Optimization of Robot Grasping Forces and Worst Case Loading

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Abstract: We consider the optimization of the vector of grasping forces that support a known generalized force acting on the grasped object—a rigid body or a mechanism. Working in the framework of finite dimensional normed vector spaces and their dual spaces, the cost function to be minimized is assumed to be a norm on the space of grasping forces. We present an expression for the optimum which depends on the external force and the kinematics of the grasping system. Next, assuming that optimal grasping forces are applied using force control, and assuming that there is a bound on the norm of the admissible grasping forces, we characterize the largest norm of an external force that the grasping system may support, that is, the norm of the worst case loading that may be applied and still be supported. A few simple examples are given for the sake of illustration.

1. Introduction

The mechanical analysis of robot grasping is under ongoing active research. See, for example, the comprehensive review of the subject in [BR19], published recently, and references cited therein. Within this general area of research, various optimization problems may be considered. In [BR19, Chapter 9], the minimal number of fingers in the grasping mechanism is studied. In [BR19, Chapter 12], the problem of “gentlest grasp” is defined, where the maximum grasping force is minimized. Another study which is relevant to the present work is [Bo88], where the author considers a planar rigid body and searches for the optimal 3 points of grasping normal
1. Introduction

and friction forces so that the maximal external force may be supported. Additional work on problems related to optimization of grasping may found in [YO10, FS12].

This article is concerned with the optimization of grasping forces that robots apply to objects they carry, where the objects may be rigid bodies or mechanisms. Assume that the external forces on an object are given, and that there are more grasping force components than the number of degrees of freedom of the object. In such cases, the grasping forces cannot be determined by the equations of mechanics alone—there will be an infinite number of solutions to the equations. Thus, it might be desirable, to find among all grasping forces that satisfy the equations of mechanics, those forces that satisfy some optimality conditions.

For example, a fragile object may be held at a number of points, so that one may wish to minimize the largest applied grasping force—gentlest grasp. In the general case, considered in Section 3 the optimality condition is given in terms of a norm on the space grasping forces. The main result, given in Proposition 3.1 describes the optimum in terms of the applied external force and the linear mapping, \( J \), that gives the generalized velocities at the points where the object is held in terms of the generalized velocities of the object.

Once the result on optimal reactions is at hand, one may consider the following problem. Conceivably, one could use a force controlled grasping system in order to realize optimal grasping of an object for any given external load. However, due to limitations of the force control system and in order to prevent damage to the object, which may be caused by grasping, the norm of admissible grasping forces may be limited to some value \( M \). The question then arises as to the nature of the set of admissible external forces on the object. That is, how can we characterize the external forces that may be supported by grasping forces, the norms of which do not exceed \( M \)? A related problem is concerned with worst case loading. How can one characterize forces that pose the greatest risk to grasping?

These issues are considered in Section 4 and the main result is given in Proposition 4.6. Simply put, there is a number \( R > 0 \), the grasping system sensitivity, such that the admissible forces, \( f \), are those satisfying the condition

\[
\|f\| \leq \frac{M}{R}.
\]  

The system’s sensitivity, \( R \), is determined by the linear mapping \( J \), mentioned above. In particular, \( R \) depends only on the geometry of the grasping.

The methods we use in this paper are analogous to those used in previous work [Seg07, FS09] in the context of continuum mechanics. Section 2 below overviews the framework for the mechanics of grasping within which we work. Of special importance is the natural assumption that the mapping \( J \), relating the generalized velocities of the object to the generalized velocities of the supported points, is injective. Section 3 presents the optimization problem in terms of a norm on the space of grasping forces. The main result of this section, Proposition 3.1 is based on the finite dimensional version of the Hahn-Banach theorem. Bounded grasping forces and worst case loadings are studied in Section 4. The notion of sensitivity of the grasping to a certain external force is defined, and is related to the factor of safety of the grasping. These objects are intimately related to the Minkowski functional for a certain convex set, \( K \), associated with the admissible grasping forces. The main result, as described above, is presented in Proposition 4.6. Since one of the methods of computing the optimal reactions is based on the construction
2. The mechanical framework for the analysis

This section will present the mechanical and mathematical settings for the grasping problem that we wish to optimize. It relies on the geometric point of view of the mechanical system involved.

2.1. Objects, configurations and constraints

Our analysis is focused on an object—a mechanical system that is grasped by the robot. In most applications, this mechanical system will be modeled as a rigid body. However, the following analysis holds also when the grasped object is a mechanism. The kinematics of the object is modeled by an $n$-dimensional differentiable manifold, the configuration space, $\mathcal{Q}$, where $n$ is the number of degrees of freedom of the object. A typical configuration of the object will be denoted by $\kappa \in \mathcal{Q}$. The physical space will be represented by $\mathbb{R}^3$.

The object is supported by constraints that determine its configurations. As a typical example, the locations in space of a fixed number of points belonging to the objects may be constrained by the robot grippers. (See illustration in Figure 2.1.)

It is natural, therefore, to assume that a given configuration $\kappa \in \mathcal{Q}$ determines the corresponding state of the constraints. For example, if the object is a rigid body which is constrained by the locations of $m_c$ points in it, any configuration of the object will determine $m_c$ points in

Figure 2.1: Illustrating an object and constraints. In this simple example, $\mathcal{Q} = \mathbb{R}^2 \times S^1 = \{(x, y, \theta)\}$, $\mathcal{P} = (\mathbb{R}^2 \times S^1)^2 = \{(x_A, y_A, \theta_A, x_B, y_B, \theta_B)\}$, $\mathcal{V} = \mathbb{R}^3 = \{(\dot{x}, \dot{y}, \dot{\theta})\}$ and $\mathcal{U} = \mathbb{R}^6 = \{\dot{x}_A, \dot{y}_A, \dot{\theta}_A, \dot{x}_B, \dot{y}_B, \dot{\theta}_B\}$.

of the proof of the finite dimensional Helly-Hahn-Banach theorem (Helly [Hel12] was the first to prove the finite-dimensional version), and in order to make the manuscript self-contained, we overview this construction in Appendix A.

In order to illustrate the approach and results presented in text, we consider in Section 3 and 4 a number of very simple examples.
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space, an element of $\mathbb{R}^{3m}$. We will denote a typical state of the constraints by $p$ and will denote the collection of all states of the constraints—the constraint space—by $\mathcal{P}$. Thus, for the example, $\mathcal{P} = \mathbb{R}^{3m}$. In the general case, the constraint space $\mathcal{P}$ is assumed to be a differentiable manifold of dimension $m$. Since fixing the state of the constraints should determine a unique state of the object, it is assumed that $\dim \mathcal{P} = m \geq \dim \mathcal{Q} = n$, so that the constraints can eliminate all degrees of freedom the object might have. Thus, the constraint state determined by a configuration of the object is given through a mapping—the constraint mapping,

$$\varphi : \mathcal{Q} \longrightarrow \mathcal{P}. \quad (2.1)$$

It is assumed that the mapping $\varphi$ is differentiable. Moreover, the assumption that a state of the constraints determines a unique configuration of the object, implies that the mapping $\varphi$ is injective. It is observed that the assumption that $\varphi$ is injective, is different from what one expects from the forward kinematics mapping in robotics, where it is accepted that the inverse kinematics problem does not have a unique solution.

We note that the dimension of the constraint space, $m$ may be strictly greater than the dimension $n$ of the configurations space. In such a case, the object is over-constrained by the grasping. If $p \in \text{Image } \varphi$, that is, there is some configuration $\kappa \in \mathcal{Q}$ such that $p = \varphi(\kappa)$, we will say that $p$ is a compatible constraint state and that $p$ is compatible with $\kappa$. For the case where $m > n$, strictly, it is clear that not every constraint state is compatible.

**Example 2.1.** Consider the two-dimensional system considered in Figure 2.1. Here, $\mathcal{Q} = \mathbb{R}^2 \times S^1 = \{ (x, y, \theta) \}$ and $\mathcal{P} = (\mathbb{R}^2 \times S^1)^2 = \{ (x_A, y_A, \theta_A, x_B, y_B, \theta_B) \}$. It is noted that for a generic element of $\mathcal{P}$, there need not be any relation between $(x_A, y_A, \theta_A)$ and $(x_B, y_B, \theta_B)$. Evidently, kinematics of rigid bodies provide the relation between the components of compatible elements of $\mathcal{P}$.

2.2. Velocities, external forces and supporting forces

Modeling the configuration space of the object as a differentiable manifold offers a natural representation of generalized velocities at any configuration $\kappa \in \mathcal{Q}$, as elements of the tangent vector space $T_\kappa \mathcal{Q}$ to the configuration manifold at the point $\kappa$. Similarly, elements of $T_p \mathcal{P}$ are interpreted as constraint or support velocities.

The differentiability assumption for the constraint mapping implies that for any $\kappa \in \mathcal{Q}$, we have a linear mapping—the tangent mapping

$$T_\kappa \varphi : T_\kappa \mathcal{Q} \longrightarrow T_{\varphi(\kappa)} \mathcal{P} \quad (2.2)$$

that assigns a constraint velocity to each generalized velocity of the object. The mapping $T_\kappa \varphi$ is represented in coordinates by the Jacobian of the mapping representing $\varphi$ (as we describe below). Henceforth, we strengthen the assumption that $\varphi$ is injective and assume further that the linear $T_\kappa \varphi$ is injective. The property that $T_\kappa \varphi$ is injective will be referred to as kinematic determinacy.

In what follows, we will be concerned with the grasping problem for one particular configuration of the object. Hence, to simplify the notation we will use $\mathcal{V}$ for $T_\kappa \mathcal{Q}$, $\mathcal{U}$ for $T_{\varphi(\kappa)} \mathcal{P}$, and...
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\[ J: \mathcal{V} \rightarrow \mathcal{U}. \]  

(2.3)

In analogy with the terminology for configurations, we say that a generalized velocity of the constraints \( u \) is compatible if \( u \in \text{Image } J \). We say that \( u \) is compatible with the generalized velocity \( v \in \mathcal{V} \) of the object if \( u = J(v) \).

In general, we adopt here the geometric point of view of statics whereby for a mechanical system having a configuration space \( \mathcal{M} \), a generalized force \( f \) at the configuration \( q \in \mathcal{M} \) is defined as a linear functional—a covector

\[ f: T_q \mathcal{M} \rightarrow \mathbb{R}. \]  

(2.4)

Thus, generalized forces act linearly on generalized velocities to produce real numbers. The action \( f(v) \) of the force \( f \) on the generalized velocity \( v \), is interpreted as the power produced by the force for the velocity \( v \). Thus, forces at the configuration \( q \) belong to the dual space

\[ (T_q \mathcal{M})^* := \{ f: T_q \mathcal{M} \rightarrow \mathbb{R} \mid f \text{ is linear} \}. \]  

(2.5)

This applies, in particular, to forces acting on the object. Thus, a generalized force on the object at a configuration \( \kappa \in \mathcal{G} \) is an element \( f \in (T_\kappa \mathcal{G})^* = \mathcal{V}^* \). Similarly, a generalized force \( g \) acting on the constraints (for example, due to forces exerted by the object) at a constrain state \( p \in \mathcal{P} \) is an element of the dual space \( (T_p \mathcal{P})^* \). In case \( p = \varphi(\kappa) \), \( (T_p \mathcal{P})^* = \mathcal{U}^* \) and \( g \in \mathcal{U}^* \).

Given a force \( g \in \mathcal{U}^* \), it is natural to interpret its opposite, \(-g\), as the force exerted by the constraints on the object.

The mapping \( J = T_\kappa \mathcal{G} \) induces the dual (adjoint) mapping

\[ J^*: \mathcal{U}^* \rightarrow \mathcal{V}^*, \]  

(2.6)

whereby a generalized force \( g \) on the supports determines an external force \( f = J^*(g) \) on the object, which is defined by

\[ f(v) = J^*(g)(v) = g(J(v)), \quad \text{or equivalently, } f = J^*(g) = g \circ J. \]  

(2.7)

Let \( f \in \mathcal{V}^* \) be a generalized force acting on the object and let \( g \in \mathcal{U}^* \) be a generalized force acting on the supports. We will say that \( g \) is consistent with \( f \) if

\[ f = J^*(g). \]  

(2.8)

Thus, \( g \) is consistent with \( f \) if the virtual powers that they produce are equal for pairs of compatible virtual velocities—the principle of virtual work. This is equivalent to the statement that \( f \) is in equilibrium with \(-g\) and we will refer to Equation (2.8) as the consistency condition or the equilibrium equation.

It is noted that the foregoing geometric framework applies to the supports and reactions of any fully constrained mechanism.
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2.3. Representation by components

Let $\kappa^1, \ldots, \kappa^n$ be a coordinate system in a neighborhood of $\kappa_0$ in $\mathcal{Q}$ and let $p^1, \ldots, p^m$ be a coordinate system in a neighborhood of $p_0 = \varphi(\kappa_0)$ in $\mathcal{P}$. In particular, $\kappa_0$ is represented by the coordinates $\kappa_0^1, \ldots, \kappa_0^n$ and $p_0$ is represented by the coordinates $p_0^1, \ldots, p_0^m$. Then, in a neighborhood of $\kappa_0$, the constraint mapping may be represented in the form

$$p^i = p^i(\kappa^1, \ldots, \kappa^n) = p^i(\kappa^\alpha), \quad \alpha = 1, \ldots, n, \ i = 1, \ldots, m. \tag{2.9}$$

(Note that in the last equation we mix the notation for the real valued functions $\varphi^i$ with the notation for the variables $p^i$.) In particular, $p_0^i = p^i(\kappa_0^\alpha) = \varphi^i(\kappa_0^\alpha)$.

The coordinate neighborhoods described above make it possible to specify tangent vectors and virtual velocities by components. Thus, a generalized velocity $v \in \mathcal{V}$ of the object will be represented by $(v^1, \ldots, v^n) \in \mathbb{R}^n$, and a generalized velocity of the constraints, $u \in \mathcal{U}$, will be represented by $(u^1, \ldots, u^m) \in \mathbb{R}^m$.

Using this coordinate representation, the tangent mapping, $J = T_{\kappa_0}\varphi$, of the constraint mapping at $\kappa_0$ is represented by Jacobian matrix

$$J^i_\alpha := \frac{\partial p^i}{\partial \kappa^\alpha}(\kappa_0^\beta) = \frac{\partial \varphi^i}{\partial \kappa^\alpha}(\kappa_0^\beta), \quad \alpha, \beta = 1, \ldots, n, \ i = 1, \ldots, m. \tag{2.10}$$

It follows that the condition that $v \in \mathcal{V}$ and $u \in \mathcal{U}$ are compatible may be expressed in the form

$$u^i = J^i_\alpha v^\alpha, \quad i = 1, \ldots, n, \tag{2.11}$$

where the summation convention is used for the repeated index $\alpha = 1, \ldots, n$.

As an element in the dual space, a force $f$ is represented by $(f_1, \ldots, f_n) \in \mathbb{R}^n$ and its action on a generalized velocity $v \in \mathcal{V}$ is given by

$$f(v) = f^\alpha v^\alpha. \tag{2.12}$$

Similarly, $g \in \mathcal{U}^*$ is represented by an element $(g_1, \ldots, g_m) \in \mathbb{R}^m$ and its action on a generalized velocity of the constraints $u$ is given by

$$g(u) = g^i u^i. \tag{2.13}$$

Finally the consistency condition for a force $f \in \mathcal{V}^*$ and a generalized force on the constraints $g \in \mathcal{U}^*$ is

$$f^\alpha v^\alpha = g^i J^i_\alpha v^\alpha \tag{2.14}$$

for every vector $v \in \mathcal{V}$. As expected, the resulting consistency (or equilibrium) condition is

$$f^\alpha = g^i J^i_\alpha, \quad \alpha = 1, \ldots, n. \tag{2.15}$$
3. Static indeterminacy and optimization

As mentioned above, \( m = \dim \mathcal{U} \geq n = \dim \mathcal{V} \). Since the dimension of a dual space is equal to the dimension of the primal space \( \dim \mathcal{U}^* = \dim \mathcal{V}^* \). This implies that the consistency condition, as represented in Equation (2.15), may be viewed, for a given \( f \in \mathcal{V}^* \), as a system of linear \( n \) equations with \( m \geq n \) unknowns. Thus, for the case where \( m > n \), the force \( f \) cannot determine a unique constraint force \( g \in \mathcal{U}^* \). This is the origin of what is usually referred to as static indeterminacy. It is noted that static indeterminacy is a result of the kinematic determinacy with \( m > n \), strictly.

The foregoing observation implies that for a given force \( f \in \mathcal{V}^* \) on the object, there is a collection
\[
J^{-1}(f) = \{ g \in \mathcal{U}^* \mid J^*(g) = f \}
\]
of forces on the constraints with which \( f \) is in equilibrium. It is meaningful, therefore, to look for an element in the collection \( J^{-1}(f) \), which is optimal in some sense.

In this paper, the optimization criterion, the value which one wishes to minimize, is assumed to be represented by a norm on the space of constraint forces \( \mathcal{U}^* \). Thus, denoting the norm of the constraint force \( g \in \mathcal{U}^* \) by \( \| g \| \), the properties of norm require that \( \| g \| > 0 \), \( \| a g \| = |a| \| g \| \), \( \| g_1 + g_2 \| \leq \| g_1 \| + \| g_2 \| \), for all \( g, g_1, g_2 \in \mathcal{U}^* \), \( a \in \mathbb{R} \), and that \( \| g \| = 0 \) if and only if \( g = 0 \).

Thus, the grasping optimization problem may be formulated as follows. Given \( J : \mathcal{V} \to \mathcal{U} \), and the force \( f \in \mathcal{V}^* \) acting on the object, find
\[
\Omega_f := \inf \{ \| g \| \mid g \in \mathcal{U}^*, J^*(g) = f \}.
\]

We show below some properties of the optimum \( \Omega_f \) below and in particular, we show the existence of a minimizer \( g_0 \) (not necessarily unique) such that \( \Omega_f = \| g_0 \| \). For this purpose and in order to make the text self-contained, we review below some properties of normed vector spaces.

3.1. Primal norms and dual norms

We consider a generic vector space \( \mathcal{V} \) with a norm \( \| \cdot \| \). (The definitions and results will be applied to both \( \mathcal{V} \) and \( \mathcal{U} \) above.) Since this is our starting point, we will refer to \( \mathcal{V} \) as the primal vector space and to the prescribed norm as the primal norm.

The norm on the vector space \( \mathcal{V} \) induces naturally a norm, the dual norm on the dual space \( \mathcal{V}^* \). The dual norm of an element \( f \in \mathcal{V}^* \) is defined by
\[
\| f \| = \sup_{\| v \| = 1} \left| \frac{f(v)}{\| v \|} \right| = \sup_{\| f \| = 1} |f(v)|.
\]

Note that we use the same notation for both primal and dual norms when no confusion may arise.

It is recalled that for the finite-dimensional case \( \dim \mathcal{V}^* = \dim \mathcal{V} \) and for any \( v \in \mathcal{V} \)
\[
\| v \| = \sup_{f \neq 0} \left| \frac{f(v)}{\| f \|} \right| = \sup_{\| f \| = 1} |f(v)|.
\]
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For the optimization problem considered, it is natural to assume that the norm for the constraint forces, which one wishes to minimize, is chosen on the basis of considerations related to the design of the gripping mechanism. This implies that in for the optimization problem, one starts with the dual norm in \( \mathcal{U}^* \). Thus, Equation (3.4) provides the corresponding primal norm on \( \mathcal{U} \).

3.2. Extension of covectors and static indeterminacy

Consider the following diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{f} & \mathcal{V} & \xrightarrow{J^{-1}} & \text{Image } J & \xrightarrow{i} & \mathcal{U} \\
& & & \downarrow g_0 = f \circ J^{-1} & & \downarrow j & & \downarrow \text{Goal} \end{array}
\]

(3.5)

where

\[
\tilde{J} : \mathcal{V} \to \text{Image } J, \quad \text{and} \quad i : \text{Image } J \to \mathcal{U}
\]

(3.6)

are the mapping between \( \mathcal{V} \) and its image under \( J \), and the natural inclusion of the vector subspace, so that \( J = i \circ \tilde{J} \). Since \( J \) is injective, \( \tilde{J} : \mathcal{V} \to \text{Image } J \) is an isomorphism, and \( \tilde{J}^{-1} : \text{Image } J \to \mathcal{V} \) is a well-defined isomorphism. A generalized force, \( f \in \mathcal{V}^* \), acting on the object, induces a unique element \( g_0 := f \circ \tilde{J}^{-1} \in (\text{Image } J)^* \). Conversely, any \( g_0 \in (\text{Image } J)^* \) determined a unique \( f \in \mathcal{V}^* \) by \( f = g_0 \circ \tilde{J} \), or \( f(v) = g_0(J(v)) \), for all \( v \in \mathcal{V} \). In other words, \( \tilde{J}^* : (\text{Image } J)^* \to \mathcal{V}^* \)

(3.7)

is an isomorphism.

Note however that for a given \( f \in \mathcal{V}^* \), \( g_0 = \tilde{J}^* (f) \) is not a constraint force. It is an element of \( (\text{Image } J)^* \) and not an element of \( \mathcal{U}^* \). In other words, we know how \( g_0 \) acts only on all compatible velocities of the constraints and not on all generalized constraint velocities. While a constraint force \( g \in \mathcal{U}^* \) may be restricted uniquely to an element \( g_0 \in (\text{Image } J)^* \) by \( g_0(u) = g(u) \), for the situation where \( m > n \), the extension of \( g_0 \) to the whole of \( \mathcal{U} \) is not unique. This observation is another aspect of static indeterminacy. It follows that our optimization problem is equivalent to minimizing the extension \( g \in \mathcal{U}^* \) of \( g_0 = \tilde{J}^* (f) \in (\text{Image } J)^* \) from \( \text{Image } J \) to \( \mathcal{U} \).

3.3. Norm-minimizing extensions and the Hahn-Banach theorem

In view of the conclusion of the preceding paragraph, we consider norm-minimizing extensions of covectors. Thus, consider \( \mathcal{W} = \text{Image } J \subset \mathcal{U} \). (The same construction applies to any other vector subspace \( \mathcal{W} \subset \mathcal{U} \).) It is assumed that dual norms are given as above on \( \mathcal{U} \) and \( \mathcal{U}^* \). The norm on \( \mathcal{U} \) induces, by restriction, a norm on \( \mathcal{W} \). In turn, a norm on \( \mathcal{W}^* \) is induced by setting

\[
\|g_0\| = \sup_{w \neq 0} \frac{|g_0(w)|}{\|w\|} = \sup_{\|w\| = 1} |g_0(w)|, \quad w \in \mathcal{W}.
\]

(3.8)
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It is noted that in case \( g_0 \in \mathcal{W}^* \) is the restriction of some constraint force \( g \in \mathcal{U}^* \), then,

\[
\|g_0\| = \sup_{w \in \mathcal{W}} \frac{|g_0(w)|}{\|w\|} = \sup_{w \in \mathcal{W}} \frac{|g(w)|}{\|w\|} \leq \sup_{u \in \mathcal{U}} \frac{|g(u)|}{\|u\|} = \|g\|. \tag{3.9}
\]

Thus, the norm of a constraint force is bounded from below by the norm of its restriction to compatible velocities of the constraints. The isomorphism \( \tilde{J}^* \) implies that the optimum \( \Omega_f \) is bounded below by the norm of \( g_0 = \tilde{J}^* (f) \).

The Hahn-Banach theorem, in fact, the theorem proved by Helly for the finite dimensional case in \cite{Hel12}, asserts that any \( g_0 \in \mathcal{W}^* \) may be extended to some \( g_H \in \mathcal{U}^* \) without increasing its norm. The proof of the theorem is given in Appendix A both for the sake of completeness, and, in particular, because we will use the construction in the following examples.

It is concluded that the lower bound \( \|g_0\| \) on \( \|g\| \) is attained by some \( g_H \in \mathcal{U}^* \) and the optimal extension is given by

\[
\|g_0\| = \inf \{ \|g\| \mid g \in \mathcal{U}^*, g(w) = g_0(w), \forall w \in \mathcal{W} \} = \|g_H\|. \tag{3.10}
\]

3.4. Norm minimizing grasping

Once optimal extensions are at our disposal, we may continue our study of optimal grasping.

Let a norm be given on the vector space \( \mathcal{V} := T_k \mathcal{G} \) of generalized velocities of the object.

**Proposition 3.1.** Using the notation introduced above, for a fixed configuration of the object, \( k \in \mathcal{G} \), let

\[
J := T_k \varphi : \mathcal{V} := T_k \mathcal{G} \rightarrow \mathcal{U} := T_{\varphi(k)} \mathcal{P}, \tag{3.11}
\]

be the tangent of the constraint mapping.

1. Assume that \( J \) is injective. Then, given a force \( f \in \mathcal{V}^* \) on the object there exists some \( g \in \mathcal{U}^* \) that satisfy the equilibrium equations, that is

\[
f = J^* (g).
\]

2. Assume that \( \mathcal{V}^* \) and \( \mathcal{U}^* \) are normed and for a given \( f \in \mathcal{V}^* \) let the optimal grasping problem be defined as

\[
\Omega_f := \inf \{ \|g\| \mid g \in \mathcal{U}^*, J^* (g) = f \}. \tag{3.12}
\]

Then,

\[
\Omega_f = \sup_{v \in \mathcal{V}} \frac{f(v)}{\|J(v)\|}. \tag{3.13}
\]

where the norms on \( \mathcal{U} \) is the primal norm for that given on \( \mathcal{U}^* \). In particular, for the case where \( \mathcal{V} \) is a normed space, \( \mathcal{V}^* \) is equipped with the dual norm, and \( J \) is norm preserving,

\[
\Omega_f = \|f\|. \tag{3.14}
\]
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3. The optimal grasping is attained for some \( g_H \in \mathcal{U}^* \), that is, there is some \( g_H \in \mathcal{U}^* \), such that

\[
\Omega_f = \|g_H\|. \tag{3.15}
\]

4. The optimal grasping force is positive homogeneous, that is, for every \( a \in \mathbb{R} \),

\[
\Omega_{af} = |a| \Omega_f. \tag{3.16}
\]

In particular,

\[
\Omega_f \Omega_f = 1. \tag{3.17}
\]

**Proof.** Using the foregoing notation and observations we proceed as follows.

Let \( \bar{f} : \mathcal{V} \rightarrow \text{Image} \, J \) be the restricted isomorphism. Then, \( g_0 \in (\text{Image} \, J)^* \), given by \( g_0 = f \circ \bar{f}^{-1} \) satisfies

\[
f = g_0 \circ \bar{f} = J^*(g_0), \quad \text{or explicitly,} \quad f(v) = g_0(J(v)), \quad \text{for all } v \in \mathcal{V}. \tag{3.18}
\]

The covector \( g_0 \in (\text{Image} \, J)^* \) may be extended to some \( g \in \mathcal{U}^* \) satisfying \( g(w) = g_0(w) \) for all \( w \in \text{Image} \, J \). Hence,

\[
f(v) = g(J(v)), \quad \text{for all } v \in \mathcal{V} \tag{3.19}
\]

and we conclude that \( f = J^*(g) \). This proves the first assertion above.

By definition,

\[
\|g_0\| := \sup_{w \in \text{Image} \, J} \frac{g_0(w)}{\|w\|}. \tag{3.20}
\]

Since \( \bar{f} \) is an isomorphism, we may write

\[
\|g_0\| = \sup_{v \in \mathcal{V}} \frac{g_0(f(v))}{\|f(v)\|} = \sup_{v \in \mathcal{V}} \frac{f(v)}{\|f(v)\|}, \tag{3.21}
\]

where it is observed that \( \|g_0\| \) is uniquely determined by \( f \). As a consequence of the Hahn-Banach theorem as discussed in Section 3.3, there is some \( g_H \in \mathcal{U}^* \) extending \( g_0 \) such that \( \|g_H\| = \|g_0\| \)–the minimal norm of all possible extensions. Thus,

\[
\Omega_f = \|g_H\| = \sup_{v \in \mathcal{V}} \frac{f(v)}{\|f(v)\|}, \tag{3.22}
\]

which proves (2) and (3) above.

Evidently, if \( J \) is norm preserving so that \( \|J(v)\| = \|v\| \), one has

\[
\Omega_f = \sup_{v \in \mathcal{V}} \frac{f(v)}{\|v\|} = \|f\|. \tag{3.23}
\]

Assertion (4) follows immediately from Equation (3.13). □
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Given a force \( f \) acting on the object, the sensitivity ratio for \( f \), \( R_f \), is defined by

\[
R_f := \inf_{f \neq f(\phi) \in \mathbb{W}} \frac{\|g\|}{\|f\|}. \tag{3.24}
\]

It immediately follows from the foregoing result that

\[
R_f = \frac{\Omega_f}{\|f\|} = \frac{1}{\|f\|} \sup_{v \in \mathcal{V}} \frac{f(v)}{\|J(v)\|}. \tag{3.25}
\]

From the practical point of view, \( R_f \) indicates how much the magnitude of \( f \) is amplified for optimal grasping forces.

**Remark 3.2.** Equation (3.16) implies that the sensitivity ratio for \( f \) does not depend on \( k_f \). It depends only on its normalized counterpart, \( f/\|f\| \), that is, for any \( a \in \mathbb{R}^+ \),

\[
R_{af} = R_f = R_f/\|f\|. \tag{3.26}
\]

It is natural to refer to \( 1/R_f \) as the attenuation of the force \( f \). One has,

\[
\frac{1}{R_f} = \frac{\|f\|}{\Omega_f},
\]

\[
= \frac{\|f\|}{\sup_{v \in \mathcal{V}} \{f(v)/\|J(v)\|\}},
\]

\[
= \|f\| \inf_{v \in \mathcal{V}} \frac{\|J(v)\|}{f(v)}. \tag{3.27}
\]

3.5. Example: grasping a 2-dimensional object at two points

Figure 3.1 illustrates two robotic hands holding a silicon wafer at two points at the ends of a vertical diameter. The offset of the weight vector from the middle of the vertical diameter is denoted by \( a \). In a 2-dimensional analysis, it is assumed that each gripper can exert two force components in the plane. Thus, using the generalized coordinates \((x, y, \theta)\) we may identify the configuration space \( \mathcal{Q} \) with \( \mathbb{R}^2 \times S^1 \) and the configuration shown is \( \kappa = (0, 0, 0) \). The configuration space of the constrained points is \( \mathcal{P} = \mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\} \) (see Figure 3.1).

The constraint mapping \( \phi : \mathcal{Q} \to \mathcal{P} \) may be written explicitly as

\[
\phi(x, y, \theta) = \begin{pmatrix}
x + r \sin \theta \\
y + r(1 - \cos \theta) \\
x - r \sin \theta \\
y - r(1 - \cos \theta)
\end{pmatrix}, \tag{3.28}
\]

and differentiating at the configuration \((0, 0, 0)\) gives

\[
J = T_\kappa \phi = \begin{pmatrix}
1 & 0 & r \\
0 & 1 & 0 \\
1 & 0 & -r \\
0 & 1 & 0
\end{pmatrix}, \quad
J(v_x, v_y, \omega) = \begin{pmatrix}
v_x + r\omega \\
v_y \\
v_x - r\omega \\
v_y
\end{pmatrix}, \quad
J^* = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -r & 0
\end{pmatrix}. \tag{3.29}
\]
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where \((v_x, v_y, \omega) \in \mathbb{R}^3\) represent an element in \(T_x Q\) and the dual mapping is obtained by transposition.

The constraint force vector is of the form \(g = (g_{x_1}, g_{y_1}, g_{x_2}, g_{y_2})\), and we consider first the sup norm so that

\[
\|g\| = \max\{g_{x_1}, g_{y_1}, g_{x_2}, g_{y_2}\}. \tag{3.30}
\]

It follows that the primal norm on \(\mathcal{U} := T_{p(x_0)} \mathcal{P} \cong \mathbb{R}^4\) is

\[
\|u\| = |u_{x_1}| + |u_{y_1}| + |u_{x_2}| + |u_{y_2}|. \tag{3.31}
\]

3.5.1. The case \(a = 0\), no offset

For simplicity, we consider first the case where \(a = 0\) so that the weight acts in the center. By Equation \(3.13\), the optimum is given by

\[
\Omega_f = \sup_{v \in \mathcal{V}} \frac{f(v)}{\|f(v)\|} = \sup_{v \in \mathcal{V}} \frac{-mgv_y}{|v_x| + |v_y| + |v_x - r\omega| + |v_y|}.
\]

Let \(\alpha, \beta \in \mathbb{R}\) satisfy \(v_x = \alpha v_y\) and \(r\omega = \beta v_y\), the optimum may be rewritten as

\[
\Omega_f = \sup_{\alpha, \beta} \frac{mg}{|\alpha + \beta| + |\alpha - \beta| + 2} = \frac{mg}{2}. \tag{3.33}
\]

Next, we construct the optimal grasping forces having this norm. The covector \(g_0 = f \circ \hat{f}^{-1} \in (\text{Image } f)^*\) satisfies, by Equation \(3.29\),

\[
g_0(u) = f \circ \hat{f}^{-1}(u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2}) = -mgv_y = -mgu_{y_1}. \tag{3.34}
\]
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In order to extend \( g_0 \) to \( g_H \) as in Proposition 3.1, we use the Helly construction of the Hahn-Banach theorem, as in Appendix A. As a basis for \( \text{Image } J \) we use

\[
\begin{pmatrix}
0 \\
1 \\
0 \\
1 \\
1 \\
1 \\
0 \\
0
\end{pmatrix},
\]

and complete it to a basis of \( \mathcal{U} \) by adding the vector \((0, 0, 0, 1) \in \mathbb{R}^4\).

By Equation (A.10), we calculate

\[
\sup_{\text{Image } J} \left\{ g_0(u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2}) - \| g_0 \| (u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2}) - (0, 0, 0, 1) \right\} = \sup_{\text{Image } J} \left\{ \| g_0 \| (u, u_{y_1}, u_{x_2}, u_{y_2}) + (0, 0, 0, 1) \right\} - g_0(u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2})
\]

and

\[
\inf_{\text{Image } J} \left\{ \| g_0 \| (u, u_{y_1}, u_{x_2}, u_{y_2}) + (0, 0, 0, 1) \right\} - g_0(u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2})
\]

and we should set \( g_H (0, 0, 0, 1) \) to be between those values.

\[
\begin{pmatrix}
g_0(u_{x_1}) \\
g_0(u_{y_1}) \\
g_0(u_{x_2}) \\
g_0(u_{y_2})
\end{pmatrix}
- \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

\[
= \sup \left\{ -mg u_{y_1} - \frac{mg}{2} (|u_{x_1}| + |u_{y_1}| + |u_{x_2}| + |u_{y_2} - 1|) \right\},
\]

\[
= mg \sup \left\{ -u_{y_1} - \frac{1}{2} |u_{x_1}| - \frac{1}{2} |u_{y_1}| - \frac{1}{2} |u_{x_2}| - \frac{1}{2} |u_{y_2} - 1| \right\},
\]

\[
= mg \sup \left\{ -u_{y_1} - \frac{1}{2} |u_{y_1}| - \frac{1}{2} |u_{y_1} - 1| \right\},
\]

\[
= -\frac{mg}{2},
\]

where in the second line we used \( \| g_0 \| = \Omega_f = mg/2 \), in the third line we omitted non-positive term and used \( u_{y_1} = u_{y_2} \) in \( \text{Image } J \). The last line is obtained by considering all possible values of \( u_{y_1} \).
Similarly,

\[
\inf_{\text{Image} f} \left\{ \begin{array}{c}
\|g_0\| \\
\left( \begin{array}{c}
u_x \\
u_{y_1} \\
u_{x_2} \\
u_{y_2}
\end{array} \right)
\end{array} \right\} + \begin{array}{c}
0 \\
0 \\
0 \\
1
\end{array} - g_0 \left( \begin{array}{c}
u_x \\
u_{y_1} \\
u_{x_2} \\
u_{y_2}
\end{array} \right)
\]

\[
= \inf \left\{ \frac{mg}{2} \left( |u_x| + |u_{y_1}| + |u_{x_2}| + |u_{y_2} + 1| \right) + mg u_{y_1} \right\},
\]

\[
= mg \inf \left\{ \frac{1}{2} \left( |u_x| + |u_{y_1}| + |u_{x_2}| + |u_{y_2} + 1| \right) + u_{y_1} \right\},
\]

\[
= mg \inf \left\{ \frac{1}{2} \left( |u_{y_1}| + |u_{y_2} + 1| \right) + u_{y_1} \right\},
\]

\[
= -\frac{mg}{2},
\]

Therefore, we must choose

\[
g_{H}(0, 0, 0, 1) = -\frac{mg}{2}.
\]

Finally, we may compute \(g_{H}\) by applying it to each base vector to obtain

\[
g_{H} \left( \begin{array}{c}
u_{x_1} \\
u_{y_1} \\
u_{x_2} \\
u_{y_2}
\end{array} \right) = u_{y_1} g_{H} \left( \begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array} \right) + \frac{u_{x_2} + u_{x_1}}{2} g_{H} \left( \begin{array}{c}
0 \\
1 \\
0 \\
0
\end{array} \right) - \frac{u_{x_1} - u_{x_2}}{2} g_{H} \left( \begin{array}{c}
0 \\
0 \\
1 \\
1
\end{array} \right) + \frac{u_{y_2} - u_{y_1}}{2} g_{H} \left( \begin{array}{c}
0 \\
0 \\
0 \\
1
\end{array} \right),
\]

\[
= -mg u_{y_1} - \left( u_{y_2} - u_{y_1} \right) \frac{mg}{2},
\]

\[
= -\frac{mg}{2} u_{y_1} - \frac{mg}{2} u_{y_2},
\]

so that

\[
g_{H} = \left( \begin{array}{c}
0 \\
-mg/2 \\
0 \\
-mg/2
\end{array} \right).
\]

Thus, the general procedure proposed yields indeed the expected result.

### 3.5.2. The case \(a \neq 0\), eccentric loading

Next, we consider the eccentric situation where \(a \neq 0\) and \(a \leq r\) as illustrated in Figure 3.1. Evidently, the generalized external force on the object is \(f = (0, -mg, -ma)\). Using the same norms as above, Equation (3.13) implies that

\[
\Omega_f = \sup_{v \in \mathbb{R}} \frac{f(v)}{\|f(v)\|} = \sup_{v \in \mathbb{R}} \frac{-mg v_y - am v_{y_1}}{\|v_x + r \omega\| + \|v_y\| + \|v_x - r \omega\| + \|v_y\|}.
\]

Once again, we define \(\alpha, \beta \in \mathbb{R}\), by \(v_x = \alpha v_{y_1}, \omega = \beta v_{y_1}\), so that the expression for the optimum assumes the form
3. Static indeterminacy and optimization

\[ \begin{align*}
\Omega_f &= \sup_{a, \beta} \frac{mg(1 + a\beta)}{|\alpha + r\beta| + |\alpha - r\beta| + 2}, \\
&= \sup_{\beta} \frac{mg(1 + a\beta)}{2r |\beta| + 2},
\end{align*} \]

(3.44)

where the triangle inequality was used in order to arrive at the second line.

Setting

\[ h(\beta) := \frac{mg(1 + a\beta)}{2r |\beta| + 2}. \]

(3.45)

analyzing \( h \) by differentiation for the cases \( \beta \geq 0 \) and \( \beta \leq 0 \), and using the assumption that \( a \leq r \), one concludes that the supremum above is attained for \( \beta = 0 \) as for the case \( a = 0 \). Thus,

\[ \Omega_f = \sup_{\beta} h(\beta) = \frac{mg}{2}. \]

To construct optimal grasping forces, we note first that \( g_0 : \text{Image } J \to \mathbb{R} \) is given by

\[ g_0(u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2}) = f \circ \tilde{J}^{-1}(u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2}), \]

\[ = -mg u_{y_1} - \frac{u_{x_1} - u_{x_2}}{2r} amg. \]

(3.46)

Next, we use a slightly different method, also based on the Helly construction, for the extension of the covector \( g_0 \in (\text{Image } J)^* \) to some \( g_H \in \mathcal{U}^* \). Using the same basis for the grasping forces as above, we set \( g_\Omega(0, 0, 1) = c \), where \( c \in \mathbb{R} \) is yet undetermined. Thus,

\[ g_\Omega \begin{pmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \end{pmatrix} = u_{y_1} g_\Omega \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{u_{x_2} + u_{x_1}}{2} g_\Omega \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{u_{x_1} - u_{x_2}}{2} g_\Omega \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + (u_{y_1} - u_{y_2}) g_\Omega \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ = -mg u_{y_1} - \frac{u_{x_1} - u_{x_2}}{2r} amg + (u_{y_1} - u_{y_2}) c, \]

\[ = -\frac{amg}{2r} u_{x_1} + (-mg - c) u_{y_1} + \frac{amg}{2r} u_{x_2} + u_{y_2} c. \]

(3.47)

Since, by the Hahn-Banach theorem \( \|g_\Omega\| = \|g_0\| = mg/2 \), one has

\[ \max \left\{ \frac{amg}{2r}, |mg - c|, \frac{-amg}{2r}, |c| \right\} = \frac{mg}{2}. \]

(3.48)

We now make an ansatz that \( c = -mg/2 \). This will imply that

\[ g_\Omega = \begin{pmatrix} \frac{-amg}{2r} & \frac{mg}{2r} & \frac{amg}{2r} & \frac{mg}{2r} \end{pmatrix}, \]

(3.49)

which justifies our ansatz as long as \( a \leq r \). (One can follow the analogous reasoning for the case \( a > r \).)

It is observed that in the Helly construction in Appendix A, the number \( c \) that determines the extension uniquely, might be, in general, in an interval in \( \mathbb{R} \). However, in our case, this interval collapses to one point. Hence, the extension is unique in these examples.
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3.5.3. Using the Euclidean norm for the grasping forces

A more realistic norm for the space of grasping forces \( U = \{ (g_{x1}, g_{y1}, g_{x2}, g_{y2}) \} \) is

\[
\|g\| = \max \left\{ \sqrt{g_{x1}^2 + g_{y1}^2}, \sqrt{g_{x2}^2 + g_{y2}^2} \right\}.
\] (3.50)

The corresponding primal on \( U \) norm is given by

\[
\|u\| = \sqrt{u_{x1}^2 + u_{y1}^2} + \sqrt{u_{x2}^2 + u_{y2}^2}.
\] (3.51)

Equation (3.13) assumes the form

\[
\Omega_f = \sup_{\alpha, \beta} \frac{mg(1 + \alpha \beta)}{\sqrt{(\alpha + r \beta)^2 + \alpha^2} + \sqrt{(\alpha - r \beta)^2 + \beta^2}} = \sup_{\beta} \frac{mg(1 + \alpha \beta)}{2 \sqrt{r^2 \beta^2 + 1}}.
\] (3.53)

By differentiating the function

\[
h(\beta) := \frac{mg(1 + \alpha \beta)}{2 \sqrt{r^2 \beta^2 + 1}}
\] (3.54)

one obtains that the maximum is attained at \( \beta = a/r^2 \), and that

\[
\Omega_f = \sup \ h(\beta) = \frac{mg \sqrt{a^2 + r^2}}{2r}.
\] (3.55)

Hence, Equation (3.47) assumes the form

\[
\max \left\{ \sqrt{\frac{amg}{2r}} \right\}^2 + (-mg - c)^2, \sqrt{\left(\frac{-amg}{2r}\right)^2 + c^2} = \frac{mg \sqrt{a^2 + r^2}}{2r}.
\] (3.56)

We make the ansatz

\[
\sqrt{\left(\frac{-amg}{2r}\right)^2 + c^2} = \frac{mg \sqrt{a^2 + r^2}}{2r}, \quad c = \pm \frac{mg}{2}.
\] (3.57)

which indeed gives the maximum above. The corresponding optimal grasping forces are given by

\[
g_{H} = \left( -\frac{amg}{2r}, \frac{mg}{2}, \frac{amg}{2r}, \frac{mg}{2} \right)
\] (3.58)

as for the supremum norm.
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3.6. Example: Supporting the weight of a 2-D body at three points

Consider the 2-dimensional analysis of rigid body supported by three robotic hands. Only vertical forces are applied by the grippers, and we ignore horizontal displacement and forces. See Figure 3.2.

Here,

\[ V \equiv \{ (v_y, \omega) \} \quad \text{and} \quad U \equiv \{ (u_1, u_2, u_3) \} \tag{3.59} \]

were \( v_y \) is the vertical velocity of the point in the body that is located at the origin in the configuration considered, \( \Omega \) is the angular velocity of the body, and \( u_i \) are the vertical velocities of the corresponding grippers as enumerated in the illustration. Assuming for simplicity that \( A = B = l \), one has

\[ J = T_{\kappa} \varphi = \begin{pmatrix} 1 & -l \\ 0 & 1 \\ 1 & l \end{pmatrix}, \quad J(v_y, \omega) = \begin{pmatrix} v_y - l\omega \\ v_y \\ v_y + l\omega \end{pmatrix}. \tag{3.60} \]

A generalized force on the body is of the form

\[ f = (-mg, M) = (-mg, -mga), \tag{3.61} \]

where \( a \), indicating the location of the center of gravity, serves to parametrize the moment acting on the body relative to the origin. A generalized supporting force will be of the form \( g = (g_1, g_2, g_3) \), where \( g_i \) indicates the vertical force exerted on the \( i \)-th gripper.

Wishing to minimize the maximal vertical force in the grippers, we use on \( U^* = \{ g \} \) the norm

\[ \| g \| := \max \{ g_1, g_2, g_3 \} \tag{3.62} \]
for which the primal norm on $\mathcal{U}$ is
\[
\|u\| = |u_1| + |u_2| + |u_3|.
\] (3.63)

Using Equation (3.13), the optimum is given by
\[
\Omega_f = \sup_{v \in V} \frac{f(v)}{\|f(v)\|} = \sup_{v \in V} \frac{-mg_2 - amg_\omega}{|v_y - l\omega| + |v_y| + |v_y + l\omega|}.
\] (3.64)

Setting $\alpha = \omega = av_y$, we rewrite,
\[
\Omega_f = \sup_{\alpha} \frac{mg + amg_\alpha}{|1 - l\alpha| + 1 + |1 + l\alpha|}.
\] (3.65)

Analyzing the function
\[
h(\alpha) := \frac{mg + amg_\alpha}{|1 - l\alpha| + 1 + |1 + l\alpha|}
\] (3.66)

for the various relative values of $l\alpha$, one obtains
\[
\Omega_f = \sup_{\alpha} h(\alpha) = \frac{mg(l + a)}{3l}.
\] (3.67)

To compute an optimal generalized grasping force, we first note that $f$ induces the element $g_0 \in (\text{Image } J)^*$ by
\[
g_0(u_1, u_2, u_3) = f \circ \widehat{f}^{-1}(u_1, u_2, u_3) = -mg_2 - mga\frac{u_3 - u_1}{2l}.
\]

Next, as a basis for Image $J$ we choose $\{(1, 1, 1), (-1, 0, 1)\}$, and we complete it to a basis of $\mathcal{U}$ by adding the vector $(0, 0, 1)$. Following the Helly-Hahn-Banach construction, we set $g_1(0, 0, 1) = c$ for a yet unspecified number $c$. Thus, in terms of $c$,
\[
g_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_1 g_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (u_2 - u_1) g_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + (u_1 - 2u_2 + u_3) g_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\] (3.68)
\[
= -mg_1 - (u_2 - u_1)\frac{mg_\alpha}{l} + (u_1 - 2u_2 + u_3),
\]
\[
= \left(\frac{mg_\alpha}{l} + c\right) u_1 + \left(-mg - \frac{mg_\alpha}{l} - 2c\right) u_2 + cu_3.
\]

It follows that $c$ should satisfy the condition
\[
\max \left\{ \left| \frac{mg_\alpha}{l} + c \right|, \left| -mg - \frac{mg_\alpha}{l} - 2c \right|, |c| \right\} = \frac{mg(l + a)}{3l},
\] (3.69)

which gives the value
\[
c = \frac{-mg(l + a)}{3l}.
\] (3.70)

We conclude that the optimal grasping forces are given by
\[
g_1 = \left( \frac{mg(2a - l)}{3l}, \frac{-mg(l + a)}{3l}, \frac{-mg(l + a)}{3l} \right).
\] (3.71)

It is observed that the forces in grippers #2 and #3 are equal, while the force in gripper #1 is different. In fact, for $a > l/2$, $g_1 > 0$.  

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3. **Static indeterminacy and optimization**

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4. Bounded Grasping Forces and Worst Case Loadings

Conceivably, optimal grasping can be realized if the grasping mechanism has a force control system. When an external force \( f \) is acting on the object and some optimal constraint forces are computed, the force control system will apply the corresponding grasping forces. This may be desirable so as to prevent damage to the object and to limit forces exerted by the gripping mechanism and stresses in it.

In this section we consider the situation where the force-controlled grasping mechanism can apply all grasping forces \( g \in \mathcal{U}^* \) provided that \( \|g\| \leq M \) and will not apply forces whose norm is greater than \( M \). We will refer to such a situation as bounded grasping and to the given bound, \( M \), as the grasping bound. In other words, the set of admissible grasping forces is exactly \( \mathcal{B}_M(0) \), the closed ball of radius \( M \) centered at the origin of \( \mathcal{U}^* \). A load \( f \) acting on the object is admissible if it can be supported by admissible grasping forces. That is, \( f \) if admissible if \( f = J^*(g) \) for some \( g \in \mathcal{B}_M(0) \).

4.1. The Minkowski functional for \( K = J^*(\mathcal{B}_M(0)) \)

The image
\[
K := J^*(\mathcal{B}_M(0))
\]

of \( \mathcal{B}_M(0) \) under \( J^* \) consists of the collection of external forces on the object that may be supported by admissible grasping forces. Since a ball is a convex set and since the image of a convex set under a linear mapping is convex, \( K \) is a convex set. It is recalled that for a convex set \( K \) whose interior contains the origin, the Minkowski functional for \( K \)
\[
p : \mathcal{V}^* \rightarrow \mathbb{R}^+ \quad \text{(4.2)}
\]
is defined by
\[
p(f) = \inf \left\{ r \mid r \in \mathbb{R}^+, \frac{f}{r} \in K \right\}. \quad \text{(4.3)}
\]

Thus, in order that the Minkowski functional \( p \) for \( K \) be well defined, we have to show that there is some \( \varepsilon > 0 \) such that \( B_\varepsilon(0) \subset K \). In other words, we have to show there is some \( \varepsilon > 0 \) such that \( \|f\| \leq \varepsilon \) implies \( \Omega_f \leq M \), independently of \( f \). From the mechanical point of view this means that every force on the object may be scaled as to be supported by admissible grasping forces, and there is a lower bound to these scaling factors.

To show the existence of \( \varepsilon \) as above, we first define the sensitivity of the grasping as
\[
R := \sup_{f \in \mathcal{V}^*} \left\{ \|R_f\| = \sup_{f \in \mathcal{V}^*} \left\{ \frac{\Omega_f}{\|f\|} \right\} \right\}. \quad \text{(4.4)}
\]

Thus, \( R \) indicates the worst-case sensitivity of the grasping mechanism.

**Proposition 4.1.** Let \( \hat{J} : \mathcal{V} \rightarrow \text{Image } J \) be the isomorphism between \( \mathcal{V} \) and its image under \( J \). Then,
\[
R = \sup_{v \in \mathcal{V}} \left\{ \frac{\|v\|}{\|J(v)\|} \right\} = \|\hat{J}^{-1}\|. \quad \text{(4.5)}
\]
4. Bounded Grasping Forces and Worst Case Loadings

Figure 4.1: Illustrating the Minkowski functional for $K = J^*(\mathcal{B}_M(0))$

**Proof.** Using Equation (3.13) of Proposition 3.1, we have

$$R = \sup_{f \in \mathcal{V}^*} \left\{ \frac{1}{\| f \|_{\mathcal{V}}} \sup_{v \in \mathcal{V}} \left\{ f(v) \right\} \right\},$$

$$= \sup_{v \in \mathcal{V}} \left\{ \frac{1}{\| J(v) \|} \sup_{f \in \mathcal{V}^*} \left\{ f(v) \right\} \right\},$$

$$= \sup_{v \in \mathcal{V}} \left\{ \frac{\| v \|}{\| J(v) \|} \right\},$$

$$= \sup_{w \in \text{Image} J} \left\{ \frac{\| \tilde{J}^{-1}(w) \|}{\| w \|} \right\},$$

$$= \| \tilde{J}^{-1} \|. \tag{4.6}$$

**Remark 4.2.** By Equation (4.4),

$$R = \sup \{ \Omega_f \mid \| f \| = 1, f \in \mathcal{V}^* \}. \tag{4.7}$$

Since $\mathcal{V}^*$ is finite dimensional, the supremum above is attained for some $f_0$ with $\| f_0 \| = 1$—a worst case loading distribution satisfying $R = \Omega_{f_0}$. Since, on the other hand, there is a generalized velocity $v_0$ such that

$$R = \sup_{v \in \mathcal{V}} \left\{ \frac{\| v \|}{\| J(v) \|} \right\} = \frac{\| v_0 \|}{\| J(v_0) \|}. \tag{4.8}$$

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the worst case loading can be obtained from \( v_0 \) by the optimization problem

\[
\sup \left\{ \frac{f(v_0)}{\|f(v_0)\|} \left| \|f\| = 1 \right. \right\}. 
\]  
\[ (4.9) \]

**Corollary 4.3.** One has

\[
\inf_{f \in V^*} \left\{ \frac{\|f\|}{\Omega_f} \right\} = \frac{1}{R} > 0.
\]  
\[ (4.10) \]

**Proof.** It follows from the foregoing computation that

\[
\inf_{f \in V^*} \left\{ \frac{\|f\|}{\Omega_f} \right\} = \frac{1}{\sup_{f \in V^*} \left\{ \frac{\Omega_f}{\|f\|} \right\}} = \frac{1}{R} = \frac{1}{\|\hat{J}^{-1}\|}
\]  
\[ (4.11) \]

Since \( \hat{J} \) is a continuous isomorphism, \( \|\hat{J}^{-1}\| < \infty \) and does not vanish. \( \square \)

**Corollary 4.4.** Let

\[
\varepsilon := \frac{M}{R}.
\]  
\[ (4.12) \]

Then,

\[
\inf\{\|f\| \mid f \in V^*, \Omega_f = M\} = \varepsilon.
\]  
\[ (4.13) \]

**Proof.** It follows from the foregoing corollary that

\[
\varepsilon = \frac{M}{R} = \inf_{f' \in V^*} \left\{ \frac{\|\Omega_{f'}\|}{\Omega_{f'}} \right\},
\]  
\[ (4.14) \]

Observing that

\[
\Omega_{Mf'}/\Omega_{f'} = M,
\]  
\[ (4.15) \]

independently of \( f' \), and writing \( f = Mf'/\Omega_{f'} \), one has

\[
\varepsilon = \inf\{\|f\| \mid \Omega_f = M\}.
\]  
\[ (4.16) \]

**Corollary 4.5.** A ball of radius \( \varepsilon = M/R \) is contained in \( K \).

It follows from the analysis presented above that indeed \( K \) contains an open neighborhood of the zero element in \( V^* \) and the Minkowski functional \( p \) is well defined for \( K \).
4. Bounded Grasping Forces and Worst Case Loadings

4.2. The sensitivity of the grasping and the load capacity ratio

Once the validity of the Minkowski functional has been established, we can turn to its applications in the study of bounded grasping reactions and worst case loadings.

It is first noted that for a given force $f$, under scaling by $1/p(f)$, the force $f/p(f)$ is located on the boundary of $K$. Hence,

$$\Omega_{f/p(f)} = M. \quad (4.17)$$

However, as we have already seen that $\Omega_{Mf/\Omega_f} = M$,

$$p(f) = \frac{\Omega_f}{M}. \quad (4.18)$$

Let

$$S(f) := \frac{1}{p(f)} = \frac{M}{\Omega_f}. \quad (4.19)$$

Then, if $f \neq 0$ so that $p(f) \neq 0$,

$$S(f) = \frac{1}{p(f)},$$

$$= \frac{1}{\inf \{r \in \mathbb{R}^+ \mid f/r \in K\}},$$

$$= \sup \{1/r \mid r \in \mathbb{R}^+, f/r \in K\},$$

$$= \sup \{s \in \mathbb{R}^+ \mid sf \in K\}. \quad (4.20)$$

Thus, $S(f)$ is interpreted naturally as the factor of safety, the largest positive number by which we can multiply $f$ so that it can be supported by admissible grasping forces.

In various situations there are uncertainties as to the nature of forces that will act on the object. In particular, one would like to characterize the forces that the grasping mechanism will be able to support. This is relevant especially in the case of bounded grasping. For example, one would like to identify the loading conditions which are most likely to cause inadmissible grasping forces so that grasping fails. The following proposition gives a bound on the norm of a force on the object that will ensure its admissibility, independently of its distribution. In addition to the obvious dependence of this bound on the value of $M$, it turns out that the bound depends only on the geometry of the grasping mechanism as reflected by the mapping $J$.

**Proposition 4.6.** Any force $f$ acting on the object is admissible if (we use the foregoing notation)

$$\|f\| \leq \frac{M}{R} = \frac{M}{\|f^{-1}\|}. \quad (4.21)$$

The condition above is optimal in the sense that for $\delta > M/R$, there is some force $f$ with $\|f\| = \delta$ that cannot be supported by the grasping mechanism. Hence, we will refer to $M/R$ as the load capacity of the grasping.
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![Diagram illustrating various objects considered in Proposition 4.6](image)

Figure 4.2: Illustrating the various objects considered in Proposition 4.6

**Proof.** To prove the claim, we should show that

$$\frac{M}{R} = \sup \{ \delta \mid B_\delta(0) \subset K, \delta > 0 \} = \sup \{ \delta \mid \Omega_f \leq M \text{ if } \|f\| \leq \delta \}$$  \hspace{1cm} (4.22)

(see illustration in Figure 4.2).

Given $\delta > 0$, consider the condition,

$$\Omega_f \leq M \text{ if } \|f\| \leq \delta.$$  \hspace{1cm} (4.23)

Since $\Omega_f = R_f \|f\|$ and $R_f > 0$, condition (4.23) is equivalent to

$$M \geq \sup \{ \Omega_f \mid \|f\| \leq \delta \},$$

$$= \sup \{ R_f \|f\| \mid \|f\| \leq \delta \},$$

$$= \sup \{ R_f \delta \mid \|f\| = \delta \},$$

$$= \delta R,$$  \hspace{1cm} (4.24)

where we have used the fact that $R_f$ is independent of $\|f\|$ as in Remark 3.2 and the definition of $R$ in Equation (4.4).

It follows that

$$\sup \{ \delta \mid B_\delta(0) \subset K, \delta > 0 \} = \sup \{ \delta \mid \delta R \leq M \},$$

$$= \frac{M}{R}.$$  \hspace{1cm} (4.25)

$\square$

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4.3. Coulomb friction as bounded grasping

As an example of bounded grasping, consider the case where a gripper applies known normal forces to an object and we are concerned with constraining the motion of the object by the resulting friction forces in the plane perpendicular to the applied forces. That is, using sliding, approach, and normal vectors as a basis for a coordinate system attached to the gripper, as for example in [Wol87, p. 72], the normal force acts along the sliding direction and the friction force acts in the approach – normal plane. Thus, under the assumption of Coulomb friction, the friction grasping force is bounded. Some further remarks follow.

4.3.1. A general framework

We start by setting up a general framework that is suitable for the analysis of various examples where Coulomb friction plays an important role. Assume that the vector space \( \mathcal{U} \) has a structure of a Cartesian product \( \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \). We denote the projections on the two subspaces by

\[
\pi_a : \mathcal{U} \rightarrow \mathcal{U}_a, \quad a = 1, 2. \tag{4.26}
\]

We will also use the notation

\[
J_a = \pi_a \circ J : \mathcal{V} \rightarrow \mathcal{U}_a, \quad a = 1, 2. \tag{4.27}
\]

We recall that \( \mathcal{U}^* = \mathcal{U}_1^* \times \mathcal{U}_2^* \), and that for \( (g_1, g_2) \in \mathcal{U}^* \),

\[
J^*(g_1, g_2) = J_1^*(g_1) + J_2^*(g_2). \tag{4.28}
\]

For a pair \( (g_1, g_2) \in \mathcal{U}^* \), \( g_1 \in \mathcal{U}_1^* \) is interpreted as a known fixed grasping force vector, such as the force in the sliding direction applied by a gripper, and so \( \mathcal{U}_1^* \) is the vector space of known fixed grasping forces. The friction grasping forces are elements \( g_2 \in \mathcal{U}_2^* \). Thus, as long as there are no friction grasping forces, \( g = (g_1, 0) \) and these grasping forces equilibrate the initial external force

\[
f_0 = J^*(g_1, 0) = J_1^*(g_1), \tag{4.29}
\]

which may be zero in many examples. It is noted that while \( J \) is assumed to be injective, the component \( J_1 : \mathcal{V} \rightarrow \mathcal{U}_1 \) is not assumed to be injective necessarily for this may mean that the constraints preventing motion in \( \mathcal{U}_1 \) are sufficient to fix the object.

The assumption that we have Coulomb friction at the supports, implies that we have a norm \( ||u_2||_2 \) on \( \mathcal{U}_2 \) induced by a norm \( ||g_2||_2 \) on \( \mathcal{U}_2^* \). The norm \( ||g_2||_2 \) depends in general on the values of the fixed reaction \( g_1 \).

Assume now that an additional external force \( f \) is applied on the mechanism so that the total external force is \( f_0 + f \). Thus, the condition that the reactions \( g = (g_1, g_2) \) equilibrate the total external force is

\[
f_0 + f = J^*(g_1, g_2). \tag{4.30}
\]

However, using \( J^* \)

\[
f_0 + f = J_1^*(g_1) + J_2^*(g_2), \tag{4.31}
\]
and the assumption that the reactions $g_1$ are given and equilibrate $f_0$ gives

$$f = J_2^T(g_2).$$

(4.32)

Thus, all the observations made in Sections 3.4 and 4.2 apply to the situation described above in case $J_2$ is injective. Note that the assumption that $J_2$ is injective implies that any force $f$ may be supported by friction alone (under the given normal forces $g_1$). This is summarized by the following.

**Proposition 4.7.** Assume that the mapping $J_2 : V \rightarrow U_2$ is injective. There is an optimal reaction $g_2H \in U_2^*$ (not unique) for which (4.32) holds. It satisfies

$$\Omega_f := \|g_2H\| = \inf_{f=J_2^T(g_2)} \|g_2\|$$

(4.33)

for which (4.32) holds. It satisfies

$$\Omega_f = \|g_2H\| = \sup_{v \in V} \frac{f(v)}{\|J_2(v)\|}.$$  

(4.34)

In addition, let

$$R := \sup_{f \in V^*} \frac{\|\Omega_f\|}{\|f\|}.$$  

(4.35)

Then,

$$R = \sup_{v \in V} \frac{\|v\|}{\|J_2(v)\|}.$$  

(4.36)

4.3.2. A constitutive model for friction

We assume that at a generic constraint point, say $\alpha$, where the reaction is some given $g_{1\alpha} > 0$, the Euclidean norm of the maximal friction grasping force is given by $\|g_{2\alpha}\|$ which satisfies

$$\|g_{2\alpha}\| \leq \mu_\alpha g_{1\alpha}.$$  

(4.37)

for some known coefficient $\mu_\alpha$ of static friction. As $\mu_\alpha$ and $g_{1\alpha}$ are assumed to be given, we may write this condition as

$$\|g_{2\alpha}\| \leq \frac{\mu_\alpha g_{1\alpha}}{\mu_\alpha g_{1\alpha}}.$$  

(4.38)

Grasping with friction forces is admissible if the last condition holds for every value of $\alpha$, or equivalently, if

$$\sup_\alpha \left\| \frac{g_{2\alpha}}{\mu_\alpha g_{1\alpha}} \right\| \leq 1.$$  

(4.39)

The condition suggests that for the norm of the friction forces we use the weighted norm

$$\|g_2\| = \sup_\alpha \{v_\alpha \|g_{1\alpha}\|\}, \quad v_\alpha = \frac{1}{\mu_\alpha g_{1\alpha}}.$$  

(4.40)

In view of Equation (4.32), the notions introduced in Section 4.2 apply to the case of friction with $M_2 = 1$, where $M_2$ is the grasping bound for admissible $g_2 \in U_2^*$ in analogy with the grasping bound $M$ introduced in Section 4.
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Remark 4.8. It follows from the setting described above, that in case
\[
\Omega_f := \inf_{f^*(g_2) = f} \|g_2\| > 1, \tag{4.41}
\]
then, \(f\) cannot be supported by friction grasping.

For the factor of safety, as in (4.19) and (4.20), we therefore have to the case of friction
\[
S(f) = \sup\{s > 0 \mid \Omega_{sf} = s\Omega_f \leq 1\}, \tag{4.42}
\]
and
\[
S(f) = \frac{1}{\Omega_f}. \tag{4.43}
\]

The main assumption regarding static friction is:

**Constitutive assumption for Coulomb friction:** A force \(f\) is supported by (not necessarily unique) friction forces if
\[
\Omega_f \leq 1. \tag{4.44}
\]

Thus, in terms of safety factor, a force \(f\) is supported by friction reactions if and only if
\[
S(f) \geq 1. \tag{4.45}
\]

Remark 4.9. Consider an external force \(f\) such that \(\Omega_f < 1\), strictly. The constitutive assumption above implies that there are some friction reactions \(g_2\), with \(\|g_2\| \leq 1\) and \(f^*(g_2) = f\). Thus, in general, \(\|g_2\| \geq \Omega_f\) and \(g_2\) is not necessarily optimal. For a force \(f\) with \(\Omega_f = 1\), any friction forces \(g_2\) with \(f^*(g_2) = f\), must satisfy \(\|g_2\| = 1 = \Omega_f\) and hence, \(g_2\) is optimal. We conclude that the friction forces for a force \(f\) with \(\Omega_f = 1\) are optimal.

Next, we consider the implications of the constitutive assumption for friction forces for the sensitivity ratio \(R\). It follows from Equation (4.4) that for each external loading \(f\),
\[
\frac{\|g_{2H}\|}{\|f\|} = \frac{\Omega_f}{\|f\|} \leq R, \quad \text{or,} \quad \|f\| R \geq \|g_{2H}\| = \Omega_f. \tag{4.46}
\]

**Corollary 4.10.** For any external force \(f\) such that
\[
\|f\| \leq \frac{1}{R} \quad \text{we have,} \quad \Omega_f \leq 1. \tag{4.47}
\]

It then follows from the constitutive assumption for friction that any external force \(f\) such that \(\|f\| \leq 1/R\) may be supported by friction grasp, independently of its distribution, direction, etc.
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4.4. Example: The sensitivity for a 2-dimensional object held at two points

We return to system considered in Example 3.5 and Figure 3.1, and we want to compute the sensitivity $R$ of the grasping as given by Equation (4.5), Proposition 4.1. This will enable us to evaluate the capacity of the system to support arbitrary external forces.

The sensitivity $R$ is defined relative to a norm on the space, $\mathcal{V}^* = \{(f_x, f_y, M)\}$, of all external forces on the object which is the dual norm of some norm on the space of generalized velocities $\mathcal{V} = \{(v_x, v_y)\}$. Thus, the norm chosen in $\mathcal{V}^*$ is

$$
||f|| = ||(f_x, f_y, M)|| = \sqrt{f_x^2 + f_y^2 + (M/r)^2}.
$$

(4.48)

The corresponding primal norm on $\mathcal{V}$ is

$$
||v|| = ||(v_x, v_y, \omega)|| = \sqrt{v_x^2 + v_y^2 + (r\omega)^2}.
$$

(4.49)

Thus,

$$
R = \sup_{v \in \mathcal{V}} \left( \frac{||v||}{||J(v)||} \right) = \sup_{v \in \mathcal{V}} \frac{\sqrt{v_x^2 + v_y^2 + (r\omega)^2}}{\sqrt{(v_x + r\omega)^2 + v_y^2 + (v_x - r\omega)^2 + v_y^2}}.
$$

(4.50)

Using the notation

$$
\alpha = v_x^2 + v_y^2 + (r\omega)^2, \quad \beta = 2v_xr\omega,
$$

(4.51)

one obtains,

$$
R = \sup_{\alpha, \beta} \frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta + \sqrt{\alpha + \beta}}},
$$

$$
= \sup_{\alpha, \beta} \frac{\alpha}{2\alpha + 2\sqrt{\alpha + \beta} \sqrt{\alpha - \beta}},
$$

(4.52)

and it follows that

$$
R = \frac{1}{\sqrt{2}}.
$$

(4.53)

We conclude from Proposition 4.6, that under a force controlled grasping bounded by $M$, the system can support an arbitrary combination of an external force and a moment, $f = (f_x, f_y, M)$, as long as

$$
\sqrt{f_x^2 + f_y^2 + (M/r)^2} \leq \frac{M}{R} = \sqrt{2}M.
$$

(4.54)
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4.5. Example: Grasping sensitivity for a 2-D object supported at three points

We return to the system considered in Section 3.6, Figure 3.2, where the weight of a 2-dimensional object is supported at three points, and we calculate the sensitivity of the grasping. As a norm for the external load, we choose

\[ k_f = \sqrt{f_y^2 + (M/l)^2} = mg\sqrt{1 + (a/l)^2}. \]  

(4.55)

The corresponding primal norm is

\[ k_v = \sqrt{v_y^2 + (l\omega)^2}. \]  

(4.56)

Thus,

\[ R = \sup_{v \in V} \left\{ \frac{\|v\|}{\|J(v)\|} \right\} = \sup_{v \in V} \frac{\sqrt{v_y^2 + (l\omega)^2}}{|v_y - l\omega| + |v_y| + |v_y + l\omega|}. \]  

(4.57)

Analyzing the function in the argument of the supremum for various relative values of \( v_y \) and \( l\omega \), the value \( R = 1/2 \) is obtained.

4.6. Example: The sensitivity of grasping for a mechanism

This example demonstrates the application of the foregoing analysis for the case where the object is a mechanism rather than a rigid body. Consider the system illustrated in Figure 4.3 where the object is a two-dimensional, two-link mechanism. It is assumed that the gripping force (in the \( z \)-direction) exerted by the left gripper is given. This implies that the Coulomb friction force in the \( x-y \) plane, which the gripper applies to the mechanism on the left, is bounded. We denote this maximal friction force by \( g_M \). The arm on the right end of the mechanism is assumed to apply a fixed force in the \( x \)-direction. Thus, considering only the mechanics of the mechanism in the \( x-y \) plane, the vertical friction force on the right hand side is bounded as well. It is assumed that the vertical force on the right is bounded by \( g_M \), also. In addition, it is assumed that that there is friction in the joint located at the origin. The friction at the joint enables friction moment \( L_M \) that the links can apply to one another, and it is assumed that it is bounded by \( |L_M| \leq 10g_Ml \).

In the notation for this example, we do not use the decomposition \( g = (g_1, g_2) \) as we simply ignore the applied normal forces which cancel one another. These normal forces just determine the maximal friction forces so we will simply write \( g \) for \( g_2 \).

Letting \( \mathcal{Q} \) denote the configuration space of the two-link mechanism—the object in this example—and considering the configuration \( x \) for which the mechanism is flat (see Figure 4.3), we have

\[ \mathcal{Q} = T_x \mathcal{Q} \cong \{(v_1 := v_x, v_2 := v_y, v_3 := v_y, v_4 := v_y)\}, \]  

(4.58)

where \( v_1, v_2 \) are the horizontal and vertical component of the velocity of the left end, respectively, \( v_3 \) is the angular velocity of the left link, and \( v_4 \) is the angular velocity of the right link relative that of the left link. Elements of the configuration space for the collection of the supports, \( \mathcal{P} \), contain in addition the location of the right end of the right-hand link, and so

\[ \mathcal{U} = T_{\psi(x)} \mathcal{P} \cong \{(u_1, u_2 := v_y, u_3 := v_x, u_4 := v_y, u_5 := v_y)\}, \]  

(4.59)
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where \( u_1 \) is the vertical velocity of the right end (see Figure 4.3).

Correspondingly, the spaces of generalized external forces and generalized forces at the supports may be written as

\[
\mathcal{V}^* = \{ (f_x, f_y, M_1, M_2) \}, \quad \mathcal{U}^* = \{ (g_1, \ldots, g_5) \}
\]  

(4.60)

where in particular, \( g_1 \) is the vertical friction force acting on the right end.

A simple kinematic analysis of the mechanism gives

\[
J = \begin{pmatrix}
0 & 1 & 3l & 2l \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{Image } J = \begin{pmatrix}
u_3 = u_4 \\
u_2 = u_2 \\
u_4 = u_5 \\
u_1 = v_2 \\
u_3 = v_4 = u_5
\end{pmatrix}, \quad \begin{pmatrix}
u_1 = u_1 \\
u_2 = v_y + 3l\omega_1 + 2l\omega_2 \\
u_3 = v_x \\
u_4 = \omega_1 \\
u_5 = \omega_2
\end{pmatrix}.
\]  

(4.61)

As a norm on the space of grasping forces we use

\[
\|g\| = \frac{1}{g_M} \max \left\{ g_1, g_2, g_3, \frac{g_4}{I}, \frac{g_5}{10l} \right\}.
\]  

(4.62)

(It is noted that it would be more physical to take \( \sqrt{g_2^2 + g_3^2} \), rather than the maximum between these two components. However, our choice is motivated by simplicity and by the fact that the Euclidean norm is bounded by the maximum.) For the norm of the external loading we chose

\[
\|f\| = \sqrt{f_x^2 + f_y^2 + \left( \frac{M_1}{l} \right)^2 + \left( \frac{M_2}{l} \right)^2}.
\]  

(4.63)

The corresponding primal norms are

\[
\|u\| = g_M(|u_1| + |u_2| + |u_3| + |u_4| + |10u_5|) \quad \text{and} \quad \|v\| = \sqrt{v_x^2 + v_y^2 + (l\omega_1)^2 + (l\omega_2)^2}.
\]  

(4.64)
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![Figure 4.4: Illustrating a wineglass held at four points](image)

We can now write down the expression for the sensitivity as

$$R = \sup_{v \neq 0} \frac{\|v\|}{\|J(v)\|}$$

$$\sup_{v \neq 0} \frac{\sqrt{v_x^2 + v_y^2 + (l_1 \omega_1)^2 + (l_2 \omega_2)^2}}{g_M |v_y| + 3l_1 \omega_1 + 2l_2 \omega_2 + g_M |v_x| + g_M |\omega_1| + 10g_M l |\omega_2|}$$

(4.65)

We were not able to compute the supremum analytically. A numerical computation using the commercial platform modeFronnier resulted the value $R = 1.00$. It was also observed that the result is independent of $l$ and the same result was obtained for links having equal lengths.

4.7. Example: Grasping sensitivity for holding a wineglass

As a 3-dimensional example, we consider next the grasping sensitivity of a wineglass held at four points as shown in Figure 4.4. The four fingers hold the glass on the same horizontal circle of maximal radius, $r$. The two pairs of diametrically opposed fingers are along the $x$-axis and $y$-axis, respectively.

It is assumed that all four fingers apply equal normal grasping forces on the glass which determine the maximal Coulomb friction, $g_M$, at these points. The generalized velocity of the glass are described by vectors $v \in \mathcal{V}$ of the form $v = (v_x, v_y, v_z, \omega_x, \omega_y, \omega_z)$ consisting of the components of the linear velocity of the center of the glass (located at the origin) and the angular velocity.

Since we will be interested in the components of friction forces at the four contact points, there are two relevant components at each point. Thus, adopting the enumeration of the fingers as shown on the right of Figure 4.4, the grasping generalized force $g \in \mathcal{U}$ is of the form $g =$
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\((g_{1x}, g_{1z}, g_{2y}, g_{2z}, g_{3x}, g_{3z}, g_{4y}, g_{4z})\). It follows that the constraints generalized velocities \(u \in \mathcal{U}\) are represented in the form \(u = (u_{1x}, u_{1z}, u_{2y}, u_{2z}, u_{3x}, u_{3z}, u_{4y}, u_{4z})\). Kinematics implies that

\[
J = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -r \\
0 & 1 & 0 & 0 & 0 & r \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -r & 0 \\
1 & 0 & 0 & 0 & 0 & r \\
0 & 0 & 1 & -r & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -r \\
0 & 0 & 1 & 0 & r & 0 \\
\end{pmatrix}
\]

(4.66)

As in the preceding example, we will not use the decomposition \(g = (g_1, g_2)\) in the notation and we will simply write \(g\) for \(g_2\). Next, we have to specify the norms chosen for \(\mathcal{U}^*\) and \(\mathcal{V}^*\) and the corresponding primal vector spaces. Thus, denoting the maximal admissible friction component by \(g_M\), we use the norm

\[
\|g\| = \frac{1}{g_M} \max\{|g_{1x}|, |g_{1z}|, |g_{2y}|, |g_{2z}|, |g_{3x}|, |g_{3z}|, |g_{4y}|, |g_{4z}|\},
\]

(4.67)

for elements of \(\mathcal{U}^*\) and the primal norm on \(\mathcal{U}\) is

\[
\|u\| = g_M (|u_{1x}| + |u_{1z}| + |u_{2y}| + |u_{2z}| + |u_{3x}| + |u_{3z}| + |u_{4y}| + |u_{4z}|).
\]

(4.68)

(The max-norm is used instead of the Euclidean for the friction at each point in order to simplify the computations and because the Euclidean norm is bounded by the max-norm.) For the generalized external forces, represented in the form \(f = (f_x, f_y, f_z, M_x, M_y, M_z)\), we use the norm

\[
\|f\| = \sqrt{f_x^2 + f_y^2 + f_z^2 + (M_x/r)^2 + (M_y/r)^2 + (M_z/r)^2},
\]

(4.69)

which gives the primal norm

\[
\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2 + (r\omega_x)^2 + (r\omega_y)^2 + (r\omega_z)^2}.
\]

(4.70)

It is concluded that the sensitivity of the grasping system is given by

\[
R = \sup_{v \neq 0} \frac{\|v\|}{\|J(v)\|} = \frac{1}{g_M} \sup_{v \neq 0} \frac{\sqrt{v_x^2 + v_y^2 + v_z^2 + (r\omega_x)^2 + (r\omega_y)^2 + (r\omega_z)^2}}{D},
\]

(4.71)

where

\[
D = |v_x - \omega_z r| + |v_x + \omega_z r| + |v_y + \omega_x r| + |v_y - \omega_x r|
+ |v_x + \omega_z r| + |v_z - \omega_y r| + |v_y - \omega_z r| + |v_z + \omega_y r|.
\]

(4.72)

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Numerical computations using modeFrontier give the approximate value \( R = 0.500/g_M \). We obtained the same result for a number of values for \( r \). Thus finally, the grasping can support a combination of external forces and external moments as long as their norms satisfy \( \|f\| \leq 2.00g_M \).

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### A. The Helly Construction for the Extension of Linear Functionals

We review here the construction of the Hahn-Banach theorem (in the finite dimensional case) for the extension of covectors. This construction will be applied in some of the examples.

Consider a normed finite dimensional vector space \( \mathcal{U} \), \( \dim \mathcal{U} = m \) and a proper vector subspace \( \mathcal{W} \subset \mathcal{U} \), \( \dim \mathcal{W} = n < m \). Let \( g_0 \in \mathcal{W}^\ast \) be a linear functional with

\[
\|g_0\| = \sup_{w \neq 0 \atop w \in \mathcal{W}} \frac{|g_0(w)|}{\|w\|}, \tag{A.1}
\]

be given. An extension \( g \in \mathcal{U}^\ast \) of \( g_0 \) is a covector that satisfies the condition \( g(w) = g_0(w) \) for all \( w \in \mathcal{W} \). Some extension \( g \) of \( g_0 \) that preserves the norm, that is

\[
\|g\| := \sup_{u \neq 0 \atop u \in \mathcal{U}} \frac{|g(u)|}{\|u\|} = \|g_0\|, \tag{A.2}
\]

is constructed as follows.

Since we remain in the finite dimensional setting, it is sufficient to provide the construction for the case where \( \mathcal{W} \) is a subspace of co-dimension \( m - n = 1 \). For the general case, one can repeat the process \((m - n)\)-times. Thus, let \( u_1 \in \mathcal{U} \setminus \mathcal{W} \) be chosen and so every vector \( u \in \mathcal{U} \) may be expressed as

\[
u = w + tu_1, \tag{A.3}
\]

for unique \( w \in \mathcal{W} \) and \( t \in \mathbb{R} \). It follows from linearity, that every extension \( g \in \mathcal{U}^\ast \) satisfies

\[
g(u) = g(w + tu_1) = g_0(w) + tg(u_1). \tag{A.4}
\]

This condition implies that every extension \( g \) is uniquely determined by a single number \( g(u_1) \). One has to show, therefore, that \( g(u_1) \) may be chosen so that \( \|g\| = \|g_0\| \).
It is observed first, that
\[
\|g\| = \sup_{w \in W} \frac{|g_0(w) + tg(u_1)|}{\|w + tu_1\|}, \quad t \neq 0,
\]
\[
= \sup_{w \in W} \frac{|t| |g_0(\frac{w}{t}) + g(u_1)|}{|t| \|\frac{w}{t} + u_1\|}, \quad t \neq 0, \tag{A.5}
\]
\[
= \sup_{w \in W} \frac{|g_0(w) + g(u_1)|}{\|w + u_1\|}.
\]

Thus, the condition that \(\|g\| = \|g_0\|\) will be satisfied if for all \(w \in W\),
\[
\|g_0\| \|w + u_1\| > |g_0(w) + g(u_1)|. \tag{A.6}
\]

This gives the following conditions on \(g(u_1)\):
\[
\|g_0\| \|w + u_1\| - g_0(w) \geq g(u_1), \quad \text{for all } w \in W, \tag{A.7}
\]
and
\[
- \|g_0\| \|w + u_1\| - g_0(w) \leq g(u_1), \quad \text{for all } w \in W, \tag{A.8}
\]
which replacing \(w\) with \(-w\) is equivalent to
\[
- \|g_0\| \|w + u_1\| + g_0(w) \leq g(u_1), \quad \text{for all } w \in W. \tag{A.9}
\]

Together, the conditions are
\[
g_0(w) - \|g_0\|\|u_1 - w\| \leq g(u_1) \leq \|g_0\|\|u_1 + w\| - g_0(w). \tag{A.10}
\]

To show that the two regions are indeed separated, so that such a number \(g(u_1)\) exists, it is noted that for any \(w, w' \in W\),
\[
g_0(w) + g_0(w') = g_0(w + w') \leq \|g_0\| \|w + w'\| \leq \|g_0\| \|w + u\| + \|g_0\| \|w' - u\|, \tag{A.11}
\]
where on the right, \(u\) is an arbitrary element of \(\mathcal{U}\). It follows that
\[
g_0(w') - \|g_0\| \|w' - u\| \leq \|g_0\| \|w + u\| - g_0(w), \quad \text{for all } w, w' \in W, u \in \mathcal{U}. \tag{A.12}
\]

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