Abstract. We study dynamics and bifurcations of two-dimensional reversible maps having a symmetric saddle fixed point with an asymmetric pair of non-transversal homoclinic orbits (a symmetric nontransversal homoclinic figure-eight). We consider one-parameter families of reversible maps unfolding the initial homoclinic tangency and prove the existence of infinitely many sequences (cascades) of bifurcations related to the birth of asymptotically stable, unstable and elliptic periodic orbits.

Preamble

We say that a dynamical system exhibits the so-called phenomenon of mixed dynamics when:

(i) it has simultaneously infinitely many hyperbolic periodic orbits of all possible types (stable, completely unstable and saddle), and

(ii) these orbits are not separated as a whole, that is, the closures of sets of orbits of different types have non-empty intersections.

Note that the property of mixed dynamics can be generic property, i.e., it holds for residual subsets of open regions of systems. In particular, it was proved in [1] that such regions, Newhouse regions with mixed dynamics, exist near any two-dimensional diffeomorphism with a non-transversal heteroclinic cycle containing at least two saddle periodic points $O_1$ and $O_2$ satisfying that $|J(O_1)| > 1$ and $|J(O_2)| < 1$, where $J(O_i)$ is the Jacobian of the Poincaré map at the point $O_i$, $i = 1, 2$. We will call such cycles contracting-expanding.

Let us recall that a heteroclinic cycle (contour) is a set consisting of (ordered) saddle hyperbolic periodic orbits $O_1, \ldots, O_n$, as well as several heteroclinic orbits $\Gamma_{i,j} \subset W^u(O_i) \cap W^s(O_j)$ containing, at least, the orbits $\Gamma_{i,i+1}$, for $i = 1, \ldots, n-1$, and $\Gamma_{n,1}$. In general, cycles can include also homoclinic orbits $\Gamma_{i,i} \subset W^u(O_i) \cap W^s(O_i)$. A heteroclinic cycle is called non-transversal (or non-rough) if at least one of the intersections $W^u(O_i) \cap W^s(O_j)$ is non-transversal. Remind also that Newhouse regions are open (in $C^2$-topology) domains in the space of dynamical systems in which systems with homoclinic tangencies are dense. It is known that they exist in any
neighbourhood of any system having a homoclinic tangency to a saddle periodic orbit \(2, 3\). Such regions appear in parameter families unfolding, generally, the initial homoclinic tangency. In the parameter space, the values of the parameters corresponding to the existence of homoclinic tangencies are dense in open domains (see, for instance, \(2, 3, 4, 5\) and \(6\) for area-preserving maps), called Newhouse intervals in the case of one-parameter families. The existence of Newhouse regions near systems with non-transversal heteroclinic cycles follows from these works. Moreover, as it was proved in \(1\), Newhouse intervals with heteroclinic tangencies (of the same type as the initial one) exist in any general one-parameter unfolding.

In this paper we consider questions of mixed dynamics for reversible two-dimensional diffeomorphisms. Recall that a diffeomorphism \(f\) is reversible if it is smoothly conjugated to its inverse by means of an involution, i.e. the conjugacy diagram \(f \circ h = h \circ f^{-1}\) holds where \(h\) is a specific diffeomorphism of the phase space such that \(h^2 = \text{Id}\). Usually, it is assumed that the conjugation \(h\) has at least the same smoothness as \(f\) (or \(h\) is analytic, linear etc).

It is worth mentioning that contracting-expanding heteroclinic cycles are rather usual among reversible maps, as shown in Figure 1(a). In that example the reversible map has two saddle fixed points \(O_1\) and \(O_2\) and two heteroclinic orbits \(\Gamma_{12} \subset W^u(O_1) \cap W^s(O_2)\) and \(\Gamma_{21} \subset W^u(O_2) \cap W^s(O_1)\) such that \(R(O_1) = O_2\) and \(R(\Gamma_{21}) = \Gamma_{21}, R(\Gamma_{12}) = \Gamma_{12}\). Besides, the orbit \(\Gamma_{12}\) is non-transversal, so that the manifolds \(W^u(O_1)\) and \(W^u(O_2)\) have a quadratic tangency along \(\Gamma_{12}\). Since \(R(O_1) = O_2\) it turns out that \(J(O_1) = J^{-1}(O_2)\). If \(J(O_i) \neq \pm 1, i = 1, 2\), then the heteroclinic cycle is contracting-expanding. This condition is robust and is perfectly compatible with reversibility.

Certainly, results of \(1\) can be applied to reversible maps with such heteroclinic cycles. However, reversible systems are sharply different from general ones by the fact that they can possess robust non-hyperbolic symmetric periodic orbits, e.g. elliptic symmetric periodic points. This leads to the idea that the phenomenon of mixed dynamics in the case of two-dimensional reversible maps should be connected with the coexistence of infinitely many attracting, repelling, saddle and elliptic periodic orbits. Moreover, we can say now that the phenomenon of mixed dynamics is universal for reversible (two-dimensional) maps with complicated dynamics when symmetric structures (symmetric periodic, homoclinic and heteroclinic orbits) are involved. This universality can be formulated as follows:

**Reversible Mixed Dynamics Conjecture** Two-dimensional reversible maps with mixed dynamics are generic in Newhouse regions where maps with symmetric homoclinic or/and heteroclinic tangencies are dense.

This conjecture is true when the Newhouse regions with \(C^r\)-topology, where \(2 \leq r \leq \infty\), are considered \(7\). However, in the real analytic case, it has been proved now only for Newhouse regions existing near reversible maps with nontransversal heteroclinic cycles of two specific types presented in Fig. 1. In the first case the cycle is contracting-expanding and it contains a symmetric couple of saddle periodic orbits (with Jacobians less and greater than 1) and a pair of symmetric transverse and nontransversal heteroclinic orbits, see Fig. 1(a). This case was studied by Lamb and Stenkin \(8\). The second case relates again to a nontransversal heteroclinic cycle but of another type: here a symmetric couple of non-transversal heteroclinic orbits to symmetric saddle points exists, see Fig. 1(b). This case was studied in \(9\).
that, both in [8] and [9], the RMD-conjecture was proved for one parameter general unfoldings, i.e. the existence of Newhouse intervals with mixed dynamics was proved.

However, probably the most interesting cases are those where the initial reversible map has a homoclinic tangency associated to a symmetric saddle point. If the point is fixed, then three main cases of homoclinic tangencies can be selected, see Fig. 2. The cases (a) and (b) are connected with the a priori presence of only two symmetric orbits, periodic (the fixed point) and homoclinic ones. Therefore, we call these cases, as well as the heteroclinic case from Fig. 1(b), “a priori conservative”. In these cases we do not known “a priori” bifurcation mechanisms of symmetry breaking, i.e. (global)
bifurcations in one parameter unfoldings leading to the appearance a pair “sink-source” of periodic orbits. Note that for the heteroclinic a priori conservative case (as in Fig. 1(b)) the corresponding symmetry breaking bifurcations were found and studied in [9]. However, for a priori conservative homoclinic cases (as in Fig. 2 (a) and (b)) the problem is still open and we are going to study it in the nearest time. At the same time, bifurcations leading here to the appearance of symmetric elliptic periodic orbits should be well known, since they are the same as in the corresponding conservative case, see [10, 11, 12, 9].

The case (c) of Fig. 2 is connected with the a priori presence of both symmetric orbits, periodic one (the fixed point), and a symmetric couple of orbits, the homoclinic ones. Evidently, the orbit behavior near every homoclinic is nonconservative, typically. Thus, in general, bifurcations of individual homoclinic tangency can lead to attractive (repelling) periodic orbits and, hence, the appearance of pairs “sink-source” of periodic orbits should be an a priori anticipated effect. Therefore, we call this cases, as well as the heteroclinic case from Fig. 1(a), “a priori nonconservative”. Moreover, in these cases the main problem becomes a clarification of bifurcation mechanisms of appearance of symmetric periodic orbits and, first of all, elliptic ones.

We will assume, from now on, that the involution $R$ is not trivial and leaves fixed a curve, that is, it satisfies

$$R^2 = \text{Id}, \quad \dim \text{Fix } R = 1,$$

where $\text{Fix } R = \{(x, y) \in \mathbb{R}^2 \mid R(x, y) = (x, y)\}$. We say that an object $\Lambda$ is symmetric when $R(\Lambda) = \Lambda$. To put more emphasis, the notation self-symmetric may be used. By a symmetric couple of objects $\Lambda_1, \Lambda_2$, we mean two different objects which are symmetric one to each other, i.e., $R(\Lambda_1) = \Lambda_2$. Note that Symmetric homoclinic (heteroclinic) tangencies can be divided into two main types, namely, when: (i) there is a non-transversal symmetric heteroclinic orbit to a symmetric couple of saddle points, or (ii) there is a symmetric homoclinic tangency or a symmetric couple of non-transversal homo/heteroclinic orbits to symmetric saddle points.

Thus, in this paper we consider the case of a symmetric couple of non-transversal homoclinic orbits to a symmetric saddle point. We assume that the multipliers $\lambda$ and $\lambda^{-1}$ are such that $0 < \lambda < 1$. Then the point $O$ divides its the stable and unstable manifolds into two connected parts called, resp., stable and unstable separatrices. If the homoclinic orbits belongs to the intersections of different pairs of the separatrices, as in Fig. 3(a), we will also say about a figure-8 homoclinic tangencies. If If the homoclinic orbits belongs to the intersections of the same pair of the separatrices, as in Fig. 4 we will also say about a figure-fish homoclinic tangencies.

Our main goal in this paper is to prove the RMD-Conjecture for general one-parameter reversible unfoldings of a couple homoclinic tangencies. For this goal, we consider a one-parameter family $\{f_\mu\}$ of $R$-reversible maps such that the map $f_0$ has a couple of symmetric non-transversal homoclinic orbits $\Gamma_1$ and $\Gamma_2$ to a symmetric saddle fixed point $O$. Thus, we have that $R(O) = O$ and $R(\Gamma_1) = \Gamma_2$, $R(\Gamma_2) = \Gamma_1$. Since $O \in \text{Fix } R$, the condition $J(O) = 1$ holds always. Therefore, our genericity condition, the condition [C] from Section 1 should be related to nonconservative properties of orbit behavior near homoclinics $\Gamma_1$ and, symmetrically, $\Gamma_2$.

We consider a small fixed neighbourhood $U$ of the contour $O \cup \Gamma_1 \cup \Gamma_2$. Note that $U$ is represented as a union of a small neighbourhood $U_0$ of the point $O$ and a number of small neighbourhoods $U_j$, $j = 1, \ldots, n$, of those points of the orbits $\Gamma_1$ and
Attracting, repelling and elliptic orbits in reversible maps

Figure 3. (a) an example of reversible map with a couple of symmetric homoclinic tangencies (homoclinic figure-8); (b) a neighbourhood of the contour \( O \cup \Gamma_1 \cup \Gamma_2 \).

Figure 4. (a) a reversible diffeomorphism with a symmetric transversal homoclinic orbit; (b) creation of a symmetric couple of nontransversal homoclinic orbits \( \Gamma_1 \) and \( \Gamma_2 \). Here the nontransversal homoclinic figure-fish is created by one pair of separatrices of the point \( O \).

\( \Gamma_2 \) which do not lie in \( U_0 \), see Fig. 3(b). We can assume that \( U \) is self-symmetric, i.e. \( R(U_0) = U_0 \) and \( R(U_1^1) = U_2^1 \), \( R(U_2^1) = U_1^1 \).
Note that we can also consider a homoclinic “figure-fish” configurations, where only one pair of separatrices of the saddle forms a couple of symmetric homoclinic tangencies, see Fig. 4. The proofs are the same, therefore, we do not consider this case separately and illustrate only certain geometric constructions in comparing with the figure-8 case (see Figs. 5 and 6).

In the family \( \{ f_\mu \} \) we study bifurcations of single round periodic orbits, that is, orbits which pass just one time along \( U \). We select three types of such orbits: the 1-orbits that pass through the neighbourhoods \( U_0 \) and \( U_1 \); the 2-orbits that pass through the neighbourhoods \( U_0 \) and \( U_2 \); and the 12-orbits that pass through the neighbourhoods \( U_0, U_1^j \) and \( U_2^j, j = 1, ..., n \). Note that the orbits of the first two types are not symmetric but they compose always (when exist) a symmetric couple; while the orbits of the third type can be self-symmetric.

In this paper we show that, in the family \( \{ f_\mu \} \) bifurcations of 1- and 2-orbits lead to the appearance of a couple “sink-source” periodic orbits, while bifurcations of 12-orbits can lead to the appearance of symmetric elliptic periodic orbits. These results imply, almost automatically, the existence of Newhouse regions (intervals) with reversible mixed dynamics for close reversible maps.

For such orbits we construct the corresponding first return (Poincaré) maps and study bifurcations of their fixed points. In Theorem 1 it is stated that two series (cascades) of infinitely many bifurcations of birth of single-round periodic orbits occur when varying \( \mu \) near 0. The first series includes bifurcations of birth a pair “sink-source” of 1- and 2-orbits, the second series includes bifurcations of birth of symmetric elliptic 12-orbits. In the first case, in the corresponding Poincaré maps (\( R \)-symmetrical each other) the fixed attracting and repelling points appear under non-degenerate saddle-node bifurcations, exist for some intervals of values of \( \mu \) and undergo non-degenerate period doubling bifurcations. In the second case, in the corresponding Poincaré map (which is \( R \)-self-symmetric), a fixed elliptic point is born as result of a nondegenerate reversible fold bifurcation.

The main result of this paper ....

1. Symmetry breaking bifurcations in the case of reversible maps with non-transversal homoclinic figure-8.

Let \( f_0 \) be a \( C^r \)-smooth, \( r \geq 4 \), two-dimensional map, reversible with respect to an involution \( R \) with \( \dim \text{Fix}(R) = 1 \). Let us assume that \( f_0 \) satisfies the following two conditions:

[A] \( f_0 \) has a saddle fixed point \( O \) belonging to the line \( \text{Fix}(R) \) and \( O \) has multipliers \( \lambda, \lambda^{-1} \) with \( 0 < \lambda < 1 \).

[B] \( f_0 \) has a symmetric couple of homoclinic orbits \( \Gamma_1 \) and \( \Gamma_2 \) such that \( \Gamma_2 = R(\Gamma_1) \) and the invariant manifolds \( W^u(O) \) and \( W^s(O) \) have quadratic tangencies at the points of \( \Gamma_1 \) and \( \Gamma_2 \).

Hypotheses [A]-[B] define reversible maps with symmetric non-transversal homoclinic figure-8 like in Figure 3(a) or homoclinic figure-fish shown in Figure 4(b). We ask them to satisfy one more condition. Namely, consider two points \( M^-_1 \in W^u_{\text{loc}}(O) \) and \( M^+_1 \in W^s_{\text{loc}}(O) \) belonging to the same homoclinic orbit \( \Gamma_1 \) and suppose \( f_0^q(M^-_1) = M^+_1 \) for a suitable integer \( q \). Let \( T_1 \) denote the restriction of the map \( f_0^q \) onto a small neighbourhood of the homoclinic point \( M^-_1 \). Then, we assume that
Attracting, repelling and elliptic orbits in reversible maps

[C] the Jacobian \( J_1 \) of the map \( T_1 \) at the point \( M_1^- \) is positive and not equal +1.

Without loss of generality, we assume that \( 0 < J_1 < 1 \). We consider also two points \( M_2^- \in W_{loc}^u(O) \) and \( M_2^+ \in W_{loc}^s(O) \) belonging to \( \Gamma_2 \) such that \( R(M_2^+) = M_1^+ \), \( R(M_2^-) = M_1^- \) and \( f_0^*(M_2^+) = M_2^- \). Let \( T_2 \) denote the restriction of the map \( f_0^* \) onto a small neighborhood of the point \( M_2^- \). Since \( T_2 = R(T_1^{-1}) \), we have that \( J(T_2)|_{M_2^-} = J_1^{-1} > 1 \). The generic condition \([C]\) implies that the global maps \( T_1 \) and \( T_2 \) defined near the corresponding homoclinic points are not conservative maps. 

Note that the point \( O \) divides the invariant manifolds \( W^u(O) \) and \( W^s(O) \) into two parts \( W_i^u(O) \) and \( W_i^s(O) \), \( i = 1, 2 \), such that \( R(W_i^u) = W_2^u \) and \( R(W_i^s) = W_1^s \) (and also \( R(W_i^u) = W_2^u \) and \( R(W_i^s) = W_1^s \)). Then, in the homoclinic figure-8 case, we have that \( M_1^+, M_1^- \in W_1^s \cap W_1^u \) and \( M_2^+, M_2^- \in W_2^s \cap W_2^u \), whereas, in the homoclinic figure-fish case, we will assume that \( M_1^+, M_1^- \in W_i^s \cap W_i^u \), \( i = 1, 2 \).

Once stated the general conditions for \( f_0 \), let us embed it into a one-parameter family \( \{ f_\mu \} \) of reversible maps that unfolds generally at \( \mu = 0 \) the initial homoclinic tangencies at the points of \( \Gamma_1 \) and \( \Gamma_2 \). Then, without loss of generality, we can take \( \mu \) as the corresponding splitting parameter. By reversibility, the invariant manifolds \( W_1^u(O) \) and \( W_1^s(O) \) split as \( W_2^u(O) \) and \( W_2^s(O) \) do when \( \mu \) varies. Therefore, since these homoclinic tangencies are quadratic, only one governing parameter is needed to control this splitting.

We study first bifurcations of single-round 1- and 2-orbits as well as symmetric 12-orbits. Any point of such an orbit is a fixed point of the corresponding first-return map \( T_{1k} \) or \( T_{2k} \), that is constructed by orbits of \( f_\mu \) with \( k \) iterations (of \( f_\mu \)) staying entirely in \( U_0 \cup U_1^j \) and \( U_0 \cup U_2^j \), \( j = 1, ..., n \), respectively.

The first main result is as follows:

**Theorem 1** Let \( \{ f_\mu \} \) be a one-parameter family of reversible diffeomorphisms that unfolds, generally, at \( \mu = 0 \) the initial homoclinic tangencies. Assume that \( f_0 \) satisfies conditions \([A]-[C]\). Then, in any segment \([ -\epsilon, \epsilon ] \) with \( \epsilon > 0 \) small, there are infinitely many intervals \( \delta_k = (\mu_k^{sn}, \mu_k^{pd}) \) and \( \delta_k^{pdC} = (\mu_k^{pd}, \mu_k^{pdC}) \) accumulating at \( \mu = 0 \) as \( k \to \infty \) and such that the following holds.

1. The value \( \mu = \mu_k^{sn} \) and \( \mu = \mu_k^{pd} \) correspond to simultaneous non-degenerate saddle-node and period doubling bifurcations of single-round 1- and 2-orbits of period \( k \). Thus, the first-return map \( T_{1k} \) and \( T_{2k} \) has at \( \mu \in \delta_k \) by two fixed points: sink and saddle for \( T_{1k} \) and source and saddle for \( T_{2k} \). At \( \mu = \mu_k^{pd} \), the sink and source undergo simultaneously a non-degenerate (soft) period doubling bifurcations.

2. The value \( \mu = \mu_k^{pd} \) and \( \mu = \mu_k^{pdC} \) correspond to symmetric and conservative nondegenerate fold and period doubling bifurcations of single-round 12-orbit of period \( 2k \). Thus, the corresponding first-return map \( T_{12k} \) has at \( \mu \in \delta_k^{pdC} \) two symmetric fixed points, elliptic and saddle ones. At \( \mu = \mu_k^{pdC} \) the elliptic point undergoes a symmetric period doubling bifurcation.

\( \dagger \) However, the property of a symmetric saddle periodic point to be a priori area-preserving is more delicate. It is well-known (see, for instance, [13]) that a symmetric reversible saddle map is “almost conservative” since its (local) analytical normal form and its \( C^\infty \) formal normal form (up to “flat terms”) are conservative.
1.1. Preliminary geometric and analytic constructions

Consider the map $f_\mu$. Denote $T_0 \equiv f_\mu|_{U_0}$. The $\mu$-dependent map $T_0$ is called the local map. We introduce also the so-called global maps $T_1$ and $T_2$ by the following relations: $T_1 \equiv f_\mu^{q_1} : \Pi^+_1 \rightarrow \Pi^+_1$ and $T_2 \equiv f_\mu^{q_2} : \Pi^+_2 \rightarrow \Pi^+_2$ which are correctly defined for all small $\mu$, since $f_\mu^{q_1}(M^-_1) = M^+_1$ and $f_\mu^{q_2}(M^-_2) = M^+_2$. Then the first-return maps $T_{1k} : \Pi^+_1 \rightarrow \Pi^+_1$, $T_{2k} : \Pi^+_2 \rightarrow \Pi^+_2$ and $T_{12km} : \Pi^+_1 \rightarrow \Pi^+_1$ are defined by the following composition of maps and neighbourhoods:

$$
\begin{align*}
\Pi^+_1 & \xrightarrow{T_0^k} \Pi_1 \xrightarrow{T_1} \Pi^+_1, \\
\Pi^+_2 & \xrightarrow{T_0^k} \Pi_2 \xrightarrow{T_2} \Pi^+_2, \\
\Pi^+_1 & \xrightarrow{T_0^k} \Pi_1 \xrightarrow{T_{10}^m} \Pi^+_1 \xrightarrow{T_2} \Pi^+_2,
\end{align*}
$$

(1.1)

see Fig. 5 and 6. Accordingly, $T_{1k} = T_1 T_0^k$, $T_{2k} = T_2 T_0^k$ and $T_{12km} = T_2 T_0^m T_1 T_0^k$.

Figure 5. A geometric structure of the homoclinic points $M^+_1, M^-_1, M^+_2$ and $M^-_2$ and their neighbourhoods in the case of figure-8 homoclinic configuration. Schematic actions of the first return maps: (a) case of map $T_{1k} = T_1 T_0^k$, (b) case of map $T_{2k} = T_2 T_0^k$ and (c) case of map $T_{12km} = T_2 T_0^m T_1 T_0^k$.

As usual, we need such local coordinates on $U_0$ in which the map $T_0$ has the simplest form. We can not assume the map $T_0$ is linear, since by condition [A], only $C^1$-linearization is possible here. Therefore, we consider such $C^{r-1}$-coordinates in which the local maps have the so-called main normal form or first order normal form. This form is given by the following lemma.

**Lemma 1** [9] Let a $C^r$-smooth map $T_0$ be reversible with $\dim \text{Fix} T_0 = 1$. Suppose that $T_0$ has a saddle fixed (periodic) point $O$ belonging to the line $\text{Fix} T_0$ and having multipliers $\lambda$ and $\lambda^{-1}$, with $|\lambda| < 1$. Then there exist $C^{r-1}$-smooth local coordinates
near $O$ in which the map $T_0$ (or $T_0^n$, where $n$ is the period of $O$) can be written in the following form:

$$T_0: \begin{cases} \bar{x} = \lambda x(1 + h_1(x, y)xy) \\ \bar{y} = \lambda^{-1}y(1 + h_2(x, y)xy), \end{cases}$$

where $h_1(0) = -h_2(0)$. The map (1.2) is reversible with respect to the standard linear involution $(x, y) \mapsto (-y, x)$. In fact, it can be expressed in the so-called cross-form:

$$T_0: \begin{cases} \bar{x} = \lambda x + \hat{h}(x, \bar{y})x^2\bar{y}, \\ \bar{y} = \lambda \bar{y} + \hat{h}(\bar{y}, x)xy^2. \end{cases}$$

In the case that $T_0$ is linear, i.e. of the form $\bar{x} = \lambda x$, $\bar{y} = \lambda^{-1}y$, its $j$-th iterates $(x_j, y_j) = T_0^j(x_0, y_0)$ are given simply by $x_j = \lambda^j x_0$, $y_j = \lambda^{-j}y_0$ in standard explicit form or by $x_j = \lambda^j x_0$, $y_j = \lambda^j y_0$ in cross-form. If $T_0$ is nonlinear such cross-form expression for $T_0^j$ exists too. Precisely, the following result holds:

**Lemma 2** \[9\] Let $T_0$ be a saddle map written in the main normal form (1.2) (or (1.3)) in a small neighbourhood $V$ of $O$. Let us consider points $(x_0, y_0), \ldots, (x_j, y_j)$ from $V$ such that $(x_{l+1}, y_{l+1}) = T_0(x_l, y_l)$, $l = 0, \ldots, j - 1$. Then one has

$$x_j = \lambda^j x_0 \left(1 + j\lambda^j h_j(x_0, y_0)\right), \quad y_j = \lambda^j y_0 \left(1 + j\lambda^j h_j(y_0, x_0)\right),$$

where the functions $h_j(y_j, x_0)$ are uniformly bounded with respect to $j$ as well as all their derivatives up to order $r - 2$.

**Remark 1** (a) Both lemmas \[1\] and \[2\] are true if $T_0$ depends on parameters. Moreover, if the initial $T_0$ is $C^r$ with respect to coordinates and parameters, then the normal form (1.2) is $C^{r-1}$ with respect to coordinates and $C^{r-2}$ with respect to parameters (see [14], Lemmas 6 and 7).
(b) Bohrcher Theorem (see [12]) ensures that any involution $R$ with $\dim \text{Fix} R = 1$ is locally smoothly conjugated to its linear part around a symmetric point. It is not a loss of generality to assume that maps $f_\mu$ are reversible under an involution $R$ with linear part given by $L(x, y) = (y, x)$. As it will be shown, this fact will be very convenient in the construction of the local maps $T_{01}$ and $T_{02}$.

(c) Similar results related to finite-smooth normal forms of saddle maps were established in [16, 17, 18, 19] for general, near-conservative and conservative maps. The proof of lemmas 7 and 8 is just an adapted version to the reversible setting.

1.2. Construction of the local and global maps

We choose in $U_0$ local coordinates $(x, y)$ given by Lemma [1]. In these coordinates, the local stable and unstable invariant manifolds of the point $O$ are straightened: $x = 0$ is the equation of $W^s_{loc}(O)$ and $y = 0$ is the equation of $W^u_{loc}(O)$. Then, we can write the $(x, y)$-coordinates of the chosen homoclinic points as follows: $M_1^+ = (x_1^+, 0), M_1^- = (0, y_1^-), M_2^+ = (x_2^+, 0)$ and $M_2^- = (0, y_2^-)$. Since $R(M_1^+) = M_2^-$ and $R(M_1^-) = M_2^+$, where $R = (x, y) \mapsto (y, x)$, then we have that $y_1^- = x_1^+$ and $x_2^+ = y_2^-$. Taking into account the homoclinic geometry of the figure-8 case according to Fig. 5, we can assume that

$$x_1^+ = y_1^- = -\alpha_1^* < 0, \quad y_1^- = -x_2^+ = \alpha_2^* > 0$$

In the homoclinic figure-fish case we obtain that $x_1^+ = y_1^- > 0$, $y_1^- = x_2^+ > 0$ according to Fig. 5.

We assume that $T_0(P_i^+) \cap P_i^+ = \emptyset$ and $T_0^{-1}(P_i^-) \cap P_i^- = \emptyset$, $i = 1, 2$. Then the domains of definition of the successor map from $P_i^+$ into $P_j^-$, $i, j = 1, 2$, under iterations of $T_0$ consists of infinitely many non-intersecting strips $\sigma_k^{ij}$ which belong to $P_i^+$ and accumulate at $W^u_{loc}(O) \cap P_i^+$ as $k \to \infty$. In turn, the range of the successor map consists of infinitely many strips $\sigma_k^{ij} = T_0^k(\sigma_k^{0ij})$ belonging to $P_i^-$ and accumulating at $W^s_{loc}(O) \cap P_i^-$ as $k \to \infty$ (see Figure 7).

Thus, the first return maps under consideration are defined as follows:

$$T_{1k} = T_1T_0^k : \sigma_k^{011} \xrightarrow{T_0^k} \sigma_k^{111} \xrightarrow{T_1} \sigma_k^{011},$$
$$T_{2k} = T_2T_0^k : \sigma_k^{022} \xrightarrow{T_0^k} \sigma_k^{222} \xrightarrow{T_2} \sigma_k^{022},$$
$$T_{12km} = T_2T_0^mT_1T_0^k : \sigma_k^{021} \xrightarrow{T_0^k} \sigma_k^{121} \xrightarrow{T_1} \sigma_m^{012} \xrightarrow{T_0^m} \sigma_m^{112} \xrightarrow{T_2} \sigma_k^{021}.$$  

By Lemma 2 the map $T_0^k : \sigma_k^{0ij} \mapsto \sigma_k^{1ij}$ can be written in the following form (for large enough values of $k$)

$$T_0^k : \begin{cases} x_k = \lambda^k x_0 \left( 1 + k\lambda^k h_k(x_0, y_k) \right), \\ y_k = \lambda^k y_k \left( 1 + k\lambda^k h_k(y_k, x_0) \right) \end{cases}$$

(1.7)

where $(x_0, y_0) \in \sigma_k^{0ij}, (x_1, y_1) \in \sigma_k^{1ij}, i, j = 1, 2$.

We write now the global map $T_1 : \Pi_i^+ \to \Pi_i^+$ in the following form

$$T_1 \begin{cases} x_01 - x_1^+ = F_1(x_1, y_1, y_1^- - y_1^-), \mu \equiv ax_11 + b(y_11 - y_1^-) + \varphi_1(x_1, y_11, \mu), \\ y_01 = G_1(x_1, y_1, y_1^- - y_1^-), \mu \equiv \mu + cx_11 + d(y_11 - y_1^-)^2 + \varphi_2(x_1, y_11, \mu), \end{cases}$$

(1.8)

where $F_1(0) = G_1(0) = 0$ since $T_1(M_1^-) = M_1^+$ at $\mu = 0$ and

$$\varphi_1 = O \left( (y_11 - y_1^-)^2 + x_1^2 \right),$$
$$\varphi_2 = O \left( x_1^2 + |y_11 - y_1^-|^3 + |x_11||y_11 - y_1^-| \right).$$
Figure 7. The domains of definition and range of the successor map from $\Pi_i^+$ into $\Pi_j^-$, $i, j = 1, 2$, under iterations of $T_0$ in the cases of (a) homoclinic figure-8; (b) homoclinic figure-fish.

Since the curves $T_1(W_{loc}^u(O) : \{x_{11} = 0\})$ and $W_{loc}^s(O) : \{y_{01} = 0\}$ have a quadratic tangency at $\mu = 0$, it implies that

\[
\frac{\partial G_1(0)}{\partial y_{11}} = 0, \quad \frac{\partial^2 G_1(0)}{\partial y_{11}^2} = 2d \neq 0.
\]

The Jacobian $J(T_1)$ has, obviously, the following form:

\[
J(T_1) = -bc + O \left( |x_{11}| + |y_{11} - y_1| \right).
\]

Thus, $J_1 = -bc$ and we assume (condition C) that $0 < J_1 < 1$.

Concerning the global map $T_2$, we cannot write it now in an arbitrary form. The point is that after written a formula for the map $T_1$ it is necessary to use the reversibility relations to get the one associated to it:

\[
T_2 = RT_1^{-1}R^{-1}, \quad T_1 = RT_2^{-1}R^{-1}
\]

for constructing $T_2$. Then, by (1.8), we obtain that the map $T_2^{-1} : \Pi_2^+ \{ (x_{02}, y_{02}) \} \mapsto \Pi_2^- \{ (x_{12}, y_{12}) \}$ must be written as follows

\[
T_2^{-1} \left\{ \begin{array}{l}
x_{12} = G_1(y_{02}, x_{02} - x_2^+, \mu) = \\
y_{12} - y_2 = F_1(y_{02}, x_{02} - x_2^+, \mu) = \\
\end{array} \right.
\]

Relation (1.9) allows to define the map $T_2 : \Pi_2^- \{ (x_{12}, y_{12}) \} \mapsto \Pi_2^+ \{ (x_{02}, y_{02}) \}$, but in the implicit form: $x_{12} = G_1(y_{02}, x_{02} - x_2^+, \mu), y_{12} - y_2 = F_1(y_{02}, x_{02} - x_2^+, \mu)$.
1.3. Proof of item 1 of Theorem 1

**Proposition 1** Let \( f_c \) reden the family under consideration redsatisfying conditions A–D. Then, for every sufficiently large \( k \), the first return map \( T_{ik} : \sigma_k^i \rightarrow \sigma_k^i \) can be brought, by a linear transformation of coordinates and parameters, to the following form

\[
\begin{align*}
\bar{X} &= Y + O(\lambda^k), \quad \bar{Y} = M_1 + M_2 X - Y^2 + O(\lambda^k), \\
M &= -d\lambda^{-2k}(\mu + 3\lambda^k (cx^+ - y^-) + \ldots), \quad M_2 = bc < 1.
\end{align*}
\] (1.10)

Proof. In virtue of (1.3) and (1.8) the first return maps \( T_{ik} \) is written in the form

\[
\begin{align*}
\bar{x} - x^+ &= b(y - y^-) + |\lambda|^k O(|\bar{x}| + (y - y^-)^2), \\
\lambda^k \bar{y}(1 + \lambda^k O(\bar{x}^2 + \bar{y}^2 + |\bar{x}\bar{y}|)) &= \mu + c\lambda^k x + d(y - y^-)^2 + O(|y - y^-|^3) + \lambda^k O(|\bar{x}| |y - y^-| + \lambda^k x^2).
\end{align*}
\] (1.11)

After a series of changes of coordinates it can be brought to the required form (1.10).

There exists a shift \( \xi = x - x^+ + O(\lambda^k), \eta = y - y^- + O(\lambda^k) \) which cancels the constant term in the first equation and the term linear in \( y - y^- \) in the second equation of (1.11), and the map takes the form

\[
\begin{align*}
\bar{\xi} &= b\eta + \lambda^k O(|\xi| + \eta^2), \quad \bar{\eta} = M_3 \lambda^{-k} + c\xi + d\lambda^{-k} y^2 + O(|\xi| + \lambda^k \xi^2) + \lambda^{-k} O(|\eta|^3),
\end{align*}
\]

where

\[
M_3 = \mu + \lambda^k (cx^+ - y^-) + O(\lambda^{2k}).
\]

Rescaling the variables

\[
\begin{align*}
\xi &= -\frac{b}{d} \lambda^k u, \quad \eta = -\frac{1}{d} \lambda^k v,
\end{align*}
\]

we get the required form (1.10) being \( M_1 = -dM_3 \lambda^{-2k} \).

Proposition 1 shows that the limit form for the first return maps \( T_{ik} \) is the standard Hénon map

\[
\begin{align*}
\bar{x} &= y, \quad \bar{y} = M_1 + M_2 x - y^2,
\end{align*}
\]

with the Jacobian \( J = -M_2 = -bc \). Recall that by assumption in Section 1??? we have \( 0 < J < 1 \). Bifurcations of fixed points of the standard Hénon map are well known. In the \( (M_1, M_2) \)-parameter plane, there are two bifurcation curves

\[
\begin{align*}
L^+ := \{(M_1, M_2) : 4M_1 = -(1 + M_2)^2\}, \quad L^- := \{(M_1, M_2) : 4M_1 = 3(1 + M_2)^2\}
\]

Corresponding to the existence of a fixed point with a multiplier +1 (saddle-node fixed point) and a fixed point with a multiplier −1 (period doubling bifurcation), respectively. For \( -1 < M_2 < 0 \), the has no fixed points below the curve \( L^+ \); has a stable (sink) fixed point in the region between the bifurcation curves \( L^+ \) and \( L^- \), while at \( L^- \) a period doubling bifurcation takes place and a stable 2-periodic orbit appears above the curve \( L^- \).
1.4. Proof of item 2 of Theorem 1

We take in $U_0$ the local $C^{r-1}$-coordinates $(x, y)$ from lemma 1.

We consider the homoclinic points $M_1^+ = (x_1^+, 0)$, $M_1^- = (0, y_1^-)$ of $\Gamma_1$ and $M_2^+ = (x_2^+, 0)$ and $M_2^- = (0, y_2^-)$ of $\Gamma_2$ such that $R(M_1^+) = M_2^-$ and $R(M_1^-) = M_2^+$ and, moreover, the equalities (1.5) hold. Now we will construct the first return map $T_{12km} = T_3 T_0^m T_1^k T_0$.

Briefly, the method we use - based on a rescaling technique - will allow us to prove that the first-return map $T_{12km}$ can be written asymptotically close (as $k, m \to \infty$) to the area-preserving (symplectic) map of the form (see also [9]):

$$H : \begin{cases} \ddot{x} = \dot{M} + \dot{c}x - y^2, \\ \dot{\bar{y}} = -\dot{M} + y + \bar{x}^2, \end{cases} \tag{1.12}$$

in which the coordinates $(x, y)$ and the parameter $\dot{M}$ can take arbitrary values. Note that $\bar{c} = -\frac{b}{c}$ is a constant and $\bar{c} < 0$, since $bc < 0$ by condition C. The bifurcation diagram of the map (1.12) is shown in Figure 8. We represent it on the $(\bar{c}, \dot{M})$-plane, since a character of bifurcations depend on the value of $\bar{c}$.

![Bifurcation diagram](image)

Figure 8. (a) Elements of the bifurcation diagram for the map $H$: the hitched regions correspond to the existence of fixed elliptic point at $H$; (b) a map of dynamical regimes for the map $H$: regions with different numbers of fixed and periodic orbits are painted with different colors.

Equations of the bifurcation curves $F_0$ (fold bifurcation), $PD_1$ and $PD_2$ (period doubling), $PF$ (pitch-fork) and $PD_{(asym)}$ (period doubling of a couple of symmetric

---

Footnote:

§ The rescaling method [20] appears to be very efficient to study homoclinic bifurcations.
fixed points) are as follows:

\[
F_0 : \quad \tilde{M} = -\frac{1}{4}(\tilde{c} - 1)^2, \quad \tilde{c} < 0,
\]

\[
PD_1 : \quad \tilde{M} = 1 - \frac{1}{4}(\tilde{c} - 1)^2, \quad \tilde{c} < 0,
\]

\[
PD_2 : \quad \tilde{M} = \left(\tilde{c} + 1\right)/(3\tilde{c} - 1), \quad \tilde{c} < 0,
\]

\[
PF : \quad \tilde{M} = \frac{3}{4}(\tilde{c} - 1)^2, \quad \tilde{c} < 0,
\]

\[
PD(\text{asym}) : \quad \tilde{M} = \frac{(1 - 3\tilde{c})(3 - \tilde{c})}{4}, \quad \tilde{c} < 0.
\]

We denote the coordinates \((x, y)\) on \(\Pi_1^+\) as \((x_0, y_0)\) and on \(\Pi_1^-\) as \((x_1, y_1)\), \(i = 1, 2\). Then, by Lemma \[2\] the map \(T_0^\sigma: \Pi_1^\sigma \to \Pi_1^\sigma\) will be defined on the strip \(\sigma_k^{021} \subset \Pi_2^+\) and \(T_0^k(\sigma_k^{021}) = \sigma_k^{121} \subset \Pi_1^+\). Analogously, the strips \(\sigma_k^{011} \subset \Pi_1^+\), \(\sigma_k^{012} \subset \Pi_1^+\), \(\sigma_k^{022} \subset \Pi_2^+\) exist such that \(T_0^k(\sigma_k^{011}) = \sigma_k^{111} \subset \Pi_1^-, \quad T_0^k(\sigma_k^{012}) = \sigma_k^{112} \subset \Pi_2^-, \quad T_0^k(\sigma_k^{022}) = \sigma_k^{122} \subset \Pi_2^-.\) See Fig. \[\tilde{7}\] Then the first return map \(T_{12k}\) can be constructed as the following composition

\[
\sigma_k^{021} \overset{T_0^k}{\longrightarrow} \sigma_k^{121} \overset{T_1}{\longrightarrow} \sigma_k^{102} \overset{T_2^m}{\longrightarrow} \sigma_m \overset{T_1}{\longrightarrow} \sigma_k^{021}
\]

We can write these relations by the coordinates as follows:

\[
x_{11} = \lambda^k x_{12} (1 + k\lambda^kh_k(x_{02}, y_{11})), \quad y_{02} = \lambda^k y_{11} (1 + k\lambda^kh_k(y_{11}, x_{02})),
\]

\[
x_{01} - x_1^+ = F_1(x_{11}, y_{11} - y_1^-; \mu) = ax_{11} + b(y_{11} - y_1^-) + \varphi_1(x_{11}, y_{11}, \mu),
\]

\[
y_{01} = G_1(x_{11}, y_{11} - y_1^-; \mu) = \mu + cx_{11} + d(y_{11} - y_1^-)^2 + \varphi_2(x_{11}, y_{11}, \mu),
\]

\[
x_{12} = \lambda^m x_{01} (1 + m\lambda^mh_m(x_{01}, y_{12})), \quad y_{01} = \lambda^m y_{12} (1 + m\lambda^mh_m(y_{12}, x_{01})),
\]

\[
x_{12} = G_1(\tilde{y}_{02}, \tilde{x}_{02} - x_2^+; \mu) = \mu + c\tilde{y}_{02} + d(\tilde{x}_{02} - x_2^+)^2 + \varphi_2(\tilde{y}_{02}, \tilde{x}_{02}, \mu),
\]

\[
y_{12} - y_2^- = F_1(\tilde{y}_{02}, \tilde{x}_{02} - x_2^+; \mu) = a\tilde{y}_{02} + b(\tilde{x}_{02} - x_2^+) + \varphi_1(\tilde{y}_{02}, \tilde{x}_{02}, \mu).
\]

In these formulas we use the cross-coordinates \((x_0, y_1)\) on the strips \(\sigma_k^0\), since by Lemma \[4\] the coordinates \((x_0, y_0)\) of any point of \(\sigma_k^0\) are recalculated via the cross-coordinates as \((x_0, \lambda^k y_1 (1 + k\lambda^kh_k(y_{12}, x_{01})))\). Accordingly, the coordinates \(y_0\) appear in \[1.14\] only as auxiliary ones. On the other hand, the coordinates \((x_{02}, y_{11})\) are dynamical ones, since formulas \[1.14\] define the map \((x_{02}, y_{11}) \mapsto (\tilde{x}_{02}, \tilde{y}_{11})\), where \(\tilde{y}_{02} = \lambda^k y_{11} (1 + k\lambda^kh_k(y_{12}, x_{01}))\). Thus, the coordinates \((x_{01}, y_{12})\) are only intermediate.

Introduce the new variables \(x_1 = x_{01} - x_1^+, \quad x_2 = x_{02} - x_2^+, \quad y_1 = y_{11} - y_1^-, \quad y_2 = y_{12} - y_2^-\). Then the system \[1.13\] is rewritten as

\[
x_1 = by_1 + O(\lambda^k) + O(y_1^2),
\]

\[
\lambda^m y_2 = (\mu + c\lambda^k x_2 - \lambda^m y_2) + dy_1^2 + c\lambda^k x_2 + O(\lambda^k |x_2| + \lambda^k x_2 y_1 + |y_1|^3),
\]

\[
\lambda^m x_1 = (\mu + c\lambda^k y_1 - \lambda^m x_1^+ + c\lambda^k \tilde{y}_1 + d\tilde{x}_2^2 + O(\lambda^k |\tilde{x}_2| + \lambda^k |\tilde{x}_2 \tilde{y}_1| + |\tilde{y}_1|^3)
\]

\[
y_2 = b\tilde{x}_2 + O(\lambda^k) + O(x_2^2),
\]

We express \(x_1\) and \(y_2\) from the first and fourth equations of \[1.15\] and put them in the second and third equations. After this, we obtain the map \(T_{12km}: (x_{02}, y_{11}) \mapsto \)
Attracting, repelling and elliptic orbits in reversible maps

\[(x_{02}, y_{11})\] in the following (implicit) form.

\[
\begin{align*}
\lambda^m b x_2 &= M + dy_1^2 + c\lambda^k x_2 + O(\lambda^{2k}|x_2| + \lambda^k |x_2 y_1| + |y_1|^3), \\
\lambda^m b y_1 &= M + c\lambda^k y_1 + dx_2^2 + O(\lambda^{2k} |x_2| + \lambda^k |x_2 y_1| + |y_1|^3)
\end{align*}
\] (1.16)

where, by virtue of (1.5),

\[M = \mu + c\lambda^k \alpha_2^* + \lambda^m \alpha_1^* .\]

Now we rescale coordinates in (1.16) as follows

\[x_2 = -b d^{-1} \lambda^m x, \ y_1 = -b d^{-1} \lambda^m y\]

Then we obtain the first return map \(T_{12k}\) in the following rescaling form

\[
\begin{align*}
\tilde{x} &= \tilde{M} + \tilde{c} x - y^2 + O(\lambda^k), \\
\tilde{y} &= \tilde{M} + \tilde{c} y - \tilde{x}^2 + O(\lambda^k),
\end{align*}
\]

where

\[
\tilde{c} = b^{-1} \lambda^{-k-m}, \ \tilde{M} = -\frac{d}{b^2} \lambda^{-2m} (\mu + c\lambda^k \alpha_2^* + \lambda^m \alpha_1^* ) .
\]

Q.E.D.

1.5. Remark to the case of nonorientable reversible maps with a symmetric couple of quadratic homoclinic tangencies.

Our consideration covers formally also the case of nonorientable reversible maps. Indeed, in this case we assume that \(f_0\) satisfies the conditions A and B and now the condition

[\(\mathcal{C}'\)] the Jacobian \(J_1\) of the map \(T_1\) at the point \(M_1^-\) is negative and not equal \(-1\).

This case is possible if \(f_0\) is defined on an non-orientable surface like Möbius band, Klein bottle etc.

Note that the necessary calculation for the first return maps are obtained the same as in the orientable case. Only the result in the case of the map \(T_{12km}\) will be different. Here, the first-return map \(T_{12km}\) can be written asymptotically close (as \(k, m \to \infty\)) to the area-preserving (symplectic) map of the form (1.12) again. However, \(\tilde{c} = \frac{b}{\lambda} \) is positive here, since \(bc > 0\) by condition \(\mathcal{C}'\). The bifurcation diagram of the map (1.12) on the \((\tilde{c}, \tilde{M})\)-half-plane, where \(\tilde{c} > 0\) is shown in Figure 9.

Equations of the bifurcation curves \(F_0\) (fold bifurcation), \(PD_1\), \(PD_2\) (period doubling) and \(PF\) (pitch-fork) are as follows:

\[
\begin{align*}
F_0: \quad \tilde{M} &= -\frac{1}{4} (\tilde{c} - 1)^2, \ \tilde{c} > 0, \\
PD_1: \quad \tilde{M} &= 1 - \frac{1}{4} (\tilde{c} - 1)^2, \ \tilde{c} > 0, \\
PD_2: \quad \tilde{M} &= \frac{1}{4} (\tilde{c} + 1)(3\tilde{c} - 1), \ \tilde{c} > 0, \\
PF: \quad \tilde{M} &= \frac{3}{4} (\tilde{c} - 1)^2, \ \tilde{c} > 0,
\end{align*}
\] (1.17)

Note that in this case the pitch-fork bifurcation leads to the appearance of a couple of symmetric saddle fixed points that do not bifurcate at further varying values of parameters. Accordingly, the bifurcation curve \(PD(symp)\) as in case \(\tilde{c} < 0\) is absent here.
Figure 9. (a) Elements of the bifurcation diagram for the map $H$ with $\tilde{c} > 0$: the painted region corresponds to the existence of fixed elliptic point at $H$; (b) a map of dynamical regimes for the map $H$: regions with different numbers of fixed and periodic orbits are painted with different colors.

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