Covariant Symmetry Classifications for Observables of Cosmological Birefringence

John P. Ralston
Department of Physics and Astronomy, Kansas University, Lawrence, KS-66045, USA
ralston@kuphsz.phsz.ukans.edu

and

Pankaj Jain
Physics Department, I.I.T. Kanpur, India 208016
pkjain@iitk.ac.in

Polarizations of electromagnetic waves from distant galaxies are known to be correlated with the source orientations. These quantities have been used to search for signals of cosmological birefringence. We review and classify transformation properties of the polarization and source orientation observables. The classifications give a firm foundation to certain practices which have sprung up informally in the literature. Transformations under parity play a central role, showing that parity violation in emission or in the subsequent propagation is an observable phenomenon. We also discuss statistical measures, correlations and distributions which transform properly and which can be used for systematic data analysis.

I. INTRODUCTION

Galaxies at cosmological distances are known to emit polarized electromagnetic radiation according to certain statistical regularities. The processes range from linear polarization from radio galaxies due to synchrotron radiation, to centro-symmetric polarizations sometimes seen in optical measurements of QSO’s. The observables can include the percentage of polarization, a linear polarization angle, a galaxy symmetry axis, and the redshift of the galaxy, as well as the angular position coordinates of the galaxy on the dome of the sky. One typical correlation is an observed alignment of the bisector of a radio galaxy’s long axis and linearly polarized radiation in the “FR1” class of objects. This has been interpreted as probing the source’s magnetic field in conventional models. While this effect is in the context of conventional physics, other polarization correlations can test extremely small effects that might indicate new physics—sensitive to “cosmological birefringence” – that might occur in the subsequent propagation. This has led to several interesting studies, from seminal early work on propagation in parity-violating electro-weak media by Leo Stodolsky [1], to the discussion of “optical activity of the universe” by Gabriel Karl and collaborators [2], to recent studies of possible cosmological anisotropy in propagation [3,4].

Galaxy and polarization axes are labeled by angles, typically measured in “degrees East of North” in a spherical-polar system of angular coordinates. The galaxy major axis projected into the tangent plane is measured with an angle $\psi$. A linearly polarized electric field orientation in the same coordinates is assigned polarization angle $\chi$. The angles, however, do not represent vectors in the rigorous sense, because a rotation of $\pi$ radians returns an axis to its original orientation. These observables are described by elements of the real projective group, which is not very familiar in physics. In this paper we will investigate and then automate proper transformation properties of such observables, and list quantities which are invariant under different symmetries. This provides a firm foundation for certain traditions and informal uses of variables which have sprung up in the literature. Rather interestingly, the split-up of the three dimensional orthogonal group into disconnected subgroups of different parity also occurs for observables connected with polarization. One then has quantities which are even or odd under parity, which extends the list of symmetries one can test using polarization data.

The physical motivation for this is as follows. Consider a linear polarization correlated with an emission axis. From “symmetry”, statistics would naively be expected to show equal offsets of polarization axis relative to galaxy axis for either a positive or a negative sense of rotation. The particular “symmetry” assumed here is parity symmetry. While the emergence of parity as an observable may be surprising, it should be intuitively clear that observation of the converse would allow one to extract a definition of right–versus –left handed senses of rotation from a data set, breaking parity symmetry. An even more obvious violation of parity symmetry would be preference for one kind of circular polarization over another. While one may not expect an object the size of a galaxy to break parity
symmetry in its emission characteristics, the possibility constitutes a valid observable, subject to classification and study. Another more interesting possibility, moreover, is to assume that objects do respect parity symmetry in a statistical sense upon emission, while the intervening medium can be tested for parity violation to an extremely fine degree. Many models predict just such effects. We will proceed by setting up the transformation properties, listing covariant and invariant quantities, and illustrating the remarks by some statistical quantities which embody the ideas for data analysis.

II. SETUP

Let us discuss transformation properties. We are not concerned with Lorentz boosts, nor the usual Lorentz covariant characterization of radiation in terms of the energy momentum tensor, etc. Our focus is the “little group” of angular variables in the rest frame of the observer, and the peculiar fact that the angles $\chi$ and $\psi$ actually observed give the orientation of “sticks”. A rotation of just $\pi$ radians about the axis of propagation makes no observable effect. Due to this unusual transformation property, both the electric field and the galaxy axis are conveniently represented by certain tensors.

Let $\hat{p}$ be the radial unit vector pointing toward a particular source. Construct a local Cartesian frame with its $z$-axis oriented along $\hat{p}$ (Fig 1). The 2-dimensional subspace orthogonal to $\hat{p}$ is a plane tangent to the unit sphere of the sky. Any consistent orientation of the local $\hat{x}$ and $\hat{y}$ axes spanning the tangent plane can be used; one can always take these to be in the directions of increasing azimuthal and polar angles ($\phi$ and $\theta$) relative to a global North pole (hence the astronomer’s “East of North”). The electric field ($E_x, E_y, 0$) is transverse to $\hat{p}$ and conventionally represented by a 2 component complex doublet, a convention hearkening back to optics. In the circular polarization basis, $|E| = \sqrt{2}(E_x + iE_y, E_x - iE_y)$. The doublet notation is a hybrid and the transformation properties need examination.

First, in averaging over many cycles of the wave, the observer does not actually measure the electric field. Instead what can be measured is an Hermitian density matrix $\begin{pmatrix} E_x & E_y \\ E_y & -E_x \end{pmatrix} = J$, where $J$ is defined to be the rest of the matrix, “as if” it were a pure state. Thus $\text{det}(J) = 0$. This matrix is conventionally expanded in Pauli matrices:

$$J = \begin{pmatrix} P_{\text{max}}(1 + \hat{\chi} \cdot \hat{\sigma}) \\ 0 \end{pmatrix}$$

where $P_{\text{max}}$ is the degree of polarization, $\hat{\chi}$ is a unit–3 vector, and $\hat{\sigma}$ are the Pauli matrices. Equivalent are the “Poincarè” coordinates $\vartheta_p$ and $\chi$ (for describing elliptical and circular polarization):

$$\hat{\chi} = \begin{pmatrix} \cos \vartheta_p e^{2i\chi} \\ \sin \vartheta_p e^{-2i\chi} \end{pmatrix}$$

We have introduced the common “slash” notation for contraction of a vector with the Pauli matrix. We can also use $\hat{\chi}$ for the $3 \times 3$ matrix when no confusion between the two is possible. The polarization parameters have now been organized in a form convenient for transformations.

Consider the important case of a rotation of tangent plane coordinates about the propagation axis, that is the local $z$-axis, $|E| \rightarrow U|E|$, with $U 3 \times 3$ and in $SO(3)$. $U$ takes the block form $U = (u_\vartheta, 1)$ where $u_\vartheta$ is $2 \times 2$ and unitary, in fact an element of $SO(2)$. Of course $\hat{\chi} \rightarrow u_\vartheta \hat{\chi} u_\vartheta^\dagger$. This is the same rule as the rotations in the spin 1/2 representation, but in this case operating on vector components, not spinors. It follows that the parameters $\hat{\chi}$ transform with the angle of rotation doubled. As a check, a short calculation shows that a linear polarization oriented at angle $\chi$ relative to the local $x$-axis gives $\hat{\chi} = (\cos 2\chi, \sin 2\chi, 0)$ in the local basis. The factor of “2” is just right to account for the periodicity of the observable plane of the electric field under rotations by $\pi$ (as opposed to $2\pi$). This supports the informal use of $\hat{\chi}$ in literature ranging from biology to astronomy, where the “2” is inserted intuitively and to make things come out right while making it clear that $\hat{\chi}$ is not a true vector.

The result is elucidated by the following general argument. Recall the familiar decomposition of angular momentum ($\hat{j}$) states for the product $|E| \times |E| = 1 \times 1 = 2 + 1 + 0$. The states are labeled by $(m, j)$ representing the
eigenvalues of $j_z$ and $\hat{j}^2$, respectively. However, since $E$ is transverse, all tensor products made from $m = 0$ are absent. From the addition rule for $m$, these absent representations are the $m = 0, m = \pm 1$ states of $j = 2$, and the $m = \pm 1$ states of $j = 1$. This eliminates 5 of 9 possible combinations, leaving 4 total which are $(2, 2), (-2, 2), (0, 1), (0, 0)$. The 4 possibilities are the 4 Stokes parameters, with the total power $\vec{E} \cdot \vec{E}^*$ being the singlet $(0, 0)$. The object $\hat{\chi}$, a deceptively vector–like position on the Poincaré sphere, is made from the peculiar combination $(2, 2), (-2, 2), (0, 1)$. Making a short calculation in the helicity basis, $iP_{\max} \cos(\theta_p) = 1/2Tr[\sigma_z J] = i(E_+ E^- - E_- E^+) = (\vec{E} \times \vec{E}^*)_z$ is the $(0, 1)$ component, showing that the ellipticity transforms like a 3-vector attached to the photon’s direction of motion.

A similar treatment is needed for a galaxy axis. An observed galaxy axis is the projection of the physical major axis (a signless eigenvector of a $3 \times 3$ tensor of intensity distribution, say) onto the tangent plane. We can again define in a helicity basis a 2 component $|\psi> = (\cos \psi + i \sin \psi, \cos \psi - i \sin \psi)/\sqrt{2}$. Again $|\psi>$ is not a good variable, because the observable is actually projective, and we must identify $|\psi> \leftrightarrow -|\psi>$. The associated $2 \times 2$ matrix, $|\psi><\psi| = \hat{1} + i\hat{\psi}$, is part of a tensor embedded in a $3 \times 3$ matrix. In our coordinates it can be expanded in Pauli matrices

$$|\psi><\psi| = 1/2 \begin{pmatrix} 1 & e^{2i\psi} \\ i\text{h.c.} & 1 \end{pmatrix} = \frac{1}{2}1 + \frac{1}{2}\hat{\psi}$$

The matrix $\hat{\psi}$ transforms like $u_{\psi} \hat{\psi} u_{\psi}^\dagger$, and defines a 3 component thing, $\hat{\psi} = (\cos 2\psi, \sin 2\psi, 0)$. Note again the doubling of angles in $\psi$, representing the information that $|\psi>$ and $-|\psi>$ are now identified, just as informally practiced in the literature. (It is interesting that if one could deduce information on the “pitch” of a galaxy, then $\psi$ could point out of the plane: such an axis might be “elliptically polarized”, otherwise $\psi$ is in the local tangent plane.)

We also have the position on the sky $\hat{\rho}$, which is a true vector and not to transform with doubled angles.

Finally we should classify the objects under parity: by definition we have that $\hat{\rho} \rightarrow -\hat{\rho}$, $\hat{\psi}$ and $\hat{\chi}$ are only a little more work: since $\hat{\psi}_j = 1/2Tr[\hat{\psi}\sigma_j]$, with $\hat{\psi}$ invariant, then $\hat{\psi}$ and $\hat{\chi}$ are invariant under parity (although transforming in a complicated way under the full $O(3)$).

Now one can transform away from the special coordinate system, using general $3 \times 3$ elements of $SO(3)$, with the three objects $\hat{\chi}, \hat{\psi}$, and $\hat{\rho}$ all transforming properly.

### III. COVARIANTS AND INVARIANTS

We now turn to invariants one can make from the matrices $\hat{\psi}, \hat{\chi}$ and $\hat{\rho}$. A “local” quantity will be one made from a single source, or (if possible) different sources at the same location on the dome of the sky; a “non-local” quantity anything made from sources at different locations.

**Local quantities:** Since for any galaxy $i$ a coordinate system exists where all the matrix elements are in the upper left, then we have the covariant identity for such $3 \times 3$ matrices

$$A B \equiv A \cdot B \ 1 + i\mathbf{C}; \ C = A \times B \quad (1)$$

The “1” of course means the $2 \times 2$ unit matrix on the upper left, covariantly written as $\delta_{ij} - \hat{p}_i\hat{p}_j$. As a consequence of this identity an obvious invariant is reduced to a simpler form:

$$s_1 = \frac{1}{2}Tr[\hat{\chi}\hat{\psi}] = \frac{1}{2}Tr[\hat{\psi}\hat{\chi}] = \hat{\chi} \cdot \hat{\psi}.$$ 

It follows that $s_1 = \cos(2(\chi - \psi))$, which is clearly invariant under the local rotation $\chi \rightarrow \chi + \delta, \psi \rightarrow \psi + \delta$. It also follows that $s_1$ is even under parity.

Another useful quantity is the anti-symmetric $3 \times 3$ commutator $[\hat{\chi}, \hat{\psi}]$. This is dual to a pseudo-vector:

$$\vec{A}_i = 1/2Tr[\epsilon_i \hat{\chi}\hat{\psi}] \quad (2)$$

where $\epsilon_i$ is the completely anti-symmetric matrix with $j, k$ elements $\epsilon_i^{jk}$. In our local coordinate frame, $\vec{A}_i$ points in the direction of $\hat{p}$ and is proportional to $\sin(2(\chi - \psi))$. The sign of proportionality depends on the right–handed convention for angles. This remains true however the coordinate system is rotated. The epsilon-tensor is even under parity, making it clear that $\vec{A}$ is even and therefore a pseudo-vector by construction. Since any quantity which is odd
in \((\chi - \psi)\) reverses when a positive “sense” of rotation is reversed to a negative one, such quantities are \textit{parity-odd} on general grounds.

This is not the only odd-parity observable. The helicity \(h\) of the wave is defined in a Lorentz-covariant manner as the projection of its spin along its direction of propagation. This is a pseudo-scalar: it must therefore be equal (up to a constant) to the product \(\hat{\chi} \cdot \hat{p}\). The other invariants which can be made by contracting \(\hat{p}\) and the matrices are trivially zero. Consulting the identity (1) above, there are 4 real-valued quantities in the products of the 2 matrices, which have now been classified into one scalar and one pseudo-vector, exhausting the possibilities for local bilinears of the two tensors. One can go further, and add another unit vector \(\hat{\lambda}\) to the problem. Such a vector is needed to quantify asymmetries of angular distribution. Then one can make a scalar \(s_2\) and a pseudoscalar \(p_1\):

\[
s_2 = \hat{p} \cdot \hat{\lambda} = \cos(\theta); \quad p_1 = \hat{A} \cdot \hat{\lambda} = \sin(2(\chi - \psi)) \cos(\theta)
\]

where \(\theta\) is the angle between \(\hat{\lambda}\) and the position of the source on the dome of the sky. The pseudovector combination \(\hat{p} \times \hat{\lambda}\) can also be considered, which makes the usual unit vector \(\hat{\theta}\) sitting in the tangent plane, transforming like \((\pm 1, 1)\). By the \(j_x\) addition rule for angular momentum, this cannot be combined with the \((2, 2), (-2, 2), (0, 1)\) representations available from \(\hat{\chi}\) and \(\hat{\psi}\) to make new invariants. Using \(\hat{p}\), which transforms like \((0, 1)\), we can take the \((0, 1)\) part of \(\hat{\chi}\) and make a pseudo-scalar, but this is the helicity \(h\) already discussed.

Continuing in this way, products of higher order can always be reduced to sums of smaller dimensional representations, much like the usual decomposition of angular momentum. Some care is needed, however, because the polarization variables are made from incomplete representations because the fields are transverse. One generally then has fewer invariants than straightforward counting using rotation group methods might indicate.

**Non-Local Invariants:** Combining different sources leads to some interesting quantities at low order. There is for example the familiar angle \(\theta_{ij}\) between galaxy \(i\) and galaxy \(j\) on the sky, given in terms of the true scalar \(\cos(\theta_{ij}) = \hat{p}_i \cdot \hat{p}_j\).

More interesting are symmetric and antisymmetric combinations of \(\hat{\chi}_i\) and \(\hat{\chi}_j\), or \(\hat{\psi}_i\) and \(\hat{\psi}_j\). It is straightforward to treat these by standard methods: thus the symmetric combinations

\[
Tr[\hat{\chi}_i \hat{\chi}_j]; \quad Tr[\hat{\chi}_i \hat{\psi}_j]; \quad Tr[\hat{\psi}_i \hat{\chi}_j];
\]

are scalars; while the antisymmetric combinations or commutators

\[
[\hat{\chi}_i, \hat{\chi}_j]; \quad [\hat{\chi}_i, \hat{\psi}_j]; \quad [\hat{\psi}_i, \hat{\psi}_j];
\]

are dual to pseudo-vectors.

The quantities above are involved in questions of angular coherence: for example, if one wants to smooth the angular distribution of sources at different locations, while it would be a bit sloppy to add the parameters such as \(\hat{\chi}\), in practice we are aware only of averaging polarization parameters over single sources of small angular dimensions, in a quasi-local way, so that this should not cause a problem. However if one were making a more ambitious study—for example examining the degree of coherent polarization over larger regions of the sky—then more formal care would be needed. One can, for example, create correlation functions of the matrices in a consistent basis and evaluated at different angular positions, which are then reduced to scalars by taking traces. Another interesting quantity is the \(h_i h_j\), or helicity-helicity correlation function, and obvious generalizations removing the means. This quantity does not need a galaxy axis for its evaluation, and might probe parity-violating effects of the medium evaluated as a function of angular scales. This may be useful for optical polarizations, for example, which often are not associated with axial structure. An application of interest is the cosmic micro-wave background, for which polarization measurements can be expected in the future.

**Distributions**

The classifications are useful in constructing statistical distributions or correlations which can be used to quantify observations. We will use a bracket \((<>\)\) symbol to denote an expectation value in a normalized distribution.

Consider the problem of quantifying the correlations between \(\chi\) and \(\psi\) mentioned in the Introduction. From our results, one would naturally assume the distribution to be a function of the rotationally invariant quantity \(\beta = \chi - \psi\). Given that \(\chi\) and \(\psi\) are defined up to multiples of \(\pi\), the difference \(\beta\) ranges in the most general case over \(2\pi\), and not \(\pi\), as sometimes assumed \([1]\) in the literature. This is seen very simply by making sketches of some trial distributions. Analytically one can expand the distribution \(f(\chi, \psi)\) in Fourier series for integer \(n, m\):

\[
f(\chi, \psi) = \sum f_{nm} \exp(i2n\chi + i2m\psi) = \sum f_{nm} \exp[i(n - m)(\chi - \psi) + i(n + m)(\chi + \psi)]
\]
Reality of $f$ must also be imposed; it prescribes the negative integer values of $f_{nm}$, but does not restrict whether $n \pm m$ is even or odd. The coefficients $f_{nm}$ for odd values of $n - m$ make $\beta = \chi - \psi$ periodic on the interval of $2\pi$, as claimed. Restricting non-zero coefficients to even values of $n - m$ can be motivated by extra assumptions. One sufficient condition is that the distribution obeys overall rotational symmetry, giving $f_{nm}$ going like $\delta(n + m)$. Such an assumption might seem very general but in fact it is not. It is an interesting fact of optics that even a perfectly transmitting (unitary) medium can treat different polarizations dissimilarly, leading to a non-trivial distribution of $\chi + \psi$ as a signal. The same mechanism can create an anisotropic distribution of linear polarizations along a pencil through a medium from an random uncorrelated distributed set of linear polarization emitters. To simplify the analysis we will assume here, however, that the distribution of $\beta$ is periodic on the interval of $\pi$.

To make invariant distributions from invariants, this leaves us with the quantities constructed earlier, namely $s_1 = \cos(2\beta)$ and $p_1 = \sin(2\beta)$. A simple, and indeed well-known distribution that follows is the von Mises form $f_{vM}(\beta) = \text{const} \cdot \exp(k \cos(2\beta)) = \text{const} \cdot \exp(k s_1)$. This distribution has often been used in likelihood tests, but unfortunately without discussion of parity symmetry. By construction, the $vM$ distribution embodies a physical assumption that the twist of one angle relative to another has no preferred parity, which may be unsuitable in some cases. The “shifted” von Mises distribution is similar: $f_{\text{shifted-vM}}(\beta) = \text{const} \cdot \exp(k \cos(2(\beta - \alpha)))$. The parameter $\alpha$ shifts the origin of $\beta$ and might seem to be free. However if parity symmetry is assumed, then $\alpha$ is quite restricted. The distribution is a function of $\cos(2(\beta - 2\alpha)) = \cos(2\beta) \cos(2\alpha) + \sin(2\beta) \sin(2\alpha)$. Parity symmetry requires $f(\beta) = f(-\beta)$, yielding $\sin(2\alpha) = 0$, or $\alpha = 0, \pi/2$. Parity symmetry, then, is sufficient to predict that marginal distributions of linear polarizations tend to be oriented either along galaxy major axes, or perpendicular. This has been a misunderstood point, because of assertions that only the single choice of angular origin relative to the perpendicular is sensible. For any statistic which is covariant under a shift of origin, however, the matter is irrelevant, making no difference. Transforming $\beta \to \beta - \pi/2$ is equivalent to transforming $k \to -k$ in the $vM$ or shifted $vM$ distributions, showing that either choice is equally well described automatically. Similarly, $\sin(2\beta)$ is odd in both $\beta$ and $\beta - \pi/2$, a pseudoscalar no matter how angles are measured. Another example of the issue is underscored by the paper of Loredo et al. using the shifted $vM$ distribution in a detailed likelihood analysis. The paper responded to an odd-parity statistic used in Ref. but exclusively used an even parity shifted $vM$ model in its analysis. In replacing the original statistic by one with opposite transformation properties, conclusions were drawn on a false basis. This illustrates the problem that can occur when using functions from a different symmetry class than the idea being tested.

There is in fact a long history of mix-ups from lumping together observables of different parity, which has led to interesting contention in the literature. Birch [14] in 1982 empirically observed a pseudovector correlation in radio polarizations, odd in the difference ($\chi - \psi$). Kendall and Young (KY) made a model distribution to explore this [15], choosing $vM$ for the null distribution of $\beta$. This is an implicit assumption of parity conservation in the null, which is quite physical (but which should be stated explicitly). KY then set up a conditional correlation function $C_{KY}(\beta, \hat{p}) = \exp(\mu \cdot \hat{p} \sin(2\beta))$ which involved odd-parity to test Birch’s pseudovector result. Unfortunately Kendall and Young did not explain the reason for their ansatz, which in retrospect was entirely appropriate. The overall $KY$ distribution finally can be written in a nicely compact form

$$f_{KY}(\chi, \psi, \hat{p}) = \exp(k \cos(2\beta) + \mu \cdot \hat{p} \sin(2\beta)) = \exp(k s_1 + \mu p_1)$$

where $s_1$ and $p_1$ are our scalar and pseudoscalars from the first section, making it clear that $\mu$ is a parity-violation anisotropic correlation parameter. Likelihood analysis, for example, can be used in a perfectly objective way to see whether any parity-violating effects might be present or not in a data set, and entirely separate from the need to quantify the marginal distribution of $\beta$. It is interesting that Kendall and Young’s data analysis along such lines then indicated a high statistical significance for the parity-violating effect Birch had observed; more recent work in this regard can also be cited [13]. However, other independent studies which used invariant correlations lumping together different parity seemed to contradict the result, a topic to which we now turn.

**Correlations**

One of the best-known statistical correlation tests comes from an influential paper by Jupp and Mardia [16]. Their prescription for correlation between 2 angular quantities involves mapping them into ”vectors” $v, w$, that is covariantly transforming elements, of which $\hat{\chi}$ and $\hat{\psi}$ are examples we have already seen. To test for correlations, $JM$ use canonical $p \times q$ correlation matrices

$$\Sigma_{ij}^{vw} = \langle (v_i - < v_i >)(w_j - < w_j >) \rangle$$

which behave under the separate transformations of $v, w$ as the indices would indicate. Then the $JM$ correlation test is to calculate

$$n \rho_{v,w}^2 = n \cdot T_{\chi,\psi} \Sigma_{ij}^{vw} \Sigma_{ij}^{vw} = n \cdot T_{\chi,\psi} \Sigma_{ij}^{vw} \Sigma_{ij}^{vw}$$

where $\Sigma_{ij}^{vw}$ is our scalar and pseudoscalars from the first section, making it clear that $\mu$ is a parity-violation anisotropic correlation parameter. Likelihood analysis, for example, can be used in a perfectly objective way to see whether any parity-violating effects might be present or not in a data set, and entirely separate from the need to quantify the marginal distribution of $\beta$. It is interesting that Kendall and Young’s data analysis along such lines then indicated a high statistical significance for the parity-violating effect Birch had observed; more recent work in this regard can also be cited [13]. However, other independent studies which used invariant correlations lumping together different parity seemed to contradict the result, a topic to which we now turn.

**Correlations**

One of the best-known statistical correlation tests comes from an influential paper by Jupp and Mardia [16]. Their prescription for correlation between 2 angular quantities involves mapping them into ”vectors” $v, w$, that is covariantly transforming elements, of which $\hat{\chi}$ and $\hat{\psi}$ are examples we have already seen. To test for correlations, $JM$ use canonical $p \times q$ correlation matrices

$$\Sigma_{ij}^{vw} = \langle (v_i - < v_i >)(w_j - < w_j >) \rangle$$

which behave under the separate transformations of $v, w$ as the indices would indicate. Then the $JM$ correlation test is to calculate

$$n \rho_{v,w}^2 = n \cdot T_{\chi,\psi} \Sigma_{ij}^{vw} \Sigma_{ij}^{vw} = n \cdot T_{\chi,\psi} \Sigma_{ij}^{vw} \Sigma_{ij}^{vw}$$

where $\Sigma_{ij}^{vw}$ is our scalar and pseudoscalars from the first section, making it clear that $\mu$ is a parity-violation anisotropic correlation parameter. Likelihood analysis, for example, can be used in a perfectly objective way to see whether any parity-violating effects might be present or not in a data set, and entirely separate from the need to quantify the marginal distribution of $\beta$. It is interesting that Kendall and Young’s data analysis along such lines then indicated a high statistical significance for the parity-violating effect Birch had observed; more recent work in this regard can also be cited [13]. However, other independent studies which used invariant correlations lumping together different parity seemed to contradict the result, a topic to which we now turn.
with matrix products indicated, and where \( n \) is the number of data points in the sample. The step of dividing by auto-correlation matrices is used to make the quantity scale-invariant. One easily finds that \( n\rho^2_{vw} = 0 \) if \( v \) and \( w \) are independent, and the distribution of fluctuations in an uncorrelated null distribution has also be obtained \[16\]. Thus \( n\rho^2_{v,w} \) has served for many years as a useful simple test for independence.

Unfortunately Jupp and Mardia, in discussing their correlation test, did not discuss parity and other discrete symmetries. In the problem at hand, to test whether the difference \( \beta \) is correlated with sky position, one might consider the \( JM \) correlation of a “natural vector” \( v = (\cos(2\beta), \sin(2\beta)) \) and \( w = \hat{p} \). This particular combination \( v \) was used to test Birch’s correlation by Bietenholz and Kronberg \[17\]. Recall, however, the even-odd rule of parity for those quantities even-or odd in \( \beta \). It is clear that the above “natural” vectorlike combination \( v \) needlessly mixes two quantities which are of opposite parity. The two pieces are also separately invariant under rotations, and should not be combined into a vector. Ironically, history shows that the mixed parity combination was used while at the same time citing that Birch had a correlation of pseudovector character \[17\]; without recognizing the parity properties of Kendall and Young’s procedure, both their and Birch’s results were then rejected. From our analysis it is sufficient to use \( n\rho^2_{v1,w} \) and \( n\rho^2_{v2,w} \), which allows separate tests of scalar or pseudoscalar kind. One can, of course, also work directly with scalar and pseudoscalar measures such as \( < s_1 > \) and \( < p_1 > \) to explore certain features of data, and create any number of statistics, once the proper transformation properties have been respected.

IV. SUMMARY

We have classified several combinations of the polarization and angular correlation observables under parity and angular momentum. The features of local observables, and their simple appearance in classic distributions and correlation analysis, should be helpful to those interested in the area. While not exploring very far into the non-local quantities, they appear to offer many possibilities for interesting studies. The possibility of exploring parity violation when there are no obvious emission axis variables, as in the case of upcoming polarized cosmic-microwave background observations from ground-based and satellite facilities, seems intriguing and bears further investigation.

ACKNOWLEDGMENTS: This work was supported in part under DOE grant number 85ER40214, by the University of Kansas General Research Fund, and the Kansas Institute for Theoretical and Computational Science/ K*STAR program.
FIG. 1. Schematic diagram of observables. The variables $\chi$ and $\psi$ are orientation angles of the plane of electric field linear polarization, and of a source symmetry axis, respectively. A unit vector pointing towards the source on the dome of the sky (big sphere) is denoted by $\hat{p}$. The polarization density matrix assigns a standard quantity $\vec{s}$ on the Poincaré sphere (small sphere). The projection of $\vec{s}$ in the tangent plane transforms like spin-2 under rotations about the local $\hat{p}$ axis; the component of $\vec{s}$ along $\hat{p}$ is a measure of ellipticity, and transforms under a spin-1 representation.

[1] L. Stodolsky, Phys. Rev. Lett. 34, 110, 1975; Erratum-ibid. 34, 508, 1975.
[2] Gabriel Karl, Can. J. Phys. 54, 568, 1976; J.N. Clarke, G. Karl, and P.J.S. Watson, Can. J. Phys. 60, 1561, 1982.
[3] B. Nodland and J. P. Ralston, Physical Review Letters 78, 3043 (1997).
[4] P. Jain and J. P. Ralston, Kansas preprint 98/07, astro-ph/9803164.
[5] R. D. Peccei and H. Quinn, Phys. Rev. Lett. 38, 1440 (1977); S. Weinberg, Phys. Rev. Lett. 40, 223 (1978); F. Wilczek, Phys. Rev. Lett. 40, 279 (1978); P. Sikivie, Phys. Lett. B137, 353 (1984); D. Harari and P. Sikivie, Phys. Lett. B289, 67 (1992).
[6] W.-T. Ni, Phys. Rev. Lett. 38, 301 (1977), C. Wolf, Phys. Lett. A132, 151,(1988).
[7] M. Mandle and E. Wolf, *Coherence and Quantum Optics*, Cambridge University Press, 1985; L. D. Landau, E. M. Lifshitz, and L.P. Piatevskii, *Electrodynamics of Continuous Media*, 2nd Edition (Pergamon Press 1984). 1995.

[8] E. Batschelet, *Circular Statistics in Biology*, (London: Academic Press, 1981).

[9] N. I. Fisher, *Statistics of Circular Data*, (Cambridge, 1993).

[10] K. Rohls and T. L. Wilson, *Tools of Radio Astronomy*, 2nd Edition (Springer Verlag, 1996).

[11] S. M. Carroll and G. B. Field, *Phys. Rev. Lett.* 79, 2934 (1997); S. M. Carroll, G. B. Field and R. Jackiw, *Phys. Rev. D* 41, 1231 (1990).

[12] K. V. Mardia, *Statistics of Directional Data*, (London: Academic Press, 1972).

[13] T. J. Loredo, E. E. Flanagan, and I. M. Wasserman, astro-ph/9706258 (1997).

[14] P. Birch, *Nature* 298, 451 (1982).

[15] D. G. Kendall and A. G. Young, *Mon. Not. Roy. Ast. Soc.* 207, 637, (1984).

[16] P. E. Jupp and K. V. Mardia, *Biometrika* 67, 163 (1980).

[17] M. F. Bietenholz and P. P. Kronberg, *Astrophys J.* 287, L1-L2 (1984).