A Proposal to Measure Photon-Photon Scattering

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We discuss a possibility to measure the photon-photon scattering cross section at low energy in a theoretical standpoint. The cross section of photon-photon scattering at low energy can be written as

$$\frac{d\sigma}{d\Omega} \approx \alpha^4 \left( \frac{12}{\pi^2} \right) \omega^2 \left( 3 + 2 \cos^2 \theta + \cos^4 \theta \right)$$

(with $\omega \ll m$) (1.1)

where $\omega$ and $m$ denote the photon energy and the mass of electron, respectively. This is calculated in the center of system of two photons. This cross section is quite reasonable in that it is described in terms of the energy of photon only. There appears no internal energy scale such as electron mass which should appear in the box diagrams, and this is because the cross section is evaluated in the low energy limit.

On the other hand, the photon-photon cross section was obtained by the classical picture of scattering procedure long time ago, and it is written as

$$\frac{d\sigma}{d\Omega} \approx \frac{139\alpha^4}{(180\pi)^2 m^2} \left( \frac{\omega}{m} \right)^6 \left( 3 + \cos^2 \theta \right)^2$$

(with $\omega \ll m$) (1.2)

which was calculated by Heisenberg and Euler in 1936 [2], and later the result is confirmed by Karplus and Neuman [4]. Since then, it is believed that this photon-photon cross section is the correct one, even though the quantum evaluation of the Feynman diagram gives the cross section of eq.(1.1) unless one should put some additional but unphysical conditions. In addition, the effective Lagrangian method proposed by Heisenberg and Euler for the vacuum polarization effects is physically incorrect since it disagrees with the observation that photon is always massless [5].

Up to now, the measurements of the photon-photon scattering cross section have been made by Moulin et al. [6, 7], but they found no evidence of the photon-photon scattering. However the measurements with the sufficient accuracy must be very difficult since photon cannot be at rest but always at the speed of light. Most of the scattering experiments are the collision experiment of the incident particle with some fixed targets which are basically taken to be at rest. In this respect, the photon-photon scattering must be quite new to the conventional experiments, and therefore this experiment must be one of the most important experiments in particle physics, which is left almost untouched until now. In this sense, the main difficulty must be connected to the initial condition of the scattering experiments in which one should control the time of photon-photon collision and the focusing of the photon-photon beams. In addition, the photon-photon scattering is the reaction process arising from the particle nature of photon, in contrast to the wave nature of photon such as diffraction or interference phenomena.

In this respect, we believe that the $\gamma + \gamma \rightarrow e^+ + e^-$ experiment should be first used as the monitor of the initial state condition check of the photon-photon scattering experiment. The reaction cross section of $\gamma + \gamma \rightarrow e^+ + e^-$ is

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I. INTRODUCTION

It is well known that photon interacts with photon via the box diagrams where fermions and anti-fermions are created from the vacuum state. This calculation of the Feynman diagrams can be carried out in a straightforward fashion, and one can find the cross section of photon-photon scattering in the low energy region [1]

$$\frac{d\sigma}{d\Omega} \approx \frac{\alpha^4}{(12\pi)^2 \omega^2} \left( 3 + 2 \cos^2 \theta + \cos^4 \theta \right)$$

(1.1)

where $\omega$ and $m$ denote the photon energy and the mass of electron, respectively. This is calculated in the center of system of two photons. This cross section is quite reasonable in that it is described in terms of the energy of photon only. There appears no internal energy scale such as electron mass which should appear in the box diagrams, and this is because the cross section is evaluated in the low energy limit.

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In this respect, we believe that the $\gamma + \gamma \rightarrow e^+ + e^-$ experiment should be first used as the monitor of the initial state condition check of the photon-photon scattering experiment. The reaction cross section of $\gamma + \gamma \rightarrow e^+ + e^-$ is

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the same order of magnitude as the Compton scattering when the incident photon energy is larger than a few MeV. Therefore, it is crucial that the experimental setup should be able to reproduce the cross section of $\gamma + \gamma \rightarrow e^+ + e^-$ process. Even though the photon-photon cross section should be smaller than the $\gamma + \gamma \rightarrow e^+ + e^-$ by several order of magnitudes in a few MeV region, it should be possible to measure the photon-photon elastic cross section once the problem of the initial condition is resolved. We believe that the measurement of two photons should not be very difficult indeed, even though a very small number.

**II. QUALITATIVE BEHAVIOR OF LOOP DIAGRAMS**

Here, we should clarify why the photon-photon cross section should have the shape of eq.(1.1) which does not depend on the ratio of $(\omega/M)^2$ in contrast to eq.(1.2). In order to understand the situation clearly, we first compare the box diagram of the photon-photon scattering with the triangle diagram of $\pi^0 \rightarrow 2\gamma$ since there is a good similarity between them. Namely, both diagrams contain one loop of fermions.

### A. $\pi^0 \rightarrow 2\gamma$ Decay

Now, the T-matrix of $\pi^0 \rightarrow 2\gamma$ can be evaluated to be

$$
T_{\pi^0 \rightarrow 2\gamma} \simeq e^2 g \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \frac{1}{p-M+i\varepsilon} (\gamma\epsilon_1) \frac{1}{p-k_1-M+i\varepsilon} (\gamma\epsilon_2) \frac{1}{p+k_2-M+i\varepsilon} \gamma_5 \frac{1}{p+k_1-M+i\varepsilon} \right] \simeq \frac{e^2 g}{M} \varepsilon_{\mu\nu\rho\sigma} \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \epsilon_4^\sigma (k_1^\rho k_2^\sigma) $$

(2.1)

where $M$ denotes the nucleon mass, and $g$ is the coupling constant of $\pi N$ interaction as described by $\mathcal{L}_t = ig\bar{\psi}\gamma_5\psi\phi$. The energy of two photons can be written as $\omega_1 = |k_1|, \omega_2 = |k_2|$. Under the condition of $M >> \omega_1, \omega_2$, one finds the leading behavior of the finite terms, apart from some numerical factors

$$
T_{\pi^0 \rightarrow 2\gamma} \sim e^2 g \left( \frac{\mu^2}{M} \right) \left( 1 + O \left( \frac{\mu}{M} \right)^2 + .. \right)
$$

(2.2)

where $\mu$ denotes the mass of $\pi^0$. It should be noted that the apparent divergences of the T-matrix are kinematically cancelled out in an exact fashion, and thus it is not due to the regularization. Even though this problem is well explained in detail in the textbook of Nishijima [8], it may be better to make a comment on the relation between the $\pi^0 \rightarrow 2\gamma$ and the chiral anomaly, in order to avoid any confusions which are sometimes found in the literatures. If one calculates the $\pi^0 \rightarrow 2\gamma$ case, then one sees immediately that there is no divergence. However, educated people may invoke a triangle anomaly in which there is an apparent linear divergence when the vertex is the axial vector current, instead of pseudoscalar interaction. However, this process, the origin of the chiral anomaly, has no divergence either, if one calculates the Feynman diagrams properly, without referring to the textbook description. Therefore, one can convince oneself that any physically observable processes have no divergence at all, and this is very reasonable indeed.

### B. Photon-Photon Scattering

Now, a typical T-matrix of the box diagrams in the photon-photon scattering can be written as

$$
T_{\gamma-\gamma} \simeq e^4 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ (\gamma\epsilon_1) \frac{1}{p-k_1-k_2-m} (\gamma\epsilon_2) \frac{1}{p-k_3-m} (\gamma\epsilon_3) \frac{1}{p-k_4-m} (\gamma\epsilon_4) \right] \simeq \frac{e^4}{M} \varepsilon_{\mu\nu\rho\sigma} \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \epsilon_4^\sigma (k_1^\rho k_2^\sigma)
$$

(2.3)

where the energy of photon can be written as $\omega = |k_1| = |k_2| = |k_3| = |k_4|$ at the center of mass system of two photons. The leading behavior of the finite terms in this T-matrix can be easily evaluated under the condition of $m >> \omega$ as

$$
T_{\gamma-\gamma} \sim \alpha^2 \left[ 1 + c_1 \left( \frac{\omega}{m} \right)^2 + c_2 \left( \frac{\omega}{m} \right)^4 + .. \right]
$$

(2.4)

where $c_1$ and $c_2$ denote some numerical constants. It should be important to note that the apparent divergences can be completely cancelled out due to the kinematical cancellation by adding up three independent Feynman diagrams together, and the disappearance of the divergences is not due to the regularization [1]. As can be seen from eq.(2.4),
the photon-photon cross section of Heisenberg-Euler in eq.(1.2) can be reproduced when one picks up the third term ($c_2$ term) in eq.(2.4), and this is very strange from the point of view of the Feynman diagram evaluation.

In terms of the reaction process, the total energy scale of the photon-photon cross section should be given by $2\omega$ which, in fact, appears in the denominator of eq.(1.1). Therefore, eq.(1.1) can be well understood since the T-matrix is given as eq.(2.4) at the low energy limit. On the other hand, the photon-photon cross section of eq.(1.2) can be obtained only when one picks up the third terms in eq.(2.4). The physical reason as to why eq.(1.2) cannot be justified is well explained in detail in [1] in terms of the modern field theory terminology. Basically, one can see that the expression of eq.(1.2) can be obtained only when one picks up the smallest piece in eq.(2.4), and this is obtained only when one requires some unphysical conditions for the box diagrams in terms of the gauge conditions [9], which will be clarified in Appendix A.

III. POSSIBLE EXPERIMENTS

If the photon-photon cross section of eq.(1.2) were correct, then there was no chance to observe it in terms of the laser-laser scattering experiment since the energy of the laser is mostly lower than a few tens of eV. However, the photon-photon scattering experiment at high energy must have a serious difficulty since the experiment should be done as the head-on collision in the center of mass system of two photons, and the control of the high energy photon flux must be non-trivially difficult.

On the other hand, the situation is completely different if one should observe the photon-photon cross section of eq.(1.1). In this case, we should consider the laser-laser scattering where the energy of the laser is around $\theta$ or lower. If the measurement is carried out at $\theta = 90$ degree, then the cross section of eq.(1.1) at $\omega \simeq 1$ eV becomes

$$
\frac{d\sigma}{d\Omega} \sim \frac{3\alpha^4}{(2\pi)^2 \omega^2} \simeq 2.3 \times 10^{-21} \text{ cm}^2 \simeq 2.3 \times 10^6 \text{ mb/st}
$$

(3.1)

which should be well detectable. On the other hand, the cross section of eq.(1.2) at $\omega \simeq 1$ eV becomes

$$
\frac{d\sigma}{d\Omega} \sim \frac{3 \times 417\alpha^4}{(180\pi)^2 m^2} \left( \frac{\omega}{m} \right)^6 \simeq 9.3 \times 10^{-67} \text{ cm}^2 \simeq 9.3 \times 10^{-40} \text{ mb/st}
$$

(3.2)

which is extremely small, and it is impossible to detect in any of the experiments. We note that the above cross section of eq.(1.2) becomes larger for larger photon energy. In fact, the cross section at $\omega \simeq 1$ MeV becomes

$$
\frac{d\sigma}{d\Omega} \sim \frac{417\alpha^4}{(180\pi)^2 m^2} \left( \frac{\omega}{m} \right)^6 \simeq 9.3 \times 10^{-31} \text{ cm}^2 \simeq 9.3 \times 10^{-4} \text{ mb/st}.
$$

(3.3)

However, in this energy region, the low energy approximation is not well satisfied even though, we believe, the order of magnitude estimation must be correct. Unfortunately, there is no such high energy laser available at present.

A. Comparison with $e^+ + e^- \rightarrow e^+ + e^-$ Scattering

In terms of the reaction process, the photon-photon scattering must be similar to the $e^+e^-$ elastic scattering. The cross section of the $e^+e^-$ elastic scattering process at high energy limit is given as

$$
\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2}{8E^2} \left[ \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{1 + \cos^2 \theta}{2} - \frac{2 \cos^4(\theta/2)}{\sin^2 \theta/2} \right].
$$

(3.4)

The typical number of this cross section can be seen from experiment at $E = 17$ GeV and $\theta = 90$ degree, and it becomes $\frac{d\sigma}{d\Omega} \simeq 1.6 \times 10^{-7}$ mb/st. In order to find a naive estimation of the cross section at around $E \sim 2$ MeV, we extrapolate the energy dependence of eq.(3.4), and thus we find $\frac{d\sigma}{d\Omega} \sim 12$ mb/st. On the other hand, the photon-photon cross section of the present estimation at $\omega \sim 2$ MeV becomes $\frac{d\sigma}{d\Omega} \sim 6 \times 10^{-7}$ mb/st which is smaller than the $e^+e^-$ elastic scattering cross section by 7 order of magnitudes. This naive estimation of the photon-photon cross section indicates that the cross section becomes quite large at low energy. However, in comparison with the head-on collisions between the $e^+e^-$ elastic scattering cross section and the photon-photon cross section at 1 MeV incident energy, the photon-photon cross section is smaller than the $e^+e^-$ elastic scattering cross section by several orders of magnitude.
B. Comparison with $\gamma + \gamma \rightarrow e^+ + e^-$ Scattering

If the energy of photon is larger than a few MeV, then we have to consider the scattering process in which the photon-photon scattering can produce the electron positron pair, that is, $\gamma + \gamma \rightarrow e^+ + e^-$. This cross section is the same order as the $e^+e^-$ elastic scattering cross section, and therefore, at higher energy than a few MeV, the photon-photon scattering process must be dominated by the $\gamma + \gamma \rightarrow e^+ + e^-$ cross section.

In this respect, the $\gamma + \gamma \rightarrow e^+ + e^-$ should be used as the monitor of the reaction process before carrying out the photon-photon elastic scattering. This is clear since the main difficulty of the photon-photon scattering should be concerned with the initial conditions of photon-photon reaction, and therefore one should examine the validity of the reaction process first by carrying out the $\gamma + \gamma \rightarrow e^+ + e^-$ experiment. It should be noted that the photon-photon elastic cross section must be smaller than the $\gamma + \gamma \rightarrow e^+ + e^-$ reaction cross section by several orders of magnitudes. However, we believe it should be observed as long as we can judge from the magnitude of the cross section.

IV. DISCUSSIONS

The real photon-photon cross section is much larger than the old expression which was obtained in the classical picture of the field theory. However, one can see that the reaction process can be observed only when the scattering should occur. The cross section we discuss is related to the probability of the scattering process when two photons collide. The basic difficulty of this scattering problem is indeed related to the fact that this scattering process is only possible for the head-on collision. Namely, the initial condition of the scattering process must be most difficult when setting up the photon-photon scattering experiment. In the case of Compton scattering, photon can scatter with electrons which are almost at rest while photon can interact with another photon only at the head-on collision since there is no photon at rest. In addition, both photons collide with each other at the speed of light. In this case, the focusing procedure must be made only in terms of the mechanical tools, in contrast to the $e^+e^-$ scattering process in which electrons can be controlled by the magnetic fields.

What should be any realistic effects of the photon-photon scattering in nature? It is most likely true that the possibility of photon-photon head-on collision which may happen in nature must be extremely small, and the only possible example may be found in the center of star where very many photons are created during the nuclear fusion processes, and these photons may collide with each other.

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Appendix A: Comments on Gauge Invariance

There is a serious misunderstanding among some of the educated physicists concerning the gauge invariance of the calculated amplitudes which involve the external photon lines. Their argument is as follows. The polarization vector $e^\mu$ is gauge dependent and therefore the calculated results must be kept invariant under the transformation of $e^\mu \rightarrow e^\mu + ck^\mu$. However, this condition is unphysical since we already fixed a gauge (for example, Lorentz gauge fixing of $k_\mu e^\mu = 0$) before the field quantization. The gauge invariance of the S-matrix evaluation is guaranteed as far as the fermion current is conserved, which is always satisfied in the perturbation calculation. Therefore, we briefly explain the gauge conditions by presenting a few examples.

1. Vacuum Polarization Tensor

The best example can be found in the vacuum polarization tensor $\Pi^{\mu\nu}$. People believe that the following gauge condition should be satisfied

$$k_\mu \Pi^{\mu\nu} = 0 \quad \text{(A.1)}$$

which is required from the above argument of the gauge condition as well as some incorrect mathematical identity equation where the mistake is simply due to the wrong replacement of the integration variables in the infinite integrals. However, as one can easily examine it, this is a wrong equation \[5\]. In fact we can write the result of the standard calculation of the vacuum polarization tensor as

$$\Pi^{\mu\nu}(k) = ie^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma^\mu \frac{1}{\hat{p} - m + i\varepsilon} \gamma^\nu \frac{1}{\hat{p} - \hat{k} - m + i\varepsilon} \right] = \frac{\alpha}{2\pi} \left( \Lambda^2 + m^2 - \frac{k^2}{6} \right) g^{\mu\nu} + \frac{\alpha}{3\pi} (k^\mu k^\nu - k^2 g^{\mu\nu}) \left[ \ln \left( \frac{\Lambda^2}{m^2 e} \right) - 6 \int_0^1 dz z(1 - z) \ln \left( 1 - \frac{k^2}{m^2 z} z(1 - z) \right) \right] \quad \text{(A.2)}$$

where $\Lambda$ denotes the cutoff momentum. There is no way that the first term of the right hand side can satisfy the gauge condition of eq.(A.1) \[3, 9\].

Since then, however, people impose the gauge conditions by hand on the amplitudes which have some external photon lines. The gauge condition of $e^\mu \rightarrow e^\mu + ck^\mu$ is not based on the solid physical principle, and therefore one can only say that, in some cases, the gauge condition can be satisfied. Among the reactions that favour the gauge condition, there are two cases, the Compton scattering and $\pi^0 \rightarrow 2\gamma$. But other reaction processes cannot satisfy the gauge condition.

2. Compton Scattering

As is well known, the Compton scattering satisfies the gauge condition, and this is mainly because there is no loop in this reaction and the external fermion line can satisfy the free Dirac equation. The Feynman amplitude of the Compton scattering can be written as

$$\mathcal{M}^{\mu\nu} = -ie^2 \left[ \bar{u}(p') \gamma^\nu \frac{1}{\hat{p}' + \hat{k}' - m + i\varepsilon} \gamma^\mu u(p) + \bar{u}(p') \gamma^\mu \frac{1}{\hat{p}' - \hat{k}' - m + i\varepsilon} \gamma^\nu u(p) \right] $$

Therefore, we can check

$$k_\mu \mathcal{M}^{\mu\nu} = -ie^2 \left[ \bar{u}(p') \gamma^\nu \frac{1}{\hat{p}' + \hat{k}' - m + i\varepsilon} \gamma^\mu \bar{u}(p) + \bar{u}(p') \gamma^\mu \frac{1}{\hat{p}' - \hat{k}' - m + i\varepsilon} \gamma^\nu \bar{u}(p) \right]. \quad \text{(A.4)}$$

Now, using some identities

$$\hat{k} = \hat{p} + \hat{k} - m + (\hat{p} - m), \quad \hat{k}' = (\hat{p}' - \hat{k}' - m) + (\hat{p}' - m)$$

and the Dirac equations of

$$(\hat{p} - m)u(p) = 0, \quad \bar{u}(p')(\hat{p}' - m) = 0$$
we can easily prove
\[ k_\mu M^{\mu\nu} = -ie^2 [\bar{u}(p')\gamma^\nu u(p) - \bar{u}(p')\gamma^\nu u(p)] = 0. \] (A.5)

Therefore, the Compton scattering can satisfy the gauge condition, but this is, of course, clear since the diagram contains no loop. Thus, the gauge condition, \( \epsilon^\mu \rightarrow \epsilon^\mu + ck^\mu \) just corresponds to the conservation of the fermion current which is guaranteed by the free Dirac equation, and this can be easily seen since the initial and final fermion in the Compton scattering can satisfy the free Dirac equation. On the other hand, if the Feynman diagrams involve the fermion loop, then there is no reason that the gauge condition can directly correspond to the current conservation of fermions.

3. Decay of \( \pi^0 \rightarrow 2\gamma \)

In addition to the Compton scattering, the amplitude of \( \pi^0 \rightarrow 2\gamma \) decay can satisfy the above type of the gauge condition since it can be written as
\[ M^{\mu\nu} = \frac{\pi g e^2}{M} \varepsilon^{\mu\nu\rho\sigma} k_\rho k'_\sigma. \] (A.6)

In this case, it is easy to prove that
\[ k_\mu M^{\mu\nu} = \frac{\pi g e^2}{M} \varepsilon^{\mu\nu\rho\sigma} k_\rho k'_\sigma = 0 \] (A.7)

which is due to the anti-symmetric character of the \( \varepsilon^{\mu\nu\rho\sigma} \) tensor. This property is basically due to the \( \gamma_5 \) interaction which generates the anti-symmetric nature of the invariant amplitude. In this respect, it is very special that the \( \pi^0 \rightarrow 2\gamma \) decay process satisfies the gauge condition even though it has a fermion loop. However, it is not due to the nature of the electromagnetic interactions. In fact, this point can be clearly seen if we examine the following reaction process of the scalar meson decay into two photons which cannot satisfy the gauge condition.

4. Decay of Scalar Boson \( \Phi \) into \( 2\gamma \)

Now, the T-matrix of \( \Phi \rightarrow 2\gamma \) which is based on the triangle diagrams can be evaluated to be
\[ T_{\Phi \rightarrow 2\gamma} \approx e^2 g_0 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ (\gamma \epsilon) \frac{1}{\not{p} - M + i\varepsilon} (\gamma \epsilon') \frac{1}{\not{p}' + \not{k} - M + i\varepsilon} (\gamma \epsilon') \frac{1}{\not{p} - \not{k}' - M + i\varepsilon} \right] \approx e^2 g_0 M (\epsilon \epsilon') \] (A.8)

where \( M \) denotes the nucleon mass, and \( g_0 \) is the coupling constant of \( \Phi N \) interaction as described by \( L_{II} = g_0 \bar{\psi}\Phi \psi \). Defining the amplitude \( M^{\mu\nu} \) as \( T_{\Phi \rightarrow 2\gamma} = M^{\mu\nu} \epsilon_\mu \epsilon'_\nu \), we can now prove
\[ k_\mu M^{\mu\nu} = e^2 g_0 M k_\mu g^{\mu\nu} \neq 0. \] (A.9)

Therefore, the gauge condition is not satisfied in the case of \( \Phi \rightarrow 2\gamma \) decay process. This is, of course, clear since the scalar interaction has a symmetric nature and therefore it is just opposite to the \( \gamma_5 \) interaction. It should be noted that there is no scalar meson in nature which decays into two photons. However, the similar type of the Feynman diagram becomes important when we consider the photon-gravity interaction. In fact, photon can interact with the gravitational field via loop diagrams which are essentially the same as the T-matrix given in eq.(A.8) [9]. In this respect, the T-matrix given in eq.(A.8) can be considered to be a real physical process.

5. Summary of Gauge Conditions

To summarize, we see that the amplitudes of \( \pi^0 \rightarrow 2\gamma \) and the Compton scattering happen to satisfy the gauge condition, while other examples of the photon self-energy, scalar meson decay into two photons and photon-photon scattering diagrams do not satisfy the gauge condition. The \( \pi^0 \rightarrow 2\gamma \) case satisfies the gauge condition due to the anti-symmetric nature of the pion-nucleon interactions while the Compton scattering diagrams can satisfy the gauge condition because they contain no fermion loops.
In general, the problem of the gauge conditions can be better understood if we give one example in terms of the Lorentz invariance. Now, the S-matrix in QED is formulated in a covariant fashion, and therefore the cross section defined from the T-matrix is, for sure, Lorentz invariant. When calculating the cross section, one should choose the system such as the laboratory system or the center of mass system, depending on the experimental or theoretical situations. Then, one can evaluate the cross section and can reliably obtain the calculated result since the Lorentz invariant quantity can be calculated at any system one wishes. At this point, a physics student may ask a question as to what should be the Lorentz invariance of the cross section. Of course, one knows that this is a meaningless question since the calculation is done by fixing the system. However, as one sees by now, the requirement of the gauge condition of $\epsilon^\mu \rightarrow \epsilon^\mu + \epsilon k^\mu$ is just the same level as the above question.