A REFORMULATION OF THE SIEGEL SERIES AND INTERSECTION NUMBERS

SUNGMUN CHO AND TAKUYA YAMAUCHI

Abstract. In this paper, we will explain a conceptual reformulation and inductive formula of the Siegel series. Using this, we will explain that both sides of the local intersection multiplicities of [GK93] and the Siegel series have the same inherent structures, beyond matching values.

As an application, we will prove a new identity between the intersection number of two modular correspondences over $\mathbb{F}_p$ and the sum of the Fourier coefficients of the Siegel-Eisenstein series for $\text{Sp}_4/\mathbb{Q}$ of weight 2, which is independent of $p (> 2)$. In addition, we will explain a description of the local intersection multiplicities of the special cycles over $\mathbb{F}_p$ on the supersingular locus of the ‘special fiber’ of the Shimura varieties for $\text{GSpin}(n, 2), n \leq 3$ in terms of the Siegel series directly.

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1. INTRODUCTION

1.1. On the Gross-Keating’s formula. In [GK93] Gross and Keating studied the arithmetic intersection number of three modular correspondences which are regarded as cycles in the self-product $Y_0(1) \times Y_0(1)/\mathbb{Z}$ of the moduli stack $Y_0(1)$ of elliptic curves over any scheme. They described it purely in terms of certain invariants of a ternary quadratic form created by themselves to formulate it. This invariant has been generalized to quadratic forms of any degree over a local field, and is nowadays called the Gross-Keating invariant. They already expected in the introduction of their paper that the arithmetic intersection number coincides with the sum of the Fourier coefficients of the derivative of the Siegel-Eisenstein series of weight 2 and of degree 3, which has been studied thoroughly in [ARGO07]. Kudla in [Kud97] later proposed a general program (local version) to make a connection between the local intersection multiplicity of special cycles on an integral model of the Shimura varieties for $GSpin(n, 2)$ and the derivative of the local factor of a Fourier coefficient of the Siegel-Eisenstein series of weight $(n + 2)/2$ and of degree $n + 1$. In the latter object, such local factor is called the Siegel series.

The program has been vastly studied by a series of papers [Kud97], [KRY99], [KRY06], [KR99], [KR00] for $0 \leq n \leq 3$, and [BY] for general $n$, when the dimension of the arithmetic intersection is zero. A strategy to compute the local intersection multiplicities which had been taken in these papers is to reduce them to the case of Gross-Keating. On the other hand, a computation of the Siegel series side is based on Katsurada’s paper [Kat99]. The relation between both sides then follows by a direct comparison. Note that beyond direct comparison, there had not been known an evidence or structure to conceptually yield the equality between them.

Therefore, in order to understand Kudla’s program in conceptual way toward the higher dimensional case of the arithmetic intersection, it would be important to have a better understanding on the relation between the local intersection multiplicity of Gross and Keating in [GK93] and the Siegel series. In [GK93], a key step is to derive an inductive formula (Lemma 5.6 of [GK93]) for the local intersection multiplicities over $W(\mathbb{F}_p)$ at a prime $p$, which involves the local intersection multiplicity on the special fiber at $p$ (Lemma 5.7 of [GK93]). As we will compare this inductive formula with our result in the next subsection, we describe the precise form of their inductive formula.

Let $(L, q_L)$ be an anisotropic quadratic lattice over $\mathbb{Z}_p$ of rank 3. Then the Gross-Keating invariant of $(L, q_L)$ consists of three integers $\text{GK}(L) = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$. If we denote by $\alpha_p(a_1, a_2, a_3)$ the local intersection multiplicity associated to $(L, q_L)$ (see (3.18) of [GK93]), then it satisfies the following inductive formula, with respect to the Gross-Keating invariant: (cf. the proof of Theorem 6.8):

\[
\alpha_p(a_1, a_2, a_3) = \alpha_p(a_1, a_2, a_3 - 2) + \tau_{a_1, a_2}.
\]

Here, $\tau_{a_1, a_2}$ is the local intersection multiplicity of two cycles on the special fiber of the setting of Gross-Keating.

1.2. On the Siegel series. On the other hand, the Siegel series has of great importance in automorphic forms, such as the study of conjectures related to automorphic $L$-functions and the construction of automorphic forms of level 1, so-called Ikeda lift. We refer to the first several paragraphs of [IK2] for more introductory discussion about the important usages of the Siegel series in this context. Theories of the Siegel series have been developed by many people such as Kitaoka and Shimura. It was Katsurada in [Kat99] who firstly found the explicit formula of the Siegel series for
\( \mathbb{Z}_p \). However, as mentioned in the introduction of [IK2], his formula is complicated and it is not clear which invariant of a quadratic form determines the Siegel series.

Recently, Ikeda and Katsurada in [IK2] obtained the formula of the Siegel series over any finite extension of \( \mathbb{Z}_p \). Furthermore, they show that the Siegel series is completely determined by the Gross-Keating invariant with extra data, called the Extended Gross-Keating datum, for any quadratic form over any finite extension of \( \mathbb{Z}_p \).

The Siegel series is usually described as a polynomial. An explicit formula of the Siegel series given in [Kats99] and [IK2] is to determine the coefficients of this polynomial. On the other hand, theoretical interpretation of these coefficients had not been known yet.

1.3. Reformulation of the Siegel series. Our first main result is to reformulate the Siegel series over any finite extension of \( \mathbb{Z}_p \) in Theorem 3.9 and Corollary 3.11. This gives a conceptual and theoretical interpretation of the coefficients as a weighted sum of certain number of quadratic lattices. The method used in this work is based on another geometric nature of the Siegel series involving the stratification of a \( p \)-adic scheme, geometric description of each stratum, Grassmannian, and lattice counting argument. This is largely different from the known techniques in this context.

Using the result of [IK2], we then obtain an inductive formula of the Siegel series, with respect to the Gross-Keating invariant, under Conjecture 4.4 concerning about quadratic forms (which is verified to be true when \( p \) is odd or when \( (L, q_L) \) is anisotropic over \( \mathbb{Z}_2 \) in Lemmas 4.5-4.6).

We describe our inductive formula more precisely. The Siegel series can be defined as an integral of certain volume form on a \( p \)-adic manifold (cf. Definitions 3.1 and 3.10) associated to a quadratic lattice \( (L, q_L) \) over \( \mathfrak{o} \), where \( \mathfrak{o} \) is the ring of integers of a finite field extension of \( \mathbb{Q}_p \) (for any \( p \)). It is usually denoted by \( \mathcal{F}_L(X) \), as a polynomial of \( X \). Let \( n \) be the rank of \( L \) so that the Gross-Keating invariant consists of \( n \)-integers \( \text{GK}(L) = (a_1, \cdots, a_n) \) satisfying \( a_i \leq a_j \) with \( i \leq j \). If we choose the integer \( d \) characterized by the condition \( a_{n-d} < a_{n-d+1} = \cdots = a_n \), then we can associate certain lattice \( L^{(d,n)} \) containing \( L \) whose rank is also \( n \). To be more precise, for a reduced basis \((e_1, \cdots, e_n)\) of \( L \) given in Definition 2.5, the lattice \( L^{(d,n)} \) is spanned by \((e_1, \cdots, e_{n-d}, \frac{1}{\pi} \cdot e_{n-d+1}, \cdots, \frac{1}{\pi} \cdot e_n)\).

Here, \( \pi \) is a uniformizer in \( \mathfrak{o} \). A second main theorem of the current paper is the following:

**Theorem 1.1.** (Theorem 4.3) Assume that \( p \) is odd or that \( (L, q_L) \) is anisotropic over \( \mathbb{Z}_2 \). Assume that \( L^{(d,n)} \) is an integral quadratic lattice. Then we have the following inductive formula, with respect to the Gross-Keating invariant, of the Siegel series \( \mathcal{F}_L(X) \):

\[
\mathcal{F}_L(X) = \sum_{m=1}^{d} \left( c_m \cdot f^{(n+1)m} \cdot X^{2m} \cdot \sum_{L' \in \mathcal{G}_{L,d,m}^{L}} \mathcal{F}_{L'}(X) \right) + (1 - X)(1 - f^d X)^{-1} \cdot \left( \prod_{i=1}^{d} (1 - f^{2i} X^2) \right) \cdot \mathcal{F}_{L_0^{(d,n)}}(f^d X),
\]

Remark in page 444 of [Kats99] says ‘it seems very interesting problem to prove Theorems 4.1 and 4.2 directly from the local theory of quadratic forms’. Here, Theorems 4.1 and 4.2 are main results of [Kats99], which give an explicit formula of the Siegel series over \( \mathbb{Z}_p \). Our method can be understood in the spirit of the problem proposed by Katsurada.
where $c_m = - \left( \binom{m}{1} f \cdot c_1 + \binom{m}{2} f \cdot c_2 + \cdots + \binom{m}{m-1} f \cdot c_{m-1} \right) + 1$ if $m > 1$ and $c_1 = 1$. Here, $f$ is the cardinality of the residue field of $\mathfrak{o}$. For $L' \in \mathcal{G}_{L,d,m}$,

$$
\begin{align*}
\text{GK}(L) &> \text{GK}(L'); \\
|\text{GK}(L')| & = |\text{GK}(L)| - 2m; \\
\text{GK}(L^{(d,n)}_0) & = \text{GK}(L^{(n-d)}).
\end{align*}
$$

Note that notion of $L^{(d,n)}$, $\mathcal{G}_{L,d,m}$, and $\binom{m}{k} f$ can be found at the beginning of Section 4. Notion of $L^{(d,n)}_0$ can be found at Remark 4.4.1.

Here, $\mathcal{G}_{L,d,m}$ is identified with Grassmannian to classify the set of $m$-dimensional subspaces of the vector space of dimension $d$ (given by $L^{(d,n)}/L$) over a finite field $\mathfrak{o}/(\pi)$, whose order is $\binom{d}{m} f$.

1.4. **The comparison between Gross-Keating’s formula and the Siegel series.** Since both sides, Gross-Keating’s formula and the Siegel series, have inductive formulas, it is natural to ask whether or not there is a relation between them.

If we restrict ourselves to an anisotropic quadratic lattice over $\mathbb{Z}_p$ of rank $n$, which covers the case of Gross-Keating, then we have more refined and simpler inductive formula (Theorem 5.10) as follows:

$$
(1.2) \quad \mathcal{F}_L(X) = \begin{cases} 
    p^{n+1} \cdot X^2 \cdot \mathcal{F}_{L^{(0)}}(X) + (1 - X)(1 + pX) \cdot \mathcal{F}_{L^{(n)}}(pX) & \text{if } 2 \leq n \leq 4; \\
    p^2 \cdot X^2 \cdot \mathcal{F}_{L^{(1)}}(X) + (1 - X)(1 + pX) & \text{if } n = 1.
\end{cases}
$$

After comparing both inductive formulas, we obtain the following result:

**Theorem 1.2.** (cf. Theorems 6.7 and 6.8) The inductive formula of Gross-Keating in Equation (L.1) and the derivative of Equation (L.2) at $1/p^2$ with $n = 3$ do match each other. As a direct consequence, we have

$$
\begin{align*}
    \alpha_p(a_1, a_2, a_3) & = c_1 \cdot \mathcal{F}'_L(1/p^2); \\
    \mathcal{T}_{a_1, a_2} & = c_2 \cdot \mathcal{F}'_{L^{(3)}}(1/p),
\end{align*}
$$

for explicitly computed constants $c_1$ and $c_2$. Here $\text{GK}(L^{(3)}_0) = (a_1, a_2)$.

This theorem shows that both sides of the local intersection multiplicity and the Siegel series have the same inherent structures, that is, the same inductive formula, beyond matching their values. In addition, it gives us a new observation that the local intersection multiplicity on the special fiber can also be described in terms of the derivative of the Siegel series.

1.5. **Applications to intersection numbers over a finite field.** Since the above theorem matches both sides on the special fiber, we can naturally consider their applications over a finite field. We explain two consequences in this line: intersection numbers over a finite field and the local intersection multiplicities on the special fiber of $\text{GSpin}(n, 2)$ Shimura varieties with $n \leq 3$ (in the case of zero dimension of the arithmetic intersection).

1.5.1. **Intersection numbers over a finite field.** Since we obtained a new description of the local intersection multiplicity on the special fiber in the setting of Gross-Keating in terms of the derivative of the Siegel series, it is natural to compute the intersection number of two modular correspondences on a finite field.

More precisely, let $\varphi_m$ be the modular polynomial in $\mathbb{Z}[x, y]$ of degree $m$ whose irreducible factor corresponds to an affine model of the modular curves $Y_0(m/n^2)$ for some $n^2|m$ in $Y_0(1) \times Y_0(1)$ (cf. Vog07). Then for positive integers $m_1, m_2$ and a prime $p$, we define the intersection number over a finite field or over the complex field as follows:

$$
(1.3) \quad (T_{m_1, p}, T_{m_2, p}) := \text{length}_{\mathcal{F}_p} \mathbb{P}[x, y]/(\varphi_{m_1}, \varphi_{m_2}), \quad (T_{m_1, c}, T_{m_2, c}) := \text{length}_{\mathbb{C}} \mathbb{C}[x, y]/(\varphi_{m_1}, \varphi_{m_2}).
$$
We compare the above two intersection numbers by using Theorem 1.2 on the supersingular locus and the theory of quasi-canonical lifts on the ordinary locus. The following theorem is our result:

**Theorem 1.3.** (Proposition 7.1 and Theorem 7.3) The intersection number \((T_{m_1,p}, T_{m_2,p})\) is finite if and only if \(m_1 m_2\) is not a square. In addition, if \(m_1 m_2\) is not a square and \(p\) is odd, then
\[
(T_{m_1,p}, T_{m_2,p}) = (T_{m_1,c}, T_{m_2,c}),
\]

We refer to Remark 7.17 for a discussion with \(p = 2\).

Since \((T_{m_1,c}, T_{m_2,c})\) is the sum of the Fourier coefficients of the Siegel-Eisenstein series for \(\text{Sp}_4 / \mathbb{Q}\) by Proposition 2.4 of \[\text{GK93}\], the intersection number on the special fiber is also the sum of the Fourier coefficients of the Siegel-Eisenstein series for \(\text{Sp}_4 / \mathbb{Q}\). Furthermore the intersection number is independent of the characteristic of a finite field with \(p > 2\), whereas the local intersection multiplicities highly depend on \(p\).

The above theorem yields a new interpretation on a classical object \(\mathbb{Z}[\frac{1}{2}] [x,y] / (\varphi_{m_1}, \varphi_{m_2})\). Note that the two main objects to be analyzed in \[\text{GK93}\] are geometric interpretations of
\[
\mathbb{C}[x,y] / (\varphi_{m_1}, \varphi_{m_2}) \quad \text{and} \quad \mathbb{Z}[x,y] / (\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3}).
\]

Namely, the dimension of the first object is the intersection number of two modular correspondences over \(\mathbb{C}\) and \((\log \text{ of})\) the cardinality of the second object is the arithmetic intersection number of three modular correspondences over \(\mathbb{Z}\). In this context, the \(\mathbb{Z}\)-module \(\mathbb{Z}[\frac{1}{2}] [x,y] / (\varphi_{m_1}, \varphi_{m_2})\) has the following interesting interpretations:

**Theorem 1.4.** (Theorem 7.4)

1. \(\mathbb{Z}[\frac{1}{2}] [x,y] / (\varphi_{m_1}, \varphi_{m_2})\) is a free \(\mathbb{Z}[\frac{1}{2}]\)-module.
2. The rank of \(\mathbb{Z}[\frac{1}{2}] [x,y] / (\varphi_{m_1}, \varphi_{m_2})\), as a \(\mathbb{Z}[\frac{1}{2}]\)-module, is equal to
\[
\frac{1}{288} \sum_{T \in \text{Sym}_2(\mathbb{Z}) > 0 \atop \text{diag}(T) = (m_1, m_2)} c(T).
\]

Here, \(c(T)\) is the Fourier coefficient of the Siegel-Eisenstein series for \(\text{Sp}_4(\mathbb{Z})\) of weight 2 with respect to the \((2 \times 2)\)-half-integral symmetric matrix \(T\).

1.5.2. The local intersection multiplicities on the special fiber in orthogonal Shimura varieties. As explained in subsection 1.1, the local intersection multiplicity on an integral model of Shimura varieties for \(\text{GSpin}(n,2)\), when the dimension of the arithmetic intersection is zero, is reduced to that of Gross-Keating. Since we have a better understanding on the latter object, it is natural to ask if our comparison argument between two inductive formulas in Gross-Keating's case can be extended to the general case. In this context, we obtain the following result:

**Theorem 1.5.** (Proposition 8.5 and Theorem 8.4) Assume that \(p\) is odd and \(0 \leq n \leq 3\).

1. Both sides of the local intersection multiplicity on \(\text{GSpin}(n,2)\) Shimura varieties (in the case of zero dimension of the arithmetic intersection) and the derivative of the Siegel series satisfy the same inductive formula induced from Equation 1.2.
2. The local intersection multiplicity on the special fiber in the supersingular locus is described in terms of the derivative of the Siegel series for a suitable quadratic lattice.

1.6. Speculation.

1.6.1. As Kudla expected, the local intersection multiplicity in higher dimensional case of the arithmetic intersection on \(\text{GSpin}(n,2)\) Shimura varieties is believed to match with the derivative of the Siegel series for a suitable quadratic lattice. The comparison results of Theorem 1.2 and Theorem 1.5(1) seem to imply that there should be an inductive formula in geometric side which...
is parallel to that on the Siegel series side. Thus our inductive formula of the Siegel series given in Theorem 1.1 would be the inductive formula that the local intersection multiplicity is expected to satisfy with.

1.6.2. Since we have an interpretation of the local intersection multiplicity on (the supersingular locus of) the special fiber of GSpin\((n, 2)\) Shimura varieties in terms of the Siegel series in Theorem 1.5, we will be able to relate the intersection numbers on the special fiber of GSpin\((n, 2)\) Shimura varieties with the sum of the Fourier coefficients of the Siegel-Eisenstein series with suitable weight and degree. We expect that the intersection number of the special cycles at the special fiber of GSpin\((n, 2)\)-Shimura variety is independent of \(p\) (possibly away from bad primes), which turns to be the sum of the Fourier coefficients of the Siegel-Eisenstein series. This observation is parallel to Theorems 1.3-1.4. This would imply that an associated arithmetic intersection is flat over \(\mathbb{Z}\) (possibly away from bad primes).

1.7. **Organizations.** We will organize this paper as follows. After fixing notations in Section 2, we will derive a conceptual study of the Siegel series in Sections 3-4. In Section 5, we will explain a refined formulation of the Siegel series for anisotropic quadratic lattices over \(\mathbb{Z}_p\). Section 6 is devoted to compare both sides of the local intersection multiplicity of [GK93] and the Siegel series. In Sections 7-8, we will explain two applications in the context of intersection numbers (or multiplicities) over a finite field. In Appendix, we list up explicit examples for the intersection numbers related to Section 7.

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### 2. Notations

- Let \(F\) be a finite field extension of \(\mathbb{Q}_p\) with \(\mathfrak{o}\) its ring of integers and \(\kappa\) its residue field. Let \(\pi\) be a uniformizer in \(\mathfrak{o}\). Let \(f\) be the cardinality of the finite field \(\kappa\).
- For an element \(x \in F\), the exponential order of \(x\) with respect to the maximal ideal in \(\mathfrak{o}\) is written by \(\text{ord}(x)\).
- Let \(e = \text{ord}(2)\). Thus if \(p\) is odd, then \(e = 0\).
- We consider an \(\mathfrak{o}\)-lattice \(L\) of rank \(n\). Then a quadratic form \(q_L\) defined on \(L\) is called an integral quadratic form if \(q_L(L) \subseteq \mathfrak{o}\). For an integral quadratic form \(q_L\) with a lattice \(L\), a pair \((L, q_L)\) is called a quadratic lattice. We sometimes say that \(L\) is a quadratic lattice, if there is no confusion. We assume that \(V = L \otimes_F \mathbb{Z}_p\) is nondegenerate with respect to \(q_L\). Similarly, we define a quadratic space \((V, q_L \otimes_F F)\).
- For a given quadratic lattice \((L, q_L)\) over \(\mathfrak{o}\), the quadratic form \(\bar{q}_L\) on \(L \otimes_F \kappa\) is defined to be \(q_L \text{ mod } \pi\).
- Let \(L\) and \(M\) be two lattices over \(\mathfrak{o}\). Assume that \(L\) and \(M\) have the same rank and that \(L \supseteq M\). Then we denote by \([L : M]\) the length of a torsion module \(L/M\) so that the cardinality of \(L/M\) is \(f^{[L:M]}\).
- The fractional ideal generated by \(q_L(X)\) as \(X\) runs through \(L\) will be called the norm of \(L\) and written \(N(L)\).
- Let \(X, Y\) be matrices with entries in \(F\). Then we denote \(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\) by \(X \perp Y\).
Let \((a_1, \cdots, a_m)\) and \((b_1, \cdots, b_n)\) be non-decreasing sequences consisting of non-negative integers. Then \((a_1, \cdots, a_m) \cup (b_1, \cdots, b_n)\) is defined as the non-decreasing sequence \((c_1, \cdots, c_{n+m})\) such that \(\{c_1, \cdots, c_{n+m}\} = \{a_1, \cdots, a_m\} \cup \{b_1, \cdots, b_n\}\) as sets.

- For \(a = (a_1, \cdots, a_n)\) with an integer \(a_i\), the sum \(a_1 + \cdots + a_n\) is denoted by \(|a|\).
- For \(a = (a_1, \cdots, a_n)\) with an integer \(a_i\), the first \(m\)-tuple \((a_1, \cdots, a_m)\) with \(m \leq n\) is denoted by \(a(m)\).

- Let \(H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\) and let \(H_k\) be the orthogonal sum of the \(k\)-copies of \(H\). We denote by \((H_k, q_k)\) the associated quadratic lattice. Let \(W = H_k \otimes_o F\). In this paper, we always assume that \(2k \geq n\), where \(n\) is the rank of \(L\).

- Let \(B\) be a non-degenerate half-integral symmetric matrix over \(\mathfrak{o}\) of size \(n \times n\). Here by an half-integral matrix over \(\mathfrak{o}\), we mean that each non-diagonal entry multiplied by 2 and each diagonal entry of \(B\) are in \(\mathfrak{o}\). ‘Non-degenerate’ means that the determinant of \(B\) is nonzero.

- Let \(U \in \text{GL}_n(\mathfrak{o})\). Then the matrix product \(^tU B U\) is denoted by \(B[U]\). Here, \(^tU\) is the matrix transpose of \(U\).

- For given two quadratic \(R\)-lattices \(L\) and \(L'\), where \(R\) is a commutative \(\mathfrak{o}\)-algebra, we say that an \(R\)-linear map \(f : L \to L'\) is isometry if it is injective and preserves the associated quadratic forms, i.e. \(q_L(x) = q_{L'}(f(x))\) for any \(x \in L\).

- We define \(O_\mathfrak{o}(L, H_k)\) to be the affine scheme defined over \(\mathfrak{o}\) such that \(O_\mathfrak{o}(L, H_k)(R)\), the set of \(R\)-points of \(O_\mathfrak{o}(L, H_k)\) for any commutative \(\mathfrak{o}\)-algebra \(R\), is defined to be the set of \(R\)-linear maps (not necessarily injective) from \(L \otimes R\) to \(H_k \otimes R\) preserving the associated quadratic forms. If \(R\) is a flat \(\mathfrak{o}\)-domain, then \(O_\mathfrak{o}(L, H_k)(R)\) is the set of isometries (i.e. injective) from the quadratic space \(L \otimes R\) to the quadratic space \(H_k \otimes R\), which will be proved in the next lemma.

**Lemma 2.1.** If \(R\) is a flat \(\mathfrak{o}\)-domain, then an \(R\)-linear map from \(L \otimes R\) to \(H_k \otimes R\) preserving the associated quadratic forms is injective. Thus the generic fiber of \(O_\mathfrak{o}(L, H_k)\), denoted by \(O\mathfrak{p}(V, W)\) with \(V = L \otimes_o F\) and \(W = H_k \otimes_o F\), represents the set of isometries from the quadratic space \(V\) to the quadratic space \(W\).

**Proof.** Let \(\varphi : L \otimes R \to H_k \otimes R\) be an \(R\)-linear map preserving the associated quadratic forms for a flat \(\mathfrak{o}\)-domain \(R\). We choose \(v \in L \otimes R\) such that \(\varphi(v) = 0\). Then it suffices to show that \(v = 0\).

Assume that \(v \neq 0\) in \(L \otimes R\). Let \(R_0\) be the quotient field of \(R\). Then the characteristic of \(R_0\) is \(0\) since \(R\) is flat over \(\mathfrak{o}\).

If we let \(\tilde{v} = v \otimes 1 \in L \otimes R_0\), then \(\tilde{v}\) is nonzero. Thus we can choose a basis, say \(\mathcal{B}\), of an \(R_0\)-vector space \(L \otimes R_0\) of dimension \(n\) involving \(\tilde{v}\). We may assume that the last vector in \(\mathcal{B}\) is \(\tilde{v}\).

We write \(\tilde{\varphi} = \varphi \otimes 1 : L \otimes R_0 \to H_k \otimes R_0\) so that \(\tilde{\varphi}(\tilde{v}) = 0\). If we express an \(R_0\)-linear map \(\tilde{\varphi}\) as a matrix \(T\) of size \((2k \times n)\) with respect to a basis \(\mathcal{B}\), then the last column vector of \(T\) is zero. We now consider the following matrix equation:

\[ q_L = ^tT \cdot q_k \cdot T. \]

Here, \(q_L\) (respectively \(q_k\)) is the symmetric matrix associated to \(L \otimes R_0\) (respectively \(H_k \otimes R_0\)) with suitable sets of basis. Thus both \(q_L\) and \(q_k\) are nondegenerate. This contracts to the given setting since the determinant of \(q_L\) is nonzero, whereas that of the right hand side is \(0\).

Thus we conclude that \(\tilde{v} = 0\) so that \(v = 0\). This implies that \(\varphi\) is injective.

Let \(B\) be a non-degenerate half-integral symmetric matrix over \(\mathfrak{o}\) of size \(n \times n\). We will define the Gross-Keating invariant for \(B\) below. The definition is taken from [IK1].
Definition 2.2 (Definitions 0.1 and 0.2 in \[IK1\]).  
(1) We express \(B = (b_{ij})\). Let \(S(B)\) be the set of all non-decreasing sequences \((a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}\) such that
\[
\begin{align*}
\text{ord}(b_{ij}) &\geq a_i \\
\text{ord}(2b_{ij}) &\geq (a_i + a_j)/2 \\
&\quad (1 \leq i \leq n, 1 \leq j \leq n).
\end{align*}
\]
Put
\[
S(\{B\}) = \bigcup_{U \in \text{GL}_n(\sigma)} S(B[U]).
\]

The Gross-Keating invariant \(\text{GK}(B)\) of \(B\) is the greatest element of \(S(\{B\})\) with respect to the lexicographic order \(\succeq\) on \(\mathbb{Z}^n_{\geq 0}\). Here, the lexicographic order \(\succeq\) on \(\mathbb{Z}^n_{\geq 0}\) is the following (cf. the paragraph following Definition 0.1 of \[IK1\]). Choose two elements \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) in \(\mathbb{Z}^n_{\geq 0}\). Let \(i\) be the first integer over which \(a_i\) differs from \(b_i\) (so that \(a_j = b_j\) for any \(j < i\)). If \(a_i > b_i\), then we say that \((a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)\). Otherwise, we say that \((a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)\).

(2) The symmetric matrix \(B\) is called optimal if \(\text{GK}(B) \in S(B)\).

(3) If \(B\) is a symmetric matrix associated to a quadratic lattice \((L, q_L)\), then \(\text{GK}(L)\), called the Gross-Keating invariant of \((L, q_L)\), is defined by \(\text{GK}(B)\). \(\text{GK}(L)\) is independent of the choice of a matrix \(B\).

It is known that the set \(S(\{B\})\) is finite (cf. \[IK1\]), which explains well-definedness of \(\text{GK}(B)\). We can also see that \(\text{GK}(B)\) depends on the equivalence class of \(B\). In general, it is a difficult question to check whether or not a given matrix \(B\) is optimal. Ikeda and Katsurada introduced so-called ‘reduced form’ associated to \(B\) and showed that it is optimal. We use a reduced form several times in this paper and thus provide its detailed definition through the following series of definitions \[IK1\] (Definition 3.1). They are taken from \[IK1\] and \[IK2\] for synchronization.

In \[IK1\], they define a reduced form for \(p = 2\). However, their definition and main theorems hold for any \(p\), which are explained in the initial version of their paper posted on arXiv. Thus, we explain relevant concepts and theorems without restriction on \(p\). When the assumption of \(p = 2\) is necessary, we will mention it.

Definition 2.3 (Definition 3.1 in \[IK2\]). Let \(\underline{a} = (a_1, \ldots, a_n)\) be a non-decreasing sequence of non-negative integers. Write \(\underline{a}\) as
\[
\underline{a} = (m_{11}, \ldots, m_{1r}; m_{21}, \ldots, m_{2r}; \ldots; m_{n1}, \ldots, m_{nr})
\]
with \(m_1 < \cdots < m_r\) and \(n = n_1 + \cdots + n_r\). For \(s = 1, 2, \cdots, r\), put
\[
n_s = \sum_{u=1}^{s} n_u,
\]
and
\[
I_s = \{n_{s-1}^*, n_{s-1}^* + 2, \ldots, n_s^*\}.
\]
Here, we let \(n_0^* = 0\).

Let \(\mathfrak{S}_n\) be the symmetric group of degree \(n\). Let \(\sigma \in \mathfrak{S}_n\) be an involution i.e. \(\sigma^2 = id\).

Definition 2.4 (Definition 3.1 in \[IK1\]). For a non-decreasing sequence of non-negative integers \(\underline{a} = (a_1, \ldots, a_n)\), we set
\[
\begin{align*}
\mathcal{P}_0^c &\quad = \mathcal{P}_0^c(\sigma) = \{i | 1 \leq i \leq n, i = \sigma(i)\}, \\
\mathcal{P}_+ &\quad = \mathcal{P}_+^c = \{i | 1 \leq i \leq n, a_i > a_{\sigma(i)}\}, \\
\mathcal{P}_- &\quad = \mathcal{P}_-^c = \{i | 1 \leq i \leq n, a_i < a_{\sigma(i)}\}.
\end{align*}
\]

We say that an involution \(\sigma \in \mathfrak{S}_n\) is an \(\underline{a}\)-admissible involution if the following three conditions are satisfied:
(1) \( \mathcal{P}^0 \) has at most two elements. If \( \mathcal{P}^0 \) has two distinct elements \( i \) and \( j \), then \( a_i \neq a_j \) mod 2, and
\[
a_i = \max\{a_j | j \in \mathcal{P}^0 \cup \mathcal{P}^+, a_j \equiv a_i \text{ mod } 2\}.
\]
(2) For \( s = 1, \cdots, r \), we have
\[
\#(I_s \cap \mathcal{P}^+) \leq 1, \quad \#(I_s \cap \mathcal{P}^-) + \#(I_s \cap \mathcal{P}^0) \leq 1.
\]
(3) If \( i \in \mathcal{P}^- \), then
\[
a_{\sigma(i)} = \min\{a_j | j \in \mathcal{P}^+, a_j > a_i, a_j \equiv a_i \text{ mod } 2\}.
\]
Similarly, if \( i \in \mathcal{P}^+ \), then
\[
a_{\sigma(i)} = \max\{a_j | j \in \mathcal{P}^-, a_j < a_i, a_j \equiv a_i \text{ mod } 2\}.
\]

**Definition 2.5** (Definition 3.2 in [IK1]). Write \( B = (b_{ij}) \). Let \( a \in S(B) \). Let \( \sigma \in \mathfrak{S}_n \) be an \( a \)-admissible involution. We say that \( B \) is a reduced form of GK-type \((a, \sigma)\) if the following conditions are satisfied:

(1) If \( i \notin \mathcal{P}^0, j = \sigma(i), \) and \( a_i \leq a_j \), then
\[
\text{GK}\left(\begin{pmatrix}
  b_{ii} & b_{ij} \\
  b_{ij} & b_{jj}
\end{pmatrix}\right) = (a_i, a_j).
\]

Note that if \( p = 2 \) then this condition is equivalent to the following condition (by Proposition 2.3 of [IK1]).
\[
\begin{aligned}
\text{ord}(2b_{ij}) &= \frac{a_i + a_j}{2} \quad \text{if } i \notin \mathcal{P}^0, j = \sigma(i); \\
\text{ord}(b_{ii}) &= a_i \quad \text{if } i \in \mathcal{P}^-.
\end{aligned}
\]
(2) if \( i \in \mathcal{P}^0 \), then
\[
\text{ord}(b_{ii}) = a_i.
\]
(3) If \( j \neq i, \sigma(i) \), then
\[
\text{ord}(2b_{ij}) > \frac{a_i + a_j}{2}.
\]

**Theorem 2.6** (Corollary 5.1 in [IK1]). A reduced form is optimal. More precisely, if \( B \) is a reduced form of GK-type \((a, \sigma)\), then
\[
\text{GK}(B) = a.
\]

In the following remark, we will explain the existence of a reduced form and the uniqueness of an involution up to equivalence when \( p = 2 \).

**Remark 2.7.** In this remark, we assume that \( p = 2 \).

(1) For any given non-decreasing sequence of non-negative integers \( a = (a_1, \cdots, a_n) \), there always exists an \( a \)-admissible involution (cf. the paragraph following Definition 3.1 of [IK1]).

(2) For any given non-degenerate half-integral symmetric matrix \( B \) over \( a \), there always exist a \( \text{GK}(B) \)-admissible involution \( \sigma \) and a reduced form of GK type \((\text{GK}(B), \sigma)\) which is equivalent to \( B \) (cf. Theorem 4.1 of [IK1]).

(3) Using the notation introduced in Definition 2.3, we say that two \( a \)-admissible involutions are equivalent if they are conjugate by an element \( \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r} \). If \( \sigma \) is an \( a \)-admissible involution, then the equivalence class of \( \sigma \) is determined by
\[
\#(\mathcal{P}^+ \cap I_s), \quad \#(\mathcal{P}^- \cap I_s), \quad \#(\mathcal{P}^0 \cap I_s)
\]
for \( 1 \leq s \leq r \) (cf. the paragraph following Remark 4.1 in [IK1]).
Let $\sigma$ and $\tau$ be $\text{GK}(B)$-admissible involutions associated to reduced forms of $\text{GK}$ types $(\text{GK}(B), \sigma)$ and $(\text{GK}(B), \tau)$, respectively, which are equivalent to a given symmetric matrix $B$. Then $\sigma$ and $\tau$ are equivalent (cf. Theorem 4.2 of [IK1]). Therefore, the above sets in (2.1) for $B$ are independent of the choice of a $\text{GK}(B)$-admissible involution with a reduced form.

For example, let $a = (0, 0, 2)$ be the $\text{GK}$-invariant of symmetric matrices $B$ and $B'$. Let $\sigma$ (resp. $\tau$) be an associated $a$-admissible involution to $B$ (resp. $B'$) such that $\sigma(1) = 2$, $\sigma(3) = 3$ (resp. $\tau(1) = 1$, $\tau(2) = 3$). Since $\sigma$ is not equivalent to $\tau$, we can find that $B$ and $B'$ are not equivalent.

We list a few facts about the Gross-Keating invariant below.

**Remark 2.8.**

1. If $p$ is odd, then a diagonal matrix, whose diagonal entries are $u_i\sigma^{a_i}$ where $u_i$ is a unit in $\mathfrak{o}$ and $a_i < a_j$ if $i < j$, is a reduced form (cf. Remark 1.1 of [IK1]) and the Gross-Keating invariant is $(a_1, \ldots, a_n)$. Note that any half-integral symmetric matrix is isometric to such diagonal matrix, if $p > 2$.

2. For the half-integral symmetric matrix $H_k$ of rank $2k$, we have

$$\text{GK}(H_k) = (0, \cdots, 0).$$

3. If there is an isometry from $(L, q_L)$ of rank $n$ to $(H_k, q_k)$, then

$$\text{GK}(L) \succeq \text{GK}(H_k)^{(n)} = (0, \cdots, 0)$$

by Lemma 1.2 of [IK1].

4. The first integer of $\text{GK}(L)$ is the exponential order of a generator of $N(L)$ (cf. Lemma B.1 of [Yan04]).

5. Let $L \subseteq L' \subseteq V$ and $[L' : L] = b$. Then

$$|\text{GK}(L')| = |\text{GK}(L)| - 2b$$

by Theorem 0.1 of [IK1]. Here, $|\text{GK}(L)| = a_1 + \cdots + a_n$ for $\text{GK}(L) = (a_1, \cdots, a_n)$.

### 3. Local densities

The Siegel series of a quadratic lattice $(L, q_L)$ can be defined in terms of the local density associated to two quadratic lattices $(L, q_L)$ and $(H_k, q_k)$ (cf. Definition 3.10). The purpose of this section is to reformulate the local density (and the Siegel series) in terms of certain lattice counting problem conceptually, whose explicit form is given in Theorem 3.9 and Corollary 3.11.

#### 3.1. Local density and primitive local density

We define the following notions:

$$\begin{align*}
\mathcal{Q} & : \text{the } F\text{-vector space of quadratic forms on } V; \\
\mathcal{M} & : \text{the set of linear maps from } V \to W; \\
\mathcal{M}_L & : \text{the set of linear maps from } L \to H_k; \\
\mathcal{Q}_L & = \{ f : f \text{ is an integral quadratic form on } L \}.
\end{align*}$$

Here, $V = L \otimes_o F$ and $W = H_k \otimes_o F$.

Regarding $\mathcal{M}$ and $\mathcal{Q}$ as varieties over $F$, let $\omega_{\mathcal{M}, L}$ and $\omega_{\mathcal{Q}, L}$ be nonzero, translation-invariant forms on $\mathcal{M}$ and $\mathcal{Q}$, respectively, with normalizations

$$\int_{\mathcal{M}_L} |\omega_{\mathcal{M}, L}| = 1 \text{ and } \int_{\mathcal{Q}_L} |\omega_{\mathcal{Q}, L}| = 1.$$

Let $\mathcal{M}^*$ be the set of injective linear maps from $V$ to $W$, which can also be viewed as a nonsingular variety over $F$. Define a map $\rho : \mathcal{M}^* \to \mathcal{Q}$ by $\rho(m) = q_k \circ m$. Here $q_k$ is the quadratic form associated to $H_k$. We fix an integral quadratic form $q_L$ on $L$. Then the inverse image of $q_L \otimes_o F$, along the map $\rho$, is $O_F(V, W)$, which represents the set of $F$-linear maps from $V$ to $W$ preserving the associated quadratic forms (cf. Lemma 2.1). One can also show that the morphism $\rho$ is representable as a
morphism of schemes over $F$ and smooth by showing the surjectivity of the differential of $\rho$ over the Zariski tangent space on any closed point. Put $\omega^d_L = \omega_{W^d,L}/\rho^*\omega_{W,L}$. For a detailed explanation of what $\omega_{W^d,L}/\rho^*\omega_{W,L}$ means, we refer to Section 3.2 of [GY00].

**Definition 3.1.** Assume that the set $O_o(L, H_k)(\sigma)$ is nonempty. Then the local density associated to the pair of two quadratic lattices $L$ and $H_k$, denoted by $\alpha(L, H_k)$, is defined as

$$\alpha(L, H_k) = \int_{O_o(L, H_k)(\sigma)} |\omega^d_L| = \lim_{N \to \infty} f^{-N \cdot \dim O_F(V, W)} \# O_o(L, H_k)(\sigma/\pi^N o).$$

Here, $\dim O_F(V, W) = \dim O(W) - \dim O(V^⊥) = 2k(2k - 1)/2 - (2k - n)(2k - n - 1)/2 = 2kn - (n^2 + n)/2$.

We define the subfunctor $O_{o}^{\text{prim}}(L, H_k)$ of $O_o(L, H_k)$ such that $O_{o}^{\text{prim}}(L, H_k)(R)$, the set of $R$-points for a commutative $\sigma$-algebra $R$, is the set of elements in $O_o(L, H_k)(R)$ whose at least one $n \times n$-minor, as a linear map from $L \otimes_o R$ to $H_k \otimes_o R$, is a unit in $R$. In particular, if $R = \sigma$, then $O_{o}^{\text{prim}}(L, H_k)(\sigma)$ is the set of elements in $O_o(L, H_k)(\sigma)$ whose reduction modulo $\sigma$ is injective from $L \otimes_o \kappa$ to $H_k \otimes_o \kappa$. Each element in $O_{o}^{\text{prim}}(L, H_k)(\sigma)$ is called a primitive isometry. In the next section, we will show that $O_{o}^{\text{prim}}(L, H_k)$ is an open (not necessarily affine) subscheme of $O_o(L, H_k)$ (cf. the paragraph just before Theorem 3.7). Furthermore, we can also see that $O_{o}^{\text{prim}}(L, H_k)(\sigma)$ is open in $O_o(L, H_k)(\sigma)$ in terms of inherent $p$-adic topology.

Let $L'$ be a lattice in $V$ containing $L$ and let $q_{L'}$ be the quadratic form attached to $L'$, whose reduction on $L$ is the same as $q_L$. We identify the set $O_{o}^{\text{prim}}(L', H_k)(\sigma)$ with a suitable subset of $O_o(L, H_k)(\sigma)$, which is naturally induced by the restriction to $L$. Since any linear map in $O_o(L, H_k)(\sigma)$ is injective (i.e. isometry) by Lemma 3.2, we have the following stratification on $O_o(L, H_k)(\sigma)$:

$$O_o(L, H_k)(\sigma) = \bigsqcup_{L \subseteq L' \subseteq V} O_{o}^{\text{prim}}(L', H_k)(\sigma).$$

It is easy to see that the above disjoint union is finite since the norm $N(L')$ of $L'$ should be contained in the ring $\sigma$, in order that $O_{o}^{\text{prim}}(L', H_k)(\sigma)$ is nonempty, which is proved in the next lemma.

**Lemma 3.2.** The condition $N(L') \subseteq \sigma$ is equivalent to the existence of a primitive isometry from $(L', q_{L'})$ to $(H_k, q_k)$.

**Proof.** If there is a primitive isometry from $(L', q_{L'})$ to $(H_k, q_k)$, then it is clear that $N(L') \subseteq \sigma$ since $N(H_k) = \sigma$.

Assume that $N(L') \subseteq \sigma$. Then we can choose an half-integral symmetric matrix $B'$ associated to $L'$. We consider the matrix $\begin{pmatrix} 0 & \frac{1}{2} \cdot id_n & B' \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n & 0 \\ B' & 0 & \frac{1}{2} \cdot id_n \end{pmatrix}$. Here, $id_n$ is the $(n \times n)$-identity matrix. Note that the matrix $\begin{pmatrix} 0 & \frac{1}{2} \cdot id_n \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \\ B' & 0 \end{pmatrix}$ is a symmetric matrix associated to the quadratic lattice $H_n$. Thus it is enough to show that $\begin{pmatrix} 0 & \frac{1}{2} \cdot id_n \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \\ B' & 0 \end{pmatrix}$ is equivalent to the matrix $\begin{pmatrix} 0 & \frac{1}{2} \cdot id_n \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \\ B' & 0 \end{pmatrix}$, which follows from the matrix equation:

$$\begin{pmatrix} 0 & \frac{1}{2} \cdot id_n \\ \frac{1}{2} \cdot id_n & 0 \end{pmatrix} = \begin{pmatrix} id_n & 0 \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \end{pmatrix} \begin{pmatrix} \text{id}_n & 0 \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \cdot id_n \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \end{pmatrix} \begin{pmatrix} \text{id}_n & 0 \\ \frac{1}{2} \cdot id_n & \frac{1}{2} \cdot id_n \end{pmatrix}.$$ 

Here, $X$ is a matrix with entries in $\sigma$ such that $X + id_n X + 2B' = 0$. □

This lemma, combined with Remark 3.3 induces the following description of the existence of a primitive isometry in terms of the Gross-Keating invariant:
Corollary 3.3. There exists a primitive isometry from \((L', q_{L'})\) to \((H_k, q_k)\) if and only if

\[
\text{GK}(L') \geq \text{GK}(H_k)^{(n)} = (0, \ldots, 0).
\]

Since the set \(O^\text{prim}_o(L', H_k)(\mathfrak{o})\) is open in \(O_o(L, H_k)(\mathfrak{o})\) in terms of \(p\)-adic topology, we have the following identity on \(\alpha(L, H_k)\) by Equation (3.1):

\[
\alpha(L, H_k) = \sum_{L \subseteq L' \subseteq V} \int_{O^\text{prim}_o(L', H_k)(\mathfrak{o})} \omega^\text{ld}_L = \sum_{L \subseteq L' \subseteq V} f[L';L]^{(n+1-2k)} \int_{O^\text{prim}_o(L', H_k)(\mathfrak{o})} \omega^\text{ld}_L.
\]

Definition 3.4. We define the primitive local density associated to the quadratic lattices \(L\) and \(H_k\), denoted by \(\alpha^\text{prim}(L, H_k)\), as follows:

\[
\alpha^\text{prim}(L, H_k) = \int_{O^\text{prim}_o(L, H_k)(\mathfrak{o})} |\omega^\text{ld}_L|.
\]

Thus Equation (3.2) is written as follows:

\[
\alpha(L, H_k) = \sum_{L \subseteq L' \subseteq V} f[L';L]^{(n+1-2k)} \cdot \alpha^\text{prim}(L', H_k).
\]

Here, the sum is finite. This formula is indeed well-known in classical literatures (cf. Lemma 3 of \[Kil83\]).

3.2. A formula of the primitive local density. We assume that \(N(L) \subseteq \mathfrak{o}\) and thus the set \(O^\text{prim}_o(L, H_k)(\mathfrak{o})\) is nonempty. In this subsection, we describe a formula of the primitive local density \(\alpha^\text{prim}(L, H_k)\) by showing that \(O^\text{prim}_o(L, H_k)\) is smooth as a scheme defined over \(\mathfrak{o}\).

Before proving smoothness of \(O^\text{prim}_o(L, H_k)\), we describe the scheme \(O^\text{prim}_o(L, H_k)\) in terms of the fiber of certain morphism between two smooth schemes over \(\mathfrak{o}\).

Let \(M_o(L, H_k)\) be the functor from the category of flat \(\mathfrak{o}\)-algebras to the category of sets such that \(M_o(L, H_k)(R)\), the set of \(R\)-points for a flat \(\mathfrak{o}\)-algebra \(R\), is the set of linear maps from \(L \otimes \mathfrak{o} R\) to \(H_k \otimes \mathfrak{o} R\) by ignoring the associated quadratic forms. Then the functor \(M_o(L, H_k)\) is uniquely represented by a flat \(\mathfrak{o}\)-algebra which is a polynomial ring over \(\mathfrak{o}\) of \(2kn\) variables. Thus we can now talk of \(M_o(L, H_k)(R)\) for any (not necessarily flat) \(\mathfrak{o}\)-algebra \(R\).

Let \(M^*_o(L, H_k)\) be the subfunctor of \(M_o(L, H_k)\) such that \(M^*_o(L, H_k)(R)\), the set of \(R\)-points for a commutative \(\mathfrak{o}\)-algebra \(R\), is the set of linear maps from \(L \otimes \mathfrak{o} R\) to \(H_k \otimes \mathfrak{o} R\) whose at least one \(n \times n\)-minor is a unit in \(R\). Then it is easy to see that \(M^*_o(L, H_k)\) is an open subscheme of \(M_o(L, H_k)\) which yields smoothness of \(M^*_o(L, H_k)\) (cf. Section 3.2 of [Cho15]). Note that \(M^*_o(L, H_k)\) is not necessarily affine, but has a finite affine covers (given by each \(n \times n\)-minor). In particular, if \(R = \mathfrak{o}\), then \(M^*_o(L, H_k)(\mathfrak{o})\) is the set of linear maps from \(L\) to \(H_k\) whose reduction modulo \(\pi\) is injective from \(L \otimes \mathfrak{o} \kappa\) to \(H_k \otimes \mathfrak{o} \kappa\).

Let \(Q^*_L\) be the affine space of dimension \(n(n+1)/2\) defined over \(\mathfrak{o}\) such that \(Q^*_L(R)\), the set of \(R\)-points for a commutative \(\mathfrak{o}\)-algebra \(R\), is the set of quadratic forms on \(L \otimes \mathfrak{o} R\) whose coefficients are in \(R\).

Let \(R\) be a flat \(\mathfrak{o}\)-algebra. As a matrix, each element of \(Q^*_L\) is given by a symmetric matrix \((a_{ij})\) of size \(n \times n\) such that each non-diagonal entry \(a_{ij}\) with \(i \neq j\) is of the form \(1/2 \cdot a'_{ij}\) for \(a'_{ij} \in R\) and each diagonal entry \(a_{ii}\) is contained in \(R\).

Then we consider the following morphism

\[
Q^*_H_k \times M^*_o(L, H_k) \longrightarrow Q^*_L, \ (q, m) \mapsto q \circ m.
\]

Here, \(q \in Q^*_H_k\) for a flat \(\mathfrak{o}\)-algebra \(R\). It is easy to see that the above morphism is well-defined and represented by an action morphism of schemes over \(\mathfrak{o}\) (cf. the last paragraph of the proof of Theorem 3.4 in [Cho16]).
The above action morphism induces the morphism
\[ \rho : M^\ast_o(L, H_k) \rightarrow Q_L, \quad m \mapsto q_k \circ m. \]

**Theorem 3.5.** The morphism \( \rho : M^\ast_o(L, H_k) \rightarrow Q_L \) is smooth.

**Proof.** The theorem follows from Lemma 5.5.1 of [GY00] and the following lemma. \( \square \)

**Lemma 3.6.** The morphism \( \rho \otimes \kappa : M^\ast_o(L, H_k) \otimes \kappa \rightarrow Q_L \otimes \kappa \) is smooth.

**Proof.** The proof is based on Lemma 5.5.2 in [GY00]. It suffices to show that, for any \( m \in M^\ast_o(L, H_k)(\kappa) \), the induced map on the Zariski tangent space \( \rho_{*,m} : T_m \rightarrow T_{\rho(m)} \) is surjective. Here, \( \kappa \) is the algebraic closure of \( \kappa \).

We introduce another functor on the category of flat \( \mathfrak{a} \)-algebras. Define \( T(R) \) to be the set of \((n \times n)\)-matrices \( y \) such that each entry is of the form \( 1/2 \cdot y_{ij} \) with \( y_{ij} \in R \). Then this functor is represented by an affine space over \( \mathfrak{a} \).

We now compute the map \( \rho_{*,m} \) explicitly. If we identify \( T_m \) with \( M^\ast_o(L, H_k)(\kappa) \) and \( T_{\rho(m)} \) with \( Q_L(\kappa) \), then
\[ \rho_{*,m} : T_m \rightarrow T_{\rho(m)}, \quad X \mapsto m^t \cdot q_k \cdot X + X^t \cdot q_k \cdot m. \]
We explain how to compute \( X \mapsto m^t \cdot q_k \cdot X + X^t \cdot q_k \cdot m \) explicitly. Firstly, we formally compute \( X \mapsto m^t \cdot q_k \cdot X \). It is of the form \( 1/2 \cdot Y \), where \( Y \) is an \( n \times n \)-matrix with entries in \( \kappa \). Then we formally compute \( 1/2 \cdot Y + 1/2 \cdot Y \). It is of the form \( Z \), whose diagonal entries are in \( \kappa \) and whose non-diagonal entries are of the form \( 1/2 \cdot z_{ij} \) with \( z_{ij} \in \kappa \) such that \( z_{ij} = z_{ji} \). Thus \( Z \) is an element of \( Q_L(\kappa) \).

To prove the surjectivity of \( \rho_{*,m} : T_m \rightarrow T_{\rho(m)} \), it suffices to show the following two statements:

1. \( X \mapsto X \mapsto m^t \cdot q_k \cdot X \) defines a surjection \( M^\ast_o(L, H_k)(\kappa) \rightarrow T(\kappa) \);
2. \( Y \mapsto \frac{1}{2} \cdot Y + \frac{1}{2} \cdot Y \) defines a surjection \( T(\kappa) \rightarrow Q_L(\kappa) \).

These two arguments are direct from the construction of \( T(\kappa) \). \( \square \)

Then the scheme \( O^\text{prim}_o(L, H_k) \) is defined as the fiber of \( q_L \) along the smooth morphism \( \rho \), which shows that \( O^\text{prim}_o(L, H_k) \) is an open (not necessarily affine) subscheme of \( O_o(L, H_k) \). We note that \( O^\text{prim}_o(L, H_k) \) has finite affine covers given by \( n \times n \)-minors, each of which is an open subscheme of an affine scheme \( O_o(L, H_k) \) as well. Since smoothness is stable under base change, \( O^\text{prim}_o(L, H_k) \) is a smooth scheme over \( \mathfrak{a} \). The special fiber of \( O^\text{prim}_o(L, H_k) \) is \( O^\text{prim}_o(q_L, q_k) \), where \( q_L = q_L \mod \pi \) defined on \( L \otimes \kappa \) and \( q_k = q_k \mod \pi \) defined on \( H_k \otimes \kappa \). In particular, \( O^\text{prim}_o(L, H_k)(\kappa) \) is the set of isometries from the quadratic space \((L \otimes \kappa, q_L)\) to the quadratic space \((H_k \otimes \kappa, q_k)\).

We finally have the following formula of the primitive local density by Section 3.9 of [GY00]:

**Theorem 3.7.** Assume that \( N(L) \subseteq \mathfrak{a} \). Then the primitive local density \( \alpha^\text{prim}(L, H_k) \) is given by the following formula:
\[ \alpha^\text{prim}(L, H_k) = f^{-\dim_{F}(V, W)} \cdot \# O^\text{prim}_o(q_L, q_k)(\kappa). \]
Here, \( \dim_{F}(V, W) = 2kn - (n^2 + n)/2 \) and \( \# O^\text{prim}_o(q_L, q_k)(\kappa) \) stands for the cardinality of the set \( O^\text{prim}_o(q_L, q_k)(\kappa) \).

**3.3. Reformulation of the local density.** Using Theorem 3.7, we can reformulate Equation (3.2) of the local density as follows:

\[ \alpha(L, H_k) = f^{-2kn + (n^2 + n)/2} \cdot \sum_{\substack{L \subseteq L' \subseteq V, \\ G_{K(L')} \geq \{0, \ldots, 0\}}} f[L' : L]^{-(n+1-2k)} \cdot \# O^\text{prim}_o(q_{L'}, q_k)(\kappa). \]

Recall that \( O^\text{prim}_o(q_{L'}, q_k)(\kappa) \) is the set of isometries from the quadratic space \((L' \otimes \kappa, q_{L'})\) to the quadratic space \((H_k \otimes \kappa, q_k)\). Let \( L' \otimes \kappa = \overline{L}_0 \perp \overline{L}_1 \), where \( \overline{L}_1 = \text{Rad}(L' \otimes \kappa) \) so that the restriction
of the quadratic form $\bar{q}_L'$ on $\bar{L}'_0$ is nonsingular. We assign the following notion $a^\pm$ for an integer $a$ to $L'$ according to $\bar{L}'_0$:

\[
\begin{cases}
  a^+ & \text{if } a = \dim \bar{L}'_0 \text{ is even and } \bar{L}'_0 \text{ is split;} \\
  a^- & \text{if } a = \dim \bar{L}'_0 \text{ is even and } \bar{L}'_0 \text{ is nonsplit;} \\
  a = a^+ = a^- & \text{if } a = \dim \bar{L}'_0 \text{ is odd.}
\end{cases}
\]

Here, $\dim \bar{L}'_0$ is the dimension of $\bar{L}'_0$ as a $\kappa$-vector space. If $a = 0$, then we say $0^+ = 0^-$. By Exercise 4 in Section 5.6 of [Kit93], we can see that $\#O^{prim}_κ(\bar{q}_L', \bar{q}_κ)(κ)$ is completely determined by three ingredients, $a^\pm$, $n$, and $2k$. Thus we can denote it by $\#O^{prim}_κ(a^\pm, n, 2k)$. Here,

\[
\begin{align*}
2k & \text{ is the dimension of the nondegenerate split quadratic space } (H_k \otimes κ, q_n); \\
n & \text{ is the dimension of the (possibly degenerate) quadratic space } (L' \otimes κ, \bar{a}_L'); \\
a^\pm & \text{ is as explained above such that } a \text{ is the dimension of a maximal nonsingular subspace of } L' \otimes κ.
\end{align*}
\]

Note that $n$ is the rank of $L$. The integer $a$ can be described in terms of $\text{GK}(L)$, which will be stated below.

**Proposition 3.8.** The integer $a$, which is defined as the dimension of $\bar{L}'_0$ as a $κ$-vector space, is the same as the number of $0$’s in $\text{GK}(L')$.

**Proof.** This directly follows by observing a reduced form of $L$. \hfill \square

Since each direct summand in Equation (3.4) is determined by the number of $0$’s in $\text{GK}(L')$ (with the signature $\pm$) and $[L' : L]$, we analyze bounds of these two objects in this paragraph. Let $[L' : L] = b$ and let $n_0$ be the number of $0$’s in $\text{GK}(L) = (a_1, \cdots, a_n)$. Remark 2.8 yields that $|\text{GK}(L')| = |\text{GK}(L)| - 2b$. The integer $b$ is then nonnegative and at most $|\text{GK}(L)|/2$ since $\text{GK}(L') \geq (0, \cdots, 0)$ (cf. Remark 2.8 and Lemma 3.2). Let $\text{GK}(L') = (a'_1, \cdots, a'_n)$ and let $a$ be the number of $0$’s in $\text{GK}(L')$ (cf. Proposition 3.3). If $b$ is positive, then the integer $a$ is at least $\max \{n_0, n - |\text{GK}(L')|\}$, denoted by $n_0$, and at most the integer $t$ such that $a_{t+1} + \cdots + a_n \geq |\text{GK}(L')| > a_{t+2} + \cdots + a_n$, denoted by $m_b$, since $(0, \cdots, 0) \preceq (a'_1, \cdots, a'_n) \preceq (a_1, \cdots, a_n)$. If there is no such $t$, then $|\text{GK}(L')| = 0$ and in this case, we say $m_b = n$. We summarize notations introduced in this paragraph as follows:

For an integer $b$ such that $0 \leq b \leq \frac{|\text{GK}(L)|}{2}$, we define two integers $n_b$ and $m_b$ depending on $b$ as follows:

\[
\begin{align*}
  n_b & = \max \{n_0, n - (|\text{GK}(L)| - 2b)\} \text{ if } b > 0, \text{ here, } n_0 = \#\{a_i | a_i = 0\}; \\
  m_b & = \text{the integer } t \text{ such that } a_{t+1} + \cdots + a_n \geq |\text{GK}(L)| - 2b > a_{t+2} + \cdots + a_n, \text{ if exists}; \\
  n_b = m_b & = n \text{ if there is no such } t \text{ described above.}
\end{align*}
\]

We introduce a new notion $S_{(L,a^\pm,b)}$ as the set of all quadratic lattices $L'$ including $L$ whose associated direct summands in Equation (3.4) are equal. More precisely,

\[
\begin{align*}
  & \quad S_{(L,a^+,b)} = \{L' \supseteq L \mid \text{GK}(L') \geq (0, \cdots, 0), [L' : L] = b, a^+ \text{ is assigned to } L'\}; \\
  S_{(L,a^-,b)} & = \{L' \supseteq L \mid \text{GK}(L') \geq (0, \cdots, 0), [L' : L] = b, a^- \text{ is assigned to } L'\}.
\end{align*}
\]

If $a$ is odd or $0$, then $S_{(L,a^+,b)} = S_{(L,a^-,b)}$. Note that $S_{(L,a^\pm,b)}$ is empty if $b > \frac{|\text{GK}(L)|}{2}$ by Remark 2.8. Therefore Equation (3.4) is now reformulated as follows:

\[
\alpha(L, H_k) = f^{-2kn+(n^2+n)/2} \cdot \sum_{0 \leq b \leq \frac{|\text{GK}(L)|}{2}, \atop n_0 \leq a \leq m_b} f^{b(n+1-2k)} \cdot \#S_{(L,a^\pm,b)} \cdot \#O^{prim}_κ(a^\pm, n, 2k).
\]

Here, if $a$ is odd or $0$, then we ignore one of $S_{(L,a^+,b)}$ or $S_{(L,a^-,b)}$. If $a$ is even and positive, then we count the summands involving $S_{(L,a^+,b)}$ and $S_{(L,a^-,b)}$ separately.
In the above equation, the number \( \#O_k^{prim}(a^\pm, n, 2k) \) is already well-known as follows (cf. Exercise 4 in Section 5.6 of [Kit93]):
\[
\#O_k^{prim}(a^\pm, n, 2k) = f^{2kn-(n^2+n)/2}(1-f^{-k})(1+\chi(a^\pm)f^{n-a/2-k})\prod_{1\leq i<n-a/2}(1-f^{2i-2k}).
\]
Here,
\[
\chi(a^\pm) = \begin{cases} 
0 & \text{if } a \text{ is odd;} \\
1 & \text{if } a \text{ is even and } a^\pm \text{ is assigned or if } a = 0; \\
-1 & \text{if } a(>0) \text{ is even and } a^- \text{ is assigned.}
\end{cases}
\]

Using Equation (3.6) combined with the above description of \( \#O_k^{prim}(a^\pm, n, 2k) \), we have the following local density formula:

**Theorem 3.9.** For a quadratic lattice \( L \), we have
\[
\alpha(L, H_k) = (1-f^{-k}) \cdot \sum_{0 \leq b \leq [GK(L)], n_b \leq a \leq m_b} \left( \#S(L,a^\pm,b) \cdot f^{b-(n+1-2k)} \cdot (1+\chi(a^\pm)f^{n-a/2-k}) \prod_{1\leq i<n-a/2}(1-f^{2i-2k}) \right).
\]
Here, if \( a \) is odd or 0, then we ignore one of \( S(L,a^+,b) \) or \( S(L,a^-,b) \). If \( a \) is even and positive, then we count the summands involving \( S(L,a^+,b) \) and \( S(L,a^-,b) \) separately.

Let \( X = f^{-k} \). Then the local density \( \alpha(L, H_k) \) is a polynomial of \( X \), as \( k \) varies. We denote it by \( F_L(X) \).

**Definition 3.10.** For a given quadratic lattice \( (L, q_L) \), the Siegel series is defined to be the polynomial \( F_L(X) \) of \( X \) such that
\[
F_L(f^{-k}) = \alpha(L, H_k).
\]

**Corollary 3.11.** From the formula of Theorem 3.9, we have the following description of the Siegel series:
\[
F_L(X) = (1-X) \cdot \sum_{0 \leq b \leq [GK(L)], n_b \leq a \leq m_b} \left( \#S(L,a^\pm,b) \cdot f^{b-n+1}X^{2b} \cdot (1+\chi(a^\pm)f^{n-a/2}X) \prod_{1\leq i<n-a/2}(1-f^{2i}X^2) \right).
\]

Thus, each coefficient of the polynomial \( F_L(X) \) is determined by the set \( S(L,a^\pm,b) \). In the next section, we will investigate \( S(L,a^\pm,b) \) more precisely and get an inductive formula of the Siegel series \( F_L(X) \).

4. **Inductive Formulas of the Siegel Series**

In the previous section, we reformulated the Siegel series, \( F_L(X) \), in terms of a lattice counting problem for \( \#S(L,a^\pm,b) \) (cf. Corollary 3.11). In this section, we will explain an inductive formula of the Siegel series, by careful investigation of the set of lattices \( S(L,a^\pm,b) \).

Let \( L \) be a quadratic lattice with \( GK(L) = (a_1, \cdots, a_n) \neq (0, \cdots, 0) \). We choose a basis \( (e_1, \cdots, e_n) \) of an optimal form of \( L \). Let \( d \) be the integer such that \( a_{n-d} < a_{n-d+1} = \cdots = a_n \). If \( a_1 = \cdots = a_n \), then we let \( d = n \). We denote the lattice \( (\subseteq L) \) having a basis
\[
(e_1, \cdots, e_{n-d}, \frac{1}{\pi} \cdot e_{n-d+1}, \cdots, \frac{1}{\pi} \cdot e_n)
\]

by \[
\begin{cases}
L^{(d,n)} & \text{if } d > 1; \\
L^{(n)} (\text{or } L^{(1,n)}) & \text{if } d = 1.
\end{cases}
\]

We assume that the quadratic form on \(L^{(d,n)}\), naturally induced by the quadratic form on \(L\), is an integral quadratic form, that is, \(L^{(d,n)}\) is a quadratic lattice. For example, if \(a_n \geq 2\), then \(L^{(d,n)}\) is a quadratic lattice. Although it is not required that \(a_n \geq 2\), the integers \(a_i\)'s always satisfy the condition that \(a_n = \cdots = a_{n-d+1} \geq 1\), in order that \(L^{(d,n)}\) is a quadratic lattice.

Let \(\tilde{V}_{L,d} = L^{(d,n)}/L\) be a \(\kappa\)-vector space of dimension \(d\). Then each lattice between \(L\) and \(L^{(d,n)}\) bijectively corresponds to each subspace of \(\tilde{V}_{L,d}\). More precisely, the set of all lattices \(L'\) between \(L\) and \(L^{(d,n)}\) with degree \([L':L] = m\), where \(0 \leq m \leq d\), equals the set of subspaces of \(\tilde{V}_{L,d}\) of dimension \(m\). We denote the former set of lattices by \(\mathcal{G}_{L,d,m}\). For example, \(\mathcal{G}_{L,d,0} = \{L\}\) and \(\mathcal{G}_{L,d,d} = \{L^{(d,n)}\}\). The latter set is the Grassmannian, denoted by \(G(m,d)\), whose cardinality is well known to be \(\binom{d}{m}\). Here, \[
\binom{d}{m}_f = \frac{[d]!_f}{[m]!_f [d-m]!_f},
\]
where \([m]!_f = \prod_{t=1}^{m} \frac{f^t - 1}{f - 1}\), for any positive integer \(m\).

We write that \([0]!_f = 1\) so that \(\binom{d}{0}_f = \binom{d}{d}_f = 1\). For example, if \(m = 1\), then \(\binom{d}{1}_f = \frac{f^d - 1}{f - 1}\). Thus we have the following formula:

\[(4.1) \quad \#\mathcal{G}_{L,d,m} = \binom{d}{m}_f.\]

In the following lemma, we explain a property of a lattice including \(L\), but not an element of \(\mathcal{G}_{L,d,m}\).

**Lemma 4.1.** Let \(L'\) be a lattice in \(V\) containing \(L\). If \(L'\) does not contain any lattice in \(\mathcal{G}_{L,d,m}\) for \(1 \leq m \leq d\) (equivalently \(L'\) does not contain any lattice in \(\mathcal{G}_{L,d,1}\)), then there exists a direct summand \(M'\) of \(L'\) such that \(L' = M' \oplus \mathfrak{o}_{e_{n-d+1}} \oplus \cdots \oplus \mathfrak{o}_{e_n}\) and \(L = M \oplus \mathfrak{o}_{e_{n-d+1}} \oplus \cdots \oplus \mathfrak{o}_{e_n}\),

where \(M = L \cap M'\).

**Proof.** Let \(l = n - d\). For such \(L'\), we denote the image of \(e_i\), with \(l + 1 \leq i \leq n\), in \(L'/\pi L'\) by \(\bar{e}_i\). Let \(\tilde{V}_{d}'\) be the subspace of \(L'/\pi L'\), as \(\kappa\)-vector space, spanned by \(\bar{e}_{l+1}, \ldots, \bar{e}_n\). Since \(L'\) does not contain any lattice in \(\mathcal{G}_{L,d,m}\) for \(1 \leq m \leq d\), the vectors \((\bar{e}_{l+1}, \ldots, \bar{e}_n)\) are linearly independent and thus the dimension of the vector space \(\tilde{V}_{d}'\) is \(d\).

Thus there are \(l\)-vectors \((\bar{e}'_1, \ldots, \bar{e}'_l)\) in \(L'/\pi L'\) having \((\bar{e}'_1, \ldots, \bar{e}'_l, \bar{e}_{l+1}, \ldots, \bar{e}_n)\) as a basis. We choose \((e'_1, \ldots, e'_l)\) in \(L'\) as preimages of \((\bar{e}'_1, \ldots, \bar{e}'_l)\), respectively. Then by Nakayama’s lemma, \((e'_1, \ldots, e'_l, e_{l+1}, \ldots, e_n)\) is a basis of \(L'\) as an \(\mathfrak{o}\)-module.

Let \(M'\) be the submodule of \(L'\) spanned by \((e'_1, \ldots, e'_l)\) so that \(L'/M' = \mathfrak{o}_{e_{l+1}} \oplus \cdots \oplus \mathfrak{o}_{e_n}\). We consider the following short exact sequence:

\[1 \to L \cap M' \to L \to L'/M' \to 1.\]

This short exact sequence splits since there exists a section from \(L'/M'\) to \(L\) such that \(e_i\) maps to \(e_i\), with \(l + 1 \leq i \leq n\). Thus \(L \cong (L \cap M') \oplus (\mathfrak{o}_{e_{l+1}} \oplus \cdots \oplus \mathfrak{o}_{e_n})\) as \(\mathfrak{o}\)-modules. Since this isomorphism is induced from the inclusions, we can identify \(L\) with \((L \cap M') \oplus (\mathfrak{o}_{e_{l+1}} \oplus \cdots \oplus \mathfrak{o}_{e_n})\) as submodules of \(L'\). We let \(M = L \cap M'\). This completes the proof. \(\square\)
Lemma 4.2. Let \((e_1, \cdots, e_n)\) be a basis of a lattice \(L\). Let \(M\) be a direct summand of \(L\) such that \(L = M \oplus (\mathcal{O}e_{n-d+1} \oplus \cdots \oplus \mathcal{O}e_n)\). Then there is a basis of \(M\) consisting of the column vectors of the matrix \(\begin{pmatrix} id_{n-d} \\ x \end{pmatrix}\), where \(id_{n-d}\) is the identity matrix of size \(n - d\) and \(x \in M_{d \times (n-d)}(\mathcal{O})\).

Proof. A basis of \(M \oplus (\mathcal{O}e_{n-d+1} \oplus \cdots \oplus \mathcal{O}e_n)\) is given by the column vectors of a matrix \(\begin{pmatrix} x_1 & 0 \\ x_2 & id_d \end{pmatrix}\) with entries in \(\mathcal{O}\), where \(x_1\) is a square matrix of size \(n - d\). Since this matrix is invertible, \(x_1\) is invertible over \(\mathcal{O}\) as well. Thus we can choose another basis for \(L = M \oplus (\mathcal{O}e_{n-d+1} \oplus \cdots \oplus \mathcal{O}e_n)\), given by the column vectors of the matrix \(\begin{pmatrix} x_1 & 0 \\ x_2 & id_d \end{pmatrix}^{-1} = \begin{pmatrix} x_1^{-1} & 0 \\ 0 & id_d \end{pmatrix}\) \(id_{n-d}\). Let \(x = x_2x_1^{-1}\). This completes the proof. \(\square\)

Remark 4.3. (1) In the situation of Lemma 4.2, a direct summand \(M\) of \(L\) has a basis given by the column vectors of a matrix \(\begin{pmatrix} id_{n-d} \\ x \end{pmatrix}\) with \(x \in M_{d \times (n-d)}(\mathcal{O})\) by Lemma 4.2. We denote the lattice \(M\) of Lemma 4.2 by \(L^{(d,n)}_x\), in order to emphasize both roles of \(x\) and \(\mathcal{O}e_{n-d+1} \oplus \cdots \oplus \mathcal{O}e_n\), so that

\[
L = L^{(d,n)}_x \oplus (\mathcal{O}e_{n-d+1} \oplus \cdots \oplus \mathcal{O}e_n).
\]

If \(d = 1\), then we write \(L^{(n)}_x\), instead of \(L^{(1,n)}_x\). We note that the choice of \(x\) for each \(M\) is not unique.

The simplest case of \(L^{(d,n)}_x\) is when \(x\) is the zero vector. In this case,

\[
L^{(d,n)}_0 = \mathcal{O}e_1 \oplus \cdots \oplus \mathcal{O}e_{n-d}.
\]

The lattice \(L^{(d,n)}_0\) will be crucially used in our inductive formula of the Siegel series (cf. Theorems 4.9 and 5.10).

(2) A symmetric matrix for \(L\) with respect to a basis consisting of the column vectors of a matrix \(\begin{pmatrix} id_{n-d} \\ x \end{pmatrix}\) is an optimal form by Theorem 0.2 of [IK1]. In this case, we call such basis an optimal basis for \(L\).

(3) Ikeda and Katsurada impose extra invariant to a quadratic lattice \(L\) in addition to the Gross-Keating invariant and call it ‘Extended Gross-Keating datum’, denoted by \(\text{EGK}(L)\) in [IK1]. They define \(\text{EGK}(L)\) based on an optimal form of \(L\), which turns to be independent of the choice of an optimal form (cf. Theorem 0.4 and Definition 6.3 of [IK1]). For an explicit description of \(\text{EGK}(L)\), we refer to Definition 6.3 of [IK1]. A main contribution of their another paper [IK2] is to prove that the Siegel series \(\mathcal{F}_L(X)\) is completely determined by \(\text{EGK}(L)\) (cf. Theorem 1.1 of [IK2]).

(4) Since a basis consisting of the column vectors of a matrix \(\begin{pmatrix} id_{n-d} \\ x \end{pmatrix}\) is an optimal basis for \(L\), we can see that \(\text{GK}(L^{(d,n)}_0) = \text{GK}(L^{(d,n)}_x)\) by Theorem 0.3 of [IK1]. In addition, by Definition 6.3 of [IK1], one can easily see that \(\text{EGK}(L^{(d,n)}_0) = \text{EGK}(L^{(d,n)}_x)\) since \(\text{EGK}(L)\) is independent from the choice of an optimal basis. Therefore, the argument of the above (3) yields that

\[
\mathcal{F}_{L^{(d,n)}_0}(X) = \mathcal{F}_{L^{(d,n)}_x}(X).
\]
From now on until the end of this section, we work with the following choice of a basis of $L$:
\[(e_1, \cdots, e_n) \text{ is } \begin{cases} 
\text{an optimal basis} & \text{if } p \text{ is odd;} \\
\text{a reduced basis} & \text{if } p \text{ is even.}
\end{cases}\]

Before analyzing $\#S_{(L,a,b)}$ in Proposition 4.3, we will introduce one conjecture regarding quadratic forms modulo $\pi$ in Conjecture 4.4 and prove it when $p$ is odd or when $L$ is anisotropic over $\mathbb{Z}_2$ in Lemmas 4.5/4.6.

We write $L = M \oplus N$, where $M$ (respectively $N$) is spanned by $(e_1, \cdots, e_{n-d})$ (respectively $(e_{n-d+1}, \cdots, e_n)$). We consider a lattice $L'$ containing $L$ on $L \otimes_\mathfrak{q} F$ of the form $L' = M' \oplus N$, where $M'$ is a lattice containing $M$ on $M \otimes_\mathfrak{q} F$. Let $q_{L'}$ be the quadratic form on $L'$ which is naturally induced from $q_L$ on $L$ so that $q_{L'|L} = q_L$. Similarly, we define a quadratic form $q_{M'}$ defined on $M'$.

Assume that the quadratic form $q_{L'}$ on $L'$ is integral (equivalently, $GK(L') > (0, \cdots, 0)$). Let $\bar{L}' = L'/\pi L'$ (respectively $\bar{M}' = M'/\pi M'$) be the quadratic space over $\kappa$ given by $q_{L'}$ (respectively $q_{M'}$) modulo $\pi$.

**Conjecture 4.4.** The dimension of $\bar{L}'$ modulo the radical, is the same as the dimension of $\bar{M}'$ modulo the radical. In other words, the number of $0$’s in $GK(\bar{L}')$ is the same as the number of $0$’s in $GK(\bar{M}')$ (cf. Proposition 3.3).

We think that the conjecture is true in the general case. In the following, we will prove it in two cases, when $p$ is odd and when $(L, q_L)$ is an anisotropic $\mathbb{Z}_2$-lattice.

**Lemma 4.5.** The conjecture is true for an odd prime $p$.

**Proof.** Let $B = \begin{pmatrix} a & b \\
               t_b & c \end{pmatrix}$ be an optimal form with respect to $(e_1, \cdots, e_n)$, where the size of $a$ is $(n-d) \times (n-d)$ and the size of $c$ is $d \times d$. Then the symmetric matrix associated to $L' = M' \oplus N$ is of the form $\begin{pmatrix} t'x & 0 \\
                       0 & id \end{pmatrix} \cdot \begin{pmatrix} a & b \\
                                           t_b & c \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\
                                           0 & id \end{pmatrix} = \begin{pmatrix} t'x \cdot a \cdot x & t'x \cdot b \\
                                           t'x \cdot b & c \end{pmatrix}$ for certain $x \in GL_{n-d}(F)$. Here, $t'x \cdot a \cdot x$ is a symmetric matrix associated to $M'$. Thus it suffices to show that the exponential order of each entry of $2 \cdot t'x \cdot b$ is at least 1.

Since $p$ is odd, we can choose another basis for $M$ given by $x \in GL_{n-d}(\mathfrak{q})$ such that $t'x \cdot a \cdot x$ is an optimal and diagonal matrix by Remark 2.3 (1). Then by Theorems 0.2-0.3 of [IK], the symmetric matrix $\begin{pmatrix} t'x \cdot a \cdot x & t'x \cdot b \\
                                           t'x \cdot b & c \end{pmatrix}$ is also optimal with respect to the decomposition $L = M \oplus N$. Thus we may and do assume that $a$ is optimal and diagonal with diagonal entries $u_i \pi^{a_i}$’s, where $u_i$ is a unit in $\mathfrak{q}$.

Let $M'$ be an integral quadratic lattice containing $M$ on $M \otimes_\mathfrak{q} F$. Then we can easily show that $M'$ is contained in the dual lattice of $M$, which is defined as $\{v \in M \otimes_\mathfrak{q} F | b_M(v, L) \in \mathfrak{q} \}$. Here, $b_M$ is the symmetric bilinear form associated to the quadratic form $q_M$ such that $b_M(v, v) = q_M(v)$. The dual lattice of $M$ is spanned by $(\pi^{-a_1} e_1, \cdots, \pi^{-a_{n-d}} e_{n-d})$. Thus, if $x_{n-d}$ is the diagonal matrix whose diagonal entries are $\pi^{-a_i}$’s with $1 \leq i \leq n-d$, then a matrix $x$ determining $M'$ is of the form $x_{n-d} \cdot x'$, where $x' \in GL_{n-d}(F) \cap M_{n-d}(\mathfrak{q})$.

On the other hand, the exponential order of each entry of $2 \cdot t'x_{n-d} \cdot b$ is at least 1 since $B$ is optimal and $a_i < a_{n-d+1} = a_n$ for any $i (\leq n-d)$. Therefore, the exponential order of each entry of $2 \cdot t'(x') \cdot t'x_{n-d} \cdot b$ is at least 1 since $x' \in M_{n-d}(\mathfrak{q})$. This completes the proof. \qed

**Lemma 4.6.** The conjecture is true for an anisotropic quadratic $\mathbb{Z}_p$-lattice with any $p$.

**Proof.** Let $L$ be an anisotropic quadratic lattice over $\mathbb{Z}_p$ and let $GK(L) = (a_1, \cdots, a_n)$ so that $n$ is at most 4. It is well known that there exists a unique maximal quadratic lattice on $L \otimes_\mathfrak{q} F$ by Theorem 91:1 of [O'Me00]. Thus it suffices to prove the conjecture when $M'$ is the maximal quadratic lattice inside $M \otimes_\mathfrak{q} F$.\qed
We can easily prove that any anisotropic quadratic lattice whose Gross-Keating invariant consists of 0 and 1 is maximal since only two among \(a_i\)'s have the same parity (cf. Proposition 5.3).

As in the proof of Lemma 4.1 let \(B = \begin{pmatrix} a & b \\ t_b & c \end{pmatrix}\) be a reduced form with respect to \((e_1, \ldots, e_n)\), where the size of \(a\) is \(m \times m\) and the size of \(c\) is \((n - m) \times (n - m)\). Let \(x_m\) be the diagonal matrix whose diagonal entries are \(\pi^{-|a_i/2|}\), where \(0 \leq i \leq m\). Then \(t \cdot x_m \cdot a \cdot x_m\) is a reduced form whose Gross-Keating invariant consists of 0 and 1 so that the associated quadratic lattice is maximal. Then the exponential order of each entry of \(2 \cdot t \cdot x_m \cdot b\) is at least 1. This completes the proof. \(\square\)

Remark 4.7. In the above lemma, the only place to use the assumption of \(\mathfrak{o} = \mathbb{Z}_2\) is that any anisotropic quadratic lattice whose Gross-Keating invariant consists of 0 and 1 is maximal. If this is true for a general \(\mathfrak{o}\), then the proof of the lemma works so that the conjecture is true for an anisotropic quadratic lattice over \(\mathfrak{o}\).

Using Lemmas 4.1-4.2 and Conjecture 4.4, we will explain a formula of \(\# \mathcal{S}(L, a^\pm, b)\) in the following proposition. This will be used to make an inductive formula of the Siegel series in Theorem 4.9.

**Proposition 4.8.** Assume that Conjecture 4.4 is true. Let \(b\) be an integer such that \(0 \leq b \leq \frac{\text{GK}(L)}{2}\). Then for any integer \(b'\) with \(b' \geq b\), we have the following formula:

\[
\# \mathcal{S}(L, a^\pm, b) = \sum_{m=1}^{d} \left( c_m \cdot \sum_{L' \in \mathcal{G}_{L,d,m}} \# \mathcal{S}(L', a^\pm, b-m) \right) + f^{d(b-(n-d)b')} \sum_{x \in M_{d \times (n-d)}(A/\pi^{b'}A)} \# \mathcal{S}(L_x, a^\pm, b),
\]

where \(c_m = -\left( \binom{m}{1} f \cdot c_1 + \binom{m}{2} f \cdot c_2 + \cdots + \binom{m}{m-1} f \cdot c_{m-1} \right) + 1\) if \(m > 1\) and \(c_1 = 1\). Here, if \(b-m < 0\), then we understand \(\# \mathcal{S}(L', a^\pm, b-m) = 0\).

**Proof.** Since \(\mathcal{S}(L', a^\pm, b-m) \subseteq \mathcal{S}(L, a^\pm, b)\) for \(L' \in \mathcal{G}_{L,d,m}\), we can choose a lattice

\[
L^\dagger \in \mathcal{S}(L, a^\pm, b) \setminus \bigcup_{L' \in \mathcal{G}_{L,d,1}} \mathcal{S}(L', a^\pm, b-1).
\]

By Lemmas 4.1-4.2 and Remark 4.3 (1), there exists a direct summand \(L_x^{(d,n)}\) of \(L\) such that

\[
L = L_x^{(d,n)} \oplus \mathfrak{o} e_{n-d+1} \oplus \cdots \oplus \mathfrak{o} e_n;
\]

\[
L^\dagger = (L_x^{(d,n)})^\dagger \oplus \mathfrak{o} e_{n-d+1} \oplus \cdots \oplus \mathfrak{o} e_n.
\]

as an \(\mathfrak{o}\)-lattice (not as a quadratic \(\mathfrak{o}\)-lattice). Here, \((L_x^{(d,n)})^\dagger\) is a direct summand of \(L^\dagger\) satisfying the condition that \(L \cap (L_x^{(d,n)})^\dagger = L_x^{(d,n)}\). Since \([L^\dagger : L] = [(L_x^{(d,n)})^\dagger : L_x^{(d,n)}] = b\), the lattice \((L_x^{(d,n)})^\dagger\) is contained in \(\mathcal{S}(L_x^{(d,n)}, a^\pm, b)\) by Conjecture 4.4. Thus we have that

\[
\mathcal{S}(L, a^\pm, b) \setminus \bigcup_{L' \in \mathcal{G}_{L,d,1}} \mathcal{S}(L', a^\pm, b-1) = \bigcup_{x \in M_{d \times (n-d)}(\mathfrak{o})} \mathcal{S}(L_x^{(d,n)}, a^\pm, b) \oplus \mathfrak{o} e_{n-d+1} \oplus \cdots \oplus \mathfrak{o} e_n.
\]

Here, \(\mathcal{S}(L_x^{(d,n)}, a^\pm, b) \oplus \mathfrak{o} e_{n-d+1} \oplus \cdots \oplus \mathfrak{o} e_n\) is the set of lattices \(\{M' \oplus \mathfrak{o} e_{n-d+1} \oplus \cdots \oplus \mathfrak{o} e_n | M' \in \mathcal{S}(L_x^{(d,n)}, a^\pm, b)\}\).
Since \([M' : L_x^{(d,n)}] = b\) for \(M' \in S_{(L_x^{(d,n)}, \alpha \pm b)}\), we can see that

\[
S_{(L_x^{(d,n)}, \alpha \pm b)} \oplus \frac{\alpha e_{n-d+1} + \alpha e_{n-d+2} + \cdots + \alpha e_n}{d} = S_{(L_y^{(d,n)}, \alpha \pm b)} \oplus \frac{\alpha e_{n-d+1} + \alpha e_{n-d+2} + \cdots + \alpha e_n}{d}
\]

if \(x \equiv y \mod \pi^b\).

Thus Equation (4.2) can be expressed as follows:

\[
S_{(L,a \pm b)} \setminus \bigcup_{L' \in G_{L,a \pm b}} S_{(L',a \pm b-1)} = \bigcup_{x \in M_{d \times (n-d)}(\alpha/\pi^b \mathfrak{a})} S_{(L_x^{(d,n)}, a \pm b)} \oplus \frac{\alpha e_{n-d+1} + \alpha e_{n-d+2} + \cdots + \alpha e_n}{d}
\]

for any integer \(b' \geq b\).

In order to compute the cardinality of the right hand side of Equation (4.3), we compare it with

\[
\sum_{x \in M_{d \times (n-d)}(\alpha/\pi^b \mathfrak{a})} \# \left( S_{(L_x^{(d,n)}, a \pm b)} \oplus \frac{\alpha e_{n-d+1} + \alpha e_{n-d+2} + \cdots + \alpha e_n}{d} \right).
\]

In the following, we will see how many times a given lattice is counted in this sum.

Let \(L_x^{(d,n)} \in S_{(L_x^{(d,n)}, a \pm b)} \oplus \alpha e_{n-d+1} + \cdots + \alpha e_n\). The cardinality of the set \(\{ y \in M_{d \times (n-d)}(\alpha/\pi^b \mathfrak{a}) | L_x^{(d,n)} \in S_{(L_y^{(d,n)}, a \pm b)} \oplus \alpha e_{n-d+1} + \cdots + \alpha e_n \}\) is then

\[
\#(\alpha/\pi^b \mathfrak{a}) \cdot \#(\alpha/\pi^b \mathfrak{a}) \cdot \cdots \cdot \#(\alpha/\pi^b \mathfrak{a}) = f^{d(n-d)b} - db.
\]

Note that the above number is independent of the choice of \(x\). Since

\[
\# \left( S_{(L_x^{(d,n)}, a \pm b)} \oplus \frac{\alpha e_{n-d+1} + \alpha e_{n-d+2} + \cdots + \alpha e_n}{d} \right) = \# \left( S_{(L_x^{(d,n)}, a \pm b)} \right),
\]

we have the following equation:

\[
\# \left( S_{(L,a \pm b)} \setminus \bigcup_{L' \in G_{L,a \pm b}} S_{(L',a \pm b-1)} \right) = f^{db - (n-d)b} \cdot \sum_{x \in M_{d \times (n-d)}(A/\pi^b \mathfrak{a})} \# S_{(L_{(d,n)}, a \pm b)}.
\]

Thus to complete the proof, it suffices to show that

\[
\# \left( \bigcup_{L' \in G_{L,a \pm b-1}} S_{(L',a \pm b-1)} \right) = \sum_{m=1}^{d} \left( \sum_{L' \in G_{L,a \pm b-m}} \# S_{(L',a \pm b-m)} \right),
\]

where \(c_m = -\left( \binom{m}{1} f \cdot c_1 + \binom{m}{2} f \cdot c_2 + \cdots + \binom{m}{m-1} f \cdot c_{m-1} \right) + 1\) if \(m > 1\) and \(c_1 = 1\).

This follows from inclusion-exclusion principle using the counting argument of the Grassmannian given in the beginning of this section.

We now state our main theorem of this section, an inductive formula of the Siegel series \(F_L(X)\).

**Theorem 4.9.** Assume that Conjecture 4.4 is true. Assume that \(L^{(d,n)}\) is an integral quadratic lattice. Then we have the following inductive formula, with respect to the Gross-Keating invariant, of the Siegel series \(F_L(X)\):

\[
\text{(4.4)}
\]
\[ \mathcal{F}_L(X) = \sum_{m=1}^{d} \left( c_m \cdot f^{(n+1)m} \cdot X^{2m} \cdot \sum_{L' \in \mathcal{G}_{L,d,m}} \mathcal{F}_{L'}(X) \right) + (1 - X)(1 - f^d X)^{-1} \cdot \left( \prod_{i=1}^{d} (1 - f^{2i} X^2) \right) \cdot \mathcal{F}_{L_0}^{(d,n)}(f^d X), \]

where \( c_m = - \left( \binom{m}{1} f \cdot c_1 + \binom{m}{2} f \cdot c_2 + \cdots + \binom{m}{m-1} f \cdot c_{m-1} \right) + 1 \) if \( m > 1 \) and \( c_1 = 1 \).

Here, for \( L' \in \mathcal{G}_{L,d,m} \),

\[ \begin{cases} 
\text{GK}(L) > \text{GK}(L') ; \\
|\text{GK}(L')| = |\text{GK}(L)| - 2m; \\
\text{GK}(L_0^{(d,n)}) = \text{GK}(L)^{(n-d)}. 
\end{cases} \]

Note that notion of \( L^{(d,n)} \), \( \mathcal{G}_{L,d,m} \), and \( \binom{m}{k} f \) can be found at the beginning of this section. Notion of \( L_0^{(d,n)} \) can be found at Remark 4.3(1).

**Proof.** If we plug the formula of Proposition 4.8 into the formula of Theorem 3.9, then we obtain

\[ \alpha(L, H_k) = \sum_{m=1}^{d} \left( c_m \cdot f^{(n-2k+1)m} \cdot \sum_{L' \in \mathcal{G}_{L,d,m}} \alpha(L', H_k) \right) + f^{-d(n-d)b_f} (1 - f^{-k})(1 - f^{-(k-d)})^{-1} \cdot \left( \prod_{i=1}^{d} (1 - f^{2i-2k}) \right) \cdot \sum_{x \in M_{d \times (n-d)}(A/\pi b_f A)} \alpha(L_x^{(d,n)}, H_{k-d}), \]

where \( c_m = - \left( \binom{m}{1} f \cdot c_1 + \binom{m}{2} f \cdot c_2 + \cdots + \binom{m}{m-1} f \cdot c_{m-1} \right) + 1 \) if \( m > 1 \) and \( c_1 = 1 \).

On the other hand, as mentioned in Remark 4.3(4), we have that

\[ \alpha(L_x^{(d,n)}, H_{k-d}) = \alpha(L_0^{(d,n)}, H_{k-d}) \]

for any \( x \in M_{d \times (n-d)}(A/\pi b_f A) \). This completes the proof of the inductive formula. The rest follows from Theorems 0.1 and 0.3 of [IK].

**Corollary 4.10.** Assume that Conjecture 4.4 is true. If \( a_{n-1} < a_n \) for \( \text{GK}(L) = (a_1, \cdots, a_n) \) so that \( d = 1 \), then the above inductive formula turns to be

\[ \mathcal{F}_L(X) = f^{n+1} \cdot X^2 \cdot \mathcal{F}_{L_0}^{(n)}(X) + (1 - X)(1 + f X) \cdot \mathcal{F}_{L_0}^{(n)}(f X). \]

Note that notion of \( L^{(n)} \) can be found at the beginning of this section and that notion of \( L_0^{(n)} \) can be found at Remark 4.3(1).

The following lemma is used in the main theorems 5.9-5.10 of the next section.

**Lemma 4.11.** Let \( B_1 \) be a reduced form with \( \text{GK}(B_1) = (a_1, \cdots, a_{n-1}) \). Let \( B = \begin{pmatrix} B_1 & C \\ C & d \end{pmatrix} \), where \( ^tC = (c_1, \cdots, c_{n-1}) \). Here, \( B_1 \) is of size \((n - 1) \times (n - 1)\) and \( d \in \mathfrak{o} \). We assume that \( B \) satisfies the following conditions:

\[ \begin{cases} 
\text{ord}(2c_i) > (a_i + a_{n-1})/2 \text{ for } i < n - 1; \\
\text{ord}(2c_{n-1}) \geq a_{n-1}; \\
\text{ord}(d) \geq a_{n-1}; \\
\text{ord}(2c_{n-1} + d) \geq a_{n-1} + 1. 
\end{cases} \]
For example, $B$ satisfies the above assumption if $\text{GK}(B) = (a_1, \cdots, a_{n-1}, a_n)$ and $B$ is a reduced form satisfying one of the followings:

$$\begin{cases}
a_n > a_{n-1}; \\
a_n = a_{n-1} > a_{n-2}, \sigma(n-1) = n, \text{ord}(d) = a_n, \text{and} \, \mathfrak{a} = \mathbb{Z}_2.
\end{cases}$$

Let $B_x = t \begin{pmatrix} id_{n-1} \\ x \end{pmatrix} \cdot B \cdot \begin{pmatrix} id_{n-1} \\ x \end{pmatrix}$, where $x \in M_{1 \times (n-1)}(\mathfrak{a})$. Let $\mathcal{F}_{B_x}(X)$ be the Siegel series of the quadratic lattice associated to the symmetric matrix $B_x$. Then we have

$$\mathcal{F}_{B_x}(X) = \mathcal{F}_{B_{\mathfrak{a}}}(X)$$

for any $x$.

**Proof.** We write

$$B_x = (id_{n-1} \cdot t x) \cdot \begin{pmatrix} B_1 & C \\ tC & d \end{pmatrix} \cdot (id_{n-1} \cdot t x) = B_1 + (Cx + t(Cx) + d \cdot t xx).$$

The $(i, j)$-th entry of $(Cx + t(Cx) + d \cdot t xx)$ is $c_i x_j + c_j x_i + dx_i x_j$. Then we have the following:

$$\begin{cases}
\text{ord}(c_i x_j + c_j x_i + dx_i x_j) > a_i; \\
\text{ord}(2(c_i x_j + c_j x_i + dx_i x_j)) > \frac{a_i + a_j}{2} \text{ if } i < j.
\end{cases}$$

This, combined with Theorems 3.3 and 1.1 of [IK2], completes the proof.

5. **On the Siegel series of anisotropic quadratic lattices**

In this section, we will explain a more precise inductive formula of the Siegel series of anisotropic quadratic lattices defined over $\mathbb{Z}_p$, so as to compare it with an inductive formula of local intersection multiplicities of [GK93] in Sections 6-7. We first list a few necessary facts about anisotropic quadratic lattices over $\mathbb{Z}_p$.

**Remark 5.1.** Let $(L, q_L)$ be an anisotropic quadratic lattice over $\mathbb{Z}_p$. Here, we say that $(L, q_L)$ is anisotropic if $q_L(x) \neq 0$ for any nonzero element $x \in L$. Then the rank of $L$ is at most 4.

1. Let $D$ be the unique quaternion division algebra over $\mathbb{Q}_p$. Let $q_D$ be the associated quadratic form, which is defined to be the reduced norm on $D$. Let $O_D$ be the maximal order of $D$, characterized as follows:

$$O_D = \{ v \in D | q_D(v) \in \mathbb{Z}_p \}. $$

2. $O_D$ is a free $\mathbb{Z}_p$-module of rank 4. Then the pair $(O_D, q_D)$ is an anisotropic quadratic lattice over $\mathbb{Z}_p$ of rank 4. By Exercise 3 of Section 5.2 in [Kit93], the quadratic lattice $O_D$ is isometric to the quadratic lattice associated to the following half-integral symmetric matrix:

$$\begin{cases}
\begin{pmatrix} 1 & -\delta \\ \delta & p \end{pmatrix} & \text{if } p \neq 2; \\
\begin{pmatrix} 1/2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } p = 2.
\end{cases}$$

Here, $\delta$ is a unit in $\mathbb{Z}_p$ (with $p \neq 2$) such that $\delta$ modulo $p$ is not a square. These symmetric matrices are reduced forms. Thus we can see that

$$\text{GK}(O_D) = (0, 0, 1, 1).$$

3. Any anisotropic quadratic lattice over $\mathbb{Z}_p$ of rank $n (\leq 4)$ is always embedded into $(O_D, q_D)$, which can be shown by using Theorem 3.5.1 and Corollary 3.5.4 of [Kit93]. Thus we may regard an anisotropic quadratic $\mathbb{Z}_p$-lattice as a sublattice of $(O_D, q_D)$.

Indeed Ikeda and Katsurada assume that $p = 2$ in Theorem 3.3, loc. cit. But this theorem also holds for $p > 2$ and it was explained in the initial version of their paper posted on arXiv.
The Gross-Keating invariants of anisotropic quadratic lattices over $\mathbb{Z}_p$ with rank $\leq 3$ are well explained in [Bou07]. In the next subsection, we will explain the Gross-Keating invariants and some properties of anisotropic quadratic lattices of rank 4.

5.1. On an anisotropic quadratic lattice of rank 4. Let $(L, q_L)$ be an anisotropic quadratic lattice defined over $\mathbb{Z}_p$ of rank 4. Let $\text{GK}(L) = (a_1, a_2, a_3, a_4)$. We first study an anisotropic quadratic lattice of rank 2 in the following lemma.

**Lemma 5.2.** Let $X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, with a basis $(e_1, e_2)$, be an anisotropic quadratic lattice over $\mathbb{Z}_2$ with $\text{GK}(X) = (a_1, a_1 + 2t)$ for $t \geq 0$. If $X$ is an optimal form, then we have the following:

$$\text{ord}(a) = a_1, \text{ord}(2b) = a_1 + t, \text{ and } \text{ord}(c) = a_1 + 2t.$$ 

Thus any optimal form of an anisotropic quadratic lattice of rank 2 over $\mathbb{Z}_2$ is a reduced form.

**Proof.** If $t = 0$, then $X$ is not diagonalizable. The lemma then follows from Lemma 3.2.(b) of [Bou07].

For $t > 0$, the symmetric matrix with respect to the basis $(e_1, \frac{1}{2} - e_2)$ is an optimal form with $\text{GK} = (a_1, a_1)$. Using the result of the case $t = 0$ completes the proof. $\square$

If we write $(a_1, a_2, a_3)$ to be the Gross-Keating invariant of an anisotropic quadratic lattice of rank 3 over $\mathbb{Z}_p$ for any $p$, then only two of $a_i$’s have the same parity by Lemma 5.3 of [Bou07]. In the following proposition, we prove the same statement in the case of $n = 4$.

**Proposition 5.3.** Only two of $a_i$’s have the same parity. This holds for any prime $p$.

**Proof.** Since $L$ is a sublattice of $O_D$ with the same rank 4, the parity of $|\text{GK}(L)|$ is the same as that of $|\text{GK}(O_D)|$ by Remark 2.8(5). Since $\text{GK}(O_D) = (0, 0, 1, 1)$ and $|\text{GK}(O_D)| = 2$, either only two of $a_i$’s have the same parity or all of $a_i$’s have the same parity.

Assume that all of $a_i$’s have the same parity. Let $(e_1, e_2, e_3, e_4)$ be a basis of a reduced form of $(L, q_L)$. If $a_3 < a_4$, then the Gross-Keating invariant of the sublattice $L'$ spanned by $(e_1, e_2, e_3)$ is $(a_1, a_2, a_3)$ by Theorem 0.3 of [IK1] so that all three have the same parity. This is a contradiction since $L'$ is anisotropic so that only two should have the same parity.

If $p \neq 2$ and $a_3 = a_4$, then we choose a diagonal basis $(e_1, e_2, e_3, e_4)$ for $L$, whose associated symmetric matrix is diagonal having $u_i p^m$ as the $i$-th diagonal entry, where $u_i$ is a unit in $\mathbb{Z}_p$ (cf. by Remark 2.8(5)). Thus the Gross-Keating invariant of the sublattice $L'$ spanned by $(e_1, e_2, e_3)$ is $(a_1, a_2, a_3)$. All these have the same parity and so it is a contradiction.

We finally treat the remaining case, when $p = 2$ and $a_3 = a_4$. By Theorem 3.1 of [CIKY2] (or Theorems 3.6-3.8 of [CIKY1]), the quadratic lattice $(L, q_L)$ is not diagonalizable. In addition, the sublattice $L''$ spanned by $e_3$ and $e_4$ is not diagonalizable as well. Since $L''$ is anisotropic of rank 2, it is equal to $2^{a_3}(uw^2 + vx_1x_2 + wx_2^2)$ for units $u, v, w \in \mathbb{Z}_2$ by Lemma 5.2. Based on Theorem 3.1 of [CIKY2] (or Lemma 2.8 of [CIKY1]), we write a reduced form of $L$ as follows:

$$\begin{pmatrix} B_1 \\ tB_2 \end{pmatrix} 2^{a_3} \begin{pmatrix} B_2 \\ v/2 \\ v/2 \\ w \end{pmatrix}.$$ 

Here, $B_1$ is a reduced form with $\text{GK}(B_1) = (a_1, a_2)$. Then it is easy to show that the symmetric matrix associated to the sublattice $L'$ spanned by $(e_1, e_2, e_3)$ is a reduced form with $\text{GK}(L') = (a_1, a_2, a_3)$. Since $a_1, a_2, a_3$ have the same parity, $L'$ is isotropic by Lemma 5.3 of [Bou07]. This contradicts the assumption that $(L, q_L)$ is anisotropic.

By combining all the above cases, we can conclude that only two of $a_i$’s have the same parity. $\square$
Proposition 5.4. Let $B$ be a reduced form of an anisotropic quadratic lattice $L$ of rank 4 with $\text{GK}(L) = (a_1, a_2, a_3, a_4)$. Let $(e_1, e_2, e_3, e_4)$ be a basis of a reduced form $B$. If $p$ is odd, then we consider $B$ as a diagonal matrix.

Then any $(3 \times 3)$-submatrix of the matrix $B$, with respect to a basis $(e_i, e_j, e_k)$ among $(e_1, e_2, e_3, e_4)$, is a reduced form whose Gross-Keating invariant is $(a_i, a_j, a_k)$, where $i < j < k$.

Similarly, any $(2 \times 2)$-submatrix of the matrix $B$, with respect to a basis $(e_i, e_j)$ among $(e_1, e_2, e_3, e_4)$, is a reduced form whose Gross-Keating invariant is $(a_i, a_j)$, where $i < j$.

Proof. If $p$ is odd, then it is clear.

Assume that $p = 2$. Let $b_{ij}$ be the $(i, i)$-th entry of $B$. By Lemma 5.2, we have that $\text{ord}(b_{ij}) = a_i$. Using this, we can show that the $(3 \times 3)$-submatrix of the matrix $B$, with respect to a basis $(e_i, e_j, e_k)$, is a reduced form whose Gross-Keating invariant is $(a_i, a_j, a_k)$, by Definition 2.5. \hfill \square

If $p$ is odd, then we can choose a diagonal matrix as a reduced form of $L$. Then the Gross-Keating invariant consists of the order of each diagonal entry (cf. Remark 2.8 (1)).

We assume that $p = 2$. Let $B$ be an half-integral symmetric matrix associated to $(L, q_L)$. By using Theorem 2.4 of [Cho15] and Lemma 3.2.(c) of [Bou07], there are three types of $B$, up to equivalence, as follows:

\[
\begin{cases}
\text{Case (I)} : B = 2^i \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \perp 2^j \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}; \\
\text{Case (II)} : B = 2^i \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \perp (u_12^{\mu_1}) \perp (u_22^{\mu_2}); \\
\text{Case (III)} : B = (u_12^{\mu_1}) \perp (u_22^{\mu_2}) \perp (u_32^{\mu_3}) \perp (u_42^{\mu_4}).
\end{cases}
\]

Here,

\[
\begin{cases}
in \text{Case (I)}, i \leq j; \\
in \text{Case (II)}, u_i \equiv 1 \mod 2 \text{ and } \mu_1 \leq \mu_2; \\
in \text{Case (III)}, u_i \equiv 1 \mod 2, \mu_i \leq \mu_j \text{ if } i < j, \mu_1 < \mu_3, \text{ and } \mu_2 < \mu_4.
\end{cases}
\]

In the following theorem, we explain the Gross-Keating invariant of each case.

Theorem 5.5. The determinant of $B$ is a square. The Gross-Keating invariant of $B$ is described as follows:

1. In Case (I), we have $\text{GK}(B) = (i, i, j, j)$, where $i$ and $j$ have different parities.
2. In case (II), $\mu_1$ and $\mu_2$ have the same parity, $u_1 + u_2 \equiv 0 \mod 4$, and $\text{GK}(B) = (i, i) \cup (\mu_1, \mu_2 + 2)$.

Here, $i$ and $\mu_1$ have different parities.
3. In case (III),
   a. Assume $\mu_1 \neq \mu_2 \mod 2$. Then $\mu_3$ and $\mu_4$ have different parities and $\text{GK}(B) = (\mu_1, \mu_2, \mu_3 + 2, \mu_4 + 2)$.
   b. Assume that $\mu_1 \equiv \mu_2 \mod 2$ and that $u_1 + u_2 \equiv 2 \mod 4$ or $\mu_2 = \mu_3$. Then $\mu_3$ and $\mu_4$ have the same parity and $\text{GK}(B) = (\mu_1, \mu_2 + 1, \mu_3 + 1, \mu_4 + 2)$.
   c. Assume that $\mu_1 \equiv \mu_2 \mod 2$, $u_1 + u_2 \equiv 0 \mod 4$, and $\mu_3 \geq \mu_2 + 1$. Then $\text{GK}(B) = (\mu_1, \mu_2 + 2) \cup (\mu_3, \mu_4 + 2)$.

Here, $\mu_3$ and $\mu_4$ have the same parity, $\mu_1$ and $\mu_3$ have different parities, and $u_3 + u_4 \equiv 0 \mod 4$. 

Proof. The determinant of \( B \) is a square since the determinant of the matrix associated to \( O_D \) given at the beginning of this section is a square. Using this, the proof easily follows from Lemma 2.8 and Theorem 3.1 of [CIKY2] (or Theorems 3.5-3.8 of [CIKY1]) and Proposition 5.3.

In the next proposition, we compute the local density \( \alpha(L, O_D) \) for an anisotropic quadratic lattice \((L, q_L)\) of rank 4. The definition (and normalization) of the local density follows from Section 5 of [IK2]. We fix an anisotropic quadratic lattice \((L, q_L)\) of rank 4.

**Proposition 5.6.** The local density \( \alpha(L, O_D) \) is
\[
\alpha(L, O_D) = [O_D : L] \cdot p^3 \cdot p^{-2}(2(p + 1))^2.
\]
Here, \([O_D : L]\) equals to \((\text{GK}(L) - 2)/2\).

**Proof.** By Hilfssatz 17 of [Sie35], we have
\[
\alpha(L, O_D) = [O_D : L] \cdot \alpha(O_D, O_D),
\]
where \([O_D : L]\) is the index of \( L \) in \( O_D \). The local density \( \alpha(O_D, O_D) \) for a single quadratic lattice \( O_D \) is fully studied in [Cho15] \((p = 2)\) and [GY00] \((p \neq 2)\).

By using the matrix description of the quadratic lattice \((O_D, q_D)\) given at the notation, the local density \( \alpha(O_D, O_D) \) (cf. Proposition 6.2.3 and Theorem 7.3 of [GY00] when \( p \neq 2 \), Theorems 4.12 and 5.2 of [Cho15] when \( p = 2 \)) is
\[
\alpha(O_D, O_D) = p^3 \cdot (p^{-1}(2(p + 1))^2).
\]
This completes the proof.

**Remark 5.7.** In the above proof, indeed Theorems 4.12 and 5.2 of [Cho15] when \( p = 2 \) yield that
\[
\alpha(O_D, O_D) = 1/2 \cdot p^{-3} \cdot p^{-6} \cdot p^4(2(p + 1))^2.
\]
But, using the normalization of the local density explained in Section 5 of [IK2], we need to multiply \( p^6 \) and to ignore \( 1/2 \) in the above formula.

**5.2. The Siegel series of anisotropic quadratic lattice.** In this subsection, we will explain a more refined inductive formula of the Siegel series of anisotropic quadratic lattices over \( \mathbb{Z}_p \). Let \((L, q_L)\) be an anisotropic quadratic lattice over \( \mathbb{Z}_p \) of rank 4. By Remark 5.1(3), we may consider it as a sublattice of \((O_D, q_D)\).

We will work with an exclusively chosen basis \((e_1, e_2, e_3, e_4)\) of \( L \) as follows until the end of this subsection:
\[
(e_1, \cdots, e_n) \begin{cases} \text{diagonal and optimal} & \text{if } p \text{ is odd}; \\ \text{reduced} & \text{if } p = 2. \end{cases}
\]

Let \( L^{(4)} \) be the lattice spanned by \((e_1, e_2, e_3, 1/p \cdot e_4)\) and let \( L^{(4)}_0 \) be the lattice spanned by \((e_1, e_2, e_3)\). Here, we allow the case of \( a_3 = a_4 \).

**Lemma 5.8.** Assume that \((L, q_L)\) is not equal to \((O_D, q_D)\). Then \( L^{(4)} \) is contained in \((O_D, q_D)\).

**Proof.** It suffices to show that \( L^{(4)} \) is an integral quadratic lattice by Remark 5.1(3). Let \( \text{GK}(L) = (a_1, a_2, a_3, a_4) \). Since only two of \( a_i \)'s have the same parity and \( \text{GK}(O_D) = (0, 0, 1, 1) \), we have that \( a_4 \geq 2 \). Since \((e_1, e_2, e_3, e_4)\) is a basis of a reduced form, \((e_1, e_2, e_3, 1/p \cdot e_4)\) is also a basis of a reduced form (up to permutation) for \( L^{(4)} \) such that \( \text{GK}(L^{(4)}) = (a_1, a_2, a_3) \cup (a_4 - 2) \geq (0, 0, 0, 0) \). This completes the proof by Remark 2.3(4).

Recall that \( \mathcal{F}_L(X) \) is the Siegel series associated to the quadratic lattice \((L, q_L)\) such that \( \mathcal{F}_L(f^{-k}) = \alpha(L, H_k) \) (cf. Definition 3.10). In the following theorem, we will explain an inductive formula of the Siegel series \( \mathcal{F}_L(X) \) for an anisotropic quadratic lattice of rank 4 over \( \mathbb{Z}_2 \). This formula is much simpler than that of Theorem 4.9.
Theorem 5.9. Assume that $L^{(4)}$ is contained in $O_D$. Let $\text{GK}(L) = (a_1, a_2, a_3, a_4)$. Then we have the following inductive formula:

$$F_L(X) = p^5 \cdot X^2 \cdot F_{L^{(4)}}(X) + (1 - X)(1 + pX) \cdot F_{L^{(4)}}(pX).$$

Here,

$$\begin{cases} 
\text{GK}(L^{(4)}) = (a_1, a_2, a_3) \cup (a_4 - 2); \\
\text{GK}(L_0^{(4)}) = (a_1, a_2, a_3).
\end{cases}$$

Proof. If $a_3 < a_4$, then the formula follows from Theorem 4.9. Assume that $a_3 = a_4$. Since $a_2$ and $a_3$ should have different parities, we have that $a_2 < a_3$.

From our choice of $(e_1, e_2, e_3, e_4)$, the symmetric matrix with a basis $(e_1, e_2, e_3, \frac{1}{p} \cdot e_4)$ (resp. $(e_1, e_2, e_3)$) is a reduced form whose associated Gross-Keating invariant is $(a_1, a_2, a_3) \cup (a_4 - 2)$ (resp. $(a_1, a_2, a_3)$) (cf. Proposition 5.4). Using the argument used in the proofs of Lemma 4.6 and Proposition 4.8, we have the following formula:

$$\#S_{(L,a^\pm,b)} = \#S_{(L^{(4)},a^\pm,b-1)} + \sum_{x \in M_{1 \times 3}(A/\pi A)} \#S_{(L^{(4)},a^\pm,b)}.$$ 

Assume that $p = 2$. We consider a reduced form $B$ with respect to a basis $(e_1, e_2, e_3, e_4)$ of $(L, q_L)$. Then $B$ satisfies the assumption of Lemma 4.11 by Proposition 5.4. Thus, if we plug the above formula into the formula of Theorem 3.9 using the result of Lemma 4.11, then we obtain the desired inductive formula.

We now assume that $p \neq 2$. We consider a diagonal matrix $B = (u_1p^{a_3}) \perp (u_2p^{a_2}) \perp (u_3p^{a_1}) \perp (u_4p^{a_4})$ with respect to a basis $(e_1, e_2, e_3, e_4)$, where $u_i \in \mathbb{Z}_p$ is a unit. Then for $x = (x_1, x_2, x_3)$, we have

$$B_x = (id_3 \cdot x) \cdot B \cdot (id_3 \cdot x)^t = \begin{pmatrix} u_1p^{a_1} & 0 & 0 \\ 0 & u_2p^{a_2} & 0 \\ 0 & 0 & u_3p^{a_3} \end{pmatrix} + u_4p^{a_4}\begin{pmatrix} x_1 & x_1x_2 & x_1x_3 \\ x_2 & x_2x_2 & x_2x_3 \\ x_3 & x_3x_2 & x_3x_3 \end{pmatrix}.$$

If $\text{ord}\((u_3 + u_4x_3^2)p^{a_3}) = a_3$, where $(u_3 + u_4x_3^2)p^{a_3}$ is the $(3, 3)$-th entry of $B_x$, then the matrix $B_x$ is a reduced form with $\text{GK}(B_x) = (a_1, a_2, a_3)$ by using Theorem 3.3 of [IK2] for the $2 \times 2$-minor involving $u_1p^{a_1}$ and $u_2p^{a_2}$ and the definition of a reduced form given in Definition 2.5.

As explained in Remark 4.3 (3), the Siegel series is completely determined by the Extended Gross-Keating datum. In our case of anisotropic quadratic lattices, the extended Gross-Keating datum is the same as the Gross-Keating invariant. Thus if $\text{GK}(B_x) = (a_1, a_2, a_3)$ for any $x$, then the associated Siegel series’s are all equal. Using a similar argument used in the case $p = 2$, we have the desired formula.

Thus, it suffices to prove that $\text{ord}\((u_3 + u_4x_3^2)p^{a_3}) = a_3$, equivalently that $u_3 + u_4x_3^2$ is a unit. For $a \in \mathbb{Z}_p$, let $(\frac{a}{p})$ be the Legendre symbol. If $u_3 + u_4x_3^2$ is not a unit, then $(\frac{-u_3}{p}) = (\frac{u_4x_3^2}{p})$ so that $(\frac{u_3 + u_4x_3^2}{p}) = (\frac{u_4x_3^2}{p})^2 = 1$. Since the lattice spanned by $e_3$ and $e_4$ is anisotropic, we have that $(\frac{u_3 + u_4x_3^2}{p}) = -1$ by Lemma 2.8 of [Bou07]. This is a contradiction. Thus we conclude that $u_3 + u_4x_3^2$ is a unit.

The proof of the above theorem also holds for any anisotropic quadratic lattice of rank $n \leq 3$. We state it as the following theorem:
**Theorem 5.10.** Let \((L, q_L)\) be an anisotropic quadratic lattice over \(\mathbb{Z}_p\) of rank \(n\). Let \(\text{GK}(L) = (a_1, \ldots, a_n)\). Then we have the following inductive formula for \(F_L(X)\):

\[
F_L(X) = \begin{cases} 
  p^{n+1} \cdot X^2 \cdot F_{L^{(n)}}(X) + (1 - X)(1 + pX) \cdot F_{L_0^{(n)}}(pX) & \text{if } 2 \leq n \leq 4; \\
  p^2 \cdot X^2 \cdot F_{L_0^{(1)}}(X) + (1 - X)(1 + pX) & \text{if } n = 1.
\end{cases}
\]

Here, \(L^{(n)}\) is spanned by \((e_1, \ldots, e_{n-1}, 1/p \cdot e_n)\) and \(L_0^{(n)}\) is spanned by \((e_1, \ldots, e_{n-1})\) so that

\[
\begin{cases} 
  \text{GK}(L^{(n)}) = (a_1, \ldots, a_{n-1}) \cup (a_n - 2); \\
  \text{GK}(L_0^{(n)}) = (a_1, \ldots, a_{n-1}).
\end{cases}
\]

6. **Comparison: the Siegel series and the local intersection multiplicities**

Gross and Keating computed the local intersection multiplicities in [GK93] and Kudla confirmed that it is the same as the derivative of the Siegel series of an anisotropic quadratic lattice of rank 3 at \(p^{-2}\) (cf. [ARGOS07]). The method used to show the equality between these two objects is to compute both sides independently, and then to compare them directly. The calculation of the local intersection multiplicities in [GK93] is based on an inductive formula given in Lemma 5.6 in loc. cit.

In this section, we will compare the inductive formula of [GK93] with our inductive formula of Theorem 5.10. Then we will show that these two are essentially equal, beyond matching values. In addition to that, we will explain a newly discovered equality between the local intersection multiplicity on the special fiber and the derivative of another Siegel series in Theorem 5.7. This observation had been missed in both of Siegel series and intersection numbers.

Let us restrict the following situation exclusively in this section:

\[
\begin{cases} 
  L : \text{anisotropic quadratic lattice over } \mathbb{Z}_p \text{ of rank } 3; \\
  M : \text{anisotropic quadratic lattice over } \mathbb{Z}_p \text{ of rank } 2; \\
  N : \text{anisotropic quadratic lattice over } \mathbb{Z}_p \text{ of rank } 1.
\end{cases}
\]

As in Section 5.2, a basis of each lattice, consisting of \(e_i\)'s, is chosen to be

\[
\begin{cases} 
  \text{diagonal and optimal} & \text{if } p \text{ is odd}; \\
  \text{reduced} & \text{if } p = 2.
\end{cases}
\]

In the following two lemmas, we list the initial values of the Siegel series and its derivative, in order to compare both inductive formulas.

**Lemma 6.1.** We have

\(F_L(1/p^2) = F_M(1/p) = F_N(1) = 0.\)

**Proof.** It is clear from Theorem 3.9. \(\square\)

**Lemma 6.2.** Special values of the derivative of the Siegel series are as follows:

\[
F_M'(1/p) = \begin{cases} 
  -(p - 1) & \text{if } (a_1, a_2) = (0, 0); \\
  -2(p - 1)(p + 1) & \text{if } (a_1, a_2) = (1, 1); \\
  -2(p - 1) & \text{if } (a_1, a_2) = (0, 1),
\end{cases}
\]

\[
F_N'(1) = \begin{cases} 
  -1 & \text{if } (a_1) = (0); \\
  -(p + 1) & \text{if } (a_1) = (1).
\end{cases}
\]

**Proof.** Let \(F_{(a_1, a_2)}(X)\) be \(F_M(X)\) such that \(M\) is an anisotropic quadratic lattice with \(\text{GK}(M) = (a_1, a_2)\). Then \(F_{(0, 0)}(X) = (1 - X)(1 - pX)\), \(F_{(1, 1)}(X) = (1 - X)(1 + p^2X)(1 - p^2X^2)\), and \(F_{(0, 1)}(X) = (1 - X)(1 - p^2X^2)\). Thus

\[
F_{(0, 0)}'(1/p) = -(p - 1), F_{(1, 1)}'(1/p) = -2(p - 1)(p + 1), \text{ and } F_{(0, 1)}'(1/p) = -2(p - 1).
\]
Similarly, Let \( F_{(a_1)}(X) \) be \( F_N(X) \) such that \( N \) is an anisotropic quadratic lattice with \( \text{GK}(N) = (a_1) \). Then \( F_{(0)}(X) = 1 - X \) and \( F_{(1)}(X) = (1 - X)(1 + pX) \). Thus
\[
F_{(0)}'(1) = -1 \text{ and } F_{(1)}'(1) = -(p + 1).
\]

Let \( \text{GK}(L) = (a_1, a_2, a_3) \). If \((e_1, e_2, e_3)\) is a chosen basis of \( L \), then \( M \left( := L_0^{(3)} \right) \) is spanned by \((e_1, e_2)\) so that \( \text{GK}(M) = (a_1, a_2) \). We have the following inductive formula:

**Proposition 6.3.** Let \( \alpha(L, O_D) \) be the local density of the pair of quadratic lattices \((L, q_L)\) and \((O_D, q_D)\). Then we have the following inductive formula:
\[
\frac{F_{L}'(1/p^2)}{\alpha(L, O_D)} = \frac{F_{L_3}'(1/p^2)}{\alpha(L^{(3)}, O_D)} + \frac{p - 1}{2p} \cdot F_{M}'(1/p).
\]

**Proof.** By differentiating the formula of Theorem 5.10 at \( p^{-2} \) using Lemma 6.1 we have
\[
F_{L}'(1/p^2) = F_{L_3}'(1/p^2) + (1 - \frac{1}{p^2})(p + 1) \cdot F_{M}'(1/p).
\]

On the other hand, for any anisotropic quadratic lattice \( L \) of rank 3, we have
\[
\alpha(L, O_D) = 2(p + 1)^2 p^{-1}
\]
by Theorem 1.1 of [Wed07-1]. Combining two, we obtain the desired inductive formula. \( \Box \)

Let \( N = M_0^{(2)} \) so that \( N \) is spanned by \((e_1)\) and \( \text{GK}(N) = (a_1) \). Then we have the following inductive formulas:

**Proposition 6.4.** Let \( M \) be an anisotropic quadratic lattice of rank 2 with \( \text{GK}(M) = (a_1, a_2) \). Then
\[
\left\{ \begin{array}{l}
F_{M}'(1/p) = p \cdot F_{M_2}'(1/p) + 2(p - 1) \cdot F_{N}'(1); \\
F_{N}'(1) = p^2 \cdot F_{N_3}'(1) - (p + 1).
\end{array} \right.
\]

**Proof.** The formulas directly follow by differentiating the formulas of Theorem 5.10 using Lemma 6.1. \( \Box \)

Let \( F_{(a_1)}'(1) = F_{N}'(1) \), where \( \text{GK}(N) = (a_1) \). If we combine the formula of Proposition 6.4 for \( N \) with Lemma 6.2 then we get the following value of \( F_{(a_1)}'(1) \).

**Lemma 6.5.** We have that
\[
F_{(a_1)}'(1) = -(1 + p + p^2 + \cdots + p^{a_1}).
\]

**Proof.** By Proposition 6.4 and Lemma 6.2 we have that
\[
F_{(a_1)}'(1) = \left\{ \begin{array}{ll}
-p^{a_1} - (p + 1)(1 + p^2 + \cdots + p^{a_1-2}) & \text{if } a_1 \text{ is even}; \\
-(p + 1)p^{a_1-1} - (p + 1)(1 + p^2 + \cdots + p^{a_1-3}) & \text{if } a_1 \text{ is odd}.
\end{array} \right.
\]
This completes the proof. \( \Box \)

For an anisotropic lattice \( M \) with \( \text{GK}(M) = (a_1, a_2) \), we define
\[
T_{a_1, a_2} = \sum_{x=0}^{a_1} \sum_{y=0}^{a_2} p^{\min\{a_1 - x + y, a_2 - y + x\}}.
\]

The number \( T_{a_1, a_2} \) is indeed the local intersection multiplicity on the special fiber, defined by Equations (5.3) and (5.16) and Lemma 5.6 of [GK93]. In the following proposition, we will explain an inductive formula of \( T_{a_1, a_2} \), motivated by an inductive formula of \( F_{M}'(1/p) \) in Proposition 6.4 as they are supposed to match each other. Later in Theorem 6.7 \( T_{a_1, a_2} \) will be compared with the derivative of the Siegel series associated to the lattice \( M \).
Proposition 6.6. If \(a_2 \geq 2\), then
\[
\mathcal{T}_{a_1, a_2} = p \cdot \mathcal{T}_{a_1, a_2} - 2F'_{(a_1)}(1).
\]

Proof. We write
\[
\mathcal{T}_{a_1, a_2} = \sum_{x=0}^{a_1} \sum_{y=0}^{a_2} p^{|a_1-x+y,a_2-y+x|},
\]
\[
p \cdot \mathcal{T}_{a_1, a_2} - 2 = \sum_{x=0}^{a_1} \sum_{y=0}^{a_2} p^{|a_1-(y+1),a_2-(y+1)+x|}.
\]

We rewrite the above sums as follows:
\[
\mathcal{T}_{a_1, a_2} = \sum_{x=0}^{a_1} \sum_{y=0}^{a_2} p^{a_1-x+y} + \sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1} p^{a_2-y+x},
\]
\[
p \cdot \mathcal{T}_{a_1, a_2} - 2 = \sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1} p^{a_1-x+y+1} + \sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1-1} p^{a_2-(y+1)+x}.
\]

Then we have
\[
\sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1} p^{a_1-x+y} - \sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1-1} p^{a_1-x+y+1} = \sum_{x=0}^{a_1} p^{a_1-x},
\]
\[
\sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1} p^{a_2-y+x} - \sum_{x=0}^{a_1} \sum_{y=0}^{a_2-a_1-1} p^{a_2-(y+1)+x} = \sum_{x=0}^{a_1} p^{a_2-a_2-x}.
\]

Thus we have
\[
\mathcal{T}_{a_1, a_2} - p \cdot \mathcal{T}_{a_1, a_2} - 2 = 2 \sum_{x=0}^{a_1} p^x = -2F'_{(a_1)}(1).
\]

We now compare the local intersection multiplicity \(\mathcal{T}_{a_1, a_2}\) on the special fiber, with the derivative of the Siegel series associated to the lattice \(M\) in the following theorem.

Theorem 6.7. We have the following equality:
\[
\mathcal{T}_{a_1, a_2} = \frac{-1}{p-1} \cdot F'_M(1/p).
\]

In addition, both sides satisfy the same inductive formula.

Proof. By Propositions 5.4 and 6.6, it suffices to prove that both sides have the same initial values.

The initial values of \(\mathcal{T}_{a_1, a_2}\) can be computed directly from its definition as follows:
\[
\mathcal{T}_{a_1, a_2} = \begin{cases} 
1 & \text{if } (a_1, a_2) = (0, 0); \\
2(p + 1) & \text{if } (a_1, a_2) = (1, 1); \\
2 & \text{if } (a_1, a_2) = (0, 1), 
\end{cases}
\]

On the other hand, by Lemma 5.7, we have
\[
F'_M(1/p) = \begin{cases} 
-(p - 1) & \text{if } (a_1, a_2) = (0, 0); \\
-2(p - 1)(p + 1) & \text{if } (a_1, a_2) = (1, 1); \\
-2(p - 1) & \text{if } (a_1, a_2) = (0, 1), 
\end{cases}
\]

Therefore, both \(\mathcal{T}_{a_1, a_2}\) and \(\frac{-1}{p-1} \cdot F'_M(1/p)\) have the same initial values. \(\square\)
For an anisotropic quadratic $\mathbb{Z}_p$-lattice $L$ of rank 3 with $\text{GK}(L) = (a_1, a_2, a_3)$, put $\alpha_p(a_1, a_2, a_3) := \alpha_p(L)$ which is the local intersection multiplicity defined in Equation (5.3) of [GK93]. We finally compare it with the derivative of the Siegel series associated to the quadratic lattice $L$ in the following theorem.

**Theorem 6.8.** Let $\text{GK}(L) = (a_1, a_2, a_3)$. Then we have the following equality:

$$\alpha_p(a_1, a_2, a_3) = -\frac{2p}{(p-1)^2} \cdot \frac{F'_L(1/p^2)}{\alpha(L, O_D)}.$$

Moreover, both sides satisfy the same inductive formula.

**Proof.** Let $\tilde{T}_{a_1, a_2, a_3} = -\frac{2p}{(p-1)^2} \cdot \frac{F'_L(1/p^2)}{\alpha(L, O_D)}$. Then by Theorem 6.7, the formula of Proposition 6.3 turns to be

$$\tilde{T}_{a_1, a_2, a_3} = \tilde{T}_{a_1, a_2, a_3 - 2} + \tilde{T}_{a_1, a_2}.$$

On the other hand, by Lemma 5.6 and Equations (5.16) and (5.18) of [GK93], the local intersection multiplicity $\alpha_p(a_1, a_2, a_3)$ satisfies the following inductive formula:

$$\alpha_p(a_1, a_2, a_3) = \alpha_p(a_1, a_2, a_3 - 2) + \tilde{T}_{a_1, a_2}.$$

Therefore, it suffices to show that both $\tilde{T}_{a_1, a_2, a_3}$ and $\alpha_p(a_1, a_2, a_3)$ have the same initial values. Therefore, both sides have the same initial values by Proposition 1 of [Rap07].

7. **Application 1: The intersection number over a finite field**

In this section we revisit the results of Gross-Keating [GK93] and give a new identity between certain intersection numbers of cycles over a finite field and the sum of the Fourier coefficients of the Siegel-Eisenstein series for $\text{Sp}$.

### 7.1. Main results

For a positive integer $m$, we denote by $T_m$ the modular correspondence of degree $m$ defined in [Gör07-2]. It can be regarded as a flat scheme $T_m$ over $\mathbb{Z}$ which is explicitly given by $T_m = \text{Spec } \mathbb{Z}[x, y]/(\varphi_m) \subset \text{Spec } \mathbb{Z}[x, y] =: S$, where we think the latter scheme $S$ as the product of two copies of the coarse moduli space $Y_0(1)$ of elliptic curves. Here $\varphi_m$ is the modular polynomial of degree $m$ (see [Vog07] and [Gör07-2]). We consider $T_{m, p} := T_m \otimes \text{Spec } \overline{\mathbb{F}}_p$ and $T_{m, \mathbb{C}} := T_m \otimes \text{Spec } \mathbb{C}$. Two cycles $T_{m_1, \mathbb{C}}$ and $T_{m_2, \mathbb{C}}$ intersect properly if and only if $m_1 m_2$ is not a square (equivalently, $T_{m_1, \mathbb{C}}$ and $T_{m_2, \mathbb{C}}$ intersect properly inside $S \otimes \text{Spec } \mathbb{C}$ by Proposition 2.4 of [GK93]).

We first state the following proposition to explain exactly when two cycles over a finite field intersect properly. It turns out to be the same as the situation over $\mathbb{C}$.

**Proposition 7.1.** For given two positive integers $m_1$ and $m_2$, the cycles $T_{m_1, p}$ and $T_{m_2, p}$ intersect properly inside $S \otimes \text{Spec } \overline{\mathbb{F}}_p$ if and only if $m_1 m_2$ is not a square (equivalently, $T_{m_1, \mathbb{C}}$ and $T_{m_2, \mathbb{C}}$ intersect properly inside $S \otimes \text{Spec } \mathbb{C}$ by Proposition 2.4 of [GK93]).

Applying Proposition 7.1, we define the following proposition.

**Application 1**

Let $\varphi_p$ be the modular polynomial of degree $p$ (see [Vog07] and [Gör07-2]). We consider $T_{m_1, p}$ and $T_{m_2, p}$ in $S \otimes \text{Spec } \overline{\mathbb{F}}_p$. We follow the notation and results in Chapters 3-5 of [ARGOS07] (cf. [Gör07-1], [Gör07-2], [Wed07-2]).

We first state the following proposition to explain exactly when two cycles over a finite field intersect properly. It turns out to be the same as the situation over $\mathbb{C}$.

**Proposition 7.1.** For given two positive integers $m_1$ and $m_2$, the cycles $T_{m_1, p}$ and $T_{m_2, p}$ intersect properly inside $S \otimes \text{Spec } \overline{\mathbb{F}}_p$ if and only if $m_1 m_2$ is not a square (equivalently, $T_{m_1, \mathbb{C}}$ and $T_{m_2, \mathbb{C}}$ intersect properly inside $S \otimes \text{Spec } \mathbb{C}$ by Proposition 2.4 of [GK93]).
Proof. ‘Only if’ part follows from the proof of Theorem 2.1 of \cite{Vog}. Assume that $m_1m_2$ is not a square. If $T_{m_1,p}$ and $T_{m_2,p}$ do not intersect properly, then there is an open subscheme $U$ of dimension one which is included in an irreducible component of the intersection. We may assume that $U$ is contained in the ordinary locus of $S \otimes \text{Spec } \mathbb{F}_p$ since the supersingular locus is of dimension zero. For each geometric point $x$ in $U$, written by $x = ((E, E'), f_1, f_2)$ with two endomorphisms $f_1, f_2$ of degree $m_1, m_2$ between ordinary elliptic curves $E, E'$, there exists a lift $\tilde{x}$ of $x$ (e.g. canonical lift) to characteristic zero which has complex multiplication. By Proposition 2.4 of \cite{GK93} and the assumption on $m_1$ and $m_2$, the set $U(\mathbb{F}_p)$ then turns to be finite, which is a contradiction to the fact that the dimension of $U$ is one.

\begin{corollary}
If $m_1m_2$ is not a square, then $\varphi_{m_1}$ and $\varphi_{m_2}$ make up a regular sequence in $\mathbb{F}_p[x, y]$ and also in $\mathbb{F}_p[[x, y]]$.
\end{corollary}

Proof. It suffices to prove that $\varphi_{m_1}$ is not a non-zero divisor in $A := \mathbb{F}_p[x, y]/(\varphi_{m_2})$. Suppose the converse. Then there exists $\alpha \in \mathbb{F}_p[x, y]$ which is not divided by $\varphi_{m_2}$ such that $\varphi_{m_1}$ divides $\alpha \varphi_{m_1}$. Since $\mathbb{F}_p[x, y]$ is UFD, there exists an irreducible common factor $h$ of $\varphi_{m_2}$ and $\varphi_{m_1}$. Proposition \text{[a]} implies that $\mathbb{F}_p(x, y)/(\varphi_{m_1}, \varphi_{m_2})$ is Artinian but this gives a contradiction with the existence of $h$. The same argument works for $\mathbb{F}_p[[x, y]]$. \hfill \Box

Let us take two positive integers $m_1$ and $m_2$ such that $m_1m_2$ is not a square. We write $(T_{m_1,p}, T_{m_2,p})$, the intersection number over a finite field, which is explicitly defined as follows:

\begin{equation}
(T_{m_1,p}, T_{m_2,p}) := \text{length}_{\mathbb{F}_p} \mathbb{F}_p[x, y]/(\varphi_{m_1}, \varphi_{m_2}).
\end{equation}

Our goal is to compute $(T_{m_1,p}, T_{m_2,p})$ explicitly.

\begin{theorem}
Assume that $p$ is odd. Then for any two positive integers $m_1$ and $m_2$ such that $m_1m_2$ is not a square, the intersection number $(T_{m_1,p}, T_{m_2,p})$ is independent of the choice of a prime number $p$ and its explicit value is given as follows:

\begin{equation}
(T_{m_1,p}, T_{m_2,p}) = \frac{1}{288} \sum_{T \in \text{Sym}^2(\mathbb{Z}) \geq 0 \atop \text{diag}(T) = (m_1, m_2)} c(T) = (T_{m_1,\mathbb{C}}, T_{m_2,\mathbb{C}}).
\end{equation}

Here, $c(T)$ is the Fourier coefficient of the Siegel-Eisenstein series for $\text{Sp}_4(\mathbb{Z})$ of weight 2 with respect to the $(2 \times 2)$-half-integral symmetric matrix $T$ (cf. \cite{Nag92}).

If we reinterpret Theorem \text{[7.3]} in terms of modular polynomials in Equation \text{[7.2]}, then the $\mathbb{Z}[\frac{1}{2}]$-module $\mathbb{Z}\left[\frac{1}{2}\right][x, y]/(\varphi_{m_1}, \varphi_{m_2})$ satisfies the following interesting properties.

\begin{theorem}
We have the following interpretation about $\mathbb{Z}\left[\frac{1}{2}\right][x, y]/(\varphi_{m_1}, \varphi_{m_2})$:
\begin{enumerate}
\item $\mathbb{Z}\left[\frac{1}{2}\right][x, y]/(\varphi_{m_1}, \varphi_{m_2})$ is a free $\mathbb{Z}[\frac{1}{2}]$-module.
\item The rank of $\mathbb{Z}\left[\frac{1}{2}\right][x, y]/(\varphi_{m_1}, \varphi_{m_2})$, as a $\mathbb{Z}[\frac{1}{2}]$-module, is equal to

\begin{equation}
\frac{1}{288} \sum_{T \in \text{Sym}^2(\mathbb{Z}) \geq 0 \atop \text{diag}(T) = (m_1, m_2)} c(T).
\end{equation}

Here, $c(T)$ is as described in the above theorem.
\end{enumerate}
\end{theorem}

7.2. Decomposition of the intersection number over a finite field. In what follows let us go into the proof of Theorem \text{[7.3]}. Let $m_1, m_2$ be positive integers such that $m_1m_2$ is not a square. We denote by $\text{CLN}_{\mathbb{F}_p}$ (resp. $\text{CLN}_{W(\mathbb{F}_p)}$) the category of complete local Noetherian $\mathbb{F}_p$ (resp. $W(\mathbb{F}_p)$)-algebras with the residue field $\mathbb{F}_p$. The local deformation functor for a pair of elliptic curves $x := (E, E')$ over $\mathbb{F}_p$ on $\text{CLN}_{\mathbb{F}_p}$ is pro-represented by $R_p := \mathbb{F}_p[[t, t']]$, which is called the universal deformation ring of $x$ on $\text{CLN}_{\mathbb{F}_p}$. Similarly we have the universal deformation ring...
for any formal scheme $S$ curves over a finite field, it holds that 

$$f, \text{ deformation functor } \text{Def}_{\mathcal{O}_{S_p,x}}$$

and that $\text{Def}_{\mathcal{O}_{S_p,x}} \simeq \mathbb{F}_p[[j - j(E), j' - j(E')]]$. Here, $j(E)$ and $j(E')$ are the j-invariants of E and $E'$, respectively. Since $\text{Aut}(x) = \text{Aut}(E) \times \text{Aut}(E')$ acts naturally on the deformation datum (cf. (8.2) of [KMS5]) we have that

$$\hat{\mathcal{O}}_{S_p,x} \simeq R_{p,\text{Aut}(x)}$$

and that $R_p$ is a free $\hat{\mathcal{O}}_{S_p,x}$-module of rank $\frac{\# \text{Aut}(E) \cdot \# \text{Aut}(E')}{4}$ (cf. page 33 of [Gör07-2]). It is better to work on $R_p$ instead of $\hat{\mathcal{O}}_{S_p,x}$ because the moduli space of elliptic curves is not a fine moduli space.

The difference between these two objects in the computation of local intersection multiplicity is understood as below. We first consider the decomposition of the modular polynomial $\varphi_m$ over $R_p$ in terms of the local deformation theory.

**Proposition 7.5.** For a positive integer $m$, let $(\varphi_m)$ be the ideal of $R_p$ generated by the modular polynomial $\varphi_m$. Then

$$(\varphi_m) = \prod_{f, E \rightarrow E' \text{ isogeny of degree } m, \text{ mod } 1} I_{m,f,p},$$

where $I_{m,f,p} = (\varphi_{m,f,p})$ with $\varphi_{m,f,p} \in R_p$ is the minimal ideal of $R_p$ such that $f$ lifts to an isogeny over $R_p/I_{m,f,p}$.

**Proof.** We imitate the proof of Lemma 4.1 of [Gör07-2] for $R_p$. Let $\text{Def}_{f,*}$ for $* \in \{\mathbb{F}_p, W(\mathbb{F}_p)\}$ be the deformation functor on CLN$_a$ for $f$. As is similar to the proof of Lemma 4.1 of [Gör07-2], the deformation functor $\text{Def}_{f,\mathbb{F}_p}$ is represented by the divisor $\text{div}(\varphi_{f,p})$ in $\text{Spf } R_p$ for some $\varphi_{f,p} \in R_p$. It is easy to see that for two isogenies $f, g : E \rightarrow E'$ of degree $m$, $\varphi_{f,p}$ and $\varphi_{g,p}$ are coprime unless $f = \pm g$.

By Lemma 4.1 of [Gör07-2], which is in the situation with $R$, we also decompose the ideal $\varphi_m R$ of $R$ generated by $\varphi_m$ as follows:

$$\varphi_m R = \prod_{f, E \rightarrow E' \text{ isog. of degree } m, \text{ mod } 1} I_{m,f},$$

where $I_{m,f}$ is the minimal ideal of $R$ such that $f$ lifts to an isogeny over $R/I_{m,f}$. The ideal $I_{m,f}$ is generated by a single element $\varphi_{m,f}$ in $R$ which cannot be divisible by $p$.

Let $\varphi_{m,f,p}$ be the image of $\varphi_{m,f}$ under the natural projection $R \rightarrow R_p$. Then we have

$$\text{div}(\varphi_{f,p})(S) = \text{Def}_{f,\mathbb{F}_p}(S) = \text{Def}_{f,\mathbb{F}_p}(S) = \text{div}(\varphi_{m,f,p})(S) = \text{div}(\varphi_{m,f,p})(S)$$

for any formal scheme $S$ over $\text{Spf } R_p$. Hence we have $I_{m,f,p} = (\varphi_{f,p}) = (\varphi_{m,f,p})$.

**Lemma 7.6.** For positive integers $m_1, m_2$ with $m_1 m_2$ non-square and a pair $(E, E')$ of elliptic curves over a finite field, it holds that

$$\sum_{f_1} \sum_{f_2} \frac{\text{length}_{\mathbb{F}_p} \mathbb{F}_p[[j, j']][j - j(E), j' - j(E')]/(\varphi_{m_1}, \varphi_{m_2})}{\# \text{Aut}(E) \cdot \# \text{Aut}(E') \cdot \text{length}_{\mathbb{F}_p} \mathbb{F}_p[[t, t']]/(\varphi_{m_1, f_1, p}, \varphi_{m_2, f_2, p})}$$

where $\varphi_{m_1, f_1, p}$ is a factor of $\varphi_m$, given in the previous proposition. Here, the sums are over isogenies $f_i : E \rightarrow E'$ of degree $m_i$ up to $\pm 1$.

**Proof.** By Corollary 7.2, $\varphi_{m_1}$ and $\varphi_{m_2}$ make up a regular sequence. Using a similar argument of Equation (4.1) in page 34 of [Gör07-2], the claim follows from Lemma 4.2 of [Gör07-2], Proposition 7.5 and the fact that $R_p$ is a free $\hat{\mathcal{O}}_{S_p}$-module of rank $\frac{\# \text{Aut}(E) \cdot \# \text{Aut}(E')}{4}$. 

\[\square\]
We denote by \((E, E')\) the universal pair of elliptic curves over \(R_p\). Let \(y = ((E, E'), f_1, f_2)\) be a pair of \((E, E')\) and two isogenies \(f_1, f_2 : E \to E'\) with \(\deg(f_i) = m_i, i = 1, 2\). We define \(I_y\) as the minimal ideal of \(R_p\) such that both \(f_i\)’s lift to isogenies \(E \to E'\) of degree \(m_i\)’s modulo \(I_y\) for \(i = 1, 2\), respectively. Put

\[
\text{IM}_{p,y} := \text{length}_{\mathbb{F}_p} R_p/I_y,
\]

which is exactly the same as the local contribution in the summation of Lemma 7.6.

From now on, for a pair of two elliptic curves \((E, E')\) defined over \(\mathbb{F}_p\), we use the following notation:

\[
\left\{ \begin{array}{ll}
(E, E') : \text{(ord)} & \text{if both } E \text{ and } E' \text{ are ordinary;} \\
(E, E') : \text{(ss)} & \text{if both } E \text{ and } E' \text{ are supersingular.}
\end{array} \right.
\]

The intersection number \((T_{m_1,p}, T_{m_2,p})\) over a finite field is described as follows:

**Proposition 7.7.** Assume that \(m_1 m_2\) is non-square. Then we have

\[
(T_{m_1,p}, T_{m_2,p}) = \sum_{y = ((E, E'), f_1, f_2)} \text{IM}_{p,y} \frac{\sharp \text{Aut}(E) \sharp \text{Aut}(E')}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')}
\]

**Proof.** LHS is decomposed as follows:

\[
(T_{m_1,p}, T_{m_2,p}) = \sum_{(E,E')} (T_{m_1,p}, T_{m_2,p})(j(E), j(E'))
\]

\[
= \sum_{(E,E') : \text{ord}} (T_{m_1,p}, T_{m_2,p})(j(E), j(E')) + \sum_{(E,E') : \text{ss}} (T_{m_1,p}, T_{m_2,p})(j(E), j(E'))
\]

Using Lemma 7.6 we have that

\[
(T_{m_1,p}, T_{m_2,p})(j(E), j(E')) = \frac{4}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')} \sum_{f_1 : E \to E' \text{ iso. of degree } m_1, \text{ mod } \mathbb{Z}_p} \text{IM}_{p,(E, E'), f_1, f_2}
\]

\[
= \sum_{f_1 : E \to E' \text{ iso. of degree } m_1} \frac{\text{IM}_{p,(E, E'), f_1, f_2}}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')}.
\]

This completes the claim. \(\square\)

### 7.3. The local intersection multiplicity over a finite field.

Based on Proposition 7.7, we compute \(\text{IM}_{p,y}\) by using Serre-Tate theory for the ordinary case and [ARGOS07] for the supersingular case. Let us start with a series of the following lemmas.

**Lemma 7.8.** Let \(m_1, m_2\) be two integers with \(m_1 m_2\) non-square and let \(E, E'\) be two elliptic curves over a finite field. Then two isogenies \(f_1, f_2 : E \to E'\) with \(\deg(f_i) = m_i\) are linearly independent in the \(\mathbb{Z}\)-module \(\text{Hom}(E, E')\) and also in \(\text{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_p\).

**Proof.** Assume that \(f_1\) and \(f_2\) are linearly dependent. Then \(n_1 f_1 = n_2 f_2\) for some integers \(n_1, n_2\). We may assume that \(n_1\) and \(n_2\) are coprime since \(\text{Hom}(E, E')\) is a free \(\mathbb{Z}\)-module (III, Proposition 4.2 of [SI09]). By comparing the degree, we have \(n_1^2 m_1 = n_2^2 m_2\). Observe \((n_1 m_1)^2 = n_2^2 m_1 m_2\). This implies that \(m_1 m_2\) is a square, which is a contradiction. The latter claim follows from the \(\mathbb{Z}\)-freeness of \(\text{Hom}(E, E')\). \(\square\)

For a \(p\)-adic integer \(z = \sum_{i=0}^{\infty} a_i p^i\) with \(a_i \in \{0, \ldots, p-1\}\) and the indeterminant \(t\), we abusively define

\[
(1 + t)^z := \sum_{n=0}^{\infty} \binom{z}{n} t^n = \prod_{i=0}^{\infty} (1 + t^{p^i})^{a_i} \text{ modulo } p
\]
as an element in $\mathbb{F}_p[[t]]$, rather than $\mathbb{Z}_p[[t]]$. Here \( \left( \begin{array}{c} z \\ n \end{array} \right) := \frac{z(z-1) \cdots (z-n+1)}{n!} \) and set \( \left( \begin{array}{c} z \\ 0 \end{array} \right) := 1. \)

**Lemma 7.9.** For $z, w \in \mathbb{Z}_p^\times$ and $r \in \mathbb{Z}_{\geq 0}$ define an element $H(t, t') = (1 + t)^p - (1 + t')^w$ in $\mathbb{F}_p[[t, t']]$. Then there exists an element $f(t)$ in $\mathbb{F}_p[[t]]$ such that $H(t, f(t)) = 0$ and $f(t) \equiv zw^{-1}t^r$ mod $(t^{p+1})$.

**Proof.** Apply IV, Lemma 1.2 (Hensel’s lemma) of [Sil09] with $R = \mathbb{F}_p[[t]]$ (here $R$ is the notation there), $I = (t)$, $F(t') = H(t, t')$, $a = zw^{-1}t^r$, and $a = -w$. Notice that $F(a) \in I^{p+1}$ and $F'(a) = -w(1 + zw^{-1}t^r)^{w-1} \in R^\times = \mathbb{F}_p[[t]]^\times$. \( \square \)

**Lemma 7.10.** Let $e_1, e_2$ be two non-negative integers. For any two elements $f, g \in \mathbb{F}_p[[t, t']]$ which are coprime, it holds that
\[
\text{length}_{\mathbb{F}_p} R_p/\langle f^{e_1}, g^{e_2} \rangle = p^{e_1+e_2} \cdot \text{length}_{\mathbb{F}_p} R_p/(f, g).
\]

**Proof.** It follows from Lemma 4.2 of [Gör07-2]. \( \square \)

For an isogeny $f: E \to E'$ between ordinary elliptic curves over a finite field $k$, by functoriality we can associate a unique element of
\[
\text{Hom}(\hat{E}, \hat{E}') \times \text{HOM}_{\mathbb{Z}_p}(T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p, T_p(E') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p) = \text{End}(\hat{G}_m/k) \times \text{End}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p) \simeq \mathbb{Z}_p^2
\]
where $\hat{E}, \hat{E}'$ are formal groups associated to $E, E'$ respectively and $\hat{G}_m$ is the multiplicative formal group over $k$. We write $(z(f), w(f)) \in \mathbb{Z}_p^2$ for the element corresponding to $f$ via the above identification.

**Proposition 7.11.** Let $y = ((E, E'), f, 1, 2)$ as explained in Lemma 7.8. Let $T$ be the half-integral symmetric matrix associated to the quadratic lattice spanned by $(1, f, 2)$ so that $\text{diag}(T) = (m_1, m_2)$. Then
\begin{enumerate}
\item (ord) if $(E, E')$ is a pair of ordinary elliptic curves, then $\text{ord}_p(\text{det}(2T))$ is an even integer and $\text{IM}_{p, y} = p^{r_T}$, where $r_T := \frac{1}{4} \text{ord}_p(\text{det}(2T))$;
\item (ss) if $(E, E')$ is a pair of supersingular elliptic curves, then $\text{IM}_{p, y} = \mathcal{T}_{a_1, a_2}$. Here, $(a_1, a_2) = \text{GK}(T \otimes \mathbb{Z}_p)$. For the definition of $\mathcal{T}_{a_1, a_2}$, see Equation 6.11.
\end{enumerate}

**Proof.** The second claim follows from (4.1), p. 158 of [Rap07]. Assume the first case. As in Equation (2.1) in page 22 of [Gör07-1] there exists $d \in \mathbb{Z}$ such that $d^2d(E) = \text{det}(2T)$, where $d(E)$ is the discriminant of $\text{End}(E)$. Since $E$ is ordinary, we see that $p \nmid d(E)$ by Theorem 12, p.182 of [La87]. We express the matrix $T$ in terms of $(z_i, w_i) := (z(f_i), w(f_i)), i = 1, 2$ as follows:
\begin{equation}
\text{(7.6)}
\begin{array}{rcl}
(1) & \left( \begin{array}{c} \text{deg}(f_1) \\ \frac{1}{2} (\text{deg}(f_1 + f_2) - \text{deg}(f_1) - \text{deg}(f_2)) \\ \frac{1}{2} z_1 w_1 \\ \frac{1}{2} z_2 w_2 
\end{array} \right) = & \frac{1}{2} (\text{det}(f_1 + f_2) - \text{det}(f_1) - \text{det}(f_2)) \\
& = & \text{deg}(f_2) \\
& = & \text{deg}(f_2)
\end{array}
\end{equation}
Then we have
\[
\text{ord}_p(\text{det}(2T)) = 2\text{ord}_p(z_1w_1 - z_2w_2).
\]
It follows from Serre-Tate theory, Theorem 2.1-3), p. 148 of [Kat81] (see also the observation in p. 148 of [Rap07]) that the minimal ideal $I_y$ is given by $I_y = (H_1, H_2)$, where
\[
H_i = H_i(t, t') := (1 + t)^{z_i} - (1 + t')^{w_i}, i = 1, 2.
\]
By Lemma 7.10 and changing $z_i$ and $w_i$ if necessary, we may assume that $w_i \in \mathbb{Z}_p^\times$ for $i = 1, 2$. Then we have that
\[
R_p/(H_1, H_2) = R_p/((1+t)^{z_1 w_2} - (1+t')^{w_1 w_2}, (1+t)^{z_2 w_1} - (1+t')^{w_1 w_2}) = R_p/((1+t)^{z_1 w_2} - (1+t)^{z_2 w_1}, H_1).
\]
Lemma 7.3 yields an existence of \( f(t) \in \mathbb{F}_p[[t]] \) such that \( H_1(t, f(t)) = 0 \). This implies that

\[
R_p/(H_1, H_2) = \mathbb{F}_p[[t]]/((1 + t)z_1w_2 - (1 + t)^2z_2w_1).
\]

Write \( z_1w_2 - z_2w_1 = pt\alpha \) for \( \alpha \in \mathbb{Z}_p^\times \). Then \( R_p/(H_1, H_2) = \mathbb{F}_p[[t]]/(tp^T) \). Hence we have the claim.

\[ \square \]

### 7.4. Comparison in the ordinary locus using quasi-canonical lifts: over \( \mathbb{C} \) and over \( \mathbb{F}_p \)

Let us recall the canonical lift and quasi-canonical lifts of an ordinary elliptic curve \( E \) over \( \mathbb{F}_p \).

By Serre-Tate’s theorem ([Kat81]), there exists a unique canonical lift \( \widetilde{E}^{\text{can}} \) to \( W = W(\mathbb{F}_p) \) such that the reduction map induces an isomorphism \( \text{End}_W(\widetilde{E}^{\text{can}}) \cong \text{End}(E) \).

For the canonical lift, it is known that \( \text{End}_C(\widetilde{E}^{\text{can}}) = \text{End}_W(\mathbb{F}_p) = \text{End}_{\mathbb{F}_p}(E) = \mathcal{O}_{K,n} \), where \( K := \mathbb{F}_p(E) \otimes \mathbb{Z} \mathbb{Q} \) is an imaginary quadratic field for which \( p \) is split. Therefore the discriminant \( d(\widetilde{E}^{\text{can}}) \) of \( \text{End}_C(\widetilde{E}^{\text{can}}) \) (which equals that for \( \text{End}_{\mathbb{F}_p}(E) \)) is given by \( nD_K \), where \( D_K \) is the discriminant, and importantly we see that \( p \mid nD_K \) (see Theorem 12, p.182 of [La87]). Furthermore for two ordinary elliptic curves \( E, E' \), the reduction map induces an isomorphism

\[
\text{Hom}_C(\widetilde{E}^{\text{can}}, \widetilde{E'}^{\text{can}}) \cong \text{Hom}_{W(\mathbb{F}_p)}(\widetilde{E}^{\text{can}}, \widetilde{E'}^{\text{can}}) \cong \text{Hom}(E, E')
\]

since the Serre-Tate local coordinates \( q(E), q(E') \) for \( E, E' \) satisfy \( q(E) = q(E') = 1 \) respectively (cf. Theorem 2.1 and also last a few lines in p.180 of [Kat81]).

On the other hand if an elliptic curve \( \widetilde{E} \) over \( \mathbb{C} \) has CM by an order in an imaginary quadratic field \( K \), then \( \text{End}_C(\widetilde{E}) = \mathcal{O}_{K,mp} \) for some positive integer \( n \) which is coprime to \( p \) and that \( K := \mathbb{F}_p(E) \otimes \mathbb{Z} \mathbb{Q} \) is an imaginary quadratic field for which \( p \) is split. Assume that \( p \) is split in \( K \) or equivalently that \( \widetilde{E} \) has a good \( p \)-ordinary reduction \( E/\mathbb{F}_p \) (notice that one can take a smooth integral model over the ring \( \mathcal{O}_L \) of integers of some number field \( L \) among the isomorphism class of \( E \) since its \( j \)-invariant is an algebraic integer). Then we see that

\[
\mathcal{O}_{K,mp} = \text{End}_C(\widetilde{E}) = \text{End}_{\mathcal{O}_L}(\widetilde{E}) \hookrightarrow \text{End}_{\mathbb{F}_p}(E) = \mathcal{O}_{K,n}.
\]

by Theorem 12, p.182 of [La87] again. The elliptic curve \( \widetilde{E} \) is also a lift of an ordinary elliptic curve \( E \) and it is called a quasi-canonical lift of level \( s \) for \( E \) (cf. [Gro86], [Yu95]). It is known by Section 6 of [Gro86] or Proposition 3.5 of [Meu07] (see also p.106 of [Meu07]) that the number of isomorphism classes of quasi-canonical lifts of level \( s \) for an ordinary elliptic curve \( E \) is given as follows:

\[
\frac{\#\mathcal{O}_{K,n}^\times}{\#\mathcal{O}_{K,mp}^\times} \mathbb{Z}/p^s\mathbb{Z}^\times \rightarrow \frac{\#\text{Aut}(\widetilde{E})}{\#\text{Aut}(E)}(p^s - p^{s-1}).
\]

Let \( t \) be the local parameter for the local deformation of quasi-canonical lift of \( p \)-divisible group of \( E \) and let \( j \) be the parameter of the coarse moduli space \( \mathcal{A}_j \) defined by \( j \)-invariant. Then we have the relation \( j - j(E) = t - \frac{\text{Aut}(E)}{2} \). This explains the appearance of \( \frac{\#\text{Aut}(E)}{\#\text{Aut}(E)} \) in (7.8).

From now on, we will count the number of lifts to quasi-canonical lifts for given two isogenies \( f_1, f_2 : E \rightarrow E' \) where \( E \) and \( E' \) are ordinary elliptic curves. Let \( (z_i, w_i) = (z(f_i), w(f_i)) \) be an element of \( \mathbb{Z}_p^2 \) for \( f_i \)'s with \( i = 1, 2 \), as explained in the paragraph following Lemma 7.10. We write \( z_i = u_ip^{a_i} \) and \( w_i = v_ip^{b_i} \), where \( u_i, v_i \) are units in \( \mathbb{Z}_p \). Let \( T \) be the symmetric matrix associated to \( (f_1, f_2) \) as Equation (7.6). Note that Lemma 7.8 confirms that \( T \) is nonsingular in our situation. Put \( r = r_T \). For two isogenies \( f_1, f_2 \) given as above we denote by \( N(s, s'; f_1, f_2) \) the number of isomorphism classes of quasi-canonical lifts \( (\widetilde{E}_s, \widetilde{E}_{s'}) \) of level \( s, s' \) respectively \( (0 \leq s, s' \leq r) \) such that \( f_1, f_2 \) lift to isogenies from \( E_s \) to \( E_{s'} \).

**Proposition 7.12.** Keep the notation being as above. Assume that \( a_1 = \min\{a_1, a_2, b_1, b_2\} \).
(1) If $b_1 < b_2$ and $a_2 + b_1 \leq a_1 + b_2$, then we have that

$$N(s,s'; f_1, f_2) = \begin{cases} 
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times & \text{if } 0 \leq s \leq a_1 \text{ and } 0 \leq s' \leq b_1; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{b_1} & \text{if } a_1 < s \leq a_2 \text{ and } s' = s + b_1 - a_1; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{b_1} & \text{if } a_2 < s \leq r - b_1 \\
& \text{and } s' = s + b_1 - a_1 = s + b_2 - a_2; \\
0 & \text{otherwise.}
\end{cases}$$

(2) If $b_1 < b_2$ and $a_2 + b_1 > a_1 + b_2$, then $r = a_1 + b_2$ and we have that

$$N(s,s'; f_1, f_2) = \begin{cases} 
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times & \text{if } 0 \leq s \leq a_1 \text{ and } 0 \leq s' \leq b_1; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{a_1} & \text{if } b_2 < s' \leq b_1 \text{ and } s = s' - b_2 + a_2; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{a_1} & \text{if } b_1 < s' \leq r - a_1 \\
& \text{and } s = s' - b_1 + a_1 = s' - b_2 + a_2; \\
0 & \text{otherwise.}
\end{cases}$$

(3) If $b_1 \geq b_2$ and $a_2 + b_1 \leq a_1 + b_2$, then we have that

$$N(s,s'; f_1, f_2) = \begin{cases} 
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times & \text{if } 0 \leq s' \leq b_2 \text{ and } 0 \leq s \leq a_1; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{a_1} & \text{if } b_2 < s' \leq b_1 \text{ and } s = s' - b_2 + a_2; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{a_1} & \text{if } b_1 < s' \leq r - a_1 \\
& \text{and } s = s' - b_1 + a_1 = s' - b_2 + a_2; \\
0 & \text{otherwise.}
\end{cases}$$

(4) If $b_1 \geq b_2$ and $a_2 + b_1 > a_1 + b_2$, then $r = a_1 + b_2$ and we have that

$$N(s,s'; f_1, f_2) = \begin{cases} 
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times & \text{if } 0 \leq s' \leq b_2 \text{ and } 0 \leq s \leq a_1; \\
\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot p^{a_1} & \text{if } b_2 < s' \leq r - a_1 \text{ and } s = s' - b_2 + a_2; \\
0 & \text{otherwise.}
\end{cases}$$

Proof. We choose quasi-canonical lifts $\hat{E}_s$ and $\hat{E}_{s'}$ of level $s$ and $s'$ for $E$ and $E'$ respectively. Let $q_s := q(\hat{E}_s)$ and $q_{s'} := (\hat{E}_{s'})$ be the Serre-Tate coordinates for $\hat{E}_s$ and $\hat{E}_{s'}$ respectively. Then $q_s$ (resp. $q_{s'}$) is a primitive $p^s$ (resp. $p^{s'}$)-th root of unity in $\mathbb{Q}_p$ by Proposition 3.5-(3) of [Meu07]. The condition to lift $f_1$ and $f_2$ as an element of Hom($\hat{E}_s, \hat{E}_{s'}$) is given by the following two equations:

$$q_s^{z_1} = q_{s'}^{w_1}, \quad q_s^{z_2} = q_{s'}^{w_2}.$$  

Here, we follow notations of page 148 of [Rap07].

Firstly we consider the case when $b_1 \leq b_2$ and $a_2 + b_1 \leq a_1 + b_2$. If $s \leq a_1$, then $q_s^{z_1} = q_s^{z_2} = 1$ since $a_1 \leq a_2$ by the assumption. Thus any $q_{s'}$ with $0 \leq s' \leq \min\{b_1, b_2\} = b_1$ also satisfies the second equation in (7.9). In addition, such a $q_{s'}$ runs over all primitive $p^{b_1}$-roots of unity. Thus we have

$$N(s,s'; f_1, f_2) = \frac{\mathfrak{O}_{K,n}^\times \cdot \mathfrak{O}_{K,n}^\times}{\mathfrak{O}_{K,n}^\times} \cdot (\mathbb{Z}/p^s\mathbb{Z})^\times \cdot (\mathbb{Z}/p^{s'}\mathbb{Z})^\times$$

for $s, s'$ satisfying $0 \leq s \leq a_1$ and $0 \leq s' \leq b_1$. 


If $a_1 < s \leq a_2$, then the first equation in (7.9) gives the equality $s' = s + b_1 - a_1$ but there are $p^{b_1}$ numbers of $q_s'$ since $q_s^{z_1} = q_s^{z_1} = (q_{s' - b_1})^{p^{b_1}}$. Notice that $q_s^{z_2} = 1$ since $s \leq a_2$. On the other hand $s' = s + b_1 - a_1 \leq a_2 + b_1 - a_1 \leq b_2$ since we have assumed that $a_2 + b_1 \leq a_1 + b_1$. Hence the second equation in (7.9) is automatically fulfilled. Thus

$$N(s, s'; f_1, f_2) = \frac{\sharp O_{K,n}^\times}{\sharp O_{K,n}^\times} \frac{\sharp O_{K,n}^\times}{\sharp O_{K,n}^\times} \sharp (\mathbb{Z}/p^s \mathbb{Z})^\times \cdot p^{b_1}$$

where $s' = s + b_1 - a_1$.

Assume that $s > a_2$. Equation (7.9) implies $s = s' - b_1 + a_1 = s' - b_2 + a_2$. In particular $a_2 + b_1 = a_1 + b_2$. In this case $a_2 + b_1 \leq r$ since $r = \text{ord}_p(z_1 w_2 - z_2 w_1)$. As discussed in the previous case there are $p^{b_1}$ numbers of $q_s'$. With the notation fixed we may write $u_1 v_2 - u_2 v_1 = p^{r - (a_2 + b_1)} \alpha$ for some $\alpha \in \mathbb{Z}_p^\times$. For each $a \frac{\sharp v_1}{\sharp \text{Aut}(E_1)} \sharp \text{Aut}(E'_1) = \frac{p^r}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')}$

Proof. By exchanging $f_1$ and $f_2$, if necessary, we may assume that either $a_1$ or $b_1$ is the minimum among $a_1, a_2, b_1, b_2$. We first assume that $a_1$ is the minimum. Let us treat the case when $b_1 < b_2$ and $a_2 + b_1 \leq a_1 + b_2$. The other cases can be handled similarly so that we may skip them. Using Proposition 7.12 the left hand side turns to be

$$= \frac{1}{\sharp O_{K,n}^\times \sharp O_{K,n}^\times} \cdot \frac{\sharp (\mathbb{Z}/p^s \mathbb{Z})^\times}{\sharp (\mathbb{Z}/p^s \mathbb{Z})^\times} \cdot p^{b_1}$$

We next assume that $b_1$ is the minimum. Let $f_i'$ be the dual isogeny of $f_i$. Then we can see that

$$N(s, s'; f_1, f_2) = N(s', s; f_1', f_2').$$

Thus the argument used in the above case gives the desired formula. □
7.5. **The intersection number in the ordinary locus.** The intersection number \((T_{m_1}, C, T_{m_2}, C)\) is described in Proposition 2.4 of [GK93] and it turns out to be the sum of the Fourier coefficients of the Siegel-Eisenstein series for \(\text{Sp}_4/\mathbb{Q}\). We first consider the contribution coming from the ordinary part. For a symmetric positive definite \((2 \times 2)\)-half-integral matrix \(T\) (namely diagonal entries are integer and anti-diagonal entries are elements in \(\frac{1}{2} \mathbb{Z}\)), we define \(\chi_T(p)\) by

\[
\chi_T(p) = \begin{cases} 
1 & \text{if } p \text{ is split in } \mathbb{Q}(\sqrt{-\det(2T)}) \\
0 & \text{if } p \text{ is ramified in } \mathbb{Q}(\sqrt{-\det(2T)}) \\
-1 & \text{if } p \text{ is inert in } \mathbb{Q}(\sqrt{-\det(2T)})
\end{cases}
\]

**Theorem 7.14.** Assume that \(m_1 m_2\) is not a square. Then for any prime number \(p\), we have that

\[
\sum_{y = ((E, E'), f_1, f_2)} \frac{1}{\# \text{Aut}(E) \# \text{Aut}(E')} = \frac{1}{288} \sum_{T \in \text{Sym}_2(\mathbb{Z}) \cap \mathbb{Z}} \frac{1}{\text{deg}(T) = (m_1, m_2), \chi_T(p) = 1} \text{c}(T).
\]

Here \(c(T)\) is the Fourier coefficient of the Siegel-Eisenstein series of weight 2 with respect to \(\text{Sp}_4(\mathbb{Z})\) (cf. [Nag92]).

**Proof.** With the observation in CM elliptic curves having good ordinary reduction at \(p\) (which we call it \(p\)-ordinary and denote it by \(p\)-ord throughout this proof), we first obtain

\[
\sum_{y = ((E, E'), f_1, f_2)} \frac{1}{\# \text{Aut}(E) \# \text{Aut}(E')} = \sum_{T \in \text{Sym}_2(\mathbb{Z}) \cap \mathbb{Z}} \frac{1}{\text{deg}(T) = (m_1, m_2), \chi_T(p) = 1} \text{c}(T)
\]

by (2.19), p. 231 of [GK93].

For a pair of \(p\)-ordinary elliptic curves \((\tilde{E}, \tilde{E}')\) having two isogenies \(f_1\) and \(f_2\) as in LHS of the above equation, we already observed in Section 7.4 that \(\tilde{E} = \tilde{E}_s\) and \(\tilde{E}' = \tilde{E}'_s\), for suitable \(s\) and \(s'\), where \(\tilde{E}_s\) and \(\tilde{E}_s'\) are quasi-canonical lifts of level \(s\) and \(s'\) for ordinary elliptic curves \(E\) and \(E'\) respectively. Since the reduction of endomorphism groups is injective, the reductions of \(f_1\) and \(f_2\) (which are also denoted by \(f_1\) and \(f_2\) respectively) are isogenies from \(E\) to \(E'\) with the same degrees. On the other hand, any two isogenies \(f_1, f_2 : E \to E'\) over \(\overline{\mathbb{F}}_p\) can be lifted to those having the same degree defined over \(\mathbb{C}\) by choosing canonical lifts (cf. Section 7.4). Therefore, using Corollary 7.13 LHS of Equation (7.10) turns to be

\[
\sum_{y = ((E, E'), f_1, f_2)} \frac{1}{\# \text{Aut}(E) \# \text{Aut}(E')} = \sum_{s, s' \geq 0} \frac{1}{\# \text{Aut}(E_s) \# \text{Aut}(E'_s)} \sum_{\text{deg}(f_1) = m_i} N(s, s'; f_1, f_2) \frac{1}{\# \text{Aut}(E_s) \# \text{Aut}(E'_s)}
\]

\[
= \sum_{y = ((E, E'), f_1, f_2)} \frac{1}{\# \text{Aut}(E) \# \text{Aut}(E')} \sum_{\text{deg}(f_1) = m_i} \frac{1}{\# \text{Aut}(E) \# \text{Aut}(E')}
\]

Here, \(a_i\) and \(b_i\) are as explained in the paragraph just before Proposition 7.12 associated to \((f_1, f_2)\). This, combined with Equation (7.10), completes the proof.}\]
7.6. The intersection number in the supersingular locus. We next consider the contribution coming from the supersingular part of Equation (7.3).

**Theorem 7.15.** Assume that \( p \) is odd and that \( m_1 m_2 \) is not a square. Then we have that

\[
\sum_{y=\{(E,E'), f_1, f_2\} \atop \deg(f_i) = m_i} \frac{\text{IM}_{p,y}}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')} = \frac{1}{288} \cdot \sum_{T \in \text{Sym}^2(\mathbb{Z}) > 0} c(T), \quad \text{for } \text{diag}(T) = (m_1, m_2), \chi_T(p) = -1, 0
\]

**Proof.** We will proceed our proof without assuming \( p > 2 \). The assumption will be made later when it is needed.

Let \( (E, E') \) be a pair of two supersingular elliptic curves and \( f_1, f_2 : E \rightarrow E' \) be isogenies with \( \deg(f_i) = m_i \). Let \( T \) be the \((2 \times 2)\) half-integral symmetric matrix associated to \((f_1, f_2)\). Since \( (E, E') \) are supersingular elliptic curves and \( D := \text{Hom}(E) \otimes \mathbb{Q} = \text{End}(E') \otimes \mathbb{Q} \) is ramified at \( p \), the prime \( p \) has to be inert or ramified in \( \mathbb{Q}(\sqrt{-\det T}) \) (cf. Theorem 12, page 182 of [La87]). Hence \( \chi_T(p) = -1 \) or 0.

By the argument explained in p. 34-35 of [Göö07-2] and Proposition 7.11(2), we see that

\[
\sum_{y=\{(E,E'), f_1, f_2\} \atop \deg(f_i) = m_i} \frac{\text{IM}_{p,y}}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')} = \sum_{T \in \text{Sym}^2(\mathbb{Z}) > 0} \left( \sum_{(E,E'): (ss)} \frac{R_{\text{Hom}(E, E')}(T)}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')} \mathcal{T}_{a_1, a_2} \right),
\]

where \((a_1, a_2) = \text{GK}(T \otimes \mathbb{Z}_p)\).

Let \( \mathcal{F}_{T,l}(X) \) be the Siegel series associated to the local completion \( T \otimes \mathbb{Z}_l \) for any finite place \( l \).

Firstly, by Theorem 6.7 we have

\[
\mathcal{T}_{a_1, a_2} = \frac{-1}{p-1} \cdot \mathcal{F}_{T,p}(1/p).
\]

Secondly, by using the argument used in the proof of Theorem 4.3 in [Wed07-2], we have

\[
\sum_{(E,E'): (ss)} \frac{R_{\text{Hom}(E, E')}(T)}{\sharp \text{Aut}(E) \sharp \text{Aut}(E')} = \frac{1}{3^2 \cdot 2^3 \left( \frac{p-1}{p} \right)^2} \cdot \frac{\pi \gamma}{\Gamma(2) \Gamma(3/2)} \cdot \text{det}(T)^{1/2} \cdot \prod_{l < \infty} \alpha_l(T, O_D).
\]

Note that the left hand side of this equation is a rational number since each summand is a rational number. Thus the right hand side is convergent. Here \( O_D \) is a maximal order in \( D \). Then \( O_D \otimes \mathbb{Z}_p \) is the maximal order in the quaternion division algebra over \( \mathbb{Q}_p \) and \( O_D \otimes \mathbb{Z}_l \) with \( l \neq p \) is isomorphic to \( H_2 \), the hyperbolic space of rank 4.

Thus we have

\[
\prod_{l < \infty} \alpha_l(T, O_D) = \alpha_{T,p}(T, O_D) \cdot \prod_{l < \infty, l \neq p} \mathcal{F}_{T,l}(1/l^2),
\]

which is convergent.

We now plug in Equations (7.13)-(7.15) into Equation (7.12) then we have the following:
Let \( L \) be an anisotropic quadratic \( \mathbb{Q}_p \)-lattice of rank 2. Assume that \( (L \otimes \mathbb{Q}_p, q_L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p) \) is nondegenerate. Let \((O_D, q_D)\) be the quadratic lattice, where \(O_D\) is the maximal order of the quaternion division algebra \( D\) over \( \mathbb{Q}_p\) with \( q_D\) the reduced norm on \( D\). Then the local density \( \alpha_p(L, O_D) \) is given as follows:

\[
\alpha_p(L, O_D) = \begin{cases} 
\frac{p^{d'}}{2} \cdot 2(1 + \frac{1}{p}) & \text{if } d \text{ is even}; \\
\frac{p^{d'}}{2(d-1)} \cdot (1 + \frac{1}{p})^2 & \text{if } d = 2d' + 1 \text{ is odd}.
\end{cases}
\]

Here, \( d = |\text{GK}(L)|\).
Proof. Let $L'$ be a sublattice of $L$ of rank 2. Then By Theorem 5.6.4.(d) of \cite{Kit93}, we have, for any prime $p$ including $p = 2$,

$$\alpha_p(L', O_D) = p^{-|L:L'|} \cdot \alpha_p(L, O_D).$$

Indeed Theorem 5.6.4.(d) of \cite{Kit93} says inequality but in our case of anisotropic lattice, the inequality turns to be the equality.

Thus we may assume that $L$ is a maximal lattice so that $G_K(L) = (a_1, a_2)$ has only three possibilities: $(a_1, a_2) = (0, 0), (0, 1), (1, 1)$. We will use the local density formula of Yang in \cite{Yan98} $(p > 2)$.

From now on we assume that $p$ is odd. Then $L$ is diagonalizable so that the exponential order of each diagonal entry is $a_i$. Based on Theorem 7.1 of \cite{Yan98}, we have

$$\alpha_p(L, O_D) = \begin{cases} 
2 \left(1 + \frac{1}{p}\right) & \text{if } (a_1, a_2) = (0, 0); \\
\frac{2}{p} \left(1 + \frac{1}{p}\right) & \text{if } (a_1, a_2) = (1, 1); \\
\left(1 + \frac{1}{p}\right)^2 & \text{if } (a_1, a_2) = (0, 1).
\end{cases}$$

This completes the proof. \hfill \Box

Remark 7.17. (1) B. Conrad informed us that Theorem 7.3 is true when $(m_1, m_2) = 1$ with any prime $p$ (cf. \cite{Con17}), by proving that the scheme $\text{Spec} Z[x, y]/(\varphi_{m_1}, \varphi_{m_2})$ is flat.

On the other hand, Theorem 7.3 is true when $m_1, m_2 \leq 9$ with any prime $p$ by numerical calculation given in Appendix A.

(2) We note that Theorem 7.3 is a combination of Theorems 7.14 and 7.15 and the identity of Theorem 7.14 holds for $p = 2$. In the proof of Theorem 7.15 the only place we make the assumption $p > 2$ is the usage of Lemma 7.16.

On the other hand, one can also compute $\alpha_p(L, O_D)$ of Lemma 7.16 with $p = 2$ by using the local density formula given in \cite{Yan04}, which is more complicated than that of \cite{Yan98} $(p > 2)$. Consequently the explicit computation of $\alpha_p(L, O_D)$ when $p = 2$ would directly yield the identity of Theorem 7.3.

8. Application 2: Local intersection multiplicities on the special fiber

In this section we recall the special cycles on the Shimura variety for $\text{GSpin}(n, 2)$, $0 \leq n \leq 3$ defined by Kudla and Rapoport with a collaborator Yang. We refer the articles \cite{KRY99}, \cite{KRY06}, \cite{KR99}, \cite{KR00} for $n = 0, 1, 2, 3$ respectively and the readers are supposed to be familiar with these references. Let $(n_1, \ldots, n_r)$ be a division of $n + 1$, hence $n_i \geq 1$, $n_1 + \cdots + n_r = n + 1$. We always consider $r = 1$ when $n \leq 1$.

Let $V$ be a quadratic form over $Z$ with the signature $(n, 2)$ over $\mathbb{R}$ considered in each paper. Let $G = \text{GSpin}(V)$ be the generalized spinor group associated to $V$. Let $p$ be an odd prime so that $G$ is smooth over $Z_p$. Then for any neat open compact subgroup $K^p \subseteq G(\mathbb{A}_f^p)$ and a hyperspecial open compact subgroup $K_p$ of $G(\mathbb{Q}_p)$ defined by a suitable structure on $G(\mathbb{Q}_p)$ in each paper (cf. p. 704, line 9 of \cite{KR00} for $n = 3$), $d_i \in \text{Sym}_{n_i}(\mathbb{Q})$ and open compact subgroups $\omega_i \subseteq V(\mathbb{A}_f)^{n_i}$ which are invariant under $K^p$, one can associate the special cycles $Z(d_i, \omega_i)$ and consider the intersection of them:

$$Z = Z(d_1, \omega_1) \times_M \cdots \times_M Z(d_r, \omega_r) = \bigcap_{T \in \text{Sym}_{n+1}(\mathbb{Z}(p))_{\geq 0}} Z(T, \omega)$$

where $M$ is the integral model over $Z(p)$ for the Shimura variety associated to $(G, K^p K_p)$ and $\omega = \{\omega_i\}_{i=1}^r$ (cf. Section 2.3 of \cite{KR00} for $n = 3$). Since $n \leq 3$, the Shimura variety $M$ is of PEL type, namely a moduli space of abelian varieties with endomorphism structure by $O$ which is a maximal order of $M_2(B_\mathbb{Q})$, $B_F$, $B_\mathbb{Q}$, or $K$ for $n = 3, 2, 1, 0$ respectively. Here $B_\mathbb{Q}$ (resp. $B_F$) is a.
quaternion algebra over \( \mathbb{Q} \) (resp. over a real quadratic field \( F \)) and \( K \) is an imaginary quadratic field. Any geometric point \( \xi = \text{Spec} k \) on \( Z \) in characteristic \( p \) consists of quintuple \( (A, \iota, \lambda, \mathfrak{P}, \mathfrak{j}) \) satisfying the following conditions:

1. \( A \) is an abelian variety of dimension \( 2^n \) over \( k \) considered up to prime to \( p \) isogeny;
2. \( \iota: \mathcal{O} \otimes \mathbb{Z}_{(p)} \hookrightarrow \text{End}_k(A) \otimes \mathbb{Z}_{(p)} \) is a ring homomorphism such that
   \[
   \det(\iota(e) : \text{Lie}(A)) = N^0(e)^2
   \]
   for any \( c \in \mathcal{O} \) where \( N^0 \) is the reduced norm of \( \mathcal{O} \).
3. \( \lambda \) is a \( \mathbb{Z}_{(p)}^* \)-class of a prime to \( p \) isogeny \( A \to A^* \) such that \( n'\lambda \) comes from an ample line bundle on \( A \) for some \( n' \in \mathbb{Z} \). Here \( A^* \) is the dual abelian variety of \( A \);
4. \( \mathfrak{j} = (\mathfrak{j}_1, \ldots, \mathfrak{j}_r) \) and \( \mathfrak{j}_i \in \text{End}_k(A)^{n_i} \) is a vector of special endomorphisms for \( i = 1, \ldots, r \) such that \( q(\mathfrak{j}_i) = d_i \), where the quadratic form \( q \) is defined by the Rosati-involution \( * \) with \( q(x)\text{id}_A = x \circ x^* \) for any \( x \in \text{End}_k(A) \) (cf. Lemma 2.2 of [KR00] for \( n = 3 \)). Here a special endomorphism is defined to be an endomorphism \( f \) on \( A \) which satisfies \( f^* = f \) and \( \text{tr}^0(f) = 0 \) where \( * \) stands for the Rosati-involution with respect to \( \lambda \) and \( \text{tr}^0 \) means the reduced trace of \( \text{End}_k((A, \iota))^\text{op} \) (note that special endomorphisms can be regarded as elements in \( \text{End}_k((A, \iota))^\text{op} \) (cf. (2.13) of [KR00] for \( n = 3 \));
5. \( \mathfrak{P} = \{np^k \mid k \in K^p \} \) is a \( K^p \)-class of a \( \mathcal{O} \)-linear isomorphism \( p^\beta : V^p(A) := \prod_{\ell \nmid p} T_\ell(A) \otimes \mathbb{Q}_\ell \tilde{\to} \mathcal{O} \otimes k^p_\ell \). Here the action of \( K^p \) on \( \mathcal{O} \) is defined by the Clifford structure of \( \text{GSpin}(n, 2) \) in each case. It is known that \( \text{End}_\mathbb{O}(\mathcal{O} \otimes \mathbb{A}_f) \) contains \( V(\mathbb{A}_f) \). We require that \( (p^\beta)^*(\mathfrak{j}_i) := p^\beta \circ \mathfrak{j}_i |_{V(\mathbb{A}_f)} \) for any \( \mathfrak{j}_i \).

By Theorem 0.1 in [KR00] for \( n = 3 \), Theorem 6.1 in [KR99] for \( n = 2 \), Theorem 3.6.1 in [KRY06] for \( n = 1 \), and Proposition 5.9 in [KR99] for \( n = 0 \) we know a criterion for \( \mathcal{O}(T) := \mathcal{O}(T, \omega) \) is isolated. From now on we assume this condition. For any geometric point \( \xi \) on \( \mathcal{O}(T) \) we define the local intersection multiplicity of \( \mathcal{O}(T) \) at \( \xi \) by

\[
e(\mathcal{O}(T), \xi) := \text{length}_{\mathcal{O}(\mathcal{O}(T), \xi)} \mathcal{O}(\mathcal{O}(T), \xi)
\]

where \( \mathcal{O}(\mathcal{O}(T), \xi) \) is the localization of \( \mathcal{O}(\mathcal{O}(T)) \) at \( \xi \). By the assumption \( e(\mathcal{O}(T), \xi) \) is finite.

To compute \( e(\mathcal{O}(T), \xi) \), we need to consider the formal completion of it along \( \xi \) to apply the deformation theory. Put \( \mathfrak{j}' = (f_1, \ldots, f_{n+1}) \) for simplicity. By the assumption of each reference as above, we see that \( A \) is isomorphic to a product of supersingular elliptic curves \( E \) over \( \mathbb{F}_p \).

Let \( G = \mathbb{A} \) be the formal group associated to \( A \) and we denote by \( \mathfrak{f}_i \) the corresponding special endomorphism on \( G \) for each special endomorphism \( f_i \) via a natural algebra homomorphism

\[
(8.1) \quad \text{End}_k((A, \iota))^\text{op} \otimes \mathbb{Z}_p \hookrightarrow \text{End}((G, F)) \subset \mathcal{O} \otimes \mathbb{Z}_p \mathbb{Q}_p
\]

where \( F \) is the Frobenius endomorphism and \( \text{End}((G, F)) \) stands for the set of endomorphisms on \( G \) commuting \( F \) (cf. (5.13), p. 726 of [KR00] for \( n = 3 \)). The \( \mathbb{Z}_p \) submodule \( \mathcal{L} \) spanned by \( \{\mathfrak{f}_i\} \) in \( \text{End}(G) \) endows with the structure as a quadratic space \( \mathcal{L}' = (L, q\mathcal{L}') \) which comes from the Clifford structure. For instance \( xy + yx = (x, y)\mathcal{L}' \) for any \( x, y \in \mathcal{L}' \).

Since \( p \) is odd, there exists a basis of \( \mathcal{L}' \) such that \( q\mathcal{L}' \) is isometric to \( T' = \text{diag}(u_1 p^{a_1}, \ldots, u_{n+1} p^{a_{n+1}}) \) over \( \mathbb{Z}_p \) with integers \( a_i \leq \cdots \leq a_{n+1} \) and with units \( u_i, 1 \leq i \leq n+1 \) in \( \mathbb{Z}_p \). Then the Gross-Keating invariant of \( T' \) is given simply by \( \text{GK}(T') = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \). Accordingly we can take an optimal basis \( \varphi_1, \ldots, \varphi_{n+1} \) in \( \text{End}(G) \) such that \( \varphi_i^2 = q\mathcal{L}'(\varphi_i) = u_i p^{a_i} \) (1 \leq i \leq n+1).

Let \( R = W(\mathbb{F}_p)[[t_1, \ldots, t_n]] \) be the universal deformation ring of \( G \) on \( \text{CLN}_W(\mathbb{F}_p) \) which is isomorphic to the strict completion of \( \mathcal{O}_M \) at \( \xi \). Let \( \mathbb{G} \) be its universal family. Here we make the convention that \( R = W(\mathbb{F}_p) \) when \( n = 0 \).
We denote by $I = I(\varphi_1, \ldots, \varphi_{n+1})$ the minimal ideal of $R$ such that all $\varphi_i$’s are liftable to special endomorphisms on $G$ modulo $I$. By the theorem of Serre and Tate it is easy to see that

$$e(\mathcal{Z}(T'), \xi) = \text{length}_{W(\mathcal{P})} R/I$$

(cf. (6.1) of [KR00]). By Section 6 of [KR00] for $n = 3$, the proof of Theorem 6.1 of [KR99] for $n = 2$, Theorem 3.61 of [KRY06] for $n = 1$, and Theorem 5.11 of [KRY99] for $n = 0$, it turns out that $e(\mathcal{Z}(T'), \xi)$ depends only on $\text{GK}(T') = (a_1, \ldots, a_{n+1})$. Hence we may write it for

$$e(a_1, \ldots, a_{n+1}) := e(\mathcal{Z}(T'), \xi) = \text{length}_{W(\mathcal{P})} R/I(\varphi_1, \ldots, \varphi_{n+1}).$$

This will be checked in Theorem 8.3 below. When $a_{n+1} \geq 2$, we see that there exist $\varphi'_{n+1} \in \text{End}(G)$ such that $\varphi_{n+1} = \varphi'_{n+1}$ and that $L' = (\varphi_1, \ldots, \varphi_n, \varphi'_{n+1})$ makes a sublattice of $L$ with $\text{GK}(L') = (a_1, \ldots, a_n) \cup (a_{n+1} - 2)$ where $(a_1, \ldots, a_n) \cup (a_{n+1} - 2)$ means the re-ordering of $\{a_1, \ldots, a_n, a_{n+1} - 2\}$ to be the non-decreasing sequence. Therefore we have

$$\text{length}_{W(\mathcal{P})} R/I(\varphi_1, \ldots, \varphi_{n+1}) = e((a_1, \ldots, a_{n+1} + 2))).$$

Our interest is to understand the difference between

$$\text{length}_{W(\mathcal{P})} R/I(\varphi_1, \ldots, \varphi_{n+1})$$

in terms of the local intersection multiplicity of special cycles over a finite field. This motivates us to consider the following situation in special cycles in the special fiber $\mathcal{M}^\phi_{\mathcal{P}}$. Assume that $1 \leq n \leq 3$. Let $(n_1, \ldots, n_r)$ be a division of $n$, hence $n_i \geq 1$, $n_1 + \cdots + n_r = n$. We always consider $r = 1$ when $n = 1$. For $d_i \in \text{Sym}_n (\mathbb{Z}(\mathcal{P}))$ $(1 \leq i \leq r)$ let us consider the closed subscheme in $\mathcal{M}^\phi_{\mathcal{P}}$ given by

$$\mathcal{Z}(d_1, \omega_1)_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}} \cdots \times \mathcal{M}_{\mathcal{P}} \mathcal{Z}(d_r, \omega_r)_{\mathcal{P}} = \prod_{T \in \text{Sym}_n(\mathcal{P}) > 0, \text{diag}(T) = (d_1, \ldots, d_r)} \mathcal{Z}(T, \omega)_{\mathcal{P}}$$

for $1 \leq n \leq 3$. Any geometric point on $\mathcal{Z}(T, \omega)_{\mathcal{P}}$ is similarly a quintuple $(A, \iota, \lambda, \bar{t}, \bar{m})$ as before but in this case we replace $n + 1$ endomorphisms with $n$ endomorphisms $\bar{m}$ which is related to the fourth condition (4). In this section, when $n \leq 3$ and the underlying space $A = A_x$ of a geometric point $x$ is superspecial, we will study that the multiplicity

$$e(\mathcal{Z}(T, \omega)_{\mathcal{P}} x) := \text{length}_{\mathcal{P}} \mathcal{O}_{\mathcal{Z}(T, \omega)_{\mathcal{P}} x}$$

is finite and that it depends only on $\text{GK}(T \otimes \mathbb{Z}_p)$ under some conditions. Let us confirm this as follows.

Recall that $G$ is the formal group of $A$. Let $R_p = \mathbb{F}_p[[t_1, \ldots, t_n]]$ be the universal deformation of $G$ on $\text{CLN}_{\mathcal{P}}$ which is isomorphic to the completion of $\mathcal{O}_{\mathcal{M}_{\mathcal{P}}}$ at $x$. We put $R_p = \mathbb{F}_p$ when $n = 0$. Clearly $\mathcal{G}_p := \mathcal{G} \otimes \mathbb{F}_p$ is the local deformation of $G$ over $R_p$. Let $\varphi_1, \ldots, \varphi_n$ be an optimal basis of the lattice consisting of $n$ special endomorphisms $\bar{m}$ in the data of $x$ via (8.1).

We define the minimal ideal $I_p = I_p(\varphi_1, \ldots, \varphi_n)$ of $R_p$ such that all $\varphi_i$’s $(1 \leq i \leq n)$ are liftable to special endomorphisms on $\mathcal{G}_p$ modulo $I_p$. As before we put $I_p = \{0\}$ when $n = 0$. By definition we see that

$$e(\mathcal{Z}(T, \omega)_{\mathcal{P}} x) = \text{length}_{\mathcal{P}} R_p/I_p.$$
where
\[(b_1, b_2) = \begin{cases} (a_1, a_2) & \text{if } n = 3 \\ (0, a_{n-1}) & \text{if } n = 1, 2 \\ (0, 0) & \text{if } n = 0 \end{cases}.\]

Proof. Let us first consider the case of \(n = 3\). We follow the argument in page 733 of [KR00]. By our assumption \(q_L\) represents 1 over \(\mathbb{Z}_p\). This implies \(a_0 = 0\). In this case the formal group \(G\) of \(A_x\) decomposes into \(\hat{A}_0^1\) where \(\hat{A}_0\) is a formal group of dimension 2 and height 4 equipped with a principal quasi polarization \(\lambda_{\hat{A}_0}\) (cf. Section 4 of [KR00]). As explained right after (6.3) of [KR00] there exists \(x_0 \in L\) such that \(q_L(x_0) = 1\). Then the idempotents \(e_0 = \frac{1}{2}(1 + x_0), e_1 = \frac{1}{2}(1 - x_0)\) induce the further decomposition of \(\hat{A}_0\) as
\[\hat{A}_0 \simeq e_0\hat{A}_0 \times e_1\hat{A}_0 \simeq G^2_0\]
where \(G_0\) is a formal group of dimension 1 and height 2. Put \(M_0 = (x_0)^{\perp}\) in \(L\) which is of rank 2 and \(\text{GK}(M_0) = (a_1, a_2)\). By the Clifford structure for the special endomorphisms we see that \(xx_0 + x_0x = (x, x_0) = 0\) for any \(x \in M_0\). It follows from this that \(xe_0 = e_1x\). Therefore \(M_0\) can be regarded as a sublattice in \(\text{Hom}(e_0\hat{A}_0, e_1\hat{A}_0) \simeq \text{End}(G_0)\) which is the maximal order of a unique quaternion division algebra over \(\mathbb{Q}_p\). Since any principal quasi-polarization deforms automatically the deformation problem on \(\text{CLN}_{\mathbb{F}_p}\) of \(x\) is same as one of \((G_0, M_0)\). This shows that
\[(8.3) \quad e(Z(T, \omega)_{\mathbb{F}_p}, x) = \mathcal{T}_{a_1, a_2}.\]
Hence we have the claim.

The remaining cases will be done similarly in which the arguments of Theorem 6.1 of [KR99], Theorem 3.6.1 of [KRY06], and Proposition 5.9 of [KRY06] for \(n = 2, 1\) and \(n = 0\) should be consulted respectively. \(\square\)

Corollary 8.2. Keep the notation and the assumptions in Proposition 8.2. Then the geometric point \(x\) is isolated in \(Z(T)_{\mathbb{F}_p}\).

**Proposition 8.3.** Let \(T'\) be an element in \(\text{Sym}_{n+1}(\mathbb{Z}_p)\). For any isolated geometric point \(\xi = (A_\xi, \nu, \lambda, \mathcal{F}_T, \mathcal{F}')\) of \(Z(T')\) let \(L'\) be the \(\mathbb{Z}_p\)-lattice corresponding to the special endomorphisms \(\mathcal{F}'\) with \(\text{GK}(L') = \text{GK}(T' \otimes \mathbb{Z}_p) = (a_0, \ldots, a_n)\). Then for any \(T \in \text{Sym}_n(\mathbb{Z}_p)\) with \(\text{GK}(T \otimes \mathbb{Z}_p) = (a_0, \ldots, a_{n-1})\) such that \(T \otimes \mathbb{Z}_p\) comes from a sublattice of \(L'\) and a geometric point \(x\) of \(Z(T, \omega)_{\mathbb{F}_p}\) whose underlying abelian variety \(A_x\) is \(A_\xi\), it holds that
\[e(Z(T, \omega)_{\mathbb{F}_p}, x) = e(a_0, \ldots, a_n) - e(a_0, \ldots, a_n - 2).\]

**Proof.** When \(n = 3\), by Theorem 0.1 or Corollary 5.15 of [KR00], we see that \(T' \otimes \mathbb{Z}_p\) represents 1 over \(\mathbb{Z}_p\). This implies \(a_0 = 0\). Let \(L'\) be the lattice in \(\mathbb{Q}_p\) corresponding to \(T'\) over \(\mathbb{Z}_p\). Then it turns out that its Gross-Keating invariant becomes \(\text{GK}(L') = (0, a_1, a_2, a_3)\) with nondecreasing integers \(0 \leq a_1 \leq a_2 \leq a_3\) which satisfy the condition that the parities of the three integers never be same. Then the argument in p. 733 of [KR00] tells us that
\[(8.4) \quad e(0, a_1, a_2, a_3) = e(0, a_1, a_2, a_3) = \alpha_p(a_1, a_2, a_3)
where \(\alpha_p\) is the intersection number in Proposition 5.4 of [GK93] as mentioned before. By Lemma 5.6 of [GK93] and Proposition 8.1
\[e(0, a_1, a_2, a_3) - e(0, a_1, a_2, a_3 - 2) = \mathcal{T}_{a_1, a_2} = e(Z(T, \omega)_{\mathbb{F}_p}, x).\]

When \(n = 2\), by Theorem 6.1 of [KR99], we see that \(T' \otimes \mathbb{Z}_p\) is isometric to \(\text{diag}(1, u_1p^{a_1}, u_2p^{a_2})\) over \(\mathbb{Z}_p\) with units \(u_1, u_2\) in \(\mathbb{Z}_p\). Let \(L'\) be the lattice in \(\mathbb{Q}_p\) corresponding to \(T'\) over \(\mathbb{Z}_p\). Then it turns out that its Gross-Keating invariant becomes \(\text{GK}(L') = (0, a_1, a_2)\) with nondecreasing
integers $0 \leq a_1 \leq a_2$ which satisfy the condition that the parities of $0, a_2, a_3$ never be same. Then the argument in p. 195 loc.cit. shows that

$$e(0, a_1, a_2) = \alpha_p(0, a_1, a_2)$$

and similarly we have $e(0, a_1, a_2) - e(0, a_1, a_2 - 2) = T_{0,a_1} = e(\mathcal{Z}(T, \omega)_{\mathcal{F}_p}, x)$.

When $n = 1$, by Theorem 3.6.1 of [KRY06], we see that $T' \otimes \mathbb{Z}_p$ is isometric to $\text{diag}(u_0 p^{a_0}, u_1 p^{a_1})$ over $\mathbb{Z}_p$ with units $u_0, u_1$ in $\mathbb{Z}_p$. Let $L$ be the lattice of rank three in $\mathbb{Q}_p$ corresponding to $\text{diag}(1, T')$ over $\mathbb{Z}_p$. Then it turns out that its Gross-Keating invariant becomes $\text{GK}(L) = (0, a_0, a_1)$ with nondecreasing integers $0 \leq a_0 \leq a_1$ which satisfy the condition that the parities of $0, a_0, a_1$ never be same. Then Theorem 3.6.1 of loc.cit. shows that

$$e(a_0, a_1) = \alpha_p(0, a_0, a_1)$$

and similarly we have $e(a_0, a_1) - e(a_0, a_1 - 2) = T_{0,a_0} = e(\mathcal{Z}(T, \omega)_{\mathcal{F}_p}, x)$.

Finally we consider the case when $n = 0$. By Proposition 5.9 of [KRY06], we see that $T'$ (it is denoted by $t$ in the reference) satisfies that $a_0 := \text{ord}_p(t) \equiv 1 \mod 2$.

Let $L$ be the lattice of rank three in $\mathbb{Q}_p$ corresponding to $\text{diag}(1, 1, T')$ over $\mathbb{Z}_p$. Then it turns out that its Gross-Keating invariant becomes $\text{GK}(L) = (0, 0, a_3)$ and it satisfies that $a_0$ is not even. Then Theorem 5.11 of loc.cit. shows that

$$e(a_0) = \alpha_p(0, 0, a_0)$$

Then $e(a_0) - e(a_0 - 2) = 1 = e(\mathcal{Z}(T, \omega)_{\mathcal{F}_p}, x)$ by Lemma 5.6 of [GK93] and the claim follows with the convention made when $n = 0$. \hfill $\square$

Assume that $T \in \text{Sym}_n(\mathbb{Z}_p)$ satisfies the condition in Proposition 8.3. For $n$ and Gross-Keating invariant for $T$ in Equations (8.3)- (8.7) we take an anisotropic lattice $M$ of rank 2 with

$$\text{GK}(M) = \begin{cases} (a_1, a_2) & \text{if } n = 3 \\ (0, a_{n-1}) & \text{if } n = 1, 2 \\ (0, 0) & \text{if } n = 0 \end{cases}$$

Note that in our situation $T \otimes \mathbb{Z}_p$ is always anisotropic and hence we can apply the results in Section 6 to $T \otimes \mathbb{Z}_p$. Then plugging Proposition 8.3 with Section 6 (cf. Theorem 6.7) we have

**Theorem 8.4.** Keep the assumption in Proposition 8.3. Then

$$e(\mathcal{Z}(T, \omega)_{\mathcal{F}_p}, x) = -\frac{1}{p-1} \cdot \mathcal{F}_M''(\frac{1}{p}).$$

**APPENDIX A. THE TABLE OF INTERSECTION NUMBERS**

In this appendix we give a table for the intersection numbers $(T_{m_1,p}, T_{m_2,p})$ and $(T_{m_1,c}, T_{m_2,c})$ for $2 \leq m_1 \leq m_2 \leq 9$ such that $m_1 m_2$ is not a square and $2 \leq p < 50$. S. Yokoyama (cf. [Yok17]) kindly computed both of intersection numbers and checked

$$\text{IN}(m_1, m_2) := (T_{m_1,p}, T_{m_2,p}) = (T_{m_1,c}, T_{m_2,c})$$

directly including the case where $p = 2$. We list up all of them as below:

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| $(m_1, m_2)$ | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, 7) | (2, 9) | (3, 4) | (3, 5) | (3, 6) | (3, 7) | (3, 8) | (3, 9) |
|-------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\text{IN}(m_1, m_2)$ | 18     | 28     | 30     | 56     | 42     | 62     | 38     | 40     | 78     | 56     | 82     | 84     |
| $(m_1, m_2)$ | (4, 5) | (4, 6) | (4, 7) | (4, 8) | (5, 6) | (5, 7) | (5, 8) | (5, 9) | (6, 7) | (6, 8) | (6, 9) | (7, 8) |
| $\text{IN}(m_1, m_2)$ | 60     | 118    | 84     | 124    | 122    | 84     | 126    | 128    | 168    | 248    | 252    | 170    |
| $(m_1, m_2)$ | (7, 9) | (8, 9) |        |        |        |        |        |        |        |        |        |        |
| $\text{IN}(m_1, m_2)$ | 172    | 256    |        |        |        |        |        |        |        |        |        |        |

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