DERIVED CATEGORY OF FINITE SPACES AND GROTHENDIECK DUALITY

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Abstract. We obtain some fundamental results, as Bokstedt-Neeman Theorem and Grothendieck duality, about the derived category of modules on a finite ringed space. Then we see how these results are transferred to schemes in a simple way and generalized to other ringed spaces.

Introduction

Finite ringed spaces are a especially simple example of ringed space and they appear in a natural way as finite models of more general ringed spaces. In the topological context, the use of finite models for the study of general topological spaces goes back to McCord ([8]) and it is still a useful tool nowadays (see for example [2], [3] and [4]). In a more algebraic context, as the theory of schemes, finite ringed spaces (or more generally quivers with a representation) have been used for the study of the category of quasi-coherent modules on a scheme (for example, in [8], [12]). Schemes admitting a finite model are precisely quasi-compact and quasi-separated schemes. The finite models of these schemes (resp. of quasi-compact and semi-separated schemes) are an example of schematic finite spaces (resp. of semi-separated finite spaces), introduced in [12], but there are schematic finite spaces that are not a finite model of a scheme. Our point of view is that a lot of concepts and results on schemes can be generalized to schematic finite spaces, recovering the results on schemes in a more essential and - generally - simpler way, and allowing a further generalization to other ringed spaces. For example, the concept of affine scheme and its different characterizations (as Serre’s cohomological criterion of affineness) led in [12] and [14] to an analysis of the concept of affineness in the context of finite ringed spaces, schematic finite spaces and then to arbitrary ringed spaces.

In this paper we continue this point of view: we obtain some fundamental results concerning the derived category of modules on a finite ringed space, recovering then the analogous results about the derived category of modules on a quasi-compact and quasi-separated scheme and then obtaining new results in other ringed spaces. More specifically:

Let \((X, \mathcal{O})\) be a semi-separated finite space, \(\text{Qcoh}(X)\) the category of quasi-coherent \(\mathcal{O}\)-modules and \(D\text{Qcoh}(X)\) its derived category. Let us denote by \(D(X)\) the derived category of complexes of \(\mathcal{O}\)-modules and by \(D_{qc}(X)\) the full subcategory of complexes of \(\mathcal{O}\)-modules with quasi-coherent cohomology. Then we prove:

**Theorem 3.2** (Bokstedt-Neeman theorem for semi-separated finite spaces). The natural functor \(D\text{Qcoh}(X) \to D_{qc}(X)\) is an equivalence.

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**Theorem 3.5** Let $f : X \to Y$ be a schematic morphism between semi-separated finite spaces. The diagram

$$
\begin{array}{ccc}
D \operatorname{Qcoh}(X) & \xrightarrow{\mathbb{R}f_*} & D \operatorname{Qcoh}(Y) \\
\downarrow & & \downarrow \\
D(X) & \xrightarrow{\mathbb{R}f_*} & D(Y)
\end{array}
$$

is commutative, where $\mathbb{R}f_*$ is the right derived functor of $f_* : \operatorname{Qcoh}(X) \to \operatorname{Qcoh}(Y)$.

**Theorem 3.6** The category $\operatorname{Qcoh}(X)$ has enough flats: any quasi-coherent module $M$ admits a resolution

$$
\cdots \to \mathcal{P}^{-2} \to \mathcal{P}^{-1} \to \mathcal{P}^0 \to M \to 0
$$

by quasi-coherent and flat $\mathcal{O}$-modules.

Regarding Grothendieck duality we shall first prove the following theorem for any morphism between finite ringed spaces:

**Theorem 4.1** Let $f : X \to Y$ be a morphism between finite ringed spaces. The functor $\mathbb{R}f_* : D(X) \to D(Y)$ has a right adjoint.

This theorem generalizes the duality theorem of [10] from finite topological spaces to arbitrary finite ringed spaces. In the quasi-coherent context we shall obtain:

**Theorem 4.2** Let $f : X \to Y$ be a morphism between finite spaces (Definition 1.3). One has:

1. The functor $\mathbb{R}f_* : D \operatorname{Qcoh}(X) \to D \operatorname{Qcoh}(Y)$, composition of $D \operatorname{Qcoh}(X) \to D(X)$ and $D(X) \xrightarrow{\mathbb{R}f_*} D(Y)$, has a right adjoint.
2. If $f$ is schematic, the functor $\mathbb{R}f_* : D \operatorname{Qcoh}(X) \to D \operatorname{qc}(Y)$ has a right adjoint.
3. If $X$ and $f$ are schematic, the functor $\mathbb{R}f_* : D \operatorname{qc}(X) \to D \operatorname{qc}(Y)$ has a right adjoint.
4. If $f$ is schematic and $X, Y$ are semi-separated, the functor $\mathbb{R}f_* : D \operatorname{Qcoh}(X) \to D \operatorname{Qcoh}(Y)$ has a right adjoint.

In order to obtain all these results for schemes and morphisms of schemes (in Theorem 5.6), we shall prove:

**Theorem 5.4** Let $S$ be a quasi-compact and quasi-separated scheme and $\pi : S \to X$ a finite model. For any $\mathcal{M} \in D \operatorname{qc}(S)$, $\mathbb{R}\pi_* \mathcal{M}$ belongs to $D \operatorname{qc}(X)$ and the functors

$$
\mathbb{R}\pi_* : D \operatorname{qc}(S) \to D \operatorname{qc}(X), \quad \pi^* : D \operatorname{qc}(X) \to D \operatorname{qc}(S)
$$

are mutually inverse.

Finally, let us see how to obtain new results for general ringed spaces. The notion of affine ringed space was introduced in [12] (see also [14]). This notion includes homotopically trivial topological spaces on the one side and affine schemes on the other side. Here we introduce the notion of quasi-compact and quasi-separated ringed space (Definition 6.5). In the topological case we obtain topological spaces that admit a finite model, as finite simplical complexes and $h$-regular finite CW-complexes. In the context of schemes, we recover the notion of a quasi-compact and quasi-separated scheme. Using the results of [6] we shall obtain (Theorem 6.7) that the category $\operatorname{Qcoh}(S)$ of quasi-coherent modules on a quasi-compact and quasi-separated ringed space is a Grothendieck abelian category and admits flat covers and cotorsion envelopes.
Regarding Grothendieck duality, we introduce the notion of a semi-separable ringed space, which essentially means those ringed spaces that admit a semi-separated finite model and then we give a Grothendieck duality theorem for quasi-coherent modules for a morphism (with certain “schematic” conditions) between semi-separable ringed spaces; the precise statement is:

**Theorem 6.11.** Let \( f: T \to S \) be a morphism of ringed spaces. Assume that there exist an open covering \( U = \{ U_1, \ldots, U_n \} \) of \( T \) and an open covering \( V = \{ V_1, \ldots, V_m \} \) of \( S \) satisfying: (for each \( t \in T, s \in S \), we shall denote \( U^t := \bigcap_{i \in t} U_i, \ V^s := \bigcap_{j \in s} V_j \)

1. \( U^t \) (resp. \( V^s \)) is affine, for any \( t \in T \) (resp. \( s \in S \)).
2. For any \( U^t \supseteq U^{t'} \) (resp. \( V^s \supseteq V^{s'} \)), the morphisms
   \[ \mathcal{O}_T(U^t) \to \mathcal{O}_T(U^{t'}) \quad \text{(resp.} \mathcal{O}_S(V^s) \to \mathcal{O}_S(V^{s'})) \]
   are flat.
3. \( H^i(U^t \cap U^{t'}, \mathcal{O}_T) = 0 \) for any \( t, t' \in T \) and any \( i > 0 \) (and analogously for \( S, V \)).
4. \( \mathcal{O}_T(U^t \cap U^{t'}) \otimes_{\mathcal{O}_T(U^t)} \mathcal{O}_T(U^{t'}) \to \mathcal{O}_T(U^t \cap U^{t'}) \) is an isomorphism for any \( t, q \in T \) and any \( U^t \supseteq U^q \) (and analogously for \( S, V \)).
5. \( U^t \subseteq f^{-1}(V^{f(t)}) \) for any \( t \in T \).
6. For any \( U^t \supseteq U^{t'} \), any \( V^s \supseteq V^{s'} \) and any \( i \geq 0 \) the natural morphisms
   \[ H^i(U^t \cap f^{-1}(V^s), \mathcal{O}_T) \otimes_{\mathcal{O}_T(U^t)} \mathcal{O}_T(U^{t'}) \to H^i(U^{t'} \cap f^{-1}(V^{s'}), \mathcal{O}_T) \]
   \[ H^i(U^t \cap f^{-1}(V^s), \mathcal{O}_T) \otimes_{\mathcal{O}_S(V^{s'})} \mathcal{O}_S(V^{s'}) \to H^i(U^t \cap f^{-1}(V^{s'}), \mathcal{O}_T) \]
   are isomorphisms.

Then: the functor \( \mathbb{R}q_* f^{qc} : D \text{Qcoh}(T) \to D \text{Qcoh}(S) \) has a right adjoint, where \( f^{qc} : \text{Qcoh}(T) \to \text{Qcoh}(S) \) is the right adjoint of \( f^* : \text{Qcoh}(S) \to \text{Qcoh}(T) \), and \( \mathbb{R}q_* f^{qc} \) is its right derived functor.

This theorem may be understood as the essentialization of which properties of schemes are involved in order to have a Grothendieck duality theorem for quasi-coherent modules.

1. Basics

Let \( X \) be a finite topological space. It is well known, since Alexandroff, that the topology of \( X \) is equivalent to a preorder relation on \( X \): \( p \leq q \) iff \( \bar{p} \subseteq \bar{q} \), where \( \bar{p}, \bar{q} \) are the closures of \( p \) and \( q \). For each point \( p \in X \) we shall denote

\[ U_p = \text{smallest open subset containing } p. \]

In other words \( U_p = \{ q \in X : q \geq p \} \). Thus, \( p \leq q \Leftrightarrow U_p \supseteq U_q \). A map \( f: X \to Y \) between finite topological spaces is continuous if and only if it is monotone, i.e., \( p \leq q \) implies \( f(p) \leq f(q) \).

**Definition 1.1.** A finite ringed space is a ringed space \((X, \mathcal{O})\) whose underlying topological space \( X \) is finite. The sheaf of rings \( \mathcal{O} \) is always assumed to be a sheaf of commutative rings with unity.

A sheaf of rings \( \mathcal{O} \) on a finite topological space \( X \) is equivalent to the following data: a ring \( \mathcal{O}_p \) for each \( p \in X \) and a morphism of rings \( r_{pq} : \mathcal{O}_p \to \mathcal{O}_q \) for each \( p \leq q \), such that \( r_{pp} = \text{Id}_{\mathcal{O}_p} \) for any \( p \in X \) and \( r_{ql} \circ r_{pq} = r_{pl} \) for any \( p \leq q \leq l \). One has that

\[ \mathcal{O}_p = \text{stalk of } \mathcal{O} \text{ at } p = \mathcal{O}(U_p) \]
and \( r_{pq} \) is the restriction morphism \( \mathcal{O}(U_p) \to \mathcal{O}(U_q) \).

A sheaf \( M \) of \( \mathcal{O} \)-modules (or an \( \mathcal{O} \)-module) is equivalent to the following data: an \( \mathcal{O}_p \)-module \( M_p \) for each \( p \in X \) and a morphism of \( \mathcal{O}_p \)-modules \( r_{pq}: M_p \to M_q \) for each \( p \leq q \), such that \( r_{pp} = \text{Id}_{\mathcal{O}_p} \) for any \( p \in X \) and \( r_{q} \circ r_{pq} = r_{pq} \) for any \( p \leq q \leq l \). Again, one has that

\[
M_p = \text{stalk at } p \text{ of } M = M(U_p)
\]

and \( r_{pq} \) is the restriction morphism \( M(U_p) \to M(U_q) \). A morphism of \( \mathcal{O} \)-modules \( f: \mathcal{M} \to \mathcal{N} \) is equivalent to giving, for each \( p \in X \), a morphism of \( \mathcal{O}_p \)-modules \( f_p: \mathcal{M}_p \to \mathcal{N}_p \), which are compatible with the restriction morphisms \( r_{pq} \).

If \( \mathcal{M} \) is an \( \mathcal{O} \)-module, for each \( p \leq q \) the morphism \( r_{pq} \) induces a morphism of \( \mathcal{O}_q \)-modules \( \overline{r}_{pq}: \mathcal{M}_p \otimes_{\mathcal{O}_p} \mathcal{O}_q \to \mathcal{M}_q \). It is proved in [13] that \( \mathcal{M} \) is a quasi-coherent \( \mathcal{O} \)-module if and only if \( \overline{r}_{pq} \) is an isomorphism for any \( p \leq q \).

We shall denote by \( \text{Mod}(X) \) the category of \( \mathcal{O} \)-modules on a ringed space \((X, \mathcal{O})\) and by \( \text{Qcoh}(X) \) the subcategory of quasi-coherent modules. For any ring \( A \), \( \text{Mod}(A) \) denotes the category of \( A \)-modules.

**Example 1.2.** The topological space with one element shall be denoted by \{\( \ast \}\}. Thus, \((\ast, A)\) denotes the finite ringed space whose underlying topological space is \{\( \ast \}\} and the sheaf of rings is a ring \( \mathcal{O}_\ast = A \). For any ringed space \((X, \mathcal{O})\) there is a natural morphism of ringed spaces \((X, \mathcal{O}) \to (\ast, A)\), with \( A = \mathcal{O}(X) \).

**Definition 1.3.** A **finite space** is a finite ringed space \((X, \mathcal{O})\) whose restriction morphisms \( r_{pq}: \mathcal{O}_p \to \mathcal{O}_q \) are flat.

The main properties of the category \( \text{Qcoh}(X) \) over a finite space are:

1. \( \text{Qcoh}(X) \) is an abelian subcategory of \( \text{Mod}(X) \).
2. \( \text{Qcoh}(X) \) is a Grothendieck category (see [6]).

For any \( \mathcal{O}_p \)-module \( M \), we shall denote by \( \widetilde{M} \) the quasi-coherent module on \( U_p \) defined by \( \widetilde{M}_x = M \otimes_{\mathcal{O}_p} \mathcal{O}_x \). In other words, \( \widetilde{M} = \pi^* M \), where \( \pi: (U_p, \mathcal{O}_{U_p}) \to (\ast, \mathcal{O}_p) \) is the natural morphism of ringed spaces. The functors

\[
\text{Qcoh}(U_p) \to \text{Mod}(\mathcal{O}_p), \quad \text{Mod}(\mathcal{O}_p) \to \text{Qcoh}(U_p)
\]

\[
\mathcal{M} \leadsto \mathcal{M}_p, \quad \mathcal{M} \leadsto \widetilde{M}
\]

are mutually inverse.

**Definition 1.4.** A **schematic** finite space is a finite space \((X, \mathcal{O})\) such that \( R^i \delta_* \mathcal{O} \) is quasi-coherent for any \( i \geq 0 \), where \( \delta: X \to X \times X \) is the diagonal morphism. If in addition \( R^i \delta_* \mathcal{O} = 0 \) for any \( i > 0 \), then we say that \((X, \mathcal{O})\) is a semi-separated finite space.

**Definition 1.5.** A morphism \( f: X \to Y \) is said to be **schematic** if \( R^i \Gamma_* \mathcal{O}_X \) is quasi-coherent for any \( i \geq 0 \), where \( \Gamma: X \to X \times Y \) is the graphic of \( f \). For any schematic morphism \( f: X \to Y \), the spaces \( X \) and \( Y \) are always assumed to be finite spaces (Definition 1.3).

The following basic properties of schematic spaces, semi-separated spaces and schematic morphisms may be found in [12].

**Proposition 1.6.** (1) A morphism \( f: X \to Y \) is schematic if and only if: for any \( x \leq x' \in X \), any \( y \leq y' \in Y \), and any \( i \geq 0 \), the natural morphisms
Definition 2.1. We say that an ordered chain $x_0 < \cdots < x_i$ of points of $X$ belongs to an open subset $U$ (denoted by $(x_0 < \cdots < x_i) \in U$) if all the $x_i$ belong to $U$; since $U$ is open, it suffices that $x_0 \in U$.

Definition 2.2. The standard complex of $\mathcal{M}$ is the complex of $\mathcal{O}$-modules

$$
\mathcal{C}^\bullet \mathcal{M} := 0 \to \mathcal{C}^0 \mathcal{M} \to \mathcal{C}^1 \mathcal{M} \to \cdots \to \mathcal{C}^n \mathcal{M} \to 0 \quad (n = \dim X)
$$

defined as follows: for each open set $U$ of $X$,

$$(\mathcal{C}^i \mathcal{M})(U) = \prod_{(x_0 < \cdots < x_i) \in U} \mathcal{M}_{x_i}
$$

and the restriction morphisms $(\mathcal{C}^i \mathcal{M})(U) \to (\mathcal{C}^j \mathcal{M})(V)$ are the natural projections (the set of chains belonging to $U$ is the disjoint union of the set of chains belonging to $V$ and the set of chains belonging to $U$ but not belonging to $V$).
The differential \(d: \mathcal{C}^i \mathcal{M} \to \mathcal{C}^{i+1} \mathcal{M}\) is defined as follows: for each element \(s = (s_{x_0 < \cdots < x_i}) \in (\mathcal{C}^i \mathcal{M})(U)\), the element \(ds \in (\mathcal{C}^{i+1} \mathcal{M})(U)\) is given by the formula:

\[
(ds)_{x_0 < \cdots < x_{i+1}} = \sum_{k=0}^{i} (-1)^k s_{x_0 < \cdots < x_k < x_{i+1}} + (-1)^{i+1} s_{x_0 < \cdots < x_i} \in \mathcal{M}_{x_{i+1}},
\]

where the notation \(\widehat{x}_k\) means that we omit the element \(x_k\) and \(\overline{s}_{x_0 < \cdots < x_i}\) is the image of the element \(s_{x_0 < \cdots < x_i} \in \mathcal{M}_{x_i}\) by the restriction morphism \(\mathcal{M}_{x_i} \to \mathcal{M}_{x_{i+1}}\).

One easily checks that \(d \circ d = 0\). There is also a natural morphism \(\mathcal{M} \to \mathcal{C}^0 \mathcal{M}\), which is injective.

**Remark 2.3.** A morphism of modules \(\mathcal{M} \to \mathcal{M}'\) induces a morphism of modules \(\mathcal{C}^i \mathcal{M} \to \mathcal{C}^i \mathcal{M}'\) and then a morphism of complexes \(\mathcal{C}^* \mathcal{M} \to \mathcal{C}^* \mathcal{M}'\). It is clear that \(\mathcal{M} \rightsquigarrow \mathcal{C}^i \mathcal{M}\) is an exact functor.

**Remark 2.4.** Let us denote \(\beta^j X = \{(x_0, \ldots, x_i) \in X \times \cdots \times X : x_0 < \cdots < x_i\}\) with the discrete topology and let \(O_{\beta^j X}\) be the sheaf of rings on \(\beta^j X\) defined by

\[
(O_{\beta^j X})_{x_0 < \cdots < x_i} = O_{x_i}.
\]

We have two natural morphisms of ringed spaces \(\pi_0, \pi_i: (\beta^j X, O_{\beta^j X}) \to (X, O)\), defined as \(\pi_0(x_0 < \cdots < x_i) = x_0\) and \(\pi_i(x_0 < \cdots < x_i) = x_i\) (and the obvious morphisms between the sheaves of rings). Then

\[
\mathcal{C}^i \mathcal{M} = \pi_0^\ast (\pi_i^\ast \mathcal{M}).
\]

**Definition 2.5.** For each open subset \(U\) of \(X\), we shall denote

\[
\mathcal{M}_U := j_\ast \mathcal{M}|_U,
\]

where \(j: U \hookrightarrow X\) is the natural inclusion.

**Definition 2.6.** The pseudo-Cech complex of \(\mathcal{M}\) is the complex of \(\mathcal{O}\)-modules

\[
\check{\mathcal{C}}^* \mathcal{M} := 0 \to \check{\mathcal{C}}^0 \mathcal{M} \to \check{\mathcal{C}}^1 \mathcal{M} \to \cdots \to \check{\mathcal{C}}^n \mathcal{M} \to 0 \quad (n = \dim X)
\]

defined by

\[
\check{\mathcal{C}}^i \mathcal{M} = \prod_{x_0 < \cdots < x_i} \mathcal{M}_{U_{x_i}}
\]

and the differential \(\check{d}: \check{\mathcal{C}}^i \mathcal{M} \to \check{\mathcal{C}}^{i+1} \mathcal{M}\) is defined as follows: for each element \(s = (s_{x_0 < \cdots < x_i}) \in (\check{\mathcal{C}}^i \mathcal{M})(U)\), the element \(\check{ds} \in (\check{\mathcal{C}}^{i+1} \mathcal{M})(U)\) is given by the formula:

\[
(\check{ds})_{x_0 < \cdots < x_{i+1}} = \sum_{k=0}^{i} (-1)^k s_{x_0 < \cdots < x_k < x_{i+1}} + (-1)^{i+1} s_{x_0 < \cdots < x_i} \in \mathcal{M}_{U_{x_{i+1}}}(U),
\]

where the notation \(\check{x}_k\) means we omit the element \(x_k\) and \(\check{s}_{x_0 < \cdots < x_i}\) is the image of the element \(s_{x_0 < \cdots < x_i} \in \mathcal{M}_{U_{x_i}}(U)\) by the natural morphism \(\mathcal{M}_{U_{x_i}}(U) \to \mathcal{M}_{U_{x_{i+1}}}(U)\).

One easily checks that \(\check{d} \circ \check{d} = 0\). There is a natural morphism \(\mathcal{M} \to \check{\mathcal{C}}^0 \mathcal{M}\) which is injective. A morphism of modules \(\mathcal{M} \to \mathcal{M}'\) induces a morphism of modules \(\check{\mathcal{C}}^i \mathcal{M} \to \check{\mathcal{C}}^i \mathcal{M}'\) and then a morphism of complexes \(\check{\mathcal{C}}^* \mathcal{M} \to \check{\mathcal{C}}^* \mathcal{M}'\).
2.1. Quasicoherentation. Let \((X, \mathcal{O})\) be a finite space. The inclusion \(\text{Qcoh}(X) \hookrightarrow \text{Mod}(X)\) commutes with direct limits. Since \(\text{Qcoh}(X)\) is a Grothendieck category, it has a right adjoint \(\text{Qc}: \text{Mod}(X) \to \text{Qcoh}(X)\).

Remark 2.7. One easily checks that \(\text{Qc}\) is additive and left exact. Its restriction to quasi-coherent \(\mathcal{O}\)-modules is the identity.

The right derived functor \(\mathbb{R}\text{Qc}: D(X) \to D\text{Qcoh}(X)\) is a right adjoint of the natural functor \(D\text{Qcoh}(X) \to D(X)\).

Proposition 2.8. Let \((X, \mathcal{O})\) be a finite ringed space and let \(\mathcal{M}, \mathcal{N}\) be \(\mathcal{O}\)-modules.

1. For any \(i \geq 0\),
   \[
   \text{Hom}_\mathcal{O}(\mathcal{N}, \check{\mathcal{C}}^i \mathcal{M}) = \Gamma(X, \mathcal{C}^i \text{Hom}_\mathcal{O}(\mathcal{N}, \mathcal{M})).
   \]

2. For any \(i \geq 0\),
   \[
   \text{Hom}_\mathcal{O}(\mathcal{N}, \mathcal{C}^i \mathcal{M}) = \prod_{x_0 < \cdots < x_i} \text{Hom}_\mathcal{O}_{x_i}(\mathcal{N}_{x_0} \otimes \mathcal{O}_{x_0} \mathcal{O}_{x_i}, \mathcal{M}_{x_i}).
   \]

(2') If \(\mathcal{N}\) is quasi-coherent, then
   \[
   \text{Hom}_\mathcal{O}(\mathcal{N}, \mathcal{C}^i \mathcal{M}) = \prod_{x_0 < \cdots < x_i} \text{Hom}_\mathcal{O}_{x_i}(\mathcal{N}_{x_i}, \mathcal{M}_{x_i}).
   \]

3. If \((X, \mathcal{O})\) is schematic, then
   \[
   \text{Qc}(\mathcal{C}^i \mathcal{M}) = \prod_{x_0 < \cdots < x_i} j_* \overline{\mathcal{M}_{x_i}}
   \]
   where \(j: U_{x_i} \hookrightarrow X\) is the natural inclusion.

(3') If \(X\) is schematic and \(\mathcal{M}\) is quasi-coherent, then
   \[
   \text{Qc}(\mathcal{C}^i \mathcal{M}) = \check{\mathcal{C}}^i \mathcal{M}.
   \]

4. If \((X, \mathcal{O})\) is semi-separated, the functor
   \[
   \text{Qc} \circ \mathcal{C}^i: \text{Mod}(X) \to \text{Qcoh}(X)
   \]
   is exact.

5. If \(X\) is a finite space and \(\mathcal{M}\) is injective, then \(\mathcal{C}^i \mathcal{M}\) is also injective.

6. If \(X\) is semi-separated and \(\mathcal{M}\) is flat, then \(\text{Qc}(\mathcal{C}^i \mathcal{M})\) is also flat.

Proof. (1) follows from the definitions and the isomorphism
   \[
   \text{Hom}_\mathcal{O}(\mathcal{N}, \mathcal{M}_{U_x}) = \text{Hom}_\mathcal{O}_{U_x}(\mathcal{N}|_{U_x}, \mathcal{M}|_{U_x}) = \overline{\text{Hom}_\mathcal{O}(\mathcal{N}, \mathcal{M})}_x
   \]
   for any \(x \in X\).

(2) follows from the equality \(\mathcal{C}^i \mathcal{M} = \pi_{0*}(\pi_i^* \mathcal{M})\) (Remark 2.4), taking into account that \(\beta^i X\) is discrete. If \(\mathcal{N}\) is quasi-coherent, then \(\mathcal{N}_{x_0} \otimes \mathcal{O}_{x_0} \mathcal{O}_{x_i} = \mathcal{N}_{x_i}\) and we obtain (2'). If \(X\) is schematic and \(\mathcal{N}\) is quasi-coherent, then \(j_* \overline{\mathcal{M}_{x_i}}\) is quasi-coherent and \(\text{Hom}_\mathcal{O}_{x_i}(\mathcal{N}_{x_i}, \mathcal{M}_{x_i}) = \text{Hom}_\mathcal{O}(\mathcal{N}, j_* \overline{\mathcal{M}_{x_i}})\). Thus (3) follows from (2'). Moreover, if \(\mathcal{M}\) is quasi-coherent, then \(j_* \overline{\mathcal{M}_{x_i}} = \mathcal{M}_{U_{x_i}}\) and we obtain (3'). If \(X\) is semi-separated, then the functor \(\mathcal{M} \sim j_* \overline{\mathcal{M}_{x_i}}\) is exact. Thus (4) follows from (3).
(5) Let \( \mathcal{I} \) be an injective \( \mathcal{O} \)-module. In order to prove that \( \mathcal{C}^{\mathcal{I}} \) is injective, it suffices to see, by (2), that \( \mathcal{I}_x \) is an injective \( \mathcal{O}_x \)-module for any \( x \in X \) (notice that the morphisms \( \mathcal{O}_{x_0} \to \mathcal{O}_{x_1} \) are flat by hypothesis). For any \( \mathcal{O}_x \)-module \( N \), one has

\[
\text{Hom}_{\mathcal{O}_x}(N, \mathcal{I}_x) = \text{Hom}_{\mathcal{O}_{|U_x}}(\bar{N}, \mathcal{I}_{|U_x})
\]

and one concludes because \( \mathcal{I}_{|U_x} \) is an injective \( \mathcal{O}_{|U_x} \)-module.

(6) If \( \mathcal{M} \) is flat, then \( \mathcal{M}_x \) is a flat \( \mathcal{O}_x \)-module for any \( x \in X \) and then \( \tilde{\mathcal{M}}_x \) is a flat \( \mathcal{O}_{|U_x} \)-module. We conclude by (3) and the following

**Lemma 2.9.** Let \( X \) be a semi-separated finite space, \( j: U_x \hookrightarrow X \) the natural inclusion and \( \mathcal{P} \) a quasi-coherent and flat module on \( U_x \). Then \( j_* \mathcal{P} \) is flat.

**Proof.** For any \( p \in X \), one has that \( (j_* \mathcal{P})_p = \mathcal{P}(U_x \cap U_p) \). Since \( X \) is semi-separated, one has an isomorphism \( \mathcal{P}_x \otimes_{\mathcal{O}_x} \mathcal{O}(U_x \cap U_p) \cong \mathcal{P}(U_x \cap U_p) \) which is a flat \( \mathcal{O}_p \)-module because \( \mathcal{P}_x \) is a flat \( \mathcal{O}_x \)-module and \( \mathcal{O}_p \to \mathcal{O}(U_x \cap U_p) \) is flat.

\( \square \)

**Theorem 2.10.** Let \( (X, \mathcal{O}) \) be a finite ringed space, \( \mathcal{M} \) an \( \mathcal{O} \)-module.

1. \( \mathcal{C}^* \mathcal{M} \) is a finite and flasque resolution of \( \mathcal{M} \).
2. If \( (X, \mathcal{O}) \) is semi-separated and \( \mathcal{M} \) is quasi-coherent, then \( \mathcal{C}^* \mathcal{M} \) is a resolution of \( \mathcal{M} \) by acyclic quasi-coherent \( \mathcal{O} \)-modules.

**Proof.** 1. See [12, Theorem 2.15].

2. \( \mathcal{C}^* \mathcal{M} \) is acyclic, since \( \mathcal{M}_{|U_x} \) is acyclic for any \( x \in X \) because \( X \) is semi-separated. Let us see that \( \mathcal{C}^* \mathcal{M} \) is a resolution of \( \mathcal{M} \). For any open subset \( U \) of \( X \), let us denote \( \mathcal{O}^U \) the sheaf \( \mathcal{O} \) supported on \( U \). For any \( \mathcal{O} \)-module \( \mathcal{L} \), one has:

\[
\text{Hom}_{\mathcal{O}}(\mathcal{O}^U, \mathcal{L}) = \Gamma(U, \mathcal{L}), \quad \text{Hom}_{\mathcal{O}}(\mathcal{O}^U, \mathcal{L}) = L_U.
\]

Then

\[
(\mathcal{C}^* \mathcal{M})_x = \text{Hom}_{\mathcal{O}}(\mathcal{O}^U_x, \mathcal{C}^* \mathcal{M}) = \Gamma(X, \mathcal{C}^* (\text{Hom}_{\mathcal{O}}(\mathcal{O}^U_x, \mathcal{M}))) = \Gamma(X, \mathcal{C}^* (\mathcal{M}_{|U_x})),
\]

where the second equality is due to Proposition [2.8] (4). Thus, \( H^i[(\mathcal{C}^* \mathcal{M})_x] = H^i(X, \mathcal{M}_{|U_x}) \). Since \( X \) is semi-separated, \( H^i(X, \mathcal{M}_{|U_x}) = 0 \) for \( i > 0 \) and we are done.

\( \square \)

**Remark 2.11.** If \( \mathcal{M} \) is a complex of \( \mathcal{O} \)-modules, then \( \mathcal{C}^* \mathcal{M} \) denotes the simple (or total) complex associated to the bicomplex \( \mathcal{C}^p \mathcal{M}^q \). Analogously, \( \mathcal{C}^* \mathcal{M} \) denotes the simple complex associated to the bicomplex \( \mathcal{C}^p \mathcal{M}^q \). Taking into account the boundedness of the complexes \( C^* \) and \( \mathcal{C}^* \), Theorem [2.10] yields that \( \mathcal{M} \to \mathcal{C}^* \mathcal{M} \) is a quasi-isomorphism and so is \( \mathcal{M} \to \mathcal{C}^* \mathcal{M} \) if \( X \) is semi-separated and \( \mathcal{M} \) is a complex of quasi-coherent modules.

### 3. Semi-separated finite spaces

Let \( (X, \mathcal{O}) \) be a finite space. We shall denote by \( D(X) \) the derived category of complexes of \( \mathcal{O} \)-modules and by \( D \text{Qcoh}(X) \) the derived category of complexes of quasi-coherent \( \mathcal{O} \)-modules. We shall denote by \( D_{qc}(X) \) the full subcategory of \( D(X) \) whose objects are the complexes of \( \mathcal{O} \)-modules with quasi-coherent cohomology. The objects of these categories have a very simple description:
A complex \( M \) of \( \mathcal{O} \)-modules is the same as giving: a complex \( M_x \) of \( \mathcal{O}_x \)-modules, for each \( x \in X \), and a morphism of complexes of \( \mathcal{O}_x \)-modules

\[
r_{xx'} : M_x \to M_{x'}
\]

for each \( x \leq x' \), such that \( r_{xx} = 1d \) for any \( x \) and \( r_{xx''} = r_{x'x''} \circ r_{xx'} \) for any \( x \leq x' \leq x'' \). If we denote

\[
\tilde{r}_{xx'} : M_x \otimes \mathcal{O}_x \mathcal{O}_{x'} \to M_{x'}
\]

the morphism of \( \mathcal{O}_{x'} \)-modules induced by \( r_{xx'} \) then:

(1) \( M \) is a complex of quasi-coherent modules if and only if \( \tilde{r}_{xx'} \) is an isomorphism for any \( x \leq x' \).

(2) \( M \) is a complex with quasi-coherent cohomology if and only if \( \tilde{r}_{xx'} \) is a quasi-isomorphism for any \( x \leq x' \).

**Theorem 3.1.** Let \((X, \mathcal{O})\) be a semi-separated finite space. Then \( \mathcal{C}^i M \) is \( \mathcal{Qc} \)-acyclic, for any \( \mathcal{O} \)-module \( M \) and any \( i \geq 0 \). Consequently,

(1) \( R^i \mathcal{Qc}(M) = 0 \) for any \( i > \dim X \) and any \( \mathcal{O} \)-module \( M \).

(2) For any complex \( M \) of \( \mathcal{O} \)-modules

\[
\mathbb{R} \mathcal{Qc}(M) \simeq \mathcal{Qc}(\mathcal{C}^\bullet M).
\]

In particular, any complex \( M \) of quasi-coherent modules is \( \mathcal{Qc} \)-acyclic and \( \mathcal{M} \simeq \mathbb{R} \mathcal{Qc}(\mathcal{M}) \).

**Proof.** Let \( M \to I^\bullet \) be an injective resolution. By Proposition 2.8 (4), \( \mathcal{Qc}(\mathcal{C}^\bullet M) \to \mathcal{Qc}(\mathcal{C}^i I^\bullet) \) is still a resolution. We conclude because \( \mathcal{C}^i I^\bullet \) is a resolution of \( \mathcal{C}^i M \) (\( \mathcal{C}^i \) is exact) by injectives (Proposition 2.8 (5)).

If \( M \) is a complex of quasi-coherent modules, then \( \mathcal{Qc}(\mathcal{C}^\bullet M) \overset{2.8 (3')}{=} \check{\mathcal{C}}^\bullet M \overset{2.10}{\simeq} M \).

□

**Theorem 3.2** (Bokstedt-Neeman Theorem for semi-separated finite spaces). Let \((X, \mathcal{O})\) be a semi-separated finite space. The functor \( D \mathcal{Qcoh}(X) \to D_{\mathcal{Qc}}(X) \) is an equivalence.

**Proof.** By Theorem 3.1 \( \mathbb{R} \mathcal{Qc}(M) \simeq \mathcal{Qc}(\mathcal{C}^\bullet M) \). The key point is to prove that the natural morphism

\[
\mathcal{Qc}(\mathcal{C}^\bullet M) \to \mathcal{C}^\bullet M
\]

is a quasi-isomorphism for any complex \( M \) with quasi-coherent cohomology. It suffices to prove that

\[
H^i_{d_{\mathcal{M}}}([\mathcal{Qc}(\mathcal{C}^\bullet M)]) \to H^i_{d_{\mathcal{M}}}([\mathcal{C}^\bullet M])
\]

is a quasi-isomorphism, where \( d_{\mathcal{M}} : \mathcal{C}^p M^q \to \mathcal{C}^p M^{q+1} \) is the “vertical” differential of the bicomplex \( \mathcal{C}^p M^q \) (and analogously for the bicomplex \( \mathcal{Qc}(\mathcal{C}^p M^q) \)). Since \( \mathcal{Qc} \circ \mathcal{C}^\bullet \) and \( \mathcal{C}^\bullet \) are exact, this amounts to prove that

\[
\mathcal{Qc}(\mathcal{C}^i H^i(\mathcal{M})) \to \mathcal{C}^i H^i(\mathcal{M})
\]

is a quasi-isomorphism; that is, we may assume that \( M \) is a quasi-coherent module. In this case, \( \mathcal{Qc}(\mathcal{C}^\bullet M) = \check{\mathcal{C}}^\bullet M \) by Proposition 2.8 (3'). We conclude because \( M \to \check{\mathcal{C}}^\bullet M \) and \( M \to \mathcal{C}^\bullet M \) are quasi-isomorphisms (Theorem 2.10).

Now let us conclude the proof of the theorem. If \( M \in D_{\mathcal{Qc}}(X) \), the quasi-isomorphism \( \mathcal{Qc}(\mathcal{C}^\bullet M) \to \mathcal{C}^\bullet M \) gives an isomorphism in \( D_{\mathcal{Qc}}(X) \), \( \mathbb{R} \mathcal{Qc}(M) \simeq M \), i.e., the composition
$D_{qc}(X) \xrightarrow{\mathbb{R}^q c} D\text{Qcoh}(X) \to D_{qc}(X)$ is isomorphic to the identity. If $\mathcal{M} \in D\text{Qcoh}(X)$, the natural morphism $\mathcal{M} = Qc(\mathcal{M}) \to Qc(\mathcal{C}^\bullet \mathcal{M})$ is a quasi-isomorphism, because its composition with $Qc(\mathcal{C}^\bullet \mathcal{M}) \to \mathcal{C}^\bullet \mathcal{M}$ is a quasi-isomorphism. Thus, we have obtained an isomorphism $\mathcal{M} \simeq \mathbb{R}^q c(\mathcal{M})$ in $D\text{Qcoh}(X)$, so the composition $D\text{Qcoh}(X) \to D_{qc}(X) \to D\text{Qcoh}(X)$ is isomorphic to the identity. □

**Corollary 3.3.** Let $X$ be a schematic finite space. The functor $D_{qc}(X) \rightarrow D(X)$ has a right adjoint.

**Proof.** We have to prove:

(*) for any $N \in D(X)$ there exists an object $N_{qc} \in D_{qc}$ and a morphism $N_{qc} \to N$ such that $\text{Hom}_{D(X)}(\mathcal{M}, N_{qc}) \to \text{Hom}_{D(X)}(\mathcal{M}, N)$ is an isomorphism for any $\mathcal{M} \in D_{qc}(X)$.

We proceed by induction on $\#X =$ number of elements of $X$. If $\#X = 1$, then $X$ is semi-separated and then $\mathbb{R}^q c: D(X) \to D\text{Qcoh}(X)$ is a right adjoint. Now let $\#X$ be greater than 1. If $X$ has a minimum, $X = U$, then $X$ is semi-separated and we conclude as before. If $X$ has not a minimum, then $X = U \cup V$, with $U,V$ open subsets different from $X$. Let us denote $i: U \hookrightarrow X$, $j: V \hookrightarrow X$ and $h: U \cap V \hookrightarrow X$ the inclusions. By induction, there exist $(\mathcal{N}_i(U))_{qc} \in D_{qc}(U)$, $(\mathcal{N}_j(V))_{qc} \in D_{qc}(V)$, $(\mathcal{N}_i(U \cap V))_{qc} \in D_{qc}(U \cap V)$ and morphisms $\alpha: (\mathcal{N}_i(U))_{qc} \to \mathcal{N}_i(U)$, $\beta: (\mathcal{N}_j(V))_{qc} \to \mathcal{N}_j(V)$ and $\gamma: (\mathcal{N}_i(U \cap V))_{qc} \to \mathcal{N}_i(U \cap V)$ satisfying (*). Then $\alpha_{\cup V}$ and $\beta_{\cup V}$ factor through $\gamma$. Hence we obtain morphisms

$$\mathbb{R}i_*(\mathcal{N}_i(U))_{qc} \xrightarrow{\phi} \mathbb{R}h_*(\mathcal{N}_i(U \cap V))_{qc}, \quad \mathbb{R}j_*(\mathcal{N}_j(V))_{qc} \xrightarrow{\psi} \mathbb{R}h_*(\mathcal{N}_j(U \cap V))_{qc}$$

and commutative diagrams

$$\begin{array}{ccc}
\mathbb{R}i_* & \downarrow & \mathbb{R}h_* \\
\mathbb{R}i_*\mathcal{N}_i(U) & \to & \mathbb{R}h_*\mathcal{N}_i(U \cap V) \\
\downarrow & & \downarrow \\
\mathbb{R}j_* & \downarrow & \mathbb{R}h_* \\
\mathbb{R}j_*\mathcal{N}_j(V) & \to & \mathbb{R}h_*\mathcal{N}_j(U \cap V) \\
\downarrow & & \downarrow \\
\mathbb{R}i_* & \downarrow & \mathbb{R}h_* \\
\mathbb{R}i_*\mathcal{N}_i(U) \oplus \mathbb{R}j_*\mathcal{N}_j(V) & \to & \mathbb{R}h_*\mathcal{N}_i(U \cap V) \\
\downarrow & & \downarrow \\
\mathbb{R}i_*\mathcal{N}_i(U) \oplus \mathbb{R}j_*\mathcal{N}_j(V) & \xrightarrow{\phi - \psi} & \mathbb{R}h_*\mathcal{N}_i(U \cap V)
\end{array}$$

Let us consider the triangle $N \to \mathbb{R}i_*\mathcal{N}_i(U) \oplus \mathbb{R}j_*\mathcal{N}_j(V) \to \mathbb{R}h_*\mathcal{N}_i(U \cap V)$ and the commutative diagram

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\mathbb{R}i_*} & \mathbb{R}i_*\mathcal{N}_i(U) \oplus \mathbb{R}j_*\mathcal{N}_j(V) \\
\downarrow & & \downarrow \\
\mathbb{R}i_*\mathcal{N}_i(U) \oplus \mathbb{R}j_*\mathcal{N}_j(V) & \xrightarrow{\phi - \psi} & \mathbb{R}h_*\mathcal{N}_i(U \cap V) \\
\downarrow & & \downarrow \\
\mathbb{R}h_*\mathcal{N}_i(U \cap V) & \to & \mathbb{R}h_*\mathcal{N}_i(U \cap V)
\end{array}$$

Let us define $\mathcal{N}_{qc} := \text{Cone}(\phi - \psi)[-1]$ and let $\mathcal{N}_{qc} \to N$ be a morphism completing the above diagram to a morphism of exact triangles. This morphism satisfies (*). Indeed, for any $\mathcal{M} \in D_{qc}(X)$ one has that

$\text{Hom}(\mathcal{M}, \mathbb{R}i_*\mathcal{N}_i(U)) = \text{Hom}(\mathcal{M}|_U, \mathcal{N}_i(U)) = \text{Hom}(\mathcal{M}|_U, (\mathcal{N}_i(U))_{qc}) = \text{Hom}(\mathcal{M}, \mathbb{R}i_*(\mathcal{N}_i(U))_{qc})$

and analogously for $V$ and $U \cap V$. One concludes by applying $\text{Hom}(\mathcal{M}, \quad)$ to the morphism of exact triangles. □

**Remark 3.4.** There are non-schematic finite spaces where Bokstedt-Neeman theorem holds. For example, for any affine finite space $X$, one has an equivalence $D_{qc}(X) \simeq D\text{Qcoh}(X)$ and they
are both equivalent to $D(A)$, with $A = \mathcal{O}(X)$ (see [12, Proposition 3.17]). Thus, for any finite space $X$, Bokstedt-Neeman holds locally: $D_{qc}(U_x) \simeq D\text{Qcoh}(U_x) \simeq D(\mathcal{O}_x)$ for any $x \in X$.

Let $f : X \to Y$ be a schematic morphism between finite spaces. Since $\text{Qcoh}(X)$ is a Grothendieck abelian category, the functor $f_* : \text{Qcoh}(X) \to \text{Qcoh}(Y)$ has a right derived functor

$$R_{qc}f_* : D\text{Qcoh}(X) \to D\text{Qcoh}(Y).$$

**Theorem 3.5.** Let $f : X \to Y$ be a schematic morphism between semi-separated finite spaces. The diagram

$$
\begin{array}{ccc}
D\text{Qcoh}(X) & \xrightarrow{R_{qc}f_*} & D\text{Qcoh}(Y) \\
\downarrow & & \downarrow \\
D(X) & \xrightarrow{Rf_*} & D(Y)
\end{array}
$$

is commutative.

**Proof.** The proof consists on proving that the pseudo-Cech resolution $\mathcal{M} \to \check{\mathcal{C}}^\bullet \mathcal{M}$ allows us to derive both functors, and this reduces to prove that for any $x \in X$ and any quasi-coherent module $\mathcal{M}$ on $X$ one has:

1. $Rf_*\mathcal{M}_{U_x} = f_*\mathcal{M}_{U_x}$.
2. $R_{qc}f_*\mathcal{M}_{U_x} = f_*\mathcal{M}_{U_x}$.

Let us consider the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow j \\
U_x & \xrightarrow{\bar{f}} & U_{f(x)}
\end{array}
$$

Let us prove (1). One has: $\mathcal{M}_{U_x} = i_*\mathcal{M}_{|U_x} = Rf_*i_*\mathcal{M}_{|U_x}$, because $\mathcal{M}_{|U_x}$ is quasicoherent and $X$ is semi-separated. Thus:

$$Rf_*\mathcal{M}_{U_x} = f_*\mathcal{M}_{U_x} = \mathcal{Rj}_*(R\bar{f}_*\mathcal{M}_{|U_x}).$$

Now, $\bar{f}_*\mathcal{M}_{|U_x} = \mathcal{R}\bar{f}_*\mathcal{M}_{|U_x}$ because $f$ is schematic (see Proposition [16]) and $j_\bar{f}_*\mathcal{M}_{|U_x} = \mathcal{R}\bar{j}_*\mathcal{M}_{|U_x}$ because $Y$ is semi-separated. Hence, $Rf_*\mathcal{M}_{U_x} = j_*\bar{f}_*\mathcal{M}_{U_x} = f_*\mathcal{M}_{U_x}$.

Now let us prove (2). Let $\mathcal{I}^\bullet$ be a resolution of $\mathcal{M}_{|U_x}$ by injective quasi-coherent modules. Since $i_* : \text{Qcoh}(U_x) \to \text{Qcoh}(X)$ is exact and takes injectives into injectives, we have that $i_*\mathcal{I}^\bullet$ is a resolution of $\mathcal{M}_{U_x}$ by quasi-coherent injective $\mathcal{O}_X$-modules. Then

$$R_{qc}f_*\mathcal{M}_{U_x} = f_*i_*\mathcal{I}^\bullet = j_*\bar{f}_*\mathcal{I}^\bullet.$$

Now, $j_*\bar{f}_*\mathcal{I}^\bullet$ is a resolution of $j_*\bar{f}_*\mathcal{M}_{|U_x} = f_*\mathcal{M}_{U_x}$, because $\bar{f}_* : \text{Qcoh}(U_x) \to \text{Qcoh}(U_{f(x)})$ and $j_* : \text{Qcoh}(U_{f(x)}) \to \text{Qcoh}(Y)$ are exact functors. □

**Theorem 3.6.** The category of quasi-coherent modules on a semi-separated finite space has enough flats.
Proof. Let $\mathcal{M}$ be a quasi-coherent module on a semi-separated finite space $(X, \mathcal{O})$. Let
\[ \mathcal{P} = \cdots \to \mathcal{P}^{-1} \to \mathcal{P}^{0} \to 0 \]
be a resolution of $\mathcal{M}$ by flat (non quasi-coherent) modules. Let $Q := \text{Qc}(C^* \mathcal{P})$. By Proposition 2.8 (6), $Q$ is a bounded above complex of flat quasi-coherent modules. Moreover, $Q \simeq R \text{Qc}(\mathcal{P}) \simeq R \text{Qc}(\mathcal{M}) \simeq \mathcal{M}$, hence $H^i(Q) = 0$ for $i \neq 0$ and $H^0(Q) = \mathcal{M}$. Thus,
\[ \tau_{\leq 0} Q = \cdots \to Q^{-2} \to Q^{-1} \to Z^0 \to 0 \]
with $Z^0$ the 0-cycles of $Q$, is a resolution of $\mathcal{M}$ by quasi-coherent modules and we conclude if we prove that $Z^0$ is flat. This follows from the fact that
\[ 0 \to Z^0 \to Q^0 \to Q^1 \to \cdots \to Q^n \to 0 \]
is an exact sequence and $Q^i$ are flat.

\section{Grothendieck duality}

**Theorem 4.1.** Let $f : X \to Y$ be a morphism between finite ringed spaces. The functor $R f_* : D(X) \to D(Y)$ has a right adjoint $f^! : D(Y) \to D(X)$.

Proof. Let $n = \dim X$. For each $0 \leq p \leq n$ the functor $f_* C^p : \text{Mod}(X) \to \text{Mod}(Y)$ is exact and commutes with filtered direct limits. Thus, for each $N \in \text{Mod}(Y)$ the functor
\[ \text{Mod}(X) \to \text{Abelian groups} \]
\[ \mathcal{M} \rightsquigarrow \text{Hom}(f_* C^p(\mathcal{M}), N) \]
is representable by an $\mathcal{O}_X$-module $f^{-p} N$. A morphism of $\mathcal{O}_Y$-modules $N \to N'$ induces a morphism of $\mathcal{O}_X$-modules $f^{-p} N \to f^{-p} N'$. Moreover, the natural morphism $f_* C^p \to f_* C^{p+1}$ induces a morphism $f^{-p} N \to f^{-p} N$. Thus, if $N$ is a complex of $\mathcal{O}_Y$-modules, we obtain a bicomplex $f^{-p} N'$ (which is zero whenever $p < 0$ or $p > n$) whose associated simple complex shall be denoted by $f^! N$. For any complex $\mathcal{M}$ of $\mathcal{O}_X$-modules, the isomorphism
\[ \text{Hom}_{\mathcal{O}_Y}(f_* C^p(\mathcal{M}^i), N^q) = \text{Hom}_{\mathcal{O}_X}(\mathcal{M}^i, f^{-p} N^q) \]
extends to a complex isomorphism
\[ \text{Hom}^*(f_* C^*(\mathcal{M}), N) = \text{Hom}^*(\mathcal{M}, f^! N). \]
Thus, if $N$ is $K$-injective, $f^! N$ is $K$-injective too. For any $N \in D(Y)$, we define $f^! N := f^! \mathcal{I}$, with $\mathcal{N} \to \mathcal{I}$ a $K$-injective resolution, and one has
\[ \text{Hom}^*(f_* C^*(\mathcal{M}), \mathcal{I}) = \text{Hom}^*(\mathcal{M}, f^! \mathcal{I}) \]
and then
\[ R \text{Hom}^*(R f_*(\mathcal{M}), N) = R \text{Hom}^*(\mathcal{M}, f^! N). \]

\section{Theorem 4.2.} Let $f : X \to Y$ be a morphism between finite spaces (Definition 1.3). One has:

1. The functor $R f_* : D \text{Qcoh}(X) \to D(Y)$, composition of $D \text{Qcoh}(X) \to D(X)$ and $D(X) \xrightarrow{R f_*} D(Y)$, has a right adjoint.
2. If $f$ is schematic, the functor $R f_* : D \text{Qcoh}(X) \to D_{qc}(Y)$ has a right adjoint.
3. If $X$ and $f$ are schematic, the functor $R f_* : D_{qc}(X) \to D_{qc}(Y)$ has a right adjoint.
If \( f \) is schematic and \( X, Y \) are semi-separated, the functor \( \mathbb{R}f_* : D\text{Qcoh}(X) \to D\text{Qcoh}(Y) \) has a right adjoint.

Proof. (1) follows from Theorem 4.1 and from the fact that \( D\text{Qcoh}(X) \to D(X) \) has a right adjoint (in fact, \( \mathbb{R}\text{Qc} \)).

(2) Since \( f \) is schematic, \( \mathbb{R}f_* \) maps \( D\text{qc}(X) \) into \( D\text{qc}(Y) \). We conclude by (1).

(3) follows from Theorem 4.1 and Corollary 3.3.

(4) follows from (3) and Theorems 3.2, 3.5.

\[ \square \]

5. Schemes

Let \( S \) be a quasi-compact and quasi-separated scheme. It is proved in [13] that there exists a schematic finite space \((X, \mathcal{O})\) and a morphism of ringed spaces \( \pi : (S, \mathcal{O}_S) \to (X, \mathcal{O}) \) such that, for any \( x \in X \), the preimage \( U_x := \pi^{-1}(U_x) \) is an affine scheme. Moreover, \( S \) is a semi-separated scheme if and only if \( (X, \mathcal{O}) \) is a semi-separated finite space. We say that \( S \to X \) is a finite model of \( S \). If \( f : T \to S \) is a morphism of schemes between quasi-compact and quasi-separated schemes, one can find finite models \( X \) and \( Y \) of \( T \) and \( S \) and a (schematic) morphism \( \bar{f} : X \to Y \) making the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
X & \xrightarrow{\bar{f}} & Y
\end{array}
\]

commutative, and we say that \( \bar{f} \) is a finite model of \( f \).

**Theorem 5.1.** ([13] Thm. 3.15) Let \( S \) be a quasi-compact and quasi-separated scheme and let \( \pi : S \to X \) be a finite model. The functors

\[
\pi_* : \text{Qcoh}(S) \to \text{Qcoh}(X), \quad \pi^* : \text{Qcoh}(X) \to \text{Qcoh}(S)
\]

are exact and mutually inverse. In particular, we obtain an equivalence \( D\text{Qcoh}(S) \simeq D\text{Qcoh}(X) \).

Let us see now that the same happens with the derived categories of complexes with quasi-coherent cohomology. We shall need the following:

**Theorem 5.2.** ([5] Thm. 5.1) Let \( S = \text{Spec } A \) be an affine scheme and \( \pi : \text{Spec } A \to (*, A) \) the natural morphism. The functor

\[
\pi^* : D(A) \to D_{qc}(S)
\]

is an equivalence.

**Remark 5.3.**

1. Since \( \mathbb{R}\pi_* : D_{qc}(S) \to D(A) \) is a right adjoint of \( \pi^* \), it is its inverse. Thus the natural morphisms \( M \to \mathbb{R}\pi_* \pi^* M \) and \( \pi^* \mathbb{R}\pi_* M \to M \) are isomorphisms, for any \( M \in D(A) \), \( \mathcal{M} \in D_{qc}(S) \). Notice also that \( \mathbb{R}\pi_* = \mathbb{R}\Gamma(S, \mathcal{M}) \).

2. If \( S' = \text{Spec } A' \) is an affine open subscheme of \( S \), then \( j^* \pi^* M \simeq \pi'^*(M \otimes_A A') \), where \( j : S' \hookrightarrow S \) is the natural immersion and \( \pi' : S' \to (*, A') \) is the natural morphism. It follows that the natural morphism

\[
\mathbb{R}\Gamma(S, \mathcal{M}) \otimes_A A' \to \mathbb{R}\Gamma(S', \mathcal{M})
\]

is an isomorphism.
Theorem 5.4. Let $S$ be a quasi-compact and quasi-separated scheme and $\pi: S \to X$ a finite model. For any $\mathcal{M} \in D_{qc}(S)$, $\mathbb{R}\pi_*\mathcal{M}$ belongs to $D_{qc}(X)$ and the functors

$$\mathbb{R}\pi_*: D_{qc}(S) \to D_{qc}(X), \quad \pi^*: D_{qc}(X) \to D_{qc}(S)$$

are mutually inverse.

Proof. Let $\mathcal{M} \in D_{qc}(S)$. In order to prove that $\mathbb{R}\pi_*\mathcal{M}$ has quasi-coherent cohomology, we have to see that for any $x \leq x'$ the natural morphism

$$(\mathbb{R}\pi_*\mathcal{M})_x \otimes_{O_x} O_{x'} \to (\mathbb{R}\pi_*\mathcal{M})_{x'}$$

is an isomorphism. Since $(\mathbb{R}\pi_*\mathcal{M})_x = \mathbb{R}\Gamma(U^x, \mathcal{M})$, this follows from Remark 5.3. Now, for any $x \in X$, the functors

$$\mathbb{R}\pi_*: D_{qc}(U^x) \to D_{qc}(U^x), \quad \pi^*: D_{qc}(U_x) \to D_{qc}(U^x)$$

are mutually inverse, since both categories are equivalent to $D(O_x)$ by Theorems 5.2 and Remark 3.4. For any $\mathcal{M} \in D_{qc}(S)$, the natural morphism $\pi^* R\pi_* \mathcal{M} \to \mathcal{M}$ is an isomorphism, because it is so after restricting to each $U^x$. For any $\mathcal{N} \in D_{qc}(X)$ the natural morphism $\mathcal{N} \to R\pi_* R\pi_* \mathcal{N}$ is an isomorphism because it is so after restricting to each $U_x$. □

Remark 5.5. (1) Let $S$ be a quasi-compact and quasi-separated scheme and $\pi: S \to X$ a finite model. The diagram

$$D_{Qcoh}(S) \longrightarrow D_{qc}(S) \quad \bigg\downarrow \pi_* \quad \bigg\downarrow \mathbb{R}\pi_*$$

$$D_{Qcoh}(X) \longrightarrow D_{qc}(X)$$

(whose vertical morphisms are isomorphisms by Theorems 5.1 and 5.4) is commutative. Indeed, let us denote $i: D_{Qcoh}(S) \to D_{qc}(S)$ and $j: D_{Qcoh}(X) \to D_{qc}(X)$ the natural functors. For any $\mathcal{N} \in D_{Qcoh}(X)$, one has $\pi^* j(\mathcal{N}) = i(\pi^* \mathcal{N})$. One concludes because $\pi^*$ is the inverse of $\pi_*$ and $\mathbb{R}\pi_*$.  

(2) Let $f: T \to S$ be a morphism of schemes between quasi-compact and quasi-separated schemes and let $\begin{array}{ccc} T & \longrightarrow & S \\ p & \bigg\downarrow & q \\ X & \longrightarrow & Y \end{array}$ be a finite model.

The diagram (whose vertical morphisms are isomorphisms by Theorem 5.4)

$$D_{qc}(T) \longrightarrow D_{qc}(S) \quad \bigg\downarrow Rf_* \quad \bigg\downarrow \mathbb{R}f_*$$

$$D_{qc}(X) \longrightarrow D_{qc}(Y)$$

is commutative. This is clear: $\mathbb{R}q_* \circ \mathbb{R}f_* = \mathbb{R}(q \circ f)_* = \mathbb{R}(f \circ p)_* = \mathbb{R}f_* \circ \mathbb{R}p_*$.  

Now, Theorems 5.1, 5.4 and Remark 5.5 allow to transport the results obtained on finite spaces (Theorems 3.2, 3.3, 3.6 and 4.2) to schemes:
Theorem 5.6. Let $S$ be a quasi-compact and semi-separated scheme. Then
(1.a) The natural functor $D \text{Qcoh}(S) \to D_{qc}(S)$ is an equivalence ([1]).
(1.b) The category $\text{Qcoh}(S)$ has enough flats ([9]).

Let $f: T \to S$ be a morphism of schemes between quasi-compact and quasi-separated schemes. Then
(2.a) The functor $Rf_*: D_{qc}(T) \to D_{qc}(S)$ has a right adjoint ([11],[7]).
(2.b) If $T,S$ are semi-separated, then the diagram
\[
\begin{array}{ccc}
D \text{Qcoh}(T) & \xrightarrow{Rf_*} & D \text{Qcoh}(S) \\
\downarrow & & \downarrow \\
D(T) & \xrightarrow{Rf_*} & D(S)
\end{array}
\]
is commutative ([7]) and the functor $R_{qc}f_*: D \text{Qcoh}(T) \to D \text{Qcoh}(S)$ has a right adjoint ([11]).

6. Ringed Spaces

Let us first recall the notion of an affine ringed space introduced in [12] (see also [14]).

Definition 6.1. Let $(S, \mathcal{O}_S)$ be a ringed space and $A = \mathcal{O}_S(S)$. We say that $(S, \mathcal{O}_S)$ is an affine ringed space if:
(1) It is acyclic: $H^i(S, \mathcal{O}_S) = 0$ for any $i > 0$.
(2) The functor $\text{Qcoh}(S) \to \{A-\text{modules}\}$
$\mathcal{M} \sim \Gamma(S, \mathcal{M})$
is an equivalence.

In the topological case (i.e., $\mathcal{O}_S = \mathbb{Z}$), $(S,\mathbb{Z})$ is an affine ringed space if and only if $S$ is homotopically trivial (where $S$ is assumed to be path-connected, locally path-connected, and locally simply connected). If $(S, \mathcal{O}_S)$ is a scheme, then $(S, \mathcal{O}_S)$ is an affine ringed space if and only if $S$ is an affine scheme, i.e., $S = \text{Spec} A$. See [14] for the proofs.

Definition 6.2. (See [13 Section. 2.2]) Let $S$ be a topological space and let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a finite open covering. For each $s \in S$, let us denote
$U^s = \bigcap_{s \in U_i}$.
Let $\sim$ be the equivalence relation on $S$ defined as: $s \sim s' \iff U^s = U^{s'}$, and let $X = S/\sim$ the (finite) quotient set, with the topology associated to the partial order: $[s] \leq [s']$ iff $U^s \supseteq U^{s'}$. The quotient map $\pi: X \to S$ is continuous and $\pi^{-1}(U_x) = U^s$ for any $s \in S$, with $x = \pi(s)$. We say that $X$ is the finite topological space associated to $\mathcal{U}$. If $\mathcal{O}_S$ is a sheaf of rings on $S$, then $\mathcal{O} := \pi_*\mathcal{O}_S$ is a sheaf of rings on $X$ and $(S,\mathcal{O}_S) \to (X,\mathcal{O})$ is a morphism of ringed spaces. We say that $(X,\mathcal{O})$ is the finite ringed space associated to the covering $\mathcal{U}$ of $(S,\mathcal{O}_S)$.

Definition 6.3. Let $(S, \mathcal{O}_S)$ be ringed space and $\mathcal{U}$ a finite open covering. We say that $\mathcal{U}$ is locally affine if $U^s$ is an affine ringed space for any $s \in S$. This is equivalent to say that the associated map $\pi: S \to X$ is “affine”: $\pi^{-1}(U_x)$ is affine for any $x \in X$. In this case we say that $(X,\mathcal{O})$ is a finite model of $(S,\mathcal{O}_S)$. 
Remark 6.4. It is proved in [14] that if \( \mathcal{U} \) is a locally affine finite covering of \((S, \mathcal{O}_S)\) and \( \pi: (S, \mathcal{O}_S) \rightarrow (X, \mathcal{O}) \) is the associated finite ringed space, then the direct image \( \pi_* \) takes quasi-coherent modules on \( S \) into quasi-coherent modules on \( X \) and the functors
\[
\pi_*: \text{Qcoh}(S) \rightarrow \text{Qcoh}(X), \quad \pi^*: \text{Qcoh}(X) \rightarrow \text{Qcoh}(S)
\]
are mutually inverse.

Definition 6.5. We say that a ringed space \((S, \mathcal{O}_S)\) is quasi-compact and quasi-separated if it admits a locally affine finite covering \( \mathcal{U} \) such that for any \( U^s \supseteq U^{s'} \) the ring homomorphism \( \mathcal{O}_S(U^s) \rightarrow \mathcal{O}_S(U^{s'}) \) is flat. This is equivalent to say that \((S, \mathcal{O}_S)\) admits a finite model \((X, \mathcal{O})\) which is a finite space (Definition 1.3).

Examples 6.6.  
1. Any finite simplicial complex \( S \) is quasi-compact and quasi-separated (i.e., \((S, \mathcal{Z})\) is a quasi-compact and quasi-separated ringed space). This is due to Mc Cord ([8]).
2. Any finite \( h \)-regular CW-complex is quasi-compact and quasi-separated (see [3]).
3. If \((S, \mathcal{O}_S)\) is a scheme, then \((S, \mathcal{O}_S)\) is a quasi-compact and quasi-separated ringed space if and only if it is a quasi-compact and quasi-separated scheme (see [12, Proposition 2.4]).

Theorem 6.7. If \((S, \mathcal{O}_S)\) is a quasi-compact and quasi-separated ringed space, then \( \text{Qcoh}(S) \) is a Grothendieck abelian category. Moreover, \( \text{Qcoh}(S) \) admits flat covers and cotorsion envelopes.

Proof. By definition \( S \) admits a locally affine finite covering \( \mathcal{U} \) such that the associated finite ringed space \((X, \mathcal{O})\) is a finite space. Since \( \text{Qcoh}(X) \) is a Grothendieck abelian category, one concludes by Remark 6.4 that \( \text{Qcoh}(S) \) is also a Grothendieck abelian category. Moreover, the equivalence \( \pi^*: \text{Qcoh}(X) \rightarrow \text{Qcoh}(S) \) preserves tensor products and hence flatness. Since \( \text{Qcoh}(X) \) admits flat covers and cotorsion envelopes ([6]), \( \text{Qcoh}(S) \) also does.

Definition 6.8. Let \( f: (T, \mathcal{O}_T) \rightarrow (S, \mathcal{O}_S) \) be a morphism of ringed spaces between quasi-compact and quasi-separated ringed spaces. The inverse image \( f^*: \text{Qcoh}(S) \rightarrow \text{Qcoh}(T) \) has a right adjoint, because \( \text{Qcoh}(S) \) is a Grothendieck abelian category. This right adjoint shall be denoted by
\[
f^{qc}: \text{Qcoh}(T) \rightarrow \text{Qcoh}(S)
\]
and named quasi-coherent direct image. If \( {}_f^*\mathcal{M}: \text{Mod}(T) \rightarrow \text{Mod}(S) \) is the ordinary direct image functor, then for any quasi-coherent module \( \mathcal{M} \) on \( T \) one has: \( f^{qc}_*\mathcal{M} = {}_f^*\mathcal{M} \), where \( \text{Qc}: \text{Mod}(S) \rightarrow \text{Qcoh}(S) \) is the quasi-coherator functor (i.e., the right adjoint of the inclusion functor \( \text{Qcoh}(S) \hookrightarrow \text{Mod}(S) \)). Since \( \text{Qcoh}(T) \) is a Grothendieck abelian category, \( f^{qc} \) has a right derived functor, which shall be denoted by
\[
\mathbb{R}f^{qc}_*: D(\text{Qcoh}(T)) \rightarrow D(\text{Qcoh}(S)).
\]

Notation. Let \( \mathcal{U} \) be a finite open covering of \( S \). For any \( s, s' \in S \) we shall denote
\[
U^{ss'} = U^s \cap U^{s'}.
\]

Definition 6.9. Let \( \mathcal{U} \) be a finite open covering of a ringed space \((S, \mathcal{O}_S)\). We say that \( \mathcal{U} \) is a semi-separating covering of \((S, \mathcal{O}_S)\) if:
1. \( \mathcal{U} \) is locally affine and \( \mathcal{O}_S(U^s) \rightarrow \mathcal{O}_S(U^t) \) is flat for any \( U^s \supseteq U^t \) (hence \((S, \mathcal{O}_S)\) is quasi-compact and quasi-separated).
2. \( H^i(U^{ss'}, \mathcal{O}_S) = 0 \) for any \( i > 0 \) and any \( s, s' \in S \).
(3) For any \( s, s' \in S \) and any \( U^s \supseteq U^t \), the natural morphism
\[
H^0(U^{ss'}, \mathcal{O}_S) \otimes_{\mathcal{O}_S(U^s)} \mathcal{O}_S(U^t) \to H^0(U^{ts'}, \mathcal{O}_S)
\]
is an isomorphism.

A ringed space \( (S, \mathcal{O}_S) \) is called semi-separable if it admits a semi-separating covering.

**Remark 6.10.**
1. A locally affine covering \( U \) is semi-separating if and only if the associated finite ringed space \( (X, \mathcal{O}) \) is semi-separated. Thus a semi-separable ringed space is a ringed space that admits a semi-separated finite model.
2. A scheme \( (S, \mathcal{O}_S) \) is semi-separable if and only if it is a semi-separated scheme.
3. Any semi-separated finite space is semi-separable, but the converse is not true. For example, a finite topological space is semi-separable if and only if it is homotopically trivial and it is semi-separated if and only if it is contractible.

**Theorem 6.11** (Grothendieck duality for semi-separable ringed spaces). Let \( f: (T, \mathcal{O}_T) \to (S, \mathcal{O}_S) \) be a morphism of ringed spaces between semi-separable ringed spaces. Assume that there exist a semi-separating covering \( U = \{U_1, \ldots, U_n\} \) of \( T \) and a semi-separating covering \( V = \{V_1, \ldots, V_m\} \) of \( S \) such that:

1. \( f \) is compatible with \( U \) and \( V \): for any \( t \in T \) one has that \( U^t \subseteq f^{-1}(V^{f(t)}) \).
2. For any \( U^t \supseteq U^{t'} \) and any \( V^s \supseteq V^{s'} \) the natural morphisms
   \[
   H^i(U^t \cap f^{-1}(V^s), \mathcal{O}_T) = H^i(U^{t'} \cap f^{-1}(V^{s'}), \mathcal{O}_T)
   \]
   are isomorphisms for any \( i \geq 0 \).

Then \( \mathbb{R}_{qc}f^!: D \text{Qcoh}(T) \to D \text{Qcoh}(S) \) has a right adjoint.

**Proof.** Let \( (X, \mathcal{O}_X) \) (resp. \( (Y, \mathcal{O}_Y) \)) be the finite ringed space associated to \( U \) (resp. to \( V \)). They are semi-separated finite spaces, because \( U \) and \( V \) are semi-separating. Condition (1) yields that there exists a continuous map \( \bar{f}: X \to Y \) such that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f} & S \\
| & & | \\
X & \xrightarrow{\bar{f}} & Y
\end{array}
\]
is commutative. Moreover, \( \bar{f} \) is a morphism of ringed spaces. Now, condition (2) implies that \( \bar{f} \) is a schematic morphism (and then \( f^{qc}_{\bar{f}} = \bar{f}_{*} \)). Let us consider the diagram

\[
\begin{array}{ccc}
D \text{Qcoh}(T) & \xrightarrow{\mathbb{R}_{qc}f^!} & D \text{Qcoh}(S) \\
\downarrow \rho_q \downarrow \rho_q & & \downarrow \rho_q \downarrow \rho_q \\
D \text{Qcoh}(X) & \xrightarrow{\mathbb{R}_{qc}\bar{f}_{*}} & D \text{Qcoh}(Y) \\
\downarrow \rho_q \downarrow \rho_q & & \downarrow \rho_q \downarrow \rho_q \\
D_{qc}(X) & \xrightarrow{\mathbb{R}_{qc}\bar{f}_{*}} & D_{qc}(Y).
\end{array}
\]
The vertical functors are equivalences, by Remark 6.4 and Theorem 3.2 and the squares are commutative (the higher one is immediate and the lower one is due to Theorem 3.5). We conclude by Theorem 4.2.

\[ \square \]

**Example 6.12.** Let \( f : (T, \mathcal{O}_T) \to (S, \mathcal{O}_S) \) be a morphism of ringed spaces between semi-separable ringed spaces. Assume that \( f \) is “affine”, i.e., there exists a semi-separating covering \( V = \{ V_1, \ldots, V_m \} \) of \( S \) such that \( f^{-1}(V_s) \) is affine for any \( s \in S \). Then \( \mathbb{R}qcf^*: D\text{Qcoh}(T) \to D\text{Qcoh}(S) \) has a right adjoint.

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