Geometry of Symplectic Intersections

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Abstract

In this paper we survey several intersection and non-intersection phenomena appearing in the realm of symplectic topology. We discuss their implications and finally outline some new relations of the subject to algebraic geometry.

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1. Introduction

Symplectic geometry exhibits a range of intersection phenomena that cannot be predicted nor explained on the level of pure topology or differential geometry. The main players in this game are certain pairs of subspaces (e.g. Lagrangian submanifolds, domains, or a mixture of both) whose mutual intersections cannot be removed (or reduced) via the group of Hamiltonian or symplectic diffeomorphisms. The very first examples of such phenomena were conjectures by Arnold in the 1960’s, and eventually established and further explored by Gromov, Floer and others starting from the mid 1980s.

The first part of the paper will survey several intersection phenomena and the mathematical tools leading to their discovery. We shall not attempt to present the most general results and since the literature is vast the exposition will be far from complete. Rather we shall concentrate on various intersection phenomena trying to understand their nature and whether there is any relations between them.

The second part is dedicated to “non-intersections”, namely to situations where the principles of symplectic intersections break down. In the case of Lagrangian submanifolds this absence of intersections is reflected in the vanishing of a symplectic invariant called Floer homology. This vanishing when interpreted algebraically leads to restrictions on the topology of Lagrangian submanifolds. As a byproduct we shall explain how these restrictions can be used to study some problems in algebraic geometry concerning hyperplane sections and degenerations.

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2. Various intersection phenomena

In this section we shall make a brief tour through the zoo of symplectic intersections, encountering three different species.

Before we start let us recall two important notions from symplectic geometry. Let \((M, \omega)\) be a symplectic manifold. A submanifold \(L \subset M\) is called **Lagrangian** if 
\[ \dim L = \frac{1}{2} \dim M \quad \text{and} \quad \omega \text{ vanishes on } T(L). \]
From now on we assume all Lagrangian submanifolds to be closed. The second notion is of **Hamiltonian isotopies**. An isotopy of diffeomorphisms \(\{ h_t : M \to M \}_{0 \leq t \leq 1}\), starting with \(h_0 = \text{id}\) is called Hamiltonian if the (time-dependent) vector field \(\xi_t\) generating it satisfies that the 1-forms \(i_{\xi_t} \omega\) are exact for all \(0 \leq t \leq 1\). Note that Hamiltonian isotopies preserve the symplectic structure: \(h^* \omega = \omega\) for all \(t\). Finally, two subsets \(A, B \subset M\) are said to be Hamiltonianly isotopic if there exists a Hamiltonian isotopy \(h_t\) such that \(h_1(A) = B\). We refer the reader to [28] for the foundations of symplectic geometry.

2.1. Lagrangians intersect Lagrangians

The most fundamental **Lagrangian intersection** phenomenon occurs in cotangent bundles. Let \(X\) be a closed manifold and \(T^*(X)\) be its cotangent bundle endowed with the canonical symplectic structure \(\omega_{\text{can}} = \sum dp_i \wedge dq_i\). Denote by \(\lambda_{\text{can}} = \sum p_i dq_i\) the Liouville form (so that \(\omega_{\text{can}} = d\lambda_{\text{can}}\)). Recall that a Lagrangian submanifold \(L \subset T^*(X)\) is called exact if the restriction \(\lambda_{\text{can}}|_{T(L)}\) is exact. Note that the property of exactness is preserved by Hamiltonian isotopies. Denote by \(O_X \subset T^*(X)\) the zero-section. The following theorem was proved by Gromov in [22]:

**Theorem A.** Let \(L \subset T^*(X)\) be an exact Lagrangian submanifold. Then:
1) For every Lagrangian \(L'\) which is Hamiltonianly isotopic to \(L\) we have \(L \cap L' \neq \emptyset\).
2) \(L \cap O_X \neq \emptyset\). In particular, \(L\) cannot be separated from the zero-section by any Hamiltonian isotopy.

If one assumes \(L\) to be a Hamiltonian image of the zero-section a more quantitative version of Theorem A holds:

**Theorem B.** Let \(L \subset T^*(X)\) be a Lagrangian submanifold which is Hamiltonianly isotopic to the zero-section and intersects it transversely. Then
\[ \# L \cap O_X \geq \sum_{j=0}^{\dim X} b_j(X), \]
where \(b_j(X)\) are the Betti numbers of \(X\).

Chronologically Theorem B preceded Theorem A. It was conjectured by Arnold (see [3] for the history), first proved for \(X = \mathbb{T}^n\) by Chaperon [12] and generalized to all cotangent bundles by Hofer [23] and by Laudenbach and Sikorav [26]. Now a days it can be viewed as a special case of Floer theory (see Section 2.4 below).

Note that the intersections described by both theorems above cannot in general be understood on a purely topological level. Indeed, in general topology predicts less than \(\sum b_j(X)\) intersection points, and sometimes even none. Finally, note that in general the statement of Theorem B fails if one assumes \(L\) to be only symplectically isotopic to \(O_X\), as the example \(X = \mathbb{T}^n\) shows.
2.2. Balls intersect balls

Denote by $B^{2n}(R)$ the closed Euclidean ball of radius $R$, endowed with the standard symplectic structure induced from $\mathbb{R}^{2n}$. Denote by $\mathbb{C}P^n$ the complex projective space, endowed with its standard Kähler form $\sigma$, normalized so that $\int_{\mathbb{C}P^n} \sigma = \pi$. The following obstruction for symplectic packing was discovered by Gromov [22]:

**Theorem C.** Let $M$ be either $B^{2n}(1)$ or $\mathbb{C}P^n$. Let $B_{\varphi_1}, B_{\varphi_2} \subset M$ be the images of two symplectic embeddings $\varphi_1 : B^{2n}(R_1) \to M$, $\varphi_2 : B^{2n}(R_2) \to M$. If $R_1^2 + R_2^2 \geq 1$ then $B_{\varphi_1} \cap B_{\varphi_2} \neq \emptyset$.

Since symplectic embeddings are also volume preserving there is an obvious volume obstruction for having $B_{\varphi_1} \cap B_{\varphi_2} = \emptyset$. However, volume considerations predict an intersection only if $R_1^2 + R_2^2 \geq 1$ (moreover for volume preserving embeddings the latter inequality is sharp).

When one considers embeddings of several balls things become more complicated and interesting. Here results are currently available only in dimension 4.

**Theorem D.** Let $M$ be either $B^4(1)$ or $\mathbb{C}P^2$, and let $B_{\varphi_1}, \ldots, B_{\varphi_N} \subset M$ be the images of symplectic embeddings $\varphi_k : B^4(R) \to M$, $k = 1, \ldots, N$, of $N$ balls of the same radius $R$. Then there exist $i \neq j$ such that $B_{\varphi_i} \cap B_{\varphi_j} \neq \emptyset$ in each of the following cases:

1. $N = 2$ or $3$ and $R^2 \geq 1/2$.
2. $N = 5$ or $6$ and $R^2 \geq 2/5$.
3. $N = 7$ and $R^2 \geq 3/8$.
4. $N = 8$ and $R^2 \geq 6/17$.

Moreover all the above inequalities are sharp in the sense that in each case if the inequality on $R$ is not satisfied then there exist symplectic embeddings $\varphi_1, \ldots, \varphi_N$ as above with disjoint images $B_{\varphi_1}, \ldots, B_{\varphi_N} \subset M$.

Statement 2 for $N = 5$ was proved by Gromov [22]. The rest was established by McDuff and Polterovich [27]. Let us mention that for $N = 4$ and any $N \geq 9$ this intersection phenomenon completely disappears in the sense that an arbitrarily large portion of the volume of $M$ can be filled by a disjoint union of $N$ equal balls (see [27] for $N = 4$ and $N = k^2$, and [5] [6] for the remaining cases).

2.3. Balls intersect Lagrangians

It turns out that there exist (symplectically) irremovable intersections also between contractible domains (e.g. balls) and Lagrangian submanifolds.

Denote by $\mathbb{R}P^n \subset \mathbb{C}P^n$ the Lagrangian $n$-dimensional real projective space (embedded as the fixed point set of the standard conjugation of $\mathbb{C}P^n$). The following was proved in [7]:

**Theorem E.** Let $B_{\varphi} \subset \mathbb{C}P^n$ be the image of a symplectic embedding $\varphi : B^{2n}(R) \to \mathbb{C}P^n$. If $R^2 \geq 1/2$ then $B_{\varphi} \cap \mathbb{R}P^n \neq \emptyset$. Moreover the inequality is sharp, namely for every $R^2 < 1/2$ there exists a symplectic embedding $\varphi : B^{2n}(R) \to \mathbb{C}P^n$ whose image avoids $\mathbb{R}P^n$.

In fact this pattern of intersections occurs in a wide class of examples (see [7]):
Theorem E'. Let $(M, \omega)$ be a closed Kähler manifold with $[\omega] \in H^2(M; \mathbb{Q})$ and $\pi_2(M) = 0$. Then for every $\epsilon > 0$ there exists a Lagrangian CW-complex $\Delta_\epsilon \subset (M, \omega)$ with the following property: every symplectic embedding $\varphi : B^{2n}(\epsilon) \to (M, \omega)$ must satisfy $\text{Image} (\varphi) \cap \Delta_\epsilon \neq \emptyset$.

By a Lagrangian CW-complex we mean a subspace $\Delta_\epsilon \subset M$ which topologically is a CW-complex and the interior of each of its cells is a smoothly embedded disc of $M$ on which $\omega$ vanishes.

2.4. Methods for studying intersections

Lagrangian intersections. The first systematic study of Lagrangian intersections was based on the theory of generating function [12, 26] (an equivalent theory was independently developed in contact geometry [13]). Gromov’s theory of pseudo-holomorphic curves [22] gave rise to an alternative approach which culminated in what is now called Floer theory. Each of these theories has its own advantage. Floer theory works in larger generality and seems to have a richer algebraic structure, on the other hand the theory of generating functions leads in some cases to sharper results (see [20]).

Since Floer theory will appear in the sequel, let us outline a few facts about it (the reader is referred to the works of Floer [16] and of Oh [29, 30] for details). Let $(M, \omega)$ be a symplectic manifold and $L_0, L_1 \subset (M, \omega)$ two Lagrangian submanifolds. In “ideal” situations Floer theory assigns to this data an invariant $HF(L_0, L_1)$. This is a $\mathbb{Z}_2$-vector space obtained through an infinite dimensional version of Morse-Novikov homology performed on the space of paths connecting $L_0$ to $L_1$. The result of this theory is a chain complex $CF(L_0, L_1)$ whose underlying vector space is generated by the intersection points $L_0 \cap L_1$ (one perturbs $L_0, L_1$ so their intersection becomes transverse). The homology of this complex $HF(L_0, L_1)$ is called the Floer homology of the pair $(L_0, L_1)$. The most important feature of $HF(L_0, L_1)$ is its invariance under Hamiltonian isotopies: if $L_0', L_1'$ are Hamiltonianly isotopic to $L_0, L_1$ respectively, then $HF(L_0', L_1') \cong HF(L_0, L_1)$. From this point of view $HF(L_0, L_1)$ can be regarded as a quantitative obstruction for Hamiltonianly separating $L_0$ from $L_1$. Indeed, the rank of $HF(L_0, L_1)$ is a lower bound on the number of intersection points of any pair of transversally intersecting Lagrangians $L_0', L_1'$ in the Hamiltonian deformation classes of $L_0, L_1$ respectively.

Let us explain the “ideal situations” in which Floer homology is defined. First of all there are restrictions on $M$: due to analytic difficulties manifolds are required to be either closed or to have symplectically convex ends (e.g. $\mathbb{C}^n$, cotangent bundles or any Stein manifold). More serious restrictions are posed on the Lagrangians. For simplicity we describe them only for the case when $L_1$ is Hamiltonianly isotopic to $L_0$. From now on we shall write $L = L_0$ and $L' = L_1$. In Floer’s original setting [16] the theory was defined under the assumption that the homomorphism $A_\omega : \pi_2(M, L) \to \mathbb{R}$, defined by $D \mapsto \int_D \omega$, vanishes. The reason for this comes from the construction of the differential of the Floer complex: the main obstruction for defining a meaningful differential turns out to be existence of holomorphic discs with boundary on $L$ or $L'$. These discs appear as a source of non-compactness of the space of solutions of the PDEs involved in the construction. Since holomorphic
discs must have positive symplectic area the assumption \( A_\omega = 0 \) rules out their existence. Under this assumption Floer defined \( HF(L, L') \) and proved its invariance under Hamiltonian isotopies. Moreover he showed that \( HF(L, L) \) is isomorphic to the singular cohomology \( H^*(L; \mathbb{Z}_2) \) of \( L \). This together with the invariance give:

**Theorem F.** Let \((M, \omega)\) be a symplectic manifold, either compact or with symplectically convex ends. Let \( L \subset (M, \omega) \) be a Lagrangian submanifold with \( A_\omega = 0 \). Then for every Lagrangian \( L' \) which is Hamiltonianly isotopic to \( L \) and intersects \( L \) transversally we have: \# \( L \cap L' \geq \text{rank} \, HF(L, L') = \text{rank} \, H^*(L; \mathbb{Z}_2) \). In particular \( L \) cannot be separated from itself by a Hamiltonian isotopy.

Floer theory was extended by Oh \[30\] to cases when \( A_\omega \neq 0 \). There are two assumptions needed for this extension to work: the Maslov homomorphism \( \mu : \pi_2(M, L) \to \mathbb{Z} \) should be positively proportional to \( A_\omega \) (such Lagrangians are called monotone). The second assumption is that the positive generator \( N_L \) of the subgroup \( \text{Image} \mu \subset \mathbb{Z} \) is at least 2. In this setting Oh defined \( HF(L, L') \) and proved its invariance under Hamiltonian isotopies. It is however no longer true in general that \( HF(L, L) \) is isomorphic to \( H^*(L; \mathbb{Z}_2) \). Still, Oh proved \[29\] that \( HF(L, L) \) is related to \( H^*(L; \mathbb{Z}_2) \) through a spectral sequence. Recently the theory was considerably generalized by Fukaya, Oh, Ohta and Ono \[21\].

**Intersections of balls.** Theorems C and D were obtained using Gromov’s theory of pseudo-holomorphic curves. The hard-core of the proofs consists of existence of pseudo-holomorphic curves of specified degrees that pass through a prescribed number of points in the manifold (see \[22, 27\]) for the details). From a more modern perspective it can be viewed as an early application of Gromov-Witten invariants.

Finally, Theorems E and E’ are proved by a decomposition technique introduced in \[7\] which enables to decompose symplectic manifolds as a disjoint union of a symplectic disc bundle and a Lagrangian \( CW \)-complex. A variation on the proof of Gromov’s non-squeezing theorem \[22\] gives an upper bound on the radius of a symplectic ball that can be squeezed inside that disc bundle. Hence, a larger ball must always intersect this \( CW \)-complex. For \( M = \mathbb{CP}^n \), the corresponding \( CW \)-complex turns out to be a smooth copy of \( \mathbb{RP}^n \). See \[7\] for the details.

### 3. Some questions and speculations

**Cotangent bundles.** The following questions show that even in the case of cotangent bundles the most fundamental invariants are not completely understood.

1. Let \( L \subset T^*(X) \) be an exact Lagrangian (not necessarily Hamiltonianly isotopic to \( O_X \)). By Theorem A, \( L \cap O_X \neq \emptyset \). Is it true that \( HF(L, O_X) \neq 0 \)?
2. Let \( L_0, L_1 \subset T^*(X) \) be two exact Lagrangians (again, not necessarily Hamiltonianly isotopic neither to \( O_X \) nor to each other). Is it true that \( L_0 \cap L_1 \neq \emptyset \)? Is it true that \( HF(L_0, L_1) \neq 0 \)?

These questions are of a theoretical importance, since the zero section and its Hamiltonian images are the only known examples of exact Lagrangians in \( T^*(X) \).

**Symplectic packing.** Lack of tools (or new ideas) prevent us from understanding
symplectic packings in dimension higher than 4. The only packing obstructions known in these dimensions are described in Theorem C. Note that $\mathbb{C}P^n$ admits full packing by $N = k^n$ equal balls \[^{27}\], but it is unclear what happens for other values of $N$. In view of this and Theorem C, the first unknown case (for $n \geq 3$) is of $N = 2^n + 1$ equal balls.

The situation in dimension 4 is only slightly better. Except of $\mathbb{C}P^2$ and a few other rational surfaces no packing obstructions are known. It is known that for every symplectic 4-manifold $(M, \omega)$ with $[\omega] \in H^2(M; \mathbb{Q})$ packing obstruction (for equal balls) disappear once the number of balls is large enough (see \[^{6}\]), but nothing is known when the number of balls is small. In fact even the case of one ball is poorly understood (namely, what is the maximal radius of a ball that can be symplectically embedded in $M$). The reason here is that the methods yielding packing obstructions strongly rely on the geometry of algebraic and pseudo-holomorphic curves in the manifold. The problem is that most symplectic manifolds have very few (or none at all) $J$-holomorphic curves for a generic choice of the almost complex structure. Thus, even in dimension 4 it is unknown whether or not packing obstructions is a phenomenon particular to a sporadic class of manifolds such as $\mathbb{C}P^2$.

**Is everything Lagrangian?** Weinstein’s famous saying could be relevant for the intersection described in Theorems C,D and E. In other words, it could be that these intersections are in fact Lagrangian intersections under disguise. To be more concrete, let $\frac{1}{2} < R^2 < \frac{1}{\sqrt{n}}$ and consider a Lagrangian $L_R$ lying on the boundary $\partial B^{2n}(R)$. Is it possible to Hamiltonianly separate $L_R$ from itself inside $B^{2n}(1)$?

If we can find a Lagrangian $L_R$ for which the answer is negative then this would strongly indicate that Theorem C is in fact a Lagrangian intersections result. Namely it would imply Theorem C for $R_1 = R_2$ under the additional assumption that $\varphi_1, \varphi_2$ are symplectically isotopic. A good candidate for $L_R$ seems to be the split torus $\partial B^2(\sqrt{R/n}) \times \cdots \times \partial B^2(\sqrt{R/n}) \subset \partial B^{2n}(R)$, but one could try other Lagrangians as well.

Attempts to approach this question with traditional Floer homology fail. The reason is that Floer homology is blind to sizes: both to the “size” of the Lagrangian $L_R$ as well as to the “size” of the domain in which we work $B^{2n}(1)$. Indeed it is easy to see that $HF(L_R, L_R)$ whether computed inside $B^{2n}(1)$ or in $\mathbb{R}^{2n}$ is the same, hence vanishes. The meaning of “sizes” can be made precise: the size of $L_R$ is encoded in its Liouville class, and the size of $B^{2n}(1)$ could be encoded here by the action spectrum of its boundary.

It would be interesting to try a mixture of symplectic field theory \[^{19}\] with Floer homology. This would require a sophisticated counting of holomorphic discs with $k$ punctures (for all $k \geq 0$), where the boundary of the discs go to $L_R$ and the punctures to periodic orbits on $\partial B^{2n}(1)$.

It is interesting to note that when the radii of the balls are not equal things become more complicated. Indeed suppose that $R_1^2 + R_2^2 > 1$ and consider two Lagrangian submanifolds $L_{R_1}, L_{R_2}$ lying on the boundaries of the balls $B_{\varphi_1}, B_{\varphi_2}$. Then clearly $L_{R_1}$ and $L_{R_2}$ can be disjoint even though the balls $B_{\varphi_1}, B_{\varphi_2}$ do intersect (e.g. two concentric balls $B_{\varphi_1} \subset B_{\varphi_2}$, where $R_1 < R_2$). It would be interesting
to see to which extent this mutual position can be detected on the level of the Lagrangians $L_{R_1}$ and $L_{R_2}$ alone. Or, in more pictorial (but less mathematical) terms, do the Lagrangians $L_{R_1}$ and $L_{R_2}$ know that they lie one “inside” the other?

Returning to the case of equal balls, if the above plan is feasible, it would be interesting to try similar approaches for more than two balls as described in Theorem D. A similar approach could be tried in the situation of Theorem E. Here one could expect an irremovable intersection between a Lagrangian submanifold $L_{R} \subset \partial B_{\psi}$ and $\mathbb{R}P^n$.

Quantitative intersections. In contrast to the quantitative version of Lagrangian intersections given by Theorems B and F, Theorems C–E provide only existence of intersections. Is it possible to measure the size of these intersections?

More concretely, consider two balls $B_{\phi_1}, B_{\phi_2} \subset B^{2n}(1)$ with $R_1^2 + R_2^2 > 1$ but with $R_1^{2n} + R_2^{2n} < 1$ (so that $\text{Vol}(B_{\phi_1}) + \text{Vol}(B_{\phi_2}) < 1$). Is it possible to bound from below the size of $B_{\phi_1} \cap B_{\phi_2}$?

It is not hard to see that volume is a wrong candidate for the size since for every $\epsilon > 0$ there exist two such balls with $\text{Vol}(B_{\phi_1} \cap B_{\phi_2}) < \epsilon$. Symplectic capacities seem also to be inappropriate for this task. It could be that “size” should be replaced here by a kind of “complexity” or a trade-off between capacity and complexity: namely if the intersection has large capacity (e.g. when $B_{\phi_1} \subset B_{\phi_2}$) the complexity is low, and vice-versa. Note that in dimension 2 a possible notion of complexity of a set is the number of connected components of its interior.

A related problem is the following. Consider two symplectic balls $B_{\phi_1}, B_{\phi_2} \subset \mathbb{C}P^n$ of radii $R_1, R_2$, where $R_1^2 + R_2^2 = 1$. Assume further that $\text{Int}(B_{\phi_1}) \cap \text{Int}(B_{\phi_2}) = \emptyset$. Theorem C implies that the balls must intersect hence the intersection occurs on the boundaries: $\partial B_{\phi_1} \cap \partial B_{\phi_2} \neq \emptyset$. What can be said about the intersection $\partial B_{\phi_1} \cap \partial B_{\phi_2} \neq \emptyset$, in terms of size, dynamical properties etc. ?

It is easy to see that this intersection cannot be discrete. Moreover, an argument based on the work of Sullivan [37] shows that the intersection must contain at least one entire (closed) orbit of the characteristic foliation of the boundaries of the balls (see [33] for a discussion on this point). Looking at examples however suggests that the number of orbits in the intersection should be much larger.

The same problem can be considered also for (some of) the extremal cases described in Theorem D. Similarly one can study the intersection $\partial B_{\varphi} \cap \mathbb{R}P^n$ where $B_{\varphi} \subset \mathbb{C}P^n$ is a symplectic ball of radius $R^2 = 1/2$ whose interior is disjoint from $\mathbb{R}P^n$. It is likely that methods of symplectic field theory [19] could shed some light on this circle of problems.

Stable intersections. The problems described here come from Polterovich [32]. Let $(M, \omega)$ be a symplectic manifold and $A \subset M$ a subset. We say that $A$ has the Hamiltonian intersection property if for every Hamiltonian diffeomorphism $f$ we have $f(A) \cap A \neq \emptyset$. We say that $A$ has the stable Hamiltonian intersection property if $O_{S^1} \times A \subset T^*(S^1) \times M$ has the Hamiltonian intersection property. Polterovich discovered in [32] that if there exists a subset $A \subset M$ with open non-empty complement and with the stable Hamiltonian property then the universal cover $\tilde{\text{Ham}}(M, \omega)$ of the
group of Hamiltonian diffeomorphisms has infinite diameter with respect to Hofer’s metric. Note that when \( \pi_1(\mathrm{Ham}(M, \omega)) \) is finite the same holds also for \( \mathrm{Ham}(M, \omega) \) itself. (See [32] for the details and references for other results on the diameter of \( \mathrm{Ham} \)). This is applicable when \((M, \omega)\) contains a Lagrangian submanifold \( A \) with \( HF(A, A) \neq 0 \), since then \( HF(O_S \times A, O_S \times A) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \otimes HF(A, A) \neq 0 \). For example, taking \( A = \mathbb{RP}^n \subset \mathbb{CP}^n \) Polterovich proved that \( \text{diam} \tilde{\mathrm{Ham}}(\mathbb{CP}^n) = \infty \) (for \( n = 1, 2 \) the same holds for \( \text{diam} \mathrm{Ham}(\mathbb{CP}^n) \)).

In view of the above the following question seems natural: does every closed symplectic manifold contain a subset \( A \) with open non-empty complement and with the stable Hamiltonian intersection property? Note that besides Lagrangian submanifolds (with \( HF \neq 0 \)) no other stable Hamiltonian intersection phenomena are known. It would also be interesting to find out whether the intersections described in Theorems C,D,E and especially \( E' \) continue to hold after stabilization.

4. Intersections versus non-intersections

In contrast to cotangent bundles there are manifolds in which every compact subset can be separated from itself by a Hamiltonian isotopy. The simplest example is \( \mathbb{C}^n \): indeed linear translations are Hamiltonian, and any compact subset can be translated away from itself. Clearly the same also holds for every symplectic manifold of the type \( M \times \mathbb{C} \) by applying translations on the \( \mathbb{C} \) factor. Note that manifolds of the type \( M \times \mathbb{C} \) sometime appear in “disguised” forms (e.g. as subcritical Stein manifolds, see Cieliebak [14]).

The “non-intersections” property has quite strong consequences on the topology of Lagrangian submanifolds already in \( \mathbb{C}^n \). Denote by \( \omega_{\text{std}} \) the standard symplectic structure of \( \mathbb{C}^n \) and let \( \lambda \) be any primitive of \( \omega_{\text{std}} \). Note that the restriction \( \lambda|_{T(L)} \) of \( \lambda \) to any Lagrangian submanifold \( L \subset \mathbb{C}^n \) is closed. The following was proved by Gromov in [22]:

**Theorem G.** Let \( L \subset \mathbb{C}^n \) be a Lagrangian submanifold. Then the restriction of \( \lambda \) to \( L \) is not exact. In particular \( H^1(L; \mathbb{R}) \neq 0 \).

Indeed if \( \lambda \) were exact on \( L \) then \( A_\omega : \pi_2(\mathbb{C}^n, L) \to \mathbb{R} \) must vanish, hence by Theorem F it is impossible to separate \( L \) from itself by a Hamiltonian isotopy. On the other hand, as discussed above, in \( \mathbb{C}^n \) this is always possible. We thus get a contradiction. (Gromov’s original proof is somewhat different, however a careful inspection shows it uses the failure of Lagrangian intersections in an indirect way). Arguments exploiting non-intersections were further used in clever ways by Lalonde and Sikorav [25] to obtain information on the topology of exact Lagrangians in cotangent bundles (see also Viterbo [42] for further results).

An important property of symplectic manifolds \( W \) having the “non-intersections” property is the following vanishing principle: for every Lagrangian submanifold \( L \subset W \) with well defined Floer homology we have \( HF(L, L) = 0 \). Applying this vanishing to \( \mathbb{C}^n \) yields restrictions on the possible Maslov class of Lagrangian submanifolds of \( \mathbb{C}^n \). (Conjectures about the Maslov class due to Audin appear already in [1]. First results in this directions are due to Polterovich [31] and to Viterbo [41]. The interpretation in Floer-homological terms is due to Oh [29]. Generalizations
to other manifolds appear in [2] and [11]. Finally, consult [21] for recent results answering old questions on the Maslov class).

4.1. Lagrangian embeddings in closed manifolds

The ideas described above can be applied to obtain information on the topology of Lagrangian submanifolds of some closed manifolds. Note that in comparison to closed manifolds the case of $\mathbb{C}^n$ can be regarded as local (Darboux Theorem). Of course, “local” should by no means be interpreted as easy. On the contrary, characterization of manifolds that admit Lagrangian embeddings into $\mathbb{C}^n$ is completely out of reach with the currently available tools.

Below we shall deal with the “global” case, namely with Lagrangians in closed manifolds. One (coarse) way to “mod out” local Lagrangians is to restrict to Lagrangians $L$ with $H_1(L;\mathbb{Z})$ zero or torsion (so that by Theorem G they cannot lie in a Darboux chart). The pattern arising in the theorems below is that under such assumptions in some closed symplectic manifolds we have homological uniqueness of Lagrangian submanifolds. Let us view some examples.

We start with $\mathbb{C}P^n$. It is known that a Lagrangian submanifold $L \subset \mathbb{C}P^n$ cannot have $H_1(L;\mathbb{Z}) = 0$ (see Seidel [39], see also [10] for an alternative proof). However, $L \subset \mathbb{C}P^n$ may have torsion $H_1(L;\mathbb{Z})$ as the example $\mathbb{R}P^n \subset \mathbb{C}P^n$ shows.

**Theorem H.** Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z})$ a $2$-torsion group (namely, $2H_1(L;\mathbb{Z}) = 0$). Then:

1. $H^*(L;\mathbb{Z}_2) \cong H^*(\mathbb{R}P^n;\mathbb{Z}_2)$ as graded vector spaces.
2. Let $a \in H^2(\mathbb{C}P^n;\mathbb{Z}_2)$ be the generator. Then $a|_L \in H^2(L;\mathbb{Z}_2)$ generates the subalgebra $H^\text{even}(L;\mathbb{Z}_2)$. Moreover if $n$ is even the isomorphism in 1 is of graded algebras.

Statement 1 of the theorem was first proved by Seidel [39]. An alternative proof based on “non-intersections” can be found in [8]. Let us outline the main ideas from [8]. Consider $\mathbb{C}P^n$ as a hypersurface of $\mathbb{C}P^{n+1}$. Let $U$ be a small tubular neighbourhood of $\mathbb{C}P^n$ inside $\mathbb{C}P^{n+1}$. The boundary $\partial U$ looks like a circle bundle over $\mathbb{C}P^n$ (in this case it is just the Hopf fibration). Denote by $\Gamma_L \to L$ the restriction of this circle bundle to $L \subset \mathbb{C}P^n$. A local computation shows that $U$ can be chosen so that $\Gamma_L \subset \mathbb{C}P^{n+1} \setminus \mathbb{C}P^n$ becomes a Lagrangian submanifold. (This procedure works whenever we have a symplectic manifold $\Sigma$ embedded as a hyperplane section in some other symplectic manifolds $M$. The next observation is that $\Gamma_L \subset \mathbb{C}P^{n+1} \setminus \mathbb{C}P^n$ is monotone and moreover its minimal Maslov number $N_{\Gamma_L}$ is the same as the one of $L$. Due to our assumptions on $H_1(L;\mathbb{Z})$ this number turns out to satisfy $N_{\Gamma_L} \geq n + 1$. The crucial point now is that $HF(\Gamma_L, \Gamma_L) = 0$. Indeed, the symplectic manifold $\mathbb{C}P^{n+1} \setminus \mathbb{C}P^n$ can be completed to be $\mathbb{C}^{n+1}$ where Floer homology vanishes.

Having this vanishing we turn to an alternative computation of $HF(\Gamma_L, \Gamma_L)$. This computation is based on the theory developed by Oh [29] for monotone Lagrangian submanifolds. According to [29] Floer homology can be computed via a spectral sequence whose first stage is the singular cohomology of the Lagrangian. The minimal Maslov number has an influence both on the grading as well as on the
5. Relations to algebraic geometry

The purpose of this section is to show how ideas from Section 4 are related to algebraic geometry. We shall not present new results here but rather try to outline a new direction in which symplectic methods can be used in algebraic geometry.

5.1. Hyperplane sections

Let $\Sigma$ be a smooth projective variety. The classical Lefschetz theorem provides restrictions on smooth varieties $X$ that may contain $\Sigma$ as their hyperplane section. It was discovered by Sommese [35] that there exist projective varieties $\Sigma$ that cannot be hyperplane sections (or even ample divisors) in any smooth variety $X$. For
example, Sommese proved that Abelian varieties of (complex) dimension $\geq 2$ have this property (see [35] for more examples).

Let us outline an alternative approach to this problem using symplectic geometry. Let $X \subset \mathbb{C}P^N$ be a smooth variety. Denote by $X^\vee \subset (\mathbb{C}P^N)^*$ the dual variety (namely, the variety of all hyperplanes $H \in (\mathbb{C}P^N)^*$ that are non-transverse to $X$).

**Theorem K.** Suppose that $\Sigma = X \cap H_0 \subset X$ is a smooth hyperplane section of $X$ obtained from a projective embedding $X \subset \mathbb{C}P^N$. Then either $\Sigma$ has a Lagrangian sphere (for the symplectic structure induced from $\mathbb{C}P^N$), or $\text{codim}_C(X^\vee) > 1$.

Here is an outline of the proof. Suppose that $\text{codim}_C(X^\vee) = 1$. Choose a generic line $\ell \subset (\mathbb{C}P^N)^*$ intersecting $X^\vee$ transversely (and only at smooth points of $X^\vee$). Consider the pencil $\{X \cap H\}_{H \in \ell}$ parametrized by $\ell$. Passing to the blow-up $\tilde{X}$ of $X$ along the base locus of the pencil we obtain a holomorphic map $\pi : \tilde{X} \to \ell \approx \mathbb{C}P^1$. The critical values of $\pi$ are in 1-1 correspondence with the point of $\ell \cap X^\vee$.

Moreover, the fact that $\ell$ intersects $X^\vee$ transversely implies that $\pi$ is a so called Lefschetz fibration, namely each critical point of $\pi$ has non-degenerate (complex) Hessian (in other words, locally $\pi$ looks like a holomorphic Morse function). The condition $\text{codim}_C(X^\vee) = 1$ ensures that $\ell \cap X^\vee \neq \emptyset$ hence at least one of the fibres of $\pi$ is singular. Let $X_0$ be such a fibre and $p \in X_0$ a critical point of $\pi$. The important point now is that the vanishing cycle (corresponding to $p$) that lies in the nearby smooth fibre $X_\epsilon$ can be represented by a (smooth) Lagrangian sphere. By Moser argument all the smooth divisors in the linear system $\{X \cap H\}_{H \in (\mathbb{C}P^N)^*}$ are symplectomorphic. In particular $\Sigma$ has a Lagrangian sphere too.

The existence of Lagrangian vanishing cycles was known folklorically for long time. Its importance to symplectic geometry was realized by Arnold [4], Donaldson [15] and by Seidel [38].

Theorem K can be applied as follows: given a smooth variety $\Sigma$, use methods of symplectic geometry to prove that $\Sigma$ contains no Lagrangian spheres, say for any symplectic structure compatible with the complex structure of $\Sigma$. Then by Theorem K the only chance for $\Sigma$ to be a hyperplane section is inside a variety $X$ with “small dual”, namely $\text{codim}_C(X^\vee) > 1$. Let us remark that smooth varieties $X \subset \mathbb{C}P^N$ with $\text{codim}_C(X^\vee) > 1$ are quite rare, and have very restricted geometry (see e.g. Zak [42] and Ein [17, 18]). Using the theory of “small dual varieties” we can either rule out this case or get strong restrictions on the pair $(X, \Sigma)$.

Let us illustrate this on the example mentioned at the beginning of the section. Let $\Sigma$ be an Abelian variety of complex dimension $n \geq 2$. Note that $\Sigma$ cannot have a Lagrangian sphere for any Kähler form. Indeed, if $\Sigma$ had such a sphere then the same would hold also for the universal cover of $\Sigma$ which is symplectomorphic to $\mathbb{C}^n$.

But this is impossible in view of Theorem G. Thus if $\Sigma$ is a hyperplane section of $X \subset \mathbb{C}P^N$ then $\text{codim}_C(X^\vee) > 1$. It is well known [24] that in this case $X$ must have rational curves (in fact lots of them). In particular $\pi_2(X) \neq 0$. By Lefschetz’s theorem we get $\pi_2(\Sigma) \neq 0$. But this is impossible since $\Sigma$ is an Abelian variety. We therefore conclude that $\Sigma$ cannot be a hyperplane section in any smooth variety $X$.

An analogous (though symplectically more involved) argument should apply also to any algebraic variety $\Sigma$ with $c_1 = 0$ and $b_1(\Sigma) \neq 0$ (see [4]). An application of more refined symplectic tools (e.g. methods described in Section 4.1 above) can
be used to obtain many more examples.

Here is another typical application: let $C$ be a projective curve of genus $> 0$. It was observed by Silva [34] that $C \times \mathbb{C}P^n$ can be realized as a hyperplane section in various smooth varieties. Note that by Theorem J, $C \times \mathbb{C}P^n$ cannot have any Lagrangian spheres. It immediately follows that the only smooth varieties $X$ that support $C \times \mathbb{C}P^n$ as their hyperplane section must have small dual. For $n \leq 5$ results of Ein [17, 18] make it even possible to list all such $X$’s.

We conclude with a remark on the methods. The symplectic approach outlined above gives coarser results. Indeed Sommese [35] provides examples of varieties that cannot be ample divisors whereas the methods above only rule out the possibility of being very ample. On the other hand the symplectic approach has an advantage in its robustness with respect to small deformations (see [9], c.f. [36]).

5.2. Degenerations of algebraic varieties

The methods of the previous section can also be used to study degenerations of algebraic varieties. Let $Y$ be a smooth projective variety. We say that $Y$ admits a Kähler degeneration with isolated singularities if there exists a Kähler manifold $X$ and a proper holomorphic map $\pi : X \to D$ to the unit disc $D \subset \mathbb{C}$ with the following properties:

1. Every $0 \neq t \in D$ is a regular value of $\pi$ (hence, all the fibres $X_t = \pi^{-1}(t)$, $t \neq 0$, are smooth Kähler manifolds).
2. $0$ is a critical value of $\pi$ and all the critical points of $\pi$ are isolated.
3. $Y$ is isomorphic (as a complex manifold) to one of the smooth fibres of $\pi$, say $X_{t_0}$, $t_0 \neq 0$.

As in the previous section this situation is related to symplectic geometry through the Lagrangian vanishing cycle construction. As pointed out by Seidel [39] one can locally morsify each of the critical points in $X_0 = \pi^{-1}(0)$ and then by applying Moser’s argument obtain for each critical point of $\pi$ at least one Lagrangian sphere in the nearby fibre $X_t$. Since all the smooth fibres are symplectomorphic we obtain Lagrangian spheres also in $Y$.

Applying results from Section 4 to this situation we obtain examples of projective varieties that do not admit any degeneration with isolated singularities. For example, let $Y$ be any of the following:

- $\mathbb{C}P^n$, $n \geq 2$. Or more generally $\mathbb{C}P^n \times M$, where $M$ is a smooth variety with $\pi_2(M) = 0$ and $\dim_\mathbb{C} M \not\equiv n + 1 \pmod{n + 1}$.
- Any variety whose universal cover is $\mathbb{C}^n$, $(n \geq 2)$, or a domain in $\mathbb{C}^n$.

Then by the results in Section 4, $Y$ has no Lagrangian spheres, hence does not admit any degeneration as above. More examples can be found in [9].

This point of view seems non-trivial especially when $H_\ast(Y; \mathbb{Z}) = 0$, where $n = \dim_\mathbb{C} Y$. In these cases the vanishing cycles are zero in homology and it seems that there are no obvious topological obstructions for degenerating $Y$ as above. From the list above, the first non-trivial example should be $\mathbb{C}P^n$ with $n = \text{odd} \geq 3$. It would be interesting to figure out to which extent the above statement could be
proved within the tools of pure algebraic geometry. Note that Lagrangian spheres are a non-algebraic object and it seems that their existence/non-existence cannot be formalized in purely algebro-geometric terms.

Another direction of applications should be to find an upper bound on the number of singular points of an algebraic variety \( X_0 \) that can be obtained from a degeneration of \( Y \). Note that the vanishing cycles of different singular points of \( X_0 \) are disjoint. Thus the idea here is to obtain an upper bound on the number of possible disjoint Lagrangian spheres that can be embedded in \( Y \). The simplest test case here should be the quadric \( Q = \{ z_0^2 + \cdots + z_{n+1}^2 = 0 \} \subset \mathbb{CP}^{n+1} \), where \( n \geq 2 \). Clearly \( Q \) can be degenerated to a variety \( X_0 \) with isolated singularities (e.g. to a cone over a smaller dimensional quadric). It seems reasonable to expect that in every such degeneration the singular fibre \( X_0 \) will have only one singular point. Note that for \( n = \text{even} \) this easily follows from topological reason but it may not be so when \( n = \text{odd} \geq 3 \) because \( H_\ast(Q; \mathbb{Z}) = 0 \). From a symplectic point of view the above statement would follow if we could prove that every two Lagrangian spheres in \( Q \) must intersect. This is currently still unknown but there are evidences supporting this conjecture [8]. It is likely that a refinement of the methods from [10] would be useful for this purpose. More generally, one could try to bound the number of singular fibres in a degeneration of other hypersurfaces \( \Sigma \subset \mathbb{CP}^{n+1} \) (in terms of \( \text{deg}(\Sigma) \) and \( n \)). See [8, 9] for the conjectured bounds.

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