time goes on, the yellow curve approaches zero and the corresponding relaxation time $t_c = t_f - t_o$. As $t_f \to \infty$, the hysteresis loop becomes a rectangle (Fig. 3(c)) of width 2 and length $2t_f$. The formation of the hysteresis loop in the non-Hermitian spin model is based on the time-reversal breaking induced by local complex field. The area enclosed in the hysteresis loop whose area is constant to the traditional hysteresis loop whose area is constant with the existence of a reversible magnetization phase\[12\] and the corresponding Hamiltonian can be obtained by taking $J_{ij} = J$ and $H_{ij} = J$.

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Appendix A: Non-Hermitian 1D XXZ model

We consider a 1D non-Hermitian XXZ model with nearest neighbour homogeneous spin-spin interaction, the corresponding Hamiltonian can be obtained by taking $J_{ij} = J$ and $H_{ij} = J$:

$$H_0 = - \sum_j \frac{J}{2} (s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+) + \Delta s_j^z s_{j+1}^z. \quad (A1)$$

We confine our discussion to $J > 0$ without loss of generality. The XXZ chain is in the ferromagnetic Ising phase when $\Delta > 0$ the ground state is the saturated state with all spins aligned in either the +z or −z direction, i.e., the classical ground state with magnetization $s_j^z = \pm N/2$, where $N$ is the number of sites. We denote the two degenerate ground states by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively; in this phase, the spin reflection symmetry $s_j^z \to -s_j^z$ of the XXZ model breaks. When the external magnetic field is switched on, the superposition of the lowest geometric multiplicity of one; for any initial state with an arbitrary number of spin flips, the final states always approaches the ground state. This discovery opens an avenue for magnetizing the non-Hermitian quantum spin system. A hysteresis loop is obtained in the time domain, the local complex field from the effective coupling between the system and the environment is associated with the irreversible thermodynamic change. This unique feature is insensitive to both the interaction range and the initial state.

VI. CONCLUSIONS

We have demonstrated that the local complex magnetic field can induce dramatic changes in the spectral and dynamics of the quantum spin systems. The form of interaction significantly matters in the cooperation between the non-Hermiticity and interaction. The inhomogeneous spin-exchange interaction assisted a local complex field can affect globally and homogeneously. Specifically, the ground state of the isotropic Heisenberg model subjected to a critical local magnetic field is a high-order of coalescent ferromagnetic state that has the
two ground states in the direction of the complex field $|\psi\rangle_{xxz,\pm}$ are the unnormalized eigenstates of $H_I$,

$$|\psi\rangle_{xxz,\pm} = \pm \sqrt{1-\gamma} |\eta\rangle + \sqrt{1+\gamma} |\psi\rangle. \quad (A2)$$

We can check that

$$(H_0 + H_I) |\psi\rangle_{xxz,\pm} = \left( -\frac{N\Delta}{4} \pm \sqrt{1-\gamma^2} \right) |\psi\rangle_{xxz,\pm}. \quad (A3)$$

Therefore the states $|\psi\rangle_{xxz,\pm}$ are the common eigenstates of both $H_0$ and $H_I$. It should be noted that when $|\gamma| = 1$, two such common states coalesce to either $|\eta\rangle$ or $|\psi\rangle$. This indicates that $|\gamma| = 1$ is the EP for $H_I$ and $H$.

**Appendix B: Non-Hermitian 1D Ising model**

The non-Hermitian 1D Ising model is

$$H = \sum_{j=1}^{N} s_{j}^x s_{j+1}^x + g \left( s_{j}^y + i\gamma s_{j}^y \right). \quad (B1)$$

The periodic boundary condition $s_{j+N}^x = s_{j+1}^x$ is assumed. Using the similar transformation, the Hamiltonian $[B1]$ can be transformed to

$$H = \sum_{j} \tau_j^x \tau_{j+1}^x + g \sqrt{1-\gamma^2} \tau_j^z, \quad (B2)$$

which is a standard Ising model with modulated transverse field $g\sqrt{1-\gamma^2}$. Note that such the transformation holds if and only if $|\gamma| \neq 1$.

To obtain $\overline{H}$, we first perform the Jordan-Wigner transformation

$$\tau_j^x = \frac{1}{2} - \overline{d}_j \overline{d}_j, \quad (B3)$$

$$\tau_j^y = \frac{i}{2} \sum_{j<l} (1-2\overline{d}_j \overline{d}_l) (\overline{d}_j - \overline{d}_l), \quad (B4)$$

$$\tau_j^z = -\frac{1}{2} \sum_{j<l} (1-2\overline{d}_j \overline{d}_l) (\overline{d}_j + \overline{d}_l), \quad (B5)$$

to replace the quasi spin operators by the new non-Hermitian operators $\overline{d}_j$ and $\overline{d}_j$, where $\overline{d}_j = S_j c_j S_j^{-1}$ ($d_j = S_j c_j S_j^{-1}$ and $c_j$ represents the creation (annihilation) operator of the spinless fermion. The new operators satisfy the fermionic anticommutation relation

$$[\overline{d}_j, d_{j'}]_+ = \delta_{jj'}. \quad (B6)$$

We note that the parity of the number of such fermions is a conservative quantity such that the Hamiltonian can be expressed as

$$\overline{H} = \begin{pmatrix} \overline{H}_+ & 0 \\ 0 & \overline{H}_- \end{pmatrix}, \quad (B7)$$

where

$$\overline{H}_+ = \overline{H}_- - 2 \overline{d}_N \overline{d}_1 + \overline{d}_N d_1 + \overline{d}_1 d_N + d_1 d_N, \quad (B8)$$

and

$$\overline{H}_- = \frac{1}{4} \sum_{j=1}^{N} \left[ 2g \sqrt{1-\gamma^2} (1-2\overline{d}_j d_j) + (\overline{d}_j \overline{d}_{j+1} + \overline{d}_{j+1} \overline{d}_j + d_{j+1} d_j + d_j d_{j+1}) \right]. \quad (B9)$$

are the corresponding reduced Hamiltonians in the invariant subspaces with even and odd parity. $\overline{H}_+$ represents a fermionic ring threaded by a half of the flux quantum. The single-particle energy in two subspaces can be obtained by the same procedures and will have the same value when the system approaches the thermodynamic limit. In the following, we only focus on the even parity subspace. Taking the Fourier transformation

$$d_j = \frac{1}{\sqrt{N}} \sum_{k} d_k e^{ikj}, \quad \overline{d}_j = \frac{1}{\sqrt{N}} \sum_{k} \overline{d}_k e^{-ikj}, \quad (B10)$$

where $k = 2\pi (m + 1/2) / N$, $m = 0, 1, 2, ..., N-1$. In the Nambu representation, the Hamiltonian can be written as a compact form

$$\overline{H}_+ = \sum_{0<k<\pi} \overline{\eta}_k \overline{H}_+^{k} \eta_k, \quad (B11)$$

with $\overline{\eta}_k = (\overline{d}_k, \overline{d}_{-k})$, $\eta_k = (d_k, d_{-k})^T$ and

$$\overline{H}_+^{k} = \frac{1}{2} \begin{pmatrix} \cos (k - \lambda) & i \sin k \\ -i \sin k & -\cos (k - \lambda) \end{pmatrix}, \quad (B12)$$

where $\lambda = 2g \sqrt{1-\gamma^2}$ and the Hamiltonian $\overline{H}_+^{k}$ satisfies the commutation relation $[\overline{H}_+^{k}, \overline{H}_+^{k'}] = 0$ ensuring the $\overline{H}_+$ can be diagonalized within each $k$ subspace. To this end, we introduce the non-Hermitian Bogoliubov transformation

$$\overline{b}_k = \cos \frac{\beta_k}{2} \overline{d}_k + i \sin \frac{\beta_k}{2} \overline{d}_{-k}, \quad (B13)$$

$$b_k = \cos \frac{\beta_k}{2} d_k - i \sin \frac{\beta_k}{2} d_{-k}, \quad (B14)$$

where $\beta_k = \tan^{-1} [\frac{\sin k}{\lambda - \cos k}]$ and $\overline{b}_k$ is the quasi fermionic operator obeying the anticommutation relation $[\overline{b}_k, b_{k'}]_+ = \delta_{kk'}$. It results in the diagonal form of the Hamiltonian

$$\overline{H}_+ = \sum_{k} \varepsilon_k \left( \overline{b}_k b_k - \frac{1}{2} \right), \quad (B15)$$

with the single-particle eigen energy $\varepsilon_k = \sqrt{\lambda^2 + 1 - 2\lambda \cos k}/2$. Evidently, it is a non-interacting Hamiltonian and hence the corresponding spectrum is fully determined by the single-particle energy. If $|\gamma| < 1$, then the single-particle energy is real. Correspondingly, the system respects complex single-particle spectrum regardless of $k$ when $|\gamma| > 1$. On the other hand, the level repulsion $\lim_{|\gamma| \to 1} (\partial \varepsilon_k / \partial \gamma) = \infty$ is observed as $|\gamma|$ approaches 1, which is a typical feature of EP. In this sense, $|\gamma| = 1$ is the EP of both $H_I$ and $H$. 

Appendix C: Non-Hermitian 1D isotropic Heisenberg model

The Hamiltonian of the non-Hermitian Heisenberg model under the external field is

\[ H_0 = -\frac{1}{2} \sum_{i,j \neq i} J_{ij} \left( s_i^+ s_j^- + s_i^- s_j^+ + 2s_i^z s_j^z \right), \]  

(C1)

\[ H_I = \sum_{i} g_i \mathbf{h} \cdot \mathbf{s}_i, \]  

(C2)

where \( \{i\} \) represents \( n \in [1, N] \) random numbers denoting that \( n \) spins are subjected to the local complex fields, respectively. The presence of inhomogeneous magnetic fields breaks the \( SU(2) \) symmetry, that is \( [s^\pm, H] \neq 0 \). However, the two Hamiltonians \( H_0 \) and \( H_I \) commute with each other when the index \( i \) runs over all the spins and the critical local fields are applied so that \( H_I \) can be treated as either \( s^+ \) or \( s^- \). Although the two Hamiltonians share the common eigenstates, the property of the ground states is not clear since \( s^\pm \) is non-Hermitian rather than Hermitian operator, which cannot guarantee the validity of the perturbation theory. In the following, we first use the transformation \( \mathcal{S} \) to obtain a Hermitian matrix \( \mathcal{H} \) and then demonstrate that the existence of the high-order EP neither depend on how many local fields are applied nor on the spin configuration of \( H \). Applying the spin-rotation \( \mathcal{S} \), the considered Hamiltonian can be transformed as

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \]  

(C3)

\[ \mathcal{H}_0 = -\frac{1}{2} \sum_{(i,j)} J_{ij} \left( \tau_i^+ \tau_j^- + \tau_i^- \tau_j^+ + 2 \tau_i^z \tau_j^z \right), \]  

(C4)

\[ \mathcal{H}_I = \sqrt{1 - \gamma^2} \sum_{(i)} g_i \tau_i^z. \]  

(C5)

In the basis of \( \tau^z = \sum_i \tau_i^z \), the matrix form of \( \mathcal{H} \) is Hermitian such that all the approximation method in quantum mechanics can be applied. When \( |\gamma| \to 1, \sqrt{1 - \gamma^2} \) is a small number indicating the weak coupling between the spin and magnetic field. Therefore, \( \mathcal{H}_I \) in the new frame can be treated as weak perturbations. We focus on the effect of \( \mathcal{H}_I \) on the ground state \( \{|G_n\} \) of \( \mathcal{H}_0 \). \( \mathcal{H}_0 \) is a standard isotropic Heisenberg model and hence the ground state is \( (N+1) \) fold-degeneracy which can be expressed as

\[ |G_n\rangle = (\sum_i \tau_i^z)^{n-1} |\uparrow\rangle \quad (n = 1, 2 \ldots N + 1), \]  

(C6)

where

\[ |\uparrow\rangle = \mathcal{S} |\uparrow\rangle, \]  

(C7)

\[ |\uparrow\rangle = \prod_{i=1}^{N} |\uparrow\rangle_i. \]  

(C7)

\( |G_n\rangle \) is also the eigenstate of \( \tau^z = \sum_i \tau_i^z \) with \( \tau = N/2 \). Notice that the existence of the degenerate ground states are independent of spin-configuration\[^{21,27}\]. With the spirit of degenerate perturbation theory, the eigenvalues up to the first order are determined by the matrix form of \( \mathcal{H}_I \) in the subspace spanned by \( \{|G_n\} \). For simplicity, the corresponding perturbed matrix is referred to as \( W' \) whose elements are given as \( W'_{m,n} = \langle G_m' | \mathcal{H}_I | G_n' \rangle \). \( \{|G_m'\} \) are the biorthogonal left eigenvectors in the form of

\[ \langle G_m' | = \langle \uparrow | U^{-1} (\sum_i \tau_i^z)^{m-1} (m = 1, 2 \ldots N + 1). \]  

(C8)

Here we stress two points: (i) One can always safely throw away high-order correction when \(|\gamma| \to 1 \) due to the Hermiticity of matrix \( W' \). (ii) When homogenous magnetic filed is applied, that is \( [\mathcal{H}_I, \mathcal{H}_0] = 0 \), \( \mathcal{H} \) can be decomposed into block matrix in light of the eigenvector of \( \tau^2 \) and hence the eigenvalues of \( W' \) are the energies of groundstate and \( N \) excited states of \( \mathcal{H} \) in the unbroken region. After straightforward algebras, one can readily obtain the entry of matrix \( W'_{m,n} = \sqrt{1 - \gamma^2} \sum_{(i)} g_i \sqrt{(N+1-m)} (\delta_{m+1,n} + \delta_{m,n+1})/2N \), where \( 1/N \) stems from the translation symmetry of the groundstate \( \{|G_n'\} \). Performing the inverse transformation \( W = \mathcal{S}^{-1} W' \mathcal{S} \) (\( W_{m,n} = \langle G_m | \mathcal{H}_I | G_n \rangle \) with \( |G_n\rangle = \mathcal{S}^{-1} |G_n'\rangle \)), the element of matrix \( W \) can be given as \( W_{m,n} = \sum_{(i)} g_i \sqrt{(N+1-m)} (1 + \gamma^2) \delta_{m+1,n} + (1 - \gamma) \delta_{m,n+1})/2N \). It is a non-Hermitian hypercub\[^{22}\], the EPN will exhibit when \( \gamma = 1 \). Therefore, a local magnetic will lead to a high-order of EP, the order of which is determined by the degeneracy of ground state energy of \( H_0 \).

Appendix D: Dynamics at the EPN

1. Generating the saturated ferromagnetic state

We show how to generate a saturated ferromagnetic state, where all local spins (or conduction electron spins) are aligned parallel to the \( z \)-axis. The non-Hermitian Heisenberg model Hamiltonian is given by Eqs. (1) with \( \Delta_{ij} = J_{ij} \) in the main text. We assume the magnetic field is applied to spin at site number 1 unless stated otherwise, that is \( i = 1 \). When the single magnetic field is at the critical value \( \gamma = 1 \), the matrix form of \( W \) can be given as \( W_{m,n} = g_1 \sqrt{(N+1-m)} m \delta_{m+1,n}/N \) that is a Jordan block form of dimension \( N + 1 \). The corresponding coalescent eigenstate is \( |\uparrow\rangle = \prod_{i=1}^{N} |\uparrow\rangle_i \). Notice that \( W \) is a nilpotent matrix with order \((N+1)\) such that \( (W)^{N+1} = 0 \). The element of matrix \( W^k \) can be expressed as

\[ (W^k)_{m,n} = \prod_{p=m}^{m+k-1} p (N+1-p)^{1/2} g_1 \sqrt{N} \delta_{m+k,n}. \]  

(D1)

where \( k < N+1 \). We focus on the dynamics of the critical system \( W \). The evolved state in this subspace is governed
by the propagator $U = e^{-i W t}$. With the aid of Eq. \[D1\], one can readily obtain the element of propagator

$$
U_{m,n} = \delta_{mn} + \left( \frac{-i t g_1}{N} \right)^{n-m} \frac{h(n-m)}{(n-m)!} \prod_{p=m}^{n-1} p (N + 1 - p)^{1/2}, \tag{D2}
$$

where $h(x)$ is a step function with the form of $h(x) = 1$ $(x > 0)$, and $h(x) = 0$ $(x < 0)$. Considering an arbitrary initial state $\sum_n c_n(0) |G_n\rangle$, the coefficient $c_m(t)$ of evolved state is

$$
c_m(t) = c_m(0) + \sum_{n \neq m} \left( \frac{-i t g_1}{N} \right)^{n-m} \frac{h(n-m)}{(n-m)!} \prod_{p=m}^{n-1} p (N + 1 - p)^{1/2} c_n(0). \tag{D3}
$$

It clearly shows that no matter what the initial state is selected, the coefficient $c_1(t)$ of evolved state always includes the highest power of time $t$. As time goes on, the component $c_1(t)$ of the evolved state overwhelms the other components ensuring the final state is coalescent state $\langle \tilde{\eta} \rangle$ under the Dirac normalization. The different types of the initial state just determine how the total Dirac probability of the evolved state increases over time and the relaxation time for it evolves to the coalescent state. To measure the similarity between evolved state and target ferromagnetic state $\langle \tilde{\eta} \rangle$, we introduce the normalized fidelity as

$$
F_n(t) = \frac{|\langle G_1| U |G_n\rangle|^2}{\langle G_n| U^* U |G_n\rangle}. \tag{D4}
$$

The quantity $F_n(t)$ also reflects how fast the evolved state approaches the final state. Using the Eq. \[D3\], one can give directly the following expression

$$
F_n(t) = \frac{\delta_{1,n} + (\eta^2)^{n-1} C_n^{-1} C_{n-1}^{-1} C_{n-2}^{-1} \cdots C_1^{-1} C_{n-1}^N}{1 + \sum_{m=1}^{N} (\eta^2)^{n-m} C_m^{-1} C_{m-1}^{-1} C_{m-2}^{-1} \cdots C_{n-m}^{-1} C_{n-m+1}^{-1} \cdots C_{n+1}^{-1} C_{n+2}^{-1} \cdots C_N^{-1} C_{N+1}^{-1} \cdots C_{N+1}^{-1}} \tag{D5}
$$

where $\eta(t) = t/t_o$ with $t_o = N/g_1$. For $n = 1$, the initial state is the eigenstate of $W$ and hence does not evolve. We plot \[4\] to compare the numerical result obtained by driving the interacting Hamiltonian $H$ and analytical result based on the perturbation matrix $W$. The initial state will evolve to the target state and excellently agrees with our prediction. On the other hand, as the system dimension increases, we observe that the initial state takes more time to arrive at the final ferromagnetic state. Although the system with larger sizes can host higher-order EPs, the relaxation time is inversely proportional to the order of EP that can be readily understood by analytical formula $F_{N+1}(t) = [1 + 1/\eta^2(t)]^{-N}$. This again confirms the validity of the perturbation treatment of the non-Hermitian external field.

![FIG. 4: Plots of the normalized fidelity $F(t)$ as functions of time $t$ for the critical systems with $N = 3, 5, 7, 9$ denoted by the blue circle, red square, yellow diamond, and purple cross, respectively. The circle denotes the corresponding analytical result obtained by Eq. \[D5\]. $F(t)$ means the fidelity driven by either $H$ (solid lines) or $W$ (hollow markers).](image)

2. The hysteresis loop in time domain

Another interesting dynamical phenomenon is the hysteresis loop in the time domain. The hysteresis loop can be obtained through measuring the average magnetization $M(t)$ [Eq. (6) in the main text]. Here we demonstrate that two components of the loop (red and yellow line in Fig. 3 of the main text) can be derived analytically based on the aforementioned mechanism. When the initial state is magnetized, the final state is coalescent state in the subspace $W$ such that all the inverse magnetization process is solely determined by the effective Hamiltonian $W$. Correspondingly, $M(t)$ is given as

$$
M(\sigma,t) = \frac{1}{N} \sum_{m} U_{m,\sigma}^* U_{m,\sigma} (N - 2m + 2), \tag{D6}
$$

where $\sigma$ is either $N + 1$ or 1 and denotes the magnetization (yellow) or inverse magnetization (red) process. Straightforward algebras show that $M(N + 1, t) = [1 - \eta^2(t)]/[1 + \eta^2(t)]$ and $M(1, t) = [\eta^2(t) - 1]/[1 + \eta^2(t)]$. In the main text, Eqs. (7)-(8) are obtained by replacing $\eta(t)$ with $\eta(t + t_f)$ according to the starting point in the time-evolution process. Consequently, the physical quantities coercive time $t_c$ and retentivity $M_r$ defined in main text can be given as

$$
t_c = t_f - t_o, M_r = 1 - \frac{2}{1 + t_o^2/t_f^2}, \tag{D7}
$$

where $t_f$ is the relaxation time that the initial state being magnetized (end point of the blue line). The area enclosed in the hysteresis loop is

$$
S_{hl} = 4[t_f - 2t_o \tan^{-1}(t_f/t_o)] \tag{D8}
$$

We find that all the quantities of the hysteresis loop are associated with $t_f$. It is a unique feature of the considered non-Hermitian spin model and is distinct from the
traditional hysteresis loop. In the context of magnetism, there exists a reversible magnetization phase in the whole magnetization process so that the area of the hysteresis loop is independent of relaxation time. This suggests that no matter how one increases the external field strength, the area surrounded by the loop is always the same. However, the presence of local complex field spoils the time-reversal symmetry and hence induce a hysteresis loop depending on $t_f$. When $t_f/t_0 > 1$, the hysteresis loop tends to a rectangle whose width and length are 2 and $2t_f$. Such a graph may inspire further interest in the experiment.

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