LIMITING WEAK-TYPE BEHAVIORS OF SOME INTEGRAL OPERATORS

WEICHAO GUO, JIANXUN HE, AND HUOXIONG WU*

ABSTRACT. In this paper, we explore the limiting weak-type behaviors of some integral operators including maximal operators, singular and fractional integral operators and maximal truncated singular integrals et al. Some optimal limiting weak-type behaviors are given, which essentially improve and extend the previous results in this topics.

1. INTRODUCTION

Let $V$ be a probability measure, and $V_t$ be the dilation of $V$, defined by

$$V_t(E) := V(E/t).$$ (1.1)

As $t$ tends to zero, $V_t$ concentrates its mass at the origin. Consider the convolution of $V_t$ and a function $\phi$, namely,

$$\phi * V_t(x) := \int_{\mathbb{R}^n} \phi(x - y) dV_t(y).$$ (1.2)

Note that this convolution can be interpreted as a weighted averages of $\phi$. Since the concentration of $V_t$, hence in the above integral, the value $g(x)$ is assigned the full mass as $t \to 0$. More precisely, for example if $\phi \in L^p(\mathbb{R}^n)$ and $V$ is an absolutely continuous measure (respect to Lebesgue measure), the classical property of approximation to the identity shows that

$$\phi * V_t \to \phi$$ (1.3)

in the sense of $L^p$.

For the linear operator $T_\phi(V) := \phi * V$, the above argument shows that if $\phi \in L^p$, then $\phi$ can be approached by $T_\phi(V_t)$ as $t \to 0^+$ in the sense of $L^p$. However, for the nonlinear case, for instance the corresponding maximal operator defined by

$$M_\phi^\alpha(\mu)(x) := \sup_{r > 0} (\phi_\alpha^r * \mu)(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi(\frac{x - y}{r}) d\mu(y),$$

where $\phi_\alpha^r(x) = \frac{1}{r^{n-\alpha}} \phi(\frac{x}{r})$, the limiting behavior become confused as $t \to 0^+$, since it can no longer be concluded or implied by the classical theory of approximation to the identity. If $\phi = \chi_{B(0,1)}$, we write $M_\phi := M_{\phi_\alpha}^\alpha$ for short. For $\alpha = 0$, we write $M_\phi := M_\phi^0$. We also use $M := M_\phi^0$ for $\alpha = 0$ and $\phi = \chi_{B(0,1)}$, which is the classical Hardy-Littlewood maximal operator.

For the special case $\alpha = 0$ and $\phi = \chi_{B(0,1)}$, the following theorem was firstly obtained by Janakiraman[4].

**Theorem A ([3] Theorem 3.1).** Let $V$ be a positive measure with finite variation. Then for any fixed $\lambda > 0$,

$$\lim_{t \to 0^+} \left| \{x \in \mathbb{R}^n : M V_t(x) > \lambda \} \right| = \left| \{x \in \mathbb{R}^n : \frac{V(\mathbb{R}^n)}{|x|^n} > \lambda \} \right|.$$

2010 Mathematics Subject Classification. 42B20; 42B25.

Key words and phrases. Limiting behaviors, weak type estimates, maximal operators, singular integrals, fractional integrals.

*Corresponding author.

Supported by the NSF of China (Nos.11771358, 11471041, 11701112, 11671414), the NSF of Fujian Province of China (No.2015J01025) and the China postdoctoral Science Foundation (No. 2017M612628).
We remark that Theorem A above is actually the essential part of [4, Theorem 3.1]. In order to better compare with the results in this paper, we would like to describe the previous known results in a unified form, without any change of their essence.

A stronger conclusion was also obtained in the same paper, see [4, Corollary 3.3] or the following.

**Theorem A∗** ([4, Corollary 3.3]). Let \( V \) be a positive measure with finite variation. Then

\[
\lim_{t \to 0^+} \left\{ x \in \mathbb{R}^n : \left| M_V(x) - \frac{V(\mathbb{R}^n)}{|x|^n} \right| > \lambda \right\} = 0
\]

for every fixed \( \lambda > 0 \).

One of the main purposes of this paper is to improve and extend Theorems A and A∗. More precisely, in Section 2, we will show a stronger limiting behavior for the more general maximal operator \( M^n_0 \), which is an essential improvement of Theorem A even for the special case \( M \).

**Theorem 1.1.** Let \( \alpha \in [0, n) \), \( \phi(x) = \Phi(|x|) \) be a radial function such that \( \sup_{r > 0} \phi^\alpha(\varepsilon_1) < \infty \), where \( \Phi : [0, \infty) \to [0, \infty) \) is decreasing, \( \varepsilon_1 = (1, 0, \ldots, 0) \) is the vector on the unit sphere \( \mathbb{S}^{n-1} \). Suppose that \( \|M^n_0 \mu\|_{L^\infty(\mathbb{R}^n \setminus B(0, \rho))} \leq \mu(\mathbb{R}^n) \) holds for all positive measure \( \mu \). Then, for any fixed \( \rho > 0 \),

\[
\lim_{t \to 0^+} \left\| M^n_0(V_t)(\cdot) - \sup_{r > 0} \phi^{\alpha}(\cdot)V(\mathbb{R}^n) \right\|_{L^{n, \infty}(\mathbb{R}^n \setminus B(0, \rho))} = 0
\]

for all positive measure \( V \).

**Corollary 1.2.** Let \( \alpha \in [0, n) \), \( \phi(x) = \Phi(|x|) \) be a radial function, where \( \Phi : [0, \infty) \to [0, \infty) \) is decreasing, bounded and compact supported. Then, for any fixed \( \rho > 0 \),

\[
\lim_{t \to 0^+} \left\| M^n_0(V_t)(\cdot) - \sup_{r > 0} \phi^{\alpha}(\cdot)V(\mathbb{R}^n) \right\|_{L^{n, \infty}(\mathbb{R}^n \setminus B(0, \rho))} = 0
\]

for any fixed finite positive measure \( V \).

**Remark 1.3.** We would like to make some comparisons between Corollary 1.2 and the previous results in Theorems A and A∗. Choosing \( \alpha = 0 \) and \( \phi = \chi_{B(0,1)} \) in Corollary 1.2 and by the fact \( \sup_{r > 0} \phi^{\alpha}(x) = \frac{1}{|x|^n} \) for \( \phi = \chi_{B(0,1)} \), we immediately obtain

\[
\lim_{t \to 0^+} \left\| M(V_t)(\cdot) - \frac{V(\mathbb{R}^n)}{|\cdot|^n} \right\|_{L^{1, \infty}(\mathbb{R}^n \setminus B(0, \rho))} = 0 \tag{1.4}
\]

for any fixed positive constant \( \rho \). Then Proposition 2.1 shows that this limiting is stronger than that in Theorems A and A∗.

In fact, the convergence in (1.4) is optimal, in the sense that there is some \( V \) such that following limiting behavior is negative:

\[
\lim_{t \to 0^+} \left\| M(V_t)(\cdot) - \frac{V(\mathbb{R}^n)}{|\cdot|^n} \right\|_{L^{1, \infty}(\mathbb{R}^n)} = 0. \tag{1.5}
\]

More precisely, we take \( dV(x) = \chi_{B(0,1)}(x)dx \), where \( dx \) denotes the Lebesgue measure. Then

\[
M_V(x) = \sup_{r > 0} \frac{1}{r^n} \int_{B(x, r)} \chi_{B(0,1)}(y)dy = \frac{1}{r^n} \int_{B(0,1)} \chi_{B(0,t)}(y)dy = \frac{1}{r^n} \cdot \frac{|B(0,t) \cap B(x,r)|}{t^n}.
\]

If \( |x| \leq t/2 \), we get

\[
M_V(x) = \frac{1}{r^n} \cdot \frac{|B(0,t) \cap B(x,r)|}{t^n} \leq \frac{1}{r^n} \cdot \frac{|B(0,1)|}{t^n} = \frac{V(\mathbb{R}^n)}{t^n}.
\]

Also, for \( |x| \leq t/2 \), we have

\[
\frac{V(\mathbb{R}^n)}{|x|^n} \geq \frac{4V(\mathbb{R}^n)}{t^n}.
\]
Combination of the above two estimates yields that
\[
\left| \mathcal{M}V_t(x) - \frac{V(\mathbb{R}^n)}{|x|^n} \right| \geq \frac{3V(\mathbb{R}^n)}{t^n} \quad \text{for } x \in B(0, t/2).
\]
Choose \( \lambda_0 = \frac{2V(\mathbb{R}^n)}{t^n} \). We have
\[
\lambda_0 \left\{ x \in \mathbb{R}^n : \left| \mathcal{M}V_t(x) - \frac{V(\mathbb{R}^n)}{|x|^n} \right| > \lambda_0 \right\} \geq \lambda_0 \left\{ x \in B(0, t/2) : \left| \mathcal{M}V_t(x) - \frac{V(\mathbb{R}^n)}{|x|^n} \right| > \lambda_0 \right\}
\]
\[
= \lambda_0 |B(0, t/2)| \geq \frac{2}{t^n} \cdot V(\mathbb{R}^n) \frac{t^n}{2^n} |B(0, 1)| = \frac{|B(0, 1)|^2}{2^{n-1}}.
\]
Thus,
\[
\lim_{t \to 0^+} \left\| \mathcal{M}(V_t)(\cdot) - \frac{V(\mathbb{R}^n)}{|x|^n} \right\|_{L^{1, \infty}(\mathbb{R}^n)} \geq \lambda_0 \left\{ x \in \mathbb{R}^n : |\mathcal{M}V_t(x) - \mathcal{M}0(x) V(\mathbb{R}^n)| > \lambda_0 \right\} \geq \frac{|B(0, 1)|^2}{2^{n-1}}.
\]
This shows that the limiting in (1.5) is negative, and (1.4) is optimal.

On the other hand, the limiting behavior of singular integral with homogeneous kernel was also considered in [4]. Subsequently, it was improved by Ding and Lai in [2]. Moreover, the weak limiting behavior of maximal operator associated with homogeneous kernel was also considered in [1]. To state the relevant previous results, we first recall several definitions and notations. The integral operator we are interested in this paper are of the form
\[
T_{\alpha}^\alpha \mu(x) := \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n - \alpha}} d\mu(y),
\]
where \( \alpha \in [0, n) \), \( \Omega \) is a homogeneous function of degree zero and satisfies the following mean value zero property when \( \alpha = 0 \):
\[
\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0.
\]
As usual for \( \alpha = 0 \), the right side of the equation (1.6) should be represented in the sense of principal value. We write \( T_{\alpha} \) := \( T_{\alpha}^\alpha \) for short.

We also consider the corresponding maximal operator associated with homogeneous kernel \( \Omega \) defined by
\[
M_{\Omega}^\alpha \mu(x) := \sup \frac{1}{r^{n - \alpha}} \int_{B(x, r)} |\Omega(x - y)| d\mu(y)
\]
Note that if we take \( \phi = |\Omega| \chi_{B(0, 1)} \), then
\[
\mathcal{M}^\alpha V(x) = \sup \frac{1}{r^{n - \alpha}} \int_{\mathbb{R}^n} |\Omega(\frac{x - y}{r})| \chi_{B(0, 1)}(\frac{x - y}{r}) dV(y)
\]
\[
= \sup \frac{1}{r^{n - \alpha}} \int_{B(x, r)} |\Omega(x - y)| dV(y) = M_{\Omega}^\alpha V(x).
\]
In this case, \( \phi = |\Omega| \chi_{B(0, 1)} \) is not a radial function anymore. To keep the limiting behavior still valid in this case, we need to add some angular regularity to \( \Omega \), which can be viewed as the alternative condition of radial property. Now, we give the definition of angular regularity, namely, the Dini-condition.

**Definition 1.4** (**\( L^q \)-Dini condition**). Suppose \( \Omega \) is a homogeneous function of degree zero. Let \( 1 \leq q \leq \infty, 0 \leq s < n \). We say that \( \Omega \) satisfies \( L^q \)-Dini condition if
\begin{enumerate}
\item \( \Omega \in L^q(S^{n-1}) \),
\item \( \int_0^1 \frac{\omega(t)}{t^{n+s}} dt < \infty \),
\end{enumerate}
where $\omega_q$ is called the (modified) integral continuous modulus of $\Omega$ of degree $q$, defined by
\[
\omega_q(t) := \left( \sup_{|h| \leq t} \int_{\mathbb{S}^{n-1}} |\Omega(x' + h) - \Omega(x')|^q \, d\sigma(x') \right)^{1/q}.
\] (1.8)

For brevity, we use $L^q$-Dini instead of $L^q_{\alpha}$-Dini. Note that for $q > 1$ the above definition is a little different from [2] Definition 2.7 and [1] Definition 4.1.

In order to compare our results with the relevant previous results, we now list the main theorems in [1] as follows.

**Theorem B** (cf. [1]). Let $\Omega$ be a homogeneous function of degree 0, satisfying (1.7) and $L^1$-Dini condition. Then for any fixed $\lambda > 0$,
\[
\lim_{t \to 0^+} \left| \left\{ x \in \mathbb{R}^n : M_\Omega V_t(x) > \lambda \right\} \right| = \left| \left\{ x \in \mathbb{R}^n : \frac{\|\Omega(x)||V(\mathbb{R}^n)|}{|x|^n} > \lambda \right\} \right|
\]
for all finite positive absolutely continuous measure $V$.

**Theorem C** (cf. [1]). Let $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ be a homogeneous function of degree 0, satisfying $L^1_{\alpha}$-Dini condition. Then for any fixed $\lambda > 0$,
\[
\lim_{t \to 0^+} \left| \left\{ x \in \mathbb{R}^n : M_\Omega^\alpha V_t(x) > \lambda \right\} \right| = \left| \left\{ x \in \mathbb{R}^n : \frac{\|\Omega(x)||V(\mathbb{R}^n)|}{|x|^{n-\alpha}} > \lambda \right\} \right|
\]
for all finite positive absolutely continuous measure $V$.

**Theorem D** (cf. [2]). Let $\Omega$ be a homogeneous function of degree 0, satisfying (1.7) and $L^1$-Dini condition. Then for any fixed $\lambda > 0$,
\[
\lim_{t \to 0^+} \left| \left\{ x \in \mathbb{R}^n : T_\Omega V_t(x) > \lambda \right\} \right| = \left| \left\{ x \in \mathbb{R}^n : \frac{\|\Omega(x)||V(\mathbb{R}^n)|}{|x|^n} > \lambda \right\} \right|
\]
for all finite positive absolutely continuous measure $V$.

**Theorem E** (cf. [2]). Let $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ be a homogeneous function of degree 0, satisfying $L^1_{\alpha}$-Dini condition. Then for any fixed $\lambda > 0$,
\[
\lim_{t \to 0^+} \left| \left\{ x \in \mathbb{R}^n : T_\Omega^\alpha V_t(x) > \lambda \right\} \right| = \left| \left\{ x \in \mathbb{R}^n : \frac{\|\Omega(x)||V(\mathbb{R}^n)|}{|x|^{n-\alpha}} > \lambda \right\} \right|
\]
for all finite positive absolutely continuous measure $V$.

The second purpose of this paper is to improve and extend the above results in Theorems B-E. $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ will be proved to be necessary if the corresponding operator $M_\Omega^\alpha$ or $T_\Omega^\alpha$ is bounded with $\Omega$ satisfying $L^1_{\alpha}$-Dini condition. The limiting behaviors in the above four theorems will be improved (see Remarks 1.7 and 1.10 below). Our main results in this part can be formulated as follows.

**Theorem 1.5.** Let $\alpha \in [0, n)$, $V$ be an absolutely continuous positive measure. Suppose that $\Omega$ is a homogeneous function of degree zero and satisfies the $L^1_{\alpha}$-Dini condition. If the maximal operator $M_\Omega^\alpha$ is bounded from $L^1$ to $L^{\frac{n}{n-\alpha}}$, then
\begin{enumerate}
\item $\frac{\Omega(x)}{|x|^n} \in L^{\frac{n}{n-\alpha}}$, $\left\| \frac{\Omega(x)}{|x|^n} \right\|_{L^{\frac{n}{n-\alpha}}} \lesssim \left\| M_\Omega^\alpha \right\|_{L^1 \to L^{\frac{n}{n-\alpha}}}$;
\item $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$, $\left\| M_\Omega^\alpha \right\|_{L^1 \to L^{\frac{n}{n-\alpha}}} \lesssim \left\| \Omega \right\|_{L^1 \to L^{\frac{n}{n-\alpha}}}$;
\item $\lim_{t \to 0^+} \left| \left\{ x \in \mathbb{R}^n : M_\Omega^\alpha V_t(x) > \lambda \right\} \right| = 0$, $\forall \lambda > 0$.
\end{enumerate}

**Theorem 1.6.** Let $\alpha \in [0, n)$, $V$ be a absolutely continuous positive measure. Suppose that $\Omega$ is a homogeneous function of degree zero and satisfies the $L^{\frac{n}{n-\alpha}}$-Dini condition. If the maximal operator $M_\Omega^\alpha$ is bounded from $L^1$ to $L^{\frac{n}{n-\alpha}}$, then
\begin{enumerate}
\item $\frac{\Omega(x)}{|x|^n} \in L^{\frac{n}{n-\alpha}}$, $\left\| \frac{\Omega(x)}{|x|^n} \right\|_{L^{\frac{n}{n-\alpha}}} \lesssim \left\| M_\Omega^\alpha \right\|_{L^1 \to L^{\frac{n}{n-\alpha}}}$;
\end{enumerate}
Theorem 1.8. In Theorem 1.5, we show that \( \Omega \) type-1 convergence if \( \Omega \) satisfies the sense, which is better than the previous results in Theorems D and E. Furthermore, we establish the

\[ \sup_{r > 0} \phi_r^\alpha (x) = \Omega(x) \left| \frac{x}{|x|} \right|^{-\alpha} \]  

if \( \phi = \chi_{B(0,1)} \). Thus, Theorem 1.8 actually has the same form as Theorem 1.1.

Furthermore, as corollary, the following result gives a partial answer for why the integral index \( \frac{\alpha}{n-\alpha} \) is optimal in the study of boundedness of the fractional integral operators with homogeneous kernel.

Corollary 1.13. Suppose \( \alpha \in (0, n) \), \( \Omega \) satisfies the \( L^1_\alpha \)-Dini condition. Then the following three statements are equivalent:

1. \( \Omega \) is \( L^{\frac{\alpha}{n-\alpha}}(S^{n-1}) \).
2. \( M^\alpha_{\Omega}(V_t) \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^{\frac{\alpha}{n-\alpha}}(\mathbb{R}^n) \).
3. \( T^\alpha_{\Omega} \) is bounded from \( L^1(\mathbb{R}^n) \) to \( L^{\frac{\alpha}{n-\alpha}}(\mathbb{R}^n) \).

This paper is organized as follows. In Section 2, we deal with the limiting behaviors for a wide class of maximal functions. The limiting behaviors for the maximal operators associated with homogeneous...
Using the assumption (2), we obtain that

\[ \nu \]

Letting

\[ \varepsilon \]

The combination of (2.1) and (2.2) yields the desired conclusion (3).

2. Maximal operator associated with radial functions

In order to distinguish the various kinds of limiting behaviors, we first establish the following proposition. In this paper, all the limiting behaviors can be compared each other in the framework of this proposition.

**Proposition 2.1.** Let \( 0 < p < \infty \). Suppose that \( f \in L^{p, \infty}(\mathbb{R}^n) \), and \(|\{x \in \mathbb{R}^n : |f(x)| = \lambda\}| = 0\) for all \( \lambda > 0 \). Let \( \{f(t)\}_{t > 0} \) be a sequence of measurable functions. Then for the following three statements:

1. \( \forall \varepsilon > 0, \exists A_\varepsilon \subset \mathbb{R}^n, \text{s.t.}, \ |A_\varepsilon| < \varepsilon \quad \text{and} \quad \lim_{t \to 0^+} \|f - f(t)\|_{L^{p, \infty}(\mathbb{R}^n \setminus A_\varepsilon)} = 0. \)
2. \( \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x) - f(x)| > \lambda\}| = 0, \forall \lambda > 0, \)
3. \( \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|, \forall \lambda > 0, \)

we have

\[ (1) \Rightarrow (2) \Rightarrow (3), \quad (3) \nRightarrow (2), \quad (2) \nRightarrow (1). \]

**Proof.** We first verify \( (1) \Rightarrow (2) \). For any \( \varepsilon > 0 \), there exists a set \( |A_\varepsilon| < \varepsilon \) such that

\[ \lim_{t \to 0^+} \|f - f(t)\|_{L^{p, \infty}(\mathbb{R}^n \setminus A_\varepsilon)} = 0. \]

This implies that

\[ |\{x \in \mathbb{R}^n : |f(t)(x) - f(x)| > \lambda\}| \leq |\{x \in A^c_\varepsilon : |f(t)(x) - f(x)| > \lambda\}| + |A_\varepsilon| \]

\[ \leq \frac{\|f(t) - f\|_{L^{p, \infty}(\mathbb{R}^n \setminus A_\varepsilon)}^p}{\lambda^p} + |A_\varepsilon|. \]

Thus,

\[ \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x) - f(x)| > \lambda\}| \leq |A_\varepsilon| < \varepsilon. \]

By the arbitrary of \( \varepsilon \), we obtain conclusion (2).

Next, we show that \( (2) \Rightarrow (3) \). For a small constant \( \nu \in (0, 1) \), we have

\[ |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| \leq |\{x \in \mathbb{R}^n : |f(x)| > (1 - \nu)\lambda\}| + |\{x \in \mathbb{R}^n : |f(t)(x) - f(x)| > \nu \lambda\}|. \]

By (2), we have \( \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x) - f(x)| > \nu \lambda\}| = 0 \). This implies that

\[ \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| \leq |\{x \in \mathbb{R}^n : |f(x)| > (1 - \nu)\lambda\}|. \]

Letting \( \nu \to 0 \), we have

\[ \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| \leq |\{x \in \mathbb{R}^n : |f(x)| \geq \lambda\}| \]

\[ = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|, \quad (2.1) \]

where in the last equality we use the fact \( |\{x \in \mathbb{R}^n : |f(x)| = \lambda\}| = 0 \).

On the other hand,

\[ |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| \geq |\{x \in \mathbb{R}^n : |f(x)| > (1 + \nu)\lambda\}| - |\{x \in \mathbb{R}^n : |f(t)(x) - f(x)| > \nu \lambda\}| \]

Using the assumption (2), we obtain that

\[ \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| \geq |\{x \in \mathbb{R}^n : |f(x)| > (1 + \nu)\lambda\}|. \]

Letting \( \nu \to 0 \), we have

\[ \lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |f(t)(x)| > \lambda\}| \geq |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|. \quad (2.2) \]

The combination of (2.1) and (2.2) yields the desired conclusion (3).
The proof of that (3) \(\Rightarrow\) (2) is simple, we omit the detail here.

Finally, we show that (2) \(\Rightarrow\) (1). Let \(g(x) = |x|^{-n/p}\), \(g(t)(x) = g(x)\chi_{B(0,t-1)}(x)\). Obviously, \(g \in L^{p,\infty}(\mathbb{R}^n)\). For any fixed \(\lambda > 0\), we have

\[
\{x \in \mathbb{R}^n : |g(x) - g(t)(x)| > \lambda\} = \emptyset \text{ for sufficient small } t.
\]

Thus,

\[
\lim_{t \to 0^+} |\{x \in \mathbb{R}^n : |g(t)(x) - g(x)| > \lambda\}| = 0.
\]

However, for any \(\epsilon > 0\), \(|A_\epsilon| < \epsilon, t > 0\), if we take \(\lambda\) sufficiently small, then

\[
\{x \in \mathbb{R}^n : |g(x) - g(t)(x)| > \lambda\} = \{x \in B^c(0, t^{-1}) : |x|^{-n/p} > \lambda\} \sim \lambda^{-p}.
\]

Consequently, as \(\lambda \to 0\),

\[
|\{x \in A_\epsilon^c : |g(x) - g(t)(x)| > \lambda\}| \gtrsim |\{x \in \mathbb{R}^n : |g(x) - g(t)(x)| > \lambda\}| - |A_\epsilon| \sim \lambda^{-p}.
\]

This implies that

\[
\|g - g(t)\|_{L^{p,\infty}(\mathbb{R}^n \setminus A_\epsilon)} = \sup_{\lambda > 0} \lambda \cdot |\{x \in A_\epsilon^c : |g(x) - g(t)(x)| > \lambda\}|^{1/p} \gtrsim 1 \text{ for every } t > 0,
\]

which leads to a contradiction with (1). Proposition 2.1 is proved. \(\square\)

**Remark 2.2.** For the sake of brevity, for \(i = 1, 2, 3\), we say that a sequence of functions \(f_{(t)}\) tends to \(f\) in the type-\(i\) sense, if (i) is valid as in Proposition 2.1.

**Proof of Theorem 1.1.** Without loss of generality, we may assume \(V\) is a probability measure, that is, \(V(\mathbb{R}^n) = 1\). For \(t > 0\), denote

\[
dV_1^t := dV_1 \chi_{B(0,r_t)}, \quad dV_2^t := dV_2 \chi_{B^c(0,r_t)}
\]

where \(r_t = \sqrt{t}\). By the definition of \(V_t\), we obtain

\[
V_t^2(\mathbb{R}^n) = V_t(B^c(0,r_t)) = 1 - V_t(B(0,r_t)) = 1 - V(B(0,t^{-1/2})) = = \epsilon_t \to 0^+
\]
as \(t \to 0^+\). Then \(V_1^t(\mathbb{R}^n) = 1 - \epsilon_t\). By the quasi-triangle inequality for \(L^{n/\alpha,\infty}\), and using the boundedness of \(M^\alpha_{\phi}\), we deduce that

\[
\|M^\alpha_{\phi}(V_1^t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)\|_{L^{n/\alpha,\infty}(\mathbb{R}^n \setminus B(0,\rho))} \lesssim \|M^\alpha_{\phi}(V_1^t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)\|_{L^{n/\alpha,\infty}(\mathbb{R}^n \setminus B(0,\rho))} + \|M^\alpha_{\phi}(V_2^t)(\cdot)\|_{L^{n/\alpha,\infty}(\mathbb{R}^n \setminus B(0,\rho))} + \|V_2^t(\mathbb{R}^n)\|
\]

\[
\leq \|M^\alpha_{\phi}(V_1^t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)\|_{L^{n/\alpha,\infty}(\mathbb{R}^n \setminus B(0,\rho))} + \epsilon_t.
\]

Set

\[
A_{t,\rho} := \{x \in B^c(0,\rho) : \left|\mathcal{M}^\alpha_{\phi}(V_1^t)(x) - \sup_{r > 0} \phi^\alpha_r(x)\right| > \lambda\}.
\]

For \(x \in A_{t,\rho}\) and sufficient small \(t\) such that \(r_t < \rho/2\), we have

\[
\mathcal{M}^\alpha_{\phi}(V_1^t)(x) - \sup_{r > 0} \phi^\alpha_r(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi(x-y) dV_1^t(y) - \sup_{r > 0} \phi^\alpha_r(x)
\]

\[
\leq \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi(x-y) dV_1^t(y) - \sup_{r > 0} \phi^\alpha_r(x)
\]

\[
= \left(\frac{\rho}{\rho - r_t}\right)^{-1} \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi(x-y) dV_1^t(y) - \sup_{r > 0} \phi^\alpha_r(x)
\]

\[
\leq \left(\frac{\rho}{\rho - r_t}\right)^{-1} \sup_{r > 0} \phi^\alpha_r(x).
\]
Also, for the opposite direction we have
\[
M_\phi(V^1_r)(x) - \sup_{r > 0} \phi^\alpha_r(x) = \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{r}\right) dV^1_t(y) - \sup_{r > 0} \phi^\alpha_r(x)
\geq \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi\left(\frac{|x|+r_1}{r}\right) dV^1_t(y) - \sup_{r > 0} \phi^\alpha_r(x)
= \left(\frac{|x|}{|x|+r_1}\right)^{n-\alpha} \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi\left(\frac{x'}{r}\right) dV^1_t(y) - \sup_{r > 0} \phi^\alpha_r(x) \tag{2.4}
\geq \left(\frac{\rho}{\rho+r_1}\right)^{n-\alpha} V^1_t(\mathbb{R}^n) - 1 \sup_{r > 0} \phi^\alpha_r(x)
= \left(\frac{\rho}{\rho+r_1}\right)^{n-\alpha} (1-\epsilon_t) - 1 \sup_{r > 0} \phi^\alpha_r(x)
\]
Thus, for \(x \in A^\lambda_{t,\rho} \)
\[
|M_\phi(V_t)(x) - \sup_{r > 0} \phi^\alpha_r(x)| \leq \beta_t \sup_{r > 0} \phi^\alpha_r(x),
\]
where \(\beta_t \to 0^+ \) as \(t \to 0^+ \). This implies that
\[
\left\|M_\phi^\alpha(V^1_t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)\right\|_{L_{\alpha-\alpha}^\infty(\mathbb{R}^n \setminus B(0,\rho))} \leq \beta_t \left\|\sup_{r > 0} \phi^\alpha_r\right\|_{L_{\alpha-\alpha}^\infty}. \tag{2.5}
\]
Note that
\[
\sup_{r > 0} \phi^\alpha_r(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \phi\left(\frac{x}{r}\right) = \frac{1}{|x|^{n-\alpha}} \sup_{r > 0} \frac{1}{(r/|x|)^{n-\alpha}} \phi(\frac{x'}{r/|x|})
= \frac{1}{|x|^{n-\alpha}} \sup_{r > 0} \phi^\alpha_r(x') = \frac{1}{|x|^{n-\alpha}} \sup_{r > 0} \phi^\alpha_r(e_1).
\]
We have
\[
\beta_t \left\|\sup_{r > 0} \phi^\alpha_r\right\|_{L_{\alpha-\alpha}^\infty} = \beta_t \sup_{r > 0} \phi^\alpha_r(e_1) \left\|\frac{1}{|\cdot|^{n-\alpha}}\right\|_{L_{\alpha-\alpha}^\infty} \approx \beta_t.
\]
This together with (2.3) and (2.5) yields that
\[
\left\|M_\phi^\alpha(V^1_t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)\right\|_{L_{\alpha-\alpha}^\infty(\mathbb{R}^n \setminus B(0,\rho))} \lesssim \left\|M_\phi^\alpha(V^1_t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)\right\|_{L_{\alpha-\alpha}^\infty(\mathbb{R}^n \setminus B(0,\rho))} + \epsilon_t \approx \beta_t + \epsilon_t \to 0 \text{ as } t \to 0^+,
\]
which is the desired conclusion and completes the proof of Theorem [1.1].

\textbf{Proof of Corollary [1.2]} Without loss of generality, we assume supp\(\phi \subset B(0,1)\) and \(\|\phi\|_{L^\infty} = 1\). Then
\[
\sup_{r > 0} \phi^\alpha_r(e_1) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \phi\left(\frac{e_1}{r}\right) = \sup_{r > 1} \frac{1}{r^{n-\alpha}} \phi\left(\frac{e_1}{r}\right) \leq 1.
\]
In order to use Theorem [1.1] we only need to verify \(\|M_\phi^\alpha \mu\|_{L_{\alpha-\alpha}^\infty} \lesssim \mu(\mathbb{R}^n)\) for all positive measure \(\mu\). Since \(M_\phi \mu \lesssim M^\alpha \mu\), it suffices to show that
\[
\|M^\alpha \mu\|_{L_{\alpha-\alpha}^\infty} \lesssim \mu(\mathbb{R}^n). \tag{2.6}
\]
Write \(A_\lambda := \{x \in \mathbb{R}^n : M^\alpha \mu(x) > \lambda\}\). For any \(x \in A_\lambda\), we can find a ball \(B(x, r_x)\), satisfying that
\[
\frac{1}{r_x^{n-\alpha}} \int_{B(x, r_x)} d\mu(y) > \lambda.
\]
This implies that $|B(x, r_x)|^{\frac{n-\alpha}{n}} \lesssim \frac{1}{\lambda} \int_{B(x, r_x)} d\mu(y)$. Obviously, $A_\lambda \subset \bigcup_{x \in A_\lambda} B(x, r_x)$. Using the Wiener Covering Lemma, there exists disjoint collection of such balls $B_i = B(x_i, r_{x_i})$ such that $A_\lambda \subset \bigcup 5B(x_i, r_{x_i})$. Therefore,

$$
|A_\lambda| \lesssim \sum_i |B_i| \lesssim \left( \sum_i |B_i|^{\frac{n-\alpha}{n}} \right)^{\frac{n}{n-\alpha}} \lesssim \left( \sum_i \frac{1}{\lambda} \int_{B_i} d\mu(y) \right)^{\frac{n}{n-\alpha}} = \left( \frac{1}{\lambda} \int_{\bigcup_i B_i} d\mu(y) \right)^{\frac{n}{n-\alpha}} \lesssim \left( \frac{\mu(\mathbb{R}^n)}{\lambda} \right)^{\frac{n}{n-\alpha}}.
$$

This implies that $\lambda |A_\lambda|^{\frac{n-\alpha}{n}} \lesssim \mu(\mathbb{R}^n)$.

By the arbitrary of $\lambda > 0$, we get (2.6) and then the desired conclusion follows from Theorem 1.2. Corollary 1.2 is proved.

Furthermore, if the measure $V$ is assumed to be absolutely continuous (with respect to Lebesgue measure), we have following corollaries.

**Corollary 2.3.** Let $\phi(x) = \Phi(|x|)$ be a radial function, where $\Phi : [0, \infty) \to [0, \infty)$ is decreasing. Suppose that $\phi$ has a continuous integrable radially decreasing majorant. Then, for any fixed $\rho > 0$, we have

$$
\lim_{t \to 0^+} \left\| \mathcal{M}_\phi(V_t)(\cdot) - \sup_{r > 0} \phi_r(\cdot)V(\mathbb{R}^n) \right\|_{L^{1, \infty}(\mathbb{R}^n \setminus B(0, \rho))} = 0
$$

for all finite positive absolutely continuous measure $V$.

**Proof.** Since $\phi$ has a continuous integrable radially decreasing majorant, we have $\mathcal{M}_\phi(\mu)(x) \lesssim \mathcal{M}(\mu)(x)$ for all absolutely continuous measure, thanks to [3] Corollary 2.1.12. Then

$$
\|\mathcal{M}_\phi \mu\|_{L^{1, \infty}} \lesssim \|\mathcal{M} \mu\|_{L^{1, \infty}} \lesssim \mu(\mathbb{R}^n).
$$

Denote by $K$ the continuous integrable radially decreasing majorant, then $\phi(x) \leq K(x), x \in \mathbb{R}^n$.

Thus,

$$
\sup_{r > 0} \phi_r(e_1) = \sup_{r > 0} \frac{1}{r^n} \phi_r(e_1) \sim \int_{B(0, r^{-1})} \phi_r(e_1) dx \lesssim \int_{B(0, r^{-1})} \phi(x) dx \lesssim \|K\|_{L^1} < \infty.
$$

The desired conclusion then follows from Theorem 1.4.

**Corollary 2.4.** Let $\alpha \in (0, n), \phi(x) = \Phi(|x|)$ be a radial function such that $\sup_{r > 0} \phi^\alpha_r(e_1) < \infty$, where $\Phi : [0, \infty) \to [0, \infty)$ is decreasing, $e_1 = (1, 0, \cdots, 0)$ is the vector on the unit sphere $\mathbb{S}^{n-1}$. Then, for any fixed $\rho > 0$, we have

$$
\lim_{t \to 0^+} \left\| \mathcal{M}_\phi^\alpha(V_t)(\cdot) - \sup_{r > 0} \phi^\alpha_r(\cdot)V(\mathbb{R}^n) \right\|_{L^{\frac{n}{n-\alpha}, \infty}(\mathbb{R}^n \setminus B(0, \rho))} = 0
$$

for all finite positive absolutely continuous measure $V$.

**Proof.** Since $\sup_{r > 0} \phi^\alpha_r(e_1) < \infty$, we have

$$
\phi(x) = \phi(\frac{e_1}{|x|}) = \frac{1}{|x|^{n-\alpha}} \frac{1}{|x|^{n-\alpha}} \phi(\frac{e_1}{|x|}) \lesssim \frac{1}{|x|^{n-\alpha}} \sup_{r > 0} \phi^\alpha_r(e_1).
$$

Then,

$$
\mathcal{M}_\phi^\alpha(\mu)(x) = \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{\mathbb{R}^n} \phi_r(x-y) d\mu(y) \lesssim \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} d\mu(y) \sim \mathcal{M}(\mu)(x)
$$

for all absolutely continuous measure. Thus,

$$
\|\mathcal{M}_\phi^\alpha(\mu)(\cdot)\|_{L^{\frac{n}{n-\alpha}, \infty}} \lesssim \|\mathcal{M}(\mu)(\cdot)\|_{L^{\frac{n}{n-\alpha}, \infty}} \lesssim \mu(\mathbb{R}^n).
$$

This leads to the desired conclusion by Theorem 1.4. \qed
Next, we give two specific applications.

**Corollary 2.5.** Let \( P(x) = \frac{c_n}{(1+|x|^2)^\frac{n+4}{2}} \), where \( c_n \) is the constant such that \( \int_{\mathbb{R}^n} P(x)dx = 1 \). The function \( P \) is called Possion Kernel. We define the maximal function with possion kernel by

\[
\mathcal{M}_P(\mu)(x) := \sup_{r>0} P_r * \mu(x),
\]

where \( \mu \) is any fixed positive measure. Then, for any fixed \( \rho > 0 \), we have

\[
\lim_{t \to 0^+} \left\| \mathcal{M}_P(V_t)(\cdot) - \frac{c_n n^\frac{2}{n}}{(1+n)^{\frac{n+2}{2}}} V(\mathbb{R}^n) \right\|_{L^{\frac{\infty}{n}}(\mathbb{R}^n \setminus B(0,\rho))} = 0 \tag{2.9}
\]

for all finite positive absolutely continuous measure \( V \).

**Proof.** Since Possion kernel itself can be treated as a continuous integrable radially decreasing majorant, in order to use Theorem 1.1, we only need to calculate \( \sup_{r>0} P_r(e_1) \). In fact

\[
\sup_{r>0} P_r(e_1) = \sup_{r>0} \frac{c_n}{r^n(1+r^{-2})^{\frac{2n+4}{2}}} = \frac{c_n}{\inf_{r>0} r^n(1+r^{-2})^{\frac{2n+4}{2}}}. 
\]

A direct calculation yields that

\[
[n^n(1+r^{-2})^{\frac{n+4}{2}}]' = n(n-1)(1+n)^{-1} + (n+1)(n-2)(1+n)^{-\frac{n+4}{2}} \\
= n(n-3)(1+n)^{-\frac{n+4}{2}} (r^{-2} - 1/n). 
\]

Thus, the function \( n^n(1+r^{-2})^{\frac{n+4}{2}} \) takes its minimal value at \( r = \frac{1}{\sqrt{n}} \).

\[
\sup_{r>0} P_r(e_1) = \frac{c_n}{r^n(1+r^{-2})^{\frac{n+4}{2}}} \bigg|_{r=1/\sqrt{n}} = \frac{c_n n^{\frac{2}{n}}}{(1+n)^{\frac{n+2}{2}}}. 
\]

\[\square\]

**Corollary 2.6.** The function \( G(x) = e^{-\pi|x|^2} \) is called heat kernel. We define the maximal function with heat kernel by

\[
\mathcal{M}_G(\mu)(x) := \sup_{r>0} G_r * \mu(x),
\]

where \( \mu \) is any fixed positive measure. Then, for any fixed \( \rho > 0 \), we have

\[
\lim_{t \to 0^+} \left\| \mathcal{M}_G(V_t)(\cdot) - \frac{n}{(2\pi)^{\frac{n}{2}}} V(\mathbb{R}^n) \right\|_{L^{\frac{n}{n}(\mathbb{R}^n \setminus B(0,\rho))}} = 0 \tag{2.10}
\]

for all finite positive absolutely continuous measure \( V \).

**Proof.** As in the proof of the above corollary, we only need to calculate \( \sup_{r>0} G_r(e_1) \). Note that

\[
\sup_{r>0} G_r(e_1) = \sup_{r>0} \frac{e^{-\pi/r^2}}{r^n} = \frac{1}{\inf_{r>0} r^n e^{\pi/r^2}}. 
\]

A direct calculation yields that

\[
[r^n e^{\pi/r^2}]' = nr^{n-1} e^{\pi/r^2} + r^n (-2\pi r^{-3}) e^{\pi/r^2} \\
= r^{n-3} e^{\pi/r^2} (nr^2 - 2\pi). 
\]

Thus, the function \( r^n e^{\pi/r^2} \) takes its minimal value at \( r = \sqrt{\frac{2\pi}{n}} \).

\[
\sup_{r>0} G_r(e_1) = \frac{1}{r^n e^{\pi/r^2}} \bigg|_{r=\sqrt{\frac{2\pi}{n}}} = \frac{n^{n/2}}{(2\pi)^{n/2} e^{n/2}} = \left( \frac{n}{2\pi e} \right)^{n/2}. 
\]

\[\square\]
Remark 2.7. In the proofs of this section, the radial decreasing property is important. In fact, we can take a function \( \phi \) without radial decreasing property such that \( \mathcal{M}_\phi V_t(x) \to \sup_{r>0} \phi_r(x) V(\mathbb{R}^n) \) is negative, even in the type-3 sense. Let \( \phi(x) = 1 \) for \( |x| = 1 \), and disappear otherwise. Let \( dV(x) = \chi_{B(0,1)}(x)dx \), where \( dx \) is the Lebesgue measure. Note that \( \sup_{r>0} \phi_r(x) = \frac{1}{|x|^n} \) and

\[
\mathcal{M}_\phi V_t(x) = \sup_{r>0} \frac{1}{r^n} \int_{\mathbb{R}^n} \phi \left( \frac{x-y}{r} \right) dV(y) = \sup_{r>0} \frac{1}{r^n} \int_{B(0,t)} \phi \left( \frac{x-y}{r} \right) dy = 0.
\]

Hence, for every fixed \( \lambda > 0 \), \( \{x : \mathcal{M}_\phi V_t(x) > \lambda\} = 0 \), but \( \{x : \sup_{r>0} \phi_r(x) V(\mathbb{R}^n) > \lambda\} \neq 0 \). We get the desired conclusion.

3. Maximal operator associated with homogeneous functions

This section is concerned with the maximal operator \( M^\alpha_{\Omega} \), where \( \Omega \) is a homogeneous function of degree zero. Firstly, we list some basic properties of \( \Omega \) as follows:

(A) \( \left\{ x \in \mathbb{R}^n : \frac{\Omega(x)}{|x|^n} > \lambda \right\} = \lambda^{\frac{n}{n-\alpha}} \left\{ x \in \mathbb{R}^n : \frac{\Omega(x)}{|x|^n} > 1 \right\} \);

(B) \( \|\Omega_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \| = n\lambda^{\frac{n}{n-\alpha}} \left\{ x \in \mathbb{R}^n : \frac{\Omega(x)}{|x|^n} > \lambda \right\} \).

Proof of Theorem 1.5. Without loss of generality, we assume that \( V \) is a probability measure. For \( t > 0 \), let \( V_t^1, V_t^2, r_t, \varepsilon_t \) be as in the proof of Theorem 1.1. For \( \lambda > 0 \), denote

\( E^\lambda_t := \{x \in \mathbb{R}^n : M^\alpha_{\Omega} V_t(x) > \lambda\} \)

and

\( E^\lambda_{t,1} := \{x \in \mathbb{R}^n : M^\alpha_{\Omega} V_t^1(x) > \lambda\}, \quad E^\lambda_{t,2} := \{x \in \mathbb{R}^n : M^\alpha_{\Omega} V_t^2(x) > \lambda\} \).

For fixed \( \nu > 0 \), recalling that \( M^\alpha_{\Omega} \) is boundedness from \( L^1 \) to \( L^{\frac{n}{n-\alpha}} \), we obtain that

\( \nu \lambda|E^\lambda_{t,2}|^{\frac{n}{n-\alpha}} \leq V_t^2(\mathbb{R}^n) = \varepsilon_t \to 0^+ \) as \( t \to 0^+ \).

For fixed \( \rho > 2r_t \), \( x \in B^*(0, \rho) \),

\[
M^\alpha_{\Omega} V_t^1(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r) \cap B(0,r_t)} \frac{|\Omega(x-y)|}{|x-y|^n} |dV_t^1(y)|
\]

\[
\geq \frac{1}{(|x|+r_t)^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y)| |dV_t^1(y)|
\]

\[
\geq \frac{1}{(|x|+r_t)^{n-\alpha}} \int_{B(0,r_t)} \left[ |\Omega(x)| |dV_t^1(y)| - \frac{1}{(|x|+r_t)^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y)| |dV_t^1(y)| \right]
\]

\[
= \frac{(1-\varepsilon_t)|\Omega(x)|}{(|x|+r_t)^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x)-\Omega(x)| |dV_t^1(y)|
\]

\[
\geq \frac{(1-\varepsilon_t)\rho^{n-\alpha}}{(\rho+r_t)^{n-\alpha}} \frac{|\Omega(x)|}{|x|^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y)| |dV_t^1(y)|.
\]

This implies that

\[
M^\alpha_{\Omega} V_t^1(x) \geq \left( \frac{1-\varepsilon_t}{\rho+r_t} \right)^{n-\alpha} \frac{|\Omega(x)|}{|x|^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y)| |dV_t^1(y)|. \quad (3.1)
\]

Set

\( F^\lambda_{t,\rho} := \left\{ x \in B^*(0, \rho) : \frac{1}{(\rho+r_t)^{n-\alpha}} \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \)

and

\( G^\lambda_{t,\rho} := \left\{ x \in B^*(0, \rho) : \frac{1}{|x|^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y)| |dV_t^1(y)| > \lambda \right\} \).
Now, we estimate \( |G_{t,\rho}^{\nu,1}| \) for fixed \( \nu > 0 \). Using Chebychev’s inequality, we conclude that
\[
|G_{t,\rho}^{\nu,1}| \leq \frac{1}{\nu \lambda} \int_{B^c(0, \rho)} \frac{1}{|x|^{n-\alpha}} \left| \int_{B(0, r(t))} |\Omega(x - y) - \Omega(x)| dV_t^1(y) dx \right|
\]
\[
= \frac{1}{\nu \lambda} \int_{B(0, r(t))} \frac{1}{|x|^{n-\alpha}} \int_{B^c(0, \rho)} |\Omega(x - y) - \Omega(x)| dV_t^1(y) dx,
\]
where
\[
\int_{B^c(0, \rho)} \frac{|\Omega(x - y) - \Omega(x)|}{|x|^{n-\alpha}} dx = \int_{\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^n} |\Omega(x' - y/r) - \Omega(x')| y^{1-\alpha} ds}{r^{n-\alpha}} dr
\]
\[
\leq \int_{\mathbb{R}^{n-1}} \left( \int_{0}^{1} \frac{|y|^{\alpha}}{s^{1+\alpha}} ds \right) \to 0
\]
as \( t \to 0^+ \).

Observing \( E_t^{\nu} \supset E_t^{(1+\nu),1} \setminus E_t^{\nu,2} \) and \( E_t^{(1+\nu),1} \supset F_{t,\rho}^{(1+2\nu)} \setminus G_{t,\rho}^{\nu,1} \), we deduce that
\[
|E_t^{\nu,1}| \geq |E_t^{(1+\nu),1}| - |E_t^{\nu,2}| \geq |F_{t,\rho}^{(1+2\nu)}| - |G_{t,\rho}^{\nu,1}|.
\]

Noting that \( |E_t^{\nu,2}|, |G_{t,\rho}^{\nu,1}| \to 0 \) as \( t \to 0^+ \), we have
\[
\lim_{t \to 0^+} |E_t^{\nu,1}| \geq \lim_{t \to 0^+} |F_{t,\rho}^{(1+2\nu)}|
\]
\[
= \lim_{t \to 0^+} \left( \frac{(\rho + r(t))^{n-\alpha}(1 + 2\nu)}{\rho^{n-\alpha}(1 - \epsilon(t))} \right)^{\frac{n}{n-\alpha}} \left| \left\{ x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \right| - |B(0, \rho)|
\]
\[
= (1 + 2\nu)^{\frac{n}{n-\alpha}} \left| \left\{ x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \right| - |B(0, \rho)|,
\]
where we use Property (A). Letting \( \nu \to 0 \) and \( \rho \to 0 \), we obtain
\[
\lim_{t \to 0^+} |E_t^{\nu,1}| \geq \left| \left\{ x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \right|.
\] (3.2)

Recalling the definition of \( E_t^{\nu} \) and the boundedness of \( M^2_\Omega \), we obtain
\[
\lambda \left| \left\{ x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \right|^{\frac{n}{n-\alpha}} \leq \lim_{t \to 0^+} \lambda |E_t^{\nu,1}|^{\frac{n}{n-\alpha}} \leq \lim_{t \to 0^+} \| M_\Omega^2 V_t \|_{L^{\frac{n}{n-\alpha}} \to L^{\frac{n}{n-\alpha}}} \leq \| M_\Omega^2 \|_{L^{1} \to L^{\frac{n}{n-\alpha}}}.
\]
By the arbitrary of \( \lambda \), we actually have
\[
\frac{|\Omega(x)|}{|x|^{n-\alpha}} \in L^{\frac{n}{n-\alpha}} \to \infty, \quad \text{and} \quad \left| \frac{\Omega(.)}{|x|^{n-\alpha}} \right|_{L^{\frac{n}{n-\alpha}} \to \infty} \leq \| M_\Omega^2 \|_{L^{1} \to L^{\frac{n}{n-\alpha}}} \to \infty.
\]
This completes the proof of conclusion (1).

On the other hand, by Property (B), we have
\[
\| \Omega \|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^{n-1})} = n \lambda^{\frac{n}{n-\alpha}} \left| \left\{ x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \right|,
\]
and then $\Omega \in L^{\frac{n}{2}-\alpha}(\mathbb{S}^{n-1})$, $\|\Omega\|_{L^{\frac{n}{2}-\alpha}(\mathbb{S}^{n-1})} \leq \|M^\nu_{\Omega}\|_{L^1-L^{\frac{n}{2}-\alpha}_{\infty}}$. The conclusion (2) is proved.

Next, we turn to verify conclusion (3). For $\rho > 2r_t$, $x \in B^c(0, \rho)$, we have

$$M^\nu_{\Omega}V^1_t(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)\cap B(0,r_t)} |\Omega(x-y)|dV^1_t(y)$$

$$= \sup_{|x|-r_t \leq r \leq |x|+r_t} \frac{1}{r^{n-\alpha}} \int_{B(x,r)\cap B(0,r_t)} |\Omega(x-y)|dV^1_t(y)$$

$$\leq \frac{1}{(|x|-r_t)^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y) - \Omega(x)|dV^1_t(y) + \frac{1}{(|x|-r_t)^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x)|dV^1_t(y)$$

$$\leq \frac{1}{|x|^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y) - \Omega(x)|dV^1_t(y) + \frac{\rho^{n-\alpha}}{(|x|-r_t)^{n-\alpha}} \cdot \frac{|\Omega(x)|}{|x|^{n-\alpha}}.$$  

This implies that for $x \in B(0, \rho)^c$,

$$M^\nu_{\Omega}V^1_t(x) - \frac{|\Omega(x)|}{|x|^{n-\alpha}} \leq \left( \frac{\rho^{n-\alpha}}{|x|^{n-\alpha}} - 1 \right) \frac{|\Omega(x)|}{|x|^{n-\alpha}} + \frac{2}{|x|^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y) - \Omega(x)|dV^1_t(y).$$

The above inequality together with (3.1) yields that there exists a sequence $\beta_t \to 0^+$ as $t \to 0^+$, such that

$$\left| M^\nu_{\Omega}V^1_t(x) - \frac{|\Omega(x)|}{|x|^{n-\alpha}} \right| \leq \beta_t - \frac{|\Omega(x)|}{|x|^{n-\alpha}} + \frac{2}{|x|^{n-\alpha}} \int_{B(0,r_t)} |\Omega(x-y) - \Omega(x)|dV^1_t(y).$$

Denote

$$A^t_\rho := \left\{ x \in \mathbb{R}^n : \left| M^\nu_{\Omega}V^1_t(x) - \frac{|\Omega(x)|}{|x|^{n-\alpha}} \right| > \lambda \right\},$$

$$A^{\lambda,1}_{t,\rho} := \left\{ x \in \mathbb{R}^n : \left| M^\nu_{\Omega}V^1_t(x) - \frac{|\Omega(x)|}{|x|^{n-\alpha}} \right| > \lambda \right\},$$

and $A^{\lambda,1}_{t,\rho} = F^\lambda_{t,\rho} \cap B^c(0, \rho)$. Then

$$A^t_{\rho} \subset A^{(1-\nu)\lambda,1}_{t,\rho} \cup E^\nu_{t,\rho} \subset B(0, \rho) \cup A^{(1-\nu)\lambda,1}_{t,\rho} \cup E^\nu_{t,\rho},$$

which implies that $|A^t_\rho| \leq |B(0, \rho)| + |A^{(1-\nu)\lambda,1}_{t,\rho}| + |E^\nu_{t,\rho}|$. A direct calculation yields that

$$|A^{(1-\nu)\lambda,1}_{t,\rho}| \leq \left\{ x \in B^c(0, \rho) : \beta_t - \frac{|\Omega(x)|}{|x|^{n-\alpha}} + \frac{2}{|x|^{n-\alpha}} \int_{B(0,r_t)} \Omega(x-y) - \Omega(x)|dV^1_t(y) > (1-\nu)\lambda \right\}$$

$$\leq \left\{ x \in \mathbb{R}^n : \beta_t - \frac{|\Omega(x)|}{|x|^{n-\alpha}} > (1-\nu)\lambda/2 \right\}$$

$$+ \left\{ x \in B^c(0, \rho) : \frac{2}{|x|^{n-\alpha}} \int_{B(0,r_t)} \Omega(x-y) - \Omega(x)|dV^1_t(y) > (1-\nu)\lambda/2 \right\}$$

$$\leq \left( \frac{2}{(1-\nu)\lambda} \beta_t \right) \left( \frac{\Omega(.)}{|\cdot|^{n-\alpha}} \right)_{L^{n-\alpha,\infty}} \left( \frac{n}{n-\alpha} \right) + |E^\nu_{t,\rho}| \to 0 \text{ as } t \to 0^+.$$

Recalling $E^\nu_{t,\rho} \to 0$ as $t \to 0^+$, we deduce that

$$\lim_{t \to 0^+} |A^t_\rho| \leq |B(0, \rho)| + \lim_{t \to 0^+} |A^{(1-\nu)\lambda,1}_{t,\rho}| + \lim_{t \to 0^+} |E^\nu_{t,\rho}| \leq |B(0, \rho)|, \tag{3.3}$$

which yields the desired conclusion by letting $\rho \to 0$.  

**Proof of Theorem 1.6** Since $\Omega$ satisfies the $L^{\frac{n}{2}-\alpha}$-Dini condition, we have $\Omega \in L^{\frac{n}{2}-\alpha}(\mathbb{S}^{n-1})$. By Property (B), we conclude that $\frac{|\Omega(x)|}{|x|^{n-\alpha}} \in L^{\frac{n}{2}-\alpha}_{\infty}$. 

As in the proof of Theorem 1.3, denote
\[
G_{t,\rho}^{\alpha,1} := \left\{ x \in B^c(0, \rho) : \frac{1}{|x|^{n-\alpha}} \int_{B(0, r_t)} |\Omega(x) - \Omega(x)| dV_t^1(y) > \lambda \right\}.
\]
By Minkowski’s inequality and the embedding \( L^{n/(n-\alpha)} \subset L^{\infty} \), we conclude that
\[
\sup_{\lambda > 0} \lambda |G_{t,\rho}^{\alpha,1}|^{\frac{n}{n-\alpha}} = \left\| \frac{1}{|x|^{n-\alpha}} \int_{B(0, r_t)} |\Omega(\cdot - y) - \Omega(\cdot)| dV_t^1(y) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))}
\leq \int_{B(0, r_t)} \left( \int_{B^c(0, \rho)} \frac{|\Omega(x) - \Omega(x)|}{|x|^{n-\alpha}} dx \right)^{\frac{n}{n-\alpha}} dV_t^1(y)
\leq \sup_{y \in B(0, r_t)} \left( \int_{B^c(0, \rho)} \frac{|\Omega(x) - \Omega(x)|}{|x|^{n-\alpha}} dx \right)^{\frac{n}{n-\alpha}},
\]
where
\[
\int_{B^c(0, \rho)} \frac{|\Omega(x) - \Omega(x)|}{|x|^{n-\alpha}} dx = \int_{\rho}^{\infty} \int_{S^{n-1}} \frac{|\Omega(x') - \Omega(x')|}{r^{n-\alpha}} d\sigma(x') r^{n-1} dr
\leq \int_{\rho}^{\infty} \frac{\omega_{\frac{n}{n-\alpha}}(|y|/r)}{r} \frac{n}{r^{n-\alpha}} dr = \int_{0}^{1} \frac{\omega_{\frac{n}{n-\alpha}}(s)}{s} ds.
\]
Since \( \Omega \) satisfies the \( L^{\frac{n}{n-\alpha}} \)-Dini condition, we have
\[
\int_{0}^{1} \frac{\omega_{\frac{n}{n-\alpha}}(s)}{s} ds < \infty,
\]
which implies that \( \omega_{\frac{n}{n-\alpha}}(s) \to 0 \) as \( s \to 0^+ \). Thus,
\[
\sup_{\lambda > 0} \lambda |G_{t,\rho}^{\alpha,1}|^{\frac{n}{n-\alpha}} \leq \sup_{y \in B(0, r_t)} \left( \int_{B^c(0, \rho)} \frac{|\Omega(x) - \Omega(x)|}{|x|} dx \right)^{\frac{n}{n-\alpha}}
\leq \sup_{y \in B(0, r_t)} \left( \int_{0}^{1} \frac{\omega_{\frac{n}{n-\alpha}}(s)}{s} ds \right)^{\frac{n}{n-\alpha}} \omega_{\frac{n}{n-\alpha}}(s) \to 0.
\]
as \( t \to 0^+ \). So we have
\[
\left\| \frac{1}{|x|^{n-\alpha}} \int_{B(0, r_t)} |\Omega(\cdot - y) - \Omega(\cdot)| dV_t^1(y) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))}
= \sup_{\lambda > 0} \lambda |G_{t,\rho}^{\alpha,1}|^{\frac{n}{n-\alpha}} \to 0 \text{ as } t \to 0^+.\]
Using the same method as in the proof of Theorem 1.3, there exist \( \beta_t \to 0^+ \) as \( t \to 0^+ \), such that
\[
\left| M_{t} \Omega^1(x) - \frac{\Omega(x)}{|x|^{n-\alpha}} \right| \leq \beta_t \frac{\Omega(x)}{|x|^{n-\alpha}} + \frac{2}{|x|^{n-\alpha}} \int_{B(0, r_t)} |\Omega(x) - \Omega(x)| dV_t^1(y).
\]
Then,
\[
\left\| M_{t} \Omega^1(\cdot) - \frac{\Omega(\cdot)}{|x|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} \leq \beta_t \left\| \frac{\Omega(\cdot)}{|x|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}
+ \frac{2}{|x|^{n-\alpha}} \int_{B(0, r_t)} |\Omega(\cdot - y) - \Omega(\cdot)| dV_t^1(y) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} \to 0 \text{ as } t \to 0^+.
\]
Recalling
\[ \| M_{\tilde{\Omega}^1} V_t^2(\cdot) \|_{L^{n/\alpha}_t(\mathbb{R}^n \setminus B(0, r_1))} \lesssim V_t^2(\mathbb{R}^n) = \epsilon_t \to 0 \] as \( t \to 0^+ \),
we conclude that
\[ \lim_{t \to 0^+} \left\| M_{\tilde{\Omega}}^n V_t^1(\cdot) - \frac{\Omega(\cdot)}{|x|^{n-\alpha}} \right\|_{L^{n/\alpha}_t(\mathbb{R}^n \setminus B(0, \rho))} \lesssim \lim_{t \to 0^+} \left\| M_{\tilde{\Omega}^1} V_t^1(\cdot) - \frac{\Omega(\cdot)}{|x|^{n-\alpha}} \right\|_{L^{n/\alpha}_t(\mathbb{R}^n \setminus B(0, \rho))} + \lim_{t \to 0^+} \left\| M_{\tilde{\Omega}}^n V_t^2(\cdot) \right\|_{L^{n/\alpha}_t(\mathbb{R}^n)} = 0. \]
This implies that (3) holds.

Next, note that for any fixed \( \lambda > 0 \), \( |G_{t, \rho}^{\lambda, 1}| \to 0 \) as \( t \to 0 \). By the same arguments used in the proof of Theorem 1.5, we can verify that
\[ \| \Omega \|_{L^{n/\alpha}_t(\mathbb{R}^n)} \sim \| \frac{\Omega(x)}{|x|^{n-\alpha}} \|_{L^{n/\alpha}_t(\mathbb{R}^n)} \lesssim \| M_{\tilde{\Omega}}^n \|_{L^1 \to L^{n/\alpha}_t}. \]
This completes the proof of conclusion (1) and (2). Theorem 1.6 is proved.

4. LIMITING WEAK-TYPE BEHAVIORS OF THE SINGULAR AND FRACTIONAL INTEGRAL OPERATORS

This section is devoted to the proofs of the limiting weak-type behaviors of \( T_{\tilde{\Omega}}^n \).

Proof of Theorem 1.8. Without loss of generality, we assume that \( V \) is a probability measure. For \( t > 0 \), let \( V_t^1, V_t^2, r_t, \epsilon_t \) be as in the proof of Theorem 1.1. For \( \rho > 2 r_t, x \in B^c(0, \rho) \), we have
\[ \left| T_{\tilde{\Omega}}^n V_t^1(x) - \frac{\Omega(x)}{|x|^{n-\alpha}} \right| \leq \int_{\mathbb{R}^n} \left( \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right) dV_t^1(y) + \epsilon_t |\Omega(x)| \left| \frac{x}{|x|^{n-\alpha}} \right| \]
\[ \leq \int_{\mathbb{R}^n} \left( \frac{\Omega(x-y) - \Omega(x)}{|x-y|^{n-\alpha}} \right) dV_t^1(y) + \int_{\mathbb{R}^n} \left( \frac{\Omega(x)}{|x|^{n-\alpha}} - \frac{\Omega(x-y)}{|x|^{n-\alpha}} \right) dV_t^1(y) + \epsilon_t |\Omega(x)| \left| \frac{x}{|x|^{n-\alpha}} \right| \]
\[ \leq \frac{2^{n-\alpha}}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} |\Omega(x-y) - \Omega(x)| dV_t^1(y) + \beta_t |\Omega(x)| \left| \frac{x}{|x|^{n-\alpha}} \right| + |T_{\tilde{\Omega}}^n V_t^1(x)|. \]
(4.1)
where \( \beta_t \to 0 \) as \( t \to 0^+ \). Thus, for \( x \in B^c(0, \rho) \),
\[ \left| \frac{\Omega(x)}{|x|^{n-\alpha}} \right| \leq \left| T_{\tilde{\Omega}}^n V_t^1(x) - \frac{\Omega(x)}{|x|^{n-\alpha}} \right| + |T_{\tilde{\Omega}}^n V_t^1(x)| \]
\[ \leq \frac{2^{n-\alpha}}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} |\Omega(x-y) - \Omega(x)| dV_t^1(y) + \beta_t |\Omega(x)| \left| \frac{x}{|x|^{n-\alpha}} \right| + |T_{\tilde{\Omega}}^n V_t^1(x)|. \]
This implies that
\[ \frac{(1 - \beta_t) |\Omega(x)|}{|x|^{n-\alpha}} \leq \frac{2}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} |\Omega(x-y) - \Omega(x)| dV_t^1(y) + |T_{\tilde{\Omega}}^n V_t^1(x)|. \]
Set
\[ \tilde{E}_{t, \rho}^\lambda := \{ x \in \mathbb{R}^n : T_{\tilde{\Omega}}^n V_t^1(x) > \lambda \}, \]
\[ \tilde{E}_{t, \rho}^{\lambda, 1} := \{ x \in \mathbb{R}^n : T_{\tilde{\Omega}}^n V_t^1(x) > \lambda \}, \]
\[ \tilde{E}_{t, \rho}^{\lambda, 2} := \{ x \in \mathbb{R}^n : T_{\tilde{\Omega}}^n V_t^2(x) > \lambda \}, \]
\[ \tilde{F}_{t, \rho}^\lambda = \left\{ x \in B^c(0, \rho) : \frac{(1 - \beta_t) |\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\}, \]
\[ \tilde{F}_{t, \rho}^{\lambda, 1} := \left\{ x \in B^c(0, \rho) : \frac{1}{|x|^{n-\alpha}} \int_{B(0, r_t)} |\Omega(x-y) - \Omega(x)| dV_t^1(y) > \lambda \right\}. \]
Then
\[ \tilde{F}_{t, \rho}^\lambda \subset \tilde{F}_{t, \rho}^{\lambda/2n+1-\alpha, 1} \cup \tilde{E}_{t, \rho}^{\lambda/2, 1}. \]
Consequently,

\[
\left\{ x \in \mathbb{R}^n : \frac{(1 - \beta_t)|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \leq |B(0, \rho)| + |\tilde{F}_t^\lambda| + |\tilde{G}_{t, \rho}^{1/2}| + |G_{t, \rho}^{1/2}|, \tag{4.2}
\]

where in the last inequality we use the fact

\[
\lambda \left( \frac{E_t^{\lambda/2}}{L} \right)^{n-\alpha} \leq \|T_{t, \rho}^\lambda\|_{L^1 \to L^{\frac{n-\alpha}{n-\alpha}}} V_t^1(\mathbb{R}^n) \leq \|T_{t, \rho}^\lambda\|_{L^1 \to L^{\frac{n-\alpha}{n-\alpha}}}.
\]

Recall \(|\tilde{G}_{t, \rho}^{1/2}| \to 0\) as \(t \to 0^+\) (see the proof of Theorem 1.5). Letting \(t \to \infty\) in (4.2), and then letting \(\rho \to 0\), we get

\[
\left\{ x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\} \leq \frac{2}{2} \|T_{t, \rho}^\lambda\|_{L^1 \to L^{\frac{n-\alpha}{n-\alpha}}} V_t^1(\mathbb{R}^n),
\]

which yields the desired conclusion (1). Then, conclusion (2) follows immediately from property (B) mentioned in Section 3.

Now, we turn to the proof of conclusion (3). For a fixed \(\nu > 0\), recalling that \(T_{t, \rho}^\lambda\) is boundedness from \(L^1\) to \(L^{\frac{n}{n-\alpha}}\), we obtain that

\[
\nu \lambda |\tilde{E}_t^{\nu, \lambda}|^{\frac{n-\alpha}{n-\alpha}} \lesssim V_t^2(\mathbb{R}^n) = \epsilon_t \to 0^+ \text{ as } t \to 0^+.
\]

Set

\[
\tilde{A}_t^{\lambda, 1} := \left\{ x \in \mathbb{R}^n : |T_{t, \rho}^\lambda V_t(x) - \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\},
\]

\[
\tilde{A}_t^{\lambda, 1, 1} := \left\{ x \in \mathbb{R}^n : |T_{t, \rho}^{1/2} V_t(x) - \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda \right\},
\]

and \(\tilde{A}_{t, \rho}^{\lambda, 1} := \tilde{A}_t^{\lambda, 1} \cap B^c(0, \rho)\). Observe that

\[
\tilde{A}_t^{\lambda, 1} \subset \tilde{A}_t^{(1-\nu)\lambda, 1} \cup \tilde{E}_t^{\nu, \lambda} \subset B(0, \rho) \cup \tilde{A}_t^{(1-\nu)\lambda, 1} \cup \tilde{E}_t^{\nu, \lambda}.
\]

Recalling \(\tilde{G}_{t, \rho}^{\lambda, 1} \to 0\) as \(t \to 0^+\), and recalling \(\frac{\Omega(x)}{|x|^{n-\alpha}} \in L^{\frac{n}{n-\alpha}}\) proved in conclusion (1), we use (4.1) to deduce that

\[
|\tilde{A}_{t, \rho}^{(1-\nu)\lambda, 1}| \leq \frac{2}{(1 - \nu)\lambda} \left\{ x \in \mathbb{R}^n : |\Omega(x)|^{\frac{1}{n-\alpha}} \right\}^{\frac{n}{n-\alpha}} + |\tilde{G}_{t, \rho}^{1/2}|^{\frac{n}{n-\alpha}} \to 0 \text{ as } t \to 0^+.
\]

Recalling \(\tilde{E}_t^{\nu, \lambda} \to 0\) as \(t \to 0^+\), we deduce that

\[
\lim_{t \to 0^+} |\tilde{A}_t^{\lambda}| \leq |B(0, \rho)| + \lim_{t \to 0^+} |\tilde{A}_{t, \rho}^{(1-\nu)\lambda, 1}| + \lim_{t \to 0^+} |\tilde{E}_t^{\nu, \lambda}| \leq |B(0, \rho)|,
\]

which yields the desired conclusion by letting \(\rho \to 0\). \(\square\)

**Proof of Theorem 1.5** Without loss of generality, we assume that \(V\) is a probability measure. The conclusions (1) and (2) can be verified by the same method as in proof of Theorem 1.6.

To prove the conclusion (3), we employ the same notations \(V_t^1, V_t^2, r_t, \epsilon_t\) for \(t > 0, \rho > 0, \lambda > 0\) as in the proof of Theorem 1.8.
Note that in the proof of Theorem 1.8 we have verified that
\[
\left| T_{\Omega}^*V_t^1(x) - \frac{\Omega(x)}{|x|^{n-\alpha}} \right| \leq \frac{2^{n-\alpha}}{|x|^{n-\alpha}} \int_{\mathbb{R}^n} |\Omega(x-y) - \Omega(x)| dV_t^1(y) + \frac{\beta t |\Omega(x)|}{|x|^{n-\alpha}},
\]
where \( \beta_t \to 0 \) as \( t \to 0^+ \), for \( x \in B^c(0, \rho) \).

Using the \( L^{\frac{n}{n-\alpha}} \)-Dini condition as in the proof of Theorem 1.6 we obtain
\[
\left\| T_{\Omega}^*V_t^1(\cdot) - \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} \to 0 \quad \text{as} \quad t \to 0^+.
\]
Thus,
\[
\left\| T_{\Omega}^*V_t^1(\cdot) - \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} \leq \beta_t \left\| \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \frac{2^{n-\alpha}}{|\cdot|^{n-\alpha}} \int_{B(0, \rho)} |\Omega(x-y) - \Omega(\cdot)| dV_t^1(y) \left\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} \to 0 \quad \text{as} \quad t \to 0^+.
\]

Recalling
\[
\left\| T_{\Omega}^*V_t^2(\cdot) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq \epsilon_4 \to 0 \quad \text{as} \quad t \to 0^+,
\]
we conclude that
\[
\lim_{t \to 0^+} \left\| T_{\Omega}^*V_t^1(\cdot) - \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} \leq \lim_{t \to 0^+} \left\| T_{\Omega}^*V_t^1(\cdot) - \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} + \lim_{t \to 0^+} \left\| T_{\Omega}^*V_t^2(\cdot) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} = 0,
\]
which completes the proof of (3). Theorem 1.8 is proved. \( \square \)

Proof of Corollary 1.5. If \( \Omega \in L^{\frac{n}{n-\alpha}}(S^{n-1}) \), we deduce \( \frac{\Omega(x)}{|x|^{n-\alpha}} \in L^{\frac{n}{n-\alpha}} \) by Property (B). The conclusions (2) and (3) follow from the weak Young's inequality (see Lemma 5.1) and the fact \( M_{\Omega}^r \leq T_{\Omega}^r \). The rest part of this proof follows directly from Theorems 1.5 and 1.8. \( \square \)

By the similar method, one can also deal with the maximal truncated singular integral operator defined by
\[
T_{\Omega}^*(f)(x) := \sup_{\epsilon > 0} \left| \int_{|x-y| > \epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|.
\]
(4.3)

We have the following two theorems.

**Theorem 4.1.** Let \( \alpha \in [0, n) \), \( V \) be a absolutely continuous positive measure. Suppose that \( \Omega \) is a homogeneous function of degree zero and satisfies the \( L^1 \)-Dini condition. If the maximal truncated singular integral operator \( T_{\Omega}^* \) is bounded from \( L^1 \) to \( L^{\frac{n}{n-\alpha}} \), then
\[
\text{(1)} \quad \frac{\Omega(x)}{|x|^{n-\alpha}} \in L^{\frac{n}{n-\alpha}}, \quad \left\| \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}} \lesssim \| T_{\Omega}^* \|_{L^1 \to L^{\frac{n}{n-\alpha}}} ;
\]
\[
\text{(2)} \quad \Omega \in L^{\frac{n}{n-\alpha}}(S^{n-1}), \quad \left\| \Omega \right\|_{L^{\frac{n}{n-\alpha}}(S^{n-1})} \lesssim \| T_{\Omega}^* \|_{L^1 \to L^{\frac{n}{n-\alpha}}} ;
\]
\[
\text{(3)} \quad \lim_{t \to 0^+} \left\{ x \in \mathbb{R}^n : \left| T_{\Omega}^*(V_t(x)) - \frac{\Omega(x)}{|x|^{n-\alpha}} \right| V(\mathbb{R}^n) > \lambda \right\} = 0, \quad \forall \lambda > 0.
\]

**Theorem 4.2.** Let \( \alpha \in [0, n) \), \( V \) be a absolutely continuous positive measure. Suppose that \( \Omega \) is a homogeneous function of degree zero and satisfies the \( L^{\frac{n}{n-\alpha}} \)-Dini condition. If the maximal truncated singular integral operator \( T_{\Omega}^* \) is bounded from \( L^1 \) to \( L^{\frac{n}{n-\alpha}} \), we have
\[
\text{(1)} \quad \frac{\Omega(x)}{|x|^{n-\alpha}} \in L^{\frac{n}{n-\alpha}}, \quad \left\| \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right\|_{L^{\frac{n}{n-\alpha}}} \lesssim \| T_{\Omega}^* \|_{L^1 \to L^{\frac{n}{n-\alpha}}} ;
\]
\[
\text{(2)} \quad \Omega \in L^{\frac{n}{n-\alpha}}(S^{n-1}), \quad \left\| \Omega \right\|_{L^{\frac{n}{n-\alpha}}(S^{n-1})} \lesssim \| T_{\Omega}^* \|_{L^1 \to L^{\frac{n}{n-\alpha}}} ;
\]
\[
\text{(3)} \quad \lim_{t \to 0^+} \left\| T_{\Omega}^*(V_t) - \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} V(\mathbb{R}^n) \right\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B(0, \rho))} = 0 \quad \text{for every} \quad \rho > 0.
\]
Proofs of Theorems 4.1 and 4.2. These two theorems can be proved by the similar argument as in the proofs of Theorems 1.8 and 1.9. Here, we only give the following key estimates:

Without loss of generality, we assume that $V$ is a probability measure. For $t > 0$, let $V^1_t$, $V^2_t$, $\epsilon_t$ be as in the proof of Theorem 1.8. For every $\epsilon > 0$, $x \in B(0, \rho)$, we have

\[
\left| \frac{\Omega(x-y)}{|x-y|^n} \right| dV^1_t(y) \leq \int_{|x-y| > \epsilon} \frac{\Omega(x)}{|x|^n} dV^1_t(y) + \int_{|x-y| \leq \epsilon} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dV^1_t(y)
\]

and then

\[
|T^\alpha_t(V_t)(x)| \leq \frac{\Omega(x)}{|x|^n} + \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dV^1_t(y).
\]

Also,

\[
\left| \frac{\Omega(x-y)}{|x-y|^n} \right| dV^1_t(y) \geq \int_{|x-y| > \epsilon} \frac{\Omega(x)}{|x|^n} dV^1_t(y) - \int_{|x-y| \leq \epsilon} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dV^1_t(y)
\]

and then

\[
|T^\alpha_t(V_t)(x)| \geq \sup_{\epsilon > 0} \left[ \int_{|x-y| > \epsilon} \frac{\Omega(x)}{|x|^n} dV^1_t(y) - \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dV^1_t(y) \right]
\]

\[
= \frac{\Omega(x)}{|x|^n} \int_{\mathbb{R}^n} dV^1_t(y) - \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dV^1_t(y)
\]

\[
= (1 - \epsilon_t) \Omega(x) \frac{1}{|x|^n} - \int_{\mathbb{R}^n} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dV^1_t(y).
\]

The other details are omitted here. \hfill \Box

5. Weak Young’s inequality

As we know, if $\Omega \in L^{\frac{n}{\alpha}}(\mathbb{S}^{n-1})$ for $\alpha \in (0, n)$, then the $L^1 \to L^{\frac{n}{\alpha}}$ boundedness of $T^\alpha_\Omega$ can be deduced by the following weak Young’s inequality (see [3, Theorem 1.2.13]).

Lemma 5.1. (cf. [3]) Let $1 \leq p < \infty$ and $1 < q, r < \infty$ satisfy

\[
1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}
\]

Then for $f \in L^p(\mathbb{R}^n)$, $g \in L^{r, \infty}(\mathbb{R}^n)$,

\[
\|f * g\|_{L^1} \lesssim \|g\|_{L^{r, \infty}} \|f\|_{L^p}.
\]

Suppose $\Omega \in L^{\frac{n}{\alpha}}(\mathbb{S}^{n-1})$, then $\frac{\Omega(x)}{|x|^n} \in L^\frac{n}{\alpha - \alpha} \infty(\mathbb{R}^n)$. Take $g(x) = \frac{\Omega(x)}{|x|^n}$ in Lemma 5.1 then the $L^1 \to L^{\frac{n}{\alpha - \alpha} \infty}$ boundedness of $T^\alpha_\Omega$ follows. Inspired by Theorem 1.8, a natural question is: if we replace $\frac{\Omega(x)}{|x|^n}$ by a more general function $g$ and keep the boundedness for the map $T^\alpha_g(V)(x) := \int_{\mathbb{R}^n} g(x-y) dV(y)$, what can we say for $g$, and how about the limiting behavior of $T^\alpha_g(V_t)$?

The main purpose of this section is to answer the above question. Firstly, we consider the case of that $g$ is a radial function.

Theorem 5.2. Let $\alpha \in [0, n)$, $r \in (1, \infty)$, $V$ be a absolutely continuous positive measure. Suppose that $g$ is a positive radial function and is decreasing in the radial direction. If $T^\alpha_g$ is bounded from $L^1$ to $L^{r, \infty}$, then

1. $g \in L^{r, \infty}$, $\|g\|_{L^{r, \infty}} \lesssim \|T^\alpha_g\|_{L^1 \to L^{r, \infty}}$;
Thus, for $\rho > 0$ and $\lambda > 0$, recalling that $t, \rho > 0, \epsilon_t \leq \nu \lambda$.

**Proof.** Without loss of generality, we assume that $V(\mathbb{R}^n) = 1$. For $t > 0$, let $V_t^1, V_t^2, r_t, \epsilon_t$ be as before. For $\lambda > 0$, denote

$$K_t^\lambda := \{ x \in \mathbb{R}^n : T_g V_t(x) > \lambda \},$$

and

$$K_t^{\lambda,1} := \{ x \in \mathbb{R}^n : |T_g V_t^1(x)| > \lambda \}, \quad K_t^{\lambda,2} := \{ x \in \mathbb{R}^n : |T_g V_t^2(x)| > \lambda \}.$$

For a fixed $\nu > 0$, recalling that $T_g$ is boundedness from $L^1$ to $L^{r,\infty}$, we obtain that

$$(\nu \lambda) |K_t^{\nu \lambda,2}|^{\frac{1}{r}} \lesssim V_t^2(\mathbb{R}^n) = \epsilon_t \to 0^+ \text{ as } t \to 0^+,$$

which implies $|K_t^{\nu \lambda,2}| \to 0$ as $t \to 0^+$. For a fixed $\rho > 0$, set

$$K_t^{\lambda,1,\rho} = K_t^{\lambda,1} \cap B^c(0, \rho).$$

Then

$$K_t^{\lambda} \supset K_t^{(1+\nu)\lambda,1,\rho} \supset K_t^{(1+\nu)\lambda,1} \setminus K_t^{\nu \lambda,2}.$$

For $\rho > 2r_t$, $x \in B^c(0, \rho)$,

$$T_t V_t^1(x) = \int_{B(0,r_t)} g(x-y) dV_t^1(y) \geq \int_{B(0,r_t)} g\left(\frac{x(|x| + r_t)}{|x|}\right) dV_t^1(y) \geq \int_{B(0,r_t)} g\left(\frac{x(\rho + r_t)}{\rho}\right) dV_t^1(y) = (1-\epsilon_t)g\left(\frac{x(\rho + r_t)}{\rho}\right),$$

and

$$T_t V_t^1(x) \leq \int_{B(0,r_t)} g\left(\frac{x(|x| - r_t)}{|x|}\right) dV_t^1(y) \leq \int_{B(0,r_t)} g\left(\frac{x(\rho - r_t)}{\rho}\right) dV_t^1(y) = (1-\epsilon_t)g\left(\frac{x(\rho - r_t)}{\rho}\right).$$

Thus, for $\rho > 2r_t$, $x \in B^c(0, \rho)$,

$$|T_t V_t^1(x)| \leq (1-\epsilon_t) \max\left\{ \left| g\left(\frac{x(\rho + r_t)}{\rho}\right) \right|, \left| g\left(\frac{x(\rho - r_t)}{\rho}\right) \right| \right\} = (1-\epsilon_t)g\left(\frac{x(\rho - r_t)}{\rho}\right)$$

and

$$|T_t V_t^1(x)| \geq (1-\epsilon_t) \min\left\{ \left| g\left(\frac{x(\rho + r_t)}{\rho}\right) \right|, \left| g\left(\frac{x(\rho - r_t)}{\rho}\right) \right| \right\} = (1-\epsilon_t)g\left(\frac{x(\rho + r_t)}{\rho}\right).$$

Denote

$$L^\lambda_{t,\rho} := \left\{ x \in B^c(0, \rho) : (1-\epsilon_t)g\left(\frac{x(\rho + r_t)}{\rho}\right) > \lambda \right\}. \tag{5.1}$$

Observing $L^\lambda_{t,\rho} \subset K_t^{\lambda,1,\rho}$ and recalling $K_t^{\lambda} \supset K_t^{(1+\nu)\lambda,1,\rho} \setminus K_t^{\nu \lambda,2}$, we deduce that

$$|K_t^\lambda| \geq |K_t^{(1+\nu)\lambda,1,\rho} - K_t^{(1+\nu)\lambda}| - |L^\lambda_{t,\rho}|.$$

Recalling $|K_t^{\nu \lambda,2}| \to 0$ as $t \to 0^+$, we have

$$\lim_{t \to 0^+} |K_t^\lambda| \geq \lim_{t \to 0^+} |L_{t,\rho}^{(1+\nu)\lambda}| \geq \lim_{t \to 0^+} \left\{ \left\{ x \in \mathbb{R}^n : (1-\epsilon_t)g\left(\frac{x(\rho + r_t)}{\rho}\right) > (1+\nu)\lambda \right\} - |B(0, \rho)| \right\} = \left\{ x \in \mathbb{R}^n : |g(x)| > (1+\nu)\lambda \right\} - |B(0, \rho)|.$$
Letting \( \nu \to 0 \) and \( \rho \to 0 \), we obtain
\[
\lim_{t \to 0^+} |K_t^\lambda| \geq |\{ x \in \mathbb{R}^n : |g(x)| > \lambda \}|. \tag{5.2}
\]
Recalling the definition of \( K_t^\lambda \) and the boundedness of \( T_g \), we obtain
\[
\lambda|\{ x \in \mathbb{R}^n : |g(x)| > \lambda \}| \leq \lim_{t \to 0^+} \lambda|K_t^\lambda| \leq \lim_{t \to 0^+} \|T_gV_t\|_{L^r,\infty} \leq \|T_g\|_{L^1 \to L^{r,\infty}}.
\]
By the arbitrary of \( \lambda \), we deduce
\[
g \in L^{r,\infty}, \quad \text{and} \quad \|g\|_{L^{r,\infty}} \leq \|T_g\|_{L^1 \to L^{r,\infty}},
\]
which completes the proof of conclusion (1).

Set
\[
\tilde{L}_{t,\rho}^\lambda = \left\{ x \in B^r(0, \rho) : (1 - \epsilon_t)\left| g\left(\frac{x(\rho - r_t)}{\rho}\right) \right| > \lambda \right\}.
\]
Observing \( \tilde{L}_{t,\rho}^\lambda \subseteq K_{t,\rho}^\lambda \cap K_{t,\rho}^{(1-\nu)\lambda,1} \cup K_{t,\rho}^{\nu\lambda,2} \cup B(0, \rho) \), we deduce that
\[
|K_t^\lambda| \leq |K_{t,\rho}^{(1-\nu)\lambda,1}| + |K_{t,\rho}^{\nu\lambda,2}| + |B(0, \rho)| \leq |\tilde{L}_{t,\rho}^{(1-\nu)\lambda}| + |K_{t,\rho}^{\nu\lambda,2}| + |B(0, \rho)|.
\]
Recalling \( |K_{t,\rho}^{\nu\lambda,2}| \to 0 \) as \( t \to 0^+ \), we have
\[
\lim_{t \to 0^+} |K_t^\lambda| \leq \lim_{t \to 0^+} |\tilde{L}_{t,\rho}^{(1-\nu)\lambda}| + |B(0, \rho)|
\]
\[
\leq \lim_{t \to 0^+} \left| \left\{ x \in \mathbb{R}^n : (1 - \epsilon_t)\left| g\left(\frac{x(\rho - r_t)}{\rho}\right) \right| > (1 - \nu)\lambda \right\} \right| + |B(0, \rho)|
\]
\[
= \left| \left\{ x \in \mathbb{R}^n : |g(x)| > (1 - \nu)\lambda \right\} \right| + |B(0, \rho)|.
\]
Letting \( \nu \to 0 \) and \( \rho \to 0 \), we obtain
\[
\lim_{t \to 0^+} |K_t^\lambda| \leq |\{ x \in \mathbb{R}^n : |g(x)| \geq \lambda \}|.
\]
This together with (5.2) yields the conclusions (2) and (3). Theorem 5.2 is proved. \( \square \)

The following result can be deduced by weak Young’s inequality (see Lemma 5.1) and Theorem 5.2.

**Corollary 5.3.** Let \( r \in (1, \infty) \). Suppose that \( g \) is a positive radial function which is decreasing in the radial direction. Then the following two statements are equivalent.

1. \( g \in L^{r,\infty} \).
2. \( T_g \) is bounded from \( L^1 \) to \( L^{r,\infty} \).

If we drop the assumption of radial and deceasing, add some integrability, the limiting behavior can be established as follows.

**Theorem 5.4.** Let \( r \in (1, \infty) \), \( V \) be an absolutely continuous positive measure. If \( T_g \) is bounded from \( L^1 \) to \( L^{r,\infty} \) for some \( g \in L^1(\mathbb{R}^n) \), then

1. \( g \in L^{r,\infty} \), \( \|g\|_{L^{r,\infty}} \lesssim \|T_g\|_{L^1 \to L^{r,\infty}} \);
2. \( \lim_{t \to 0^+} \left| \{ x \in \mathbb{R}^n : |T_g(V_t)(x) - g(x)| > \lambda \} \right| = 0 \) for every \( \lambda > 0 \).

**Proof.** Without loss of generality, we assume that \( V(\mathbb{R}^n) = 1 \). For \( t > 0 \), \( \lambda > 0 \), let \( V_t^1, V_t^2, \epsilon_t, K_t^\lambda, K_t^{(1+\nu)\lambda,1}, K_t^{\nu\lambda,2} \) be as in the proof of Theorem 5.2. Then
\[
K_t^\lambda \supset K_t^{(1+\nu)\lambda,1} \setminus K_t^{\nu\lambda,2}.
\]
For a fixed \( \nu > 0 \), recalling that \( T_g \) is boundedness from \( L^1 \) to \( L^{r,\infty} \), we obtain that
\[
(\nu \lambda)|K_t^{\nu\lambda,2}| \lesssim V_t^2(\mathbb{R}^n) = \epsilon_t \to 0^+ \text{ as } t \to 0^+.
\]
For $\rho > 2r_t$, $x \in B^c(0, \rho)$,

$$|T_g V_i^1(x)| = \left| \int_{B(0, r_t)} g(x - y) dV_i^1(y) \right|$$

$$= \left| \int_{B(0, r_t)} g(x - y) - g(x) dV_i^1(y) + (1 - \epsilon_t) g(x) \right|$$

$$\geq (1 - \epsilon_t) |g(x)| - \left| \int_{B(0, r_t)} g(x - y) - g(x) dV_i^1(y) \right|.$$  

Set

$$R_t^\lambda = \{ x \in \mathbb{R}^n : (1 - \epsilon_t) |g(x)| > \lambda \}$$

and

$$S_t^{\nu, 1} = \left\{ x \in \mathbb{R}^n : \left| \int_{B(0, r_t)} (g(x - y) - g(x)) dV_i^1(y) \right| > \lambda \right\}.$$  

Now, we estimate $|S_t^{\nu, 1}|$ for fixed $\nu > 0$. Using Chebychev’s inequality, we conclude that

$$|S_t^{\nu, 1}| \leq \frac{1}{\nu \lambda} \int_{\mathbb{R}^n} \left| \int_{B(0, r_t)} (g(x - y) - g(x)) dV_i^1(y) \right| dx$$

$$\leq \frac{1}{\nu \lambda} \int_{B(0, r_t)} \int_{\mathbb{R}^n} |g(x - y) - g(x)| dx dV_i^1(y)$$

$$\leq \frac{1}{\nu \lambda} \sup_{y \in B(0, r_t)} \int_{\mathbb{R}^n} |g(x - y) - g(x)| dx \to 0 \text{ as } t \to 0^+,$$

where we use the average continuous of for $g \in L^1$.

Observing $K_t^{(1+\nu)\lambda} \supset R_t^{(1+2\nu)\lambda} \setminus S_t^{\nu, 1}$ and recalling $K_t^\lambda \supset K_t^{(1+\nu)\lambda} \setminus K_t^{\nu, 2}$, we deduce that

$$|K_t^\lambda| \geq |K_t^{(1+\nu)\lambda}| - |K_t^{\nu, 2}| \geq |R_t^{(1+2\nu)\lambda}| - |S_t^{\nu, 1}| = | \{ x \in \mathbb{R}^n : (1 - \epsilon_t) |g(x)| > (1 + 2\nu) \lambda \} |.$$

Recalling $|K_t^{\nu, 2}|, |S_t^{\nu, 1}| \to 0$ as $t \to 0^+$, we have

$$\lim_{t \to 0^+} \frac{|K_t^\lambda|}{|K_t^{(1+\nu)\lambda}|} \geq \lim_{t \to 0^+} \frac{|R_t^{(1+2\nu)\lambda}|}{|R_t^{(1+2\nu)\lambda}|} = \lim_{t \to 0^+} \frac{| \{ x \in \mathbb{R}^n : (1 - \epsilon_t) |g(x)| > (1 + 2\nu) \lambda \} |}{| \{ x \in \mathbb{R}^n : |g(x)| > (1 + 2\nu) \lambda \} |}.$$

Letting $\nu \to 0$, we obtain

$$\lim_{t \to 0^+} \frac{|K_t^\lambda|}{|K_t^{(1+\nu)\lambda}|} \geq | \{ x \in \mathbb{R}^n : |g(x)| > \lambda \} |. \quad (5.3)$$

Recalling the definition of $K_t^\lambda$ and the boundedness of $T_g$, we obtain

$$\lambda \{ x \in \mathbb{R}^n : |g(x)| > \lambda \} \leq \lim_{t \to 0^+} \frac{\lambda |K_t^\lambda|}{|K_t^{(1+\nu)\lambda}|} = \lim_{t \to 0^+} \| T_g V_i \|_{L^r, \infty} \leq \| T_g \|_{L^1 \to L^r, \infty}.$$  

Since $\lambda > 0$ is arbitrary, we deduce $g \in L^{r, \infty}$ and $\|g\|_{L^r, \infty} \leq \|T_g\|_{L^1 \to L^r, \infty}$, which completes the proof of conclusion (1).

Next, we turn to verify conclusion (2). For $x \in \mathbb{R}^n$, we have

$$|T_g V_i^1(x) - g(x)| = \left| \int_{B(0, r_t)} (g(x - y) - g(x)) dV_i^1(y) - \epsilon_t g(x) \right|$$

$$\leq \epsilon_t |g(x)| + \left| \int_{B(0, r_t)} (g(x - y) - g(x)) dV_i^1(y) \right|$$

Denote

$$Q_t^\lambda = \{ x \in \mathbb{R}^n : |T_g V_i(x) - g(x)| > \lambda \},$$

$$Q_t^{\nu, 1} = \{ x \in \mathbb{R}^n : |T_g V_i^1(x) - g(x)| > \lambda \}.$$
Then

\[ Q_t^\lambda \subset Q_t^{(1-\nu)\lambda,1} \cup K_t^{\nu,2}. \]

A direct calculation yields that

\[
|Q_t^{(1-\nu)\lambda,1}| \leq \left| \left\{ x \in \mathbb{R}^n : |T_gV_1^t(x) - g(x)| > (1 - \nu)\lambda \right\} \right|
\leq \left| \left\{ x \in \mathbb{R}^n : \epsilon_t g(x) > (1 - \nu)\lambda/2 \right\} \right|
+ \left| \left\{ x \in \mathbb{R}^n : \int_{B(0, r_t)} g(x - y) - g(x) dV_1^t(y) > (1 - \nu)\lambda/2 \right\} \right|
\leq \left( \frac{2}{(1 - \nu)\lambda} \| g \|_{L^r, \infty} \right)^r + |S_t^{(1-\nu)\lambda/2,1}| \to 0 \text{ as } t \to 0^+.
\]

Recalling \( |K_t^{\nu,2}| \to 0 \text{ as } t \to 0^+ \), we deduce that

\[
\lim_{t \to 0^+} |Q_t^\lambda| \leq \lim_{t \to 0^+} |Q_t^{(1-\nu)\lambda,1}| + \lim_{t \to 0^+} |K_t^{\nu,2}| = 0,
\]

which is the desired conclusion. \( \square \)

**Remark 5.5.** Note that Proposition 2.1 does not work in the case of Theorem 5.4, since the set \( \{ x \in \mathbb{R}^n : |g(x)| = \lambda \} \) may have positive measure for some \( \lambda > 0 \). Thus, we can not deduce type-3 convergence from the conclusion of type-2 convergence obtained in Theorem 5.4.

**Corollary 5.6.** Let \( r \in (1, \infty) \). Suppose \( g \in L^1(\mathbb{R}^n) \). Then the following two statements are equivalent.

1. \( g \in L^{r, \infty} \).
2. \( T_g \) is bounded from \( L^1 \) to \( L^{r, \infty} \).

This corollary follows immediately from the weak Young’s inequality (see Lemma 5.1) and Theorem 5.4. We omit the details here.

**References**

[1] Y. Ding and X. Lai, Weak type \((1, 1)\) behavior for the maximal operator with \( L^1\)-Dini kernel, Potential Anal. 47(2) (2017), 169-187.
[2] Y. Ding and X. Lai, \( L^1\)-Dini conditions and limiting behavior of weak type estimates for singular integrals, Rev. Mat. Iberoam. (to appear).
[3] L. Grafakos, Classical Fourier Analysis, Graduate Texts in Mathematics, 2008, 249.
[4] P. Janakiraman, Limiting weak-type behavior for singular integral and maximal operators, Trans. Amer. Math. Soc. 358 (2006), 1937-1952.