Well-Posedness and Comparison Principle for Option Pricing with Switching Liquidity

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Abstract

We consider an integro-differential equation derived from a system of coupled parabolic PDE and an ODE which describes an European option pricing with liquidity shocks. We study the well-posedness and prove comparison principle for the corresponding initial value problem.

1 Introduction

This work is devoted to the study of an initial value problem of the following form

$$\begin{align*}
\frac{\partial u}{\partial \tau} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} &= -\nu_{01} e^{\gamma h(S, \tau)} \left( \nu_{10} \int_0^\tau e^{-u(S, s)} ds + e^{-\gamma h(S)} \right) + \kappa, \\
u(S, 0) &= \gamma h(S).
\end{align*}$$

Here \( \tau \in [0, T], S \in (0, +\infty), h(S) \) is a given function and \( \sigma, \nu_{01}, \nu_{10}, \kappa \) and \( \gamma \) are constants.

The integro-differential equation in (1) is derived from a system of coupled parabolic PDE and ODE which is suggested by M. Ludkovski and Q. Shen [6] in European option pricing in a financial market switching between two states - a liquid state \( (0) \) and an illiquid \( (1) \) one. We briefly describe their model. First, it is assumed that the dynamics of the liquidity is represented by a continuous-time Markov chain \( (M_t) \) with intensity rates of the transitions \( 0 \to 1 \) and \( 1 \to 0 \) and determined by the constants \( \nu_{01} \) and \( \nu_{10} \), respectively. During the liquid phase \( (M_t = 0) \) the market dynamics follows the classical Black-Scholes model. More precisely, the price \( S_t \) of a stock is modelled by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with drift \( \mu \) and volatility \( \sigma \) and a standard one-dimensional Brownian motion \( (W_t) \) which is independent of the Markov chain \( (M_t) \) (under the “real world” probability \( P \)). Then the wealth process \( (X_t) \) satisfies

$$dX_t = \mu \pi_t X_t dt + \sigma \pi_t X_t dW_t,$$

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where $\pi_t$ denotes the proportion of stock holdings in the total wealth $X_t$. For simplicity, it is assumed that the interest rate of the riskless asset is zero.

Respectively, in the illiquid phase ($M_t = 1$), the market is static and trading in stock is not permitted, i.e., $dS_t = dX_t = 0$.

The presence of liquidity shocks is a source of non-traded risk and makes the market incomplete. Ludkovski and Shen investigate expected utility maximization with exponential utility function:

$$u(x) = -e^{-\gamma x},$$

where $\gamma > 0$ is the investor’s risk aversion parameter. The value functions $\hat{U}^i(t, X, S)$, $i = 0, 1$ for the optimal investment problem are defined as follows:

$$\hat{U}^i(t, X, S) := \sup_{\pi_t} \mathbb{E}^p_{t, X, S, i} \left[ -e^{-\gamma (X_T + h(S_T))} \right], \quad i = 0, 1,$$

where $\mathbb{E}^p_{t, X, S, i}$ is the expectation under the measure $\mathbb{P}$ with starting values $S_t = S$, $X_t = X$ and $M_t = i$. The supremum above is taken over all admissible trading strategies ($\pi_t$) and the function $h(S)$ denotes the terminal payoff of a contingent claim. Standard stochastic control methods and the properties of the exponential utility function imply that the value functions can be presented by

$$\hat{U}^i(t, X, S) = -e^{-\gamma X} e^{-\gamma R^i(t, S)}, \quad i = 0, 1,$$

where $R^i(t, S)$ are the unique viscosity solutions of the system (2)

$$\left\{ \begin{array}{l}
  R^0_1 + \frac{1}{2} \sigma^2 S^2 R^0_S - \frac{\mu_0}{\gamma} e^{-\gamma (R^1 - R^0)} + \frac{d_0 + \nu_0}{\gamma} = 0, \\
  R^1_1 - \frac{\nu_1}{\gamma} e^{-\gamma (R^2 - R^1)} + \frac{d_0 + \nu_1}{\gamma} = 0,
\end{array} \right. $$

with the terminal condition $R^i(T, S) = h(S)$, $i = 0, 1$. Here $d_0 := \mu_2 / 2\sigma^2$.

Let $p$ and $q$ denote the buyer’s indifference prices corresponding to liquid and illiquid initial state respectively. They are defined as follows: $\hat{U}^0(t, X - p, S) = \hat{V}^0(t, X)$ and $\hat{U}^1(t, X - q, S) = \hat{V}^1(t, X)$ where $\hat{V}^i$, $i = 0, 1$ are the value functions of the Merton optimal investment problem (i.e. the case when $h(S) \equiv 0$). It can be shown that $p$ and $q$ satisfy a system of differential equations which is quite similar to (2) (see [15]). In fact, 

$$p = R^0 + \gamma^{-1} \ln F_0(t) \quad \text{and} \quad q = R^1 + \gamma^{-1} \ln F_1(t)$$

where

$$F_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$F_1(t) = \frac{1}{\nu_0} \left( c_1 \left( d_0 + \nu_0 - \lambda_1 \right) e^{\lambda_1 t} + c_2 \left( d_0 + \nu_0 - \lambda_2 \right) e^{\lambda_2 t} \right)$$

$$\lambda_{1,2} = \frac{d_0 + \nu_0 + \nu_1 \pm \sqrt{(d_0 + \nu_0 + \nu_1)^2 - 4d_0 \nu_1}}{2},$$

$$c_1 = \frac{\lambda_2 - d_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 T}, \quad \text{and} \quad c_2 = \frac{\lambda_1 - d_0}{\lambda_1 - \lambda_2} e^{-\lambda_2 T}. $$

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Indifference pricing was first used in the pioneering paper of Hodges and Neuberger [3]. We refer also to [2] for further applications (see [4] and [8] as well).

The existence of classical solutions was proved in [6] when the payoff function $h(S)$ is bounded. This case is restrictive since it does not include such typical example as the call option $h = \max\{S - K, 0\}$ with strike price $K$. We investigate the solvability of the problem and prove the existence and uniqueness of a weak solution in suitable Sobolev weighted spaces which allows unbounded terminal payoff functions.

The integro-differential equation (1) is derived from (2) as follows. Denote $r^0 := \gamma R^0$, $r^1 = \gamma R^1$. The system of differential equations for $r^0$ and $r^1$ has the following form:

\[
\begin{align*}
\frac{d^2}{dt^2} r^0_S - \frac{1}{2} \sigma^2 S^2 r^0_{SS} &= -\nu_{01} e^{-(r^1 - r^0)} + d_0 + \nu_{01} \\
\frac{d}{dt} r^1_S &= -\nu_{10} e^{-(r^0 - r^1)} + \nu_{10}
\end{align*}
\]

where $\tau = T - t$. The ODE in (3) can be solved explicitly with respect to $r^1$. Then we obtain the initial value problem (1) under the substitution $u := r^0 - \nu_{10} \tau$ and $\kappa := d_0 + \nu_{01} - \nu_{10}$.

The paper is organized as follows. In Section 2 we prove a comparison principle (Theorem 2.1) for classical solutions to the problem (1). Then, in Section 3 we prove a comparison principle (Theorem 3.4) for weak sub/super solutions. In addition, we study the existence and uniqueness of weak solutions in a suitable weighted Sobolev space (see Theorem 3.7).

2 Comparison principle for classical solutions

In this section we consider solutions of (1) satisfying

\[ |u|, |h| \leq A \exp (\alpha \ln S) = AS^\alpha \ln S, \]

for some positive constants $A$ and $\alpha$. Note that conditions (4) include for example linear growth, polynomial and powers of $S$ with arbitrary exponent.

We prove the following comparison principle:

**Theorem 2.1.** Let $u_1, u_0 \in C((0, +\infty) \times [0, T)) \cap C^{2,1}((0, +\infty) \times (0, T))$ be two classical solutions of (1) corresponding to the initial data $h = h_1$ and $h = h_0$, respectively and such that the conditions (4) hold. Then

\[ \gamma \inf (h_1 - h_0) \leq u_1 - u_0 \leq \gamma \sup (h_1 - h_0). \]

We will only prove the lower bound in (5) since the upper one follows immediately from it. In addition, we can assume that $\bar{h} := \gamma \inf (h_1 - h_0) > -\infty$,

otherwise the left inequality in (5) is trivial. We will use the following auxiliary lemma
Lemma 2.2. Let \( u_1 \) and \( u_0 \) be as in Theorem 2.1 and \( \tau_1 \geq 0 \) be such that \( u_1(S, \tau) - u_0(S, \tau) \geq \bar{h} \) for any \( \tau \in [0, \tau_1] \). Then, there exists a constant \( \bar{\tau} > 0 \) such that \( u_1(S, \tau) - u_0(S, \tau) \geq \bar{h} \) for any \( \tau \in [0, \tau_1 + \bar{\tau}] \). In addition, \( \bar{\tau} \) depends only on \( \alpha \) defined in 4 and \( \sigma \).

Proof. Let \( u_1 \) and \( u_0 \) be two solutions of (4) corresponding to the initial conditions \( u_1(S, 0) = \gamma h_1(S) \) and \( u_0(S, 0) = \gamma h_0(S) \). Denote \( \bar{u} = u_1 - u_0 \), \( \bar{h} = \gamma (h_1 - h_0) \), \( u_\xi = \xi u_1 + (1 - \xi) u_0 \), \( h_\xi = \xi h_1 + (1 - \xi) h_0 \), for \( \xi \in [0, 1] \) and define

\[
F[\tau; u, g] := -\nu_0 e^{u(\tau)} \left( \nu_1 \int_0^\tau e^{-u(s)} ds + e^{-g} \right) + \kappa.
\]

Then

\[
F[\tau; u_1, \gamma h_1] - F[\tau; u_0, \gamma h_0] = \int_0^1 \frac{d}{d\xi} \left( F[\tau; u_\xi, \gamma h_\xi] \right) d\xi \tag{6}
\]

\[
= -\nu_0 \bar{u} \int_0^1 e^{u_\xi(\tau)} \left( \nu_1 \int_0^\tau e^{-u_\xi(s)} ds + e^{-\gamma h_\xi} \right) d\xi \tag{7}
\]

\[
+ \nu_0 \int_0^1 e^{u_\xi(\tau)} \left( \nu_1 \int_0^\tau e^{-u_\xi(s)} \bar{u}(s) ds + e^{-\gamma h_\xi} \bar{h} \right) d\xi
\]

\[
= -\nu_0 \nu_1 \int_0^1 \int_0^\tau e^{u_\xi(\tau)-u_\xi(s)} (\bar{u}(\tau) - \bar{u}(s)) ds d\xi \tag{8}
\]

\[
- \nu_0 \left( \bar{u}(\tau) - \bar{h} \right) \int_0^1 e^{u_\xi(\tau)-\gamma h_\xi} d\xi
\]

and

\[
\bar{u}_\tau - \frac{1}{2} \sigma^2 S^2 \bar{u}_{SS} = -\nu_0 \nu_1 \int_0^\tau (\bar{u}(\tau) - \bar{u}(s)) ds \int_0^1 e^{u_\xi(\tau)-u_\xi(s)} d\xi \tag{9}
\]

\[
- \nu_0 \left( \bar{u}(\tau) - \bar{h} \right) \int_0^1 e^{u_\xi(\tau)-\gamma h_\xi} d\xi
\]

Next, define

\[
\omega(S, \tau) := \frac{1}{\sqrt{T_1 - \tau}} \exp \left( \frac{(\ln S - \frac{1}{2} \sigma^2 (T_1 - \tau))^2}{2 \sigma^2 (T_1 - \tau)} \right), \tag{10}
\]

where \( T_1 > 0 \) and \( (S, \tau) \in (0, +\infty) \times [0, T_1] \). Note that \( \mathcal{L}_{BS} \omega = \omega_\tau - \frac{1}{2} \sigma^2 S^2 \omega_{SS} = 0 \) and \( \omega \) is increasing with respect to \( \tau \) in the interval \( \tau \in [T_1 - 4/\sigma^2, T_1] \). Choose \( T_1 > \tau_1 \) in (10) such that the inequality

\[
\alpha < \frac{1}{2\sigma^2 (T_1 - \tau)},
\]

holds for all \( \tau \in [\tau_1, T_1] \) and \( T_1 - 4/\sigma^2 < \tau_1 \). It is enough to define \( T_1 := \tau_1 + \bar{\tau} \),

\[4\]
where $0 < \bar{\tau} < \min \left\{ \left(2\sigma^2\alpha\right)^{-1}, 4/\sigma^2 \right\}$. Next, let $\varphi_\epsilon = \bar{u} + \epsilon \omega$. Then

$$
(\varphi_\epsilon)_\tau = -\frac{1}{2} \sigma^2 S \varphi_\epsilon_S S - \nu_0 \nu_1 \int_0^\tau (\bar{u}(\tau) - \bar{u}(s)) ds \int_0^1 e^{u_\tau(s) - u_\epsilon(s)} d\xi \\
- \nu_0 \left(\bar{u}(\tau) - \bar{h}\right) \int_0^1 e^{u_\tau(s) - h} d\xi \tag{11}
$$

$$
\geq -\nu_0 \nu_1 \int_0^\tau ds \int_0^1 e^{u_\tau(s) - u_\epsilon(s)} d\xi \tag{12}
$$

$$
- \nu_0 \left(\bar{u}(\tau) - \bar{h}\right) \int_0^1 e^{u_\tau(s) - h} d\xi
$$

We will prove that $\varphi_\epsilon \geq \bar{h}$ for any $\tau \in [\tau_1, T_1]$. Indeed, assume by contradiction that $\inf \varphi_\epsilon < \bar{h}$. Note that $\varphi_\epsilon |_{\tau = \tau_1} > \bar{h}$ and there exist $S$ and $\bar{S}$ such that $\varphi_\epsilon > \bar{h}$ if either $S \leq \bar{S}$ or $S > \bar{S}$. In fact, $\varphi_\epsilon \rightarrow +\infty$ uniformly when either $\ln S \rightarrow +\infty$ or $\tau \rightarrow T_1$. The last observations imply that $\varphi_\epsilon$ attains minimum in an interior point $(S_*, \tau_*) \in (S, \bar{S}) \times (\tau_1, T_1)$ and $\varphi_\epsilon(S_*, \tau_*) < \bar{h}$. Then, $(\varphi_\epsilon)_\tau(S_*, \tau_*) = 0$, $(\varphi_\epsilon)_S(S_*, \tau_*) \geq 0$ and

$$
\bar{u}(S_*, \tau_*) - \bar{h} \leq \bar{u}(S_*, \tau_*) - \bar{h} = \varphi_\epsilon(S_*, \tau_*) - \bar{h} - \epsilon \omega(S_*, \tau_*) < 0 \tag{13}
$$

$$
\bar{u}(S_*, \tau_*) - \bar{u}(S_*, s) = \varphi_\epsilon(S_*, \tau_*) - \varphi_\epsilon(S_*, s) - \epsilon (\omega(S_*, \tau_*) - \omega(S_*, s)) < 0, \quad \forall s \in [\tau_1, \tau_*], \tag{14}
$$

since $\omega$ is increasing in $\tau$. Thus the right hand side of (12) is positive, a contradiction. Hence $\varphi_\epsilon = \bar{u} + \epsilon \omega \geq \bar{h}$ for any $\tau \in [\tau_1, T_1]$. Let $\epsilon \rightarrow 0$. Then $\bar{u} = u_1 - u_0 \geq \bar{h}$ for any $\tau \in [\tau_1, T_1]$. \qed

**Proof.** (of Theorem 2.1) The comparison principle follows by induction and the auxiliary Lemma 2.2 we first take $\tau_1 = 0$ and prove it in the interval $[0, 1/2\bar{\tau}]$, then let $\tau_1 = 1/2\bar{\tau}$ and consider the interval $[1/2\bar{\tau}, \bar{\tau}]$ and etc.

Now, as a corollary we formulate comparison principle for the buyer’s indifference prices $p(S, t)$, $q(S, t)$ which satisfy the terminal value problem

$$
\begin{align*}
& p_t + \frac{1}{2} \sigma^2 S^2 p_{SS} - \frac{\nu_0}{\gamma} F_0 e^{-\gamma(p)} + \frac{d_0}{\gamma} + \frac{\nu_1}{\gamma} F_1 e^{-\gamma(q)} = 0 \\
& q_t - \frac{\nu_0}{\gamma} F_0 e^{-\gamma(p)} + \frac{\nu_1}{\gamma} F_1 e^{-\gamma(q)} = 0 \\
& p(S, T) = q(S, T) = h(S)
\end{align*}
\tag{15}
$$

By classical solutions of (15) we mean functions such that $p \in C((0, +\infty) \times (0, T]) \cap C^2((0, +\infty) \times (0, T))$, $q \in C((0, +\infty) \times (0, T])$, $q_t \in C((0, +\infty) \times (0, T))$.

Note that

$$
\gamma p = \nu_0 (T - t) + \ln F_0(t) + u(S, T - t), \tag{16}
$$

$$
\gamma q = \nu_0 (T - t) + \ln F_1(t) - \ln \left( \nu_0 \int_0^{T-t} e^{-u(S,s)} ds + e^{-\gamma h(S)} \right), \tag{17}
$$

5
since \( p(t) = \gamma^{-1} \left( r^0 + \ln F_0(t) \right) \) and \( q(t) = \gamma^{-1} \left( r^1 + \ln F_1(t) \right) \). Then, a comparison principle in \((p, q)\) solutions will be equivalent to a comparison principle for the \((r^0, r^1)\) variables.

We consider growth conditions analogous to (14)

\[
|p|, |q| \leq A \exp (\alpha \ln^2 S) = AS^\alpha \ln S, \quad (18)
\]

for some positive constants \( A \) and \( \alpha \).

**Corollary 2.3.** Let \((p_1, q_1)\) and \((p_0, q_0)\) be two classical solutions of the system (15) corresponding to terminal data \( h \equiv h_1(S) \) and \( h \equiv h_0(S) \), respectively. If there exist some positive constants \( A \) and \( \alpha \) such that \( p_i(S, t) \) and \( h_i(S) \), \( i = 0, 1 \) satisfy the conditions (18), then

\[
\inf (h_1 - h_0) \leq p_1(S, t) - p_0(S, t) \leq \sup (h_1 - h_0), \quad (19)
\]

\[
\inf (h_1 - h_0) \leq q_1(S, t) - q_0(S, t) \leq \sup (h_1 - h_0). \quad (20)
\]

In particular, let \( h(S) \) be bounded from below (or from above) by a constant, i.e. \( h(S) \geq h_* \) (resp. \( h(S) \leq h^* \)) and \( p(S, t), q(S, t) \), be a classical solutions of the terminal value problem (15) satisfying (18). Then

\[
p(S, t) \geq h_* \text{ and } q(S, t) \geq h_* \text{ (respectively } p(S, t) \leq h^* \text{ and } q(S, t) \leq h^*),
\]

for any \( S \in (0, +\infty) \) and any \( t \in (0, T] \).

**Proof.** The inequalities (19) follow immediately from Theorem 2.1 and representation (18). In order to prove (20) we will use (17), i.e.

\[
q_i(\cdot, t) = \gamma^{-1} \left[ \nu_{10} (T - t) + \ln F_1(t) - \ln \left( \nu_{10} \int_0^{T-t} e^{-u(\cdot, s)} ds + e^{-\gamma h_i(\cdot)} \right) \right],
\]

for \( i = 0, 1 \). Similarly to the proof of Lemma 2.2 we derive

\[
q_1(\cdot, t) - q_0(\cdot, t) = -\gamma^{-1} \int_0^t \frac{d}{d\xi} \left[ \ln \left( \nu_{10} \int_0^{T-t} e^{-u(\cdot, s)} ds + e^{-\gamma h_i(\cdot)} \right) \right] d\xi
\]

\[
= \gamma^{-1} \int_0^t \nu_{10} \frac{\int_0^{T-t} e^{-u(\cdot, s)} (u_1(\cdot, s) - u_0(\cdot, s)) ds}{\nu_{10} \int_0^{T-t} e^{-u(\cdot, s)} ds + e^{-\gamma h_i(\cdot)}} d\xi
\]

\[
+ (h_1(\cdot) - h_0(\cdot)) \int_0^t \frac{e^{-\gamma h_i(\cdot)}}{\nu_{10} \int_0^{T-t} e^{-u(\cdot, s)} ds + e^{-\gamma h_i(\cdot)}} d\xi
\]

Now, (3) implies the estimates (20).

The second part follows immediately due to the fact that \( p_*(S, t) \equiv h_* \) and \( q_*(S, t) \equiv h_* \) are the solutions of the problem (15) with constant terminal condition \( h \equiv h_* \). Indeed, if we formally substitute \( p_*(S, t) \equiv h_* \) and \( q_*(S, t) \equiv h_* \) in (15), then we arrive at the conclusion that it is sufficient to check the following identities

\[
- \frac{\nu_{10} F_1}{\gamma F_0} + \frac{d_0 + \nu_{10}}{\gamma} - \frac{1}{\gamma} \frac{F_0'}{F_0} = 0, \quad (21)
\]

\[
- \frac{\nu_{10} F_0}{\gamma F_1} + \frac{\nu_{10}}{\gamma} - \frac{1}{\gamma} \frac{F'_1}{F_1} = 0, \quad (22)
\]
or equivalently

\[
\begin{align*}
F'_0 &= -\nu_{01} F_1 + (d_0 + \nu_{01}) F_0, \\
F'_1 &= -\nu_{10} F_0 + \nu_{10} F_1,
\end{align*}
\]

which follow directly from the definition of \( F_0 \) and \( F_1 \).

## 3 Existence of weak solutions

In this section we study the existence and uniqueness of weak solutions in suitable function spaces. First we introduce the weighted \( L^2 \) space

\[
L^2_w := \left\{ u : \|u\|_0^2 := \int_0^{+\infty} u^2(S) w(S) dS < \infty \right\},
\]

given a weight function \( w > 0 \). Then we define a weighted Sobolev space as follows

\[
H^1_w := \left\{ u : u \in L^2_w \text{ s.t. } Su'(S) \in L^2_w \right\},
\]

with norm \( \|\cdot\|_1 \) such that

\[
\|u\|_1^2 = \|u\|_0^2 + \|Su'\|_0^2.
\]

Let \( \xi : [0, +\infty) \to [0, 1] \) be increasing, infinitely continuously differentiable function and such that \( \xi \equiv 0 \) on \([0, 1/2]\) and \( \xi \equiv 1 \) on \([1, +\infty)\). We will use \( \xi \) to construct a sequence \( \{u_\epsilon\} \) of compactly supported functions converging in \( H^1_w \) to a given element \( u \in H^1_w \). More precisely, the following auxiliary result holds.

**Lemma 3.1.** Let \( \xi_\epsilon(x) := \xi(x/\epsilon) \left[ 1 - \xi(x\epsilon/2) \right] \), \( 0 < \epsilon < 1 \) and \( u_\epsilon := \xi_\epsilon u \). Then \( u_\epsilon \to u \) in \( H^1_w \), as \( \epsilon \to 0 \).

**Proof.** Note that

\[
(u - u_\epsilon)' = (1 - \xi_\epsilon) u' - \xi_\epsilon' u,
\]

is uniformly bounded with respect to \( \epsilon \) and \( 1 - \xi_\epsilon \to 0 \) as well as \( S\xi_\epsilon(S) \to 0 \) as \( \epsilon \to 0 \). Then the Lebesgue's dominated convergence theorem implies that \( \|u - u_\epsilon\| \to 0 \) as \( \epsilon \to 0 \).

Next, let \( u(S) \) be twice continuously differentiable on \((0, +\infty)\) and denote the operator \( \mathcal{L}u := -\frac{1}{2} \sigma^2 S^2 u'' \). Then after integration by parts we formally obtain:

\[
(\mathcal{L}u, v)_{L^2_w} = -\frac{1}{2} \sigma^2 \int_0^{+\infty} w S^2 u'' v dS
= \frac{1}{2} \sigma^2 \int_0^{+\infty} \left[ w S^2 u' v' + \left( \frac{S u'}{w} + 2 \right) w S u' v \right] dS,
\]

provided that the integrals above are well-defined, \( w \) is continuously differentiable and \( w S^2 u' v \to S \to 0 \) and \( S \to +\infty \). For example, the above holds when \( v \) is continuously differentiable and with compact support.

Following the above observations we introduce the bilinear form:

\[
a(u, v) := \frac{1}{2} \sigma^2 \int_0^{+\infty} w S u' \left[ S v' + \left( \frac{S u'}{w} + 2 \right) v \right] dS.
\]
If the weight function $w$ is twice continuously differentiable, and there exists a constant $C > 0$, such that

$$\left| \frac{S w'(S)}{w(S)} \right| , \left| \frac{S^2 w''(S)}{w(S)} \right| \leq C, \forall S \in (0, +\infty). \tag{26}$$

then the bilinear form $a(u, v)$ is continuous and semi-coercive on $H^1_w$, i.e.,

$$|a(u, v)| \leq c \|u\|_1 \|v\|_1, \quad \forall u, v \in H^1_w \tag{27}$$

$$a(u, u) \geq \alpha \|u\|_2^2 - \beta \|u\|_0^2, \quad \forall u \in H^1_w \tag{28}$$

for some suitable constants $c > 0$, $\alpha > 0$ and $\beta > 0$ which are independent of $u$ and $v$.

We can choose such weight function that the call option payoff function $h = \max\{S - K, 0\}$ belongs to the space $H^1_w$, for example, take $w := (1 + S)\gamma$, where $\gamma < -3$.

In addition, we assume that

$$\theta := \int_0^{+\infty} w(S)dS < +\infty. \tag{29}$$

This assumption guarantees that any bounded and measurable function belongs to $L^2_w$.

**Lemma 3.2.** There exists a constant $c_0 > 0$ such that

$$|u(S)|^2 \leq c_0 \|u\|_1^2 \frac{1}{S} \exp(C|\ln S|), \quad \forall u \in H^1_w, \tag{30}$$

where $C$ satisfies (26).

**Proof.** Note that there exists a constant $c_0$ such that

$$|u(1)|^2 \leq c_0 \|u\|_1^2, \quad \forall u \in H^1_w, \tag{31}$$

due to the Sobolev embedding theorem.

Let $S$ be fixed and denote $v(\zeta) := u(\zeta S)$. We have

$$\|v\|_1^2 = \int_0^{+\infty} w(\zeta) \left( \zeta^2 S^2 (u'(\zeta S))^2 + u^2(\zeta S) \right) d\zeta \tag{32}$$

$$= \int_0^{+\infty} \frac{w(\zeta)}{Sw(\zeta S)} w(\zeta S) \left( \zeta^2 S^2 (u'(\zeta S))^2 + u^2(\zeta S) \right) d(\zeta S) \tag{33}$$

$$\leq \frac{1}{S} \exp(C|\ln S|) \|u\|_1^2, \tag{34}$$

since

$$\frac{w(\zeta)}{Sw(\zeta S)} = \frac{1}{S} \exp \left( \int_{\zeta S}^\zeta \frac{w'(\xi)}{w(\xi)} d\xi \right) \leq \frac{1}{S} \exp(C|\ln S|).$$

Then (30) follows from (31) since $v(1) = u(S)$. \qed
The space $H^1_w$ is densely and continuously embedded in $L^2_w$. We consider the Gelfand triples

$$H^1_w \subset L^2_w \subset H^*_w,$$

and

$$L^2(0, T; H^1_w) \subset L^2(0, T; L^2_w) \subset L^2(0, T; H^*_w),$$

where $H^*_w$ is the dual of $H^1_w$. Next, we define the set

$$W(0, T) := \{u \in L^2(0, T; H^1_w), \dot{u} \in L^2(0, T; H^*_w)\},$$

where $\dot{u}$ is the distributional derivative of $u$. It is well known (see Lions and Magenes[5]) that

$$W(0, T) \subset C([0, T], L^2_w).$$

For simplicity we will further write $u(\tau)$ instead of $u(S, \tau)$ when this does not lead to misunderstanding. Recall that

$$\mathcal{F}[\tau; u, \gamma h] := -\nu_{10} e^{u(\tau)} \left( \int_0^\tau e^{-u(s)} ds + e^{-\gamma h} \right) + \kappa.$$  \hspace{1cm} (35)

**Definition 3.3.** A function $u \in W(0, T)$ is called weak supersolution (subsolution) of the initial value problem (1) if $u(0) \geq \gamma h$ (resp. $u(0) \leq \gamma h$) and for a.a. $\tau \in (0, T)$ the inequality

$$\langle \dot{u}, v \rangle + a(u, v) \geq (\leq) \int_0^{+\infty} \mathcal{F}[\tau; u, \gamma h] v dS,$$  \hspace{1cm} (36)

holds for any nonnegative $v \in H^1_w$. Respectively, the function $u \in W(0, T)$ is called weak solution of the initial value problem (1) if $u(0) = \gamma h$ and for a.a. $\tau \in (0, T)$ the equality

$$\langle \dot{u}, v \rangle + a(u, v) = \int_0^{+\infty} \mathcal{F}[\tau; u, \gamma h] v dS, \quad \forall v \in H^1_w,$$  \hspace{1cm} (37)

holds.

Next, we prove the following comparison principle for weak super/subsolutions satisfying growth conditions of type (4).

**Theorem 3.4.** Let $\overline{u}$ be a weak supersolution of the initial value problem (1) with initial data $h(S) \equiv \overline{h}$ and $u$ be a weak subsolution corresponding to the initial data $h(S) \equiv \underline{h}$ where $\underline{h}$ and $\overline{h}$ are given and $\underline{h} \leq \overline{h}$. Assume in addition, that there exist positive constants $A$ and $\alpha$ such that

$$|\underline{u}|, |\overline{h}|, |u|, |\overline{u}| \leq A \exp \left( \alpha \ln^2 S \right) = A S^{\alpha \ln S},$$  \hspace{1cm} (38)

for a.a. $(S, t) \in (0, +\infty) \times [0, T]$.

Then $\underline{u} \leq \overline{u}$ for a.a. $(S, t) \in (0, +\infty) \times [0, T]$.

Denote $u := \overline{u} - \underline{u}$. We will prove that $u_+ := \max \{-u, 0\} = 0$ almost everywhere. Similarly to (38), we obtain that the following inequality holds for a.a. $\tau \in (0, T)$ and for any nonnegative $v \in H^1_w$ with compact support in $(0, +\infty)$:

$$\langle \dot{u}, v \rangle + a(u, v) \geq -\nu_{10} \nu_{10} \int_0^{+\infty} \left( \int_0^\tau \delta(\tau, s) (u(S, \tau) - u(S, s)) ds \right) v(S) w dS$$

$$- \nu_{10} \int_0^{+\infty} (u(S, \tau) - \tilde{h}(S)) v(S) \delta(\tau) w dS,$$  \hspace{1cm} (39)
where
\[ \delta(\tau, s) := \int_0^1 e^{\mu_c(\tau) - \mu_c(s)} d\xi, \quad \delta(\tau) := \int_0^1 e^{\mu_c(\tau)} - \gamma_{\xi} d\xi, \]
\[ u_\xi := \xi(1 - \xi) w, \quad u(\cdot, 0) \geq \tilde{h} := \gamma(\overline{h} - \overline{h}) \geq 0 \text{ and } h_\xi := \xi(\overline{h} + (1 - \xi) \overline{h}). \]

It is sufficient to prove the following auxiliary result:

**Lemma 3.5.** Assume that \( \tau_1 \geq 0 \) is such that for any \( \tau \in [0, \tau_1] \) the inequality \( \overline{\pi}(\tau) - u(\tau) \geq 0 \) holds a.e. on \( (0, +\infty) \). Then the same inequality holds for any \( \tau \in [0, \tau_1 + \bar{\tau}] \), where \( \bar{\tau} > 0 \) is a constant which depends only on \( \alpha \) and \( \sigma \).

**Proof.** Let \( \omega \) be defined by (10) and \( u_\epsilon := u + \epsilon \omega \) where \( u = \overline{\pi} - \overline{\eta} \). Then, assume that \( \bar{\tau} \) is chosen as in the proof of Lemma 2.2. We will prove that \( u_{\epsilon -} := \max \{ -u_\epsilon, 0 \} \equiv 0 \) for a.a. \((S, l) \in (0, +\infty) \times [\tau_1, \tau_1 + \bar{\tau}]\). Note that there exist a closed interval \( I_\epsilon \subset (0, +\infty) \) such that \( u_{\epsilon -} = 0 \) on the set \((0, +\infty) \backslash I_\epsilon \times [\tau_1, \tau_1 + \bar{\tau}]\) due to the conditions (38). Now, let \( \varphi(S) \) be a smooth function with compact support in \((0, +\infty)\) such that \( \varphi(S) = 1 \) on the interval \( I_\epsilon \). Then \( u_\epsilon \varphi \in L^2(\tau_1, \tau_1 + \bar{\tau}; H_w^1) \) and \( (u_\epsilon \varphi)_- = u_{\epsilon -} \). Next, for any nonnegative \( v \in H_w^1 \) with compact support \( \supp v \subset I_\epsilon \) we have \( \varphi v = v, \quad a(u \varphi, v) = a(u, v) \) and then

\[
\left\langle \frac{d}{d\tau} (u_\epsilon \varphi), v \right\rangle + a(u_\epsilon \varphi, v) = \langle \dot{u}, \varphi v \rangle + \epsilon \langle \varphi \dot{\omega}, v \rangle + a(u \varphi, v) + \epsilon a(\omega \varphi, v) \tag{40}
\]
\[
= \langle \dot{u}, v \rangle + a(u, v) - \frac{1}{2} \sigma^2 (2\omega' \varphi' + \omega \varphi'', v)_{L^2_w} = -\nu_0 \int_0^\infty \left( \int_0^\tau \delta(\tau, s) (u(S, \tau) - u(S, s)) ds \right) v(S)wdS \tag{41}
\]
\[
- \nu_0 \int_0^\infty \left( u(S, \tau) - \tilde{h}(S) \right) v(S)\delta(\tau)wdS 
\]
\[
\geq -\nu_0 \int_0^\infty \left( \int_0^\tau \delta(\tau, s) ds \right) u(S, \tau) v(S)wdS - \nu_0 \int_0^\infty \left( \int_0^\tau \delta(\tau, s) (u(S, \tau) - u(S, s)) ds \right) v(S)wdS 
\]
\[
- \nu_0 \int_0^\infty u(S, \tau) v(S)\delta(\tau)wdS, \tag{42}
\]

i.e.,
\[
\left\langle \frac{d}{d\tau} (u_\epsilon \varphi), v \right\rangle + a(u_\epsilon \varphi, v) \geq -\nu_0 \int_0^\infty \left( \int_0^\tau \delta(\tau, s) ds \right) u_\epsilon (S, \tau) v(S)wdS \tag{43}
\]
\[
- \nu_0 \int_0^\infty \left( \int_0^\tau \delta(\tau, s) (u_\epsilon (S, \tau) - u_\epsilon (S, s)) ds \right) v(S)wdS 
\]
\[
- \nu_0 \int_0^\infty u_\epsilon (S, \tau) v(S)\delta(\tau)wdS,
\]

where we have used the fact that \( u_\epsilon > u \) and \( u_\epsilon (S, \tau) - u_\epsilon (S, s) > u (S, \tau) - u (S, s) \) for any \( s \in [\tau_1, \tau] \) since \( \omega(S, \cdot) \) is increasing on that interval. Now, take \( v = u_{\epsilon -} \) and note that \( u_\epsilon = u_{\epsilon +} - u_{\epsilon -}, \quad a(u_\epsilon \varphi, u_{\epsilon -}) = -a(u_{\epsilon -}, u_{\epsilon -}) \) and
\[
\begin{align*}
    u_\epsilon (S, s) u_{\epsilon -} (S, \tau) &\geq -u_{\epsilon -} (S, s) u_{\epsilon -} (S, \tau) 
    \geq -\frac{1}{2} (u_{\epsilon -}^2 (S, s) + u_{\epsilon -}^2 (S, \tau)).
\end{align*}
\]
After integration with respect to $\tau$ form $\tau_1$ to $t \in [\tau_1, \tau_1 + \bar{\tau}]$ the inequality \ref{44} implies
\[ \frac{1}{2} \| u_{e-}(t) \|^2_0 + a(u_{e-}, u_{e-}) \leq - \int_{\tau_1}^t \left( \int_{0}^{\tau_1} \Sigma(S, \tau) u_{e-}^2 (S, \tau) v dS \right) d\tau, \tag{44} \]
where
\[ \Sigma(S, \tau) := \nu_0 \nu_1 \left( \int_{0}^{\tau_1} \delta(\tau, s) ds + \frac{1}{2} \int_{\tau_1}^{\tau} \delta(\tau, s) ds - \frac{1}{2} \int_{0}^{\tau} \delta(s, \tau) ds \right) + \nu_0 \delta(\tau). \]
$|Sigma(S, \tau)|$ is bounded from above by a constant, say $C > 0$, when $S \in I_t$ and due to the semi-coercivity of the bilinear form $a(\cdot, \cdot)$ (see \ref{28}) we obtain:
\[ \frac{1}{2} \| u_{e-}(t) \|^2_0 \leq (C + \beta) \int_{\tau_1}^t \| u_{e-}(\tau) \|^2_0 d\tau. \tag{45} \]
Hence the Gronwall inequality implies $\| u_{e-}(t) \|^2_0 = 0$ for any $t \in [\tau_1, \tau_1 + \bar{\tau}]$ since $\| u_{e-}(\tau_1) \|^2_0 = 0$. Then $u + \epsilon \omega \geq 0$ a.e. Thus $u \geq 0$ a.e. since $\epsilon > 0$ is arbitrary.

We further prove another useful estimate.

**Lemma 3.6.** There exists a constant $C > 0$ such that
\[ \max_{t \in [0, T]} \| u(t) \|_0 + \| u \|_{L^2(0, T; H^1_\omega) \} \leq C \left( \| u(0) \|_0 + \| \hat{u} \|_{W(0, T)} + \gamma \| h \|_0 + 1 \right) \tag{46} \]
for any weak subsolution $u$ and any function $\hat{u} \in W(0, T)$ satisfying $u \geq \hat{u}$.

**Proof.** Let $v \in H^1_\omega$ be some nonnegative function. We have
\[ \langle \hat{u}, v \rangle + a(u, v) \leq \int_{0}^{+\infty} \omega F \left[ \tau; u, \gamma h \right] vdS, \]
\[ \leq -\nu_0 \nu_1 \left( \int_{0}^{\tau} [u(\tau) - u(s)] ds \right)_{L^2_\omega} \]
\[ - \nu_0 (u(\tau) - \gamma h, v)_{L^2_\omega} \]
\[ + (\kappa - \nu_0 \nu_1 \tau - \nu_0) (1, v)_{L^2_\omega}. \]
Take $v = u - \hat{u}$ and integrate \ref{17} with respect to $\tau$ from 0 to $t$.
\[ \frac{1}{2} \| u(t) \|^2_0 + a(u, u) \leq \frac{1}{2} \| u(0) \|^2_0 + (u, \hat{u})_{L^2_0} |_{t}^t + a(u, \hat{u}) - \int_{0}^{t} \langle \hat{u}, u \rangle d\tau \tag{48} \]
\[ - \nu_0 \int_{0}^{t} (\nu_0 \tau + 1) \| u(\tau) \|^2_0 d\tau + \nu_0 \nu_1 \frac{1}{2} \left\| \int_{0}^{t} \omega(\tau) d\tau \right\|^2 \]
\[ + C_1 \left( \| \hat{u} \|_{L^2(0, T; L^2_\omega)} + \gamma \| h \|_0 + 1 \right) \| u \|_{L^2(0, T; L^2_\omega)} \]
\[ + C_2 (\gamma \| h \|_0 + 1) \| \hat{u} \|_{L^2(0, T; L^2_\omega)}. \]
Then a standard argument implies the estimate \ref{16}. \qed
Now, we prove the existence of weak solutions, provided that \( h \in H^1_w \). The proof is based on the lower and upper solution method (cf. \([14]\)). However, the exponential nonlinearity in \([1]\) causes some very technical difficulties which have to be overcome.

**Theorem 3.7.** Assume that \( h \in H^1_w \). Then there exist a weak solution \( u \) to the initial value problem \([1]\). Moreover, there exists a constant \( C > 0 \) independent of \( u \) such that

\[
\|\dot{u}\|_{L^2(0,T,L^2_w)} + \|u\|_{L^\infty(0,T,H^1_w)} \leq C \left( \|u(0)\|_1 + 1 \right)
\]

**Proof.** We will present the proof in several steps. 

**Step 1.** Let \( h \in L^2_w \) be bounded. Then there exists a weak solution \( u \) to the initial value problem \([1]\). In addition, if \( u(0) = \gamma h \in H^1_w \), then the inequality \([19]\) holds with a constant \( C \) independent of \( u(0) \).

Note that we can construct appropriate couple of a supersolution \( \overline{u} \) and a subsolution \( \underline{u} \). Indeed, let the constant \( c_0 \) be such that \( |\gamma h| \leq c_0 \) and take \( \underline{u} := -c_0 - Mt \) for some positive constant \( M \). If \( M \) is great enough then \( \underline{u} \) is a subsolution. Analogously, \( \overline{u} := c_0 + Mt \) is a supersolution provided that \( M \geq \kappa \).

Next, according to \([5]\) we can choose a constant \( N > 0 \) such that

\[
Nu(\tau) + \mathcal{F}[\tau; u, \gamma h] = Nu(\tau) - v_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} ds + e^{-\gamma h} \right) + \kappa
\]

is increasing in \( u \), i.e.

\[
Nu_1(\tau) + \mathcal{F}[\tau; u_1, \gamma h] \geq Nu_0(\tau) + \mathcal{F}[\tau; u_0, \gamma h],
\]

for all \( u_0 \) and \( u_1 \) such that \( \underline{u} \leq u_0 \leq u_1 \leq \overline{u} \). Now, we can construct a decreasing sequence of supersolutions \( u_0 := \overline{u}, u_1, u_2, \ldots \) such that \( u_{n+1} \) is the solution of the initial value problem

\[
\begin{aligned}
\dot{u}_{n+1} - \frac{1}{2} \sigma^2 S^2 u''_{n+1,SS} + Nu_{n+1} &= Nu_n + \mathcal{F}[\tau; u_n, \gamma h], \\
u_{n+1}(S,0) &= \gamma h(S)
\end{aligned}
\]

and \( \underline{u} \leq u_n \leq \overline{u} \). A standard argument implies that \( u_n \) converges to a weak solution of the problem \([1]\). We omit the details.

Next, assume in addition that \( h \in H^1_w \). Then \( \dot{u} \in L^2(0,T;L^2_w) \) and \( u \in L^\infty(0,T;H^1_w) \) (see, e.g., Bonnans \([1]\)) and the following parabolic estimate holds:

\[
\|\dot{u}\|_{L^2(0,T,L^2_w)} + \|u\|_{L^\infty(0,T,H^1_w)} \leq c_0 \left( \|u(0)\|_1 + \|\mathcal{F}[\cdot; u, \gamma h]\|_{L^2(0,T,L^2_w)} \right)
\]

We will prove the stronger estimate \([19]\). First, we have

\[
-\frac{1}{2} \sigma^2 S^2 u''_{SS} = \mathcal{F}[\tau; u, \gamma h] - \dot{u} \in L^2(0,T,L^2_w),
\]

\[
-\frac{1}{2} \sigma^2 \int_0^\tau \langle S^2 u''_{SS}, \dot{u} \rangle_{L^2_w} d\tau = \frac{1}{2} \sigma^2 \left( \frac{1}{2} \|u(t)\|_1^2 - \frac{1}{2} \|u(0)\|_1^2 \right)
\]

\[
+ \frac{1}{2} \sigma^2 \int_0^\tau \left( S \left( \frac{w'}{w} + 2 \right) u'_S - u, \dot{u} \right)_{L^2_w} d\tau
\]
\[
\int_0^t (F[\tau; u, \gamma h], \dot{u})_{L^2} \, d\tau = \int_0^{+\infty} \left( \int_0^t \frac{d}{d\tau} (F[\tau; u, \gamma h]) \, d\tau \right) \, \omega S + \int_0^{+\infty} \left( \int_0^t (\kappa \dot{u} + \nu_{01} \nu_{10}) \, d\tau \right) \, \omega S \leq |\kappa| \theta^{1/2} \int_0^t ||\dot{u}(\tau)||_0 \, d\tau + \nu_{01} (1 + \nu_{10} t) \theta
\]

since
\[
\frac{d}{d\tau} (F[\tau; u, \gamma h]) = \frac{d}{d\tau} \left[ -\nu_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} \, ds + e^{-\gamma h} \right) + \kappa \right] \\
= -\nu_{01} e^{u(\tau)} \left( \nu_{10} \int_0^\tau e^{-u(s)} \, ds + e^{-\gamma h} \right) \dot{u} - \nu_{01} \nu_{10}
\]
and
\[
\int_0^t \frac{d}{d\tau} (F[\tau; u, \gamma h]) \, d\tau = F[t; u, \gamma h] - F[0; u, \gamma h] \leq \nu_{01}
\]
We multiply both sides of the equation \( \dot{u} - 1/2\sigma^2 S^2 u_{SS} = F[\tau; u, \gamma h] \) with \( \dot{u} \) in \( L^2 \) and integrate from 0 to \( T \). Then (53) and (55) imply
\[
\int_0^t ||\dot{u}\|^2_0 \, d\tau + \frac{1}{4} \sigma^2 \|u(t)\|^2_2 \leq -\frac{1}{2} \sigma^2 \int_0^t \left( S \left( S u'_w + 2 \right) u'_S - u, \ddot{u} \right)_{L^2} \, d\tau \\
+ |\kappa| \theta^{1/2} \int_0^t ||\dot{u}(\tau)||_0 \, d\tau + \frac{1}{4} \sigma^2 \|u(0)\|^2_2 \\
+ \nu_{01} (1 + \nu_{10} t) \theta \\
\leq \tilde{C} \left[ \int_0^t (\|u(\tau)\|_1 + 1) ||\dot{u}(\tau)||_0 \, d\tau + \|u(0)\|^2_2 + 1 \right]
\]
for some constant \( \tilde{C} > 0 \). Now, a technical, but standard argument implies that (10) holds.

**Step 2.** Let \( h \in H^1_w \) be bounded from below, i.e., \( u(0) = \gamma h \geq c \). Then there exists a weak solution \( u \) to the initial value problem (11). In addition, the inequality (10) holds.

Let \( \xi(x) \) be defined as in Lemma (5.1), i.e., \( \xi(x) := \xi(x/\epsilon) \left[ 1 - \xi(x/2) \right] \). Step 1 implies that there exists a solution \( u_\epsilon \) corresponding to the initial condition \( u_\epsilon(0) = \xi(\gamma h - c) + c = \xi \gamma h + (1 - \xi)c \) which is bounded. Moreover, \( \xi \gamma h + (1 - \xi)c \leq \gamma h \) increases as \( \epsilon \downarrow 0 \) and converges to \( H^1_w \) to \( \gamma h \). Then the comparison principle from Theorem (5.4) implies that the sequence \( u_\epsilon \) is increasing as \( \epsilon \downarrow 0 \). Next, the estimate (10) and Lemma (5.2) imply that \( u_\epsilon(S, \tau) \) converges to a finite limit \( u(S, \tau) \) for any \( (S, \tau) \in (0, +\infty) \times [0, T] \). What is more, \( u_\epsilon \) is weakly convergent to \( \dot{u}(S, \tau) \) in \( L^2(0, T; L^2) \), \( u_\epsilon \) is weakly*- convergent to \( u \) in \( L^\infty(0, T, H^1_w) \) and \( u \) satisfies the estimate (10). Then it is sufficient to prove that \( F[\tau; u_\epsilon, \xi \gamma h + (1 - \xi)c] \) is weakly convergent to \( F[\tau; u, \gamma h] \) in \( L^2(0, T; H^m_w) \).

First, note that
\[
F[\tau; u_\epsilon, \xi \gamma h + (1 - \xi)c] = \dot{u}_\epsilon - \frac{1}{2} \sigma^2 S^2 u''_{\epsilon, SS}
\]
is bounded in $L^2(0,T; H^1_w)$ and then there exists an element $\tilde{F} \in L^2(0,T; H^1_w)$ such that
\[
\mathcal{F}[\tau; u, \xi] = (1 - \xi_\epsilon) c \quad \text{in} \quad L^2(0,T; H^1_w)
\]
\[
\implies \mathcal{F}[\tau; u, \xi] \gamma h \quad \text{in} \quad L^2(0,T; L^2_w)
\]
On the other hand, $\mathcal{F}[\tau; u, \xi] \gamma h + (1 - \xi_\epsilon) c$ is bounded from above by the constant function $\kappa$. Let $v \in L^2(0,T; H^1_w)$ be some arbitrary nonnegative function. Then Fatou’s lemma implies
\[
\mathcal{F}[\mathcal{F}[\tau; u, \xi] \gamma h, v] = \lim_{\epsilon \to 0} \mathcal{F}[\tau; u, \xi] \gamma h + (1 - \xi_\epsilon) c, v]_{L^2(0,T; L^2_w)} \nonumber
\]
i.e.
\[
\mathcal{F}[\mathcal{F}[\tau; u, \xi] \gamma h, v] = \lim_{\epsilon \to 0} \mathcal{F}[\tau; u, \xi] \gamma h + (1 - \xi_\epsilon) c, v] \geq 0,
\]
Finally, we prove that in fact
\[
\mathcal{F}[\mathcal{F}[\tau; u, \xi] \gamma h, v] = \lim_{\epsilon \to 0} \mathcal{F}[\tau; u, \xi] \gamma h + (1 - \xi_\epsilon) c, v] \geq 0.
\]

**Step 3.** Let $h \in H^1_w$. Then there exists a weak solution $u$ to the initial value problem. In addition, the inequality holds.

Consider a sequence of problems with initial condition
\[
u_N(S,0) = \max \{ \gamma h(S), -N \}, \quad N = 1, 2, \ldots
\]
Then the corresponding solutions $u_N$ form a decreasing sequence due to the comparison principle and Lemma. Moreover, the pointwise limit $\lim_{N \to \infty} u_N(S, \tau)$ is finite for any $(S, \tau)$ since the inequality holds for each function $u_N$. Then the proof follows similar arguments as in Step 2.

Finally, note that the uniqueness of the weak solution is a consequence of the comparison principle. More precisely, we have the following corollary.

**Corollary 3.8.** Assume that $h \in H^1_w$. Then there exists a unique weak solution $u \in W(0,T) \cap L^\infty(0,T; H^1_w)$ to the initial value problem. Moreover, the estimate holds with a constant $C > 0$ independent of $u$.

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