A NONEQUILIBRIUM THERMODYNAMIC APPROACH TO GENERALIZED STATISTICS FOR BROWNIAN MOTION

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Abstract

We analyze the dynamics of a Brownian gas in contact with a heat bath in which large temperature fluctuations occur. There are two distinct time scales present, one describes the decay of the fluctuations in the temperature and the other one is associated with the establishment of local equilibrium. Although the gas has reached local equilibrium, there exist large fluctuations in an intensive parameter (temperature) which break the thermodynamic equilibrium with the heat bath. Thus the decay of the fluctuations in the intensive parameter is larger than the characteristic time for the establishment of local equilibrium. We show that the dynamics of such large and intensive fluctuations may be described by adopting a nonequilibrium thermodynamics approach with an adequate formulation of local equilibrium. A coarsening procedure is then used to contract the space of mesoscopic variables needed to describe the dynamics of the gas and the extensive character of the description is lost. This procedure allows us to derive an effective Maxwell-Boltzmann factor (EMBF) for the Brownian gas, as has been recently proposed in the literature [13]. Furthermore,
we use this local equilibrium distribution and an entropy functional to derive a nonequilibrium probability distribution and a hydrodynamic description for the Brownian gas which contains fluctuating transport coefficients. The ensuing description is nonextensive and our analysis shows that the coarse-graining procedure is responsible for the nonextensivity property.

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1 Introduction

Complex systems where long range interactions, large external gradients or large fluctuations of their intensive variables are present, show peculiar behaviors which have attracted a good deal of attention in recent years [2, 3, 5, 6, 4]. The theoretical description of these systems constitutes a nowadays challenge and is far beyond the limit of applicability of the theory of fluctuations near equilibrium as developed by Onsager and Machlup [7] and subsequently generalized by Fox and Uhlenbeck [8]. This is essentially due to the fact that these theories are restricted to describe fluctuations of extensive variables only.

Several theories have been recently proposed to describe the special properties of simple and complex fluids in nonequilibrium stationary states [2, 3, 4, 5]. However, theories describing physical situations where fluctuations of the intensive variables, such as temperature or chemical potential among others, may be important are scarce, [11, 10]. These issues have been mostly investigated within the framework of the so called nonextensive statistical mechanics (NEM) [11], [12], which according to Beck and Cohen [13], is only one of the possible more general statistics that can deal with the physical situations mentioned above. NEM proposes that for systems with a sufficiently complex dynamics, it is necessary to introduce a more general statistics than the usual Boltzmann-Gibbs description. Such statistics have been discussed for a variety of systems [13] and have led to the introduction of an effective Boltzmann factor (EMBF) which reduces to the ordinary one in the appropriate limit.

These concepts have been explored in more detail within the model of Brownian motion, for which Beck and Cohen have postulated a modification of the Boltzmann factor whose explicit form depends on the statistics of the intensive parameters fluctuations. They show that if it is assumed that the inverse temperature of ordinary statistical mechanics fluctuates with the Gamma distribution, then the ensuing Brownian dynamics can be correctly described by Tsallis statistics, [13, 10]. One of the main purposes of this paper is to use this same model of nonequilibrium Brownian motion to examine how the generalization of the Boltzmann factor may be justified in terms of well established concepts used in irreversible thermodynamics, such as Einstein’s formula for the probability distribution and a Kullback-like entropy functional [15, 16, 17, 19]. By adopting this point of view, we show that the modified Boltzmann’s factor has its origin in the contraction of the space of mesoscopic variables that describe the dynamics of the system, and that this contraction is responsible for the non-extensive character of the new description.

Usually, the dynamics of a Brownian gas is analyzed by taking into account only the equilibrium fluctuations of the extensive variables of the heat bath, and by assuming thermodynamic equilibrium between the Brownian gas and the bath. Since the characteristic decaying time of these fluctuations is short as compared with the time characterizing the establishment of local equilibrium, they become independent from those of the velocity of the Brownian particles which, in this case, are well described by the Maxwell-Boltzmann factor [14]. However, a more general case occurs when intensive parameters (temperature) may fluctuate and are large enough to break its thermodynamic equilibrium with the heat bath. In this case, the Brownian gas can be assumed to be in local thermodynamic equilibrium, but not with the entire heat bath. In this situation the work performed on the gas will be different from that corresponding to the equilibrium case. These conditions can be met when the system is constrained to be far away from equilibrium and the time scale and wavelength of the intensive fluctuations are larger than those characterizing the establishment of local equilibrium. As a consequence, the proba-
bility distribution for the fluctuating velocities of the Brownian particles depends, in general, on the
distribution of the temperature fluctuations and is no longer described by the usual Maxwell-Boltzmann
factor.

The paper is organized as follows. In Sec. II we define the system and obtain a probability
distribution for the velocities of the Brownian particles that incorporates the effects of nonequilibrium
temperature fluctuations; from it we derive the effective Maxwell-Boltzmann factor. Then in Sec. III
we derive a Fokker-Planck equation describing the relaxation mesoscopic dynamics of the Brownian gas
and we use it to derive a hydrodynamic description with fluctuating transport coefficients. A generalized
Onsager-Machlup formalism is presented in section IV and the last section V is devoted to discuss our
main results.

2 Model and Basic Equations

As mentioned above, the dynamics of a diluted gas of Brownian particles in contact with a heat bath
in the presence of two distinct time scales, one related to the decay of the fluctuations and the other
to the establishment of local equilibrium, can be interpreted by assuming that each Brownian particle
is embedded in a system in contact with a heat bath at temperature \( T_B(t) \). We want to analyze the
dynamics of the Brownian particle when nonequilibrium temperature fluctuations \( \delta T(t) \) occur in the
system. In particular, we will assume that the characteristic length and time scales of these fluctuations
are sufficiently larger than those characterizing the establishment of local equilibrium. The relation
between the system and heat bath temperatures is then \( T(t) = T_B(t) + \delta T(t) \).

When a fluctuation occurs, the system only performs the amount of work \( W \) on the Brownian
particle. This modifies its kinetic energy by \( W/m = \delta e_{\text{kin}} = -\frac{\delta^2}{2} \), where \( m \) is the mass of the particle
and the sign indicates that the system performs the work. Moreover, the system exerts the work \( p\delta V \)
and transfers the amount of heat \( T\delta S \) to the bath. Assuming that the variation of the total entropy
of the system plus bath is \( \delta S_T = \delta S + \delta S_B \), and by using the Gibbs equation, after dividing by \( m \) we
obtain the relation [14, 17]

\[
\delta s_T = \frac{1}{T(t)} \left[ \delta e_{\text{kin}} + \delta e + p\delta \rho^{-1} \right] + \frac{1}{T_B(t)} \left[ \delta e_B + p_B\delta \rho_B^{-1} \right],
\]

(1)

where \( \delta e \) and \( \delta e_B \) are the variations of the internal energy of the system and the bath, \( p \) and \( p_B \) the
system and bath pressures, and \( \rho \) and \( \rho_B \) the corresponding specific volumes.

Eq. (1) can be used to obtain the minimum amount of work performed by the system to modify the
state of the Brownian particle [14]. This situation occurs when the processes taking place between the
system and the heat bath are reversible. Since in that case \( \delta e_B = -\delta e \) and \( \delta \rho_B^{-1} = -\delta \rho^{-1} \), and given
that \( \delta T = T - T_B \), then Eq. (1) becomes

\[
\delta s_T = \frac{1}{T(t)} \delta e_{\text{kin}} - \frac{1}{T B(t)} \delta T \delta e + \left( \frac{p}{T} - \frac{p_B}{T_B} \right) \delta \rho^{-1},
\]

(2)

which is in agreement with the Le Chatelier-Braun principle [14, 17]. In order to write equation (2) in
a more convenient form, we expand \( \delta e(T, \rho) \) up to first order in \( \delta T \) and \( \delta \rho^{-1} \). After using the relation
\[ \left( \frac{\partial e}{\partial \rho} \right)_T = -p + T(t) \left( \frac{\partial p}{\partial T} \right)_\rho \]

we arrive at

\[ \delta s_T = \frac{1}{T(t)} \delta e_{\text{kin}} - \frac{1}{TT_B} \left( \frac{\partial e}{\partial T} \right)_\rho (\delta T)^2 + \left[ \frac{p}{TT_B} - \frac{1}{T_B} \left( \frac{\partial p}{\partial T} \right)_\rho \right] \delta T \delta \rho^{-1} + \left( \frac{p}{T_B} - \frac{p_B}{T_B} \right) \delta \rho^{-1}, \]  

which can still be simplified if we use again the relation \( T = T_B + \delta T \) in order to make the approximations

\[ \frac{p}{TT_B} \simeq \frac{p}{T_B^2} \left( 1 - \frac{\delta T}{T_B} \right), \quad \frac{p}{T} - \frac{p_B}{T_B} = \frac{\delta p}{T_B} - \frac{\delta T}{T_B}. \]  

where \( \delta p \equiv p - p_B \). After substitution of Eq. (4) into Eq. (3), keeping terms up to second order in the fluctuations and using the expansion \( \delta p \simeq \left( \frac{\partial p}{\partial \rho} \right)_T \delta \rho + \left( \frac{\partial p}{\partial T} \right)_\rho \delta T \), we arrive at the following expression for the variation of the total entropy \( \delta s_T \)

\[ \delta s_T = \frac{1}{T(t)} \delta e_{\text{kin}} - \frac{1}{TT_B} \left( \frac{\partial e}{\partial T} \right)_\rho (\delta T)^2 - \frac{1}{T_B} \left( \frac{\partial p}{\partial \rho} \right)_\rho \rho^{-2} (\delta \rho)^2. \]  

However, since our interest in this paper is to analyze the effects of the large temperature fluctuations on the dynamics of the Brownian gas, we shall adopt a statistical description point of view and assume that the probability \( P_{le} \) of occurrence of a fluctuation in the composite system may be expressed through Einstein’s formula [14]

\[ P_{le} = P_0 e^{-\frac{m \delta s_T}{k_B T_B}}, \]

where \( k_B \) is Boltzmann’s constant and \( P_0 \) normalizes \( P_{le} \). Thus, by substituting Eq. (5) into (6) and using \( \delta e_{\text{kin}} = -\frac{u^2}{2} \), \( P_{le} \) can be rewritten in the more convenient form

\[ P_{le} = P_0 e^{-\frac{m}{2k_B T_B(t)} h(\delta T) \left\{ \delta \rho^2 + \left( \frac{\partial p}{\partial \rho} \right)_T (\delta T)^2 + h^{-1}(\delta T) \left( \frac{\partial p}{\partial \rho} \right)_\rho \rho^{-2} (\delta \rho)^2 \right\}}. \]  

which is the probability that a fluctuation occurs. To arrive at Eq. (7) we have expanded \( \frac{1}{T(t)} \simeq \frac{1}{T_B(t)} h(\delta T) \) with \( h(\delta T) \simeq \left[ 1 - \frac{\delta T}{T_B(t)} - \left( \frac{\delta T}{T_B(t)} \right)^2 \right] \). This last equation allows us to analyze how the distribution function accounting for the fluctuations of the velocities of the Brownian particle becomes modified in the presence of large temperature fluctuations \( \delta T \) occuring in the system. Integrating Eq. (7) over \( \delta T \) leads to the effective Maxwell-Boltzmann factor \( B_{le} \)

\[ B_{le} = P_0 \int e^{-\frac{m}{2k_B T_B(t)} h(\delta T) \left[ \delta \rho^2 + 2 \left( \frac{\partial p}{\partial \rho} \right)_T (\delta T)^2 \right]} d(\delta T), \]

where we have assumed a constant volume processes for simplicity. This probability distribution is an effective or modified Boltzmann factor (EMBF) for the nonequilibrium Brownian system and it contains corrections arising from a non-Gaussian distribution. However, in the usual case where system and bath are in thermal equilibrium, \( h(\delta T) \to 1 \), and the temperature fluctuations become independent of velocity fluctuations, as expected [14].
2.1 Connection with superstatistics

Equation (8) implies that when system and bath are not in thermal equilibrium, temperature fluctuations modify the local equilibrium distribution of the Brownian particle in the sense discussed in Ref. [13]. To illustrate this point more explicitly, let us rewrite (8) as

\[ B(e_B) = \int f(\delta T) e^{-h(\delta T)e_B} d(\delta T) = \langle e^{-h(\delta T)e_B} \rangle f, \]

where \( e_B \equiv \frac{m \delta r^2}{2k_B T_B(t)} \) and \( \langle .. \rangle_f \) denotes the average over the distribution function of the temperature fluctuations \( f(\delta T) \) given by

\[ f(\delta T) = P_0 e^{-\frac{m h(\delta T)}{k_B T_B(t)}} \left[ \left( \frac{\partial e}{\partial T} \right)_p \frac{(\delta T)^2}{T_B(t)} \right]. \]

If we now follow a procedure similar to that sketched in Ref. [13], we may write the relation

\[ B(e_B) = e^{-\langle h \rangle_f} e_B \langle e^{-\langle h \rangle_f} e_B \rangle_f, \]

where \( \langle h \rangle_f \) is the average of \( h(\delta T) \) over the distribution \( f(\delta T) \). By expanding the exponential within the angular bracket up to second order in its argument and after evaluating the average we obtain

\[ B(e_B) \simeq e^{-\langle h \rangle_f} e_B \left[ 1 - \frac{1}{2} \Sigma^2 e_B^2 \right], \]

where we have introduced the abbreviation \( \Sigma^2 \equiv \frac{\langle \delta T^2 \rangle_f}{T_B^2} \). It should be stressed that this result was obtained from a nonequilibrium thermodynamic analysis and expresses the effective Maxwell-Boltzmann factor in an universal form which only depends on the second moment, \( \langle \delta T^2 \rangle_f \), of the temperature fluctuations. This corresponds to the low-E universal behavior discussed in Ref. [13], where this universality allows for the definition of a universal parameter for any superstatistics and not only for Tsallis statistics.

3 Hydrodynamics with fluctuating transport coefficients

As mentioned in the introduction, fluctuations in intensive parameters such as the temperature, can not be described by the usual Onsager-Machlup theory, since it is devised to describe fluctuations in the extensive parameters only [7]. For this reason, here we will describe the dynamics of a Brownian particle in the presence of large local temperature fluctuations \( \delta T(t) \) in its \((\vec{r}, \vec{u})\)-space, where \( \vec{r} \) and \( \vec{u} \) stand for the position and the velocity of a Brownian particle. More precisely, the state of the particle is defined not only by \( \vec{r} \) and \( \vec{u} \), but also by the value of the parameter \( \delta T(t) \) entering in the single particle nonequilibrium probability distribution in the form \( P(\vec{r}, \vec{u}; t, \delta T) \). To accomplish the description, note that at the mesoscale the nonequilibrium probability distribution \( P \) obeys the continuity equation

\[ \frac{\partial}{\partial t} P(\vec{r}, \vec{u}; t, \delta T) + \nabla \cdot [\vec{u} P] = - \frac{\partial}{\partial \vec{u}} \cdot (P \vec{v}_u), \]

(13)
where $\nabla \equiv \frac{\partial}{\partial \vec{r}}$, $\vec{v}_u(\vec{r}, \vec{u}; t, \delta T)$ is the corresponding streaming velocity in $\vec{u}$-space. In this case, the explicit form of Eq. (13) can be constructed by using Eq. (7) and by using the following functional for the local nonequilibrium Kullback-like entropy $s(\vec{r}, t)$.

$$s(\vec{r}, t) = -k_B \int P(\vec{r}, \vec{u}; t, \delta T) \ln \frac{P}{P_{le}} \, d\vec{u}, \quad (14)$$

where $P_{le}$ characterizes a (local) equilibrium reference state, and is given by Eq. (7) for $\delta \rho = 0$. It is important to stress that the entropy functional (14) will be used here as an irreversibility criterion that reduces to the Gibbs entropy in the case of equilibrium.

By taking the time derivative of Eq. (14) and by substitution of the result into Eq. (13), one may calculate the entropy production $\sigma(\vec{r}, t)$ of the Brownian gas through its natural evolution in time. After integrating by parts and by assuming that the currents vanish at the boundaries, one arrives at

$$\frac{\partial s}{\partial t} + \nabla \cdot \vec{J}_s = -k_B \int P\vec{v}_u \cdot \frac{\partial}{\partial \vec{u}} \ln \frac{P}{P_{le}} \, d\vec{u}, \quad (15)$$

where we have defined the entropy flow by

$$\vec{J}_s \equiv -k_B \int P\vec{u} \left( \ln \frac{P}{P_{le}} - 1 \right) \, d\vec{u}. \quad (16)$$

For convenience we have assumed that the average temperature of the bath is constant. The right hand side of Eq. (15) is the entropy production in the space of variables determining the state of the gas in the mesoscale.

Given the entropy production in (15), one may impose the condition of a (locally) positive entropy production in the $(\vec{r}, \vec{u})$-space and by following a scheme similar to that of nonequilibrium thermodynamics, in first approximation formulate linear laws between the current $P\vec{v}_u$ and its conjugated force $\frac{\partial}{\partial \vec{u}} \ln \frac{P}{P_{le}}$. After taking into account Eq. (7), the linear law takes the form

$$P\vec{v}_u = \beta h(\delta T)\vec{u}P + \frac{k_BT_B}{m} \beta \frac{\partial P}{\partial \vec{u}}, \quad (17)$$

where $\beta$ is the corresponding coupling coefficient which has been assumed to be a scalar function for simplicity. Near equilibrium, this coefficient must reduce to the corresponding Onsager coefficient. Now, by substituting Eq. (17) into (13), we finally arrive at the following Fokker-Planck equation

$$\frac{\partial}{\partial t} P(\vec{r}, \vec{u}; t, \delta T) + \nabla \cdot [\vec{u}P] = \frac{\partial}{\partial \vec{u}} \cdot \left( \beta h(\delta T)\vec{u}P + \frac{k_BT_B}{m} \beta \frac{\partial P}{\partial \vec{u}} \right), \quad (18)$$

which describes the evolution in time of the nonequilibrium probability distribution of the Brownian particle. It is important to point out that the factor $h(\delta T)$ breaks the equilibrium form of the fluctuation-dissipation theorem (FDT), thus implying that in the presence of large temperature fluctuations the usual form of the FDT is not valid. This result is due to the lack of thermal equilibrium between the system and the heat bath.
Once obtained the Fokker-Planck equation (18) it is now possible to derive an hydrodynamic-like hierarchy of equations for the moments of the probability distribution function \( P(\vec{r}, \vec{u}; t, \delta T) \). As shown below, these equations constitute a fluctuating hydrodynamic description since they contain fluctuating transport coefficients. The evolution equation for the mass density field \( \rho(\vec{r}; t, \delta T) = m \int P d\vec{u} \) is simply obtained by directly integrating Eq. (18) over the velocity \( \vec{u} \), yielding

\[
\frac{\partial}{\partial t} \rho(\vec{r}; t, \delta T) = -\nabla \cdot [\rho \vec{v}(\vec{r}; t, \delta T)].
\]

In a similar form, the evolution equation for the momentum field \( \rho \vec{v}(\vec{r}; t, \delta T) \equiv m \int \vec{u} P d\vec{u} \) can be shown to be

\[
\rho \frac{d}{dt} \vec{v}(\vec{r}; t, \delta T) + \nabla \cdot \vec{P}_k = -\rho \beta^* \cdot \vec{v},
\]

where \( \beta^* = \beta h(\delta T(t)) \) is a fluctuating friction coefficient. Here we have introduced the kinetic part of the pressure tensor \( \vec{P}_k \)

\[
\vec{P}_k(\vec{r}; t, \delta T) = m \int (\vec{u} - \vec{v})(\vec{u} - \vec{v}) P d\vec{u},
\]

which obeys the following relaxation equation

\[
\frac{d}{dt} \vec{P}_k + 2 \left[ \vec{P}_k \cdot \left( \beta^* \vec{1} + \nabla \vec{v} + \frac{1}{2}(\nabla \cdot \vec{v}) \vec{1} \right) \right]^s = \frac{2k_B T_B}{m} \beta \rho \vec{1},
\]

obtained by taking the time derivative of Eq. (21) and using (18). Here \( \vec{1} \) is the unit tensor, the superscript \( s \) denotes the symmetric part of a tensor and we have neglected the contribution of the third and higher order moments (see Ref. [23]). Eqs. (19)-(22) constitute a generalized fluctuating hydrodynamic equations of the Brownian phase when large temperature fluctuations drive the system out of equilibrium, breaking its thermal equilibrium with the bath [27]. It should be remarked that if the usual internal thermal fluctuations associated with the term proportional to \( k_B T_B \) were to be taken into account, a random heat and stress flows should be added to the elements of the pressure tensor \( \vec{P}_k \). Then Eqs. (20) and (22) may be rewritten in the usual form of the Landau and Lifshitz fluctuating hydrodynamic equations, [22].

3.1 The limit of long times

At sufficiently long times \( t \beta^* >> 1 \), the description of the Brownian motion can be performed in terms of a diffusion equation for \( \rho(\vec{r}; t, \delta T) \). This equation can be derived by neglecting the time derivatives in Eqs. (20) and (22). Then, Eq. (22) leads to the following constitutive relation for the pressure tensor

\[
\vec{P}_k \simeq \frac{k_B T_B}{m} \rho h^{-1}(\delta T) \vec{1},
\]

where, for simplicity, the Brownian contribution to the viscosity has been neglected [23, 24]. It should be noted in passing, that precisely the same first two terms on the right hand side of Eq. (18), which were
responsible for the breaking of the fluctuation-dissipation theorem in its usual form, are also responsible for the appearance of the correction $h^{-1}(\delta T)$ on the right hand side of Eq. (23). Substitution of (23) into Eq. (20) leads to the following expression for the diffusion current $\rho \vec{v}$

$$\rho \vec{v} \simeq -D_{\text{fluc}}(\delta T)\nabla \rho,$$  \hspace{1cm} (24)

where the fluctuating diffusion coefficient has been defined as

$$D_{\text{fluc}}(\delta T) \equiv \frac{k_B T_B}{m\beta} h^{-2}(\delta T).$$  \hspace{1cm} (25)

From Eqs. (23) and (24), one may expect that when the fluctuations $\delta T$ are spatially non-homogeneous, then a fluctuating heat flow contributes to the diffusion flow. These results imply that in the long time limit under consideration and when large temperature fluctuations occur in the system, the diffusion coefficient of the Brownian particle is converted into a fluctuating quantity. The evolution equation for the mass density $\rho(\vec{r}; t, \delta T)$ in this limit is the obtained by substituting Eq. (24) into (19) yielding the fluctuating Smoluchowski equation

$$\frac{\partial}{\partial t} \rho(\vec{r}; t, \delta T) = D_{\text{fluc}}(\delta T)\nabla^2 \rho,$$  \hspace{1cm} (26)

which describes the evolution of the Brownian particle for long times. Eq. (26) reinforces the fact that the fluctuation-dissipation theorem in its usual form is no longer valid.

## 4 Regression laws in Brownian dynamics

We can also derive the regression laws for both, the conditional averages of the velocity of the Brownian particles, and of the temperature fluctuations over a given volume element $dV = d\vec{r}$. To accomplish this objective we must take the average of the corresponding Fokker-Planck equation that also incorporates the temperature fluctuation $\delta T$, over $d\vec{r}$ and then use the definitions of the conditional averages for the velocity and for the temperature

$$\overline{\vec{u}}^0(t) = \int \vec{u} P_u(\vec{u}, \delta T; t) d\vec{u} d(\delta T)$$  \hspace{1cm} (27)

and

$$\overline{\delta T}^0(t) = \int \delta T P_u(\vec{u}, \delta T; t) d\vec{u} d(\delta T).$$  \hspace{1cm} (28)

The upper bar denotes a given initial state and

$$P_u(\vec{u}; t, \delta T) = \int P(\vec{r}, \vec{u}, \delta T; t) d\vec{r}.  \hspace{1cm} (29)$$
Using this definition, from the corresponding Fokker-Planck equation a time evolution equation for \( P_u(\vec{u}, \delta T; t) \) is readily obtained. If we insert the resulting equation into the time derivatives of Eqs. (27) and (28), after a partial integration similar to that of Sec. III, we arrive at the equations

\[
\frac{d\vec{u}}{dt} = -\tilde{\beta}(\delta T^0) \vec{u}^0(t),
\]

(30)

and

\[
\frac{d\delta T^0}{dt} = -\epsilon \chi_0^0 \delta T(t),
\]

(31)

where \( \epsilon \) is an Onsager coefficient and we have assumed that \( \beta \) and \( \epsilon \) are independent of position. The nonlinear generalized force \( \chi_0^0(t) \) has been defined as

\[
\chi_0^0(t) = \int \frac{\partial}{\partial \delta T} \left( \frac{h u^2}{2} + \left( \frac{\partial e}{\partial T}(t) \right) \frac{(\delta T)^2}{T_B} \right) P_u d\vec{u} d(\delta T),
\]

(32)

and the modified friction coefficient \( \tilde{\beta} \) through the relations

\[
\tilde{\beta}(\delta T^0) \vec{u}^0(t) = \int \vec{u} \beta h(\delta T) P_u d\vec{u} d(\delta T)
\]

\[
= \int \vec{u} \left\{ \int \beta h(\delta T) P_u d(\delta T) \right\} d\vec{u} = \tilde{\beta}(\delta T^0) \int \vec{u} f(\vec{u}, t) d\vec{u},
\]

(33)

where \( f(\vec{u}, t) = \int P_u(\vec{u}, \delta T; t) d(\delta T) \) is a reduced distribution function. By taking into account the explicit expression for \( h(\delta T) \) and the fact that the average of \( \delta T \) vanishes, one obtains \( \tilde{\beta} = \beta \left( 1 - \frac{1}{T_B} (\delta T^0)^2 \right) \).

Note that the regression law for the temperature fluctuations is a non-linear one, and that it is coupled with that for \( \vec{u}^0(t) \).

From Eqs. (30) and (31), it follows that mean regression laws are obeyed by the time-dependent value of the fluctuations occurring in the system after being averaged over an elemental volume. It is interesting to note that in the present case, when external factors induce temperature fluctuations, the average coefficients characterizing the dissipation in the system become modified by a term proportional to the square of the amplitude of the fluctuation. Eqs. (30) and (31) may be interpreted as the generalization of the Onsager regression hypothesis in which the presence of large temperature fluctuations introduce corrections to the dissipation coefficients.

5 DISCUSSION

By using an irreversible thermodynamic approach we have derived an effective Maxwell-Boltzmann factor (EMBF) for a gas of Brownian particles, when there occur large fluctuations in the temperature of the heat bath in which they are suspended. The derived EMBF has the same form as the one the proposed in the literature [13]. Our derivation was carried out by using irreversible thermodynamic
arguments of the type used to evaluate the minimum amount of work that a system embedded into a heat bath needs to perform in order to modify the state of an external body [14]. The EMBF has been expressed in a form that relates it to the superstatistics discussed in Ref. [13]. The general point of view we have presented here indicates that this EMBF had its origin in a contraction of the mesoscopic space of variables necessary to completely describe the dynamics of the system. In this case, the superstatistics present in this mesoscopic stochastic variables have been eliminated by averaging over them. As a consequence of this contraction the extensive character of the description is lost.

A hydrodynamic description was derived from a Fokker-Planck equation for the nonequilibrium distribution function of the Brownian system. The last equation was obtained by using a Kullback-like entropy functional which incorporates the complete distribution function (7) as a reference state, and then averaging over the fluctuating velocities of the Brownian particle. The resulting description can be considered as a generalized fluctuating hydrodynamics scheme for the Brownian gas, that incorporates fluctuations of the intensive variables through random transport coefficients. Finally, by studying the appropriate space averages, we have derived non-linear regression equations for the time dependent average of the variables of the Brownian system.

The results we have obtained in this paper, show that the description of far from equilibrium systems given in ($\vec{r}, \vec{u}, \delta T$)-space lead, after contraction, to a non-extensive description. They also lead to a generalized hydrodynamics with fluctuating transport coefficients. Actually, since the transport coefficients depend on $\delta T$, the linear relation between fluxes and forces in the ($\vec{r}, \delta T$)-space is in general lost, implying again a nonextensive description [9].

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