ON (STRONGLY) GORENSTEIN (SEMI)HEREDITARY RINGS

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Abstract. In this paper, we introduce and study the rings of Gorenstein homological dimensions small or equal than 1, which we call Gorenstein (semi)hereditary rings, specially particular cases of these rings, which we call strongly Gorenstein (semi)hereditary rings.

1. Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Setup and Notation: Let \( R \) be a ring, and let \( M \) be an \( R \)-module. As usual we use \( pd_R(M) \), \( id_R(M) \) and \( fd_R(M) \) to denote, respectively, the classical projective, injective and flat dimensions of \( M \). By \( gldim(R) \) and \( wdim(R) \) we denote, respectively, the classical global and weak dimensions of \( R \).

It is by now a well-established fact that even if \( R \) to be non-Noetherian, there exists Gorenstein projective, injective and flat dimensions of \( M \), which are usually denoted by \( Gpd_R(M) \), \( Gid_R(M) \) and \( Gfd_R(M) \), respectively. Some references are \([2, 3, 8, 9, 11, 12, 14, 16]\).

Recently in \([3]\), the authors started the study of global Gorenstein dimensions of rings, which are called, for a commutative ring \( R \), Gorenstein global projective, injective, and weak dimensions of \( R \), denoted by \( GPD(R) \), \( GID(R) \), and \( G.wdim(R) \), respectively; and respectively, defined as follows:

1) \( GPD(R) = \sup \{ Gpd_R(M) \mid M \text{ is } R\text{-module} \} \)
2) \( GID(R) = \sup \{ Gid_R(M) \mid M \text{ is } R\text{-module} \} \)
3) \( G.wdim(R) = \sup \{ Gfd_R(M) \mid M \text{ is } R\text{-module} \} \)

They proved that, for any ring \( R \), \( G.wdim(R) \leq GID(R) = GPD(R) \) (\([3\) Theorem 1.1 and Corollary 1.2(1)])). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of \( GPD(R) \) and \( GID(R) \) is called Gorenstein global dimension of \( R \), and denoted by \( G.gldim(R) \).

They also proved that the Gorenstein global and weak dimensions are refinement of the classical global and weak dimensions of rings. That is: \( G.gldim(R) \leq gldim(R) \) and \( G.wdim(R) \leq wdim(R) \) with equality if \( wdim(R) \) is finite (\([3\) Corollary 1.2(2 and 3)])).

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In [2], the authors studied particular cases of Gorenstein projective, injective and flat modules which they call strongly Gorenstein projective, injective and flat modules respectively, and defined as follows:

**Definitions 1.1.**

(1) A module $M$ is said to be strongly Gorenstein projective ($SG$-projective for short), if there exists an exact sequence of projective modules of the form:

$$
P = \cdots \rightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \rightarrow \cdots$$

such that $M \cong \ker(f)$ and such that $\text{Hom}(-, P)$ leaves the sequence $P$ exact whenever $P$ is projective. The exact sequence $P$ is called a strongly complete projective resolution.

(2) The strongly Gorenstein injective modules are defined dually.

(3) A module $M$ is said to be strongly Gorenstein flat ($SG$-flat for short), if there exists an exact sequence of flat module of the form:

$$
F = \cdots \rightarrow F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \rightarrow \cdots$$

such that $M \cong \ker(f)$ and such that $- \otimes I$ leaves $F$ exact whenever $I$ is injective. The exact sequence $F$ is called a strongly complete flat resolution.

The principal role of the strongly Gorenstein projective and injective modules is to give a simple characterization of Gorenstein projective and injective modules, respectively, as follows:

**Theorem 1.2** ([2], Theorem 2.7). A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.

Using [2, Theorem 3.5] together with [16, Theorem 3.7], we have the next result:

**Proposition 1.3.** Let $R$ be a coherent ring. A module is Gorenstein flat if, and only if, it is a direct summand of a strongly Gorenstein flat module.

In this paper, we often use the following Lemma:

**Lemma 1.4.** Consider the following diagram of modules over a ring $R$.

$$
\begin{array}{ccccc}
0 & \rightarrow & M & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & M & \rightarrow & 0 \\
\uparrow{u} & & \downarrow{u} & & \downarrow{u} & & \downarrow{u} & & \\
0 & \rightarrow & Q & \xrightarrow{i} & Q \oplus Q & \xrightarrow{j} & Q & \rightarrow & 0 \\
\end{array}
$$

where $M$ is a Gorenstein projective module, $P$ and $Q$ are projective and $i$ and $j$ are the canonical injection and projection respectively. Then, there is a morphism $\gamma : P \rightarrow Q \oplus Q$ which complete $(\star)$ and make it commutative.

**Proof.** If we apply the functor $\text{Hom}(-, Q)$ to the short exact sequence

$$
(\star) \quad 0 \rightarrow M \xrightarrow{\alpha} P \xrightarrow{\beta} M \rightarrow 0
$$

we obtain the short exact sequence:

$$
(\star\star) \quad 0 \rightarrow \text{Hom}(M, Q) \xrightarrow{\alpha} \text{Hom}(P, Q) \xrightarrow{\beta} \text{Hom}(M, Q) \rightarrow 0
$$
since $\text{Ext}(M, Q) = 0$ ([16, Proposition 2.3]). On the other hand, $u \in \text{Hom}(M, Q)$. Then, from the exactness of (**), there is a morphism $v : P \to Q$ such that $v \circ \alpha = u$. Consequently, we can verify that the morphism $\gamma : P \to Q \oplus Q$ defined by $\gamma(p) := (v(p), u \circ \beta(p))$ whenever $p \in P$ is the desired morphism. □

Dually, we obtain easily the injective version of Lemma 1.4.

In section 2 and 3, motivated by the important role of the rings of global and weak dimensions smaller or equal to one in several areas of algebra, we study the rings of Gorenstein homological dimensions smaller or equal to one which we call, by analogy to the classical ones, Gorenstein hereditary, semihereditary rings, specially particular cases of these rings which we call strongly Gorenstein hereditary, semihereditary rings.

2. On (strongly) Gorenstein hereditary rings

The aim of this section is to characterize the rings of Gorenstein global dimension smaller or equal than 1, specially a particular case of them over which every Gorenstein projective module is strongly Gorenstein projective.

Definitions 2.1.

(1) A ring $R$ is called a Gorenstein hereditary ring ($G$-hereditary for short) if every submodule of a projective module is $G$-projective (i.e., $G - \text{gldim}(R) \leq 1$) and $R$ is called a Gorenstein Dedekind ring ($G$-Dedekind for short), if it is a $G$-hereditary domain.

(2) A ring $R$ is called a strongly Gorenstein hereditary ring ($SG$-hereditary for short) if every submodule of a projective module is $SG$-projective and $R$ is called strongly Gorenstein Dedekind ($SG$-Dedekind for short), if it is an $SG$-hereditary domain.

Remark 2.2. It is easy to see that the definition of a strongly Gorenstein hereditary ring is equivalent to say that every submodule of a strongly Gorenstein projective module is strongly Gorenstein projective.

In the next, we give a characterization of the $G$-hereditary rings.

Proposition 2.3. Let $R$ be a ring with finite Gorenstein global dimension. The following assertions are equivalent:

(1) $R$ is $G$-hereditary.
(2) $Gpd_R(M) \leq 1$ for all finitely generated $R$-modules $M$.
(3) Every ideal of $R$ is Gorenstein projective.
(4) $id_R(P) \leq 1$ for all $R$-modules $P$ with finite $pd_R(P)$.
(5) $id_R(P') \leq 1$ for all projective $R$-modules $P'$.
(6) $pd_R(E) \leq 1$ for all $R$-modules $E$ with finite $id_R(E)$.
(7) $pd(E') \leq 1$ for all injective $R$-modules $E'$.

Proof. All no obvious implications follow immediately from [3, Theorem 1.1], [16, Theorems 2.20 and 2.22] and [17, Lemma 9.11]. □

The main result, in this section, is the following characterization of the $SG$-hereditary rings.

Theorem 2.4. Let $R$ be a ring. The following assertions are equivalent:

(1) $R$ is $SG$-hereditary.
(2) For every $R$-module $M$, there exists a short exact sequence:

$$0 \to M \to Q \to M \to 0$$

where $\text{pd}_R(Q) \leq 1$, and for every projective module $P$ there is an integer $i > 1$ such that $\text{Ext}_R^i(M, P) = 0$.

(3) For every $R$-module $M$, there exists a short exact sequence:

$$0 \to M \to E \to M \to 0$$

where $\text{id}_R(E) \leq 1$, and for every injective module $I$ there is an integer $i > 1$ such that $\text{Ext}_R^i(I, M) = 0$.

Proof. $1 \Rightarrow 2$. Assume (1) and we claim (2). Let $M$ be an arbitrary $R$-module. Pick a short exact sequence $0 \to G \to Q \to M \to 0$ where $Q$ is a projective module. The module $G$ is immediately strongly Gorenstein projective by the hypothesis conditions. Hence, from [2, Proposition 2.9], there exists a short exact sequence $0 \to G \to P \to G \to 0$ where $P$ is projective. Consider the following diagram:

$$
\begin{array}{ccc}
\text{0} & \text{0} & \text{0} \\
\downarrow & \downarrow & \downarrow \\
0 & G & P \\
\downarrow & \downarrow & \downarrow \\
0 & Q & Q \oplus Q \\
\downarrow & \downarrow & \downarrow \\
0 & Q & \text{0}
\end{array}
$$

By Lemma 1.4, the above diagram can be completed and so applying the Snake Lemma and the fact that $M \cong \text{coker}(G \to Q)$, we can construct a short exact sequence $0 \to G \to P \to G \to 0$ where $P$ is projective. Consider the following diagram:

$$
\begin{array}{ccc}
\text{0} & \text{0} & \text{0} \\
\downarrow & \downarrow & \downarrow \\
0 & G & P \\
\downarrow & \downarrow & \downarrow \\
0 & Q & \text{0}
\end{array}
$$

From the middle vertical short exact sequence, we deduce that $X$ is projective. Moreover, by the hypothesis conditions again (applying to the module $M$), for every projective module $F$ there is an integer $i > 1$ such that $\text{Ext}_R^i(M, F) = 0$. Then, from the short exact sequence $0 \to M \to X \to M \to 0$ we get $\text{Ext}_R^i(M, F) = 0$. So, from [2, Proposition 2.9], $M$ is an $SG$-projective module, as desired.

$2 \Rightarrow 1$. Assume the second assertion and let $M$ be a submodule of a projective $R$-module $P$. We claim that $M$ is strongly Gorenstein projective. Applying the hypothesis conditions to the module $P/M$, there exists a short exact sequence $0 \to P/M \to Q \to P/M \to 0$ where $pd_R(Q) \leq 1$. Now, consider the following diagram with exact rows and columns:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & M & X \\
\downarrow & \downarrow & \downarrow \\
0 & P & P \oplus P \\
\downarrow & \downarrow & \downarrow \\
0 & P/M & Q \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
$$

From the middle vertical short exact sequence, we deduce that $X$ is projective. Moreover, by the hypothesis conditions again (applying to the module $M$), for every projective module $F$ there is an integer $i > 1$ such that $\text{Ext}_R^i(M, F) = 0$. Then, from the short exact sequence $0 \to M \to X \to M \to 0$ we get $\text{Ext}_R^i(M, F) = 0$. So, from [2, Proposition 2.9], $M$ is an $SG$-projective module, as desired.

$1 \Rightarrow 3$. Using the dual of the results in the proof of the implication $1 \Rightarrow 2$, the proof of the present implication is similar to the one of $1 \Rightarrow 2$. 
3 ⇒ 1. Assume (3). By a dual argument to the one of the first part of the implication 2 ⇒ 1 we can prove that \( I/M \) is a strongly Gorenstein injective module for an arbitrary module \( M \) and an injective module \( I \) which contains \( M \). Hence, \( Gid(M) \leq 1 \). Consequently, by [3, Theorem 1.1], \( G.gldim(R) = GID(R) \leq 1 \), and so, \( R \) is \( G \)-hereditary. Let \( M \) be a submodule of a projective module. We claim that \( M \) is strongly Gorenstein projective. Hence, \( M \) is Gorenstein projective since \( R \) is a Gorenstein hereditary ring. Thus, from [16, Theorem 2.20], \( Ext(M, P) = 0 \) for any projective module \( P \). Moreover, by the hypothesis conditions, there is a short exact sequence (\( \star \)) \( 0 \to M \to E \to M \to 0 \) where \( id(E) \leq 1 \). Hence, \( pd(E) \leq 1 \) (by Proposition 2.3). Applying [16, Theorem 2.5] to (\( \star \)), we conclude that \( E \) is Gorenstein projective. Thus, from [16, Proposition 2.27], \( E \) is projective. Consequently, from [2, Proposition 2.9], \( M \) is strongly Gorenstein projective, as desired. □

**Corollary 2.5.** Let \( R \) be a ring with finite Gorenstein global dimension. The following assertions are equivalent:

1. \( R \) is \( S \)-\( G \)-hereditary.
2. For every \( R \)-module \( M \), there exists a short exact sequence
   \[ 0 \to M \to Q \to M \to 0 \]
   such that \( pd_R(Q) \leq 1 \).
3. For every \( R \)-module \( M \), there exists a short exact sequence
   \[ 0 \to M \to E \to M \to 0 \]
   such that \( id_R(E) \leq 1 \).

**Proof.** From [16, Theorems 2.20 and 2.22] and the definition of \( G.gldim(-) \) (see [3, page 1]), for every \( R \)-module and every projective \( R \)-module \( P \) we have \( Ext^n_R(M, P) = 0 \) where \( n = G.gldim(R) \); and similarly for every injective \( R \)-module \( I \) we have \( Ext^n_R(I, M) = 0 \). Then, this Corollary follows directly from Theorem 2.4. □

It’s well-known that the hereditary rings (resp. Dedekind domains) are coherent (resp. Noetherian). Now, it is natural to ask what about the \( G \)-hereditary rings. Now, we can give an affirmative answer just in the strongly Gorenstein hereditary case.

**Theorem 2.6.**

1. Every \( S \)-\( G \)-hereditary ring is coherent.
2. Every coherent \( G \)-Dedekind domain, in particular every \( S \)-\( G \)-Dedekind domain, is Noetherian.

**Proof of Theorem 2.6.**

(1) Assume that \( R \) is an \( S \)-\( G \)-hereditary ring and let \( I \) be a finitely generated ideal of \( R \). Then, \( I \) is an \( S \)-\( G \)-projective \( R \)-module, since \( I \) is a submodule of the projective \( R \)-module \( R \). Therefore, \( I \) is a finitely presented \( R \)-module (by [2, Theorem 3.9]). So, \( R \) is coherent, as desired.

(2) Let \( R \) be a \( G \)-Dedekind coherent domain. Then, \( G.gldim(R) \leq 1 \). If \( G.gldim(R) = 0 \), then \( R \) is a quasi-Frobenius ring (by [3, Proposition 2.6]) and so is Noetherian. Now, suppose that \( G.gldim(R) = 1 \). The finitistic Gorenstein projective dimension of \( R \), denoted by \( FGPDR \) is finite (See [16, p.182]). Namely, \( FGPDR = G.gldim(R) \). From [16, Theorem 2.28], it is equal to the finitistic projective dimension of \( R \), denoted by \( FPD(R) \). Then, \( FPD(R) = 1 \). Therefore, \( R \) is Noetherian (from [15, Theorems 2.5.14]).
Taking the consideration (1) above, the particular case (i.e; where $R$ is $SG$-Dedekind domain) is immediate. □

**Proposition 2.7.** Let $R$ be a Gorenstein hereditary ring. The following statements are equivalents:

1. $R$ is a strongly Gorenstein hereditary ring;
2. Every $G$-projective $R$-module is strongly Gorenstein projective;
3. $R$ is coherent and every Gorenstein flat $R$-module is strongly Gorenstein flat module.

**Proof.** 1 $\iff$ 2. Obvious since every $G$-projective module is a submodule of a projective module.

2 $\implies$ 3. The coherence of $R$ is guarantied by Theorem 2.6(1). Now, let $M$ be a $G$-flat module. By hypothesis $Gpd_R(M) \leq 1$. Thus, from Theorem 2.4 there is an exact sequence $0 \to M \to X \to M \to 0$ where $pd(X) \leq 1$. Then, $id(\text{Hom}_R(X, \mathbb{Q}/\mathbb{Z})) = fd(X) \leq pd(X) \leq 1$. Furthermore, from [16, Theorem 3.7], $X$ is a Gorenstein flat module since $G^f(R)$ is projectively resolving. Hence, by [16, Proposition 3.11], $\text{Hom}_R(X, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective. Consequently, by the dual of [16, Proposition 2.27], $\text{Hom}_R(X, \mathbb{Q}/\mathbb{Z})$ is injective. Then, from [15, Theorem 1.2.1]), $X$ is flat. Hence, $M$ is immediately $SG$-flat (by [2, Proposition 3.6] and since for any injective module $I$, we have $Tor(M, I) = 0$ as $M$ is $G$-flat).

3 $\implies$ 2. Let $I$ be an injective $R$-module. From [3, Corollary 2.7], $fd_R(I) \leq 1$. Then, from [10, Theorem 3.8], $R$ is an 1 – $FC$ ring (i.e., coherent ring with $Ext^2_R(P, R) = 0$ for each finitely presented $R$-module $P$). Now, let $M$ be a $G$-projective $R$-module. Then, $M$ embeds in a projective $R$-module. So, from [7, Theorem 7], $M$ is $G$-flat. Then, by hypothesis $M$ becomes $SG$-flat. Hence, there exists a short exact sequence $0 \to M \to F \to M \to 0$ where $F$ is flat. By the resolving of the class $G^f(R)$ and from the short exact sequence above we deduce that $F$ is $G$-projective (since $M$ is $G$-projective). On the other hand, $pd_R(F) < \infty$ (by [3, Corollary 2.7] and since $F$ is flat). Therefore, $F$ is projective by [16, Proposition 2.27]. So $M$ is $SG$-projective (by [2, Proposition 2.9] and since $Ext(M, P) = 0$ for every projective module $P$ as $M$ is $G$-projective).

□

**Remark 2.8.** Using [3, Corollary 1.2(2)], we say clearly that a $G$-hereditary ring (in particular an $SG$-hereditary ring) is hereditary if, and only if, $wdim(R)$ is finite.

In what follows we give an example of non $G$-semisimple $SG$-hereditary ring and a $G$-hereditary ring which is not $SG$-hereditary.

**Example 2.9.** Consider a non-semisimple quasi-Frobenius rings $R = K[X]/(X^2)$, $R' = K[X]/(X^3)$ where $K$ is a field, and a non-Noetherian hereditary ring $S$. Then,

1. $R \times S$ is a strongly Gorenstein hereditary ring which is not hereditary.
2. $R' \times S$ is a Gorenstein hereditary ring which is not strongly Gorenstein hereditary.

**Proof.** From [4, Example 3.4], the rings $R \times S$ and $R' \times S$ are both Gorenstein hereditary rings with infinite weak dimension.

(1) We have to prove that $R \times S$ is strongly Gorenstein hereditary. From Proposition 2.7 it remains to prove that every Gorenstein projective module is strongly Gorenstein projective. Let $M$ be a Gorenstein projective $R \times S$-module. We claim that $M$ is an $SG$-projective module. We have the isomorphism of $R \times S$-modules:

$$M \cong M \otimes_{R \times S} (R \times 0) \cong M \otimes_{R \times S} (R \times 0 \oplus 0 	imes S) \cong M_1 \times M_2$$
where $M_1 = M \otimes_{R \times S} R$ and $M_2 = M \otimes_{R \times S} S$ (for more details see [6, p.102]). By [4, Lemma 3.2], $M_1$ (resp. $M_2$) is a $G$-projective $R$-module (resp. $S$-module). Then, since $R$ is strongly Gorenstein semisimple and $S$ is hereditary, $M_1$ (resp. $M_2$) is an $SG$-projective $R$-module (resp. $S$-module) (precisely $M_2$ is a projective $S$-module). On the other hand, the family $\{R, S\}$ of rings satisfies the conditions of $[4, \text{Lemma 3.3}]$ (by $[3, \text{Corollary 2.7}]$ since $G.gldim(R)$ and $G.gldim(S) = gldim(S)$ are finite). Thus, $M = M_1 \times M_2$ is an $SG$-projective $R \times S$-module, as desired.

(2) We have to prove that $R \times S$ is not strongly Gorenstein module. By $[5, \text{Corollary 3.10}]$, there exists a Gorenstein projective $R'$-module $M$ which is not strongly Gorenstein projective. And by, $[5, \text{Lemma 3.2}]$, $M \times S$ is Gorenstein projective $R' \times S$-module which is not strongly Gorenstein projective. Thus, from Proposition $[7, 2]$ $R' \times S$ is not strongly Gorenstein hereditary.

3. On (strongly) Gorenstein semihereditary rings

The aim of this section is to characterize the rings of Gorenstein weak dimension smaller or equal than 1, specially a particular case of them over which every Gorenstein flat module is strongly Gorenstein flat.

**Definition 3.1.**

(1) A ring $R$ is called Gorenstein semihereditary ($G$-semihereditary fort short) if $R$ is coherent and every submodule of flat module is $G$-flat (i.e., $R$ is coherent and $G.wdim(R) \leq 1$).

(2) A ring $R$ is called strongly Gorenstein semihereditary ($SG$-semihereditary for short) if $R$ is coherent and every submodule of flat module is $SG$-flat.

**Remarks 3.2.** (i) Clearly we have the followings equivalences:

(1) A $G$-semihereditary ring $R$ is semihereditary if, and only if, $G.wdim(R)$ is finite ($[3, \text{Corollary 1.2(3)}]$).

(2) A $G$-semihereditary ring $R$ is $SG$-semihereditary if, and only if, every $G$-flat module is $SG$-flat.

(ii) Every $SG$-hereditary ring is $SG$-semihereditary (by $[3, \text{Corollary 1.2(1)}]$). Proposition $[7, 2]$ and (2) above).

(iii) Every Noetherian $G$-semihereditary (resp. $SG$-semihereditary) ring is $G$-hereditary (resp. $SG$-hereditary) (From $[13, \text{Theorem 12.3.1}]$ and also Proposition $[7, 2]$ and (ii2) above in the strongly case).

Recall that we say that an $R$-module $M$ has $FP$-injective dimension at most $n$ (for some $n \geq 0$) over a ring $R$, denoted by $FP - id_R(M) \leq n$, if $\text{Ext}_R^{n+1}(P, M) = 0$ for all finitely presented $R$-modules $P$. Recall also that $R$ is called $n - FC$ (for some $n \geq 0$), if it is coherent and it has self-$FP$-injective dimension at most $n$ (i.e., $FP - id_R(R) \leq n$). Now, we give a characterization of the $G$-semihereditary rings.

**Proposition 3.3.** Let $R$ be a coherent ring. then the following statements are equivalent:

(1) $R$ is $G$-semihereditary.

(2) $Gfd_R(M) \leq 1$ for all finitely presented $R$-modules $M$.

(3) $Gpd_R(M) \leq 1$ for all finitely presented $R$-modules $M$.

(4) Every finitely generated ideal of $R$ is Gorenstein flat.

(5) $fd_R(I) \leq 1$ for all injective $R$-modules $I$.

(6) $FP - id_R(F) \leq 1$ for all flat $R$-modules $F$.

(7) $fd_R(E) \leq 1$ for all $FP$-injective $R$-modules $E$. 

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (5). Follows from [16, Theorem 3.14] and [15, Theorem 1.3.8].
(5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1). Follows from [10, Theorem 3.8].
(1) $\Rightarrow$ (3). Follows from [7, Theorem 7].

Now, we give the main two results in this section.

**Theorem 3.4.** Let $R$ be a ring such that every direct limit of $SG$-flat $R$-modules is $SG$-flat. Then, the following statements are equivalent:

1. $R$ is $SG$-semihereditary.
2. Every finitely generated submodule of a projective module is $SG$-projective.

**Proof.** We assume that $R$ is an $SG$-semihereditary ring and let $M$ be a finitely generated submodule of a projective module $P$. Then, $M$ is a finitely presented $SG$-flat module (since $R$ is $SG$-semihereditary and $R$ is coherent). Then, $M$ is $SG$-projective (By [2, Proposition 3.9]), as desired.

Conversely, we assume that every finitely generated submodule of a projective module is $SG$-projective. Our aim is to show that $R$ is $SG$-semihereditary. Let $I$ be a finitely generated ideal of $R$. By hypothesis, $I$ is a finitely generated $SG$-projective $R$-module. Then, by [2, Proposition 3.9], $I$ is a finitely presented $SG$-flat $R$-module. So, by Proposition 3.3, $R$ is a $G$-semihereditary ring (see that $R$ is coherent). Now, to prove that $R$ is $SG$-semihereditary ring, it suffices, by Remark 3.2, to prove that every $G$-flat $R$-module is $SG$-flat. So, let $M$ be a $G$-flat $R$-module. Then, $M$ embeds in a flat $R$-module $F$. By Lazard’s theorem ([15, Theorem 1.2.6]), there is a direct system $(L_i, \varphi_{ij})_{i\in I}$ of finitely generated free $R$-modules such that $\lim L_i \simeq F$. On the other hand $\lim (L_i, \varphi_{ij}) = \oplus L_i/S$ where $S$ is the submodule generated by all elements $\lambda_i \circ \varphi_{ij}(a_i) - \lambda_j(a_i)$, where $a_i \in L_i$ and $i \leq j$, and for each $i \in I$ the homomorphism $\lambda_i$ is the injection of $L_i$ into the sum $\oplus L_i$ (for more details see [1] pages 32, 33 and 34).

We can identify $M$ to a submodule of $\oplus L_i/S$ and we consider an $R$-module $A$ and an homomorphism $\alpha$ of $R$-modules such that the short sequence of $R$-modules

$$0 \rightarrow M \rightarrow \oplus L_i/S \overset{\alpha}{\rightarrow} A \rightarrow 0$$

is exact.

Now, consider the family of exacts sequences $0 \rightarrow M_i \rightarrow L_i \overset{\alpha \circ \lambda_i}{\rightarrow} A_i \rightarrow 0$ where $M_i = \ker(\alpha \circ \lambda_i)$, $A_i = \Im(\alpha \circ \lambda_i)$ and the homomorphism $\lambda_i : L_i \rightarrow \oplus L_i/S$ is such that for each $a \in L_i$, $\lambda_i(a) = \lambda_i(x)$. 

a) First, we claim that $A = \lim(A_i, \subseteq)$. For each $x \in A_i$, there exists an element $y \in L_i$ such that $\alpha \circ \lambda_i(y) = x$. By definition of direct system, we have $\lambda_i = \lambda_j \circ \varphi_{ij}$ (for $i \leq j; i, j \in I$). Thus, we deduce that

$$x = \alpha \circ \lambda_i(y) = \alpha \circ \lambda_j(\varphi_{ij}(y)) \in \alpha \circ \lambda_j(L_j) = A_j$$

Consequently, for $i \leq j$, we have $A_i \subseteq A_j$. So, we conclude that $\lim(A_i, \subseteq) = \sum A_i$. On the other hand, for every $i \in I$,

$$A_i = \alpha \circ \lambda_i(L_i) = \alpha(\lambda_i(L_i)) \subseteq \alpha(\oplus L_i/S) = A$$
This implies that \( \lim(A_i) = \sum A_i \subseteq A \).

Conversely, for each \( x \in A \), there exists \( y \in \frac{\oplus L_i}{S} \) such that \( \alpha(y) = x \). We have \( y = (\chi_i)_{i \in I} = \sum \lambda_i(x_i) = \sum \lambda_i(x) \) such that \( x_i = 0 \) except for a finite elements of \( I \). Then, \( x = \alpha(y) = \sum \alpha \lambda_i(x) \in \sum A_i \). Thus, we conclude that: \( A = \lim(A_i) \).

b) For each \( x \in M_i = \ker(\alpha \circ \lambda_i) \), we have \( \alpha \circ \lambda_i(\varphi_{i,j}(x)) = \alpha \circ \lambda_i(x) = 0 \). So, \( \varphi_{i,j}(x) \in M_j \). Then, for \( i \leq j \), the family of homomorphisms: \( \varphi_{i,j} : M_i \mapsto M_j \), such that for each \( x \in M_i \subseteq L_i \), \( \varphi_{i,j}(x) = \varphi_{i,j}(x) \) are well defined and the system \( (M_i, \varphi_{i,j})_{i \in I} \) is direct.

More thus, the following diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & M_i & \leftarrow & L_i & \xrightarrow{\alpha_{i,j}} & A_j & \rightarrow & 0 \\
\phi_{i,j} & \downarrow & \phi_{i,j} & \downarrow & \phi_{i,j} & \downarrow & \phi_{i,j} & \downarrow & \phi_{i,j}
\end{array}
\]

where \( \mu_{i,j} \) is the embedding of \( A_i \) in \( A_j \), is commutative. So, the short sequence of direct system over \( I \) induced from (\( \ast \)):

\[
0 \rightarrow (M_i, \varphi_{i,j})_{i \in I} \rightarrow (L_i, \varphi_{i,j})_{i \in I} \rightarrow (A_i, \subseteq)_{i \in I} \rightarrow 0
\]

is exact and by [1, Exercise 18, p.33], \( \lim(\alpha \circ \lambda_i) = \alpha \). Consequently, the short exact sequence:

\[
0 \rightarrow \lim(M_i, \varphi_{i,j}) \xrightarrow{\oplus L_i / A} 0.
\]

is exact and so \( \lim(M_i) \equiv \ker(\alpha) = M \).

c) For each \( i \in I, M_i \) is direct limit of his finitely generated submodules \( (M_i')_j \), and for each \( j, M'_i \subseteq M_i \subseteq L_i \). Then, by hypothesis, \( M'_i \) is \( S,G \)-projective. So, \( M_i \) is direct limit of a finitely generated \( S,G \)-projective modules (then \( S,G \)-flat modules by [2, Proposition 3.9]). Then, by hypothesis, \( M_i \) is \( S,G \)-flat.

**Conclusion**: By (b) and (c) we conclude that \( M \) is a direct limit of \( S,G \)-flat modules. Thus, by hypothesis, \( M \) is \( S,G \)-flat, as desired.

\[\square\]

**Theorem 3.5.** Let \( R \) be a ring such that every \( G \)-flat module is \( S,G \)-flat. Then, \( R \) is \( S,G \)-semihereditary if, and only if, every finitely generated ideal is \( S,G \)-projective.

**Proof.** We assume that \( R \) is \( S,G \)-semihereditary and let \( I \) be a finitely generated ideal of \( R \). Then, \( I \) is a finitely presented \( S,G \)-flat \( R \)-module (since \( R \) is coherent). Thus, by [2, Proposition 3.9], \( I \) is \( S,G \)-projective.

Conversely, we assume that every finitely generated ideal of \( R \) is \( S,G \)-projective and every \( G \)-flat module is \( S,G \)-flat. It is clear that \( R \) is coherent (by [2, Proposition 3.9], we deduce that every finitely generated ideal is finitely presented \( G \)-flat). We have also that \( GfR(I) \leq 1 \) for every finitely generated ideal \( I \). Then, for every injective module \( E \) we have \( Tor^2(E, R/I) = 0 \) (by [16, Theorem 3.14]). Therefore, by [15, Theorem 1.3.8], \( fR(E) \leq 1 \). Using [10, Theorem 3.8], \( FP - id(R) \leq 1 \) and so \( R \) is \( 1 - FC \) since it is coherent. Then, by [7, Theorem 7], \( Gf(M) \leq 1 \) for every module \( M \) and then \( Gwdim(R) \leq 1 \). Moreover, every Gorenstein flat module is strongly Gorenstein flat. Then \( R \) is \( S,G \)-semihereditary (by Remarks 3.3[2]). \[\square\]
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