SMOOTH HYPERSURFACE SECTIONS CONTAINING A GIVEN SUBSCHEME OVER A FINITE FIELD

BJORN POONEN

1. Introduction

Let $\mathbb{F}_q$ be a finite field of $q = p^n$ elements. Let $X$ be a smooth quasi-projective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. N. Katz asked for a finite field analogue of the Bertini smoothness theorem, and in particular asked whether one could always find a hypersurface $H$ in $\mathbb{P}^n$ such that $H \cap X$ is smooth of dimension $m - 1$. A positive answer was proved in [Gab01] and [Poo04] independently. The latter paper proved also that in a precise sense, a positive fraction of hypersurfaces have the required property.

The classical Bertini theorem was extended in [Blo70, KA79] to show that the hypersurface can be chosen so as to contain a prescribed closed smooth subscheme $Z$, provided that the condition $\dim X > 2 \dim Z$ is satisfied. (The condition arises naturally from a dimension-counting argument.) The goal of the current paper is to prove an analogous result over finite fields. In fact, our result is stronger than that of [KA79] in that we do not require $Z \subseteq X$, but weaker in that we assume that $Z \cap X$ be smooth. (With a little more work and complexity, we could prove a version for a non-smooth intersection as well, but we restrict to the smooth case for simplicity.) One reason for proving our result is that it is used by [SS07].

Let $S = \mathbb{F}_q[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Let $S_d \subseteq S$ be the $\mathbb{F}_q$-subspace of homogeneous polynomials of degree $d$. For each $f \in S_d$, let $H_f$ be the subscheme $\text{Proj}(S/(f)) \subseteq \mathbb{P}^n$. For the rest of this paper, we fix a closed subscheme $Z \subseteq \mathbb{P}^n$. For $d \in \mathbb{Z}_{\geq 0}$, let $I_d$ be the $\mathbb{F}_q$-subspace of $f \in S_d$ that vanish on $Z$. Let $I_{\text{homog}} = \bigcup_{d \geq 0} I_d$. We want to measure the density of subsets of $I_{\text{homog}}$, but under the definition in [Poo04], the set $I_{\text{homog}}$ itself has density 0 whenever $\dim Z > 0$; therefore we use a new definition of density, relative to $I_{\text{homog}}$. Namely, we define the density of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ by

$$\mu_Z(\mathcal{P}) := \lim_{d \to \infty} \frac{\#(\mathcal{P} \cap I_d)}{\#I_d};$$

if the limit exists. For a scheme $X$ of finite type over $\mathbb{F}_q$, define the zeta function [Wei49]

$$\zeta_X(s) = Z_X(q^{-s}) := \prod_{\text{closed } P \in X} (1 - q^{-s \deg P})^{-1} = \exp \left( \sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{q^r})}{r} q^{-rs} \right);$$

the product and sum converge when $\text{Re}(s) > \dim X$.

Date: June 29, 2007.

1991 Mathematics Subject Classification. Primary 14J70; Secondary 11M38, 11M41, 14G40, 14N05.

This article has appeared in Math. Research Letters 15 (2008), no. 2, 265–271. This research was supported by NSF grant DMS-0301280.
Theorem 1.1. Let $X$ be a smooth quasi-projective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. Let $Z$ be a closed subscheme of $\mathbb{P}^n$. Assume that the scheme-theoretic intersection $V := Z \cap X$ is smooth of dimension $\ell$. (If $V$ is empty, take $\ell = -1$.) Define

$$\mathcal{P} := \{ f \in I_{\text{homog}} : H_f \cap X \text{ is smooth of dimension } m - 1 \}.$$ 

(i) If $m > 2\ell$, then

$$\mu_Z(\mathcal{P}) = \frac{\zeta_V(m+1)}{\zeta_V(m-\ell) \zeta_X(m+1)} = \frac{1}{\zeta_V(m-\ell) \zeta_{X-V}(m+1)}.$$ 

In this case, in particular, for $d \gg 1$, there exists a degree-$d$ hypersurface $H$ containing $Z$ such that $H \cap X$ is smooth of dimension $m - 1$.

(ii) If $m \leq 2\ell$, then $\mu_Z(\mathcal{P}) = 0$.

The proof will use the closed point sieve introduced in [Poo04]. In fact, the proof is parallel to the one in that paper, but changes are required in almost every line.

2. Singular points of low degree

Let $\mathcal{I}_Z \subseteq \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of $Z$, so $I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d))$. Tensoring the surjection

$$\mathcal{O}^{\oplus (n+1)} \rightarrow \mathcal{O}$$

$$(f_0, \ldots, f_n) \mapsto x_0 f_0 + \cdots + x_n f_n$$

with $\mathcal{I}_Z$, twisting by $\mathcal{O}(d)$, and taking global sections shows that $S_1 I_d = I_{d+1}$ for $d \geq 1$. Fix $c$ such that $S_1 I_d = I_{d+1}$ for all $d \geq c$.

Before proving the main result of this section (Lemma 2.3), we need two lemmas.

Lemma 2.1. Let $Y$ be a finite subscheme of $\mathbb{P}^n$. Let

$$\phi_d : I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \rightarrow H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$$

be the map induced by the map of sheaves $\mathcal{I}_Z \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_Y$ on $\mathbb{P}^n$. Then $\phi_d$ is surjective for $d \geq c + \dim H^0(Y, \mathcal{O}_Y)$.

Proof. The map of sheaves $\mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_Y$ on $\mathbb{P}^n$ is surjective so $\mathcal{I}_Z \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_Y$ is surjective too. Thus $\phi_d$ is surjective for $d \gg 1$.

Enlarging $\mathbb{F}_q$ if necessary, we can perform a linear change of variable to assume $Y \subseteq \mathbb{A}^n := \{x_0 \neq 0\}$. Dehomogenization (setting $x_0 = 1$) identifies $S_d$ with the space $S_d' \subseteq \mathbb{F}_q[x_1, \ldots, x_n]$ of total degree $\leq d$. and identifies $\phi_d$ with a map

$$I_d' \rightarrow B := H^0(\mathbb{P}^n, \mathcal{I}_Z \cdot \mathcal{O}_Y).$$

By definition of $c$, we have $S_1' I_d' = I_{d+1}'$ for $d \geq c$. For $d \geq b$, let $B_d$ be the image of $I_d'$ in $B$, so $S_1' B_d = B_{d+1}$ for $d \geq c$. Since $1 \in S_1'$, we have $I_d' \subseteq I_{d+1}'$, so

$$B_c \subseteq B_{c+1} \subseteq \cdots.$$ 

But $b := \dim B < \infty$, so $B_j = B_{j+1}$ for some $j \in [c, c+b]$. Then

$$B_{j+2} = S_1' B_{j+1} = S_1' B_j = B_{j+1}.$$ 

Similarly $B_j = B_{j+1} = B_{j+2} = \ldots$, and these eventually equal $B$ by the previous paragraph. Hence $\phi_d$ is surjective for $d \geq j$, and in particular for $d \geq c + b$. \[\Box\]
Lemma 2.2. Suppose \( m \subseteq \mathcal{O}_X \) is the ideal sheaf of a closed point \( P \in X \). Let \( Y \subseteq X \) be the closed subscheme whose ideal sheaf is \( m^2 \subseteq \mathcal{O}_X \). Then for any \( d \in \mathbb{Z}_{\geq 0} \),

\[
\# H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) = \begin{cases} q^{(m-\ell)\deg P}, & \text{if } P \in V, \\ q^{(m+1)\deg P}, & \text{if } P \notin V. \end{cases}
\]

Proof. Since \( Y \) is finite, we may now ignore the twisting by \( \mathcal{O}(d) \). The space \( H^0(Y, \mathcal{O}_Y) \) has a two-step filtration whose quotients have dimensions 1 and \( m \) over the residue field \( \kappa \) of \( P \). Thus \( \# H^0(Y, \mathcal{O}_Y) = (\# \kappa)^{m+1} = q^{(m+1)\deg P} \). If \( P \in V \) (or equivalently \( P \in Z \)), then \( H^0(Y, \mathcal{O}_{Z \cap Y}) \) has a filtration whose quotients have dimensions 1 and \( \ell \) over \( \kappa \); if \( P \notin V \), then \( H^0(Y, \mathcal{O}_{Z \cap Y}) = 0 \). Taking cohomology of

\[
0 \to \mathcal{I}_Z \cdot \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_{Z \cap Y} \to 0
\]
on the 0-dimensional scheme \( Y \) yields

\[
\# H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) = \frac{\# H^0(Y, \mathcal{O}_Y)}{\# H^0(Y, \mathcal{O}_{Z \cap Y})} = \begin{cases} q^{(m+1)\deg P}/q^{(\ell+1)\deg P}, & \text{if } P \in V, \\ q^{(m+1)\deg P}, & \text{if } P \notin V. \end{cases}
\]

□

If \( U \) is a scheme of finite type over \( \mathbb{F}_q \), let \( U_{<r} \) be the set of closed points of \( U \) of degree \( < r \). Similarly define \( U_{>r} \).

Lemma 2.3 (Singulaties of low degree). Let notation and hypotheses be as in Theorem 1.1, and define

\[
\mathcal{P}_r := \{ f \in I_{\text{homog}} : H_f \cap X \text{ is smooth of dimension } m-1 \text{ at all } P \in X_{<r} \}.
\]

Then

\[
\mu_Z(\mathcal{P}_r) = \prod_{P \in V_{<r}} (1 - q^{-(m-\ell)\deg P}) \cdot \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}).
\]

Proof. Let \( X_{<r} = \{ P_1, \ldots, P_s \} \). Let \( m_i \) be the ideal sheaf of \( P_i \) on \( X \). let \( Y_i \) be the closed subscheme of \( X \) with ideal sheaf \( m_i^2 \subseteq \mathcal{O}_X \), and let \( Y = \bigcup Y_i \). Then \( H_f \cap X \) is singular at \( P_i \) (more precisely, not smooth of dimension \( m-1 \) at \( P_i \)) if and only if the restriction of \( f \) to a section of \( \mathcal{O}_{Y_i}(d) \) is zero.

By Lemma 2.2, \( \mu_Z(\mathcal{P}) \) equals the fraction of elements in \( H^0(\mathcal{I}_Z \cdot \mathcal{O}_Y(d)) \) whose restriction to a section of \( \mathcal{O}_{Y_i}(d) \) is nonzero for every \( i \). Thus

\[
\mu_Z(\mathcal{P}_r) = \prod_{i=1}^s \frac{\# H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) - 1}{\# H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})} = \prod_{P \in V_{<r}} (1 - q^{-(m-\ell)\deg P}) \cdot \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}),
\]

by Lemma 2.2. □

3
Corollary 2.4. If $m > 2\ell$, then
\[
\lim_{r \to \infty} \mu_Z(\mathcal{P}_r) = \frac{\zeta_V(m+1)}{\zeta_X(m+1) \zeta_V(m-\ell)}.
\]

Proof. The products in Lemma 2.3 are the partial products in the definition of the zeta functions. For convergence, we need $m - \ell > \dim V = \ell$, which is equivalent to $m > 2\ell$. \qed

Proof of Theorem 1.1(ii). We have $\mathcal{P} \subseteq \mathcal{P}_r$. By Lemma 2.3,
\[
\mu_Z(\mathcal{P}_r) \leq \prod_{P \in \mathcal{V}_r} (1 - q^{-(m-\ell) \deg P}),
\]
which tends to 0 as $r \to \infty$ if $m \leq 2\ell$. Thus $\mu_Z(\mathcal{P}) = 0$ in this case. \qed

From now on, we assume $m > 2\ell$.

3. Singular points of medium degree

Lemma 3.1. Let $P \in X$ is a closed point of degree $e$, where $e \leq \frac{d-b}{m+1}$. Then the fraction of $f \in I_d$ such that $H_f \cap X$ is not smooth of dimension $m - 1$ at $P$ equals
\[
\begin{cases}
q^{-(m-\ell)e}, & \text{if } P \in \mathcal{V}, \\
q^{-(m+1)e}, & \text{if } P \notin \mathcal{V}.
\end{cases}
\]

Proof. This follows by applying Lemma 2.1 to the $Y$ in Lemma 2.2 and then applying Lemma 2.2. \qed

Define the upper and lower densities $\mu_Z(\mathcal{P}), \mu^Z(\mathcal{P})$ of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ as $\mu_Z(\mathcal{P})$ was defined, but using $\limsup$ and $\liminf$ in place of $\lim$.

Lemma 3.2 (Singularities of medium degree). Define
\[
Q_r^{\text{medium}} := \bigcup_{d \geq 0} \{ f \in I_d : \exists P \in X \text{ with } r \leq \deg P \leq \frac{d-b}{m+1} \text{ such that } H_f \cap X \text{ is not smooth of dimension } m - 1 \text{ at } P \}.
\]

Then $\lim_{r \to \infty} \mu_Z(Q_r^{\text{medium}}) = 0$.

Proof. By Lemma 3.1 we have
\[
\frac{\#(Q_r^{\text{medium}} \cap I_d)}{\#I_d} \leq \sum_{\substack{P \in \mathcal{Z} \cap \mathcal{V} \cap I_d \ni P : \deg P \leq \frac{d-b}{m+1}}} q^{-(m-\ell) \deg P} + \sum_{\substack{P \in \mathcal{Z} \cap \mathcal{V} \cap I_d \ni P : \deg P \leq \frac{d-b}{m+1}}} q^{-(m+1) \deg P}
\]
\[
\leq \sum_{\substack{P \in \mathcal{Z} \cap \mathcal{V} \cap I_d \ni P : \deg P \leq \frac{d-b}{m+1}}} q^{-(m-\ell) \deg P} + \sum_{\substack{P \in (\mathcal{X} \cap \mathcal{Z}) \cap I_d \ni P : \deg P \leq \frac{d-b}{m+1}}} q^{-(m+1) \deg P}.
\]

Using the trivial bound that an $m$-dimensional variety has at most $O(q^m)$ closed points of degree $e$, as in the proof of \[\text{[Poo04, Lemma 2.4]}\], we show that each of the two sums converges to a value that is $O(q^{-r})$ as $r \to \infty$, under our assumption $m > 2\ell$. \qed
4. Singular points of high degree

Lemma 4.1. Let $P$ be a closed point of degree $e$ in $\mathbb{P}^n - Z$. For $d \geq c$, the fraction of $f \in I_d$ that vanish at $P$ is at most $q^{-\min(d-c,e)}$.

Proof. Equivalently, we must show that the image of $\phi_d$ in Lemma 2.1 for $Y = P$ has $\mathbb{F}_q$-dimension at least $\min(d - c, e)$. The proof of Lemma 2.1 shows that as $d$ runs through the integers $c, c + 1, \ldots$, this dimension increases by at least 1 until it reaches its maximum, which is $e$. □

Lemma 4.2 (Singularities of high degree off $V$). Define

$$Q^\text{high}_{X - V} := \bigcup_{d \geq 0} \{ f \in I_d : \exists P \in (X - V)_{d-c\over m+1} \text{ such that } H_f \cap X \text{ is not smooth of dimension } m-1 \text{ at } P \}$$

Then $\overline{\mu}_Z(Q^\text{high}_{X - V}) = 0$.

Proof. It suffices to prove the lemma with $X$ replaced by each of the sets in an open covering of $X - V$, so we may assume $X$ is contained in $\mathbb{A}^n = \{ x_0 \neq 0 \} \subseteq \mathbb{P}^n$, and that $V = \emptyset$. Dehomogenize by setting $x_0 = 1$, to identify $I_d \subseteq S_d$ with subspaces of $I'_d \subseteq S'_d \subseteq A := \mathbb{F}_q[x_1, \ldots, x_n]$.

Given a closed point $x \in X$, choose a system of local parameters $t_1, \ldots, t_n \in A$ at $x$ on $\mathbb{A}^n$ such that $t_{m+1} = t_{m+2} = \cdots = t_n = 0$ defines $X$ locally at $x$. Multiplying all the $t_i$ by an element of $A$ vanishing on $Z$ but nonvanishing at $x$, we may assume in addition that all the $t_i$ vanish on $Z$. Now $dt_1, \ldots, dt_n$ are a $\mathcal{O}_{\mathbb{A}^n,x}$-basis for the stalk $\mathcal{O}^1_{\mathbb{A}^n, x}$ of the tangent sheaf. Let $\partial_1, \ldots, \partial_n$ be the dual basis of the stalk $\mathcal{T}_{\mathbb{A}^n, \mathbb{F}_q,x}$ of the tangent sheaf. Choose $s \in A$ with $s(x) \neq 0$ to clear denominators so that $D_i := s\partial_i$ gives a global derivation $A \to A$ for $i = 1, \ldots, n$. Then there is a neighborhood $N_x$ of $x$ in $\mathbb{A}^n$ such that $N_x \cap \{ t_{m+1} = t_{m+2} = \cdots = t_n = 0 \} = N_x \cap X$, $\mathcal{O}^1_{N_x, \mathbb{F}_q} = \oplus_{i=1}^n \mathcal{O}_{N_x,dt_i}$, and $s \in \mathcal{O}(N_x)^*$. We may cover $X$ with finitely many $N_x$, so we may reduce to the case where $X \subseteq N_x$ for a single $x$. For $f \in I'_d \simeq I_d$, $H_f \cap X$ fails to be smooth of dimension $m-1$ at a point $P \in U$ if and only if $f(P) = (D_1 f)(P) = \cdots = (D_m f)(P) = 0$.

Let $\tau = \max(x(\deg t_i), \gamma) = [(d - \tau)/p]$ and $\eta = [d/p]$. If $f_0 \in I'_d$, $g_1 \in S'_1$, $\ldots$, $g_m \in S'_m$, and $h \in I'_n$ are selected uniformly and independently at random, then the distribution of

$$f := f_0 + g_1^\tau t_1 + \cdots + g_m^\eta t_m + h^p$$

is uniform over $I'_d$, because of $f_0$. We will bound the probability that an $f$ constructed in this way has a point $P \in X_{d-c\over m+1}$ where $f(P) = (D_1 f)(P) = \cdots = (D_m f)(P) = 0$. We have $D_i f = (D_i f_0) + g_i^\tau s$ for $i = 1, \ldots, m$. We will select $f_0, g_1, \ldots, g_m, h$ one at a time. For $0 \leq i \leq m$, define

$$W_i := X \cap \{ D_1 f = \cdots = D_i f = 0 \}.$$

Claim 1: For $0 \leq i \leq m-1$, conditioned on a choice of $f_0, g_1, \ldots, g_i$ for which $\dim(W_i) \leq m - i$, the probability that $\dim(W_{i+1}) \leq m - i - 1$ is $1 - o(1)$ as $d \to \infty$. (The function of $d$ represented by the $o(1)$ depends on $X$ and the $D_i$.)

Proof of Claim 1: This is completely analogous to the corresponding proof in [Poo04].

Claim 2: Conditioned on a choice of $f_0, g_1, \ldots, g_m$ for which $W_m$ is finite, $\Prob(H_f \cap W_m \cap X_{d-c\over m+1} = \emptyset) = 1 - o(1)$ as $d \to \infty$.  

5
Proof of Claim 2: By Bézout’s theorem as in [Ful84, p. 10], we have $\#W_m = O(d^m)$. For a given point $P \in W_m$, the set $H_{\text{bad}}$ of $h \in I'_{\eta}$ for which $H_f$ passes through $P$ is either $\emptyset$ or a coset of $\ker(ev_P : I'_{\eta} \to \kappa(P))$, where $\kappa(P)$ is the residue field of $P$, and $ev_P$ is the evaluation-at-$P$ map. If moreover $\deg P > \frac{d-c}{m+1}$, then Lemma 4.1 implies $\#H_{\text{bad}}/\#I'_{\eta} \leq q^{-\nu}$ where $\nu = \min(\eta, \frac{d-c}{m+1})$. Hence

$$\Prob(H_f \cap W_m \cap X_{\geq \frac{d-c}{m+1}} \neq \emptyset) \leq \#W_m q^{-\nu} = O(d^m q^{-\nu}) = o(1)$$
as $d \to \infty$, since $\nu$ eventually grows linearly in $d$. This proves Claim 2.

End of proof: Choose $f \in I_d$ uniformly at random. Claims 1 and 2 show that with probability $\prod_{i=0}^{m-1}(1-o(1)) \cdot (1-o(1)) = 1-o(1)$ as $d \to \infty$, $\dim W_i = m-i$ for $i = 0, 1, \ldots, m$ and $H_f \cap W_m \cap X_{\geq \frac{d-c}{m+1}} = \emptyset$. But $H_f \cap W_m$ is the subvariety of $X$ cut out by the equations $f(P) = (D_1f)(P) = \cdots = (D_mf)(P) = 0$, so $H_f \cap W_m \cap X_{\geq \frac{d-c}{m+1}}$ is exactly the set of points of $H_f \cap X$ of degree $\frac{d-c}{m+1}$ where $H_f \cap X$ is not smooth of dimension $m-1$. Thus $\mathbb{P}_Z(Q^\text{high}_X) = 0$.

**Lemma 4.3 (Singularities of high degree on $V$).** Define

$$Q^\text{high}_V := \bigcup_{d \geq 0} \{ f \in I_d : \exists P \in V_{\geq \frac{d-c}{m+1}} \text{ such that } H_f \cap X \text{ is not smooth of dimension } m-1 \text{ at } P \}.$$

Then $\mathbb{P}_Z(Q^\text{high}_V) = 0$.

**Proof.** As before, we may assume $X \subseteq \mathbb{A}^n$ and we may dehomogenize. Given a closed point $x \in X$, choose a system of local parameters $t_1, \ldots, t_n \in A$ at $x$ on $A^n$ such that $t_{m+1} = t_{m+2} = \cdots = t_n = 0$ defines $X$ locally at $x$, and $t_1 = t_2 = \cdots = t_{m-\ell} = t_{m+1} = t_{m+2} = \cdots = t_n = 0$ defines $V$ locally at $x$. If $m_w$ is the ideal sheaf of $w$ on $\mathbb{P}^n$, then $I_Z \to m_w$ is surjective, so we may adjust $t_1, \ldots, t_{m-\ell}$ to assume that they vanish not only on $V$ but also on $Z$.

Define $\partial_i$ and $D_i$ as in the proof of Lemma 4.2. Then there is a neighborhood $N_x$ of $x$ in $\mathbb{A}^n$ such that $N_x \cap \{t_{m+1} = t_{m+2} = \cdots = t_n = 0\} = N_x \cap X$, $\Omega^1_{N_x/F_q} = \otimes_{i=1}^n O_{N_x} dt_i$, and $s \in O(N_x)^*$. Again we may assume $X \subseteq N_x$ for a single $x$. For $f \in I'_d \simeq I_d$, $H_f \cap X$ fails to be smooth of dimension $m-1$ at a point $P \in V$ if and only if $f(P) = (D_1f)(P) = \cdots = (D_mf)(P) = 0$.

Again let $\tau = \max_i (\deg t_i)$, $\gamma = \lfloor (d-\tau)/p \rfloor$, and $\eta = \lfloor d/p \rfloor$. If $f_0 \in I'_d$, $g_1 \in S'_\gamma$, $\ldots$, $g_{\ell+1} \in S'_\gamma$, are chosen uniformly at random, then

$$f := f_0 + g_1 t_1 + \cdots + g_{\ell+1} t_{\ell+1}$$
is a random element of $I'_d$, since $\ell + 1 \leq m - \ell$.

For $i = 0, \ldots, \ell + 1$, the subscheme

$$W_i := V \cap \{ D_1f = \cdots = D_{i}f = 0 \}$$
depends only on the choices of $f_0, g_1, \ldots, g_i$. The same argument as in the previous proof shows that for $i = 0, \ldots, \ell$, we have

$$\Prob(\dim W_i \leq \ell - i) = 1 - o(1)$$
as $d \to \infty$. In particular, $W_\ell$ is finite with probability $1 - o(1)$.
To prove that $\mu_{Z}(Q_{\text{high}}^\ell) = 0$, it remains to prove that conditioned on choices of $f_0, g_1, \ldots, g_\ell$ making $\dim W_\ell$ finite,

$$\text{Prob}(W_{\ell+1} \cap V_{d/d+c_m+1} > 0) = 1 - o(1).$$

By Bézout’s theorem, $\# W_\ell = O(d^n)$. The set $H_{\text{bad}}$ of choices of $g_{\ell+1}$ making $D_{\ell+1}f$ vanish at a given point $P \in W_\ell$ is either empty or a coset of ker$(ev_P : S'_\gamma \rightarrow \kappa(P))$. Lemma 2.5 of [Poo04] implies that the size of this kernel (or its coset) as a fraction of $\# S'_\gamma$ is at most $q^{-\nu}$ where $\nu := \min(\gamma, d - c_m + 1)$. Since $\# W_\ell q^{\nu} = o(1)$ as $d \to \infty$, we are done. □

5. Conclusion

Proof of Theorem 1.1(i). We have

$$P \subseteq P_r \subseteq P \cup Q_{\text{medium}} \cup Q_{X-V} \cup Q_{V},$$

so $\mu_{Z}(P)$ and $\mu_{Z}(P_r)$ each differ from $\mu_{Z}(Q_{r})$ by at most $\mu_{Z}(Q_{\text{medium}}) + \mu_{Z}(Q_{X-V}^\ell) + \mu_{Z}(Q_{V})$. Applying Corollary 2.4 and Lemmas 3.2, 4.2, and 4.3, we obtain

$$\mu_{Z}(P) = \lim_{r \to \infty} \mu_{Z}(P_r) = \frac{\zeta_V(m + 1)}{\zeta_V(m - \ell) \zeta_X(m + 1)}.$$

□

Acknowledgements

I thank Shuji Saito for asking the question answered by this paper, and for pointing out [KA79].

References

[Bl07] Spencer Bloch, 1970. Ph.D. thesis, Columbia University. [1]

[Ful84] William Fulton, Introduction to intersection theory in algebraic geometry, CBMS Regional Conference Series in Mathematics, vol. 54, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1984.MR735435 (85j:14008) [1]

[Gab01] O. Gabber, On space filling curves and Albanese varieties, Geom. Funct. Anal. 11 (2001), no. 6, 1192–1200.MR1878318 (2003g:14034) [1]

[KA79] Steven L. Kleiman and Allen B. Altman, Bertini theorems for hypersurface sections containing a subscheme, Comm. Algebra 7 (1979), no. 8, 775–790.MR529493 (81i:14007) [1]

[Poo04] Bjorn Poonen, Bertini theorems over finite fields, Ann. of Math. (2) 160 (2004), no. 3, 1099–1127. MR2144974 (2006a:14035) [1]

[SS07] Shuji Saito and Kanetomo Sato, Finiteness theorem on zero-cycles over p-adic fields (April 11, 2007). arXiv:math.AG/0605165. [1]

[Wei49] André Weil, Numbers of solutions of equations in finite fields, Bull. Amer. Math. Soc. 55 (1949), 497–508.MR0029393 (10,592e) [1]