ON REGULARITY THEOREMS FOR LINEARLY INVARIANT FAMILIES OF HARMONIC FUNCTIONS

Abstract. The classical theorem of growth regularity in the class $S$ of analytic and univalent in the unit disc $\Delta$ functions $f$ describes the growth character of different functionals of $f \in S$ and $z \in \Delta$ as $z$ tends to $\partial \Delta$. Earlier the authors proved the theorems of growth and decrease regularity for harmonic and sense-preserving in $\Delta$ functions which generalized the classical result for the class $S$. In the presented paper we establish new properties of harmonic sense-preserving functions, connected with the regularity theorems. The effects both common for analytic and harmonic case and specific for harmonic functions are displayed.

Key words: regularity theorem, linearly invariant family, harmonic function

2010 Mathematical Subject Classification: 30C55

1. Introduction. For a function $u(z)$, continuous in the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, we denote

$$M(r, u) = \max_{|z| \leq r} |u(z)| \quad \text{and} \quad m(r, u) = \min_{|z| \leq r} |u(z)|.$$

Let $S$ be the class of all univalent analytic functions $f(z) = z + \ldots$ in $\Delta$. The theorem of growth regularity asserts that functions having the maximal growth in the given class, grows smoothly (regularly).

Theorem A. [1], [2], [3] pp. 104, 105], [4] pp. 8–9] Let $f \in S$. Then there exist a $\delta_0 \in [0, 1]$ with

$$\lim_{r \to 1^-} \left[ M(r, f) \frac{(1-r)^2}{r} \right] = \lim_{r \to 1^-} \left[ M(r, f') \frac{(1-r)^3}{1+r} \right] = \delta^0,$$
\[ \delta^0 = 1 \text{ iff } f(z) = z(1 - ze^{-i\theta})^{-2}. \] If \( \delta^0 \neq 1 \), then the functions under the sign of the limit increase on \( r \).

If \( \delta^0 \neq 0 \), then there exists \( \varphi^0 \in [0; 2\pi) \) such that
\[
\lim_{r \to 1^-} \left| f(re^{i\varphi}) \right| \left( \frac{(1-r)^2}{r} \right) = \lim_{r \to 1^-} \left| f'(re^{i\varphi}) \right| \left( \frac{(1-r)^3}{1+r} \right) = \begin{cases} \delta^0, & \varphi = \varphi^0 \\ 0, & \varphi \neq \varphi^0. \end{cases}
\]
Here the functions under the sign of the limit are also increasing on \( r \in (0, 1) \).

In [5], Ch. Pommerenke showed that many properties of functions from the class \( S \) can be extended to linearly invariant families (LIFs) of locally univalent analytic functions in \( \Delta \) of finite order. In [6] and [7], the theorem of growth regularity was obtained for such LIFs.

In [8], [9], the authors introduced the notion of LIF for complex-valued harmonic functions \( f \) in \( \Delta \). Every such function can be presented, using analytic functions \( h \) and \( g \) in \( \Delta \) in the following way:
\[ f(z) = h(z) + g(z), \] (1)
where
\[ h(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_{-n}(f)z^n. \]
As in [5], L. E. Shaubroek considered locally univalent functions in \( \Delta \). Moreover, these functions are sense-preserving in \( \Delta \), i.e. the Jacobian \( J_f(z) \) satisfies
\[ J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \quad \forall z \in \Delta. \]

**Definition 1.** [8], [9] A set \( \mathcal{M}_H \) of harmonic sense-preserving functions \( f \) in \( \Delta \) of form (1) is called the linearly invariant family (LIF) if for all \( f \in \mathcal{M}_H \) and for any conformal automorphism \( \phi(z) = \frac{z+a}{1+\bar{a}z}, a \in \Delta \), the function \( e^{-i\theta}f_a(ze^{i\theta}) \) belongs to \( \mathcal{M}_H \), where
\[ f_a(z) = \frac{f(\phi(z)) - f(\phi(0))}{h'(\phi(0))\phi'(0)}. \] (2)

It is assumed that the order of a family \( \mathcal{M}_H \)
\[ \text{ord} \mathcal{M}_H = \sup_{f \in \mathcal{M}_H} |a_2(f)| \]
is finite.
In the analytic case (when \(g(z) \equiv 0\)), the definitions of LIF and \(\text{ord}\mathcal{M}_H\) coincide with the definitions of Pommerenke [5].

In [10], for LIFs of harmonic functions, the strong order

\[
\overline{\text{ord}}\mathcal{M}_H = \sup_{f \in \mathcal{M}_H} \frac{|a_2(f) - a_{-1}(f)a_{-2}(f)|}{1 - |a_{-1}(f)|^2}
\]

was defined. The strong order proved to be convenient for investigation of LIFs, because it is not necessary to assume the affine invariance of a family. Moreover, for an affine LIF \(\mathcal{M}_H\) the strong order does not exceed the old order:

\[
\text{ord}\mathcal{M}_H - \frac{1}{2} \leq \overline{\text{ord}}\mathcal{M}_H \leq \text{ord}\mathcal{M}_H.
\]

This fact allows to describe properties of affine LIFs more precisely. For a LIF \(\mathcal{M}\) of analytic functions, \(\text{ord}\mathcal{M}_H = \overline{\text{ord}}\mathcal{M}_H\). Analogously to the analytic case in [10] the universal LIF \(\mathcal{U}^H_\alpha\) was introduced and studied. The family \(\mathcal{U}^H_\alpha\) is defined as the union of all LIFs \(\mathcal{M}_H\) such that \(\overline{\text{ord}}\mathcal{M}_H \leq \alpha\). Equivalently, \(\mathcal{U}^H_\alpha\) is the set of all harmonic sense-preserving functions \(f\) in \(\Delta\) of the form (1) such that

\[
\overline{\text{ord}} f \overset{\text{def}}{=} \overline{\text{ord}} \{e^{-i\theta} f_a(ze^{i\theta}) : a \in \Delta, \theta \in \mathbb{R}\} \leq \alpha.
\]

It was shown in [10] that \(\overline{\text{ord}}\mathcal{U}^H_\alpha \geq 1\).

In [11] and [12], the following regularity theorems for harmonic functions were proved:

**Theorem B. (regularity of growth)** Let \(f \in \mathcal{U}^H_\alpha\). Set

\[
\Phi_1(r) = \int_0^r M(\rho, J_f) \, d\rho, \quad \Psi_1(r, \varphi) = \int_0^r J_f(\rho e^{i\varphi}) \, d\rho, \quad \text{and}
\]

\[
F_1(r) = \int_0^r \frac{(1 + \rho)^{2\alpha - 2}}{(1 - \rho)^{2\alpha + 2}} \, d\rho.
\]

For each \(n \geq 2\) successively denote

\[
\Phi_n(r) = \int_0^r \Phi_{n-1}(\rho) \, d\rho, \quad \Psi_n(r, \varphi) = \int_0^r \Psi_{n-1}(\rho, \varphi) \, d\rho, \quad \text{and}
\]
$$F_n(r) = \int_0^r F_{n-1}(\rho) d\rho.$$ 

Then

a) for every $\varphi \in [0; 2\pi)$ and $n \in \mathbb{N}$, the functions

$$J_f(re^{i\varphi})(1 - r)^{2\alpha+2} \left( 1 + r \right)^{2\alpha-2}, \quad M(r, J_f)(1 - r)^{2\alpha+2} \left( 1 + r \right)^{2\alpha-2},$$

$$\frac{\Phi_n(r)}{F_n(r)}, \quad \frac{\Psi_n(r, \varphi)}{F_n(r)}, \quad \text{and} \quad \max_{\varphi} \frac{\Psi_n(r, \varphi)}{F_n(r)}$$

are non-increasing on $r \in (0; 1)$;

b) there exist constants $\delta^0 \in [0; 1]$ and $\varphi^0 \in [0; 2\pi)$ such that for $1 \leq n \leq 2\alpha+2,

$$\delta^0 = \lim_{r \to 1^-} \left[ \frac{M(r, J_f)(1 - r)^{2\alpha+2}}{J_f(0)4(\alpha+1)(1 + r)^{2\alpha-3}} \right] = \lim_{r \to 1^-} \left[ \frac{M(r, \partial_r J_f)}{J_f(0)4(\alpha+1)(1 + r)^{2\alpha-3}} \right] = \lim_{r \to 1^-} \left[ \frac{\left| \frac{\partial}{\partial r} J_f(re^{i\varphi^0}) \right|}{J_f(0)4(\alpha+1)(1 + r)^{2\alpha-3}} \right] = \lim_{r \to 1^-} \left[ \frac{\int_0^r M(\rho, \partial_\rho J_f) d\rho (1 - r)^{2\alpha+2}}{J_f(0)(1 + r)^{2\alpha-2}} \right] = \lim_{r \to 1^-} \left[ \frac{\int_0^r \left| \frac{\partial}{\partial \rho} J_f(\rho e^{i\varphi^0}) \right| d\rho (1 - r)^{2\alpha+2}}{J_f(0)(1 + r)^{2\alpha-2}} \right] = \lim_{r \to 1^-} \Phi_n(r) = \lim_{r \to 1^-} \frac{\Psi_n(r, \varphi^0)}{J_f(0)F_n(r)} = \lim_{r \to 1^-} \frac{\Psi_n(r, \varphi^0)}{J_f(0)F_n(r)} = \lim_{r \to 1^-} \frac{\max_{\varphi} \Psi_n(r, \varphi)}{J_f(0)F_n(r)};$$

c) $\delta^0 = 1$ for functions $q_\theta(z) = e^{i\theta}k_\alpha(ze^{-i\theta}) + \sigma e^{i\theta}k_\alpha(ze^{-i\theta})$, where $\sigma \in \Delta, \theta \in \mathbb{R}$, and

$$k_\alpha(z) = \frac{1}{2\alpha} \left[ \left( \frac{1 + z}{1 - z} \right)^\alpha - 1 \right].$$
Theorem C. (regularity of decrease) Let $f \in \mathcal{U}_\alpha^H$. Set
\begin{align*}
Q_1(r) &= \int_r^1 m(\rho, J_f) \, d\rho, \quad E_1(r) = \int_r^1 \frac{(1 - \rho)^{2\alpha - 2}}{(1 + \rho)^{2\alpha + 2}} \, d\rho.
\end{align*}
For each $n \geq 2$ successively denote
\begin{align*}
Q_n(r) &= \int_r^1 Q_{n-1}(\rho) \, d\rho, \quad \text{and} \quad E_n(r) = \int_r^1 E_{n-1}(\rho) \, d\rho.
\end{align*}
Then
\begin{enumerate}
\item[a)] for every $\varphi \in [0; 2\pi)$ and $n \in \mathbb{N}$ the functions
\begin{align*}
J_f(re^{i\varphi}) \frac{(1 + r)^{2\alpha + 2}}{(1 - r)^{2\alpha - 2}}, \quad m(r, J_f) \frac{(1 + r)^{2\alpha + 2}}{(1 - r)^{2\alpha - 2}}, \quad \text{and} \quad \frac{Q_n(r)}{E_n(r)}
\end{align*}
are non-decreasing on $r \in (0; 1)$;
\item[b)] there exist constants $\delta_0 \in [1; \infty)$ and $\varphi_0 \in [0; 2\pi)$ such that
\begin{align*}
\delta_0 &= \lim_{r \to 1^-} \left[ \frac{m(\rho, J_f)}{J_f(0)} \frac{(1 + r)^{2\alpha + 2}}{(1 - r)^{2\alpha - 2}} \right] = \lim_{r \to 1^-} \left[ \frac{J_f(re^{i\varphi_0})}{J_f(0)} \frac{(1 + r)^{2\alpha + 2}}{(1 - r)^{2\alpha - 2}} \right] = \lim_{r \to 1^-} \frac{Q_n(r)}{J_f(0)E_n(r)};
\end{align*}
\item[c)] for $\varphi \in [0; 2\pi)$ denote
\begin{align*}
R_1(r, \varphi) &= \int_r^1 J_f(\rho e^{i\varphi}) \, d\rho,
\end{align*}
and for $n \geq 2$, set
\begin{align*}
R_n(r, \varphi) &= \int_r^1 R_{n-1}(\rho, \varphi) \, d\rho
\end{align*}
(under the assumptions of Theorem C the integrals converge). If $\delta_0 < \infty$ then for $n \geq 1$ the function $\frac{R_n(r, \varphi_0)}{E_n(r)}$ is non-decreasing on $r \in (0; 1)$. Moreover,
\begin{align*}
\delta_0 &= \lim_{r \to 1^-} \frac{R_n(r, \varphi_0)}{J_f(0)E_n(r)};
\end{align*}
d) if $J_f(z)$ is bounded in $\Delta$, then for every $n \in \mathbb{N}$ and every $\varphi \in [0; 2\pi)$, the functions
\[
\frac{R_n(r, \varphi)}{E_n(r)} \quad \text{and} \quad \frac{\min R_n(r, \varphi)}{E_n(r)}
\]
are non-decreasing on $r \in (0; 1)$ and
\[
\delta_0 = \lim_{r \to 1^-} \frac{\min R_n(r, \varphi)}{J_f(0)E_n(r)};
\]
e) $\delta_0 = 1$ for functions $q_\theta(z) = e^{i\theta}k_\alpha(ze^{-i\theta}) + \sigma e^{i\theta}k_\alpha(ze^{-i\theta})$, where $\sigma \in \Delta$, $\theta \in \mathbb{R}$, and $k_\alpha(z)$ is the function defined by (3).

**Definition 2.** We say that the constant $\varphi^0$ from Theorem B is a direction of maximal growth (d.m.g.) of a function $f(z)$. The constant $\varphi_0^*$ from Theorem C is a direction of maximal decrease (d.m.d.) of $f(z)$.

**Definition 3.** The numbers $\delta^0$ from Theorem B and $\delta_0$ from Theorem C are called the Hayman numbers of a function $f(z)$.

In the presented paper we establish new properties of $U^H_\alpha$, connected with the regularity theorems.

2. Main results. For fixed $c \in [0; 1)$ introduce the class $U^H_{\alpha,c}$, consisting of all functions $f = h + \overline{g} \in U^H_\alpha$ such that $|g'(0)| \leq c$. That is, $J_f(0) \geq 1 - c^2 > 0$ for all $f \in U^H_{\alpha,c}$. The class $U^H_{\alpha,c}$ is not a LIF. Note that the family $U^H_\alpha$ is not compact in the topology induced by locally uniform convergence in $\Delta$, but for $U^H_{\alpha,c}$ the following theorem takes place.

**Theorem 1.** The family $U^H_{\alpha,c}$ is compact in the topology induced by locally uniform convergence in $\Delta$.

**Proof.** Let $f_n \in U^H_{\alpha,c}$, $f_n = h_n + \overline{g}_n$, $n \in \mathbb{N}$, $h_n$ and $g_n$ be analytic functions in $\Delta$. By $A_\alpha$ denote the set of all analytic functions $h$ in $\Delta$ such that there exists an analytic function $g$ in $\Delta$ and $f = h + \overline{g} \in U^H_\alpha$. In other words, $A_\alpha$ is the set of analytic parts of functions $f \in U^H_\alpha$. The lineary invariance of $U^H_\alpha$ implies that $A_\alpha$ is a LIF of analytic functions. But for LIFs of analytic functions $\text{ord} A_\alpha = \text{ord} A_\alpha$. Therefore for all $h \in A_\alpha$
\[
|h'(z)| \leq \frac{(1 + r)^{\alpha - 1}}{(1 - r)^{\alpha + 1}}, \quad |z| = r;
\]
see [5]. Since $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ for all $z \in \Delta$ and all $f \in \mathcal{U}^H_\alpha$, we have

$$|g'(z)| \leq \frac{(1 + r)^{\alpha - 1}}{(1 - r)^{\alpha + 1}},$$

for all $f = h + \bar{g} \in \mathcal{U}^H_\alpha$ and $z \in \Delta$, $|z| = r$. Consequently, $\mathcal{U}^H_{\alpha, c} \subset \mathcal{U}^H_\alpha$ is uniformly bounded on compact subsets of $\Delta$. According to the compactness principle, there exists a subsequence of $f_n$ (let us save the notation) which converges locally uniformly in $\Delta$ to a harmonic function $f_0$. Let us show that $f_0 \in \mathcal{U}^H_{\alpha, c}$.

For $f \in \mathcal{U}^H_\alpha$ the following inequality holds (see [10])

$$\frac{(1 - r)^{2\alpha - 2}}{(1 + r)^{2\alpha + 2}} \leq J_f(z) \leq \frac{(1 + r)^{2\alpha - 2}}{(1 - r)^{2\alpha + 2}}, \quad |z| = r.$$ 

Therefore for $f_n \in \mathcal{U}^H_{\alpha, c}$ we have

$$J_{f_n}(z) \geq \frac{(1 - r)^{2\alpha - 2}}{(1 + r)^{2\alpha + 2}}(1 - c^2) > 0.$$ 

This implies $J_{f_0}(z) > 0$ for all $z \in \Delta$. This means that the harmonic in $\Delta$ function $f_0$ is sense-preserving.

Next, we prove that $\text{ord} f_0 \leq \alpha$. Suppose not. Then, we may let $\text{ord} f_0 = \beta > \alpha$. Then, by the definition of the strong order, there exist a conformal automorphism $\varphi(z) = \frac{z + a}{1 + \bar{a}z}$ of $\Delta$ and $\theta \in \mathbb{R}$ such that for harmonic function

$$e^{-i\theta}(f_0)_a(ze^{i\theta}) = \frac{f_0(\varphi(ze^{i\theta})) - f_0(\varphi(0))}{h'_0(\varphi(0))\varphi'(0)e^{i\theta}} = \sum_{k=1}^{\infty} (A_k z^k + A_{-k} \bar{z}^k),$$

$(A_1 = 1, \ f_0 = h_0 + \bar{g}_0)$ the inequality

$$\frac{|A_2 - A_{-1}A_{-2}|}{1 - |A_{-1}|^2} > \alpha + \frac{\beta - \alpha}{2} \quad (4)$$

is valid.

For the automorphism $\varphi$ and the number $\theta$ denote

$$e^{-i\theta}(f_n)_a(ze^{i\theta}) = \sum_{k=1}^{\infty} (A_k^{(n)} z^k + A_{-k}^{(n)} \bar{z}^k), \quad (A_1^{(n)} = 1).$$
From locally uniform convergence of $f_n$ to $f_0$, the Weierstrass theorem on series of analytic functions, and inequality [4] it follows that for sufficiently large $n > N$

$$\frac{|A_2^{(n)} - A_{-1}^{(n)}A_{-2}^{(n)}|}{1 - |A_{-1}^{(n)}|^2} > \alpha + \frac{\beta - \alpha}{2}.$$ 

Hence if $n > N$ we have $\text{ord}_f f_n > \alpha + \frac{\beta - \alpha}{2}$ and $f_n \notin U_{\alpha,c}$. This contradiction proves the theorem. □

In claim c) of Theorem B and claim e) of Theorem C some set of functions with the Hayman number $\delta^0 = 1$ (or $\delta_0 = 1$ for the theorem of decrease regularity) is described. These claims differ from the analytic case. In the analytic case $\delta^0 = 1$ and $\delta_0 = 1$ only for the functions $e^{i\theta}k_\alpha(ze^{-i\theta})$, where $\theta \in \mathbb{R}$, $k_\alpha(z)$ is the function defined by [3], [7], [13], [14]. The following example shows that in the harmonic case this set has more complicated structure. We construct the example of functions $f$ of arbitrary strong order $\beta \geq 3/2$ with $\delta_0 = 1$. These functions are not equal to the function $q_\theta(z)$ from Theorem B. We use the Clunie and Sheil-Small shear construction [15] (see also [16, ch. 3.4]) to give our example. Let us note that our construction is not stable. As one can show, if we multiply the coanalytic part $g$ of the function from our example by constant $k \in (0, 1)$, then the strong order of the function changes stepwise and $\delta_0 \neq 1$ for this function.

**Example.** Put $h'(z) = \frac{(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}}$, $g'(z) = zh'(z)$, $z \in \Delta$. Let $\alpha \in [1, \infty)$ be fixed. If $\varphi(z) = \frac{z+a}{1+az}$, $a \in \Delta$, is an automorphism of $\Delta$, then for $f = h + \tilde{g}$ we have

$$f_a(z) =: F(z) = H(z) + \overline{G(z)} = \frac{h(\varphi(z)) - h(\varphi(0))}{h'(\varphi(0))\varphi'(0)} + \left(\frac{g(\varphi(z)) - g(\varphi(0))}{h'(\varphi(0))\varphi'(0)}\right),$$

where $H$ and $G$ are functions analytic in $\Delta$,

$$H'(z) = \frac{h'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)}$$

and

$$G'(z) = \frac{g'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)} = \frac{\varphi(z)h'(\varphi(z))\varphi'(z)}{h'(\varphi(0))\varphi'(0)}.$$
Note that

\[ J_F(z) = |H'(z)|^2 - |G'(z)|^2 = \frac{|h'(\varphi(z))|^2|\varphi'(z)|^2(1 - |\varphi(z)|^2)}{|h'(\varphi(0))|^2|\varphi'(0)|^2}, \]

and, in particular,

\[ J_F(0) = 1 - |\varphi(0)|^2. \]

Therefore,

\[
\frac{J_F(z)}{J_F(0)} = \left| 1 + \frac{z + a}{\overline{1+\overline{a}z}} \right|^{2\alpha-2} \cdot \left| 1 - \frac{a}{\overline{1+\overline{a}z}} \right|^{2\alpha+4} \cdot \frac{(1 - |a|^2)^2}{|1 + \overline{a}z|^4} \times \]

\[
\left( 1 - \left| \frac{z + a}{\overline{1+\overline{a}z}} \right|^2 \right) \frac{1}{(1 - |a|^2)^3} = \]

\[
\left| \frac{1 + \overline{1+\overline{a}z}}{1 - \overline{z}} \right|^{2\alpha-2} \cdot \left| 1 + \overline{a}z \right|^2 - \left| z + a \right|^2 = \left| \frac{1 + \overline{1+\overline{a}z}}{1 - \overline{z}} \right|^{2\alpha+4} (1 - |z|^2), \]

by generalized Schwarz’s lemma. Consequently, for \( r \in (0, 1) \)

\[
\sup_{a \in \Delta, \ |z| = r} \frac{J_F(z)}{J_F(0)} = \frac{(1 + r)^{2\alpha-1}}{(1 - r)^{2\alpha+3}}. \]

Therefore for \( \beta = \alpha + \frac{1}{2} \), all \( a \in \Delta \), and \( |z| = r \) we get

\[
\frac{J_F(z)}{J_F(0)} \leq \frac{(1 + r)^{2\beta-2}}{(1 - r)^{2\beta+2}}. \tag{5} \]

In \[10\] it was shown that for functions \( f \) harmonic and sense-preserving in \( \Delta \),

\[
\overline{\text{ord}} f = \inf \left\{ \beta : \frac{J_F(z)}{J_F(0)} \leq \frac{(1 + |z|)^{2\beta-2}}{(1 - |z|)^{2\beta+2}}, \ \forall F = f_a, \forall z \in \Delta \right\}. \tag{6} \]

From (5) and (6) we conclude that \( \overline{\text{ord}} f \leq \beta = \alpha + \frac{1}{2} \). From Theorem B it follows that if for a function \( f \) harmonic and sense-preserving in \( \Delta \)

\[
\lim_{r \to 1^-} \left[ \frac{J_f(z)}{J_f(0)} \left( \frac{1 - r}{1 + r} \right)^{2\beta+2} \right] > 0, \tag{7} \]
then $\text{ord} f \geq \beta$. For the considered function $f$ the limit in \[7\] equals 1. Therefore, $\text{ord} f = \beta$ and

$$\delta^0 = \lim_{r \to 1^-} \left[ \frac{J_f(r)}{J_f(0)} \frac{(1-r)^{2\beta+2}}{(1+r)^{2\beta-2}} \right] = 1.$$  

It is interesting to find out if there exist functions with $\delta^0 = 1$ which are not equal to the function from the example and the functions $q_\theta(z)$.

Definition 4. A direction of intensive growth (d.i.g.) of a function $f(z)$ is a constant $\varphi \in [0; 2\pi)$ such that

$$\lim_{r \to 1^-} \left[ \frac{J_f(re^{i\varphi})}{J_f(0)} \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \right] = \delta(f, \varphi) > 0.$$  

A direction of intensive decrease (d.i.d) of a function $f(z)$ is a constant $\varphi \in [0; 2\pi)$ such that

$$\lim_{r \to 1^-} \left[ \frac{J_f(re^{i\varphi})}{J_f(0)} \frac{(1+r)^{2\alpha+2}}{(1-r)^{2\alpha-2}} \right] = \delta'(f, \varphi) < \infty.$$  

Since we study LIFs, it is important to know how d.i.g.-’s and d.i.d.-’s of a function $f(z)$ are changed under the transformation $e^{-i\theta} f_a(ze^{i\theta})$. The case $a = 0$ is trivial: a d.i.g. (d.i.d.) $\varphi - \theta$ of the function $e^{-i\theta} f(ze^{i\theta})$ corresponds to the d.i.g. (d.i.d.) $\varphi$ of $f(z)$. In this situation $\delta(f(z), \varphi) = \delta(f(ze^{i\theta}), \varphi - \theta)$ (and $\delta'(f(z), \varphi) = \delta'(f(ze^{i\theta}), \varphi - \theta)$). It is also interesting to find out the relationship between the Hayman numbers of the functions $f$ and $f_a$ in general case. The following theorem concerns the non-obvious case $a \neq 0$.

Theorem 2. Let $f \in U^H_\alpha$. Denote

$$R(r) = \left| \frac{re^{i\varphi} + a}{1 + \overline{a}re^{i\varphi}} \right|, \quad \gamma(r) = \arg \frac{re^{i\varphi} + a}{1 + \overline{a}re^{i\varphi}}, \quad a \in \Delta, \quad re^{i\varphi} \neq -a.$$  

1) $\varphi$ is a d.i.g. (d.i.d.) of the function $f_a(z)$ iff $\gamma$ is a d.i.g. (d.i.d.) of $f(z)$ and

$$e^{i\varphi} = \frac{e^{i\gamma} - a}{1 - \overline{a}e^{i\gamma}}.$$  

(8)
2) for all $\gamma \in [0, 2\pi)$

$$
\lim_{r \to 1^-} \left[ \frac{J_f(re^{i\gamma})}{J_f(0)} \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \right] = \lim_{r \to 1^-} \left[ \frac{J_f(R(r)e^{i\gamma(r)})}{J_f(0)} \frac{(1 - R(r))^{2\alpha + 2}}{(1 + R(r))^{2\alpha - 2}} \right],
$$

and

$$
\lim_{r \to 1^-} \left[ \frac{J_f(re^{i\gamma})}{J_f(0)} \frac{(1 + r)^{2\alpha + 2}}{(1 - r)^{2\alpha - 2}} \right] = \lim_{r \to 1^-} \left[ \frac{J_f(R(r)e^{i\gamma(r)})}{J_f(0)} \frac{(1 + R(r))^{2\alpha + 2}}{(1 - R(r))^{2\alpha - 2}} \right].
$$

Here $\varphi$ and $\gamma$ are connected by (8).

3) if $\varphi$ is a d.i.g. of $f_a(z)$, $\gamma$ is a d.i.g. of $f(z)$, and $\varphi$ is connected with $\gamma$ by (8), then

$$
\delta(f, \gamma) = \delta(f_a, \varphi) \frac{J_f(a)}{J_f(0)} \frac{1 - |a|^2}{|1 + \alpha e^{i\varphi}|^4};
$$

if $\varphi$ is a d.i.d. of $f_a(z)$, $\gamma$ is a d.i.d. of $f(z)$, and $\varphi$ is connected with $\gamma$ by (8), then

$$
\delta'(f, \gamma) = \delta'(f_a, \varphi) \frac{J_f(a)}{J_f(0)} \frac{|1 + \alpha e^{i\varphi}|^4}{|1 + \alpha z|^4};
$$

Proof. 1) Let $\varphi$ be a d.i.g. of $f_a(z)$. This means that there exists the limit

$$
\delta(f_a, \varphi) = \lim_{r \to 1^-} \left[ \frac{J_{f_a}(re^{i\varphi})}{J_{f_a}(0)} \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \right] > 0.
$$

Note that

$$
J_{f_a}(z) = \frac{J_f \left( \frac{z + a}{1 + \alpha z} \right)}{|h'(a)|^2|1 + \alpha z|^4}, \quad (9)
$$

and

$$
J_{f_a}(0) = \frac{J_f(a)}{|h'(a)|^2}. \quad (10)
$$

Let us calculate the following limit, using (9) and (10),

$$
\delta \overset{\text{def}}{=} \lim_{r \to 1^-} \left[ \frac{J_f(R(r)e^{i\gamma(r)})}{J_f(0)} \frac{(1 - R(r))^{2\alpha + 2}}{(1 + R(r))^{2\alpha - 2}} \right] =
$$

$$
= \lim_{r \to 1^-} \left[ \frac{J_{f_a}(re^{i\varphi})}{J_{f_a}(0)} |h'(a)|^2|1 + \alpha e^{i\varphi}|^4 \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \left( \frac{1 - R(r)}{1 - r} \right)^{2\alpha + 2} \right].
$$
We have
\[
\lim_{r \to 1^-} \frac{1 - R(r)}{1 - r} = \lim_{r \to 1^-} R'(r) = \frac{1 - |a|^2}{|1 + \overline{a}e^{i\varphi}|^2}.
\] (11)

Using (11), we obtain
\[
\delta = \delta(f, \varphi) \frac{J_f(a)}{J_f(0)} |1 + \overline{a}e^{i\varphi}|^4 \left( \frac{1 - |a|^2}{|1 + \overline{a}e^{i\varphi}|^2} \right)^{2\alpha + 2} > 0.
\] (12)

By (11), \( r \to 1^- \), therefore the function \( R(r) \) increases on an interval \((r_0, 1)\). By Theorem B, for \( r_0 < r < r_1 < 1 \)
\[
\frac{J_f(R(r_1)e^{i\gamma(r_1)})}{J_f(0)} (1 - R(r_1))^{2\alpha + 2} \leq \frac{J_f(R(r)e^{i\gamma(r_1)})}{J_f(0)} (1 - R(r))^{2\alpha + 2} \frac{(1 + R(r_1))^{2\alpha - 2}}{(1 + R(r))^{2\alpha - 2}}.
\]

Passing to the limit as \( r_1 \to 1^- \) and using (8), we get
\[
\delta \leq \frac{J_f(R(r)e^{i\gamma})}{J_f(0)} (1 - R(r))^{2\alpha + 2} \frac{(1 + R(r_1))^{2\alpha - 2}}{(1 + R(r))^{2\alpha - 2}}.
\]

Thus,
\[
\delta(f, \gamma) = \lim_{r \to 1^-} \left[ \frac{J_f(R(r)e^{i\gamma})}{J_f(0)} (1 - R(r))^{2\alpha + 2} \frac{(1 + R(r_1))^{2\alpha - 2}}{(1 + R(r))^{2\alpha - 2}} \right] \geq \delta.
\] (13)

Taking into account (12), we conclude that \( \gamma \) is a d.i.g. of \( f(z) \).

Now let us consider the sets
\[
A = \{e^{i\gamma} : \gamma \text{ is a d.i.g. of } f(z)\},
\]
\[
B = \left\{ \frac{e^{i\varphi} + a}{1 + \overline{a}e^{i\varphi}} : \varphi \text{ is a d.i.g. of } f_a(z) \right\},
\]
\[
C = \{e^{i\eta} : \eta \text{ is a d.i.g. of } [f_a]_{(-a)}(z)\}.
\]

Here \([f_a]_{(-a)}(z)\) is the transformation (2) of the function \( f_a \) with the parameter \(-a\). If \( \eta \) is a d.i.g. of \([f_a]_{(-a)}(z)\), then, as it was proved above,
\[
e^{i\eta} = \frac{e^{i\varphi} + a}{1 + \overline{a}e^{i\varphi}},
\]
where \( \varphi \) is a d.i.g. of \( f_a(z) \). This implies that \( C \subset B \). Let \( \varphi \) be a d.i.g. of \( f_a(z) \). Then
\[
e^{i\varphi} = \frac{e^{i\varphi} + a}{1 + \overline{a}e^{i\varphi}},
\]
where \( \gamma \) is a d.i.g. of \( f(z) \). Thus \( B \subset A \). Since \([f_a](-a)(z) = f(z)\), we have \( A = C \) and, consequently, \( A = B \). This completes the proof of the statement about d.i.g.-’s.

The statement about d.i.d.-’s is proved analogously.

2) Let us prove the first equality. If \( \gamma \) is not a d.i.g. of \( f(z) \), then

\[
\lim_{r \to 1^-} \left[ \frac{J_f(re^{i\gamma})}{J_f(0)} \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \right] = 0.
\]

Thus, by (13),

\[
\delta \leq \lim_{r \to 1^-} \left[ \frac{J_f(R(r)e^{i\gamma})}{J_f(0)} \frac{(1 - R(r))^{2\alpha + 2}}{(1 + R(r))^{2\alpha - 2}} \right] = 0.
\]

This implies \( \delta = 0 \).

Now let us consider the case when \( \gamma \) is a d.i.g. of \( f(z) \). We have proved above that \( \delta(f, \gamma) \geq \delta \) (see (13)). It remains to show that \( \delta(f, \gamma) \leq \delta \).

Denote

\[
R_1(r) = \left| \frac{re^{i\gamma} - a}{1 - \overline{a}re^{i\gamma}} \right|.
\]

Since \([f_a](-a)(z) = f(z)\), \( \gamma \) is a d.i.g. of \([f_a](-a)(z)\), i.e.

\[
\delta([f_a](-a), \gamma) = \delta(f, \gamma) = \lim_{r \to 1^-} \left[ \frac{J_{[f_a](-a)}(re^{i\gamma})}{J_f(0)} \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \right] > 0.
\]

Arguing as in the proof of claim 1), one can note that there exists

\[
\delta^* \overset{\text{def}}{=} \lim_{r \to 1^-} \left[ \frac{J_{[f_a]}(re^{i\varphi} - a)}{J_{[f_a]}(0)} \frac{(1 - R_1(r))^{2\alpha + 2}}{(1 + R_1(r))^{2\alpha - 2}} \right] = 0.
\]

Apply (13) to the function \( f_a(z) \), using (9), (10), and (11):

\[
\delta^* \leq \lim_{r \to 1^-} \left[ \frac{J_{f_a}(re^{i\varphi})}{J_{f_a}(0)} \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \right] = \lim_{r \to 1^-} \left[ \frac{J_f(re^{i\varphi} + a)}{J_f(a)|1 + \overline{a}re^{i\varphi}|^4} \frac{(1 - R(r))^{2\alpha + 2}}{(1 + R(r))^{2\alpha - 2}} \right] \cdot \lim_{r \to 1^-} \left( \frac{1 - r}{1 - R(r)} \right)^{2\alpha + 2} \]
\[
\frac{\delta J_f(0)}{J_f(a)|1 + \bar{a}e^{i\varphi}|^4} \left( \frac{|1 + \bar{a}e^{i\varphi}|^2}{1 - |a|^2} \right)^{2\alpha + 2} = \frac{\delta J_f(0)}{J_f(a)} \frac{|1 + \bar{a}e^{i\varphi}|^{4\alpha}}{(1 - |a|^2)^{2\alpha + 2}}. \tag{14}
\]

On the other hand, by (9),
\[
J_f \left( \frac{z - a}{1 - \bar{a}z} \right) = \frac{J_f(z)}{|h'(a)|^2 \left| 1 + \bar{a} \frac{z - a}{1 - \bar{a}z} \right|^4}.
\]

Thus, using (8), (10), and (11), we can write \(\delta^*\) in the form
\[
\delta^* = \lim_{r \to 1^-} \left[ \frac{J_f(\rho e^{i\gamma})}{J_f(a)} \left| 1 + \bar{a} \frac{\rho e^{i\gamma} - a}{1 - \bar{a}e^{i\gamma}} \right|^4 \frac{(1 - r)^{2\alpha + 2}}{(1 + r)^{2\alpha - 2}} \right] \times \\
\times \lim_{r \to 1^-} \left( \frac{1 - R_1(r)}{1 - r} \right)^{2\alpha + 2} = \\
= \delta(f, \gamma) \frac{J_f(0)}{J_f(a)|1 + \bar{a}e^{i\varphi}|^4} \left( \frac{1 - |a|^2}{|1 - \bar{a}e^{i\gamma}|^2} \right)^{2\alpha + 2} = \\
= \delta(f, \gamma) \frac{J_f(0)}{J_f(a)} \frac{|1 + \bar{a}e^{i\varphi}|^{4\alpha}}{(1 - |a|^2)^{2\alpha + 2}}.
\]

Substituting
\[
\delta^* = \delta(f, \gamma) \frac{J_f(0)}{J_f(a)} \frac{|1 + \bar{a}e^{i\varphi}|^{4\alpha}}{(1 - |a|^2)^{2\alpha + 2}}
\]
in (14), we get \(\delta(f, \gamma) \leq \delta\). Therefore, \(\delta(f, \gamma) = \delta\).

The second equality of claim 2) is proved analogously.

3) The formula, connected \(\delta(f, \gamma)\) and \(\delta(f_a, \varphi)\) is obtained from (12), using \(\delta = \delta(f, \gamma)\).

The second equality is proved analogously. \(\square\)

Theorem 2 implies the following

Remark. Let \(f \in \mathcal{U}^H_\alpha\). For every \(\varphi \in [0; 2\pi)\) there exist \(\delta(f, \varphi) \in [0; 1]\) and \(\delta'(f, \varphi) \in [1; \infty]\) such that for any circle or straight line \(\Gamma \subset \Delta\), orthogonal to \(\partial \Delta\) at the point \(e^{i\varphi}\), we have
\[
\lim_{\Gamma \ni z \to e^{i\varphi}} \left[ \frac{J_f(z) (1 - |z|)^{2\alpha + 2}}{J_f(0) (1 + |z|)^{2\alpha - 2}} \right] = \delta(f, \varphi),
\]
By Theorem B, for any \( a \in [0; 2\pi) \) there exists

\[
\Delta \ni \phi \quad \text{such that} \quad f(z)(1 + |z|)^{2\alpha+2} [J_f(0)/(1 - |z|)^{2\alpha-2}] = \phi.
\]

From Theorem C.

Theorem 3. 1) If \( f \in U^H_\alpha(\delta_0) \), \( \delta_0 \in (0; 1) \), then for every \( \delta \in [\delta_0, 1) \) there exists \( a \in \Delta \) such that \( f_a(z) \in U^H_\alpha(\delta) \).

2) If \( f \in U^H_\alpha(\delta_0) \), \( \delta_0 \in (1; \infty) \), then for every \( \delta' \in (1, \delta_0] \) there exists \( a \in \Delta \) such that \( f_a(z) \in U^H_\alpha(\delta') \).

Proof. By Theorem B, for any \( \varphi \in [0; 2\pi) \) there exists

\[
\lim_{r \to 1-} \left[ \frac{J_f(re^{i\varphi})}{J_f(0)} \right] (1 - r)^{2\alpha+2} = \delta(f, \varphi).
\]

Let us fix \( a \in \Delta \) \( \varphi \in [0; 2\pi) \). Denote \( z = \frac{re^{i\varphi} - a}{1 - ae^{i\varphi}}, |z| = R(r) \) and consider the limit

\[
\delta^*(\varphi) \overset{\text{def}}{=} \lim_{r \to 1-} \left[ \frac{J_f_a(z)}{J_f_a(0)} \right] (1 - R(r))^{2\alpha+2}.
\]

Let us calculate \( \delta^*(\varphi) \), using (9) and (10)

\[
\delta^*(\varphi) = \lim_{r \to 1-} \left[ \frac{J_f(re^{i\varphi})}{J_f(a)} \right] \left| 1 + \frac{re^{i\varphi} - a}{1 - ae^{i\varphi}} \right|^{2\alpha+2} = \frac{J_f(0)(1 - r)^{2\alpha+2}}{J_f(0)(1 + r)^{2\alpha-2}} \left( \frac{1 - R(r)}{1 - r} \right)^{2\alpha+2}.
\]

By (11),

\[
\delta^*(\varphi) = \delta(f, \varphi) \frac{J_f(0)(1 - |a|^2)^{2\alpha+2}}{|1 - ae^{i\varphi}|^{4\alpha+4}} \frac{|1 - ae^{i\varphi}|^4}{(1 - |a|^2)^4} = \frac{J_f(0)(1 - |a|^2)^{2\alpha-2}}{|1 - ae^{i\varphi}|^{4\alpha}} \leq \delta(f, \varphi).
\]
Then, using (11) for $R$

\[ \lim_{R(r) \to 1} - \left[ \frac{M(R(r), J_{f_a}) (1 - R(r))^{2\alpha + 2}}{J_{f_a}(0)} (1 + R(r))^{2\alpha - 2} \right] \text{def} = \delta_a. \]

Let $\varphi$ be equal to d.m.g. $\varphi^0$ of $f(z)$ and $a = re^{i\varphi^0}$. Then $\delta(f, \varphi) = \delta^0$ and

\[ \delta^0 \frac{J_f(0)}{J_f(re^{i\varphi^0})} \left( \frac{1 - \rho^2}{1 - \rho} \right)^{2\alpha - 2} = \delta^0 \frac{J_f(0)}{J_f(re^{i\varphi^0})} \left( 1 + \rho \right)^{2\alpha - 2} \leq \delta_a. \tag{15} \]

By Theorem B, there exists a d.m.g. $\varphi_1 \in [0; 2\pi)$ of $f_a(z)$ such that

\[ \delta_a = \lim_{r \to 1^-} \left[ \frac{J_{f_a}(re^{i\varphi^0}) (1 - r)^{2\alpha + 2}}{J_{f_a}(0) (1 + r)^{2\alpha - 2}} \right] = \]

\[ = \lim_{r \to 1^-} \left[ \frac{J_f \left( \frac{re^{i\varphi^0} + a}{1 + are^{i\varphi^0}} \right) (1 - r)^{2\alpha + 2}}{J_f(a) |1 + are^{i\varphi^0}|^4 (1 + r)^{2\alpha - 2}} \right]. \]

Denote $R_1(r)e^{i\gamma_1(r)} = \frac{re^{i\varphi^0} + a}{1 + are^{i\varphi^0}}$, where $\gamma_1(r)$ is a real-valued function. Then, using (11) for $R(r) = R_1(r)$, we obtain

\[ \delta_a \leq \lim_{r \to 1^-} \left[ \frac{M(R_1(r), J_f) (1 - r)^{2\alpha + 2}}{J_f(a) |1 + are^{i\varphi^0}|^4 (1 + r)^{2\alpha - 2}} \right] = \]

\[ = \lim_{r \to 1^-} \left[ \frac{M(R_1(r), J_f) (1 - R_1(r))^{2\alpha + 2}}{J_f(0) (1 + R_1(r))^{2\alpha - 2}} \right] \times \]

\[ \times \frac{J_f(0)}{J_f(a) |1 + are^{i\varphi^0}|^4} \cdot \lim_{r \to 1^-} \left( \frac{1 - r}{1 - R_1(r)} \right)^{2\alpha + 2} = \]

\[ = \delta^0 \frac{J_f(0)}{J_f(a) |1 + are^{i\varphi^0}|^4} \left( \frac{|1 + are^{i\varphi^0}|^2}{1 - |a|^2} \right)^{2\alpha + 2} \leq \delta^0 \frac{J_f(0) |1 + are^{i\varphi^0}|^{4\alpha}}{J_f(a) (1 - |a|^2)^{2\alpha + 2}} \leq \]

\[ \leq \delta^0 \frac{J_f(0) (1 + \rho)^{4\alpha}}{J_f(a) (1 - \rho^2)^{2\alpha + 2}} = \delta^0 \frac{J_f(0) (1 + \rho)^{2\alpha - 2}}{J_f(a) (1 - \rho)^{2\alpha + 2}}. \]

Taking into account inequality (15), we get

\[ \delta^0 \frac{J_f(0) (1 + \rho)^{2\alpha - 2}}{J_f(re^{i\varphi^0}) (1 - \rho)^{2\alpha + 2}} = \delta_a. \]
Since the continuous function \( \frac{J_f(0)}{J_f(\rho e^{i\phi})} \frac{(1+\rho)^{2\alpha-2}}{(1-\rho)^{2\alpha+2}} \) decreases on \( \rho \), equals 1 as \( \rho = 0 \), and tends to zero as \( \rho \to 1^- \), then we can find \( \rho \in [0; 1) \) such that \( \delta_a \) takes preassigned value from \([\delta^0; 1)\).

Claim 2 of the theorem is proved analogously. □

In [7] (see also [17], [14]) it was proved that the set of all d.i.g.-’s and d.i.d.-’s of a given analytic function is at most countable. The following theorem shows that this statement is true for set of d.i.g.-’s of harmonic function too. But we don’t know whether this fact is true for set of d.i.d.-’s.

**Theorem 4.** Let \( f \in \mathcal{U}^H_{\alpha} \). Then the set of all d.i.g.-’s of \( f \) is at most countable.

**Proof.** If \( f = h + \bar{g} \in \mathcal{U}^H_{\alpha} \), then \( \overline{\text{ord}} \ h \leq \alpha \). Since

\[
J_f(z) = |h'(z)|^2 - |g'(z)|^2 \leq |h'(z)|^2
\]

for all \( z \in \Delta \), then for \( \varphi \in [0, 2\pi) \) and \( r \in [0, 1) \)

\[
\frac{J_f(re^{i\varphi})}{J_f(0)} \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \leq \left[ |h'(re^{i\varphi})| \frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}} \right]^2 \frac{1}{J_f(0)}, \tag{16}
\]

By Theorem B and theorem of growth regularity from [7], there exist the limits

\[
\delta(f, \varphi) = \lim_{r \to 1^-} \left[ \frac{J_f(re^{i\varphi})}{J_f(0)} \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \right],
\]

and

\[
\tilde{\delta}(h, \varphi) = \lim_{r \to 1^-} \left[ |h'(re^{i\varphi})| \frac{(1-r)^{2\alpha+2}}{(1+r)^{2\alpha-2}} \right];
\]

From (16) we get \( \delta(f, \varphi) \leq \frac{\tilde{\delta}^2(h, \varphi)}{J_f(0)} \). If \( \varphi \) is a d.i.g. of \( f \), then \( \delta(f, \varphi) > 0 \). Consequently, \( \tilde{\delta}(h, \varphi) > 0 \) and \( \varphi \) is a d.i.g. of \( h \). Therefore the set \( V \) of all d.i.g.-’s of \( f \) is contained in the set \( W \) of all d.i.g.-’s of \( h \). As it was proved in [7], \( W \) is at most countable. Hence \( V \) is at most countable too. □

**Acknowledgment.** This work was supported by RFBR (projects N 14-01-00510f, N 14-01-92692). The authors thank S. Yu. Graf and S. Ponnusamy for valuable comments on improving the paper.
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Received May 14, 2015.
In revised form, September 3, 2015.

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