First passage time distribution of active thermal particles in potentials

Benjamin Walter, Gunnar Pruessner and Guillaume Salbreux

1Department of Mathematics, Imperial College London, 180 Queen’s Gate, SW7 2AZ London, United Kingdom
2Centre for Complexity & Networks, Imperial College London, SW7 2AZ London, United Kingdom
3The Francis Crick Institute, 1 Midland Road, NW1 1AT London, United Kingdom

(Dated: June 2, 2020)

We introduce a perturbative method to calculate all moments of the first-passage time distribution in stochastic one-dimensional processes which are subject to both white and coloured noise. This class of non-Markovian processes is at the centre of the study of thermal active matter, that is self-propelled particles subject to diffusion. The perturbation theory about the Markov process considers the effect of self-propulsion to be small compared to that of thermal fluctuations. To illustrate our method, we apply it to the case of active thermal particles (i) in a harmonic trap (ii) on a ring. For both we calculate the first-order correction of the moment-generating function of first-passage times, and thus to all its moments. Our analytical results are compared to numerics.

I. INTRODUCTION AND MAIN RESULTS

A. Introduction

Understanding the statistical properties of first-passage times (FPTs), the time a stochastic process takes to first reach a prescribed target, has fuelled research in stochastic dynamics for over a century. Thanks to its wide-ranging applications it has enjoyed increased attention over the last decade [1–3]. First-passage times are often used as a key characteristic of complex systems, such as chemical reactions [4], polymer-synthesis [5], intra-cellular events [6], neuronal activity [7] or financial systems [8]. Besides their dynamical information, FPTs are helpful to understand spatial properties of complex networks [9], extreme values of stochastic processes [10] and characteristic observables in out-of-equilibrium statistical physics [11].

Historically, the first settings in which FPT-problems were studied were Markovian processes in one spatial dimension. Schrödinger approached this problem first by integrating over the probability density with absorbing boundary conditions [12]. Pontryagin et al. introduced a method which casts the mean first-passage time (MFPT) into an ODE derived from the Kolmogorov backward equation [13, 14]. Further, Siegert and Darling introduced a method to obtain the moment-generating function of the FPT starting from a renewal equation [15, 16]. These three key advances of the first “classical” period suggest that, despite the innocent looking simplicity of the problem, even Markovian processes do generally not allow for a closed form expression of the full distribution of FPT. Expectations alone, the MFPT, are difficult enough to compute in very simple systems.

In a parallel development, and sparked by the work of Kramers [17], much effort was put into investigating the rate with which fluctuating particles escape from metastable potentials. The forces the surrounding heat bath exerts on the particle, usually inter-molecular collisions, were modelled as white noise $\xi_t$ with correlator $\langle \xi_s \xi_t \rangle = 2D_\xi \delta(s-t)$. This correlator implies that the timescale on which the particle evolves is infinitely separated from the heat bath’s correlations [1]. In this setup, the fluctuation-dissipation theorem (FDT) [18, 19] identifies the diffusivity $D_\xi$ with a fixed temperature $T$ of the surrounding heat bath. Particles subject to white noise follow Markovian (memoryless) trajectories and are immersed in a heat bath in thermodynamical equilibrium (e.g. [20]).

In many systems, the paradigm of white noise is a drastic over-simplification. Coloured noise, i.e. noise with non-uniform power spectrum, was introduced to account for correlations due to the heat bath [21, 22]. Coloured noise is usually assumed to have a correlator that decays exponentially with some characteristic inverse rate $\beta$
\[ \beta^{-1}, \text{ sometimes referred to as the “colour” of the noise.} \]

White noise corresponds to the limit of \( \beta \to \infty \). Escape rate problems with coloured noise were a highly active area of research in the late 1980s and early 1990s, when various approximation methods to calculate the MFPT were developed [23–28]. There contradictory predictions initially led to a considerable degree of confusion [29]. Many of these developments are reviewed in [21].

More recently, coloured noise was suggested as a model for “active” swimmers which are self-propelled and whose energy-consumption is fuelled by the environment [30]. In these systems, the fluctuation-dissipation theorem does not hold (e.g. [31]) such that no thermodynamic equilibrium can be assumed. Ensembles of non-interacting “active swimmers” have been intensively studied over the last five years in the light of their non-equilibrium features [32–35]. Notably, their first-passage time distributions have been recognised as one of their characterising properties [36].

Over the last twenty years or so, first-passage times have regained interest independently of escape problems, and new methods have been developed to extend older techniques into the realm of non-Markovian stochastic processes. The classic approaches of Schrödinger (cf. [37–39]), Pontryagin (cf. [40, 41]) and Siegert (cf. [5]) have been successfully employed to a plethora of non-Markovian systems such as generalised Langevin and Fokker-Planck equations and coupled oscillator chains. Further approaches have been found in [42–46]. The vast majority of that work is concerned with MFPT, also because in many cases higher moments or the full distribution contain prohibitively complicated expressions. And yet, the MFPT has recently been criticised as insufficient or misleading in characterising the timescale of a dynamics. Therefore a more precise understanding of the full distribution poses a pressing current challenge [47, 48].

In this work, we address this challenge and compute the full moment-generating function of a class of non-Markovian stochastic processes perturbatively. In doing so, we obtain all moments of the distribution to the same order in the perturbative expansion. To our knowledge, this is the first time the full distribution is obtained systematically in the presence of correlated driving noise and white thermal noise for a wide range of settings, including a potential. The formulas we obtain order by order are exact, and the results we obtain for two systems, as an illustration, are in excellent agreement with numerical simulation.

B. Outline

In the following, we study the first-passage time distribution of non-interacting active particles subject to thermal noise in an external potential. Particles which are driven by coloured noise alone, such as those studied in [31, 49], are used as a model for self-propelled particles, but do not have a coupling to a thermal heat bath; Such systems are referred to as active and athermal [50]. More recent models contain an active force term to capture self-propulsion in addition to thermal white noise which represents the heat bath, e.g. [35, 51, 52]. Additionally, a conservative force may be considered stemming from an external potential.

Together, for a particle moving in a 1D space, this class of thermal and active models is characterised by a single degree of freedom \( x_t \) satisfying a Langevin-type equation [1, 52] (see Fig. 1 for illustration)

\[
\dot{x}_t = -V'(x_t) + \xi_t + \varepsilon y_t; \tag{1}
\]

Here, \( V(x) \) denotes a potential, \( \xi_t \) a white noise with \( \langle \xi_t \xi_s \rangle = 2D_\xi \delta(t - s) \) and \( D_\xi \) a diffusivity fixed by the FDT. The second, stochastic term \( y_t \) in Eq. (1) denotes an additional coloured noise, in the following referred to as driving noise, which we assume to be stationary and of zero mean. Since there are two noise-terms, we introduce the driving average \( \langle \cdot \rangle \) over all realisations of \( y_t \), as opposed to the thermal average \( \langle \cdot \rangle \) over realisations of \( \xi_t \). The central result of this work consists in finding the doubly averaged moment-generating function of first-passage times \( \langle \exp(-s\tau_{x_0,x_1}) \rangle \) perturbatively in \( \varepsilon \).

Since \( y_t \) is self-correlated, \( x_t \) by itself is no longer Markovian; Our framework therefore provides a tool to study FPT-distributions of non-Markovian processes. On the other hand, this work stands in the context of recent efforts to approach active thermal systems with field-theoretic methods. As will be reported elsewhere, the process Eq. (1) can be mapped to a non-equilibrium field-theory from which a variety of observables can be deduced. In the present work, we focus on first-passage times, for which a simpler setup is sufficient in which we use functional perturbation theory instead of a fully-fledged field theory.

The perturbation takes place in the regime where \( \varepsilon y_t \) is small compared to \( \xi_t \), that means where thermal fluctuations dominate. In our perturbation theory, we will emphasise this point by controlling the amplitude of the driving noise term via \( \varepsilon \) which is chosen small. A priori no assumption is made about the noise-colour \( \beta^{-1} \) and we recover all moments at once to equal perturbative order.

Our framework is based on the established renewal approach [15, 16], but uses functional expansions to circumvent the problem of non-Markovianity of Eq. (1).

C. Main results

The central result of the present work concerns the moment-generating function of first-passage times of \( x_t \),

\[
F(s) = \langle e^{-s\tau_{x_0,x_1}} \rangle = 1 - s \langle \tau_{x_0,x_1} \rangle + \frac{s^2}{2} \langle \tau_{x_0,x_1}^2 \rangle + \ldots \tag{2}
\]
where \( \tau_{x_0,x_1} \) is the first-passage time of \( x_t \) defined as
\[
\tau_{x_0,x_1} := \inf_{t \geq 0} \{ t : x_t = x_1 | x_{t=0} = x_0 \} .
\] (3)

The following argument is based on two assumptions. First, we assume the moment-generating function \( F(s) \) is known for \( \varepsilon = 0 \). The case of \( \varepsilon = 0 \) corresponds to the case of a purely “passive” particle with no self propulsion. The particle behaves Markovian in this case, and already established techniques can be applied (see Introduction). We refer to this state as in equilibrium. Secondly, we assume that around this state \( F(s) \) is analytic in \( \varepsilon \) or, to be more precise, in a dimensionless parameter \( \nu \) which is of order \( O(\varepsilon^2) \). This means that the moment generating function of first-passage times has an expansion in \( \nu \) of the form
\[
F(s) = \left< \exp\left( -s\tau_{x_0,x_1} \right) \right> = M_0(x_0,x_1;s) + \nu M_1(x_0,x_1;s) + O(\nu^2) .
\] (4)

where \( M_0 \) and \( M_1 \) are the coefficients of expansion of \( F(s) \) in \( \nu \) around \( \nu = 0 \). The equilibrium component, \( M_0 \), can be found by classical methods such as the Darling-Siegert method which is discussed in the next subsection. The first-order contribution, \( M_1 \), requires some deeper analysis. Much of what follows is dedicated to the calculation of \( M_1 \), which to our knowledge is new in the literature. In principle, the method we present here is capable of calculating coefficients \( M_2, M_3, \ldots \) of arbitrarily high order of \( \nu \) for arbitrarily coloured noise (cf. Eq. (67)) as long as the autocorrelations can be integrated suitably. Further the potentials \( V(x) \) are arbitrary, as long as an associated differential operator can be diagonalised (Sec. III and [53]), otherwise it needs to be treated perturbatively as well.

We further illustrate our framework by explicitly computing \( M_0 \) and \( M_1 \) for two cases each of which are additionally driven by coloured Gaussian noise, i.e. \( \nu_i \) is Gaussian and has correlator \( \langle \eta_{i1}\eta_{i2} \rangle = D_g\beta \exp(-\beta|t_1 - t_2|) \) with some diffusivity \( D_g \) and correlation time \( \beta^{-1} \). In the first case, the particle is placed in a harmonic potential, \( V(x) = \frac{1}{2}x^2 \). This particular model has been studied in, e.g. [52], we refer to this model as active thermal Ornstein Uhlenbeck process (ATOU). While \( M_0 \) (see Eq. (98)) has been long known, \( M_1 \) (see Eq. (99)) is a new result. By numerically computing its inverse Laplace transform, we obtain the full probability distribution of first-passage times to first order in \( \nu \). In Fig. 1a we compare this to the numerically obtained probability density function. Secondly, we calculate \( M_0 \) and \( M_1 \) for the case of a Brownian Motion on a ring of radius \( r \) driven by coloured Gaussian noise which we refer to as active
thermal Brownian Motion (ATBM). Again, through numerically computing its inverse Laplace transform we obtain the corrected probability distribution of first-passage times which is compared to numerics in Fig. 2b.

Our method is systematic since it allows its user to calculate in principle corrections to arbitrary order, and it is controllable in the sense that the error can be made arbitrarily small. Further all moments are available at once. It is also valid for arbitrary noise colours \( \beta^{-1} \).

The paper is structured as follows. In Sec. II we give detailed account of how to calculate \( F(s) \) for small \( \nu \). First, we reproduce the Darling-Siegert argument in the equilibrium case \((\nu = \varepsilon = 0)\). Next, we introduce a perturbative version of the Darling-Siegert equation. Then, we obtain, as an intermediate result, a formula for \( F(s) \) which is still a functional of the colored noise \( y_t \). In the last step, we need to average over the stationary distribution of \( y_t \) to arrive at the explicit formula Eq. (74) which is the main result of our work. In the subsequent section III, we calculate all quantities required for the case of a harmonic potential and a Brownian motion with periodic boundary conditions and arrive at the first-order correction to the moment-generating function of first-passage times Eq. (99). Section IV concludes with a discussion of our findings.

II. PERTURBATION THEORY

As outlined above, in this work we present a way to calculate the moment-generating function of first-passage times of stochastic processes which are close to an equilibrium state. The underlying assumption is that moment-generating function varies smoothly as \( \varepsilon \), the coupling to the self-propelling force, is switched on. The moment-generating function of the equilibrium version of the process \((\varepsilon = 0)\) is assumed to be known in closed form, as is for instance justified for the Ornstein-Uhlenbeck process or Brownian Motion \([53]\). This exact form is then corrected by terms in the spirit of a perturbative expansion which is controlled by powers of a dimensionless parameter describing the distance to equilibrium. First, we revise the arguments given by Darling and Siegert for the equilibrium case \([15, 16]\). Next, we outline our perturbative approach to the active case.

Before going further, we introduce some notations. The transition probability density of progressing from \( x_0 \) at \( t_0 \) to \( x_1 \) at \( t_1 \) is denoted by

\[
T(x_0, x_1; t_0, t_1) = T(t_0, t_1)
\]

where the subscripts \( x_0 \) and \( x_1 \) are dropped wherever confusion can be avoided for the sake of easier notation. Analogously the return probability at \( x_1 \), \( R(x_1, t_0, t_1) \) is denoted by

\[
R(x_1, t_0, t_1) = R(t_0, t_1) = T(x_1, x_1; t_0, t_1)
\]

Further, the first-passage time density to first reach \( x_1 \) starting from \( x_0 \) is denoted by

\[
F(x_0, x_1; t_0, t_1) = F(t_0, t_1)
\]

In the following, we denote the Fourier transform of a function \( f(t) \) by

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} f(t),
\]

where \( F \) denotes the Fourier transform operation, with inverse

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \hat{f}(\omega),
\]

where \( d\omega = \frac{d\omega}{2\pi} \).

In the following, we consider functions \( f(t_0, t_1) \) which depend on the difference \( t_1 - t_0 \) only, i.e. \( f(t_0, t_1) = f(t_1 - t_0) \). We refer to functions of this type as diagonal. The Fourier transform in two variables of these functions read

\[
\hat{f}(\omega_0, \omega_1) = \int_{-\infty}^{\infty} dt_0 \int_{-\infty}^{\infty} dt_1 e^{-i\omega_0 t_0 - i\omega_1 t_1} f(t_0, t_1)
\]

\[
= \hat{f}(\omega_1) \delta(\omega_0 + \omega_1).
\]

1. Equilibrium case: The Darling-Siegert solution

We here consider the equilibrium case of Eq. (1) defined by setting \( \varepsilon = 0 \). As \( x_t \) is Markovian, the functions \( \hat{F} \) and \( \hat{T} \) satisfy the following renewal equation:

\[
T(x_0, x_1; t_0, t_1) = \int_{t_0}^{t_1} dt' F(x_0, x_2; t_0, t')T(x_2, x_1; t', t_1)
\]

for all \( x_2 \in (x_0, x_1] \).

Applying a Fourier transform to Eq. (11), the time-homogeneity of both \( T(t_0, t_1) \) and \( F(t_0, t_1) \) translates into diagonality in frequencies and turns the convolution into a product, such that the result can be stated at the level of the amplitudes alone. Rearranging the terms and choosing \( x_2 = x_1 \) results in

\[
\hat{F}(\omega) = \frac{\hat{T}(\omega)}{\hat{R}(\omega)}
\]

Since the Fourier transform of a probability density equals its characteristic function, Eq. (12) recovers all moments of the first passage time provided \( \hat{T} \) is known. Further, setting \( \omega = -is \) for some \( s \in \mathbb{R}_+ \) turns the Fourier transforms into Laplace transforms and the characteristic function into the moment generating function. This recovers the Darling-Siegert equation in its original form in which the Laplace transform of transition and return probabilities is linked to the moment-generating function of first-passage times.
2. Out-of-equilibrium: A perturbative approach

The argument made by Darling and Siegert breaks down when \( \varepsilon \neq 0 \): indeed when averaging over the driving noise \( y(t) \), the renewal equation Eq. (11) no longer is true. The approach we take in this paper, consists of three steps:

1. Fix a particular realisation \( y(t) \), and expand transition and return probabilities of \( x_t \) as functional expansion around \( y = 0 \) of the form

\[
\bar{T}(\omega_0, \omega_1, [\bar{y}] = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int d\omega_1 \ldots d\omega_n \delta \left( \omega_0 + \omega_1 + \sum_{i=1}^{n} \bar{\omega}_i \right) \\
\times \bar{T}^{(n)}(\omega_1, \bar{\omega}_1, \ldots, \bar{\omega}_n) \bar{y}(\bar{\omega}_1) \ldots \bar{y}(\bar{\omega}_n). \tag{13}
\]

2. As long as \( y(t) \) is fixed, the process, when understood as conditioned on this particular driving, satisfies a renewal equation of the type Eq. (11). Inserting the perturbative transition and return probabilities from the previous step, gives a perturbative series for the first-passage time density \( \bar{F}(\omega_0, \omega_1; [\bar{y}] \) of \( x_t \) conditioned on a particular \( y_t \).

3. Averaging over the ensemble of driving noises. For simplicity, we here assume that the correlation function of the driving noise is given by

\[
\overline{y_t y_r} = D_y \beta \exp(-\beta |t_r - t_t|) \tag{14}
\]

where \( \beta \) is the inverse correlation time. Generally, when computing the term of order \( \varepsilon^n \), the first \( n \) moments of \( y_t \) need to be known.

This procedure leads to the central result of this work: the moment-generating function of first-passage times to second order in \( \varepsilon \) reads

\[
F(\omega) = \frac{\bar{T}^{(0)}(\omega)}{R^{(0)}(\omega)} + \frac{\varepsilon^2 D_y \beta}{2} \left[ \frac{\bar{T}^{(2)}(\omega; i\beta, -i\beta)}{R^{(0)}(\omega)} - 2 \frac{\bar{T}^{(1)}(\omega - i\beta, i\beta)}{R^{(0)}(\omega - i\beta)} \frac{\bar{R}^{(1)}(\omega, i\beta)}{R^{(0)}(\omega)} \right] + O(\varepsilon^4). \tag{15}
\]

In the next sections, we derive this relation in more details.

A. Perturbative Darling-Siegert equation

1. Expression for the first-passage time distribution

By imposing an additional driving noise \( y_t \), the transition probability and FPT probability density of \( x_t \) depend on a particular realisation of \( y_t \). Accordingly, we introduce the transition probability density \( \bar{T}(t_0, t_1, [\bar{y}] \) and FPT probability density \( \bar{F}(t_0, t_1, [\bar{y}] \) as the densities of the process \( x_t \) conditioned on \( y_t \). The conditional densities are explicitly dependent on \( t_0 \) and \( t_1 \) rather than their difference \( t_1 - t_0 \) because \( y(t) \) is an explicit function of time.

For \( y \) fixed, the process remains Markovian and therefore Eq. (11) still applies and gives rise to:

\[
T(x_0, x_1; t_0, t_1; [y]) = \int_{-\infty}^{\infty} dt' F(x_0, x_1; t_0, t'; [y]) \\
\times R(x_1; t', t_1; [y]) \tag{16}
\]

where the dependency of the functions on the spatial values \( x_0, x_1 \) has been made explicit for clarity, and we have used the fact that \( F(t_0, t'; [y]) \) vanishes for \( t' < t_0 \) and \( R(t', t_1) \) vanishes for \( t' > t_1 \) to integrate over the full real axis.

It is no longer possible to directly invert this equation in Fourier space to solve for \( \bar{F} \), as done in Eq. (12), since neither terms in the integral are diagonal, i.e. they depend explicitly on both \( t_0, t' \) and \( t', t_1 \). However, introducing the inverse functional \( R^{-1} \) which is defined by the implicit equation:

\[
\int_{-\infty}^{\infty} dt R^{-1}(t_0, t; [y]) R(t, t_1; [y]) = \delta(t_1 - t_0) \tag{17}
\]

the renewal equation can be formally solved by the relation:

\[
F(x_0, x_1; t_0, t_1; [y]) = \int_{-\infty}^{\infty} dt T(x_0, x_1; t_0, t; [y]) R^{-1}(x_1; t, t_1; [y]) \tag{18}
\]

Our approach is then to perform a functional expansion of the quantities involved in Eq. (18) in the function \( y \), around the Markovian case \( y = 0 \).
2. Functional expansion of the transition and return probability densities

We start by performing a Taylor expansion of the transition probability $T(t_0, t_1)$:

$$T(t_0, t_1; [y]) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tilde{t}_1 \ldots d\tilde{t}_n \delta^n T \left( t_0; t_1; [y(t_1) \ldots y(t_n)] \right)$$

(19)

We now argue that this functional expansion can be rewritten in a simpler form, using the time-invariance properties of the dynamic equation (1). We introduce the time-shifting operator which transforms a function $y(t)$ into a time-shifted function $\tilde{y}(t)$, according to:

$$S_{t_0} [y](t) = y(t + t_0) \equiv \tilde{y}(t)$$

(20)

and we will use the fact that:

$$T(t_0, t_1; [y]) = T(t_0 + \tau, t_1 + \tau; [S_{-\tau} [y]])$$

(21)

for any time-shift $\tau$. This relation results from the invariance of time shifting of Eq. (1), when the function $y(t)$ is shifted in time accordingly. As a result we introduce the shifted function $T_0$, defined by:

$$T_0(\tau; [\tilde{y}]) = T(0, \tau; [\tilde{y}])$$

(22)

such that Eq. (21) gives, choosing $\tau = -t_0$:

$$T(t_0, t_1; [y]) = T_0(t_1 - t_0; [S_{t_0} [y]])$$

(23)

Using Eq. (23), the functional derivatives of $T$ can be related to the functional derivatives of $T_0$ through

$$\frac{\delta^n T}{\delta y(t_1) \ldots \delta y(t_n)}(t_0, t_1; [y]) = \frac{\delta^n T_0}{\delta \tilde{y}(t_1 - t_0) \ldots \delta \tilde{y}(t_n - t_0)}(t_1 - t_0; [S_{t_0} [y]])$$

(24)

and the functional expansion of $T$ can now be written

$$T(t_0, t_1; [y]) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tilde{t}_1 \ldots d\tilde{t}_n \delta^n T_0 \left( \tilde{y}(t_1 - t_0) \ldots \tilde{y}(t_n - t_0); [\tilde{y}(t_1) \ldots \tilde{y}(t_n)] \right)$$

(25)

where we have used that $S_{t_0}[y = 0] = [0]$. Going in Fourier space, we associate to a functional $F[y]$ the functional $\hat{F}[\tilde{y}] = F[y]$. The functional derivatives of these two functionals are related by:

$$\frac{\delta^n F[y]}{\delta y(t_1) \ldots \delta y(t_n)} = \int d\tilde{\omega}_1 \ldots d\tilde{\omega}_n \delta^n F[\tilde{y}] \delta^{n\hat{F}[\tilde{y}]} \delta^{n\hat{y}(\omega_1)} \ldots \delta^{n\hat{y}(\omega_n)} e^{-i\tilde{\omega}_1 \tilde{t}_1} \ldots e^{-i\tilde{\omega}_n \tilde{t}_n}.$$ 

(26)

As a result, calculating the Fourier transform of Eq. (25) with respect to the variables $t_0, t_1$, we obtain the functional expansion:

$$\hat{T}(\omega_0, \omega_1, [\tilde{y}]) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tilde{\omega}_1 \ldots d\tilde{\omega}_n \delta \left( \omega_0 + \omega_1 - \sum_{i=1}^{n} \tilde{\omega}_i \right)$$

$$\times \delta^n \hat{T}_0 \left( \tilde{y}(\omega_1) \ldots \tilde{y}(\omega_n); [\tilde{y}(0)] \right)$$

(27)

where $1/\varepsilon^n$ has been introduced for convenience, to make the expansion in small $\varepsilon$ explicit in Eq. (27). We then arrive at the following form:

$$\hat{T}(\omega_0, \omega_1; [\tilde{y}]) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int d\tilde{\omega}_1 \ldots d\tilde{\omega}_n \delta \left( \omega_0 + \omega_1 - \sum_{i=1}^{n} \tilde{\omega}_i \right)$$

$$\times \hat{T}^{(n)}(\omega_1, \omega_1, \ldots, \omega_n) \hat{y}(\tilde{\omega}_1) \ldots \hat{y}(\tilde{\omega}_n)$$

(28)

$$\hat{R}(\omega_0, \omega_1; [\tilde{y}]) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int d\tilde{\omega}_1 \ldots d\tilde{\omega}_n \delta \left( \omega_0 + \omega_1 + \sum_{i=1}^{n} \tilde{\omega}_i \right)$$

$$\times \hat{R}^{(n)}(\omega_1, \omega_1, \ldots, \omega_n) \hat{y}(\tilde{\omega}_1) \ldots \hat{y}(\tilde{\omega}_n)$$

(29)

where we have made a choice of signs of $\tilde{\omega}_1, \ldots, \tilde{\omega}_n$ for convenience.

3. Functional expansion of the inverse of the return probability density

We now turn to the expansion of $R^{-1}$ in Eq. (18). As $T$ and $R$, $R^{-1}$ satisfy the time-shift invariance relation (21). This can be seen by time shifting Eq. (17) by the time $\tau$:

$$\int_{-\infty}^{\infty} dt R^{-1}(t_0 + \tau, t_1 + \tau; S_{-\tau} [y])$$

$$\times R(t_1, t_1, \tau, t_1; S_{-\tau} [y]) = \delta(t_1 - t_0)$$

(31)

and, since $R(t_1, t_1, \tau, t_1; S_{-\tau} [y]) = R(t, t_1, [y])$, one also obtains $R^{-1}(t_0 + \tau, t_1 + \tau; S_{-\tau} [y]) = R^{-1}(t_0, t_1; [y])$ since

$$\int_{-\infty}^{\infty} dt R^{-1}(t_0 + \tau, t_1 + \tau; S_{-\tau} [y])$$

$$\times R(t_1, t_1, \tau, t_1; S_{-\tau} [y]) = \delta(t_1 - t_0)$$

(32)

$$\int_{-\infty}^{\infty} dt R^{-1}(t_0 + \tau, t_1 + \tau; S_{-\tau} [y])$$

$$\times R(t_1, t_1, \tau, t_1; S_{-\tau} [y]) = \delta(t_1 - t_0)$$

(33)
Eq. (17) defines $R^{-1}$. The analysis of the previous subsection still applies and one obtains the expansion

$$
\hat{R}^{-1}(\omega_0, \omega_1, [\hat{y}]) = \sum_{n=0}^{\infty} \varepsilon^n \int d\omega_1 ... d\omega_n \delta \left( \omega_0 + \omega_1 + \sum_{i=1}^{n} \omega_i \right) 
$$

\begin{equation}
\times (\hat{R}^{-1})^{(n)}(\omega_1, \omega_2, ..., \omega_n) \hat{y}(\omega_1)...\hat{y}(\omega_n).
\end{equation}

We then obtain by applying a Fourier transform to Eq. (17):

$$
\int d\omega \hat{R}^{-1}(\omega_0, \omega; [\hat{y}]) \hat{R}(-\omega, \omega_1; [\hat{y}]) = \delta(\omega_0 + \omega_1).
\end{equation}

This relation allows to relate the functions $(R^{-1})^{(n)}$ to the functions $R^{(n)}$, by plugging Eqs. (30) and (32) and identifying order by order in $\varepsilon$. Limiting ourselves up to order 2, we obtain:

$$
(\hat{R}^{-1})^{(0)}(\omega) = \frac{1}{R^{(0)}(\omega)}
\end{equation}

$$
(\hat{R}^{-1})^{(1)}(\omega, \omega_1) = -\frac{\hat{R}^{(1)}(\omega, \omega_1)}{R^{(0)}(\omega) R^{(0)}(\omega + \omega_1)}
\end{equation}

$$
(\hat{R}^{-1})^{(2)}(\omega, \omega_1, \omega_2) = \frac{1}{R^{(0)}(\omega + \omega_1 + \omega_2) R^{(0)}(\omega)} \left\{ \frac{2\hat{R}^{(1)}(\omega + \omega_2, \omega_1) \hat{R}^{(1)}(\omega, \omega_2)}{R^{(0)}(\omega + \omega_2)} - \hat{R}^{(2)}(\omega, \omega_1, \omega_2) \right\}.
\end{equation}

4. Second-order expansion of the first passage time distribution

Equipped with these expansions, one now can expand the first-passage time density expression (18) to obtain a functional expansion of the first-passage density in $\varepsilon$, involving the functions $\hat{T}^{(n)}$ and $\hat{R}^{(n)}$ which are simpler to calculate. We first need to write down the Fourier-transformed version of Eq. (18):

$$
\hat{F}(\omega_0, \omega_1; [\hat{y}]) = \int_{-\infty}^{\infty} d\omega \hat{T}(\omega_0, \omega; [\hat{y}]) \hat{R}^{-1}(-\omega, \omega_1; [\hat{y}])
\end{equation}

where the dependency on $x_0, x_1$ is here implicit. Performing an expansion of this relation in $\varepsilon$, the result reads to second order:

$$
\hat{F}(\omega_0, \omega_1; [\hat{y}]) = \hat{T}^{(0)}(\omega_1) \delta(\omega_0 + \omega_1) + \varepsilon \left[ -\hat{T}^{(0)}(-\omega_0) \hat{R}^{(1)}(\omega_1, -\omega_0 - \omega_1) + \hat{T}^{(1)}(\omega_1, -\omega_0 - \omega_1) \frac{1}{R^{(0)}(\omega_1)} \hat{y}(\omega_0 + \omega_1) 
\right] 
\end{equation}

$$
+ \frac{1}{2} \varepsilon^2 \int d\omega \left[ \hat{T}^{(0)}(-\omega_0) \hat{R}^{(1)}(\omega_1, -\omega_0 - \omega_1) - \hat{T}^{(2)}(\omega_1, -\omega_0 - \omega_1) \hat{R}^{(0)}(-\omega_0 - \omega_1) \right] \hat{y}(\omega_0 + \omega_1 + \omega_1) + ...
\end{equation}

\]
for $\varepsilon = 0$, depends on the time-difference only and can therefore be written $T(x_0, x_1; t_0, t_1) = T^{(0)}(x_0, x_1; t_1 - t_0)$; following definitions given in Eqs. (22) and (28).

The forward operator $T^{(0)}$ solves the Kolmogorov forward equation

\[
\begin{cases}
\partial_t T^{(0)}(x_0, x_1; t) = \mathcal{L}_x T^{(0)}(x_0, x_1; t) & t > 0 \\
T^{(0)}(x_0, x_1; t = 0) = \delta(x_1 - x_0) \\
T^{(0)}(x_0, x_1; t) = 0 & t < 0
\end{cases}
\]

where we introduce the forward evolution operator $\mathcal{L}_x$ as

\[
\mathcal{L}_x f = \partial_t (V'(x)f) + D_x \partial^2_x f,
\]

where $f$ is a twice differentiable test function. In Eq. (39), we denote the forward operator as $L_x$ to indicate that its gradient terms are acting on the $x_1$ dependency of $T^{(0)}$. Correspondingly, the $L^2$-adjoint operator $\mathcal{L}^T_x$, also referred to as backward operator, is

\[
\mathcal{L}^T_x f = -V'(x)\partial_t f + D_x \partial^2_x f.
\]

The forward operator $\mathcal{L}_x$ has a countable set of eigenfunctions $\{u_n(x)\}$ and a non-positive spectrum $0 \geq -\lambda_0 > -\lambda_1 > ... [54]$, but is a priori not self-adjoint in $L^2(\mathbb{R})$. It is straightforward to show [55] however that the operator

\[
\mathcal{L}_x = \exp \left( \frac{V(x)}{2D_x} \right) \mathcal{L}_x \exp \left( -\frac{V(x)}{2D_x} \right)
\]

is self-adjoint and that therefore the family of

\[
\left\{ \exp \left( \frac{V(x)}{2D_x} \right) u_n(x) \right\}, \ n \geq 0
\]

as eigenfunctions of $\mathcal{L}_x$, form an orthogonal set of eigenfunctions spanning $L^2(\mathbb{R})$. We further impose a choice of normalization of the functions $\{u_n\}$, such that the set

\[
\left\{ \exp \left( \frac{V(x)}{2D_x} \right) u_n(x) \right\}
\]

is orthonormal. Defining $u_n(x)$ as right eigenfunctions, and

\[
u_n(x) = e^{\frac{V(x)}{2D_x}} u_n(x)
\]

as left eigenfunctions, satisfying

\[
\mathcal{L}^T_x v_n(x) = -\lambda_n v_n(x),
\]

we obtain a biorthogonal system for the pair $\{u_n(x)\}$, $\{v_n(x)\}$:

\[
\int dx \ v_m(x) u_n(x) = \delta_{m,n}.
\]

This is useful to solve the forward equation: taking the Fourier transform in time of Eq. (40), one obtains

\[
i\omega \hat{T}^{(0)}(x_0, x_1; \omega) = \mathcal{L}_x \hat{T}^{(0)}(x_0, x_1; \omega) + \delta(x_1 - x_0).
\]

Inserting the ansatz

\[
\hat{T}^{(0)}(x_0, x_1; \omega) = \sum_{n \geq 0} \hat{u}_n(x_1) u_n(x_0)
\]

into Eq. (48) and using Eq. (42) leads to

\[
\sum_{n \geq 0} (i\omega + \lambda_n) \hat{u}_n(x_1) u_n(x_0) = \sum_{n \geq 0} v_n(x_0) u_n(x_1),
\]

where we made use of the decomposition of unity,

\[
\delta(x_1 - x_0) = \sum_{n \geq 0} v_n(x_0) u_n(x_1).
\]

Since the $u_n(x)$ are linearly independent, their prefactors in Eq. (50) need to agree. Therefore,

\[
\hat{T}^{(0)}(x_0, x_1; \omega) = \sum_{n \geq 0} \frac{v_n(x_0)}{i\omega + \lambda_n} u_n(x_1).
\]

Turning to the case of $\varepsilon \neq 0$, the transition probability of the driven Langevin equation (1) solves the forward equation

\[
\partial_t T(x_0, x_1; t_0, t_1; [y]) = (\mathcal{L}_{x_1} + e y(t_1) \partial_{x_1}) T(x_0, x_1; t_0, t_1; [y]), \quad t_1 > t_0
\]

\[
T(x_0, x_1; t_0, t_1; [y]) = \delta(x_1 - x_0)
\]

\[
T(x_0, x_1; t_0, t_1; [y]) = 0 \quad t_1 < t_0
\]

where time-homogeneity can no longer be assumed. Under Fourier transform, this forward equation becomes

\[
(i\omega_1 - \mathcal{L}_{x_1}) \hat{T}(x_0, x_1, \omega_0, \omega_1; [\hat{y}]) = \delta(x_1 - x_0) \hat{T}_{\omega_1 + \omega_0} + \varepsilon \int d\tilde{\omega}_1 \hat{y}(\tilde{\omega}_1) \partial_{x_1} \hat{T}(x_0, x_1, \omega_0, \omega_1 - \tilde{\omega}_1; [\hat{y}])
\]

where the $y$-dependent term turns from a product into a convolution under the Fourier transform. We develop a perturbative solution of $\hat{T}(x_0, x_1, \omega_0, \omega_1; [\hat{y}])$ in powers of $\varepsilon$.

1 Instead, it is self-adjoint in the weighted space $L^2(u_0(x))$, [54].
order result (53), we obtain:
\[
\hat{T}(x_0, x_1, \omega_0, \omega_1, [\dot{y}]) = \sum_{n \geq 0} \frac{v_n(x_0)u_n(x_1)}{\omega_1 + \lambda_n} \delta(\omega_0 + \omega_1) \\
+ \varepsilon \int d\tilde{\omega}_1 \hat{T}^{(1)}(x_0, x_1, \omega_1, \tilde{\omega}_1) \tilde{y}(\tilde{\omega}_1) \delta(\omega_0 + \omega_1 + \tilde{\omega}_1) + \ldots
\]
and we wish to determine the function \( \hat{T}^{(1)} \). Since the \( u_n(x) \) span the \( L^2 \)-space, and in analogy to the ansatz (49), the first-order correction too can be written as a sum
\[
\hat{T}^{(1)}(x_0, x_1, \omega_1, \tilde{\omega}_1) = \sum_{n \geq 0} \frac{T^{(1)}(x_0; \omega_1, \tilde{\omega}_1)}{\omega_1 + \lambda_n}
\]
Re-inserting Eqs. (56) and (57) into Eq. (55) causes all terms to zeroth order in \( y \) to cancel, and one obtains an equation relating the contributions proportional to \( \varepsilon \),
\[
\sum_{n \geq 0} (i\omega_1 + \lambda_n) \int d\tilde{\omega}_1 \hat{T}^{(1)}_n(x_0; \omega_1, \tilde{\omega}_1) \tilde{y}(-\tilde{\omega}_1) u_n(x_1) = \int d\tilde{\omega}_1 \sum_{n \geq 0} \frac{v_n(x_0)\partial_{x_1} u_n(x_1)}{i(\omega_1 - \tilde{\omega}_1) + \lambda_n} \tilde{y}(\tilde{\omega}_1).
\]
The right hand side, which is the convolution of \( \hat{T}^{(0)}(\omega) \) and \( \tilde{y}(\omega) \), no longer sums over \( u_n(x_1) \) but their derivative \( \partial_{x_1} u_n(x_1) \). In order to compare both left and right terms, we need to express this sum as a sum over the linearly independent \( u_n(x_1) \) again. The decomposition of the derivative in terms of \( u_n(x_1) \) is given by
\[
\partial_{x_1} u_n(x_1) = \sum_k \Delta_{nk} u_k(x_1)
\]
where we refer to the \( \Delta_{nk} \) as derivative coupling matrix whose entries, as follows from bi-orthogonality, are
\[
\Delta_{nk} = \int dx v_k(x) \partial_{x_1} u_n(x).
\]
Further, Eq. (58) holds true for arbitrary \( \tilde{y}(\tilde{\omega}_1) \). In order to compare both integrations over \( d\tilde{\omega}_1 \), we relabel \( \tilde{\omega}_1 \rightarrow -\tilde{\omega}_1 \) in the left-hand side of the equation. Using this notation, inserting the sum (59) into Eq. (58), and resolving the ansatz (57), one obtains:
\[
\hat{T}^{(1)}(x_0, x_1, \omega_1, \tilde{\omega}_1) = \sum_{n, k \geq 0} \frac{v_k(x_0)\Delta_{kn} u_n(x_1)}{i(\omega_1 + \tilde{\omega}_1) + \lambda_k (i\omega_1 + \lambda_n)}.
\]
In a similar way, the second order correction can be found; using the functional expansion (30) to second order:
\[
\hat{T}(x_0, x_1, \omega_0, \omega_1, [\dot{y}]) = \sum_{n \geq 0} \frac{v_n(x_0)u_n(x_1)}{\omega_1 + \lambda_n} \delta(\omega_0 + \omega_1) \\
+ \varepsilon \sum_{n, k \geq 0} \frac{v_k(x_0)\Delta_{kn} u_n(x_1)}{(-i\omega_0 + \lambda_k (i\omega_1 + \lambda_n)} \tilde{y}(\omega_0 + \omega_1) \\
+ \frac{\varepsilon^2}{2} \int d\tilde{\omega}_1 d\tilde{\omega}_2 \hat{T}^{(2)}(x_0, x_1, \omega_1, \tilde{\omega}_1, \tilde{\omega}_2) \tilde{y}(-\tilde{\omega}_1) \tilde{y}(-\tilde{\omega}_2) \delta(\omega_0 + \omega_1 + \tilde{\omega}_1 + \tilde{\omega}_2) + \cdots,
\]
with the results from Eqs. (53) and (61) to zeroth and first order, inserting this ansatz into the forward equation (55) gives, following in complete analogy to the previous steps,
\[
\hat{T}^{(2)}(x_0, x_1, \omega_1, \tilde{\omega}_1, \tilde{\omega}_2) = \sum_{n, k, m \geq 0} \frac{2v_n(x_0)\Delta_{nk} \Delta_{km} u_m(x_1)}{(i(\omega_1 + \tilde{\omega}_1 + \tilde{\omega}_2) + \lambda_n)(i(\omega_1 + \tilde{\omega}_1) + \lambda_k)(i\omega_1 + \lambda_m)}.
\]
Following this method, it is straightforward to generate the perturbative terms of \( \hat{T}(n) \) to arbitrary order in \( n \),
\[
\hat{T}^{(n)}(x_0, x_1, \omega_1, \tilde{\omega}_1, \ldots, \tilde{\omega}_n) = n! \sum_{k_0, \ldots, k_n \geq 0} \prod_{j=0}^n \frac{v_{k_j}(x_0)\Delta_{k_{j-k_{j-1}} k_j} \cdots \Delta_{k_{n-k_n}} u_{k_n}(x_1)}{(i(\omega_1 + \sum_{1 \leq \ell \leq n-j} \tilde{\omega}_k) + \lambda_{k_j})}.
\]
Finally, choosing \( x_0 = x_1 \) in any of the expressions (53), (61), (63) and (64) gives the corresponding terms for the return probability coefficients \( \hat{R}^{(0)}(\omega), \hat{R}^{(1)}(\omega, \tilde{\omega}_1), \ldots \). Equipped with these expressions, we are able to compute the relevant integrals in the formula for the \( y \)-averaged first-passage time density Eq. (66).

C. Driving noise averaging

As was set out initially, the quantity of interest is the first-passage time density when averaged over all driving noises (even if the quantity given above might be of interest in itself). The average over driving noise realisations \( y_t \) is an average different to the average over the underlying stochastic process \( x_t \), in that sense akin to “quenched disorder averages” which replace a stochastic background field by an effective deterministic correction to observables. The expansion in Eq. (38) is a power series in orders of \( \varepsilon \), where contributions of order \( \varepsilon^n \) contain an internal integration over \( n - 1 \) free frequencies. The expansion terms which stand in front of the \( y \) terms, those denoted within square brackets, are independent of \( y \). They may be interpreted as the nth order response functionals of the first-passage time distribution (in \( s \)) to perturbations in the driving noise \( y \). To calculate the
y-average of \( \hat{F}(\omega_0, \omega_2; [y]) \), each term in Eq. (38) is integrated over the path-measure of \( y \), \( \mathcal{P}[y] \). The order of internal integration and y-averaging can be swapped. Consequently, since \( \langle y \rangle = 0 \) by assumption, all terms in first order in \( y \) vanish. To second order, correlations of \( y \) come into play. We introduce the correlation function

\[ \bar{y}(\omega_1) \bar{y}(\omega_2) = \int D[y] \mathcal{P}[y] \bar{y}(\omega_1) \bar{y}(\omega_2) = \tilde{C}_2(\omega_2) \delta(\omega_1 + \omega_2), \]

(65)

where the last equality arises from the stationary in time of the random process \( y(t) \). Stationary in time also implies that \( \tilde{C}_2(\omega_2) \) is symmetric in \( \omega \rightarrow -\omega \). Performing an averaging over \( \mathcal{P}[\bar{y}] \) of Eq. (38), we obtain:

\[
\begin{align*}
\bar{F}(\omega_0, \omega_1; [y]) &= \frac{\hat{T}^{(0)}(\omega_1)}{\hat{R}^{(0)}(\omega_1)} \delta(\omega_0 + \omega_1) \\
&+ \frac{1}{2} \varepsilon^2 \left[ - \int d\omega \frac{\hat{T}^{(0)}(\omega_1) \hat{R}^{(2)}(\omega_1, \omega_2, -\tilde{\omega})}{\left( \hat{R}^{(0)}(\omega_1) \right)^2} \tilde{C}_2(\omega) + 2 \int d\omega \frac{\hat{T}^{(0)}(\omega_1) \hat{R}^{(1)}(\omega_1, \omega_2, \omega) \hat{R}^{(1)}(\omega_1, -\omega)}{\left( \hat{R}^{(0)}(\omega_1) \right)^2} \tilde{C}_2(\omega) } \right] \\
&\phantom{=} : = \text{(I)} \\
- 2 \int d\omega \frac{\hat{T}^{(1)}(\omega_1, \omega_2) \hat{R}^{(1)}(\omega_1, -\tilde{\omega})}{\left( \hat{R}^{(0)}(\omega_1) \right)^2} \tilde{C}_2(\omega) + \int d\omega \frac{\hat{T}^{(2)}(\omega_1, \omega_2) \hat{R}^{(0)}(\omega_1)}{\left( \hat{R}^{(0)}(\omega_1) \right)^2} \tilde{C}_2(\omega) \right] \delta(\omega_0 + \omega_1) + \ldots \quad (66)
\end{align*}
\]

The first term, of zeroth order, represents the Darling-Siegert solution (12). This is consistent with our expansion around the base-point of no driving noise (fully Markovian process). Once averaged, the second-order contribution is again diagonal (i.e. proportional to \( \delta(\omega_0 + \omega_2) \)) indicating that the y-averaged first-passage distribution is again invariant under time-shifts. The four correction terms featuring in the second order expansion in \( \varepsilon \) are labelled (I) to (IV), and need to be calculated explicitly. The corresponding expressions are derived in the following section, for the case of Gaussian coloured noise.

D. Second-order correction with Gaussian coloured noise

So far, in our derivation of the second-order correction to the first-passage time density, Eq. (66), we only demanded the active driving noise to be stationary, with finite correlations and vanishing mean. In what follows, we specify \( y_t \) to be Gaussian coloured noise. This choice is almost canonical in the study of coloured noise [21]. In our case it greatly simplifies the necessary integrals. It is, however, possible to use any other correlation functions as long as the integrals remain manageable. Generally, to compute the perturbative contribution of nth order in \( y \), the n-point correlation function of \( y_t \) needs to be known.

For Gaussian processes all higher moments follow from the two-point correlation function which simplifies the calculation of potential higher order corrections. Since the correlation function of coloured noise is an exponential, the results obtained in this section to order \( \mathcal{O}(\varepsilon^2) \) hold for any noise with such autocorrelation, in particular telegraphic noise as used in run-and-tumble processes [56].

Gaussian coloured noise is defined by its exponential correlator,

\[ y_{t_1} y_{t_2} = D_y \beta \exp \left( -\beta |t_1 - t_2| \right), \quad (67) \]

which in Fourier space reads (Eq. (65))

\[ \tilde{C}_2(\omega) = \frac{2D_y \beta^2}{\omega^2 + \beta^2}, \quad (68) \]

With the explicit expressions (53),(61), (63), we perform the integration in (I) (see Eq. (66) for notation)
in eigenfunction-representation,

$$
(I) = \int d\omega \frac{\hat{T}^{(0)}(\omega) \hat{R}^{(2)}(\omega_1, \hat{\omega}, -\hat{\omega})}{\left(\hat{R}^{(0)}(\omega_1)\right)^2} C_2(\hat{\omega})
$$

$$
= 2D_y\beta^2 \frac{\hat{T}^{(0)}(\omega_1)}{\left(\hat{R}^{(0)}(\omega_1)\right)^2} \int d\omega \frac{\hat{R}^{(2)}(\omega_1, \hat{\omega}, -\hat{\omega})}{\beta^2 + \omega^2}
$$

$$
= 2D_y\beta^2 \frac{\hat{T}^{(0)}(\omega_1)}{\left(\hat{R}^{(0)}(\omega_1)\right)^2} \sum_{n,k,m \geq 0} \frac{\nu_n(x_1)\Delta_{mk}\Delta_{km}u_m(x_1)}{(i\omega_1 + \lambda_n)(i(\omega_1 + \hat{\omega}) + \lambda_k)(i\omega_1 + \lambda_m)} \frac{1}{\omega^2 + \beta^2}
$$

$$
= D_y\beta \frac{\hat{T}^{(0)}(\omega_1) \hat{R}^{(2)}(\omega_1, -i\beta, i\beta)}{\left(\hat{R}^{(0)}(\omega_1)\right)^2}.
$$

where in the last equality we employed Cauchy’s residue theorem closing the contour in the lower half-plane containing the simple pole at $\hat{\omega} = -i\beta$. Likewise, we find

$$
(IV) = \int d\omega \frac{\hat{T}^{(2)}(\omega_1, \hat{\omega}, -\hat{\omega})}{\hat{R}^{(0)}(\omega_1)} C_2(\hat{\omega})
$$

$$
= 2D_y\beta^2 \frac{1}{\hat{R}^{(0)}(\omega_1)} \sum_{n,k,m \geq 0} \frac{\nu_n(x_1)\Delta_{mk}\Delta_{km}u_m(x_1)}{(i\omega_1 + \lambda_n)(i(\omega_1 + \hat{\omega}) + \lambda_k)(i\omega_1 + \lambda_m)} \frac{1}{\omega^2 + \beta^2}
$$

$$
= D_y\beta \frac{\hat{T}^{(2)}(\omega_1, -i\beta, i\beta)}{\hat{R}^{(0)}(\omega_1)}
$$

where again the integral is evaluated by closing the contour in the lower half-plane enclosing the pole at $\hat{\omega} = -i\beta$.

The integrals (II) and (III), featuring $\omega$-dependent denominators, require some more careful analysis. We have

$$
(II) = 2 \int d\omega \frac{\hat{T}^{(0)}(\omega) \hat{R}^{(1)}(\omega_1, \hat{\omega}, -\hat{\omega}) \hat{R}^{(1)}(\omega_1, -\hat{\omega}) C_2(\hat{\omega})}{\left(\hat{R}^{(0)}(\omega_1)\right)^2\hat{R}^{(0)}(\omega_1 - \hat{\omega})}
$$

As before we calculate this integral by application of Cauchy’s residue theorem, but now closing the contour in the upper half-plane. The numerator’s poles all lie in the lower half-plane with the exception of the pole at $\hat{\omega} = i\beta$ stemming from the correlator, as can be confirmed by inspection of Eq. (61). Besides, the denominator $\hat{R}^{(0)}(\omega_1 - \hat{\omega})$ does not have any roots for $\Im(\hat{\omega}) > 0$. Indeed, using the relations Eq. (53) and (45), $\hat{R}^{(0)}(\omega_1 - \hat{\omega}) = \sum_{n \geq 0} \frac{v_{n}(x_1)u_n(x_1)}{i(\omega_1 - \hat{\omega}) + \lambda_n}$. Since the $\{u_n(x)\}$ span $L^2$, there cannot be a $x_1$ for which all $v_n(x_1) = 0$. Besides the real part of the denominator in the sum is strictly positive, $\lambda_n + \Im(\hat{\omega}) > 0$, assuming $\omega_1 \in \mathbb{R}$. In the upper half plane the sum therefore contains terms whose real part is non-negative and at least once strictly positive; hence the sum is free of roots in the upper half plane. Invoking Cauchy’s residue formula the integral is then given by

$$
(II) = 2D_y\beta \frac{\hat{T}^{(0)}(\omega_1) \hat{R}^{(1)}(\omega_1 - i\beta, i\beta) \hat{R}^{(1)}(\omega_1, -i\beta)}{\left(\hat{R}^{(0)}(\omega_1)\right)^2\hat{R}^{(0)}(\omega_1 - i\beta)}.
$$

By analogous reasoning one obtains

$$
(III) = 2 \int d\omega \frac{\hat{T}^{(1)}(\omega_1, -\hat{\omega}, \hat{\omega}) \hat{R}^{(1)}(\omega_1, -\hat{\omega}) C_2(\hat{\omega})}{\hat{R}^{(0)}(\omega_1) \hat{R}^{(1)}(\omega_1 - \hat{\omega})}
$$

$$
= 2D_y\beta \frac{\hat{T}^{(1)}(\omega_1, -i\beta, i\beta) \hat{R}^{(1)}(\omega_1, -i\beta)}{\hat{R}^{(0)}(\omega_1) \hat{R}^{(1)}(\omega_1 - i\beta)}.
$$

All four terms together give a general formula for the moment generating function of first-passage times for arbitrary underlying processes and driving noises provided the eigenfunctions and correlators are known. In the case of driving noise with exponentially decaying auto-correlation, the full formula for $\tilde{F}(\omega, \omega_1) = F(\omega_1) \delta(\omega_1 + \omega)$, reads

$$
F(\omega) = \frac{\hat{T}^{(0)}(\omega)}{\hat{R}^{(0)}(\omega)} + \frac{\epsilon^2 D_y\beta}{2\hat{R}^{(0)}(\omega)} \left[ \frac{\hat{T}^{(0)}(\omega)}{\hat{R}^{(0)}(\omega)} \left(2 \hat{R}^{(1)}(\omega, -i\beta, i\beta) \hat{R}^{(1)}(\omega, -i\beta) - \hat{R}^{(2)}(\omega, -i\beta, i\beta) \right) + \hat{T}^{(2)}(\omega, -i\beta, i\beta) - 2 \frac{\hat{T}^{(1)}(\omega, -i\beta, i\beta) \hat{R}^{(1)}(\omega, -i\beta)}{\hat{R}^{(0)}(\omega - i\beta)} \right]
$$

This general result concludes this section. In the next section, we consider two concrete examples to demonstrate how this perturbation theory can be turned into analytical results.
III. RESULTS FOR SIMPLE POTENTIALS

A. Active Thermal Ornstein Uhlenbeck Process (ATOU)

In this example we study the case of a particle in a harmonic potential driven by white and coloured noise described by the Langevin Equation

$$\dot{x}_t = -\alpha x_t + \xi_t + \varepsilon y_t$$

(75)

with driving noise correlator (see Eq. (67))

$$\overline{y_t y_{t'}} = D_y \beta e^{-\beta |t_1 - t_2|}.$$  (76)

This process reduces to the simple Ornstein Uhlenbeck process when $\varepsilon = 0$ which models a particle in a harmonic potential $V(x) = \frac{1}{2} x^2$ within a thermal bath. We consider, however, the process driven by an additional “active” term $\varepsilon y_t$. We therefore refer to this process as active thermal Ornstein Uhlenbeck process (ATOU). In the un-driven case ($\varepsilon = 0$), the dynamics are characterised by the time and length-scales $\alpha^{-1}$ and $\ell = \sqrt{D_x \alpha^{-1}}$.

1. From eigenfunctions to the moment generating function of first-passage times

The Fokker-Planck equation associated (cf. Eq. (40)) to the Langevin Equation (75) is

$$\mathcal{L} = D \partial_x^2 + x \partial_x + 1.$$  (77)

It has eigenvalues $\lambda_n = \alpha n$

(78)

and is diagonalised by the (normalised) eigenfunctions

$$v_n(x) = \frac{1}{\sqrt{2\pi \ell n!}} \text{He}_n \left( \frac{x}{\ell} \right)$$

(79)

$$u_n(x) = \frac{1}{\sqrt{2\pi \ell n!}} \text{He}_n \left( \frac{x}{\ell} \right) \cdot \exp \left( - \frac{x^2}{2\ell^2} \right)$$

(80)

where we introduced Hermite polynomials using the convention

$$\text{He}_n(x') = (-1)^n e^{x'^2/2} \frac{d^n}{dx^n e^{-x'^2/2}}$$

(81)

which satisfy the relation

$$\partial_{x'} \text{He}_n(x') = x' \text{He}_n(x') - \text{He}_{n+1}(x')$$

(82)

such that the coupling matrix (cf. Eq. (60)) resolves to

$$\Delta_{mn} = -\frac{\sqrt{\pi}}{\ell} \delta_{m+1,n}.$$  (83)

The coupling matrix is not diagonal: incoming momentum $n$ is upgraded to outgoing momentum $n+1$ by the noise coupling. The amplitude of $\Delta_{m,n}$ grows like $|\Delta_{m,n}| \sim \sqrt{\lambda_n}$. For later use, we also note that

$$\left( \frac{x}{\ell} - \ell \partial_x \right) v_n(x) = \sqrt{n+1} v_{n+1}(x)$$  (84)

$$\ell \partial_x v_n(x) = \sqrt{n} v_{n-1}(x)$$  (85)

By $L^2$-adjointness, it follows that the adjoint creation and annihilation operators are

$$-\ell \partial_x u_n(x) = \sqrt{n+1} u_{n+1}(x)$$  (86)

$$\left( \frac{x}{\ell} + \ell \partial_x \right) u_n(x) = \sqrt{n} u_{n-1}(x)$$  (87)

In order to compute the transition and return probabilities, the following identity [57]

$$\sum_{k=0}^{\infty} \text{He}_k(x) \text{He}_k(y) e^{-z^2/2} \frac{z^k}{k!} = \frac{1}{\sqrt{1-z^2}} \exp \left( -\frac{1}{2} \left( \frac{y-zx}{1-z^2} \right)^2 \right)$$

(88)

proves to be useful.

We introduce all quantities in dimensionless form, and take $\omega$ to be imaginary, such as the reduced frequency $\sigma$ and the reduced autocorrelation time $\beta'$ as

$$\sigma = i \alpha^{-1} \omega \quad \beta' = \alpha^{-1} \beta$$

(89)

By considering the Fourier transform at $\sigma = i \alpha^{-1} \omega$, we are effectively studying the Laplace transform. This is intended since our final observable is the moment generating function, the Laplace transform of the first-passage time distribution. Further, we denoted the lengths rescale by $\ell$ as

$$x'_1 = \ell^{-1} x_1 \quad x'_0 = \ell^{-1} x_0.$$  (90)

In the discussion that follows, we analyse all densities as densities in these dimensionless quantities to simplify notation and calculations. Following Eq. (53), one obtains for the transition probability

$$\hat{T}^{(0)} (\omega = -i \sigma \alpha)$$

(91)

$$= \frac{1}{\sqrt{2\pi \ell^2 \alpha}} \sum_{n=0}^{\infty} \frac{\text{He}_n(x'_0) \text{He}_n(x'_1) e^{-z'^2/2}}{n! (\sigma \alpha + n)}$$

$$= \frac{1}{\sqrt{2\pi \ell^2 \alpha}} \int_0^\infty dt e^{-\sigma t} \sum_{n=0}^{\infty} \frac{\text{He}_n(x'_0) \text{He}_n(x'_1) e^{-z'^2/2}}{n!} \left( e^{-t} \right)^n$$

$$= \frac{1}{\sqrt{2\pi \ell^2 \alpha}} \int_0^\infty dt \frac{e^{-\sigma t}}{\sqrt{1-e^{-2t}}} \exp \left[ -\frac{(x'_1 - x'_0 e^{-t})^2}{2(1-e^{-2t})} \right]$$

(92)

where we used identity (88) setting $z = e^{-t}$. This integral is the Laplace transform of the Ornstein-Uhlenbeck
propagator (in \(t_{} \leftrightarrow \sigma\)), and its value is known in the literature to be ([15])

\[
\hat{T}^{(0)}(-i\sigma) = \sum_{n=0}^{\infty} \frac{\Gamma(\sigma)}{\sqrt{\pi} \tau^{\sigma}} \int_{-\infty}^{\infty} D_{\sigma}(x) D_{\sigma}(x') \left[ \frac{1}{\sigma + \beta' + n + 1} - \frac{1}{\sigma + n + 1} \right] \partial_{x} x_{n}(x) x_{n}(x') \]  

(92)

where we introduced the parabolic cylinder functions \(D_{\sigma}(x)\) ([58]). By continuity, for \(x_{0} \rightarrow x_{1}'\) it follows that

\[
\hat{R}^{(0)}(-i\sigma) = \frac{\Gamma(\sigma)}{\sqrt{2\pi} \tau^{\sigma}} \int_{-\infty}^{\infty} D_{\sigma}(x) D_{\sigma}(x') D_{\sigma}(x) \]  

(93)

In order to compute \(\hat{T}^{(1)}(-i\sigma, -i\beta')\), we use Eq. (61) and the derivative coupling matrix computed in (83) to find

\[
\hat{T}^{(1)}(-i\sigma, -i\beta') = \frac{1}{\tau^{\sigma} (1 - \beta')} \int_{-\infty}^{\infty} D_{\sigma}(x) D_{\sigma}(x') \left[ \frac{1}{\sigma + \beta' + n + 1} - \frac{1}{\sigma + n + 1} \right] \partial_{x} x_{n}(x) x_{n}(x') \]  

(94)

\[
= \frac{\partial_{x} x_{n}(x)}{\tau^{\sigma} (1 - \beta')} \hat{T}^{(0)}(-i\sigma) - \hat{T}^{(0)}(-i(\sigma + 1)\alpha) \]  

(95)

These terms can be explicitly calculated and simplified. The rather lengthy but explicit expressions are given in appendix C.

For the second order derivative term, using formula (63), one finds

\[
\hat{T}^{(2)}(-i\sigma\alpha, -i\beta, \nu) = 2 \sum_{n=0}^{\infty} \frac{\Gamma(\sigma) x_{n}(x) x_{n}(x') \left[ \frac{1}{\sigma + \beta' + n + 1} - \frac{1}{\sigma + n + 1} \right]}{\sigma + 1} \int_{-\infty}^{\infty} D_{\sigma}(x) D_{\sigma}(x') \]  

(96)

Using for instance a computer algebra system like [60], all moments can be obtained by differentiation and eval-
uating the limit of $\sigma \to 0$ at which all derivatives have a removable singularity.

2. Numerical Validation

In order to corroborate the closed form result of the first-order correction to the moment generating function of first-passage times, Eq. (99), we employ Monte Carlo simulations integrating the driven Langevin equation (75) $N \approx 10^6$ times, numerically find the first-passage time $\tilde{\tau}_i$, and average the moment generating function $\mathcal{M} = \frac{1}{N} \sum_{i=1}^{N} \exp(-s\tilde{\tau}_i)$ for values of $s$ within the range $s \in [0, 5]$. In the following, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ denote quantities that have been numerically obtained. Since we assume an expansion of the form $\mathcal{M}(\nu) = \mathcal{M}_0 + \nu \mathcal{M}_1 + \nu^2 \mathcal{M}_2 + \ldots$, we take the numerical first derivative

$$\tilde{\mathcal{M}}_1(\nu) = \frac{\mathcal{M}(\nu) - \mathcal{M}(\nu = 0)}{\nu} \tag{100}$$

to verify our analytic prediction of $\mathcal{M}_1$. In Fig. 4, the numerical estimate $\tilde{\mathcal{M}}_1$ is shown for various values of $\nu$, together with the analytic expression Eq. (99) of the scaling function $\mathcal{M}_1$. For small $\nu$, the agreement is excellent. For larger values of $\nu$, higher-order corrections become more visible. The next-higher contribution, which we did not calculate analytically but which can be found by following the framework to second order in $\nu$, is numerically estimated by taking the second numerical derivative, and is shown in the inset of Fig. 4. For $0.2 \leq \nu \leq 0.8$, the second-order corrections collapse, indicating that the deviations in the main figure are well accounted for by second-order corrections. For $\nu = 0.1$, $\tilde{\mathcal{M}}_2$ deviates slightly due to the statistical noise, since the second order correction is very small.

For further physical intuition of the process, we numerically compute the inverse Laplace transform of $\mathcal{M}_0$ and $\mathcal{M}_1$ to obtain the first-passage time distribution (cf. Eq. (7))

$$F(\tau_{x_0,x_1}) = F_0(\tau_{x_0,x_1}) + \nu F_1(\tau_{x_0,x_1}) \tag{102}$$

neglecting, as usual, terms of higher order in $\nu$. In Fig. 2a, we show the first-order corrected distribution compared to a numerical sampling of the probability function, $\tilde{F}(\tau_{x_0,x_1})$, for values of $0 \leq \nu \leq 3.2$. The agreement for $\nu \lesssim 1$ is excellent, and plotting the rescaled deviation

$$\tilde{F}_1(\nu) = \frac{\bar{F}(\nu) - \tilde{F}(\nu = 0)}{\nu} \tag{103}$$

against the theoretical result of $F_1(\tau_{x_0,x_1})$ (see inset of Fig. 2a) shows systematic higher-order corrections consistent with our framework. Since for small $\nu$ the deviation to the undriven case is small, $\bar{F}(\nu)$ suffers from statistical noise, therefore we omit showing $\bar{F}(\nu = 0.1)$ in the inset.

These numerical results therefore confirm the analytically obtained first-order correction to the moment-generating function; consequently, the correction to all moments has been gained. As an illustration, we further show the first and second moment of the Ornstein-Uhlenbeck process driven by coloured noise in Fig. 3a and Fig. 3b. In analogy to the moment-generating function, we measure the mean and mean square first passage times $\tilde{T}^1 = \frac{1}{N} \sum_{i=1}^{N} \tilde{\tau}_i$ and $\tilde{T}^2 = \frac{1}{N} \sum_{i=1}^{N} \tilde{\tau}_i^2$, which we assume to expand in $\nu$ as $\tilde{T}^1(\nu) = \tilde{T}_0 + \nu\tilde{T}_1 + \nu^2\tilde{T}_2 + \ldots$ and $\tilde{T}^2(\nu) = \tilde{T}_0^2 + \nu\tilde{T}_1^2 + \nu^2\tilde{T}_2^2 + \ldots$. The first-order corrections introduced are obtained by differentiation wrt $\sigma$

$$\tilde{T}^n = \frac{1}{n!} \left. \frac{d^n}{d(-\sigma)^n} \right|_{\sigma = 0} \mathcal{M}_1 \tag{104}$$

using the result of Eq. (99) which is performed by a computer algebra system and evaluated exactly. Due to their lengthiness, we do not give their full expression here. In order to numerically confirm these predictions, we measure the first order derivatives

$$\tilde{T}_1^1(\nu) = \frac{\tilde{T}^1(\nu) - \tilde{T}^1(\nu = 0)}{\nu} \tag{105}$$
$$\tilde{T}_1^2(\nu) = \frac{\tilde{T}^2(\nu) - \tilde{T}^2(\nu = 0)}{\nu} \tag{106}$$

and compare it to the result obtained from Eq. (104). In Fig. 3a and Fig. 3b, we show the resulting moments of first-passage times obtained for fixed start position $x_0 = 0$ (at the minimum of the potential) but varied $x_1 \in [0, 0.5, 2]$. The figures show a clear agreement with the theoretical result and systematic deviations for larger $\nu$. Further, we observe that in this setting the coloured noise increases the mean-first passage time for smaller distances ($x_1 \lesssim 1.6$) and decreases it for larger distances. This also holds true for the mean square first-passage time. This example therefore further illustrates that the effect of coloured driving (or memory) on the Langevin dynamics is highly non-trivial, yet our framework is able to capture this effect. The insets in both figures show the measured moments $\tilde{T}^1, \tilde{T}^2$.

B. Active Thermal Brownian Motion on a ring (ATBM)

In this subsection, we consider the case of Brownian Motion $x_t$ driven by coloured noise with periodic boundary conditions ($x = x + 2\pi \rho$). The position of the particle satisfies the Langevin equation

$$\dot{x}_t = \xi_t + \varepsilon y_t \tag{107}$$

with

$$\langle \xi_t, \xi_{t+} \rangle = 2D_x \delta(t_1 - t_2) \tag{108}$$
$$\langle y_t, y_{t+} \rangle = D_y \beta e^{-\beta|t_1 - t_2|} \tag{109}$$
Figure 3. First order correction to first and second moment of ATOU. Simulation parameters are $x_0 = 0, D_x = 1, D_y = 1, \alpha = 1, \beta = \frac{1}{2}$ and $\epsilon$ suitably chosen to fix $\nu = D_y \beta \epsilon^2/(D_x \alpha)$. Averages were taken over $10^6$ samples.

Figure 4. Numerical validation of first order correction to FPT moment generating function $M_1$ (cf. Eq. (4)) of ATOU (see Eq. (75) and Sec. IIIA for discussion). The result is calculated in Eq. (99). Numerical Simulations are shown for various values of $0.1 \leq \nu \leq 0.8$ (plot marks). The moment generating functions were sampled for $x_0 = 0, x_1 = 1, D_x = 1, D_y = 1, \alpha = 1, \beta = \frac{1}{2}$ and $\epsilon$ suitably chosen to fix $\nu = D_y \beta \epsilon^2/(D_x \alpha)$. For small values of $\nu$ agreement with theoretical first-order correction (black line) is excellent. For larger values of $\nu$ the deviation increases. The rescaled deviation, $M_2$ (see Eq. (101)), (inset) collapse and thus confirm that these deviations are systematic higher-order corrections. See (IIIA.2) for further results and discussion.

Figure 5. A Particle on a circle of radius $r$ is driven by both white (thermal) and coloured (active) noise (cf. Eq. (107)). We study the first passage time distribution from $x_0$ to $x_1$ as a function of the angle $\theta \in [0, 2\pi)$.
that the noise-coupling matrix defined in Eq. (60), $L$, plays a role in diffusion, cf. where by allowing complex conjugate of the left eigenfunctions and obtain

$$u_n(x) = \frac{1}{\sqrt{2\pi \tau}} e^{i\pi x}$$

$$v_n^*(x) = \frac{1}{\sqrt{2\pi \tau}} e^{-i\pi x}$$

with corresponding eigenvalues

$$\lambda_n = D_x r^{-2} n^2 \quad (n \in \mathbb{Z}).$$

In what follows, we therefore consider the complex conjugate of the left eigenfunctions and obtain

$$u_n(x) = \frac{1}{\sqrt{2\pi \tau}} e^{i\pi x}$$

$$v_n^*(x) = \frac{1}{\sqrt{2\pi \tau}} e^{-i\pi x}$$

with corresponding eigenvalues

$$\lambda_n = D_x r^{-2} n^2 \quad (n \in \mathbb{Z}).$$

where by allowing $n \in \mathbb{Z}$, we enumerate the eigenfunctions efficiently (cf. Eq. (42)). The eigenfunctions are conjugate to each other since the forward operator of simple diffusion, $L = D_x \partial_x^2$, is self-adjoint. From this follows that the noise-coupling matrix defined in Eq. (60),

$$\Delta_{mn} = \int dx \, u_n(x) \partial_x u_m(x) = \frac{1}{2\pi \tau} \int_0^{2\pi \tau} dx \, \frac{n}{r} e^{i(n-m)x} = \frac{n}{r} \delta_{mn}$$

is diagonal and purely imaginary. Because of scale-invariance and rotational symmetry, we simplify the following discussion by introducing the dimensionless angle

$$\theta := \frac{x_1 - x_0}{r},$$

where we restrict ourselves to $\theta \in [0, 2\pi)$, and the diffusive timescale

$$\alpha^{-1} = \frac{r^2}{D_x}$$

with which we rescale the Fourier-frequency $\omega$ to

$$\sigma = i\alpha^{-1} \omega,$$

again effectively evaluating the Laplace transform (cf. Eq. (8)) of the respective probability densities. With this simplified notation, the transition density to zeroth order reads

$$\hat{T}^{(0)}(-i\sigma \alpha) = \frac{1}{2\pi \alpha r} \sum_{n=-\infty}^{\infty} \frac{e^{-i n \theta}}{\sigma + n^2} = \frac{1}{2\pi r} \frac{\cosh ((\theta - \pi)\sqrt{\sigma})}{\sqrt{\sigma \sinh (\pi \sqrt{\sigma})}},$$

and the return probability, setting $\theta = 0$, is

$$\hat{R}^{(0)}(-i\sigma \alpha) = \frac{1}{2\pi \alpha r} \frac{\cosh (\pi \sqrt{\sigma})}{\sqrt{\sigma \sinh (\pi \sqrt{\sigma})}}.$$

Assuming the $\nu$-averaged moment generating function has an expansion of

$$F(s) = M_0 \left( \frac{x_1 - x_0}{r} \frac{r^2 s}{D_x} \right) + \frac{D_x \nu^2 \beta}{2} M_1 \left( \frac{x_1 - x_0}{r} \frac{r^2 s}{D_x} \right) + O(\nu^2),$$

where we use as in the Ornstein-Uhlenbeck case the dimensionless correlation time-scale of the coloured noise.

Being a moment generating function, the prefactors in front of $\sigma^n$ correspond to the (rescaled) moments of the first-passage time $\langle x_0 \rightarrow x_1 \rangle \left( \frac{D_x}{r} \right)^n$.

We turn to higher orders in $\varepsilon$. To first order, the transition density

$$\hat{T}^{(1)}(-i\sigma \alpha, -i\beta) = \frac{1}{2\pi \alpha^2 r^2} \sum_{n=-\infty}^{\infty} \frac{\sin e^{-i n \theta}}{\sigma + \beta' + n^2} \frac{\sin e^{i n \theta}}{\sigma + n^2}$$

$$= \frac{1}{2\pi \alpha^2 r^2} \sum_{n=0}^{\infty} \frac{\sin e^{-i n \theta}}{\sigma + \beta' + n^2} \frac{\sin e^{i n \theta}}{\sigma + n^2}$$

Here, we introduced as before $\beta' = \alpha^{-1} \beta$ as the dimensionless correlation time-scale of the coloured noise. The function $\hat{R}^{(1)}(-i\sigma \alpha, -i\beta)$, which corresponds to $\hat{T}^{(1)}(-i\sigma \alpha, -i\beta)$ evaluated at $\theta = 0$, vanishes. This implies that the contribution of integrals (II) and (III), as given in Eq. (72) and (73), vanish, leaving only (I) and (IV) as correction terms.

To second order, the transition density is
\[ \hat{T}^{(2)}(-i\sigma \alpha, -i\beta, i\beta) = -\frac{1}{\pi \alpha^3 r^3} \sum_{n=-\infty}^{\infty} \frac{n^2 e^{-in\theta}}{(\sigma + n^2)^2(\sigma + \beta'^2 + n^2)} \]

\[ = -\frac{1}{\pi \alpha^3 r^3} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\beta'^2 \sigma + \beta' + n^2} - \frac{1}{\beta'^2 \sigma + n^2} + \frac{1}{\beta' \sigma + n^2} \right] \]

\[ = \frac{2}{\alpha^2 r^2} \frac{1}{\beta'^2} \hat{\theta}^2 \left[ \hat{T}^{(0)}(\sigma + \beta') - \hat{T}^{(0)}(\sigma) - \beta' \partial_\sigma \hat{T}^{(0)}(\sigma) \right] \]

\[ = \frac{2}{\alpha^3 r^3} \frac{1}{4\beta'^2 \sqrt{\sigma}} \left[ \cosh ((\theta - \pi) \sqrt{\sigma}) \left( \frac{\beta' \sqrt{\sigma}}{\sinh (\pi \sqrt{\sigma})} \cosh \left( \frac{\pi \sqrt{\sigma}}{\beta' + \sigma} \right) - 2\sigma - \beta' \right) \right. \]

\[ + \sqrt{\sigma} \left\{ 2\sqrt{\beta'} + \frac{\cosh ((\theta - \pi) \sqrt{\sigma})}{\sinh (\pi \sqrt{\beta' + \sigma})} + \beta(\pi - \theta) \frac{\sinh ((\theta - \pi) \sqrt{\sigma})}{\sinh (\pi \sqrt{\sigma})} \right\} \] \hspace{1cm} (125)

where \( \hat{\theta} = r \hat{\theta}_{x_1} \). Setting \( \theta = 0 \), one obtains the second order response of the return probability,

\[ \hat{R}^{(2)}(-i\sigma \alpha, -i\beta, i\beta) = \frac{1}{2\alpha^3 \beta'^2 r^3} \left[ \pi \beta' \coth (\pi \sqrt{\sigma}) - \left( \frac{\beta' + 2\sigma}{\sqrt{\sigma}} \right) \coth (\pi \sqrt{\sigma}) + 2\sqrt{\beta'} + \sigma \coth (\pi \sqrt{\beta' + \sigma}) - \pi \beta' \right], \hspace{1cm} (126) \]

Inserting these quantities into the FPT correction (74), gives

\[ \hat{F}(s = \alpha \sigma) = \left\langle e^{-\pi \sigma x_0 \to \tau_{x_1}} \right\rangle \]

\[ = \cosh ((\theta - \pi) \sqrt{\sigma}) \cosh (\pi \sqrt{\sigma}) + \frac{D_\epsilon \sqrt{\sigma}}{2D_\epsilon} \tan (\pi \sqrt{\sigma}) + \frac{\cosh ((\theta - \pi) \sqrt{\beta' + \sigma})}{\sinh (\pi \sqrt{\beta' + \sigma})} \frac{2\sqrt{\beta' + \sigma}}{\beta'^2} \]

\[ + \frac{\cosh ((\theta - \pi) \sqrt{\beta' + \sigma})}{\cosh (\pi \sqrt{\sigma})} \pi \beta' - 2\sqrt{\beta' + \sigma} \coth (\pi \sqrt{\beta' + \sigma}) + \frac{\sinh ((\theta - \pi) \sqrt{\sigma})}{\sinh (\pi \sqrt{\sigma})} \beta'(\pi - \theta) \] \hspace{1cm} (127)

This expression is the full moment generating function up to second order in \( \epsilon \) in elementary functions. In line with our previous notation, we identify the dimensionless scaling functions

\[ M_{0 \text{BM}}^0(\theta, \sigma) = \frac{\cosh ((\theta - \pi) \sqrt{\sigma})}{\cosh (\pi \sqrt{\sigma})} \] \hspace{1cm} (128)

\[ M_{1 \text{BM}}^1(\theta, \sigma) = \frac{\sqrt{\sigma} \tan \left( \frac{\sqrt{\sigma}}{2\beta'^2} \right) \cosh \left( \frac{(\theta - \pi) \sqrt{\beta' + \sigma}}{\sinh (\pi \sqrt{\beta' + \sigma})} \right) 2\sqrt{\beta' + \sigma}}{\beta'^2} \]

\[ + \beta'(\pi - \theta) \frac{\sinh ((\theta - \pi) \sqrt{\sigma})}{\sinh (\pi \sqrt{\sigma})} + \left( \pi \beta' - 2\sqrt{\beta' + \sigma} \coth (\pi \sqrt{\beta' + \sigma}) \right) \frac{\cosh ((\theta - \pi) \sqrt{\sigma})}{\cosh (\pi \sqrt{\sigma})} \] \hspace{1cm} (129)

which together form the first-order correction of the moment-generating function

\[ \left\langle \exp(-s\tau_{x_0 \to \tau_{x_1}}) \right\rangle = M_{0 \text{BM}}^0 \left( \frac{x_1 - x_0}{r} \right) + \frac{D_\epsilon \sqrt{\sigma}}{2D_\epsilon} \beta' \left( \frac{x_1 - x_0}{r} \right) + \frac{r^2 s}{D_\epsilon} \beta' \] \hspace{1cm} (130)

An expansion around \( \sigma = 0 \) gives corrections to all moments. The correction to first order in \( \nu \) of the mean first-passage time over an angle of \( \theta \), \( \left\langle \tau_{0 \to \theta} \right\rangle \), for instance reads

\[ \alpha \left\langle \tau_{0 \to \theta} \right\rangle = \pi \theta - \frac{\theta^2}{2} - \nu \left[ \frac{1}{2\beta'^2} \sqrt{\beta'(2\pi - \theta) \pi \sqrt{\beta'}} + 2\pi \cos \left( \frac{\sqrt{\beta'(\theta - \pi)}}{\sinh (\pi \sqrt{\beta'})} \right) \right] \] \hspace{1cm} (131)

where we indicated that the result can be written as the classical contribution \( \left\langle T_{0 \to 1}^1 \right\rangle \) times the dimensionless per-
turbative coefficient $\nu$ and a dimensionless scaling function $\mathcal{T}_1^1(\theta, \beta')$. By successive derivation, any higher order moment may be obtained from Eq. (127).

2. Numerical Validation

In order to validate the analytic result of the moment generating function of first-passage times Eq. (127) and the mean first-passage time Eq. (131), we follow the same steps as in the previous section III A 2. Using Monte-Carlo simulations, we sample the first-passage times $\tau_i$ of the integrated stochastic equation (107). To validate the moment-generating function, we average over $N \approx 10^6$ to $10^7$ iterations to sample

$$\hat{\mathcal{M}} = \frac{1}{N} \sum_{i=1}^{N} \exp(-s\tau_i)$$

(132)

for various values of $\nu$ and across $x'_1 - x'_0 \in (0, \pi]$ (with $r = 1$). Again, symbols with a tilde denote quantities which are measured numerically. To validate the theoretically predicted first-order correction Eq. (129), we subtract the $\nu = 0$ contribution, $\mathcal{M}_0$ (cf. Eq. (128)), and rescale by $\nu$,

$$\hat{\mathcal{M}}_1(\nu) = \frac{\hat{\mathcal{M}}(\nu) - \hat{\mathcal{M}}(\nu = 0)}{\nu}$$

(133)

where we introduced, in analogy to Eq. (101),

$$\hat{\mathcal{M}}_2 = \frac{\hat{\mathcal{M}}_1(\nu) - \hat{\mathcal{M}}_1(\nu = 0)}{\nu}$$

(134)

as shorthand for numerically measured higher-order corrections. In Fig. 7, we show the analytic result $\mathcal{M}_1$ (cf. Eq. (129)) and numerically obtained $\hat{\mathcal{M}}_1(\nu)$ (cf. Eq. (133)) for various values of $0 \leq \nu \leq 0.8$. The agreement is again excellent, and the discrepancy between simulated result and theoretical first order correction grows, as expected, with larger values of $\nu$. The rescaled discrepancy, $\hat{\mathcal{M}}_2$, to leading order the second-order correction $\mathcal{M}_2$ (cf. Eq. (134)), is plotted in the inset and collapses, indicating that the discrepancy is systematic and confirming the validity of the result in Eq. (129). By numerically computing the inverse Laplace transform, we obtain the perturbative correction to the first-passage time distribution (cf. Eq. (102)) which in Fig. 2b is compared to the numerically sampled distribution. The inset shows the numerical estimate of the first order correction (cf. Eq. (103)) which agrees well for small $\nu$. For larger values of $\nu$, second-order corrections play an increasing role as indicated in Fig. 7.

In Figs. 6a and 6b, we show the first and second moment of first-passage times and how its deviation to the $\nu = 0$ case is captured by the first-order correction obtained using Eq. (105) and Eq. (106) with the result of Eq. (129). For the first and second moment, the agreement is again excellent, showing that the correction induced by the active driving noise is accurately captured to leading order. The insets of Figs. 6a and 6b show the respective moments of the FPT for various $\nu$, indicating the systematic decrease for increased values of $\nu$.

C. Limit Cases

The framework we introduce here allows to study coloured driving noises at any noise-colour $\beta$. In particular, this includes two limit cases of $\beta \to 0$ and $\beta \to \infty$. For appropriate re-scaling of $D_0$, the former limit corresponds to a particular quenched disorder model, and the latter to additional white noise. In what follows we discuss these limit cases in more detail.

1. The white noise limit

For very small autocorrelation times $\beta^{-1}$ the driving noise $y_t$ appears more and more as white noise. The correlator of $y_t$ tends towards

$$\langle y_s y_t \rangle = D_0 \beta e^{-|t-s|} \to 2D_0 \delta(t - s), \quad \beta \to \infty.$$ 

(135)

In this limit, the driving noise features in the Langevin equation (1) as additional white noise and is absorbed as

$$\dot{x} = -V'(x_t) + \sqrt{2(D_x + \varepsilon^2 D_y)} \xi,$$

(136)

such that effectively the diffusion constant is shifted by $D_x \to D_x + \varepsilon^2 D_y$. In the white noise limit, the theory is Markovian and Eq. (12) may be applied using the shifted diffusivity to obtain exact results. The perturbation theory presented here then correspond to an expansion of the Markovian result in a perturbation to the diffusion constant $D_x$; and as a result the following identity must hold:

$$\lim_{\beta \to \infty} \beta \mathcal{M}_1(x_0, x_1; s) = \alpha D_x \frac{\partial}{\partial D_x} \mathcal{M}_0(x_0, x_1; s).$$

(137)

One can verify that this relation is indeed satisfied for the Brownian motion case (Eqs. (128) and (129)) as well as the Ornstein-Uhlenbeck case (Eqs. (98) and (99)).

2. Quenched Disorder limit

In the opposite limit of $\beta \to 0$, provided $D_0 \beta = w^2$ remains fixed, the driving noise “freezes” to a random constant since

$$\langle y_s y_t \rangle = D_0 \beta e^{-|t-s|} \to w^2, \quad \beta \to 0.$$ 

(138)

Effectively, the Langevin equation (1) therefore turns into

$$\dot{x} = -V'(x_t) + \xi_t + \nu$$

(139)
(a) Correction to mean first passage time of ATBM on a ring (cf. Eq. (107)) as obtained from Eq. (105) versus target position \( x_1 \) and various values of \( \nu \) (plot marks). This is compared to theoretical result of Eq. (131) (solid blue line).

(b) Correction to mean squared first passage time of ATBM on a ring (cf. Eq. (107)) as obtained from Eq. (106) versus target position \( x_1 \) and various values of \( \nu \) (plot marks). This is compared to theoretical result of twice differentiating Eq. (129) (solid blue line).

Figure 6. First order correction to first and second moment of first-passage times of ATBM with periodic boundary conditions. Simulation parameters are \( x_0 = 0, D_x = 1, r = 1, D_y = 1, \beta = \frac{1}{2} \) and \( \varepsilon \) suitably chosen to fix \( \nu = D_y r^2 \beta \varepsilon^2 / (D_x^2) \).

![Graph](image1)

![Graph](image2)

Figure 7. Numerical validation of first order correction to FPT moment generating function \( M_1 \) of ATBM (cf. Eq. (107) and Sec. III.B for discussion) for various values of \( 0.1 \leq \nu \leq 0.8 \). The moment generating functions were sampled for \( x_0 = 0, x_1 = \pi, D_x = 1, \alpha = 1, D_y = 1, \beta = \frac{1}{2} \) and \( \varepsilon \) suitably chosen to fix \( \nu = D_y r^2 \beta \varepsilon^2 / (D_x^2) \) (plot marks).

For small values of \( \nu \) agreement with theoretical first-order correction (black line) is very good. For larger values of \( \nu \) the deviation increases. The rescaled deviations, \( M_2 \) (see Eq. (134)), (inset) collapse and thus confirm that these deviations are systematic higher-order corrections. See (III.B.2) for further results and discussion.

![Graph](image3)

where \( v \) is a constant driving velocity which is normal distributed according to \( v \sim \mathcal{N}(0, \varepsilon^2 w^2) \). The driving noise average \( e^{-\beta \varepsilon^2 w^2} \), then corresponds to a quenched average over the ensemble of normal distributed velocities

\[
\alpha \langle \tau_{0,0} \rangle = \frac{\theta (2\pi - \theta)}{2} - \frac{\varepsilon^2 w^2}{\alpha^2 r^2} \cdot \frac{\theta^2 (\theta - 2\pi)^2}{24} + \ldots
\]
such that quenched disorder lowers the mean first-passage time for any choice of parameters. Further, for the mean squared first-passage time we obtain
\[
\alpha^2 \left\langle \tau_{0,0}^2 \right\rangle = \frac{1}{12} \theta (\theta - 2\pi)(\theta - 2\pi + 2\varphi \pi) \left( \theta - 2\pi + 2\varphi \pi \right) \\
+ \frac{e^2 \omega^2}{\alpha^2 \tau} \left( \theta - \pi(1 + \psi^+)(\theta - \pi(1 - \psi^+)) \right) \\
\cdot (\theta - \pi(1 + \psi^-))(\theta - \pi(1 - \psi^-)),
\]
where \( \varphi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio, and \( \psi^\pm = \frac{1 \pm \sqrt{10}}{\sqrt{4}} \).

IV. CONCLUSION

In the present work, we introduce a perturbative approach to study the first-passage time distribution of stochastic processes which are driven both by white and coloured noise. This class of stochastic processes lies at the heart of the study of self-propelled particles in a thermal environment. The self-propulsion is modelled by a white noise with diffusion constant \( D_x \) and an external periodic driving force. This framework therefore would allow to study piece-wise combinations of the potentials at the boundary between two heat baths at different temperatures (e.g. \( D(x) = D_0 + \text{sgn}(x)\Delta D \)).

 Furthermore, the functional expansion in \( \tilde{y}(\omega) \) (cf. Eq. (27)) drastically simplifies in the case of \( y_t \) being a periodic driving force. This framework therefore would not only be able to capture stochastic \( y_t \) but also oscillating deterministic driving forces. This will possibly be addressed in future work.

To first order in \( \nu \), the corrections, as given in Eqs. (69) to (73), involve simple complex integrals which can be solved by the residue theorem. For higher-order corrections, however, the integration runs over more than one free internal variable and will require more work. This corresponds to the problems of typical Feynman-diagrams of higher order in statistical field theories which often involve non-trivial integrals. To study higher orders, field theory therefore would provide the necessary toolbox to solve the required correction terms.

The results obtained in this work can be derived alternatively using field-theoretic methods. In fact, the field theoretic treatment allows for the study of a broader class of extreme events and will be reported in [61].
ACKNOWLEDGMENTS

BW and GP would like to thank the Francis Crick Institute for hospitality and Andy Thomas for invaluable computing support. BW thanks Kay Wiese, Grigorios Pavliotis, Gregory Schehr and Raphael Voituriez for insightful discussions. GS was supported by the Francis Crick Institute which receives its core funding from Cancer Research UK (FC001317), the UK Medical Research Council (FC001317), and the Wellcome Trust (FC001317).

[1] N. G. v. Kampen, *Stochastic processes in physics and chemistry*, 3rd ed. (Elsevier, 2007).
[2] S. Redner, *A guide to first-passage processes* (Cambridge University Press, 2001).
[3] R. Metzler, G. Oshanin, and S. Redner, *First-Passage Phenomena and Their Applications* (World Scientific, 2014).
[4] A. Szabo, K. Schulten, and Z. Schulten, First passage time approach to diffusion controlled reactions, *J. Chem. Phys.* **72**, 4350 (1980).
[5] I. M. Sokolov, Cyclization of a Polymer: First-Passage Problem for a Non-Markovian Process, *Phys. Rev. Lett.* **90**, 080601 (2003).
[6] K. R. Ghusinga, J. J. Dennehy, and A. Singh, First-passage time approach to controlling noise in the timing of intracellular events, *Proc. Natl. Acad. Sci.* **114**, 693 (2017).
[7] T. Taillefermier and M. O. Magnasco, A phase transition in the first passage of a Brownian process through a fluctuating boundary with implications for neural coding, *Proc. Natl. Acad. Sci.* **110**, E1438 (2013).
[8] R. Chicheportiche and J.-P. Bouchaud, Some Applications of First-Passage Ideas to Finance, in *First-Passage Phenomena and Their Applications*, edited by R. Metzler, G. Oshanin, and S. Redner (World Scientific, 2014) pp. 447–476.
[9] V. Tejedor, O. Bénichou, and R. Voituriez, Global mean first-passage times of random walks on complex networks, *Phys. Rev. E* **80**, 065104 (2009).
[10] D. Hartich and A. Godec, Extreme value statistics of ergodic Markov processes from first passage times in the large deviation limit, *J. Phys. A** 52**, 244001 (2019).
[11] A. J. Bray, S. N. Majumdar, and G. Schehr, Persistence and first-passage properties in nonequilibrium systems, *Adv. Phys.* **62**, 225 (2013).
[12] E. Schrödinger, Zur Theorie der Fall- und Steigversuche an Teilchen mit Brownscher Bewegung, *Z. Phys.* **16**, 289 (1915).
[13] L. Pontryagin, A. Andronov, and A. Vitt, Appendix: On the statistical treatment of dynamical systems, in *Noise in Nonlinear Dynamical Systems*, Vol. 1, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, 1989) pp. 329–348.
[14] P. Hänggi, P. Talkner, and M. Borkovec, Reaction-rate theory: fifty years after Kramers, *Rev. Mod. Phys.* **62**, 251 (1990).
[15] A. J. F. Siegert, On the First Passage Time Probability Problem, *Phys. Rev.* **81**, 617 (1951).
[16] D. A. Darling and A. J. F. Siegert, The First Passage Problem for a Continuous Markov Process, *Ann. Math. Stat.* **24**, 624 (1953).
[17] H. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica* **7**, 284 (1940).
[18] H. B. Callen and T. A. Welton, Irreversibility and generalized noise, *Phys. Rev.* **83**, 34 (1951).
[19] R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.* **29**, 255 (1966).
[20] J. Luczka, Non-Markovian stochastic processes: Colored noise, *Chaos* **15**, 026107 (2005).
[21] P. Hanggi and P. Jung, Colored Noise in Dynamical Systems, *Adv. Chem. Phys.* **89**, 239 (1995).
[22] R. Kubo, Note on the Stochastic Theory of Resonance Absorption, *J. Phys. Soc. Jpn.* **9**, 935 (1954).
[23] J. Masoliver, K. Lindenberg, and B. J. West, First-passage times for non-Markovian processes: Correlated impacts on bound processes, *Phys. Rev. A** 34**, 2351 (1986).
[24] M. Kuś, E. Wajnryb, and K. Wódkiewicz, Mean first-passage time in the presence of colored noise: An exact theory, *Phys. Rev. A* **42**, 7500 (1990).
[25] M. Kuś, E. Wajnryb, and K. Wódkiewicz, Mean first-passage time in the presence of colored noise: A random-telegraph-signal approach, *Phys. Rev. A* **43**, 4167 (1991).
[26] L. Ramirez-Piscina and J. M. Sancho, First-passage times for a marginal state driven by colored noise, *Phys. Rev. A** 43**, 663 (1991).
[27] M. I. Dykman, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, Probability distributions and escape rates for systems driven by quasimonochromatic noise, *Phys. Rev. E* **47**, 3996 (1993).
[28] P. Jung, Colored noise in dynamical systems: Some exact solutions, in *Stochastic Dynamics*, Lecture Notes in Physics, Vol. 484, edited by L. Schimansky-Geier and T. Föschel (Springer Berlin Heidelberg, 1997) pp. 23–31.
[29] R. F. Fox, Mean first-passage times and colored noise, *Phys. Rev. A* **37**, 911 (1988).
[30] G. Szamel, Self-propelled particle in an external potential: Existence of an effective temperature, *Phys. Rev. E* **90**, 012111 (2014).
[31] D. Mandal, K. Klymko, and M. R. DeWeese, Entropy production and fluctuation theorems for active matter, *Phys. Rev. Lett.* **119**, 258001 (2017).
[32] C. Maggi, U. M. B. Marconi, N. Gnan, and R. Di Leonardo, Multidimensional stationary probability distribution for interacting active particles, *Sci. Rep.* **5**, 10742 (2015).
[33] C. Sandford, A. Y. Grosberg, and J.-F. Joanny, Pressure and flow of exponentially self-correlated active particles, *Phys. Rev. E* **96**, 052605 (2017).
[34] L. Caprini and M. B. U. Marconi, Active particles under confinement and effective force generation among surfaces, *Soft Matter* **14**, 9044 (2018).
[35] L. Caprini, U. M. B. Marconi, A. Puglisi, and A. Vulpiani, The entropy production of Ornstein-Uhlenbeck active particles: a path integral method for correlations, *J.
Appendix A: First passage times of Brownian Motion on a ring with drift

We calculate the first-passage time distribution of a Brownian Motion with drift \( v \) on a ring of radius \( r \) departing from the methods outlined in [62, Chp. V] although the explicit formula is not given there. Instead of considering the first passage event of transition \( x_0 \to x_1 \), we calculate the exit probability of a Brownian Motion on the real line over the absorbing boundaries at \( x_1 \) and \( x_1 - 2\pi r \) where without loss of generality we chose \( x_1 - 2\pi r < x_0 < x_1 \). The transition density \( T(t) \equiv T(x_0, x_1, t) \) satisfies the Fokker-Planck Equation

\[
\partial_t T(t) = D_x \partial_{x_1}^2 T(t) - v \partial_{x_1} T(t) \quad (A1)
\]

and the Kolmogorov backward equation

\[
\partial_t T(t) = D_x \partial_{x_0}^2 T(t) + v \partial_{x_0} T(t) \quad (A2)
\]

with absorbing boundary conditions for \( t \geq 0 \)

\[
T(x_0, x_1; t) = 0 \quad (A3)
\]

\[
T(x_0, x_1 - 2\pi r; t) = 0 \quad (A4)
\]
and initial condition

\[ T(x_0, x, t = 0) = \delta(x - x_0) \]  

The net flux of the surviving probability over the boundary is related to the first-passage time distribution via

\[ F(x_0, x_1, t) = -\partial_t \int_{x_1 - 2\pi r}^x dx \, T(x_0, x, t). \]  

Using the backward equation (A2), and applying a further derivative in time one finds

\[ \partial_t F(x_0, x_1, t) = -(D_x \partial_x^2 + \nu \partial_x) \partial_t \int_{x_1 - 2\pi r}^x dx \, T(x_0, x, t) \]

\[ = (D_x \partial_x^2 + \nu \partial_x) F(x_0, x_1, t). \]  

At the boundary, the first-passage time distribution is given by

\[ F(x_1, x_1, t) = F(x_1 - 2\pi r, x_1, t) = \delta(0^+) \]  

since the particle is immediately absorbed, while at time \( t = 0 \) for \( x_0 \) away from the boundary

\[ F(x_0, x_1, t = 0) = 0 \]  

since the particle has not yet begun to diffuse. Under Laplace transform in \( t \), the differential equation (A8), together with these boundary and initial conditions returns

\[ \begin{align*}
  s F(x_0, x_1, s) & = (D_x \partial_x^2 + \nu \partial_x) F(x_0, x_1, s) \quad x_1 - 2\pi r < x_0 < x_1 \\
  F(x_0, x_1, s) & = F(x_0, x_1 - 2\pi r, s) = 1 \quad x_0 = x_1 - 2\pi r \text{ or } x_0 = x_1. 
\end{align*} \]  

The ordinary differential equation (away from the boundary) is solved by the exponential ansatz

\[ F(x_0, x_1, s) = Ae^{\omega_1 x_0} + Be^{\omega_2 x_0} \]  

Inserting this ansatz into (A11) enforces

\[ \omega_{1,2} = -\frac{\nu}{2D_x} \pm \frac{\sqrt{\nu^2 + 4D_x s}}{2D_x} \]  

The boundary conditions (A11) fix the normalising constants \( A, B \) to

\[ A = -e^{\pi r \omega_1 - \omega_1 x_1} \frac{\sinh(\pi r \omega_2)}{\sinh(\pi r(\omega_1 - \omega_2))} \]  

\[ B = \sinh(\pi r \omega_1) \frac{e^{\pi r \omega_2 - \omega_2 x_1}}{\sinh(\pi r(\omega_1 - \omega_2))} \]  

After some further simplifications one arrives at the \( v \)-dependent moment generating function

\[ F(x_0, x_1, s; v) = \exp \left( \frac{rv}{2D_x} (\theta - 2\pi) \right) \sinh \left( \frac{r\sqrt{v^2 + 4D_x s}}{2D_x} \theta \right) - \exp \left( \frac{-rv}{2D_x} \right) \sinh \left( \frac{r\sqrt{v^2 + 4D_x s}}{2D_x} (\theta - 2\pi) \right) \]  

\[ \sinh \left( \frac{r\sqrt{v^2 + 4D_x s}}{2D_x} \right) \]  

with \( \theta = (x_1 - x_0)/r \) following the notation from the main text. We note that for \( v \to 0 \),

\[ \frac{r\sqrt{v^2 + 4D_x s}}{2D_x} \to \sqrt{\sigma} \]  

and one recovers the undriven moment generating function

\[ F(x_0, x_1, s; v = 0) = F(x_0, x_1, s) = \frac{\sinh(\theta \sqrt{\sigma}) - \sinh((\theta - 2\pi)\sqrt{\sigma})}{\sinh(2\pi \sqrt{\sigma})} = \frac{\cosh((\theta - \pi)\sqrt{\sigma})}{\cosh(\pi \sqrt{\sigma})} \]  

in agreement with the independently found expression (122). In App. B, we show that to second order in \( v \) this result is identical to the first-order correction \( \mathcal{M}_1 \) from Eq. (127) in the limit of \( \beta \to 0 \).
Appendix B: Equivalence of quenched averages

In this section, we provide a more detailed proof showing that Eq. (141) indeed holds for the case of Brownian Motion driven by coloured noise, i.e. that our framework perturbatively gives the correct moment generating function of first-passage times when taking the quenched average over Eq. (139) with normal distributed drift \( v \) and \( \beta \). To that end, we take the limit \( \beta \to 0 \) of the analytically found \( \mathcal{M}_1 \) (cf. Eq. (129)),

\[
\lim_{\beta \to 0} \mathcal{M}_1 = \frac{1}{8\sqrt{\sigma}} \left\{ \theta \cosh(\theta \sqrt{\sigma}) \left( \theta \sqrt{\sigma} - \tanh(\pi \sqrt{\sigma}) \right) \\
+ \sinh(\theta \sqrt{\sigma}) \left[ - (\theta^2 - 2\pi \theta + 2\pi^2) \sqrt{\sigma} \tanh(\pi \sqrt{\sigma}) + 2\pi(\pi - \theta) \sqrt{\sigma} \coth(\pi \sqrt{\sigma}) + \pi \tanh^2(\pi \sqrt{\sigma}) + \theta - \pi \right] \right\}.
\]

In Eq. (141), it is claimed that this equals

\[
\frac{D_x \alpha}{2\sqrt{\sigma}} \int_{-\infty}^{\infty} dv \frac{e^{-\frac{v^2}{2\sigma^2}}}{\sqrt{2\pi \sigma^2} v^2} \partial_v^2 \left|_{v=0} F(x_0, x_1; v) \right.
\]

Evaluating this expression using the result from Eq. (A16) and setting \( v = 0 \) results in

\[
\frac{D_x \alpha}{2} \partial_v^2 \left|_{v=0} \right. F(x_0, x_1; s; v) = \frac{1}{8\sqrt{\sigma}} \left\{ \sinh(2\sqrt{\sigma}) \left[ \sinh((\theta - 2\pi) \sqrt{\sigma}) (2\pi \coth(2\sqrt{\sigma}) - 2\pi^2 \sqrt{\sigma}) + \theta \cosh(\theta \sqrt{\sigma}) \right] \\
+ (2\pi - \theta) \cosh((\theta - 2\pi) \sqrt{\sigma}) + \sinh(\theta \sqrt{\sigma}) \left( (\theta - 2\pi^2 \sqrt{\sigma} - 2\pi \coth(2\sqrt{\sigma}) \right) \right\}.
\]

The expressions in (B1) and (C1) are indeed equal as can be shown with the help of Mathematica [60].

Appendix C: Explicit expressions for functional derivatives of transition probability densities of Ornstein-Uhlenbeck processes

Following the notation from Sec. III A 1, we here give the explicit expressions of \( \hat{T}^{(1)}, \hat{T}^{(2)} \) as implicitly given in Eqs. (94)-(96). We confine ourselves to the case of \( x_0' < x_1' \). Starting from the Fourier-transformed transition probability density (cf. Eq. (92)), all other functional derivatives are given as partial derivatives of this density. From formula (94), one obtains

\[
\hat{T}^{(1)} (-i\sigma \alpha, -i\beta) = -\frac{e^{\frac{\nu^2}{2} - \frac{x_1'^2}{2}}}{\sqrt{2\pi t^2 \sigma^2 (1 - \beta')}} \left[ \Gamma(\beta' + \sigma) D_{-\beta' - \sigma}(-x_0') D_{-\beta' - \sigma + 1}(x_1') - \Gamma(\sigma + 1) D_{-\sigma - 1}(-x_0') D_{-\sigma}(x_1') \right]
\]

where \( \Gamma(\sigma) \) is the usual Gamma-function. Letting \( x_0' \to x_1' \), gives the first functional derivative of the return probability at \( x_1' \),

\[
\hat{R}^{(1)} (-i\sigma \alpha, -i\beta) = -\frac{\Gamma(\beta' + \sigma) D_{-\beta' - \sigma}(-x_0') D_{-\beta' - \sigma + 1}(x_1') - \Gamma(\sigma + 1) D_{-\sigma - 1}(-x_0') D_{-\sigma}(x_1')}{\sqrt{2\pi t^2 \sigma^2 (1 - \beta')}}
\]

These results further imply

\[
\hat{T}^{(1)} (-i\sigma \alpha - i\beta, i\beta) = -\frac{e^{\frac{\nu^2}{2} - \frac{x_1'^2}{2}}}{\sqrt{2\pi (1 + \beta') \sigma^2}} \Gamma(\sigma) D_{-\sigma}(-x_0') D_{1-\sigma}(x_1') - \Gamma(\beta' + \sigma + 1) D_{-\beta' - \sigma - 1}(-x_0') D_{-\beta' - \sigma}(x_1')
\]

and

\[
\hat{R}^{(1)} (-i\sigma \alpha - i\beta, i\beta) = -\frac{\Gamma(\sigma) D_{-\sigma}(-x_1') D_{1-\sigma}(x_1') - \Gamma(\beta' + \sigma + 1) D_{-\beta' - \sigma - 1}(-x_1') D_{-\beta' - \sigma}(x_1')}{\sqrt{2\pi (1 + \beta') \sigma^2}}
\]

The second order derivative of the transition probability is

\[
\hat{T}^{(2)} (-i\sigma \alpha, -i\beta, i\beta) = \frac{e^{\frac{\nu^2}{2} - \frac{x_1'^2}{2}}}{\sqrt{2\pi (\beta'^2 - 1) \beta^3 \sigma^3}} \left[ 2\Gamma(\beta' + \sigma + 1) D_{-\beta' - \sigma - 1}(-x_0') D_{-\beta' - \sigma + 1}(x_1') \\
+ (\beta' - 1) \Gamma(\sigma) D_{-\sigma}(-x_0') D_{2-\sigma}(x_1') - (\beta' + 1) \Gamma(\sigma + 2) D_{-\sigma - 2}(-x_0') D_{-\sigma}(x_1') \right]
\]
and of the return probability

\[ \hat{R}^{(2)}(-i\sigma \alpha, -i\beta, i\beta) = \frac{1}{\sqrt{2\pi (\beta'^2 - 1)}} \frac{1}{\ell^3 \alpha^3} \left[ 2\Gamma(\beta' + \sigma + 1)D_{-\beta' - \sigma - 1}(-x'_1)D_{-\beta' - \sigma + 1}(x'_1) \\
+ (\beta' - 1)\Gamma(\sigma)D_{-\sigma}(-x'_1)D_{2-\sigma}(x'_1) - (\beta' + 1)\Gamma(\sigma + 2)D_{-\sigma - 2}(-x'_1)D_{-\sigma}(x'_1) \right] \]

(C6)