Examining the dual of an algebraic quantum group.

J. Kustermans
Institut for Matematik og Datalogi
Odense Universitet
Campusvej 55
5230 Odense M
Denmark
April 1997

Abstract
In the first part of this paper, we implement the multiplier algebra of the dual of an algebraic quantum group \((A, \Delta)\) (see [13]) as a subset of the space of linear functionals on \(A\).

In a second part, we construct the universal corepresentation and use it to prove a bijective correspondence between corepresentations of \((A, \Delta)\) and homomorphisms on its dual.

Introduction
In [13], A. Van Daele introduces and investigates a class of algebraic quantum groups. These algebraic quantum groups are, loosely speaking, non-unital Hopf algebras which have a left invariant functional. These algebraic quantum groups behave very well, as can be found in [13]. The left invariant functional is faithful and unique. It also satisfies some weak KMS property. You can also introduce the modular function of such an algebraic quantum group.

It is also possible to construct the dual of an algebraic quantum group and get again an algebraic quantum group. An overview of the most important properties can be found in the first section.

The canonical examples of algebraic quantum groups are the compact and algebraic quantum groups. However, the double construction of Drinfel’d guarantees the existence of non-discrete non-compact algebraic quantum groups.

If an algebraic quantum group has a compatible *-structure and a positive left Haar functional, it is possible to construct C*-algebraic quantum groups out of them in the sense of Masuda, Nakagami & Woronowicz (see [6] and [9]).

Consider an algebraic quantum group \((A, \Delta)\). Then we get a dual algebraic quantum group \((\hat{A}, \hat{\Delta})\) where \(\hat{A}\) is a certain subspace of \(A'\). We prove in section 2 that the multiplier algebra \(M(\hat{A})\) can also be considered as a certain subspace of \(A'\). This is then used in section 3 to get a familiar implementation of the comultiplication \(\hat{\Delta}\).

In section 5, we prove some useful properties about corepresentations on algebraic quantum groups. In section 6, we construct the universal corepresentation of \((A, \Delta)\). We get a bijective correspondence between non-degenerate corepresentations of \((A, \Delta)\) and non-degenerate homomorphisms on \(\hat{A}\) where this universal corepresentation serves as the linking bridge.

All vector spaces in this paper are considered over the complex numbers. For every vector space \(V\), we will denote the algebraic dual by \(V'\) and the set of linear operators on \(V\) by \(L(V)\).

If \(V, W\) are two vector spaces, then \(\chi\) will denote the flip map \(\chi : V \odot W \rightarrow W \odot V\).

1Research Assistant of the National Fund for Scientific Research (Belgium)
1 Algebraic quantum groups

In this section, we will introduce the notion of an algebraic quantum group as can be found in [13]. Moreover, we will give an overview of the properties of this algebraic quantum group. The proofs of these results can be found in the same paper [13]. After this section, we will prove some further properties about these algebraic quantum groups. We will first introduce some terminology.

We call an algebra $A$ non-degenerate if and only if we have for every $a \in A$ that:

$$(\forall b \in A : ab = 0) \Rightarrow a = 0 \quad \text{and} \quad (\forall b \in A : ba = 0) \Rightarrow a = 0.$$ 

For a non-degenerate algebra $A$, you can define the multiplier algebra $M(A)$. This is a unital algebra in which $A$ sits as a two-sided ideal.

If you have two non-degenerate algebras $A, B$ and a multiplicative linear mapping $\pi$ from $A$ to $M(B)$, we call $\pi$ non-degenerate if and only if the vector spaces $\pi(A)B$ and $B\pi(A)$ are equal to $B$. Such a non-degenerate multiplicative linear map has a unique multiplicative linear extension to $M(A)$, this extension will be denoted $\pi$. For every $a \in M(A)$, we define $\pi(a) = \pi(a)$.

We have of course similar definitions and results for anti-multiplicative mappings.

For a linear functional $\omega$ on a non-degenerate algebra $A$ and any $a \in M(A)$ we define the linear functionals $\omega a$ and $a\omega$ on $A$ such that $(\omega a)(x) = \omega(xa)$ and $(\omega a)(x) = \omega(ax)$ for every $x \in A$.

You can find some more information about non-degenerate algebras in the appendix of [14].

Let $\omega$ be a linear functional on an algebra $A$. Then $\omega$ is said to be faithful if and only if we have for every $a \in A$ that

$$(\forall b \in A : \omega(ab) = 0) \Rightarrow a = 0 \quad \text{and} \quad (\forall b \in A : \omega(ba) = 0) \Rightarrow a = 0.$$ 

We have now gathered the necessary information to understand the following definition.

**Definition 1.1** Consider a non-degenerate algebra $A$ and a non-degenerate homomorphism $\Delta$ from $A$ into $M(A \odot A)$ such that

1. $(\Delta \odot \iota)\Delta = (\iota \odot \Delta)\Delta$.

2. The linear mappings $T_1$, $T_2$, $T_3$, $T_4$ from $A \odot A$ into $M(A \odot A)$ such that

$$T_1(a \odot b) = \Delta(a)(b \odot 1) \quad \text{and} \quad T_2(a \odot b) = \Delta(a)(1 \odot b)$$

$$T_3(a \odot b) = (b \odot 1)\Delta(a) \quad \text{and} \quad T_4(a \odot b) = (1 \odot b)\Delta(a)$$

for all $a, b \in A$, are bijections from $A \odot A$ to $A \odot A$.

Then we call $(A, \Delta)$ a regular Multiplier Hopf algebra.

**Remark 1.2** Let $a \in A$ and $\omega \in A'$, then $(\omega \odot \iota)\Delta(a)$ will be by definition the element in $M(A)$ such that

$$[(\omega \odot \iota)\Delta(a)]b = (\omega \odot \iota)(\Delta(a)(1 \odot b)) \quad \text{and} \quad b[(\omega \odot \iota)\Delta(a)] = (\omega \odot \iota)(1 \odot b)\Delta(a)$$

for every $b \in B$ (if $\Delta(a)$ would belong to $A \odot A$, this definition of $(\omega \odot \iota)\Delta(a)$ would be equal to the usual notion of $(\omega \odot \iota)\Delta(a)$ ).
• It is not difficult to check that we have for $\omega \in A'$ and $a, b \in A$ that $(\omega b \circ \iota)\Delta(a)$ belongs to $A$ and $(\omega b \circ \iota)\Delta(a) = (\omega \circ \iota)((b \otimes 1)\Delta(a))$. A similar remark applies for $b \omega$.

• Of course, we will use a similar notation for $(\iota \circ \omega)\Delta(a)$ if $\omega \in A'$ and $a \in A$

In [16], A. Van Daele proves the existence of a unique non-zero homomorphism $\varepsilon$ from $A$ to $\mathbb{C}$ such that

$$(\varepsilon \circ \iota)\Delta = (\iota \circ \varepsilon)\Delta = \iota.$$ 

He proves moreover the existence of a unique anti-automorphism $S$ on $A$ such that

$$m(S \circ \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b \quad \text{and} \quad m(\iota \circ S)((b \otimes 1)\Delta(a)) = \varepsilon(a)b$$

for every $a, b \in A$ (here, $m$ denotes the multiplication map from $A \otimes A$ to $A$).

As usual, $\varepsilon$ is called the counit and $S$ the antipode of $(A, \Delta)$. Furthermore, $\chi(S \circ S)\Delta = \Delta S$.

Let $\omega$ be a linear functional on $A$. We call $\omega$ left invariant (with respect to $(A, \Delta)$), if and only if $$(\iota \circ \omega)\Delta(a) = \omega(a)1$$ for every $a \in A$. Right invariance is defined in a similar way.

**Definition 1.3** Consider a regular Multiplier Hopf algebra $(A, \Delta)$ such that there exists a non-zero linear functional $\varphi$ on $A$ which is left invariant. Then we call $(A, \Delta)$ an algebraic quantum group.

Such a non-zero left invariant linear functional $\varphi$ will be called a left Haar functional on $(A, \Delta)$.

For the rest of this section, we will fix an algebraic quantum group with a left Haar functional $\varphi$ on it. We define $\psi$ as the linear functional $\varphi S$. It is clear that $\psi$ is a non-zero right invariant linear functional on $A$.

An important feature of such an algebraic quantum group is the faithfulness and uniqueness of left invariant functionals:

1. Consider a left invariant linear functional $\omega$ on $A$, then there exists a unique element $c \in \mathbb{C}$ such that $\omega = c\varphi$.

2. Consider a non-zero left invariant linear functional $\omega$ on $A$, then $\omega$ is faithful.

In particular, $\varphi$ is faithful.

We have of course similar faithfulness and uniqueness results about right invariant linear functionals.

A first application of this uniqueness result concerns the antipode: Because $\varphi S^2$ is left invariant, there exists a unique complex number $\mu$ such that $\varphi S^2 = \mu \varphi$ (in [13], our $\mu$ is denoted by $\tau!$).

In this paper, we will need the following formula:

$$(\iota \circ \varphi)((1 \otimes a)\Delta(b)) = S((\iota \circ \varphi)(\Delta(a)(1 \otimes b)))$$ (1)

for all $a, b \in A$. A proof of this result can be found in proposition 3.11 of [13].

Using the formula $\chi(S \circ S)\Delta = \Delta S$, we get the following form of the formula above.

$$(\psi \circ \iota)((a \otimes 1)\Delta(b)) = S^{-1}((\psi \circ \iota)(\Delta(a)(b \otimes 1)))$$ (2)
Another non-trivial property about \( \varphi \) is the existence of a unique automorphism \( \rho \) on \( A \) such that \( \varphi(ab) = \varphi(b\rho(a)) \) for every \( a, b \in A \). We call this the weak KMS-property of \( \varphi \) (In \[13\], our mapping \( \rho \) is denoted by \( \sigma' \)).

As usual there exists a similar object \( \rho' \) for the right invariant functional \( \psi \), i.e. \( \rho' \) is an automorphism on \( A \) such that \( \psi(ab) = \psi(b\rho'(a)) \) for every \( a, b \in A \).

Using the antipode, we can connect \( \rho \) and \( \rho' \) via the formula \( S\rho' = \rho S \). Furthermore, we have that \( S^2 \) commutes with \( \rho \) and \( \rho' \). The interplay between \( \rho, \rho' \) and \( \Delta \) is given by the following formulas:

\[
\Delta \rho = (S^2 \circ \rho) \Delta \quad \text{and} \quad \Delta \rho' = (\rho' \circ S^2) \Delta.
\]

It is also possible to introduce the modular function of our algebraic quantum group. This is an invertible element \( \delta \) in \( M(A) \) such that \( (\varphi \circ \iota)(\Delta(a)(1 \otimes b)) = \varphi(a) \delta b \) for every \( a, b \in A \).

Concerning the right invariant functional, we have that \( (i \circ \psi)(\Delta(a)(b \otimes 1)) = \psi(a) \delta^{-1} b \) for every \( a, b \in A \).

This modular function is, like in the classical group case, a one dimensional (generally unbounded) corepresentation of our algebraic quantum group:

\[
\Delta(\delta) = \delta \otimes \delta \quad \varepsilon(\delta) = 1 \quad S(\delta) = \delta^{-1}.
\]

As in the classical case, we can relate the left invariant functional to our right invariant functional via the modular function: we have for every \( a \in A \) that

\[
\varphi(S(a)) = \varphi(a\delta) = \mu \varphi(\delta a).
\]

Not surprisingly, we have also that \( \rho(\delta) = \rho'(\delta) = \mu^{-1} \delta \).

Another connection between \( \rho \) and \( \rho' \) is given by the equality \( \rho'(a) = \delta \rho(a) \delta^{-1} \) for all \( a \in A \).

We have also a property which says, loosely speaking, that every element of \( A \) has compact support. This result was first proven by myself but the (simpler) proof here is due to A. Van Daele.

**Proposition 1.4** Consider \( a_1, \ldots, a_n \in A \). Then there exists an element \( c \in A \) such that \( ca_i = a_i c = a_i \) for every \( i \in \{1, \ldots, n\} \).

**Proof**: Define the following subspace \( K \) of \( A^{2n} \):

\[
K = \{ (ba_1, \ldots, ba_n, a_1b_1, \ldots, a_nb) \mid b \in A \}.
\]

We have to prove that \( (a_1, \ldots, a_n, a_1, \ldots, a_n) \) belongs to \( K \).

Therefore, choose a linear functional \( \omega \) on \( A^{2n} \) which is 0 on \( K \). For every \( i \in \{1, \ldots, 2n\} \), we define \( \omega_i \in A' \) such that \( \omega_i(x) = \omega(0, \ldots, \hat{x}, \ldots, 0) \) for every \( x \in A \). It is clear that \( \omega(y) = \sum_{i=1}^{2n} \omega_i(y_i) \) for every \( y \in A^{2n} \).

Take \( d \in A \) such that \( \varphi(d) = 1 \). We have for every \( e \in A \) that

\[
e \left[ \sum_{i=1}^{n} (\omega_i \circ \iota)(\Delta(d)(a_{i} \otimes 1)) + \sum_{i=1}^{n} (\omega_{n+i} \circ \iota)((a_{i} \otimes 1)\Delta(d)) \right]
\]

\[
= \sum_{i=1}^{n} (\omega_i \circ \iota)((1 \otimes e)\Delta(d)(a_{i} \otimes 1)) + \sum_{i=1}^{n} (\omega_{n+i} \circ \iota)((a_{i} \otimes 1)(1 \otimes e)\Delta(d))
\]

which is 0 because \( \omega = 0 \) on \( K \) and \( (1 \otimes e)\Delta(d) \) belongs to \( A \). So we get that

\[
\sum_{i=1}^{n} (\omega_i \circ \iota)(\Delta(d)(a_{i} \otimes 1)) + \sum_{i=1}^{n} (\omega_{n+i} \circ \iota)((a_{i} \otimes 1)\Delta(d)) = 0.
\]
Applying \( \varphi \) to this equality gives that
\[
0 = \sum_{i=1}^{n} \omega_i((\iota \circ \varphi)((\Delta(d))(a_i \otimes 1))) + \sum_{i=1}^{n} \omega_{n+i}((\iota \circ \varphi)((a_i \otimes 1)\Delta(d)))
\]
\[
= \sum_{i=1}^{n} \omega_i(a_i) + \sum_{i=1}^{n} \omega_{n+i}(a_i) = \omega(a_1, \ldots, a_n, a_1, \ldots, a_n).
\]

From this all, we infer that \((a_1, \ldots, a_n, a_1, \ldots, a_n)\) belongs to \(K\).

In a last part of this section, we are going to say something about duality.

**Definition 1.5** We define the subspace \( \hat{A} \) of \( A' \) as follows:
\[
\hat{A} = \{ \varphi a \mid a \in A \} = \{ a\varphi \mid a \in A \}.
\]
The last equality results from the fact that \( \varphi a = \rho(a)\varphi \) for every \( a \in A \).

Because \( \psi = \delta\varphi = \mu\varphi\delta \), we have also that
\[
\hat{A} = \{ \psi a \mid a \in A \} = \{ a\psi \mid a \in A \}.
\]
The faithfulness of \( \varphi \) implies that \( \hat{A} \) separates \( A \).

Like in the theory of Hopf-algebras, we want to turn \( \hat{A} \) into a non-degenerate algebra:

- We know already that \((\iota \circ \omega)\Delta(a)\) and \((\omega \circ \iota)\Delta(a)\) belong to \(A\) for every \(a \in A\) and \(\omega \in \hat{A}\).
- Choose \(\omega_1, \omega_2 \in \hat{A}\). Then there exist \(a_1, a_2 \in A\) such that \(\omega_1 = \varphi a_1\) and \(\omega_2 = \varphi a_2\). Then it is not difficult to see that
\[
\omega_1((\iota \circ \omega_2)\Delta(x)) = (\varphi \circ \varphi)((\Delta(x))(a_1 \otimes a_2)) = \omega_2((\omega_1 \circ \iota)\Delta(x))
\]
for every \(x \in A\).

We have the following definition (see propositions 4.2 and 4.3 of [13]).

**Definition 1.6** We can turn \( \hat{A} \) into a non-degenerate algebra such that \((\omega_1\omega_2)(a) = \omega_1((\iota \circ \omega_2)\Delta(a)) = \omega_2((\omega_1 \circ \iota)\Delta(a))\) for every \(\omega_1, \omega_2 \in \hat{A}\) and \(a \in A\).

In a next step, a comultiplication is introduced on the level of \( \hat{A} \). Because \( \hat{A} \) is a subspace of \( A' \), we regard \( \hat{A} \circ \hat{A} \) in the obvious way as a subspace of \((A \circ A)'\). Therefore we can formulate the next proposition (proposition 4.7 of [13]).

**Proposition 1.7** There exists a unique non-degenerate homomorphism \( \hat{\Delta} \) from \( \hat{A} \) into \( M(\hat{A} \circ \hat{A}) \) such that we have for every \( \omega_1, \omega_2 \) in \( \hat{A} \) that

1. The element \( \hat{\Delta}(\omega_1)(1 \otimes \omega_2) \) belongs to \( \hat{A} \circ \hat{A} \) and \( [\hat{\Delta}(\omega_1)(1 \otimes \omega_2)](x \otimes y) = (\omega_1 \circ \omega_2)((x \otimes 1)\Delta(y)) \) for every \(x, y \in A\).
2. The element \( (\omega_1 \otimes 1)\hat{\Delta}(\omega_2) \) belongs to \( \hat{A} \circ \hat{A} \) and \( [(\omega_1 \otimes 1)\hat{\Delta}(\omega_2)](x \otimes y) = (\omega_1 \circ \omega_2)((\Delta(x))(1 \otimes y)) \) for every \(x, y \in A\).
Next we define the linear functionals \( \hat{\varphi} \) and \( \hat{\psi} \) on \( \hat{A} \) which will play the role of left and right invariant functionals on \((\hat{A}, \hat{\Delta})\).

**Definition 1.8** We define the linear functionals \( \hat{\varphi} \) and \( \hat{\psi} \) on \( \hat{A} \) such that we have for every \( a \in A \) that 
\[
\hat{\varphi}(\psi a) = \varepsilon(a) \text{ and } \hat{\psi}(\varphi a) = \varepsilon(a).
\]

Now we can formulate a major result of \([13]\).

**Theorem 1.9** We have that \((\hat{A}, \hat{\Delta})\) is an algebraic quantum group with left Haar functional \( \hat{\varphi} \) and right Haar functional \( \hat{\psi} \).

**Remark 1.10** The counit and antipode on \((\hat{A}, \hat{\Delta})\) are determined by the following formulas:
- We have for every \( a \in A \) that \( \varepsilon(\varphi a) = \varepsilon(a \varphi) = \varphi(a) \) and \( \varepsilon(\psi a) = \varepsilon(a \psi) = \psi(a) \).
- We have for every \( \omega \in \hat{A} \) that \( \hat{S}(\omega) = \omega \circ S \).

The proof of these results can be found in \([13]\).

2 Implementing the multiplier algebra of the dual as a space of linear functionals

Consider an algebraic quantum group \((A, \Delta)\) with left Haar functional \( \varphi \). We will use notations an conventions of the previous section.

The non-degenerate algebra \( \hat{A} \) was introduced as a subset of \( A' \). In this section we will show that \( \mathcal{M}(\hat{A}) \) can also be implemented as a subset of \( A' \).

**Definition 2.1** We define left and right actions of \( \hat{A} \) on \( A' \) as follows. Consider \( \omega \in \hat{A} \) and \( \theta \in A' \). Then we define \( \omega\theta \) and \( \theta\omega \) in \( A' \) such that 
\[
(\omega\theta)(x) = \theta((\omega \circ \iota)\Delta(x)) \text{ and } (\theta\omega)(x) = \theta((\iota \circ \omega)\Delta(x)) \text{ for every } x \in A.
\]

If \( \theta \) belongs to \( \hat{A} \), these definitions of \( \omega\theta \) and \( \theta\omega \) correspond to the ones given before.

The associativity of the product on \( \hat{A} \) will be essential in the proof of the following lemma.

**Lemma 2.2** Consider \( \omega_1, \omega_2 \in \hat{A} \) and \( \theta \in A' \). Then

1. \( (\omega_1 \omega_2)\theta = \omega_1(\omega_2\theta) \)
2. \( \theta(\omega_1 \omega_2) = (\theta\omega_1)\omega_2 \)
3. \( (\omega_1)\omega_2 = \omega_1(\theta\omega_2) \)

**Proof:** Choose \( x \in A \). We have for every \( \omega \in \hat{A} \) that
\[
\omega((\omega_1 \omega_2 \circ \iota)\Delta(x)) = ((\omega_1 \omega_2)\omega)(x) = (\omega_1(\omega_2\omega))(x) = (\omega_2\omega)((\omega_1 \circ \iota)\Delta(x)) = \omega((\omega_2 \circ \iota)\Delta((\omega_1 \circ \iota)\Delta(x))) .
\]

Because \( \hat{A} \) separates \( A \), this implies that 
\[
(\omega_1 \omega_2 \circ \iota)\Delta(x) = (\omega_2 \circ \iota)\Delta((\omega_1 \circ \iota)\Delta(x)) .
\]

Applying \( \theta \) to this equation gives in a similar way as above that 
\[
((\omega_1 \omega_2)\theta)(x) = (\omega_1(\omega_2\theta))(x) .
\]
So we see that 
\[
(\omega_1 \omega_2)\theta = \omega_1(\omega_2\theta) .
\]

The other equalities are proven in an analogous way.
Lemma 2.3 Consider \( \theta \in A' \) such that \( \theta \omega = 0 \) for every \( \omega \in \hat{A} \). Then \( \theta = 0 \).

This follows easily because we can write every element of \( A \) as a sum of elements \((\iota \circ \varphi)a\Delta(b)\) with \(a, b \in A\). Of course, a similar separation result applies for left multiplication with elements of \( A\).

We will use the temporary notation

\[ \hat{A} = \{ \theta \in A' \mid \text{We have for every } \omega \in \hat{A} \text{ that } \omega \theta \text{ and } \theta \omega \text{ belong to } \hat{A} \} . \]

It is clear that \( \hat{A} \) is a subset of \( \hat{A} \).

Let \( a \) be an element in \( A \). For the purpose of clarity, we will use the notations \( \varphi[a] = \varphi a \) and \([a]\psi = a\psi\).

Lemma 2.4 Consider \( a \in A \) and \( \theta \in A' \). Then the following equalities hold:

\[
\begin{align*}
\bullet \quad \theta(\varphi a) &= \varphi [S^{-1}(\iota \circ \theta)\Delta(S(a))] & \bullet \quad \theta(a\varphi) &= \varphi [S((\iota \circ \theta)\Delta(S^{-1}(a)))] \\
\bullet \quad (\psi a)\theta &= \psi [S((\iota \circ \iota)\Delta(S^{-1}(a)))] & \bullet \quad (a\psi)\theta &= [S^{-1}((\theta \circ \iota)\Delta(S(a)))\psi]
\end{align*}
\]

Proof: We have for every \( x \in A \) that

\[
\begin{align*}
(\theta(\varphi a))(x) &= \theta((\iota \circ \varphi a)\Delta(x)) = \theta((\iota \circ \varphi)((1 \circ a)\Delta(x))) \\
&= \varphi((\theta S \circ \iota)(\Delta(a)(1 \circ x))) = \varphi((\theta S \circ \iota)(\Delta(a)) x),
\end{align*}
\]

where we used equation \([\text{III}]\) in (*). So

\[
\theta(\varphi a) = \varphi [(\theta S \circ \iota)\Delta(a)] = \varphi [S^{-1}((\iota \circ \theta)\Delta(S(a)))],
\]

where we used the equality \((S \circ S)\Delta = S\Delta S\) in the last equation.

Using also equation \([\text{IV}]\), we find in a similar way the other results.

Lemma 2.5 Let \( \theta \) be a linear functional on \( A \). Then \( \theta \) belongs to \( \hat{A} \) \( \iff \) We have for every \( x \in A \) that \((\theta \circ \iota)\Delta(x)\) and \((\iota \circ \theta)\Delta(x)\) belong to \( A \).

Proof:

\(\Rightarrow\) Choose \( a \in A \). By assumption, there exists \( b \in A \) such that \( \theta(\varphi a) = \varphi b \).

This implies by the previous lemma that \( \varphi [S^{-1}((\iota \circ \theta)\Delta(S(a)))] = \varphi b \). So the faithfulness of \( \varphi \) implies that \( S^{-1}((\iota \circ \theta)\Delta(S(a))) = b \), so \( (\iota \circ \theta)\Delta(S(a)) = S(b) \) which belongs to \( A \).

Using \( \psi \) instead of \( \varphi \), we prove in a similar way that \((\theta \circ \iota)\Delta(S(a))\) belongs to \( A \).

\(\Leftarrow\) Choose \( a \in A \). We have by assumption that \( S^{-1}((\iota \circ \theta)\Delta(S(a))) \) belongs to \( A \).

By the previous lemma, we get that \( \theta(\varphi a) = \varphi [S^{-1}((\iota \circ \theta)\Delta(S(a)))\psi] \) which belongs to \( \hat{A} \).

Similarly, we get that \((\psi a)\theta\) belongs to \( \hat{A} \).

Because of lemma \([\text{2.2}]\), we can define a mapping \( F \) from \( \hat{A} \) into \( M(\hat{A}) \) such that we have for every \( \theta \in \hat{A} \) and \( \omega \in A \) that \( F(\theta)\omega = \theta \omega \) and \( \omega F(\theta) = \omega \theta \). Then we have that \( F(\theta) = \theta \) for every \( \theta \in \hat{A} \).

It is easy to check that \( F \) is linear and injective (injectivity follows from lemma \([\text{2.3}]\)). We will prove that \( F \) is also surjective.

Remember that we defined the non-zero linear functional \( \hat{\varphi} \) on \( \hat{A} \) such that \( \hat{\varphi}(\psi a) = \varepsilon(a) \) for every \( a \in A \) (definition \([\text{1.8}]\) ).
Lemma 2.6 Consider \( a \in A \) and \( \theta \in A' \). Then \( \varepsilon((\iota \otimes \theta)\Delta(a)) = \varepsilon((\theta \otimes \iota)\Delta(a)) = \theta(a) \).

Proof: There exist \( c \in A \) such that \( \varepsilon(c) = 1 \). Then

\[
\varepsilon((\iota \otimes \theta)\Delta(a)) = \varepsilon((\iota \otimes \theta)\Delta(a)) \varepsilon(c) = \varepsilon((\iota \otimes \theta)(\Delta(a))c) = \varepsilon((\iota \otimes \theta)(\Delta(a)c \otimes 1)) = \theta(\varepsilon(c)a) = \theta(a) .
\]

The other equality is proven in a similar way.

Lemma 2.7 Consider \( a \in A \) and \( \omega \in \hat{A} \). Then \( \hat{\varphi}((\psi a) \omega) = \omega(S^{-1}(a)) \).

Proof: By lemmas 2.4 and 2.5, we know that \( (\psi a) \omega = \psi[S((\omega \otimes \iota)\Delta(S^{-1}(a)))] \) and that \( S((\omega \otimes \iota)\Delta(S^{-1}(a))) \) belongs to \( A \). Therefore

\[
\hat{\varphi}((\psi a) \omega) = \varepsilon(S((\omega \otimes \iota)\Delta(S^{-1}(a)))) = \varepsilon((\omega \otimes \iota)\Delta(S^{-1}(a)))
\]

which by the previous lemma equals \( \omega(S^{-1}(a)) \).

This lemma suggests a solution for the proof of the surjectivity of \( F \). First, we need another lemmas.

Lemma 2.8 Consider \( b \in A \) and \( \omega \in \hat{A} \). Then \( \hat{\varphi}((\psi \delta S^2(b)) \omega) = \hat{\varphi}(\omega(\psi b)) \).

Proof: Take \( a \in A \) such that \( \omega = \psi a \). By the previous lemma we have that

\[
\hat{\varphi}((\psi \delta S^2(b))(\psi a)) = (\psi a)(S^{-1}(\delta S^2(b))) = (\psi a)(S(b)\delta^{-1}) = \psi(aS(b)\delta^{-1}) = \varphi(aS(b)) = \psi(S^{-1}(a)) = (\psi b)(S^{-1}(a)) = \hat{\varphi}((\psi a)(\psi b)),
\]

where we used the previous lemma for a second time in the last equality.

Lemma 2.9 Consider \( b, x \in A \). Then \( \psi[S((\iota \otimes \psi b)\Delta(x)) ] = (\psi \delta S^2(b))(\psi S(x)) \).

Proof: By lemma 2.4, we know that

\[
(\psi \delta S^2(b))(\psi S(x)) = \psi[S((\psi S(x) \otimes \iota)\Delta(S^{-1}(\delta S^2(b)))]) = \psi[S((\psi S(x) \otimes \iota)\Delta(S(b)\delta^{-1}))].
\]

But we have that

\[
S((\psi S(x) \otimes \iota)\Delta(S(b)\delta^{-1})) = S((\psi \otimes \iota)((S(x) \otimes 1)\Delta(S(b)\delta^{-1})))
\]

\[
= (\varphi \otimes \iota)(\Delta(S(x))(S(b)\delta^{-1} \otimes 1)) = (\delta^{-1}\psi \otimes \iota)(\Delta(S(x))(S(b) \otimes 1))
\]

\[
= (\varphi \otimes \iota)(\chi(S \otimes S)((1 \otimes b)\Delta(x))) = S((\iota \otimes \varphi S)((1 \otimes b)\Delta(x)))
\]

\[
S((\iota \otimes \varphi S)((1 \otimes b)\Delta(x))) = S((\iota \otimes \varphi S)(\delta^{-1} \psi \otimes \iota)(\Delta(S(x))(S(b) \otimes 1)))
\]

where we used equation 2.4 in equality (*). Combining these two results, the lemma follows.

We get now to a proposition of which the proof is rather straightforward.

Proposition 2.10 The mapping \( F \) is a isomorphism of vector spaces from \( \hat{A} \) to \( M(\hat{A}) \).
Proof: Choose $T \in M(\hat{A})$. Define the element $\theta \in A'$ such that $\theta(x) = \hat{\varphi}(\psi S(x))T$ for every $x \in A$ (This is suggested by lemma 2.7). Choose $b \in A$.

1. First we prove that $(\psi b)\theta = (\psi b)T$

Choose $x \in A$. Using definition 2.1, we get that

$$(\psi b)\theta(x) = \theta((\psi b \circ \iota)\Delta(x)) = \hat{\varphi}(\psi S((\psi b \circ \iota)\Delta(x)))T.$$ 

By lemma 2.4, we have that $\psi [S((\psi b \circ \iota)\Delta(x))] = (\psi S(x))(\psi b)$. Therefore, the associativity of the product in $M(\hat{A})$ and lemma 2.7 imply that

$$(\psi b)\theta(x) = \hat{\varphi}((\psi S(x))((\psi b)T)) = (\psi S(x))((\psi b)T) = ((\psi b)T)(x).$$

So we see that $(\psi b)\theta = (\psi b)T$.

2. Next we prove that $\theta(\psi b) = T(\psi b)$.

Choose $x \in A$. Then definition 2.1 implies that

$$\theta(\psi b)(x) = \hat{\varphi}((\psi S((\psi \delta S^2(b)))(\psi S(x))).$$

The previous lemma implies that $\psi S((\psi \delta S^2(b))(\psi S(x))) = (\psi \delta S^2(b))(\psi S(x))$.

So we see that

$$\theta(\psi b)(x) = \hat{\varphi}((\psi S(x))T(\psi b)) = (\psi S(x))[T(\psi b)] = (T(\psi b))(x).$$

Using lemma 2.8 and 2.7, this equation implies that

$$\theta(\psi b)(x) = \hat{\varphi}((\psi S(x))T) = \hat{\varphi}(\psi S(x))[T(\psi b)] = (T(\psi b))(x).$$

So we arrive at the conclusion that $\theta(\psi b) = T(\psi b)$

Looking at the definition of $\hat{A}$ and $F$, we get from these two equations easily that $\theta$ belongs to $\hat{A}$ and that $F(\theta) = T.$

Now we will use the mapping $F$ to turn $\hat{A}$ into an algebra. We define on $\hat{A}$ a product operation such that $F(\theta_1 \theta_2) = F(\theta_1)F(\theta_2)$ for every $\theta_1, \theta_2 \in \hat{A}$.

**Proposition 2.11** We have for every $\theta_1, \theta_2 \in \hat{A}$ and $x \in A$ that $(\theta_1 \theta_2)(x) = \theta_1((\iota \circ \theta_2)\Delta(x)) = \theta_2((\iota \circ \theta_1)\Delta(x))$.

**Proof:** Choose $a \in A$. Using lemma 2.4 twice, we get that

$$F(\theta_1 \theta_2)(\varphi a) = (F(\theta_1)F(\theta_2))(\varphi a) = F(\theta_1)(F(\theta_2)(\varphi a))$$

$$= F(\theta_1)(\theta_2(\varphi a)) = F(\theta_1)(\varphi [S^{-1}((\iota \circ \theta_2)\Delta(S(a)))])$$

$$= \theta_1 \varphi [S^{-1}((\iota \circ \theta_2)\Delta(S(a)))]$$

$$= \varphi [S^{-1}((\iota \circ \theta_1)\Delta((\iota \circ \theta_2)\Delta(S(a)))).$$

On the other hand, lemma 2.4 implies also that

$$F(\theta_1 \theta_2)(\varphi a) = (\theta_1 \theta_2)(\varphi a) = \varphi [S^{-1}((\iota \circ \theta_1 \theta_2)\Delta(S(a)))] .$$
Comparing these two different expressions for $F(\theta_1\theta_2)$ and remembering that $\varphi$ is faithful, we get that

$$S^{-1}( (\iota \circ \theta_1)\Delta((\iota \circ \theta_2)\Delta(S(a)))) = S^{-1}( (\iota \circ \theta_1\theta_2)\Delta(S(a))) ,$$

which implies that

$$(\iota \circ \theta_1)\Delta((\iota \circ \theta_2)\Delta(S(a))) = (\iota \circ \theta_1\theta_2)\Delta(S(a)) .$$

Applying $\varepsilon$ to this equation and using lemma 2.6, we see that $\theta_1( (\iota \circ \theta_2)\Delta(S(a))) = (\theta_1\theta_2)(S(a))$. The other equality is proven in a similar way. 

Until now, we used $M(\hat{A})$ as an abstract object. From now on, we will use $\hat{A}$ as a concrete realization of $M(\hat{A})$ and want to forget about the mapping $F$. This will be formulated in the next theorem.

**Theorem 2.12** As a vector space, $M(\hat{A})$ is equal to

$$\{ \theta \in A' \mid \text{ We have for every } a \in A \text{ that } (\theta \circ \iota)\Delta(a) \text{ and } (\iota \circ \theta)\Delta(a) \text{ belong to } A \} .$$

The product operation on $M(\hat{A})$ is defined in such a way that $(\theta_1\theta_2)(a) = \theta_1((\iota \circ \theta_2)\Delta(a)) = \theta_2((\iota \circ \theta_1)\Delta(a))$ for every $\theta_1, \theta_2 \in M(\hat{A})$ and $a \in A$. Furthermore, $\varepsilon$ is the unit of $M(\hat{A})$.

The last statement of the proposition is easy to check. The others have been proven in the previous part of the section.

**Remark 2.13** Consider $a \in A$ and $\omega \in A$. Using the previous theorem, it is easy to check that $\omega a$ and $a\omega$ belong to $M(\hat{A})$.

From this section, we only need to remember the previous theorem and the following results. The rest of this section was intended to prove these results.

We can also extend the left and right actions of $\hat{A}$ on $A'$ to left and right actions of $M(\hat{A})$ on $A'$ in the obvious way.

**Definition 2.14** Consider $\omega \in M(\hat{A})$ and $\theta \in A'$. We define $\omega\theta$ and $\theta\omega$ in $A'$ such that $(\omega\theta)(a) = \theta((\omega \circ \iota)\Delta(a))$ and $(\theta\omega)(a) = \theta((\iota \circ \omega)\Delta(a))$ for every $a \in A$.

This definition implies for instance that $((\omega b)\theta)(x) = (\omega \circ \theta)((b \circ 1)\Delta(a))$ for every $\omega, \theta \in A'$ and $a, b \in A$. Another consequence is the equality $\theta\varepsilon = \varepsilon\theta = \theta$ for every $\theta \in A'$.

The proof of the following proposition is the same as the proof of lemma 2.2.

**Proposition 2.15** Consider $\omega_1, \omega_2 \in M(\hat{A})$ and $\theta \in A'$. Then

1. $(\omega_1\omega_2)\theta = \omega_1(\omega_2\theta)$
2. $\theta(\omega_1\omega_2) = (\theta\omega_1)\omega_2$
3. $(\omega_1\theta)\omega_2 = \omega_1(\theta\omega_2)$

Remember that we have also the following result.

**Proposition 2.16** Consider $\theta \in A'$. Then $\theta$ belongs to $M(\hat{A})$ if and only if we have for every $\omega \in \hat{A}$ that $\theta\omega$ and $\omega\theta$ belong to $\hat{A}$.
3 A nice implementation of the comultiplication on the dual

Consider an algebraic quantum group \((A, \Delta)\) with a left Haar functional \(\varphi\) (we will use the notations for counit, antipode, ... as in the previous sections). The results of the previous section will be used to get nice formulas for the comultiplication, counit and antipode on \(\hat{A}\).

As can be expected, we put on \(A \otimes A\) a comultiplication \(\Delta\) such that \(\Delta(a \otimes b) = (\iota \otimes \chi \otimes \iota)(\Delta(a) \otimes \Delta(b))\) for every \(a, b \in A\). (The use of the same symbol for the comultiplication on \(A\) as for the comultiplication on \(A \otimes A\) should not cause any confusion because we always know on which elements they work.)

It is not so hard to check that \((\hat{A}, \Delta, \hat{\varphi}, \hat{\epsilon})\) is again an algebraic quantum group with counit \(\iota \otimes \epsilon\), antipode \(S \otimes S\) and left Haar functional \(\varphi \otimes \varphi\). Moreover, \((\hat{A} \otimes \hat{A})'\) will be equal to \(\hat{A} \otimes \hat{A}\).

By the results of the previous section, we know that \(M(\hat{A} \otimes \hat{A})\) is a subset of \((A \otimes A)'\) and this will be essential to the next proposition.

**Lemma 3.1** Let \(a, b\) be elements of \(A\) and \(\omega, \theta\) elements in \(A'\). Then \((\iota \otimes \iota \otimes \omega \otimes \theta)\Delta(a \otimes b) = (\iota \otimes \omega)\Delta(a) \otimes (\iota \otimes \theta)\Delta(b)\) and \((\omega \otimes \theta \otimes \iota \otimes \iota)\Delta(a \otimes b) = (\omega \otimes \iota)\Delta(a) \otimes (\theta \otimes \iota)\Delta(b)\).

**Proof:** Choose \(c, d \in A\). Then we have by definition that

\[
(c \otimes d) (\iota \otimes \iota \otimes \omega \otimes \theta)\Delta(a \otimes b) = (\iota \otimes \iota \otimes \omega \otimes \theta)((c \otimes d \otimes 1 \otimes 1)\Delta(a \otimes b)) .
\]

We have also that

\[
(c \otimes d) (\iota \otimes \iota \otimes \omega \otimes \theta)\Delta(a \otimes b) = (\iota \otimes \iota \otimes \omega \otimes \theta)((c \otimes 1 \otimes (d \otimes 1)\Delta(b)) .
\]

Notice that \((c \otimes 1)\Delta(a)\) and \((d \otimes 1)\Delta(b)\) belong to \(A \otimes A\). So we get that

\[
(c \otimes d) (\iota \otimes \iota \otimes \omega \otimes \theta)\Delta(a \otimes b) = (\iota \otimes \iota \otimes \omega \otimes \theta)((c \otimes 1)\Delta(a) \otimes (d \otimes 1)\Delta(b)) .
\]

So we get that \((\iota \otimes \iota \otimes \omega \otimes \theta)\Delta(a \otimes b) = (\iota \otimes \omega)\Delta(a) \otimes (\iota \otimes \theta)\Delta(b)\). The other equality is proven in a similar way.

**Proposition 3.2** Consider \(\omega \in \hat{A}\). Then we have for every \(a, b \in A\) that \(\hat{\Delta}(\omega)(a \otimes b) = \omega(ab)\).

**Proof:** Define \(\theta \in (A \otimes A)'\) such that \(\theta(a \otimes b) = \omega(ab)\) for every \(a, b \in A\).

Take \(\eta_1, \eta_2 \in \hat{A}\). Choose \(x, y \in A\). We know that \((\iota \otimes \eta_2)\Delta(y)\) belongs to \(A\). By definition \((2.14)\) we get that

\[
(\theta (\iota \otimes \eta_2))(x \otimes y) = \theta((\iota \otimes \iota \otimes \iota \otimes \eta_2)\Delta(x \otimes y)) = \theta((\iota \otimes \iota \otimes \iota \otimes \eta_2)\Delta(x \otimes y)) = \theta((\iota \otimes \eta_2)\Delta(y)) = \theta((\iota \otimes \eta_2)\Delta(y)) = \omega((\iota \otimes \eta_2)((x \otimes 1)\Delta(y))) = (\omega \otimes \eta_2)((x \otimes 1)\Delta(y)) .
\]

Hence, proposition \((2.7)\) implies that

\[
(\theta (\iota \otimes \eta_2))(x \otimes y) = (\hat{\Delta}(\omega)(\iota \otimes \eta_2))(x \otimes y) .
\]

Therefore, we get that \(\theta (\iota \otimes \eta_2) = \hat{\Delta}(\omega)(\iota \otimes \eta_2)\). Multiplying this equation with \(\eta_1 \otimes \epsilon\) to the right and using proposition \((2.13)\), we see that

\[
\theta (\eta_1 \otimes \eta_2) = \hat{\Delta}(\omega)(\eta_1 \otimes \eta_2) .
\]

By lemma \((2.3)\), this implies that \(\theta = \hat{\Delta}(\omega)\).

We can even do better:
Proposition 3.3 Consider \( \omega \in M(\hat{A}) \). Then we have for every \( a, b \in A \) that \( \hat{\Delta}(\omega)(a \otimes b) = \omega(ab) \).

Proof: Define \( \theta \in (A \otimes A)' \) such that \( \theta(a \otimes b) = \omega(ab) \) for every \( a, b \in A \).
Choose \( y \in A \) and \( \omega_1, \omega_2 \in \hat{A} \). Then we have for \( a, b \in A \) that

\[
(\theta \hat{\Delta}(\varphi y))(a \otimes b) = \theta(\iota \otimes \iota \otimes \hat{\Delta}(\varphi y))\Delta(a \otimes b) .
\] (a)

Notice that \((1 \otimes 1 \otimes y)\Delta(a)_{13}\Delta(b)_{23}\) belongs to \(A \otimes A \otimes A\). We have for every \( c, d \in A \) that

\[
[(\iota \otimes \iota \otimes \hat{\Delta}(\varphi y))\Delta(a \otimes b)](c \otimes d) = (\iota \otimes \iota \otimes \hat{\Delta}(\varphi y))(\Delta(a \otimes b)(c \otimes d \otimes 1 \otimes 1))
= (\iota \otimes \iota \otimes \hat{\Delta}(\varphi y))(\iota \otimes \chi \otimes \iota)(\Delta(a)(c \otimes 1) \otimes \Delta(b)(d \otimes 1))
= (\iota \otimes \iota \varphi)((1 \otimes 1 \otimes y)(\Delta(a)(c \otimes 1))\Delta(b)(d \otimes 1))_{13}\Delta(b)(d \otimes 1)_{23}
= (\iota \otimes \iota \varphi)((1 \otimes 1 \otimes y)\Delta(a)_{13}\Delta(b)_{23}(c \otimes d) .
\]

Here we used the previous lemma in equality (*).
So we see that

\[
(\iota \otimes \iota \otimes \hat{\Delta}(\varphi y))\Delta(a \otimes b) = (\iota \otimes \iota \varphi)((1 \otimes 1 \otimes y)\Delta(a)_{13}\Delta(b)_{23}) .
\]

Plugging this equality in equality (a) gives

\[
(\theta \hat{\Delta}(\varphi y))(a \otimes b) = \varphi((1 \otimes 1 \otimes y)\Delta(a)_{13}\Delta(b)_{23})) .
\] (b)

The definition of \( \theta \) implies for every \( e \in A \) that

\[
(\theta \otimes \iota)((1 \otimes 1 \otimes y)\Delta(a)_{13}\Delta(b)_{23})e = (\theta \otimes \iota)([(1 \otimes y)\Delta(a)]_{13}[\Delta(b)(1 \otimes e)]_{23})
= (\omega \otimes \iota \otimes \iota)([(1 \otimes y)\Delta(a)] [\Delta(b)(1 \otimes e)])
= (\omega \otimes \iota \otimes \iota)((1 \otimes y)\Delta(ab))e .
\]

So we get that \((\theta \otimes \iota)((1 \otimes 1 \otimes y)\Delta(a)_{13}\Delta(b)_{23}) = (\omega \otimes \iota)((1 \otimes y)\Delta(ab))\). Using this in equality (b), gives us that

\[
(\theta \hat{\Delta}(\varphi y))(a \otimes b) = (\omega \otimes \varphi)((1 \otimes y)\Delta(ab)) = (\omega(\varphi y))(ab) .
\]

Therefore, the previous lemma implies that

\[
(\theta \hat{\Delta}(\varphi y))(a \otimes b) = \hat{\Delta}(\omega(\varphi y))(a \otimes b) .
\]

From this all we get that

\[
\theta \hat{\Delta}(\varphi y) = \hat{\Delta}(\omega(\varphi y)) = \hat{\Delta}(\omega)\hat{\Delta}(\varphi y) .
\]

Multiplying this equation from the right by \( \omega_1 \otimes \omega_2 \) and using proposition 3.3, results in the equality

\[
\theta [\hat{\Delta}(\varphi y)(\omega_1 \otimes \omega_2)] = \hat{\Delta}(\omega)[\hat{\Delta}(\varphi y)(\omega_1 \otimes \omega_2)] .
\]

Using the non-degeneracy of \( \hat{\Delta} \) and lemma 2.14, we infer from this all that \( \hat{\Delta}(\omega) = \theta \).

This proposition implies easily the following result.

Result 3.4 Consider \( \omega \in M(\hat{A}) \) and \( \omega_1, \omega_2 \in M(\hat{A}) \). Then \([\hat{\Delta}(\omega)(\omega_1 \otimes \omega_2)](x \otimes y) = \omega(\iota \otimes \omega_1)\Delta(x)(\iota \otimes \omega_2)\Delta(y) \) for every \( x, y \in A \).
Proof: Remember that \((\iota \otimes \omega_1)\Delta(x)\) and \((\iota \otimes \omega_2)\Delta(y)\) belong to \(A\). So we get that
\[
[\hat{\Delta}(\omega)(\omega_1 \otimes \omega_2)](x \otimes y) = \hat{\Delta}(\omega)((\iota \otimes \iota \otimes \omega_1 \otimes \omega_2)\Delta(x \otimes y))
= \hat{\Delta}(\omega)((\iota \otimes \omega_1)\Delta(x) \otimes (\iota \otimes \omega_2)\Delta(y)) = \omega((\iota \otimes \omega_1)\Delta(x) \otimes (\iota \otimes \omega_2)\Delta(y)) ,
\]
where we used the previous proposition in the last equality.

Remark 3.5 A special case of the foregoing result is the case where \(\omega_1 = \varepsilon\) or \(\omega_2 = \varepsilon\). This gives rise to the following equalities:

Consider \(\omega_1, \omega_2 \in M(\hat{A})\). Then we have for every \(x, y \in A\) that
\[
[\hat{\Delta}(\omega_1)\varepsilon \otimes \omega_2)](x \otimes y) = (\omega_1 \otimes \omega_2)((x \otimes 1)\Delta(y)) \text{ and } [\hat{\Delta}(\omega_2)\omega_1 \otimes \varepsilon)](x \otimes y) = (\omega_1 \otimes \omega_2)(\Delta(x)(y \otimes 1)).
\]

Of course, a similar calculation as in the proof gives rise to formulas for \((\omega_1 \otimes \omega_2)\hat{\Delta}(\omega)\) for every \(\omega, \omega_1, \omega_2 \in M(\hat{A})\).

In the last propositions of this section, we prove the usual formulas for the counit and antipode on \(M(\hat{A})\).

Proposition 3.6 We have for every \(\omega \in A'\) and \(a \in A\) that \(\hat{\varepsilon}(\omega a) = \omega(a)\) and \(\hat{\varepsilon}(a \omega) = \omega(a)\).

Proof: Take \(b \in A\) such that \(\varphi(b) = 1\). Then \(\hat{\varepsilon}(\varphi b) = \varphi(b) = 1\.

By lemma 2.4, we have that \((\omega a)\varphi b = \varphi[S^{-1}(\iota \varphi a)\Delta(S(b))]\). This implies that
\[
\hat{\varepsilon}(\omega a) = \hat{\varepsilon}((\omega a)(\varphi b)) = \hat{\varepsilon}(\varphi[S^{-1}(\iota \varphi a)\Delta(S(b))]) = \varphi(S^{-1}(\iota \varphi a)\Delta(S(b)')) = (\varphi S^{-1})(\iota \varphi a)(1 \otimes a)\Delta(S(b'))) = \omega((\varphi S^{-1} \iota)(1 \otimes a)\Delta(S(b'))) .
\]

Using the right invariance of \(\varphi S^{-1}\), this implies that
\[
\hat{\varepsilon}(\omega a) = \omega(a (\varphi S^{-1}(\omega(S(b)))) = \omega(a \varphi b) = \omega(a) .
\]

The other equality is proven in a similar way.

Proposition 3.7 Consider \(\omega \in M(\hat{A})\). Then \(\hat{S}(\omega) = \omega \circ S\) and \(\hat{S}^{-1}(\omega) = \omega \circ S^{-1}\).

Proof: It is not difficult to check that \(\hat{S}(\varphi c) = (\varphi c) \circ S = S^{-1}(c)\psi\) for every \(c \in A\).

Take \(b \in A\).

Using lemma 2.4 once again, we see that \(\omega(\varphi b) = \varphi[S^{-1}(\iota \varphi a)\Delta(S(b))]\). By the remark in the beginning of this proof, we see that
\[
\hat{S}(\varphi b) \hat{S}(\omega) = \hat{S}(\omega(b)) = [S^{-2}((\iota \varphi a)\Delta(S(b)))] \psi .
\]

On the other hand, \(\hat{S}(\varphi b) \hat{S}(\omega) = (S^{-1}(b)\psi) \hat{S}(\omega)\) which by lemma 2.4 implies that
\[
\hat{S}(\varphi b) \hat{S}(\omega) = [S^{-1}((\hat{S}(\omega) \varphi a)\Delta(b))] \psi .
\]

Comparing these two expressions for \(\hat{S}(\varphi b) \hat{S}(\omega)\) and using the faithfulness of \(\psi\), we arrive at the conclusion that
\[
S^{-2}((\iota \varphi a)\Delta(S(b))) = S^{-1}((\hat{S}(\omega) \varphi a)\Delta(b)) .
\]

If we apply \(\varepsilon\) to this equation and use lemma 2.4, we see that \(\omega(S(b)) = \hat{S}(\omega)(b)\).

So we see that \(\hat{S}(\omega) = \omega \circ S\).

The result concerning \(\hat{S}^{-1}\) follows immediately from the result concerning \(\hat{S}\).
4 Slicing with certain functionals

In this section, we will consider an algebraic quantum group \((A, \Delta)\) with left Haar functional \(\varphi\).

Let \(B\) be a non-degenerate algebra and \(V\) an element of \(M(A \odot B)\). In this section, we want to define a left multiplier \((\omega \odot \iota)(V)\) for certain functionals \(\omega\) on \(A\) and prove some familiar calculation rules about this kind of slicings.

**Notation 4.1** We define the set \(A^o\) as the vector space \(\langle a\omega | a \in A \rangle\). So \(A^o\) is a subspace of \(A'\).

By theorem 2.12, it is clear that \(A^o\) is a subspace of \(M(\hat{A})\). It is not so difficult to check that \(A^o\) is a subalgebra of \(M(\hat{A})\). We have also immediately that \(\hat{A}\) is a subset of \(A^o\).

We will use these functionals to make slicings. First we need an easy but useful result.

**Lemma 4.2** Consider a non-degenerate algebra \(B\) and \(V \in M(A \odot B)\). Let \(a_1, \ldots, a_n \in A\) and \(\omega_1, \ldots, \omega_n \in A'\) such that \(\sum_{i=1}^{n} a_i \omega_i = 0\). Then \(\sum_{i=1}^{n} (\omega_i \odot \iota)(V(a_i \otimes x)) = 0\) for every \(x \in B\).

**Proof:** Choose \(x \in B\). There exists \(e \in A\) such that \(ea_i = a_i\) for every \(i \in \{1, \ldots, n\}\). Then we have that \(V(e \otimes x)\) belongs to \(A \odot B\). This implies that

\[
\sum_{i=1}^{n} (\omega_i \odot \iota)(V(a_i \otimes x)) = \sum_{i=1}^{n} (\omega_i \odot \iota)(V(ea_i \otimes x)) = \sum_{i=1}^{n} (\omega_i \odot \iota)(V(e \otimes x)) = 0 .
\]

**Proposition 4.3** Consider a non-degenerate algebra \(B\) and \(V \in M(A \odot B)\). There exists a unique linear map \(G\) from \(A^o\) into the set of left multipliers on \(B\) such that \(G(\omega x) = (\omega \odot \iota)(V(a \otimes x))\) for every \(\omega \in A^o\), \(a \in A\) and \(x \in B\).

For every \(\theta \in A^o\), we put \((\theta \odot \iota)(V) = G(\theta)\), so \((\theta \odot \iota)(V)\) is a left multiplier on \(B\).

**Proof:** Using the definition of left multipliers on \(B\) it is not difficult to check for every \(a \in A\) and \(\omega \in A\) the existence of a unique left multiplier \(T(a, \omega)\) on \(B\) such that \(T(a, \omega) x = (\omega \odot \iota)(V(a \otimes x))\) for every \(x \in B\).

Let \(a_1, \ldots, a_m, b_1, \ldots, b_n \in A\), \(\omega_1, \ldots, \omega_m, \theta_1, \ldots, \theta_n \in A'\) such that \(\sum_{i=1}^{m} a_i \omega_i = \sum_{j=1}^{n} b_j \theta_j\). Then the previous lemma implies that \(\sum_{i=1}^{m} T(a_i, \omega_i) = \sum_{j=1}^{n} T(b_j, \theta_j)\).

So we can define a mapping \(G\) from \(A^o\) into the set of left multipliers on \(B\) such that \(G(\sum_{i=1}^{m} a_i \omega_i) = \sum_{i=1}^{m} T(a_i, \omega_i)\) for every \(a_1, \ldots, a_m \in A\) and \(\omega_1, \ldots, \omega_n \in A'\). It is easy to check that \(G\) satisfies the requirements of the proposition.

It is easy to see that this notation \((\theta \odot \iota)(V)\) is consistent with the usual notation in the case that \(V\) is an element of \(A \odot B\).

**Remark 4.4** Consider a non-degenerate algebra \(B\). We only need to remember the following obvious rules:

- The mapping \(A^o \times M(A \odot B) \rightarrow \text{the set of left multipliers of } B : (V, \omega) \mapsto (\omega \odot \iota)(V)\) is bilinear.
Consider $V \in M(A \otimes B)$, $a \in A$ and $\omega \in A'$. Then $(a\omega \circ \iota)(V)x = (\omega \otimes \iota)(V(a \otimes x))$ for every $x \in B$.

The usual separation results stay valid.

**Result 4.5** Consider a non-degenerate algebra $B$ and $V \in M(A \otimes B)$. Then $V = 0 \iff (\omega \circ \iota)(V) = 0$ for every $\omega \in \hat{A}$.

**Proof**: Suppose that $(\omega \circ \iota)(V) = 0$ for every $\omega \in \hat{A}$.

Take $a \in A$, $b \in B$. Choose $\eta \in B'$. Then we have for every $c \in A$ that

$$\varphi((\iota \circ \eta)(V(a \otimes b))c) = \varphi((\iota \circ \eta)(V(ac \otimes b)))$$

$$= \eta(\varphi(\iota \circ \eta)(V(ac \otimes b))) = (\iota \circ \eta)((\varphi \circ \iota)(V)b) = 0$$

The faithfulness of $\varphi$ implies that $(\iota \circ \eta)(V(a \otimes b)) = 0$. Hence, $V(a \otimes b) = 0$.

So we see that $V = 0$.

We will be mainly interested in the cases where $(\omega \circ \iota)(V)$ becomes an element in $M(B)$. The following result deals with a natural case. The proof of this result is straightforward and will therefore be left out.

**Result 4.6** Consider a non-degenerate algebra $B$ and $V \in M(A \otimes B)$. Let $a,b$ be elements in $A$ and $\omega$ an element in $A'$. Then $(a\omega b \circ \iota)(V)$ is an element of $M(B)$ and we have for every $x \in B$ that

$$x(a\omega b \circ \iota)(V) = (\omega \circ \iota)((b \otimes x)V(a \otimes 1))$$ and

$$(a\omega b \circ \iota)(V)x = (\omega \circ \iota)((b \otimes 1)V(a \otimes x)).$$

**Corollary 4.7** Consider a non-degenerate algebra $B$ and $V \in M(A \otimes B)$. Then we have for every $\omega \in \hat{A}$ that $(\omega \circ \iota)(V)$ belongs to $M(B)$.

**Result 4.8** Consider a non-degenerate algebra $B$ and $V \in M(A \otimes B)$. Then we have for every $a \in A$ and $b \in B$ that

$$(a \varphi \circ \iota)(V)b = (\varphi \circ \iota)(V(a \otimes b))$$ and

$$(b \varphi \circ \iota)(V) = (\varphi \circ \iota)((a \otimes b)V).$$

**Proof**: The first result is true by definition. Let us turn to the second one.

There exist $e \in A$ such that $a = ea$, then $\varphi a = \varphi ea = \rho(e)\varphi a$. By result 4.6, this implies for every $b \in B$ that

$$b(\varphi a \circ \iota)(V) = b(\rho(e)\varphi a \circ \iota)(V) = (\varphi \circ \iota)((a \otimes b)V(\rho(e) \otimes 1))$$

$$= (\varphi \circ \iota)((ea \otimes b)V) = (\varphi \circ \iota)((a \otimes b)V).$$

In the next result, we prove a natural calculation rule.

**Result 4.9** Consider non-degenerate algebras $B,C$ and a non-degenerate homomorphism $\pi$ from $B$ into $M(C)$. Let $V$ be an element in $M(A \otimes B)$ and $\omega \in A'$. Then $\pi((\omega \circ \iota)(V)) = (\omega \circ \iota)((\iota \circ \pi)(V))$.

**Proof**: Choose $a \in A$ and $\theta \in A'$. Then we have for every $x \in B$ that

$$\pi((a\theta \circ \iota)(V))(x)y = \pi((a\theta \circ \iota)(V)x)y = \pi((\theta \circ \iota)(V(a \otimes x)))y$$

$$= (\theta \circ \iota)((\iota \circ \pi)(V)(a \otimes x))(1 \otimes y) = (\theta \circ \iota)((\iota \circ \pi)(V)(a \otimes \pi(x)y))$$

$$= (a\theta \circ \iota)((\iota \circ \pi)(V))\pi(x)y.$$

Therefore, the non-degeneracy of $\pi$ implies that $\pi((a\theta \circ \iota)(V)) = (a\theta \circ \iota)((\iota \circ \pi)(V))$. 


5 Corepresentations of algebraic quantum groups

Consider an algebraic quantum group \((A, \Delta)\) with a left Haar functional \(\varphi\). We will introduce the notion of a corepresentation of \((A, \Delta)\) and prove some familiar results about them.

We start with the usual definition of a corepresentation.

**Definition 5.1** Consider a non-degenerate algebra \(B\). A corepresentation of \((A, \Delta)\) on \(B\) is by definition an element \(V\) in \(M(A \odot B)\) such that \((\Delta \otimes \iota)(V) = V_{13} V_{23}\).

We call \(V\) non-degenerate \(\Leftrightarrow V\) is invertible in \(M(A \odot B)\).

Remember that \(A \odot A\) is again an algebraic quantum group with \((A \odot A)^\circ = \hat{A} \odot \hat{A}\). It is also easy to check that \(A^\circ \circ A^\circ \subseteq (A \odot A)^\circ\).

Let \(B\) be a non-degenerate algebra and \(V \in M(A \odot A \odot B)\). The remark above implies that we can define the left multiplier \((\omega \otimes \iota)(V)\) on \(B\) for every \(\omega \in A^\circ \circ A^\circ\) (using the definitions of the previous section).

**Lemma 5.2** Consider a non-degenerate algebra \(B\) and \(V \in M(A \odot B)\). Then we have for every \(\omega_1, \omega_2 \in A^\circ\) that \((\omega_1 \omega_2 \otimes \iota)(V) = (\omega_1 \otimes \omega_2 \otimes \iota)((\Delta \otimes \iota)(V))\).

**Proof:** Take \(\theta_1, \theta_2 \in A'\) and \(a_1, a_2 \in A\). Then there exist \(p_1, \ldots, p_n, q_1, \ldots, q_n, r_1, \ldots, r_n \in A\) such that

\[
\begin{align*}
\sum_{i=1}^n \eta_i(x r_i) &= \sum_{i=1}^n (\theta_1 \otimes \theta_2) \Delta (r_i) p_i q_i) = (\theta_1 \otimes \theta_2) \Delta (\sum p_i q_i) = ((a_1 \theta_1)(a_2 \theta_2))(x)
\end{align*}
\]

which implies that \(\sum_{i=1}^n r_i \eta_i = (a_1 \theta_1)(a_2 \theta_2)\). Hence, we get by definition for every \(b \in B\) that

\[
\begin{align*}
((a_1 \theta_1)(a_2 \theta_2) \otimes \iota)(V) b &= \sum_{i=1}^n \eta_i(V(r_i \otimes b)) = \sum_{i=1}^n ((\theta_1 \otimes \theta_2) \Delta \otimes \iota)p_i q_i \otimes 1)
\end{align*}
\]

So we see that \(((a_1 \theta_1)(a_2 \theta_2) \otimes \iota)(V)) = ((a_1 \theta_1 \otimes a_2 \theta_2 \otimes \iota)((\Delta \otimes \iota)(V)). The result follows by linearity. \(\blacksquare\)

**Lemma 5.3** Consider a non-degenerate algebra \(B\) and \(V, W \in M(A \odot B)\). Then we have for every \(\omega_1, \omega_2 \in A^\circ\) that \((\omega_1 \otimes \iota)(V)(\omega_2 \otimes \iota)(W) = (\omega_1 \otimes \omega_2 \otimes \iota)(V_{13} W_{23})\).

**Proof:** Choose \(a_1, a_2 \in A\) and \(\theta_1, \theta_2 \in A'\). Take \(b \in B\). Then we get by definition that

\[
\begin{align*}
(a_1 \theta_1 \otimes \iota)(V)(a_2 \theta_2 \otimes \iota)(W) b &= (a_1 \theta_1 \otimes \iota)(V)(a_2 \theta_2 \otimes \iota)(W(b))
\end{align*}
\]

We have for every \(p, q \in A\) that

\[
\begin{align*}
V(a_1 \otimes (\theta_2 \otimes \iota)(p \otimes q)) &= \theta_2(p) V(a_1 \otimes q) = (\iota \otimes \theta_2 \otimes \iota)(V(a_1 \otimes q))_{13}(1 \otimes p \otimes 1)
\end{align*}
\]

\[
\begin{align*}
 &= (\iota \otimes \theta_2 \otimes \iota)(V_{13}(a_1 \otimes p \otimes q)).
\end{align*}
\]
Using this in equality (*), we get that
\[
(a_1\theta_1 \circ \iota)(V) (a_2\theta_2 \circ \iota)(W) b = (\theta_1 \circ \iota)((\iota \circ \theta_2 \circ \iota)(V_{13}(a_1 \otimes W(a_2 \otimes b))))
\]
\[
= (\theta_1 \circ \theta_2 \circ \iota)(V_{13}W_{23}(a_1 \otimes a_2 \otimes b)) = (a_1\theta_1 \circ a_2\theta_2 \circ \iota)(V_{13}W_{23}) b.
\]
Hence, we see that \((a_1\theta_1 \circ \iota)(V) (a_2\theta_2 \circ \iota)(W) = (a_1\theta_1 \circ a_2\theta_2 \circ \iota)(V_{13}W_{23}).\)

The lemma follows. \hfill \blacksquare

**Definition 5.4** Consider a non-degenerate algebra \(A\) and a corepresentation \(V\) of \((A, \Delta)\) on \(B\). Then we define the mapping \(\pi_V\) from \(\hat{A}\) into \(M(B)\) such that \(\pi_V(\omega) = (\omega \circ \iota)(V)\) for every \(\omega \in \hat{A}\). Then \(\pi_V\) is an algebra homomorphism.

**Remark 5.5**

- The multiplicativity follows immediately from lemmas 5.2 and 5.3. Combining these lemmas with result 4.3, we have also a converse:

  Consider a non-degenerate algebra \(B\) and an element \(V \in M(A \otimes B)\). Then \(V\) is a corepresentation \(\iff\) We have for every \(\omega_1, \omega_2 \in \hat{A}\) that \((\omega_1 \circ \iota)(V) (\omega_2 \circ \iota)(V) = (\omega_1\omega_2 \circ \iota)(V)\).

- Let \(B\) be a non-degenerate algebra and \(V, W\) corepresentations of \((A, \Delta)\) on \(B\). Then result 4.3 implies immediately that \(V = W\) if and only if \(\pi_V = \pi_W\).

**Proposition 5.6** Consider a non-degenerate algebra \(B\) and a corepresentation \(V\) of \((A, \Delta)\) on \(B\) such that \(A \otimes B = V(A \otimes B) = (A \otimes B)V\). Then \(\pi_V\) is non-degenerate.

**Proof:** Choose \(b \in B\). There exists \(a \in A\) such that \(\varphi(a) = 1\). By assumption, there exist \(p_1, \ldots, p_n \in A\) and \(q_1, \ldots, q_n \in B\) such that \(a \otimes b = \sum_{i=1}^n V(p_i \otimes q_i)\). This implies that
\[
\sum_{i=1}^n \pi_V(p_i \varphi) q_i = \sum_{i=1}^n (p_i \varphi \circ \iota)(V) q_i = \sum_{i=1}^n (\varphi \circ \iota)(V(p_i \otimes q_i))
\]
\[
= (\varphi \circ \iota)(a \otimes b) = b.
\]
Hence, we see that \(\pi_V(\hat{A})B = B\). Similarly, we get that \(B \pi_V(\hat{A}) = A\). \hfill \blacksquare

Notice that the conclusion of this proposition is especially true for a non-degenerate corepresentation. Later on, we will prove that the non-degeneracy of \(V\) is equivalent with the non-degeneracy of \(\pi_V\).

Next we prove some results about the behaviour of a corepresentation with respect to the counit and the antipode.

**Proposition 5.7** Consider a non-degenerate algebra \(B\) and a non-degenerate corepresentation \(V\) of \((A, \Delta)\) on \(B\). Then \((\varepsilon \circ \iota)(V) = (\iota \circ \varepsilon)(V) = 1\).

**Proof:** We have that \((\Delta \circ \iota)(V) = V_{13}V_{23}\). Applying \(\varepsilon \circ \iota \circ \iota\) to this equation implies that
\[
(\varepsilon \circ \iota)(\Delta \circ \iota)(V) = (1 \otimes (\varepsilon \circ \iota)(V)) V.
\]
Using the fact that \((\varepsilon \circ \iota)\Delta = \iota\), this implies that \(V = (1 \otimes (\varepsilon \circ \iota)(V)) V\). Hence, the invertibility of \(V\) implies that \(1 \otimes (\varepsilon \circ \iota)(V) = 1 \otimes 1\), so \((\varepsilon \circ \iota)(V) = 1\). Similarly, one proves that \((\iota \circ \varepsilon)(V) = 1\). \hfill \blacksquare
Proposition 5.8 Consider a non-degenerate algebra \( B \) and a non-degenerate corepresentation \( V \) of \( (A, \Delta) \) on \( B \). Then we have the following equalities:

1. \( A \odot B = \langle (a \otimes 1)V(1 \otimes b) \mid a \in A, b \in B \rangle \)
2. \( A \odot B = \langle (a \otimes 1)V^{-1}(1 \otimes b) \mid a \in A, b \in B \rangle \)
3. \( A \odot B = \langle (1 \otimes b)V(a \otimes 1) \mid a \in A, b \in B \rangle \)
4. \( A \odot B = \langle (1 \otimes b)V^{-1}(a \otimes 1) \mid a \in A, b \in B \rangle \)

**Proof:** Choose \( a, c \in A \) and \( b \in B \). We are going to prove the equality

\[
(\psi \circ \iota \circ \iota)((V^{-1})_{13}((1 \otimes a)\Delta(c) \otimes b)) = (a \otimes 1)V(1 \otimes (\psi \circ \iota)(V^{-1}(c \otimes b)))
\]

Choose \( d \in B \). Then we have that

\[
(1 \otimes d)(\psi \circ \iota \circ \iota)((V^{-1})_{13}((1 \otimes a)\Delta(c) \otimes b)) = (\psi \circ \iota \circ \iota)((1 \otimes 1 \otimes d)(V^{-1})_{13}(1 \otimes a \otimes 1)(\Delta(c) \otimes b))
\]

\[
= (\psi \circ \iota \circ \iota)((1 \otimes a \otimes d)(V^{-1})_{13}(\Delta(c) \otimes b)) = (\psi \circ \iota \circ \iota)((1 \otimes a \otimes d)V_{23}(\Delta \circ \iota)(V^{-1})(\Delta \circ \iota)(c \otimes b))
\]

where in the last equality, we used the fact that \( V \) is a corepresentation. So we get that

\[
(1 \otimes d)(\psi \circ \iota \circ \iota)((V^{-1})_{13}((1 \otimes a)\Delta(c) \otimes b)) = (\psi \circ \iota \circ \iota)((1 \otimes (a \otimes d)V(\Delta \circ \iota)(V^{-1}(c \otimes b)))) \quad (*)
\]

We have for every \( p, r \in A \) and \( q, s \in B \) that

\[
(\psi \circ \iota \circ \iota)((1 \otimes p \otimes q)(\Delta \circ \iota)(r \otimes s)) = (\psi \circ \iota \circ \iota)((1 \otimes p)\Delta(r \otimes qs)) = \psi(r)p \otimes qs = (p \otimes q)(1 \otimes (\psi \circ \iota)(r \otimes s)).
\]

where we used the right invariance of \( \psi \) in the second last equality.

Combining this with equality (*), we arrive at the conclusion that

\[
(1 \otimes d)(\psi \circ \iota \circ \iota)((V^{-1})_{13}((1 \otimes a)\Delta(c) \otimes b)) = (a \otimes d)V (1 \otimes (\psi \circ \iota)(V^{-1}(c \otimes b))).
\]

So we get that

\[
(\psi \circ \iota \circ \iota)((V^{-1})_{13}((1 \otimes a)\Delta(c) \otimes b)) = (a \otimes 1)V (1 \otimes (\psi \circ \iota)(V^{-1}(c \otimes b))).
\]

Using this equality, it is not difficult to check that \( A \odot B = \langle (a \otimes 1)V(1 \otimes b) \mid a \in A, b \in B \rangle \). The other equalities are proven in a similar way (The last two, by using the left Haar functional).

Now we get easily the following result:

**Result 5.9** Consider a non-degenerate algebra \( B \) and a non-degenerate corepresentation \( V \) of \( (A, \Delta) \) on \( B \). Let \( a \in A \) and \( \omega \) be an element in \( A' \). Then \( (a \omega \circ \iota)(V) \) belongs to \( M(B) \) and

\[
b(a \omega \circ \iota)(V) = (\omega \circ \iota)(1 \otimes b)V(a \otimes 1)
\]

for every \( b \in B \).

Proposition 5.8 makes it also possible to give the following definition.

**Definition 5.10** Consider a non-degenerate algebra \( B \) and a non-degenerate corepresentation \( V \) of \( (A, \Delta) \) on \( B \). Then \( (S \circ \iota)(V) \) is defined to be the unique element in \( M(A \odot B) \) such that

\[
(S \circ \iota)(V)(a \otimes b) = (S \circ \iota)((S^{-1}(a) \otimes 1)V(1 \otimes b)) \quad \text{and} \quad (a \otimes b)(S \circ \iota)(V) = (S \circ \iota)((1 \otimes b)V(S^{-1}(a) \otimes 1))
\]

for every \( a \in A \) and \( b \in B \).
It is not difficult to check that \((S \circ \iota)(V)\) is really an element of \(M(A \otimes B)\).

In this terminology, we get the usual result:

**Proposition 5.11** Consider a non-degenerate algebra \(B\) and a non-degenerate corepresentation \(V\) of \((A, \Delta)\) on \(B\). Then \((S \circ \iota)(V) = V^{-1}\).

**Proof:** Choose \(b \in B\) and \(c, d \in A\). Take \(a \in A\) such that \(\varepsilon(a) = 1\).

Because \((\Delta \circ \iota)(V) = V_{13}V_{23}\), we have the equality

\[
(\Delta \circ \iota)((a \otimes 1)V(1 \otimes b))(c \otimes d \otimes 1) = (\Delta(a) \otimes 1)V_{13}V_{23}(c \otimes d \otimes b) = (\Delta(a) \otimes 1)[V(c \otimes 1)]_{13}[V(d \otimes b)]_{23} \quad (a)
\]

By proposition 5.8 we know that \((a \otimes 1)V(1 \otimes b)\) belongs to \(A \circ B\). It is not difficult to see that \([V(c \otimes 1)]_{13}[V(d \otimes b)]_{23}\) belongs to \(A \circ A \circ B\).

It is rather easy to check that \(m(S \circ \iota)(\Delta(x)y) = \varepsilon(x)m(S \circ \iota)(y)\) for every \(x, y \in A \circ A\).

Therefore, if we apply \(m(S \circ \iota) \circ \iota\) to equation (a), we get the equality

\[
m(S \circ \iota)(c \otimes d) \otimes (\iota \circ \iota)((a \otimes 1)V(1 \otimes b)) = \varepsilon(a)(m(S \circ \iota) \circ \iota)([V(c \otimes 1)]_{13}[V(d \otimes b)]_{23}) . \quad (b)
\]

Using the fact that \((\iota \circ \iota)(V) = 1\), the left hand side of equation (b) is equal to \(S(c)d \otimes b\).

Now we are going to work on the right hand side. We have for every \(p, q \in A\) that

\[
(m(S \circ \iota) \circ \iota)([V(c \otimes 1)]_{13}[p \otimes q]_{23}) = (m(S \circ \iota) \circ \iota)([V(c \otimes q)]_{13}(1 \otimes p \otimes 1)) = (S \circ \iota)(V(c \otimes q))(p \otimes 1) = (S(c) \otimes 1)(S \circ \iota)(V)(p \otimes q) .
\]

This implies that the right hand side of (b) is equal to \((S(c) \otimes 1)(S \circ \iota)(V)V(d \otimes b)\).

So we get the equality \(S(c)d \otimes b = (S(c) \otimes 1)(S \circ \iota)(V)V(d \otimes b)\).

Consequently, we have proven that \((S \circ \iota)(V) = 1\) which implies that \((S \circ \iota)(V) = V^{-1}\). ■

If \(\pi_V\) is non-degenerate, we can extend it to \(M(\hat{A})\) which contains \(A'\). For these elements, \(\pi_V\) will act in the obvious way:

**Proposition 5.12** Consider a non-degenerate algebra \(B\) and a non-degenerate corepresentation \(V\) of \((A, \Delta)\) on \(B\). Then we have for every \(a \in A\), \(\omega \in A'\) and \(b \in B\) that \(b\pi_V(\omega a) = (\omega \circ \iota)((1 \otimes b)V(a \otimes 1))\) and \(\pi_V(\omega a)b = (\omega \circ \iota)(V(a \otimes b))\).

**Proof:** Remember that \(\omega a\) is an element of \(A'\), so we have already defined the element \((a \omega \circ \iota)(V)\).

Choose \(\theta \in \hat{A}\) and \(b \in B\). Using lemmas 5.2 and 5.3, we get that

\[
\pi_V(\omega a)(\pi_V(\theta)b) = \pi_V((a \omega)(\theta)b) = ((a \omega)(\theta \circ \iota)(V)b = ((a \omega) \circ \theta \circ \iota)((\Delta \circ \iota)(V))b = (a \omega \circ \theta \circ \iota)(V_{13}V_{23})b = (a \omega \circ \iota)(V)(\theta \circ \iota)(V)b = (a \omega \circ \iota)(V)\pi_V(\theta)b .
\]

Hence, the non-degeneracy of \(\pi_V\) implies that \(\pi_V(\omega a) = (a \omega \circ \iota)(V)\). ■

Of course, a similar result is true for linear functionals of the form \(\omega a\). In this case, the results in the previous section would have to be stated in terms of right multipliers.
Proposition 5.13 Consider non-degenerate algebras $B, C$ and a non-degenerate homomorphism $\theta$ from $B$ into $M(C)$. Let $V$ be a corepresentation of $(A, \Delta)$ on $B$. Then $(\iota \circ \theta)(V)$ is a corepresentation of $(A, \Delta)$ on $C$ such that $\pi_{(\iota \circ \theta)(V)} = \theta \circ \pi_V$. If $V$ is non-degenerate, then $(\iota \circ \theta)(V)$ is non-degenerate.

Proof : We have that

$$(\iota \circ \iota)(((\iota \circ \theta)\iota)(V)) = (\iota \circ \iota \circ \theta)(\iota \iota \iota)(V) = (\iota \circ \iota \circ \theta)(V_{13}V_{23})$$

$$= (\iota \circ \iota \circ \theta)(V_{13}) (\iota \circ \iota \circ \theta)(V_{23}) = (\iota \circ \theta)(V_{13}) (\iota \circ \theta)(V_{23})$$

which implies that $(\iota \circ \theta)(V)$ is a corepresentation. Using Lemma 4.9, we have for every $\omega \in \hat{A}$ that

$$\pi_{(\iota \circ \theta)(V)}(\omega) = (\omega \circ \iota)(((\iota \circ \theta)\iota)(V)) = \theta((\omega \circ \iota)(V)) = \theta(\pi_V(\omega)).$$

In the following section, we show the existence of a special corepresentation such that every corepresentation can be constructed from this special one using the method described in the proposition above.

6 The universal corepresentation of an algebraic quantum group

Consider an algebraic quantum group $(A, \Delta)$ with a left Haar functional $\varphi$. We will construct the universal corepresentation of $(A, \Delta)$ and use it to prove that there is a natural bijection between corepresentations of $(A, \Delta)$ and representations of $\hat{A}$.

Because $\hat{A}$ is a subset of $A'$, we can regard $A \odot \hat{A}$ in a natural way as a subspace of the linear operators on $A$.

Notation 6.1 We define linear mappings $U_l, U_r$ from $A \odot \hat{A}$ into $L(A)$ such that $[U_l(x \odot \omega)](y) = (\iota \circ \omega)(\Delta(y)(x \odot 1))$ and $[U_r(x \odot \omega)](y) = (\omega \circ \iota)(\Delta(1 \odot x))$ for every $x, y \in A$ and $\omega \in \hat{A}$.

Proposition 6.2 We have for every $a \in A \odot \hat{A}$ that $U_l(a)$ and $U_r(a)$ belong to $A \odot \hat{A}$.

Proof : Choose $x \in A$ and $\omega \in \hat{A}$. Then there exists $z \in A$ such that $\omega = z\varphi$. Furthermore, there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in A$ such that $x \odot z = \sum_{i=1}^n \Delta(p_i)(q_i \odot 1)$.

Then we have for every $y \in A$ that

$$[U_l(x \odot \omega)](y) = (\iota \circ \omega)(\Delta(y)(x \odot 1)) = (\iota \circ \varphi)(\Delta(y)(x \odot z))$$

$$= \sum_{i=1}^n (\iota \circ \varphi)(\Delta(y)(p_i)(q_i \odot 1)) = \sum_{i=1}^n \varphi(y p_i) q_i$$

Where in the last equality, we used the left invariance of $\varphi$. So we get that $U_l(x \odot \omega) = \sum_{i=1}^n q_i \otimes p_i \varphi$. Similarly, we get that $U_r(x \odot \omega)$ belongs to $A \odot \hat{A}$. 

As usual, $A \odot \hat{A}$ has a natural algebra structure. In a next step, we want to prove that $(U_l, U_r)$ is a two-sided multiplier on $A \odot \hat{A}$. First we need an easy lemma.

Lemma 6.3 Consider $\omega \in A \odot \hat{A}$, $a, b \in A$ and $x \in A$. Then we have for every $x \in A$ that

$$((b \otimes \varphi a)\omega)(x) = m((b \otimes \varphi \otimes \omega)((a \otimes 1)\Delta(x)))$$

and

$$\omega((b \otimes \varphi a))(x) = m(\omega \otimes (b \otimes \varphi))((1 \otimes a)\Delta(x)).$$
Proof: Choose \( c \in A \) and \( \theta \in \hat{A} \). Take \( x \in A \), then we have that

\[
[(b \otimes \varphi a)(c \otimes \theta)](x) = [(bc) \otimes ((\varphi a)\theta)](x) = bc ((\varphi a)\theta)(x) = bc (\varphi \otimes \theta)((a \otimes 1)\Delta(x)) .
\]

We have for every \( p, q \in A \) that

\[
bc (\varphi \otimes \theta)(p \otimes q) = b \varphi(p) c \theta(q) = (b \otimes \varphi)(p) (c \otimes \theta)(q) = m((b \otimes \varphi) \otimes (c \otimes \theta))(p \otimes q) .
\]

This implies that

\[
[(b \otimes \varphi a)(c \otimes \theta)](x) = m((b \otimes \varphi) \otimes (c \otimes \theta))(a \otimes 1)\Delta(x)) .
\]

The result follows by linearity. The other equality is proven in the same way. 

Lemma 6.4 Consider \( \omega, \theta \in A \otimes \hat{A} \). Then

1. \( U_l(\omega) \theta = U_l(\omega \theta) \)
2. \( \omega U_r(\theta) = U_r(\omega \theta) \)
3. \( U_l(\omega) \theta = \omega U_r(\theta) \)

Proof: Choose \( a_1, a_2, b_1, b_2 \in A \).

1. Take \( x \in A \). Then we have by the previous lemma that

\[
[U_l(b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2)](x) = m(U_l(b_1 \otimes \varphi a_1) \otimes (b_2 \otimes \varphi))((1 \otimes a_2)\Delta(x)) . \tag{a}
\]

We have for every \( p, q \in A \) that

\[
m(U_l(b_1 \otimes \varphi a_1) \otimes (b_2 \otimes \varphi))(p \otimes q) = [U_l(b_1 \otimes \varphi a_1)](p) (b_2 \otimes \varphi)(q) \\
= (\iota \otimes \varphi a_1)((\Delta(p)(b_1 \otimes 1)) b_2 \varphi(q)) = (\iota \otimes \varphi a_1 \otimes \varphi)((\Delta(p)(b_1 b_2 \otimes 1) \otimes q) \\
= (\iota \otimes \varphi a_1 \otimes \varphi)(\Delta(\iota)(p \otimes q)(b_1 b_2 \otimes 1)) .
\]

Using this with equation \( \text{(a)} \), we get that

\[
[U_l(b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2)](x) = (\iota \otimes \varphi a_1 \otimes \varphi)((\Delta \otimes \iota)(((1 \otimes a_2)\Delta(x))(b_1 b_2 \otimes 1 \otimes 1)) \\
= (\iota \otimes \varphi a_1 \otimes \varphi)(((1 \otimes a_2)\Delta(\iota)(b_1 b_2 \otimes 1 \otimes 1))((\Delta \otimes \iota)(\Delta(x))(b_1 b_2 \otimes 1 \otimes 1)) \\
= (\iota \otimes \varphi a_1 \otimes \varphi)(((1 \otimes a_2)(\iota \otimes \Delta)(\Delta(x))(b_1 b_2 \otimes 1 \otimes 1)) \\
= (\iota \otimes \varphi a_1 \otimes \varphi)(((1 \otimes a_2)(\iota \otimes \Delta)(\Delta(x))(b_1 b_2 \otimes 1 \otimes 1)) .
\]

Using \text{def. } 6.3, this implies that

\[
[U_l(b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2)](x) = (\iota \otimes (\varphi a_1)(\varphi a_2))(\Delta(x)(b_1 b_2 \otimes 1)) \\
= [U_l(b_1 b_2 \otimes (\varphi a_1)(\varphi a_2)](x) = [U_l((b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2)](x) .
\]

So we get that \( U_l(b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2) = U_l(((b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2)) .
\]

2. In a similar way, one proves that \( (b_1 \otimes \varphi a_1) U_r(b_2 \otimes \varphi a_2) = U_r(((b_1 \otimes \varphi a_1)(b_2 \otimes \varphi a_2)).
\]
3. Take \( x \in A \). Again, the previous lemma implies that
\[
[U_r(b_1 \otimes \varphi a_1) (b_2 \otimes \varphi a_2)](x) = m(U_r(b_1 \otimes \varphi a_1) \otimes (b_2 \otimes \varphi))(1 \otimes a_2)\Delta(x) .
\] (b)

We have for every \( p, q \in A \) that
\[
m(U_r(b_1 \otimes \varphi a_1) \otimes (b_2 \otimes \varphi))(p \otimes q) = [U_r(b_1 \otimes \varphi a_1)](p) (b_2 \otimes \varphi)(q) \\
= (\varphi a_1 \otimes \iota)((1 \otimes b_1)\Delta(p)) (1 \otimes b_2) \varphi(q) \\
= (\varphi a_1 \otimes \iota \otimes \varphi)((1 \otimes b_1)\Delta(p)(1 \otimes b_2) \otimes q) \\
= (\varphi a_1 \otimes \iota \otimes \varphi)((1 \otimes b_1)(\Delta \otimes \iota)(p \otimes q)(1 \otimes b_2 \otimes 1)) .
\]

Using this, equation (b) implies that
\[
[U_r(b_1 \otimes \varphi a_1) (b_2 \otimes \varphi a_2)](x) = (\varphi a_1 \otimes \iota \otimes \varphi)((1 \otimes b_1 \otimes 1)(\Delta \otimes \iota)(1 \otimes a_2)\Delta(x))(1 \otimes b_2 \otimes 1)) ,
\]
which implies that
\[
[U_r(b_1 \otimes \varphi a_1) (b_2 \otimes \varphi a_2)](x) = (\varphi \otimes \iota \otimes \varphi)((a_1 \otimes b_1 \otimes a_2)(\Delta \otimes \iota)(\Delta(x))(1 \otimes b_2 \otimes 1)) .
\] (c)

In a similar way, we get that
\[
[(b_1 \otimes \varphi a_1) U_l(b_2 \otimes \varphi a_2)](x) = (\varphi \otimes \iota \otimes \varphi)((1 \otimes b_1 \otimes 1)(\iota \otimes \Delta)((a_1 \otimes 1)\Delta(x))(1 \otimes b_2 \otimes 1)) ,
\]
which implies that
\[
[(b_1 \otimes \varphi a_1) U_l(b_2 \otimes \varphi a_2)](x) = (\varphi \otimes \iota \otimes \varphi)((a_1 \otimes b_1 \otimes a_2)(\iota \otimes \Delta)(\Delta(x))(1 \otimes b_2 \otimes 1)) .
\] (d)

Hence, using the coassociativity of \( \Delta \) and looking at equalities (c) and (d), we arrive at the conclusion that
\[
[U_r(b_1 \otimes \varphi a_1) (b_2 \otimes \varphi a_2)](x) = [(b_1 \otimes \varphi a_1) U_l(b_2 \otimes \varphi a_2)](x) .
\]
So we have proven that \( U_r(b_1 \otimes \varphi a_1) (b_2 \otimes \varphi a_2) = (b_1 \otimes \varphi a_1) U_l(b_2 \otimes \varphi a_2) .\)

This lemma justifies the following definition:

**Definition 6.5** There exists a unique element \( U \in M(A \otimes \hat{A}) \) such that \( U(x \otimes \omega)(y) = (\iota \otimes \omega)(\Delta(y)(x \otimes 1)) \) and \( [(x \otimes \omega)U](y) = (\omega \otimes \iota)((1 \otimes x)\Delta(y)) \) for every \( x, y \in A \) and \( \omega \in \hat{A} \).

Later on, we will prove that \( U \) is a corepresentation of \((A, \Delta)\) on \( \hat{A} \). It is called the universal corepresentation of \((A, \Delta)\).

**Proposition 6.6** The element \( U \) is invertible in \( M(A \otimes \hat{A}) \).

**Proof:** We prove that \( U_l \) is bijective:
• First we prove that \( U_l \) is injective.

Choose \( x_1, \ldots, x_n \in A \) and \( \omega_1, \ldots, \omega_n \in \hat{A} \) such that \( U_l(\sum_{i=1}^{n} x_i \otimes \omega_i) = 0 \).

Take \( y \in A \). Choose \( b \in A \). Then there exist \( p_1, \ldots, p_m, q_1, \ldots, q_m \in A \) such that \( b \otimes y = \sum_{j=1}^{m} (p_j \otimes 1)\Delta(q_j) \). So we get that

\[
0 = \sum_{j=1}^{m} p_j [U_l(\sum_{i=1}^{n} x_i \otimes \omega_i)](q_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} p_j (\iota \otimes \omega_i)(\Delta(q_j)(x_i \otimes 1))
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} (\iota \otimes \omega_i)([(p_j \otimes 1)\Delta(q_j)](x_i \otimes 1)) = \sum_{i=1}^{n} (\iota \otimes \omega_i)((b \otimes y)(x_i \otimes 1))
\]

\[
= bx_i \omega_i(y) = b (\sum_{i=1}^{n} x_i \otimes \omega_i)(y) .
\]

So we get that \( (\sum_{i=1}^{n} x_i \otimes \omega_i)(y) = 0 \). Consequently \( \sum_{i=1}^{n} x_i \otimes \omega_i = 0 \).

• Next we prove that \( U_l \) is surjective.

Choose \( a, b \in A \). Then there exist \( p_1, \ldots, p_n, q_1, \ldots, q_n \in A \) such that \( \Delta(b)(a \otimes 1) = \sum_{i=1}^{n} p_i \otimes q_i \).

Then we have for every \( x \in A \) that

\[
[U_l(\sum_{i=1}^{n} p_i \otimes q_i \varphi)](x) = \sum_{i=1}^{n} (\iota \otimes q_i \varphi)(\Delta(x)(p_i \otimes 1)) = \sum_{i=1}^{n} (\iota \otimes \varphi)(\Delta(x)(p_i \otimes q_i))
\]

\[
= (\iota \otimes \varphi)(\Delta(xb)(a \otimes 1)) = \varphi(xb)a = (a \otimes b \varphi)(x) .
\]

Consequently \( a \otimes b \varphi = U_l(\sum_{i=1}^{n} p_i \otimes q_i \varphi) \).

In a similar way, one proves that \( U_r \) is bijective. These two facts imply that \( U \) is invertible in \( M(A \otimes \hat{A}) \).

\[\blacksquare\]

**Lemma 6.7** Consider \( x \in A \otimes \hat{A} \) and \( \omega \in A' \). Then \( [\omega \otimes \iota](x)(a) = \omega(x(a)) \) for every \( a \in A \).

**Proof:** Choose \( b \in A \) and \( \theta \in \hat{A} \). Then we have for every \( a \in A \) that

\[
[(\omega \otimes \iota)(b \otimes \theta)](a) = [\omega(b)\theta](a) = \omega(b)\theta(a) = \omega((b \otimes \theta)(a)) .
\]

The lemma follows. \[\blacksquare\]

**Result 6.8** We have for every \( \omega \in \hat{A} \) that \( (\omega \otimes \iota)(U) = \omega \).

**Proof:** There exists \( a \in A \) such that \( \omega = \varphi a \).

Take \( \theta \in \hat{A} \). Then we have for every \( x \in A \) that

\[
(\theta(\varphi a \otimes \iota)(U))(x) = (\varphi \otimes \iota)((a \otimes \theta)U)(x) (\varphi(\Delta(a \otimes \theta))(U)(x))
\]

\[
= \varphi((\theta \otimes \iota)((1 \otimes a)\Delta(x)))(\theta(\varphi a))(x)
\]

where the previous lemma was used in the equality \( (\varphi(\Delta(a \otimes \theta))(U)(x)) \). So we get that \( \theta(\varphi a \otimes \iota)(U) = \theta(\varphi a) \). Therefore, lemma 6.3 implies that \( (\varphi a \otimes \iota)(U) = \varphi a \).

\[\blacksquare\]

**Corollary 6.9** The element \( U \) is a non-degenerate corepresentation of \( (A, \Delta) \) on \( \hat{A} \) such that \( \pi_U = \iota \).
This follows easily from remark 5.5 and the previous result.

**Proposition 6.10** Consider a non-degenerate algebra $B$ and a corepresentation $V$ of $(A, \Delta)$ on $B$. Then $V$ is non-degenerate $\iff \pi_V$ is non-degenerate $\iff V(A \odot B) = (A \odot B)V = A \odot B$

**Proof**:
- If $V$ is non-degenerate, we get easily that $V(A \odot B) = (A \odot B)V = A \odot B$.
- If $V(A \odot B) = (A \odot B)V = A \odot B$, we get by 5.6 that $\pi_V$ is non-degenerate.
- Suppose that $\pi_V$ is non-degenerate. By result 4.9, we have for every $\omega \in \hat{A}$ that 
  
  $$(\omega \otimes \iota)((\iota \otimes \pi_V)(U)) = \pi_V((\omega \otimes \iota)(U)) = \pi_V(\omega) = (\omega \otimes \iota)(V) ,$$

  which, by result 4.8, implies that $(\iota \otimes \pi_V)(U) = V$.

  Because $U$ is invertible and $\pi_V$ is non-degenerate, this implies that $V$ is invertible.

In fact, we have even proven the following proposition:

**Proposition 6.11** Consider a non-degenerate algebra $B$ and a non-degenerate corepresentation $V$ of $(A, \Delta)$ on $B$. Then $(\iota \otimes \pi_V)(U) = V$.

Hence, $U$ satisfies the following universal property (which was introduced for compact quantum groups in [11]).

**Proposition 6.12** Consider a non-degenerate algebra $B$ and a non-degenerate corepresentation $V$ of $(A, \Delta)$ on $B$. Then there exists a unique non-degenerate homomorphism $\theta$ from $\hat{A}$ into $M(B)$ such that $(\iota \otimes \theta)(U) = V$.

**Proof** : We have already proven the existence. We prove now quickly the uniqueness.

Therefore, let $\theta$ be a non-degenerate homomorphism from $\hat{A}$ into $M(B)$ such that $(\iota \otimes \theta)(U) = V$.

Then we have for every $\omega \in \hat{A}$ that 

$$\pi_V(\omega) = (\omega \otimes \iota)(V) = (\omega \otimes \iota)((\iota \otimes \theta)(U)) = \theta((\omega \otimes \iota)(U)) = \theta(\omega)$$

So we see that $\pi_V = \theta$.

Using proposition 5.13, we have also a converse of the previous result.

**Proposition 6.13** Consider a non-degenerate algebra $B$ and a non-degenerate homomorphism $\theta$ from $\hat{A}$ into $M(B)$. Then $(\iota \otimes \theta)(U)$ is a non-degenerate corepresentation of $(A, \Delta)$ on $B$ and $\pi_{(\iota \otimes \theta)(U)} = \theta$.

Consequently, we have proven that there is a bijective correspondence between non-degenerate corepresentations of $(A, \Delta)$ and homomorphisms on $\hat{A}$. The universal corepresentation serves as a linking mechanism.

**Result 6.14** Consider a non-degenerate algebra $B$ and a non-degenerate corepresentation $V$ of $(A, \Delta)$ on $B$. Then $(\omega \otimes \iota)(V^{-1}) = \pi_V(S(\omega))$ for every $\omega \in \hat{A}$.

24
Proof: Choose \( a \in A \). Using proposition 5.11, we have for every \( b \in B \) that
\[
(a \varphi \circ i)(V^{-1}) b = (\varphi \circ i)((S \circ i)((S^{-1}(a) \otimes 1)V(1 \otimes b))) \\
= (\psi \circ i)((S^{-1}(a) \otimes 1)V(1 \otimes b)) = (\psi S^{-1}(a) \otimes i)(V) b = \pi_V(\psi S^{-1}(a)) b .
\]
Because \( \psi S^{-1}(a) = (a \varphi) \circ S = \hat{S}(a \varphi) \), we get that
\[
(a \varphi \circ i)(V^{-1}) = \pi_V(\psi S^{-1}(a)) = \pi_V(\hat{S}(a \varphi)) .
\]

7 Unitary corepresentations of *-algebraic quantum groups

We start this section with the definition of a *-algebraic quantum group.

Definition 7.1 Consider a non-degenerate *-algebra \( A \) and a non-degenerate *-homomorphism \( \Delta \) from \( A \) into \( M(A \otimes A) \) such that

1. \( (\Delta \circ i)\Delta = (i \circ \Delta)\Delta \).

2. The linear mappings \( T_1, T_2 \) from \( A \otimes A \) into \( M(A \otimes A) \) such that
\[
T_1(a \otimes b) = \Delta(a)(b \otimes 1) \quad \text{and} \quad T_2(a \otimes b) = \Delta(a)(1 \otimes b)
\]
for all \( a, b \in A \), are bijections from \( A \otimes A \) to \( A \otimes A \).

Then we call \( (A, \Delta) \) a Multiplier Hopf *-algebra.

It is easy to see that a Multiplier Hopf *-algebra is a regular Multiplier Hopf algebra.

Definition 7.2 Consider a Multiplier Hopf *-algebra \( (A, \Delta) \) such that there exists a non-zero linear functional \( \varphi \) on \( A \) which is left invariant. Then we call \( (A, \Delta) \) a *-algebraic quantum group.

So every *-algebraic quantum group is an algebraic quantum group.

Remark 7.3 Consider a *-algebraic quantum group \( (A, \Delta) \). Let \( \varphi \) be a left Haar functional of \( (A, \Delta) \). Because \( \Delta \) is a *-homomorphism, \( \overline{\varphi} \) will also be left invariant. So \( \frac{\varphi}{\varphi} \) and \( \frac{\varphi}{\varphi} \) are self adjoint left invariant functionals. Because their sum is equal to \( \varphi \), one of them has to be non zero.

Hence, we get the existence of a self adjoint left Haar functional on \( (A, \Delta) \).

For the rest of this section, we fix a *-algebraic quantum group \( (A, \Delta) \) with a self adjoint left Haar functional \( \varphi \). In this case \( M(\hat{A}) \) is also a *-algebra. First we prove the usual formula for the adjoint operation on \( M(\hat{A}) \).

Proposition 7.4 Let \( \omega \) be an element of \( M(\hat{A}) \). Then \( \omega^*(x) = \overline{\omega(S(x))^*} \) for every \( x \in A \).
Proof: It is not very difficult to check that \((\varphi b)^* = \psi S(b)^*\) and \((\psi b)^* = \varphi S(b)^*\) for every \(b \in A\).

Choose \(a \in A\). Then lemma 2.4 implies that

\[\omega^* (\varphi a) = \varphi [S^{-1}((\iota \otimes \omega^*)\Delta(S(a)))]\]

On the other hand, we have that

\[\omega^* (\varphi a) = ((\varphi a)^*)^* = ((\psi S(a))^*)^*\]

By lemma 2.4, we know that \((\psi S(a))^*\omega = \psi [S((\omega \circ \iota)\Delta(S^2(a)^*))]\), so

\[\omega^*(\varphi a) = (\psi [S((\omega \circ \iota)\Delta(S^2(a)^*))])^* = \varphi [S^2((\omega \circ \iota)\Delta(S^2(a)^*))]^*\].

Again, the faithfulness of \(\varphi\) implies that

\[S^{-1}((\iota \otimes \omega^*)\Delta(S(a))) = S^2((\omega \circ \iota)\Delta(S^2(a)^*))^*\].

Applying \(\varepsilon\) to this equation and using lemma 2.4, we get that

\[\omega^*(S(a)) = \overline{\omega(S^2(a)^*)} = \overline{\omega(S(S(a))^*)}.\]

\[\square\]

Definition 7.5 Consider a non-degenerate \(*\)-algebra \(B\). A unitary corepresentation of \((A, \Delta)\) on \(B\) is by definition a corepresentation of \((A, \Delta)\) on \(B\) which is a unitary element in the \(*\)-algebra \(M(A \otimes B)\).

It is clear that a unitary corepresentation is automatically non-degenerate.

Remark 7.6 Consider non-degenerate \(*\)-algebras \(B, C\) and a non-degenerate \(*\)-homomorphism \(\theta\) from\(B\) into \(M(C)\). If \(V\) is a unitary corepresentation of \((A, \Delta)\) on \(B\), then it is clear that \((\iota \otimes \theta)(V)\) is a unitary corepresentation of \((A, \Delta)\) on \(C\).

Proposition 7.7 Consider a non-degenerate \(*\)-algebra \(B\) and a non-degenerate corepresentation \(V\) of \((A, \Delta)\) on \(B\). Then \((\omega \circ \iota)(V^*) = \pi_V(\overline{\omega})^*\) for every \(\omega \in \hat{A}\).

Proof: Choose \(a \in A\). Then we have for every \(b \in B\) that

\[(a\varphi \circ \iota)(V^*)b = (\varphi \circ \iota)(V^*(a \otimes b)) = (\varphi \circ \iota)((a^* \otimes b^*)V)^*\]

\[= [b^* (\varphi a^* \circ \iota)(V)]^* = (\overline{\varphi a^*} \circ \iota)(V)^*b = \pi_V(\overline{\varphi a^*})^*b.\]

So we get that \((a\varphi \circ \iota)(V^*) = \pi_V(\overline{\varphi a^*})^*\).

\[\square\]

Proposition 7.8 Consider a non-degenerate \(*\)-algebra \(B\) and a non-degenerate corepresentation \(V\) of \((A, \Delta)\) on \(B\). Then \(V\) is unitary \(\iff\) \(\pi_V\) is a \(*\)-homomorphism.

Proof:

- First suppose that \(V\) is unitary. Choose \(\omega \in \hat{A}\). Then

\[\pi_V(\omega)^* = (\overline{\omega} \circ \iota)(V^*) = (\overline{\omega} \circ \iota)(V^{-1}) = \pi_V(\overline{\Delta(\omega)}) = \pi_V(\omega^*).\]

So we see that \(\pi_V\) is a \(*\)-homomorphism.
Next suppose that $\pi_V$ is a $^*\text{-homomorphism}$. Choose $\omega \in \hat{A}$. Then
\[(\omega \circ i)(V^*) = \pi_V(\omega)(V^*) = \pi_V(S(\omega)) = (\omega \circ i)(V^{-1}).\]
Therefore, result 4.5 implies that $V^* = V^{-1}$.

**Corollary 7.9** The universal corepresentation $U$ is a unitary corepresentation of $(A, \Delta)$ on $\hat{A}$.

This follows immediately from the fact that $\pi_U$ is the identity mapping which is obviously a $^*\text{-homomorphism}$.

This result together with the results from the previous section imply that there is a bijective correspondence between unitary corepresentations on $(A, \Delta)$ and non-degenerate $^*\text{-homomorphisms}$ on $\hat{A}$.

### 8 The universal corepresentation of the dual

Consider an algebraic quantum group $(A, \Delta)$ with a left Haar functional $\varphi$. Then $(\hat{A}, \hat{\Delta})$ is again an algebraic quantum group for which we can construct the dual $(\hat{A}, \hat{\Delta})$. Theorem 4.12 of [13] guarantees that $(\hat{A}, \hat{\Delta})$ is isomorphic to $(A, \Delta)$:

**Theorem 8.1** There exists an isomorphism of algebras $\Upsilon$ from $A$ to $\hat{A}$ such that $\Upsilon(x)(\omega) = \omega(x)$ for every $x \in A$ and $\omega \in \hat{A}$. We have moreover that $\Delta \Upsilon = (\Upsilon \circ \Upsilon)\Delta$.

In the rest of this section, we use this theorem to identify $(A, \Delta)$ and $(\hat{A}, \hat{\Delta})$.

**Proposition 8.2** Denote the universal corepresentation of $(A, \Delta)$ by $U$. Then $\chi(U)$ is the universal corepresentation of $(\hat{A}, \hat{\Delta})$.

**Proof:** Choose $a, b \in A$. Then there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in A$ such that $b \otimes a = \sum_{i=1}^n \Delta(p_i)(q_i \otimes 1)$.

Looking at the proof of proposition 6.2, we see that $U(b \otimes a\varphi) = \sum_{i=1}^n q_i \otimes p_i\varphi$.

This implies that $\chi(U)(a\varphi \otimes b) = \sum_{i=1}^n p_i\varphi \otimes q_i$.  \((*)\)

Denote the universal corepresentation of $(\hat{A}, \hat{\Delta})$ by $V$. Using the identification mentioned above, we get that $V$ is an element of $M(\hat{A} \otimes A)$. Looking at section 3 we consider $\hat{A} \otimes A$ in this case as a subspace of $L(\hat{A})$. So we consider $V(a\varphi \otimes b)$ as an element of $L(\hat{A})$.

Therefore choose $\omega \in \hat{A}$. Definition 7.3 implies that $(V(a\varphi \otimes b))(\omega) = (b \otimes \omega)(\Delta(\omega)(a\varphi \otimes 1))$ (where $b$ is considered as an element of $\hat{A}'$).

It is now easy to check for every $x \in A$ that
\[
[(V(a\varphi \otimes b))(\omega))(x) = [(b \otimes \omega)(\hat{\Delta}(\omega)(a\varphi \otimes 1))](x) = [\hat{\Delta}(\omega)(a\varphi \otimes 1)](x \otimes b).
\]

Hence, remark 5.3 implies that
\[
[(V(a\varphi \otimes b))(\omega))(x) = (\omega \circ a\varphi)(\Delta(x)(b \otimes 1)) = (\omega \circ a\varphi)(\Delta(x)(b \otimes a))
\]
\[
= \sum_{i=1}^n (\omega \circ a\varphi)(\Delta(x)p_i)(q_i \otimes 1) = \sum_{i=1}^n \varphi(\Delta(x)p_i)\omega(q_i)
\]
where the left invariance was used once again in the last equality. So
\[
(V(a\varphi \otimes b))(\omega) = \sum_{i=1}^n \omega(q_i)p_i\varphi = [\sum_{i=1}^n p_i\varphi \otimes q_i](\omega).
\]

Therefore we get that $V(a\varphi \otimes b) = \sum_{i=1}^n p_i\varphi \otimes q_i$ which is equal to $\chi(U)(a\varphi \otimes b)$ by equation \((*)\).  \(\blacksquare\)
Corollary 8.3 We have the equality $(\iota \circ \hat{\Delta})(U) = U_{12} U_{13}$.

**Proof**: Because $\chi(U)$ is the universal corepresentation of $(\hat{\Delta}, \hat{\Delta})$, corollary 6.9 implies that $(\hat{\Delta} \circ \iota) (\chi(U)) = \chi(U)_{13} \chi(U)_{23}$. The corollary follows easily from this equality. 

**References**

[1] E. Abe, Hopf Algebras. *Cambridge University Press* (1977).

[2] M. S. Dijkhuizen & T. H. Koornwinder, CQG algebras: a direct algebraic approach to compact quantum groups. *Lett. Math. Phys.* 32 (1994), 315–330.

[3] B. Drabant & A. Van Daele, Pairing and Quantum Double of Multiplier Hopf Algebras. *Preprint K.U.Leuven* (1996)

[4] E.G. Effros & Z.-J. Ruan, Discrete Quantum Groups I. The Haar Measure. *Int. J. of Math.* (1994).

[5] M. Enock & J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups. *Springer-Verlag, Berlin* (1992).

[6] E.C. Gootman & A.J. Lazar, Quantum Groups and Duality. *Reviews in Math. Physics* 5 No. 2 (1993), 417–451

[7] J. Kustermans & A. Van Daele, C*-algebraic quantum groups arising from algebraic quantum groups.(1996) To appear in *International Journal of Mathematics*.

[8] J. Kustermans, Two Notions of Discrete Quantum Groups. *Preprint K.U. Leuwen* (1994).

[9] J. Kustermans, Universal C*-algebraic quantum groups arising from algebraic quantum groups. *Preprint Odense Universitet* (1997)

[10] T. Masuda & Y. Nakagami A von Neumann Algebra Framework for the Duality of Quantum Groups. *Publications of the RIMS Kyoto University* 30 (1994), 799–850

[11] P. Podleś & S.L. Woronowicz, Quantum Deformation of the Lorentz Group. *Commun. Math. Phys.* 130 (1990), 381–431.

[12] J.M. Vallin, C*-algèbres de Hopf et C*-algèbres de Kac. *London Math. Soc.* 50(3) (1985), 131–174.

[13] A. Van Daele, An Algebraic Framework for Group Duality. (1996) To appear in *Advances of Mathematics*.

[14] A. Van Daele, Dual Pairs of Hopf *-algebras. *Bull. London Math. Soc.* 25 (1993), 209–230.

[15] A. Van Daele, Discrete Quantum Groups. *Journal of Algebra* 180 (1996), 431–444.

[16] A. Van Daele, Multiplier Hopf Algebras. *Trans. Am. Math. Soc.* 342 (1994), 917–932.

[17] A. Van Daele & Y. Zhang, Multiplier Hopf Algebras of Discrete Type. *Preprint K.U. Leuven & University of Antwerp* (1996)

[18] S.L. Woronowicz, Compact matrix pseudogroups. *Commun. Math. Phys.* 111 (1987), 613–665.

[19] S.L. Woronowicz, Pseudospaces, pseudogroups and Pontriagin duality. *Proceedings of the International Conference on Mathematical Physics, Lausanne* (1979), 407–412.