GEODESIC BI-ANGLES AND FOURIER COEFFICIENTS OF
RESTRICTIONS OF EIGENFUNCTIONS

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Abstract. This article concerns joint asymptotics of Fourier coefficients of restrictions of Laplace eigenfunctions \( \varphi_j \) of a compact Riemannian manifold to a submanifold \( H \subset M \). We fix a number \( c \in (0,1) \) and study the asymptotics of the thin sums,

\[
N_{c,H}^c(\lambda) := \sum_{j, \lambda_j \leq \lambda, k : |\mu_k - c\lambda_j| < \epsilon} \left| \int_H \varphi_j \overline{\psi_k} dV_H \right|^2
\]

where \( \{\lambda_j\} \) are the eigenvalues of \( \sqrt{-\Delta}_M \), and \( \{(\mu_k, \psi_k)\} \) are the eigenvalues, resp. eigenfunctions, of \( \sqrt{-\Delta}_H \). The inner sums represent the ‘jumps’ of \( N_{c,H}^c(\lambda) \) and reflect the geometry of geodesic \( c \)-bi-angles with one leg on \( H \) and a second leg on \( M \) with the same endpoints and compatible initial tangent vectors \( \xi \in S^c_H M, \pi_H \xi \in B^* H \), where \( \pi_H \xi \) is the orthogonal projection of \( \xi \) to \( H \). A \( c \)-bi-angle occurs when \( |\pi_H \xi| / |\xi| = c \). Smoothed sums in \( \mu_k \) are also studied, and give sharp estimates on the jumps. The jumps themselves may jump as \( \epsilon \) varies, at certain values of \( \epsilon \) related to periodicities in the \( c \)-bi-angle geometry. Subspheres of spheres and certain subtori of tori illustrate these jumps. The results refine those of the previous article [WXZ20] where the inner sums run over \( k : |\mu_k / \lambda_j - c| \leq \epsilon \) and where geodesic bi-angles do not play a role.

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1. Introduction

Let \((M, g)\) be a compact, connected Riemannian manifold of dimension \( n \) without boundary, let \( \Delta_M = \Delta_g \) denote its Laplacian, and let \( \{\varphi_j\}_{j=1}^{\infty} \) be an orthonormal basis of its eigenfunctions,

\[
(\Delta_M + \lambda_j^2)\varphi_j = 0, \quad \int_M \varphi_j \overline{\varphi_k} dV_M = \delta_{jk},
\]
where $dV_M$ is the volume form of $g$, and where the eigenvalues are enumerated in increasing order (with multiplicity), $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \cdots \uparrow \infty$. Let $H \subset M$ be an embedded submanifold of dimension $d \leq n - 1$ with induced metric $g|_H$, let $\Delta_H$ denote the Laplacian of $(H, g|_H)$, and let $\{\psi_k\}_{k=1}^\infty$ be an orthonormal basis of its eigenfunctions on $H$,

$$(\Delta_H + \mu_k^2)\psi_k = 0, \quad \int_H \psi_j \overline{\psi_k} dV_H = \delta_{jk},$$

where $dV_H$ is the volume form of $g|_H$. We denote the restriction operator to $H$ by

$$\gamma_H : C(M) \to C(H), \quad \gamma_H f = f|_H.$$

This article is concerned with the Fourier coefficients

$$a_{\mu_k}(\lambda_j) := \langle \gamma_H \varphi_j, \psi_k \rangle_H = \int_H (\gamma_H \varphi_j) \psi_k dV_H \quad (1.1)$$

in the $L^2(H)$-orthonormal expansion,

$$\gamma_H \varphi_j(y) = \sum_{k=1}^\infty \langle \gamma_H \varphi_j, \psi_k \rangle_H \psi_k(y) \quad (1.2)$$

of the restriction of $\varphi_j$ to $H$. Our goal is to understand the joint asymptotic behavior of the Fourier coefficients $\Box{1.1}$ as the eigenvalue pair $(\lambda_j, \mu_k)$ tends to infinity along a ‘ray’ or ‘ladder’ in the joint spectrum of $(\sqrt{-\Delta_M} \otimes I, I \otimes \sqrt{-\Delta_H})$ on $M \times H$. The motivating problem is to determine estimates or asymptotics for all of the Fourier coefficients $\Box{1.1}$ of an individual eigenfunction $\varphi_j$ as $\lambda_j \to \infty$.

This problem originates in the spectral theory of automorphic forms, where $(M, g)$ is a compact (or finite area, cusped) hyperbolic surface, and where $H$ is a closed geodesic, or a distance circle, or a closed horocycle (in the cusped case). In this case, $\Box{1.1}$ takes the form,

$$\gamma_H \varphi_j(s) = \sum_{n \in \mathbb{Z}} a_n(\lambda_j) e^{2\pi i ns/L}, \quad (1.3)$$

where $L$ is the length of $H$ and $s$ is the arc-length parameter. See Section $\Box{2.7}$ for a more precise statement when $H$ is a closed horocycle. Classical estimates and asymptotics of Fourier coefficients of modular forms are studied by Rankin $\Box{Ra77}$, Selberg $\Box{Se65}$, Bruggeman $\Box{Br81}$, Kuznecov $\Box{K80}$, and many others. Systematic expositions in this context may be found in $\Box{Br81}$ $\Box{G83}$, $\Box{102}$. It is ‘folklore’, and proved also in this article, that $|a_n(\lambda_k)| = O_{k, \epsilon}(n^{-N})$ for all $N \geq 0$ for $n \geq \lambda_k + \epsilon$, and for any $\epsilon > 0$; see also $\Box{Wo04}$ $\Box{Xi19}$ for some estimates of this type. If one thinks of $\lambda_k$ as the ‘energy’ and $n$ as the angular momentum, then rapid decay occurs in the “forbidden region” where the angular momentum exceeds the energy see Section $\Box{1.2}$. Hence, we restrict to the case where $|n| \leq \lambda_j$. The motivating question is, how does the dynamics of the geodesic flow of $(M, g)$ and of $(H, g|_H)$ determine the equidistribution properties of the restricted Fourier coefficients?

Studying the joint asymptotics of the Fourier coefficients $\Box{1.1}$ for individual eigenfunctions of general compact Riemannian manifolds $(M, g)$ and submanifolds $H \subset M$ is a very difficult problem and all but intractable except in special cases such as subspheres of standard spheres. Asymptotic knowledge of the coefficients $\Box{1.1}$ would a fortiori imply asymptotic knowledge of the $L^2$ norms of restrictions of eigenfunctions,

$$\int_H |\gamma_H \varphi_j|^2 dS_H = \sum_k |a_{\mu_k}(\lambda_j)|^2, \quad (1.4)$$
which are themselves only known in special cases. General estimates of (1.4) are proved in [BGT], but are only sharp in special cases. Knowledge of the distribution of Fourier coefficients and their relations to the geodesic flow are yet (much) more complicated. In the case where \((M, g)\) has ergodic geodesic flow and \(H \subset M\) is a hypersurface, the \(L^2\) norms of restrictions is known as the QER (quantum ergodic restriction) problem, i.e. to determine when \(\{\gamma H \varphi_j\}_{j=1}^\infty\) (or at least a subsequence of density one) is quantum ergodic along \(H\). For instance, if \(H\) is a hypersurface of a compact negatively curved manifold, it is shown in [TZ13] that there exists a full density subsequence of the restrictions \(\{\gamma H \varphi_{jk}\}\) such that
\[\int_H |\gamma H \varphi_{jk}|^2 dS_H = \sum_{m: \mu_m \leq \lambda_{jk}} |a_{\mu_m}(\lambda_{jk})|^2 \simeq 1, \quad (k \to \infty). \quad (1.5)\]

The Fourier coefficient distribution problem is to determine how the ‘mass’ of the Fourier coefficients are distributed. It has been conjectured that when the geodesic flow is ‘chaotic’ (e.g. for a negatively curved surface), the Fourier coefficients should exhibit ‘equipartition of energy’, i.e. all be roughly of the same size. By comparison, for standard spherical harmonics \(Y_m^m\) on \(S^2\), the restrictions have only one non-zero Fourier coefficient. The general Fourier coefficient problem is to find conditions on \((M, g, H)\) under which the Fourier coefficients \(a_\mu(\lambda_j)\) concentrate at one particular frequency (tangential eigenvalue), or under which they exhibit equipartition of energy.

In this article, we take the first step in this program by studying the averages in \(\lambda_j\) of sums of squares of a localized set of Fourier coefficients. Equivalently, we study the joint asymptotics of (1.4) of the thinnest possible sums over the joint spectrum \(\{(\lambda_j, \mu_k)\}_{j,k=1}^\infty\). Here, ‘thin’ refers to both the width of the window in the spectral averages in \(\lambda_j\), and also the width of the window of Fourier modes \(\mu_k\). In comparison, conic or wedge windows are studied in [WXZ20].

**Kuznecov-Weyl sums.** The Kuznecov sum formula in the sense of [Ze92] refers to the asymptotics of the Weyl-type sums,
\[N_H(\lambda; f) := \sum_{j: \lambda_j \leq \lambda} \left| \int_H f \varphi_j dV_H \right|^2, \quad (1.6)\]
where \(f \in C^\infty(H)\). Here, and henceforth, we drop the restriction operator \(\gamma_H\) from \(\gamma_H \varphi_j\) if it is obvious from the context that \(\varphi_j\) is being restricted. The asymptotics are controlled by the structure of the ‘common orthogonals’, i.e. the set of geodesic arcs which hit \(H\) orthogonally at both endpoints. The original Kuznecov formulae pertained to special curves on arithmetic hyperbolic surfaces [K80], but the Weyl asymptotics have been generalized to submanifolds of general Riemannian manifolds in [Ze92]. Recent improvements of spectral projection estimates under various conditions appear in [ChST15, CGT17, SXZh17, Wy19, CG19, WX18]. Fine asymptotics in the arithmetic setting are given in [M16]. We refer to these articles for many further references.

In this article, we are interested in the joint asymptotics of \(|\langle \gamma_H \varphi_j, \psi_k \rangle_{L^2(H)}|^2\) as the pair \((\lambda_k, \mu_j)\) tends to infinity along a ray in \(\mathbb{R}^2\). Except for exceptional \((M, g, H)\) it is not possible to obtain asymptotics along a single ray. Instead, we study the joint asymptotics in a thin strip around the ray. Our main result deploys a smooth version of such a strip, sometimes called ‘fuzzy ladder asymptotics’ in the sense of [GU89]. Let \(\psi \in \mathcal{S}(\mathbb{R})\) (Schwartz space)
with \( \hat{\psi} \in C_0^\infty(\mathbb{R}) \) a positive test function. We then define the fuzzy-ladder sums,

\[
N_{\psi,H}^c(\lambda) := \sum_{j : \lambda_j \leq \lambda} \sum_{k=0}^{\infty} \psi(\mu_k - c\lambda_j) \left| \int_H \varphi_j \psi k dV_H \right|^2.
\]  

(1.7)

The asymptotics of (1.7) depend on the geometry of what we term \((c,s,t)\) bi-angles in Section 1.3. Roughly speaking, such a bi-angle consists of two geodesic arcs, one a geodesic of \(H\) of length \(s\), the other a geodesic of \(M\) of length \(cs + t\), with common endpoints and making a common angle \(\arccos c\) with \(H\). When \(t = 0\), the set of such bi-angles is defined by,

\[
G_c^0 = \{(q,\xi) \in S_n^*M : |\pi_H \xi| = c|\xi|, G_H^{-s} \circ \pi_H \circ G_M^{cs}(q,\xi) = (q,\pi_H(\xi))\},
\]

(1.8)

where \(\pi_H : T_q^*M \rightarrow T_q^*H\) is the orthogonal projection at \(q \in H\).

It turns out that the “spectral edge” case \(c = 1\) requires different techniques from the case \(c < 1\). It is the edge or ‘interface’ between the allowed interval \([0, 1]\) and the complementary ‘forbidden’ intervals of \(c\)-values (see Section 1.2 for discussion). As often happens at interfaces, there is a variety of possible behaviors when \(c = 1\). As discussed below, the case \(c = 1\) corresponds to tangential intersections of geodesics with \(H\), which depend on whether \(H\) is totally geodesic or whether it has non-degenerate second fundamental form. For this reason, we separate out the cases \(0 < c < 1\) and \(c = 1\) and only study \(0 < c < 1\) in this article. The case of \(c = 1\) and \(H\) totally geodesic is studied in [Z+]. The case of \(c = 1\) and \(H\) with a non-degenerate second fundamental form requires quite different techniques from the totally geodesic case, and is currently under investigation.

The first result is a ladder refinement of the main result of [WXZ20]. For technical convenience, we assume the test function \(\psi\) is even and non-negative.

**Theorem 1.1.** Let \(\dim M = n\) and let \(\dim H = d\). Let \(0 < c < 1\) and assume that \(G_c^0\) is clean in the sense of Definition 1.12. Then, if \(\psi \geq 0\) is even, \(\hat{\psi} \in C_0^\infty(\mathbb{R})\), and \(\text{supp} \hat{\psi}\) is contained in a sufficiently small interval around \(s = 0\), there exist universal constants \(C_{n,d}\) such that

\[
N_{\psi,H}^c(\lambda) = C_{n,d} a_c^0(H,\psi) \lambda^{n-1} + R_{\psi,H}^c(\lambda), \quad \text{with } R_{\psi,H}^c(\lambda) = O(\lambda^{n-2}),
\]

where the leading coefficient is given by,

\[
a_c^0(H,\psi) := \hat{\psi}(0) c^{d-1}(1 - c^2)^{n-d-2} H^d(H),
\]

(1.9)

The definition of “\(\text{supp} \hat{\psi}\) is sufficiently small” is given in Definition 1.11 below. For the moment, we say that when \(0 < c < 1\), it means that the only component of (1.8) with \(s \in \text{supp} \hat{\psi}\) is the component with \(s = 0\). It is straightforward to remove the condition that \(\text{supp} \hat{\psi}\) is small, but then there are further contributions from other components of \(G_c^0\), which require further definitions and notations. We state the generalization in Theorem 1.18 below. The coefficient \(a_c^0(H,\psi)\) is discussed in Section 1.9.

Under additional dynamical assumptions on the geodesic flow \(G_M^t\) of \((M,g)\) and the geodesic flow \(G_H^t\) of \((H,g|_H)\), one can improve the remainder estimate of Theorem 1.1 (see Theorem 1.20 and Theorem 1.22). The improvement requires the inclusion of all components of \(G_c^0\) and therefore requires the generalization Theorem 1.18 of Theorem 1.1. Since it requires further notation and definitions, we postpone it until later in the introduction.
1.1. **Jumps in the Kuznecov-Weyl sums.** To obtain results for individual eigenfunctions (more precisely, for individual eigenvalues) by the techniques of this article, we study the jumps,

\[ J_{c,H}^\ell (\lambda_j) := \sum_{\ell: \lambda\ell = \lambda_j} \sum_{k=0}^\infty \psi(\mu_k - c\lambda_j) \left| \int_H \varphi_k \psi H dV_H \right|^2, \quad (1.10) \]

in the Kuznecov-Weyl sums \((1.7)\) at the eigenvalues \(\lambda_j\). The sum over \(\ell\) is a sum over an orthonormal basis for the eigenspace \(\mathcal{H}(\lambda_j)\) of \(-\Delta_M\) of eigenvalue \(\lambda_j^2\). Since the leading term is continuous, the jumps are jumps of the remainder,

\[ J_{c,H}^\ell (\lambda_j) = R_{c,H}^\ell (\lambda_j + 0) - R_{c,H}^\ell (\lambda_j - 0). \quad (1.11) \]

By \((1.11)\) and Theorem 1.1, we get

**Corollary 1.2.** With the same assumptions and notations as in Theorem 1.1, for any positive even test function \(\psi\) with \(\hat{\psi} \in C_\infty^0(\mathbb{R})\), there exists a constant \(C_{c,\psi} > 0\) such that

\[ J_{c,H}^\ell (\lambda_j) \leq C_{c,\psi} c n^{-2}, \quad 0 < c < 1. \]

Corollary 1.2 has a further implication on the ‘sharp jumps’ where we replace \(\psi\) by an indicator function, \(1_{\left[-\epsilon, \epsilon\right]}\).

\[ J_{c,H}^\ell (\lambda_j) := \sum_{\ell: \lambda\ell = \lambda_j} \sum_{k: |\mu_k - c\lambda_j| \leq \epsilon} \left| \int_H \varphi_k \psi H dV_H \right|^2. \quad (1.12) \]

By choosing \(\psi\) carefully in Corollary 1.2 we prove,

**Corollary 1.3.** With the above notation and assumptions,

\[ J_{c,H}^\ell (\lambda_j) = O_{c,\epsilon}(\lambda_j^{n-2}), \quad 0 < c < 1. \]

This estimate is shown to be sharp for closed geodesics in general spheres \(S^n\) in Section 2.5. The sharpness of the jump estimates is intimately related to the periodicity properties of both \(G_M^t\) and \(G_H^t\), and is discussed in detail in Section 2.1. The dependence on \(\epsilon\) in Corollary 1.3 is complicated in general, as illustrated in Section 2.4.2 and Section 2.5. This is because of potential concentration of the Fourier coefficients at joint eigenvalues \((\lambda_j, \mu_k)\) at the ‘edges’ or endpoints of the strip of width \(\epsilon\) around a ray. See Lemma 2.1 for the explicit formula in the case of subspheres of spheres. When \(\dim M = 2, \dim H = 1\), the eigenvalues of \(\frac{d}{ds}\) of \(H\) are of course multiples of \(n \in \mathbb{N}\) involving the length of \(H\) and are therefore separated by gaps of size depending on the length. The jumps \((1.12)\) can themselves jump as \(\epsilon\) varies, as illustrated by restrictions from spheres to subspheres in the same section. See also Section 1.7 for further discussion. These edge effects arise again in Theorem 1.22. For \(\epsilon\) sufficiently small, only one eigenvalue will occur in the sum over \(k: |\mu_k - c\lambda_j| \leq \epsilon\) and therefore Corollary 1.3 gives a bound on the size of an individual Fourier coefficient of an individual eigenfunction under the constraints on \((c, H)\) in the corollary. We refer to Section 2.4.1 for examples of curves on \(S^2\). We also refer to the second author’s work [Xi19] and the first two authors’ work [WX18] for prior results on bounds on Fourier coefficients.

**Remark 1.4.** For a generic metric \(g\) on any manifold \(M\), the spectrum of the Laplacian is simple, i.e. all eigenvalues have multiplicity one [U]. In this case, the \(\lambda_j\) sum of \((1.12)\)
reduces to a single term and one gets, hence,
\[
\left| \int_H \varphi_j \overline{\psi_k} dV_H \right|^2 \leq J^c_{\epsilon,H}(\lambda_j) := \sum_{k: |\mu_k - c\lambda_j| \leq \epsilon} \left| \int_H \varphi_j \overline{\psi_k} dV_H \right|^2, \quad \forall k : |\mu_k - c\lambda_j| \leq \epsilon. \tag{1.13}
\]

### 1.2. Allowed and forbidden joint eigenvalue regions

We briefly indicate why we restrict to \(c \in [0,1]\). The reason is that the asymptotics are trivial outside this range.

**Lemma 1.5.** If \(\mu_k/\lambda_j \geq 1 + \epsilon\), then for any \(N \geq 1\), \(\int_H \varphi_j \overline{\psi_k} dV_H \) is rapidly decaying in \(\lambda\).

The proof is contained in the proof of Theorem 1.22. The main point of the proof is that, if one uses a cutoff \(\psi(\mu_k - c\lambda_j)\) with \(c > 1\), then there \(N^c_{H,\psi}(\lambda)\) is rapidly decaying. This is also proved in detail in [Xi19] for the case \(\dim H = 1\).

We will not comment on this issue further in this article. As Lemma 1.5 indicates, \(c = 1\) is an ‘interface’ in the spectral asymptotics, separating an allowed and a forbidden region. It would be interesting to study the interface asymptotics around \(c = 1\), i.e. the transition from the asymptotics of Theorem 1.1 to trivial asymptotics for \(c > 1\); we plan to study the scaling asymptotics around \(c = 1\) in future work.

### 1.3. Geometric objects: \((c,s,t)\) bi-angles

We now give the geometric background to Theorem 1.1 and Theorem 1.22 in particular defining the relevant notion of ‘cleanliness’ and explaining the coefficients. We will need some further notation (see [TZ13] for further details). Let \(G^s_M\), resp. \(G^s_H\) denote the geodesic flow of \((M,g)\) on \(\dot{T}^* M\), resp. \(\dot{T}^* H\) (since the flow is homogeneous, and \(S^* M\) is invariant, it is sufficient to consider the flow on \(S^* M\)).

**Remark 1.6.** Notational conventions: For any manifold \(X\) we denote by \(\dot{T}^* X = T^* X \setminus 0\) the punctured cotangent bundle. All of the canonical relations in this paper are homogeneous and are subsets of \(\dot{T}^* X\).

We denote by \(S^* H, M\) the covectors \((y, \xi) \in S^* M\) with footpoint \(y \in H\), and by \(\pi_H(y, \xi) = (y, \eta) \in B^* H\) the orthogonal projection of \(\xi\) to the unit coball \(B^* H\) of \(H\) at \(y\). We also denote by \(T^*_c H\) the cone of covectors in \(\dot{T}^*_H M\) making an angle \(\theta\) to \(T^* H\) with \(\cos \theta = c\). It is the cone through
\[
S^* H, M = \{(y, \xi) \in S^* H, M : |\pi_H \xi| = c\}. \tag{1.14}
\]

**Definition 1.7.** By a \((c,s,t)\)-bi-angle through \((q, \xi) \in S^* H, M\), we mean a periodic, once-broken orbit of the composite geodesic flow solving the equations,
\[
G^s_H \circ \pi_H \circ G^s_M(q, \xi) = \pi_H(q, \xi), \quad (q, \xi) \in S^* H, M \tag{1.15}
\]
Thus, a bi-angle is a pair of geodesic arcs with common endpoints, for which the \(M\)-length is \(cs + t\) and whose \(H\) length is \(s\).

Note that the geometry changes considerably when \(c = 1\) and \(H\) is totally geodesic. In this case, when \(t = 0\), a \((1,s,0)\) bi-angle is a geodesic arc of \(H\), traced forward for time \(s\) and backwards for time \(s\). The consequences are explored in [Z+].

The projection of a \((c,s,t)\) bi-angle to \(M\) consists of a geodesic arc of \(M\) of length \(ct + s\), with both endpoints on \(H\), making the angle \(\theta\) with \(\cos \theta = c\) at both endpoints \(q\) resp. \(q'\), and an \(H\)-geodesic arc of oriented \(H\)-length \(-s\) with initial velocity \((q, \pi_H \xi)\) and with terminal velocity \((q', \pi_H G^s_M(q, \xi))\). Equivalently, let \(\gamma^M_{x,\xi}\) denote the geodesic of \(M\) with initial
data \((x, \xi) \in S^*_H M\), and let \(\gamma^H_{y, \eta}\) denotes the geodesic of \(H\) with initial data \((y, \eta) \in S^*H\). Then the defining property of a \((c, S, T)\)-bi-angle is that it consists of a geodesic arc \(\gamma^M_{x, \xi}\) of \(M\) of length \(T\), and a geodesic arc \(\gamma^H_{x, \eta}\) of \(H\) of length \(S\), such that:

\[
\begin{align*}
\gamma^M_{x, \xi}(0) &= x = \gamma^H_{x, \eta}(0) \in H, \quad \pi_H \gamma^M_{x, \xi}(0) = \dot{\gamma}^H_{x, \eta}(0); \\
|\dot{\gamma}^M_{x, \xi}(0)| &= 1, |\dot{\gamma}^H_{x, \eta}(0)| = c; \\
\gamma^M_{x, \xi}(T) &= \gamma^H_{x, \eta}(S), \quad \pi_H \gamma^M_{x, \xi}(T) = \dot{\gamma}^H_{y, \eta}(S).
\end{align*}
\]

Some examples on the sphere \(S^2\) are given in Section 2.

We denote the set of all solutions with a fixed \(c\) by

\[G_c : \{ (s, t, q, \xi) \in \mathbb{R} \times \mathbb{R} \times S^*_H M : (1.15) \text{ is satisfied} \}. \tag{1.16} \]

For \(t\) fixed, we also define

\[G'_c : \{ (s, y, \xi) \in \mathbb{R} \times S^*_H M : (s, t, y, \xi) \in G_c \}. \tag{1.17} \]

We decompose \(G_c\) into the subsets,

\[G_c = G^0_c \bigcup G^\neq 0_c, \quad G^0_c = G^{0,0}_c \bigcup G^{0, \neq 0}_c, \]

where (cf. (1.8))

\[
\begin{align*}
G^0_c &= \{ (s, 0, y, \xi) \in \mathbb{R} \times S^*_H M : (1.15) \text{ is satisfied} \}, \\
G^{0,0}_c &\simeq S^*_H M = \{ (0, 0, y, \xi) \in S^*_H M : (1.15) \text{ is satisfied} \} \tag{1.18} \\
G^{0, \neq 0}_c &= \{ (s, t, y, \xi) \in \mathbb{R} \times \mathbb{R} \times S^*_H M : t \neq 0, (1.15) \text{ is satisfied} \}.
\end{align*}
\]

The principal term of (1.7) only involves the set \(G^0_c\). When \(c < 1\), or when \(c = 1\) and \(H\) has non-degenerate second fundamental form, the equation cannot hold for small enough \(s\) because the \(M\)-geodesic between the endpoints must be shorter than the \(H\) geodesic. More formally, we state,

**Lemma 1.8.** When \(0 < c < 1\), or if \(c = 1\) and \(H\) has non-degenerate second fundamental form, \(G^{0,0}_c\) is a connected component of \(G^0_c\). If \(H\) is totally geodesic and \(0 < c < 1\), \(G^{0,0}_c\) is also a connected component, i.e. there do not exist any \((c, s, 0)\)-bi-angles when \(s \neq 0\) is sufficiently small (depending on \(c\)). When \(c = 1\) and \(H\) is totally geodesic, \(G^{0,0}_c\) is not a connected component of \(G^0_c\).

In general, the order of magnitude of \(N^c_{\psi, H}(\lambda)\) depends on the dimension of (1.18) and on its symplectic volume measure.

**Definition 1.9.** For \(0 < c < 1\) the symplectic volume \(\text{Vol}(G^0_c)\) of (1.18) is the Euclidean measure of \(G^0_c\). The pushforward volume form acquires an extra factor \((1-c^2)^{-\frac{1}{2}}\) (see [WXZ20] for more details).

The additional components only contribute to the main term of the asymptotics, when they have the same dimension as \(G^{0,0}_c\).

**Definition 1.10.** For \(0 < c < 1\) define the ‘principal component’ of \(G^0_c\) as \(G^{0,0}_c\). We say that the principal component is dominant if \(\dim G^{0,0}_c\) is strictly greater than any other component of \(G^0_c\).
Examples on spheres illustrating the various scenarios are given in Section 2.3. If \( c \in \mathbb{Q}, c < 1 \), then \( \mathcal{G}^{1,0}_c \) and \( \mathcal{G}^{2,0}_c \) have an arithmetic progression of common periods, hence there are infinitely many components of the fixed point set (1.8) of the same dimension as \( \mathcal{G}^{0,0}_c \), all contributing to the main term of the asymptotics. For instance, if \( H \subset \mathbb{S}^2 \) is a latitude circle, then for \( 0 < c < 1 \), \( \mathcal{G}^{0,0}_c \) is not dominant, due to periodicity of the geodesic flow and of rotations (see Section 2). On a negatively curved surface, \( \mathcal{G}^{0,0}_c \) is dominant. Note that \( \mathcal{G}^{0,0}_c \cap \{0\} \times \mathbb{R}_s \times S^t H \) is a single component when \( c = 1 \) and \( H \) is totally geodesic so that the notion of principal and dominant is vacuous then. When \( c = 1 \) (1.16) consists of bi-angles formed by an \( M \) geodesic hitting \( H \) tangentially at both endpoints, closed up by an \( H \)-geodesic arc through the same endpoint velocities. When \( c = 0 \), \( \xi = \nu_t \) is the unit (co-)normal and (1.16) consists of \( M \)-geodesic arcs hitting \( H \) orthogonally at both endpoints. These are the arcs relevant to the original Kuznecov formula of [Zel92].

To eliminate the contribution of non-principal components when \( 0 < c < 1 \), we assumed in Theorem 1.1 that supp \( \hat{\psi} \) is ‘sufficiently small’. We now define the term more precisely.

**Definition 1.11.** When \( 0 < c < 1 \), we say that supp \( \hat{\psi} \) is sufficiently small if \( \mathcal{G}^{0,0}_c \) is the only component of \( \mathcal{G}^0_c \) with values \( s \in \text{supp} \hat{\psi} \).

1.4. **Cleanliness and Jacobi fields.** As in most asymptotics problems, we will be using the method of stationary phase, mainly implicitly when composing canonical relations. This requires the standard notion of cleanliness from [DG75].

**Definition 1.12.** We say that (1.16), resp. (1.18), is clean if \( \mathcal{G}^c \), resp. \( \mathcal{G}_c^0 \), is a submanifold of \( \mathbb{R} \times \mathbb{R} \times S^t M^H \), resp. \( \mathbb{R} \times S^t M^H \), and if its tangent space at each point is the subspace fixed by \( D_\xi G^{-s} \circ \pi_H \circ C^c_{s,t} \) (resp. the same with \( s = 0 \)), where \( \zeta \) denotes the \( S^c M^H \) variables.

Some examples are given in Section 1.2. It is often not very difficult to determine whether \( \mathcal{G}^c, \mathcal{G}^0_c \) are manifolds. The tangent cleanliness condition is often difficult to check. It may be stated in terms of bi-angle Jacobi fields, as follows. A Jacobi field along a geodesic arc \( \gamma \) is the vector field \( Y(t) \) along \( \gamma(t) \) arising from a 1-parameter variation \( \alpha(t,s) \) of geodesics, with \( \alpha(t,0) = \gamma(t) \). An \((S,T)\) bi-angle consists of an \( H \)-geodesic arc \( \gamma^H(t) \) of length \( S \) and an \( M \) geodesic arc \( \gamma^M(t) \) of length \( T \) which have the same initial and terminal point, and such that the projection of the initial and terminal velocities to \( \gamma^M(T) \) equal those of \( \gamma^H(S) \). A bi-angle Jacobi field \( Y(t) \) arises as the variation vector field of a one-parameter variation \((\gamma^M_\epsilon(t), \gamma^H_\epsilon(t))\) of \((S(\epsilon), T(\epsilon))\)-bi-angles. It consists of a Jacobi field \( J_M(t) \) along \( \gamma^M(t) \) and a Jacobi field \( J_H(t) \) along \( \gamma^H(t) \) which are compatible at the endpoints. Namely, if we differentiate in \( \epsilon \) the equations

\[
\gamma^H_\epsilon(0) = \gamma^M_\epsilon(0), \quad \gamma^H_\epsilon(S(\epsilon)) = \gamma^M_\epsilon(T(\epsilon)),
\]

\[
\pi_* \dot{\gamma}^M_\epsilon(0) = \dot{\gamma}^M_\epsilon(0), \quad \pi_* \dot{\gamma}^M(S(\epsilon)) = \dot{\gamma}^M(T(\epsilon)), \quad |\gamma^H_\epsilon(0)| = c
\]

at \( \epsilon = 0 \) we get

\[
J_H(0) = J_M(0), \quad J_H(t_0) \dot{S} = J_M(T) \dot{T},
\]

together with two equations for \( \frac{\partial}{\partial T} J_H, \frac{\partial}{\partial T} J_M \).

**Definition 1.13.** A bi-angle Jacobi field is a pair \((J_H(t), J_M(t))\) of Jacobi fields, as above, along the arcs of the bi-angle which satisfy the above compatibility conditions at the endpoints.

In defining variations of bi-angles, we may allow the angle parameter \( c \) to vary with \( \epsilon \) in a variation, or to hold it fixed. In Definition 1.12 the parameter \( c \) is held fixed, and therefore
all variations must have the same value of \( c \). As with standard Jacobi fields, the bi-angle Jacobi field arises by varying the initial tangent vector \((x, \xi) \in S^*_x M, x \in H, x \in H \) along \( S^* H M \). A ‘horizontal’ bi-angle Jacobi field is one that arises by varying \( x \in H \), and a vertical bi-angle Jacobi field arises by fixing \( x \) but varying the initial direction \( \xi \).

**Remark 1.14.** The vectors in the subspace fixed by \( DG_H^{-k} \circ \pi_H \circ G_M^{s+1}(\zeta) = \pi_H(\zeta) \) always correspond to bi-angle Jacobi fields along the associated bi-angle. Lack of cleanliness occurs if there exists a bi-angle Jacobi field which does not arise as the variation vector field of a variation of \((c, S, T)\)-bi-angles of \((M, g, H)\).

Cleanliness is often difficult to verify. We refer to Section 4.2 for further discussion and un-clean examples.

1.5. **Outline of the proof of Theorem 1.1 and further results.** To prove Theorem 1.1 we first prove the related result where the indicator functions are replaced by smooth test functions. The exposition will set the stage for the further results (Theorem 1.18 and Theorem 1.22).

The sharp sums over \( \lambda_k \) in (1.23) and half-sharp sums (1.7) may be replaced by ‘twice-smoothed’ sums with shorter windows if we introduce a second eigenvalue cutoff \( \rho \in S(\mathbb{R}) \) with \( \hat{\rho} \in C_0^\infty(\mathbb{R}) \) and define,

\[
N_{\psi, \rho, H}^c(\lambda) := \sum_{j,k} \rho(\lambda - \lambda_k) \psi(\mu_j - c \lambda_k) \left| \int_H \varphi_j \varphi_k dV_H \right|^2. \tag{1.19}
\]

Under standard ‘clean intersection hypotheses,’ the sums (1.19) admit complete asymptotic expansions. They are the raw data of the problem, to which the methods of Fourier integral operators apply. To obtain the sharper results, we apply some Tauberian theorems, first (when \( \psi \geq 0 \)) to replace \( \rho \) by the corresponding indicator function to obtain two-term asymptotics for (1.7). Then, in Theorem 1.1 we apply a second Tauberian theorem to replace \( \psi \) (when possible) by the corresponding indicator function. In the following, we assume that \( \text{supp} \psi \) is ‘sufficiently small’ as defined in Definition 1.11.

To study the asymptotics of (1.19), we express the left side as the oscillatory integral,

\[
N_{\psi, \rho, H}^c(\lambda) = \int_{\mathbb{R}} \hat{\rho}(t) e^{-it \lambda} S^c(t, \psi) dt, \quad \text{where}
\]

\[
S^c(t, \psi) := \sum_{j,k} e^{it \lambda_j} \psi(\mu_j - c \lambda_j) \left| \int_H \varphi_j \varphi_k(x, x) dV_H(x) \right|^2,
\]

show that \( S^c(t, \psi) \) is a Lagrangian distribution on \( \mathbb{R} \), and determine its singularities (Proposition 5.2).

**Definition 1.15.** Let \( \Sigma^c(\psi) = \text{sing supp} S^c(t, \psi) \) be the set of singular points of \( S^c(t, \psi) \). \( \Sigma^c(\psi) \) is called the set of ‘sojourn times’, and consists of \( t \) for which there exist solutions of (1.15) with \( s \in \text{supp} \psi \).

We then have,

**Theorem 1.16.** Let \( \dim M = n, \dim H = d \). Let \( \psi, \rho \in S(\mathbb{R}) \) with \( \hat{\psi}, \hat{\rho} \in C_0^\infty(\mathbb{R}) \). Assume that \( \text{supp} \hat{\rho} \cap \Sigma^c(\psi) = \{0\} \) and \( \hat{\rho}(0) = 1 \). Also assume that \( \text{supp} \hat{\psi} \) is sufficiently small in the sense of Definition 1.17. Let \( c \in (0, 1) \) and assume that \( G_0^c \) is clean in the sense of Definition 1.12. Then, there exists a complete asymptotic expansion of \( N_{\psi, \rho, H}^c(\lambda) \) with principal terms,

\[
N_{\psi, \rho, H}^c(\lambda) = C_{n,d} a_0^c(H, \psi) \lambda^{-2} + O(\lambda^{-3}), \quad (0 < c < 1).
\]
where the leading coefficient is given by (1.9).

While this theorem follows as a corollary to [WXZ20 Theorem 1.4], our proof here also admits a more general statement (see Theorem 1.18). Theorem 1.16 is proved in Section 6. Note that the order of asymptotics is 1 less than in Theorem 1.1 and Theorem 1.22. This is due to the fact that the $\lambda$ intervals are ‘thin’ in Theorem 1.16 and ‘wide’ in the previous theorems.

To obtain the ‘sharp’ Weyl asymptotics of Theorem 1.1 in the $\lambda$-variable, we need to replace $\rho$ by an indicator function. This is done in Section 6.1 by a standard Tauberian theorem.

1.6. Asymptotics when supp $\hat{\psi}$ is arbitrarily large. Theorem 1.1 and Theorem 1.16 both assume that supp $\hat{\psi}$ is sufficiently small in the sense of Definition 1.12. They readily admit generalizations to any $\hat{\psi} \in C_0^\infty$ where we take into account the additional components of $G^0_c$ coming from large $s$. Only the components of $G^0_c$ of the same maximal dimension as the principal component contribute to the principal term of the asymptotics.

In the notation of [DG75 Theorem 4.5], we are assuming that the set of $(c, s, 0)$-bi-angles with $t = 0$ is a union of clean components $Z_j(0)$ of dimension $d_j$. In our situation $Z_j(0)$ is a component of $G^0_c$. Then, for $t$ sufficiently close to 0, each $Z_j(0)$ gives rise to a Lagrangian distribution $\beta_j(t)$ on $\mathbb{R}$ with singularities only at $t = 0$, such that,

$$S^c(t, \psi) = \sum_j \beta_j(t), \quad \text{where } \beta_j(t) = \int_{\mathbb{R}} \alpha_j(s)e^{-ist}ds,$$

(1.21)

with $\alpha_j(s) \sim \left(\frac{s}{2\pi}\right)^{-1+\frac{1}{2}(n-d)+\frac{d_j}{2}} i^{-\sigma_j} \sum_{k=0}^{\infty} \alpha_{j,k}s^{-k}$, where $d_j$ is the dimension of the component $Z_j(0) \subset G^0_c$.

**Definition 1.17.** We say that a connected component $Z_j(0)$ of $G^0_c$ is maximal if its dimension is the same as $G^0_{c,0}$. We denote the set of maximal components by $\{Z_j^m(0) = G^0_{c,j} \}_{j=1}^{\infty}$. When $0 < c < 1$, $s$ is constant on each maximal component and we denote these values of $s$ by $s_j^m$.

Existence of a maximal component other than $G^0_c$ implies that there exists periodicity in the geodesic flows $G^M_t$ and $G^t_H$ (see Section 2.1), and that $c$ is a special value where there exists periodicity in the flows $G^H_{t,cs}$. For instance, when $H = S^d \subset S^n$, and when $c = \frac{p}{q} \in \mathbb{Q}$, there exist maximal components when $s = 2k\pi q$ with $k = 1, 2, \ldots, \text{since then the left side of (1.8) is the identity, } G^c_{s, \mathbb{Q}} \circ S^c_{\mathbb{Q}} = G^c_{s, \mathbb{Q}} \circ S^c_{\mathbb{Q}} = S^c_{\mathbb{Q}} = \text{Id} \times \text{Id}$. This is the geometric reason behind the discontinuous behavior of the jumps in Section 2.4.2. Recall $\Sigma^c(\psi)$ from Definition 1.15 for the following theorem.

**Theorem 1.18.** Let $\dim M = n, \dim H = d$. Let $\psi, \rho \in S(\mathbb{R})$ with $\hat{\psi}, \hat{\rho} \in C^\infty_0(\mathbb{R})$. Assume that $\hat{\rho}(0) = 1$ and that supp $\rho \cap \Sigma^c(\psi) = \{0\}$. Let $0 < c < 1$ and assume that $G^c_c$ is clean in the sense of Definition 1.12. Then, there exists a complete asymptotic expansion of $N^c_{\psi, \rho, H}(\lambda)$ with principal terms,

$$N^c_{\psi, \rho, H}(\lambda) = C_{n,d} a^0_c(H, \psi)\lambda^{-2} + O^\psi(\lambda^{n-5/2}), \quad 0 < c < 1$$

where the leading coefficient is given by the following sum over the maximal components $Z_j^m(0)$ of $G^0_c$,

$$a^0_c(H, \psi) := \left(\sum_j \hat{\psi}(s_j^m)\right) c^{d-1}(1 - c^2)^{\frac{n-d-2}{2}} \mathcal{H}(H).$$
Moreover, if we replace ρ by 1_{[0,λ]}, then
\[ N_{ψ,H}^c(λ) = C_{n,d} a^0_c(H, ψ)λ^{n-1} + O_ψ(λ^{n-3/2}). \]

The exponent \( n - 5/2 \) in the second term of the asymptotics can arise from components \( Z_j(0) \) of dimension \( d_j \) possibly one less than maximal. This drops the exponent of the top term of \( \beta_j \) of (1.21) by 1/2. If no such components occur, the remainder is of order \( λ^{n-3} \).

See Section 5.2 for the proofs of Theorem 1.16 and Theorem 1.18.

Remark 1.19. The formula for \( a^0_c(H, ψ) \) reflects the fact that, under the cleanliness hypothesis of Definition 1.12, each maximal component \( Z_j^m(0) \simeq C^0 \) must be essentially the same as the principal component modulo the change in the \( s \) parameter, hence the corresponding leading coefficient \( α_{j,0} \) in (1.21) is the same as for the principal component. The only essential change to the principal coefficient in Theorem 1.10 is that it is now necessary to compute the sum of the canonical symplectic volumes of all maximal components with \( s_j^m \in \text{supp } \hat{ψ} \).

When \( c < 1 \), the set of such \( s_j^m \) is discrete and thus we sum \( \hat{ψ} \) over this finite set of parameters. See Section 2.1 for a discussion of the periodicity properties necessary to have maximal components.

1.7. Two term asymptotics with small oh remainder. Theorem 1.11 and Theorem 1.18 assume that supp \( \hat{ρ} \cap Σ^c(ψ) = \{0\} \) and \( \hat{ρ}(0) = 1 \). By allowing general \( ρ \in C^∞(R) \) with \( \hat{ρ} \in C^c \), and by studying long-time asymptotics of \( G^t_M \) and \( G^t_H \), it is possible under favorable circumstances to prove two-term asymptotics of the type initiated by Y. Safarov (see [SV] for background). From Theorem 1.18 we note that the order of the second term depends on the geometry of \( G_c \). When the second term of Theorem 1.18 has order \( λ^{n-3} \), the two term asymptotics have the following form: for any \( ε > 0 \), we have as \( λ \to ∞ \),
\[
C_{n,d} a^0_c(H, ψ)λ^{n-1} + Q_{H,ψ}(λ - ε)λ^{n-2} - o(λ^{n-2}) \leq N_{ψ,H}^c(λ) \leq C_{n,d} a^0_c(H, ψ)λ^{n-1} + Q_{H,ψ}(λ + ε)λ^{n-2} + o(λ^{n-2}),
\]
where \( Q_{H,ψ}(λ) \) is an oscillatory function determined by the singularities for all \( t \) (see Section 2.8).

The rather unusual notion of second term asymptotics is necessary and was first proved by Safarov for the pointwise Weyl law or the fully integrated (over \( M \) Weyl law; see [SV] and also [Hod] (29.2.16)-(29.2.17)). Depending on the periodicity properties of \( G^t_M \) and \( G^t_H \), \( Q_{H,ψ}(λ) \) can be continuous or have jumps. Existence of jumps in \( Q_{H,ψ}(λ) \) is not a sufficient condition for jumps of size \( λ^{n-2} \) in \( N_{ψ,H}^c(λ) \) but it is a necessary one and can be used to analyze situations where maximal jumps occur. In view of the many possible types of phenomena discussed in Section 2, it is a lengthy additional problem to prove such two term asymptotics and in particular to calculate \( Q_{H,ψ}(λ) \) explicitly, and we defer their study to a future article. In this section, we state some results on situations where \( Q_{H,ψ} = 0 \).

The next result improves the remainder estimate in Theorem 1.11 under the dynamical hypothesis that the \( t = 0 \) singularity is dominant in the sense of Definition 1.10.

Theorem 1.20. With the notation and assumptions as in Theorem 1.11, we assume \( \hat{ψ} \) has small support. We further assume that the singularity at \( t = 0 \) is dominant, i.e. there do not exist maximal components \( Z_1(T) \) for \( T ≠ 0 \). Then,
\[ N_{\psi,H}^c(\lambda) = C_{n,d} a_0^c(H,\psi) \lambda^{n-1} + R_{\psi,H}^c(\lambda), \]

where

\[ R_{\psi,H}^c(\lambda) = o_\psi(\lambda^{n-2}), \quad J_{\psi,H}^c(\lambda) = o_\psi(\lambda^{n-2}). \]

To prove Theorem 1.20 we first need to prove that the coefficient in the expansion of \( N_{\psi,H}^c(\lambda) \) of the term of order \( \lambda^{n-2} \) is zero. This is shown in Section 5.3. From Theorem 1.20 we obtain an improved estimate on the jumps in the case where both \( G_M^t \) and \( G_H^t \) are ‘aperiodic’, i.e. where the Liouville measure in \( S^*M \), resp. \( S^*H \) of the periodic orbits of \( G_M^t \), resp. \( G_H^t \) are zero. The following estimate follows directly from Theorem 1.20 and the same technique of bounding \( 1_{[-\epsilon,\epsilon]} \) by a test function \( \psi \in C_0^\infty \) used in Corollary 1.3.

**Corollary 1.21.** If both \( G_M^t \) and \( G_H^t \) are aperiodic (see Section 2.7), then for any \( \epsilon > 0 \),

\[ J_{\psi,H}^c(\lambda_j) = o_\psi(\lambda_j^{n-2}). \]

The proof of Corollary 1.21 from Theorem 1.20 follows a well-known path and is sketched in Section 6.4.

### 1.8. Sharp-sharp asymptotics.

Instead of the fuzzy ladder sums (1.7), it may seem preferable to study the sharp Weyl-Kuznecov sums,

\[ N_{\psi,H}^c(\lambda) := \sum_{j,k:\lambda_j \leq \lambda, |\mu_k - c\lambda_j| \leq \epsilon} \left| \int_H \varphi_j \bar{\psi}_k dV_H \right|^2, \tag{1.23} \]

in which we constrain the tangential modes \( \mu_k \) to lie in an \( \epsilon \)-window around \( c\lambda_j \) for \( 0 < c < 1 \).

**Theorem 1.22.** \( \dim M = n, \dim H = d \). Let \( 0 < c < 1 \), and assume that \( G_0^c \) is clean in the sense of Definition 1.12. Assume that no component of \( G_0^c \) has maximal dimension except for the principal component (cf. Definition 1.10). Then, in the notation of Theorem 1.17 - Theorem 1.18 the Weyl-Kuznecov sums (1.23) satisfy:

\[ N_{\psi,H}^c(\lambda) = C_{n,d} a_0^c(H,\epsilon) \lambda^{n-1} + o(\lambda^{n-1}), \]

where

\[ a_0^c(H,\epsilon) := c c^{d-1} (1 - c^2)^{n-d} \mathcal{H}^d(H), \tag{1.24} \]

where \( C_{n,d} \) is some constant depending only on the dimensions \( n \) and \( d \).

Note that the remainder estimate is much weaker than for Theorem 1.20, which has a hypothesis on the \( t \neq 0 \) singularities (i.e. on the “\( \rho \) aspect”), and has a smooth test function \( \psi \) with \( \hat{\psi} \in C_0^\infty(\mathbb{R}) \) instead of \( 1_{[-\epsilon,\epsilon]} \). Indeed, by Corollary 1.3 we get the jump estimate,

\[ N_{\psi,H}^c(\lambda_j) - N_{\psi,H}^c(\lambda_j - 0) = J_{\psi,H}^c(\lambda_j) = O_\epsilon(\lambda_j^{n-2}). \tag{1.25} \]

The worse remainder in Theorem 1.22 is due to the explicit dependence on \( \epsilon \) of the main term \( a_0^c(H,\epsilon) \), which results in two parameters having possible jumps: \( \lambda_j \) and \( \epsilon \). The \( \epsilon \)-dependence of the coefficient is discussed in Section 1.9. The large jump size reflects the new ‘layer’ of eigenvalues one gets when \( \epsilon \) hits its critical values, in cases where each ‘layer’ has the same order of magnitude as the principal layer. The layers correspond to connected components of \( G_0^c \). In the case \( c < 1 \) the components indexed by certain values of \( s \). As illustrated in the case of spheres in Section 2.4.2 there can exist large contributions from the edges (endpoints) of the interval \( \mu_k - c\lambda_j \in [-\epsilon,\epsilon] \) for special values of \( \epsilon \), i.e. \( J_{\psi,H}^c(\lambda_j) \).
itself can jump as \( \epsilon \) increases by the amount \( \lambda_j^{-2} \). But by Corollary 1.21 again, this cannot happen if \( G_M^t \) and \( G_H^s \) are aperiodic. Hence the principal component condition is necessary.

To obtain the ‘doubly sharp’ Weyl asymptotics of Theorem 1.22, we need to replace \( \psi \) by an indicator function. This is done in Section 7 by a Tauberian argument of semi-classical type adapted from Petkov-Robert [PR85].

### 1.9. The Principal Coefficient \( a^0_j(H, \psi) \) in Theorems 1.1 - 1.22

There are two aspects (roughly speaking) to the principal coefficient [1.9]: the volume aspect and the \( \psi \)-aspect.

In the case where \( \psi \) and \( \hat{\rho} \) have small support, the symbol calculations in Theorem 1.22 are contained in [WXZ20]. In this article, we use the symbol calculus of Fourier integral operators under pullback and pushforward as in [GS77, GS13] to calculate symbols.

We note that for \( 0 < c < 1 \), \( c^{d-1}(1-c^2)^{-\frac{d-2}{2}} \mathcal{H}^d(H) \) is the \( (n-2) \)-dimensional volume of the set in \( S_{H,c} \subset S_H \) projecting to \( G_{c,0}^0 \). Indeed, for each \( x \in H \), \( \pi_H \xi \in G_{c,0}^0 \) if and only if the components \( \xi = \xi^\perp + \xi^T \) of \( \xi \in S_H^c \) of the orthogonal decomposition \( T_x M = T_x H \oplus (N_x H) \) have norms \( \sqrt{1-c^2} \), resp. \( c \), so that \( S_{H,c} \simeq S_{c}^{d-1} \times S_{1-c^2} \) where \( S_r \) is the Euclidean \( k \)-sphere of radius \( r \); the \( n-2 \) dimensional surface measure of \( S_{H,c} \) is \( c^{d-1}(1-c^2)^{-\frac{d-2}{2}} \) times the \( n-2 \)-volume of \( S_{d-1} \times S_{n-d-1} \). The extra factor of \( (1-c^2)^{-\frac{d}{2}} \) is due to the density \( \frac{1}{\det D\pi_H} \) of the Leray measure relative to the Euclidean volume measure. The projection \( \pi_H : S_q^c M \to S_q^c H \) has a fold singularity where \( c = 1 \) along \( S_q^c H \) with a one-dimensional kernel, so the Leray density vanishes to order 1 when \( c = 1 \), hence the difficulties at the \( c = 1 \) interface.

Now let us check the \( \epsilon \)-dependence of \( a_{H,c}^0 \) in Theorem 1.22. This is different from the analysis above, because the spectral weight \( 1_{[-\epsilon, \epsilon]} \) has a non-compactly supported Fourier transform and indeed it has very long tails. The hypothesis of Theorem 1.22 rules out the case of \( S^d \subset S^n \) when \( c < 1 \) because in the rational case the latter has many maximal dimensional components due to fixed point sets of the periodic Hamilton flow. In the proof, we replace \( 1_{[-\epsilon, \epsilon]} \) with \( \psi_T \epsilon := \theta_T \ast 1_{[-\epsilon, \epsilon]} \) and then the principal coefficient is \( a_c(H, \psi_T, \epsilon) \), with remainder of order \( \frac{1}{\epsilon} \). To obtain the remainder estimate, we take the limit as \( T \to \infty \). By Theorem 1.18 we get \( \hat{\psi}_{T, \epsilon}(0) \) for \( c < 1 \). As \( T \to \infty \), \( \psi_T \epsilon \to 1_{[-\epsilon, \epsilon]} \), so \( \hat{\psi}_{T, \epsilon}(0) \to 2\epsilon \).

### 1.10. Remarks on Tauberian Theorems

If we think of \( \lambda_j^{-1} =: h_j \) as the Planck constant, then we are considering eigenvalues of the zeroth order operator \( h\sqrt{-\Delta_H} \) in the semi-classical thin interval \( |h_j \mu_k - c| \leq \epsilon h_j \) (whose width is one lower order than that of the operator). The Tauberian theorem of Section 7 is indeed modeled on a semi-classical Tauberian theorem. However, in several essential respects, the sharp jumps and the sharp sums in \( \lambda_j \) do not behave in a semi-classical way and the results are homogeneous rather than semi-classical. For instance, the jumps rarely have asymptotic expansions.

### 1.11. Related Results and Problems

We first compare the results to those of [WXZ20]. In comparison to this article, ‘conic’ sums

\[
N_{\epsilon,H}^{mc}(\lambda) := \sum_{j,k : \lambda_k \leq \lambda, |\psi_j| \leq \epsilon} \left| \int_H \varphi_j \overline{\psi}_k dV_H \right|^2
\]

are emphasized in [WXZ20]. The sums \( N_{\epsilon,H}^{mc} \) are wide in \( \lambda \) and ‘conic’ in \( \mu \), so they are wide in every sense. Hence, we expect to have asymptotics of these sums with no dynamical hypotheses. In [Zel92], the special (and singular) case \( c = 0 \) was studied, and indeed, the
Fourier coefficient along $H$ as fixed at 0. The $\lambda$-sums were wide and sharp. In more recent work, [SXZh17, CGT17, CG19, Wy19], short and sharp sums were considered under various geometric and dynamical hypotheses.

Some of the symbol calculations needed for this article are contained in [WXZ20]. We refer there for the calculations and do not duplicate them here.

1.11.1. Two-term asymptotics. As discussed in Section 1.7, a significant refinement of the results of this article is to prove a two-term asymptotic expansion for $N^c_{\psi,H}(\lambda)$ and to calculate $Q^c_{\psi,H}(\lambda)$. The existence of the two-term asymptotics is sketched in Section 2.8 but the calculation of $Q^c_{\psi,H}(\lambda)$ is deferred to later work.

1.11.2. $c = 1$ and $H$ totally geodesic. As indicated above, the case $c = 1$ is the edge case, and there are several different types of ‘singular’ or extremal behavior in this case. When $H$ is totally geodesic, the asymptotics for $c = 1$ are determined in the subsequent article [Z+]. It turns out that the power of $\lambda$ in Theorem 1.1 is $\lambda^{\frac{n+d}{2}}$, hence depends on $d$ as well as on $n$.

1.11.3. Submanifolds with non-degenerate second fundamental form when $c = 1$. When $c = 1$ and $H$ has non-degenerate second fundamental form, one expects diffractive or Airy type effects when $H$ is a caustic hypersurface for the geodesic flow. This type of eigenfunction concentration seems to require Airy integral operator techniques that are beyond the scope of this article. Simple examples of eigenfunctions in this sense on caustic latitude circles of $S^2$ are presented in Section 2.4.1. The example of closed horocycles of cusped hyperbolic surfaces is studied in [Wo04]. It would be interesting to see if there exist more exotic examples involving more general singularities than fold singularities.

1.11.4. Equipartition of energy among Fourier coefficients. This problem is mentioned around (1.5).

**Problem 1.** How is the mass of the Fourier coefficients distributed among the $a_n(\lambda_j)$?

In the case of ‘chaotic’ geodesic flow, it is plausible that the distribution of mass among the Fourier coefficients should be roughly constant, at least away from the endpoints of the allowed interval. It is a very difficult problem to determine if and when the Fourier coefficients are equidistributed, or when they have singular concentration, but the problem guides many of the studies of averaged Fourier coefficients.

1.11.5. Estimates for individual eigenfunctions. It would be most desirable to obtain estimates on individual Fourier coefficients of individual eigenfunctions, but again it is usually only feasible to study averages over thin windows of the Fourier coefficients when $\dim M > 2$.

**Problem 2.** What are sharp upper bounds on the individual terms (1.13). Which are the quadruples $(M, g, H, \varphi_j)$ for which the individual Fourier coefficients (1.13) are maximal? Do they have the property that $\gamma_H \varphi_j$ is an eigenfunction of $H$? In that case, (1.13) are the same as $L^2$ norms, for which sharp estimates are given in [BGT] (see Section 2.2). When does (1.12) have the same order of magnitude as (1.13)?

It may be expected that the maximal case occurs when $H$ has many almost orthogonal Gaussian beams. This occurs when $H$ is a totally geodesic subsphere of a standard sphere. Gaussian beams may be constructed around elliptic periodic geodesics, but usually as quasi-modes rather than modes. The quasi-modes give rise to jumps in the second term but not jumps in the Weyl-Kuznecov sums.
1.11.6. When is the restriction of an eigenfunction an eigenfunction? It is plausible that restrictions of individual eigenfunctions of $M$ with maximal Fourier coefficients are eigenfunctions of $H$. Otherwise its Fourier coefficients are spread out too much.

**Problem 3.** What are necessary and sufficient conditions that the restriction of a $\Delta_M$ eigenfunction is a $\Delta_H$ eigenfunction?

It is likely that this problem has been studied before, but the authors were unable to find a reference. The known examples all seem to involve separation of variables.

Examples where $\gamma_H \varphi_j$ is an eigenfunction of $H$ are standard exponentials $e^{ix \cdot \xi}$ on a flat torus, where $H$ is a totally geodesic sub-torus. Other examples are the standard spherical harmonics $Y^m_n$ on the $n$-sphere $S^n$, where $H$ is a ‘latitude sphere’ i.e. an orbit of a point under $SO(k)$ for some $k \leq n$. In Section 2 we consider various examples on the standard spheres $S^n$.

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2. Sharpness of the remainder estimates

In this section, we give examples illustrating the sharpness of the remainder estimate of Theorem 1.22 and the behavior of the jumps (1.12) and the estimate of Corollary 1.3. We also illustrate the cleanliness issues in Definition 1.12 with some examples that explain the reasons for assuming that $0 < c < 1$. The main example of jump behavior is that of totally geodesic or of latitude spheres $S^d \subset S^n$ in standard spheres. We postpone the discussion of these examples to [Z+].

As mentioned above, the behavior of remainders and jumps depends on the periodicity properties of $G^t_M$ and $G^t_H$. We begin by discussing the role of periodicities in spectral asymptotics.

2.1. Spectral clustering, jump behavior and periodicity of geodesic flows. There is a well-known dichotomy among geodesic flows and Laplace spectra which plays an important, if implicit, role in the Kuznetsov-Weyl asymptotics. Namely, if the geodesic flow $G^t_M$ of $(M, g)$ is periodic in the sense that $G^T = Id$ then the spectrum of $\sqrt{-\Delta_M}$ clusters along an arithmetic progression $\{\frac{2\pi}{T} k + \frac{T}{4}, k \in \mathbb{N}\}$ where $T$ is the minimal period and $\beta$ is the common Morse index of the closed geodesics. On the other hand, if the geodesic flow is “aperiodic” in the sense that the set of closed geodesics has Liouville measure zero in $S^* M$, then the spectrum is uniformly distributed modulo zero. We refer to [DG75] for the original theorem of this kind and to [Zel17] for further background. There also exist intermediate cases with a positive measure but not a full measure of closed geodesics.

The principal term of Theorem 1.4 (and subsequent theorems) does not depend on whether the eigenvalues cluster or are uniformly distributed, but the remainder terms and jump formulae do. In the examples of subspheres $H = S^d \subset M = S^n$, both $G^t_M$ and $G^t_H$ are periodic and both Laplacians have spectral clustering. Indeed, the eigenvalues of $\sqrt{-\Delta_{S^n}}$ concentrate along the arithmetic progression $\{N + \frac{n-1}{2}\}$ and have multiplicities of order $N^{n-1}$.
As discussed in detail in Section 2, this causes huge jumps in the \( \lambda \) aspect at eigenvalues of \( \sqrt{-\Delta_{\text{ren}}} \). Furthermore, the equation \( \mu_k = c \lambda_j \) for fixed \( (\lambda_j, c) \) can have many solutions when \( \sqrt{-\Delta_H} \) has spectral clustering, i.e. when \( G^t_H \) is periodic. The number of solutions depends on the relation between the periods and therefore on \( c \).

To be more precise, the Kuznecov-Weyl asymptotics are determined by the dimension of the ‘fixed point set’ \((1.8)\). For general \( (n,d) \) this is not literally the fixed point set of a flow, since the relevant ‘joint flow’ at \( t \) is periodic. Suppose that \( G^t_H \) is the identity map at time \( t \), \( \Phi : B^*H \to B^*H \) is the family of maps (depending on \( s \)),

\[
G^{-s}_H \circ \pi_H \circ G^s_M(q, \xi) : S^*_HM \to B^*H, \quad B^*H = \{(q, \eta) \in T^*H : |\eta|_H \leq 1\}
\]

between different spaces. In the special case where \( \dim H = \dim M - 1 \) is an oriented hypersurface, this joint flow may be considered a double-valued flow on \( B^*H \). As in \([TZ13]\), we can define lifts \( \xi_\pm : B^*M \to S^*_H M \) where \( \xi_\pm(q, \eta) \) are the two unit covectors (on opposite sides of \( T^*H \)) that lift \( (q, \eta) \) in the sense that \( \pi_H \xi_\pm(q, \eta) = (q, \eta) \). Then \((1.8)\) is the fixed point equation at \( t = 0 \) of the double-valued flow,

\[
G^{-s}_H \circ \pi_H \circ G^s_M \xi_\pm(q, \eta) : B^*H \to B^*H.
\]

When studying the singularities at \( t \neq 0 \) one has the equation \((1.15)\), which in the hypersurface case is the equation,

\[
G^{-s}_H \circ \pi_H \circ G^{s+t}_M \xi_\pm(q, \eta) = (q, \eta),
\]

\[(2.1)\]

When \( c < 1 \) and \( t = 0 \), the fixed point set has maximal dimension for \( s \neq 0 \) (i.e. if the principal component is non-dominant in the sense of Definition \((1.10)\) only if the double-valued flow is the identity map at time \( s \). The main example is when both \( G^s_H \) and \( G^s_M \) are both periodic and \( c \) is such that they have a common period. When \( c = 1 \) and \( H \) is totally geodesic, this equation holds trivially for all \( s \) and all \((q, \eta)\). Periodicity of \( G^s_H \) is a necessary and sufficient condition to obtain singularities at times \( t \neq 0 \) as strong as the one at \( t = 0 \) in the case \( c = 1 \) and \( H \) totally geodesic. In the language of \([TZ13]\), there is a first return map \( \Phi^c : S^*_HM \to S^*_HM \) defined by following geodesics with initial data in \( S^*_HM \) until they return to \( S^*_HM \). The first return time to \( S^*_HM \) is denoted by \( T^*_H \) and \( \Phi^c = G^{T^*_H}_M \). Since \( S^*_HM \) has codimension \( > 1 \) in \( B^*H \), the first return may be infinite on a large subset of \( S^*_HM \).

We have not formulated the results in terms of \( T^*_H \) or \( S^*_HM \), but they are implicitly relevant in the main results. We refer to Section 2.4 for the example of convex surfaces of revolution, where \( T^*_H \) is a constant when \( H \) is an orbit of the rotation group.

For submanifolds \( H \) of codimension \( > 1 \), the double valued lift generalizes to a correspondence taking \( \eta \in B^*_yH \) to a sphere \( S^{n-2} \) of possible covectors \( \eta + \sqrt{1 - |\eta|^2} \nu \in S^*_HM \) projecting to \( \eta \), as \( \eta \) varies over \( SN^*H \). One may still think of \((2.1)\) as the fixed point equation for a symplectic correspondence rather than a flow.

Flat tori also exhibit certain kinds of periodicities. Suppose that \( H = \{x_1 = 0\} \) is a totally geodesic coordinate slice of the flat torus \( \mathbb{R}^n/\mathbb{Z}^n \). The geodesic flow is \( G^t(x, \xi) = (x + t\frac{\xi}{\|\xi\|}, \xi) \) and it leaves invariant the tori \( T_\xi = \{(x, \xi) : x \in \mathbb{R}^n/\mathbb{Z}^n\} \subset T^*\mathbb{R}^n/\mathbb{Z}^n \). The coordinate slice defines a transversal to the Kronecker flow on each \( T_\xi \). Fixing \( |\pi_H\xi| = c \) forces \( |\xi_1| = \sqrt{1 - c^2} \). The return time of \((x, \xi) \in S^*\mathbb{R}_x^1 = \mathbb{R}^n/\mathbb{Z}^n \) to the slice on \( T_\xi \) is the time \( t \) so that \( t\xi_1 = 0 \), or \( t = (1 - c^2)^{-\frac{1}{2}} \). Thus, the return time is independent of the invariant torus and one has periodicity of the return to \( S^*_HM \) even though the geodesic flow of \( M \) fails to be periodic.

It is plausible from \((1.12)\) that \( J^c_H(\lambda_j) \) should attain its maximal size when the the multiplicity of \( \lambda_j \) is maximal and when there is clustering of the \( \sqrt{-\Delta_H} \)-spectrum \( \{\mu_k\} \).
around $\lambda_j$, forcing both geodesic flows $G_t^M$ and $G_t^H$ to be periodic. But the jump depends on the sizes of the Fourier coefficients as well as the spectrum.

To understand the general picture of jumps and remainder estimates, the reader may keep in mind some examples in the simplest case where $\dim M = 2$ and $H$ is a geodesic. Periodicity of $G_t^H$ coincides with $H$ being a closed geodesic. Let us consider four examples (see Section 2 for further discussion): (i) $M = S^2$ and $H = \gamma$ is the equator; (ii) $M$ is a convex surface of revolution and $H = \gamma$ is the equator (see Section 2.4); (iii) $M$ is a non-Zoll surface of revolution in the shape of a ‘peanut’, i.e. has a periodic hyperbolic geodesic ‘waist’ $\gamma$ and a top and bottom convex parts, each with a unique elliptic periodic geodesic; (iv) $\gamma$ is a closed geodesic of a hyperbolic surface. In all cases, the geodesic flow of $H$ is periodic. In case (i) the geodesic flow of $M$ is periodic, while in case (ii) it is not. In case (i) the multiplicity of the $N$th eigenvalue is $2N - 1$ while in case (ii) all eigenvalues have multiplicity $\leq 2$. Yet both (i)-(ii) have Gaussian beams along $\gamma$, and when $c = 1$ they are the only eigenfunctions contributing to the Kuznecov-Weyl asymptotics; hence the asymptotics are the same in both cases. Case (iii) is different in that $\gamma$ is now hyperbolic and there do not exist standard Gaussian beams along it but there does exist an eigenfunction which concentrates on $\gamma$ due to the fact that this example is quantum completely integrable. To our knowledge, the $L^2$ norm of its restriction (or equivalently, its Fourier coefficient with $|n| = \lambda$) have not been determined; In case (iv) there should not exist any such concentrating eigenfunctions. One would expect at least logarithmic improvements on the Fourier coefficient bounds, as in the case where $c = 0$ (see [WX18, SXZh17, CG19]).

2.2. Review of results on $L^2(H)$ norms of restrictions. Before discussing examples, we compare the results of Theorem 1.22 with prior results on $L^2$ norms of restrictions [BGT]. In the notation of [BGT], the estimates take the form\footnote{The notation in [BGT] is $\dim M = d, \dim H = k$.}

$$\|\varphi_\lambda\|_{L^2(H)} \leq C(1 + \lambda)^{\rho(d,n)} \sqrt{\log \lambda}\|\varphi_\lambda\|_{L^2(M)}$$

where

$$\rho(d,n) = \begin{cases} \frac{n-1}{4} - \frac{n-2}{4} = \frac{1}{4}, & d = n - 1, \\ \frac{1}{2}, & d = n - 2, \\ \frac{n-1}{2} - \frac{d}{2}, & 1 \leq d \leq n - 3 \end{cases}$$

and where the $\sqrt{\log \lambda}$ in the bound can be removed if $d \neq n - 2$. (See also [HM].)

The problem of finding extremals for restricted $L^2$ norms on submanifolds is studied in [BGT]. It is shown that extremals vary between Gaussian beams and zonal spherical harmonics depending on the pair $(n,d)$. The most difficult case is where $d = n - 2$.

When $\dim M = 2$, $\dim H = 1$, $c = 1$ and $H$ is totally geodesic, the estimates on $\|\gamma_H \varphi_j\|_{L^2(H)}^2$ and (1.12) can be the same (see [Z+]). But for $\dim M > 2$, the estimates on individual norms are significantly smaller, illustrating that (1.12) is an average and that, when $\gamma_H \varphi_j$ is not an eigenfunction of $H$ for every $j$, the Kuznecov-Weyl sums are of a different nature from $L^2$-norms of restrictions. In general, the sum (1.12) is a very thin sub-sum of (1.4) and (1.23) is a very thin sub-sum of the Weyl type function for restricted $L^2$ norms,

$$N_{L^2(H)}(\lambda) := \sum_{j: \lambda_j \leq \lambda} \int_H |\gamma_H \varphi_j|^2 dV_H. \quad (2.2)$$

Further details are given in the examples below.
2.3. Examples illustrating different types of Fourier coefficient behavior. As mentioned above, the jumps (1.12) are averages over modes of $H$, and also involve sums over repeated eigenvalues of $M$ and $H$ when there exist multiple eigenvalues. We now list some of the issues involved in relating remainder estimates on (1.12) to estimates on individual Fourier coefficients of individual eigenfunctions. The issues are illustrated on standard spheres $S^n$ in Section 2.4.1.

- **Multiplicity issues:** The eigenspace $H(\lambda_j)$ may have a large dimension $m(\lambda_j)$, so that (1.12) is an $m(\lambda_j)$-fold sum over an orthonormal basis of eigenfunctions of $H(\lambda_j)$. See Section 2.4.1 for the example of standard spheres $S^n$.

- The $\sqrt{-\Delta_H}$-eigenspaces may have large dimension, so that for each $\lambda_j$, the $\mu_k$ in (1.12) sum is over many 'Fourier coefficients.' This again is illustrated by sub-spheres of spheres (Section 2.4.1).

- **Fourier-sparsity of restricted eigenfunctions:** It may occur (and does in the case of latitude circles of $S^2$) that $\Delta_M$ has a sequence of eigenspaces of high multiplicity but, for each mode $\psi_k$ of $H$ and $\lambda$ in the spectrum of $\sqrt{-\Delta_M}$, there exists a single eigenfunction $\varphi_j$ in a given orthonormal basis of the $\lambda$-eigenspace with a non-zero $k$th Fourier coefficient $\langle \varphi_j, \psi_k \rangle$. Alternatively, for each eigenfunction $\varphi_j$ in the eigenbasis for $L^2(M)$, there might exist a single $\psi_k$ for which the Fourier coefficient is non-zero. An extreme (and interesting) case occurs when the restriction $\gamma_H \varphi_j$ of an eigenfunction of $M$ is an eigenfunction of $H$ (it is unknown when this occurs; see Section 1.11.6.)

- **Codimension of $H$.** The higher the dimension of $H$, the higher the number of eigenvalues $\mu_k : |\mu_k - c\lambda_j| \leq \epsilon$, hence the greater amount of averaging in (1.12) for fixed $\varphi_j$. In the extreme case of curves, $\dim H = 1$, the $\sqrt{-\Delta_H}$-spectrum is an arithmetic progression with large gaps, and for $\epsilon$ sufficiently small, the sum over $k$ might have just one element $\mu_k$. This eliminates the $H$-multiplicity aspect. However, there can be many $\Delta_M$-eigenfunctions which restrict to the same (up to scalar multiple) eigenfunction of $H$; see Section 2.4.1.

  The results of [BGT] reviewed in Section 2.2 show that the $L^2$ norms of restrictions of eigenfunctions decrease linearly with the dimension of $H$. This in some sense balances the additional growth rate of eigenvalues as the dimension of $H$ increases. This issue is only relevant for $c = 1$.

- **Uniformity of Fourier coefficients.** Another interesting scenario, which probably holds for compact hyperbolic surfaces at least, is where the Fourier coefficients $|\langle \varphi_j, \psi_k \rangle_{L^2(H)}|$ are uniform in size as $\mu_k$ varies in the 'allowed window' where $|\mu_k| < \lambda_j$. This is the opposite scenario from Fourier sparsity.

The sparsity phenomenon is illustrated in Section 2.4.1 for the standard 2-spheres $S^2$. In the case where $\gamma_H \varphi_j = c_{j,k} \psi_k$ for some $(j, k)$, $|\langle \gamma_H \varphi_j, \psi_k \rangle_{L^2(H)}|^2 = c_{j,k}^{-1} |\gamma_H \varphi_j||_{L^2(H)}|^2$.

2.4. Restrictions to curves in a convex surface of revolution in $\mathbb{R}^3$. In this section, we illustrate some of the possible types of Fourier coefficient behavior in the case where $H$ is a latitude circle (an orbit of the rotational action around the third axis) of a convex surface of revolution $(S^2, g)$ in $\mathbb{R}^3$ and for the joint eigenfunctions $\varphi_l^m$ of the Laplacian and of the
The geodesic flow of $(\mathbb{S}^2, g)$ is completely integrable, since rotations commute with the geodesic flow. The Hamiltonian $|\xi|_g$ of the geodesic flow Poisson commutes with the angular momentum, or Clairaut integral, $p_\theta(x, \xi) = \langle \xi, \frac{\partial}{\partial \xi} \rangle = \left| \frac{\partial}{\partial \phi} \right|_{H_{\varphi_0}} \cos \angle(\frac{\partial}{\partial \phi}, \gamma_{x, \xi}(0))$, $(x, \xi) \in T^*_x \mathbb{S}^2$. The moment map for the joint Hamiltonian action is defined by $P := \langle |\xi|, p_\theta \rangle : T^* \mathbb{S}^2 \to \mathbb{R}^2$. A level set $\Lambda_a = P^{-1}(a, 1) \subset S^* \mathbb{S}^2$ is a Lagrangian torus when $a \neq \pm 1$ and is the equatorial (phase space) geodesic when $a = \pm 1$. A ray or ladder in the image of the moment map $P$ is defined by $\{(m, E) : \frac{m}{E} = a\} \subset \mathbb{R}^2_+$, and its inverse image under $P$ is $\mathbb{R}_+ \Lambda_a \subset T^* \mathbb{S}^2$.

If $(\theta, \varphi)$ denote spherical coordinates with respect to $(M, g)$ (i.e. $\varphi$ is the distance from the north pole, $\theta$ is the angle of rotation from a fixed meridian), then an orbit of the rotation action is a latitude circle $H_{\varphi_0}$ with fixed $\varphi = \varphi_0$. We denote by $\frac{\partial}{\partial \varphi}$ the unit vector field tangent to the meridians.

The parameter $c$ is related to the values of $p_\theta$ by the formula,
\[
\frac{|p_\theta(x, \xi)|}{|\xi|} = c \left| \frac{\partial}{\partial \theta} \right|_{H_{\varphi_0}}, \quad (x \in H_{\varphi_0}). \tag{2.3}
\]

To see this, let $u_\theta(\theta, \varphi) := \frac{\partial}{\partial \theta}^{-1} \frac{\partial}{\partial \varphi}$ and let $u^*_\theta, u^*_\varphi$ be the dual unit coframe field. The orthogonal projection from $T_{H_{\varphi_0}} \mathbb{S}^2 \to T^* H_{\varphi_0}$ is given by $\pi_{H_{\varphi_0}}(x, \xi) = \langle \xi, u^*_\varphi \rangle u^*_\theta$, and (2.3) follows. The reason that the parameter $c$ is not the usual ratio $\frac{p_{\theta}(x, \xi)}{|\xi|}$ is because we choose the operator on $H$ to be $\sqrt{\Delta_H}$ rather than $\frac{\partial}{\partial \theta}$.

We now show that the first return time $T^c_H$ defined in Section 2.1 is a constant when $H$ is a latitude circle of a surface of revolution. This is because the rational in (2.3) between $c$ and $p_{\theta}$ is constant on a latitude circle. Since $p_{\theta}$ is constant along geodesics, an initial vector in $S^*_{H_{\varphi_0}} \mathbb{S}^2$ at time zero will return to $S^*_{H_{\varphi_0}} \mathbb{S}^2$ each time the geodesic returns to $H$. Moreover, in the setting of curves on surfaces, $S^*_{H_{\varphi_0}} \mathbb{S}^2$ is a cross section to the geodesic flow, hence the first return time to $S^*_{H_{\varphi_0}} \mathbb{S}^2$ is finite almost surely. Given that $p_{\theta}$ is constant on orbits, it follows that $T^c_{H_{\varphi_0}}$ is constant too.

The flow on $H_{\varphi_0}$ is of course periodic as well of period $\frac{2\pi}{L}$ where $L$ is the length of $H_{\varphi_0}$. It follows that the equation (2.1) can have fixed point sets of maximal dimension when $t \neq 0$ on a convex surface of revolution, despite the fact that the geodesic flow itself is not periodic. Indeed, $G^{-s}_H \circ \pi_H \circ C^{s+t}_M(q, \xi) = (q, \xi)$ for any $(q, \eta) \in S^*_{H_{\varphi_0}}$ if $t = T^c_{H_{\varphi_0}}$.

We now introduce notation for quantum ladders. Let $\varphi^m_\ell$ be the standard orthonormal basis of joint eigenfunctions of $\Delta$ and of the generator $\frac{\partial}{\partial \varphi}$ of rotations around the third axis. The orthonormal eigenfunctions of $H_{\varphi_0}$ are given by $\psi_m(\theta) = C_{\varphi_0} e^{im\theta}$ where $C_{\varphi_0} = \frac{1}{L(\varphi_0)}$. Hence, the Fourier coefficients (1.1) are constant multiples of the Fourier coefficients relative to $\{e^{im\theta}\}$. It follows that the $m$th Fourier coefficient of $\varphi^m_\ell$ is its only non-zero Fourier coefficient along any latitude circle $H_{\varphi_0}$, and that $\int_{H_{\varphi_0}} \varphi^m_\ell e^{-im\theta} d\theta|^2 = |C_{\varphi_0}|^2 |\ell|^2$.

On the quantum level, a ray corresponds to a ‘ladder’ $\{\varphi^m_\ell\}_{m=a}$ of eigenfunctions. The possible Weyl-Kuznecov sum formulae for latitude circles $H = H_{\varphi_0}$ thus depend on the two parameters $(\varphi_0, \frac{\ell}{L})$. The first corresponds to a latitude circle, the second to a ladder in the joint spectrum. It is better to parametrize the ladder as $\frac{\ell_{im}}{L} = c$ as discussed above.
2.4.1. The standard $S^2$. The standard sphere $(S^2, g_0)$ is of course a special case of a surface of revolution, and the joint eigenfunctions $\varphi^m_\ell$ are denoted by $Y^m_\ell$. The special feature of the standard sphere is that its geodesic flow is periodic and the eigenvalues of the Laplacian have dimensions $2N+1$. This gives it the special properties discussed in the next subsection.

2.4.2. Fourier sparsity phenomena. In the case of $S^2$, we slightly re-adjust the definition of $\sqrt{-\Delta}$ to $\sqrt{-\Delta + \frac{1}{4} - \frac{1}{2}}$, whose eigenvalues are $\lambda_N = N$. Also, $\mu_m = m \in \mathbb{Z}$. Suppose that $c \in \mathbb{Q}_+$ and write it in lowest terms as $c = \frac{p}{q}$ with $(p, q) = 1$. Then let $\epsilon > 0$ and consider the set $\{ (m, N) : |m - \frac{p}{q} N| < \epsilon \} = \{ (m, N) : |\frac{m}{N} - \frac{p}{q}| < \frac{\epsilon}{N} \}$. Roughly, this is the set of lattice points inside a strip of width $\frac{1}{n}$ around the ray through $(0, 0)$ of slope $\frac{p}{q}$. Of course, the lattice points $\{ k(p, q), k \in \mathbb{N} \}$ lie in the strip. But for other lattice points, $|\frac{m}{N} - \frac{p}{q}| = |\frac{mq-Np}{Nq}| \geq \frac{1}{Nq}$, so there are no solutions aside from the lattice points on the rational ray if $\epsilon < \frac{1}{q}$. Moreover, the possible ‘gaps’ $\{ m - \frac{p}{q} N \}$ in this example are $\geq \frac{1}{q}$. Hence, when $n = 2, d = 1$ the remainder terms in Theorem 1.1 and elsewhere only sum over one eigenvalue of $H$ and the magnitude of $(1.12)$ is the magnitude of the extremal Fourier coefficient of a restricted eigenfunction.

2.4.3. $H$ is a closed geodesic of $S^2$ and $c < 1$. Let $M = S^2$ and let $H$ be a closed geodesic $S^2$. It is always the case that $\dim G^0_\ell = \dim S^\ell H = 1$ in the case of $S^2$. In the rest of this section, we assume $0 < c < 1$.

For concreteness suppose that $H$ is a meridian through the north pole $p$. Then for any $\xi \in S^*_p S^2$, $G^\sigma_s(p, \xi) \in S^*_H S^2$ and $\exp_p(\pi \xi) = -p$. The same holds for any $p$ on the meridian geodesic. In this case, $G^\sigma = G^0_\ell$ if $H$ is totally geodesic and $c < 1$. There exist $(c,s,0)$ bi-angles if $s$ is a common period for $G^s_H$ and $G^s_{0,\mathbb{M}}$. For fixed $c < 1$, the $L^2$ norms of the restrictions of spherical harmonics $Y^m_\ell$ with $\frac{m}{N} \simeq c < 1$ to $H$ are uniformly bounded above, and therefore so are their $m$th Fourier coefficients. This is consistent with Corollary 1.3. On the other hand, $G^{0,0}$ is non-dominant due to periodicity of the geodesic flow, and one cannot improve the remainder estimates. We refer to [Geis] for a recent study of how the restricted $L^2$ norms vary with $c$.

On the other hand, if we restrict $Y^m_\ell$ to a meridian geodesic, then all the Fourier coefficients in the range $[-\ell, \ell]$ can be non-zero. We now show that the squares $\left| \int_H Y^0_N(\varphi) e^{-i \frac{N}{T} \varphi} d\varphi \right|^2$ of the Fourier coefficients of the zonal spherical harmonic along a meridian geodesic are bounded above and below by positive constants, proving that Corollary 1.3 is sharp for $(n = 2, d = 1)$.

Let $Y^0_N(\theta, \varphi) = \sqrt{(2N+1)} P^0_N(\cos \varphi)$. be the zonal spherical harmonic on $S^2$. Here, $P^0_N(\cos \varphi) = \bar{P}_N(\cos \varphi)$ is a normalized Legendre polynomial. Let $H$ be a meridian geodesic through the poles of $Y^0_N$. The Fourier coefficients of $\gamma_HY^0_N$ are known explicitly [HP] (2.6)-(2.7a)-(2.7b)]. To quote one special value,

$$P_N(\cos \varphi) = \sum_{k=0}^{N} \bar{p}_k p_{N-k} \cos(n-2k) \varphi,$$

where $p_j = 4^{-j} (\frac{2j}{j})$. Multiplying by $2N + 1$ shows that the $L^2$ norm square of $Y^0_N$ is the the partial sum of the harmonic series and equals $\log N + \gamma$, where $\gamma$ is Euler’s constant (this
calculation was first done in [T09] by a different method. On the other hand,
\[
\sum_{k:|k-cN|<\epsilon} \left| \int_H Y^0_N(\varphi)e^{-ik\varphi}d\varphi \right|^2 = (2N + 1) \sum_{k:|k-cN|<\epsilon} |p_{N-k}p_{N+k}|^2, \tag{2.4}
\]
for any \( \epsilon > 0 \) and \( c \in (0, 1] \). Since the number of terms in the sum is bounded for fixed \( \epsilon > 0 \), it suffices to calculate one term \(|p_{N-k}p_{N+k}|^2\) asymptotically by Stirling’s formula, and for simplicity of exposition we only calculate the middle case with \( c = \frac{1}{2} \) and for \( N = 2n \) even. Using that \((\frac{2}{j}) \simeq 2\frac{2^j}{\sqrt{j}}\)
\[
p_{2n-n}p_{2n+n} = p_n p_{3n} = C_0 n^{-1} 4^{-n} 2^{2n} 4^{-3n} 2^{6n} = C_0 n^{-1} > 0
\]
for a certain constant \( C_0 > 0 \). Multiplying by \((2N + 1)\) shows that \( \left| \int_H Y^0_N(\varphi)e^{-i\frac{2\pi}{N}\varphi}d\varphi \right|^2 \)
is asymptotically a positive constant, corroborating Corollary 1.3. Essentially the same calculation is valid for any \( 0 < c < 1 \) (see [Stan] for the relevant binomial asymptotics).

2.4.4. Gaussian beam sequences: \( c = 1 \) and \( H \) is totally geodesic. Another extremal scenario occurs when \( c = 1 \) and \( H \) is totally geodesic, where the classical ray occurs on the boundary of the moment map image. The corresponding ladder of eigenfunctions consists of the Gaussian beams, \( C_0 N^\frac{1}{4} (x + ix) N^N \), around the equator \( \gamma \). The standard Gaussian beams (highest weight spherical harmonics) \( \{Y^N_N\}_{N=0}^\infty \) are then a semi-classical sequence of extremals. Their restrictions to the equator \( \varphi = \frac{\pi}{2} \) are equal to \( C_0 N^{1/4} e^{iN\theta} \). The only non-zero Fourier mode is the \( N \)th, and that Fourier coefficient is of magnitude \( N^{1/4} \). The growth rate of the Kuznecov-Fourier sum (1.23) is that of \( \sum_{N:N\leq N} N^{1/2} \simeq \frac{N^{3/2}}{N} \) with remainder of order \( N^{-1/2} \). This situation is studied systematically in [Z+].

2.4.5. Caustic sequences. We briefly mention the case where where \( c = 1 \) and \( H \) has non-degenerate second fundamental form, although they are not studied in this article. In this case, there exist caustic effects which dominate the estimate of Fourier coefficients. The simplest example is the restriction of the standard spherical harmonics \( Y^N_N \) to non-geodesic latitude circles, where the Fourier coefficients of certain sequences blow up at the rate \( N^{1/6} \). Such caustic effects on restrictions of eigenfunctions will be investigated systematically in a future article.

2.5. Higher dimensional spheres \( S^n \). We denote by \( \Pi_N^S(x, y) \) the degree \( N \) spectral projections kernel on \( S^n \). To verify the sharpness of Corollary 1.3 we will need to use some explicit formulae for this kernel. We follow [AH] for notation and refer there for the proofs.

In a well-known way, we slightly change the definition of \( \sqrt{-\Delta} \) to obtain operators \( A^S_n \) with positive integer eigenvalues:
\[
A^S_n = \sqrt{-\Delta^S_n + \frac{(n - 1)^2}{4} + \frac{n - 1}{2}}.
\]

We replace \( \{\lambda_j\}_{j=1}^\infty \) and \( \{\mu_k\}_{k=1}^\infty \) by \( \mathbb{N} \), denoting the eigenvalues of \( A^S_n \) by \( \{N\}_{N=0}^\infty \) and those of \( A^d \) by \( \{M\}_{M=0}^\infty \). Let \( \Pi_N^S \) denote the spectral projections onto the \( N \)th eigenspace \( \mathcal{H}_N^S \) of \( A^S_n \), i.e. the orthogonal projection onto the space of spherical harmonics of \( S^n \) of degree \( N \).

The submanifolds \( H \) we consider are sub-spheres (with standard metrics) of \( S^n \). The notation for totally geodesic and latitude subspheres is as follows. In standard Euclidean

\footnote{The pair \((d, n)\) in [AH] corresponds to \((n + 1, N)\) in this article.}
coordinates on $\mathbb{R}^{n+1}$, the unit sphere is defined by $\sum_{j=1}^{n+1} x_j^2 = 1$. We define a latitude $d$-sphere of height $a$ to be the subspaces

$$S^{n,d}(a) := \{ \vec{x} \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1, \sum_{j=d+2}^{n+1} x_j^2 = a^2 \}.$$ 

Henceforth, we drop the superscript $n$ when the dimension is understood. $SO(d + 1)$ act by isometries on $S^n$ and all latitude sub-spheres $H = S^d$ are orbits of the action.

The sup-spheres $S^d(a)$ are totally geodesic when $a = 0$ and are not totally geodesic if $a > 0$. In particular, if $d = 1$ we obtain the closed geodesic,

$$\gamma := S^{n,1}(0) := \{ \vec{x} \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1, \sum_{j=3}^{n+1} x_j^2 = 0 \},$$

which is the intersection $S^n \cap \mathbb{R}^2_{x_1,x_2}$ where $\mathbb{R}^2_{x_1,x_2} \subset \mathbb{R}^{n+1}$ is the plane $x_3 = \cdots = x_{n+1} = 0$.

The eigenspaces $\mathcal{H}^{S^n}_{N}$ of $\Delta$ on $S^n$ are spaces of degree $N$ spherical harmonics, i.e. restrictions to $S^n$ of homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$. We denote the orthogonal projection onto $\mathcal{H}^{S^n}_{N}$ by

$$\Pi^{S^n}_{N} : L^2(S^n) \to \mathcal{H}^{S^n}_{N}.$$ 

As is well-known, $D^n_N := \dim \mathcal{H}^{S^n}_{N} \sim C_n N^{n-1}$ (see e.g. \cite{StW}.) The more interesting calculation occurs when we restrict $\Pi^{S^n}_{N}(x,y)$ to the $SO(d + 1)$ invariant coordinate sub-sphere $S^d$ in $x, y$ and and sift out one ‘Fourier coefficient’ of the sub-sphere, i.e. one degree of spherical harmonic. We denote the restriction (in both variables) of (2.5) to $S^d \times S^d$ by

$$\gamma_{S^d(a)} \Pi^{S^n}_{N} \gamma_{S^d(a)}(x,y) = [\gamma_{S^d} \otimes \gamma_{S^d} \Pi^{S^n}_{N}](x,y), \quad (x,y \in S^d).$$ 

When integrating over $S^d$ it is clear that the variables are restricted to this submanifold and we drop the restriction operators $\gamma_{S^d}$ for simplicity of notation.

For any $0 < c \leq 1$ and $N$, and for $\epsilon$ sufficiently small (depending on $c$), there exists at most one $M = M(N, c)$ satisfying $|M - cN| < \epsilon$. We always assume that $\epsilon$ is chosen this way. In fact, as discussed in Section 2.4.2 there might not exist any such $M$ for small $\epsilon$. To illustrate this, suppose $c = \frac{1}{2}$. Then only even $N$ will contribute and $M = N/2$. If $N$ is odd, there does not exist any $M$ within $\epsilon$ of $cN$, so that (1.12) is zero for odd $N$ and non-zero for even $N$.

We now give an explicit formula for the jumps (1.12) when $S^d \subset S^n$ is any latitude subsphere.

**Lemma 2.1.** Fix $c \in (0, 1]$ and $(M,N,\epsilon)$ that one (and only one) degree $M(c,N)$ satisfies $|M - cN| < \epsilon$, one has

$$J^c_{S^d}(N) := \int_{S^d} \int_{S^d} \Pi^{S^n}_{N}(x,y) \Pi^{S^d}_{M(c,N)}(x,y)dV_{S^d}(x)dV_{S^d}(y).$$

**Proof.** For each fixed tangential mode $\psi_k$, the sum over $\ell$ in (1.12) takes the form,

$$\sum_{\ell, \lambda = N} \left| \int_{H} \varphi_{\ell,\lambda} \overline{\psi_k} dV_{H} \right|^2 = \int_{S^d} \int_{S^d} \Pi^{S^n}_{N}(x,y) \overline{\psi_k(x)} \psi_k(y) dV_{S^d}(x)dV_{S^d}(y),$$

and the sum over all $k$ giving the jump (1.12) is given by (2.1). 

\[\square\]
Thus, there exists at most one full projector $\Pi_{M(c,N)}^d$ of $S^d$ contributing to the sum if $\epsilon < \frac{\xi}{2}$ is sufficiently small that each term of the arithmetic progression $\{cN\}_{N=1}^\infty$ is $\epsilon$-close to only one integer $M(N,c)$. Note that $\gamma_{S^d(\alpha)}\Pi_{N}^{\alpha}\gamma_{S^d(\alpha)}(x,y)$ is not a spherical harmonic on $S^d$ in either variable, but is a sum of harmonics of degrees 0, \ldots, $N$. The integrals in Lemma 2.1 are special kinds of Clebsch-Gordan integrals. For $n = 2$, the integrals are evaluated in (2.4) when $H$ is a geodesic. To our knowledge, the integrals have not been studied for subspheres of higher dimensional spheres, although they are not hard if $d = 1$ (see [Z+]).

For general $n$ and $d = 1$, we restrict to $x_3 = x_4 = \cdots = x_{n+1} = 0$. When $0 < c < 1$ is rational, we choose $M < N$ with $\frac{M}{N} = c$ and assume $\epsilon$ small enough so it is the unique integer satisfying $|M - cN| < \epsilon$. The restricted Fourier coefficient is then,

$$\int_{S^1}\int_{S^1} \Pi_N^{S_\alpha}(\langle x(\theta_1), y(\theta_2) \rangle))e^{-iN(\theta_1 - \theta_2)}d\theta_1d\theta_2.$$

Let $SO(2) \simeq S^1 \subset SO(d)$ denote the 1-parameter subgroup so that the orbit of $\epsilon_1$ is the circle above. In the space $H_N^d$, the integral sifts out the orthogonal projection $\Pi^M_N$ to the subspace $H^d_N$ of spherical harmonics which transform by $e^{iM\theta}$ when translated by the circle action. The integral equals $\gamma_{\Pi^d_N}^{\alpha}\gamma_{S^d(\alpha)}^*(x,x)$ where $x \in S^1$ is any point. Since $\dim H^d_N(S^d) = C_nN^{-1}$, its order of magnitude is $N^{-1}$. This proves that the statement of Theorem 1.22 is sharp for general $n$ and $d = 1$.

For general $(n,d)$ with $d > 1$, explicit estimates for the Fourier coefficient sums of restrictions to latitude spheres for $S^n$ when $n > 3$ are much more difficult by means of classical analysis (see Section 2.4.3 for $n = 2$).

2.6. Flat tori. Let $H$ be a $d$-dimensional coordinate plane in the $n$-dimensional torus with the usual eigenfunctions $\varphi_j(x) = \exp(ij \cdot x)$ and $\psi_k(x') = \exp(ik \cdot x')$. Here, $x = (x_1, \ldots, x_n) = (x', x'')$ with $x' = (x_1, \ldots, x_d)$ and $x'' = (x_{d+1}, \ldots, x_n)$. Then each $|\langle H^d\varphi_j, \psi_k \rangle|_{L^2(H)}$ is a power of $2\pi$ times $\delta_{j'k}$. The ladder sum now just counts the lattice points $j \in \mathbb{Z}^n$ such that $|\epsilon_j - j'| < \epsilon$ and $|j| \leq \lambda$. When $0 < c < 1$, this region is asymptotic to an $\epsilon$ thickening of a (codimension 1) cone in $\mathbb{R}^n$ and has volume to the order of $\lambda^{n-1}$. Now run this construction again except replace the sharp cutoff by a fuzzy cutoff. The main term of the ladder sum agrees (up to a constant and a lower order term) with the volume of this thickened cone, $\lambda^{n-1}$, which does not depend on $d$.

To illustrate the necessity of the hypotheses of Theorem 1.22, we consider the two-dimensional case. We write $j = (j_1, j_2) \in \mathbb{Z}^2$ and observe

$$N_{\epsilon,H}(\lambda) = (2\pi)^{-1}\# \{j \in \mathbb{Z}^2 : |j| \leq \lambda, \|j_1| - c|j_2| \leq \epsilon \}.$$

The region capturing the lattice points in the set above,

$$\{\xi \in \mathbb{R}^2 : \|\xi_1| - c|\xi_2| \leq \epsilon \}, \quad (2.8)$$

has hyperbolas for boundaries and is asymptotically

$$\left\{\xi \in \mathbb{R}^2 : \sqrt{1 - c^2}|\xi_1| - c|\xi_2| \leq \frac{\epsilon \sqrt{1 - c^2}}{\sqrt{1 - c^2}}\right\},$$

the union of two strips of slope $\pm\frac{\sqrt{1-c^2}}{2\epsilon}$ and thickness $\frac{2\epsilon}{\sqrt{1-c^2}}$. We conclude that the area of the region (2.8) within the the ball of radius $\lambda$ is asymptotic to

$$8\epsilon(1 - c^2)^{-1/2}\lambda,$$
which is consistent with the main term of Theorem 1.22. However, if the slope $c/\sqrt{1-c^2}$ is rational, we may carefully select two different values of $\epsilon$ which yield exactly the same count of points for $N_{c,H}^*(\lambda)$. The difference in the main terms must be absorbed into the remainder. Hence, the improved remainder in Theorem 1.22 may not be obtained in this setting.

2.7. Hyperbolic quotients. Fourier coefficients of restrictions of eigenfunctions on hyperbolic surfaces to closed geodesics, to horocycles or distance circles is a classical problem in automorphic forms. To our knowledge, the only case studied rigorously to date is that of cuspidal eigenfunctions $\psi$ of the modular curve $\mathbb{H}^2/SL(2,\mathbb{Z})$ to a closed horocycle $H_y$ of ‘height’ $y$ [Wo04, Page 428]. This is a case where $c = 1$ and caustic effects occur, and the estimates of [Wo04] are of the same nature as the $N^{1/6}$ estimate for spherical harmonics in the caustic case. It seems likely that in the negatively curved case, one can improve such estimates by powers of $\log N$. Such effects will be studied in future work.

2.8. Singularities for $t \neq 0$. In this section, we discuss the existence of two-term asymptotics discussed in Section 1.7.

To determine two-term asymptotics, it is necessary to allow the support of $\hat{\rho}$ to be any finite interval and to calculate the contribution of all sojourn times $t \in \Sigma_c(\psi)$ with fixed point sets of maximal dimension. Thus, we need to determine the connected components of all $(c,s,t)$ bi-angles for general $t$, i.e. solve (1.7) for all $(s,t)$ with $s \in \text{supp} \hat{\psi}$ and to locate the maximal components with $t \neq 0$ of the same dimension as the principal component at $t = 0$.

We now assume further that the set of $(c, s, t)$-bi-angles with $t = T \in \text{singsupp } S^c(t, \psi) \{0\}$ is a union of clean components $Z_j(T)$ of dimension $d_j(T)$, where $Z_j(T)$ is a component of $G^c_t$. Then, for $t$ sufficiently close to $T$, there exist Lagrangian distributions $\beta_j$ on $\mathbb{R}$ with [2.6] singularities only at $t = 0$ such that,

$$S^c(t, \psi) = \sum_j \beta_j(t - T), \quad \beta_j(t) = \int_{\mathbb{R}} \alpha_j(s) e^{-ist} ds,$$

with $\alpha_j(s) \sim \frac{(\pi)}{2\pi i}^{-1+\frac{1}{2}(n-d)+\frac{d_j(T)}{2}} i^{-\sigma_j} \sum_{k=0}^{\infty} \alpha_j,k s^{-k},$

where $d_j(T)$ is the dimension of the component $Z_j(T)$. We refer to Section 2.1 for background on the role of periodicity properties of $G^c_M$ and $G^c_H$ in the existence of maximal components $Z_j(T)$ for $T \neq 0$, i.e. in whether “fixed point sets” defined by (1.16) and (2.1) can be of the same full dimension as for the principal component at $t = 0$. When $c < 1$, $G^{cs+t}_M$ must take $S^c_q M \to S^c_q M$ for every $q \in H$, and moreover must map each set $S^c_{q,\eta} M := \{ \xi \in S^c_q M : \pi_H \xi = \eta \}$ to $B^s H$ into itself.

**Proposition 2.2.** Let $\rho \in \mathcal{S}(\mathbb{R})$ with $\hat{\rho} \in C^\infty_c$ and with $0 \notin \text{supp} \hat{\rho}$. Assume that the bi-angle equation is clean in the sense of Definition 1.12 and let $S_\psi = \text{singsupp } S^c(t, \psi) \{0\}$. Denote by $d_j$ the dimension of a component $Z_j$ of $G^c_t$ where $t$ is a non-zero period. Then, there exists $\beta_j \in \mathbb{R}$ and a complete asymptotic expansion,

$$N_{c,\psi,H}^c(\lambda) \sim \lambda^{-1+\frac{1}{2}(n-d)} \sum_{T \in S(\psi)} \sum_{\ell=0}^{\infty} \beta_\ell(t - T) \lambda^\frac{-d_j(T)}{2} - \ell,$$

The asymptotics are of lower order than the principal term of Theorem 1.1 (resp. Theorem 1.18) unless there exists a maximal component.
To obtain two-term asymptotics of the type discussed in Section 1.7, one needs to specify the maximal components, and to calculate the associated $\beta_\ell$ and $\alpha_j$ in geometric terms. In effect, the function $Q_{\psi,H}(\lambda)$ is a sum over the maximal components for all $t \neq 0$. Its calculation is postponed to a subsequent study.

3. Fuzzy ladder projectors and Kuznecov formulae

In this section, we set up the main objects in the proof Theorem 1.16. We use the terminology of the ‘fuzzy ladders’ of [GU89] to describe the main operators and their canonical relations. However, there are some significant differences in that we consider ‘fuzzy’ ladders with respect to two elliptic operators with erratically distributed eigenvalues, rather than with respect to a compact group such as $S^1$ in [GU89] with a lattice of eigenvalues.

3.1. Notation. Since we are often dealing with operators on product spaces, we use the notation $f \otimes g$ for a function on $X \times Y$ of the product form $f(x)g(y)$. Linear combinations of such functions are of course dense in $L^2(X \times Y)$ and it suffices to define operators on product spaces on such product functions.

We introduce the two commuting operators on $M \times H$,
\[
P_M = \sqrt{-\Delta_M} \otimes I \quad \text{and} \quad P_H = I \otimes \sqrt{-\Delta_H}.
\]
and denote an orthonormal basis of their joint eigenfunctions by
\[
\varphi_{j,k} = \psi_k \otimes \varphi_j.
\]
(3.1)

Thus, we have
\[
(P_M, P_H)\varphi_{j,k} = (\lambda_j, \mu_k)\varphi_{j,k}.
\]
As discussed in [GU89], $P_M$ and $P_H$ are not quite pseudodifferential operators due to singularities in their symbols on $0_H \times T^*M \cup T^*H \times 0_M$, where $0_M$ denotes the zero section. These singularities lie far from the canonical relations determining the asymptotics and therefore may be handled by suitable cutoffs as in [GU89].

We then introduce the operators on $C^\infty(M \times H)$ defined by,
\[
\begin{cases}
P := P_M := \sqrt{-\Delta_M} \otimes I, \\
Q_c := c\sqrt{-\Delta_M} \otimes I - I \otimes \sqrt{-\Delta_H} = cP_M - P_H.
\end{cases}
\]
(3.2)
The system $(P, Q_c)$ is elliptic; $Q_c$ is a non-elliptic first order pseudo-differential operator of real principal type with characteristic variety,
\[
\text{Char}(Q_c) : \{(x, \xi, q, \eta) \in T^*M \times T^*H : c|\xi|_g - |\eta|_{g_H} = 0\}.
\]
(3.3)

As in [GU89] we are interested in the “nullspace” of $Q_c$, i.e. its 0-eigenspace. The corresponding pairs of eigenvalues of $(P_M, P_H)$ would concentrate along a ray of slope $c$ in $\mathbb{R}^2$. Except in rare situations with symmetry, there are at most finitely many such eigenvalue pairs, but there always exist an approximate or ‘fuzzy’ null-space, intuitively defined by a strip around the ray or ladder in the joint spectrum of the pair $(P_M, P_H)$. A ‘ladder strip’ as in [GU89] is defined by a strip around a ray in the joint spectrum,
\[
\{(\lambda_j, \mu_k) : |\mu_k - c\lambda_j| \leq \epsilon\} \subset \mathbb{R}^2_+.
\]
It is usually difficult to study a strip directly, and in place of the indicator function of a strip one constructs Schwartz test functions which concentrate in the strip and are rapidly
decaying outside of it in \( \mathbb{R}^2 \). When one uses such a test function rather than an indicator function one gets a ’fuzzy ladder.’

Such ladders, sharp or truly fuzzy, arise when one studies eigenvalue ratios \( \frac{\mu_k}{\lambda_j} \) in intervals of width \( O(\lambda_j^{-1}) \). These are very short intervals and it is much more difficult to obtain asymptotics for such short intervals than for intervals of constant width. To this end, one can study wedged (or coned) Weyl sums. Asymptotics for cones and some slowly thickening ladders were obtained in [WXZ20], as well as ladders which are sufficiently fuzzy as in Theorem 1.1. In this article, the aim is to obtain improved asymptotics for both sharp and fuzzy ladders.

3.2. Fuzzy ladder projectors. To prove Theorem 1.16 by Fourier integral operator methods, we need to smooth out the projection operators onto the spectral subspaces corresponding to these ladders. We therefore introduce \( \psi \in \mathcal{S}(\mathbb{R}) \) with \( \hat{\psi} \in C_0^\infty(\mathbb{R}) \) and define,

\[
\psi(Q_c) : L^2(M \times H) \to L^2(M \times H),
\]

\[
\psi(Q_c) = \int_{\mathbb{R}} \hat{\psi}(s)e^{isQ_c}ds = \sum_{j,k} \psi(\mu_k - c\lambda_j)\varphi_{j,k} \otimes \varphi_{j,k}^*.
\]

(3.4)

Here, \( \varphi_{j,k}^* \) denotes the linear functional dual to \( \varphi_{j,k} \) in \( L^2(M \times H) \), and hence \( \varphi_{j,k} \otimes \varphi_{j,k}^* \) is the rank-1 projector onto the line spanned by \( \varphi_{j,k} \). \( \psi(Q_c) \) is sometimes denoted \( \Pi_c \) to emphasize that it is an approximate projector.

The operator \( (3.4) \) is a smoothing of the sharp fuzzy ladder projection of \( L^2(M \times H) \) onto the span of the joint eigenfunctions for which \( |\mu_k - c\lambda_j| \leq \epsilon \) for some \( \epsilon > 0 \). It is a Fourier integral operator of real principal type, whose properties we now review. The characteristic variety of \( Q_c \) is the hypersurface \( T^*M \times T^*H \) defined by \( (3.3) \). Its characteristic (null) foliation is given by the integral curves of the Hamiltonian \( c|\xi|_g - |g|_{gh} \), i.e. the orbits of the flow \( G^c_{M} \otimes G^{-s}_{H} \) on \( T^*M \times T^*H \) restricted to the level set \( \text{Char}(Q_c) \). The next Lemma is similar to calculations in [GU89] and [TU92, Proposition 2.1]:

**Lemma 3.1.** \( \psi(Q_c) \) of \( (3.4) \) is a Fourier integral operator in the class \( \mathcal{F}_1^{\frac{1}{2}}((M \times H) \times (M \times H)) \), \( \mathcal{I}_\psi^{(s)} \) with canonical relation

\[
\mathcal{I}_\psi^{(s)} := \{(x, \xi, q, \eta; x', \xi', q', \eta') \in \text{Char}(Q_c) \times \text{Char}(Q_c) : \exists s \in \text{supp} \psi \text{ such that } G^c_{M} \times G^{-s}_{H}(x, \xi, q, \eta) = (x', \xi', q', \eta') \}.
\]

The symbol of \( \psi(Q_c) \) is the transport of \( (2\pi)^{-\frac{1}{2}}\hat{\psi}(s)|ds|^{\frac{1}{2}} \otimes |d\mu_L|^{\frac{1}{2}} \) via the implied parametrization \( (s, \zeta) \mapsto (\zeta, G^c_{M} \times G^{-s}_{H}(\zeta)) \), where \( \mu_L \) is Liouville surface measure on \( \text{Char}(Q_c) \).

Since we often use the term Liouville measure we give the general definition.

**Definition 3.2.** Let \( Y \subset T^*M \) be a hypersurface defined by \( \{f = 0\} \). By the Liouville measure on \( Y \), we mean the Leray form \( \frac{\Omega}{\det f} \) on \( Y \), where \( \Omega \) is the symplectic volume form of \( T^*M \).

We now prove Lemma 3.1.
Proof. Let \( \zeta = (x, \xi, q, \eta) \in T^*(M \times H) \). Since \( Q_c \) is of real principal type, we may apply [29] Proposition 2.1 to obtain that \( \psi(Q_c) \in I^{-\frac{1}{2}}((M \times H) \times (M \times H), \mathcal{I}_\psi^c) \), where

\[
\mathcal{I}_\psi^c = \{(\zeta_1, \zeta_2) \in \dot{T}^*(M \times H) \times \dot{T}^*(M \times H) : \sigma_{Q_c}(\zeta_1) = 0, \exists s \in \text{supp} \hat{\psi}, \exp sH_{\sigma_{Q_c}}(\zeta_1) = \zeta_2 \}. \tag{3.5}
\]

The Hamiltonian flow \( \exp sH_{\sigma_{Q_c}} \) on the characteristic variety is given by

\[
G_M^{cs} \times G_H^{-s} : \text{Char}(Q_c) \to \text{Char}(Q_c).
\]

By [29] Lemma 2.6, the principal symbol of \( \psi(Q_c) \) is \( (2\pi)^{-\frac{1}{2}}\hat{\psi}(s)|ds|^{\frac{1}{2}} \otimes |d\mu_L|^{\frac{1}{2}} \). \( \square \)

Remark 3.3. Without the support condition, the wave front relation is an equivalence relation on points of \( \text{Char}(Q_c) \), namely \( (x, \xi, q, \eta) \sim (x', \xi', q', \eta') \) if they lie on the null bicharacteristic.

3.3. Elliptic cutoff. Since \( \psi(Q_c) \) is not elliptic, we introduce a second smooth cutoff \( \rho \in \mathcal{S}(\mathbb{R}) \), with \( \hat{\rho} \in C_0^\infty \) and define

\[
\rho(P - \lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(t)e^{-it\lambda}e^{itP}dt.
\]

Thus, \( \rho(P - \lambda)\psi(Q_c) : L^2(M \times H) \to L^2(M \times H) \) is the operator,

\[
\rho(P - \lambda)\psi(Q_c) = \sum_{j,k} \rho(\lambda_j - \lambda)\psi(\mu_k - c\lambda_j)\varphi_{j,k} \otimes \varphi_{j,k}^* \tag{3.6}
\]

To understand its purpose, we note that if the cutoff \( \rho \) were the indicator function of an interval \([−1, 1]\), it would restrict the \( \lambda_j \) to \([\lambda - 1, \lambda + 1]\), while if \( \psi \) were an indicator function, it would restrict the \( \mu_k \) to \(|\mu_k - c\lambda_j| \leq A \). Hence, the pair of cutoffs would restrict the joint spectrum to a rectangle. The smooth cutoffs \( \psi, \rho \) should be thought of as smoothings of such indicator functions.

By Fourier inversion, (3.6) is given by

\[
\rho(P - \lambda)\psi(Q_c) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\rho}(t)e^{-it\lambda}e^{itP}\psi(Q_c)dt
\]

and the next step is to elucidate the integrand. For simplicity of notation we denote \( \zeta = (\zeta_1, \zeta_2) \in T^*(M \times H) \). Since the canonical relation of \( e^{itP} \) is the graph of the bicharacteristic flow of \( \sigma_P \) on \( T^*(M \times H) \), the composition theorem for Fourier integral operators gives,

Lemma 3.4. \( e^{itP}\psi(Q_c) : L^2(M \times H) \to L^2(\mathbb{R} \times M \times H) \) is a Fourier integral operator in the class \( I^{-\frac{1}{2}}((\mathbb{R} \times M \times H) \times (M \times H), \mathcal{C}_\psi^c) \), with canonical relation

\[
\mathcal{C}_\psi^c := \{(t, \tau, G_M^{cs} \times G_H^{-s}(\zeta), \zeta) \in \dot{T}^*\mathbb{R} \times \text{Char}(Q_c) \times \text{Char}(Q_c) : s \in \text{supp} \hat{\psi}, \tau + |\zeta_M|_g = 0 \}
\]

In the natural parametrization of \( \mathcal{C}_\psi^c \) by \((s, t, \zeta) \in \text{supp} \hat{\psi} \times \mathbb{R} \times \text{Char}(Q_c) \) given by

\[
(t, -|\zeta_M|_g, G_M^{cs} \times G_H^{-s}(\zeta), \zeta),
\]

the symbol of \( e^{itP}\psi(Q_c) \) is \( (2\pi)^{-\frac{1}{2}}\hat{\psi}(s)|ds|^{\frac{1}{2}} \otimes |dt|^{\frac{1}{2}} \otimes |d\mu_L|^{\frac{1}{2}} \), where \( \mu_L \) is Liouville surface measure on \( \text{Char}(Q_c) \).
Proof. We recall that if $\chi : \dot{T}^*M \rightarrow \dot{T}^*M$ is a homogeneous canonical transformation and $\Gamma_\chi \subset \dot{T}^*M \times \dot{T}^*M$ is its graph, and if $\Lambda \subset T^*M \times T^*M$ is any homogeneous Lagrangian submanifold with no elements of the form $(0, \lambda_2)$, then $\Gamma_\chi \circ \Lambda$ is a transversal composition with composed relation $\{ (\chi(\lambda_1), \lambda_2) : (\lambda_1, \lambda_2) \in \Lambda \}$. The condition that $\lambda_1 \neq 0$ is so that $\chi(\lambda_1)$ is well-defined.

It follows that $e^{itP} \psi(Q_c)$ is a transversal composition, and therefore its order is the sum of the order $\frac{1}{2}$ of $e^{itP}$ DG75 and the order $-\frac{1}{2}$ of $\psi(Q_c)$ (Lemma 3.1).

4. Reduction to $H$

The Schwartz kernel of $e^{itP} \psi(Q_c)$ lies in $\mathcal{D}'((\mathbb{R} \times M \times H) \times (M \times H))$. To study sums of squares of inner products, $|\int_H \varphi_j \overline{\psi_k} dV_H|^2$, we need to restrict the Schwartz kernels to $(\mathbb{R} \times H \times H) \times (H \times H)$. To this end, we introduce the restriction operator,

$$
\gamma_H \otimes I : C(M \times H) \rightarrow C(H \times H).
$$

For instance (in the notation of (3.1)),

$$(\gamma_H \otimes I)(\varphi_{j,k}) \in C(H \times H), \quad (\gamma_H \otimes I)\varphi_{j,k}(y_1,y_2) = (\gamma_H \varphi_j)(y_1)\psi_k(y_2).
$$

We are interested in the operator with Schwartz kernel in $C((H \times H) \times (H \times H))$ given by,

$$(\gamma_H \otimes I)(\varphi_{j,k}) \otimes [(\gamma_H \otimes I)(\varphi_{j,k})]^*.
$$

This is an operator from $L^2(H \times H) \rightarrow L^2(H \times H)$. We may construct this operator as the composition of the rank one projection $\varphi_{j,k} \otimes \varphi_{*,j,k} : L^2(M \times H) \rightarrow L^2(M \times H)$ with the restriction operator and its adjoint,

$$(\gamma_H \otimes I)^* : C(H \times H) \rightarrow \mathcal{D}'(M \times H).
$$

Note the the adjoint is an extension as a kind of delta-function and does not preserve continuous functions; see e.g. [D73 Proposition 4.4.6] or [TZ13] for background. The relevant composition is, $(\gamma_H \otimes I) \circ (\varphi_{j,k} \otimes \varphi_{*,j,k}) \circ (\gamma_H \otimes I)^*$ given by the map

$$
C(H \times H) \rightarrow C(H \times H),
$$

$$
K \rightarrow \left( \int_H \int_H K(q_1,q_2)\varphi_{j,k}(q_1,q_2) dV_H(q_1) dV_H(q_2) \right) \cdot (\gamma_H \otimes I)\varphi_{j,k}.
$$

Extending to the operator $e^{itP} \psi(Q_c)$, we define

$$(\gamma_H \otimes I) \circ e^{itP} \psi(Q_c) \circ (\gamma_H \otimes I)^* := \sum_{j,k} e^{it\lambda_j} \psi(\mu_k - c\lambda_j) (\gamma_H \otimes I)\varphi_{j,k} \otimes ((\gamma_H \otimes I)\varphi_{j,k})^*.
$$

If $X \subset M$ is a submanifold, we refer to the composition $\gamma_X F \gamma_X^*$ of a Fourier integral operator as its reduction to $L^2(X)$. Such operators are studied in many articles; we refer to [TZ13, Si18] for background. We will need Sipailo’s Theorem 3.1 Si18, stated below for convenience. In the following, $\pi_X : T_X^*M \rightarrow T^*X$ is the natural restriction (or projection) of covectors to $T^*X$.

Lemma 4.1. [Si18 Theorem 3.1] Let $F \in \mathcal{I}^m(M \times M, \Lambda)$ be a Fourier integral operator of order $m$ associated to the canonical relation $\Lambda' \subset T^*M \times T^*M$. If

- (i) $\Lambda \cap (T^*_X M \times T^*_Y M)$ is a clean intersection, and
- (ii) $\Lambda \cap N^*(X \times X) = \emptyset$,
It follows that also the spacetime graph \( \tilde{\Lambda} \), its intersection with the zero section. The graph of \( g \) is always clean.

Remark 4.2. We recall that \( X \cap Y \) is a clean intersection of two submanifolds of \( Z \) if \( X \cap Y \) is a submanifold of \( Z \) and \( T_p(X \cap Y) = T_pX \cap T_pY \) at all points \( p \in X \cap Y \). Failure of clean intersection can happen in two ways: (i) \( X \cap Y \) fails to be a submanifold of \( Z \), or (ii) it is a submanifold of \( Z \) but \( T_p(X \cap Y) \neq T_pX \cap T_pY \). Usually, (i) is easy to check, but (ii) can be hard to check when (i) holds.

We also quote the result in the simplest case, when \( \Lambda \) is the graph of a symplectic diffeomorphism \( g \) of \( T^*M \). Here and henceforth, for any subset \( V \subset T^*M \), \( \tilde{V} = V \setminus \{0\} \) is \( V \) minus its intersection with the zero section. The graph of \( g \) is denoted by \( \text{Graph}(g) := \{(g(\zeta), \zeta) : \zeta \in \tilde{T}^*M\} \).

Corollary 4.3. \cite[Corollary 3.9]{Si18} Let \( \Phi \) be a Fourier integral operator of order \( m \) quantizing a canonical transformation \( g \). Assume that \( g \) satisfies the conditions:

- (i) the intersection \( \tilde{T}_X^*M \cap g(T_X^*M) \) is clean.
- (ii) \( \tilde{N}_X^*X \cap g(\tilde{N}_X(X)) = \emptyset. \)

Then, \((\pi_X \times \pi_X)(\text{Graph}(g) \cap (T^*_X M \times T^*_X M)) \) is a Lagrangian submanifold of \( T^*(X \times X) \) and \( \gamma_X \Phi \gamma_X^* \) is a Fourier integral operator in the class \( I^m(X \times X, \pi_X \times \pi_X)(\text{Graph}(g)) \), where \( m^* \) is defined in Lemma 4.1.

4.1. Generalization to \( F(t) := e^{itP}(Q_c) \). In fact, we need to extend Lemma 4.1 to the case where \( F \) is replaced by \( F(t) := e^{itP}(Q_c) \) as in Lemma 3.4. We state the result in the generality of Lemma 4.1 where \( F \) is replaced by \( F(t) = e^{itP}F \), where (for each \( t \)) \( e^{itP} : L^2(M) \to L^2(M) \) is the unitary group generated by a first order elliptic pseudodifferential operator \( P \). Note that \( F : \mathcal{D}'(M) \to \mathcal{D}'(\mathbb{R} \times M) \), so the domain and range are not the same, and Lemma 4.1 does not apply as stated, although it applies for each fixed \( t \).

As is well-known \cite{DG75}, \( e^{itP} \in I^{-\frac{1}{4}}(\mathbb{R} \times M \times M, \text{Graph}(g^t)) \) where \( \text{Graph}(g^t) = \{(t, \tau, g^t(\zeta), \zeta) : \tau + \sigma_P(\zeta) = 0\} \) is the space-time graph of the flow. Let \( F \in I^m(M \times M, \Lambda) \) be a Fourier integral operator as in Lemma 4.1 let \( e^{itP} \) be as in the preceding paragraph and let \( F(t) = e^{itP}F \). Assume:

\[ (**) \quad \text{Graph}(g^t) \circ \Lambda' \subset \tilde{T}^*\mathbb{R} \times \tilde{T}^*M \times \tilde{T}^*M \quad \text{is a clean composition}. \]

Then by the composition theorem for Fourier integral operators, \( F(t, x, y) \in I^{-\frac{1}{4}}(\mathbb{R} \times M \times M, \tilde{\Lambda}) \) is a Fourier integral operator associated to the canonical relation,

\[ \tilde{\Lambda} := \{(t, \tau, g^t(\zeta), \zeta') : \tau + p(\zeta) = 0, (\zeta, \zeta') \in \Lambda\} \subset \tilde{T}^*\mathbb{R} \times \tilde{T}^*M \times \tilde{T}^*M. \]

We define the t-slice of \( \tilde{\Lambda} \) by,

\[ \tilde{\Lambda}_t := \{(g^t(\zeta), \zeta') : (\zeta, \zeta') \in \Lambda\} \subset \tilde{T}^*M \times \tilde{T}^*M. \]

Since \( g^t \) is a diffeomorphism, it is evident that if \( \Lambda \) is a manifold, then so is \( \tilde{\Lambda}_t \) for every \( t \). It follows that also the spacetime graph \( \tilde{\Lambda} \) of \( \tilde{\Lambda}_t \) is a manifold and that the composition is always clean.
Lemma 4.4. With the notation and assumptions of Lemma 4.1, let \(F(t) := e^{itF} : C^\infty(M) \to C^\infty(\mathbb{R} \times M)\). Let \(X \subset M\) be a submanifold and assume:

(i) For each \(t\), \(\Lambda_t \cap \hat{T}_X^* M \times \hat{T}_X^* M\) is a clean intersection;
(ii) For each \((\zeta, \zeta') \in \Lambda\), the curve \(t \mapsto g'(\zeta)\) intersects \(T_X^* M\) cleanly;
(iii) For each \(t\), \(\Lambda_t \cap \hat{N}^*(X \times X) = \emptyset\).

Then, \(\gamma_{\mathbb{R} \times X} F(t) \gamma_X^* \in I^{m^*}(\mathbb{R} \times X \times X, \tilde{\Lambda}_X)\) where

\[
\tilde{\Lambda}_X = (I \times \pi_X \times \pi_X)(\tilde{\Lambda} \cap (\hat{T}^* \mathbb{R} \times \hat{T}_X^* M \times \hat{T}_X^* M)),
\]

and where

\[
m^* = \text{ord}\text{F}(t) + \frac{1}{2} \text{codim}\ X + \frac{1}{2} \text{dim}\ \tilde{\Lambda} \cap (\hat{T}^* \mathbb{R} \times \hat{T}_X^* M \times \hat{T}_X^* M) - \frac{1}{2} (2 \text{dim}\ X + 1)
\]

Here, \(\text{ord}\text{F}(t)\) denotes the order of \(F(t)\) as a Fourier integral kernel in \(I^*(\mathbb{R} \times M \times M; \tilde{\Lambda})\).

Proof. (Sketch) Since the proof is almost the same as for Lemma 4.1 we only provide a brief sketch of the proof, emphasizing the new aspects.

We claim first that the conditions (i) - (iii) of the Lemma are equivalent to:

(i)’ \(\tilde{\Lambda} \cap (\hat{T}^* \mathbb{R} \times \hat{T}_X^* M \times \hat{T}_X^* M)\) is a clean intersection, and
(ii)’ \(\tilde{\Lambda} \cap \hat{T}^* \mathbb{R} \times \hat{N}^*(X \times X) = \emptyset\).

It is obvious that (iii) and (ii)’ are equivalent, so we only show that (i)’ is equivalent to (i) - (ii). In fact, it is clear that (i)’ implies (i)-(ii) since \(dt \neq 0\) and all \(t\) slices of the intersection are submanifolds if (i)’ is a clean intersection and the tangent space condition is satisfied. The non-trivial statement is the converse, that (i)-(ii) implies (i)’. To prove it, let \(f_X : M \to \mathbb{R}^k\) be a local defining function for the codimension \(k\) submanifold \(X\), i.e. locally \(X = \{f_X = 0\}\) and \(df_X\) has full rank on \(X\). For instance, one may use the normal variables \(y\) of local Fermi normal coordinates. Then, \(\tilde{\Lambda} \cap T^* \mathbb{R} \times T_X^* M \times T_X^* M\) is the set of points in \(\tilde{\Lambda}\) where \(\pi^* f_X \times \pi^* f_X = 0\) (here, we use the notation \(\pi^* f_X\) for its pullback to \(T^* M\)), and the intersection in (i)’ is a submanifold if \(\pi^* f_X \times \pi^* f_X\) (the pullback to \(T^* \mathbb{R} \times T_X^* M \times T_X^* M\)) is non-singular on \(\tilde{\Lambda}\), that is, if \(\pi^* f_X \times \pi^* f_X : \tilde{\Lambda} \to \mathbb{R}^k \times \mathbb{R}^k\) has a surjective differential at each point \((t, \tau, g^l(\zeta), \zeta')\). Since \(\pi^* f_X \times \pi^* f_X((t, \tau, g^l(\zeta), \zeta')) = (f_X(\pi g^l(\zeta)), f_X(\zeta'))\) we can first calculate \(D \pi^* f_X \times \pi^* f_X : T \tilde{\Lambda} \rightarrow \mathbb{R}^k \times \mathbb{R}^k\) on tangent vectors to curves in \(\zeta\) and \(\zeta'\) for fixed \(t\) and then for for tangent vectors as \(t\) varies. If \(t\) is fixed, then the calculation is the same as for the \(t\)-slice and by assumption (i) the derivative is already surjective. A fortiori, it is surjective if \(t\) is allowed to vary.

The more difficult condition is that the tangent space to the intersection equals the intersection of the tangent spaces. As mentioned in the introduction, the tangent space of the intersection always contains the intersection of the tangent spaces but may possibly be larger. If we decompose the tangent space into the \(\frac{\partial}{\partial t}\) direction and the tangent vectors to the slices, we find that the only condition not contained in (i) is that the tangent vectors to an orbit \(t \mapsto g^l(\zeta)\) may be tangent to the intersection but not in the intersection of the tangent spaces. Since this vector lies only in one component, the condition that there are no such additional tangent vectors is precisely that the curve \(g^l(\zeta)\) intersects \(T_X^* M\) cleanly.
Once it is proved that the composition is clean, the order can be calculated just using the standard calculus of Fourier integral operators under clean composition [HoIV, DG75, D73]. As in [DG75, (1.20)] or [D73, Example, page 111], restriction $\gamma_X$ to a submanifold is a Fourier integral operator of order $\frac{1}{2}\text{codim } X$ (after cutting away normal directions, which are irrelevant to our application). The adjoint $\gamma^*_X$ has the same order.

In our application, the domain and range are different, creating an asymmetry in the order calculation. The domain (incoming variables) is $M$ and the range (outgoing variables) is $\mathbb{R} \times M$. The left restriction $\gamma_{\mathbb{R} \times X}$ acts on the outgoing variables $\mathbb{R} \times M \to \mathbb{R} \times X$ and is of order $\frac{1}{4}(\text{dim } M - \text{dim } X)$. The right restriction $\gamma^*_X$ acts on the incoming variables, Hence the order of $\gamma^*_X$ is $\frac{1}{4}(\text{dim } M - \text{dim } X)$.

The clean composition $\gamma_{\mathbb{R} \times X} F(t) \gamma^*_X$ has order

$$\text{ord} \gamma_{\mathbb{R} \times X} + \text{ord} F(t) + \text{ord}(\gamma^*_X) + \frac{e}{2} = \text{ord} F(t) + \frac{1}{2}\text{codim } X + \frac{e}{2}$$

where $e$ is the excess (see [5,7]). This is the formula for the composition of two operators, but here we extend it to three operators by successively computing excesses and adding the two excesses. To complete the proof, we need to show that

$$\frac{e}{2} = \frac{1}{2} \dim \tilde{\Lambda} \cap (T^*\mathbb{R} \times T^*_X M \times T^*_X M) - \frac{1}{2} (\text{dim } X + \text{dim } X + \text{dim } \mathbb{R}). \quad (4.2)$$

The excess of a general clean composition $A_1 \circ A_2$ of Fourier integral operators, with respective canonical relations $C_1 \subset T^*X \times T^*Y$, $C_2 \subset T^*Y \times T^*Z$, is defined as follows (cf. [HoIV] Page 18). The composition is defined in terms of the clean intersection $\tilde{C} := C_1 \times C_2 \cap T^*X \times \text{Diag}(T^*Y \times T^*Y) \times T^*Z$. Denote by $C$ the range of the map $\pi_{T^*X \times T^*Z} : \tilde{C} \to T^*X \times T^*Z$, $(x, \xi; (y, \eta, y, \eta), z, \zeta) \mapsto (x, \xi, z, \zeta)$. Cleanliness implies that the map $\tilde{C} \to C$ has constant rank. The excess is the dimension of the fiber $C_\gamma$ over a point $\gamma \in C$.

The composition $\gamma_{\mathbb{R} \times X} F(t) \gamma^*_X$ involves three canonical relations, but in the case of restriction operators there is a convenient way to summarize the above two sided composition into a single operation $\gamma_{\mathbb{R} \times X} \circ \gamma^*_X$ (where $\gamma(A)$ is the canonical relation of $A$). Namely, for the right composition, the submanifold $\tilde{C}_R$ is $\tilde{\Lambda} \cap T^*\mathbb{R} \times T^*_X M \times T^*_X M$. It projects to $\pi_{T^*\mathbb{R} \times T^*_X M \times T^*_X}(\tilde{C}_R)$. Similarly, the left composition produces the submanifold $\tilde{C}_L := \tilde{\Lambda} \cap T^*\mathbb{R} \times T^*_X M \times T^*_X M$. The double composition produces the intersection $\tilde{C} = \tilde{C}_L \cap \tilde{C}_R = \tilde{\Lambda} \cap (T^*\mathbb{R} \times T^*_X M \times T^*_X M)$ and the projection map

$$\pi_{T^*\mathbb{R} \times T^*_X M \times T^*_X} : \tilde{\Lambda} \cap (T^*\mathbb{R} \times T^*_X M \times T^*_X M) \to C \subset T^*\mathbb{R} \times T^*X \times T^*X$$

projects the $T^*_X M$ components to $T^*X$. The combined excess of the double composition is the dimension of the fiber of this map over its image. We know that the map has constant rank, since $C$ is a Lagrangian submanifold, $\dim C = (2 \dim X + 1)$ and therefore the dimension of the fiber is

$$\dim \tilde{\Lambda} \cap (T^*\mathbb{R} \times T^*_X M \times T^*_X M) - (2 \dim X + 1),$$

agreeing with (1.2).

4.2. Geometry of submanifolds, transversality to the geodesic flow and cleanliness. We further review the geometry of unclean intersections in restriction problems from [TZ13]. In that article, $H$ was assumed to be a hypersurface, whereas here $H \subset M$ can be any submanifold. We briefly generalize the statements accordingly. In the following, we assume that $H$ is locally defined in an open set $U$ by $\{ f_H = 0 \}$ with $df_H \neq 0$ on $H$. 


and with \( f_H : U \rightarrow \mathbb{R}^k \). Then \((f_H, df_H) : TM \rightarrow T\mathbb{R}^k\) is a local defining function of \( TH \). A natural choice is to use Fermi-normal coordinates along \( H \), the coordinates defined by \( \exp^1 : NH \rightarrow M \). We let \((s_1, \ldots, s_d)\) be a choice of coordinates on \( H \) and let \( f_H = (y_1, \ldots, y_{n-d})\) be normal coordinates. We also let \((\sigma_1, \ldots, \sigma_d, \eta_1, \ldots, \eta_{n-d})\) be the symplectically dual coordinates on \( T^*M \). The following Lemma explains why geodesics tangent to \( H \) may cause a lack of cleanliness.

**Lemma 4.5.** Let \( H \subset M \) be a submanifold. Then, \( S^*H \) is the set of points of \( S^*HM \) where \( S^*HM \) fails to be transverse to the geodesic flow \( G^t \), i.e. where the Hamilton vector field \( H_{pM} \) of \( pM = |\xi|_M \) is tangent to \( S^*HM \).

This is proved in [TZ13] for hypersurfaces.

**Proof.** The generator \( H_{pM} \) of the geodesic flow of \( M \) is the vector field on \( S^*M \) obtained by horizontally lifting \( q \mapsto H^t \) a covector \((q, \eta) \in S^*M\) to \( T(q, \eta)S^*M \) with respect to the Riemannian connection on \( S^*M \); here, we freely identify covectors and vectors by the metric. Lack of transversality \( G^t \) and \( H \) occurs when \( \eta^t \in T_{(q, \eta)}(S^*HM) \). The latter is the kernel of \( df_H \). Since \( f_H \) is a pullback to \( S^*M \), \( df_H(\eta^t) = df_H(\eta) = 0 \) if and only if \( \eta \in TH \).

When \( H \) is totally geodesic, the orbits of \( G^t \) starting with initial data \((s, \xi) \in S^*H\) remain in \( S^*H \), proving the last statement. \( \square \)

We now consider an example of unclean bi-angle sets \( G_c \), resp. \( G^0_c \), in the sense of Definition 1.12 of Section 1.4, which arise when \( c < 1 \) and the \( M \)-geodesic of the \((c, s, t)\) bi-angles have transverse intersection with \( H \), i.e. examples where the solution set of \( G^s_H \circ \pi_H \circ G^{cs+t}_M(q, \xi) = \pi_H(q, \xi) \) is unclean. In these examples \( t \neq 0 \).

Suppose that \( \gamma_0 \) is a closed geodesic of \( S^d \), which we envision as a meridian through the poles. Let \( SO(2) \) be the one-parameter subgroup of rotations fixing \( \gamma_0 \). Let \( p \in \gamma_0 \). Then for any \( \xi \in S^*_p S^2 \), \( G^\pi_{S^2}(p, \xi) \in S^*_H S^2 \) and \( \exp_p(\pi \xi) = -p \).

We then define \( H \) to be the bumped geodesic in which we add a small ‘bump’ on some proper subinterval of \( \gamma_0 \). For instance, we deform \( \gamma_0 \) by a nearly rectangular bump centered along the equator which is \( \epsilon \) in length along \( \gamma_0 \) and comes away from the geodesic \( \gamma_0 \) by \( \epsilon^2 \).

Then we smooth it out near the corners.

We then consider \( c \)-bi-angles i.e. solutions of \( G^{-s}_H \circ \pi_H \circ G^{cs+t}_M(q, \xi) = \pi_H(q, \xi) \) for some \((s, t)\) and some \((q, \xi)\) with \( q \in H \).

First, assume that \( q \) is not on the bump, i.e. \( q \in \gamma_0 \), and let \( \xi \in S^c_q H \). If \( q \) is sufficiently far from the equator and \( \epsilon \) is small enough, then the \( M \)-geodesic \( \gamma_{q, \xi}(\sigma) = G^\sigma(q, \xi) \) does not intersect the bump. It will produce a \( c \)-bi-angle in which the geodesic \( \gamma_{q, \xi}(\sigma) \) hits \( H \) at \( \sigma = \pi \). Let \( s \) be the arc-length on \( H \) between the two intersection points and define \( t \) by \( \pi = cs + t \).

We now move the initial data \((q, \xi)\) under \( g_0 \in SO(2) \). Let \( d(g_0 \theta) \) be the distance from \( g_0q \in \gamma_0 \) to the equator, where the bump lies. As \( d(g_0 \theta) \rightarrow 0 \), the geodesic \( \gamma_{g_0 q, \xi} \) first intersects the bump when \( \theta = \theta_0 \). The angle of intersection of \( \gamma_{g_0 q, \xi} \) and \( H \) ceases to be constant at \( \theta_0 \) and then depends on \( \theta \). In particular, the angle ceases to be the one corresponding to \( c \). Thus, the \( c \)-bi-angle set has a boundary and therefore the solution set is not even a manifold. For \( \theta \geq \theta_0 \), a \( c \)-bi-angle with footpoint on \( H \) no longer lies in the 1-parameter family obtained from \( g_0q \) where \( q \in \gamma_0 \).

A related example is to put two bumps into \( \gamma_0 \). An extreme case is that the bumps touch at a point \( q_0 \) where they are tangent to \( \gamma_0 \) to high order. If \( c, \epsilon \) are chosen so that \( \gamma_{q_0, \xi}(t) \)
does not intersect the bumps, then one can find \((s,t)\) to have a \(c\)-bi-angle. The bi-angle cannot be deformed preserving \(c\), i.e it is an isolated bi-angle.

4.3. Microlocal cutoffs. The hypotheses of Lemma 4.1 and Lemma 4.4 will not be satisfied in all the cases for which we wish to prove Theorem 1.22 and Theorem 1.16. As discussed in Section 2, non-clean intersections arise due to tangential intersections of geodesics with \(H\). This problem is discussed at length in [TZ13], to which we refer for much of the background. As in [TZ13], we introduce some cutoff operators supported away from glancing and conormal directions to \(H\). For fixed \(\epsilon > 0\), let \(\chi^{(\text{tan})}_\epsilon(x,D) = Op(\chi^{(\text{tan})}_\epsilon) \in \Psi^0(M)\), with homogeneous symbol \(\chi^{(\text{tan})}_\epsilon(x,\xi)\) supported in an \(\epsilon\)-aperture conic neighbourhood of \(T^*H \subset T^*M\) with \(\chi^{(\text{tan})}_\epsilon \equiv 1\) in an \(\frac{\epsilon}{2}\)-aperture subcone. The second cutoff operator \(\chi^{(n)}_\epsilon(x,D) = Op(\chi^{(n)}_\epsilon) \in Op(S^0_c(T^*M))\) has its homogeneous symbol \(\chi^{(n)}_\epsilon(x,\xi)\) supported in an \(\epsilon\)-conic neighbourhood of \(N^*H\) with \(\chi^{(n)}_\epsilon \equiv 1\) in an \(\frac{\epsilon}{2}\) subcone. Both \(\chi^{(\text{tan})}_\epsilon\) and \(\chi^{(n)}_\epsilon\) have spatial support in the tube \(T_\epsilon(H)\), the tube of radius \(\epsilon\) around \(H\) (see [TZ13] (5.1) and (5.2)). To simplify notation, define the total cutoff operator

\[
\chi_\epsilon(x,D) := \chi^{(\text{tan})}_\epsilon(x,D) + \chi^{(n)}_\epsilon(x,D).
\]

(4.3)

We put,

\[
B_\epsilon(x,D) = I - \chi_\epsilon(x,D).
\]

We use these cutoff operators to ensure that the relevant cleanliness conditions are satisfied. However, our ultimate goal is to prove singularity results for the traces (1.20), which involve diagonal pullbacks and pushforwards that will erase some of the uncleanliness problems, and make it possible to remove the cutoffs. Moreover, the cutoff is unnecessary when \(H\) is totally geodesic.

4.4. Application to fuzzy ladder propagators. We now apply Lemma 4.4 to the operator \(e^{itP}\psi(Q_\epsilon)\) of Lemma 3.4, where \(M\) in the lemma is replaced with \(M \times H\), and where \(X = H \times H\), so \(\dim X = 2 \dim H, \frac{1}{2} \codim X = \frac{1}{2} \codim H\). We use the following notation:

\[
\zeta = (\zeta_M, \zeta_H) \in T^*_H M \times T^* H \text{ and } \pi_H(\zeta) = (\pi_H(\zeta_M), \zeta_H).
\]

Since tangential intersections of geodesics with \(H\) and conormal vectors to \(H\) apriori cause problems, we will apply Lemma 4.4 to the reduction of the cutoff operator,

\[
(B_\epsilon(x,D) \otimes I)e^{itP}\psi(Q_\epsilon) : L^2(M \times H) \to L^2(\mathbb{R} \times M \times H).
\]

(4.4)

The tensor product notation means, as usual, that the cutoff is applied only in the \(M\) variables (we omit tensor product with \(I\) for the time variables). The cutoff has the effect of cutting down the canonical relation \(C_\psi\) of \(e^{itP}\psi(Q_\epsilon)\) in Lemma 3.4 to,

\[
C^c_\psi,\epsilon := \{(t,\tau, G^{cs+}_M \times G^{cs}_H(\zeta,\zeta)) \in C_\psi^c : (1 - \chi_\epsilon)(G^{cs+}_M(\zeta_M)) \neq 0\}\]  

(4.5)

Here, we use that if \(F\) is any Fourier integral operator with canonical relation \(C_F = \{(x,\xi,y,\eta)\} \subset T^*X \times T^*Y\) and symbol \(\sigma_F\), and if \(a(x,D)\) is a pseudo-differential operator, then the symbol of the left composition \(a(x,D)F\) at \((x,\xi,y,\eta)\) is \(a(x,\xi)\sigma_F(x,\xi,y,\eta)\).

Lemma 4.6. Let \(C^c_\psi,\epsilon\) be the canonical relation of (4.5). Then, for \(c < 1\), the intersection

\[
C^c_\psi,\epsilon \cap (T^*\mathbb{R} \times T^*_H M \times T^* H \times T^*_H M \times T^* H)
\]

is always clean.
Proof. Denote the codimension of $H$ by $\text{codim } H = k$. By Lemma 4.4, the cleanliness of \((4.6)\) holds as long as it holds for (i) fixed $t$-slices, and (ii) for $t$-curves with fixed $\zeta$.

In the case of fixed $t$ slices, we claim that the intersection is clean if and only if
\[ G^{t+cs}(S^*_H M) \cap S^*_H M \tag{4.7} \]
is a clean intersection for all $s \in \text{supp } \hat{\psi}$ and $\zeta_M$ in the support of the above cutoff. Indeed, the intersection \((4.6)\) at time $t$ is parametrized by the points $(\zeta, s) \in S^*_H M \times \text{supp } \hat{\psi}$ such that
\[ G^{cs+t}_M \times G^{-s}_H(\zeta) \in S^*_H M \times B^*H. \]
Since cleanliness in the $\zeta_H$ component is automatic, the cleanliness of this intersection is equivalent to cleanliness of the intersection \((4.7)\). By Lemma 4.3, the intersection is necessarily clean unless there exists $\zeta_M \in S^*_H M$ such that $G^{t+cs}(\zeta_M) \in S^*H$. However, in this case \((1 - \chi_e)(G^{t+cs}(\zeta_M)) = 0\) and the point is not in the canonical relation.

In addition, we need to check that the orbits $t \to G^{cs+t}_M(\zeta)$ intersect $S^*_H M$ cleanly. But this case is again covered by Lemma 4.5 for the same reasons as above. □

Proposition 4.7. Let
\[ \Gamma^c_{\psi, \epsilon} := (\pi_{xH \times H} \times \pi_{H \times H}) G^c_{\psi, \epsilon} \cap (T^*R \times T^*_H M \times T^*H \times T^*_H M \times T^*H) \]
\[ = \{(t, \tau, \pi_{H \times H}\zeta, \pi_{H \times H}(G^{cs+t}_M \times G^{-s}_H)(\zeta)) : |\zeta|_g + \tau = 0, \]
\[ \zeta \in \text{Char } Q_c \cap T^*_H \times H(M \times H), \ G^{cs+t}_M(\zeta_M) \notin T^*_H M \]
\[ (1 - \chi_e)(G^{cs+t}_M(\zeta_M)) \neq 0, \ s \in \text{supp } \hat{\psi} \}
\[ \subset T^*R \times (T^*H \times T^*H \times T^*H \times T^*H). \]

For $0 < c < 1$, $\Gamma^c_{\psi, \epsilon}$ is a Lagrangian submanifold and the ‘reduced’ Fourier integral operator
\[ \gamma_{\pi \times H \times H} \circ (B_\epsilon(x, D) \otimes I) e^{itP} \psi(Q_c) \circ \gamma_H \times H \]
belongs to the class
\[ I^{\rho(m, d)}(R \times (H \times H) \times (H \times H), \Gamma^c_{\psi, \epsilon}), \]
with
\[ \rho(m, d) = \text{order}_e e^{itP} \psi(Q_c) + \frac{1}{2}(n - d) + 2d + \frac{1}{2} - 2(4d + 1) = \text{order}_e e^{itP} \psi(Q_c) + \frac{1}{2}(n - d). \]

The principal symbol of this kind of composition is calculated in [TZ13] using symbol calculus and in [SIS] using oscillatory integrals. We postpone the calculation until the end, since it involves two different restrictions and a pushforward, which can be done in one step rather than in three steps. The ultimate symbol is obtained by a sequence of canonical pushforward and pullback operations.

Proof. By Lemma 4.6, \((4.6)\) is a clean intersection and it follows from Lemma 4.4 that $\Gamma^c_{\psi, \epsilon}$ is a homogeneous Lagrangian submanifold of $T^*R \times T^*H \times T^*H \times T^*H \times T^*H$ and that \((4.8)\) is a Fourier integral operator with canonical relation $\Gamma^c_{\psi, \epsilon}$.

To complete the proof, we compute the order of \((4.8)\) when $\text{dim } M = n$ and $\text{dim } H = d$. We have, \(\text{dim}(R \times H \times H) = 2d + 1\) and $\frac{1}{2} \text{codim}(R \times H \times H \subset R \times M \times H) = \frac{1}{2} \text{codim } H = \frac{1}{2}(n - d)$. The main problem is to calculate the dimension,
\[ D^c(n, d) := \text{dim}(C^c_{\psi, \epsilon} \cap (T^*R \times T^*_H M \times T^*H \times T^*_H M \times T^*H)), \]

\[ \boxed{\text{Dimension calculation}} \]
and how it depends on \(c\) and on whether or not \(H\) is totally geodesic. Note that \(\tilde{L} = \mathcal{C}^c_{\psi, \epsilon}\) in the notation of Lemma 4.4, where

\[
\mathcal{C}^c_{\psi, \epsilon} := \{(t, \tau, G_M^{cs+t} \times G_H^s(\zeta), \zeta) \in T^* \mathbb{R} \times \text{Char}(Q_c) \times \text{Char}(Q_c) : s \in \text{supp}(\dot{\psi}), \tau + |\zeta|_g = 0\}.
\]

**Lemma 4.8.** If \(c < 1\) and \(H\) is any submanifold of dimension \(d\), we have,

\[
\frac{1}{2} D^c(n, d) = 2d + \frac{1}{2}.
\]

**Proof.** The equation for \(\mathcal{C}^c_{\psi, \epsilon}\) involves \(2 + 2n + 2d\) parameters \((t, s, \zeta_M, \zeta_H) \in \mathbb{R} \times \mathbb{R} \times \dot{T}^* M \times \dot{T}^* H\), and \(1 + 2(n - d)\) constraints. One is that \(\zeta \in \dot{T}^*_H M\), so we may regard the parameters as \((t, s, \zeta_M, \zeta_H) \in \mathbb{R} \times \mathbb{R} \times \dot{T}^*_H M \times \dot{T}^* H\), and then we have \(1 + (n - d)\) further constraints,

- \(\sigma_Q(\zeta) = 0\) (which implies \(G_M^{cs+t} \times G_H^s(\zeta) \in \text{Char}(Q_c)\));
- \(G_M^{c+cs}(\zeta_M) \in T^*_H M\). If \(f_H : U \to \mathbb{R}^{n-d}\) is a local defining function of \(H\) in \(U \subset M\), then we may write the \((n - d)\) constraints as \(\pi^* f_H(G_M^{c+cs}(\zeta_M)) = 0\) (see Section 4.2 and Lemma 4.5).

The first return time from \(S^*_H M\) to itself is a smooth function on the support of \(1 - \chi_\epsilon\) (see [TZ13 Section 2.3]). Hence, the solutions \((\sigma, \zeta_M)\) of \(\pi^* f_H(G_M^{c+cs}(\zeta_M)) = 0\) is a smooth submanifold of \(T^*_H M\) of codimension \(n - d\).

Hence, the dimension \((4.9)\) is given by

\[D^c(n, d) = 2 + d + n + 2d - (n - d) - 1 = 4d + 1.\]

This completes the proof of Lemma 4.8. \(\Box\)

We now complete the proof of Proposition 4.7.

We now apply Lemma 3.4, Lemma 4.4 and Lemma 4.8 with \(X = H \times H \subset M \times H\). The intersection has dimension \(4d + 1\). The fiber dimension over \(\mathbb{R} \times (H \times H) \times (H \times H)\) is \(4 \dim H + 1\), so we subtract \(\frac{1}{2}(4d + 1)\). By Lemma 4.4

\[\text{ord} e^{it^p} \psi(Q_c) + \frac{1}{2}(n - d) + 2d + \frac{1}{2} - \frac{1}{2}(4d + 1) = \text{ord} e^{it^p} \psi(Q_c) + \frac{1}{2}(n - d).\]

\(\Box\)

5. **Asymptotics of \(N^c_{\psi, p, \rho, H}(\lambda)\) : Proof of Theorem 1.16 and Theorem 1.18**

5.1. **Diagonal pullback and pushforward to \(\mathbb{R}\).** The next (and final) step is to compose with the diagonal pullback and to integrate over \(H\). By the diagonal embedding \(\Delta_H \times \Delta_H\) we mean the partial diagonal embedding

\[(\Delta_H \times \Delta_H)(x, y) \in H \times H \to (x, x, y, y) \in (H \times H) \times (H \times H),\]

and let \((\Delta_H \times \Delta_H)^*\) be the corresponding pull back operator. For instance, when applied to the rank one orthogonal projections onto the joint eigenfunctions, the Schwartz kernels satisfy

\[(\Delta_H \times \Delta_H)^* (\gamma_H \otimes I)(\varphi_{j,k}) \otimes [(\gamma_H \otimes I)(\varphi_{j,k})]^*(x, y)\]

\[ = (\gamma_H \varphi_j(x) \psi_k(x)) \otimes (\gamma_H \varphi_j(y) \psi_k(y)) \in C(H \times H).\]
We then compose with the pushforward under the projection, \( \Pi : \mathbb{R} \times H \times H \to \mathbb{R} \). The pushforward of the eigenfunctions is given by,

\[
\Pi_* \left( (\Delta_H \times \Delta_H)^* (\gamma_H \otimes I)(\varphi_{j,k}) \otimes [\gamma_H \otimes I](\varphi_{j,k})^* \right) = \int_H \gamma_H \varphi_j(x) \psi_k(x) \, dV_H(x)^2.
\]

We then apply the pushforward-pullback operation to the Fourier integral operator \( \text{(4.8)} \). To keep track of which components are being paired by the diagonal embedding, we note that \( \text{(4.8)} \) is

\[
V_H(t, \psi) = \sum_{j,k} e^{it\lambda_j} \psi(\mu_k - c\lambda_j)(\gamma_H \otimes I)(\varphi_{j,k}) \otimes [\gamma_H \otimes I](\varphi_{j,k})^*
\]

and its pushforward-pullback is given by the following: The \( S_c(t, \psi) \) defined in \( \text{(1.20)} \) is given by,

\[
S_c(t, \psi) = \Pi_* (\Delta_H \times \Delta_H)^* (\gamma_H \otimes I) e^{itP} \psi(Q_c)(\gamma_H \otimes I)^* = \sum_{j,k} e^{it\lambda_j} \psi(\mu_k - c\lambda_j) \left| \int_H \varphi_{j,k}(x, x) \, dV_H(x) \right|^2.
\]

Of course,

\[
F_{\lambda \to t} dN_{\psi,H}^c(t) = \sum_{j,k} e^{it\lambda_j} \psi(\mu_k - c\lambda_j) \left| \int_H \varphi_{j,k} \, dV_H \right|^2 = S_c(t, \psi).
\]

**Definition 5.1.** Recall \( \mathcal{G}_c \) from \( \text{(1.16)} \). Let

\[
\mathcal{S}_{c,\psi} := \{ t \in \mathbb{R} : \exists s \in \text{supp} \psi, \gamma \in \mathcal{G}_c : \gamma \text{ is a } (c, s, t)\text{-bi-angle} \},
\]

\[
\mathcal{S}_c := \{ t \in \mathbb{R} : \exists s \in \mathbb{R}, \gamma \in \mathcal{G}_c : \gamma \text{ is a } (c, s, t)\text{-bi-angle} \},
\]

\( S_c(t, \psi) \) is the distribution trace of the restriction of the Schwartz kernel of \( e^{itP} \psi(Q_c) \), and \( \text{(1.19)} \) is its integral in \( dt \) against \( \hat{\rho}(t) e^{it\lambda} \). Thus, in \( \text{(5.1)} \), we have expressed the smoothed Kuznecov sums as compositions of Fourier integral operators, specifically as the composition of the diagonal pullback to a suitable diagonal in \( H \times H \times H \times H \) and then the pushforward over the diagonal. These operations are also Fourier integral operators, and as reviewed in Section 8.1, we can use the calculus of Lagrangian distribution under pullback and pushforward to determine the singularities of \( S_c(t, \psi) \) and then the asymptotics as \( \lambda \to \infty \) of \( \text{(1.19)} \) (see also \[DG75\] Proposition 1.2 and \[GU89\] and many subsequent articles).

The next step is to prove that \( \text{(5.1)} \) is Lagrangian distribution and to use Proposition 4.6 to calculate the singular set \( \mathcal{S}_c \) of Definition 5.1 and the order and principal symbol of \( \text{(5.1)} \) at the singular points.

We will need to recall the definition of a homogeneous Lagrangian distribution. By definition, \( I^{2-\frac{1}{2}}(\Lambda_T) \), with \( \Lambda_T = \hat{T}_+^* \mathbb{R} \), consists of scalar multiples of the distribution

\[
\int_0^\infty s^{\frac{\nu-1}{2}} e^{-i(t-T)} \, ds
\]

plus similar distributions of lower order and perhaps a smooth function.
Proposition 5.2. Assume that $\Gamma_{\psi,\epsilon}^c \subset \hat{T}^s(\mathbb{R} \times H \times H \times H \times H)$ (cf. Proposition 4.7) is a Lagrangian submanifold, let $(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c$ be its pullback under the diagonal embedding $\Delta_H(y, y') = (y, y, y, y')$ in the sense of (8.2) and let $\Pi : \mathbb{R} \times H \times H \to \mathbb{R}$ be the natural projection. Let

$$\Lambda_{\psi}^c := \Pi_*(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c$$

be the pushforward-pullback of $\Gamma_{\psi,\epsilon}^c$. Then,

$$\Lambda_{\psi} = \{ (t, \tau) \in T^*H : \exists (s, \sigma, q, \eta) : s \in \text{supp} (\hat{\psi}) : (1.15) \text{ is satisfied} \}$$

(5.4)

$$= \bigcup_{t \in \mathcal{S}_{\psi,\epsilon}} \mathcal{G}_c(t), \text{ (see (1.17)).}$$

Moreover, (5.5) is a clean composition if and only if the equation (1.15) is clean in the sense of condition of Definition 1.10, and then

$$S^c(t, \psi) \in \Gamma^{n-\frac{c}{2}}(\mathbb{R}, \Lambda_{\psi}^c), (0 < c < 1).$$

(5.6)

The displayed order occurs at $t = 0$ and the symbol $\sigma(S^c(t, \psi))$ at $t = 0$ equals,

$$\sigma(S^c(t, \psi))|_{t=0} = C_{n,d} \alpha_{\psi}^0(H, \psi) \tau^{n-2} \frac{d\tau}{2}.$$  

(5.7)

$C_{n,d}$ is a dimensional constant depending only on $n = \dim M, d = \dim H$.

Remark 5.3. $S^c(t, \psi)$ is a sum of (translates of) homogeneous distributions with singularities at the discrete set $\mathcal{S}_c$. The order displayed above is the order of the singularity at $t = 0$. The order of the singularity of (5.1) at any $t$ is less than or equal to the order at $t = 0$. In the dominant case of Definition 1.10, the order displayed above only occurs at $t = 0$.

Proof. The first step is to calculate the wave front relation of the pullback $(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c$ using the pullback formula (8.2). The calculation is similar to that of [DG73, (1.20)] for the pullback to the ‘single diagonal’ in $M \times M$. The pullback to the ‘double-diagonal’ $\Delta_{H \times H} \subset H \times H \times H \times H$ subtracts the two covectors at the same base points in the double-diagonal, i.e.

$$(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c = \{(t, \tau, (q, \eta - \pi_H \xi); (q', \eta' - \pi_H \xi')) \in T^* \mathbb{R} \times T^* H \times T^* H : \exists s$$

$$ (t, \tau, (q, \eta), (q, \pi_H \xi), (G_H^s(q, \eta), \pi_H G_M^{c+\alpha}(q, \xi))) \in \Gamma_{\psi,\epsilon}^c, \}$$

or in the notation $\zeta = (\zeta_H, \zeta_M) = (x, \xi, y, \eta)$ $\in \text{Char}(Q_c)$ such that $(x, \xi) \in T^*_H M, G_M^{c+a}(x, \xi) \in T^*_H M, \zeta_H = (q, \eta), \zeta_M' = (q', \eta')$, and with $\pi : T^* X \to X$ the natural projection,

$$(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c = \{(t, \tau, \zeta_H, \zeta_M') : \exists (s, \pi_H \times H(x, \xi, y, \eta)), \pi_H \times H(G_M^{c+a}(x, \xi, y, \eta)) \in \Gamma_{\psi,\epsilon}^c,$$

$$(q, q', q) = (x, \pi G^{s+\alpha}_M(x, \xi), \pi G^{s-a}_H(y, \eta)), \pi_H^{\zeta_H} G^{c+\alpha}(x, \zeta_H, \zeta_H', \zeta) \}$$

Remark 5.4. For the sake of clarity, we note that $(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c$ consists of analogues for $c$-bi-angles of geodesic loops. Unlike a closed geodesic, the initial and terminal directions of a geodesic loop do not have to be the same. A bi-angle is the analogue of a closed geodesic but the ‘bi-angle-loop’ consists of two geodesic arcs, an $M$-arc and an $H$-arc from $q$ to $q'$, with no constraint that the projection of the initial or terminal directions of the $M$ arc agree with those of the $H$ arc.
Next, we pushforward the canonical relation \((\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c\) under the projection \(\Pi_t : \mathbb{R} \times H \times H \rightarrow \mathbb{R}\), \(\Pi_t(t, \xi_H, \xi_H') = t\). As in \((8.3)\) (cf. \[DG75\] (1.21)), the pushforward operation erases points of
\[
(\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c = \{(t, \tau, (q, \eta - \pi_H \xi); (q', \eta' - \pi_H \xi'))\}
\]
unless \(\eta - \pi_H \xi = \eta' - \pi_H \xi' = 0\). Equivalently, the pushforward relation only retains covectors normal to the fiber, which results in ‘closing’ the bi-angle-loop wave front set to the set of ‘closed bi-angles’. Hence, the pushed forward Lagrangian is,
\[
\Pi_t^* (\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c = \{(t, \tau); (t, \tau, (q, \eta - \pi_H \xi); (q', \eta' - \pi_H \xi')) \in (\Delta_H \times \Delta_H)^* \Gamma_{\psi,\epsilon}^c : \eta - \pi_H \xi = \eta' - \pi_H \xi' = 0\} = \{(t, \tau) : \mathcal{G}_c(t) \neq \emptyset\} = \Lambda_{\psi}^c, \quad (\text{cf. } (1.16))
\]

The pushforward Lagrangian is the stage in the sequence of compositions where closed geodesic bi-angles first occur. \(\zeta_M\) is constrained to make an angle of \(\arccos c\) with \(H\). At this state, the cutoffs \(\chi\) away from tangential and normal directions become unnecessary and \(B_\epsilon\) may be removed. Cleanliness of the diagonal pullback is equivalent to the cleanliness conditions on geodesic bi-angles of Definition \((1.12)\) completing the proof of \((5.5)\).

The next step is to calculate the order \((5.6)\) of \(S^c(t, \psi)\) at its singularities. As in \[DG75\] Lemma 6.3, \(\Pi_t \Delta_H^*\) is an operator of order 0 from \(\mathbb{R} \times H \times H \rightarrow \mathbb{R}\) with the Schwartz kernel of the identity operator on \(\mathbb{R} \times H \times H \times \mathbb{R}\). The excess of its composition with \(\Gamma_{\psi,\epsilon}^c\) is by definition (and by \((1.16)\))
\[
e(t) = \dim \{(s, \xi) \in \text{supp } \hat{\psi} \times S_H^c : G^{c,s} \circ \pi_H G^{c,s} = \pi_H (\xi)\} = \dim \mathcal{G}_c(t). \quad (5.8)
\]

We refer to Section 8.1 for background (see \((8.1)\) and \((8.7)\)). The calculation of the excess is parallel to the calculation of the excess of \((\pi_\epsilon \Delta^*) U(t)\) in \[DG75\], where the excess is the dimension of the fixed point set of \(G^t\) on \(S^c M\). Another description of the excess is given in \[HoIV\] Page 18. It is the fiber dimension of the fiber of the intersection over the composition (see Section 8.1 for the precise statement); that explains why we may restrict to \(S^c H^t M\) and not \(T^c H^t M\) and the fiber is \(\dim \mathcal{G}_c^t\). By the order calculation at the end of the proof of Proposition \((4.7)\) the order of the singularity at each \(t \in S_c\) in the singular support is given by,
\[
\text{ord } S^c(t, \psi) = \text{ord } e^{it^p} \psi(Q_c) + \frac{1}{2}(n - d) + \frac{e(t)}{2} = -\frac{3}{4} + \frac{1}{2}(n - d) + \frac{\dim \mathcal{G}_c^t}{2}. \quad (5.9)
\]

In the notation \((1.17)\),
\[
e(0) = \dim \mathcal{G}_c^0 = \dim \mathcal{G}_c^{(0,0)} = \dim S^c H^0 M = n + d - 2, \quad (0 < c < 1).
\]
Indeed, at each point \(y \in H\) and for \(c \leq 1\),
\[
S^c_y M = \{\xi \in S^c_y H, \xi = \eta + \nu, \eta \in T^c_y H, \nu \in N^c_y H, |\eta| = c, |\nu| = \sqrt{1 - c^2}\}.
\]
For \(0 < c < 1\), \(S^c_y M\) is a hypersurface in \(S^c M\) and has dimension \(n - 2\). Combining with \((5.9)\), it follows that the order of \((5.1)\) at \(t = 0\) equals,
\[
\text{ord } S^c(t, \psi)|_{t=0} = -\frac{3}{4} + \frac{1}{2}(n - d) + \frac{1}{2}(n + d - 2) = n - 1 - \frac{3}{4} \quad (0 < c < 1). \quad (5.10)
\]
For general \( T \in S_c \),
\[
\text{ord } S^c(t, \psi)|_{t=T} = -\frac{3}{4} + \frac{1}{2}(n-d) + \frac{1}{2} \dim G^c_T.
\]
(5.11)

To complete the proof of Proposition 5.2, we need to calculate the symbol of (5.1) at each singularity, and particular to show that the symbol at \( t = 0 \) is given by (5.7). Since the coefficient at \( t = 0 \) was calculated in great detail in the earlier paper \[WXZ20\], we only briefly sketch the argument.

The symbol is calculated using iterated pushforward and pullback formulae as reviewed in (8.8) (see also (8.2) - (8.3)). The pushforward is by the canonical projection \( \Pi_t \), and the pull-back is under the canonical embedding \( \Delta_H \times \Delta_H \). In the terminology of \[GS13\], these maps must be ‘enhanced’ with natural half-densities to make them ‘morphisms’ on half-densities. As reviewed in Section 8.2, the embedding must be enhanced by a half-density on the conormal bundle of the image, and the projection must be enhanced by a half-density along the fiber. As discussed in \[DG75\], Page 66, there is a natural half-density on the conormal bundle of the diagonal. In fact, the enhancement construction in this case is quite simple, since the pull-back under \( \Delta_H \times \Delta_H \) of the half-density symbol on \( \Gamma^c_{\psi,\epsilon} \) produces a density on the fibers of the projection \( \Pi_t \). One integrates this density over the fibers to obtain the symbol in (5.7) at any \( t \). The half-density symbols are always volume half-densities on their respective canonical relations. In particular when \( c \in (0,1) \) the coefficient of the singularity at \( t = 0 \) is a dimensional constant times \( \text{Vol}(S^c_{\epsilon}M) \). □

5.2. Proof of Theorem 1.16, Theorem 1.18 and Proposition 2.2. By (5.2), \( F_{\lambda \to t} dN^c_{\psi,H} = S^c(t, \psi) \) and Proposition 5.2 shows that \( S^c(t, \psi) \) is a discrete sum of translated polyhomogeneous distributions in \( t \). It follows from Remark 5.3 that if a distribution lies in \( I_{\frac{7}{2} - \frac{1}{4}}(\Lambda_T) \) then its inverse Fourier transform is asymptotic to \( s^{-\frac{1}{2}} \). This suggests that the singularity at \( t = 0 \) produces an asymptotic expansion of order \( n - 2 \) equal to \( n - 2 \) for \( 0 < c < 1 \).

The next Lemma gives the precise statement and concludes the proof of Theorem 1.16 for test functions whose support contains only the singularity at \( t = 0 \). If one uses general test functions with Fourier transforms in \( C^\infty_0(\mathbb{R}) \), one adds similar contributions from the non-zero \( t \in S_c \). As mentioned above, the order of these singularities is no larger than the order at \( t = 0 \). If they are less than that order \( t = 0 \) is called dominant (Definition 1.10).

Lemma 5.5. Assume \( 0 < c < 1 \). Let \( \rho \in \mathcal{S}(\mathbb{R}) \) with \( \hat{\rho} \in C^\infty_0 \), \( \int \rho = 1 \), and with \( \text{supp} \hat{\rho} \) in a sufficiently small interval around 0. Then, there exists \( \beta_j \in \mathbb{R} \) and a complete asymptotic expansion,
\[
N^c_{\rho,\psi,H}(\lambda) \sim \lambda^{n-2} \sum_{j=0}^{\infty} \beta_j \lambda^{-j},
\]
with \( \beta_0 = C_{n,d} a^0_c(H, \psi) \) (see Theorem 1.18 for the general formula).

Proof. The asymptotic expansions follow immediately from Proposition 5.2. By definition (see 5.11),
\[
N^c_{\rho,\psi,H}(\lambda) = \sum_{j,k} \rho(\lambda - \lambda_k) \psi(\mu_j - c\lambda_k) \left| \int_H \varphi_j \overline{\psi_k} dV_H \right|^2
= \int_\mathbb{R} \hat{\rho}(t)e^{it\lambda} S^c(t, \psi) dt.
\]
(5.12)
If $G_c^{0,0}$ is the only component with $t = 0$, then by Proposition 5.2 and by definition of Lagrangian distributions (5.3), for sufficient small $|t|$, 

\[
S_c^c (t, \psi) = \sum_{j=0}^{\infty} \alpha_j \int_0^\infty s^{n-2-j} e^{-ist} ds \mod C^\infty. \tag{5.13}
\]

\[
\int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} S_c^c (t, \psi) dt = \sum_{j=0}^{\infty} \alpha_j \int_0^\infty s^{n-2-j} \left( \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} e^{-ist} dt \right) ds + O(\lambda^{-\infty})
\]

\[
= \sum_{j=0}^{\infty} \alpha_j \int_0^\infty s^{n-2-j} \rho(\lambda - s) ds + O(\lambda^{-\infty})
\]

\[
= \sum_{j=0}^{\infty} \alpha_j \int_{-\infty}^{\lambda} (\lambda - s)^{n-2-j} \rho(s) ds + O(\lambda^{-\infty})
\]

\[
= \sum_{j=0}^{\infty} \alpha_j \int_{-\infty}^{\lambda} (\lambda - s)^{n-2-j} \rho(s) ds + o(\lambda^{-\infty})
\]

where $\alpha_0 = \alpha_0$ is the principal symbol of $S_c^c (t, \psi)$ at $t = 0$ (given in Proposition 5.2), and $\tilde{\alpha}_j$ are obtained in part by expanding

\[
\int_{-\infty}^{\lambda} (\lambda - s)^{n-2-j} \rho(s) ds
\]

in $\lambda$.

As discussed in Section 1.6, when $\hat{\psi}$ is arbitrarily large, the principal symbol is changed by summing over the components of $G_c^0$. As in that section, the $(c, s, 0)$-bi-angles with $t = 0$ is assumed to be a union of clean components $Z_j(0)$ of dimension $d_j$. In our situation $Z_j$ is a component of $G_c^0$. Then, for $t$ sufficiently close to $0$,

\[
S_c^c (t, \psi) = \sum_j \beta_j(t),
\]

with

\[
\beta_j(t) = \int_{\mathbb{R}} \alpha_j(s) e^{-ist} ds, \quad \text{with} \quad \alpha_j(s) \sim \left( \frac{s}{2\pi i} \right)^{-1 + \frac{1}{2}(n-d) + \frac{d_j}{2}} i^{-\sigma_j} \sum_{k=0}^{\infty} \alpha_{j,k} s^{-k}, \tag{5.14}
\]

where $d_j$ is the dimension of the component $Z_j(0) \subset G_c^0$. \hfill $\Box$

5.2.1. Proof of Proposition 2.2.

Proof. By Proposition 4.7, at a non-zero period, the exponents are calculated from (5.9), with the excess (5.8) given by $\dim G_c^t$. Hence, the exponents at a non-zero period are now

\[
\frac{1}{2}(n-d) - 1 + \frac{\dim G_c^t}{2}. \tag{5.9}
\]

In the notation of [DG75, Theorem 4.5], we are assuming that the set of $(c, s, t)$-bi-angles with $t = T \in \text{singsupp} S_c^c (t, \psi) \setminus \{0\}$ is a union of clean components $Z_j$ of dimension $d_j$. In our situation $Z_j$ is a component of $G_c^t$. Then, for $t$ sufficiently close to $T$,

\[
S_c^c (t, \psi) = \sum_j \beta_j(t - T),
\]
with
\[ \beta_j(t) = \int_\mathbb{R} \alpha_j(s)e^{-ist}ds, \quad \text{with} \quad \alpha_j(s) \sim \left(\frac{s}{2\pi i}\right)^{-1+\frac{1}{2}(n-d)+\frac{d_j}{2}}i^{-\sigma_j}\sum_{k=0}^{\infty} \alpha_{j,k}s^{-k}, \]
where \( d_j \) is the dimension of the component \( Z_j \) of \( G^l \).

5.3. **Sub-principal term of \( N_{\rho,\psi}(\lambda) \) when \( \hat{\psi} \) has small support and both \( \hat{\psi} \) and \( \hat{\rho} \) are even.** The subprincipal term may be calculated by the stationary phase method, but even the subprincipal term is a sum of a large number of terms with up to six derivatives on the phase and two derivatives of the amplitude. It is easily seen (for instance, on a flat torus) that the subprincipal term contains sums with the derivatives of \( \hat{\rho}(t) \) and \( \hat{\psi}(s) \) at \( t = s = 0 \). It is assumed in [DG75, Proposition 2.1] (which only involves \( \hat{\rho} \)) that \( \hat{\rho} \equiv 1 \) near 0, hence none of its derivatives contribute to the subprincipal term. We are making the same assumption.

In addition, derivatives of the amplitude of the wave kernel may contribute to the subprincipal term. We use a parity argument as in [DG75, Page 48] to show that the contribution of degree \( k \) in \( \sigma_p(x, \xi) \) is even, resp. odd if \( k \) is even, resp. odd. By induction with respect to \( r \) it follows that \( (\hat{\rho}_{\mathcal{M}})^r a_{-j} \) is an even, resp. odd. if \( r - j \) is even, resp. odd.

The amplitude of \( e^{itP_{\psi}e^{-isQ_c}} \) is obtained by integrating the parametrix formula for \( e^{itP_{\psi}e^{-isQ_c}} \) as a tensor product \( e^{i(t-c)sP_{M}} \otimes e^{isP_{H}} \). The parities of the terms in the amplitude are thus determined as in [DG75], for \( s = t = 0 \). One has to restrict the \( M \) amplitudes to \( H \) but they still have the same parity. One further has to restrict them to the diagonals in \( H \times H \), which seems to multiply the amplitudes. But the subprincipal term can only be obtained as the product of the principal symbol and the subprincipal symbol. Hence it is odd and its integrals over cospheres vanishes.

**Remark 5.6.** We opt not to calculate the Maslov indices \( \sigma_j \) for the sake of brevity, and absorb them into the constants \( \beta \).

6. **Proof of Theorem 1.1 and Theorem 1.20**

In this section we apply a cosine Tauberian theorem to deduce Theorem 1.1 from Theorem 1.16.

6.1. **Tauberian Theorems.** For the reader’s convenience we quote the statements of two Fourier Tauberian theorem from [SV]. In what follows, \( \rho \) will be a strictly positive even Schwartz-class function on \( \mathbb{R} \) with compact Fourier support satisfying that \( \hat{\rho}(0) = 1 \). \( N \) will be a tempered, monotone increasing function with \( N(\lambda) = 0 \) for \( \lambda < 0 \), and \( N' \) its distributional derivative as a nonnegative measure on \( \mathbb{R} \).

**Proposition 6.1** (Corollary B.2.2 in [SV]). Fix \( \nu \geq 0 \). If \( N' \ast \rho(\lambda) = O(\lambda^\nu) \), then
\[ N(\lambda) = (N \ast \rho)(\lambda) + O(\lambda^\nu). \]
This estimate holds uniformly for a set of such \( N \) provided \( N' \ast \rho(\lambda) = O(\lambda^\nu) \) holds uniformly.

**Proposition 6.2** (Theorem B.5.1 in [SV]). Fix \( \nu \geq 0 \). If \( N' \ast \rho(\lambda) = O(\lambda^\nu) \) and additionally
\[ N' \ast \chi(\lambda) = o(\lambda^\nu) \]
for every Schwartz-class $\chi$ on $\mathbb{R}$ whose Fourier support is contained in a compact subset of $(0, \infty)$. Then,

$$N(\lambda) = N \ast \rho(\lambda) + o(\lambda^{n-2}).$$

6.2. **Proof of Theorem 1.1**

**Proof.** Theorem 1.1 pertains to the Weyl function $N_{c,\psi,H}(\lambda)$ of (1.7),

$$N_{c,\psi,H}(\lambda) := \sum_{j,k} \psi(\mu_j - c\lambda_k) \left| \int_{\mathcal{H}} \varphi_j \bar{\psi}_k dV_H \right|^2.$$

Recall our assumption that $\psi \geq 0$. Then, $N_{c,\psi,H}(\lambda)$ is monotone non-decreasing and has Fourier transform $S_{c,\psi}(t,\phi)$ (5.1).

We apply Proposition 6.1 with $\hat{\rho} \cap \text{singsupp} S_{c,\psi}(t,\phi) = \{0\}$ and to $dN_{c,\psi,H}(\lambda)$. By Lemma 5.5, $\rho \ast dN_{c,\psi,H}(\lambda) = \beta_0 \lambda^{n-1} + O(\lambda^{n-2})$, and therefore,

$$N_{c,\psi,H}(\lambda) = \rho \ast N_{c,\psi,H}(\lambda) + O(\lambda^{n-2})$$

$$= \frac{\beta_0}{n-1} \lambda^{n-1} + O(\lambda^{n-2}),$$

concluding the proof of Theorem 1.1. \qed

6.3. **Proof of Corollary 1.3**. To prove Corollary 1.3 it suffices to prove that, for any $\epsilon > 0$ there exists a test function $\psi \geq 0, \hat{\psi} \in C_0^\infty(\mathbb{R}), \hat{\psi}(0) = 1$ and a universal constant $C(\epsilon, \delta)$ depending only on $(\epsilon, \delta)$ so that for all $\lambda_j$,

$$J_{c,\psi,H}(\lambda_j) \geq C(\epsilon, \delta) J_{c,\psi,H}(\lambda_j). \quad (6.1)$$

Then the upper bound for $J_{c,\psi,H}(\lambda_j)$ given in Corollary 1.2 provides the upper bound for $J_{c,\psi,H}(\lambda_j)$.

**Proof.** We have

$$\sum_{k:|\mu_k - c\lambda_j| \leq \epsilon} \left| \int_{\mathcal{H}} \varphi_j \bar{\psi}_k dV_H \right|^2 \leq \sum_k \psi(\mu_k - c\lambda_j) \left| \int_{\mathcal{H}} \varphi_j \bar{\psi}_k dV_H \right|^2$$

provided $\psi$ is chosen to be a nonnegative Schwartz function with small Fourier support, with $\hat{\psi}(0) > 1$, and perhaps scaled wider so that $\psi \geq 1_{[-\epsilon,\epsilon]}$. By Corollary 1.2, the right side is $O(\lambda_j^{n-2})$, concluding the proof. \qed

6.4. **Proof of Theorem 1.20**

**Proof.** By Proposition 6.2 it suffices to check the additional condition,

$$\chi \ast dN_{c,\psi,H}(\lambda) = o(\lambda^{n-2}) \quad (6.2)$$

for every Schwartz-class $\chi$ on $\mathbb{R}$ whose Fourier support is contained in a compact subset of $(0, \infty)$.

To prove this, we consider the expansions given in Theorem 1.16 and Theorem 1.18 and especially Proposition 2.2. In addition to the assumptions of Theorem 1.16 the assumption of Theorem 1.20 is that $d_j(T) < d_j(0)$ for $T \neq 0$, i.e. that $\dim Z_j(0) > \dim Z_j(T)$ for all $T \neq 0$. This assumption together with Proposition 2.2 shows that (6.2) holds; then Proposition 6.2 implies the first claim in Theorem 1.20.
Finally, we bound the jumps in the Weyl function of Theorem 1.20 by the remainder as before and obtain \( J_{\psi,H}^c(\lambda) = o_{\psi}(\lambda^{n-2}) \).

7. Proof of Theorem 1.22

In this section we deduce Theorem 1.22 from Theorem 1.18 and an additional Tauberian theorem, which allows us to replace \( \psi \) in the inner sum of (1.19) (or (5.22)), by an indicator function \( 1_{[-\epsilon,\epsilon]} \). Throughout this section, we assume that \( c < 1 \).

For simplicity of notation, when \( \psi = 1_{[-\epsilon,\epsilon]} \), we write,

\[
N_{\epsilon,H}^c(\lambda) := \sum_{j: \lambda_j \leq \lambda} J_{1_{[-\epsilon,\epsilon]}(\lambda_j)}^c
\]

where as in (1.12)

\[
J_{1_{[-\epsilon,\epsilon]}(\lambda_j)}^c := \sum_{\ell: \lambda_\ell = \lambda_j} \sum_{k: |\mu_k - c\lambda_j| \leq \epsilon} \left| \int_H \varphi_{\ell \psi_k} dV_H \right|^2
\]

The Tauberian theorem is not used in the traditional way, i.e. to replace a monotone increasing function with jumps by a smoothly varying sum. The relevant monotone function is the Weyl-Kuznecov sum (7.1), and as with (1.7), its jump discontinuities occur only at the points \( \lambda = \lambda_j \), with jumps \( J_{1_{[-\epsilon,\epsilon]}(\lambda_j)}^c \).

We treat the sum (7.2) as a semi-classical Weyl function with semi-classical parameter \( \lambda_j^{-1} \) and deploy the semi-classical Tauberian theorem of [PR85, R87]. The main complication is that the terms are weighted by the unbounded and non-uniform weights \( \left| \int_H \varphi_j \psi_k dV_H \right|^2 \) (in \( (\lambda_j, \mu_k) \)). Moreover, \( J_{1_{[-\epsilon,\epsilon]}^c(\lambda_j)}^c \) does not usually have an asymptotic expansion as \( \lambda_j \to \infty \) due to lack of asymptotics for the individual eigenfunctions \( \varphi_j \). We need to sum in \( \lambda_j \) as well to obtain asymptotics.

To set things up for the Tauberian arguments in [PR85, R87], we define

\[
\begin{aligned}
&\left\{ d\mu_\lambda^c(x) := \sum_{j: \lambda_j \leq \lambda} \sum_k \left| \int_H \varphi_j \psi_k dV_H \right|^2 \delta_{\mu_k - c\lambda_j}(x). \\
&\sigma_\lambda^c(x) = \int_{-\infty}^x d\mu_\lambda^c(y).
\end{aligned}
\]

By (1.23),

\[
N_{\epsilon,H}^c(\lambda) := \int_{-\epsilon}^\epsilon d\mu_\lambda^c = \sigma_\lambda^c(\epsilon) - \sigma_\lambda^c(-\epsilon) = \sum_{j: \lambda_j \leq \lambda} J_{1_{[-\epsilon,\epsilon]}^c(\lambda_j)}^c.
\]

Note that \( \text{supp} \mu_\lambda^c \subset \{ x \geq -c\lambda \} \) and that,

\[
\int_{-\infty}^\infty d\mu_\lambda^c = \sum_{j: \lambda_j \leq \lambda} \| \gamma_H \varphi_j \|_{L^2(H)}^2.
\]

In comparison with the proofs of Theorems 1.1 - 1.20, we do not have a complete, or even two-term, asymptotic expansion for either of the once-smoothed sums, (1.7) or \( \sum_j \rho(\lambda - \lambda_j)J_{1_{[-\epsilon,\epsilon]}^c(\lambda_j)}^c \).

The Tauberian strategy is to smooth out the indicator functions \( 1_{[-\epsilon,\epsilon]} \) and apply Theorem 1.18 to this kind of mollified sum. Our aim is to show that, under the assumption that \( G_{\epsilon,0}^c \) is dominant, we can sharpen the sum at the expense of weakening the remainder to \( O_{c,\epsilon}(\lambda^{n-1}) \).
The smoothing error is an ‘edge effect’ due to the sum of the terms $|\int_H \varphi_j \overline{\psi}_k dV_H|^2$ near the endpoints $|\mu_k - c\lambda_j| = \epsilon$ of the interval $|\mu_k - c\lambda_j| \leq \epsilon$. The Tauberian theorem is used to show that the eigenvalues and the quantities $|\mu_k - c\lambda_j| = \epsilon$ are sufficiently uniform, i.e. do not concentrate near the endpoints.

7.1. The proof of Theorem 1.22. Following [PR85] [R87], we denote by $\rho_1 \in C^\infty_0(-1,1)$ a smooth cutoff satisfying $\rho_1(0) = 1$, $\rho_1(-t) = \rho_1(t)$. With no loss of generality, we assume $\hat{\rho}_1(\tau) \geq 0$ and $\hat{\rho}_1(\tau) \geq \delta_0 > 0$ for $|\tau| \leq \epsilon_0$. Then set,

$$\rho_T(\tau) = \rho_1\left(\frac{\tau}{T}\right), \quad \theta_T(x) := \hat{\rho}_T(x) = T\hat{\rho}_1(Tx). \tag{7.5}$$

In particular, $\int \theta_T(x) dx = 1$ and $\theta_T(x) > T\delta_0$ for $|x| < \epsilon_0/T$. Note that $\theta_T * d\mu_\lambda^c$ is by definition the measure,

$$\theta_T * d\mu_\lambda^c(x) = \sum_{j: \lambda_j \leq \lambda} \sum_k \theta_T(\mu_k - c\lambda_j - x) \left| \int_H \varphi_j \overline{\psi}_k dV_H \right|^2.$$

Of course, $\theta_T * d\mu_\lambda^c(x) \to d\mu_\lambda^c(x)$ as $T \to \infty$.

Let us record the relation between the various relevant quantities.

**Lemma 7.1.** We have,

$$\int_{-\epsilon}^{\epsilon} \theta_T * d\mu_\lambda^c(x) = N_{\theta_T * 1_{[-\epsilon,\epsilon]}, H}(\lambda) = \sigma_\lambda^c * \theta_T(\epsilon) - \sigma_\lambda^c * \theta_T(-\epsilon)$$

**Proof.** This follows from the definitions. \[\square\]

The asymptotics of $N_{\theta_T * 1_{[-\epsilon,\epsilon]}, H}(\lambda)$ are given in Theorem 1.18 with

$$\psi = \psi_{T,\epsilon} := \theta_T * 1_{[-\epsilon,\epsilon]}.$$ We use the notation $\psi_{T,\epsilon}$ henceforth to simplify the notation. Putting things together,

$$N^c_{\epsilon, H}(\lambda) = N^c_{\psi_{T,\epsilon} H}(\lambda) + N^c_{(\psi_\infty - \psi_{T,\epsilon}), H}(\lambda), \tag{7.6}$$

where $\psi_\infty = 1_{[-\epsilon,\epsilon]}$. Here, $\epsilon$ is fixed. The hard step is to estimate the error in the smoothing approximation,

$$N^c_{(\psi_\infty - \psi_{T,\epsilon}), H}(\lambda) = (\sigma_\lambda^c(\epsilon) - \sigma_\lambda^c(-\epsilon)) - (\sigma_\lambda^c * \theta_T(\epsilon) - \sigma_\lambda^c * \theta_T(-\epsilon)), \tag{7.7}$$

in terms of $(\lambda, T)$.

**Proposition 7.2.** With the same notation and assumptions as in Theorem 1.22, for any $\epsilon > 0$ and $c \in (0,1)$, there exist constants $\gamma(c, \epsilon)$ such that, for any $T > 0$,

$$|N^c_{(\psi_\infty - \psi_{T,\epsilon}), H}(\lambda)| = \left| \int_{-\epsilon}^{\epsilon} (\theta_T * d\mu_\lambda^c - d\mu_\lambda^c) \right| \leq \frac{\gamma(c, \epsilon)}{T} \lambda^{n-1} + O_T \lambda^{n-3/2}.$$

Before giving the proof, we verify that Proposition 7.2 implies Theorem 1.22. By Theorem 1.18 and the hypothesis that $G_{0,0}^c$ is dominant, we have,

$$N^c_{\psi_{T,\epsilon} H}(\lambda) = \hat{\psi}_{T,\epsilon}(0) A^c_n \mathcal{H}^d(H) \lambda^{n-1} + R_{\psi_{T,\epsilon}}(\lambda),$$
where $A_{n,d}^c$ is the leading coefficient, e.g. $A_{n,d}^c = C_{n,d}c^{d-1}(1 - c^2)^{\frac{n-d-2}{2}}$ for $0 < c < 1$, and where $R_{\psi_T, \lambda}(\lambda) = O_{T, \epsilon}(\lambda^{n-3/2})$. Moreover, $\hat{\psi}_{T, \epsilon}(0) = 2\epsilon$. The full error term in (7.6) is therefore,

$$
\tilde{R}_{T, \epsilon}(\lambda) = N_{\psi_{\infty}, \epsilon - \psi_{T, \epsilon}, \lambda}(\lambda) + R_{\psi_T, \lambda}(\lambda)
= O\left(\frac{\gamma(c, \epsilon)}{T}\lambda^{n-1}\right) + O_{T, \epsilon}(\lambda^{n-3/2}) + O_{T, \epsilon}(\lambda^{n-3/2}).
$$

The bound $\tilde{R}_{T, \epsilon}(\lambda) = o_\epsilon(\lambda^{n-1})$ follows by taking $T = T(\lambda)$ as a function of $\lambda$ increasing $T(\lambda) \not\to \infty$ sufficiently slowly.

### 7.2. Proof of Proposition 7.2

We have,

$$
\int_{-\epsilon}^{\epsilon}(\theta_T * d\mu^\epsilon_{T, \lambda} - d\mu^\epsilon_{\lambda}) = \int_{\mathbb{R}}(\mu^\epsilon_{T, \lambda}([\epsilon, \epsilon] - \tau) - \mu^\epsilon_{\lambda}([-\epsilon, \epsilon])) \theta_T(\tau)d\tau
= T\int_{\mathbb{R}}(\mu^\epsilon_{T, \lambda}([\epsilon, \epsilon] - \tau) - \mu^\epsilon_{\lambda}([-\epsilon, \epsilon])) \hat{\rho}_1(\tau T)d\tau
= T\int_{|\tau| \leq \frac{\tau}{T}}(\mu^\epsilon_{T, \lambda}([\epsilon, \epsilon] - \tau) - \mu^\epsilon_{\lambda}([-\epsilon, \epsilon])) \hat{\rho}_1(\tau T)d\tau
+ T\int_{|\tau| > \frac{\tau}{T}}(\mu^\epsilon_{T, \lambda}([\epsilon, \epsilon] - \tau) - \mu^\epsilon_{\lambda}([-\epsilon, \epsilon])) \hat{\rho}_1(\tau T)d\tau
=: I_1 + I_2.
$$

The key point is to prove the analogue of [PR85 Proposition 3.2].

### Proposition 7.3

With the same notation and assumptions as in Theorem 1.22, and for any $0 < c < 1$ here exist constants $\gamma_1(c, \epsilon)$ such that, for any $T > 0$,

$$
|\mu^\epsilon_{T, \lambda}([\epsilon, \epsilon] - \tau) - \mu^\epsilon_{\lambda}([-\epsilon, \epsilon])| \leq \gamma_1(c, \epsilon)(\frac{1}{T} + |\tau|)\lambda^{n-1} + C_1(T, c)O(\lambda^{n-3/2}).
$$

We first show that Proposition 7.3 implies Proposition 7.2.

**Proof.** We only verify this in the case $0 < c < 1$, since the second case is proved in the same way. First, observe that Proposition 7.3 implies,

$$
I_1 \leq \sup_{|\tau| \leq \frac{1}{T}} |\mu^\epsilon_{T, \lambda}([-\epsilon, \epsilon] - \tau) - \mu^\epsilon_{\lambda}([-\epsilon, \epsilon])|,
$$

and Proposition 7.3 immediately implies the desired bound for $|\tau| \leq \frac{1}{T}$. For $I_2$ one uses that $\hat{\rho}_1 \in S(\mathbb{R})$. Since $T\int_{|\tau| \geq \frac{\tau}{T}} \hat{\rho}_1(\tau T)d\tau \leq 1$, Proposition 7.3 implies that there exist constants $A > 0$, $C_1(T, c)$ so that

$$
I_2 \leq A\lambda^{n-1}\gamma_1(c, \epsilon)T\int_{|\tau| \geq \frac{\tau}{T}}(\frac{1}{T} + |\tau|)\hat{\rho}_1(\tau T)d\tau
+C_1(T, c)O(\lambda^{n-3/2})T\int_{|\tau| \geq \frac{\tau}{T}} \hat{\rho}_1(\tau T)d\tau.
$$

If one changes variables to $r = T\tau$ one also gets the estimate of the Tauberian Lemma. □

We now prove Proposition 7.3.
Proof. Since we are studying the increments $\mu^c(\epsilon, \eta, -\tau) - \mu^c(\epsilon, \eta, -\tau)$ and since the integral is a sum where $|r| \leq \frac{1}{T}$ and $|r| \geq \frac{1}{T}$, the proof is broken up into 3 cases: (1) $|r| \leq \frac{1}{T}$, (2) $r = \frac{1}{T} \epsilon_0$, for some $\ell \in \mathbb{Z}$, and (3) $\frac{1}{T} \epsilon_0 \leq \tau \leq \frac{1}{T} \epsilon_0$, for some $\ell \in \mathbb{Z}$.

The key assumption that the only maximal component is the principal component is used to obtain the factor of $\frac{1}{T}$, which is responsible for the small oh of the remainder. We use the exact formula for the leading coefficient when $\hat{\psi}$ has arbitrarily large compact support in Theorem [1,18]. When $0 < c < 1$ and when the only maximal component is the principal component, the sum over $s_j^m$ is merely the value of $s = 0$. When $\hat{\psi}(0) = 1$, $a^c_0(H, \psi)$ is independent of $supp \hat{\psi}$. When there do exist many maximal components, as in the case of subspheres of spheres, the sum $\sum_j \hat{\psi}(s_j^m)$ essentially counts the number of the components with $s$-parameter in $supp \psi$, and that can cancel the $\frac{1}{T}$.

(1) Assume $|\tau| \leq \frac{1}{T}$. Also assume $\tau > 0$ since the case $\tau < 0$ is similar. We claim that,

$$|\mu^c(\epsilon, \eta, -\tau) - \mu^c(\epsilon, \eta, -\tau)| \leq \frac{2\gamma_0(c, \epsilon)}{T\delta_0} \lambda^{n-1}.$$

Write

$$\mu^c(\epsilon, \eta, -\tau) - \mu^c(\epsilon, \eta, -\tau) = \int_\mathbb{R} [1_{[-\epsilon - \tau, -\tau]} - 1_{[-\epsilon, -\epsilon]}] (x) d\mu^c(\epsilon, \eta, -\tau).$$

For $T$ sufficiently large so that $\tau \ll 2\epsilon$,

$$[1_{[-\epsilon - \tau, -\tau]} - 1_{[-\epsilon, -\epsilon]}] (x) = 1_{[-\epsilon - \tau, -\epsilon]} - 1_{[-\epsilon, -\epsilon]}.$$

Since they are similar we only consider the $[-\epsilon - \tau, -\epsilon]$ interval. Since for $|\tau| < \epsilon_0/T$, we have $\theta_T(\tau) > T\delta_0$, it follows from Theorem [1,18] that,

$$\mu^c(\epsilon, \eta, -\tau, -\epsilon) \leq \frac{1}{T\delta_0} \int_\mathbb{R} \theta_T(\epsilon - x) d\mu^c(\epsilon, \eta, -\epsilon) \leq \frac{2\gamma_0(c, \epsilon)}{T\delta_0} \lambda^{n-1} + O_T(\lambda^{n-3/2}).$$

The estimate in the third line uses the formula for the leading coefficient of Theorem [1,18] with $\psi(s) = \theta_T(\epsilon + s)$. Under the hypotheses of the Proposition, the sum of $\hat{\psi}(s_j^m)$ is just equal to $\hat{\psi}(0)$ and is independent of $T$, as explained above. This completes the proof of the claim.

(2) Assume $\tau = \ell \frac{1}{T} \epsilon_0$, $\ell \in \mathbb{Z}$. With no loss of generality, we may assume $\ell \geq 1$. Write

$$\mu^c(\epsilon, \eta, \eta, -\tau) - \mu^c(\epsilon, \eta, \eta, -\tau) = \sum_{j=1}^\ell \mu^c(\epsilon, \eta, \eta, -\tau) = \mu^c(\epsilon, \eta, \eta, -\tau) - \mu^c(\epsilon, \eta, \eta, -\tau)$$

and apply the estimate of (1) to upper bound the sum by

$$\frac{2\ell \gamma_0(c, \epsilon)}{T\delta_0} \lambda^{n-1} + O_T(\lambda^{n-3/2}) = \frac{2\gamma_0(c, \epsilon)}{\delta_0} \tau^{n-1} + O_T(\tau^{n-3/2}).$$

(3) Assume $\frac{\epsilon}{T} \epsilon_0 \leq \tau \leq \frac{\epsilon}{T} + 1 \epsilon_0$ with $\ell \in \mathbb{Z}$. Write

$$\mu^c(\epsilon, \eta, \eta, \tau) - \mu^c(\epsilon, \eta, \eta, \tau) = \mu^c(\epsilon, \eta, \eta, \tau) - \mu^c(\epsilon, \eta, \eta, \tau) + \mu^c(\epsilon, \eta, \eta, \tau) - \mu^c(\epsilon, \eta, \eta, \tau).$$

$$+ \mu^c(\epsilon, \eta, \eta, \tau) - \mu^c(\epsilon, \eta, \eta, \tau).$$
Applying (1) and (2), it follows that
\[|\mu_c^c([-\epsilon, \epsilon] + \tau) - \mu_c^c([-\epsilon, \epsilon])| \leq \frac{2\gamma_0(c, \epsilon)}{\delta_0} \left( \frac{\tau}{\epsilon_0} + \frac{1}{T} \right) \lambda^{n-1} + O_T, (\lambda^{n-3/2}).\]

This completes the proof of Proposition 7.3, hence also Proposition 7.2 and therefore Theorem 1.22. □

8. Appendix

8.1. Background on Fourier integral operators and their symbols. The advantage of expressing \( Y^{(1)}(t) \) in terms of pullback and pushforward is the symbol calculus of Lagrangian distributions is more elementary to describe for such compositions. We refer to [HoIV, GS77, D73] for background but quickly review the basic definitions.

The space of Fourier integral operators of order \( \mu \) associated to a canonical relation \( C \) is denoted by \( K_A \in I^\mu(M \times M, C') \).

If \( A_1 \in I^{\mu_1}(X \times Y, C'_1) \), \( A_2 \in I^{\mu_2}(Y \times Z, C'_2) \), and if \( C_1 \circ C_2 \) is a ‘clean’ composition, then by [HoIV, Theorem 25.2.3],
\[ A_1 \circ A_2 \in I^{\mu_1+\mu_2+e/2}(X \times Z, C'), \quad C = C_1 \circ C_2, \quad (8.1) \]
where \( e \) is the ‘excess’ of the composition, i.e. if \( \gamma \in C \), then \( e = \dim C_\gamma \), the dimension of the fiber of \( C_1 \times C_2 \cap T^*X \times \Delta_{T^*Y} \times T^*Z \) over \( \gamma \) (see (8.7) below).

Pullback and pushforward of half-densities on Lagrangian submanifolds are more difficult to describe. They depend on the map \( f \) being a morphism in the language of [GS77, page 349]. Namely, if \( f : X \to Y \) is a smooth map, we say it is a morphism on half-densities if it is augmented by a section \( r(x) \in \Hom(|\Lambda|^1/2(TY_{f(x)}, |\Lambda|^1/2 T_x X)) \), that is, a linear transformation mapping densities on \( TY_{f(x)} \) to densities on \( T_x X \). As pointed out in [GS77, page 349], such a map is equivalent to augmenting \( f \) with a special kind of half-density on the co-normal bundle \( N^*(\text{graph}(f)) \) to the graph of \( f \), which is constant along the fibers of the co-normal bundle. In our application, the maps are all restriction maps or pushforwards under canonical maps, and they are morphisms in quite obvious ways. Note that under pullback by an immersion, or under a restriction, the number \( n \) of independent variables is decreased by the codimension \( k \) and therefore the order goes up by \( k/4 \). Pullbacks under submersions increase the number \( n \). Pushforward is adjoint to pullback and therefore also decreases the order by the same amount.

Assume that \( F : M \to N \) is a smooth map between manifolds. Let \( \Lambda \subset \hat{T}^*N \) be a Lagrangian submanifold. Then its pullback is defined by,
\[ f^*\Lambda = \{(m, \xi) \in T^*M \mid \exists (n, \eta) \in \Lambda, f(m) = n, f^*\eta = \xi \}. \quad (8.2) \]

On the other hand, let \( \Lambda \subset \hat{T}^*M \). Then its pushforward is defined by,
\[ f_*\Lambda = \{(y, \eta) \in T^*N \mid y = f(x), (x, f^*\eta) \in \Lambda \}. \quad (8.3) \]

The principal symbol of a Fourier integral operator associated to a canonical relation \( C \) is a half-density times a section of the Maslov line bundle on \( C \). We refer to [HoIV, Section 25.2] and to [D73, Definition 4.1.1] for the definition; see also [DG75, GU89] for further
expositions and for several calculations of principal symbols closely related to those of this article.

The order of a homogeneous Fourier integral operator $A: L^2(X) \to L^2(Y)$ in the non-degenerate case is given in terms of a local oscillatory integral formula

$$K_A(x, y) = \frac{1}{(2\pi)^{n/4+N/2}} \int_{\mathbb{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta) d\theta$$

by

$$\text{ord} A = m + \frac{N}{2} - \frac{n}{4}, \quad \text{where } n = \dim X + \dim Y, \quad m = \text{ord } a$$

(8.4)

where the order of the amplitude $a(x, y, \theta)$ is the degree of the top order term of the polyhomogeneous expansion of $a$ in $\theta$, and $N$ is the number of phase variables $\theta$ in the local Fourier integral representation (see [HoIV, Proposition 25.1.5]); in the general clean case with excess $e$, the order goes up by $\frac{e}{2}$ (see [HoIV, Proposition 25.1.5']). The order is designed to be independent of the specific representation of $K_A$ as an oscillatory integral.

Further, the principal symbol of a Fourier integral distribution $I(x, y) = \int_{\mathbb{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta) d\theta$ with non-degenerate homogeneous phase function $\varphi$ and amplitude $a \in S^0_{cl}(M \times M \times \mathbb{R}^N)$, is the transport to the Lagrangian $\Lambda_\varphi = \iota_\varphi(C_\varphi)$ of $a(\lambda)\sqrt{d_{C_\varphi}}$ where $\sqrt{d_{C_\varphi}}$ is the half density given by the square root of

$$d_{C_\varphi} := \left|\frac{\partial(\lambda, \varphi'_\theta)}{\partial(x, y, \theta)}\right|^{-1} |d\lambda|$$

(8.5)

on $C_\varphi$, where $\lambda = (\lambda_1, ..., \lambda_n)$ are local coordinates on the critical manifold $C_\varphi = \{(x, y, \theta); d_{\theta}\varphi(x, y, \theta) = 0\}$.

We next review the definition of the excess in a fiber product diagram. Let $F = \{(x, y) \in X \times Y, f(x) = g(y)\}$,

$$\begin{align*}
X & \xleftarrow{f} \quad F \\
F & \quad \downarrow \\
Z & \xleftarrow{g} \quad Y
\end{align*}$$

(8.6)

The maps $f : X \to Z$ and $g : Y \to Z$ are said to intersect cleanly if the fiber product $F$ is a submanifold of $X \times Y$ and if the tangent diagram is a fiber product diagram. The excess is

$$e = \dim F + \dim Z - (\dim X + \dim Y).$$

(8.7)

Then $e = 0$ if and only if the diagram is transversal. Above $d = \dim \text{Fix}(G^T)$ is the excess of the diagram.

8.2. Enhancement, morphisms and pullbacks and pushforward of symbols. The behavior of symbols under pushforwards and pullbacks of Lagrangian submanifolds is described in [GS77], Chapter IV. 5 (page 345). The main statement (Theorem 5.1, loc. cit.) states that the symbol map $\sigma : I^m(X, \Lambda) \to S^m(\Lambda)$ has the following pullback-pushforward
properties under maps \( f : X \to Y \) satisfying appropriate transversality conditions,

\[
\begin{align*}
\sigma(f^*\nu) &= f^*\sigma(\nu), \\
\sigma(f_*\mu) &= f_*\sigma(\mu),
\end{align*}
\tag{8.8}
\]

To be precise, \( f \) must be “enhanced” as defined in [GS13, Chapter 7] in order to define a pullback or pushforward on symbols. This is because the pullback/pushforward of a half-density on \( \Lambda \) is often not a half-density on \( f^*\Lambda \).

The enhancement of a smooth map \( f : X \to Y \) is a map \((f, r)\) with \( r : |fr^*T Y|^{\frac{1}{2}} \to |TX|^{\frac{1}{2}} \). Thus if \( \rho \) is a half-density on \( Y \)

\[(f, r)^*\rho = r(\rho(f(x)) \in |T_x X|^{\frac{1}{2}}.
\]

If \( f : X \to Y \) is an immersion, then \( N_*^r X \) consists of covectors \( \xi \in T_{\xi(x)}Y : d\iota_\xi^*\xi = 0 \).

Enhancing an immersion is giving a section of \( |N_*^r X|^{\frac{1}{2}} \).

If \( \pi : Z \to X \) is a submersion and \( V_x \) is the tangent space to \( \pi^{-1}(x) \). Then enhancing the fibration is giving a section of \( |V_x|^{\frac{1}{2}} \).

In [GS13, p. 349], the authors explain that enhancement with \( r \) is to define a half-density on \( N^r(\Gamma_f) \) which is constant along the fibers of \( N^r(\Gamma_f) \to \Gamma_f \). As a result, a morphism \( f : X \to Y \) induces a pushforward

\[f_* : \Omega^{\frac{1}{2}}(\Lambda_X) \to \Omega^{\frac{1}{2}}(f_*\Lambda_X).
\]

It also induces a pullback operation

\[f^* : \Omega^{\frac{1}{2}}(\Lambda_Y) \to \Omega^{\frac{1}{2}}f^*(\Lambda_Y).
\]

Under appropriate clean or transversal assumptions, if \( f : X \to Y \) is a morphism of half-densities, then \( f_* \) and \( f^* \) are morphisms of half-densities on Lagrangian submanifolds.

**Remark 8.1.** If \( f : X \to Y \) is a submersion then \( f^* \) is injective. Indeed if \( f^*\eta = 0 \) then \( \eta \perp f_*TX = TY \). If \( f \) is an immersion, then \( f_* \) is injective.

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