Statistical analysis of the non-ergodic fractional Ornstein–Uhlenbeck process with periodic mean

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Abstract
Consider a periodic, mean-reverting Ornstein–Uhlenbeck process $X = \{X_t, t \geq 0\}$ of the form $dX_t = (L(t) + \alpha X_t) \, dt + dB^H_t$, $t \geq 0$, where $L(t) = \sum_{i=1}^{p} \mu_i \phi_i(t)$ is a periodic parametric function, and $\{B^H_t, t \geq 0\}$ is a fractional Brownian motion of Hurst parameter $\frac{1}{2} \leq H < 1$. In the “ergodic” case $\alpha < 0$, the parametric estimation of $(\mu_1, \ldots, \mu_p, \alpha)$ based on continuous-time observation of $X$ has been considered in Dehling et al. (Stat Inference Stoch Process 13:175–192, 2010; Stat Inference Stoch Process 20:1–14, 2016) for $H = \frac{1}{2}$, and $\frac{1}{2} < H < 1$, respectively. In this paper we consider the “non-ergodic” case $\alpha > 0$, and for all $\frac{1}{2} \leq H < 1$. We analyze the strong consistency and the asymptotic distribution for the estimator of $(\mu_1, \ldots, \mu_p, \alpha)$ when the whole trajectory of $X$ is observed.

Keywords Parameter estimation · Strong consistency · Joint asymptotic distribution · Fractional Ornstein–Uhlenbeck process · Periodic mean function · Young integral

Mathematics Subject Classification 60G15 · 60G22 · 62F12 · 62M09 · 62M86

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1 Introduction

While the statistical inference of Itô type diffusions has a long history (see e.g. Basawa and Scott (1983), Kutoyants (2004), Liptser and Shiryaev (2001) and the references therein), the statistical analysis for equations driven by fractional Brownian motion (fBm) is relatively recent. The development of stochastic calculus with respect to the fBm allowed to study such models. Estimation of the drift parameters in fractional-noise-driven Ornstein–Uhlenbeck processes is a problem that is both well-motivated by practical needs and theoretically challenging. In the finance context, a practical motivation to study this estimation problem is to provide tools to understand volatility modeling in finance. Indeed, any mean-reverting model in discrete or continuous time can be taken as a model for stochastic volatility. Recently, several researchers have been interested in studying statistical estimation problems when the volatility exhibits long-memory, which means that the volatility today is correlated to past volatility values with a dependence that decays very slowly. We refer the reader to Chronopoulou and Viens (2012), Comte et al. (2012) for further details.

Let \( B^H := \{ B^H_t, \ t \geq 0 \} \) be a fractional Brownian motion (fBm) with Hurst parameter \( H \in (0, 1) \), that is a centered Gaussian process with covariance

\[
E (B^H_s B^H_t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \ s, t \geq 0.
\]

Note that, when \( H = \frac{1}{2} \), \( B^{\frac{1}{2}} \) is a standard Brownian motion.

In the present paper we consider a fractional Ornstein–Uhlenbeck (fOU) process with periodic mean \( X := \{ X_t, \ t \geq 0 \} \), defined as solution of the following linear stochastic differential equation

\[
X_0 = 0, \ dX_t = (L(t) + \alpha X_t)dt + dB^H_t, \ t \geq 0, \quad (1.1)
\]

where \( B^H := \{ B^H_t, \ t \geq 0 \} \) is a fractional Brownian motion with Hurst index \( \frac{1}{2} \leq H < 1, \alpha > 0 \) and \( L(t) = \sum_{i=1}^{p} \mu_i \phi_i(t) \) for all \( t \in [0, 1] \) with \((\mu_1, \ldots, \mu_p) \in \mathbb{R}^p\), and the functions \( \phi_i, i = 1, \ldots, p \), are bounded 1-periodic \( L^2([0, 1]) \)-orthonormal functions. Let \( \mu := (\mu_1, \ldots, \mu_p) \), and \( \theta := (\mu, \alpha) \).

Let us recall some results on parameter estimation related to the process (1.1):

- **“Ergodic” case** \( \alpha < 0 \): When \( H = \frac{1}{2} \), the maximum likelihood estimator (MLE) of \( \theta \) has been studied by Dehling et al. (2010) based on continuous-time observation of process \( X \), defined by (1.1). The authors proved the strong consistency and asymptotic normality of the MLE. Moreover, at this stage it is worth noticing that the MLE coincides with the least squares estimator (LSE) as well. On the other hand, the consistency for \( \frac{1}{2} < H < 1 \), and asymptotic normality for \( \frac{1}{2} < H < \frac{3}{4} \), of the LSE of \( \theta \) based on continuous-time observation of \( X \), have been studied in Dehling et al. (2016), Bajja et al. (2017). Also, some non-Gaussian extensions of the model (1.1) have been considered by several authors (see e.g. Nourdin and Tran 2019; Shevchenko and Tudor 2019), by replacing the fBm in (1.1) by a Hermite process. On the other hand, for \( p = 1 \) and \( \phi_1 = 1 \), a large number of research
articles considered the problem of drift parameter estimation for various fractional diffusions and in particular for the fOU process, we refer among many others to Kleptsyna and Le Breton (2002), Hu et al. (2017), El Onsy et al. (2017), Es-Sebaiy and Viens (2019), Douissi et al. (2019), Sottinen and Viitasaari (2018).

- “Non-ergodic” case $\alpha > 0$ : In the case when $L = 0$ (respectively $p = 1$ and $\phi_1 = 1$), several researchers have been interested in studying parameter estimation problems in various fractional Gaussian models related to (1.1), we refer among many others to Belfadli et al. (2011), El Machkouri et al. (2016), Es-Sebaiy et al. (2019) (respectively Es-Sebaiy and Sebaiy 2021; Alazemi et al. 2020).

However, in the “non-ergodic” case corresponding to $\alpha > 0$, no authors as far as we know have ever studied the drift parameter estimation for the model (1.1) in its general form. So, our goal in the present paper is to consider the MLE and the LSE for $\theta$, respectively, when $H = \frac{1}{2}$ and $\frac{1}{2} < H < 1$, based on continuous-time observation of $X$. We study the strong consistency and the asymptotic distribution for those estimators. Although the formulation of the problem appears rather similar to the one studied in Dehling et al. (2010, 2016), the proofs and the results are quite different. There are several points that make our approach different: first, the behavior of the solution to (1.1) in the non-ergodic case is not the same as in the ergodic case, the covariance and the memory properties of these processes being significantly different; second, in contrast to the method used in Dehling et al. (2010, 2016) based on a divergence-type (or Itô–Skorohod) integral with respect to the fbm and which corresponds to the classical Itô integral in the case $H = 1/2$, we use a method based on a pathwise integral with respect to the noise and this makes in principle our estimators easier to be simulated; a third significant difference is the behavior of the estimators. While in Dehling et al. (2010, 2016), the estimators $\hat{\mu}_n$ and $\hat{\alpha}_n$, defined by (2.7) are asymptotically normal, in our case we prove that $\hat{\mu}_n$ is still asymptotically normal but the limit distribution of $\hat{\alpha}_n$ is a standard Cauchy distribution.

Let us also recall several approaches to estimate the parameters in the fractional Ornstein Uhlenbeck process $X^H := \{ X^H_t, t \geq 0 \}$ defined by

$$X^H_0 = 0, \quad dX_t = \alpha X^H_t dt + dB^H_t, \quad t \geq 0,$$

(1.2)
corresponding to the case $L = 0$ in the Eq. (1.1). We mention that the below list is not exhaustive:

- For the maximum likelihood estimation approach, the techniques used to construct maximum likelihood estimators (MLEs) for the drift parameter of the fOU (1.2) are based on Girsanov transforms for fBm and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In general, the MLE is not easily computable. In particular, it relies on being able to constitute a discretization of an MLE. The MLE of $\alpha$ in (1.2) was first introduced by Kleptsyna and Le Breton (2002) as follows:

$$\tilde{\alpha}_t = \frac{\int_0^t Q_H(s)dz_H(s)}{\int_0^t Q^2_H(s) dw_H(s)},$$

(1.3)
where, for every $0 < r < s$,

$$w_H(s) = \lambda_H^{-1} s^{2-2H},$$

$$\lambda_H = 2H \Gamma(3-2H) \Gamma(H + \frac{1}{2}) / \Gamma(\frac{3}{2} - H),$$

$$k_H(s, r) = \kappa_H^{-1} (r(s-r))^{\frac{1}{2}-H},$$

$$\kappa_H = 2H \Gamma(\frac{3}{2} - H) \Gamma(H + \frac{1}{2}),$$

$$Q_H(s) = \frac{d}{dw_H(s)} \int_0^s k_H(s, r) X_r^H dr,$$

$$Z_H(s) = \int_0^s k_H(s, r) dX_r^H.$$

For a more recent comprehensive discussion via this method, we refer to Kleptsyna and Le Breton (2002), Liu et al. (2019), Tanaka et al. (2019).

- A least squares approach has been also considered by several researchers. The least squares estimator (LSE) of $\alpha$ in (1.2) was first introduced by Hu and Nualart (2010) as follows:

$$\hat{\alpha}_t = \left( \int_0^t X_s^H dX_s^H \right) / \left( \int_0^t (X_s^H)^2 ds \right). \quad (1.4)$$

At this stage it is worth noticing that the MLE $\tilde{\alpha}_t$ and the LSE $\tilde{\alpha}_t$, given respectively by (1.3) and (1.4), coincide when $H = \frac{1}{2}$. Indeed, if $H = \frac{1}{2}$, we have $X_t^H = Q_H(t) = Z_H(t)$ and $w_H(t) = t$ and so that

$$\tilde{\alpha}_t = \hat{\alpha}_t$$

which implies $\tilde{\alpha}_t = \hat{\alpha}_t$ when $H = \frac{1}{2}$. On the other hand, when $H \neq \frac{1}{2}$ the LSE $\tilde{\alpha}_t$ is no longer the MLE $\tilde{\alpha}_t$. Nevertheless, there is a major difference with respect to the standard Brownian motion case. Indeed, the process $X^H$ being no longer a semimartingale, in (1.4) one cannot utilize the Itô integral to integrate with respect to it. However, because $X^H$ has $\gamma$-Hölder continuous paths for all $\gamma \in (\frac{1}{2}, H)$, one can choose, instead, the Young integral for the case $\alpha > 0$, and Skorohod integral for the case $\alpha < 0$. The case of ergodic-type fOU processes, corresponding to $\alpha < 0$, has been considered in El Onsy et al. (2017), Hu and Nualart (2010), Hu et al. (2017). In this case one interprets the integral $\int_0^t X_s^H dX_s^H$ in (1.4) as a divergence-type (or Skorohod) integral to ensure the consistency of the LSE $\tilde{\alpha}_t$ as $t \to \infty$. In this situation the LSE $\tilde{\alpha}_t$ is asymptotically normal, see Hu and Nualart (2010). However in the case of non-ergodic fOU processes, corresponding to $\alpha > 0$, the integral $\int_0^t X_s^H dX_s^H$ in (1.4) is understood in the Young sense. In this situation the LSE $\tilde{\alpha}_t$ is asymptotically Cauchy, see El Machkouri et al. (2016) and Es-Sebaiy et al. (2019). Recently, the paper Tanaka (2020) has compared the LSE $\tilde{\alpha}_t$ to the MLE $\tilde{\alpha}_t$ for the fOU process (1.2).

- Finally, using a method of moments, the work Es-Sebaiy and Viens (2019) used Malliavin-calculus advances to provide new techniques to statistical inference for stochastic differential equations related to stationary Gaussian processes, and its result has been used to study drift parameter estimation problems for some
stochastic differential equations driven by fractional Brownian motion with fixed-
time-step observations, in particular for the fractional Ornstein–Uhlenbeck given
in (1.1), where \( L = 0 \) and \( \alpha < 0 \). This latter case has been extended by Douissi
et al. (2019), where the fractional Brownian motion is replaced with a general
Gaussian process.

The rest of the paper is structured as follows. In Sect. 2 we analyze some
properties of our model. In Sect. 3 we prove the strong consistency for estimator
\( \hat{\theta}_n = (\hat{\mu}_n, \hat{\alpha}_n) \), defined by (2.7). Section 4 is devoted to the joint asymptotic distribu-
tion of \( (e^{\alpha n}(\hat{\alpha}_n - \alpha), n^{1-H}(\hat{\mu}_n - \mu)) \), as \( n \to \infty \).

In what follows, \( C \) denotes a generic positive constant (perhaps depending on \( H \)
and \( \alpha \), but not on anything else), which may change from line to line.

\section{Some almost sure convergence properties}

In this section, we study some properties of our model (1.1), which rely on the peri-
odicity of the mean function and the fact that \( \alpha > 0 \). These properties will be needed
in order to analyze the asymptotic behavior of the LSE.

Consider the fOU process with periodic mean that is defined as the solution to the
Langevin equation whose drift is a periodic function, given by (1.1).
We assume that the parameter \( \theta = (\mu, \alpha) \) is unknown and our aim is to estimate it
by using maximum likelihood method when \( H = \frac{1}{2} \), and least squares method when
\( \frac{1}{2} < H < 1 \), which are introduced in Dehling et al. (2010, 2016), respectively.

More precisely, we consider the following LSE \( \hat{\theta}_n \) for \( \theta \), which coincides with the
MLE when \( H = \frac{1}{2} \):
\[
\hat{\theta}_n := Q_n^{-1} P_n, \tag{2.1}
\]
where \( P_n \) is a vector with \((p + 1)\) components, and \( Q_n \) is a \((p + 1) \times (p + 1)\) matrix,
defined by
\[
P_n := \left( \int_0^n \phi_1(s) dX_s, \ldots, \int_0^n \phi_p(s) dX_s, \int_0^n X_s dX_s \right)^T, \quad Q_n := \begin{pmatrix} A_{p,n} & U_n \\ U_n^T & V_n \end{pmatrix},
\]
with
\[
U_n = \left( \int_0^n \phi_1(s) X_s ds, \ldots, \int_0^n \phi_p(s) X_s ds \right)^T, \quad V_n = \int_0^n X_s^2 ds,
\]
and \( A_{p,n} \) is a \( p \times p \) matrix given by
\[
A_{p,n} := \left( \int_0^n \phi_i(s) \phi_j(s) ds \right)_{1 \leq i, j \leq p} = n I_p,
\]
since \( \int_0^1 \phi_i(s)\phi_j(s)ds = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \), where \( I_p \) is the identity matrix of size \( p \).

This enables us to write

\[
Q_n = \begin{pmatrix} nI_p & U_n \\ U_n^T & V_n \end{pmatrix}.
\]

Straightforward calculation yields that for every \( a \in \mathbb{R}^p \) and \( b \in \mathbb{R} \),

\[
\left( I_p \begin{pmatrix} a \\ a^T \end{pmatrix} \right)^{-1} = \begin{pmatrix} I_p + \frac{aa^T}{b-\|a\|^2} & -\frac{a}{b-\|a\|^2} \\ -\frac{a^T}{b-\|a\|^2} & \frac{1}{b-\|a\|^2} \end{pmatrix}.
\]

Thus the explicit expression of the inverse matrix \( Q_n^{-1} \) can be written as:

\[
Q_n^{-1} = \frac{1}{n} \begin{pmatrix} I_p + \gamma_n \Lambda_n \Lambda_n^T & -\gamma_n \Lambda_n \\ -\gamma_n \Lambda_n^T & \gamma_n \end{pmatrix},
\]

where

\[
\Lambda_n := (\Lambda_n,1,\ldots,\Lambda_n,p)^T := \frac{1}{n} U_n, \quad \gamma_n := \left( \frac{1}{n} \int_0^n X_s^2 ds - \sum_{i=1}^p \Lambda_{n,i}^2 \right)^{-1}.
\]

On the other hand, the explicit strong solution of the Langevin equation (1.1) is given by

\[
X_t = e^{\alpha t} \left( \int_0^t e^{-\alpha s} L(s)ds + \int_0^t e^{-\alpha s} dB_s^H \right)
= e^{\alpha t} (A_t + \zeta_t), \quad t \geq 0,
\]

where, for every \( t \geq 0 \),

\[
A_t := \int_0^t e^{-\alpha s} L(s)ds; \quad \zeta_t := \int_0^t e^{-\alpha s} dB_s^H = e^{-\alpha t} B_t^H + \alpha Z_t, \quad (2.2)
\]

with

\[
Z_t := \int_0^t e^{-\alpha s} B_s^H ds, \quad t \geq 0. \quad (2.3)
\]

Therefore, the process \( X \) can be rewritten as

\[
X_t = e^{\alpha t} A_t + \alpha e^{\alpha t} \zeta_t + B_t^H, \quad t \geq 0. \quad (2.4)
\]
On the other hand, from Eq. (1.1), we can also write
\[ X_t = \tilde{L}(t) + \alpha \Sigma_t + B^H_t, \quad t \geq 0, \]  
where
\[ \tilde{L}(t) := \int_0^t L(s)ds, \quad \Sigma_t := \int_0^t X_sds, \quad t \geq 0. \]  
Note that almost surely all paths of \( X \) are \( \gamma \)-Hölder continuous with \( \gamma < H \). Indeed, it is well known that the trajectories of \( B^H \) are \( \gamma \)-Hölder continuous with \( \gamma < H \) (see, for example, Nualart 2006, p. 274). Moreover, since \( L(t) \) is bounded, we have, for every \( s, t \geq 0 \),
\[ |A_t - A_s| \leq \int_s^t e^{-\alpha r} |L(r)|dr \leq C|t - s|, \]  
which implies that the \( A_t \) 1-Hölder continuous. Furthermore, we have for every \( s, t \in [0, T] \),
\[ |Z_t - Z_s| = \left| \int_s^t e^{-\alpha r} B^H_t dr \right| \leq \sup_{r \in [0, T]} |B^H_r||t - s|, \]  
which proves that the trajectories of \( Z \) are 1-Hölder continuous. Combining these facts together with (2.4), the desired result is obtained.

Therefore, the estimator (2.1) is well defined in the following sense:

- For \( H = \frac{1}{2} \), the stochastic integrals in the expression (2.1) of \( \hat{\theta}_n \) are understood in the Itô sense. Moreover the integral, that is the pth component of \( P_n \) above,
\[ \int_0^n X_s dX_s = \frac{1}{2} (X^2_n - n), \]  
by Itô formula.
- For \( \frac{1}{2} < H < 1 \), the stochastic integrals in the expression (2.1) of \( \hat{\theta}_n \) are understood in the Young sense (see Appendix). In addition, the integral \( \int_0^n X_s dX_s \), that is the pth component of \( P_n \), can be expressed as \( \int_0^n X_s dX_s = \frac{1}{2} X^2_n \), by using the integration by parts formula (5.1).

Thus, in the rest of the paper, we will use \( \frac{1}{2} (X^2_n - n) \) instead of \( \int_0^n X_s dX_s \) when \( H = \frac{1}{2} \), and \( \frac{1}{2} X^2_n \) instead of \( \int_0^n X_s dX_s \) when \( \frac{1}{2} < H < 1 \). We will also make use of the following form of \( \hat{\theta}_n \):
\[ \hat{\theta}_n := (\hat{\mu}_n, \hat{\alpha}_n), \]  
where
\[ \hat{\alpha}_n := \frac{\gamma n}{n} \left( \int_0^n X_s dX_s - \sum_{k=1}^p \Lambda_{n,k} \int_0^n \phi_k(s) dX_s \right), \]  
and \( \hat{\mu}_n := (\hat{\mu}_{n,1}, \ldots, \hat{\mu}_{n,p}) \) such that, for all \( i = 1, \ldots, p \),
\[ \mu_{n,i} := \frac{1}{n} \left( \int_0^n \phi_i(s) dX_s + \gamma_n \Lambda_{n,i} \sum_{k=1}^p \Lambda_{n,k} \int_0^n \phi_k(s) dX_s - \gamma_n \Lambda_{n,i} \int_0^n X_s dX_s \right). \] (2.9)

To prove our main results we will need the following lemmas.

**Lemma 2.1** Assume \( \frac{1}{2} \leq H < 1 \). Let \( \{\zeta_t, t \geq 0\} \) and \( \{Z_t, t \geq 0\} \) be given by (2.2) and (2.3). Then,

\[ \lim_{t \to \infty} B_H^t = 0 \quad \text{almost surely for all } \delta > H, \] (2.10)

\[ \lim_{t \to \infty} Z_t = Z_\infty := \int_{\infty}^{\infty} e^{-\alpha s} B_s^H ds \quad \text{almost surely and in } L^2(\Omega), \] (2.11)

\[ \lim_{t \to \infty} e^{-2\alpha t} \int_0^t e^{2\alpha s} \zeta_s^2 ds = \frac{\alpha}{2} Z_\infty^2 \quad \text{almost surely}, \] (2.12)

\[ \lim_{n \to \infty} \frac{1}{n} \int_0^n \phi_i(s) dB_s^H = 0 \quad \text{almost surely for every } i = 1, \ldots, p. \] (2.13)

**Proof** The proofs of the convergences (2.10), (2.11) and (2.12) are given in Es-Sebaiy and Sebaiy (2021, Lemma 2.2), El Machkouri et al. (2016, Lemma 2.1) and (2016, Lemma 2.2), respectively. For (2.13), since the functions \( \phi_i, i = 1, \ldots, p \), are bounded, we have if \( \frac{1}{2} < H < 1 \),

\[ E \left[ \left( \frac{1}{n} \int_0^n \phi_i(s) dB_s^H \right)^2 \right] = \frac{H(2H-1)}{n^2} \int_0^n \phi_i(u)\phi_i(v)|u-v|^{2H-2}dudv \]

\[ \leq C \frac{H(2H-1)}{n^2} \int_0^n |u-v|^{2H-2}dudv = Cn^{-2(1-H)}, \]

if \( H = \frac{1}{2} \),

\[ E \left[ \left( \frac{1}{n} \int_0^n \phi_i(s) dB_s^H \right)^2 \right] = \frac{1}{n^2} \int_0^n \phi_i^2(u)du \leq Cn^{-1}. \]

Combining these facts together with the Gaussian property of \( \frac{1}{n} \int_0^n \phi_i(s) dB_s^H \) and Lemma 5.1, the proof is finished. \( \square \)

**Lemma 2.2** Let \( \{A_t, t \geq 0\} \) be given by (2.2). Then, for every \( t \geq 1 \),

\[ A_t = \frac{A_1}{1-e^{-\alpha}} + R_t, \] (2.14)

where

\[ R_t := \frac{A_1e^{-\alpha[t]}}{1-e^{-\alpha}} + e^{-\alpha[t]} \int_0^{t-[t]} e^{-\alpha s} L(s)ds, \text{ with } [t] \text{ is the integer part of } t. \]
Moreover, there is a positive constant $C$ depending only on $\alpha$ such that, for every $t \geq 1$,

$$|R_t| \leq Ce^{-\alpha t}.$$  

(2.15)

As a consequence, as $t \to \infty$,

$$A_t \longrightarrow A_\infty := \frac{A_1}{1 - e^{-\alpha}}.$$  

(2.16)

Also, for every $i = 1, \ldots, p$,

$$\lim_{n \to \infty} e^{-\alpha n} \int_0^n \phi_i(s) e^{\alpha s} A_s ds = \lambda_{\phi_i} A_\infty, \text{ where } \lambda_{\phi_i} := \frac{\int_0^1 \phi_i(s) e^{\alpha s} ds}{e^\alpha - 1}.$$  

(2.17)

**Proof** Notice that for every positive integer $n \geq 1$,

$$A_n = \int_0^n e^{-\alpha s} L(s) ds$$

$$= \sum_{k=0}^{n-1} \int_k^{k+1} e^{-\alpha s} L(s) ds$$

$$= \sum_{k=0}^{n-1} e^{-\alpha k} \int_0^1 e^{-\alpha s} L(s) ds$$

$$= A_1 \times \frac{1 - e^{-\alpha n}}{1 - e^{-\alpha}},$$

where in the third equality we used the fact that $L$ is 1-periodic. Hence, for every $t \geq 1$,

$$A_t = \int_0^t e^{-\alpha s} L(s) ds$$

$$= A_{[t]} + \int_{[t]}^t e^{-\alpha s} L(s) ds$$

$$= A_1 \times \frac{1 - e^{-\alpha [t]}}{1 - e^{-\alpha}} + e^{-\alpha [t]} \int_0^{t-[t]} e^{-\alpha s} L(s) ds$$

$$= \frac{A_1}{1 - e^{-\alpha}} + R_t,$$

which implies (2.14). On the other hand, since $L$ is bounded and $0 \leq t - [t] < 1$, we obtain
\[ |R_t| = \left| \frac{A_1 e^{-\alpha [t]}}{1 - e^{-\alpha}} + e^{-\alpha [t]} \int_0^{t-[t]} e^{-\alpha s} L(s) ds \right| \]
\[ = e^{-\alpha [t]} \left| - \int_0^1 e^{-\alpha s} L(s) ds \frac{1}{1 - e^{-\alpha}} + \int_0^{t-[t]} e^{-\alpha s} L(s) ds \right| \]
\[ \leq C e^{-\alpha [t]} \left( 1 + \int_0^{t-[t]} e^{-\alpha s} ds \right) \]
\[ \leq C e^{-\alpha t} \left( 1 + \int_0^1 e^{-\alpha s} ds \right) \]
\[ \leq C e^{-\alpha t}, \]

which proves (2.15). Furthermore, the convergence (2.16) is immediately obtained from (2.14) and (2.15).

Let us now prove (2.17). According to (2.14), we have, for every \( i = 1, \ldots, p \),

\[ e^{-\alpha n} \int_0^n \phi_i(s) e^{\alpha s} A_s ds = A_{\infty} e^{-\alpha n} \int_0^n \phi_i(s) e^{\alpha s} ds + e^{\alpha n} \int_0^n e^{\alpha s} \phi_i(s) R(s) ds. \]

Using (2.15) and the boundedness of the functions \( \phi_i, i = 1, \ldots, p \), we deduce that, as \( n \to \infty \),

\[ e^{-\alpha n} \int_0^n e^{\alpha s} \phi_i(s) R(s) ds \to 0. \]

Moreover, since the functions \( \phi_i, i = 1, \ldots, p \), are 1-periodic,

\[ \lim_{n \to \infty} e^{-\alpha n} \int_0^n \phi_i(s) e^{\alpha s} ds = \lim_{n \to \infty} e^{-\alpha n} \sum_{k=0}^{n-1} \int_k^{k+1} \phi_i(s) e^{\alpha s} ds \]
\[ = \lim_{n \to \infty} e^{-\alpha n} \sum_{k=0}^{n-1} e^{\alpha k} \int_0^1 \phi_i(s) e^{\alpha s} ds \]
\[ = \lim_{n \to \infty} \frac{1 - e^{-\alpha n}}{e^\alpha - 1} \int_0^1 \phi_i(s) e^{\alpha s} ds \]
\[ = \int_0^1 \phi_i(s) e^{\alpha s} ds \left( e^{\alpha} - 1 \right). \] (2.18)

\[ \Box \]
Lemma 2.3 Let \( \{X_t \geq 0\} \) and \( \{Z_t \geq 0\} \) be given by (1.1) and (2.3), respectively. Then, for every \( i = 1, \ldots, p \), we have almost surely, as \( n \to \infty \),

\[
e^{-an} \int_0^n \phi_i(s) e^{as} Z_s ds \to \lambda \phi_i Z_\infty, \tag{2.19}
\]

\[
n e^{-an} \Lambda_{n,i} = e^{-an} \int_0^n \phi_i(s) X_s ds \to \lambda \phi_i (A_\infty + \alpha Z_\infty), \tag{2.20}
\]

\[
e^{-an} \int_0^n \phi_i(s) dX_s \to \alpha \lambda \phi_i (A_\infty + \alpha Z_\infty), \tag{2.21}
\]

where \( Z_\infty, A_\infty \) and \( \lambda \phi_i \) are defined by (2.11), (2.16) and (2.17), respectively.

**Proof** First we prove (2.19). We have, for every \( i = 1, \ldots, p \),

\[
e^{-an} \int_0^n \phi_i(s) e^{as} Z_s ds = Z_\infty e^{-an} \int_0^n \phi_i(s) e^{as} ds + e^{-an} \int_0^n e^{as} \phi_i(s)(Z_s - Z_\infty) ds
= Z_\infty I_{i,n} + J_{i,n}.
\]

Using (2.18), we obtain \( I_{i,n} \to \lambda \phi_i \) as \( n \to \infty \). Further, using the fact that for all \( s \leq r \), \( e^{-\frac{a}{2}r} \leq e^{-\frac{a}{2}s} \), and (2.10), we get

\[
|J_{i,n}| \leq e^{-an} \int_0^n e^{as} |\phi_i(s)||Z_s - Z_\infty| ds
\leq C e^{-an} \int_0^n e^{as} \int_s^\infty e^{-ar} |B_r^H| dr ds
\leq C e^{-an} \int_0^n e^{as} \int_s^\infty e^{-\frac{a}{2}r} |B_r^H| dr ds
\leq C e^{-\frac{an}{2}} \int_0^\infty e^{-\frac{a}{2}r} |B_r^H| dr ds
\to 0
\]

almost surely, as \( n \to \infty \). This completes the proof of (2.19).

On the other hand, by (2.4), we can write, for every \( i = 1, \ldots, p \),

\[
e^{-an} \int_0^n \phi_i(s) X_s ds = e^{-an} \int_0^n \phi_i(s) e^{as} A_s ds + \alpha e^{-an} \int_0^n \phi_i(s) e^{as} Z_s ds + e^{-an} \int_0^n \phi_i(s) B_s^H ds
= A_{i,n,1} + A_{i,n,2} + A_{i,n,3}.
\]

By (2.17) and (2.19), we obtain, for every \( i = 1, \ldots, p \), \( A_{i,n,1} + A_{i,n,2} \to \lambda \phi_i (A_\infty + \alpha Z_\infty) \) almost surely, as \( n \to \infty \). Moreover, for every \( i = 1, \ldots, p \), \( A_{i,n,3} \to 0 \) almost surely, as \( n \to \infty \), thanks to (2.10) and the fact that the functions \( \phi_i, i = 1, \ldots, p \), are bounded. Thus, the convergence (2.20) is obtained.
For (2.21), we have, for every \( i = 1, \ldots, p \),
\[
e^{-\alpha n} \int_0^n \phi_i(s) dX_s = e^{-\alpha n} \sum_{i=1}^p \mu_i \int_0^n \phi_i^2(s) ds + \alpha e^{-\alpha n} \int_0^n \phi_i(s) X_s ds + e^{-\alpha n} \int_0^n \phi_i(s) dB_s^H,
\]
which converges to \( \alpha \lambda \phi_i (A_{\infty} + \alpha Z_{\infty}) \) almost surely, as \( n \to \infty \), according to (2.13) and (2.20). Then the proof of Lemma 2.3 is completed.

**Lemma 2.4** The following convergences hold almost surely, as \( n \to \infty \),
\[
e^{-\alpha n} X_n \to A_{\infty} + \alpha Z_{\infty}, \tag{2.22}
\]
\[
e^{-\alpha n} \int_0^n X_s ds \to \frac{A_{\infty}}{\alpha} + Z_{\infty}, \tag{2.23}
\]
\[
e^{-2\alpha n} \int_0^n X_s^2 ds \to \frac{1}{2\alpha} (A_{\infty} + \alpha Z_{\infty})^2, \tag{2.24}
\]
\[
\frac{e^{-\alpha n}}{n} \int_0^n |B_s^H X_s| ds \to 0, \tag{2.25}
\]
\[
\frac{e^{-\alpha n}}{n} \int_0^n \tilde{L}(s) X_s ds \to \left( \frac{A_{\infty}}{\alpha} + Z_{\infty} \right) \int_0^1 L(s) ds, \tag{2.26}
\]
where \( \tilde{L}(t) \) is defined by (2.6).

**Proof** The statement (2.22) can be immediately obtained from (2.4), (2.10), (2.11) and (2.16). Let us prove (2.23). From (2.4) we can write
\[
e^{-\alpha n} \int_0^n X_s ds = e^{-\alpha n} \int_0^n e^{\alpha s} (A_s + \alpha Z_s) ds + e^{-\alpha n} \int_0^n B_s^H ds
\]
\[
= e^{-\alpha n} \int_0^n e^{\alpha s} \left( L(r) + \alpha B_r^H \right) dr ds + e^{-\alpha n} \int_0^n B_s^H ds
\]
\[
= e^{-\alpha n} \int_0^n e^{-\alpha r} \left( L(r) + \alpha B_r^H \right) \int_r^n e^{\alpha s} ds dr + e^{-\alpha n} \int_0^n B_s^H ds
\]
\[
= \frac{1}{\alpha} \int_0^n e^{-\alpha r} \left( L(r) + \alpha B_r^H \right) dr + \frac{e^{-\alpha n}}{\alpha} \int_0^n \left( L(r) + \alpha B_r^H \right) dr + e^{-\alpha n} \int_0^n B_s^H ds
\]
\[
= \frac{A_t}{\alpha} + Z_t + \frac{e^{-\alpha n}}{\alpha} \int_0^n \left( L(r) + \alpha B_r^H \right) dr + e^{-\alpha n} \int_0^n B_s^H ds.
\]
According to (2.10) and the fact that the function \( L(t) \) is bounded, we can easily see that
\[
\frac{e^{-\alpha n}}{\alpha} \int_0^n \left( L(r) + \alpha B_r^H \right) dr + e^{-\alpha n} \int_0^n B_s^H ds \to 0,
\]
as \( n \to \infty \). Combining these facts with (2.11) and (2.16) proves (2.23).
Now we prove (2.24). From (2.4) we can write \(X_t = e^{\alpha t} Y_t, \ t \geq 0\), where \(Y_t := A_t + \alpha Z_t + e^{-\alpha t} B^H_t\). Notice that \(Z_\infty \sim N \left(0, E \left[ Z^2_\infty \right] \right)\), where

\[
E \left[ Z^2_\infty \right] = \int_0^\infty \int_0^\infty e^{-\alpha r} e^{-\alpha s} E \left( B^H_r B^H_s \right) dr ds \\
\leq \int_0^\infty \int_0^\infty e^{-\alpha r} e^{-\alpha s} (rs)^H dr ds \\
= \left( \frac{\Gamma(H+1)}{\alpha^{H+1}} \right)^2 < \infty.
\]

This combined with the fact that \(A_\infty\) is a deterministic constant, imply that

\[
P(A_\infty + \alpha Z_\infty = 0) = P \left( Z_\infty = -\frac{A_\infty}{\alpha} \right) = 0. \tag{2.27}
\]

The continuity of \(Y\) entails that, for every \(n \geq 1\),

\[
\int_0^n X_s^2 ds = \int_0^n e^{2\alpha s} Y_s^2 ds \geq \int_0^n e^{2\alpha s} Y_s^2 ds \geq \frac{n}{2} e^{\alpha n} \left( \inf_{\frac{n}{2} \leq s \leq n} Y_s^2 \right) \text{ almost surely.} \tag{2.28}
\]

Furthermore, applying the Intermediate Value Theorem to the continuous function \(Y\), and using (2.10), (2.11) and (2.16), we obtain

\[
\lim_{n \to \infty} \left( \inf_{\frac{n}{2} \leq s \leq n} Y_s^2 \right) = (A_\infty + \alpha Z_\infty)^2 \text{ almost surely.}
\]

Combining this last convergence with (2.27) and (2.28), we deduce that

\[
\lim_{n \to \infty} \int_0^n e^{2\alpha s} Y_s^2 ds = \infty \text{ almost surely.}
\]

Hence, we can use L’Hôpital’s rule to obtain

\[
\lim_{n \to \infty} \frac{\int_0^n X_s^2 ds}{e^{2\alpha n}} = \lim_{n \to \infty} \frac{\int_0^n e^{2\alpha s} Y_s^2 ds}{e^{2\alpha n}} = \lim_{n \to \infty} \frac{Y_n^2}{2\alpha} = \frac{(A_\infty + \alpha Z_\infty)^2}{2\alpha} \text{ almost surely,}
\]

which completes the proof of (2.24).

The convergence (2.25) can be obtained from (2.10), (2.24), and the fact that

\[
e^{-\alpha n} n^{-1} \int_0^n |B^H_s X_s| ds \leq \left( \frac{1}{n^2} \int_0^n |B^H_s|^2 ds \right)^{1/2} \left( e^{-2\alpha n} \int_0^n X_s^2 ds \right)^{1/2}.
\]
It remains to prove (2.26). Notice that

\[ \tilde{L}(t) = \int_{0}^{t} L(s)ds + \int_{[t]}^{t} L(s)ds \]

\[ = \sum_{k=0}^{[t]-1} \int_{k}^{k+1} L(s)ds + \int_{[t]}^{t} L(s)ds \]

\[ = [t] \int_{0}^{1} L(s)ds + \int_{[t]}^{t} L(s)ds \]

\[ = t \int_{0}^{1} L(s)ds + \left( [t] - t \right) \int_{0}^{1} L(s)ds + \int_{[t]}^{t} L(s)ds \]

\[ = t \int_{0}^{1} L(s)ds + l(t). \]

Moreover, it is clear that the function \( l(t) \) is bounded. So,

\[ e^{-\alpha n} \int_{0}^{n} \tilde{L}(s)X_s ds = e^{-\alpha n} \int_{0}^{n} sX_s ds \left( \int_{0}^{1} L(r)dr \right) + e^{-\alpha n} \int_{0}^{n} l(s)X_s ds. \]

Furthermore, using similar arguments as in the proof of (2.23), we get

\[ \frac{e^{-\alpha n}}{n} \left| \int_{0}^{n} l(s)X_s ds \right| \leq C \frac{e^{-\alpha n}}{n} \int_{0}^{n} |X_s| ds \rightarrow 0, \]

and

\[ \frac{e^{-\alpha n}}{n} \int_{0}^{n} sX_s ds \rightarrow \left( \frac{A_{\infty}}{\alpha} + Z_{\infty} \right) \]

almost surely, as \( n \rightarrow \infty \). Then the desired result is obtained.

\[ \Box \]

3 Strong consistency

Here we prove the strong consistency of the LSE \( \hat{\theta}_n \), as \( n \rightarrow \infty \).

**Theorem 3.1** Assume \( \frac{1}{2} \leq H < 1 \). Then, almost surely, as \( n \rightarrow \infty \),

\[ \hat{\theta}_n = (\hat{\mu}_n, \hat{\alpha}_n) \rightarrow \theta = (\mu, \alpha). \]
**Proof** Suppose \( \frac{1}{2} < H < 1 \). Then, using \( \int_0^n X_s \, dX_s = \frac{1}{2} X_n^2 \) and (2.8), we can write
\[
\hat{\alpha}_n = \frac{-e^{-2an} \sum_{k=1}^p \Lambda_{n,k} \int_0^n \phi_k(t) \, dX_t + \frac{e^{-2an}}{2} X_n^2}{e^{-2an} \int_0^n X_s^2 \, ds - n \sum_{k=1}^p e^{-2an} \Lambda_{k,n}^2} = \frac{- \sum_{k=1}^p \frac{1}{n} \left( e^{-an} \int_0^n \phi_k(s) \, dX_s \right) \left( e^{-an} \int_0^n \phi_k(s) \, dX_s \right) + \frac{e^{-2an}}{2} X_n^2}{e^{-2an} \int_0^n X_s^2 \, ds - \frac{1}{n} \sum_{k=1}^p \left( e^{-an} \int_0^n \phi_k(s) \, dX_s \right)^2}.
\]

Combining this with (2.20), (2.21), (2.22) and (2.24), we deduce that \( \hat{\alpha}_n \) converges to \( \alpha \), almost surely as \( n \to \infty \). Hence, it remains to prove the strong consistency for each \( \hat{\mu}_{n,i}, i = 1, \ldots, p \). It follows from (2.9) that \( \hat{\mu}_{n,i}, i = 1, \ldots, p \) can be written as
\[
\hat{\mu}_{n,i} = \frac{\frac{1}{n} I_{i,n,1} + I_{i,n,2}}{I_{n,3}},
\]  
with
\[
I_{i,n,1} = \int_0^n \phi_i(s) \, dX_s \int_0^n X_s^2 \, ds - \frac{X_n^2}{2} \int_0^n \phi_i(s) \, dX_s,
I_{i,n,2} = \Lambda_{n,i} \sum_{k=1}^p \Lambda_{n,k} \int_0^n \phi_k(s) \, dX_s - \left( \int_0^n \phi_i(s) \, dX_s \right) \sum_{k=1}^p \Lambda_{n,k}^2,
I_{n,3} = \int_0^n X_s^2 \, ds - \sum_{k=1}^p \Lambda_{n,k}^2.
\]

According to the convergences (2.20) and (2.24), we have almost surely, as \( n \to \infty \),
\[
e^{-2an} I_{n,3} = e^{-2an} \int_0^n X_s^2 \, ds - n \sum_{k=1}^p e^{-2an} \Lambda_{n,k}^2
= e^{-2an} \int_0^n X_s^2 \, ds - \frac{1}{n} \sum_{k=1}^p \left( e^{-an} \int_0^n \phi_i(s) \, dX_s \right)^2
\to \frac{(A_\infty + \alpha Z_\infty)^2}{2\alpha}.
\]

Then, it remains to prove the following convergence, for every \( i = 1, \ldots, p \),
\[
e^{-2an} \frac{1}{n} I_{i,n,1} + e^{-2an} I_{i,n,2} \to \frac{1}{2\alpha} (A_\infty + \alpha Z_\infty)^2 \mu_i.
\]
almost surely, as \( n \to \infty \). Using (2.5), we get for \( i = 1, \ldots, p \),

\[
I_{i,n,1} = \int_{0}^{n} \phi(s) dX_s \left[ \int_{0}^{n} \tilde{L}(s) d\Sigma_s + \frac{\alpha}{2} \Sigma_n^2 + \int_{0}^{n} B_s^H d\Sigma_s \right] - \frac{1}{2} \left( \tilde{L}(n) + \alpha \Sigma_n + B_n^H \right)^2 \int_{0}^{n} \phi_i(s) d\Sigma_s.
\]

Moreover, since

\[
\int_{0}^{n} \phi_i(s) d\Sigma_s = \frac{1}{\alpha} \int_{0}^{n} \phi_i(s) dX_s - \frac{1}{\alpha} \int_{0}^{n} \phi_i(s) L(s) ds - \frac{1}{\alpha} \int_{0}^{n} \phi_i(s) dB_s^H,
\]

we obtain

\[
I_{i,n,1} = \int_{0}^{n} \phi_i(s) dX_s \left( \int_{0}^{n} \tilde{L}(s) d\Sigma_s + \int_{0}^{n} B_s^H d\Sigma_s \right) - \frac{1}{2} \left( \tilde{L}(n) + B_n^H \right)^2 + 2\alpha \Sigma_n (\tilde{L}(n) + B_n^H) \int_{0}^{n} \phi_i(s) d\Sigma_s - \frac{1}{2} (\alpha \Sigma_n)^2 \left( -\frac{1}{\alpha} \int_{0}^{n} \phi_i(s) L(s) ds - \frac{1}{\alpha} \int_{0}^{n} \phi_i(s) dB_s^H \right)
\]

\[
= J_{i,n,1} + J_{i,n,2} + J_{i,n,3} + J_{i,n,4} + J_{i,n,5}.
\]

Using (2.21) and (2.26), we have almost surely, as \( n \to \infty \),

\[
e^{-2\alpha n} \frac{j_{i,n,1}}{n} = e^{-\alpha n} \int_{0}^{n} \phi_i(s) dX_s \left( e^{-\alpha n} \int_{0}^{n} \tilde{L}(s) X_s ds \right) \to \alpha \lambda \phi_i (A_{\infty} + \alpha Z_{\infty}) \left( \frac{A_{\infty}}{\alpha} + Z_{\infty} \right) \int_{0}^{1} L(s) ds.
\]

By (2.21) and (2.25), we get almost surely, as \( n \to \infty \),

\[
e^{-2\alpha n} \frac{j_{i,n,2}}{n} = e^{-\alpha n} \int_{0}^{n} \phi(s) dX_s \left( e^{-\alpha n} \int_{0}^{n} B_s^H X_s ds \right) \to 0.
\]

The convergences (2.10) and (2.20) imply that almost surely, as \( n \to \infty \),

\[
e^{-2\alpha n} \frac{j_{i,n,3}}{n} = -\frac{1}{2} \frac{e^{-\alpha n}}{n} (\tilde{L}(n) + B_n^H)^2 e^{-\alpha n} \int_{0}^{n} \phi_i(s) X_s ds \to 0.
\]
According to (2.23), (2.21) and (2.10), we get almost surely, as $n \to \infty$,

$$
e^{-2\alpha} J_{i,n,4} = -\alpha \left( e^{-\alpha} \int_0^n X_s \left( \frac{1}{n} L(n) \right) \right) \left( e^{-\alpha} \int_0^n \phi_i(s) X_s ds \right)$$

$$- \alpha \left( e^{-\alpha} \frac{B_n H}{n} \right) \left( e^{-\alpha} \int_0^n \phi_i(s) X_s ds \right)$$

$$\to -\alpha \lambda_{i} \left( A_{\infty} + \alpha Z_{\infty} \right) \int_0^1 L(s) ds.$$

Furthermore, (2.23) and the periodicity property yield almost surely, as $n \to \infty$,

$$
e^{-2\alpha} J_{i,n,5} = \alpha \left( e^{-\alpha} \int_0^n X_s ds \right)^2 \frac{1}{n} \int_0^n \phi_i(s) L(s) ds$$

$$\to \frac{1}{2\alpha} \left( A_{\infty} + \alpha Z_{\infty} \right)^2 \mu_i.$$

Also, (2.23) and (2.13) imply that almost surely, as $n \to \infty$,

$$
e^{-2\alpha} J_{i,n,6} = \frac{\alpha}{2} \left( e^{-\alpha} \int_0^n X_s ds \right)^2 \frac{1}{n} \int_0^n \phi_i(s) dB_s^H$$

$$\to 0.$$

Consequently, as $n \to \infty$,

$$
e^{-2\alpha} I_{i,n,1} \to \frac{1}{2\alpha} \left( A_{\infty} + \alpha Z_{\infty} \right)^2 \mu_i \quad (3.3)$$

almost surely for every $i = 1, \ldots, p$.

Finally, since \(\int_0^n \phi_i^2(s) ds = n\) for $i = 1, \ldots, p$, then

$$I_{i,n,2} = \Lambda_{n,i} \sum_{k=1}^p \Lambda_{n,k} \left( n \mu_k + \alpha n \Lambda_{n,k} + \int_0^n \phi_k(s) dB_s^H \right)$$

$$- \left( n \mu_i + \alpha n \Lambda_{n,i} + \int_0^n \phi_i(s) dB_s^H \right) \sum_{k=1}^n \Lambda_{n,k}^2$$

$$= \Lambda_{n,i} \sum_{k=1}^p \Lambda_{n,k} \left( n \mu_k + \int_0^n \phi_k(s) dB_s^H \right) - \left( n \mu_i + \int_0^n \phi_i(s) dB_s^H \right) \sum_{k=1}^n \Lambda_{n,k}^2,$$

where we used

$$\int_0^n \phi_i(s) dX_s = \int_0^n \mu_i \phi_i(s)^2 ds + \alpha n \Lambda_{n,i} + \int_0^n \phi_i(s) dB_s^H.$$
By (2.13) and (2.20), we get for every \(i = 1, \ldots, p\), almost surely, as \(n \to \infty\),

\[ e^{-2\alpha n} I_{i,n,2} \to 0. \quad (3.4) \]

Therefore, the facts (3.1), (3.2), (3.3) and (3.4) achieve the proof of desired result. For the case \(H = \frac{1}{2}\), it suffices to use \(\int_0^n X_s dX_s = \frac{1}{2} \left( X_n^2 - n \right)\) and the same arguments as above. \(\square\)

4 Asymptotic distribution

In order to investigate the asymptotic behavior in distribution of the estimator \(\hat{\theta}_n\), as \(n \to \infty\), we will need the following lemmas.

Lemma 4.1 Assume that \(\frac{1}{2} \leq H < 1\). Then, for every \(n \geq 0\),

\[
\frac{1}{2} X_n^2 - \frac{p}{\Lambda_{n,k}} \sum_{k=1}^n \Lambda_{n,k} \int_0^n \phi_k(s)dX_s = n\alpha\gamma_n^{-1} + (A_\infty + \alpha Z_n) \int_0^n e^{\alpha s} dB^H_s + S_n, 
\]

(4.1)

where \(Z_n\) is given by (2.3), and the sequence \(S_n\) is defined by

\[
S_n := \frac{1}{2}(B_n^H)^2 + e^{\alpha s} R_n B_n^H - \int_0^n L(s)B_s^H ds - \alpha \int_0^n (B_s^H)^2 ds - \sum_{k=1}^p \Lambda_{n,k} \int_0^n \phi_k(s)dX_s \\
- \sum_{k=1}^p \Lambda_{n,k} \int_0^n \phi_k(s)dB^H_s - \alpha \int_0^n e^{\alpha s} B_s^H R_s ds + \alpha^2 \int_0^n e^{-\alpha s} B_s^H \int_0^s e^{\alpha r} B_r^H dr ds.
\]

Moreover, as \(n \to \infty\),

\[ e^{-\alpha n} S_n \to 0 \text{ almost surely}. \quad (4.2) \]

Proof Let us prove (4.1). First define the process

\[ M_t := \int_0^t e^{\alpha s} B_s^H ds, \quad t \geq 0. \]

According to (2.5), we have

\[ \frac{1}{2} X_t^2 = \frac{1}{2}(\tilde{L}(t))^2 + \frac{1}{2}(B_t^H)^2 + \tilde{L}(t)B_t^H + \frac{1}{2}\alpha^2 \Sigma_t^2 + \alpha \tilde{L}(t) \Sigma_t + \alpha \Sigma_t B_t^H, \quad (4.3) \]

where \(\tilde{L}(t)\) and \(\Sigma_t\) are defined in (2.6). \(\square\) Springer
Furthermore, using (5.1) and (2.5)

\[
\frac{\alpha^2}{2} \Sigma_t^2 = \alpha^2 \int_0^t \Sigma_s d\Sigma_s \\
= \alpha^2 \int_0^t \Sigma_s X_s ds \\
= \alpha \int_0^t X_s^2 ds - \alpha \int_0^t \tilde{L}(s)d\Sigma_s - \alpha \int_0^t B_s^H X_s ds \\
= \alpha \int_0^t X_s^2 ds - \alpha \tilde{L}(t)\Sigma_t + \alpha \int_0^t \Sigma_s L(s)ds - \alpha \int_0^t B_s^H X_s ds. \tag{4.4}
\]

Moreover, it follows from (2.4) that

\[
-\alpha \int_0^t B_s^H X_s ds = -\alpha \int_0^t B_s^H e^{\alpha s} A_s ds - \alpha^2 \int_0^t B_s^H e^{\alpha s} Z_s ds - \alpha \int_0^t (B_s^H)^2 ds \\
= -\alpha \int_0^t B_s^H e^{\alpha s} A_s ds - \alpha \int_0^t (B_s^H)^2 ds - \alpha^2 \left[ M_t Z_t - \int_0^t M_s dZ_s \right] \\
= -\alpha \int_0^t B_s^H e^{\alpha s} A_s ds - \alpha \int_0^t (B_s^H)^2 ds - \alpha^2 M_t Z_t \\
+ \alpha^2 \int_0^t e^{-\alpha s} B_s^H B^*_s \int_0^s e^{\alpha r} B_r^H drds. \tag{4.5}
\]

On the other hand, by (2.5) and (2.4), we have

\[
\alpha \Sigma_t B^*_t = B^*_t (X_t - \tilde{L}(t) - B^*_t) = B^*_t \left( -\tilde{L}(t) + e^{\alpha t} A_t + \alpha e^{\alpha t} Z_t \right). \tag{4.6}
\]

Combining (4.3), (4.4), (4.5) and (4.6), we obtain

\[
\begin{align*}
\frac{1}{2} X_t^2 &= \frac{1}{2} (\tilde{L}(t))^2 + \frac{1}{2} (B_t^H)^2 + \alpha \int_0^t X_s^2 ds + \alpha \int_0^t \Sigma_s L(s)ds - \alpha \int_0^t B_s^H e^{\alpha s} A_s ds \\
&\quad - \alpha \int_0^t (B_s^H)^2 ds - \alpha^2 M_t Z_t + \alpha^2 \int_0^t e^{-\alpha s} B_s^H B^*_s \int_0^s e^{\alpha r} B_r^H drds + e^{\alpha t} B_t^H A_t + \alpha e^{\alpha t} B_t^H Z_t. \\
&= \alpha Z_t \int_0^t e^{\alpha s} dB^*_s. \tag{4.7}
\end{align*}
\]

Further,

\[
- \alpha^2 M_t Z_t + \alpha e^{\alpha t} B_t^H Z_t = -\alpha Z_t \left( \alpha M_t - e^{\alpha t} B_t^H \right) \\
= \alpha Z_t \int_0^t e^{\alpha s} dB^*_s. \tag{4.8}
\]
Also, using \((2.14)\),
\[
e^{αt} B_t^H A_t - α \int_0^t B_s^H e^{αs} A_s ds = A_∞ e^{αt} B_t^H + e^{αt} B_t^H R_t - α A_∞ \int_0^t B_s^H e^{αs} R_s ds
\]
\[
= A_∞ \int_0^t e^{αs} B_s^H ds + e^{αt} B_t^H R_t - α \int_0^t B_s^H e^{αs} R_s ds.
\]
\[(4.9)\]
Combining \((4.7)\), \((4.8)\) and \((4.9)\), we get
\[
\frac{1}{2} X_t^2 = \frac{1}{2} \langle L(t) \rangle^2 + \frac{1}{2} (B_t^H)^2 + α \int_0^t X_s^2 ds + α \int_0^t Σ_s L(s) ds - α \int_0^t (B_s^H)^2 ds
\]
\[
+ α^2 \int_0^t e^{-αs} B_s^H \int_0^s e^{αr} B_r^H dr ds + α Z_t \int_0^t e^{αs} dB_s^H + A_∞ \int_0^t e^{αs} B_s^H ds + e^{αt} B_t^H R_t
\]
\[
- α \int_0^t B_s^H e^{αs} R_s ds.
\]
\[(4.10)\]
On the other hand, using \((1.1)\),
\[
\sum_{k=1}^p Λ_{n,k} \int_0^n \phi_k(s) dX_s = \sum_{k=1}^p μ_k n Λ_{n,k} + \sum_{k=1}^p α n Λ_{n,k}^2 + \sum_{k=1}^p Λ_{n,k} \int_0^n \phi_k(s) dB_s^H
\]
\[
= \int_0^n L(s) X_s ds + \sum_{k=1}^p α n Λ_{n,k}^2 + \sum_{k=1}^p Λ_{n,k} \int_0^n \phi_k(s) dB_s^H.
\]
\[(4.11)\]
Further, by \((2.5)\) and \((5.1)\),
\[
α \int_0^t Σ_s L(s) ds - \int_0^n L(s) X_s ds = \int_0^t (α Σ_s - X_s) ds
\]
\[
= \int_0^t L(s) \widehat{L}(s) ds - \int_0^t L(s) B_s^H ds
\]
\[
= \frac{1}{2} \langle \widehat{L}(t) \rangle^2 - \int_0^t L(s) B_s^H ds.
\]
\[(4.12)\]
Consequently, the equalities \((4.10)\), \((4.11)\) and \((4.12)\) lead to \((4.1)\).
Finally, the convergence \((4.2)\) is a direct consequence of \((2.10)\), \((2.15)\), \((2.20)\) and \((2.13)\).
\[
\square
\]

**Lemma 4.2** Assume \(\frac{1}{2} \leq H < 1\). Then, we have
\[
\hat{α}_n - α = \frac{(A_∞ + α Z_n) \int_0^n e^{αs} dB_s^H + S_n H}{n^{γ_n}}.
\]
\[(4.13)\]
where \( S_{n,H} = S_n - \frac{n^H}{H} \) if \( H = \frac{1}{2} \), and \( S_{n,H} = S_n \) if \( \frac{1}{2} < H < 1 \), with \( S_n \) defined in Lemma 4.1. Moreover, we have

\[
\hat{\mu}_n - \mu = (\alpha - \hat{\alpha}_n)(\Lambda_{n,1}, \ldots, \Lambda_{n,p}) + \frac{G_n}{n}, \tag{4.14}
\]

where

\[
G_n := \left( \int_0^n \phi_1(s) dB_s^H, \ldots, \int_0^n \phi_p(s) dB_s^H \right). \tag{4.15}
\]

**Proof** The representation \((4.13)\) follows directly from \((2.8)\) and \((4.1)\). For \((4.14)\), according to \((2.8)\) and \((2.9)\), we can write, for every \( i = 1, \ldots, p \),

\[
\hat{\mu}_{n,i} = \frac{1}{n} \int_0^n \phi_i(s) dX_s - \Lambda_{n,i} \hat{\alpha}_n.
\]

This combined with \((1.1)\) imply that, for every \( i = 1, \ldots, p \),

\[
\hat{\mu}_{n,i} = \frac{1}{n} \int_0^n \mu_i \phi_i^2(s) ds + \alpha \int_0^n \phi_i(s) X_s ds + \frac{1}{n} \int_0^n \phi_i(s) dB_s^H - \Lambda_{n,i} \hat{\alpha}_n
\]

\[
= \mu_i + \alpha \Lambda_{n,i} + \frac{1}{n} \int_0^n \phi_i(s) dB_s^H - \Lambda_{n,i} \hat{\alpha}_n,
\]

which completes the proof. \(\Box\)

**Lemma 4.3** Assume \( \frac{1}{2} \leq H < 1 \), and let \( G_n \) be given by \((4.15)\). Let \( F \) be any \( \sigma\{B_t^H, t \geq 0\} \)-measurable random variable such that \( P(F < \infty) = 1 \). Then, as \( n \to \infty \),

\[
\left( e^{-\theta T} \int_0^T e^{\alpha t} dB_t^H, F, \frac{G_n}{n^H} \right) \xrightarrow{\text{Law}} \left( N_1, F, N_2 \right), \tag{4.16}
\]

where \( N_1 \sim \mathcal{N}(0, \sigma_H^2) \), \( N_2 \sim \mathcal{N}(0, D) \) and \( B^H \) are independent, with the variance \( \sigma_H^2 = \frac{H \Gamma(2H)}{\alpha^2 n^H} \) and the covariance matrix \( D \) is given by

\[
D = \begin{cases} 
I_p & \text{if } H = \frac{1}{2}, \\
\left( \int_0^1 \int_0^1 \phi_i(x) \phi_j(y) dx dy \right)_{1 \leq i,j \leq p} & \text{if } \frac{1}{2} < H < 1,
\end{cases}
\]

with \( I_p \) is the identity matrix of size \( p \).

**Proof** Using similar arguments as in the proof of Es-Sebaiy and Nourdin (2013, Lemma 7) it suffices to prove that for every positive integer \( d \), and positive constants \( s_1, \ldots, s_d \),

\[
\left( e^{-\alpha n} \int_0^n e^{\alpha s} dB_s^H, B_{s_1}^H, \ldots, B_{s_d}^H, \frac{G_n}{n^H} \right) \xrightarrow{\text{Law}} \left( N_1, B_{s_1}^H, \ldots, B_{s_d}^H, N_2 \right)
\]
as $n \to \infty$. Moreover, since the left-hand side in this latter convergence is a Gaussian vector, it is sufficient to establish the convergence of its covariance matrix. From Belfadli et al. (Belfadli et al. (2011), Lemma 6) we have, for every $1/2 \leq H < 1$,

$$
\lim_{n \to \infty} E \left[ \left( e^{-\alpha n} \int_0^n e^{\alpha s} dB_s^H \right)^2 \right] = \frac{H \Gamma(2H)}{\alpha^{2H}},
$$

and for all fixed $s \geq 0$,

$$
\lim_{n \to \infty} E \left( B_s^H \times e^{-\alpha n} \int_0^n e^{\alpha r} dB_r^H \right) = 0.
$$

If $1/2 < H < 1$, then it follows from Bajja et al. (2017) that, for every $1/2 < H < 1$, as $n \to \infty$

$$
n^{-H} G_n \xrightarrow{\text{law}} \mathcal{N} (0, D). \quad (4.17)
$$

Hence, to finish the proof of the case $1/2 < H < 1$ it remains to check that, for all fixed $s \geq 0$,

$$
\lim_{n \to \infty} E \left( B_s^H \times \frac{1}{n^H} \int_0^n \phi_i(s) dB_s^H \right) = 0, \quad i = 1, \ldots, p, \quad (4.18)
$$

and

$$
\lim_{n \to \infty} E \left( e^{-\alpha n} \int_0^n e^{\alpha r} dB_r^H \times \frac{1}{n^H} \int_0^n \phi_i(s) dB_s^H \right) = 0, \quad i = 1, \ldots, p. \quad (4.19)
$$

Let us prove (4.18). Suppose $1/2 < H < 1$. Fix $s \geq 0$. We have, for every $i = 1, \ldots, p$,

$$
\left| E \left( B_s^H \times \frac{1}{n^H} \int_0^n \phi_i(s) dB_s^H \right) \right|
\leq \frac{H(2H-1)}{n^H} \int_0^s dv \left| \phi_i(v) \right| \int_0^s du |u-v|^{2H-2}
\leq \frac{H(2H-1)}{n^H} \left( \int_0^s dv \left| \phi_i(v) \right| \int_0^s du |u-v|^{2H-2} + \int_s^n dv \left| \phi_i(v) \right| \int_0^s du (v-u)^{2H-2} \right)
\leq \frac{C}{n^H} \left( s^{2H} + s \int_0^s dv \int_0^s du (v-u)^{2H-2} \right)
\leq \frac{C}{n^H} \left( s^{2H} + s \int_s^n dv (v-s)^{2H-2} \right)
= \frac{C}{n^H} \left( s^{2H} + \frac{s}{(2H-1)n^H} (n-s)^{2H-2} \right)
\xrightarrow{n \to \infty} 0,
$$

as $n \to \infty$, since $H < 1$. 
For (4.19), we have, for every $i = 1, \ldots, p$,
\[
\begin{align*}
&\left| E \left( e^{-an} \int_0^n e^{ar} dB_t^H \times \frac{1}{n^H} \int_0^n \phi_i(s) dB_s^H \right) \right| \\
&\leq \frac{H(2H-1)e^{-an}}{n^H} \int_0^n d\alpha^v \int_0^n d\alpha u |\phi_i(u)||u - v|^{2H-2} \\
&\leq C e^{-an} \int_0^n d\alpha^v \int_0^n d\alpha u |u - v|^{2H-2} \\
&= C \left[ e^{-an} \int_0^n d\alpha^v \int_0^v d\alpha (v - u)^{2H-2} \\
&\quad + e^{-an} \int_0^n d\alpha^v \int_0^n d\alpha (u - v)^{2H-2} \right].
\end{align*}
\]

Further, by L’Hôpital’s rule,
\[
\lim_{n \to \infty} \frac{e^{-an}}{n^H} \int_0^n d\alpha^v \int_0^v d\alpha (v - u)^{2H-2} = \lim_{n \to \infty} \frac{e^{-an}}{(2H-1)n^H} \int_0^n d\alpha^v v^{2H-1} \\
= \lim_{n \to \infty} \frac{(2H-1)(n^H \alpha e^{an} + Hn^H e^{an})}{n^H} \\
= \lim_{n \to \infty} (2H-1)(\alpha + H/n) \\
= 0.
\]

Moreover, making the change of variables $x = n - v$, we obtain
\[
\begin{align*}
\frac{e^{-an}}{n^H} \int_0^n d\alpha^v \int_0^n d\alpha (u - v)^{2H-2} &= \frac{e^{-an}}{(2H-1)n^H} \int_0^n d\alpha^v (n - v)^{2H-1} d\alpha \\
&= \frac{1}{(2H-1)n^H} \int_0^n d\alpha x^{2H-1} d\alpha \\
&\leq \frac{\Gamma(2H)}{(2H-1)n^H \alpha^{2H}} \\
&\to 0,
\end{align*}
\]
as $n \to \infty$. The proof of (4.18) and (4.19), when $H = \frac{1}{2}$, is quite similar to the proof above. For (4.17), we have if $H = \frac{1}{2}$,
\[
\lim_{n \to \infty} n^{-2H} \mathbb{E} \left( \int_0^n \phi_i(s) dB_s^H \int_0^n \phi_j(s) dB_s^H \right) = \lim_{n \to \infty} n^{-1} \int_0^n \phi_i(s) \phi_j(s) ds \\
= \int_0^1 \phi_i(s) \phi_j(s) ds
\]
which vanishes for $i \neq j$ and equals 1 for $i = j$. Thus the desired result is obtained. \qed
Recall that if $X \sim \mathcal{N}(m_1, \sigma_1)$ and $Y \sim \mathcal{N}(m_2, \sigma_2)$ are two independent random variables, then $X/Y$ follows a Cauchy-type distribution. For a motivation and further references, we refer the reader to Pham-Gia et al. (2006), as well as Marsaglia (1965). Notice also that if $N \sim \mathcal{N}(0, 1)$ is independent of $B^H$, then $N$ is independent of $Z_{\infty}$, since $Z_{\infty} = \int_{0}^{\infty} e^{-\theta s} B^H_s ds$ is a functional of $B^H$.

**Theorem 4.1** Assume $\frac{1}{2} \leq H < 1$, and let $N_1 \sim \mathcal{N}(0, \sigma_1^2)$, $N_2 \sim \mathcal{N}(0, D)$, where the variance $\sigma_1^2$ and the covariance matrix $D$ are given in Lemma 4.3. Suppose that $N_1, N_2$ and $B^H$ are independent. Then, as $n \to \infty$,

$$e^{\alpha n} (\hat{\alpha}_n - \alpha) \xrightarrow{law} \frac{2\alpha N_1}{A_{\infty} + \alpha Z_{\infty}}, \quad (4.20)$$

$$(e^{\alpha n} (\hat{\alpha}_n - \alpha), n^{1-H}(\hat{\mu}_n - \mu)) \xrightarrow{law} \left(\frac{2\alpha N_1}{A_{\infty} + \alpha Z_{\infty}}, N_2\right). \quad (4.21)$$

Consequently, as $n \to \infty$,

$$n^{1-H}(\hat{\mu}_T - \mu) \xrightarrow{law} N_2. \quad (4.22)$$

**Proof** Before proceeding to the proof of (4.21) we need first to prove (4.20). From (4.13) we can write

$$e^{\alpha n} (\hat{\alpha}_n - \alpha) = \frac{e^{-\alpha n} \int_{0}^{n} e^{\alpha s} dB^H_s}{A_{\infty} + \alpha Z_{\infty}} \times \frac{(A_{\infty} + \alpha Z_{\infty})(A_{\infty} + \alpha Z_n)}{e^{-2\alpha n n_\gamma^{-1}}} + \frac{e^{-\alpha n} S_{n, H}}{e^{-2\alpha n n_\gamma^{-1}}} \quad (4.23)$$

Using Lemma 4.3 and the fact that $A_{\infty} + \alpha Z_{\infty}$ is $\sigma\{B^H_t, t \geq 0\}$-measurable, we obtain, as $n \to \infty$,

$$a_n \xrightarrow{law} \frac{N_1}{A_{\infty} + \alpha Z_{\infty}}, \quad (4.24)$$

whereas (2.11), (2.20) and (2.24) imply that $b_n \to 2\alpha$ almost surely, as $n \to \infty$. On the other hand, by (4.2), (2.20) and (2.24), we obtain that $c_n \to 0$ almost surely as $n \to \infty$.

Combining all these facts together with (4.16) and Slutsky’s theorem, we deduce (4.20).

According to (4.14), we can write

$$n^{1-H}(\hat{\mu}_n - \mu) = e^{\alpha n} (\alpha - \hat{\alpha}_n) \times n^{1-H} e^{-\alpha n} (\Lambda_{n,1}, \ldots, \Lambda_{n,p}) + \frac{G_n}{n^H}$$

$$=: d_n + \frac{G_n}{n^H}. \quad (4.23)$$
This combined with (4.23) allow to write
\[(e^{\alpha_n}(\hat{\alpha}_n - \alpha), n^{1-H}(\hat{\mu}_n - \mu)) = \left( a_n \times b_n + c_n, d_n + \frac{G_n}{n^H} \right) = \left( a_n \times b_n, \frac{G_n}{n^H} \right) + (c_n, d_n) \]
\[= \left( 2\alpha a_n, \frac{G_n}{n^H} \right) + (a_n(b_n - 2\alpha), 0) + (c_n, d_n). \]

Hence, from Lemma 4.3 and the fact that \( A_\infty + \alpha Z_\infty \) is \( \sigma \{ B_t^H, t \geq 0 \} \)-measurable, we get, as \( n \to \infty \),
\[
\left( 2\alpha a_n, \frac{G_n}{n^H} \right) \overset{\text{Law}}{\longrightarrow} \left( \frac{2\alpha N_1}{A_\infty + \alpha Z_\infty}, N_2 \right) .
\]

Furthermore, combining (4.24) and \( b_n \to 2\alpha \) almost surely, together with Slutsky’s theorem, we obtain \((a_n(b_n - 2\alpha), 0) \to 0\) in probability as \( n \to \infty \). Also, by (2.20), (4.20) and Slutsky’s theorem, we have \( d_n \to 0\) in probability as \( n \to \infty \). Then, since \( c_n \to 0 \) almost surely, we get \((c_n, d_n) \to 0\) in probability as \( n \to \infty \). Therefore, applying Slutsky’s theorem, the convergence in law (4.21) is obtained.

\[ \square \]

**Remark 4.1** In the case when \( L = 0 \) in (1.1), Lemma 4.3 and Theorem 4.1 are proved in El Machkouri et al. (2016).

## 5 Appendix

As a direct consequence of the Borel-Cantelli Lemma, we found the following result, see Kloeden and Neuenkirch (Kloeden and Neuenkirch 2007, Lemma 2.1), which ensures the almost sure convergence from \( L^p \) convergence.

**Lemma 5.1** Let \( \gamma > 0 \). Let \((Z_n)_{n \in \mathbb{N}}\) be a sequence of random variables. If for every \( p \geq 1 \) there exists a constant \( c_p > 0 \) such that for all \( n \in \mathbb{N} \),
\[(E|Z_n|^p)^{1/p} \leq c_p n^{-\gamma},\]
then for all \( \epsilon > 0 \) there exists a random variable \( \beta_\epsilon \) such that
\[|Z_n| \leq \beta_\epsilon n^{-\gamma + \epsilon}, \text{ almost surely},\]
for all \( n \in \mathbb{N} \). Moreover, \( E|\beta_\epsilon|^p < \infty \) for all \( p \geq 1 \).

In what follows, we briefly recall some basic elements of Young integral (see Young 1936), which are helpful for some of the arguments we use in this paper. For any \( \alpha \in [0, 1] \), we denote by \( \mathcal{H}^\alpha([0, T]) \) the set of \( \alpha \)-Hölder continuous functions, that is, the set of functions \( f: [0, T] \to \mathbb{R} \) such that
\[ |f|_\alpha := \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty. \]

We also set \( |f|_\infty = \sup_{t \in [0,T]} |f(t)| \), and we equip \( \mathcal{H}^\alpha ([0, T]) \) with the norm \( \|f\|_\alpha := |f|_\alpha + |f|_\infty \).

Let \( f \in \mathcal{H}^\alpha ([0, T]) \), and consider the operator \( T_f : C^1 ([0, T]) \rightarrow C^0 ([0, T]) \) defined as

\[ T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0, T]. \]

It can be shown (see, e.g., Nourdin 2012, Section 3.1) that, for any \( \beta \in (1 - \alpha, 1) \), there exists a constant \( C_{\alpha, \beta, T} > 0 \) depending only on \( \alpha \), \( \beta \) and \( T \) such that, for any \( g \in C^1 ([0, T]) \),

\[ \left\| \int_0^t f(u)g'(u)du \right\|_\beta \leq C_{\alpha, \beta, T} \|f\|_\alpha \|g\|_\beta. \]

We deduce that, for any \( \alpha \in (0, 1) \), any \( f \in \mathcal{H}^\alpha ([0, T]) \) and any \( \beta \in (1 - \alpha, 1) \), the linear operator \( T_f : C^1 ([0, T]) \subset \mathcal{H}^\beta ([0, T]) \rightarrow \mathcal{H}^\beta ([0, T]) \), defined as \( T_f(g) = \int_0^tu f(u)g'(u)du \), is continuous with respect to the norm \( \| \cdot \|_\beta \). By density, it extends (in an unique way) to an operator defined on \( \mathcal{H}^\beta \). As consequence, if \( f \in \mathcal{H}^\alpha ([0, T]) \), if \( g \in \mathcal{H}^\beta ([0, T]) \) and if \( \alpha + \beta > 1 \), then the (so-called) Young integral \( \int_0^t f(u)dg(u) \) is (well) defined as being \( T_f(g) \).

The Young integral obeys the following formula. Let \( f \in \mathcal{H}^\alpha ([0, T]) \) with \( \alpha \in (0, 1) \), and \( g \in \mathcal{H}^\beta ([0, T]) \) with \( \beta \in (0, 1) \). If \( \alpha + \beta > 1 \), then \( \int_0^tg_uf_u\) and \( \int_0^tf_udg_u \) are well-defined as Young integrals, and for all \( t \in [0, T] \),

\[ f_tg_t = f_0g_0 + \int_0^tg_udf_u + \int_0^tf_udg_u. \quad (5.1) \]

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