Multiscale Finite Element Method for scattering problem in heterogeneous domain

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Abstract. In this paper, we consider the wave scattering problem in heterogeneous domain. The mathematical model is described by the Helmholtz equation for time-harmonic wave propagation with absorbing boundary condition. We construct a coarse grid approximation for the efficient numerical solution on coarse grids using the Multiscale Finite Element Method (MsFEM), where we construct a multiscale space using solution of some local problems in each local domain. Numerical results are presented to illustrate the performance of the method.

1. Introduction

In this paper, we consider time harmonic wave scattering in heterogeneous media. These problems occur in many real world applications, for example, in engineering design, where the wave propagation through composite materials is used to understand particular microstructure behavior and construct materials with special properties [3]. For problems in periodic media, macroscale mathematical model is constructed by the homogenization theory [4, 3, 2, 1]. For the wave propagation in non-periodic media, macroscale models are developed based on the numerical homogenization or using multiscale methods [6, 7, 8, 10, 5, 9]. In this work, we present a Multiscale Finite Element Method (MsFEM) for solution of wave propagation in heterogeneous media. For the construction of the coarse grid approximation, we solve local problems to construct multiscale basis functions. We test two approaches for construction of the local basis functions based on the two types of local problems using elliptic part of operator and using the Helmholtz operator. We present numerical results for two-dimensional model problem and study the performance of the proposed method for various wave numbers.
The paper is organized as follows. In Section 2, we present the mathematical model and fine grid approximation of the wave propagation problem in heterogeneous media. In Section 3, we present a coarse grid approximation based on the Multiscale Finite Element Method. Numerical results for two-dimensional model problem is presented in Section 4.

2. Problem formulation and fine grid approximation

We consider the complex-valued Helmholtz problem in heterogeneous domain \( \Omega \)
\[
\nabla \cdot (A(x) \nabla u_s) + k^2 n(x) u_s = f(x), \quad x \in \Omega,
\]
where the wave number \( k \) is constant and assumed to be positive, \( u = u_i + u_s \) is the total field, \( u_s \) is the scattered field and

\[
\Omega = \Omega_0 \cup \sum_{j=1,2,...} D_j
\]

with

\[
n(x) = \begin{cases} 1 & x \in \Omega_0 \\ \alpha_j & x \in D_j, j = 1,2,.. \end{cases}, \quad A(x) = \begin{cases} I & x \in \Omega_0 \\ A_j & x \in D_j, j = 1,2,.. \end{cases}
\]

and non-zero right hand side in subdomains \( D_j \)

\[
f(x) = \nabla \cdot ((I - A(x)) \nabla u_i) + k^2 (1 - n(x)) u_i,
\]
where \( u_i = e^{ikx \cdot d} = \cos(kx \cdot d) + i \sin(kx \cdot d) \) is the given incident field and \( d \) is the incident wave direction.

We consider equation with Sommerfield radiation condition

\[
\frac{\partial u_s}{\partial n_b} - iku_s = 0, \quad x \in \partial \Omega,
\]
where \( n_b \) is the unit normal on boundary.

For approximation of the Helmholtz equation on the fine grid, we use finite element method and have following variational formulation: find \( u \in V_h = H_1(\Omega) \) such that

\[
a(u_s, v) - k^2 m(u_s, v) - ib(u_s, v) = l(v), \quad v \in V_h,
\]
where

\[
a(u, v) = \int_\Omega A \nabla u \cdot \nabla v \, dx, \quad m(u, v) = \int_\Omega n \, u \, v \, dx,
\]

\[
b(u, v) = \int_{\partial \Omega} k \, u \, v \, ds, \quad l(v) = -\int_\Omega f \, v \, dx,
\]
and \( u_s = Re(u_s) + i Im(u_s) \).

We can write the complex valued problem in matrix form

\[
(K_h - k^2 M_h - i B_h) U = F_h,
\]
where \( U = Re(U) + i Im(U) \), \( (F_h)_j = l(\phi_j) \), and complex valued matrices \( M_h, B_h \) and \( K_h \) are given by

\[
(M_h)_{rj} = m(\phi_r, \phi_j), \quad (B_h)_{rj} = b(\phi_r, \phi_j), \quad (K_h)_{rj} = a(\phi_r, \phi_j),
\]
and \( u_s = \sum_j U_j \phi_j \), \( \phi_j \) is the linear basis functions for fine grid approximation.
3. Coarse grid approximation using MsFEM

For coarse-scale approximation, we use Multiscale Finite Element Method with following computational algorithm:

- the construction of the multiscale basis functions in local domain \( \omega_i \) and
- the construction and solution of the coarse grid system on multiscale space.

We define the coarse grid \( T_H \) in domain \( \Omega \), \( T_H = \bigcup_{i=1}^{N_c} \omega_i \) with mesh sizes \( H \gg h > 0 \) where \( \omega_i \) is obtained by the combining all the coarse cells \( K \) around one vertex of the coarse grid (local domain), \( N_c \) is number of cells. Let \( V_H \) be a finite dimensional function space, which consists of functions that are smooth on each local domain related to the coarse mesh cell. We construct a local multiscale space \( V_{\omega_i}^H \) by solution of the local problems. The global multiscale space \( V_H \) is then defined as the linear span of all \( V_{\omega_i}^H \), \( \omega_i \in T_H \).

For approximation on the coarse grid, we use a continuous Galerkin (CG) approach and have following variational formulation: find \( u_H \in V_H \):

\[
a(u_H, v) - k^2 m(u_H, v) - ib(u_H, v) = l(v), \quad v \in V_H.
\]  

(4)

For construction of the multiscale basis functions in local domains, we consider following local problems in \( K_j \in \omega_i \):

- Case 1 with elliptic operator
  \[
  \nabla \cdot (A(x) \nabla \phi_i) = 0, \quad x \in K_j, \quad \forall K_j \in \omega_i,
  \]
  with linear boundary condition on \( \partial K_j \).

- Case 2 with full operator
  \[
  \nabla \cdot (A(x) \nabla \phi_i) + k^2 n(x) \phi_i = 0, \quad x \in K_j, \quad \forall K_j \in \omega_i,
  \]
  with linear boundary condition on \( \partial K_j \).

Then multiscale space \( V_H \) is defined as follow

\[
V_H = \text{span} \{ \phi_i \}.
\]
For construction of the coarse grid system, we use a projection approach and define the projection operator

\[ R = (\phi_1, ..., \phi_{N_c})^T, \]

and therefore Galerkin projection 4 can be written in the following matrix form

\[ (K_H - k^2 M_H - iB_H)U_H = F_H, \]

where

\[ K_H = RK_hR^T, \quad M_H = RM_hR^T, \quad B_H = RB_hR^T, \quad F_H = RF_h, \]

and the fine-scale representation of the multiscale solution is given by \( U_{ms} = R^T U_H \).

4. Numerical results

We present a numerical results for the fine-scale solution and the coarse-scale solution using multiscale finite element method. Computational domain with heterogeneous obstacles is presented in Figure 1 and have dimensions \( \Omega = [-2, 2] \times [-2, 2] \). For the computational mesh we consider fine mesh, \( 10 \times 10 \) and \( 20 \times 20 \) coarse meshes. Fine mesh contains 100802 vertices and 200482 triangle elements, structured coarse \( 10 \times 10 \) mesh with 121 vertices, 100 triangle elements and \( 20 \times 20 \) mesh 441 vertices, 400 triangle elements. We set \( k = 0.1, 1, 2 \) and \( d = (1,0) \). For background media, we set \( \alpha = 1 \) and \( n = 1 \) and for circle inclusions \( \alpha = 10 \) and \( n = 0.2 \).

![Figure 1](image.png)

**Figure 1.** Heterogeneous domain with fine mesh that contains 100802 vertices and 200482 triangle elements. Coarse mesh with 121 vertices and 100 triangle elements (middle). Coarse mesh with 441 vertices and 400 triangle elements (right).

In Figure 2, we present the fine-scale and coarse-scale solution for the real part, imaginary part and absolute value with wave number \( k = 1 \). We obtain solution with good accuracy for reduced order model using MsFEM. In Table 1, we show a multiscale solver errors for Case 1 and Case 2 for the coarse grids with 121 and 441 local domains. We observe error reduction for a finer coarse grid. Furthermore, we have larger errors for higher values of the wave number \( k \). Comparing the errors from Case 1 and Case 2, we see that the case with full operator provide smaller errors. Our results show that the presented method give good approximation of the solution and reduce size of system for scattering problems with the small wave numbers.

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Figure 2. Fine-scale (top) and coarse-scale (bottom) solutions for heterogeneous domain with $k = 1.0$. Re(u), Im(u), Abs(u) (from left to right).

Table 1. Relative $L_2$ errors in %. Coarse mesh with 121 ($10 \times 10$) and 441 local domains ($20 \times 20$). Case 1 (left) and Case 2 (right).

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