Existence of global solutions to a semilinear pseudo-parabolic equation

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Abstract

In this article, we consider a semilinear pseudo parabolic heat equation with the nonlinearity which is the product of logarithmic and polynomial functions. Here we prove the global existence of solution to the problem for arbitrary dimension \( n \geq 1 \) and power index \( p > 1 \). Asymptotic behaviour of the solution has been addressed at different energy levels. Moreover, we prove that the global solution indeed decays with an exponential rate. Finally, sufficient conditions are provided under which blow up of solutions take place.

Keywords— Global existence; potential well; decay estimates; finite time blow-up
Mathematics Subject Classification (2020) — 35A01, 35B40, 35K61, 35S16

1 Introduction

In this article, we are interested to study the global existence and the longtime behaviour of the following semilinear pseudo parabolic heat equation

\[
\begin{aligned}
&v_t - \Delta v_t - \Delta v = v|v|^{p-1} \log |v|, \quad t \in \mathbb{R}^+, \quad x \in U, \\
v(x, t) = 0, \quad t \in \mathbb{R}^+, \quad x \in \partial U, \\
v(x, 0) = v_0(x), \quad x \in U,
\end{aligned}
\]

where \( U \subset \mathbb{R}^n (n \geq 1) \) is a smooth bounded domain and the power index \( p > 1 \).

Pseudo equations are characterized by the appearance of a Laplacian of a time derivative of the unknown. Specially, pseudo-parabolic equations describe various important physical phenomena, such as the heat conduction involving two temperatures [7], the seepage of homogeneous fluids in fissured rock [4], unidirectional propagation of long waves in nonlinear dispersive media [6] [29], the aggregation of populations [23], etc. In particular, problem (1.1) also describes several natural phenomena (see [15] [16] [23] [24] and the references there in). For example, in the analysis of nonstationary processes in semiconductors the presence of source term, \( v_t - \Delta v_t \) denotes the free electron density rate, \( \Delta v \) denotes the linear dissipation of free charge current, and the source term represents a source of free electron current (see [17]). On the other hand, partial differential equations (PDEs) with polynomial and logarithmic nonlinearities are studied widely due to many applications in physics and other applied science such as theory of superfluidity, nuclear physics, transport phenomenon and diffusion phenomenon (see [1] [9] [3] [14] [18] [37] [40]).

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Let us consider the following pseudo parabolic equation

\[
\begin{aligned}
v_t - \Delta v_t - \Delta v &= f(v), & t \in \mathbb{R}^+, & x \in U, \\
v(x, t) &= 0, & t \in \mathbb{R}^+, & x \in \partial U, \\
v(x, 0) &= v_0(x), & x \in U.
\end{aligned}
\]  

(1.2)

The authors in \[30\] considered (1.2) when \(f = 0\) and proved the existence and uniqueness of solutions. After this precursory result, many authors investigated the global existence, uniqueness, longtime behavior, regularity and blow-up phenomena of the solutions to (1.2) for different choices of \(f\) using various methods (see \[11, 13, 28, 31, 32\] and the references therein). In \[29, 27\], Sattinger proposed a powerful technique (popularly known as the "potential well method") to study the existence and longtime behaviour of the solution to (1.2). Sattinger and Payne’s work [26] was the most influential one which was followed by many mathematicians. In [26], the authors introduced a potential well \(W\), an outer potential well \(V\) in terms of a Nehari functional \(I(u)\) and a potential energy functional \(J(v)\). Under few technical assumptions on \(f\), they studied the properties of \(W, V, J(v)\). Using the invariance of the potential well, they proved the finite time blow-up of the solutions to the nonlinear heat equation. After that, the method was improved by many mathematicians and physicists to study different types of PDEs such as nonlinear heat, and nonlinear wave equations (see \[11, 14, 33, 21, 18, 34, 12, 5, 10\]). The authors of \[35\] considered (1.2) when \(f(v) = v^p\) and under some assumptions on \(p\) and \(n\) they established existence and studied asymptotic behaviour of the solution. Using the properties of potential wells (PWs) the authors in \[35\] proved the global existence and investigated the longtime behavior of solutions. However in \[28, 34\], the authors amend the proofs of \[35\]. In particular, the authors of \[34\] generalized and extended the results of \[35\] using a family of PWs. Moreover, the authors of \[3\] investigated the global existence, exponential decay and finite time blow-up of solutions to (1.2) when the source term is given by \(v \log |v|\). Recently, many authors studied (1.2) using method of potential wells for different choice of \(f(v)\) (see \[19, 22, 33\]). In particular, the authors of \[35\] considered (1.2) and studied the global existence, blow-up of solution when the source term is given by \(|x|^n v|v|^{p-1}, 1 < p < \infty, \sigma > -n\), if \(n = 1, 2; 1 < p \leq \frac{n+2}{n-2}, \sigma > \frac{(1+\sigma)(n-2)}{2} - n\), if \(n \geq 3\).

Moreover, when \(f(v)\) in (1.2) is solely polynomial type i.e., \(f(v) = |v|^p\) or \(v|v|^{p-1}\), then the authors of \[11\] established the global existence, and blow up with restrictions on the power index viz., \(1 < p \leq \frac{n+2}{n-2}\) if \(n \geq 3\). The main objective of this paper is to study the global existence, and blow up phenomenon of solutions to (1.1) with any restriction on \(p\) and \(n\). We also discuss the asymptotic behavior of solution to (1.1) at all three energy levels (supercritical, critical and sub-critical).

The paper is organized as follows. We give some definitions and introduce the family of PWs in Section 2. We recall the results which are helpful to prove the global existence and longtime behavior of the solution to (1.1) in Section 2. In Section 3, we prove the global existence of a weak solution to the problem (1.1) under some conditions for an arbitrary \(n\) and \(p\). A decay estimate on the solution in \(H_0^1\)-norm is discussed in Section 4. In the last section, we obtain the finite time blow-up of weak solutions to (1.1) under the cases \(J(v_0) < d, J(v_0) = d\) and \(J(v_0) > 0\).

2 Preliminaries

Henceforth, we denote the norm \(\|\cdot\|_p\) by \(\|g\|_p = (\int_U |g|^p dx)^{1/p}, \forall g \in L^p(U), 1 \leq p < \infty\) and the inner product \((h, g)_2 = \int_U h g dx\). When \(p = 2\), we simply write \(\|\cdot\|\) instead of \(\|\cdot\|_2\).

Definition 2.1. (Weak solution) A function

\[
v \in L^\infty(0, T); H_0^1(U)\]  

with \(v_t \in L^2((0, T); L^2(U)), v(x, 0) = v_0(x)\)

is called a weak solution to (1.1) if it satisfies (1.1) in weak sense i.e.,

\[
(v_t, w)_2 + (\nabla v_t, \nabla w)_2 + (\nabla v, \nabla w)_2 = (v|v|^{p-1} \log |v|, w)_2, \forall w \in H_0^1(U), \quad t \in (0, T).
\]  

(2.1)

For completeness, we recall here the definitions of the maximal existence time, and the notion of the finite time blow-up which are quite standard.
Definition 2.2. Let \( v := v(x,t) \) be a weak solution to (1.1). The maximal existence time \( T \) of \( v \) is defined as follows:

(i) If \( v \) exists for every \( t > 0 \), then \( T = \infty \).
(ii) If there exists \( \hat{t} \in (0, \infty) \) such that \( v \) exists for \( 0 \leq t < \hat{t} \), but does not exist at \( t = \hat{t} \), then \( T = \hat{t} \).

Definition 2.3. A weak solution \( v \) of (1.1) is said to blow-up in finite time if the maximal existence time \( T \) is finite and

\[
\lim_{t \to T^-} \|v(\cdot, t)\|_{H^1_0(U)} = \infty.
\]

2.1 Potential wells

In this subsection, we define the energy functionals associated to the nonlinear term \( v|v|^{p-1} \log |v| \) and the corresponding family of PWs and state few results which are useful for analysis in the subsequent sections.

First, we define the potential energy functional \( J \) as

\[
J(v) = \frac{1}{2} \|\nabla v\|^2 - \frac{1}{1+p} \int_U |v|^{1+p} \log |v| dx + \frac{1}{(1+p)^2} \|v\|^{1+p}_{1+p},
\]

and the Nehari functional \( I \) is defined as

\[
I(v) = \|\nabla v\|^2 - \int_U |v|^{1+p} \log |v| dx.
\]

From (2.2) and (2.3), observe that

\[
J(v) = \frac{p-1}{2(1+p)} \|\nabla v\|^2 + \frac{1}{(1+p)^2} \|v\|^{1+p}_{1+p} + \frac{1}{1+p} I(v).
\]

The Nehari manifold \( \mathcal{N} \) is defined as

\[
\mathcal{N}(v) = \{ v \in H^1_0(U) : I(v) = 0 \text{ and } \|\nabla v\|^2 \neq 0 \},
\]

and the depth of the well \( d \) is defined as

\[
d = \inf_{v \in \mathcal{N}} J(v).
\]

We now introduce the potential well \( W \) and the outer potential well \( V \) as follows:

\[
W = \{ 0 \} \cup \{ v \in H^1_0(U) : I(v) > 0, J(v) < d \},
\]

\[
V = \{ v \in H^1_0(U) : J(v) < d, I(v) < 0 \}.
\]

If \( v \) is a weak solution to (1.1) then multiplying \( v_t \) to (1.1) and integrating over \( U \times [0,t) \), we get

\[
\int_0^t \|v_t(\cdot, \tau)\|^2_{H^1_0(U)} d\tau + J(v(\cdot, t)) = J(v_0(\cdot)), \quad t \in [0,T).
\]

Set the energy functional \( E \) associated to (1.1) by

\[
E(v(\cdot, t)) = J(v(\cdot, t)) + \int_0^t \|v_t\|^2_{H^1_0(U)} d\tau.
\]

The equation (2.5) immediately gives \( E(v(\cdot, t)) = E(v_0) \). Thus (2.5) is referred as the conservation of energy.

Next, we extend the notion of a “single potential well to the family of PWs” by defining

\[
J_\delta(v) = \frac{\delta}{2} \|\nabla v\|^2 - \frac{1}{1+p} \int_U |v|^{1+p} \log |v| dx + \frac{1}{(1+p)^2} \|v\|^{1+p}_{1+p},
\]

with \( \delta > 0 \).
Lemma 2.4. \( \text{ Cf. [18], Lemma 2.3} \) The function \( d(\beta) \) attains local maximum and \( \text{Cf. [18], Lemma 2.2} \) Assume \( r > 0 \) and the outer of the family of PWs \( W \)

\[ \delta > 0. \text{ We define the corresponding Nehari manifolds } N_\delta \text{ as } \]

\[ N_\delta(v) = \{ v \in H_0^1(U) : I_\delta(v) = 0, \|\nabla v\|^2 \neq 0 \}, \]

and “the depth of family of PWs” \( d(\delta) \) as

\[ d(\delta) = \inf_{v \in N_\delta} J(v). \] \hfill (2.8)

Moreover, we define the family of PWs \( W_\delta \) as

\[ W_\delta = \{ v \in H_0^1(U) : J(v) < d(\delta), I_\delta(v) > 0 \} \cup \{0\}, \]

and the outer of the family of PWs \( V_\delta \) as

\[ V_\delta = \{ v \in H_0^1(U) : J(v) < d(\delta), I_\delta(v) < 0 \}. \]

Since the potential energy \( J \) and \( I \) defined in (2.2)–(2.3) are the same as those in [18] (in the context of semilinear wave equation). Now, “we recall few results from [18] and [6]:

Lemma 2.1. \( \text{(Cf. [18], Lemma 2.1)} \) Let \( v \in H_0^1(U) \) and \( \|v\| \neq 0 \). Then the following hold true.

(i) \( \lim_{\beta \to 0} J(\beta v) = 0 \), \( \lim_{\beta \to \infty} J(\beta v) = -\infty. \)

(ii) There exists \( \beta^* = \beta^*(v) \) in \( (0, \infty) \) such that

\[ \frac{d}{d\beta} J(\beta v) \big|_{\beta = \beta^*} = 0. \]

(iii) The function \( \beta \mapsto J(\beta v) \) is decreasing on \( \beta^* \leq \beta < \infty \), increasing on \( 0 \leq \beta \leq \beta^* \) and attains its maximum at \( \beta = \beta^*. \)

(iv) The function \( I \) satisfies \( I(\beta^* v) = 0 \), \( I(\beta v) = \beta \frac{d}{d\beta} J(\beta v) > 0 \) for \( 0 < \beta < \beta^* \), and \( I(\beta v) < 0 \) for \( \beta^* < \beta < \infty \).

Lemma 2.2. \( \text{(Cf. [18], Lemma 2.2)} \) Assume \( \delta > 0 \) and \( \phi(r) = C^{p+2} r^p \), where \( C = \sup_{v \in H_0^1(U)} \frac{\|v\|^{p+2}}{\|\nabla v\|} \) and \( r(\delta) \) is the unique real root of the equation \( \phi(r) = \delta \). Then we have:

(i) \( I_\delta(v) > 0 \), provided \( 0 < \|\nabla v\| \leq r(\delta) \).

(ii) \( \|\nabla v\| > r(\delta) \), provided \( I_\delta(v) < 0 \).

(iii) \( \|\nabla v\| > r(\delta) \) or \( \|\nabla v\| = 0 \), provided \( I_\delta(v) = 0 \).

Lemma 2.3. \( \text{(Cf. [18], Lemma 2.3)} \) The function \( \delta \mapsto d(\delta) \) defined in (2.8) has the following properties:

(i) \( d(\delta) = \frac{1}{2} - \frac{1}{p+2} r(\delta)^2 > 0 \) for \( 0 < \delta < \frac{1}{1+p} \).

(ii) There exists a unique \( \delta_0 > \frac{1+p}{2} \) such that \( d(\delta_0) = 0 \), and \( d(\delta) > 0 \) for \( 0 < \delta < \delta_0 \).

(iii) The function \( \delta \mapsto d(\delta) \) is decreasing on \( 1 \leq \delta \leq \delta_0 \), strictly increasing on \( 0 < \delta \leq 1 \), at \( \delta = 1 \) this function attains local maximum and \( d(1) = 0 \).

Lemma 2.4. \( \text{(Cf. [18], Lemma 2.4)} \) Assume \( 0 < J(v) < d \) for some \( v \in H_0^1(U) \) and \( \delta_1, \delta_2 \) are two roots of the equation \( d(\delta) = J(v) \) such that \( \delta_1 < 1 < \delta_2 \), then the sign of \( I_\delta(v) \) does not change in \( \delta_1 < \delta < \delta_2 \).
We now seek a sequence of functions \( v \) the initial data on \( J \).

We have the following possibilities:

(i) All solutions to problem (1.1) with \( 0 < J(v_0) \leq \eta \) belong to \( W_\delta \) for \( \delta_1 < \delta < \delta_2 \), provided \( I(v_0) > 0 \) or \( \| \nabla v_0 \| = 0 \).

(ii) All solutions to problem (1.1) with \( 0 < J(v_0) \leq \eta \) belong to \( V_\delta \) for \( \delta_1 < \delta < \delta_2 \), provided \( I(v_0) < 0 \).

Proposition 2.1. (Cf. [9], Theorem 2.5) Let \( v_0 \in H_{0}^{1}(U) \), \( 0 < J(v_0) \leq d \) and \( v(x,t) \) is a weak solution to the problem (1.1). Then we have the following conclusions:

1. If \( I(v_0) > 0 \) then \( I(v) > 0 \), \( \forall t \in [0,T) \)

2. If \( I(v_0) < 0 \) then \( I(v) < 0 \), \( \forall t \in [0,T) \),

where \( T \) is the maximal existence time of \( v \).

The proof’s of the above results follows from the similar lines given in [18] and [9]. So we omit the details.”

3 Global existence

In this section, we prove global existence of solution to (1.1) when \( I(v_0) \geq 0 \). We use the Galarkin method to find a sequence of approximate solution to (1.1). Using compactness arguments, we can find a subsequence of this approximate solution whose limit turns out to be a global solution to (1.1). The details are given in the following theorems.

Theorem 3.1. Let \( v_0 \in H_{0}^{1}(U) \). Assume \( I(v_0) \geq 0 \) and \( J(v_0) \leq d \), then there exists a global weak solution \( v \) to problem (1.1) with \( v \in L^\infty(0,\infty;H_{0}^{1}(U)) \) and \( v_t \in L^\infty(0,\infty;L^2(U)) \).

Proof. We divide the proof into two cases. In Case 1, we consider the situation where \( I(v_0) \geq 0 \), and \( J(v_0) < d \) and the next case deals with \( J(v_0) = d \) and \( I(v_0) \geq 0 \).

Case 1: \( I(v_0) \geq 0 \), \( J(v_0) < d \).

We have the following possibilities:

(a) Suppose \( I(v_0) \geq 0 \) and \( J(v_0) < 0 \). Then we get a contradiction to (2.4). Therefore this situation does not arise.

(b) Assume that \( I(v_0) \geq 0 \) and \( J(v_0) = 0 \). From (2.4) it readily implies that \( v_0 = 0 \). Thus \( v \equiv 0 \) is a global solution to (1.1).

(c) If \( 0 < J(v_0) < d \) and \( I(v_0) = 0 \), then we have \( J(v_0) \geq d \) (from the definition of \( d \)) which is a contradiction.

Hence the remaining possibility in this case is \( I(v_0) > 0 \) and \( 0 < J(v_0) < d \) and we focus only on this possibility. Let \( \{\psi_i\} \) be an orthogonal basis of \( H_{0}^{1}(U) \). Let \( V_m = \text{span}\{\psi_1,\psi_2,\ldots,\psi_m\} \) and the projection of the initial data on \( V_m \) be given by

\[
v_m(x,0) = \sum_{j=1}^{m} a_j \psi_j(x),
\]

i.e.,

\[
v_m(\cdot,0) \to v_0(\cdot), \quad \text{in} \quad H_{0}^{1}(U). \tag{3.1}
\]

We now seek a sequence of functions \( (v_m) \) of the form

\[
v_m(x,t) = \sum_{j=1}^{m} q_j^m(t) \psi_j(x), \quad m = 1,2,3,\ldots,
\]
where each $v_m$ satisfies the following approximated problem

$$
\begin{cases}
\left( \frac{\partial v_m}{\partial t}, \psi \right)_2 + \left( \frac{\partial \nabla v_m}{\partial t}, \nabla \psi \right) + \left( \nabla v_m, \nabla \psi \right)_2 = \left( v_m |v_m|^{p-1} \log |v_m|, \psi \right)_2, \quad \psi \in V_m,
\vspace{1mm}

v_m(x, 0) = \sum_{j=1}^{m} (v_0, \psi_j) \psi_j,
\vspace{1mm}

q_j^m(0) = a_j.
\end{cases}
$$

(3.2)

Observe that (3.2) is a system of nonlinear ordinary differential equations (ODEs) for the unknown functions $q_j^m(t)$. Using the standard existence theory of ODEs, we obtain

$$q_j : [0, t_m) \rightarrow \mathbb{R}, \quad j = 1, 2, 3, \ldots, m,$$

and $q_j$ satisfy (3.2) in a maximal interval $[0, t_m)$, where $t_m \in (0, T]$. We next show that for sufficiently large $m$, $v_m(., t) \in W$ for $0 < t < t_m$. For, we first notice that $v_m(., 0) \in W$ for sufficiently large $m$. If $v_m(., \tilde{t}) \in \partial W$ for some $\tilde{t}$, then we have either $v_m(., \tilde{t}) \in \mathcal{N}$ or $J(v_m(., \tilde{t})) = d$. In both cases we get $J(v_m(., \tilde{t})) \geq d$, which is contradiction. This shows that there exists $M \in \mathbb{N}$ such that $v_m(., t) \in W$ for $m \geq M, \ t \in [0, t_m)$.

On the other hand, on multiplying equation (3.2) with $v_m$, we get

$$
\int_0^t \left[ \frac{\partial |v_m|^2}{\partial t} + \frac{\partial \nabla |v_m|^2}{\partial t} \right] dt + J(v_m(., t)) = J(v_m(., 0)), \ m \in \mathbb{N}, \ t \in (0, t_m).
$$

(3.3)

This readily implies $J(v_m(., t)) < d$. Note that

$$
J(v_m(., t)) = \frac{1}{2} \| \nabla v_m \|^2 - \frac{1}{1 + p} \int_U |v_m|^{1+p} \log |v_m| dx + \frac{1}{(1+p)^2} \| v_m \|^{1+p}_{1+p} \\
\geq \frac{1}{1 + p} I(v_m) + \left( - \frac{1}{1 + p} + \frac{1}{2} \right) \| \nabla v_m \|^2 + \frac{1}{(1+p)^2} \| v_m \|^{1+p}_{1+p} \\
\geq \frac{p-1}{2p+2} \| \nabla v_m \|^2 + \frac{1}{(1+p)^2} \| v_m \|^{1+p}_{1+p}.
$$

Hence we find

$$
\| \nabla v_m \|^2 \leq \frac{2(1+p)}{p-1} J(v_m) \leq \frac{2(1+p)d}{p-1},
$$

(3.4)

and

$$
\| v_m \|^{1+p}_{1+p} \leq (1+p)^2 J(v_m) \leq (1+p)^2 d.
$$

(3.5)

From (3.3) and using $J(v_m) > 0$, we have

$$
\int_0^t \left[ \frac{\partial |v_m|^2}{\partial t} + \frac{\partial \nabla |v_m|^2}{\partial t} \right] dt < d.
$$

So the sequence of approximate solutions is uniformly bounded and it is independent of $m$ and $t$. Thus, we can extend $q_m$ to $[0, T]$. Let $\gamma = \frac{p+2}{2p+1}$, observe that

$$
\int_U (|v_m|^p \log |v_m|)^\gamma dx = \left( \int_{\{x \in U: v_m \leq 1\}} + \int_{\{x \in U: v_m > 1\}} \right) (|v_m|^p \log |v_m|)^\gamma dx,
$$

(3.6)

$$
\leq (ep)^{-\gamma} |U| + 2\gamma \int_U |v_m|^{\gamma(p+\frac{1}{2})} dx,
$$

$$
= (ep)^{-\gamma} |U| + 2\gamma \int_U |v_m|^{1+p} dx,
$$

$$
\leq (ep)^{-\gamma} |U| + 2(1+p)^2 d.
$$
where the last inequality follows from (3.5). Moreover, we obtain
\[
\begin{align*}
\frac{\partial v_m}{\partial t} & \text{ is uniformly bounded in } L^2((0, T); H^1_0(U)), \\
\{ \partial w_m \} & \text{ uniformly bounded in } L^2((0, T); H^1_0(U)), \\
v_m|v_m|^{p-1} \log |v_m| & \text{ uniformly bounded in } L^\infty((0, T); L^\gamma(U)).
\end{align*}
\]
From (3.7), it is easy to see that there exists a subsequence of \((v_m)\) which is still denoted by \((v_m)\) and \(v\) such that
\[
\begin{align*}
v_m & \overset{w^*}{\longrightarrow} v \text{ in } L^\infty((0, T); H^1_0(U)), \\
\frac{\partial v_m}{\partial t} & \overset{w^*}{\longrightarrow} \frac{\partial v}{\partial t} \text{ in } L^2((0, T); H^1_0(U)), \\
v_m|v_m|^{p-1} \log |v_m| & \overset{w^*}{\longrightarrow} v|v|^{p-1} \log |v| \text{ in } L^\infty((0, T); L^\gamma(U)).
\end{align*}
\]
In view of (3.8) and using the standard arguments, we can show that \(v\) satisfies the weak formulation (3.2). Repeating the same argument from \([T, 2T]\), \([2T, 3T]\), ..., we obtain that \(v\) is a global weak solution to (1.1), and \(v \in W\).

**Case 2:** Denote \(v_{0m} = \mu_m v_0\) where \(\mu_m = 1 - \frac{1}{m}\), \(m \geq 2\). Consider problem (1.1) with initial data \(v(x, 0) = v_{0m}(x)\) i.e.,
\[
\begin{align*}
w_t - \Delta w_t + \Delta w & = |w| |w|^{p-1} \log |w|, \quad t \in \mathbb{R}^+, \ x \in U, \\
w(x, t) & = 0, \quad t \in \mathbb{R}^+, \ x \in \partial U, \\
w(x, 0) & = v_{0m}(x), \quad x \in U.
\end{align*}
\]
Now observe that
\[
J(v_{0m}) = J(\mu_m v_0) < J(v_0) = d \quad \text{ and } \quad I(v_{0m}) > 0.
\]
Following the similar steps from Case 1, there exists a global solution to (3.9), say \(w_m\). Moreover, we have
\[
\int_0^t \left[ \left| \frac{\partial w_m}{\partial t} \right|^2 + \left( \frac{\partial \nabla w_m}{\partial t} \right)^2 \right] dt + J(w_m(., t)) = J(w_m(., 0)) < d.
\]
Also, by the invariance of \(W\), we get \(I(w_m(., t)) > 0\). Therefore there exists a subsequence of \(w_m\), which is still denoted with \(w_m\), such that
\[
\begin{align*}
w_m & \overset{w^*}{\longrightarrow} w \text{ in } L^\infty((0, T); H^1_0(U)), \\
\frac{\partial w_m}{\partial t} & \overset{w^*}{\longrightarrow} \frac{\partial w}{\partial t} \text{ in } L^2((0, T); H^1_0(U)),
\end{align*}
\]
for each \(T > 0\). Using the arguments presented in Case 1, we conclude that \(w \in W\) is a global weak solution to (1.1). Hence the theorem is proved.

Next we discuss the global existence of the solutions to (1.1) under the following restriction on \(p\) and dimension \(n\)
\[
1 < p < \begin{cases} 
\infty, & \text{if } n \leq 2, \\
\frac{4}{n-2}, & \text{if } 3 \leq n \leq 5.
\end{cases}
\]
For, we define
\[
\mathcal{N}_\alpha = \{ v \in \mathcal{N} : J(v) < \alpha \},
\]
\[
\Lambda_\alpha = \inf_{v \in \mathcal{N}_\alpha} \| v \|^2_{H^1_0(U)},
\]
where \(\alpha > d\).

**Remark 3.1.** For any \(\alpha > d\), if power index \(p\) satisfies (3.11) and \(\| \nabla v \| \neq 0\), then \(\Lambda_\alpha > 0\).
Proof. Let \( v \in \mathcal{N}_\alpha \) and \( \| \nabla v \| \neq 0 \). From the definition of \( I \), we get
\[
\| \nabla v \|^2 = \int_U |v|^{1+p} \log |v| dx
= |v|^p + 2
= C \| \nabla v \|^2 + 2,
\]
where \( C \) is the Sobolev embedding constant. From (3.12), one can easily get \( \| v \|^2_{H^1_0(U)} > \left( \frac{1}{\gamma} \right)^2 \). This completes the proof.

**Theorem 3.2.** Let \( v_0 \in H^1_0(U) \) and power index \( p \) satisfies (3.11). For a given \( \alpha > d \), assume that \( J(v_0) < \alpha \), \( \| v_0 \|^2_{H^1_0(U)} < \Lambda_\alpha \) and \( I(v_0) > 0 \), then there exists a global weak solution \( v \) to problem (1.1) with \( v \in L^\infty(0, \infty; H^1_0(U)) \) and \( v_t \in L^\infty(0, \infty; L^2(U)) \).

Proof. Using the same argument employed in Theorem 3.1, we obtain a sequence \( v_m \) which satisfies (3.2). From (3.1), it is clear \( I(v_m(\cdot,0)) > 0 \), for sufficiently large \( m \). Next we show \( I(v_m(\cdot, t)) > 0 \), for sufficiently large \( m \). On the contrary, we assume that there exists a \( t > 0 \) such that \( I(v_m(\cdot, \tilde{t})) = 0 \) and \( I(v_m(\cdot, t)) > 0 \), \( 0 < t < \tilde{t} \). From the fact
\[
\frac{d}{dt} \| v_m(\cdot, t) \|^2_{H^1_0(U)} = -2I(v_m(\cdot, t)),
\]
we deduce that \( t \mapsto \| v_m(\cdot, t) \|^2_{H^1_0(U)} \) is a decreasing function, for \( 0 < t < \tilde{t} \). Hence, we conclude that
\[
\| v_m(\cdot, \tilde{t}) \|^2_{H^1_0(U)} < \| v_m(\cdot, 0) \|^2_{H^1_0(U)} < \Lambda_\alpha.
\]
On the other hand, (2.5) gives us \( J(v_m(\cdot, \tilde{t})) < J(v_m(\cdot, 0)) \), which is a contradiction to (3.14). Therefore for sufficiently large \( m \), we obtain \( I(v_m(\cdot, t)) > 0, t \geq 0 \). The rest of the proof follows along the same lines of that Theorem 3.1 and the only difference is \( d \) needs to be replaced by \( \alpha \).

## 4 Decay estimate

In this section, we prove a few decay estimates of the global solutions to (1.1) in the \( H^1_0 \)-norm whenever \( J(v_0) \leq d \), \( I(v_0) > 0 \). As in the proof of Theorem 3.1, we consider the cases \( 0 < J(v_0) < d \), \( I(v_0) > 0 \); \( J(v_0) = d \), \( I(v_0) > 0 \) separately. We begin with the subcritical case, i.e., \( 0 < J(v_0) < d \), \( I(v_0) > 0 \). In this case we take advantage of the invariance of the family of potential wells described in Theorem 2.1 to prove the exponential decay of solutions to (1.1) in the \( H^1_0 \) norm.

**Theorem 4.1.** Assume that \( I(v_0) > 0 \), \( J(v_0) < d \) and \( v \) is a global solution to (1.1). Then there exist constants \( \mu > 0 \) and \( C > 0 \) such that
\[
\| v(\cdot, t) \|^2_{H^1_0(U)} \leq Ce^{-\mu t}, \quad 0 \leq t < \infty.
\]

Proof. From the proof of Theorem 3.1 we get \( I(v(\cdot, t)) > 0 \), and \( 0 < J(v(\cdot, t)) < d \). Therefore from Theorem 2.1 we deduce \( I_\delta(v(\cdot, t)) > 0 \) for \( 0 < \delta < 1 \). Set \( \beta = 1 - \delta \) and observe that
\[
\int_U |v(x,t)|^{p+1} \log |v(x,t)| dx < (1 - \beta) \| \nabla v(\cdot, t) \|^2,
\]
or
\[
\beta \| \nabla v(\cdot, t) \|^2 < I(v(\cdot, t)).
\]
On the other hand, it follows immediately that
\[
\frac{d}{dt} \| v(\cdot, t) \|^2_{H^1_0(U)} = -2I(v(\cdot, t)).
\]
Now using (4.2)–(4.3), we obtain
\[
\frac{d}{dt} \|v(\cdot, t)\|^2_{H_0^1(U)} = -2I(v(\cdot, t)) < -2\beta \|\nabla v(\cdot, t)\|^2 \leq -2\beta \frac{\lambda_1}{1 + \lambda_1} \|v(\cdot, t)\|^2_{H_0^1(U)},
\]
where \(\lambda_1\) is the optimal constant in the Poincaré inequality. Finally, Gronwall’s lemma gives
\[
\|v(\cdot, t)\|_{H_0^1(U)} \leq \|v_0\|_{H_0^1(U)} e^{-\mu t}, \quad t \geq 0,
\]
where \(\mu = \frac{1}{1 + \lambda_1}\). This completes the proof. \(\square\)

We now turn our attention towards the critical case, i.e., \(J(v_0) = d\), \(I(v_0) > 0\).

**Theorem 4.2.** Assume \(I(v_0) \geq 0\), \(J(v_0) = d\), and \(v\) is a global solution to (1.1). Then there exist constants \(C > 0\), \(\mu > 0\) such that
\[
\|v(\cdot, t)\|_{H_0^1(U)} \leq C e^{-\mu t}, \quad 0 \leq t < \infty.
\]

**Proof.** From the proof of Theorem 5.1, it follows that \(I(v(\cdot, t)) \geq 0\) for \(0 \leq t < \infty\). We complete the proof by considering following two cases.

**Case 1.** Assume that \(I(v(\cdot, t)) > 0\) for \(t \geq 0\). Then from the relation
\[
\langle v_1, v \rangle + \langle \nabla v_1, \nabla v \rangle_2 = -I(v(\cdot, t)) < 0,
\]
it follows that \(\|v_1\|_{H_0^1(U)} > 0\) and \(\int_0^t \|v(\cdot, \tau)\|^2_{H_0^1(U)} d\tau\) is strictly increasing in \([0, \infty)\). Therefore from (2.5), we get
\[
J(v(\cdot, t_1)) = -\int_0^{t_1} \|v(\cdot, \tau)\|^2_{H_0^1(U)} d\tau + J(v_0) < d.
\]
Using the arguments that are employed in the proof of the decay estimate in Theorem 5.1, it is easy to obtain the exponential decay (4.5).

**Case 2.** Let if possible there exists \(t_1 > 0\) such that \(I(v(\cdot, t)) > 0\) for \(0 \leq t < t_1\) and \(I(v(\cdot, t_1)) = 0\). Now two possibilities can arise, they are: (i) \(\|\nabla v(\cdot, t_1)\| = 0\), (ii) \(\|\nabla v(\cdot, t_1)\| > 0\).

We now prove that \(\|\nabla v(\cdot, t_1)\| > 0\) can not hold. For, it is enough to show that if \(\|\nabla v(\cdot, t_1)\| > 0\) then \(I(v(\cdot, t_1)) > 0\).

**Claim.** If \(\|\nabla v(\cdot, t_1)\| > 0\) then \(I(v(\cdot, t_1)) > 0\).

For \(0 \leq t < t_1\), since \(\langle v_1, v \rangle + \langle \nabla v_1, \nabla v \rangle_2 = -I(v(\cdot, t))\), it follows that \(t \mapsto \int_0^t \|v_1\|^2_{H_0^1(U)} dt\) is strictly increasing. Owing to (2.5), we get
\[
J(v(\cdot, t_1)) = -\int_0^{t_1} \|v(\cdot, \tau)\|^2_{H_0^1(U)} d\tau + J(v_0) < d.
\]
Since \(\|\nabla v(\cdot, t_1)\| > 0\) and \(I(v(\cdot, t_1)) = 0\), from the definition of \(d\) we get \(J(v(\cdot, t_1)) \geq d\), which is contradiction to (4.4). This proves Claim.

Therefore we have \(\|\nabla v(\cdot, t_1)\| = 0\). Hence one can easily deduce that \(v\) satisfies (1.3).

\(\square\)

# 5 Finite time blowup

In this section, we prove some blowup results when \(I(v_0) < 0\). In fact, we use different tools to prove that the solutions exhibits finite time blow up depending on the value of \(J(v_0)\).

**Theorem 5.1.** Let \(v_0 \in H_0^1(U)\). Assume \(I(v_0) < 0\) and \(J(v_0) < d\), then the weak solution to problem (1.1) blows up in finite time, i.e., there exists \(T > 0\) such that
\[
\lim_{t \to T^-} \|v\|_{H_0^1(U)} = \infty.
\]
Proof. Let \( v(x,t) \) be any solution to problem (1.1) with \( I(v_0) < 0 \) and \( J(v_0) < d \).

Define the function \( N : [0, \infty) \to \mathbb{R}^+ \) by \( N(t) = \int_0^t \| v \|_{H^1_0(U)}^2 \, d\tau \). Then an easy computation yields

\[
\dot{N}(t) = \| v \|_{H^1_0(U)}^2, \quad \ddot{N}(t) = -2I(v).
\]

(5.1)

From (2.4), (2.5), and the Poincaré inequality there exists \( \lambda_1 > 0 \) such that

\[
\dot{N}(t) = -2(1 + p)J(v) + (p - 1)\| \nabla v \|^2 + \frac{2}{1 + p} \| v \|_{1 + p}^4
\]

\geq 2(1 + p) \int_0^t \| v \|_{H^1_0(U)}^2 \, d\tau + (p - 1)\frac{\lambda_1}{\lambda_1 + 1} \dot{N}(t) - 2(1 + p)J(v_0).
\]

(5.2)

Since

\[
\left( \int_0^t (v_\tau, v)_{H^1_0(U)} \, d\tau \right)^2 = \left( \frac{1}{2} \int_0^t \| v \|_{H^1_0(U)}^2 \, d\tau \right)^2
\]

\[
= \frac{1}{4} \left( \dot{N}^2(t) - 2\| v_0 \|_{H^1_0(U)}^2 \dot{N}(t) + \| v_0 \|_{H^1_0(U)}^4 \right),
\]

(5.3)

we obtain

\[
N\ddot{N} - \left( \frac{1 + p}{2} \right) \dot{N}^2 \geq (p - 1)\frac{\lambda_1}{\lambda_1 + 1} N\dot{N} - 2(1 + p)J(v_0)N - (1 + p)\| v_0 \|_{H^1_0(U)}^2 \dot{N} + \frac{1 + p}{2} \| v_0 \|_{H^1_0(U)}^4
\]

\[+ 2(1 + p) \left\{ \int_0^t \| v \|_{H^1_0(U)}^2 \, d\tau \right\} \int_0^t \| v \|_{H^1_0(U)}^2 \, d\tau - \left( \int_0^t (v_\tau, v)_{H^1_0(U)} \, d\tau \right)^2 \}
\]

By Hölder’s inequality, we get

\[
N\ddot{N} - \left( \frac{1 + p}{2} \right) \dot{N}^2 \geq (p - 1)\frac{\lambda_1}{\lambda_1 + 1} N\dot{N} - 2(1 + p)J(v_0)N - (1 + p)\| v_0 \|_{H^1_0(U)}^2 \dot{N}.
\]

(5.4)

Claim: For large \( t \), it follows that

\[
N\ddot{N} - \left( \frac{1 + p}{2} \right) \dot{N}^2 > 0.
\]

(5.5)

To prove this claim we consider two cases and discuss separately.

Case-1: Assume \( J(v_0) \leq 0 \). From (5.1) and (5.2), we get \( \ddot{N} \geq 0 \). Since, \( \dot{N}(t) = \| v \|_{H^1_0(U)}^2 \geq 0 \), then there exists \( t_0 \geq 0 \) such that \( \ddot{N}(t_0) > 0 \) and

\[
N(t) \geq N(t_0) + \dot{N}(t_0)(t - t_0) > \dot{N}(t_0)(t - t_0), \quad t \geq t_0.
\]

Thus for \( t \) large, we have \( (p - 1)\lambda_1 N > (1 + p) \| v_0 \|_{H^1_0(U)}^2 \) and (5.5) holds.

Case-2: Assume that \( 0 < J(v_0) < d \). From Theorem 2.4 we get \( v(\cdot, t) \in V_\delta \) for \( 1 \leq \delta < \delta_2 \) and \( t > 0 \), where \( \delta_2 \) is the same as the one introduced in Theorem 2.4. Thus \( I_\delta(v(\cdot, t)) < 0 \), for \( 1 \leq \delta < \delta_2 \), \( t \geq 0 \). Next we prove that \( \| \nabla v \|^2 \geq \frac{\lambda_1}{1 + \lambda_1} \| v_0 \|_{H^1_0(U)}^2 \), \( t \geq 0 \). For, since \( \frac{\lambda_1}{1 + \lambda_1} \| v_0 \|_{H^1_0(U)}^2 = -2I(v) > 0 \), we obtain \( t \to \| v \|_{H^1_0(U)}^2 \), \( t \geq 0 \) is a strictly increasing function. On the other hand, the Poincaré inequality gives

\[
\| \nabla v \|^2 \geq \frac{\lambda_1}{1 + \lambda_1} \| v_0 \|_{H^1_0(U)}^2 > \frac{\lambda_1}{1 + \lambda_1} \| v_0 \|_{H^1_0(U)}^2.
\]

Therefore it is easy to get that \( \| \nabla v \|^2 > \frac{\lambda_1}{1 + \lambda_1} \| v_0 \|_{H^1_0(U)}^2 > 0 \), \( t \geq 0 \). Now from (5.1) and the definition of \( I_\delta \), we find that

\[
\dot{N}(t) \geq -2I_\delta(v) + 2(\delta_2 - 1)\| \nabla v \|^2 \geq 2(\delta_2 - 1)^2 \delta_2 > 0,
\]

\[
\dot{N}(t) \geq \dot{N}(0) + 2(\delta_2 - 1)^2 \delta_2 t \geq 2(\delta_2 - 1)^2 \delta_2 t,
\]
and
\[ N(t) \geq N(0) + (\delta_2 - 1)r^2(\delta_2)t^2 \geq (\delta_2 - 1)r^2(\delta_2)t^2. \]
Thus for \( t \) large, we have
\[
\begin{cases}
(p - 1)\frac{\lambda_1}{\lambda_1 + 1}N(t) > 2(1 + p)\|v_0\|_{H_1^0(U)}^2, \\
(p - 1)\frac{\lambda_1}{\lambda_1 + 1}N(t) > 4(1 + p)J(v_0).
\end{cases}
\tag{5.6}
\]
On substituting (5.6) in (5.4), we get (5.5) for sufficiently large \( t \).
On the other hand, a straightforward calculation gives
\[ (N^{-\alpha})'' = -\alpha N^{-\alpha-2}(N\dot{N} - (\alpha + 1)\ddot{N}). \]
For \( \alpha = \frac{p-1}{2} \), (5.6) implies \((N^{-\alpha})'' < 0\) for sufficiently large \( t > 0 \). Hence, for \( t > \tilde{t} \), we can write
\[ N^{-\frac{p-1}{2}}(t) < N^{-\frac{p-1}{2}}(\tilde{t}) \left( 1 - \left( \frac{p-1}{2} \right) \frac{\dot{N}(\tilde{t})}{\ddot{N}(\tilde{t})}(t - \tilde{t}) \right), \]
which implies that there exists \( T > 0 \) such that
\[ \lim_{t \to T^-} N^{-\frac{p-1}{2}}(t) = 0. \]
This completes the proof.

We now focus on the critical case, i.e., \( J(v_0) = d, I(v_0) < 0 \).

**Theorem 5.2.** Let \( v_0 \in H_0^1(U) \). Assume that \( I(v_0) < 0 \) and \( J(v_0) = d \), then the weak solution \( v \) to equation (1.1) blows up in finite time.

**Proof.** Assume that \( T \) is the existence time of \( v \). We need to prove that \( T < \infty \). From the continuity of \( I(v) \) and \( J(v) \), there exists \( t_1 \in (0, T) \) small enough such that \( I(v(\cdot, t)) < 0 \) and \( J(v(\cdot, t)) > 0 \) for \( t \in [0, t_1] \).

Therefore \( \int_0^t \|v_r\|_{H_1^0(U)}^2 d\tau \) is strictly increasing for \( t \in [0, t_1] \). From (2.5), we can choose \( t_1 \) such that
\[ 0 < J(v(\cdot, t_1)) = J(v_0) - \int_0^{t_1} \|v_r\|_{H_1^0(U)}^2 d\tau < J(v_0) = d. \tag{5.7} \]
If we take \( t_1 \) as the initial time in Theorem 5.1, it follows that the maximal existence time \( T \) of \( v \) is finite, i.e.,
\[ \lim_{t \to T^-} \|v(t)\|_{H_1^0(U)} = \infty. \]
This completes the proof.

Finally, we discuss the finite time blow-up property when the initial energy is high.

**Theorem 5.3.** Let \( v_0 \in H_0^1(U) \). Moreover assume that the initial data satisfies
(i) \( J(v_0) > 0 \),
(ii) \( \|v_0\|_{H_1^0(U)}^2 > \frac{2(\lambda_1 + 1)(1 + p)}{\lambda_1 (p-1)} J(v_0) \),
(iii) \( I(v_0) < 0 \),
where \( \lambda_1 \) is the optimal constant in the Poincaré inequality. Then the weak solution \( v \) to equation (1.1) blows up in finite time.
Proof. As in the proof of Theorem 5.1, we work with the quantity \( N(t) = \int_0^t \|v\|^2_{H_0^1(U)} \, d\tau \). We prove the theorem in two steps: (i) \( I(v) < 0 \) and \( \|v(t)\|^2_{H_0^1(U)} > \frac{2(\lambda_1 + 1)(1 + p)}{\lambda_1(p - 1)} J(v_0) \), for every \( t \in (0, T) \), (ii) for sufficiently large \( t \), \( \dot{N}N - \frac{1 + p}{2} \dot{N}^2 > 0 \).

Step 1. Let if possible, there exists \( t_0 \in (0, T) \) such that \( I(v(\cdot, t_0)) = 0 \) and \( I(v(\cdot, t)) < 0 \) for \( 0 \leq t < t_0 \). Again consider the function \( N(t) \) as in Theorem 5.1. Since \( \dot{N}(t) = -2I(v) > 0 \) for \( t \in [0, t_0] \), \( \dot{N} \) is increasing in \([0, t_0]\), we deduce that

\[
\dot{N}(t_0) > \dot{N}(0) = \|v_0\|^2_{H_0^1(U)} > \frac{2(\lambda_1 + 1)(1 + p)}{\lambda_1(p - 1)} J(v_0).
\]  

(5.8)

Since \( I(v(\cdot, t_0)) = 0 \), the conservation of energy gives

\[
J(v_0) \geq J(v(\cdot, t_0)) = \frac{p - 1}{2(1 + p)} \|\nabla v(\cdot, t_0)\|^2 + \frac{1}{(1 + p)^2} \|v(\cdot, t_0)\|^2_{L^2(U)} \geq 0.
\]  

(5.9)

Hence we get \( \dot{N}(t_0) = \|v(\cdot, t_0)\|^2 \leq \frac{2(\lambda_1 + 1)(1 + p)}{\lambda_1(p - 1)} J(v_0) \), which is a contradiction to (5.8). Thus we obtain

\[
I(v) < 0, \ t \in (0, T),
\]

and (5.8) holds for every \( t \in (0, T) \). Hence \( N \) is strictly increasing. Therefore for large \( t \), we obtain

\[
N(t) > \frac{2(\lambda_1 + 1)(1 + p)}{\lambda_1(p - 1)} \|v_0\|^2_{H_0^1(U)}.
\]  

(5.10)

Step 2. From (5.4), (5.8) and (5.10), we deduce that for sufficiently large \( t \),

\[
\dot{N}N - \frac{1 + p}{2} \dot{N}^2 \geq \frac{N}{\lambda_1 + 1} N \dot{N} - 2(1 + p) J(v_0) N - (1 + p) \|v_0\|^2_{H_0^1(U)} \dot{N} > 0.
\]  

(5.11)

Using the arguments employed in Theorem 5.1 we conclude the proof of the theorem.

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