Abstract

In this paper, we discuss tensegrity from the perspective of nonlinear algebra in a manner accessible to undergraduates. We compute explicit examples and include the SAGE code so that readers can continue their own experiments and computations. The entire framework is a natural extension of linear equations of equilibrium, however, to describe the space of solutions will require (nonlinear) polynomials. The maximal minors of a certain matrix $A$ will cut out the algebraic variety of prestress solutions from the configuration space of a given structure. Imposing an additional list of inequalities defines the semi-algebraic set of tensegrity solutions for a specific choice of cables and bars. These inequalities will come as $b_i \geq 0$ for the entries of a vector $b$ in the left nullspace of $A$. Tools from algebraic geometry and commutative algebra, such as primary decomposition, can be used to single out certain spaces of configurations. Although at first it is all linear algebra, the examples will motivate the study of systems of polynomial equations, including the algebraic geometry and commutative algebra used to solve them. In particular, we will see the importance of varieties cut out by determinants of matrices.

1 Introduction

In 1948, the artist and sculptor Kenneth Snelson created a surprisingly stable structure from rigid bars and almost invisible wires. It looked like it should collapse, but it didn’t. He showed Buckminster Fuller, who then spread similar ideas across the world. These structures are fascinating to look at, simply because their existence seems an impossibility. Bars appear to float in midair, and yet cables and bars together can sometimes create a remarkably rigid object complete with both tension and structural integrity.

Figure 1: NASA has a robot named Super Ball Bot, inspired by tensegrity structures, adapted for landing on other planets.

Figure 2: Snelson’s Needle Tower, located in Washington D.C.

Figure 3: Kurilpa Bridge in Brisbane, Australia, is the world’s largest tensegrity bridge.

Although we will describe the situation in general, we first consider an example called the 3-prism. This structure consists of 6 nodes, 9 red cables, and 3 green bars. The cables are sometimes called strings, and can sustain stretching, but if compressed they will collapse. Like a child building a toy telephone out of a string and two cups, our strings/cables are useless unless they are under tension, stretched. The 3 bars
are able to sustain both stretching and compression. We will start by analyzing this structure, viewing it through the eyes of nonlinear algebra and applied algebraic geometry.

When someone says tensegrity structure they may mean different things. But one common thread, sufficient for the purposes of this article, is a balancing of forces at the nodes. Consider the following picture of the tensegrity 3-prism.

Thinking about one of the corners, what if the green bar did not exist? All the red cables are stretched, in tension, and so exert forces along their individual directions. When you stretch a rubber band, it wants to contract. Your fingers feel that force. If you felt the three blue vectors shown above, the overall force would be the vector sum, and hence look something like the pink force arrow. But the green bar in our tensegrity 3-prism is compressed. It wants to expand, and so exerts a force \textit{exactly and perfectly} opposite to the pink arrow. In fact, the exact locations of the 6 blue nodes of our tensegrity 3-prism are delicate. It turns out that the top equilateral triangle must be rotated at exactly $7\pi/6$ with respect to the bottom equilateral triangle in order for such a perfect balancing of forces to exist! If you search for videos of people building tensegrity structures, you will get a sense for their tricky behavior.

Consider the following graph, which is simply a set of nodes and another set of edges. We do not have to consider the nodes as having a \textit{location} or position in $\mathbb{R}^2$ or $\mathbb{R}^3$ or in fact any $\mathbb{R}^n$. We could simply consider the combinatorial object.

But we won’t. We will now endow our nodes with positions, meaning they are located somewhere in $\mathbb{R}^3$, for instance. There are many ways to do this. For example, we could place the nodes at the loca-
tions \((0,0,-1),(0,0,1),(0,1,0),(0,-1,0),(1,0,0),(-1,0,0)\). This would create a structure that looks like an octahedron. However, this structure would not admit any tensegrity solutions. This configuration of the 6 nodes of our graph inside \(\mathbb{R}^3\) makes it impossible for the kind of perfect cancellation of prestressing forces that we will require in this paper. However, do not despair. If instead we place our nodes at locations \((1,0,0),\left(\frac{-1}{\sqrt{2}},\frac{\sqrt{3}}{\sqrt{2}},0\right),\left(\frac{-1}{\sqrt{2}},\frac{-\sqrt{3}}{\sqrt{2}},0\right),\left(\frac{\sqrt{3}}{\sqrt{2}},\frac{-1}{\sqrt{2}},1\right),\left(\frac{-\sqrt{3}}{\sqrt{2}},\frac{-1}{\sqrt{2}},1\right),\left(0,1,1\right)\), then we get exactly the tensegrity 3-prism pictured above. This brings us to the main

**Question:** Given our favorite graph, what are all the ways to place its nodes into space, so that we can endow the edges with tensions and compressions but still obtain a perfect balancing of forces at each node?

We call a configuration (=placement of nodes) which allows balancing of forces a solution. We will actually be interested in two related questions. First, if any edge can sustain both compression and stretching, what configurations are allowed? We call these prestress solutions rather than tensegrity solutions because they do not differentiate between cables and bars. In this case, we will be able to describe explicit equations which give us all possible configurations. These solutions form an algebraic variety.

As a separate (harder) question, we can decide beforehand which edges can stretch and which can compress. For example, if we decide that we want several edges to be strings, then strings cannot compress. Strings/cables can only stretch. In this case, we will need both equations and inequalities. Entries of a certain vector will be required \(\geq 0\) whenever that entry comes from a string. Configurations which satisfy these additional requirements are called tensegrity solutions. These solutions form a semialgebraic set.

## 2 From electrical networks to tensegrity structures

Consider modelling an electrical network as a graph. We imagine voltages located on the nodes, driving currents of electricity to flow through the edges. Current flows from high voltage to low voltage, so what we really need to know are the voltage differences. Study the following picture, which includes also the matrix which gathers the voltage differences for us.

The matrix is the incidence matrix \(A\) and we will create one for our tensegrity structures as well. Each row comes from an edge, and each row has a 1 and a \(-1\). This corresponds to giving the edge a direction in our minds. At this point we note that our green flows along each edge are in the vector \(Av\), which we can then scale by a diagonal matrix of conductances (not pictured above) to produce the actual currents of electricity which would flow through the edges of our graph. This brings us to a vector of currents \(CAv\) produced from our voltages \(v\). At this point we would like to record how much current flows in and out of each individual node. A miracle of nature occurs. Because each column of our incidence matrix corresponds to a specific node in our graph, we can add up all the currents flowing in and out of each node simply by multiplying with \(A^T\). In this case we obtain:
The first entry tells us that 1 unit of current would leave node 1, while the second entry tells us 9 units of current would enter node 2. Analogous interpretations exist for the other entries.

Of course, to start analyzing actual electrical networks, we would need to use a few laws (Kirchoff and Ohm). Since our goal is to briefly mention electrical networks as an entry-way into understanding tensegrity structures, we will not venture farther in this direction. For an introduction, we recommend [10]. The main take-away is that we created an incidence matrix $A$ which took voltages concentrated at nodes, and turned their voltage differences into flows along the edges of our graph. $A$ also played another role as $A^T$. $A^T$ sums the net flows into each particular node. While $A$ took things from nodes to edges, $A^T$ took things from edges to nodes. The basic equations can be written as follows, where $f$ stands for forces or flows.

$$A^T C A v = f$$

In fact, for tensegrity structures the same equations will be relevant. We will build the matrix $A$ by realizing how it should take forces pulling on the nodes of our graph and turn them into stresses along the bars and cables of our structure. Finally, we will understand $A^T$ as taking any prestressed cables or bars, and summing up their effects on each node. Imagine building our structure for the first time, and as we are connecting our components, we prestress the cables and bars by various amounts that we collect in a vector $b$, one component of the vector for each bar and cable. Our structure will be be in a state of prestress if $A^T$ takes those prestresses to zero. $A^T b = 0$ gives a balance of forces at each node, so that our structure stays still. To this end, let’s examine the entries $-1$ and $1$ which appeared in our incidence matrix $A$.

3 What is special about -1 and 1?
To build the incidence matrix for an electrical network, it’s easiest to build it one row at a time. Each row corresponds to an edge, and the entries of the row are all zeros except for a 1 and −1. These are located at the entries corresponding to the nodes which are endpoints of your edge.

To build the incidence matrix \( A \) of a structure in the plane \( \mathbb{R}^2 \) or in space \( \mathbb{R}^3 \), or even in any higher dimensional space \( \mathbb{R}^n \), we will proceed similarly. We build \( A \) one row at a time, and almost all entries of our row will be zeros. The only change is to replace 1 and −1 with antipodal points on the relevant sphere of correct dimension. If our structure lives in the plane then 1 and −1 become antipodal points on the unit circle instead. If our structure lives in space, we replace 1 and −1 with points on the colloquial sphere. If our structure lives in \( \mathbb{R}^4 \), we can replace 1 and −1 by unit vectors on the sphere \( S^3 \subset \mathbb{R}^4 \). Consider building our matrix \( A \) for the following picture in the plane:

To build the row of our matrix corresponding to the red edge between nodes 1 and 2 we would use the red unit vectors in the picture. Although they are drawn on different circles, all the circles are unit circles and the two red vectors are antipodal points on the unit sphere \( S^1 \subset \mathbb{R}^2 \). Explicitly, the row of our matrix would be

\[
\frac{1}{\sqrt{(x_{11}-x_{21})^2 + (x_{12}-x_{22})^2}} \begin{bmatrix} x_{11}-x_{21}, & x_{12}-x_{22}, & x_{21}-x_{21}, & x_{22}-x_{12}, & 0, & 0, & 0, & 0 \\ \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{bmatrix}
\]

To be clear, \( x_{i1} \) is the \( x \)-coordinate for the \( i \)th node. There are now two entries of our row corresponding to each node. I think you can see how it goes. In \( \mathbb{R}^3 \) we will have antipodal unit vectors on \( S^2 \subset \mathbb{R}^3 \), and so each node will need 3 entries in the row of our matrix.

Let’s build another row for our picture above. The green edge connects nodes 1 and 3, and so our unit vectors will go in the first two entries, skip 2 entries of zeros, and then fill the next two entries. Here we go.

\[
\frac{1}{\sqrt{(x_{11}-x_{31})^2 + (x_{12}-x_{32})^2}} \begin{bmatrix} x_{11}-x_{31}, & x_{12}-x_{32}, & 0, & 0, & x_{31}-x_{11}, & x_{32}-x_{12}, & 0, & 0 \\ \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{bmatrix}
\]

In this way, you can easily build the matrix \( A \) one row/edge at a time. But why is this the case? Why should we build the incidence matrix \( A \) in this way? Consider the following blue force vector we have added to our picture.
Imagine our poor, unsuspecting structure is standing still. Then, a large blue force vector pulls on node 1. At least at the beginning, this blue force vector will induce tensions in the red and green edges connected to node 1. These tensions are sometimes called internal forces. The (internal) force induced on the green edge will be proportional to the dot product of the blue vector with the green unit vector. The (internal) force induced on the red edge will likewise come from the dot product of the blue force with the red unit vector. Remember in electrical networks, we imagined voltages sitting on nodes. The voltage differences drove current through the edge, or wire. Will this happen again? Consider yet another, light-blue force vector, this time sitting on node 2.

How can we imagine our red edge will react to two blue force vectors? If we had voltages, we would look at their difference, and that would drive a current in our red edge from high voltage to low. Here, we will consider the difference of dot products. The light-blue vector has a negative dot product with its red unit vector, causing the red edge to want to compress. The dark-blue vector has a positive dot product with its red unit vector, causing the red edge to want to stretch. Now we see the importance of taking antipodal points on the unit sphere, in order to keep everything straight. Suppose that the dark-blue vector has a larger dot product, then our red edge will overall feel a stretching (internal) force. It’s all working out nicely. In fact, it should be clear that the story will be the same in any dimension. Even if we have structures in $\mathbb{R}^5$ feeling 5-dimensional force vectors, antipodal points on the sphere $S^4$ should cover it.

Recall the tensegrity 3-prism from the introduction. If we don’t know where to position its 6 nodes inside $\mathbb{R}^3$, we could find out by first building the matrix $A$ and leaving the positions of the 6 nodes as variables $x_{ij}$. For example, the 4th node will eventually be located somewhere in 3 dimensional space, but for now we leave its location variable as $(x_{41}, x_{42}, x_{43})$. Consider the following matrix of size 12 by 18 (which we would only let a computer write down for us since even to print it here we had to split it into the first 9 columns and then the next 9 columns below!). Spoiler: We will eventually use this matrix to find the angle $\theta = 7\pi/6$. 


by solving a single polynomial equation in one variable coming from this matrix. But let’s take the scenic route, admiring matrices along the way.

\[
\begin{pmatrix}
  x_{11} - x_{21} & x_{12} - x_{22} & x_{13} - x_{23} & -x_{11} + x_{21} & -x_{12} + x_{22} & -x_{13} + x_{23} & 0 & 0 & 0 \\
  x_{11} - x_{31} & x_{12} - x_{32} & x_{13} - x_{33} & 0 & 0 & 0 & -x_{11} + x_{31} & -x_{12} + x_{32} & -x_{13} + x_{33} \\
  x_{11} - x_{41} & x_{12} - x_{42} & x_{13} - x_{43} & 0 & 0 & 0 & 0 & x_{11} - x_{41} & x_{12} - x_{42} & -x_{13} + x_{43} \\
  x_{11} - x_{51} & x_{12} - x_{52} & x_{13} - x_{53} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  x_{11} - x_{61} & x_{12} - x_{62} & x_{13} - x_{63} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  -x_{41} + x_{51} & -x_{42} + x_{52} & x_{43} - x_{53} & 0 & 0 & 0 & 0 & x_{41} - x_{51} & -x_{42} + x_{52} & -x_{43} + x_{53} \\
  -x_{41} + x_{61} & -x_{42} + x_{62} & x_{43} - x_{63} & 0 & 0 & 0 & 0 & -x_{41} + x_{61} & -x_{42} + x_{62} & -x_{43} + x_{63} \\
  -x_{51} + x_{61} & -x_{52} + x_{62} & -x_{53} + x_{63} & 0 & 0 & 0 & 0 & -x_{51} + x_{61} & -x_{52} + x_{62} & -x_{53} + x_{63} \\
  -x_{51} + x_{61} & -x_{52} + x_{62} & -x_{53} + x_{63} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  -x_{61} + x_{61} & -x_{62} + x_{62} & -x_{63} + x_{63} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \end{pmatrix}
\]

This matrix (which we split in half to print) contains a lot of information. We know there are 12 edges because there are 12 rows. With more thought, we see 6 nodes since each comes with 3 dimensions of space, giving the 18 columns. It is perhaps worth your time to pick one or two rows of this matrix, track what’s happening in that row, and understand which edge it corresponds to in the graph with nodes 1, 2, 3, 4, 5, 6. This matrix gives our graph embedded in all possible ways into 3 dimensional space, since the node locations are variables. Using the SAGE code provided on the author’s website, you can take your favorite graph and immediately print the corresponding matrix \( A \) embedding it in \( \mathbb{R}^n \) for whatever dimension you desire. You could print it out and create a (large) poster. For example, the code which produced the matrix above goes as follows:

```python
1 nodes = [1, 2, 3, 4, 5, 6]  
2 edges = [(1, 2), (2, 3), (3, 1), (4, 5), (5, 6), (6, 4),  
3 (1, 5), (4, 3), (2, 6), (1, 4), (2, 5), (3, 6)]  
4 t = VarTruss(nodes, edges, dim=3)  
5 show(t.A0)
```

If we want to set node 2 precisely at the location \( \left( \frac{2}{3}, \frac{5}{7}, \frac{11}{13} \right) \), we simply replace \( x_{21} \) by \( \frac{2}{3} \), \( x_{22} \) by \( \frac{5}{7} \), and \( x_{23} \) by \( \frac{11}{13} \). If instead we want to only consider situations where node 2 is located along the \( x \) axis, we would leave \( x_{21} \) variable, but replace \( x_{22} \) by 0 and \( x_{23} \) by 0. We can refer to this kind of substitution or specializing of variables by the name setting locations. We could also say that we are slicing configuration space. The entire configuration space of 6 nodes in \( \mathbb{R}^3 \) is 18 dimensional because we have 18 variables \( x_{ij} \). Just as the line \( (2 + t, 3 - 4t) \) is a one-dimensional slice of the plane, replacing two variables of freedom \( x, y \) with only one variable of freedom \( t \), whenever we restrict our attention to configurations which require fewer than 18 variables, we are slicing configuration space. Easy ways to accomplish this include simply removing a variable like \( x_{21} \) by setting it to the value 2/3, but we will see other ways soon.

This notion of dimension reduction has been used in very interesting ways. For example, in the article Auxetic deformations and elliptic curves \[1\], the authors study structures similar to ours, but they are not interested in tensegrity. At one point, they replace a cubic polynomial in 6 variables with one in only 3 variables, creating a ternary cubic. This allowed them to use invariants of elliptic curves to decide whether their structures have counter-intuitive expansions called auxetic deformations, relevant for materials called zeolites, among others. In the articles \[2, 3\] the authors study our 3-prism and many other tensegrity structures by using symmetry. The dihedral group and other finite subgroups of \( S_3 \) can be used to algorithmically produce tensegrity structures. In fact, new tensegrity structures were discovered using this method. Clearly, there are many ways to study tensegrity. The most hands-on approach is to create them yourself, from strings and bars, by trial-and-error. In this paper we produce polynomial equations and inequalities which give us all possible tensegrity structures for a specific graph. This approach is elucidated
in [12], where they call our matrix \( A \) the *rigidity matrix*. The authors find a single polynomial whose zeros pick out configurations where certain structures fail to be *infinitesimally rigid*, which they explain. They also show why *projective space* is the natural setting for many of our questions. You may have noticed that we ignored the factors like

\[
\frac{1}{\sqrt{(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2 + (x_{13} - x_{23})^2}}
\]

which, by all rights, should appear as multipliers of each row. But polynomials do not have squareroots or denominators. This brings us to the next section.

## 4 Towards an algebraic variety

As we analyze tensegrity structures, we will find a wonderful situation. We will have equations. Often, when we try to understand some phenomenon, say, the stock market, we run into a problem: we cannot describe the phenomenon very well. However, when our space of solutions is an *algebraic variety*, this means we can describe our space as all the solutions to a finite list of *polynomial equations*. For an introduction to this subject, we recommend [4] or [11]. Often in mathematics we resort to approximate solutions. For example, the article [5] explains a numerical method of finding tensegrity solutions, and they demonstrate it on the 3-prism, our example. But in the case of algebraic varieties, we can sometimes hope for exact solutions. And, instead of finding one solution, we can hope to describe *all solutions*. This is because we have equations, and because those equations are polynomials.

As a quick example, a parabola is an algebraic variety because it is the set of all solutions to the polynomial equation \( y - x^2 = 0 \). As another example, what if the space of solutions we want was given by the equations

\[
\{ xy - x^3 = 0, \quad xz - x^4 = 0, \quad x^2 yz - x^4 z - x^5 y + x^7 = 0 \}?
\]

By this we mean to consider all the points \((x,y,z)\) which simultaneously satisfy all three equations. In fact, (although it’s not obvious) this *algebraic variety* has two irreducible components, drawn below.

There is the plane \( x = 0 \), whose points \((0,y,z)\) are all part of our *space of solutions*, and there is the *twisted cubic* coming out of the plane at \((0,0,0)\) and extending (sort of) upwards and downwards along all the points \((t,t^2,t^3)\), which are also all part of our *space of solutions* (we recommend plugging in to check).

I choose my favorite graph and ask, what are all the configurations which allow some kind of self-balancing equilibrium? The edges are stretched or compressed, but miraculously all the forces at each node cancel. The answer is in fact an algebraic variety. That means it is some *space* like the one in the picture above. It could have pieces that are 7-dimensional, and other pieces that are 17-dimensional, and another piece that is 4-dimensional. The algebraic variety in the picture above had a 2-dimensional piece (the plane) and a 1-dimensional piece (the twisted cubic). Of course, when we have 6 nodes in \( \mathbb{R}^3 \) our algebraic variety lives
Let’s record the tensions in our edges as numbers in a vector $b$. The vector will have one entry $b_i$ for each edge, and if that entry is positive $b_i > 0$, it will mean a stretching of that edge, and if the entry is negative $b_i < 0$, it will mean a compression. Perhaps our vector looks like

$$b^T = (1.000, 1.000, 1.000, 1.000, 1.000, 1.000, 1.732, 1.732, 1.732, -1.732, -1.732, -1.732).$$

This would simply mean we are imagining 1 unit of stretching tension on each of the first 6 edges of our graph, then the next 3 edges are endowed with 1.732 units of stretching tension, while the last 3 edges are compressed with 1.732 units of compression tension. Perhaps the last 3 edges would be green bars, and the first 9 edges would be red strings.

We say that a vector is in the **left nullspace** of a matrix $A$ if $b^T A = 0$. Alternatively, we could say that $A^T b = 0$, it does not matter. In fact, reading this paper might convince you of the importance of becoming fluid and familiar with linear algebra, and in particular the four ways to multiply matrices. What would this mean for our matrix $A$ and our vector $b$? Think about the first column of $A$. It contains the $x$ values for the unit vectors attached to node 1. When $b^T$ hits our matrix $A$, it becomes some row vector named $b^T A$. If $b$ was the zero vector, then $b^T A$ will also be the zero vector. But if $b$ is not the zero vector, then it is somewhat amazing that $b^T A$ would be. In particular, looking at the first column of $A$, this means that $b^T$ contained exactly the correct values so that all the $x$ components of the unit vectors attached to node 1 cancel each other out, producing a balance of the $x$ coordinates at least, which shows up as the first entry of the vector $b^T A$ being 0 (you will probably need to look back at how we create $A$ and think about this for a while). But if $b^T A$ creates an entire vector of zeros, this means $b^T$ contained a delicate balance of quantities which made the $x$ components of the unit vectors at node 1 cancel out, and the $y$ components of the unit vectors at node 1 cancel out (column 2 of $A$), and the $z$ components too (column 3 of $A$). And then the same for node 2 which starts at column 4 of $A$. And so on and so forth.

At this point, it should be clear that the existence of a vector $b^T$ in the left nullspace of $A$ means we can **prestress** the edges in a way that perfectly balances at every node. Now we can understand why most of our 18-dimensional space of configurations will not work. Consider the 12 by 18 matrix filled with $x_{ij}$ for the 3-prism. If we make the substitutions

$$(0, 0, -1), (0, 0, 1), (0, 1, 0), (0, -1, 0), (1, 0, 0), (-1, 0, 0),$$

replacing all our variables $x_{ij}$ by these numbers, the resulting matrix will not have a left nullspace (other than the zero vector $b^T = 0$). However, if instead we make the substitutions

$$(1, 0, 0), (-1/2, \sqrt{3}/2, 0), (-1/2, -\sqrt{3}/2, 0), (-\sqrt{3}/2, -1/2, 1), (\sqrt{3}/2, -1/2, 1), (0, 1, 1)$$

we do get a left nullspace! In fact, the left nullspace is 1-dimensional, spanned by exactly the vector $b^T$ we wrote above. Here is the code required to make this calculation.

```python
nodes = [1, 2, 3, 4, 5, 6]
edges = [(1, 2), (2, 3), (3, 1), (4, 5), (5, 6), (6, 4),
(1, 5), (4, 3), (2, 6), (1, 4), (2, 5), (3, 6)]
t = VarTruss(nodes, edges, dim=3)
t.set_location(1, loc=(1,0,0))
t.set_location(2, loc=(-1/2, sqrt(3)/2,0))
t.set_location(3, loc=(-1/2,-sqrt(3)/2,0))
t.set_location(4, loc=(-sqrt(3)/2,-1/2,1))
t.set_location(5, loc=(sqrt(3)/2,-1/2,1))
t.set_location(6, loc=(0,1,1))
```
We hope you are asking yourself: But how can we find such a solution? First, we find the equations. Recall that all possible configurations of our graph are encoded in a matrix \( A \) full of variables \( x_{ij} \). Finding the space of solutions means finding all the configurations \( x_{ij} \) that admit a nonzero left nullspace. What are all the numbers \( x_{ij} \) such that when we substitute, our matrix \( A \) admits some nonzero vector \( b^T A = 0 \)? Fortunately, there is a condition on the matrix which alerts us to the existence of a left nullspace: the vanishing of all maximal minors.

If our matrix \( A \) is taller than it is wide, like a \( 9 \times 4 \) matrix, then we are guaranteed nonzero \( b^T \) in the left nullspace. However, for most of our graphs, there will be fewer rows than columns. This means \( A \) will look short and wide, like our \( 12 \times 18 \) matrix \( A \) above. In every case, a left nullspace will exist exactly when the rows of \( A \) are linearly dependent. But linear dependence is detected by zero determinants. If we have 12 rows which are each 18 entries long, in order to make sure they are dependent rows, we must verify that all 18 choose 12 = 18564 maximal minors are zero (choose 12 of the 18 columns, that gives you a square matrix, take its determinant, get the number zero, that’s what we want). If even one of those determinant minors is nonzero, that means that our 12 vectors stick out from each other inside 18-dimensional space. There is no way to combine them and get the zero vector, unless we take the zero combination \( b^T = 0 \).

In fact, this discussion of minors being zero or nonzero explains why we can disregard the factors like

\[
\frac{1}{\sqrt{(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2 + (x_{13} - x_{23})^2}}.
\]

If we don’t put our nodes on top of each other, these factors will be nonzero positive scalars. Multiplying a row by a scalar does not change the determinant from being zero or nonzero. Thus, in our search for solutions is one we have seen: \( x_{ij} \) which simultaneously solve all 18564 equations are exactly our solutions. One such solution is one we have seen:

\[
(1, 0, 0), (-1/2, \sqrt{3}/2, 0), (-1/2, -\sqrt{3}/2, 0), (-\sqrt{3}/2, -1/2, 1), (\sqrt{3}/2, -1/2, 1), (0, 1, 1)
\]

Here is the beginning of one such equation taken from columns 1 through 12 of our 18-column matrix \( A \):

\[
\begin{vmatrix}
\begin{array}{cccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{vmatrix}
\]
Are there any others? Will the solutions form a line like solutions to \(2x + 5y = 7\)? Will the solutions have pieces of different dimensions and shapes like our example above? What does our algebraic variety cut out by 18564 equations look like? We will obviously need tools to help us deal with these equations.

5 What to do with all these equations?

Now that we have a list of equations, we would like to better understand their space of solutions, an algebraic variety. We won’t introduce the theory required, but instead we demonstrate some of the tools that are available for dealing with polynomial equations. We hope this serves as motivation to open \([4]\) or \([11]\). Let’s reconsider the example of the plane and twisted cubic. From now on, when we list something like \(xy - x^3\) and call it an equation, what we mean implicitly is \(xy - x^3 = 0\). You should really get used to this, as its very helpful and often speeds up your thinking. Consider the following code:

```python
1 R.<x,y,z> = PolynomialRing(QQ,3)
2 eqns = [x*y-x^3, x+z-x^4, x^2*y*z - x^4*z - x^5*y + x^7]
3 I = R.ideal(eqns)
4 AP = I.associated_primes()
5 for p in AP:
  print p.gens()
```

The output (shown below) tells us that there are two irreducible components (pieces) of our variety, and gives us (new but related) generating equations for each component separately. In this case, we find a 2 dimensional plane, and a 1 dimensional twisted cubic, although it is not obvious that the 3 equations in the second list cut out the curve that we drew earlier. By using Geogebra.org/3d you can try to confirm it on your own. This is highly recommended!

```plaintext
1 [x]
2 [y^2 - x*z, x*y - z, x^2 - y]
```

However, since our ultimate goal is to better understand the algebraic variety of solutions which lives somewhere in 18 dimensional space, we must give up on the desire to draw them. Instead, we should learn to appreciate explicit computational tools like those above. Although in small examples we might be able to work directly with our original equations (perhaps realizing that one of them is the product of the other two then factoring out an \(x\)) that is because there are only three equations, and they only involve three variables. The point is that we have tools to deal with complicated polynomial equations of many variables, automatically.

Now we try a bigger example discussed in \([7]\), although their approach is different. We take our 3-prism graph and embed it into space by creating the matrix \(A\). Then we set locations for all its nodes except node 6. This means that we replace 18 variables by only 3, the coordinates of node 6. We print out the corresponding 12 by 18 matrix, and then gather all of its 18654 minors of size 12 \(\times\) 12. In fact only 8454 of these minors will be nonzero. These minors are polynomials in the only remaining variables \(x_{61}, x_{62}, x_{63}\). Then, we calculate something called the associated primes for the ideal generated by these 8454 equations. Later you can learn what this means from \([4]\) or \([11]\), but for now its enough to know that we are asking the computer to make sense of the 8454 nonzero equations we are handing it. In this case, it succeeds. It tells us that the algebraic variety we have described by these 8454 equations is equivalent to specifying only 1 equation. This is quite amazing. See the code below.

```python
1 nodes = [1,2,3,4,5,6]
2 edges = [(1,2),(2,3),(3,1),(4,5),(5,6),(6,4),
3 (1,5),(4,3),(2,6),(1,4),(2,5),(3,6)]
4 t = VarTruss(nodes, edges, dim=3)
5 t.set_location(1, loc=(0, 0, 0))
6 t.set_location(2, loc=(1, 1, 1))
7 t.set_location(3, loc=(0, 1, 0))
8 t.set_location(4, loc=(1, 0, 0))
```

11
t.set_location(5, loc=(0, 0, 1))
t.set_location(6, loc=(x61, x62, x63))

\[ A = t.A0 \]

show(A)

\[ AP = \text{associated_primes}(A, t.X.list()) \]

for p in AP:
    print p.gens()

Below you see the output. The single equation listed generates the one associated prime ideal corresponding to our algebraic variety of solutions.

There are 8454 nonzero equations to start with.

\[ [x61^2 - x62^2 - x63^2 - x61 + x62 + x63] \]

Because we only left 3 variables out of the 18, we can actually draw the algebraic variety this single equation cuts out.

\[ r = 2 \]

plt = implicit_plot3d(x61^2 - x62^2 - x63^2 - x61 + x62 + x63, (x61,-r,r), (x62,-r,r), (x63,-r,r))

show(plt)

Maybe there is still hope to understand the algebraic variety of solutions.
6 The 3-prism and its special angle

Earlier, we mentioned that the 3-prism must be rotated at exactly $\theta = 7\pi/6$ in order for a solution to exist. We will examine this now. First, we recall our analogy with electrical networks. Often, it is necessary to ground a node in your electrical network. This means removing the corresponding column from your incidence matrix. Often when you build structures, you attach them to things. Perhaps your structure is attached to the ground. Maybe the wall. When we say attached, we mean that you treat those parts of your structure as glued or affixed or nailed into the ground. They do not move. For the 3-prism, it’s natural to attach the bottom triangle (of 3 nodes) to the ground. Carrying this out, we achieve a great simplification.

We can delete the first 9 columns of $A$, which come from the first 3 nodes. We do not worry about achieving a balance of forces at those nodes, because they aren’t going anywhere. They are fixed to the ground. This first matrix with 9 columns deleted is the following:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{41} - x_{51} & x_{42} - x_{52} & x_{43} - x_{53} & -x_{51} + x_{61} & x_{52} - x_{62} & x_{53} - x_{63} \\
0 & 0 & 0 & x_{41} - x_{61} & x_{42} - x_{62} & x_{43} - x_{63} & 0 & 0 & 0 \\
0 & 0 & 0 & -x_{51} + x_{61} & x_{52} - x_{62} & x_{53} - x_{63} & 0 & 0 & 0 \\
-x_{11} + x_{41} & -x_{12} + x_{42} & -x_{13} + x_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

We notice the first three rows are entirely comprised of zeros. Since we have fixed nodes 1 through 3 to the ground, the 3 red strings connecting them are now irrelevant. Of course, those three basis vectors are part of the left nullspace, but we are interested in tensions on the other strings and bars now! The other strings and bars are connected to nodes which are not fixed to the ground. These are the ones we still need to balance. Hence we delete those 3 rows, obtaining the following $9 \times 9$ matrix:

\[
\begin{bmatrix}
x_{41} - x_{51} & x_{42} - x_{52} & x_{43} - x_{53} & -x_{51} + x_{61} & x_{52} - x_{62} & x_{53} - x_{63} & 0 & 0 & 0 \\
x_{41} - x_{61} & x_{42} - x_{62} & x_{43} - x_{63} & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_{51} + x_{61} & x_{52} - x_{62} & x_{53} - x_{63} & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_{11} + x_{41} & -x_{12} + x_{42} & -x_{13} + x_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Finally, here is the code that accomplishes this.

```python
# Get rid of first 3 rows
A = t.A.matrix_from_rows([3, 11])
show(A)
```

The output counts the number of terms in our determinant, and gives its degree.

```
10944
```

By fixing 3 nodes to the ground (which is an entirely reasonable thing to do!) we have greatly decreased the complexity of our situation. Instead of 18564 equations, we have just one.

At this point, we can slice configuration space, reducing 18 variables to 1 variable $\theta$. We accomplish this by setting locations for the nodes in the bottom and top triangles, but leaving a rotation angle $\theta$ of freedom.
in the top nodes. In doing this, think geometrically. We are taking a one-parameter family of configurations (a curve) within 18-dimensional space, and asking, does it touch our algebraic variety of solutions? If it touches, that is a solution. This is what the equations mean.

```python
1  nodes = [1, 2, 3, 4, 5, 6]
2  edges = [(1, 2), (2, 3), (3, 1), (4, 5), (5, 6), (6, 4),
3            (1, 5), (4, 3), (2, 6), (1, 4), (2, 5), (3, 6)]
4  t = VarTruss(nodes, edges, dim=3)
5
6  # the bottom triangle
7  t.set_location(1, loc=(1, 0, 0))
8  t.set_location(2, loc=(-1/2, sqrt(3)/2, 0))
9  t.set_location(3, loc=(-1/2, -sqrt(3)/2, 0))
10
11  # the top triangle
12  theta = var('theta')
13  t.set_location(4, loc=(cos(theta), sin(theta), 1))
14  t.set_location(5, loc=(cos(theta + 2*pi/3), sin(theta + 2*pi/3), 1))
15  t.set_location(6, loc=(cos(theta + 4*pi/3), sin(theta + 4*pi/3), 1))
16
17  t.fix_nodes([(1, 2, 3)])
18  A = t.A.matrix_from_rows([3, ..11])
19  print A.det()
```

After slicing, our determinant is a single equation in one variable \( \theta \):

\[
\begin{align*}
9/4*sqrt(3)cos(2/3*pi + theta)*sin(4/3*pi + theta) & - 9/4*sqrt(3)cos(theta)*sin(2/3*pi + theta) - 9/4*sqrt(3)cos(4/3*pi + theta)*sin(2/3*pi + theta) - 9/4*sqrt(3)cos(3)cos(2/3*pi + theta)*sin(4/3*pi + theta) - 9/4*sqrt(3)cos(4/3*pi + theta)^2 + 9/4*sqrt(3)cos(theta)*sin(2/3*pi + theta)^2 \quad - 9/4*sqrt(3)cos(2/3*pi + theta)*sin(4/3*pi + theta)*sin(theta) + 9/4*sqrt(3)cos(4/3*pi + theta)*sin(2/3*pi + theta)*sin(theta) + 9/4*sqrt(3)cos(4/3*pi + theta)*sin(2/3*pi + theta)sin(theta) - 9/4*sqrt(3)cos(3)cos(theta)*sin(2/3*pi + theta)sin(theta) - 9/4*sqrt(3)cos(3)cos(2/3*pi + theta)*sin(4/3*pi + theta)sin(theta) - 9/4*sqrt(3)cos(4/3*pi + theta)^2 - 9/2*cos(4/3*pi + theta)*cos(2/3*pi + theta)*sin(4/3*pi + theta) + 9/4*cos(2/3*pi + theta)^2*sin(4/3*pi + theta) + 9/2*cos(4/3*pi + theta)*cos(2/3*pi + theta)*sin(4/3*pi + theta) - 9/4*cos(4/3*pi + theta)^2*sin(4/3*pi + theta) + 9/2*cos(4/3*pi + theta)*cos(2/3*pi + theta)*sin(4/3*pi + theta) - 9/4*cos(2/3*pi + theta)^2*sin(4/3*pi + theta) + 9/4*cos(4/3*pi + theta)*cos(2/3*pi + theta)*sin(4/3*pi + theta) - 9/4*cos(4/3*pi + theta)*cos(2/3*pi + theta)*sin(4/3*pi + theta) - 9/4*cos(4/3*pi + theta)*cos(2/3*pi + theta)*sin(4/3*pi + theta)
\end{align*}
\]

Solving this one equation for \( \theta \) we obtain:

```python
1  sol = solve(A.det(), theta, to_poly_solve=True)
2  show(sol)
```

\[
\left[ \theta = -\frac{5}{6}\pi + 2\pi z_1, \quad \theta = \frac{1}{6}\pi + 2\pi z_2 \right]
\]

If we plug in each of these values for \( \theta \) and then ask for the left nullspace of the resulting matrix (which no longer has any variable entries) we find the following two vectors.

```python
1  [ 1.0000  1.0000  1.0000  1.7320  1.7320  1.7320  -1.7320  -1.7320  -1.7320  -1.7320]
2  [ 1.0000  1.0000  1.0000  -1.7320  -1.7320  -1.7320  1.7320  1.7320  1.7320  1.7320]
```

After a moment’s thought we see that one solution \( \theta = -5\pi/6 = 7\pi/6 \) gives us what we expected: the last 3 edges compressed, since they correspond to green bars. However, the other solution \( \theta = \pi/6 \) means the computer wants us to compress the 3 red (vertical) strings and stretch the 3 green bars. Of course, the computer did not know which edges of our graph we wanted to be strings, and which edges we wanted to be bars. So it considered both options. Exactly this issue brings us to the next section.
7 From prestress to tensegrity (and equations to inequalities)

Say we have decided beforehand which edges we want to be cables (lightweight and inexpensive) and which we want to be bars (heavier and perhaps more costly). This means some left nullspace vectors $b^T$ will be unacceptable. In turn, the configurations $x_{ij}$ which produced them are also unacceptable. We will have fewer solutions. If we want all of our edges to be strings/cables except for the last 3 edges, which we allow to be green bars, then the second vector from the end of the previous section is unacceptable. It asks us to compress strings. This will not do.

The requirement here is that any entry of $b^T$ corresponding to a string/cable must be nonnegative. Once we have decided ahead of time which edges are strings and which are bars, our algebraic variety of configurations $x_{ij}$ cut out by the maximal minors of the matrix $A$ includes such unacceptable non-solutions. It turns out that equations alone are insufficient. We will also need inequalities. However, they will be polynomial inequalities. The corresponding sets of configurations they determine are called semi-algebraic sets. Let’s see an example.

Picture a single, lonesome, green bar, standing upright. We tie three cables from the top of the bar stretching down to the ground, reaching the ground at 3 points $(1,0,0), (0,1,0), (0,0,0)$. Then, we consider moving the bottom of the bar to various locations $(x,y,0)$ along the ground, keeping the top of the bar where it is. If we started the bottom of the bar at the barycenter of the triangle below, we would have the following picture:

This tent-like structure comes from a graph with 5 nodes and 4 edges. We have decided that the edge connecting node 4 and node 5 should be a green bar. The other edges connecting node 5 with node 1, 2, and 3 will be red strings. We imagine fixing nodes 1, 2, 3, 4 to the ground. We know nodes 1, 2, 3 should form a triangle at $(0,0,0), (1,0,0), (0,1,0)$, and we know node 5 should hover above its barycenter at height $1, (1/3,1/3,1)$. However, we are not sure where to fix node 4. (Of course, we pick this example because we know exactly what locations will work for node 4. Hopefully you see that our structure will be stable as long as node 4 is fixed into the ground somewhere inside the triangle.) We will test our understanding by seeing this result come directly from the equations and inequalities we will build.

```
1 nodes = [1,2,3,4,5]
2 edges = [(1,5),(2,5),(3,5),(4,5)] # create 3 cables and 1 bar
3 t = VarTruss(nodes, edges, dim=3)
4 # the bottom triangle
5 t.set_location(1, loc=(1,0,0))
6 t.set_location(2, loc=(0,1,0))
7 t.set_location(3, loc=(0,0,0))
```
# the endpoints of the one bar

```
x, y = var(‘x y’)  
t.set_location(4, loc=(x, y, 0))  
t.set_location(5, loc=(1/3, 1/3, 1))  
t.fix_nodes([1, 2, 3, 4])  
show(t.A)
```

$$
\begin{pmatrix}
  -\frac{2}{3} & -\frac{1}{3} & 1  
  \frac{1}{3} & 1 & 1 
  -x + \frac{1}{3} & -y + \frac{1}{3} & 1 
\end{pmatrix}
$$

For once, our matrix fits comfortably on the page. Notice that we are fixing nodes 1-4 to the ground. This made the matrix much smaller. In fact, it deleted so many columns that we are guaranteed a left nullspace. Any 4 by 3 matrix has a nonzero left nullspace. However, because we have decided that only one of our edges can be compressed (the green bar), we have to be careful about what we call solutions.

```
A = t.A  
LK = A.left_kernel()  
b = LK.basis_matrix()  
show(b)
```

$$
\begin{pmatrix}
  1 & y & \frac{x+y-1}{x} & -\frac{1}{x}
\end{pmatrix}
$$

You’ll notice that in the process of gaussian elimination, it is necessary to divide by the pivot entries of your matrix. In effect, this produces entries of our nullspace vector which are *rational functions* in our variables \(x, y\). However, when you have an equation of the form

$$
\frac{p(x, y)}{q(x, y)} \geq 0
$$

you can *clear denominators*, obtaining simply

$$
p(x, y) \geq 0.
$$

which is now a *polynomial inequality*.

```
bprime = b[0]*x  # clear denominators  
ineqs = [bprime[i] >=0 for i in [0, 1, 2]]  # the first three entries come from strings , >= 0  
print(ineqs)  
region_plot(ineqs, (x, -1, 2), (y, -1, 2))
```

```
[x >= 0, y >= 0, -x - y + 1 >= 0]
```

We can clearly see that this process produced the space of acceptable *tensegrity solutions*, confirming our intuition that we can place the bottom of the tent pole anywhere within the triangle.
8 Conclusion

We hope to have shown you one possible entrance into the world of algebraic varieties. You can use the SAGE code that accompanies this paper to embed your favorite graph in $\mathbb{R}^n$ and perhaps start making slices of configuration space. You can test out the tools of primary decomposition and Groebner bases to describe irreducible pieces of your solution space. And, you can get a feel for when equations become too large for even a computer to handle. Finally, you will start to understand that equations have meaning, and sometimes that meaning is hidden. But it is there nonetheless, waiting to be discovered. Sometimes it’s only when you find your current toolkit inadequate to solve some of these problems that you realize the need for better understanding. Perhaps now you have motivation to learn more.

The SAGE code used in this article will be posted to the author’s website. Explorations using this code could be suitable for an extended project near the end of a course on Linear Algebra or afterwards.

Acknowledgements: The author would like to thank Andrew Frohmader, Istvan Lauko, and Gabriella Pinter for discussions about tensegrity and applied math throughout the Spring of 2019.

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