Integral transforms related to Nevanlinna-Pick functions from an analytic, probabilistic and free-probability point of view

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Abstract We establish a new connection between the class of Nevanlinna-Pick functions and the one of the exponents associated to spectrally negative Lévy processes. As a consequence, we compute the characteristics related to some hyperbolic functions and we show a property of temporal complete monotonicity, similar to the one obtained via the Lamperti transformation by Bertoin & Yor (On subordinators, self-similar Markov processes and some factorizations of the exponential variable, Elect. Comm. in Probab., vol. 6, pp. 95–106, 2001) for self-similar Markov processes. More precisely, we show the remarkable fact that for a subordinator \( \xi \), the function \( t \mapsto t^n \mathbb{E}[\xi_t^{-p}] \) is, depending on the values of the exponents \( n = 0, 1, 2, \ p > -1 \), or a Bernstein function or a completely monotone function. In particular, \( \xi \) is the inverse time subordinator of a spectrally negative Lévy process, if, and only if, for some \( p \geq 1 \), the function \( t \mapsto t \mathbb{E}[\xi_t^{-p}] \) is a Stieltjes transform. Finally, we clarify to which extent Nevanlinna-Pick functions are related to free-probability and to Voiculescu transforms, and we provide an inversion procedure.

Key words: Characteristic functions; Laplace Transform; Cauchy transform; Voiculescu transform; Convolution; Infinite divisibility; Free-infinite divisibility; Lévy processes; Subordinators; Analytic functions; Nevanlinna-Pick functions; Bernstein functions; Complete Bernstein functions; Hyperbolic characteristic functions.

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1 Introduction

In complex analysis, analytic functions that preserve the upper complex half-plane play an important role. According to Nevanlinna-Pick theorem, these are precisely functions $F$ that admit the (unique) representation:

$$F(z) = a + b z + \int \frac{zx-1}{z+x} \rho(dx), \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}, \text{ where } a \in \mathbb{R}, \ b \geq 0, \text{ and } \rho \text{ is a finite Borel measure.}$$

We will denote by $\mathcal{N}$ the set of all Nevanlinna-Pick functions while $\mathcal{N}(A)$, for a Borel set $A$, stands for

$$\mathcal{N}(A) := \{ F \in \mathcal{N} \text{ such that in } (5), \ \text{support} (\rho) \subset A \}.$$

On the other hand, there are complete Bernstein functions, cf. Appendix, i.e. those functions $f : (0, \infty) \to \mathbb{R}$ that analytically extend to $z \in \mathbb{C} \setminus (-\infty, 0]$ and have a form as Nevanlinna-Pick functions as in (1). Furthermore, in the free-probability theory (cf. Appendix) an infinitely divisible probability measure $\nu$ has its Voiculescu transform $V_\nu(z)$ as follows

$$V_\nu(z) = a + \int \frac{zx-1}{z+x} m(dx), \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R},$$

for a uniquely determined finite measure $m$. In this paper, we will not use the analyticity property of Nevanlinna-Pick functions or Voiculescu transforms. In fact, from the values of $F(it)$ or $V_\nu(it)$, $t \in \mathbb{R}$, we can retrieve the representing parameters, in particular, the $\rho$-measures, cf. (7) below.

Recall that in the (classical) probability we have that a measure $\mu$ is infinitely divisible if and only if its Fourier transform (or characteristic function) $\hat{\mu}(t)$ is of a form

$$\hat{\mu}(s) := \int e^{i sx} \mu(dx) = \exp \left[ iu a - b^2 u^2 + \int \left( e^{i sx} - 1 - \frac{ix}{1+x^2} \right) \frac{1+x^2}{x^2} m(dx) \right], \quad \text{(3)}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $m$ is a finite Borel measure. The above is the famous Lévy-Khintchine formula for infinitely divisible characteristic functions and those are closely related to Lévy stochastic process. Note that (2) and (3) give a bijection between the classical infinite divisibility and the free-probability infinite divisibility. Namely, $\nu$ in (2) is a free-probability analogue of $\mu$ in the classical probability (when $b = 0$).

In this paper, Theorem 1, gives some equivalent characterizations of $F \in \mathcal{N}(A)$, among them, there is a way to associate $F$ with a Cauchy transform. In Proposition and Theorem 2, we establish a temporal monotonicity property for subordinators. Finally, in Propositions 3 and 5, are found Voiculescu transforms of free-probability analogues of the classical infinitely divisible hyperbolic functions. In particular, the
\( \rho \)-measures in (2) are given as a composition of Laplace and Fourier transforms.

In order to precise our main results, we need to formalize our objects. The right, left, upper and lower complex half-plane are denoted by

\[
\mathbb{C}^+ := \{ z \in \mathbb{C} : \Re(z) > 0 \}, \quad \mathbb{C}^- := -\mathbb{C}^+, \quad \mathbb{H}^+ := \{ z \in \mathbb{C} : \Im(z) > 0 \}, \quad \mathbb{H}^- := -\mathbb{H}^+.
\]

The Alexandrov compactification of \( \mathbb{C} \cup \{ \infty \} \) is denoted by \( \overline{\mathbb{C}} \). For given \( \alpha, N > 0 \) and \( M \in \mathbb{R} \), we introduce the angular sectors and cones in \( \mathbb{H}^+ \):

\[
\Gamma_{\alpha}(M) := \{ z = a + ib \in \mathbb{H}^+ : \alpha b > |a-M| \}, \quad \Gamma_{\alpha} := \Gamma_{\alpha}(0) = \Gamma_{\alpha}(M) - M, \quad \Gamma_{\alpha,N} := \{ z = a + ib \in \Gamma_{\alpha} : b > N \}.
\]

In classical complex analysis, one of the fundamental results is the integral representation of analytic functions defined on \( \mathbb{H}^+ \), or \( \mathbb{C} \setminus \mathbb{R} = \mathbb{H}^+ \cup \mathbb{H}^- \), that preserve \( \mathbb{H}^+ \). If such an analytic function, say \( F \), is only defined on \( \mathbb{H}^+ \), then we can extend it onto \( \mathbb{C} \setminus \mathbb{R} \) by setting \( F(z) = F(\overline{z}) \), if \( z \in \mathbb{H}^- \). Then, \( F \) admits the (unique) canonical form: for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
F(z) = a + bz + \int_{\mathbb{R}} \frac{ze^{-1}}{z+x} \rho(dx)
\]

\[
= a + bz + \int_{\mathbb{R}} \left( \frac{x}{1+x^2} - \frac{1}{x+z} \right) (1+x^2) \rho(dx).
\]

where \( a \in \mathbb{R} \) is a real number, \( b \geq 0 \) and \( \rho \) is a finite (Borel) measure on \( \mathbb{R} \). Note that

\[
a = \Re(F(i)), \quad b = \lim_{u \to \infty} F(au)/u \quad \text{and} \quad \rho(\mathbb{R}) = \Im(F(i)) - b,
\]

that for every measure \( \rho \) we have the inversion formula, cf. [1, p.126] or Lang [20, p.380]:

\[
\int_{[u,v]} (1+x^2) \rho(dx) = \lim_{t \to 0^+} \frac{1}{\pi} \int_{[u,v]} \Im(F(-x+it)) \ dx, \quad \text{whenever } \rho([u,v]) = 0,
\]

and that the half-plane preservation property could be seen from property (P1) in Section 2, of the function \( f \) in (14) below. Those function \( F \) are coined in the literature as Pick or Nevanlinna-Pick functions. cf. Akhiezer [1], Bondesson [8, p.21], or Schilling, Song and Vondraček [25, Chapter 6]. The (unique) representing measure \( \rho \) appearing in (5) and (6) is called the Nevanlinna measure.

In a couple of last decades, representations of the form (5) or (6), with \( b = 0 \), appeared in the free-probability as the free-probability analog of the classical Lévy-Khintchine formula for infinitely divisible characteristic functions (Fourier transforms). More precisely, they are defined as

\[
V(z) := a + \int_{\mathbb{R}} \frac{1-ze^x}{z+x} \rho(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}
\]
and are called Voiculescu transforms, cf. Bercovici and Voiculescu [4, Section 5, in particular Theorem 5.10]. Observe that if \( \text{support}(\rho) = A \subset \mathbb{R} \), then

- \( -V \in \mathcal{N}(A) \);
- if \( \tilde{\rho} \) is the image of \( \rho \) by the mapping \( x \mapsto -x \), then
  
  \[
  z \mapsto V\left(\frac{1}{z}\right) = a + \rho(\{0\}) + \int_{\tilde{A}} \frac{z x - 1}{z + x} \tilde{\rho}(dx) \in \mathcal{N}(\tilde{A}).
  \]

Note that in (8), we need to know \( F \) in some strips of the complex plane to retrieve a measure \( \rho \). A natural question is:

- What can be said about a measure \( \rho \) if we only have values \( F(iw) \), for \( w \neq 0 \), and we don’t know if it is a restriction of an analytic function to the imaginary axis?

The answer was given by Jankowski and Jurek [11, Theorem 1], where there is an inversion procedure, which allows to identify the measure \( \rho \), or more precisely its Fourier transform \( \mathcal{F}[\rho, s] \), \( s \in \mathbb{R} \), and which is justified as follows: let us, for an index \( X \) (where \( X \) can be a random variable, or a stochastic process or a measure), define the function \( F_X \) on the imaginary axis \( i(\mathbb{R} \setminus \{0\}) \) by

\[
F_X(iw) := a_X + ib_X w + \int_{\mathbb{R}} \frac{iwx - 1}{iw + x} \rho_X(dx), \quad w \neq 0,
\]

where \( a_X \in \mathbb{R} \), \( b_X \geq 0 \) and \( \rho_X \) is a non-negative, finite Borel measure. Furthermore, if

\[
\mathcal{F}[^{\rho_X}; s] := \int_{\mathbb{R}} e^{iws} \rho_X(dx), \quad s \in \mathbb{R},
\]

is the Fourier transform (respectively the characteristic function) of a finite measure (respectively probability measure) \( \rho_X \), then the Laplace transform of \( \mathcal{F}[^{\rho_X}; s] \) satisfies the equality

\[
\mathcal{L}[\mathcal{F}[^{\rho_X}; s]; w] := \int_{\mathbb{R}} e^{-ws} \mathcal{F}[^{\rho_X}; s] ds = \int_{\mathbb{R}} \frac{1}{w - iw} \rho_X(dx), \quad w > 0
\]

\[
= \begin{cases} 
\frac{1}{w - 1} \left[iF_X(iw) - i\Re\{F_X(iw)\} + w\Im\{F_X(iw)\}\right] & \text{if } w \neq 1 \\
\int_{\mathbb{R}} \frac{1 + iw}{1 + x^2} \rho_X(dx) & \text{if } w = 1.
\end{cases}
\]

Motivated by the special structure of the Nevanlinna-Pick functions appearing in (11), we focus in Section 2 on their analytical aspect and provide additional clarification to their relation to classical and free probability context. An appendix is
Our main results are obtained in sections 3 and 4, where we study two aspects of Nevanlinna-Pick functions:

- The first studied aspect, is mainly probabilistic. In Section 3 we generalize the connection described by Schilling, Song & Vondraček [25, Theorem 6.9], between the subclass of nonnegative functions in \( N \) and the one of Bernstein functions associated to subordinators. We provide an connection of the same nature between a subclass of Nevanlinna-Pick functions and spectrally negative Lévy processes. This connection is also of same nature than the one obtained right after (30) below when replacing Bernstein functions by Lévy-Laplace exponents. We also provide a property of temporal complete monotonicity for subordinators as Bertoin and Yor [6] did for self-similar Markov processes obtained via the Lamperti transform, cf. (34) below. During several discussions of the first author with Loïc Chaumont, a recurrent question was:

Let \( \Phi \) be the Bernstein function of some subordinator \( \xi \) such that is impossible to check if \( \Psi = \Phi^{-1} \) is a Lévy-Laplace exponent by standard calculations. Are there any other clue indicating whether \( \xi \) is the first passage time of some spectrally negative Lévy process? (12)

Motivated by this question, we provide in Theorem 2 a temporal complete monotonicity property: Any subordinator \( \xi = (\xi_t)_{t \geq 0} \), such that its \( \Psi \)-function satisfies some integrability conditions, satisfies one the following:

(i) \( t \mapsto \mathbb{E}[\xi_{ut}^{-p}] \) is a Bernstein function, for some \( p \in (-1, 0) \);
(ii) \( t \mapsto \mathbb{E}[\xi_{ut}^{-p}] \) is a completely monotone function, for some \( p \in [0, 1) \);
(iii) \( t \mapsto \mathbb{E}[\xi_{ut}^{-p}] \) is a completely monotone function, for some \( p \geq 1 \).

In Corollary 1, we answer to question (12) by showing that a subordinator \( \xi \) is the inverse times of a spectrally negative Lévy process, if, and only if,

\( t \mapsto t \mathbb{E}[\xi_{ut}^{-p}] \) is a Stieltjes transform, for some \( p \geq 1 \).

The latter is also equivalent to:

\( t \mapsto t^2 \mathbb{E}[\xi_{ut}^{-p}] \) is a complete Bernstein function, for some \( p \geq 1 \).

Of course, one can argue that the calculus of the moments \( \mathbb{E}[\xi_{ut}^{-p}], t > 0, p > -1 \), are also not always feasible, but, when they are, we immediately have a conclusion, see Example 2 and especially Example 3. To the best of our knowledge, the temporal properties that we obtain are new.

- The second aspect, studied in Section 4, is related to the free-probability context, and consists in the application of the inversion formula (32). In Propositions 3, 4, 5 and Corollary 2, below, we show that for some related index \( X \), the functions
in (11), are indeed, Laplace transforms of some functions or measures that, in principle, enables us to identify the corresponding measure \( \rho \). Let \( X = C, S \) and \( T \) be random variables defined by their hyperbolic characteristic functions

\[
\widehat{\gamma}_X(s) := \mathbb{E}[e^{isX}] = \int_{\mathbb{R}} e^{isx} \mathbb{P}(X \in dx), \quad s \in \mathbb{R},
\]

\[
\widehat{\gamma}_C(s) := \frac{1}{\cosh(s)}, \quad \widehat{\gamma}_S(s) := \frac{s}{\sinh(s)}, \quad \widehat{\gamma}_T(s) := \frac{\tanh(s)}{s}.
\]

and \( \tilde{X} \) be the free-analog of the of \( X \), obtained by the procedure (P4) in Section 2.

We emphasize that in our approach to free-probability theory, we only use purely imaginary numbers \( z = iw, w > 1 \); cf. (32) and Jurek [16, Corollaries 3, 4 and 5] and we recall that the hyperbolic characteristic functions were studied:

(i) from an infinite divisibility point of view, in Pitman and Yor [24];
(ii) from a self-decomposability point of view in Jurek [15], as infinite series of independent exponentially distributed variables;
(iii) from stochastic representations of their background driving Lévy processes (BDLP) in Jurek-Vervaat [17, Theorem 3.2]. The last can be done since all hyperbolic characteristic functions are self-decomposable ones and therefore, they admit a representation by random integrals

\[
X = \int_0^\infty e^{-t} dY_t, \quad \text{where } Y := Y_X \text{ is the BDLP,} \tag{13}
\]

see also Jurek and Mason [14, Chapter 3, Theorem 3.6.8].

2 Various interpretations of Nevanlinna-Pick functions

We first list several interesting properties of Nevanlinna-Pick functions.

(P1) Every Nevanlinna-Pick function \( F \) in (5) is represented by

\[
F(z) = a + bz + \int_{\mathbb{R}} f(x, z) \rho(dx), \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

where the functions

\[
z \mapsto f(x, z) = \frac{xz - 1}{z + x}, \quad x \in \mathbb{R}, \ z \in \mathbb{C} \setminus \{x\} \tag{14}
\]
are the it Möbius mappings, which are conformal (i.e analytic with non-null derivative) on any region contained in \( \mathbb{C} \setminus \{x\} \) and then, preserve the angles in this region. Roughly speaking, \( f \) preserve the angles means that for any two rays \( L \) and \( L' \), starting at a point \( z_0 \), the angle which their images \( f(L) \) and \( f(L') \) make at \( z_0 \) is the same as that made by \( L \) and \( L' \), in size as well as in orientation. These mappings enjoy the following properties:

(a) \( f(x,i) = i \);

(b) \( \Im(f(x,z)) = \frac{1 + x^2}{|x + z|^2} \Im(z) \implies f(x, \mathbb{H}^+) \subset \mathbb{H}^+ \) and \( f(x, \mathbb{H}^-) \subset \mathbb{H}^- \);

(c) \( f(x, \overline{z}) = f(x, z) \);

(d) \( f(x, -z) = -f(-x, z) \);

(e) \( f \left( \frac{1}{x}, \frac{1}{z} \right) = -f(x, z), \quad x, z \neq 0. \)

(P2) For every \( \alpha > 0 \) there exists \( N_\alpha \) such that \( F \) has a left inverse \( F^{-1} \) defined on the region \( \Gamma_{\alpha, N_\alpha} \) given by (4), cf. [4, Proposition 5.4 and Corollary 5.5].

(P3) Any Cauchy transforms, i.e., a function of the form

\[
G_\mu(z) = c + \int_{\mathbb{R}} \frac{1}{z + x} \mu(dx), \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

where \( c \) is a real number and \( \mu(dx) = (1 + x^2) \rho(dx) \) is a finite measure, satisfies \( -G \in \mathcal{N}(\mathbb{R}) \). To see that, it is enough to take \( b = 0 \) in (6) and to replace \( a + \int_{\mathbb{R}} x \rho(dx) \) by \( -c \). In other terms, \( G \) swaps the upper and lower complex half planes. In this case, \( z \mapsto G_\mu(1/z) \in \mathcal{N}(\mathbb{R}) \). To see this, use property (c) of the function \( f \) in (14) or the half-plane swapping property, which also yields \( H_\mu = 1/G_\mu \in \mathcal{N}(\mathbb{R}) \). Now, assume \( c = 0 \) and observe that

\[
d := \mu(\mathbb{R}) = \lim_{|z| \to \infty, z \in \Gamma_{\alpha, N}} z G_\mu(z) > 0, \quad \text{for every } \alpha, N > 0.
\]

The finite measure \( \mu \) is called (additive)-free infinitely divisible if, and only if, the function \( H_\mu = 1/G_\mu \) is such that its inverse \( H_\mu^{-1} \) (which, by (P2)) in Section (2), always exists) satisfies

\[
z \mapsto F_\mu(z) := dz - H_\mu^{-1}(z) \in \mathcal{N}(\mathbb{R}).
\]

The function \( V_\mu(z) = -F_\mu \) is commonly called the Voiculescu transform of \( \mu \), cf. (9) and Bercovici and Voiculescu [4]. The free probability context corresponds to probability measures \( \mu \) and the parameter \( d \) in (15) is set to be equal 1.

(P4) Let \( \Psi \) be a Lévy-Laplace exponent, i.e., a function represented by

\[
\Psi(\lambda) = \alpha + \beta \lambda + \frac{\lambda^2}{2} + \int_{\mathbb{R} \setminus \{0\}} \left( e^{-\lambda x} - 1 + \lambda \frac{x}{1 + x^2} \right) v(dx), \quad \lambda \in i \cdot \mathbb{R},
\]
where \( \alpha, \gamma \geq 0, \beta \in \mathbb{R} \), and the so-called Lévy measure \( \nu \) is supported by \( \mathbb{R} \setminus \{0\} \) and such that

\[
\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty \iff \int_{\mathbb{R}} \frac{x^2}{1 + x^2} \nu(dx) < \infty.
\]

As we did for the Nevanlinna class, we denote

\[
\mathcal{LE}(A) := \{ \text{Laplace exponents } \Psi \text{ represented by (18), s.t. } \text{support}(\nu) \subset A \}, \quad A \subset \mathbb{R}.
\]

For the set \( \mathcal{LE}(0, \infty) \), we have the following stochastic interpretation: there is a bijection between the class of (killed) Lévy process, i.e. a process \( Z = (Z_t)_{t \geq 0}, Z_0 = 0 \), with stationary and independent increments, and the class of Lévy-Laplace exponent \( \Psi_X \), represented by (18), via the celebrated Lévy-Khintchine formula, i.e. the Laplace representation: if \( X := Z_1 \), then

\[
\mathbb{E}[e^{\lambda Z_t}] = e^{\Psi_X(\lambda t)}, \quad \text{for } t \geq 0, \ \lambda \in i.\mathbb{R}.
\]

Actually, the distributions of \( Z_t, t > 0 \), are entirely determined by the one of the infinitely divisible random variable \( X = Z_1 \). In (18), the usage is to call the quantity \( \alpha_X \) the killing rate, \( \beta_X \) the drift term and \( \gamma_X \) the Brownian coefficient.

To every function Nevanlinna-Pick function \( F \), represented by (10), one can associate a free-infinitely divisible random variable \( \tilde{X} \) that we call free-analog of \( X \) by these means:

- Pick a Voiculescu transform of some free-infinitely divisible random variable \( \tilde{X} \) whose distribution is the probability measure \( \mu \) obtained by (17).

- Barndorff-Nielsen and Thorbjørnsen [2, Theorem 4.1] and Jurek [16, Theorem 1], clarified, in a more involved way, the so-called Bergovici-Pata bijection from the class of Lévy processes to the one of free-infinitely divisible random variables, cf. Bercovici, Pata & Biane [3]. More precisely, there exists an infinitely divisible random variable \( X \) with Lévy-Laplace exponent \( \Psi_X \), represented by (18), such that the function

\[
F_X(iw) = iw^2 \int_0^\infty e^{-wu} \Psi_X(-iu) du, \quad w > 0.
\]

- Observe that the Laplace transform of \( \Psi_X(-iu) \) always exists since it is a continuous function which satisfies \( \lim_{|u| \to \infty} \Psi_X(-iu)/u^2 = -\gamma^2/2 \). The latter is justified by the control of the kernel: for \( \Re(\lambda) \geq 0, x \neq 0 \), we have

\[
e^{-\lambda x} - 1 + \lambda \frac{x}{1 + x^2} = (e^{-\lambda x} - 1 + \lambda x) \mathbb{I}_{|x| \leq 1} + (e^{-\lambda x} - 1) \mathbb{I}_{|x| > 1} + \lambda \left( \frac{x^2}{1 + x^2} \mathbb{I}_{|x| \leq 1} - \frac{x}{1 + x^2} \mathbb{I}_{|x| > 1} \right).
\]

and for \( \lambda \geq 0 \) or \( \lambda = iu, u \in \mathbb{R} \), we have
\[ \left| e^{-\lambda x} - 1 + \lambda \frac{x}{1 + x^2} \right| \leq \frac{1}{2} \lambda^2 |x|^2 \mathbb{1}_{|x| \leq 1} + 2 \mathbb{1}_{|x| \geq 1} + |\lambda| \left( \frac{x^2}{1 + x^2} \mathbb{1}_{|x| \leq 1} + \frac{x}{1 + x^2} \mathbb{1}_{|x| \geq 1} \right) \leq C(\lambda) (x^2 \wedge 1), \]

where \( C(\lambda) := 3 + |\lambda| + \frac{|\lambda|^2}{2} \). The latter insures that \( \Psi_X \) is well defined due to the condition (19) on its Lévy measure. Thus, the triplet of characteristics of \( \hat{F}_X \) are given by

\[ a_{\hat{X}} = a_X, \quad b_{\hat{X}} = a_X \quad \text{and} \quad \rho_{\hat{X}}(dx) = \gamma_X^2 \delta_0(dx) + \frac{x^2}{1 + x^2} \nu_X(dx), \quad (23) \]

\( \rho_{\hat{X}} \) is a finite measure and

the operator \( \Delta : \Psi \mapsto F \) defined by (21), induces a bijection from \( \mathcal{L}\mathcal{E}(0, \infty) \) to \( \mathcal{N}(\mathbb{R}) \), or equivalently, from the class of Lévy processes to the one of free-infinitely divisible random variables.

**Remark 1.** Assume \( X \overset{d}{=} X_1 + X_2 \) where \( X_1 \) and \( X_2 \) are independent infinitely divisible random variables. On the level of the Lévy measures, we have \( \nu_X = \nu_{X_1} + \nu_{X_2} \) and (23) gives \( \rho_{\hat{X}} = \rho_{\hat{X}_1} + \rho_{\hat{X}_2} \) on the level of the Nevanlinna measures.

**P5** If \( F \) has a continuous extension with \( F : (0, \infty) \to \mathbb{R} \), then \( F \in \mathcal{N}(0, \infty) \). Further, if

\[ F(0+) = a - \int_{(0,1]} \frac{1}{x} \rho(dx) \]

is finite, then \( F \in \mathcal{N}(0, \infty) \), cf. [25, Theorem 6.9].

**P6** The class \( \mathcal{S} \) of *Stieltjes transforms* is formed by functions with a represented similar to (15): \( f \in \mathcal{S} \), if

\[ f(\lambda) = d + \frac{q}{z} + \int_{(0,\infty)} \frac{1}{z + u} \Delta(du), \quad z \in \mathbb{C}_+, \quad (24) \]

where \( d, q \geq 0 \) are constants and \( \int_{(0,\infty)} (1 + u)^{-1} \Delta(du) < \infty \), cf. [25, Definition 2.1]. The class \( \mathcal{S} \) is simply seen as the class of double iterated Laplace transforms, due to the representation

\[ \frac{1}{1+z} \int_0^\infty e^{-zx} e^{-x} dx, \quad z \geq 0. \]

Using the the *exponential integral function* \( \text{Ei} \) defined by (82) in the Appendix, an example is given by the function \( \mathcal{E}(z) := -e^z \text{Ei}(z) \) represented by

\[ \mathcal{E}(z) := \int_0^\infty \frac{e^{-x}}{z + x} dx, \quad z > 0. \]

The function

\[ \left| e^{-\lambda x} - 1 + \lambda \frac{x}{1 + x^2} \right| \leq \frac{1}{2} \lambda^2 |x|^2 \mathbb{1}_{|x| \leq 1} + 2 \mathbb{1}_{|x| \geq 1} + |\lambda| \left( \frac{x^2}{1 + x^2} \mathbb{1}_{|x| \leq 1} + \frac{x}{1 + x^2} \mathbb{1}_{|x| \geq 1} \right) \leq C(\lambda) (x^2 \wedge 1), \]

where \( C(\lambda) := 3 + |\lambda| + \frac{|\lambda|^2}{2} \). The latter insures that \( \Psi_X \) is well defined due to the condition (19) on its Lévy measure. Thus, the triplet of characteristics of \( \hat{F}_X \) are given by

\[ a_{\hat{X}} = a_X, \quad b_{\hat{X}} = a_X \quad \text{and} \quad \rho_{\hat{X}}(dx) = \gamma_X^2 \delta_0(dx) + \frac{x^2}{1 + x^2} \nu_X(dx), \quad (23) \]

\( \rho_{\hat{X}} \) is a finite measure and

the operator \( \Delta : \Psi \mapsto F \) defined by (21), induces a bijection from \( \mathcal{L}\mathcal{E}(0, \infty) \) to \( \mathcal{N}(\mathbb{R}) \), or equivalently, from the class of Lévy processes to the one of free-infinitely divisible random variables.

**Remark 1.** Assume \( X \overset{d}{=} X_1 + X_2 \) where \( X_1 \) and \( X_2 \) are independent infinitely divisible random variables. On the level of the Lévy measures, we have \( \nu_X = \nu_{X_1} + \nu_{X_2} \) and (23) gives \( \rho_{\hat{X}} = \rho_{\hat{X}_1} + \rho_{\hat{X}_2} \) on the level of the Nevanlinna measures.

**P5** If \( F \) has a continuous extension with \( F : (0, \infty) \to \mathbb{R} \), then \( F \in \mathcal{N}(0, \infty) \). Further, if

\[ F(0+) = a - \int_{(0,1]} \frac{1}{x} \rho(dx) \]

is finite, then \( F \in \mathcal{N}(0, \infty) \), cf. [25, Theorem 6.9].

**P6** The class \( \mathcal{S} \) of *Stieltjes transforms* is formed by functions with a represented similar to (15): \( f \in \mathcal{S} \), if

\[ f(\lambda) = d + \frac{q}{z} + \int_{(0,\infty)} \frac{1}{z + u} \Delta(du), \quad z \in \mathbb{C}_+, \quad (24) \]

where \( d, q \geq 0 \) are constants and \( \int_{(0,\infty)} (1 + u)^{-1} \Delta(du) < \infty \), cf. [25, Definition 2.1]. The class \( \mathcal{S} \) is simply seen as the class of double iterated Laplace transforms, due to the representation

\[ \frac{1}{1+z} \int_0^\infty e^{-zx} e^{-x} dx, \quad z \geq 0. \]

Using the the *exponential integral function* \( \text{Ei} \) defined by (82) in the Appendix, an example is given by the function \( \mathcal{E}(z) := -e^z \text{Ei}(z) \) represented by

\[ \mathcal{E}(z) := \int_0^\infty \frac{e^{-x}}{z + x} dx, \quad z > 0. \]

The function

\[ \left| e^{-\lambda x} - 1 + \lambda \frac{x}{1 + x^2} \right| \leq \frac{1}{2} \lambda^2 |x|^2 \mathbb{1}_{|x| \leq 1} + 2 \mathbb{1}_{|x| \geq 1} + |\lambda| \left( \frac{x^2}{1 + x^2} \mathbb{1}_{|x| \leq 1} + \frac{x}{1 + x^2} \mathbb{1}_{|x| \geq 1} \right) \leq C(\lambda) (x^2 \wedge 1), \]
is another important example of a Stieltjes function and, actually, it is a double iterated Stieltjes function. As for the class $S$, it appears as the kernel of the class $S_2$, of double iterated Stieltjes transforms of functions, whose several properties were provided by Yakubovich and Martins [27]. This class could be extended to iterated Stieltjes transforms of measures $S_2 := \left\{ f(z) = \int_{0,\infty} \mathcal{G}(x) \Delta(dx), \; z > 0 \right\}$.

We believe that it might be interesting to study the properties of the class of iterated Laplace transform of any order, and we hope that this will be the scope of a future work.

By Theorem [25, Theorem 6.2, Proposition 7.1, Theorem 7.3], the class $\mathcal{CBF}$ of complete Bernstein functions is formed by those functions $f$ of one the following forms

$$f(z) = z g(z) = h \left( \frac{1}{z} \right) = \frac{1}{k(z)}, \quad \text{where} \quad g, h, k \in S.$$ 

Notice that $\mathcal{CBF}$ is contained into the class $\mathcal{BF}$ of Bernstein functions formed by functions $\phi$ of the form:

$$\phi(z) = q + dz + \int_{0,\infty} (1 - e^{-zx}) \Pi(dx), \quad z \geq 0 \quad (25)$$ 

where $q \geq 0$ is the so-called killing rate, $d \in \mathbb{R}$ is the drift and $\Pi$ is the Lévy measure of $\phi$, i.e. a positive measure on $(0, \infty)$ which satisfies $\int_{(0,\infty)} (x \wedge 1) \Pi(dx) < \infty$ and that $\phi'(z)$ is a completely monotone function. We recall that a function $f$ is completely monotone function on $(0, \infty)$ if $f$ in infinitely often differentiable and $(-1)^n f^{(n)} \geq 0$, for all $n = 0, 1, 2, \ldots$, equivalently $f$ is the Laplace transform of some measure, represented by

$$f(\lambda) = \int_{\{0,\infty\}} e^{-\lambda x} \mu(dx) = \lambda \int_{0}^{\infty} e^{-\lambda x} \mu([0,x)) dx. \quad (26)$$

Cf. [25, Chapter 1]. Complete Bernstein function were defined in [25, p. 69] as those Bernstein function such that their Lévy measure $\Pi$ have the form

$$\Pi(dx) = m(x) dx, \; m > 0, \quad \text{where} \; m \; \text{is a completely monotone function}, \quad (27)$$

i.e. $m(x) = \mathcal{L}[\sigma; x]$. Thus, every $f \in \mathcal{CBF}$ takes the $\mathcal{N}(0, \infty)$ form:
\[ f(\lambda) = q + d\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda + x} \sigma(dx), \quad \lambda \geq 0, \quad (28) \]

where \( q, d \geq 0 \) and \( \sigma \) is some measure on \((0, \infty)\) that integrates \( 1/(1 + x) \).

From the proof of [25, Theorem 6.9], one can extract that the Nevanlinna-characteristics of \( f \) are

\[ a_f = q + \int_{(0,\infty)} \frac{\sigma(dx)}{1 + x^2}, \quad b_f = d \quad \text{and} \quad \rho_f(dx) = \frac{x}{1 + x^2} \sigma(dx). \quad (29) \]

By [25, Theorem 6.2], the extension of \( f \) is also represented by a Nevanlinna representation analogous of (21):

\[ f(z) = z^2 \int_{0}^{\infty} e^{-zx} \phi(x) dx = z^2 \mathcal{L}[\phi, z], \quad z \in \mathbb{C}^+, \quad (30) \]

where \( \phi \) is a some Bernstein function. The latter induces that

\textit{the operator} \( \phi \mapsto f \) \textit{given by} (30), \( \text{is a bijection from} \ \mathcal{B}\mathcal{F} \ \text{to} \ \mathcal{CBF}, \ \text{or}
\textit{equivalently, from the class of subordinators to the one of complete subordinators.}

By [25, Theorem 6.9], we extract that actually, the analytic extension on \( \mathbb{C} \setminus \mathbb{R}_+ \) of \( \mathcal{CBF} \)-functions corresponds to those functions \( F \) in \( \mathcal{N}(0, \infty) \) which are nonnegative-valued on \((0, \infty)\). A non trivial example of a \( \mathcal{CBF} \)-function is given by the following: let \( \text{Li}_s \) be the \textit{polylogarithm functions} given by (87) and define

\[ \mathcal{L}_s(z) := -\text{Li}_s(-z), \quad z \geq 0, \quad s > 0. \]

By (89), \( \mathcal{L}_s \) meets the \( \mathcal{CBF} \)-form (28):

\[ \mathcal{L}_s(z) = \frac{1}{\Gamma(s)} \int_{1}^{\infty} \frac{z}{x + z} \log^{s-1}(x) \frac{dx}{x}, \quad z \geq 0. \]

If furthermore \( s \geq 1, \mathcal{L}_s \) is in the class \( \mathcal{TBF} \) of \textit{Thorin Bernstein functions}, the subclass of \( \mathcal{CBF} \)-functions, such that the \( m \)-function in (27)

\[ \Pi(dx) = m(x) dx, \quad x > 0, \quad \text{where} \quad x m(x) \quad \text{is a completely monotone function,} \quad (31) \]

cf. [25, Theorem 8.2 (v)]. In general, \( \mathcal{L}_s, s > 0 \), meets the Nevanlinna \( \mathcal{N}[1, \infty) \)-form (5) on \( \mathbb{C} \setminus [-1, \infty) \) and its characteristics are given by (29):

\[ a = \frac{1}{\Gamma(s)} \int_{1}^{\infty} \frac{\log^{s-1}(x)}{x(1 + x^2)} dx, \quad b = 0 \quad \text{and} \quad \rho(dx) = \frac{1}{\Gamma(s)} \frac{\log^{s-1}(x)}{1 + x^2} dx, \quad x \geq 1. \]

By formula [21, (1.110) p. 27], observe that \( \mathcal{L}_2 \) and \( \mathcal{Ei} \) are linked by a formula of type (30):

\[ z \mapsto z^2 \int_{0}^{\infty} e^{-zx} \mathcal{L}_2(x) dx = z \int_{0}^{\infty} \frac{\mathcal{Ei}(y)}{y} dy \in \mathcal{CBF}. \]
The class $\mathcal{CBF}_e$ of extended complete Bernstein functions is formed by those functions $f$ of the form $f = g - h$, where $g \in \mathcal{CBF}$ and $h \in \mathcal{S}$. Actually, the analytic extension on $\mathbb{H}^+$ of $\mathcal{CBF}_e$-functions corresponds to those functions $F$ in $\mathcal{N}([0, \infty))$ which are real-valued on $(0, \infty)$. See [25, Proposition 6.12] and Remark 2 below.

After these preliminaries, we are able to state our first result which is a generalisation of the inversion procedure (11) and of [25, Corollary 6.13] stated for $\mathcal{CBF}_e$-functions.

**Theorem 1.** Let $A \subset \mathbb{R}$ and $F : \mathbb{H}^+ \to \mathbb{H}^+$ be an analytic function and recall $\mathcal{L}$ and $\mathcal{F}$ stand of the Laplace and Fourier transforms. Then, the following statements are equivalent:

1) $F \in \mathcal{N}(A)$;

2) $G(z) := \frac{1}{1 + z} \left[ \Re(F(i)) + i \Im(F(i)) z - F(z) \right]$, $z \in \mathbb{H}^+$, is the Cauchy transform of a finite measure on $A$.

3) The function $H(w) := \frac{iF(iw)}{w^2 - 1}$, $w > 1$, is of the form

$$H(w) = \mathcal{L}\left[ \mathcal{F}[\rho; s] + i \Re(F(i)) \sinh(s) - \Im(F(i)) \cosh(s); w \right].$$

for some finite measure $\rho$ on $A$.

In particular $F \in \mathcal{N}(\mathbb{R}_+)$, if, and only if the function $G$ has continuous extension $G : (0, \infty) \to \mathbb{R}_+$ which is the Stieltjes transform of some finite measure on $\mathbb{R}_+$.

### 3 The $\Theta$-transform, Nevanlinna-Pick functions and Lévy processes

A (possibly killed) subordinator $\xi = (\xi_t)_{t \geq 0}$ started from 0 is an increasing Lévy process whose distribution are also obtained via a Lévy-Khintchine formula

$$\mathbb{E}[e^{-\lambda \xi_t}] = \int_{(0, \infty)} e^{-\lambda x} \mathbb{P}(\xi_t \in dx) = e^{-t \phi(\lambda)} , \quad \text{for } t, \lambda \geq 0. \quad (33)$$

where $\phi$ is Bernstein function, i.e. a function represented by (25). See the monograph of Bertoin [5] for background on subordinators and the book of Schilling, Song & Vondraček [25] for Bernstein functions. Lamperti’s transformation uses the implicit time-change $\tau_t$, $t \geq 0$, defined by the identity

$$t = \int_0^\tau e^{\xi_s} ds,$$

where $\xi$ is a general subordinator. The process $E = (E_t)_{t \geq 0} := (e^{\xi_t})_{t \geq 0}$ is a strong Markov process started from 1 which enjoys the scaling property: if $\mathbb{P}_x$, $x > 0$, is
the distribution of the process $x E_{t/x}$, $t \geq 0$, then $\mathbb{P}_x$ coincides with the law of the process $E$ started from $x$ (that is, when the subordinator $\xi$ is replaced by $\xi + \log x$ in Lamperti’s transform). In [6], it was shown that for all $p > 0$, the function $t \mapsto \mathbb{E}[E^{-p}_t]$, $t \geq 0$, is completely monotone with Laplace transform representation

$$
\mathbb{E}[E^{-p}_t] = \int_{(0, \infty)} e^{-tx} \sigma_p(dx),
$$

where the entire moments of the probability measure $\sigma_p$ are given by

$$
\int_{(0, \infty)} x^k \sigma_p(dx) = \phi(p) \phi(p+1) \cdots \phi(p+k-1), \quad k = 1, 2, \ldots
$$

A proper subclass of subordinators is formed by first passage times of spectrally negative Lévy processes $Z = (Z_t)_{t \geq 0}$. That means that $Z$ is a Lévy process with no positive jumps and the distribution of $Z_t$ is obtained by a Lévy-Laplace exponent $\Psi$ such that $\mathbb{E}[e^{\lambda Z_1}] = e^{\Psi(\lambda)}$ and in (18), the Lévy measure $\nu$ of $\Psi$ is supported by $(0, \infty)$. This entails that $\Psi$ has the continuous extension on $\mathbb{R}_+$:

$$
\Psi(\lambda) = \alpha + \beta \lambda + \frac{x^2}{2} \lambda^2 + \int_{(0, \infty)} \left( e^{-\lambda x} - 1 + \lambda \frac{x}{1+x^2} \right) \nu(dx), \quad \lambda \geq 0. \tag{35}
$$

i.e. $\Psi \in \mathcal{LE}(0, \infty)$. Observe that if in (35), $\nu$ satisfies

$$
\bar{\nu} := \beta + \int_{(0, \infty)} \frac{x}{1+x^2} \nu(dx) \quad \text{is a finite quantity}, \tag{36}
$$

then $\Psi$ satisfies

$$
\lambda \mapsto \phi(\lambda) := \alpha + \bar{\nu} \lambda + \frac{x^2}{2} \lambda^2 - \Psi(\lambda) = \int_{(0, \infty)} (1 - e^{-\lambda x}) \nu(dx) \in \mathcal{BF}. \tag{37}
$$

Conversely, every Bernstein function $\phi$ satisfies

$$
\lambda \mapsto \alpha + \beta \lambda + \frac{x^2}{2} \lambda^2 - \phi(\lambda) \in \mathcal{LE}(0, \infty), \quad \text{for all } \alpha, \gamma \geq 0, \beta \in \mathbb{R}.
$$

By the proof of [19, Theorem 3.12 and Section 8.1], under the assumption that $Z$ is non-killed, (i.e. $\alpha = \Psi(0) = 0$) and $\mathbb{E}[Z_1] = \Psi'(0+) = \beta - \int_{(0, \infty)} \frac{x^3}{1+x^2} \nu(dx) \geq 0,$

we have unique (nonnegative) solution $\phi(\mu)$ to the equation $\Psi(\lambda) = \mu$, given by the right inverse

$$
\phi(\mu) = \sup \{ \lambda \geq 0 : \Psi(\lambda) = \mu \}. \tag{39}
$$

Necessarily $\phi(0) = 0$, and $\phi$ is a Bernstein function represented as in (25). Actually $\phi$ corresponds to a subordinator $\xi$ which is the first passage of $Z$ time above a level $s > 0$, i.e.
\[ \xi_s = \inf\{t > 0 : Z_t > s\}. \quad (40) \]

Due to (22), observe that for every \( \Psi \in \mathcal{LE}(\mathbb{R}_+) \), there exists \( A, B, C \geq 0 \), such that
\[ |\Psi(\lambda)| \leq A + B\lambda + C\lambda^2, \quad \text{for all } \lambda \geq 0, \]
and then the Laplace transform of \( \Psi \) is well defined. With this observation together with (37) and even if the integrability condition (36) is not satisfied, we still be able to establish a correspondence between the Nevanlinna class \( \mathcal{N}(\mathbb{R}_+) \) with the class \( \mathcal{LE}(\mathbb{R}_+) \) given by (20) as obtained the one obtained in (30):

**Proposition 1 (The \( \Theta \)-transform).** To every pair \((b, \Psi) \in \mathbb{R}_+ \times \mathcal{LE}(0, \infty)\), is associated a (unique) Nevanlinna \( \mathcal{N}(\mathbb{R}_+) \)-function obtained by the operator \( \Theta \) given, for all \( z \in \mathbb{C}^+ \), by
\[ \Theta(b, \Psi)(z) = F(z) := z^2 \int_0^\infty e^{-zu} (b + \Psi(0) - \Psi(u)) du = a + bz + \int_{(0, \infty)} \frac{zu - 1}{z + x} \rho(dx), \quad (41) \]
and the characteristics \((a, \rho)\) of \( F \) are related to the one’s of \( \Psi \), \((\alpha, \beta, \gamma, \nu)\), by
\[ a = -\beta, \quad \text{and} \quad \rho(dx) = \gamma^2 \delta_0(dx) + \frac{x^2}{1 + x^2} \nu(dx). \]

Furthermore, the Nevanlinna class of functions
\[ \mathcal{N}^*(\mathbb{R}_+) := \left\{ F \in \mathcal{N}(\mathbb{R}_+) \; \text{s.t.} \; b = \lim_{|z| \to \infty, z \in \mathbb{C}^+} \frac{F(z)}{z} = 0 \right\}, \]
is obtained as the image of \( \{0\} \times \mathcal{LE}(0, \infty) \) by the operator \( \Theta \).

**Remark 2.** As in (P7) in Section 2, one can see that the class
\[ \mathcal{CBF}_e^* := \{ f \in \mathcal{CBF}_e, \; \text{such that} \; \lim_{\lambda \to \infty} f(\lambda)/\lambda = 0 \} \]
is such that the extension on \( \mathbb{H}^+ \) of all its members \( f \) coincides with the class \( \mathcal{N}^*(\mathbb{R}_+) \) given in last proposition.

**Example 1 (The \( \Theta \)-transform for spectrally negative stable processes).** Let \( \vartheta \in (1, 2] \). The function \( \Psi_\vartheta(\lambda) := \lambda^\vartheta, \; \lambda \geq 0 \), multiplied by positive constants, corresponds to the Lévy-Laplace exponents of spectrally negative stable processes, see [19, Section 6.5.3]. Observe that \( \Psi_2 \) corresponds to the Gaussian distribution and that \( \Theta(0, y \mapsto u^2)(z) = -2z^2 \). For \( \vartheta \in (1, 2) \), let
\[ c_\vartheta = \frac{\vartheta(\vartheta - 1)}{\Gamma(2 - \vartheta)} \quad \text{and} \quad \rho_\vartheta(x) = \frac{c_\vartheta x^{1-\vartheta}}{1 + x^2}, \quad x > 0. \]
Using [10, Formula F1 II 718 p. 319] and Euler’s reflection formula, we easily get that
\[ \beta_{\vartheta} = c_{\vartheta} \int_0^{\infty} \frac{x^{2-\vartheta}}{1+x^2} \, dx = \frac{\vartheta}{\Gamma(2-\vartheta)} \Gamma\left(\frac{\vartheta+1}{2}\right) \Gamma\left(\frac{3-\vartheta}{2}\right). \]

The function \( \Psi_{\vartheta} \) is represented by
\[
\Psi_{\vartheta}(\lambda) = \int_0^{\infty} \left( e^{\lambda x} - 1 - \lambda x \right) \frac{c_{\vartheta}}{x^{\vartheta+1}} \, dx = \beta_{\vartheta} \lambda + \int_0^{\infty} \left( e^{\lambda x} - 1 - \lambda \frac{x}{1+x^2} \right) \frac{c_{\vartheta}}{x^{\vartheta+1}} \, dx.
\]

The Nevanlinna-Pick function \( F_{\vartheta}(z) = -\Gamma(\vartheta+1) z^{1-\vartheta}, \ z \in \mathbb{C}^+ \), is the CBF obtained by the operator \( \Theta \) in (41). Actually \( -F_{\vartheta} \) is a Stieltjes function in the sense of (P6) in Section 2 and without effort, we get the Nevanlinna representation
\[
F_{\vartheta}(z) = -\beta_{\vartheta} + \int_0^{\infty} \frac{zx^{1-\vartheta}}{z} \rho_{\vartheta}(x) \, dx, \quad z \in \mathbb{C}^+.
\]

Lemma 1 in Subsection 5.1 below, is the key result for the following temporal completely monotone property for subordinators.

**Proposition 2 (Temporal completely monotonicity property for subordinators).**
Let \( \phi: \mathbb{R}_+ \to \mathbb{R}_+ \) be a strictly increasing function such that \( \phi(0) = 0 \) and \( \phi(\infty) = \infty \). Let \( \Psi \) be its inverse function and \( e \) denotes a r.v. with standard exponential distribution. Then the following statements are equivalent.

1) \( \phi \) is a Bernstein function;
2) There exists a subordinator \( \xi = (\xi_t)_{t \geq 0} \) satisfying the identities in law
\[
\int_0^t \psi(t) \, dt \overset{d}{=} \psi \left( \int_0^t e \, dt \right), \quad \forall t > 0, \quad \text{where } e \text{ in the l.h.s. is assumed to be independent of } \xi_t;
\]
\[
(42)
\]
3) There exists a family of positive r.v.'s \( \eta = (\eta_t)_{t \geq 0} \) such that (42) holds true with \( \xi \) replaced by \( \eta \);
4) There exists a subordinator \( \xi = (\xi_t)_{t \geq 0} \) such that the representation
\[
E[\xi_t^{-p}] = \frac{t}{\Gamma(p+1)} \int_0^{\infty} e^{-tx} \psi(t) \, dx
\]
holds true for every \( t > 0 \) and \( p > -1 \), such that \( \mathbb{L}[\psi^p; t] < \infty \).

Under the latter, \( \phi \) is necessarily the Bernstein function of \( \xi \).

**Remark 3.** Observe that the temporal completely monotone property in (43) involves the Laplace representation in the right hand side term similar to the one obtained in (34) with \( p = 1 \). Also observe that \( \phi(\lambda) = \lambda \) is associated to the trivial subordinator \( \xi_t = t \) , then replacing by \( \psi(\lambda) = \lambda \) in (43), we retrieve the integral representation of the Gamma function (81).
Example 2 (Moments of positive stable distributions). If \( \xi^{(\alpha)} = (\xi^{(\alpha)}_t)_{t \geq 0} \) is a standard \( \alpha \)-stable subordinator, i.e., with Bernstein function \( \phi_{\alpha} = \lambda_{\alpha} \), \( 0 < \alpha < 1 \), then (42) gives the following well known results, valid for all \( t > 0 \), \( p > -\alpha \): by (42), we have

\[
\frac{e}{\xi^{(\alpha)}_t} = (\frac{e}{t})^{\frac{\alpha}{\lambda}} \quad \Rightarrow \quad \frac{e}{\xi^{(\alpha)}_1} = e^{\frac{\lambda}{\alpha}} \quad \text{and} \quad (\xi^{(\alpha)}_t)^{\frac{\lambda}{\alpha}} = t^{\frac{\lambda}{\alpha}} \xi^{(\alpha)}_1 \quad \text{(scaling property for stable processes)},
\]

and by (43), we have

\[
E[(\xi^{(\alpha)}_t)^{-p}] = \frac{t}{\Gamma(p+1)} \int_0^\infty e^{-tx^{\frac{\lambda}{\alpha}}} dx = \frac{t}{\Gamma(p+1)} \frac{\Gamma(1 + \frac{p}{\alpha})}{t^{\frac{\lambda}{\alpha}}} = t^{\frac{\lambda}{\alpha}} \frac{\Gamma(1 + \frac{p}{\alpha})}{\Gamma(1 + p)},
\]

a formula that in [9, Exercise 4.17], whereas the second identity in (44) is in [9, Exercise 4.19]. See [13, Theorem 1] for all moments of all stable distributions.

Remark 4. Proposition 2 shows that if \( p > -1 \) and \( \Psi \) is the inverse of a Bernstein function \( \phi \) associated to the subordinator \( \xi \), then the following assertions are equivalent.

(i) \( t \mapsto E[\xi^{-p}/t] \) is completely monotone on \( (0, \infty) \)
(ii) \( t \mapsto E[\xi^{-p}/t] \) is a Bernstein function.

This simple remark is improved as follows:

Theorem 2 (Temporal completely monotonicity property for subordinators improved). Let \( p > -1 \) and \( \phi \) be a non-trivial Bernstein function (i.e. \( \phi \) is not affine) such that \( \phi(0) = 0 \), \( \phi(\infty) = \infty \). Let \( \Psi \) and \( \xi = (\xi_t)_{t \geq 0} \) be respectively, the inverse function and the subordinator, associated to \( \phi \). We have the following results.

1) If \( p \in (-1, 0) \), then \( e^{-tx^{\frac{\lambda}{\alpha}}} \) is integrable near \( \infty \) and the following assertions are equivalent.

(i) \( \Psi^{p} \) is integrable at the neighborhood of \( 0 \);
(ii) \( t \mapsto E[\xi^{-p}] \) is a Bernstein function.

We then have the representation

\[
E[\xi^{-p}] = \frac{1}{\Gamma(p+1)} \int_0^\infty (1 - e^{-tx^{\frac{\lambda}{\alpha}}}) (-\Psi^{p})'(x) dx \quad (45)
\]

2) If \( p \geq 0 \), then \( \Psi^{p} \) is integrable near \( 0 \) and the following assertions are equivalent.

(i) \( x \mapsto e^{-tx^{\frac{\lambda}{\alpha}}} \Psi^{p}(x) \) is integrable near \( \infty \) for all \( t > 0 \);
(ii) \( t \mapsto E[\xi^{-p}] \) is completely monotone on \( (0, \infty) \).

We then have the representation

\[
E[\xi^{-p}] = \frac{1}{\Gamma(p+1)} \int_0^\infty e^{-tx^{\frac{\lambda}{\alpha}}} \Psi^{p}(x) dx, \quad t > 0. \quad (46)
\]
The function \( t \mapsto t \mathbb{E}[\xi^{-p}_t] \) is Bernstein if, and only if, \( \lambda \mapsto \lambda^{1-p} \varphi'(\lambda) \) is non-decreasing, hence \( p \in [0, 1) \). In this case, we have the representation

\[
t \mathbb{E}[\xi^{-p}_t] = \frac{1}{\Gamma(p+1)} \int_0^\infty (1 - e^{-t\xi}) (\Psi^p)'(x) \, dx, \quad t > 0. \tag{47}
\]

3) If \( p \geq 1 \), then \( \Psi^p \) is convex, \( (\Psi^p)'(0^+) \in [0, \infty) \), and the following two assertions are equivalent.

(i) \( x \mapsto e^{-t\xi} \Psi^p(x) \) is integrable near \( \infty \) for all \( t > 0 \);
(ii) \( t \mapsto t \mathbb{E}[\xi^{-p}_t] \) is completely monotone on \( (0, \infty) \).

In this case, we have the representation

\[
t \mathbb{E}[\xi^{-p}_t] = \frac{1}{\Gamma(p+1)} \left[ (\Psi^p)'(0^+) + \int_0^\infty e^{-t\xi} (\Psi^p)'(x) \, dx \right], \quad t > 0, \tag{48}
\]

and the following holds true:

(a) The function \( t \mapsto t^2 \mathbb{E}[\xi^{-p}_t] \) is Bernstein if, and only if, \( (\Psi^p)' \) is concave;
(b) If further \( \Psi^p \in L^2(0, \infty) \) with quadruple of characteristics \( (0, \beta_p, \gamma_p, v_p) \) in its representation \( \Phi \), then

\[
(\Psi^p)'(0^+) = \beta_p - \int_{(0, \infty)} \frac{x^3}{1 + x^2} v_p(dx) \in [0, \infty), \tag{49}
\]

and the function \( t \mapsto t \mathbb{E}[\xi^{-p}_t] \) has the Stieltjes transform form \( \Phi \), i.e.

\[
t \mathbb{E}[\xi^{-p}_t] = \frac{1}{\Gamma(p+1)} \left[ (\Psi^p)'(0^+) \frac{\gamma_p^2}{t} + \int_{(0, \infty)} \frac{u^2}{t + u} v_p(du) \right], \quad t > 0. \tag{50}
\]

Remark 5. Under the assumptions of point 3) in Theorem 2, we have

\[
\lim_{t \to \infty} t \mathbb{E}[\xi^{-p}_t] = \frac{(\Psi^p)'(0^+)}{\Gamma(p+1)},
\]

and under the assumptions of point 4) in Theorem 2, we have

\[
\lim_{t \to \infty} t^2 \mathbb{E}[\xi^{-p}_t] - \frac{(\Psi^p)'(0^+)}{\Gamma(p+1)} = (\Psi^p)''(0^+) = \frac{\gamma_p^2}{\Gamma(p+1)} + \int_{(0, \infty)} u^2 v_p(du).
\]

Example 3 (Moments of Lambert distributions). Besides the example of stable subordinators given in Example 2, we propose the following less trivial one: in Pakes [22], the principal Lambert function \( \phi \), defined as the unique real-valued concave increasing solution to the functional equation \( \lambda = \phi(\lambda) e^{\lambda \varphi(\lambda)} \), \( \lambda \geq 0 \), is shown to be a Thorin Bernstein, i.e., a Bernstein function with Lévy measure as in \( \mathcal{L} \). Let \( \xi_t = (\xi^t_x)_{t \geq 0} \) be its associated subordinator. Since \( \mathcal{L}(x) = x e^x \), \( x \geq 0 \), then r.v.’s \( \xi^t_x \) satisfy the identity \( \mathbb{E}\left[\xi^{-p}_t\right] = \frac{\gamma_p^2}{\Gamma(p+1)} + \int_{(0, \infty)} u^2 v_p(du) \).
\[
\frac{e^{\xi_L t}}{\xi_L t} \overset{d}{=} \frac{e^{-t u}}{t u} \overset{d}{=} \frac{1}{t} u^{-1} \log u, \quad \text{for all } t > 0,
\]

where \( u \) has the uniform distribution on \((0, 1)\), and also (43):

\[
\mathbb{E}[(\xi_L t)^{-p}] = \frac{t}{\Gamma(p+1)} \int_0^\infty e^{-tx} (xe^x)^p dx = \frac{t}{(t-p)^{p+1}}, \quad \text{for all } t > 0, \ t > p > -1.
\]

Thus, for all \( t > 0 \), the function

\[
q \mapsto M_t(q) := t (t+q)^{q-1}, \quad -t < q < 1,
\]

is a Mellin transform whose domain of definition can be extended to the interval \((-q, \infty)\), i.e., the representation

\[
M_t(q) = \mathbb{E}[(\xi_L^{1+})^q] = t (t+q)^{q-1}
\]

remains valid if \( q > -t \).

For more arguments justifying this extension on the Mellin transform, we refer to Jedidi et al. [12] for instance. Note that Pakes [22, Theorem 3.4], calculated only the moments of natural numbers order for \( \xi_L^1 \):

\[
\mathbb{E}[(\xi_L^1)^q] = (1+q)^{q-1}, \quad \text{for } q = 0, 1, 2, \ldots
\]

We can go beyond by observing that condition 3)(i) in Theorem 2 fails for \( \Psi \). Nevertheless, since \( \mathbb{E}[(\Psi L)^p, p+t] \) is finite for every \( p, t > 0 \), then (51) can be restated

\[
t \mapsto (t+p) \mathbb{E}[\xi_{p+t}^{-p}] = \frac{t+p}{tp+1} - \frac{1}{tp} + \frac{p}{tp+1}
\]

is a completely monotone function.

Let \( \phi \) be a Bernstein function, with inverse \( \Psi \) and associated subordinator \( \xi \). Observe that it is not always possible to have \( \Psi \) explicitly, for instance, take the Bernstein function \( \phi(\lambda) = \lambda / \log(1 + \lambda), \ \lambda \geq 0 \). Even one is lucky enough to have \( \Psi \) explicitly, it might be impossible to easily check whether it is a Lévy-Laplace exponent! For instance, no standard calculation would exhibit that \( \Psi(\lambda) = \lambda / [(1 + \lambda) \log(1 + 1/\lambda)], \ \lambda \geq 0 \), is genuinely a Lévy-Laplace exponent, cf. [25, Example 40 p. 320].

An answer to question (12) might be a consequence of Theorem 2: we show in next result that we only need to strengthen the equivalence in Remark 4, \( \xi \) needs to have a temporal Stieltjes property! Representation (53) below is actually connected to the Laplace and Nevanlinna representations in (41). For

**Corollary 1 (On temporally Stieltjes property for spectrally negative Lévy processes).**

1) Let \( \Psi \) be a Lévy-Laplace exponent represented by (35) and satisfying (49) wiz. \( \psi'(0+) \in [0, \infty) \). Let the inverse function of (the non-negative) function \( \Psi \) be the
Bernstein function $\phi$ associated to the subordinator $\xi = (\xi_t)_{t \geq 0}$ given by (39) and (40) respectively. Then, the following holds:

(i) For all $t > 0$, $\xi_t$, satisfies the identity in law (42).
(ii) If $p \geq 0$, or if either $0 > p > -1$ and $\Psi^p$ is integrable at 0, then the function $t \mapsto E[\xi_t^{-p}]$, $t > 0$, is well defined, is completely monotone, and is represented by (43).
(iii) If $p \geq 1$ and $\Psi^p \in \mathcal{LE}(0, \infty)$, with quadruple of characteristics $(0, \beta_p, \gamma_p, \nu_p)$ in its representation (35) and satisfying (49), then

$$t \mapsto \frac{t^2}{\Gamma(p+1)} \Omega[\Psi, t] = t \mathbb{E}[\xi_t^{-p}] \text{ is a Stieltjes transform,}$$

and we have the representation

$$\Gamma(p+1) t \mathbb{E}[\xi_t^{-p}] = \Psi'(0+) + \frac{\gamma_p^2}{t} + \int_{(0,\infty)} \frac{x^2}{t + x} \nu_p(dx).$$

The latter is equivalent to

$$t \mapsto \frac{t^3}{\Gamma(p+1)} \Omega[\Psi, t] = t^2 \mathbb{E}[\xi_t^{-p}] \text{ is a complete Bernstein function.}$$

2) The converse is stated as follows. Let $p \geq 1$ and $(\xi_t^{(1/p)})_{t \geq 0}$ be a $1/p$-standard stable subordinator, (see Example 2, with the convention $\xi_t^{(1)} = t$). If a subordinator $\xi$ is such that the function $t \mapsto t \mathbb{E}[\xi_t^{-p}]$ is a Stieltjes transform, then the subordinated process $(\xi_t^{(1/p)} \circ \xi_t)_{t \geq 0}$ is the first passage time process of a spectrally negative Lévy processes with Lévy-Laplace exponent $\Psi^p \in \mathcal{LE}(0, \infty)$.

**Remark 6.** Last results merit the following comments:

(i) The equivalence between (53) and (54) is immediate by property (P6) above.
(ii) Recall the Lambert Bernstein function $\phi_\lambda$ given in Example 3 and its associated subordinator $\xi^\lambda$. Since the inverse function $\Psi^\lambda(x) = xe^x$ is not a Lévy-Laplace exponent, then $\xi^\lambda$ is certainly not the inverse time process of a Lévy process. Thus, Corollary 1 does not apply.
(iii) Assume we know that a subordinator $\xi = (\xi_t)_{t \geq 0}$ is such that $\mathbb{E}[\xi_t^{-p}]$ is finite and is explicit for all $t > 0$ and some $p > 0$, for example $\xi = \xi^{(\alpha)}$ where is an $\alpha$-stable subordinator. A “good test” to check whether $\xi$ is the inverse time of some spectrally negative Lévy process with Lévy exponent $\Psi$, is “just check if the temporal property (52) or (54) of $\xi$ holds true. If it fails, then $\xi$ is certainly not an inverse time”.
(iv) Additionally, the operator applied on $\Psi$ in (54) has to be compared with the one obtained by Schilling, Song and Vondraček [25, Theorem 6.2.] and illustrated by (30). Recall for instance the function $\Psi_\theta(\lambda) := \lambda^\theta$, $\lambda \geq 0$ used in Example (1). $\Psi_\theta$ is a Lévy-Laplace exponent if, and only if, $\theta \in [1,2]$ and then, $\Psi_\theta$
corresponds to a (non-trivial) spectrally negative stable processes $Z^\vartheta = (Z^\vartheta_t)_{t \geq 0}$ if, and only if $\vartheta \in (1, 2]$. Denoting $\alpha = 1/\vartheta$, the subordinator $\xi^{(\alpha)} = (\xi^{(\alpha)}_t)_{t \geq 0}$ is the inverse in time of $Z^\vartheta$ and is associated to the Bernstein function $\lambda \mapsto \lambda^{\alpha}$, $\alpha \in [1/2, 1)$. Observe that $\xi^{(\alpha)}$ has the so-called temporal scaling property: $\xi^{(\alpha)}_t d \equiv t^{1/\alpha} \xi^{(\alpha)}_1$. Example 2 gave, that for $p > -\alpha$,

$$
E \left[ (\xi^{(\alpha)}_t)^{-p} \right] = ct^{-\frac{p}{\alpha}}, \quad c := \frac{\Gamma \left( \frac{p}{\alpha} + 1 \right)}{\Gamma (p + 1)}
$$

and then

$$
t \mapsto t^2 E \left[ (\xi^{(\alpha)}_t)^{-p} \right] = ct^{-\frac{p}{\alpha}} \in \mathcal{CBF} \iff \lambda \mapsto (\Psi_{\vartheta})^{p}(\lambda) = \lambda^{p \vartheta} \in \mathcal{LE}(\mathbb{R}^+)$$

and the latter is equivalent to $1 \leq p \vartheta = p/\alpha \leq 2$.

4 Some results on Voiculescu transforms related to Hyperbolic functions

In the following, for Voiculescu (Nevanlinna) transforms found in Jurek [16], we compute the triplet of characteristics $(a_X, b_X, \mathcal{F}[(\rho_X)])$ in their corresponding representations (10) via the inversion formula (32). For more facts and formulas, we refer to the Appendix at the end of this article. We recall that $\tilde{X}$ indicates the free-probability analog, in the sense of the procedure (P4) in Section 2, of the classical hyperbolic characteristic function $X = C, S, T$ which are self-decomposable, that

$$
\begin{align*}
\tilde{\mathcal{F}}_C(s) &= \frac{1}{\cosh(s)}, \\
\tilde{\mathcal{F}}_S(s) &= \frac{s}{\sinh(s)}, \\
\tilde{\mathcal{F}}_T(s) &= \frac{\tanh(s)}{s}, \quad s \in \mathbb{R}.
\end{align*}
$$

and that the Voiculescu transforms of propositions 3, 4 and Corollary 2 correspond to the free-probability analogs $\tilde{X}$.

**Proposition 3.** Recall the $\beta$ function is defined in [10, 8.371(2)] by

$$
\beta(z) = \int_0^{\infty} e^{-z x} \frac{dx}{1 + e^{-x}}, \quad z \in \mathbb{C}^+.
$$

For the free-infinitely divisible Voiculescu transform

$$
V_{\vartheta}(iw) = -F_{\vartheta}(iw) = i \left[ 1 - \frac{w}{\vartheta} \right]^{\frac{w}{\vartheta}} - 1, \quad w > 0,
$$

the characteristics are given by $a_{\vartheta} = b_{\vartheta} = 0$, and the measure $\rho_{\vartheta}$ is a such that $\rho_{\vartheta}(\mathbb{R}) = \frac{\vartheta}{2} - 1$ and
\[ \tilde{\gamma}[\rho_\varepsilon; s] = 2 \sinh(s) \arctan(e^{-s}) + \frac{\pi}{2} e^{-s} - 1 = \int_0^\infty \frac{\cos(sx)}{(1 + x^2) \sinh\left(\frac{x}{2}\right)} dx, \quad s \in \mathbb{R}, \]  

(57)

where \( \beta \) is given by (56) in the Appendix.

**Proposition 4.** Recall the digamma function is given by

\[ \psi(z) = \log z + \int_1^{\infty} \left( \frac{1}{s} - \frac{1}{1 - e^{-s}} \right) e^{-ys} ds, \quad z \in \mathbb{C}^+, \quad [10, 8.361(8)] \]  

(58)

and the Euler-Mascheroni constant corresponds to \( \gamma := -\psi(1) \). For the free-infinitely divisible Voiculescu transform

\[ V_\beta(iw) = -F_\beta(iw) = i \left[ w \psi\left(\frac{w}{2}\right) - \log\left(\frac{w}{2}\right) + 1 \right], \quad w > 0, \]

the characteristics are given by \( a_\beta = b_\beta = 0 \) and the measure \( \rho_\beta \) is a such that \( \rho_\beta(\mathbb{R}) = \gamma + \log 2 - 1 \) and

\[ \tilde{\gamma}[\rho_\beta; s] = \frac{e^{-s} \text{Ei}(s) + e^s \text{Ei}(-s)}{2} + \cosh(s) \log\left(\frac{1 + e^{-|s|}}{1 - e^{-|s|}}\right) - 1 \]

\[ = 2 \int_0^\infty \frac{\cos(sx)}{(1 + x^2) (e^{sx} - 1)} dx, \quad s \in \mathbb{R}. \]  

(59)

where \( \text{Ei} \) is the exponential integral function given by (82) in the Appendix.

By the elementary relation \( \tilde{\gamma}_c = \tilde{\gamma}_s \cdot \tilde{\gamma}_r \), we have \( C \overset{d}{=} S + T \), where \( S \) and \( T \) independent versions. Using Remark 1, we can state:

**Corollary 2.** For the free-infinitely divisible Voiculescu transform,

\[ V_\beta(iw) = -F_\beta(iw) = F_\beta(iw) - F_\beta(iw) = iw \left[ \log\left(\frac{w}{2}\right) - \beta\left(\frac{w}{2}\right) - \psi\left(\frac{w}{2}\right) \right], \quad w > 0, \]

the characteristics are given by \( a_\beta = b_\beta = 0 \), and the measure \( \rho_\beta \) is a such that \( \rho_\beta(\mathbb{R}) = \frac{\pi}{2} - \gamma - \log 2 \) and

\[ \tilde{\gamma}[\rho_\beta; s] = \frac{\pi}{2} e^{-|s|} + 2 \sinh(s) \arctan(e^{-|s|}) + \cosh(s) \log\left(\frac{1 - e^{-|s|}}{1 + e^{-|s|}}\right) - \frac{e^{-s} \text{Ei}(s) + e^s \text{Ei}(-s)}{2} \]

\[ = \int_0^\infty \frac{\cos(sx)}{(1 + x^2) (e^{sx} + 1)} dx, \quad s \in \mathbb{R}. \]  

(60)

In general, if \( X \) is a self-decomposable random variable, then the corresponding Lévy measure \( \nu_x \) has the form

\[ \nu_x(dx) = h_x(x) dx, \quad x \neq 0 \]

\( h \) is a measurable function such that \( x \mapsto xh_x(x) \) is increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\). Therefore, there exists a BDLP \( Y = Y_x \), from the corre-
sponding random integral representations (13), such that the Lévy measures is 
\[ \nu_Y(dx) = -(xh_x)'(x) \, dx \] 
whenever \( h_x \) is differentiable, cf. Steutel and van Harn [26, Proposition 6.11 and Theorem 6.12, Chapter V] and also Jurek [15, Corollary 1.1, p.97], [16, Section 2.1] or Jurek and Yor [18, p. 183, formulae (d) and (e)]. Consequently, on the level of Nevanlinna measures, we have, by (23)

\[ \rho_{\tilde{Y}_X}(dx) := \frac{x^2}{1+x^2} \nu_Y(dx) = -\frac{x^2}{1+x^2} (h_x(x) + xh_x'(x)) \, dx = -\rho_{\tilde{X}}(dx) \frac{x^3 h_x'(x)}{1+x^2} \, dx. \]

(61)

where \( \tilde{X} \) and \( \tilde{Y}_X \) are the free analogues of \( X \) and \( Y_X \), respectively. As in the previous propositions we have a similar results for the BDLP's as well, although we computed it for \( \tilde{Y}_C \), only:

**Proposition 5.** Let \( K \) stands for the Catalan constant, and \( \zeta, \text{Li}_n \) be the Riemann’s zeta function and the polylogarithm functions given in (86) and (87) in the Appendix. For the free-infinitely divisible Voiculescu transform

\[ V_{\tilde{Y}_C}(iw) = -F_{\tilde{Y}_C}(iw) = i \left[ 1 + \frac{w^2}{2} \zeta \left( 2, \frac{w}{2} \right) - \frac{w^2}{4} \zeta \left( 2, \frac{w}{4} \right) \right], \quad w > 0, \]

the characteristics are given by \( a_{\tilde{Y}_C} = b_{\tilde{Y}_C} = 0 \) and the measure \( \rho_{\tilde{Y}_C} \) is such that \( \rho_{\tilde{Y}_C}(\mathbb{R}) = 2K - 1 \) and

\[
\mathfrak{F}[\rho_{\tilde{Y}_C}; s] = 2 \cosh(s) \left( K - \int_0^s \frac{x}{\cosh(x)} \, dx \right) - s \tanh(s) - 1 = \frac{\pi}{2} \int_0^\infty \cos(sx) \frac{x^2}{1+x^2} \frac{\cosh(x)}{\sinh^2\left( \frac{x}{2} \right)} \, dx, \quad s \in \mathbb{R}. \]

(62)

As a by-product of our Propositions 3 and 5, we have

**Corollary 3.** With the notations of Proposition 5, we have

\[
\mathfrak{F}[\rho_{\tilde{Y}_C}; s] + \mathfrak{F}[\rho_{\tilde{X}}; s] = 2 \int_0^\infty \cos(sx) \frac{x^3}{1+x^2} \left( -h_c(x) \right)' \, dx, \quad s \in \mathbb{R},
\]

where the function \( h_c(x) := 1/(2x \sinh\left( \frac{x}{2} \right)) \) is the density of the Lévy measure of the hyperbolic cosine function \( \mathfrak{F}_C \).

**Remark 7.** Formula (57) is confirmed by [10, 4.113(8)] and (59) was confirmed numerically for \( s = 0.5, 1, 2 \). Formulae (60) and (62) seem to be new and might be of some interest.
5 Proofs

5.1 Useful results on cumulant functions

Consider the class of cumulant functions denoted by

\[ C \mathcal{F} = \{ \lambda \mapsto \phi_X(\lambda) = -\log \mathbb{E}[e^{-\lambda X}], \ \lambda \geq 0, \ \text{where } X \text{ is a non-negative r.v.} \} \]

Remark 8. The class \( C \mathcal{F} \) contains the class of Bernstein functions \( \mathcal{B} \mathcal{F} \). By injectivity of the Laplace transform, it is seen that to every \( \phi \in C \mathcal{F} \) corresponds a unique non-negative r.v. \( X \) such that \( \phi = \phi_X \), i.e.

\[ \phi(\lambda) = -\log \mathbb{E}[e^{-\lambda X}] = -\log \left( \mathbb{P}(X = 0) + \mathbb{P}(X > 0) \mathbb{E}[e^{-\lambda X} | X > 0] \right), \ \lambda \geq 0. \] (63)

Observe that \( \phi \) is linear if, and only if, \( X \) is deterministic. Also observe that

- \( \phi \) is infinitely differentiable \((0, \infty)\) and is a strictly increasing bijection \([0, \infty) \mapsto [0, l_\phi)\). We denote from now on, by

\[ \Psi : [0, l_\phi) \to \mathbb{R}_+, \text{ the inverse function of } \phi; \]

- we necessarily have

\[ l_\phi := \lim_{\lambda \to \infty} \phi(\lambda) = -\log \mathbb{P}(X = 0); \] (64)

- if \( e \) denotes a r.v. with standard exponential distribution, then

\[ \mathbb{P}\left( \frac{e}{X} > x \right) = \mathbb{P}(X = 0) + \mathbb{P}(X > 0) \mathbb{E}[e^{-\lambda X} | X > 0], \ \text{for all } x \geq 0 \]

and

\[ \mathbb{P}\left( \Psi\left( \min(e, l_\phi) \right) > x \right) = \mathbb{P}\left( \min(e, l_\phi) > \phi(x) \right) = \mathbb{P}(e > l_\phi) + \mathbb{P}(e < l_\phi, e > \phi(x)) = e^{-\phi(x)}. \]

Exploiting last remarks, we propose the following lemma that will be useful in the sequel:

Lemma 1 (Some results on cumulant functions). Let \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) a differentiable function on \((0, \infty)\), \( \phi(0) = 0 \). Denote \( l_\phi := \lim_{\lambda \to \infty} \phi(\lambda) \) and by \( e \) a r.v. with standard exponential distribution.

1) Assume \( \phi \) is a non-linear cumulant function, hence is associated to a non-negative and non-deterministic r.v. \( X \). Then,

(i) \( \phi' \) is a (strictly) decreasing bijection on \((0, \infty)\), hence \( \phi \) is strictly concave and increases to \( l_\phi \);
(ii) The inverse function \(\Psi\) of \(\phi\), is such that \(\Psi': (0,l_\phi) \to (0,\infty)\) is a (strictly) increasing bijection, hence \(\Psi\) is strictly convex;

(iii) As \(\lambda \uparrow \infty\), the function \(\phi(\lambda)/\lambda\) decreases to \(L_X := \inf\{x \geq 0, \text{s.t. } P(X \leq x) > 0\}\).

Hence, on \((0,\infty)\), the function \(\Psi(x)/x\) increases to \(1/L_X\) as \(x \uparrow l_\phi\). Further,

\[L_X = \lim_{x \to \infty} \phi'(x) \in [0,\infty); \quad (65)\]

(iv) If \(p > 0\), then \((\Psi^p)'\) is positive on \((0,\infty)\) and we have the equivalence

\[(\Psi^p)'(0+) := \lim_{\lambda \to 0+} (\Psi^p)'(\lambda) < \infty \iff \lim_{\lambda \to 0+} \frac{\lambda^{p-1}}{\phi'(\lambda)}; \quad (66)\]

(v) The function \(\Psi^p\), is convex (respectively concave) if, and only if,

\[\lambda \mapsto \lambda^{1-p} \phi'\text{ is non-increasing (respectively non-decreasing ).}\]

In particular, if \(p \geq 1\), then \(\Psi^p\) is strictly convex and \((\Psi^p)'(0+) \in [0,\infty)\).

2) Assume \(\phi\) is a strictly increasing bijection with inverse \(\Psi\). Then, the following statements are equivalent.

i) \(\phi\) is a cumulant function;

ii) For some non-negative r.v. \(X\), we have the identity in law

\[e^X \overset{d}{=} \Psi(\min(e,l_\phi)), \quad \text{(where } e \text{ in the l.h.s. is assumed to be independent of } X; \quad (67)\]

iii) For some non-negative r.v. \(X\), we have the representations

\[\mathbb{E}[X^{-p}] = \frac{M(p)}{\Gamma(p+1)}, \quad M(p) := \int_0^{l_\phi} e^{-x} \Psi^p(x) \, dx + e^{-l_\phi} \Psi^p(l_\phi) \mathbb{I}_{(l_\phi < \infty)}, \quad (68)\]

for every \(p > -1\), such that \(e^{-x} \Psi^p(x)\) is integrable on \((0,l_\phi)\).

Remark 9. Observe that formula (68) reminds the one involving the Gamma function in (81). Actually, under the condition of integrability of \(\Psi^p\), the injectivity of the Mellin transform insures that the function \(M(p)\), \(p > -1\) is the Mellin transform of a (unique) positive random variable, namely

\[M(p) = \mathbb{E}\left[\Psi^p(\min(e,l_\phi))\right], \quad (69)\]

thus, (68) reads

\[\mathbb{E}\left[\left(\frac{e}{X}\right)^p\right] = \mathbb{E}[e^p] \mathbb{E}[X^{-p}] = \Gamma(p+1) \mathbb{E}[X^{-p}] = \mathbb{E}\left[\Psi^p(\min(e,l_\phi))\right].\]
**Proof of Lemma 1.** 1) (i): The first and the second derivative of $\phi$ are given by
\[
\phi'(\lambda) = \frac{E[Xe^{-\lambda X}]}{E[e^{-\lambda X}]} \quad \text{and} \quad \phi''(\lambda) = \frac{E[X^2e^{-\lambda X}] - E[Xe^{-\lambda X}]^2}{E[e^{-\lambda X}]^2}, \quad \lambda > 0.
\]

By Cauchy-Schwarz inequality, every pair of nonnegative random variable $Y,Z$ satisfy the inequality $E[YZ]^2 \leq E[Y^2]E[Z^2]$ and the equality holds if and only if there exists $c \geq 0$ such that $Y = cZ$ a.s. Choosing $Y = Xe^{-\frac{\lambda}{2}X}$ and $Z = e^{-\frac{\lambda}{2}X}$ in (70), we recover the negativity of $\phi''$ and the claim on $\phi'$ follows.

1) (ii): The assertion is due to the observation $\Psi' = 1/\phi'(\Psi)$ and $\Psi'' = -\phi''(\Psi)/\phi'((\Psi))^3$.

1) (iii) The assertion is an adaptation of the result of Pakes [23, Theorem 2.1] by taking the Mellin transform $M(\lambda)^{1/\lambda} = E[(e^{-X})^\lambda]^{1/\lambda} = e^{-\theta(\lambda)/\lambda}$ there. The limit in (65) is obtained by Karamata’s Theorem [7, Theorem 1.5.11], the monotone density theorem [7, Theorem 1.7.2] and the concavity of a cumulant function.

1) (iv): We have
\[
(\Psi^p)' = p \frac{\Psi^{p-1}}{\phi'(\Psi)}, \quad (\Psi^p)'' = p \frac{\Psi^{p-2}}{\phi'(\Psi)^2}\left[(p-1)\phi'(\Psi) - \Psi''(\Psi)\right]
\]

Positivity of $(\Psi^p)'$ is immediate and since $\Psi(0) = 0$, then by the change of variable $x = \phi(\lambda)$, we have
\[
\lim_{x \to 0^+} x (\Psi^p)'(x) = p \lim_{x \to 0^+} \frac{\Psi^{p-1}(x)}{\phi'(\Psi(x))} = p \lim_{\lambda \to 0^+} \frac{\lambda^{p-1}}{\phi'(\lambda)}.
\]

For a later use, one can observe that
\[
p \geq 1 \implies \lim_{x \to 0^+} x (\Psi^p)'(x) = p \lim_{\lambda \to 0^+} \frac{\lambda^{p-1}}{\phi'(\lambda)} \in [0, \infty).
\]

1) (v): By (71), we have
\[
Sign\left((\Psi^p)''\right) = Sign\left[(p-1)\phi' - \lambda \phi''\right] = \left[\left(\frac{\lambda^{-1} \phi'}{\phi''}\right)'\right]
\]

and if $p \geq 1$, then $\lambda^{p-1}/\phi'$ is increasing, viz. $\Psi^p$ is convex and last assertion is then obtained with the help of (72).

2) (i) $\iff$ (ii): Is obtained from the discussion in Remark 8.

2) (ii) $\implies$ (iii): The case $I_\phi = \infty$ is easy to prove due to the facts, that $\Psi(0) = \phi(0) = 0$, $\Psi$ is an increasing bijection from $\mathbb{R}_+ \to \mathbb{R}_+$ (with inverse $\phi$), and that
\[
\frac{d}{dy}E[e^{-\lambda X}] = -\frac{d}{dy}e^{-\phi(y)} = \phi'(y)e^{-\phi(y)} = E[Xe^{-\lambda X}].
\]

Indeed, making the change of variable $x = \phi(y)$, using Tonelli-Fubini’s theorem and the representation of the gamma function (81), these facts entail
\[
\int_0^\infty e^{-x} \frac{\psi^p(x)}{\Gamma(p+1)} \, dx = \int_0^\infty \phi'(y) e^{-\phi(y)} \frac{y^p}{\Gamma(p+1)} \, dy = \int_0^\infty \mathbb{E}[X e^{-yX}] \frac{y^p}{\Gamma(p+1)} \, dy
\]

The case \( l_{\phi} < \infty \) is proved similarly.

2) \((iii) \implies (ii)\): This has been noticed in Remark 9. □

### 5.2 The proofs

**Proof of Theorem 1.** Observe that if \( F \) is a Nevanlinna-Pick function represented by (5), then \( a, b \) and \( \rho(R) \) are given by (7).

1) \( \implies 2) \): It is enough to reproduce the steps of the proof [25, Corollary 6.13] taking into account that \( \rho \) is supported by \( A \).

2) \( \implies 3) \): Use Jankowski and Jurek inversion’s procedure given [11, Theorem 1] and explained before (10) and observe that \( H(w) = G(iw), \ w > 0 \). Also observe that

\[
\mathcal{L}[\sinh(x);w] = \frac{1}{w^2 - 1} \quad \text{and} \quad \mathcal{L}[\cosh(x);w] = \frac{w}{w^2 - 1}, \quad w > 1,
\]

which yields (32).

3) \( \implies 1) \): Observe that knowing the function \( w \mapsto F(iw) = i(1 - w^2)H(w) \) on the interval \((1, \infty)\) or even on any interval \((a,b) \subset (0, \infty)\) is sufficient to fully recover \( F \), cf. Jedidi et al. [12, Lemma 4.1].

The last claim is a straightforward adaptation of [25, Corollary 6.13]. □

**Proof of Proposition 1.** In order to get (41), just use the representation (35) of \( \Psi \), the properties (81) of the Gamma function and the identity

\[
-\frac{x}{z^2} \left( \frac{1}{z+x} - \frac{1}{z} + \frac{1}{z^2} \frac{x}{1+x^2} \right) = \frac{xz - 1}{z+x} \frac{x^2}{1+x^2},
\]

in order to write: for \( z \in \mathbb{C}^+ \)
\[
F(z) = z^2 \int_0^\infty e^{-zu} \left( b - \beta u - \frac{\gamma^2}{2} u^2 - \int_{(0,\infty)} \left( e^{-ux} - 1 + u \frac{x}{1+x^2} \right) v(dx) \right) du \\
= -\beta + b\gamma - \frac{\gamma^2}{2} \int_{(0,\infty)} \left( 1 + \frac{1}{z} + \frac{x}{z^2} + \frac{x}{1+x^2} \right) v(dx) \\
= -\beta + b\gamma + \int_{(0,\infty)} \frac{x^2 - 1}{z + x} \phi(dx) \\
= a + b\gamma + \int_{(0,\infty)} \frac{x^2 - 1}{z + x} \rho(dx).
\]

The controls (22) and Tonelli-Fubini’s theorem allow the reversal of order in the integrals. □

**Proof of Proposition 2.** 1) \(\implies\) 2): If \(\phi \in BF\), then it is associated to a subordinator \(\tilde{\xi} = (\tilde{\xi}_t)_{t \geq 0}\), such that each \(\tilde{\xi}_t\), \(t > 0\), has the cumulant (Bernstein) function \(\Phi_t = t\Phi\) whose inverse is \(\Psi_t(x) = \Psi(x/t)\). Since \(\Phi(\infty) = \infty\), then assertion (42) immediately follows from (67).

2) \(\implies\) 3): is trivially satisfied with \(\eta = \tilde{\xi}\).

3) \(\implies\) 1)): By 2) in Lemma 1, necessarily the inverse \(t\Phi\) of \(\Psi_t(x) = \Psi(x/t)\) is the cumulant of \(\eta_t\) for all \(t > 0\), i.e., \(e^{-\Phi}\) is a completely monotone, or equivalently \((1 - e^{-\Phi})/t\) is a Bernstein function for all \(t > 0\). Passing to the limit as \(t \to 0\) and using the fact that the class BF is closed under pointwise limits, cf. [25, Corollary 3.8], we deduce that \(\phi \in BF\).

3) \(\implies\) 4): By the implication 2) \(\implies\) 3), point 3) is equivalent to the existence of the subordinator \(\tilde{\xi}\) satisfying (42), then (43) follows from of (68) and from the change of variable \(x \to tx\) there.

4) \(\implies\) 1): By Lemma 1, necessarily the inverse \(t\Phi\) of \(\Psi_t(x) = \Psi(x/t)\) is the cumulant of \(\tilde{\xi}_t\) and \(\tilde{\xi}\) is a subordinator. Then, \(\phi\) is a Bernstein function. □

**Proof of Theorem 2.** 1) Property 1) (iii) in Lemma 1 gives the implication

\[
x \mapsto \frac{\Psi(x)}{x}
\quad\text{is non-decreasing,}\quad p \in (-1,0) \implies e^{-tx}\Psi^p(x)\text{ is integrable at }\infty, \quad \forall t > 0.
\]

Then, the definiteness of \(E[\Psi^p;\cdot]\) on \((0,\infty)\) is equivalent to 1)(i). Since \(\Psi^p\) is a non-increasing function, the latter is equivalent to assertion 1)(ii), because representation (43) of \(E[\tilde{\xi}_t^\nu]\) meets the form [25, (3.3) p. 23] of a Bernstein function. Representation (45) is obtained by integration by parts form (43).

2) Similarly,

\[
x \mapsto \frac{\Psi(x)}{x}
\quad\text{is non-decreasing,}\quad p \geq 0 \implies \Psi^p(x)\text{ is integrable at }0.
\]

Then, the definiteness of \(E[\Psi^p;\cdot]\) \((0,\infty)\) is then equivalent to 2)(i). Since \(\Psi^p\) is an non-decreasing function, null at 0, and by an integration by parts, the latter is
equivalent to representation (46), hence $\mathbb{E}[\xi_{t}^{-p}]$ meets the form of a completely monotone function. The last claim is due to 1) (v) in Lemma 1 that insures the concavity of $\Psi^p$, and to the form (43) that gives

$$t \mathbb{E}[\xi_{t}^{-p}] = \frac{t^2}{p+1} \int_{0}^{\infty} e^{-tx} \Psi^p(x) \, dx,$$

(75)

which meets the form [25, (3.4) p. 23] of a Bernstein function. Representation (47) is obtained by integration by parts in (46). 3) The convexity of $\Psi^p$ and $(\Psi^p)'(0^+) \in [0, \infty)$ are guaranteed by 1) (v) in Lemma 1. The integrability of $\Psi^p$ near zero is guaranteed by $p \geq 1$. If further $\mathbb{E}[\Psi^p; t] < \infty$, for some $t > 0$, then two integration by parts in (75), give representation

$$t \mathbb{E}[\xi_{t}^{-p}] = \frac{1}{p+1} \left( (\Psi^p)'(0) + \int_{0}^{\infty} e^{-tx} (\Psi^p)''(x) \, dx \right)$$

(76)

Differentiation

$$\frac{d}{dt}(t \mathbb{E}[\xi_{t}^{-p}]) = \frac{1}{p+1} \int_{0}^{\infty} e^{-tx} (\Psi^p)''(x) \, dx.$$

The function $t \mathbb{E}[\xi_{t}^{-p}]$ being positive, deduce the equivalences 3) (i) $\iff$ 3) (ii) from 2) (i) $\iff$ 2) (ii).

The last claims in 3) are obtained as follows:

3) (a) Representation (46), when multiplied by $t^2$ meets with the form [25, (3.4) p. 23] of a Bernstein function, if, and only if $(\Psi^p)'$ is concave.

3) (b) Representation (50) is a simple reformulation of (48), when differentiating twice the expression of $\Psi^p$ in (35). $\square$

**Proof of Corollary 1.** 1)(i) and 1)(ii) are a direct consequence of Proposition 2 because $\xi$ is a subordinator.

1)(iii): If $\Psi^p \in \mathcal{LE}(0, \infty)$, then, by Proposition 2, the function $f(t) := t \mathbb{E}[\xi_{t}^{-p}]$, $t > 0$, has the representation

$$f(t) = \frac{t^2}{p+1} \int_{0}^{\infty} e^{-tx} \Psi^p(x) \, dx.$$

By Proposition 1, the extension on $\mathbb{H}^+$ of $f$ is such that $-f$ is a non-positive Nevanlinna $\mathcal{N}(\mathbb{R}_+)$-function, which is continuous on $(0, \infty)$. By [25, Corollary 6.14] this equivalent to say that $f$ is a Stieltjes transform and by [25, Theorem 6.2], this also equivalent to $t \mapsto t f(t) \in \mathcal{CBF}$. 2) Since $\xi$ is a subordinator, then, by Proposition 2, the function $t \mapsto f(t) := t \mathbb{E}[\xi_{t}^{-p}]$ is represented by

$$-f(t) = -t^2 \int_{0}^{\infty} e^{-tx} \Psi^p(x) \, dx.$$
Since \( f \) is a Stieltjes function, then, using [25, Corollary 6.14] again, we deduce the extension on \( \mathbb{H}^+ \) of \(-f\) is a Nevanlinna \( \mathcal{N}^+ (\mathbb{R}^+) \)-function because it satisfies \( \lim_{|z| \to \infty, z \in \mathbb{C}^+} -f(z)/z = 0 \). Finally, by Proposition 1, we necessarily have \( \Psi^p \in \mathcal{LE}(0, \infty) \). To conclude, observe that the inverse function of \( \Psi^p \) is the Bernstein function \( (\Psi^p)^{-1} (\lambda) = \phi (\lambda^{1/p}) \) corresponding to the subordinator \( \xi (1/p) \circ \xi \).

In all the following proofs, the symbol \( \ast \) denotes the usual additive convolution of functions on the positive half-line. The calculi where also checked by WolframAlpha or Mathematica.

**Proof of Proposition 3.** Firstly, note that \( F_c (i) = -i(1 - \beta (1/2)) = i(\pi/2 - 1) \) and therefore \( \rho_c (\mathbb{R}) = \frac{\pi}{2} - 1 \). Secondly, by (56), we have \( \beta (z) = \mathcal{L}[(1 + e^{-x})^{-1}; z], z \in \mathbb{C}^+ \). Consequently,

\[
\frac{iF_c (iw)}{w^2 - 1} = \frac{1 - w \beta (\frac{w}{w^2 - 1})}{w^2 - 1} = \mathcal{L}[\sinh(s); w] - \mathcal{L}[(\cosh(s); w)] \mathcal{L}\left[ \frac{2}{1 + e^{-2s}}; w \right] = \mathcal{L}[\sinh(s); w] - \mathcal{L}\left[ (\cosh(u) \ast \frac{2}{1 + e^{-2u}}) (s); w \right], \text{ for } w > 0, w \neq (77)
\]

Thirdly, by a differentiation, one checks that, for \( s > 0 \),

\[
\left( \cosh(u) \ast \frac{2}{1 + e^{-2u}} \right) (s) := \int_0^s \cosh (s - u) \frac{2}{1 + e^{-2(s-u)}} du = 2 \sinh(s) \left[ \frac{\pi}{4} - \arctan(e^{-s}) \right] e^{-s} + 1,
\]

and inserting the latter into (77), we get

\[
\frac{iF_c (iw)}{w^2 - 1} = \mathcal{L} \left[ \sinh(s) - 2 \sinh(s) \left( \frac{\pi}{4} - \arctan(e^{-s}) \right) + e^{-s} - 1 \right]; w \right]. \quad (78)
\]

Finally, since \( \rho_c (\mathbb{R}) = \frac{\pi}{2} - 1 \) and using (74), we arrive to

\[
\mathcal{F} [\rho_c; s] = \left( \frac{\pi}{2} - 1 \right) \cosh(s) + \sinh(s) - 2 \sinh(s) \left( \frac{\pi}{4} - \arctan(e^{-s}) \right) + e^{-s} - 1 \\
= 2 \sinh(s) \arctan(e^{-s}) + \frac{\pi}{2} e^{-s} - 1,
\]

which gives the first equality in (57). On the other hand, from Jurek [16, Corollary 2], we know that \( V_c (iw) = -F_c (iw) \) is a free-probability analog of the classical hyperbolic characteristic function \( 1/\cosh(t) \) whose Nevanlinna measure \( \rho_c \) has a density

\[
\frac{|x|}{2(1 + x^2) \sinh(\frac{\pi}{2}|x|)} \quad x \in \mathbb{R}.
\]

Thus,

\[
\mathcal{F} [\rho_c; s] = \int_{\mathbb{R}} e^{ixs} \frac{|x|}{2(1 + x^2) \sinh(\frac{\pi}{2}|x|)} dx = \int_0^\infty \cos (sx) \frac{x}{(1 + x^2) \sinh(\frac{\pi}{2}x)} dx,
\]
which shows the second equality in (57). □

**Proof of Proposition 4.** By [10, 8.366(2)], we have \( \psi(1/2) = -\gamma - 2\log 2 \), then \( F_3(i) = i(\gamma + \log 2 - 1) \). Hence, in (10), we have \( a_3 = b_3 = 0 \) and the measure \( \rho_3 \) has a finite mass \( \rho_3(\mathbb{R}) = \gamma + \log 2 - 1 \). Using the integral formula (58) for the \( \psi \)-function, we get

\[
\frac{iF_3(iw)}{w^2 - 1} = \frac{1 + w(\psi(w/2) - \log(w/2))}{w^2 - 1} = \mathcal{L}[\sinh(s); w] + \mathcal{L}[\cosh(s); w] \mathcal{L}\left[\frac{1}{s} \cdot \frac{2}{1 - e^{-2s}}; w\right]
\]

\[
= \mathcal{L}[\sinh(s) + g(s); w], \text{ where } g(s) := \left( \cosh(u) * \left( \frac{1}{u} - \frac{2}{1 - e^{-2u}} \right) \right)(s).
\]

By a direct differentiation in identities (F3) in the Appendix of the Ei function, one checks that, for \( s > 0 \), we have

\[
g(s) = \int_0^s \cosh(s - u) \left[ \frac{1}{u} - \frac{2}{1 - e^{-2u}} \right] du = \left[ \frac{e^{-s}}{2} \text{Ei}(u) + \frac{e^s}{2} \text{Ei}(-u) - \cosh(s) \log \frac{1 + e^{-s}}{1 - e^{-s}} \right]_{u=0}^{u=s}
\]

\[= \frac{1}{2} e^{-s} \text{Ei}(s) + \frac{e^s}{2} \text{Ei}(-s) - 1 + \cosh(s) \log \frac{1 + e^{-s}}{1 - e^{-s}} - \lim_{u \to 0^+} \left[ \frac{e^{-s}}{2} (\text{Ei}(u) - \log(1 - e^{-u})) \right]
\]

\[+ \frac{e^s}{2} (\text{Ei}(-u) - \log(1 - e^{-u})) - e^{u-s} + \cosh(u) \log(1 + e^{-u}) \]

\[= \frac{1}{2} e^{-s} \text{Ei}(s) + \frac{1}{2} e^s \text{Ei}(-s) - 1 + e^{-s} + \cosh(s) \left( \log \frac{1 + e^{-s}}{1 - e^{-s}} - \gamma - 2 \right).
\]

Inserting the above equality into (79) and using (74) with \( \rho_3(\mathbb{R}) = \gamma + \log 2 - 1 \), we get for \( s \in \mathbb{R} \),

\[
\mathfrak{W}[\rho; s] = (\gamma + \log 2 - 1) \cosh(s) + \sinh(s) + \frac{e^{-s}}{2} \text{Ei}(s) + \frac{e^s}{2} \text{Ei}(-s) - 1 + e^{-s} + \cosh(s) \left( \log \frac{1 + e^{-|s|}}{1 - e^{-|s|}} - \gamma - 2 \right)
\]

\[= \frac{1}{2} e^{-s} \text{Ei}(s) + \frac{e^s}{2} \text{Ei}(-s) - 1 + \cosh(s) \log \frac{1 + e^{-|s|}}{1 - e^{-|s|}},
\]

which proves the equality (59). From Jurek [16, Corollary 3], we know that \( V_3(iw) = -F_3(iw) \) is the free-analog of the classical hyperbolic sine characteristic function \( \tilde{\phi}_s(t) = t/\sinh(t) \) whose Nevanlinna measure equals to

\[
\rho_s(dx) = \frac{1}{2} \frac{|x|}{1 + x^2 \sinh((\frac{\pi}{2}) |x|)} dx = \frac{1}{2} \frac{|x|}{1 + x^2 e^{\pi |x|}} dx, \quad x \in \mathbb{R},
\]

and we get the equality (59). □

**Proof of Proposition 5.** Since \( F_{\zeta}(i) = i(2K - 1) \), then in (10), \( a_{K_0} = b_{K_0} = 0 \), and for the measure \( \rho_{K_0} \), we have \( \rho_{K_0}(\mathbb{R}) = 2K - 1 \). Using (74) and the integral representation (88) for the \( \zeta(2, s) \) function, we have
\[ \mathcal{L}[\delta[p_{\xi_c}](s) - (2K - 1) \cosh(s); w] = \frac{1}{w^2 - 1} \left[ \frac{w^2}{2} \left( \zeta(2, \frac{w}{2}) - \frac{1}{2} \zeta(2, \frac{w}{4}) \right) + 1 \right] \]
\[ = \frac{1}{w^2 - 1} + 2 \frac{w^2}{w^2 - 1} \left[ \int_0^\infty e^{-wu} \frac{u}{1 - e^{-2u}} du - 2 \int_0^\infty e^{-wu} \frac{u}{1 - e^{-4u}} du \right] \]
\[ = \mathcal{L}[\sinh(s); w] + 2 \left( 1 + \frac{1}{w^2 - 1} \right) \int_0^\infty e^{-wu} \frac{u}{1 - e^{-2u}} du \]
\[ = \mathcal{L}[\sinh(s); w] + (1 + \frac{1}{w^2 - 1}) \int_0^\infty e^{-wu} \frac{1 - e^{2u}}{\sinh(2u)} du \]
\[ = \mathcal{L} \left[ \sinh(s) + s \frac{1 - e^{2s}}{\sinh(2s)}; w \right] + 2 \mathcal{L}[\sinh(s); w] \mathcal{L} \left[ s \frac{1 - e^{2s}}{\sinh(2s)}; w \right] \]

where,
\[ h(s) := (\sinh u + u \frac{1 - e^{2u}}{\sinh(2u)}) (s) = \int_0^s \sinh(s - u) u \frac{1 - e^{2u}}{\sinh(2u)} du. \quad (80) \]

Using elementary computations, together with (85), (90) in the Appendix, one checks that

\[ h(s) = \left( e^{-s}(u - 1) e^{u} - i \cosh(s) \left( \text{Li}_2(ie^u) - \text{Li}_2(-ie^u) + u \log \frac{1 - ie^u}{1 + ie^u} \right) \right) \]_{u=0}^{s} \]
\[ = \left( e^{-s}(u - 1) e^{u} - i \cosh(s) \left( -2i \int_0^u \frac{x}{\cosh(x)} dx \right) \right) \]_{u=0}^{s} \]
\[ = e^{-s} - 1 + s - 2 \cosh(s) \int_0^s \frac{x}{\cosh(x)} dx. \]

From the above calculation and from the equality \((e^{2s} - 1)/\sinh(2s) = 2e^{2s}/(e^{2s} + 1)\), we arrive at the representation of \( \delta[p_{\xi_c}], s \geq 0:\)

\[ \delta[p_{\xi_c}](s) = (2K - 1) \cosh(s) + \sinh(s) - 2s \frac{e^{2s}}{e^{2s} + 1} + h(s) \]
\[ = 2 \cosh(s) \left( K - \int_0^s \frac{x}{\cosh(x)} dx \right) + s \left( 1 - 2 \frac{e^{2s}}{e^{2s} + 1} \right) + \left( \frac{\sinh(s) - \cosh(s) + e^{-s}}{e^{2s} + 1} \right) - 1 \]
\[ = 2 \cosh(s) \left( K - \int_0^s \frac{x}{\cosh(x)} dx \right) - s \tanh(s) - 1 \]

which gives the equality (62). For the equality (62), we just need to recall that from Jurek and Yor [18, Corollary 1 and a formula (7) p. 186], that the Nevanlinna measure corresponding to the BDLP \( Y_c \), given by (13) is represented by
\[ \rho_{\hat{C}}(dx) = \frac{\pi}{4} \frac{x^2}{1 + x^2} \frac{\cosh \left( \frac{2x}{\pi} \right)}{\sinh^2 \left( \frac{2x}{\pi} \right)} dx, \quad x \in \mathbb{R}. \]

\[ \square \]

**Proof of Corollary 3.** It is a simple application of Remark (61) applied to \( X = C \).

\[ \square \]

**Remark 10.** Since the hyperbolic sine and the hyperbolic tangent are self-decomposable as well, we may have a statement about \( \tilde{S} \) and \( \tilde{T} \), analogous to the one in Corollary 3 for the hyperbolic cosine function.

### 6 Appendix

For a convenience of reading, we collect some facts on special functions. Boldface numbers below refer to formulae from [10].

**F1** Many of those functions we use are derived from Euler’s \( \Gamma \) gamma function and the digamma function \( \psi \) used in Proposition 4:
\[
\Gamma'(p) := \int_0^\infty e^{-tx} t^{p-1} dx, \quad \psi(p) := \frac{\Gamma'(p)}{\Gamma(p)} \quad z, \quad p \in \mathbb{C}^+. \quad (81)
\]

**F2** In Proposition 3, we used the \( \beta \)-function which was originally derived from the digamma function via the formula
\[
\beta(x) := \frac{1}{2} \left[ \psi \left( \frac{x+1}{2} \right) - \psi \left( \frac{x}{2} \right) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}, \quad -x \notin \mathbb{N} \quad [10, 8.372(1)].
\]

**F3** To formulate Proposition 4, we need another special function, namely, the exponential integral function \( \text{Ei} \), defined by
\[
\text{Ei}(x) := -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \quad x < 0 \quad [10, 8.211(1)].
\]

For \( x > 0 \), \( \text{Ei} \) is defined by the Cauchy principal value (P.V.) method:
\[
\text{Ei}(x) = -\lim_{\epsilon \to 0} \left[ \int_{-x}^{-\epsilon} \frac{e^{-t}}{t} dt + \int_{\epsilon}^{\infty} \frac{e^{-t}}{t} dt \right], \quad x > 0 \quad [10, 8.211(2)].
\]

Other useful representation is
\[
\text{Ei}(x) = \gamma + \log |x| + \sum_{k=1}^{\infty} \frac{x^k}{k!}, \quad x \neq 0 \quad [10, 8.214(1,2)].
\]
For Proposition \((F5)\), we need another two special functions, namely, the Riemann’s zeta function, \(\zeta\), and the polylogarithm functions \(\text{Li}_n\), defined as follows:

\[
\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \quad \Re(s) > 1, \quad -a \notin \mathbb{N};
\]

\[
\text{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| < 1, \quad s \in \mathbb{C},
\]

with the properties, \(\text{Li}_s(1) = \zeta(s, 1)\), and the representations

\[
\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-ax} \frac{x^{s-1}}{1-e^{-x}} \, dx, \quad \Re(a) > 0,
\]

\[
\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_1^{\infty} \frac{\log^{s-1}(x)}{x(x-z)} \, dx, \quad z \in \mathbb{C} \setminus [1, \infty). \quad [21, (7.188), p. 236(89)]
\]

where (88) is clearly a consequence of (81). The name of the polylogarithm functions is due to the fact that \(\text{Li}_n, n = 1, 2, \ldots\), can be defined recursively by

\[
\text{Li}_n(z) = \int_0^z \frac{\text{Li}_{n-1}(y)}{y} \, dy, \quad \text{Li}_0(z) = \frac{z}{1-z}, \quad \text{Li}_1(z) = -\log(1-z), \quad z < 1.
\]

By (85) and [21, Formulae (2.3) p. 38 and (4.29) p. 106], we have

\[
\text{Li}_2(ie^s) - \text{Li}_2(-ie^s) + s \log \frac{1-ie^s}{1+ie^s} = -2i \int_0^{\frac{x}{\cosh(x)}} x \, dx, \quad s \in \mathbb{R},
\]

where \(K = (\text{Li}_2(i) - \text{Li}_2(-i))/2i\) stands for the Catalan constant \(\approx 0.9159\). In particular, we have:

\[
\text{Li}_2(-ie^s) + \text{Li}_2(ie^s) = 2i \sum_{k=1}^{\infty} (-1)^k - \frac{e^{2k-1}s}{(2k-1)^2}.
\]
References

1. N.I. Akhiezer, *The Classical Moment Problem and some related questions in analysis*, Oliver & Boyd, Edinburgh and London, 1965.
2. O.E. Barndorff-Nielsen and S. Thorbjørnsen (2004), A connection between free and classical infinite divisibility, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, Vol. 7 (4), pp. 573–590.
3. H. Bercovici, V. Pata and P. Biane (1999), Stable laws and domains of attraction in free probability theory, *Ann. Math.*, 149, 1023–1060.
4. H. Bercovici and D. Voiculescu (1993), Free convolution of measures with unbounded support, *Indiana U. Math. J.*, Vol. 42, No. 3, pp. 733–773.
5. J. Bertoin, *Subordinators: Examples and Applications*, Lectures on Probability Theory and Statistics: Ecole d’Eté de Probabilités de Saint-Flour XXVII - 1997”, Springer Berlin Heidelberg, 1999.
6. J. Bertoin and M. Yor (2001), On subordinators, self-similar markov processes and some factorizations of the exponential variable, *Elect. Comm. in Probab.*, vol. 6, pp. 95–106.
7. Bingham, N.H.; Goldie, C.M.; J.L. Teugels: *Regular variation*. Encyclopedia of Mathematics and its Applications 27. Cambridge University Press (1987).
8. L. Bondesson, *Generalized Gamma Convolutions and Related Classes of Distributions and Densities*, Springer-Verlag, New York; Lecture Notes in Statistics, vol. 76, 1992.
9. Chaumont, L.; Yor, M.: *Exercises in probability*. A guided tour from measure theory to random processes, via conditioning. Cambridge Series in Statistical and Probabilistic Mathematics, 13. Cambridge University Press, Cambridge, 2003.
10. I.S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 7th edition, Academic Press, New York, 2007.
11. L. Jankowski and Z. J. Jurek (2012), Remarks on restricted Nevalinna transforms, *Demonstratio. Math.*, vol. XLV (2), pp. 297–307.
12. W. Jedidi, F. Bouzeffour, N. Harthi (2020), Contribution to an open problem of Harkness and Shantaram. *Math. Meth. Appl. Sci.*, Special Issue Paper, pp. 1–20 (2020).
13. W. Jedidi (2009), Stable Processes, mixing, and distributional properties, II, *Theory Probab. its Appl.*, vol. 53 (1) pp. 81–105.
14. Z. J. Jurek and J. D. Mason, *Operator-Limit Distributions in Probability Theory*, J. Wiley, New York, 1993.
15. Z. J. Jurek (1997), Self-decomposability: an exception or a rule?, *Ann. Univ. Marie Curie-Skłodowska Sect. A.*, vol. LI (1), pp. 93-107.
16. Z. J. Jurek (2020), On a relation between classical and free infinitely divisible transforms, *Probab. Math. Statist.*, vol. 40(2), pp. 349, Ái 367. [Also: math.arXiv:1707.02540 [math.PR], 9 July 2017.]
17. Z.J. Jurek and W. Vervaat (1983), An integral representation for self-decomposable Banach space valued random variables, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, vol. 62, pp. 247–262.
18. Z. J. Jurek and M. Yor (2004), Self-decomposable laws associated with hyperbolic functions, *Probab. Math. Statist.*, vol. 24 (1), pp. 181–190.
19. A. E. Kyprianou, *Fluctuations of Lévy Processes with Applications*, Introductory Lectures, Second Edition, Springer-Verlag, Berlin Heidelberg, 2006.
20. S. Lang, *SL2(R)*, Addison-Wesley, Reading Massachusetts, 1975.
21. L. Lewin, *Polylogarithms and Associated Functions*, Elsevier/North-Holland, New York, 1981.
22. A. G. Pakes (2011), Lambert’s W, infinite divisibility and Poisson mixtures, *J. Math. Anal. Appl.*, 378, pp. 480–492.
23. A. G. Pakes (1997), Characterization by invariance under length-biasing and random scaling, *J. Statist. Planning Inf.*, 63, pp. 285–310.
24. J. Pitman and M. Yor (2003), Infinitely divisible laws associated with hyperbolic functions, *Canad. J. Math.* 55 (2), pp. 292–330.
25. R.L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions. Theory and applications*, second edition, de Gruyter Studies in Mathematics, 37. Walter de Gruyter & Co., Berlin, 2012.
26. L. Steutel and K. Van Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, Marcel Dekker, New York, Basel, 2004.
27. Yakubovich, S. and Martins, M. (2014): On the iterated Stieltjes transform and its convolution with applications to singular integral equations. *Integral Transforms Spec. Funct.*, 25(5), pp. 398–411.