Stability of the
Rotating Ellipsoidal D0-brane System

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Abstract
In this note we prove the complete stability of the classical fluctuation modes of
the rotating ellipsoidal membrane. The analysis is carried out in the full $SU(N)$
setting, with the conclusion that the fluctuation matrix has only positive eigenvalues.
This proves that the solution will remain close to the original one for all time, under
arbitrary infinitesimal perturbations of the gauge fields.

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1 Introduction

The purpose of this note is to make a complete analysis of the gauge field fluctuations in the neighbourhood of the rotating ellipsoidal membrane solution of [2]. We extend the previous treatment in [2] whereby only perturbations that do not modify the original $SU(2)$ ansatz to the case when perturbations are in the full $SU(N)$ algebra. The results indicate that in the case of $SU(2)$ most of the modes display the enhanced symmetry of the original solution, i.e. after the imposition of the constraint most of the additional degrees of freedom are zero-modes. All the other modes, for the totality of all possible gauge field perturbations in $SU(N)$, are completely stable and execute harmonic oscillations around the original trajectory.

The effective action of $N$ D0-branes for weak and slowly varying fields is the non-abelian $SU(N)$ Yang-Mills action plus the Chern-Simons action (for the bosonic part). For weak fields the action is gotten by dimensionally reducing the action of 9+1 dimensional $U(N)$ Super Yang-Mills theory to 0+1 dimensions [1]. Up to a constant term it is

$$ S = - T_0 (2\pi l_s)^2 \int dt \; \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), $$

where $F_{\mu\nu}$ is the non-abelian $U(N)$ field strength in the adjoint representation and $T_0 = (g_s l_s)^{-1}$ is the D0-brane mass. To write this action in terms of coordinate matrices $X^i$, one has to use the dictionary

$$ A_i = \frac{1}{2\pi l_s^2} X^i, \quad F_{0i} = \frac{1}{2\pi l_s^2} \dot{X}^i, \quad F_{ij} = \frac{-i}{(2\pi l_s^2)^2} [X^i, X^j] $$

with $i, j = 1, 2, ..., 9$, giving

$$ S = T_0 \int dt \; \text{Tr} \left( \frac{1}{2} \dot{X}^i \dot{X}^i + \frac{1}{4 (2\pi l_s^2)^2} [X^i, X^j][X^i, X^j] \right) $$

To derive this it is necessary to gauge the $A_0$ potential away, which is possible for a non-compact time. The equations of motion

$$ \ddot{X}^i = - [X^j, [X^j, X^i]] $$

should be taken together with the Gauss constraint,

$$ \left[ \dot{X}^i, X^i \right] = 0 $$
which is preserved by (4).

Introduce the $N \times N$ matrices $L_1, L_2, L_3$ as the generators of the $N$ dimensional irreducible representation of $SU(2)$, with algebra

$$[L_i, L_j] = i \epsilon_{ijk} L_k.$$  

(6)

The rotating spherical membrane solution was constructed using this subalgebra of the $SU(N)$. It is also the only finite dimensional subalgebra of the group of diffeomorphisms of $S^2$, the Sdiff($S^2$). That is why the $SU(2)$ ansatz is in some sense unique: it is the only type of solution that carries over to the supermembrane without modification

$$X^i(t) = L_i R \cos(\omega t)$$

$$\tilde{X}^i(t) = X^{i+3}(t) = L_i R \sin(\omega t)$$

(7) (8)

In what follows we will set $R = 1$, with $\omega$ being determined through the equations of motion, and equal to $\omega^2 = 2R^2 = 2$.

In order to treat the perturbations of the system within the full $SU(N)$ we will need a convenient basis, provided [4, 6] by the spherical operators $Y^l_m$,

$$[L_z, Y^l_m] = m Y^l_m$$

for $l = 1, \ldots, N - 1$

$$[L_{\pm}, Y^l_m] = \sqrt{(l \mp m)(l \pm m + 1)} Y^l_{m \pm 1}$$

(9)

It is immediately clear that these are not hermitian, neither they are anti-hermitian. For example, at $l = 1$ these should coincide with the $L_i$’s:

$$Y^1_0 = L_z , \quad Y^1_{\pm 1} = \mp \frac{1}{\sqrt{2}} L_{\pm}.$$  

(10)

For $l = 2$ these are the five linear independent symmetric traceless products of pairs of $L$’s, and with correct normalization read:

$$Y^2_0 = \sqrt{\frac{2}{3}} (L_x L_x + L_y L_y - L_z L_z)$$

$$Y^2_{\pm 1} = \pm \{L_{\pm}, L_z\}$$

$$Y^2_{\pm 2} = (L_x L_x + L_y L_y) \mp 2i \{L_x, L_y\}$$

(11)
The general explicit construction of the $Y^l_m$ is due to Schwinger, and was used to show the correspondence at $N \mapsto \infty$ between the relativistic membrane and $SU(N)$ YM in [4, 5, 6]. We will not use the explicit form of these matrices, as the defining relations (8) is all that is needed. The properties under Hermitian conjugation can be summed up as

$$Y^\dagger_{l,m} = (-1)^m Y_{l,-m}$$

(12)

2 Internal Perturbations

In this section perturbations will be considered that are parallel in space to one of the directions of the system. It is a generalization of our previous treatment, in [2], where perturbations of the same structure as the ansatz were considered. The case of completely transverse perturbations is considered in the next section. The details are rather technical, but completely straightforward.

Let us decompose the fluctuation fields in the basis defined by $Y^l_m$, where $l$ runs from 1 to $N - 1$

$$\delta X^i = \sum_{m=-l}^l Y^l_m \xi^i_m , \quad \delta \tilde{X}^i = \sum_{m=-l}^l Y^l_m \eta^i_m \quad i = 1, 2, 3.$$  

(13)

The total number of modes is then $\sum_{l=1}^{N-1} (2l + 1) = N^2 - 1$ as it should for $SU(N)$.

We do not introduce an $l$ index on the $\eta, \xi$ because it will be shown below, and also suggested in [7], that the fluctuations with different $l$ do not couple at linear order. The behaviour of linear perturbations is sufficient to establish the correct phase portrait of the dynamical system in the neighbourhood of a periodic trajectory.

Even though the basis is not Hermitian, the gauge field should still be real, and so we should impose

$$\xi^*_m = (-1)^m \xi_{-m} \quad \text{and} \quad \eta^*_m = (-1)^m \eta_{-m} \quad \text{for all} \quad m = -l, \ldots, l .$$  

(14)

The variational equations of motion are

$$-\delta \ddot{X}^i = \left[ \delta X^j, [X^j, X^i] \right] + [X^j, [\delta X^j, X^i]] + [X^j, [X^j, \delta X^i]] + [\delta \tilde{X}^j, [\tilde{X}^j, X^i]] + [\tilde{X}^j, [\delta \tilde{X}^j, X^i]] + [\tilde{X}^j, [\tilde{X}^j, \delta X^i]] .$$  

(15)
The linearized constraint equation looks like
\[
\sum_{i,m} \left[ \delta \dot{X}^i, X^i \right] + \left[ \dot{X}^i, \delta X^i \right] + \left[ \delta \ddot{X}^i, X^i \right] + \left[ \ddot{X}^i, \delta X^i \right] = 0 \quad (16)
\]

Using the commutation relations (9) we get for the constraint
\[
\sum_i L^i_{nm} \left( \cos(\omega t) \dot{\xi}^i_m + \omega \sin(\omega t) \xi^i_m + \sin(\omega t) \dot{\eta}^i_m - \omega \cos(\omega t) \eta^i_m \right) = 0 \quad , \quad (17)
\]
where \( L^i_{nm} \) are now the \( SU(2) \) generators in the \((2l + 1) \times (2l + 1)\) representation. In the co-moving coordinates
\[
u^i_m = \cos(\omega t) \xi^i_m + \sin(\omega t) \eta^i_m \quad \text{and} \quad \nu^i_m = -\sin(\omega t) \xi^i_m + \cos(\omega t) \eta^i_m \quad (18)
\]
the constraint looks simpler,
\[
\sum_{i,m} L^i_{nm} \left( \ddot{u}^i_m - 2\omega v^i_m \right) = 0 \quad . \quad (19)
\]
The variational equation of motion (15) after substituting the fields (13) is
\[
- \sum_m Y^l_{m} \dot{\xi}^i_m = \sum_{j,k,m} - \cos(\omega t) \cos(\omega t) i \epsilon_{ijk} \left[ Y^l_{m}, L^j_{k} \right] \xi^j_m
+ \cos(\omega t) \cos(\omega t) \left[ L^j_{j} \left[ Y^l_{m}, L^j_{i} \right] \right] \xi^j_m
+ \cos(\omega t) \cos(\omega t) \left[ L^j_{j} \left[ L^l_{j}, L^j_{i} \right] \right] \xi^j_m
- \cos(\omega t) \sin(\omega t) i \epsilon_{ijk} \left[ Y^l_{m}, L^j_{k} \right] \eta^j_m
+ \cos(\omega t) \sin(\omega t) \left[ L^j_{j} \left[ Y^l_{m}, L^j_{i} \right] \right] \eta^j_m
+ \sin(\omega t) \sin(\omega t) l(l + 1) Y^l_{m} \eta^l_m \quad (20)
\]
and in component form,
\[
- \ddot{\xi}^i_n = \cos(\omega t) \left( i \epsilon_{ijk} L^k_{nm} - L^j_{nm}, L^j_{nm} \right) \left( \cos(\omega t) \xi^j_m + \sin(\omega t) \eta^j_m \right) + l(l + 1) \xi^i_n \quad . \quad (21)
\]
The decoupling of the modes with different \( l \) is seen to be a direct consequence of (9), and more fundamentally, of the pure \( SU(2) \) structure of the original background solution.

The above can be conveniently rewritten as
\[
\ddot{\xi}^i_n + l(l + 1) \xi^i_n = \cos(\omega t) \left( L^j_{nm}, L^j_{nm} + i \epsilon_{ijk} L^k_{nm} \right) \left( \cos(\omega t) \xi^j_m + \sin(\omega t) \eta^j_m \right) \quad . \quad (22)
\]
The equation for $\eta$ is gotten by exchanging cosines for sines and $\xi$ for $\eta$

$$\eta_n^i + l(l + 1) \eta_n^i = \sin(\omega t) \left( L_{nn}^j L_{n'm}^i + i \epsilon_{jik} L_{n'mm}^k \right) \left( \cos(\omega t) \xi_{nm}^i + \sin(\omega t) \eta_{nm}^i \right). \quad (23)$$

In the co-moving coordinates $\ref{18}$ the time dependency drops out, and the equation becomes a linear system with constant coefficients:

$$\ddot{u}_n^i + (l(l + 1) - 2) u_n^i - 2\omega \dot{v}_n^i = \left( L_{nn}^j L_{n'm}^i + i \epsilon_{jik} L_{n'mm}^k \right) u_m^j, \quad (24)$$

$$\ddot{v}_n^i + (l(l + 1) - 2) v_n^i + 2\omega \dot{u}_n^i = 0. \quad (25)$$

Thus we shall analyze the system of equations $\ref{19}$ $\ref{24}$, $\ref{25}$. In order to display explicitly the constant matrix structure of the equation, one has to write the rhs of $\ref{24}$ as a matrix acting on a $3(2l+1)$ component vector

$$\begin{pmatrix}
L_1^3 L_1^1 & L_2^3 L_1^1 - i L_3 L_1^3 & L_3^3 L_1^1 + i L_2^3 \\
L_1^3 L_2^1 + i L_3^3 & L_2^3 L_2^1 & L_3^3 L_2^1 - i L_1^3 \\
L_1^3 L_3^1 - i L_2^3 & L_2^3 L_3^1 + i L_1^3 & L_3^3 L_3^1
\end{pmatrix}
\begin{pmatrix}
u_1^i \\
u_2^i \\
u_3^i
\end{pmatrix}. \quad (26)$$

The eigenvalues $\Lambda$ of this size $3(2l+1)$ block matrix, where each block is of size $(2l+1)$, are given in the table, together with their multiplicity

| $\Lambda$       | multiplicity |
|-----------------|--------------|
| $-2l - 2$       | $2l - 1$     |
| $l(l + 1) - 2$  | $2l + 1$     |
| $2l$            | $2l + 3$     |

One can check that the trace of the matrix matches with the weighted sum of the eigenvalues given above for arbitrary values of $l$. The proof will be published elsewhere.

Since the second equation $\ref{25}$ is completely diagonal with respect to $i, l, m$, we can now solve the complete system. Choose a fixed frequency ansatz

$$u_n^i(t) = e^{i\Omega t} u_n^i, \quad v_n^i(t) = e^{i\Omega t} v_n^i. \quad (28)$$

The second equation $\ref{25}$ can be solved as

$$v_n^i = \frac{-2\sqrt{2} i \Omega^2}{l(l + 1) - 2 - \Omega^2} u_n^i. \quad (29)$$
and substituted back into the first equation (24),

$$- \Omega^2 u_n + (l(l + 1) - 2)u_n - \frac{8 \Omega^2}{l(l + 1) - 2 - \Omega^2} u_n = (L^{ij}_{nm} L^{i}_{nm} + i \epsilon_{jik} L^{k}_{nm}) u_m^i . \quad (30)$$

In the basis in which the matrix on the rhs is diagonalized, it can be replaced with its respective eigenvalue $\Lambda$, resulting in an algebraic equation for the $\Omega$

$$(l(l + 1) - 2 - \Omega^2)^2 - 8 \Omega^2 = \Lambda (l(l + 1) - 2 - \Omega^2) \ . \quad (31)$$

Finally, this quadratic equation can be solved,

$$\Omega_{1,2}^2 = -\frac{1}{2} \Lambda + l(l + 1) + 2 \pm \frac{1}{2} \sqrt{\Lambda^2 - 16 \Lambda + 32 l(l + 1)} . \quad (32)$$

For the values of $\Lambda$, taken from table (27), the modes are:

| $\Lambda$ | $\Omega_1^2$ | $\Omega_2^2$ | multiplicity |
|-----------|--------------|--------------|--------------|
| $l(l + 1) - 2$ | 0 | $l^2 + l + 6$ | $2l + 1$ |
| $2l$ | $l^2 - 3l + 2$ | $l^2 + 3l + 2$ | $2l + 3$ |
| $-(2l + 2)$ | $l^2 - l$ | $l^2 + 5l + 6$ | $2l - 1$ |

(33)

Note that the number of zero modes changes from 9 for the case $l = 1$, and 12 for $l = 2$, to $2l + 1$ for arbitrary $l > 2$. This is connected with the fact that the original solution is based on an $l = 1$ ansatz, and so the symmetries of the equations are manifested as zero-modes under $l = 1$ perturbations. Thus, for $SU(N)$, the total number of zero modes is the sum for all $l$ up to $N - 1$

$$9 + 12 + \sum_{l=3}^{N-1} (2l + 1) = N^2 + 12 \quad (34)$$

where $l$ runs up to $N - 1$.

Note that because the frequencies are real, we can indeed satisfy the gauge field reality conditions (14) by choosing initial vectors that satisfy reality. In addition the constraint conditions should be imposed, with the result that the $2l + 1$ modes with frequency $l^2 + l + 6$ are projected out at each $l$. 

7
3 Transverse perturbations

In addition to the already considered perturbations there are also those that are completely transverse to the system. That is the directions 789, if we had oriented the original system along 123456. The analysis is considerably simpler that the previous case. The perturbations

$$\delta X^k = \sum_m Y^l_m \zeta^k_m \quad \text{for} \quad k = 7, 8, 9$$

(35)

satisfy the simple harmonic equation

$$\ddot{\zeta}^k_m + l(l + 1) \zeta^k_m = 0.$$  

(36)

This clearly has only positive frequencies and is therefore stable. For $l = 1$ all the 9 modes have the same frequency as the original solution, corresponding to infinitesimal global rotations of the system into the 789 hyperplane. The counting goes as follows, there are $9 \times 2 = 18$ first order degrees of freedom here, which coincides with the dimensionality of the grassmanian manifold of embeddings of a 6-hyperplane into $\mathbb{R}^9$, i.e. $SO(9)/SO(6) \times SO(3)$.

4 Conclusion

From the results of the two previous sections it follows that, zero-modes notwithstanding, all the frequencies in the system are positive, and arbitrary small perturbation will remain bounded for all times. We have learned also of the paper [7] where the same problem is considered in the membrane language. However the authors of [7] in their approach to the same problem arrive at the Mathieu equation instead of the equations (24), (25), (19) and therefore to the opposite conclusion, namely that there exist solutions to the linearized perturbation equations which grow exponentially. We hope that the present work will contribute to the clarification of the question.
5 Appendix

Here we shall present stability analysis of the $l = 1$ system made in [] and shall compare it with the general consideration in the main text. In addition we shall find new solutions. The Hamiltonian of the system is

$$H = \frac{1}{2} \sum_{i=1}^{6} \dot{r}_i^2 + \frac{1}{2} [(r_1^2 + r_2^2)(r_3^2 + r_4^2) + (r_3^2 + r_4^2)(r_5^2 + r_6^2) + (r_5^2 + r_6^2)(r_1^2 + r_2^2)]$$  \hspace{1cm} (37)

where

$$X_{i+1} = L_{1+i/2} r_{i+1}, \quad X_{i+2} = L_{1+i/2} r_{i+2}, \quad i = 0, 2, 4.$$

It is convenient to introduce new coordinates

$$r_1 = \rho_1 \cos \phi_1, \quad r_2 = \rho_1 \sin \phi_1,$$
$$r_3 = \rho_2 \cos \phi_2, \quad r_4 = \rho_2 \sin \phi_2,$$
$$r_5 = \rho_3 \cos \phi_3, \quad r_6 = \rho_3 \sin \phi_3,$$

so that the Hamiltonian take the form:

$$H = \frac{1}{2} \sum_{i=1}^{3} \left[ \ddot{\rho}_i^2 + \dot{\rho}_i^2 \dot{\phi}_i^2 \right] + \frac{1}{2} [\rho_1^2 \dot{\rho}_2^2 + \rho_2^2 \dot{\rho}_3^2 + \rho_3^2 \dot{\rho}_1^2].$$  \hspace{1cm} (38)

The conservation integrals are:

$$\dot{\rho}_1 \dot{\phi}_1 = M_1, \quad \dot{\rho}_2 \dot{\phi}_2 = M_2, \quad \dot{\rho}_3 \dot{\phi}_3 = M_3,$$

and the effective Hamiltonian take the form:

$$H = \frac{1}{2} \sum_{i=1}^{3} \left[ \ddot{\rho}_i^2 + \frac{M_i^2}{\rho_i^2} \right] + \frac{1}{2} [\rho_1^2 \dot{\rho}_2^2 + \rho_2^2 \dot{\rho}_3^2 + \rho_3^2 \dot{\rho}_1^2].$$  \hspace{1cm} (39)

The effective potential is equal to

$$U = \frac{1}{2} \left[ \frac{M_1^2}{\rho_1^2} + \frac{M_2^2}{\rho_2^2} + \frac{M_3^2}{\rho_3^2} \right] + \frac{1}{2} [\rho_1^2 \dot{\rho}_2^2 + \rho_2^2 \dot{\rho}_3^2 + \rho_3^2 \dot{\rho}_1^2].$$

The equations of motion are:

$$\ddot{\rho}_1 = -\rho_1^2 (\rho_2^2 + \rho_3^2) + \frac{M_1^2}{\rho_1^2}$$  \hspace{1cm} (40)
$$\ddot{\rho}_2 = -\rho_2^2 (\rho_1^2 + \rho_3^2) + \frac{M_2^2}{\rho_2^2}$$  \hspace{1cm} (41)
$$\ddot{\rho}_3 = -\rho_3^2 (\rho_1^2 + \rho_2^2) + \frac{M_3^2}{\rho_3^2}$$  \hspace{1cm} (42)
and the previous solution \( \rho_i = R_i = \text{Const}, i = 1, 2, 3 \) and \( \dot{\phi}_i \equiv \omega_i = M_i^2 / R_i^4 = R_{i+1}^2 + R_{i+2}^2 \). Let us now consider the special case when all coordinates are equal to each other \( \rho_1 = \rho_2 = \rho_3 = \rho(t) \) and can depend on time, then

\[
H = \frac{3}{2}[\dot{\rho}^2 + \frac{M^2}{\rho^2} + \rho^4]
\]

and corresponding equation can be integrated. The new solution is elliptic function \( \rho = \rho(t) \)

\[
t = \int_{\rho_{\text{min}}}^{\rho(t)} \frac{d\rho}{\sqrt{2E/3 - \rho^4 - M^2/\rho^2}}
\]

with period

\[
T = 2 \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} \frac{d\rho}{\sqrt{2E/3 - \rho^4 - M^2/\rho^2}},
\]

where \( \rho_{\text{min}}, \rho_{\text{max}} \) are the solutions of the equation \( 2E/3 - \rho^4 - M^2/\rho^2 = 0 \).

Let us now turn to a stability analysis of the solution \( \rho_1 = \rho_2 = \rho_3 = R = \text{Const} \) considered in the main text. The equations of variation are (\( M_i^2 = 2R_i^4 \)):

\[
\delta \ddot{\rho}_1 = -2R^2(\delta \rho_1 + \delta \rho_2 + \delta \rho_3) - \frac{3M_i^2}{R_i^4} \delta \rho_1 = -2R^2(8\delta \rho_1 + 2\delta \rho_2 + 2\delta \rho_3) \tag{43}
\]

\[
\delta \ddot{\rho}_2 = -2R^2(\delta \rho_1 + \delta \rho_2 + \delta \rho_3) - \frac{3M_i^2}{R_i^4} \delta \rho_2 = -2R^2(2\delta \rho_1 + 8\delta \rho_2 + 2\delta \rho_3) \tag{44}
\]

\[
\delta \ddot{\rho}_3 = -2R^2(\delta \rho_1 + \delta \rho_2 + \delta \rho_3) - \frac{3M_i^2}{R_i^4} \delta \rho_3 = -2R^2(2\delta \rho_1 + 2\delta \rho_2 + 8\delta \rho_3) \tag{45}
\]

with three stable modes 12, 6, 6. They coincide with the ones in (33). For the more general case when \( R_1 \neq R_2 \neq R_3 \) we have:

\[
\delta \ddot{\rho}_1 = -4(R_1^2 + R_3^2) \delta \rho_1 - 2R_1R_2\delta \rho_2 - 2R_1R_3\delta \rho_3 \tag{46}
\]

\[
\delta \ddot{\rho}_2 = -2R_2R_1 \delta \rho_1 - 4(R_1^2 + R_3^2) \delta \rho_2 - 2R_2R_3\delta \rho_3 \tag{47}
\]

\[
\delta \ddot{\rho}_3 = -2R_3R_1 \delta \rho_1 - 2R_3R_2\delta \rho_2 - 4(R_1^2 + R_2^2)\delta \rho_3, \tag{48}
\]

which also has only positive modes [3].

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