Entanglement entropy of dispersive media from thermodynamic entropy in one higher dimension

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A dispersive medium becomes entangled with zero-point fluctuations in the vacuum. We consider an arbitrary array of material bodies weakly interacting with a quantum field and compute the quantum mutual information between them. It is shown that the mutual information in D dimensions can be mapped to classical thermodynamic entropy in (D + 1) dimensions. As a specific example, we compute the mutual information both analytically and numerically for a range of separation distances between two bodies in D = 2 dimensions and find a logarithmic correction to the area law at short separations. A key advantage of our method is that it allows the strong subadditivity property—notoriously difficult to prove for quantum systems—to be easily verified.

Entanglement as a purely quantum phenomenon has become a central subject in many fields, ranging from high-energy physics to many-body systems and quantum information. In particular, the characteristic entropy associated with entanglement has proved a powerful tool for describing the quantum nature of a system [1–4] as well as for quantifying resources available for quantum computation [5]. Inspired by formal similarities with thermodynamics, parallels between quantum and classical notions of entropy have been suggested in the literature [6].

While many studies of entanglement entropy have considered geometrical regions in vacuum [4] or domains in many-body systems [7], rapid advances in quantum computation and atomic and optical physics have prompted studies of entanglement between atoms and light [8, 9]. Although most studies have addressed the case of one or a few bodies, recent attention has turned to the problem of quantifying the entanglement of a medium interacting with a quantum field, both described by (infinitely) many modes. In particular, Ref. [10] took key first steps in this direction, but simple conceptual pictures and efficient computational tools have remained elusive.

In this letter, we consider two or more dispersive media in the presence of a fluctuating field and compute their quantum mutual information, a characteristic measure of quantum correlations between parts of a system, and closely related to the entanglement entropy. Our model is a simplified electromagnetism in which a free scalar field couples weakly to “dielectric” bodies. We relate the mutual information in D dimensions to classical thermodynamic entropy in (D + 1) dimensions, where the dispersive media become planar disk-like regions (Fig. 1). Inspired by an electrostatic analogy, our formalism relates quantum information to classical scattering theory, thus allowing practical computations by any number of techniques including multipole expansion or numerical methods. As an example, we compute the mutual information between two bodies at all separations in D = 2 dimensions and observe area-law-violating logarithmic behavior at short separations.

Model. Our starting point is the action describing a free scalar field in D spatial dimensions in the presence of dispersive material bodies,

\[ I_{\text{cl}}[\phi] = \int_0^\infty d\omega \int \frac{d^D x}{(2\pi)^D} \phi_\omega^*(x) \left[ \epsilon(\omega, x) \omega^2 + \nabla^2 \right] \phi_\omega(x), \] (1)

where \( \epsilon = 1 + 4\pi \chi \) is a frequency- and position-dependent dielectric function. This model offers a simplified theoretical framework for exploring quantum electrodynamics and has proven useful in studies of Casimir [11, 12] and entanglement [10] phenomena. Specifically, the limit \( \epsilon \rightarrow 1 \) is the basis of a vast literature on entanglement entropy. We take the susceptibility \( \chi \) to be nonvanishing in a region \( \Omega \) (the material bodies), within which it assumes the spatially constant form \( \chi(\omega) = \frac{\omega^2}{\omega_p^2 - \omega^2} \) with \( \omega_0 \) and \( \omega_p \) the resonant and plasma frequencies of the dispersive material, respectively. Equation (1) may be viewed as an effective action arising upon integrating out the matter degrees of freedom from a parent action describing
both vacuum and matter fields [10], which (upon Wick rotation, \( \omega \to i \xi \)) reads

\[
I_E[\phi, \psi] = \int_0^\infty \frac{d \xi}{2\pi} \left\{ \int d \Omega \phi_\xi(x) (\xi^2 - \nabla^2) \phi_\xi(x) \right. \\
+ \left. \int d \Omega \left[ \frac{\xi^2 + \omega^2}{4\pi} \psi_\xi^2(x) + 2\omega \xi \phi_\xi(x) \psi_\xi(x) \right] \right\}, \tag{2}
\]

where \( \psi \) represents the degrees of freedom within the material bodies. Note that the action in the first line is that of a free scalar field, while the first term in the second line describes harmonic oscillator modes localized in \( \Omega \) [31]. Integrating out the field \( \psi \) from Eq. (2) recovers the action (1). On the other hand, we are interested in finding an effective action for the matter field \( \psi \). Integrating out the scalar field \( \phi \) yields

\[
I_{\text{eff}}[\psi] = \int_0^\infty \frac{d \xi}{2\pi} \left\{ \int d \Omega \xi^2 + \omega^2 \psi_\xi^2(x) \right. \\
+ \left. \omega^2 \frac{e^2}{4\pi} \int d \Omega d \Omega' \psi_\xi(x) G_\xi(x, x') \psi_{\xi'}(x') \right\}, \tag{3}
\]

where \( G_\xi = (\xi^2 - \nabla^2)^{-1} \) is the Green’s function of the Euclidean Helmholtz operator. The above effective action will allow us to study the entanglement between different parts of medium (or between two media). To this end, we briefly introduce the main tools to compute the entanglement entropy.

**Entanglement from covariance matrices.** For a mixed state \( \rho \), the von Neumann entropy, a widely used measure of entanglement, is defined as \(-\text{Tr}[\rho \log \rho]\). Let us consider the local field \( \{\psi(t, x)\} \) and its conjugate momentum \( \{\pi(t, x)\} \) for the local degrees of freedom in the region \( \Omega \). For a quadratic action, the ground state is a Gaussian functional of \( \{\psi\} \) (or \( \{\pi\} \)). Therefore, the quantum state can be fully represented by equal-time two-point functions \( \Xi(x, x') = \langle \psi(t, x)\psi(t, x') \rangle \) and \( \Pi(x, x') = \langle \pi(t, x)\pi(t, x') \rangle \). The von Neumann entropy is directly related to the two-point functions via \( S_\Omega = \text{Tr}[\log(\Delta_\Omega + 1)] \), where \( \Delta_\Omega \) is the identity matrix, and \( \Delta_\Omega = \sqrt{\Pi_{\Xi} - 1/2} \) is the covariance matrix [13, 14] with the subscript \( \Omega \) emphasizing its zero support outside \( \Omega \). We find it more convenient to recast the von Neumann entropy as

\[
S_\Omega = \int_0^1 d \lambda s_\Omega(\lambda),
\]

with \( s_\Omega(\lambda) = \text{Tr}[\log(\lambda^{-1}\Delta_\Omega + 1)] \), \( \lambda \in [0, 1] \) is an auxiliary parameter. The last expression is reminiscent of a one-loop effective action. In fact, we shall see that, by using the above identity, the entanglement entropy finds a convenient path-integral form.

Next we compute the covariance matrix \( \Delta_\Omega \). The correlation functions in imaginary frequency are given by \( \Xi(x, x') = \int \frac{d \xi}{2\pi} \langle \psi_\xi(x)\psi_\xi(x') \rangle \) and \( \Pi(x, x') = -\int \frac{d \xi}{2\pi} \langle \psi_\xi(x)\psi_\xi(x') \rangle \) [32]; the two-point function in the integrands can be derived from Eq. (3). In general, this is a difficult task which requires knowledge of the full (off-shell) \( T \)-matrix—an object that appears in the Lippmann-Schwinger equation—of the material bodies in position space [11]. Nevertheless, we consider a weak-coupling limit in which \( \omega_{\Omega, p} \ll 1 \) with \( L \) the linear size of the media. In other words, the bodies are sufficiently small that their coupling to the background scalar field [the second line in Eq. (3)] can be treated perturbatively, a condition well satisfied for sub-micron-scale bodies with typical plasma and resonant frequencies. One can then expand the correlation functions perturbatively: \( \Xi \approx (4\pi/\omega_0)(1/2 + \delta \Xi) \) and \( \Pi \approx (\omega_0/4\pi)(1/2 + \delta \Pi) \). In the absence of interaction with the scalar field, one recovers the standard variances of the field and its conjugate momentum as those of a harmonic oscillator, whereupon \( \Delta_\Omega \to 0 \) and the entanglement vanishes. To leading order, we have \( \Delta_\Omega \approx (\delta \Xi + \delta \Pi)/2 \). Within this approximation, we find that \( \Delta_\Omega = P_\Omega \Delta P_\Omega \), with \( P_\Omega \) the projection operator onto the spatial domain of \( \Omega \), and

\[
\Delta(x, x') = \omega_c G_0^{(D+1)}(x, x'),
\]

where we have put \( \omega_c = 2\pi \omega_p^2/\omega_0 \). This equation relates the covariance matrix \( \Delta \) to the Green’s function for the Laplacian in \( D + 1 \)-dimensions, \( G_0^{(D+1)}(x, x') \sim 1/|x - x'|^{D+1} \); see the Supplemental Material. It is perhaps surprising that \( \Delta(x, x') \) depends only on \( x \) and \( x' \), and not the specific geometry of \( \Omega \); however, this is an approximation, and the geometric dependence appears in higher orders in \( \omega_{\Omega, p} L \).

Now we are in a position to write Eq. (4) in a familiar form. First note that, with \( \Delta_\Omega = P_\Omega \Delta P_\Omega \), the second line of Eq. (4) can be written as \( \text{Tr}[\log(\lambda^{-1}\Delta_\Omega + 1)/\Delta_\Omega] \). The inverse of \( \Delta \) finds an awkward nonlocal form in \( D \) dimensions, while it is simply proportional to the Laplacian (hence, local) operator in \( D + 1 \) dimensions. Therefore, we can cast the above expression as a functional integral over a real-valued field \( \theta(x, x_{D+1}) \) living in \( D + 1 \) dimensions \( (x, x_{D+1}) \in \mathbb{R}^{D+1} \),

\[
s_\Omega(\lambda) = -2\log \int D\theta \exp \left[ -\omega_c^{-1} \int_{D+1} (\nabla \theta)^2 - \lambda^{-1} \int_\Omega \theta^2 \right] \int D\theta \exp \left[ -\omega_c^{-1} \int_{D+1} (\nabla \theta)^2 \right],
\]

which is easily seen to reproduce Eq. (4) upon functional integration. We emphasize that \( \theta \) is an auxiliary field, and is not related to the original fields in the model. The exponent in the numerator inside the logarithm can be interpreted as a Hamiltonian which is a sum of the free and potential terms: The first term is the usual gradient term in \( D + 1 \) dimensions, while the latter is the local potential felt only in the \( D \)-dimensional region \( \Omega \) where \( x_{D+1} = 0 \) (see Fig. 1 with \( \Omega = \Lambda \cup \Lambda' \)). Thus the terms inside the logarithm take the form of a partition function.
of the classical field \( \theta \) at a finite temperature; we choose \( k_B T = 1 \) for convenience. The above expression then characterizes the change of the classical free energy in the presence of the potential term, that is \(-F_\Omega(\lambda) = \log \langle \exp [-\lambda^{-1} \int_\Omega \theta^2] \rangle_0 \) where \( \langle \cdots \rangle_0 \) denotes the average with respect to the Gaussian fluctuations of the gradient term.

We will argue later that the change of the free energy is purely entropic, and make the connection to the thermodynamic entropy. For now, we point out another simplification: \( \lambda \) can be scaled by \( \omega_c^{-1} \) to eliminate the latter in the gradient term. This gives rise to an overall factor of \( \omega_c \) to the integral in Eq. (4) and changes its upper bound to \( \omega_c^{-1} \), which we extend to infinity as \( \omega_c L \ll 1 \). The von Neumann entropy is then obtained from the free energy of the thermodynamic model described above as

\[
S_\Omega = 2 \omega_c \int_0^\infty d\lambda \; F_\Omega(\lambda).
\]

The degrees of freedom in \( \Omega \) are uncountably infinite and thus the entropy depends on a UV cutoff [15]; such dependence may exhibit some form of universality in one dimension [4, 16, 17]. We, on the other hand, consider the quantum mutual information (QMI) between two subsystems \( A \) and \( B \) defined as \( \mathcal{I}_{A,B} = S_A + S_B - S_{A\cup B} \), which is inherently cutoff-independent, and measures quantum correlations between \( A \) and \( B \). Equation (6) then relates the QMI to the change of the free energy with the local potential turned on in one or both regions, \( \Delta F_{A,B}(\lambda) = F_{A\cup B}(\lambda) - F_A(\lambda) - F_B(\lambda) \),

\[
\mathcal{I}_{A,B} = -2 \omega_c \int_0^\infty d\lambda \; \Delta F_{A,B}(\lambda).
\]

The latter, however, is a familiar quantity in the context of thermal Casimir effect that gives rise to an interaction between two objects due to thermal, rather than quantum, fluctuations in the space between them [18]. The relation in Eq. (7) enables us to use the powerful tools available to study the (thermal or quantum) Casimir effect. It is well-known that one can compute the Casimir energy between two objects from the knowledge of their classical scattering matrices, i.e. their response to incoming classical waves [11, 19]. In the case of the thermal Casimir effect, one should compute generalized capacitance matrices, or the multipole (monopole, dipole, etc.) response of the objects to an external electrostatic potential. The interested reader can find more details in Refs. [11, 19]; we quote the final expression for \( \Delta F_{A,B}(\lambda) \),

\[
\Delta F_{A,B}(\lambda) = \frac{1}{2} \text{Tr}(D+1) \log [I - C_A(\lambda) G_0 C_B(\lambda) G_0],
\]

where \( G_0 \) and \( C_1 \) denote the \( D + 1 \)-dimensional Green’s function and capacitance matrix elements, respectively. To understand the \( \lambda \)-dependent \( C_A \), we note that, in the absence of fluctuations, the field \( \theta \) satisfies

\[
[-\nabla^2 + V_A] \theta = 0; \quad \text{the derivatives in this equation act in } D + 1 \text{ dimensions, and the potential is defined as } V_A(x, x_{D+1}) = \lambda^{-1} (x_{D+1}) \text{ when } x \in A, \text{ and } 0 \text{ otherwise. As the potential is confined to the plane } x_{D+1} = 0, \text{ its effect is simply to impose the boundary condition } [\zeta \equiv x_{D+1}]
\]

\[
- \partial_\zeta \theta \big|_{0^+} + \lambda^{-1} \theta(x, \zeta = 0) = 0, \quad x \in A,
\]

on the otherwise free electrostatic field satisfying \( \nabla^2 \theta = 0 \). For example, \( \lambda = 0 \) corresponds to a perfect conductor, while in the limit \( \lambda \to \infty \) the object is completely transparent. The generalized capacitance matrix elements in \( C_A(\lambda) \) then characterize the electrostatic response of the object due to the \( \lambda \)-dependent boundary condition in Eq. (9). For the sake of completeness, we mention that the capacitance can be expressed as \( C_A = V_A \frac{1}{1 - G_A N^\dagger} \), where \( V_A \) is the operator-valued potential corresponding to \( V_A \). This equation is reminiscent of the Lippmann-Schwinger equation, which is yet another route to scattering theory.

In short, Eqs. (7,8) are the central results of this paper, and relate quantum mutual information in \( D \) dimensions to classical (generalized) capacitance elements in \( D + 1 \) dimensions, which allows exploitation of myriad analytical and numerical methods for computing \( C \)-matrix elements. Next we consider specific examples to showcase the efficiency of our method.

We start with two material bodies (Fig. 1) separated by a large distance \( d \) compared to their sizes (yet small compared to \( \omega_c^{-1} \)). The \( C \)-matrix elements may be computed analytically for simple shapes via techniques reminiscent of electrostatics. For large \( d \), we can keep only the monopole coefficient in Eq. (8), combined with Eq. (7), to find

\[
\mathcal{I}_{A,B} = \frac{\omega_c}{A_{D+1}} \frac{\omega_c R_A^2 R_B^2}{d(2D-1)} \int_0^\infty d\lambda C_A^0(\lambda) C_B^0(\lambda),
\]

where \( A_D = [\Omega_D(D - 2)] \) with \( \Omega_D \) the area of the unit \( D \)-sphere, and \( C^0(\lambda) = \frac{b}{4\pi^{D/2}} \); see the Supplemental Material. The QMI is then given by

\[
\mathcal{I}_{A,B} = \frac{\omega_c R_A^2 R_B^2}{32\pi^3 (R_B - R_A)d^2} \frac{d^2}{R_A \to R_B = R} \xrightarrow{R_A \to R_B = R} \frac{\omega_c R^3}{32\pi^3 d^3}.
\]
bodies evaluated numerically for a range of separations. The predictions of Eq. (8) for disc-shaped and square-shaped bodies are shown in Fig. 2 and introduce a basis of \( N \times N \)-dimensional matrix elements of \( C^{-1} \) with \( \alpha \beta \) indicated. The operator \( C^{-1} \) discretized into small triangles (upper insets). At large separations, the QMI obeys a power law, while at short separations the square–square data suggest a logarithmic dependence on \( d \). The lower inset shows an enlarged view of the short-distance data for squares, together with \( a_0 = 0.050 \).

Note that, even at large separations, the mutual information depends nontrivially on the radii, reflecting the difficulty of computations in general.

At short separations \( d \ll R \), on the other hand, the multipole expansion becomes inconvenient due to the need to retain large numbers of multipoles. In this case it is useful to note that, while the \( C \) operator itself is unwieldy to describe in real space, the inverse of this operator has a simple real-space representation [28], namely \( C^{-1}(x,x') = \lambda d(x-x') + G_0(x,x') \). It is then tempting to discretize regions \( A \) and \( B \) into small finite elements (Fig. 2) and introduce a basis of \( N \) expansion functions \( \{ b_\alpha(x) \} \) localized on these elements; in this basis, we easily compute matrix elements of \( C^{-1} \) directly according to \( C^{-1}\alpha\beta(\lambda) \equiv \langle b_\alpha | C^{-1}(\lambda) | b_\beta \rangle \), then simply invert the matrix \( C^{-1} \) numerically to yield the \( C \) matrix for arbitrary \( \lambda \) (with \( G \) computed similarly). The operator products and the trace in Eq. (8) become \((N \times N)\)-dimensional matrix operations and the \( \lambda \) integral is evaluated by numerical quadrature. As two examples, Fig. 2 plots the predictions of Eq. (8) for disc-shaped and square-shaped bodies evaluated numerically for a range of separations \( d \).

In the long-distance regime, the QMI follows the power-law in Eq. (11) and increases monotonically as the separation becomes shorter. For small separations \( d \ll R \), the numerical data suggests a logarithmic dependence on \( d \),

\[
\mathcal{I}_{A,B} \sim \omega_c R \log \frac{R}{d},
\]  

(12)

Note that the mutual information is proportional to the “area” of the common boundary of the two media up to a logarithm, in this case the area being the common edge of the two-dimensional regions. The area law can be proved under certain conditions [24–26]; for a review, see Ref. [15]. Logarithmic violations of the area law also appear in several contexts [16, 17, 27].

**Mutual information from thermodynamic entropy**—Finally, we provide a geometrical interpretation of the thermodynamic model described above in terms of the random-chain polymer ensemble [28]: this approach was inspired by the worldline formalism [29]. We first note that the functional integral of the gradient term \( \int D\theta \exp \left[ -\int (\nabla \theta)^2 \right] \) has an entropic origin as it sums over configurations of phantom chain polymers in free space [33]. In the presence of boundaries [Eq. (9)], this sum is further weighted when a polymer intersects a boundary. In fact, to compute \( \Delta F_{A,B}(\lambda) \), one must sum over chain polymers that intersect both \( A \) and \( B \) regions, weighted by their size and intersections, the latter through \( \lambda \) (see Fig. 1). In this sense, the change in free energy is nothing but the change of the entropy \( (k_B T = 1) \), that is \( -\Delta F = \Delta S_{\text{th}} \) where \( S_{\text{th}} \) denotes the purely thermodynamic entropy of a fluctuating polymer. We thus arrive at the fundamental conclusion of this Letter: The QMI in \( D \) dimensions is obtained from the thermodynamic entropy in \( D + 1 \) dimensions as

\[
\mathcal{I}^{(D)} = 2\omega_c \int_0^{\infty} d\lambda \Delta S_{\text{th}}^{(D+1)}(\lambda),
\]  

(13)

where we dropped the dependence on \( A, B \), but made the dependence on the dimension explicit. We stress that Eq. (13) and the above geometrical picture is easily generalized to three or more regions. For example, the quantum tripartite information is due to polymer chains that intersect all three regions. In fact, the strong subadditivity property for quantum systems [30] expressed in terms of mutual and tripartite information becomes rather trivial once formulated in the geometrical picture above; see the Supplemental Material for details. The thermodynamic analogy (sum over classical configurations) is crucial here.

**Discussion and outlook**—We have studied the entanglement of dispersive media, and shown that their mutual (and tripartite, etc.) information maps to the thermodynamic entropy in a higher dimension. Both analytical and numerical computations are illustrated based on an electrostatic analogy. Extending the results of this paper to realistic models such as electromagnetism is worthwhile. The response function can also be generalized to a sum of Lorentzians, and thus a general \( \epsilon(\omega) \), at the expense of introducing several copies of the matter field for each term in the sum. Including finite temperature should be of interest from both fundamental and practical perspectives. Finally, lifting the assumption of weak coupling as well as generalizing...
to non-linear and conformal field theories are worthwhile but more challenging.

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Supplemental Material

I. THE COVARIANCE MATRIX $\Delta$

Computing the covariance matrix $\Delta$ requires two-point functions. The latter should be computed by inverting the operator (cf. Eq. (3) of the main text)

$$\frac{\xi^2 + \omega_0^2}{4\pi} \delta(x - x') + \omega_0^2 \xi^2 G_\xi(x, x')$$

(S1)

defined on the support of the media $V$. To invert this operator, we will treat the second term perturbatively justified in the weak-coupling limit where $\omega_0 p L \ll 1$ with $L$ the linear size of the media. Therefore, to the first order, we find

$$\langle \psi_\xi(x) \psi_\xi(x') \rangle \approx \frac{4\pi}{\xi^2 + \omega_0^2} \delta(x - x') - \left( \frac{4\pi \omega_0^2 \xi}{\xi^2 + \omega_0^2} \right)^2 G_\xi(x, x').$$

(S2)

The two-point functions $\Xi$ and $\Pi$ defined in the manuscript can be computed accordingly. The field correlation function reads

$$\Xi(x, x') = 2 \int_0^\infty \frac{d\xi}{2\pi} \langle \psi_\xi(x) \psi_\xi(x') \rangle \approx \frac{2\pi}{\omega_0^2} \delta(x - x') - 16\omega_0^2 \int_0^\infty \frac{d\xi}{\xi^2 + \omega_0^2} G_\xi(x, x').$$

(S3)

where the factor of 2 in the first equality comes from the restriction to $\xi \in [0, \infty)$. Similarly, the correlator of the conjugate momentum is given by

$$\Pi(x, x') = -2 \int_0^\infty \frac{d\xi}{2\pi} \langle \psi_\xi(x) \psi_\xi(x') \rangle \approx \frac{\omega_0^2}{8\pi} \delta(x - x') + \frac{\omega_0^2}{\pi} \int_0^\infty \frac{d\xi}{\xi^2 + \omega_0^2} G_\xi(x, x'),$$

(S4)

with the minus sign in the first equality due to $\omega \to i\xi$. Superficially, the integral over the first term in Eq. (S2) is divergent; however, a proper regularization is to compute $\lim_{\xi \to 0^+} \langle p(t)p(0) \rangle$ ($p$ representing the conjugate momentum) which yields the first term in the last equality above [S1]. The last term in this equality can be simplified by dropping $\omega_0$ in the denominator as $\omega_0 L \ll 1$. It is useful to note that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^{(D)}_\omega(x) = G^{(D+1)}_0(x, 0),$$

(S5)

where the superscript denotes the dimension, $(x, x_{D+1})$ represents coordinates in one higher dimension, and $G_0$ is the Green’s function corresponding to the Laplacian. Defining $\delta\Xi$ and $\delta\Pi$ as in the manuscript, the above equations yield $\delta\Pi(x, x') \approx (4\pi \omega_0^2/\omega_0) G^{(D+1)}_0(\tilde{x}, \tilde{x}')$ with $\tilde{x} = (x, 0)$, while $\delta\Xi$ can be neglected compared to $\delta\Pi$. With $\Delta \approx (\delta\Xi + \delta\Pi)/2$ (cf. the manuscript), we recover the expression for $\Delta(x, x')$ in Eq. (5) of the manuscript.

II. THE CAPACITANCE OF A DISC IN $D + 1 = 3$ DIMENSIONS

In this section, we consider a modified electrostatic problem in 3 dimensions. The ‘potential’ $\theta(x, y, z)$ satisfies the Laplace equation, $\nabla^2 \theta = 0$, except on the surface of a disc of radius $R$ sitting at the $z = 0$ plane. The disk imposes the $\lambda$-dependent boundary condition (7) on $\theta$ (whose dependence on $\lambda$ is implicit). It is convenient to consider the oblate spheroidal coordinates [S2],

$$x = R \sqrt{\xi^2 + 1}(1 - \eta^2) \cos \phi,$$

$$y = R \sqrt{\xi^2 + 1}(1 - \eta^2) \sin \phi,$$

$$z = R \xi \eta,$$

(S6)

where $0 \leq \xi < \infty$, $-1 \leq \eta \leq 1$, and $0 \leq \phi < 2\pi$. With the reflection symmetry $z \to -z$, we separately consider even and odd solutions with respect to the $z$ coordinate. Odd solutions become trivial as $\theta(x, y, z = 0) = 0$ whereby the boundary condition simply requires the continuity of the first derivative. For even solutions, the boundary condition
becomes \(-2\partial_n \theta + \lambda^{-1} \theta(x, y, z = 0) = 0\) where \(\partial_n\) is the normal derivative on the surface. In oblate spherical coordinates, the latter condition takes the form

\[
\left[-\partial_\xi \theta + \frac{R}{2\lambda} |\eta| \frac{\partial}{\partial \theta} \right]_{\xi=0} = 0. \tag{S7}
\]

The regular solutions (decaying at infinity as \(\xi \to \infty\)) of the Laplace equation are given by

\[
Q_n(i\xi)P_n^m(\eta)e^{im\phi}, \tag{S8}
\]

where \(Q_n(ix) = Q_n(i\xi) - \frac{i\pi}{Q_n(i\xi)}P_n(ix)\) with \(P(Q)\) the Legendre polynomial of the first (second) kind. To find the capacitance, we consider a constant potential of magnitude 1 at \(\xi = 0\) with \(\sim = (x, y, z)\) and \(0\) in Eq. (S9), we find that, at large radius \(r = \sqrt{x^2 + y^2 + z^2}\), the potential behaves as \(\theta \sim 1 - \frac{iR_0}{R}\) (using \(Q_0(i\xi) \sim -i/\xi\) and \(\xi \sim r/R\) for large \(\xi\)). The capacitance is then given by \(C = iR_0 \approx \frac{R}{4\lambda/R + \pi/2}\). This formula correctly reproduces the Dirichlet case for \(\lambda = 0\) [S2].

### III. INTERPRETATION IN TERMS OF RANDOM CHAIN POLYMERS

In this section, we outline an alternative approach to computing the free energy of a thermally fluctuating field based on the worldline formalism [S3]; this technique was first applied to computing the Casimir interaction energy between two objects. We shall consider the objects as infinitely thin surfaces represented by a \(\delta\)-function potential \([\delta = (x, x_{D+1})\) denotes the coordinates in \(D + 1\) dimensions]}

\[
V_A(\vec{x}) = \lambda^{-1} \int_A d\Sigma \delta(\vec{x} - \vec{x}_\Sigma), \tag{S13}
\]

defined on a \(D\) dimensional hyper-surface \(A\) in \(D + 1\) dimensions. Note that the limit \(\lambda \to 0\) corresponds to Dirichlet boundary conditions. Now consider a worldline \(\vec{x}(\tau)\) parameterized by the variable \(\tau\) which is normalized as the Euclidean length of the worldline trajectory, and define

\[
T_A[\vec{x}(\tau)] = \int d\tau V_A(\vec{x}(\tau)). \tag{S14}
\]
We are interested in finding the thermodynamic entropy in the presence of two hyper-surfaces \( A \) and \( B \). The change of the thermodynamic entropy \( \Delta S_{th} \) (cf. the manuscript for the definition) can be expressed as a sum over worldline loops, or closed phantom polymers, as

\[
\Delta S_{th}(A, B; \lambda) \propto \int \frac{dl}{l^{1+\frac{D}{2}}} \int d\hat{x}_{CM} \left\langle 1 - e^{-T_A} - e^{-T_B} + e^{-T_{A\cup B}} \right\rangle,
\]

where \( l \) and \( \hat{x}_{CM} \) are the length and the center of mass of a worldline trajectory, respectively, and the average \( \left\langle . \right\rangle \) is taken over all loops with a fixed \( l \) and \( \hat{x}_{CM} \); see Ref. [S3] for more details. We also made explicit the dependence of thermodynamic entropy on the hyper-surfaces \( A \) and \( B \). Finally, the proportionality coefficient is a \( D \)-dependent constant, which will not be important for our purposes. The above equation has a simple interpretation: Only worldlines that “see” both surfaces contribute to \( \Delta S_{th} \). For the special case of Dirichlet boundary conditions, \( \lambda = 0 \), the integrand is 1 when the worldline intersects both \( A \) and \( B \), and 0 otherwise, since

\[
e^{-T_A} = \begin{cases} 0, & \text{polymer does not intersect } A \\ 1, & \text{polymer intersects } A \end{cases}
\]

IV. QUANTUM TRIPARTITE INFORMATION

In the main text, we derived an expression for the mutual information between two material bodies in terms of the thermodynamic free energy, and consequently the thermodynamic entropy, in one higher dimension. Here, we generalize this result to three bodies. Further generalization to more than three regions is straightforward. First we define the quantum tripartite information \( I(A, B, C) \) for three systems \( A, B \) and \( C \) as \[S4\]

\[
I(A, B, C) = S(A) + S(B) + S(C) - S(A \cup B) - S(A \cup C) - S(B \cup C) + S(A \cup B \cup C),
\]

where \( S \) represents the von-Neumann entropy. Again this definition is free of UV divergences. Therefore, the tripartite quantum information maps to the thermodynamic entropy as (also making the dependence on the dimension \( D \) explicit)

\[
I^{(D)}(A, B, C) = 2\omega_c \int_0^\infty d\lambda \Delta_3 S^{(D+1)}_{th}(A, B, C; \lambda).
\]

where we have defined (making the dependence on \( \lambda \) implicit) \( \Delta_3 S_{th}(A, B, C) \equiv S_{th}(A) + S_{th}(B) + S_{th}(C) - S_{th}(A \cup B) - S_{th}(A \cup C) - S_{th}(B \cup C) + S_{th}(A \cup B \cup C) \). A trace expression in terms of capacitance matrices similar to Eq. (8) of the main text, but for three objects, can be found in Ref. [S5].

V. STRONG SUBADDITIVITY PROPERTY

For any tripartite quantum system comprising disjoint subsystems \( A, B, \) and \( C \), the strong subadditivity theorem states that \[S6\]

\[
S(A \cup B \cup C) + S(B) \leq S(A \cup B) + S(B \cup C).
\]

This inequality can be recast in terms of the mutual and tripartite information (cf. the manuscript and the previous section for the definitions) as

\[
I(A, B, C) \leq I(A, C).
\]

The proof of the above theorem is highly nontrivial. Nevertheless, using the thermodynamic analogy, Eq. (S18), the subadditivity property can be easily verified in our model. More precisely, we show that

\[
\Delta_3 S_{th}(A, B, C; \lambda) \leq \Delta S_{th}(A, C; \lambda)
\]

for each value of \( \lambda \). The quantum strong subadditivity property follows immediately. To this end, we cast the change of the thermodynamic entropy for three objects in terms of polymer chains:

\[
\Delta S_{th}(A, B, C; \lambda) \propto \int \frac{dl}{l^{1+\frac{D}{2}}} \int d\hat{x}_{CM} \left\langle 1 - e^{-T_A} - e^{-T_B} - e^{-T_C} + e^{-T_{A\cup B}} + e^{-T_{A\cup C}} + e^{-T_{B\cup C}} - e^{-T_{A\cup B\cup C}} \right\rangle.
\]
Similar to the mutual information, only worldlines that see all three surfaces contribute to $\Delta S_{ab}$. Equation (S21) then finds a pictorial meaning: The contribution of polymer configurations that intersect all three regions $A$, $B$ and $C$ is less than or equal to those that intersect $A$ and $C$, but not necessarily $B$. In fact, for the special case of $\lambda = 0$, the inequality in Eq. (S21) becomes rather trivial as it simply means [cf. Eq. (S16)] that the number of polymers that intersect all three regions is less than or equal to the number of those which intersect any two regions, but not necessarily the third one. To prove the inequality in general, we define

$$e^{-T_A} = x, \quad e^{-T_B} = y, \quad e^{-T_C} = z,$$

(S23)

hence, $e^{-T_{AB}} = xy$ and $e^{-T_{ABC}} = xyz$, etc. To derive Eq. (S21), it is sufficient to show that

$$1 - x - y - z + xy + yz + xz - xyz \leq 1 - x - z + xz$$

(S24)

but this is simply equivalent to $(-1 + x)(-1 + z) \geq 0$ which is trivially satisfied (note that $0 \leq x, y, z \leq 1$).

VI. NUMERICAL COMPUTATIONS

As noted in the main text, an advantage of our method is that it expresses quantum mutual information in terms of classical scattering data, thus opening the door to myriad existing methods for practical computations. Indeed, in principle any method of computational electrostatics could be used to calculate the quantities $C(\lambda)$ and $G_0$ that enter Eq. (8) of the main text; this includes both analytical techniques such as multipole expansions and numerical methods such as finite-difference or boundary-element approaches.

The choice of a computational method may be viewed as the choice of a discrete basis in which to evaluate matrix elements of the $C(\lambda)$ and $G_0$ operators in Eq. (8) of the main text. Many different choices of basis are possible; we here consider two in particular:

1. For highly symmetric shapes at medium or long distances, it is convenient to introduce a multipole basis that takes advantage of symmetries to encode scattering data in compact form. In some cases the representation is so concise that Eq. (8) of the main text may be evaluated analytically, as in the case of the discs at large separation discussed in the main text.

2. For more general shapes, or in the short-distance limit, it is convenient to introduce a basis of localized functions supported only on the material bodies. This allows treatment of essentially arbitrary shapes—such as the squares of Figure 2, for which a multipole expansion would be cumbersome—and calculations at essentially arbitrarily short distances.

Before describing each of these possibilities in turn, we pause to note that this dichotomy mirrors the recent evolution of theoretical approaches to electromagnetic fluctuation phenomena (Casimir forces and near-field radiative heat transfer), in which geometry-specific multipole bases have been employed to yield rapidly convergent or even analytically tractable series expansions of Casimir quantities and heat-transfer rates in high-symmetry geometries [S7, S8], while localized basis functions have been used for numerical Casimir or heat-transfer calculations in arbitrary geometries [S9, S10].

**Multipole basis**

For objects of high symmetry—such as the disc-shaped material bodies considered in the main text—it is convenient to work in a multipole basis. For example, in a basis of spherical multipoles one expands scalar fields in solutions of the Laplace equation indexed by the usual spherical indices $\ell, m$:

$$\phi_{\ell m}^\text{int}(r) = r^\ell Y_{\ell m}(\theta, \phi), \quad \phi_{\ell m}^\text{ext}(r) = \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \phi)$$

(S25)

where $Y_{\ell m}$ are the usual spherical harmonics. In this case,

- the $G_0$ matrices in Eq. (8) of the main text become translation matrices, which relate spherical multipoles centered at the origin of one body to multipoles centered at the origin of another body, while
the elements of the $C(\lambda)$ matrices in Eq. (8) of the main text describe the multipole sources induced on (or, equivalently, the outgoing multipole fields emitted by) material bodies by an incident multipole field.

More specifically, the definition of the spherical-wave $C$ matrix elements for a body is as follows: If the body is exposed to an “incident” field consisting of a single incoming multipole field, i.e. $\phi^{\text{inc}} = \phi_{\ell,m}^{\text{int}}$, then the “scattered” field produced by the body is

$$
\phi^{\text{scat}} = \sum_{\ell',m'} C_{\ell,m;\ell',m'} \phi_{\ell',m'}^{\text{ext}}(r).
$$

Thus, to compute one full column of the $C$-matrix we solve a modified electrostatics problem in which the body [on whose surface we have the $\lambda$-dependent boundary condition (7)] is exposed to an external $(\ell, m)$ field; after solving the scattering problem we extract $C_{\ell,m;\ell',m'}$ as the $(\ell', m')$ source multipole induced by this field on the body.

**Localized basis**

As noted in the main text, the multipole basis becomes unwieldy in the short-distance limit due to the need to retain large numbers of multipoles to obtain sufficient accuracy. Moreover, even at long distances the analytical usefulness of the multipole basis is restricted to highly symmetric objects for which the $C$-matrix elements may be computed in closed form. For shorter distances or less symmetric objects (such as the squares of Figure 2), it is convenient to work with position-space representations of the $C$ and $G_0$ operators in Eq. (8) of the main text.

A direct position-space interpretation of the operator trace in Eq. (8) of the main text is ill-defined for at least two reasons. One difficulty is that the operator $G_0$, whose position-space representation is

$$
G_0(x, x') = \frac{1}{4\pi|x - x'|} \tag{S26}
$$

is singular “on the diagonal,” i.e. for $x = x'$. This singularity needs somehow to be regularized before the operator products and trace in Eq. (8) of the main text may be performed. A second difficulty is that the $C(\lambda)$ operators are unwieldy in position space. Indeed, for e.g. body $A$ the position-space matrix elements $C_A(\lambda; x, x')$ satisfy the following mixed definition:

1. If either of the points $x, x'$ lies outside of object $A$, then we have an explicit representation for $C_A(\lambda; x, x')$, namely

$$
C_A(\lambda; x, x') = 0 \quad \text{if} \quad x \notin A \quad \text{or} \quad x' \notin A.
$$

2. On the other hand, if both points $x, x'$ lie inside body $A$ then we have only an explicit representation for the inverse of the $C$ operator,

$$
C_A^{-1}(\lambda; x, x') = \lambda \delta(x - x') + G_0(x, x') \tag{S27}
$$

Both of these difficulties are alleviated by introducing a finite basis of $N_A$ localized functions $\{b_{An}(x)\}, n = 1, \cdots, N_A$, whose supports are restricted to a finite subregion of body $A$. For example, we might discretize the body into $N_A$ small triangles (as depicted in the inset of Figure 2); to the $n$th triangle $T_{An}$ we associate a basis function $b_{An}(x)$ defined to be unity for $x \in T_{An}$ and vanishing elsewhere. [Similarly, for body $B$ we introduce $N_B$ basis functions $\{b_{Bn}(x)\}$ supported on subregions of body $B$.

With this choice of basis functions, the elements of the $G$ operator take the form of four-dimensional integrals over triangle-product domains:

$$
G_{\alpha\beta} = \left< b_{\alpha} \right| G \left| b_{\beta} \right> = \int \int b_{\alpha}(x)b_{\beta}(x') \frac{4\pi}{4\pi|x - x'|} \, dx \, dx' = \int_{T_{\alpha}} \int_{T_{\beta}} \frac{1}{4\pi|x - x'|} \, dx \, dx'. \tag{S28}
$$

(Note that the $G$ matrix in the localized basis is nonsingular even on the diagonal, i.e. $G_{\alpha\alpha} < \infty$; the finite extent of the basis functions regularizes the position-space divergence of equation (S26).) The $G$ operators in Eq. (8) of the
main text become $N_A \times N_B$ or $N_B \times N_A$ matrices describing the interactions of triangles on body $A$ with triangles on body $B$.

The $C_A$ and $C_B$ operators in Eq. (8) of the main text become $N_A \times N_A$ and $N_B \times N_B$ matrices. To compute these it is easiest to use equation (S27) to compute first the inverse matrices $C_{A,B}^{-1}$. The elements of e.g. $C_A^{-1}$ read

\[ C_{A,mn}^{-1}(\lambda) = \langle b_{Am} | C_A^{-1}(\lambda) | b_{An} \rangle \]

\[ = \lambda N_m \delta_{mn} + \langle b_{Am} | G_0 | b_{An} \rangle \]

(S29)

(S30)

where $N_m = \int b_m^2(x) \, dx$ is the norm of basis function $b_m$ and the $G_0$ matrix element is computed as in (S28). Having computed the full numerical matrix $C^{-1}(\lambda)$, we then simply invert this matrix numerically to obtain the C matrices that enter Eq. (8) of the main text. Matrix products are computed using standard numerical linear algebra software, and the logarithm is computed using the identity $\text{Tr} \log M = \sum \log \lambda_n$ where \{\lambda_n\} are the eigenvalues of $M$. Finally, the $\lambda$ integral in Eq. (8) of the main text is evaluated using an adaptive numerical quadrature method.

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