ON THE EQUIVALENCE BETWEEN $\Theta_n$-SPACES AND ITERATED SEGAL SPACES

RUNE HAU GSENG

Abstract. We give a new proof of the equivalence between two of the main models for $(\infty, n)$-categories, namely the $n$-fold Segal spaces of Barwick and the $\Theta_n$-spaces of Rezk, by proving that these are algebras for the same monad on the $\infty$-category of $n$-globular spaces. The proof works for a broad class of $\infty$-categories that includes all $\infty$-topoi.

1. Introduction

Just as $\infty$-categories (or $(\infty, 1)$-categories) are a homotopical version of categories, where in addition to objects and morphisms we have homotopies between morphisms, homotopies of homotopies, and so forth, $(\infty, n)$-categories are a homotopical version of $n$-categories. This means that they have $i$-morphisms between $(i-1)$-morphisms for $i = 1, \ldots, n$ and also homotopies between $n$-morphisms, homotopies of homotopies, etc. (or in other words, invertible $i$-morphisms for $i > n$), with composition of $i$-morphisms only associative up to a coherent choice of higher homotopies. There are now a number of good models for $(\infty, n)$-categories, but the two that have seen the most use so far are $n$-fold Segal spaces and $\Theta_n$-spaces.

Iterated Segal spaces were first defined in Barwick’s thesis [Bar05], building on Rezk’s work on Segal spaces [Rez01], and were later generalized by Lurie [Lur09b, §1] to the setting of $\infty$-topoi; they are presheaves of spaces on the category $\Delta^n$ satisfying iteratively defined “Segal conditions” and constancy conditions. $\Theta_n$-spaces, which were introduced by Rezk [Rez10] (no doubt influenced by Joyal’s unpublished work on $\Theta_n$-sets and Berger’s description of $n$-fold loop spaces [Ber07]), are similarly presheaves of spaces on categories $\Theta_n$ that satisfy appropriate Segal conditions; in this paper we consider their natural generalization to $\infty$-topoi, which we will refer to as Segal $\Theta_n$-objects for clarity.

In [BSP11] Barwick and Schommer-Pries give axioms that characterize the homotopy theory of $(\infty, n)$-categories. They also prove that these axioms are satisfied in the case of $n$-fold Segal spaces and $\Theta_n$-spaces, which implies that these two models are equivalent. Another comparison, which relates the two models directly in the setting of model categories, has been given more recently by Bergner and Rezk [BR14].

The goal of this short paper is to give a new, conceptual proof of this equivalence: we will show that both models are the $\infty$-categories of algebras for a monad on the $\infty$-category of $n$-globular spaces (i.e. presheaves of spaces on the $n$-globular category, cf. Definition 2.5), and that these two monads are equivalent. This also brings out the relation between $(\infty, n)$-categories and $n$-categories: strict $n$-categories are the algebras for the analogous monad on the category of $n$-globular sets.

Our proof only makes use of formal properties of the $\infty$-category of spaces that hold for all $\infty$-topoi, so we obtain a comparison between iterated Segal objects and Segal $\Theta_n$-objects in any $\infty$-topos $\mathcal{X}$. In fact, our comparison works for a general
class of ∞-categories (equipped with a full subcategory of “constant” objects), which allows us to apply the comparison iteratively and conclude that Segal Ω_n × ⋯ × Ω_n-k-objects in X that are reduced (i.e., satisfy certain constancy conditions) are equivalent to Segal Ω_{n+1+⋯+nk}-objects.

**Remark 1.1.** In this paper we are concerned with the “algebraic” theory of (∞, n)-categories, i.e., we will not invert the fully faithful and essentially surjective morphisms. However, it is easy to see that the fully faithful and essentially surjective morphisms are preserved under our equivalence, so the two models remain equivalent after this localization. For iterated Segal spaces this localization corresponds, by results of Rezk (for n = 1), Barwick and Lurie (for the generalization to ∞-topoi), to the full subcategory of complete objects, and it is also easy to show that these correspond under our equivalence to the complete Ω_n-spaces of Rezk.

1.1. **Notation.** This paper is written in the language of ∞-categories, and we reuse some of the terminology and notation of [Lur09a, Lur14] without comment. If C and X are ∞-categories we will write P(C, X) for the ∞-category Fun(C, X) of presheaves on C valued in X.

1.2. **Overview.** In §2 we introduce the objects we will be concerned with in this paper, namely reduced Segal Ω_n-objects in presentable ∞-categories with good constants. Then in §3 we show that these are the algebras for a monad, and describe this monad explicitly. Finally, in §4 we prove our comparison result, Theorem 4.1.

2. **Ω_n-Objects and Segal Conditions**

In this section we will define our main objects of study in this paper: reduced Segal Ω_n-objects, which are certain presheaves on the categories Ω_n. We begin by recalling the definition of the categories Ω_n; these categories were originally introduced by Joyal, but here we make use of the inductive reformulation of the definition due to Berger [Ber07].

**Definition 2.1.** The category Ω_n is defined inductively as follows: First set Ω_0 to be the final category 1. Then define Ω_n to be the category with objects [n](I_1, . . . , I_n) with [n] ∈ Δ and I_i ∈ Ω_{n-1}; a morphism [n](I_1, . . . , I_n) → [m](J_1, . . . , J_m) is given by a morphism φ: [n] → [m] in Δ and morphisms ψ_{ij}: I_i → J_j in Ω_{n-1} where 0 < i < n and φ(i) < j ≤ φ(i). If X is an ∞-category, we will refer to presheaves Ω_n^{op} → X as Ω_n-objects in X.

The objects of Ω_n can be thought of as n-dimensional pasting diagrams for compositions in n-categories. We now wish to define the appropriate Segal conditions for Ω_n-objects that make their values at such a pasting diagram decompose appropriately as a limit of the values at the basic i-morphisms (i = 0, . . . , n). These were originally specified by Rezk [Rez10], but we will use an alternative formulation influenced by the work of Barwick on operator categories [Bar13], starting with the observation that the category Ω_n has a useful factorization system:

**Definition 2.2.** Recall that a morphism φ: [n] → [m] in Δ is inert if it is the inclusion of a subinterval in [m], i.e., φ(i) = φ(0) + i for all i, and active if it preserves the endpoints, i.e., φ(0) = 0 and φ(n) = m. We then inductively say a morphism (φ, ψ_{ij}) in Ω_n is inert if φ is inert in Δ and each ψ_{ij} is inert in Ω_{n-1}, and active if φ is active in Δ and each ψ_{ij} is active in Ω_{n-1}. We write Ω_n for the subcategory of Ω_n containing only the inert maps and i_n: Ω_{n,i} → Ω_n for the inclusion.

**Lemma 2.3.** The active and inert morphisms in Ω_n form a factorization system.
Proof. This is a special case of [Bar13, Lemma 8.3]; it is also easy to check by hand. \qed

Remark 2.4. Since objects of $\Theta_n$ have no non-trivial automorphisms, the active-inert factorizations are necessarily strictly unique.

Definition 2.5. Let $G_n$, the $n$-globular category, be the category with objects $C_i$, $i = 0, \ldots, n$, and morphisms generated by $s_i, t_i: C_i \to C_i$ with relations $s_is_i = t_is_i$ and $s_it_i = t_it_i$. We can informally depict this category as

$$C_0 \Rightarrow C_1 \Rightarrow \cdots \Rightarrow C_n.$$  

We refer to the object $C_k$ as the $k$-cell. There is a fully faithful inclusion $\gamma_n: G_n \to \Theta_{n,i}$, which is defined inductively by

$$\gamma_n(C_i) = \begin{cases} [1](\gamma_{n-1}(C_{i-1})), & i > 0 \\ [0](\cdot), & i = 0. \end{cases}$$

$$\gamma_n(s_i) = \begin{cases} [1](\gamma_{n-1}(s_{i-1})), & i > 0 \\ d_i: [0]() \to [1][0]() & i = 0, \end{cases}$$

$$\gamma_n(t_i) = \begin{cases} [1](\gamma_{n-1}(t_{i-1})), & i > 0 \\ d_0: [0]() \to [1][0]() & i = 0. \end{cases}$$

We abusively write $C_i$ also for $\gamma_n(C_i) \in \Theta_n$. Given $I \in \Theta_n$, we will write $G_{n/I}$ for the category $G_n \times_{\Theta_{n,i}} (\Theta_{n,i})/I$, and refer to its objects as the cells of $I$.

Definition 2.6. Suppose $X$ is a presentable infinite-category. A presheaf $F: \Theta_n^{op} \to X$ is a Segal $\Theta_n$-object if its restriction $F|_{\Theta_n^{op}}$ is the right Kan extension along $\gamma_n$ of its restriction to $G_n^{op}$—in other words, for $I$ in $\Theta_n$ the natural map $F(I) \to \lim_{C \in G_n^{op}} F(C)$ is an equivalence. We write $P_{\text{Seg}}(\Theta_n; X)$ for the full subcategory of $P(\Theta_n; X)$ spanned by the Segal $\Theta_n$-objects, and $P_{\text{Seg}}(\Theta_{n,i}; X)$ for the analogous subcategory of $P(\Theta_{n,i}; X)$; these are accessible localizations of $P(\Theta_n; X)$ and $P(\Theta_{n,i}; X)$, respectively.

For later use, we note that the Segal conditions imply more general decompositions of $F(I)$ as limits:

Definition 2.7. Suppose $f: I \to J$ is an active morphism in $\Theta_n$. For $\alpha: C \to I$ in $G_{n/I}$, let $C \xrightarrow{\xi} J_\alpha \xrightarrow{\alpha'} J$ be the (unique) active-inert factorization of $f \circ \alpha: C \to J$. Given a morphism

$$\begin{array}{ccc} C & \xrightarrow{\xi} & C' \\ \alpha \downarrow & & \alpha' \downarrow \\ I & & \end{array}$$

in $G_{n/I}$, the composite $C \to C' \to J_{\alpha'}$ has an active-inert factorization $C \to X \to J_{\alpha'}$. Since this also gives an active-inert factorization of $C \to J_{\alpha'} \to J$ we see that $X = J_\alpha$, and so $\xi$ determines an inert morphism $J_\alpha \to J_{\alpha'}$. We thus get a functor $G_{n/I} \to \text{Cat}$ by sending $\alpha$ to $G_{n,J_\alpha}$ and a morphism in $G_{n/I}$ to the functor given by composition with the associated inert morphism $J_\alpha \to J_{\alpha'}$. Let $G_{n/I} \to G_{n/J}$ denote the corresponding coCartesian fibration. Composition with the inert morphisms $J_\alpha \to J$ gives a functor $G_{n/I} \to G_{n/J}$.

Lemma 2.8. For any active morphism $f: I \to J$ in $\Theta_n$, the functor $G_{n/I} \to G_{n/J}$ is cofinal.
Proof. By [Lur09a, Theorem 4.1.3.1] it suffices to show that for every $\epsilon: C \to J \in G_{n/J}$, the category $(G_{n/J})_{\epsilon/J} := G_{n/J} \times G_{n/J} (G_{n/J})_{\epsilon/J}$ is weakly contractible. But by inspection this category always has an initial object, corresponding to the cell of $I$ that is the intersection of all the cells whose image contains $\epsilon$. \hfill $\square$

Lemma 2.9. Suppose $F \in \mathcal{P}(\mathcal{F}_n; X)$ is a Segal object. Then for any active morphism $f: I \to J$, the natural map

$$F(J) \to \lim_{\alpha \in G_{n/J}} F(J_{\alpha})$$

is an equivalence.

Proof. Using the Segal conditions for $J_{\alpha}$ we have

$$\lim_{\alpha \in G_{n/J}} F(J_{\alpha}) \simeq \lim_{\alpha \in G_{n/J}} \lim_{C \to J_{\alpha} \in G_{n/J}} F(C).$$

By [Hau16, Corollary 5.7] we can rewrite this limit as $\lim_{C \in G_{n/J}} F(C)$, and by Lemma 2.8 this limit is equivalent to $\lim_{C \to J \in G_{n/J}} F(C)$, which we know by the Segal condition for $J$ is equivalent to $F(J)$. \hfill $\square$

For the $\infty$-category $\mathcal{S}$ of spaces, $\mathcal{P}_{\text{Seg}}(\mathcal{F}_n; \mathcal{S})$ is the $\infty$-category underlying Rezk’s model category of $\mathcal{F}_n$-spaces from [Rez10]. More generally, if $X$ is, say, an $\infty$-topos, the $\infty$-category $\mathcal{P}_{\text{Seg}}(\mathcal{F}_n; X)$ gives the (algebraic) $\infty$-category of internal $(\infty, n)$-categories in $X$. We would like to be able to iterate this definition, so that we get a good definition of Segal $\mathcal{F}_n$-objects in $\mathcal{P}_{\text{Seg}}(\mathcal{F}_n; X)$. Just as in Barwick’s definition of $n$-fold Segal spaces, this requires forcing some of the images to be constant; to formalize this notion, it is convenient to introduce the following technical definition:

Definition 2.10. A presentable $\infty$-category with good constants is a pair $(X, \mathcal{U})$ consisting of an $\infty$-category $X$ together with a full subcategory $\mathcal{U}$ satisfying the following requirements:

(a) $X$ and $\mathcal{U}$ are both presentable.

(b) The inclusion $\mathcal{U} \to X$ preserves all limits and colimits (and hence, by the adjoint functor theorem, has both a left and a right adjoint).

(c) Coproducts in $\mathcal{U}$ are disjoint, i.e. for any two objects $U, U' \in \mathcal{U}$, the commutative square

$$\begin{array}{ccc}
\emptyset & \to & U \\
\downarrow & & \downarrow \\
U' & \to & U \amalg U'
\end{array}$$

is Cartesian.

(d) Coproducts over $\mathcal{U}$ are universal, i.e. for any morphism $f: X \to U$ in $X$ with $U \in \mathcal{U}$, the functor $f^*: X/U \to X/X$, given by pullback along $f$, preserves the initial object and arbitrary coproducts.

Example 2.11. If $X$ is an $\infty$-topos, then $(X, X)$ is a presentable $\infty$-category with good constants by [Lur09a, Theorem 6.1.0.6].

Remark 2.12. Since we are requiring pullbacks over $\mathcal{U}$ to preserve all coproducts in $X$, not just coproducts in $\mathcal{U}$, a distributor in the sense of Lurie [Lur09b, Definition 1.2.1] is not necessarily a presentable $\infty$-category with good constants. However, the key examples — $\infty$-topoi and iterated $\mathcal{F}_n$-objects in $\infty$-topoi — are both distributors and presentable $\infty$-categories with good constants.
Definition 2.13. Suppose \((\mathcal{X}, \mathcal{U})\) is a presentable \(\infty\)-category with good constants. We say a presheaf \(X \in \mathcal{P}(\Omega_n; \mathcal{X})\) is reduced if \(X(C_i)\) is in \(\mathcal{U}\) for all \(i < n\); we write \(\mathcal{P}_r(\Omega_n; \mathcal{X}, \mathcal{U})\) for the full subcategory of \(\mathcal{P}(\Omega_n; \mathcal{X})\) spanned by the reduced objects. A presheaf \(X\) in \(\mathcal{P}(\Omega_n; \mathcal{X})\) or \(\mathcal{P}(\Omega_n; \mathcal{X}, \mathcal{U})\) is then called reduced if \(X|_{\Omega_n^{\omega}}\) is reduced, and a reduced Segal \(\Omega_n\) (or \(\Omega_n, i\)-object) if it is both reduced and a Segal \(\Omega_n\) (or \(\Omega_n, i\)-)object. We write \(\mathcal{P}_{rSeg}(\mathcal{X}, \mathcal{U})\) and \(\mathcal{P}_{rSeg}(\Omega_n; \mathcal{X}, \mathcal{U})\) for the full subcategories of \(\mathcal{P}(\Omega_n; \mathcal{X})\) and \(\mathcal{P}(\Omega_n; \mathcal{X}, \mathcal{U})\), respectively, spanned by the reduced Segal objects.

Proposition 2.14. Suppose \((\mathcal{X}, \mathcal{U})\) is a presentable \(\infty\)-category with good constants.

(i) The \(\infty\)-category \(\mathcal{P}_{rSeg}(\Omega_n; \mathcal{X}, \mathcal{U})\) is presentable, and the inclusion
\[
\mathcal{P}_{rSeg}(\Omega_n; \mathcal{X}, \mathcal{U}) \hookrightarrow \mathcal{P}(\Omega_n; \mathcal{X})
\]
admits a left adjoint \(L_n\).

(ii) The functor \(c^*: \mathcal{U} \to \mathcal{P}(\Omega_n; \mathcal{X})\) that takes an object in \(\mathcal{U}\) to the constant presheaf with that value is fully faithful and takes values in \(\mathcal{P}_{rSeg}(\Omega_n; \mathcal{X}, \mathcal{U})\).

(iii) The pair \((\mathcal{P}_{rSeg}(\Omega_n, \mathcal{X}, \mathcal{U}), \mathcal{U})\), with \(\mathcal{U}\) viewed as the full subcategory of constant presheaves, is a presentable \(\infty\)-category with good constants.

Before we prove the proposition, we need some technical lemmas:

Lemma 2.15. Let \((\mathcal{X}, \mathcal{U})\) be a presentable \(\infty\)-category with good constants. Suppose given maps of sets \(f: A \to B\) and \(g: C \to B\), objects \(X_a \in \mathcal{X}\) for \(a \in A\), \(Y_c \in \mathcal{X}\) for \(c \in C\), \(U_b \in \mathcal{U}\) for \(b \in B\), and morphisms \(\phi_a: X_a \to U_{f(a)}\) and \(\psi_c: Y_c \to U_{g(c)}\) in \(\mathcal{X}\) for all \(a \in A\) and \(c \in C\). Then the natural map
\[
\left(\prod_{(a, b, c) \in A \times B \times C} X_a \times_{U_b} Y_c\right) \to \left(\prod_{a \in A} X_a\right) \times_{\left(\prod_{b \in B} U_b\right)} \left(\prod_{c \in C} Y_c\right)
\]
is an equivalence in \(\mathcal{X}\).

Proof. We first consider the case where \(f\) and \(g\) are both \(\text{id}_B\). Then we wish to prove that the natural map
\[
\left(\prod_{b \in B} X_b \times_{U_b} Y_b\right) \to \left(\prod_{b \in B} X_b\right) \times_{\left(\prod_{b \in B} U_b\right)} \left(\prod_{b \in B} Y_b\right)
\]
is an equivalence. The map
\[
\left(\prod_{i, j \in B} X_i \times_{\left(\prod_{b \in B} U_b\right)} Y_j\right) \to \left(\prod_{b \in B} X_b\right) \times_{\left(\prod_{b \in B} U_b\right)} \left(\prod_{b \in B} Y_b\right)
\]
is an equivalence by condition (d) in Definition 2.10. To complete the proof in this case it therefore suffices to show that
\[
X_i \times_{\left(\prod_{b \in B} U_b\right)} Y_j \simeq \begin{cases} \emptyset, & i \neq j, \\ X_i \times_{U_i} Y_j, & i = j. \end{cases}
\]
Since \(X_i \times_{\left(\prod_{b \in B} U_b\right)} Y_j \simeq X_i \times_{U_i} U_i \times_{\left(\prod_{b \in B} U_b\right)} U_j \times_{U_j} Y_j\) it is enough to show that
\[
U_i \times_{\left(\prod_{b \in B} U_b\right)} U_j \simeq \begin{cases} \emptyset, & i \neq j, \\ U_i, & i = j \end{cases}
\]
(in the case \(i \neq j\) this is sufficient since pullbacks over objects in \(\mathcal{U}\) preserve the initial object). To see this we observe that, setting \(V := \prod_{b \neq i} U_b\), for \(i \neq j\) we have
\[
U_i \times_{\left(\prod_{b \in B} U_b\right)} U_j \simeq U_i \times_{U_i, \mathcal{U}V} V \times V U_j \simeq \emptyset \times V U_j \simeq \emptyset
\]
using that coproducts in \( \mathcal{U} \) are disjoint and pullbacks in \( \mathcal{U} \) preserve the initial object, and for \( i = j \) we have

\[
U_i \simeq U_i \times_{U_j} (U_i \amalg U_j) \simeq (U_i \times_{U_j} U_i) \amalg (U_i \times_{U_j} U_j) \simeq U_i \times_{U_j} U_i.
\]

We now turn to the general case. For each \( b \in B \), let \( A_b \) and \( C_b \) denote the fibres of \( f \) and \( g \) at \( b \). Then the natural map

\[
\prod_{(a,c) \in A_b \times C_b} X_a \times_{U_a} Y_c \to \left( \prod_{a \in A_b} X_a \right) \times_{U_b} \left( \prod_{c \in C_b} Y_c \right)
\]

is an equivalence by condition (d) in Definition 2.10. Since \( A \simeq \amalg_{b \in B} A_b \) etc., taking the coproduct of these equivalences over \( b \in B \) we can complete the proof by applying the previous case.

\[\square\]

**Lemma 2.16.** For each \( J \) in \( \Omega_n \), the category \( G_{n/J} \) is weakly contractible.

**Proof.** Suppose \( J = [j](\ldots) \); then there is an obvious active map

\[
f : I = [j](C_{n-1}, \ldots, C_{n-1}) \to J.
\]

By Lemma 2.8 the map \( G_{n/J} \to G_{n/J} \) is cofinal and hence in particular a weak homotopy equivalence, so it suffices to show that \( G_{n/J} \) is weakly contractible.

Let \( \Lambda_j \) denote the partially ordered set of pairs \((a,b)\) with \( 0 \leq a \leq b \leq j \) and \( b - a \leq 1 \), with \((a,b) \leq (a',b')\) when \( a \leq a' \leq b' \leq b \). This is weakly contractible since it’s just a wedge of \( \Delta^j \)’s. Moreover, the inclusion \( \Lambda_j \to G_{n/J} \) that takes the objects \((i,i)\) to the \( n \)-cells of \( I \) and the objects \((i,i + 1)\) to the \( 0 \)-cells connecting them, is cofinal. Thus \( G_{n/J} \) is weakly contractible; to complete the proof we will show that the coCartesian fibration \( G_{n/J} \to G_{n/J} \) is cofinal and hence a weak homotopy equivalence. Since this is a coCartesian fibration, using [Lur09a, Theorem 4.1.3.1] we see that this is equivalent to the fibres \( G_{n/J} \) being weakly contractible. Pulling back to \( \Lambda_j \) we conclude that it suffices to check this for \( 0 \)-cells, where the fibre is a point, and \( n \)-cells. If \( \alpha \) is an \( n \)-cell, then \( J_\alpha = [1](I_\alpha) \), and \( G_{n/J_\alpha} \) consists of \( G_{n-1/J_\alpha} \) together with two \( 0 \)-cells that map to everything else. Appealing to [Lur09a, Theorem 4.1.3.1] again we see that \( G_{n-1/J_\alpha} \to G_{n/J_\alpha} \) is cofinal, which allows us to finish the proof by induction.

\[\square\]

**Proof of Proposition 2.14.** The \( \infty \)-category \( \mathcal{P}_{rSeg}(\Omega_n; X, \mathcal{U}) \) fits in a commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}_{rSeg}(\Omega_n; X, \mathcal{U}) & \longrightarrow & \mathcal{P}_{Seg}(\Omega_n; X) \\
\downarrow & & \downarrow \\
\mathcal{P}(\Omega_n; X, \mathcal{U}) & \longrightarrow & \mathcal{P}(\Omega_n; X) \\
\downarrow & & \downarrow \\
\mathcal{P}(G_{n-1}; \mathcal{U}) & \longrightarrow & \mathcal{P}(G_{n-1}; X).
\end{array}
\]

where both squares are Cartesian. Moreover, the bottom horizontal and the two right vertical functors are right adjoints between presentable \( \infty \)-categories. By [Lur09a, Theorem 5.5.3.18] limits in the \( \infty \)-category \( \mathcal{P}_{rSeg} \) of presentable \( \infty \)-categories and right adjoints are computed in that of large \( \infty \)-categories, hence all \( \infty \)-categories in this diagram are presentable and all functors are right adjoints. This proves (i).

Since \( \Omega_n \) is weakly contractible (as it has an initial object) the image of the constant presheaf functor \( c^* : \mathcal{U} \to \mathcal{P}(\Omega_n; \mathcal{U}) \to \mathcal{P}(\Omega_n; X) \) is fully faithful. Constant presheaves on objects in \( \mathcal{U} \) satisfy the Segal condition by Lemma 2.16, so this functor factors through \( \mathcal{P}_{rSeg}(\Omega_n; X, \mathcal{U}) \), which gives (ii).
The Definition 3.2. Define $U$ is an equivalence. This limit can be written as an iterated pullback over objects in $Y$ to check condition (d), i.e. given maps $Y_i \to c^* U$ for $i \in S$ we need to show that the natural map

$$\prod_i X \times_{c^* U} Y_i \to X \times_{c^* U} \prod_i Y_i$$

is an equivalence. To see this we will first check that the inclusion $\mathcal{P}_{rSeg}(\mathcal{O}_n; X, \mathcal{U}) \hookrightarrow \mathcal{P}(\mathcal{O}_n; X)$ preserves the initial object and arbitrary coproducts. It suffices to show that for objects $Y_i \in \mathcal{P}_{rSeg}(\mathcal{O}_n; X, \mathcal{U})$ for $i \in S$, the coproduct $Y := \coprod_{i \in S} Y_i$ in $\mathcal{P}(\mathcal{O}_n; X)$ is a reduced Segal $\mathcal{O}_n$-object. Since $\mathcal{U}$ is closed under colimits in $X$, the object $Y$ is reduced, and it remains to show that for $J \in \mathcal{O}_n$, the map

$$\prod_{i \in S} Y_i(I) \to \lim_{C \to \mathcal{U}_n/I, i \in S} \prod_i Y_i(C)$$

is an equivalence. This limit can be written as an iterated pullback over objects in $\mathcal{U}$, so this follows from Lemma 2.15. This means that it suffices to show that for $I \in \mathcal{O}_n$ we have that

$$\prod_i X(I) \times_{U} Y_i(I) \to X(I) \times_{U} \prod_i Y_i(I)$$

is an equivalence, which is true since $U$ is in $\mathcal{U}$.

Definition 2.17. For $(X, \mathcal{U})$ a presentable $\infty$-category with good constants, we write $\mathcal{P}_{rSeg}(\mathcal{O}_n \times \mathcal{O}_m; X, \mathcal{U})$ for the full subcategory of $\mathcal{P}(\mathcal{O}_n \times \mathcal{O}_m; X)$ corresponding to $\mathcal{P}_{rSeg}(\mathcal{O}_n; \mathcal{P}_{rSeg}(\mathcal{O}_m; X, \mathcal{U}), \mathcal{U})$. Similarly, we (inductively) define $\mathcal{P}_{rSeg}(\mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k}; X, \mathcal{U})$ and $\mathcal{P}_{rSeg}(\mathcal{O}_{n_1,i} \times \cdots \times \mathcal{O}_{n_k,i}; X, \mathcal{U})$.

Example 2.18. The $\infty$-category $\mathcal{P}_{rSeg}(\Delta^n; \mathcal{U})$ is the $\infty$-category of Barwick’s $n$-fold Segal spaces [Bar05]. More generally, $\mathcal{P}_{rSeg}(\Delta^n, X, \mathcal{U})$ gives Lurie’s $n$-fold $\mathcal{U}$-Segal spaces from [Lur09b].

3. The Free Reduced Segal $\mathcal{O}_n$-Object Monad

Our goal in this section is to show that the $\infty$-category $\mathcal{P}_{rSeg}(\mathcal{O}_n; X, \mathcal{U})$ is the $\infty$-category of algebras for a monad on $\mathcal{P}_{r}(\mathcal{O}_n; X, \mathcal{U})$, and to understand this monad explicitly. Before we state our precise result, we must introduce some notation:

Definition 3.1. For $I \in \mathcal{O}_n$, let $\text{Act}(I)$ denote the set of active morphisms $I \to J$ in $\mathcal{O}_n$. A morphism $f: I' \to I$ determines a map of sets $f^*: \text{Act}(I) \to \text{Act}(I')$ by taking $\phi: I \to J$ to the active morphism $\phi': I' \to J'$ that gives the (unique) active-inert factorization of $I' \to I \to J$. Since this factorization is unique, it is easy to see that this determines a functor $\text{Act}: \mathcal{O}_n^{op} \to \text{Set}$.

Definition 3.2. Define $\iota_n: \mathcal{O}_{n-1} \to \mathcal{O}_n$ inductively by taking $\iota_1: \ast = \mathcal{O}_0 \to \mathcal{O}_1 = \Delta$ to be the inclusion of $[0]$ and setting

$$\iota_n([m](I_1, \ldots, I_m)) = [m](\iota_{n-1}(I_1), \ldots, \iota_{n-1}(I_m))$$

Notice that $\iota_n$ is fully faithful. We write $\iota_k^n := \iota_n \circ \cdots \circ \iota_k^1: \mathcal{O}_k \to \mathcal{O}_n$.

Proposition 3.3. Let $(X, \mathcal{U})$ be a presentable $\infty$-category with good constants.

(i) The functor

$$i_n^*: \mathcal{P}_{rSeg}(\mathcal{O}_n; X, \mathcal{U}) \to \mathcal{P}_{rSeg}(\mathcal{O}_{n,i}; X, \mathcal{U})$$

has a left adjoint $F_n$.

(ii) The adjunction $F_n \dashv i_n^*$ is monadic.
The monad $T_n := i_n^* F_n$ on $\mathcal{P}_{\text{Seg}}(\Theta_n; X, U)$ satisfies
\[ T_n X(I) \cong \coprod_{I \to J \in \text{Act}(I)} X(J). \]

In particular,
\[ T_n X(C_k) \cong \coprod_{J \in \Theta_k} X(i_k^n J). \]

The proof relies on a simple description of the left Kan extension functor $i_{n,!}$, which we prove first:

**Lemma 3.4.** The functor $i_{n,!} : \mathcal{P}(\Theta_n; X) \to \mathcal{P}(\Theta_n; X)$ can be described explicitly as
\[ i_{n,!} F(I) \cong \coprod_{I \to J \in \text{Act}(I)} F(J). \]

In particular, $i_{n,!} F(C_k) \cong \coprod_{I \in \text{cob} \Theta_k} F(i_k^n(I))$.

**Proof.** Since the active and inert maps form a factorization system on $\Theta_n$, for every object $X = (J, f : I \to J)$ in $(\Theta_n)_J$ the category $(\text{Act}(I))_X$, which consists of active-inert factorizations of $f$, is contractible. The inclusion $\text{Act}(I) \to (\Theta_n)_I$ is therefore cofinal by [Lur09a, Theorem 4.1.3.1], hence the left Kan extension $i_{n,!} F$ is indeed given by
\[ i_{n,!} F(I) \cong \colim_{(I \to J) \in (\Theta_n)_I} F(J) \cong \coprod_{I \to J \in \text{Act}(I)} F(J). \]

If $I = C_k$ then the only objects of $\Theta_n$ that admit an active map from $C_k$ are those in the image of the fully faithful functor $i_k^n : \Theta_k \to \Theta_n$ (and these active maps are unique), which gives the expression for $i_{n,!} F(C_k)$.

We need one more observation:

**Lemma 3.5.** Given $I \in \Theta_n$, the natural map of sets
\[ \text{Act}(I) \to \lim_{C \to J \in G_{n-1,I}} \text{Act}(C) \]
is an isomorphism.

**Proof.** We will prove this by induction on $n$ (starting with the case $n = 1$, which is trivial). First suppose $I = [1](J)$ for some $J \in \Theta_{n-1}$. Then it is immediate from the definition of active maps in $\Theta_n$ that
\[ \text{Act}(I) \cong \coprod_{i=0}^\infty \text{Act}(J)^\times n. \]

By assumption we have $\text{Act}(J) \cong \lim_{C \to J \in G_{n-1,J}} \text{Act}(C)$, hence
\[ \text{Act}(J) \cong \coprod_{i=0}^\infty \left( \lim_{C \to J \in G_{n-1,J}} \text{Act}(C) \right)^\times n \cong \lim_{C \to J \in G_{n-1,J}} \left( \prod_{i=0}^\infty \text{Act}(C)^\times n \right), \]

where the second isomorphism holds since limits commute and the argument of Lemma 2.15 is also valid for the category of sets, and the final isomorphism follows by cofinality as in the proof of Lemma 2.16.

For a general $I = [n](J_1, \ldots, J_n)$, let $I_i = [1](J_i)$, then the definition of active morphisms in $\Theta_n$ immediately implies that $\text{Act}(I) \cong \text{Act}(I_1) \times \cdots \times \text{Act}(I_n)$ (and $\text{Act}(C_0) \cong *)$. If $f : K := [n](C_{n-1}, \ldots, C_{n-1}) \to I$ denotes the active map given by
id_{[n]} and the unique active maps $C_{n-1} \to J_i$, then by the same cofinality argument as in the proof of Lemma 2.16 we get isomorphisms

$$\text{Act}(I) \cong \lim_{\alpha \in G_{n/2}^I} \lim_{\tau_{\alpha} \in C_{n/2}} \text{Act}(C) \cong \lim_{\tau_{\alpha} \in C_{n/2}} \text{Act}(C),$$

where the last isomorphism follows from Lemma 2.8.

\[ \square \]

**Proof of Proposition 3.3.** Let $L_n$ denote the localization functor from $\mathcal{P}(\Theta_n; X)$ to $\mathcal{P}_{rSeg}(\Theta_n; X, U)$; then $L_n i_n !$, clearly restricts to a left adjoint to $i_n ^*$, which gives (i).

To see that the adjunction is monadic it suffices by [Lur14, Theorem 4.74.4.5] to prove that $i_n ^*$ detects equivalences and that colimits of $i_n ^*$-split simplicial objects exist in $\mathcal{P}_{rSeg}(\Theta_n; X, U)$ and are preserved by $i_n ^*$. Since $\Theta_n,i$ is a subcategory of $\Theta_n$ containing all the objects it is clear that $i_n ^*$ detects equivalences. Suppose we have an $i_n ^*$-split simplicial object $X_\bullet$ in $\mathcal{P}_{rSeg}(\Theta_n; X, U)$, i.e. $i_n ^* X_\bullet$ extends to a split simplicial object $X'_\bullet: \Delta^\omega \to \mathcal{P}_{rSeg}(\Theta_n;i, X, U)$. If we consider $X_\bullet$ as a diagram in $\mathcal{P}(\Theta_n; X)$ with colimit $X$, then this colimit is preserved by $i_n ^*: \mathcal{P}(\Theta_n; X) \to \mathcal{P}(\Theta_n;i, X)$ (since this functor is a left adjoint). But by [Lur14, Remark 4.7.3.3], the diagram $X'_\bullet$ is a colimit diagram also when viewed as a diagram in $\mathcal{P}(\Theta_n;i, X)$, so $i_n ^* X \simeq X'_\omega$. This means that $X$ is a reduced Segal $\Theta_n$-object, and so it is also the colimit of $X'_\bullet$ in $\mathcal{P}_{rSeg}(\Theta_n;i, X, U)$, and its image in $\mathcal{P}_{rSeg}(\Theta_n;i, X, U)$ is $X'_\omega$, as required. This proves (ii).

To prove (iii), we will show that if $X \in \mathcal{P}_{rSeg}(\Theta_n;i, X, U)$ then $i_n ! X$ is a reduced Segal $\Theta_n$-object, hence $F_n X$ is just given by the left Kan extension $i_n ! X$:

To see that $i_n !$ is reduced, we observe that for $i < n$ the expression for $i_n ! F(C_i)$ in Lemma 3.4 is a coproduct of limits of objects in $U$, and hence is also in $U$ since this is closed in $X$ under all limits and colimits.

Now since $X$ is a Segal $\Theta_n,i$-object we have, using Lemma 2.9,

$$i_n ! X(I) \simeq \coprod_{I \to J \in \text{Act}(I)} X(J) \simeq \coprod_{I \to J \in \text{Act}(I)} \lim_{\alpha \in G_{n/2}} X(J_\alpha).$$

These limits over $G_{n/2}$ can be rewritten as iterated pullbacks over objects in $U$, and by Lemma 3.5 we have that $\text{Act}(I)$ is equivalent to $\lim_{\alpha: C \to I \in G_{n/2}} \text{Act}(C)$. Applying Lemma 2.15 iteratively we can then conclude that the natural map

$$\coprod_{I \to J \in \text{Act}(I)} \lim_{\alpha \in G_{n/2}} X(J_\alpha) \to \lim_{\alpha: C \to I \in G_{n/2}} \coprod_{C \to J_n} X(J_n)$$

is an equivalence. Here the target is equivalent to $\lim_{\alpha: C \to I \in G_{n/2}} i_n ! X(C)$, i.e. $i_n ! X$ satisfies the Segal condition. The expression for $F_n X(C_i)$ is then immediate from Lemma 3.4.

\[ \square \]

4. Comparison

Our goal in this section is to prove our comparison result. More precisely, we will show:

**Theorem 4.1.** Let $\tau_{1,n}: \Delta \times \Theta_n \to \Theta_{n+1}$ be the functor determined by sending $([n], I)$ to $[n](I, \ldots, I)$. Then composition with $\tau_{1,n}$ induces, for $(X, U)$ a presentable $\infty$-category with good constants, an equivalence

$$\tau_{1,n} ^*: \mathcal{P}_{rSeg}(\Theta_{n+1}; X, U) \simeq \mathcal{P}_{rSeg}(\Delta \times \Theta_n; X, U).$$

Iterating this result, we get:

**Corollary 4.2.** Let $\tau_{k,n}: \Delta^k \times \Theta_n \to \Theta_{n+k}$ be defined inductively as

$$\Delta^k \times \Theta_n \xrightarrow{id_k \times \tau_{k-1,n}} \Delta \times \Theta_{n+k-1} \xrightarrow{\tau_{1,n+k-1}} \Theta_{n+k}.$$
Then for \((X, U)\) any presentable \(\infty\)-category with good constants the functor

\[
\tau^*_n: \mathcal{P}_{rsSeg}(\Delta^k \times \Theta_n; X, U) \to \mathcal{P}_{rsSeg}(\Theta_{n+k}; X, U)
\]

is an equivalence.

In particular, taking \(X\) to be an \(\infty\)-topos and \(k = 0\) we get an equivalence between the \(\infty\)-category \(\mathcal{P}_{rsSeg}(\Delta^n; X)\) of \(n\)-fold Segal spaces in \(X\) and the \(\infty\)-category \(\mathcal{P}_{rsSeg}(\Theta_n; X)\) of Segal \(\Theta_n\)-objects in \(X\).

**Remark 4.3.** Similarly, applying Theorem 4.1 inductively we get for any sequence of positive integers \((n_1, \ldots, n_k)\) an equivalence between \(\mathcal{P}_{rsSeg}(\Theta_{n_1} \times \cdots \times \Theta_{n_k}; X, U)\) and \(\mathcal{P}_{rsSeg}(\Theta_{n_1+\cdots+n_k}; X, U)\).

To prove Theorem 4.1, we will use the following analogue of Proposition 3.3:

**Proposition 4.4.**

(i) Let \(i_{1,n} := i_1 \times i_n: \Delta_1 \times \Theta_{n-1} \to \Delta \times \Theta_n\). The functor

\[
i_{1,n}^*: \mathcal{P}_{rsSeg}(\Delta \times \Theta_n; X, U) \to \mathcal{P}_{rsSeg}(\Delta_1 \times \Theta_{n-1}; X, U)
\]

has a left adjoint \(F_{1,n}\).

(ii) The adjunction \(F_{1,n} \dashv i_{1,n}^*\) is monadic.

(iii) The monad \(T_{1,n} := i_{1,n}^* F_{1,n}\) on \(\mathcal{P}_{rsSeg}(\Delta^n \times \Theta_{n-1}; X, U)\) satisfies

\[
T_{1,n} X([0], C_k) \simeq X([0], C_k),
\]

\[
T_{1,n} X([1], C_k) \simeq \prod_{j=0}^\infty F_n X(C_k) \times X([0], C_0) \cdots \times X([0], C_0) F_n \tilde{X}(C_k),
\]

where \(\tilde{X} := X([1], \cdot)\) and the factor \(F_n \tilde{X}(C_k)\) occurs \(j\) times.

For the proof we need the following observation:

**Lemma 4.5.** Suppose \(L: \mathcal{C} \rightleftarrows \mathcal{D}: R\) is an adjunction. Then for any \(d \in \mathcal{D}\) there is an adjunction

\[
L_d: \mathcal{C}/Rd \rightleftarrows \mathcal{D}/d: R_d,
\]

where \(L_d(x \to Rd)\) is the composite \(Lx \to LRd \to d\) using the counit, and \(R_d(y \to d)\) is \(Ry \to Rd\).

**Proof.** Let \(\eta: LR \to id\) be the counit for the adjunction. This determines a natural transformation \(\eta_d: LdRd \to id\), and the map

\[
\text{Map}_{\mathcal{D}/d}(x, Rdy) \to \text{Map}_{\mathcal{C}/Rd}(Ldx, LdRdy) \to \text{Map}_{\mathcal{D}/d}(Ldx, y)
\]

is the map on fibres at \(x \to Rd\) of the commutative square

\[
\begin{array}{ccc}
\text{Map}_{\mathcal{D}}(x, Ry) & \xrightarrow{id} & \text{Map}_{\mathcal{C}}(Lx, y) \\
\downarrow & & \downarrow \\
\text{Map}_{\mathcal{D}}(x, Rd) & \xrightarrow{\eta_d} & \text{Map}_{\mathcal{C}}(Lx, d)
\end{array}
\]

induced by \(\eta\). Here both horizontal maps are equivalences, since \(\eta\) is the counit of the adjunction \(L \dashv R\), hence so is the map on fibres. The natural transformation \(\eta_d\) is therefore the counit of an adjunction \(L_d \dashv R_d\) by [Lur09a, Proposition 5.2.2.8].

\[\square\]
Proof of Proposition 4.4. The functor $i_{1,n} : \Delta_i \times \Theta_{n,i} \to \Delta \times \Theta_n$ factors as the composite of inclusions $i'_{1,n} := \text{id} \times i_n : \Delta_i \times \Theta_{n,i} \to \Delta_i \times \Theta_n$ and $i''_{1,n} := i_1 \times \text{id} : \Delta_1 \times \Theta_n \to \Delta \times \Theta_n$. Here $(i''_{1,n})^*$ is just $i_1^*$ applied to the presentable category with good constants $(\mathcal{P}_{rSeg}(\Theta_n; X, \mathcal{U}))_n$, so by Proposition 3.3 it has a left adjoint, given by $i_{1,\ast}$.

In the diagram

$$\begin{array}{ccc}
\mathcal{P}_{rSeg}(G_1 \times \Theta_n; X, \mathcal{U}) & \xrightarrow{(i'_{1,n})^*} & \mathcal{P}_{rSeg}(G_1 \times \Theta_{n,i}; X, \mathcal{U}) \\
\downarrow & & \downarrow \\
\mathcal{U}, & & \mathcal{U},
\end{array}$$

the diagonal maps, given by evaluation at $C_0$, are Cartesian fibrations by [Hau14b, Corollary 4.52], the functor $(i'_{1,n})^*$ preserves Cartesian morphisms, and by [Hau14a, Lemma 6.4] the morphism on fibres at $U \in \mathcal{U}$ is the functor

$$\mathcal{P}_{rSeg}(\Theta_n; X, \mathcal{U})_{/U^n \times U} \to \mathcal{P}_{rSeg}(\Theta_{n,i}; X, \mathcal{U})_{/U^n \times U}$$
given by composing with $i_n$, where $U \times U$ denotes the constant presheaf with this value.

By Lemma 4.5 the functor $(i'_{1,n})^*$ therefore has a left adjoint on the fibre over each $U \in \mathcal{U}$, given by applying $F_n$ and composing with the counit map to the constant presheaf. By [Lur14, Proposition 7.3.2.5] this implies that $(i''_{1,n})^*$ has a left adjoint globally, giving (i).

(ii) now follows by the same argument as in the proof of Proposition 3.3(ii), and (iii) by Proposition 3.3 and our description of the left adjoints to $(i'_{1,n})^*$ and $(i''_{1,n})^*$.

Proof of Theorem 4.1. We first check that $\tau_{1,n}$ takes reduced Segal $\Theta_{n+1}$-objects to reduced Segal $\Delta \times \Theta_n$-objects. For $X \in \mathcal{P}_{rSeg}(\Theta_{n+1}; X, \mathcal{U})$, we want to show

1. $(\tau_{1,n}^*X)(\{0\}, -)$ is constant and lies in $\mathcal{U}$,
2. $(\tau_{1,n}^*X)(\{1\}, -) \times (\tau_{1,n}^*X)(\{0\}, -) \times \cdots \times (\tau_{1,n}^*X)(\{0\}, -)$ is an equivalence,
3. $(\tau_{1,n}^*X)(\{0\}, -)$ is a reduced Segal $\Theta_n$-object.

The functor $\tau_{1,n}(\{0\}, -)$ is constant at $[0]$, which proves (1). Next, the Segal condition for $X$ implies that $X([n](I_1, \ldots, I_n)) \simeq X([n](1)) \times X([n](0)) \times \cdots \times X([n](0)) X([1](I_n))$, which in particular proves (2). Finally, $(\tau_{1,n}^*X)(\{1\}, -)$ is a reduced Segal $\Theta_{n-1}$-object. We then have a commutative square

$$\begin{array}{ccc}
\mathcal{P}_{rSeg}(\Theta_{n+1}; X, \mathcal{U}) & \xrightarrow{\tau_{1,n}} & \mathcal{P}_{rSeg}(\Delta \times \Theta_n; X, \mathcal{U}) \\
\downarrow \quad \tau_{n+1}^* & & \downarrow \quad \tau_{1,n}^* \\
\mathcal{P}_{rSeg}(\Theta_{n+1,i}; X, \mathcal{U}) & \xrightarrow{\tau_{1,n,i}^*} & \mathcal{P}_{rSeg}(\Delta_i \times \Theta_{n,i}; X, \mathcal{U}).
\end{array}$$

Let us next show that the bottom horizontal map here is an equivalence. The functor $\tau_{1,n}$ restricts to a functor $\beta_n : G_1 \times G_n \to G_{n+1}$ which sends $(C_0, C_i)$ to $C_0$ and $(C_1, C_i)$ to $C_{i+1}$. We also define a functor $\alpha_n : G_{n+1} \to G_1 \times G_n$ by sending $C_i$ to $(C_1, C_{i-1})$ for $i \geq 1$ and $C_0$ to $(C_0, C_0)$, with $s_i$ and $t_i$ going to $s_{i-1}$ and $t_{i-1}$.
on the second factor for \(i > 1\) and to \(s_1\) and \(t_1\) on the first factor for \(i = 1\); then \(\beta_n \circ \alpha_n \simeq \text{id}\). If we let \(\gamma_{1,n} : G_{n+1} \to \Delta \times \Theta_n\) denote the composite \((\gamma_1 \times \gamma_n) \circ \alpha_n\), then we have a commutative triangle

\[
\begin{array}{ccc}
P_{\text{Seg}}(\Theta_{n+1}; X, U) & \xrightarrow{\tau_{1,n,1}} & P_{\text{Seg}}(\Delta_1 \times \Theta_n; X, U) \\
\gamma_{n+1}^* & & \gamma_{1,n}^* \\
P_{1}(G_{n+1}; X, U). & & 
\end{array}
\]

The inclusion \(\gamma_{n+1} : G_n \to \Theta_{n+1; i}\) induces an equivalence \(P_{\text{Seg}}(\Theta_{n+1}; X, U) \xrightarrow{\sim} P_{1}(G_{n+1}; X, U)\), since by definition \(P_{\text{Seg}}(\Theta_{n+1}; X, U)\) is the subcategory of \(P(\Theta_{n+1}; X)\) spanned by the presheaves that are right Kan extensions along \(\gamma_{n+1}\) of the objects of \(P_{1}(G_{n+1}; X, U)\). By the 2-out-of-3 property to see that \(\tau_{1,n,1}\) is an equivalence it then suffices to show that the restriction

\(\gamma_{1,n}^* : P_{\text{Seg}}(\Delta_1 \times \Theta_n; X, U) \to P_{1}(G_{n+1}; X, U)\)

is an equivalence. Observe that \(P_{\text{Seg}}(\Delta_1 \times \Theta_n; X)\) is the subcategory of \(P(\Delta_1 \times \Theta_n; X)\) spanned by those presheaves that are right Kan extensions along \(\gamma_1 \times \gamma_n\) of presheaves on \(G_1 \times G_n\). Thus, the restriction \((\gamma_1 \times \gamma_n)^* : P_{\text{Seg}}(\Delta_1 \times \Theta_n; X) \to P(G_1 \times G_n; X)\) is an equivalence. Now note that \(\beta_n^* : P(G_{n+1}; X) \to P(G_1 \times G_n; X)\) is fully faithful (since \(\alpha_n^* \circ \beta_n^* \simeq \text{id}\)) and \(P_{\text{Seg}}(\Delta_1 \times \Theta_n; X, U)\) is precisely the full subcategory of \(P_{\text{Seg}}(\Delta_1 \times \Theta_n; X)\) whose image in \(P(G_1 \times G_n; X)\) lies in the image under \(\beta_n^*\) of \(P_{1}(G_{n+1}; X, U)\).

The vertical maps in the commutative square above are monadic right adjoints by Proposition 3.3 and Proposition 4.4. To see that \(\tau^*\) is an equivalence it then suffices, by [Lur14, Corollary 4.7.4.16], to show that for every \(X \in P_{1}(G_{n}) \simeq P_{\text{Seg}}(\Theta_{n+1,i})\) the unit map \(X \to \iota_{n+1}^* F_{n+1} \simeq \iota_{1,n}^* F_{n+1}\) induces an equivalence \(F_{n+1}^* X \xrightarrow{\sim} \tau_{1,n}^* F_{n+1} X\), or (since \(\iota_{1,n}^*\) detects equivalences) the induced map \(\left(F_{1,n} X\right)(C_k) \to \left(F_{n+1}^* X\right)(C_k)\) is an equivalence for \(k = 0, \ldots, n + 1\).

To prove this we will rewrite our expression for \((F_{n+1} X)(C_k)\) from Proposition 3.3, which says

\[F_{n+1} X(C_k) \simeq \prod_{I \in \text{ob} \Theta_k} i_k^{n+1,*} X(I).
\]

Let \((\text{ob} \Theta_k)_j\) denote the subset of \(\text{ob} \Theta_k\) consisting of objects of the form \([j](\cdots)\). Every object \(I \in (\text{ob} \Theta_k)_j\) admits an obvious active map \([j]([C_1, \ldots, C_{k-1}]) \to I\), and using this we have by Lemma 2.9 an equivalence

\[i_k^{n+1,*} X(I) \simeq i_k^{n+1,*} X(I_0) \times i_k^{n+1,*} X(C_0) \times \cdots \times i_k^{n+1,*} X(C_{k-1}) \simeq i_k^{n+1,*} X(I_j).
\]

Here each \(I_j\) is of the form \(\sigma_k I_j\), where \(\sigma_k : \Theta_{k-1} \to \Theta_k\) is the functor \([1](\cdot)\), and using \(\sigma_k\) we get a bijection \((\text{ob} \Theta_k) \cong (\text{ob} \Theta_{k-1})^j\). Since coproducts over \(U\) are universal, we can rewrite our expression for \(F_{n} X(C_k)\) as

\[
\prod_{j=0}^{\infty} \left( \prod_{I \in \Theta_{k-1}} i_k^{n+1,*} X(\sigma_k I_j) \right) \times X(C_0) \times \cdots \times X(C_{k-1}) \left( \prod_{I \in \Theta_{k-1}} i_k^{n+1,*} X(\sigma_k I_j) \right).
\]

Here, as \(i_k^{n+1} \circ \sigma_k = \sigma_{n+1} i_{k-1}^{n+1}\), we have equivalences

\[
\prod_{I \in \Theta_{k-1}} i_k^{n+1,*} X(\sigma_k I_j) \simeq \prod_{I \in \Theta_{k-1}} i_k^{n,*} (\sigma_{n+1} X(\sigma_k I_j)) \simeq F_n \left( \sigma_{n+1} X(\sigma_k I_j) \right) \simeq F_{n+1} X(C_k).
\]

Comparing this to the expression for \(F_{1,n}\) in Proposition 4.4 then completes the proof. \(\Box\)
References

[Bar05] Clark Barwick, \((\infty, n)\)-Cat as a closed model category, 2005. Thesis (Ph.D.)–University of Pennsylvania.
[Bar13] , From operator categories to topological operads (2013), available at arXiv:1302.5756.
[BSP11] Clark Barwick and Christopher Schommer-Pries, On the unicity of the homotopy theory of higher categories (2011), available at arXiv:1112.0040.
[Ber07] Clemens Berger, Iterated wreath product of the simplex category and iterated loop spaces, Adv. Math. 213 (2007), no. 1, 230–270.
[BR14] Julia E. Bergner and Charles Rezk, Comparison of models for \((\infty, n)\)-categories II (2014), available at arXiv:1406.4182.
[Hau14a] Rune Haugseng, Iterated spans and “classical” topological field theories (2014), available at arXiv:1409.0837.
[Hau14b] , The higher Morita category of \(E_n\)-algebras (2014), available at arXiv:1412.8459.
[Hau16] , Bimodules and natural transformations for enriched \(\infty\)-categories, Homology Homotopy Appl. 18 (2016), 71–98, available at arXiv:1506.07341.
[Lur09a] Jacob Lurie, Higher Topos Theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available at http://math.harvard.edu/~lurie/papers/highertopoi.pdf.
[Lur09b] , \((\infty, 2)\)-Categories and the Goodwillie Calculus I (2009), available at http://math.harvard.edu/~lurie/papers/GoodwillieI.pdf.
[Lur14] , Higher Algebra, 2014. Available at http://math.harvard.edu/~lurie/papers/higheralgebra.pdf.
[Rez01] Charles Rezk, A model for the homotopy theory of homotopy theory, Trans. Amer. Math. Soc. 353 (2001), no. 3, 973–1007 (electronic).
[Rez10] , A Cartesian presentation of weak \(n\)-categories, Geom. Topol. 14 (2010), no. 1, 521–571.

Københavns Universitet, Copenhagen, Denmark
E-mail address: haugseng@math.ku.dk
URL: http://sites.google.com/site/runehaugseng/