Integer Sequences Realized by the Subgroup Pattern of the Symmetric Group

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Abstract

The subgroup pattern of a finite group $G$ is the table of marks of $G$ together with a list of representatives of the conjugacy classes of subgroups of $G$. In this article we describe a collection of sequences realized by the subgroup pattern of the symmetric group.

1 Introduction

The table of marks of a finite group $G$ was introduced by Burnside [2]. It is a matrix whose rows and columns are indexed by a list of representatives of the conjugacy classes of subgroups of $G$, where, for two subgroups $H, K \leq G$ the $(H, K)$ entry in the table of marks of $G$ is the number of fixed points of $K$ in the transitive action of $G$ on the cosets of $H$, $(\beta_{G/H}(K))$. If $H_1, \ldots, H_r$ is a list of representatives of the conjugacy classes of subgroups of $G$, the table of marks is then the $(r \times r)$-matrix

$$M(G) = (\beta_{G/H_i}(H_j))_{i,j=1,\ldots,r}.$$

In much the same fashion as the character table of $G$ classifies matrix representations of $G$ up to isomorphism, the table of marks of $G$ classifies permutation representations of $G$ up to equivalence. It also encodes a wealth of information about the subgroup lattice of $G$ in a compact way. The GAP [4] library of tables of marks Tomlib [11] provides ready access to the tables of marks and conjugacy classes of subgroups of some 400 groups. These tables have been produced using the methods described in [8] and [7]. The data exhibited in later sections has been computed using this library. The purpose of this article is to illustrate how interesting integer sequences related to the subgroup structure of the symmetric group $S_n$, and the alternating group $A_n$, can be computed from this data. This paper is organized as follows. In Section 2 we study the conjugacy classes of subgroups of $S_n$ for $n \leq 13$. In Section 3 we examine the tables of marks of $S_n$ for $n \leq 13$ and describe how much more information regarding the subgroup structure of $S_n$ can be obtained. In Section 4 we discuss the Euler Transform and its applications in counting subgroups of $S_n$. 

1
2 Counting Subgroups

Given a list of representatives \{H_1, \ldots, H_r\} of Sub(G)/G, the conjugacy classes of subgroups of G, we can enumerate those subgroups which satisfy particular properties. The numbers of conjugacy classes of subgroups of \(S_n\) and \(A_n\) are sequences \(A000638\) and \(A029726\) respectively in Sloane’s encyclopedia \([9]\). The GAP table of marks library Tomlib provides access to the conjugacy classes of subgroups of the symmetric and alternating groups for \(n \leq 13\). Table 1 records the number of conjugacy classes of subgroups of \(S_n\) which are abelian, cyclic, nilpotent, solvable and supersolvable (SupSol). A similar table for the conjugacy classes of subgroups of the alternating groups can be found in Appendix A.

| n | Sub(\(S_n\))/\(S_n\) | Abelian | Cyclic | Nilpotent | Solvable | SupSol |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 4 | 3 | 3 | 4 | 4 | 4 |
| 4 | 11 | 7 | 5 | 8 | 11 | 9 |
| 5 | 19 | 9 | 7 | 10 | 17 | 15 |
| 6 | 56 | 20 | 11 | 25 | 50 | 38 |
| 7 | 96 | 26 | 15 | 32 | 84 | 65 |
| 8 | 296 | 61 | 22 | 127 | 268 | 187 |
| 9 | 554 | 82 | 30 | 156 | 485 | 341 |
| 10 | 1593 | 180 | 42 | 531 | 1418 | 923 |
| 11 | 3094 | 236 | 56 | 648 | 2691 | 1789 |
| 12 | 10723 | 594 | 77 | 3727 | 9725 | 6118 |
| 13 | 20832 | 762 | 101 | 4221 | 18286 | 11616 |

Table 1: Sequences in \(S_n\)

2.1 Subgroup Orders

A question of historical interest concerns the orders of subgroups of \(S_n\). In \([3]\) Cameron writes: *The Grand Prix question of the Academie des Sciences, Paris, in 1860 asked “How many distinct values can a function of \(n\) variables take?” In other words what are the possible indices of subgroups of \(S_n\).* For \(n \leq 13\), Table 2 records the numbers of different orders \(\mathcal{O}(S_n), \mathcal{O}(A_n)\) of subgroups of \(S_n\) and \(A_n\). One might as well also enumerate the number of “missing” subgroup orders, that is, the number, \(d(S_n)\), of divisors \(d\) such that \(d \mid |S_n|\) but \(S_n\) has no subgroup of order \(d\). Table 3 records the number of missing subgroup orders of \(S_n\) and \(A_n\) for \(n \leq 13\).
### 3 Counting Using the Table of Marks

If in addition to a list of conjugacy classes of subgroups of $G$, the table of marks of $G$ is also available, or can be computed, one can say quite a lot about the structure of the lattice of subgroups of $G$. We begin this section by giving some basic information about tables of marks and then go on to describe how we can count incidences and edges in the lattice of subgroups.

#### 3.1 About Tables of Marks

Let $G$ be a finite group and let $\text{Sub}(G)$ denote the set of subgroups of $G$. By $\text{Sub}(G)/G$ we denote the set of conjugacy classes of subgroups of $G$. For $H, K \in \text{Sub}(G)$ let

$$\beta_{G/H}(K) = \#\{Hg \in G/H : (Hg)k = Hg \text{ for all } k \in K\} = \#\{g \in G : K \leq H^g\}/|H|$$

denote the mark of $K$ on $H$. This number depends only on the $G$-conjugacy classes of $H$ and $K$. Note that $\beta_{G/H}(K) \neq 0 \Rightarrow |K| \leq |H|$. If $H_1, \ldots, H_r$ is a list of representatives of the conjugacy classes of subgroups of $G$, the table of marks of $G$ is then the $(r \times r)$-matrix

$$M(G) = (\beta_{G/H_i}(H_j))_{i,j=1,\ldots,r}.$$ 

If the subgroups in the transversal are listed by increasing group order the table of marks is a lower triangular matrix. The table of marks $M(S_4)$ of the symmetric group $S_4$ is shown in Figure 1.
As a matrix, we can extract a variety of sequences from the table of marks, the most obvious of which is the sum of the entries. The sum of the entries of $M(S_n)$ for $n \leq 13$ is shown in Figure 4. We can also sum the entries on the diagonal to obtain the sequences in Figure 5.

Table 4: Sum of $M(G)$

| $n$ | $S_n$ | $A_n$ |
|-----|------|------|
| 1   | 1    | 1    |
| 2   | 4    | 1    |
| 3   | 18   | 5    |
| 4   | 146  | 39   |
| 5   | 681  | 192  |
| 6   | 7518 | 1717 |
| 7   | 58633| 13946|
| 8   | 952826| 243391|
| 9   | 11168496| 2693043|
| 10  | 232255571| 38343715|
| 11  | 3476965896| 545787051|
| 12  | 108673489373| 15787210045|
| 13  | 1951392769558| 268796141406|

Table 5: Sum of the Diagonal

| $n$ | $S_n$ | $A_n$ |
|-----|------|------|
| 1   | 1    | 1    |
| 2   | 3    | 1    |
| 3   | 10   | 4    |
| 4   | 47   | 19   |
| 5   | 165  | 73   |
| 6   | 950  | 412  |
| 7   | 5632 | 2660 |
| 8   | 43772| 21449|
| 9   | 376586| 184541|
| 10  | 3717663| 1827841|
| 11  | 40555909| 20043736|
| 12  | 484838080| 240206213|
| 13  | 6286289685| 3119816216|

We will now collect some elementary properties of tables of marks in Lemma 1.

**Lemma 1.** Let $H, K \leq G$. Then the following hold:

(i) The first entry of every row of $M(G)$ is the index of the corresponding subgroup,

$$\beta_{G/H}(1) = |G : H|.$$
(ii) The entry on the diagonal is,
\[ \beta_{G/H}(H) = |N_{G}(H) : H|. \]

(iii) The length of the conjugacy class \(|H|\) of \(H\) is given by,
\[ |H| = |G : N_{G}(H)| = \frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}. \]

(iv) The number of conjugates of \(H\) which contain \(K\) is given by,
\[ |\{H^a : a \in G, K \leq H^a\}| = \frac{\beta_{G/H}(K)}{\beta_{G/H}(H)}. \]

The following formula which follows trivially from Lemma 1 (iv) relates marks to incidences in the subgroup lattice of \(G\).
\[ \beta_{G/H}(K) = |N_{G}(H) : H| \cdot \#\{H^g : K \leq H^g, g \in G\}. \quad (1) \]

As a first application of Formula 1 we obtain the following lemma which enables us to count the total number of subgroups of \(G\).

**Lemma 2.** Given a list \(\{H_1, \ldots, H_r\}\) of representatives of the conjugacy classes of subgroups of \(G\), the total number of subgroups of \(G\) is
\[ |\text{Sub}(G)| = \sum_{i=1}^{r} \frac{\beta_{G/H_i}(1)}{\beta_{G/H_i}(H_i)}. \]

**Proof.** It follows from Formula 1 that for any subgroup \(H \leq G\), \(\frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}\) is the length of the conjugacy class of \(H\) in \(G\). \(\square\)

Table 6 lists the total number of subgroups of \(S_n\) and \(A_n\) for \(n \leq 13\).

| n  | A029725 | A005432 |
|----|--------|--------|
| 1  | 1      | 1      |
| 2  | 1      | 2      |
| 3  | 2      | 6      |
| 4  | 10     | 30     |
| 5  | 59     | 156    |
| 6  | 501    | 1455   |
| 7  | 3786   | 11300  |
| 8  | 48337  | 151221 |
| 9  | 508402 | 1694723|
| 10 | 6469142| 29594446|
| 11 | 81711572| 404126228|
| 12 | 2019160542| 10594925360|
| 13 | 31945830446| 175238308453|

**Table 6:** Total Number of Subgroups of \(A_n\) and \(S_n\)
3.2 Counting Incidences

Another immediate consequence of Formula 1 is that by dividing each row of the table of marks of $G$ by its diagonal entry $\beta_{G/H}(H)$ we obtain a matrix $C(G)$ describing containments in the subgroup lattice of $G$, where the $(H, K)$-entry is

$$C(H, K) = \#\{K^g : H \leq K^g, g \in G\}.$$  \hspace{1cm} (2)

Figure 2 illustrates the containment matrix of the symmetric group $S_4$.

```
1 1
2 3 1
2 6 . 1
3 4 . . 1
2^2 1 1 . . 1
2^2 3 1 1 . . 1
4 3 1 . . . . 1
S_3 4 . 2 1 . . . 1
D_8 3 3 1 . 3 1 1 . 1
A_4 1 1 . 1 1 . . . . 1
S_4 1 1 1 1 1 1 1 1 1
```

Figure 2: Containment Matrix : $C(S_4)$

The conjugacy classes of subgroups of $G$ are partially ordered by $[H] \leq [K]$ if $H \leq K^g$ for some $g \in G$ i.e. if $C(H, K) \neq 0$. Therefore we can easily obtain the incidence matrix, $I(G)$, of the poset of conjugacy classes of subgroups of $G$ by replacing each nonzero entry in $C(G)$, (or $M(G)$ ) by an entry 1. Figure 3 shows the incidence matrix $I(S_4)$ of the poset of conjugacy classes of subgroups of $S_4$. 
Figure 3: Incidence Matrix : $\mathcal{I}(S_4)$

For comparison with Figure 3 we illustrate the poset of conjugacy classes of subgroups of $S_4$ in Figure 4.

Figure 4: Poset of Conjugacy Classes of Subgroups of $S_4$

**Lemma 3.** The number of incidences in the poset of conjugacy classes of subgroups of $G$ is given by

$$\sum \mathcal{I}(G).$$
Proof. The incidence matrix $I(G)$ is obtained by replacing every nonzero entry in the table of marks by an entry 1. By Formula 1 $I(H, K) = 1$ if and only if $K$ is subconjugate to $H$ in $G$, i.e. if and only if $H$ and $K$ are incident in the poset of conjugacy classes of subgroups of $G$. \[\square\]

Table 7 lists the number of incidences in the poset of conjugacy classes of subgroups of $A_n$ and $S_n$ for $n \leq 13$.

**Lemma 4.** The total number of incidences in the entire subgroup lattice of $G$ is given by

$$\sum \mathcal{C}(G).$$

Proof. For $H, K \in \text{Sub}(G)/G$ the $H, K$ entry in $\mathcal{C}(G)$ is the number of incidences between $H, K$ in the subgroup lattice of $G$. Thus summing over the entries in $\mathcal{C}(G)$ yields the total number of incidences in the entire subgroup lattice of $G$. \[\square\]

Table 8 records the number of incidences in the subgroup lattices of $S_n$ and $A_n$ for $n \leq 13$.

| $n$ | $A_218921$ | $A_218922$ |
|-----|------------|------------|
| 1   | 1          | 1          |
| 2   | 1          | 1          |
| 3   | 3          | 3          |
| 4   | 44         | 13         |
| 5   | 101        | 32         |
| 6   | 523        | 128        |
| 7   | 1195       | 330        |
| 8   | 6751       | 2309       |
| 9   | 16986      | 4271       |
| 10  | 87884      | 12468      |
| 11  | 248635     | 33329      |
| 12  | 1709781    | 196182     |
| 13  | 4665651    | 490137     |

| $n$ | $A_218924$ | $A_218923$ |
|-----|------------|------------|
| 1   | 1          | 1          |
| 2   | 1          | 3          |
| 3   | 3          | 11         |
| 4   | 18         | 68         |
| 5   | 85         | 262        |
| 6   | 657        | 2261       |
| 7   | 4374       | 14032      |
| 8   | 55711      | 176245     |
| 9   | 530502     | 1821103    |
| 10  | 6603007    | 30883491   |
| 11  | 82736601   | 415843982  |
| 12  | 2032940127 | 10779423937|
| 13  | 32102236563| 177718085432|

Table 7: Incidences in Poset          Table 8: Incidences in Subgroup Lattice

### 3.3 Counting Edges in Hasse Diagrams

The table of marks also allows us to count the number of edges in both the Hasse diagrams of the poset of conjugacy classes of subgroups and the subgroup lattice of $G$. Computing such data requires careful analysis of maximal subgroups in the subgroup lattice.

Formula 1 describes containments in the poset of conjugacy classes of subgroups looking upward through the subgroup lattice of $G$. But we can also view marks as containments looking downward through the subgroup lattice of $G$.  

8
Lemma 5. Let $H, K \in \text{Sub}(G)/G$. Then the number of conjugates of $H$ contained in $K$ is given by

$$E^\uparrow(H, K) = |\{H^g, g \in G : H^g \leq K\}| = \frac{\beta_{G/K}(H)\beta_{G/H}(1)}{\beta_{G/H}(H)\beta_{G/K}(1)}$$

Proof. The total number of edges between the classes $[H]_G$ and $[K]_G$ can be counted in two different ways, as the length of the class times the number of edges leaving one member of the class. Thus

$$|[H]_G| \cdot |\{H^g, g \in G : H^g \leq K\}| = |[K]_G| \cdot |\{K^g, g \in G : K^g \geq H\}|.$$ 

By Formula 1 $|[H]_G| = \frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}$ and $|[K]_G| = \frac{\beta_{G/K}(1)}{\beta_{G/K}(K)}$. Thus $E^\uparrow(H, K)$ can be expressed in terms of marks. \(\square\)

3.3.1 Identifying Maximal Subgroups

It will be necessary, for the sections that follow, to identify for $H_i \in \text{Sub}(G)/G$ which classes $H_j \in \text{Sub}(G)/G$ are maximal in $H_i$.

Lemma 6. Let $H_i \in \text{Sub}(G)/G = \{H_1, \ldots, H_r\}$. Denote by $\rho_i = \{j : H_j <_G H_i\}$ the set of indices in $\{1, \ldots, r\}$ of proper subgroups of $H_i$ up to conjugacy in $G$. Then the positions of all maximal subgroups of $H_i$ are given by

$$\text{Max}(H_i) = \rho_i \setminus \bigcup_{j \in \rho_i} \rho_j$$ (3)

The set of values $\rho_i$ are easily read off the table of marks of $G$ by simply identifying the nonzero entries in the row corresponding to $G/H_i$. Formula 3 is implemented in GAP via the function MaximalSubgroupsTom.

Lemma 7. Let $\text{Sub}(G)/G = \{H_1, \ldots, H_r\}$ be a list of representatives of the conjugacy classes of subgroups of $G$. The number of edges in the Hasse diagram of the poset of conjugacy classes of subgroups of $G$ is given by

$$|E(\text{Sub}(G)/G)| = \sum_{i=1}^r |\text{Max}(H_i)|.$$

Proof. By Lemma 6, $\text{Max}(H_i)$ is a list of the positions of the maximal subgroups of $H_i$ up to conjugacy in $G$. In the Hasse diagram of the poset $\text{Sub}(G)/G$ each edge corresponds to a maximal subgroup. \(\square\)

Table 9 records the number of edges in the Hasse diagram of the poset of conjugacy classes of subgroups of $S_n$ and $A_n$ for $n \leq 13$. In order to count the number of edges in the Hasse diagram of the entire subgroup lattice of $G$ we appeal to Formula 1 and Lemma 5.
Lemma 8. Let $\text{Sub}(G)/G = \{H_1, \ldots, H_r\}$ be as above. The total number of edges $E(L(G))$ in the Hasse diagram of the subgroup lattice of $G$ is given by

$$E(L(G)) = \sum_{i=1}^{r} \sum_{j \in \text{Max}(H_i)} E(H_i, H_j).$$

Proof. By restricting $E(H_i, H_j)$ to those classes $H_i, H_j$ which are maximal we obtain the number of edges connecting maximal subgroups of $G$. \qed

Table 10 records the total number of edges in the Hasse diagram of the subgroup lattice of $S_n$ and $A_n$ for $n \leq 13$.

|       | \text{A218925} | \text{A218926} |
|-------|----------------|----------------|
| $n$   | $S_n$          | $A_n$          |
| 1     | 0              | 0              |
| 2     | 1              | 0              |
| 3     | 4              | 1              |
| 4     | 17             | 5              |
| 5     | 37             | 13             |
| 6     | 149            | 44             |
| 7     | 290            | 98             |
| 8     | 1080           | 419            |
| 9     | 2267           | 722            |
| 10    | 8023           | 1592           |
| 11    | 17249          | 3304           |
| 12    | 72390          | 12645          |
| 13    | 153419         | 24792          |

Table 9: Edges in Poset

|       | \text{A218928} | \text{A218927} | \text{A218928} | \text{A218927} |
|-------|----------------|----------------|----------------|----------------|
| $n$   | $A_n$          | $S_n$          | $A_n$          | $S_n$          |
| 1     | 0              | 0              | 0              | 0              |
| 2     | 0              | 1              | 0              | 1              |
| 3     | 1              | 8              | 1              | 8              |
| 4     | 15             | 66             | 15             | 66             |
| 5     | 168            | 501            | 168            | 501            |
| 6     | 2051           | 6469           | 2051           | 6469           |
| 7     | 19305          | 60428          | 19305          | 60428          |
| 8     | 283258         | 926743         | 283258         | 926743         |
| 9     | 3255913        | 11902600       | 3255913        | 11902600       |
| 10    | 4664854        | 240066343      | 4664854        | 240066343      |
| 11    | 670282962      | 3677270225     | 670282962      | 3677270225     |
| 12    | 18723796793    | 108748156239   | 18723796793    | 108748156239   |
| 13    | 321480817412   | 1980478458627  | 321480817412   | 1980478458627  |

Table 10: Edges in Subgroup Lattice

### 3.4 Maximal Property-P Subgroups

For any property $P$ which is inherited by subgroups of $G$ we can use the table of marks of $G$ to enumerate the maximal property $P$ subgroups of $G$.

Lemma 9. Let $\text{Sub}(G)/G = \{H_1, \ldots, H_r\}$ and let $\rho = \{i \in [1, \ldots, r] : H_i \text{ is a property } P \text{ subgroup}\}$. Then the positions of the maximal property $P$ subgroups of $G$ are given by

$$P(G) = \rho \setminus \bigcup_{j \in \rho} \text{Max}(H_j) \quad (4)$$

In Figure 5 the classes of maximal abelian subgroups of $S_4$ are boxed.
Table 11 records, for each of the properties listed across the first row of the table, the numbers of maximal property P classes of subgroups of $S_n$. A similar table for the alternating groups can be found in the Appendix.

| n | Solvable | SupSol | Abelian | Cyclic | Nilpotent |
|---|---------|--------|---------|--------|-----------|
| 1 | 1       | 1      | 1       | 1      | 1         |
| 2 | 1       | 1      | 1       | 1      | 1         |
| 3 | 1       | 2      | 2       | 2      | 2         |
| 4 | 1       | 4      | 3       | 2      | 2         |
| 5 | 3       | 5      | 3       | 3      | 3         |
| 6 | 4       | 7      | 5       | 5      | 5         |
| 7 | 5       | 10     | 6       | 6      | 6         |
| 8 | 6       | 17     | 11      | 7      |           |
| 9 | 9       | 23     | 15      | 9      |           |
| 10| 12      | 30     | 20      | 12     |           |
| 11| 14      | 41     | 24      | 15     |           |
| 12| 17      | 61     | 34      | 20     |           |
| 13| 24      | 80     | 43      | 25     |           |

Table 11: Maximal Property-P Subgroups of $S_n$
4 Connected Subgroups and the Euler Transform

The conjugacy classes of subgroups of the symmetric group play an important role in the theory of combinatorial species as described in [6]. Permutation groups have been used to answer many questions about species. Every species is the sum of its molecular subspecies. These molecular species correspond to conjugacy classes of subgroups of $\text{Sym}(n)$. Molecular species decompose as products of atomic species which in turn correspond to connected subgroups of $\text{Sym}(n)$ in the following sense. It will be convenient to denote the symmetric group on a finite set $X$ by $\text{Sym}(X)$.

**Definition 1.** For each $H \leq \text{Sym}(X)$ there is a finest partition of $X = \bigsqcup Y_i$ such that $H = \prod H_i$ with $H_i \leq \text{Sym}(Y_i)$. We allow $H_i = 1$ when $|Y_i| = 1$. We say that $H$ is a connected subgroup of $\text{Sym}(X)$ if the finest partition is $X$.

**Example 1.** Let $X = \{1, 2, 3, 4\}$ and consider $H = \langle (1, 2), (3, 4) \rangle$ and $H' = \langle (1, 2)(3, 4) \rangle$. Partitioning $X$ into $Y_1 = \{1, 2\}, Y_2 = \{3, 4\}$ gives $H = H_1 \times H_2$ where $H_1 = \langle (1, 2) \rangle \leq \text{Sym}((1, 2)), H_2 = \langle (3, 4) \rangle \leq \text{Sym}((3, 4))$, hence $H$ is not connected. On the other hand, $H'$ is connected since there is no finer partition of $X$ which permits us to write $H'$ as a product of connected $H_i$.

An algorithm to test a group $H$ acting on a set $X$ for connectedness checks each non-trivial $H$-stable subset $Y$ of $X$. If $H$ is the direct product of its action on $Y$ and its action on $X \setminus Y$ then $H$ is not connected.

In general a subgroup $H \leq \text{Sym}(X)$ is a product of connected subgroups $H_i \leq \text{Sym}(Y_i)$. Sequence $A000638$ records the number of molecular species of degree $n$ or equivalently the number of conjugacy classes of subgroups of $\text{Sym}(n)$. Sequence $A005226$ records the number of atomic species of degree $n$ or equivalently the number of conjugacy classes of connected subgroups of $\text{Sym}(n)$. These sequences are related by the Euler Transform.

### 4.1 The Euler Transform

If two sequences of integers $\{c_k\} = (c_1, c_2, c_3, \ldots)$ and $\{m_n\} = (m_1, m_2, m_3, \ldots)$ are related by

$$1 + \sum_{n \geq 1} m_n x^n = \prod_{k \geq 1} \left( \frac{1}{1-x^k} \right)^{c_k}. \quad (5)$$

Then we say that $\{m_n\}$ is the Euler transform of $\{c_k\}$ and that $\{c_k\}$ is the inverse Euler transform of $\{m_n\}$ (see [1]). One sequence can be computed from the other by introducing the intermediate sequence $\{b_n\}$ defined by

$$b_n = \sum_{d|n} dc_d = nm_n - \sum_{k=1}^{n-1} b_km_{n-k}. \quad (6)$$
Then
\[ m_n = \frac{1}{n} \left( b_n + \sum_{k=1}^{n-1} b_k m_{n-k} \right), \quad c_n = \frac{1}{n} \sum_{d|n} \mu(n/d) b_d, \]

(7)

where \( \mu \) is the number-theoretic Möbius function.

There are many applications of this pair of transforms (see [10]). For example, the inverse Euler transform applied to the sequence of numbers of unlabeled graphs on \( n \) nodes \((A000088)\) yields the sequence of numbers of connected graphs on \( n \) nodes \((A001349)\). To understand how Formula 5 can be used to count connected graphs we note that the coefficient of \( x^n \) in the expansion of the product on the right hand side of Formula 5 is

\[ m_n = \sum_{1^{a_1} 2^{a_2} \ldots n^{a_n} \vdash n} \prod_{i} \binom{c_i}{a_i} \]

(8)

where \( \binom{c_i}{a_i} \) denotes the number of \( a_i \)-element multisets chosen from a set of \( c_i \) objects. On the other hand, an unlabeled graph on \( n \) nodes, as a collection of connected components, can be characterized by a pair \( (\lambda, (C_1, \ldots, C_n)) \) where \( \lambda = 1^{a_1}, 2^{a_2}, \ldots, n^{a_n} \) is a partition of \( n \) and \( C_i \) is a multiset of \( a_i \) connected unlabeled graphs on \( i \) nodes, for \( 1 \leq i \leq n \). If \( c_i \) is the number of connected unlabeled graphs on \( i \) nodes then, by Formula 8, \( m_n \) is the total number of unlabeled graphs on \( n \) nodes.

In the same way, the inverse Euler transform of \( A000638 \) (the number of conjugacy classes of subgroups of \( S_n \)) is \( A005226 \), (the number of connected conjugacy classes of subgroups of \( S_n \)) as formalized in the following Lemma.

**Lemma 10.** There is a bijection between the conjugacy classes of subgroups of \( S_n \) and the set of pairs of the form \( (\lambda, (C_1, \ldots, C_n)) \) where \( \lambda = 1^{a_1}, 2^{a_2}, \ldots, n^{a_n} \) is a partition of \( n \) and \( C_i \) is a multiset of \( a_i \) conjugacy classes of connected subgroups of \( S_i \) for \( i = 1, \ldots, n \).

**Proof.** Given a representative \( H \) of the conjugacy class of subgroups \([H] \in \text{Sub}(S_n)/S_n\) we associate a pair \( (\lambda, (C_1, \ldots, C_n)) \) to \( H \) as follows. Write \( H = \prod H_k \) where each \( H_k \) is a connected subgroup of \( \text{Sym}(Y_k) \). Then \( X = \{1, \ldots, n\} = \bigsqcup Y_k \). Recording the size of each \( Y_k \) yields a partition \( \lambda = 1^{a_1}, 2^{a_2}, \ldots, n^{a_n} \). For \( 1 \leq i \leq n, C_i \) is the multiset of \( S_i \)-classes of subgroups \( H_k \) with \( |Y_k| = i \). Bijectivity follows from the fact that conjugate subgroups yield the same \( \lambda \) and since \( H^g = \prod H_k^g \), conjugate subgroups yield conjugate \( C_i \). \( \square \)

### 4.2 Counting Connected Subgroups of the Alternating Group

In Section 4 we noted that molecular species correspond to conjugacy classes of subgroups of \( \text{Sym}(n) \) and that atomic species correspond to conjugacy classes of connected subgroups of \( \text{Sym}(n) \) in the sense of Definition 1. In this Section we will count connected conjugacy classes of subgroups of the alternating group, up to \( S_n \) conjugacy and \( A_n \) conjugacy.

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13
4.2.1 $S_n$-Orbits of Subgroups of the Alternating Group

In order to count the number of $S_n$-conjugacy classes of subgroups of the alternating group we introduce the following notation. Let

$$B = \{ H \leq S_n : H \leq A_n \text{ and } R = \{ H \leq S_n : H \not\leq A_n \} \}.$$

Then $\text{Sub}(S_n)/S_n = B/S_n \sqcup R/S_n$ and $B/S_n$ is the set of $S_n$ conjugacy classes of subgroups of the alternating group. The set $R/S_n$ is the set of conjugacy classes of subgroups of $S_n$ which are not contained in $A_n$. Table 12 illustrates both of these sequences together with the numbers of conjugacy classes of subgroups of $S_n$ and $A_n$. In order to count the number of connected $S_n$-conjugacy classes of subgroups of $A_n$ we apply the inverse Euler transform to the sequence $|B/S_n|$ in Table 12, to obtain

$$A218968 : 1, 0, 1, 3, 4, 12, 12, 65, 198, 1207, 1178.$$

We can also count the number of connected conjugacy classes of subgroups of $S_n$ not contained in $A_n$ (i.e. corresponding to $R/S_n$) by subtracting the sequence above from $A005226$ to obtain

$$A218969 : 0, 1, 1, 3, 2, 15, 8, 65, 66, 431, 443, 3643, 3594.$$

|          | A000638 | A029726 | A218966 | A218965 |
|----------|---------|---------|---------|---------|
| n        | Sub(S_n)/S_n | Sub(A_n)/A_n | | |
| 1        | 1       | 1       | 1       | 0       |
| 2        | 2       | 1       | 1       | 1       |
| 3        | 4       | 2       | 2       | 2       |
| 4        | 11      | 5       | 5       | 6       |
| 5        | 19      | 9       | 9       | 10      |
| 6        | 56      | 22      | 22      | 34      |
| 7        | 96      | 40      | 37      | 59      |
| 8        | 296     | 137     | 112     | 184     |
| 9        | 554     | 223     | 195     | 359     |
| 10       | 1593    | 430     | 423     | 1170    |
| 11       | 3094    | 788     | 780     | 2314    |
| 12       | 10723   | 2537    | 2401    | 8322    |
| 13       | 20832   | 4558    | 4409    | 16423   |

Table 12: Red and Blue Subgroups of $S_n$

4.2.2 Connected Subgroups of the Alternating Group

Every subgroup of $A_n$ is either connected or not connected with respect to the set $\{1, \ldots, n\}$ and shares this property with all subgroups in its $A_n$-conjugacy class. So we wish to count the number of $A_n$-conjugacy classes of connected subgroups of $A_n$. Unfortunately, the Euler
transform does not apply to $A_n$-orbits. We test for connectedness a list of representatives of \( \text{Sub}(A_n)/A_n \) in GAP and obtain

\[
A218967 : \ 1, 0, 1, 3, 4, 12, 15, 87, 64, 168, 205, 1336, 1198.
\]

**Remark 1.** There is a sequence in the encyclopedia, \( \text{A116653} \), which currently claims to count both the number of atomic species based on conjugacy classes of subgroups of the alternating group (i.e. the number of \( S_n \)-conjugacy classes of connected subgroups of \( A_n \)) and the number of \( A_n \)-conjugacy classes of connected subgroups of \( A_n \). However this sequence is merely the inverse Euler transform of sequence \( \text{A029726} \), the number of conjugacy classes of subgroups of the alternating group. The number of \( S_n \)-conjugacy classes of connected subgroups of \( A_n \) is sequence \( \text{A218968} \) and the number of \( A_n \)-conjugacy classes of connected subgroups of \( A_n \) is \( \text{A216967} \).

### 4.3 Connected Subgroups with Additional Properties

Appealing to Definition 1 we can count the connected subgroups of \( S_n \) which possess additional group theoretic properties. If the property of interest is compatible with taking direct products we can apply the inverse Euler transform to the sequence of numbers of all conjugacy classes of subgroups of \( S_n \) with this property to obtain the sequence of numbers of conjugacy classes of connected subgroups of \( S_n \) with this property. Table 13 records the number of connected subgroups of \( S_n \) which additionally possess the properties listed in the first row of the table. Each of the sequences in Table 13 is the inverse Euler transform of the corresponding sequence in Table 1.

| n | $|\text{Sub}(S_n)/S_n|$ | \( \text{A000638} \) | \( \text{A218971} \) | \( \text{A218972} \) | \( \text{A218973} \) | \( \text{A218974} \) |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 1 | 1 | 1 | 1 |
| 3 | 4 | 1 | 1 | 2 | 2 |
| 4 | 11 | 3 | 4 | 6 | 4 |
| 5 | 19 | 1 | 1 | 4 | 4 |
| 6 | 56 | 6 | 9 | 23 | 15 |
| 7 | 96 | 1 | 1 | 16 | 13 |
| 8 | 296 | 17 | 69 | 122 | 81 |
| 9 | 554 | 5 | 8 | 109 | 77 |
| 10 | 1593 | 40 | 238 | 551 | 352 |
| 11 | 3094 | 2 | 2 | 570 | 406 |
| 12 | 10723 | 162 | 2339 | 4633 | 2995 |
| 13 | 20832 | 5 | 8 | 4224 | 2866 |

Table 13: Connected Subgroups of \( S_n \)
4.4 Connected Partitions

The number of conjugacy classes of cyclic subgroups of $S_n$ equals the number of partitions of $n$. The inverse Euler transform of this sequence yields the all ones sequence. This does not count the number of conjugacy classes of connected cyclic subgroups of $S_n$ since the direct product of cyclic groups is not necessarily cyclic.

Definition 2. Let $\lambda = [x_1, \ldots, x_l]$ be a partition of $n$. Let $G_\lambda$ be the simple graph with $l$ vertices labeled by $x_1, \ldots, x_l$ where two vertices are connected by an edge if and only if their labels $x_i, x_j$ are not coprime. We call the partition $\lambda$ connected if the graph $G_\lambda$ is connected.

Proposition 1. The number of conjugacy classes of connected cyclic subgroups of $S_n$ equals the number of connected partitions of $n$.

Proof. Let $C = \langle c \rangle \leq S_n$ be cyclic. Then the cycle lengths of the permutation $c$ form a partition $\lambda$ of $n$. For simplicity, we assume $\lambda = [x_1, x_2]$. Then $c = ab$ where $a$ is an $x_1$ cycle and $b$ is an $x_2$ cycle. Let $d = \gcd(x_1, x_2)$. If $d = 1$ then $G_\lambda$ is not connected and $1 = yx_1 + zx_2$, for some $y, z \in \mathbb{Z}$. Then $c^{y x_1} = a, c^{z x_2} = b$ and $C = \langle a \rangle \times \langle b \rangle$ is not connected. If $d > 1$ then $C$ does not contain a generator of $\langle a \rangle$ or of $\langle b \rangle$. The case for general $\lambda$ follows by a similar argument.

Example 2. There are 3 connected partitions $\lambda$ of 13. Their graphs $G_\lambda$ are shown in Figure 6.

![Figure 6: The Graphs of the Connected Partitions of 13](image)

Using Proposition 1 we obtain the sequence of numbers of conjugacy classes of connected cyclic subgroups of $S_n$

$$A218970: 1, 1, 1, 2, 1, 4, 1, 5, 3, 8, 2, 14, 3.$$  

Remark 2. There are two sequences in the encyclopedia which are quite similar to this sequence. Sequence $A018783$ counts the number of partitions of $n$ into parts all of which have a common factor greater than 1. Sequence $A200976$ counts the number of partitions of $n$ such that each pair of parts (if any) has a common factor greater than 1. For $n \leq 13$, sequence $A218970$ above differs from both of these sequences when $n = 1, 11, 13$. 

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5 Concluding Remarks

The sequences presented in this article have been computed using GAP. A GAP file containing the programs can be found at www.maths.nuigalway.ie/~liam/CountingSubgroups.g. The GAP table of marks library Tomlib can be found here [11] and is a requirement for computing many of the sequences presented. It is worth pointing out that Holt has determined all conjugacy classes of subgroups of $S_n$ for values of $n$ up to and including $n = 18$, (see [5]). The majority of the sequences presented in this article rely on the availability of the table of marks of $S_n$ and so we restrict our attention to $n \leq 13$. We are grateful to Des MacHale for suggesting many of the sequences that we compute in this article. We thank the anonymous referees for helpful suggestions.
Appendix A  Additional Sequences

Using the methods described in this article the following additional sequences have been computed.

| n | |Sub(A_n)/A_n| A029726 | A218934 | A218935 | A218936 | A218937 | A218938 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 4 | 5 | 4 | 3 | 4 | 5 | 4 | 4 | 4 |
| 5 | 9 | 5 | 4 | 5 | 8 | 7 | 7 | 7 |
| 6 | 22 | 9 | 6 | 10 | 19 | 14 | 14 | 14 |
| 7 | 40 | 128 | 8 | 13 | 33 | 22 | 22 | 22 |
| 8 | 137 | 30 | 12 | 53 | 122 | 70 | 70 | 70 |
| 9 | 223 | 41 | 17 | 69 | 192 | 122 | 122 | 122 |
| 10 | 430 | 60 | 23 | 122 | 364 | 225 | 225 | 225 |
| 11 | 788 | 81 | 29 | 160 | 650 | 395 | 395 | 395 |
| 12 | 2537 | 193 | 40 | 734 | 2194 | 1240 | 1240 | 1240 |
| 13 | 4558 | 243 | 52 | 848 | 3845 | 2185 | 2185 | 2185 |

Table 14: Conjugacy classes of subgroups of A_n

| n | |Sub(S_n)| A005432 | A062297 | A051625 | A218939 | A218940 | A218941 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 6 | 5 | 5 | 5 | 6 | 6 | 6 | 6 |
| 4 | 30 | 21 | 17 | 24 | 30 | 28 | 28 | 28 |
| 5 | 156 | 87 | 67 | 102 | 154 | 144 | 144 | 144 |
| 6 | 1455 | 612 | 362 | 837 | 1429 | 1259 | 1259 | 1259 |
| 7 | 11300 | 3649 | 2039 | 5119 | 11065 | 9560 | 9560 | 9560 |
| 8 | 151221 | 35515 | 14170 | 78670 | 148817 | 123102 | 123102 | 123102 |
| 9 | 1694723 | 289927 | 109694 | 664658 | 166797 | 1371022 | 1371022 | 1371022 |
| 10 | 29594446 | 3771118 | 976412 | 13514453 | 29103894 | 23449585 | 23449585 | 23449585 |
| 11 | 404126228 | 36947363 | 8921002 | 137272713 | 396571224 | 317178020 | 317178020 | 317178020 |
| 12 | 10594925360 | 657510251 | 101134244 | 4919721831 | 10450152905 | 8296640115 | 8296640115 | 8296640115 |
| 13 | 175238308453 | 7736272845 | 1104940280 | 60598902665 | 172658168937 | 136245390535 | 136245390535 | 136245390535 |

Table 15: Total number of subgroups of S_n
| n  | A218955 | A218956 | A218957 | A218958 | A218959 |
|----|---------|---------|---------|---------|---------|
| 1  | 1       | 1       | 1       | 1       | 1       |
| 2  | 1       | 1       | 1       | 1       | 1       |
| 3  | 1       | 1       | 4       | 4       | 4       |
| 4  | 1       | 7       | 11      | 13      | 7       |
| 5  | 21      | 31      | 51      | 31      | 31      |
| 6  | 76      | 101     | 241     | 246     | 211     |
| 7  | 456     | 491     | 1506    | 1296    | 1156    |
| 8  | 1956    | 3011    | 9649    | 10774   | 5419    |
| 9  | 12136   | 18467   | 80281   | 83238   | 40027   |
| 10 | 80836   | 114983  | 640741  | 788820  | 348331  |
| 11 | 807676  | 1283723 | 6196576 | 6835170 | 3204796 |
| 12 | 8779816 | 13380643| 66883411| 81364944| 38422891|
| 13 | 104127596| 14832163| 775421219| 848378532| 467645179|

Table 16: Total number of maximal property-P subgroups of $S_n$

| n  | A029725 | A218942 | A051636 | A218943 | A218944 | A218945 |
|----|---------|---------|---------|---------|---------|---------|
| 1  | 1       | 1       | 1       | 1       | 1       | 1       |
| 2  | 1       | 1       | 1       | 1       | 1       | 1       |
| 3  | 2       | 2       | 2       | 2       | 2       | 2       |
| 4  | 10      | 9       | 8       | 9       | 10      | 9       |
| 5  | 59      | 37      | 32      | 37      | 58      | 53      |
| 6  | 501     | 207     | 167     | 252     | 488     | 418     |
| 7  | 3786    | 1192    | 947     | 1507    | 3664    | 2894    |
| 8  | 48337   | 11449   | 6974    | 21739   | 47210   | 33675   |
| 9  | 508402  | 93673   | 53426   | 186983  | 498102  | 369763  |
| 10 | 6469142 | 892783  | 454682  | 2369258 | 6293475 | 4769542 |
| 11 | 81711572| 8534308 | 4303532 | 22872863| 78805290| 58853842|
| 12 | 2019160542| 148561283| 50366912| 746597568| 1960342409| 1395051100|
| 13 | 31945830446| 1740198891| 553031624| 9157758326| 31130243721| 21847262156|

Table 17: Total no of subgroups of $A_n$
Table 18: Maximal property-P subgroups of $A_n$

| \(n\) | Solvable | SupSol | Abelian | Cyclic | Nilpotent |
|-------|----------|--------|---------|--------|-----------|
| 1     | 1        | 1      | 1       | 1      | 1         |
| 2     | 1        | 1      | 1       | 1      | 1         |
| 3     | 1        | 1      | 1       | 1      | 1         |
| 4     | 1        | 2      | 2       | 2      | 2         |
| 5     | 3        | 3      | 3       | 3      | 3         |
| 6     | 4        | 3      | 5       | 4      | 3         |
| 7     | 5        | 4      | 6       | 5      | 5         |
| 8     | 6        | 6      | 13      | 6      | 6         |
| 9     | 10       | 8      | 19      | 8      | 7         |
| 10    | 12       | 10     | 22      | 10     | 9         |
| 11    | 14       | 13     | 27      | 14     | 12        |
| 12    | 17       | 18     | 40      | 20     | 17        |
| 13    | 24       | 22     | 54      | 24     | 20        |

Table 19: Connected subgroups of $A_n$

| \(n\) | \(|\text{Sub}(A_n)/A_n|\) | Abelian | Nilpotent | Solvable | SupSol |
|-------|-----------------------------|---------|-----------|----------|--------|
| 1     | 1                           | 1       | 1         | 1        | 1      |
| 2     | 1                           | 0       | 0         | 0        | 0      |
| 3     | 2                           | 1       | 1         | 1        | 1      |
| 4     | 5                           | 2       | 2         | 3        | 2      |
| 5     | 9                           | 1       | 1         | 3        | 3      |
| 6     | 22                          | 3       | 4         | 10       | 6      |
| 7     | 40                          | 1       | 1         | 11       | 6      |
| 8     | 137                         | 14      | 36        | 80       | 42     |
| 9     | 223                         | 5       | 9         | 52       | 39     |
| 10    | 430                         | 12      | 49        | 145      | 85     |
| 11    | 788                         | 2       | 2         | 165      | 104    |
| 12    | 2537                        | 69      | 489       | 1208     | 686    |
| 13    | 4558                        | 3       | 4         | 1033     | 617    |

The number of connected even partitions of $n$

\(A_{218975} : 1, 0, 1, 1, 1, 2, 1, 3, 3, 4, 2, 8, 2.\)
| n  | Solvable | SupSol | Abelian | Cyclic | Nilpotent |
|----|----------|--------|---------|--------|-----------|
| 1  | 1        | 1      | 1       | 1      | 1         |
| 2  | 1        | 1      | 1       | 1      | 1         |
| 3  | 3        | 3      | 3       | 3      | 3         |
| 4  | 1        | 10     | 10      | 9      | 10        |
| 5  | 36       | 40     | 30      | 30     | 30        |
| 6  | 225      | 110    | 115     | 100    | 110       |
| 7  | 686      | 645    | 861     | 665    | 1001      |
| 8  | 4655     | 5670   | 10536   | 3885   | 4005      |
| 9  | 28728    | 47754  | 78474   | 33093  | 45696     |
| 10 | 397005   | 311850 | 1008000 | 371700 | 379155    |
| 11 | 2210890  | 3014550| 9302964 | 3790875| 4913040   |
| 12 | 26975025 | 24022845| 73024380| 37839285| 36701280 |
| 13 | 26121667 | 46950904| 563291872| 350984414| 158538380|

Table 20: Total number of maximal property-P subgroups of $A_n$

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2010 Mathematics Subject Classification: Primary 20B40; Secondary 20D30, 19A22.

*Keywords:* symmetric group, alternating group, table of marks, subgroup pattern.

(Concerned with sequences A218909, A218910, A218911, A218912, A218913, A218914, A218915, A218916, A218917, A218918, A218919, A218920, A218921, A218922, A218923, A218924, A218925, A218926, A218927, A218928, A218929, A218930, A218931, A218932, A218933, A218934, A218935, A218936, A218937, A218938, A218939, A218940, A218941, A218942, A218943, A218944, A218945, A218946, A218947, A218948, A218949, A218950, A218951, A218952, A218953, A218954, A218955, A218956, A218957, A218958, A218959, A218960, A218961, A218962, A218963, A218964, A218965, A218966, A218967, A218968, A218969, A218970, A218971, A218972, A218973, A218974, A000638, A029726, A005432, A029725, A005226, A116653, A018783 and A200976).