DOUBLE COSET PROBLEM FOR PARABOLIC SUBGROUPS OF BRAID GROUPS

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ABSTRACT. We provide the first solution to the double coset problem (DCP) for a large class of natural subgroups of braid groups, namely for all parabolic subgroups which have a connected associated Coxeter graph.

1. Outline

Section 2 recalls basic decision problems, defines the DCP and related problems, and states the main result. The Main Theorem is proven in section 3. Some open questions are discussed in the last section.

2. Double coset problem and other decision problems

Decision problems, like the word problem, the conjugacy problem and the subgroup membership problem, play an important role in combinatorial group theory. The subgroup membership problem, given a subgroup \( H \leq G \), decide whether an element \( g \in G \) lies in \( H \), is also referred to as occurrence problem. In braid groups the word problem and the conjugacy problem were solved by Artin \([Ar26, Ar47]\) and Garside \([Ga69]\), respectively. The occurrence problem for subgroups of \( B_n \) was shown to be unsolvable for \( n \geq 5 \) in \([Ma81]\). There, unsolvability was proved by means of a famous theorem of Mikhailova \([Mi58, Mi66]\), which states that a sufficient condition for the unsolvability of the occurrence problem is the presence of a subgroup isomorphic to \( F_2 \times F_2 \). In \([Ak91]\) it is stated (without proof) that the occurrence problem in \( B_3 \) is solvable. Mikhailova’s sufficient unsolvability criterion fails to work in \( B_4 \). Indeed, it is shown in \([Ak91]\) that \( B_4 \) does not contain any subgroups isomorphic to \( F_2 \times F_2 \).

Another important problem is the double coset problem.

Definition 2.1. Let \( G \) be a group and \((A, B)\) a pair of proper subgroups of \( G \). The double coset problem for \((A, B)\) in \( G \) is, given any pair \((g, g')\) in \( G^2 \), decide whether \( g' \) lies in the double coset \( AgB \), i.e., decide whether there exist elements \( a \in A \) and \( b \in B \) such that \( g' = agb \).

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Theorem 2.2. The double coset problem for subgroups of the $n$-strand braid group is unsolvable for $n \geq 5$.

Proof. This is a trivial reduction from the subgroup membership problem. An algorithm that solves the DCP for any pair of subgroups of $B_n$ will solve it in particular for $(H, \{1\})$. Let $x \in G$ be our instance element for the occurrence problem of $H$ in $B_n$. Since $x = x \cdot 1 \cdot 1$, our $(H, \{1\})$-DCP oracle provides a solution $(x, 1) \in H \times \{1\}$ to the instance pair $(1, x) \in B_n^2$ if and only if $x \in H$.

The unsolvability result for $n \geq 5$ is a simple corollary from [Mi58]. □

Nevertheless, it is interesting to consider the double coset problem for particular natural subgroups of $B_n$. Indeed, even the DCP for the natural embedded subgroups $B_m \leq B_n$ for $m < n$, has been an open problem. Our main result is the solution of the DCP for this class of subgroups. We may extend this result to parabolic subgroups of $B_n$ with a connected associated Coxeter graph, in the following sense of Paris [Pa97]:

Definition 2.3. A subgroup $H$ of the braid group $B_n$ is called parabolic with a connected associated Coxeter graph if it is conjugate to $B_{[k,m]} = \langle \sigma_k, \sigma_{k+1}, \ldots, \sigma_{m-1} \rangle$ for some $1 \leq k < m \leq n$.

Main Theorem 2.4. Let $A, B$ be parabolic subgroups of $B_n$ with a connected associated Coxeter graph. Then the double coset problem for $(A, B)$ in $B_n$ is solvable.

We solve the problem by reduction to an instance of the simultaneous conjugacy problem. The simultaneous conjugacy problem in braid groups was solved in [LL02]. Nevertheless, the invariant subsets of the simultaneous conjugacy class involved in Lee and Lee’s solution [LL02] are relatively big. In [KTV13], we introduce new much smaller invariant subsets of the simultaneous conjugacy class, namely the so-called Lexicographic Super Summit Sets.

A central ingredient of our proof is a result on the centralizer of the centralizer of such parabolic subgroups which was proven in [GKL13].

Main Lemma 2.5. Let $H$ be parabolic subgroup of $B_n$ with a connected associated Coxeter graph. Then the centralizer of the centralizer of $H$ is given by:

$$C_{B_n}(C_{B_n}(H)) = \langle \Delta^2_n \rangle \cdot H.$$  

Recall that, for $n \geq 3$, the center $Z(B_n)$ is infinite cyclic, generated by $\Delta^2_n$, where $\Delta_n$ denotes the so-called fundamental element.

Remark 2.6. A related problem which lies somehow between the double coset problem and the conjugacy problem is the subgroup conjugacy problem. More
precisely, the subgroup conjugacy problem for $H \leq G$ is, given $a, b \in G$, decide whether there exists a $c \in H$ such that $b = c^{-1}ac$.

In [KLT09], the subgroup conjugacy problem for $B_{n-1} \leq B_n$ was transformed to an equivalent bisimultaneous conjugacy problem. Then, in [KLT10], the subgroup conjugacy problem for all parabolic subgroups of braid groups, even for all so-called Garside subgroups [Go07] of Garside groups, was solved completely, and deterministic algorithms were provided. The solution in [KLT10] does not resort to a detour via a simultaneous conjugacy problem. [GKLT13] provides a second solution of the subgroup conjugacy problem for parabolic subgroups of $B_n$ with a connected Coxeter graph. This solution is a generalization of the approach developed in [KLT09], namely we reduce the problem to an instance of a simultaneous conjugacy problem. Though less general, this second solution is of relevance, because it provides possibly smaller invariant subsets, and therefore leads to a more efficient solution. Furthermore, the Main Lemma is proven in [GKLT13], which open the path to the solution of the DCP for these parabolic subgroups of $B_n$.

### 3. Proof of Main Theorem

In this section we prove the Main Theorem, i.e., we show a reduction of DCP to simultaneous conjugacy. Our solvability result for DCP (for the considered class of parabolic subgroups) then follows from the solvability of simultaneous conjugacy in braid groups. First we transform the parabolic subgroups into some standard form.

We need the following notion. Let $\partial : B_\infty \rightarrow B_\infty$ be the injective shift homomorphism, defined by $\sigma_i \mapsto \sigma_{i+1}$.

**Definition 3.1.** ([De00, Definition I.4.6.]) For $n \geq 2$, define $\delta_n = \sigma_{n-1} \cdots \sigma_2 \sigma_1$. For $p, q \geq 1$, we set:

$$\tau_{p,q} = \delta_{p+1} \partial(\delta_{p+1}) \cdots \partial^{q-1}(\delta_{p+1}),$$

i.e. the strands $p+1, \ldots, p+q$ cross over the strands $1, \ldots, p$.

**Lemma 3.2.** Let $A, B$ be parabolic subgroups of $B_n$ with a connected associated Coxeter graph. The $(A, B)$-DCP-instance $(g, g')$ is solvable if and only if a $(B_{m_A}, B_{m_B})$-instance $(g_1, g'_1)$ is solvable, where $m_A, m_B \in \mathbb{N}$ are solely given by $A, B$, respectively.

More precisely, write $B_{[k,l]}$ for the standard parabolic subgroup $\langle \sigma_k, \ldots, \sigma_{l-1} \rangle$. Let $\alpha, \beta \in B_n$ be fixed such that $\alpha^{-1}A\alpha = B_{[k_A, l_A]}$ and $\beta^{-1}B\beta = B_{[k_B, l_B]}$ for some $k_A, l_A, k_B, l_B$ with $k_A < l_A$ and $k_B < l_B$. Then the $(A, B)$-DCP-instance $(g, g')$ is solvable if and only if the $(B_{m_A}, B_{m_B})$-instance $(g_1, g'_1)$ is solvable, where
\[ m_A = l_A - k_A + 1, \quad m_B = l_B - k_B + 1, \quad g_1 = \tau_A \alpha^{-1} g \beta \tau_B^{-1} \text{ and } g'_1 = \tau_A \alpha^{-1} g' \beta \tau_B^{-1} \]

for some fixed braids \( \tau_A = \tau_{m_A,k_A-1} \) and \( \tau_B = \tau_{m_B,k_B-1} \).

**Proof.** Observe that \( a \in A \) if and only if \( \alpha^{-1} a \alpha \in B_{[k_A,l_A]} \) if and only if \( a_1 := \tau_A \alpha^{-1} a \alpha \tau_A^{-1} \in B_{m_A} \). Analogously, we have \( b \in B \) if and only if \( b_1 := \tau_B \beta^{-1} b \beta \tau_B^{-1} \in B_{m_B} \). If we write \( g' = agb \), then we may also write

\[ g'_1 = \tau_A \alpha^{-1} g' \beta \tau_B^{-1} = \tau_A \alpha^{-1} (agb) \beta \tau_B^{-1} = \tau_A \alpha^{-1} a \alpha \tau_A^{-1} (\tau_A \alpha^{-1} g \beta \tau_B^{-1}) \tau_B \beta^{-1} b \beta \tau_B^{-1} = a_1 g_1 b_1. \]

We conclude that the \((A,B)\)-DCP-instance \((g, g')\) has solution \((a, b) \in A \times B\) if and only if the \((B_{m_A}, B_{m_B})\)-DCP-instance \((g_1, g'_1)\) has solution \((a_1, b_1) \in B_{m_A} \times B_{m_B}\). \( \square \)

**Theorem 3.3.** Let \( A, B \) be parabolic subgroups of \( B_n \) with a connected associated Coxeter graph. The DCP for \((A,B)\) in \( B_n \) reduces in polynomial time to a simultaneous conjugacy problem.

More precisely, consider the \((A,B)\)-DCP-instance \((g, g') \in B_n^2\), i.e., decide whether \( g' = agb \) for some \( a \in A \) and \( b \in B \). The centralizers of \( A \) and \( B \) in \( B_n \) are finitely generated by \( c_1, \ldots, c_{k_A} \) and \( d_1, \ldots, d_{k_B} \) for some \( k_A, k_B \in \mathbb{N} \), respectively. Then this DCP-instance for \((A,B)\) in \( B_n \) can be solved by using the solution (if it exists) of the following simultaneous conjugacy problem (simCP).

\[
\begin{align*}
\left( * \right) \quad c_i &= x c_i x^{-1} \quad \forall \ i = 1, \ldots, k_A, \\
\quad g'd_j(g')^{-1} &= x(gd_jg^{-1})x^{-1} \quad \forall \ j = 1, \ldots, k_B.
\end{align*}
\]

**Proof.** The centralizer of a parabolic subgroup of \( B_n \) with a connected associated Coxeter graph is finitely generated. It was explicitly computed in \cite{FRZ96}. This result was extended to parabolic subgroups with connected associated Coxeter graph of Artin groups of type \( B \) and \( D \) in \cite{Pa97}.

According to Lemma 3.2 we may assume that \( A = B_{m_A} \) and \( B = B_{m_B} \) for some \( m_A, m_B < n \). If \( m_A \) or \( m_B \) equals \( n \), then the DCP becomes easy. It amounts to checking membership in a standard parabolic subgroup which can be accomplished, e.g., by means of Thurston normal forms \cite{EC+92}.

First, assume that \( g' \in AgB \), i.e., the DCP-instance \((g, g')\) has a solution \( a \in A \), \( b \in B \). Then \( a \in A \subset B_n \) solves the simCP-instance \((\ast)\), because \( a \) commutes with all \( c_i \) and we have

\[ g'd_j(g')^{-1} = (agb)d_j(b^{-1}g^{-1}a^{-1}) = agd_jg^{-1}a^{-1} \quad \forall \ j = 1, \ldots, k_B. \]

Therefore, our simCP-oracle will return some solution \( \hat{a} \in G = B_n \) on input \((\ast)\). Since \( \hat{a} \) commutes with all \( c_i \), we conclude that \( \hat{a} \) lies in \( C_G(C_G(A)) \) MainLemma

\[ A \cdot Z(G). \] Since \( m_A < n \), \( A \) is a proper parabolic subgroup of \( B_n \). Therefore, there exists a unique decomposition \( \hat{a} = \Delta^k \hat{a} \) with \( k \in \mathbb{Z} \) and \( \hat{a} \in A \).
Also, since

\[ ac_i a^{-1} = \bar{a} c_i \bar{a}^{-1} \iff [\bar{a}^{-1} a, c_i] = 1 \quad \forall i \leq k_A, \]

we conclude that

\[ a^{-1} \bar{a} \in \bigcap_{i=1}^{k_A} C_G(c_i) = C_G(C_G(A)) \]

Remark 3.5 that

\[ \Delta^2 \]

Clearly, by definition of \( \hat{a} \), we have \( a \). It remains to show that \( \hat{b} \in B \). A straightforward computation yields

\[ \hat{b} = g^{-1} \Delta^2 \bar{a}^{-1} g' = g^{-1} \Delta^2 g^{-1} \Delta^2 \bar{a}^{-1} g = \Delta^2 \bar{a}^{-1} b. \]

Therefore, it suffices to show that \( k = l \). To establish that, we identify the two equations (1), (2) for \( \hat{a} \). We obtain \( \Delta^2 = gb^{-1} \bar{a}^{-1} \), and conclude by the following Lemma 3.5 that 2(k − l) = 0 \iff k = l. \]

**Remark 3.4.** Given a proper standard parabolic subgroup \( H \) of \( B_n \) and an element \( a \in Z(B_n) \cdot H \), one may find the unique decomposition \( a = \Delta^2 \hat{a} \) for some \( q \in \mathbb{Z} \) and \( \hat{a} \in H \) as follows. Let \( \eta_H \) be the map that removes all but one strand from \( H \). This trivializes \( \hat{a} \) without trivializing the \( \Delta^2 \)-power. More precisely, if we remove \( m - 1 \) strands in \( H \), we get \( \eta_H(a) = \Delta^{q_{m+1}} \). Now, one may compute the Garside normal form of \( \eta_H(a) \) to determine \( q \in \mathbb{Z} \). Hence, we get \( \hat{a} = \Delta^{q} a \in H \). **Lemma 3.5.** Let \( H_1, H_2 \) be proper parabolic subgroups of the \( n \)-strand braid group \( B_n \). Consider elements \( h_1 \in H_1, h_2 \in H_2 \) and \( g \in G = B_n \). If \( gh_1 g^{-1} h_2 = \Delta^2 \) then \( k = 0 \).
Proof. (i) The reduction from parabolic subgroups to standard parabolic subgroups of $B_{m_i}$ ($i = 1, 2$) is again straightforward. Therefore, and since $H_1, H_2$ are proper subgroups, we may assume that $H_1 \leq B_{m_1}$ and $H_2 \leq B_{m_2}$ for some $m_1, m_2 < \eta$. Wlog we may also assume that $m_1 \geq m_2$. If not, we conjugate (from the right) by $h_2^{-1} g$ to obtain instance elements that satisfy that condition.

(ii) Recall that the pure braid group $P_n$ is the kernel of the homomorphism $\nu : B_n \rightarrow S_n$ that maps each braid to its induced permutation on the strands. Here we consider a right action of the braid group on the set $[\eta] := \{1, \ldots, \eta\}$ which enumerates the left end points of the strands. If $g$ is a pure braid, or more generally, if $\nu(g)$ maps $[m_1]$ onto itself, then we apply the map (not a homomorphism!) $\eta_{H_1} : B_n \rightarrow B_{n-m_1}$ which erases (or removes) all but one of the first $m_1$ strands (labelled by their left end points), say all but strand $m_1$. The $B_n$-equation $gh_1g^{-1}h_2 = \Delta_n^{2k}$ transforms to the following identity in $B_{n-m_1}$.

$$\eta_{H_1}(g) \cdot 1 \cdot \eta_{H_1}(g^{-1}) \cdot 1 = \Delta_n^{2k}.$$

Since $m_1 < \eta$ which implies $n - m_1 + 1 \geq 2$, we conclude from $\Delta_{n-m_1+2} \neq 1$ that $k = 0$. Note that, since $\Delta_1 = 1$, we removed all but one strand from $B_{m_1}$ (which affects the case $m_1 = n - 1$).

(iii) Now, assume there exists an $i \in [m_1]$ such that $\nu(g)(i) = \tilde{i} \geq m_1 + 1$. Then there also exists a $j \geq m_1 + 1$ such that $\nu(g)(j) =: \tilde{j} \in [m_1]$. Since $gh_1g^{-1}h_2 = \Delta_n^{2k}$ is pure, we have $\nu(gh_1g^{-1}h_2)(j) = j$. From $m_2 \leq m_1 < j$ and $h_2 \in B_{m_2}$ we get $\nu(gh_1g^{-1})(j) = j$ and $\nu(g)(j) = \tilde{i}$ implies $\nu(h_1)(\tilde{i})$.

Now, we may evaluate the algebraic crossing number of the strand pair $(i, j)$ in the braid $gh_1g^{-1}h_2 = \Delta_n^{2k}$.

$$cr(gh_1g^{-1}h_2)(i, j) = \begin{cases} cr(g)(i, j) + cr(h_1)(j, \tilde{i}) + cr(g^{-1})(\tilde{j}, \tilde{i}) + cr(h_2)(i, j) \\ cr(g)(i, j) + 0 - cr(g)(i, j) + 0 = 0 = cr(\Delta_n^{2k})(i, j) = 2k. \end{cases}$$

\[\square\]

4. Open problems

We list possibilities for further work, generalizations and some open questions.

- Extend the Main Theorem to all parabolic subgroups. This relies solely on an extension of the Main Lemma [GKLT13] to all parabolic subgroups of $B_n$ - not only those with connected associated Coxeter graph.

- Extend the Main Theorem to parabolic subgroups of Artin groups of type B and D. To accomplish that one has to algebraize our proof which still relies on some geometric ideas notions like erasing strands and crossing number. This remark also applies to the Main Lemma.
• According to [Ak91] the occurrence problem is solvable for subgroups of $B_3$. Is the double coset problem also solvable for all subgroup pairs of $B_3$?
• Can one establish also an unsolvability (or hardness) result for the occurrence problem and the DCP for subgroups of $B_3$?
• Is it possible to reduce the occurrence problem to the subgroup conjugacy problem? This would establish the unsolvability of the subgroup conjugacy problem for $B_n, n \geq 5$.

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