Compressed frictional granular matter cannot flow without dilation. Upon forced shearing to generate flow, the amount of dilation may depend on the initial preparation and a host of material variables. Here we show that as a result of training by repeated compression-decompression cycles the amount of dilation induced by shearing the system depends only on the shear rate and on the packing fraction. Relating the rheological response to structural properties allows us to derive a scaling law for the amount of dilation after \( n \) cycles of compression-decompression. The resulting scaling law has a universal exponent that for trained systems is independent of the inter-granules force laws, friction parameters and strain rate. The amplitude of the scaling law is analytically computable, and it depends only on the shear rate and the asymptotic packing fraction.

I. INTRODUCTION

Compressed granular media, with or without friction, are jammed, and cannot flow without dilation [1, 2]. Subjected to shear rate by external forces, such media dilate, reducing the packing fraction in regions that participate in flows. The dilation may be very inhomogeneous, and may depend on a host of parameters that characterize the granular assembly. Understanding the resulting rheology is complicated due to the inherent properties of granular matter, like frictional losses, arching, segregation and thixotropy [3]. These complications result in a paucity of universal results, and the literature of frictional granular rheology at finite strain rate offers a bewildering array of particular examples that are not easy to comprehend, resisting attempts to organize and systematize [4–7].

In recent studies it became apparent that some universal results can be gleaned by training the system under repeated cycles of compression-decompression [8–10], building a memory that “cleans” the system from random effects present in “as compressed” frictional granular systems. For example it was shown that the packing fraction converges under repeated cycles to an asymptotic value following a universal law [11]. Another example is the universal giant friction slip event that occurs when the pressure goes to zero upon unjamming [12]. Here we follow on this line of reasoning and study the dilation induced by shear rate after training the system by \( n \) compression-decompression cycles. Indeed we find an enormous simplification resulting in a universal power law that characterizes the amount of dilation observed after training with \( n \) cycles. The power law indicates that training and memory result in the amount of dilation becoming a function of the strain rate and the packing fraction only. The exponent of the scaling law is independent of the working pressure, the strain rate, the friction parameters and the force laws between granules. To introduce these results we review briefly some recent results on training and memory formation.

II. TRAINING BY COMPRESSION-DECOMPRESSION CYCLES

When frictional granular media are trained by cyclic loading and unloading [10, 11, 12, 13, 14] memory is introduced in the system. Here we refer to training by uniaxial compression until the pressure reaches a maximal value \( P_{\text{max}} \) after which the the system is decompressed back to zero pressure, and then compressed again. The compression and decompression are achieved by one moving wall (upper wall in our simulations) and “pressure” always refer to the external pressure on this moving wall. In each cycle the packing fraction is increased until it reaches an asymptotic limit. During compression and decompression dissipation leads to hysteresis, but with repeated cycles the dissipation diminishes to a finite limit and the system retains memory of an asymptotic loaded state that is not forgotten even under complete unloading. An example of such training protocol as observed in numerical simulations [11] is shown in Fig. 1.

Associated with the reduced dissipation and the increase in memory one finds a universal power law in the packing fraction \( \Phi_n \) after the \( n \)-th cycle. This scalings is expected to hold irrespective of the details of the microscopic interactions. In every compression leg of the cycle the system compactifies, until a limit \( \Phi \) value is reached for the chosen maximal pressure. To quantify this process we can measure the volume fraction \( \Phi_n(P_{\text{max}}) \) at the highest value of the pressure in the \( n \)-th cycle. Define then a new variable

\[
X_n = \Phi_n(P_{\text{max}}) - \Phi_0(P_{\text{max}}).
\]

This new variable is history dependent in the sense that

\[
X_{n+1} = g(X_n) \text{ where the function } g(x) \text{ is unknown at this point. This function must have a fixed point } g(x=0) = 0 \text{ since the series } \sum_n X_n \text{ must converge; for any given chosen maximal pressure there is a limit volume fraction that cannot be exceeded. Near the fixed point, assuming analyticity, we expect the form}
\]

\[
X_{n+1} = g(X_n) = X_n - CX_n^2 + \cdots .
\]
apply a small pressure boundary condition in the horizontal direction we now ters in the continuous range of \([0, d]\). Here an assembly of disks with binary sizes are compressed from above in the presence of gravity. The pressure \(P\) is on the upper piston, measured in dimensionless units of \(mg/d\), see text for details. The packing fraction \(\Phi\) is dimensionless. Compression legs are made of particles of identical properties and diameter \(d\). We begin with a “box” of fixed horizontal length (in the \(x\) direction) of \(40d\) and a height (in the \(y\) direction) of \(100d\). To start, 1000 small and 1000 large disks are placed randomly without overlaps. The upper wall has a mass \(100m\)

The solution of this equation for \(n\) large is

\[
X_n = \frac{C}{n}.
\]

A direct measurement of \(X_n\) as a function of \(n\) in the present simulations which are recorded below is shown in the log-log plot presented in Fig. 2. In Ref. [11] one can find arguments and evidence for the generality of this power law.

III. DILATION UNDER SHEAR

A. preparation

The granular system that we simulate consists of disks of mass \(m = 1\) and diameters \(d = 1\) and \(1.4d\) in equal numbers. To prepare the system for shear and dilation we begin with a “box” of fixed horizontal length (in the \(x\) direction) of \(40d\) and a height (in the \(y\) direction) of \(100d\). To start, 1000 small and 1000 large disks are placed randomly without overlaps. The upper wall has a mass \(M = 100\) that is free to move; gravity is chosen such that \(g = 1\). The moving upper wall and the fixed lower wall are made of particles of identical properties and diameters in the continuous range of \([d, 2d]\). Applying periodic boundary condition in the horizontal direction we now apply a small pressure \(P\) on the upper wall. We simulate the system using molecular dynamics with Hertz normal forces and Mindlin tangential forces as described below. We solve Newton’s equations of motion with linear damping in the velocities of the disks. For a given pressure the simulation continues until mechanical equilibrium is reached. The pressure is then increased in small steps followed by equilibration until the desired final pressure is obtained. An example of an initial configuration is shown in Fig. 3.

The contact forces (both the normal and tangential forces which arises due to friction) are modeled according to the DEM (discrete element method) developed by Cundall and Strack [15]. Implementation of static friction is done via tracking the elastic part of the shear displacement from the time contact was first formed. When the disks are compressed they interact via both normal and tangential forces. Particles \(i\) and \(j\), at positions \(r_i, r_j\) with velocities \(v_i, v_j\) and angular velocities \(\omega_i, \omega_j\) will experience a relative normal compression on contact given by \(\Delta_{ij} = |r_{ij} - D_{ij}|\), where \(r_{ij}\) is the vector joining the centers of mass and \(D_{ij} = R_i + R_j\); this gives rise to a normal force \(F_{ij}^{(n)}\). The normal force is modeled as a Hertzian contact, whereas the tangential force is given by a Mindlin force \([16]\). Defining \(R_{ij}^{-1} \equiv R_i^{-1} + R_j^{-1}\), the force magnitudes are,

\[
F_{ij}^{(n)} = k_n \Delta_{ij} n_{ij} - \frac{\gamma_n}{2} v_{n_{ij}}, \quad F_{ij}^{(t)} = -k_t t_{ij} - \frac{\gamma_t}{2} v_{t_{ij}}
\]

\[
k_n = k_n \sqrt{\Delta_{ij} R_{ij}^3}, \quad k_t = k_t \sqrt{\Delta_{ij} R_{ij}}
\]

\[
\gamma_n = \gamma_n \sqrt{\Delta_{ij} R_{ij}}, \quad \gamma_t = \gamma_t \sqrt{\Delta_{ij} R_{ij}}
\]

Here \(\delta_{ij}\) and \(t_{ij}\) are normal and tangential displacement; \(n_{ij}\) is the normal unit vector. \(k_n = 2 \times 10^5\) and
$k_i = 2k_i'/7$ are spring stiffness for normal and tangential mode of deformation: $\gamma_n = 50$ and $\gamma_t = 50$ are viscoelastic damping constant for normal and tangential deformation. $v_{n_{ij}}$ and $v_{t_{ij}}$ are respectively normal and tangential component of the relative velocity between two particles. The relative normal and tangential velocity are given by

$$
v_{n_{ij}} = (v_{ij} \cdot n_{ij}) n_{ij}
$$

$$
v_{t_{ij}} = v_{ij} - v_{n_{ij}} - \frac{1}{2}(\omega_i + \omega_j) \times r_{ij}.
$$

where $v_{ij} = v_i - v_j$. Elastic tangential displacement is set to zero when the contact is first made and is calculated using $\frac{dt}{dt} = v_{t_{ij}}$ and also the rigid body rotation around the contact point is accounted for to ensure that $t_{ij}$ always remains in the local tangent plane of the contact plane.

The translational and rotational acceleration of particles are calculated from Newton’s second law; total forces and torques on particle $i$ are given by

$$
F_{i}^{(tot)} = \sum_j F_{ij}^{(n)} + F_{ij}^{(t)}
$$

$$
\tau_{i}^{(tot)} = -\frac{1}{2} \sum_j r_{ij} \times F_{ij}^{(t)}.
$$

The tangential force varies linearly with the relative tangential displacement at the contact point as long as the tangential force does not exceed the Coulomb limit

$$
F_{ij}^{(t)} \leq \mu F_{ij}^{(n)},
$$

where $\mu$ is a material dependent coefficient. When this limit is exceeded the contact slips in a dissipative fashion. In our simulations we keep the magnitude of $t_{ij}$ so that $F_{ij}^{(t)} = \mu F_{ij}^{(n)}$. The direction of $t_{ij}$ is allowed to change if further slip takes place.

### B. Shearing and Dilating

Having compacted the granular medium through a certain number of cycles, we next examine what happens if this same medium is subjected to a shear strain at a rate $\dot{\gamma}$ on its upper surface. Flow is possible only by dilating the material especially close to the upper moving wall. Denoting the rest height of the box by $L_y(0)$ we measure the actual height of the upper wall which is a function of time and the shear rate, denoted as $L_y(t, \dot{\gamma})$. The dilation is now denoted by $\delta(t, \dot{\gamma})$ where

$$
\delta(t, \dot{\gamma}) \equiv L_y(t) - L_y(0).
$$

The time dependence of $\delta(t)$ is quite complex. A typical trajectory of this quantity is shown in Fig. 4. Obviously, the trajectory indicates some noisy periodicity around some average. To extract the dominant frequency of the response of the upper wall we can compute the Fourier transform of this trajectory,

$$
S(f) \equiv \frac{1}{500} \int_{t=200}^{700} dt [\delta(t, \dot{\gamma}) - \langle \delta(\dot{\gamma}) \rangle] e^{i2\pi ft},
$$

$$
\langle \delta(\dot{\gamma}) \rangle \equiv \frac{1}{500} \int_{t=200}^{700} dt \delta(t, \dot{\gamma}).
$$

The limits of integration were chosen to eliminate the initial rise to a ‘steady state’ and to ensure convergence of the result. Averaging such spectra over 50 independent initial configurations results in a typical spectrum as seen in Fig. 5. The spectrum is dominated by one typical frequency. The nature of this frequency and its dependence on shear rate are interesting by themselves, but they fall outside the scope of the present paper. We only note in passing that the principal frequency (the main peak in Fig. 5) is fully understandable as a result...
IV. DATA COLLAPSE AND UNIVERSAL LAWS

A. Cyclic Training

Motivated by the universal scaling laws for the packing fraction as described in Sect. III and Fig. 2 we study next the physics of dilation in cyclically trained systems. The cyclic training is achieved by uniaxial straining such that the pressure is increased by pushing down the upper wall in quasistatic fashion until we reach a maximal chosen pressure; in the present simulations this pressure is $P_{\text{max}} = 100$. After each compression step, the system is allowed to relax to reach a new mechanical equilibrium. After a full compression leg, a cycle is completed by decompressing back to zero pressure, where the next compression cycle begins. The packing fraction $\Phi$ is monitored throughout this process. Each such cycle traces a hysteresis loop in the $P - \Phi$ plane, see Fig. 3 as an example.

The measurements of average dilation will be made now after $n - 1$ cycles. The system is decompressed to zero at the $n - 1$’th cycle, and then compressed again to a chosen value of the pressure $P_w$. At that pressure we then strain the system at a given strain rate $\dot{\gamma}$ to measure the average dilation. To get better statistics we repeat the whole procedure to obtain $\langle \delta_n \rangle (\dot{\gamma})$ averaged over many realizations.

To achieve universal results it is always prudent to work with dimensionless quantities. Thus instead of working with $\langle \delta_n \rangle (\dot{\gamma})$ we opt to define a new, related quantity which is dimensionless. To define this dimensionless quantity denote by $\Phi_n(P_w)$ the packing fraction associated with the unstrained systems in the $n$th cycle. After settling into the steady state with a given shear rate $\dot{\gamma}$ the asymptotic average packing fraction is denoted as $\Phi^*_{n} (\Phi_n(P_w), \dot{\gamma})$. The dimensionless dilation is then

$$D(\Phi_n(P_w), \dot{\gamma}) \equiv \frac{\Phi_n(P_w)}{\Phi^*_{n}(\Phi_n(P_w), \dot{\gamma})} - 1. \quad (15)$$

Needless to say, besides being dimensionless the dependence of this measure on the shear rate and on the friction coefficient remains identical to the data shown in Fig. 3. To simplify the notation we use below $D_n(\dot{\gamma}) \equiv D(\Phi_n(P_w), \dot{\gamma})$.

B. Universal scaling law

Having at our disposal the universal scaling law for the series $X_n$ it is natural to consider the series of differences in dimensionless dilations $D_{n+1} - D_n$. The main result of this subsection will be the series of these differences can be re-scaled to become (for large $n$) independent of $\dot{\gamma}$, the initial pressure, the friction coefficient etc. To see how to achieve this simplification we note that after many cycles, when $\Phi_n \to \Phi_\infty$, we can write

$$D_{n+1}(\dot{\gamma}) - D_n(\dot{\gamma}) \approx D'(\Phi_\infty, \dot{\gamma}) X_n \approx \frac{D'(\Phi_\infty, \dot{\gamma}) C}{n}. \quad (16)$$

where $D'(\Phi_\infty, \dot{\gamma}) = dD(\Phi, \dot{\gamma})/d\Phi|_{\Phi=\Phi_\infty}$. Besides the immediate consequence that dilation difference tends to zero as $1/n$, we also predict that the amplitude of this scaling appears to be a universal coefficient $D'(\Phi_\infty, \dot{\gamma}) / C$. Since $C$ is known from the data on the packing fraction itself, we need here to examine the coefficient $D'(\Phi_\infty, \dot{\gamma})$.

To compute the coefficient we start from Eq. (16) and write

$$\Phi_n D'(\Phi, \dot{\gamma}) = [1 + D(\Phi_n, \dot{\gamma})] - [1 + D(\Phi_n, \dot{\gamma})]^2 d\Phi_n / d\Phi_n. \quad (17)$$

Now if we assume that as $n \to \infty$ the granular medium loses its memory of its initial condition then we would
expect that \( d\Phi_n^*/d\Phi_n \to 0 \) and asymptotically we will find
\[
D'(\Phi_\infty, \gamma) = \frac{[1 + D(\Phi_\infty, \gamma)]}{\Phi_\infty}, \tag{18}
\]
and the asymptotic scaling of the dilation can be written as
\[
D_{n+1}(\gamma) - D_n(\gamma) \approx \frac{[1 + D(\Phi_\infty, \gamma)]}{n}(\Phi_\infty C). \tag{19}
\]
Finally, denoting
\[
A^{-1} \equiv \frac{[1 + D(\Phi_\infty, \gamma)]}{(\Phi_\infty C)}, \tag{20}
\]
we expect that \( A[D_n+1 - D_n] \) should become independent of any parameter in the problem yielding a universal power law \( 1/n \). This prediction is tested against the numerical simulations and the results are shown in Fig. 7. We find that instead of unity the pre-factor is close to 1.2. Of course constants of the order of unity are permissible in this theory.

V. CONCLUDING REMARKS

The main point of this paper is that training a frictional granular system by compression-decompression cycles can “clean” an “as compressed” system from random effects that complicate the interpretation of rheological properties. In the present example we examined the amount of dilation caused by shearing the system with a give shear rate \( \dot{\gamma} \). After training with \( n \) compression-decompression cycles the scaled dilation \( D_n(\dot{\gamma}) \) could be predicted since the series converges with a universal scaling exponent \( n^{-\frac{1}{2}} \). The pre-factors could be also estimated from the knowledge of the equally generic \( n^{-1} \) dependence of the associated series \( \Phi_\infty \) of the packing fraction after \( n \) cycles. One should note that in our simulations finite size effects are significant, introducing errors of the order of unity in the coefficient \( C \) of Eq. (8), in the value of \( \Phi_\infty \) and in the values of \( D(\Phi_\infty, \dot{\gamma}) \). All these enter the coefficient \( A \) of Eq. (20). Together they contribute to the scatter seen in Fig. 7. Taking all this into account we consider the agreement between theory and measurement quite satisfactory. It appears quite worthwhile to continue in the future to examine the effects of training and memory on disordered systems and their rheology.

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