EXAMPLES AND PROPERTIES OF STAR SURGERIES

LAURA STARKSTON

Abstract. Cut-and-paste operations on symplectic manifolds called star surgery were introduced and shown to be an effective tool in constructing examples of exotic 4-manifolds in [11]. Here we provide more examples of star surgery operations, and define an operation called sprouting which generates an infinite family of star surgeries from a single star surgery. Through these examples we begin to establish a broader correspondence between two different methods of finding and classifying symplectic fillings. We also prove that star surgeries always decrease Euler characteristic, and that there are examples of star surgeries which cannot be realized by a sequence of symplectic rational blow-downs.

1. Introduction

The rational blow-down operation on 4-manifolds was first defined by Fintushel and Stern in [6] and generalized in [18] and [22]. The operation cuts out a neighborhood of spheres intersecting in a particular way and glues in a rational homology ball along the common boundary. Fintushel and Stern showed that this operation behaves nicely with respect to the Seiberg-Witten invariants and is therefore an effective tool to construct 4-manifolds where an exotic smooth structure can be detected. The Fintushel-Stern rational blow-downs and their generalizations by Park were shown to be symplectic operations by Symington in [23] and [24] in the case that the spheres to be cut out are symplectic submanifolds. This resulted from the fact that both the neighborhood of these symplectic spheres and the corresponding rational homology balls supported symplectic structures with convex boundary inducing the same contact structures. Stipsicz, Szabó, and Wahl, introduced new families of spheres intersecting according to star-shaped graphs, such that convex neighborhoods of such spheres can be symplectically replaced with rational homology balls [22]. Michalogiorgaki proved that these types of rational blow-downs affect the Seiberg-Witten invariants in a similar way, and showed these operations can be used to produce exotic 4-manifolds [14].

More general operations, called star surgery, were introduced in [11]. These operations similarly cut out a convex neighborhood of spheres intersecting according to a star shaped graph, but the piece which is glued in no longer need be a rational homology ball. Instead, this piece is another convex symplectic filling of the corresponding contact manifold. The classification of such fillings was studied in [21], using a generalization of techniques of Lisca [12]. In the classification of fillings problem, a combinatorial problem appears in classifying homological embeddings of the “dual graph.” Different homological embeddings (solutions to this combinatorial problem) give rise to different possible diffeomorphism types of symplectic fillings. In certain examples, each of the combinatorial solutions suggested a certain Lefschetz fibration inducing the correct contact boundary. In other words, the solutions to the combinatorial problem, suggested actual symplectic fillings of the correct contact structure. Such a symplectic filling gives rise to a star surgery operation which can be described in terms of a monodromy substitution - two different factorizations into positive Dehn twists of the monodromy of an open book decomposition supporting the boundary contact structure.

The benefit of working with homological embeddings instead of directly with factorizations of the monodromy, is that everything is abelian, and all possible homological embeddings can be determined algorithmically by a finite process. Therefore it is easier to narrow down which
configurations of spheres could have interesting symplectic replacements. The downside is that a homological embedding does not automatically ensure the existence of a corresponding symplectic filling. However, if the homological embedding data can be translated into a monodromy substitution then the corresponding Lefschetz fibration yields the appropriate symplectic filling. In this paper, we will provide new examples of useful star surgery operations, extend the dictionary between homological embeddings and monodromy substitutions to these examples, and leverage the strengths of the homological embedding approach to prove some properties of star surgeries. In particular, we prove in section 3 that star surgeries (with sufficiently negative weight on the central vertex) can never increase Euler characteristic of the 4-manifold. In section 5, we will provide an example of star surgery which is not equivalent to any sequence of symplectic rational blow-downs (including all of the generalized versions). This is in contrast to results of Bhupal and Ozbagci [1] showing that all symplectic fillings of lens spaces with their canonical contact structures, are obtained by a sequence of rational blowdowns (in the Fintushel-Stern or generalized Park sense), from a plumbing of spheres.

ACKNOWLEDGMENTS

I am grateful for many valuable conversations with Çağrı Karakurt, the support of my advisor Bob Gompf, and helpful correspondence with Tom Mark and András Stipsicz. I appreciatively acknowledge support from the National Science Foundation under Grant No. DGE-1110007.

2. HOMOLOGICAL EMBDIEEDS OF DUAL PLUMBINGS AND LEFSCHETZ FIBRATIONS

In this section, we will discuss two main strategies which can be used to find and classify star-surgery operations, or equivalently symplectic fillings of the contact boundary of symplectic plumbings. We also start to establish some correspondences between these two strategies. Since these different strategies have different strengths, by understanding the correspondence between them we can utilize the strengths of both techniques.

The first strategy is that used in [13], [12], [20] and [21]. The idea is to construct a concave cap for the symplectic plumbing, which has well-understood topology, and contains a particular symplectic surface on its interior (which in this paper will be a symplectic sphere of square +1). Then any alternate convex filling can be glued to this concave cap to give a closed symplectic manifold, which by a classification theorem of McDuff [13], is necessarily a blow-up of $\mathbb{C}P^2$. (When the symplectic surface in the cap is not a +1 sphere, there could be other possibilities. See [20] for examples of classification statements which can be made in this case.) Therefore all convex fillings embed into $\mathbb{C}P^2 \# N \mathbb{C}P^2$ (for some $N$) as the complement of the cap. The strategy is to classify all symplectic embeddings of the cap into $\mathbb{C}P^2 \# N \mathbb{C}P^2$ up to smooth isotopy to obtain a list of possible diffeomorphism types of symplectic fillings by looking at the complement of such embeddings.

Classifying symplectic embeddings of the cap into $\mathbb{C}P^2 \# N \mathbb{C}P^2$ can be split into two steps. The first is classification of homological embeddings, where one uses the adjunction formula and intersection form on the cap to obtain a list of possibilities for the maps on second homology induced by such embeddings. The second step is to determine the isotopy classes of embeddings inducing a given homological embedding. In many cases, one can prove there is a unique such isotopy class for each homological embedding using pseudoholomorphic curve arguments. To show existence, one constructs a handlebody decomposition for a smooth 4-manifold which is the complement of a smooth embedding of the cap representing the given homological embedding, then shows (after performing Kirby moves on the diagram) that these can be made to be Stein handles (in the sense of [2] and [9]), or that the handlebody naturally
supports a Lefschetz fibration. Establishing a correspondence between this strategy and the next would provide a more systematic way to construct a symplectic filling corresponding to a given homological embedding of the cap.

The second strategy which is used in [25], [19], and [10], relies on a theorem of Wendl which states that if a contact structure is supported by a planar open book decomposition, then all symplectic fillings of that contact structure are supported by Lefschetz fibrations whose boundary open book is that particular planar open book decomposition. Each Lefschetz fibration thus corresponds to a different factorization of the monodromy of this open book into a product of positive Dehn twists (about curves corresponding to the vanishing cycles of the Lefschetz fibration). Two products of positive Dehn twists which are equal in the mapping class group yield a monodromy substitution. The strategy is to understand enough about the mapping class group to classify all such factorizations of a specific monodromy. The non-abelian nature of the mapping class group makes this classification difficult in general. Specifically, it is hard to obtain upper bounds on the number of such factorizations. On the other hand, any such factorization automatically yields a corresponding symplectic filling. Note that two different factorizations of this monodromy into positive Dehn twists, do not always yield non-diffeomorphic manifolds. One needs invariants to distinguish the corresponding 4-manifolds (sometimes simple algebraic invariants like Euler characteristic suffice), or show that they are in fact diffeomorphic.

2.1. Star-shaped graphs and dual graphs. A star-shaped graph is a weighted tree where at most one vertex has valence greater than two. There are $k$ arms emanating from this central vertex. We say that a star-shaped graph is *dually positive* if the weight on its central vertex is at most $-k-1$, and the weights on the vertices in the arms are at most $-2$. For every dually positive star-shaped graph, there is a weighted dual graph which is star-shaped and the weight on the central vertex is $+1$. Such dual graphs were first described in [22].

The dual graph construction provides an embedding of a dually positive star-shaped plumbing into $\mathbb{C}P^2 \# N \mathbb{C}P^2$ whose complement is a plumbing for the dual graph. The core spheres for both plumbings are symplectic submanifolds. To build this construction, we start with $\mathbb{C}P^2 \# \mathbb{C}P^2$, viewed as a sphere bundle over a sphere. Keep track of two sections - the 0-section which has square $+1$ and is a complex projective line representing the homology class $h$, and the $\infty$-section which has square $-1$ and is an exceptional sphere representing the homology class $e_1$. Also keep track of a number of fiber spheres (necessarily of square 0) which represent $h - e_1$. Then blow-up at intersections between the spheres being tracked (though never on the +1 sphere). After blowing up we add the new exceptional spheres into the set of spheres we track along with the proper transforms of the spheres we were tracking before. The blow-ups are performed to create singular fibers which can each be cut along the most recently introduced exceptional sphere so that on one side we have the arms of the plumbing graph emanating from the proper transform of the $\infty$-section, and on the other side we have the arms of the dual graph emanating from the 0-section sphere of square $+1$ (see section 2.1 of [21] for further discussion and examples of the dual graph construction in this context).

In [8] Gay and Stipsicz show that the union of symplectic spheres which intersect according to a negative definite graph have small convex neighborhoods (the negative definite hypothesis is met for dually positive star-shaped plumbings). Therefore $\mathbb{C}P^2 \# N \mathbb{C}P^2$ decomposes into a convex and a concave piece where the convex piece is a plumbing according to the dually positive star-shaped graph and the concave piece is a plumbing according to the dual graph. The contact manifold which the two are glued along is a Seifert fibered space $M$ with its canonical contact structure $\xi_{can}$ by [16]. The dual plumbing is a concave cap for any symplectic filling of $(M, \xi_{can})$.

To understand fillings of $(M, \xi_{can})$ we want to understand symplectic embeddings of the dual plumbing into $\mathbb{C}P^2 \# N \mathbb{C}P^2$. The dual graph construction provides one such embedding,
where the complement is the original plumbing. The homological embedding corresponding to this embedding can be easily computed. The spheres adjacent to the central +1 sphere represent the class $h - e_1 - e_{i_1} - \cdots - e_{i_n}$. They all share $e_1$ with coefficient $-1$, but the other $e_x$’s that appear with nonzero coefficient are all distinct. After blowing up enough times, the most recently introduced exceptional sphere in each singular fiber becomes the next sphere in the arm of the dual graph (instead of becoming part of the corresponding arm in the plumbing graph). Then we blow up at points at its intersection with the subsequent sphere until its proper transform has the necessary self-intersection number, so that its proper transform represents $e_1 - e_{x_1} - \cdots - e_{x_n}$. Repeating this process, the spheres in the $j^{th}$ arm represent homology classes as follows.

$$h - e_1 - e_{1,j} - \cdots - e_{n_{1,j}}$$

$$e_{1,j} - e_1 - e_{2,j} - \cdots - e_{n_{2,j}}$$

$$e_{n_{2,j}} - e_1 - e_{3,j} - \cdots - e_{n_{3,j}}$$

$$\vdots$$

$$e_{n_{m-1,j}} - e_1 - e_{m,j} - \cdots - e_{n_{m,j}}$$

Here all $e_{i,j}$ are exceptional classes distinct from each other and from $e_1$. There are no exceptional sphere classes which appear with nonzero coefficient in more than one arm because the blow-ups are all done in distinct singular fibers which each correspond to distinct arms. The only exceptional classes besides $e_1$ that appear with nonzero coefficient in two different spheres are in adjacent spheres, and appear with coefficient $-1$ in the inner-more sphere and with coefficient $+1$ in the outer-more sphere.

### 2.2. Canonical Lefschetz fibrations on plumbings.

The results of [8] were refined by Gay and Mark in [7] by describing an explicit Lefschetz fibration on the small neighborhoods of the union of the symplectic spheres which induces the convex symplectic structure. In this paper, we will refer to this as the canonical Lefschetz fibration for the plumbing. Note that this provides an explicit open book decomposition supporting the contact structure induced on the boundary. The canonical Lefschetz fibration and open book decomposition of the plumbing are described explicitly in terms of the graph in [7].

We now provide a procedure for constructing the canonical Lefschetz fibration for a dually positive star-shaped plumbing of spheres, completely in terms of the dual graph. Let $DT$ denote the dual graph of a dually positive star-shaped graph $\Gamma$. The fibers of the Lefschetz fibration will be disks with $|DT| - 1$ holes where $|DT|$ denotes the number of vertices of $DT$. The vanishing cycles will be disjoint curves on this $|DT| - 1$ holed disk (so the order of the vanishing cycles will not matter). Label each of the vertices in the dual graph $\nu_j$ such that $j$ specifies the arm that the vertex is in, and $i$ specifies the distance (number of edges) between $\nu_j^i$ and the central vertex. The canonical Lefschetz fibration for $\Gamma$ is then given by the following procedure using the data of $DT$.

Start with a disk with a single boundary parallel vanishing cycle. For each arm of the dual graph choose a subdisk such that these subdisks are all disjoint from each other. In arm $j$, start with $\nu_1^j$ (which is adjacent to the central vertex), and suppose it has weight $-n_1^j \leq -1$. Place $n_1^j$ vanishing cycles on concentric curves which are parallel to the boundary of the subdisk and place a hole at the center of the disk labelled $h_1^j$. If there are more vertices in the dual graph in arm $j$, choose a subdisk which lies between the two innermost vanishing cycles just added (or if only one vanishing cycle was added, in a neighborhood just outside of it). If the weight of $\nu_j^j$ is $-n_2^j$, place $n_2^j - 1$ concentric vanishing cycles parallel to the boundary of this new subdisk. Add a new hole labelled $h_2^j$ at the center. If there are more vertices in arm $j$, identify a subdisk between the two innermost vanishing cycles just added (or just outside of the single vanishing cycle just added if $n_2^j = 2$) and repeat the procedure for $\nu_3^j$ (adding
Figure 1. An example of a dual graph (top left), its corresponding graph (bottom left), and the corresponding canonical Lefschetz fibration for the plumbing of the lower graph, represented by the fiber with vanishing cycles drawn in blue in the center, and as a handlebody on the right (complete the braid trivially).

This procedure associates to each vertex in the dual graph, $v^i_j$, a hole $h^i_j$ and a set of vanishing cycles (the number of which is determined by the weight on the vertex and whether or not $i = 1$). Of these vanishing cycles, one is boundary parallel to the hole and the others are parallel to each other and contain on their interior $h^j_\ell$ for $\ell \geq i$.

The regions of the fiber in the complement of the vanishing cycles have two types: either all of the boundary components of the region lie on vanishing cycles or the region is an annulus with one boundary component on the boundary of the fiber. Deforming the Lefschetz fibration so that all singularities occur in the same fiber, the regions with all boundary components on vanishing cycles appear as closed spheres in this singular fiber where the self-intersection number of the sphere is negative the number of vanishing cycle boundary components of the region. These spheres correspond to the core spheres of the plumbing for the graph $\Gamma$.

Note that the outer boundary of the disk is distinguished in this case so the interior of a simple closed curve on the disk is well-defined. In the construction of the Lefschetz fibration for a plumbing in [7], there is not an automatically distinguished outer boundary component in the fiber. However in a star-shaped plumbing of spheres, the central vertex is distinguished, and in a dually positive star-shaped plumbing the weight on the central vertex is strictly less than the number of arms. Therefore there is necessarily at least one annular region on the fiber of the canonical Lefschetz fibration with one boundary component on the boundary of the surface, and the other on a vanishing cycle which bounds the region corresponding to the central vertex sphere on the other side. The holes whose boundary lies on such annular regions are distinguished holes, and we choose one of them as the distinguished outer boundary component for the disk (all choices among these are equivalent).

We formulate this correspondence between the dual graph and the canonical Lefschetz fibration in order to describe the way that the natural handlebody representing this Lefschetz fibration for the plumbing glues to the natural handlebody representing the dual plumbing in the standard embedding into $\mathbb{C}P^2 \# N \mathbb{C}P^2$ given by the dual graph construction. Such a gluing can be specified by a sequence of Kirby calculus moves starting with the surgery diagram on the boundary of the Lefschetz fibration handlebody, but with reversed orientation,
and ending with the standard surgery diagram for the dual plumbing where each vertex corresponds to an unknotted surgery curve with linking specified by edges and framings specified by weights. Turning the handlebody decomposition for the Lefschetz fibration upsidedown and then performing this Kirby calculus sequence on the lower boundary makes visible the embedding of the dual graph into $\mathbb{C}P^2 \# N \mathbb{C}P^2$.

There is a natural handlebody diagram corresponding to a Lefschetz fibration whose fibers are disks with holes. It has one 1-handle for each hole, represented by dotted circles forming a trivial braid where we view the disk with holes as transverse to this braid, such that neighborhoods of the dotted circles correspond to the holes. Additionally, there is one 2-handle for each vanishing cycle, attached along the vanishing cycle curve in the disk with framing $-1$ (see figure [1] for an example). The induced surgery diagram for the boundary with reversed orientation is correspondingly given by one 0-framed unknotted surgery curve for each hole, forming a trivial braid, and one $+1$-framed surgery curve at each vanishing cycle curve in the disk, where we continue to view the disk with holes as orthogonal to the trivial braid.

Now we can describe the desired Kirby calculus sequence. We will refer to the 0-framed surgery curve corresponding to hole $h^j_i$ as $c^j_i$, and to the weight on a vertex $v^j_i$ of the dual graph, by $-w^j_i$. We heavily rely on the description of the canonical Lefschetz fibration in terms of the dual graph given above.

1. In each arm, slide the curve corresponding to the last vertex in that arm, $c^j_{\text{last}}$, over the curve corresponding to the adjacent previous vertex, $c^j_{\text{last} - 1}$. Then slide $c^j_{\text{last} - 1}$ over $c^j_{\text{last} - 2}$ and so on until reaching $c^j_1$. Note that before these slides, each $c^j_i$ except the last in each arm has exactly one 1-framed meridian. After the slides, these 1-framed surgery curves link two consecutive $c^j_i$’s, and every other 1-framed surgery curve becomes a meridian of some $c^j_i$.

2. Blow down all 1-framed surgery curves except the one corresponding to the outermost vanishing cycle which was parallel to the outer boundary of the disk. Due to the above description of the canonical Lefschetz fibration in terms of the dual graph, there are precisely $w^j_i$ 1-framed surgery curves linking $c^j_i$. For each edge in the dual graph, there is one 1-framed surgery curve linking both of the corresponding $c^j_i$’s, so blowing this down links those curves together. The rest of the 1-framed surgery curves are meridians of the $c^j_i$ so blowing them down only reduces the framings on the $c^j_i$ until they agree with the weights of the dual graph.

At the end of this process, all of the $+1$ surgery curves except the one corresponding to the vanishing cycle parallel to the outer boundary component of the disk have been blown down. The curves $c^j_i$ have framings $-w^j_i$ and are linked ordered by $i$ in a chain for each arm $j$. In other words, this is a standard surgery diagram for the dual plumbing.

Starting with the standard surgery diagram for the dual plumbing, keeping track additionally of meridional curves for each surgery curve, and reversing this sequence shows one how this dual graph cap glues onto the canonical Lefschetz fibration, (specifically, where the meridional curves to each dual graph sphere lie). Therefore finding these sequences of moves, connecting the surgery diagram where a Lefschetz fibration structure is apparent on one side to the surgery diagram where the dual graph plumbing is apparent on the other side, is the key connection between the monodromy substitution perspective and the concave cap embedding perspective of understanding convex fillings.

Let $L$ be a Lefschetz fibration inducing the same contact boundary as a dually-positive star-shaped plumbing $S$, plumbed according to a graph $\Gamma$, whose dual graph is $D\Gamma$. Let $H$ be the homological embedding data of the embedding of the plumbing of $D\Gamma$ into $\mathbb{C}P^2 \# N \mathbb{C}P^2$ obtained by gluing $L$ to the dual plumbing. We define a translation between $L$ and $H$ to be a
sequence of Kirby calculus moves starting with the boundary surgery diagram of the natural
handlebody decomposition for $L$ with reversed orientation and ending with the standard
surgery diagram for the boundary of the plumbing $D'$ such that the first move of the sequence
is given by step (1) as in the above translation between the canonical Lefschetz fibration and
the canonical homology embedding, and such that the 0-framed surgery curve corresponding
to hole $h_i$ is sent to the surgery curve induced by the 2-handle corresponding to the dual
graph vertex $v_i$ after blowing up and down in the Kirby calculus sequence. All the examples
in this paper will have translations.

2.3. Conventions on monodromy factorizations and Lefschetz fibrations. A Lef-
schetz fibration naturally induces an open book decomposition on the boundary where the
fibers of the open book are the same as the fibers of the Lefschetz fibration, and the mon-
odromy is given by a product of positive (right-handed) Dehn twists about the vanishing
cycles. Since mapping class groups of surfaces are non-abelian, the order of the vanishing
cycles generally matters. For this reason, we will briefly mention the conventions that we will
use throughout this paper.

Suppose $c_1, \ldots, c_n$ are simple closed curves on the fiber. Denote by $\phi_{c_i}$ a positive Dehn twist
around $c_i$. The product $\phi_{c_1}\phi_{c_2}\cdots\phi_{c_n}$ means first Dehn twist along $c_1$, then $c_2$, and so on until
finally along $c_n$. When the fiber is a disk with holes, we can place the holes along a circle
concentric with the boundary of the disk. Labelling the holes $\{1, \ldots, m\}$ counterclockwise, we
use the notation $\phi_{i_1, \ldots, i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots, m\}$ to indicate a positive Dehn twist about a
curve which convexly contains the holes $i_1, \ldots, i_k$.

Any factorization of the monodromy of an open book decomposition into a product of positive
Dehn twists corresponds to a Lefschetz fibration. When the fibers are disks with holes, we have the natural handlebody decomposition for this Lefschetz fibration where the holes are represented by dotted circles forming a trivial braid corresponding to 1-handles and the vanishing cycles correspond to 2-handles. We view the holed-disk fibers as orthogonal to the dotted circles, oriented so that the outward normal points downward (i.e. turn the holed-disk upside-down). Then the monodromy factorization $\phi_{c_1} \cdots \phi_{c_n}$ corresponds to the Lefschetz fibration where the vanishing cycle $c_1$ appears at the top and $c_n$ at the bottom (though these vanishing cycles lie on the upside-down disk). Flipping the entire diagram $180^\circ$ around the horizontal axis in the page gives a handlebody decomposition where the vanishing cycle $c_1$ appears at the bottom and $c_n$ appears at the top, but now these vanishing cycles are viewed as living on the disk without turning it upside-down. We will typically use the convention where the disk fibers are oriented with a downward normal, and the vanishing cycles are ordered top to bottom.

Typically, to draw the handlebody, we will isotope the holes on the disk so that they all lie
on the bottom half of the disk along a circle concentric to the boundary. An example, using
the top to bottom convention where the outward normal to the disk points downward, is in
figure 2.

The mapping class group on a disk with holes is generated by Dehn twists. Dehn twists about
disjoint curves commute. If we place the holes on a circle concentric with the boundary, we
can order them counter-clockwise. Suppose $A$, $B$, and $C$ are collections of holes such that all
the holes of $A$ precede all the holes of $B$ which preced all the holes of $C$ going around the
circle counterclockwise. Then the lantern relation states

$$\phi_{A\cup B\cup C} \phi_A \phi_B \phi_C = \phi_{A\cup B} \phi_{A\cup C} \phi_{B\cup C}.$$

The Dehn twists on the right-hand side can be cyclically permuted.

We mention one other relation here that we will use. Let $B_0, B_1, \ldots, B_p$ be collections of
holes ordered counterclockwise. The daisy relation (defined in [5]), which can be obtained by
Figure 2. The Lefschetz fibration corresponding to the monodromy factorization $\phi_1 \phi_2 \phi_4 \phi_7 = \phi_{c_1} \phi_{c_2} \phi_{c_3}$.

iteratively using the lantern relation, is as follows

$$\phi_{B_0 \cup B_1 \cup \cdots \cup B_p} \phi_{B_0}^{p-1} \phi_{B_1} \cdots \phi_{B_p} = \phi_{B_0 \cup B_1 \cup B_0 \cup B_2} \cdots \phi_{B_0 \cup B_p} \phi_{B_1 \cup \cdots \cup B_p}.$$

2.4. Minimal fillings with small Euler characteristic. The star surgery operations replace a convex symplectic neighborhood of star-shaped intersecting spheres in a closed symplectic 4-manifold with an alternate minimal symplectic filling. We will prove in section 3 that such alternate minimal symplectic fillings always have smaller Euler characteristic than the original plumbing. Constructions of many symplectic 4-manifolds with small Euler characteristic have historically proven to be more difficult to find than those with large Euler characteristic. Therefore operations which reduce Euler characteristic and can be used to produce exotic 4-manifolds are of particular interest to 4-manifold topologists.

Note that the Euler characteristic of any symplectic filling which arises as the complement of an embedding of the dual graph into $\mathbb{C}P^2 \# N\overline{\mathbb{C}P^2}$ is $2 + N - |DG|$ where $|DG|$ is the number of vertices in the dual graph. To find minimal fillings, $N$ is the minimal number of distinct exceptional classes appearing in the homological embedding. This follows from Lemma 4.5 of [12], which proves that there is a $J$-holomorphic exceptional sphere disjoint from the dual plumbing spheres whenever there is an exceptional sphere with algebraic intersection 0 with each sphere in the dual plumbing. Therefore to search for minimal symplectic fillings with small Euler characteristic, we want to look for homological embeddings of the dual graph which use a small number of exceptional classes relative to the size of the dual graph.

There are significant restrictions on the homology classes represented by the images of the dual plumbing spheres under a symplectic embedding. McDuff’s theorem identifies the +1 sphere in the dual graph with the complex projective line, so the image of its homology class is necessarily $h$. The adjunction formula and the intersection form determined by the dual graph imply the following properties which are proven in [12] section 4 and [21] section 2.4.

- The homology classes of embedded dual graph spheres which intersect the central +1 sphere once have coefficient 1 for $h$, and coefficients 0 or $-1$ for each $e_i$. The number of exceptional classes which appear with coefficient $-1$ is $w+1$ where $-w$ is the square of the corresponding dual graph sphere specified by the weight.

- The homology classes of embedded dual graph spheres which do not intersect the central +1 sphere have coefficient 1 for exactly one exceptional class $e_{i_0}$, and have coefficient 0 or $-1$ for all other exceptional classes $e_i$. The number of exceptional classes which appear with coefficient $-1$ is $w - 1$ where $-w$ is the square of the corresponding dual graph sphere specified by the weight.
Additional restrictions can be deduced by comparing the intersection pairings of the embedded homology classes with the intersection relations of the dual graph spheres specified by the edges of the graph.

3. Euler characteristic upper bounds

The main goal of this section is to prove the following theorem.

**Theorem 3.1.** Let $\Gamma$ be a dually-positive star-shaped graph. Let $X$ be the corresponding symplectic plumbing of spheres with convex boundary inducing the canonical contact structure, and let $(Y, \xi) = \partial(X, \omega)$. Then the Euler characteristic of any convex symplectic fillings of $(Y, \xi)$ is bounded above by the Euler characteristic of $X$.

Recall from section 2.4 that the Euler characteristic of a minimal symplectic filling is $2 + N - |DG|$ where $N$ is the minimal number such that the dual plumbing embeds into $\mathbb{CP}^2 \# N \mathbb{CP}^2$ as the complement of the filling. All $N$ exceptional homology classes appear with non-zero coefficient in at least one of the homology classes of the dual spheres. The homology classes of the embedded dual spheres, written in terms of the standard basis for $H_2(\mathbb{CP}^2 \# N \mathbb{CP}^2)$, must have the form specified at the end of section 2.4. The following lemmas are easy algebraic consequences of the intersection relations. Denote by $C_i^j$ the dual graph sphere in the $j^{th}$ arm separated from the central vertex by $i$ edges.

**Lemma 3.2.** For each distinct pair $j, j'$, there is exactly one $e_x$ which appears with coefficient $-1$ in both $[C_i^j]$ and $[C_i^{j'}]$.

**Lemma 3.3.** The class of the exceptional sphere which appears with coefficient $+1$ in $[C_2^j]$ appears with coefficient $-1$ in sphere $[C_1^j]$.

**Lemma 3.4.** For $i > 1$, either the exceptional class with coefficient $+1$ in $[C_i^j]$ appears with coefficient $-1$ in $[C_{i+1}^j]$ or the exceptional class with coefficient $+1$ in $[C_i^{j+1}]$ appears with coefficient $-1$ in $[C_{i+1}^j]$, or both. In particular, they do not share the same exceptional class with $+1$ coefficient. Furthermore, the exceptional classes which appear with coefficients $-1$ in $[C_i^j]$ are disjoint from those which appear with coefficient $-1$ in $[C_{i+1}^j]$ unless both conditions in the first sentence are satisfied, in which case they share exactly one exceptional class with $-1$ coefficient in common.

**Lemma 3.5.** If $e_x$ appears with coefficient $+1$ in $[C_i^j]$ then it does not appear with coefficient $+1$ in $[C_i^{j'}]$ for any $(i', j') \neq (i, j)$.

**Lemma 3.6.** If $e_x$ appears with nonzero coefficient in distinct classes $[C_i^j]$ and $[C_i^{j'}]$ and we do not have that $(i, j) = (i' \pm 1, j')$ or that $i, i' = 1$, then we have one or both of the following two possibilities.

1. the exceptional class with coefficient $+1$ in $[C_i^j]$ appears with coefficient $-1$ in $[C_i^{j'}]$

2. the exceptional class with coefficient $+1$ in $[C_i^{j'}]$ appears with coefficient $-1$ in $[C_i^j]$

If only one of these possibilities holds then there is exactly one exceptional class which appears with coefficient $-1$ in both $[C_i^j]$ and $[C_i^{j'}]$. If both (1) and (2) hold, then there are exactly two exceptional classes which appear with coefficient $-1$ in both.

With all of these restrictions on the homology embeddings, the different possibilities for the Euler characteristics of the corresponding fillings are determined by the varying ways that exceptional classes appear with non-zero coefficients in the homology classes of distinct spheres.
Proof of theorem 3.1. We described the homological embedding of the dual graph such that its complement is the plumbing in section 2.1. It suffices to show that any other homological embedding of the dual graph uses no more distinct exceptional classes the one specified there.

By lemma 3.2 each pair of spheres adjacent to the central +1 sphere in the dual graph, must have exactly one shared $e_i$ appearing with $-1$ coefficient in both. In the plumbing embedding, they all share the same class, $e_1$. If they did not all share the same class, there would necessarily be at least one sphere $C_1^j$ in which two exceptional classes $e_x$ and $e_y$ appear with $-1$ coefficient, where $e_x$ appears with $-1$ coefficient in $C_1^j$, $C_1^{j+1}$, and $C_1^{n+1}$ and $e_y$ appears with $-1$ coefficients in a disjoint set of spheres $C_1^j, C_1^{j+1}, C_1^{n+1}$. There is a third exceptional class $e_z$ which appears with $-1$ coefficient in $C_1^j$ and $C_1^{n+1}$. Now consider the homology embedding where $C_1^j, C_1^{j+1}, C_1^{n+1}$ all share the same $e_x$ with non-zero coefficient. Then in order to keep the squares of the homology classes of $C_1^j, C_1^{j+1}, C_1^{n+1}$ the same, there must be new distinct exceptional classes $e_a, e_b, e_c$, with one appearing with coefficient $-1$ in each of these classes. Therefore decreasing the number of distinct $e_i$’s which are shared between the spheres adjacent to the central vertex, increases the total number of distinct exceptional classes appearing with nonzero coefficients in the embedding.

By lemma 3.3 spheres which are adjacent in the same arm must share at least one exceptional class with nonzero coefficients. In the plumbing embedding, the only sharing of exceptional classes (other than $e_1$ which was discussed in the previous paragraph) is a single exceptional class shared between consecutive spheres within an arm. Therefore this homology embedding has the minimal possible amount of sharing of exceptional classes away from the central vertex. If more of these exceptional classes were shared amongst multiple different spheres, the total number of exceptional classes appearing with non-zero coefficient would decrease. This implies that the plumbing homology embedding maximizes the number of exceptional classes which appear with nonzero coefficient.

Therefore, the original plumbing has the maximal Euler characteristic of any minimal convex filling, since all convex fillings appears as the complement of the symplectic embedding of the dual graph into a blow-up of $\mathbb{CP}^2$. 

\[\square\]

4. Fillings from arm length one dual graphs

The simplest case of classifying homological embeddings is where the length of each dual graph arm is one. The corresponding graphs are those where the weights on the vertices in the arms are all $-2$. These examples were studied in [21] section 5, but the focus there was on making complete classification statements for graphs with four or five arms. The infinite family of star surgeries $S_i, T_i$ described in [11] are also of this type. Here we will show how to find other examples of this type that are particularly useful for star surgery.

Suppose we have a dual graph $D\Gamma$ whose arms are length one. Any symplectic embedding of the corresponding plumbing into $\mathbb{CP}^2 \# N \overline{\mathbb{CP}^2}$ which sends the central +1 sphere to $\mathbb{CP}^1$, sends the spheres in the $n$ arms to symplectic spheres which represent homology classes $[C_1], \ldots, [C_n]$. By lemma 3.2 $[C_j] = h - e^j_1 - \cdots - e^j_i$ where the $e^j_i$ are homology classes represented by exceptional spheres ($e^j_i = e^j_k$ if and only if $k = k'$) and $\{|e^j_1, \ldots, e^j_i| \cap \{e^j_1, \ldots, e^j_n\}\} = 1$.

Remark 4.1 (Avoiding extra vertices). Suppose we have any dual graph $D\Gamma$ dual to a graph $\Gamma$. Adding an extra exceptional class into one of the dual graph spheres, $[C_x]$, decreases the weight on that vertex yielding a new dual graph $D\Gamma'$ and dually adds an extra $-2$-weighted vertex into the graph yielding a modified graph $\Gamma'$. By symplectically resolving the intersection between this extra $-2$ sphere with an adjacent sphere in the graph, we obtain a configuration of spheres represented by the original graph $\Gamma$. Any star-surgery on $\Gamma$ cutting out a neighborhood of the
spheres and replacing it with another symplectic filling, yields a star-surgery on $\Gamma'$ since the symplectic resolution can be performed in arbitrarily small neighborhoods of the intersection. Furthermore the homological embeddings of $\mathcal{D}'$ are in one-to-one correspondence with the homological embeddings of $\mathcal{D} \Gamma'$ in which there is an exceptional class which appears with coefficient $-1$ in only $[C_2]$. Therefore the most useful star surgery operations of this type come from homological embeddings of dual graphs where each exceptional class appears at least twice with non-zero coefficient, since otherwise the star-surgery does not utilize all of the symplectic spheres specified by the graph.

For a dual graph with $n$ arms of length 1, homological embeddings where each exceptional class appears at least twice with non-zero coefficient are in correspondence with paired groupings of $\{1, \cdots, n\}$ defined as follows.

**Definition 4.2.** A paired grouping of $\{1, \cdots, n\}$ is a set of subsets of $\{1, \cdots, n\}$ such that each pair of distinct indices appears together in a subset exactly once, and each subset has more than one element.

For example when $n = 3$, the paired groupings are $\{(1, 2, 3)\}$ and $\{(1, 2), (1, 3), (2, 3)\}$. From one of these paired groupings, we obtain a homological embedding by assigning a distinct exceptional class to each subset in the grouping. The two paired groupings for the $n = 3$ case yield the homological embeddings

$$[C_1] = h - e_1 \quad [C_2] = h - e_1 \quad [C_3] = h - e_1$$
$$[C_1] = h - e_1 - e_2 \quad [C_2] = h - e_1 - e_3 \quad [C_3] = h - e_2 - e_3$$

The second homological embedding is realized by a symplectic embedding into $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, such that the complement of this embedded dual plumbing is the rational homology ball that can replace a neighborhood of a $-4$ sphere. In the first homological embedding, the weights on the arms are all 0, and this situation does not arise as the dual graph of any dually positive star shaped graph. By blowing up three additional times, once along each vertex in the three arms, we find the standard homological embedding represented by the dual graph construction.

$$[C_1] = h - e_1 - e_2 \quad [C_2] = h - e_1 - e_3 \quad [C_3] = h - e_1 - e_4$$

The corresponding embeddings into $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ and the complements of these embeddings are shown in figure 3. The relation between these two Lefschetz fibrations (monodromy substitution) is a single lantern relation. The rational blow-down of the $-4$ sphere was first shown to correspond to this monodromy substitution in [4].

When the number of arms in the dual graph is larger, there are more interesting homological possibilities. For any $n$, there is a paired grouping where each subset contains exactly two indices (the grouping of all distinct pairs). In the corresponding homological embedding, for each pair of arms, there is a unique exceptional class which appears with coefficient $-1$ in the vertex in those two arms. These embeddings were studied in [11], where the complements of these embeddings were fillings denoted $\mathcal{F}_{n-2}$. These complements were shown to have natural Lefschetz fibration structures whose fibers are $n$-holed disks and whose vanishing cycles are the set of curves which convexly enclose each pair of holes. The corresponding monodromy substitutions are

$$\phi_1, \cdots, n \phi_1^{n-2} \cdots \phi_n^{n-2} = \phi_1, \cdots, \phi_1, \phi_2, \phi_3, \phi_2, \cdots, \phi_2, \phi_n, \cdots, \phi_n, \phi_{n-2}, \cdots, \phi_{n-2}, \phi_{n-1}, \phi_{n-1}.$$  

The paired grouping of $\{1, \cdots, n\}$ which contains a subset of size $n - 1$ and $n - 1$ subsets of size two corresponds to the Fintushel-Stern rational blow-down, where the monodromy substitution is a daisy relation.

Now we describe two examples of star surgery operations of this type, which were not explicitly described in the literature previously, but are particularly effective at reducing Euler
Figure 3. Embeddings of the dual graph $DG$ and Lefschetz fibrations $Lf$ on the complements. The Lefschetz fibration on the left is for the $-4$ disk bundle over the sphere. The Lefschetz fibration on the right is for a rational homology ball. The embeddings are represented by 2-handles attached along the completion of the depicted braids, with three additional 3-handles and a 4-handle. By blowing down the $-1$ curves representing exceptional spheres, then cancelling the three 3-handles with three of the $+1$-framed 2-handles, one obtains the standard handlebody decomposition of $\mathbb{C}P^2$. The Lefschetz fibrations on the complements of the embeddings are obtained by cutting out the 0-handle and the 2-handles that correspond to dual graph spheres, and turning the resulting handlebody upside-down and isotoping the diagram (see section 3 of [21]).

![Figure 3](image)

**Figure 4.** Graph and dual graph for the paired grouping \{\{1, 2, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4, 5\}\}.

characteristic. These examples are meant to be illustrative of a more general strategy of searching for these types of star-surgery operations.

Consider the dual graph with 5 arms of length one coming from the paired grouping \{\{1, 2, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4, 5\}\}.

This corresponds to the dual graph given in figure 4b and graph given in figure 4a where the homological embedding of the dual graph is as follows.

$$
\begin{align*}
[C_1] &= h - e_1 - e_2 - e_3 \\
[C_2] &= h - e_1 - e_4 - e_5 \\
[C_3] &= h - e_1 - e_6 \\
[C_4] &= h - e_2 - e_4 - e_6 \\
[C_5] &= h - e_3 - e_5 - e_6
\end{align*}
$$

This homological embedding can be realized by a smooth embedding into $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$ as in figure 5. The complement of this embedding will have Euler characteristic 2 since the dual graph has six vertices. This complement has a handlebody diagram shown on the right of figure 5. This handlebody diagram naturally supports a Lefschetz fibration whose fibers are five holed disks and whose vanishing cycles are given by the attaching curves for the $-1$ framed 2-handles. The boundary open book decomposition can be shown to agree with the boundary open book decomposition of the canonical Lefschetz fibration for the plumbing graph of figure 4a through the following relations in the mapping class group which show the monodromies
Figure 5. The leftmost diagram represents a handlebody decomposition for $\mathbb{C}P^2$ with five 3-handles cancelling five of the 2-handles. The center diagram (together with five 3-handles) is a handlebody diagram for $\mathbb{C}P^2 \# 6 \mathbb{C}P^2$ such that the cores of the 2-handles attached along the blue curves together with Seifert surfaces form spheres in the configuration of the dual graph representing the homology classes specified. The final diagram is a handlebody for the complement of these embedded spheres (with no 3- or 4-handles). It is obtained by removing the blue 2-handles and the 0-handle, turning the resulting relative handlebody upside-down, and simplifying the diagram by blowing down surgery curves on the lower boundary (a translation). Note that in this process the five 3-handles become the five 1-handles represented by dotted circles.

$$\phi_{123} \phi_{14} \phi_{15} \phi_{24} \phi_{25} \phi_{345} = \phi_{123} \phi_{14} \phi_{15} \phi_{23}^{-1} \phi_2^2 \phi_3 \phi_4 \phi_5 \phi_{2345}$$

$$= \phi_{123} \phi_{14} \phi_{15} \phi_{2345} \phi_{23}^{-1} \phi_2^2 \phi_3 \phi_4 \phi_5$$

$$= \phi_1^2 \phi_{23} \phi_4 \phi_5 \phi_{12345} \phi_{23}^{-1} \phi_2^2 \phi_3 \phi_4 \phi_5$$

$$= \phi_1^2 \phi_2^2 \phi_3 \phi_4^2 \phi_5^2 \phi_{12345}$$

A more complicated example is given by a dual graph with 6 length one dual graph arms coming from the paired grouping $\{\{1, 2, 3\}, \{1, 5, 6\}, \{1, 4\}, \{2, 5\}, \{2, 4, 6\}, \{3, 4, 5\}, \{3, 6\}\}$. This corresponds to the dual graph in figure 6(b) where the homological embedding of the dual graph is as follows.

$[C_1] = h - e_1 - e_2 - e_3$  
$[C_2] = h - e_1 - e_4 - e_5$  
$[C_3] = h - e_1 - e_6 - e_7$  
$[C_4] = h - e_3 - e_5 - e_6$  
$[C_5] = h - e_2 - e_4 - e_6$  
$[C_6] = h - e_2 - e_5 - e_7$

Note that there is a configuration of seven complex projective lines labelled $\{1, \cdots, 7\}$ such that their intersections are specified by the grouping. By blowing up at each of the intersection
points corresponding to the sets in the grouping, we will obtain a symplectic embedding of the positive Dehn twists around curves which enclose the holes specified by the grouping. Using substitution using the data of the grouping. The monodromy should factor into a product of a Lefschetz fibration structure on the complement, or we can directly search for a monodromy Lefschetz fibration we can look for a handlebody diagram representing the embedding and find of the dual graph into $\mathbb{C}P^2 \# 7 \mathbb{C}P^2$.

![Plumbing graph](image)

Figure 6. Graph and dual graph for the paired grouping $\{1, 2, 3\}, \{1, 5, 6\}, \{1, 4\}, \{2, 5\}, \{2, 4, 6\}, \{3, 4, 5\}, \{3, 6\}$.

The final line is a factorization of the monodromy into positive Dehn twists (the only negative Dehn twists which appear are part of a conjugation). The corresponding Lefschetz fibration is shown in figure[8] which is the complement of an embedding of the dual graph with the correct homological information as shown in figure[8]. The handlebody diagrams demonstrating this embedding can be obtained by working backwards from the Lefschetz fibration (though this is unnecessary to show the existence of a star-surgery operation), but they are more unwieldy than in previous examples because not all of the vanishing cycles enclose holes convexly.

Alternatively, one could directly search for an embedding representing this homological data and work to put a Lefschetz fibration structure on the complement. A somewhat simpler
handlebody for the pair \((\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}}^2, D\Gamma)\) representing the specified homology embedding is given in figure 8a and the Lefschetz fibration on the complement can be found as in the previous simpler example by turning the complementary handlebody upside-down. This yields a slightly different monodromy substitution, as the vanishing cycles are different. The product of Dehn twists corresponding to this Lefschetz fibration is

\[
\phi_{1,2,3,4,5,6}(\phi_{5,6}^{-1}\phi_{2,4,6}\phi_{5,6})\phi_{2,5}(\phi_{4,5,6}\phi_{3,6}\phi_{4,5,6})\phi_{3,4,5}
\]

This monodromy can be shown to equal the monodromy induced by the canonical Lefschetz fibration for the plumbing of figure 6a as follows. Using the lantern relation to split all of the Dehn twists about curves containing three holes into Dehn twists about curves containing two holes we get that this monodromy is equal to

\[
(\phi_{1,2,3,4,5,6}(\phi_{5,6}^{-1}\phi_{2,4,6}\phi_{5,6})\phi_{2,5}(\phi_{4,5,6}\phi_{3,6}\phi_{4,5,6})\phi_{3,4,5}
\]

Observing that \(\phi_{5,6}^{-1}\phi_{2,5}\) is a positive Dehn twist about the image of the curve convexly containing 2 and 5 under a negative Dehn twist about \(\phi_{5,6}\), and that this image is disjoint from the curve convexly containing 4 and 6, and further that \(\phi_{4,5}\) and \(\phi_{3,6}\) commute, we can
Embedding of dual graph into $\mathbb{C}P^2 \# 7 \mathbb{C}P^2$

Lefschetz fibration on complement of embedding

Commuting Dehn twists about disjoint curves, and cyclically permuting lantern relation triples we can simplify this to

$$(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6)^{-2} \phi_{1,2} \phi_{1,3} \phi_{1,4} \phi_{1,5} \phi_{1,6} \phi_{2,4} \phi_{2,5} \phi_{2,6} \phi_{3,5} \phi_{3,6} \phi_{4,5} \phi_{4,6} \phi_{5,6} \phi_{3,4} \phi_{3,5} \phi_{4,5}$$

A final cyclic permutation of a lantern triple gives

$$(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6)^{-2} \phi_{1,2} \phi_{1,3} \phi_{1,4} \phi_{1,5} \phi_{1,6} \phi_{2,3} \phi_{2,4} \phi_{2,5} \phi_{2,6} \phi_{3,6} \phi_{4,5} \phi_{4,6} \phi_{5,6} \phi_{3,4} \phi_{3,5} \phi_{4,5} \phi_{4,6} \phi_{5,6} \phi_{3,4} \phi_{3,5}$$

The last six Dehn twists form the left hand side of equation 1 and can thus be cyclically permuted to give

$$(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6)^{-2} \phi_{1,2} \phi_{1,3} \phi_{1,4} \phi_{1,5} \phi_{1,6} \phi_{2,3} \phi_{2,4} \phi_{2,5} \phi_{2,6} \phi_{3,6} \phi_{4,3} \phi_{4,5} \phi_{4,6} \phi_{5,6} \phi_{3,4}$$

Finally using equation 1 with $n = 6$, this is equal to the factorization of the monodromy induced by the canonical Lefschetz fibration of the plumbing, $\phi_{1,2,3,4,5,6} \phi_{1} \phi_{2} \phi_{3} \phi_{4} \phi_{5} \phi_{6}$.

Although the Lefschetz fibration of figure 9b differs from that of figure 8d, they necessarily represent diffeomorphic 4-manifolds. In fact, the diffeomorphism type of the complement of a dual graph embedding with this fixed homological embedding data is unique. This can be shown using analogous results to Lemma 2.7 and 2.8 of [21] because the dual graph configuration descends to a configuration of pseudo-holomorphic $+1$ spheres which intersect as in figure 7 with an additional generically intersecting $+1$ curve. In this configuration, no curve contains more than two intersection points with the others of multiplicity greater than two, so the proof of Lemma 2.7 of [21] applies to show that these pseudo-holomorphic curves can be isotoped while maintaining the intersection configuration to complex projective lines. The space of complex projective lines intersecting as in figure 7 is isomorphic to the space of four distinct points in $\mathbb{C}P^2$ such that no three of them are collinear, which is a
connected, non-empty space. The space of configurations of seven lines intersecting such that six of the lines are as in figure 7 and the seventh intersects these six generically in double points is a subset of this space obtained by cutting out pieces of real codimension at least two, and is thus still connected and non-empty, giving the result of Lemma 2.8 of [21] in this situation. Alternatively, one could show these 4-manifolds are diffeomorphic directly via Kirby calculus.

5. Star surgeries unobtainable from rational blowdowns

In this section, we will show that the plumbing according to the graph in figure 4a has a star-surgery operation which is not equivalent to any sequence of symplectic rational blow-downs (where here symplectic rational blow-down includes Fintushel and Stern’s original family, Park’s generalization, as well as the further negative definite examples classified in [22] and [2]). This is in contrast to the results of [1] which show that all fillings of lens spaces are obtained from a linear plumbing by a sequence of rational blow-downs.

**Theorem 5.1.** The boundary \((Y, \xi_{can})\) of the plumbing of spheres \(P\), plumbed according to the graph in figure 4a, has exactly two minimal strong symplectic fillings. One is the plumbing itself and the other has Euler characteristic 2. The filling of Euler characteristic 2 cannot be obtained from the plumbing by any single symplectic rational blow-down, or any sequence of symplectic rational blow-downs.

**Proof.** The dual graph is given in figure 4b. The only possible homological embeddings satisfying the adjunction formula and lemma 3.2 are given below.

|      | \(C_1\) | \(C_2\) | \(C_3\) | \(C_4\) | \(C_5\) |
|------|--------|--------|--------|--------|--------|
| Emb 1 | \(h-e_1-e_2-e_3\) | \(h-e_1-e_4-e_5\) | \(h-e_1-e_6-e_7\) | \(h-e_1-e_8-e_9\) | \(h-e_1-e_{10}\) |
| Emb 2 | \(h-e_1-e_3-e_4\) | \(h-e_1-e_5-e_6\) | \(h-e_2-e_3-e_5\) | \(h-e_2-e_4-e_6\) | \(h-e_1-e_2\) |

By the results of [21] section 2.5, there is at most one isotopy class of smooth embeddings for each of these two homological embeddings. Therefore there are at most two diffeomorphism types of convex symplectic fillings of the canonical contact boundary of this plumbing.

The Euler characteristic of the two potential complementary fillings is \(\chi(\mathbb{C}P^2 \# N \mathbb{C}P^2) - \chi(\text{Dual Graph}) = 3 + N - 7 = N - 4\). So here we see there are at most two minimal strong symplectic fillings, one of Euler characteristic 6 and the other of Euler characteristic 2. The first homological embedding is that of the dual graph construction, so the original plumbing graph is realized as a complementary filling to such an embedding. The second filling of Euler characteristic two was realized by a Lefschetz fibration in section 4 shown in figure 5.

Any sequence of symplectic rational blow-downs of spheres contained inside the plumbing \(P\) would produce a sequence of symplectic fillings of the same contact boundary (since the rational blow-downs are symplectic operations that are performed on the interior). There are only two minimal symplectic fillings of this particular contact manifold, the plumbing and the smaller filling. Therefore, the only way it would be possible to obtain the smaller filling from the plumbing by a sequence of rational blow-downs is by a single rational blow-down; there can be no intermediate steps. A rational blow-down replaces a neighborhood of a set of embedded symplectic spheres (whose union is simply connected), by a rational homology ball. Therefore the change in the Euler characteristic of a manifold before and after a rational blowdown, is precisely the number of spheres in the rational blow-down. The Euler characteristic of the smaller filling in this case is 2, and the Euler characteristic of the plumbing is 6. Therefore we need to check which plumbing graphs with 4 vertices can be rationally blown down.

Such graphs are either linear, or have 3 arms of length one. The classification in [2] describes explicitly which graphs with 3 arms can be rationally blown down. The three of these where each arm has length one are shown in figure 9. The linear graphs which can be rationally
blown include the examples of Fintushel and Stern and the more general examples of Park, where the continued fraction expansion of the weights is $-\frac{p}{2pq-1}$ for $\gcd(p,q) = 1$. There is an inductive procedure to build all such plumbings (described in [22] section 4), where the induction increases the length of the plumbing by one each time. This shows there are four different linear plumbings of length 4 which have rational homology ball fillings. The self-intersection numbers of the spheres in the plumbings are: $(-7, -2, -2, -2)$ (the original Fintushel-Stern rational blow-down with $p = 5$), $(-3, -5, -3, -2)$ (Park’s generalization with $p = 8$, $q = 3$), $(-2, -2, -5, -4)$ (Park’s generalization with $p = 7$, $q = 5$), and $(-2, -6, -2, -3)$ (Park’s generalization with $p = 7$, $q = 4$).

If the plumbing $P$ could be symplectically rationally blown down to obtain the smaller filling, that would mean that there exist four symplectic spheres in $P$ whose intersection data is specified by one of these seven graphs. We can use the adjunction formula to rule out this possibility. Note that $H_2(P; \mathbb{Z})$ is generated by the five spheres which are the cores of the disk bundles that are plumbed together. We denote the sphere corresponding to the central vertex by $C_0$, and the spheres corresponding to the vertices in the four arms by $C_1$, $C_2$, $C_3$, and $C_4$. The $C_i$ are a symplectic spheres, so the adjunction formula holds: $\langle c_1(\omega), C_i \rangle = C_i^2 + 2$. Therefore, $\langle c_1(\omega), [C_0] \rangle = -4$ and $\langle c_1(\omega), [C_i] \rangle = 0$ for $i = 1, 2, 3, 4$. Now for any other symplectic sphere $S$ embedded in $P$, we can write $[S] = \sum_{i=0}^{4} a_i[C_i]$. Then $[S]^2 + 2 = \langle c_1(\omega), [S] \rangle = -4a_0$. In particular, $[S]^2 + 2$ must be divisible by 4 since $a_0$ must be an integer.

All of the graphs with four vertices representing plumbings that can be rationally blown down contain at least one sphere whose self-intersection number $+2$ is not divisible by 4. Therefore none of these symplectic plumbings which can be rationally blown down, can embed into $P$. □

Remark 5.2. In fact none of these rational blow-downs can be done smoothly either because the spheres to be rationally blown down have odd intersection form whereas $P$ has even intersection form. However, it is not clear that there is no sequence of smooth (but not symplectic) rational blow-downs/ups which results in the diffeomorphism type of the smaller filling, since the intermediate steps need not be symplectic in this case. However, this result shows that as a symplectic operation, star surgery is strictly more general than rational blowdowns.

6. Sprouting

In this section we will introduce an operation called “sprouting” which produces an infinite family of star surgery operations from a single star surgery operation. We will define this operation in terms of homological embeddings and in terms of monodromy substitutions, and then show that they are equivalent operations on star surgeries. Sprouting is a simple operation under both interpretations, but it is useful for producing many more examples of star surgery with well-controlled properties. Specifically, sprouting increases the Euler characteristic of the plumbing of spheres which is removed under star surgery, but keeps fixed the Euler characteristic of the piece that is glued in during star surgery. Therefore a sprouted star surgery decreases Euler characteristic more than the original star surgery. However, the sprouted plumbing may be more difficult to embed into a small closed symplectic manifold than the original plumbing.
6.1. Homology embedding interpretation.

**Definition 6.1.** Suppose we have a homological embedding $\mathcal{H}$ of a dual graph $D\Gamma$ corresponding to a symplectic filling $T$, such that a certain exceptional class $e_{b_0}$ appears with coefficient $-1$ in the classes of exactly two distinct spheres, $[C_a]$ and $[C_b]$, such that $[C_a]$ is on the end of the $j^{th}$ arm of the dual graph. Let $N$ be the largest index of the exceptional classes appearing in $\mathcal{H}$. Then the length $n$ sprout of $\mathcal{H}$ along $(C_a, C_b)$, $Spr^n_{a,b}(\mathcal{H})$, is a homological embedding into $\mathbb{CP}^2 \# (N+n) \mathbb{CP}^2$ of a different dual graph, $Spr^n_{a,b}(D\Gamma)$, obtained from $D\Gamma$ by appending $n$ spheres of square $-2$ to the $j^{th}$ arm in homology classes $e_{b_0} - e_{N+1}, e_{N+1} - e_{N+2}, \ldots, e_{N+n-1} - e_{N+n}$, and by changing the homology class of $[C_b]$ to $[C_b] - e_{N+1} - \cdots - e_{N+n}$ and thus reducing its weight by $n$.

Observe that $Spr^n_{a,b}(\mathcal{H})$ is a homological embedding of a star-shaped dual graph respecting the star-shaped intersection data. Furthermore, the number of the vertices in the dual graph $Spr^n_{a,b}(D\Gamma)$ is $n$ greater than the number of vertices of $D\Gamma$, and the number of exceptional classes used in the homology embedding also increases by $n$. Therefore, the Euler characteristic of a complementary filling to an embedding with homological data given by $Spr^n_{a,b}(\mathcal{H})$ is the same as the Euler characteristic of $T$, the complementary filling corresponding to $\mathcal{H}$.

The sprouted dual graph $Spr^n_{a,b}(D\Gamma)$ differs from the original dual graph $D\Gamma$ in two places. First it has $n$ additional $-2$ weighted spheres appended to the end of the $j^{th}$ arm. Second, the weight on the vertex corresponding to $[C_b]$ is reduced by $n$. By redualizing, we see that $Spr^n_{a,b}(D\Gamma)$ is dual to a graph $Spr^n_{a,b}(\Gamma)$, which is obtained from the original graph $\Gamma$ by decreasing the weight on the last vertex in the $j^{th}$ arm by $n$, and inserting $n$ vertices of weight $-2$ into $\Gamma$ in the position dual to $C_b$. Therefore $Spr^n(\Gamma)$ has $n$ additional vertices (and thus the corresponding plumbing of spheres has Euler characteristic increased by $n$). Since the filling of the sprouted graph corresponding to $Spr^n_{a,b}(\mathcal{H})$ has fixed Euler characteristic equal to $\chi(T)$, the star surgeries corresponding to the sprouted homology embeddings reduce Euler characteristic by a larger amount.

The definition becomes more clear in a specific example. Let $\mathcal{H}$ be the homological embedding described below of the dual graph $D\Gamma$ (shown on the right in figure 10) which corresponds to the filling $T_2$ used in [11] (which has Euler characteristic 3).

$$
\begin{align*}
[C_0] &= h \\
[C_1] &= h - e_1 - e_2 - e_3 \\
[C_2] &= h - e_1 - e_4 - e_5 \\
[C_3] &= h - e_2 - e_4 - e_6 \\
[C_4] &= h - e_3 - e_5 - e_6
\end{align*}
$$
Figure 11. A graph corresponding to $Spr^n_{1,2}(\mathcal{H})$ and the dual graph whose homology embedding is given by $Spr^n_{1,2}(\mathcal{H})$.

Figure 12. Graphs $Spr^q_{4,1}(Spr^p_{3,4}(Spr^m_{2,3}(Spr^n_{1,2}(\Gamma))))$ and their dual graphs.

Then $Spr^n_{1,2}(\mathcal{H})$ is a homological embedding of the dual graph $Spr^n_{1,2}(D\Gamma)$ on the right in figure 11 where the homological embedding is

- $[C_0] = h$
- $[C_1] = h - e_1 - e_2 - e_3$
- $[C_2] = h - e_1 - e_4 - e_5 - e_7 - \cdots - e_{n+6}$
- $[C_3] = h - e_2 - e_4 - e_6$
- $[C_4] = h - e_3 - e_5 - e_6$
- $[C_5] = e_1 - e_7$
- $[C_6] = e_7 - e_8$
- $\vdots$
- $[C_{n+4}] = e_{n+5} - e_{n+6}$

Note that one can iteratively sprout along different pairs of vertices. For example, the graphs and dual graphs corresponding to $Spr^q_{4,1}(Spr^p_{3,4}(Spr^m_{2,3}(Spr^n_{1,2}(\Gamma))))$ are given in figure 12, and the corresponding homology embeddings suggest these graphs can be symplectically replaced by a filling of Euler characteristic 3.

6.2. Monodromy substitution interpretation. Now we will define sprouting in terms of monodromy substitutions. Dually positive star-shaped graphs have canonical Lefschetz fibrations with planar fibers (genus 0). It follows from a theorem of Wendl [25] that any convex symplectic filling inducing the same contact boundary as such a plumbing is supported by a Lefschetz fibration whose fibers are the same as those in the canonical Lefschetz fibration, and whose boundary open book decomposition is the same as that of the canonical Lefschetz fibration (the vanishing cycles correspond to different factorizations of the monodromy of the open book decomposition into positive Dehn twists).

Definition 6.2. Suppose that a star surgery operation is defined by replacing a dually positive star-shaped plumbing $\mathcal{S}$ with an alternate convex filling $\mathcal{T}$. Choose a Lefschetz fibration $\mathcal{L}$ whose fibers are the same as those for the canonical Lefschetz fibration for $\mathcal{S}$, and the boundary open book decompositions for these two Lefschetz fibrations are the same. Assume that there is a vanishing cycle $c_{xy}$ of $\mathcal{L}$ which convexly encloses only $x$ and $y$ where $x$ is a single hole and $y$ is a collection of holes disjoint from $x$ which is convexly enclosed by a curve $c_y$ disjoint
from the other vanishing cycles of \( \mathcal{L} \), where \( c_y \) appears in the canonical Lefschetz fibration \( \mathcal{L}_{\text{can}} \) for \( S \) (e.g. \( y \) could be a single hole and \( c_y \) a boundary parallel curve).

Then define the length \( n \) sprout of \( \mathcal{L} \) along \((x, y)\), \( \text{Spr}^n_{x,y}(\mathcal{L}) \), to be the Lefschetz fibration obtained from \( \mathcal{L} \) as follows. Modify the fiber by replacing the hole \( x \) by \( n + 1 \) smaller holes \( x_1, \cdots, x_{n+1} \) obtained by splitting up \( x \) with parallel bands. Modify the vanishing cycles by replacing \( c_{xy} \) by \( n + 1 \) vanishing cycles \( c_{x_1y}, \cdots, c_{x_{n+1}y} \) (which convexly enclose the holes specified in the subscripts) in that order, and keep all of the other vanishing cycles as they were (so any curve which went around \( x \) now goes around a convex disk containing \( x_1, \cdots, x_{n+1} \)).

Observe under sprouting the number of vanishing cycles (corresponding to 2-handles in the 4-manifold) increases by \( n \) and the number of holes in the fiber (corresponding to 1-handles in the 4-manifold) also increases by \( n \). Therefore the Euler characteristic of the sprouted Lefschetz fibration is equal to the Euler characteristic of the original Lefschetz fibration.

Although this modification resembles a stabilization, it is not a stabilization of the Lefschetz fibration (see figure 13). Sprouting generally changes the diffeomorphism types of both the 4-manifold represented by the Lefschetz fibration and the 3-manifold on its boundary represented by the corresponding open book decomposition.

In the example used in the previous subsection, the Lefschetz fibration for \( T_2 \) has fibers which are 4-holed disks, and the vanishing cycles are six curves which convexly contain each distinct pair of holes. The monodromy of the boundary open book, written as a factorization of positive Dehn twists about these vanishing cycles is

\[
\phi_{AB}\phi_{AC}\phi_{AD}\phi_{BC}\phi_{BD}\phi_{CD}
\]

where the holes are labelled \( A, B, C, D \) counter-clockwise.

A length \( n \) sprout of this Lefschetz fibration along \((A, B)\) is a Lefschetz fibration with \( n + 4 \) holes \( A_1, \cdots, A_{n+1}, B, C, D \) ordered counter-clockwise. The vanishing cycles are curves convexly containing \( \{A_1, B\}, \{A_2, B\}, \cdots, \{A_{n+1}, B\}, \{A_1, \cdots, A_{n+1}, C\}, \{A_1, \cdots, A_{n+1}, D\}, \{B, C\}, \{B, D\}, \{C, D\} \) (in that order).

We would like to show that if the Lefschetz fibration \( \mathcal{L} \) represents an alternate filling of the contact boundary of a plumbing \( S \) with graph \( \Gamma \), then the sprouted Lefschetz fibration \( \text{Spr}^n_{x,y}(\mathcal{L}) \) represents an alternate filling of the contact boundary of \( \text{Spr}^n_{x,y}(\Gamma) \). Recall from section [22] that there is an association between the holes of the fiber of the canonical Lefschetz fibration for a dually-positive star-shaped plumbing, and the vertices of the dual graph \( \Gamma \). In this context, \( y \) may be a collection of holes, convexly contained in a vanishing cycle \( c_y \) which appears in the canonical Lefschetz fibration. In this case, the dual graph vertex corresponding to \( y \) is the one corresponding to the outermost hole contained in \( c_y \).

\( \text{Spr}^n_{x,y}(\Gamma) \) is obtained from \( \Gamma \) by decreasing by \( n \) the coefficient on the last sphere in the arm corresponding to the dual sphere associated with the hole \( x \) and by inserting \( n \) vertices of
weight $-2$ into the graph in the position dual to $C_y$, the dual vertex associated to $y$. Therefore the canonical Lefschetz fibration for the plumbing for $Spr^n_{x,y}(\Gamma)$ is obtained from $\mathcal{L}_{can}(\Gamma)$ by

replacing the hole $x$ and its innermost boundary parallel vanishing cycle by $n+1$ holes, each with a boundary parallel vanishing cycle, and by adding $n$ vanishing cycles parallel to $c_y$. The following lemma shows that given a monodromy substitution relating the canonical Lefschetz fibration for $S$ to the Lefschetz fibration $\mathcal{L}$ for the alternate filling, there is a monodromy substitution relating the canonical Lefschetz fibration for a graph sprouted from $S$ as described above, to the sprout of $\mathcal{L}$.

**Lemma 6.3.** Suppose we have a relation in the mapping class group of a disk with $m$ holes placed counterclockwise on a circle concentric to the boundary which has the form $\phi_x \phi_y \alpha = \beta \phi_{xy} \gamma$ where $\alpha, \beta, \gamma$ are products of positive Dehn twists, $x$ is a single hole and $y$ is a collection of holes disjoint from $x$ which can be convexly contained by a curve $c_y$ such that $\phi_y$ commutes with $\beta$. Then the following relation holds on the disk with $m+n$ holes placed counterclockwise on a circle such that $x_1, \ldots, x_{n+1}$ are obtained by splitting the hole $x$ into $n+1$ holes by attaching parallel bands, and all other holes remain in the same position.

$$\phi_{x_1} \cdots \phi_{x_{n+1}} \phi_{y}^{n+1} \alpha = \tilde{\beta} \phi_{x_1 y} \cdots \phi_{x_{n+1} y} \gamma$$

Here $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are the products of positive Dehn twists along the images of the curves used for $\alpha, \beta, \gamma$ in the disk in which the hole $x$ has been split into $n+1$ holes.

**Proof.** The relation $\phi_x \phi_y \alpha = \beta \phi_{xy} \gamma$ induces a relation on the mapping class group of the disk with $m+n$ holes:

$$\phi_{x_1} \cdots \phi_{x_{n+1}} \phi_y \alpha = \tilde{\beta} \phi_{x_1 \cdots x_{n+1} y} \gamma$$

Since boundary parallel curves can be made disjoint from any other curve, Dehn twists about these curves commute with any other Dehn twists. Furthermore Dehn twists about curves which convexly enclose $x_1, \ldots, x_{n+1}$ can be made disjoint from all other curves that are being Dehn twisted along in this decomposition since $\tilde{\beta}, \tilde{\gamma}$ decompose into Dehn twists about curves which do not intersect a convex disk containing $x_1, \ldots, x_{n+1}$. Therefore the following relation holds.

$$\phi_{x_1} \cdots \phi_{x_{n+1}} \phi_y^{n+1} \alpha = \tilde{\beta} \phi_{x_1 \cdots x_{n+1}} \phi_y \phi_{x_1} \cdots \phi_{x_{n+1}} \phi_{x_1 \cdots x_{n+1} y} \gamma$$

Using the daisy relation on the right hand side gives the relation stated in the lemma. \qed

In the example we have been working with, the graph corresponding to the length $n$ sprout along the vanishing cycle $\phi_{AB}$ has graph shown on the left of figure 11. The canonical Lefschetz fibration for the corresponding plumbing has fibers and vanishing cycles as in figure 14. The corresponding decomposition for the monodromy of the open book into positive Dehn twists is $\phi_{A_1} \cdots \phi_{A_{n+1}} \phi_{A_1 \cdots A_{n+1}} \phi_{B}^{2} \phi_{C}^{2} \phi_{D}^{2} \phi_{A_1 \cdots A_{n+1} BCD}$. The monodromy substitution corresponding to this sprouted star surgery is

$$\phi_{A_1} \cdots \phi_{A_{n+1}} \phi_{A_1 \cdots A_{n+1}} \phi_B^{n+2} \phi_{C}^{2} \phi_{D}^{2} \phi_{A_1 \cdots A_{n+1} BCD}$$

$$= \phi_{B}^{n} \phi_{A_1} \cdots \phi_{A_{n+1}} \phi_{A_1 \cdots A_{n+1}} B \phi_{A_1 \cdots A_{n+1}} C \phi_{A_1 \cdots A_{n+1}} D \phi_{BC} \phi_{BD} \phi_{CD}$$

$$= \phi_{A_1} \phi_{B} \cdots \phi_{A_{n+1}} B \phi_{A_1 \cdots A_{n+1}} C \phi_{A_1 \cdots A_{n+1}} O \phi_{BC} \phi_{BD} \phi_{CD}$$

**6.3. Equivalence of homology embedding and monodromy substitution interpretations.** Suppose we have a star surgery operation which replaces a dually positive star-shaped plumbing $S$, plumbed according to a graph $\Gamma$, with an alternate convex filling $T$ supported by a Lefschetz fibration $\mathcal{L}$, and appearing as the complement of a symplectic embedding of the dual plumbing $DT$ into $\mathbb{CP}^2 \# N \overline{\mathbb{CP}^2}$ with homological embedding $\mathcal{H}$. Furthermore, assume that there is a translation between $\mathcal{L}$ and the embedding of $DT$. Under these conditions we have the following result.
Proposition 6.4. There is an embedding of $Spr_{x,y}^n(D\Gamma)$ into $\mathbb{C}P^2 \# (N + n) \mathbb{C}P^2$ representing the homological embedding $Spr_{x,y}^n(H)$ such that the complement has the structure of the Lefschetz fibration $Spr_{x,y}^n(L)$.

Proof. We need to provide a translation between $Spr_{x,y}^n(L)$ and $Spr_{x,y}^n(H)$, using the translation between $L$ and $H$. In fact, since length $n$ sprouting is equivalent to $n$ consecutive length 1 sprouts, it suffices to provide a translation between $Spr_{x,y}^1(L)$ and $Spr_{x,y}^1(H)$. The proof is essentially contained in figure 15. Start with the standard handlebody diagram for $Spr_{x,y}^1(L)$, and turn this handlebody diagram upside-down. There is one extra 1-handle in the diagram for $Spr_{x,y}^1(L)$ than for $L$ corresponding to the splitting of the hole $x$ into two holes $x_1$ and $x_2$. Introduce a cancelling $+1$ and $-1$ pair of surgery curves along $c_{x,y}$. Consider $x_2$ as corresponding to the last vertex in the sprouted dual graph. Perform handle-slides on the 0-framed surgery curves coming from the dotted circles in order for each arm sliding the curves corresponding to the last vertex over the curve corresponding to the previous vertex, and then sliding that curve over the previous curve, and so on until reaching the curve corresponding to the first vertex in the arm. In particular, the hole corresponding to $x_2$ is slid over the curve corresponding to $x_1$ first. Then there is a 1-framed surgery curve for each vanishing cycle in $L$, so we can perform the translation between $L$ and $H$ resulting in the diagram in figure 15 with additional 2-handles attached according to $L$. Blow down the $\pm 1$-framed surgery curves in the boxed region where the diagram for the sprout differs from the original diagram, and then isotope as shown in figure 15. Gluing in the sprouted dual plumbing to the lower boundary, we obtain the embedding of the sprouted dual plumbing into $\mathbb{C}P^2 \# (N + 1) \mathbb{C}P^2$. The extra $-2$ sphere is given by the core and Seifert surface of the 2-handle attached along the green curve, and it intersects the two blue exceptional spheres involved with opposite signs. The image of $C_x$ is given by the core and Seifert-surface for the 2-handle attached along the red curve, and it intersects only one of the relevant exceptional spheres positively. The image of $C_y$ is given by the core and Seifert-surface for the 2-handle attached along the purple curve, whose framing is reduced by 1. This sphere intersects both of the relevant exceptional spheres positively. Therefore the embedding indicated by the diagram represents the sprouted homology embedding.

\[\square\]

6.4. Examples of sprouting. Sprouting yields both familiar and new examples of star-surgeries.
The first examples we mention here are families of rational blow-downs which arise as sprouts of the rational blow-down of the \(-4\) sphere. Plumbings which can be symplectically rationally blown down have been studied extensively (see [6], [17], [22], [2], [15]) and are fully classified, so it is no surprise that these operations are well-understood already.

The homology embeddings and Lefschetz fibrations of the rational blow-down of the \(-4\) sphere were discussed in section 4, figure 3. Recall that the dual graph has three length one arms, so the Lefschetz fibration fibers are disks with three holes, which we will label 1, 2, 3. Denote the homology embedding and Lefschetz fibrations corresponding to the rational blow-down by \(H\) and \(L\) respectively. Length \(n\) sprouting along a single pair of vertices, \(Spr^1_{1,2}(L)\) results in the family of rational blow-downs introduced by Fintushel and Stern [6]: see figure 16 for the dual graphs and corresponding graphs. By iteratively sprouting along the other two arms, one can obtain all of the rational blow-downs of linear plumbings whose boundary is \(L(p^2, pq - 1)\).

This can be seen most easily using the description of these plumbings as the family \(G_r\) defined in [22] as the smallest family satisfying both

1. \(G_r\) contains \((-4)\), and
2. if \((-a_1, \ldots, -a_n) \in G_r\) then \((-2, -a_1, \ldots, -a_n-1, -a_n-1)\) and \((-a_1-1, -a_2, \ldots, -a_n, -2)\) are also in \(G_r\).

Finally, the family \(W_{p,q,r}\) defined in [22] and shown there to bound rational homology balls, corresponds to \(Spr^1_{1,2}Spr^2_{2,3}Spr^3_{3,1}(H)\) (see the right side of figure 16 for the graphs and dual graphs). The corresponding monodromy substitutions for all of these rational blow-downs (which can be determined by sprouting the lantern relation), were already found by Endo, Mark, and Van Horn-Morris in [5].

By sprouting other well understood star-surgeries, we can obtain operations which can be more useful in constructing exotic 4-manifolds of small Euler characteristic. For example, sprouts of the star-surgery replacing \(S_2\) with \(T_2\) as in figure 17 can be used to construct exotic copies of \(\mathbb{CP}^2 \# 6\mathbb{CP}^2\) and \(\mathbb{CP}^2 \# 7\mathbb{CP}^2\) using the techniques of [11].
Examples and Properties of Star Surgeries

Figure 16. Dual graphs and graphs sprouted from the $-4$ sphere graph.

Figure 17. Dual graphs and graphs of $S_2$, and two types of sprouts. Sprouting the star-surgery replacing $S_2$ by $T_2$ yields star-surgeries replacing the lower graph plumblings by fillings of Euler characteristic 2, which can be used to construct exotic copies of $\mathbb{CP}^2 \# 7 \mathbb{CP}^2$ and $\mathbb{CP}^2 \# 6 \mathbb{CP}^2$.

References

[1] Mohan Bhupal and Burak Ozbagci. Symplectic fillings of lens spaces as lefschetz fibrations. arxiv:1307.6935.
[2] Mohan Bhupal and András I. Stipsicz. Weighted homogeneous singularities and rational homology disk smoothings. Amer. J. Math., 133(5):1259–1297, 2011.
[3] Yakov Eliashberg. Topological characterization of Stein manifolds of dimension $> 2$. Internat. J. Math., 1(1):29–46, 1990.
[4] Hisaaki Endo and Yusuf Z. Gurtas. Lantern relations and rational blowdowns. Proc. Amer. Math. Soc., 138(3):1131–1142, 2010.
[5] Hisaaki Endo, Thomas E. Mark, and Jeremy Van Horn-Morris. Monodromy substitutions and rational blowdowns. J. Topol., 4(1):227–253, 2011.
[6] Ronald Fintushel and Ronald J. Stern. Rational blowdowns of smooth 4-manifolds. J. Differential Geom., 46(2):181–235, 1997.
[7] David Gay and Thomas E. Mark. Convex plumbings and Lefschetz fibrations. J. Symplectic Geom., 11(3):363–375, 2013.
[8] David T. Gay and András I. Stipsicz. Symplectic surgeries and normal surface singularities. Algebr. Geom. Topol., 9(4):2203–2223, 2009.
[9] Robert E. Gompf. Handlebody construction of Stein surfaces. Ann. of Math. (2), 148(2):619–693, 1998.
[10] Amey Kaloti. Stein fillings of planar open books. arxiv:1311.0208.
[11] Çağrı Karakurt and Laura Starkston. Surgery along star shaped plumblings and exotic smooth structures on 4-manifolds. arxiv:1402.0801.
[12] Paolo Lisca. On symplectic fillings of lens spaces. Trans. Amer. Math. Soc., 360(2):765–799 (electronic), 2008.
[13] Dusa McDuff. The structure of rational and ruled symplectic 4-manifolds. J. Amer. Math. Soc., 3(3):679–712, 1990.
[14] Maria Michalogiorgaki. Rational blow-down along Wahl type plumbing trees of spheres. *Algebr. Geom. Topol.*, 7:1327–1345, 2007.

[15] Heesang Park, Dongsoo Shin, and András I. Stipsicz. Normal complex surface singularities with rational homology disk smoothings. arxiv:1311.1929.

[16] Heesang Park and András I. Stipsicz. Smoothings of singularities and symplectic surgery, 2012. arXiv:1211.6830v1 [math.GT].

[17] Jongil Park. Seiberg-Witten invariants of generalised rational blow-downs. *Bull. Austral. Math. Soc.*, 56(3):363–384, 1997.

[18] Jongil Park. Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$. *Invent. Math.*, 159(3):657–667, 2005.

[19] Olga Plamenevskaya and Jeremy Van Horn-Morris. Planar open books, monodromy factorizations and symplectic fillings. *Geom. Topol.*, 14(4):2077–2101, 2010.

[20] Stephan Schönberger. Determining symplectic fillings from planar open books. *J. Symplectic Geom.*, 5(1):19–41, 2007.

[21] Laura Starkston. Symplectic fillings of Seifert fibered spaces. arxiv:1304.2420. to appear in Trans. Amer. Math. Soc.

[22] András I. Stipsicz, Zoltán Szabó, and Jonathan Wahl. Rational blowdowns and smoothings of surface singularities. *J. Topol.*, 1(2):477–517, 2008.

[23] Margaret Symington. Symplectic rational blowdowns. *J. Differential Geom.*, 50(3):505–518, 1998.

[24] Margaret Symington. Generalized symplectic rational blowdowns. *Algebr. Geom. Topol.*, 1:503–518 (electronic), 2001.

[25] Chris Wendl. Strongly fillable contact manifolds and $J$-holomorphic foliations. *Duke Math. J.*, 151(3):337–384, 2010.