Special values of Gauss’s hypergeometric series derived from Appell’s series $F_1$ with closed forms

Akihito Ebisu

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Abstract

In a previous work ([Eb]), the author proposed a method employing contiguity relations to derive hypergeometric series in closed form. In [Eb], this method was used to derive Gauss’s hypergeometric series $2F_1$ possessing closed forms. Here, we consider the application of this method to Appell’s hypergeometric series $F_1$ and derive several $F_1$ possessing closed forms. Moreover, analyzing these $F_1$, we obtain values of $2F_1$ with no free parameters. Some of these results provide new examples of algebraic values of $2F_1$.

Key Words and Phrases: Gauss’s hypergeometric series, algebraic value, Appell’s hypergeometric series, hypergeometric identity.

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1 Introduction

The study of special values of Gauss’s hypergeometric series

$$2F_1\left(\frac{a, b}{c}; x\right) := \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} x^n,$$

where $(a, n) := \alpha(a+1) \cdots (a+n-1)$, has a long history. The oldest and most well-known formula is

$$2F_1\left(\frac{a, b}{c}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (1.1)$$

where $\Re(c-a-b) > 0$. The formula (1.1), due to Gauss, is called “Gauss’s summation formula”. Since this formula was derived by Gauss, many other identities for Gauss’s hypergeometric series have been obtained by many other people. For example, we have

$$2F_1\left(\frac{2a, 2b}{a+b+1/2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)}$$

(see formula (50) in Section 2.8 of [Erd1]),

$$2F_1\left(\frac{1/2, -a}{2a+5/2}; \frac{1}{4}\right) = \frac{1}{3 \cdot 2^{2a}} \frac{\sqrt{\pi}\Gamma(2a+5/2)}{\Gamma(a+3/2)^2} \quad (1.2)$$

(see formula (1/4.2) in [Gos] and formula (1,3,3-1)(xvi) in [Eb]). There are many other known identities for $2F_1$ similar to the above, containing one or more free parameters. These identities can be found by demonstrating that the corresponding hypergeometric
series possess closed forms. For instance, the formula (1.2) is derived by showing that
\[ F(a) := 2F_1(1/2, -a; 2a + 5/2; 1/4) \]
has a closed form, that is, that \( F(a) \) satisfies the closed-form relation

\[ \frac{F(a+1)}{F(a)} = \frac{(2a + 5/2, 2)}{2^2(a + 3/2)^2}. \] (1.3)

The relation (1.3) can be obtained by employing Gosper’s algorithm, the W-Z method, Zeilberger’s algorithm (see [Ko] and [PWZ]), and the method of contiguity relations, which was recently introduced in [EB]. Thus, using these methods, we are able to find numerous identities for \( 2F_1 \) with one or more free parameters. Most of the identities that can be derived with these methods are listed in [EB].

There are also many known identities for \( 2F_1 \) with no free parameters. The following are some examples:

\[ 2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{80}{81}\right) = \frac{9}{5} \] (1.4)

(see formula (1.6) in [JZ2]),

\[ 2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1323}{1331}\right) = \frac{3\sqrt{11}}{4} \] (1.5)

(see Theorem 3 in [BW]). Unfortunately, it seems that such identities can not be obtained by direct application of the above methods. Indeed, if we could find (1.4) directly with one of the above methods, then there must exist \( p, q, r \in \mathbb{Q} \) satisfying

\[ \frac{F(a+1)}{F(a)} \in \mathbb{Q}(a), \]

where

\[ F(a) := 2F_1\left(\frac{pa + 1/4, qa + 1/2}{ra + 3/4}; \frac{80}{81}\right). \]

However, no such parameters \( p, q, r \) have yet been identified. The same holds for other identities, including (1.5). To obtain these formulae, other methods have been used, including methods employing modular forms and elliptic integrals (see [Ar], [BG], [BW], [JZ1], [JZ2], [JZ3]).

As another approach, if we could find Appell’s hypergeometric series

\[ F_1\left(\alpha; \beta_1, \beta_2; x, y\right) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha, m+n)(\beta_1, m)(\beta_2, n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n \]
in closed forms, then it may be possible to obtain identities for \( 2F_1 \) with no free parameters by analyzing the corresponding closed-form relations for \( F_1 \). Below, we consider two examples.

As the First example, we consider the following (see Example 1 in Section 3.1 for details). Using the method of contiguity relations, which is effective for deriving hypergeometric series in closed form, we find the closed-form relation

\[ \frac{F(a+1)}{F(a)} = \frac{3^8(2a + 1/2, 2)}{2^2 5^5 (a + 1/2)^2}, \] (1.6)
where

\[ F(a) := F_1\left(\frac{2a; a + 1/2, 4a - 1}{2a + 1/2}; \frac{1}{81}, \frac{1}{6}\right). \]

From (1.6), we obtain

\[
\frac{F(a + n)}{F(a)} = \frac{3^{8n}}{2^{2n} 5^{5n}} \frac{(2a + 1/2, 2n)}{(a + 1/2, n)^2},
\]

and

\[
F_1\left(\frac{2a; a + 1/2, 4a - 1}{2a + 1/2}; \frac{1}{81}, \frac{1}{6}\right) = F(a) = \frac{3^{8a}}{2^{2a} 5^{5a}} \frac{\sqrt{\pi} \Gamma(2a + 1/2)}{\Gamma(a + 1/2)^2}.
\]

Then, from the relation

\[
F_1\left(\alpha; \beta_1, 0; \gamma, x, y\right) = 2F_1\left(\alpha; \beta_1; \gamma; x\right),
\]

which follows from the definition of Appell’s hypergeometric series \( F_1 \), we have

\[
2F_1\left(\frac{1/2, 3/4}{1}; \frac{1}{81}\right) = \frac{9}{100} \frac{\sqrt{2} 5^{3/4} \Gamma(1/4)^2}{\pi^{3/2}},
\]

by substituting \( a = 1/4 \) into (1.8).

As the second example, we consider the following (see Example 2 in Section 3.2 for details). From the method of contiguity relations, we are also able to derive the closed-form relation

\[
\frac{F(a + 1)}{F(a)} = \frac{3^4}{5^4}
\]

and, from this,

\[
F(a) = \frac{5^{4n}}{3^{4n}} F(a + n),
\]

where

\[
F(a) := F_1\left(a; 2a, 1 - 4a; \frac{80}{81}, \frac{16}{15}\right).
\]

It is known that, in general, \( F_1(\alpha; \beta_1, \beta_2; \gamma; x, y) \) is absolutely convergent when \(|x| < 1\) and \(|y| < 1\), and divergent when \(|x| > 1\) or \(|y| > 1\) (for example, see Section 9.1 in [Ba]). Therefore, \( F(a) \) is not meaningful for a parameter \( a \) with unrestricted value. However, \( F(a) \) is meaningful if we restrict \( a \) to values satisfying \( a = 1/4 + n \), where \( n \) is any non-negative integer. Thus, the relation (1.12) with \( a = 1/4 \) is meaningful. Then, investigating the asymptotic behavior of the right-hand side of this relation by taking the limit \( n \to +\infty \), we are able to deduce (1.14). In this way, analyzing closed-form relations for \( F_1 \), we can derive values of \( 2F_1 \) with no free parameters.

In this article, making use of the method of contiguity relations, we derive several \( F_1 \) in closed forms. These are listed in Table 1. Moreover, in the cases that these \( F_1 \) are convergent, we evaluate them with (1.8). These hypergeometric identities for \( F_1 \) are tabulated in Table 2. In addition, analyzing the closed-form relations for \( F_1 \) listed in Table 1, we derive values of \( 2F_1 \) with no free parameters. These values are listed in Table 3.
Tables 4 and 5, we present several complicated identities obtained by applying algebraic transformations to the identities listed in Table 3. As seen from these, we are able to obtain several new identities using our approach. In particular, $(B''\cdot3)$, $(C''\cdot1)$, $(C''\cdot4)$ and $(C''\cdot1)$ provide new examples of algebraic values of Gauss’s hypergeometric series. However, it seems that we are not able to derive the beautiful formula (1.5) obtained by Beukers and Wolfart with our approach.

Remark 1.1. In [Si], Siegel posed the problem of determining the nature of the following set:

$$E(a, b, c) := \{ x \in \bar{\mathbb{Q}} \; ; \; _2F_1(a, b; c; x) \in \bar{\mathbb{Q}} \}.$$  

This set is a so-called “exceptional set”. Since Wolfart’s celebrated work [Wo] investigating $E(a, b, c)$, much progress has been made in the study of exceptional sets. In those studies, Gauss’s hypergeometric series $_2F_1(a, b; c; x)$ corresponding to the arithmetic triangular group and satisfying the relations

$$c < 1, \; 0 < a < c, \; 0 < b < c, \; |1 - c| + |a - b| + |c - a - b| < 1,$$

$$\frac{1}{|1 - c|}, \frac{1}{|a - b|}, \frac{1}{|c - a - b|} \in \mathbb{Z}_{>0}$$

is a focus of investigation. The formulae $(B''\cdot3)$, $(C''\cdot1)$, $(C''\cdot4)$ and $(C''\cdot1)$ provide new examples of elements of these exceptional sets. The formulae $(B''\cdot3)$, $(C''\cdot1)$, $(C''\cdot4)$ and $(C''\cdot1)$ arise from Gauss’s hypergeometric equations corresponding to the $(3,6,6)$, $(2,5,10)$ and $(2,3,10)$ triangular groups, respectively.

2 The method of contiguity relations

In this section, we review the method of contiguity relations, which was introduced in [Eb]. Using this method, we obtain the examples of Appell’s hypergeometric series $F_1$ possessing closed forms listed in Table II.

First, for simplicity, we write the parameters $(\alpha; \beta_1, \beta_2; \gamma)$ as $\alpha$. Then, Appell’s hypergeometric series

$$F_1(\alpha; \beta_1, \beta_2; \gamma; x, y)$$

is expressed as $F_1(\alpha; x, y)$. We also define

$$e_{10} := (1; 1, 0; 1), \; e_{01} := (1; 0, 1; 1), \; k := (k; l_1, l_2; m) \in \mathbb{Z}^4.$$  

Now, we review the method of contiguity relations. It is known that for a given quadruple of integers $k \in \mathbb{Z}^4$, there exists a unique triple of rational functions $(Q_{10}, Q_{01}, Q_{00}) \in (\mathbb{Q}(\alpha; \beta_1, \beta_2; \gamma, x, y))^3$ satisfying

$$F_1(\alpha + k; x, y) = Q_{10} F_1(\alpha + e_{10}; x, y) + Q_{01} F_1(\alpha + e_{01}; x, y) + Q_{00} F_1(\alpha; x, y). \quad (2.1)$$

The relation (2.1) is called the “contiguity relation” for $F_1$ (or the “four-term relation” for $F_1$). Note that it is possible to compute $Q_{10}$, $Q_{01}$ and $Q_{00}$ exactly by using the method introduced in [1a1] (see also [1a2]). Next, we define

$$Q^{(n)}_{ij} := Q_{ij}|_{\alpha \to \alpha + nk},$$
where each $Q^{(n)}_{ij}$ is a rational expression in $n$ whose coefficients belong to $\mathbb{Z} [\alpha, \beta_1, \beta_2, \gamma, x, y]$. Now, let the sextuple $(\alpha; \beta_1, \beta_2; \gamma, x, y)$ satisfy
\[
\begin{cases}
Q^{(n)}_{10} \equiv 0, \\
Q^{(n)}_{01} \equiv 0
\end{cases}
\] for every integer $n$. \hfill (2.2)

Such a sextuple can be found by solving the polynomial system in $(\alpha, \beta_1, \beta_2, \gamma, x, y)$ arising from (2.2). Then, from (2.1), we find that the relation
\[
F_1(\alpha + (n + 1)k; x, y) = Q^{(n)}_{00} F_1(\alpha + nk; x, y)
\] holds for such a sextuple. The relation (2.3) implies that $F(n) := F_1(\alpha + nk; x, y)$ has a closed form. Thus, we are able to find $F_1$ in closed form. The above method is called the “method of contiguity relations”.

As an example, we consider the case $k = (2, 1, 4, 2)$. Applying the method of contiguity relations to this case, we find that
\[
(\alpha, \beta_1, \beta_2, \gamma, x, y) = (2a, a + 1/2, 4a - 1, 2a + 1/2, 1, 8a, 1/6)
\] satisfies (2.2). For this sextuple, $Q_{00}$ becomes
\[
\frac{3^8}{2^2 5^5} \left(\frac{a + 1/2, 2}{a + 1/2} \right).
\]

Hence, we have the following relation:
\[
\frac{F_1(\alpha + (n + 1)k; x, y)}{F_1(\alpha + nk; x, y)} = \frac{F_1\left(\frac{2a + 2n + 2a + n + 3/2, 4a + 4n + 3}{2a + 2n + 5/2}; \frac{1}{81}; \frac{1}{6}\right)}{F_1\left(\frac{2a + 2n + 1/2, 4a + 4n - 1}{2a + 2n + 1/2}; \frac{1}{81}; \frac{1}{6}\right)} = \frac{3^8}{2^2 5^5} \left(\frac{2a + 2n + 1/2, 2}{a + n + 1/2} \right) = Q^{(n)}_{00}.
\]

Thus, with
\[
F(a) := F_1\left(\frac{2a; a + 1/2, 4a - 1}{2a + 1/2}; \frac{1}{81}; \frac{1}{6}\right);
\]
we see that $F(a)$ has a closed form, and $F(a)$ satisfies the closed-form relation
\[
\frac{F(a + 1)}{F(a)} = \frac{3^8}{2^2 5^5} \left(\frac{2a + 1/2, 2}{a + 1/2} \right).
\]

As seen in the above example, if $k \in \mathbb{Z}^4$ is given, then we can obtain $F_1$ in closed form by using the method of contiguity relations. Such examples are tabulated in Table 1.

### 3 Special values of $2F_1$ derived from $F_1$ with closed forms

In the previous section, we presented examples of $F_1$ possessing closed forms. These are listed in Table 1. In this section, from among these $F_1$, we evaluate those that are convergent. The derived hypergeometric identities for $F_1$ are listed in Table 2. In addition, analyzing the closed-form relations for $F_1$ listed in Table 1, we obtain values of $2F_1$ with no free parameters. These values are presented in Table 3.
Table 1: Some examples of $F_1$ possessing closed forms.

| No. | $k$                | $F(a)$                                           | $F(a + 1)/F(a)$ |
|-----|--------------------|--------------------------------------------------|-----------------|
| (A.1) | (2, 1, 4, 2)       | $F_1 \left( \frac{2a; a + 1/2, 4a - 1}{a + 1/2}; \frac{1}{81}; \frac{1}{6} \right)$ | $\frac{3^8}{2^2 5^6} \frac{(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (A.2) | (2, 1, 4, 5)       | $F_1 \left( \frac{2a; a + 1/2, 4a - 1}{5a}; \frac{80}{81}; \frac{5}{6} \right)$ | $\frac{3^8}{2^2 5^6} \frac{(a + 1/2)^2(3a, 3)}{(a + 1/2)^2}$ |
| (A.3) | (1, 2, −4, 1)      | $F_1 \left( \frac{a; 2a, 1 - 4a, 80}{a + 1/2}; \frac{16}{81}; \frac{15}{15} \right)$ | $\frac{3^4}{2}$ |
| (B.1) | (2, −3, 4, 2)      | $F_1 \left( \frac{2a; 1 - 3a, 4a - 1}{2a + 1/2}; \frac{1}{80}; \frac{5}{32} \right)$ | $\frac{2^6}{2^3} \frac{(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (B.2) | (1, 3, 0, 5)       | $F_1 \left( \frac{a; 3a - 1/2, 1/2}{5a}; \frac{25}{2}; \frac{5}{32} \right)$ | $\frac{2^3}{2} \frac{a(5a, 5)}{(a + 1/2)^2}$ |
| (B.3) | (2, −3, 4, 1)      | $F_1 \left( \frac{2a; 1 - 3a, 4a - 1}{a + 1/2}; \frac{81}{80}; \frac{27}{32} \right)$ | $\frac{2^6}{2^3} \frac{(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (B.4) | (0, −1, 3, 4)      | $F_1 \left( \frac{1/2; -a, 3a + 5/2}{4a + 9/2}; \frac{27}{80}; \frac{5}{32} \right)$ | $\frac{(4a + 9/2, 4)}{2^6} \frac{(a + 3/2)^2(2a + 5/2, 2)}{(a + 1/2)^2}$ |
| (C.1) | (2, 1, 5, 3)       | $F_1 \left( \frac{2a; a + 1/2, 5a - 3/2}{3a}; \frac{3}{128}; \frac{3}{8} \right)$ | $\frac{2^5}{2^5} \frac{a(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (C.2) | (3, −1, 5, 2)      | $F_1 \left( \frac{3a; 1 - a, 5a - 3/2}{2a + 1/2}; \frac{3}{128}; \frac{1}{16} \right)$ | $\frac{2^5}{2^5} \frac{a(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (C.3) | (2, 1, 5, 5)       | $F_1 \left( \frac{2a; a + 1/2, 5a - 3/2}{5a}; \frac{125}{128}; \frac{5}{8} \right)$ | $\frac{2^5}{2^5} \frac{a(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (C.4) | (1, −3, 5, 0)      | $F_1 \left( \frac{a; 1 - 3a, 5a - 3/2}{1/2}; \frac{125}{128}; \frac{25}{16} \right)$ | $\frac{2}{3}$ |
| (D.1) | (2, −3, 5, 3)      | $F_1 \left( \frac{2a; 1 - 3a, 5a - 3/2}{3a}; \frac{9}{125}; \frac{25}{25} \right)$ | $\frac{2^5}{2^5} \frac{a(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (D.2) | (1, 1, 3, −1)      | $F_1 \left( \frac{a; a + 1/2, 3a - 1/2}{1 - a}; \frac{16}{25}; \frac{16}{16} \right)$ | $\frac{2^5}{2^5} \frac{a(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (D.3) | (5, 3, 2, 4)       | $F_1 \left( \frac{5a; 3a - 1/2, 2a}{4a + 1/2}; \frac{16}{25}; \frac{16}{16} \right)$ | $\frac{2^5}{2^5} \frac{a(2a + 1/2, 2)}{(a + 1/2)^2}$ |
| (E.1) | (0, 2, 3, 3)       | $F_1 \left( \frac{1/2; 2a, 3a - 1/2}{3a + 1/2}; \frac{4}{5}; \frac{4}{5} \right)$ | $\frac{(a + 1/6)(a + 5/6)}{(a + 1/3)(a + 2/3)}$ |

3.1 Hypergeometric identities for $F_1$ obtained from Table I

In this subsection, we derive hypergeometric identities from the closed-form relations given in Table I.

**Example 1.** As seen in Table I setting $k = (2, 1, 4, 2)$, we find that

$$F(a) := F_1 \left( \frac{2a; a + 1/2, 4a - 1}{2a + 1/2}; \frac{1}{81}; \frac{1}{6} \right)$$

possesses a closed form, and it satisfies the closed-form relation

$$\frac{F(a + 1)}{F(a)} = \frac{3^8}{2^2 5^6} \frac{(2a + 1/2, 2)}{(a + 1/2)^2}. \quad (3.1)$$
This implies
\[
\frac{F(a + n)}{F(a)} = \frac{3^{8n}}{2^{2n}5^{5n}} (2a + 1/2, 2n)_{0}^{a + 1/2, n} \cdot (a + 1/2, n)^2. \tag{3.2}
\]

Then, noting that
\[
F_1\left(0; \beta_1, \beta_2; x, y\right) = 1,
\]
and, substituting \(a = 0\) into \(3.2\), we obtain
\[
F(n) = G(n) \tag{3.3}
\]
for any integer \(n\), where
\[
G(n) := \frac{3^{8n}}{2^{2n}5^{5n}} \frac{(1/2, 2n)}{(1/2, n)^2} = \frac{3^{8n}}{2^{2n}5^{5n}} \frac{\Gamma(1/2)\Gamma(2n + 1/2)}{\Gamma(n + 1/2)^2}.
\]

Now, we show that the identity \(3.3\) holds for any complex number \(a\), except for \(a = -1/4, -3/4, -5/4, \ldots\). For this purpose, we use the following lemma proved by Carlson (see Section 3.1 [13a]).

**Lemma 3.1. (Carlson’s theorem)** We assume that \(f(a)\) and \(g(a)\) are regular and of the form \(O(e^{k|a|})\), where \(k < \pi\), for \(\Re(a) \geq 0\), and that \(f(a) = g(a)\) for \(a = 0, 1, 2, \ldots\). Then, we have \(f(a) = g(a)\) on \(\{a \mid \Re(a) \geq 0\}\).

Obviously, both sides of \(3.3\) are regular for \(\Re(a) \geq 0\). Also, we have the following:

\[
|F(a)| \leq \sum_{m, n=0}^{\infty} \frac{|2a||2a + 1| \cdots 2a + m + n - 1||(a|1/2, m + 1, n)}{|2a + 1/2|2a + 3/2 \cdots 2a - 1/2 + m + n(1, m)(1, n)} \left(\frac{1}{81}\right)^m \left(\frac{1}{6}\right)^n
\]
\[
\leq \sum_{m, n=0}^{\infty} \frac{(a|1/2, m)(4a|1, n)}{(1, m)(1, n)} \left(\frac{1}{81}\right)^m \left(\frac{1}{6}\right)^n
\]
\[
= \sum_{m=0}^{\infty} \frac{(a|1/2, m)}{(1, m)} \left(\frac{1}{81}\right)^m \left(\frac{1}{6}\right)^n
\]
\[
= \left(1 - \frac{1}{81}\right)^{-|a| - 1/2} \left(1 - \frac{1}{6}\right)^{-4|a| - 1}
\]
\[
= O\left(\exp\left(|a| \log \left(\frac{3^8}{55}\right)\right)\right).
\]

Note that we can also compute the asymptotic behavior of \(F(a)\) exactly by making use of Laplace’s method (also, see Section 3 in [13a]). In any case, we see that \(|F(a)|\) is of the form \(O(e^{k|a|})\), whereas it is easily demonstrated using Stirling’s formula that \(|G(a)|\) is of the form \(O(e^{k|a|})\). Of course, we know that from \(3.3\),
\[
F(a) = G(a) \tag{3.4}
\]
holds for any non-negative integer \(a\). Therefore, it follows from Carlson’s theorem that the identity \(3.4\) is valid for \(\Re(a) \geq 0\). Also, by analytic continuation, we find that \(3.4\) holds for any complex number \(a\), except for \(a = -1/4, -3/4, -5/4, \ldots\). Thus, we derive \(1.8\), which appears as \((A’1)\) in Table 2.

In this way, it is possible to obtain hypergeometric identities for \(F_1\) from the closed-form relations in Table 1. These are listed in Table 2.
identities given in Tables 1 and 2.

2 \( F \) has a closed form, and it satisfies the following closed-form relation: formula (1.9), we obtain (1.10). This appears as formula (A'').

This implies that

3.2 Special values of \( _2F_1 \) derived from closed-form relations for \( F_1 \)

In this subsection, we derive identities for \( _2F_1 \) with no free parameters, using the closed-form relations listed in Table 1 and the hypergeometric identities for \( F_1 \) listed in Table 2.

For example, let us substitute \( a = 1/4 \) into (A'.1). Then, recalling the reduction formula (1.9), we obtain (1.10). This appears as formula (A''1) in Table 3. In a similar way, reducing the hypergeometric identities for \( F_1 \) appearing in Table 2 to identities for \( _2F_1 \), we derive (A''2), (B''1), (B''4), (C''1), (C''2), (C''3), (D''1), (D''3) and (E''1).

We now derive the remaining identities that can be obtained from the relations and identities given in Tables 1 and 2.

**Example 2.** From Table 1 choosing \( k = (1, 2, -4, 1) \), we find that

\[
F(a) := F_1\left( \frac{a; 2a, 1 - 4a}{a + 1/2; \frac{80}{81}; \frac{16}{15}} \right)
\]

has a closed form, and it satisfies the following closed-form relation:

\[
\frac{F(a + 1)}{F(a)} = \frac{3^4}{5^4}.
\]

This implies that

\[
F(a) = \frac{5^{4a}}{3^{4a}} F(a + n)
\]
holds for any non-negative integer $n$. Although (3.6) (and (3.5)) is valid by virtue of analytic continuation, $F(a)$ and $F(a+n)$, which are regarded as infinite double series expressions, are meaningless. For this reason, we carry out a reduction of each of these to a finite sum of a single series expression that is meaningful. This is done, for example, by substituting $a = 1/4$ into (3.6). Then, it becomes

$$2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{80}{81}\right) = \frac{5^{4n}}{3^{4n}} F_1\left(\frac{1}{4} + n; \frac{1}{2} + 2n, -4n; \frac{80}{81}, \frac{16}{15}\right).$$

(3.7)

Next, we evaluate the left-hand side of (3.7) by determining the asymptotic behavior of the right-hand side of (3.7) in the limit $n \to +\infty$. Because $F_1(\alpha; \beta_1, \beta_2; \gamma; x, y)$ has the integral representation

$$F_1\left(\alpha; \beta_1, \beta_2; \gamma; x, y\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta_1}(1-yt)^{-\beta_2} dt,$$

the right-hand side of (3.7) can be expressed as $A \cdot B$, where

$$A := \frac{5^{4n}}{3^{4n}} \frac{\Gamma(3/4 + n)}{\Gamma(1/4 + n)\Gamma(1/2)}, \quad B := \int_0^1 g(t)e^{nh(t)} dt,$$

and, here,

$$g(t) := t^{-3/4}(1-t)^{-1/2} \left(1 - \frac{80}{81}t\right)^{-1/2}, \quad h(t) := \log \left(\left(1 - \frac{80}{81}t\right)^{-2} \left(1 - \frac{16}{15}t\right)^4\right).$$

The function $h(t)$ is plotted in Figure 1. From the relation

$$h'(t) = \frac{15(256t^2 - 352t + 81)}{t(15 - 16t)(81 - 80t)}.$$
we find that $h'(t)$ becomes zero at

$$t_0 := \frac{11}{16} - \frac{\sqrt{10}}{8}.$$  

We also obtain the relations

$$h(t_0) = h(1) = \log \left( \frac{3^4}{5^4} \right).$$

From Figure 1, it can be seen that the major contribution to the value of $B$ arises from the neighborhoods of the points $t = t_0, 1$. So, using Laplace’s method (see Section 2.4 in [Erd2]), we find that $B$ takes the form

$$B \sim \frac{9}{5} \left( \frac{3^4}{5^4} \right)^n \sqrt{\frac{n}{\pi}}$$

in the limit $n \to +\infty$. We can also compute the asymptotic behavior of $A$ using Stirling’s formula; we find that

$$A \sim \left( \frac{5^4}{3^4} \right)^n \sqrt{\frac{n}{\pi}}$$

in the limit $n \to +\infty$. The formulae (3.8) and (3.9) yield

$$\lim_{n \to +\infty} A \cdot B = \frac{9}{5}.$$ 

Thus, we obtain (1.4). This appears as $(A''\cdot 3)$ in Table 3.

Similarly, some of the remaining identities can be obtained. However, among the relations given in Table 1, there are some cases in which it is difficult to obtain values of $\, \!_2F_1$ by direct application of Laplace’s method. For such cases, we must use connection formulae for $\, \!_2F_1$.

For example, it is apparently difficult to compute the asymptotic behavior of (B.3) with $a = 1/3 + n$ in the limit $n \to +\infty$, by direct use of Laplace’s method. For this reason, we use a connection formula for $\, \!_2F_1$ to obtain (B'\''\cdot 3) in Table 3. As seen in the beginning of this subsection, we already know (B'\''\cdot 1) and (B'\''\cdot 4). Also, the following is a known connection formula for $\, \!_2F_1$:

$$u_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} u_2 + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} u_6,$$  

(3.10)

where

$$u_1 := \, \!_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; x \right),$$  

$$u_2 := \, \!_2F_1 \left( \begin{array}{c} a, b \\ a+b+1-c ; 1-x \end{array} \right),$$  

$$u_6 := (1-x)^{c-a-b} \, \!_2F_1 \left( \begin{array}{c} c-a, c-b \\ c+1-a-b ; 1-x \end{array} \right).$$

(see formula (33) in Section 2.9 in [Erd2]). Then, substituting $(a, b, c, x) = (1/3, 2/3, 7/6, 5/32)$ into (3.10), we deduce (B'\''\cdot 3). In this way, all the remaining identities are derived. All of the identities we have been able to derive are presented in Table 3. There, “type” refers to the type of Schwarz triangle for the Schwarz map of the corresponding hypergeometric equation. Explicitly, for a given $\, \!_2F_1(a, b; c; x)$, “type” is given by $(1/|1-c|, 1/|c-a-b|, 1/|a-b|)$. 
| No.  | Formula | Type | Ref. |
|------|---------|------|-----|
| (A")1 | \(2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{1}{81}\right)\) | \(9 \left[\frac{2^{5/4}}{100} \Gamma(1/4)^2\right]\) | (4,4,\(\infty\)) |
| (A")2 | \(2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{80}{5/4}\right)\) | \(9 \left[rac{2^{5/4}}{200} \Gamma(1/4)^4\right]\) | (4,4,\(\infty\)) *1 |
| (A")3 | \(2F_1\left(\frac{1/4}{2}; \frac{3/4}{81}\right)\) | \(9 \left[\frac{2^{5/4}}{\pi^2}\right]\) | (4,4,\(\infty\)) *1 |
| (B")1 | \(2F_1\left(\frac{1}{3}; \frac{2/3; 5}{7/6}; \frac{32}{32}\right)\) | \(1 \left[\frac{2^{1/3}}{5} \Gamma(1/3)^6\right]\) | (3,6,6) |
| (B")2 | \(2F_1\left(\frac{1}{6}; \frac{1/2; 5}{5/6}; \frac{32}{32}\right)\) | \(2 \left[\frac{2^{1/3}}{3} \sqrt[3]{5}\right]\) | (3,6,6) *3 |
| (B")3 | \(2F_1\left(\frac{1/3}{2}; \frac{3/27}{5/6}; \frac{32}{32}\right)\) | \(8 \left[\frac{2^{1/3}}{5}\right]\) | (3,6,6) |
| (B")4 | \(2F_1\left(\frac{1/2}{5}; \frac{6/32}{7/6}; \frac{32}{32}\right)\) | \(1 \left[\frac{2^{1/3}}{5} \Gamma(1/3)^6\right]\) | (3,6,6) |
| (C")1 | \(2F_1\left(\frac{3/5}{4}; \frac{4/5; 3}{9/10}; \frac{128}{128}\right)\) | \(8 \left[\frac{2^{1/3}}{75} \sqrt[3]{5}\right]\) | (2,5,10) |
| (C")2 | \(2F_1\left(\frac{7/10}{9/10}; \frac{10; 3}{11/10}; \frac{128}{128}\right)\) | \(8 \left[\frac{2^{1/3}}{75} \Gamma(1/5)^3 \Gamma(2/5)\right]\) | (2,5,10) |
| (C")3 | \(2F_1\left(\frac{3/2}{5/2}; \frac{4/5; 3}{125}; \frac{125}{128}\right)\) | \(4 \left[\frac{2^{1/3}}{3} \Gamma(1/5)^3 \Gamma(2/5)\right]\) | (2,5,10) |
| (C")4 | \(2F_1\left(\frac{1/10}{3/10}; \frac{10; 125}{12/12}; \frac{125}{128}\right)\) | \(1 \left[\frac{2^{3/10}}{6} \sqrt[3]{5}\right]\) | (2,5,10) |
| (D")1 | \(2F_1\left(\frac{1/6}{2/3}; \frac{3/9}{1}; \frac{25}{125}\right)\) | \(1 \left[\frac{2^{1/3}}{4} \sqrt[3]{5}\Gamma(1/3)^3\right]\) | (2,6,\(\infty\)) *4 |
| (D")2 | \(2F_1\left(\frac{1/6}{2/3}; \frac{16/5/6; 25}{25}\right)\) | \(2 \left[\frac{2^{1/3}}{3}\right]\) | (2,6,\(\infty\)) *5 |
| (D")3 | \(2F_1\left(\frac{1/3}{7/6}; \frac{6/25; 25}{25}\right)\) | \(1 \left[\frac{2^{1/3}}{48} \sqrt[3]{5}\Gamma(1/3)^3\right]\) | (2,6,\(\infty\)) *6 |
| (E")1 | \(2F_1\left(\frac{1/3}{1/2}; \frac{1/1; 5}{5}\right)\) | \(3 \left[\frac{2^{2/3}}{20} \sqrt[3]{5}\Gamma(1/3)^3\right]\) | (6,6,\(\infty\)) |

*1: (7.10) in [JZ2].
*2: (1.6) in [JZ2].
*3: See *3 in Table 4 and *1 in Table 5.
*4: See *4 in Table 3.
*5: See *4 and *5 in Table 3.
*6: See *4 and *6 in Table 3.
4 Algebraic transformations of identities in Table 3

In this section, applying algebraic transformations to the identities in Table 3, we have some complicated identities for $2F_1$. These identities are tabulated in Tables 4 and 5.

As an example, here we derive $(A'''.1)$ in Table 4. As seen in Table 3, $(A'''.1)$ is known. Then, applying Goursat’s algebraic transformation

$$2F_1\left(\frac{a}{2b}; x\right) = (1-x)^{b-a} \left(1 - \frac{x}{2}\right)^{a-2b} 2F_1\left(\frac{b-a/2}{b+1/2}; \frac{x}{2-x}\right)$$

(see formula (45) in [Gour]) with $(a, b) = (3/4, 1/2)$ to the left-hand side of $(A'''.1)$, $(A'''.1)$ is easily deduced. In a similar way, we derive the following:

- $(A''''.2)$ from $(A'''.2)$ using Goursat’s algebraic transformation (51) in [Gour],
- $(A''''.3)$ from $(A''''.3)$ using Goursat’s algebraic transformation (51) in [Gour],
- $(B''''.1)$ from $(B''''.1)$ using Goursat’s algebraic transformation (41) in [Gour],
- $(B''''.2)$ from $(B''''.2)$ using Goursat’s algebraic transformation (38) in [Gour],
- $(B''''.3)$ using a connection formula for $2F_1$ and $(B''''.1)$ and $(B''''.2)$,
- $(B''''.4)$ using a connection formula for $2F_1$ and $(B''''.1)$ and $(B''''.2)$,
- $(C''''.1)$ from $(C''''.1)$ using Goursat’s algebraic transformation (121) in [Gour],
- $(C''''.2)$ from $(C''''.2)$ using Goursat’s algebraic transformation (121) in [Gour],
- $(C''''.3)$ using a connection formula for $2F_1$ and $(C''''.1)$ and $(C''''.2)$,
- $(C''''.4)$ using a connection formula for $2F_1$ and $(C''''.1)$ and $(C''''.2)$,
- $(D''''.1)$ from $(D''''.1)$ using Goursat’s algebraic transformation (50) in [Gour],
- $(D''''.2)$ from $(D''''.2)$ using Goursat’s algebraic transformation (45) in [Gour],
- $(D''''.3)$ from $(D''''.3)$ using Goursat’s algebraic transformation (45) in [Gour],
- $(E''''.1)$ from $(E''''.1)$ using Goursat’s algebraic transformation (45) in [Gour].

The identities in Table 5 are derived as follows:

- $(B''''.1)$ from $(B''''.1)$ using Goursat’s algebraic transformation (50) in [Gour],
- $(B''''.2)$ from $(B''''.2)$ using Goursat’s algebraic transformation (50) in [Gour],
- $(B''''.3)$ from $(B''''.3)$ using Goursat’s algebraic transformation (45) in [Gour],
- $(B''''.4)$ from $(B''''.4)$ using Goursat’s algebraic transformation (45) in [Gour].
This is a special case of (1,2,3-1)(xxiii) in [Eb], (8/9.2) in [Gos], (5.24) in [DS], (3.3) in [Ka].

This is a special case of (1,2,3-1)(vii) in [Eb], (8/9.1) in [Gos] and (3.2) in [Ka].

(7.10) in [JZ3].

Formula 171 in Section 8.1.1 of [Br].

Table 4: Algebraic transformations of identities in Table 3.

| No. | Formula | Type | Ref. |
|-----|---------|------|------|
| (A''''1) | \( _2F_1 \left( \frac{1/8, 5/8}{1}; \frac{1}{25921} \right) = \frac{1}{10} \frac{\sqrt{2} \sqrt[8]{161} \Gamma(1/4)^2}{\pi^{3/2}}. \) | (2,4,∞) | |
| (A''''2) | \( _2F_1 \left( \frac{3/8, 7/8}{5/4}; \frac{25920}{25921} \right) = \frac{1}{600} \frac{5^{3/4}161^{3/4} \Gamma(1/4)^4}{\pi^2}. \) | (2,4,∞) | *1 |
| (A''''3) | \( _2F_1 \left( \frac{1/8, 5/8}{3/4}; \frac{25920}{25921} \right) = \frac{3}{5} \sqrt{161}. \) | (2,4,∞) | *2 |
| (B''''1) | \( _2F_1 \left( \frac{1/4, 5/12}{7/6}; \frac{135}{256} \right) = \frac{1}{20} \frac{2^{5/6} \Gamma(1/3)^6}{\pi^3}. \) | (2,6,6) | |
| (B''''2) | \( _2F_1 \left( \frac{1/12, 1/4}{5/6}; \frac{135}{256} \right) = \frac{2}{5} \sqrt{2} \sqrt{3} \sqrt{5}. \) | (2,6,6) | *3 |
| (B''''3) | \( _2F_1 \left( \frac{1/4, 5/12}{1/2}; \frac{121}{256} \right) = \frac{1}{5} \frac{2^{5/12} \sqrt{3} \Gamma(1/3)^3}{\Gamma(1/4)^2}. \) | (2,6,6) | |
| (B''''4) | \( _2F_1 \left( \frac{3/4, 11/12}{3/2}; \frac{121}{256} \right) = \frac{8}{55} \frac{2^{11/12} 3^{3/8} \Gamma(1/3)^3 \Gamma(1/4)^2}{(1 + \sqrt{3})^{3/2} \pi^{5/2}}. \) | (2,6,6) | |
| (C''''1) | \( _2F_1 \left( \frac{1/30, 8/15}{9/10}; \frac{20736}{45125} \right) = \frac{1}{6} \frac{\sqrt[10]{3} \sqrt[5]{3} \sqrt{19} \sqrt{5 + \sqrt{5}}}{\sqrt{5 + \sqrt{5}}}. \) | (2,3,10) | |
| (C''''2) | \( _2F_1 \left( \frac{2/15, 19/30}{11/10}; \frac{20736}{45125} \right) = \frac{1}{12} \frac{5^{3/5} 19^{1/15} \Gamma(1/5)^3 \Gamma(2/5)}{(5 + \sqrt{5}) \pi^2}. \) | (2,3,10) | |
| (C''''3) | \( _2F_1 \left( \frac{1/30, 8/15}{2/3}; \frac{24389}{45125} \right) = \frac{1}{36} \frac{2^{17/30} 3^{19/15} \Gamma(1/5)^2}{\Gamma(1/3)^3 \Gamma(1/15)} \times \Gamma(1/3) \Gamma(1/15). \) | (2,3,10) | |
| (C''''4) | \( _2F_1 \left( \frac{11/30, 13/15}{4/3}; \frac{24389}{45125} \right) = \frac{5}{522} \frac{\sqrt{5} - \sqrt{15} - 6 \sqrt{5}}{\sqrt{5} + \sqrt{15} - 6 \sqrt{5}} \times \frac{2^{2/30} \sqrt[5]{3}^{10/3} 19^{11/15}}{\sqrt{5 + \sqrt{5}}} \times \frac{\Gamma(1/5) \Gamma(2/5) \Gamma(1/3) \Gamma(1/15)}{\pi^2}. \) | (2,3,10) | |
| (D''''1) | \( _2F_1 \left( \frac{1/3, 1/3}{1}; \frac{1}{9} \right) = \frac{1}{4} \frac{3^{3/3} \Gamma(1/3)^3}{\pi^2}. \) | (3,∞,∞) | *4 |
| (D''''2) | \( _2F_1 \left( \frac{1/3, 1/3, 8}{2/3}; \frac{8}{9} \right) = \frac{2}{3} \frac{3^{2/3}}{\pi^2}. \) | (3,∞,∞) | *5 |
| (D''''3) | \( _2F_1 \left( \frac{2/3, 2/3}{4/3}; \frac{8}{9} \right) = \frac{1}{16} \frac{3^{5/6} \Gamma(1/3)^6}{\pi^3}. \) | (3,∞,∞) | *6 |
| (E''''1) | \( _2F_1 \left( \frac{1/6, 2/3}{1}; \frac{1}{81} \right) = \frac{3}{20} \frac{\sqrt[2]{3}^{2/3} \sqrt[5]{5} \Gamma(1/3)^3}{\pi^2}. \) | (2,6,∞) | |

*1: Formula 171 in Section 8.1.1 of [Br].

*2: (7.10) in [JZ3].

*3: See *1 in Table 5.

*4: This is a special case of (1,2,3-1)(ix) in [Eb], (1/9.4) in [Gos] and (1.2) in [Ka]. Because (D''''1) can be obtained by applying an algebraic transformation to the left-hand side of (D''''1), it is regarded as a previously known formula. The same holds for (D''''2) and (D''''3).

*5: This is a special case of (1,2,3-1)(vii) in [Eb], (8/9.1) in [Gos] and (3.2) in [Ka].

*6: This is a special case of (1,2,3-1)(xxiii) in [Eb], (8/9.2) in [Gos], (5.24) in [DS], (3.3) in [Ka].
Table 5: Algebraic transformations of identities in Table 4.

| No. | Formula | Type | Ref. |
|-----|---------|------|------|
| (B′′′′.1) | \( _2F_1 \left( \frac{5}{24}, \frac{17}{24}; \frac{138240}{152881} \right) = \frac{1}{160} \sqrt{2} \frac{17^{5/12} 23^{5/12} \Gamma(1/3)^6}{\pi^3} \) | (2.4.6) |
| (B′′′′.2) | \( _2F_1 \left( \frac{1}{24}, \frac{13}{24}; \frac{138240}{152881} \right) = \frac{1}{5} \sqrt{2} \frac{\sqrt{3} \sqrt{5} \sqrt{17} \sqrt{23}}{\sqrt{23}} \) | (2.4.6) *1 |
| (B′′′′.3) | \( _2F_1 \left( \frac{5}{24}, \frac{17}{24}; \frac{14641}{152881} \right) = \frac{1}{80} \frac{2^{2/3} \sqrt{3} \sqrt{17}^{5/12} \sqrt{23}^{5/12}}{(1 + \sqrt{3})^{3/2}} \times \frac{\Gamma(1/3)^3}{\sqrt{\pi} \Gamma(1/4)^2} \) | (2.4.6) |
| (B′′′′.4) | \( _2F_1 \left( \frac{11}{24}, \frac{23}{24}; \frac{14641}{152881} \right) = \frac{1}{1760} \frac{2^{2/3} \sqrt{3} \sqrt{17}^{11/12} \sqrt{23}^{11/12}}{(1 + \sqrt{3})^{3/2}} \times \frac{\Gamma(1/3)^3 \Gamma(1/4)^2}{\pi^{5/2}} \) | |

*1: We see that (B′′′′.2) is identical to (13) in [BG] using the Pfaff transformation. Therefore, because (B′′.2) and (B′′′.2) can be obtained by applying algebraic transformations to the left-hand side of (B′′′.2), these are regarded as previously known formulae.

5 Concluding remarks

Applying the method of contiguity relations to Appell’s hypergeometric series \( _1F_1 \), we have obtained several identities for \( _2F_1 \) with no free parameters. In addition to the identities derived above, there are a number of cases in which the same method allows us to conjecture values of \( _2F_1 \) with no free parameters, as we now discuss.

For example, for \( k = (-2, 3, 0, 1) \), we have the closed-form relation

\[
\frac{F(a + 1)}{F(a)} = \frac{5 (a + 1/2)(a + 3/2)}{3^3 (a + 5/6)(a + 7/6)},
\]

where

\[
F(a) := F_1 \left( \begin{array}{c}
-2a; 3a + 1.1/2, 16/4 \\
a + 3/2; 25/5
\end{array} \right).
\]

From this relation, we can conjecture

\[
F(a) = F_1 \left( \begin{array}{c}
-2a; 3a + 1.1/2, 16/4 \\
a + 3/2; 25/5
\end{array} \right) = \left( \frac{5}{3^3} \right)^a \frac{\cos(\pi a) \Gamma(5/6) \Gamma(7/6) \Gamma(a + 1/2) \Gamma(a + 3/2)}{\Gamma(1/2) \Gamma(3/2) \Gamma(a + 5/6) \Gamma(a + 7/6)}
\]

\[
= \frac{2}{3} \left( \frac{5}{3^3} \right)^a \frac{\cos(\pi a) \Gamma(a + 1/2) \Gamma(a + 3/2)}{\Gamma(a + 5/6) \Gamma(a + 7/6)}.
\]

Unfortunately, however, it seems difficult to compute the asymptotic behavior of \( F(a) \) in the limit \( |a| \to +\infty \) from a direct application of Laplace’s method. For this reason, we have not been able to prove (5.1). However, with numerical calculations, we have obtained results consistent with this identity. If indeed this identity does hold, then we have

\[
_2F_1 \left( \begin{array}{c}
1/2, 2/3; 4/5 \\
7/6; 5/5
\end{array} \right) = \frac{1}{40} \frac{\sqrt{3}^{5/3} \Gamma(1/3)^6}{\pi^3}
\]

(5.2)
by substituting $a = -1/3$ into (5.1). Moreover, using the connection formula (3.10), (E’’.1) and (5.2), we obtain

$$\frac{2F1}{\left(\begin{array}{c}
1/3, 1/2; 4/5
\end{array}\right)} = \frac{3}{\sqrt{5}} \text{ (5.3)}$$

Although this relation follows from a conjecture, if indeed it does hold, it provides a new example of an algebraic value of $2F1$. In addition, by applying algebraic transformations to (5.2) and (5.3), we can similarly conjecture the relations

$$\frac{2F1}{\left(\begin{array}{c}
1/3, 5/6; 80/81
\end{array}\right)} = \frac{3}{40} \frac{3^{5/6} \Gamma(1/3)^6}{\pi^3} \text{ (5.4)}$$

and

$$\frac{2F1}{\left(\begin{array}{c}
1/6, 2/3; 80/81
\end{array}\right)} = \frac{3}{5} \frac{3^{2/3} \sqrt{5}}{\sqrt{3}}, \text{ (5.5)}$$

respectively.

We can also obtain non-trivial algebraic values of $F1$ from our approach. For instance, substituting $a = -2/3$ into (B’’.4), we have

$$\frac{F1}{\left(\begin{array}{c}
1/2; 2/3, 1/2; 27/32, 5/6
\end{array}\right)} = \frac{2}{3} \sqrt{2} \sqrt{3}. \text{ (5.6)}$$

We thus conclude that for the purpose of finding algebraic values of Gauss’s hypergeometric series $2F1$ and Appell’s hypergeometric series $F1$, it is worthwhile studying $F1$ possessing closed forms.

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Akihito Ebisu  
Department of Mathematics  
Hokkaido University  
Kita 10, Nishi 8, Kita-ku, Sapporo, 060-0810  
Japan  
a-ebisu@math.sci.hokudai.ac.jp

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