The equations governing the gravitational and electromagnetic perturbations of Kerr-Newman spacetime are here derived. They generalize the Teukolsky equation in Kerr and the Teukolsky-like system of equations in Reissner-Nordström spacetime. Through their Chandrasekhar transformation, we obtain a system of physical-space coupled wave equations. In particular, the physical-space analysis of this system will solve the issue of the “apparent indissolubility of the coupling between the spin-1 and spin-2 fields in the perturbed spacetime”, as put by Chandrasekhar. The derivation of the equations here obtained makes use of the formalism introduced in [13] for Kerr, and represents the first step towards an analytical proof of the stability of the Kerr-Newman black hole.

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1 Introduction

One of the fundamental problems in General Relativity is to understand the final state of evolution of initial data for the Einstein equation. Through gravitational collapse and dispersion of gravitational waves, the geometry to which solutions to the Einstein equation are expected to relax outside the event horizon of a black hole is expected to be the one given by the known explicit solutions: the Kerr and the Kerr-Newman black hole.

The Kerr-Newman metric is the most general known explicit black hole solution. It is a 3-parameter family of solutions to the Einstein-Maxwell equation and describes the gravitational field around an isolated black hole of mass $m$, angular momentum $ma$ and electric charge $Q$. Its expression in Boyer-Lindquist coordinates $(t, r, \theta, \varphi)$ is given by

$$g_{m,a,Q} = -\frac{\Delta}{|q|^2} (dt - a \sin^2 \theta d\varphi)^2 + \frac{|q|^2}{\Delta} dr^2 + |q|^2 d\theta^2 + \frac{\sin^2 \theta}{|q|^2} (adt - (r^2 + a^2)d\varphi)^2,$$

where

$$\Delta = r^2 - 2mr + a^2 + Q^2,$$

$$|q|^2 = r^2 + a^2(\cos \theta)^2.$$

The Kerr-Newman metric generalizes the Kerr solution (for $Q = 0$), the Reissner-Nordström solution (for $a = 0$) and the Schwarzschild solution (for $Q = a = 0$), and therefore plays a fundamental role in describing the final state of evolution in General Relativity.

Being the most general explicit black hole solution of the Einstein equation coupled with matter, the Kerr-Newman spacetime has been at the center of analytical and numerical research for decades. Starting with the works in the black hole perturbation theory community, masterfully summarized by Chandrasekhar [6], the Kerr-Newman case has stood up as genuinely different from the Kerr and Reissner-Nordström solutions which generalizes. As stated by Chandrasekhar in Section 111 of [6], “the methods that have proved to be so successful in treating the gravitational perturbations of the Kerr spacetime do not seem to be applicable (nor susceptible to easy generalizations) for treating the coupled electromagnetic-gravitational perturbations of the Kerr-Newman spacetime.” In particular, the mode stability for gravitational perturbations of Schwarzschild and Kerr solution, as well as the gravitational and electromagnetic perturbations of Reissner-Nordström, have been previously solved using metric perturbations and Newman-Penrose formalism by several authors between the 70s and 80s (see for example [4], [5], [26], [28], [30], [31], [33], [34]).

The techniques applied in those early works, which relied on decomposition in frequency modes of perturbations of the solutions, failed to be extended to the case of Kerr-Newman spacetime, despite the manifest similarity of the metric to the Kerr case. Again as pointed out by Chandrasekhar in Section 111 of [6], “the principal obstacle is in finding separated equation” and in the “apparent indissolubility of the coupling between the spin-1 and spin-2 fields in the perturbed spacetime”. Following the same procedure as in the case of Kerr or Reissner-Nordström, one reaches a point where the equations can not be decoupled or separated any further. In page 583 of [6], Chandrasekhar gives an explanation of “why the system of equations..."
proves intractable in contrast to apparently similar system of equations encountered in the treatment of the perturbations of the Reissner-Nordström and Kerr spacetimes. The reason has to do with the interaction of the spin-1 and spin-2 fields in a non-spherically symmetric background. We summarize his argument here and describe how we intend to overcome such difficulties towards an analytical proof of the stability of the Kerr-Newman black hole.

1.1 Why the analytical proof of mode stability for Kerr-Newman fails

Numerical works strongly support the mode stability of Kerr-Newman spacetime (see for example [12], [29]), and the Kerr-Newman metric is expected to be stable as a solution to the fully non-linear Einstein-Maxwell equation. Nevertheless, an analytical proof of even its mode stability is missing, and the state of the art on this problem is pretty much the same as described by Chandrasekhar [6] in 1983. We now explain what are the main issues.

In the mode stability analysis, one does not study general solutions of linearized gravity in Kerr or Kerr-Newman, but rather individual modes. In the case of the linear wave equation on the Kerr or Kerr-Newman background

$$\Box_{g_{\text{m.a.q}}} \psi = 0$$

(1)

the symmetries of the Kerr(-Newman) metric allow for a definition of modes, which separates the wave operator. A mode is a solution of (1) of the form

$$\psi = \psi_{m\ell}(r) S_{m\ell}(a\omega, \varphi) e^{im\varphi} e^{-i\omega t}$$

where

$$S_{-2m\ell}(a\omega, \varphi) \text{ and } S_{-1m\ell}(a\omega, \varphi)$$

are the oblate spherical harmonics associated to the Laplacian operator for 2-tensors and 1-tensors respectively. In a spherically symmetric background, like the case of Reissner-Nordström, the oblate spherical harmonics $S_{m\ell}^2(a\omega, \varphi)$ reduce to the standard spherical harmonics $Y_{m\ell}^2(\theta, \varphi)$ and $Y_{m\ell}^1(\theta, \varphi)$. The standard spherical harmonics associated to different tensors are simply related through the Hodge decomposition in spheres and their eigenvalues are the same. This is the key to the solvability of the mode stability for electromagnetic-gravitational perturbations of Reissner-Nordström. Suppose that the Einstein-Maxwell equation (2) can be translated into an equation of the form

$$\mathcal{T}(\phi) = \text{div} \psi,$$

(3)

for an operator $\mathcal{T}$ (for instance the Teukolsky operator) applied to the spin-1 field $\phi$, and an angular operator (for instance the divergence) applied to the spin-2 field $\psi$. Equation (3) is an identity between 1-tensors,
and therefore its decomposition in modes involves the $Y_{\ell m}^{\pm 1}(\theta, \varphi)$. On the other hand, the relation between $Y_{\ell m}^{\pm 2}(\theta, \varphi)$ and $Y_{\ell m}^{\pm 1}(\theta, \varphi)$ allows to write the right hand side of $\mathcal{E}$ with respect to the decomposition of the spin-2 field $\psi$. Denoting by $(\cdot)^{\pm 1}$ and $(\cdot)^{\pm 2}$ the decomposition in modes using the spherical harmonics for 2- or 1-tensors respectively, we can write schematically

$$(\mathcal{T}(\phi))^{\pm 1} = (\text{div} \psi)^{\pm 1} = \text{div} ((\psi)^{\pm 2}),$$

where the last equality holds true because of the spherical symmetry of the background, which assures that the decomposition in modes of $\mathcal{E}$ passes through.

In the general axially symmetric case, as for Kerr or Kerr-Newman, the oblate spherical harmonics relative to different tensors are not simply related, and this is the main issue in dealing with the interaction of spin-2 and spin-1 fields in a non-spherically symmetric background. In the case of gravitational perturbations of Kerr, the Einstein vacuum equation reduces to an equation of the form $\mathcal{T}(\psi) = 0$, i.e. the Teukolsky operator for the spin-2 field $\psi$. In particular, its decomposition in modes only involves the oblate spherical harmonics $S_{\ell m}^{\pm 2}(a\omega, \varphi)$ for spin-2 fields, so the decomposition in modes passes through, and the problem of the interaction between spin-2 and spin-1 fields does not arise.

It is now clear what is the main issue in obtaining separated equations for gravitational and electromagnetic perturbations of Kerr-Newman. In the axisymmetric background, the interaction between the spin-2 and spin-1 fields prevents the separability in modes. Suppose again that the Einstein-Maxwell equation translates into an equation of the form $\mathcal{E}$. Then its decomposition with respect to the oblate spherical harmonics $S_{\ell m}^{\pm 1}(a\omega, \varphi)$ for spin-1 fields does not pass through, and the decomposition of $(\text{div} \psi)^{\pm 1}$ cannot be written in terms of the decomposition of $\psi$ given by $(\cdot)^{\pm 2}$.

These are the main obstacles to the separability of the equations in the case of gravitational and electromagnetic perturbations of Kerr-Newman spacetime. As Chandrasekhar ends at page 583 of [6], “one might be inclined to conclude that a decoupling of the system of equations and a separation of the variables will be possible, if at all, only by contemplating equations of order 4 or higher”.

1.1.1 Beyond the mode stability of Kerr-Newman

In treating the coupled gravitational and electromagnetic perturbations of Kerr-Newman spacetime, the decomposition in modes of the equations, which had the objective of simplifying the analysis of the perturbations, actually makes them unsolvable as consequence of the discussion in the previous section. Observe that such failure is explicitly related to the fact that the equations as analyzed in [6] required the decomposition in spherical harmonics, which yields the problem of non-commutativity of the decomposition. There is no reason to believe that if one does not decompose in modes nor separate the equations using the oblate spherical harmonics such problems could be circumvented.

The key to solve this issue is to abandon the decomposition in modes, and perform a physical-space analysis of the equations. Following the road map that mathematicians have taken in interpreting in physical space the mode analysis done by the physics community, the Kerr-Newman solution may be the case where a physical-space approach could succeed where the mode analysis in physics failed.

The analysis of the wave equation in black hole backgrounds has been object of extensive study in the last fifteen years (see for example [9], [10], [11]), and quantitative decay statements for solutions to the wave equations have been obtained in the case of Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman. Physical-space analysis for the gravitational perturbations of Schwarzschild and Kerr, and coupled electromagnetic and gravitational perturbations of Reissner-Nordström, represent a fundamental guideline in the study of the most general known black hole solution, the Kerr-Newman black hole.

1.2 The case of Schwarzschild, Kerr and Reissner-Nordström

The study of gravitational perturbations of black hole solutions relies on the analysis of the so-called Teukolsky equations, which govern the radiation. Those are wave-like equations satisfied by gauge-invariant quantities of the perturbations. We recall here the most important steps in the resolution of linear stability of Schwarzschild, Kerr and Reissner-Nordström spacetime. In the case of positive cosmological constant, the non-linear stability of Kerr-Newman-de Sitter spacetime was proved in [10].
1.2.1 Schwarzschild spacetime

The Schwarzschild metric

$$g_{m} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$$

represents the gravitational field outside a static spherically symmetric black hole of mass $m$, and is a solution to the Einstein vacuum equation $\text{Ric}(g) = 0$.

The linear stability of the Schwarzschild solution to gravitational perturbation was first proved by Dafermos-Holzegel-Rodnianski [7] using curvature perturbations. Other proof have followed: using metric perturbations [20], wave coordinates [21] [22] and generalized wave coordinates [23]. The first proof of non-linear stability of Schwarzschild was obtained by Klainerman-Szeftel [24], within the class of symmetry of axially symmetric polarized perturbations.

The gravitational perturbations of Schwarzschild can be described through the extreme null component of the Weyl curvature. Let $L$ and $L$ be a pair of null vectors, with $L$ outgoing and $L$ ingoing, and $e_a$ (for $a = 1, 2$) be orthogonal vectors on the sphere. The 2-tensor on the sphere $\alpha$ defined as

$$\alpha_{ab} = W(e_a, L, e_b, L)$$

where $W$ is the Weyl curvature of the perturbation, is a gauge-invariant quantity, i.e. it is invariant to coordinate transformations at linear order. This is a tensorial version of the $\Phi_0$ component in Newman-Penrose formalism. The quantity $\alpha$ satisfies a wave equation which is decoupled at the linear level from all other curvature components [3], known as Teukolsky equation of spin 2, of the schematic form:

$$\Box g_{m} \alpha + c_1(r) \partial_r \alpha + c_2(r) \partial_t \alpha + V(r) \alpha = 0,$$

where $\Box g_{m}$ is the D’Alembertian operator for the Schwarzschild metric $g_{m}$, and $c_i$, $V$ are smooth functions of the coordinate $r$.

The decomposition in modes of the above equation is the starting point for the proof of mode stability as obtained by the black hole perturbation community [6]. On the other hand, a physical space analysis of the Teukolsky equation presents a fundamental obstacle to overcome: even if the Teukolsky equation is a wave-like equation, the presence of the first order terms $\partial_r \alpha$ and $\partial_t \alpha$ prevents one to apply directly the techniques known for the standard wave equations on black hole backgrounds.

The way to unlock the above issue is to transform the Teukolsky equation into a well-behaved wave equation, which does not have first order terms. Such transformation is now known as Chandrasekhar transformation, who introduced it in fixed frequency mode decomposition [6] and connected the Teukolsky equation for curvature perturbations to the Regge-Wheeler equation for metric perturbations (see also [22]). The physical space version of the Chandrasekhar transformation was first introduced in [7], and consists of taking two derivatives along the ingoing null direction of the Teukolsky equation. By schematically defining $q = L(L(\alpha))$, the transformed quantity $q$ verifies the Regge-Wheeler equation, of the schematic form:

$$\Box g_{m} q + V(r) q = 0.$$ (4)

Observe that the above transformation has as a result the cancellation of the first order terms in the equation, and therefore the techniques known for the standard wave equation can be applied to the above Regge-Wheeler equation to obtain quantitative decay for $q$. This unlocks the analysis of the gauge-invariant quantities in Schwarzschild which govern the gravitational radiation. After a choice of gauge, such quantity controls the whole perturbation [7].

1.2.2 Kerr spacetime

The Kerr metric

$$g_{m,a} = -\frac{\Delta}{|q|^2} \left(dt - a \sin^2 \theta d\varphi\right)^2 + \frac{|q|^2}{\Delta} dr^2 + |q|^2 d\theta^2 + \frac{\sin^2 \theta}{|q|^2} \left(a dt - (r^2 + a^2) d\varphi\right)^2,$$
with
\[
\Delta = r^2 - 2mr + a^2, \\
|q|^2 = r^2 + a^2(\cos \theta)^2,
\]
represents the gravitational field outside an axially symmetric rotating black hole of mass \( m \) and angular momentum \( ma \), and is a solution to the Einstein vacuum equation. The quantitative decay for solutions to the Teukolsky equation in slowly rotating Kerr has been obtained by Ma \[25\] and Dafermos-Holzegel-Rodnianski \[8\]. The linear stability of slowly rotating Kerr has been obtained in \[2\] and \[18\].

The tensor \( \alpha \) can be defined in the case of Kerr as above, but since in Kerr the distribution orthogonal to the principal null frame \( L \) and \( \mathcal{L} \) is not integrable, the tensor \( \alpha \) is not a 2-tensor on the sphere as in Schwarzschild but rather an orthogonal tensor in the distribution (see Section 2 for the relevant definitions).

We briefly describe here the general formalism for treating the fully non-linear gravitational perturbations of Kerr spacetime as introduced in \[13\]. Denote the complexified version of \( \alpha \) as
\[
\mathcal{A}_{ab} = \alpha_{ab} + i \ast \alpha_{ab}, \quad \text{with } \ast \alpha_{ab} = \varepsilon_{ac} \alpha_{cb}
\]
where \( \varepsilon_{ab} \) denotes the volume form on the horizontal distribution. The Newman-Penrose complex scalar \( \Psi_0 \) is the projection of the self-antidual tensor \( A \), i.e. \( \Psi_0 = A_{11} = \alpha_{11} + i \ast \alpha_{11} \). The tensor \( A \) verifies the following Teukolsky equation \[31\] of spin 2:
\[
\Box_{g_{m,Q}} A + c_1(r,\theta) \partial_r A + c_2(r,\theta) \partial_t A + a \cdot c_3(r,\theta) \partial_{\varphi} A + V(r,\theta) A = 0,
\]
for which a Chandrasekhar transformation can be generalized (see \[23\], \[8\]). By again schematically defining \( q = \mathcal{L}(\mathcal{L}(\alpha)) \), one obtains an equation of the schematic form
\[
\Box_{g_{m,Q}} q + V(r,\theta) q - i \frac{4a \cos \theta}{|\rho|^2} \partial_t q = a \cdot \left( c_1(r,\theta) \partial_{\varphi} \mathcal{L}(A) + c_2(r,\theta) \partial_{r} A + c_3(r,\theta) \mathcal{L}(A) + c_4(r,\theta) A \right)
\]
\[
= a \cdot \text{l.o.t.}
\]
Observe that the above left hand side generalizes the Regge-Wheeler equation in Schwarzschild \[11\], but contains a first order term which vanishes for \( a = 0 \). Such a term is potentially as dangerous as the first order terms appearing in Teukolsky, but it turns out that the presence of a term of this form (i.e. \( i \partial_t \)) does not change the good divergence properties of the left hand side of the Regge-Wheeler equation. The right hand side of the above equation contains up to two derivatives of \( A \), and since \( q \) is at the level of two derivatives of \( A \), the right hand side can be interpreted as lower order terms, and analyzed through transport equations (see \[25\], \[8\]). Quantitative decay obtained for \( q \) and therefore for \( A \) can be used to control the remaining components of the gravitational perturbations of Kerr \[2\].

### 1.2.3 Reissner-Nordström spacetime

The Reissner-Nordström metric
\[
g_{m,Q} = - \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)
\]
represents the electromagnetic and gravitational field outside a static spherically symmetric black hole of mass \( m \) and charge \( Q \), and is a solution to the Einstein-Maxwell equation \[2\].

The linear stability of Reissner-Nordström solution to coupled electromagnetic and gravitational perturbations was first solved in \[16\] for very small charge \( |Q| \ll m \), and then in \[17\] in the full subextremal range \( |Q| < m \). As in the case of Schwarzschild and Kerr, a fundamental step in the proof is the quantitative decay for solutions to the Teukolsky equations of spin 1 \[15\] and spin 2 \[14\].

In the electromagnetic and gravitational perturbations of Reissner-Nordström, the gauge-invariant quantity \( \alpha \) is not sufficient to describe the full perturbation. Additional gauge-invariant quantities are coupled to it, and both spin-1 (for the electromagnetic radiation) and spin-2 (for the gravitational radiation) quantities have to be identified.
In the electromagnetic-gravitational perturbations of Reissner-Nordström, the tensor $\alpha_{ab}$ satisfies a Teukolsky like equation \([14]\) which is coupled to an electromagnetic component $\mathcal{J}$ in the following schematic way:

$$
\Box_{g_{m,Q}} \alpha + c_1(r) \partial_r \alpha + c_2(r) \partial_t \alpha + V_1(r) \alpha = Q \cdot c_3(r) L(f) \tag{6}
$$

The above reduces to the Teukolsky equation for $\alpha$ in Schwarzschild for $Q = 0$. The 2-tensor on the sphere $f$ is defined as

$$
f_{ab} = \nabla \otimes (F\beta_{ab}) + \frac{1}{2} F(L,L) \chi_{ab}, \quad \text{where } (F\beta)_{a} = F(e_a, L)
$$

and is gauge-invariant and satisfies a Teukolsky-type equation \([14]\) which is coupled back to $\alpha$:

$$
\Box_{g_{m,Q}} f + d_1(r) \partial_t f + d_2(r) \partial_r f + V_2(r) f = Q \cdot d_3(r) L(\alpha) \tag{7}
$$

These are Teukolsky equations of spin 2. To fully describe the electromagnetic perturbations of Reissner-Nordström we need to take into account an equation for a spin-1 gauge-invariant quantity, which transport electromagnetic radiation. The 1-tensor on the sphere $\beta$ defined as

$$
\beta_a = \frac{1}{2} F(L,L) W(e_a, L, L, L) - \frac{3}{4} W(L, L, L, L) F(e_a, L)
$$

is gauge-invariant and satisfies a Teukolsky-type equation \([15]\) which is coupled to $f$ and to another gauge-invariant 1-tensor $\gamma$:

$$
\Box_{g_{m,Q}} \beta + f_1(r) \partial_t \beta + f_2(r) \partial_r \beta + V_3(r) \beta = Q \cdot f_3(r) \text{div} f + Q \cdot f_4(r) \gamma \tag{8}
$$

This is a Teukolsky equation of spin 1, where the interaction between the spin-1 and spin-2 fields is manifest. By denoting $T(\Psi) = \Box_{g_{m,Q}} \Psi + c(r) \partial_r \Psi + c(r) \partial_t \Psi + V(r) \Psi$ a Teukolsky-type operator, equations \([6], \tag{7}, \tag{8}\) can be summarized as the following mixed spin-1 and spin-2 Teukolsky system:

$$
\begin{align*}
T_{t-2}(\alpha) &= Q \cdot c_3(r) L(f) \\
T_{t-2}(f) &= Q \cdot d_3(r) L(\alpha) \\
T_{t-1}(\beta) &= Q \cdot f_3(r) \text{div} f + Q \cdot f_4(r) \gamma
\end{align*}
$$

In \([14]\) and \([15]\), the Chandrasekhar transformation has been extended to the above system. It consists in applying two incoming null derivatives to the first equation (which generalizes the equation in Schwarzschild) and one incoming null derivative to the second and third equation. Defining schematically

$$
q = L(L(\alpha)), \quad q^F = L(f), \quad p = L(\beta),
$$

the result of the Chandrasekhar transformation is the following system of coupled wave equations:

$$
\begin{align*}
\Box_{g_{m,Q}} q + V(r) q &= Q \left( \frac{4}{r^2} \Delta q^F + \frac{16}{r^2} \partial_r (q^F) + W(r) q^F + \text{l.o.t.} \right), \\
\Box_{g_{m,Q}} q^F + V(r) q^F &= Q \left( \frac{1}{r^2} q^F + \text{l.o.t.} \right), \\
\Box_{g_{m,Q}} p + V(r) p &= Q \left( \text{div} q^F \right)
\end{align*}
$$

where l.o.t. denotes lower order terms in order of differentiability with respect to $q$, $q^F$ and $p$. Observe that in the process of applying the Chandrasekhar transformation the auxiliary quantity $f$ has disappeared, since its $L$ derivatives can be written in terms of $f$ and $\beta$.

In addition, there is a relation between the three quantities $q$, $q^F$ and $p$ which holds true, of the schematic form:

$$
Q \cdot q = h_1(r) q^F + h_2(r) \mathcal{D} q^F + \text{l.o.t.} \tag{9}
$$

\footnote{See Section 2 for the relevant definitions.}

\footnote{Notice that in particular the Teukolsky equation \([5]\) is of the schematic form \([3]\).}
where $\mathcal{D}^2$ is a symmetric traceless angular derivative on the sphere, and $h_i$ are smooth functions of $r$. Considering the above equations for $q, q^F$ and $p$ and the relation (11) between them, it is clear that the above system of three equations is equivalent to a system of two equations, which is to say that one of the equations is redundant. By neglecting the equation for $q$, we schematically obtain

$$
\begin{align*}
\Box_{g_{m,Q}} p + V_1(r) \ p &= Q \cdot a_1(r) \nabla q^F \\
\Box_{g_{m,Q}} q^F + V_2(r) \ q^F &= a_2(r) \mathcal{D}^2 p
\end{align*}
$$

This system governs the coupled gravitational and electromagnetic perturbations of the Reissner–Nordström black hole. The system is symmetric, due to fact that the operators $\nabla$ and $\mathcal{D}^*/2$ are adjoint operator on the sphere, and this allows to obtain quantitative decay for it in the full subextremal range $|Q| < m$ [17].

1.3 The system of equations in Kerr-Newman

The equations describing the electromagnetic-gravitational perturbations of the Kerr-Newman metric $g_{m,a,Q}$ should generalize both

- the equation governing the gravitational perturbation of Kerr:

$$
\Box_{g_{m,a}} q + V(r,\theta) \ q - i \frac{4a \cos \theta}{|\rho|^2} \partial_t q = a \cdot 1.o.t. \quad (10)
$$

- and the equations governing the electromagnetic-gravitational perturbations of Reissner-Nordström:

$$
\begin{align*}
\Box_{g_{m,Q}} p + V_1(r) \ p &= Q \cdot a_1(r) \nabla q^F \\
\Box_{g_{m,Q}} q^F + V_2(r) \ q^F &= a_2(r) \mathcal{D}^2 p
\end{align*}
$$

(11)

More precisely, our goal is to obtain, through a Chandrasekhar transformation, a system of equations which maintain the good properties which allowed for a quantitative physical-space analysis in the case of Kerr and Reissner-Nordström, i.e.

- good divergence properties on the left hand side as in (10) (e.g. a first order term of the form $i\partial_t$)
- symmetry properties on the right hand side of the system as in (11) (e.g. adjoint operators)

In this paper, we show that both properties can be generalized to the case of Kerr-Newman, opening the way for an analytical proof of the stability of Kerr-Newman black hole to coupled electromagnetic and gravitational perturbations. The generalization to Kerr-Newman is more closely related to the system of equations obtained for Reissner-Nordström, and their gauge-invariant quantities, while presenting the same structure as the Regge-Wheeler equation in Kerr.

Recall the gauge-invariant quantities $\tilde{\beta}$ and $f$ as defined above for Reissner-Nordström. We define their complexified anti-self dual versions $\mathfrak{B}$ and $\mathfrak{F}$ in Kerr-Newman as

$$
\mathfrak{B} = \tilde{\beta} + i \ast \tilde{\beta} + a \cdot \text{terms}, \quad \mathfrak{F} = f + i \ast f + a \cdot \text{terms}
$$

where $a \cdot \text{terms}$ denotes additional terms which vanish in Reissner-Nordström but are needed to obtain gauge-invariant quantities $\mathfrak{B}$ and $\mathfrak{F}$ in Kerr-Newman. Those quantities are the ones which transport electromagnetic (the 1-tensor $\mathfrak{B}$) and gravitational (the 2-tensor $\mathfrak{F}$) radiation, and they verify a coupled system of Teukolsky-type equations of the form

$$
\mathcal{T}_1(\mathfrak{B}) = Q \cdot (f_1(r,\theta) \ D \cdot \mathfrak{F} + f_2(r,\theta) \mathfrak{F} + a \cdot \text{terms})
$$

$$
\mathcal{T}_2(\mathfrak{F}) = Q \cdot d_1(r,\theta) L(A) + a \cdot \text{terms}
$$

3See (42) and (105) for the definition.

4See equations (111) and (112) for the exact form of the equations.
where $A$ is the complexified version of $\alpha$ as in (5), $\mathfrak{X}$ is the complexified version of $\mathfrak{x}$ and $\overline{\mathcal{D}}$ is the complexified version of the divergence operator$^3$. More precisely, for an anti-self dual symmetric 2-tensor $U = u + i \ast u$

$$\frac{1}{2} \overline{\mathcal{D}} \cdot U = \frac{1}{2} \overline{\mathcal{D}} \cdot (u + i \ast u) = \text{div} u + i \ast (\text{div} u)$$

Observe that the above Teukolsky equations directly generalize equations (7) and (8) in Reissner-Nordström, maintaining the same highest order terms.

We now generalize the Chandrasekhar transformation to the case of Kerr-Newman. Once again, the case of Reissner-Nordström is a particularly useful guidance for the more general case of Kerr-Newman. As reminded earlier in Section 1.2.3, the two out of three equations which contain all the information about the perturbations of Reissner-Nordström are the transformed for $\beta$ and $f$, while the transformed for $\alpha$ can be neglected in the analysis. The same feature holds in Kerr-Newman, where in particular we can avoid computing the Chandrasekhar transformed of the Teukolsky equation for $A$ (which involves two derivative), and it is instead sufficient to compute the Chandrasekhar transformed of $\mathfrak{X}$ and $\mathfrak{B}$ (which involves one derivative only).

We are now ready to state the main theorem of this paper, concerning the final system of equations for the Chandrasekhar transformed of the Teukolsky-type equations in electromagnetic-gravitational perturbations of Kerr-Newman spacetime. We define

$$q = r + ia \cos \theta, \quad \overline{q} = r - ia \cos \theta.$$

Also, the complexified version of the symmetric traceless angular derivative $\nabla \hat{\otimes}$ is defined for an anti-self dual 1-tensor $F = f + i \ast f$ as

$$\frac{1}{2} \overline{\mathcal{D}} \hat{\otimes} F = \frac{1}{2} \overline{\mathcal{D}} \hat{\otimes} (f + i \ast f) = \nabla \hat{\otimes} f + i \ast (\nabla \hat{\otimes} f).$$

We state here a rough version of the main theorem. See Theorem 7.1 for the precise version.

**Theorem 1.1.** Consider a linear gravitational and electromagnetic perturbations of Kerr–Newman spacetime, and its associated complex tensors and gauge invariant quantities $\mathfrak{X}$ and $\mathfrak{B}$. Then, their Chandrasekhar-transformed quantities schematically defined as $p = L(\mathfrak{B})$ and $q^F = L(\mathfrak{X})$ satisfy the following coupled system of wave equations:

$$\Box_{g_{m.a.,q}} p + V_1(r, \theta) p - \frac{2a \cos \theta}{|q|^2} \partial_t p = \frac{4Q^2}{|q|^5} \overline{\mathcal{D}} \cdot q^F + a \cdot \text{l.o.t.}$$

$$\Box_{g_{m.a.,q}} q^F + V_2(r, \theta) q^F - \frac{4a \cos \theta}{|q|^2} \partial_t q^F = -\frac{q^3}{2|q|^5} \overline{\mathcal{D}} \hat{\otimes} p + a \cdot \text{l.o.t.}$$

where $a \cdot \text{l.o.t.}$ denotes terms which vanish in Reissner-Nordström and are lower order terms with respect to $p$ or $q^F$.

As a consequence of the above theorem, the good properties of the equations obtained in Reissner-Nordström and in Kerr can be generalized to the case of Kerr-Newman. This strikingly compares with the equations in separated modes as described at the beginning of this introduction, which could not be generalized from the Kerr and Reissner-Nordström case. By avoiding the decomposition in modes, and maintaining the above equations for the 1-tensor $p$ and the 2-tensor $q^F$, the issue of non-commutativity of the decomposition is not present and a physical space analysis of the above system is in principle possible. More precisely, such analysis would have to avoid decomposition in modes for the solutions, for example in the spirit of (11) for small angular momentum.

This paper is organized as follows. In Section 2 we recall the general formalism introduced in [13] in the case of Kerr and in Section 3 we derive the Einstein-Maxwell equations in their full generality (in particular,
we do not assume integrability of the horizontal structure or any gauge conditions). In Section 4 we recall the complex notation as defined in [13] and make use of them to rewrite the Einstein-Maxwell equations. In Section 5, we introduce the Kerr-Newman spacetime and its representation within the complex notation. We also define the main gauge-invariant quantities in the gravitational and electromagnetic perturbations of the Kerr-Newman spacetime and the differential relations between them. In Section 6, we derive the Teukolsky-type equations satisfied by the gauge-invariant quantities. Finally, in Section 7 we define the Chandrasekhar transformation in Kerr-Newman and derive the Regge-Wheeler-type equations for the perturbations, proving the main theorem of the paper.

To facilitate the reading of the paper, we diverted most of the proofs (involving lengthy computations) to the Appendix. In Appendix A we collect the explicit computations needed in the first five sections of the paper. In Appendix B we derive the Teukolsky-type equations and in Appendix C we derive the system of Regge-Wheeler equations.

2 Preliminaries

In this section we collect the main definitions and preliminaries to the formalism introduced in [13]. Such formalism is adapted to the study of stability of the Kerr(-Newman) spacetime, whose horizontal distribution relative to the principal null frame is not integrable.

Consider an arbitrary null pair $L$ and $L$, i.e. $g(L, L) = g(L, L) = 0$ and $g(L, L) = -2$. We say that a vectorfield $X$ is horizontal if $g(L, X) = g(L, X) = 0$.

On the set of horizontal vectors, we define the induced volume form by $\in(X, Y) := \frac{1}{2} \in(X, Y, L, L)$. The commutator $[X, Y]$ of two horizontal vectorfields may fail to be horizontal. We say that the pair $(L, L)$ is integrable if the set of horizontal vectorfields forms an integrable distribution, i.e. $X, Y$ horizontal implies that $[X, Y]$ is horizontal.

**Definition 2.1.** For any horizontal $X, Y$ we define the null second fundamental forms

$$\chi(X, Y) = g(D_X L, Y), \quad \bar{\chi}(X, Y) = g(D_X L, Y).$$

Observe that $\chi$ and $\bar{\chi}$ are symmetric if and only if the horizontal structure is integrable. We define the trace and anti-trace of $\chi$ according to the following definition.

**Definition 2.2.** The trace of a horizontal 2-tensor $U$ is defined by

$$\text{tr}(U) := \delta^{ab} U_{ab} = \delta^{ab} (s) U_{ab}.$$  

where $(s) U_{ab} = \frac{1}{2} (U_{ab} + U_{ba})$. We define the anti-trace of $U$ by,

$$\text{(a)tr}(U) := \epsilon^{ab} U_{ab} = \epsilon^{ab} (a) U_{ab},$$

where $(a) U_{ab} = \frac{1}{2} (U_{ab} - U_{ba})$.

A general horizontal, 2-tensor $U$ can be decomposed according to,

$$U_{ab} = (s) U_{ab} + (a) U_{ab} = \bar{U}_{ab} + \frac{1}{2} \delta_{ab} \text{tr}(U) + \frac{1}{2} \epsilon_{ab} (a) \text{tr}(U).$$

In what follows we fix a null pair $e_3, e_4$ and an orientation on $O(M)$.

**Definition 2.3.** We define the left and right duals of horizontal 1-forms $\omega$ and 2-covariant tensor-fields $U$,

$$\omega = \epsilon_{ab} \bar{\omega}_a = \epsilon_{ab} \bar{\omega}_b, \quad (a) U_{ab} = U_{ac} \epsilon_{cb}.$$  

($^* U)_{ab} = U_{ac} \epsilon_{cb}$,  

$$(U^*)_{ab} = U_{ac} \epsilon_{cb}.$$
See [13] for properties of the duals.

Given $X, Y$ horizontal vectors, the covariant derivative $D_X Y$ fails in general to be horizontal. We thus define,

$$\nabla_X Y := D_X Y - \frac{1}{2}\chi(X, Y)L - \frac{1}{2}\chi(Y, X)L.$$ 

In what follows we define Ricci coefficients, electromagnetic and curvature components of a general spacetime $(\mathcal{M}, g)$.

### 2.1 Ricci coefficients

**Definition 2.4.** We define the horizontal 1-forms,

$$\eta(X) := \frac{1}{2}g(X, D_L L), \quad \eta(X) := \frac{1}{2}g(X, D_L L),$$

$$\xi(X) := \frac{1}{2}g(X, D_L L), \quad \xi(X) := \frac{1}{2}g(X, D_L L),$$

$$\zeta(X) := \frac{1}{2}g(D_X L, L).$$

and the scalars

$$\omega := \frac{1}{4}g(D_L L, L), \quad \omega := \frac{1}{4}g(D_L L, L).$$

**Definition 2.5.** The horizontal tensor-fields $\chi, \chi, \eta, \eta, \zeta, \xi, \omega, \omega$ are called the connection coefficients of the null pair $(L, \underline{L})$. Given an arbitrary basis of horizontal vectorfields $e_1, e_2$, we write using the short hand notation $D_a = D_{e_a}, a = 1, 2$,

$$\chi_{ab} = g(D_a L, e_b), \quad \chi_{ab} = g(D_a L, e_b),$$

$$\xi_a = \frac{1}{2}g(D_L L, e_a), \quad \xi_a = \frac{1}{2}g(D_L L, e_a),$$

$$\omega = \frac{1}{4}g(D_L L, L), \quad \omega = \frac{1}{4}g(D_L L, L),$$

$$\eta_a = \frac{1}{2}(D_L L, e_a), \quad \eta_a = \frac{1}{2}(D_L L, e_a),$$

$$\zeta_a = \frac{1}{2}(D_a L, L).$$

We easily derive the Ricci formulae,

$$D_a e_b = \nabla_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\chi_{ab} e_4,$$

$$D_a e_4 = \chi_{ab} e_b - \zeta_a e_4,$$

$$D_a e_3 = \chi_{ab} e_b + \zeta_a e_3,$$

$$D_a e_a = \nabla_a e_a + \eta_a e_3 + \xi_a e_4,$$

$$D_3 e_3 = -2\omega e_3 + 2\omega e_b,$$

$$D_3 e_4 = 2\omega e_4 + 2\eta e_b,$$

$$D_4 e_a = \nabla_4 e_a + \eta e_4 + \xi_a e_3,$$

$$D_4 e_4 = -2\omega e_4 + 2\xi e_b,$$

$$D_4 e_3 = 2\omega e_3 + 2\eta e_b.$$

**Definition 2.6.** We introduce the notation

$$tr\chi := tr(\chi), \quad (a)tr\chi := (a)tr(\chi), \quad tr\chi := tr(\chi), \quad (a)tr\chi := (a)tr(\chi).$$
\( \hat{\chi}, \text{tr} \chi \) and \( \hat{\chi}, \text{tr} \chi \) are called, respectively, the shear and expansion of the horizontal distribution. The scalars \( \alpha \text{tr} \chi \) and \( \alpha \text{tr} \chi \) measure the integrability defects of the distribution.

In particular we can write

\[
\chi_{ab} = \hat{\chi}_{ab} + \frac{1}{2} \delta_{ab} \text{tr} \chi + \frac{1}{2} \epsilon_{ab} \alpha \text{tr} \chi. \tag{13}
\]

**Definition 2.7.** For a given horizontal 1-form \( \omega \), we define the frame dependent operators,

\[
\begin{align*}
\text{div} \omega &= \delta^{ab} \nabla_b \omega_a, \\
\text{curl} \omega &= \epsilon^{ab} \nabla_a \omega_b, \\
(\nabla \otimes \omega)_{ba} &= \frac{1}{2} (\nabla_b \omega_a + \nabla_a \omega_b - \delta_{ab} (\text{div} \omega)).
\end{align*}
\]

### 2.2 The electromagnetic components

Assume that \( F \) is a two form on \( M \). We define the null components of the two form \( F \) as the horizontal vectors \( \beta(F), \alpha(F) \) by the formulas

\[
\begin{align*}
\alpha(F)(X,Y) &= W(L,X,L,Y), \\
\alpha(F)(X,Y) &= W(L,X,L,Y), \\
\beta(F)(X,Y) &= \frac{1}{2} F(L,L,X,Y), \\
\beta(F)(X,Y) &= \frac{1}{2} F(L,L,X,Y), \\
\rho(F)(X,Y) &= \frac{1}{2} F(L,L,X,Y).
\end{align*}
\]

If \( F \) is a two form, then \( *F \) denotes the Hodge dual on \((M,g)\) of \( F \), defined by \( *F = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} \).

It is convenient to express in terms of the following two scalar quantities

\[
\begin{align*}
\rho(F) = \rho(F) &= \frac{1}{4} F(L,L,X,Y), \\
\rho(F) = \rho(F) &= \frac{1}{4} F(L,L,X,Y).
\end{align*}
\]

Thus, \( \beta(F)(X,Y) = -\epsilon_{ab} * \rho(F) \in (X,Y) \), \( \forall X,Y \in O(M) \).

i.e. \( F_{ab} = -\epsilon_{ab} * \rho \).

### 2.3 The curvature components

Assume that \( W \in T^0_4(M) \) is a Weyl field, i.e.

\[
\begin{align*}
W_{\alpha\beta\mu\nu} &= -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu} = W_{\mu\nu\alpha\beta}, \\
W_{\alpha\beta\mu\nu} &= W_{\alpha\beta\mu\nu} + W_{\alpha\beta\nu\mu} + W_{\nu\alpha\beta\mu} = 0, \\
g^{\alpha\beta} W_{\alpha\beta\mu\nu} &= 0.
\end{align*}
\]

We define the null components of the Weyl field \( W \), horizontal 2-tensors \( \alpha(W), \alpha(W) \), \( \rho(W) \) and horizontal 1-tensors \( \beta(W), \beta(W) \) by the formulas

\[
\begin{align*}
\alpha(W)(X,Y) &= W(L,X,L,Y), \\
\alpha(W)(X,Y) &= W(L,X,L,Y), \\
\beta(W)(X) &= \frac{1}{2} W(X,L,L,L), \\
\beta(W)(X) &= \frac{1}{2} W(X,L,L,L), \\
\rho(W)(X) &= W(X,L,L,L), \\
\rho(W)(X) &= W(X,L,L,L).
\end{align*}
\]

Recall that if \( W \) is a Weyl field its Hodge dual \( *W \), defined by \( *W_{\alpha\beta\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} W_{\alpha\beta\rho\sigma} \), is also a Weyl field. It is convenient to express it in terms of the following two scalar quantities,

\[
\rho(W) = \frac{1}{4} W(L,L,L,L), \quad *\rho(W) = \frac{1}{4} *W(L,L,L,L). \tag{15}
\]
Thus,
\[ q(X,Y) = -\rho \gamma(X,Y) + ^*\rho \in (X,Y), \quad \forall X,Y \in \mathbf{O}(\mathcal{M}). \]

We have
\[
\begin{align*}
W_{a3b4} &= g_{ab} = (-\rho \delta_{ab} + ^*\rho \epsilon_{ab}), \\
W_{ab34} &= 2 \epsilon_{ab} ^*\rho, \\
W_{abcd} &= - \epsilon_{ab} \epsilon_{cd} \rho, \\
W_{ab3} &= \epsilon_{ab} \beta_c, \\
W_{abc4} &= - \epsilon_{ab} \beta_c.
\end{align*}
\]

3 The Einstein-Maxwell equations

In this section we derive the Einstein-Maxwell equations in full generality, for a spacetime with a non-integrable null frame.

3.1 The Maxwell equations

We say that the 2-form \( F \) satisfies the Maxwell equations if
\[
D_{[\mu} F_{\nu\lambda]} = 0, \quad D^{\mu} F_{\mu\nu} = 0. \tag{16}
\]

We derive the Maxwell equations by paying particular attention to the symmetric and antisymmetric part of \( \chi \) and \( \chi \).

The equation \( D_{[\mu} F_{\nu\lambda]} = 0 \) gives three independent equations. The first one is obtained in the following way, using \( [12] \):
\[
0 = D_a F_{34} + D_3 F_{4a} + D_4 F_{3a} = \nabla_a (F_{34}) - F(\chi_{ab} e_b + \zeta_a e_3, e_4) - \nabla_b (F_{34}) - F(e_3, -\zeta_a e_4 + \chi_{ab} e_b) \\
+ \nabla_3 (F_{4a}) - F(2\omega a e_4 + 2\eta_b e_b, e_a) - F(e_a, \eta_a e_3 + \xi_a e_4) \\
+ \nabla_4 (F_{3a}) - F(2\omega_a e_4 + \chi_{ab} e_b, e_a) - F(e_a, 2\omega_a e_3 + 2\eta_a e_b) \\
= 2\nabla_a (^{(F)}p) - \chi_{ab} ^{(F)}\beta_b + \chi_{ab} ^{(F)}\beta_b - \nabla_3 (^{(F)}\beta_a) + 2\omega a ^{(F)}\beta_a - 2 \epsilon_{ab} \eta_b ^{(F)}p + 2 \eta_a ^{(F)}p \\
+ \nabla_4 ^{(F)}\beta_a + 2\eta_a ^{(F)}p - 2\omega ^{(F)}\beta_a + 2 \epsilon_{ab} \eta_b ^{(F)}p
\]

By writing
\[
\chi_{ab} ^{(F)}\beta_b = \left( \hat{\chi}_{ab} + \frac{1}{2} \delta_{ab} \text{tr} \chi + \frac{1}{2} \epsilon_{ab} ^{(a)} \text{tr} \chi \right) ^{(F)}\beta_a = \hat{\chi}_{ab} ^{(F)}\beta_a + \frac{1}{2} \text{tr} \chi ^{(F)}\beta_a + \frac{1}{2} (a) \text{tr} \chi ^{(F)}\beta_a
\]

and using the definition of Hodge duals, we obtain
\[
\nabla_3 ^{(F)}\beta_a - \nabla_4 ^{(F)}\beta_a = - \frac{1}{2} \text{tr} \chi ^{(F)}\beta_a + 2\omega a ^{(F)}\beta_a - \frac{1}{2} (a) \text{tr} \chi ^{(F)}\beta_a \\
+ \frac{1}{2} \text{tr} \chi ^{(F)}\beta_a - 2\omega ^{(F)}\beta_a + \frac{1}{2} (a) \text{tr} \chi ^{(F)}\beta_a \\
+ 2\nabla_a (^{(F)}p) + 2 \left( \eta_a + \eta_a \right) ^{(F)}p + 2 \left( ^*\eta_a - ^*\eta_a \right) ^{(F)}p \\
- \hat{\chi}_{ab} ^{(F)}\beta_b + \hat{\chi}_{ab} ^{(F)}\beta_a
\tag{17}
\]
The second equation is obtained in the following way:

\[
0 = \mathbf{D}_a F_{b3} + \mathbf{D}_b F_{3a} + \mathbf{D}_3 F_{ab}
\]

\[
= \nabla_a (F_{b3}) - F(\frac{1}{2} \chi_{ab} e_3 + \frac{1}{2} \chi_{ba} e_4, e_3) - F(e_b, \chi_{ac} e_c + \zeta_a e_3)
\]

\[
+ \nabla_b (F_{3a}) - F(\chi_{bc} e_c + \zeta_b e_3, e_a) - F(e_3, \frac{1}{2} \chi_{ba} e_3 + \frac{1}{2} \chi_{ab} e_4)
\]

\[
+ \nabla_3 (F_{ab}) - F(\eta_a e_3 + \xi_4 e_4, e_b) - F(e_a, \eta_b e_3 + \xi_4 e_4)
\]

\[
= \nabla_a (F_{b3}) - \nabla_b (F_{3a}) + \left(\chi_{ab} - \chi_{ba}\right) (F)^p + \left(\chi_{bc} \epsilon_{bc} \right) (F)^p - \chi_{ac} \epsilon_{a c} (F)^p + \chi_{bc} \epsilon_{ca} (F)^p
\]

Contracting the above with \(e^a\) and recalling that \(\text{curl} (F)^p = \epsilon^{ab} \nabla_a (F)^p\), we have

\[
\nabla^* (F)^p = \nabla^* (F)^p - \nabla_3 (F)^p = - \left(\text{tr} \chi (F)^p - (a) \text{tr} (F)^p\right) + (\eta - \zeta) \cdot (F)^p
\]

The third equation is obtained from symmetrization of the above:

\[
\nabla_4 (F)^p = \nabla^* (F)^p - \nabla_3 (F)^p = - \left(\text{tr} \chi (F)^p + (a) \text{tr} (F)^p\right) + (\eta + \zeta) \cdot (F)^p
\]

The equation \(\mathbf{D}^\mu F_{\mu
u} = \delta_{ab} \mathbf{D}_b F_{ca} - \frac{1}{2} \mathbf{D}_4 F_{3a} - \frac{1}{2} \mathbf{D}_3 F_{4a} = 0\) gives three additional independent equations. The first one is obtained in the following way:

\[
0 = \delta_{bc} \mathbf{D}_c F_{ca} - \frac{1}{2} \mathbf{D}_4 F_{3a} - \frac{1}{2} \mathbf{D}_3 F_{4a}
\]

\[
= \delta_{bc} \left(\epsilon_{ac} \nabla_b (F)^p - F(\frac{1}{2} \chi_{bc} e_4 + \frac{1}{2} \chi_{bc} e_4, e_a) - F(e_c, \frac{1}{2} \chi_{bc} e_4 + \frac{1}{2} \chi_{bc} e_4)\right)
\]

\[
+ \frac{1}{2} \nabla_4 (F)^p + \frac{1}{2} F(2 \omega e_3 + 2 \eta e_c, e_a) + \frac{1}{2} F(e_3, \eta e_4 + \chi e_3)
\]

\[
+ \frac{1}{2} \nabla_3 (F)^p + \frac{1}{2} F(2 \omega e_4 + 2 \eta e_3, e_a) + \frac{1}{2} F(e_4, \eta e_3 + \chi e_4)
\]

\[
= \epsilon_{ac} \nabla_c (F)^p + \frac{1}{2} \text{tr} \chi (F)^p + \frac{1}{2} \text{tr} \chi (F)^p - \frac{1}{2} \chi_{ca} (F)^p - \frac{1}{2} \chi_{ca} (F)^p
\]

By writing

\[
\chi_{ca} (F)^p = \left(\chi_{ca} + \frac{1}{2} \delta_{ca} \text{tr} \chi + \frac{1}{2} \epsilon_{ca} (a) \text{tr} (F)^p\right)
\]

we obtain

\[
\nabla_4 (F)^p + \nabla_3 (F)^p = - \frac{1}{2} \text{tr} \chi (F)^p - \frac{1}{2} (a) \text{tr} (F)^p + 2 \omega (F)^p
\]

\[
- \frac{1}{2} \text{tr} \chi (F)^p - \frac{1}{2} (a) \text{tr} (F)^p + 2 \omega (F)^p
\]

\[
- 2 \epsilon_{ac} \nabla_c (F)^p - 2 \left(\eta_{a c} + \eta_a\right) (F)^p + 2 \left(\eta_{a c} - \eta_a\right) (F)^p
\]

\[
+ \chi_{ca} (F)^p + \chi_{ca} (F)^p
\]

Summing and subtracting \(17\) and \(18\) we obtain

\[
\nabla_4 (F)^p + \nabla_3 (F)^p + \nabla_3 (F)^p + \nabla_4 (F)^p = - \frac{1}{2} \left(\text{tr} \chi (F)^p + (a) \text{tr} (F)^p\right) + 2 \omega (F)^p
\]

\[
+ 2 \left(\eta (F)^p - \eta (F)^p\right) + \chi_{ca} (F)^p
\]

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and

\[ \nabla_4 (F) \beta + \nabla (F) \rho + \ast \nabla \ast (F) \rho = - \frac{1}{2} \left( \text{tr} \chi (F) \beta + (a) \text{tr} \chi \ast (F) \beta \right) + 2 \omega (F) \beta \\
\quad - 2 \left( \eta (F) \rho + \ast \eta \ast (F) \rho \right) + \chi \ast (F) \beta \]

The last equation is obtained by

\[ 0 = \delta_{bc} D_b F_{c4} - \frac{1}{2} D_4 F_{34} \]

\[ = \delta_{bc} \left( \nabla_b (F) \beta c - F \left( \frac{1}{2} \delta_{ba} e_4 + \frac{1}{2} \delta_{ba} e_3, e_4 \right) - F(e_c, -\zeta_0 e_4 + \chi_{ba} e_a) \right) \]

\[ - \frac{1}{2} \left( 2 \nabla_4 (F) \rho - F(2 \omega e_3 + 2 \rho, e_4, e_4) - F(e_3, -2 \omega e_4 + 2 \xi_a e_a) \right) \]

\[ = \text{div} (F) \beta - \text{tr} \chi (F) \rho + \zeta \ast (F) \beta + (a) \text{tr} \chi \ast (F) \rho - \nabla_4 (F) \rho + \eta \ast (F) \beta - \xi \ast (F) \beta \]

which gives

\[ \nabla_4 (F) \rho - \text{div} (F) \beta = - \left( \text{tr} \chi (F) \rho - (a) \text{tr} \chi \ast (F) \rho \right) + (\zeta + \eta) \ast (F) \beta - (\xi + \eta) \ast (F) \beta \]

\[ \nabla_3 (F) \rho + \text{div} (F) \beta = - \left( \text{tr} \chi (F) \rho + (a) \text{tr} \chi \ast (F) \rho \right) + (\zeta - \eta) \ast (F) \beta + (\xi - \eta) \ast (F) \beta \]

(19)

We summarize the Maxwell equations in the following Proposition.

**Proposition 3.1.** We have

\[ \nabla_3 (F) \beta - \nabla (F) \rho + \ast \nabla \ast (F) \rho = - \frac{1}{2} \left( \text{tr} \chi (F) \beta + (a) \text{tr} \chi \ast (F) \beta \right) + 2 \omega (F) \beta \\
\quad + 2 \left( \eta (F) \rho + \ast \eta \ast (F) \rho \right) \]

\[ \nabla_4 (F) \rho - \text{div} (F) \beta = - \left( \text{tr} \chi (F) \rho - (a) \text{tr} \chi \ast (F) \rho \right) + (\zeta + \eta) \ast (F) \beta - (\xi + \eta) \ast (F) \beta \]

\[ \nabla_4 \ast (F) \rho - \text{curl} (F) \beta = - \left( \text{tr} \chi \ast (F) \rho + (a) \text{tr} \chi \ast (F) \rho \right) + (\eta + \zeta) \ast (F) \beta + (\xi + \eta) \ast (F) \beta \]

and the others are obtained by symmetrization.

### 3.2 The null structure equations

We have the following null structure equations [13]:

\[ \nabla_3 \chi_{ba} = 2 \nabla_3 b \xi_a - 2 \omega \chi_{ba} - \chi_{bc} \chi_{ca} + 2 \left( -2 \zeta_0 \xi_a + \eta \xi_0 + \eta \xi_0 \right) + R_{b34a}, \]

\[ \nabla_3 \chi_{ba} = 2 \nabla_3 b \eta_a + 2 \omega \chi_{ba} - \chi_{bc} \chi_{ca} + 2 \left( \xi_0 \xi_a + \eta \eta_a + \eta \eta_a \right) + R_{a43b}, \]

\[ \nabla_4 \chi_{ba} = 2 \nabla_4 b \xi_a + 2 \omega \chi_{ba} - \chi_{bc} \chi_{ca} + 2 \left( \xi_0 \xi_a + \eta \eta_a \right) + R_{a34b}, \]

\[ \nabla_4 \chi_{ba} = 2 \nabla_4 b \xi_a + 2 \omega \chi_{ba} - \chi_{bc} \chi_{ca} + 2 \left( \xi_0 \xi_a + \eta \eta_a + \eta e_0 \right) + R_{b444a}. \]

Also,

\[ \nabla_3 \zeta_a + 2 \nabla_3 \omega_a = - \chi_{ba} \left( \zeta_0 + \eta_0 \right) + 2 \omega (\zeta - \eta) a + \chi_{ab} \zeta_a + 2 \omega \zeta_a - \frac{1}{2} R_{a344}. \]

\[ \nabla_4 \zeta_a - 2 \nabla_4 \omega_a = \chi_{ab} (-\zeta_0 + \eta_0) + 2 \omega (\zeta + \eta) a - \chi_{ba} \zeta_a - 2 \omega \zeta_a + \frac{1}{2} R_{a434}, \]

\[ \nabla_3 \omega_a - \nabla_4 \xi_a = - \chi_{ba} (\eta - \eta_0) b - 4 \omega \xi_a + \frac{1}{2} R_{a344}, \]

\[ \nabla_4 \eta_a - \nabla_3 \zeta_a = - \chi_{ba} (\eta - \eta_0) b - 4 \omega \xi_a + \frac{1}{2} R_{a444}, \]
The Ricci tensor of \((\mathcal{M}, g)\) can be expressed in terms of the electromagnetic null decomposition according to Einstein equation [20]. Observe that

\[
F^{\alpha\beta} F_{\alpha\beta} = -\frac{1}{2} (F_{34})^2 - 2 \delta_{cd} F_{d4} F_{c3} + \delta_{cd}\delta_{ab} F_{db} F_{ca} \\
= -\frac{1}{2} (F_{a3} F_{b4} + F_{b3} F_{a4}) + \delta_{cd}\delta_{ab} (F_{d3} F_{b4} - F_{b3} F_{d4}) = -\frac{1}{2} (F_{ab} F_{cd} + F_{cd} F_{ab}) - \frac{1}{2} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}
\]

We compute the following components of the Ricci tensor:

\[
R_{a3} = 2 F_{a\lambda} F_{3\lambda} - \frac{1}{2} \delta_{ab} F^{\alpha\beta} F_{\alpha\beta} \\
= 2 F_{a3} F_{3\lambda} - \frac{1}{2} \delta_{ab} F^{\alpha\beta} F_{\alpha\beta} = 2 F_{a3} F_{3\lambda} - \frac{1}{2} \delta_{ab} F^{\alpha\beta} F_{\alpha\beta}
\]

\[
R_{a4} = 2 * (F_{b\lambda})^2 + 2 * (F_{b\lambda})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2 = 2 * (F_{b\lambda})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2
\]

\[
R_{33} = 2 g^{\lambda\mu} F_{3\lambda} F_{\mu\nu} = 2 \delta_{ab} F_{3a} F_{3b} = 2 F_{a3} F_{3b} = 2 (F_{a3} F_{3b}) = 2 (F_{a3} F_{3b})
\]

\[
R_{44} = 2 (F_{b\lambda})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2 = 2 (F_{b\lambda})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2
\]

We also have

\[
R_{34} = 2 F_{3\lambda} F_{4\lambda} - \frac{1}{2} \delta_{ab} F^{\alpha\beta} F_{\alpha\beta} \\
= (F_{34})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2 = (F_{34})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2
\]

and

\[
R_{ab} = 2 F_{a\lambda} F_{b\lambda} - \frac{1}{2} \delta_{ab} F^{\alpha\beta} F_{\alpha\beta} \\
= - F_{a3} F_{b4} - F_{a4} F_{b3} + 2 \delta_{cd} F_{ac} F_{bd} - \frac{1}{2} \delta_{ab} (F_{b\lambda})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2 = - \frac{1}{2} \delta_{ab} (F_{b\lambda})^2 + \frac{1}{2} \delta_{ab} (F_{b\lambda})^2
\]

Using the decomposition of the Riemann curvature in Weyl curvature and Ricci tensor:

\[
R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{2} (g_{\beta\delta} R_{\alpha\gamma} + g_{\alpha\gamma} R_{\beta\delta} - g_{\beta\gamma} R_{\alpha\delta} - g_{\alpha\delta} R_{\beta\gamma}),
\]

Note that this equation follows from the \(R_{34ab}\).
we can express the full Riemann tensor of \((M, g)\) in terms of the above decompositions. We compute the following components of the Riemann tensor.

\[
\begin{align*}
R_{a33b} &= W_{a33b} - \frac{1}{2} \delta_{ab} R_{33} = -\omega_{ab} - \left( (F)_{ab} \cdot (F)_{ab} \right) \delta_{ab}, \\
R_{a34b} &= W_{a34b} + R_{ab} - \frac{1}{2} \delta_{ab} R_{34} = \rho \delta_{ab} - \left( (F) \cdot (F) \right)_{ab}, \\
R_{a334} &= W_{a334} - R_{a3} = 2 \beta_{a} - 2 \left( (F) \cdot (F) \right)_{a} + 2 \left( (F) \cdot (F) \right)_{a}, \\
R_{3434} &= W_{3434} + 2 R_{34} = 4 \rho + 4 \left( (F) \cdot (F) \right), \\
R_{a3c6} &= W_{a3c6} + \frac{1}{2} \left( \delta_{ac} R_{3b} - \delta_{ab} R_{3c} \right) \\
&= \epsilon_{cb} \left( (F) \cdot (F) \right) - \left( (F) \cdot (F) \right)_{b} - \delta_{ab} \left( (F) \cdot (F) \right)_{a} - \left( (F) \cdot (F) \right)_{b}.
\end{align*}
\]

The previous proposition immediately implies the following.

**Proposition 3.2.**

\[
\begin{align*}
\nabla_3 \text{tr}_\chi &= -|\chi|^2 - \frac{1}{2} \left( \text{tr}_\chi^2 - (\text{tr}_\chi)^2 \right) + 2 \text{div} \xi + 2 \omega \text{tr}_\chi + 2 \xi \cdot (\eta + \eta + 2 \xi) - 2 \left( (F) \cdot (F) \right), \\
\nabla_3 (\text{tr}_\chi) &= -\text{tr}_\chi (\text{tr}_\chi + 2 \text{curl} \xi - 2 \omega (\text{tr}_\chi + 2 \xi \wedge (\eta + \eta + 2 \xi)), \\
\nabla_3 \hat{\chi} &= -\text{tr}_\chi \hat{\chi} + 2 \nabla \hat{\xi} - 2 \omega \hat{\chi} + 2 \xi \wedge (\eta + \eta + 2 \xi), \\
\nabla_4 \text{tr}_\chi &= -\hat{\chi} \cdot \hat{\chi} - \frac{1}{2} \left( \text{tr}_\chi^2 - (\text{tr}_\chi)^2 \right) + 2 \text{div} \eta + 2 \omega \text{tr}_\chi + 2 (\xi \cdot (\eta + \eta + 2 \xi) + 2 \rho, \\
\nabla_4 (\text{tr}_\chi) &= -\hat{\chi} \wedge \hat{\chi} - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega (\text{tr}_\chi + 2 \xi \wedge (\eta + \eta + 2 \xi) - 2 \hat{\chi}, \\
\nabla_4 \hat{\chi} &= -\frac{1}{2} \left( \text{tr}_\chi \hat{\chi} + \text{tr}_\chi \hat{\chi} \right) - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega \hat{\chi} + 2 \xi \wedge \eta - 2 \left( (F) \cdot (F) \right) + 2 \rho, \\
\nabla_4 (\text{tr}_\chi) &= -\hat{\chi} \cdot \hat{\chi} - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega (\text{tr}_\chi + 2 \xi \wedge (\eta + \eta + 2 \xi) + 2 \rho, \\
\nabla_4 (\text{tr}_\chi) &= -\hat{\chi} \wedge \hat{\chi} - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega (\text{tr}_\chi + 2 \xi \wedge (\eta + \eta + 2 \xi) + 2 \rho, \\
\nabla_4 \hat{\chi} &= -\frac{1}{2} \left( \text{tr}_\chi \hat{\chi} + \text{tr}_\chi \hat{\chi} \right) - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega \hat{\chi} + 2 \xi \wedge \eta - 2 \left( (F) \cdot (F) \right) + 2 \rho, \\
\nabla_4 (\text{tr}_\chi) &= -\hat{\chi} \cdot \hat{\chi} - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega (\text{tr}_\chi + 2 \xi \wedge (\eta + \eta + 2 \xi) + 2 \rho, \\
\nabla_4 (\text{tr}_\chi) &= -\hat{\chi} \wedge \hat{\chi} - \frac{1}{2} \left( \text{tr}_\chi \text{tr}_\chi + 2 \text{curl} \eta + 2 \omega (\text{tr}_\chi + 2 \xi \wedge (\eta + \eta + 2 \xi) + 2 \rho.
\end{align*}
\]
Also,
\[
\nabla_3 \zeta + 2 \nabla \omega = -\hat{\chi} \cdot (\zeta + \eta) - \frac{1}{2} \text{tr} \chi (\zeta + \eta) - \frac{1}{2} (a) \text{tr} \chi (\ast \zeta + \ast \eta) + 2 \omega (\zeta - \eta) \\
+ \hat{\chi} \cdot \xi + \frac{1}{2} \text{tr} \chi \xi + \frac{1}{2} (a) \text{tr} \chi \ast \xi + 2 \omega \xi - \beta + (F) \rho \ast (F) \beta - (F) \rho (F) \beta.
\]
\[
\nabla_4 \zeta - 2 \nabla \omega = \hat{\chi} \cdot (-\zeta + \eta) + \frac{1}{2} \text{tr} \chi (-\zeta + \eta) + \frac{1}{2} (a) \text{tr} \chi (-\ast \zeta + \ast \eta) + 2 \omega (\zeta + \eta) \\
- \hat{\chi} \cdot \xi - \frac{1}{2} \text{tr} \chi \xi - \frac{1}{2} (a) \text{tr} \chi \ast \xi - 2 \omega \xi - \beta - (F) \rho \ast (F) \beta - (F) \rho (F) \beta.
\]
\[
\nabla_3 \eta - \nabla 4 \xi = -\hat{\chi} \cdot (\eta - \eta) - \frac{1}{2} \text{tr} \chi (\eta - \eta) + \frac{1}{2} (a) \text{tr} \chi (\ast \eta - \ast \eta) - 4 \omega \xi + \beta - (F) \rho \ast (F) \beta + (F) \rho (F) \beta.
\]
\[
\nabla_4 \eta - \nabla 3 \xi = -\hat{\chi} \cdot (\eta - \eta) - \frac{1}{2} \text{tr} \chi (\eta - \eta) + \frac{1}{2} (a) \text{tr} \chi (\ast \eta - \ast \eta) - 4 \omega \xi - \beta - (F) \rho \ast (F) \beta - (F) \rho (F) \beta.
\]

and
\[
\nabla_3 \omega + \nabla_4 \omega - 4 \omega - \xi \cdot (\eta - \eta) \cdot \zeta + \eta \cdot \eta = \rho + (F) \rho^2 + \ast (F) \rho^2.
\]

Also,
\[
\text{div} \ \hat{\chi} + \zeta = \frac{1}{2} \nabla \text{tr} \chi + \frac{1}{2} \text{tr} \chi \zeta - \frac{1}{2} \nabla (a) \text{tr} \chi - \frac{1}{2} (a) \text{tr} \chi \ast \zeta - (a) \text{tr} \chi \ast \eta - (a) \text{tr} \chi \ast \xi \\
- \beta + (F) \rho \ast (F) \beta + (F) \rho (F) \beta,
\]
\[
\text{div} \ \hat{\chi} - \zeta = \frac{1}{2} \nabla \text{tr} \chi - \frac{1}{2} \text{tr} \chi \zeta - \frac{1}{2} \nabla (a) \text{tr} \chi + \frac{1}{2} (a) \text{tr} \chi \ast \zeta - (a) \text{tr} \chi \ast \eta - (a) \text{tr} \chi \ast \xi \\
+ \beta + (F) \rho \ast (F) \beta + (F) \rho (F) \beta.
\]

### 3.2.1 The Gauss equation

In the general framework of a spacetime with non-integrable horizontal structure, there is no surface orthogonal to the null frames for which the Gauss curvature can be computed. Nevertheless it is possible to define the curvature of the horizontal structure and to relate it in terms of the spacetime curvature. Such relation gives new Gauss equations, which we summarize in the following Lemma. See also [13].

**Proposition 3.3.** We have

1. for a 1-tensor $\psi$:
   \[
   \left( \nabla_a \nabla_b - \nabla_b \nabla_a \right) \psi_c = \frac{1}{2} \epsilon_{ab} (a) \text{tr} \chi \nabla_3 + (a) \text{tr} \chi \nabla_4) \psi_c - \frac{1}{2} E_{cdab} \psi_d \\
   + \left( - \epsilon_{cd} \epsilon_{ab} \rho + (\delta_{db} \epsilon_{ca} - \delta_{da} \delta_{cb}) \right) (F) \rho^2 + \ast (F) \rho^2 \right) \psi_d + N \tag{22}
   \]

2. for a 2-tensor $\Psi$:
   \[
   \left( \nabla_a \nabla_b - \nabla_b \nabla_a \right) \Psi_{st} = \frac{1}{2} \epsilon_{ab} (a) \text{tr} \chi \nabla_3 + (a) \text{tr} \chi \nabla_4) \Psi_{st} - \frac{1}{2} E_{sdab} \Psi_{dt} - \frac{1}{2} E_{cdab} \Psi_{sd} \\
   + \left( - \epsilon_{sd} \epsilon_{ab} \rho + (\delta_{db} \epsilon_{ca} - \delta_{da} \delta_{cb}) \right) (F) \rho^2 + \ast (F) \rho^2 \right) \Psi_{dt} + N \\
   + \left( - \epsilon_{td} \epsilon_{ab} \rho + (\delta_{db} \epsilon_{ca} - \delta_{da} \delta_{cb}) \right) (F) \rho^2 + \ast (F) \rho^2 \right) \Psi_{sd} + N
   \]

where
\[
E_{cdab} = \frac{1}{2} \text{tr} \chi \text{tr} \chi (\delta_{ac} \delta_{bd} - \delta_{bc} \delta_{ad}) + \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi (\epsilon_{ac} \epsilon_{bd} - \epsilon_{bc} \epsilon_{ad}) \\
+ \frac{1}{4} (\delta_{ac} \epsilon_{bd} + \epsilon_{ac} \delta_{bd} - \delta_{bc} \epsilon_{ad} - \epsilon_{bc} \delta_{ad}) \text{tr} \chi (a) \text{tr} \chi \\
+ \frac{1}{4} (\epsilon_{ac} \delta_{bd} + \epsilon_{ac} \delta_{bd} - \epsilon_{bc} \delta_{ad} - \delta_{bc} \epsilon_{ad}) (a) \text{tr} \chi \text{tr} \chi + N \tag{23}
\]
and $N$ denotes non-linear terms in the context of linear perturbations of Kerr-Newman spacetime.

Proof. See Appendix A.1.

### 3.3 The Bianchi identities

The Bianchi identities for the Weyl curvature are given by

\[
D^a W_{a\beta\gamma\delta} = \frac{1}{2} (D_\gamma R_{\beta\delta} - D_\delta R_{\beta\gamma}) =: J_{\beta\delta}
\]

\[
D_\gamma W_{\alpha\beta\gamma\delta} = g_{\beta\delta} J_{\alpha\gamma\sigma} + g_{\gamma\sigma} J_{\beta\delta\gamma} + g_{\sigma\gamma} J_{\alpha\delta\gamma} + g_{\delta\alpha} J_{\beta\gamma\sigma} + g_{\sigma\alpha} J_{\beta\gamma\delta} := \tilde{J}_{\gamma\delta\alpha\beta}
\]

We have the following Bianchi identities [12].

**Proposition 3.4.** We have,

\[
\nabla_\beta \alpha - 2 \nabla_\beta \beta = -\frac{1}{2} \left( \text{tr} \, \chi \alpha + (a) \text{tr} \, \chi \ast \alpha \right) + 4 \omega \alpha + 2 (\zeta + 4 \eta) \hat{\otimes} \beta - 3 (\rho \hat{\chi} + \ast \rho \ast \hat{\chi}) + a,
\]

\[
\nabla_\alpha + 2 \nabla_\beta \beta = -\frac{1}{2} \left( \text{tr} \, \chi \alpha - (a) \text{tr} \, \chi \ast \alpha \right) + 4 \omega \alpha + 2 (\zeta - 4 \eta) \hat{\otimes} \beta - 3 (\rho \hat{\chi} + \ast \rho \ast \hat{\chi}) + a,
\]

where

\[
a_{ab} = J_{ba4} + J_{ab4} - \frac{1}{2} \delta_{ab} J_{434}, \quad a_{ab} = J_{ba3} + J_{ab3} - \frac{1}{2} \delta_{ab} J_{343}
\]

We also have

\[
\nabla_\beta \beta - \text{div} \, \alpha = -2 (\text{tr} \, \chi \beta - (a) \text{tr} \, \chi \ast \beta) - 2 \omega \beta + \alpha \cdot (2 \zeta + \eta) + \alpha \cdot (2 \zeta - \eta) - 3 (\rho \hat{\chi} + \ast \rho \ast \hat{\chi}) + 2 a,
\]

\[
\nabla_\beta + \text{div} \, \alpha = -2 (\text{tr} \, \chi \beta - (a) \text{tr} \, \chi \ast \beta) - 2 \omega \beta - \alpha \cdot (-2 \zeta + \eta) - 3 (\rho \hat{\chi} + \ast \rho \ast \hat{\chi}) + 3 a,
\]

\[
\nabla_\beta + \text{div} \, \rho = -(\text{tr} \, \chi \beta + (a) \text{tr} \, \chi \ast \beta) + 2 \omega \beta + 2 \beta \cdot \hat{\chi} + 3 (\rho \eta + \ast \rho \ast \eta) + \alpha \cdot \xi + J_{3a3},
\]

\[
\nabla_\beta - \text{div} \, \hat{\rho} = -(\text{tr} \, \chi \beta + (a) \text{tr} \, \chi \ast \beta) + 2 \omega \beta + 2 \beta \cdot \hat{\chi} - 3 (\rho \eta + \ast \rho \ast \eta) - \alpha \cdot \xi - J_{4a3}
\]

where

\[
\text{div} \, \rho = - (\nabla \rho + \ast \nabla \ast \rho), \quad \text{div} \, \hat{\rho} = - (\nabla \rho - \ast \nabla \ast \rho).
\]

We also have

\[
\nabla_\beta \rho - \text{div} \, \beta = -\frac{3}{2} (\text{tr} \, \chi \rho + (a) \text{tr} \, \chi \ast \rho) + (2 \eta + \zeta) \cdot \beta - 2 \xi \cdot \beta - \frac{1}{2} \hat{\chi} \cdot \beta - 3 a
\]

\[
\nabla_\beta \ast \rho + \text{curl} \, \beta = -\frac{3}{2} (\text{tr} \, \chi \ast \rho - (a) \text{tr} \, \chi \ast \rho) - (2 \eta + \zeta) \cdot \ast \beta + 2 \xi \cdot \ast \beta + \frac{1}{2} \hat{\chi} \cdot \ast \beta - 3 a
\]

\[
\nabla_\beta \ast \rho + \text{curl} \, \beta = -\frac{3}{2} (\text{tr} \, \chi \ast \rho - (a) \text{tr} \, \chi \ast \rho) - (2 \eta - \zeta) \cdot \ast \beta + 2 \xi \cdot \ast \beta - \frac{1}{2} \hat{\chi} \cdot \ast \beta - 3 a
\]

We have the following formulas for the $J$s on the right hand side of the equations.

**Lemma 3.5.** The following formulas hold:

\[
a = 2 \nabla \otimes (\ast (F) p \ast (F) \beta + (F) p (F) \beta) + 2 (\zeta + 2 \eta) \hat{\otimes} (\ast (F) p \ast (F) \beta + (F) p (F) \beta)
\]

\[
- 2 (\ast (F) p^2 + \ast (F) \rho^2) \hat{\chi} + N
\]

\[
J_{4a4} = - \nabla_4 (\ast (F) p \ast (F) \beta_a + (F) p (F) \beta_a) - 2 \omega (\ast (F) p \ast (F) \beta_a + (F) p (F) \beta_a)
\]

\[
- \text{tr} \chi (\ast (F) p \ast (F) \beta_a + (F) p (F) \beta_a) + (a) \text{tr} \chi (\ast (F) p (F) \beta_a - (F) p (F) \beta_a)
\]

\[
+ 2 \left( (F) p^2 + \ast (F) \rho^2 \right) \xi_a + N
\]
We define the following complexified versions of the curvature components

\[ J_{34} = \nabla_a (F) \rho^2 + * (F) \rho^2 \]
\[ -\frac{1}{2} \text{tr} \chi (\ast (F) \rho * (F) \beta) - (F) \rho (F) \beta) - \frac{1}{2} \text{tr} \chi (\ast (F) \rho * (F) \alpha + (F) \rho (F) \alpha) \]
\[ -\frac{1}{2} (a) \text{tr} \chi (\ast (F) \rho * (F) \beta) - (F) \rho (F) \beta) - \frac{1}{2} (a) \text{tr} \chi (\ast (F) \rho * (F) \alpha + (F) \rho (F) \alpha) \]
\[ -\nabla_4 (\ast (F) \rho * (F) \beta) - (F) \rho (F) \beta) + 2 \omega (\ast (F) \rho * (F) \beta) - (F) \rho (F) \beta) \]
\[ + 2 (\ast (F) \rho^2 + * (F) \rho^2) \eta + N \]

\[ J_{434} = -2 \text{div} (\ast (F) \rho * (F) \beta + (F) \rho (F) \beta) - 2 (\zeta + 4 \eta) \cdot (\ast (F) \rho (F) \beta + * (F) \rho * (F) \beta) \]
\[ + 2 \text{tr} \chi (\ast (F) \rho^2 + * (F) \rho^2) + N \]
\[ * J_{434} = 2 \text{curl} (\ast (F) \rho * (F) \beta + (F) \rho (F) \beta) + 2 \zeta \cdot (\ast (F) \rho * (F) \beta + (F) \rho (F) \beta) \]
\[ - 2 (a) \text{tr} \chi (\ast (F) \rho^2 + * (F) \rho^2) + N \]

where the \( N \) indicates terms which are quadratic in \( (F) \beta, (F) \beta, \hat{\chi}, \hat{\chi}, \hat{\xi}, \hat{\xi} \). The other quantities are obtained by symmetrization.

**Proof.** See Appendix A.2 \[ \square \]

### 4 Complex notations

We recall here the complex notations introduced in [13]. Their use is motivated by the fact that the equations in the previous section can be simplified through the following complex notations.

**Definition 4.1.** We define the following complexified versions of the curvature components

- \( A := \alpha + i \ast \alpha \), \( B := \beta + i \ast \beta \), \( P := \rho + i \ast \rho \), \( \overline{B} := \overline{\beta} + i \ast \overline{\beta} \), \( \overline{A} := \overline{\alpha} + i \ast \overline{\alpha} \),
- of the electromagnetic components
  - \((F)B := (F)\beta + i \ast (F)\beta\), \((F)P := (F)\rho + i \ast (F)\rho\), \((F)\overline{B} := (F)\overline{\beta} + i \ast (F)\overline{\beta}\)
- and of the Ricci components
  - \(X = \chi + i \ast \chi\), \(\overline{X} = \overline{\chi} + i \ast \overline{\chi}\), \(H = \eta + i \ast \eta\), \(\overline{H} = \overline{\eta} + i \ast \overline{\eta}\), \(Z = \zeta + i \ast \zeta\), \(\overline{Z} = \overline{\zeta} + i \ast \overline{\zeta}\).

In particular, note that

\[ \text{tr} X = \text{tr} \chi - i (a) \text{tr} \chi, \quad \overline{X} = \overline{\chi} + i \ast \overline{\chi}, \quad \text{tr} \overline{X} = \text{tr} \overline{\chi} - i (a) \text{tr} \overline{\chi}, \quad \overline{X} = \overline{\chi} + i \ast \overline{\chi}. \]

We denote by \( S_1(\mathbb{C}) \) the set of complex 1-tensors \( F \) given by \( F = f + i \ast f \) for a real 1-tensor \( f \). Similarly, we denote by \( S_2(\mathbb{C}) \) the set of complex symmetric traceless 2-tensors \( U \) given by \( U = u + i \ast u \) for a real symmetric traceless 2-tensor \( u \). Note that for \( F \in S_1(\mathbb{C}), U \in S_2(\mathbb{C}) \),

\[ \ast F = -i F, \quad \ast U = -i U, \]

i.e., such \( F \) and \( U \) are self-anti dual. In particular \( F_2 = -i F_1 \) and \( U_{12} = -i U_{11}, U_{22} = -U_{11} \).

**Definition 4.2.** We define derivatives of complex quantities as follows
For \((a,b)\) a pair of scalars, we define
\[
\mathcal{D}(a + ib) := (\nabla + i \star \nabla)(a + ib), \quad \mathcal{D}^*(a + ib) := (\nabla - i \star \nabla)(a + ib)
\]

For \(F = f + i \star f \in \mathcal{S}_1(\mathbb{C})\), we define
\[
\mathcal{D} \cdot (f + i \star f) := (\nabla + i \star \nabla) \cdot (f + i \star f) = 0, \\
\mathcal{D} \cdot (f + i \star f) := (\nabla - i \star \nabla) \cdot (f + i \star f), \\
\mathcal{D} \hat{\otimes} (f + i \star f) := (\nabla + i \star \nabla) \hat{\otimes} (f + i \star f).
\]

For \(U = u + i \star u \in \mathcal{S}_2(\mathbb{C})\), we define
\[
\mathcal{D}(u + i \star u) := (\nabla + i \star \nabla)(u + i \star u) = 0, \\
\mathcal{D}(u + i \star u) := (\nabla - i \star \nabla)(u + i \star u).
\]

**Lemma 4.3.** The following holds

- **If \(\xi, \eta\) are 1-tensors**
  \[
  \xi \cdot \eta + i \star \xi \cdot \eta = \frac{1}{2} \left( (\xi + i \star \xi) \cdot (\eta + i \star \eta) \right), \\
  \xi \hat{\otimes} \eta + i \star (\xi \hat{\otimes} \eta) = \frac{1}{2} \left( (\xi + i \star \xi) \hat{\otimes} (\eta + i \star \eta) \right).
  \]

- **If \(\eta\) is a 1-tensor and \(u\) is a 2-tensor**
  \[
  u \cdot \eta + i \star u \cdot \eta = \frac{1}{2} (u + i \star u) \cdot (\eta + i \star \eta), \\
  u \cdot \eta + i \star (u \cdot \eta) = \frac{1}{2} (u + i \star u) \cdot (\eta + i \star \eta).
  \]

- **If \(u, v\) are 2-tensors**
  \[
  u \cdot v + i \star u \cdot v = \frac{1}{2} (u + i \star u) \cdot (v + i \star v).
  \]

- **If \((a, b)\) are scalars**
  \[
  \nabla a - \star \nabla b + i(\star \nabla a + \nabla b) = \mathcal{D}(a + ib).
  \]

- **If \(\xi\) is a 1-tensor**
  \[
  \text{div} \xi + \text{curl} \xi = \frac{1}{2} \mathcal{D} \cdot (\xi + i \star \xi) \\
  \nabla \hat{\otimes} \xi + i \star (\nabla \hat{\otimes} \xi) = \frac{1}{2} \mathcal{D} \hat{\otimes} (\xi + i \star \xi).
  \]

- **If \(u\) is a 2-tensor**
  \[
  \text{div} u + i \star (\text{div} u) = \frac{1}{2} \mathcal{D} \cdot (u + i \star u).
  \]

We also have the following Leibniz rules.
Lemma 4.4. Let $E, F, G \in \mathcal{S}_1(\mathbb{C})$ and $U \in \mathcal{S}_2(\mathbb{C})$, then the following holds.

$$F(\overline{E} \cdot G) = (F \cdot \overline{E})G$$  \hspace{1cm} (29)
$$\overline{G} \cdot (E \otimes F) = (\overline{G} \cdot E)F = (\overline{G} \cdot F)E$$  \hspace{1cm} (30)
$$E \otimes (\overline{F} \cdot U) + F \otimes (\overline{E} \cdot U) = (E \cdot \overline{F} + \overline{E} \cdot F)U$$  \hspace{1cm} (31)

$$F(\overline{D} \cdot G) = F \cdot \overline{D}G$$  \hspace{1cm} (32)
$$\overline{D} \cdot (F \otimes G) = (\overline{D} \cdot F)G + (\overline{D} \cdot G)F$$  \hspace{1cm} (33)
$$F \cdot (D \otimes G) = F \cdot \overline{D}G$$  \hspace{1cm} (34)
$$D(F \cdot G) = (D \cdot F)G + F \cdot DG$$  \hspace{1cm} (35)
$$D \otimes (F \cdot U) = (D \cdot F)U + (F \cdot D)U$$  \hspace{1cm} (36)
$$F \otimes (\overline{D} \cdot U) = (F \cdot \overline{D})U$$  \hspace{1cm} (37)

4.1 Main equations in complex notation

We now translate the Einstein-Maxwell equations into complex notations.

Proposition 4.5. We have

$$\nabla_3 (F \beta) - D(F \beta) = -\frac{1}{2} \text{tr} X(F \beta) + 2 \omega(F \beta) + 2 \beta \bar{\beta}$$  \hspace{1cm} (38)
$$\nabla_4 (F \beta) - \frac{1}{2} \beta \bar{\beta} = -\frac{1}{2} \text{tr} Y(F \beta) + \frac{1}{2} \text{tr} Z(F \beta)$$  \hspace{1cm} (39)
$$\nabla_3 (F \beta) + \frac{1}{2} D(F \beta) = -\frac{1}{2} \text{tr} X(F \beta) + \frac{1}{2} \text{tr} Z(F \beta) + 2 \omega(F \beta)$$  \hspace{1cm} (40)
$$\nabla_4 (F \beta) + D(F \beta) = -\frac{1}{2} \text{tr} X(F \beta) + \frac{1}{2} \text{tr} Z(F \beta) - 2 \omega(F \beta) + 2 \bar{\beta} \beta$$  \hspace{1cm} (41)

Proof. We derive the equation for $(F \beta)$. Observe that from

$$\nabla_3 (F \beta) = \nabla (F \beta) - \frac{1}{2} (\text{tr} X(F \beta) + \text{tr} Y(F \beta)) + 2 \omega(F \beta)$$  \hspace{1cm} (42)
$$= \nabla (F \beta) - \frac{1}{2} (\text{tr} X(F \beta) + \text{tr} Y(F \beta)) + 2 \omega(F \beta)$$  \hspace{1cm} (43)

we obtain

$$\nabla_3 (F \beta) = \nabla (F \beta) - \frac{1}{2} (\text{tr} X(F \beta) + \text{tr} Y(F \beta)) + 2 \omega(F \beta)$$  \hspace{1cm} (44)

This implies

$$\nabla_3 (F \beta) = \nabla (F \beta) - \frac{1}{2} \text{tr} X(F \beta) + \frac{1}{2} \text{tr} Y(F \beta) + 2 \omega(F \beta)$$  \hspace{1cm} (45)

which gives

$$\nabla_3 (F \beta) - D(F \beta) = -\frac{1}{2} \text{tr} X(F \beta) + 2 \omega(F \beta) + 2 \beta \bar{\beta}$$  \hspace{1cm} (46)
We derive the equation for \( (F)P \). From
\[
\nabla_4 (F)P - \text{div} (F)\beta = - \left( \text{tr} \chi (F)P - (a) \text{tr} \chi \ast (F)P \right) + \left( \zeta + \eta \right) \cdot (F)\beta - \xi \cdot (F)\beta,
\]
\[
\nabla_4 \ast (F)P - \text{curl} (F)\beta = - \left( \text{tr} \chi \ast (F)P + (a) \text{tr} \chi (F)P \right) + \left( \eta + \zeta \right) \cdot (F)\beta + \xi \cdot \ast (F)\beta,
\]
we obtain
\[
\nabla_4 (F)P = \nabla_4 (F)P + i \ast (F)P = \nabla_4 (F)P + i \ast (F)P
\]
\[
+ \left( \zeta + \eta \right) \cdot (F)\beta + i \left( \eta + \zeta \right) \cdot (F)\beta - \xi \cdot (F)\beta + i \xi \cdot \ast (F)\beta
\]
which gives
\[
\nabla_4 (F)P - \frac{1}{2} \nabla \cdot (F)B = - \text{tr} \chi (F)P + \frac{1}{2} (Z + \bar{H}) \cdot (F)B - \frac{1}{2} \bar{\xi} \cdot (F)B
\]
as desired. The other equations are obtained by symmetrization. □

The complex notations allow us to rewrite the Ricci equations in the following form [13].

**Proposition 4.6.**
\[
\nabla_3 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 + 2 \omega \text{tr} X = D \cdot \Xi + \Xi \cdot \bar{\Pi} + \Xi \cdot (H - 2Z) - \frac{1}{2} \hat{X} \cdot \bar{X} - (F)B \cdot (F)B,
\]
\[
\nabla_3 \hat{X} + \Re(\text{tr} X) \hat{X} + 2 \omega \hat{X} = D \otimes \Xi + \Xi \otimes (H + \bar{H} - 2Z) - A.
\]
\[
\nabla_4 \text{tr} X + \frac{1}{2} \text{tr} X \text{tr} X - 2 \omega \text{tr} X = D \cdot \bar{H} + \bar{H} \cdot \bar{H} + 2P + \Xi \cdot \bar{\Xi} - \frac{1}{2} \hat{X} \cdot \bar{X},
\]
\[
\nabla_4 \hat{X} + \frac{1}{2} \text{tr} X \hat{X} - 2 \omega \hat{X} = D \otimes \bar{H} + \bar{H} \otimes \bar{H} - \frac{1}{2} \text{tr} \Xi \hat{X} + \frac{1}{2} \Xi \otimes \Xi - \frac{1}{2} (F)B \otimes (F)B.
\]
\[
\nabla_4 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 + 2 \omega \text{tr} X = D \cdot \Xi + \Xi \cdot \bar{H} + \Xi \cdot (H + 2Z) - \frac{1}{2} \hat{X} \cdot \bar{X} - (F)B \cdot (F)B,
\]
\[
\nabla_4 \hat{X} + \Re(\text{tr} X) \hat{X} + 2 \omega \hat{X} = D \otimes \Xi + \Xi \otimes (H + 2Z) - A.
\]

Also,
\[
\nabla_3 Z + \frac{1}{2} \text{tr} X (Z + \bar{H}) - 2 \omega (Z - H) = -2D \omega - \frac{1}{2} \hat{X} \cdot (Z + \bar{H}) + \frac{1}{2} \text{tr} X \Xi + 2 \omega \Xi - B - (F)P (F)B + \frac{1}{2} \Xi \cdot \hat{X},
\]
\[
\nabla_4 Z + \frac{1}{2} \text{tr} X (Z - H) - 2 \omega (Z + H) = 2D \omega + \frac{1}{2} \hat{X} \cdot (Z + \bar{H}) - \frac{1}{2} \text{tr} X \Xi - 2 \omega \Xi - B - (F)P (F)B + \frac{1}{2} \Xi \cdot \hat{X},
\]
\[
\nabla_3 H - \nabla_4 \Xi = -\frac{1}{2} \text{tr} X (H - H) - \frac{1}{2} \hat{X} \cdot (\bar{H} - \bar{H}) - 4 \omega \Xi + B + (F)P (F)B,
\]
\[
\nabla_4 H - \nabla_3 \Xi = -\frac{1}{2} \text{tr} X (H - H) - \frac{1}{2} \hat{X} \cdot (\bar{H} - \bar{H}) - 4 \omega \Xi - B - (F)P (F)B,
\]

Also,
\[
\frac{1}{2} \bar{D} \cdot \hat{X} + \frac{1}{2} \hat{X} \cdot \bar{Z} = \frac{1}{2} D \text{tr} X + \frac{1}{2} \text{tr} X Z - i \Xi (\text{tr} X H - i \Xi (\text{tr} X) \Xi - B + (F)P (F)B,
\]
\[
\frac{1}{2} \bar{D} \cdot \hat{X} - \frac{1}{2} \hat{X} \cdot \bar{Z} = \frac{1}{2} D \text{tr} X - \frac{1}{2} \text{tr} X Z - i \Xi (\text{tr} X H + i \Xi (\text{tr} X) \Xi + B - (F)P (F)B.
\]
We can specialize Proposition 3.3 to the case of \( \psi \in S_1(\mathbb{C}) \) and \( \Psi \in S_2(\mathbb{C}) \). We obtain the following Gauss equation.

**Proposition 4.7.** We have

1. For \( \psi \in S_1 \):

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi = \frac{1}{2} (\text{tr} \chi \nabla_3 + (a) \text{tr} \chi \nabla_4) \psi \\
+ i \left( \frac{1}{4} \text{tr} \chi \nabla_3 + \frac{1}{4} (a) \text{tr} \chi (a) \cdot \text{tr} \chi + \rho - (F) \rho^2 - * (F) \rho^2 \right) \psi
\]  

or also:

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi = \frac{1}{2} (\text{tr} \chi \nabla_3 + (a) \text{tr} \chi \nabla_4) \psi \\
+ \frac{1}{2} i \left( \frac{1}{4} \text{tr} \chi \nabla_3 + \frac{1}{4} (a) \text{tr} \chi (a) \cdot \text{tr} \chi + P + \overline{P} - 2 (F) \rho \overline{(F) \rho} \right) \psi
\]

2. For \( \Psi \in S_2 \):

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi = \frac{1}{2} (\text{tr} \chi \nabla_3 + (a) \text{tr} \chi \nabla_4) \Psi \\
+ 2i \left( \frac{1}{4} \text{tr} \chi \nabla_3 + \frac{1}{4} (a) \text{tr} \chi (a) \cdot \text{tr} \chi + \rho - (F) \rho^2 - * (F) \rho^2 \right) \Psi
\]  

or also:

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi = \frac{1}{2} (\text{tr} \chi \nabla_3 + (a) \text{tr} \chi \nabla_4) \Psi \\
+ i \left( \frac{1}{4} \text{tr} \chi \nabla_3 + \frac{1}{4} (a) \text{tr} \chi (a) \cdot \text{tr} \chi + P + \overline{P} - 2 (F) \rho \overline{(F) \rho} \right) \Psi
\]

**Proof.** See Appendix A.1

The complex notations allow us to rewrite the Bianchi identities as follows [13].

**Proposition 4.8.** We have,

\[
\nabla_3 A - D \otimes B = -\frac{1}{2} \text{tr} X A + 4 \omega A + (Z + 4 H) \otimes B - 3 (F) \rho H \otimes (F) B - 3 \overline{P} \nabla X + 2 (F) \rho \overline{X} + N
\]

\[
\nabla_4 A + \frac{1}{2} D \otimes B = -\frac{1}{2} \text{tr} X A + 4 \omega A + \frac{1}{2} (Z - 4 H) \otimes B + 3 (F) \rho H \otimes (F) B - 3 \overline{P} \nabla X + 2 (F) \rho \overline{X} + N
\]

where

\[
\nabla_3 B - \frac{1}{2} \nabla \cdot A = -2 \text{tr} X B - 2 \omega B + \frac{1}{2} A \cdot (2Z + H) + \left( 3 \overline{P} - 2 (F) \rho \overline{(F) \rho} \right) \Xi
\]

\[
+ \overline{(F) \rho} \left( \nabla_4 (F) B + 2 \omega (F) B \right) + N,
\]

\[
\nabla_4 B + \frac{1}{2} \nabla \cdot A = -2 \text{tr} X B - 2 \omega B - \frac{1}{2} A \cdot (-2Z + H) - \left( 3 P - 2 (F) \rho \overline{(F) \rho} \right) \Xi
\]

\[
+ (F) \rho \left( \nabla_3 (F) B + 2 \omega (F) B \right) + N
\]

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\[ \nabla_3 B - D\hat{P} = -\text{tr}XB + 2\omega B + \hat{B} \cdot \hat{X} + 3\hat{P}H + \frac{1}{2}A \cdot \Xi + \frac{1}{2}(P \cdot D)\hat{(F)}P - \frac{1}{2}\text{tr}X(F)P(F)B - \frac{1}{2}\text{tr}X(F)P(F)B + N, \]
\[ \nabla_4 B + DP = -\text{tr}XB + 2\omega B + \hat{B} \cdot \hat{X} - 3PH - \frac{1}{2}A \cdot \Xi - (F)P \cdot D\hat{(F)P} - \frac{1}{2}\text{tr}X(F)P(F)B - \frac{1}{2}\text{tr}X(F)P(F)B + N \]
and
\[ \nabla_4 P - \frac{1}{2}D \cdot B = -\frac{3}{2}\text{tr}XP - \frac{1}{2}\text{tr}X(F)P(F)P + \frac{1}{2}(2H + Z) \cdot \hat{B} - \Xi \cdot B - \frac{1}{4}\hat{X} \cdot \hat{A} + \frac{1}{2}(F)P \cdot D\hat{(F)P} + \frac{1}{2}Z \cdot (F)P(F)B + \frac{1}{2}(F)P(F)B + N \]
\[ \nabla_3 P + \frac{1}{2}D \cdot B = -\frac{3}{2}\text{tr}XP - \frac{1}{2}\text{tr}X(F)P(F)P - \frac{1}{2}(2H - Z) \cdot \hat{B} + \Xi \cdot B - \frac{1}{4}\hat{X} \cdot \hat{A} - \frac{1}{2}(F)P \cdot D\hat{(F)P} - \frac{1}{2}Z \cdot (F)P(F)B - \frac{1}{2}(F)P(F)B + N \]

where \( N \) denotes quadratic terms for perturbations of Kerr-Newman spacetime.

Proof. See Appendix A.3 \( \square \)

4.2 Conformal invariance

We recall here the notion of conformal invariance and define conformal invariant derivatives \[13\] to further simplify the Einstein-Maxwell equations.

A general null frame transformation can be written at the linear level in the form:

\[ e'_4 = \lambda (e_4 + f_a e_a) \]
\[ e'_3 = \lambda^{-1} (e_3 + f_a e_a) \]
\[ e'_a = e_a + \frac{1}{2}f_a e_4 + \frac{1}{2}f_a e_3 \]

for a scalar function \( \lambda \) and 1-tensors \( f \) and \( f_a \). The transformations for which \( f = f_a = 0 \) are called conformal transformations, i.e.

\[ e'_3 = \lambda^{-1} e_3, \quad e'_4 = \lambda e_4, \quad e'_a = e_a. \]

Note that under the above mentioned frame transformation we have

\[ \text{tr} \chi' = \lambda^{-1} \text{tr} \chi, \quad (a) \text{tr} \chi' = \lambda^{-1} (a) \text{tr} \chi, \quad \text{tr} \chi' = \lambda \text{tr} \chi, \quad (a) \text{tr} \chi' = \lambda (a) \text{tr} \chi \]
\[ \xi' = \lambda \xi, \quad \eta' = \eta, \quad \eta' = \eta, \quad \xi' = \lambda^{-1} \xi \]
\[ \alpha' = \lambda^2 \alpha, \quad \beta' = \lambda \beta, \quad \rho' = \rho, \quad \rho' = \rho, \quad \beta' = \lambda^{-1} \beta, \quad \alpha' = \lambda^{-2} \alpha \]

and

\[ \omega' = \lambda^{-1} \left( \omega + \frac{1}{2} e_3 (\log \lambda) \right), \quad \omega' = \lambda \left( \omega - \frac{1}{2} e_4 (\log \lambda) \right), \quad \zeta' = \zeta - \nabla (\log \lambda). \]

Remark 4.9. If \( f \) verifies \( f' = \lambda^* f \), then \( \nabla_3 f, \nabla_4 f, \nabla_a f \) are not conformal invariant.

We correct the lacking of being conformal invariant by making the following definition \[13\].

Lemma 4.10. If \( f \) verifies \( f' = \lambda^* f \), then
1. \( (c)\nabla_3 f := \nabla_3 f - 2s\omega f \) is \((s - 1)\)-conformally invariant.

2. \( (c)\nabla_4 f := \nabla_4 f + 2s\omega f \) is \((s + 1)\)-conformally invariant.

3. \( (c)\nabla_A f := \nabla_A f + s\zeta_A f \) is \(s\)-conformally invariant.

We also correct the complex derivatives to make them conformally invariant.

**Definition 4.11.** We define the following conformal angular derivatives in the complex notation:

- For \((a, b)\) scalars we define
  \[
  (c)\mathcal{D}(a + ib) := ( (c)\nabla + i (c)\nabla ) (a + ib)
  \]

- For \(f\) a 1-tensor we define
  \[
  (c)\mathcal{D}(f + i \ast f) := ( (c)\nabla + i (c)\nabla ) \cdot (f + i \ast f)
  \]
  \[
  (c)\mathcal{D}(\hat{f} + i \ast f) := ( (c)\nabla + i (c)\nabla ) \hat{f} + i \ast (f + i \ast f)
  \]

- For \(u\) a 2-tensor we define
  \[
  (c)\mathcal{D}(c u + i \ast u) := ( (c)\nabla + i (c)\nabla ) (c u + i \ast u).
  \]

- In all the above cases we set
  \[
  (c)\mathcal{D} := (c)\nabla - i (c)\nabla
  \]

**4.3 Main equations in complex form using conformal operators**

Using the conformal operators, we rewrite all the equations. Observe that the quantities \(\omega, \bar{\omega}\) and \(Z\) do not appear anymore explicitly in the equations. We have

\[
\begin{align*}
(3)\nabla_3 (F) B - (c)\mathcal{D}(F) P &= -\frac{1}{2} \text{tr}(X) (F) B + 2 (F) \mathcal{P} H + \frac{1}{2} \hat{X} \cdot \overline{(F) B} \\
(3)\nabla_4 (F) P - \frac{1}{2} (c)\mathcal{D}(F) B &= -\text{tr}(X) (F) P + \frac{1}{2} H \cdot \overline{(F) B} - \frac{1}{2} \overline{\hat{X} \cdot \overline{(F) B}} \\
(3)\nabla_3 (F) P + \frac{1}{2} (c)\mathcal{D}(P) \overline{(F) B} &= -\text{tr}(X) (F) P - \frac{1}{2} H \cdot \overline{(F) B} + \frac{1}{2} \overline{\hat{X} \cdot \overline{(F) B}} \\
(3)\nabla_4 (F) \overline{B} + (c)\mathcal{D}(F) P &= -\frac{1}{2} \text{tr}(X) (F) \overline{B} - 2 (F) \mathcal{P} H + \frac{1}{2} \hat{X} \cdot \overline{(F) B}
\end{align*}
\]

We have

\[
\begin{align*}
(3)\nabla_3 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 &= (c)\mathcal{D} \cdot \Xi + \Xi \cdot \overline{H} + \Xi \cdot H - \frac{1}{2} \hat{X} \cdot \overline{X} - (F) B \cdot \overline{(F) B} \\
(3)\nabla_4 \hat{X} + \text{tr} (H) \hat{X} &= (c)\mathcal{D} \hat{\Xi} + \Xi \hat{\Xi} (H + H) - \mathcal{A} \\
(3)\nabla_3 \text{tr} X + \frac{1}{2} \text{tr} X \text{tr} X &= (c)\mathcal{D} \cdot \overline{H} + \overline{H} \cdot H + 2 P + \Xi \cdot \Xi - \frac{1}{2} \hat{X} \cdot \overline{X}, \\
(3)\nabla_4 \hat{X} + \frac{1}{2} \text{tr} X \hat{X} &= (c)\mathcal{D} \hat{H} + H \cdot \hat{H} - \frac{1}{2} \text{tr} X \hat{X} + \frac{1}{2} \Xi \hat{\Xi} - \frac{1}{2} (F) B \hat{(F) B}, \\
(3)\nabla_4 \text{tr} X + \frac{1}{2} \text{tr} X \text{tr} X &= (c)\mathcal{D} \cdot \overline{H} + \overline{H} \cdot H + 2 P + \Xi \cdot \Xi - \frac{1}{2} \hat{X} \cdot \overline{X}, \\
(3)\nabla_4 \hat{X} + \frac{1}{2} \text{tr} X \hat{X} &= (c)\mathcal{D} \hat{H} + H \cdot \hat{H} - \frac{1}{2} \text{tr} X \hat{X} + \frac{1}{2} \Xi \hat{\Xi} - \frac{1}{2} (F) B \hat{(F) B}.
\end{align*}
\]
\[(c) \nabla_4 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 = (c) \mathcal{D} \cdot \Xi + \Xi \cdot H + \Xi \cdot H - \frac{1}{2} \hat{X} \cdot \hat{X} - (F) B \cdot (F) B, \]
\[(c) \nabla_4 \hat{X} + \nabla_4 \Xi = (c) \mathcal{D} \otimes \Xi + \Xi \otimes (H + H) - A. \]

\[(c) \nabla_4 H - (c) \nabla_4 \Xi = - \frac{1}{2} \text{tr} \hat{X} (H - H) - \frac{1}{2} \hat{X} \cdot (H - H) + B + (F) P (F) B, \]
\[(c) \nabla_4 H - (c) \nabla_3 \Xi = - \frac{1}{2} \text{tr} \hat{X} (H - H) - \frac{1}{2} \hat{X} \cdot (H - H) - B - (F) P (F) B, \]

\[\frac{1}{2} (c) \mathcal{D} \cdot \hat{X} = - \frac{1}{2} (c) \text{Dtr} \hat{X} - i \Xi (\text{tr} X) H - i \Xi (\text{tr} X) \Xi - B + (F) P (F) B, \]
\[\frac{1}{2} (c) \mathcal{D} \cdot \hat{X} = - \frac{1}{2} (c) \text{Dtr} \hat{X} - i \Xi (\text{tr} X) H - i \Xi (\text{tr} X) \Xi + B - (F) P (F) B. \]

We have
\[(c) \nabla_3 A + \frac{1}{2} \text{tr} X A = (c) \mathcal{D} \otimes B + H \otimes \left( 4B - 3(F) P (F) B \right) - 3P \hat{X} - 2(F) P \hat{X} + N, \]
\[(c) \nabla_4 A + \frac{1}{2} \text{tr} X A = - (c) \mathcal{D} \otimes B - H \otimes \left( 4B - 3(F) P (F) B \right) - 3P \hat{X} + 2(F) P \hat{X} + N \]

with
\[\mathcal{F} = - \frac{1}{2} (c) \mathcal{D} \otimes (F) B - \frac{3}{2} (F) P (F) \hat{X}, \]
\[\mathcal{G} = - \frac{1}{2} (c) \mathcal{D} \otimes (F) B - \frac{3}{2} (F) P (F) \hat{X}. \]

Also
\[(c) \nabla_4 B - \frac{1}{2} (c) \mathcal{D} \cdot A \]
\[= - 2 \text{tr} X B + \frac{1}{2} A \cdot H + \left( 3P - 2(F) P (F) P \right) \Xi + (F) P (c) \nabla_4 ((F) B) + N, \]
\[(c) \nabla_4 B + \frac{1}{2} (c) \mathcal{D} \cdot A \]
\[= - 2 \text{tr} X B - \frac{1}{2} A \cdot H - \left( 3P - 2(F) P (F) P \right) \Xi + (F) P (c) \nabla_4 ((F) B) + N \]

\[(c) \nabla_3 B - (c) \mathcal{D} P \]
\[= - \text{tr} X B + B \cdot \hat{X} + 3P H + \frac{1}{2} A \cdot \Xi \]
\[+ (F) P (c) \mathcal{D} ((F) P) - \frac{1}{2} \text{tr} X (F) P (F) B - \text{tr} X (F) P (F) B + N, \]
\[(c) \nabla_4 B + (c) \mathcal{D} P \]
\[= - \text{tr} X B + B \cdot \hat{X} - 3P H - \frac{1}{2} A \cdot \Xi \]
\[+ (F) P (c) \mathcal{D} ((F) P) - \frac{1}{2} \text{tr} X (F) P (F) B - \text{tr} X (F) P (F) B + N \]

\[(c) \nabla_4 P - \frac{1}{2} (c) \mathcal{D} \cdot B \]
\[= \frac{3}{2} \text{tr} X P - \text{tr} X (F) P (F) P + H \cdot B - \Xi \cdot B - \frac{1}{4} \hat{X} \cdot A \]
\[+ \frac{1}{2} (F) P (c) \mathcal{D} \cdot (F) B + H \cdot (F) P (F) B + N, \]
\[(c) \nabla_3 P + \frac{1}{2} (c) \mathcal{D} \cdot B \]
\[= \frac{3}{2} \text{tr} X P - \text{tr} X (F) P (F) P - H \cdot B + \Xi \cdot B - \frac{1}{4} \hat{X} \cdot A \]
\[+ \frac{1}{2} (F) P (c) \mathcal{D} \cdot (F) B - H \cdot (F) P (F) B + N \]

### 4.4 Commutator formulas

We collect here some commutator formulas for conformal derivatives.
Lemma 4.12. Let $G = g_1 + ig_2$ be a scalar function of conformal type 0. Then

$$\left[ (c)\nabla_3, (c)\nabla_4 \right] G = -\frac{1}{2} \text{tr} X (c)DG + H (c)\nabla_4 G + \Xi (c)\nabla_3 G - \frac{1}{2} \hat{\nabla} \cdot (c)\nabla G$$

(44)

Let $F = f + i*s f \in S_1$ be of conformal type $s$. Then

$$\left[ (c)\nabla_3, (c)\nabla_4 \right] F = \frac{1}{2} (H - \overline{H}) \cdot (c)\nabla F + \frac{1}{2} (\overline{H} - H) \cdot (c)\nabla F$$

$$+ \left( (s - 1)P + (s + 1)\overline{P} + 2s (F)\overline{P} (F)\overline{P} - \frac{s + 1}{2} (H \cdot \overline{H}) + \frac{1 - s}{2} (\overline{H} \cdot H) \right) F$$

$$+ N$$

(45)

We also have

$$E \cdot \left[ (c)\nabla_3, (c)\nabla_4 \right] F = -\frac{1}{2} \text{tr} X E \cdot (c)\nabla F + \frac{1}{2} \text{tr} X (1 - s) (\overline{H} \cdot E) F + (E \cdot \overline{H}) (c)\nabla_3 F$$

(46)

and

$$\overline{E} \cdot \left[ (c)\nabla_3, (c)\nabla_4 \right] F = -\frac{1}{2} \text{tr} X \overline{E} \cdot (c)\nabla F - \frac{1}{2} \text{tr} X (s + 1) (E \cdot H) F$$

$$- \frac{1}{2} \text{tr} X \overline{H} (E \cdot F) + (E \cdot H) (c)\nabla_3 F$$

(47)

If $U = u + i*s u \in S_2$ of conformal type $s$. Then

$$\left[ (c)\nabla_3, (c)\nabla_4 \right] U = \frac{1}{2} (H - \overline{H}) \cdot (c)\nabla U + \frac{1}{2} (\overline{H} - H) \cdot (c)\nabla U$$

$$+ \left( (s - 2)P + (s + 2)\overline{P} + 2s (F)\overline{P} (F)\overline{P} - \frac{s + 1}{2} H \cdot \overline{H} - \frac{s + 2}{2} \overline{H} \cdot H \right) U$$

$$+ 2 \overline{H} \overline{\nabla} (\overline{H} \cdot U) + N$$

(50)

$$\left[ (c)\nabla_3, (c)\nabla_4 \right] U = -\frac{1}{2} \text{tr} X (c)\nabla U + (s - 2)\overline{H} \cdot U + H \cdot \nabla U + N$$

(51)

Proof. See Appendix A.4

4.5 Laplacian operators

For $\Psi \in S_k(\mathbb{C})$ we define the Laplacian with respect to the horizontal structure as

$$\Delta_k \Psi = \nabla_1 \nabla_1 \Psi + \nabla_2 \nabla_2 \Psi$$

We now relate the Laplacian to the angular derivatives $D$ for a 1-tensor in $S_1(\mathbb{C})$ and for a 2-tensor in $S_2(\mathbb{C})$ using the Gauss equations of Proposition 4.7. We generalize here the known relations in the case of an integrable distribution, like

$$\mathcal{P}_1 \mathcal{P}_1 = -\Delta_1 + K, \quad \mathcal{P}_2 \mathcal{P}_2 = \frac{1}{2} \Delta_1 - \frac{1}{2} K, \quad \mathcal{P}_2 \mathcal{P}_1 = -\frac{1}{2} \Delta_2 + K$$

We have the following lemma.
We first consider the case of a 1-tensor $\Psi$. Using (52) to write

$$\Box_1 \Psi = -\nabla_3 \nabla_4 \Psi + \Delta_1 \Psi + \left(2\omega - \frac{1}{2} tr \chi \right) \nabla_4 \Psi - \frac{1}{2} tr \chi \nabla_3 \Psi + 2\eta \cdot \nabla \Psi - k \cdot \mathbf{\rho} \Psi. \quad (56)$$

We specialize the above expression to the case of $S_1(\mathbb{C})$ and $S_2(\mathbb{C})$.

### 4.6 Invariant wave operators

We make use of the complex notation and derivatives to rewrite the wave operator. For $\Psi_k \in S_k(\mathbb{C})$ we have

$$\Box_k \Psi = -\frac{1}{2} \left( \nabla_3 \nabla_4 \Psi + \nabla_4 \nabla_3 \Psi \right) + \Delta_k \Psi + \left(\omega - \frac{1}{2} tr \chi \right) \nabla_4 \Psi + \left(\omega - \frac{1}{2} tr \chi \right) \nabla_3 \Psi + (\eta + \frac{1}{2}) \cdot \nabla \Psi. \quad (57)$$

or

$$\Box_k \Psi = -\nabla_3 \nabla_4 \Psi + \Delta_k \Psi + \left(2\omega - \frac{1}{2} tr \chi \right) \nabla_4 \Psi - \frac{1}{2} tr \chi \nabla_3 \Psi + 2\eta \cdot \nabla \Psi - k \cdot \mathbf{\rho} \Psi. \quad (56)$$

We specialize the above expression to the case of $S_1(\mathbb{C})$ and $S_2(\mathbb{C})$.

#### 4.6.1 Wave operator of a $S_1(\mathbb{C})$-tensor

We first consider the case of a 1-tensor $\psi \in S_1(\mathbb{C})$. Applying (56) in this case gives:

$$\Box_1 \psi = -\nabla_3 \nabla_4 \psi + \Delta_1 \psi + \left(2\omega - \frac{1}{2} tr \chi \right) \nabla_4 \psi - \frac{1}{2} tr \chi \nabla_3 \psi + 2\eta \cdot \nabla \psi + \left(\frac{1}{2} P - \frac{1}{2} \bar{P} \right) \psi$$

where $\Delta_1$ is the Laplacian on the horizontal structure for a 1-tensor. Using (55) to write

$$\Delta_1 \psi = \frac{1}{2} \nabla \cdot (D \nabla \psi) - \frac{1}{2} i \left( (\omega \ nabla_3 + \bar{\omega} \ n\nabla_4) \psi \right)$$

$$+ \left( \frac{1}{8} tr X tr \Xi + \frac{1}{8} tr \Xi tr X + \frac{1}{2} P + \frac{1}{2} \bar{P} - (F \bar{P})(P \bar{F}) \right) \psi$$

we obtain

$$\Box_1 \psi = -\nabla_3 \nabla_4 \psi + \frac{1}{2} \nabla \cdot (D \nabla \psi) + \left(2\omega - \frac{1}{2} tr \chi \right) \nabla_4 \psi - \frac{1}{2} tr \chi \nabla_3 \psi + 2\eta \cdot \nabla \psi$$

$$+ \left( \frac{1}{8} tr X tr \Xi + \frac{1}{8} tr \Xi tr X + \frac{1}{2} P + (F \bar{P})(P \bar{F}) \right) \psi$$

Lemma 4.13. Let $\psi \in S_1(\mathbb{C})$ and $\Psi \in S_2(\mathbb{C})$. The following relations hold true.

$$D(\nabla \cdot \psi) = 2\Delta_1 \psi - i(\omega \ nabla_3 + \bar{\omega} \ n\nabla_4) \psi$$

$$+ \left( \frac{1}{4} tr X tr \Xi + \frac{1}{4} tr \Xi tr X + P + \bar{P} - 2(F \bar{P})(P \bar{F}) \right) \psi \quad (52)$$

$$\nabla \cdot (D \nabla \psi) = 2\Delta_1 \psi + i(\omega \ nabla_3 + \bar{\omega} \ n\nabla_4) \psi$$

$$- \left( \frac{1}{4} tr X tr \Xi + \frac{1}{4} tr \Xi tr X + P + \bar{P} - 2(F \bar{P})(P \bar{F}) \right) \psi \quad (53)$$

$$D \nabla (\nabla \cdot \psi) = 2\Delta_2 \Psi - i(\omega \ nabla_3 + \bar{\omega} \ n\nabla_4) \Psi$$

$$+ \left( \frac{1}{2} tr X tr \Xi + \frac{1}{2} tr \Xi tr X + 2P + 2\bar{P} - 4(F \bar{P})(P \bar{F}) \right) \Psi \quad (54)$$

In particular,

$$\nabla \cdot (D \nabla \psi) = D(\nabla \cdot \psi) + (tr X - tr X) \nabla_3 \psi + (tr \Xi - tr \Xi) \nabla_4 \psi$$

$$- \left( \frac{1}{4} tr X tr \Xi + \frac{1}{4} tr \Xi tr X + 2P + 2\bar{P} - 4(F \bar{P})(P \bar{F}) \right) \psi \quad (55)$$

Proof. See Appendix A.5.
Using \((H \cdot D)\psi + (\overline{H} \cdot D)\psi = 4\eta \cdot \nabla \psi\), we can write
\[
\Box_1 \psi = -\nabla_3 \nabla_4 \psi + \frac{1}{2} D \cdot (D \otimes \psi) + \left(2\omega - \frac{1}{2} \text{tr} X\right) \nabla_4 \psi - \frac{1}{2} \text{tr} X \nabla_3 \psi + \frac{1}{2} (H \cdot D)\psi + \frac{1}{2} (\overline{H} \cdot D)\psi \\
+ \left(\frac{1}{8} \text{tr} X \text{tr} X + \frac{1}{8} \text{tr} X \text{tr} X + P - (F)P(FP)\right) \psi \tag{57}
\]

Suppose \(\psi \in \mathcal{S}_1(\mathbb{C})\) is of conformal type 0. Then we can write \(\nabla_4 = (c) \nabla_4\), \(\nabla_3 = (c) \nabla_3\), \(D = (c) D\), and \(\nabla_3 \nabla_4 = (c) \nabla_3 (c) \nabla_4 + 2\omega (c) \nabla_4\). We can therefore write
\[
\Box_1 \psi = -(c) \nabla_3 (c) \nabla_4 \psi + \frac{1}{2} (c) D \cdot (c) D \otimes \psi - \frac{1}{2} \text{tr} X (c) \nabla_4 \psi - \frac{1}{2} \text{tr} X (c) \nabla_3 \psi \\
+ \frac{1}{2} (H \cdot (c) D)\psi + \frac{1}{2} (\overline{H} \cdot (c) D)\psi + \left(\frac{1}{8} \text{tr} X \text{tr} X + \frac{1}{8} \text{tr} X \text{tr} X + P - (F)P(FP)\right) \psi \tag{58}
\]
which is the invariant conformal wave operator for \(\psi \in \mathcal{S}_1(\mathbb{C})\) of conformal type 0.

### 4.6.2 Wave operator of a \(S_2(\mathbb{C})\)-tensor

Consider the case of \(\Psi \in \mathcal{S}_2(\mathbb{C})\). Applying (59) in this case gives:
\[
\Box_2 \Psi = -\nabla_3 \nabla_4 \Psi + \Delta_2 \Psi + \left(2\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_4 \Psi - \frac{1}{2} \text{tr} \chi \nabla_3 \Psi + 2\eta \cdot \nabla \Psi + (\overline{P} - P) \Psi.
\]
where \(\Delta_2\) is the Laplacian on the horizontal structure for a 2-tensor. Using (53) to write
\[
\Delta_2 \Psi = \frac{1}{2} D \otimes (D \cdot \Psi) + i \left(\text{tr} \chi \nabla_3 + \text{tr} \chi \nabla_4\right) \Psi - \left(\frac{1}{4} \text{tr} X \text{tr} X + \frac{1}{4} \text{tr} X \text{tr} X + P + \overline{P} - 2(F)P(FP)\right) \Psi
\]
we obtain
\[
\Box_2 \Psi = -\nabla_3 \nabla_4 \Psi + \frac{1}{2} D \otimes (D \cdot \Psi) + \left(2\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_4 \Psi - \frac{1}{2} \text{tr} X \nabla_3 \Psi \\
+ \frac{1}{2} (H \cdot D)\Psi + \frac{1}{2} (\overline{H} \cdot D)\Psi + \left(-\frac{1}{4} \text{tr} X \text{tr} X - \frac{1}{4} \text{tr} X \text{tr} X - 2P + 2(F)P(FP)\right) \Psi \tag{59}
\]
For \(\Psi \in \mathcal{S}_2(\mathbb{C})\) of conformal type 0 we can write
\[
\Box_2 \Psi = -(c) \nabla_3 (c) \nabla_4 \Psi + \frac{1}{2} (c) D \otimes (c) D \cdot \Psi - \frac{1}{2} \text{tr} X (c) \nabla_4 \Psi - \frac{1}{2} \text{tr} X (c) \nabla_3 \Psi \\
+ \frac{1}{2} (H \cdot (c) D)\Psi + \frac{1}{2} (\overline{H} \cdot (c) D)\Psi + \left(-\frac{1}{4} \text{tr} X \text{tr} X - \frac{1}{4} \text{tr} X \text{tr} X - 2P + 2(F)P(FP)\right) \Psi \tag{60}
\]
which is the invariant conformal wave operator for \(\Psi \in \mathcal{S}_2(\mathbb{C})\) of conformal type 0.

### 5 The Kerr-Newman spacetime

We here introduce the Kerr-Newman spacetime and its representation within the formalism above introduced.

#### 5.1 The metric

We consider the Kerr-Newman metric in standard Boyer-Lindquist coordinates
\[
g = -\frac{|q|^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{|q|^2} \left(d\varphi - \frac{2amr}{\Sigma^2} dt\right)^2 + \frac{|q|^2}{\Delta} (dr)^2 + |q|^2 (d\theta)^2,
\]

30
where

\[ q = r + ia \cos \theta \]  \tag{61}

and

\[
\begin{align*}
\Delta &= r^2 - 2mr + a^2 + Q^2, \\
|q|^2 &= r^2 + a^2 (\cos \theta)^2, \\
\Sigma^2 &= (r^2 + a^2)|q|^2 + 2mra^2 (\sin \theta)^2 = (r^2 + a^2) - a^2 (\sin \theta)^2 \Delta.
\end{align*}
\]

The null frame is given by

\[
\begin{align*}
e_4 &= \frac{r^2 + a^2}{\Delta} \partial_t + \frac{a}{\Delta} \partial_r + \frac{\Delta}{|q|^2} \partial_\varphi, \\
e_3 &= \frac{r^2 + a^2}{|q|^2} \partial_t - \frac{\Delta}{|q|^2} \partial_r + \frac{a}{|q|^2} \partial_\varphi, \\
e_1 &= \frac{1}{\sqrt{|q|^2}} \partial_\theta, \\
e_2 &= \frac{a \sin \theta}{\sqrt{|q|^2}} \partial_t + \frac{1}{\sqrt{|q|^2} \sin \theta} \partial_\varphi.
\end{align*}
\]

The complex Ricci coefficients are given by

\[
\begin{align*}
\hat{X} &= \hat{X} = \Xi = \Xi = \omega = 0, \\
\hat{H} &= -Z, \quad H_1 = \frac{2(2r + ia \cos \theta)}{|q|^2} = \frac{2\Delta}{|q|^3}, \quad tr X = \frac{2\Delta}{|q|^3}, \\
\omega &= \frac{a^2 \cos^2 \theta (r - m) + mr^2 - a^2 r - Q^2 r}{|q|^3}, \\
H_1 &= \frac{ai \sin \theta (r + ia \cos \theta)}{|q|^3}, \quad H_2 = \frac{ai \sin \theta (r - ia \cos \theta)}{|q|^3}, \\
Z_1 &= \frac{ai \sin \theta (r - ia \cos \theta)}{|q|^3}, \\
Z_2 &= \frac{ai \sin \theta (r - ia \cos \theta)}{|q|^3}.
\end{align*}
\]

**Remark 5.1.** Note the identities

\[
\begin{align*}
H_1 &= -Z_1, \quad H_2 = Z_2, \quad H_1 = H_1, \quad H_2 = -H_2.
\end{align*}
\]

The complex curvature components are given by

\[
\begin{align*}
A &= B = \overline{B} = A = 0, \quad P = -\frac{2m}{q^3} + \frac{2Q^2}{q^3 q},
\end{align*}
\]

The complex electromagnetic components are given by

\[
\begin{align*}
(F)B &= (F)\overline{B} = 0, \quad (F)P = \frac{Q}{q^3}
\end{align*}
\]

From the above we deduce (see also [13]) the following identities:

\[
\begin{align*}
\nabla_3 q &= \frac{1}{2} tr X q, \quad \nabla_4 q = \frac{1}{2} tr X q, \quad \overline{\nabla} q = q H, \quad D q = q H \tag{62}
\end{align*}
\]

**Lemma 5.2.** In Kerr-Newman spacetime we have for \( \Psi \in S_k(\mathbb{C}) \)

\[
\begin{align*}
\frac{1}{2} (tr X - tr X) \nabla_3 \Psi + \frac{1}{2} (tr X - tr X) \nabla_4 \Psi + (-H + H - H + H) \cdot \nabla \Psi &= i \frac{4a \cos \theta}{|q|^2} \partial_t \Psi
\end{align*}
\]

where \( \partial_t \Psi \) denotes the Lie derivative of \( \Psi \) with respect to the vectorfield \( \partial_t \).
Proof. From the above values of tr\(X\) and tr\(X\) we deduce

\[
^{(a)}\text{tr}X = \frac{2a \cos \theta}{|q|^2}, \quad ^{(a)}\text{tr}\bar{X} = \frac{2a \Delta \cos \theta}{|q|^4}
\]

This implies

\[
\begin{align*}
\frac{1}{2} \left( (\text{tr}X - \text{tr}X) \nabla_3 \Psi + \frac{1}{2} (\text{tr}X - \text{tr}X) \nabla_4 \Psi \right) &= i \left( (^{(a)}\text{tr}X \nabla_3 \Psi + ^{(a)}\text{tr}\bar{X} \nabla_4 \Psi) \right) \\
&= i \left( \frac{2a \cos \theta}{|q|^2} \nabla_3 \Psi + \frac{2a \Delta \cos \theta}{|q|^4} \nabla_4 \Psi \right) \\
&= i \left( 2a \cos \theta \left( \left( \frac{1}{|q|^2} \partial_t - \frac{\Delta}{|q|^2} \partial_r + \frac{a}{|q|^2} \partial_\varphi \right) \Psi + \frac{\Delta}{|q|^2} \left( \frac{1}{\Delta} \partial_t + \frac{a}{\Delta} \partial_\varphi \right) \Psi \right) \right) \\
&= i \frac{4a \cos \theta (r^2 + a^2)}{|q|^4} \partial_t \Psi + i \frac{4a^2 \cos \theta}{|q|^4} \partial_\varphi \Psi
\end{align*}
\]

On the other hand, using that \(H_1 = \frac{\Xi}{\bar{p}}, \quad H_2 = -\frac{\Pi}{\Pi},\) we have

\[
(-H + \Pi - H + \Pi) \cdot \nabla \Psi = (-H_1 + \Pi_1 - H_1 + \Pi_1) \nabla_1 \Psi + (-H_2 + \Pi_2 - H_2 + \Pi_2) \nabla_2 \Psi = 2 \left( \Pi_1 - H_1 \right) \nabla_2 \Psi
\]

\[
= 2 \left( \frac{a \sin \theta (r + ia \cos \theta)}{|q|^3} + \frac{a \sin \theta (r - ia \cos \theta)}{|q|^3} \right) \nabla_2 \Psi
\]

\[
= -\frac{4ia^2 \sin \theta \cos \theta}{|q|^3} \left( \frac{a \sin \theta}{|q|} \partial_t + \frac{1}{|q| \sin \theta} \partial_\varphi \right) \Psi
\]

\[
= -\frac{4ia^3 \sin^2 \theta \cos \theta}{|q|^4} \partial_t \Psi - 4i \frac{a^2 \cos \theta}{|q|^4} \partial_\varphi \Psi
\]

Summing the above, we obtain the desired expression. \(\Box\)

### 5.2 The linear equations in Kerr-Newman

Consider a linear gravitational and electromagnetic perturbation of Kerr-Newman spacetime. We neglect the quadratic terms in the Einstein-Maxwell equations, i.e. terms which involve products of quantities which vanish in the background (e.g. \(A, \bar{A}, B, \bar{B}, \ (F)B, \ (F)\bar{B}, \ X, \ \bar{X}, \ \Xi, \ \bar{\Xi}\)).

**Proposition 5.3.** The linear Maxwell equations are given by

\[
\begin{align*}
^{(c)}\nabla_3 (F)B -^{(c)}D (F)P &= -\frac{1}{2} tr X (F)B + 2 (F)PH \quad \text{(63)} \\
^{(c)}\nabla_4 (F)P - \frac{1}{2}^{(c)}D (F)B &= -\frac{1}{2} tr X (F)P + \frac{1}{2} \Pi (F)B \quad \text{(64)} \\
^{(c)}\nabla_3 (F)P + \frac{1}{2}^{(c)}D (F)\bar{B} &= -\frac{1}{2} tr X (F)P - \frac{1}{2} H (F)\bar{B} \quad \text{(65)} \text{ and } H \equiv 2 \Delta \Psi \nabla_4 (F)B +^{(c)}D (F)P &= -\frac{1}{2} tr X (F)B - 2 (F)PH \quad \text{(66)}
\end{align*}
\]

The linear null structure equations are given by

\[
\begin{align*}
^{(c)}\nabla_3 tr X + \frac{1}{2} (tr X)^2 &=^{(c)}D \cdot \Xi + \Xi \cdot \Pi + \bar{\Xi} \cdot H, \quad \text{(67)} \\
^{(c)}\nabla_3 \bar{X} + R (tr X) \bar{X} &=^{(c)}D \hat{\Xi} + \Xi \hat{\bar{A}} (H + \bar{H}) - A, \quad \text{(68)} \\
^{(c)}\nabla_3 tr X + \frac{1}{2} tr X tr X &=^{(c)}D \hat{\Pi} + H \cdot \Pi + 2 P, \quad \text{(69)} \text{ and } P \equiv 2 \Delta \Psi \nabla_3 \bar{X} + \frac{1}{2} tr X \bar{X} &=^{(c)}D \hat{\bar{H}} + \bar{H} \hat{H} - \frac{1}{2} \bar{X} \bar{X}, \quad \text{(70)}
\end{align*}
\]
Neglecting the terms which are zero in Kerr-Newman, we obtain the reduced equations.

The linear Bianchi identities are given by

\[
(c)\nabla_3 A + \frac{1}{2} tr X A = (c) D \bar{H} B + H \otimes \left( 4 B \bar{F} - 3 (F \bar{F} B) \right) - 3 P \bar{X} - 2 (F \bar{F} B),
\]

\[
(c)\nabla_4 A + \frac{1}{2} tr X A = -(c) D \bar{H} B - H \otimes \left( 4 B \bar{F} - 3 (F \bar{F} B) \right) - 3 P \bar{X} + 2 (F \bar{F} B),
\]

\[
(c)\nabla_4 B - \frac{1}{2}(c) D \cdot A = -2 tr X B + \frac{1}{2} A \cdot \bar{H} + \left( 3 P - 2 (F \bar{F} B) \right) \Xi + (F \bar{F} B) (c) \nabla_4 (F \bar{F} B),
\]

\[
(c)\nabla_4 B + \frac{1}{2}(c) D \cdot A = -2 tr X B - \frac{1}{2} A \cdot \bar{H} - \left( 3 P - 2 (F \bar{F} B) \right) \Xi + (F \bar{F} B) (c) \nabla_3 (F \bar{F} B).
\]

\[
(c)\nabla_3 B - (c) D \bar{F} = - tr X B + 3 P H + (F \bar{F} B) (c) D (F \bar{F} B) - \frac{1}{2} tr X (F \bar{F} B) B - tr X (F \bar{F} B) B,
\]

\[
(c)\nabla_4 B + (c) D P = - tr X B - 3 P H - (F \bar{F} B) (c) D (F \bar{F} B) - \frac{1}{2} tr X (F \bar{F} B) B - tr X (F \bar{F} B) B
\]

\[
(c)\nabla_4 P - \frac{1}{2}(c) D \cdot \bar{B} = - \frac{3}{2} tr XP - tr X (F \bar{F} B) - \bar{H} \cdot \bar{B} + \frac{1}{2} (F \bar{F} B) (c) D \cdot \bar{F} B + \bar{H} \cdot (F \bar{F} B)
\]

\[
(c)\nabla_3 P + \frac{1}{2}(c) D \cdot \bar{B} = - \frac{3}{2} tr XP - tr X (F \bar{F} B) - \bar{H} \cdot \bar{B} - \frac{1}{2} (F \bar{F} B) (c) D \cdot \bar{F} B - \bar{H} \cdot (F \bar{F} B)
\]

5.3 The reduced equations in Kerr-Newman

Neglecting the terms which are zero in Kerr-Newman, we obtain the reduced equations.

**Proposition 5.4.** The reduced equations are

\[
(c)\nabla_3 tr X + \frac{1}{2} (tr X)^2 = 0,
\]

\[
(c)\nabla_3 tr X + \frac{1}{2} tr X tr X = (c) D \cdot \bar{H} + H \cdot \bar{H} + 2 P,
\]

\[
(c)\nabla_4 tr X + \frac{1}{2} tr X tr X = (c) D \cdot \bar{H} + H \cdot \bar{H} + 2 P,
\]

\[
(c)\nabla_4 tr X + \frac{1}{2} (tr X)^2 = 0.
\]
\[
\begin{align*}
(\mathcal{C}) \mathcal{D} \text{tr} X &= (\text{tr} X - \text{tr} \mathcal{X}) H, \\
(\mathcal{C}) \mathcal{D} \text{tr} \mathcal{X} &= (\text{tr} \mathcal{X} - \text{tr} \mathcal{X}) H, \\
(\mathcal{C}) \nabla_3 H &= -\frac{1}{2} \text{tr} \mathcal{X} (H - H), \\
(\mathcal{C}) \nabla_4 H &= -\frac{1}{2} \text{tr} \mathcal{X} (H - H), \\
(\mathcal{C}) \mathcal{D} \hat{\otimes} H + H \hat{\otimes} H &= 0, \\
(\mathcal{C}) \mathcal{D} H + H \hat{\otimes} H &= 0, \\
(\mathcal{C}) \nabla_4 (FP) &= -\text{tr} \mathcal{X} (FP) \\
(\mathcal{C}) \nabla_3 (FP) &= -\text{tr} \mathcal{X} (FP) \\
(\mathcal{C}) \mathcal{D} (FP) &= -2 (FP) H \\
(\mathcal{C}) \mathcal{D} (FP) &= -2 (FP) H \\
(\mathcal{C}) \mathcal{D}(\overline{P}) &= -\left(3\overline{P} - 2 (FP) \overline{FP} P\right) H, \\
(\mathcal{C}) \mathcal{D} P &= -\left(3P - 2 (FP) \overline{FP} P\right) H \\
(\mathcal{C}) \nabla_4 P &= -\frac{3}{2} \text{tr} XP - \text{tr} X (FP) \overline{FP} P \\
(\mathcal{C}) \nabla_3 P &= -\frac{3}{2} \text{tr} XP - \text{tr} X (FP) \overline{FP} P
\end{align*}
\]

5.4 Gauge invariant quantities for perturbations of Kerr-Newman

In this section we identify the gauge-invariant quantities in linear gravitational and electromagnetic perturbations of Kerr-Newman spacetime. Those quantities will play a fundamental role in the derivation of their Teukolsky equation and in the resolution of the stability of Kerr-Newman black hole.

To identify the gauge-invariant quantities, we recall the transformations of null frames as introduced in Section 4.2. The non-conformal transformation (i.e. for which \( \lambda = 1 \)) are given by

\[
\begin{align*}
\epsilon_4' &= \epsilon_4 + f_4 e_a \\
\epsilon_3' &= \epsilon_3 + f_3 e_a \\
\epsilon_a' &= \epsilon_a + \frac{1}{2} f_4 e_4 + \frac{1}{2} f_a e_3
\end{align*}
\]

for \( f \) and \( f \) 1-tensors. Define \( F = f + i * f \) and \( \overline{F} = f + i * f \) their complexified version. Observe that \( F \) has signature 1 and \( \overline{F} \) has signature \(-1\). Then we have the following linear transformations for the curvature components [21]:

\[
\begin{align*}
A' &= A \\
B' &= B + \frac{3}{2} \overline{PF} \\
P' &= P \\
B' &= B - \frac{3}{2} PE \\
A' &= A
\end{align*}
\]

\[\text{We ignore the quadratic terms involving } F \text{ and } \overline{F}.\]
For the electromagnetic components:

\[ \begin{align*}
(F) B' &= (F) B + (F) P F \\
(F) P' &= (F) P \\
(F) P' &= (F) B - (F) P F
\end{align*} \]

For the connection components:

\[ \begin{align*}
\text{tr} X' &= \text{tr} X + \frac{1}{2} (c) D \cdot \hat{F} + \frac{1}{2} H \cdot F \\
\text{tr} X' &= \text{tr} X + \frac{1}{2} (c) D \cdot \hat{F} + \frac{1}{2} H \cdot F \\
\hat{X}' &= \hat{X} + \frac{1}{2} (c) D \otimes F + \frac{1}{2} H \otimes F \\
\hat{X}' &= \hat{X} + \frac{1}{2} (c) D \otimes F + \frac{1}{2} H \otimes F \\
H' &= H + \frac{1}{2} (c) \nabla_3 F + \frac{1}{4} \text{tr} \hat{X} F \\
\hat{H}' &= H + \frac{1}{2} (c) \nabla_4 F + \frac{1}{4} \text{tr} \hat{X} F \\
\Xi' &= \Xi + \frac{1}{2} (c) \nabla_4 F + \frac{1}{4} \text{tr} \hat{X} F \\
\Xi' &= \Xi + \frac{1}{2} (c) \nabla_4 F + \frac{1}{4} \text{tr} \hat{X} F
\end{align*} \]

The above transformations are fundamental to identify the gauge-invariant quantities in gravitational and electromagnetic perturbations of Kerr–Newman.

Recall the quantity $\mathfrak{B}$ appearing in the Bianchi identity, as defined in (12). Also define

\[ \mathfrak{B} = 2 (F) P B - 3 \hat{F} (F) B \quad (105) \]

\[ \mathfrak{X} = (c) \nabla_4 (F) B + \frac{3}{2} \text{tr} X (F) B - 2 (F) P \Xi \quad (106) \]

**Lemma 5.5.** For a linear gravitational and electromagnetic perturbation of the Kerr–Newman spacetime, the quantities $A$, $\mathfrak{B}$, $\mathfrak{X}$ and $\Xi$ are gauge-invariant.

**Proof.** The invariance of $A$ is straightforward from the gauge transformations. We check the invariance of $\mathfrak{B}$. We have

\[ \begin{align*}
\mathfrak{B}' &= 2 (F) P' B' - 3 \hat{F}' (F) B' \\
&= 2 (F) P \left( B + \frac{3}{2} \hat{F} F \right) - 3 \hat{F} \left( (F) B + (F) P F \right) \\
&= \mathfrak{B} + 2 (F) P \left( \frac{3}{2} \hat{F} F \right) - 3 \hat{F} \left( (F) P F \right) = \mathfrak{B}
\end{align*} \]

where we used (99). We check the invariance of $\mathfrak{B}$. We have

\[ \begin{align*}
\mathfrak{B}' &= 2 (F) P' B' - 3 \hat{F}' (F) B' \\
&= 2 (F) P \left( B + \frac{3}{2} \hat{F} F \right) - 3 \hat{F} \left( (F) B + (F) P F \right) \\
&= \mathfrak{B} + 2 (F) P \left( \frac{3}{2} \hat{F} F \right) - 3 \hat{F} \left( (F) P F \right) = \mathfrak{B}
\end{align*} \]
We check the invariance of $\mathcal{X}$. We have
\[
\mathcal{X}' = \nabla_4^2 (F)B + (F)PF + \frac{3}{2} \text{tr} X \left( (F)B + (F)PF \right) - 2 (F)P(\Xi + \frac{1}{2} \nabla_4 F + \frac{1}{4} \text{tr} X F)
\]
\[
= \nabla_4^2 (F)B + \frac{3}{2} \text{tr} X (F)B - 2 (F)P \Xi
\]
\[- \text{tr} X (F)PF + (F)P (\nabla_4 (F)PF) + \frac{3}{2} \text{tr} X ((F)PF) - 2 (F)P (\frac{1}{2} \nabla_4 F + \frac{1}{4} \text{tr} X F)
\]
\[
= \nabla_4^2 (F)B + \frac{3}{2} \text{tr} X (F)B - 2 (F)P \Xi = \mathcal{X}
\]
This proves the lemma.

Remark 5.6. Observe that the quantities $A$, $\tilde{\mathfrak{f}}$, $\mathfrak{B}$ and $\mathfrak{X}$ respectively generalize the quantities $\alpha$, $\tilde{f}$, $\beta$ and $\chi$ in Reissner-Nordström appearing in [14] and [15].

We summarize here some fundamental relations between the above gauge invariant quantities $A$, $\tilde{\mathfrak{f}}$, $\mathfrak{B}$ and $\mathcal{X}$ obtained as consequence of the Einstein-Maxwell equation.

Proposition 5.7. The following relations among the gauge invariant quantities $A$, $\tilde{\mathfrak{f}}$, $\mathfrak{B}$ and $\mathcal{X}$ hold true for a linear gravitational and electromagnetic perturbation of Kerr-Newman spacetime.

- The following relation between the $(c)\nabla_3$ derivative of $A$, the $(c)\mathcal{D}$ derivative of $\mathfrak{B}$ and $\tilde{\mathfrak{f}}$:
\[
(F)P \left( (c)\nabla_3 A + \frac{1}{2} \text{tr} \mathcal{X} A \right) = \frac{1}{2} (c)\mathcal{D} \otimes \mathfrak{B} + 3 \mathfrak{H} \otimes \mathfrak{B} - \left( 3 \mathcal{P} + 2 (F)P(F)P \right) \tilde{\mathfrak{f}}
\] (107)

- The following relation between the $(c)\nabla_4$ derivative of $\tilde{\mathfrak{f}}$, the $(c)\mathcal{D}$ derivative of $\mathcal{X}$ and $A$:
\[
(c)\nabla_4 \tilde{\mathfrak{f}} + \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} \mathcal{X} \right) \mathcal{X} = - \frac{1}{2} (c)\mathcal{D} \otimes \mathcal{X} - \frac{1}{2} (3 \mathfrak{H} + \mathfrak{H}) \otimes \mathcal{X} - (F)PA
\] (108)

- The following relation between the $(c)\nabla_4$ derivative of $\mathfrak{B}$, the $(c)\mathcal{D}$ derivative of $A$ and $\mathcal{X}$:
\[
(c)\nabla_4 \mathfrak{B} + 3 \text{tr} \mathcal{X} \mathfrak{B} = (F)P \left( (c)\mathcal{D} \cdot A + \mathcal{H} \cdot A \right) - \left( 3 \mathcal{P} - 2 (F)P(F)P \right) \mathcal{X}
\] (109)

- The following relation between the $(c)\nabla_3$ derivative of $\mathcal{X}$, the $(c)\mathcal{D}$ derivative of $\tilde{\mathfrak{f}}$ and $\mathfrak{B}$:
\[
(c)\nabla_3 \mathcal{X} + \frac{1}{2} \text{tr} \mathcal{X} \mathcal{X} = - (c)\mathcal{D} \cdot \tilde{\mathfrak{f}} - \mathcal{H} \cdot \mathcal{X} - 2 \mathfrak{B}
\] (110)

Proof. See Appendix A.6.

Remark 5.8. Similar relations between the gauge-invariant quantities $\alpha$, $\tilde{f}$, $\beta$ and $\chi$ hold for linear gravitational and electromagnetic perturbations of Reissner-Nordström spacetime [14] [15].

6 The Teukolsky equations for the gauge invariant quantities

In this section, we derive the Teukolsky-type equations which govern the coupled electromagnetic and gravitational perturbations of Kerr-Newman spacetime.

Among the four gauge-invariant quantities $A$, $\tilde{\mathfrak{f}}$, $\mathfrak{B}$ and $\mathcal{X}$ identified in the previous section, we pay special attention to $\mathfrak{B}$ and $\mathcal{X}$. These are the spin-1 and spin-2 quantities which transport electromagnetic and gravitational radiation in the perturbations of the Kerr-Newman spacetime. They satisfy coupled Teukolsky-like equations, which involve the gauge-invariant quantities $A$ and $\mathcal{X}$ as well.

We summarize in the following theorem the main result for the Teukolsky equations for gravitational and electromagnetic perturbations of the Kerr-Newman spacetime.

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Theorem 6.1. Consider a linear gravitational and electromagnetic perturbation of Kerr–Newman spacetime, and its associated complex tensors and gauge-invariant quantities \( A, \mathcal{F}, \mathcal{B} \) and \( \mathcal{X} \), as defined in Definition \([7, 12, 105]\) and \([106]\). Then \( \mathcal{B} \) and \( \mathcal{F} \) verify the following system of coupled Teukolsky-like equations:

\[
\mathcal{T}_1(\mathcal{B}) = 2(F)P(F)P\left(2(\mathcal{C})D \cdot \mathcal{F} + 4(\mathcal{C})D \cdot \mathcal{B} - \left(2trX - \overline{2trX}\right) \mathcal{X}\right)
\]

\[
\mathcal{T}_2(\mathcal{F}) = -(F)P\left(\left(\mathcal{C}\right) \nabla_3 A + \frac{1}{2} \left(3trX - \overline{trX}\right) A\right) + \left(\frac{3}{2} (\mathcal{C})trH\right) \mathcal{X} + (2H - \overline{H}) \mathcal{B}
\]

where the Teukolsky operator \( \mathcal{T}_1 \) for a one form \( \psi \in S_1(\mathbb{C}) \) is defined as

\[
\mathcal{T}_1(\psi) = -\mathcal{C} \nabla_4 \mathcal{C} \nabla_2 \psi + \frac{1}{2} \mathcal{C} \mathcal{D} \mathcal{D} \mathcal{C} \psi - 3\overline{trX} \mathcal{C} \nabla_3 \psi - \left(\frac{3}{2} trX + \frac{1}{2} tr\mathcal{X}\right) \mathcal{C} \nabla_4 \psi
\]

\[
+ 3H \cdot \mathcal{C} \mathcal{D} \psi + \left(\frac{1}{2} H + \frac{3}{2} \overline{H}\right) \cdot \mathcal{C} \mathcal{D} \psi + \left(-\frac{9}{2} trX tr\mathcal{X} - 4(F)P(F)P + 9(\mathcal{C})trH\right) \psi
\]

and the Teukolsky operator \( \mathcal{T}_2 \) for a symmetric traceless two tensor \( \Psi \in S_2(\mathbb{C}) \) is defined as

\[
\mathcal{T}_2(\Psi) = -\mathcal{C} \nabla_3 \mathcal{C} \nabla_4 \Psi + \frac{1}{2} \mathcal{C} \mathcal{D} \mathcal{D} \mathcal{C} \Psi - \left(\frac{3}{2} trX + \frac{1}{2} tr\mathcal{X}\right) \mathcal{C} \nabla_3 \Psi
\]

\[
- \frac{1}{2} \left(trX + \overline{trX}\right) \mathcal{C} \nabla_4 \Psi + \frac{1}{2} \mathcal{C} \mathcal{D} \mathcal{D} \Psi - \left(2H + \frac{1}{2} H\right) \cdot \mathcal{C} \mathcal{D} \Psi
\]

\[
+ \left(-\frac{3}{4} trX tr\mathcal{X} - \frac{1}{4} \overline{trX tr\mathcal{X}} + 2P - 4(F)P(F)P - \frac{3}{2} (\mathcal{C})trH\right) \Psi + \frac{1}{2} H \mathcal{D} (\mathcal{D} \cdot \Psi)
\]

Proof. The derivation of the above Teukolsky equations relies on Proposition \([5, 7]\), and is obtained in Appendix \([18]\).

We have the following remarks on Theorem 6.1:

1. Observe that the highest order terms of the Teukolsky operators defined in \([113]\) and \([114]\) are wave-like operators, as it is clear from comparing them to the expression \([57]\) for \( \Box_1 \) and \([59]\) for \( \Box_2 \). They importantly differ from the wave operator in their first order terms, \( \mathcal{C} \nabla_3, \mathcal{C} \nabla_4, \mathcal{C} \mathcal{D} \) and \( \mathcal{C} \mathcal{D} \), just like the standard Teukolsky equation in Kerr.

2. The gauge-invariant quantity \( A \) also satisfies a Teukolsky-type equation, which is given by

\[
\mathcal{T}_3(A) = 2(F)P(F)P\left(2(F) \mathcal{F} + 2\overline{trX} \mathcal{F} + (H + H) \mathcal{X}\right)
\]

where the operator \( \mathcal{T}_3(A) \) is given by

\[
\mathcal{T}_3(A) = -\mathcal{C} \nabla_4 \mathcal{C} \nabla_3 A + \frac{1}{2} \mathcal{C} \mathcal{D} \mathcal{D} \mathcal{C} \cdot A - \left(\frac{1}{2} trX + \overline{2trX}\right) \mathcal{C} \nabla_3 A - \frac{1}{2} tr\mathcal{X} \mathcal{C} \nabla_4 A
\]

\[
+ \frac{1}{2} \mathcal{C} \mathcal{D} \cdot \mathcal{D} \mathcal{C} \cdot A + \left(\frac{1}{2} H + 2H\right) \cdot \mathcal{C} \mathcal{D} \mathcal{C} \cdot A
\]

\[
+ \left(-trX tr\mathcal{X} + 2P - 2(F)P(F)P\right) A + 2H \mathcal{D} (A \cdot \mathcal{H})
\]

In the case of perturbations of Kerr spacetimes (which corresponds to Kerr-Newman spacetimes with \( (F)P = 0 \)), the above equation reduces to the Teukolsky equation of spin 2 in Kerr, as obtained in \([13]\). Equations \([112]\) and \([113]\) have no equivalent in the gravitational perturbations of Kerr spacetime.

3. In the case of perturbations of Reissner-Nordström spacetime (which corresponds to Kerr-Newman spacetimes with \( H = \mathcal{H} = 0 \)), the above system of Teukolsky equations reduce to the one obtained
in [14] and [15]. More precisely, for perturbations of Reissner-Nordström spacetime, the Teukolsky equation for $\alpha$ is given by
\[
 - (c^2) \nabla^2 \chi + 2 \partial^2 \text{div} \chi - \frac{1}{2} \text{tr} \chi (c^2) \nabla^2 \chi - \frac{5}{2} \text{tr} \chi (c^2) \nabla^2 \chi + \left( - \text{tr} \chi + 2 \rho - 2 (F \rho)^2 \right) \chi
 = 4 (F \rho) \left( (c^2) \nabla^2 \chi + \text{tr} \chi \right)
\]

The Teukolsky equation for $\chi$ is given by
\[
 - (c^2) \nabla^2 \chi - 2 \partial^2 \text{div} \chi - \text{tr} \chi (c^2) \nabla^2 \chi + \left( - \text{tr} \chi + 2 \rho + 4 (F \rho)^2 \right) \chi
 = - (F \rho) \left( (c^2) \nabla^2 \chi + \text{tr} \chi \right)
\]

The Teukolsky equation for $\bar{\beta}$ is given by
\[
 - (c^2) \nabla^2 \bar{\beta} - 2 \partial^2 \text{div} \bar{\beta} - 3 \text{tr} \chi (c^2) \nabla^2 \bar{\beta} - 2 \text{tr} \chi (c^2) \nabla^2 \bar{\beta} + \left( - \frac{9}{2} \text{tr} \chi + 4 (F \rho)^2 \right) \bar{\beta}
 = 2 (F \rho)^2 \left( 4 \text{div} \bar{\chi} + \text{tr} \chi \right)
\]

The comparison between these equations and the equations in Kerr-Newman obtained in Theorem 6.1 is immediate, by defining $A = \alpha + i * \alpha$, $\tilde{\beta} = \beta + i * \beta$, $\bar{\chi} = \chi + i * \chi$.

In what follows, we write the Teukolsky system (111) and (112) in the following schematic form:
\[
\begin{align*}
\mathcal{T}_1(\mathcal{B}) &= \mathcal{M}_1[\tilde{\beta}, \bar{\chi}] \\
\mathcal{T}_2(\tilde{\beta}) &= \mathcal{M}_2[A, \chi, \mathcal{B}]
\end{align*}
\]

where $\mathcal{M}_1[\tilde{\beta}, \bar{\chi}]$ denotes the expression involving $\tilde{\beta}$ and $\bar{\chi}$ on the right hand side of (111), and $\mathcal{M}_2[A, \chi, \mathcal{B}]$ denotes the expression involving $A$, $\chi$ and $\mathcal{B}$ on the right hand side of (112), i.e.
\[
\begin{align*}
\mathcal{M}_1[\tilde{\beta}, \bar{\chi}] &:= 2 (F \rho)^2 \left( 2 (c^2) \tilde{\beta} + 4 \tilde{\mathcal{H}} \cdot \bar{\chi} - (2 \text{tr} \bar{\chi} - \text{tr} \bar{\chi}) \bar{\chi} \right) \\
\mathcal{M}_2[A, \chi, \mathcal{B}] &:= - (F \rho) \left( (c^2) \nabla^2 A + \frac{1}{2} (3 \text{tr} \bar{\chi} - \text{tr} \bar{\chi}) A \right) + \left( \frac{3}{2} (c^2) \nabla^2 \tilde{\mathcal{H}} \right) \bar{\chi} \\
&\quad + (2 \mathcal{H} - \tilde{\mathcal{H}}) \mathcal{B}
\end{align*}
\]

The equations in system (113) are the Teukolsky equations governing the gravitational and electromagnetic perturbations of the Kerr-Newman spacetime.

7 The Chandrasekhar transformation in Kerr-Newman spacetime

In this section, we describe the Chandrasekhar transformation applied to the equations of system (114), with the goal of obtaining a system of coupled wave equations for which quantitative decay statements can be obtained in physical space. In particular, we want to retain the good properties enjoyed by the Regge-Wheeler equation in Kerr and the system of Regge-Wheeler-type equations in Reissner-Nordström.

7.1 Statement of the main theorem

We summarize in the following theorem the main result of this paper, which contains the system of Regge-Wheeler-type equations which govern the gravitational and electromagnetic perturbation of the Kerr-Newman spacetime.

Main Theorem. Consider a linear gravitational and electromagnetic perturbations of the Kerr–Newman spacetime $\mathfrak{g}_{m,a,Q}$ and its associated complex tensors and gauge-invariant quantities $\mathcal{B}$ and $\mathfrak{g}$.
Then, their Chandrasekhar-transformed quantities \( p \) and \( q^F \) defined as

\[
p = \frac{i}{4}\eta_0^{\frac{3}{2}} \left( (c)\nabla_3 B + (2tr\chi + ip_1^{(a)}tr\chi) B \right), \quad q^F = \eta_1^{\frac{1}{2}} \left( (c)\nabla_3 \tilde{\Psi} + (tr\chi + ip_2^{(a)}tr\chi) \tilde{\Psi} \right)
\]

for any real numbers \( p_1 \) and \( p_2 \) satisfy the following coupled system of wave equations:

\[
\begin{align*}
\Box_1 p + V_1(r, \theta) p - \frac{2a\cos\theta}{|q|^2} \partial_r p &= 4Q^2 \frac{7t}{|q|^5} (c)D \cdot q^F + a \cdot \text{l.o.t.}_1 \quad (118) \\
\Box_2 q^F + V_2(r, \theta) q^F - i\frac{4a\cos\theta}{|q|^2} \partial_r q^F &= -\frac{q^3}{2|q|^5} (c)D\otimes p + a \cdot \text{l.o.t.}_2 
\end{align*}
\]

where

- \( \Box_1 \) and \( \Box_2 \) denote the wave operators for the tensors \( p \in S_1 \) and \( q^F \in S_2 \) (of conformal type 0) as in (58) and (60) respectively,
- the potentials \( V_1(r, \theta) \) and \( V_2(r, \theta) \) are given by
  \[
  V_1(r, \theta) = \frac{1}{4} tr\chi tr\chi - 5^{(F)}\rho^2, \quad V_2(r, \theta) = tr\chi tr\chi - 2^{(F)}\rho^2
  \]
- the first order terms \( \partial_t p \) and \( \partial_t q^F \) denote the Lie derivative with respect to the vectorfield \( \partial_t \) of \( p \) and \( q^F \) respectively,
- \( a \cdot \text{l.o.t.}_1 \) and \( a \cdot \text{l.o.t.}_2 \) denote terms which vanish for vanishing angular momentum and are lower order terms with respect to \( p \) and \( q^F \) respectively.

\[
a \cdot \text{l.o.t.}_1 = \frac{i}{4}\eta_0^{\frac{3}{2}} \left( (2^{(F)}p)(c)D \cdot (d_{(\cdot)D}(r, \theta) (c)D \cdot \tilde{\Psi}) + d_1(r, \theta) (c)\nabla_3 \tilde{\Psi} + d_0(r, \theta) \tilde{\Psi} + c^{(a)}_1(r, \theta) X \right) \\
a \cdot \text{l.o.t.}_2 = \eta_1^{\frac{1}{2}} \left( c_{(\cdot)D\otimes}(r, \theta) (c)D\otimes B + c_1(r, \theta) (c)\nabla_3 B + c_0(r, \theta) B + d_0(r, \theta) (c)\nabla_3 \tilde{\Psi} \\
- \frac{3}{2} (c)\nabla_3 H \cdot (c)D \cdot \tilde{\Psi} - d_1(r, \theta) (c)D \cdot \tilde{\Psi} - d_0(r, \theta), (c)\nabla_3 \tilde{\Psi} - d_1(r, \theta) (c)\nabla_3 \tilde{\Psi} \\
+ d_0(r, \theta) (c)\nabla_3 \tilde{\Psi} + \hat{d}_0(r, \theta) (c)D\otimes X + d_{-1}(r, \theta) X \right)
\]

where the coefficients \( c_{(\cdot)D}(r, \theta) \), \( c_{(\cdot)D\otimes}(r, \theta) \), \( d_{(\cdot)D}(r, \theta) \), \( d_{(\cdot)D\otimes}(r, \theta) \), \( c_{(\cdot)D\otimes}(r, \theta) \), \( \hat{d}_1(r, \theta) \) are given by (163), (164), (165), (166), (167), (168), (169), (168).

Observe that the structure of the above equations retains the good properties of the Regge-Wheeler type equations which allowed for a quantitative physical-space analysis in the case of Kerr and Reissner-Nordström, like the good divergence properties on the left hand side of (10) in Kerr (e.g. a first order term of the form \( i\partial_t \)) and symmetry properties on the right hand side of the system (11) in Reissner-Nordström (e.g. formally adjoint operators).

### 7.2 Proof of the Main Theorem

In this section, we present the proof to the Theorem. The main steps of the proof are the following.

1. The Chandrasekhar transformation consists in applying one incoming null derivatives to the equations for the gauge-invariant quantities \( B \) and \( \tilde{\Psi} \). We represent such derivatives with the following conformal differential operator \( P_C : S_k \rightarrow S_k \) defined as

\[
P_C(\Psi) := (c)\nabla_3 \Psi + C \Psi
\]

for a complex scalar function \( C \) of signature 1. More precisely, \( C \) is taken to be a linear combination of \( tr\chi \) and \( tr\chi \).
We apply such operator to $\mathfrak{B}$ and $\mathfrak{F}$ for a different choice of $C$. In particular we define

$$\mathfrak{P} := \mathcal{P}_{C_1}(\mathfrak{B}) = (c)\nabla_3 \mathfrak{B} + C_1 \mathfrak{B}, \quad \mathfrak{Q} := \mathcal{P}_{C_2}(\mathfrak{F}) = (c)\nabla_3 \mathfrak{F} + C_2 \mathfrak{F}$$

(121)

for $C_1$ and $C_2$ to be determined.

2. We apply the operator $\mathcal{P}_{C_1}$ to the equation for $\mathfrak{B}$, and the operator $\mathcal{P}_{C_2}$ to the equation for $\mathfrak{F}$ in (115), and by writing the commutator between $\mathfrak{P}$ and $\mathcal{T}$, the left hand sides reduce to

$$\mathcal{P}_{C_1}(\mathcal{T}_1(\mathfrak{B})) = \mathcal{T}_1(\mathcal{P}_{C_1}(\mathfrak{B})) + [\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) = \mathcal{T}_1(\mathfrak{P}) + [\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B})$$

$$\mathcal{P}_{C_2}(\mathcal{T}_2(\mathfrak{F})) = \mathcal{T}_2(\mathcal{P}_{C_2}(\mathfrak{F})) + [\mathcal{P}_{C_2}, \mathcal{T}_2](\mathfrak{F}) = \mathcal{T}_2(\mathfrak{Q}) + [\mathcal{P}_{C_2}, \mathcal{T}_2](\mathfrak{F})$$

From (115) we therefore obtain

$$\mathcal{T}_1(\mathfrak{P}) + [\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) = \mathcal{C}_{C_1}\left(M_1[\mathfrak{F}, \mathfrak{X}]\right)$$

(122)

$$\mathcal{T}_2(\mathfrak{Q}) + [\mathcal{P}_{C_2}, \mathcal{T}_2](\mathfrak{F}) = \mathcal{C}_{C_2}\left(M_2[\mathfrak{A}, \mathfrak{X}, \mathfrak{B}]\right)$$

(123)

3. In Proposition 7.1 we compute the commutators $[\mathcal{P}_{C}, \mathcal{T}]$. The proof relies on commutation lemmas which are presented in the Appendix.

At this stage we impose conditions on the real parts of both $C_1$ and $C_2$ in order to eliminate the first order term in $(c)\nabla_3 \mathfrak{B}$ and $(c)\nabla_3 \mathfrak{F}$ from the left hand sides of (122) and (123). This is done in Section 7.2.1.

4. By using the expressions for $\mathcal{T}_1$ and $\mathcal{T}_2$ in (113) and (114) together with the commutators computed in the previous step, we are able to obtain the left hand sides of the equations for $\mathfrak{P}$ and $\mathfrak{Q}$. Those equations still contain first order terms in both $(c)\nabla_3 \mathfrak{P}$ (and $(c)\nabla_3 \mathfrak{Q}$) and $(c)\nabla_4 \mathfrak{P}$ (and $(c)\nabla_4 \mathfrak{Q}$).

We now show that there is a rescaling of $\mathfrak{P}$ and $\mathfrak{Q}$ in terms of the functions $q$ and $\mathfrak{q}$ such that the first order terms can be all reduced to only terms of the form $i\partial_i$. More precisely, defining

$$p = q^3 \mathfrak{q}^2 \mathfrak{P}, \quad q^F = q^2 \mathfrak{Q}$$

the left hand side of the equations for $p$ and $q^F$ reduce to

$$\square_1 p + V_1(r, \theta) \ p - i \frac{2a \cos \theta}{|q|^2} \partial_\theta p, \quad \square_2 q^F + V_2(r, \theta) \ q^F - i \frac{4a \cos \theta}{|q|^2} \partial_\theta q^F$$

respectively. This is done in Section 7.2.2 and Section 7.2.3.

5. Finally, we compute the result of the Chandrasekhar operator $\mathcal{P}_{C_1}\left(M_1[\mathfrak{F}, \mathfrak{X}]\right)$ and $\mathcal{P}_{C_2}\left(M_2[\mathfrak{A}, \mathfrak{X}, \mathfrak{B}]\right)$ applied to the left hand side of equations (122) and (123). By writing those left hand side in terms of the above defined $p$ and $q^F$, the right hand side of the equations reduce to

$$4Q^2 \frac{\mathfrak{F}^3}{|q|^5} (c)\mathcal{D} \cdot q^F + a \cdot \text{l.o.t.}_1, \quad -\frac{q^3}{2|q|^3} (c)\mathcal{D} \mathfrak{P} + a \cdot \text{l.o.t.}_2$$

respectively. This is done in Section 7.2.4 and concludes the proof of the Main Theorem.

We separate the proof of the above steps in the next few subsections.

7.2.1 The commutators

Recall that after applying the Chandrasekhar operator to the system (115) and commuting with the Telnikov operators, we obtain equations (122) and (123) for $\mathfrak{P}$ and $\mathfrak{Q}$ respectively. We here compute the commutators $[\mathcal{P}_{C_1}, \mathcal{T}_1]$ and $[\mathcal{P}_{C_2}, \mathcal{T}_2]$.

To eliminate the terms in $(c)\nabla_3 \mathfrak{B}$ and $(c)\nabla_3 \mathfrak{F}$, which can not be written in terms of $\mathfrak{P}$ and $\mathfrak{Q}$, we obtain a constraint on the real parts of $C_1$ and $C_2$. We summarize it in the following Proposition.
Proposition 7.1. Suppose that $C_1$ and $C_2$ are complex scalar functions which satisfy respectively:

\[
(c) \nabla_3 C_1 + \frac{C_1}{2} (\text{tr}X + \text{tr}\bar{X}) - \text{tr}X \nabla_3 \bar{X} = 0
\]  \hspace{1cm} (124)

\[
(c) \nabla_3 C_2 + \frac{C_2}{2} (\text{tr}X + \text{tr}\bar{X}) - \frac{1}{2} \text{tr}X \nabla_3 \bar{X} = 0
\]  \hspace{1cm} (125)

Then the commutators between the Chandrasekhar operator $P$ and the Teukolsky operator $T$ are respectively given by

\[
[P_{C_1}, T_1](\mathfrak{g}) = \frac{1}{2} H \cdot \nabla_3 \bar{X} \nabla_3 P + \frac{1}{2} H P - \frac{1}{2} (\text{tr}X + \text{tr}\bar{X}) (c) \nabla_4 \bar{X} + W_1 \bar{X}
\]

\[
= -\frac{1}{2} (\text{tr}X + \text{tr}\bar{X}) M_1[A, X, \mathfrak{g}] - \text{l.o.t.} LHS[\mathfrak{g}],
\]

and

\[
[P_{C_2}, T_2](\mathfrak{g}) = \frac{1}{2} H \cdot \nabla_3 \bar{X} \nabla_3 P + \frac{1}{2} H P - \frac{1}{2} (\text{tr}X + \text{tr}\bar{X}) (c) \nabla_4 \bar{X} + W_2 \bar{X}
\]

\[
= -\frac{1}{2} (\text{tr}X + \text{tr}\bar{X}) M_2[A, X, \mathfrak{g}] - \text{l.o.t.} LHS[\bar{X}],
\]

where

- the potential $W_1$ and $W_2$ are given by

\[
W_1 = (c) \nabla_4 C_1 - \frac{3}{2} trX tr\bar{X} - 8 P - 2 (F) P (F) P - 3 (c) \nabla_3 H + H \cdot \bar{H}
\]

\[
W_2 = (c) \nabla_4 C_2 - \frac{3}{4} trX tr\bar{X} - \frac{1}{4} trX trX - 6 P - 2 (F) P (F) P
\]

\[
= -\frac{3}{2} (c) \nabla_3 H + H \cdot \bar{H}) - \frac{1}{2} (c) \nabla_3 \bar{X} + H \cdot \bar{H})
\]

- $\text{l.o.t.} LHS[\mathfrak{g}]$ and $\text{l.o.t.} LHS[\bar{X}]$ denote lower order terms with respect to $\mathfrak{g}$ and $\bar{X}$ respectively. More precisely, they are given by

\[
\text{l.o.t.} LHS[\mathfrak{g}] = -2 (F) P (F) P (\text{tr}X + \text{tr}\bar{X}) \mathfrak{g} + a \cdot \text{l.o.t.} LHS[\mathfrak{g}]
\]

\[
\text{l.o.t.} LHS[\bar{X}] = 2 (F) P (F) P (\text{tr}X + \text{tr}\bar{X}) \bar{X} + a \cdot \text{l.o.t.} LHS[\bar{X}]
\]

where $a \cdot \text{l.o.t.} LHS$ and $a \cdot \text{l.o.t.} LHS$ denote terms which vanish for vanishing angular momentum, and are of the form

\[
a \cdot \text{l.o.t.} LHS = -c_{\mathfrak{g}}(r, \theta) \cdot (c) \nabla_3 \bar{X} - c_{\mathfrak{g}}(r, \theta) \cdot (c) \nabla_3 \bar{X} + c_{\mathfrak{g}}(r, \theta) \bar{X}
\]

\[
a \cdot \text{l.o.t.} LHS = -d_{\mathfrak{g}}(r, \theta) \cdot (c) \nabla_3 \bar{X} - d_{\mathfrak{g}}(r, \theta) \cdot (c) \nabla_3 \bar{X} + d_{\mathfrak{g}}(r, \theta) \bar{X}
\]

where the coefficients $c_{\mathfrak{g}}(r, \theta)$, $c_{\mathfrak{g}}(r, \theta)$, $c_{\mathfrak{g}}(r, \theta)$, $d_{\mathfrak{g}}(r, \theta)$, $d_{\mathfrak{g}}(r, \theta)$ are given by $[163]$, $[164]$, $[165]$, $[166]$. $[167]$.

Proof. See Appendix C.1.

Observe that the highest order terms of the commutators $[P_{C_1}, T_1](\mathfrak{g})$ and $[P_{C_2}, T_2](\mathfrak{g})$ are the same, while the zero-th order term of the potentials $W_1$ and $W_2$ and the lower order terms differ in the two cases. The conditions on the scalar functions $C_1$ and $C_2$ given by \[(124)\] and \[(125)\] differ in the two cases, but are both conditions on the real part of $C_1$ and $C_2$. They imply that their real parts coincide with the rescaling used in the case of Reissner-Nordström, i.e.

\[
C_1 = 2 \text{tr} \chi + ip_1 (a) \text{tr} \chi, \hspace{1cm} C_2 = \text{tr} \chi + ip_2 (a) \text{tr} \chi \hspace{1cm} \text{for any real numbers } p_1, p_2
\]
We can use this restriction on the $C_1$ and $C_2$ to simplify the expressions for the potentials. Notice that any term in the potential which vanish for vanishing angular momentum can be diverted to the terms $c_1(r, \theta)$ $(c)\nabla_3 \mathfrak W$ and $d_1(r, \theta)$ $(c)\nabla_3 \mathfrak Y$ in the lower order terms. We therefore write
\[
W_1 = (c)\nabla_4 C_1 - \frac{3}{2}tr X X - 8 P - 2 (F) P (F) - 3 (c) D \cdot H + H \cdot H
\]
\[
= (c)\nabla_4 (2tr X) - \frac{3}{2}tr X X - 8\rho - 2 (F) P^2 + a \cdot \text{terms}
\]
and
\[
W_2 = (c)\nabla_4 C_2 - \frac{3}{4}tr X X - \frac{1}{4}tr X X - 6 P - 2 (F) P (F)
\]
\[
= (c)\nabla_4 (tr X) - tr X X - 6\rho - 2 (F) P^2 + a \cdot \text{terms}
\]
as a consequence of conditions (124) and (125).

7.2.2 The left hand side of the equation for $p$

We now consider the left hand side of the equation (122) for $\mathfrak P$. Using (113) to write
\[
\mathcal T_1(\mathfrak P) = -(c)\nabla_3 (c)\nabla_4 \mathfrak P - 3 tr X (c)\nabla_3 \mathfrak P - \left(\frac{3}{2} tr X + \frac{1}{2} tr X\right) \quad (c)\nabla_4 \mathfrak P
\]
\[
+ \left(-\frac{9}{2} tr X X - 4 (F) P (F) P + 9 \mathfrak P \cdot H\right) \quad \mathfrak P
\]
\[
+ \frac{1}{2} (c) D \cdot (c) D \otimes \mathfrak P + 3 H \cdot (c) D \otimes \mathfrak P + \left(\frac{1}{2} H + \frac{3}{2} \mathfrak P\right) \cdot (c) D \mathfrak P
\]
and Proposition (14) to write $[P_{C_1}, \mathcal T_1](\mathfrak P)$, equation (122) becomes
\[
-(c)\nabla_3 (c)\nabla_4 \mathfrak P - 3 tr X (c)\nabla_3 \mathfrak P - (2tr X + tr X) \quad (c)\nabla_4 \mathfrak P
\]
\[
+ \left(-\frac{9}{2} tr X X - 4 (F) P (F) P + 9 \mathfrak P \cdot H + W_1\right) \quad \mathfrak P
\]
\[
+ \frac{1}{2} (c) D \cdot (c) D \otimes \mathfrak P + \left(3 H + \frac{1}{2} H\right) \cdot (c) D \otimes \mathfrak P + \left(\frac{1}{2} H + 2 \mathfrak P\right) \cdot (c) D \mathfrak P
\]
\[
= P_{C_1} \left(\mathcal M_1[\mathfrak S, \mathfrak X]\right) + \frac{1}{2} \left(\text{tr} X + \frac{1}{2} \text{tr} X\right) \mathcal M_1[\mathfrak S, \mathfrak X] + 1. \quad \text{o.t.} \quad LHS[\mathfrak P]
\]
Since $\mathfrak P$ is a one-tensor in $S_1(\mathbb C)$ of conformal type 0, we can use (58) to write
\[
\Box \mathfrak P = -(c)\nabla_3 (c)\nabla_4 \mathfrak P + \frac{1}{2} (c) D \cdot (c) D \otimes \mathfrak P - \frac{1}{2} tr X (c)\nabla_4 \mathfrak P - \frac{1}{2} tr X (c)\nabla_3 \mathfrak P
\]
\[
+ \frac{1}{2} (H \cdot (c) D) \mathfrak P + \frac{1}{2} (H \cdot (c) D) \mathfrak P + \left(\frac{1}{8} tr X tr X + \frac{1}{8} tr X tr X + P - (F) P (F) P\right) \quad \mathfrak P
\]
From the above, we then obtain
\[
\Box \mathfrak P = \frac{5}{2} tr X tr X + \left(2tr X + \frac{1}{2} tr X\right) \nabla_4 \mathfrak P - \left(\frac{5}{2} H + \frac{1}{2} H\right) \cdot (c) D \mathfrak P + 2 (c) D \otimes \mathfrak P
\]
\[
+ \left(\frac{1}{8} tr X tr X + \frac{37}{8} tr X tr X + P + 3 (F) P (F) P - 9 H \cdot H \cdot H - W_1\right) \quad \mathfrak P
\]
\[
+ P_{C_1} \left(\mathcal M_1[\mathfrak S, \mathfrak X]\right) + \frac{1}{2} \left(\text{tr} X + \frac{1}{2} \text{tr} X\right) \mathcal M_1[\mathfrak S, \mathfrak X] + 1. \quad \text{o.t.} \quad LHS[\mathfrak P]
\]
(126)
We now define \( p \) to be a rescaled version of \( \Psi \) so that it absorbs the first order terms of the above equation. Define

\[ p = q^\frac{1}{2}q^\frac{2}{4} \Psi, \quad \text{where} \quad q = r + ia \cos \theta, \quad q = r - ia \cos \theta \]  

(127)

Using (126), we have

\[ \nabla_3(q^\frac{3}{2}q^\frac{2}{4}) = \left( \frac{1}{4} \nabla_X + \frac{9}{4} \nabla_X \right) q^\frac{1}{2}q^\frac{2}{4} \]

\[ \nabla_4(q^\frac{3}{2}q^\frac{2}{4}) = \left( \frac{1}{4} \nabla_X + \frac{9}{4} \nabla_X \right) q^\frac{1}{2}q^\frac{2}{4} \]

\[ \nabla(q^\frac{3}{2}q^\frac{2}{4}) = \frac{1}{2} \nabla(q^\frac{3}{2}q^\frac{2}{4}) + \frac{1}{2} \nabla(q^\frac{3}{2}q^\frac{2}{4}) = \left( \frac{9}{4} H + \frac{1}{4} \Pi + \frac{1}{4} H + \frac{9}{4} \Pi \right) q^\frac{1}{2}q^\frac{2}{4} \]

Observe that for a scalar function \( f \), we have

\[ \Box_1(f\Psi) = \Box(f\Psi) + f\Box_1(\Psi) - \nabla_3 f \nabla_4 \Psi - \nabla_4 f \nabla_3 \Psi + 2 \nabla f \cdot \nabla \Psi \]

(128)

Using (126), one computes the wave equation for the rescaled \( p \), for \( f = q^\frac{1}{2}q^\frac{2}{4} \):

\[ \Box_1 p = \Box(f\Psi) + f\Box_1(\Psi) - \nabla_3 f \nabla_4 \Psi - \nabla_4 f \nabla_3 \Psi + 2 \nabla f \cdot \nabla \Psi \\ = \left( \frac{1}{4} \nabla_X + \frac{9}{4} \nabla_X \right) \nabla_3 \Psi + \left( \frac{1}{4} \nabla_X + \frac{9}{4} \nabla_X \right) \nabla_4 \Psi \\ + \left( \frac{9}{4} H + \frac{1}{4} \Pi + \frac{1}{4} H + \frac{9}{4} \Pi \right) f \nabla_3 \Psi - \nabla_3 f \nabla_4 \Psi - \nabla_4 f \nabla_3 \Psi + 2 \nabla f \cdot \nabla \Psi \\ + f \Delta_{C_1}(M_1[\delta, \xi]) + \frac{1}{2} (\nabla_X + \nabla_X) f M_1[\delta, \xi] + f \text{l.o.t.}_LHS[\Psi] \]

Writing \( (F \cdot \nabla)U = 2F \cdot \nabla U \), this gives

\[ \Box_1 p = \frac{1}{4} \left( \nabla_X + \nabla_X \right) f \nabla_3 \Psi + \frac{1}{4} \left( \nabla_X + \nabla_X \right) f \nabla_4 \Psi + \left( \frac{9}{4} H + \frac{1}{4} \Pi + \frac{1}{4} H + \frac{9}{4} \Pi \right) \cdot f \nabla_3 \Psi \\ + \left( \frac{9}{4} H + \frac{1}{4} \Pi + \frac{1}{4} H + \frac{9}{4} \Pi \right) f \nabla_3 \Psi + \left( \frac{9}{4} H + \frac{1}{4} \Pi + \frac{1}{4} H + \frac{9}{4} \Pi \right) f \nabla_3 \Psi \\ + f \Delta_{C_1}(M_1[\delta, \xi]) + \frac{1}{2} (\nabla_X + \nabla_X) M_1[\delta, \xi] + \text{l.o.t.}_LHS[\Psi] \]

Recall that from the previous subsection \( W_1 = -\frac{a}{2} \text{tr} \chi \nabla_3 \Psi - 4 \rho - 2 (F)^2 + a \cdot \text{terms} \), and the terms which vanish for vanishing angular momentum can be diverted to the \( \text{l.o.t.}_LHS[\Psi] \). We can therefore write

\[ f \nabla_3 \Psi = \nabla_3 p + \frac{5}{2} \text{tr} \chi p + c_1(r, \theta) (c) \nabla_3 \mathcal{B} + c_0(r, \theta) \mathcal{B} \]

\[ f \nabla_4 \Psi = \nabla_4 p + \frac{5}{2} \text{tr} \chi p + c_1(r, \theta) (c) \nabla_3 \mathcal{B} + c_0(r, \theta) \mathcal{B} \]

\[ f \nabla_5 \Psi = \nabla p + c_1(r, \theta) (c) \nabla_3 \mathcal{B} + c_0(r, \theta) \mathcal{B} \]

and

\[ \Box(f) = \Box(q^\frac{1}{2}q^\frac{2}{4}) = \Box(r^5) + a \cdot \text{terms} = r^5 \left( -\frac{15}{2} \kappa \mathcal{L} - 5 \rho \right) + a \cdot \text{terms} \]
The above gives
\[
\Box_4 p = \left( -\frac{1}{4} \text{tr}X + \frac{1}{4} \text{tr} \chi \right) \nabla_3 p + \left( -\frac{1}{4} \text{tr}X + \frac{1}{4} \text{tr} \chi \right) \nabla_4 p + \left( -\frac{1}{2} H + \frac{1}{2} \hat{H} - \frac{1}{2} H + \frac{1}{2} \hat{H} \right) \cdot \nabla p
\]
\[+ \left( \frac{19}{4} \text{tr} \chi \text{tr}X + \rho + 3 (F)^2 - \left( -\frac{5}{2} \text{tr} \chi \text{tr}X - 4 \rho - 2 (F)^2 \right) \right) p
\]
\[+ q \frac{2}{7} \left( \mathcal{P}_{C_1} (M_1 [\hat{\tau}], X) + \frac{1}{2} (\text{tr}X + \text{tr} \chi) M_1 [\hat{\tau}, X] + 1 \text{.o.t.} LHS [\Omega] \right)
\]

Using Lemma 5.2 we finally obtain
\[
\Box_4 p + V_1 (r, \theta) - \frac{2 \alpha \cos \theta}{|q|^2} \partial \nu p
\]
\[= q \frac{2}{7} \left( \mathcal{P}_{C_1} (M_1 [\hat{\tau}], X) + \frac{1}{2} (\text{tr}X + \text{tr} \chi) M_1 [\hat{\tau}, X] + 1 \text{.o.t.} LHS [\Omega] \right)
\]

where the potential reduces to
\[
V_1 (r, \theta) = \frac{1}{4} \text{tr} \chi \text{tr}X - 5 (F)^2
\]
as stated in the Theorem.

7.2.3 The left hand side of the equation for \(q\)

We now consider equation (128) for \(q\). Using (114) to write
\[
\mathcal{T}_2 (\Omega) = - \nabla_3 \nabla_4 \Omega - \left( \frac{3}{4} \text{tr}X + \frac{1}{2} \text{tr}X \right) \nabla_3 \Omega - \frac{1}{2} \left( \text{tr}X + \text{tr} \chi \right) \nabla_4 \Omega
\]
\[+ \left( \frac{3}{4} \text{tr}X \text{tr}X - \frac{1}{4} \text{tr}X \text{tr}X + 3 \hat{H} - P + 4 (F)^2 \right) \Omega + \frac{1}{2} \hat{H} \hat{\Omega} \cdot \Omega
\]
\[+ \frac{1}{2} \nabla_3 \nabla_4 \Omega = \mathcal{P}_{C_2} (M_2 [A, \hat{\tau}, \mathfrak{B}]) + \frac{1}{2} \left( \text{tr}X + \text{tr} \chi \right) M_2 [A, \hat{\tau}, \mathfrak{B}] + 1 \text{.o.t.} LHS [\Omega]
\]

Since \(\Omega\) is a symmetric traceless two tensor in \(S_2 (\mathbb{C})\) of conformal type 0, we can use (60) to write
\[
\Box_2 \Omega = - \nabla_3 \nabla_4 \Omega + \frac{1}{2} \nabla_3 \nabla_4 \Omega - \frac{1}{2} \text{tr}X \nabla_3 \Omega - \frac{1}{2} \text{tr}X \nabla_4 \Omega
\]
\[+ \frac{1}{2} \left( H \cdot \nabla_3 \nabla_4 \Omega \right) + \frac{1}{2} \left( \hat{H} \cdot \nabla_3 \nabla_4 \Omega \right) + \left( -\frac{1}{4} \text{tr}X \text{tr}X - 4 \rho - 2 (F)^2 \right) \Omega + \frac{1}{2} \hat{H} \hat{\Omega} \cdot \Omega
\]

From the above we then obtain
\[
\Box_2 \Omega = \frac{3}{2} \text{tr}X \nabla_3 \nabla_4 \Omega + \frac{1}{2} \left( \text{tr}X + \text{tr} \chi \right) \nabla_3 \nabla_4 \Omega - \frac{1}{2} H \cdot \nabla_3 \Omega - \left( \frac{3}{2} H + \frac{1}{2} \hat{H} \right) \cdot \nabla_3 \Omega
\]
\[+ \left( \frac{1}{2} \text{tr}X \text{tr}X - 3 \hat{H} - P + 2 (F)^2 \right) \Omega - \frac{1}{2} \hat{H} \hat{\Omega} \cdot \Omega
\]
\[+ \mathcal{P}_{C_2} (M_2 [A, \hat{\tau}, \mathfrak{B}]) + \frac{1}{2} \left( \text{tr}X + \text{tr} \chi \right) M_2 [A, \hat{\tau}, \mathfrak{B}] + 1 \text{.o.t.} LHS [\Omega]
\]
We now define $q^F$ to be a rescaled version of $\Omega$ so that it absorbs the first order terms of the above equation. Define

$$q^F = q^2 \Omega$$

where $q = r + ia \cos \theta$, $\eta = r - ia \cos \theta$ (131)

Using (122), we have

$$\nabla_3(q^2) = \left( \frac{1}{2} \text{tr} \chi + \text{tr} X \right) q^2$$

$$\nabla_4(q^2) = \left( \frac{1}{2} \text{tr} X + \text{tr} X \right) q^2$$

$$\nabla(q^2) = \left( H + \frac{1}{2} \eta + \frac{1}{2} H + \eta \right) q^2$$

Using (128) and (130) for $f = q^2$, we obtain the wave equation for the rescaled $q$ to be:

$$\Box_2 q^F = \Box(f) \Omega + f \Box_2(\Omega) - \nabla_3 f \nabla_4 \Omega - \nabla_4 f \nabla_3 \Omega + 2 \nabla f \cdot \nabla \Omega$$

$$= \left( \frac{1}{2} \text{tr} X + \frac{3}{2} - 1 \text{tr} X \right) f^{(c)} \nabla_3 \Omega + \left( \frac{1}{2} - 1 \text{tr} X + (1 - \frac{1}{2} \text{tr} X) \right) f^{(c)} \nabla_4 \Omega$$

$$+ \left( \frac{1}{2} \text{tr} X - 3 \eta - P - 2 (P^p P^p + \frac{3}{2} \text{tr} D) \cdot H - W_2 \right) f \Omega - \frac{1}{2} H \nabla \nabla (H) + \Box(f) \Omega$$

$$- \frac{1}{2} f \nabla \nabla (H \cdot \Omega) - \left( \frac{1}{2} \eta \right) \cdot f^{(c)} D \Omega - \left( \frac{3}{2} H + H \right) \cdot f^{(c)} D \Omega + 2 \left( H + \frac{1}{2} \eta + \frac{1}{2} H + \eta \right) f \cdot \nabla \Omega$$

$$+ f \left( \mathcal{P}_{C_2} (M_2[A, \chi, \mathfrak{B}] \right) + \frac{1}{2} (\text{tr} \chi + \text{tr} X) M_2[A, \chi, \mathfrak{B}] + \text{l.o.t.}LH^S |\Omega|)$$

Writing $(F \cdot D)U = 2F \cdot \nabla U$, this gives

$$\Box_2 q^F = \frac{1}{2} \left( \text{tr} X - \text{tr} X \right) f \nabla_3 \Omega + \frac{1}{2} \left( \text{tr} X - \text{tr} X \right) f \nabla_4 \Omega + \left( - H + \eta - H + \eta \right) f \nabla \Omega$$

$$+ \left( \frac{1}{2} \text{tr} X - 3 \eta - P - 2 (P^p P^p + \frac{3}{2} \text{tr} D) \cdot H - W_2 + f^{-1} \Box(f) \Omega$$

$$+ q^2 \left( \mathcal{P}_{C_2} (M_2[A, \chi, \mathfrak{B}] \right) + \frac{1}{2} (\text{tr} X + \text{tr} X) M_2[A, \chi, \mathfrak{B}] + \text{l.o.t.}LH^S |\Omega|)$$

Recall that from the previous subsection $W_2 = - \frac{3}{2} \text{tr} \chi r X - 4 \rho - 2 (P^p)^2 + a \cdot \text{terms}$, and the terms which vanish for vanishing angular momentum can be diverted to the $\text{l.o.t.}LH^S |\Omega|$. We can therefore write

$$f \nabla_3 \Omega = \nabla_3 q^F - \frac{3}{2} \text{tr} \chi f^F + c_1(r, \theta) \nabla_3 \mathfrak{F}_3 + c_0(r, \theta) \mathfrak{S}$$

$$f \nabla_4 \Omega = \nabla_4 q^F - \frac{3}{2} \text{tr} \chi f^F + c_1(r, \theta) \nabla_3 \mathfrak{F}_3 + c_0(r, \theta) \mathfrak{S}$$

$$f \nabla \Omega = \nabla q^F + c_1(r, \theta) \nabla_3 \mathfrak{F}_3 + c_0(r, \theta) \mathfrak{S}$$

and

$$\Box(f) = \Box(q^2) = \Box(r^3) + a \cdot \text{terms} = r^3 (-3 \kappa - 3 \rho) + a \cdot \text{terms}$$

The above gives

$$\Box_2 q^F = \frac{1}{2} \left( \text{tr} X - \text{tr} X \right) \nabla_3 q^F + \frac{1}{2} \left( \text{tr} X - \text{tr} X \right) \nabla_4 q^F + \left( - H + \eta - H + \eta \right) f \nabla q^F$$

$$+ \left( \frac{1}{2} \text{tr} \chi r X - 4 \rho - 2 (P )^2 - \left( - \frac{3}{2} \text{tr} \chi r X - 4 \rho - 2 (P )^2 - 3 \kappa - 3 \rho \right) q^F$$

$$+ q^2 \left( \mathcal{P}_{C_2} (M_2[A, \chi, \mathfrak{B}] \right) + \frac{1}{2} (\text{tr} X + \text{tr} X) M_2[A, \chi, \mathfrak{B}] + \text{l.o.t.}LH^S |\Omega|)$$

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Using Lemma \ref{Lemma 5.2} we finally obtain

\[
\Box q^F + \widetilde{V}_2(r, \theta) q^F - i \frac{A_0 \cos \theta}{|q|^2} \partial_\theta q^F = q^2 \left( P_{C_2} \left( M_2[A, \mathcal{X}, \mathfrak{B}] \right) + \frac{1}{2} (\mathrm{tr} X + \overline{\mathrm{tr} X}) M_2[A, \mathcal{X}, \mathfrak{B}] + \mathrm{l.o.t.} \right)
\]  

(132)

where

\[
\widetilde{V}_2(r, \theta) = \mathrm{tr} \chi \mathrm{tr} X + 3 \rho
\]

The right hand side of (132) contains a term in \( q^F \) which does not vanish for vanishing angular momentum, and which will modify the above potential.

7.2.4 The right hand side of the equations and the conclusion of the proof

Equations (129) and (132) constitute the system of equations. We summarize in the following lemma the computation of their right hand side.

Lemma 7.2. The right hand side of the equations are respectively given by

\[
P_{C_1} \left( M_1[\mathfrak{S}, \mathcal{X}] \right) + \frac{1}{2} (\mathrm{tr} X + \overline{\mathrm{tr} X}) M_1[\mathfrak{S}, \mathcal{X}] = 4 \frac{F}{P} (\overline{F} P) (c \overline{D} \cdot \Omega) + \mathrm{l.o.t.} \quad \text{LHS}[\mathfrak{P}, \Omega]
\]

and

\[
P_{C_2} \left( M_2[A, \mathcal{X}, \mathfrak{B}] \right) + \frac{1}{2} (\mathrm{tr} X + \overline{\mathrm{tr} X}) M_2[A, \mathcal{X}, \mathfrak{B}] = - \frac{1}{2} (c \overline{D} \otimes \mathfrak{P} + (3 \overline{F} + 2 F P) \overline{P}) \Omega + \mathrm{l.o.t.} \quad \text{RHS}[\mathfrak{P}, \Omega]
\]

where \( \mathrm{l.o.t.} \) and \( \text{RHS} \) denote lower order terms with respect to \( \mathfrak{P} \) or \( \Omega \). More precisely, they are given by

\[
\begin{align*}
\mathrm{l.o.t.} \quad \text{LHS}[\mathfrak{P}, \Omega] &= 2 \frac{F}{P} \overline{F} P (4 \mathrm{tr} X - 2 \overline{\mathrm{tr} X}) \mathfrak{B} + a \cdot \mathrm{l.o.t.} \quad \text{RHS} \\
\mathrm{l.o.t.} \quad \text{RHS}[\mathfrak{P}, \Omega] &= -2 \frac{F}{P} \overline{F} P (\mathrm{tr} X + \overline{\mathrm{tr} X}) \mathfrak{S} + a \cdot \mathrm{l.o.t.} \quad \text{RHS}
\end{align*}
\]

where \( a \cdot \mathrm{l.o.t.} \quad \text{RHS} \) and \( \mathrm{l.o.t.} \quad \text{RHS} \) denote terms which vanish for vanishing angular momentum, and are of the form

\[
\begin{align*}
a \cdot \mathrm{l.o.t.} \quad \text{LHS} &= 2 \frac{F}{P} \overline{F} P (d_{(c \overline{D} \cdot \mathfrak{P})} (r, \theta)^{(c) \overline{D} \cdot \mathfrak{P}} + d_1 (r, \theta)^{(c) \overline{D} \cdot \mathfrak{P}} + c_0 (r, \theta)^{(c) \overline{D} \cdot \mathfrak{P}}) \\
a \cdot \mathrm{l.o.t.} \quad \text{RHS} &= (c) \overline{D} \otimes \mathfrak{P} + c_1 (r, \theta)^{(c) \overline{D} \otimes \mathfrak{P}} + c_0 (r, \theta)^{(c) \overline{D} \otimes \mathfrak{P}}
\end{align*}
\]

where the coefficients \( d_{(c \overline{D} \cdot \mathfrak{P})} (r, \theta) \), \( c_{(c) \overline{D} \otimes \mathfrak{P}} (r, \theta) \), \( d_1 (r, \theta)^{(c) \overline{D} \cdot \mathfrak{P}} \) and \( c_0 (r, \theta)^{(c) \overline{D} \cdot \mathfrak{P}} \) are given by (167) – (168).

Proof. See Appendix C.2

We can finally conclude the proof of the Main Theorem. Consider the equation (129) for \( p \). Using the above lemma and Proposition \ref{Proposition 7.1} the right hand side can be written as

\[
P_{C_1} \left( M_1[\mathfrak{S}, \mathcal{X}] \right) + \frac{1}{2} (\mathrm{tr} X + \overline{\mathrm{tr} X}) M_1[\mathfrak{S}, \mathcal{X}] + \mathrm{l.o.t.} \quad \text{LHS}[\mathfrak{P}]
\]

\[
= 4 \frac{F}{P} \overline{F} P (c \overline{D} \cdot \Omega) + \mathrm{l.o.t.} \quad \text{RHS}[\mathfrak{P}, \Omega] + \mathrm{l.o.t.} \quad \text{LHS}[\mathfrak{P}]
\]

\[
= 4 \frac{F}{P} \overline{F} P (c \overline{D} \cdot \Omega) + 2 \frac{F}{P} \overline{F} P (4 \mathrm{tr} X - 2 \overline{\mathrm{tr} X}) \mathfrak{B} + a \cdot \mathrm{l.o.t.} \quad \text{RHS}
\]

\[
- 2 \frac{F}{P} \overline{F} P (\mathrm{tr} X + \overline{\mathrm{tr} X}) \mathfrak{B} + a \cdot \mathrm{l.o.t.} \quad \text{LHS}
\]
which gives
\[ P_{C_1}\left(\mathcal{M}_1[\mathfrak{g}, \mathfrak{x}]\right) + \frac{1}{2} (\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}) \mathcal{M}_1[\mathfrak{g}, \mathfrak{x}] + \text{l.o.t.}_{LHS}[\mathfrak{p}] \]
\[ = 4\left(\frac{F}{P}\right)^2 P \mathcal{A}, \mathcal{F}_d \cdot \Omega + a \cdot \text{l.o.t.}_1^{RHS} + a \cdot \text{l.o.t.}_1^{LHS} \]

where we write \(2\left(\frac{F}{P}\right)^2 P\) \((\text{tr} \mathfrak{X} - 2 \text{tr} \mathfrak{X}) \mathfrak{B} = 2\left(\frac{F}{P}\right)^2 P\) \((\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}) \mathfrak{B} = \mathfrak{c}_0(r, \theta) \mathfrak{B}\). By writing
\[ q\frac{q^2}{|q|^2}(\frac{\mathcal{C}}{D}) \cdot \Omega = \frac{\frac{\mathcal{C}}{D}}{(\frac{\mathcal{C}}{D}) \cdot q + d_1(r, \theta)(\frac{\mathcal{C}}{D}) \cdot \mathfrak{X}} + \text{l.o.t.}_{LHS}[\mathfrak{p}] \]

the right hand side of (129) can be finally written as
\[ q\frac{q^2}{|q|^2}\left(P_{C_1}\left(\mathcal{M}_1[\mathfrak{g}, \mathfrak{x}]\right) + \frac{1}{2} (\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}) \mathcal{M}_1[\mathfrak{g}, \mathfrak{x}] + \text{l.o.t.}_{LHS}[\mathfrak{p}]\right) \]
\[ = 4\left(\frac{F}{P}\right)^2 P \left(\frac{q^2}{|q|^2}\frac{\mathcal{C}}{D} \cdot q + d_1(r, \theta)\left(\frac{\mathcal{C}}{D} \cdot q + d_1(r, \theta)\right)\right) + \frac{\mathcal{C}}{D} \cdot q + d_1(r, \theta)(\frac{\mathcal{C}}{D}) \cdot \mathfrak{X} + \text{l.o.t.}_{LHS}[\mathfrak{p}] \]

where we used that \(\left(\frac{F}{P}\right)^2 = \frac{\mathcal{C}}{D}\). This finally proves (133).

Consider the equation (132) for \(q^2\). Using Lemma 7.2 and Proposition 7.1, the right hand side can be written as
\[ P_{C_2}\left(\mathcal{M}_2[A, \mathfrak{x}, \mathfrak{B}]\right) + \frac{1}{2} (\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}) \mathcal{M}_2[A, \mathfrak{x}, \mathfrak{B}] + \text{l.o.t.}_{LHS}[\mathfrak{p}] \]
\[ = -\frac{1}{2}(\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} + \left(3\mathcal{F}_d + 2\left(\frac{F}{P}\right)^2 P\right) \Omega + \text{l.o.t.}_2^{RHS}[\mathfrak{p}, \Omega] + \text{l.o.t.}_{LHS}[\mathfrak{p}] \]
\[ = -\frac{1}{2}(\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} + \left(3\mathcal{F}_d + 2\left(\frac{F}{P}\right)^2 P\right) \Omega + 2\left(\frac{F}{P}\right)^2 P \left(\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}\right) \mathfrak{g} + a \cdot \text{l.o.t.}_2^{RHS} \]
\[ + 2\left(\frac{F}{P}\right)^2 P \left(\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}\right) \mathfrak{g} + a \cdot \text{l.o.t.}_2^{LHS} \]

which gives
\[ P_{C_2}\left(\mathcal{M}_2[A, \mathfrak{x}, \mathfrak{B}]\right) + \frac{1}{2} (\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}) \mathcal{M}_2[A, \mathfrak{x}, \mathfrak{B}] + \text{l.o.t.}_{LHS}[\mathfrak{p}] \]
\[ = -\frac{1}{2}(\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} + \left(3\mathcal{F}_d + 2\left(\frac{F}{P}\right)^2 P\right) \Omega + a \cdot \text{l.o.t.}_2^{RHS} + a \cdot \text{l.o.t.}_2^{LHS} \]

By writing
\[ q\frac{q^2}{|q|^2}(\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} = (\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} + c_1(r, \theta)\frac{\mathcal{C}}{D} \cdot \mathfrak{p} + c_1(r, \theta)\frac{\mathcal{C}}{D} \cdot \mathfrak{p} \]
\[ = (\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} + c_1(r, \theta)\frac{\mathcal{C}}{D} \cdot \mathfrak{p} \]

the right hand side of (132) can be finally written as
\[ q\frac{q^2}{|q|^2}\left(P_{C_2}\left(\mathcal{M}_2[A, \mathfrak{x}, \mathfrak{B}]\right) + \frac{1}{2} (\text{tr} \mathfrak{X} + \text{tr} \mathfrak{X}) \mathcal{M}_2[A, \mathfrak{x}, \mathfrak{B}] + \text{l.o.t.}_{LHS}[\mathfrak{p}]\right) \]
\[ = -\frac{1}{2}q\frac{q^2}{|q|^2}\frac{\mathcal{C}}{D} \cdot \mathfrak{p} + \left(3\mathcal{F}_d + 2\left(\frac{F}{P}\right)^2 P\right) \mathfrak{p} + q\frac{q^2}{|q|^2}(a \cdot \text{l.o.t.}_2^{RHS} + a \cdot \text{l.o.t.}_2^{LHS}) \]
\[ = -\frac{q^3}{2|q|^2}(\frac{\mathcal{C}}{D}) \cdot \mathfrak{p} + \left(3\mathcal{F}_d + 2\left(\frac{F}{P}\right)^2 P\right) \mathfrak{p} + q\frac{q^2}{|q|^2}(a \cdot \text{l.o.t.}_2^{RHS} + a \cdot \text{l.o.t.}_2^{LHS}) \]
From (132) we finally obtain
\[ \Box_2 q^F + \left( \mathcal{V}_2 (r, \theta) - 3 \rho - 2 F^2 \right) q^F - i \frac{4 \alpha \cos \theta}{|q|^2} \partial_t q^F = - \frac{q^3}{2|q|^5} \mathcal{D} \mathcal{D} \mathcal{p} + q^2 \left( a \cdot \text{RHS}^{\text{RHS}} + a \cdot \text{RHS}^{\text{LHS}} \right) \]
which proves (139). This concludes the proof of the Main Theorem.

A. Explicit computations

We collect here some explicit computations needed in Section 2–5.

A.1 The Gauss equation - Proof of Proposition 3.3 and Proposition 4.7

Let \( \psi \) be a horizontal 1-tensor. We compute
\[
\begin{align*}
D_b \psi_c & = \nabla_b \psi_c \\
D^a \psi_c & = \nabla^a \psi_c \\
D_b \psi_3 & = - \chi_{bd} \psi_d \\
D_b \psi_4 & = - \chi_{bd} \psi_d \\
D_a D_b \psi_c & = \nabla_a (\nabla_b \psi_c) - \frac{1}{2} \chi_{ab} D_3 \psi_c - \frac{1}{2} \chi_{ac} D_b \psi_3 - \frac{1}{2} \chi_{bc} D_a \psi_4 \\
D_b D_a \psi_c & = \nabla_b (\nabla_a \psi_c) - \frac{1}{2} \chi_{ab} \nabla_3 \psi_c - \frac{1}{2} \chi_{ac} \nabla_b \psi_3 - \frac{1}{2} \chi_{bc} \nabla_a \psi_4 \\
D_b D_a \psi_c & = \nabla_b (\nabla_a \psi_c) - \frac{1}{2} \chi_{ab} \nabla_3 \psi_c - \frac{1}{2} \chi_{ac} \nabla_b \psi_3 - \frac{1}{2} \chi_{bc} \nabla_a \psi_4 \\
D_b D_a \psi_c & = \nabla_b (\nabla_a \psi_c) - \frac{1}{2} \chi_{ab} \nabla_3 \psi_c - \frac{1}{2} \chi_{ac} \nabla_b \psi_3 - \frac{1}{2} \chi_{bc} \nabla_a \psi_4
\end{align*}
\]
We deduce,
\[
\begin{align*}
R_{cdab} \psi^d & = D_a D_b \psi_c - D_b D_a \psi_c \\
& = \nabla_a (\nabla_b \psi_c) - \nabla_b (\nabla_a \psi_c) - \frac{1}{2} \chi_{ab} \nabla_3 \psi_c - \frac{1}{2} \chi_{ac} \nabla_b \psi_3 - \frac{1}{2} \chi_{bc} \nabla_a \psi_4 + \frac{1}{2} \left( \chi_{ac} \chi_{bd} \psi_d + \chi_{ac} \chi_{bd} \psi_d - \chi_{bd} \psi_d \right)
\end{align*}
\]
We deduce,
\[
\begin{align*}
\nabla_a (\nabla_b \psi_c) - \nabla_b (\nabla_a \psi_c) & = \frac{1}{2} \chi_{ab} \nabla_3 \psi_c + \frac{1}{2} \chi_{ac} \nabla_b \psi_3 - \frac{1}{2} \chi_{bc} \nabla_a \psi_4 + \frac{1}{2} \left( \chi_{ac} \chi_{bd} \psi_d + \chi_{ac} \chi_{bd} \psi_d - \chi_{bd} \psi_d \right) \\
E_{cdab} & = \chi_{ac} \chi_{bd} + \chi_{ac} \chi_{bd} - \chi_{bd} \psi_d
\end{align*}
\]
\[ (133) \]
we deduce,
\[
\begin{align*}
\nabla_a \nabla_b - \nabla_b \nabla_a \psi_c & = \frac{1}{2} \epsilon_{ab} \left( (a) \text{tr} \nabla^3 + (a) \text{tr} \nabla^4 \right) \psi_c - \frac{1}{2} E_{cdab} \psi^d + R_{cdab} \psi^d
\end{align*}
\]
which proves the expression for a 1-tensor \( \psi \). Similarly, applying the above procedure to a 2-tensor \( \Psi \) we obtain
\[
\begin{align*}
\nabla_a \nabla_b - \nabla_b \nabla_a \Psi_{st} & = \frac{1}{2} \epsilon_{ab} \left( (a) \text{tr} \nabla^3 + (a) \text{tr} \nabla^4 \right) \Psi_{st} - \frac{1}{2} E_{cdab} \Psi_{dt} - \frac{1}{2} E_{cdab} \Psi_{sd} + R_{cdab} \chi_{st}
\end{align*}
\]
We now compute the \( E \) as given by (133). Neglecting the terms \( \tilde{\chi} \) and \( \chi \) which are \( O(\epsilon) \) terms in perturbations of Kerr-Newman, we have:
\[
\begin{align*}
\chi_{ac} \chi_{bd} & = \frac{1}{4} \left( \text{tr} \chi \delta_{ac} + (a) \text{tr} \chi \epsilon_{ac} \right) \left( \text{tr} \chi \delta_{bd} + (a) \text{tr} \chi \epsilon_{bd} \right) \\
& = \frac{1}{4} \left( \text{tr} \chi \text{tr} \chi \delta_{ac} \delta_{bd} + (a) \text{tr} \chi \text{tr} \chi \epsilon_{ac} \epsilon_{bd} + \delta_{ac} \epsilon_{bd} \text{tr} \chi (a) \text{tr} \chi + \epsilon_{ac} \delta_{bd} (a) \text{tr} \chi \text{tr} \chi \right) \\
\chi_{ac} \chi_{bd} & = \frac{1}{4} \left( \text{tr} \chi \text{tr} \chi \delta_{ac} \delta_{bd} + (a) \text{tr} \chi \text{tr} \chi \epsilon_{ac} \epsilon_{bd} + \delta_{ac} \epsilon_{bd} \text{tr} \chi (a) \text{tr} \chi + \epsilon_{ac} \delta_{bd} (a) \text{tr} \chi \text{tr} \chi \right) \\
\chi_{bc} \chi_{ad} & = \frac{1}{4} \left( \text{tr} \chi \text{tr} \chi \delta_{bc} \delta_{ad} + (a) \text{tr} \chi \text{tr} \chi \epsilon_{bc} \epsilon_{ad} + \delta_{bc} \epsilon_{ad} \text{tr} \chi (a) \text{tr} \chi + \epsilon_{bc} \delta_{ad} (a) \text{tr} \chi \text{tr} \chi \right) \\
\chi_{bc} \chi_{ad} & = \frac{1}{4} \left( \text{tr} \chi \text{tr} \chi \delta_{bc} \delta_{ad} + (a) \text{tr} \chi \text{tr} \chi \epsilon_{bc} \epsilon_{ad} + \delta_{bc} \epsilon_{ad} \text{tr} \chi (a) \text{tr} \chi + \epsilon_{bc} \delta_{ad} (a) \text{tr} \chi \text{tr} \chi \right)
\end{align*}
\]
Thus,

\[ E_{cdab} = X_{ac}X_{bd} + X_{bc}X_{ad} - X_{bd}X_{ac} - X_{ac}X_{bd} \]

\[ = \frac{1}{2} \tau_{tr} \chi (d_{ac}d_{bd} - d_{bc}d_{ad}) + \frac{1}{2} (a) tr_{X} (a) tr_{X} (\epsilon_{ac} \epsilon_{bd} - \epsilon_{bc} \epsilon_{ad}) \]

\[ + \frac{1}{4} (\epsilon_{ac} \epsilon_{bd} + \epsilon_{ac} \epsilon_{bd} - \epsilon_{bc} \delta_{ad} - \epsilon_{bc} \delta_{ad}) (a) tr_{X} tr_{X} \]

which proves (23). The Riemann tensor term is given by

\[ R_{cdab} = W_{cdab} + \frac{1}{2} (\delta_{db}R_{ca} - \delta_{ca}R_{db} - \delta_{db}R_{ca}) \]

\[ = - \epsilon_{cd} R_{ab} + (\delta_{db} \delta_{ca} - \delta_{ca} \delta_{db}) (F)^{b} + * (F)^{b} \]

This proves Proposition 5.3. To prove Proposition 4.7, we specialize the above to the case of \( \psi \in \mathcal{S}_{1} \) and \( \Psi \in \mathcal{S}_{2} \). For \( a = 1 \) and \( b = 2 \), the above relations become

\[ (\nabla_{1} \nabla_{2} - \nabla_{2} \nabla_{1}) \psi_{c} = \frac{1}{2} (a) tr_{X} \nabla_{3} + (a) tr_{X} \nabla_{4}) \psi_{c} - \frac{1}{2} E_{cd12} \psi^{d} + R_{cd12} \psi^{d} \]

\[ (\nabla_{1} \nabla_{2} - \nabla_{2} \nabla_{1}) \Psi_{st} = \frac{1}{2} (a) tr_{X} \nabla_{3} + (a) tr_{X} \nabla_{4}) \Psi_{st} - \frac{1}{2} E_{sd12} \Psi_{dt} - \frac{1}{2} E_{td12} \Psi_{st} + R_{sd12} \Psi_{dt} + R_{td12} \Psi_{sd} \]

Define \( X_{c} = - \frac{1}{2} E_{cd12} \psi^{d} \). Since \( \psi \in \mathcal{S}_{1} \), we have that \( \psi_{2} = - i \psi_{1} \) and \( \psi_{1} = i \psi_{2} \) and therefore evaluating \( X \) in coordinates:

\[ X_{1} = - \frac{1}{2} E_{1d12} \psi^{d} = - \frac{1}{2} E_{1112} \psi_{1} - \frac{1}{2} E_{1212} \psi_{2} = - \frac{1}{2} (E_{1112} - i E_{1212}) \psi_{1} \]

\[ X_{2} = - \frac{1}{2} E_{2d12} \psi^{d} = - \frac{1}{2} E_{2112} \psi_{1} - \frac{1}{2} E_{2212} \psi_{2} = - \frac{1}{2} (E_{2212} + i E_{2212}) \psi_{2} \]

Observe that

\[ E_{1112} = \frac{1}{4} (\delta_{11} \epsilon_{21} - \epsilon_{21} \delta_{11}) tr_{X} (a) tr_{X} = \frac{1}{4} (\delta_{11} \epsilon_{21} - \epsilon_{21} \delta_{11}) (a) tr_{X} tr_{X} = 0 \]

\[ E_{2212} = \frac{1}{4} (\epsilon_{12} \delta_{22} - \delta_{22} \epsilon_{12}) tr_{X} (a) tr_{X} = \frac{1}{4} (\epsilon_{12} \delta_{22} - \delta_{22} \epsilon_{12}) (a) tr_{X} tr_{X} = 0 \]

and

\[ E_{1212} = \frac{1}{2} tr_{X} tr_{X} (\delta_{12} \epsilon_{21}) + \frac{1}{2} (a) tr_{X} (a) tr_{X} (- \epsilon_{21} \epsilon_{12}) = \frac{1}{2} tr_{X} tr_{X} + \frac{1}{2} (a) tr_{X} (a) tr_{X} \]

\[ E_{2112} = \frac{1}{2} tr_{X} tr_{X} (- \delta_{21} \epsilon_{21}) + \frac{1}{2} (a) tr_{X} (a) tr_{X} (\epsilon_{12} \epsilon_{12}) = - \frac{1}{2} tr_{X} tr_{X} + \frac{1}{2} (a) tr_{X} (a) tr_{X} \]

This implies

\[ - \frac{1}{2} E_{cd12} \psi^{d} = \frac{1}{4} i (tr_{X} tr_{X} (a) tr_{X}) \psi_{c} \]

Similarly, define \( Y_{st} = - \frac{1}{2} E_{sd12} \Psi_{dt} - \frac{1}{2} E_{td12} \Psi_{sd} \). Since \( \Psi \in \mathcal{S}_{2} \), we have that \( \Psi_{12} = - i \Psi_{11} \) and \( \Psi_{11} = i \Psi_{12} \) and therefore evaluating \( Y \) in coordinates:

\[ Y_{11} = - \frac{1}{2} E_{1d12} \Psi_{dt} = - \frac{1}{2} E_{1112} \Psi_{11} = \frac{1}{2} E_{1212} \Psi_{22} - \frac{1}{2} E_{2112} \Psi_{11} - \frac{1}{2} E_{2212} \Psi_{12} = (- E_{1112} + i E_{1212}) \Psi_{11} \]

\[ Y_{12} = - \frac{1}{2} E_{1112} \Psi_{12} - \frac{1}{2} E_{1212} \Psi_{22} - \frac{1}{2} E_{2112} \Psi_{11} - \frac{1}{2} E_{2212} \Psi_{12} = (- E_{1112} + i E_{1212}) \Psi_{12} \]

This implies

\[ - \frac{1}{2} E_{sd12} \Psi_{dt} - \frac{1}{2} E_{td12} \Psi_{sd} = \frac{1}{2} i (tr_{X} tr_{X} (a) tr_{X}) \Psi_{st} \]

We finally compute the Riemann tensor part. Define \( Z_{c} = R_{cd12} \psi^{d} \). We have

\[ Z_{1} = R_{1d12} \psi^{d} = (- \epsilon_{12} \epsilon_{12} + (\delta_{22} \delta_{22}) (F)^{2} + * (F)^{2}) \psi_{1} = i (\rho - (F)^{2} - * (F)^{2}) \psi_{1} \]

\[ Z_{2} = R_{2d12} \psi^{d} = (- \epsilon_{21} \epsilon_{21} + (\delta_{11} \delta_{11}) (F)^{2} + * (F)^{2}) \psi_{1} = i (\rho - (F)^{2} - * (F)^{2}) \psi_{2} \]

This implies

\[ R_{cd12} \psi^{d} = i (\rho - (F)^{2} - * (F)^{2}) \psi_{c} \]

Similarly, define \( W_{st} = R_{sd12} \Psi_{dt} + R_{td12} \Psi_{sd} \). Evaluating \( W \) in coordinates we have

\[ W_{11} = R_{1d12} \Psi_{dt} + R_{1d12} \Psi_{1d} = 2 R_{1212} \Psi_{12} = - 2 i R_{1212} \Psi_{11} \]

\[ = 2 i (\rho - (F)^{2} - * (F)^{2}) \Psi_{11} \]

We obtain

\[ R_{sd12} \Psi_{dt} + R_{td12} \Psi_{sd} = 2 i (\rho - (F)^{2} - * (F)^{2}) \Psi_{st} \]

Putting the above together we obtain the final formula.
A.2 Proof of Lemma 3.5

We compute the $J_\nu$ which are needed in the above Bianchi identities. In what follows we indicate by $N$ terms which are quadratic in $(F_\beta \gamma)^2$, $(F_\beta \gamma)\xi, \xi, \xi, \xi$. We compute $J_{444}$.

\[2J_{444} = D_3 R_{44} - D_4 R_{44} = \nabla_3 (R_{44}) - 2\mathbf{R} (D_4 e_4, e_4) - \nabla_4 (R_{44}) + \mathbf{R} (D_4 e_4, e_4) + \mathbf{R} (e_4, D_4 e_4) \]
\[= 2\nabla_4 (F_\beta \gamma)^2 - \nabla_4 (F_\beta \gamma)^2 + 2 (F_\beta \gamma)^2 - 2\mathbf{R} (2\mathbf{R} e_4 + 2 \xi_4, e_4) + \mathbf{R} (-2 \mathbf{R} e_4 + 2 \xi_4, e_4) + \mathbf{R} (e_4, 2 \mathbf{R} e_4 + 2 \xi_4) \]
\[= \nabla_4 (F_\beta \gamma)^2 + (F_\beta \gamma)^2 - 2\mathbf{R} (\eta_4 - 2 \eta_4) R_{44} + 2 \xi_4 R_{44} + 2 \xi_4 R_{44} + 2 \nabla_3 (F_\beta \gamma) \]
\[\nabla_3 (F_\beta \gamma) = (F_\beta \gamma)^2 + 2 \nabla_3 (F_\beta \gamma) = \nabla_3 (F_\beta \gamma) = \frac{1}{2} \mathbf{R} \nabla_3 (F_\beta \gamma) \]
\[This gives \]
\[J_{444} = -\nabla_4 (F_\beta \gamma)^2 + (F_\beta \gamma)^2 + 2 \left( \eta_4 - 2 \eta_4 \right) \cdot \left( (F_\beta \gamma)^2 + (F_\beta \gamma)^2 + N \right) \quad (134) \]

We compute $J_{ab4}$.

\[2J_{ab4} = D_3 R_{4a} - D_4 R_{4a} = \nabla_3 (R_{4a}) - 2\mathbf{R} (D_4 e_a, e_a) - \nabla_4 (R_{4a}) + \mathbf{R} (D_4 e_a, e_a) + \mathbf{R} (e_a, D_4 e_a) \]
\[= \nabla_4 (F_\beta \gamma)^2 - 2\mathbf{R} (\mathbf{R} e_a, e_a) - \nabla_4 (F_\beta \gamma)^2 + 2 \mathbf{R} (\mathbf{R} e_a, e_a) + \mathbf{R} (e_a, \mathbf{R} e_a) \]
\[= \nabla_4 (F_\beta \gamma)^2 + 2 \mathbf{R} (\mathbf{R} e_a, e_a) - \nabla_4 (2 \mathbf{R} e_a, e_a) + (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} + \]
\[-R(-\mathbf{R} e_a, e_a) + \mathbf{R} (e_a, \mathbf{R} e_a) + \mathbf{R} (e_a, \mathbf{R} e_a) \]
\[= \nabla_4 (F_\beta \gamma)^2 + 2 \mathbf{R} (\mathbf{R} e_a, e_a) - \nabla_4 (2 \mathbf{R} e_a, e_a) + (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} + \]
\[+ \xi_4 R_{4a} - \xi_4 R_{4a} - \frac{1}{2} \chi_4 R_{44} - \frac{1}{2} \chi_4 R_{44} + \eta_4 R_{4a} + \eta_4 R_{4a} + \xi_4 R_{4a} \]
\[This gives \]
\[J_{ab4} = \nabla_4 (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} - \chi_4 (F_\beta \gamma)^2 + (F_\beta \gamma)^2 + N \quad (135) \]

We therefore compute $a_{\nu}$ using (134) and (135).

\[a_{ab} = J_{ab4} + J_{444} - \frac{1}{2} \delta_{ab} J_{444} \]
\[= \nabla_a (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} + \nabla_b (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} + \nabla_a (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} + \nabla_b (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} \]
\[+ (\xi_4 + 2 \eta_4) (F_\beta \gamma)^2 + (F_\beta \gamma)^2 + (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} \]
\[\delta_{ab} (2 \mathbf{R} e_a, e_a) + (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} \]
\[\frac{1}{2} \mathbf{R} (F_\beta \gamma)^2 + (F_\beta \gamma)^2 \delta_{ab} - (2 \mathbf{R} e_a, e_a) (F_\beta \gamma)^2 + (F_\beta \gamma)^2 + N \]

We use the Maxwell equations in Proposition 3.3 to compute

\[\nabla_4 (F_\beta \gamma)^2 + (F_\beta \gamma)^2 = (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
\[= 2 (F_\beta \gamma)^2 (F_\beta \gamma)^2 + 2 \mathbf{R} (F_\beta \gamma)^2 \]
We now compute \( \text{div} \left( * (F_p) \ast (F_p) + (F_p) \ast (F_p) \right) \). It is given by
\[
\text{div} \left( * (F_p) \ast (F_p) + (F_p) \ast (F_p) \right) = \nabla \ast (F_p) - * (F_p) + * (F_p) \text{div} \ast (F_p) + \nabla (F_p) \ast (F_p) + (F_p) \text{div} (F_p)
\]
Using the Maxwell equation
\[
\nabla_3 (F_p) - \nabla (F_p) + * \nabla (F_p) = - \frac{1}{2} \left( \text{tr} \left( \nabla (F_p) \ast (F_p) \right) + (a) \text{tr} \left( \ast (F_p) \right) \right) + 2 \Omega (F_p) \ast (2 \Omega (F_p) + \ast (F_p)) + \chi \left( (F_p) + \ast (F_p) \right)
\]
we obtain
\[
\text{div} \left( * (F_p) \ast (F_p) + (F_p) \ast (F_p) \right) = (F_p) \text{div} (F_p) + * (F_p) \text{curl} (F_p) + 2 \eta \left( (F_p) + \ast (F_p) \right) + \chi \left( (F_p) + \ast (F_p) \right)
\]
which prove (24).

Using (25) and the computation for \( \nabla_4 (F_p) \ast (F_p) \) and (130) we obtain
\[
\nabla_4 (F_p) \ast (F_p) + 2 \Omega (F_p) + (F_p) \ Jailbreak (131):
\]
This proves (26).

Using (132) and the computation for \( \nabla_4 (F_p) \ast (F_p) \) and (130) we obtain
\[
* J_{434} = -2 (F_p) \text{div} (F_p) \ast (F_p) + 2 \Omega (F_p) \ast (F_p) + (F_p) \ast (F_p)
\]
which prove (27).

We also compute using (135):
\[
* J_{434} = -2 (F_p) \text{div} (F_p) \ast (F_p) + 2 \Omega (F_p) \ast (F_p) + (F_p) \ast (F_p)
\]
This proves (28).

### A.3 Proof of Proposition 4.8

The complexified Bianchi identity for \( \alpha \) is given by
\[
\nabla A - D \hat{\bigcirc} B = - \frac{1}{2} \text{tr} \nabla A + 4 a A + (Z + 4 H) \hat{\bigcirc} B - 3 \hat{\bigcirc} X + a + i \ast a
\]
Using (24), we have

\[ a + i \cdot a = 2\nabla \hat{B}(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + i 2 \cdot \nabla \hat{B}(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + 2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g)) - 2(\* (F)_p^2 + \* (F)_b^2) a + i \cdot a^N + N \]

\[
= D \hat{B}(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + i * (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g))
\]

\[
+ (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g)) + i * (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g))
\]

Using (25), we have

\[
= -2(\* (F)_p^2 + \* (F)_b^2) (\hat{\chi} + i \cdot \hat{\chi}) + N
\]

Observe that

\[
\hat{B}(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g)
\]

and \((F)_p \hat{B}(F)_p^2 + \* (F)_p^2\). We can therefore write

\[
a + i \cdot a = -2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - 2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g)
\]

Using (100), we obtain

\[
a + i \cdot a = -2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - 2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g)
\]

\[
= -2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + N
\]

From (139), we obtain

\[
\nabla_3 A - D \hat{B} = \frac{1}{2} \nabla \hat{B} A + 4 \omega A + (Z + 4H) \hat{B} B - 3 \hat{B} \nabla - 2 \hat{B} F - \frac{1}{2}(D + Z) \hat{B} F + (F)_p \hat{B} + N
\]

By adding and subtracting \(3(\* F) \hat{B} \hat{B} \hat{B} \) we have

\[
\nabla_3 A - D \hat{B} = \frac{1}{2} \nabla \hat{B} A + 4 \omega A + (Z + 4H) \hat{B} B - 3(\* F) \hat{B} \hat{B} - 3 \hat{B} \nabla
\]

\[
= -2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + N
\]

By recalling the definition of \( \xi \), this proves the first equation. The complexified Bianchi identity for \( \beta \) is given by

\[
\nabla_4 B - \frac{1}{2} D \cdot A = -\frac{1}{2} \nabla \hat{\beta} B + 2 \omega B + \frac{1}{2} A \cdot (Z + \frac{H}{2}) + 3 \hat{B} \Xi - (J_{4a4} + i \cdot J_{4a4})
\]

Using (23), we have

\[
* J_{4a4} = -\nabla_4 \* (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - \chi \* \nabla \* (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - (\omega) tr \chi (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - 2(\* (F)_p^2 + \* (F)_b^2) \* \xi_a + N
\]

and therefore

\[
J_{4a4} + i \cdot J_{4a4} = -\nabla_4 (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - \chi \* (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + \frac{1}{2} (\* (F)_p^2 + \* (F)_b^2) \cdot \xi_a + N
\]

Using (D1), we obtain

\[
J_{4a4} + i \cdot J_{4a4} = -\nabla_4 (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - \chi \* (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + \frac{1}{2} (\* (F)_p^2 + \* (F)_b^2) \cdot \xi_a + N
\]

which proves the second equation. The other complexified Bianchi identity for \( \beta \) is given by

\[
\nabla_3 B - D \hat{F} = -\nabla X (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - 2(\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + N
\]

Using (23), we have

\[
* J_{3a4} = \nabla_3 (\* (F)_p^2 + \* (F)_b^2) - \chi \* (\* (F)_p^2 + \* (F)_b^2) - \frac{1}{2} \nabla \chi (\* (F)_p^2 + \* (F)_b^2) - \frac{1}{2} \chi \* \nabla (\* (F)_p^2 + \* (F)_b^2)
\]

\[
+ \frac{1}{2} (\omega) tr \chi (\* (F)_p^2 + \* (F)_b^2) + \frac{1}{2} (\omega) tr \chi (\* (F)_p^2 + \* (F)_b^2) + \frac{1}{2} (\* (F)_p^2 + \* (F)_b^2) \cdot \xi_a + N
\]

and therefore

\[
J_{3a4} + i \cdot J_{3a4} = -\nabla_4 (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) - \chi \* (\* (F)_p \* (F)_b + (F)_p^b (F)_b^g) + \frac{1}{2} (\* (F)_p^2 + \* (F)_b^2) \cdot \xi_a + N
\]
Using the Maxwell equations, we obtain
\[ J_{a4} + i J_{a4} = D\nabla F + \frac{1}{2} \nabla X F - \frac{1}{2} \nabla X F + \frac{1}{2} \nabla X F \]
which gives the desired formula. We prove (45). From
\[ \frac{1}{2} \nabla X F = \frac{1}{2} \nabla X F + \frac{1}{2} \nabla X F \]
which gives the desired formula.

Using the Maxwell equations, we obtain
\[ J_{a4} + i J_{a4} = D\nabla F + \frac{1}{2} \nabla X F - \frac{1}{2} \nabla X F + \frac{1}{2} \nabla X F \]
which gives the desired formula. The complexified Bianchi identity for \( F \) is given by
\[ \nabla_4 P - \frac{1}{2} D \cdot F = - \frac{1}{2} \nabla X P - \frac{1}{2} \nabla X P - \frac{1}{2} \nabla X P - \frac{1}{2} \nabla X P + \frac{1}{2} \nabla X P + N \]
We compute using (27) and (28).
\[ J_{a4} + i J_{a4} = -2\text{div} \{ (F)F * (F) + (F)F * (F) \} + 2i\text{curl} \{ (F)F * (F) + (F)F * (F) \}
- 2(\xi + i \xi + 4\eta) \cdot \left( (F)F + (F)F \right) + 2(\xi + i \xi + 4\eta) \cdot \left( (F)F + (F)F \right) + N \]
Observe that
\[ (F)F * (F) = \left( (F)F * (F) + (F)F * (F) \right) - i \left( (F)F * (F) + (F)F * (F) \right) \]
and therefore
\[ J_{a4} + i J_{a4} = -D \cdot (F)F * (F) - (F)P \cdot (F)F - (F)F \cdot (F)F - 2(\xi + \xi)^2 \cdot (F)F * (F)F + N \]
Using (29), we obtain
\[ J_{a4} + i J_{a4} = 2\nabla_4 F \cdot (F)F - (F)F \cdot (F)F - (F)F \cdot (F)F - 2(\xi + \xi)^2 \cdot (F)F * (F)F + N \]
Since for any complex one forms \( F \) and \( G \), we have \( F \cdot G + \overline{F} \cdot \overline{G} = 0 \), we can write the above as
\[ J_{a4} + i J_{a4} = 2\nabla_4 F \cdot (F)F - (F)F \cdot (F)F - (F)F \cdot (F)F + N \]
which gives the desired formula.

### A.4 Proof of Lemma 4.12

We prove (43). From the commutators (as obtained in (33))
\[ \nabla_4 \nabla_4 [G] = -\frac{1}{2} \left( \nabla \nabla_4 G + (a) \nabla_4 \nabla_4 G \right) \]
\[ \nabla_4 * [G] = -\frac{1}{2} \left( \nabla \nabla_4 G - (a) \nabla_4 \nabla_4 G \right) \]
we obtain
\[ \nabla_4 \nabla_4 [G] = -\frac{1}{2} \left( \nabla \nabla_4 G + (a) \nabla_4 \nabla_4 G \right) + \frac{1}{2} \left( \nabla H + Z \right) \nabla_4 G - \nabla_4 \nabla_4 G \]
Writing \( \nabla = \frac{1}{2} D + \frac{1}{2} \overline{D} \), we obtain the desired formula with non conformal derivatives. Using conformal derivatives we have
\[ \nabla_4 \nabla_4 [G] = \nabla_4 \nabla_4 [G] = \nabla_4 \nabla_4 [G] \]
which gives the desired formula. We prove (44). From
\[ \nabla_4 \nabla_4 [f] = 2\omega \nabla_4 f - 2\omega \nabla_4 f + 2(\eta_0 - \eta_0) \nabla_4 f + 2(\eta \cdot f) \eta_0 - 2(\eta \cdot f) \eta_0 - 2(\eta \cdot f) \eta_0 + N \]
and adding the corresponding formula for the dual
\[ \nabla_4 \nabla_4 [f] = 2\omega \nabla_4 f - 2\omega \nabla_4 f + 2(\eta_0 - \eta_0) \nabla_4 f + 2(\eta \cdot f) \eta_0 - 2(\eta \cdot f) \eta_0 - 2(\eta \cdot f) \eta_0 + N \]
we derive for \( F = f + i \ast f \),
\[ \nabla_4 \nabla_4 [F] = 2\omega \nabla_4 F - 2\omega \nabla_4 F + 2(\eta_0 - \eta_0) \nabla_4 F + 2(\eta \cdot f) \eta_0 - 2(\eta \cdot f) \eta_0 - 2(\eta \cdot f) \eta_0 + N \]
We have
\[ \nabla_4 \nabla_4 [F] = \nabla_4 \nabla_4 [F] = \nabla_4 \nabla_4 [F] \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
\[ = \frac{1}{2} \nabla_4 \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F + 2\omega \nabla_4 \nabla_4 F - 2\omega \nabla_4 \nabla_4 F \]
Using the above and
\[ \nabla \omega + \nabla \omega^* - 4 \omega \omega^* = \rho + (F)_1^2 + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N \]
we obtain
\[ |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N \]
\[ = 2(\eta - \eta^*) \cdot (i \nabla F + (s + 1)P + 2s(P \cdot F) F) = \eta \cdot (\eta - \eta^*) \cdot (i \nabla F + (s + 1)P + 2s(P \cdot F) F) \]
\[ - \frac{s}{2} \nabla (H \cdot F) + \frac{s}{2} \nabla (H \cdot F) \]
Observe that \(2(\eta - \eta^*) \cdot (i \nabla F + (s + 1)P + 2s(P \cdot F) F = \eta \cdot (\eta - \eta^*) \cdot (i \nabla F + (s + 1)P + 2s(P \cdot F) F) \)

Starting with
\[ |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N \]
and recalling that \( \eta F = iF \), we have
\[ |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N \]
\[ = - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) + (\eta - \eta^*) \nabla F \]
\[ + (H - Z) \nabla F + N \]
This gives
\[ E \cdot (3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N \]
We also have
\[ (\eta - \eta^*) \cdot (\eta \cdot \eta^* + N |(3(P \cdot H) + (\eta - \eta^* \cdot \chi) + \eta \cdot \eta^* + N \]
\[ = - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) + (\eta - \eta^*) \nabla F \]
\[ + (H - Z) \nabla F + N \]
This gives
\[ E \cdot (3(P \cdot H) + (\eta - \eta^*) \cdot (\eta \cdot \eta^* + N |(3(P \cdot H) + (\eta - \eta^*) \cdot (\eta \cdot \eta^* + N \]
\[ = - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) + (\eta - \eta^*) \nabla F \]
\[ + (H - Z) \nabla F + N \]
In particular we have
\[ E \cdot (\eta \cdot \eta^*) |(3(P \cdot H) + (\eta - \eta^*) \cdot (\eta \cdot \eta^*) |(3(P \cdot H) + (\eta - \eta^*) \cdot (\eta \cdot \eta^*) \]
\[ = E \cdot (\eta \cdot \eta^*) + sE \cdot \nabla F + s \nabla F - 2 \omega (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) \]
Using that
\[ \nabla \nabla Z = - \frac{1}{2} \nabla \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) + \nabla (H - Z) \nabla F \]
and the above commutator we have
\[ E \cdot (\eta \cdot \eta^*) \nabla F = E \cdot (\nabla F 
\[ + sE \cdot (\nabla F + \eta \cdot \eta \cdot \eta \cdot F) + \nabla (H - Z) \nabla F \]
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We can rewrite the above as

\[ E \cdot (c)\nabla_3 (c)\nabla_3 F = E \cdot (c)\nabla_3 (c)\nabla_3 F - \frac{1}{2} \tr (E \cdot (c)\nabla_3 (c)\nabla_3 F) + \frac{1}{2} \tr (E \cdot (c)\nabla_3 (c)\nabla_3 F)
\]

which gives the commutator \[ \text{[47]} \]. Similarly we have

\[ E \cdot (c)\nabla_3 (c)\nabla_3 F = E \cdot (c)\nabla_3 (DF + sZF) = E \cdot (c)\nabla_3 (DF + sZF) - E \cdot 2s\omega (DF + sZF) \]

\[ = E \cdot \nabla_3 (DF) + sE \cdot \nabla_3 ZF + E \cdot sZ \nabla_3 F - 2s\omega (E \cdot DF + sE \cdot ZF)
\]

which gives

\[ E \cdot (c)\nabla_3 (c)\nabla_3 F = E \cdot D\nabla_3 F - \frac{1}{2} \tr (E \cdot D\nabla_3 F) - \frac{1}{2} \tr (E \cdot (c)\nabla_3 (c)\nabla_3 F) + \frac{1}{2} \tr (E \cdot (c)\nabla_3 (c)\nabla_3 F) + (H - Z) \nabla_3 F
\]

which can be rewritten as

\[ E \cdot (c)\nabla_3 (c)\nabla_3 F = E \cdot (c)\nabla_3 (c)\nabla_3 F - \frac{1}{2} \tr (E \cdot D\nabla_3 F) - \frac{1}{2} \tr (E \cdot (c)\nabla_3 (c)\nabla_3 F) + (H - Z) \nabla_3 F
\]

which gives \[ \text{[47]} \]. Formulas \[ \text{[48]} \] and \[ \text{[49]} \] are proved in \[ \text{[13]} \]. We prove \( \text{[50]} \). From

\[ [\nabla_4, \nabla_3] u_ab = 2\omega_{\nabla_4} u_{ab} - 2\omega_{\nabla_4} u_{ab} + 2(\eta_i - \eta_c)\nabla_c u_{ab} + 4\eta \circ (\eta \cdot u) - 4\eta \circ (\eta \cdot u) - 4 \rho \circ u_{ab} + N\]

and adding the corresponding formula for the dual

\[ [\nabla_4, \nabla_3]^* u_ab = 2\omega_{\nabla_4} u_{ab} - 2\omega_{\nabla_4} u_{ab} + 2(\eta_i - \eta_c)\nabla_c u_{ab} + 4\eta \circ (\eta \cdot u) - 4\eta \circ (\eta \cdot u) - 4 \rho \circ u_{ab} + N \]

we derive for \( U = u + i \star u \),

\[ [\nabla_4, \nabla_3] U_{ab} = 2\omega_{\nabla_4} U_{ab} - 2\omega_{\nabla_4} U_{ab} + 2(\eta_i - \eta_c)\nabla_c u_{ab} + 4\eta \circ (\eta \cdot U) - 4\eta \circ (\eta \cdot U) - 2P - \nabla U_{ab} \]

\[ = 2\omega_{\nabla_4} U_{ab} - 2\omega_{\nabla_4} U_{ab} + 2(\eta_i - \eta_c)\nabla_c u_{ab} + 2P - \nabla U_{ab} \]

from which we derive the commutator between the conformal derivatives as above. Equation \[ \text{[51]} \] is proved in \[ \text{[13]} \].

### A.5 Proof of Lemma \[ \text{[4.13]} \]

Define \( Z_a := (D(D \cdot \psi))_a = D_a D^\dagger \psi \) and evaluate it in coordinates. We have

\[ Z_1 = D_1 D_1 \psi_1 + D_1 D_2 \psi_2 = (\nabla_1 + i \psi_1) (\nabla_1 - i \psi_1) + (\nabla_1 + i \psi_1) (\nabla_2 - i \psi_2) \psi_2 = (\nabla_1 + i \psi_2) (\nabla_1 - i \psi_2) \psi_1 + (\nabla_1 + i \psi_2) (\nabla_2 + i \psi_2) \psi_2 = (\nabla_1 + i \psi_2) (\nabla_1 - i \psi_2) \psi_1 + (\nabla_1 \nabla_2 - \nabla_2 \nabla_1 + i(\nabla_1 + \nabla_2 \nabla_2)) \psi_2 \psi_2 \]

Using that \( \psi_2 = -i \psi_1 \), we obtain \( Z_1 = 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_1 \). Also,

\[ Z_2 = D_2 D_1 \psi_1 + D_2 D_2 \psi_2 = (\nabla_2 + i \psi_2) (\nabla_2 - i \psi_2) (\nabla_2 - i \psi_2) \psi_1 + (\nabla_2 - i \psi_2) (\nabla_2 + i \psi_2) \psi_2 = (\nabla_2 - i \psi_2) (\nabla_2 - i \psi_2) (\nabla_2 - i \psi_2) \psi_1 + (\nabla_2 \nabla_1 + \nabla_2 \nabla_2) \psi_1 + (\nabla_2 \nabla_1 + \nabla_2 \nabla_2) \psi_1 + (\nabla_2 \nabla_1 - \nabla_1 \nabla_2) \psi_2 \psi_2 \]

which gives

\[ (D(D \cdot \psi))_a = 2\Delta \psi_a - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_a \]

Using the Gauss equation \[ \text{[39]} \] we obtain \( \text{[39]} \). Define \( Y_a := (DF \cdot (D \circ F))_a \), and evaluate it in coordinates, i.e. \( Y_a = \nabla_1 D_1 \psi_b + \nabla_1 D_2 \psi_b - \delta_{ab} \nabla_1 D^d \psi_d \). We have

\[ Y_1 = \nabla_1 D_1 \psi_b + \nabla_1 D_2 \psi_b - \delta_{ab} \nabla_1 D^d \psi_d = (\nabla_1 D_1 + \nabla_1 D_2) \psi_1 + (\nabla_1 D_1 + \nabla_1 D_2) \psi_2 \]

\[ = ((\nabla_1 - i \psi_1) (\nabla_1 + i \psi_1) + (\nabla_2 - i \psi_2) (\nabla_2 + i \psi_2)) \psi_1 + ((\nabla_2 - i \psi_2) (\nabla_2 + i \psi_2)) \psi_2 = ((\nabla_1 - i \psi_1) (\nabla_1 + i \psi_1)) \psi_1 + (\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_2 \psi_2 \]

Using that \( \psi_2 = -i \psi_1 \), we obtain \( Y_1 = 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_1 \). Also,

\[ Y_2 = \nabla_2 D_2 \psi_b + \nabla_2 D_2 \psi_b - \delta_{ab} \nabla_2 D^d \psi_d = (\nabla_2 D_1 + \nabla_2 D_2) \psi_1 + (\nabla_2 D_1 + \nabla_2 D_2) \psi_2 \]

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which gives
\[ 2(\nabla \cdot (D \otimes \psi)) = 4\Delta \psi + 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_a \]

Using the Gauss equation (39) we obtain (39). Define \( W_{ab} := 2(\nabla \otimes (D \cdot \psi))_{ab} \), and evaluate it in coordinates, i.e. \( W_{ab} = D_a \nabla_b \psi_{cb} + D_b \nabla_a \psi_{ca} - \delta_{ab} D^c D_c \psi_{dc} \). We have
\[
W_{11} = D_1 \nabla_1 \psi_{c1} + D_1 \nabla_2 \psi_{c1} - \delta_{11} D^c \nabla_1 \psi_{dc}
\]
\[
= 2D_1 \nabla_1 \psi_{c1} - D^c \nabla_1 \psi_{dc}
\]
\[
= 2D_1 \nabla_1 \psi_{11} + 2D_2 D_2 \psi_{21} - D^c \nabla_1 \psi_{dc}
\]
\[
= 2D_1 \nabla_1 \psi_{11} + D_1 D_2 \psi_{21} - D_1 D_1 \psi_{11} - D_2 D_1 \psi_{21} - D_1 D_2 \psi_{12} - D_2 D_2 \psi_{22}
\]
\[
= D_1 \nabla_1 \psi_{11} + D_2 \nabla_2 \psi_{21} - D_1 D_2 \psi_{21} - D_2 D_2 \psi_{22}
\]

Writing \( \psi_{22} = -\psi_{11} \), we have
\[
W_{11} = (D_1 \nabla_1 + D_2 \nabla_2) \psi_{11} + (D_1 D_2 - D_2 D_1) \psi_{12}
\]
\[
= ((\nabla_1 + i \nabla_1)(\nabla_1 + i \nabla_1) + (\nabla_2 + i \nabla_2)(\nabla_2 + i \nabla_2)) \psi_{11}
\]
\[
+ ((\nabla_1 + i \nabla_1)(\nabla_2 - i \nabla_2) - (\nabla_2 + i \nabla_2)(\nabla_1 - i \nabla_1)) \psi_{12}
\]
\[
= ((\nabla_1 + i \nabla_1)(\nabla_1 + i \nabla_1) + (\nabla_2 - i \nabla_2)(\nabla_2 + i \nabla_2) + (\nabla_1 - i \nabla_1)(\nabla_1 - i \nabla_1)) \psi_{11}
\]
\[
+ ((\nabla_1 + i \nabla_1)(\nabla_2 - i \nabla_2) - (\nabla_2 + i \nabla_2)(\nabla_1 - i \nabla_1)) \psi_{12}
\]
\[
= 2\Delta \psi_{11} - i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{11} + 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{12} + 2i\Delta \psi_{12}
\]

Using that \( \psi_{12} = -i\psi_{11} \), we obtain
\[
W_{11} = 2\Delta \psi_{11} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{11} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{12} + 2\Delta \psi_{12}
\]
\[
= 4\Delta \psi_{11} - 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{ab}
\]

Also,
\[
W_{12} = D_1 \nabla_1 \psi_{c2} + D_2 \nabla_2 \psi_{c1} - \delta_{12} D^c \nabla_1 \psi_{dc}
\]
\[
= D_1 D_1 \psi_{12} + D_2 D_2 \psi_{22} + D_2 D_1 \psi_{11} + D_2 D_2 \psi_{21}
\]
\[
= (D_1 \nabla_1 + D_2 \nabla_2) \psi_{12} + (D_2 D_1 - D_1 D_2) \psi_{11}
\]
\[
= 2\Delta \psi_{12} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{12} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{11} - 2i\Delta \psi_{11}
\]

This implies
\[
2(\nabla \otimes (D \cdot \psi))_{ab} = 4\Delta \psi_{ab} - 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \psi_{ab}
\]

Using the Gauss equation (41), we obtain (43).

### A.6 Proof of Proposition 5.7

We start by deriving relation (102). Multiply the Bianchi identity (79) by \((F \cdot P)\):
\[
(F \cdot P) \nabla F_{\alpha \beta} + \frac{1}{2} (F \cdot P) tr X_{\alpha \beta} = \left( F \cdot (P \cdot D) B + H \otimes \left( 4(F \cdot P B - 3(F \cdot P) P (F \cdot B) - 3(F \cdot P) P X - 2(F \cdot P) P X \right) \right)
\]

Multiply the definition of \( 3 \) by \( 3 P \): 
\[
3 P \psi = -\frac{3}{2} (F \cdot P) D B - \frac{9}{2} H \otimes (F \cdot P) B + 3 \otimes (F \cdot P) X
\]

Summing the above we obtain the cancellation of \( 3 \otimes (F \cdot P) X \):
\[
\left( 3 P + 2(F \cdot P) P (F \cdot P) \right) \psi + (F \cdot P) \nabla F_{\alpha \beta} + \frac{1}{2} (F \cdot P) tr X_{\alpha \beta}
\]
\[
= \frac{1}{2} \left( 2(F \cdot P) D B - 3(F \cdot P) D \otimes (F \cdot B) \right) + H \otimes \left( 4(F \cdot P B - 3(F \cdot P) P (F \cdot B) - \frac{9}{2} (F \cdot P) B \right)
\]

On the other hand
\[
(F \cdot D) B = \left( F \cdot D \right) \left( 2(F \cdot P) B - 3(F \cdot P) B \right) = \left( 2(F \cdot P) D B - 3(F \cdot P) D \otimes (F \cdot B) \right) + 2(F \cdot P) D B - 3(F \cdot P) D \otimes (F \cdot B)
\]
\[
= \left( 2(F \cdot P) D B - 3(F \cdot P) D \otimes (F \cdot B) \right) - 4H \otimes (F \cdot P B) + \left( 6(F \cdot P) B - 6(F \cdot P) P (F \cdot B) \right) H \otimes (F \cdot B)
\]

Therefore
\[
\left( 3 P + 2(F \cdot P) P (F \cdot P) \right) \psi + (F \cdot P) \nabla F_{\alpha \beta} + \frac{1}{2} (F \cdot P) tr X_{\alpha \beta}
\]
\[
= \frac{1}{2} \left( 2(F \cdot D) B + 4H \otimes (F \cdot P B) - 6(F \cdot P) P (F \cdot B) H \otimes (F \cdot B) \right) + H \otimes \left( 4(F \cdot P B - 3(F \cdot P) P (F \cdot B) - \frac{9}{2} (F \cdot P) B \right)
\]
\[
= \frac{1}{2} (F \cdot D) B + 3H \otimes B
\]
which proves (107). Using the definition of \( \text{\( X \)} \), we compute
\[
(c) \nabla_4 \text{\( \tilde{\text{\( H \)}} \)) = \nabla_4 \left( \frac{1}{2} (c) \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{B} \otimes \mathcal{F} \right) + \frac{3}{2} H \otimes (c) \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{B} \otimes \mathcal{F} + (c) \nabla_4 \left( \frac{1}{2} (c) \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{F} \right)
\]
Using the commutator formula (19) for \( F = (c) \mathcal{B} \) and \( s = 1 \), using (93), (97) and (74) we obtain
\[
(c) D \otimes X = (c) D \otimes \left( (c) \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{B} \otimes \mathcal{F} + \frac{3}{2} trX (c) \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{B} \otimes \mathcal{F} - \frac{3}{2} (3H + \hat{H}) \otimes (c) \nabla_4 \mathcal{F} \right)
\]
On the other hand, using the definition of \( \mathcal{X} \) (108), we compute using (93) and (99):
which, using the definition of $X$ (106), proves (107). We now derive relation (110). Using the definition of $X$ (105), we compute

$$(c)\nabla_3 X = (c)\nabla_3 \left( (c)\nabla_4 (F)B + \frac{3}{2} (\nabla X (F)B - 2 (F)P) \right)$$

$$= (c)\nabla_4 (c)\nabla_3 (F)B + [ (c)\nabla_3, (c)\nabla_4 ] (F)B + \frac{3}{2} (c)\nabla_3 \nabla_3 (F)B - \frac{3}{2} (c)\nabla_3 \nabla_3 (F)B$$

We compute each term. Using (63) and (80), we obtain

$$(c)\nabla_4 (c)\nabla_3 (F)B = (c)\nabla_4 \left( -\frac{1}{2} (c)\nabla_3 (F)B + (c)D (F)P + 2 (F)P \right)$$

$$= -\frac{1}{2} (c)\nabla_3 (c)\nabla_4 (F)B - \frac{1}{2} (c)\nabla_4 (F)B + (c)D (c)\nabla_4 (F)P$$

$$= -\frac{1}{2} \nabla X (c)\nabla_4 (F)B + \left( \frac{1}{4} \nabla X \nabla X \left( \frac{3}{4} \nabla X \nabla X - \frac{1}{2} (c)D \cdot \nabla X + \frac{1}{2} H \cdot \nabla X - \nabla X \right) \right) (F)B$$

We simplify $L_1$ by making use of the definition of $X$ and writing $(c)\nabla_4 (F)B = X - \frac{3}{2} \nabla X (F)B + 2 (F)P$. We obtain

$$L_1 = -\frac{1}{2} \nabla X (c)\nabla_4 (F)B + \left( \frac{1}{4} \nabla X \nabla X \left( \frac{3}{4} \nabla X \nabla X - \frac{1}{2} (c)D \cdot \nabla X + \frac{1}{2} H \cdot \nabla X - \nabla X \right) \right) (F)B$$

which gives

$$L_1 = -\frac{1}{2} \nabla X (c)\nabla_4 (F)B + \left( \frac{1}{4} \nabla X \nabla X \left( \frac{3}{4} \nabla X \nabla X - \frac{1}{2} (c)D \cdot \nabla X + \frac{1}{2} H \cdot \nabla X - \nabla X \right) \right) (F)B + (F)P (-\nabla X \nabla X) \quad (138)$$

We compute $L_2$ using (63):

$$L_2 = (c)D \nabla_4 (c)\nabla_3 (F)B = \left( (c)D \cdot \nabla X \nabla X \left( \frac{3}{4} \nabla X \nabla X - \frac{1}{2} (c)D \cdot \nabla X + \frac{1}{2} H \cdot \nabla X - \nabla X \right) \right) (F)B$$

Using (63) to write

$$(c)D \nabla X = (c)D \cdot \nabla X = (c)D \cdot \nabla X$$

and using (80) to write

$$(c)D (\nabla X \cdot (F)B) = \left( (c)D \cdot \nabla X \nabla X \left( \frac{3}{4} \nabla X \nabla X - \frac{1}{2} (c)D \cdot \nabla X + \frac{1}{2} H \cdot \nabla X - \nabla X \right) \right) (F)B$$

we obtain

$$L_2 = \left( \frac{1}{2} (c)D \nabla_4 (c)\nabla_3 (F)B \right) (F)B + \left( \frac{1}{2} (c)D \nabla_4 (c)\nabla_3 (F)B \right) (F)B + \left( \frac{1}{2} (c)D \nabla_4 (c)\nabla_3 (F)B \right) (F)B$$

Using the definition of $\mathfrak{B}$ to write

$$2 (F)P = 2 (F)P + 1 (c)D \cdot \nabla X \nabla X \left( \frac{3}{4} \nabla X \nabla X - \frac{1}{2} (c)D \cdot \nabla X + \frac{1}{2} H \cdot \nabla X - \nabla X \right)$$

we finally obtain

$$L_2 = \left( \frac{1}{2} (c)D \nabla_4 (c)\nabla_3 (F)B \right) (F)B + \left( \frac{1}{2} (c)D \nabla_4 (c)\nabla_3 (F)B \right) (F)B + \left( \frac{1}{2} (c)D \nabla_4 (c)\nabla_3 (F)B \right) (F)B$$

We compute $L_3$ by applying (111) to $(F)P$ (which is of conformal type 0) and using (64), (93), and (100), we obtain

$$L_3 = \left( [ (c)\nabla_4, (c)D ] (F)P \right)$$

$$= \frac{1}{2} \nabla X (c)D (F)P + \frac{1}{2} (c)D \nabla_4 (F)P + \nabla (c)\nabla_3 (F)P - \frac{1}{2} \hat{X} \cdot (c)D \nabla_4 (F)P$$

which gives, using (80) and (83)

$$L_3 = \frac{1}{2} \nabla X (c)D (F)P - \nabla (c)\nabla_3 (F)P + \frac{1}{2} (c)D \nabla_4 (F)P + \nabla (c)\nabla_3 (F)P - \frac{1}{2} \hat{X} \cdot (c)D \nabla_4 (F)P$$

We compute $L_4$ using (76)

$$L_4 = 2 (F)P (c)\nabla_4 H = 2 (F)P \left( \frac{1}{2} (c)\nabla_4 (H - H) + (c)\nabla_3 \Xi - \frac{1}{2} \hat{X} \cdot (c)D \nabla_4 (F)P \right)$$

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which can be written as
\[ L_4 = -\mathfrak{g}_3 - F P \begin{pmatrix} H & -H \end{pmatrix} + \left( -3 P - 2 F P \begin{pmatrix} F P \end{pmatrix} \right) (F P) B + (F P) \left( 2 (c) \nabla_3 \Xi - \dot{X} \cdot (\dot{H} - \dot{P}) \right) \] (141)

We compute \( L_5 \) using (94),
\[ L_5 = 2 \begin{pmatrix} c \end{pmatrix} \nabla_4 (F P) P H = 2 \left( -\begin{pmatrix} \text{tr} X \end{pmatrix} (F P) P + \frac{1}{2} \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi - \dot{X} \cdot (\dot{H} - \dot{P}) \right) \] (142)
and using (93) and (95) it can be written as
\[ L_5 = -2 \begin{pmatrix} \text{tr} X \end{pmatrix} (F P) P H + H \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi + \left( H \cdot \dot{H} \right) (F P) B \]

Putting together (138), (139), (140), (144) and (142) we obtain
\[ \begin{pmatrix} c \end{pmatrix} \nabla_4 (\begin{pmatrix} c \end{pmatrix} \nabla_3 \Xi) \] (143)

Using (138) to substitute \( F \) by \( (F P) B \) and \( s = 1 \), we compute
\[ \begin{pmatrix} c \end{pmatrix} \nabla_3, (\nabla_4) \Xi (F P) B = \frac{1}{2} \left( H - H \right) \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi + \frac{1}{2} \begin{pmatrix} H \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi + \frac{1}{2} \begin{pmatrix} H \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi \] (144)

We also compute using (99)
\[ \frac{3}{2} \begin{pmatrix} c \end{pmatrix} \nabla_3 \begin{pmatrix} c \end{pmatrix} \nabla_4 \dot{X} (F P) B = \frac{1}{2} \begin{pmatrix} \text{tr} X \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi + \frac{3}{2} \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi + \frac{3}{2} \begin{pmatrix} H \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_5 \Xi \] (145)
and using (99)
\[ -2 \begin{pmatrix} c \end{pmatrix} \nabla_3 \Xi \nabla_4 \Xi = 2 \begin{pmatrix} \text{tr} X \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_4 \Xi \] (146)

Using (126), (144), (140), (146) we obtain
\[ \begin{pmatrix} c \end{pmatrix} \nabla_3 \dot{X} = \begin{pmatrix} c \end{pmatrix} \nabla_4 \nabla_3 \Xi + \begin{pmatrix} c \end{pmatrix} \nabla_3 (\begin{pmatrix} c \end{pmatrix} \nabla_4 \Xi) + \frac{3}{2} \begin{pmatrix} \text{tr} X \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_3 \Xi + \frac{3}{2} \begin{pmatrix} c \end{pmatrix} \nabla_3 \nabla_3 \Xi - 2 \begin{pmatrix} c \end{pmatrix} \nabla_3 \Xi \nabla_3 \Xi \] (147)

which gives
\[ \begin{pmatrix} c \end{pmatrix} \nabla_3 \dot{X} + \frac{1}{2} \begin{pmatrix} \text{tr} X \end{pmatrix} \begin{pmatrix} c \end{pmatrix} \nabla_4 \Xi + 2 \begin{pmatrix} c \end{pmatrix} \nabla_3 \Xi \nabla_3 \Xi \] (148)

We now write the right hand side of the equation in terms of \( \tilde{a} \). We have the following lemma.
Lemma A.1. The following formula for the divergence of $\mathcal{F}$ holds:

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} + \mathcal{H} \cdot \mathcal{F} &= -\frac{1}{2} (\partial X - tr X) X - \frac{1}{2} (c) \mathcal{D} (c) \mathcal{D} \cdot (F) B - \frac{3}{2} H \cdot \mathcal{D} (c) F B - \frac{1}{2} \mathcal{H} \cdot (c) \mathcal{D} (F) B - \frac{1}{2} (tr X - tr X) (c) \nabla_3 (F) B \\
&\quad + \frac{1}{4} tr X tr X + \frac{3}{2} tr X tr X - \frac{1}{2} tr X tr X - \omega (tr X - tr X) + \frac{1}{2} \mathcal{D} \cdot Z - \frac{1}{2} \mathcal{D} \cdot Z \\
&\quad + P + \mathcal{F} - 2 (F) p (F) \mathcal{P} - \frac{3}{2} \mathcal{D} \cdot H - \frac{3}{2} \mathcal{H} \cdot (F) B \\
&\quad + (F) p \left( \mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} + (\mathcal{H} - 2 \mathcal{H}) \cdot \mathcal{F} \right) \mathcal{P} \mathcal{E}
\end{align*}
\]

Proof. Using the definition of $\mathcal{F}$ we compute

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} + \mathcal{H} \cdot \mathcal{F} &= \mathcal{D} \left( \frac{1}{2} \mathcal{D} \cdot \mathcal{F} - \frac{3}{2} H \cdot \mathcal{D} \cdot (F) B - \frac{1}{2} \mathcal{H} \cdot \mathcal{D} \cdot (F) B \right) \\
&\quad + \frac{1}{2} \mathcal{H} \cdot \mathcal{D} \cdot (F) B - \frac{3}{2} H \cdot \mathcal{D} \cdot (F) B - \frac{1}{2} \mathcal{H} \cdot \mathcal{D} \cdot (F) B \\
&\quad + (-\frac{3}{2} \mathcal{D} \cdot H - \frac{3}{2} \mathcal{H} \cdot (F) B \\
&\quad + (F) p \left( \mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} + (\mathcal{H} - 2 \mathcal{H}) \cdot \mathcal{F} \right) \mathcal{P} \mathcal{E}
\end{align*}
\]

Using (33), (34), (30) the above becomes

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} + \mathcal{H} \cdot \mathcal{F} &= \frac{1}{2} \mathcal{D} \cdot \mathcal{F} - \frac{3}{2} H \cdot \mathcal{D} \cdot (F) B - \frac{1}{2} \mathcal{H} \cdot \mathcal{D} \cdot (F) B \\
&\quad + (F) p \left( \mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} + (\mathcal{H} - 2 \mathcal{H}) \cdot \mathcal{F} \right) \mathcal{P} \mathcal{E}
\end{align*}
\]

Since $(F) B$ is of conformal type 1, we have

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} &= (D + Z) \cdot (D + Z) \mathcal{F} B = D \cdot (\mathcal{D} \mathcal{F} B) + Z \cdot (D B) + Z \cdot D (F) B + (D \cdot Z + Z \cdot D) (F) B \\
\end{align*}
\]

Using the relation (55) of Lemma 4.13 applied to $(F) B$:

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} &= \mathcal{D} \cdot (D \mathcal{F} B) + (D \cdot Z + Z \cdot D \cdot Z) (F) B \\
&\quad + (D \cdot Z + Z \cdot D) (F) B + (D \cdot Z + Z \cdot D) (F) B
\end{align*}
\]

we obtain

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} &= \mathcal{D} \cdot (c) \mathcal{D} \cdot (F) B + (c) \mathcal{D} \cdot (F) B + (c) \mathcal{D} \cdot (F) B - \frac{1}{2} (tr X tr X + 2 P + 2 \mathcal{F} - 4 (F) p \mathcal{P} \mathcal{F}) (F) B \\
&\quad - \frac{1}{2} (tr X tr X + 2 P + 2 \mathcal{F} - 4 (F) p \mathcal{F}) (F) B \\
&\quad - \frac{1}{2} (tr X tr X + 2 P + 2 \mathcal{F} - 4 (F) p \mathcal{F}) (F) B
\end{align*}
\]

This finally gives

\[
\begin{align*}
\mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} &= \mathcal{D} \cdot (c) \mathcal{D} \cdot (F) B - \frac{3}{2} H \cdot (c) \mathcal{D} \cdot (F) B - \frac{1}{2} \mathcal{H} \cdot (c) \mathcal{D} \cdot (F) B \\
&\quad + \frac{1}{2} (tr X tr X) \mathcal{D} \cdot (F) B - \frac{1}{2} (tr X tr X) \mathcal{D} \cdot (F) B \\
&\quad + (F) p \left( \mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} + (\mathcal{H} - 2 \mathcal{H}) \cdot \mathcal{F} \right) \mathcal{P} \mathcal{E}
\end{align*}
\]

Using the definition of $\mathcal{X}$ to write $(c) \mathcal{D} \cdot (F) B = \mathcal{X} - \frac{3}{2} tr X (F) B + 2 (F) p \mathcal{E}$ we obtain the final expression. 

We can therefore write the right hand side of (147) as

\[
(c) \nabla_3 \mathcal{X} + \frac{1}{2} tr X \mathcal{X} + 2 P
\]

\[
= -\mathcal{D} \cdot \mathcal{F} \cdot \mathcal{H} \cdot \mathcal{F} - \frac{1}{2} (tr X tr X) X + \frac{1}{4} tr X tr X - \frac{1}{4} tr X tr X - \omega (tr X tr X) + \frac{1}{2} \mathcal{D} \cdot Z - \frac{1}{2} \mathcal{D} \cdot Z + P - \mathcal{F}
\]

Observe that the coefficient of $(F) B$ vanishes. We therefore obtain (149).
B Derivation of the Teukolsky equations - Proof of Theorem 6.1

In this section, we derive the system of Teukolsky equations for $\mathcal{B}$ and $\mathcal{H}$.

B.1 The Teukolsky equation for $\mathcal{B}$

Recall the relation \([c] \nabla_3 \mathcal{B} + 3 \nabla X \mathcal{B} = (F)^{P} \left( (c)^{D} \cdot A + \nabla \nabla \right) + \left( 2 \left( F \right)^{P} \left( \left( F \right)^{P} \cdot F \right) - 3 F \right) \mathcal{X} \)

We apply \((c) \nabla_3\) to the above, and using \([a] [b] [c] [d] [e] \) we obtain

\((c) \nabla_3 \mathcal{B} = -3 \nabla X \mathcal{B} + \left( \frac{3}{2} \nabla X \nabla_3 - 3 \nabla \nabla - 3 H \cdot \nabla - 6 F \right) \mathcal{B} + I_1 + I_2 + I_3 \)

where

\[ I_1 = \left( c \right) \mathcal{B} \left( (F)^{P} \left( (c)^{D} \cdot A + \nabla \nabla \right) \right), \quad I_2 = \left( c \right) \mathcal{B} \left( 2 \left( F \right)^{P} \left( \left( F \right)^{P} \cdot F \right) - 3 F \right) \mathcal{X} \]

\[ I_3 = \left( 2 \left( F \right)^{P} \left( \left( F \right)^{P} \cdot F \right) - 3 F \right) \left( c \right) \mathcal{B} \mathcal{X} \]

We compute $I_1$. Using \([a] [b] [c] [d] [e] \) we obtain

\[ I_1 = \left( c \right) \mathcal{B} \left( (F)^{P} \left( (c)^{D} \cdot A + \nabla \nabla \right) \right) + \left( F \right)^{P} \left( (c)^{D} \cdot A + \nabla \nabla \right) \]

\[ = -3 \nabla \nabla \left( c \right) \mathcal{B} + \left( F \right)^{P} \left( c \right) \nabla_3 \mathcal{B} + \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

Using \([a] [b] [c] [d] [e] \) applied to $A$ with $s = 2$ we obtain

\[ I_1 = -3 \nabla \nabla \left( c \right) \mathcal{B} + \frac{1}{2} \left( F \right)^{P} \left( (c)^{D} \cdot A + \nabla \nabla \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

We write the last term of the above using \([a] [b] [c] [d] [e] \) as

\[ (F)^{P} \left( (c)^{D} \cdot \nabla_3 \mathcal{B} + (\nabla \nabla) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

Using \([a] [b] [c] [d] [e] \) to write

\[ (F)^{P} \left( (c)^{D} \cdot \nabla_3 \mathcal{B} + (\nabla \nabla) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

we compute

\[ (F)^{P} \left( (c)^{D} \cdot \nabla_3 \mathcal{B} + (\nabla \nabla) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

\[ = -\frac{1}{2} \left( F \right)^{P} \left( \nabla_3 \mathcal{B} + \nabla \nabla \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + \left( c \right) \mathcal{B} \mathcal{X} \]

\[ = \left( c \right) \mathcal{B} \left( \left( F \right)^{P} \right) \left( \left( F \right)^{P} \cdot F \right) \mathcal{X} \]

Using \([a] [b] [c] [d] [e] \), the above becomes

\[ \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

\[ = \left( c \right) \mathcal{B} \left( \left( F \right)^{P} \right) \left( \left( F \right)^{P} \cdot F \right) \mathcal{X} \]

We therefore obtain

\[ I_1 = -\left( \frac{3}{2} \nabla X + \frac{1}{2} \nabla X \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

\[ = -\left( \frac{3}{2} \nabla X + \frac{1}{2} \nabla X \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

\[ = \left( c \right) \mathcal{B} \left( \left( F \right)^{P} \right) \left( \left( F \right)^{P} \cdot F \right) \mathcal{X} \]

which finally gives

\[ I_1 = -\left( \frac{3}{2} \nabla X + \frac{1}{2} \nabla X \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \left( F \right)^{P} \left( \left( c \right) \nabla_3 \mathcal{B} + \left( \nabla \nabla \right) \right) \mathcal{B} + (c) \mathcal{B} \mathcal{X} \]

\[ = \left( c \right) \mathcal{B} \left( \left( F \right)^{P} \right) \left( \left( F \right)^{P} \cdot F \right) \mathcal{X} \]
We compute $I_2$. Using (109) and (110) we have

$$I_2 = (2^{(c)} \nabla_3 (F)p(\overline{F}Fp + 2(F)p(c)\nabla_3 (\overline{F}Fp) - 3^{(c)} \nabla_4 F)p)X = \left(\frac{9}{2} tr X + \frac{1}{2} tr X\right) (F)p(\overline{F}Fp)X$$

We use (109) again to write

$$3\overline{F}X = (\overline{c})\nabla_4 \mathcal{B} - 3tr \overline{X} \mathcal{B} + (F)p \left(\frac{1}{2} \nabla^{(c)} D \cdot A + \frac{3}{2} \nabla \cdot \mathcal{B}\right) + 2(F)p(\overline{F}Fp)X$$

and substituting in the above we obtain

$$I_2 = -3\frac{1}{2} tr X (\overline{c})\nabla_4 \mathcal{B} + \frac{9}{2} tr X tr \overline{X} \mathcal{B} + \frac{3}{2} tr \overline{X} (F)p \left(\frac{1}{2} \nabla^{(c)} D \cdot A + \frac{3}{2} \nabla \cdot \mathcal{B}\right) + (4tr X - 2\frac{1}{2} tr \overline{X}) (F)p(\overline{F}Fp)X$$

We compute $I_3$. Using (110) we obtain

$$I_3 = (2(F)p(\overline{F}Fp - 3\overline{F}) (\overline{c})\nabla \mathcal{X}$$

Putting the above together we obtain

$$\begin{gather*}
(\overline{c})\nabla_3 (\overline{c})\nabla_4 \mathcal{B} = -3tr X (\overline{c})\nabla_3 \mathcal{B} + \left(\frac{9}{2} tr X + \frac{1}{2} tr X\right) (F)p \left(\frac{1}{2} \nabla^{(c)} D \cdot A + \frac{3}{2} \nabla \cdot \mathcal{B}\right) + \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (F)p \left(\frac{1}{2} \nabla^{(c)} D \cdot A + \frac{3}{2} \nabla \cdot \mathcal{B}\right) + \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (F)p(\overline{F}Fp)X
\end{gather*}$$

which gives

$$\begin{gather*}
(\overline{c})\nabla_3 (\overline{c})\nabla_4 \mathcal{B} = -3tr X (\overline{c})\nabla_3 \mathcal{B} - \frac{9}{2} tr X (\overline{c})\nabla_4 \mathcal{B} + \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (F)p \left(\frac{1}{2} \nabla^{(c)} D \cdot A + \frac{3}{2} \nabla \cdot \mathcal{B}\right) + \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (F)p(\overline{F}Fp)X
\end{gather*}$$

Finally using (109) to write

$$\left(\frac{1}{2} \nabla^{(c)} D \cdot A + \frac{3}{2} \nabla \cdot \mathcal{B}\right) = (\overline{c})\nabla_4 \mathcal{B} + 3tr X \mathcal{B} - \left(2(F)p(\overline{F}Fp - 3\overline{F})\right) X$$

we obtain

$$\begin{gather*}
(\overline{c})\nabla_3 (\overline{c})\nabla_4 \mathcal{B} = -3tr X (\overline{c})\nabla_4 \mathcal{B} - \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (\overline{c})\nabla_4 \mathcal{B} + \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (F)p(\overline{F}p)X
\end{gather*}$$

Consider now the operator

$$T_1(\mathcal{B}) = (\overline{c})\nabla_3 (\overline{c})\nabla_4 \mathcal{B} - 3tr \overline{X} (\overline{c})\nabla_4 \mathcal{B} - \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (\overline{c})\nabla_4 \mathcal{B} + \left(\frac{9}{2} tr X + \frac{3}{2} tr \overline{X}\right) (F)p(\overline{F}Fp)X$$

Using (102), (114), (109) the operator $T_1$ reduces to (110). This completes the derivation of the Teukolsky equation for $\mathcal{B}$. 

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B.2 The Teukolsky equation for $\mathfrak{g}$

Recall the relation (108)

$$(c) \nabla_4 \mathfrak{g} + \left( \frac{3}{2} \nabla X + \frac{1}{2} \nabla X \right) \mathfrak{g} = - \frac{1}{2} (c) \delta \otimes \mathfrak{g} - \left( \frac{3}{2} \mathfrak{H} + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} - (c) P A$$

We apply $(c) \nabla_3$ to the above, and using (99) we obtain

$$(c) \nabla_3 (c) \nabla_4 \mathfrak{g} = - \left( \frac{3}{2} \nabla X + \frac{1}{2} \nabla X \right) (c) \nabla_3 \mathfrak{g} + \left( \frac{1}{4} \nabla X \nabla X + \frac{2}{4} \nabla X \nabla X - 3 \mathfrak{F} - P - \frac{3}{2} (c) D \cdot \mathfrak{H} - H - \frac{1}{2} (c) D \cdot \mathfrak{H} - 2 H \cdot \mathfrak{F} \right) \mathfrak{g} + K_1 + K_2 + K_3$$

where

$$K_1 = - \frac{1}{2} (c) \nabla_3 (c) D \otimes \mathfrak{g}, \quad K_2 = - (c) \nabla_3 \left( \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} \right), \quad K_3 = - (c) \nabla_3 (c) P A$$

We compute $K_1$. Using (110) and (111) for $s = 2$, we have

$$K_1 = - \frac{1}{2} (c) D \otimes (c) D \cdot \mathfrak{g} + \frac{1}{2} H \otimes (c) D \cdot \mathfrak{g}$$

$$+ (c) D \otimes tX + (c) D \otimes tX + \left( \frac{1}{4} (tX + tX) \right) (c) D \otimes \mathfrak{g} + \left( \frac{1}{4} (tX + tX) \mathfrak{H} + \frac{3}{4} tX \mathfrak{H} + \frac{1}{4} tX \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g}$$

Using (105) and (110) to write

$$(c) D \otimes \mathfrak{g} = - (c) \nabla_4 \mathfrak{g} - (3tX + tX) \mathfrak{g} - (3 \mathfrak{H} + \mathfrak{H}) \mathfrak{g} - (c) P A$$

$$(c) D \otimes tX = 2 (c) P (c) \nabla_4 A + (c) P \nabla X - 6 (c) \mathfrak{H} + 2 (c) P (c) P$$

We obtain

$$K_1 = - \frac{1}{2} \left( tX + tX \right) (c) \nabla_4 \mathfrak{g} - \frac{1}{4} \left( tX + tX + 3 \nabla X \nabla X + 3 \nabla X \nabla X \right) \mathfrak{g}$$

$$+ \frac{1}{2} (c) D \otimes (c) D \cdot \mathfrak{g} + \frac{1}{2} H \otimes (c) D \cdot \mathfrak{g} + (c) D \otimes tX + H \otimes tX$$

$$- \frac{1}{2} \left( tX + tX \right) (F P) A - \frac{1}{2} P (c) A$$

$$+ \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g} - \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g}$$

We compute $K_2$. Using (99) and (110)

$$K_2 = - (c) \nabla_3 \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} - \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g}$$

$$- \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g} - \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g}$$

We compute $K_3$. Using (99) we obtain

$$K_3 = - (c) \nabla_3 (c) P A - (c) P (c) A + \mathfrak{X} (c) P A$$

We therefore obtain

$$(c) \nabla_4 (c) \nabla_4 \mathfrak{g} = - \left( \frac{3}{2} tX + \frac{1}{2} tX \right) (c) \nabla_3 \mathfrak{g} - \frac{1}{2} (tX + tX) (c) \nabla_4 \mathfrak{g}$$

$$+ \left( \frac{3}{4} tX tX - \frac{1}{4} \nabla X \nabla X \right) \mathfrak{g} - P - 2 (c) P (c) P - \frac{3}{2} (c) D \cdot \mathfrak{H} - H - \frac{1}{2} (c) D \cdot \mathfrak{H} - 2 H \cdot \mathfrak{F}$$

$$+ (c) D \otimes (c) D \cdot \mathfrak{g} + \left( \frac{3}{2} H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g} + \left( \frac{3}{2} \mathfrak{H} + \frac{1}{2} tX \mathfrak{H} \right) \mathfrak{g}$$

$$+ (c) P (c) \nabla X A + \frac{1}{2} (3 tX - tX) (c) P A + \left( \frac{3}{2} \mathfrak{H} + \frac{1}{2} tX \mathfrak{H} \right) \mathfrak{g} \otimes \mathfrak{g} - \left( - 2 H + \frac{1}{2} \mathfrak{H} \right) \mathfrak{g}$$
We summarize here the commutators for a general operator $P$. Using (46), we obtain the stated expression. We compute

$$\frac{1}{2} \left( \hat{D} \hat{\Box} \left( \hat{c} \hat{D} \cdot \hat{\delta} + \hat{H} \cdot \hat{\delta} \right) \right) + \left( 2H + \frac{1}{2} \hat{H} \right) \hat{\Box} \left( \hat{c} \hat{D} \cdot \hat{\delta} + \hat{H} \cdot \hat{\delta} \right)$$

Using (50), (31), and (27) the above operator reduces to (44) and this ends the derivation of the equation for $\hat{\delta}$.

C Derivation of the system of Regge–Wheeler equations

We show here the derivation of the main computations in the derivation of the Regge–Wheeler equations.

C.1 The commutator of $P_C$ - Proof of Proposition 7.1

We summarize here the commutators for a general operator $P_C$, defined as in (40), i.e. $P_C(\Psi) = (c)\nabla_3 \Psi + C \Psi$, for either $F \in S_1$ or $U \in S_2$.

**Lemma C.1.** Let $F$ be a one form of conformal type $s$. The following commutators hold.

$$[P_C, (c)\nabla_3 F] = (- (c)\nabla_3 C) F$$

$$[P_C, (c)\nabla_4 F] = \frac{1}{2} (H - H) \cdot (c)\hat{D} F + \frac{1}{2} \hat{H} - \hat{H} \cdot (c)\hat{D} F$$

$$+ \left( -(c)\nabla_4 C + (s - 1)P + (s + 1)\nabla + 2sF P + (s + 1) \hat{D} F - \frac{1}{2} (H - \hat{H}) \right) \left( c \hat{D} C - 1 \right)$$

$$[P_C, (c)\hat{D} \hat{\Box}] = \frac{1}{2} tr \left( c \hat{D} \hat{\Box} F + H \hat{\Box} (c)\nabla_3 F + \left( -(c)\nabla_3 C - 1 \right) \left( c \hat{D} C - 1 \right) \right)$$

$$E \cdot [P_C, (c)\hat{D} F] = \frac{1}{2} tr \left( c \hat{D} E F + \left( c \hat{D} E - \nabla_3 \right) \nabla_3 F + \left( - \nabla_3 C - 1 \right) \left( c \hat{D} C - 1 \right) \right)$$

Let $U$ be a symmetric traceless two tensor of conformal type $s$. The following commutators hold.

$$[P_C, (c)\nabla_3 U] = (- (c)\nabla_3 C) U$$

$$[P_C, (c)\nabla_4 U] = \frac{1}{2} (H - H) \cdot (c)\hat{D} U + \frac{1}{2} \hat{H} - \hat{H} \cdot (c)\hat{D} U$$

$$+ \left( -(c)\nabla_4 C + (s - 2)P + (s + 2)\nabla + 2sF P - \frac{1}{2} (H - \hat{H}) \right) \left( c \hat{D} C - 1 \right) U$$

$$[P_C, (c)\hat{D}] U = \frac{1}{2} tr \left( c \hat{D} \hat{D} U + \hat{H} \cdot (c)\nabla_3 U + \left( -(c)\nabla_3 C - 1 \right) \left( c \hat{D} C - 1 \right) \right)$$

Proof. We compute

$$[P_C, (c)\nabla_3 F] = \left( (c)\nabla + C \right) \left( (c)\nabla_3 F + U \right) \left( c \nabla_3 F + U \right)$$

The proof for the symmetric traceless two tensor is identical. Also,

$$[P_C, (c)\nabla_4 F] = \left( (c)\nabla + C \right) (c)\nabla_4 F - (c)\nabla_4 (c)\nabla_3 F + C F = \left( (c)\nabla_3, (c)\nabla_4 \right) (c)\nabla_3 F + C F$$

Using (45), we obtain the stated expression. Similarly for $U$, we compute

$$[P_C, (c)\hat{D} \hat{\Box}] = \left( (c)\nabla + C \right) (c)\hat{D} \hat{\Box} F + (c)\hat{D} \hat{\Box} (c)\nabla_3 F + C F = \left( (c)\nabla_3, (c)\hat{D} \hat{\Box} \right) (c)\nabla_3 F + C F$$

Using (43), we obtain the stated expression. We compute

$$E \cdot [P_C, (c)\hat{D} F] = \left( (c)\nabla + C \right) \left( (c)\hat{D} F - (c)\hat{D} \hat{\Box} \right) \left( (c)\nabla_3 F + C F \right)$$

$$= \left( (c)\nabla_3, (c)\hat{D} \hat{\Box} \right) \left( (c)\nabla_3 F + C F \right)$$

Using (46), we obtain the stated expression. We compute

$$E \cdot [P_C, (c)\hat{D} F] = \left( (c)\nabla + C \right) \left( (c)\hat{D} F - (c)\hat{D} \hat{\Box} \right) \left( (c)\nabla_3 F + C F \right)$$

$$= \left( (c)\nabla_3, (c)\hat{D} \hat{\Box} \right) \left( (c)\nabla_3 F + C F \right)$$

Using (47), we obtain the stated expression. We compute

$$[P_C, (c)\hat{D}] U = \left( (c)\nabla + C \right) \left( (c)\hat{D} \hat{D} U + \hat{H} \cdot (c)\nabla_3 U + \left( -(c)\nabla_3 C - 1 \right) \left( c \hat{D} C - 1 \right) \right)$$

Using (48), we obtain the stated expression. Using (44) and (140), we separate the computations of $[P_{C1}, T_1]$ and $[P_{C2}, T_2]$ into the following terms. In what follows, we keep track of the exact lower order terms which are vanishing in Reissner–Nordström only for the angular derivatives $(c)\hat{D}$ and $(c)\hat{D}$. **64**
C.1.1 The term \((c)\nabla_3 (c)\nabla_4\)

We have

\[
I = -[\mathcal{P}_C, (c)\nabla_3 (c)\nabla_4]\Psi = -[\mathcal{P}_C, (c)\nabla_3] (c)\nabla_4\Psi - (c)\nabla_3 ([\mathcal{P}_C, (c)\nabla_4]\Psi)
\]

In the case of \(\Psi = \mathfrak{B}\), for \(s = 1\):

\[
[\mathcal{P}_{C_1}, (c)\nabla_4]\mathfrak{B} = \frac{1}{2}(H - \mathfrak{H}) \cdot \mathcal{D}\mathfrak{B} + \frac{1}{2}(\mathfrak{H} - \mathfrak{T}) \cdot \mathcal{D}\mathfrak{B} + \left(- (c)\nabla_4\mathfrak{C}_1 + 2\mathfrak{T} + 2(F)^{\mathfrak{P}(\mathfrak{P}\mathfrak{P}) - H \cdot \mathfrak{H}}\right) \mathfrak{B}
\]

We therefore obtain

\[
-(c)\nabla_3 ([\mathcal{P}_C, (c)\nabla_4]\mathfrak{B}) = -\frac{1}{2}(H - \mathfrak{H}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B} - \frac{1}{2}(\mathfrak{H} - \mathfrak{T}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B}
\]

We therefore obtain

\[
I_3^\mathfrak{B} = -\frac{1}{2}(H - \mathfrak{H}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B} - \frac{1}{2}(\mathfrak{H} - \mathfrak{T}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B}
\]

where \(c_0(r, \theta)\) denotes a function of \(r\) and \(\theta\) which vanishes in Reissner-Nordström. Observe that the commutators \([\mathcal{D}, \mathfrak{B}]\) and \([\mathcal{D}, \mathfrak{B}]\) can be written as

\[
(H - \mathfrak{H}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B} = -\frac{1}{2}X_\mathfrak{H} (H - \mathfrak{H}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B} + c_1(r, \theta) (c)\nabla_3 (c)\nabla_4\mathfrak{B} + c_0(r, \theta) (c)\nabla_3 (c)\nabla_4\mathfrak{B}
\]

\[
(H - \mathfrak{H}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B} = -\frac{1}{2}X_\mathfrak{H} (H - \mathfrak{H}) \cdot (c)\nabla_3 (c)\nabla_4\mathfrak{B} + c_1(r, \theta) (c)\nabla_3 (c)\nabla_4\mathfrak{B} + c_0(r, \theta) (c)\nabla_3 (c)\nabla_4\mathfrak{B}
\]

C.1.2 The Laplacian term

We compute here the commutator with \((c)\mathcal{D}\cdot(c)\mathcal{D}\) and with \((c)\mathcal{D}\cdot(c)\mathcal{D}\).

For \(\mathfrak{B}\), we have

\[
J^\mathfrak{B} = \frac{1}{2}[\mathcal{P}_C, (c)\mathcal{D}\mathfrak{B}] + \frac{1}{2} \mathcal{D}(c)\mathcal{D}\mathfrak{B}
\]

Using Lemma [CL1], we have

\[
[\mathcal{P}_C, (c)\mathcal{D}\mathfrak{B}] = -\frac{1}{2}X_\mathfrak{H} (c)\mathcal{D}\mathfrak{B} + \mathfrak{H} \cdot \nabla_3 (c)\mathcal{D}\mathfrak{B} + \left( - (c)\mathcal{D}\mathfrak{C}_1 + \frac{1}{2}X_\mathfrak{H} \mathfrak{H} \right) \cdot (c)\mathcal{D}\mathfrak{B}
\]

\[
= -\frac{1}{2}X_\mathfrak{H} (c)\mathcal{D}\mathfrak{B} + \mathfrak{H} \cdot (c)\mathcal{D}\mathfrak{B} + \left( - (c)\mathcal{D}\mathfrak{C}_1 + \frac{1}{2}X_\mathfrak{H} \mathfrak{H} \right) \cdot (c)\mathcal{D}\mathfrak{B}
\]

\[+ c_1(r, \theta) (c)\nabla_3 \mathfrak{B} + c_0(r, \theta) \mathfrak{B} \]
We also have
\[
\left( c \right) D \cdot \left[ \left( P_{C_1} \right), \left( c \right) D \otimes \mathfrak{B} \right] = \left( c \right) D \cdot \left( \frac{1}{2} \text{tr} X \left( c \right) D \otimes \mathfrak{B} + H \left( c \right) \nabla_3 \mathfrak{B} + \left( - \left( c \right) D C_1 - \text{tr} X H \right) \otimes \mathfrak{B} \right)
\]
\[
= - \frac{1}{2} \text{tr} X \left( c \right) D \otimes \mathfrak{B} + H \cdot \left( c \right) D \otimes \mathfrak{B} - \frac{1}{2} \left( \text{tr} X - \text{tr} X H \right) \cdot \left( c \right) D \otimes \mathfrak{B} + \left( - \left( c \right) D C_1 - \text{tr} X H \right) \cdot \left( c \right) D \otimes \mathfrak{B} + c_1 (r, \theta) \left( c \right) \nabla_3 \mathfrak{B} + c_0 (r, \theta) \mathfrak{B}
\]
where we used \( \left( B \right) \). This gives
\[
J^\mathfrak{B} = - \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) \left( \frac{1}{2} \left( c \right) D \otimes \mathfrak{B} \right) + \frac{1}{2} H \cdot \left( c \right) D \otimes \mathfrak{B} + \frac{1}{2} \left( - \text{tr} X - \text{tr} X H \right) \cdot \left( c \right) D \otimes \mathfrak{B} + \left( - \left( c \right) D C_1 - \text{tr} X H \right) \cdot \left( c \right) D \otimes \mathfrak{B} + c_1 (r, \theta) \left( c \right) \nabla_3 \mathfrak{B} + c_0 (r, \theta) \mathfrak{B}
\]
Using the Teukolsky equation for \( \mathfrak{B} \) given by \( T_1 (\mathfrak{B}) = M_1 [3, \mathfrak{X}] \) and the expression for the Teukolsky operator \( \left( 13 \right) \) we can write
\[
\frac{1}{2} \left( c \right) D \cdot \left( c \right) D \otimes \mathfrak{B} = \left( c \right) \nabla_4 \mathfrak{B} + \left( 3 \right) \text{tr} X \left( c \right) \nabla_3 \mathfrak{B} + \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \left( c \right) \nabla_4 \mathfrak{B} + \left( \frac{9}{2} \text{tr} X \text{tr} X + 4 (F) (F) H - 9 H^3 \right) \left( c \right) \mathfrak{B}
\]
\[
- 3 H \cdot \left( c \right) D \otimes \mathfrak{B} - \left( \frac{1}{2} \left( \text{tr} X + \frac{1}{2} \text{tr} X \right) \right) \cdot \left( c \right) D \otimes \mathfrak{B} - \left( M_1 [3, \mathfrak{X}] \right)
\]
\[
= \left( c \right) \nabla_4 \mathfrak{B} + \frac{1}{2} H \cdot \left( c \right) D \otimes \mathfrak{B} + \frac{1}{2} \left( - \text{tr} X - \text{tr} X H \right) \cdot \left( c \right) D \otimes \mathfrak{B} + \left( 2 \left( F \right) (F) H + 4 (F) (F) H - 9 H^3 \right) \mathfrak{B}
\]
\[
+ \left( 9 \frac{1}{2} \text{tr} X \text{tr} X + 3 \text{tr} X \left( c \right) \nabla_3 \mathfrak{B} + \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \left( c \right) \nabla_4 \mathfrak{B} + \left( \frac{9}{2} \text{tr} X \text{tr} X + 4 (F) (F) H - 9 H^3 \right) \mathfrak{B}
\]
\[
- 3 H \cdot \left( c \right) D \otimes \mathfrak{B} - \left( \frac{1}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \cdot \left( c \right) D \otimes \mathfrak{B} + \left( M_1 [3, \mathfrak{X}] \right)
\]
\[
= \left( c \right) \nabla_4 \mathfrak{B} + \frac{1}{2} H \cdot \left( c \right) D \otimes \mathfrak{B} + \frac{1}{2} \left( - \text{tr} X - \text{tr} X H \right) \cdot \left( c \right) D \otimes \mathfrak{B}
\]
\[
+ \left( \frac{9}{2} \text{tr} X \text{tr} X + 2 (F) (F) H + 6 (F) (F) H + c_1 (r, \theta) \left( c \right) \nabla_3 \mathfrak{B} + c_0 (r, \theta) \mathfrak{B} + M_1 [3, \mathfrak{X}] \right)
\]
where we used \( \left( 15 \right) \). We therefore obtain
\[
J^\mathfrak{B} = - \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) \left( c \right) \nabla_4 \mathfrak{B} + \frac{1}{2} \left( c \right) D \otimes \mathfrak{B} + \frac{1}{2} H \cdot \left( c \right) D \otimes \mathfrak{B} + \left( c \right) \nabla_3 \mathfrak{B} + J_3^\mathfrak{B} \left( c \right) \nabla_2 \mathfrak{B}
\]
\[
+ J_4^\mathfrak{B} \left( c \right) \nabla_3 \mathfrak{B} + J_5^\mathfrak{B} \left( c \right) \nabla_2 \mathfrak{B} + J_{\nu}^\mathfrak{B} \left( c \right) D \otimes \mathfrak{B} + J_{\psi}^\mathfrak{B} \left( c \right) D \otimes \mathfrak{B} + c_1 (r, \theta) \left( c \right) \nabla_3 \mathfrak{B} + c_0 (r, \theta) \mathfrak{B} - \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) M_1 [3, \mathfrak{X}]
\]
where
\[
J_3^\mathfrak{B} = - \frac{3}{2} \text{tr} X \text{tr} X \text{tr} X + \frac{1}{2} \text{tr} X \text{tr} X \text{tr} X
\]
\[
J_4^\mathfrak{B} = - \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right)
\]
\[
J_5^\mathfrak{B} = - \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) \left( \frac{9}{2} \text{tr} X \text{tr} X + 2 (F) (F) H \right)
\]
and
\[
J_{\mu}^\mathfrak{B} = \frac{1}{2} \left( \left( - \left( c \right) D C_1 - \text{tr} X H \right) - \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) \left( - 5 H - H \right) \right)
\]
For \( \mathfrak{B} \), we have
\[
J^\mathfrak{B} = \frac{1}{2} \left( c \right) \left( P_{C_2} \right) \left( c \right) D \otimes \left( \mathfrak{B} \right) + \frac{1}{2} \left( c \right) D \otimes \left( \left( P_{C_2} \right) \left( c \right) \mathfrak{B} \right)
\]
Using Lemma \( \left( 11 \right) \) we have
\[
\left[ P_{C_2} \right] \left( c \right) D \otimes \left( \mathfrak{B} \right) = \frac{1}{2} \text{tr} X \left( c \right) D \otimes \left( \mathfrak{B} \right) + H \otimes \left( c \right) \nabla_3 \mathfrak{B} + \left( - \left( c \right) D C_2 - \text{tr} X \mathfrak{B} \right) \otimes \left( c \right) D \otimes \mathfrak{B}
\]
\[
= \frac{1}{2} \text{tr} X \left( c \right) D \otimes \left( \mathfrak{B} \right) + H \cdot \left( c \right) D \otimes \mathfrak{B} + \left( - \left( c \right) D C_2 - \frac{1}{2} \text{tr} X \right) \mathfrak{B} + d_1 (r, \theta) \left( c \right) \nabla_3 \mathfrak{B} + d_0 (r, \theta) \mathfrak{B}
\]

where we used \((31)\). We also have
\[
(c)D\hat{\mathcal{S}}([P_{C2}, (c)D\hat{\mathcal{S}}]) = (c)D\hat{\mathcal{S}}(\frac{1}{2} tr X^T (c)D \cdot \hat{\mathcal{S}} + \hat{\mathcal{H}} \cdot (c)\nabla_3 \hat{\mathcal{S}} + \left(-\frac{1}{2}(c)DC_2 + \frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}})
\]
\[
= \frac{1}{2} tr X^T (c)D\hat{\mathcal{S}}(\frac{1}{2} (c)D \cdot \hat{\mathcal{S}}) + \hat{\mathcal{H}} \cdot (c)D\hat{\mathcal{S}} - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
where we used \((32)\). This gives
\[
J^\hat{\mathcal{S}} = -\frac{1}{2} tr X^T + \frac{1}{2} tr X \left(\frac{1}{2} tr X^T H \cdot \hat{\mathcal{S}}\right)
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
Using the Teukolsky equation for \(\hat{\mathcal{S}}\) given by \(T_2(\hat{\mathcal{S}}) = M_2[A, X, \mathcal{B}]\) and the expression for the Teukolsky operator \([14]\) we can write
\[
\frac{1}{2} (c)D\hat{\mathcal{S}}(\frac{1}{2} (c)D \cdot \hat{\mathcal{S}}) = (c)\nabla_3 (c)\nabla_4 \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H \cdot \hat{\mathcal{S}}\right)
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
We therefore obtain
\[
J^\hat{\mathcal{S}} = -\frac{1}{2} tr X^T + \frac{1}{2} tr X \left(\frac{1}{2} tr X^T H \cdot \hat{\mathcal{S}}\right)
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
+ \frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
\text{where we used \((33)\).}
\]
\[
J_3^\hat{\mathcal{S}} = -\frac{1}{2} tr X^T + \frac{1}{2} tr X \left(\frac{1}{2} tr X^T H \cdot \hat{\mathcal{S}}\right)
\]
\[
J_4^\hat{\mathcal{S}} = -\frac{1}{2} \left(\frac{1}{2} tr X^T H - \frac{1}{2} \left(\frac{1}{2} tr X^T H\right)\right) \cdot \hat{\mathcal{S}} + \frac{1}{2} \left(\frac{1}{2} tr X^T H\right) \cdot \hat{\mathcal{S}}
\]
\[
\text{C.1.3 \ The term \((c)\nabla_3\)}
\]
\[
\text{Observe that}
\]
\[
P_C(gF) = (c)\nabla_3 + C(gF) = (c)\nabla_3 gF + g^{(c)} \nabla_3 F + C g F = g P_C(F) + (c)\nabla_3 g F
\]
We therefore compute
\[ K = [P_{C, g} (c) \nabla_3] \Psi = P_{C, g} (c) \nabla_3 \Psi - g (c) \nabla_3 (P_{C, g} \Psi) = g [P_{C, g} \Psi, \nabla_3 \Psi] + (c) \nabla_3 g (c) \nabla_3 \Psi = (c) \nabla_3 g (c) \nabla_3 \Psi - g (c) \nabla_3 C \Psi \]
For \( \mathfrak{B} \) we have \( g = -\text{tr} X \) and therefore
\[ K^{(c)}  = K^{(c)}_3 \nabla_3 \mathfrak{B} + K^{(c)}_3 \mathfrak{B} \]
where
\[ K^{(c)}_3 = -3 (c) \nabla_3 (\text{tr} X) \]
For \( \mathfrak{D} \) we have \( g = -\left( \frac{2}{3} \text{tr} X + \frac{1}{2} \text{tr} X \right) \) and therefore
\[ K^{(c)} = K^{(c)}_3 (c) \nabla_3 \mathfrak{D} + K^{(c)}_3 \mathfrak{D} \]
where
\[ K^{(c)}_3 = - (c) \nabla_3 \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \]
\[ K^{(c)}_3 = \frac{3}{2} \text{tr} X - \frac{1}{2} \text{tr} X \]

### C.1.4 The term \((c) \nabla_4\)

Using (154), we compute
\[ L = [P_{C, g} (c) \nabla_4] \Psi = P_{C, g} (c) \nabla_4 \Psi - g (c) \nabla_4 (P_{C, g} \Psi) = g [P_{C, g} \Psi, (c) \nabla_4 \Psi] + (c) \nabla_4 g (c) \nabla_4 \Psi \]
For \( \mathfrak{B} \), we have \( g = -\left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \) and therefore using (153) we obtain
\[ L^{(c)} = L^{(c)}_1 (c) \nabla_4 \mathfrak{B} + L^{(c)}_0 (c) \nabla_4 \mathfrak{D} + L^{(c)}_0 \mathfrak{D} \]
where
\[ L^{(c)}_1 = - (c) \nabla_3 \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \]
\[ L^{(c)}_0 = \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \]

For \( \mathfrak{D} \), we have \( g = -\frac{1}{4} (\text{tr} X + \text{tr} X) \) and therefore using (150) we obtain
\[ L^{(c)} = L^{(c)}_1 (c) \nabla_4 \mathfrak{D} + L^{(c)}_0 \mathfrak{D} \]
where
\[ L^{(c)}_1 = - \frac{1}{2} (c) \nabla_3 (\text{tr} X + \text{tr} X) \]
\[ L^{(c)}_0 = - \frac{1}{2} (\text{tr} X + \text{tr} X) \]

### C.1.5 The terms \((c) \mathcal{D}\) and \((-c) \mathcal{D})

Observe that
\[ P_{C_1} (F \cdot U) = (c) \nabla_3 + C_1 (F \cdot U) = (c) \nabla_3 F \cdot U + F \cdot (c) \nabla_3 U + C_1 F \cdot U = F \cdot P_{C_1} (U) + (c) \nabla_3 F \cdot U \]
We compute
\[ M = [P_{C, F} (c) \mathcal{D}] \Psi + [P_{C, F} (c) \mathcal{D}] \Psi = P_{C, F} (c) \mathcal{D} \Psi + (c) \mathcal{D} \Psi = P_{C, F} (c) \mathcal{D} \Psi \]
\[ = F \cdot [P_{C, (c) \mathcal{D}} \Psi + \mathcal{D} \Psi] + (c) \mathcal{D} \Psi = F \cdot [P_{C, (c) \mathcal{D}} \Psi + (c) \nabla_3 F \cdot (c) \mathcal{D} \Psi + (c) \nabla_3 (c) \mathcal{D} \Psi \]
\[ = (c) \nabla_3 F - \frac{1}{2} (c) \nabla_3 F \]
\[ = (c) \nabla_3 F - \frac{1}{2} (c) \nabla_3 F \]
For \( \mathfrak{B} \), we have \( F = 3H \) and \( \mathcal{D} = \frac{1}{2} H + \frac{3}{2} H \), and therefore we have
\[ M^{(c)} = M^{(c)}_0 \mathfrak{B} + (c) \mathcal{D} \mathfrak{B} + c_1 (r, \theta) (c) \nabla_3 \mathfrak{B} + c_0 (r, \theta) \mathfrak{B} \]
where
\[ M^{(c)}_0 = 3 (c) \nabla_3 H - \frac{2}{2} (c) \nabla_3 H \]
\[ M^{(c)}_0 = \frac{1}{2} (c) \nabla_3 \left( \frac{1}{2} H + \frac{3}{2} H - \frac{1}{2} \text{tr} X \right) - \frac{1}{2} (c) \nabla_3 \left( \frac{1}{2} H + \frac{3}{2} H \right) \]
For \( \mathfrak{D} \), we have \( (2H + \frac{1}{2} H) \) and \( \mathcal{D} = \frac{1}{2} H \), and therefore we have
\[ M^{(c)} = M^{(c)}_0 \mathfrak{D} + (c) \mathcal{D} \mathfrak{D} + d_1 (r, \theta) (c) \nabla_3 \mathfrak{D} + d_0 (r, \theta) \mathfrak{D} \]
where
\[ M^{(c)}_0 = \frac{1}{2} (c) \nabla_3 (\frac{1}{2} H) - \frac{1}{4} (c) \nabla_3 \left( \frac{1}{2} H \right) \]

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C.1.6 The potential

Using (164) we have

\[ N = [P_C, g] \Psi = (c) \nabla g \Psi \]

For \( \mathcal{B} \), we have

\[ N^\mathcal{B} = N_0^\mathcal{B} R^\mathcal{B} + c_0(r, \theta) R^\mathcal{B} \]

where

\[ N_0^\mathcal{B} = (c) \nabla \left( -\frac{9}{2} r X + \frac{9}{4} X^2 \right) \]

For \( \mathfrak{S} \) we have

\[ N^\mathfrak{S} = N_0^\mathfrak{S} R^\mathfrak{S} + d_0(r, \theta) \mathfrak{S} \]

where

\[ N_0^\mathfrak{S} = (c) \nabla \left( -\frac{3}{4} r X + \frac{3}{4} X^2 \right) \]

C.1.7 The final expression

Using (113) and (114), we obtain

\[ [P_{C_1}, T_1](\mathcal{B}) = I^\mathcal{B} + J^\mathcal{B} + K^\mathcal{B} + L^\mathcal{B} + M^\mathcal{B} + N^\mathcal{B}, \quad [P_{C_2}, T_2](\mathfrak{S}) = I^\mathfrak{S} + J^\mathfrak{S} + K^\mathfrak{S} + L^\mathfrak{S} + M^\mathfrak{S} + N^\mathfrak{S} \]

Putting the above expressions together we obtain

\[ [P_{C_1}, T_1](\mathcal{B}) = \frac{1}{2} H \cdot (c) D (c) \nabla \mathcal{B} + \frac{1}{2} H \cdot (c) D (c) \nabla \mathfrak{S} - \frac{1}{2} (r X + \frac{1}{4} X^2) (c) \nabla \mathfrak{S} - (c) \nabla \mathfrak{S} \]

Using (124) to write

\[ (c) \nabla \mathcal{B} = \mathfrak{P} - C_1 \mathcal{B}, \quad (c) \nabla \mathfrak{S} = \Omega - C_2 \mathfrak{S} \]

we rewrite

\[ H \cdot (c) D (c) \nabla \mathcal{B} = H \cdot (c) D (\mathfrak{P} - C_1 \mathcal{B}) = H \cdot (c) D (\mathfrak{P}) - C_1 H \cdot (c) D \mathcal{B} + c_0(r, \theta) \mathcal{B} \]

\[ (c) \nabla_4 (c) \nabla \mathfrak{S} = (c) \nabla_4 (\mathfrak{P} - C_1 \mathcal{B}) = (c) \nabla_4 \mathfrak{P} - C_1 (c) \nabla_4 \mathcal{B} - (c) \nabla_4 C_1 \mathcal{B} \]

and

\[ H \cdot (c) D (c) \nabla \mathfrak{S} = H \cdot (c) D (\Omega) - C_2 H \cdot (c) D \mathfrak{S} + d_0(r, \theta) \mathfrak{S} \]

\[ (c) \nabla_4 (c) \nabla \mathfrak{S} = (c) \nabla_4 (\Omega - C_2 \mathfrak{S}) = (c) \nabla_4 \mathfrak{P} - C_2 (c) \nabla_4 \mathcal{B} - (c) \nabla_4 C_2 \mathcal{B} \]

Hence we obtain

\[ [P_{C_1}, T_1](\mathcal{B}) = \frac{1}{2} H \cdot (c) D \mathfrak{P} + \frac{1}{2} H \cdot (c) D \mathfrak{S} - \frac{1}{2} (r X + \frac{1}{4} X^2) (c) \nabla \mathfrak{P} \]

\[ + (I_3^\mathcal{B} + J_3^\mathcal{B} + K_3^\mathcal{B}) \mathfrak{P} + (I_3^\mathfrak{S} + J_3^\mathfrak{S} + K_3^\mathfrak{S}) \mathfrak{S} \]

\[ + (I_3^\mathcal{B} + J_3^\mathcal{B} + K_3^\mathcal{B} + L_3^\mathcal{B} + M_3^\mathcal{B} + N_3^\mathcal{B}) \mathcal{B} \]

\[ + (I_3^\mathfrak{S} + J_3^\mathfrak{S} + L_3^\mathfrak{S} + M_3^\mathfrak{S} + N_3^\mathfrak{S}) \mathfrak{S} + (I_3^\mathcal{B} + J_3^\mathcal{B} + L_3^\mathcal{B} + M_3^\mathcal{B} + N_3^\mathcal{B}) \mathcal{B} \]

\[ + (I_3^\mathfrak{S} + J_3^\mathfrak{S} + L_3^\mathfrak{S} + M_3^\mathfrak{S} + N_3^\mathfrak{S}) \mathfrak{S} + (I_3^\mathcal{B} + J_3^\mathcal{B} + L_3^\mathcal{B} + M_3^\mathcal{B} + N_3^\mathcal{B}) \mathcal{B} \]

\[ + c_1 (r, \theta) (c) \nabla_4 \mathcal{B} + c_0 (r, \theta) \mathcal{B} - \frac{1}{2} (r X + \frac{1}{4} X^2) M_1 [\mathfrak{S}, \mathcal{X}] \]
and
\[ [PC_2, T_2](\delta) = \frac{1}{2} H \cdot (c) \partial_\Omega + \frac{1}{2} \nabla C_2 \cdot (c) \partial_\Omega - \frac{1}{2} (\text{tr} X + \text{tr} X) (c) \nabla C_2 \]
\[ + (I_3^\delta + J_3^\delta + K_3^\delta) \cdot (c) \partial_\Omega + (I_3^\delta + J_3^\delta + L_3^\delta + \frac{1}{2} (\text{tr} X + \text{tr} X) C_2) (c) \nabla \delta \]
\[ + (I_0^\delta + J_0^\delta + K_0^\delta + L_0^\delta + N_0^\delta + \frac{1}{2} (\text{tr} X + \text{tr} X) (c) \nabla C_2 - C_2 (I_3^\delta + J_3^\delta + K_3^\delta)) \cdot \nabla \delta + (I_0^\delta + J_0^\delta + L_0^\delta + M_0^\delta - \frac{1}{2} C_2 H) \cdot (c) \partial_\delta \]
\[ + d_1 (r, \theta) (c) \nabla \delta + d_0 (r, \theta) \delta - \frac{1}{2} (\text{tr} X + \text{tr} X) \Lambda_2 [A, X, \delta] \]

C.1.8 The coefficients of \((c) \nabla_4 B\) and \((c) \nabla_4 \delta\)

The coefficient of \((c) \nabla_4 B\) is given by
\[ I_4^B + J_4^B + L_4^B + \frac{1}{2} (\text{tr} X + \text{tr} X) C_1 \]
\[ = (c) \nabla_4 C_1 - \frac{1}{2} (\text{tr} X + \text{tr} X) (\frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X) - (c) \nabla_3 (\frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X) + \frac{1}{2} (\text{tr} X + \text{tr} X) C_1 \]

Observe that the coefficient of \((c) \nabla_4 \delta\) is given by
\[ I_4^\delta + J_4^\delta + L_4^\delta + \frac{1}{2} (\text{tr} X + \text{tr} X) C_2 \]
\[ = (c) \nabla_4 C_2 - \frac{1}{2} (\text{tr} X + \text{tr} X) (\text{tr} X + \frac{1}{2} \text{tr} X) - \frac{1}{2} (c) \nabla_3 (\text{tr} X + \text{tr} X) + \frac{1}{2} (\text{tr} X + \text{tr} X) C_2 \]

In particular, the conditions (122) and (126) in the Lemma provide the vanishing of the coefficients of \((c) \nabla_4 B\) and \((c) \nabla_4 \delta\).

C.1.9 The coefficients of \(\psi\) and \(\Omega\)

The coefficient of \(\mathcal{B}\) is given by
\[ I_3^B + J_3^B + K_3^B = (c) \nabla_4 C_1 - 2 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} - \frac{3}{2} \text{tr} X (\text{tr} X + \frac{1}{2} \text{tr} X) - 3 (c) \nabla_3 (\text{tr} X) \]
\[ = (c) \nabla_4 C_1 - \frac{3}{2} \text{tr} X \text{tr} X - 8 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} - 3 (\partial_\mathcal{F} \cdot \mathcal{H} + \mathcal{H} \cdot \mathcal{H}) \]

The coefficient of \(\Omega\) is given by
\[ I_3^\Omega + J_3^\Omega + K_3^\Omega = (c) \nabla_4 C_2 + P - 3 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} - \frac{1}{2} (\text{tr} X + \text{tr} X) (\frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X) - (c) \nabla_3 \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \]
\[ = (c) \nabla_4 C_2 + \frac{3}{4} \text{tr} X \text{tr} X - \frac{1}{4} \text{tr} X \text{tr} X - 6 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} - \frac{3}{2} (\partial_\mathcal{F} \cdot \mathcal{H} + \mathcal{H} \cdot \mathcal{H}) - \frac{1}{2} (c) \partial_\mathcal{F} \cdot \mathcal{H} + \mathcal{H} \cdot \mathcal{H} \]
as obtained in the Theorem.

C.1.10 The coefficients of \(\mathcal{B}\) and \(\delta\)

We now show that conditions (123) and (126) imply the following structure for the coefficient of \(\mathcal{B}\) and \(\delta\), i.e. that
\[ I_3^B + J_3^B + K_3^B + L_3^B + N_3^B + \frac{1}{2} (\text{tr} X + \text{tr} X) (c) \nabla_4 C_1 - C_1 (I_3^B + J_3^B + K_3^B) = 2 (\text{tr} X + \text{tr} X) (\mathcal{F})^T \mathcal{F} \mathcal{P} + c_0 (r, \theta) \]
and
\[ I_3^\delta + J_3^\delta + K_3^\delta + L_3^\delta + N_3^\delta + \frac{1}{2} (\text{tr} X + \text{tr} X) (c) \nabla_4 C_2 - C_2 (I_3^\delta + J_3^\delta + K_3^\delta) = -2 (\text{tr} X + \text{tr} X) (\mathcal{F})^T \mathcal{F} \mathcal{P} + d_0 (r, \theta) \delta \]

We compute the first one. We have
\[ I_3^B + J_3^B + K_3^B + L_3^B + N_3^B + \frac{1}{2} (\text{tr} X + \text{tr} X) (c) \nabla_4 C_1 - C_1 (I_3^B + J_3^B + K_3^B) \]
\[ = (c) \nabla_3 \left( (c) \nabla_4 C_1 - 2 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} - \frac{1}{2} (\text{tr} X + \text{tr} X) \left( \frac{9}{2} \text{tr} X \text{tr} X + 2 \mathcal{B} + 6 (\mathcal{F})^T \mathcal{F} \mathcal{P} \right) + 3 \text{tr} X (c) \nabla_4 C_1 \right) \]
\[ + \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) (c) \nabla_4 C_1 - 2 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} + (c) \nabla_3 \left( - \frac{9}{2} \text{tr} X \text{tr} X - 4 (\mathcal{F})^T \mathcal{F} \mathcal{P} \right) + \frac{1}{2} (\text{tr} X + \text{tr} X) (c) \nabla_4 C_1 \]
\[ - C_1 \left( (c) \nabla_4 C_1 - \frac{3}{2} \text{tr} X \text{tr} X - 8 \mathcal{B} - 2 (\mathcal{F})^T \mathcal{F} \mathcal{P} \right) + c_0 (r, \theta) \]

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We compute

\[ (c) \nabla_3 \mathcal{P} = -\frac{3}{2} \text{tr} \mathcal{P}^2 - \text{tr} \mathcal{P} \mathcal{P}^2 \]

\[ (c) \nabla_3 (\mathcal{P} \mathcal{P}^2 \mathcal{P}^2) = - (\text{tr} \mathcal{P}^2 + \text{tr} \mathcal{P}) \mathcal{P} \mathcal{P}^2 \]

\[ (c) \nabla_3 \left( \frac{9}{2} \text{tr} \mathcal{P}^2 - 4 \mathcal{P} \mathcal{P}^2 \right) = \frac{9}{4} \text{tr} \mathcal{P}^2 \mathcal{P} + \frac{9}{4} \text{tr} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P}^2 - 9 \text{tr} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + 4 \text{tr} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P}^2 + \mathcal{C}_0 (r, \theta) \]

We then obtain

\[ = (c) \nabla_3 (c) \nabla_4 \mathcal{C}_1 - 2 \left( -\frac{3}{2} \text{tr} \mathcal{P}^2 - \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 + 2 (\text{tr} \mathcal{P}^2 + \text{tr} \mathcal{P}) \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 - \frac{1}{2} (\text{tr} \mathcal{P}^2 + \text{tr} \mathcal{P}) \left( \frac{1}{2} \text{tr} \mathcal{P}^2 + 2 \mathcal{P} + 6 \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \right) \right) 
+ 3 \text{tr} \mathcal{P} \nabla_3 (c) \nabla_4 \mathcal{C}_1 + \frac{3}{4} \text{tr} \mathcal{P} + \frac{1}{4} \text{tr} \mathcal{P} \left( (c) \nabla_4 \mathcal{C}_1 - 2 \mathcal{P} - 2 \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \right) + \frac{9}{4} \text{tr} \mathcal{P}^2 \mathcal{P} + \frac{9}{4} \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} - 9 \text{tr} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + 4 \text{tr} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + \mathcal{C}_0 (r, \theta) \]

By writing

\[ \text{tr} \mathcal{P} = \text{tr} \mathcal{P} \mathcal{P}^2 = \text{tr} \mathcal{P} + \text{tr} \mathcal{P} \mathcal{P}^2 = \text{tr} \mathcal{P} \mathcal{P}^2 = \mathcal{C}_0 (r, \theta), \quad \text{tr} \mathcal{P} \mathcal{P}^2 = \mathcal{C}_1 (r, \theta), \quad \text{tr} \mathcal{P} \mathcal{P}^2 = \mathcal{C}_2 (r, \theta), \quad \text{tr} \mathcal{P} \mathcal{P}^2 = \mathcal{C}_3 (r, \theta) \]

the above becomes

\[ = \left( c \nabla_3 (-c \text{tr} \mathcal{P}) - 6 \text{tr} \mathcal{P} - 3 \text{tr} \mathcal{P} \mathcal{P}^2 - \text{tr} \mathcal{P} \left( \frac{9}{2} \text{tr} \mathcal{P} + 2 \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 + 2 \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + \mathcal{C}_0 (r, \theta) \right) \right) 
+ 3 \text{tr} \mathcal{P} \nabla_3 \nabla_4 \mathcal{C}_1 + \frac{3}{4} \text{tr} \mathcal{P} + \frac{1}{4} \text{tr} \mathcal{P} \left( (c) \nabla_4 \mathcal{C}_1 - 2 \mathcal{P} - 2 \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + \frac{9}{4} \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} - 9 \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + 4 \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + \mathcal{C}_0 (r, \theta) \right) \]

Similarly, we compute the coefficients of \( \mathcal{P} \). We have

\[ \mathcal{P}_0 + \mathcal{P}_0 + \mathcal{K}_0 + \mathcal{L}_0 + \mathcal{N}_0 + \frac{1}{2} (\text{tr} \mathcal{P} + \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + \mathcal{C}_0 (r, \theta) \right) = -2 \text{tr} \mathcal{P} \mathcal{P}^2 \mathcal{P}^2 \mathcal{P}^2 \mathcal{P} + \mathcal{C}_0 (r, \theta) \]

C.1.11 The coefficients of \( (c) \mathcal{D} \mathcal{B} \) and \( (c) \mathcal{D} \mathcal{G} \)

We compute the coefficients of \( (c) \mathcal{D} \mathcal{B} \) and \( (c) \mathcal{D} \mathcal{G} \).

The angular derivatives of \( \mathcal{B} \) are given by

\[ \mathcal{G}(r, \theta) \cdot (c) \mathcal{D} \mathcal{B} = \mathcal{G}(r, \theta) \cdot (c) \mathcal{D} \mathcal{B} \]

where

\[ \mathcal{G}(r, \theta) = \mathcal{I}_0 + \mathcal{J}_0 + \mathcal{K}_0 + \mathcal{L}_0 + \mathcal{M}_0 - \frac{1}{2} \mathcal{C}_1 \mathcal{H} \]

\[ - \frac{1}{2} \left( (c) \nabla_3 (H - \mathcal{H}) - \frac{1}{2} \text{tr} \mathcal{P} (H - \mathcal{H}) \right) + \frac{1}{2} \left( (c) \mathcal{C}_1 - \text{tr} \mathcal{P} - \frac{1}{2} \text{tr} \mathcal{P} \mathcal{P} \mathcal{P} - \frac{1}{2} (H - \mathcal{H}) \right) \]

(163)
and
\[
c_D(r, \theta) = I_\psi + J_\psi + L_\psi + M_\psi - \frac{1}{2} C_1 H
\]
\[
= \frac{1}{2} \left( c \nabla_3 (H - H) - \frac{1}{2} \nabla_3 (H - H) \right) + \frac{1}{2} \left( -c D C_2 - (\text{tr}_X + \frac{1}{2} \text{tr}_X) H - \frac{1}{2} (\text{tr}_X - \text{tr}_X) H + (\text{tr}_X + \text{tr}_X) \left( \frac{3}{2} H + H \right) \right) - \frac{3}{4} (\text{tr}_X + \frac{1}{4} \text{tr}_X) (H - H) + c \nabla_3 \left( 2H + \frac{1}{2} H \right) - \frac{1}{2} \text{tr}_X \left( 2H + \frac{1}{2} H \right) - \frac{1}{2} C_1 H
\]
\[(164)\]

The angular derivatives of $\tilde{\delta}$ are given by
\[
d_D(r, \theta) \cdot (c) D \tilde{\delta} + d_D(r, \theta) \cdot (c) D \tilde{\delta}
\]
where
\[
d_D(r, \theta) = I_\psi + J_\psi + L_\psi + M_\psi - \frac{1}{2} C_1 H
\]
\[
= \frac{1}{2} \left( c \nabla_3 (H - H) - \frac{1}{2} \nabla_3 (H - H) \right) + \frac{1}{2} \left( -c D C_2 - (\text{tr}_X + \frac{1}{2} \text{tr}_X) H - \frac{1}{2} (\text{tr}_X - \text{tr}_X) H + (\text{tr}_X + \text{tr}_X) \left( \frac{3}{2} H + H \right) \right) - \frac{3}{4} (\text{tr}_X + \frac{1}{4} \text{tr}_X) (H - H) + c \nabla_3 \left( 2H + \frac{1}{2} H \right) - \frac{1}{2} \text{tr}_X \left( 2H + \frac{1}{2} H \right) - \frac{1}{2} C_1 H
\]
\[(165)\]

and
\[
d_D(r, \theta) = I_\psi + J_\psi + L_\psi + M_\psi - \frac{1}{2} C_1 H
\]
\[
= \frac{1}{2} \left( c \nabla_3 (H - H) - \frac{1}{2} \nabla_3 (H - H) \right) + \frac{1}{2} \left( -c D C_2 + \frac{1}{2} \text{tr}_X H + \frac{1}{2} (\text{tr}_X + \text{tr}_X) H \right) - \frac{1}{4} (\text{tr}_X + \text{tr}_X) (H - H) + \frac{1}{2} (c) \nabla_3 \left( H - H \right) - \frac{1}{2} \text{tr}_X H - \frac{1}{2} C_1 H
\]
\[(166)\]

Putting all the above parts together, we prove Proposition 7.1.

**C.2 The right hand side of the equations - Proof of Lemma 7.2**

We compute here the right hand side of the equations. We first compute
\[
\mathcal{P}_C_1 \left( A_1 [\tilde{\delta}, X] \right) + \frac{1}{2} (\text{tr}_X + \text{tr}_X) A_1 [\tilde{\delta}, X]
\]

Using the definition of $A_1 [\tilde{\delta}, X]$ \[(113)\] and \[(134)\], we have
\[
\mathcal{P}_C_1 \left( A_1 [\tilde{\delta}, X] \right) = \mathcal{P}_C_1 \left( 2\langle F, F \rangle \left( 2(c) D \cdot \tilde{\delta} + 4 H \cdot \tilde{\delta} - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right) \right)
\]
\[
= 2\langle F, F \rangle \left( 2(c) D \cdot \tilde{\delta} + 4 H \cdot \tilde{\delta} - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right)\]
\[
+ \frac{1}{2} (c) \nabla_3 \left( 2\langle F, F \rangle \left( 2(c) D \cdot \tilde{\delta} + 4 H \cdot \tilde{\delta} - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right) \right)
\]

We first compute the first line of the above:
\[
\mathcal{P}_C_1 \left( 2\langle c) D \cdot \tilde{\delta} + 4 H \cdot \tilde{\delta} - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right) = 2\mathcal{P}_C_1 \left( (c) D \cdot \tilde{\delta} + 4 H - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right) = 2\mathcal{P}_C_1 \left( (c) D \cdot \tilde{\delta} + 4 H - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right)
\]
\[
= (2\langle c) D \cdot \tilde{\delta} + 4 H \cdot \tilde{\delta} - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right) = 2\mathcal{P}_C_1 \left( (c) D \cdot \tilde{\delta} + 4 H - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right)
\]

Using Lemma \[(111)\] we have
\[
= \frac{2\langle c) D \cdot \mathcal{P}_C_1 \tilde{\delta} + (\text{tr}_X - \text{tr}_X) \mathcal{P}_C_1 \tilde{\delta} + 4 H \cdot \tilde{\delta} - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right) = 2\mathcal{P}_C_1 \left( (c) D \cdot \tilde{\delta} + 4 H - (\text{tr}_X - \text{tr}_X) \tilde{\delta} \right)
\]

Writing $\mathcal{P}_C_1 \tilde{\delta} = (c) \nabla_3 \tilde{\delta} + C_1 \tilde{\delta} = \Omega + (C_1 - C_2) \tilde{\delta}$ and
\[
\mathcal{P}_C_1 \tilde{\delta} = \nabla_3 \tilde{\delta} + C_1 \tilde{\delta} = \left( -\frac{1}{2} \text{tr}_X + C_1 \right) \tilde{\delta} - \frac{1}{2} \text{tr}_X \tilde{\delta} - 2B
\]
we obtain:

\[
\begin{align*}
2^{(c)}D \cdot (\Omega + (C_1 - C_2) \delta) - \mathrm{tr}X \left(2^{(c)}D \cdot \delta + 2\tilde{H} \cdot (\Omega - C_2 \delta) + \left(-2^{(c)}DC_1 + \mathrm{tr}X\tilde{H}\right) \cdot \delta + 4\tilde{H} \cdot (\Omega + (C_1 - C_2) \delta)\right) \\
+ 4^{(c)}\nabla_3 \tilde{H} \cdot \delta - (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \left(\left(-\frac{1}{2}\mathrm{tr}X + C_1\right) \chi - \left(2^{(c)}D \cdot \delta + \tilde{H} \cdot \delta - 2\mathbb{B}\right)\right) - ^{(c)}\nabla_3 \left(2\mathrm{tr}X - \mathrm{tr}\tilde{X}\right) \chi \\
= 2^{(c)}D \cdot \Omega + (2\tilde{H} + 4\mathbb{B}) \cdot (\Omega + (2\mathrm{tr}X - 2\mathrm{tr}\tilde{X}) + 2(C_1 - C_2)) ^{(c)}D \cdot \delta \\
+ (-2^{(c)}DC_2 - 2\tilde{H}C_2 + 4\mathbb{B} \cdot (C_1 - C_2) + \mathrm{tr}X\tilde{H} + 4^{(c)}\nabla_3 \tilde{H} + (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \tilde{H}) \cdot \delta \\
- \left(\left(\nabla_3 \left(2\mathrm{tr}X - \mathrm{tr}\tilde{X}\right) + (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \left(-\frac{1}{2}\mathrm{tr}X + C_1\right)\right) \chi + (4\mathrm{tr}X - 2\mathrm{tr}\tilde{X}) \mathbb{B}\right)
\end{align*}
\]

As above, we write the coefficient of \( \delta \), which vanishes for vanishing angular momentum, as a coefficient \( d_0(r, \theta) \). We therefore have:

\[
\begin{align*}
n \partial C_1 \left(M_M[\delta, \chi]\right) + \frac{1}{2} \left(\chi + \tilde{X}\right) \left(M_M[\delta, \chi]\right) \\
= 2^{(F)}p^{(F)p} \left(2^{(c)}D \cdot \delta + 4\tilde{H} \cdot \delta - (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \chi\right) \\
- \left(2^{(F)}p^{(F)p} \left(2^{(c)}DC_1 + \mathrm{tr}X\tilde{H}\right) \cdot \delta + \left(-2^{(c)}DC_1 + \mathrm{tr}X\tilde{H}\right) \cdot \delta + 4\tilde{H} \cdot (\Omega + (C_1 - C_2) \delta)\right) \\
+ 4^{(c)}\nabla_3 \left(2\mathrm{tr}X - \mathrm{tr}\tilde{X}\right) + (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \left(-\frac{1}{2}\mathrm{tr}X + C_1\right)\right) \chi + (4\mathrm{tr}X - 2\mathrm{tr}\tilde{X}) \mathbb{B}\right)
\end{align*}
\]

which gives:

\[
\begin{align*}
2^{(F)}p^{(F)p} \left(2^{(c)}D \cdot \Omega + (2\tilde{H} + 4\mathbb{B}) \cdot \Omega + (2\mathrm{tr}X - 2\mathrm{tr}\tilde{X}) + 2(C_1 - C_2)\right) ^{(c)}D \cdot \delta \\
- \left(\left(\nabla_3 \left(2\mathrm{tr}X - \mathrm{tr}\tilde{X}\right) + (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \left(-\frac{1}{2}\mathrm{tr}X + C_1\right)\right) \chi + (4\mathrm{tr}X - 2\mathrm{tr}\tilde{X}) \mathbb{B}\right)
\end{align*}
\]

The coefficient of \( \chi \) simplifies to:

\[
\begin{align*}
2^{(F)}p^{(F)p} \left(2^{(c)}DC_1 + \mathrm{tr}X\tilde{H}\right) \cdot \delta + \left(-2^{(c)}DC_1 + \mathrm{tr}X\tilde{H}\right) \cdot \delta + 4\tilde{H} \cdot (\Omega + (C_1 - C_2) \delta)\right) \\
+ 4^{(c)}\nabla_3 \left(2\mathrm{tr}X - \mathrm{tr}\tilde{X}\right) + (2\mathrm{tr}X - \mathrm{tr}\tilde{X}) \left(-\frac{1}{2}\mathrm{tr}X + C_1\right)\right) \chi + (4\mathrm{tr}X - 2\mathrm{tr}\tilde{X}) \mathbb{B}\right)
\end{align*}
\]

For \( C_1 = n_1\mathrm{tr}X + (2 - 2n_1)\mathrm{tr}\tilde{X} \), the above coefficient of \( \chi \) becomes:

\[
\begin{align*}
(2n_1 + 2)\mathrm{tr}X^2 + \left(-\frac{1}{2} + n_1\right)\tilde{X} \chi^2 + \left(-3n_1 + \frac{5}{2}\right) \mathrm{tr}X\tilde{X} = O(a) = c_{-1}(r, \theta)
\end{align*}
\]

On the other hand the coefficient of \( ^{(c)}D \cdot \delta \) becomes:

\[
\begin{align*}
\mathrm{tr}X - 3\mathrm{tr}\tilde{X} + 2(C_1 - C_2) \Rightarrow \mathrm{tr}X - 3\mathrm{tr}\tilde{X} + 2(n_1\mathrm{tr}X + (2 - 2n_1)\tilde{X} - (n_2\mathrm{tr}X + (1 - n_2)\tilde{X})) \\
= (1 + 2(n_1 - n_2)) \mathrm{tr}X - \mathrm{tr}\tilde{X} = \delta_{c_{-1}}(r, \theta)
\end{align*}
\]

This proves the first formula. We now compute:

\[
\nabla_3 \left(M_2[A, X, \mathbb{B}]\right) + \frac{1}{2} \left(\chi + \tilde{X}\right) M_2[A, X, \mathbb{B}] = ^{(c)}\nabla_3 \left(M_2[A, X, \mathbb{B}]\right) + \left(C_2 + \frac{1}{2} \left(\chi + \tilde{X}\right)\right) M_2[A, X, \mathbb{B}]
\]

Using the definition of \( M_2[A, X, \mathbb{B}] \) [117], we compute:

\[
\begin{align*}
^{(c)}\nabla_3 \left(M_2[A, X, \mathbb{B}]\right) = & ^{(c)}\nabla_3 \left(\left(\mathrm{tr}X^2 + \mu X^2\right) + \left(-\frac{1}{2} + n_1\right)\tilde{X} \chi^2 + \left(-3n_1 + \frac{5}{2}\right) \mathrm{tr}X\tilde{X} = O(a) = c_{-1}(r, \theta)\right)
\end{align*}
\]

This proves the first formula.
Computing \((c) \nabla_3 (3 \text{tr} X - \overline{\text{tr}} Y) = -\frac{3}{2} \text{tr} X^2 + \frac{1}{2} \overline{\text{tr}} X^2\), the above gives

\[
(c) \nabla_3 \left( M_2[A, X, \mathfrak{B}] \right) = -(F)p \left( (c) \nabla_3 (c) \nabla_3 A + \frac{1}{2} (\text{tr} X - \overline{\text{tr}} X) (c) \nabla_3 A + \left( -\frac{9}{4} \text{tr} X^2 + \frac{1}{2} \overline{\text{tr}} X \text{tr} X + \frac{1}{4} \overline{\text{tr}} X^2 \right) A \right)
\]

\[-\frac{3}{2} \nabla_3 H \cdot (c) \overline{\nabla} \mathfrak{B} + c_1(r, \theta) (c) \nabla_3 \mathfrak{B} + c_0(r, \theta) \mathfrak{B} + d_0(r, \theta) \mathfrak{B} + d_{-1}(r, \theta) X \]

We therefore obtain

\[
P_{C_2} \left( M_2[A, X, \mathfrak{B}] \right) + \frac{1}{2} \left( \text{tr} X + \text{tr} X \right) M_2[A, X, \mathfrak{B}]
\]

\[
= -(F)p \left( (c) \nabla_3 (c) \nabla_3 A + (C_2 + \text{tr} X) (c) \nabla_3 A + \left( -\frac{3}{2} \text{tr} X^2 + \overline{\text{tr}} X \text{tr} X + \frac{C_2}{2} \left( 3 \text{tr} X - \overline{\text{tr}} X \right) A \right) \right)
\]

\[-\frac{3}{2} \nabla_3 H \cdot (c) \overline{\nabla} \mathfrak{B} + c_1(r, \theta) (c) \nabla_3 \mathfrak{B} + c_0(r, \theta) \mathfrak{B} + d_0(r, \theta) \mathfrak{B} + d_{-1}(r, \theta) X \]

We now use relation (107) to eliminate from the above the dependence on \( A \), using the fact that \((F)p \ (c) \nabla_3 A + \frac{1}{2} \text{tr} X A \) can be written in terms of \( \mathfrak{B} \) and \( \mathfrak{A} \), as

\[
(F)p \left( (c) \nabla_3 A + \frac{1}{2} \text{tr} X A \right) = \frac{1}{2} (c) \nabla_3 \mathfrak{B} + 3 H \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B}
\]

\[
\frac{1}{2} (c) \nabla_3 \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B}
\]

We also compute

\[
(c) \nabla_3 \left( (F)p \left( (c) \nabla_3 A + \frac{1}{2} \text{tr} X A \right) \right)
\]

\[
= (F)p \left( (c) \nabla_3 (c) \nabla_3 A + \frac{1}{2} \text{tr} X (c) \nabla_3 A - \frac{1}{4} \text{tr} X^2 A \right) - \text{tr} X (F)p \left( (c) \nabla_3 A + \frac{1}{2} \text{tr} X A \right)
\]

\[
= (F)p \left( (c) \nabla_3 (c) \nabla_3 A - \frac{1}{4} \text{tr} X^2 A \right)
\]

The first line of the above can be written as

\[
- (F)p \left( (c) \nabla_3 (c) \nabla_3 A + (C_2 + \text{tr} X) (c) \nabla_3 A + \left( -\frac{3}{2} \text{tr} X^2 + \overline{\text{tr}} X \text{tr} X + \frac{C_2}{2} \left( 3 \text{tr} X - \overline{\text{tr}} X \right) A \right) \right)
\]

\[
= -(F)p \left[ (c) \nabla_3 (c) \nabla_3 A - \frac{3}{2} \text{tr} X (c) \nabla_3 A - \frac{3}{4} \text{tr} X^2 A + \left( C_2 + \frac{3}{2} \text{tr} X \right) (c) \nabla_3 A \right]
\]

\[
+ \left( -\frac{3}{2} \text{tr} X^2 + \overline{\text{tr}} X \text{tr} X + \frac{C_2}{2} \left( 3 \text{tr} X - \overline{\text{tr}} X \right) A \right)
\]

\[
= -\left( c \overline{\nabla} \left( \frac{1}{2} (c) D \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} \right) \right.
\]

\[
\left. \left( C_2 + \frac{3}{2} \text{tr} X \right) \left( \frac{1}{2} (c) D \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} \right) \right)
\]

\[
- (F)p \left( \frac{C_2}{2} \left( 2 \text{tr} X - \overline{\text{tr}} X \right) - \frac{3}{4} \text{tr} X^2 + \overline{\text{tr}} X \text{tr} X \right) A
\]

Observe that for \( C_2 = n_2 \text{tr} X + (1 - n_2) \overline{\text{tr}} X \), the coefficient of \( A \) becomes

\[
\frac{C_2}{2} \left( 2 \text{tr} X - \overline{\text{tr}} X \right) - \frac{3}{4} \text{tr} X^2 + \overline{\text{tr}} X \text{tr} X
\]

\[
= \left( n_2 - \frac{3}{2} \right) \text{tr} X^2 + \left( 2 - \frac{3}{2} \right) \text{tr} X \overline{\text{tr}} X + \left( n_2 - \frac{1}{2} \right) \overline{\text{tr}} X^2 = O(a) =: \tilde{d}_1(r, \theta)
\]

(168)

Using (105), we obtain

\[
- \tilde{d}_1(r, \theta) (F)p A = \tilde{d}_1(r, \theta) \left( (c) \nabla_3 \mathfrak{B} + \left( \frac{3}{2} \text{tr} X + \frac{1}{2} \text{tr} X \right) \mathfrak{B} + \frac{1}{2} (c) D \nabla \mathfrak{B} + \frac{1}{2} (3 \text{tr} X + \overline{\text{tr}} X) \mathfrak{B} \right)
\]

\[
\tilde{d}_1(r, \theta) ( (c) \nabla_3 \mathfrak{B} + d_0(r, \theta) \mathfrak{B} + \frac{1}{4} \text{tr} X \overline{\text{tr}} X \mathfrak{B} + \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} + d_{-1}(r, \theta) X \mathfrak{B}
\]

We now compute the first line of the above.

\[
- (c) \nabla_3 \left( \frac{1}{2} (c) D \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} \right)
\]

\[
- \left( C_2 + \frac{3}{2} \text{tr} X \right) \left( \frac{1}{2} (c) D \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} \right)
\]

\[
= -\left( C_2 + \frac{3}{2} \text{tr} X \right) \left( \frac{1}{2} (c) D \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} \right)
\]

\[
- \left( C_2 + \frac{3}{2} \text{tr} X \right) \left( \frac{1}{2} (c) D \nabla \mathfrak{B} - \left( 3 \overline{F} + 2 \overline{(F)p(Fp)} \right) \mathfrak{B} + c_0(r, \theta) \mathfrak{B} + c_1(r, \theta) (c) \nabla_3 \mathfrak{B}
\]

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Writing \((c)\nabla_3 \mathfrak{B} = \mathfrak{B} - C_1 \mathfrak{B}\) and \((c)\nabla_3 \hat{\mathfrak{B}} = \Omega - C_2 \hat{\mathfrak{B}}\), we obtain
\[
= -\frac{1}{2} (c)D \hat{\mathfrak{B}} \mathfrak{P} + (C_1 - C_2 - \text{tr} \hat{\mathfrak{X}}) \frac{1}{2} (c)D \mathfrak{B} + (3 \mathfrak{P} + 2 (F) \mathfrak{P} (F) \mathfrak{P}) \Omega - C_2 \left( 3 \mathfrak{P} + 2 (F) \mathfrak{P} (F) \mathfrak{P} \right) \hat{\mathfrak{B}} \\
+ \left( \frac{9}{2} \text{tr} X \mathfrak{P} - (5 \text{tr} X + 2 \text{tr} \mathfrak{X}) (F) \mathfrak{P} (F) \mathfrak{P} \right) \hat{\mathfrak{B}} - \left( C_2 + \frac{3}{2} \text{tr} \mathfrak{X} \right) \left( - \left( 3 \mathfrak{P} + 2 (F) \mathfrak{P} (F) \mathfrak{P} \right) \hat{\mathfrak{B}} + c_0 (r, \theta) \mathfrak{B} + c_1 (r, \theta) (c)\nabla_3 \mathfrak{B} \\
+ \alpha (r, \theta) \mathfrak{B} + \gamma (r, \theta) (c)\nabla_3 \mathfrak{B} \right)
\]
The coefficient of \((c)D \hat{\mathfrak{B}} \mathfrak{P}\) is given by
\[
\frac{1}{2}(C_1 - C_2 - \text{tr} \hat{\mathfrak{X}}) = (n_1 - n_2 - 1) \frac{1}{2} (\text{tr} \hat{\mathfrak{X}} - \text{tr} \mathfrak{X}) := c_{(c)D \hat{\mathfrak{B}}} (r, \theta)
\]
This proves the second formula.

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