Energy functionals and Kähler-Ricci solitons

Haozhao Li

June 30, 2009

Abstract In this paper, we generalize Chen-Tian energy functionals to Kähler-Ricci solitons and prove that the properness of these functionals is equivalent to the existence of Kähler-Ricci solitons. We also discuss the equivalence of the lower boundedness of these functionals and their relation with Tian-Zhu’s holomorphic invariant.

1 Introduction

In [6], a series of energy functionals \( E_k \) \((k = 0, 1, \cdots, n)\) were introduced by X.X. Chen and G. Tian which were used to prove the convergence of the Kähler Ricci flow under some curvature assumptions. The first energy functional \( E_0 \) of this series is exactly the \( K \)-energy introduced by Mabuchi in [12], which can be defined for any Kähler potential \( \varphi(t) \) on a Kähler manifold \((M, \omega)\) as follows:

\[
\frac{d}{dt} E_0(\varphi(t)) = -\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (R_\varphi - r) \omega^n_\varphi.
\]

Here \( R_\varphi \) is the scalar curvature with respect to the Kähler metric \( \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \), \( r = \frac{c_1(M) \| \omega \|^{n-1}}{\| \omega \|_n} \) is the average of \( R_\varphi \) and \( V = \| \omega \|_n \) is the volume.

It is well-known that the behavior of the \( K \)-energy plays a central role on the existence of Kähler-Einstein metrics and constant scalar curvature metrics. In [1], Bando-Mabuchi proved that the \( K \)-energy is bounded from below on a Kähler-Einstein manifold with \( c_1(M) > 0 \). It has been shown by G. Tian in [17][18] that \( M \) admits a Kähler-Einstein metric if and only if the \( K \)-energy or \( F \) functional defined by Ding-Tian [8] is proper. Thus, it is natural to study the relation between \( E_k \) functionals and Kähler-Einstein metrics. Following a question posed by Chen in [3], Song-Weinkove studied the lower bound of energy functionals \( E_k \) on Kähler-Einstein manifolds. Shortly afterwards, N. Pali [13] gave a formula between \( E_1 \) and the \( K \)-energy \( E_0 \), which implies \( E_1 \) has a lower bound if the \( K \)-energy is bounded from below. Inspired by Song-Weinkove and Pali’s work, we proved that the lower boundedness of \( F \) functional, the \( K \)-energy and \( E_1 \) are equivalent in the canonical Kähler class in [5][10], and we proved a general formula which gives the relations of all energy functionals \( E_k \) in [2]. In [14] Y. Rubinstein extended these results and proved all the lower boundedness and properness of \( E_k \) functionals are equivalent under some natural restrictions.
For the case of Kähler-Ricci solitons, Tian-Zhu generalized the $K$-energy and $F$ functional in [19] and proved that these generalized energy are bounded from below on a Kähler manifold which admits a Kähler-Ricci soliton. In [2] Cao-Tian-Zhu proved the properness of the generalized energy functionals. Inspired by these work, we will define the generalized Chen-Tian energy functionals $\tilde{E}_k$ in Section 2 and prove the following result:

**Theorem 1.1.** Let $(M, \omega)$ be a compact Kähler manifold with $c_1(M) > 0$ and $\omega \in 2\pi c_1(M)$. For any $k \in \{0, 1, \cdots, n\}$ we have

(a) If $\tilde{E}_k$ is proper on $\mathcal{M}_X^+(\omega)$, then $M$ admits a Kähler-Ricci soliton with respect to $X$;

(b) If $M$ admits a Kähler-Ricci soliton $\omega_{KS}$, then $\tilde{E}_k$ is proper on $\mathcal{M}_{G}(\omega_{KS})' \cap \mathcal{M}_{X,k}^+(\omega_{KS})$.

where $\mathcal{M}_{X,k}^+(\omega_{KS})$ and $\mathcal{M}_{G}(\omega_{KS})'$ are some subspaces of Kähler potentials defined in Section 2.

The idea of the proof is more or less standard. We follow the continuity method from [14] and [2] to prove this. The crucial point is that by the construction of the generalized energy functionals $\tilde{E}_k$, all the arguments for the Kähler-Einstein case work very well for our situation. Following the results in [5] and [10] we discuss the lower bound of these energy functionals:

**Theorem 1.2.** Let $(M, \omega)$ be a compact Kähler manifold with $c_1(M) > 0$ and $\omega \in 2\pi c_1(M)$. Then for any $k \in \{0, 1, \cdots, n\}$, $\tilde{E}_k$ is bounded from below on $\mathcal{M}_X^+(\omega)$ if and only if $\tilde{F}$ is bounded from below on $\mathcal{M}_X^+(\omega)$. Moreover, we have

$$\inf_{\omega' \in \mathcal{M}_X^+(\omega)} \tilde{E}_{k, \omega}(\omega') = (k + 1) \inf_{\omega' \in \mathcal{M}_X^+(\omega)} \tilde{F}_{\omega}(\omega') + C_{\omega, X, k} - \frac{k + 1}{V} \int_M u_0 e^{\theta_X} \omega^n, \quad (1.1)$$

where $u_0 = -h_{\omega} + \theta_X$ and $C_{\omega, X, k}$ is given by

$$C_{\omega, X, k} = \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k + 1}{i} \frac{1}{V} \int_M \sqrt{-1}\partial u_0 \wedge \bar{\partial} u_0 \wedge (\sqrt{-1}\bar{\partial}\bar{\partial} u_0)^{k-1} \wedge e^{\theta_X} \omega_n^{n-k}. \quad (1.2)$$

Here we take the ideas from [8] to prove Theorem 1.2. For the energy functionals $E_k$, there are two different ways to prove their equivalence. In [5] we use the Kähler-Ricci flow to prove the equivalence of the lower boundedness of the $K$ energy and $E_1$ energy, and in [10] we use Perelman’s estimates to prove the equivalence of the $K$ energy and $F$ functional. The flow method is very tricky and we lack some crucial estimates here. In [14] Y. Rubinstein proved that the equivalence of the lower boundedness of the energy functionals $\tilde{E}_k$ and $\tilde{F}$, which relies on an interesting observation on the relation of $E_n$ and $F$ (cf. Lemma 2.4 in [14]). Here it seems difficult to find such a relation in the case of generalized energy functionals. Fortunately, we can use the continuity method in [8] to overcome these difficulties.

As a by-product of Theorem 1.2, we have the following result:
Theorem 1.3. Let $M$ be a compact Kähler manifold with $c_1(M) > 0$ and $\omega$ be any given Kähler metric in $2\pi c_1(M)$. If $\tilde{F}$ is bounded from below for the solution $\varphi_t$ of the equation

$$ (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\omega - \theta_X(\varphi) - t\varphi}, $$

(1.3)

then $\tilde{F}$ is bounded from below in the class $2\pi c_1(M)$. Moreover, we have

$$ \inf_{\omega' \in 2\pi c_1(M)} \tilde{F}_\omega(\omega') = \inf_{t \in [0,1)} \tilde{F}_\omega(\varphi_t). $$

Similar results also hold for $\tilde{E}_k$ on $\mathcal{M}_{X,k}^+(\omega)$ with $k = 0, 1, \ldots, n$.

This result is inspired by the beautiful work [4]. In [4] X. X. Chen proved that if the $K$-energy is bounded from below and the infimum of the Calabi energy vanishes along a particular geodesic ray, then the $K$-energy is bounded from below in the Kähler class. As an application, he essentially proved that for any Kähler class admits constant scalar curvature metric, the $K$-energy in a nearby Kähler class with possibly different complex structure is bounded from below. We remark that under the assumption of Theorem 1.3, the solution $\varphi_t$ of (1.3) will exist for all $t \in [0,1)$. Theorem 1.3 shows that if the $K$-energy is bounded along one solution $\varphi_t$, then the $K$-energy is bounded from below in the whole Kähler class. It is interesting to know whether there is a similar phenomenon for the Kähler-Ricci flow:

Question 1.4. If the $K$-energy is bounded from below along a certain Kähler-Ricci flow, is the $K$-energy bounded from below in the class $2\pi c_1(M)$?

In [19], Tian-Zhu introduced a new holomorphic invariant $\mathcal{F}_X(\cdot)$ from the space of holomorphic vector fields $\eta(M)$ into $\mathbb{C}$:

$$ \mathcal{F}_X(Y) = \int_M Y(h_g - \theta_X(g))e^{\theta_X(g)}\omega_g^n, \quad Y \in \eta(M). $$

The invariant $\mathcal{F}_X(\cdot)$ is defined for any holomorphic vector fields $X \in \eta(M)$ and it is independent of the choice of $g$ with the Kähler class $\omega_g \in 2\pi c_1(M)$. When $X = 0$, $\mathcal{F}_X(\cdot)$ is exactly the Futaki invariant. By the definition, we see that $\mathcal{F}_X$ is an obstruction to the existence of Kähler-Ricci solitons. The next result shows that the holomorphic invariants defined by $\tilde{E}_k$ are scalar multiples of $\mathcal{F}_X(\cdot)$, which generalized the results for energy functionals $E_k$ (cf. [11] [9]).

Theorem 1.5. Let $Y$ be a holomorphic vector field and $\{\Phi(t)\}_{|t|<\infty}$ the one-parameter subgroup of automorphisms induced by $\text{Re}(Y)$, we have

$$ \frac{d}{dt} \tilde{E}_k(\varphi) = \frac{(k+1)n}{V} \mathcal{F}_X(Y), $$

where $\varphi$ is given by $\Phi^*_t \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$.

Acknowledgements: The author would like to thank Professor X. X. Chen, W. Y. Ding and F. Pacard for their constant, warm encouragements over the past several years.
2 Energy functionals

In this section, we recall some energy functionals introduced by Tian-Zhu in [19] and give the definition of the generalized Chen-Tian energy function.

Let $M$ be an $n$-dimensional compact Kähler manifold with positive first Chern class, and $\omega$ be a fixed Kähler metric in the Kähler class $2\pi c_1(M)$. Then there is a smooth real-valued function $h_\omega$ such that

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega, \quad \int_M (e^{h_\omega} - 1) \omega^n = 0.$$ 

Suppose that $X$ is a holomorphic vector field on $M$ so that the integral curve of $K_X$ of the imaginary part $\text{Im}(X)$ of $X$ consists of isometries of $\omega$. By the Hodge decomposition theorem, there exists a unique smooth real-valued function $\theta_X$ on $M$ such that

$$i_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X, \quad \int_M (e^{\theta_X} - 1) \omega^n = 0.$$ 

Now we define the space of Kähler potentials which are invariant under $\text{Im}(X)$:

$$\mathcal{M}_X(\omega) = \{ \varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \quad \text{Im}(X)(\varphi) = 0 \}.$$ 

Tian-Zhu in [19] introduced the following functional, which can be regarded as a generalization of Mabuchi’s $K$ energy,

$$\tilde{E}_{0,\omega}(\varphi) = \frac{n}{V} \int_0^1 \int_M \sqrt{-1} \partial \frac{\partial \varphi_t}{\partial t} \wedge \bar{\partial} (h_{\varphi_t} - \theta_X(\varphi_t)) \wedge e^{\theta_X(\varphi_t)} \omega_{\varphi_t}^{n-1} \wedge dt$$

where $\varphi_t(t \in [0, 1])$ is a path connecting $0$ and $\varphi$ in $\mathcal{M}_X(\omega)$ and

$$\theta_X(\varphi_t) = \theta_X + X(\varphi_t)$$

is the potential function of $X$ with respect to the metric $\omega_{\varphi_t}$. We define the following functionals on $\mathcal{M}_X(\omega)$:

$$\tilde{I}_\omega(\varphi) = \frac{1}{V} \int_M \varphi (e^{\theta_X} \omega^n - e^{\theta_X(\varphi)} \omega_{\varphi}^n),$$

and

$$\tilde{J}_\omega(\varphi) = \frac{1}{V} \int_0^1 \int_M \frac{\partial \varphi_t}{\partial t} (e^{\theta_X} \omega^n - e^{\theta_X(\varphi_t)} \omega_{\varphi_t}^n) \wedge dt.$$ 

As before, $\varphi_t(t \in [0, 1])$ is a path connecting $0$ and $\varphi$ in $\mathcal{M}_X(\omega)$. Then for any path $\varphi_t$, we have

$$\frac{d}{dt} (\tilde{I}_\omega(\varphi_t) - \tilde{J}_\omega(\varphi_t)) = -\frac{1}{V} \int_M \frac{\partial \varphi_t}{\partial t} (\Delta \varphi_t + X) \varphi_t e^{\theta_X(\varphi_t)} \omega_{\varphi_t}^n.$$ 

Then, by Lemma 3.1 in [2] there exist two positive constants $c_1(n)$ and $c_2(n)$ such that

$$c_1 I_\omega(\varphi) \leq \tilde{I}_\omega(\varphi) - \tilde{J}_\omega(\varphi) \leq c_2 I_\omega(\varphi),$$

(2.2)
where
\[ I_\omega(\varphi) = \frac{1}{V} \int_M \varphi(\omega^n - \omega^n_\varphi). \] (2.3)

In [19] Tian-Zhu defined the generalized \( F \) functional which is defined by Ding-Tian in [8] as follows
\[ \tilde{F}_\omega(\varphi) = \tilde{J}_\omega(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta_X} \omega^n - \log \left( \frac{1}{V} \int_M e^{h_{\omega - \varphi}} \omega^n \right). \] (2.4)

The generalized \( F \) functional has exactly the same behavior as in the Kähler-Einstein case. For example, the generalized \( K \)-energy and \( \tilde{F} \) functional are related by the identity (cf. [19])
\[ \tilde{E}_0(\varphi) = \tilde{F}(\varphi) - \frac{1}{V} \int_M u e^{\theta_X} \omega^n + \log \left( \frac{1}{V} \int_M e^{h_{\omega - \varphi}} \omega^n \right), \] (2.5)

where \( u \) is defined by
\[ u(\varphi) = -h_{\varphi} + \theta_X(\varphi) = \log \frac{\omega^n_\varphi}{\omega^n} + \varphi - h_{\omega} + \theta_X(\varphi), \] (2.6)

and \( u_0 = -h_{\omega} + \theta_X \). It follows that \( \tilde{E}_0 \) is always bigger than \( \tilde{F} \) up to a constant:
\[ \tilde{E}_0(\varphi) \geq \tilde{F}(\varphi) - \frac{1}{V} \int_M u_0 e^{\theta_X} \omega^n. \] (2.7)

Now we recall some results in [2]. Let \( K_0(\supseteq K_X) \) be a maximum compact subgroup of the automorphisms group of \( M \) such that \( \sigma \cdot \eta = \eta \cdot \sigma \) for any \( \eta \in K_0 \) and any \( \sigma \in K_X \). If \( \omega_{KS} \) is a Kähler-Ricci soliton with respect to the holomorphic vector field \( X \), we define the inner product by
\[ (\varphi, \psi) = \int_M \varphi \psi e^{\theta_X(\omega_{KS})} \omega_{KS}^n, \]
and denote by
\[ \Lambda_1(\omega_{KS}) = \{ u \in C^\infty \mid \Delta_{K_0} u + X(u) = -u \}. \]
For any compact subgroup \( G \supset K_X \) of \( K_0 \) with \( \sigma \cdot \eta = \eta \cdot \sigma \) for any \( \eta \in G \) and any \( \sigma \in K_X \), we denote by \( \mathcal{M}_G(\omega_{KS})' \) the space of \( G \)-invariant Kähler potentials perpendicular to \( \Lambda_1(\omega_{KS}) \). We call a functional \( F(\varphi) \) proper on \( \mathcal{M}_X(\omega) \), if there exists an increasing function \( \rho : \mathbb{R} \to \mathbb{R} \) satisfying \( \lim_{t \to +\infty} \rho(t) = +\infty \) such that for any \( \varphi \in \mathcal{M}_X(\omega) \), \( F'(\varphi) \geq \rho(I_\omega(\varphi)) \), where \( I_\omega(\varphi) \) is given by (2.3).

In [2], Cao-Tian-Zhu proved the following result, which is crucial in the proof of Theorem 1.1.

Theorem 2.1. (cf. [2]) If \( M \) admits a Kähler-Ricci soliton, then \( \tilde{F} \) is proper on \( \mathcal{M}_G(\omega_{KS})' \).

Inspired by the work in [19] and [9], we define the generalized Chen-Tian energy functionals \( \tilde{E}_k \) as follows:
Definition 2.2. We define the generalized Chen-Tian energy functionals for any \( k = 1, 2, \cdots, n \),
\[
\tilde{E}_{k,\omega}(\varphi) = \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi) + (k+1)\tilde{E}_{0}(\varphi),
\]
where
\[
\tilde{G}_{k,\omega}(\varphi) = -\frac{1}{V} \int_{M} \sqrt{-1}\partial u \wedge \bar{\partial} u \wedge (\sqrt{-1}\partial \bar{\partial} u)^{k-1} \wedge e^{\theta_{X}(\varphi)} \omega_{\varphi}^{n-k} + \frac{1}{V} \int_{M} \sqrt{-1}\partial u_{0} \wedge \bar{\partial} u_{0} \wedge (\sqrt{-1}\partial \bar{\partial} u_{0})^{k-1} \wedge e^{\theta_{X}} \omega_{\varphi}^{n-k},
\]
where \( u = -h_{\varphi} + \theta_{X}(\varphi) \) and \( u_{0} = -h_{\omega} + \theta_{X} \).

Remark 2.3. For Chen-Tian energy functionals \( E_{k} \), there are many different expressions as in [6], [15] and [14]. It is interesting how to write the generalized functionals \( \tilde{E}_{k} \) as similar expressions.

By the definition, it is easy to check that all of \( \tilde{E}_{k} \) satisfy the following cocycle condition
\[
\tilde{E}_{k,\omega}(\varphi) + \tilde{E}_{k,\omega}(\psi - \varphi) = \tilde{E}_{k,\omega}(\psi),
\]
for any \( \varphi, \psi \in \mathcal{M}_{X}(\omega) \). Let \( k = 1 \), we have the generalized Pali’s formula:
\[
\tilde{E}_{1}(\varphi) = 2\tilde{E}_{0}(\varphi) + \frac{1}{V} \int_{M} \sqrt{-1}\partial u \wedge \bar{\partial} u \wedge e^{\theta_{X}(\varphi)} \omega_{\varphi}^{n-1} - \frac{1}{V} \int_{M} \sqrt{-1}\partial u_{0} \wedge \bar{\partial} u_{0} \wedge e^{\theta_{X}} \omega_{\varphi}^{n-1}
\geq 2\tilde{E}_{0}(\varphi) - C_{\omega,X},
\]
(2.8)

Now we define the subspace of Kähler potentials for \( k = 2, 3, \cdots, n \)
\[
\mathcal{M}_{X,k}^{+}(\omega) = \{ \varphi \in \mathcal{M}_{X}(\omega) \mid \text{Ric}_{\varphi} - L_{X} \omega_{\varphi} \geq -\frac{2}{k-1} \omega_{\varphi} \},
\]
(2.9)
and let \( \mathcal{M}_{X,0}^{+}(\omega) = \mathcal{M}_{X,1}^{+}(\omega) = \mathcal{M}_{X}(\omega) \). The definition of (2.9) is inspired by the result in [9].

With these notations, we have the result:

Lemma 2.4. For any \( \omega_{\varphi} \in \mathcal{M}_{X,k}(\omega)(k \geq 2) \), we have
\[
\tilde{E}_{k}(\varphi) \geq (k+1)\tilde{E}_{0}(\varphi) - C_{\omega,X,k},
\]
where \( C_{\omega,X,k} \) is given by (1.2).

Proof. The argument is the same as in [9] and here we give the details for completeness. By the definition of \( \tilde{E}_{k} \), we have
\[
\tilde{E}_{k} - (k+1)\tilde{E}_{0}
= \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i}
= \frac{1}{V} \int_{M} \sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \left( \sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k+1}{i} \sqrt{-1}\partial \bar{\partial} u \wedge \omega_{\varphi}^{i} \right) \wedge e^{\theta_{X}(\varphi)} \omega_{\varphi}^{n-k} + C_{\omega,X,k}
= \frac{1}{V} \int_{M} \sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \left( \sum_{i=0}^{k-1} \binom{k+1}{i} (\text{Ric}_{\varphi} - L_{X} \omega_{\varphi} - \omega_{\varphi})^{k-i-1} \wedge \omega_{\varphi}^{i} \right) \wedge e^{\theta_{X}(\varphi)} \omega_{\varphi}^{n-k} + C_{\omega,X,k}.
\]
Observe that
\[
\sum_{i=0}^{k-1} \binom{k+1}{i} (Ric_\varphi - L_X \omega_\varphi - \omega_\varphi)^{k-i-1} \wedge \omega_\varphi^i = \sum_{i=1}^k i(Ric_\varphi - L_X \omega_\varphi)^{k-i} \wedge \omega_\varphi^{i-1}. \tag{2.10}
\]

For \(k \geq 2\), let
\[
P(x) = \sum_{i=1}^k ix^{k-i} = (x + \frac{2}{k-1})^{k-1} + \sum_{i=2}^k a_i(x + \frac{2}{k-1})^{k-i},
\]
where \(a_i\) are the constants defined by
\[
a_i = \frac{1}{(k-i)!} P^{(k-i)}(- \frac{2}{k-1}).
\]

By Lemma A.1 in the appendix of [9], \(a_i \geq 0\). Thus, for any \(\varphi \in M_{X,k}^+(\omega)\) we have
\[
\sum_{i=1}^k i(Ric_\varphi - L_X \omega_\varphi)^{k-i} \wedge \omega_\varphi^{i-1} = \left( Ric_\varphi - L_X \omega_\varphi + \frac{2}{k-1} \omega_\varphi \right)^{k-1} + \sum_{i=2}^k a_i \left( Ric_\varphi - L_X \omega_\varphi + \frac{2}{k-1} \omega_\varphi \right)^{k-i} \wedge \omega_\varphi^{i-1} \geq 0.
\]

Therefore, \(\tilde{E}_k \geq (k+1)\tilde{E}_0 + C_{\omega,X,k}\) and the lemma is proved.

\[\Box\]

### 3 Proof of Theorem 1.1

Suppose that \(M\) admits a Kähler-Ricci soliton in the class \(M_X(\omega)\). This implies that \(\tilde{F}\) functional is proper on \(M_G(\omega_{KS})'\) by Theorem 2.1, and also \(\tilde{E}_0\) is proper on \(M_G(\omega_{KS})'\) by (2.7). Thus, \(\tilde{E}_1\) is also proper on \(M_G(\omega_{KS})'\) by (2.8) and so is \(\tilde{E}_k\) on \(M_{X,k}^+(\omega) \cap M_G(\omega_{KS})'\) for any \(k \in \{2, \cdots, n\}\) by Lemma 2.4. Thus, part (b) of Theorem 1.1 is proved.

To finish part (a) of Theorem 1.1 it suffices to prove:

**Lemma 3.1.** If \(\tilde{E}_k\) is proper on \(M_{X,k}^+(\omega)\) for any \(k \in \{2, 3, \cdots, n\}\), then there exists a Kähler-Ricci soliton on \(M\).

**Proof.** We consider the complex Monge-Ampere equations with parameter \(t \in [0, 1]\)
\[
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{h - \theta_X(\varphi) - t \varphi} \omega^n. \tag{3.1}
\]

There exists a unique solution at \(t = 0\) modulo constants by [21], and the set of \(t \in [0, 1]\) such that (3.1) has a solution is open by the implicit function theorem(cf. [20]). Therefore, to prove that there is a solution for \(t = 1\), it suffices to prove that \(I_\omega(\varphi)\) is uniformly bounded for \(0 \leq t < 1\).
Note that the solution \( \varphi_t \in M_{X,k}(\omega) \), since the equation (3.1) can be written as
\[
Ric_{\varphi} - L_X \varphi = t \omega + (1 - t) \omega > 0.
\]
Since \( \tilde{E}_k \) is proper on \( M_{X,k}(\omega) \), there exists an increasing function \( \rho : \mathbb{R} \to \mathbb{R} \) satisfying
\[
\lim_{s \to +\infty} \rho(s) = +\infty \quad \text{such that} \quad \tilde{E}_k(\varphi(t)) \geq \rho(I_\omega(\varphi(t))).
\]
Now we show that \( \tilde{E}_k \) is uniformly bounded from above for \( t \in [0, 1) \). In fact,
\[
\frac{\partial}{\partial t} \tilde{E}_0(\varphi_t) = \frac{n}{V} \int_M \sqrt{-1} \partial \varphi \partial t \wedge \partial(h_{\varphi} - \theta_X(\varphi)) \wedge e^{\theta_X(\varphi)} \omega^{n-1}_{\varphi}
\]
\[
= -\frac{n}{V} \int_M \sqrt{-1} \partial \varphi \partial t \wedge \partial u \wedge e^{\theta_X(\varphi)} \omega^{n-1}_{\varphi}.
\]
Thus, for the solution \( \varphi_t (0 \leq t \leq \tau \leq 1) \) we have
\[
\tilde{E}_0(\varphi_\tau) - \tilde{E}_0(\varphi_0) = -\frac{n}{V} \int_0^\tau \int_M (1 - t) \sqrt{-1} \partial \varphi \partial t \wedge \partial \varphi \wedge e^{\theta_X(\varphi)} \omega^{n-1}_{\varphi} \wedge dt
\]
\[
= \frac{1}{V} \int_0^\tau \int_M (1 - t) \varphi(\Delta_{\varphi} + X) \frac{\partial \varphi}{\partial t} e^{\theta_X(\varphi)} \omega^n_{\varphi} \wedge dt
\]
\[
= \int_0^\tau (1 - t) \frac{d}{dt}(\tilde{I} - \tilde{J}) dt
\]
\[
\leq -c(n)(1 - \tau) L_\omega(\varphi, \tau) + (\tilde{I} - \tilde{J})(\varphi_0) - \int_0^\tau (\tilde{I} - \tilde{J}) dt
\]
\[
(3.2)
\]
where we have used the inequality (2.2). Hence, by the definition of \( \tilde{E}_k \) we have
\[
\tilde{E}_k(\varphi_\tau) = (k + 1) E_0(\varphi_\tau) + \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_\tau)
\]
\[
\leq (k + 1) \tilde{E}_0(\varphi_0) - c(n)(k + 1)(1 - \tau) L_\omega(\varphi_\tau) + (k + 1)(\tilde{I} - \tilde{J})(\varphi_0)
\]
\[
- (k + 1) \int_0^\tau (\tilde{I} - \tilde{J}) dt + \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_\tau).
\]
\[
(3.3)
\]
Note that \( \varphi_\tau \) satisfies the equation (3.1) and we have
\[
u(\tau) = log \frac{\varphi_\tau}{\omega^{n}_{\varphi}} + \varphi_\tau - h_{\omega} + \theta_X(\varphi_\tau)
\]
\[
= (1 - \tau) \varphi_\tau.
\]
Thus, we have

\[
\sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i, \omega}(\varphi_{\tau})
\]

\[
= \frac{1}{V} \int_{M} \sqrt{-1} \partial \bar{\partial} u \wedge \varphi_{\tau} \wedge \left( \sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k+1}{i} \right) \left( \sqrt{-1} \partial \bar{\partial} u \right)^{k-i} \wedge \omega_{\varphi_{\tau}}^{i} \wedge e^{\theta_{X}(\varphi_{\tau})} \omega_{\varphi_{\tau}}^{n-k} + C_{\omega, X, k}
\]

\[
= \frac{1}{V} \int_{M} (1 - \tau)^{2} \sqrt{-1} \partial \bar{\partial} \varphi_{\tau} \wedge \varphi_{\tau} \wedge \\
\left( \sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k+1}{i} \right) \cdot \left( (1 - \tau) \sqrt{-1} \partial \bar{\partial} \varphi_{\tau} \right)^{k-i} \wedge \omega_{\varphi_{\tau}}^{i} \wedge e^{\theta_{X}(\varphi_{\tau})} \omega_{\varphi_{\tau}}^{n-k} + C_{\omega, X, k}
\]

\[
\leq (1 - \tau)^{2} c(n) I(\varphi_{\tau}) + C_{\omega, X, k}.
\]

Combining this with inequality (3.3), for any \( \tau \) sufficiently close to 1 we have

\[\rho(I(\varphi_{\tau})) \leq \tilde{E}_{k}(\varphi_{\tau}) \leq C(\omega, \varphi_{0}).\]

Hence \( I(\varphi_{t}) \) is uniformly bounded from above for \( t \in [0, 1) \). Thus, \( |\varphi_{t}|_{C^{0}} \) and all higher order estimates are uniformly bounded for any \( t \in [0, 1) \) and the solution \( \varphi_{t}(t \in [0, 1)) \) can be extended to \( t = 1 \) smoothly. This concludes that \( M \) admits a Kähler-Ricci soliton.

\[\square\]

### 4 Proof of Theorem [1.2](#)

Suppose \( \tilde{F} \) is bounded from below on \( M_{X}(\omega) \). Then by [22], \( \tilde{E}_{0} \) is bounded from below and so is \( \tilde{E}_{k} \) on \( M_{k}^{+}(\omega) \) by Lemma [2.4](#) for any \( k \in \{1, 2, \cdots, n\} \). Thus, it suffices to prove the following

**Lemma 4.1.** If \( \tilde{E}_{k} \) is bounded from below on \( M_{k}^{+}(\omega) \) for any \( k \in \{1, 2, \cdots, n\} \), then \( \tilde{F} \) is bounded from below on \( M(\omega) \).

**Proof.** For any \( \psi \in M_{X}(\omega) \), we set \( \omega_{s} = \omega + s \sqrt{-1} \partial \bar{\partial} \psi \), and let \( \varphi_{s, t} \) be the solution of the equation

\[
(\omega_{s} + \sqrt{-1} \partial \bar{\partial} \varphi)^{n} = e^{h_{s} - \theta_{s}(\varphi) - t \varphi} \omega_{s}^{n},
\]

where \( h_{s} \) satisfies

\[
Ric(\omega_{s}) - \omega_{s} = \sqrt{-1} \partial \bar{\partial} h_{s}, \quad \int_{M} e^{h_{s}} \omega_{s}^{n} = V,
\]

and \( \theta_{s} \) is defined by

\[
\theta_{s} = \theta_{X} + X(s\psi), \quad \int_{M} e^{\theta_{s}} \omega_{s}^{n} = V.
\]
Since $\tilde{E}_k$ is bounded from below, from the proof of Lemma 3.1 the solution $\varphi_{s,t}$ of (4.1) exists for any $t \in [0, 1)$ and each $s \in [0, 1]$. Now we have the following

**Claim 4.2.** For any $s \in [0, 1]$ we have

$$-\infty < \lim_{t \to 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t}) \leq 0. \quad (4.3)$$

**Proof.** By the proof of Lemma 3.1, for any $t \in [0, 1)$ we have

$$\tilde{E}_{k,\omega_s}(\varphi_{s,t}) \leq -c(n, k)(1 - t) I_\omega(\varphi_{s,t}) + C(\omega_s) - (k + 1) \int_0^t (\tilde{I}_\omega(\varphi_{s,\tau}) - \tilde{J}_\omega(\varphi_{s,\tau})) d\tau. \quad (4.4)$$

By the assumption that $\tilde{E}_k$ is bounded from below, we have

$$c(n, k)(1 - t) I_\omega(\varphi_{s,t}) + (k + 1) \int_0^1 (\tilde{I}_\omega(\varphi_{s,\tau}) - \tilde{J}_\omega(\varphi_{s,\tau})) d\tau \leq C(\omega_s). \quad (4.5)$$

Note that $\tilde{I}_\omega(\varphi_{s,\tau}) - \tilde{J}_\omega(\varphi_{s,\tau})$ is increasing with respect to $\tau$, we have

$$0 \leq \tilde{I}_\omega(\varphi_{s,t}) - \tilde{J}_\omega(\varphi_{s,t}) \leq \frac{1}{1 - t} \int_t^1 (\tilde{I}_\omega(\varphi_{s,\tau}) - \tilde{J}_\omega(\varphi_{s,\tau})) d\tau. \quad (4.6)$$

and

$$\lim_{t \to 1^-} (1 - t)(\tilde{I}_\omega(\varphi_{s,t}) - \tilde{J}_\omega(\varphi_{s,t})) = 0. \quad (4.7)$$

By Proposition 3.1 in [2], there exists two constants $c_1 = c_1(X, \omega)$ and $c_2 = c_2(X, \omega)$ such that for any $t \in [\frac{1}{2}, 1)$

$$\|\varphi_{s,t}\|_{C^0} \leq c_1 I_\omega(\varphi_{s,t}) + c_2.$$ 

Combining this with (4.5)(2.2), for any $s \in [0, 1]$ we have

$$\lim_{t \to 1^-} (1 - t) \|\varphi_{s,t}\|_{C^0} = 0. \quad (4.8)$$

Note that

$$\frac{d}{dt} \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = \int_M (\Delta_{s,t} + X) \frac{\partial}{\partial t} e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = 0,$$

we infer that

$$\int_M e^{h_k - t\varphi_{s,t}} \omega^n = \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = \int_M e^{\theta_s(\varphi_{s,t})} \omega_{\varphi_{s,t}}^n = V.$$ 

Combining this with (4.6), for any $s \in [0, 1]$ we have

$$\lim_{t \to 1^-} \int_M e^{h_k - \varphi_{s,t}} \omega_s^n = \lim_{t \to 1^-} \int_M e^{h_k - \varphi_{s,t}} \cdot e^{(1 - t)\varphi_{s,t}} \omega_s^n = V, \quad (4.7)$$

and we can infer that

$$\lim_{t \to 1^-} \tilde{F}_{\omega_s}(\varphi_{s,t}) = \lim_{t \to 1^-} \left( \tilde{J}_\omega(\varphi_{s,t}) - \frac{1}{V} \int_M \varphi_{s,t} e^{\theta_s(\varphi_{s,t})} \omega_s^n \right) = -\int_0^1 (\tilde{I}_\omega(\varphi_{s,\tau}) - \tilde{J}_\omega(\varphi_{s,\tau})) d\tau \leq 0, \quad (4.8)$$
where we used (4.4) and the fact that (cf. Proposition 1.1 in [2])
\[
\tilde{J}_\omega(\varphi_{s,t}) - \frac{1}{V} \int_M \varphi_{s,t} e^{\theta_s} \omega_s^n = - \frac{1}{t} \int_0^t (\tilde{I}_{\omega_s}(\varphi_{s,\tau}) - \tilde{J}_\omega(\varphi_{s,\tau})) d\tau.
\]
Thus, the claim is proved.

Claim 4.3. For any \( s \in [0, 1] \), we have
\[
\lim_{t \to 1^-} \tilde{F}_\omega(s, \varphi_{s,t}) = \lim_{t \to 1^-} \tilde{F}_\omega(s, \varphi_{0,t}).
\]
In other words, the limit \( \lim_{t \to 1^-} \tilde{F}_\omega(s, \varphi_{s,t}) \) is independent of \( s \).

Proof. By (4.2) we have
\[
h_s = -\log \frac{\omega_s^n}{\omega_n} - s\psi + h_\omega + c_s
\]
where \( c_s \) is a constant given by
\[
\int_M e^{h_s - \psi + c_s} \omega_s^n = V.
\]
Thus, (4.1) can be written as
\[
(\omega + \sqrt{-1} \partial \bar{\partial} (s\psi + \varphi_{s,t}))^n = e^{h_\omega - \theta X - X(s\psi + \varphi_{s,t}) - t\varphi_{s,t} - s\psi + c_s} \omega^n.
\]
Let \( \hat{\varphi}_{s,t} = s\psi + \varphi_{s,t} - c_s \), we have
\[
(\omega + \sqrt{-1} \partial \bar{\partial} \hat{\varphi}_{s,t})^n = e^{h_\omega - \theta X(\hat{\varphi}_{s,t}) - t\hat{\varphi}_{s,t} - (1-t)(s\psi - c_s)} \omega^n.
\]
Taking derivative with respect to \( s \), we have
\[
(\Delta_{s,t} + X) \frac{\partial \hat{\varphi}_{s,t}}{\partial s} = -t \frac{\partial \hat{\varphi}_{s,t}}{\partial s} - (1-t)(\psi - \frac{d c_s}{d s}).
\]
Direct calculation shows that
\[
\frac{\partial}{\partial s} \left( \tilde{J}_\omega(\hat{\varphi}_{s,t}) - \frac{1}{V} \int_M \hat{\varphi}_{s,t} e^{\theta_s} \omega^n \right) = - \frac{1}{V} \int_M \frac{\partial \hat{\varphi}_{s,t}}{\partial s} e^{\theta_s(\hat{\varphi}_{s,t})} \omega^n_{s,t} = - \frac{1-t}{tV} \int_M (\psi - \frac{d c_s}{d s}) \omega^n_{s,t},
\]
where we used (4.11). Note that for any \( s \in [0, 1] \) we have
\[
\lim_{t \to 1^-} \int_M e^{h_\omega - \hat{\varphi}_{s,t} \omega^n} = \lim_{t \to 1^-} \int_M e^{h_\omega - \varphi_{s,t} \omega^n} = V,
\]
where we used (4.7) and (4.9). Combining (4.12) with (4.13), for any \( s \in [0, 1] \) we have
\[
\lim_{t \to 1^-} \left| \tilde{F}_\omega(\hat{\varphi}_{s,t}) - \tilde{F}_\omega(\hat{\varphi}_{0,t}) \right| = \lim_{t \to 1^-} \left| \int_0^s \frac{\partial}{\partial \tau} \left( \tilde{J}_\omega(\hat{\varphi}_{\tau,t}) - \frac{1}{V} \int_M \hat{\varphi}_{\tau,t} e^{\theta_s} \omega^n \right) d\tau \right| = \lim_{t \to 1^-} \left| \frac{1-t}{tV} \int_0^s d\tau \int_M (\psi - \frac{d c_s}{d s}) \omega^n_{\tau,t} \right| = 0.
\]
The claim is proved.
By Claim 4.2 we have
\[
\lim_{t \to 1^{-}} (\tilde{F}_\omega(\omega_{s,t}) - \tilde{F}_\omega(\omega_s)) = \lim_{t \to 1^{-}} \tilde{F}_\omega(\omega_{s,t}) \leq 0.
\]
and by Claim 4.3
\[
\tilde{F}_\omega(\psi) = \tilde{F}_\omega(\omega_1) \geq \lim_{t \to 1^{-}} \tilde{F}_\omega(\omega_{1,t}) = \lim_{t \to 1^{-}} \tilde{F}_\omega(\omega_{0,t}).
\]
Thus, \( \tilde{F} \) is uniformly bounded from below on \( \mathcal{M}_X(\omega) \).

Now we prove the equality (1.1). Suppose that one of the energy functionals \( \tilde{E}_k \) and \( \tilde{F} \) is bounded from below on \( \mathcal{M}_{X,k}(\omega) \), by Lemma 2.4 and the inequality (2.7) we have
\[
\inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') \geq (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{E}_{0,\omega}(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta X} \omega^n. \tag{4.14}
\]
On the other hand, for the solution \( \varphi_t (t \in [0,1)) \) of (3.1) the inequality (3.4) implies that
\[
\lim_{t \to 1^{-}} \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} \tilde{G}_{k-i,\omega}(\varphi_t) \leq C_{\omega,X,k}.
\]
Combining this with the definition of \( \tilde{E}_k \) and the equality (3.2) (4.8) we have
\[
\inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') \leq (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{F}_\omega(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta X} \omega^n, \tag{4.15}
\]
where \( \varphi_t (t \in [0,1)) \) is the solution of (3.1) and we have used
\[
\tilde{E}_{0,\omega}(\varphi_0) + (\tilde{I} - \tilde{J})_\omega(\varphi_0) = \frac{1}{V} \int_M u_0 e^{\theta X} \omega^n
\]
since \( \varphi_0 \) is a solution of (3.1) when \( t = 0 \). Combining (4.14) (4.15), we have
\[
\inf_{\omega' \in \mathcal{M}_{X,k}(\omega)} \tilde{E}_{k,\omega}(\omega') = (k+1) \inf_{\omega' \in \mathcal{M}_X(\omega)} \tilde{F}_\omega(\omega') + C_{\omega,X,k} - \frac{k+1}{V} \int_M u_0 e^{\theta X} \omega^n.
\]
The theorem is proved. 

Following the ideas of the previous proof, we can finish Theorem 1.3.
Proof of Theorem 1.3. For any $\psi \in \mathcal{M}_X(\omega)$, we consider the solution $\varphi_{s,t}$ of the equation (4.1). Suppose that $\tilde{F}$ is bounded from below for the solution $\varphi_{0,t}$, then $\varphi_{0,t}$ exists for all $t \in [0, 1)$ and

$$\lim_{t \to 1^-} \tilde{F}_\omega(\varphi_{0,t}) = - \int_0^1 (\tilde{I}_\omega - \tilde{J}_\omega)(\varphi_{0,t}) d\tau > -\infty.$$  

For simplicity, we set

$$\tilde{F}_0^0(\varphi) = \tilde{J}_\omega(\varphi) - \frac{1}{V} \int_M \varphi e^{\theta X} \omega^n.$$ 

Then by (4.12) for any $s \in [0, 1]$ we have

$$\tilde{F}_\omega^0(\hat{\varphi}_{s,t}) - \tilde{F}_\omega^0(\varphi_{0,t}) = - \int_0^s \frac{1-t}{tV} \int_M \left( \psi - \frac{dc}{dt} \right) \omega^n d\tau,$$  

(4.16)

which implies that

$$\tilde{F}_\omega^0(\varphi_{s,t}) = \tilde{F}_\omega^0(\hat{\varphi}_{s,t}) - \tilde{F}_\omega^0(s\psi)$$

is uniformly bounded from below for any $t \in [\frac{1}{2}, 1)$ and $s \in [0, 1]$. Thus, the solution $\varphi_{s,t}$ exists for all $t \in [0, 1)$ when $s \in [0, 1]$ and

$$\lim_{t \to 1^-} \tilde{F}_\omega^0(\varphi_{s,t}) = \lim_{t \to 1^-} \tilde{F}_\omega^0(\varphi_{0,t}) - \tilde{F}_\omega^0(s\psi) > -\infty.$$

On the other hand, we have

$$(1 - t)(\tilde{I}_\omega(\varphi_{s,t}) - \tilde{J}_\omega(\varphi_{s,t})) \leq \int_0^1 (\tilde{I}_\omega(\varphi_{s,t}) - \tilde{J}_\omega(\varphi_{s,t})) dt \leq - \lim_{t \to 1^-} \tilde{F}_\omega^0(\varphi_{s,t})$$

and thus we have

$$\lim_{t \to 1^-} (1 - t)|\varphi_{s,t}|_{C^0} = 0.$$ 

We can argue as (4.7) to derive

$$\lim_{t \to 1^-} \int_M e^{h_s - \varphi_{s,t}} \omega^n_s = V.$$  

(4.17)

Thus, by the definition of $\tilde{F}$ and (4.17) we have

$$-\infty < \lim_{t \to 1^-} \tilde{F}_\omega(\varphi_{s,t}) = \lim_{t \to 1^-} \tilde{F}_\omega^0(\varphi_{s,t}) \leq 0,$$

and

$$\tilde{F}_\omega(\psi) = \tilde{F}_\omega(\omega_1) \geq \lim_{t \to 1^-} \tilde{F}_\omega(\omega_{1,t}) = \lim_{t \to 1^-} \tilde{F}_\omega(\omega_{0,t}),$$

where we used (4.16) in the last equality. This shows that $\tilde{F}$ is bounded from below in the Kähler class $2\pi c_1(M)$.

Suppose that $\tilde{E}_k$ is bounded from below for $\varphi_t$, we can see from the proof of Lemma 3.1 that $\tilde{F}$ is also bounded from below along $\varphi_t$. Thus, $\tilde{F}$ is bounded from below in the Kähler class $2\pi c_1(M)$ and by Theorem 1.2 $\tilde{E}_k$ is bounded from below on $\mathcal{M}_k^+(\omega)$. The theorem is proved. $\square$
5 The holomorphic invariant

Recall that Tian-Zhu defined the holomorphic invariant by
\[
\mathcal{F}_X(Y) = \int_M Y(h_g - \theta_X(g))e^{\theta_X(g)}\omega_g^n, \quad Y \in \eta(M),
\]
which generalized the Futaki invariant. Here \(\eta(M)\) denotes the space of holomorphic vector fields on \(M\). Let \(\{\Phi(t)\}_{|t|<\infty}\) be the one-parameter subgroup of automorphisms induced by \(Re(Y)\), and \(\varphi(x, t)\) be the Kähler potential satisfying
\[
\omega_\varphi = \Phi_t^* \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi. \tag{5.1}
\]
Differentiating (5.1), we have
\[
L_{Re(Y)}\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\varphi. \tag{5.2}
\]
On the other hand, we have \(L_Y\omega_\varphi = \sqrt{-1}\partial\bar{\partial}Y_\varphi\). Thus,
\[
\frac{\partial \varphi}{\partial t} = Re(Y_\varphi) + c \tag{5.3}
\]
for some constant \(c\). Recall that \(u\) satisfies
\[
\sqrt{-1}\partial\bar{\partial}u = -Ric(\omega_\varphi) + \omega_\varphi + \sqrt{-1}\partial\bar{\partial}X_\varphi.
\]
Taking the interior product on both sides, we have
\[
Y(u) = \Delta X_\varphi + \theta_X(\varphi) + Y\theta_X(\varphi).
\]
By the definition of \(u\), we have
\[
\frac{\partial u}{\partial t} = \Delta \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t} + X\left(\frac{\partial \varphi}{\partial t}\right) + c = Re(Yu) + c, \tag{5.4}
\]
where we used the fact that \(Y\theta_X(\varphi) = X\theta_Y(\varphi)\). Following a direct calculation, we have the lemma:

**Lemma 5.1.** Let \(\{\Phi(t)\}_{|t|<\infty}\) be the one-parameter subgroup of automorphisms induced by \(Re(Y)\), we have
\[
\frac{d}{dt} \tilde{E}_0(\varphi) = \frac{n}{V} \mathcal{F}_X(Y), \tag{5.5}
\]
where \(\varphi\) is given by \(\Phi_t^* \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi\).

The main result in this section is

**Theorem 5.2.** Let \(\{\Phi(t)\}_{|t|<\infty}\) be the one-parameter subgroup of automorphisms induced by \(Re(Y)\), we have
\[
\frac{d}{dt} \tilde{E}_k(\varphi) = \frac{(k+1)n}{V} \mathcal{F}_X(Y),
\]
where \(\varphi\) is given by \(\Phi_t^* \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi\).
Proof. By the definition of $\tilde{E}_k$ and Lemma 5.1, it suffices to check that for any $k = 0, \cdots, n$

$$\frac{d}{dt} \tilde{G}_k(\varphi) = 0. \quad (5.6)$$

Direct calculation shows that

$$\frac{d}{dt} \tilde{G}_k(\varphi) = -\frac{1}{V} \frac{d}{dt} \int_M \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi$$

$$= -\frac{1}{V} \text{Re} \int_M 2 \sqrt{-1} \partial Y u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi$$

$$- \frac{1}{V} \text{Re} \int_M (k-1) \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-2} \wedge \sqrt{-1} \partial \bar{\partial} Y u \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi$$

$$- \frac{1}{V} \text{Re} \int_M J \theta_Y(\varphi) \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi$$

$$- \frac{1}{V} \text{Re} \int_M (n-k) \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k-1}_\varphi$$

$$= \text{Re}(I_1 + I_2 + I_3 + I_4),$$

where $I_i (1 \leq i \leq 4)$ denote the integrations on the right hand side respectively. On the other hand, we have

0 &= \frac{1}{V} \int_M i Y (\partial u \wedge (\sqrt{-1} \partial \bar{\partial} u)^k \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi) \\
&= \frac{1}{V} \int_M Y u(\sqrt{-1} \partial \bar{\partial} u)^k \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi \\
&- \frac{1}{V} \int_M k \sqrt{-1} \partial u \wedge \bar{\partial} Y u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi \\
&- \frac{1}{V} \int_M (n-k) \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^k \wedge e^{\theta_X(\varphi)} \omega^{n-k-1}_\varphi \\
&= J_1 + J_2 + J_3.$

Note that

$$J_1 = \frac{1}{V} \int_M -\sqrt{-1} \partial Y u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi$$

$$- \frac{1}{V} \int_M Y u(\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k}_\varphi$$

$$= J_{1a} + J_{1b}$$

and

$$J_3 = \frac{1}{V} \int_M -(n-k) \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} Y u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k-1}_\varphi$$

$$+ \frac{1}{V} \int_M -(n-k) \partial Y u \wedge \bar{\partial} Y u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega^{n-k-1}_\varphi$$

$$= J_{3a} + J_{3b}.$$
Now we calculate

\[
0 = \frac{1}{V} \int_M -i_Y \left( \partial \theta_X(\varphi) \wedge \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k} \right)
\]

\[
= \frac{1}{V} \int_M -Y \theta_X(\varphi) \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k}
\]

\[
+ \frac{1}{V} \int_M Y u \sqrt{-1} \partial \theta_X(\varphi) \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k}
\]

\[
+ \frac{1}{V} \int_M (k-1) \partial \theta_X(\varphi) \wedge \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-2} \wedge \sqrt{-1} \partial Y u \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k}
\]

\[
+ \frac{1}{V} \int_M (n-k) \partial \theta_X(\varphi) \wedge \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k-1} \wedge \sqrt{-1} \partial Y(\varphi)
\]

\[= K_1 + K_2 + K_3 + K_4. \]

Note that

\[ K_3 = \frac{1}{V} \int_M (k-1) \sqrt{-1} \partial u \wedge \bar{\partial} Y u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-1} \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k} \]

\[- \frac{1}{V} \int_M (k-1) \sqrt{-1} \partial u \wedge \bar{\partial} u \wedge (\sqrt{-1} \partial \bar{\partial} u)^{k-2} \wedge \sqrt{-1} \partial \bar{\partial} Y u \wedge e^{\theta_X(\varphi)} \omega_{\varphi}^{n-k} \]

\[= K_{3a} + K_{3b}. \]

Combining these equalities, we have the following relations:

\[ I_1 = 2J_{1a}, \quad I_2 = K_{3b}, \quad I_3 = K_1, \quad I_4 = J_{3a} \]

and

\[ J_{1b} + K_2 = 0, \quad J_{3b} + K_4 = 0, \quad J_2 + K_{3a} = J_{1a}. \]

Thus, we have

\[ \sum_{i=1}^{4} I_i = \sum_{i=1}^{3} J_i + \sum_{i=1}^{4} K_i = 0. \]

The theorem is proved. \( \square \)

**References**

[1] S. Bando, T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, 11–40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[2] H. D. Cao, G. Tian, X. H. Zhu. Kähler Ricci solitons on compact complex manifolds with \( c_1(M) > 0 \), Geom. Funct. Anal., 15(2005), 697-719.

[3] X. X. Chen. On the lower bound of energy functional \( E_1(I) \)– a stability theorem on the Kähler Ricci flow. J. Geometric Analysis. 16 (2006) 23-38.
[4] X. X. Chen, Space of Kähler metrics (IV)–On the lower bound of the $K$-energy. arXiv:0809.4081.

[5] X. X. Chen, H. Li, B. Wang. Kahler-Ricci flow with small initial energy, Geom. Func. Anal., Vol 18, No 5 (2009), 1525-1563.

[6] X. X. Chen, G. Tian. Ricci flow on Kähler-Einstein surfaces. Invent. Math. 147 (2002), no. 3, 487–544.

[7] X. X. Chen, G. Tian. Ricci flow on Kähler-Einstein manifolds. Duke. Math. J. 131, (2006), no. 1, 17-73.

[8] W. Y. Ding and G. Tian, The generalized Moser-Trudinger inequality, in Nonlinear Analysis and Microlocal Analysis: Proceedings of the International Conference at Nankai Institute of Mathematics (K.-C. Chang et al., Eds.), World Scientific, 1992, 57-70. ISBN 9810209134.

[9] H. Z. Li, A new formula for the Chen-Tian energy functionals $E_k$ and its applications, International Mathematics Research Notices, Vol. 2007, Article ID rnm033, 17 pages, 2007 SCI

[10] H. Z. Li, On the lower bound of $F$ functional and $K$ energy, Osaka J. Math. Volume 45, Number 1 (2008), 253-264.

[11] C. J. Liu. Bando-Futaki Invariants on Hypersurfaces. math.DG/0406029.

[12] T. Mabuchi. $K$-energy maps integrating Futaki invariants. Tohoku Math. J. (2) 38(1986), no. 4, 575-593.

[13] N. Pali. A consequence of a lower bound of the $K$-energy. Int. Math. Res. Not. 2005, no. 50, 3081–3090.

[14] Y. Rubinstein, On energy functionals, Kähler-Einstein metrics, and the Moser-Trudinger-Onofri neighborhood, J. Func. Anal. 255(2008), no. 1, 641-2660.

[15] J. Song, B. Weinkove. Energy functionals and canonical Kahler metrics. Duke Math. J. 137(2007), no. 1, 159-184.

[16] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. Invent. Math. 89 (1987), no. 2, 225–246.

[17] G. Tian. Kähler-Einstein metrics with positive scalar curvature. Invent. Math. 130 (1997), no. 1, 1–37.

[18] G. Tian. Canonical metrics in Kähler geometry. Notes taken by Meike Akveld. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000.
[19] G. Tian, X. H. Zhu, A new holomorphic invariant and uniqueness of Kähler-Ricci solitons. Comment. Math. Helv. 77(2002), 297-325.

[20] G. Tian, X. H. Zhu, Uniqueness of Kähler-Ricci solitons, Acta Math. 184 (2000), 271-305.

[21] X. H. Zhu, Kähler-Ricci soliton typed equations on compact complex manifolds with $C_1(M) > 0$, J. Geometric Analysis 10 (2000), 747-762.

Department of Mathematics,
East China Normal University, Shanghai, 200241, China.
Email: lihaozhao@gmail.com