Converting exhausters and coexhausters

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Abstract

Exhausters and coexhausters are notions of constructive nonsmooth analysis which are used to study extremal properties of functions. An upper exhauster (coexhauster) is used to get an approximation of a considered function in the neighborhood of a point in the form of min max of linear (affine) functions. A lower exhauster (coexhauster) is used to represent the approximation in the form of max min of linear (affine) functions. Conditions for a minimum in a most simple way are expressed by means of upper exhausters and coexhausters, while conditions for a maximum are described in terms of lower exhausters and coexhausters. Thus the problem of obtaining an upper exhauster or coexhauster when the lower one is given and vice versa arises. We study this problem in the paper and propose new method for its solution which allows one to pass easily between min max and max min representations.

1 Introduction

To study extremal properties of a function we usually approximate it with some function from a certain class. In terms of the approximations optimality conditions and optimization algorithms are built. This is a natural way which is used for smooth functions. Therefore it is logical to apply the same idea in a nonsmooth case, however there exist other approaches that do not use this concept. For example we can mention Shor [1], Clarke [2,3], Mordukhovich [4,5], Michel–Penot [6] subdifferentials, approximate and geometric Ioffe subdifferentials [7], Varga containers [8]. Detailed analysis of various approaches to the problem is given in [9].

In 1980-th Demyanov Rubinov and Polyakova introduced notion of quasidifferentials [10]. These are pairs of convex compact sets which is used to represent directional derivative and approximation of a function at a point. The emergence of quasidifferentials laid the foundation for constructive nonsmooth analysis. A calculus was developed, which formulas let researchers to obtain quasidifferentials for a wide class of nonsmooth
functions. Optimality conditions were formulated in terms of quasidifferentials as well as the procedures of finding directions of a steepest descent and ascent when these conditions are not satisfied [16]. Thereafter the theory of quasidifferentials progressed rapidly due to the many significant studies in the area [17–24].

Subsequently exhausters notion appeared as an attempt to expand the class of studied functions. It was introduced by Demyanov [25, 26] and is based on the ideas of Pshenichny [27] and Rubinov [28, 29]. Lower and upper exhausters are families of convex compact sets which are used to describe the directional derivative and approximation of a function at a point in the form of maxmin and minmax of linear functions correspondingly. Since calculus and optimality conditions have been derived in terms of exhausters too, this concept retains the constructiveness of quasidifferentials. At the same time the class of exhausterable functions is wider than the class of quasidifferentiable ones. Any quasidifferentiable functions has exhausters but the opposite is not true.

It turned out that conditions for the minimum most organically are expressed via an upper exhauster while conditions for a maximum via a lower one. Therefore an upper exhauster is called proper for the minimization problems and adjoint for the maximization ones while a lower exhauster is called proper for the maximization and adjoint for the minimization problems. Therefore when having adjoint exhauster we can either convert it to get a proper exhauster or work with the adjoint exhauster itself.

The latter requires that optimality conditions to be derived in terms of adjoint exhausters. Roshchina was the first who obtained these results [30, 31]. Later this problem was considered in works of Abbasov [32, 33] where the conditions were stated and proved in a simple and geometrically transparent form, which allows one to get directions of the steepest descent and ascent by means of adjoint exhausters.

The procedure of exhausters converting was described in [26]. It is applicable only to two-dimensional cases. So the problem of describing more general procedure for exhausters converting is still opened. The main aim of this paper is to solve this problem.

The fact that exhauster set-valued mappings are not, in general, continuous in Hausdorff metric leads to the convergence problems of the algorithms which employ this objects. To overcome these drawbacks coexhausters notion was introduced in [26]. These are families of convex compact sets which describe nonhomogeneous approximations of a function at a point in the form of maxmin and minmax of affine functions. All the results obtained for exhausters were generalized and described for coexhausters, but the problem of converting coexhausters is opened too. One can work with continuous coexhauster set-valued mappings which guarantee stability and convergence of numerical algorithms.

The paper is organized as follows: in Section 2 we give definitions of exhausters and coexhausters and discuss existing procedure for converting of these families; in Section 3 we state and prove the main results and describe new converting procedure; illustrative examples are provided in Section 4; concluding remarks are presented in Section 5.
2 Exhausters and coexhausters. Converting procedure.

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a directionally differentiable function and \( h(\Delta) = f'(x, \Delta) \) be the derivative of the function \( f \) at a point \( x \) in a direction \( \Delta \).

Fix \( x \in \mathbb{R}^n \). M. Castellani (see [35]) proved that, if \( h \) is Lipschitz, then there exist families of convex and compact sets \( E^* \) and \( E_* \) in the space \( \mathbb{R}^n \) such that \( h(\Delta) \) can be written in the form

\[
h(\Delta) = h_1(\Delta) = \min_{C \in E^*} \max_{w \in C} \langle w, \Delta \rangle, \quad \forall \Delta \in \mathbb{R}^n, \quad (1)
\]

and in the form

\[
h(\Delta) = h_2(\Delta) = \max_{C \in E_*} \min_{w \in C} \langle w, \Delta \rangle, \quad \forall \Delta \in \mathbb{R}^n. \quad (2)
\]

The family of sets \( E^* \) is called an upper exhauster of the function \( f \) at the point \( x \), while the family \( E_* \) is called a lower exhauster of the function \( f \) at the point \( x \).

For an arbitrary p.h. function \( h \) represented in the form (1), the family \( E_* \) is called an upper exhauster of the function \( h \). If (2) holds then the family \( E_* \) is called a lower exhauster of \( h \).

Exhausters were introduced in [25, 26, 29]. The converting procedure was suggested by Demyanov in [26]. Let \( E \subset \mathbb{R}^n \) be a family of convex compacts in \( \mathbb{R}^n \), which is totally bounded, i.e. there exists an \( r < \infty \), such that

\[
C \subset B_r(0) \quad \forall C \in E.
\]

For any \( \Delta \in \mathbb{R}^n \) such that \( \|\Delta\| = 1 \) build

\[
\tilde{C}(\Delta) = \mathrm{cl} \, \mathrm{co} \left\{ w(C) \in C \mid \langle w(C), \Delta \rangle = \min_{w \in C} \langle w, \Delta \rangle, \ C \in E \right\}.
\]

Then the family

\[
\tilde{E} = \left\{ C = \tilde{C}(\Delta) \mid \Delta \in \mathbb{R}^n, \ |\Delta| = 1 \right\}
\]

is converted family, i.e. if \( E \) were a lower exhauster of the function \( h \), then \( \tilde{E} \) is an upper exhauster of \( h \) and vise versa.

Some problems addressing converting procedure were considered in [36, 37].

Let a function \( f \) be continuous at a point \( x \in X \). We say that at the point \( x \) the function \( f \) has an upper coexhauster in the sense iff the following expansion holds:

\[
f(x + \Delta) = f(x) + \min_{C \in \overline{E}(x)} \max_{(a, v) \in C} [a + \langle v, \Delta \rangle] + o_x(\Delta),
\]

where \( \overline{E}(x) \) is a family of convex compact sets in \( \mathbb{R}^{n+1} \), and \( o_x(\Delta) \) satisfies

\[
\lim_{\alpha \to 0} \frac{o_x(\alpha \Delta)}{\alpha} = 0 \quad \forall \Delta \in \mathbb{R}^n. \quad (3)
\]

The set \( \overline{E}(x) \) is called an upper coexhauster of \( f \) at the point \( x \).

We say that at the point \( x \) the function \( f \) has a lower coexhauster iff the following expansion holds:

\[
f(x + \Delta) = f(x) + \max_{C \in \overline{E}(x)} \min_{(b, w) \in C} [b + \langle w, \Delta \rangle] + o_x(\Delta),
\]
where \( E(x) \) is a family of convex compact sets in \( \mathbb{R}^{n+1} \), and \( o_x(\Delta) \) satisfies (3).

The set \( E(x) \) is called a lower coexhauster of the function \( f \) at the point \( x \).

Due to the fact that \( f \) is continuous it is obvious that the following equality holds

\[
\min_{C \in \overline{E}(x)} \max_{[a, v] \in C} a = \max_{C \in \overline{E}(x)} \min_{[b, w] \in C} b = 0
\]

for an upper and a lower coexhauster at any \( x \). Therefore we can deal with the approximations of \( f \) itself, i.e.

\[
h(\Delta) = h_3(\Delta) = \min_{C \in \overline{E}(x)} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle], \quad \forall \Delta \in \mathbb{R}^n,
\]

and

\[
h(\Delta) = h_4(\Delta) = \max_{C \in \overline{E}(x)} \min_{[b, w] \in C} [b + \langle w, \Delta \rangle], \quad \forall \Delta \in \mathbb{R}^n,
\]

Hereinafter we consider the families \( \overline{E} \) and \( E \) as an upper and lower coexhausters of the function \( h \) correspondingly.

The notion of coexhauster was introduced in [25, 26].

If a family \( \overline{E} \subset 2^{\mathbb{R}^{n+1}} \) of convex compact sets is a totally bounded lower coexhauster of \( h \), then the family

\[
\tilde{E} = \{ C = C^o(g) \mid g \in \mathbb{R}^{n+1}, \| g \| = 1, \ g_i \geq 0 \},
\]

where

\[
C^o(g) = \text{cl co} \left\{ w(C) \in C \mid \langle w(C), g \rangle = \min_{[b, w] \in C} \langle [b, w], g \rangle, \ C \in \overline{E} \right\}.
\]

is an upper coexhauster of the function \( h \).

If a family \( E \subset 2^{\mathbb{R}^n} \) of convex compact sets is a totally bounded upper coexhauster of \( h \), then the family

\[
\bar{E} = \{ C = C_o(g) \mid g \in \mathbb{R}^{n+1}, \| g \| = 1, \ g_i \geq 0 \},
\]

where

\[
C_o(g) = \text{cl co} \left\{ v(C) \in C \mid \langle v(C), g \rangle = \max_{[a, v] \in C} \langle [a, v], g \rangle, \ C \in \overline{E} \right\}.
\]

is a lower coexhauster of \( h \).

For a wide class of functions which have exhausters and coexhausters these families consists of finite number of convex polytopes. In what follows we will consider only this case.

The described converting methods imply only graphical use, that is, the construction of converted families by means of visual geometric illustrations. However, this is possible only for low-dimensional problems.

### 3 New method for converting exhausters and coexhausters

We will need the following auxiliary result.
Theorem 1. Let $D = \{d_{ij}\}_{k \times p}$ be a matrix in $\mathbb{R}^{k \times p}$ and there exists $\vec{j} \in 1, \ldots, p$ such that for all $i \in 1, \ldots, k$ we have

$$d_{i\vec{j}} = \max_{j \in 1, \ldots, p} d_{ij}.$$  

Then the following equation

$$\min_{i \in 1, \ldots, k} \max_{j \in 1, \ldots, p} d_{ij} = \max_{j \in 1, \ldots, p} \min_{i \in 1, \ldots, k} d_{ij}.$$  

holds.

Proof. Denote by $i$ the index on which the minimum $\min_{i \in 1, \ldots, k} d_{ij}$ is attained, i.e. $\min_{i \in 1, \ldots, k} d_{ij} = d_{i\vec{j}}$. Then we have

$$\min_{i \in 1, \ldots, k} \max_{j \in 1, \ldots, p} d_{ij} = d_{i\vec{j}}.$$  

Considering condition (4) we get the chain of inequalities

$$\min_{i \in 1, \ldots, k} d_{ij} \leq d_{i\vec{j}} \leq \max_{j \in 1, \ldots, p} d_{ij} = d_{i\vec{j}}$$

which are true for any $j \in 1, \ldots, p$ and therefore we conclude that

$$\max_{j \in 1, \ldots, p} \min_{i \in 1, \ldots, k} d_{ij} \leq d_{i\vec{j}}.$$  

Since $\min_{i \in 1, \ldots, k} d_{ij} = d_{i\vec{j}}$ we obtain

$$\max_{j \in 1, \ldots, p} \min_{i \in 1, \ldots, k} d_{ij} \geq d_{i\vec{j}}.$$  

From (7) and (8) we get that

$$\max_{j \in 1, \ldots, p} \min_{i \in 1, \ldots, k} d_{ij} = d_{i\vec{j}},$$

whence recalling (6) we obtain (5). □

Similarly we can state and prove the following result.

Theorem 2. Let $D = \{d_{ij}\}_{k \times p}$ be a matrix in $\mathbb{R}^{k \times p}$ and there exists $\vec{j} \in 1, \ldots, p$ such that for all $i \in 1, \ldots, k$ we have

$$d_{i\vec{j}} = \min_{j \in 1, \ldots, p} d_{ij}.$$  

Then the following equation

$$\max_{i \in 1, \ldots, k} \min_{j \in 1, \ldots, p} d_{ij} = \min_{j \in 1, \ldots, p} \max_{i \in 1, \ldots, k} d_{ij}.$$  

holds.

Theorems 1 and 2 can be used in procedure of converting exhausters and coexhausters.

First consider the problem of obtaining a lower exhauster from an upper one.

Theorem 3. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function such that $h(\Delta) = \min_{C \in E^*} \max_{v \in C} \langle v, \Delta \rangle$ and $E^*$ is a finite family of convex compact sets from $\mathbb{R}^n$, where $E^* = \{C_i | i = 1, \ldots, k\}$, $C_i = \text{co}\{v_{ij} | j = 1, \ldots, m_i\}$. Then the family $E^*$ which contains $p = m_1 m_2 \ldots m_k$ sets of the form

$$C = \text{co}\{v_{ij} | i = 1, \ldots, k, j_i \in 1, \ldots, m_i\}$$

is a lower exhauster of the function $h$. 

Proof. It is obvious that \( \bar{E}^* = \{ \bar{C}_j \mid j = 1, \ldots, p \} \), where \( \bar{C}_j = \text{co}\{ \bar{v}_{ij} \mid i \in 1, \ldots, k \} \) and \( \bar{v}_{ij} \) is some vertex of the set \( C_i \).

Choose an arbitrary \( \Delta \in \mathbb{R}^n \). Consider the matrix \( D(\Delta) = \{ d_{ij} \}_{k \times p} \), which is composed of columns consisting of inner products of vertices of all of the sets of the family \( E^* \) and \( \Delta \), i.e. \( d_{ij} = [\bar{v}_{ij}, \Delta] \). From the way we constructed the family \( \bar{E}^* \) it is obvious that there exists \( \bar{\eta} \in 1, \ldots, p \) such that the following condition holds

\[
d_{ij} = \max_{j \in 1, \ldots, p} d_{ij}, \quad \forall i \in 1, \ldots, k.
\]

Hence due to Theorem 5.4 we have

\[
\min_{i \in 1, \ldots, k} \max_{j \in 1, \ldots, p} d_{ij} = \max_{j \in 1, \ldots, p} \min_{i \in 1, \ldots, k} d_{ij}. \tag{11}
\]

Since

\[
\min_{i \in 1, \ldots, k} \max_{j \in 1, \ldots, p} d_{ij} = \min_{C \in E^*} \max_{v \in \bar{C}} \langle v, \Delta \rangle
\]

and

\[
\max_{j \in 1, \ldots, p} \min_{i \in 1, \ldots, k} d_{ij} = \max_{C \in \bar{E}, v \in C} \min_{C} \langle v, \Delta \rangle
\]

equality (11) implies that the family \( \bar{E}^* \) is a lower exhauster of the function \( h \).

Similarly we can state and prove the following results for converting a lower exhauster and upper and lower coexhausters.

**Theorem 4.** Let \( h: \mathbb{R}^n \to \mathbb{R} \) be a function such that \( h(\Delta) = \max_{C \in E} \min_{(v, \bar{C})} \langle v, \Delta \rangle \) and \( E \) is a finite family of convex compact sets from \( \mathbb{R}^n \), where \( E = \{ C_i \mid i = 1, \ldots, k \} \), \( C_i = \text{co}\{ w_{ij} \mid j = 1, \ldots, m_i \} \). Then the family \( E^* \) which contains \( p = m_1m_2 \ldots m_k \) sets of the form

\[
C = \text{co}\{ w_{ij} \mid i \in 1, \ldots, k, \; j_i \in 1, \ldots, m_i \}
\]

is an upper exhauster of the function \( h \).

**Theorem 5.** Let \( h: \mathbb{R}^n \to \mathbb{R} \) be a function such that \( h(\Delta) = \min_{C \in E} \max_{(v, \bar{C})} \langle v, \Delta \rangle \) and \( E \) is a finite family of convex compact sets from \( \mathbb{R}^{n+1} \), where \( E = \{ C_i \mid i = 1, \ldots, k \} \), \( C_i = \text{co}\{ (a_{ij}, v_{ij}) \mid j = 1, \ldots, m_i \} \). Then the family \( E^* \) which contains \( p = m_1m_2 \ldots m_k \) sets of the form

\[
C = \text{co}\{ (a_{ij}, v_{ij}) \mid i \in 1, \ldots, k, \; j_i \in 1, \ldots, m_i \}
\]

is a lower coexhauster of the function \( h \).

**Theorem 6.** Let \( h: \mathbb{R}^n \to \mathbb{R} \) be a function such that \( h(\Delta) = \max_{C \in E} \min_{(v, \bar{C})} \langle v, \Delta \rangle \) and \( E \) is a finite family of convex compact sets from \( \mathbb{R}^{n+1} \), where \( E = \{ C_i \mid i = 1, \ldots, k \} \), \( C_i = \text{co}\{ (b_{ij}, w_{ij}) \mid j = 1, \ldots, m_i \} \). Then the family \( E^* \) which contains \( p = m_1m_2 \ldots m_k \) sets of the form

\[
C = \text{co}\{ (b_{ij}, w_{ij}) \mid i \in 1, \ldots, k, \; j_i \in 1, \ldots, m_i \}
\]

is an upper coexhauster of the function \( h \).
4 Examples

Let us consider some illustrative examples.

**Example 4.1.** Let the function \( h : \mathbb{R}^4 \to \mathbb{R} \), such that

\[
h(\Delta) = \max_{C \in E, \nu \in C} \min(w, \Delta)
\]

be given, where \( E = \{C_1, C_2\} \) and

\[
C_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad C_2 = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}.
\]

Using Theorem 4 we get an upper exhauster of the form \( \tilde{E}_* = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\} \),

\[
\tilde{C}_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad \tilde{C}_2 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\},
\]

\[
\tilde{C}_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}, \quad \tilde{C}_4 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}.
\]

Thus \( h \) can be represented as

\[
h(\Delta) = \min_{C \in E, \nu \in C} \max(v, \Delta).
\]

**Example 4.2.** Consider the function \( h : \mathbb{R}^4 \to \mathbb{R} \),

\[
h(\Delta) = \min_{C \in \Xi, \nu \in C} |a + \langle v, \Delta \rangle|,
\]

where \( \Xi = \{C_1, C_2\} \)

\[
C_1 = \text{co}\{[a_i, v_i] | i = 1, 2, 3\}, \quad C_2 = \{0_4\},
\]

\( a_i = 1 \) for all \( i = 1, 2, 3 \), and \( v_i \) is \( i \)-th standard basis vector, i.e. \( v_i = e_i \) for all \( i = 1, 2, 3 \).

Via Theorem 5 we obtain a lower coexhauster of the form \( \tilde{\Xi} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3\} \),

where

\[
\tilde{C}_i = \text{co}\{[a_i, e_i], 0_4\} \quad \forall i = 1, 2, 3,
\]

and therefore can represent the function \( h \) as

\[
h(\Delta) = \max_{C \in \Xi, \nu \in C} \min \left\{ |b + \langle v, \Delta \rangle| \right\}.
\]

5 Conclusion

Obtained results give a solution for the problem of converting exhausters and coexhausters in cases when these families consist of finite number of convex polytopes. After applying the developed procedure we usually get a family with many redundant sets. These sets can be discarded via various reduction techniques and methods presented in \([35][31]\).
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