LÉVY APPROXIMATION OF IMPULSIVE RECURRENT PROCESS WITH SEMI-MARKOV SWITCHING

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Abstract

In this paper, the weak convergence of impulsive recurrent process with semi-Markov switching in the scheme of Lévy approximation is proved. Singular perturbation problem for the compensating operator of the extended Markov renewal process is used to prove the relative compactness.

Key Words: Lévy approximation, semimartingale, semi-Markov process, impulsive recurrent process, piecewise deterministic Markov process, weak convergence, singular perturbation.

1 Introduction

Lévy approximation is still an active area of research in several theoretical and applied directions. Since Lévy processes are now standard, Lévy approximation is quite useful for analyzing complex systems (see, e.g. \[1\] [8]). Moreover they are involved in many applications, e.g., risk theory, finance, queueing, physics, etc. For a background on Lévy process see, e.g. \[1\] [8] [3].

In particular in \[3\] it has been studied the following impulsive process as partial sums in a series scheme

\begin{equation}
\xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t)} \alpha_k^\varepsilon(x_{k-1}^\varepsilon), \quad t \geq 0,
\end{equation}
the random variables \( \alpha_k^\varepsilon(x), k \geq 1 \) are supposed to be independent and perturbed by the jump Markov process \( x(t), t \geq 0 \).

We propose to study generalization of the problem (1):

\[
(2) \quad \xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t)} \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon), \quad t \geq 0.
\]

Here the random variables \( \alpha_k^\varepsilon(u, x), k \geq 1 \) depend on the process \( \xi(t) \).

We propose to study convergence of (2) using a combination of two methods. The one, based on semimartingales theory, is combined with a singular perturbation problem for the compensating operator of the extended Markov renewal process. So, the method includes two steps.

In the first step we prove the relative compactness of the semimartingales representation of the family \( \xi^\varepsilon, \varepsilon > 0 \), by proving the following two facts [2]:

\[
\lim_{\varepsilon \to 0} \sup_{\varepsilon \leq \varepsilon_0} \mathbb{P}\{\sup_{t \leq T} |\xi^\varepsilon(t)| > c\} = 0,
\]

known as the compact containment condition, and

\[
\mathbb{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t - s|,
\]

for some positive constant \( k \).

In the second step we prove convergence of the extended Markov renewal process \( \xi_n^\varepsilon, x_n^\varepsilon, \tau_n^\varepsilon, n \geq 0 \) by using singular perturbation technique as presented in [5].

Finally, we apply Theorem 6.3 from [5].

The paper is organized as follows. In Section 2 we present the time-scaled impulsive process (2) and the switching semi-Markov process. In the same section we present the main results of Lévy approximation. In Section 3 we present the proof of the theorem.

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### 2 Main results

Let us consider the space \( \mathbb{R}^d \) endowed with a norm \( |\cdot| \) (\( d \geq 1 \)), and \((E, \mathcal{E})\), a standard phase space, (i.e., \( E \) is a Polish space and \( \mathcal{E} \) its Borel \( \sigma \)-algebra). For a vector \( v \in \mathbb{R}^d \) and a matrix \( c \in \mathbb{R}^{d \times d} \), \( v^* \) and \( c^* \) denote their transpose respectively. Let \( C_3(\mathbb{R}^d) \) be a measure-determining class of real-valued bounded functions, such that \( g(u)/|u|^2 \to 0 \), as \( |u| \to 0 \) for \( g \in C_3(\mathbb{R}^d) \) (see [4, 5]).

The impulsive processes \( \xi^\varepsilon(t), t \geq 0, \varepsilon > 0 \) on \( \mathbb{R}^d \) in the series scheme with small series parameter \( \varepsilon \to 0, (\varepsilon > 0) \) are defined by the sum ([5 Section 9.2.1])

\[
(3) \quad \xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t/\varepsilon^2)} \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon), \quad t \geq 0.
\]
For any $\varepsilon > 0$, and any sequence $z_k, k \geq 0$, of elements of $\mathbb{R}^d \times E$, the random variables $\alpha_k^\varepsilon(z_k-1), k \geq 1$ are supposed to be independent. Let us denote by $G_{u,x}^\varepsilon$ the distribution function of $\alpha_k^\varepsilon(x)$, that is,

$$G_{u,x}^\varepsilon(dv) := P(\alpha_k^\varepsilon(u, x) \in dv), k \geq 0, \varepsilon > 0, x \in E, u \in \mathbb{R}^d.$$ 

It is worth noticing that the coupled process $\xi^\varepsilon(t), x^\varepsilon(t), t \geq 0$, is a Markov additive process (see, e.g., [5, Section 2.5]).

We make natural assumptions for the counting process $\nu(t)$, namely:

$$\int_0^t E[\varphi(s)d\nu(s)] < l_1 \int_0^t E(\varphi(s))ds$$

for any nonnegative, increasing $\varphi(s)$ and $l_1 > 0$.

The switching semi-Markov process $x(t), t \geq 0$ on the standard phase space $(E, \mathcal{E})$, is defined by the semi-Markov kernel

$$Q(x, t) = P(x, B)F_x(t), x \in E, B \in \mathcal{E}, t \geq 0,$$

which defines the associated Markov renewal process $x_n, \tau_n, n \geq 0$:

$$Q(x, t) = P(x_{n+1} \in B, \theta_{n+1} \leq t|x_n = x) = P(x_{n+1} \in B|x_n = x)P(\theta_{n+1} \leq t|x_n = x).$$

Finally we should denote $\xi_n^\varepsilon$ in (3):

$$\xi_n^\varepsilon := \xi(\varepsilon^2 \tau_n) = \xi_0^\varepsilon + \sum_{k=1}^n \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon).$$

The Lévy approximation of Markov impulsive process (3) is considered under the following conditions.

**C1:** The semi-Markov process $x(t), t \geq 0$ is uniformly ergodic with the stationary distribution

$$\pi(dx)q(x) = q\rho(dx), q(x) := 1/m(x), q := 1/m,$$

$$m(x) := \mathbb{E}\theta_x = \int_0^\infty T_x(t)dt, m := \int_E \rho(dx)m(x),$$

$$\rho(B) = \int_E \rho(dx)P(x, B), \rho(E) = 1.$$ 

**C2:** Lévy approximation. The family of impulsive processes $\xi^\varepsilon(t), t \geq 0$ satisfies the Lévy approximation conditions [5, Section 9.2].

**L1:** Initial value condition

$$\sup_{\varepsilon > 0} E|\xi_0^\varepsilon| \leq C < \infty$$

and

$$\xi_0^\varepsilon \Rightarrow \xi_0.$$
L2: Approximation of the mean values:

\[ a^\varepsilon(u; x) = \int_{\mathbb{R}^d} vG_{u,x}^\varepsilon(dv) = \varepsilon a_1(u; x) + \varepsilon^2[a(u; x) + \theta^\varepsilon_a(u; x)], \]

and

\[ c^\varepsilon(u; x) = \int_{\mathbb{R}^d} vv^*G_{u,x}^\varepsilon(dv) = \varepsilon^2[c(u; x) + \theta^\varepsilon_c(u; x)], \]

where functions \( a_1, a \) and \( c \) are bounded.

L3: Poisson approximation condition for intensity kernel (see [4])

\[ G_{g}^\varepsilon(u; x) = \int_{\mathbb{R}^d} g(v)G_{u,x}^\varepsilon(dv) = \varepsilon^2[G_g(u; x) + \theta_g^\varepsilon(u; x)] \]

for all \( g \in C_3(\mathbb{R}^d) \), and the kernel \( G_g(u; x) \) is bounded for all \( g \in C_3(\mathbb{R}^d) \), that is,

\[ |G_g(u; x)| \leq G_g \quad \text{a constant depending on } g. \]

Here

\[ G_g(u; x) = \int_{\mathbb{R}^d} g(v)G_{u,x}(dv), \quad g \in C_3(\mathbb{R}^d). \]

The above negligible terms \( \theta^\varepsilon_a, \theta^\varepsilon_c, \theta_g^\varepsilon \) satisfy the condition

\[ \sup_{x \in E} |\theta^\varepsilon(u; x)| \to 0, \quad \varepsilon \to 0. \]

L4: Balance condition.

\[ \int_E \rho(dx)a_1(u; x) = 0. \]

In addition the following conditions are used:

C3: Uniform square-integrability:

\[ \lim_{c \to \infty} \sup_{x \in E} \int_{|v| > c} vv^*G_{u,x}(dv) = 0. \]

C4: Linear growth: there exists a positive constant \( L \) such that

\[ |a(u; x)| \leq L(1 + |u|), \quad \text{and} \quad |c(u; x)| \leq L(1 + |u|^2), \]

and for any real-valued non-negative function \( f(v), v \in \mathbb{R}^d \), such that \( \int_{\mathbb{R}^d \setminus \{0\}} (1 + f(v)) |v|^2 \, dv < \infty \), we have

\[ |G_{u,x}(v)| \leq Lf(v)(1 + |u|). \]

The main result of our work is the following.
**THEOREM 1** Under conditions C1 – C4 the weak convergence
\[ \xi^\varepsilon(t) \Rightarrow \xi^0(t), \quad \varepsilon \to 0 \]
takes place.

The limit process \( \xi^0(t), t \geq 0 \) is a Lévy process defined by the generator \( \mathbf{L} \) as follows
\[ (6) \mathbf{L}_c \varphi(u) = (\hat{a}(u) - \tilde{a}_0(u)) \varphi'(u) + \frac{1}{2} \sigma^2(u) \varphi''(u) + \lambda(u) \int_{\mathbb{R}^d} [\varphi(u + v) - \varphi(u)] G^0_u(dv), \]
where:
\[ \hat{a}(u) = q \int_E \rho(dx) a(u; x), \quad \tilde{a}_0(u) = \int_E v G_u(dv), \quad \sigma^2(u) = q \int_E \rho(dx) G_{u,x}(dv), \]
\[ \hat{a}_1^2(u) = q \int_E \rho(dx) a_1^2(u; x), \quad \tilde{a}_1(u; x) = q(x) \int_E P(x, dy) a_1(u; x), \quad \gamma_0(u; x) = \int_E vv^* G_{u,x}(dv) \]
\[ \lambda(u) = G_u(\mathbb{R}^d), \quad G^0_u(dv) = G_u(dv)/\lambda(u), \]
here \( \tilde{R}_0 \) is the potential operator of embedded Markov chain.

**Remark 1.** The limit Lévy process consists of three parts: deterministic drift, diffusion part and Poisson part.

There are some possible cases:

1). If \( \hat{b}(u) - \tilde{b}_0(u) = 0 \) then the limit process does not have deterministic drift.

2). If \( \sigma^2(u) = 0 \) then the limit process does not have diffusion part. As a variant of this case we note that if \( c(u; x) = c_0(u; x) \) then also \( b_1(u; x) = 0 \) and we obtain the conditions of Poisson approximation after re-normation \( \varepsilon^2 = \bar{\varepsilon} \) (see, for example Chapter 7 in [5]).

**Remark 2.** In the work [5] (Theorem 9.3) an analagous result was obtained for impulsive process with Markov switching. If we study an ordinary impulsive process without switching, we should obtain \( \sigma^2 = E(\alpha_k^2) - (E(\alpha_k^2))^2 = (c - c_0) - a_1^2 \). This result correlates with the similar results from [4]. In case of our Theorem this may be easily shown, but in [5] (Theorem 9.3) it is not obvious.

The difference is that we used \( \tilde{R}_0 \) – the potential operator of embedded Markov chain instead of \( R_0 \) – the potential operator of Markov process. Due to this, our result obviously correlates with other well-known result.

**Remark 3.** Asymptotic of the second moment in the condition L1 contains second modified characteristics \( c(u; x) \) (see correlation 4.2 at page 555 in [4]). This characteristics in limit contains both second moment of Poisson part and dispersion of diffusion part, namely \( c = c_0 + \sigma^2 \).
3 Proof of Theorem 1

The proof of Theorem 1 is based on the semimartingale representation of the impulsive process (3).

We split the proof of Theorem 1 in the following two steps.

**STEP 1.** In this step we establish the relative compactness of the family of processes \( \xi^\varepsilon(t), t \geq 0, \varepsilon > 0 \) by using the approach developed in [6]. Let us remind that the space of all probability measures defined on the standard space \((E, \mathcal{E})\) is also a Polish space; so the relative compactness and tightness are equivalent.

First we need the following lemma.

**LEMMA 1** Under assumption **C4** there exists a constant \( k > 0 \), independent of \( \varepsilon \) and dependent on \( T \), such that

\[
E \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq kT.
\]

**COROLLARY 1** Under assumption **C4**, the following compact containment condition (CCC) holds:

\[
\lim_{c \to \infty} \sup_{\varepsilon \leq \varepsilon_0} \sup_{t \leq T} P\{\sup_{t \leq T} |\xi^\varepsilon(t)| > c\} = 0.
\]

**Proof:** The proof of this corollary follows from Kolmogorov’s inequality.

**Proof of Lemma 1:** (following [6]). The impulsive process (3) has the following semimartingale representation

\[
\xi^\varepsilon(t) = u + B^\varepsilon_t + M^\varepsilon_t,
\]

where \( u = \xi_0^\varepsilon \); \( B^\varepsilon_t \) is the predictable drift

\[
B^\varepsilon_t = \sum_{k=1}^{\nu(t/\varepsilon^2)} \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} \int_0^1 \Phi^\varepsilon_k \, d\mu^\varepsilon_k(t) = A^\varepsilon(t) + A^\varepsilon(t) + \theta^\varepsilon(t),
\]

where

\[
A^\varepsilon(t) := \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} \int_{\mathbf{R}^d} \nu^\varepsilon_k \Phi^\varepsilon_k \, d\mu^\varepsilon_k(t),
\]

\[
A^\varepsilon(t) := \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} a(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}) + \theta^\varepsilon(t) =
\]

\[
\langle M^\varepsilon \rangle_t = \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} \int_{\mathbf{R}^d \setminus \{0\}} \nu^\varepsilon_k \Phi^\varepsilon_k \, d\mu^\varepsilon_k(t) + \theta^\varepsilon(t) =
\]

\[
\varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} c(\xi^\varepsilon_{k-1}; x^\varepsilon_{k-1}) + \theta^\varepsilon(t),
\]
and for every finite $T > 0$

$$\sup_{0 \leq t \leq T} |\theta^\varepsilon(t)| \to 0, \varepsilon \to 0.$$ 

To verify compactness of the process $\xi^\varepsilon(t)$ we split it at two parts. The first part of order $\varepsilon$

$$A^\varepsilon_1(t) = \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} a_1(\xi^\varepsilon_{k-1}; x^\varepsilon_{k-1}),$$

can be characterized by the compensating operator

$$L^\varepsilon \varphi(u; x) = \varepsilon^{-2} q(x) [A^\varepsilon_1(x) P - I] \varphi(u; x),$$

where $A^\varepsilon_1(x) \varphi(u) = \varphi(u + \varepsilon a_1(u; x)) = \varepsilon a_1(u; x) \varphi'(u) + \varepsilon \theta^\varepsilon \varphi(u)$. After simple calculations we may rewrite the operator:

$$L^\varepsilon = \varepsilon^{-2} Q + \varepsilon^{-1} A_1(x) P + \theta^\varepsilon,$$

here $A_1(x) \varphi(u) = \varepsilon a_1(u; x) \varphi'(u)$.

Corresponding martingale characterization is the following

$$\mu^\varepsilon_{n+1} = \varphi(A^\varepsilon_{1,n+1}, x^\varepsilon_{n+1}) - \varphi(A^\varepsilon_{1,0}, x^\varepsilon_{0}) - \sum_{m=0}^{\nu_n} \theta^\varepsilon_{m+1} L^\varepsilon \varphi(A^\varepsilon_{1,m}, x^\varepsilon_{m}).$$

Using the results from [5], Section 1 we obtain the last martingale in the form

$$\tilde{\mu}^\varepsilon_t = \varphi^\varepsilon(A^\varepsilon_1(t), x^\varepsilon_t) + \varphi^\varepsilon(A^\varepsilon_1(0), x^\varepsilon_0) - \int_0^t L^\varepsilon \varphi^\varepsilon(A^\varepsilon_1(s), x^\varepsilon_s) ds,$$

where $x^\varepsilon_t := x(t/\varepsilon^2)$.

Thus (see, for example Theorem 1.2 in [5]), it has quadratic characteristic

$$<\tilde{\mu}^\varepsilon>_t = \int_0^t \left[ L^\varepsilon (\varphi^\varepsilon(A^\varepsilon_1(s), x^\varepsilon_s))^2 - 2 \varphi^\varepsilon(A^\varepsilon_1(s), x^\varepsilon_s) L^\varepsilon \varphi^\varepsilon(A^\varepsilon_1(s), x^\varepsilon_s) \right] ds.$$

Applying the operator $L^\varepsilon = \varepsilon^{-2} Q + \varepsilon^{-1} A_1(x) P + \theta^\varepsilon$ to test-function $\varphi^\varepsilon = \varphi + \varepsilon \varphi_1$ we obtain the integrand of the view

$$Q \varphi_1^2 - 2 \varphi_1 Q \varphi + \theta^\varepsilon \varphi^\varepsilon.$$

Thus the integrand is limited. The boundedness of the quadratic characteristic provides $\tilde{\mu}^\varepsilon_t$ is compact. Thus, $\varphi(A^\varepsilon_1(t))$ is compact too and bounded uniformly by $\varepsilon$. By the results from [2] we obtain compactness of $A^\varepsilon_1(t)$, because the test-function $\varphi(u)$ belongs to the measure-determining class.

Now we should study the second part of order $\varepsilon^2$. 
For a process \( y(t), t \geq 0 \), let us define the process \( y^\dagger(t) = \sup_{s \leq t} |y(s)| \), then from (7) we have

\[
((\xi^\varepsilon(t))^\dagger)^2 \leq 4[u^2 + ((A^\varepsilon(t))^\dagger)^2 + ((M^\varepsilon)^\dagger)^2].
\]

(9)

Now we may apply the result of Section 2.3 [5], namely

\[
\sum_{k=1}^{\nu(t)} a(\xi^{\varepsilon}_{k-1}, x^{\varepsilon}_{k-1}) = \int_0^t a(\xi^{\varepsilon}(s), x^{\varepsilon}(s))d\nu(s).
\]

Condition C4 implies that for sufficiently large \( \varepsilon \)

\[
(A^\varepsilon(t))^\dagger = \varepsilon^2 \int_0^{t/\varepsilon^2} a(\xi^{\varepsilon}(s), x^{\varepsilon}(s))d\nu(s) \leq L\varepsilon^2 \int_0^{t/\varepsilon^2} (1 + (\xi^{\varepsilon}(s))^\dagger) d\nu(s)
\]

(10)

Now, by Doob’s inequality (see, e.g., [7, Theorem 1.9.2]),

\[
E((M^\varepsilon(t))^\dagger)^2 \leq 4|E(M^\varepsilon)|,
\]

(8) and condition C4 we obtain

\[
|E(M^\varepsilon)_t| = \varepsilon^2 \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d \setminus \{0\}} vu^* G(\xi^\varepsilon(s), dv; x^\varepsilon(s), d\nu(s)) = \varepsilon^2 \int_0^{t/\varepsilon^2} c(\xi^\varepsilon(s), x^\varepsilon(s))d\nu(s) \leq
\]

\[
L\varepsilon^2 \int_0^{t/\varepsilon^2} [1 + ((\xi^{\varepsilon}(s))^\dagger)^2] d\nu(s).
\]

(11)

Inequalities (9)-(11), condition (4) and Cauchy-Bunyakovsky-Schwarz inequality, \( ([\int_0^t \varphi(s)ds]^2 \leq t \int_0^t \varphi^2(s)ds) \), imply

\[
E((\xi^\varepsilon(t))^\dagger)^2 \leq k_1 + k_2\varepsilon^2 \int_0^{t/\varepsilon^2} E((\xi^{\varepsilon}(s))^\dagger)^2 d\nu(s) \leq k_1 + k_2l_1 \varepsilon^2 \int_0^{t/\varepsilon^2} E((\xi^{\varepsilon}(s))^\dagger)^2 ds =
\]

\[
 k_1 + k_2l_1 \int_0^t E((\xi^\varepsilon(s))^\dagger)^2 ds,
\]

where \( k_1, k_2 \) and \( l_1 \) are positive constants independent of \( \varepsilon \).

By Gronwall inequality (see, e.g., [2, p. 498]), we obtain

\[
E((\xi^\varepsilon(t))^\dagger)^2 \leq k_1 \exp(k_2l_1 t).
\]

Thus, both parts of \( \xi^\varepsilon(t) \) are compact and bounded, so

\[
E \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq k_T.
\]

Hence the lemma is proved.

\[\square\]
LEMA 2 Under assumption C4 there exists a constant $k > 0$, independent of $\varepsilon$ such that

$$E|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t - s|.$$ 

Proof: In the same manner with (9), we may write

$$|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq 2|B_t^\varepsilon - B_s^\varepsilon|^2 + 2|M_t^\varepsilon - M_s^\varepsilon|^2.$$ 

By using Doob’s inequality, we obtain

$$E|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq 2E\{|B_t^\varepsilon - B_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s|\}.$$ 

Now (11) and condition (11) and assumption C4 imply

$$|B_t^\varepsilon - B_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s| \leq k_3[1 + ((\xi^\varepsilon(T))^2)]|t - s|,$$

where $k_3$ is a positive constant independent of $\varepsilon$.

From the last inequality and Lemma 1 the desired conclusion is obtained.

Step 2. At the next step of proof we apply the problem of singular perturbation to the generator of the process $\xi^\varepsilon(t)$. To do this, we mention the following theorem. $C^2_0(\mathbb{R}^d \times E)$ is the space of real-valued twice continuously differentiable functions on the first argument, defined on $\mathbb{R}^d \times E$ and vanishing at infinity, and $C(\mathbb{R}^d \times E)$ is the space of real-valued continuous bounded functions defined on $\mathbb{R}^d \times E$.

THEOREM 2 (Theorem 6.3) Let the following conditions hold for a family of Markov processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$:

CD1: There exists a family of test functions $\varphi^\varepsilon(u, x)$ in $C^2_0(\mathbb{R}^d \times E)$, such that

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(u, x) = \varphi(u),$$

uniformly on $u, x$.

CD2: The following convergence holds

$$\lim_{\varepsilon \to 0} L^\varepsilon \varphi^\varepsilon(u, x) = L\varphi(u),$$

uniformly on $u, x$. The family of functions $L^\varepsilon \varphi^\varepsilon, \varepsilon > 0$ is uniformly bounded, and $L\varphi(u)$ and $L^\varepsilon \varphi^\varepsilon$ belong to $C(\mathbb{R}^d \times E)$. 


CD3: The quadratic characteristics of the martingales that characterize a coupled Markov process \( \xi^\varepsilon(t), x^\varepsilon(t), t \geq 0, \varepsilon > 0 \) have the representation
\[
\langle \mu^\varepsilon \rangle_t = \int_0^t \zeta^\varepsilon(s) ds,
\]
where the random functions \( \zeta^\varepsilon, \varepsilon > 0 \), satisfy the condition
\[
\sup_{0 \leq s \leq T} E|\zeta^\varepsilon(s)| \leq c < +\infty.
\]

CD4: The convergence of the initial values holds and
\[
\sup_{\varepsilon > 0} E|\zeta^\varepsilon(0)| \leq C < +\infty.
\]

Then the weak convergence
\[
\xi^\varepsilon(t) \Rightarrow \xi(t), \quad \varepsilon \to 0,
\]
takes place.

We consider the the extended Markov renewal process
\[
(12) \quad \xi^\varepsilon_n, x^\varepsilon_n, \tau^\varepsilon_n, n \geq 0,
\]
where \( x^\varepsilon_n = x^\varepsilon(\tau^\varepsilon_n), x^\varepsilon(t) := x(t/\varepsilon^2), \xi^\varepsilon_n = \xi^\varepsilon(\tau^\varepsilon_n) \) and \( \tau^\varepsilon_{n+1} = \tau^\varepsilon_n + \varepsilon^2 \theta^\varepsilon_n, n \geq 0, \) and
\[
P(\theta^\varepsilon_{n+1} \leq t | x^\varepsilon_n = x) = F_x(t) = P(\theta_x \leq t).
\]

**DEFINITION 1** [9] The compensating operator \( L^\varepsilon \) of the Markov renewal process (12) is defined by the following relation
\[
L^\varepsilon \varphi(\xi^\varepsilon_0, x_0, \tau_0) = q(x_0) E[\varphi(\xi^\varepsilon_1, x_1, \tau_1) - \varphi(\xi^\varepsilon_0, x_0, \tau_0)|\mathcal{F}_0],
\]
where
\[
\mathcal{F}_t := \sigma(\xi^\varepsilon(s), x^\varepsilon(s), \tau^\varepsilon(s); 0 \leq s \leq t).
\]

Using Lemma 9.1 from [5] we obtain that the compensating operator of the extended Markov renewal process from Definition 1 can be defined by the relation (see also Section 2.8 in [5])
\[
(13) \quad L^\varepsilon \varphi(u, v; x) = \varepsilon^{-2} q(x) \left[ \int_E P(x, dy) \int_{\mathbb{R}^d} G^\varepsilon_{u,z}(dz) \varphi(u + z, v; y) - \varphi(u, v; x) \right].
\]

By analogy with [5] Lemma 9.2] we may prove the following result:
**Lemma 3** The main part in the asymptotic representation of the compensating operator \( (13) \) is as follows

\[
L^\varepsilon \varphi(u, v, x) = \varepsilon^{-2}Q\varphi(\cdot, \cdot, x) + \varepsilon^{-1}a_1(u; x)Q_0\varphi(u, \cdot, \cdot) + \left[ a(u; x) - a_0(u; x) \right]Q_0\varphi'(u, \cdot, \cdot) + \frac{1}{2}[c(u; x) - c_0(u; x)]Q_0\varphi''(u, \cdot, \cdot) + G_{u,v}Q_0\varphi(u, \cdot, \cdot)
\]

where:

\[
Q_0\varphi(x) := q(x) \int_E P(x, dy)\varphi(y), \quad G_{u,v}\varphi(u) := \int_{\mathbb{R}^d} [\varphi(u + z) - \varphi(u)]G_{u,v}(dz),
\]

\[
a_0(u; x) = \int_E v G_{u,v}(dv), \quad c_0(u; x) = \int_E vv^* G_{u,v}(dv).
\]

**Proof** of this Lemma is analogical to the proof of [5, Lemma 9.2].

The solution of the singular perturbation problem at the test functions \( \varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon\varphi_1(u, x) + \varepsilon^2\varphi_2(u, x) \) in the form

\[
(14) \quad L^\varepsilon \varphi^\varepsilon = L\varphi + \theta^\varepsilon \varphi
\]

can be found in the same manner with Lemma 9.3 in [5].

To simplify the formula, we refer to the embedded Markov chain. Corresponding generator \( \tilde{Q} := P - I \), and the potential operator satisfies the correlation \( \tilde{R}_0(P - I) = \tilde{\Pi} - I \).

From (14) we obtain

\[
\tilde{Q}\varphi = 0,
\]

\[
\tilde{Q}\varphi_1 + A_1(x)P\varphi = 0,
\]

\[
\tilde{Q}\varphi_2 + A_1(x)P\varphi_1 + (A(x) + C(x) + G_{u,v})P_\varphi = m(x)L\varphi,
\]

where

\[
A(x)\varphi(u) := [a(u; x) - a_0(u; x)]\varphi'(u), \quad A_1(x)\varphi(u) := a_1(u; x)\varphi'(u),
\]

\[
C(x) := \frac{1}{2}[c(u; x) - c_0(u; x)]\varphi''(u).
\]

From the second equation we obtain \( \varphi_1 = \tilde{R}_0 A_1(x)\varphi \), and substituting it into the last equation we have:

\[
\tilde{Q}\varphi_2 + A_1(x)P\tilde{R}_0 A_1(x)\varphi + (A(x) + C(x) + G_{u,v})\varphi = m(x)L\varphi.
\]

As soon as \( P\tilde{R}_0 = \tilde{R}_0 + \tilde{\Pi} - I \) we finally obtain

\[
(15) \quad q^{-1}L = \tilde{\Pi}[A(x) + C(x) + G_{u,v} + A_1(x)\tilde{R}_0 A_1(x) - A_1^2(x)]\tilde{\Pi}.
\]

Simple calculations give us (6) from (15).

Now Theorem 2 can be applied.

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We see from (13) and (15) that the solution of singular perturbation problem for \( L^\varepsilon \varphi^\varepsilon(u,v;x) \) satisfies the conditions CD1, CD2. Condition CD3 of this theorem implies that the quadratic characteristics of the martingale, corresponding to a coupled Markov process, is relatively compact. The same result follows from the CCC (see Corollary 2 and Lemma 2) by [4]. Thus, the condition CD3 follows from the Corollary 2 and Lemma 2. Due to L1 the condition CD4 is also satisfied. Thus, all the conditions of above Theorem 2 are satisfied, so the weak convergence \( \xi^\varepsilon(t) \Rightarrow \xi^0(t) \) takes place.

Theorem 1 is proved.

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