Comparing $f(R)$ modified gravity and noncommutative geometry in the context of dark matter and traversable wormholes: a survey

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Abstract

Noncommutative geometry, as conceptualized by Nicolini, Smailagic, and Spallucci, may be viewed as a slight modification of Einstein’s theory. The same can be said for $f(R)$ modified gravity for an appropriate choice of the function $f(R)$. Since such an $f(R)$ could be determined from the noncommutative-geometry background, these gravitational theories make very similar predictions in the discussion of (a) dark matter and (b) traversable wormholes; they can therefore be taken as equally viable models.

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1 Introduction

Both $f(R)$ gravity and noncommutative geometry may be viewed as modifications of Einstein’s theory. For the former, the field equations can be derived from the action

$$S_{f(R)} = \frac{1}{2\kappa} \int \sqrt{-g} f(R) d^4x, \quad (1)$$

where $R$ is the Ricci scalar, $f(R)$ is a nonlinear function of $R$, and $\kappa = 8\pi G$. If $f(R) \equiv R$, we recover the Hilbert-Einstein action

$$S_{\text{HE}} = \frac{1}{2\kappa} \int \sqrt{-g} R d^4x, \quad (2)$$

showing that the effect of modifying Einstein’s theory can be quite small. In other words, a proper choice of $f(R)$ can lead to what will be referred to as a “slightly modified gravitational theory.”

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Another example of a small effect is noncommutative geometry, as described in Refs. [1, 2, 3]. Noncommutativity replaces point-like objects by smeared objects with the aim of eliminating the divergences that normally appear in general relativity. The need for spacetime quantization was first proposed by Snyder [4]. (For more recent developments, see Refs. [5, 6].) Further motivation is provided by an important outcome of string theory, the realization that coordinates may become noncommutative operators on a D-brane [7, 8]. As a consequence, spacetime can be encoded in the commutator $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck’s constant discretizes phase space [2]. The physical consequences of a noncommutative-geometry background appear to be quite small, although the effect on the Einstein field equations remains to be seen (Section 2.2).

The purpose of this survey is to compare the two gravitational theories in the context of dark matter and traversable wormholes. Both theories are characterized by small effects, ultimately resulting in very similar predictions.

2 Dark matter

2.1 $f(R)$ modified gravity

Returning now to $f(R)$ modified gravity, renewed interest in the theory arose from attempts to explain dark matter, as well as the late-time accelerated expansion of the Universe. For further discussion, see Refs. [9, 10, 11, 12, 13]. Since we are primarily interested in small effects, we need to define “slightly modified gravitational theory,” as proposed in Ref. [14]. To that end, we start with a static and spherically symmetric line element using units in which $c = G = 1$:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - m(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $m(r)$ is the effective mass inside a sphere of radius $r$ with $m(0) = 0$ and $\lim_{r \to \infty} m(r)/r = 0$ [15]. Next, we list the gravitational field equations in the form used by Lobo and Oliveira [16] (replacing $b(r)$ by $m(r)$ for notational convenience). We also assume that $\Phi'(r) \equiv 0$; otherwise, according to Ref. [16], the analysis becomes intractable. The equations are

$$\rho(r) = F(r)\frac{m'(r)}{r^2},$$

$$p_r(r) = -F(r)\frac{m(r)}{r^3} + F'(r)\frac{rm'(r) - m(r)}{2r^2} - F''(r) \left(1 - \frac{m(r)}{r}\right),$$

and

$$p_t(r) = -\frac{F'(r)}{r} \left(1 - \frac{m(r)}{r}\right) + \frac{F(r)}{2r^3} [m(r) - rm'(r)],$$

where $F = \frac{df}{dR}$. The Ricci curvature scalar is given by

$$R(r) = \frac{2m'(r)}{r^2}.$$
Observe that the field equations reduce to the Einstein field equations for \( \Phi'(r) \equiv 0 \) whenever \( F \equiv 1 \). We can now readily define the notion of slightly modified gravity by assuming that \( F(r) \) remains close to unity and relatively flat; in other words, both \( F'(r) \) and \( F''(r) \) remain relatively small in absolute value [14].

Before continuing, we need to recall that one objective in \( f(R) \) modified gravity is to explain the flat galactic rotation curves without assuming the existence of dark matter. It is known that in the outer region of the halo, test particles move with constant velocity in a circular path \([17, 18, 19, 20, 21]\). According to Ref. \([18]\), for a test particle with four-velocity \( U^\alpha = dx^\alpha/d\tau \), we have

\[
\left( \frac{dr}{d\tau} \right)^2 = E^2 + V(r),
\]

where \( E \) is the relativistic energy. An orbit \( r = a \) is stable if

\[
\frac{d^2V}{dr^2} \bigg|_{r=a} < 0
\]

and unstable if

\[
\frac{d^2V}{dr^2} \bigg|_{r=a} > 0.
\]

It is shown in Ref. \([22]\) that a slightly modified gravitational theory as defined in Ref. \([14]\) is enough to yield stable galactic orbits without the need for dark matter. That dark matter is a geometric effect in \( f(R) \) gravity had already been shown in Ref. \([10]\).

### 2.2 Noncommutative geometry

As noted in Sec. \([1]\) noncommutative geometry replaces point-like structures by smeared objects. A direct way to model this behavior is to use a Gaussian distribution of minimal length \( \sqrt{\beta} \) rather than the Dirac delta function \([2, 23]\). An equivalent but simpler approach is to assume that the energy density of a static and spherically symmetric and particle-like gravitational source has the form \([24, 25]\)

\[
\rho(r) = \frac{M\sqrt{\beta}}{\pi^2(r^2 + \beta)^2},
\]

showing that the mass \( M \) of a particle is diffused throughout the region of linear dimension \( \sqrt{\beta} \) due to the uncertainty. An important observation in noncommutative geometry is that whenever we make use of Eq. \((11)\), we can keep the standard forms of the Einstein field equations for the simple reason that the Einstein tensor retains its original form; only the stress-energy tensor is modified \([2]\). It follows that the length scales can be macroscopic. It is also brought out in Ref. \([2]\) that noncommutative geometry is an intrinsic property of spacetime and does not depend on any particular feature such as curvature.

Returning to the dark-matter hypothesis, although first introduced in the 1930’s, the implications thereof were not fully recognized until the 1970’s when it was observed that galaxies exhibit flat rotation curves, i.e., constant tangential velocities sufficiently far
from the galactic center [26]. To recall why, suppose $m_1$ is the mass of a star, $v$ its constant velocity, and $m_2$ the mass of everything else. Multiplying $m_1$ by the centripetal acceleration yields

$$m_1 \frac{v^2}{r} = m_1 m_2 \frac{G}{r^2},$$

(12)

where $G$ is Newton’s gravitational constant. Reverting to geometrized units ($c = G = 1$), we obtain the linear form

$$m_2 = rv^2.$$  

(13)

This form essentially characterizes the dark-matter hypothesis. For the purpose of this survey, the most important conclusion is drawn in Ref. [27]: the linear relationship can be attributed to the noncommutative-geometry background. That noncommutative geometry can account for the flat rotation curves without the need for dark matter is also shown in Refs. [28, 29].

### 3 Traversable wormholes

Wormholes are handles or tunnels that link widely separated regions of our Universe or different universes altogether. Morris and Thorne [30] proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(14)

Here $b = b(r)$ is called the shape function and $\Phi = \Phi(r)$ is called the redshift function, which must be everywhere finite to prevent the occurrence of an event horizon. For the shape function, we must also have $b(r_0) = r_0$, where $r = r_0$ is the radius of the throat of the wormhole. An important geometric requirement is the flare-out condition at the throat: $b'(r_0) < 1$, while $b(r) < r$ near the throat. The flare-out condition can only be met by violating the null energy condition (NEC)

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0$$

(15)

for all null vectors $k^\alpha$, where $T_{\alpha\beta}$ is the stress-energy tensor. Matter that violates the NEC is referred to as “exotic” in Ref. [30]. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation has the form

$$T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0.$$  

(16)

Here $T^t_t = -\rho(r)$, where $\rho(r)$ is the energy density, $T^r_r = p_r(r)$ is the radial pressure, and $T^{\theta\theta} = T^{\phi\phi} = p_l(r)$ is the lateral pressure. For the wormhole spacetime, another important physical property is asymptotic flatness, which demands that $\lim_{r \to \infty} \Phi(r) = 0$ and $\lim_{r \to \infty} b(r)/r = 0$. 

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4
3.1 $f(R)$ modified gravity

To apply $f(R)$ modified gravity to wormholes, we need to replace $m(r)$ in Eqs. (4)-(6) by $b(r)$, as in Ref. [16]. While the shape function $b(r)$ has the usual properties, the flare-out condition no longer implies that the NEC has been violated, as in classical general relativity. In fact, it is asserted by Lobo and Oliveira in Ref. [16] that for the material threading the throat of a wormhole, the NEC can be met, thereby allowing the use of ordinary (nonexotic) matter in the modified theory. Imposing the conditions $\rho + p_r \geq 0$ and $\rho \geq 0$, it now follows from Eqs. (4) and (5) that the function $F$ must be positive and must satisfy the following conditions at the throat:

$$\frac{Fb'}{r^2} \geq 0 \quad (17)$$

and

$$\frac{(2F + rF')(b'r - b)}{2r^2} - F'' \left(1 - \frac{b}{r}\right) \geq 0. \quad (18)$$

To complete the proof of the above assertion, we also need to show that the NEC is met for all null vectors

$$(1, a, b, c), \text{ where } 0 \leq a, b, c \leq 1 \text{ and } a^2 + b^2 + c^2 = 1. \quad (19)$$

First, from Eq. (6) we have, at the throat, $\rho + p_t \geq 0$, $F$ being positive. Since $\rho = (a^2 + b^2 + c^2)\rho$, we can write $\rho + a^2p_r + b^2p_t + c^2p_t$ in the form

$$a^2(1, 0, 0, 0) + b^2(1, 0, 0, 0) + c^2(1, 0, 0, 0) + a^2(0, 1, 0, 0) + b^2(0, 0, 1, 0) + c^2(0, 0, 0, 1)
= a^2(1, 1, 0, 0) + b^2(1, 0, 1, 0) + c^2(1, 0, 0, 1)
= a^2(\rho + p_r) + b^2(\rho + p_t) + c^2(\rho + p_t) \geq 0,$$

showing that the NEC is satisfied for all null vectors.

Addressing an apparent conflict with the classical theory, it is shown in Ref. [31] that to sustain a wormhole, a violation of the NEC is indeed unavoidable by referring back to the Raychaudhury equation. According to Ref. [16], such a violation can be attributed to the higher-order curvature terms, interpreted as a gravitational fluid.

Having established that $f(R)$ modified gravity can support traversable wormholes, we need to recall that we are primarily interested in slightly modified gravitational theories, as defined in Sec. 2.1. This issue is addressed in Ref. [14] by showing that even a slight modification is sufficient for sustaining a wormhole, while avoiding a violation of the NEC for matter threading the throat. The analysis assumes zero tidal forces.

3.2 Noncommutative geometry

In this section we return to Eq. (11) and line element (14) describing a wormhole space-time. Here we need to list the Einstein field equations in classical general relativity:

$$\rho(r) = \frac{b'}{8\pi r^2}, \quad (20)$$
\[ p_r(r) = \frac{1}{8\pi} \left[ \frac{-b}{r^3} + 2 \left( \frac{1 - \frac{b}{r}}{r} \right) \Phi' \right], \quad (21) \]

\[ p_t(r) = \frac{1}{8\pi} \left( 1 - \frac{b}{r} \right) \left[ \Phi'' - \frac{b'r - b}{2r(r - b)} \Phi' + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)} \right]. \quad (22) \]

Eqs. (11) and (20) yield the shape function [25]

\[ b(r) = \frac{4M\sqrt{\beta}}{\pi} \left( \frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} - \frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r_0}{\sqrt{\beta}} + \frac{r_0}{r_0^2 + \beta} \right) + r_0, \quad (23) \]

where \( M \) is now the mass of the wormhole. To check the flare-out condition, we first observe that

\[ b'(r) = \frac{4M\sqrt{\beta}}{\pi} \frac{2r^2}{(r^2 + \beta)^2} > 0, \quad (24) \]

as required by Eq. (20), but it also follows that \( b'(r) < 1 \) as long as \( \sqrt{\beta} \ll M \). So the flare-out condition is met and the NEC is automatically violated, as we saw earlier. As another check,

\[ \rho + p_r = \frac{M\sqrt{\beta}}{\pi^2(r^2 + \beta)^2} - \frac{1}{8\pi} \frac{b(r)}{r^3} < 0 \quad (25) \]

at or near the throat since \( \sqrt{\beta} \ll 1 \). Finally, \( \lim_{r \to \infty} b(r)/r = 0 \), showing that the wormhole spacetime is asymptotically flat, provided that \( \lim_{r \to \infty} \Phi(r) = 0 \).

Higher-dimensional wormholes, given a noncommutative-geometry background, are discussed in Ref. [32]. Charged wormholes with low tidal forces that are also inspired by noncommutative geometry are analyzed in Ref. [33]. Wormholes that take into account both gravitational theories, i.e., noncommutative wormholes in \( f(R) \) gravity, are discussed in Ref. [34]. The effects of the two gravitational theories are independent without being mutually exclusive. As a result, the combined effects may differ from the individual effects.

The possible avoidance of exotic matter at the throat is discussed in the next section.

### 4 Connecting noncommutative geometry to \( f(R) \) modified gravity

Both \( f(R) \) modified gravity and noncommutative geometry can help sustain traversable wormholes and both can account for the flat galactic rotation curves and hence for dark matter, even if the effect of \( f(R) \) gravity is small. The effect of noncommutative geometry is small to begin with, thereby suggesting a connection between the two: according to Ref. [35], a noncommutative-geometry background can determine the corresponding function \( f(R) \) for modeling dark matter and may even provide a motivation for the choice of \( f(R) \). So it follows from Sec. 3.1 that a noncommutative-geometry wormhole can in principle be constructed without exotic matter. This conclusion is made explicit in Ref. [36] by determining the form of \( f(R) \) for the wormhole spacetime:

\[ f(R) = \frac{2M\sqrt{\beta}(\beta R + 2b')\ln(\beta R + 2b') - \beta R}{\beta^2(\beta R + 2b')} + C. \quad (26) \]
The zero-tidal force assumption is retained for the technical reason given in Sec. 2.1 but it does not constitute a necessary condition.

5 Summary

This survey has pointed out that noncommutative geometry and $f(R)$ modified gravity are equally viable models for discussing dark matter and traversable wormholes, in part because the function $f(R)$ can be determined from the noncommutative-geometry background.

Even a slightly modified $f(R)$ gravitational theory can account for the galactic rotation curves and hence for dark matter. A noncommutative-geometry background with its characteristic small effect can account for the same phenomenon. So dark matter can be viewed as a geometric effect in modified gravity. It is also noted that both gravitational theories are not only able to support traversable wormholes, they allow the throat to be threaded with nonexotic matter. The unavoidable violation of the NEC can be attributed to the higher-order curvature terms in the modified theory.

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