LAX PAIRS OF DISCRETE PAINLEVÉ EQUATIONS: \((A_2 + A_1)^{(1)}\) CASE

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ABSTRACT. In this paper, we provide a comprehensive method for constructing Lax pairs of discrete Painlevé equations. In particular, we consider the \(A_2^{(1)}\)-surface \(q\)-Painlevé system which has the affine Weyl group symmetry of type \((A_2 + A_1)^{(1)}\) and show two new Lax pairs.

1. INTRODUCTION

The purpose of this paper is to provide a comprehensive method for constructing Lax pairs of discrete Painlevé equations. As an example, we consider the \(q\)-Painlevé equations of \(A_2^{(1)}\)-surface type in Sakai’s classification [37], which are called \(q\)-Painlevé IV [25], \(q\)-Painlevé III [25, 37] and \(q\)-Painlevé II [36] equations, respectively given by Equations (1.1), (1.2) and (1.3). These equations are collectively called the \(A_2^{(1)}\)-surface \(q\)-Painlevé equations.

This work is motivated by our previous findings in [20], where quad-equations were observed on what is called the \(\omega\)-lattice, constructed from the \(\tau\) functions of \(A_2^{(1)}\)-surface \(q\)-Painlevé equations, and those in [21], where the 4-dimensional integer lattice, can be reduced to the \(\omega\)-lattice via a periodic reduction, and its extended lattices are investigated. Combining these results enables us to systematically construct the Lax pairs for any discrete Painlevé equations on the \(A_2^{(1)}\)-surface.

Our Lax pairs for the \(q\)-Painlevé IV equation: (1.4) and \(q\)-Painlevé III equation: (1.5) are new, while the one for the \(q\)-Painlevé II equation: (1.6) coincides with that provided in [15]. They all satisfy Carmichael’s hypotheses [9] for existence of solutions around singular points at the origin and infinity. Not all existing Lax pairs (see §1.2) satisfy these conditions. Moreover, some known Lax pairs hold only for restricted parameters or modified cases of the discrete Painlevé equations. We remark that multiple Lax pairs are known for each of the continuous Painlevé equations [10, 18, 19].

1.1. Main result. Our main results are Lax pairs for the following \(q\)-Painlevé equations:

\[
\begin{align*}
\text{\(q\)-P_{IV}:} & \quad \left\{ \begin{array}{l}
  f(qt) &= ab \ g(t) \left( 1 + c h(t) (a f(t) + 1) \right) \\
  g(qt) &= bc h(t) \left( 1 + a f(t) (b g(t) + 1) \right) \\
  h(qt) &= ca f(t) \left( 1 + b g(t) (c h(t) + 1) \right)
\end{array} \right. \\
\text{\(q\)-P_{III}:} & \quad \left\{ \begin{array}{l}
  g(qt) &= \frac{a}{t + f(t)} \left( f(t) g(qt) + bt + g(qt) \right) \\
  f(qt) &= \frac{1 + tf(t)}{t + f(t)} \left( f(t) g(qt) + bt + g(qt) \right)
\end{array} \right. \\
\text{\(q\)-P_{II}:} & \quad f(pt) = \frac{a}{1 + tf(t)} \left( f(t) p^{-1} f(t) + tf(t) \right)
\end{align*}
\]
where $t \in \mathbb{C}$ is an independent variable, $f(t), g(t), h(t)$ are dependent variables and $a, b, c, q, p \in \mathbb{C}$ are parameters. In the case of $q$-P$_{IIV}$ we have $f(t)g(t)h(t) = t^2$ and $abc = q$.

**Theorem 1.1.** The following statements hold:

(i): The following system is the $2 \times 2$ Lax pair of $q$-P$_{IIV}$ (1.1):

$$\phi(qx, t) = \begin{pmatrix} qt \frac{h(t)}{f(t)} & 1 \\ -1 & \frac{qf(t)}{f(t)} \end{pmatrix} \begin{pmatrix} \frac{a}{t} & 1 \\ \frac{ac}{t} & \frac{a}{t} \end{pmatrix} \begin{pmatrix} 
\frac{at}{g(t)} & 1 \\
-1 & \frac{ag(t)}{t} \end{pmatrix} \phi(x, t), \quad (1.4a)$$

$$\phi(x, qt) = \begin{pmatrix} (qt^2 - 1)h(t) \\
(1 + b + bch(t))tg(t) \end{pmatrix} \begin{pmatrix} -1 \\
0 \end{pmatrix} \phi(x, t). \quad (1.4b)$$

(ii): The following system is the $2 \times 2$ Lax pair of $q$-P$_{III}$ (1.2):

$$\phi(qx, t) = \begin{pmatrix} \frac{a^{1/2}bt}{f(t)g(t)} \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{g(t)}{f(t)} \end{pmatrix} \begin{pmatrix} a^{1/2}t \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{tg(t)}{a^{1/2}} \end{pmatrix} \phi(x, t), \quad (1.5a)$$

$$\phi(x, qt) = \begin{pmatrix} \frac{a^{1/2}bt}{f(t)} \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{btf(t)}{a^{1/2}} \end{pmatrix} \begin{pmatrix} a^{1/2}t \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{tg(t)}{a^{1/2}} \end{pmatrix} \phi(x, t). \quad (1.5b)$$

(iii): The following system is the $2 \times 2$ Lax pair of $q$-P$_{II}$ (1.3):

$$\phi(p^2x, t) = \begin{pmatrix} \frac{p^2f(t)}{f(p^{-1}t)} \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{p^2a^{1/2}}{f(t)f(p^{-1}t)} \end{pmatrix} \begin{pmatrix} \frac{pa^{1/2}}{f(t)} \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{ptf(t)}{a^{1/2}} \end{pmatrix} \phi(x, t), \quad (1.6a)$$

$$\phi(x, pt) = \begin{pmatrix} \frac{a^{1/2}t}{f(p^{-1}t)} \\
-1 \end{pmatrix} \begin{pmatrix} 1 \\
\frac{tf(p^{-1}t)}{a^{1/2}} \end{pmatrix} \phi(x, t). \quad (1.6b)$$

Here, $\phi(x, t)$ is a 2-column vector called a wave function and $x$ is a complex parameter called a spectral parameter.

This theorem is proved by extending the 4-dimensional setting of the periodically reduced partial difference equation in §2 to a 5-dimensional setting in §4.

1.2. **Background.** Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which include discrete analogs of the six Painlevé equations: P$_1$, . . . , P$_{IIV}$. The geometric classification of discrete Painlevé equations, based on types of rational surfaces connected to affine Weyl groups, is well known [37]. Together with the Painlevé equations, they are now regarded as one of the most important classes of equations in the theory of integrable systems (see, e.g., [13]).

Another important class is given by integrable partial difference equations (PΔEs), including discrete versions of integrable PDEs such as Korteweg-de Vries equation (KdV...
equation). They are usually constructed as relations on vertices of a quadrilateral on a lattice and are often called quad-equations or lattice equations. In [1, 2, 7], integrable PΔEs were constructed and classified by a property known as consistency around the cube (3D consistency). The resulting list of equations are now collectively referred to as the ABS equations.

These equations are called integrable because they arise as compatibility conditions for associated linear problems called Lax pairs. The search for and construction of Lax pairs of discrete Painlevé equations has been a very active research area and the investigations have been carried out through many approaches. Noteworthy approaches include extensions of Birkhoff’s study of linear $q$-difference equations [17, 38, 39], periodic-type reductions from ABS equations or the discrete KP/UC hierarchy [14, 15, 24, 31–33, 35, 41], extensions of Schlesinger transformations [4, 11, 12], search for linearizable curves in initial-value space [45, 46], padé approximation or interpolation [16, 27, 30] and the theory of orthogonal polynomials [3, 6, 8, 34, 43, 44].

There are possibly infinitely many discrete Painlevé equations on each rational surface in Sakai’s theory [22, 37]. In the case we study, $q$-P$_V$ (1.1), $q$-P$_{II}$ (1.2) and $q$-P$_{III}$ (1.3) are all obtained on the $A^{(1)}_5$-surface. However, the construction of Lax pairs for each case has been carried out in different ways for different equations on the same surface.

The Lax pair constructed for the $q$-Painlevé VI equation [17] satisfies Carmichael’s hypotheses [9] for existence of solutions around singular points at the origin and infinity. However, other Lax pairs [26] do not, because the coefficient matrix of either the highest degree or the lowest degree term in the associated plane is singular. We note that the Lax pairs we provided in Equations (1.4)–(1.6) avoid this difficulty.

1.3. Plan of the paper. The plan of this paper is as follows. In §2, we construct the Lax pairs of the PΔEs on the 4-dimensional integer lattice. Moreover, we apply the periodic reduction to the PΔEs together with their Lax pairs. In §3, we reconstruct the periodically reduced PΔEs and their Lax pairs given in §2 from the $\tau$ functions of $A^{(1)}_5$-surface $q$-Painlevé equations. In §4, we construct the Lax pairs of the $A^{(1)}_4$-surface $q$-Painlevé equations from the PΔEs on the 5-dimensional integer lattice including those on the 4-dimensional integer lattice defined in §2. Some concluding remarks are given in §5.

2. The partial difference equations on the 4-dimensional integer lattice and their Lax pairs

2.1. The partial difference equations. In this section, we consider the function on the 4-dimensional integer lattice $\mathbb{Z}^4$ generated by a standard basis for $\mathbb{R}^4$, $\{\epsilon_1, \ldots, \epsilon_4\}$. The shift operators $\hat{T}_i$, $i = 1, \ldots, 4$, on the lattice $\mathbb{Z}^4$ are defined by

$$\hat{T}_i : \mathbb{Z}^4 \ni l \mapsto l + \epsilon_i \in \mathbb{Z}^4. \quad (2.1)$$

It is well known that the lattice $\mathbb{Z}^4$ can be constructed from a 4-dimensional hypercube (4-cube) by the space filling. Assigning the quad-equations of ABS type to the faces of
each 4-cube (see Appendix A for details), we define the PDEs on the lattice $\mathbb{Z}^4$ as follows:

\[
\begin{align*}
\frac{u(l + e_1 + e_2)}{u(l)} &= \frac{\alpha_i u(l + e_1) - \beta_i u(l + e_2)}{\alpha_i u(l) - \beta_i u(l + e_1)}, \\
\frac{u(l + e_2 + e_3)}{u(l)} &= \frac{\beta_i u(l + e_2) - \gamma_i u(l + e_3)}{\beta_i u(l) - \gamma_i u(l + e_2)}, \\
\frac{u(l + e_3 + e_4)}{u(l)} &= \frac{\gamma_i u(l + e_3) - \alpha_i u(l + e_4)}{\gamma_i u(l) - \alpha_i u(l + e_3)}, \\
\frac{u(l + e_1 + e_3)}{u(l + e_1)} &= \frac{u(l + e_1)}{u(l)} - \alpha_i K_{l_1}, \\
\frac{u(l + e_2 + e_4)}{u(l + e_2)} &= \frac{u(l + e_2)}{u(l)} - \beta_i K_{l_2}, \\
\frac{u(l + e_3 + e_4)}{u(l + e_3)} &= \frac{u(l + e_3)}{u(l)} - \gamma_i K_{l_3},
\end{align*}
\]

(2.2a) \(\quad\) (2.2b) \(\quad\) (2.2c) \(\quad\) (2.2d) \(\quad\) (2.2e) \(\quad\) (2.2f)

where \(l = \sum_{i=1}^4 l_i e_i \in \mathbb{Z}^4\). Here, \(u(l)\) is the function on the lattice $\mathbb{Z}^4$ and \(\{\alpha_i\}_{i \in \mathbb{Z}}, \{\beta_i\}_{i \in \mathbb{Z}}, \{\gamma_i\}_{i \in \mathbb{Z}}\) and \(\{K_i\}_{i \in \mathbb{Z}}\) are complex parameters. We define the actions of $\hat{T}_i, i = 1, \ldots, 4$, on the infinite extension field of the complex field $\mathbb{C}$, generated by \(u(l)_{l \in \mathbb{Z}^2}, \{\alpha_i\}_{i \in \mathbb{Z}}, \{\beta_i\}_{i \in \mathbb{Z}}, \{\gamma_i\}_{i \in \mathbb{Z}}\) and \(\{K_i\}_{i \in \mathbb{Z}}\), as the automorphisms by the following actions:

\[
\begin{align*}
\hat{T}_1 : (u(l), \alpha_i, \beta_i, \gamma_i, K_i) &\mapsto (u(l + e_1), \alpha_{i+1}, \beta_i, \gamma_i, K_i), \\
\hat{T}_2 : (u(l), \alpha_i, \beta_i, \gamma_i, K_i) &\mapsto (u(l + e_2), \alpha_i, \beta_{i+1}, \gamma_i, K_i), \\
\hat{T}_3 : (u(l), \alpha_i, \beta_i, \gamma_i, K_i) &\mapsto (u(l + e_3), \alpha_i, \beta_i, \gamma_{i+1}, K_i), \\
\hat{T}_4 : (u(l), \alpha_i, \beta_i, \gamma_i, K_i) &\mapsto (u(l + e_4), \alpha_i, \beta_i, \gamma_i, K_{i+1}).
\end{align*}
\]

(2.3a) \(\quad\) (2.3b) \(\quad\) (2.3c) \(\quad\) (2.3d)

Therefore, it is obvious that $\hat{T}_i, i = 1, \ldots, 4$, commute with each other under the actions on the lattice $\mathbb{Z}^4, \{u(l)\}_{l \in \mathbb{Z}^2}, \{\alpha_i\}_{i \in \mathbb{Z}}, \{\beta_i\}_{i \in \mathbb{Z}}, \{\gamma_i\}_{i \in \mathbb{Z}}$ and \(\{K_i\}_{i \in \mathbb{Z}}\).

For convenience, throughout this paper we use the following notation for arbitrary mappings $w_i, i = 1, \ldots, n$:

\[
w_1 \cdots w_n := w_1 \circ \cdots \circ w_n,
\]

(2.4)

and we in particular use the following notation for the composition of the shift operators $\hat{T}_i$:

\[
\hat{T}_{k_1} \cdots \hat{T}_{k_n} := \hat{T}_{k_1} \cdots \hat{T}_{k_n},
\]

(2.5)

where $k_1, \ldots, k_n \in \mathbb{Z}$.

In addition, we consider the sublattice

\[
\mathcal{R} = \mathcal{V}^{(1)} \cup \mathcal{V}^{(2)} \subset \mathbb{Z}^4,
\]

(2.6)

where

\[
\mathcal{V}^{(1)} = \left\{ \sum_{i=1}^4 l_i e_i \bigg| l_i \in \mathbb{Z}, l_3 = l_2 = 1 \right\}, \quad \mathcal{V}^{(2)} = \left\{ \sum_{i=1}^4 l_i e_i \bigg| l_i \in \mathbb{Z}, l_3 = l_2 = 0 \right\},
\]

(2.7)

with the zigzag-shift operator $\hat{R}_1$ (see Figure 1) of which action on the lattice $\mathcal{R}$ is defined by

\[
\hat{R}_1(l) = \begin{cases} \hat{T}_2^{-1}(l) & \text{if } l \in \mathcal{V}^{(1)}, \\ \hat{T}_3^{-1}(l) & \text{if } l \in \mathcal{V}^{(2)}, \end{cases}
\]

(2.8)
The function \( u(t) \) on the lattice \( \mathcal{R} \) satisfying the following P\( \Delta \)Es:

\[
\begin{align*}
\frac{u(t + \varepsilon_1 + \varepsilon_2) - u(t)}{u(t)} &= -\frac{\alpha_1 u(t + \varepsilon_1) - \beta_1 u(t + \varepsilon_2)}{\alpha_1 u(t) - \beta_1 u(t)}, \\
\frac{u(t + \varepsilon_1) - u(t)}{u(t + \varepsilon_1)} &= -\frac{\gamma_{t-1} u(t) - \alpha_{t-1} u(t + \varepsilon_1 - \varepsilon_3)}{\gamma_{t-1} u(t + \varepsilon_1 - \varepsilon_3)}, \\
\frac{u(t + \varepsilon_1) + u(t + \varepsilon_2)}{u(t) + u(t + \varepsilon_1)} &= \alpha_{t-1} K_{t-1}, \\
\frac{u(t + \varepsilon_1) + u(t + \varepsilon_2)}{u(t) + u(t + \varepsilon_2)} &= -\beta_{t-1} K_{t-1}, \\
\frac{u(t + \varepsilon_1) + u(t + \varepsilon_2)}{u(t) + u(t + \varepsilon_1)} &= -\gamma_{t-1} K_{t-1}, \\
\frac{u(t + \varepsilon_1) + u(t + \varepsilon_2)}{u(t) + u(t + \varepsilon_2)} &= -\beta_{t-1} K_{t-1}, \\
\frac{u(t + \varepsilon_1) + u(t + \varepsilon_2)}{u(t) + u(t + \varepsilon_1)} &= -\gamma_{t-1} K_{t-1},
\end{align*}
\]

where

\[ I = l_1 \varepsilon_1 + l_2 (\varepsilon_2 + \varepsilon_3) + l_4 \varepsilon_4 \in \mathcal{V}^2, \]

which can be obtained from the P\( \Delta \)Es (2.2a) \( |_{l_3 = l_4} \), (2.2c) \( |_{l_3 = l_4} \), (2.2d) \( |_{l_3 = l_4} \), (2.2e) \( |_{l_3 = l_4} \), (2.2f) \( |_{l_3 = l_4} \), respectively. We lift the action of \( \hat{\mathcal{R}}_1 \) upon the infinite extension field of the complex field \( \mathbb{C} \), generated by \( \{ u(t) \}_{l_0 \in \mathbb{Z}}, \{ \alpha_{l} \}_{l \in \mathbb{Z}}, \{ \beta_{l} \}_{l \in \mathbb{Z}}, \{ \gamma_{l} \}_{l \in \mathbb{Z}} \) and \( \{ K_l \}_{l \in \mathbb{Z}} \), as the automorphism with the following action:

\[
\hat{R}_1(u(t)) = u(\hat{R}_1(t)).
\]

Since the mapping \( \hat{R}_1 \) acts on the lattice \( \mathcal{R} \) as (2.8) and on \( \{ u(t) \}_{l_0 \in \mathbb{Z}} \) as (2.11), we can presume that \( \hat{R}_1 \) shifts Equations (2.9) as follows:

\[
\begin{align*}
\cdots \xrightarrow{\hat{R}_1} \text{(2.9a)} \xrightarrow{\hat{R}_1} \text{(2.9b)} \xrightarrow{\hat{R}_1} \text{(2.9a)} \xrightarrow{\hat{R}_1} \text{(2.9b)} \xrightarrow{\hat{R}_1} \cdots, \\
\cdots \xrightarrow{\hat{R}_1} \text{(2.9d)} \xrightarrow{\hat{R}_1} \text{(2.9c)} \xrightarrow{\hat{R}_1} \text{(2.9d)} \xrightarrow{\hat{R}_1} \text{(2.9c)} \xrightarrow{\hat{R}_1} \cdots, \\
\cdots \xrightarrow{\hat{R}_1} \text{(2.9f)} \xrightarrow{\hat{R}_1} \text{(2.9e)} \xrightarrow{\hat{R}_1} \text{(2.9f)} \xrightarrow{\hat{R}_1} \text{(2.9e)} \xrightarrow{\hat{R}_1} \cdots,
\end{align*}
\]

which lead to the action of \( \hat{R}_1 \) on the parameters: \( \{ \alpha_{l} \}_{l \in \mathbb{Z}}, \{ \beta_{l} \}_{l \in \mathbb{Z}}, \{ \gamma_{l} \}_{l \in \mathbb{Z}} \) and \( \{ K_l \}_{l \in \mathbb{Z}} \), as

\[
\begin{align*}
\hat{R}_1 : (\alpha_{l}, \beta_{l}, \gamma_{l}, K_l) &\mapsto (\alpha_{l}, \gamma_{l-1}, \beta_{l}, K_l), \\
\hat{R}_1^{-1} : (\alpha_{l}, \beta_{l}, \gamma_{l}, K_l) &\mapsto (\alpha_{l}, \gamma_{l}, \beta_{l+1}, K_l).
\end{align*}
\]

Note that \( \hat{R}_1 \) commutes with \( \hat{T}_1 \) and \( \hat{T}_2 \) and satisfies \( \hat{R}_1^2 = \hat{T}_2^{-1} \hat{T}_1^{-1} \) under the actions on the lattice \( \mathcal{R} \), \( \{ u(t) \}_{l_0 \in \mathbb{Z}}, \{ \alpha_{l} \}_{l \in \mathbb{Z}}, \{ \beta_{l} \}_{l \in \mathbb{Z}}, \{ \gamma_{l} \}_{l \in \mathbb{Z}} \) and \( \{ K_l \}_{l \in \mathbb{Z}} \). Therefore, the P\( \Delta \)Es (2.9) can be expressed by the following three P\( \Delta \)Es:

\[
\begin{align*}
\frac{u_{l+1,l_1-1} - u_{l_1,l_1-1} - \hat{R}_1^4(\beta_{l_1}) u_{l_1,l_1-1}}{u_{l_1,l_1-1}} &= -\alpha_{l_1} u_{l_1+1,l_1-1} - \hat{R}_1^4(\beta_{l_1}) u_{l_1+1,l_1-1}, \\
\frac{u_{l_1+1,l_1+1} + u_{l_1,l_1+1} + \hat{R}_1^4(\beta_{l_1}) u_{l_1,l_1+1}}{u_{l_1+1,l_1+1}} &= -\alpha_{l_1} K_{l_1}, \\
\frac{u_{l_1,l_1+1} + u_{l_1,l_1+1} - \hat{R}_1^4(\beta_{l_1}) K_{l_1}}{u_{l_1+1,l_1+1}} &= -\alpha_{l_1} K_{l_1},
\end{align*}
\]

where

\[
u_{l_1,l_1+1} = \hat{T}_1^{-1} \hat{R}_1^4(u_0), \quad u_0 = u(0).
\]

Note that \( 0 \in \mathbb{Z}^4 \) is the origin of the lattice \( \mathbb{Z}^4 \).

Henceforth, if new quantity \( x \) is added, we extend the field on which \( \hat{T}_i, i = 1, \ldots, 4 \), (or, \( \hat{T}_1, \hat{R}_1 \) and \( \hat{T}_2 \)) act as the automorphisms by adding the generator \( x \). Note that throughout
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this paper every field is of characteristic zero. Moreover, when the field is generated by \
\{x_1, \ldots, x_k\}, the mappings also act on an arbitrary function F = F(x_1, \ldots, x_k) as follows:

\[
\hat{T}_i(F) = F(\hat{T}_i(x_1), \ldots, \hat{T}_i(x_k)), \quad i = 1, \ldots, 4, \tag{2.15}
\]

\[
\hat{R}_1(F) = F(\hat{R}_1(x_1), \ldots, \hat{R}_1(x_k)). \tag{2.16}
\]

2.2. The Lax pairs of the P\(\Delta\)Es. First, we consider the Lax pairs of the P\(\Delta\)Es (2.2).

Define the variables \(F^{(i)}\) and \(G^{(i)}\), \(i = 1, \ldots, 4\), by

\[
u(\epsilon_i) = \frac{F^{(i)}}{G^{(i)}}, \tag{2.17}
\]

and the vectors \(\Psi^{(i)}\), \(i = 1, \ldots, 4\), by

\[
\Psi^{(i)} = \begin{pmatrix} F^{(i)} \\ G^{(i)} \end{pmatrix}. \tag{2.18}
\]

We assume that \(\hat{T}_i\), \(i = 1, \ldots, 4\), commute with each other under the actions on \(F^{(j)}\) and \(G^{(j)}\), \(j = 1, \ldots, 4\). The transformation from \(u\) to \(\Psi\) provides a Hopf-like linearization of Equations (2.2), obtained by identifying numerators and denominators, and introduces decoupling factors \(\delta^{(i,j)}\) (see Lemma A.1).

We approach the construction of the Lax pairs by starting with a local 4-cube that has a vertex at the origin (see Appendix A for details). All other 4-cubes in the lattice \(\mathbb{Z}^4\) are obtained by translation from this local 4-cube. Therefore, letting

\[
\Psi^{(i)}_{h_1,h_2,h_3,h_4} = \hat{T}_1^{h_1} \hat{T}_2^{h_2} \hat{T}_3^{h_3} \hat{T}_4^{h_4}(\Psi^{(i)}) = \begin{pmatrix} \hat{T}_1^{h_1} \hat{T}_2^{h_2} \hat{T}_3^{h_3} \hat{T}_4^{h_4}(F^{(i)}) \\ \hat{T}_1^{h_1} \hat{T}_2^{h_2} \hat{T}_3^{h_3} \hat{T}_4^{h_4}(G^{(i)}) \end{pmatrix}, \quad i = 1, \ldots, 4, \tag{2.19a}
\]

\[
\delta^{(i,j)}_{h_1,h_2,h_3} = \hat{T}_1^{h_1} \hat{T}_2^{h_2} \hat{T}_3^{h_3}(\delta^{(i,j)})_{h_1} = 1, \ldots, 4, \quad i \neq j, \tag{2.19b}
\]

and using Lemma A.1, we obtain the following lemma. For conciseness, we have provided the explicit equations for each Lax pair in Lemma B.1.

**Lemma 2.1.** Each P\(\Delta\)E in Equations (2.2) is the compatibility condition of two Lax pairs provided by the following table:
Here, the decoupling factors satisfy the following relations:

$$\hat{T}_k (\delta_{i,j,k,l}) \hat{\delta}_{i,j,k,l} = \hat{T}_j (\delta_{i,j,k,l}) \hat{\delta}_{i,j,k,l},$$  
(2.20)

where $i \in \{1, \ldots, 4\}$, $j \in \{1, \ldots, 4\} - [i]$, $k \in \{1, \ldots, 4\} - [i, j]$.

**Proof.** From the compatibility condition of the Lax pair (B.1g) and (B.1h):

$$\hat{T}_2 \circ \hat{T}_1 \left( \Psi^{(3)}_{h, j, d, i} \right) = \hat{T}_1 \circ \hat{T}_2 \left( \Psi^{(3)}_{h, j, d, i} \right),$$  
(2.21)

we obtain

$$\hat{T}_2 \left( \delta^{(3,1)}_{h, j, d, i} \right) \delta^{(2,2)}_{h, j, d, i} = \hat{T}_1 \left( \delta^{(3,1)}_{h, j, d, i} \right) \delta^{(2,2)}_{h, j, d, i},$$  
(2.22)

where $I = \sum_{i=1}^{4} l_i \epsilon_i$. Considering the determinant of both sides of Equation (2.22), we obtain

$$\hat{T}_2 \left( \delta^{(3,1)}_{h, j, d, i} \right) \delta^{(3,1)}_{h, j, d, i} = \hat{T}_1 \left( \delta^{(3,1)}_{h, j, d, i} \right) \delta^{(3,1)}_{h, j, d, i}.$$  
(2.23)

Therefore, Equation (2.22) can be rewritten as

$$\left( \frac{\hat{T}_2(u(I))}{\alpha_1} + \frac{\beta_1}{\beta_2} \hat{T}_2(u(I)) \right) \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -\hat{T}_2(u(I)) \\ \hat{T}_2(u(I)) \end{pmatrix},$$  
(2.24)

which gives Equation (2.2a). In a similar manner, we can also verify the others. Therefore, we complete the proof. 

Next, we consider the Lax pairs of the P\AEs (2.13). Define the variables $F^{(R)}$ and $G^{(R)}$ by

$$u_{0_1} = u(\epsilon_1) = \frac{F^{(R)}}{G^{(R)}},$$  
(2.25)

and the vectors $\Psi^{(R)}$ and $\Psi^{(R)}_{i, j, d, i}$, $i = 1, \mathcal{R}, 4$, by

$$\Psi^{(R)} = \begin{pmatrix} F^{(R)} \\ G^{(R)} \end{pmatrix}, \quad \Psi^{(R)}_{i, j, d, i} = \hat{T}_i \hat{R}_i \hat{T}_j \hat{R}_j \Psi^{(R)} = \begin{pmatrix} \hat{T}_i \hat{R}_i \hat{T}_j \hat{R}_j (F^{(R)}) \\ \hat{T}_i \hat{R}_i \hat{T}_j \hat{R}_j (G^{(R)}) \end{pmatrix}.$$  
(2.26)

We assume that $\hat{T}_i$, $\hat{R}_i$ and $\hat{T}_4$ commute with each other under the actions on $F^{(i)}$ and $G^{(i)}$, $i = 1, \mathcal{R}, 4$. Then, in a similar manner as Lemma 2.1, we can prove the following lemma. For conciseness, we have provided the explicit equations for each Lax pair in Lemma 2.2.

**Lemma 2.2.** The P\AEs (2.13a), (2.13b) and (2.13c) are the compatibility conditions of the Lax pairs: Equations (B.2c) and (B.2f), Equations (B.2c) and (B.2d) and Equations...
2.3. The Lax pairs of the periodically reduced P\AE s. First, we consider the (1, 1, 1)-
reduction of the P\AE s (2.2) investigated in [21]. Let

\[ u(I) = h_{i_1,i_2,i_3} \omega(I), \]  

where \( I = \sum_{i=1}^{4} t_i e_i \in \mathbb{Z}^4 \). Here, the gauge factor \( h_{i_1,i_2,i_3} \) is defined by

\[ h_{i_1,i_2,i_3} = \exp i \log(q^{\lambda_i} \gamma^{\lambda_4}/\alpha^{\lambda_4} \gamma^{\lambda_1} - q^{\lambda_4} \gamma^{\lambda_1} \alpha^{\lambda_4} \gamma^{\lambda_1}), \]

where \( i = \sqrt{-1} \) and \( \alpha, \beta, \gamma, \lambda, q \in \mathbb{C} \) are parameters. Moreover, by imposing the following
(1, 1, 1)-periodic condition for \( \{\omega(I)\}_{I \in \mathbb{Z}^4} \):

\[ \omega(I + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \omega(I), \]

with the following condition of the parameters \( \{\alpha_i\}_{I \in \mathbb{Z}^4}, \{\beta_i\}_{I \in \mathbb{Z}^4}, \{\gamma_i\}_{I \in \mathbb{Z}^4} \) and \( \{K_i\}_{I \in \mathbb{Z}^4} \):

\[ \alpha_i = q^i \alpha, \quad \beta_i = q^i \beta, \quad \gamma_i = q^i \gamma, \quad K_i = q^{2i+1} \lambda^2 - 1, \]

the P\AE s (2.2) can be reduced to

\[ \frac{\omega(I + \varepsilon_1 + \varepsilon_2)}{\omega(I)} = \frac{\omega(I) - q^{-l_1+l_2+l_3} \frac{\lambda_3}{\lambda_1} \omega(I + \varepsilon_2)}{q^i \lambda \omega(I + \varepsilon_2) - q^{-l_1+l_2+l_3} \frac{\lambda_3}{\lambda_1} \omega(I + \varepsilon_1)}, \]

\[ \frac{\omega(I + \varepsilon_2 + \varepsilon_3)}{\omega(I)} = \frac{q^i \lambda \omega(I + \varepsilon_2) - q^{-l_1+l_2+l_3} \frac{\lambda_3}{\lambda_1} \omega(I + \varepsilon_3)}{q^i \lambda \omega(I + \varepsilon_2) - q^{-l_1+l_2+l_3} \frac{\lambda_3}{\lambda_1} \omega(I + \varepsilon_3)}, \]

\[ \frac{\omega(I + \varepsilon_3 + \varepsilon_1)}{\omega(I)} = \frac{\omega(I + \varepsilon_3) - q^{l_3-l_1} \frac{\lambda_4}{\lambda_3} \omega(I + \varepsilon_3)}{\omega(I + \varepsilon_3) - q^{l_3-l_1} \frac{\lambda_4}{\lambda_3} \omega(I + \varepsilon_3)}, \]

\[ \frac{\omega(I + \varepsilon_1 + \varepsilon_3)}{\omega(I)} = \frac{\omega(I + \varepsilon_1) - q^{l_1-l_3} \frac{\lambda_1}{\lambda_2} \omega(I + \varepsilon_3)}{\omega(I + \varepsilon_1) - q^{l_1-l_3} \frac{\lambda_1}{\lambda_2} \omega(I + \varepsilon_3)}, \]

\[ \frac{\omega(I + \varepsilon_1 + \varepsilon_3)}{\omega(I)} = \frac{\omega(I + \varepsilon_1) - q^{l_1-l_3} \frac{\lambda_1}{\lambda_2} \omega(I + \varepsilon_3)}{\omega(I + \varepsilon_1) - q^{l_1-l_3} \frac{\lambda_1}{\lambda_2} \omega(I + \varepsilon_3)}, \]

\[ \frac{\omega(I + \varepsilon_2 + \varepsilon_3)}{\omega(I)} = \frac{\omega(I + \varepsilon_2) - q^{2l_3} \lambda^2 \omega(I + \varepsilon_3)}{\omega(I + \varepsilon_2) - q^{2l_3} \lambda^2 \omega(I + \varepsilon_3)}, \]

\[ \frac{\omega(I + \varepsilon_1 + \varepsilon_3)}{\omega(I)} = \frac{\omega(I + \varepsilon_1) - q^{l_1-l_3} \frac{\lambda_1}{\lambda_2} \omega(I + \varepsilon_3)}{\omega(I + \varepsilon_1) - q^{l_1-l_3} \frac{\lambda_1}{\lambda_2} \omega(I + \varepsilon_3)}, \]

From the actions (2.3) and the condition (2.31), we obtain the following lemma.
**Lemma 2.3.** The following actions hold:

\[ \hat{T}_1 : (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q) \mapsto (q\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q), \]  
(2.33a)

\[ \hat{T}_2 : (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q) \mapsto (\hat{\alpha}, q\hat{\beta}, \hat{\gamma}, \lambda, q), \]  
(2.33b)

\[ \hat{T}_3 : (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q) \mapsto (\hat{\alpha}, \hat{\beta}, q\hat{\gamma}, \lambda, q), \]  
(2.33c)

\[ \hat{T}_4 : (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q) \mapsto (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, q\lambda, q). \]  
(2.33d)

**Proof.** Since \( \hat{T}_1(\lambda) \) must be uniquely-determined, we have \( \hat{T}_1(\lambda) = \lambda \). Therefore, the action (2.33a) hold. In a similar manner, we can prove the others. Therefore, we have completed the proof.

Therefore, the mappings \( \hat{T}_i, i = 1, \ldots, 4 \), act on the gauge factor \( h_{1,i,j,i,j} \) as

\[ \hat{T}_1(h_{1,i,j,i,j}) = h_{1,i+1,j,j,i} \]  
(2.41a)

\[ \hat{T}_2(h_{1,i,j,i,j}) = h_{1,j,i,j+1,i} \]  
(2.41b)

From the actions (2.3) and (2.41) and the relation (2.28), we obtain the following actions on \( \{\omega(I)\}_{i \leq 2^r} \):

\[ \hat{T}_i(\omega(I)) = \omega(\hat{T}_i(I)), \]  
(2.42)

where \( i = 1, \ldots, 4 \). We can verify that \( \hat{T}_i, i = 1, \ldots, 4 \), commute with each other under the actions on \( \{\omega(I)\}_{i \leq 2^r}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda \) and \( q \).

**Remark 2.4.** If we consider the actions of the mappings \( \hat{T}_i, i = 1, \ldots, 4 \), only on the field generated by \( \{\omega(I)\}_{i \leq 2^r}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda \) and \( q \), then the mapping \( \hat{T}_{123} \) can be regarded as the identity mapping:

\[ \hat{T}_{123} = \text{Id}. \]  
(2.43)

In this situation, we can consider as if the reduction acts on the lattice \( \mathbb{Z}^4 \), that is,

\[ I = I + \epsilon_1 + \epsilon_2 + \epsilon_3, \]  
(2.44)

for \( I \in \mathbb{Z}^4 \). Then, the lattice \( \mathbb{Z}^4 \) is reduced to the \((A_2 + A_1)^{(1)}\)-lattice (see Figure 2). The reduction from the lattice \( \mathbb{Z}^4 \) to the \((A_2 + A_1)^{(1)}\)-lattice with the P\( \Delta \)Es is referred to as the geometric reduction [21].

**Lemma 2.5.** Each periodically reduced P\( \Delta \)E in Equations (2.32) is the compatibility condition of two Lax pairs provided by the following table:
We can verify that

\[ \hat{L} \] factors satisfy the relations

\[
\begin{align*}
R_1 & \equiv \hat{L}^{-1} \hat{L}
\end{align*}
\]

Here, the explicit equations for each Lax pair are given in Appendix B.3 and the decoupling factors satisfy the relations (2.20).

**Proof.** In a similar manner as the proof of Lemma 2.1, we can prove this lemma.

Next, we consider the reduction of the PΔEs (2.13). In a similar manner as the proof of Lemma 2.3, from the action (2.12) and the condition (2.31) we obtain the following action:

\[
\begin{align*}
\hat{R}_1 : & (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q) \mapsto (\hat{\alpha}, q^{-1} \hat{\gamma}, \hat{\beta}, \lambda, q), \\
\hat{R}^{-1}_1 : & (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda, q) \mapsto (\hat{\alpha}, \hat{\gamma} \hat{\beta}, \lambda, q),
\end{align*}
\]

which implies that the mapping \( \hat{R}_1 \) act on the gauge factor \( h_{i_1,j_1,i_2,j_2} \) as the following:

\[
\begin{align*}
\hat{R}_1(h_{i_1,j_1,i_2,j_2}) & = -i(q^{i_1,j_1})^{-\log \hat{\beta} + \log \hat{\gamma}}/\log q^{-1} h_{i_1,j_1,i_2,j_2}, \\
\hat{R}^{-1}_1(h_{i_1,j_1,i_2,j_2}) & = i(q^{i_1,j_1})^{-\log \hat{\beta} + \log \hat{\gamma}}/\log q h_{i_1,j_1,i_2,j_2}.
\end{align*}
\]

From the actions (2.11) and (2.46) and the relation (2.28), we obtain the action of \( \hat{R}_1 \) on the function \( \omega = \omega(I) \) where \( I \in \mathcal{R} \) as the following:

\[
\begin{align*}
\hat{R}_1(\omega) & = \begin{cases} 
(q^{i_1,j_1})^{(\log \hat{\beta} - \log \hat{\gamma})/\log q} \omega(\hat{R}_1(I)) & \text{if } I = I_1 e_1 + I_2 (e_2 + e_3) - e_3 + I_4 e_4 \in \mathcal{V}^{(1)} , \\
(q^{i_1,j_1})^{(\log \hat{\beta} - \log \hat{\gamma})/\log q + q^{-1}} \omega(\hat{R}_1(I)) & \text{if } I = I_1 e_1 + I_2 (e_2 + e_3) + I_4 e_4 \in \mathcal{V}^{(2)} ,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\hat{R}^{-1}_1(\omega) & = \begin{cases} 
(q^{i_1,j_1})^{(\log \hat{\beta} - \log \hat{\gamma})/\log q} \omega(\hat{R}_1^{-1}(I)) & \text{if } I = I_1 e_1 + I_2 (e_2 + e_3) - e_3 + I_4 e_4 \in \mathcal{V}^{(1)} , \\
(q^{i_1,j_1})^{(\log \hat{\beta} - \log \hat{\gamma})/\log q + q^{-1}} \omega(\hat{R}_1^{-1}(I)) & \text{if } I = I_1 e_1 + I_2 (e_2 + e_3) + I_4 e_4 \in \mathcal{V}^{(2)} ,
\end{cases}
\end{align*}
\]

We can verify that \( \hat{R}_1 \) commutes with \( \hat{T}_1 \) and \( \hat{T}_4 \) and satisfies

\[
\hat{R}_1^2 = \hat{T}_2^{-1} \hat{T}_2.
\]

Figure 2. \((A_2 + A_1)^{(1)}\)-lattice

Here is the table:

| PΔE | Lax pair |
|-----|----------|
| (2.32a) | (B.7g) and (B.7h), (B.7j) and (B.7k) |
| (2.32b) | (B.7a) and (B.7b), (B.7k) and (B.7l) |
| (2.32c) | (B.7d) and (B.7e), (B.7j) and (B.7l) |
| (2.32d) | (B.7d) and (B.7f), (B.7g) and (B.7l) |
| (2.32e) | (B.7a) and (B.7c), (B.7h) and (B.7l) |
| (2.32f) | (B.7b) and (B.7c), (B.7e) and (B.7f) |
under the action on \(\{\omega(t)\}_{t \in \mathbb{R}}\), \(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda\) and \(q\), and the following relation holds:
\[
\hat{R}_1^3 = \hat{T}^{-1}_{23} = \hat{T}_1,
\]
(2.49)
under the action on \(\{\omega(t)\}_{t \in \mathbb{R}}\), \(\hat{\alpha}/\hat{\gamma}, \hat{\beta}/\hat{\beta}, \hat{\gamma}/\hat{\gamma}, \lambda\) and \(q\). Therefore, the PDEs (2.13) can be reduced to the following PDEs:
\[
\frac{\omega_{j+3,i}}{\omega_{j,i}} = \frac{q^{2j+1} \lambda}{\hat{R}_1^i(T_{14}^i(k))} \omega_{j+1,i} + q^{2j+1} \lambda \hat{R}_4^i \left( \frac{q^{\hat{\alpha}}}{\hat{\gamma}} \right) \omega_{j+2,i}.
\]
(2.50a)
\[
\frac{\omega_{j+2,i+1}}{\omega_{j,i}} = \frac{q^{2j+1} \lambda^2 - 1}{q^{4j} \lambda}.
\]
(2.50b)
\[
\frac{\omega_{j+1,i+1}}{\omega_{j+1,i}} = \hat{R}_1^i \left( \frac{\hat{\alpha}}{\hat{\beta}} \right) \frac{q^{2j+1} \lambda^2 - 1}{q^{4j+3} \lambda},
\]
(2.50c)
where
\[
\hat{k} = \lambda^{(\log \hat{\beta}/\log \hat{\gamma})/\log q}, \quad \omega_{j,i} = \hat{R}_1^i \left( T_{14}^i(\omega(0)) \right).
\]
(2.51)

**Lemma 2.6.** The PDEs (2.50a), (2.50b) and (2.50c) are the compatibility conditions of the Lax pairs: Equations (B.11e) and (B.11f). Equations (B.11c) and (B.11d) and Equations (B.11a) and (B.11b), respectively. Here, the explicit equations for each Lax pair are given in Appendix B.4 and the decoupling factors \(\delta_{0,i,i}, i = 1, \ldots, 6\), satisfy the relations (2.27).

**Proof.** In a similar manner as the proof of Lemma 2.1, we can prove this lemma. \(\square\)

3. **The linear systems arising from the \(\tau\)-functions of \(A^{(1)}_2\)-surface \(q\)-Painlevé equations**

In this section, we reconstruct the periodically reduced PDEs (2.32) and (2.50), the linear systems (B.7) and (B.11) and \(q\)-Painlevé equations (1.1)–(1.3) from the \(\tau\)-functions of \(A^{(1)}_2\)-surface \(q\)-Painlevé equations.

3.1. **The \(\tau\) functions.** The transformation group \(\hat{W}((A_2 + A_1)^{(1)})\) has seven generators \(s_0, s_1, s_2, \pi, w_0, w_1, \) and \(r\). Below, we describe their actions on parameters: \(a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}\), and on variables: \(\tau_i, \tilde{\tau}_i, i = 0, 1, 2\). The actions of the transformations on the parameters are given by:

\[
\begin{align*}
\gamma: & \quad (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \rightarrow (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \\
\pi: & \quad (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \rightarrow (a_{1}^{1/6}, a_{2}^{1/6}, a_{0}^{1/6}, c^{1/3}) \\
w_0: & \quad (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \rightarrow (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \\
w_1: & \quad (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \rightarrow (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \\
r: & \quad (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3}) \rightarrow (a_{0}^{1/6}, a_{1}^{1/6}, a_{2}^{1/6}, c^{1/3})
\end{align*}
\]

while their actions on variables are given by:

\[
\begin{align*}
s_i(\tau_i) &= \frac{\mu_i \tau_{i+1} \tau_{i+1} \tau_{i+1} + \tau_{i+1} \tau_{i+1} \tau_{i+1}}{\mu_i \tau_{i+1} \tau_{i+1}}. \\
s_i(\tilde{\tau}_i) &= \frac{v_i \tilde{\tau}_{i+1} \tau_{i+1} + \tau_{i+1} \tau_{i+1} \tau_{i+1}}{v_i \tau_{i+1} \tau_{i+1}}. \\
\pi(\tau_i) &= \tau_{i+1}. \\
\pi(\tilde{\tau}_i) &= \tilde{\tau}_{i+1}. \\
w_0(\tau_i) &= \frac{a_{1/3}^{-1}(\tau_{i+1} \tau_{i+1} \tau_{i+1} + \mu_{i-1} \tau_{i+1} \tau_{i+1} \tau_{i+1} + \mu_{i+1}^{-1} \tau_{i+1} \tau_{i+1} \tau_{i+1})}{a_{1/3}^{1/3} \tau_{i+1} \tau_{i+1} \tau_{i+1}}. \\
w_0(\tilde{\tau}_i) &= \frac{a_{1/3}^{-1}(\tau_{i+1} \tau_{i+1} \tau_{i+1} + \mu_{i-1} \tau_{i+1} \tau_{i+1} \tau_{i+1} + \mu_{i+1}^{-1} \tau_{i+1} \tau_{i+1} \tau_{i+1})}{a_{1/3}^{1/3} \tau_{i+1} \tau_{i+1} \tau_{i+1}}. \\
w_1(\tau_i) &= \frac{a_{1/3}^{-1}(\tau_{i+1} \tau_{i+1} \tau_{i+1} + \mu_{i-1} \tau_{i+1} \tau_{i+1} \tau_{i+1} + \mu_{i+1}^{-1} \tau_{i+1} \tau_{i+1} \tau_{i+1})}{a_{1/3}^{1/3} \tau_{i+1} \tau_{i+1} \tau_{i+1}}. \\
r(\tau_i) &= \tilde{\tau}_{i+1}. \\
r(\tilde{\tau}_i) &= \tilde{\tau}_{i+1}.
\end{align*}
\]
Letting $u_i^{1/2} = q^{-1/6}c^{-1/3}a_i^{1/2}$, $v_i^{1/2} = q^{1/6}c^{1/3}a_i^{1/2}$, $q^{1/6} = a_0^{1/6}a_1^{1/6}a_2^{1/6}$, (3.1) and $i, j \in \mathbb{Z}/3\mathbb{Z}$. Note that each element $w \in \tilde{W}((A_2 + A_1)^{(1)})$ acts on an arbitrary function $F = F(a_0^{1/6}, c^{1/3}, \tau_j, \tilde{\tau}_k)$ as $w(F) = F(w(a_0^{1/6}), w(c^{1/3}), w(\tau_j), w(\tilde{\tau}_k))$.

**Remark 3.1.** Notations in this paper are related to those in [40] by the following correspondence:

\[
\begin{align*}
(s_0, s_1, s_2, \pi, w_0, w_1, r) &\rightarrow (s_0, s_1, s_2, \pi^2, r, r_0, \pi^3), \\
(a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, c^{1/3}, q^{1/6}) &\rightarrow (a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, q^{-1/6}a_0^{1/6}a_1^{1/6}a_2^{1/6}, q^{1/6}), \\
(r_0, r_1, r_2, \tilde{r}_0, \tilde{r}_1, \tilde{r}_2) &\rightarrow (r_3, r_1, r_5, r_6, r_4, r_2). 
\end{align*}
\]

We also note that in [40] each element $w \in \tilde{W}((A_2 + A_1)^{(1)})$ acts on the arguments from the right, whereas in the present paper it acts from the left.

The following proposition shows that $\tilde{W}((A_2 + A_1)^{(1)})$ gives a representation of an affine Weyl group of type $(A_2 + A_1)^{(1)}$.

**Proposition 3.2 ([40]).** The group of transformations $\tilde{W}((A_2 + A_1)^{(1)}) = \langle s_0, s_1, s_2, \pi, w_0, w_1, r \rangle$ forms the extended affine Weyl group of type $(A_2 + A_1)^{(1)}$. Namely, the transformations satisfy the fundamental relations

\[
\begin{align*}
s_i^2 = (s_is_{i+1})^3 = 1, & \quad \pi s_i = s_i\pi, \quad (i \in \mathbb{Z}/3\mathbb{Z}), \\
w_0^3 = w_1^2 = r^2 = 1, & \quad rw_0 = w_1r,
\end{align*}
\]

and the action of $\tilde{W}(A_2^{(1)})$ is $(s_0, s_1, s_2, \pi)$ and that of $\tilde{W}(A_1^{(1)})$ is $(w_0, w_1, r)$ commute. Note that $q^{1/6}$ and $c^{1/3}$ are invariant under the action of $\tilde{W}((A_2 + A_1)^{(1)})$ and $\tilde{W}(A_2^{(1)})$, respectively.

To iterate each variable $\tau_j, \tilde{\tau}_k$, we need the translations $T_j, i = 1, \ldots, 4$, defined by

\[
T_1 = \pi s_2 s_1, \quad T_2 = \pi s_0 s_2, \quad T_3 = \pi s_1 s_0, \quad T_4 = rw_0.
\]

The actions of the translations on the parameters are given by

\[
\begin{align*}
T_1 : (a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, c^{1/3}, q^{1/6}) &\rightarrow (q^{1/6}a_0^{1/6}a_1^{1/6}a_2^{1/6}, c^{1/3}), \\
T_2 : (a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, c^{1/3}, q^{1/6}) &\rightarrow (a_0^{1/6}a_1^{1/6}a_2^{1/6}, c^{1/3}), \\
T_3 : (a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, c^{1/3}, q^{1/6}) &\rightarrow (q^{1/6}a_0^{1/6}a_1^{1/6}a_2^{1/6}, c^{1/3}), \\
T_4 : (a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, c^{1/3}, q^{1/6}) &\rightarrow (a_0^{1/6}a_1^{1/6}a_2^{1/6}, q^{1/6}c^{1/3}).
\end{align*}
\]

Note that $T_j, i = 1, \ldots, 4$, commute with each other and

\[
T_1T_2T_3 = 1.
\]

We define $\tau$ functions by

\[
\tau_{n,m}^N = T_1^nT_2^mT_4^N(\tau_1),
\]

where $n, m, N \in \mathbb{Z}$. We note that

\[
\tau_0 = \tau_0^{-1,0}, \quad \tau_1 = \tau_0^{0,0}, \quad \tau_2 = \tau_0^{0,1}, \quad \tilde{\tau}_0 = \tau_1^{-1,0}, \quad \tilde{\tau}_1 = \tau_0^{0,0}, \quad \tilde{\tau}_2 = \tau_1^{0,1}.
\]

We also define the half-translation

\[
R_1 = \pi^2 s_1
\]

satisfying

\[
R_1^2 = T_1.
\]

Note that $R_1$ commutes with $T_1$ and $T_2$. The action of $R_1$ on the parameters is given by

\[
R_1 : (a_0^{1/6}, a_1^{1/6}, a_2^{1/6}, c^{1/3}) \rightarrow (a_2^{1/6}a_0^{1/6}a_1^{1/6}, q^{1/6}a_0^{1/6}a_1^{1/6}a_2^{1/6}, q^{1/6}a_2^{1/6}a_1^{1/6}c^{1/3}).
\]

Letting

\[
\tau_{N}^j = R_1^jT_4^N(\tau_1),
\]

(3.15)
where \( l, N \in \mathbb{Z} \), we obtain the following relation
\[
\tau_{2l-1}^N = \tau_{1}^N, \quad \tau_{2l}^N = \tau_{0}^N.
\] (3.16)

3.2. The partial differential equations. Let
\[
k_0^{1/3} = \lambda^{\log \omega_0/(3 \log q)}, \quad k_1^{1/3} = \lambda^{\log \omega_1/(3 \log q)}, \quad k_2^{1/3} = \lambda^{\log \omega_2/(3 \log q)},
\] (3.17)
where
\[
\lambda^{1/3} = q^{1/6} \kappa^{1/3}.
\] (3.18)

The action of \( \bar{W}(A_2 + A_1)^{(1)} \) on the parameters \( k_i^{1/3} \) is given by
\[
s_1 : (k_i^{1/3}, k_{i+1}^{1/3}, k_{i+2}^{1/3}) \mapsto (k_i^{-1/3}, k_{i+1}^{1/3} k_{i+1}^{1/3}, k_{i+2}^{1/3} k_{i+2}^{1/3}),
\]
\[
\pi : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (k_1^{1/3}, k_2^{1/3}, k_0^{1/3}),
\]
\[
w_0 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (a_0 k_0^{-1/3}, a_1 k_1^{-1/3}, a_2 k_2^{-1/3}),
\]
\[
w_1 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (a_0 k_0^{-1/3}, a_1 k_1^{-1/3} k_1^{1/3}, a_2 k_2^{-1/3}),
\]
\[
r : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (k_0^{-1/3}, k_1^{-1/3} k_1^{1/3}, k_2^{-1/3}),
\]
where \( i \in \mathbb{Z}/3\mathbb{Z} \). From the definitions (3.4) and (3.12), it follows that the actions of \( T_i \) for \( i = 1, \ldots, 4 \), and \( R_l \) on the parameters \( k_i^{1/3}, j = 0, 1, 2 \), are given by the following:
\[
T_1 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (\lambda^{1/3} k_0^{1/3}, \lambda^{-1/3} k_1^{1/3}, k_2^{1/3}),
\] (3.19)
\[
T_2 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (k_0^{1/3}, \lambda^{1/3} k_1^{1/3}, \lambda^{-1/3} k_2^{1/3}),
\] (3.20)
\[
T_3 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (\lambda^{-1/3} k_0^{1/3}, k_1^{1/3}, \lambda^{1/3} k_2^{1/3}),
\] (3.21)
\[
T_4 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (a_0 k_0^{1/3}, a_1 k_1^{1/3} k_1^{1/3}, a_2 k_2^{1/3}),
\] (3.22)
\[
R_1 : (k_0^{1/3}, k_1^{1/3}, k_2^{1/3}) \mapsto (k_2^{1/3} k_0^{1/3}, k_1^{1/3} k_0^{1/3}, \lambda^{-1/3} k_1^{1/3} k_2^{1/3}).
\] (3.23)

Now we are in a position to define the three variables
\[
\omega_0 = \frac{k_2^{1/3}}{k_1^{1/3}} \frac{\tau_0}{\tau_1}, \quad \omega_1 = \frac{k_1^{1/3}}{k_2^{1/3}} \frac{\tau_1}{\tau_2}, \quad \omega_2 = \frac{k_2^{1/3}}{k_0^{1/3}} \frac{\tau_2}{\tau_0}.
\] (3.24)

The action of \( \bar{W}(A_2 + A_1)^{(1)} \) on these variables \( \omega_i \) is given by the following, which follows from the above definitions:
\[
s(\omega_i) = \omega_i \left[ a_i \omega_{i+1} + K_i \omega_{i+2} \right], \quad s'(\omega_i) = k_i^{-1} \omega_{i+1}, \quad s'(\omega_{i+2}) = k_i \omega_{i+2},
\]
\[
\pi(\omega_i) = \omega_{i+1}, \quad w_0(\omega_i) = \frac{a_i k_i k_{i+1} \omega_i + a_{i+1} k_{i+2} \omega_{i+1} + k_i \omega_{i+2}}{a_i k_i k_{i+2} \omega_i + a_{i+1} k_{i+2} \omega_{i+1} + k_i \omega_{i+2}},
\]
\[
w_1(\omega_i) = \frac{a_i k_i k_{i+2} \omega_i + a_{i+1} k_{i+2} \omega_{i+1} + k_i \omega_{i+2}}{a_i k_i k_{i+2} \omega_i + a_{i+1} k_{i+2} \omega_{i+1} + k_i \omega_{i+2}}, \quad r(\omega_i) = \omega_i^{-1},
\]
where \( i \in \mathbb{Z}/3\mathbb{Z} \).

We define \( \omega \)-functions by
\[
\omega_{l_1 l_2 l_3 l_4} = T_1^{l_4} T_2^{l_3} T_3^{l_2} T_4^{l_1} (\omega_0),
\] (3.25)
where \( l_1, \ldots, l_4 \in \mathbb{Z} \) and the \( \omega \)-lattice is as shown in Figure 3. We note that
\[
\omega_0 = \omega_{0,0,0,0}, \quad \omega_1 = k_2^{-1} \omega_{1,0,0,0}, \quad \omega_2 = k_1 \omega_{1,1,0,0}.
\] (3.26)
The following PΔEs hold on the $\omega$-lattice [20]:

\[
\begin{align*}
\omega_{i_1,i_2,i_3,i_4} & = \frac{q^{i_1+i_2+i_3+i_4}A_1\omega_{i_1+1,i_2,i_3,i_4} - \omega_{i_1,i_2+1,i_3,i_4} - \omega_{i_1,i_2,i_3+1,i_4} + \omega_{i_1,i_2,i_3,i_4+1}}{q^{i_1+i_2+i_3+i_4}A_1(q^{i_1+i_2+i_3+i_4}A_1 - q^{i_1+i_2+i_3+i_4}A_1)} \quad & (3.27a) \\
\omega_{i_1,i_2+1,i_3,i_4} & = \frac{q^{i_1+i_2+i_3+i_4}A_2\omega_{i_1,i_2+1,i_3+1,i_4} - \omega_{i_1,i_2,i_3+1,i_4} - \omega_{i_1,i_2+1,i_3,i_4+1} + \omega_{i_1+1,i_2,i_3,i_4}}{q^{i_1+i_2+i_3+i_4}A_2(q^{i_1+i_2+i_3+i_4}A_1 - q^{i_1+i_2+i_3+i_4}A_1)} \quad & (3.27b) \\
\omega_{i_1+1,i_2,i_3,i_4} & = \frac{q^{i_1+i_2+i_3+i_4}A_3\omega_{i_1+1,i_2+1,i_3,i_4} - \omega_{i_1,i_2+1,i_3,i_4} - \omega_{i_1+1,i_2,i_3+1,i_4} + \omega_{i_1+1,i_2,i_3,i_4+1}}{q^{i_1+i_2+i_3+i_4}A_3(q^{i_1+i_2+i_3+i_4}A_1 - q^{i_1+i_2+i_3+i_4}A_1)} \quad & (3.27c) \\
\omega_{i_1,i_2,i_3+1,i_4} & = \frac{q^{i_1+i_2+i_3+i_4}A_4\omega_{i_1,i_2,i_3+1,i_4+1} - \omega_{i_1,i_2,i_3,i_4} - \omega_{i_1+1,i_2,i_3,i_4+1} + \omega_{i_1,i_2+1,i_3,i_4}}{q^{i_1+i_2+i_3+i_4}A_4(q^{i_1+i_2+i_3+i_4}A_1 - q^{i_1+i_2+i_3+i_4}A_1)} \quad & (3.27d) \\
\omega_{i_1,i_2,i_3,i_4+1} & = \frac{q^{i_1+i_2+i_3+i_4}A_5\omega_{i_1,i_2+1,i_3+1,i_4} - \omega_{i_1,i_2,i_3+1,i_4} - \omega_{i_1+1,i_2+1,i_3,i_4} + \omega_{i_1,i_2+1,i_3,i_4+1}}{q^{i_1+i_2+i_3+i_4}A_5(q^{i_1+i_2+i_3+i_4}A_1 - q^{i_1+i_2+i_3+i_4}A_1)} \quad & (3.27e) \\
\omega_{i_1+1,i_2,i_3,i_4} & = \frac{q^{i_1+i_2+i_3+i_4}A_6\omega_{i_1+1,i_2,i_3+1,i_4+1} - \omega_{i_1,i_2,i_3+1,i_4} - \omega_{i_1+1,i_2+1,i_3+1,i_4} + \omega_{i_1+1,i_2+1,i_3,i_4+1}}{q^{i_1+i_2+i_3+i_4}A_6(q^{i_1+i_2+i_3+i_4}A_1 - q^{i_1+i_2+i_3+i_4}A_1)} \quad & (3.27f)
\end{align*}
\]

Lemma 3.3. The PΔEs (3.27) are equivalent to the periodically reduced PΔEs (3.32) by the following correspondence:

\[
\begin{align*}
a_0 & = q^{\hat{\alpha}}, \quad a_1 = \frac{\beta}{\hat{\alpha}}, \quad a_2 = \frac{\hat{\gamma}}{\beta}, \quad a_3 = \frac{1}{\hat{\lambda}} \quad \omega_{0,0,0,0} = \omega(0), \quad & (3.28) \\
T_i & = \hat{T}_i, \quad i = 1, \ldots, 4. \quad & (3.29)
\end{align*}
\]

Furthermore, by letting

\[
\omega_{l_1} = R_1^l T_4^4(\omega_0), \quad & (3.30)
\]

where $l, l_4 \in \mathbb{Z}$, the following relation holds

\[
\omega_{l_2-l_1} = q^{l_2-l_1} \omega_{l_1,0,l_4}, \quad \omega_{l_2} = \omega_{0,0,l_4}. \quad & (3.31)
\]
Lemma 3.4. The PΔEs decided uniquely, which implies that the expression of action of the following correspondence:

We can easily verify that the following relation holds:

\[
\Phi_i = \begin{pmatrix} 0 & \omega_i \\ 1 & \frac{1}{\omega_i} \end{pmatrix} \Phi_i,
\]

which implies that the expression of action of \( \tilde{W}(A_2 + A_1)^{(1)} \) on the vectors \( \Phi_i \) cannot be decided uniquely.
Lemma 3.5. An action of \( \tilde{W}((A_2 + A_1)^{(1)}) \) on the vectors \( \Phi_i, i = 0, 1, 2 \), is given by

\[
s_i(\Phi_{i+1}) = \frac{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2})}{1 - a_i^{1/2} \lambda^{1/3} T_{i+1}(\omega_{i+2})} \left( \frac{1 - a_i^{1/2} \lambda^{1/3} T_{i+1}(\omega_{i+2})}{\lambda^{1/3} T_{i+1}(\omega_{i+2})} \right) \Phi_i, \tag{3.37a}
\]

\[
s_i(\Phi_{i+2}) = \frac{1}{\lambda^{1/3} T_{i+1}(\omega_{i+2})} \left( \begin{array}{c} \kappa_i^{1/3} \\ 0 \\ 1 \end{array} \right) \Phi_{i+2}, \quad \pi(\Phi_i) = \Phi_{i+1}, \tag{3.37b}
\]

\[
w_0(\Phi_i) = \frac{a_i^{1/3} \kappa_{i+1}^{1/3} \lambda^{1/3} T_{i+1}^{-1}(\omega_{i+2})}{\omega_{i+2}} \left( \begin{array}{c} \kappa_{i+1}^{1/3} \\ a_i \lambda^{1/3} T_{i+1}^{-1}(\omega_{i+2}) \end{array} \right) \Phi_i, \tag{3.37c}
\]

\[
r(\Phi_i) = \frac{\lambda^{1/3} T_{i+1}^{-1}(\omega_{i+2})}{\lambda^{1/3} T_{i+1}^{-1}(\omega_{i+2})} \left( \begin{array}{c} \kappa_i^{1/3} \\ 0 \\ 1 \end{array} \right) \Phi_i, \tag{3.37d}
\]

where \( i \in \mathbb{Z}/3\mathbb{Z} \).

**Proof.** From the action of \( T_{i+1} \) on \( \bar{\tau}_{i+2} \) and \( \tau_{i+2} \):

\[
\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2}) = \lambda^{2/3} a_i^{1/2} \bar{\tau}_{i+2} \tau_{i+1} + \bar{\tau}_{i+2} \tau_{i+1}, \tag{3.38}
\]

\[
\lambda^{1/3} a_i^{-1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2}) = \lambda^{2/3} a_i^{-1/2} \tau_{i+2} \bar{\tau}_{i+1} + \bar{\tau}_{i+2} \tau_{i+1}, \tag{3.39}
\]

where \( i \in \mathbb{Z}/3\mathbb{Z} \), we obtain the following two equations:

\[
a_i^{1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2}) - a_i^{-1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2}) = \lambda^{1/3} (a_i - a_i^{-1}) \bar{\tau}_{i+2} \tau_{i+1}, \tag{3.40}
\]

\[
\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2}) - \lambda^{1/3} a_i^{-1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2}) = (1 - a_i^2) \bar{\tau}_{i+2} \tau_{i+1}, \tag{3.41}
\]

which lead to

\[
s_i(\bar{\tau}_{i+1}) = \bar{\tau}_{i+1} = \lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2}) \left( \frac{1 - a_i^{1/2} \lambda^{1/3} T_{i+1}(\omega_{i+2})}{\lambda^{1/3} T_{i+1}(\omega_{i+2})} \right) \bar{\tau}_{i+1}, \tag{3.42}
\]

\[
s_i(\tau_{i+1}) = \tau_{i+1} = \lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2}) \left( \frac{a_i^{1/2} \lambda^{1/3} T_{i+1}(\omega_{i+2})}{\lambda^{1/3} T_{i+1}(\omega_{i+2})} \right) \tau_{i+1}, \tag{3.43}
\]

respectively. Therefore, Equation (3.37a) holds.

Moreover, from the actions of \( w_0 \) on \( \bar{\tau}_i \) and \( T_{i+1}^{-1} \) on \( \tau_{i+2} \):

\[
w_0(\bar{\tau}_i) = \frac{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2})}{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2})} \bar{\tau}_i \tau_{i+1} \tau_{i+2} + \lambda^{2/3} a_i^{-1} \lambda^{1/3} \bar{\tau}_i \tau_{i+1} \tau_{i+2}, \tag{3.44}
\]

\[
T_{i+1}^{-1}(\tau_{i+2}) = \frac{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2})}{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2})} \bar{\tau}_i \tau_{i+1} \tau_{i+2} + \lambda^{2/3} a_i^{-1} \lambda^{1/3} \bar{\tau}_i \tau_{i+1} \tau_{i+2} + \lambda^{2/3} a_i^{-1} \lambda^{1/3} \bar{\tau}_i \tau_{i+1} \tau_{i+2}, \tag{3.45}
\]

where \( i \in \mathbb{Z}/3\mathbb{Z} \), the following relation holds:

\[
w_0(\bar{\tau}_i) = \frac{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\bar{\tau}_{i+2})}{\lambda^{1/3} a_i^{1/2} \lambda^{1/3} T_{i+1}(\tau_{i+2})} \bar{\tau}_i \tau_{i+1} \tau_{i+2} + \lambda^{2/3} a_i^{-1} \lambda^{1/3} \bar{\tau}_i \tau_{i+1} \tau_{i+2}, \tag{3.46}
\]

which leads to Equation (3.37c) with

\[
w_0(\bar{\tau}_i) = \tau_i. \tag{3.47}
\]

The others are obvious. Therefore, we have completed the proof. \( \square \)
Lemma 3.6. The following linear systems hold:

\begin{align*}
T_2 (\Phi_{l_1,l_2,l_3,l_4}^{(1)}) &= \frac{q^{3(-l_1+l_2)/2+4l_4/3} a_1^{3/2}}{q^{2l_1+6l_2} a_1^2 - 1} \\
& \quad \times \left( \begin{array}{c}
q^{l_1-l_2+l_4} a_1 \\
q^{l_1-l_3+l_4} a_1 \\
\omega_{l_1,l_2,l_3,l_4} \\
\omega_{l_1,l_2,l_3,l_4} \end{array} \right) \Phi_{l_1,l_2,l_3,l_4}^{(1)}, & (3.49a)
\end{align*}

\begin{align*}
T_3 (\Phi_{l_1,l_2,l_3,l_4}^{(1)}) &= \frac{q^{(-l_1+l_2)/2+2l_4/3} a_1^{1/2}}{q^{2l_1+6l_2} a_1^2 - 1} \\
& \quad \times \left( \begin{array}{c}
-q^{l_1-l_2+l_4} a_0 \\
-q^{l_1-l_3+l_4} a_0 \\
\omega_{l_1,l_2,l_3,l_4} \\
\omega_{l_1,l_2,l_3,l_4} \end{array} \right) \Phi_{l_1,l_2,l_3,l_4}^{(1)}, & (3.49b)
\end{align*}

\begin{align*}
T_4 (\Phi_{l_1,l_2,l_3,l_4}^{(1)}) &= \frac{a_1^{1/3}}{q^{(-l_1+l_2)/2+2l_4/3} a_1^{1/2}} \\
& \quad \times \left( \begin{array}{c}
q^{2l_1+1} a_1^2 - 1 \\
q^{l_1-l_2+l_4} a_1 \\
\omega_{l_1,l_2,l_3,l_4} \\
\omega_{l_1,l_2,l_3,l_4} \end{array} \right) \Phi_{l_1,l_2,l_3,l_4}^{(1)}, & (3.49c)
\end{align*}

\begin{align*}
T_1 (\Phi_{l_1,l_2,l_3,l_4}^{(2)}) &= \frac{1}{q^{(-l_1+l_2)/2+2l_4/3} a_1^{1/2}} \\
& \quad \times \left( \begin{array}{c}
-a_1 \\
q^{l_1-l_2+l_4} a_1 \\
\omega_{l_1,l_2,l_3,l_4} \\
\omega_{l_1,l_2,l_3,l_4} \end{array} \right) \Phi_{l_1,l_2,l_3,l_4}^{(2)}, & (3.49d)
\end{align*}

\begin{align*}
T_3 (\Phi_{l_1,l_2,l_3,l_4}^{(2)}) &= \frac{q^{(-l_1+l_2+2l_4)/3} a_2^{1/2}}{q^{2l_1+6l_2} a_2^2 - 1} \\
& \quad \times \left( \begin{array}{c}
-q^{l_1-l_2+l_4} a_2 \\
-q^{l_1-l_3+l_4} a_2 \\
\omega_{l_1,l_2,l_3,l_4} \\
\omega_{l_1,l_2,l_3,l_4} \end{array} \right) \Phi_{l_1,l_2,l_3,l_4}^{(2)}, & (3.49e)
\end{align*}

\begin{align*}
T_4 (\Phi_{l_1,l_2,l_3,l_4}^{(2)}) &= q^{(-l_1+l_2+2l_4+1)/3} a_1^{1/3}
\end{align*}
\[
T_1 \left( \Phi_{l_1,l_2,l_3}^{(3)} \right) = \frac{q^{3(l_1-l_2-l_3)/2}a_0^{-3/2}}{A^{1/3}(q^{2l_1-l_2-l_3}a_0^2 - 1)} \begin{pmatrix}
\Phi_{l_1,l_2,l_3,l_4}^{(2)} \\
\omega_{l_1,l_2,\lambda_4}
\end{pmatrix}
\]
\[
T_2 \left( \Phi_{l_1,l_2,l_3}^{(3)} \right) = \frac{q^{3-l_1+l_2+l_3}A^{1/3}a_2}{1 - q^{2l_2+l_1}a_2^2} \begin{pmatrix}
a_2 \\
q^{-l_1+l_2+l_3}A
\end{pmatrix}
\begin{pmatrix}
\omega_{l_1,l_2,\lambda_4} \\
\omega_{l_1,l_2,\lambda_4}
\end{pmatrix}
\]
\[
T_4 \left( \Phi_{l_1,l_2,l_3}^{(3)} \right) = \frac{q^{3-l_1+l_2-l_3}a_1^{1/3}}{A^{1/3}} \begin{pmatrix}
(q^{2l_1+1}A^2 - 1)a_2 \\
q^{-l_1+l_2+l_3}A
\end{pmatrix}
\begin{pmatrix}
\omega_{l_1,l_2,\lambda_4} \\
\omega_{l_1,l_2,\lambda_4}
\end{pmatrix}
\]
\[
T_1 \left( \Phi_{l_1,l_2,l_3}^{(4)} \right) = \frac{1}{q^{2l_1+1}A^{1/3}} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\begin{pmatrix}
q^{2l_1+1}A^2 - 1 \\
q^{-l_1+l_2+l_3}Aa_1
\end{pmatrix}
\begin{pmatrix}
\omega_{l_1,l_2,\lambda_4} \\
\omega_{l_1,l_2,\lambda_4}
\end{pmatrix}
\]
\[
T_2 \left( \Phi_{l_1,l_2,l_3}^{(4)} \right) = \frac{1}{q^{2l_1+1}A^{1/3}} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\begin{pmatrix}
(q^{2l_1+1}A^2 - 1)\omega_{l_1,l_2,\lambda_4} \\
q^{2l_1+1}A^2 \omega_{l_1,l_2,\lambda_4}
\end{pmatrix}
\]
\[
T_3 \left( \Phi_{l_1,l_2,l_3}^{(4)} \right) = \frac{1}{q^{2l_1+1}A^{1/3}} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\begin{pmatrix}
(q^{2l_1+1}A^2 - 1)a_2 \\
q^{-l_1+l_2+l_3}Aa_1
\end{pmatrix}
\begin{pmatrix}
\omega_{l_1,l_2,\lambda_4} \\
\omega_{l_1,l_2,\lambda_4}
\end{pmatrix}
\]
\[
(3.49f)
\]
\[
(3.49g)
\]
\[
(3.49h)
\]
\[
(3.49i)
\]
\[
(3.49j)
\]
\[
(3.49k)
\]
\[
(3.49l)
\]
Proof. First, we prove Equations (3.49a)–(3.49i). From Lemma 3.5 and the definition (3.4), the following relations hold:

\[
T_{i+1}(\Phi_i) = \frac{a_i^{3/2}}{A_{i+1}^3(a_i^2 - 1)} t_{i+1}(\tau_{i+2}) \begin{pmatrix} 1 & \frac{1}{a_i} & 0 \\ \frac{1}{a_i} & 0 & 0 \\ \frac{1}{a_i} & 0 & 0 \end{pmatrix} \Phi_i, \quad (3.50a)
\]

\[
T_{i+2}(\Phi_i) = q\frac{a_{i+2}^{3/2}}{q^2 - a_{i+2}^2} t_{i+2}(\tau_{i+2}) \begin{pmatrix} -a_{i+2} & \frac{1}{a_{i+1}} & 0 \\ \frac{1}{a_{i+1}} & 0 & 0 \\ \frac{1}{a_{i+1}} & 0 & 0 \end{pmatrix} \Phi_i, \quad (3.50b)
\]

\[
T_3(\Phi_i) = \frac{a_{i+1}^{1/3}}{A_{i+1}^{1/3}a_{i+2}^{1/2}} t_{i+3}(\tau_{i+2}) \begin{pmatrix} (qA^2 - 1)a_{i+2} & -q & 0 \\ -q & 0 & 0 \\ -q & 0 & 0 \end{pmatrix} \Phi_i, \quad (3.50c)
\]

where \(i \in \mathbb{Z}/3\mathbb{Z}\). Moreover, from (3.11), (3.35) and (3.48), we obtain

\[
\Phi^{(1)}_{0,0,0} = \frac{\kappa_{1/3}^{1/3} A_{1/3}}{A_{1/3}^{1/3}a_1^{1/2}} \begin{pmatrix} \kappa_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi_i, \quad (3.51a)
\]

\[
\Phi^{(2)}_{0,0,0} = \frac{\kappa_2^{1/3} A_{1/3}}{A_{1/3}^{1/3}a_1^{1/2}} \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi_i, \quad (3.51b)
\]

\[
\Phi^{(3)}_{0,0,0} = \frac{1}{A_{1/3}^{1/3}} \Phi_0. \quad (3.51c)
\]

Therefore, from Equations (3.50) and (3.51), we obtain

\[
\Phi^{(1)}_{0,1,0} = \frac{A_{1/3}^{1/3}a_1^{1/2}}{a_1^2 - 1} \begin{pmatrix} -\frac{1}{Aa_1} & \frac{1}{a_1} & 0 \\ \frac{1}{a_1} & 0 & 0 \\ \frac{1}{a_1} & 0 & 0 \end{pmatrix} \Phi^{(1)}_{0,0,0}, \quad (3.52a)
\]

\[
\Phi^{(2)}_{0,1,0} = q\frac{a_0^{1/2}}{q^2 - a_0^2} \begin{pmatrix} -a_0 & \frac{1}{a_1} & 0 \\ \frac{1}{a_1} & 0 & 0 \\ \frac{1}{a_1} & 0 & 0 \end{pmatrix} \Phi^{(1)}_{0,0,0}, \quad (3.52b)
\]

\[
\Phi^{(3)}_{0,0,1} = \frac{a_1^{1/3}}{q^{1/3}A_{1/3}} t_{i+1}(\tau_{i+2}) \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi^{(1)}_{0,0,0}, \quad (3.52c)
\]

\[
\Phi^{(2)}_{1,0,0} = \frac{A_{1/3}^{1/3}a_1^{1/2}}{a_1^2 - 1} \begin{pmatrix} -a_1 & \frac{1}{a_1} & 0 \\ \frac{1}{a_1} & 0 & 0 \\ \frac{1}{a_1} & 0 & 0 \end{pmatrix} \Phi^{(2)}_{0,0,0}, \quad (3.52d)
\]

\[
\Phi^{(3)}_{0,1,0} = \frac{A_{1/3}^{1/3}a_1^{1/2}}{a_1^2 - 1} \begin{pmatrix} -a_1 & \frac{1}{a_1} & 0 \\ \frac{1}{a_1} & 0 & 0 \\ \frac{1}{a_1} & 0 & 0 \end{pmatrix} \Phi^{(3)}_{0,0,0}, \quad (3.52e)
\]

\[
\Phi^{(2)}_{0,0,1} = q\frac{a_0^{1/2}}{q^2 - a_0^2} \begin{pmatrix} -a_0 & \frac{1}{a_1} & 0 \\ \frac{1}{a_1} & 0 & 0 \\ \frac{1}{a_1} & 0 & 0 \end{pmatrix} \Phi^{(2)}_{0,0,0}, \quad (3.52f)
\]
Lemma 3.7. The linear systems (B.7) are equivalent to the linear systems (3.49) by the following correspondences:

\[
\frac{\dot{x}}{y} = a_0, \quad \frac{\dot{\beta}}{\gamma} = a_1, \quad \frac{\dot{y}}{\beta} = a_2, \quad \tilde{T}_i = T_i, \quad i = 1, \ldots, 4, \quad \omega(0) = \omega_{0,0,0,0},
\]

\[
\delta_{h,l,j,k}^{(1,2)} = -\frac{1}{q^{3R+2l+3j+2k}A^{1/3}A^1} a_1^{1/2}, \quad \delta_{h,l,j,k}^{(1,3)} = -\frac{1}{q^{3R+2l+3j+2k}A^{1/3}A^1} a_0^{1/2},
\]

\[
\delta_{h,l,j,k}^{(2,3)} = -\frac{1}{q^{3R+2l+3j+2k}A^{1/3}A^1} a_2^{3/2}, \quad \delta_{h,l,j,k}^{(2,4)} = -\frac{1}{q^{3R+2l+3j+2k}A^{1/3}A^1} a_1^{1/2}.
\]
Lemma 3.8. The following linear systems hold:

\[
\delta^{(3,1)}_{l; k_3, l_3, a_4} = -i \frac{q^{2l_3+1}}{a_{l_3}^3} \frac{a_{l_3}^{3/2}}{1 - q^{2l_3+1}} \frac{a_{l_3}^{1/2}}{1 - q^{2l_3+1}}
\]

\[
\delta^{(3,2)}_{l; k_3, l_3, a_4} = -i \frac{q^{2l_3+1} a_{l_3}^{1/3}}{1 - q^{2l_3+1}} \frac{a_{l_3}^{1/2}}{1 - q^{2l_3+1}}
\]

\[
\delta^{(4,1)}_{l; k_3, l_3, a_4} = -i \frac{q^{2l_3+1} a_{l_3}^{1/3}}{1 - q^{2l_3+1}} \frac{a_{l_3}^{1/2}}{1 - q^{2l_3+1}}
\]

\[
\Phi^{(4)}_{l; l_3} = \frac{1}{t_{l_3}} R_1^{l+2} T_4^{i}(\Phi_0),
\]

\[
\Phi^{(5)}_{l; l_3} = \frac{1}{t_{l_3}} R_1^{l+1} T_4^{i}(\Phi_0),
\]

we obtain the following lemma.

Lemma 3.8. The following linear systems hold:

\[
R_1^{-1} \left( \Phi^{(1)}_{l; l_3} \right) = \frac{R_1^{l} T_4^{i}(k_3^{1/3}) R_1^{i}(a_1^{1/2})}{R_1^{i}(a_2^{1/2}) - 1}
\]

\[
T_4 \left( \Phi^{(1)}_{l; l_3} \right) = \frac{R_1^{l} T_4^{i}(k_3^{1/3}) R_1^{i}(a_1^{1/2})}{R_1^{i}(a_2^{1/2}) - 1}
\]

\[
T_1 \left( \Phi^{(2)}_{l; l_3} \right) = -\frac{R_1^{l} T_4^{i}(k_2) R_1^{i}(a_1^{1/2})}{R_1^{i}(a_2^{1/2}) - 1}
\]

\[
T_4 \left( \Phi^{(2)}_{l; l_3} \right) = \frac{R_1^{l} T_4^{i}(k_2)}{R_1^{i}(a_2^{1/2}) - 1}
\]

\[
T_1 \left( \Phi^{(3)}_{l; l_3} \right) = \frac{R_1^{l} T_4^{i}(k_3^{1/3}) R_1^{i}(a_1^{1/2})}{R_1^{i}(a_2^{1/2}) - 1}
\]

\[
T_4 \left( \Phi^{(3)}_{l; l_3} \right) = \frac{R_1^{l} T_4^{i}(k_3^{1/3}) R_1^{i}(a_2^{1/2})}{R_1^{i}(a_2^{1/2}) - 1}
\]

The following systems hold:

\[
F^{(i)}(T_{i}(h_0,0,0)) = T_i(k_2^{1/3}) T_i(\tau_0),
\]

\[
G^{(i)} = T_i(k_1^{1/3}) T_i(\tau_0),
\]

\[
i = 1, \ldots, 4.
\]
Lemma 3.9. The linear systems (B.11) are equivalent to the linear systems (3.59) by the following correspondence:

\[
\begin{align*}
\dot{q} &= a_0, & \dot{\bar{q}} &= a_1, & \dot{\bar{q}} &= a_2, & k &= \kappa_2, & \hat{T}_1 &= T_1, & \hat{R}_1 &= R_1, & \hat{T}_2 &= T_4, & \omega(0) &= \omega_0
\end{align*}
\]
The action of $\delta_{0, l_{\mu_4}}^{(1)}$ and $\delta_{0, l_{\mu_4}}^{(2)}$ is given by

$$\delta_{0, l_{\mu_4}}^{(1)} = -\frac{R_i^1 T_{l_{\mu_4}}^h(k_{1/3}) R_i^1(a_{1/3})}{R_i^1(a_{1/2}) - 1}, \quad \delta_{0, l_{\mu_4}}^{(2)} = -\frac{R_i^1(a_{1/3})}{q^{d_{+1}(3/3, 1/3)}},$$

(3.64b)

$$\delta_{0, l_{\mu_4}}^{(3)} = \frac{1}{q^{d_{+1}(1/3, 2/3)}}, \quad \delta_{0, l_{\mu_4}}^{(4)} = -\frac{1}{q^{d_{+1}(3/3, 1/3)} R_i^1(a_{0/1})}.$$  

(3.64c)

$$\delta_{0, l_{\mu_4}}^{(5)} = \frac{i}{q^{2d_{+1}(1/3, 2/3)}}, \quad \delta_{0, l_{\mu_4}}^{(6)} = \frac{R_i^1 T_{l_{\mu_4}}^h(k_{1/3}) R_i^1(a_{2/3})}{q^{d_{+1}(1/3, 2/3)}}.$$  

(3.64d)

$$\frac{F_i^{(i)}}{T_i(h_{0,0,0,0})} = T_i(k_{1/3}) T_i(\tau_0) \frac{L_i}{\tau_0}, \quad \frac{G_i^{(i)}}{R_i^{1-1}(h_{0,0,0,0})} = R_i^{1-1}(k_{1/3}) R_i^{1-1}(\tau_0) \frac{L_i}{\tau_0}, \quad i, \ 1, 4,$$

(3.64e)

$$f_0 = \lambda^{2/3} \frac{\tilde{\tau}_1 \tilde{\tau}_2}{\tau_1 \tau_2}, \quad f_1 = \lambda^{2/3} \frac{\tilde{\tau}_2 \tau_0}{\tau_2 \tau_0}, \quad f_2 = \lambda^{2/3} \frac{\tilde{\tau}_0 \tau_1}{\tau_0 \tau_1}, \quad (3.65)$$

3.4. The discrete Painlevé equations. Let

The action of $\tilde{W}((A_2 + A_1)^1)$ on the variables $f_i, \ i = 0, 1, 2,$ is given by

$$s_i(f_{i-1}) = f_{i-1} + a_{i+1} f_i / a_{i} f_i, \quad s_i(f_i) = f_i, \quad s_i(f_{i+1}) = f_{i+1} + a_{i} f_i / (1 + a_{i} f_i), \quad \pi(f_i) = f_{i+1},$$

$$w_{0i}(f_i) = \frac{a_{i+1}(a_{i-1} a_{i} f_{i-1} + a_{i} f_i + f_{i-1})}{f_{i-1}(a_{i+1} a_{i} f_{i-1} + f_{i-1} f_{i+1})}, \quad \frac{1 + a_{i} f_i + a_{i+1} f_{i+1}}{a_{i+1} f_{i+1} f_{i+1} f_{i+1}}, \quad \tau(f_i) = f_{i+1},$$

where $i \in \mathbb{Z}/3\mathbb{Z}$. We note that the variables $f_i$ satisfy the condition

$$f_0 f_1 f_2 = \lambda^2.$$  

(3.66)

Define $f$-functions by

$$f_0^{m,n} = T_0^m T_2^N T_4^N (f_0), \quad f_1^{m,n} = T_0^m T_2^N T_4^N (f_1), \quad f_2^{m,n} = T_0^m T_2^N T_4^N (f_2),$$

where $n, m, N \in \mathbb{Z}$. The relations between $f$-functions and $\omega$-functions are given as follows:

$$f_0 = k_{1/2} \frac{\omega_1}{\omega_2}, \quad f_1 = k_{1/2} \frac{\omega_2}{\omega_0}, \quad f_2 = k_{1/2} \frac{\omega_0}{\omega_1},$$

or

$$f_0^{1-h, 1-h, -h} = \frac{\omega_{1/2+1/2, 1/2, 1/2}}{\omega_{1/2+1/2, 1/2, 1/2}}.$$  

(3.68a)

$$f_1^{1-h, 1-h, -h} = q^h \frac{\omega_{1/2+1/2, 1/2, 1/2}}{\omega_{1/2, 1/2, 1/2}}.$$  

(3.68b)

$$f_2^{1-h, 1-h, -h} = q^h \frac{\omega_{1/2+1/2, 1/2, 1/2}}{\omega_{1/2, 1/2, 1/2}}.$$  

(3.68c)

The action of $T_4$ on the variables $f_i$ can be expressed as

$$T_4(f_0) = a_{0} a_{1} f_{1} \frac{1 + a_{2} f_{2} (a_{0} f_{0} + 1)}{1 + a_{0} f_{0} (a_{1} f_{1} + 1)},$$

(3.69a)

$$T_4(f_1) = a_{1} a_{2} f_{2} \frac{1 + a_{0} f_{0} (a_{1} f_{1} + 1)}{1 + a_{1} f_{1} (a_{2} f_{2} + 1)}.$$  

(3.69b)

$$T_4(f_2) = a_{2} a_{0} f_{0} \frac{1 + a_{1} f_{1} (a_{2} f_{2} + 1)}{1 + a_{2} f_{2} (a_{0} f_{0} + 1)}.$$  

(3.69c)
or applying $T_1^a T_2^m T_4^N$ on System (3.69) and using (3.67), we obtain

\[ f_{0, N+1}^{m} = q^m a_0 a_1 f_{1, N}^{m} + 1 + q^{-m} a_2 f_{2, N}^{m} (q^m a_0 f_{0, N}^{m} + 1) \]

(3.70a)

\[ f_{1, N+1}^{m} = q^{-m} a_1 a_2 f_{2, N}^{m} + 1 + q^{m} a_0 f_{0, N}^{m} (q^{-m} a_1 f_{1, N}^{m} + 1) \]

(3.70b)

\[ f_{2, N+1}^{m} = q^{m} a_2 a_0 f_{0, N}^{m} + 1 + q^{-m} a_1 f_{1, N}^{m} (q^m a_2 f_{2, N}^{m} + 1) \]

(3.70c)

which is equivalent to \( q^m A = t, \) \( q^m a_0 = a, \) \( q^{-m} a_1 = b, \) \( q^{-m} a_2 = c, \)

(3.71)

\[ f_{0, N}^{m} = f(t), \quad f_{1, N}^{m} = g(t), \quad f_{2, N}^{m} = h(t). \]

(3.72)

In contrast, the action of $T_1$:

\[ T_1(f_1) = \frac{x^2}{f_0} f_0 \quad T_1(f_0) = \frac{x^2}{f_0} f_1 + a_0 + f_0 \]

(3.73)

leads to a system of first-order ordinary difference equations

\[ f_{1, N}^{m+1} = q^{2N} x^2 \quad f_{0, N}^{m+1} = q^{2N} x^2 f_{0, N}^{m} + q^{-m} a_0 a_2 f_{1, N}^{m+1} \]

(3.74)

which is equivalent to \( q^m A = t, \) \( q^m a_0 = a, \) \( q^{-m} a_1 = b, \) \( q^{-m} a_2 = c, \)

(3.75)

In a similar manner, in each of the $T_2$- and $T_3$-directions, we also obtain $q$-P$_{III}$.

Let

\[ f_N^{1} = R_1^N T_4^N (f_0), \]

(3.76)

where

\[ f_N^{2i-1} = f_{1, N}^{2i-1}, \quad f_N^{2i} = f_{0, N}^{2i}. \]

(3.77)

By considering the case in which $f_N^{l}$ are defined, System (3.74) becomes the following system:

\[ f_{1, N}^{m+1, 0} = q^{2N} x^2 \quad f_{0, N}^{m+1, 0} = q^{2N} x^2 f_{0, N}^{m, 0} + q^{-m} a_0 a_2 f_{1, N}^{m+1, 0} \]

(3.78)

which is equivalent to the following single equation (symmetric $q$-P$_{III}$):

\[ f_N^{l+1} = \frac{q^{2N} x^2}{f_N^{l}} f_N^{l} + R_1^N (a_0) + f_N^{l}. \]

(3.79)

In addition, by assuming $a_2 = q^{1/2}$, transformation $R_1$ becomes the translational motion in the parameter subspace:

\[ R_1 : (a_0, a_1) \rightarrow (q^{1/2} a_0, q^{-1/2} a_1), \]

(3.80)

then Equation (3.79) can be regarded as the single second-order ordinary difference equation:

\[ f_N^{l+1} = q^{2N} x^2 \quad f_N^{l+1} = q^{2N} x^2 f_N^{l} + a_0 q^{1/2} f_N^{l} \]

(3.81)

which is equivalent to \( q^m A = t, \) \( q^m a_0 = a, \) \( q^{m} a_1 = b, \) \( q^{-m} a_2 = c, \)

(3.82)
Remark 3.10. For a Lax pair of a discrete Painlevé equation, an additional parameter called a spectral parameter and an additional function called a wave function are necessary. However, since the linear systems (B.7) and (B.11) can be obtained from the τ functions of $A^{(1)}_3$-surface $q$-Painlevé equations, we cannot construct the Lax pairs of $A^{(1)}_3$-surface $q$-Painlevé equations from the lattice $\mathbb{Z}^4$.

In the next section, we shall construct the Lax pairs of the $A^{(1)}_3$-surface $q$-Painlevé equations from the 5-dimensional integer lattice.

4. The Lax pairs of the $A^{(1)}_3$-surface $q$-Painlevé equations

4.1. The PΔEs on the lattice $\mathbb{Z}^5$. To get the Lax pairs of the $q$-Painlevé equations (3.70), (3.74), (3.79) and (3.81), we consider the 5-dimensional integer lattice $\mathbb{Z}^5$ generated by a standard basis for $\mathbb{R}^5$, $\{e_1, \ldots, e_5\}$, which includes the lattice $\mathbb{Z}^4$ defined in §2 as the sublattice. We define the shift operators $T_i, i = 1, \ldots, 5$, on the lattice $\mathbb{Z}^5$ by

$$ T_i : \mathbb{Z}^5 \ni l \mapsto l + e_i \in \mathbb{Z}^5. $$

(4.1)

Let $u(l)$ be the function on the lattice $\mathbb{Z}^5$ satisfying the following PΔEs:

$$
\begin{align*}
\frac{u(l + e_1 + e_2)}{u(l)} &= \frac{\alpha_{ij}u(l + e_1) - \beta_{ij}u(l + e_2)}{\alpha_{ij}u(l + e_1) - \beta_{ij}u(l + e_2)}, \\
\frac{u(l + e_3 + e_4)}{u(l)} &= \frac{\beta_{ij}u(l + e_3) - \gamma_{ij}u(l + e_4)}{\beta_{ij}u(l + e_3) - \gamma_{ij}u(l + e_4)}, \\
\frac{u(l + e_4 + e_5)}{u(l)} &= \frac{\gamma_{ij}u(l + e_4) - \alpha_{ij}u(l + e_5)}{\gamma_{ij}u(l + e_4) - \alpha_{ij}u(l + e_5)}, \\
\frac{u(l + e_1 + e_4)}{u(l)} &= \frac{\alpha_{ij}u(l + e_1) - \mu_{ij}u(l + e_4)}{\alpha_{ij}u(l + e_1) - \mu_{ij}u(l + e_4)}, \\
\frac{u(l + e_1 + e_5)}{u(l)} &= \frac{\alpha_{ij}u(l + e_1) - \mu_{ij}u(l + e_5)}{\alpha_{ij}u(l + e_1) - \mu_{ij}u(l + e_5)}, \\
\frac{u(l + e_2 + e_5)}{u(l)} &= \frac{\beta_{ij}u(l + e_2) - \mu_{ij}u(l + e_5)}{\beta_{ij}u(l + e_2) - \mu_{ij}u(l + e_5)}.
\end{align*}
$$

(4.2)

where $l = \sum_{i=1}^5 t_i e_i \in \mathbb{Z}^5$ and $\{\alpha_{ij}\}_{i, j \in \mathbb{Z}}, \{\beta_{ij}\}_{i, j \in \mathbb{Z}}, \{\gamma_{ij}\}_{i, j \in \mathbb{Z}}, \{\mu_{ij}\}_{i, j \in \mathbb{Z}}$ are complex parameters. We also define the actions of $T_i, i = 1, \ldots, 5$, on the field, generated by $\{u(l)\}_{l \in \mathbb{Z}^5}$, $\{\alpha_{ij}\}_{i, j \in \mathbb{Z}}, \{\beta_{ij}\}_{i, j \in \mathbb{Z}}, \{\gamma_{ij}\}_{i, j \in \mathbb{Z}}, \{\mu_{ij}\}_{i, j \in \mathbb{Z}}$, as the automorphisms by the following actions:

$$
\begin{align*}
\hat{T}_1 : (u(l), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i) \mapsto (u(l + e_1), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i), \\
\hat{T}_2 : (u(l), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i) \mapsto (u(l + e_2), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i), \\
\hat{T}_3 : (u(l), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i) \mapsto (u(l + e_3), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i), \\
\hat{T}_4 : (u(l), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i) \mapsto (u(l + e_4), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i), \\
\hat{T}_5 : (u(l), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i) \mapsto (u(l + e_5), \alpha_{ij}, \beta_{ij}, \gamma_{ij}, K_i, \mu_i).
\end{align*}
$$

(4.3)
It is obvious that $\hat{T}_i$, $i = 1, \ldots, 5$, commute with each other under the actions on the lattice $\mathbb{Z}^5$, $(u(I)|_{\mathbb{R}^n}, (\alpha_i)|_{\mathbb{R}^n}, (\beta_i)|_{\mathbb{R}^n}, (\gamma_i)|_{\mathbb{R}^n}, (K_i)|_{\mathbb{R}^n}$ and $(\mu)|_{\mathbb{R}^n}$.

As is the case with the lattice $\mathbb{Z}^4$ in §2, we consider the sublattice

$$\mathcal{R}^S = \nabla^{(51)} \cup \nabla^{(52)} \subset \mathbb{Z}^5,$$

(4.4)

where

$$\nabla^{(51)} = \left\{ \sum_{i=1}^5 l_i e_i \mid l_i \in \mathbb{Z}, \ l_3 = l_2 - 1 \right\}, \ \nabla^{(52)} = \left\{ \sum_{i=1}^5 l_i e_i \mid l_i \in \mathbb{Z}, \ l_3 = l_2 \right\},$$

(4.5)

with the zigzag-shift operator $\hat{R}_1$ of which action on the lattice $\mathcal{R}^S$ is defined by

$$\hat{R}_1(I) = \begin{cases} \hat{T}_1^{-1}(I) & \text{if } I \in \nabla^{(51)}, \\ \hat{T}_3^{-1}(I) & \text{if } I \in \nabla^{(52)}, \end{cases}, \ \hat{R}_1^{-1}(I) = \begin{cases} \hat{T}_3(I) & \text{if } I \in \nabla^{(51)}, \\ \hat{T}_2(I) & \text{if } I \in \nabla^{(52)}. \end{cases}$$

(4.6)

Note that the lattices $\mathcal{R}, \nabla^{(1)}$ and $\nabla^{(2)}$ defined in §2 are sublattices of the lattices $\mathcal{R}^S$, $\nabla^{(51)}$ and $\nabla^{(52)}$, respectively. We also define the action of $\hat{R}_1$ on the field, generated by $(u(I)|_{\mathbb{R}^n}, (\alpha_i)|_{\mathbb{R}^n}, (\beta_i)|_{\mathbb{R}^n}, (\gamma_i)|_{\mathbb{R}^n}, (K_i)|_{\mathbb{R}^n}$ and $(\mu)|_{\mathbb{R}^n}$, as the automorphism by the following action:

$$\hat{R}_1 : (u(I), \alpha_i, \beta_i, \gamma_i, K_i, \mu) \mapsto (u(\hat{R}_1(I)), \alpha_i, \gamma_i, \beta_i, K_i, \mu),$$

(4.7a)

$$\hat{R}_1^{-1} : (u(I), \alpha_i, \beta_i, \gamma_i, K_i, \mu) \mapsto (u(\hat{R}_1^{-1}(I)), \alpha_i, \beta_i, \gamma_i, K_i, \mu).$$

(4.7b)

Then, the PDEs satisfied by the function $u(I)$ on the lattice $\mathcal{R}^S$ are given by

$$\frac{u_{i+1,j-1,k,l}}{u_{i,j,l,k}} - \frac{u_{i,j,l,k}}{u_{i,j-1,l,k}} \alpha_i = \frac{u_{i+1,j+1,l,k}}{u_{i+1,j,l,k}} - \frac{u_{i,j+1,l,k}}{u_{i,j,l,k}} \alpha_i = -\alpha_i K_i,$$

(4.8a)

$$\frac{u_{i+1,j+1,l,k}}{u_{i+1,j,l,k}} = \frac{u_{i+1,j+1,l,k}}{u_{i+1,j,l,k}} \alpha_i = \frac{u_{i,j+1,l,k}}{u_{i,j,l,k}} - \alpha_i K_i,$$

(4.8b)

$$\frac{u_{i-1,j+1,l,k}}{u_{i,j+1,l,k}} = \frac{u_{i,j+1,l,k}}{u_{i,j,l,k}} - \alpha_i K_i,$$

(4.8c)

$$\frac{u_{i+1,j+l+1,k}}{u_{i,j+l+1,k}} - \frac{u_{i,j+l+1,k}}{u_{i,j+l+1,k}} = \frac{u_{i+1,j+l+1,k}}{u_{i,j+l+1,k}} - \frac{u_{i,j+l+1,k}}{u_{i,j+l+1,k}} \alpha_i = \alpha_i K_i,$$

(4.8d)

$$\frac{u_{i+1,j+l+1,k}}{u_{i,j+l+1,k}} = \frac{u_{i,j+l+1,k}}{u_{i,j+l+1,k}} - \alpha_i K_i,$$

(4.8e)

$$\frac{u_{i,j+l+1,k}}{u_{i,j+l+1,k}} = \frac{u_{i,j+l+1,k}}{u_{i,j+l+1,k}} - \alpha_i K_i,$$

(4.8f)

where

$$u_{i,j,l,k} = \hat{T}_1 \hat{R}_1 \hat{T}_4 \hat{R}_5 u_0, \quad u_0 = u(0).$$

(4.9)

We note that $\hat{R}_1$ commutes with $\hat{T}_i$, $i = 1, 4, 5$, and satisfies $\hat{R}_1^{-1} = \hat{T}_{23}^{-1}$ under the actions on the lattice $\mathcal{R}^S$, $(u(I)|_{\mathbb{R}^n}, (\alpha_i)|_{\mathbb{R}^n}, (\beta_i)|_{\mathbb{R}^n}, (\gamma_i)|_{\mathbb{R}^n}, (K_i)|_{\mathbb{R}^n}$ and $(\mu)|_{\mathbb{R}^n}$.

Henceforth, if new quantity $x$ is added, we extend the field on which $\hat{T}_i$, $i = 1, \ldots, 5$, (or, $\hat{T}_1$, $\hat{R}_1$, $\hat{T}_4$ and $\hat{T}_5$) act as the automorphisms by adding the generator $x$. Moreover, when the field is generated by $x_1, \ldots, x_4$, the mappings also act on an arbitrary function $F = F(x_1, \ldots, x_4)$ as follows:

$$\hat{T}_i(F) = F(\hat{T}_i(x_1), \ldots, \hat{T}_i(x_4)), \quad i = 1, \ldots, 5,$$

(4.10)

$$\hat{R}_1(F) = F(\hat{R}_1(x_1), \ldots, \hat{R}_1(x_4)).$$

(4.11)
4.2. The Lax pairs of the PΔEs on the lattice $\mathbb{Z}^5$. We introduce new variables $F^{(5)}$ and $G^{(5)}$ satisfying

$$ u(e_i) = \frac{F^{(5)}}{G^{(5)}} $$

and assume that $\hat{T}_i, i = 1, \ldots, 5$, commute with each other under the actions on the variables $F^{(j)}$ and $G^{(j)}$, $j = 1, \ldots, 5$. Furthermore, let

$$ \psi^{(5)} = \begin{pmatrix} F^{(5)} \\ G^{(5)} \end{pmatrix}, \quad \psi^{(j)} = \hat{T}_1^{b_1} \hat{T}_2^{b_2} \hat{T}_3^{b_3} \hat{T}_4^{b_4} \hat{T}_5^{b_5} (\psi^{(j)}) = \begin{pmatrix} \hat{T}_1^{b_1} \hat{T}_2^{b_2} \hat{T}_3^{b_3} \hat{T}_4^{b_4} \hat{T}_5^{b_5} (F^{(j)}) \\ \hat{T}_1^{b_1} \hat{T}_2^{b_2} \hat{T}_3^{b_3} \hat{T}_4^{b_4} \hat{T}_5^{b_5} (G^{(j)}) \end{pmatrix}, $$

where $i = 1, \ldots, 5$. Then, in a similar manner as Lemma 2.1, we can prove the following lemma. For conciseness, we have provided the explicit equations for each Lax pair in Lemma B.3.

**Lemma 4.1.** Each PΔE in Equations (4.2) is the compatibility condition of three Lax pairs provided by the following table:

| PΔE   | Lax pair                                      |
|-------|-----------------------------------------------|
| (4.2a) | (B.12g) and (B.12h), (B.12j) and (B.12k), (B.12q) and (B.12r) |
| (4.2b) | (B.12a) and (B.12b), (B.12k) and (B.12i)      |
| (4.2c) | (B.12d) and (B.12e), (B.12j) and (B.12i)      |
| (4.2d) | (B.12d) and (B.12f), (B.12g) and (B.12i)      |
| (4.2e) | (B.12a) and (B.12c), (B.12h) and (B.12i)      |
| (4.2f) | (B.12b) and (B.12c), (B.12e) and (B.12i)      |
| (4.2g) | (B.12d) and (B.12e), (B.12g) and (B.12i)      |
| (4.2h) | (B.12a) and (B.12m), (B.12h) and (B.12i)      |
| (4.2i) | (B.12b) and (B.12m), (B.12e) and (B.12i)      |
| (4.2j) | (B.12c) and (B.12m), (B.12f) and (B.12i)      |

Here, the decoupling factors satisfy the following relations:

$$ \hat{T}_k (\delta^{(j)}_{i_1, \ldots, i_k}) \delta^{(k)}_{i_1, \ldots, i_k} = \hat{T}_j (\delta^{(k)}_{i_1, \ldots, i_k}) \delta^{(k)}_{i_1, \ldots, i_k}, $$

(4.14)

where $i, j, k = 1, \ldots, 5$. Then, in a similar manner as Lemma 2.1, we can prove the following lemma. For conciseness, we have provided the explicit equations for each Lax pair in Lemma B.4.

**Lemma 4.2.** Each PΔE in Equations (4.8) is the compatibility condition of two Lax pairs provided by the following table:

| PΔE   | Lax pair                                      |
|-------|-----------------------------------------------|
| (4.8a) | (B.16c) and (B.16d), (B.16j) and (B.16k)      |
| (4.8b) | (B.16c) and (B.16d), (B.16i) and (B.16l)      |
| (4.8c) | (B.16a) and (B.16b), (B.16k) and (B.16l)      |
| (4.8d) | (B.16a) and (B.16b), (B.16i) and (B.16j)      |
| (4.8e) | (B.16a) and (B.16g), (B.16f) and (B.16i)      |
| (4.8f) | (B.16b) and (B.16g), (B.16d) and (B.16h)      |

Here, the decoupling factors satisfy the following relations:

$$ \hat{R}_1^{(5)} (\delta^{(5)}_{i_1, \ldots, i_5}) \delta^{(5)}_{i_1, \ldots, i_5} = \hat{T}_1 (\delta^{(5)}_{i_1, \ldots, i_5}) \delta^{(5)}_{i_1, \ldots, i_5}, $$

(4.16a)

$$ \hat{T}_1^{(10)} (\delta^{(10)}_{i_1, \ldots, i_5}) \delta^{(10)}_{i_1, \ldots, i_5} = \hat{T}_1 (\delta^{(10)}_{i_1, \ldots, i_5}) \delta^{(10)}_{i_1, \ldots, i_5}, $$

(4.16b)
4.3. Local periodic reduction. Let us consider the \((1, 1, 1)\)-reduction of the P\(\Delta\)Es (4.2) with the relation (2.28) for \([\omega(I)]_{I \in \mathbb{Z}^2}\) and the \((1, 1, 1)\)-periodic condition (2.30) for \([\omega(I)]_{I \in \mathbb{Z}^2}\). Then, Equation (4.2g) are reduced to

\[
\frac{\omega(I + \epsilon_1 + \epsilon_3)}{\omega(I)} = \frac{\alpha_{i+1} \omega(I + \epsilon_1) + i\mu_{i+1} \omega(I + \epsilon_3)}{\alpha_i \omega(I + \epsilon_3) - i\mu_i \omega(I + \epsilon_1)}. \tag{4.17}
\]

Moreover, from the periodic condition (2.30) the following equation also holds:

\[
\frac{\omega(I + \epsilon_1 + \epsilon_3)}{\omega(I)} = \frac{\alpha_{i+1} \omega(I + \epsilon_1) + i\mu_{i+1} \omega(I + \epsilon_3)}{\alpha_{i+1} \omega(I + \epsilon_3) - i\mu_{i+1} \omega(I + \epsilon_1)}. \tag{4.18}
\]

Comparing Equations (4.17) and (4.18), we obtain the condition

\[
\alpha_{i+1} = \alpha_i, \quad I \in \mathbb{Z}. \tag{4.19}
\]

However, since the condition (2.31) holds at the same time, we obtain \(q = 1\) which means that the reduced equations become autonomous difference equations. Therefore, we need to consider another reduction.

The keys of the reduction to avoid automonisation are

(i) considering only the sublattice

\[\mathbb{Z}^4 \cup \hat{T}_3(\mathbb{Z}^4) \subset \mathbb{Z}^4,\]

where \(\hat{T}_3(\mathbb{Z}^4)\) is given by shifting the lattice \(\mathbb{Z}^4\) in the \(\hat{T}_3\)-direction:

\[
\hat{T}_3(\mathbb{Z}^4) = \left\{ \sum_{i=1}^{4} l_i \epsilon_i + \epsilon_3 \mid l_i \in \mathbb{Z} \right\}. \tag{4.21}
\]

(ii) discriminating the function \(u(I)\) on the lattice \(\mathbb{Z}^4\) from that on the lattice \(\hat{T}_3(\mathbb{Z}^4)\); (iii) imposing the periodic condition only for the function \(u(I)\) on the lattice \(\mathbb{Z}^4\).

Let

\[
u(I) = h_{l_1, l_2, l_3, l_4} \omega(I), \quad \nu(I + \epsilon_3) = h_{l_1, l_2, l_3, l_4} \omega^{(5)}(I), \tag{4.22}
\]

where \(I = \sum_{i=1}^{4} l_i \epsilon_i \in \mathbb{Z}^4\) and the gauge factor \(h_{l_1, l_2, l_3, l_4}\) is defined by (2.29). Imposing the \((1, 1, 1)\)-periodic condition (2.30) for \([\nu(I)]_{I \in \mathbb{Z}^2}\) with the condition of the parameters (2.31), from the P\(\Delta\)Es (4.2) we obtain the P\(\Delta\)Es of the function \(\omega(I)\): (2.32), the P\(\Delta\)Es of the function \(\omega^{(5)}(I)\):

\[
\frac{\omega^{(5)}(I + \epsilon_1 + \epsilon_3)}{\omega^{(5)}(I)} = \frac{\omega^{(5)}(I + \epsilon_1) - q^{l_{4-h_1+h_4}} A \frac{\partial}{\partial \epsilon_3} \omega^{(5)}(I + \epsilon_3)}{q^{l_4} A q^{l_4} \omega^{(5)}(I + \epsilon_1) - q^{l_{4-h_1+h_4}} A \frac{\partial}{\partial \epsilon_3} \omega^{(5)}(I + \epsilon_1)}. \tag{4.23a}
\]
\[
\frac{\omega^{(5)}(l + \epsilon_1 + \epsilon_3)}{\omega^{(5)}(l)} = q^{4l} \lambda \omega^{(5)}(l + \epsilon_2) - q^{-4l+12} \frac{\hat{\gamma}}{\hat{\alpha}} \omega^{(5)}(l + \epsilon_3)
\]
(4.23b)

\[
\frac{\omega^{(5)}(l + \epsilon_1 + \epsilon_3)}{\omega^{(5)}(l)} = q^{4l} \lambda \omega^{(5)}(l + \epsilon_2) - q^{-4l+12} \frac{\hat{\gamma}}{\hat{\alpha}} \omega^{(5)}(l + \epsilon_3)
\]
(4.23c)

\[
\frac{\omega^{(5)}(l + \epsilon_1 + \epsilon_3)}{\omega^{(5)}(l)} = q^{4l} \lambda \omega^{(5)}(l + \epsilon_2) - q^{-4l+12} \frac{\hat{\gamma}}{\hat{\alpha}} \omega^{(5)}(l + \epsilon_3)
\]
(4.23d)

\[
\frac{\omega^{(5)}(l + \epsilon_1 + \epsilon_3)}{\omega^{(5)}(l)} = q^{4l} \lambda \omega^{(5)}(l + \epsilon_2) - q^{-4l+12} \frac{\hat{\gamma}}{\hat{\alpha}} \omega^{(5)}(l + \epsilon_3)
\]
(4.23e)

\[
\frac{\omega^{(5)}(l + \epsilon_1 + \epsilon_3)}{\omega^{(5)}(l)} = q^{4l} \lambda \omega^{(5)}(l + \epsilon_2) - q^{-4l+12} \frac{\hat{\gamma}}{\hat{\alpha}} \omega^{(5)}(l + \epsilon_3)
\]
(4.23f)

and the relations between the functions \(\omega(l)\) and \(\omega^{(5)}(l)\):

\[
\frac{\omega^{(5)}(l + \epsilon_1)}{\omega(l)} = \frac{\mu_0 \omega^{(5)}(l) - iq^l \hat{\alpha} \omega(l + \epsilon_1)}{\mu_0 \omega(l + \epsilon_1) + iq^l \hat{\alpha} \omega^{(5)}(l)}
\]
(4.24a)

\[
\frac{\omega^{(5)}(l + \epsilon_2)}{\omega(l)} = \frac{\mu_0 \omega^{(5)}(l) - iq^{l+1} \hat{\beta} \omega(l + \epsilon_2)}{q^l \lambda \left( \mu_0 \omega(l + \epsilon_2) + iq^{l+1} \hat{\beta} \omega^{(5)}(l) \right)}
\]
(4.24b)

\[
\frac{\omega^{(5)}(l + \epsilon_3)}{\omega(l)} = \frac{\mu_0 \omega^{(5)}(l) - iq^{l+1} \hat{\gamma} \omega(l + \epsilon_3)}{\mu_0 \omega(l + \epsilon_3) + iq^{l+1} \hat{\gamma} \omega^{(5)}(l)}
\]
(4.24c)

\[
\frac{\omega^{(5)}(l + \epsilon_4)}{\omega^{(5)}(l)} = \frac{\omega(l + \epsilon_4)}{q^l \lambda \hat{\gamma} \omega^{(5)}(l)}
\]
(4.24d)

where \(l = \sum_{i=1}^{4} l_i \epsilon_i\). We call this reduction the local periodic reduction. The function \(\omega(l)\) defined in this section is exactly the same as the one defined in \(\S 2.3\). Therefore, from Lemma 3.3 and the relations (3.68) we find that letting

\[
\begin{align*}
 f_{0,1}^{l_1-l_2-l_3} &= \hat{T}_1(\omega(l)) \quad f_{1,1}^{l_1-l_2-l_3} = \hat{T}_1(\omega(l)), \quad f_{2,1}^{l_1-l_2-l_3} = \hat{T}_1(\omega(l)), \\
 a_0 &= \frac{\hat{\alpha}}{\hat{\gamma}}, \quad a_1 = \frac{\hat{T}_1(\omega(l))}{\mu_0}, \quad a_2 = \frac{\hat{\gamma}}{\hat{\beta}}.
\end{align*}
\]
(4.25)

where \(l = \sum_{i=1}^{4} l_i \epsilon_i\), we can obtain \(q\)-P\(_{IV}\) (3.70) and \(q\)-P\(_{III}\) (3.74). We note that from definition the following relation holds:

\[
 f_{0,1}^{l_1-l_2-l_3} f_{1,1}^{l_1-l_2-l_3} f_{2,1}^{l_1-l_2-l_3} = q^{2l_i} \lambda^2.
\]
(4.27)

4.4. The Lax pairs of the \(A_{3}^{(1)}\)-surface \(q\)-Painlevé equations. First, we construct the Lax pairs of \(q\)-P\(_{IV}\) (3.70) and \(q\)-P\(_{III}\) (3.74). Setting

\[
\Psi_{l_1,l_2,l_3,0}^{(5)} = \begin{pmatrix} h_{l_1,l_2,l_3}^{(5)}(\omega(l)) & 0 \\ 0 & \phi_{l_1,l_2,l_3} \end{pmatrix},
\]
(4.28)
where $I = \sum_{i=1}^{d} I_i e_i$, from the linear systems (B.12q)–(B.12i) we obtain the following linear systems:

\[
\hat{T}_1 (\phi_{l_h, l_s, l_d, l_4}) = \delta_{l_h, l_s, l_d, l_4, 0}^{(5,1)} \begin{pmatrix}
\mu_0 f_{1,0}^{l_1-l_2-l_h} & 1 & -1 & \frac{\alpha}{f_{2,0}^{l_1-l_2-l_h}} \\
q^{l_1-t_2} & 1 & -1 & \frac{1}{\beta f_{2,0}^{l_1-l_2-l_h}} \\
\end{pmatrix} \phi_{l_h, l_s, l_d, l_4}, \tag{4.29a}
\]

\[
\hat{T}_2 (\phi_{l_h, l_s, l_d, l_4}) = \delta_{l_h, l_s, l_d, l_4, 0}^{(5,2)} \begin{pmatrix}
\mu_0 f_{1,0}^{l_1-l_2-l_3} & 1 & -1 & \frac{\alpha}{f_{2,0}^{l_1-l_2-l_3}} \\
q^{l_1-t_2} & 1 & -1 & \frac{1}{\beta f_{2,0}^{l_1-l_2-l_3}} \\
\end{pmatrix} \phi_{l_h, l_s, l_d, l_4}, \tag{4.29b}
\]

\[
\hat{T}_3 (\phi_{l_h, l_s, l_d, l_4}) = \delta_{l_h, l_s, l_d, l_4, 0}^{(5,3)} \begin{pmatrix}
\mu_0 f_{1,0}^{l_1-l_2-l_3} & 1 & -1 & \frac{\alpha}{f_{2,0}^{l_1-l_2-l_3}} \\
q^{l_1-t_2} & 1 & -1 & \frac{1}{\beta f_{2,0}^{l_1-l_2-l_3}} \\
\end{pmatrix} \phi_{l_h, l_s, l_d, l_4}, \tag{4.29c}
\]

\[
\hat{T}_4 (\phi_{l_h, l_s, l_d, l_4}) = \delta_{l_h, l_s, l_d, l_4, 0}^{(5,4)} \begin{pmatrix}
\mu_0 f_{1,0}^{l_1-l_2-l_3} & 1 & -1 & \frac{\alpha}{f_{2,0}^{l_1-l_2-l_3}} \\
q^{l_1-t_2} & 1 & -1 & \frac{1}{\beta f_{2,0}^{l_1-l_2-l_3}} \\
\end{pmatrix} \phi_{l_h, l_s, l_d, l_4}, \tag{4.29d}
\]

where $I = \sum_{i=1}^{d} I_i e_i$. We define the time evolutions of $q$-$P_{IV}$ (3.70) and $q$-$P_{III}$ (3.74) by

\[
\hat{T}_{IV} = \hat{T}_4, \quad \hat{T}_{III} = \hat{T}_3^{-1} \hat{T}_2^{-1}, \tag{4.30}
\]

respectively. Note that $\hat{T}_{III} \neq \hat{T}_1$ but the relation $\hat{T}_{III} = \hat{T}_1$ holds under the actions on $a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}_{1, m, N, m \in \mathbb{Z}}$, and $f_{2,0}^{\alpha, m}_{0, N, m, N \in \mathbb{Z}}$. Moreover, we define the spectral parameter $\hat{x}$ by

\[
\hat{T}_{SP} = \hat{T}_3^{-1} \hat{T}_2^{-1} \hat{T}_1^{-1}, \quad \hat{x} = \frac{\mu_0}{\alpha}, \tag{4.31}
\]

respectively. The actions of the mappings $\hat{T}_{IV}, \hat{T}_{III}$ and $\hat{T}_{SP}$ on Painlevé parameters $a_0, a_1, a_2, \lambda$ Painlevé variables $f_{0,N}^{\alpha, m}$, $f_{1, N}^{\alpha, m}$, $f_{2, N}^{\alpha, m}$, and $f_{1,0}^{\alpha, m}$, $f_{2,0}^{\alpha, m}$ are given by

\[
\hat{T}_{IV} : (a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}, f_{1, N}^{\alpha, m}, f_{2, N}^{\alpha, m}, \hat{x}) \mapsto (a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}, f_{1, N}^{\alpha, m}, f_{2, N}^{\alpha, m}, \hat{x}),
\]

\[
\hat{T}_{III} : (a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}, f_{1, N}^{\alpha, m}, f_{2, N}^{\alpha, m}, \hat{x}) \mapsto (a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}, f_{1, N}^{\alpha, m}, f_{2, N}^{\alpha, m}, \hat{x}),
\]

\[
\hat{T}_{SP} : (a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}, f_{1, N}^{\alpha, m}, f_{2, N}^{\alpha, m}, \hat{x}) \mapsto (a_0, a_1, a_2, \lambda, f_{0,N}^{\alpha, m}, f_{1, N}^{\alpha, m}, f_{2, N}^{\alpha, m}, \hat{x}),
\]

while their actions on wave function $\phi_{l_h, l_s, l_d, l_4}$ are given by

\[
\hat{T}_{IV} (\phi_{l_h, l_s, l_d, l_4}) = \delta_{l_h, l_s, l_d, l_4}^{IV} \frac{q^{l_1-t_2}(q^{2l_1-1} - 1)f_{2,0}^{l_1-l_2-l_3}}{\lambda (1 + q^{l_1-t_2} a_1 (1 + q^{l_1-t_2} a_2 f_{1,0}^{l_1-l_2-l_3}) f_{2,0}^{l_1-l_2-l_3})} \chi^{-1} \phi_{l_h, l_s, l_d, l_4}, \tag{4.32a}
\]

\[
\hat{T}_{III} (\phi_{l_h, l_s, l_d, l_4}) = \frac{\delta_{l_h, l_s, l_d, l_4}^{III}}{(1 - q^{2l_2} a_0^2 a_2^2 \hat{x}^2)(1 - q^{-2l_2} a_0^2 \hat{x}^2)} \tag{4.32b}
\]
Theorem 4.3. The pairs of Equations (4.32a) and (4.32b) and Equations (4.32b) and (4.32c) are the Lax pairs of $q$-PV (3.70) and $q$-PIII (3.74), respectively. Here, the following relations hold:

\[
\hat{T}_SP(\phi_{t,0,b,c,\lambda}) = \left(1 - q^{-2l_1+2\hat{x}\hat{x}}\right)(1 - q^{-2l_1}a_0a_2^2\hat{x}\hat{x})(1 - q^{-2l_1}a_0^2\hat{x}\hat{x})\left(1 - q^{-2l_1+1}\right)
\]

\[
\hat{T}_SP(\phi_{t,0,b,c,\lambda}) = \left(1 - q^{-2l_1+2\hat{x}\hat{x}}\right)(1 - q^{-2l_1}a_0a_2^2\hat{x}\hat{x})(1 - q^{-2l_1}a_0^2\hat{x}\hat{x})\left(1 - q^{-2l_1+1}\right)
\]

Therefore, in a similar manner as Lemma 2.1 we can prove the following theorem.

Theorem 4.3. The pairs of Equations (4.32a) and (4.32b) and Equations (4.32b) and (4.32c) are the Lax pairs of $q$-PV (3.70) and $q$-PIII (3.74), respectively. Here, the following relations hold:

\[
\hat{T}_SP(\phi_{t,0,b,c,\lambda}) = \left(1 - q^{-2l_1+2\hat{x}\hat{x}}\right)(1 - q^{-2l_1}a_0a_2^2\hat{x}\hat{x})(1 - q^{-2l_1}a_0^2\hat{x}\hat{x})\left(1 - q^{-2l_1+1}\right)
\]

Remark 4.4. Theorem 4.3 gives the part (i) of Theorem 1.1 by the following setting:

\[
q^{4.1} = t, \quad q^{4.1} = a, \quad q^{4.1} = b, \quad q^{4.1} = c, \quad -i q^{-1} \hat{x} = x.
\]

\[
f^{1.0}_{l_0} = f(t), \quad f^{1.0}_{l_0} = g(t), \quad f^{1.0}_{l_0} = h(t), \quad \phi_{t,0,b,c} = \phi(x,t).
\]

\[
\delta^{(IV)}_{t,0,b,c} = 1, \quad \delta^{(SP)}_{t,0,b,c} = \left(1 - q^{-2l_1+2\hat{x}\hat{x}}\right)(1 - q^{-2l_1}a_0a_2^2\hat{x}\hat{x})(1 - q^{-2l_1}a_0^2\hat{x}\hat{x}).
\]
respectively. The actions of the mappings \( \hat{\text{SymIII}} \) and \( \hat{\text{SymSP}} \) on Painlevé parameters \( a_0, a_1, a_2, A, \) and Painlevé variable \( f_{l_1}^1 \) and spectral parameter \( \hat{x} \) are given by

\[
\hat{T}_{\text{SymIII}} = \hat{R}_1, \quad \hat{T}_{\text{SymSP}} = \hat{R}_1^2 \hat{T}_1^{-1}, \quad \hat{x} = \frac{\mu_0}{\hat{\alpha}}.
\]
Remark 4.6. Theorem 4.5. The pair of Equations (4.49a) and (4.49b) is the Lax pair of symmetric q-PIII (3.79). Moreover, when $a_2 = q^{1/2}$, the pair of Equations (4.49a) and (4.49b) is the Lax pair of q-PIII (3.81). Here, the following relation holds:

$$\hat{T}_{\text{SymIII}} (\phi_{l,t}) = \hat{T}_{\text{SymSP}} (\phi_{l,t}) = \hat{T}_{\text{SymIII}} (\phi_{l,t}) = \hat{T}_{\text{SymSP}} (\phi_{l,t}).$$

(4.50)

Therefore, in a similar manner as Lemma 2.1 we can prove the following theorem.

Theorem 4.5. The pair of Equations (4.49a) and (4.49b) is the Lax pair of symmetric q-PIII (3.79). Moreover, when $a_2 = q^{1/2}$, the pair of Equations (4.49a) and (4.49b) is the Lax pair of q-PIII (3.81). Here, the following relation holds:

$$\hat{T}_{\text{SymIII}} (\phi_{l,t}) = \hat{T}_{\text{SymSP}} (\phi_{l,t}) = \hat{T}_{\text{SymIII}} (\phi_{l,t}) = \hat{T}_{\text{SymSP}} (\phi_{l,t}).$$

(4.51)

Remark 4.6. Theorem 4.5 gives the part (iii) of Theorem 1.1 by the following setting:

$$q^{1/2} = p, \quad q^{1/2} = a_1^{1/2}, \quad q^{1/2} = a_0 = t, \quad q^{1/2} = a_1 = \frac{p}{t}, \quad a_2 = p, \quad -i\tilde{x} = x,$$

(4.52)

$$f_{l,t}^f = f(t), \quad \phi_{l,t} = \phi(x,t), \quad \delta_{l,t}^{(\text{SymIII})} = 1 - q^{1/2} a_0^2 \tilde{x}^2,$$

(4.53)

$$\delta_{l,t}^{(\text{SymSP})} = (1 - q^{1/2} a_0^2 \tilde{x}^2)(1 - q^{1/2} a_0^2 \tilde{x}^2)(1 - q^{1/2} a_0^2 \tilde{x}^2).$$

(4.54)

5. Concluding remarks

In this paper, we provided a comprehensive method for construction of Lax pairs of discrete Painlevé equations and new reduction called local periodic reduction. As an example, we constructed the Lax pairs of the $A_5^{(1)}$ surface-q-Painlevé equations from the 5-dimensional integer lattice. An interesting future direction is to extend our method to other types of discrete Painlevé equations classified by Sakai [37].

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which are called ABS classification, respectively. The face-equations of the 4-cube are given by the following:

\[ \hat{Q}(v_1, v_2, v_3, v_4; \alpha, \beta) = v_1 v_3 + v_2 v_4 - \frac{\alpha}{\beta} (v_1 v_2 + v_3 v_4), \]
\[ A(v_1, v_2, v_3, v_4; \alpha, \beta) = v_1 v_3 + v_2 v_4 + \alpha \beta v_1 v_2, \]

which are called \( H_{3,0} \) (or \( H_{3,0}^{(0)} \)) and \( D_4 \) and categorized as \( H^4 \) type and \( H^6 \) type in the ABS classification, respectively [1, 2, 7]. Moreover, the PDE given by tessellation of the polynomial \( Q \) is referred to as the lattice modified KdV equation [28]. The twenty-four face-equations of the 4-cube are given by the following:

\[ Q(u_0, \hat{T}_1(u_0), \hat{T}_2(u_0), \hat{T}_{12}(u_0); \alpha_0, \beta_0) = 0, \]
\[ Q(u_0, \hat{T}_1(u_0), \hat{T}_3(u_0), \hat{T}_{32}(u_0); \beta_0, \gamma_0) = 0, \]
\[ Q(u_0, \hat{T}_1(u_0), \hat{T}_4(u_0), \hat{T}_{13}(u_0); \gamma_0, \alpha_0) = 0, \]
\[ Q(\hat{T}_3(u_0), \hat{T}_{31}(u_0), \hat{T}_{32}(u_0), \hat{T}_{312}(u_0); \alpha_0, \beta_0) = 0, \]
\[ Q(\hat{T}_1(u_0), \hat{T}_{12}(u_0), \hat{T}_{13}(u_0), \hat{T}_{123}(u_0); \beta_0, \gamma_0) = 0, \]
\[ Q(\hat{T}_2(u_0), \hat{T}_{23}(u_0), \hat{T}_{21}(u_0), \hat{T}_{213}(u_0); \gamma_0, \alpha_0) = 0, \]
\[ Q(\hat{T}_4(u_0), \hat{T}_{41}(u_0), \hat{T}_{42}(u_0), \hat{T}_{412}(u_0); \alpha_0, \beta_0) = 0, \]
\[ Q(\hat{T}_4(u_0), \hat{T}_{42}(u_0), \hat{T}_{43}(u_0), \hat{T}_{423}(u_0); \beta_0, \gamma_0) = 0, \]
where \( u_0 = u(0) \).

Let \( C_{ijk} \) be a cube around the origin \( 0 \in \mathbb{Z}^4 \) given by the eight vertices

\[
\{0, \hat{T}_i(0), \hat{T}_j(0), \hat{T}_k(0), \hat{T}_{ij}(0), \hat{T}_{jk}(0), \hat{T}_{ik}(0), \hat{T}_{ijk}(0)\},
\]

and \( \hat{T}_i(C) \) for a cube \( C \) means a shift in the \( \hat{T}_i \)-direction. We consider the 4-cube by dividing it into the eight cubes. The following table describes the relations between the eight cubes which the 4-cube contains and their face-equations:

| Cube             | Face-equation |
|------------------|---------------|
| \( C_{312} \)    | (A.4a)–(A.4f) |
| \( \hat{T}_4(C_{312}) \) | (A.4g)–(A.4i) |
| \( C_{214} \)    | (A.4a), (A.4g), (A.4m), (A.4n), (A.4q), (A.4r) |
| \( \hat{T}_3(C_{214}) \) | (A.4d), (A.4j), (A.4o), (A.4p), (A.4s), (A.4i) |
| \( C_{324} \)    | (A.4b), (A.4h), (A.4q), (A.4s), (A.4u), (A.4w) |
| \( \hat{T}_1(C_{324}) \) | (A.4e), (A.4k), (A.4r), (A.4t), (A.4v), (A.4x) |
| \( C_{134} \)    | (A.4c), (A.4i), (A.4m), (A.4o), (A.4u), (A.4v) |
| \( \hat{T}_2(C_{134}) \) | (A.4f), (A.4l), (A.4n), (A.4p), (A.4w), (A.4x) |

**Lemma A.1.** From the 4-cube, the following twelve equations are derived:

\[
\hat{T}_2(\Psi^{(1)}) = \delta^{(1,2)} \begin{pmatrix} \frac{\alpha_0}{\beta_0} & -\hat{T}_3(u_0) \\ \frac{1}{u_0} & \frac{\alpha_0}{\beta_0} \hat{T}_2(u_0) \end{pmatrix} \Psi^{(1)},
\]

(A.6a)
\[ T_3(\psi^1) = \delta^{(1,3)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_3(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_3(u_0) \end{pmatrix} \psi^1, \] (A.6b)

\[ T_4(\psi^1) = \delta^{(1,4)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_4(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_4(u_0) \end{pmatrix} \psi^1, \] (A.6c)

\[ T_1(\psi^{22}) = \delta^{(2,1)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_1(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_1(u_0) \end{pmatrix} \psi^{22}, \] (A.6d)

\[ T_3(\psi^{22}) = \delta^{(2,3)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_3(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_3(u_0) \end{pmatrix} \psi^{22}, \] (A.6e)

\[ T_4(\psi^{22}) = \delta^{(2,4)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_4(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_4(u_0) \end{pmatrix} \psi^{22}, \] (A.6f)

\[ T_1(\psi^{31}) = \delta^{(3,1)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_1(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_1(u_0) \end{pmatrix} \psi^{31}, \] (A.6g)

\[ T_2(\psi^{31}) = \delta^{(3,2)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_2(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_2(u_0) \end{pmatrix} \psi^{31}, \] (A.6h)

\[ T_4(\psi^{31}) = \delta^{(3,4)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_4(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_4(u_0) \end{pmatrix} \psi^{31}, \] (A.6i)

\[ T_1(\psi^{41}) = \delta^{(4,1)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_1(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_1(u_0) \end{pmatrix} \psi^{41}, \] (A.6j)

\[ T_2(\psi^{41}) = \delta^{(4,2)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_2(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_2(u_0) \end{pmatrix} \psi^{41}, \] (A.6k)

\[ T_3(\psi^{41}) = \delta^{(4,3)} \begin{pmatrix} \frac{\alpha_0}{\gamma_0} + \frac{\beta_0}{u_0} & -\hat{T}_3(u_0) \\ 0 & 1 - \frac{\gamma_0}{\alpha_0} \hat{T}_3(u_0) \end{pmatrix} \psi^{41}, \] (A.6l)

where \( \delta^{(ij)} \) are arbitrary decoupling factors and \( \hat{T}_i, i = 1, \ldots, 4 \), commute with each other under the actions on the decoupling factors. Note that the vectors \( \psi^{(i)}, i = 1, \ldots, 4 \), are defined in Equation (2.18).

**Proof.** First, we consider the cube \( C_{312} \). The cube \( C_{312} \) has the six face-equations (A.4a)–(A.4f). Since Equations (A.4d)–(A.4f) are respectively given by (A.4a)|\( l_{i-1} \leq l_{i+1} \), (A.4b)|\( l_{i-1} \leq l_{i+1} \) and (A.4c)|\( l_{i-1} \leq l_{i+1} \), we here only use the essential equations (A.4a)–(A.4c). Moreover, we consider the pairs of these equations: Equations (A.4a) and (A.4b), Equations (A.4b) and (A.4c) and Equations (A.4c) and (A.4a).
Let us derive Equations (A.6d) and (A.6e) from Equations (A.4a) and (A.4b) following the method given in [5, 29, 42]. The key of the derivations is to regard the \( \hat{T}_2 \)-direction as the “virtual” direction and \( u(e_2) \) as new quantity. Then, Equations (A.4a) and (A.4b) can be regarded as the discrete Riccati equations of the quantity \( u(e_2) \) with the \( \hat{T}_1 \)-direction and the \( \hat{T}_3 \)-direction, respectively. By substituting (2.17) into the quantity \( u(e_2) \), Equations (A.4a) and (A.4b) can be rewritten as

\[
\frac{\hat{T}_1(F^{(2)})}{\hat{T}_1(G^{(2)})} = \frac{\beta_0}{u_0} F^{(2)} - \hat{T}_1(u_0)G^{(2)}
\]

\[
\frac{\hat{T}_3(F^{(2)})}{\hat{T}_3(G^{(2)})} = \frac{1}{u_0} F^{(2)} - \frac{\beta_0}{\gamma_0} \hat{T}_3(u_0)G^{(2)}
\]

respectively, which may be divided into the following linear equations for \( F^{(2)} \) and \( G^{(2)} \):

\[
\hat{T}_1(F^{(2)}) = \delta^{(2,1)} \left( \frac{\beta_0}{u_0} F^{(2)} - \hat{T}_1(u_0)G^{(2)} \right),
\]

\[
\hat{T}_1(G^{(2)}) = \delta^{(2,1)} \left( \frac{1}{u_0} F^{(2)} - \frac{\beta_0}{\alpha_0} \hat{T}_1(u_0)G^{(2)} \right),
\]

\[
\hat{T}_3(F^{(2)}) = \delta^{(2,3)} \left( \frac{\beta_0}{\gamma_0} F^{(2)} - \hat{T}_3(u_0)G^{(2)} \right),
\]

\[
\hat{T}_3(G^{(2)}) = \delta^{(2,3)} \left( \frac{1}{u_0} F^{(2)} - \frac{\beta_0}{\gamma_0} \hat{T}_3(u_0)G^{(2)} \right),
\]

where \( \delta^{(2,1)} \) and \( \delta^{(2,3)} \) are arbitrary decoupling factors. Therefore, from Equations (A.8) we obtain Equations (A.6d) and (A.6e). In a similar manner, from Equations (A.4b) and (A.4c) and Equations (A.4c) and (A.4a), we obtain Equations (A.6g) and (A.6h) and Equations (A.6a) and (A.6b), respectively. Therefore, we totally obtain the six equations: (A.6a), (A.6b), (A.6d), (A.6e), (A.6g), (A.6h), from the cube \( C_{312} \). Since the cube \( \hat{T}_3(C_{312}) \) can be obtained by shifting the cube \( C_{312} \) in the \( \hat{T}_1 \)-direction, from the cube \( \hat{T}_3(C_{312}) \) we can also obtain the same six equations.

Next, we consider the cube \( C_{214} \). The essential face-equations are Equations (A.4a), (A.4m) and (A.4q). In a similar manner as the case of the cube \( C_{312} \), from Equations (A.4a) and (A.4m), Equations (A.4m) and (A.4q) and Equations (A.4q) and (A.4a), we obtain Equations (A.6a) and (A.6c), Equations (A.6j) and (A.6k) and Equations (A.6d) and (A.6l), respectively. From the cube \( \hat{T}_3(C_{214}) \) we can obtain the same equations as those from the cube \( C_{214} \).

Finally, we consider the remaining cubes \( C_{324}, C_{134}, \hat{T}_1(C_{324}) \) and \( \hat{T}_2(C_{134}) \). The face-equations on the cubes \( C_{324} \) and \( C_{134} \) can be obtained from those on the cube \( C_{214} \) by the replacements \( [T_1, \alpha_0] \rightarrow [\hat{T}_1, \gamma_0] \) and \( [\hat{T}_2, \beta_0] \rightarrow [\hat{T}_3, \gamma_0] \), respectively. Therefore, from the cubes \( C_{324} \) and \( C_{134} \) we can obtain the remaining equations (A.6i) and (A.6l). From the cubes \( \hat{T}_1(C_{324}) \) and \( \hat{T}_2(C_{134}) \), we can obtain the same equations as those from the cubes \( C_{324} \) and \( C_{134} \), respectively.

Since \( \hat{T}_i, i = 1, \ldots, 4, \) commute with each other under the actions on \( \Psi^{(i)}, i = 1, \ldots, 4, u_0, \alpha_0, \beta_0, \gamma_0 \) and \( K_0 \), they also commute with each other under the actions on the factors \( \delta^{(i,j)} \). Therefore we have completed the proof.

\[ \square \]

**Appendix B. Explicit expressions for Lax pairs**

In this appendix, we provide the explicit forms of the Lax pairs used in Lemmas 2.1, 2.2, 2.5, 2.6, 4.1 and 4.2.

**B.1. For Lemma 2.1.** We here derive the linear systems used in Lemma 2.1. From Lemma A.1 and definitions (2.19), we obtain the following lemma.
Lemma B.1. The following twelve linear systems hold:

\[
\begin{align*}
\hat{T}_2(\Psi^{(1)}_{i,j,k,l}) &= \mathbf{d}^{(1,2)}_{i,j,k,l} \left( \frac{\alpha_i}{\beta_i} \frac{-\hat{T}_3(u(l))}{u(l)} - \frac{\alpha_j}{\beta_j} \frac{\hat{T}_2(u(l))}{u(l)} \right) \Psi^{(1)}_{i,j,k,l}, \\
\hat{T}_3(\Psi^{(1)}_{i,j,k,l}) &= \mathbf{d}^{(1,3)}_{i,j,k,l} \left( \frac{\alpha_i}{\gamma_i} \frac{-\hat{T}_3(u(l))}{u(l)} - \frac{\alpha_j}{\gamma_j} \frac{\hat{T}_3(u(l))}{u(l)} \right) \Psi^{(1)}_{i,j,k,l}, \\
\hat{T}_4(\Psi^{(1)}_{i,j,k,l}) &= \mathbf{d}^{(1,4)}_{i,j,k,l} \left( \frac{-\alpha_i K_i}{u(l)} \hat{T}_4(u(l)) - 0 \right) \Psi^{(1)}_{i,j,k,l}, \\
\hat{T}_1(\Psi^{(2)}_{i,j,k,l}) &= \mathbf{d}^{(2,1)}_{i,j,k,l} \left( \frac{\beta_i}{\alpha_i} \frac{-\hat{T}_1(u(l))}{u(l)} - \frac{\beta_j}{\gamma_j} \frac{\hat{T}_1(u(l))}{u(l)} \right) \Psi^{(2)}_{i,j,k,l}, \\
\hat{T}_3(\Psi^{(2)}_{i,j,k,l}) &= \mathbf{d}^{(2,3)}_{i,j,k,l} \left( \frac{\beta_i}{\gamma_i} \frac{-\hat{T}_3(u(l))}{u(l)} - \frac{\beta_j}{\gamma_j} \frac{\hat{T}_3(u(l))}{u(l)} \right) \Psi^{(2)}_{i,j,k,l}, \\
\hat{T}_4(\Psi^{(2)}_{i,j,k,l}) &= \mathbf{d}^{(2,4)}_{i,j,k,l} \left( \frac{-\beta_i K_i}{u(l)} \hat{T}_4(u(l)) - 0 \right) \Psi^{(2)}_{i,j,k,l}, \\
\hat{T}_1(\Psi^{(3)}_{i,j,k,l}) &= \mathbf{d}^{(3,1)}_{i,j,k,l} \left( \frac{\gamma_i}{\alpha_i} \frac{-\hat{T}_1(u(l))}{u(l)} - \frac{\gamma_j}{\alpha_j} \frac{\hat{T}_1(u(l))}{u(l)} \right) \Psi^{(3)}_{i,j,k,l}, \\
\hat{T}_2(\Psi^{(3)}_{i,j,k,l}) &= \mathbf{d}^{(3,2)}_{i,j,k,l} \left( \frac{\gamma_i}{\beta_i} \frac{-\hat{T}_2(u(l))}{u(l)} - \frac{\gamma_j}{\beta_j} \frac{\hat{T}_2(u(l))}{u(l)} \right) \Psi^{(3)}_{i,j,k,l}, \\
\hat{T}_4(\Psi^{(3)}_{i,j,k,l}) &= \mathbf{d}^{(3,4)}_{i,j,k,l} \left( \frac{-\gamma_i K_i}{u(l)} \hat{T}_4(u(l)) - 0 \right) \Psi^{(3)}_{i,j,k,l}, \\
\hat{T}_1(\Psi^{(4)}_{i,j,k,l}) &= \mathbf{d}^{(4,1)}_{i,j,k,l} \left( \frac{1}{\alpha_i} \frac{\alpha_j K_j}{u(l)} \hat{T}_1(u(l)) \right) \Psi^{(4)}_{i,j,k,l}, \\
\hat{T}_2(\Psi^{(4)}_{i,j,k,l}) &= \mathbf{d}^{(4,2)}_{i,j,k,l} \left( \frac{1}{\beta_i} \frac{\beta_j K_j}{u(l)} \hat{T}_2(u(l)) \right) \Psi^{(4)}_{i,j,k,l}, \\
\hat{T}_3(\Psi^{(4)}_{i,j,k,l}) &= \mathbf{d}^{(4,3)}_{i,j,k,l} \left( \frac{1}{\gamma_i} \frac{\gamma_j K_j}{u(l)} \hat{T}_3(u(l)) \right) \Psi^{(4)}_{i,j,k,l},
\end{align*}
\]

where \( l = \sum_{i=1}^{m} t_i \epsilon_i \).
B.2. **For Lemma 2.2.** We derive the linear systems used in Lemma 2.2 by using the Hopf-like linearization (2.17), (2.18), (2.25) and (2.26).

**Lemma B.2.** The following six linear systems hold:

\[
\begin{align*}
\hat{R}_1^{-1}(\Psi_{h,l,0}^{(1)}) &= \delta_{h,l,0}^{(1)} \begin{pmatrix} \frac{\alpha_i}{l_i} & -u_{l_i,l-1,0} \\ 1 & -\frac{\alpha_i}{l_i} u_{l_i,l-1,0} \end{pmatrix} \Psi_{h,l,0}^{(1)}, \\
\hat{T}_4(\Psi_{h,l,0}^{(1)}) &= \delta_{h,l,0}^{(2)} \begin{pmatrix} \frac{-\alpha_i K_i}{l_i} & -u_{l_i,l+1,0} \\ 1 & 0 \end{pmatrix} \Psi_{h,l,0}^{(1)}, \\
\hat{T}_1(\Psi_{h,l,0}^{(2)}) &= \delta_{h,l,0}^{(3)} \begin{pmatrix} \frac{\hat{R}_1^{-1}(\beta_0)}{l_i} & -u_{l_i,l+1,0} \\ 1 & -\frac{\hat{R}_1^{-1}(\beta_0)}{l_i} u_{l_i,l+1,0} \end{pmatrix} \Psi_{h,l,0}^{(2)}, \\
\hat{T}_4(\Psi_{h,l,0}^{(2)}) &= \delta_{h,l,0}^{(4)} \begin{pmatrix} \frac{\hat{R}_1^{-1}(\beta_0) K_i}{l_i} & -u_{l_i,l+1,0} \\ 1 & -\frac{\hat{R}_1^{-1}(\beta_0) K_i}{l_i} u_{l_i,l+1,0} \end{pmatrix} \Psi_{h,l,0}^{(2)}, \\
\hat{T}_1(\Psi_{h,l,0}^{(4)}) &= \delta_{h,l,0}^{(5)} \begin{pmatrix} \frac{\alpha_i K_i u_{l_i,l+1,0}}{l_i} & -u_{l_i,l+1,0} \\ 1 & 0 \end{pmatrix} \Psi_{h,l,0}^{(4)}, \\
\hat{R}_1^{-1}(\Psi_{h,l,0}^{(4)}) &= \delta_{h,l,0}^{(6)} \begin{pmatrix} \frac{\hat{R}_1^{-1}(\beta_0) K_i u_{l_i,l-1,0}}{l_i} & -u_{l_i,l-1,0} \\ 0 & \frac{\hat{R}_1^{-1}(\beta_0) K_i u_{l_i,l-1,0}}{l_i} u_{l_i,l-1,0} \end{pmatrix} \Psi_{h,l,0}^{(4)},
\end{align*}
\]

where \( \delta_{h,l,0}^{(i)} = \hat{T}_1^h \hat{R}_1^{-1} \hat{T}_4^b (\delta^{(i)}), \) \( i = 1, \ldots, 6, \) are arbitrary decoupling factors. Note that \( \hat{T}_1, \hat{R}_1 \) and \( \hat{T}_4 \) commute with each other under the actions on \( \{\delta^{(i)}\}_{i=1}^{6}. \)

**Proof.** Let us consider the cube around the origin \( \mathbf{0} \in \mathbb{Z}^4 \) defined by

\[
\{ \mathbf{0}, \hat{T}_1(\mathbf{0}), \hat{R}_1^{-1}(\mathbf{0}), \hat{T}_4(\mathbf{0}), \hat{T}_1\hat{R}_1^{-1}(\mathbf{0}), \hat{T}_4\hat{R}_1^{-1}(\mathbf{0}), \hat{T}_1\hat{R}_1^{-1}\hat{T}_4(\mathbf{0}), \hat{T}_1\hat{R}_1^{-1}\hat{T}_4\hat{R}_1^{-1}(\mathbf{0}) \}. \quad (B.3)
\]

The six face-equations of this cube are given by

\[
\begin{align*}
Q \left( u_0, \hat{T}_1(\mathbf{0}), \hat{R}_1^{-1}(\mathbf{0}), \hat{T}_1\hat{R}_1^{-1}(\mathbf{0}); \alpha_0, \beta_0 \right) &= 0, \\
A \left( u_0, \hat{T}_1(\mathbf{0}), \hat{T}_4(\mathbf{0}); \alpha_0, K_0 \right) &= 0, \\
A \left( u_0, \hat{R}_1^{-1}(\mathbf{0}), \hat{T}_4(\mathbf{0}); \beta_0, K_0 \right) &= 0, \\
Q \left( \hat{T}_4(\mathbf{0}), \hat{T}_4(\mathbf{0}); \hat{T}_4(\mathbf{0}); \beta_0, K_0 \right) &= 0, \\
Q \left( \hat{R}_1^{-1}(\mathbf{0}), \hat{R}_1^{-1}(\mathbf{0}); \hat{R}_1^{-1}(\mathbf{0}); \alpha_0, \beta_0 \right) &= 0, \\
A \left( \hat{T}_1(\mathbf{0}), \hat{T}_1\hat{R}_1^{-1}(\mathbf{0}); \hat{T}_4(\mathbf{0}); \beta_0, K_0 \right) &= 0.
\end{align*}
\]

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where the polynomials $Q$ and $A$ are given by (A.2) and (A.3), respectively. In a similar manner as the proof of Lemma A.1, we can obtain the following equations:

\[
\hat{R}_1^{-1}(\Psi^{(i)}) = \delta^{(1)} \begin{pmatrix} \frac{\alpha_0}{u_0} & -\hat{R}_1^{-1}(u_0) \\ \frac{\beta_0}{u_0} & \alpha_0 \hat{R}_1^{-1}(u_0) \end{pmatrix} \Psi^{(i)}, \tag{B.5a}
\]

\[
\hat{T}_1(\Psi^{(i)}) = \delta^{(2)} \begin{pmatrix} \frac{\beta_0}{u_0} & -\hat{T}_1(u_0) \\ 1 & 0 \end{pmatrix} \Psi^{(i)}, \tag{B.5b}
\]

\[
\hat{T}_1(\Psi^{(0)}) = \delta^{(3)} \begin{pmatrix} \frac{\beta_0}{u_0} & -\hat{T}_1(u_0) \\ 1 & 0 \end{pmatrix} \Psi^{(0)}, \tag{B.5c}
\]

\[
\hat{T}_4(\Psi^{(0)}) = \delta^{(4)} \begin{pmatrix} \frac{1}{u_0} & -\hat{T}_4(u_0) \\ 0 & 0 \end{pmatrix} \Psi^{(0)}, \tag{B.5d}
\]

\[
\hat{T}_1(\Psi^{(4)}) = \delta^{(5)} \begin{pmatrix} \frac{1}{u_0} & -\hat{T}_1(u_0) \\ 0 & 0 \end{pmatrix} \Psi^{(4)}, \tag{B.5e}
\]

\[
\hat{R}_1^{-1}(\Psi^{(4)}) = \delta^{(6)} \begin{pmatrix} \frac{1}{u_0} & -\hat{R}_1^{-1}(u_0) \\ 0 & 0 \end{pmatrix} \Psi^{(4)}, \tag{B.5f}
\]

where $\delta^{(i)}, i = 1, \ldots, 6$, are arbitrary decoupling factors. Since $\hat{T}_1$, $\hat{R}_1$, and $\hat{T}_4$ commute with each other under the actions on $\Psi^{(1)}$, $\Psi^{(0)}$, $\Psi^{(4)}$, $u_0$, $\alpha_0$, $\beta_0$, $\gamma_0$ and $K_0$, it also holds under the actions on the factors $\delta^{(i)}, i = 1, \ldots, 6$. Therefore we have completed the proof. \(\square\)

B.3. For Lemma 2.5. In this section, we obtain the linear systems used in Lemma 2.5. By applying the $(1,1,1)$-periodic condition (2.30) with the condition of the parameters (2.31) and setting (2.28) and

\[
\Psi^{(i)}_{l,h_{1,2,3},l_{1,2}} = \begin{pmatrix} \hat{T}_1(h_{1,2,3,l_{1,2}}) & 0 \\ 0 & 1 \end{pmatrix} \Phi^{(i)}_{l,h_{1,2,3},l_{1,2}}, \tag{B.6}
\]

where $i = 1, \ldots, 4$, the linear systems (B.1) can be reduced to the following linear systems:

\[
\hat{T}_2(\Psi^{(1)}_{l,h_{1,2,3},l_{1,2}}) = i\delta^{(1,2)}_{l,h_{1,2,3},l_{1,2}} \begin{pmatrix} \hat{T}_2(\omega(l)) \\ \frac{1}{\omega(l)} \hat{T}_2(\omega(l)) \end{pmatrix} \Phi^{(1)}_{l,h_{1,2,3},l_{1,2}}, \tag{B.7a}
\]

\[
\hat{T}_3(\Psi^{(1)}_{l,h_{1,2,3},l_{1,2}}) = i\delta^{(1,3)}_{l,h_{1,2,3},l_{1,2}} \begin{pmatrix} \hat{T}_3(\omega(l)) \\ \frac{1}{\omega(l)} \hat{T}_3(\omega(l)) \end{pmatrix} \Phi^{(1)}_{l,h_{1,2,3},l_{1,2}}, \tag{B.7b}
\]

\[
\hat{T}_4(\Psi^{(1)}_{l,h_{1,2,3},l_{1,2}}) = i\delta^{(1,4)}_{l,h_{1,2,3},l_{1,2}} \begin{pmatrix} \hat{T}_4(\omega(l)) \\ \frac{1}{\omega(l)} \hat{T}_4(\omega(l)) \end{pmatrix} \Phi^{(1)}_{l,h_{1,2,3},l_{1,2}}, \tag{B.7c}
\]
\[
\begin{align*}
\hat{T}_1(\Phi^{(2)}_{h,j,j_l,j_{\epsilon_l}}) &= i q^l \lambda \delta^{(2,1)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
-q^{l_3+1-l_4} \frac{\hat{\beta}}{\lambda^2} & \frac{1}{q^{l_2+l_3}} \hat{T}_1(\omega(I)) \\
1 & -q^{l_1+l_2-l_4} \frac{\hat{\beta}}{\lambda^2} \hat{T}_4(\omega(I)) 
\end{pmatrix} \Phi^{(2)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7d)} \\
\hat{T}_3(\Phi^{(2)}_{h,j,j_l,j_{\epsilon_l}}) &= i q^l \lambda \delta^{(2,3)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
-q^{l_1-l_2-l_4} \frac{\hat{\beta}}{\lambda^2} & \frac{1}{q^{l_2+l_3}} \hat{T}_3(\omega(I)) \\
1 & -q^{l_2-l_1-l_4} \frac{\hat{\beta}}{\lambda^2} \hat{T}_4(\omega(I)) 
\end{pmatrix} \Phi^{(2)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7e)} \\
\hat{T}_4(\Phi^{(2)}_{h,j,j_l,j_{\epsilon_l}}) &= i q^l \lambda \delta^{(2,4)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
\frac{q^{2l+1} \lambda^2 - 1}{q^{2l_2+l_3} \lambda^2} & \frac{1}{q^{2l_2+l_3} \lambda^2} \hat{T}_4(\omega(I)) \\
1 & 0 
\end{pmatrix} \Phi^{(2)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7f)} \\
\hat{T}_1(\Phi^{(3)}_{h,j,j_l,j_{\epsilon_l}}) &= i \delta^{(3,1)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
-q^{l_1+1-l_4} \frac{\hat{\gamma}}{\alpha} & \hat{T}_1(\omega(I)) \\
1 & -q^{l_1+1-l_4} \frac{\hat{\gamma}}{\alpha} \hat{T}_2(\omega(I)) 
\end{pmatrix} \Phi^{(3)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7g)} \\
\hat{T}_2(\Phi^{(3)}_{h,j,j_l,j_{\epsilon_l}}) &= i \delta^{(3,2)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
-q^{l_2+1-l_4} \frac{\hat{\gamma}}{\alpha} & \hat{T}_2(\omega(I)) \\
1 & -q^{l_2+1-l_4} \frac{\hat{\gamma}}{\alpha} \hat{T}_3(\omega(I)) 
\end{pmatrix} \Phi^{(3)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7h)} \\
\hat{T}_4(\Phi^{(3)}_{h,j,j_l,j_{\epsilon_l}}) &= i \delta^{(3,4)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
q^{l_3+1} \frac{\hat{\gamma}}{\lambda \beta} (q^{2l_4+1} \lambda^2 - 1) & \hat{T}_4(\omega(I)) \\
1 & 0 
\end{pmatrix} \Phi^{(3)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7i)} \\
\hat{T}_1(\Phi^{(4)}_{h,j,j_l,j_{\epsilon_l}}) &= -i \delta^{(4,1)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
1 & q^{l_1-l_2} \frac{\hat{\gamma}}{\lambda \beta} (q^{2l_3+1} \lambda^2 - 1) \hat{T}_1(\omega(I)) \\
0 & \hat{T}_1(\omega(I)) \frac{\hat{T}_1(\omega(I))}{\omega(I)} 
\end{pmatrix} \Phi^{(4)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7j)} \\
\hat{T}_2(\Phi^{(4)}_{h,j,j_l,j_{\epsilon_l}}) &= \frac{i}{q^{l_3+l_4} \lambda} \delta^{(4,2)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
0 & q^{l_3+1} \lambda \frac{\hat{T}_2(\omega(I))}{\omega(I)} \frac{\hat{T}_2(\omega(I))}{\omega(I)} \\
1 & \frac{q^{2l_3+1} \lambda^2 - 1}{q^{2l_3+1} \lambda^2} \hat{T}_2(\omega(I)) \frac{\hat{T}_2(\omega(I))}{\omega(I)} 
\end{pmatrix} \Phi^{(4)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7k)} \\
\hat{T}_3(\Phi^{(4)}_{h,j,j_l,j_{\epsilon_l}}) &= -i \delta^{(4,3)}_{h,j,j_l,j_{\epsilon_l}} \begin{pmatrix}
1 & q^{l_1-l_2} \frac{\hat{\gamma}}{\lambda \beta} (q^{2l_3+1} \lambda^2 - 1) \hat{T}_3(\omega(I)) \\
0 & \hat{T}_3(\omega(I)) \frac{\hat{T}_3(\omega(I))}{\omega(I)} 
\end{pmatrix} \Phi^{(4)}_{h,j,j_l,j_{\epsilon_l}}, \quad \text{(B.7l)}
\end{align*}
\]

where \( I = \sum_{\epsilon_l=1}^4 I_{\epsilon_l} \).

B.4. **For Lemma 2.6.** In this section, we obtain the linear systems used in Lemma 2.6. By applying the (1, 1, 1)-periodic condition (2.30) with the condition of the parameters (2.31) and setting (2.28), (2.51) and

\[
\Psi^{(1)}_{0,l_4} = \begin{pmatrix}
\hat{R}(h_{0,0,0}) & 0 \\
0 & 1
\end{pmatrix} \Phi^{(1)}_{l_4},
\quad \text{(B.8)}
\]

\[
\Psi^{(8)}_{0,l_4} = \begin{pmatrix}
\hat{R}^{-1}(h_{0,0,0}) & 0 \\
0 & 1
\end{pmatrix} \Phi^{(8)}_{l_4},
\quad \text{(B.9)}
\]

\[
\Psi^{(4)}_{0,l_4} = \begin{pmatrix}
\hat{R}(h_{0,0,0}) & 0 \\
0 & 1
\end{pmatrix} \Phi^{(4)}_{l_4},
\quad \text{(B.10)}
\]
the linear systems (B.2) can be reduced to the following linear systems:

\[
\begin{align*}
\hat{R}_1^{-1}(\Phi^{(1)}_{1k}) &= i\delta^{(1)}_{0,1k} \begin{pmatrix}
-\hat{R}_1^{(4) 4} \left( \frac{\hat{\alpha}}{\hat{\beta} k} \right) & 0 \\
1 & -\hat{R}_1^{(4) 4} \left( \frac{\hat{\alpha}}{\hat{\beta} k} \right)
\end{pmatrix} \Phi^{(1)}_{1k}, \\
\hat{T}_4(\Phi^{(1)}_{1k}) &= i\delta^{(2)}_{0,1k} \begin{pmatrix}
\hat{R}_1^{(4) 4} \left( \frac{\hat{\alpha}}{\hat{\beta} k} \right) \frac{q^{2l+1}A^2 - 1}{q^{2l+1} A} \omega_{l+1} & 0 \\
1 & -\hat{R}_1^{(4) 4} \left( \frac{\hat{\alpha}}{\hat{\beta} k} \right) \omega_{l+1}
\end{pmatrix} \Phi^{(1)}_{1k}, \\
\hat{T}_1(\Phi^{(Q)}_{1k}) &= i\hat{R}_1^{(4) 4} \left( \frac{\hat{\beta}}{\hat{\gamma} k} \right) \frac{1}{\omega_{l+2}} \omega_{l+2} \omega_{l+1} \\
\hat{R}_1^{-1}(\Phi^{(Q)}_{1k}) &= -i\delta^{(5)}_{0,1k} \begin{pmatrix}
1 & -\hat{R}_1^{(4) 4} \left( \frac{\hat{\beta}}{\hat{\gamma} k} \right) \frac{q^{2l+1}A^2 - 1}{q^{2l+1} A} \omega_{l+1} \\
0 & -\hat{R}_1^{(4) 4} \left( \frac{\hat{\beta}}{\hat{\gamma} k} \right) \omega_{l+1}
\end{pmatrix} \Phi^{(Q)}_{1k}, \\
\hat{T}_1(\Phi^{(4)}_{1k}) &= -i\delta^{(1)}_{0,1k} \begin{pmatrix}
\hat{R}_1^{(4) 4} \left( \frac{\hat{\beta}}{\hat{\gamma} k} \right) \frac{q^{2l+1}A^2 - 1}{q^{2l+1} A} \omega_{l+1} \\
0 & -\hat{R}_1^{(4) 4} \left( \frac{\hat{\beta}}{\hat{\gamma} k} \right) \omega_{l+1}
\end{pmatrix} \Phi^{(4)}_{1k}.
\end{align*}
\]

B.5. For Lemma 4.1. We derive the linear systems used in Lemma 4.1 by using the Hopf-like linearization (2.17), (2.18), (4.12) and (4.13).

Lemma B.3. The following twenty linear systems hold:

\[
\begin{align*}
\hat{T}_2(\Psi^{(1)}_{1k,l,k,l,k,l,k}) &= \delta^{(1,2)}_{1,2,1,2,1,2,1,2} \begin{pmatrix}
\alpha_{l} & 0 \\
0 & \beta_l
\end{pmatrix} \hat{T}_2(u(I)) \Psi^{(1)}_{1k,l,k,l,k,l,k}, \\
\hat{T}_3(\Psi^{(1)}_{1k,l,k,l,k,l,k}) &= \delta^{(1,3)}_{1,3,1,3,1,3,1,3} \begin{pmatrix}
\gamma_l & 0 \\
0 & -\gamma_l
\end{pmatrix} \hat{T}_3(u(I)) \Psi^{(1)}_{1k,l,k,l,k,l,k}, \\
\hat{T}_4(\Psi^{(1)}_{1k,l,k,l,k,l,k}) &= \delta^{(1,4)}_{1,4,1,4,1,4,1,4} \begin{pmatrix}
-\alpha_l & 0 \\
0 & \beta_l
\end{pmatrix} \hat{T}_4(u(I)) \Psi^{(1)}_{1k,l,k,l,k,l,k}, \\
\hat{T}_1(\Psi^{(2)}_{1k,l,k,l,k,l,k}) &= \delta^{(2,1)}_{2,1,2,1,2,1,2,1} \begin{pmatrix}
\beta_l & 0 \\
0 & -\alpha_l
\end{pmatrix} \hat{T}_1(u(I)) \Psi^{(2)}_{1k,l,k,l,k,l,k}.
\end{align*}
\]
\[
\hat{T}_3(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(2,3)} \begin{pmatrix}
\frac{\beta_{l_2}}{u(l)} & -\hat{T}_3(u(l)) \\
\frac{1}{u(l)} - \frac{\beta_{l_2}}{\gamma_{l_1}} & \frac{\beta_{l_2}}{u(l)}
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(2)},
\] (B.12e)

\[
\hat{T}_4(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(2,4)} \begin{pmatrix}
-\beta_{l_3} K_{l_2} & -\hat{T}_4(u(l)) \\
\frac{1}{u(l)} & 0
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(2)},
\] (B.12f)

\[
\hat{T}_1(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(3,1)} \begin{pmatrix}
\frac{\gamma_{l_1}}{\alpha_{l_1}} & -\hat{T}_1(u(l)) \\
\frac{1}{u(l)} - \frac{\gamma_{l_1}}{\alpha_{l_1}} & \frac{\gamma_{l_1}}{u(l)}
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(3)},
\] (B.12g)

\[
\hat{T}_2(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(3,2)} \begin{pmatrix}
\frac{\gamma_{l_1}}{\beta_{l_2}} & -\hat{T}_2(u(l)) \\
\frac{1}{u(l)} - \frac{\gamma_{l_1}}{\beta_{l_2}} & \frac{\gamma_{l_1}}{u(l)}
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(3)},
\] (B.12h)

\[
\hat{T}_4(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(3,4)} \begin{pmatrix}
\frac{-\gamma_{l_4} K_{l_5}}{u(l)} & -\hat{T}_4(u(l)) \\
\frac{1}{u(l)} & 0
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(3)},
\] (B.12i)

\[
\hat{T}_1(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(4,1)} \begin{pmatrix}
1 & \alpha_{l_1} K_{l_2} \hat{T}_1(u(l)) \\
0 & -\hat{T}_1(u(l))
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(4)},
\] (B.12j)

\[
\hat{T}_2(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(4,2)} \begin{pmatrix}
1 & \beta_{l_3} K_{l_2} \hat{T}_2(u(l)) \\
0 & -\hat{T}_2(u(l))
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(4)},
\] (B.12k)

\[
\hat{T}_3(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(4,3)} \begin{pmatrix}
1 & \gamma_{l_4} K_{l_5} \hat{T}_3(u(l)) \\
0 & -\hat{T}_3(u(l))
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(4)},
\] (B.12l)

\[
\hat{T}_5(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(1,5)} \begin{pmatrix}
\alpha_{l_1} & -\hat{T}_5(u(l)) \\
\frac{1}{u(l)} & \frac{\alpha_{l_1}}{\mu_{l_6}}
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(1)},
\] (B.12m)

\[
\hat{T}_5(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(2,5)} \begin{pmatrix}
\beta_{l_2} & -\hat{T}_5(u(l)) \\
\frac{1}{u(l)} & -\frac{\beta_{l_2}}{\mu_{l_6}}
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(2)},
\] (B.12n)

\[
\hat{T}_5(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(3,5)} \begin{pmatrix}
\gamma_{l_3} & -\hat{T}_5(u(l)) \\
\frac{1}{u(l)} & -\frac{\gamma_{l_3}}{\mu_{l_6}}
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(3)},
\] (B.12o)

\[
\hat{T}_5(\Psi_{l_1,l_2,l_3,l_4,l_5}) = \delta_{l_1,l_2,l_3,l_4,l_5}^{(4,5)} \begin{pmatrix}
1 & \mu_{l_4} K_{l_5} \hat{T}_5(u(l)) \\
0 & -\hat{T}_5(u(l))
\end{pmatrix} \Psi_{l_1,l_2,l_3,l_4,l_5}^{(4)},
\] (B.12p)
Note that the space filling. The 5-cube around the origin of Lemma 44. NALINI JOSHI, NOBUTAKA NAKAZONO, AND YANG SHI

\[
\begin{align*}
\hat{T}_1 \left( \psi^{(5)} \right)_{i1, j1, k1, l1, m1} &= \delta^{(5,1)}_{i1, j1, k1, l1, m1} \begin{pmatrix}
\frac{\mu_0}{\alpha_1} & -\hat{T}_1(u(I)) \\
1 & -\frac{\mu_0}{\alpha_1} \hat{T}_1(u(I))
\end{pmatrix} \psi^{(5)}_{i1, j1, k1, l1, m1}, \\
\hat{T}_2 \left( \psi^{(5)} \right)_{i1, j1, k1, l1, m1} &= \delta^{(5,2)}_{i1, j1, k1, l1, m1} \begin{pmatrix}
\frac{\mu_0}{\beta_1} & -\hat{T}_2(u(I)) \\
1 & -\frac{\mu_0}{\beta_1} \hat{T}_2(u(I))
\end{pmatrix} \psi^{(5)}_{i1, j1, k1, l1, m1}, \\
\hat{T}_3 \left( \psi^{(5)} \right)_{i1, j1, k1, l1, m1} &= \delta^{(5,3)}_{i1, j1, k1, l1, m1} \begin{pmatrix}
\frac{\mu_0}{\gamma_1} & -\hat{T}_3(u(I)) \\
1 & -\frac{\mu_0}{\gamma_1} \hat{T}_3(u(I))
\end{pmatrix} \psi^{(5)}_{i1, j1, k1, l1, m1}, \\
\hat{T}_4 \left( \psi^{(5)} \right)_{i1, j1, k1, l1, m1} &= \delta^{(5,4)}_{i1, j1, k1, l1, m1} \begin{pmatrix}
-\mu_0 \hat{T}_4(u(I)) & 0 \\
1 & \mu_0 \hat{T}_4(u(I))
\end{pmatrix} \psi^{(5)}_{i1, j1, k1, l1, m1},
\end{align*}
\]

where \( l = \sum_{i=1}^{5} l_i e_i \) and \( \delta^{(i,j)}_{i1, j1, k1, l1, m1} = \hat{T}_1 \hat{T}_2 \hat{T}_3 \hat{T}_4 \hat{T}_5 (\delta^{(i,j)})(i, j \in \{1, \ldots, 5\}, i \neq j) \) are arbitrary decoupling factors. Note that \( \hat{T}_i, i = 1, \ldots, 5, \) commute with each other under the actions on the factors \( \delta^{i,j}. \)

Proof. The lattice \( \mathbb{Z}^5 \) can be constructed from the 5-dimensional hypercube (5-cube) by the space filling. The 5-cube around the origin \( 0 \in \mathbb{Z}^5 \) defined by

\[
\{0, \hat{T}_1(0), \hat{T}_2(0), \hat{T}_3(0), \hat{T}_4(0), \hat{T}_{12345}(0) \mid i, j, k, l \in \{1, \ldots, 5\}, i < j < k < l \},
\]

contains the following forty cubes:

\[
\begin{align*}
C_{312}, & \quad \hat{T}_1(C_{312}), \quad \hat{T}_2(C_{312}), \quad \hat{T}_{23}(C_{312}), \quad C_{214}, \quad \hat{T}_3(C_{214}), \quad \hat{T}_5(C_{214}), \quad \hat{T}_{25}(C_{214}), \\
C_{324}, & \quad \hat{T}_1(C_{324}), \quad \hat{T}_3(C_{324}), \quad \hat{T}_{13}(C_{324}), \quad C_{134}, \quad \hat{T}_2(C_{134}), \quad \hat{T}_5(C_{134}), \quad \hat{T}_{15}(C_{134}), \\
C_{215}, & \quad \hat{T}_3(C_{215}), \quad \hat{T}_1(C_{215}), \quad \hat{T}_{13}(C_{215}), \quad C_{325}, \quad \hat{T}_2(C_{325}), \quad \hat{T}_4(C_{325}), \quad \hat{T}_{14}(C_{325}), \\
C_{135}, & \quad \hat{T}_2(C_{135}), \quad \hat{T}_4(C_{135}), \quad \hat{T}_{24}(C_{135}), \quad C_{415}, \quad \hat{T}_3(C_{415}), \quad \hat{T}_5(C_{415}), \quad \hat{T}_{25}(C_{415}), \\
C_{425}, & \quad \hat{T}_4(C_{425}), \quad \hat{T}_5(C_{425}), \quad \hat{T}_{45}(C_{425}), \quad C_{435}, \quad \hat{T}_1(C_{435}), \quad \hat{T}_3(C_{435}), \quad \hat{T}_{13}(C_{435}).
\end{align*}
\]

Note that \( C_{ijkl} \) denotes a cube around the origin given by the following eight vertices:

\[
\{0, \hat{T}_i(0), \hat{T}_j(0), \hat{T}_k(0), \hat{T}_{ij}(0), \hat{T}_{ik}(0), \hat{T}_{jk}(0), \hat{T}_{ijk}(0) \},
\]

and \( \hat{T}_i(C) \) for a cube \( C \) means a shift in the \( \hat{T}_i \)-direction. In a similar manner as the proof of Lemma A.1, we can obtain Equations (A.6) and the following linear systems:

\[
\begin{align*}
\hat{T}_5 \left( \psi^{(1)} \right) &= \delta^{(1,5)} \begin{pmatrix}
\alpha_0 & -\hat{T}_5(u_0) \\
\mu_0 & 1
\end{pmatrix} \psi^{(1)}, \\
\hat{T}_5 \left( \psi^{(2)} \right) &= \delta^{(2,5)} \begin{pmatrix}
\beta_0 & -\hat{T}_5(u_0) \\
\mu_0 & 1
\end{pmatrix} \psi^{(2)}, \\
\hat{T}_5 \left( \psi^{(3)} \right) &= \delta^{(3,5)} \begin{pmatrix}
\gamma_0 & -\hat{T}_5(u_0) \\
\mu_0 & 1
\end{pmatrix} \psi^{(3)}.
\end{align*}
\]
\[
\hat{T}_5 (\Psi^{(4)}) = \delta^{(4,5)} \begin{pmatrix}
1 & \frac{\mu_0K_0\hat{T}_5(u_0)}{u_0} \\
0 & -\frac{\hat{T}_5(u_0)}{u_0}
\end{pmatrix} \Psi^{(4)},
\]
\[(B.15d)\]
\[
\hat{T}_1 (\Psi^{(5)}) = \delta^{(5,1)} \begin{pmatrix}
\frac{\mu_0}{\alpha_0} & 1 & -\hat{T}_1(u_0) \\
\frac{\alpha_0}{\mu_0} & 0 & \hat{T}_1(u_0)
\end{pmatrix} \Psi^{(5)},
\]
\[(B.15e)\]
\[
\hat{T}_2 (\Psi^{(5)}) = \delta^{(5,2)} \begin{pmatrix}
\frac{\mu_0}{\beta_0} & 1 & -\hat{T}_2(u_0) \\
\frac{\beta_0}{\mu_0} & 0 & \hat{T}_2(u_0)
\end{pmatrix} \Psi^{(5)},
\]
\[(B.15f)\]
\[
\hat{T}_3 (\Psi^{(5)}) = \delta^{(5,3)} \begin{pmatrix}
\frac{\mu_0}{\gamma_0} & 1 & -\hat{T}_3(u_0) \\
\frac{\gamma_0}{\mu_0} & 0 & \hat{T}_3(u_0)
\end{pmatrix} \Psi^{(5)},
\]
\[(B.15g)\]
\[
\hat{T}_4 (\Psi^{(5)}) = \delta^{(5,4)} \begin{pmatrix}
1 & -\hat{T}_4(u_0) \\
0 & \hat{T}_4(u_0)
\end{pmatrix} \Psi^{(5)},
\]
\[(B.15h)\]

where \(\delta^{(i,j)}\) are arbitrary decoupling factors. Therefore, we have completed the proof. \(\square\)

**B.6. For Lemma 4.2.** We derive the linear systems used in Lemma 4.2 by using the Hopf-like linearization \((2.17), (2.18), (2.25), (2.26), (4.12), (4.13)\) and \((4.15)\).

**Lemma B.4.** The following twelve linear systems hold:

\[
\hat{R}_1^{-1} (\Psi^{(1)}_{t,\Delta t,\Delta x}) = \delta^{(1)}_{t,\Delta t,\Delta x} \begin{pmatrix}
\frac{\alpha_l}{1 + \alpha_l} & -\frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}} \\
0 & \frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}}
\end{pmatrix} \Psi^{(1)}_{t,\Delta t,\Delta x},
\]
\[(B.16a)\]
\[
\hat{T}_4 (\Psi^{(1)}_{t,\Delta t,\Delta x}) = \delta^{(2)}_{t,\Delta t,\Delta x} \begin{pmatrix}
-\alpha_lK_l & \frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}} \\
1 & 0
\end{pmatrix} \Psi^{(1)}_{t,\Delta t,\Delta x},
\]
\[(B.16b)\]
\[
\hat{T}_1 (\Psi^{(R)}_{t,\Delta t,\Delta x}) = \delta^{(3)}_{t,\Delta t,\Delta x} \begin{pmatrix}
\frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}} & -\frac{u_{l,1+l,1,1}}{u_{l,1+l,1,1}} \\
0 & \frac{u_{l,1+l,1,1}}{u_{l,1+l,1,1}}
\end{pmatrix} \Psi^{(R)}_{t,\Delta t,\Delta x},
\]
\[(B.16c)\]
\[
\hat{T}_4 (\Psi^{(R)}_{t,\Delta t,\Delta x}) = \delta^{(4)}_{t,\Delta t,\Delta x} \begin{pmatrix}
\frac{1}{u_{l,1-l,1,1}} & -\frac{u_{l,1+l,1,1}}{u_{l,1+l,1,1}} \\
0 & \frac{u_{l,1+l,1,1}}{u_{l,1+l,1,1}}
\end{pmatrix} \Psi^{(R)}_{t,\Delta t,\Delta x},
\]
\[(B.16d)\]
\[
\hat{T}_4 (\Psi^{(6)}_{t,\Delta t,\Delta x}) = \delta^{(5)}_{t,\Delta t,\Delta x} \begin{pmatrix}
\alpha_lK_l & \frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}} \\
0 & \frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}}
\end{pmatrix} \Psi^{(6)}_{t,\Delta t,\Delta x},
\]
\[(B.16e)\]
\[
\hat{R}_1^{-1} (\Psi^{(4)}_{t,\Delta t,\Delta x}) = \delta^{(6)}_{t,\Delta t,\Delta x} \begin{pmatrix}
1 & -\frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}} \\
0 & \frac{u_{l,1-l,1,1}}{u_{l,1-l,1,1}}
\end{pmatrix} \Psi^{(4)}_{t,\Delta t,\Delta x},
\]
\[(B.16f)\]
\[ \tilde{T}_5 (\Psi^{(i)})_{h,l,l,h} = \delta^{(7)}_{h,l,l,h} \left( \begin{array}{cc} \alpha_h & -u_{l,l,h,h+1} \\ \mu_l & u_{l,l,h,l} \end{array} \right) \Psi^{(i)}_{h,l,l,h}, \]

(B.16g)

\[ \tilde{T}_5 (\Psi^{(R)})_{h,l,l,h} = \delta^{(8)}_{h,l,l,h} \left( \begin{array}{cc} \tilde{R}_l (\beta_0) \\ \mu_l & u_{l,l,h,l} \end{array} \right) \Psi^{(R)}_{h,l,l,h}, \]

(B.16h)

\[ \tilde{T}_5 (\Psi^{(4)})_{h,l,l,h} = \delta^{(9)}_{h,l,l,h} \left( \begin{array}{cc} 1 & \mu_l K_l u_{l,l,h,h+1} \\ 0 & -u_{l,l,h,l} \end{array} \right) \Psi^{(4)}_{h,l,l,h}, \]

(B.16i)

\[ \tilde{T}_1 (\Psi^{(5)})_{h,l,l,h} = \delta^{(10)}_{h,l,l,h} \left( \begin{array}{cc} \mu_l & -u_{l+1,l,l,h} \\ \alpha_l & u_{l,l,h,l} \end{array} \right) \Psi^{(5)}_{h,l,l,h}, \]

(B.16j)

\[ \tilde{R}_l^{-1} (\Psi^{(5)})_{h,l,l,h} = \delta^{(11)}_{h,l,l,h} \left( \begin{array}{cc} \mu_l & -\tilde{R}_l^{-1} (u_{l,l,h,l}) \\ \tilde{R}_l (\beta_0) & 1 \\ \mu_l & u_{l,l,j+1,l} \end{array} \right) \Psi^{(5)}_{h,l,l,h}, \]

(B.16k)

\[ \tilde{T}_4 (\Psi^{(5)})_{h,l,l,h} = \delta^{(12)}_{h,l,l,h} \left( \begin{array}{cc} -\mu_l K_l & -u_{l+1,l,l,h+1} \\ \mu_l & u_{l+1,l,l,h} \end{array} \right) \Psi^{(5)}_{h,l,l,h}, \]

(B.16l)

where \( \delta^{(i)}_{h,l,l,h} = \tilde{T}_1 \tilde{T}_4 \tilde{T}_5 \tilde{T}_4 \tilde{T}_5 \tilde{T}_4 \tilde{T}_5 (\delta^{(i)}), i = 1, \ldots, 12, \) are arbitrary decoupling factors. Note that \( \tilde{T}_1, \tilde{T}_4, \tilde{T}_5, \) and \( \tilde{T}_5 \) commute with each other under the actions on \( \delta^{(i)} \) \( i = 1, \ldots, 12. \)

**Proof.** Let us consider the 4-cube around the origin \( 0 \in \mathbb{Z}^5 \) defined by

\[ \{0, 0, \tilde{T}_1 (0), \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{T}_1 \tilde{T}_4 (0), \tilde{T}_4 \tilde{T}_5 (0), \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{R}_1 (0), \tilde{R}_5 (0), \tilde{R}_4 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{T}_4 \tilde{R}_1 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{R}_1 \tilde{R}_4 \tilde{T}_5 (0)\}. \]

The 4-cube contains the following eight cubes:

\[ C_{1824}, C_{1855}, C_{1425}, C_{1525}, \tilde{T}_5 (C_{1824}), \tilde{T}_4 (C_{1855}), \tilde{T}_5 (C_{1425}), \tilde{T}_1 (C_{1525}), \]

where the cubes \( C_{1824}, C_{1855}, C_{1425}, \) and \( \tilde{T}_5 (C_{1425}) \) are defined by

\[ \{0, \tilde{T}_1 (0), \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{T}_1 \tilde{T}_4 (0), \tilde{T}_4 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{R}_1 \tilde{R}_4 \tilde{T}_5 (0)\}, \]

\[ \{0, \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{T}_1 \tilde{T}_5 (0), \tilde{T}_1 \tilde{R}_4 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{T}_4 \tilde{R}_1 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{R}_4 \tilde{T}_5 (0), \tilde{R}_1 \tilde{R}_4 \tilde{R}_5 (0)\}, \]

\[ \{0, \tilde{T}_5 (0), \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{T}_1 \tilde{T}_5 (0), \tilde{T}_1 \tilde{R}_4 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{T}_4 \tilde{R}_1 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{R}_4 \tilde{T}_5 (0), \tilde{R}_1 \tilde{R}_4 \tilde{R}_5 (0)\}, \]

\[ \{0, \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{T}_4 (0), \tilde{T}_5 (0), \tilde{T}_1 \tilde{T}_5 (0), \tilde{T}_1 \tilde{R}_4 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{T}_4 \tilde{R}_1 \tilde{T}_5 (0), \tilde{T}_4 \tilde{R}_5 (0), \tilde{R}_4 \tilde{T}_5 (0), \tilde{R}_1 \tilde{R}_4 \tilde{R}_5 (0)\}, \]

respectively. Therefore, in a similar manner as the proof of Lemma A.1, we obtain Equations (B.5) and the following linear systems:

\[ \tilde{T}_5 (\Psi^{(1)}) = \delta^{(7)} \left( \begin{array}{cc} \alpha_0 & -\tilde{T}_5 (u_0) \\ \mu_0 & u_0 \end{array} \right) \Psi^{(1)}, \]

(B.17a)
\[
\begin{align*}
\hat{T}_3 (\Psi^{(8)}) &= \delta^{(8)} \begin{pmatrix}
\frac{\beta_0}{\mu_0} & -\frac{T_3(u_0)}{u_0} \\
\mu_0 & \beta_0 & -\frac{T_3(u_0)}{u_0}
\end{pmatrix} \Psi^{(8)}, \\
\hat{T}_3 (\Psi^{(4)}) &= \delta^{(9)} \begin{pmatrix}
1 & \frac{\mu_0 K_0}{u_0} \\
0 & -\frac{T_3(u_0)}{u_0}
\end{pmatrix} \Psi^{(4)}, \\
\hat{T}_1 (\Psi^{(5)}) &= \delta^{(10)} \begin{pmatrix}
\frac{\mu_0}{\alpha_0} & -\frac{\hat{T}_1(u_0)}{u_0} \\
1 & -\frac{\alpha_0}{\beta_0} & -\frac{\hat{T}_1(u_0)}{u_0}
\end{pmatrix} \Psi^{(5)}, \\
\hat{R}_1^{-1} (\Psi^{(5)}) &= \delta^{(11)} \begin{pmatrix}
\frac{\mu_0}{\beta_0} & -\hat{R}_1^{-1}(u_0) \\
1 & -\frac{\beta_0}{\mu_0} & -\hat{R}_1^{-1}(u_0)
\end{pmatrix} \Psi^{(5)}, \\
\hat{T}_4 (\Psi^{(5)}) &= \delta^{(12)} \begin{pmatrix}
-\frac{\mu_0 K_0}{u_0} & -\frac{\hat{T}_4(u_0)}{u_0} \\
0 & 1
\end{pmatrix} \Psi^{(5)}.
\end{align*}
\]

where \(\delta^{(i)}, i = 1, \ldots, 12\), are arbitrary decoupling factors. Therefore we have completed the proof.

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