Quantum fluctuation generated vortices, dual singular gauge transformation and zero temperature transition from d-wave superconductor to underdoped regime

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By extending the original Anderson singular gauge transformation for static vortices to two mutual flux-attaching singular gauge transformations for moving vortices, we derive an effective action describing the zero temperature quantum phase transition from d-wave superconductor to underdoped regime. In this action, quantum fluctuation generated vortices couple to quasi-particles by a mutual statistical interaction with statistical angle \( \theta = 1/2 \) and a dynamic Doppler shift term, the vortices are also interacting with each other by long-range logarithmic interactions due to charge fluctuation. Neglecting the charge fluctuation first, we find that the mutual statistical interaction is exactly marginal. In the underdoped regime, the quasi-particles are described by \( 2 + 1 \) dimensional QED; in the superconducting regime, they are essentially free. However, putting back the charge fluctuation changes the physical picture dramatically: both the dynamic Doppler shift term and the mutual statistical interaction become irrelevant short-ranged interactions on both sides of the quantum critical point. There are no spin-charge separation and no dynamic gapless gauge field in the Cooper-pair picture. The formalism developed at \( T = 0 \) is applied to study thermally generated vortices in the vortex plasma regime near the finite temperature KT transition. The important effects of the AB phase scattering and the Doppler shift on ARPES data presented in Phys. Rev. Lett. 87, 227003 (2001) are also briefly reviewed.

I. INTRODUCTION

In this paper, we are trying to study the nature of zero temperature quantum phase transition from d-wave superconductor at \( x > x_c \) to the underdoped regime at \( x < x_c \) of the high temperature superconductors (Fig.1).

\[ H = \int dx d^3(r) \left( \frac{h + V(x)}{\Delta^2} - \frac{\hat{\Delta}}{h^2} - V(x) \right) d(x) \quad (1) \]

with \( d_\uparrow(x) = c_\uparrow(x), d_\downarrow(x) = c_\downarrow^\dagger(x) \). \( V(x) \) is random chemical potential due to non-magnetic impurities which will be set to be zero in the following.

Based on the earlier work on d-wave superconductor by Simon and Lee [3] and on \( t-J \) model by Ye and Sachdev [4], the author derived the gauge-invariant form in Ref. [3]:

\[ h = \frac{1}{2m}(\hat{p}^2 - \frac{c}{e} \hat{A})^2 - \epsilon_F \]

\[ \hat{\Delta} = \frac{1}{4p_F^2}[(p_x - \frac{1}{2}\partial_x \phi, p_y - \frac{1}{2}\partial_y \phi) \Delta(\hat{r}) + (p_x - \frac{1}{2}\partial_x \phi) \Delta(\hat{r})(p_y + \frac{1}{2}\partial_y \phi) + (p_y - \frac{1}{2}\partial_y \phi) \Delta(\hat{r})(p_x + \frac{1}{2}\partial_x \phi)] \]
+ \Delta(\vec{r})\{p_x + \frac{1}{2}\partial_x \phi, p_y + \frac{1}{2}\partial_y \phi\}\] (2)

where \(\phi\) is the phase of the order parameter \(\Delta(\vec{r})\) and \(p_x\) and \(p_y\) are the momenta relative to the origin.

Because high \(T_c\) cuprates are strongly type II superconductors \((\kappa = \lambda/\xi \gg 1)\), we assume the gap amplitude to be a constant \(\Delta(x) \sim \Delta_0 e^{i\phi}\) across the transition from \(x > x_c\) to \(x < x_c\). The phase \(\phi\) can be set to be zero inside the superconducting state \(x > x_c\) because its fluctuation is suppressed. However, in the underdoped regime \(x < x_c\), there are strong phase fluctuation which leads to the free \(\hbar c/2e\) vortices located at \(\vec{R}_i\) with parity \(n_i = \pm 1\), then the phase \(\phi\) satisfies \(\nabla \times \nabla \phi = 2\pi \sum \delta(\vec{r} - \vec{R}_i)\).

A lot of work has been done on a closely related problem where quasi-particles are coupled to the vortices generated by external magnetic field inside the superconducting state \([1,2,3,4,5,6,7,8,9,10,11,12,13] \). Employing semi-classical approximation, Volovik pointed out that the circulating supercurrents around vortices induce Doppler energy shift to the quasi-particle spectrum, which leads to a finite density of states at the nodes \([8]\). This effect (Volovik effect) has been employed to attempt to explain the experimental observations \([10]\) of longitudinal thermal conductivity \(\kappa_{xx}\) in \([8]\).

Starting from BCS Hamiltonian, Wang and MacDonald performed a first numerical calculation on quasi-particle spectrum in vortex lattice state \([10]\). By phenomenological scaling arguments, Simon and Lee (SL) proposed the approximate scaling forms for longitudinal and transverse thermal conductivities for dirty \(d\) wave superconductors in the mixed state \([8]\). Anderson \([11]\) employed the first single-valued singular gauge transformation to study quasi-particle dynamics in the mixed state. Unfortunately, Anderson made an incorrect mean field approximation which violates the ”Time-Reversal” symmetry \([8]\), therefore leads to the incorrect conclusion that there is Landau level quantization of energy levels of the quasi-particle. Franz and Tesanovic employed a different single-valued singular gauge transformation and studied the quasi-particle spectrum numerically \([11]\). They did not see the signature of Landau level quantization in their numerical calculations. More detailed studies in various kinds of vortex lattice states were conducted in Refs. \([12,13]\). The quasi-particle spectrum around a single vortex, paying special attention to the strong anisotropy of Fermi and gap velocities, was given in \([14,15]\).

In the vortex lattice state, the vector potential only provides a periodic potential instead of scattering quasi-particles, its effect on the energy spectrum and other physical quantities is not transparent. In Ref. \([1]\), the author studied the quasi-particle transport in the disordered vortex state where the random AB phase scattering may show its important effects. I observed that because infinite thin \(\hbar c/2e\) vortices do not break the T reversal symmetry \([3]\), there is no Landau level quantization, any correct mean field theory should respect this T symmetry (The other way around need not be necessarily true, for example, in \(\nu = 1/2\) Fractional Quantum Hall system \([27]\), the mean field state is a Fermi surface of composite fermion which respects T symmetry even the original electron system breaks the T symmetry due to the external magnetic field). I applied the Anderson singular transformation Eqn.\([3]\) to disordered vortex state and found that the long-range logarithmic interaction between vortices suppresses the fluctuation of superfluid velocity (scalar potential), but does not affect the fluctuation of the internal gauge field. Therefore the scalar field acquires a ”mass” determined by the vortex density, but the gauge field remains ”massless”. The quasi-particle scattering from the ”massless” internal gauge field dominates over those from the well-known ”massive” Volovik effect and the non-magnetic scattering at sufficient high magnetic field. This dominant scattering mechanism is a purely quantum mechanical effects which was overlooked by all the previous semi-classical treatments \([8]\). In fact, it is responsible for the behaviors of both \(\kappa_{xx}\) and \(\kappa_{xy}\) in high magnetic field observed in the experiments \([10]\).

When the vortices are generated by quantum fluctuations themselves are moving around, new physics may arise. For example, moving \(\hbar c/2e\) vortices does break T reversal symmetry and leads to Hall voltage drop, a moving \(\hbar c/2e\) vortex is different from a moving \(-\hbar c/2e\) vortex. This in-equivalence renders FT gauge in Ref. \([11]\) useless for moving vortices although it is equivalent to Anderson’s gauge for static vortices. Anderson gauge has to be employed to study moving vortices. Intuitively, from Newton’s third law, the vortices must also feel the counter-acting AB phase coming from the quasi-particles. Therefore there is a mutual AB phase scattering between vortices and quasi-particles. The dynamic Doppler shift term on the quasi-particles can also be equivalently viewed as an effective gauge field coupled to the phase fluctuation (see Eqn \([14]\)). In this paper, we enlarge Anderson singular gauge transformation to two mutual flux-attaching singular gauge transformations. We then apply them to study the physics of quasi-particles coupled to moving vortices near the zero temperature quantum critical point. In canonical quantization and in Coulomb gauge, we perform the two singular transformations (which are dual to each other) to quasi-particles(which are spinons) and moving vortices(which are holons) respectively to satisfy all the possible commutation relations. Just like a conventional singular gauge transformation leads to a conventional Chern-Simon (CS) term, the two mutual singular gauge transformations lead to a mutual CS term. This elegant mutual CS term describe the mutual AB phase scattering between vortices and quasi-particles. Alternatively, in path-integral presentation, the effective action describes the quasi-particle moving in both vector and scalar potentials due to the phase fluctuations of quantum gen-
erated vortices. By a duality transformation presented in Refs. [7,11] to a vortex representation, the quantum fluctuation generated vortices couple to quasi-particles by a mutual CS term, the vortices are also interacting with each other by long range logarithmic interactions due to charge fluctuation described by a Maxwell term. Neglecting the charge fluctuation and its associated dynamic Doppler shift term first, we find a fixed line characterized by the mutual statistical angle $\theta$ and calculate the universal spinon, vortex and mutual Hall drag activities which continuously depend on $\theta$. This transition can also be viewed as a simple example of confinement and deconfinement transition. In the disordered phase, the quasi-particles are described by $2 + 1$ dimensional QED. In the superconducting side, they are essentially free. We stressed explicitly the lack of periodicity of $U(1)$ mutual CS gauge theory. When the $U(1)$ charge fluctuation is taken into account, we treat both $U(1)$ gauge fluctuations on the same footing and find the $U(1)$ charge fluctuation turns the fixed line into a fixed point separating the d-wave superconductor and some unknown charge ordered state. Both the dynamic Doppler shift term and the mutual statistical interaction become irrelevant short-ranged interactions on both sides of the QCP. The spinon and holons are confined into electrons and Cooper pairs in the condensation of $hc/e^2$ vortices, in contrast to the condensation of double strength $hc/e$ vortices discussed in [13] and reviewed in most general form in appendix B. There are no gapless gauge field fluctuations in the final theory, in contrast to $U(1)$ and $SU(2)$ gauge theory [3]. We also discuss the properties of the two stable phases: disorder and superconducting phases around the critical point.

As stressed in Ref. [3], in weakly type-II limit $\xi < \lambda < d_v$, the superfluid velocity vanishes in the interior of the superconductor, the Doppler shift effect is completely absent, the AB phase scattering effect becomes the sole scattering mechanism [28]. In this case, the interaction between vortices become short-ranged. The discussions in [3] on high $T_c$ cuprates are limited to $H_{c1} \ll H \ll H_{c2}$ where $\xi \ll d_v \ll \lambda$. Except $H$ is extremely close to $H_{c1}$, we can safely take $\lambda \to \infty$ limit. The interaction between vortices become long-ranged logarithmic interaction and played leading roles as demonstrated in [3]. In this paper, the quantum or thermal fluctuation generated free vortex density scales as $n_f \sim \xi_c^{-2}$ ( here $\xi_c$ is the correlation length rather than the coherence length $\xi$). For high $T_c$ cuprates, $\kappa = \lambda/\xi \sim 60$, $\xi/\mu \sim 5$, then $\lambda \sim 60\xi \sim 300\mu$, so as long as $d_v \sim \xi_c < \lambda$, namely except we get extremely close to the phase boundary, we can safely set $\lambda \to \infty$. Then we have to consider the long-range interaction between the vortices which is mediated by charge gauge field, its important role will be demonstrated explicitly in Sec. IV. All the present experimental situations lie in the regime $\xi_c < \lambda$.

The paper is organized as the following. In the next section, we derive the effective action with both the mutual C-S interaction and the charge fluctuation, first in the path integral language, and then in canonical quantization representation. We stress the importance of the dual singular gauge transformation on the moving vortices to keep the mutual statistics invariant and derive the important expression of real electron operator in terms of spinon and holon (Eqn. [23]). In Sec. III, neglecting the charge fluctuation, we concentrate on the effect of the mutual C-S interaction and charge fluctuation, we show how the long-range charge fluctuation change the physical picture in Sec. III dramatically. There are only electrons and Cooper pairs in the spectrum; the fixed line is replaced by a quantum critical point (QCP) separating d-wave superconductor and some still unknown charge -ordered state; there are no spin-charge separation and no gapless gauge field fluctuations in the final theory, in contrast to $U(1)$ or $S(2)$ gauge theories investigated in Ref. [21]. We also discuss the properties of the two phases around the QCP. In section V, we apply the formalism developed for quantum generated vortices to study thermally generated vortices and random static vortex array in an external magnetic field inside d-wave superconductor and recovered the results established previously in Ref. [3]. Finally, in Sec. VI, we discuss the connection and differences of our present approach to the $Z_2$ gauge theory and point out some open problems.

II. THE DUAL SINGULAR GAUGE TRANSFORMATION, EFFECTIVE ACTION AND SINGLE ELECTRON OPERATOR

In the following, we follow the notation in Ref. [10] and introduce the spinon by performing a general single-valued Anderson singular gauge transformation $d = Ud_s$:

$$H_s = U^{-1}HU, \quad U = \begin{pmatrix} e^{i\phi_A} & 0 \\ 0 & e^{-i\phi_B} \end{pmatrix}$$ (3)

where $\phi_A = 0, \phi_B = \phi$ or vice versa and $\phi$ is the phase of the Cooper pair. In the former(latter), the spinon is electron-like ( hole-like). The original Anderson singular gauge transformation [10] is devised for static vortices. Here we extend it to moving vortices whose phase $\phi$ is also fluctuating, therefore, depends on both the space and time.
A. Path integral formulation

Expanding $H_s$ around the node 1 where $\vec{p} = (p_F, 0)$, we obtain $H_s = H_t + H_c$. The corresponding linearized quasi-particle Lagrangian $\mathcal{L}_{qp}$ in the presence of the external gauge potential $A_\mu$ in the imaginary time $\tau$ is:

$$
\mathcal{L}^\mu_{qp} = \bar{\psi}_1^\dagger \left[ (\partial_\tau - i a_\tau) + v_f (p_\sigma - a_\sigma) \right] \psi_1 + \bar{\psi}_2^\dagger \left[ (\partial_\tau + i a_\tau) + v_f (p_\sigma + a_\sigma) \right] \psi_2
$$

where $v_f, \Delta$ are Fermi and gap velocities respectively, $v_\mu = \frac{\partial}{\partial \phi} \phi = - A_\mu, \mu = x, y, \tau$ is the dynamical gauge-invariant superfluid momentum, it acts as a scalar scattering potential and provides dynamical Doppler shift to the quasi-particles (this effect can be considered as a dynamic Volovik effect), $a_\mu = \frac{\partial}{\partial \phi} \phi$ is the dynamical AB gauge field due to the phase winding of vortices. Note that the external gauge potential only appear explicitly in the superfluid momentum $v_\mu$. The fermion at node 1 (2) only couples to the $x(y)$-component of superfluid momentum, because the Fermi momentum at node 1 (2) is along the $x(y)$ direction. Because there are equal number of positive and negative vortices, on the average, the vanishing of $v_\tau$ and $a_\mu$ is automatically ensured in the Anderson gauge.

We get the corresponding expression at node $\bar{1}$ and $\bar{2}$ by changing $v_f \rightarrow -v_f, \Delta \rightarrow -\Delta$ in the above Eqn.

$$
\mathcal{L}^\mu_{qp} = \bar{\psi}_1^\dagger \left[ (\partial_\tau - i a_\tau) + v_f (p_\sigma - a_\sigma) \right] \psi_1 + \bar{\psi}_2^\dagger \left[ (\partial_\tau + i a_\tau) + v_f (p_\sigma + a_\sigma) \right] \psi_2
$$

The curvature term $H_c$ can be written as:

$$
H_c = \frac{1}{m} \left[ \{ \Pi_\alpha, v_\alpha \} + \frac{\Pi^2 + \Pi^2}{2} + \frac{\Delta_0}{2\epsilon_F} \{ \Pi_x, \Pi_y \} \right] \tau^1 \tag{6}
$$

Where $\Pi = \vec{p} + \vec{a}$ is the covariant derivative $[2]$. $H_c$ takes the same form for all the four nodes. Although $H_c$ is very important for the thermal Hall conductance in the presence of external magnetic field $[3]$, it is irrelevant near the quantum critical point, so we will not discuss it anymore in the present paper.

Performing a $P - H$ transformation $\tilde{\psi}_1 = e_{\alpha\beta}\psi_1^{1\beta}$ ($\alpha, \beta$ are p-h indices) at node $\bar{1}$ and the corresponding transformation at node 2, the above Eqn. becomes:

$$
\mathcal{L}^\mu_{qp} = \bar{\psi}_1^\dagger \left[ (\partial_\tau + i a_\tau) + v_f (p_\sigma - a_\sigma) \right] \psi_1 + \bar{\psi}_2^\dagger \left[ (\partial_\tau - i a_\tau) + v_f (p_\sigma + a_\sigma) \right] \psi_2
$$

In order to make the final expressions explicitly $SU(2)$ invariant, we perform the singular gauge transformation $\psi_{12\alpha} = e^{-i(\phi_A - \phi_{\bar{a}})} \psi_1$, and the corresponding transformation at node 2, then $a_\mu \rightarrow -a_\mu$, Eqn. takes the same form as Eqn. $[23]$. Adding the two equations leads to:

$$
\mathcal{L}_{qp} = \psi_1^\dagger \left[ (\partial_\tau - i a_\tau) + v_f (p_\sigma - a_\sigma) \right] \psi_1 + \psi_2^\dagger \left[ (\partial_\tau + i a_\tau) + v_f (p_\sigma + a_\sigma) \right] \psi_2 + \bar{\psi}_1 \psi_0 v_f \psi_2 + \bar{\psi}_2 \psi_0 v_f \psi_1
$$

where $a = 1, 2$ is the spin indices. $\tau^s$ matrices are acting on particle-hole space. As intended, the above Eqn. is explicitly $SU(2)$ spin invariant. Equivalently, we can start with the explicitly spin $SU(2)$ invariant approach advocated in Ref. $[4]$ and perform the singular gauge transformation in the p-h space.

$\mathcal{L}_{qp}$ enjoys gauge symmetry $U(1)_u \times U(1)_d$ (in fact, for static vortices, it is $U(1) \times Z_2$ ), the first being uniform and second being staggered gauge symmetry: Uniform (or external) $U_u(1)$ gauge symmetry

$$
\begin{align*}
&c_a \rightarrow c_a e^{i\chi}, \quad d_a \rightarrow d_a \\
&\phi_A \rightarrow \phi_A + \chi, \quad \phi_B \rightarrow \phi_B + \chi
\end{align*}
$$

Under this uniform $U(1)$ transformation, the corresponding fields transform as:

$$
\begin{align*}
&\phi \rightarrow \phi + 2\chi, \quad A_a \rightarrow A_a + \partial_a \chi \\
&v_a \rightarrow v_a, \quad a_a \rightarrow a_a
\end{align*}
$$

$d, v, a$ are all are invariant under this external $U(1)$ transformation. Therefore the spinon $d_a$ is charge neutral to the external magnetic field.

Staggered (or internal) $U_d(1)$ gauge symmetry

$$
\begin{align*}
&c_a \rightarrow c_a, \quad d_a \rightarrow d_a e^{-i\chi} \\
&\phi_A \rightarrow \phi_A + \chi, \quad \phi_B \rightarrow \phi_B - \chi
\end{align*}
$$

Under this internal $U(1)$ transformation, the corresponding fields transform as:

$$
\begin{align*}
&\phi \rightarrow \phi, \quad A_a \rightarrow A_a \\
&v_a \rightarrow v_a, \quad a_a \rightarrow a_a + \partial_a \chi
\end{align*}
$$

Although the spinon $d_a$ is charge neutral to the external magnetic field, it carries charge 1 to the internal gauge field $a_a$.

It is easy to realize that $U_u(1)$ acts only on the boson sector, since the spinon is charge neutral, $U_u(1)$ acts only on the fermion sector. In fact, $U_u(1)$ should be a discrete local $Z_2$ symmetry for static vortices, because up and down static $hc/2e$ vortices are equivalent and do not break $T$ symmetry $[4]$.

The phase fluctuation is simply $2 + 1$ dimensional X-Y model (For simplicity, we neglect the possible anisotropy in the spin-stiffness):

$$
\mathcal{L}_{ph} = \frac{K}{2} v_\mu^2 = \frac{K}{2} (\partial_\mu \phi - 2A_\mu)^2 \tag{13}
$$

After absorbing the scalar potential scattering part (the dynamic Doppler shift term) into $\mathcal{L}_{qp}$, we can write the total Lagrangian $\mathcal{L} = \mathcal{L}_{qp} + \mathcal{L}_{ph}$ as:
therefore also conserved.

There is no feedback effect on the superfluid momentum 

which is static and causes Doppler shift to the quasi-particle. 

It is easy to see that the dynamical Doppler shift term on the quasi-particles can also be equivalently thought of as a dynamical effective gauge field coupled to the phase fluctuation. (Note that the static vortex array generated by external magnetic field \[B\], the superfluid momentum is static and causes Doppler shift to the quasi-particle, there is no feedback effect on the superfluid momentum from the quasi-particles)

From Eqn[14] it is easy to identify the two conserved Noether currents: spinon current and electric current.

The spinon current is given by:

\[
\begin{align*}
  j^\psi_0 &= \psi_1^\dagger(x)\psi_1(x) + \psi_2^\dagger(x)\psi_2(x) \\
  j^\psi_s &= \psi_1^\dagger(x)v_\mu r^\mu\psi_1(x) + \psi_2^\dagger(x)v_\mu r^\mu\psi_2(x) \\
  j^\psi_y &= \psi_1^\dagger(x)v_\Delta r^\mu\psi_1(x) + \psi_2^\dagger(x)v_\Delta r^\mu\psi_2(x)
\end{align*}
\]

Obviously the spinon current only comes from quasiparticle. In principle, the spinon current is not conserved due to scatterings between different nodes which lead to anomalous terms not included in Eqn[14]. However, the inter-node scatterings involve large momenta transfer \(K_i - K_j\) for \(i \neq j\), so we neglect them due to momentum conservation in the long-wavelength limit of the phase fluctuation. The exact conserved spin current \(j^\psi_s\) is with \(\hat{\sigma}/2\) inserted in the above spinon currents.

The electric current is given by:

\[
  j^\mu_\text{eff} = -\frac{\partial L}{\partial A^\mu}\left(\phi - A^\mu_{\text{eff}}\right) = K(\partial_\mu\phi - A^\mu_{\text{eff}}) + J_\mu
\]

(16)

Where the first part coming from Cooper pair and the second from the quasi-particle. Although they are not separately conserved, their sum is.

In the electron-like Anderson gauge where \(a_\mu = \frac{i}{2}\partial_\mu\phi\), the Noether current due to the symmetry under \(\phi \rightarrow \phi + \chi\) can be written as:

\[
  j^\mu_\text{eff} = \frac{\partial L}{\partial A^\mu}\left(\phi - A^\mu_{\text{eff}}\right) - \frac{1}{2}j^\mu_\text{eff} = j^\mu_\text{eff} - \frac{1}{2}j^\mu_\text{eff}
\]

(17)

It is a combination of electric and spinon currents, therefore also conserved.

Following Refs. [13][18], we perform a duality transformation to Eqn[14]

\[
  K \left(\partial_\mu\phi - A^\mu_{\text{eff}}\right)^2 - \frac{1}{2}\partial_\mu\phi j^\mu_\text{eff} = j^\psi_\mu - j^\psi_\mu
\]

(18)

Where we have separated topological trivial spin-wave part and topological non-trivial vortex parts.

Integrating out the spin-wave part, we get the conservation equation for the total current \(j^\mu_\mu = j^\mu_\mu - \frac{1}{2}j^\mu_\mu\). In fact, as shown in the previous paragraphs, \(j^\mu_\mu\) and \(j^\psi_\mu\) are separately conserved. Therefore we can introduce spin and electric gauge fields by:

\[
  j^\mu_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu a^\lambda_\mu
\]

(19)

We can also define the vortex current:

\[
  j^\mu_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda a^\mu_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda a^\mu_\mu
\]

(20)

where \(a^\mu_\mu = \partial_\mu\phi/2\pi\) is the vortex gauge field.

Substituting the above expressions into Eqn[18], we reach:

\[
  \begin{align*}
    \frac{1}{4K}f^2_{\mu\nu} + i\partial_\mu\phi j^\mu_{\text{eff}} - 4iA^\mu_{\text{eff}}\epsilon_{\mu\nu\lambda}\partial_\nu a^\lambda_\mu \\
    + (1 \rightarrow 2, x \rightarrow y) \\
    \delta_{\mu\nu}\Phi_\mu_\nu + V(|\Phi|) + \frac{i}{2\pi}\Phi_\mu_\nu a^\mu_\nu a^\nu_\mu \\
    + \frac{1}{4}\epsilon^2_{\mu\nu\lambda\mu\nu\lambda\mu\nu\lambda} = 0
  \end{align*}
\]

(22)

where \(V(|\Phi|) = m_N^2|\Phi|^2 + g_N|\Phi|^4 + \cdots\) stands for the short range interaction between the vortices. The last term is due to the Berry phase in the boson representation \([13]\) which is a first order time derivative term. It can be absorbed into \(A^\mu_{\text{eff}}\) by redefining \(A^\mu_{\text{eff}} = A^\mu_{\text{eff}} + iq_\mu\partial_\mu\phi\). It acts like an external magnetic field in the \(\hat{z}\) direction in the above vortex representation. There are \(M = 1\) species of vortex and \(N = 4\) species of Dirac fermion(2 spin components at 2 nodes at the upper half plane), the mutual statistical angle \(\theta = \pm 1/2\). We also changed the notation by setting \(a^\psi = a^\nu, a^\phi = a^\nu, a^e = a\). The dynamic Doppler-shifted term \(iK^{-1}J_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\) (The dynamic Doppler shift effect) is encoded in the second to the last term in Eqn[22]. Note that this term only involves electrical gauge field. This effect was not written down explicitly in the \(Z_2\) gauge theory \([13]\).
Eqn. 22 again enjoys the gauge symmetry $U(1)_c \times U(1)_s$ which is a dynamic generalization of $U(1)_u \times U(1)_v$ in Eqns. 14, 12. The first one acting on the vortex sector is the electric $U(1)_c \sim U(1)_u$ gauge field, the second on both the vortex and fermion sectors is the $U(1)_s$ mutual CS gauge field which is a dynamic generalization of $U(1)_s$. The mutual Chern-Simons term enforces the constraints: $2\pi \theta_j^v = \epsilon_{\mu\nu\lambda} \partial_\lambda a_\mu^v$; $2\pi \theta_j^v = \epsilon_{\mu\nu\lambda} \partial_\lambda a_\nu^v$. Physically, it means that when a quasi-particle encircles a vortex, it picks up a phase $2\pi \theta$. Equivalently when a vortex moves around a quasi-particle, it also picks up a phase $2\pi \theta$. Although the conventional C-S terms have the periodicity under $\theta \to \theta + 2\pi$, the mutual C-S term does not break $T$ symmetry and does not have the periodicity under $\theta \to \theta + 1$. For example, $\theta = -1/2$ is not equivalent to $\theta = 1/2$.

B. Electron operator

The general form of electron annihilation operator is:

$$C_\alpha(x) = \sum_{i=1,2} [e^{i\phi_A} e^{iK_i} e^{i\psi_{1\alpha}} - \epsilon_{\alpha\beta} e^{i\psi_{i2\beta}}]$$ (23)

where $i$ is the node index, 1, 2 are $p-h$ indices, $\alpha, \beta$ are spin indices (in the rest of the paper, we will stick to this notation) and $\phi_A$ and $\phi_B$ are given by:

$$\phi_A = \frac{\phi}{2} + \int a^\phi dx$$

$$\phi_B = \frac{\phi}{2} - \int a^\phi dx$$ (24)

Eqn. 22 reads:

$$C_\alpha(x) = \sum_{i=1,2} [e^{i\phi/2} e^{iK_i} e^{i\psi_{1\alpha}} - \epsilon_{\alpha\beta} e^{i\psi_{i2\beta}}]$$ (25)

In principle, the electron Green function $G(x, t) = \langle C_\alpha(x) C^\dagger_\beta(0, 0) \rangle$ can be calculated from the above equation. It is easy to see that the real electron in Eqn. 22 is invariant under the internal gauge transformation $U(1)_s$.

C. Canonical quantization approach

In subsection A, we derived Eqn. 22 by path integral approach, in this subsection, we rederive this equation by canonical quantization method and stress the importance of the dual singular gauge transformation. Eqn. 29.

Substituting Eqn. 24 into Eqn. 19, we find that the singular unitary transformation $U$ takes the physically more transparent form:

$$d = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} e^{i\int a^\phi dx_\tau}$$ (26)

where, in first quantization form:

$$\bar{a}^\phi = \frac{1}{2}(\nabla \phi_A - \nabla \phi_B)$$ (27)

The first factor leads to the dynamic Doppler shift term which is not explicitly written in the $Z_2$ gauge theory. The second leads to mutual CS term with $\theta = 1/2$. To see this, in Anderson electron-like gauge, we rewrite $a^\phi$ in the second quantization form in terms of the vortex operator $\Phi$:

$$\bar{a}^\phi = \theta \int d^2 \tau' \nabla \phi(\tau' - \tau) j^v_0(\tau')$$ (28)

where $j^v_0$ is vortex number operator and $\phi(\tau' - \tau)$ is the angle the vector $(\tau' - \tau)$ makes with the axis.

Obviously, $d_s$ still satisfies fermion commutation relations. When the vortices are static, $j^v_0(\tau) = \sum_0 \delta(\tau - \tau_i)$. Eqn. 28 recovers the original Anderson singular gauge transformation and is the only necessary transformation. However, when vortices are generated by quantum fluctuations, they have their own dynamics. Eqn. 28 alone can not satisfy all the possible commutation relations. This motivated us to introduce the following dual singular gauge transformation to the vortex operator:

$$\Phi = \int a^\phi dx_\tau \Phi_s$$ (29)

where, in second quantization form:

$$\bar{a}^\phi = \theta \int d^2 \tau' \nabla \phi(\tau' - \tau) j^v_0(\tau')$$ (30)

Where $j^v_0$ is given by Eqn. 13.

Obviously, $\Phi_s$ still satisfies boson commutation relation.

Most importantly, from Eqns. 26 and 29, we can check that $d_s$ and $\Phi_s$ commute with each other, so their mutual statistics is kept intact which is our original motivation. Note that for static vortices, the dual transformation in Eqn. 28 is unnecessary, because the static vortices are not dynamic variables.

It is very instructive to compare the above transformations to the well-known singular gauge transformation leading to composite boson in $\nu = 1/3$ Fractional Quantum Hall state by Zhang, Hansen and Kivelson (ZHK) [26] and composite fermion in $\nu = 1/2$ system by Halperin, Lee and Read (HLR) [27]:

$$\psi = e^{i\int a^\phi dx} \psi$$ (31)

where, in second quantization form:
\[ \vec{a} = \theta \int d^2 \vec{r}' \nabla \phi(\vec{r}' - \vec{r}) \rho(\vec{r}') \]  

where \( \rho(\vec{r}) = \psi_1 \psi_1 \) is the fermion number operator.

The crucial difference between Eqs.\textsuperscript{21} and Eqs.\textsuperscript{22,23} is that in the former, we attach electron’s own \( \theta \) to boson and \( \theta \) where \( \rho \) \( \theta \) however, in the latter, we attach vortex’s \( \psi \) site fermion \( \theta \) right and deserves detailed investigation.

Hall drag problem in double layer Quantum Hall system superconductor “ which has not been discovered yet in can only be applied to presumed “ weakly type II d-wave ductors are all s-wave superconductors where there is no dynamic Doppler shift term also drops out. As pointed out in Sec.\textsuperscript{IV}, we will study the combined effect of charge fluctuation (therefore its associated Dynamic Doppler shift term) and concentrate on the mutual statistics term. In the following section, we will neglect the electric charge fluctuation (therefore its associated Dynamic Doppler shift term) and concentrate on the mutual statistics term. In Sec.\textsuperscript{IV}, we will study the combined effect of charge fluctuation described by the Maxwell term and the mutual C-S term.

For simplicity, we neglect all the possible anisotropy in the quasi-particle and vortex velocities. We also take the relativistic form for both fermion and boson, because the difference between the velocity of spinon and that of vortex in Eqn.\textsuperscript{22} is expected to be irrelevant near the zero temperature QCP.\textsuperscript{24}

III. THE EFFECT OF MUTUAL STATISTICS

In this section, we neglect the charge fluctuation, namely setting \( a_{\mu} = 0 \) in Eqn\textsuperscript{22}. The associated dynamic Doppler shift term also drops out. As pointed out in \( \theta \) and re-emphasized in the introduction, this is the weakly type II case. So far, weakly type II superconductors are all s-wave superconductors where there is no low energy quasi-particles. So the results in this section can only be applied to presumed “ weakly type II d-wave superconductor ” which has not been discovered yet in nature. The model is also a relativistic version of the Hall drag problem in double layer Quantum Hall system \( \textsuperscript{26} \), therefore the model itself is interesting on its own right and deserves detailed investigation.

The charge fluctuation can be suppressed by assuming the condensation of \( \hbar c/e \) vortices reviewed in appendix B. We could add the kinetic and potential terms for the \( \hbar c/e \) vortex operator \( \Phi_2 \) to Eqn.\textsuperscript{22}

\[ \mathcal{L}_{\Phi_2} = |(\partial_\mu - i 2 a_{\mu}) \Phi_2|^2 + V(|\Phi_2|) \]  

As shown in \[ \theta \] and in appendix B, there is no mutual statistical interaction between spinon and \( \hbar c/e \) vortices \( \Phi_2 \). The long range logarithmic interaction between \( \hbar c/e \) vortices \( \Phi \) and \( \hbar c/e \) vortices \( \Phi_2 \) is mediated by the electrical gauge field \( a_\mu \). Condensing \( \Phi_2 \) \( \Phi_2 \) will generate a mass term \( \frac{\Phi_2}{2} (a_\mu)^2 \) which dominates over the Maxwell term. Integrating out \( a_\mu \) leads to

\[ \frac{1}{\Phi_2} \left( \frac{|f_{\mu
u}^{eff}|^2}{4} - i A_{\mu
u}^{\alpha\beta\gamma} \epsilon_{\mu\nu\lambda} \partial_\alpha a_\lambda^{\alpha\beta\gamma} \right) + V(\Phi) \]  

All the generated terms only renormalize the short range interactions already included in \( V(\Phi) \).

A. Quantum Critical point

In order to calculate conductivities, we add two source fields \( A_\mu^\psi \) and \( A_\mu^\Phi \) for the quasi-particles and vortex respectively:

\[ \mathcal{L} = \psi_\alpha^\dagger \gamma_\mu (\partial_\mu - i a_\mu^\psi - i A_\mu^\psi) \psi_\alpha + |(\partial_\mu - i a_\mu^\Phi - i A_\mu^\Phi) \Phi|^2 + V(|\Phi|) + \frac{i}{2 \pi \theta} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^{\alpha\beta\gamma} \partial_\alpha a_\lambda^{\alpha\beta\gamma} \]  

where \( a = 1, 2, 3, 4 \) stands for \( N = 4 \) species of Dirac fermion. In fact, two Maxwell terms for \( a^\psi \) and \( a^\Phi \) can be added to the above equation, but they are subleading to the mutual CS term in the low energy limit. It is expected that there is no periodicity under \( \theta \to \theta + 1 \) in the continuum limit.

The RG calculation in Refs.\textsuperscript{29,30,31} can be used to show that \( \theta \) is exactly marginal, therefore there is a line of fixed points determined by the mutual statistical angle \( \theta \). In order to calculate the spin conductivity along this fixed line, a source field could be introduced to couple to the spin current \( j_{\mu\nu}^{78} = \psi_\alpha^\dagger \gamma_\mu (\partial_\nu \psi_\beta) \). Similar calculations follow.

Integrating out both fermion and boson leads to:

\[ \mathcal{L} = - \frac{i}{2} a_\mu^\psi (\partial_\mu - i k) \Pi_{\mu\nu}^\psi (k) a_\nu^\psi (k) - \frac{i}{2} a_\mu^\Phi (\partial_\mu - i k) \Pi_{\mu\nu}^\Phi (k) a_\nu^\Phi (k) \]

\[ - \frac{i}{2} \partial_\mu (\partial_\mu - i k) \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \partial_\nu a_\lambda^{\alpha\beta\gamma} \]  

where the exact forms of \( \Pi^\alpha \) are dictated by gauge invariance and Furry’s theorem :
\[ \Pi_{\mu\nu}(k) = \Pi_{1}(k)(\delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}) \]
\[ \Pi_{\mu\nu}^{\psi}(k) = \Pi_{1}(k)\epsilon_{\mu\nu\lambda}k_{\lambda} \]
\[ \Pi_{\mu\nu}^{\Phi}(k) = \Pi_{1}(k)\epsilon_{\mu\nu\lambda}k_{\lambda} \]

(37)

Where \( \Pi_{1}, \Pi_{2}, \Pi_{3} \) are the polarizations for \( a^\psi a^\psi, a^\psi a^\Phi, a^\Phi a^\Phi \).

If we are only interested in DC conductivities, for simplicity, we can put \( \vec{k} = 0 \). Eqn.36 becomes (for the most general form, see appendix A):

\[ \mathcal{L} = -\frac{1}{2}a_{i}^{\psi}(\omega_{n})\Pi_{1}\omega_{n}a_{i}^{\psi}(\omega_{n}) - \frac{1}{2}a_{i}^{\Phi}(\omega_{n})\Pi_{2}\omega_{n}a_{i}^{\Phi}(\omega_{n}) \]
\[ - \frac{1}{2}\theta a_{i}^{\psi}(\omega_{n}) - A_{i}^{\psi}(\omega_{n})\epsilon_{ij}\omega_{n}(a_{j}^{\psi}(\omega_{n}) - A_{j}^{\psi}(\omega_{n})) \]
\[ - \frac{1}{2}\theta a_{i}^{\Phi}(\omega_{n}) - A_{i}^{\Phi}(\omega_{n})\epsilon_{ij}\omega_{n}(a_{j}^{\Phi}(\omega_{n}) - A_{j}^{\Phi}(\omega_{n})) \]

(38)

If we define \( \tilde{a}_{i}^{\psi}(\omega_{n}) = \epsilon_{ij}a_{j}^{\psi}(\omega_{n}), \tilde{A}_{i}^{\psi}(\omega_{n}) = \epsilon_{ij}A_{j}^{\psi}(\omega_{n}) \), the above equation becomes diagonal in the spatial indices \( i = 1, 2 \). Finally, integrating out \( a_{i}^{\psi}, \tilde{a}_{i}^{\psi} \) leads to

\[ \mathcal{L} = -\frac{1}{2}(A_{i}^{\psi}, \tilde{A}_{i}^{\psi}) \left( \begin{array}{cc} \sigma^{\psi} & \sigma^{H} \\ \sigma^{H} & \sigma^{\Phi} \end{array} \right) \left( \begin{array}{c} A_{i}^{\psi} \\ \tilde{A}_{i}^{\psi} \end{array} \right) \]

(39)

Where spinon, vortex and mutual Hall drag conductivities are:

\[ \sigma^{\psi} = \frac{1}{\theta}^{2}\frac{\Pi_{1}}{\Pi_{1}\Pi_{3} + (1/\theta - \Pi_{2})^{2}} \]
\[ \sigma^{\Phi} = \frac{1}{\theta}^{2}\frac{\Pi_{2}}{\Pi_{1}\Pi_{3} + (1/\theta - \Pi_{2})^{2}} \]
\[ \sigma^{H} = \frac{1}{\theta}^{2}\frac{\Pi_{2} - \theta(\Pi_{1}\Pi_{3} + \Pi_{2})}{\Pi_{1}\Pi_{3} + (1/\theta - \Pi_{2})^{2}} \]

(40)

In fact, all the three conductivities can be written in the elegant connection formula:

\[ \rho_{ij} = (\rho_{FB})_{ij} - \theta \epsilon_{ij} \]

(41)

with the conductivity tensor of fermion and boson given by

\[ \sigma_{FB} = \left( \begin{array}{cc} \Pi_{1} & -\Pi_{2} \\ -\Pi_{2} & \Pi_{3} \end{array} \right) \]

(42)

Although its form is similar to the conventional connection formulas discussed in [27,29,31], the physical interpretations of the conductivities are quite different (see the following).

When the vortices are generated by external magnetic field and pinned by impurities as discussed in [1], the total conductivity is the same as the fermion conductivity because the static vortices do not contribute. As explained in section V, the static vortices only feel the electric gauge field \( a_{\mu} \), but not the statistical gauge field \( a_{\mu}^{\psi} \).

The equations [1,22] are exact, but \( \Pi_{1}, \Pi_{2}, \Pi_{3} \) can only be calculated perturbatively in the coupling constant \( g_{2}^{2} = 2\pi \beta \). The renormalized propagators for \( a^{\psi} \) and \( a^{\Phi} \) can be found from the following Feynman diagrams:

![Feynman diagrams](attachment:diagram.png)

Fig 2: The renormalized propagators for \( a^{\psi} \) (thick wiggle line) and \( a^{\Phi} \) (thick dashed line). The thin wiggle line stands for \( a^{\psi} \), the dashed thin line for \( a^{\psi} \), the thin solid line for fermion propagators, the thick solid line for the boson propagators.

By using the bare propagators < \( a_{\mu}^{\psi} a_{\nu}^{\psi} > = -\epsilon_{\mu\nu\lambda}k_{\lambda}/k^{2} \) and the bare fermion and boson loop results

\[ \Pi_{\mu\nu}^{0} = \Pi_{\mu\nu}^{0} = -\frac{g_{2}^{2}}{16}k(\delta_{\mu\nu} - k_{\mu}k_{\nu}/k^{2}) \]

(43)

In contrast to the conventional CS theory studied in [29,31], the propagators are even in \( k \), this is because the theory respects \( T \) symmetry. On the other hand, in contrast to the Maxwell propagators, they behave as \( 1/k \) instead of \( 1/k^{2} \).

The three loop diagrams for \( \Pi_{1} \) are given by:

![Three loop diagrams](attachment:diagram2.png)

Fig 3: The three loop diagrams of \( \Pi_{1} \). The thick wiggle line stands for the renormalized propagators of \( a^{\psi} \), the thin solid line stands for the fermion propagator. The one loop diagram is not shown.

By using \( G_{\mu\nu}^{0} \), and extracting the symmetric part of the gauge propagator in the large \( N \) results in Refs. [29,30], we are able to calculate the above three loop diagrams. Furry’s theorem can be used to eliminate large number of null diagrams. We get the following series:

\[ \Pi_{1} = MN\frac{\pi}{8}(1 + \frac{3}{16}g_{2}^{4} + g_{2}^{8} + \cdots) \]

(43)

where \( N = 4 \) ( \( M = 1 \) ) is due to the sum over 4 (1) species of Dirac fermions (bosons).

The three loop diagrams for \( \Pi_{3} \) are given by:

![Three loop diagrams](attachment:diagram3.png)
The mutual Hall drag conductivity is an odd function of $\theta$. Note that both $\Pi_1$ and $\Pi_3$ are even functions of $\theta$.

From Furry’s theorem, one of the first non-vanishing diagram for $\Pi_2$ is:

\[ \Pi_2 = MN \frac{\pi}{8} (1 + g^4 + g^8 + \cdots) \]  

Note that both $\Pi_1$ and $\Pi_3$ are even functions of $\theta$.

The series is $\Pi_2 = MN (g^0 + g^{10} + \cdots)$ which is an odd function of $\theta$.

From Eqn. 33, $\sigma^\psi$, $\sigma^\Phi$ are even functions of $\theta$, but the mutual Hall drag conductivity is an odd function of $\theta$. These are expected from $P-H$ transformation. Under the $P-H$ transformation of the vortex operator $\Phi \to \Phi^\dagger$ in Eqn. 33, it can be shown that $A^\Phi_{\mu \gamma \theta} = -A^\Phi_{\mu \gamma \theta}$. From Eqn. 31, we reach the same conclusions. Specifically, $\sigma^H$ takes opposite values for $\theta = \pm 1/2$, the periodicity under $\theta \to \theta + 1$ is not preserved. Experimentally, the Hall drag conductivity can be detected by measuring the transverse voltage drop (or transverse temperature drop for thermal conductivity) of spinons due to the longitudinal driving of vortices. The Hall drag conductivity in double layer Quantum Hall systems has been investigated by several authors [33]. In double layer systems, the electrons in different layers are treated as two different species. There is a mutual CS interaction between the two species (both are fermions) which is directly responsible for this Hall drag conductivity, although the Coulomb interaction between the two species is responsible for the Coulomb (longitudinal) drag [33]. This example shows that no external magnetic field is needed to produce a Hall effect! For example $\theta = \pm 1/2$ lead to opposite Hall drag conductivities although static $\alpha = \pm 1/2$ vortex leads to no Hall conductivity.

Again, by using the large $N$ result of [23,30], we find the anomalous dimensions of the fermion and vortex to two loops:

\[ \eta_\psi = -\frac{g^4}{48\pi^2} M \]
\[ \eta_\Phi = \eta_{XY} - \frac{g^4}{12\pi^2} N \]

where $\eta_{XY} \sim 0.038$ is the anomalous dimension for the 3d XY model [33].

The correlation length exponent can also be calculated to two loops by the insertion of the operator $\Phi^\dagger \Phi$:

\[ \nu = \nu_{XY} - \frac{g^4}{12\pi^2} N \]

where $\nu_{XY} \sim 0.672$ is correlation length exponent for the 3d XY model [33].

It is instructive to go to dual representation of Eqn. 33, namely go to the boson representation:

\[ \mathcal{L} = \psi^\dagger_\alpha \gamma_\mu \partial_\mu a^\alpha - i a^\alpha \partial_\mu \psi^\dagger_\alpha + |(\partial_\mu - ia^\alpha/\theta)|^2 + V(|\phi|) + \frac{1}{4} (\partial^\alpha a^\alpha)^2 + ia^\mu_\alpha \epsilon_{\mu\nu\lambda} \partial_\nu (a^\lambda_{\alpha} - a^\alpha_{\lambda}/\theta) - i A^\Phi_{\mu \nu \lambda} \partial_\nu a^\alpha_{\lambda} \]

where $V(|\phi|) = m^2_0 |\phi|^2 + g_0 |\phi|^4 + \cdots$.

Integrating out $a^\alpha_{\mu}$ leads to the constraint up to a pure gauge:

\[ a^\alpha_{\mu} = \theta a^\alpha_{\mu} \]

Substituting the above constraint to Eqn. 47 and setting $a^\alpha_{\mu} = a_{\mu}$, we find:

\[ \mathcal{L} = \psi^\dagger_\alpha \gamma_\mu (\partial_\mu - i a_{\alpha}) \psi^\dagger_\alpha + \frac{1}{4} f^2_{\mu \nu} \\
+ |(\partial_\mu - i a_{\mu})|\phi|^2 + V(|\phi|) - i A^\Phi_{\mu \nu \lambda} \partial_\nu a_{\lambda} \]

The above Eqn. indicates that fermions and bosons are coupled to the same gauge field whose dynamics is described by Maxwell term instead of C-S term. This Eqn. is simply $2 + 1$ dimensional combination of spinor...
QED and scalar QED. This action is similar to a relativistic analogue of the $U(1)$ gauge field theory investigated extensively in Ref. \[24\]. The RG analysis at $4 - \epsilon$ by dimensional regularization is possible, because the marginal dimension of all the relevant couplings are 4. However, a RG analysis directly at $2 + 1$ dimension is formidable. The physical meaning of $\theta$ is obscure and the exact marginality of $\theta$ is a highly non-trivial result in the boson representation, but all these become evident in the dual vortex representation Eqn. \[35\]. It is evident that there is no periodicity under $\theta \to \theta + 1$ in Eqn. \[13\].

If we perform duality transformation again on Eqn. \[19\] to go to the vortex representation, then we recover Eqn. \[24\] upon neglecting the two Maxwell terms which are subleading to the mutual C-S term.

In the next subsection, setting the two source terms vanishing, we look at the properties of the different phases on the two sides of this quantum critical point.

**B. 2 + 1 dimensional QED in the underdoped phase and free fermions in the superconducting phase**

In the disordered phase, the vortex condenses $\langle \Phi \rangle = \Phi_0$ which generates a mass term for $a^\phi_\mu$ in Eqn. \[23\]

$$\frac{\Phi_0^2}{2} (a_{\mu}^\phi)^2$$

where the subscript $t$ means transverse projection.

Integrating out the massive $a^\phi_\mu$ leads to a Maxwell term for $a_{\mu}^\psi$:

$$\mathcal{L} = \psi^\dagger_\alpha \gamma_{\mu} (\partial_{\mu} - ia_{\mu}^\psi) \psi_\alpha + \frac{m_\phi}{2} (a_{\mu}^\psi)^2$$

(51)

This is simply 2+1 dimensional spinor QED which was studied by large N expansion by the authors in \[23\]. They used four component Dirac fermions and discussed possible dynamic mass generation of the fermions which break the chiral symmetry, but preserves Parity and Time reversal symmetry. However, for 2+1 dimensional compact QED, the possible important instanton effects were not well understood yet. The possible connection between this dynamic mass generation and Anti-ferromagnetism in the context of gapless flux phase was discussed by Marston \[34\].

In fact, we reach the same description from the boson representation Eqn. \[19\]. Because in the disordered phase, the boson $\phi$ is massive, therefore can be integrated out, it generates the Maxwell term $\frac{1}{4 m_\phi} f_{\mu\nu}^2$, which dominates over the existing non-critical Maxwell term. We reach Eqn. \[23\] after identifying $m_\phi \sim \Phi_0^2$.

In the superconductor phase, the vortex $\Phi$ is massive, therefore can be integrated out, it leads to the old Maxwell term $\frac{1}{4 m_\phi} (f_{\mu\nu}^\phi)^2$. Integrating out $a^\phi$ generates a mass term:

$$\mathcal{L} = \psi^\dagger_\alpha \gamma_{\mu} (\partial_{\mu} - ia_{\mu}^\psi) \psi_\alpha + \frac{m_\phi}{2} (a_{\mu}^\psi)^2$$

(52)

where $m_\phi$ is the mass of the vortex.

The Dirac fermions become free. In fact, we reach the same description from the boson representation Eqn. \[19\]. Because in superconductor phase, the boson $\phi$ condenses, therefore generates a mass term $\frac{\phi_0^2}{4 \Phi_0^2}$ which renders the Maxwell term ineffective. We reach the same conclusion from both sides by identifying $m_\phi \sim \phi_0^2$ which is dual to the relation in the disordered phase $m_\phi \sim \Phi_0^2$.

In short, in the disordered phase, the system is described by spinor QED Eqn. \[23\] in the superconductor side, by free Dirac fermion. We can view the transition as a simple example of confinement-deconfinement transition. In the confined (disordered) phase, the boson and fermion are confined together by the fluctuating gauge field. In the deconfined (superconductor) phase, the boson condensed, the gauge field becomes massive, the fermion becomes free. There is a line of fixed point governed by the mutual statistical angle $\theta$ separating the two phases.

**IV. THE EFFECT OF CHARGE FLUCTUATION AND THE ABSENCE OF MASSLESS DYNAMIC GAUGE FIELD AT $T = 0$**

In this section, we try to investigate the effect of charge fluctuation on the fixed line characterized by the statistical angle $\theta$ discovered in the last section by considering the combined effects of $U(1)_c$ mutual statistical gauge fluctuation and $U(1)_e$ electrical gauge fluctuation and treat both $U(1)$ gauge fields on the equal footing. We find the condensation of $hc/2e$ vortex condensation indeed leads to the confinement of spinon and chargon into Cooper pairs and electrons, in contrast to the condensation of $hc/e$ vortex. The fixed line in the last section is destroyed and replaced by a quantum critical point separating the superconducting state and some sort of charge ordered state. There is no gapless gauge field left in the final theory, in sharp contrast to the $U(1)$ or $SU(2)$ gauge theory \[24\]. Both mutual C-S interaction term and the dynamic Doppler-shift term become irrelevant short-ranged interactions.

The authors in Ref. \[20\] studied the similar model. They performed a perturbative RG calculation and reached very different conclusions from ours.

**A. Quantum critical point**

Putting $a_{\mu}^\phi \to a_{\mu}^\phi - a_{\mu}$ in Eqn. \[24\], we get

$$\mathcal{L} = \psi^\dagger_\alpha \gamma_{\mu} (\partial_{\mu} - ia_{\mu}^\psi) \psi_\alpha + (|\partial_{\mu} - ia_{\mu}^\phi|^{2} + V(|\Phi|)$$

$$+ \frac{i}{2\pi} a_{\mu}^\psi \varepsilon_{\mu\nu\lambda} \partial_{\nu} a_{\lambda}^\psi - ia_{\mu}^\psi \varepsilon_{\mu\nu\lambda} \partial_{\nu} (a_{\lambda}^\psi / \theta + A_{\lambda}^{\psi / f}) + \frac{1}{4} f_{\mu\nu}^2$$

(53)
Integrating out the electric gauge field $a_{\mu}$ leads to:

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i a_\mu^\psi) \psi_\alpha + |(\partial_\mu - i a_\mu^\psi) \Phi|^2 + V(|\Phi|)$$

$$+ \frac{i}{2\pi \theta} a_\mu^\psi \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^\Phi + \frac{1}{2} (a_\mu^\psi / \theta + A_\mu^{eff})^2$$  \hspace{1cm} (54)

Comparing to Eqn. 47, it is easy to see that the charge fluctuation leads to a mass term for the gauge field $a_\mu^\psi$. Shifting $a_\mu^\psi / \theta + A_\mu^{eff} \rightarrow a_\mu$, and adding the gauge fixing term $\frac{1}{4}(\partial_\mu a_\mu)^2$, we can integrate out the massive gauge field $a_\mu$ in Lorentz gauge $\alpha = 0$ and find:

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i a_\mu^\psi) \psi_\alpha + |(\partial_\mu - i a_\mu^\psi) \Phi|^2 + V(|\Phi|)$$

$$+ \frac{1}{4} (f_{\mu\nu}^\psi)^2 - i (A_\mu^{eff} + \partial_\mu a_\mu^\psi) (a_\mu^\psi / \theta + j_\mu^\psi)^2$$  \hspace{1cm} (55)

Note the Maxwell term for $a_\mu^\psi$ is generated by the integration over the massive $a_\mu$. Setting $A_\mu = 0$, integrating out the fermions only leads to higher derivative terms than the Maxwell term:

$$\mathcal{L} = |(\partial_\mu - i a_\mu^\psi) \Phi|^2 + V(|\Phi|) + \frac{1}{4} (f_{\mu\nu}^\psi)^2 + \cdots$$  \hspace{1cm} (56)

where $\cdots$ means higher than second order derivatives. Therefore, the vortex and fermion are asymptotically decoupled. It indicates that the charge fluctuation neglected in the last section destroy the fixed line characterized by $\theta$. However, very different conclusions are reached in Ref. [24]. We think that the perturbative RG calculation in [24] may not treat the charge gauge field fluctuation correctly.

Just like the last section, it is instructive to go to dual representation of Eqn. 54, namely go to the boson representation:

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i a_\mu^\phi) \psi_\alpha + |(\partial_\mu - i a_\mu^\phi) \Phi|^2 + V(|\Phi|)$$

$$+ \frac{1}{4} (f_{\mu\nu}^\phi)^2 + i a_\mu^\phi \epsilon_{\mu\nu\lambda} \partial_\nu (a_\lambda^\phi - a_\lambda^\phi / \theta)$$

$$+ \frac{1}{2} (a_\mu^\phi / \theta + A_\mu^{eff})^2$$  \hspace{1cm} (57)

Integrating out $a_\mu^\phi$ leads to the same constraint as Eqn. 56, up to a pure gauge:

$$a_\mu^\phi = \theta a_\mu^\phi$$  \hspace{1cm} (58)

Substituting the above constraint to Eqn. 57 and setting $a_\mu^\phi = a_\mu$, we find:

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i a_\mu^\psi) \psi_\alpha + |(\partial_\mu - i a_\mu^\psi) \Phi|^2 + V(|\Phi|)$$

$$+ \frac{1}{4} (f_{\mu\nu}^\psi)^2 + \frac{1}{2} (a_\mu + A_\mu^{eff})^2$$  \hspace{1cm} (59)

Comparing to Eqn. 49, the important difference is that the gauge field acquires a mass due to the charge fluctuation which renders the Maxwell term ineffective. Note that the dynamic Doppler shift term is encoded in $A_\mu^{eff}$.

Up to irrelevant couplings, we can safely set $a_\mu = A_\mu^{eff}$ in the above equation and find

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i 2 A_\mu^\psi) \psi_\alpha + |(\partial_\mu - i 2 A_\mu^\psi) \phi|^2 + V(|\phi|) + \cdots$$  \hspace{1cm} (60)

where $\cdots$ means the irrelevant couplings between bosons and fermions.

The above Eqn. leads to the conclusion that the mutual statistical interaction between $hc/2e$ vortex and spinons leads to the confinement of spinon and charge into electron and Cooper pair. The quasi-particles carry charge $2e$ which can be identified as electrons or holes, the Cooper-pairs are described by 3D X-Y model. The mutual C-S coupling and the dynamic Doppler-shift coupling between Cooper pairs and electrons are short-ranged interactions which are irrelevant near the quantum critical point. There is no gapless gauge fields in the final effective action! However, there are several important factors left out in the above analysis: (1) The Berry phase term for the boson which is first order time derivative term is not included. This term is vanishing only at half-filling with particle-hole symmetry hopping. (2) The long-range Coulomb interaction between Cooper pairs and electrons is not included. When the both effects are taken into account, the final ground state in the underdoped side at $T = 0$ maybe an insulating state with some kind of charge order. The nature of the QCP between the d-wave superconducting state and this charge ordered state is an important unsolved problem. However, the conclusion that there is no spin-charge separation and no gapless dynamic gauge fluctuation remains robust.

### B. Disordered and superconducting phases

We follow the discussions in the previous section. In the disordered phase, the vortex condense $< \Phi >= \Phi_0$ which generates a mass term for $a_\mu^\phi$ in Eqn. 54. Integrating out the massive $a_\mu^\phi$ leads to a Maxwell term for $a_\mu^\psi$:

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i a_\mu^\psi \psi_\alpha + \frac{1}{4\Phi_0^2} (f_{\mu\nu}^\psi)^2 + \frac{1}{2} (a_\mu^\psi / \theta + A_\mu^{eff})^2$$  \hspace{1cm} (59)

Because of the mass term, the Dirac fermions become free and carry spin $1/2$ and charge $2e$. In fact, we reach the same conclusion from the boson representation Eqn. 53.

In the superconductor phase, the vortex $\Phi$ is massive, therefore can be integrated out, it leads to the old Maxwell term $\frac{1}{4\Phi_0^2} (f_{\mu\nu}^\psi)^2$. Integrating out $a_\mu^\phi$ generates a mass term for $a_\mu^\psi$:

$$\mathcal{L} = \psi_\alpha^* \gamma_\mu (\partial_\mu - i a_\mu^\psi) \psi_\alpha + m_{\phi} (a_\mu^\psi)^2 + \frac{1}{2} (a_\mu^\psi / \theta + A_\mu^{eff})^2$$  \hspace{1cm} (59)
where $m_\Phi$ is the mass of the vortex.

Diagonizing the last two mass terms leads to a continuously changing charge. This is expected, because in the superconducting state, the quasi-particle is a linear combination of electrons and holes, therefore carrying continuously changing charge. In fact, we reach the same description from the boson representation.

Because in superconductor phase, the boson condense $\phi = \phi_0$, therefore generates a mass term for $a_\mu$:

$$\mathcal{L} = \psi_\alpha^\dagger \gamma_\mu (\partial_\mu - i \theta a_\mu) \psi_\alpha + \frac{g^2}{2} (a_\mu)^2 + \frac{1}{2} (a_\mu + A_\mu f f)^2$$

(63)

Which is essentially the same as Eqn.62 after the identification $m_\Phi \sim \phi_0^2$.

In short, in the disordered phase, the quasi-particles carry charge $2e$ and spin 1/2 which can be identified as electrons or holes. When the Berry phase term and the long-range Coulomb interactions are taken into account, the electrons may form some charge ordered state. In the superconducting side, the quasi-particles carry continuously changing charge which is a linear combination of electrons and holes.

V. APPLICATION TO STATIC DISORDERED VORTEX ARRAY AND THERMALLY GENERATED VORTICES

So far, we discussed the vortices generated by quantum fluctuations in the underdoped regime. Vortices can also be generated by an external magnetic field inside superconductors or by thermal fluctuations in the pseudogap regime. They were discussed in Refs. [6] and [40] respectively. In this section, we apply the formalism developed for quantum generated vortices at $T = 0$ to study vortices generated by an external magnetic field or by thermal fluctuations. By studying the three different kinds of vortices in a unified picture, we not only recover the previous established results, but also provide additional physical insights on the internal gauge field fluctuations.

Inside the $d$-wave superconductor, although the Cooper-pair condensates dominates the electrical transport at zero frequency, the quasi-particles near the four nodes of a $d$-wave superconductor are responsible for the heat and spin transport at low temperature. Recently Ye studied the quasi-particle transport in a random vortex array with a fully quantum mechanical approach [40]. Although the previous semi-classical approach only capture the physics of the Volovik effect, fully quantum mechanical approach treats the Volovik effect (scalar field scattering) and the AB phase scattering (the gauge field scattering) on the equal footing. Ye found that the long range logarithmic interaction between vortices suppress the superfluid velocity fluctuation, but does not affect the internal gauge field fluctuation. He concluded that the quasi-particle scattering from the random gauge field dominate over that from the superfluid velocity (the Volovik effect).

Roughly speaking, the mixed state with vortex density $n_v$ corresponds to the vortex condensed phase (disordered phase). As discussed in the previous sections, the vortex condensation leads to a mass term $\frac{1}{2} n_v (a_\mu^v)^2$ for the statistical gauge field $a_\mu^v$ in Eqn.22. Adding the mass term to the equation, integrating out $a_\mu^v$ and keeping only the time component of the charge gauge field $a_\mu^v$ which mediates the long-range density-density logarithmic interaction between the static vortices, we get two static Maxwell terms for $a_\mu^v, v_\alpha$ respectively and a mass term for $v_\alpha$:

$$\mathcal{L} = \psi_\alpha^\dagger \gamma_\mu (\partial_\mu - i a_\mu^v) \psi_\alpha + \sum_{\alpha} v_\alpha \psi_\alpha^\dagger \psi_\alpha$$

$$+ \frac{p}{4 a_\alpha} (f_{\alpha\beta}^v)^2 + \frac{p}{4 a_\alpha} (f_{\alpha\beta}^v)^2 + \frac{p}{4} (v_\alpha)^2$$

(64)

where $p$ is added to stand for an unknown function of pinning potential, $f_{\alpha\beta}^v = \theta_\alpha \phi_\beta - \theta_\beta \phi_\alpha, f_{\alpha\beta}^v = \theta_\alpha v_\beta - \theta_\beta v_\alpha$.

In Eqn.64, the averages over $a_\mu^v, v_\alpha$ should be understood as quenched instead of annealed averages. Namely there is no feedback effects on the gauge field and scalar field propagators from the fermions [11]. The above equation shows that the long-range logarithmic interaction between vortices suppress the fluctuation of the superfluid velocity and leads to a mass term for the superfluid velocity and the incompressibility of the vortex system. The quasi-particle is moving in a long-range correlated random magnetic field and short-range correlated scalar potential. The distribution of the random gauge field is given by the corresponding static Maxwell term and that of the scalar field is given by the corresponding static Maxwell term and the mass term. Eqn. 64 is essentially the same as Eqn. 9 in Ref. [3].

Strictly speaking, the gauge field from the random vortex array is a random $Z_2$ gauge field. In Ref. [3], in order to perform an analytic calculation in the continuum limit, Ye made an approximation by replacing the discrete random $Z_2$ AB phase by a continuous random $U(1)$ AB phase and performed a standard diagrammatic perturbation theory.

In Ref. [10], assuming the finite temperature phase transition from $d$-wave superconductor to pseudo-gap regime is a Kosterlize and Thouless (KT) vortex and anti-vortex unbinding transition [29]. Ye studied the electron spectral function in the vortex plasma regime in Fig.1 [10]. Starting from Eqn.25, he found that the random gauge field $a^v$ missed in previous semiclassical approaches [42] destroyed the coherent spinon motion and leads to branch cut singularities and pertinent non-Fermi liquid behaviors. There are three subtle differences between thermally generated vortices from the static vortices generated by an external magnetic field inside the
The following alternative derivation of Eqn.64 may be possible:

After taking into account these subtle differences, the above procedures leading to Eqn.64 can be equally applied to thermally generated free vortices with density $n_f$. The following alternative derivation of Eqn.64 may be stimulating: the vortices being treated classically, their commutation relations can be neglected, the dual singular gauge transformation in Eqn.29. On the vortices is not necessary, the Berry phase term for the boson (the linear time derivative term) can also be neglected. More specifically, only the time component of the charge gauge field $a_0$ in Eqn.24 is kept to mediate the long-range density-density logarithmic interaction between the vortices. Obviously, $a_0$ couples to the vortices the same way as the superfluid velocity $v_\alpha$ couples to the spinon, its fluctuation leads to a mass term for the $v - v$ correlation. However, being orthogonal to the spatial component $\tilde{\alpha}^0$, $a_0 - \tilde{\alpha}^0$ correlator remains gapless. This also leads to exactly the same equation as Eqn.4 with $p$ setting equal to 1 and the same conclusions as those first reached in Ref.1.

However, at zero temperature, the vortices being treated quantum mechanically, all the components of the charge gauge field $a_\mu$ should be kept to mediate the long-range current-current logarithmic interactions between the vortices. Then the charge gauge field fluctuation leads to mass terms not only for $v$, but also for the spatial component $\tilde{\alpha}^\mu$. There is no gapless dynamic gauge fluctuations in the Cooper-pair picture as demonstrated in Eqn.5. The phenomenon that the two gapless gauge fields cancel each other is similar to the standard Higgs mechanism. In Higgs mechanism, the gapless gauge field is eaten by gapless Goldstone modes. The two gapless modes annihilate each other, the Goldstone mode disappears, while the gapless gauge field becomes massive.

VI. DISCUSSIONS AND CONCLUSIONS

Based on the earlier work by Balents, Fisher and Nayak [1], Senthil and Fisher developed $Z_2$ gauge theory [19,20] to study quasi-particles coupled to vortices generated by quantum fluctuations. By breaking electrons and Cooper pairs into smaller constitutes: chargons with charge $e$, spin 0 and spinons with charge 0,spin 1/2, SF introduced a local $Z_2$ gauge degree of freedom to constrain the Hilbert space to be the original one. The effective action describes both chargons and spinons coupled to local fluctuating $Z_2$ gauge theory with a doping dependent Berry phase term. By the combination of standard duality transformation of 3 dimensional XY model and that of $Z_2$ gauge theory, the action is mapped into a dual vortex representation where the $hc/2e$ vortices and spinons are coupled by a mutual $Z_2$ CS gauge theory. As usual vortices in XY model, the $hc/2e$ vortices also couple to a fluctuating $U(1)$ gauge field which mediates the long-range logarithmic interaction between the vortices. Starting from the dual representation, the authors in Ref.20 studied a transition from d-wave superconductor to confined Mott insulator driven by the condensing of $hc/2e$ vortices at half filling. In order to study the critical behaviors of this particular confinement and deconfinement transition, they replaced the $Z_2$ mutual CS theory on the lattice by $U(1)$ mutual CS theory in the continuum and performed renormalization Group (RG) analysis. Some of their RG analysis may not be correct as demonstrated in Sec.IV. In contrast to $U(1)$ gauge theory which has only confined phase, $Z_2$ gauge theory has both confined and deconfined phases. The topological excitation of the $Z_2$ gauge field play an important role. If $hc/2e$ vortices condense, the visons are also condensed, the system is in the confined phase, the chargon and spinon are confined into electron, Cooper-pair and magnon. If $hc/e$ vortices condense, the visons are gapped, the system is in the deconfined phase, the spinon and chargon are asymptotically decoupled in the long-wavelength limit. SF argued that the cuprates are in the deconfined phase, therefore the zero temperature transition at $x = x_c$ is driven by the condensation of $hc/e$ vortex. They further proposed the ” vison trapping experiment ” to test their spin-charge separation scenario. Several groups performed such experiments, but found no signature of visons [88].

The original Anderson singular gauge transformation [14] was proposed for static vortices. In this paper, we extend the one singular gauge transformation for static vortices to two mutual flux-attaching singular gauge transformations for moving vortices generated by quantum fluctuations. By making a close analogy to the conventional singular gauge transformation of FQH system, we perform the two singular gauge transformations attaching flux of moving vortices to quasi-particles or vice versa. Just like conventional singular gauge transformation leads to conventional CS term, the two mutual singular gauge transformations lead to mutual CS term. In this way, we propose an intuitive and physical transparent approach to bring out explicitly the underlying physics associated with the condensation of the $hc/2e$ vortices. When considering both mutual C-S interaction and long-range logarithmic interaction between the vortices, we find that there are only electrons and Cooper pairs in the spectrum and there are no gapless gauge
field fluctuation in contrast to the $U(1)$ or $SU(2)$ gauge theory\cite{21}. When taking into account both the linear time derivative Berry phase term for the bosons and the long-range Coulomb interactions between Cooper pairs and electrons, the true ground state in the underdoped side may be some unknown charge ordered state. Understanding the zero temperature transition from d-wave superconductor to this unknown charge ordered state is an important future research direction. The electron spectral weight starting from Eqn.\ref{eq:23} measured by ARPES in the pseudo-gap regime at $T > T_c$ indicated in Fig.1 was presented in a separate publication\cite{10}. The theory presented in this paper focus on the confined phase. The recent experiment\cite{38} indicates that the cuprates may be inside the confined phase. In principle, our theory should be equivalent to the $Z_2$ gauge theory in the confined phase side after tracing out the fluctuating and AB phase scattering on ARPES data was discussed\cite{16}. The electron spectrum under CS theory in the continuum does not have the periodicity of the conventional CS theory under $\theta = 0$, we recover the results calculated in Sec. III.

APPENDIX B: MOVING $HC/E$ VORTICES AND THE IRRELEVANCE OF THE DYNAMIC DOPPLER SHIFT EFFECT

For $hc/e$ vortex, we can perform a neutral-like single-valued gauge transformation by setting $\phi_A = \phi_B = \phi/2$. The double strength vortex and its stability was investigated in Ref.\cite{3}, the associated spin-charge separation has been discussed extensively in Refs.\cite{11}. In this appendix, considering a moving $hc/e$ may still scatter quasi-particles by the AB phase, in analogy to the $hc/2e$ vortices discussed in the main text, we still include a possible mutual C-S interaction with the statistical angle $\theta$ taking any integer values. For $\theta = 0$, we recover the results in Ref.\cite{3} and also get the new result on the expression of current operator $J\mu$ in the superconducting phase. Although the static Volovik effect is marginally relevant\cite{11} and cause finite density of states of the quasi-particles at zero energy. In this appendix, we show that surprisingly the dynamic Volovik effect is highly irrelevant on the both sides of the QCP.

Putting $\phi_A = \phi_B = \phi/2$, namely setting $a_\mu = 0$ in Eqn.\ref{eq:14} leads to:

$$\mathcal{L} = \bar{\psi}_\mu \left[ \partial_\tau + v_\phi p_x \tau^3 + v_\Delta p_y \tau^1 \right] \psi_\mu (1 \rightarrow 2, x \rightarrow y) + \frac{K}{2} \left( \partial_\mu \phi - A_\mu \right)^2$$

(141)
where the effective gauge field is $A^{\text{eff}}_{\mu} = A_{\mu} - K^{-1}J_{\mu}$.

There is no internal gauge field $a_{\mu}$, because static double strength vortices do not scatter the quasi-particles by AB phase.

Performing duality transformation to vortex representation leads to

$$\mathcal{L} = \bar{\psi}_{\mu} \gamma_\mu \partial_{\mu} \psi_{\alpha} + |(\partial_{\mu} - i a_{\mu})\phi|^2 + V(\phi)$$

$$+ \frac{1}{4} f_{\alpha \beta}^2 + i a_{\mu} \epsilon_{\mu \lambda \nu} \lambda_{\nu} (a_{\lambda}^{\text{eff}} - A_{\lambda}^{\text{eff}}) + \frac{1}{4} f_{\mu \nu}^2$$

where the last (Berry phase) term can be absorbed into $A^{\text{eff}}_{\mu}$ by redefining $A^{\text{eff}}_{\mu} \rightarrow A^{\text{eff}}_{\mu} + i\mu a_{\mu}$. Comparing with Eqn.22, there is only charge fluctuation. There is no internal C-S term, namely $\theta = 0$.

The duality transformation to boson representation is

$$\mathcal{L} = \bar{\psi}_{\mu} \gamma_\nu \partial_\nu \psi_{\alpha} + |(\partial_{\mu} - i a_{\mu})\phi|^2 + V(\phi)$$

$$+ \frac{1}{4} f_{\mu \nu}^2 + \frac{1}{2} (a_{\mu}^{\text{eff}} - A_{\mu}^{\text{eff}})^2$$

(B3)

Integrating out the electric gauge field leads to

$$\mathcal{L} = \bar{\psi}_{\mu} \gamma_\nu \partial_\nu \psi_{\alpha} + |(\partial_{\mu} - i a_{\mu})\phi|^2 + V(\phi)$$

$$+ \frac{1}{4} f_{\mu \nu}^2 + \frac{1}{2} (a_{\mu}^{\text{eff}} - A_{\mu}^{\text{eff}})^2$$

(B4)

Setting $a_{\mu}^{\text{eff}} = A_{\mu}^{\text{eff}}$ up to irrelevant term leads to

$$\mathcal{L} = \bar{\psi}_{\mu} \gamma_\nu \partial_\nu \psi_{\alpha} + \mu \phi \partial_\mu \phi + |(\partial_{\mu} - i A_{\mu})\phi|^2 + V(\phi) + \cdots$$

(B5)

where the second term is the Berry phase term which is a first order time derivative and the irrelevant couplings between bosons and fermions. Eqn.32 is essentially the same as Eqn.31.

However, a moving $hc/e$ vortex may still scatter quasi-particles by AB phase. In general, we should also include a possible mutual C-S term in the effective action. The most general form is again given by Eqn.31 with the statistical angle $\theta$ taking any integers and $A^{\text{eff}}_{\mu} = A_{\mu} - K^{-1}J_{\mu}$ for $hc/e$ vortex. The formulation developed in Sec.IV can be straightforwardly applied to $hc/e$ vortices. Putting the above two values for $\theta$ and $A^{\text{eff}}_{\mu}$ into Eqn.53 leads to

$$\mathcal{L} = \bar{\psi}_{\mu} \gamma_\nu \partial_\nu \psi_{\alpha} + \mu \phi \partial_\mu \phi + |(\partial_{\mu} - i A_{\mu})\phi|^2 + V(\phi) + \cdots$$

(B6)

where the second and third terms are the Berry phase terms which are first order time derivatives.

The above equation indicates that the quasi-particle carries charge $\theta$ which can take any integer numbers. Obviously, there is no periodicity in $\theta$. The most natural and simplest choice is $\theta = 0$ where the above equation reduces to Eqn.33. In this equation, bosons and fermions are asymptotically decoupled. The fermions (spinons) carry only spin 1/2, the bosons (holons) carry only charge $e$. There is spin-charge separation. The long-range Coulomb interaction could be incorporated by adding the dynamic term for the time component of the gauge field $\frac{1}{2}k|A_0(k)|^2$. Without the fermionic part, the action is the same as the superconductor to insulator transitions studied in Ref.10, except here condensed is a charge $e$ boson instead of a Cooper pair.

In the underdoped regime, the vortex condenses $\Phi = \Phi_0$ which generates a mass term for $a_{\mu}$ in Eqn.22. Integrating out the massive $a_{\mu}$ leads to a Maxwell term for $A^{\text{eff}}_{\mu} = A_{\mu} - K^{-1}J_{\mu}$ from which we can identify the automatically conserved electric current $I_{\mu} = \frac{1}{e} (\partial_\nu J_{\nu} - \partial_\nu a_{\nu}).$ The **dynamic** Doppler shift effect is $\sim \frac{1}{\Phi_0^2}(\partial J)^2$, it is a four fermion term with two derivative, therefore highly irrelevant. The static Volovik effect is marginally relevant and cause finite density of states at zero energy. Surprisingly, the dynamic Volovik effect is highly irrelevant. Note that its coefficient $\Phi_0^{-2}$ diverges as we approach the QCP from the underdoped side.

In the superconductor phase, the vortex $\Phi$ is massive, therefore can be integrated out, it leads to the old Maxwell term $\frac{1}{4\mu_0}(\partial J)^2$. Integrating out $a_{\mu}$ generates a mass term $\frac{1}{2\mu_0}(A^{\text{eff}}_{\mu})^2.$ From the mass term, we can identify the automatically conserved electric current $I_{\mu} = m_{\phi}(\partial_{\nu} J_{\nu}).$ Note that although the quasi-particle electric current is not conserved itself, the total electric current $I_{\mu} = m_{\phi}(\partial_{\nu} J_{\nu} - \frac{2e}{\Phi_0^2} J_{\nu}).$ is conserved. The dynamic Doppler shift effect due to the virtually fluctuating vortex-anti-vortex pairs is $\sim m_{\phi}(J)^2$. It is a four fermion interaction term, therefore irrelevant. Note that its coefficient vanishes as we approach the QCP from the superconducting side.

We reach the conclusion that the dynamic Doppler shift term is irrelevant on both sides of the QCP. We conjecture that it is likely also irrelevant at the QCP. Counter-intuitively, it contains two more derivatives in the vortex condensed underdoped regime than in the vortex-depleted superconducting phase, therefore even more irrelevant in the underdoped regime than in the superconducting phase. The previous scaling argument on the irrelevance of this term at the QCP presented in Ref.1 is questionable, because the authors scaled the phase $\phi$ like a free field. In fact, $\phi$ should be confined to be $0 < \phi < 2\pi$, so it is far from being a free field.

Putting $\phi_A = \phi_B = \phi/2$ in Eqn.24, we get the electron annihilation operator:

$$C_{\alpha} = \sum_{i=1,2} [e^{i\phi/2} e^{ik_\nu} \overline{\psi}_{1\alpha} - e^{-i\phi/2} e^{-ik_\nu} \overline{\psi}_{1\alpha}^{\dagger}]$$

(B7)

The above Eqn. demonstrates clearly that electron is separated into charge $e$ boson $b = e^{i\phi/2}$ and spin 1/2 spinon $\psi$. Note that the crucial difference from Eqn.27 is that there is no internal gauge field!
The electron Green function is given by:

\[
G(\vec{x}, t) = \langle b(\vec{x}, t) b^\dagger (0, 0) \rangle = \langle \psi_{1\alpha}(\vec{x}, t) \psi^\dagger_{1\alpha}(0, 0) \rangle + e^{-iK_{1\alpha} \cdot \vec{x}} \langle \psi_{2\alpha}(\vec{x}, t) \psi^\dagger_{2\alpha}(0, 0) \rangle.
\]

In the underdoped regime, boson \( b = e^{i\phi/2} \) is massive, therefore short-ranged correlated. Being the product of fermion and boson Green functions, the electron correlation function is also short-ranged. The underdoped regime is a Mott insulator as suggested in Refs. [1,19,20].

At the critical point, in real space, \( b \) decays with a power law \( 1 + \eta_{XY} \), the electron Green function decays with a power \( 2 + 1 + \eta_{XY} \).

In the superconductivity phase, \( <b> \neq 0 \), the electron Green function decays with a power 2. The anomalous Green functions \( F(\vec{x}, t) = \langle C_\uparrow(\vec{x}, t) C_\downarrow(0, 0) \rangle \) also starts to form:

\[
F(\vec{x}, t) = \langle b >^2 \langle e^{iK_{1\alpha} \cdot \vec{x}} \psi_{1\alpha}(\vec{x}, t) \psi^\dagger_{1\alpha}(0, 0) \rangle - e^{-iK_{1\alpha} \cdot \vec{x}} \langle \psi_{1\alpha}(\vec{x}, t) \psi^\dagger_{1\alpha}(0, 0) \rangle
\]

It also decays with power 2. Note that the matrix elements between different \( p - \hbar \) indices are non-vanishing.

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