Research Article

Time-Fractional Heat Conduction in a Half-Line Domain due to Boundary Value of Temperature Varying Harmonically in Time

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The Dirichlet problem for the time-fractional heat conduction equation in a half-line domain is studied with the boundary value of temperature varying harmonically in time. The Caputo fractional derivative is employed. The Laplace transform with respect to time and the sin-Fourier transform with respect to the spatial coordinate are used. Different formulations of the considered problem for the classical heat conduction equation and for the wave equation describing ballistic heat conduction are discussed.

1. Introduction

In the paper [1] and later on in the book [2] Nowacki studied the classical parabolic heat conduction equation with a heat source term varying harmonically as a function of time

\[
\frac{\partial T(x,t)}{\partial t} = a \frac{\partial^2 T(x,t)}{\partial x^2} + q_0 \delta(x) e^{i\omega t}
\]

in the domain \(-\infty < x < \infty\). Here \(a > 0\) is the thermal diffusivity coefficient, \(\delta(x)\) is the Dirac delta function, and \(\omega > 0\) denotes the frequency.

Nowacki’s solution of (1) is based on the assumption that temperature can be expressed as a product of the auxiliary function \(U(x)\) and the time harmonic term

\[
T(x,t) = U(x) e^{i\omega t}.
\]

In this case, there are no initial and boundary conditions (excepting the zero condition at \(x \to \pm \infty\)), and the problem is reduced to solving the corresponding equation for the auxiliary function \(U(x)\). The final result reads

\[
T(x,t) = \frac{q_0}{2a \sqrt{i\omega/a}} e^{-|x| \sqrt{i\omega/a}} e^{i\omega t}.
\]

The square root of the imaginary unit is defined as \(\sqrt{i} = e^{i\pi/4}\).

If the heat conduction equation

\[
\frac{\partial T(x,t)}{\partial t} = a \frac{\partial^2 T(x,t)}{\partial x^2}
\]

is considered in a half-line domain \(0 < x < \infty\), then the boundary condition at \(x = 0\) should be imposed. For example, we can assume the Dirichlet boundary condition varying harmonically in time:

\[
T(x,t) = T_0 e^{i\omega t} \quad x = 0.
\]

Similar analysis can be also carried out in the case of the boundary value of heat flux varying harmonically in time (the physical Neumann boundary condition). Boundary conditions varying harmonically in time describe various situations, in particular, thermal processing of materials using pulsed lasers or collection of solar energy [3].

Under Nowacki’s assumption (2), there is no initial condition, and for the auxiliary function \(U(x)\) we obtain

\[
i\omega U(x) = a \frac{d^2 U(x)}{dx^2}, \quad 0 < x < \infty,
\]

\[
U(x) = T_0 \quad x = 0,
\]

\[
\lim_{x \to \infty} U(x) = 0.
\]
To compare with the subsequent results it is worthwhile to solve (6) under boundary conditions (7) and (8) using the sin-Fourier transform with respect to the spatial coordinate $x$. The solution has a form
\[
U(x) = T_0 e^{-x \sqrt{i\omega/a}}. \tag{9}
\]
Hence,
\[
T(x, t) = T_0 e^{-x \sqrt{i\omega/a + i\omega t}}. \tag{10}
\]
If the surface temperature is described by the dependence
\[
T(x, t) = T_0 \sin (\omega t), \quad x = 0, \tag{11}
\]
then the solution becomes [3]
\[
T(x, t) = T_0 \exp \left(-x \sqrt{\frac{\omega}{2a}}\right) \sin \left(\omega t - x \sqrt{\frac{\omega}{2a}}\right). \tag{12}
\]

Many experimental and theoretical investigations testify that in media with complex internal structure the standard heat conduction equation is no longer sufficiently accurate. This results in formulation of nonclassical theories, in which the parabolic heat conduction equation is replaced by more general one (see [4–12] and the references therein).

For example, Green and Naghdi [7] proposed the theory of thermoelasticity without energy dissipation based on the wave equation for temperature. In the framework of this theory, the following boundary value problem can be studied:
\[
\frac{\partial^2 T(x, t)}{\partial t^2} = a \frac{\partial^2 T(x, t)}{\partial x^2}, \tag{13}
\]
\[
T(x, t) = T_0 e^{i\omega t}, \quad x = 0.
\]
Under the assumption (2), using the sin-Fourier transform, we get the solution (see (A.3) from appendix)
\[
T(x, t) = T_0 \cos \left(\frac{\omega x}{\sqrt{a}}\right) e^{i\omega t}. \tag{14}
\]

2. Time-Fractional Heat Conduction

The time-nonlocal generalization of the Fourier law with the “long-tail” power kernel [11, 13–15] can be interpreted in terms of fractional calculus (theory of integrals and derivatives of noninteger order) and results in the time-fractional heat conduction equation
\[
^C D_x^\alpha T = a \Delta T, \quad 0 < \alpha \leq 2, \tag{15}
\]
with the Caputo fractional derivative of order $\alpha$ defined as [16–18]
\[
^C D_x^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \tag{16}
\]
and having the following Laplace transform rule:
\[
L \left\{^C D_x^\alpha f(t) \right\} = s^\alpha f^+(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{n-1-k}, \tag{17}
\]
where the asterisk denotes the transform, $s$ is the transform variable, and $\Gamma(x)$ is the gamma function. The Caputo fractional derivative is a regularization in the time origin for the Riemann-Liouville fractional derivative by incorporating the relevant initial conditions [19]. The major utility of the Caputo fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the governing equation is of fractional order [17, 20]. Additional discussion on the use of the Caputo and Riemann-Liouville fractional derivatives can be found in [21] (see Section 3.4 “Which type of fractional derivative? Caputo or Riemann-Liouville?” in this book).

Equations with fractional derivatives describe many important physical phenomena in amorphous, colloidal, glassy, and porous materials, in fractals, comb structures, polymers, and random and disordered materials, in viscoelasticity and hereditary mechanics of solids, in biological systems, and in geophysical and geological processes (see, e.g., [22–30] and the references therein). Important applications of fractional calculus can be found in such fields as fractional dynamics [31–35], fractional kinetics [36–38], and fractional thermoelasticity [11, 12, 39–41].

Equation (15) describes the whole spectrum from localized heat conduction (the Helmholtz equation for $\alpha \to 0$) through the standard heat conduction ($\alpha = 1$) to the ballistic heat conduction (the wave equation when $\alpha = 2$).

The interested reader is referred to the book [15], which systematically presents solutions to different initial and boundary value problems for the time-fractional diffusion-wave equation (15) in Cartesian, cylindrical, and spherical coordinates. In [42, 43], this equation was considered in unbounded domains with the source term varying harmonically in time.

In the present paper, we study the Dirichlet problem for the time-fractional heat conduction equation in a half-line domain with the surface value of temperature varying harmonically in time. The integral transform technique is used. Different formulations of the considered problem for the classical heat conduction equation ($\alpha = 1$) and for the wave equation describing ballistic heat conduction ($\alpha = 2$) are discussed.

3. Formulation of the Problem

We consider the time-fractional heat conduction equation in a half-line:
\[
^C D_x^\alpha T = a \frac{\partial^2 T(x, t)}{\partial x^2}. \tag{18}
\]
\[0 < x < \infty, \quad 0 < t < \infty, \quad 0 < \alpha \leq 2,
\]
under the harmonic boundary condition
\[ T(x, t) = T_0 e^{i\omega t} \quad x = 0, \quad (19) \]
and zero condition at infinity
\[ \lim_{x \to \infty} T(x, t) = 0. \quad (20) \]

For the Caputo derivative of the exponential function we have
\[ C_D^\alpha e^{\lambda t} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n e^{\lambda \tau}}{d\tau^n} d\tau, \quad n-1 < \alpha < n. \quad (21) \]

Substituting \( t-\tau = \lambda^{-1} u \) (with \( \lambda > 0 \)) gives the final result
\[ C_D^\alpha e^{\lambda t} = \lambda e^{\lambda t} \frac{\gamma(n-\alpha, \lambda t)}{\Gamma(n-\alpha)}, \quad n-1 < \alpha < n, \quad (22) \]
where \( \gamma(a, x) \) is the incomplete gamma function [44]:
\[ \gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt. \quad (23) \]

Hence, for fractional (noninteger) values of the order \( \alpha \) of derivative,
\[ C_D^\alpha e^{\lambda t} \neq \lambda^\alpha e^{\lambda t}. \quad (24) \]

Therefore, the Nowacki assumption (2) cannot be used for the time-fractional heat conduction equation and the corresponding initial conditions should be imposed (see also [42, 43]). In the present paper we assume zero initial conditions:
\[ 0 < \alpha \leq 2 \quad t = 0, \quad T = 0, \]
\[ 1 < \alpha \leq 2 \quad t = 0, \quad \frac{\partial T}{\partial t} = 0. \quad (25) \]

4. Solution to the Problem

Application of the Laplace transform with respect to time \( t \) and the sin-Fourier transform with respect to the spatial coordinate \( x \) to (18) under the initial conditions (25) and the boundary conditions (19), (20) gives
\[ \tilde{T}^* (\xi, s) = aT_0 \frac{\xi}{s^2 + a\xi^2} \frac{1}{s - i\omega}, \quad (26) \]
where the tilde denotes the sin-Fourier transform and \( \xi \) is the transform variable.

At first, we analyze the standard heat conduction equation corresponding to \( \alpha = 1 \):
\[ \tilde{T}^* (\xi, s) = aT_0 \frac{\xi}{s + a\xi^2} \frac{1}{s - i\omega}. \quad (27) \]

The inverse Laplace transform gives (see (A.6) from appendix):
\[ \tilde{T} (\xi, t) = aT_0 \frac{\xi}{a\xi^2 + i\omega} \left( e^{i\omega t} - e^{-a\xi^2 t} \right). \quad (28) \]

Integrals (A.2) and (A.4) allow us to invert the sin-Fourier transform and to obtain the solution
\[ T(x, t) = T_0 e^{-x \sqrt{\omega t/\gamma t}} \]
\[ - \frac{T_0}{2} e^{-x \sqrt{\omega t/\gamma t}} \text{erfc} \left( \frac{\sqrt{i\omega t} - x}{2\sqrt{\gamma t}} \right) \]
\[ + \frac{T_0}{2} e^{x \sqrt{\omega t/\gamma t}} \text{erfc} \left( \frac{\sqrt{i\omega t} + x}{2\sqrt{\gamma t}} \right). \quad (29) \]

The first term in (29) coincides with solution (10) and describes the quasi-steady-state oscillations; the second and third ones describe the transient process.

Starting in (27) from the inversion of the sin-Fourier transform, we get
\[ T^* (x, s) = T_0 \frac{1}{s - i\omega} \exp \left( -\frac{x}{\sqrt{\alpha s}} \right). \quad (30) \]

Taking into account (A.10) and using the convolution theorem for the Laplace transform allow us to obtain an alternative form of the solution to the standard heat conduction equation:
\[ T(x, t) = \frac{T_0 x^2}{2\sqrt{\pi \gamma t}} \int_0^\infty \frac{1}{\tau^{3/2}} e^{-\frac{x^2}{4\alpha \tau}} e^{i\omega (t-\tau)} d\tau. \quad (31) \]

Another particular case of solution (26) in the transform domain corresponds to the ballistic heat conduction (\( \alpha = 2 \)):
\[ \tilde{T}^* (\xi, s) = aT_0 \frac{\xi}{s^2 + a\xi^2} \frac{1}{s - i\omega}. \quad (32) \]

Inversion of the sin-Fourier transform (see (A.2)) results in
\[ T^* (x, s) = T_0 \frac{1}{s - i\omega} \exp \left( -\frac{x}{\sqrt{\alpha s}} \right). \quad (33) \]

Taking into account (A.9), we obtain
\[ T(x, t) = \begin{cases} T_0 e^{i\omega (t-x/\sqrt{\gamma t})}, & 0 \leq x < \sqrt{\alpha t}, \\ 0, & \sqrt{\alpha t} < x < \infty. \end{cases} \quad (34) \]

It should be noted that solution (34) describes the wavefront at \( x = \sqrt{\alpha t} \).

In applications there often appears the value \( \alpha = 1/2 \) (see investigations of diffusion on fractals [45] and comb structures [46, 47]). Using (A.7), we get
\[ T(x, t) = \frac{2aT_0}{\pi} \int_0^\infty \frac{\xi \sin (x\xi)}{\alpha\xi^4 - i\omega} \left[ a\xi^2 e^{i\omega t} + \sqrt{-i\omega \xi^4 \text{erfi} (\sqrt{i\omega t})} \right] d\xi. \quad (35) \]
Now we return to the analysis of the time-fractional heat conduction equation and its solution in the transform domain (26). The inverse sin-Fourier transform gives

$$T^*(x,s) = T_0 \frac{1}{s - i\omega} \exp \left( -\frac{x}{\sqrt{a}} \right).$$

(36)

The inverse Laplace transform of $\exp(-\lambda s^\alpha)$ is expressed in terms of the Mainardi function $M(\alpha; z)$ (see (A.11) from appendix). The solution has the form

$$T(x,t) = \frac{\alpha T_0 x}{2 \sqrt{a}} \int_0^t \frac{1}{\tau^{\alpha/2-1}} M \left( \frac{\alpha}{2}, \frac{x}{\sqrt{a} \tau^{\alpha/2}} \right) e^{i\omega(t-\tau)},$$

(37)

$$0 < \alpha < 2.$$  

The particular case of the Mainardi function $M(1/2; z)$ is reduced to the exponential function (see [48, 49]):

$$M \left( \frac{1}{2}, z \right) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{z^2}{4} \right),$$

(38)

and solution (37) for $\alpha = 1$ coincides with solution (31).

Starting from the inverse Laplace transform of (26), we have

$$\tilde{T}(\xi, t) = a T_0 \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha} \left( -a \xi^2 \tau^\alpha \right) e^{i\omega(t-\tau)} d\tau,$$

(39)

where $E_{\alpha,\alpha}(z)$ is the Mittag-Leffler function in two parameters $\alpha$ and $\beta$ (see appendix). In parallel with (37), the inverse Fourier transform of (39) leads to another form of the solution

$$T(x,t) = \frac{2a T_0}{\pi} \int_0^t \int_0^\infty \tau^{\alpha-1} E_{\alpha,\alpha} \left( -a \xi^2 \tau^\alpha \right) e^{i\omega(t-\tau)} \xi \sin (x\xi) d\xi d\tau.$$

(40)

Comparison of (37) and (40) allows us to establish the relation between the Mainardi function and the Mittag-Leffler function in the form of sin-Fourier transform (see also [15, 50], where the similar relations were obtained in terms of the cos-Fourier transform).

Figures 1 and 2 present the dependence of solution on distance in the case of the boundary condition

$$T(x,t) = T_0 \cos(\omega t) \quad x = 0$$

(41)

for different values of the order $\alpha$ of fractional derivative and different values of time. In numerical calculations we have used the following nondimensional quantities:

$$\bar{T} = \frac{T}{T_0},$$

$$\bar{x} = \frac{x}{\sqrt{a t^{\alpha/2}}},$$

$$\bar{t} = \omega t.$$  

(42)

To evaluate the Mittag-Leffler function $E_{\alpha,\alpha}(z)$, the algorithm suggested in the paper [51] has been used.

5. Concluding Remarks

We have considered the Dirichlet problem for the time-fractional heat conduction equation in a half-line with the Caputo fractional derivative and with the boundary value of temperature varying harmonically in time. The solution has been obtained using the integral transform technique.
The Caputo fractional derivative of the exponential function has more complicated form than the corresponding derivative of the integer order. Hence, the Nowacki approach based on the representation of temperature as the product of a function of the spatial coordinate $U(x)$ and a function harmonic in time $e^{\omega t}$ cannot be used, and the initial conditions should be taken into account.

In such a statement of the problem, the particular cases of the general solution for integer values of the order of derivative ($\alpha = 1$ and $\alpha = 2$) describe both the quasi-steady-state oscillations and the transient process. It should be emphasized that in the case of the ballistic heat conduction equation ($\alpha = 2$) the obtained solution presents the wavefront at $x = \sqrt{a}t$ ($\alpha = 1$ in Figures 1 and 2), which does not appear in the Nowacki-type solution.

The obtained solution may also be used in constructing solutions for boundary functions varying periodically in an arbitrary manner. Expanding the boundary function in the time-Fourier series, the solution can be obtained as a result of superposition of successive harmonic terms.

Appendix

We present integrals [52, 53] used in the paper:

\begin{equation}
\int_0^\infty \frac{1}{x^2 + c^2} \cos (b x) \, d x = \frac{\pi}{2 c} e^{-bc}, \quad b > 0, \text{ Re } c > 0,
\end{equation}

\begin{equation}
\int_0^\infty \frac{x}{x^2 + c^2} \sin (b x) \, d x = \frac{\pi}{2} e^{-bc}, \quad b > 0, \text{ Re } c > 0,
\end{equation}

\begin{equation}
\int_0^\infty \frac{1}{x^2 - c^2} \sin (b x) \, d x = \frac{\pi}{2} \cos (bc), \quad b > 0, \text{ Re } c > 0,
\end{equation}

\begin{equation}
\int_0^\infty \frac{x}{x^2 + c^2} e^{-\alpha x^2} \sin (b x) \, d x = \frac{\pi}{4} \cdot e^{\frac{b^2}{4\alpha}} \left[ \text{erfc} \left( \frac{ac - \frac{b}{2a}}{\sqrt{\alpha}} \right) - e^{\frac{bc}{2}} \text{erfc} \left( \frac{ac + \frac{b}{2a}}{\sqrt{\alpha}} \right) \right], \quad b > 0, \text{ Re } a > 0, \text{ Re } c > 0,
\end{equation}

where erfc(x) is the complementary error function

\begin{equation}
\text{erfc} (x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, d t.
\end{equation}

Formulae for inverse Laplace transform (equations (A.6)–(A.10)) are borrowed from [54, 55]:

\begin{equation}
L^{-1} \left\{ \frac{1}{(s + a) (s + b)} \right\} = \frac{e^{-at} - e^{-bt}}{a - b}, \quad (A.6)
\end{equation}

\begin{equation}
L^{-1} \left\{ \frac{1}{(s + a) (\sqrt{s} + b)} \right\} = \frac{1}{a + b^2} \left[ b e^{-at} + \sqrt{a} e^{-bt} \text{erfi} \left( \sqrt{a} t \right) \right. \nonumber \\
- b e^{bt} \text{erfc} \left( b \sqrt{t} \right) \right], \quad (A.7)
\end{equation}

where erfi(z) is the error function of an imaginary argument:

\begin{equation}
\text{erfi} (z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} \, d t,
\end{equation}

\begin{equation}
L^{-1} \left\{ \frac{e^{-ax}}{s + b} \right\} = \begin{cases} \frac{e^{-bt} - e^{-ct}}{(c - b)}, & 0 < c < t, \\ 0, & 0 < t < c, \end{cases} \quad (A.9)
\end{equation}

\begin{equation}
L^{-1} \left\{ \frac{e^{-\lambda \sqrt{s}}}{\lambda s^{3/2}} \right\} = \frac{\lambda}{2 \sqrt{\pi t^{3/2}} \exp \left( -\frac{\lambda^2}{4t} \right)}, \quad (A.10)
\end{equation}

Equation (A.11) can be found in [15, 48, 49]

\begin{equation}
L^{-1} \left\{ \frac{e^{-\lambda s}}{s^{a+1}} M (\alpha; \lambda t^a) \right\} = \frac{\alpha \lambda}{\Gamma (a+1)} M (\alpha; \lambda t^{-a}), \quad 0 < a < 1, \lambda > 0. \quad (A.11)
\end{equation}

Here $M (\alpha; z)$ is the Mainardi function [17, 48, 49], being the particular case of the Wright function:

\begin{equation}
M (\alpha; z) = \sum_{k=0}^\infty \frac{(-1)^k z^k}{k! (\alpha k + (1 - \alpha))}, \quad 0 < \alpha < 1, z \in \mathbb{C}.
\end{equation}

Equation (A.13) is taken from [16, 17]

\begin{equation}
L^{-1} \left\{ \frac{s^{a-\beta}}{s^{a} + b} \right\} = t^{\beta-1} E_{a,\beta} (-bt^a), \quad (A.13)
\end{equation}

where $E_{a,\beta}(z)$ is the Mittag-Leffler function in two parameters $a$ and $\beta$ [16, 17, 56] described by the following series representation:

\begin{equation}
E_{a,\beta} (z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma (ak + \beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C}. \quad (A.14)
\end{equation}

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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