A new hierarchy of avalanches observed in Bak-Sneppen evolution model

W. Li* and X. Cai†

Institute of Particle Physics, Hua-zhong Normal University, Wuhan, 430079, P. R. China

A new quantity, $\bar{f}$, denoting the average fitness of the ecosystem, is introduced in Bak-Sneppen model. Through this new quantity, a new hierarchy of avalanches, $f_0$-avalanche, is observed in the evolution of Bak-Sneppen model. An exact gap equation governing the self-organization of the model in terms of $\bar{f}$ is presented. It is found that self-organized threshold $f_c$ of $\bar{f}$ can be exactly obtained. Two basic exponents of the new avalanche, $\tau$, avalanche distribution, and $D$, avalanche dimension are given through simulations of one- and two dimensional Bak-Sneppen models. It is suggested that $\bar{f}$ may be a good quantity in determining the emergence of criticality.

PACS number(s): 87.10.+e, 05.40.+j, 64.60.Lx

*E-mail: liw@iopp.ccnu.edu.cn
†E-mail: xcai@wuhan.cnjb.com
The term of avalanche may originate from the phenomena occurred in nature. It is referred to as a sequential events which may cause devastating catastrophe. The phenomena of avalanches are ubiquitous in nature. The canonical example of avalanche in nature is the mountain slide, during which great mass of snow and ice at a high altitude slide down a mountain side, often carrying with it thousands of tons of rock, and sometimes destroy forests, houses, etc in its path [1]. Since avalanches occur everywhere, from the sandpile or ricepile, to the Himalayan sandpiles; from the river network, to the earthquake, starquakes and even solar flares; from the biology, to the economy, etc, it is hence proposed [2] that avalanches may be the underlying mechanism of the formation of various geographical structures and complex organisms, e.g., brains, etc. Furthermore, avalanches may be the origin of fractals in the world. From this point of view, avalanches can be viewed as the immediate results of complex systems, and hence can be used as the theoretical justification for catastrophism. This is because if the real world is complex then the catastrophes are inevitable and unavoidable in biology, history, and economics. It is now even proposed by Meng et al [3] that the formation of colorless gluon clusters may be attributed to avalanches intriguied by the emission or absorption of gluons.

Plenty of patterns provided by nature exhibit coherent macroscopic structures developed at various scales and do not exhibit elementary interconnections. They immediately suggest seeking a compact description of the spatio-temporal dynamics based on the relationship among macroscopic elements rather than lingering on their inner structure [4]. That is, one needs to condense information when dealing with complex systems. Maybe only this way is efficient and turns out successful.

As known, avalanche is one kind of macroscopic phenomenon driven by local interactions. The size of avalanche, spatial and temporal as well, may be sensitive to the initial configuration, or more generally, the detailed dynamics of the system. However, the distribution of avalanches, Gutenberg-Ritcher law [5], or equivalently, power-law, does not depend on such kind of details due to the universality of complexity. Hence, in this sense avalanche study may be an appropriate tools in studying various complex phenomena. On the other hand, observation of a great variety of patterns, such as self-similar, fractal behavior in nature [6,7,8,9], noise in quasar [10], river flow [11] and brain activity [12], and many natural and social phenomena, including earthquakes, economic activity and biological evolution suggests that these phenomena are signatures of spatiotemporal complexity and can be related via scaling relations to the fractal properties of the avalanches [13]. This suggests the occurrence of these general, empirical phenomena may be attributed to the same underlying avalanche dynamics. Thus, one can see that study of avalanche is crucial in investigating the critical features of complex systems. It can be even inferred that avalanche dynamics does provide much useful information for us to understand the general features of the ubiquitous complexity around us. That is probably also why this paper focuses on such kind of topic.

Despite the fact that avalanche may provide insight into complexity, the definition of which can be vastly different for various systems, and the same kinds of systems, even the same system. Let us recall some definitions of avalanches given before. In sandpile model [2], an avalanche is intrigued by adding a grain or several grains of sand into the system at some time and causing the topple of some sites, which may later on cause some other sites to topple. The avalanche is considered over when the height of all sites are less than the critical value, say, 4. In Bak-Sneppen model [14], several kinds of avalanche [13] are presented. For instance, $f_0$-avalanche, G(s)-avalanche, forward avalanche, backward avalanche, etc. Despite the fact that these kinds of definitions of avalanches may show the various hierarchical structures they manifest the same underlying fractal feature of the ecosystem, i.e., self-organized criticality. Relating all these kinds of avalanches one can provide a general definition of the avalanche for Bak-Sneppen model: An avalanche corresponds to sequential mutations below certain threshold. One can see that this kind of definition can ensure the mutation events within a single avalanche are casually and spatially connected. In addition, with this definition there exists a hierarchy of avalanches, each defined by their respective threshold. It is the hierarchical structure of the avalanche that exhibits the fractal geometry of the system and that implies complexity.

It can be inferred from the definition of avalanche that there always exists a triggering event which initiates the avalanche and whose effect, that is, causing an avalanche to spread within the system later and later on, will disappear at the end of the avalanche. And, the observation of avalanche through the triggering event, up to now, is based on the individual level, despite that the avalanche is a macroscopic and global phe-
nomenon of the system studied, in the laboratory, and in nature as well. Take sandpile model, the triggering event is adding a grain or several grains of sand to some sites and causing them to topple, thus initiating an avalanche. Consider another model, Bak-sneppen model, in which the corresponding triggering event of an avalanche is mutation of the extremal species causing the fitness [14] of the extremal site at the next time step less than a certain threshold. One can see that in the above two models triggering events are directly concerned with the feature of individuals, e.g., the height of the site in the former model, or, the fitness of the extremal site in the latter one. It can be readily learned that the triggering events, whether those in the laboratory or those in nature, are not directly related to the global feature of the systems although avalanche can span across the whole systems. Generally speaking, the observation of avalanches is done through some feature of individuals, instead of that of the system as a whole. However, general feature of the complex system may provide insight into knowing the tendency of the evolution of the system. Specifically, global feature of a complex system may help one to understand the critical behavior of the system. That is, it is feasible that some characteristic quantities representing the corresponding global features can be employed in describing the critical behavior of the system. Furthermore, these quantities ought to be related to avalanche dynamics, and hence can be used to describe complexity emerged in a variety of complex systems. Apparently, our aim is to search for or define such kind of quantities and then to expect to observe new kind of avalanche based on these quantities. Indeed, we obtain a new quantity which can be used to define a new hierarchy of avalanches in Bak-Sneppen model. We suggest that this quantity may be used as a criterion in determining the emergence of criticality. It will be shown later that this new kind of avalanche still exhibits spatio-temporal complexity in a different context.

Consider Bak-Sneppen model [14], which is a very simple evolution model of biological ecosystem. Despite the simplicity of the model itself, it can exhibit the skeleton of species evolution, punctuated equilibrium behavior. Detailed information about this original model of evolution can be available in Ref. [14]. In Bak-Sneppen model, each species is represented by a single fitness. The fitness may represent population of a whole species or living capability of the species [15]. Hence, one can see that fitness is a vital quantity and is the only one describing the model. No other addi-
tional quantities are considered in this oversimplified model. Thus, the fitness is the most important feature of species and that of the model. So, when considering the global feature of the species ecosystem, one has to relate this general feature to the feature of individuals. That is, the general feature of the ecosystem should be associated with the fitness of the species. As previously mentioned, a corresponding quantity should be found to describe this general feature. Before presenting such kind of quantity let us briefly review Bak-Sneppen model so that the readers who are not so familiar with this model can have a rough idea of what it is about.

Bak-Sneppen model is perhaps the simplest model of self-organized criticality. In this “toy” model, random numbers, $f_i$, chosen from a flat distribution, $p(f)$, are assigned independently to each species located on a $d$-dimensional lattice of linear size $L$. At each time step, the extremal site, i.e., the species with the smallest random number, together with its $2d$ nearest neighboring sites, is chosen for updating by assigning $2d+1$ new random numbers also chosen from the same uniform distribution $p(f)$ to them. This updating process continues indefinitely. After a long transient process the system reaches a statistically stationary state where the density of random numbers in the system vanishes for $f < f_c$ and is uniform above $f_c$, the self-organized threshold.

Having briefly introduced the model, next, we will introduce a new quantity. Please note that the model we used is still Bak-Sneppen model. We observe the evolution of the model without adding anything to the model. We simply introduce the new quantity based on the fitness of the species. Define the average fitness, denoted by $\bar{f}$, as below,

$$\bar{f} = \frac{1}{L^d} \sum_{i=1}^{L^d} f_i$$

(1)

, where $f_i$ is the fitness of the $i$th species. Here, we refer to $\bar{f}$ as the average fitness of the whole system and as a global quantity. $\bar{f}$ may represent the average population or average living capability of the whole ecosystem. Large $\bar{f}$, i.e., high average fitness, may imply the total population of the system is immense or its average living capability is great, and vice versa. Initial value of $\bar{f}$, denoted by $\bar{f}(0)$, can be easily calculated. As known, at the beginning of the evolution $f_i$’s are uniformly distributed between $(0,1)$. So, for an infinite system, $\bar{f}(0)$ equals to 0.5. However, for a finite-size system $\bar{f}(0)$ will fluctuate slightly due to the finite size of the system, which is not so important.
in the latter evolution. We will simply consider the average value, 0.5. It should be pointed out that \( \bar{f}(0) \) does not reflect the correlation among species. As the evolution goes on the correlation among the species within a system will become more and more distinctive. Denote \( \bar{f}(s) \) the average fitness of the system at time step \( s \) in the evolution. Hence, in the \( s \) limit, i.e., \( s \gg L^d \), \( \bar{f}(s) \) may partly reflect information about correlation. As a global quantity, \( \bar{f}(s) \) should include information concerning the interaction between species. Hence, it is natural to expect that \( \bar{f} \) may be a good quantity in describing the feature of the system as a whole.

Before introducing the new hierarchy of avalanches it is necessary and worthwhile to know some feature of the new quantity, \( \bar{f}(s) \). Firstly, let us present some theoretical analysis. Recall the definition of \( f \) one can see that \( \Delta \bar{f}(s) = f(s + 1) - f(s) \) approaches zero in the \( L \to \infty \). An observer can not even perceived the change of \( \bar{f}(s) \) during the short time period since it is vanishingly small. However, changes at very time step are accumulated to form a relatively distinctive change after a long time, which is perceivable for the observers. This long time period is required to be much greater than the system size, i.e., \( s \gg L^d \). In other words, \( f(s + s_0) - \bar{f}(s_0) \) may only be "noticed" when \( s \gg L^d \). Thus, one can not expect \( \bar{f}(s) \) will have great variation from the current time step to the next nearest time step, which is vastly different from the variation of the fitness of extremal site. The latter can vary from one value, say, 0, to the next value, 1 between two successive time steps. It should also be expected that there exists an increasing tendency of \( \bar{f} \) versus time \( s \). This is because at each time step the least fitness is eliminated from the system the general fitness of the whole system will tend to increase. And due to the slow fluctuation of the \( \bar{f} \) the increasing behaves like a stepwise, i.e., Devil’s stepwise [2].

In order to show the feature of \( \bar{f} \) versus time \( s \) we performed simulations of Bak-Sneppen model. At each time step, in addition to the updating of the extremal sites, we also track the signals \( f(s) \). FIG. 1 shows the evolution of \( \bar{f}(s) \) versus time during a time period for a one-dimensional Bak-Sneppen model of size \( L = 200 \). This plot shows that \( \bar{f} \) varies slightly between two successive time steps but tends to increase in the long evolution process. Simulation of a two-dimensional model of size \( L = 20 \) exhibits the similar behavior of the evolution of \( \bar{f}(s) \).

Before searching for the punctuated equilibrium behavior let us first introduce another quantity, \( F(s) \), the gap of the average fitness. The definition of \( F(s) \) is given as follows: Initial value of \( F(s) \) is equal to \( \bar{f}(0) \). After \( s \) updates, a large \( F(s) > \bar{f}(0) \) opens up. The current gap \( F(s) \) is the maximum of all \( F(s') \), for all \( 0 \leq s' < s \). FIG. 2 shows the \( F(s) \) as a step-wising increasing function of \( s \) during the transient for a one-dimensional Bak-Sneppen model of size \( L = 100 \). Actually, the gap is an envelope function that tracks the increasing peaks in \( \bar{f}(s) \). Indeed, punctuated equilibrium behavior appears in terms of this new quantity, \( \bar{f}(s) \).

By definition [14], the separate instances when the gap \( F(s) \) jumps to its next higher value are separated by avalanches. Avalanches correspond to plateaus in \( F(s) \) during which \( f(s) < F(s) \), which ensures the mutation events within a single avalanche are spatially and casually connected. A new avalanche is initiated each time the gap jumps and ends up when the gap jumps again. As the gap increases, the probability for the average fitness, \( f \), to fall below the gap increases also, and larger and larger avalanches typically occur.

We can also obtain an exact gap equation of \( F(s) \), similar to the one found for Bak-sneppen model in Ref. [16]. Suppose in the system the current gap is \( F(s) \). If \( F(s) \) is to be increased by \( \Delta F \), i.e., from \( F(s) \) to \( F(s) + \Delta F \), the average number of avalanches needed is \( N_{av} = \Delta F L^d/(1 - F(s)) \). We can guarantee \( N_{av} \gg 1 \) by selecting \( \Delta F \gg L^{-d} \). In the large \( L \) limit, \( N_{av} \) can be arbitrarily large. Hence, in this limit, the average number of time steps required to increase the gap from \( F(s) \) to \( F(s) + \Delta F \) given by the interval \( \Delta s = (S)_{F(s)} N_{av} = \langle S \rangle_{F(s)} \Delta F L^d/(1 - F(s)) \), where \( \langle S \rangle_{F(s)} \) is the average avalanche size of the plateaus in the gap function. From the law of large numbers the fluctuation of this interval around its average value vanishes. In the \( \Delta F \to 0 \) limit, \( \Delta s \to 0 \). Taking the continuum limit we can obtain the differential equation for \( F(s) \),

\[
\frac{dF(s)}{ds} = \frac{1 - F(s)}{L^d \langle S \rangle_{F(s)}}.
\]

Note this equation is exact.

All SOC models, e.g., the BTW sandpile model [17], the earthquake models [18], or Bak-Sneppen model [14], exhibit self-organized criticality in terms of a power-law distribution of avalanche. It is natural to expect that we can observe SOC in terms of the hierarchical structure of \( \bar{f} \), which itself manifests complexity. Using this new quantity
to define the avalanche is simply another way of observing the same phenomenon which can be observed in other ways. As known, the emergence of complexity is independent of the tools used to observe them provided that these tools are efficient and strong enough. Similar to the ones used in Refs. [13,19], we present the definition of $f_0$-avalanche, where $f_0$ is only a parameter between 0.5 and 1 to define the avalanche. Suppose at time step $s_1$, $f(s_1)$ is larger than $f_0$. If, at time step $s_1 + 1$, $f(s_1 + 1)$ is less than $f_0$, this initiates a creation-annihilation branching process. The avalanche still continues at time step $s'$, if all the $f(s)$ are less than $f_0$ for $1 \leq s \leq s' - 1$. And the avalanche stops, say, at time step $s_1 + S$, when $f(s_1 + S) > f(s_1)$. In terms of this definition, the size of the avalanche is the number of time steps between subsequent punctuations of the barrier $f_0$ by the signal $f(s)$. In the above example, the size of the avalanche is $S$. It can be clearly seen from Fig.1 that this definition guarantees the hierarchical structure of avalanches, larger avalanches consists of smaller avalanches. As $f_0$ is lowered, bigger avalanches are subdivided into smaller ones. Hence, the statistics of $f_0$-avalanche will inevitably have a cutoff if $f_0$ is not chosen to be the value of $f(s)$ at critical state, denoted by $f_c$. We can also define the $f_c$-avalanche. Nevertheless, $f_0$-avalanche in the stationary state has the same scaling behavior as $f_c$-avalanche provided that $f_0$ is close to $f_c$. We measure $f_0$-avalanche distribution for one- and two-dimensional Bak-Sneppen models. The simulation results are given in Fig. 3. The exponent $\tau$, defined by $P(S) \sim S^{-\tau}$, is 1.80 for 1D model and 1.725 for 2D model. Another exponent, $D$, avalanche dimension [13], defined by $n_{cav} \sim S^{D/d}$, where $n_{cav}$ is the number of sites covered by an avalanche, and $d$ is the space dimension, is measured. We find $D=2.45$ for 1D model and 1.55 for 2D model.

Up to now, a question is still unsolved. It is about the critical value of $f$, $f_c$. This may be a hard bone if the system size is finite, but when the consider the L limit, everything will be smooth and can be easily accomplished. Recall the evolution of Bak-Sneppen model, or the detailed research of this model [13], the densities of sites with random numbers is uniform above $G$ [13] and vanishes below $G$ in the $L \rightarrow \infty$, where $G$ is the gap of extremal site and detailed information of it can be found, for instance, in Ref. [13]. Hence, one can obtain,

$$\lim_{L \rightarrow \infty} f(s) = \lim_{L \rightarrow \infty} \frac{1 + G(s)}{2}. \quad (3)$$

Interestingly, inserting Eq. (3) into the gap equation of $G$ found in Ref. [19], one can immediately obtain Eq. (2). Please note that Eq. (2) is also valid for finite-size systems. From Eq. (3) one can immediately obtain,

$$\lim_{L \rightarrow \infty} f_c = \lim_{L \rightarrow \infty} \frac{1 + f_c}{2}. \quad (4)$$

Hence, $f_c$ can be easily determined from Eq. (4). Using the results of $f_c$ provided by Refs. [13,20], one can obtain $f_c$, 0.83351 for 1D model and 0.66443 for 2D model. However, Eqs. (3) and (4) are not valid for a finite-size system, since one can not ensure the distribution of random numbers during a finite-size system is really uniform. Due to the fluctuation of $f(s)$ it is extremely difficult for one to determine exactly the critical value of $f$ for a finite-size system. One may estimate $f_c$ for a finite-size system using the simulation. We find that this value weakly depends on the system size. When the system size is very large, $f_c$ approaches the corresponding value for infinite systems. Actually, the value of $f_c$ itself is not so important. Fig. 4 shows the fluctuation of $f$ for a one-dimensional model of size $L = 200$ near its critical state. We note, in this figure, $f$ fluctuates slightly around some average value and does not tend to increase any more for a long time. We may say that the system approaches its stationary state. In this sense, we suggest that $f$ may be a good quantity in determining the emergence of criticality. That is, the great fluctuation of $f_{min}$ will not affect us to determine when we approach the critical state. We need only to know the feature of $\bar{f}$. This is more reasonable and easily accepted since $\bar{f}$ is a global quantity and condenses information of the system and its components.

Why we call the $f_0$-avalanche a new hierarchy of avalanches? Firstly, this kind of avalanche is defined on the global level, in terms of the new global quantity, $f$. The background of this definition is different from any one used before. This new kind of avalanche reflects the fractal geometry in terms of the global feature. Secondly, one can notice that the exponents $\tau$ obtained in our simulation are different from the ones found in Ref. [13]. From this point of view, one can judge that this kind of avalanche is totally different from any one observed before. Hence, its a new kind of avalanche.

Self-organized criticality is suggested by Bak et al to be the “fingerprint” of a large variety of complex system (they call system with variability as complex) and is represented by a scale-free line on a log-log plot. In order to know the criticality of a system one needs to know when the system reaches the stable stationary state where the
phase transition occurs. It is extremely difficult and almost impossible for one to know when a system in nature approaches, not even reaches, its critical state. One can just study the ubiquitous fractal geometrical structure carved by avalanches through thousands of millions of years. However, in laboratory experiments and computer simulations, one needs a criterion to judge when stationary state approaches, even, reaches, since statistics of avalanches may only be done under critical state. Thus, when the extremal signal, $f_{\text{min}}$, approaches to the self-organized threshold, $f_c$, the ecosystem reaches its stationary state. However, $f_{\text{min}}$ itself fluctuates greatly time to time, which brings great difficulty in determining the appearance of criticality. As shown, $\bar{f}$ is relatively stable in a short time period. Hence, when $\bar{f}$ does not tend to increase any more, we may say that the system approaches its stationary state. And, we can observe criticality in a rather long time period. Surely, the emergence of criticality is rather complex, other physical mechanism is needed, this is what we will consider in the future work.

In conclusion, a new hierarchy of avalanches is observed in Bak-Sneppen model. A new quantity, $\bar{f}$, is presented and is suggested by us to be a possible candidate in determining the emergence of criticality. An exact gap equation and simulation results are also given.

This work was supported by NSFC in China. We thank Prof. T. Meng for correspondence and helpful discussions.

---

[1] Extracted from *Oxford advanced learner’s dictionary of current English with Chinese translation*, revised third edition. Hong Kong, Oxford University Press, 1984.

[2] P. Bak, *How nature works* (Springer-Verlag, New York, 1996).

[3] Meng Ta-chung, R. Rittel, and Zhang Yang, Phys. Rev. Lett. **82**, 2044 (1999).

[4] R. Badii and A. Politi, *Complexity* (Cambridge University Press, London, 1997).

[5] B. B. Mandelbrot, *The fractal geometry of nature* (Freeman, New York, 1983).

[6] B. B. Mandelbrot, *The fractal geometry of nature* (Freeman, New York, 1983).

---

Figure Captions

FiG. 1: The variation of $\bar{f}$ versus time during a time period for a (a) one-dimensional Bak-Sneppen model of size $L = 200$ and (b) two-dimensional Bak-Sneppen model of size $L = 20$. The plots show the hierarchical structure of $\bar{f}$.

FiG. 2: Punctuated equilibrium of $\bar{f}$ for a (a) one-dimensional Bak-Sneppen model of size $L = 200$ and (b) two-dimensional Bak-Sneppen model of size $L = 20$. We track the increasing signal of $\bar{f}_s$, i.e, F(s).

FiG. 3: Distribution of $\bar{f}_0$-avalanche for a (a) one-dimensional Bak-Sneppen model of size $L = 200$ and (b) two-dimensional Bak-Sneppen model of size $L = 20$. $\bar{f}_0$ for (a) is chosen to be 0.821, and for (b), 0.648. The slopes are -1.800 and -1.725 for the two plots respectively.

FiG. 4: The fluctuation of $\bar{f}$ around the critical state of a one-dimensional Bak-Sneppen model of size $L = 200$. 

---

[7] P. H. Coleman, L. Pietronero, and R. H. Sanders, Astron. Astrophys. **200**, L32 (1988); L. Pietronero, Physica A **144**, 257 (1987).

[8] A. Rinaldo, I. Rodriguez-Iturbe, R. Rigon, E. Ijjasz-Vasquez, and R. L. Bras, Phys. Rev. Lett **70**, 822 (1993).

[9] *Fractals in the Earth Sciences*, edited by C. C. Barton and P. R. Lapointe (plenum, New York, 1994).

[10] W. H. Press, Comments Astrophys. **7**, 103 (1978).

[11] See, for example, the empirical observations of H. E. Hurst described in Refs. [6] and [8].

[12] F. Gruneis, M. Nakao, and M. Yamamato, Biol. Cybernetics **62**, 407 (1990).

[13] Maya Paczuski, Sergi Malsoc, and Per Bak, Phys. Rev. E **53** (414) 1996.

[14] Per Bak, Kim Sneppen, Phys. Rev. Lett **71**, 4083 (1993).

[15] W. Li and X. Cai, cond-mat/9904313.

[16] M. Paczuski, S. Malsoc, and P. Bak, europhys. Lett. **27**, 97 (1994).

[17] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); P. Bak and M. Paczuski, Phys. world **6**, (12), 39 (1993).

[18] P. Bak and C. Tang, J. geophys. res. B **94**, 15635 (1989); Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992); K. Christensen and Z. Olami, Phys. rev. A **46**, 1829 (1992).

[19] M. Paczuski, S. Malsoc, and P. Bak, europhys. Lett. **28**, 295 (1994).

[20] P. Grassberger, Phys. Lett. A **200**, 277 (1995).
Fig. 1

$\tilde{f}(s)$

$d=1$
$L=200$

$d=2$
$L=20$
Figure 2

$d=1$
$L=200$

$d=2$
$L=20$
