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Statistical Analysis of Multipath Fading Channels Using Generalizations of Shot-Noise

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1. INTRODUCTION

A statistical temporal model which captures the time-varying and time-spreading properties of the channel is the so-called multipath fading channel model (MFC) \[1, \text{pages 12, 13, 760, 761}, \] \[2, \text{page 146}], \[3\]. The output of such channel, when the input is the low-pass signal \(x(t)\), is given by

\[
y_L(t) = \sum_{i=1}^{N(t)} r_i(t) e^{j\Phi_i(t, t)} x_L(t - \tau_i),
\]

which corresponds to that of the so-called quasistatic channel. Here, \(r_i(t)\), \(\Phi_i(t, t)\), \(\tau_i\) denote the attenuation, phase, and propagation time delay, respectively, of the signal received in the \(i\)th path, and \(N(t)\) denotes the number of paths at time \(t\). The phase \(\Phi_i(t, t)\) is typically a function of the carrier frequency, the relative velocity between the transmitter and the receiver, and the angle of arrival of the incident on the receiver plane wave \[4–6\]. On the other hand, if \(x_B(t)\) is the low-pass equivalent representation of a bandpass signal, modulated at some carrier frequency \(\omega_i\), namely, \(x(t) = \text{Re}\{x_B(t)e^{j\omega_i t}\}\), then the received bandpass signal is \(y(t) = \text{Re}\{y_L(t)e^{j\omega_i t}\}\). In the works found in the literature, the authors often omit this explicit dependence of \(r_i\) on \(\tau_i\) during the computation of the various statistics \[2, \text{page 146}\] is an exception). Although for a deterministic or fixed sample path of \(\{N(s); 0 \leq s \leq t\}\) the computation of the statistical properties of \(y_L(t)\) is not affected by this omission, this is not the case when the ensemble statistics are analyzed. Ensemble statistics using a counting process as simple as the nonhomogeneous Poisson process reveal an additional smoothing property associated with each propagation environment, which is expressed in terms of the rate of the Poisson process and the attenuations.

The objective of this paper is to introduce a unified framework for computing the statistical properties of the received signal when \(\{\tau_i\}_{i \geq 1}\) are the points of a Poisson counting process \(N(t)\), while for fixed sample paths of the points the distribution of the instantaneous amplitude and phase, \(\{r_i(\tau_i), \Phi_i(t, \tau_i)\}, i = 1, 2, \ldots,\) is arbitrary, by performing an analysis which can be viewed as a generalization of the shot-noise analysis investigated by Rice \[7, 8\] in the mid 1940’s. This approach is similar to the one considered in \[9\] which investigates the statistical properties of cochannel interference. However, in \[9\] the authors are interested in stable distributed processes although their approach can be extended to other distributions.

In \[10, 11\], the authors questioned the accuracy of the Poisson counting processes in matching experimental data of path arrival time and number of paths, and thus a modified
Poisson process is introduced, the so-called \( \Delta - K \) model. However, the failure of the Poisson process to model path arrival times does not imply that the Poisson model will also be inappropriate when considered as part of (1) to study the statistics of the received signal. In this paper, we show that when the Poisson counting process is included in (1), then various existing properties of MFCs, such as the power delay profile, the Doppler spread, and the Gaussianity of the channel, are predicted. Due to its simplicity, the Poisson counting process is the most natural process to start the analysis with. It can form the core for subsequent generalizations in which the rate of the counting process is random. The validity of the Poisson counting process is start the analysis with. It can form the core for subsequent generalizations in which the rate of the counting process is random.

The above analysis is important when designing specific receivers as follows. Assume that (1) represents the baseband received signal which is corrupted by additive white Gaussian noise. A well-known optimal receiver is the matched filter, which maximizes the output signal-to-noise ratio [1]. The implementation of the matched filter requires the knowledge of the power spectral density of (1), which is computed in the paper. Moreover, in many applications such as filter design and interference analysis, it is important to know the precise joint distribution of the processes \( \{y(t)\}_{t \geq 0} \) and \( \{\rho(t)\}_{t \geq 0} \). This joint distribution is also computed when \( \{y(t)\}_{t \geq 0} \) are Gaussian distributed. Moreover, the results of the paper when combined with [9] can be used to analyze interference statistics of multipath fading channels.

The paper is organized as follows. Section 2 discusses correlation properties and relations to known statistical properties of \( y(t) \) and \( \rho(t) \). Section 3 presents several power spectral densities of \( y(t) \) and \( \rho(t) \) for any information signal. Section 4 establishes central limit theorems which imply Gaussianity of \( y(t) \) and \( \rho(t) \).

**Notation 1.** \( \mathbb{N}_+ \) denotes the set of positive integers; \( E \) will denote the expectation operator; \( |c|^2 \triangleq c^*c \), where \( c \in \mathbb{C} \) is complex and \( * \) denotes complex conjugation. For \( \mathcal{T} \in \mathcal{L}(\mathbb{C}^m;\mathbb{C}^n) \), a linear operator \( \mathcal{T}^\dagger \) denotes Hermitian conjugation. For \( \rho \in \mathbb{C}^n \), where \( \rho_{\mathbb{R}} \triangleq \text{Re}(\rho) \) and \( \rho_{\mathbb{I}} \triangleq \text{Im}(\rho) \), denote the real and imaginary components of \( \rho \), respectively. The complex derivatives with respect to \( \rho \) and \( \rho^* \) are defined in terms of real derivatives as follows: \( \partial \rho = \partial_{\mathbb{R}} + j \partial_{\mathbb{I}} \) and \( \partial \rho^* = \partial_{\mathbb{R}}^* + j \partial_{\mathbb{I}}^* \). For \( f, g \) real- or complex-valued functions, \( f \ast g \) denotes convolution operation of \( f \) with \( g \), and \( F_T \{ f \} \) denotes Fourier transform (FT).

**2. MEAN, VARIANCE, AND CORRELATION**

Let \( (\Omega, \mathcal{A}, \mathbb{P}) \) be a complete probability space equipped with filtration \( \{\mathcal{A}_t\}_{t \geq 0} \) and finite-time \( [0, T_s] \), \( T_s < \infty \), on which the following random variables are defined: \( r_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \), \( \tau_i : \Omega \rightarrow \mathbb{R} \), \( \omega_i : \Omega \rightarrow \mathbb{R} \), \( N \in [0, T_s) \times \Omega \rightarrow \mathbb{N}_+ \), \( m_i(t) \triangleq (r_i(t), \tau_i, \omega_i) \). This paper investigates the statistical properties of a noncausal version of (1), namely,

\[
y(t) = \sum_{i=1}^{N(T_s)} r_i(t) e^{i\omega_i} e^{-j(r_i(t)+j\omega_i)\tau_i} x(t) e^{j\omega_i(t-T)}
\]

where \( 0 \leq t \leq T_s \) and its bandpass representation is

\[
y(t) = \text{Re}\left\{ \sum_{i=1}^{N(T_s)} r_i(t) e^{i\omega_i} e^{-j(r_i(t)+j\omega_i)\tau_i} x(t) e^{j\omega_i(t-T)} \right\}
\]

in which \( h(t, \tau_i; m_i(t)) = r_i(t) e^{i\omega_i} e^{-j(r_i(t)+j\omega_i)\tau_i} x(t) e^{j\omega_i(t-T)} \), \( h(t, \tau_i; m_i(t)) = r_i(t) x(t-T) \), and \( x(t) = \text{Re}\{x(t) e^{i\omega_i(t+\tau_1)} \} \), \( r_i \) is the attenuation, \( \tau_i \) is the time delay, \( \phi_i \) is the phase, \( \omega_i \) is the Doppler spread of the \( i \)th path, and \( \omega_c \) is the carrier frequency. For fixed \( \tau_i = \tau \), the dependence of the attenuations \( \{r_i(\tau)\}_{i \geq 1} \) on \( \tau \) implies that the attenuations are random variables. Notice that each occurrence time \( t_i \) is associated with \( m_i(t_i) = (r_i(t_i), \tau_i, \omega_i) \), and \( h(t, \tau; m_i(t_i)) \) (or \( h(t, \tau; m_i(\tau)) \) may be viewed as the impulse response at time \( t \) due to the occurrence of \( \tau_i \). In the preliminary calculations, it is assumed that for a fixed occurrence time \( \tau_i = \tau \), \( h(t; \tau; m_i(t)) \) and \( h(t; \tau; m_i(\tau)) \) are independent of the counting process \( N(T_s) \). However, in obtaining explicit expressions, we will often make the following assumption.

**Assumption 1.** Let \( \lambda_T(s) \triangleq \lambda \times \lambda_s(s); 0 \leq s \leq t \) denote the nonnegative and nonrandom rate of the counting process \( \{N(s)\}; 0 \leq s \leq t \), where \( \lambda \) is constant and nonrandom and \( \lambda_s(t) \) is a time-varying nonrandom function. For fixed \( \tau_i = \tau \), the random processes \( \{h(t, \tau; m_i(t))\}_{i \geq 0} \) (resp., \( \{h(t, \tau; m_i(\tau))\}_{i \geq 0} \), \( i = 1, 2, 3, \ldots \), are mutually independent and identically distributed, having the same distribution as \( \{h(t, \tau; m_i(t))\}_{i \geq 0} \) (resp., \( \{h(t, \tau; m_i(\tau))\}_{i \geq 0} \), and are also independent of \( \{N(s)\}; 0 \leq s \leq t \).

Assumption 1 is invoked only when seeking closed-form expressions for various statistics. We note that when \( \lambda \) is a random variable, most of the subsequent results of this note remain valid provided that we include an extra integration with respect to the density of \( \lambda \). Such generalizations do not
suffer from the orderliness and the independent increment properties of the Poisson counting process; however, the analysis is more complicated and should be discussed elsewhere.

**Mean and variance**

The mean (expected value) and the variance of the received complex signal $y(t)$ are, respectively, defined by $y_\nu(t) \triangleq E \left[ \sum_{s=1}^{N(t)} h(t, \tau_s; m(\tau_s)) \right]$ and $\text{Var}(y(t)) \triangleq E \left[ y(t)^2 - y(t)^2 \right]$, where $E$ denotes expectation with respect to the joint density of $\{m(\tau), N(T), \tau_1, \ldots, \tau_s \}$. Suppose that $\{N(s); 0 \leq s \leq T\}$ is Poisson with rate $\lambda_T(t)$, and that the delay times $\{\tau_i\}_{i=1}^k$ are independent identically distributed with density $f(t) = \lambda_T(t)/\sum_{s=1}^{T} \lambda_T(t) dt$, $0 \leq t \leq T$, (see [12]). Hence, $y_\nu(t) \triangleq E \left[ \sum_{s=1}^{N(t)} h(t, \tau_s; m(\tau_s)) \right]$, where $N(T) = k = \sum_{t=1}^{k} T_0 f(\tau) E \left[ |h(t, \tau_s; m(\tau_s))^2| \right] d\tau + k \sum_{i=1}^{k} T_0 f(\tau_i) d\tau \sum_{j=1}^{k} f(\tau_j) d\tau \times E \left[ h(t, \tau_s; m(\tau_s))^2 h(t, \tau_s; m(\tau_s)) \right]$, (4)

**Lemma 1.** Consider model (2)-(3) under Assumption 1. Then,

$$E[y(t)] = \int_0^T \lambda_T(t) E[h(t, \tau; m(\tau))] d\tau$$

$$\text{Var}(y(t)) = \int_0^T \lambda_T(t) E[|h(t, \tau; m(\tau)|^2] d\tau$$

**Remark 1.** Some observations concerning the results of Lemma 1 are now in order. These observations are important because they provide additional insight regarding the role of the rate of Poisson process in modeling quasistatic channels.

(1) Clearly, the rate of the Poisson process is an important parameter which shapes the statistics of the received signal, and therefore the multipath delay profile and the Doppler spread. It models the filtering properties of the propagation environment. If the arrival times of the different paths are known (information which is obtained by sounding the channel), then the rate of the Poisson process should be replaced by a linear combination of impulses. Thus, by setting $\lambda_T(t) = \sum_{i=1}^{N} \lambda_i \delta(t - \tau_i)$, we obtain

$$\text{Var}(y(t)) = \int_0^T \sum_{i=1}^{N} \lambda_i \delta(t - \tau_i) E[|h(t, \tau; m)^2|] d\tau$$

for $0 \leq t \leq T$, which is exactly what one would obtain if the arrival times of the multipath components are known.

(2) Tapped delay channel. Consider the tapped delay channel model, which corresponds to a frequency-selective channel with transmitted signal bandwidth $W$. The rate $\lambda_T(t)$ is greater than the coherence bandwidth $B_{coh}$ of the channel, and $W \gg B_{coh}$. In this case, the sampling theorem (see [1, pages 795–797]) leads to the tapped delay line model, where $N = [(1/B_{coh})W] + 1, \tau_i = i/W, 1 \leq i \leq N$, and $N$ is the number of resolvable paths. This tapped delay model can be generated from the model presented using a Poisson process by choosing the rate of the Poisson process...
so that most points are concentrated at \( \{i/W\}_{i=1} \) (e.g., letting \( \lambda_T(t) \) be a series of mountains concentrated near \( i/W \)). That is, the orderliness effect of the Poisson process is mitigated because of the limitations of the equipment that is used to measure the received signal. In the next two statements, we present a comparison of the computation of the received power when the arrival times of the multipath components are known and when these are assumed to be the points of a homogeneous Poisson process.

(3) **Wideband transmission.** Consider the periodic transmission of a pulse \( x(t) = \pi(t) \) every \( T_s \) seconds, where \( \pi(t) = \sqrt{T_m/T_c} \) if \( 0 < t < T_c \) and \( \pi(t) = 0 \) or, otherwise, where \( T_s \gg T_m \), with \( T_m \) denoting the duration of the channel impulse response (e.g., excess delay of the channel).

Suppose that the low-pass received signal is

\[
y_{t,n}(t) = \sum_{i=1}^{N} r_i e^{-j(\omega_d t + \omega_k t_0)} \pi(t - \tau_i),
\]

where \( N, \{\tau_i\}_{i=1}^{N} \) is a realization of the Poisson process (e.g., known).

Then, the energy received over \( [0, T_m] \) at some \( t_0 \in [0, T_s] \) is defined by (see [2, pages 147–150]) \( y_{t,n}(t_0) \triangleq (1/T_m) \int_0^{T_m} y_{t,n}(t) y_{t,n}(t) dt \), which is the time average of the second moment of \( y_{t,n}(t) \) based on a single realization over the interval \( [0, T_m] \). Further, if the multipath components are assumed to be resolved by the probing signal \( \pi(t) \) (e.g., \( |\tau_i - \tau_j| > T_c \) for all \( i \neq j \)), then

\[
|y_{t,n}(t_0)|^2 = \frac{1}{T_m} \sum_{i=1}^{N} r_i^2 (t_0) \int_0^{T_m} \pi^2(t - \tau_i) dt = \frac{1}{T_m} \sum_{i=1}^{N} r_i^2 (t_0).
\]

The ensemble average power (due to a wideband signal transmission) is \( E_{WB} = \sum_{t=1}^{N} E[r_i^2(t_0)] \) if \( r_i \) are i.i.d.). Our earlier equations calculate \( E_{WB} \) using ensemble average. In particular, \( E_{WB} \) corresponds to

\[
\bar{y}_{t,n}(t) = \frac{1}{T_s} \sum_{t=1}^{T_s} E[r_i^2(\tau)] \pi^2(t - \tau) d\tau \approx \frac{T_m}{T_s} \sum_{t=1}^{N} E[r_i^2(\tau)],
\]

which is obtained under the assumption that \( N(T_s) = N \) is fixed, \( \lambda_T(t) = \lambda \) is a constant, and \( E[y_{t,n}(t)] = 0 \). On the other hand, under the assumptions of Lemma 1, assuming constant rates \( \lambda_T(t) = \lambda \) and \( E[y_{t,n}(t)] = 0 \), we have from (7) that

\[
E[|y_{t,n}(t)|^2] = \lambda \int_0^{T_s} E[r_i^2(\tau)] \pi^2(t - \tau) d\tau = \lambda T_m \sum_{i=1}^{N} E[r_i^2(t_0)]
\]

if \( r(\tau) = \sum_{i=1}^{N} r_i(t_0) \delta(\tau - t_0), \ t_0 \in [t - T_c, t], \)

(12)

which is proportional to (10) and (11).

(4) **Narrowband transmission.** Consider next the transmission into the channel (9) of a continuous-wave signal, \( x(t) = 1 \). Then, the received power, given the realization of \( N(T_s); 0 \leq t \leq T_s \), is

\[
\beta_{CW} = E \left[ \left( \sum_{i=1}^{N} r_i e^{-j(\omega_d t + \omega_k t_0)} \pi(t - \tau_i) \right)^2 \right]
\]

\[
= \sum_{i=1}^{N} E[r_i^2] + \sum_{i=1}^{N} \sum_{i \neq m} E[r_i r_m e^{-j(\phi_i - \phi_m)} e^{-j(\omega_d t_0 - \omega_{dA} t_0)}] \times e^{j(\omega_{di} - \omega_{dA}) t_0} dt_0 dt_m,
\]

(13)

On the other hand, if \( N(T_s) = N \) and \( \lambda = \text{constant} \), then by (4) letting \( x(t) = 1 \) yields

\[
\bar{y}_{t,n}(t) = \frac{1}{T_s} \int_0^{T_s} E[r_i^2(\tau)] d\tau
\]

\[
+ \frac{1}{T_s} \int_0^{T_s} \sum_{i=1}^{N} \sum_{i \neq m} E[r_i r_m e^{j(\phi_i - \phi_m)} e^{-j(\omega_d t_0 - \omega_{dA} t_0)}] e^{j(\omega_{di} - \omega_{dA}) t_0} dt_0 dt_m,
\]

(14)

which is proportional to (13). Clearly, the above comparisons indicate the consistency of the ensemble averages based on our model and analysis with respect to the analysis found in [2], even for the simple homogeneous Poisson process.

**Correlation and covariance**

The correlation of \( y_{t,n}(t) \) and \( y_{t,n}(t_2) \) is \( R_{y_{t,n}y_{t,n}}(t_1, t_2) \triangleq E[y_{t,n}^*(t_1) y_{t,n}(t_2)] = E[\sum_{i=1}^{N} h_{t,n}^*(t_1, \tau_i; m_i(t_1)) \sum_{j=1}^{N} h_{t,n}(t_2, \tau_j; m_j(t_2))] \), and the covariance is

\[
C_{y_{t,n}y_{t,n}}(t_1, t_2) \triangleq R_{y_{t,n}y_{t,n}}(t_1, t_2) - E[y_{t,n}^*(t_1)] E[y_{t,n}(t_2)].
\]

(15)

where

\[
R_{y_{t,n}y_{t,n}}(t_1, t_2) = E[y_{t,n}^*(t_1) y_{t,n}(t_2)]
\]

\[
= E \left[ \sum_{i=1}^{k} h_{t,n}^*(t_1, \tau_i; m_i(t_1)) h_{t,n}(t_2, \tau_j; m_j(t_2)) \right]
\]

\[
+ E \left[ \sum_{i,j=1}^{k} h_{t,n}^*(t_1, \tau_i; m_i(t_1)) h_{t,n}(t_2, \tau_j; m_j(t_2)) \right].
\]
The above expression is further simplified by invoking Assumption 1.

**Lemma 2.** Consider model (2)–(3) under Assumption 1. Then,

$$R_{y_t}(t_1, t_2) = \int_0^{T_y} \lambda_\tau(\tau) E[h_t^*(t_1, \tau; m(\tau)) h_t(t_2, \tau; m(\tau))] d\tau$$

$$+ \lambda \int_0^{T_y} \lambda_\tau(\tau) E[h_t^*(t_1, \tau; m(\tau)) h_t(t_2, \tau; m(\tau))] d\tau$$

$$\times \lambda \int_0^{T_y} \lambda_\tau(\tau) E[h_t(t_2, \tau; m(\tau))] d\tau$$

$$= \lambda \int_0^{T_y} \lambda_\tau(\tau) E[r^2(\tau) e^{j\omega d(t_1-t_2)}] x_t^*(t_1 - \tau) x_t(t_2 - \tau) d\tau$$

$$+ \lambda \int_0^{T_y} \lambda_\tau(\tau) E[r(\tau) e^{-j\phi} e^{-j\omega d(t_1-t_2)}] x_t^*(t_1 - \tau) x_t(t_2 - \tau) d\tau$$

$$\times \lambda \int_0^{T_y} \lambda_\tau(\tau) e^{-j\omega d} E[r(\tau) e^{j\phi} e^{j\omega d(t_2-t_1)}] x_t(t_2 - \tau) d\tau,$$

$$0 \leq t_1, t_2 \leq T_y.$$  \hspace{1cm} (17)

**Remark 2.** Next we illustrate how the rate of Poisson process affects both the Doppler power spectrum and the power delay profile. Consider the results of Lemma 2 when \( t_1 = t, t_2 = t + \Delta t, \) and \( x(t) = 1, \) for all \( t \in [0, T] \) (e.g., a narrowband signal), and for fixed \( t_1 = \tau, \omega_d(\tau) = (2\pi v(\tau)/\lambda_n) \cos \theta_i, \) where \( v(\tau) \) is the speed of the mobile, corresponding to the \( i \)-th path, \( \lambda_n \) is the wavelength, and \( \theta_i \) is uniformly distributed in \([0, 2\pi]\) [4, 5] (the dependence of \( \omega_d \) on \( \tau \) is obviously incorporated in the previous results). We will compute the autocorrelation, Doppler spread, and power delay profile of the channel.

(1) Doppler power spectrum. Under the above assumptions (and assuming \( Ey(t) = 0 \)), the autocorrelation of \( y(t) \) is

$$R_{y_t}(\Delta t) = \lambda \int_0^{T_y} \lambda_\tau(\tau) E[r^2(\tau) e^{j\omega u(\tau)\Delta t}] d\tau,$$

and its power spectral density is

$$F_{\Delta t}[R_{y_t}(\Delta t)] = \lambda \int_0^{\infty} \int_0^{T_y} \lambda_\tau(\tau) E[r^2(\tau) e^{j\omega u(\tau)\Delta t}] e^{-j2\pi f\Delta t} dt dt.$$  \hspace{1cm} (20)

Moreover, if \( r(\tau) \) and \( \omega_d(\tau) \) are independent (as commonly assumed) and \( \lambda_c(t) = \sum_{i=1}^{N} \delta(t - \tau_i), \) then \( R_{y_t}(\Delta t) = \lambda \sum_{i=1}^{N} E[r^2(\tau)] \times f_0((2\pi v(\tau)/\lambda) \Delta t), \) which is a commonly known expression, where \( f_0(\cdot) \) is a Bessel function of first kind of zero order (see [5] for \( N = 1 \)), and \( F_{\Delta t}[C_{y_t}(\Delta t)] = \lambda \sum_{i=1}^{N} E[r^2(\tau)] \times S_{D_i}(f), \) where

$$S_{D_i}(f) = \frac{1}{2\pi v(t_i)} \frac{1}{\sqrt{1 - (f\lambda_n/v(t_i))^2}}, \quad |f| \leq \frac{v(t_i)}{\lambda_n},$$

otherwise, \hspace{1cm} (21)

for \( 1 \leq i \leq N. \) Thus, \( S_{D_i}(f) \) is the Doppler spread predicted in [4, 5] for a two-dimensional propagation model. More general models such as those found in [5] can be considered as well.

(2) Power delay profile. Under the above assumptions (and assuming \( Ey(t) = 0 \)), the power delay profile of \( y(t) \), denoted by \( \phi(\tau) \), is obtained from \( (17) \) by letting \( t_1 = t_2 = \tau \) and letting \( x(t) \) be a delta function, which implies that \( \phi(\tau) = \lambda T(\tau) E[r^2(\tau)]. \) Clearly, the rate of the Poisson process determines the shape of the power delay profile as expected. Note that in practise one can obtain the rate \( \lambda_T(\cdot) \) via maximum-likelihood methods by noisy channel measurements.

However, if \( r(\tau) \) and \( \omega_d(\tau) \) are not independent, then more general expressions for the autocorrelation and Doppler spread are obtained.

### 3. Power Spectral Densities

Throughout this section, it is assumed (for simplicity) that \( \{r_\tau(t_i)\}_{i=1} \) are independent of \( t_i \)'s, and thus we denote them by \( \{r_i\}_{i=1}; \) \( N(T_y) \) is homogeneous Poisson. However, if one considers the \( \tau \)-dependent attenuations \( \{r_\tau(t)\}_{i=1} \), then as a
function of $\tau$, each $r_i : \Omega \times [0,T_i] \rightarrow [0,\infty)$, and therefore each $r_i$ is a random process as a function of $\tau$. In this case, the results will also hold provided that one assumes that $\{r_i(\tau)\}_{\tau \geq 0}$ as functions of $\tau$ are wide-sense stationary (because $E[r^2(\tau)]$ and $E[r(\tau)]$ are independent of $\tau$).

**Power spectral density**

The expressions for the correlation function and the covariance function (assuming that $\int_{t-T_i} E_r$ is denoted by $\int_{t-\infty}^\infty$) are

$$C_{r}(\tau) = \lambda E [r^2 e^{j\omega t}] \int_{-\infty}^\infty x^*_\alpha(\omega) x_\alpha(\tau + \omega) d\omega,$$

$$R_{r}(\tau) = C_{r}(\tau) + \lambda E \left[ e^{-j\omega t} \int_{-\infty}^\infty e^{-j(\omega+\omega_\tau)\tau} x^*_\alpha(\omega) d\omega \right] \times \lambda E \left[ e^{j\omega t} \int_{-\infty}^\infty e^{j(\omega+\omega_\tau)\tau} x_\alpha(\tau + \omega) d\omega \right].$$

Taking Fourier transforms, we obtain the following result.

**Theorem 1.** Consider model (2)-(3) under Assumption 1 with $r_i(\tau)_{\tau \geq 0}$ being independent of $\tau$, and consider $\lambda$ N(t) a homogeneous Poisson process with rate $\lambda \geq 0$. Define the centered processes $x(t) = x(t) - \overline{x}(t)$, $y(t) = y(t) - \overline{y}(t)$, and

$$X,Y(t) = \left\{ \begin{array}{ll}
\int_{-\infty}^\infty x(t)e^{-j\omega t} dt, & \text{for } X(t), \\
\int_{-\infty}^\infty x(t)e^{-j\omega t} dt, & \text{for } Y(t),
\end{array} \right.$$

$$X(t) = \text{Re} \left\{ x(t)e^{i(\omega_\tau + \omega_\tau)t + j\phi} \right\}.$$

The power spectral densities of the centered processes $x(t)$ and $y(t)$ are

$$S_{x}(\omega) = \mathcal{F}_x [C_{x}(\tau)],$$

$$S_{y}(\omega) = \mathcal{F}_x [R_{y}(\tau)].$$

and the power spectral densities of $x(t)$ and $y(t)$ are

$$S_{x}(\omega) = \mathcal{F}_x [R_{x}(\tau)]$$

$$S_{y}(\omega) = \mathcal{F}_x [R_{y}(\tau)].$$

Further, assuming $y(t) = \lambda \int_0^T E[h(t,\tau; \mathbf{m})] d\tau = 0$, the power spectral density of $y^2(t)$ is

$$S_{y^2}(j\omega) \triangleq \mathcal{F}_x [C_{y^2}(\tau)]$$

$$= \frac{\lambda^2}{\pi} E[r^2 |X(j\omega)|^2] E[r^2 |X(j\omega)|^2] + 2\pi\lambda^2 \mathcal{E}^2 \delta(\omega) + \frac{\lambda}{\delta \pi^2} E[|r^2 X(j\omega)\times X(j\omega)|^2],$$

where $\mathcal{E} = E[\int_{-\infty}^\infty r^2 x^2(t) dt].$

**Remark 3.** The behavior of the power spectral densities for high and low rates $\lambda$ obtained as follows.

(1) **High-rate approximation.** If $\lambda$ is sufficiently large, then the third term in (29) can be neglected and the power spectrum of $y^2(t)$ consists of only the first and second right-hand side terms of (29).

(2) **Low-rate approximation.** If $\lambda$ is small, then the probability that the terms $h(t; t; \mathbf{m})$ and $h(t; t; \mathbf{m})$ have significant overlap, for $i \neq j$, is very small, hence the approximation $y^2(t) \approx \sum_{i=1}^{N(T_i)} h(t; t; \mathbf{m})$. This is equivalent to assuming that the paths do not overlap. As described earlier, the power spectral density expressions are important in receiver design and for modeling the interference.

4. DISTRIBUTIONS AND MOMENT-GENERATING FUNCTIONS

Let $I_{[A]}$ denote the indicator function of the event $A$, which is 1 if the event $A$ occurs and zero otherwise. The probability density function and moment-generating functions of $y(t)$ and $y(t)$ are, respectively, defined by $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$, $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$, $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$, $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$, $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$. Consider the real signal $y(t)$ for fixed $N(T_i) = k$, the density of $y(t)$ is $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$, $f_y(x, t) dx \triangleq E[I_{[y(t) \in (dx)]}]$. Assuming a homogeneous Poisson process (for simplicity of presentation), we obtain $f_y(x, t) = e^{-\lambda T_i} \sum_{i=1}^{N(T_i)} f_y(x, t)((\lambda T_i)^k/k!)$. For fixed $N(T_i) = k$, the moment-generating function of $y(t)$ is

$$\Phi_{y}(s, t) = \exp \left\{ \sum_{i=1}^{N(T_i)} \left[ h(t, t; \mathbf{m}_i(1)) \right] \right\}$$

$$= \sum_{i=1}^{N(T_i)} \left[ \frac{1}{T_i} \int_0^{T_i} dt_1 \int_0^{T_i} dt_2 \cdots \int_0^{T_i} dt_k \mathcal{E} \prod_{i=1}^{k} e^{h(t, t; \mathbf{m}_i(1))} \right]$$

$$= \prod_{i=1}^{k} \frac{1}{T_i} \int_0^{T_i} dt \mathcal{E} \left[ e^{h(t; t; \mathbf{m}_i(1))} \right]$$

Clearly, the above calculations hold for the low-pass equivalent complex representation as well, leading to the following
Moreover, for 
\[ m \] and its density is

\[
f_j(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-j\omega x} \times \exp \left\{ j \int_0^T \lambda_c(\tau) E\left[ e^{j\omega(x, \tau; \mathbf{m}(\tau))} - 1 \right] d\tau \right\}.
\]

Moreover,
\[
\Psi_j(j\omega, t) \overset{\Delta}{=} \ln E\left[ e^{j\omega y(t)} \right] = \sum_{k=1}^{\infty} \frac{(j\omega)^k}{k!} \gamma_k(t) \quad \text{provided that } \gamma_k(t) < \infty,
\]

where
\[
\gamma_k(t) = \lambda \int_0^T \lambda_c(\tau) E\left[ h(t, \tau; \mathbf{m}(\tau)) \right]^{k} d\tau
\]
is the kth cumulant of \( y(t) \), and \( \gamma_1(t) = E[y(t)] \) and \( \gamma_2(t) = \text{Var}(y(t)) \).

(2) The characteristic function of \( y(t) \) is

\[
\Phi_y(\rho, t) \overset{\Delta}{=} \exp \left\{ j \int_0^T \lambda_c(\tau) E\left[ \text{Re}(\rho^* h(t, \tau; \mathbf{m}(\tau))) \right] d\tau \right\}.
\]

where \( \rho = \rho_R + j\rho_I \), and its density is

\[
f_j(x, t) = \frac{1}{(2\pi)^r} \int_{-\infty}^{\infty} d\rho_R d\rho_I e^{-j\text{Re}(\rho^* x)} \times \exp \left\{ \lambda \int_0^T \lambda_c(\tau) E\left[ e^{j\text{Re}(\rho^* h(t, \tau; \mathbf{m}(\tau)))} - 1 \right] d\tau \right\}.
\]

Moreover, for \( m, n > 0 \) integers

\[
E\left[ (y(t)^* y(t))^m \right] = (-2j)^{m} \frac{\partial^m}{\partial \rho^m} \Phi_y(\rho, t) \mid_{\rho=0},
\]

\[
\Psi_y(\rho, t) \overset{\Delta}{=} \ln E\left[ e^{j\rho^* y(t)} \right] = \sum_{k=1}^{\infty} \frac{j^k \gamma_k(t)}{k!},
\]

where

\[
y(t) = \lambda \int_0^T \lambda_c(\tau) E\left[ \text{Re}(\rho^* h(t, \tau; \mathbf{m}(\tau))) \right] d\tau,
\]

\[
E[y(t)] = (-2j) \frac{\partial \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho} y(t),
\]

\[
\text{Var}(y(t)) = (-2j)^2 \frac{\partial^2 \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho^2} \frac{\partial \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho} y(t).
\]

The above theorem gives closed-form expressions for all the moments of \( y(t) \) and \( y(t) \) along with their cumulants are obtained following the derivation of Theorem 2.

Central limit theorem

The joint characteristic functions of \( y(t_1), \ldots, y(t_n) \) and \( y(t_1), \ldots, y(t_n) \) along with their cumulants are obtained following the derivation of Theorem 2.

Corollary 1. Consider model (2)-(3) under Assumption 1.

(1) The joint characteristic function of \( y(t_1), \ldots, y(t_n) \) is

\[
\Phi_Y(s, t_1; \ldots; s_n, t_n) \overset{\Delta}{=} E\left[ \exp \left\{ \sum_{i=1}^{n} s_i y(t_i) \right\} \right],
\]

\[
\text{Var}(y(t)) = (-2j)^2 \frac{\partial^2 \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho^2} \frac{\partial \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho} y(t).
\]

(2) The joint characteristic function of \( y(t_1), \ldots, y(t_n) \) is

\[
\Phi_Y(\rho_1, t_1; \ldots; \rho_n, t_n) \overset{\Delta}{=} E\left[ \exp \left\{ \text{Re}(\rho h(t_1, \tau; \mathbf{m}(\tau))) \right\} \right],
\]

\[
\text{Var}(y(t)) = (-2j)^2 \frac{\partial^2 \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho^2} \frac{\partial \text{Re}(y(t, 1), \rho h(1, t, \tau; \mathbf{m}(\tau)))}{\partial \rho} y(t).
\]
The joint moment-generating function of the complex random variables \( y_\ell(t_1), \ldots, y_\ell(t_n) \) is

\[
\mathbb{E} \left[ \prod_{i=1}^{n} (y_\ell^*(t_i))^{k_\ell} \prod_{i=1}^{n} (y_\ell(t_i))^{m_\ell} \right] = (-2j)^{k_\ell} T_\ell \prod_{i=1}^{n} \left( \frac{\partial}{\partial \rho_i} \right)^{k_\ell} \left( \frac{\partial}{\partial \rho_i^*} \right)^{m_\ell} \Phi_{y_\ell}(\rho_1, t_1, \ldots; \rho_n, t_n)_{\rho=0}.
\]

(43)

Corollary 1 gives closed-form expressions for joint statistics of \( \{y_\ell(t)\}_{t \geq 0} \) and \( \{y(t)\}_{t \geq 0} \), including correlations and higher-order statistics. These are easily computed for the example of Remark 2.

We will show next that for large \( \lambda \), compared to the time constants of the signal \( x \), the joint distribution of \( y(t_1), \ldots, y(t_n) \) is normal, thus establishing a central limit theorem for \( \{y(t)\}_{t \geq 0} \) as a random process. Further, we will illustrate that similar results hold for the complex random variables \( y_\ell(t_1), \ldots, y_\ell(t_n) \). This is a generalization of the Gaussianity of shot noise described by Rice in [7, 8].

To this end, define the centered random variables \( y_\ell(t_i) \equiv (y(t_i) - \overline{y}(t_i))/\sigma_y(t_i) \) and \( \sigma_y(t_i) = \sqrt{\text{Var}(y(t_i))} \). In general, the above central limit result holds as \( \lambda \to \infty \) (this is consistent with the understanding that as \( T \) becomes large, more paths are present and hence the central limit theorem will hold).

The last expression shows that the joint characteristic function is quadratic in \( \{\omega_i\}_i \). Hence, \( y_\ell(t_1), \ldots, y_\ell(t_n) \) are approximately Gaussian, with zero mean and the covariance matrix identified. Moreover, \( y_\ell(t_1) \sim N(0; 1), 1 \leq i \leq n \) in the limit, as \( \lambda \to \infty \). If we consider the example of Remark 2, and let \( \lambda T \) be a constant (say \( \lambda \)), then the Gaussianity statement holds provided that \( T \to \infty \) (this is consistent with the understanding that as \( T \) becomes large, more paths are present and hence the central limit theorem will hold).

**Lemma 3.** Consider model (2)-(3) under Assumption 1.

1. The joint characteristic function of the centered random variables

\[
y_\ell(t_i) \equiv \frac{y(t_i) - \overline{y}(t_i)}{\sigma_y(t_i)}, \quad 1 \leq i \leq n, \tag{48}
\]

\[
\sigma_y(t_i) = \sqrt{\text{Var}(y(t_i))},
\]

is in the limit, as \( \lambda \to \infty \), and is Gaussian with

\[
\lim_{\lambda \to \infty} \Phi_{y_\ell}(j\omega_1, t_1; \ldots; j\omega_n, t_n) \equiv \lim_{\lambda \to \infty} \mathbb{E} \left[ \exp \left( \sum_{i=1}^{n} s_i y_\ell(t_i) \right) \right],
\]

\[
= \exp \left\{ -\frac{\lambda}{2} \int_{0}^{T} \lambda_c(r) E \left[ \sum_{i=1}^{n} \frac{\omega_i}{\sigma_y(t_i)} h(t_i, r; m(r)) \right] dr \right\},
\]

(49)

2. The joint characteristic function of the centered random variables

\[
y_{\ell c}(t_i) \equiv \frac{y_{\ell c}(t_i) - \overline{y}(t_i)}{\sigma_y(t_i)}, \quad 1 \leq i \leq n, \tag{50}
\]

\[
\sigma_y(t_i) = \sqrt{\text{Var}(y_{\ell c}(t_i))},
\]

Since \( \sigma_y(t_i) \) is proportional to \( \lambda^{1/2} \), the first term in the power series expansion is of order \( \lambda^{1/2} \), the second term is of order 1, the third term is of order \( 1/\lambda^{1/2} \), and the kth term is of order \( \lambda^{k/2} = \lambda^{-(k-2)/2} \). Hence, for large \( \lambda \), we have the following approximation (neglecting terms of order \( \lambda^{-(k-2)/2}, k \geq 3 \):

\[
\lambda \int_{0}^{T} \lambda_c(r) E \left[ \exp \left( \sum_{i=1}^{n} \frac{\omega_i}{\sigma_y(t_i)} h(t_i, r; m(r)) \right) - 1 \right] dr
\]

\[
\approx \lambda \int_{0}^{T} \lambda_c(r) E \left[ \sum_{i=1}^{n} \frac{\omega_i}{\sigma_y(t_i)} h(t_i, r; m(r)) \right] dr + \frac{1}{2} \lambda \int_{0}^{T} \lambda_c(r) E \left[ \sum_{i=1}^{n} \frac{\omega_i}{\sigma_y(t_i)} h(t_i, r; m(r)) \right]^2 dr.
\]

(46)
is in the limit, as $\lambda \to \infty$, and is complex Gaussian with

$$\lim_{\lambda \to \infty} \Phi_{\lambda}(\rho_1, t_1; \ldots; \rho_n, t_n) \triangleq \lim_{\lambda \to \infty} E\left[ \exp \left( j \Re (\rho^T y_{\ell e}(t)) \right) \right]$$

$$= \exp \left\{ -\frac{\lambda}{2} \int_0^T \lambda_c(\tau) E \left[ \sum_{i=1}^n \left[ \frac{\rho_{Ri}}{\sigma_{Ri}(t)} \Re (h_{\ell i}(t, \tau, m(\tau))) \right]^2 + \frac{\rho_{Li}}{\sigma_{Li}(t)} \Im (h_{\ell i}(t, \tau, m(\tau))) \right] \, d\tau \right\},$$

where $y_{\ell e}(t) = (y_{\ell e}(t_1), \ldots, y_{\ell e}(t_n))^T \in \mathcal{C}^n$.

**Proof.** (1) The proof follows from the above construction. (2) Equation (51) is obtained by following exactly the same procedure as in (1) (see also [13, page 157]).

**Remark 4.** Next, we discuss the implications of the previous lemma and some generalizations of the results obtained.

(1) Clearly, in (49) and (51), the exponents are quadratic functions of $\{\omega_{ij}\}_{i,j=1}^n$ and $\{\rho_{Ri}, \rho_{Li}\}_{i=1}^n$, respectively; therefore one can easily specify the correlation properties of the received Gaussian signal, irrespective of the transmitted input signal. Unlike [5] in which Gaussianity of the inphase and quadrature components is derived, the last theorem shows Gaussianity of the received signal which is multipath, and identifies one of the parameters which is responsible for such Gaussianity to hold. Further, in many places it is often conjectured that for a large number of paths the inphase and quadrature components of the received signal are Gaussian. Some authors argue that the low-pass representation of the band-limited channel impulse response is complex Gaussian. Lemma 3 establishes the above conjecture in the limit as the rate of the Poisson process tends to infinity, by identifying the mean and the covariance of the Gaussian process. Clearly, as $\lambda$ increases the number of paths received in a given observation interval increases, which then implies that resolvability of the paths is highly unlikely.

Note that Lemma 3 can be used to compute the second-order statistics of the inphase and quadrature components. The mean of the inphase component is $E[I(t)] = \lambda \int_0^T \lambda_c(\tau) E[r(r) \cos(\omega_c \tau + \omega_d(t-\tau))] d\tau$, and its covariance is $C(t_1, t_2) = \lambda \int_0^T \lambda_c(\tau) E[r^2(\tau) \cos(\omega_c \tau + \omega_d(t_1 - t) - \omega_d(t_2 - t))] d\tau$.

(2) Every result obtained also holds for random signals $x$ and $x_e$, such as CDMA signals, provided that the expectation operation $E[\cdot]$ operates on the signals $x$ and $x_e$ as well. Moreover, if the counting process is neither orderly nor independent increment, then the rate of the counting process, namely, $\lambda \times \lambda_c(t)$, should be random. This will be the case if $\lambda$ is a random variable, and the earlier results will hold provided that there is an additional expectation with respect to the distribution of the random variable $\lambda$. Finally, we point out that one may use the current paper and the methodology in [9] to derive expressions for interference signals.

## 5. CONCLUSION

This paper presents a unified framework for studying the statistical characteristics of multipath fading channels, which can be viewed as a generalization of the mathematical analysis of the shot-noise effect. These include the second-order statistics, power spectral densities, and central limit theorems which are generalizations of Campbell’s theorem. In the case of nonhomogeneous Poisson process, each propagation environment is identified with the rate $\lambda(t) = \lambda \times \lambda_c(t)$, in which $\lambda_c(\cdot)$ acts as a filter in shaping the received signal. This rate is an important parameter which needs to be identified prior to any design considerations associated with wireless channels.

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