Optimal Allocation Rule for Infectious Disease Prevention Under Partial Interference and Outcome-Oriented Targets With Repeated Cross-Sectional Surveys

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Abstract

When developing allocation policies for infectious disease prevention, policymakers often set outcome-oriented targets and measure the policy’s progress with repeated cross-sectional surveys. For example, the United Nations’ Integrated Global Action Plan for Pneumonia and Diarrhea (GAPPD) recommended increasing water, sanitation, and hygiene (WASH) resources to reduce childhood diarrhea incidence and outlined a specific level of diarrhea incidence to achieve. For policymakers, the goal is to design an allocation rule that optimally allocates WASH resources to meet the outcome targets.

The paper develops methods to estimate an optimal allocation rule that achieves these pre-defined outcome-oriented targets using repeated cross-sectional surveys. The estimated allocation rule helps policymakers optimally decide what fraction of units in a region should get the resources based on its characteristics. Critically, our estimated policies account for spillover effects within regions in the form of partial interference, and we characterize the policies’ performance in terms of excess risk. We apply our methods to design Senegal’s WASH policy to prevent diarrheal diseases using the 2014-2017 Demographic and Health Survey. We show that our policy not only outperforms competing policies but also provides new insights about how Senegal should design WASH policies to achieve its outcome targets.

Keywords: Causal inference, Continuous treatment, Demographic and Health Survey, Excess risk, Optimal treatment rule
1 Introduction

1.1 Motivation: Designing Optimal Allocation Policies to Achieve Global Policy Targets for Diarrheal Incidence

In 2013, the United Nations’ International Children’s Emergency Fund and the World Health Organization launched the Integrated Global Action Plan for Pneumonia and Diarrhea (GAPPD) to reduce under-5 mortality rates in developing countries by 2030 (World Health Organization, 2013). To achieve this goal, GAPPD laid out specific targets for countries to achieve by 2030, notably reducing the incidence of severe diarrhea by 75% compared to existing country-specific levels in 2010, and these targets are monitored using annual, cross-sectional surveys, say the annual Demographic and Health Survey (DHS) (ANSD and ICF, 2020) (see below). Also, following existing works on diarrhea prevention, GAPPD recommends increasing household access to water, sanitation, and hygiene (WASH) resources to reduce rates of diarrhea-related diseases among children. But, existing allocation policies of WASH resources are often informed by political factors, not evidence-based, and have led to sub-optimal outcomes (Easterly, 2002, Chapter 5 of Evans and Mara, 2011; UN-Water and WHO, 2017). The main theme of this paper is (i) to develop methods to estimate optimal resource allocation rules to achieve outcome-oriented targets using existing repeated cross-sectional surveys and (ii) to use these methods to inform how Senegal, one of the nations in the GAPPD coalition, should allocate WASH resources to its census blocks.

As of 2020, Senegal lags behind other sub-Saharan nations in terms of having well-maintained sanitation and handwashing facilities (UN-Water, 2021), and the lack of WASH resources in Senegal has historically contributed to high, persistent levels of risk factors associated with children’s diarrheal diseases (Okeke, 2009). For Senegal, the ideal WASH policy would be to exhaustively install WASH facilities in every household. But, this is an impossible task for budgetary concerns, and consequently, Senegalese policymakers and international development organizations need a more targeted allocation of WASH facilities. Specifically, given (a) a target goal for diarrhea incidence, say the desired, minimum diarrhea-free incidence is $T \in [0, 1]$ where $T$ is from GAPPD’s targets, and (b) characteristics about a census block and households within the census block, a targeted allocation rule would determine what fraction $\alpha \in [0, 1]$ of households in the census block needs WASH facilities to achieve the target goal. For example, a block that is economically well-off may
not need any WASH facilities (i.e., \( \alpha = 0 \)) to meet the threshold \( T \), whereas another block that is economically worse-off may need 70% of its households to have WASH facilities (i.e., \( \alpha = 0.7 \)) to meet the same threshold. Critically, installing a sufficient fraction of WASH facilities in a block is believed to have a multiplying, positive spillover effect for all households in the same census block by reducing the chance that pathogens responsible for diarrhea contaminate communal water sources (Benjamin-Chung et al., 2018). This phenomenon is akin to reaching herd immunity in vaccine studies. Our proposed allocation rule considers this last part by combining recent methods in personalized medicine and partial interference in causal inference.

To train our allocation rule, we use the data provided by Senegal’s DHS (ANSD and ICF, 2020), an annual national survey on health and demography. Among other things, Senegal’s DHS contains information about (i) the number of WASH facilities, defined as having a private water source with a flushable toilet in a house, (ii) children’s diarrhea status, and (iii) other demographic and socioeconomic characteristics, all measured at the household level. We use the data from 2014 to 2017, which consist of 13556 total households from 1027 census blocks, to train our allocation rule and test its performance on the data from 2018; see Section 5 for additional details.

### 1.2 Prior Works & Our Contribution

Our work broadly fits into the literature on optimal treatment regimes (Murphy, 2003; Robins, 2004; Hirano and Porter, 2009; Zhao et al., 2012; Qian and Murphy, 2011; Zhang et al., 2012; Kitagawa and Tetenov, 2018), notably works by Laber and Zhao (2015), Chen et al. (2016), Schulz and Moodie (2021), and Chen et al. (2022) with non-binary treatment. However, there are some important differences between the existing work on optimal treatment regimes and our work. First, as noted above, having a sufficient number of households with WASH facilities is believed to have a social multiplier effect for a region. This phenomenon is known as partial interference in causal inference (Cox, 1958; Sobel, 2006; Hudgens and Halloran, 2008) where interference occurs within clusters but not across clusters. Under partial interference, the optimal regime cannot be determined by only focusing on a single unit because other units’ treatment status may affect the outcome of the said unit. Recent works by Su et al. (2019) and Viviano (2021) considered Q- and A-learning to optimally assign treatment under interference. But, Su et al. (2019) required the response model to be linear, and both works used a binary treatment variable. In contrast, our work deals with a
continuous treatment variable (i.e., $\alpha \in [0, 1]$) and uses nonparametric, doubly robust estimators.

Second, as we elaborate in Section A.8 of the Supplementary Material, from many prior works, there are minimal, if any, negative externalities of installing WASH facilities with respect to children’s diarrhea status, and we incorporate this useful, side information as a monotonic, shape constraint on the response function. Third, often in public policy, there is a minimum, pre-defined target that policymakers want to achieve (e.g. GAPPD’s minimum targets for diarrhea incidence) and, to our knowledge, the current literature on optimal treatment regimes do not use this information even if it is available (Chen et al., 2022). Finally, our allocation rules estimate the smallest fraction of households in a census block that should get WASH facilities in order to achieve a target but do not tell which households in the census block should get WASH facilities. For example, our rule may suggest that a block needs 60% of its households to install WASH facilities in order to achieve at least 70% diarrhea-free incidence. But, the rule does not suggest which households in the block should install WASH facilities. We believe household-level allocation rules given a limited amount of resources, say Luedtke and van der Laan (2016) modified to account for interference or, more recently, Kitagawa and Wang (2021) with an existing epidemiological model can complement our more “macro-level” resource allocation rule.

The paper presents two allocation rules under partial interference with continuous treatment and a target threshold. The first allocation rule is a modest extension of the indirect method (Chakraborty and Moodie, 2013) to partial interference, where only the outcome model that incorporates partial interference is used to estimate the allocation rule indirectly; we call this the indirect rule in the paper. The second allocation rule (and our recommended rule for practice) directly estimates the allocation rule by recasting our problem as an instance of estimating an optimal dose in Chen et al. (2016) with weights derived from partial interference and a threshold term to reflect the pre-defined target objective; we call this the direct rule. Specifically, for the direct rule, we propose a novel loss function to measure the different effects of treatment under partial interference; see Section 3.3. The paper also discusses a tempting “interference-free” estimator that analyzes the data at the block level and demonstrates that this approach will almost always lead to misleading estimates of the allocation rules.

Finally, while the proposed methods are developed to address Senegal’s WASH allocation policy, the proposed methods may have broader applications beyond WASH allocation. For example, the
methods may be useful for estimating the minimal fraction of individuals that need to be vaccinated in a region to achieve a disease-free incidence of $T$. Unlike estimators based on parametric, epidemic differential equation models, say the susceptible-infectious-recovered (SIR) model (Magal and Ruan, 2014; Kitagawa and Wang, 2021), our estimators use principles from causal inference, notably confounding adjustment, interference, and double robustness, and we believe leveraging both approaches to obtain epidemiologically important statistics can better inform public health policy.

2 Setup

2.1 Review: Notation, Interference, and Overall Outcomes

Consider the study unit to be households and each household belongs to one of the census blocks in Senegal. Let $N$ be the number of census blocks and let each census block be indexed by $i = 1, \ldots, N$. For each block $i$, let $n_i$ be the number of households from block $i$ and let each household be indexed by $j = 1, \ldots, n_i$. We assume $n_i$ is bounded above by a constant $M$. Practically, this is plausible in our data as the average block size is 13.2, which is much smaller than the number of blocks (i.e., $N = 1027$). Also, the Senegal DHS uses this assumption in their stratified sampling strategy where a fixed number of houses are sampled from each block (ANSD and ICF, 2020). Theoretically, this assumption allows us to use standard asymptotics under partial interference; see Section 3.5.

For each household $j$ in block $i$, we observe $O_{ij} = (Y_{ij}, A_{ij}, X_{ij})$ where $Y_{ij}$ is a binary indicator on whether all children in household $ij$ are diarrhea-free or not, $A_{ij}$ is a binary indicator on whether the household has a WASH facility or not, and $X_{ij}$ consists of the following nine household- and block-level characteristics: block size ($n_i$), indicator on whether census block $i$ is located in an urban area, number of household members, number of children in household $i$, indicator on whether both parents do not have jobs, indicators on whether parents have ever attended schools, mother’s age, and average age of children in household $i$.

For each census block $i$, let $O_i = (Y_i, A_i, X_i)$ be the collection of all household data from block $i$. Let $S_i = \sum_{j=1}^{n_i} A_{ij}$ be the sum of treatment variables in block $i$ and let $S_i(-j) = \sum_{\ell \neq j} A_{i\ell}$ be the sum of the treatment variables in block $i$ that excludes household $j$. Also, let $A_{i(-j)}$ be the vector of treatment for all households in block $i$ that excludes household $j$. Without loss of generality, we
assume larger outcomes are preferred.

We use the potential outcomes notation of Neyman (1923) and Rubin (1974) to define causal effects. Let $\mathcal{A}(t)$ be a collection of $t$-dimensional binary vectors (e.g., $\mathcal{A}(2) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$). For a treatment vector $a_i \in \mathcal{A}(n_i)$, let $Y_{ij}^{(a_i)} = Y_{ij}^{(a_{ij}, a_{(i)(-j)})}$ be the potential outcome of household $j$ in block $i$. Critically, departing from the usual setup in optimal treatment regimes, the potential outcome $Y_{ij}^{(a_{ij}, a_{(i)(-j)})}$ allows for partial interference from other households in block $i$. Following the works on partial interference (Hudgens and Halloran, 2008; Tchetgen Tchetgen and VanderWeele, 2012), we define the expected overall potential outcome under policy $\alpha \in [0, 1]$ as

$$
\tau_{OV}(\alpha) = E\left\{\frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a_i \in \mathcal{A}(n_i)} Y_{ij}^{(a_i)} \pi(a_i; \alpha)\right\}, \pi(a_i; \alpha) = \prod_{j=1}^{n_i} \alpha^{a_{ij}}(1 - \alpha)^{1 - a_{ij}}. \tag{1}
$$

In words, $\tau_{OV}(\alpha)$ measures the effect of intervening on a census block with a policy that assigns treatment to households in the block with probability $\alpha$. VanderWeele and Tchetgen Tchetgen (2011) and Halloran (2019) show that $\tau_{OV}(\alpha)$ measures the entire effect of a policy applied at the block-level by accounting for both the direct effect and the spillover effect of treatment. Notably, following Halloran (2019)’s recommendation on using the overall potential outcome to evaluate policies under interference, we use the overall potential outcome as our primary outcome to optimize in our allocation rule.

Also, we define the following secondary outcome of interest, which we call the “spillover outcome under policy $\alpha$,” and denoted as $\tau_{SO}(\alpha)$ below:

$$
\tau_{SO}(\alpha) = E\left\{\frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a_i \in \mathcal{A}(n_i-1)} Y_{ij}^{(a_{ij}=0,a_{(i)(-j)})} \pi(a_{i(-j)}; \alpha)\right\}. \tag{2}
$$

In words, $\tau_{SO}(\alpha)$ is the expected outcome of the census block when the ego is untreated, but their peers are treated with probability $\alpha$; see Sections 3 and 4 of VanderWeele and Tchetgen Tchetgen (2011) for additional discussion of the estimand. While not the primary outcome of interest in this paper, comparing the optimal allocation rule under $\tau_{SO}(\alpha)$ with that under $\tau_{OV}(\alpha)$ can help investigators explain the mechanism behind the optimal policy to policymakers; see Section 5.2. Also, deriving the optimal allocation rule under $\tau_{SO}(\alpha)$ allows us to highlight our methods’ broad applicability to a wide range of common, interference-specific potential outcomes. In particular, we hope the derivation below is useful for researchers who are interested in outcomes beyond the
Finally, we use the following notations for norms, limits, and common functions. Let $\|v\|_q$ be the $q$-norm of a vector for $q \geq 1$ and let $\|f\|_{P,q} = \left\{ \int \|f(a_i)\|_2^q dP(a_i) \right\}^{1/q}$ be the $L_q(P)$-norm of a function $f$ for $q \geq 1$. For a set $\mathcal{D} \subset \{1, \ldots, N\}$, let $\mathcal{D}^c$ be its complement. Let $\lfloor \cdot \rfloor$ be the floor function. Let the bar notations denote block-level averages, i.e., $\bar{V}_{ij} = \sum_{n_i}^{n_i} V_{ij}$ and $\bar{V}_{i(-j)} = \sum_{t \neq j} V_{it} / (n_i - 1)$. Let $\sum_j$, $\sum_{j,s}$, and $\sum_{j,a,s}$ be shorthand for $\sum_{j=1}^{n_i}$, $\sum_{j=1}^{n_i} \sum_{s=0}^{n_i-1}$, and $\sum_{j=1}^{n_i} \sum_{a=0}^{1} \sum_{s=0}^{n_i-1}$, respectively. Finally, let $O(\cdot)$, $O_P(\cdot)$, $o(\cdot)$, and $o_P(\cdot)$ be the usual big-O and small-O notations.

## 2.2 Assumptions for Causal Identification and Estimation

To identify the overall potential outcome in (1), we make the following assumptions.

(A1) **Consistency:** $Y_{ij} = \sum_{a_i \in \mathcal{A}(n_i)} 1(A_i = a_i) Y_{ij}^{(a_i)}$.

(A2) **Conditional Ignorability:** $Y_{ij}^{(a_i)} \perp A_i \mid X_i$ for all $a_i$.

(A3) **Overlap:** For some $c > 0$, we have $c < P(A_i = a_i \mid X_i)$ for any $a_i \in \mathcal{A}(n_i)$ and $X_i$.

Assumptions (A1)-(A3) are natural extensions of consistency/Stable Unit Treatment Value Assumption (SUTVA) (Rubin, 1976, 1978), conditional ignorability, and overlap to partial interference; see Imbens and Rubin (2015) and Hernán and Robins (2020) for textbook discussions. Under Assumptions (A1)-(A3), we can identify the causal estimands in (1) and (2) from the observed data as

$$
\tau_{OV}(\alpha) = E\left\{ \frac{1}{n_i} \sum_{a_i \in \mathcal{A}(n_i)} E(Y_{ij} \mid A_i = a_i, X_i) \pi(a_i ; \alpha) \right\}
$$

$$
\tau_{SO}(\alpha) = E\left\{ \frac{1}{n_i} \sum_{a_{i(-j)} \in \mathcal{A}(n_i-1)} E(Y_{ij} \mid A_{ij} = 0, A_{i(-j)} = a_{i(-j)}, X_i) \pi(a_{i(-j)} ; \alpha) \right\}.
$$

While Assumptions (A1)-(A3) are sufficient for causal identification, actually estimating it, specifically the outcome regression $E(Y_{ij} \mid A_i, X_i)$, can be challenging due to the curse of dimensionality and some assumptions are required for estimation.

To this end, we make two additional assumptions about the outcome regression. The first assumption is a type of “exposure mapping” in Aronow and Samii (2017) where the response of a household in block $i$ is a function of $A_{ij}$ and $\overline{A}_{i(-j)}$, a lower-dimensional summary of the treatment
vector $A_i$, and all covariates in the same block.

(A4) **Conditional Stratified Interference:** There exists a function $\mu^*$ that satisfies $E(Y_{ij} \mid A_i, X_i) = \mu^*(A_{ij}, \overline{A}_{i(-j)}, X_{ij}, X_{i(-j)})$ for any $A_i$ and $X_i$.

Assumption (A4) is closely related to the stratified interference assumption of Hudgens and Halloran (2008) where after fixing ego’s treatment $A_{ij}$ and the proportion of peers’ treatment $\overline{A}_{i(-j)}$, the average response in a block is invariant to who actually received treatment. Also, Assumption (A4) allows nonparametric models with interactions between $(A_{ij}, \overline{A}_{i(-j)}, X_i)$. However, Assumption (A4) can be violated if a household’s response depends on the treatment status of a few, focal neighbors. Theoretically, variants of Assumption (A4) have been used in works on partial interference to achieve consistency, non-degenerate asymptotic limits, and/or consistent estimation of variance (Liu and Hudgens, 2014; van der Laan, 2014; Sofrygin and van der Laan, 2016; Bargagli-Stoffi et al., 2020), including the recent work on optimal treatment regimes under interference (Viviano, 2021).

The second assumption on the outcome regression formalizes the useful side information in Section 1.2 about the relationship between diarrheal incidence and WASH facilities.

(A5) **Monotonic Response:** $\sum_j \mu^*(a, a', X_{ij}, X_{i(-j)})/n_i$ is non-decreasing in $(a, a')$ for any $X_i$.

Assumption (A5) implies that the average number of diarrhea-free households in a census block is a non-decreasing function of the number of WASH facilities; note that a sufficient condition of Assumption (A5) is that $\mu^*(a, a', X_{ij}, X_{i(-j)})$ itself (and not its average) is non-decreasing in $(a, a')$. Assumption (A5) would be plausible if the treatment is harmless, as is the case for WASH facilities, but would be violated if some study units have severe side effects from the treatment. Overall, Assumptions (A4) and (A5) are used for estimating the optimal rule and unlike Assumptions (A1)-(A3), they are not required for identification.

### 2.3 Problem Statement

Under Assumptions (A1)-(A4), the expected overall outcome can be rewritten as

$$
\tau_{OV}(\alpha) = E\left\{\frac{1}{n_i} \sum_{j,a,s} \left(\frac{n_i - 1}{s}\right) \mu^*(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)}) \alpha^a (1 - \alpha)^{n_i - a - s}\right\}. 
$$

Let $\Theta = \{\theta \mid \theta(x) \in [0, 1]\}$ be a collection of functions from the support of $X_i$ to $[0, 1]$. A function
θ ∈ Θ is one possible allocation rule where it outputs the proportion of WASH facilities that block \( i \) should have based on both block- and household-level characteristics. Then, we define \( \tau_{OV}(\theta(X_i)) \) as the expected overall outcome under the allocation rule \( \theta \). Let \( T \in [0,1] \) be the target outcome informed by public policy goals, say the proportion of diarrhea-free children should be at least 70%.

Given the target \( T \) and Assumptions (A1)-(A5), the optimal allocation rule that achieves \( T \) is the rule \( \theta^*_{OV}(x_i) \in \Theta \) where

\[
\theta^*_{OV}(x_i) = \inf_{\alpha \in [0,1]} \left\{ \alpha \left| \frac{1}{n_i-1} \sum_{j,a,s} \left( n_i - 1 \right) \mu^*(a, \frac{s}{n_i-1}, x_{ij}, x_{i(j)}(\alpha)) \alpha^{a+s}(1 - \alpha)^{n_i-a-s} \geq T \right\}. \tag{4}
\]

In words, \( \theta^*_{OV}(x_i) \) is the minimum proportion of households with WASH facilities needed in census block \( i \) in order to achieve the target, diarrhea-free incidence level of \( T \); we refer to \( \theta^*_{OV} \) as the optimal minimal allocation rule (OMAR) for the overall outcome. If the set in (4) is empty, we let \( \theta^*_{OV}(x_i) = 1 \) to maximize the expected average outcome.

Similarly, we define the OMAR associated with the spillover outcome \( \tau_{SO}(\alpha) \) as

\[
\theta^*_{SO}(x_i) = \inf_{\alpha \in [0,1]} \left\{ \alpha \left| \frac{1}{n_i-1} \sum_{j,s} \left( n_i - 1 \right) \mu^*(0, \frac{s}{n_i-1}, x_{ij}, x_{i(j)}(\alpha)) \alpha^s(1 - \alpha)^{n_i-1-s} \geq T \right\}.
\]

In words, \( \theta^*_{SO}(x_i) \) is the minimum proportion of neighboring households with WASH facilities needed in order to achieve the target \( T \) when the ego’s household does not have WASH facilities. Under monotonicity (i.e., Assumption (A5)), \( \theta^*_{OV}(x_i) \) cannot be smaller than \( \theta^*_{SO}(x_i) \).

3 Estimation of the Optimal Minimum Allocation Rule

3.1 The (Mostly) Wrong Approach: Analysis With Aggregated, Block-Level Data

Before we present our solutions to the problem, we briefly discuss a tempting approach based on aggregating the data at the block-level. This aggregation approach has been discussed in the literature (e.g. Section 2.3 of Imbens and Wooldridge (2009)) as a simple way to deal with interference. While this approach will clearly not work for estimating OMARs like \( \theta^*_{SO}(x_i) \) or other OMARs targeting spillover-specific outcomes, from a practitioner’s point of view, it is worth asking whether this approach can be used to estimate, or at least approximate, OMARs like \( \theta^*_{OV}(x_i) \) which com-
bine both the direct and spillover effects of treatment in a block. Unfortunately, as we illustrate below, this aggregation approach will lead to grossly misleading estimates of $\theta^{*}_{OV}(x_i)$ except in very restrictive settings.

Formally, following the above advice from the literature, suppose an investigator attempts to bypass the problem from interference at the unit/household-level by aggregating their data at the cluster/block-level. That is, for each block $i$, the investigator can consider $O_i = (Y_i, A_i, X_i)$ to be the available data and use existing techniques in the optimal treatment regime literature for a continuous treatment, say Chen et al. (2016), to obtain the minimum proportion of WASH facilities necessary to achieve a certain target $T$. For example, given a block-level outcome model for the expected value of $Y_i$ as a function of block-level variables $A_i$ and $X_i$, the investigator can find the smallest $A_i \in [0, 1]$ where the expected outcome exceeds $T$.

Despite its simplicity, the above analysis is only appropriate in very restrictive settings, which we illustrate with an example. Suppose the treatment assignment depends on the measured covariates, and the outcome regression is given as $E\{Y_i(a_{ij}, a_{i(-j)}) \mid X_i\} = \beta_1 a_{ij} + \beta_2 a_{i(-j)} + \beta_3^T X_{ij} a_{ij} + \beta_4^T X_{ij} \vec{a}_{i(-j)}$ where $\beta_1, \ldots, \beta_4$ are non-negative coefficients to guarantee Assumption (A5). Some algebra reveals the average potential outcome at the block level is

$$E\{Y_i(a_{ij}, a_{i(-j)}) \mid X_i\} = \left( \beta_1 + \beta_2 + \frac{n_i \beta_3^T X_i}{n_i - 1} \right) \vec{a}_i + \left\{ \frac{(n_i - 1) \beta_3 + \beta_4}{n_i - 1} \right\}^T \left( \frac{1}{n_i} \sum_{j=1}^{n_i} a_{ij} X_{ij} \right).$$

If the investigator uses the aggregated, block-level data to estimate the OMAR, the resulting estimate will be biased because the block-level outcome model of $Y_i$ given $A_i$ and $X_i$ is misspecified. Or equivalently, there is an omitted variable bias because of the term $\sum_{j=1}^{n_i} a_{ij} X_{ij}$, which roughly measures the covariance between the household-level treatment variable and the household-level covariate. The magnitude and the direction of the bias will depend on (a) the magnitude of treatment effect heterogeneity, as measured by $\beta_3$ and $\beta_4$, and (b) the magnitude and the sign of the measured, household-level confounding, as measured by the covariance of $A_{ij}$ and $X_{ij}$. More generally, if the outcome model is nonlinear, which is often the case in popular epidemiological models (e.g. Magal and Ruan (2014)), no amount of modeling with aggregated, block-level data $(Y_i, A_i, X_i)$ will completely remove this bias as the block-level data cannot capture both household-level treatment heterogeneity and household-level confounding; see Section A.1 of
the Supplementary Material for details.

In summary, an analysis based on aggregated, block-level data will often lead to biased estimates of OMAR. Also, if the investigator is interested in OMARs based on spillover-specific outcomes say $\theta^*_\text{SO}(x_i)$, a block-level analysis is simply infeasible.

### 3.2 An Indirect Approach Via Outcome Modeling

Next, we present a correct, but naive solution where we indirectly use the outcome regression model to find the optimal rule, sometimes referred to as the “indirect” approach (Chakraborty and Moodie, 2013). Formally, let $\hat{\mu}$ be an estimate of $\mu^*$; see Section 3.4 on how to estimate $\mu^*$ under partial interference. Then, an estimate of $\theta$ for a given $x_i$, denoted as $\hat{\theta}_{\text{IND}}(x_i)$, is obtained by replacing $\mu$ with $\hat{\mu}$ in (4) and doing a grid search on the unit interval.

$$
\hat{\theta}_{\text{OV,IND}}(x_i) = \inf_{\alpha \in [0,1]} \left\{ \alpha \left| \frac{1}{n_i} \sum_{j,a,s} \left( n_i - 1 \right) \hat{\mu} \left( a, \frac{s}{n_i - 1}, x_{ij}, x_{i(-j)} \right) \alpha^{a+s}(1-\alpha)^{n_i-a-s} \geq T \right\},
\hat{\theta}_{\text{SO,IND}}(x_i) = \inf_{\alpha \in [0,1]} \left\{ \alpha \left| \frac{1}{n_i} \sum_{j,s} \left( n_i - 1 \right) \hat{\mu} \left( 0, \frac{s}{n_i - 1}, x_{ij}, x_{i(-j)} \right) \alpha^{s}(1-\alpha)^{n_i - 1 - s} \geq T \right\}. \tag{5}
$$

We make a few remarks about the indirect allocation rule $\hat{\theta}_{\text{IND}}$, all of which are well-known either in the literature on optimal treatment regimes. First, $\hat{\theta}_{\text{IND}}$ does not yield a closed, functional form of the allocation rule. Second, similar to prior observations on indirect methods under no interference, $\hat{\theta}_{\text{IND}}$ may have poor finite-sample properties compared to our preferred approach below that directly estimate $\theta^*$ (Zhao et al., 2012) and we reconfirm this observation under partial interference in Sections 4 and 5.

### 3.3 A Direct Approach Via Risk Minimization

In this section, we propose our preferred approach based on directly estimating the OMAR. We do this by recasting OMAR as a solution to a risk minimization problem with a specialized loss function $L(\theta, O_i)$ tailored for partial interference. Notably, by reframing the original problem as a risk minimization problem, we can use an array of methods in empirical risk minimization to directly obtain an estimate of $\theta^*$. 

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Formally, for a given \( t \in \mathbb{R} \) and data \( \mathbf{O}_i \), consider the following loss function:

\[
L_z(t, \mathbf{O}_i) = \begin{cases} 
\nu_z(0, \mathbf{O}_i) + \delta - \delta e^t & \text{if } -\infty < t < 0 \\
\nu_z(t, \mathbf{O}_i) & \text{if } 0 \leq t \leq 1 \\
\nu_z(1, \mathbf{O}_i) + \delta - \delta e^{-t+1} & \text{if } 1 < t < \infty
\end{cases}
\]

where \( \nu_z(t, \mathbf{O}_i) \) is defined as:

\[
u_z(t, \mathbf{O}_i) = \begin{cases} 
\frac{1}{n_i} \sum_{j,a,s} (n_i-1) \psi_{\text{DR}}(a, s, \mathbf{O}_{ij}, \mathbf{O}_{i(j)}) \sum_{\ell=0}^{n_i-a-s} \left( \frac{\ell}{\ell + a + s + 1} \right) - \mathcal{T} t + C_0 & \text{if } z = \text{OV} \\
\frac{1}{n_i} \sum_{j,s} (n_i-1) \psi_{\text{DR}}(0, s, \mathbf{O}_{ij}, \mathbf{O}_{i(j)}) \sum_{\ell=0}^{n_i-1-s} \left( \frac{s}{s + a + s + 1} \right) - \mathcal{T} t + C_0 & \text{if } z = \text{SO}
\end{cases}
\]

The reason behind the peculiar form of the loss function is due to the following result.

**Lemma 3.1 (Equivalence Lemma).** For \( z \in \{\text{OV}, \text{SO}\} \), the OMAR \( \theta^*_z \) is the minimizer of the risk

\[
R_z(\theta) := \mathbb{E}\{L_z(\theta(\mathbf{X}_i), \mathbf{O}_i)\}, \quad \text{i.e., } R_z(\theta^*_z) \leq R_z(\theta) \text{ for all } \theta \in \Theta.
\]

Hereafter, we suppress the subscript \( z \) for notational brevity, and the established results in the subsequent sections apply to both types of outcomes and corresponding OMARs.

Methodologically, Lemma 3.1 provides a direct approach to estimating \( \theta^* \) by solving for the minimizer of the risk function \( R \). In particular, the connection to risk minimization allows investigators to use a variety of methods under empirical risk minimization with the proposed loss function \( L \). As a concrete example, consider a support vector machine (SVM) estimator of \( \theta^* \).
using a Gaussian kernel $K(x, x’) = \exp\left(-\frac{\|x - x’\|^2}{2\gamma^2 N}\right)$ with bandwidth $\gamma_N$. Let $\mathcal{H}_K$ denote the corresponding reproducing kernel Hilbert space (RKHS). Of note, a Gaussian kernel implicitly implies that the domain has a unified dimension $d$ for all $i$; see Section 3.4 for further discussions.

If a Gaussian kernel is not appropriate, other kernel functions that are symmetric, continuous, and positive definite can be used. The representer theorem (Kimeldorf and Wahba, 1970; Schölkopf et al., 2001) states that the estimated rule can be written as

$$\tilde{\theta}(x) = \sum_{j=1}^{N} \tilde{\eta}_j K(x, x_j) + \tilde{b}$$

where $\tilde{\eta}_j$ and $\tilde{b}$ solve

$$\min_{\theta \in \mathcal{H}_K} \left\{ \frac{1}{N} \sum_{i=1}^{N} L(\theta(X_i), O_i) + \frac{\lambda_N}{2} \|\theta\|_{\mathcal{H}_K}^2 \right\} = \min_{\eta, b} \left\{ \frac{1}{N} \sum_{i=1}^{N} L(k_i^T \eta + b, O_i) + \frac{\lambda_N}{2} \eta^T K \eta \right\}. \tag{7}$$

Here, $\|\theta\|_{\mathcal{H}_K}$ is a seminorm of $\theta$ in $\mathcal{H}_K$, $\lambda_N$ is a regularization parameter, $\eta = (\eta_1, \ldots, \eta_N)^T \in \mathbb{R}^N$, $k_i = [K(x_i, x_1), \ldots, K(x_i, x_N)]^T \in \mathbb{R}^N$ for $i = 1, \ldots, N$, and $K = [k_1, \ldots, k_N] \in \mathbb{R}^{N \times N}$. Note that the optimization problem in (7) is nonconvex, but this particular nonconvex function can be efficiently solved; see Section A.4 of the Supplementary Material for details. Finally, we winsorize the estimated rule $\tilde{\theta}$ from (7) to satisfy the bound restrictions for policy $\alpha$, i.e., $\tilde{\theta} = W(\tilde{\theta})$ where $W(\tilde{\theta}) = \tilde{\theta} 1\{\tilde{\theta} \in [0, 1]\} + 1\{\tilde{\theta} > 1\}$. Investigators can then use the winsorized rule $\hat{\theta}$ to optimally allocate WASH facilities.

We conclude by briefly discussing how to choose the tuning parameters $\gamma$ and $\lambda$ in the SVM; for additional details, see Steinwart and Christmann (2008) and Hastie et al. (2009). The tuning parameter $\gamma$ controls the bandwidth of the SVM kernel whereas the tuning parameter $\lambda$ is the regularization term for the SVM coefficients $\eta$ in (7). While there are theoretically optimal values of $\gamma$ and $\lambda$ (see Theorem 3.1 below), in practice, cross-validation is often used to choose these values where we would start with a grid of values for $\gamma, \lambda$ and find the value that minimizes the cross-validated empirical risk. Our empirical analysis also uses cross-validation to choose $\gamma, \lambda$ and Section A.5 of the Supplementary Material lays out the computational details for completeness.

### 3.4 Estimating the Nuisance Functions

Some care must be exercised when estimating the two nuisance functions, the outcome regression and the propensity score, under partial interference. For example, for the outcome regression $\mu^*$, naively using some popular nonparametric (or parametric) regression methods where $Y_{ij}$ is
the dependent variable and \((A_{ij}, A_{i(-j)}, X_{ij}, X_{i(-j)})\) are the independent variables will often be infeasible because the dimension of \(X_{i(-j)}\) varies according to the number of peers, which may be different across census blocks. Thankfully, this issue is not new in modern machine learning and we briefly review some common solutions. Also, we remark that similar to recent literature on causal inference with machine learning (e.g. Chernozhukov et al. (2018)), our theoretical results are agnostic to specific estimators of the nuisance functions so long as they are estimated consistently; see Section 3.5.

We start with estimating the outcome regression \(\mu^*\). In data-rich environments where \(N\) is large and \(\mu^*\) is estimated using overparametrized neural networks, a common trick is to pad the variable length \(X_{i(-j)}\) with zeros so that all the independent variables have identical length across blocks (Goodfellow et al., 2016, Chapter 10). Alternatively, in data-scarce environments where \(N\) is small to moderate or investigators prefer simpler, nonparametric methods, it’s common to modify the original \(X_{i(-j)}\) to have a common dimension \(d\) and then use “classic” nonparametric regression estimators, say the Nadaraya-Watson kernel regression estimator (Nadaraya, 1964; Watson, 1964). Some common ways of modifying \(X_{i(-j)}\) in the literature include (a) taking a random subsample of a fixed number of households from each block, (b) using covariates from the “average” or the “extreme” households, say \(X_{i(-j)} = (X_{i(-j)}^{(\max)}, X_{i(-j)}^{(\min)})\) where \(X_{i(-j)}^{(\max)}\) and \(X_{i(-j)}^{(\min)}\) are the collection of maximum and minimum values, respectively, or (c) generating a \(d\)-dimensional, low-rank summary of \(X_{i(-j)}\). Investigators can also use an ensemble of learners based on different modifications of \(X_{i(-j)}\) via the super learner algorithm (van der Laan et al., 2007; Polley and van der Laan, 2010); see Section A.3 of the Supplementary Material for details. Given the moderate sample size of our data, we use the latter approach to train \(\mu^*\).

For the propensity score \(e(a, s \mid x_i)\), we suggest nonparametric and semiparametric approaches. The nonparametric approach is similar to fitting the outcome regression model where we use \(A_{ij}\) as the dependent variable and \(X_{ij}\) the independent variable inside the super learner to estimate the propensity score \(P(A_{ij} = a_{ij} \mid X_{ij})\). Then, we use the estimated propensity score to obtain an estimated \(e(a, s \mid X_{ij}, X_{i(-j)})\) as follows.

\[
\hat{e}(a, s \mid X_{ij}, X_{i(-j)}) = \hat{P}(A_{ij} = a \mid X_{ij}) \left\{ \sum_{a_{i(-j)}} 1 \left( \sum_{\ell \neq j} a_{i\ell} = s \right) \prod_{\ell \neq j} \hat{P}(A_{i\ell} = a_{i\ell} \mid X_{i\ell}) \right\}.
\]
Implicitly, the estimate $\hat{e}$ assumes that the propensity score $P(A_i = a_i \mid X_i)$ can be decomposed into a product of individual propensity scores $P(A_{ij} = a_{ij} \mid X_{ij})$. If this assumption is implausible or the estimated $\hat{e}$ is numerically unstable due to the product term, an alternative approach is to use a semiparametric model. Specifically, we decompose the propensity score as $e(\beta, P)$ into a product of individual propensity scores $e(\beta, P)$. Implicitly, the estimate $\hat{e}$ assumes that the propensity score $P$ can be estimated via the likelihood principle. The final estimate of $e(\beta, P)$ is evaluated in (7) where $\hat{e}$ is numerically unstable due to the product term, an alternative approach is to use $\hat{e}$.

For the second conditional probability, we bin $\bar{A}_{ij}(-j)$ into $(M + 1)$ equi-spaced bins and consider the following ordinal regression model for $\bar{A}_{ij}(-j)$ parametrized by $(\beta_{0,0}, \ldots, \beta_{0,M}, \beta_{ego}, \beta_{peer})$:

$$P\left(\bar{A}_{ij}(-j) \leq \frac{t}{M} \mid X_{ij}, \bar{A}_{ij}(-j)\right) = \expit(\beta_{0,t} + X_{ij}^\top \beta_{ego} + X_{ij}^\top \beta_{peer}) , \quad t = 0, \ldots, M .$$

Here, the intercept parameters satisfy $\beta_{0,t} \leq \beta_{0,t+1}$. We then estimate $(\hat{\beta}_{0,0}, \ldots, \hat{\beta}_{0,M}, \hat{\beta}_{ego}, \hat{\beta}_{peer})$ via the likelihood principle. The final estimate of $e(a, s \mid X_{ij}, \bar{A}_{ij}(-j))$ is simply the plug-in estimates of $\beta_t$, i.e., $\hat{e}(a, s \mid X_{ij}, \bar{A}_{ij}(-j)) = \expit(\hat{\beta}_{0,t} + X_{ij}^\top \hat{\beta}_{ego} + X_{ij}^\top \hat{\beta}_{peer}) - \expit(\hat{\beta}_{0,t'} + X_{ij}^\top \hat{\beta}_{ego} + X_{ij}^\top \hat{\beta}_{peer})$. For our dataset, we use the semiparametric approach as it was more numerically stable than the nonparametric approach, especially after taking measures to prevent overfitting; see the next paragraph.

We also discuss two common techniques that we use to prevent overfitting the nuisance functions. First, we randomly remove observations so that the number of study units among clusters used in the nuisance function estimation are roughly similar to each other. This adjustment is often referred to as undersampling in multilevel studies (Fernández et al., 2018, Chapter 5.2) and the adjustment prevents a few large, outlying clusters from having a dominant effect on the estimated nuisance function. Second, for the two nuisance functions in the loss function in (6), we use cross-fitting by Chernozhukov et al. (2018) where we train the two nuisance functions and the optimal rule $\theta^*$ from two different subsamples of the data. Specifically, suppose we split the data into two folds $\mathcal{D}_1$ and $\mathcal{D}_2$ and let $\hat{\mu}_(-\ell), \hat{\epsilon}_(-\ell), \hat{L}_(-\ell), \ell \in \{1, 2\}$, be the estimated outcome regression, propensity score, and loss function, respectively, by using $\mathcal{D}_\ell^c$. We estimate the OMAR by solving the SVM in (7) where $\hat{L}_(-\ell)$ is evaluated in $\mathcal{D}_\ell$. Again, we take the winsorized rule $\hat{\theta}_(-\ell)$ to make sure the optimal rule is bounded between 0 and 1, i.e., $\hat{\theta}_(-\ell) = \mathcal{W}(\hat{\theta}_(-\ell))$. Investigators may use either $\hat{\theta}_(-1),$.
or the winsorized average of the two rules, i.e., \( W \{ \hat{\theta}_{(-1)} + \hat{\theta}_{(-2)} \}/2 \), as the final estimated OMAR. Finally, we remark that it is common to repeat undersampling and cross-fitting multiple times and aggregate the final estimates to remove finite-sample effects from random sampling; see Section A.6 in the Supplementary Material for details.

\[ \theta_{(-2)} , \] or the winsorized average of the two rules, i.e., \[ W \{ \hat{\theta}_{(-1)} + \hat{\theta}_{(-2)} \}/2 \], as the final estimated OMAR. Finally, we remark that it is common to repeat undersampling and cross-fitting multiple times and aggregate the final estimates to remove finite-sample effects from random sampling; see Section A.6 in the Supplementary Material for details.

3.5 Theoretical Properties

To study the performance of the estimate rule \( \hat{\theta}_{(-\ell)} \) from the previous section, we introduce the following assumptions about the estimated nuisance functions \( \hat{\mu}_{(-\ell)} \) and \( \hat{e}_{(-\ell)} \) (\( \ell = 1, 2 \)).

(E1) Overlap of \( \hat{e}_{(-\ell)} \): There exists a constant \( c' > 0 \) such that \( c' < \hat{e}_{(-\ell)}(a, s | X_{ij}, X_{i(-j)}) \) for any \( (a, s, X_{ij}, X_{i(-j)}) \).

(E2) Bounded \( \hat{\mu}_{(-\ell)} \): \( \hat{\mu}_{(-\ell)}(a, s/(n_i - 1), X_{ij}, X_{i(-j)}) \) is bounded for any \( (a, s, X_{ij}, X_{i(-j)}) \).

(E3) Consistency of \( \hat{e}_{(-\ell)} \) and \( \hat{\mu}_{(-\ell)} \): As \( N \to \infty \), we have \( r_{e,N} := \| \hat{e}_{(-\ell)}(a, s | X_{ij}, X_{i(-j)}) - e^*(a, s | X_{ij}, X_{i(-j)}) \|_{p,2} = o(1) \) and \( r_{\mu,N} := \| \hat{\mu}_{(-\ell)}(a, s/(n_i - 1), X_{ij}, X_{i(-j)}) - \mu^*(a, s/(n_i - 1), X_{ij}, X_{i(-j)}) \|_{p,2} = o(1) \) for any \( s \) with probability \( 1 - \Delta_N \) and \( \Delta_N = o(1) \).

Assumption (E1) implies that the estimated propensity score satisfies overlap and Assumption (E2) implies that the estimated outcome regression is bounded. For the Senegal DHS, Assumption (E2) is satisfied because the outcome is binary. Assumption (E3) is satisfied so long as the estimated nuisance functions converge to the true nuisance functions. We remark that similar conditions are introduced in previous works that use nonparametric methods to estimate the nuisance components; see, for example, Chernozhukov et al. (2018) for assumptions under no interference with independent and identically distributed data and Sofrygin and van der Laan (2016) and Park and Kang (2022) for assumptions under partial interference with independent, but not necessarily identically distributed data.

In Theorem 3.1, we characterize the excess risk of \( \hat{\theta}_{(-\ell)} \), a common metric of performance in this literature (e.g. Zhao et al. (2012); Díaz et al. (2018); Kitagawa and Tetenov (2018)).

**Theorem 3.1.** Suppose that Assumptions (A1)-(A5) and (E1)-(E3) hold and \( \theta^* \) belongs to a Besov space on \( \mathbb{R}^d \) with \( \beta > 0 \) smoothness; see Section B.3 of the Supplementary Material for the definition of a Besov space. If \( \gamma_N \propto N^{-1/(2\beta+d)} \) and \( \lambda_N \propto N^{-(\beta+d)/(2\beta+d)} \), we have \( R(\hat{\theta}_{(-\ell)}) - R(\theta^*) = O_P(N^{-\beta/(2\beta+d)}) + O_P(r_{e,N}r_{\mu,N}) \) with high probability.
The upper bound on the excess risk is determined by two quantities, the convergence rate of the SVM (i.e., $N^{-\beta/(2\beta+d)}$) and the convergence rate of the estimated nuisance functions (i.e., $r_{e,N}\mu,N$). The convergence rate of the SVM depends on the smoothness of the true OMAR $\theta^*$ and if $\theta^*$ is very smooth (i.e., $\beta \to \infty$ with fixed $d$), the rate of the excess risk bound approximates the rate $N^{-1/2}$ from Chen et al. (2022) if the nuisance functions are estimated at $N^{-1/4}$. Also, in the same scenario, our excess risk is better than the $N^{-1/4}$ rate from Chen et al. (2016) because unlike their work, our work finds an optimal rule achieving the desired target $T$ and we have an additional monotonicity constraint (A5).

4 Simulation

We conduct a simulation study to assess the finite-sample performance of the indirect rule and the direct rule. We generate $N = 1000$ clusters, and, for each cluster $i$, we randomly generate cluster size $n_i \in \{2, \ldots, 5\}$ with equal probability. For unit $j$ in cluster $i$, we generate three unit-level covariates $(W_{ij1}, W_{ij2}, W_{ij3})$ and one cluster-level covariates $C_i$ independently from a standard Normal distribution. Also, for each cluster $i$, we generate the treatment from $A_{ij} \mid X_i \sim \text{Ber} \left( \expit \left\{ 0.25 \left( -2 + W_{ij1} + W_{ij2} + W_{ij3} + C_i \right) \right\} \right)$ and the outcome from the following model:

$$Y_{ij} \mid (A_i, X_i) \sim \text{Ber} \left( \expit \left[ -0.35 + \left\{ 0.1 + 0.25(C_i + W_{ij1})^2 \right\} A_{ij} + \left\{ 0.05 + 0.15(\overline{W}_{i(-j)2} + \overline{W}_{i(-j)3})^2 \right\} \overline{A}_{i(-j)} + 0.1(C_i + \sum_{k=1}^{3} W_{ijk}) + 0.25(C_i^2 + \sum_{k=1}^{3} W_{ijk}^2) + 0.05(\sum_{k=1}^{3} \overline{W}_{i(-j)k}) \right] \right).$$

Here, $\overline{W}_{i(-j)k} = \sum_{\ell \neq j} W_{ijk}/(n_i - 1)$. The propensity score and the outcome regression model satisfy Assumptions (A1)-(A5).

Since the estimation procedures for the two OMARs, $\theta_{OV}^*$ and $\theta_{SO}^*$, are nearly identical except for the difference in the loss functions $L_{OV}$ and $L_{SO}$ associated with each OMAR, we focus on $\theta_{OV}^*$. For the direct rule, we unify the dimension of $X_i$ with $\overline{X}_i$, use the SVM described above, run 10-fold cross-validation for the SVM parameters, and do median-adjustment from 5 cross-fitted estimators, where each cross-fitted estimator is also median-adjusted from 3 undersampled estimators. To obtain the indirect rule, we use two methods to estimate $\mu$, a linear regression with main and second-order interaction terms of $(\overline{A}_i, \overline{X}_i)$ regularized by the Lasso penalty (Tibshirani, 1996) and random forests (Breiman, 2001); these methods are referred to as Lasso and RF, respectively. For
the target $T$, we vary between 0.67 and 0.73, which matches the range in the data analysis. We evaluate the performance of each method on a test set with 200 clusters. We repeat this entire process 50 times and obtain 10000 OMAR estimates for each method and $T$.

Figure 4.1 shows the squared deviations between the estimated and the true OMARs. Our direct rule shows the best performance where its squared deviation between the predicted and the true OMARs is generally the smallest across all threshold values $T$.

Figure 4.1: Boxplots of the Squared Deviation Between the Predicted and True OMARs. The $x$ and $y$-axes show the threshold $T$ and the squared deviation between the predicted and true OMARs, respectively.

Next, we assess the methods using the following classification performance measures: accuracy, two-sided F1 score, and the Matthews correlation coefficient (MCC) (Matthews, 1975); see Section A.7 of the Supplementary Material for details. For every classification performance measure, a larger value means better performance. Figure 4.2 shows the result for each method. We also plot the true OMAR for comparison. The true OMAR performs the best across the different measures of performance and all thresholds $T$, justifying that the measures are meaningful for evaluating the performance of the rules. Among the estimated rules, the direct rule generally performs well across most $T$ and the three performance measures. Combined with Figure 4.1, we would generally recommend our direct rule in terms of bias and classification performance measures.
5 Optimal Allocation of WASH Facilities in Senegal

5.1 Background

We use the proposed methods to estimate the optimal allocation of WASH facilities in Senegal. Specifically, we use the 2014-2018 Senegal DHS, which used a two-stage stratified survey design where households are nested under census blocks. As a reminder, we treat the census blocks as clusters and households within each census block as individual study units. We restrict the sample to households with complete data on $O_{ij} = (Y_{ij}, A_{ij}, X_{ij})$; see Section 2.1 for the exact definition of the observed variables. For training the allocation rule, we use the 2014-2017 DHS, which contains $N = 1027$ census blocks with 13556 households. For testing the allocation rule, we use the 2018 DHS, which contains 213 census blocks with 2859 households. For the test set, we vary the diarrhea-free target threshold $\mathcal{T}$ from 0.67 to 0.73, which is just slightly above and below the average of the diarrhea-free incidence level of 0.689 between 2014 and 2017. Then, for each target threshold $\mathcal{T}$, the estimated OMAR for the overall potential outcome (i.e., $\hat{\theta}_{OV}$) reports the minimum proportion of WASH facilities needed to have at least $\mathcal{T}$ proportion of households in a census block to be diarrhea free. For the estimation of the direct and indirect rules, we use the same procedures as in the simulation except that, for the direct rule, we use 100 cross-fitted estimators with median adjustment and 10 rounds of undersampling. Finally, we examine the plausibility of Assumptions (A1)-(A5) in the Senegal DHS and discover no serious violations; see Section A.8 of the Supplementary Material.

Figure 4.2: Evaluation of OMARs Using Different Measures of Classification Performance. The $x$-axis shows the threshold $\mathcal{T}$. The $y$-axis shows the value of the measure that is specified at the top of each plot. A larger value in the $y$-axis implies better performance.
5.2 Results

Finding 1: Our direct rule is more accurate.

For all of our empirical results below, we use the same estimators and classification performance measures used in the simulation study in Section 4 except we cross-fit 100 times. Figure 5.1 shows the classification performance measures. Across all classification performance measures, our direct rule performs better than the indirect rules across all thresholds $T$ except for very small values of $T$. Combined with the results from the simulation, our direct rule seems to be more accurate (i.e., smaller deviance from the true OMAR) than the other two competing methods.

![Figure 5.1: Evaluation of the Estimated OMARs in the 2018 Senegal DHS. The $x$ and $y$-axes show the threshold $T$ and the magnitude of each classification performance measure, respectively. Higher $y$ values indicate better performance.](image)

To help understand the results in Figure 5.1, Figure 5.2 shows the allocation rules at four values of the thresholds. Specifically, the heatmap shows the weighted averages of the estimated OMARs aggregated to 45 Senegalese administrative regions where the weight is based on the number of households in a census block (i.e., $n_i$); see Section A.9 of the Supplementary Material for more details behind the heatmaps. For each $T$, our direct rule is generally more diffused than the other two indirect rules, implying that our direct rule generally prefers a more balanced, nation-wide approach to allocating WASH facilities across Senegal compared to the other rules. Also, since our direct rule is more accurate than the other rules, the finding broadly suggests that the optimal strategy to allocate WASH facilities in Senegal is a diffuse approach compared to a targeted, all-or-nothing approach where neighbors of an area with a high OMAR should also receive a relatively large amount of WASH facilities.
Finding 2: Our direct rule is more resource-efficient.

Figure 5.3 shows the weighted averages of estimated OMARs needed to achieve the target outcome level $\mathcal{T}$, weighted by the size of the census blocks. When $\mathcal{T}$ is greater than 0.69, which is the outcome level in the training data, our direct rule uses fewer WASH facilities than the two indirect rules to achieve $\mathcal{T}$. More importantly, by combining the first two findings, the direct rule is more resource-efficient than the indirect rules in terms of allocating the right regions of Senegal to achieve the highest accuracy. Specifically, for thresholds greater than 0.69, the direct rule is using less resources and is more accurate than competing methods, suggesting that the indirect rules are targeting the wrong regions of Senegal or overusing resources than what’s necessary to achieve the target $\mathcal{T}$. This can be seen in the last two columns of Figure 5.2 where the random forest is over-allocating WASH facilities across most of Senegal, even in areas that may not need additional WASH facilities to achieve $\mathcal{T}$. In contrast, our direct rule is generally allocating less resources throughout the nation while simultaneously achieving higher accuracy. Given that Senegalese policymakers would like to
set $T$ higher than the current level 0.69, our direct rule is not only accurate but also cost-efficient for allocating WASH facilities in Senegal.

Finding 3: Compared to current policy recommendations, our direct rule uses different characteristics to allocate WASH facilities.

In international development, one of the prevailing advice on allocation of WASH facilities is to target rural areas (ESARO, 2019; World Bank Group, 2019). Specifically, these works suggest that nations use proportion of rural areas in a region to allocate WASH facilities, where a large proportion of rural areas would indicate that a large number of WASH facilities is needed. However, as depicted in Figure 5.4, our direct rule suggests this should not be the main strategy to allocate WASH facilities. Instead, the allocation rule should focus on average household size in a census block. For example, the regions in Figure 5.4 marked with an asterisk (*) are mostly rural areas but are associated low OMAR estimates. In contrast, the same regions have higher average household size and is associated with higher OMAR estimates. Overall, our direct rule is more closely correlated with the average household size, where a census block with many large households need more WASH facilities. This correlation between the OMAR and average household size also agrees with prior works in infectious diseases where incidence of diarrheal disease is strongly correlated with household size (Yilgwan et al., 2012; Beyene and Melku, 2018; Omona et al., 2020). In short, if the target outcome is to reduce incidence of diarrhea among children, our direct rule suggests that household size may be a better indicator compared to the proportion of rural area.

Figure 5.3: Averages of Estimated OMARs in 2018 Senegal DHS. The $x$ and $y$-axes show the threshold $T$ and the average of estimated OMARs across census blocks. The dotted vertical line at $T = 0.69$ shows the outcome level in the training data (i.e., between 2014 and 2017).
Figure 5.4: Estimated OMARs (left), Household Size (middle), and Proportion of Rural Areas (right) in the 2018 Senegal DHS. Deeper (lighter) blue color indicates a larger (smaller) value. The two regions with an asterisk (*) are mostly rural, but are associated with low OMARs and small household sizes.

Finding 4: Comparing OMARs under different outcomes can provide insights about the mechanism behind the optimal policy of the primary outcome.

Using the methods developed in the paper, we can compare OMARs from different potential outcomes by simply changing the loss functions and such comparisons can reveal insights about the mechanism behind the optimal policy under the primary outcome. As an illustrate example, we compare the OMAR discussed above (i.e., \( \theta^*_\text{OV} \)) with the OMAR \( \theta^*_\text{SO} \) under the spillover potential outcome.

By Theorem 1 of VanderWeele and Tchetgen Tchetgen (2011), the difference \( \theta^*_D = \theta^*_\text{SO} - \theta^*_\text{OV} \) serves as a proxy for the magnitude of the optimal policy’s direct effect on diarrhea incidence rate. A large \( \theta^*_D \) would indicate that the optimal policy has a large direct effect on reducing diarrhea incidence whereas a small \( \theta^*_D \) would indicate the opposite. Also, similar to Finding 3, by comparing \( \theta^*_D \) as a function of an observed covariate, investigators can assess the heterogeneity of the optimal policy’s direct effect.

Figure 5.5 shows one example of such heterogeneity with respect to the average number of children per household. The y-axis plots the estimated \( \theta^*_D \) by plugging in estimates of \( \theta^*_\text{OV} \) and \( \theta^*_\text{SO} \) at \( T = 0.70 \) and the x-axis plots the average number of children per household. The estimated \( \theta^*_D \) shows a quadratic relationship where \( \theta^*_D \) is generally the largest when the average number of children per household is between 2 and 3.5. This suggests that the optimal policy’s direct effect is the largest when the average number of children per household is of moderate size. This also suggests that if policymakers seek a household-level implementation of WASH facilities instead of
the block-level implementation we derived in the paper, policymakers should prioritize households with moderate number of children as this strategy may yield the largest short-term effect. If, on the other hand, the relationship between the estimated $\theta^*_D$ and the average number of children per household is positive and linear, policymakers may consider prioritizing households with large number of children to yield the largest short-term effect.

Figure 5.5: Local Constant Regression Curves of the Estimated OMARs $\hat{\theta}_{SO}$ (black solid line in the left panel), $\hat{\theta}_{OV}$ (black dashed line in the left panel), and the Difference of the Estimated OMARs $\hat{\theta}_D = \hat{\theta}_{SO} - \hat{\theta}_{OV}$ (blue solid line in the right panel). The $x$-axis is the average number of children per household. A larger $y$ value means that more WASH facilities are required in a census block to meet the threshold $T = 0.7$. Note that the $y$-axes in the two plots are under two different scales for better visualization.

6 Discussion

In this paper, we propose methods for estimating optimal allocation rules given a target objective for the outcome and under partial interference. Specifically, if the policymaker wants a policy that achieves children diarrhea-free incidence to be least 70%, the proposed rules report the proportion of WASH facilities necessary to achieve this objective where the policy accounts for potential spillover effects of WASH facilities. We consider two potential solutions, one based on a modest extension of an indirect method and the other based on a direct method using a novel loss function that incorporates partial interference. We also show that a tempting, analysis strategy based on aggregated data at the block-level will be grossly misleading. We then use the proposed methods to train the optimal allocation of WASH facilities by using the Senegal DHS, an annual, cross-
sectional survey. We show that our direct rule is more accurate and resource-efficient than the other methods. We also show the difference between our direct rule and what’s currently recommended for allocating WASH facilities in international development.

We end by providing some guidelines on how to use our direct rule and interpret its results in practice. First, one may simply maximize the overall potential outcome without having to set a target $T$. We conjecture that our approach can be modified to find the optimal treatment rule that maximize equation (3), but the convergence rate is likely slower than the convergence rate discussed in Section 3.3. Second, as mentioned in the introduction, OMAR can complement “micro-level” allocation rules where the estimated OMAR could be used as a data-driven approach to inform the “hard” resource constraints in such rules. More generally, it would be interesting future work to combine both macro and micro approaches to obtain a unified allocation rule. Third, similar to any method in causal inference with observational data, the estimated OMAR for Senegal makes untestable assumptions that cannot be validated with the observed data. This includes not only the usual identification Assumptions (A1)-(A3), but also modeling Assumptions (A4)-(A5) to allow for consistent estimation. While the diagnostics in Section A.8 of the Supplementary Material showed no strong reason for concern with respect to our data, the diagnostics are not perfect. Nevertheless, so long as these assumptions are plausible, first-order approximations of the underlying truth, we believe our direct rule provides a useful, data-driven approach to allocate public resources related to infectious diseases for an entire nation.
Supplementary Material

This document contains supplementary materials for “Optimal Allocation Rule for Infectious Disease Prevention Under Partial Interference and Outcome-Oriented Targets With Repeated Cross-Sectional Surveys.” Section A presents additional results related to the main paper. Section B proves lemmas and theorems stated in the paper.

A Additional Details of the Main Paper

A.1 Another Example of the OMAR Based on the Aggregated Block-level Outcome Regression

We provide another example when the aggregated block-level outcome regression fails to identify the OMAR. Suppose the treatment is completely randomized and there are no interactions between the covariates and the treatment, but there exists non-linear relationship between the treatment on the outcome:

\[
E \{ Y_{ij}^{(a_{ij}, a_i(-j))} | X_i \} = \beta_0 + \beta_1 a_{ij} + \beta_2 \{ (\bar{a}_{i(-j)} - q_a)^p \}_+ \tag{8}
\]

where \( \beta_1 \) and \( \beta_2 \) are non-negative coefficients, \( q_a \in [0, 1] \), and \( p \) is a positive integer. Roughly speaking, the model states that the household’s outcome can be affected by its peer households through a non-linear function \( z \mapsto (z - q_a)^p \) if at least \( 100 \times q_a \)% of their peers are treated; see Granovetter (1978), Watts (2002), and Kempe et al. (2003) and references therein for other types of threshold models in networks. As before, Assumptions (A4) and (A5) hold for this model and some algebra will reveal that the block-level outcome model will be mis-specified when using only aggregated, block-level data \((\bar{Y}_i, \bar{A}_i, \bar{X}_i)\) due to the non-linearity of \( \bar{a}_{i(-j)} \) in the household-level outcome model. Consequently, the resulting OMAR with the block-level data will be biased.

We provide a graphical illustration of model (8). To demonstrate, we fix the cluster size \( n_i = 10 \) and the coefficients \( \beta_0 = \beta_1 = 0 \), and choose \( \beta_2 \) so that the range of the outcome regression becomes \([0, 1]\). We consider three levels for \( q_a \in \{0.4, 0.6, 0.8\} \) and \( p \in \{1, 2, 5\} \), respectively, and we choose the threshold \( T = 0.2 \). Figure A.1 visually presents the differences between the OMARs based on \( \tau_{OV} \) and the aggregated block-level outcome regression. We find that the differences vary between
0.07 and 0.17. The toy example suggests that estimating the OMAR based on the aggregated outcome regression may yield significantly biased estimates of $\theta_{OV}$ in (4).

Figure A.1: Graphical Comparison between the OMARs Based on the Aggregated Block-level Outcome Regression and $\tau_{OV}(\alpha)$. Black dotted and red dashed lines indicate the OMAR based on the aggregated block-level outcome regression and that based on $\tau_{OV}(\alpha)$ (i.e., $\theta_{OV}$), respectively.

A.2 Inverse Probability-Weighted and Outcome Regression-based Loss Functions

We introduce the IPW and outcome regression-based loss functions. Specifically, we can replace $\psi_{DR}$ in (6b) in the main paper with the following functions.

$$
\psi_{IPW}(a, s, O_{ij}, O_{i(-j)}) = \frac{Y_{ij}1(A_{ij} = a, S_{i(-j)} = s)}{e^s(a, s \mid X_i)}, \quad \psi_{OR}(a, s, O_{ij}, O_{i(-j)}) = \mu^a(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)}).
$$

A.3 Machine Learning Methods Used for the Outcome Regression Estimation

As candidate machine learning methods, we include the following methods and R packages in our super learner library: linear regression (glm), Lasso/elastic net (glmnet (Friedman et al., 2010)), spline (earth (Friedman, 1991), polspline (Kooperberg, 2020)), generalized additive model (gam (Hastie and Tibshirani, 1986)), boosting (xgboost (Chen and Guestrin, 2016), gbm (Greenwell...
A.4 Computation: Training the Support Vector Machine in (7) in the Main Paper

This section presents the computational details on training the support vector machine in (7) in the main paper. The algorithm to train SVMs under this type of nonconvex function is already discussed in prior works (An and Tao, 1997; Chen et al., 2016) and we present a summary of it for completeness. Also, to keep the notation clear, the discussion below assumes that the nuisance functions are known, but the identical computation algorithm is used to train the SVM when the nuisance functions are estimated.

To start off, we can decompose the loss function for the overall outcome case in equation (6) in the main paper into the difference of two convex function $L^+$ and $L^-$, i.e., $L(t, O_i) = L^+(t, O_i) - L^-(t, O_i)$ for any $t$ and $O_i$ where

$$L^+_i(t, O_i) = \begin{cases} 
\nu_+(0, O_i) - 2\delta t & \text{if } -\infty < t < 0 \\
\nu_+(t, O_i) & \text{if } 0 \leq t \leq 1 \\
\nu_+(1, O_i) + (\bar{\delta} + 2\delta)(t - 1) & \text{if } 1 < t < \infty 
\end{cases}$$

$$L^-_i(t, O_i) = \begin{cases} 
\nu_-(0, O_i) - 2\delta t - \delta + \delta e^t & \text{if } -\infty < t < 0 \\
\nu_-(t, O_i) & \text{if } 0 \leq t \leq 1 \\
\nu_-(1, O_i) + (\bar{\delta} + 2\delta)(t - 1) - \delta + \delta e^{1-t} & \text{if } 1 < t < \infty 
\end{cases}$$

$$\nu_\pm(t, O_i) = \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{1} \sum_{s=0}^{n_i-1} \binom{n_i - 1}{s} \psi_{DR}(a, s, O_{ij}, O_{i(-j)})$$

$$\times \sum_{\ell=0}^{n_i-a-s} \binom{n_i - a - s}{\ell} \left\{ \frac{(-1)^{\ell}}{\ell + a + s + 1} \right\}_{\pm} t^{\ell + a + s + 1} + (T)_{\pm} t$$

Here, $(a)_+ = \max(a, 0)$, $(a)_- = -\min(a, 0)$, and $\bar{\delta}$ is chosen as the maximum of the left derivatives of $\nu_+(t, O_i)$ and $\nu_-(t, O_i)$ at $t = 1$, i.e., $\bar{\delta} = \max\left\{ \lim_{\epsilon \downarrow 0} \nabla \nu_+(1 - \epsilon, O_i), \lim_{\epsilon \downarrow 0} \nabla \nu_-(1 - \epsilon, O_i) \right\}$ and $\nabla \nu_\pm(t, O_i)$ is the derivative of $\nu_\pm(t, O_i)$ with respect to $t$. Critically, the two loss functions $L_+$ and
\(L_\cdot\) are convex and non-decreasing in \(t\). For the spillover outcome case, we use
\[
\nu_{SO, \pm}(t, O_i) = \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{s=0}^{n_i-1} \left( \frac{n_i - 1}{s} \right) \psi_{\text{DR}}(0, s, O_{ij}, O_{i(-j)}) \sum_{\ell=0}^{n_i-1-s} \left( \frac{n_i - 1 - s}{\ell} \right) \left( \frac{(-1)^\ell}{\ell + s + 1} \right) \pm t^{\ell+s+1} + (T)^{\pm t}
\]

Given the decomposition of the loss function into the difference of two convex functions, we use the DC algorithm (An and Tao, 1997), which is an iterative algorithm, to solve the original non-convex optimization problem; see Algorithm 1 for details.

**Algorithm 1 DC Algorithm**

**Require:** Initialize values \(\eta^{(0)} \in \mathbb{R}^N, b^{(0)} \in \mathbb{R}\). Set iteration number to zero, \(j \leftarrow 0\).

1: Precompute the gradient \(\nabla L_\cdot(t, O_i)\) where
\[
\nabla L_\cdot(t, O_i) = \left\{ \begin{array}{ll}
\frac{\partial}{\partial t} L_\cdot(t, O_i) & t \neq 0, 1 \\
\frac{1}{2} \lim_{\epsilon \downarrow 0} \left\{ \frac{\partial}{\partial t} L_\cdot(t + \epsilon, O_i) + \frac{\partial}{\partial t} L_\cdot(t - \epsilon, O_i) \right\} & t = 0, 1
\end{array} \right.
\]

2: repeat
3: Let \(\eta^{(j+1)}\) and \(b^{(j+1)}\) be the solution to the following convex optimization problem.
\[
\left[ \eta^{(j+1)}, b^{(j+1)} \right] \in \arg\min_{\eta, b} \left[ \frac{1}{N} \sum_{i=1}^{N} \left\{ L_+(k_i^T \eta + b, O_i) - \nabla L_-(k_i^T \eta^{(j)} + b^{(j)}, O_i)(b + k_i^T \eta) \right\} + \frac{\lambda_N}{2} \eta^T K \eta \right]
\]
4: \(j \leftarrow j + 1\)
5: until convergence
6: return \((\hat{\eta}, \hat{b}) \leftarrow (\eta^{(j)}, b^{(j)})\).

To initiate the DC algorithm, we choose the initial value as follows. First, for each \(i\), let the solution be \(r_i\), i.e., \(r_i = \arg\min_{t \in [0, 1]} L(t, O_i)\) which can be obtained from a grid-search. In words, \(r_i\) is an approximate of \(\hat{\theta}(x_i)\) that are found by a grid-search. But, since \(r_i\) is bounded in the unit interval, it may not be a suitable approximate of \(\tilde{\theta}(x_i)\), the SVM solution before the winsorization. As a consequence, directly using \(r_i\) to construct initial points may lead to an estimate rule shrinking to a certain value, i.e., a rule does not reflect the heterogeneity induced by \(x_i\). To stretch \(r_i\) outside of the unit interval, we consider the following steps.

(a) Let \(\phi\) and \(\varphi\) be
\[
\phi(a, a', X_i) = \frac{1}{n_i} \sum_{j=1}^{n_i} \tilde{\mu}(a, a', X_{ij}, X_{i(-j)}) \ , \ \varphi(X_i) = \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{Y_{ij} - \tilde{\mu}(A_{ij}, A_{i(-j)}, X_{ij}, X_{i(-j)})}{\tilde{c}(A_{ij}, S_{i(-j)} | X_i)}
\]
(b) By only using the clusters with non-0 and non-1 \( r_i \)'s, i.e., \( r_i \in (0,1) \), we fit linear regression models where \( \phi(a, a', X_i) \) and \( \varphi(X_i) \) are regressed on \( r_i \)'s. We choose \( (a, a') \) from \( \{0,1\} \otimes \{0,0.2,0.4,0.6,0.8,1\} = \{(0,0), (0,0.2), \ldots, (1,0.8), (1,1)\} \), i.e., 12 levels. Let \( (\hat{\beta}_{0,\text{model } k}, \hat{\beta}_{1,\text{model } k}) \) are the estimated regression coefficients from \( k \)th model.

(c) Let \( \hat{r}_i \) be the adjusted initial points which are defined as follows.

(c-1) If \( r_i \in (0, 1) \), no adjustment is required, i.e., \( \hat{r}_i = r_i \).

(c-2) For clusters having \( r_i = 1 \), we use the largest prediction values obtained from the 13 regression models and 1, i.e.,

\[
\hat{r}_i = \max \left\{ \frac{\phi(0, 0, X_i) - \hat{\beta}_{0,\text{model } 1}}{\hat{\beta}_{1,\text{model } 1}}, \ldots, \frac{\varphi(X_i) - \hat{\beta}_{0,\text{model } 13}}{\hat{\beta}_{1,\text{model } 13}}, 1 \right\}.
\]

(c-3) Similarly, for clusters having \( r_i = 0 \), we use the smallest prediction values obtained from the 13 regression models and 0, i.e.,

\[
\hat{r}_i = \min \left\{ \frac{\phi(0, 0, X_i) - \hat{\beta}_{0,\text{model } 1}}{\hat{\beta}_{1,\text{model } 1}}, \ldots, \frac{\varphi(X_i) - \hat{\beta}_{0,\text{model } 13}}{\hat{\beta}_{1,\text{model } 13}}, 0 \right\}.
\]

Second, we take \( b^{(0)} = \sum_{i=1}^{N} \hat{r}_i / N \) and \( \eta^{(0)} \) as a vector satisfying \( \hat{r}_i = k_i^T \eta^{(0)} + b^{(0)} \) for all \( i \); i.e., \( \hat{r} = K \eta^{(0)} + b^{(0)} 1 \) where \( \hat{r} = [\hat{r}_1, \ldots, \hat{r}_N]^T \in \mathbb{R}^N \) and \( 1 = [1, \ldots, 1]^T \in \mathbb{R}^N \). Even though the kernel matrix \( K \) is invertible due to the positive definiteness of the kernel function \( \mathcal{K} \), the inverse of \( K \) cannot be obtained due to the numerical singularity. Under such case, we add a tiny value to diagonal of \( K \) until its inverse can be obtained. In line 1, \( \nabla L_- \) is a subgradient of \( L_- \) that accounts for the non-differentiability of \( L_- \) at \( t = 0 \) and \( t = 1 \).

The convex optimization in line 3 can be solved by using many standard algorithms and softwares. The iteration stops when \( \| (\eta^{(j+1)}, b^{(j+1)}) - (\eta^{(j)}, b^{(j)}) \|_2 \) drops below some threshold value.

We remark that because the objective function in (7) in the main paper is bounded below, the algorithm will always converge in finite steps (An and Tao, 1997; Chen et al., 2016).

A.5 Details of Cross-validation

We present the details on how to choose the SVM parameters \( \gamma \) and \( \lambda \). We consider a set of candidate values for \( (\gamma_\ell, \lambda_\ell) \) where \( \ell = 1, \ldots, K \). Without loss of generality, let the estimation
data fold be \( \mathcal{D}_1 = \mathcal{D}_2 \) and, as a consequence, observations in \( \mathcal{D}_2 \) is used to evaluate the estimated loss function \( \hat{L}_{(-1)}(t, \mathcal{O}_i) \) for \( i \in \mathcal{D}_2 \). We further split \( \mathcal{D}_2 \) into training and tuning sets, denoted by \( \mathcal{D}_{2, \text{train}} \) and \( \mathcal{D}_{2, \text{tuning}} \), respectively, based on the number of cross-validation folds. For each candidate parameter \((\gamma_{\ell}, \lambda_{\ell})\), we estimate the direct OMAR rule \( \hat{\theta}_{\text{train}}(X_i; \ell) \) by only using the training set \( \mathcal{D}_{2, \text{train}} \) and obtain the empirical risk using the tuning set \( \mathcal{D}_{2, \text{tuning}} \). The optimal parameters \((\gamma^*, \lambda^*)\) are the minimizer of the average of the empirical risks across the tuning sets, i.e.

\[
(\gamma^*, \lambda^*) = \arg \min_{\ell=1, \ldots, K} \frac{1}{|\mathcal{D}_{2, \text{tuning}}|} \sum_{i \in \mathcal{D}_{2, \text{tuning}}} \hat{L}_{(-1)}(\hat{\theta}_{\text{train}}(X_i; \ell), \mathcal{O}_i).
\]

### A.6 Details of Undersampling and Cross-fitting Procedures

We discuss the details on how to negate the impact of a particular realization of undersampling procedure. We randomly choose a subset of observations so that the cluster sizes are (nearly) balanced, and we repeat the undersampling for \( U \) times indexed by \( u \). Let \( \hat{\mu}^{(u)} \) and \( \hat{e}^{(u)} \) be the estimated outcome regression and propensity score obtained from \( u \)th undersample. Then, we take the median-adjusted nuisance function across \( U \) estimated functions as the final estimate of the nuisance function, i.e., \( \hat{\mu} := \text{median}_{u=1, \ldots, U} \hat{\mu}^{(u)} \) and \( \hat{e} := \text{median}_{u=1, \ldots, U} \hat{e}^{(u)} \).

Next, we discuss the median-adjustment of cross-fitting procedure. Once we split the data into two folds \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), we obtain two direct rules \( \hat{\theta}_{(-\ell)} \) for \( k = 1, 2 \) where \( \mathcal{D} \) is used as the estimation data fold and \( \mathcal{D}_\ell \) is used as the evaluation data fold. Investigators may use either \( \hat{\theta}_{(-1)} \) or \( \hat{\theta}_{(-2)} \) as the final estimate of the OMAR, denoted by \( \hat{\theta}^{(F)} \). However, we recommend to use \( \hat{\theta}^{(F)}(x) = \mathcal{W}(\{\hat{\theta}_{(-1)} + \hat{\theta}_{(-2)}\}/2)(x) \), the winsorized rule of the average of two unwinsorized rules, for the new \( \mathcal{F} \) as the estimate of the OMAR to fully use the data. If the evaluation point is one of the points in the data, i.e., \( x = x_i \) for some \( i \in \mathcal{D}_\ell \), we recommend using \( \hat{\theta}^{(F)}(x) = \hat{\theta}_{(-\ell)}(x_i) \) because \( \hat{\theta}_{(-\ell)} \) does not depend on \( i \) while \( \hat{\theta}_{(\ell)} \) depends on \( i \) which may lead to an overfitted value. Second, to construct a more robust estimate of the OMAR under cross fitting, we use the recommendation in Chernozhukov et al. (2018) to our setting by taking the mean or the median of multiple OMAR estimates. Specifically, we repeat the estimation of \( \hat{\theta}^{(F)} \) multiple times, say \( T \) times, and obtain \( \hat{\theta}_t^{(F)}(t = 1, \ldots, T) \) where the sample partitions are randomly done across splits. We define the mean-OMAR estimate \( \hat{\theta}^{(F, \text{mean})}(x) = \sum_{t=1}^{T} \hat{\theta}_t^{(F)}(x)/T \) and the median-OMAR estimate \( \hat{\theta}^{(F, \text{median})}(x) = \text{median}_{t=1, \ldots, T} \hat{\theta}_t^{(F)}(x) \).
A.7 Details of Classification Performance Measures

For an given OMAR $\theta$, we define the true positives (TP), true negatives (TN), false positives (FP), and false negatives (FN) as follows:

$$TP = \sum_{i \in D_{test}} 1\{Y_i > T, \bar{A}_i > \theta(X_i)\}, \quad TN = \sum_{i \in D_{test}} 1\{Y_i \leq T, \bar{A}_i \leq \theta(X_i)\} , \quad (9)$$
$$FP = \sum_{i \in D_{test}} 1\{Y_i \leq T, \bar{A}_i > \theta(X_i)\}, \quad FN = \sum_{i \in D_{test}} 1\{Y_i > T, \bar{A}_i \leq \theta(X_i)\} .$$

Given these definitions, we use the following classification performance measures: accuracy, two-sided F1 score, and the Matthews correlation coefficient (MCC) (Matthews, 1975) which are defined as follows:

$$\text{Accuracy} = \frac{TP + TN}{TP + TN + FP + FN}, \quad F1 = \frac{2TP}{2TP + FP + FN} + \frac{2TN}{2TN + FP + FN},$$
$$\text{MCC} = \frac{TP \times TN - FP \times FN}{\{(TP + FP) \times (TP + FN) \times (TN + FP) \times (TN + FN)\}^{1/2}} .$$

The usual F1 score does not use true negatives in its score, i.e., $2TP/(2TP+FP+FN)$, and is sensitive to the definition of a positive label. For example, if we were to define the positive label as the opposite of the definition in equation (9), i.e., positive label if $Y_i \leq T$, the F1 score changes. To avoid this, we consider the two-sided F1 score, the average of the usual F1 score and the “opposite” F1 score, $2TN/(2TN+FP+FN)$.

A.8 Assessment of Assumptions in the Main Paper

We take a moment to discuss the plausibility of Assumptions (A1)-(A5) in the Senegal DHS.

(A1) Assumption (A1) is plausible as long as households in different census blocks do not interact with each other. In the data, 99.15% of the census blocks are geographically far apart from each other. The average and median distances among 22,578 pairs of census blocks in the 2018 Senegal DHS are 245.04km and 230.17km, respectively; only 192 (0.85%) pairs of census blocks have distance smaller than 10km. Given this, we find that the partial interference assumption is plausible where interference likely occurs between households in the same census block and not across different census blocks.
(A2) To check Assumptions (A2) and (A3), we check covariate balance and overlap by using the binning approach in Hirano and Imbens (2004), Kluve et al. (2012) and Flores et al. (2012) for a continuous treatment variable. Algorithm 2 shows the details on the covariate balance assessment.

**Algorithm 2** Assessment of Covariate Balance

1: Divide \( \sum_{i=1}^{N} n_i = 13556 \) units into four groups:

\[
A_k = \left\{ (i, j) \mid (A_{ij}, \overline{A}_{i(j)}) \in R_k \right\}, \quad R_k = \ \left\{ \begin{array}{ll}
\{0\} \times [0, \alpha_0] \in \{0,1\} \times [0,1] & \text{if } k = 1 \\
\{0\} \times (\alpha_0, 1] \in \{0,1\} \times [0,1] & \text{if } k = 2 \\
\{1\} \times [0, \alpha_1] \in \{0,1\} \times [0,1] & \text{if } k = 3 \\
\{1\} \times (\alpha_1, 1] \in \{0,1\} \times [0,1] & \text{if } k = 4 
\end{array} \right.
\]

where \( \alpha_0 \) and \( \alpha_1 \) are chosen so that \( A_1, \ldots, A_4 \) have similar sizes.

2: for \( k = 1, 2, 3, 4 \) do

3: Obtain unadjusted \( t \)-statistics that compare the distribution of \( X_i \) between \( A_k \) and \( \overline{A}_k \), i.e.,

\[
\left\{ \overline{X}_{i,k} \mid \overline{X}_{i,k} = \frac{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \in A_k\} X_{ij}}{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \in A_k\}} \right\} \text{ v. } \left\{ \widetilde{X}_{i,k}\|c = \frac{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \notin A_k\} X_{ij}}{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \notin A_k\}} \right\}
\]

4: Calculate the estimated propensity score \( \hat{P}_{ij,k} = \hat{P}\{(A_{ij}, \overline{A}_{i(j)}) \in R_k \mid X_{ij}, X_{i(-j)}\} \).

5: Let \( -\infty = q_0 \leq q_1 \leq \ldots \leq q_9 \leq q_{10} = \infty \) be the deciles of \( \{\hat{P}_{ij,k} \mid (i,j) \in A_k\} \).

6: Let \( E_{b,k} = \{(i,j) \mid \hat{P}_{ij,k} \in (q_{b-1}, q_b)\} \) \( (b = 1, \ldots, 10) \).

7: Obtain \( t \)-statistics that compare the distribution of \( X_i \) between \( E_{b,k} \cap A_k \) and \( E_{b,k} \cap \overline{A}_k \), i.e.,

\[
\left\{ \overline{X}_{i,k} \mid \overline{X}_{i,k} = \frac{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \in E_{b,k} \cap A_k\} X_{ij}}{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \in E_{b,k} \cap A_k\}} \right\} \text{ v. } \left\{ \widetilde{X}_{i,k}\|c = \frac{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \notin E_{b,k} \cap A_k\} X_{ij}}{\sum_{j=1}^{n_i} \mathbb{1}\{(i,j) \notin E_{b,k} \cap A_k\}} \right\}
\]

8: Aggregate the \( t \)-statistics obtained in Step 7 with weights from the size of \( E_{1,k}, \ldots, E_{10,k} \).

9: Obtain adjusted \( t \)-statistics by taking the median of \( t \)-statistics in Step 6 across multiple cross-fitting procedures.

10: end for

We use the median of the propensity score estimates from 100 cross-fitting procedures. As a consequence, we obtain the unadjusted/adjusted \( t \)-statistics in Figure A.2, which suggests covariate balance was satisfied for all cases.

(A3) Next, we assess the overlap assumption based on Algorithm 3. Again, we use the median of the propensity score estimates from 100 cross-fitting procedures.

Figure A.3 shows histograms that visually assess the overlap assumption. Based on the histograms, the overlap assumption seems to be satisfied or to be not severely violated.
Algorithm 3 Assessment of Overlap

1: Divide $\sum_{i=1}^{N} n_i = 13556$ units into four groups:

$$A_k = \{(i,j) \mid (A_{ij}, \overline{A}_{i(-j)}) \in R_k \}, \quad R_k = \begin{cases} 
\{0\} \times [0, \alpha_0] \times \{0,1\} \times [0,1] & \text{if } k = 1 \\
\{0\} \times (\alpha_0, 1) \in \{0,1\} \times [0,1] & \text{if } k = 2 \\
\{1\} \times [0, \alpha_1] \in \{0,1\} \times [0,1] & \text{if } k = 3 \\
\{1\} \times (\alpha_1, 1) \in \{0,1\} \times [0,1] & \text{if } k = 4 
\end{cases}$$

where $\alpha_0$ and $\alpha_1$ are chosen so that $A_1, \ldots, A_4$ have similar sizes.

2: Calculate the median of the estimated propensity scores obtained from multiple cross-fitting procedures, i.e.,

$$\hat{e}_{ij,k}^{(\text{median})} = \text{median}_{s=1,\ldots,S} \hat{P}(s) \{(A_{ij}, \overline{A}_{i(-j)}) \in R_k \mid X_{ij}, X_{i(-j)}\}$$

where the conditional probability $\hat{P}(s)$ is calculated from the estimated propensity score obtained from the $s$th cross-fitting procedure.

3: Compare histograms of $\hat{e}_{ij,k}^{(\text{median})}(X_i)$ for $A_k$ and $A_k'$ for each $k$.

Overall, all 9 observed covariates are balanced across different bins of treatment values and overlap is reasonable.

(A4) Assumption (A4) is plausible if the number of diarrhea-free children in a household can be reasonably approximated by a summary of peers’ WASH status. However, the assumption may fail if a few households’ presence (or absence) of WASH facilities is driving the incidence of diarrhea in the entire block, say if a few WASH-less households are located near communal water sources and they are primarily responsible for the diarrhea in the entire block. For example, if the census block has 20 households and the true response model for each
household is $E(Y_{ij} \mid A_i, X_i) = \beta_0 + \beta_1 A_{i1} + \beta_2 A_{ij} + \beta_3^\top X_{ij}$, i.e., every household $j$’s outcome depends on household 1’s treatment status, then $E(Y_i \mid A_i, X_i) = \beta_0 + \beta_1 A_{i1} + \beta_2 \overline{A}_i + \beta_3^\top \overline{X}_i$ and Assumption (A4) is violated because the average response of block $i$ depends on the treatment status of household 1. Unfortunately, the data does not contain information about the location of households to test these hypothesized violations of Assumption (A4). Instead, we visually diagnose the assumption by using a residual plot of the predicted values of the mean block-level response versus the observed block-level response. Specifically, let $\hat{\epsilon}_{ij}^{(\text{median})} = Y_{ij} - \hat{\mu}^{(\text{median})}(A_{ij}, \overline{A}_{i(-j)}, X_{ij}, X_{i(-j)})$ be the residuals where $\hat{\mu}^{(\text{median})}$ is the median of the outcome regression from 100 cross-fitting procedures. We compare the residuals across the outcome regression estimate and the regressors $(A_{ij}, \overline{A}_{i(-j)}, X_{ij}, X_{i(-j)})$ and check whether the residuals deviate from zero in Figure A.4. Since the dimension of $X_{i(-j)}$ varies, we use the average of $X_{i(-j)}$, i.e., $\overline{X}_{i(-j)} = \sum_{\ell \neq j} X_{i\ell} / (n_i - 1)$. In general, the residuals are close to zero across the regressors, implying that the outcome regression under Assumption (A4) is not severely violated. That is, while the diagnostic is not perfect, we find the predicted means do not show trends across the $x$-axis and Assumption (A4) could be plausible, subject to inherent limitations of the diagnostic plot.

(A5) Finally, for Assumption (A5), many prior works (Esrey et al., 1985; Daniels et al., 1990; Clasen et al., 2007; Ejemot-Nwadiaro et al., 2015; McMichael, 2019) suggest that installing WASH facilities will not have a negative impact on incidence of diarrhea; however, it may have a
negative effect on other, non-health outcomes. Also, when we empirically assess Assumption (A5), we find that the monotonicity assumption is rarely violated in the Senegal DHS and if violated, the deviation from monotonicity is small. Specifically, we first consider the difference between two cluster-level outcome regressions:

\[ V(a, a', s, s') = \bar{\mu} \left( a', \frac{s'}{n_i - 1}, X_i \right) - \bar{\mu} \left( a, \frac{s}{n_i - 1}, X_i \right) , \quad \bar{\mu}(a, a', X_i) = \frac{1}{n_i} \sum_{j=1}^{n_i} \mu(a, a', X_{ij}, X_{i(-j)}) . \]

In particular, we focus on the variations contrasting two adjacent outcome regressions. As a consequence, there are 3\( n_i - 2 \) finest variations where \( n_i - 1 \) variations have the form \( V(0, 0, s, s + 1) \), \( n_i - 1 \) variations have the form \( V(1, 1, s, s + 1) \), and \( n_i \) variations have the form \( V(0, 1, s, s) \); see the diagram below.

\[
\begin{align*}
\bar{\mu}(0, \frac{0}{n_i-1}, X_i) \xrightarrow{V_i(0,0,0,1)} & \bar{\mu}(0, \frac{1}{n_i-1}, X_i) \xrightarrow{V_i(0,0,1,2)} \ldots & \bar{\mu}(0, \frac{n_i-1}{n_i-1}, X_i) \\
\bar{\mu}(0, \frac{n_i-1}{n_i-1}, X_i) \xrightarrow{V_i(0,1,0,0)} & \bar{\mu}(1, \frac{0}{n_i-1}, X_i) \xrightarrow{V_i(1,0,1,1)} \bar{\mu}(1, \frac{1}{n_i-1}, X_i) \xrightarrow{V_i(1,1,1,1)} \bar{\mu}(1, \frac{n_i-1}{n_i-1}, X_i)
\end{align*}
\]
In the Senegal DHS, we have $\sum_{i=1}^{N}(3n_i - 2) = 38614$ variations in total. Let $\hat{V}_i^{(t)}(a, a', s, s')$ be the estimated variation of cluster $i$ obtained from the $t$th cross-fitting procedure, and let $\hat{V}_i^{(m)}(a, a', s, s')$ be the median of the variation, i.e.,

$$\hat{V}_i^{(m)}(a, a', s, s') = \text{median}\{\hat{V}_i^{(1)}(a, a', s, s'), \ldots, \hat{V}_i^{(S)}(a, a', s, s')\}$$

Assumption (A5) can be empirically assessed by two means. First, out of 38614 variations, we count the number of times monotonicity is violated. Second, we measure the worst-case slope of the estimated $\mu$ as follows. Let $TV_i(a, a', s, s')$ be the absolute value of $V_i(a, a', s, s')$. Thus, the sum of 38614 $TV_i(a, a', s, s')$ is the total variation of the cluster-level outcome regression. We compute the relative magnitude of the slopes that are decreasing compared to the total variation, i.e., $\sum 1(V < 0)TV/\sum TV$. Overall, under the first assessment, we found that the monotonicity is violated 1.11% of the time and under the second assessment, the relative magnitude of decreasing slopes is $6.01 \times 10^{-4}$. In short, the empirical validations show that the monotonicity assumption is rarely violated in the Senegal DHS and if violated, the deviation from monotonicity is small.

A.9 Details of Figures 5.2-5.4 in the Main Paper

We additionally describe how we draw Figures 5.2-5.4 in the main paper. The reported estimated OMARs in Figures 5.2 and 5.4 are weighted average of the estimated OMARs in each administrative region where weights are the number of households in a census block, i.e., census block size $n_i$. That is, the values represent $\bar{\theta}_g$s, which are defined as

$$\bar{\theta}_g = \frac{\sum_{i \in D_{2018}} 1\{i \in \text{administrative area } g\} \cdot n_i \cdot \hat{\theta}(x_i)}{\sum_{i \in D_{2018}} 1\{i \in \text{administrative area } g\} \cdot n_i}$$

where $D_{2018}$ is the collection of census blocks in the 2018 Senegal DHS. In words, $\bar{\theta}_g$ is the proportion of households in administrative area $g$ that require WASH facilities. Similarly, the average
household sizes in Figure 5.4 represent
\[
\bar{x}_{g,\text{Household Size}} = \frac{\sum_{i \in D_{2018}} 1\{i \in \text{administrative area } g\} \cdot n_i \cdot \bar{x}_{i,\text{Household Size}}}{\sum_{i \in D_{2018}} 1\{i \in \text{administrative area } g\} \cdot n_i}.
\]

Again, \(\bar{x}_{g,\text{Household Size}}\) is the average household wise in administrative area \(g\). The proportions of rural area in Figure 5.4 represent
\[
\bar{c}_{g,\text{Rural}} = \frac{\sum_{i \in D_{2018}} 1\{i \in \text{administrative area } g\} \cdot c_{i,\text{Rural}}}{\sum_{i \in D_{2018}} 1\{i \in \text{administrative area } g\}}.
\]

Here \(\bar{c}_{g,\text{Rural}}\) is the proportion of the rural census blocks in administrative area \(g\). Note that \(\bar{\theta}_g\), \(\bar{x}_{g,\text{Household Size}}\), and \(\bar{c}_{g,\text{Rural}}\) do not address the geographical distance between census regions in different administrative areas. But, we believe that these statistics are geographically meaningful summaries to highlight the heterogeneity across administrative areas; see Figures 1 and 2 of Houngbonon et al. (2021) for similar summary statistics aggregated at Senegalese administrative areas.

Lastly, Figure 5.3 shows the weighted average of the estimated OMARs across all 45 administrative areas where weights are the number of households in a census block, i.e., census block size \(n_i\). That is, the \(y\)-axis represents \(\bar{\theta}\), which is defined as
\[
\bar{\theta} = \frac{\sum_{i \in D_{2018}} n_i \cdot \tilde{\theta}(x_i)}{\sum_{i \in D_{2018}} n_i}.
\]

In words, \(\bar{\theta}\) is the proportion of households in Senegal that require WASH facilities.

## B Proof of Lemmas and Theorems

### B.1 Useful Lemmas

**Lemma B.1.** Suppose that \(\theta^*\) belongs to a Besov space on \(\mathbb{R}^d\) with smoothness parameter \(\beta > 0\), i.e.,
\[
\mathcal{B}_\beta^{1,\infty}(\mathbb{R}^d) = \{ \theta \in L_\infty(\mathbb{R}^d) \mid \sup_{t > 0} t^{-\beta} \omega_r(\mathcal{L}_1(\mathbb{R}^d))(\theta, t) < \infty, r > \beta \}
\]
where \(\omega_r\) is the modulus of continuity of order \(r\). Then, for any \(\epsilon > 0\) and \(d/(d+\tau) < p < 1\) with \(\tau > 0\), we have the following...
excess risk bound of \( \hat{\theta} \) with probability not less than \( 1 - 3e^{-\tau} \).

\[
R(\hat{\theta}) - R(\theta^*) \leq c_1 \lambda_N \gamma_N^{-d} + c_2 \gamma_N + c_3 \left\{ \gamma_N^{(1-p)(1+e)d} \lambda_N^p N \right\}^{-\frac{1}{p}} + c_4 N^{-1/2} \tau^{1/2} + c_5 N^{-1} \tau
\]

**Lemma B.2.** Let \( \hat{R}_{(\epsilon)}(\theta) = E\{L_{(\epsilon)}(\theta(X_i), O_i) | D^e_i) \} \) be the estimated risk function where the expectation is taken with respect to \( O_i \) while \( L_{(\epsilon)} \) is considered as a fixed function which is clarified by denoting \( D^e_i \) in the conditioning statement. Under conditions in Assumption (A1)-(A5) and (E1)-(E3) in the main paper, we have \( |R(\theta) - \hat{R}_{(\epsilon)}(\theta)| = O_P(r_N) \) where \( r_N = r_{e,N} \) if the inverse probability-weighted loss function is used, \( r_N = r_{\mu,N} \) if the outcome regression loss function is used, and \( r_N = r_{e,N}r_{\mu,N} \) if the doubly robust loss function is used.

**B.2 Proof of Lemma 3.1**

We only show the result about the overall outcome case because the result about the spillover outcome case is obtained from a similar manner. Let \( \Theta^{[0,1]} = \{ f \mid f(x_i) \in [0,1] \} \) be the collection of rules ranging over the unit interval and \( W(\theta) \in \Theta^{[0,1]} \) be the winsorized function of \( \theta \in \Theta \) over the unit interval, i.e.

\[
W(\theta)(x_i) = 0 \cdot \mathbb{1}\{\theta(x_i) < 0\} + \theta(x_i) \cdot \mathbb{1}\{0 \leq \theta(x_i) \leq 1\} + 1 \cdot \mathbb{1}\{1 < \theta(x_i)\}.
\]

From the definition of \( L \), we find \( L(0, O_i) \leq L(t, O_i) \) for any \( t \in (-\infty, 0) \) and \( L(1, O_i) \leq L(t, O_i) \) for any \( t \in (1, \infty) \). As a consequence, for any rule \( \theta \in \Theta \) satisfies \( L(W(\theta)(X_i), O_i) \leq L(\theta(X_i), O_i) \) and \( R(W(\theta)) \leq R(\theta) \). This implies that \( \theta^* \), the minimizer of \( R \), must belong to \( \Theta^{[0,1]} \).

For any function \( \theta \in \Theta^{[0,1]} \), we find \( L(\theta(X_i), O_i) = \nu(\theta(X_i), O_i) \) and \( R(\theta) = R^{[0,1]}(\theta) \triangleq E\{\nu(\theta(X_i), O_i)\} \) due to the constructions of \( L \) and \( R \). Combining the above results, we observe the following relationship.

\[
\arg \min_{\theta \in \Theta} R(\theta) = \arg \min_{\theta \in \Theta^{[0,1]}} R(\theta) = \arg \min_{\theta \in \Theta^{[0,1]}} R^{[0,1]}(\theta)
\]

Thus, it suffices to show that \( \theta^* \) defined in (4) in the main paper minimizes \( R^{[0,1]} \), which is repre-
sent as follows.

\[ R^{[0,1]}(\theta) \]

\[ = E\{\nu(\theta(X_i), O_i)\} \]

\[ = E\left[ \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \left( \frac{n_i - 1}{s} \right) \psi_{\text{DR}}(a, s, O_{ij}, O_{i(-j)}) \sum_{\ell=0}^{n_i-a-s} \left( \frac{n_i - a - s}{\ell} \right) \frac{(-1)^{\ell} \{\theta(X_i)\}^{\ell+a+s+1}}{\ell + a + s + 1} - T \{\theta(X_i)\} \right] + C_0 \]

\[ = E\left[ \int_0^1 \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \left( \frac{n_i - 1}{s} \right) E\{\psi_{\text{DR}}(a, s, O_{ij}, O_{i(-j)}) \mid X_i\} \alpha^{a+s}(1 - \alpha)^{n_i-a-s} - T \} \{\alpha \leq \theta(X_i)\} d\alpha \right] + C_0 \]

\[ = E\left[ \int_0^1 \left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \left( \frac{n_i - 1}{s} \right) \mu^*(a, \frac{s}{n_i-1}, X_{ij}, X_{i(-j)}) \alpha^{a+s}(1 - \alpha)^{n_i-a-s} - T \} \{\alpha \leq \theta(X_i)\} d\alpha \right] + C_0 . \]

The first and second identities are trivial from the definition of \( R^{[0,1]} \) and \( \nu \). The third identity is from the law of iterated expectation and the following algebra:

\[ H(O_i) \sum_{\ell=0}^{n_i-a-s} \left( \frac{n_i - a - s}{\ell} \right) \frac{(-1)^{\ell} \alpha^{a+s}(1 - \alpha)^{n_i-a-s} - T \} \{\alpha \leq \theta(X_i)\} d\alpha \]

\[ = \int_0^t \left[ H(O_i) \alpha^{a+s}(1 - \alpha)^{n_i-a-s} - T \} \{\alpha \leq \theta(X_i)\} \right] d\alpha \]

\[ = \int_0^t \left\{ H(O_i) \alpha^{a+s}(1 - \alpha)^{n_i-a-s} - T \} \{\alpha \leq \theta(X_i)\} \right\} d\alpha , \forall H(O_i) , \forall t \in [0,1] . \]

The fourth identity is from \( E\{\psi_{\text{DR}}(a, s, O_{ij}, O_{i(-j)}) \mid X_i\} = \mu^*(a, \frac{s}{n_i-1}, X_{ij}, X_{i(-j)}) \) for \( s = 0, 1, \ldots, n_i-1 \); we remark that any \( \psi' \) satisfying \( E\{\psi'(a, s, O_{ij}, O_{i(-j)}) \mid X_i\} = \mu^*(a, \frac{s}{n_i-1}, X_{ij}, X_{i(-j)}) \) (e.g., inverse probability-weighted or outcome regression-based) can be used instead of \( \psi_{\text{DR}} \). From the last representation, it is straightforward that \( R^{[0,1]}(\theta) \) is minimized at \( \theta' \) that satisfies the following conditions.

\[ \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \left( \frac{n_i - 1}{s} \right) E\{\psi_{\text{DR}}(a, s, O_{ij}, O_{i(-j)}) \mid X_i\} \alpha^{a+s}(1 - \alpha)^{n_i-a-s} \leq T \text{ for all } \alpha \in [0, \theta'(X_i)] , \]

\[ \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \left( \frac{n_i - 1}{s} \right) E\{\psi_{\text{DR}}(a, s, O_{ij}, O_{i(-j)}) \mid X_i\} \alpha^{a+s}(1 - \alpha)^{n_i-a-s} \geq T \text{ for all } \alpha \in [\theta'(X_i), 1] . \]

As a consequence, \( \theta' \) agrees with \( \theta_{SV}^* \) defined in (4) in the main paper. We can establish the result about \( \theta_{SV}^* \) from a similar fashion. This concludes the proof.
B.3 Proof of Lemma B.1

We only show the result about the overall outcome case because the result about the spillover outcome case is obtained from a similar manner. The proof of B.1 is similar to that of Theorem 2 of Chen et al. (2016) which use Theorem 7.23 of Steinwart and Christmann (2008) and Theorem 2.2 and 2.3 of Eberts and Steinwart (2013). For completeness, we present a full exposition to our setting below. We first introduce Theorem 7.23 of Steinwart and Christmann (2008).

**Theorem 7.23.** (Oracle Inequality for SVMs Using Benign Kernels; Steinwart and Christmann (2008)) Let $L$ be a loss function having non-negative value. Also, let $H_K$ be a separable RKHS of a measurable kernel $K$ over $\mathcal{X} = \text{supp}(X_i) \subset \mathbb{R}^d$. Let $P$ be a distribution on $O_i$. Furthermore, suppose the following conditions are satisfied.

1. For all $(t, o_i)$, there exists a constant $B > 0$ satisfying $L(t, o_i) \leq B$.
2. $L(t, o_i)$ is locally Lipschitz continuous with respect to $t$.
3. For all $(t, o_i)$, we have $L(W_{c_0}(t), o_i) \leq L(t, o_i)$ where $W_{c_0}(t) = t \cdot \mathbb{1}\{|t| \leq c_0\} + \text{sign}(t)c_0 \cdot \mathbb{1}\{c_0 < |t|\}$.
4. $E[\{L(W_{c_0}(\theta)(X_i), O_i) - L(\theta^*(X_i), O_i)\}^2] \leq V \cdot [E\{L(W_{c_0}(\theta)(X_i), O_i) - L(\theta^*(X_i), O_i)\}]^v$ is satisfied for constant $v \in [0, 1]$, $V \geq B^{2-v}$, and for all $\theta \in H_K$.
5. For fixed $N \geq 1$, there exists constants $p \in (0, 1)$ and $a \geq B$ such that the dyadic entropy number $E_{D_X \sim P_N}[e_i(\text{identity map} : H_K \to L_2(D_X))] \leq a \cdot i^{-\frac{1}{p}}$ (i $\geq 1$) where $e_i(A)$ is the entropy number of $A$.

We fix $\theta_0 \in H_K$ and a constant $B_0 \geq B$ such that $L(\theta_0(x_i), o_i) \leq B_0$ for any $o_i$. Then, for all fixed $\tau > 0$ and $\lambda_N$, the SVM using $H_K$ and $L$ satisfies

$$
\lambda_N \| \theta \|^2_{H_K} + R(W_{c_0}(\theta)) - R(\theta^*)
\leq 9\{\lambda_N \| \theta_0 \|^2_{H_K} + R(\theta_0) - R(\theta^*)\} + K_0 \left( \frac{a^{2p} \tau}{N} \right)^{\frac{1}{2-p-v+w}} + 3 \left( \frac{72 V \tau}{N} \right)^{\frac{1}{2-v}} + \frac{15 B_0 \tau}{N}.
$$

with probability $P^N$ not less than $1 - 3e^{-\tau}$, where $K_0 \geq 1$ is a constant only depending on $p$, $c_0$, $B$, $v$, and $V$. 

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We verify Assumptions (C1)-(C5) as follows:

**Verification of Assumption (C1):** From Assumptions (A1)-(A5) and $Y_i \in [0, 1]$, we find that $\psi_{\text{IPW}}, \psi_{\text{OR}},$ and $\psi_{\text{DR}}$ are bounded and, as a consequence, $\nu$ in (6a) in the main paper is bounded. As a result, $L$ in (6) in the main paper is bounded as well.

**Verification of Assumption (C2):** We find the derivative of $L$ in (6) in the main paper

\[
\nabla L(t, o_i) = \begin{cases} 
\delta e^t & t \in (-\infty, 0) \\
\sum_{s=0}^{n_i} \psi_t(s, o_i)t^s(1-t)^{n_i-s} - T & t \in (0, 1) \\
\delta e^{1-t} & t \in (0, \infty)
\end{cases}
\]

and we find that $\nabla L(t, o_i)$ is bounded for all $t$ except $t = 0, 1$. Moreover, $L(t, o_i)$ is continuous at $t = 0$ and $t = 1$. Thus, $L(t, o_i)$ is locally Lipschitz continuous with Lipschitz constant $B' = \sup_{(t, o_i)} \nabla L(t, o_i)$.

**Verification of Assumption (C3):** We take $c_0 = 1$. It is trivial that $L(W_{c_0}(\theta), o_i) \leq L(\theta, o_i)$ from the form of $L$ in (6) in the main paper.

**Verification of Assumption (C4):** Note that $L(t, o_i) \leq B$. Thus, we find

\[
E\left[\{L(W_{c_0}(\theta)(X_i), O_i) - L(\theta^*(X_i), O_i)\}^2\right] \leq 2E\left[\{L(W_{c_0}(\theta)(X_i), O_i)\}^2 + \{L(\theta^*(X_i), O_i)\}^2\right] \leq 4B^2.
\]

We take $v = 0$ and $V = 4B^2$ and the condition is satisfied.

**Verification of Assumption (C5):** Since we use the Gaussian kernel, we can directly use Theorem 7.34 of Steinwart and Christmann (2008) which is given below.

**Theorem 7.34. (Entropy Numbers for Gaussian Kernels; Steinwart and Christmann (2008))** Let $\nu$ be a distribution on $\mathbb{R}^d$ having tail exponent $\tau \in (0, \infty]$. Then, for all $\epsilon > 0$ and $d/(d+\tau) < p < 1$, 

there exists a constant $c_{\epsilon,p} \geq 1$ such that
\[
e_i(\text{identity map} : \mathcal{H}_K \to L_2(\nu)) \leq c_{\epsilon,p} \gamma^{\frac{(1-p)(1+\epsilon)d}{2p} - \frac{1}{2p}}
\]
for all $i \geq 1$ and $\gamma \in (0,1]$.

Therefore, Assumption (C5) holds with $a = c_{\epsilon,p} \gamma_N^{\frac{(1-p)(1+\epsilon)d}{2p}}$.

As a consequence, the result in equation (10) holds with $c_0 = 1$, $v = 1$, $V = 4B^2$, $B_0 = B$, $a = c_{\epsilon,p} \gamma_N^{\frac{(1-p)(1+\epsilon)d}{2p}}$. Moreover, we find $L(W(\theta)(x_i), o_i) \leq L(W_{c_0 = 1}(\theta)(x_i), o_i)$ since $L(0, o_i) \leq L(t, o_i)$ for $t \in [-1, 0]$ and this leads $R(W(\theta)) \leq R(W_{c_0 = 1}(\theta))$. Thus, we find the following result holds with probability $P_N$ not less than $1 - 3e^{-\tau}$.

\[
R(\hat{\theta}) - R(\theta^*) 
\leq R(W_{c_0 = 1}(\tilde{\theta})) - R(\theta^*) 
\leq \lambda_N ||\tilde{\theta}||_{\mathcal{H}_K}^2 + R(W_{c_0 = 1}(\tilde{\theta})) - R(\theta^*) 
\leq 9\{\lambda_N ||\theta_0||_{\mathcal{H}_K}^2 + R(\theta_0) - R(\theta^*)\} + K_0\left\{\frac{\gamma_N^{-\frac{(1-p)(1+\epsilon)d}{2p}}}{\lambda_N^p} \right\}^{\frac{1}{2-p}} + 36\sqrt{2B}N + 15B N.
\]

The above result holds for any $\theta_0 \in \mathcal{H}_K$, so we can further bound the approximation error $\lambda_N ||\theta_0||_{\mathcal{H}_K}^2 + R(\theta_0) - R(\theta^*)$ by choosing $\theta_0$ in a specific way which is presented in Eberts and Steinwart (2013). We first define a function $Q_{r,\gamma} : \mathbb{R}^d \to \mathbb{R}$ as
\[
Q_{r,\gamma}(z) = \sum_{j=1}^{r-j} \frac{(-1)^{1-j}}{j^d} \left(\frac{2}{\sqrt{2}}\right)^{\frac{d}{2}} K_{\gamma/\sqrt{2}}(z), \ K_{\gamma}(z) = \exp \left\{ -\gamma^2 ||z||_2^2 \right\}
\]
for $r \in \{1,2,\ldots\}$ and $\gamma > 0$. Since the range of $\theta^*$ is bounded between $[0,1]$, we find $\theta^* \in L_2(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Thus, we can define $\theta_0$ by convolving $Q_{r,\gamma}$ with $\theta^*$ as follows (Eberts and Steinwart, 2013).

\[
\theta_0(x_i) = (Q_{r,\gamma} \ast \theta^*)(x_i) = \int_{\mathbb{R}^d} Q_{r,\gamma}(x_i - z)\theta^*(z) \, dz.
\]

Next, we introduce theorem 2.2 and 2.3 of Eberts and Steinwart (2013).

**Theorem 2.2.** Let us fix some $q \in [1,\infty)$. Furthermore, assume that $P_X$ is a distribution on
\( \mathbb{R}^d \) that has a Lebesgue density \( f_X \in L_p(\mathbb{R}^d) \) for some \( p \in [1, \infty] \). Let \( \theta : \mathbb{R}^d \to \mathbb{R} \) be such that \( \theta \in L_q(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \). Then, for \( r \in \{1, 2, \ldots\} \), \( \gamma > 0 \), and \( s \geq 1 \) with \( 1 = s^{-1} + p^{-1} \), we have

\[
\|Q_{r,\gamma} * \theta - \theta\|_{L_q(\mathbb{R}^d)}^q \leq C_{r,q} \cdot \|f_X\|_{L_p(\mathbb{R}^d)} \cdot \omega_{r,L_q(\mathbb{R}^d)}^q(\theta, \gamma/2)
\]

where \( C_{r,q} \) is a constant only depending on \( r \) and \( q \).

**Theorem 2.3.** Let \( \theta \in L_2(\mathbb{R}^d) \), \( \mathcal{H}_K \) be the RKHS of the Gaussian kernel \( K \) with parameter \( \gamma > 0 \), and \( Q_{r,\gamma} \) be defined by (12) for a fixed \( r \in \{1, 2, \ldots\} \). Then we have \( Q_{r,\gamma} * \theta \in \mathcal{H}_K \) with

\[
\|Q_{r,\gamma} * \theta\|_{\mathcal{H}_K} \leq \left( \gamma \sqrt{\pi} \right)^{-d} (2^r - 1) \|\theta\|_{L_2(\mathbb{R}^d)}.
\]

Moreover, if \( \theta \in L_\infty(\mathbb{R}^d) \), we have \( |Q_{r,\gamma} * \theta| \leq (2^r - 1) \|\theta\|_{L_\infty(\mathbb{R}^d)} \).

As a result, we obtain

\[
\lambda_N \|\theta_0\|_{\mathcal{H}_K}^2 + R(\theta_0) - R(\theta^*)
= \lambda_N \|Q_{r,\gamma_N} * \theta^*\|_{\mathcal{H}_K}^2 + R\left(Q_{r,\gamma_N} * \theta^*\right) - R(\theta^*)
\leq \lambda_N \left( \gamma_N \sqrt{\pi} \right)^{-d} (2^r - 1)^2 \|\theta^*\|_{L_2(\mathbb{R}^d)}^2 + R\left(Q_{r,\gamma_N} * \theta^*\right) - R(\theta^*)
\leq \lambda_N \left( \gamma_N \sqrt{\pi} \right)^{-d} (2^r - 1)^2 \|\theta^*\|_{L_2(\mathbb{R}^d)}^2 + B' \cdot \|Q_{r,\gamma_N} * \theta^* - \theta^*\|_{L_1(\mathbb{R}^d)}
\leq \lambda_N \left( \gamma_N \sqrt{\pi} \right)^{-d} (2^r - 1)^2 \|\theta^*\|_{L_2(\mathbb{R}^d)}^2 + B' \cdot C_{r,1} \cdot \|f_X\|_{L_\infty(\mathbb{R}^d)} \cdot \omega_{r,L_1(\mathbb{R}^d)}(\theta, \gamma_N / 2)
\leq \lambda_N \left( \gamma_N \sqrt{\pi} \right)^{-d} (2^r - 1)^2 \|\theta^*\|_{L_2(\mathbb{R}^d)}^2 + B' c' \cdot C_{r,1} \cdot \|f_X\|_{L_\infty(\mathbb{R}^d)} \gamma_N^\beta
\]

The first equality is from the construction of \( \theta_0 \). The first inequality is from Theorem 2.3 of Eberts and Steinwart (2013). The second inequality is from Lipschitz continuity of \( L \). The third inequality is from Theorem 2.2 of Eberts and Steinwart (2013) with \( q = s = 1 \) and \( p = \infty \). The last inequality holds for some constant \( c' \) since \( \theta^* \in \mathcal{B}_1^\beta(\mathbb{R}^d) \) implies \( \omega_{r,L_1(\mathbb{R}^d)}(\theta^*, \gamma_N / 2) \leq c' \gamma_N^\beta \) from the definition.
of a Besov space. Combining the results in (11) and (13), we have the following result

\[
R(\hat{\theta}) - R(\theta^*) \leq 9 \{ \lambda_N \sqrt{\tau} \}^{-d} (2^r - 1)^2 \| \theta^* \|_{L^2(\mathbb{R}^d)}^2 + B \cdot C_{t,1} \cdot \| f_X \|_{L^\infty(\mathbb{R}^d)}^{1/2} + K_0 \left\{ \frac{\gamma_N (1-p)(1+\epsilon)d}{\lambda_N N} \right\} \frac{1}{2^{p-1}} + 36 \sqrt{2} B \sqrt{\tau} N + 15 B \tau N
\]

\[
\leq c_1 \lambda_N \gamma_N^{-d} + c_2 \gamma_N^\beta + c_3 \left\{ \gamma_N (1-p)(1+\epsilon)d\lambda_N^p N \right\} \frac{1}{2^{p-1}} + c_4 N^{-1/2} \tau^{1/2} + c_5 N^{-1} \tau
\]

B.4 Proof of Lemma B.2

We only show the result about the overall outcome case because the result about the spillover outcome case is obtained from a similar manner. We find the following result for \( t \in [0, 1], \)

\[
\left| \tilde{L}_{(-\ell)}(t, O_i) - L(t, O_i) \right|
\]

\[
= \left| \tilde{\nu}_{(-\ell)}(t, O_i) - \nu(t, O_i) \right|
\]

\[
= \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \sum_{s=0}^{n_i-1} \left( n_i - 1 \right) \left\{ \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) - \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) \right\} \sum_{\ell=0}^{n_i-a-s} \left( n_i - a - s \right) \frac{(-1)\ell^{\ell+a+s+1}}{\ell + a + s + 1}
\]

\[
\leq \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a=0}^{n_i-1} \sum_{s=0}^{n_i-1} \left( n_i - 1 \right) \left| \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) - \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) \right| \sum_{\ell=0}^{n_i-a-s} \left( n_i - a - s \right) \frac{\ell^{\ell+a+s+1}}{\ell + a + s + 1}
\]

\[
\leq C' \max_{(a,s) \in \{0,1\} \times \{0,..,M\}} \left| \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) - \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) \right|
\]

for some generic constant \( C' \). The last inequality is from \( t \in [0, 1] \) and bounded \( n_i \). Also, we find the following result for \( t \in (-\infty, 0), \)

\[
\left| \tilde{L}_{(-\ell)}(t, O_i) - L(t, O_i) \right| = \left| \tilde{\nu}_{(-\ell)}(0, O_i) - \nu(0, O_i) \right| = 0.
\]

Finally, we find the following result for \( t \in (1, \infty) \) for all \( s = 0, 1, \ldots, n_i \).

\[
\left| \tilde{L}_{(-\ell)}(t, O_i) - L(t, O_i) \right| = \left| \tilde{\nu}_{(-\ell)}(1, O_i) - \nu(1, O_i) \right|
\]

\[
\leq C' \max_{(a,s) \in \{0,1\} \times \{0,..,M\}} \left| \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) - \hat{\psi}_k(a, s, O_{ij}, O_{(i-j)}) \right|
\]
where constant $C'$ is given in (14). Therefore, for any $\theta$, we find

$$\left| R(\theta) - \tilde{R}_{(-\ell)}(\theta) \right| \leq E \left[ \left| L(\theta(X_i), O_i) - \tilde{L}_{(-\ell)}(\theta(X_i), O_i) \right| D_{\ell}^c \right]$$

$$\leq E \left[ \left| L(\theta(X_i), O_i) - \tilde{L}_{(-\ell)}(\theta(X_i), O_i) \right|^2 \right]^{1/2}$$

$$\leq C'E \max_{(a,s) \in \{0,1\} \otimes \{0,...,M\}} \left| \tilde{\psi}_k(a, s, O_{ij}, O_{i(-j)}) - \psi_k(a, s, O_{ij}, O_{i(-j)}) \right|^2 \left| D_{\ell}^c \right]^{1/2}$$

$$\leq C' \max_{(a,s) \in \{0,1\} \otimes \{0,...,M\}} \left\| \tilde{\psi}_k(a, s, O_{ij}, O_{i(-j)}) - \psi_k(a, s, O_{ij}, O_{i(-j)}) \right\|_{P,2}$$

The first inequality is from the definition of $R$. The second inequality is from the Jensen’s inequality. The third inequality is from the above results. The last inequality is from the definition of $\| \cdot \|_{P,2}$. Therefore, it suffices to bound $\left\| \tilde{\psi}_k(a, s, O_{ij}, O_{i(-j)}) - \psi_k(a, s, O_{ij}, O_{i(-j)}) \right\|_{P,2}$ for all three types of $\psi_k$ where $k \in \{\text{IPW}, \text{OR}, \text{DR}\}$.

First, the difference between $\tilde{\psi}_{\text{IPW}}$ and $\psi_{\text{IPW}}(a, s, O_{ij}, O_{i(-j)})$ is

$$\left| \tilde{\psi}_{\text{IPW}}(a, s, O_{ij}, O_{i(-j)}) - \psi_{\text{IPW}}(a, s, O_{ij}, O_{i(-j)}) \right|$$

$$= \left| Y_{ij} 1(A_{ij} = a, S_{i(-j)} = s) - \hat{e}_{(-\ell)}(a, s \mid X_i) \right|$$

$$\leq \left| Y_{ij} 1(A_{ij} = a, S_{i(-j)} = s) \right| \left| e^*(a, s \mid X_i) - \hat{e}_{(-\ell)}(a, s \mid X_i) \right|_{cc'}$$

The upper bound is from the bounded outcome and Assumptions (A3) and (E1). Thus, we find

$$\left\| \tilde{\psi}_{\text{IPW}}(a, s, O_{ij}, O_{i(-j)}) - \psi_{\text{IPW}}(a, s, O_{ij}, O_{i(-j)}) \right\|_{P,2} \leq \left\| e^*(a, s \mid X_i) - \hat{e}_{(-\ell)}(a, s \mid X_i) \right\|_{P,2}/(cc') = O(r_{e,N})$$

with probability greater than $1 - \Delta_N$.

Second, we study the difference between $\tilde{\psi}_{\text{OR}}$ and $\psi_{\text{OR}}$ which is

$$\left| \tilde{\psi}_{\text{OR}}(a, s, O_{ij}, O_{i(-j)}) - \psi_{\text{OR}}(a, s, O_{ij}, O_{i(-j)}) \right|$$

$$= \left| \hat{\mu}_{(-\ell)}(a, s/\{n_i - 1\}, X_{ij}, X_{i(-j)}) - \mu^*(a, s/\{n_i - 1\}, X_{ij}, X_{i(-j)}) \right|.$$
Lastly, we prove the result when $\psi_{DR}$ is chosen. From (14), we have

$$-C'[\hat{\psi}_{DR}(a, s, O_{ij}, O_{i(-j)}) - \psi_{DR}(a, s, O_{ij}, O_{i(-j)})]$$

$$\leq \hat{L}(-\ell)(t, O_i) - L(t, O_i) \leq C'[\hat{\psi}_{DR}(a, s, O_{ij}, O_{i(-j)}) - \psi_{DR}(a, s, O_{ij}, O_{i(-j)})]$$

where the sign of $C'$ is chosen to satisfy the inequality above. The expectation of $\hat{\psi}_{DR} - \psi_{DR}$ is

$$E\{\hat{\psi}_{DR}(a, s, O_{ij}, O_{i(-j)}) - \psi_{DR}(a, s, O_{ij}, O_{i(-j)}) \mid D'_{\ell}\}$$

$$= E\left\{ \frac{Y_{ij} - \hat{\mu}(-\ell)(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)})}{\hat{c}(-\ell)(a, s \mid X_i)} - \frac{Y_{ij} - \mu(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)})}{e^*(a, s \mid X_i)} \right\} 1(A_{ij} = a, S_{i(-j)} = s)$$

$$+ \hat{\mu}(-\ell)(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)}) - \mu^*(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)}) \mid D'_{\ell}\}$$

$$= E\left\{ \frac{\{\mu^*(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)}) - \hat{\mu}(-\ell)(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)})\}}{e^*(a, s \mid X_i)} \right\} 1(A_{ij} = a, S_{i(-j)} = s)$$

The equalities are straightforward from the definition of $\psi_{DR}$ and the law of total expectation. Since $c' \leq \hat{c}(-\ell)$, we find

$$E\left\{ \frac{\mu^*(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)}) - \hat{\mu}(-\ell)(a, \frac{s}{n_i - 1}, X_{ij}, X_{i(-j)})}{\hat{c}(-\ell)(a, s \mid X_i)} \right\} 1(A_{ij} = a, S_{i(-j)} = s)$$

$$\leq \frac{1}{c'} E\left\{ \left\| \mu(s/n_i, X_i) - \hat{\mu}(-\ell)(s/n_i, X_i) \right\| E(s \mid X_i) - \hat{c}(-\ell)(s \mid X_i) \right\}$$

$$\leq \frac{1}{c'} \left\| \mu(s/n_i, X_i) - \hat{\mu}(-\ell)(s/n_i, X_i) \right\|_{p,2} E\left\{ \left\| E(s \mid X_i) - \hat{c}(-\ell)(s \mid X_i) \right\|_{p,2} \right\}$$

The first inequality is straightforward. The second inequality is from the Hölder’s inequality. From the last line, we find $E\{\hat{\psi}_{DR}(a, s, O_{ij}, O_{i(-j)}) - \psi_{DR}(a, s, O_{ij}, O_{i(-j)}) \mid D'_{\ell}\} = O(r_c N r_{\mu, N})$. As a
result, we have the following result with probability greater than \(1 - \Delta_N\).

\[
|R(\theta) - \tilde{R}_{(-\ell)}(\theta)| = \left| \mathbb{E}\left\{ L(\theta(X_i), O_i) - \tilde{L}_{(-\ell)}(\theta(X_i), O_i) \mid D_\ell^c \right\} \right|
\leq C' \max_{(a,s) \in \{0,1\} \otimes \{0,\ldots,M\}} \left| \mathbb{E}\left\{ \tilde{\psi}_{DR}(a,s,O_{ij},O_{i(-j)}) - \psi_{DR}(a,s,O_{ij},O_{i(-j)}) \mid D_\ell^c \right\} \right|
\leq C \max_{(a,s) \in \{0,1\} \otimes \{0,\ldots,M\}} \left[ \| \mu^*(a,s/n_{i-1}, X_{ij}, X_{i(-j)}) - \tilde{\mu}_{(-\ell)}(a,s/n_{i-1}, X_{ij}, X_{i(-j)}) \|_{P,2} \right]
\times \| e^*(a,s \mid X_i) - \tilde{e}_{(-\ell)}(a,s \mid X_i) \|_{P,2}
= O(r_{e,Nr_{\mu,N}}).
\]

Combining the established results, we have the following results with probability greater than \(1 - \Delta_N\).

\[
|R(\theta) - \tilde{R}_{(-\ell)}(\theta)| = \begin{cases} 
O(r_{e,N}) & \text{if } \psi_{\text{IPW}} \text{ is chosen} \\
O(r_{\mu,N}) & \text{if } \psi_{\text{OR}} \text{ is chosen} \\
O(r_{e,Nr_{\mu,N}}) & \text{if } \psi_{\text{DR}} \text{ is chosen}
\end{cases}
\]

This implies \(|R(\theta) - \tilde{R}_{(-\ell)}(\theta)| = O_P(r_N)\) where \(r_N = r_{e,N}\) if the inverse probability-weighted loss function is used, \(r_N = r_{\mu,N}\) if the outcome regression loss function is used, and \(r_N = r_{e,Nr_{\mu,N}}\) if the doubly robust loss function is used.

**B.5 Proof of Theorem 3.1 in the Main Paper**

We only show the result about the overall outcome case because the result about the spillover outcome case is obtained from a similar manner. We start with defining the risk function and the OMAR associated with the estimated loss function. Let \(\tilde{R}_{(-\ell)}(\theta) = \mathbb{E}\left\{ \tilde{L}_{(-\ell)}(\theta(X_i), O_i) \mid D_\ell^c \right\}\) be the estimated risk function where the expectation is taken with respect to \(O_i\) while \(\tilde{L}_{(-\ell)}\) is considered as a fixed function which is clarified by denoting \(D_\ell^c\) in the conditioning statement. Accordingly, let \(\theta^*_{(-\ell)}\) be the approximated OMAR which is the minimizer of \(\tilde{R}_{(-\ell)}(\theta)\), i.e., \(\tilde{R}_{(-\ell)}(\theta^*_{(-\ell)}) \leq \tilde{R}_{(-\ell)}(\theta)\) for all \(\theta \in \Theta\). Using \(\theta^*_{(-\ell)}\) as the intermediate quantities, we can establish the excess risk of \(\tilde{\theta}_{(-\ell)}\).
We decompose the excess risk as follows.

\[
\left| R(\hat{\theta}_{(-\ell)}) - R(\theta^*) \right| \\
= \left| R(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) \right| + \left| \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) \right| + \left| \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) \right| .
\]

In the rest of the proof, we bound terms \((A), (B),\) and \((C).\)

From Lemma 3.1 in the main paper, we find the upper bound of \((A).\)

\[
\left| R(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) \right| = O_P(r_N) .
\]

Next, we bound of \((B).\) Since $\hat{L}_{(-\ell)}$ satisfies Assumptions (C1)-(C5) and $\theta^*_{(-\ell)}$ belongs to a Besov space $B^\beta_{1,\infty}(\mathbb{R}^d)$, Theorem B.1 can be applied. Hence, we find

\[
\left| \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) \right| = O_P\left(N^{-\frac{\beta}{2\beta+2}}\right) .
\]

Lastly, we bound \((C).\) Since $\theta^*$ is the minimizer of $R(\theta)$, we find

\[
\hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) = R(\theta^*_{(-\ell)}) + \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*_{(-\ell)}) \geq R(\theta^*) + \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) ,
\]

and this implies $\hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) \leq \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*)$. Similarly, since $\theta^*_{(-\ell)}$ is the minimizer of $\hat{R}_{(-\ell)}(\theta)$, we find

\[
R(\theta^*) = \hat{R}_{(-\ell)}(\theta^*) + R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*) \geq \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) + R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*) ,
\]

and this implies $R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*) \leq R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*)$, i.e., $\hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) \leq \hat{R}_{(-\ell)}(\theta^*) - R(\theta^*)$. Combining two results, we have

\[
\left| \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) \right| \leq \max \left\{ \left| \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) \right| , \left| \hat{R}_{(-\ell)}(\theta^*) - R(\theta^*) \right| \right\} .
\]

From Lemma 3.1 in the main paper, the right hand side of the above term is $O_P(r_N)$. As a result, we find an upper bound of \((C),\) which is $\left| \hat{R}_{(-\ell)}(\theta^*_{(-\ell)}) - R(\theta^*) \right| = O_P(r_N)$. This concludes the desired result.
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