Learning and Efficiency in Games with Dynamic Population

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We study the quality of outcomes in repeated games when the population of players is dynamically changing and participants use learning algorithms to adapt to the changing environment. Game theory classically considers Nash equilibria of one-shot games, while in practice many games are played repeatedly, and in such games players often use algorithmic tools to learn to play in the given environment. Learning in repeated games has only been studied when the population playing the game is stable over time.

We analyze efficiency of repeated games in dynamically changing environments, motivated by application domains such as packet routing and Internet ad-auctions. We prove that, in many classes of games, if players choose their strategies in a way that guarantees low adaptive regret, then high social welfare is ensured, even under very frequent changes. This result extends previous work, which showed high welfare for learning outcomes in stable environments. A main technical tool for our analysis is the existence of a solution to the welfare maximization problem that is both close to optimal and relatively stable over time. Such a solution serves as a benchmark in the efficiency analysis of learning outcomes. We show that such stable and near-optimal solutions exist for many problems, even in cases when the exact optimal solution can be very unstable. We develop direct techniques to show the existence of a stable solution in some classes of games. Further, we show that a sufficient condition for the existence of stable solutions is the existence of a differentially private algorithm for the welfare maximization problem. We demonstrate our techniques by focusing on three classes of games as examples: simultaneous item auctions, bandwidth allocation mechanisms and congestion games.

Key words: No-regret learning, price of anarchy, auctions, congestion games, differential privacy

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1. Introduction

The goal of this paper is to understand the quality of outcomes of games and simple mechanisms in a dynamic environment. The Internet allows for the repeated strategic interaction of many entities with constantly changing parameters and participants. Primary examples of such interactions include online advertising auction platforms, packet routing and allocation of cloud computing resources. Understanding whether the constant change in these strategic environments can severely damage the efficiency of the corresponding system, as compared to the hypothetical centralized optimal, is of prime importance as these systems constitute the cornerstone of the online economy. For example, advertising provides close to 90% of Google’s revenue (Google 2015).

Classical economic analysis of the interaction of strategic agents assumes that players reach a stable outcome where all players are mutually best-responding to each others’ actions (or considers mechanisms that are dominant strategy solvable). Dynamic environments, with high volume interactions of small individual value or cost, such as packet routing or ad-auctions, are better modeled as repeated games with learning players. Nash equilibria of the one-shot game correspond to stable outcomes repeated in each iteration, where the players have no regret for their choice of strategies. Hence, analyzing the quality of outcomes in repeated games via the price of anarchy (Koutsoupias and Papadimitriou 1999) assumes that the repeated game reaches a stable, stationary outcome. Such an analysis of price of anarchy of one-shot Nash equilibria has received large attention in the past few years in both the computer science and operations research community and in a plethora of application domains such as routing games (Roughgarden and Tardos 2002, Correa et al. 2003), bandwidth allocation (Johari and Tsitsiklis 2004), strategic supply among firms (Johari and Tsitsiklis 2011) and online ad-auctions (Caragiannis et al. 2015) (see e.g. Chapters 17 to 21 of (Nisan et al. 2007) for a survey).

A more attractive model of player behavior in such repeated environments is to assume players use a form of algorithmic learning. Modeling players as learners is especially appealing in online auctions, as individual auctions provide very little value, costing only a few cents to a few dollars each, so using experimentation to learn from the data is natural. Many advertisers use sophisticated optimization tools or services to optimize their bidding, such as Bluekai\footnote{http://www.oracle.com/us/corporate/acquisitions/bluekai/index.html} or AdRoll\footnote{https://www.adroll.com/}.

It is well known that in most games natural game play does not lead to equilibria, under any definition of “natural play” (see e.g. Chapter 7 of (Hart and Mas-Colell 2012)). In fact, results on polynomial time computability of Nash equilibria of general games are mostly negative: finding equilibria is computationally hard (see (Daskalakis 2009) for a survey).
Even with computational concerns aside, the game that the participants are playing at each
time-step and the participants they are playing against, can change at any time without even the
players realizing it or being able to form any distributional belief. Hence, even the concept of a
Nash equilibrium is debatable in such an adversarially evolving setting, as the players don’t even
have the information necessary to calculate their expected utility at each time-step. Instead they
observe their utility from the action they took or from any alternative action they could have taken,
only after the fact. In such an evolving setting, players can base their actions on past experience.

A particular class of learning behaviors, no-regret learning, emerged as a nice way to capture
the intuition that players learn to play appropriate strategies over time without necessitating
convergence to a stationary equilibrium. A stationary distribution that is also a no-regret learning
outcome corresponds to a Nash equilibrium of the one shot game, and in this sense, learning
outcomes generalize Nash equilibrium. More importantly, there are several simple and natural
algorithms that achieve the no-regret property (e.g. regret matching [Hart and Mas-Colell 2000],
multiplicative weight updates [Arora et al. 2012]). However, no-regret does not preclude the use
of possibly much more sophisticated tools, including using the above learning algorithms with
more complex benchmarks. Achieving small regret is a relatively simple expectation from bid
optimization tools.

Blum et al. (2006, 2008) consider regret-minimization as a model of player behavior in repeated
games, and study the average inefficiency of the outcome, coining the term price of total anarchy
for the worst-case ratio between the optimal objective value and the average objective value when
players use a no-regret algorithm. In a sequence of play all players achieve the no-regret property, if
and only if the empirical distribution of strategy vectors is a coarse correlated equilibrium, hence the
price of total anarchy is the ratio of the socially optimal welfare to the welfare at the worst coarse
correlated equilibrium. Roughgarden (2009) observed that many of the Nash equilibrium price
of anarchy bounds are shown via a proof technique he called smoothness, and such proofs easily
extend also to show bounds on the quality of coarse correlated equilibria. Syrgkanis and Tardos
(2013) extend smoothness to simple mechanisms, such as independent item auctions.

However, this learning outcome analysis is based on the strong assumption that the underlying
environment and player population is stable. The reason for this requirement is easy to understand:
with the game and the players stable, there is a fixed optimal solution, and a fixed strategy, that
each player $i$ would need to play (action $a_i^*$) as his or her part for achieving the optimum. To
guarantee high social welfare via the smoothness approach, all we need is that each player $i$ doesn’t
regret not playing this optimal action $a_i^*$. No-regret learning guarantees exactly this; player $i$ will
not regret any fixed strategy with hindsight, including strategy $a_i^*$. However, online environments
are typically not stable.
In this paper, we study learning outcomes in games with a dynamically changing player population. As stated in [Fudenberg and Levine 1998, p. 4], the fact that players *extrapolate across games they view as similar* is an important reason learning has relevance in a real-world situation. A repeated game with an evolving population is exactly a setup where players are asked to play repeatedly in similar games. Rather than aiming to predict the exact outcome, our goal is to predict properties of outcomes, such as their efficiency, i.e. the price of anarchy.

In a changing game environment, we need a slightly stronger notion of regret minimization. No-regret learning aims to select strategies that do at least as well on the average over a sequence of steps as the best single strategy would have done in hindsight. With the game environment and population changing, a single best strategy in hindsight gives a really weak benchmark. Players, using good learning algorithms, should be able to adapt to the changing environment, and such adaptation may be very useful with the population changing over time. For example, in the context of routing games, a player with many route options, may want to adjust their route choices depending which part of the network is more congested, or in auction games, a player may want to bid for items that are less in demand.

[Hazan and Seshadhri 2007] formally introduced the stronger notion of *adaptive regret* that we will use, bounding the average regret over any sub-interval of steps \([\tau_1, \tau_2]\), compared to a single best action over this interval in hindsight. The study of adaptive learning goes back much further: the work of Lehrer (2003) and Blum and Mansour (2007) studied generalizations of adaptive regret prior to [Hazan and Seshadhri 2007]. Clearly short intervals will result in relatively high regret with any learning algorithm, but *adaptive learning algorithms* guarantee for the player that the cumulative regret grows sub-linearly with the length of the interval. Most adaptive learning algorithms are constructed by modifying classical no-regret learning algorithms to stop relying too heavily on experience from the distant past. We believe that such adaptive learning is a better model of behavior when strategic agents (such as bidders in online auctions) use sophisticated optimization tools. The current best adaptive learning algorithm is a natural adaptation of the classical Hedge algorithm, AdaNormalHedge, due to [Luo and Schapire 2015]. With this framework in mind, we ask the following main question:

*How much rate of change can a system admit to sustain approximate efficiency, when its participants are adaptive learners?*

**Our Results.** We show that in large classes of games, if players choose their strategies in a way that guarantees low adaptive regret, this ensures high social welfare, even under surprisingly high turnover. To model a changing environment we consider a dynamic player population where between every pair of iterations each player leaves independently with a (small) probability \(p\) and is replaced by an arbitrary new player, implying that in expectation a \(p\) fraction of the population
is replaced. The independent departure probability models churn in player population caused by effects that are external to the game. We make no assumptions on the sequence of arriving players, which can be chosen in an adversarial way. We use independence of departures for simplicity of presentation, and most of our results carry over to any process where the departing players are also chosen adversarially, subject to a constraint on the number of per-step replacements. This model of the environment is simple enough to allow a clean analysis, and allows arbitrary worst case shifts in player populations.

We show that learning behavior ensures high social welfare in dynamic situations with high churn for four classes of games:

- In Section 4.1 we consider an item auction game with unit demand bidders. At each period the auction sells \( m \) different items, and the bidders have value for at most one item per-period. The value of player \( i \) for each item \( j \) is different and is denoted by \( v_{ij} \). We consider a simple auction format: each item is auctioned independently (via a first or second price auction). We show that adaptive learning by players ensures high social welfare (i.e. price of anarchy close to 4), even when the probability \( p \) of player departure is close to a constant (independent of the number of items or players, and depends only on the range of values that players have).

- In Section 4.2, we consider a bandwidth allocation: a unit of bandwidth is to be divided across the players of the game and each player \( i \) has a valuation function \( v_i(x) \) for bandwidth \( x \). We consider the proportional mechanism of Kelly (1997), analyzed in Johari and Tsitsiklis (2011), and show a price of anarchy close to 4 under mild assumptions on the utility functions and even with high player turnover.

- In Section 5.2 we prove that in large dynamic congestion games learning by players ensures low social cost even with a dynamically changing player population. For example, when the costs are a linear function of the congestion, we get a price of anarchy guarantee close to the \( 5/2 \) price of anarchy of the corresponding one-shot atomic congestion game, even if a \( 1/\text{polylog}(n) \) fraction of the \( n \) players are changing at each time-step.

- In Section 5.3 we consider auction games where bidders have gross substitute valuations. Extending the results of Section 4.1, we prove that in large dynamic markets, learning by players ensures high social welfare, i.e. price of anarchy close to 2, even if a \( 1/\text{polylog}(n) \) fraction of the \( n \) players are changing at each time-step.

We achieve these results by developing a general technique (in Section 3) to show that in many games adaptive learners achieve high social welfare in dynamically changing environments. Our technique is based on the following three conditions:

1. All players are adaptive learners, i.e. they choose their strategies in a way that guarantees small adaptive regret on the outcome (for instance, using an adaptive learning algorithm). In
deriving concrete bounds, we assume that players use adaptive learning algorithm with the best known bound of (Luo and Schapire 2015) or (Blum and Mansour 2007). Our results deteriorate gracefully with weaker assumptions on the regret of learning.

2. The game repeated in each state (called stage) needs to have low price of anarchy. In particular, we need that the game satisfies a slight strengthening of the roughgarden (2009) smoothness property (or the smooth mechanisms property of (Syrgkanis and Tardos 2013)), which is typically used to prove price of anarchy guarantees.

3. There exists a sequence of solutions for the underlying optimization problem that is approximately optimal, and where on average each player’s part of the solution is stable, i.e. doesn’t change much over time.

With our model of players leaving the game independently with probability $p$ at each step, on average each player is expected to participate in $1/p$ rounds of the game, which turns out to be long enough to learn good strategies. On the other hand, players will experience dynamic population changes, and with no assumption on arriving players, they will need to adapt to the changing environment. With a player population of size $n$, and each player being replaced with turnover probability $p$, after each step we have $np$ new players in expectation, so the population is constantly changing. We use an approximately optimal solution where each player’s allocation is relatively stable as a benchmark for each player; a stable enough benchmark that will allow adaptive learners to learn how to play at least as well as this solution. We will be interested in understanding what value of $p$ is needed to guarantee high social welfare.

To apply the above outline to a game, we need to develop techniques for point 3 above: show that there exists a stable sequence of close to optimal solutions in our changing environment. We present two ways to achieve this stability. In Section 4, we consider solution sequences that are produced by greedy algorithms where a turnover in the input has only local influence in the output. In Section 5, we consider solution sequences that are produced by differentially private algorithms where a turnover in the input affects the whole output but only with a small probability.

Our first application, via the greedy algorithm approach, is the unit demand auction problem analyzed in Section 4.1. In a unit-demand auction, after a change in one player, we could recompute the optimal solution by an augmenting path algorithm. Unfortunately, a single augmenting path can change the assignment of many (or even all) players, and hence in no sense is the evolving optimal solution stable. Such major changes can happen even if the player valuations are all 0 or 1. We develop a greedy algorithm that finds stable solution sequences losing only a factor of 2 from the optimum value. To illustrate the idea, observe that in the special case of 0/1 values a greedy matching is essentially stable, and has size at least 1/2 of the optimal matching. In Section 4.1 we extend this idea beyond 0/1 valuations and give stable solution sequences to the unit demand auction problem.
auction problem. We use this algorithm to show that players using adaptive learning guarantee high social welfare in the item auction game with unit demand bidder even with a dynamically changing player population, allowing for a probability $p$ of player departure that depends logarithmically on the range of values players have, and does not depend on the number of items or players.

Another application of the greedy algorithm approach is the bandwidth allocation problem (Section 4.2), where some bandwidth is divided across players with smooth concave valuation functions. Segmenting the bandwidth in small parts and viewing each segment as an item, we provide an almost optimal greedy approximation algorithm with similar stability guarantees as in the unit-demand auction setting.

In Section 5 we develop a general method for applying our framework via the use of differential privacy. Differential privacy has been developed by Dwork et al. (2006) for (approximately) answering queries of databases of private information, while protecting the privacy of data. Consider a database of sensitive personal information (such as medical data). The framework of differential privacy has been developed to allow us to take advantage of the statistical information in the database without compromising the privacy of the individuals. A differentially private response to a database query is randomized, and it requires that if two databases differ only in the data related to one individual, the probability that the response differs is very small. In recent years many optimization problems have been shown to be solvable in a differentially private way (see the recent book of Dwork and Roth (2014)).

The requirement of differential privacy for a solution to an optimization problem is very close to what we need for our stable solution sequences: if there is a differentially private close to optimal solution, this immediately implies that the solution cannot change much as one person’s data changes. We will be using a variant of the notion of differential privacy adapted to game theoretic environments, joint differential privacy (Kearns et al. 2014). Player $i$’s share of any reasonable solution must depend on his/her own input, so a solution cannot be fully differentially private. Joint differential privacy fixes this discrepancy. In fact, the notion of marginal differential privacy of Kannan et al. (2014) seems even more appropriate, as it only requires that the output for each player $j$ is differentially private in the data of other players. In order to take advantage of differential privacy in the context of dynamically changing games, we need to overcome an important technical difficulty: with the output of the differentially private algorithm randomized, the natural measure of change in a sequence of such outputs is the sum of the total variation distances between adjacent pairs of distributions. We need to turn the sequence of output distributions with low total variation distance into a distribution of stable output sequences. We do this in Section 5 for joint differential privacy. In Appendix D.C.2 we show how to adapt our analysis to the weaker notion of marginal differential privacy.
We illustrate the differential privacy approach via two applications. In Section 5.2 we use the differentially private algorithm of Rogers et al. (2015) for congestion games to prove that in large dynamic congestion games players using adaptive learning guarantees low social cost even with a dynamically changing player population. In Section 5.3 we use differentially private algorithms of Hsu et al. (2014) for matchings and allocations with gross substitute valuations to prove that in large dynamic markets players using adaptive learning guarantees high social welfare even with a dynamically changing player population. For simplicity of presentation, we focus on first price auctions in Sections 4.1 and 5.3, but our results apply also to second price auction (assuming no overbidding) as well as any hybrids of the two auction formats. In this setting we show, roughly, that if we have a smoothness-based price of anarchy bound for the single-shot game then, in the dynamic population setting, the price of anarchy is $\epsilon$ close to the same bound assuming that $p = O(\epsilon^5 / \text{polylog}(n))$, as long as the market is large enough, in the sense that the supply of goods is large enough. The simultaneous first price auction gives a price of anarchy bound of 2. Thus even if approximately $n / \log(n)$ players are changing at each time-step, a constant inefficiency is guaranteed.

As a benchmark for the latter two results, it is interesting to consider a simpler model of dynamic player population, where the departure or arrival of a player is announced to all players. We expect $np$ new players each step, so in expectation there will be $1/(np)$ steps with no change at all. If all the changes are announced, players could be expected to restart their learning algorithms due to the change. If the stable period $1/(np)$ is long enough, we can use results for the total price of anarchy to guarantee high social welfare. Under standard no-regret learning algorithms each player will then have average regret approximately $O(\sqrt{n \cdot p})$. Hence, if we want the regret in the system to be at most an $\epsilon$ fraction of the optimal welfare and hence contribute only an $\epsilon$ to the inefficiency, we would require that $p = O(\epsilon^2 / n)$. In other words, the probability that any player changes in a period needs to be $\epsilon^2 / n$, which is a tiny rate of change for large $n$.

Our results are stronger than what is implied by this argument in two ways. First, we do not assume that change is announced, rather, we take advantage of the fact that players using learning algorithms can adjust to the changing environment even without the announcement of the change. More importantly, our results allow a probability of change much higher than the required by the above argument. The resulting dynamic game will not have long periods with no change. Multiple players will be arriving and leaving at each step. We show that in many games, despite the constant change, there exists a good benchmark of the kind mentioned in the conditions above, where each player’s individual solution or allocation is relatively stable. The rate of expected change $np$ in our applications will turn out to be high, especially as the number of players increases. Roughly speaking, if we want the regret of the players to be an $\epsilon$ fraction of the optimal welfare, we will only
require that $p = O(\text{poly}(\varepsilon) / \text{polylog}(n))$, where the constants depend on several parameters of each game at hand, but importantly depends only logarithmically in the number of players. Moreover, in some games we even give a bound that is independent of $n$. Hence, for any constant $\varepsilon$ we allow almost a constant fraction of players to be changing at each period.

Further related work on dynamic games. Dynamic games have a long history in economics, dynamical systems and operations research, see for example the survey books of Baar and Olsder (1998) and Van Long (2010). The classic approach of analyzing behavior in such dynamic games is to assume that players have prior beliefs about their competitors and that their behavior will constitute a perfect Bayesian equilibrium or refinements of it such as the sequential equilibrium of Fudenberg and Tirole (1991) or Markov perfect equilibrium of Maskin and Tirole (2001). The intractability of such equilibrium solution concepts and the informational and rationality assumptions that they impose on the players casts doubt on whether players in practice and in complex game theoretic environments such as packet routing or internet ad-auctions would behave as prescribed by such an equilibrium behavior.

Equilibrium-like behavior might be more plausible in large game approximations (see e.g. recent work of Kalai and Shmaya (2015)). A natural approximation to equilibrium behavior in large game situations that has been recently extensively analyzed in economics and in operations research (and particularly, in auction settings) is that of the mean field equilibrium (Balseiro et al. 2015, Weintraub et al. 2006, 2008, Adlakha et al. 2015, Iyer et al. 2014, Adlakha and Johari 2013). However, even these large game approximations require the players to form almost correct beliefs about the competition and exactly best-respond to these approximate large-market beliefs. Moreover, the approach requires that the environment either is stochastically stable or evolves in a known stochastic manner and in most situations the mean field approach captures behavior at a stochastically stable state of the system. On the contrary our dynamic model allows for adversarial changes and our analysis attempts to analyze even constantly evolving and never converging behavior. Moreover, our assumption that players invoke adaptive learning algorithms does not impose that players possess or form any beliefs on the competition. Most of the algorithms that achieve adaptive regret only require that the player is able to see the utility that each of his strategic options would have given in-retrospect, in past time-steps. Last our approach also applies in small markets.

There is also a large literature on truthful mechanisms in a dynamic setting analogous to our dynamic player population model, where the goal is to truthfully implement a desired outcome with dynamically changing populations of users with private value. This line of work goes back to Parkes and Singla (2003) in the computer science literature, but has been also considered much earlier with queuing models by Dolan (1978). In a more recent work Cavallo et al. (2010) offers a generalized VCG mechanism in an environment very similar to the one we are considering with
departures and arrivals, and also provides a nice overview of work in truthful mechanisms in a dynamic setting. For a more complete overview, the reader is referred to the survey on dynamic auctions by Bergemann and Said (2010).

Further related work on learning in games. There is a large literature analyzing learning in games, dating back to the work on fictitious play by Brown (1951). For an overview of this area, the reader is referred to the books of Fudenberg and Levine (1998) and Cesa-Bianchi and Lugosi (2006). A standard notion of learning in games is that of no-regret learning. The notion of no-regret against the best fixed action in hindsight dates back to the work of Hannan (1957), and is also referred as, Hannan consistency. There are many learning algorithms achieving this guarantee such as regret matching by Hart and Mas-Colell (2000) and multiplicative weights updates by Freund and Schapire (1997).

Related work on learning in dynamic environments. The notion of no-regret learning against time-varying benchmarks, as opposed to fixed actions, traces back to Herbster and Warmuth (1998) who provided guarantees compared to the best sequence of $k$ experts. The stronger notion of adaptive regret, i.e. having guarantees for every sub-interval was formalized by Hazan and Seshadhri (2007) and near-optimal adaptive regret guarantees were achieved through a series of algorithms by Lehrer (2003), Blum and Mansour (2007), Cesa-Bianchi et al. (2012), and Luo and Schapire (2015). One important trait of these algorithms is that they display some sort of recency bias, in the sense that the influence of past steps decays as time goes by. Recent experimental evidence by Fudenberg and Peysakhovich (2014) suggests that humans display such forms of recency bias when making repeated decisions.

Competing against an adaptive benchmark has also been studied in the context of online convex optimization. Besbes et al. (2013) compare to a target function that is changing from step to step. In order to guarantee some stability across steps they require that the total variation distance between subsequent target functions is bounded by some number. This is a way to capture the notion that subsequent rounds are not very different, related to our notion of turnover probability which in expectation guarantees a similar stability bound on the number of changes per step.

2. Preliminaries

Games and mechanisms. We will consider a game played repeatedly, where the population of players is drifting over time. Let $G$ be an $n$-player normal form stage game and assume that game $G$ is played repeatedly $T$ times. Each player $i$ who participates in a stage game has a strategy space $S_i$, with $\max_i |S_i| = N$, a type $v_i \in V_i$ and a cost function $c_i(s; v_i)$ that depends on the strategy profile $s \in \times_i S_i$, and on his type. We will denote with $C(s; v) = \sum_{i \in [n]} c_i(s; v_i)$ the social cost, where $s$ is a strategy profile and $v$ a type profile. We will also analyze the case when the stage game
is a utility maximization mechanism $M$, which takes as input a strategy profile and outputs an allocation $X_i(s)$ for each player and a payment $P_i(s)$. We will assume that players have quasi-linear utility $u_i(s; v_i) = v_i(X_i(s)) - P_i(s)$ and the welfare is the sum of valuations (sum of utilities of bidders and revenue of auctioneer): $W(s; v) = \sum_{i \in [n]} v_i(X_i(s))$.

In all the games that we study, the optimal social welfare problem can equivalently be defined as an optimization over a “feasible solution space” $X^n$ which involves no incentives (e.g. in network congestion games it is the set of feasible integral flows, in a combinatorial auction setting it is the set of feasible partitions of items to bidders, in the bandwidth allocation setting it is the set of valid partitions of the bandwidth). We will overload the social cost and welfare notations, and for a feasible solution (or allocation) $x \in X^n$ we will use $C(x; v)$ and $W(x; v)$ to denote the social cost or welfare of the solution. We denote the optimal social cost or welfare for a type profile $v$, as $\text{Opt}(v) = \min_{x \in X^n} C(x; v)$ and $\text{Opt}(v) = \max_{x \in X^n} W(x; v)$ respectively.

**Definition 2.1 (Repeated game/mecchanism with dynamic population).** A repeated game with dynamic population consists of a stage game $G$ played for $T$ time steps. Let $P_t$ denote the set of players at time $t$, where each player $i \in P_t$ has a private type $v_t^i$. After each step, every player independently exits the game with a (small) probability $p > 0$ and is replaced by a new player with an arbitrary type. The utility of a player is additive across steps. We denote this repeated game with $\Gamma = (G, T, p)$. Similarly, we denote with $M = (M, T, p)$ a mechanism that is played $T$ times with player replacement probability $p$.

Our model of dynamic population assumes that after each step every player independently exits the game with a probability $p > 0$, so each player is expected to play the game for $1/p$ rounds. To keep our model simple, we make the assumption that when a player exits, she is replaced by a new participant. This assumption guarantees that we will have exactly $n$ players in each iteration, with a $p$ fraction of the population changing each iteration in expectation. We make no assumption about the types of the new arriving players which can be picked adversarially. Most of our results could be extended to the case when the players that are being replaced is also chosen adversarially, subject to some constraint on the number of per-step replacements.

To simplify the notation, we will use player $i$ to denote the current $i$th player, where this player is replaced by a new $i$th player with probability $p$ each round. An alternate view of the dynamic player population is to think of players as changing types after each iteration with a small probability $p$. We will refer to such a change as *player $i$ switches* or *turns over*.

**Basic notation.** For any quantity $x$ we will denote with $x^{1:T}$ the sequence $x^1, \ldots, x^T$. For instance, $v_i^{1:T}$ will denote the sequence of types of player $i$ produced by the random choice of leaving players and by the choices of the adversary.

We will consider three special classes of games, two welfare-maximization mechanisms and one cost-minimization game:
First-price Auction Game. The auction games we consider are defined by a set of \( m \) goods, where we will assume that each good has a supply of \( s \) identical copies in each iteration. We assume for simplicity of presentation that the supply of each item is identical. The players are buyers who repeatedly participate in item auctions to buy copies of the items. Each buyer wants at most one copy of each item. The type of a buyer \( i \) is her valuation over sets of items.

We will use \( v^i_t(A) \) to denote the valuation of the \( i \)-th player in iteration \( t \), if he gets at least one copy of each item in set \( A \subset [m] \). We will assume, that valuations are non-negative and at most 1. Last we will assume that conditional on having a set \( S \), the marginal value of a player for any extra item \( j \), i.e. \( v^i_t(\{j\} \cup S) - v^i_t(S) \) is either 0 or at least some constant \( \rho \). Valuations over time are additive, which models perishable items, such as advertising opportunity, where a player will play to repeatedly win items in each period she is participating.

We will focus the presentation on first price item auctions, where players submit a bid on each item separately: if we have \( s \) copies of an item, the \( s \) highest bidders for the item get one copy each, and pay their bid (ties are broken arbitrarily). The bid on each item comes from some sufficiently fine, discrete bid space. Specifically, bids are multiples of \( \delta \cdot \rho \) for some small \( \delta \) and lie in \([0,1]\). Our results also extend to second price auctions, as well as hybrid auctions.

In our first application (in Section 4.1) we will consider unit demand buyers and the supply of each item is arbitrary. Thus a buyer’s value for any set of items in one iteration is their value for the best single item in the set they acquired. For the case of unit demand, we use \( v^i_t(j) \) to denote the value of an item \( j \) for buyer \( i \) at time \( t \), so the player’s value for a set \( A \) is \( v^i_t(A) = \max_{j \in A} v^i_t(j) \).

In this application, we will assume that players will bid for at most one item at each iteration. Thus the number of strategies available to each player is \( N = \frac{m}{\delta \rho} \).

In Section 5.3 we consider large markets of first price item auctions with players that have more complex valuations satisfying the gross substitute property. In this application, we will assume that players want at most \( d \) types of different items, i.e. for any set \( A \): \( v^i_t(A) = \max_{T \subset A: |T|=d} v^i_t(T) \) and we assume that they will bid on only \( d \) different auctions. Thus the number of strategies available to them is \( N = \binom{m}{d} \left( \frac{1}{\delta \rho} \right)^d \leq \left( \frac{m}{\delta \rho} \right)^d \).

Proportional bandwidth allocation mechanism. The proportional bandwidth allocation mechanism, introduced by Kelly (1997) and first studied for price of anarchy by Johari and Tsitsiklis (2011), is defined by a bandwidth of \( B \) and a valuation function for each player which is concave on the bandwidth she receives. At every round, each player \( i \) submits a bid \( b^i_t \), pays her bid and gets allocated bandwidth proportional to her bid, i.e. \( x_i(b^t) = \frac{b^i_t}{\sum_j b^j_t} \).

In this setting the type of the player is her valuation function. We will use \( v^i_t(x_i) \) to denote player \( i \)'s valuation for bandwidth \( x_i \). We will make the assumption for the valuation functions that their slope will be lower bounded by some \( \rho > 0 \). Player \( i \)'s utility will be again quasilinear, i.e.
\[ u_i^t(b) = v_i^t(x_i(b')) - b_i^t. \] Similarly as before, we will assume that bids will be only multiples of \( \rho \delta \) for some \( \delta > 0 \). Therefore bidding space is sufficiently discrete and the number of strategies available is at most \( N = \frac{1}{\rho \delta} \).

**Atomic congestion game.** In the atomic congestion game, we assume that we have a set of congestible elements \( E \) (and let \( m = |E| \)), each element \( e \) has a latency function \( \ell_e(x) \), or cost, that is monotone non-decreasing in the congestion \( x \). Given some selection of sets \( s_i \subseteq E \) for each player \( i \), the congestion in an element \( e \) is the number of players that have selected it: \( x_e(s) = |\{ i : e \in s_i \}| \), and the cost of player \( i \) is then the sum \( \sum_{e \in s_i} \ell_e(x_e(s)) \).

A player’s type \( v_i^t \) denotes the possible subsets of the element set she can select. For example, in the routing game on a graph, the type of a player \( i \) is a source-sink pair \((o_i, d_i)\), and her strategy is the choice of a path from \( o_i \) to \( d_i \) in the graph. We assume that a player’s cost is infinity if her solution is not one of the selected sets. Thus the number of strategies available to the player is the number of \((o_i, d_i)\), paths in the graph and thereby \( N \) is the maximum number of such possible paths across possible source-sink pairs.

**Adaptive Learning in Dynamic Environments.** We use the notion of adaptive regret introduced by [Hazan and Seshadhri (2007)](https://www.jmlr.org/papers/volume8/hazan07a/hazan07a.pdf). We start by defining no-regret learning, and then consider adaptive regret. To formally define regret, and no-regret learning, we consider an arbitrary loss function. For a cost-game, we will think of the cost the player incurs as loss. For a utility game, we define loss each step as the difference between the maximum possible utility and the player’s utility. Consider a player who has \( N \) possible choices, the \( N \) strategies that the player has to choose from. In defining regret, and no-regret learning we are focusing on a single player, and hence we will temporarily drop the index \( i \) for the player from the notation. We use \( L(s, t) \) to denote the loss (cost or lost utility) of the player if she plays strategy \( s \) at time \( t \). We can assume without loss of generality that \( L(s, t) \) is a value in \([0, 1]\), but make no assumption beyond this about the sequence of loss values. We say that a player achieves no-regret, if she does at least as well over a period of time, as the best choice \( s^* \) with hindsight. Formally, we say the regret of a strategy sequence \( s^{1:T} \) is

\[
\sum_{t=1}^{T} L(s^t, t) - \min_{s^*} L(s^*, t)
\]

Note that even with a stable set of players the value \( L(s, t) \) will vary over time, depending of the strategies chosen by other players. As mentioned in the Introduction, there are many simple algorithms (see, for example, [Arora et al. 2012](https://papers.nips.cc/paper/5044-regret-minimization-in-adversarial-environments.pdf)) that achieve regret \( O(\sqrt{T}) \) against any (adversarial) sequence of loss values \( L(s, t) \).

In dealing with changing environments, we will need a stronger assumption on the learning of the players, we need that the players adapt their strategies to the environment. We will use a notion of
adaptive regret, regret over (long) intervals of time \([\tau_1, \tau_2]\), in addition to the regret of the whole sequence, defined by Hazan and Seshadhri (2007).

**Definition 2.2 (Adaptive Regret).** The adaptive regret of strategy sequence \(s^{1:T}\) in time frame \([\tau_1, \tau_2]\) is defined as:

\[
R(\tau_1, \tau_2) = \max_{s^*} \sum_{t = \tau_1}^{\tau_2-1} (L(s^*, t) - L(s^*, t))
\]

Adaptive learning algorithms go back to the work of Lehrer (2003) and Blum and Mansour (2007) who considered more general notions of regret. We say that a player satisfies adaptive learning if her regret \(R(\tau_1, \tau_2)\) can be bounded by a function that is \(o(\tau_2 - \tau_1)\), that is, regret grows slower than linearly over time. Our results are affected by the quality of the learning algorithm players use, as with better learning we can tolerate higher turnover in the population of players. In the rest of the paper we will use the learning bounds of the recent work of Luo and Schapire (2015), who developed an adaptation of the classical Hedge algorithm, AdaNormalHedge that achieves small regret on all intervals. An alternate algorithm with a bound of the same type was also given in (Blum and Mansour 2007).

**Theorem 2.1 (Luo and Schapire 2015).** Suppose a player uses AdaNormalHedge and selected strategy sequence \(s^{1:T}\). For any time frame \([\tau_1, \tau_2]\), AdaNormalHedge achieves adaptive regret:

\[
E(R(\tau_1, \tau_2)) \leq C_R \sqrt{\tau_2 - \tau_1} \ln(N\tau_2)
\]

where \(N\) is the number of choices, \(C_R\) is a small constant less than 2, and loss is assumed to be in \([0, 1]\) for all \(s\) and \(t\).

In what follows, we will assume that all players in our repeated game use a learning algorithm with low adaptive regret, will use \(R_i(\tau_1, \tau_2)\) to denote the adaptive regret of player \(i\) over the period \([\tau_1, \tau_2]\). For simplicity of presentation, we will assume that, for some constant \(C_R\), \(E(R(\tau_1, \tau_2)) \leq C_R \sqrt{\tau_2 - \tau_1} \ln(N\tau_2)\) for all players and all time periods \([\tau_1, \tau_2]\). Throughout the paper, we will refer to this assumption as “The players use adaptive learning algorithms with constant \(C_R\).” Our results would smoothly degrade if we assumed only that players achieve adaptive regret that is some other sublinear concave function of the interval’s length \((\tau_2 - \tau_1)\).

**Solution-based Smoothness in Games and Mechanisms.** Smooth games were introduced by Roughgarden (2009) as a general framework bounding the price of anarchy in games. He also showed that smoothness based price of anarchy bounds extend to outcomes in repeated games when all players use no-regret learning.
We need a somewhat more general variant of smooth games, that compares the cost or utility resulting from a strategy choice to the social welfare of a specific solution, rather than comparing to the social optimum. For two strategy vectors \( s \) and \( s^* \) we use \((s^*_i, s_{-i})\) to denote the vector where player \( i \) uses strategy \( s^*_i \) and all other players \( j \) use their strategy \( s_j \).

**Definition 2.3 (Solution-based smooth game).** A cost-minimization game \( G \) is \((\lambda, \mu)\)-smooth with respect to a solution \( x_\ast \), if for some \( \lambda > 0 \) and \( \mu < 1 \), for any type profile \( v \), for each player \( i \) there is a strategy \( s^*_i \in S_i \) depending on his type \( v_i \) and her part of the solution \( x_i \) such that for any strategy profile \( s \)

\[
\sum_i c_i(s^*_i(v_i, x_i), s_{-i}; v_i) \leq \lambda C(x; v) + \mu C(s; v)
\]

A game \( G \) is solution-based \((\lambda, \mu)\)-smooth if it is smooth with respect to any feasible solution \( x \in X^n \).

Note that, when \( x \) is the optimal solution, we recover the traditional examples of smooth games, as the deviating strategy \( s^*_i \) usually depends on other players’ types through his part of the optimal solution \( x^*_i(v) \). A game that is \((\lambda, \mu)\)-smooth with respect to the optimal solution \( x^*(v) \) is \((\lambda, \mu)\)-smooth in the sense of [Roughgarden 2009], and the game has price of anarchy bounded by \( \lambda/(1-\mu) \), and the average social cost of no-regret learning outcomes is also bounded by \( \lambda/(1-\mu) \) \( \text{Opt} \).

More generally,

**Theorem 2.2.** If a game is \((\lambda, \mu)\)-smooth with respect to a solution \( x \), then at any Nash equilibria of the game, as well as at any no-regret learning outcome, the expected cost is at most \( \frac{\lambda}{1-\mu} C(x; v) \).

**Proof of Theorem 2.2.** We include the proof for the case of pure Nash equilibria for completeness. Consider a strategy vector \( s \) that is a Nash equilibrium. At a Nash equilibrium, no player has regret for any alternate strategy, so in particular we get that \( c_i(s^*_i(v_i, x_i), s_{-i}; v_i) \geq c_i(s; v_i) \) for all \( i \). Adding up these inequalities and using the smoothness property, we get

\[
C(s; v) = \sum_{i \in [n]} c_i(s; v_i) \leq \sum_i c_i(s^*_i(v_i, x_i), s_{-i}; v_i) \leq \lambda C(x; v) + \mu C(s; v) \quad (2.1)
\]

The claimed bound follows by rearranging the terms. The proof extends to randomized equilibria by taking expectations, including the distribution resulting in no-regret learning in the limit. \[\square\]

[Syrgkanis and Tardos 2013] give a related definition for smooth mechanisms assuming quasi-linear valuation for all players. Again, we define a mechanism smooth with respect to a solution \( x \), and allow the choice of strategy \( s^* \) to depend on the player’s part of the solution \( x_i \) and his type \( v_i \). More formally, we will use the following definition.
**Definition 2.4 (Solution-based smooth mechanism).** A mechanism \( \mathcal{M} \) is \((\lambda, \mu)\)-smooth with respect to a solution \( x \) for some \( \lambda, \mu \geq 0 \) if for any valuation profile \( v \) for each player \( i \) there exists a deviating strategy \( s^*_i \in S_i \) depending on \( v_i \) and \( x_i \) such that for all strategy vectors \( s \),
\[
\sum_i u_i(s^*_i(v_i, x_i), s_{-i}; v_i) \geq \lambda W(x; v) - \mu R(s).
\]
where \( R(s) = \sum_{i=1}^n P_i(s) \). \( \mathcal{M} \) is a solution-based \((\lambda, \mu)\)-smooth mechanism if the latter holds for any feasible solution \( x \in X^n \).

Syrgkanis and Tardos (2013) proved that a \((\lambda, \mu)\)-smooth mechanism has price of anarchy bounded by \( \max(\mu, 1)/\lambda \), and the average social welfare of no-regret learning outcome is also at least \( \frac{\lambda}{\max(\mu, 1)} \text{Opt}(v) \). Analogously we get:

**Theorem 2.3.** If a mechanism is \((\lambda, \mu)\)-smooth with respect to a solution \( x \), then at any Nash equilibria of the game, as well as at any no-regret learning outcome, the expected social welfare is at least \( \frac{\lambda}{\max(\mu, 1)} W(x; v) \).

**Differential privacy.** Differential privacy has been developed for databases storing private information for a population. A database \( D \in V^n \) is a vector of inputs, one for each player. Two databases are \( i \)-neighbors if they differ just in the \( i \)-th coordinate, i.e. differ only in the input the \( i \)-th player. If two databases are \( i \)-neighbors for some \( i \), they are called neighboring databases.

In the context of repeated games, every time a player leaves or arrives, the solution may change drastically. Instead of comparing the game outcomes to the socially optimal solution that changes with every player change, we will want to compare the outcome to a more stable but close to optimal solution. The notion of differential privacy offers a useful framework for this goal.

Dwork et al. (2006) define an algorithm as differentially private if one person’s information has little influence on the outcome. In the setting of a game or mechanism the outcome for player \( i \) clearly should depend on player \( i \)'s input (her claimed valuation, or source destination pair), so cannot be differentially private. The notion of joint differential privacy which has been developed by Kearns et al. (2014) to adapt differential privacy to settings, where the algorithm has a set of \( n \) outcomes, one for each player. We use \( \mathcal{X} \) to denote the set of possible outcomes for one player, so an algorithm in this context is a function \( \mathcal{A} : V^n \rightarrow \mathcal{X}^n \). The algorithm is jointly differentially private, if for all players \( i \), the output for all other players is differentially private in the input of player \( i \). More formally,

**Definition 2.5 ((Kearns et al., 2014)).** An algorithm \( \mathcal{A} : V^n \rightarrow \mathcal{X}^n \) is \((\epsilon, \delta)\)-jointly differentially private if for every \( i \), for every pair of \( i \)-neighbors \( D, D' \in V^n \), and for every subset of outputs \( S \subseteq \mathcal{X}^{n-1} \),
\[
\Pr[\mathcal{A}(D)_{-i} \in S] \leq \exp(\epsilon) \Pr[\mathcal{A}(D')_{-i} \in S] + \delta
\]
If \( \delta = 0 \), we say that \( \mathcal{A} \) is \( \epsilon \)-jointly differentially private.
We will see that close to optimal and jointly private solutions along with smoothness with respect to the sequence of solutions $x^t$, can be used to show the strength of learning outcomes in our setting. Over the last the years there have been a number of algorithms developed that solve problems close to optimally in a differentially private way. See the recent book of Dwork and Roth (2014) for a survey. In this paper, we will take advantage of such algorithms, including the algorithms for solving matching problems (Hsu et al. 2014) and finding socially optimal routing (Rogers et al. 2015).

Marginal privacy. A recent work of Kannan et al. (2014) introduced the weaker notion of marginal differential privacy, also in the setting when the algorithm outputs a set of $n$ outcomes, one for each player. A mechanism is marginally differentially private if the distribution of outcomes for any one player $j$ is differentially private in the input of another player $i \neq j$, but not requiring that the combined output of all players $j \neq i$ should be differentially private in $i$th input. Our main results continue to hold even under this weaker notion of privacy. However since no improved approximation algorithms are known under this notion for the settings that we study, we focus on joint privacy in the main part of the paper and present the extension in Appendix EC.2.

3. Price of Anarchy for Dynamic Games and Mechanisms

In this section we offer our two main theorems which follow the high level outline presented in section 1. Specifically, we formalize the connection between adaptive learning, solution-based smoothness and the existence of approximately optimal and stable solution sequences. We give this connection both in the context of cost-minimization games and in the context of mechanisms. In the next section we give an application of the framework to unit-demand matching markets and bandwidth allocation, and in Section 5 we provide a more canonical approach towards producing stable sequences by connecting the problem to differential privacy, along with a way we can relax the stability notion required.

Definition 3.1 (k-stable sequence). A randomized sequence of solutions $x^{1:T} = \{x^1, \ldots , x^T\}$ and types $v^{1:T} = \{v^1, \ldots , v^T\}$ is k-stable if the average (across players) expected number of changes in each individual player’s solution or type is at most $k$, i.e., if $k_i(v_i^{1:T}, x_i^{1:T})$ is the number of times that $x_i^t \neq x_i^{t+1}$ or $v_i^t \neq v_i^{t+1}$, then:

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[k_i(v_i^{1:T}, x_i^{1:T})] \leq k$$

Theorem 3.1 (Main theorem for cost-minimization games). Consider a repeated cost game with dynamic population $\Gamma = (G,T,p)$, such that the stage game $G$ is solution-based $(\lambda, \mu)$-smooth and costs are bounded in $[0,1]$. Suppose that $v^{1:T}$ and $x^{1:T}$ is a k-stable sequence,
such that \( x^t \) is feasible (pointwise) and \( \alpha \)-approximately (in-expectation) optimal for each \( t \), i.e. 
\[
\mathbb{E}[C(x^t; v^t)] \leq \alpha \cdot \mathbb{E}[\text{OPT}(v^t)].
\]
If players use an adaptive learning algorithm with constant \( C_R \) then:
\[
\sum_t \mathbb{E}[C(s^t; v^t)] \leq \frac{\lambda}{1 - \mu} \sum_t \mathbb{E}[\text{OPT}(v^t)] + \frac{n}{1 - \mu} \cdot C_R \sqrt{T \cdot (k+1) \cdot \ln(NT)}
\]

An analogue of the theorem above holds for mechanisms too.

**Theorem 3.2 (Main theorem for mechanisms).** Consider a repeated mechanism with dynamic population \( M = (M, T, p) \), such that the stage mechanism \( M \) is solution-based \( (\lambda, \mu) \)-smooth and utilities are bounded in \([0,1]\). Suppose that \( v^{1:T} \) and \( x^{1:T} \) is a \( k \)-stable sequence, such that \( x^t \) is feasible (pointwise) and \( \alpha \)-approximately optimal (in-expectation) for each \( t \), i.e. 
\[
\alpha \cdot \mathbb{E}[W(x^t; v^t)] \geq \mathbb{E}[\text{OPT}(v^t)].
\]
If players use an adaptive learning algorithm with constant \( C_R \) then:
\[
\sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{\lambda}{\alpha \max\{1, \mu\}} \sum_t \mathbb{E}[\text{OPT}(v^t)] - n \cdot C_R \sqrt{T \cdot (k+1) \cdot \ln(NT)}
\]

We also show an improved bound for some classes of mechanisms that satisfy an non-negative utility property and which we will use in our application in Section 4.1. For the case of simultaneous single-item first price auctions with unit-demand bidders it leverages the fact that by bidding only on one item at-a-time, player utilities are guaranteed to be nonnegative at all times, and only a subset of the players (e.g. at most \( m \) in the case of an \( m \) item auction) are being allocated in any feasible allocation. Under these conditions, players with no item in the feasible allocation will have no regret against a deviating strategy that attempts to “win” the empty allocation. For a general mechanism \( M \) the required Property is stated as follows:

**Property 1.** \( M \) has an empty allocation \( \emptyset \) in the allocation space. Moreover \( u_i(s^t_i(v_i, \emptyset), s_{-i}) = 0 \) and \( u_i(s; v_i) \geq 0 \) for any strategy that is used by the players.

**Theorem 3.3 (Improved bound for mechanisms).** Consider a repeated mechanism with dynamic population \( M = (M, T, p) \), such that the stage mechanism \( M \) is solution-based \( (\lambda, \mu) \)-smooth, satisfies Property 1 and utilities are in \([0,1]\). Assume that there exists a randomized sequence of solutions \( x^{1:T} = \{x^1, \ldots, x^T\} \) and types \( v^{1:T} = \{v^1, \ldots, v^T\} \), such that \( x^t \) is feasible (pointwise) and \( \alpha \)-approximately optimal (in-expectation) for each \( t \), i.e. 
\[
\alpha \cdot \mathbb{E}[W(x^t; v^t)] \geq \mathbb{E}[\text{OPT}(v^t)].
\]

For each player \( i \), let \( \kappa_i(v^{1:T}_i, x^{1:T}_i) \) be the number of times that \( x^t_i \neq x^{t+1}_i \) or \( x^t_i \neq \emptyset \) and \( v^t_i \neq v^{t+1}_i \). If the randomized sequence satisfies an analogue of \( k \)-stability:
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \kappa_i(v^{1:T}_i, x^{1:T}_i) \right] \leq k \quad (3.1)
\]

\(^3\) Observe that unlike the definition of \( k_i(v^{1:T}_i, x^{1:T}_i), \kappa_i(v^{1:T}_i, x^{1:T}_i) \) does not account for changes in the type of players that are not currently allocated an item in solution \( x^t_i \).
and players use an adaptive learning algorithm with constant $C_R$ then:

$$\sum_t \mathbb{E}[W(s^t; \mathbf{v}^t)] \geq \frac{\lambda}{\alpha \max\{1, \mu\}} \sum_t \mathbb{E}[\text{Opt}(\mathbf{v}^t)] - C_R \sqrt{T \cdot m \cdot (k \cdot n + m) \cdot \ln(NT)}$$

where $m$ is such that for any feasible allocation $x$, $|\{i : x_i \neq \emptyset\}| \leq m$.

**Removing the dependence on $T$.** In all the theorems of this section there is a logarithmic dependence of the average regret on the time horizon $T$. This will lead in the efficiency theorems throughout the paper to require that the probability of change $p$ be at most a quantity that is inversely proportional to $\log(T)$. As we want to think of $T$ as a really large quantity, one might argue that this dependence makes the requirements on $p$ very harsh. However, we note that this dependence on $T$ is not essential and is only for the simplicity of exposition. The quantity that should actually go into the regret bounds presented in this section is rather of the order of the expected lifespan of any player in the repeated game, which is of the order of $1/p$. Therefore the $\log(T)$ terms in the theorems of this section can be replaced by terms that are roughly $O(\log(1/p))$.

In Section 4.C.2 of the supplementary material we formalize this argument and provide a detailed proof of how to remove the dependence on $T$ in all our theorems.

## 4. Stable Sequences via Greedy Algorithms

In this section we offer direct arguments to show the existence of stable solution sequences and hence good efficiency results for games with dynamic population. We prove efficiency results for the case of matching markets with dynamic population and the case of proportional bandwidth allocation with dynamic population. Our method is based on a combination of using the greedy algorithm and rounding the input parameters.

### 4.1. Matching markets

As a first application we focus on a repeated mechanism with dynamic population $\Gamma = (M, T, p)$, where the stage mechanism is simultaneous first price auction with unit-demand bidders (matching markets). To apply our improved theorem, Theorem 3.3, we need two things: i) that the mechanism is allocation based $(\lambda, \mu)$-smooth, and ii) that there exists a relatively stable sequence of approximately optimal solutions for the optimization problem.

We start by showing that the mechanism is smooth. $(1/2, 1)$-smoothness of the simultaneous first price auction with submodular bidders (a super-set of unit-demand valuations) and continuous bids was known by Syrgkanis and Tardos (2013). We consider discrete bidding spaces. A simple modification of the result of Syrgkanis and Tardos (2013) shows that if the discretization is fine enough, then the mechanism is approximately $(1/2, 1)$-solution based smooth. We will present the more general result for submodular valuations, as we will re-use this fact in Section 5.3, where we consider more general valuations.
Lemma 4.1 (Smoothness of simultaneous first price auction). The simultaneous first price mechanism where players are restricted to bid on at most \(d\) items and on each item submit a bid that is a multiple of \(\delta \cdot \rho\), is a solution based \((\frac{1}{2} - \delta, 1)\)-smooth mechanism, when players have submodular valuations, such that all marginals are either 0 or at least \(\rho\) and such that each player wants at most \(d\) items, i.e. \(v_i(S) = \max_{T \subseteq S: |T| = d} v(T)\).

To get a stable and approximately optimal allocation, we use a layered version of the greedy algorithm. The greedy matching algorithm considers item valuations \(v_i(j)\) in decreasing order and assigns item \(j\) to player \(i\) if, when \(v_i(j)\) is considered, neither item \(j\) nor player \(i\) are matched.

To make this algorithm more stable we define the greedy-layered matching algorithm, which works as follows. Let \(\rho > 0\) be the smallest non-zero value that a player has for any item. For a positive \(\epsilon \leq \frac{1}{3}\), we round each player’s value down to the closest number of the form \(\rho(1 + \epsilon)^\ell\) for some integer \(\ell\), and run the greedy algorithm with these rounded values. It is well known that the greedy algorithm guarantees a solution that is within a factor of 2 to optimal. We lose an additional factor of \((1 + \epsilon)\) by working with the rounded values. The greedy algorithm will have many ties and we will resolve ties in a way to make the output stable.

Lemma 4.2 (Stability via the greedy algorithm). Consider a repeated matching market mechanism with dynamic population \(M = (M, T, p)\), with \(m\) items and \(n\) players, where \(\rho\) is the minimum possible non-zero valuation. Assuming \(T \geq \frac{1}{p}\), the greedy-layered matching algorithm with parameter \(\epsilon\) guarantees that \(W(x^t; v^t) \geq \frac{1}{2(1+\epsilon)} \cdot \text{Opt}(v^t)\) for all \(t\), and it can be implemented so that the average (over players) expected number of changes in the allocation sequence or the type for players who hold an item at the time of the change is upper bounded by

\[
\sum_i \sum_{t=1}^n E\left[ \kappa_i \left( v_i^{1:T}, x_i^{1:T} \right) \right] \leq \frac{5T \cdot m \cdot p \cdot \log(1 + \epsilon)(1/p) \cdot \ln(NT)}{n} \tag{4.1}
\]

Theorem 4.1 (Main theorem for matching markets). In the simultaneous first price auction mechanism with dynamic population and unit-demand bidders, if all players use adaptive learning algorithms with constant \(C_R\) and if \(T \geq \frac{1}{p}\) we have:

\[
\sum_t E[W(s^t; v^t)] \geq \frac{1}{4(1+\epsilon)} \sum_t E[\text{Opt}(v^t)] - mT \cdot C_R \sqrt{6 \cdot p \cdot \log((1+\epsilon)(1/p) \cdot \ln(NT)} \tag{4.2}
\]

where \(N\) is the number of different strategies considered by a player.

If in addition we assume that all items get allocated at each round for the minimum value of \(\rho\), or that the average optimal welfare in each round is at least \(m\rho\), that is \(\frac{1}{T} \sum_{t=1}^T E[\text{Opt}(v^t)] \geq m\rho\), then we can also get a purely multiplicative bound:

\[
\sum_t E[W(s^t; v^t)] \geq \frac{1}{4(1+\epsilon)} \sum_t E[\text{Opt}(v^t)] \tag{4.3}
\]

if the turnover probability \(p\) is at most \(C \cdot \frac{\epsilon^2 \rho^2}{\ln(NT)}\) for \(C = (96(1+\epsilon)^2(C_R)^2 \log((1+\epsilon)(1/p))^{-1}.\)
Remark 4.1. An interesting feature of Theorem 4.1 is that the probability $p$ is independent of the number of players $n$ and the number of items $m$, implying that the game can accommodate extremely high turnover in player population, as the number of players increases, without losing in the quality of the outcome. The probability $p$ required for the high quality solution, needs to depend only on $\log_{(1+\epsilon)}(1/\rho)$, $\log N$ and $\log T$, where $N$ is bounded by $\frac{m}{\rho^2}$ and the dependence on $T$ can be removed as presented in Section E.C.4 of the supplementary material.

The high-level intuition why the greedy algorithm can sustain such a rate of change is as follows: At any time-step the only players that incur any non-zero regret are the players to whom the greedy solution currently allocates some item. Since the optimal welfare is at least $m \cdot \rho$, if we want the efficiency to be $\epsilon$ close to what is implied by having absolutely no regret for the greedy layered algorithm we need the total regret in the system to be at most $\epsilon \cdot m \cdot \rho$. In other words, we need the regret associated with each item to be at most $\epsilon \cdot \rho$. Now observe that when an item is allocated to a player in the highest level, i.e. with a value in $[\frac{1}{(1+\epsilon)}, 1]$, then this player is never unassigned from that item until he leaves the game. Thus we can roughly view the lifetime of an item as decomposing into $p \cdot T$ cycles such that during each cycle the item transitions from level-1 players to level-$\log_{(1+\epsilon)}(1/\rho)$ players. In other words, the lifetime of an item splits in roughly $pT \log_{(1+\epsilon)}(1/\rho)$ stable allocation intervals, leading to average interval length $(p \log_{(1+\epsilon)}(1/\rho))^{-1}$ and thereby average regret at most $\sqrt{p \cdot \log_{(1+\epsilon)}(1/\rho)}$. Since we want this regret to be at most $\epsilon \cdot \rho$, we get $p \leq \frac{\rho^2}{\log_{(1+\epsilon)}(1/\rho)}$ which is essentially the bound we have in Theorem 4.1.

4.2. Bandwidth allocation

As a second application we focus on a repeated mechanism with dynamic population $\mathcal{M} = (M, T, p)$ where the stage mechanism is the proportional bandwidth allocation mechanism. Recall the bandwidth sharing mechanism, where every player $i$ submits and pays a bid $b_i$, and the available bandwidth $B$ (which we assume is $B = 1$ for notational simplicity) is divided proportionally to the player’s bid, so bidder $i$ gets bandwidth $x_i(b) = \frac{b_i}{\sum_j b_j}$ and pays $b_i$. We assume that the player’s utility is quasilinear, so if the player’s valuation function is $v_i(x)$ for $x$ amount of bandwidth, then the resulting utility is $u_i(b) = v_i(x_i(b)) - b_i$. Following [Kelly (1997) and Johari and Tsitsiklis (2011)], we will assume that the player’s valuation functions $v_i : [0, B] \rightarrow \mathbb{R}$ are increasing, concave and differentiable. Further, we will make some Lipschitz style assumptions on the rate of change of the value functions. Concretely, we will assume the following:

1. Value functions $v_i(x)$ are increasing, concave and twice differentiable, $v_i(0) = 0$ and $v_i(B) \leq 1$.
2. The rate of increase is at least $\rho$, i.e. $\forall i, x : v_i'(x) \geq \rho$.

\footnote{Not completely accurate as players can leave to other items too, but a good approximation.}
3. The gradient is \( \alpha \)-Lipschitz, i.e. \( \forall i, x : |v''_i(x)| \leq \alpha \).

Following a similar approach as in the previous section, we can derive an efficiency guarantee in this setting too.

**Theorem 4.2 (Main theorem for bandwidth allocation).** Consider the proportional bandwidth sharing game with dynamic population and with valuations satisfying the conditions listed above. If all players use adaptive learning algorithms with constant \( C_R \) and if \( T \geq \frac{1}{p} \) then we have:

\[
\sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{(2-\sqrt{3}-\epsilon)(1-\epsilon)}{1+\epsilon} \sum_t \mathbb{E}[\text{Opt}(v^t)]
\]

if the turnover probability \( p \) is at most \( C \cdot \frac{\rho^{4/2}}{\alpha^2 \ln(NT)} \) for \( C = (96(1-\epsilon^2)(C_R)^2 \log(1+\epsilon)(\alpha(1-\epsilon)/\rho^2\epsilon))^{-1} \).

The high-level outline of the proof consists of three lemmas.

- As a benchmark optimization problem, we consider the \( \delta \)-segmented bandwidth allocation problem for some \( \delta > 0 \), where all allocated bandwidths are integer multiples of \( \delta \). We show that the Lipschitz condition above ensures that for a small enough \( \delta > 0 \), the segmented optimum is not much smaller than the true optimum.

**Lemma 4.3.** The social welfare of the optimal \( \delta \)-segmented solution approximates within \( (1-\epsilon) \) the optimum if \( \delta \leq \frac{2\rho^2}{\alpha} \).

- To get a stable and approximately optimal allocation for the \( \delta \)-segmented bandwidth problem, we use a layered version of the greedy algorithm, similar to our greedy matching algorithm in Section 4.1. We divide the bandwidth in segments of length \( \delta \). The greedy bandwidth allocation algorithm greedily allocates segments based on the marginal increase in the players’ valuation function. We will denote as \( v_{i,j} \) the marginal valuation that player \( i \) has for her \( j \)-th segment. Note that, due to concavity of the valuation function \( v_{i,j} \) is a non-increasing function on \( j \) and, due to the lower bound on the gradient, it is at least \( \rho \delta \). The greedy algorithm is therefore optimal for the \( \delta \)-segmented bandwidth problem.

To make it more stable, similarly as in the matching markets, we use a layered version of the valuation functions where the layer of some marginal valuation \( v_{i,j} \) is the highest \( \ell \) such that \( \ell(v_{i,j}) \geq \rho \delta (1+\epsilon)^{\ell-1} \). We will use \( \ell^t(j) \) to denote the layer that the \( j \)-th most valued (as in marginal values) segment was assigned at step \( t \). We will again select the tie-breaking rule across marginal values of the same layer to facilitate stability, i.e. previous holders of segments are helped in the tie-breaks to keep the same number of segments as they had before. We show that the greedy layered algorithm for the \( \delta \)-segmented bandwidth allocation problem finds a solution within a \( (1+\epsilon) \) factor of the welfare of the optimal \( \delta \)-segmented solution, and that the sequence of solutions found by this greedy algorithm is stable.
Lemma 4.4. Consider a repeated $\delta$-segmented bandwidth allocation game with dynamic population $\mathcal{M} = (M, T, p)$ and $n$ players. Assuming $T \geq 1/p$, the greedy layered algorithm with parameter $\epsilon$ guarantees that $W(x^t; v^t) \geq \frac{1}{(1+\epsilon)} \text{OPT}(v^t)$ for all $t$, and it can be implemented so that the average (over players) expected number of changes in the allocation sequence or the type for players who hold an item at the time of the change is upper bounded by

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \kappa_i (v^1_i; x^1_i, T_i) \right] \leq \frac{5 \cdot T \cdot (1/\delta) \cdot p \cdot \log(1+\epsilon) \cdot (1/\rho)}{n}.$$ 

Finally, we need to show that the proportional sharing mechanism is smooth. Syrgkanis and Tardos (2013) showed that the mechanism is $(2 - \sqrt{3} - \epsilon, 1)$-smooth using a randomized deviation. To use this deviation in our framework, we want to consider a discretized bidding space. We show that for every $\epsilon > 0$, the proportional allocation mechanism is $(2 - \sqrt{3} - \epsilon, 1)$-solution based smooth with respect to any solution of the $\delta$-segmented bandwidth allocation problem, using a the discretized deviation.

Lemma 4.5. The proportional mechanism allowing only bids that are multiples of $\zeta = \epsilon \delta$ is $(2 - \sqrt{3} - \epsilon, 1)$-solution based smooth with respect to any $\delta$-segmented allocation.

Combining these lemmas, we use Theorem 3.3 to get the claimed efficiency result.

Proof of Theorem 4.2. From Lemmas 4.3, 4.4, 4.5 and Theorem 3.3, setting $\delta = \frac{\epsilon \rho}{\alpha (1-\epsilon)}$, we get that the aggregate social welfare of the proportional allocation bandwidth is $(2 - \sqrt{3} - \epsilon, 1)$ of the optimum. This is achieved for turnover probability $p$:

$$p \leq \frac{(\rho \delta)^2 \epsilon^2}{6 \cdot 16(1+\epsilon)^2 (C_R)^2 \log(1+\epsilon)(1/\rho \delta) \ln(NT)}.$$ 

Replacing $\delta$, the result follows. $\square$

5. Stable Sequences via Differential Privacy

In this section we formally connect joint differential privacy with the construction of stable sequences needed by our main Theorems 3.1 and 3.2. In Appendix EC.2 we offer a strengthening of these theorems that allows us to use marginal differential privacy. Differential privacy offers a general framework to find solutions that are close to optimal, yet more stable to changes in the input than the optimum itself. To guarantee privacy, the output of the algorithm is required to depend only minimally on any player’s input. This is exactly what we need in our framework.

Theorem 5.1 (Stable sequences via privacy). Suppose there exists an algorithm $\mathcal{A} : \mathcal{V}^n \rightarrow \Delta(\mathcal{X}^n)$ that is $(\epsilon, \delta)$-jointly differentially private, takes as input a valuation profile $v$ and outputs a distribution of solutions such that a sample from this distribution is feasible with probability $1 - \beta$, and is $\alpha$-approximately efficient in expectation (for $0 \leq \epsilon \leq 1/2$, $\alpha > 1$ and $\delta, \beta > 0$).
Consider a sample $v^{1:T}$ from the distribution of valuations produced by the adversary in a repeated cost-minimization game with dynamic population $\Gamma = (G, p, T)$. There exists a randomized sequence of solutions $x^{1:T}$ for the sequence $v^{1:T}$, such that for each $1 \leq t \leq T$, $x^t$ conditional on $v^t$ is an $\alpha$-approximation to $\text{OPT}(v^t)$ in expectation and the joint randomized sequence $(v^{1:T}, x^{1:T})$ is $pT(1+n(2\epsilon + 2\beta + \delta))$-stable (as in Definition 3.7).

We defer the proof of Theorem 5.1 to the next subsection. Combining Theorem 5.1 with Theorem 3.2, we immediately get the following corollary.

**Corollary 5.1.** Consider a repeated cost game with dynamic population $\Gamma = (G, T, p)$, such that the stage game $G$ is allocation based $(\lambda, \mu)$-smooth and $T \geq \frac{1}{p}$. Assume that there exists an $(\epsilon, \delta)$-joint differentially private algorithm $A : \mathcal{V}^n \rightarrow \mathcal{X}^n$ with error parameter $\beta$ that satisfies the conditions of Theorem 5.1. If all players use adaptive learning algorithms with constant $C_R$ in the repeated game then the overall cost of the solution is at most:

$$
\sum_t \mathbb{E}[C(s^t; v^t)] \leq \frac{\lambda}{\max\{1, \mu\}} \sum_t \text{OPT}(v^t) + \frac{nT}{\mu} \cdot C_R \sqrt{2p(1+n(\epsilon+\beta+\delta)) \ln(NT)}
$$

Similarly for a mechanism we get:

$$
\sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{\lambda}{\alpha \max\{1, \mu\}} \sum_t \mathbb{E}[\text{OPT}(v^t)] - \frac{nT}{\max\{1, \mu\}} \cdot C_R \sqrt{2p(1+n(\epsilon+\beta+\delta)) \ln(NT)}
$$

**5.1. Proof of Theorem 5.1**

We will use total variation distance to measure the distance between distributions. For two distributions $\mu$ and $\eta$ on some finite probability space $\Omega$ the following are two equivalent versions of the total variation distance:

$$
d_{tv}(\mu, \eta) = \frac{1}{2} \|\mu - \eta\|_1 = \max_{A \subseteq \Omega} \langle \mu(A) - \eta(A), \rangle
$$

where in the 1-norm in the middle we think of $\mu$ and $\eta$ as a vector of probabilities over the possible outcomes.

**Lemma 5.1.** Suppose that $A : \mathcal{V}^n \rightarrow \Delta(\mathcal{X}^n)$ is an $(\epsilon, \delta)$-joint differentially private algorithm with failure probability $\beta$ (for $0 \leq \epsilon \leq 1/2$ and $\delta, \beta > 0$) that takes as input a valuation profile $v$ and outputs a distribution over feasible solutions $\sigma$. Let $\sigma$ and $\sigma'$ be the algorithm’s outputs on two inputs $v$ and $v'$ that differ only in coordinate $i$. Then we can bound the total variation distance between $\sigma_{-i}$ and $\sigma'_{-i}$ by $d_{tv}(\sigma_{-i}, \sigma'_{-i}) \leq (2\epsilon + \delta)$.

**Proof of Lemma 5.1.** Condition (2.5) of joint differential privacy guarantees that if we let $S \subseteq \mathcal{X}_i$ be a subset of possible solutions for players other than $i$ and with $\sigma_{-i}(S)$ and $\sigma'_{-i}(S)$ the probability that the two distributions assign on $S$, then for any $S$: $\sigma_{-i}(S) \leq \exp(\epsilon)\sigma'_{-i}(S) + \delta$. 

**: Lykouris, Syrgkanis and Tardos: Learning and Efficiency in Games with Dynamic Population**
Since $\epsilon \leq 1/2$, we can use the bound $\exp(\epsilon) \leq 1 + 2\epsilon$ to get that $\sigma_{-i}(S) - \sigma'_{-i}(S) \leq 2\epsilon\sigma'_{-i}(S) + \delta \leq 2\epsilon + \delta$. Thus by the second definition of the total variation distance in Equation 5.1 we get that $d_{tv}(\sigma_{-i}, \sigma'_{-i}) \leq 2\epsilon + \delta$. □

To facilitate the proof we need a simple lemma from basic probability theory.

**Lemma 5.2 (Coupling Lemma).** Let $\mu$ and $\eta$ be two probability measures over a finite set $\Omega$. There is a coupling $\omega$ of $(\mu, \eta)$, such that if the random variable $(X, Y)$ is distributed according to $\omega$, then the marginal distribution on $X$ is $\mu$, the marginal distribution on $Y$ is $\eta$, and

$$\Pr[X \neq Y] = d_{tv}(\mu, \eta).$$

**Proof of Theorem 5.1.** Suppose that $A : V^n \rightarrow \Delta(\mathcal{X}^n)$ is an $(\epsilon, \delta)$-joint differentially private algorithm as described in the definition of the theorem. The differentially private algorithm fails with probability $\beta$. We will denote with $\sigma$ the output distribution over solutions for an input $v$, where we use the optimal solution in the low probability event that the algorithm fails. (Equivalently $A$ could be a randomized algorithm and $\sigma$ its implicit distribution over solutions).

Let $\sigma^1, \ldots, \sigma^T$, be the sequence of distributions output by the private algorithm when run on a deterministic sequence of valuation profiles $v^1, \ldots, v^T$ with the modification described in the paragraph above. To simplify the discussion we will assume that only one player changes valuation at each time-step $t$. Essentially we are breaking every transition from time-step $t$ to $t + 1$ into many sequential transitions where only one player changes at every time step, and then deleting the solutions from the resulting sequence that correspond to the added steps. Thus the number of steps within this proof should be thought as being equal to $n \cdot p \cdot T$ in expectation.

By differential privacy we know that the total variation distance of two consecutive distributions without the modification of replacing failures with the optimal solution is at most $2\epsilon + \delta$. Since, by the union bound, the probability that any of the two consecutive runs of the algorithm fail is at most $2\beta$, we can show that the total variation distance of the latter modified output is at most $2\epsilon + \delta + 2\beta$, i.e. for any $t \in \{T\}$: $d_{tv}(\sigma_{t-1}^+, \sigma'_{t-1}) \leq 2\epsilon + \delta + 2\beta$ (see Lemma 5.3 for a formal proof).

We can turn the sequence of distributions $\sigma^1, \ldots, \sigma^T$ into a distribution of sequences of allocations $x^{1:T}$ by coupling the randomness used to select the solutions in different distributions $\sigma^t$. To do this, we take advantage of the coupling lemma from probability theory, Lemma 5.2. If at step $t$ no player changes values, then $\sigma_t = \sigma_{t+1}$, and we select the same outcome from the two distributions, so we get $\Pr[x_{-i}^{t} \neq x_{-i}^{t+1}] = 0$.

Now consider a step in which a player $i$ changes her private type $v_i$. We use Lemma 5.2 to couple $x_{-i}^{t+1}$ and $x_{-i}^{t}$, so that

$$\Pr[x_{-i}^{t+1} \neq x_{-i}^{t}] = d_{tv}(\sigma_{-i}^{t+1}, \sigma_{-i}^{t}) \leq 2\epsilon + \delta + 2\beta. \quad (5.2)$$

One can think of it as sampling $x_{i}^{t+1}$ conditional on $x_{-i}^{t}$ and assuming the joint distribution of $x_{i}^{t}$ and $x_{i}^{t+1}$ is as prescribed by the coupling lemma applied to $\sigma^{t}$ and $\sigma^{t+1}$. This is to address concerns that $x_{i}^{t}$ is already coupled with $x_{i}^{t-1}$ in the previous step.
Note that this couples the \( i \)th coordinate \( x_i^{t+1} \) and \( x_i^t \) in an arbitrary manner, which is fine, as we assumed that the valuation of player \( i \) changes at this step.

We have defined a probability distribution of sequences \( x^{1:T} \) for every fixed sequence of valuations \( v^{1:T} \). We extend this definition to random sequences of valuation in the natural way adding the distribution of valuations \( v^{1:T} \).

We claim that the resulting random sequences of \((\text{valuation},\text{solution})\) pairs satisfies the statement of the theorem: the \( \alpha \)-approximation follows by the guarantees of the private algorithm and by the fact that we use the optimal solution when the algorithm fails. Next we argue about the stability of the sequence. Consider a player \( i \), and the distribution of her sequence \((v_i^{1:T}, x_i^{1:T})\).

In each step \( t \) her valuation \( v_i^t \) changes with probability \( p \) contributing \( pT \) in expectation to the number of changes. In a step \( t \) when some other value \( j \neq i \) changes, we use \((5.2)\) to bound the probability that \( x_i^t \neq x_i^{t+1} \) by \( 2\epsilon + \delta + 2\beta \). Thus any change in the value of some other player \( j \) contributes \((2\epsilon + 2\beta + \delta)\) to the expectation of the number of changes for player \( i \). The expected number of such changes in other values is \((n-1)pT\) over the sequence, showing that the sequence is \( pT + (n-1)pT(2\epsilon + 2\beta + \delta) \leq pT(1 + n(2\epsilon + 2\beta + \delta)) \) stable, as claimed. \( \square \)

**Lemma 5.3.** Let \( q \) and \( q' \) be the output of an \((\epsilon,\delta)\)-joint differentially private algorithm with failure probability \( \beta \), on two valuation profiles \( v \) and \( v' \) that differ only in coordinate \( i \). Let \( \sigma \) and \( \sigma' \) be the modified output where the outcome is replaced with optimal outcome when the algorithm fails. Then:

\[
d_{tv}(\sigma, \sigma') \leq 2\epsilon + \delta + 2\beta
\]

**Proof of Lemma 5.3.** Consider two random coupled random variables \( y, y' \) that are implied by Lemma 5.2 applied to distributions \( q \) and \( q' \), such that \( y \sim q \) and \( y' \sim q' \) and \( \Pr[y \neq y'] = d_{tv}(q, q') \leq 2\epsilon + \delta \) (by \((\epsilon,\delta)\)-joint privacy). Now consider two other random variables \( x \) and \( x' \) where \( x = y \) except for the cases where \( y \) is an outcome of a failure in which case \( x \) is equal to the welfare optimal outcome and similarly for \( x' \) and \( y' \). Obviously: \( x \sim \sigma \) and \( x' \sim \sigma' \), thus \((x, x')\) is a valid coupling for distributions \( \sigma \) and \( \sigma' \). Thus if we show that \( \Pr[x \neq x'] \leq 2\epsilon + \delta + 2\beta \), then by properties of total variation distance \( d_{tv}(\sigma, \sigma') \leq \Pr[x \neq x'] \leq 2\epsilon + \delta + 2\beta \), which is the property we want to show.

Let \( \text{fail} \) be the event that either \( y \) or \( y' \) is the outcome of a failed run of the algorithm. Then by the union bound \( \Pr[\text{fail}] \leq 2\beta \). Thus we have:

\[
\Pr[x \neq x'] = \Pr[x \neq x' \mid \neg \text{fail}] \cdot \Pr[\neg \text{fail}] + \Pr[x' \neq x \mid \text{fail}] \Pr[\text{fail}]
\leq \Pr[x \neq x' \mid \neg \text{fail}] \cdot \Pr[\neg \text{fail}] + 2\beta
\leq \Pr[y \neq y' \mid \neg \text{fail}] \cdot \Pr[\neg \text{fail}] + 2\beta
\leq \Pr[y \neq y'] + 2\beta \leq d_{tv}(q, q') + 2\beta \leq 2\epsilon + \delta + 2\beta
\]

This completes the proof of the Lemma. \( \square \)
5.2. Large Congestion Games with Dynamic Population

Our first application of differential privacy is for the atomic congestion game with dynamic population, defined in Section 2. Rogers et al. (2015) gives a jointly differentially private algorithm for finding an optimal solution in congestion games, called Private gradient descent algorithm. They focus on routing games due to the paper’s focus on tolls as mediators, but their algorithm works in full generality for any atomic congestion game.

We illustrate our technique with linear latencies \( \ell_e(x) = a_e x + b_e \). We assume latency is monotone increasing, i.e., \( a_e > 0 \) for all \( e \in E \) and that \( b_e \geq 0 \). The algorithm of Rogers et al. (2015) assumes that \( \ell_e(x) \leq 1 \) for all \( e \). To achieve this we need to scale latencies by \( \frac{n \max_e(a_e + b_e)}{a_e} \). This makes the functions \( \gamma \)-Lipschitz for \( \gamma = \frac{1}{n} \). For this case, the algorithm outputs an integer solution that satisfies \( \epsilon, \delta \) joint differential privacy, and has an error probability of \( \beta \) for parameters \( \epsilon, \delta, \beta > 0 \), and for player types \( v \) with probability \( 1 - \beta \) returns a solution \( x \) with cost in expectation over the randomization of the algorithm

\[
\mathbb{E}[C(x; v)] \leq \text{Opt}(v) + \frac{m^{3/2}n^{1/2}}{\epsilon^{1/2}} \text{polylog}(\epsilon, 1/\delta, 1/\beta, n, m). \tag{5.3}
\]

We can combine this differentially private algorithm with Corollary 5.1 for a class of latency functions \( \ell(x) \) that we have good smoothness properties. The class of linear latencies \( \ell_e(x) = a_e x + b_e \) are \((5/3, 1/3)\)-smooth (Christodoulou and Koutsoupias 2005, Awerbuch et al. 2013, Roughgarden 2009). The same proof also gives:

**Lemma 5.4.** Congestion games with linear latencies \( \ell_e(x) = a_e x + b_e \) for \( a_e, b_e \geq 0 \) are \((5/3, 1/3)\)-smooth with respect to any solution \( x \).

**Theorem 5.2 (Main theorem for large congestion games.).** Consider a repeated congestion game with dynamic population \( \Gamma = (G, T, p) \), such that \( T \geq \frac{1}{p} \), the stage game \( G \) is an atomic congestion game with affine latency functions \( \ell_e(x) = a_e x + b_e \) with \( a_e > 0 \) and \( b_e \geq 0 \) for all \( e \). For any \( \eta > 0 \), if all players use adaptive learning algorithms with constant \( C_R \), then the overall expected cost is bounded by

\[
\sum_t \mathbb{E}[C(s^t; v^t)] \leq \frac{5}{2}(1 + \eta) \sum_t \text{Opt}(v^t)
\]

assuming the probability \( p \) of departures is at most \( C \cdot \eta^4 \cdot m^{-10} \cdot (\ln T)^{-1} \) for

\[
C = \left( \frac{5}{12 \cdot 141} \right)^2 \cdot (C_R)^{-2} \left( \frac{\min_e a_e}{\max_e (a_e + b_e)} \right)^4 \cdot (\log^2 (m \cdot n) \ln(n))^{-1}
\]

**Remark 5.1.** We note that the probability \( p \) depends mainly on the number of congestible elements \( m \), but depends on \( n \) only in a polylogarithmic way. For large \( n \), almost a constant fraction of the player population can turn over at each step.

In Appendix EC.3 we generalize the bound to polynomial functions, and also give additive error results for congestion games with general latency functions.
5.3. Large Markets with Dynamic Population

Next we revisit the first price auction game, but consider a much broader class of valuations: we consider large markets with valuations that satisfy the gross substitute property. Hsu et al. (2014) give a jointly differentially private algorithm to find close to optimal allocation in markets where buyers have the gross substitute property, and there are enough copies of each item. This algorithm will allow us to derive good welfare guarantees for outcomes on adaptive learning in repeated auctions with dynamic population using Corollary 5.1.

We will assume that the valuation functions satisfy the gross substitute property, i.e., increasing prices outside a subset doesn’t decrease the player’s demand in the set.

**Definition 5.1 (Gross-substitute valuation).** For a price $p$ let $p(A) = \sum_{j \in A} p_j$ denote the total price, and let $\omega(p)$ denote the player’s most desirable set of goods, that is, let $\omega(p) = \arg \max_A v(A) - p(A)$. The valuation satisfies the gross substitutes condition if for every pair of price vectors $(p, p')$ such that $\forall$ items $j$ $p_j \leq p'_j$ and for every set of goods $S \in \omega(p)$ if $S' \subseteq S$ satisfies $p'_j = p_j$ for every $j \in S'$ then there is a set $S^* \in \omega(p')$ with $S' \subseteq S^*$.

We will make the following large market assumptions:

1. The number of items $m s$ is large, in particular $m s \geq c n$ for some constant $c \leq 1$.
2. In the optimal solution each item can be assigned for at least $\rho$ marginal gain. This implies immediately that the optimal social welfare is at least $\text{Opt}^t \geq \rho m s$ at each time $t \in [T]$.
3. The players are interested in at most $d$ types of items and want only one copy of each item (meaning that their value for any bundle $A$ of items is equal to the maximum value among any subset of this bundle with cardinality at most $d$).

We will use the $\text{PAloc}$ algorithm from Hsu et al. (2014) as our benchmark for adaptive learning. The algorithm has two additional parameters $\alpha > 0$ and $\beta > 0$, it is $\epsilon$-jointly differentially private, that is $(\epsilon, 0)$-jointly differentially private and with probability $(1 - \beta)$ it computes a feasible efficient allocation. Assuming the supply $s$ is high enough, the social value of the allocation is at least $\text{OPT} - \alpha \cdot \max(m s, n)$ in expectation, where recall that $m s$ is the total supply, as we have $s$ copies of $m$ different items each. Concretely, with supply $s$ we get

$$\alpha = O\left(\frac{1}{(s \epsilon)^{1/3}} \cdot \text{polylog}(n, m, s, 1/\beta)\right) \tag{5.4}$$

In order to be able to use this algorithm as a benchmark in Corollary 5.1, we need to show that this is an approximation algorithm with small approximation factor.

**Lemma 5.5.** For every $\eta > 1$, when the players’ valuations satisfy the gross substitute assumption, the algorithm $\text{PAloc}$ with privacy parameter $\epsilon(n)$ can be used to output an allocation, w.p. $1 -$
\(\beta(n)\), that has social welfare at least \((1 - \frac{\eta}{2})\text{OPT}\) under the large market assumptions listed above, assuming in addition that
\[
\eta = O\left(\frac{1}{\rho \cdot c \cdot (s \cdot \epsilon(n))^{1/3}}\right) \cdot \text{polylog}(n, m, s, 1/\beta(n))
\]

**Theorem 5.3 (Main theorem for large markets).** Consider a repeated large market mechanism with dynamic population \(\Gamma = (M, T, p)\), such that \(T \geq \frac{1}{p}\) where the stage mechanism \(M\) is a solution-based \((\lambda, \mu)\)-smooth mechanism, the players have gross substitute valuations and the market satisfies the large assumption. If all players use an adaptive learning algorithm with constant \(C_R\), then the overall expected social welfare is at least:
\[
\sum_t \mathbb{E}(W(x^t; v^t)) \geq \lambda \max(1, \mu) \cdot (1 - \eta) \sum_t \text{OPT}^t
\]
if the probability \(p\) of a player leaving is
\[
p \leq C \cdot \frac{\eta^5 \cdot \rho^5 \cdot c^6}{m \cdot \ln(NT)}
\]
for \(C = \Theta((\text{polylog}(n, m, s)^{-1})\) where \(N\) is the number of different strategies each player is using, which is at most \(\left(\frac{m}{\delta \cdot \rho}\right)^d\), when bids on each item are multiples of \(\delta \cdot \rho\).

There are several mechanisms for this setting that are \((\lambda, \mu)\)-smooth. As we showed in Lemma 4.1, running simultaneous first price auctions for each type of good (as described in Section 2), results in a \((\frac{1}{2} - \delta, 1)\)-solution based smooth mechanism.

**A. Proofs of Main Results**

**A.1. Proofs from Section 3**

**Proof of Theorem 5.3.** Let \(s^*_i\) be the deviation \(s^*_i(v^t_i; x^t_i)\) defined by the smoothness property and \(s_{1:T}^*_i\) the sequence of these deviations. Let \(K_i\) be the number of time steps that \(s^*_i \neq s_{i+1}^*\) and \(r_i(s_{1:T}^*_i, s_{1:T}^*; v_{1:T})\) the regret that player \(i\) has compared to selecting \(s_{1:T}^*_i\) at every step, i.e.:
\[
r_i(s_{1:T}^*_i, s_{1:T}^*; v_{1:T}) = \sum_{t=1}^{T} \left( c_i(s^t; v^t) - c_i(s_{1:T}^*_i, s_{1:T}^*; v^t) \right)
\]
(A.1)

For shorthand, we denote this with \(r_i^*\) in this proof. Observe that since \(s_{1:T}^*_i\) is uniquely determined by \(v^t_i\) and \(x^t_i\), \(K_i\) is a random variable that is equal to \(k_i(v_{1:T}^i, x_{1:T}^i)\), for each instantiation of the sequences \(v_{1:T}^i\) and \(x_{1:T}^i\).

For any period \([\tau_r, \tau_{r+1})\) that the strategy \(s_{1:T}^*_i\) is fixed, adaptive learning guarantees that the player’s regret for this strategy is bounded by
\[
R_i(\tau_r, \tau_{r+1}) \leq C_R \sqrt{(\tau_{r+1} - \tau_r) \ln(NT)},
\]
(A.2)
Summing over the \( K_i \) periods in which the strategy is fixed and using the Cauchy-Schwartz inequality, we can bound the total regret of each \( i \):

\[
r_i^* \leq C_R \sqrt{(K_i + 1) \sum_{r=1}^{K_i+1} (\tau_{r+1} - \tau_r) \ln(NT)} = C_R \sqrt{(K_i + 1)T \ln(NT)},
\]

(A.3)

Thus for each instance of \( x^{1:T} \) and \( v^{1:T} \), we have:

\[
\sum_{t=1}^{T} c_i(s^t; v^t) = \sum_{t=1}^{T} c_i(s_i^{*,t}, s_{-i}^t; v^t) + r_i^* \leq \sum_{t=1}^{T} c_i(s_i^{*,t}, s_{-i}^t; v^t) + C_R \sqrt{(K_i + 1)T \ln(NT)},
\]

(A.4)

Adding over all players, and using the smoothness property, we get that

\[
\sum_{t} C(s^t; v^t) \leq \lambda \sum_{t} C(x^t; v^t) + \mu \sum_{t} C(s^t; v^t) + \sum_{i} C_R \sqrt{(K_i + 1)T \ln(NT)}.
\]

By Cauchy-Schwartz, \( \sum_{i} \sqrt{(K_i + 1)T \ln(NT)} \leq \sqrt{n \cdot T \cdot \ln(NT) \cdot \sum_{i=1}^{n}(K_i + 1)} \). Taking expectation over the allocation and valuation sequence and using the \( \alpha \)-approximate optimality and Jensen’s inequality:

\[
\sum_{t} \mathbb{E}[C(s^t; v^t)] \leq \lambda \sum_{t} \mathbb{E}[\text{OPT}(v^t)] + \mu \sum_{t} \mathbb{E}[C(s^t; v^t)] + n \cdot C_R \sqrt{T \ln(NT)} \left(1 + \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[K_i]\right).
\]

By the \( k \)-stability of the sequence, we have that \( \sum_{i=1}^{n} E[K_i] \leq k \cdot n \). By re-arranging we get the claimed bound. \( \square \)

**Proof of Theorem 3.2** The proof follows along similar lines as the proof of Theorem 3.1 and for completeness is given in Section C.1 of the supplementary material. \( \square \)

**Proof of Theorem 3.3** Let \( s_i^{*,1:T} \), \( K_i \) and \( r_i(s_i^{*,1:T}, s_{-i}^{1:T}; v^{1:T}) \) be defined exactly as in the proof of Theorem 3.2 including the shorthand of \( r_i^* \). For any period \( [\tau_r, \tau_{r+1}) \) that the strategy \( s_i^{*,t} \) is fixed, adaptive learning guarantees that the player’s regret for this strategy is bounded by

\[
R_i(\tau_r, \tau_{r+1}) \leq C_R \sqrt{(\tau_{r+1} - \tau_r) \ln(NT)},
\]

Moreover, if in period \( r \), \( x_i^t = \emptyset \), then by Assumption 1 we have that: \( R_i(\tau_r, \tau_{r+1}) \leq 0 \). Thus, if we denote with \( X_{i,r} \) the indicator of whether in period \( r \), \( x_i^t = \emptyset \), we get:

\[
R_i(\tau_r, \tau_{r+1}) \leq C_R \sqrt{X_{i,r}^2(\tau_{r+1} - \tau_r) \ln(NT)},
\]

Summing over the \( K_i + 1 \) periods in which the strategy is fixed and using the Cauchy-Schwartz inequality, we can bound the total regret of each \( i \):

\[
r_i^* = \sum_{r=1}^{K_i} C_R \sqrt{X_{i,r} \cdot X_i \cdot (\tau_{r+1} - \tau_r) \ln(NT)} \leq C_R \sqrt{\sum_{r=1}^{K_i} X_{i,r} \cdot X_i \cdot (\tau_{r+1} - \tau_r) \ln(NT)}
\]

\[
\sum_{r=1}^{K_i+1} X_{i,r} \cdot (\tau_{r+1} - \tau_r) \ln(NT)
\]
Let $Y_i^t = 1_{\{x_i^t \neq \emptyset\}}$. Then observe that:

$$
\sum_{r=1}^{K_t+1} X_{i,r}(\tau_{r+1} - \tau_r) = \sum_{t=1}^{T} Y_i^t.
$$

Replacing in the previous inequality, summing over all players and using Cauchy-Swartz:

$$
\sum_{i=1}^{n} r_i^* \leq \sum_{i=1}^{n} C_R \sqrt{\sum_{i=1}^{K_t+1} X_{i,r} \cdot \sqrt{\sum_{t=1}^{T} Y_i^t \ln(NT)}} \leq C_R \sqrt{\sum_{i=1}^{n} \sum_{r=1}^{K_t+1} X_{i,r} \cdot \sqrt{\sum_{t=1}^{T} Y_i^t \ln(NT)}}
$$

Since each $x^t$ is a feasible allocation: $\sum_{i=1}^{n} Y_i^t \leq m$. Hence, $\sum_{i=1}^{n} \sum_{t=1}^{T} Y_i^t \leq mT$. Moreover:

$$
\sum_{i=1}^{n} \sum_{r=1}^{K_t+1} X_{i,r} \leq \sum_{i=1}^{n} \sum_{r=1}^{K_t} X_{i,r} + \sum_{i=1}^{n} X_{i,K_t+1} = \sum_{i=1}^{n} \sum_{r=1}^{K_t} X_{i,r} + \sum_{i=1}^{n} Y_i^T \leq m + \sum_{i=1}^{n} \sum_{r=1}^{K_t} X_{i,r}
$$

Now observe that for each instance of $(v^{1:T}, x^{1:T})$: $\sum_{r=1}^{K_t} X_{i,r} \leq \kappa_i(v^{1:T}, x^{1:T})$, since the latter summation sums all changes in type or allocation ranging from $r = 1$ to $K_t$, such that the allocation $x_i^r$ in the period right before the $r$-th change is non-empty. This is at most the set of changes that are accounted in $\kappa_i(v^{1:T}, x^{1:T})$. It is an inequality as there could be an index $r$ at which both a type and an allocation is changing and the summation only accounts it once, while $\kappa_i(v^{1:T}, x^{1:T})$ counts it twice, or there could be changes where $x_i^t \neq x_i^{t-1}$ and $x_i^t \neq \emptyset$, which are not accounted in the above, but are accounted in $\kappa_i(v^{1:T}, x^{1:T})$. Combining all the above we get:

$$
\sum_{i=1}^{n} r_i^* \leq C_R \sqrt{m + \sum_{i=1}^{n} \kappa_i(v^{1:T}, x^{1:T}) \cdot \sqrt{mT \ln(NT)}}
$$

By the no-regret property of each player, for each instance of $x^{1:T}$ and $v^{1:T}$, we have:

$$
\sum_{t=1}^{T} u_i(s^t; v^t) \geq \sum_{t=1}^{T} u_i(s_i^{t, t}, s_{-i}^t; v^t) - r_i^*
$$

Adding over all players, and using the smoothness property and the bound on the sum of regrets, we get that

$$
\sum_{t} \sum_{i} u_i(s^t; v^t) \geq \lambda \sum_{t} W(x^t; v^t) - \mu \sum_{t} \mathcal{R}(s^t) - C_R \sqrt{m + \sum_{i=1}^{n} \kappa_i(v^{1:T}, x^{1:T}) \cdot \sqrt{mT \ln(NT)}}
$$

Taking expectation over the allocation and valuation sequence and using the $\alpha$-approximate optimality and Jensen’s inequality:

$$
\sum_{t} \sum_{i} \mathbb{E}[u_i(s^t; v^t)] \geq \frac{\lambda}{\alpha} \sum_{t} \mathbb{E}[\text{OPT}(v^t)] - \mu \sum_{t} \mathbb{E}[\mathcal{R}(s^t)] - C_R \sqrt{m + \sum_{i=1}^{n} \mathbb{E}[\kappa_i(v^{1:T}, x^{1:T})] \cdot \sqrt{mT \ln(NT)}}.
$$
By the analogue of $k$-stability of the sequence, as defined in Equation (3.1), we have that

$$\sum_{i=1}^{n} E[\kappa_i(v^{1:T}, x^{1:T})] \leq k \cdot n.$$  

By re-arranging and using the fact that $W(s^t; v^t) = \sum_i u_i(s^t; v^t) + R(s^t)$ and that $R(s^t) \leq W(s^t; v^t)$ (since utilities are non-negative), we get the claimed bound. □

A.2. Proofs from Section 4.1

Proof of Lemma 4.1

The proof is similar to the proof of Syrgkanis and Tardos (2013) that the mechanism is $(1/2, 1)$-smoothness for continuous bids. Hence, we defer this proof to Appendix EC.5 of the supplementary material. □

Proof of Lemma 4.2

The $2(1 + \epsilon)$-approximation result holds as we lose an approximation factor of 2 due to the greedy algorithm and another approximation factor of $(1 + \epsilon)$ due to the layers.

To show the stability let $\ell(v_i(j))$ be the highest $\ell$ such that $\ell(v_i(j)) \geq \rho(1 + \epsilon)^{\ell-1}$, i.e., the rounded version of $v_i(j)$ is $\rho(1 + \epsilon)^{\ell(v_i(j))-1}$, which we call the layer of this value. For example, any value in the range $[\rho, \rho(1 + \epsilon))$ is in layer 1. Let $\ell^t(j)$ denote $\ell(v_i(j))$ if item $j$ is assigned to player $i$ at time $t$, and let $\ell^t(j) = 0$ if item $j$ is not assigned at time $t$. We will use the potential function

$$\Phi(x^t) = \sum_j \ell^t(j)$$

to show stability.

We will show that changes in assignments correspond to increases in the potential function, and the potential function can only decrease due to departures.

When a player who was assigned item $j$ leaves at time $t$, this immediately decreases the potential function by $\ell^t(j) \leq \log_{(1+\epsilon)}(1/\rho)$. Next we see how to restore the layered greedy solution after a departure and after an arrival. We will claim that each change in the solution corresponds to an increase in the potential function.

To get the desired stability, we will only reassign an item $j$ from a player $i$ to a different player $i'$ if $\ell(v_i(j)) > \ell(v_i(j))$, that is, if the rounded value is higher. If this is the case, we say that $i$ is eligible to be reassigned to item $j$, and similarly, we will say that player $i$ is eligible to be moved from an item $j$ to a different item $j'$ if $\ell(v_i(j')) > \ell(v_i(j))$.

When a new player $i$ arrives, we assign the player to her highest valued item $j$ to which she is eligible to be assigned. This increases the potential function by at least one. Now the previous owner of the item $j$ has no allocation, and again we assign this player to her highest value item to which she is eligible to be reassigned, further increasing the potential function. We continue this process till a layered greedy solution is obtained.
After a player departs, the remaining solution may have an item \( j \) that is unassigned. We reassign item \( j \) to the eligible player \( i \) of highest value. This increases the potential function, but possibly leaves a different item, one that \( i \) used to have, unassigned. Again we assign this item to the eligible player of highest value for the item, further increasing the potential function. We continue this process till a layered greedy solution is obtained.

We have shown that each change in the assignment, other than player departures, increases the potential function \( \Phi \) allowing us to bound the expected number of changes. Each step \( t \), each of the up to \( m \) players with assigned item leaves with probability \( p \), so the expected decrease in the potential function over the \( T \) steps of the algorithm is at most \( pmT \log_{(1+\epsilon)}(1/\rho) \). The potential function \( \Phi \) is nonnegative, integral, and is bounded by \( m \log_{(1+\epsilon)}(1/\rho) \). This implies that the expected increase in the potential function during the algorithm is at most \( m(1 + pT) \log_{(1+\epsilon)}(1/\rho) \). Since each change in the solution also increases the potential function by at least 1, the same expression also bounds the total number of changes in the allocation and each such change affects at most two players. Thus the aggregate number of changes in allocation across players is at most \( 2m(1 + pT) \log_{(1+\epsilon)}(1/\rho) \).

Last we also need to account for the departures (or changes in type) of players that are already allocated an item. Since there are \( m \) such players in each iteration and each is replaced with probability \( p \), there are \( mpT \) such changes in expectation. Thus the total number of changes in allocation or changes in type of players that are allocated an item is at most \( m(2 + 3pT) \log_{(1+\epsilon)}(1/\rho) \). The average change for a player is an \( n \)th fraction of this, leading to the claimed bound using that \( T \geq 1/p \).

□

Proof of Theorem 4.1. Apply Theorem 3.3 where \( x^{1:T} \) is the outcome of the greedy-layered mechanism; the fact that first price auction is \((\frac{1}{2},1)\)-smooth by Lemma 4.1; and that there is a stable close to optimal solution by Lemma 4.2 to get that:

\[
\sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{1}{4(1 + \epsilon)} \sum_t \mathbb{E}[\text{OPT}(v^t)] - C_R \sqrt{T \cdot m \cdot (5 \cdot T \cdot m \cdot p \cdot \log_{(1+\epsilon)}(1/\rho) + m) \cdot \ln(NT)}
\]

Using that \( pT > 1 \), we get the first claimed bound.

To get the multiplicative bound, it suffices to upper bound the expected aggregate regret by \( \frac{\epsilon}{4(1+\epsilon)} \sum_t \mathbb{E}[\text{OPT}(v^t)] \), which is at least \( \frac{\epsilon}{4(1+\epsilon)} Tm\rho \), by the assumptions \( \epsilon \leq 1/3 \) and that each item is allocated for a value of \( \rho \). To show that this is true, what we need to prove is the following (using the inequality [122]):

\[
mT \cdot C_R \sqrt{6 \cdot p \cdot \log_{(1+\epsilon)}(1/\rho) \cdot \ln(NT)} \leq \frac{\epsilon}{4(1+\epsilon)} Tm\rho
\]

which is true if

\[
p \leq \frac{\rho^2 \epsilon^2}{6 \cdot 16(1+\epsilon)^2 (C_R)^2 \log_{(1+\epsilon)}(1/\rho) \ln(NT)}.
\]

□
A.3. Proofs from Section 4.2

Proof of Lemma 4.3. For a given value $M$, if we allocate bandwidth to players up to the point when their marginal value for bandwidth is $M$ or more, i.e., setting $x_i$ such that $u'_i(x_i) = M$ whenever $x_i > 0$ and $u'_i(0) \leq M$ when $x_i = 0$, then the allocations $\{x_i\}$ form the optimal solution for total bandwidth $\sum_i x_i$. The idea of the proof is to consider this optimal solution to a smaller bandwidth, and then round each allocation $x_i$ up to the next multiple of $\delta$. For a value $M$ let $x_i(M) = 0$ if $u'(0) < M$, and otherwise set $x_i(M) > 0$ such that $u'_i(x_i(M)) = M$. So the optimal solution is the allocation $x_i(M)$ for an $M$ such that $\sum_i x_i(M) = 1$.

Now for an allocation $x_i$ to player $i$ let $\hat{x}_i = [x_i/\delta]\delta$, the allocation rounded up to a multiple of $\delta$. Now let $\text{seg}(M) = \sum_i \hat{x}_i(M)$. Clearly, $\text{seg}(M)$ is a monotone decreasing function of $M$, and is right-continuous. Set $M$ be the minimum value such that $\text{seg}(M) \leq 1$ (Clearly $M \leq \max_i u'_i(0)$).

Now we consider the following segmented allocation: for player $i$ such that $x_i(M) < \hat{x}_i(M)$, or $x_i(M) = 0$ and $u'_i(0) < M$, we set $y_i = \hat{x}_i(M)$. For the remaining players we have that $x_i(M)$ is an integer multiple of $\delta$ and $u'_i(x_i(M)) = M$. For these players we set $y_i$ either $x_i(M)$ or $x_i(M) + \delta$ such that $\sum_i y_i = 1$. We note that such allocation always exists, as $\text{seg}(M') > 1$ for any $M' > M$, so there must be enough players with $\hat{x}_i(M') > \hat{x}_i(M)$, using $y_i = \hat{x}_i(M') = x_i(M) + \delta$ for a subset of these players can make the total exactly 1.

Now we claim that the segmented allocation $\{y_i\}$ satisfies the claim of the lemma. Let $z_i = y_i - x_i(M)$, be the additional allocation due to rounding, and let $z = \sum_i z_i$ denote the total rounding used.

First note that the value of the optimum allocation is at most $\sum_i u_i(x_i(M)) + zM$. This is true, as at the allocation $\{x_i(M)\}$ there is $z$ amount of space left to be allocated, and all players have marginal utility at most $M$ for additional space.

To bound a player’s utility for its allocation $y_i$, we use the fact that the second derivative of the utility is at least $-\alpha$, so we get that

$$ u_i(y_i) - u_i(x_i(M)) = \int_{x_i(M)}^{y_i} u'(\xi)d\xi \geq \int_{x_i(M)}^{y_i} (M - \alpha \xi)d\xi = Mz_i - \frac{1}{2}\alpha z_i^2 $$

where the inequality used the fact that $u'_i(x_i(M)) = M$ for all players whose allocation was rounded, i.e., who have $y_i > x_i(M)$.

To bound the utility of the segmented solution, we add the above bound for all players, and use that $z_i \leq \delta$ for all $i$ to get

$$ \sum_i u_i(y_i) \geq \sum_i u_i(x_i(M)) + Mz - \frac{1}{2}\alpha z\delta $$
Now by the choice of allocation \( x_i(M) \) we have that \( \sum_i u_i(x_i(M)) + Mz \geq M \), and so we can bound the last term by

\[
\frac{1}{2} \alpha z \delta \leq \frac{1}{2} \alpha \delta \leq \epsilon (\sum_i u_i(x_i) + Mz)
\]

using the bound on \( \delta \) and the fact the first derivative is at least \( \rho \).

Combining these bounds, we get the claimed overall bound

\[
OPT \leq \sum_i u_i(x_i(M)) + Mz \leq \frac{1}{1 - \epsilon} \sum_i u_i(y_i)
\]

as claimed.  \( \square \)

**Proof of Lemma 4.4.** The \((1 + \epsilon)\)-approximation holds as at most this is lost due to the layers whilst the non-layered greedy algorithm would be optimal as the valuation functions are concave.

In order to prove stability we use the same potential function as in the matching markets:

\[
\Phi(x^t) = \sum_j \ell^t(j)
\]

We will again show that, unless some player who holds bandwidth departs, changes in the allocation correspond to equal increase in the potential function. Hence, we will show that decrease in the potential function happens only due to the departures of current holders of bandwidth.

When a player \( j \) who is assigned \( m_j \) segments of bandwidth leaves, all her segments become free. Hence, this causes a decrease in the potential function of \( m_j \log_{1+\epsilon}(1/\rho \delta) \). Summing over all the players who have items, the expected decrease in the potential function is equal to \( \sum_j p \cdot m_j \log_{1+\epsilon}(1/\rho \delta) = \frac{p}{\delta} \log_{1+\epsilon}(1/\rho \delta) \). This is the same as the expected decrease in the potential function in matching markets with a lower bound of \( \rho \delta \) instead of \( \rho \) and \( 1/\delta \) segments instead of \( m \) items.

When a player \( i \) arrives, she either gets assigned to some segments or not. If she does not then she does not affect the allocation at all. If she is assigned to some segments, given the tie-breaking rule, it means that her marginal value for the segment is higher compared to the player’s who does not get assigned to segments due to that. Hence, the potential function increases at least by the number of segments she gets and she causes no more changes in the allocation (for each segment she takes, she might affect the allocation of at most one player). The total increase is bounded by the decrease that is previously done in the potential function hence there is correspondence to the matching markets case.

The remainder of the proof follows exactly the same steps as the proof of Lemma 4.2, having \( \rho \delta \) instead of \( \rho \) as this is the minimum value of one segment (correspondingly item) and \( 1/\delta \) as this is the number of segments (correspondingly items).  \( \square \)
**Proof of Lemma 4.5** In [Syrgkanis and Tardos 2013](#), the proportional mechanism is proven \((2 - \sqrt{3}, 1)\)-smooth. The deviating bid used is a bid selected uniformly at random from \([0, \lambda v_i(x^*_i)]\) where \(x^*_i\) is the optimal allocation and \(\lambda\) is a carefully tuned parameter. If, instead of the optimal solution, we selected any other solution \(\hat{x}^*\), the same result would hold for solution-based smoothness. Letting \(B_i\) be \(i\)'s realized deviating bid and \(b_i\) be the bid played, this means that:

\[
\mathbb{E}_B \left[ \sum_i u_i(B_i, b_{-i}) \right] \geq (2 - \sqrt{3})W(\hat{x}^*) - \sum_j b_j
\]

Recall that we defined the mechanisms with a discrete action space. Hence, we will consider only bids that are multiples of \(\zeta\). The deviating bid we will use is the rounding of the bid of [Syrgkanis and Tardos 2013](#) (to multiples of \(\zeta\)). Hence the deviating bid now will be \(\bar{B}_i = \left\lceil \frac{B_i}{\zeta} \right\rceil \cdot \zeta\). Summing over all players that hold items in \(\hat{x}^*\)

\[
\mathbb{E}_B \left[ \sum_i u_i(\bar{B}_i, b_{-i}) \right] = \mathbb{E}_B \left[ \sum_i (v_i(\bar{B}_i, b_{-i}) - \bar{B}_i) \right] \geq \mathbb{E}_B \left[ \sum_i (v_i(B_i, b_{-i}) - B_i - \zeta) \right] \geq (2 - \sqrt{3})W(\hat{x}^*) - \sum_j b_j - \sum_i \zeta = (2 - \sqrt{3} - \epsilon)W(\hat{x}^*) - \sum_j b_j
\]

The first inequality holds from the monotonicity of the valuation function and the discretization of bids. The second holds from the smoothness condition of the non-discretized version. The last equality holds replacing \(\zeta = \epsilon \delta\) and by the fact that, for any \(\delta\)-segmented allocation \(x\), \(W(x) \leq 1/\delta\) (as the valuation function is upper bounded by 1 and the number of players that can hold segments are upper bounded by \(1/\delta\)).

**A.4. Proofs from Section 5.2**

**Proof of Theorem 5.2** To use the jointly differentially private algorithm of [Rogers et al. 2015](#) with a set of affine latency functions \(\ell_c(x_c) = a_c x_c + b_c\), we need to scale them by \(n \max_c (a_c + b_c)\) to guarantee that \(\ell_c(n) \leq 1\) as required. This makes the functions \(\gamma = 1/n\)-Lipschitz. We will use the jointly differentially private algorithm on the scaled problem, with privacy parameters \(\epsilon(n), \delta(n)\), and \(\beta(n)\) that will depend on the size of the population, and then rescale to the original costs, to get a solution with expected cost:

\[
\mathbb{E}[C(x; v)] \leq \text{OPT}(v) + 141 n^{1/2} \cdot m \cdot (\max(n \gamma, m))^{1/2} \cdot e^{-1/2} \cdot \log(4m \cdot n \cdot \max(n \gamma, m) \cdot \epsilon / \beta) \cdot \sqrt{\ln(1/\delta)}
\]

More precisely, the authors proved that they can find a fractional solution with cost at most \(\text{OPT} + R + 4R\) for \(R \leq \frac{(nm)^2 n (2n \gamma + 8m)}{\sqrt{4T}^2} + 2m \cdot 2 \log(2m T / \beta) \cdot \sqrt{2T \ln(1/\delta)}\) where \(T = \frac{n (n \gamma + 4m) \epsilon}{4 \sqrt{2}}\) and then lose an additional \(m \sqrt{2 \ln(m / \beta)}\) to get the integral solution. This can give an upper bound of \(141 \cdot n^{1/2} \cdot m \cdot (\max(n \gamma, m))^{1/2} \cdot e^{-1/2} \cdot \log(4m \cdot n \cdot \max(n \gamma, m) \cdot \epsilon / \beta) \cdot \sqrt{\ln(1/\delta)}\).
where $\gamma = 1/n$, and the polylog term is the actual expression in (5.3).

Corollary 5.1 is expecting an $\alpha$-approximation algorithm, so we need to bound the approximation factor of this algorithm. To claim that it is a $(1 + \frac{\eta}{n})$-approximation algorithm we need to guarantee that

$$\frac{141m^{3/2}\sqrt{n}}{\sqrt{\epsilon(n)}} \log(4m^2n\epsilon(n)/(\beta(n))\sqrt{\ln(1/\delta(n))}) \cdot n \max_e(a_e + b_e) \leq \frac{\eta}{2} \text{OPT}.$$ 

A simple lower bound on the optimal solution is $\text{OPT} \geq n \min_e a_e n/m = \frac{n^2}{m} \min_e a_e$, assuming all players are congesting at least one element. Using this lower bound, and rearranging terms, we can guarantee the desired approximation bound by assuming that

$$n \geq \left(\frac{141m^{3/2}}{\sqrt{\epsilon(n)}} \log(4m^2n\epsilon(n)/(\beta(n))\sqrt{\ln(1/\delta(n))}) \cdot \max_e(a_e + b_e) \cdot \frac{2m}{\eta \min_e a_e}\right)^2 \quad (A.5)$$

To use this solution as a benchmark in Corollary 5.1 we need a small enough $\epsilon(n)$ and $\delta(n)$ as each person leaving and arriving causes the benchmark solution to change for an $O(\epsilon(n) + \beta(n) + \delta(n))$ fraction of the population in expectation. We will let $\delta(n), \beta(n) = \epsilon(n)/3$ and set $\epsilon(n)$ as small as is allowed by Equation (A.5). Since $\epsilon(n)/\beta(n) = 3$ and $\delta(n) = \epsilon(n)/3$, we need:

$$\frac{\epsilon(n)}{\ln(3/\epsilon(n))} \geq \frac{1}{n} \left(141m^{3/2} \log(12m^2n) \cdot \max_e(a_e + b_e) \cdot \frac{2m}{\eta \min_e a_e}\right)^2$$

Let $f(n) = \left(141m^{3/2} \log(12m^2n) \cdot \max_e(a_e + b_e) \cdot \frac{2m}{\eta \min_e a_e}\right)^2 = O\left(m^5 \left(\frac{\log(m^2n)\max_e(a_e + b_e)}{\eta \min_e a_e}\right)^2\right)$, and observe that $f(n) = \text{poly}(m, \log(n))$. The latter inequality is satisfied if:

$$\epsilon(n) = \frac{1}{n} f(n) \ln(3n)$$

Moreover, by the latter parameters we also have that $\epsilon(n) + \beta(n) + \delta(n) \leq \frac{5}{3} \epsilon(n)$.

Now applying Corollary 5.1 to the problem scaled by $m \cdot n \max_e(a_e + b_e)$ to guarantee the assumption $\ell_e(x) \leq 1$, that the loss functions for every player are bounded by 1, and scaling back, we get that

$$\sum_i \mathbb{E}[C(s_i; v^i)] \leq \frac{5}{2} \left(1 + \frac{\eta}{2}\right) \sum_i \text{OPT}(v^i) + \frac{3}{2} C_R \cdot nT \sqrt{2p\left(1 + \frac{5}{3}n\epsilon(n)\right) \ln(NT) \max_e(a_e + b_e) \cdot n \cdot m}$$

\footnote{Consider the cost minimization problem assuming the latency function of all edges is replaced with the latency $\hat{\ell}(x) = x \cdot \min_e a_e$. The value of the original cost minimization problem is at least the value of this new one. The social cost in this new problem is simply: $\min_e a_e \cdot \sum_i x_i^2$. Since each player congests at least one edge the solution must satisfy the constraint: $\sum_i x_i \geq n$. By the convexity and symmetry of the objective function, the latter relaxed problem achieves a minimum when all $x_i$ are identical and equal to $n/m$ in which case the value is $\frac{m^2}{n} \min_e a_e$.

\footnote{If we set $\epsilon(n) = \frac{1}{n} f(n) \cdot \ln(3n)$ then: $\ln(3/\epsilon(n)) = \ln(3n) - \ln(f(n)) - \ln\ln(3n) \leq \ln(3n)$. Thus: $\epsilon(n) = \frac{1}{n} f(n) \cdot \ln(3/\epsilon(n))$.}
To get the desired bound, we need to make sure that the additive error is bounded by a small multiple of $\text{Opt}$. Concretely, we need:

$$
\frac{3}{2} C_R \cdot nT \sqrt{2p \left( 1 + \frac{5}{3} n\epsilon(n) \right) \ln(nT) \max_e (a_e + b_e)n \cdot m} \leq \frac{5}{2} \cdot \eta \sum_t \text{Opt}(v^t).
$$

Using again the $\frac{n^2}{m} \min_e a_e \leq \text{Opt}(v^t)$ lower bound for the cost in each step $t$, we will now show that we can guarantee this with the choice of $p$ suggested in the theorem. With no loss of generality we can assume that $\epsilon(n)n > 3$ (since it holds if $m \geq 2$ and $n \geq 2$), it suffices to show the following:

$$
\frac{3}{2} C_R \cdot nT \sqrt{2p \cdot 2n\epsilon(n) \ln(nT) \max_e (a_e + b_e)n \cdot m} \leq \frac{5}{2} \cdot \eta \cdot T \cdot \frac{n^2}{m} \min_e a_e.
$$

Finally, we use that the number of player strategies $N$ in a congestion game with $m$ elements is clearly bounded by $N \leq 2^m$, and hence $\ln(nT) \leq m \ln T$. Using this fact, we can rearrange the above inequality, and guarantee the required inequality if have

$$
p \leq \left( \frac{5}{C_R \cdot 12} \cdot \frac{\min_e a_e}{m^2 \cdot \max_e (a_e + b_e) \cdot \eta} \right)^2 \cdot \left( \frac{\epsilon(n) \cdot n}{m \ln T} \right)^{-1} \cdot \frac{1}{f(n) \ln(3n) m \ln T} \cdot \frac{1}{\log^2(12m^2 n) \ln(3n) m^{10} \ln T}
$$

$$
= \left( \frac{5}{C_R \cdot 12 \cdot 141} \cdot \frac{(\min_e a_e)^2}{(\max_e (a_e + b_e))^2 \cdot \eta^2} \right)^2 \cdot \left( \frac{(\min_e a_e)}{\max_e (a_e + b_e) \cdot \eta} \right)^4 \cdot \frac{1}{\log^2(12m^2 n) \ln(3n) m^{10} \ln T}
$$

The latter completes the proof of the theorem. □

A.5. Proofs from Section 5.3

Proof of Lemma 5.5 Algorithm $P\text{Alloc}$ with parameter $\alpha$, finds, w.p. $1 - \beta$, a feasible solution with social welfare at least $^9$:

$$W(x; v) \geq \text{Opt} - \alpha \cdot \max(m, s\cdot 2n)$$

w.p. $1 - \beta$, assuming holds. We will use the $\rho ms \geq \rho \cdot c \cdot \max\{m, s\cdot 2n\}$ lower bound on $\text{Opt}$, by the two first large market assumptions. Now setting $\alpha = \frac{\eta}{2}$, with $c$ from the first large market assumption, we get that:

$$W(x; v) \geq \text{Opt} - \alpha \max(m, s\cdot 2n) \geq \left( 1 - \frac{\eta}{2} \right) \text{Opt}$$

$^9$ The algorithm assumes $ms > n$ and gives an additive error bound of $\alpha \cdot ms$. If $ms < n$, we run $P\text{Alloc}$ with an extra $m'$ items such that $(m' + m)s = a$ for some $a \in \mathbb{N}$. For all the extra items every player has valuation 0 and, by the way the algorithm works, no user gets extra item in the algorithm’s allocation. Applying the algorithm we have an error bound of $\alpha(m + m')s \leq \alpha \cdot (n + s) \leq \alpha \cdot 2n$. 

as required.

For a given supply \( s \), the bound from Equation (5.4) required is exactly the one claimed in the lemma. □

**Proof of Theorem 5.3.** We apply Lemma 5.5 with a \( \epsilon(n) = \beta(n) \) that satisfy the condition, i.e., set

\[
\epsilon(n) = O \left( \frac{1}{\eta^3 \cdot c^3 \cdot \rho^3 \cdot s} \right) \text{polylog}(n, m, s),
\]

By Corollary 5.1 and Lemma 5.5 we have:

\[
\sum_{t} \mathbb{E}[W(s^t; \nu^t)] \geq \frac{\lambda}{\max\{1, \mu\}} \cdot \left( 1 - \frac{\eta}{2} \right) \sum_{t} \mathbb{E}[\text{OPT}(\nu^t)] - T n \cdot C_R \sqrt{2p(1 + 2n\epsilon(n)) \ln(NT)}
\]

In order to lower bound the second term by \( \frac{\lambda}{\max\{1, \mu\}} \cdot \frac{\eta}{2} \sum_{t} \text{OPT}^t \), we bound \( \text{OPT}^t \geq \rho ms \) as before, and then it suffices to prove the following:

\[
T n \cdot C_R \cdot \sqrt{2p(1 + 2n\epsilon(n)) \ln(NT)} \leq T \cdot \frac{\eta}{2} \rho ms
\]

Using the assumption that \( ms \geq cn \), and rearranging terms this is ensured by:

\[
C_R \sqrt{2p(1 + 2n\epsilon(n)) \ln(NT)} \leq \frac{\eta}{2} \rho c
\]

Assuming wlog that \( n \cdot \epsilon(n) \geq 1 \) and rearranging terms again, we get that this is ensured by

\[
p \leq \frac{\eta^2 \cdot \rho^2 \cdot c^2}{24(C_R)^2 \ln(NT)} \cdot (\epsilon(n) \cdot n)^{-1} = \Theta \left( \frac{\eta^5 \cdot \rho^5 \cdot c^5}{\ln(NT)} \cdot \frac{s}{n \cdot \text{polylog}(n, m, s)} \right).
\]

Using the assumption that \( ms \geq cn \), this is implied by the condition of the theorem assumed. □

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Supplementary Material

EC.1. Proof of Theorem 3.2

**Theorem 3.2.** Consider a repeated mechanism with dynamic population \( M = (M, T, p) \), such that the stage mechanism \( M \) is allocation-based \((\lambda, \mu)\)-smooth. Suppose that \( v^{1:T} \) and \( x^{1:T} \) is a \( k \)-stable sequence, such that \( x^t \) is feasible (pointwise) and \( \alpha \)-approximately optimal (in-expectation) for each \( t \), i.e. \( \alpha \cdot \mathbb{E}[W(x^t; v^t)] \geq \mathbb{E}[\text{OPT}(v^t)] \). If players use an adaptive learning algorithm with constant \( C_R \) then:

\[
\sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{\lambda}{\alpha \max \{1, \mu\}} \sum_t \mathbb{E}[\text{OPT}(v^t)] - n \cdot C_R \sqrt{T \cdot (k+1) \cdot \ln(NT)}
\]

**Proof of Theorem 3.2.** Let \( s^t_i \) be defined exactly as in the proof of Theorem 3.1 and \( r_i(s_i^{t}, s^t; v^{1:T}) \) be defined similarly as:

\[
r_i(s_i^{t}, s^t; v^{1:T}) = \sum_{t=1}^{T} (u_i(s_i^{t}, s^t; v^t) - u_i(s^t; v^t))
\]

For shorthand, we will denote this as \( r_i^* \) in this proof. Following exactly the same arguments as in the proof of Theorem 3.1 we can show that for each instance of \( v^{1:T} \) and \( x^{1:T} \):

\[
r_i^* \leq C_R \sqrt{(K_i + 1)T \ln(NT)},
\]

We sum the latter inequality over all players and take expectation over \( v^{1:T} \) and \( x^{1:T} \). Then we apply Cauchy-Schwartz and Jensen inequalities and the \( k \)-stability of the sequence, i.e., \( \sum_i \mathbb{E}[K_i] \leq k \cdot n :

\[
\mathbb{E} \left[ \sum_i r_i^* \right] \leq \mathbb{E} \left[ C_R \sum_i \sqrt{(K_i + 1)T \ln(NT)} \right] \leq \mathbb{E} \left[ C_R \sqrt{n \cdot T \cdot \ln(NT) \cdot \sum_{i=1}^{n} (K_i + 1)} \right]
\]

\[
\leq C_R \sqrt{n \cdot T \cdot \ln(NT) \cdot \sum_{i=1}^{n} (\mathbb{E}[K_i] + 1)} \leq n \cdot C_R \sqrt{T \cdot \ln(NT) \cdot (k+1)} \quad \text{(EC.EC.1.1)}
\]

By the definition of regret for each instance of \( x^{1:T} \) and \( v^{1:T} \), we have:

\[
\sum_{t=1}^{T} u_i(s^t; v^t) = \sum_{t=1}^{T} u_i(s_i^{t}, s^t; v^t) - r_i^*
\]

Summing over all players and using the smooth mechanism property, we get that

\[
\sum_i \sum_t u_i(s^t; v^t) \geq \lambda \sum_t W(x^t; v^t) - \mu \sum_i \mathcal{R}(s^t) - \sum_i r_i^*.
\]
By re-arranging and using the fact that $W(s^t; v^t) = \sum_i u_i(s^t; v^t) + R(s^t)$:

$$\sum_i W(s^t; v^t) + (\mu - 1) \sum_i R(s^t) \geq \lambda \sum_i W(x^t; v^t) - \sum_i r^*_i.$$  

Taking expectation over the allocation and valuation sequence and using the $\alpha$-approximate optimality and Inequality (EC.EC.1.1):

$$\sum_t \mathbb{E}[W(s^t; v^t)] + (\mu - 1) \sum_t \mathbb{E}[R(s^t)] \geq \lambda \sum_t \mathbb{E}[\text{OPT}(v^t)] - n \cdot C_R \sqrt{T \ln(NT)} (k + 1).$$

If $\mu \leq 1$ we get the Theorem, since revenue is non-negative. If $\mu > 1$, we will show that total revenue is approximately bounded from above by welfare. Specifically, we will show that:

$$\sum_t \mathbb{E}[R(s^t)] \leq \sum_t \mathbb{E}[W(s^t; v^t)] + n \cdot C_R \sqrt{T \ln(NT)} (k + 1).$$

The latter is equivalent to showing:

$$\sum_t \sum_i \mathbb{E}[u_i(s^t; v^t)] \geq - n \cdot C_R \sqrt{T \ln(NT)} (k + 1).$$

We use the fact that players can always play the empty strategy $\emptyset_i$ of exiting the mechanism and receiving zero utility. Thus it suffices to bound the expected average per player regret with respect to this empty fixed strategy. Define $\emptyset_i^{1:T}$ the sequence of fixed empty strategies and denote $r_i^\emptyset = r_i(\emptyset_i^{1:T}, s_i^{1:T}; v_i^{1:T})$. Then, using the no-regret definition with respect to this empty strategy for each player $i$:

$$\sum_t u_i(s^t; v^t) = -r_i^\emptyset$$

Hence, for what we want to show, it suffices:

$$\sum_i \mathbb{E}[r_i^\emptyset] \leq n \cdot C_R \sqrt{T \ln(NT)} (k + 1). \quad \text{(EC.EC.1.2)}$$

Observe that since this strategy and the type of each player $i$ are fixed in the intervals defined by the changes accounted for in $k_i(v_i^{1:T}; x_i^{1:T})$, from the exact same reasoning as what we used to bound $r^*_i$, we can also derive that for each instance of $v^{1:T}$ and $x^{1:T}$:

$$r_i^\emptyset \leq C_R \sqrt{(K_i + 1)T \ln(NT)},$$

and thereby similarly as in Inequality (EC.EC.1.1) we get the desired property given in Equation (EC.EC.1.2).

Hence, we get that:

$$\mu \sum_t \mathbb{E}[W(s^t; v^t)] \geq \sum_t \mathbb{E}[W(s^t; v^t)] + (\mu - 1) \sum_t \mathbb{E}[R(s^t)] - (\mu - 1)n \cdot C_R \sqrt{T \ln(NT)} (k + 1) \geq \frac{\lambda}{\alpha} \sum_t \mathbb{E}[\text{OPT}(v^t)] - \mu n \cdot C_R \sqrt{T \ln(NT)} (k + 1).$$

Dividing over by $\mu$ yields the Theorem. \qed
EC.2. Stable Sequences via Marginal Privacy

Here we extend the Corollary 5.1 to use a weaker form of privacy, marginal differential privacy, showing that results on marginal differential privacy would have sufficed for our main results from Section 5. This weaker form of privacy may make it easier to prove the existence of approximately optimal private solutions. We first state marginal privacy formally and then prove the extension of our results.

Definition EC.2.1 ([Kannan et al. 2014]). An algorithm \( M : C^n \rightarrow G^n \) is \((\epsilon, \delta)\)-marginally differentially private if for every \( i \), for every pair of \( i \)-neighbors \( D, D' \in C^n \), every other player \( j \neq i \), and for every subset of outputs \( S \subseteq G \) for player \( j \).

\[
Pr[M(D)_j \in S] \leq \exp(\epsilon) Pr[M(D')_j \in S] + \delta
\]

If \( \delta = 0 \), we say that \( M \) is \( \epsilon \)-marginally differentially private.

Similar to joint privacy, we will allow for our algorithms to have a failure probability \( \beta \), with which they either return a very inefficient solution or an infeasible solution.

Theorem EC.2.1. Consider a repeated cost game with dynamic population \( \Gamma = (G, T, p) \), such that the stage game \( G \) is allocation-based \((\lambda, \mu)\)-smooth and \( T \geq \frac{1}{p} \). Assume that there exists an \((\epsilon, \delta)\)-marginal differentially private algorithm \( A : V^n \rightarrow X^n \) with failure probability \( \beta \) that satisfies the conditions of Theorem 5.1. If all players use adaptive learning in the repeated game then the overall cost of the solution is at most:

\[
\sum_t E[C(s^t; v^t)] \leq \frac{\lambda \alpha}{1 - \mu} \sum_t \text{OPT}(v^t) + \frac{nT}{1 - \mu} C_R \sqrt{2p(1 + n(\epsilon + \beta + \delta)) \ln(NT)}
\]

Proof outline. The proof follows roughly the same outline as the proof of Corollary 5.1 (which used Lemma 5.1 and Theorem 3.1). The outline of the changes needed is as follows.

1. The notion of marginal privacy is not strong enough to allow the kind of global coupling offered by Theorem 5.1. Instead, we can couple the distributions \((v^1:T_i, x^1:T_i)\) separately for each player \( i \), while ensuring that each sequence has expected number of changes in either her solution or type at most \( p \cdot T(1 + n(2\epsilon + \delta)) \).

2. With no global coupling of solutions, we cannot directly use Theorem 3.1. Rather we need to prove that the stable coupling of distributions of each player’s value and outcome individually is strong enough to reach the same conclusion.

We note that, while we can prove Theorems 3.1 and 3.2 without the need for global coupling, Theorem 3.3 requiring Property 1 does need the global coupling used there. □

We state the claims used by the two steps, and offer a sketch of how to modify the proves used so far to prove the claims.
Lemma EC.2.1 (Stable sequences via marginal privacy). Suppose that there exists an algorithm $A : \mathcal{V}^n \rightarrow \mathcal{X}^n$ that is $(\epsilon, \delta)$-marginal differentially private algorithm, takes as input a valuation profile $v$ and outputs a distribution such that a sample from this distribution is feasible with probability $1 - \beta$, and is an $\alpha$-approximately efficient in expectation (for $0 \leq \epsilon < 1/2$, $\alpha > 1$ and $\delta, \beta > 0$).

Consider the sequence of valuations $v^1:T$ produced by the adversary in a repeated cost-minimization game with dynamic population $\Gamma = (G, p, T)$, and let $\sigma^1:T$ be the sequence of the resulting outcome distributions produced by algorithm $A$. Then there exists a randomized sequence of solutions $x^1:T_i$ for each player $i$, such that for each $1 \leq t \leq T$, conditional on $v^t_i$ for each $i$ the distribution of $(v^t_i, x^t_i)$ is the $i$th marginal distribution of an $\alpha$-approximation to $\text{Opt}(v^t)$, and the distribution of the sequences $(v^1:T_i, x^1:T_i)$ has expected number of changes in $i$’s solution or type at most $p \cdot T \left(1 + n(2\epsilon + 2\beta + \delta)\right)$ for each player $i$.

Proof of Lemma EC.2.1. This is an application of the coupling Lemma 5.2 for each distribution $\sigma_i$, where we use the optimal solution in the low probability event that the marginally differentially private algorithm fails. Using the notation from the proof of Theorem 5.1, marginal privacy bounds the effect of a change in valuation of player $j \neq i$ on the distribution $\sigma_i$. Note that there is no requirement that coupling is coordinated between the different coordinates, so the resulting distribution of sequences $(v^1:T_i, x^1:T_i)$ cannot be viewed as a distribution of global sequences $(v^1:T, x^1:T)$. □

Next we prove the analog of Theorem 3.1, which will finish our proof of Theorem EC.2.1.

Theorem EC.2.2 (Improved main theorem for cost-minimization games). Consider a repeated cost game with dynamic population $\Gamma = (G, p, T)$, such that the stage game $G$ is allocation-based $(\lambda, \mu)$-smooth. Suppose $D^1:T$ is a sequence of solution distributions, such that the solution in $D^t$ has cost at most $\alpha$ times the minimum possible cost $\text{Opt}(v^t)$ in expectation, and suppose the marginal distributions $D^1:T_i$ can be though of as a randomized sequence of solutions $x^1:T_i$ for each player $i$, such that the distribution of the sequences $(v^1:T_i, x^1:T_i)$ has expected number of changes in $i$’s solution or type at most $k$. If players use adaptive learning algorithms with constant $C_R$ then:

$$\sum_i \mathbb{E}[C(s^t_i; v^t)] \leq \frac{\lambda \alpha}{1 - \mu} \sum_i \mathbb{E}[\text{Opt}(v^t)] + \frac{n}{1 - \mu} C_R \cdot \sqrt{T \cdot (k + 1) \cdot \ln(NT)}$$

Proof of Theorem EC.2.2. We follow the outline of the proof of Theorem 3.1 till equation (A.4). Then take expectation of the resulting inequality to get

$$\sum_{t=1}^{T} \mathbb{E}(c_i(s^t_i; v^t)) \leq \sum_{t=1}^{T} \mathbb{E}(c_i(s^t_i, s^t_{-i}; v^t)) + C_R \cdot \sqrt{(k + 1)T \ln(NT)}.$$
Adding over all players, and using the smoothness property, we get that

\[ \sum_t \mathbb{E}(C(s^t; v^t)) \leq \lambda \sum_t \mathbb{E}(C(x^t; v^t)) + \mu \sum_t \mathbb{E}(C(s^t; v^t)) + n \cdot C_R \cdot \sqrt{(k+1)T \ln(NT)}, \]

which finishes the proof. \( \square \)

We can prove the analogous theorems for mechanisms as well.

**Theorem EC.2.3 (Improved main theorem for mechanisms).** Consider a repeated mechanism with dynamic population \( M = (M, T, p) \), such that the stage mechanism \( M \) is allocation-based \((\lambda, \mu)\)-smooth. Suppose \( \sigma^{1:T} \) is a sequence of solution distributions, such that the solution in \( \sigma^t \) has social welfare at least an \( \alpha \) fraction of the maximum possible value \( \text{Opt}(v^t) \) in expectation, and suppose the marginal distributions \( \sigma^{1:T}_i \) can be though of as a randomized sequence of solutions \( x^{1:T}_i \) for each player \( i \), such that the distribution of the sequences \( (v^{1:T}_i, x^{1:T}_i) \) has expected number of changes in \( i \)'s solution or type at most \( k \) for each player \( i \). If players use adaptive learning algorithms with constant \( C_R \) then:

\[ \sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{\lambda}{\alpha \max\{1, \mu\}} \sum_t \mathbb{E}[\text{Opt}(v^t)] - n \cdot C_R \cdot \sqrt{T \cdot (k+1) \cdot \ln(NT)}. \]

**Theorem EC.2.4.** Consider a repeated mechanism with dynamic population \( \Gamma = (M, T, p) \), such that the stage mechanism \( M \) is allocation-based \((\lambda, \mu)\)-smooth and \( T \geq 1/p \). Assume that there exists an \((\epsilon, \delta)\)-marginal differentially private algorithm \( A: V^n \rightarrow X^n \) with error parameter \( \beta \) that satisfies the conditions of Theorem 5.1. If all players use adaptive learning with constant \( C_R \) in the repeated mechanism then the overall welfare of the solution is at least

\[ \sum_t \mathbb{E}[W(s^t; v^t)] \geq \frac{\lambda}{\alpha \max\{1, \mu\}} \sum_t \mathbb{E}[\text{Opt}(v^t)] - n \cdot C_R \cdot \sqrt{2p(1+n(\epsilon+\beta+\delta)) \ln(NT)} \]

**EC.3. Large Congestion Games with General Latencies**

Considering congestion games more generally, Rogers et al. (2015) assume that the latency functions \( \ell_e(x) \) satisfy the following conditions:

1. The functions \( \ell_e(x) \) are non-decreasing, convex and twice differentiable.
2. Latency on each edge is bounded by 1, that is, \( \ell_e(n) \leq 1 \).
3. the functions are \( \gamma \)-Lipschitz, that is \( |\ell_e(x) - \ell_e(x')| \leq \gamma |x - x'| \) for some parameter \( 0 < \gamma < 1 \).

Under these assumptions, the algorithm outputs an integer solution that satisfies \((\epsilon, \delta)\) joint differential privacy, and has an error probability of \( \beta \) for parameters \( \epsilon, \delta, \beta > 0 \), and for player types \( v \) with probability \( 1 - \beta \) returns a solution \( x \) with close to minimum cost:

\[ C(x; v) \leq \text{Opt}(v) + 20 \frac{m^{3/2}n\sqrt{\gamma}}{\sqrt{\epsilon}} \log \left( 2m^2n^2\gamma\epsilon(n)/\beta(n) \right) \sqrt{\ln \left( 1/\delta(n) \right)} \]
Polynomial Latencies. Using this algorithm, we can extend the result for Linear Congestion games in Section 5.2 to polynomial latency functions. Consider congestion games with latency functions are polynomial of the form

$$\ell_e(x) = \sum_{j=0}^{d} a_{e,j} x^j$$

with $a_{e,d} > 0$ and $a_{e,j} \geq 0$ for all $j$. More formally:

**Theorem EC.3.1.** Consider a repeated congestion game with dynamic population $\Gamma = (G,T,p)$, such that $T \geq \frac{\eta}{p}$, the stage game $G$ is an atomic $(\lambda, \mu)$ allocation based smooth congestion game with polynomial latency functions $\ell_e(x) = \sum_{j=0}^{d} a_{e,j} x^j$ with $a_{e,d} > 0$ and $a_{e,j} \geq 0$ for all $e$ and $j \neq d$.

For any $\eta > 0$, if all players use adaptive learning algorithms with constant $C_R$ then the overall expected cost is bounded by

$$\sum_t \mathbb{E}[C(s^t; v^t)] \leq \frac{1}{1-\eta} (1+\eta) \sum_t \text{OPT}(v^t)$$

assuming the probability $p$ of departures is at most: $C \cdot \eta^4 \cdot (d \cdot m^{6d+6})^{-1} \cdot (\ln(T))^{-1}$ for some

$$C = \Theta \left( \left( \frac{\min_e a_e}{\max_e \sum_j a_{e,j}} \right)^4 \cdot (C_R)^{-2} \cdot \left( \log^2 (6m^2 n) \log(3n d) \right)^{-1} \right)$$

**Proof of Theorem EC.3.1.** The proof follows the same steps with the proof of Theorem 5.2.

Here we will illustrate just the places where the analysis differs. Similarly to there, let $\epsilon(n), \delta(n)$ and $\beta(n)$ be the privacy parameters of the algorithm.

In order to make the latency function on each edge bounded by 1 as required by the algorithm, we need to scale the latency of each edge by an upper bound on it. As upper bound, we will use $n^d (\max_e \sum_{j=0}^{d} a_{e,j})$. Recall that for affine latencies, this upper bound was $n \max_e (a_e + b_e)$ so here we are using its natural extension to polynomials of degree $d$.

This scaling down also makes the latencies $d/n$-Lipschitz, as required by the algorithm:

$$\frac{\ell_e(n) - \ell_e(n-1)}{n} \leq \frac{(n^d - (n-1)^d)}{n^d (\max_e \sum_{j=0}^{d} a_{e,j})} \leq \frac{d \cdot n^{d-1}}{n^d} = \frac{d}{n}$$

Similarly with the affine case, to claim that this is a $(1 + \frac{\eta}{2})$-approximation algorithm, we need to guarantee that

$$\frac{141 m^3/2 \sqrt{nd \ell(n)}}{\ell(n)} \log \left( 4 m^2 n \epsilon(n) / \beta(n) \right) \sqrt{\ln \left( 1 / \delta(n) \right)} \cdot n^d (\max_e \sum_{j=0}^{d} a_{e,j}) \leq \frac{\eta}{2} \text{OPT}$$

The lower bound we will use for the optimum is: $\text{OPT} \geq n \min_e a_{e,d} (\frac{n}{m})^d = n \frac{d+1}{m^2} \min_e a_{e,d}$ again assuming that each player congests at least one elements, and using the fact that all latency functions are degree $d$. Hence, the desired approximation bound is guaranteed for:

$$n \geq \left( \frac{141 m^3/2 \sqrt{d}}{\sqrt{\ell(n)}} \log \left( 4 m^2 n \epsilon(n) / \beta(n) \right) \sqrt{\ln \left( 1 / \delta(n) \right)} \cdot \frac{2m^d}{\eta \min_e a_{e,d}} \right)^2$$
The rest of the proof goes as the proof of Theorem 5.2 replacing $\epsilon(n)$ and the upper and lower bounds accordingly. □

**General Congestion Games.** We can use algorithm in the proof of Corollary 5.1 for general congestion games satisfying the conditions of (Rogers et al. 2015), and we get the following Theorem.

**Theorem EC.3.2.** Consider a repeated congestion game with dynamic population $\Gamma = (G, T, p)$, such that the stage game $G$ is allocation based $(\lambda, \mu)$-smooth and $T \geq \frac{1}{p}$. Assume the game satisfies the conditions above. For any parameters $\epsilon, \delta, \beta > 0$, if all players use adaptive learning algorithms with constant $C_R$ in the repeated game then the overall cost of the solution is at most:

$$
\sum_t E[C(s^t; v^t)] \leq \frac{\lambda}{1 - \mu} \sum_t \text{OPT}(v^t) + \frac{nmT}{1 - \mu} \tilde{O}\left(\sqrt{p(1 + n(\epsilon + \beta + \delta))} + \lambda m^{1/2} \gamma^{1/2} \epsilon^{-1/2}\right)
$$

where the $\tilde{O}$ is a polylog term in $N, T, \epsilon, 1/\delta, 1/\beta, n, m$.

**Proof of Theorem EC.3.2.** A small technical difficulty in using the proof of Corollary 5.1 in a black box form is that Corollary 5.1, as well as the main Theorem 3.1 used to prove it, are stated with multiplicative error bounds. However, using the additive error in the proof of Theorem 3.1, we get the following, where $v^t$ is the type vector of players, $s^t$ is the strategy vector played at time $t$, and $x^t$ is the allocation that the differentially private algorithm generates. The assumption for congestion games was that each individual latency is bounded by 1. Dividing each latency function by $m$, the number of edges to make the total latency bounded by 1, or equivalently scaling down the error bounds from Corollary 5.1 by a factor of $m$, we get

$$
\sum_t E[C(s^t; v^t)] \leq \frac{\lambda}{1 - \mu} \sum_t \text{OPT}(v^t) + \frac{nmT}{1 - \mu} \tilde{O}\left(\sqrt{p(1 + n(\epsilon + \beta + \delta))} + \lambda m^{1/2} \gamma^{1/2} \epsilon^{-1/2}\right)
$$

Adding the bound for the quality of the solution $x^t$, and rearranging terms we get the claimed bound. □

**EC.4. Removing the dependence on $T$**

In our results presented so far, we have a logarithmic dependence on the total time $T$ the game is played. Here we show that with a more careful analysis this dependence is not needed.

**Theorem EC.4.1.** Under the assumptions of Theorem 3.1 the bound can be replaced by:

$$
\sum_t E[C(s^t; v^t)] \leq \frac{\lambda \alpha}{1 - \mu} \sum_t E[\text{OPT}(v^t)] + \frac{1}{1 - \mu} \cdot n \cdot C_R \sqrt{T(k + 1) \ln \left(\frac{2N}{p} \ln (n/\kappa)\right) + \frac{1}{1 - \mu} \cdot \kappa T}
$$

for all $\kappa \in (0, n/e)$
Proof of Theorem EC.4.1: In the proof of Theorem 3.1, the dependence on the total time \( T \), shows up in equation (A.2) bounding the regret of a player over time. The bound on regret is derived from Theorem 2.1 of [Luo and Schapire 2015] where regret over an interval of time \( [\tau_1, \tau_2) \) is bounded with \( \tau_2 \) inside the logarithm. In equation (A.2) we used the upper bound \( \tau_2 \leq T \) for all the regret terms.

If all players in our game live at most \( T_{\text{max}} \) steps, we can bound the total regret of the players in one position \( i \) (using the shorthand \( r^*_i \) from the proof of Theorem 3.1) as:

\[
r^*_i \leq C_R \sqrt{(K_i + 1) \sum_{r=1}^{K_i+1} (\tau_{r+1} - \tau_r) \ln(NT_{\text{max}})} = C_R \sqrt{(K_i + 1)T \ln(NT_{\text{max}})}
\]

With a high enough \( T_{\text{max}} \), only a very small fraction of the players will live more than \( T_{\text{max}} \) steps. To bound the overall regret without any assumption on how long players can live, we can bound the regret of such long living players by 1 in each step.

Let \( L^t_i \) denote the random event that at time \( t \) player \( i \) has been alive for more than \( T_{\text{max}} \) steps for a value of \( T_{\text{max}} \) that we will set later. Let also \( L_{i,t} \) correspond to the indicator random variable of the event \( L^t_i \). Following the proof of Theorem 3.1, and bounding regret by 1 for each player \( i \) at any step \( t \) that \( L_{i,t} \) occurs, we get the following bound.

\[
\sum_t \mathbb{E}[C(s^t; v^t)] \leq \frac{\lambda \alpha}{1 - \mu} \sum_t \mathbb{E}[\text{OPT}(v^t)] + \frac{n}{1 - \mu} C_R \sqrt{T \cdot (k+1) \ln(NT_{\text{max}})} + \frac{1}{1 - \mu} \mathbb{E} \left[ \sum_{i,t} L_{i,t} \right].
\]

To prove the theorem, we set \( T_{\text{max}} = \frac{2\ln(n/\kappa)}{p} \), and we will show that this suffices to get \( \mathbb{E} \left[ \sum_{i,t} L_{i,t} \right] \leq \kappa T \), which finishes the proof.

To bound the expected value of the sum \( \mathbb{E} \left[ \sum_i L_{i,t} \right] \) for a given player \( i \), divide the sequence of \( T \) time steps into intervals \( \mathcal{I}_j \) of length \( T_{\text{max}}/2 \). For any interval \( \mathcal{I}_j \), let \( B_{i,j} \) denote the event that player \( i \) doesn’t change value throughout this interval, and note that the probability of this event is bounded by \( \Pr[B_{i,j}] = (1 - p)^{T_{\text{max}}/2} \). Now note that, if \( L_{i,t} = 1 \), i.e. player \( i \) has lived more than \( T_{\text{max}} \) steps at some time \( t \in \mathcal{I}_j \), there exists a sequence of at most one contiguous intervals ending at \( \mathcal{I}_{j-1} \) such that player \( i \) has not changed value. We will say that player \( i \) at time \( t \) is associated with the first interval in this sequence. Note that, with this process, every player \( i \) at some time step \( t \) with \( L_{i,t} = 1 \) is associated to at most one interval \( \mathcal{I}_j \) where a bad event occurs. Hence, \( \mathbb{E} \left[ \sum_{i,t} L_{i,t} \right] \) is at most the expected number of steps \( t \) when player \( i \) is associated with an interval where a bad event occurred.

To get the claimed bound, we note the following facts:

- there are \( n \) players (indices \( i \)) we need to consider,
• for each index $i$ we consider $2T/T_{\text{max}}$ intervals,
• the probability that this interval is associated with one particular long living player $i$ is bounded by $(1 - p)^{T_{\text{max}}/2}$,
• For every player index $i$, a bad event in an interval may incur an expected increase in $E_{i,t} L_{i,t}$ of at most the expected lifespan of the user after the interval, i.e. $(1 - p) + (1 - p)^2 + \cdots \leq 1/p$ (as every player $i$ has a probability $p$ at each step to turn over).

Combining these, we get the bound

$$E \left[ \sum_{i,t} L_{i,t}^j \right] \leq n \cdot \frac{2T}{T_{\text{max}}} \cdot (1 - p)^{T_{\text{max}}/2} \cdot \frac{1}{p}$$

Substituting $T_{\text{max}}$ and using that $(1 - p)^{1/p} \leq 1/e$ we get the following bound:

$$E \left[ \sum_{i,t} L_{i,t} \right] \leq n \cdot \frac{2T p}{2 \ln(n/\kappa)} \cdot e^{-\ln(n/\kappa)} \cdot \frac{1}{p} = n \cdot \frac{T}{\ln(n/\kappa)} \cdot \frac{\kappa}{n} \leq \kappa T$$

where the last inequality follows from the assumption that $\kappa \leq n/e$ and hence $\ln(n/\kappa) \geq 1$.

**Corollary EC.4.1.** In Theorem 5.2, it suffices to bound the probability of departures by

$$O \left( \left( \frac{\min e_{ae}}{\max e_{ae} (a_e + b_e)} \cdot \eta \right)^4 \cdot m^{-10} \left( \text{polylog} \left( n, m, \eta, \frac{\min e_{ae}}{\max e_{ae} (a_e + b_e)} \right) \right)^{-1} \right).$$

**Proof of Corollary EC.4.1.** From Theorem EC.4.1 by setting $\kappa = \frac{\eta}{2} \frac{\ln(n/\kappa)}{m \cdot \min e_{ae} (a_e + b_e)}$, together with the conditions of Theorem 5.2, we get that the approximation guarantee in the Theorem holds if the probability of departure $p$ is at most:

$$O \left( \left( \frac{\min e_{ae}}{\max e_{ae} (a_e + b_e)} \cdot \eta \right)^4 \cdot (m^{10} \log^2 (m \cdot n) \ln(n))^{-1} \cdot \frac{1}{\ln(2 \ln(n/\kappa))} \right)$$

which essentially is derived by replacing $T$ with $\frac{2\ln(n/\kappa)}{p}$ in the bound stated in Theorem 5.2.

To observe that, note that if we do the analysis in the proof of Theorem 5.2 but using $\eta/2$ wherever we used $\eta$ and replace $\kappa$ as described, the first two terms of the RHS of Theorem EC.4.1 after rescaling back with $m \cdot n \cdot \max e_{ae} (a_e + b_e)$ can be upper bounded by

$$\frac{\lambda}{1 - \mu} \left( 1 + \frac{\eta}{2} \right) E \left[ \sum_{t} \text{OPT}(v^t) \right]$$

(given that $\lambda/(1 - \mu) \geq 1$). Moreover, the last term in the RHS of Theorem EC.4.1 is also bounded by $\frac{\eta}{2} E \left[ \sum_{t} \text{OPT}(v^t) \right]$, after rescaling, by our choice of $\kappa$ and by the lower bound on the optimum of $\frac{n^2}{m} \min e_{ae}$. 
Thus the requirement on the probability $p$ is of the form:

$$p \leq \frac{A}{\ln(B/p)} \quad \text{(EC.EC.4.1)}$$

for

$$A = O\left( \left( \frac{\min_e a_e}{\max_e (a_e + b_e)} \right)^4 \cdot \left( m^{10} \log^2 (m \cdot n) \ln(n) \right)^{-1} \right)$$

and

$$B = 2 \ln(n/\kappa) > 1.$$

We argue that $p \leq A/\ln(2B/A)$ implies Inequality EC.EC.4.1 and hence is a sufficient upper bound on the probability $p$. Observe that the function $g(p) = p \log(B/p)$ is monotone increasing in the region $p \in [0, B/e]$. Wlog in this analysis assume that $p < 1/e$, hence the latter monotonicity holds in this range, since $B > 1$. Moreover, we might as well assume that $A < 1/e$, since we can always assume that $A < 1/e$. Thus if $p \leq A/\ln(2B/A)$, then:

$$p \log(B/p) = g(p) \leq g\left( \frac{A}{2 \log(2B/A)} \right) = \frac{A}{2 \log(2B/A)} \log\left( \frac{2B \log(2B/A)}{A} \right)$$

$$= \frac{A}{2 \log(2B/A)} \log\left( \frac{2B}{A} \right)$$

$$\leq \frac{A}{2 \log(2B/A)} 2 \log\left( \frac{2B}{A} \right) = A$$

Which is exactly inequality EC.EC.4.1.

Thus we conclude that $p \leq A/\ln(2B/A)$ suffices to get the efficiency guarantee we want. Replacing $A$ and $B$ in the latter gives an upper bound of the asymptotic form stated in the corollary and which concludes the proof. □

**Theorem EC.4.2.** Under the assumptions of Theorem 3.2, the bound can be replaced by:

$$\sum_t E[W(s^t; v^t)] \geq \frac{\lambda}{a_{\max} \{1, \mu \}} \sum_t E[\text{OPT}(v^t)] - n \cdot C_R \sqrt{T(k+1) \ln \left( \frac{2N}{p} \ln (n/\kappa) \right)} - \kappa T$$

for all $\kappa \in (0, n/e)$, where the term under the square root improves to an $T \cdot m(k \cdot n + m) \ln \left( \frac{2N}{p} \ln (m/\kappa) \right)$ under Property 1

**Proof of Theorem EC.4.2** The proof of the first part of the theorem has the same steps as the proof of Theorem EC.4.1 hence we omit it. For the second part, the proof is also the same albeit invoking the proof of Theorem 3.3 to replace $n$ with $m$. The main difference, for the latter result is that, under Property 1, it suffices to set $T_{\max} = 2 \ln(m/\kappa)$, as in the second term we add at most $mT_{\max}/2$ summands. Hence, we can totally remove the dependence on $n$. □
Corollary EC.4.2. Theorem 5.3 continues to hold with an extra $\eta$ multiplicative loss in the welfare, even under the weaker requirement that the probability of departure is at most:

$$O \left( \frac{\eta^6 \rho^6}{m \cdot \text{polylog}(n, m, s, n, \rho, c, N)} \right),$$

i.e. there is no dependence on $T$ at all in the upper bound.

Proof of Corollary EC.4.2. Similarly to the Proof of Corollary EC.4.1, we set $A = \frac{\eta^6 \rho^6}{m \cdot \text{polylog}(n, m, s)}$ and $B = N \ln(1/\eta \cdot \rho \cdot c)$. The claim then follows from the previous theorem by setting $\kappa = \eta \cdot \rho \cdot c \cdot n$. □

Corollary EC.4.3. Theorem 4.1 continues to hold with an extra $\epsilon$ multiplicative loss in the welfare, even under the weaker requirement that the probability of departure is at most:

$$O \left( \frac{\rho^2 \epsilon^2}{\log(1+\epsilon)(1/\rho) \cdot \text{polylog}(N, \rho, \epsilon)} \right),$$

i.e. there is no dependence on $T$ at all in the upper bound.

Proof of Corollary EC.4.3. Again similarly to the Proof of Corollary EC.4.1, we set $A = \frac{\rho^2 \epsilon^2}{96 \cdot (1+\epsilon)^2 \log(1+\epsilon)(1/\rho)}$ and $B = N \ln\left(1/(\epsilon \rho)\right)$. The claim then follows from the previous theorem by setting $\kappa = \epsilon \cdot m \rho$. □

Corollary EC.4.4. Theorem 4.2 continues to hold with an extra $\epsilon$ multiplicative loss in the welfare, even under the weaker requirement that the probability of departure is at most:

$$\frac{\rho^4 \epsilon^4}{96a^2(1-\epsilon)^2 \log(1+\epsilon)(\alpha(1-\epsilon)/\rho \epsilon) \ln(NT)},$$

i.e. there is no dependence on $T$ at all in the upper bound.

Proof of Corollary EC.4.4. Similarly to the proof of Corollary EC.4.3, we set $A = \frac{\rho^2 \delta^2 \epsilon^2}{96 \cdot (1-\epsilon)}$ and $B = N \ln\left(1/(\epsilon \rho \delta)\right)$ where $\delta = \frac{\rho}{\alpha(1-\epsilon)}$. The claim then follows from the previous theorem by setting $\kappa = \epsilon \rho$. □

EC.5. Smoothness of First Price Auction with Discrete Bid Spaces

Lemma 4.1 The simultaneous first price mechanism where players are restricted to bid on at most $d$ items and on each item submit a bid that is a multiple of $\delta \cdot \rho$, is a solution based $(\frac{1}{2} - \delta, 1)$-smooth mechanism, when players have submodular valuations, such that all marginals are either 0 or at least $\rho$ and such that each player wants at most $d$ items, i.e. $v_i(S) = \max_{T \subseteq S} v(T)$.

Proof of Lemma 4.1 Consider a valuation profile $v = (v_1, \ldots, v_n)$ for the $n$ players and a bid profile $b = (b_1, \ldots, b_n)$. Each valuation $v_i$ is submodular and thereby also falls into the class of XOS valuations [Lehmann et al. 2001], i.e. it can be expressed as a maximum over additive valuations. More formally, for some index set $L_i$:

$$v_i(S) = \max_{\ell \in L_i} \sum_{j \in S} a_{ij}^\ell$$

Moreover, by the assumption that marginals are either 0 or at least $\rho$, it can be easily shown that $a_{ij}^\ell$ is either 0 or at least $\rho$. Moreover, when the player has value for at most $d$ types of items, it can also be shown that for any $\ell \in L_i$ at most $d$ of the $(a_{ij}^\ell)_{j \in [m]}$ will be non-zero.
Consider a feasible allocation \( x = (x_1, \ldots, x_n) \) of the items to the bidders, where \( x_i \) is the set of types of items allocated to player \( i \) (the latter is feasible if each item is never allocated more than its supply). Consider the following deviation \( b^*_i(v_i, x_i) \) that is related to the valuation \( v_i \) of player \( i \) and to allocation \( x_i \): Let \( \ell^*_i(x_i) = \arg \max_{\ell \in \mathcal{L}_i} \sum_{j \in x_i} a_{ij}^\ell \). Then on each item \( j \in x_i \) with \( a_{ij}^\ell \neq 0 \), submit \( \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \). On each \( j \not\in x_i \), submit a zero bid. This will submit at most \( d \) non-zero bids.

Now we argue that this deviations imply the solution based smooth property. Let \( p_j(b) \) be the lowest winning bid on item \( j \), under bid profile \( b \). Observe that for each \( j \), if \( p_j(b) < \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \), the player wins item \( j \) and pays \( \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \). Thus we get:

\[
\begin{align*}
\mathcal{U}_i(b^*_i(v_i, x_i), b_{-i}; v_i) &\geq \sum_{j \in x_i} \left( a_{ij}^\ell - \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \right) \cdot 1 \left\{ p_j(b) < \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \right\} \\
&\geq \sum_{j \in x_i} \left( a_{ij}^\ell - \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \right) \cdot 1 \left\{ p_j(b) < \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \right\} \\
&\geq \sum_{j \in x_i} \left( a_{ij}^\ell - \left\lfloor \frac{a_{ij}^\ell}{\frac{d}{\delta \cdot \rho}} \right\rfloor \right) - p_j(b) \\
&\geq \sum_{j \in x_i} \left( a_{ij}^\ell - \frac{d}{\delta \cdot \rho} - p_j(b) \right) \\
&\geq \left( \frac{1}{2} - \delta \right) \sum_{j \in x_i} a_{ij}^\ell - \sum_{j \in x_i} p_j(b) \\
&= \left( \frac{1}{2} - \delta \right) v_i(x_i) - \sum_{j \in x_i} p_j(b)
\end{align*}
\]

Summing over all players and observing that \( \mathcal{R}(b) \geq \sum_{j \in x_i} p_j(b) \), we get the theorem. \( \square \)

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10 We denote with \( \left\lfloor x \right\rfloor_{\delta \cdot \rho} \) the closest multiple of \( \delta \cdot \rho \) that is less than or equal to \( x \).