Geometrical trinity of unimodular gravity

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Abstract
We construct a Weyl transverse diffeomorphism invariant theory of teleparallel gravity by employing the Weyl compensator formalism. The low-energy dynamics has a single spin two graviton without a scalar degree of freedom. By construction, it is equivalent to unimodular gravity (as well as Einstein’s general relativity with an adjustable cosmological constant) at the non-linear level. Combined with our earlier construction of a Weyl transverse diffeomorphism invariant theory of symmetric teleparallel gravity, unimodular gravity is represented in three alternative ways.

Keywords: unimodular gravity, symmetric teleparallel gravity, teleparallel gravity, general relativity

1. Introduction

The realization of massless particles with higher spin in field theories necessary involves redundancy in their description because the number of components in fields is much larger than the physical degrees of freedom [1]. In order to remove the redundancy, we typically assume a gauge symmetry of the action and employ the gauge principle. The necessary gauge symmetry may not be unique. In the field theory with massless spin two particles, there are two minimal possibilities: the one is to use the diffeomorphism and the other is to use the Weyl transverse diffeomorphism [2].

The former description of the massless spin-two particles is well-known. It gives Einstein’s general relativity at the non-linear level [3]. The latter description of the massless spin-two particles seems less known, but it gives unimodular gravity at the non-linear level. It turns out that, at least classically, both descriptions are equivalent as we will review in the main text (see e.g. [4, 5] for a review).

Einstein’s general relativity and unimodular gravity are based on the (pseudo) Riemannian geometry, where the diffeomorphism (as well as Weyl symmetry) is naturally defined. It

1 The Weyl transverse diffeomorphism is a combination of the Weyl transformation of the metric and the diffeomorphism whose Jacobian is unity.
is, however, possible to reformulate the gravitational theory in terms of other geometries. For example, it is possible to formulate the equivalent gravitational theory in teleparallel geometry (e.g. [6–8] and reference therein) or symmetric teleparallel geometry [9, 10], where the connection is torsional or metric incompatible respectively. While the appearance is different, the dynamical contents turn out to be equivalent by choosing an appropriate action [8, 11–14].

In this paper, we construct a Weyl transverse diffeomorphism invariant theory of teleparallel gravity by employing the Weyl compensator formalism. We show that it is equivalent to unimodular gravity (as well as Einstein’s general relativity with an adjustable cosmological constant) at the non-linear level. Combined with our earlier construction of a Weyl transverse diffeomorphism invariant theory of symmetric teleparallel gravity [15], unimodular gravity is represented in three alternative ways. Our systematic construction offers a unified framework to discuss various representations of massless spin-two particles with different gauge symmetry.

The organization of our paper is as follows. In section 2, we construct a Weyl transverse diffeomorphism invariant theory of teleparallel gravity and show it is equivalent to unimodular gravity. In section 3, we first review a Weyl transverse diffeomorphism invariant theory of symmetric teleparallel gravity and offer a unified framework to discuss these theories. In section 4, we conclude with some discussions.

2. Weyl transverse diffeomorphism invariant theory of teleparallel gravity

Teleparallel gravity is formulated in terms of a metric-compatible but torsional flat connection $\Gamma^\rho_{\mu\nu}$ (where no symmetry of the lower indices is assumed). In order to understand the dynamical degrees of freedom, it is convenient to use the tetrad formalism by introducing the tetrad $e^a_{\mu}$ and an independent spin connection $\omega^a_{\mu\nu}$. Throughout the paper, Latin indices refer to tangent space indices while Greek indices refer to space-time indices. The metric can be reconstructed from the tetrad as

$$g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu},$$

where $\eta_{ab} = \text{diag}(−1, 1, 1, 1)$.

The two connections are related through the tetrad compatibility condition:

$$\partial_{\nu} e^a_{\mu} + \omega^a_{\mu b} e^b_{\nu} - \Gamma^b_{\rho\mu} e^a_{\nu} = 0,$$

(1)

Thanks to the tetrad compatibility condition, the tensor indices are raised, lowered and converted by using the metric $g_{\mu\nu}$, the inverse metric $g^{\mu\nu}$, the tetrad $e^a_{\mu}$ and the inverse tetrad $e^a_{\mu}$ as usual in the general relativity. The curvature tensor is given by

$$R^a_{\mu b\nu} = \partial_{\nu} \omega^a_{\mu b} - \partial_{\nu} \omega^a_{b\mu} + \omega^e_{e\mu} \omega^a_{b\nu} - \omega^e_{e\nu} \omega^a_{b\mu},$$

(2)

and we assume that it vanishes in teleparallel geometry.

Instead of the curvature, in teleparallel gravity, we assume that the gravitational force is propagated by a torsion. In terms of the spin connection and the tetrad, the torsion tensor is defined by

$$T^a_{\mu b} = \partial_{\nu} e^a_{\mu} - \partial_{\nu} e^a_{\mu} + \omega^a_{b\mu} e^b_{\nu} - \omega^b_{b\nu} e^a_{\mu}.$$

(3)

Due to (1), it is equivalent to the anti-symmetric part of the connection $\Gamma^a_{\mu b} e^b_{\nu}$. If the tangent space index is replaced by the space-time index by the tetrad, the torsion tensor $T^a_{\mu b} = e^a_{\mu} T^b_{\mu b}$ is invariant under the local Lorentz transformation. We also define $T_{\nu} = e^a_{\nu} T^a_{\mu b}$ for later notational convenience. The assumption of the flat connection (i.e. vanishing of the curvature tensor) means that we can (locally) choose the Weitzenböck gauge by a local Lorentz transformation. We also define $T_{\nu} = e^a_{\nu} T^a_{\mu b}$ for later notational convenience.
Let us introduce a particular quadratic combination of the torsion tensor
\[ T_E = \frac{1}{4} T_{\mu\nu\rho} T^{\mu\nu\rho} + \frac{1}{2} T_{\mu\nu\rho} T^{\mu
u\rho} - T_\mu T^\mu. \]  
(4)

With or without imposing the Weitzenböck gauge, vanishing of the curvature tensor leads to the identity for the Ricci scalar \( \mathcal{R} \), which is constructed out of the metric through the Christoffel connection \( \Gamma^{\alpha}_{\mu\nu} \):
\[ \mathcal{R} = -g^{\mu\nu} \left( \partial_\lambda \left\{ \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right\} - \left\{ \Gamma^\sigma_{\mu\nu} \right\} \left\{ \Gamma^\lambda_{\sigma\nu} \right\} - \left\{ \Gamma^\sigma_{\nu\mu} \right\} \left\{ \Gamma^\lambda_{\sigma\mu} \right\} \right) = T_E - 2e^{-1} \partial_\mu (e T^\mu), \]
(5)
where \( e = g^{\frac{1}{2}} = \sqrt{\text{det} g_{\mu\nu}} \). In other words, \( T_E \) is independent of \( \omega_{\mu}^a \) up to total derivatives under the flatness condition, and it fixes the transformation law. In practice, we can use the decomposition of the torsion tensor (3) in the Weitzenböck gauge and study its Weyl variation:
\[ \delta_w T_{\mu\nu}^a = \partial_\mu \delta_w e^a_\nu - \partial_\nu \delta_w e^a_\mu = \sigma T_{\mu\nu}^a + (\partial_\mu \sigma) e^a_\nu - (\partial_\nu \sigma) e^a_\mu. \]
(6)
Equivalently, we have assumed \( \delta_w \omega_{\mu}^a = 0 \) in more general gauge. Note that under this assumption, the curvature tensor (2) is invariant under the Weyl transformation (unlike the Riemann curvature tensor), and the flatness condition is compatible with the Weyl symmetry.

Accordingly, the quadratic scalars constructed out of the torsion tensor transform as
\[
\begin{align*}
\delta_w (T_{\mu\nu\rho}^w T^{\mu\nu\rho}) &= -2\sigma (T_{\mu\nu\rho}^w T^{\mu\nu\rho}) - 4T^\nu \partial_\nu \sigma \\
\delta_w (T^{\mu\nu\rho} T_{\nu\mu\rho}) &= -2\sigma (T^{\mu\nu\rho} T_{\nu\mu\rho}) - 2T^\nu \partial_\nu \sigma \\
\delta_w (T^{\mu} T_{\mu}) &= -2\sigma (T^\nu T_{\nu}) - 6T^\nu \partial_\nu \sigma
\end{align*}
\]
(7)
under the infinitesimal Weyl transformation (in four dimensions). We also recall that the Ricci scalar transforms under the Weyl transformation as
\[ \delta_w \mathcal{R} = -2\sigma \mathcal{R} - 6\Box \sigma \]
(8)
in four dimensions. Here the connection used in the Laplacian \( \Box \) is the Christoffel connection.

To construct a Weyl invariant action, we introduce the Weyl compensator \( \phi \), which transforms under the infinitesimal Weyl transformation as
\[ \delta_w \phi = -\sigma \phi. \]
(9)

\[ ^2 \text{In the literature, it is also called `conformal transformation', but we preferably use the word Weyl transformation in this paper. It is also different from the scale transformation [16].} \]
The compensator is a scalar under diffeomorphism. It will be non-dynamical, and we use it to fix a part of the gauge symmetry.

### 2.1. Equivalence to Einstein’s general relativity

Let us now construct the Weyl invariant action to see the equivalence to Einstein’s general relativity. We regard the Weyl symmetry as a gauge symmetry in the sense that the solutions related by the Weyl transformation are physically equivalent. We also demand the diffeomorphism and the Local Lorentz symmetry as gauge symmetries. Up to two derivatives, the candidate action is

\[
S_W = \int d^4x e \phi^2 \left( AT^{\mu\nu T}_{\mu\nu\rho} + BT^{\mu\nu T}_{\nu\mu\rho} + CT^\nu T_\mu \right) \\
+ \eta \xi g^{\mu\nu} \left( \partial_\mu - \frac{1}{3} T_\mu \right) \phi \left( \partial_\nu - \frac{1}{3} T_\nu \right) \phi + \epsilon \phi^4.
\] (10)

The Weyl invariance demands \(2A + B + 3C = 0\), but otherwise we seem to have three arbitrary parameters (up to overall factors that can be absorbed by a definition of \(\phi\)).

In order to construct a consistent field theory of a massless spin-two particle, however, we need to choose a particular parameter set \(A = -\frac{1}{4}, B = -\frac{1}{2}, C = \frac{1}{3}, \xi = 6\) (up to overall rescaling). This choice was made in [17], and it leads to the teleparallel equivalent of general relativity by fixing the Weyl gauge symmetry with the condition \(\phi = 1\). Indeed, if we substitute \(\phi = 1\) into (10), the action becomes

\[
S_E = \int d^4x (-T_E + \lambda) \\
\int d^4x (R + \lambda)
\] (11)

due to the identity (5) (up to surface terms). If we take \(e_\mu^a\) as a dynamical variable, we reproduce the Einstein–Hilbert action with a cosmological constant.

Why do we have to take this particular parameter set among other Weyl invariant sets? There are several different viewpoints about this particular choice of action.

If we use the Weitzenböck gauge \(\omega^a_{\mu\nu} = 0\), the local Lorentz symmetry was already fixed, and there is generically no reason to expect that the action written in terms of the tetrad (in the Weitzenböck gauge) is invariant under the local Lorentz transformation any longer. Nevertheless, the particular choice above leads to the emergent local Lorentz symmetry (up to a surface term) even in the Weitzenböck gauge. This emergent local Lorentz symmetry can be used to remove unphysical degrees of freedom and achieve the equivalence to Einstein’s general relativity. It is this emergent Lorentz symmetry what we observe in our lab experiments of special relativity.

If we do not use the Weitzenböck gauge, the relevance of this particular parameter choice can be seen as a decoupling of the spin connection from the action (again up to surface terms) when the total curvature tensor (2) vanishes. Thus, the spin connection is not a dynamical degree of freedom in this particular theory, and the dynamical contents become identical to Einstein’s general relativity (by discarding the spin connection that does not appear in the action).
2.2. Equivalence to the unimodular gravity

So far, we have shown that our theory is equivalent to the Einstein gravity. From now on, we would like to construct a Weyl transverse diffeomorphism invariant theory of teleparallel gravity, which is our first main result of this paper. The transverse diffeomorphism is a subset of the diffeomorphism whose Jacobian is unity: \( \det(\partial \mu \xi^\nu) = 1 \). When it is also invariant under the Weyl symmetry, such a gravitational theory describes a massless spin-two particle at the linearized level. We will show the consistency at the non-linear level by showing the equivalence to unimodular gravity (and Einstein’s general relativity with an adjustable cosmological constant).

The starting point is the action given by (10), which is invariant under full diffeomorphism and Weyl transformation. Instead of fixing the Weyl symmetry, we are going to fix the volume-changing part of the diffeomorphism by setting \( \phi = e^{-z} \). The resulting action becomes

\[
S_U = \int d^4x e^{\frac{1}{2}} \left( -\frac{1}{4} T^\mu\nu T_{\mu\nu\rho} - \frac{1}{2} T^\mu\nu T_{\rho\mu\nu} + \frac{1}{3} T^\nu T_{\mu} + e6g^\mu\nu \left( \partial_\mu - \frac{1}{3} T_{\mu} \right) e^{-\frac{z}{2}} + \lambda \right).
\]

Note that \( \lambda \) term is just constant (without depending on \( e^\mu \), unlike the cosmological constant in the Einstein–Hilbert action) and we may simply discard it as a classical action.

By using identity (5) and integration by part, we can further rewrite the action into the form:

\[
S_U = \int d^4x e^{\frac{1}{2}} \left( \mathcal{R} + \frac{3}{32} g^{\mu\nu} \partial_\mu g \partial_\nu g \right),
\]

where we emphasize again that \( \mathcal{R} \) is the standard Ricci scalar constructed out of metric tensor through the Christoffel connection. This action is invariant under the Weyl transformation and the transverse diffeomorphism (i.e. diffeomorphism whose Jacobian is unity). One may further fix the Weyl symmetry by setting e.g. \( e = g^\frac{1}{2} = 1 \), but we opt not to (because the variation of \( g_{\mu\nu} \) becomes non-trivial after imposing the non-linear constraint such as \( e = g^\frac{1}{2} = 1 \)). It describes unimodular gravity by regarding \( e^\mu \) as dynamical variable \( \eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) (so that \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \)). Due to the (emergent) local Lorentz symmetry, only the symmetric part of \( h_{\mu\nu} \) contributes to the action. Explicitly, the linearized action is given by

\[
S_U = \int d^4x \left( -\frac{1}{4} \partial_\mu h_{\mu\nu} \partial^\nu h^{\mu\nu} + \frac{1}{2} \partial_\mu \eta_{\mu\nu} \partial^\alpha \eta_{\mu\alpha} \frac{1}{4} \partial_\mu h \partial_\nu \partial^\mu h + \frac{3}{32} \partial_\mu h \partial_\nu h \right),
\]

where \( h_{\mu\nu} = h_{\nu\mu} \) is assumed, and \( h = \eta^{\mu\nu} h_{\mu\nu} \). The indices are raised and lowered by the Minkowski metric \( \eta_{\mu\nu} \) (only) here. We see that the linearized action can be expressed only in terms of the traceless mode \( h_{\mu\nu} = h_{\mu\nu} - \frac{\eta_{\mu\nu}}{4} h \) and the linearized Weyl symmetry, a shift of the trace part of \( h_{\mu\nu} \) becomes obvious. It is invariant under the transverse diffeomorphism \( \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \) with \( \partial_\mu \xi^\mu = 0 \). As is known in the literature (e.g. see [2]), it describes a massless spin-two graviton without a scalar degree of freedom.

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3 We note that there is an alternative approach to the unimodular gravity by introducing the condition \( \sqrt{-g} = 1 \) by a Lagrange multiplier. At the end of the computation, the equations of motion agree. See e.g. [25] for the related approach in the context of teleparallel geometry.
3. Geometrical trinity of unimodular gravity

In the previous section, we have constructed the teleparallel equivalent of the unimodular gravity. In our earlier work [15], we have constructed the symmetric teleparallel equivalent of unimodular gravity. The situation is similar to the geometrical trinity of the general relativity presented in [11, 12]. Our two constructions are both equivalent to unimodular gravity, and we may declare that we have completed the geometrical trinity of unimodular gravity.

To make our presentation self-contained, let us briefly summarize the symmetric teleparallel equivalent of unimodular gravity and then offer a unified framework. In the symmetric teleparallel gravity, we assume that the connection is flat and torsion-free but is not compatible with the metric. The dynamical degrees of freedom are encoded in the non-metricity tensor $Q_{\rho\sigma} = \nabla^\mu g_{\rho\sigma}$. The analogue of the crucial identity (5) is

\[-R = \mathcal{R}_E + g^{-\frac{1}{2}} \partial_\mu \left( g^{\frac{1}{2}} Q^\mu - g^{\frac{1}{2}} \bar{Q}^\mu \right),\]

\[Q_E = \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\alpha\nu} - \frac{1}{4} Q_{\alpha} Q^{\alpha} + \frac{1}{2} Q_{\alpha} \bar{Q}^{\alpha},\]  

(15)

where $\mathcal{R}$ is the Ricci scalar constructed out of the metric tensor through the Christoffel connection. Here we have introduced $Q_{\mu} = Q^{\alpha}_{\alpha\mu}$ and $\bar{Q}_{\mu} = Q^{\alpha}_{\alpha\mu}$.

Since the symmetric teleparallel connection is flat, we can always (at least locally) choose the so-called coincident gauge [10, 26], where $Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}$. Sticking with this gauge, one can define the infinitesimal Weyl transformation [15] (see [27, 28] as well):

\[\delta_W g_{\mu\nu} = 2\sigma g_{\mu\nu},\]

\[\delta_W g = 8\sigma g,\]

\[\delta_W Q_{\alpha\mu\nu} = 2g_{\mu\nu} g^{\alpha\beta} \partial_\beta \sigma,\]

\[\delta_W Q_{\mu} = 8\partial_\mu \sigma,\]

\[\delta_W \bar{Q}_{\mu} = 2\partial_\mu \sigma,\]

\[\delta_W \phi = -\sigma \phi.\]  

(16)

With these transformations in mind, we may start with the Weyl invariant action

\[S_W = \int d^4x \sqrt{g} \left( -\phi^2 (Q_E + g^{-\frac{1}{2}} \partial_\mu \left( g^{\frac{1}{2}} Q^\mu - g^{\frac{1}{2}} \bar{Q}^\mu \right)) + 6g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda \phi^4 \right).\]  

(17)

If we fix the Weyl symmetry by setting $\phi = 1$, we obtain the Einstein–Hilbert action with a cosmological constant. Instead, we now fix the volume-changing part of the diffeomorphism by setting $\phi = g^{-\frac{1}{2}}$. The resulting Weyl transverse diffeomorphism invariant action is

\[S_U = \int d^4x g^{\frac{1}{2}} (-Q_W),\]  

(18)

where

\[Q_W = c_1 Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + c_2 Q_{\alpha\mu\nu} Q^{\mu\alpha\nu} + c_3 Q_{\alpha} Q^{\alpha} + c_4 \bar{Q}_{\alpha} \bar{Q}^{\alpha} + c_5 \bar{Q}_{\alpha} Q^{\alpha},\]  

(19)

with $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, $c_3 = -\frac{1}{2}$, $c_4 = 0$ and $c_5 = \frac{1}{2}$. As in the previous section, the $\lambda$ term is dropped because it does not affect the equations of motion. This action describes symmetric teleparallel equivalent of unimodular gravity. By construction it is equivalent to unimodular gravity or Einstein’s general relativity with an adjustable cosmological constant.

The teleparallel equivalent of unimodular gravity and the symmetric teleparallel equivalent of unimodular gravity can be unified in the following way. We now consider the most generic
connection which is flat but with torsion and non-metricity. Under the flatness condition, we have the identity

\[-\mathcal{R} = G_E + g^{-\frac{1}{2}} \partial_\mu \left( g^{\frac{1}{2}} (Q^\mu - \tilde{Q}^\mu - 2T^\mu) \right) \]

\[G_E = T_E + Q_E + Q_{\mu\nu\rho} T^{\mu\nu\rho} - \tilde{Q}_\mu T^\mu + \tilde{Q}_\mu T^\mu. \tag{20} \]

Starting with a particular form of the Weyl invariant action

\[S_W = \int d^4x \sqrt{g} \left( -\phi^2 \left( G_E + g^{-\frac{1}{2}} \partial_\mu \left( g^{\frac{1}{2}} (Q^\mu - \tilde{Q}^\mu - 2T^\mu) \right) \right) + 6g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda \phi^4 \right), \tag{21} \]

and fixing the gauge by setting \( \phi = g^{-\frac{1}{2}} \), we obtain

\[S_U = \int d^4x \sqrt{g} \left( \mathcal{R} + \frac{3}{32} g^{\mu\nu} \partial_\mu g \partial_\nu g \right). \tag{22} \]

Note that if we write the action in terms of \( g_{\mu\nu} \) (or \( e_\mu^a \)) by using (20), it is obvious that the resultant theory is unimodular gravity with the Weyl transverse diffeomorphism, but it is worthwhile mentioning what happens to the other degrees of freedom in the most generic (flat) connection. For this purpose, we can use the flatness condition to parameterize the torsion and the non-metricity as

\[T^\alpha_{\mu\nu} = (\Lambda^{-1})^\alpha_\mu (\partial_\mu \Lambda^\rho_\nu - \partial_\nu \Lambda^\rho_\mu) \]

\[Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu} - (\Lambda^{-1})^\lambda_\rho (\partial_\alpha \Lambda^\rho_{\mu\nu} + \partial_\rho \Lambda^\rho_{\mu\nu} g_{\lambda\nu}). \tag{23} \]

Importantly, \( \Lambda^\rho_\mu \) does not appear in the action given by (22) (up to surface terms), so one may claim that the degrees of freedom encoded in \( \Lambda^\rho_\mu \) is a ‘gauge’ degree of freedom. This is the enhanced symmetry, which makes the action (22) special compared with the other general quadratic action of the torsion and the non-metricity (under the additional Weyl symmetry).

One may use the enhanced symmetry to set either \( T^\alpha_{\mu\nu} = 0 \) or \( Q_{\alpha\mu\nu} = 0 \) and then we have the teleparallel equivalent of unimodular gravity and the symmetric teleparallel equivalent of unimodular gravity, and the construction here gives a unified framework. Note that the resultant theory still enjoys the Weyl transverse diffeomorphism necessary for unimodular gravity.

Let us briefly discuss the matter coupling in the (symmetric) teleparallel equivalent of unimodular gravity. It might occur to us that we can use the covariant derivative that are natural in each geometry (i.e. the connection with torsion or the connection with non-metricity). It turns out, however, that such a choice would not be equivalent to the conventional matter coupling in unimodular gravity; what is more worse, it would be inconsistent. The reason is that we have emphasized that it is important to retain emergent local Lorentz or diffeomorphism symmetry even after imposing the Weitzenböck gauge in teleparallel case or the coincident gauge in symmetric teleparallel case. Among many other possibilities, we have chosen the gravity action as (10) and (17) based on this requirement, and so should be in the matter action. We do not have to treat each case separately because the above construction gives the unified framework based on (21).

Let us take some examples motivated from the standard model of particle physics. We begin with the gauge field. The gauge kinetic term as well as theta terms are Weyl invariant in four-dimensions, so

\[S_{\text{gauge}} = \int d^4x e \left( -\frac{1}{4g^2} \text{Tr} F^{\mu\nu} F_{\mu\nu} + \theta \epsilon_{\mu\nu\rho\sigma} \text{Tr} F^{\mu\nu} F^{\rho\sigma} \right) \tag{24} \]

does not change after fixing the gauge by \( \phi = e^{-\frac{1}{2}} \). Here the field strength is defined by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \), and its Weyl weight is zero. In the Riemannian geometry, one may
replace the partial derivative with the covariant derivative in the definition of the field strength, but here we cannot. To state it more explicitly, the coupling to the connection, in particular torsional part, is not allowed.

Next, let us consider the scalar field. A canonical example is the Higgs field. We start with the Weyl invariant action

\[
S_{\text{scalar}} = - \int d^4x \left( g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{6} R \Phi^2 + \phi^4 V(\Phi/\phi) \right),
\]

where \( \delta_W \Phi = -\sigma \Phi \). Setting \( \phi = e^{-\frac{1}{4}} \) to fix the gauge, we obtain the matter action compatible with Weyl transverse diffeomorphism.

\[
S_{\text{scalar}} = - \int d^4x \left( g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{6} R \Phi^2 + e^{-1} V \left( \Phi e^{\frac{1}{4}} \right) \right).
\]

Note that the peculiar appearance of \( e \) in the potential is necessary for the consistency of the gravitational equations of motion in unimodular gravity.

One may wonder if the non-minimal coupling is fixed here, but it is not the case. The non-minimal coupling can be introduced if we add a more elaborate Weyl invariant action

\[
S_{\text{non-minimal}} = \int d^4x \xi \Phi \left( - \frac{\Phi^2}{\phi^2} \Box \phi + \frac{1}{6} R \Phi^2 \right).
\]

to the above. Clearly, if we set \( \phi = 1 \), we obtain the non-minimal coupling \( R \Phi^2 \) in Einstein’s general relativity. If we instead set \( \phi = e^{-\frac{1}{4}} \), we obtain the equivalent non-minimal coupling in unimodular gravity:

\[
S_{\text{non-minimal}} = \int d^4x \xi \Phi \left( - \Phi^2 e^{\frac{1}{4}} \Box e^{-\frac{1}{4}} + \frac{1}{6} R \Phi^2 \right).
\]

Finally, consider the Weyl invariant fermion action (where \( \delta_W \Psi = -\frac{3}{2} \sigma \Psi \))

\[
S_{\text{fermion}} = \int d^4x \left( i \bar{\psi} \gamma^\mu D_\mu \psi + \phi m \bar{\psi} \psi \right),
\]

where \( D_\mu = \partial_\mu + \omega_\mu^\nu \), and the spinor Lorentz connection here is the torsion-free metric-compatible connection constructed out of the tetrad. In the unimodular frame \( \phi = e^{-\frac{1}{4}} \), we obtain

\[
S_{\text{fermion}} = \int d^4x \left( i \bar{\psi} \gamma^\mu D_\mu \psi + e^{-\frac{1}{4}} m \bar{\psi} \psi \right).
\]

As we have emphasized, it is crucial to use the torsion-free metric-compatible connection here rather than the spin connection naturally defined in the (symmetric) teleparallel geometry.

The other interaction among various fields such as gauge interactions and Yukawa interactions can be introduced with no difficulty. We have therefore shown that all the known matter couplings of the standard model of particle physics can be realized in our geometrical trinity of unimodular gravity.

4. Discussions

In this paper, we have formulated the teleparallel equivalent of unimodular gravity and the symmetric teleparallel equivalent of unimodular gravity in a unified framework. Our starting point is the Weyl invariant action with enhanced symmetry. The enhanced symmetry was
important to guarantee that the resultant theory is equivalent to unimodular gravity or Einstein’s general relativity with an adjustable cosmological constant.

Since all of them are physically equivalent, the difference may only reside in their interpretation. We may be able to find the superiority of one formulation over the other only if it is embedded in a deeper structure. For example, it is interesting to point out that the supergeometry in superspace is naturally equipped with torsion, so the Riemannian geometry that we are used to may not be the most natural formulation of gravity from the supersymmetric viewpoint.

Another possible application of (symmetric) teleparallel formulation of unimodular gravity is to define gravitational energy. Since the variation of the action with respect to the tetrad becomes a tensor rather than pseudo-tensor, we can define a covariant gravitational energy-momentum tensor. See e.g. [29] and reference therein. Once we fix the gauge, however, since it is equivalent to Einstein’s general relativity anyway, the advantage may not be obvious. Most probably, only asymptotic energy can be defined like in Einstein’s general relativity.

Let us discuss a possible generalizations of \( f(T, Q) \) type (symmetric) teleparallel gravity in unimodular gravity. The idea is to make the torsion tensor and the non-metricity tensor Weyl covariant:

\[
\hat{T}^a_{\mu\nu} = \partial_\mu (\phi e^a_\nu) - \partial_\nu (\phi e^a_\mu) \\
= \phi T^a_{\mu\nu} + (\partial_\mu \phi) e^a_\nu - (\partial_\nu \phi) e^a_\mu \\
\hat{Q}^{\alpha\mu\nu} = \partial_\alpha (\phi^2 g_{\mu\nu}) \\
= \phi^2 Q^{\alpha\mu\nu} + 2 (\phi \partial_\alpha \phi) g_{\mu\nu} .
\]

Then the theory constructed out of \( \hat{T}^a_{\mu\nu} \) and \( \hat{Q}^{\alpha\mu\nu} \) (as well as \( \hat{g}_{\mu\nu} = \phi^2 g_{\mu\nu} \) and \( \hat{e}^a_\mu = \phi e^a_\mu \)) gives Weyl invariant action, which can be a starting point to construct the unimodular version of \( f(T, Q) \) gravity (including new general relativity [30] and newer general relativity [10]) by setting \( \phi = e^{-\frac{1}{4}} \). Generic \( f(T, Q) \) gravity lacks the enhanced symmetry and the consistency becomes non-trivial, and the unimodular version inherits the same difficulty, but there may exist consistent theories including extra propagating degrees of freedom.

In this paper, we have not discussed cosmological or astrophysical applications of these \( f(T, Q) \) theories from our viewpoint. They are actively studied e.g. in [31–63].

\[\text{4}\] Since \( f(T, Q) \) theories typically introduce extra degrees of freedom, they may be a potential source for the cosmological evolution and fluctuations of the Universe, which may be observed in cosmic microwave background or gravitational wave.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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\[\text{4}\] We have cited the papers that appeared in 2022. Earlier papers can be found in review articles such as [8].
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