Models of heterogeneous medium in medicine

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Abstract. A review of models of the blood circulation in the tissues of the human brain and lower extremities is presented. The models take into account the well-known laws of physiology and theoretical mechanics. Mathematical models are based on the representation of organ tissues as a heterogeneous continuous medium consisting of three interpenetrating continua: living cells, and circulating arterial and venous blood. The results of numerical calculations of boundary value problems of the pulse blood flow in the brain of a healthy person and cases of pathologies such as arteriovenous malformation and embolism are presented. The distribution of blood pressure in the lower extremities is studied under the action of gravity and other overloads.

1. Governing equations and formulation of problems
The organism of mammals, including human body, is a complex natural system of elements in which dozens of physicochemical and electrical processes take place. Friedrich Engels defined life as “way of existence of protein bodies”. A living organism of a mammal contains many other organisms, bacteria, viruses and other protozoa capable of reproduction. There are physiological laws that govern the “way of existence of protein bodies” in nature. Primary such laws are anabolism and catabolism, fission and atrophy of cells which are as well essentially living organisms. The blood circulation plays a decisive role in these processes. Nutrition and waste removal in cells are carried out by a directed blood flow in small capillaries. Violation of capillary blood flow leads to the death of cells.

The pattern of blood flow in a selected single element of organ can be presented in the scheme shown in figure 1. On the scheme, numerals 1 and 2 mark the parts of arterial and venous blood in single element respectively. Arrows show a direction of capillary flow from arterial network to venous one. The volume of organ cells is presented by light-coloured rectangles.

Figure 1. Blood flow scheme in single element of organ living tissue.
The mathematical model of the blood flow in muscle tissue as a heterogeneous structure consisting of interpenetrating continua includes the equations of motion in continua 1 and 2 disjoint with each other [1-3]

\[ v_i = -\frac{k_i}{\mu} \text{grad} p_i \quad (i = 1, 2) \]

and the laws of conservation of blood mass

\[ \frac{\partial (m_i \rho)}{\partial t} + \text{div}(\rho v_i) + q_i = 0. \]

Here \( v_i, \ p_i, \ k_i, \ \mu, \ m_i, \ \rho, \ t, \ q_i \) are velocity, pressure, permeability of networks, fluid viscosity, porosity of continua (specific volumes of networks), fluid density, time, and values of overflows, respectively.

The equations written above are reduced to system

\[ \frac{k_i}{\mu} \Delta p_i = \beta_i \frac{\partial p_i}{\partial t} + q_i \quad (i = 1, 2) \]

where coefficients \( \beta_i = \frac{dm_i}{dp_i} \) describe the compressibility of vessel network. \( \Delta \) is Laplace operator. The equations of the system (1) are linked to each other by cross-flows

\[ -q_1 = q_2 = q = \frac{\alpha}{\mu} (p_1 - p_2), \]

where \( \alpha \) is specific parameter of inner exchange (capillary conduction).

The system (1) contains five parameters: the conductivities of arterial and venous networks, vessel system compressibilities and capillary conduction.

The values of some of them can be estimated by statistical methods if the distribution function of the vessel sizes by their sizes is known. It is assumed that the elasticity of the vessels in the continua is identical and is actually determined by the elasticity of the cells. In addition, the total conductivity of the arterial and venous networks can be considered approximately the same.

Including the new required functions \( S^\pm(x,t) = p_1 \pm p_2 \), the system (1) can be reduced to the next system of independent equations

\[ -\beta_i \frac{\partial S^-}{\partial t} + \Delta S^- - 2\eta_i S^- = 0, \quad -\beta_i \frac{\partial S^+}{\partial t} + \Delta S^+ = 0. \]

Applying to pulse blood flow in the cerebral cortex, the problem without initial data are solved of the form

\[ z = z_w: S^\pm = A^\pm = A \cos \omega t + B^\pm, \quad z = z_p: \frac{\partial S^\pm}{\partial n} / \partial n = 0, \]

where \( B^\pm = B \pm p_0, \ A = (p_s - p_d) / 2, \ B = (p_s + p_d) / 2, \ p_s \) is systolic pressure, \( p_d \) is diastolic pressure, \( p_0 \) is suction pressure in veins, \( \omega \) is pulse regime frequency, \( z_w = x_w + y_w, z_p = x_p + y_p \) are points of the Willis circle and a brain periphery respectively, \( \partial / \partial n \) is derivative of normal to periphery. The last boundary condition means blood unflowing through external brain border (periphery).
The problem is solved with finite element method using net crowding. The calculating domain is shown in figure 2. When numerically investigating the effect of malformation or embolism in the brain on the total blood flow, the coordinates and size of the pathology zone are set. In the zone, the parameters corresponding to the nature of the pathology are specified.

In figure 3 and figure 4 are shown izolines of arterial blood pressure for cases of vessel malformation and embolism. Circles mark locations of the pathologies. Three types of pathology can be considered in a split system of equations (2). A more detailed study requires the use of a complete system of equations.

**Figure 2.** Configuration of calculating domain (brain cortex).

**Figure 3.** Malformation $\eta_1 = 1$.

**Figure 4.** Vessel embolism $\eta_1 = 100$.

**Figure 5.** Pressure surfaces in the presence of embolism ($\eta_1 = 100$).
The curves correspond to the point in time at which the maximum pressure is reached in the arteries of the Willis circle.

Figure 5 presents the surfaces of pressure distribution in arterial and venous blood flow in case of embolism in circle with center point (11, 6) where parameters are taken as $\beta = 0.01; \eta = 100$.

When modeling blood flow in the lower extremities [4-5], averaged over the cross section of muscle tissue and directed along the axis $z$ flows are considered. Let $q_a$ be flux in distributive (arterial) network, $q_v$ be flux in collector (venous) network. Then analog of equations (1) is the following system:

\[
\begin{aligned}
-\beta_1 \frac{\partial h_a}{\partial t} + \frac{1}{s(z)} \frac{\partial}{\partial z} \left( s(z) \frac{\partial h_a}{\partial z} \right) &= \eta_1 (h_a - h_v); \\
-\beta_1 \frac{\partial h_v}{\partial t} + \frac{1}{s(z)} \frac{\partial}{\partial z} \left( s(z) \frac{\partial h_v}{\partial z} \right) &= -\eta_1 (h_a - h_v).
\end{aligned}
\]

The boundary conditions at the hip and at the foot respectively must be satisfied for this system:

\[ z = 0: p_a = h_a = A \cos(\omega t) + B, \quad p_v = h_v = p_v; \quad z = z_p: \frac{\partial h_a}{\partial z} = \frac{\partial h_v}{\partial z} = 0. \]

Here we denote $h_a = p_a - \rho g z; \quad h_v = p_v - \rho g z; \quad v_a = -\frac{\partial h_a}{\partial z}; \quad v_v = -\frac{\partial h_v}{\partial z},$ and $s(z)$ is a cross-section area of leg muscle bulk.

Analytical solutions are obtained for particular model cases. In the general case, the problem is solved numerically. The numerical results demonstrate that blood pressure distribution in low limbs depends essentially on a human position relative to the vector gravity and other overloads.

2. Conclusions

New mathematical models of blood circulation in human organs are considered. They take into account well-known laws of physiology, theoretical fluid mechanics and base on conception that tissues of human organ are heterogeneous continuum consisting from interpenetrative continua: living cells as well as circulating arterial, venous and capillary blood. A branched arterial network is a distribution system that passes through the capillary system into the collector system of veins. There are no intersections between systems in a healthy body. Cases of intersections or clogging of blood vessels lead to dangerous pathologies: malformation and embolism. Calculations show that pathologies significantly affect the distribution of blood pressure in the arterial and venous channels.

The mathematical models enable one to choose a treatment strategy for the human organism.

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