CHERN-GAUSS-BONNET FORMULA FOR SINGULAR YAMABE METRICS IN DIMENSION FOUR

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Abstract. We derive a formula of Chern-Gauss-Bonnet type for the Euler characteristic of a four dimensional manifold-with-boundary in terms of the geometry of the Loewner-Nirenberg singular Yamabe metric in a prescribed conformal class. The formula involves the renormalized volume and a boundary integral. It is shown that if the boundary is umbilic, then the sum of the renormalized volume and the boundary integral is a conformal invariant. Analogous results are proved for asymptotically hyperbolic metrics in dimension four for which the second elementary symmetric function of the eigenvalues of the Schouten tensor is constant. Extensions and generalizations of these results are discussed. Finally, a general result is proved identifying the infinitesimal anomaly of the renormalized volume of an asymptotically hyperbolic metric in terms of its renormalized volume coefficients, and used to outline alternate proofs of the conformal invariance of the renormalized volume plus boundary integral.

1. Introduction

In this paper we derive a Chern-Gauss-Bonnet formula for singular Yamabe metrics in dimension 4, and also analyze related questions for metrics solving certain generalizations of the singular Yamabe problem. In our context, a singular Yamabe metric means an asymptotically hyperbolic metric $g$ on the interior of a smooth, compact, connected manifold-with-boundary $(M^{n+1}, \partial M)$ with constant scalar curvature $R = -n(n+1)$. Contained in the class of singular Yamabe metrics are the Poincaré-Einstein metrics: those asymptotically hyperbolic metrics satisfying $\text{Ric}(g) = -ng$. Clearly any Poincaré-Einstein metric is a singular Yamabe metric (ignoring here the issue of the boundary regularity which is assumed).

The basic result concerning singular Yamabe metrics ([LN], [AM], [M], [ACF]) is that for $n \geq 2$, given any smooth metric $\overline{g}$ on $M$, there exists a unique defining function $u$ for $\partial M$ so that $g = u^{-2} \overline{g}$ is a singular Yamabe metric. The metric $g$ depends only on the conformal class $[\overline{g}]$ determined by $\overline{g}$: if $\overline{g}$ is replaced by $\Omega^2 \overline{g}$ with $0 < \Omega \in C^\infty(M)$, then $u$ is replaced by $\Omega u$ so that $g$ is unchanged. If $\rho$ is any defining function for $\partial M$ (not necessarily $C^\infty$), the metric $\overline{g} = \rho^2 g$ is called a compactification of $g$. A smooth metric $\overline{g}$ is in particular a compactification of the singular Yamabe metric which it determines, realized by taking $\rho = u$. We will denote by $h = \overline{g}|_{\partial M}$ the metric on $\partial M$ induced by $\overline{g}$.

It follows from [M], [ACF] that the defining function $u$ determined by a smooth metric $\overline{g}$ has an asymptotic expansion of the form

$$u = r + u^{(2)} r^2 + \ldots + u^{(n+1)} r^{n+1} + L r^{n+2} \log r + O(r^{n+2})$$

relative to the product identification of a collar neighborhood of $\partial M$ induced by $\overline{g}$ (see the beginning of [2]). In particular $r$ is the $\overline{g}$-distance to $\partial M$. The coefficients $L$ and the indicated $u^{(j)}$ are smooth, locally determined functions on $\partial M$.

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Volume renormalization for singular Yamabe metrics was considered in [G3], [GoW], generalizing the discussion for Poincaré-Einstein metrics in [G1]. We follow the formulation in [G3]. As \( \epsilon \to 0 \),

\[
\text{(1.2)} \quad \text{Vol}_g\{ r > \epsilon \} = c_0 \epsilon^{-n} + c_1 \epsilon^{-n+1} + \cdots + c_{n-1} \epsilon^{-1} + \mathcal{E} \log \frac{1}{\epsilon} + V + o(1),
\]

where each of \( \mathcal{E} \) and the \( c_j \)'s is the integral over \( \partial M \) of a local invariant of the extrinsic geometry of \( \partial M \) with respect to \( \bar{g} \). The log coefficient \( \mathcal{E} \) can be viewed as an energy of the submanifold \( \partial M \) of \( (M, \bar{g}) \) which is invariant under conformal rescalings of \( \bar{g} \). The renormalized volume \( V = V(g, \bar{g}) \) is a globally determined quantity which in general depends on \( \bar{g} \). But its anomaly, i.e. its change under conformal rescaling \( \bar{g} = e^{2\omega} \bar{g} \), \( \omega \in C^\infty(M) \), is locally determined; it has the form

\[
\text{(1.3)} \quad V(g, \bar{g}) - V(g, \bar{g}) = \int_{\partial M} P_{\bar{g}}(\omega) \, dh,
\]

where \( P_{\bar{g}}(\omega) \) is a polynomial nonlinear differential operator determined by the local geometry of \( \partial M \) in the metric \( \bar{g} \). If \( g \) is Poincaré-Einstein and one restricts to geodesic compactifications (meaning \( \bar{g} = r^2 g \) with \( |dr|_{\bar{g}} = 1 \) near \( \partial M \)), then \( c_j = 0 \) for \( j \) odd. If in addition \( n \) is odd, then also \( \mathcal{E} = 0 \) and \( V \) is conformally invariant (see [G1]).

Recall that the Chern-Gauss-Bonnet formula for a compact Riemannian 4-manifold \( (M, g) \) reads

\[
\text{(1.4)} \quad 8\pi^2 \chi(M) = \int_M \left( \frac{1}{8} |W|^2 + 4\sigma_2(g^{-1}P) \right) \, dv_g,
\]

where \( \chi(M) \) is the Euler characteristic, \( W \) is the Weyl tensor, \( |W|^2 = W_{\alpha\beta\gamma\delta}W^{\alpha\beta\gamma\delta} \), and \( \sigma_2(g^{-1}P) \) denotes the second elementary symmetric function of the eigenvalues of the endomorphism \( g^{\alpha\gamma}P_{\gamma\beta} \). Here \( P \) is the Schouten tensor, given by

\[
(n-1)P = \text{Ric}(g) - Jg, \quad J = \text{tr} \, P = \frac{R}{2n}
\]

for a metric in dimension \( n + 1 \). For \( n = 3 \), we have

\[
\text{(1.5)} \quad 4\sigma_2(g^{-1}P) = \frac{1}{24} R^2 - \frac{1}{2} |E|^2,
\]

where \( E = \text{Ric}(g) - \frac{R}{4} g \) is the Einstein tensor. In particular, if \( g \) has \( R = -12 \), then (1.4) becomes

\[
\text{(1.6)} \quad 8\pi^2 \chi(M) = \frac{1}{4} \int_M |W|^2 \, dv_g - \frac{1}{2} \int_M |E|^2 \, dv_g + 6V
\]

with \( V \) the volume of \( (M, g) \). In case \( g \) is also Einstein, this reduces to

\[
\text{(1.7)} \quad 8\pi^2 \chi(M) = \frac{1}{4} \int_M |W|^2 \, dv_g + 6V.
\]

In [A], Anderson showed that (1.7) holds also if \( g \) is Poincaré-Einstein, where now \( V \) is the renormalized volume of \( (M, g) \). Conformal invariance of the integrand shows in this case that \( \int_M |W|^2 \, dv_g \) is convergent.

Our formula is an analogue of (1.3) for singular Yamabe metrics, but now a boundary integral appears. We denote by \( L \) the second fundamental form for \( \partial M \) relative to \( \bar{g} \) with respect to the inward pointing unit normal \( \tilde{v} \): \( L(X, Y) = \bar{g}(\nabla_X Y, \tilde{v}) \). \( \tilde{L} \) denotes its trace-free part, \( H = \text{tr}_h L \) the mean curvature, and \( |L|^2 \) and \( |\tilde{L}|^2 \) the norms with respect to the induced metric \( h \). Curvature expressions for \( \bar{g} \) carry an overline (for example the scalar
The last three terms on the right-hand side of (1.9) depend on the choice of compactification $\alpha$ index for the inward unit normal, so that a Greek index $i$ specializes either to a 0 or an $i$. Thus $\overline{\partial}_{(00)}$ is another notation for $\text{Ric}_{\overline{\mathcal{g}}}(\overline{\nu}, \overline{\nu})$, and $\overline{W}_{(0i0j)}$ denotes the section of $S^{2}T^{*}\partial M$ obtained by contracting the Weyl tensor for $\mathcal{g}$ twice into $\overline{\nu}$ and orthogonally projecting onto $T^{*}\partial M$ in the other two indices. Define $B_{\mathcal{g}} \in C^{\infty}(\partial M)$ by

\begin{equation}
(1.8) \quad 24B_{\mathcal{g}} = \partial_{\nu} \overline{R} + 124 \overline{\hat{L}}^{ij} \overline{W}_{(0i0j)} + 108 \text{tr}(\hat{L}^{3}) + 14H|\hat{L}|^{2} + 24 \overline{\hat{L}}^{ij} \overline{R}_{ij} + 6H\overline{R}_{(00)} - \frac{10}{3} H\overline{R} - \frac{16}{9} H^{3}.
\end{equation}

**Theorem 1.1.** Let $g$ be a singular Yamabe metric in dimension 4 and $\mathcal{g}$ a smooth compactification of $g$. Then

\begin{equation}
(1.9) \quad 8\pi^{2}\chi(M) = \frac{1}{4} \int_{M} |W|^{2} \, dv_{g} - \frac{1}{2} \text{fp} \int_{r>\epsilon} |E|^{2} \, dv_{g} + 6V(g, \mathcal{g}) + \int_{\partial M} B_{\mathcal{g}} \, dv_{h}.
\end{equation}

The integral $\int_{M} |E|^{2} \, dv_{g}$ typically diverges. As above, $r$ denotes the $\mathcal{g}$-distance to $\partial M$, and as will explained in more detail in [2], $\text{fp}$ denotes the finite part of the integral: this is the constant term in the expansion of $\int_{r>\epsilon} |E|^{2} \, dv_{g}$ in powers of $\epsilon^{-1}$ and $\log \epsilon$. Typically each of the last three terms on the right-hand side of (1.9) depends on the choice of compactification $\mathcal{g}$.

Recall that a hypersurface is said to be umbilic with respect to a background metric if $\hat{L} = 0$. This condition is invariant under conformal rescalings of the background metric. We will say that $\partial M$ is umbilic for a singular Yamabe metric $g$ if it is umbilic for any compactification $\mathcal{g}$. This is an important special case which includes all Poincaré-Einstein metrics. If $\partial M$ is umbilic, all the terms involving $\hat{L}$ drop out in (1.8), which therefore simplifies to

\begin{equation}
(1.10) \quad 24B_{\mathcal{g}} = \partial_{\nu} \overline{R} + 6H\overline{R}_{(00)} - \frac{10}{3} H\overline{R} - \frac{16}{9} H^{3}.
\end{equation}

Recall from [Es] that any Riemannian metric on a 4-dimensional manifold-with-boundary can be conformally rescaled to a Yamabe metric having constant scalar curvature and for which $\partial M$ is minimal, i.e. $H = 0$. Observe that (1.10) implies that $B_{\mathcal{g}} = 0$ if $\partial M$ is umbilic and $\mathcal{g}$ is chosen to be such a Yamabe representative in the conformal class. It holds also that $B_{\mathcal{g}} = 0$ in this case $\mathcal{g}$ is a geodesic compactification of a Poincaré-Einstein metric. Then $H = 0$ and $\partial_{\nu} \overline{R} = 0$, for instance by parity considerations.

We will see in [2] that the following proposition follows from an easy calculation of the leading asymptotic term in the Einstein tensor.

**Proposition 1.2.** Suppose $n \geq 2$. If $g$ is a singular Yamabe metric with $\partial M$ umbilic, then $|E|_{\mathcal{g}} \in L^{\infty}(M)$.

Since $|E|^{2} \, dv_{g} = |E|^{2} \, dv_{\mathcal{g}}$ when $n = 3$, Proposition 1.2 implies that $\int_{M} |E|^{2} \, dv_{g} < \infty$ if $\partial M$ is umbilic and $n = 3$. So in this case $\text{fp} \int_{r > \epsilon} |E|^{2} \, dv_{g} = \int_{M} |E|^{2} \, dv_{g}$. In particular, $\text{fp} \int_{r > \epsilon} |E|^{2} \, dv_{g}$ is independent of choice of $\mathcal{g}$ in the $n = 3$ umbilic case. In this case, Theorem 1.1 therefore becomes:

**Theorem 1.3.** Let $g$ be a singular Yamabe metric in dimension 4 with $\partial M$ umbilic and let $\mathcal{g}$ be a smooth compactification of $g$. Then

\begin{equation}
(1.11) \quad 8\pi^{2}\chi(M) = \frac{1}{4} \int_{M} |W|^{2} \, dv_{g} - \frac{1}{2} \int_{M} |E|^{2} \, dv_{g} + 6V(g, \mathcal{g}) + \int_{\partial M} B_{\mathcal{g}} \, dv_{h}.
\end{equation}
Let $g$ be a singular Yamabe metric with $n = 3$ and $\partial M$ umbilic and let $\overline{g}$ be a compactification of $g$. Set

$$\tilde{V}(g) = V(g, \overline{g}) + \frac{1}{6} \int_{\partial M} B_{\overline{g}} \, dv_h.$$  

The notation is justified by:

**Corollary 1.4.** If $\partial M$ is umbilic, then $\tilde{V}(g)$ is conformally invariant, i.e. it is independent of the choice of compactification $\overline{g}$.

Corollary 1.4 is an immediate consequence of Theorem 1.3 since $\chi(M)$, $\int_M |W|^2 \, dv_g$, and $\int_M |E|^2 \, dv_g$ are all independent of the choice of compactification. Note that (1.11) can be written

$$8\pi^2 \chi(M) = \frac{1}{4} \int_M |W|^2 \, dv_g - \frac{1}{2} \int_M |E|^2 \, dv_g + 6 \tilde{V}(g),$$

in which each term on the right-hand side is an invariant of $g$, i.e. is independent of choice of $\overline{g}$. By the observation above, $\tilde{V}(g) = V(g, \overline{g})$ if the representative $\overline{g}$ is chosen to be a Yamabe representative having constant scalar curvature and $H = 0$. Thus choosing such a Yamabe representative can be regarded as a sort of “conformal gauge fixing” for $\tilde{V}(g)$.

In the general, not necessarily umbilic, case, Theorem 1.1 implies instead that

$$6V(g, \overline{g}) - \frac{1}{2} \lim_{r \to 0} \int_{r > \epsilon} |E|^2 \, dv_g + \int_{\partial M} B_{\overline{g}} \, dv_h$$

is conformally invariant. A direct proof of this is outlined at the end of §4

Equation (1.12) has the following consequence.

**Proposition 1.5.** Let $g$ be a singular Yamabe metric with $\partial M$ umbilic. Then

$$8\pi^2 \chi(M) \leq \frac{1}{4} \int_M |W|^2 \, dv_g + 6 \tilde{V}(g)$$

with equality if and only if $g$ is Poincaré-Einstein. In the case of equality, $\tilde{V}(g)$ agrees with the usual renormalized volume of $g$ as a Poincaré-Einstein metric, and (1.14) reduces to Anderson’s formula (1.7).

Proposition 1.5 suggests the following variational approach to the existence problem for Poincaré-Einstein metrics with prescribed conformal infinity in dimension 4. Given $(M, \partial M)$ and a conformal class $[h]$ on $\partial M$, let $[\overline{g}]$ be a conformal class extending $[h]$ to $M$ for which $\partial M$ is umbilic and let $g$ be the associated singular Yamabe metric. Consider the following minimization problem:

$$\Phi := \inf_{[\overline{g}]} \left( \frac{1}{4} \int_M |W|^2 \, dv_g + 6 \tilde{V}(g) \right)$$

Proposition 1.5 implies that $\Phi \geq 8\pi^2 \chi(M)$. It also implies that if $\Phi > 8\pi^2 \chi(M)$, then there does not exist a Poincaré-Einstein metric on $M$ with conformal infinity $[h]$. If $\Phi = 8\pi^2 \chi(M)$ and the infimum is attained, then any minimizer is a Poincaré-Einstein metric having conformal infinity $[h]$. It is tempting to view this as a sort of Dirichlet Principle for the Poincaré-Einstein problem. But using it seems to be problematic. For starters, one must first solve for the singular Yamabe metric to evaluate the “energy” $\frac{1}{4} \int_M |W|^2 \, dv_g + 6 \tilde{V}(g)$. Moreover, this energy includes the nonlocal, renormalized contribution $\tilde{V}(g)$ which is difficult to analyze. Finally, one must compare the infimum to $8\pi^2 \chi(M)$ in order to deduce any conclusions.
The formula (1.13) suggests that in the context of the Chern-Gauss-Bonnet Theorem in dimension 4, it is natural to consider metrics for which \( \sigma_2(g^{-1}P) \) is constant. This motivates consideration here of the \( \sigma_k \)-Yamabe problem introduced in [VI]: given a closed manifold \((M, \overline{g})\) of dimension at least three, find a conformal metric \( g = u^{-2} \overline{g} \) satisfying

(1.15) \hspace{1cm} \sigma_k(g^{-1}P_g) = \text{const.}

If \( k = 1 \) this reduces to the Yamabe problem, while if \( k \geq 2 \) (1.15) is fully non-linear (as an equation for the conformal factor \( u \)), and one must impose a condition to guarantee ellipticity. To this end, a metric \( g \) is said to be \( k\)-admissible (or, if the context is clear, simply admissible) if \( \sigma_j(g^{-1}P_g) > 0 \) for all \( 1 \leq j \leq k \). If \( \overline{g} \) is \( k \)-admissible and the constant on the right-hand side is positive, then (1.15) is elliptic at any solution (see Proposition 2 of [V2]). Equation (1.15) is also elliptic if \( \sigma_j(-\overline{g}^{-1}P) > 0 \) for all \( 1 \leq j \leq k \), in which case \( \overline{g} \) is said to be negative \( k \)-admissible. For admissible metrics the existence theory for (1.15) is well developed; see [V3], [STW2] for surveys. In contrast to the Yamabe problem, existence for classical solutions in the negative admissible case is not fully understood, due to the lack of interior \( C^2 \)-estimates for solutions (see Section 3.3 of [STW1] for a discussion).

In [MP], Mazzeo-Pacard considered a singular version of the \( \sigma_k \)-Yamabe problem in connection with the existence question for Poincaré-Einstein metrics: given a compact manifold-with-boundary \((M, \partial M, \overline{g})\) of dimension \( n + 1 \), construct an asymptotically hyperbolic metric \( g = u^{-2} \overline{g} \) solving (1.15), where the constant is the value on hyperbolic space, namely \((-2)^{-k}(-1)^{n+1}\). (A continuity argument shows that \( g \) is automatically negative \( k \)-admissible, since \( g \) is asymptotically hyperbolic and satisfies (1.15).) Mazzeo-Pacard showed that the perturbation problem is never obstructed, so that given a solution \( g = u^{-2} \overline{g} \) of the singular \( \sigma_k \)-Yamabe problem, every conformal class sufficiently close to \( \overline{g} \) also admits a solution. The connection to Poincaré-Einstein metrics follows from the simple observation that an asymptotically hyperbolic metric is Poincaré-Einstein if and only if it solves the singular \( \sigma_k \)-Yamabe problem for all \( k \). In particular, given a compactification \( \rho^2 g_+ \) of a Poincaré-Einstein metric, it follows from the Mazzeo-Pacard result that every conformal class \( \overline{g} \) near \( \rho^2 g_+ \) admits metrics \( g_k = u_k^{-2} \overline{g} \), \( 1 \leq k \leq n + 1 \), where \( g_k \) is a solution of the singular \( \sigma_k \)-Yamabe problem.

Although the result of Mazzeo-Pacard gives local existence – i.e., existence of solutions in conformal classes near a given solution, the same issues arise as in the closed case when attempting to solve the singular \( \sigma_k \)-Yamabe problem in general. In fact, in [GSW], Gursky-Street-Warren gave an example of a conformal manifold-with-boundary that does not admit a solution to the singular \( \sigma_k \)-Yamabe problem for \( k = n + 1 \) (see Proposition 6.3 in [GSW]). The obstruction is easy to explain: let \((M^{n+1}, \partial M, \overline{g})\) be a locally conformally flat manifold-with-boundary, and suppose \( g = u^{-2} \overline{g} \) is a solution of the singular \( \sigma_k \)-Yamabe problem with \( k = n + 1 \). The continuity argument mentioned in the previous paragraph shows that the Schouten tensor of \( g \) is everywhere negative definite. Since \( g \) is locally conformally flat, the curvature tensor of \( g \) is given by

\[ R_{ijkt} = g_{ik}P_{j\ell} - g_{i\ell}P_{jk} - g_{jk}P_{i\ell} + g_{j\ell}P_{ik}. \]

If \( P_g \) is negative definite, it is easy to check that \( g \) has negative sectional curvature. By the Cartan-Hadamard Theorem the universal cover of \( \overline{M}^{n+1} \), the interior of \( M^{n+1} \), is diffeomorphic to \( \mathbb{R}^{n+1} \). However, it is easy to give examples where this not the case: take \( M^{n+1} = S^n \times [0, 1] \), with \( \overline{g} \) the product metric.
Interestingly, recent work of Gonzalez-Li-Nguyen [GLN] establishes the existence of a unique, Lipschitz continuous \textit{viscosity} solution of the singular $\sigma_k$-Yamabe problem for domains in Euclidean space. Although this generalizes the classical Loewner-Nirenberg result, the example of Gursky-Streets-Warren illustrates that viscosity solutions need not be classical (i.e., $C^2$) solutions.

For our considerations here, which are based on formal asymptotics, we will simply assume that we have a smooth metric $\bar{g}$ on $M$ and a defining function $u \in C^\infty(M)$ which has a polyhomogeneous expansion at the boundary, such that $g = u^{-2}\bar{g}$ satisfies

\begin{equation}
\sigma_k(-g^{-1}P_g) = 2^{-k} \left( \frac{n+1}{k} \right).
\end{equation}

Henceforth, this is what we will mean by a solution of the singular $\sigma_k$-Yamabe problem.

The form of the expansion of $u$ at the boundary is determined by the indicial roots of (1.16), viewed as an equation for $u$. The indicial roots were calculated in [MP], but that derivation contains an error. As we discuss in §3 the indicial roots are 0 and $n+2$, and in particular are independent of $k$. Thus for any $k$, the expansion of $u$ is of the form (1.1), where the coefficients $L$ and the $u^{(j)}$, $2 \leq j \leq n+1$, are locally determined, and, of course, depend on $k$. Arguing exactly as in [G3] (or see [GoW]), it follows that Vol$_g(\{r > \epsilon\})$ has an asymptotic expansion of the same form (1.2), where again each of $E$ and the $c_j$’s is the integral over $\partial M$ of a local invariant of the extrinsic geometry induced by $\bar{g}$, which depends on $k$.

The constant term $V = V(g, \bar{g})$ is the renormalized volume for $g$. We denote by $L^{\sigma_k}$, $E^{\sigma_k}$ the coefficients of the log terms in the expansions (1.1) and (1.2) for a solution of the singular $\sigma_k$-Yamabe problem. The coefficients $u^{(j)}$, $L^{\sigma_k}$, $E^{\sigma_k}$ and $c_j$ depend only on formal calculations, so are well-defined in terms of $\bar{g}$ independently of existence of actual solutions $u$. The same arguments in [G3], [GoW] show that the log coefficients $L^{\sigma_k}$ and $E^{\sigma_k}$ are conformally invariant: under conformal change $\tilde{g} = \Omega^2g$, one has $\tilde{E}^{\sigma_k} = E^{\sigma_k}$ and $\tilde{L}^{\sigma_k} = (\Omega|_\Sigma)^{-n-1} L^{\sigma_k}$.

We prove an analogue of Theorem 1.3 for solutions of the singular $\sigma_2$-Yamabe problem. There are two major simplifications as compared with the case $k = 1$: the term involving the Einstein tensor $E$ does not appear, and the general version of the formula and the conformal invariance of $\tilde{V}$ hold without the assumption of umbilicity. The boundary term $B^\sigma_g$ which enters is given by:

\begin{equation}
24B^\sigma_g = \partial_\nu R + 52 \hat{L}^{(1)}W_{000} + 36 \text{tr}(\hat{L}^{(3)}) + \frac{50}{3} H|\hat{L}|^2 + 24 \hat{L}^{(2)} \tilde{R}_{ij}^0 + 6H\tilde{R}_{00} - \frac{10}{3} H\tilde{R} - \frac{16}{3} H^3.
\end{equation}

**Theorem 1.6.** Let $g = u^{-2}\bar{g}$ be a solution of the singular $\sigma_2$-Yamabe problem in dimension 4. Then in the notation of Theorem 1.4

\begin{equation}
8\pi^2 \chi(M) = \frac{1}{4} \int_M |W|^2 dv_g + 6\tilde{V}^{\sigma_2}(g),
\end{equation}

where

\begin{equation}
\tilde{V}^{\sigma_2}(g) = V(g, \bar{g}) + \frac{1}{6} \int_{\partial M} B_g^{\sigma_2} dv_h.
\end{equation}

Moreover, $\tilde{V}^{\sigma_2}(g)$ is conformally invariant, i.e. it is independent of the choice of compactification $\bar{g}$.

Observe that when the boundary is umbilic with respect to $\bar{g}$, then $B_g^{\sigma_2} = B_g$. An immediate consequence of this fact is...
Corollary 1.7. Let \((M, \partial M)\) be a compact four-dimensional manifold-with-boundary. Suppose \(g_1\) and \(g_2\) are solutions of the singular \(\sigma_k\)-Yamabe problem in the same conformal class, for \(k = 1\) and \(k = 2\) respectively. Let \(g_1 = u_1^{-2} \overline{g}\) and \(g_2 = u_2^{-2} \overline{g}\), where \(\overline{g}\) is a smooth compactification. If \(\partial M\) is umbilic with respect to \(\overline{g}\), then

\[
V(g_2, \overline{g}) \leq V(g_1, \overline{g}),
\]

and equality holds if and only if \(g_1 = g_2\) is a Poincaré-Einstein metric.

As a by-product of our analysis of the Chern-Gauss-Bonnet formula, we will deduce that the log coefficient vanishes in the volume expansion for the \(\sigma_2\)-Yamabe problem:

**Theorem 1.8.** Let \(n = 3\). Then \(E_{\sigma_2} = 0\).

This paper is organized as follows. In §2 we prove Proposition 1.2 and Theorem 1.1. As one would anticipate, Theorem 1.1 is proved by taking a limit of the Chern-Gauss-Bonnet formula on \(\{r \geq \epsilon\}\) as \(\epsilon \to 0\). In order to identify the boundary term \(B_{\overline{g}}\), we have to compute the expansion of the solution \(u\) to one higher order than in [CS]. In §3 we discuss the singular \(\sigma_k\)-Yamabe problem and use the same sort of argument as in the case \(k = 1\) to prove Theorems 1.6 and 1.8. In the process we calculate the first few terms in the expansion of the solution for the singular \(\sigma_2\)-Yamabe problem when \(n = 3\). In §4 we discuss renormalized volume coefficients and anomalies. We derive a general result (Proposition 4.1) identifying the infinitesimal anomaly for the renormalized volume of an asymptotically hyperbolic metric in terms of the full set of its renormalized volume coefficients. We make explicit these coefficients for solutions of the singular \(\sigma_k\)-Yamabe problem for \(k = 1, 2\) when \(n = 3\). These calculations allow us to give another proof by direct calculation of the conformal invariance of \(\tilde{V}^{\sigma_k}(g)\) in these cases.

In §5 we consider two other generalizations of the singular Yamabe problem. The first is the singular \(\sigma_k(\text{Ric})\)-problem: given \(\overline{g}\), find \(g = u^{-2} \overline{g}\) asymptotically hyperbolic so that \(\sigma_k(g^{-1} \text{Ric}_g) = \text{const}\). It was shown in [CSW] that this problem always has a unique solution, just like the singular Yamabe problem. The asymptotics of the solution are studied in [W] in the case of domains in Euclidean space. We discuss a version of the Chern-Gauss-Bonnet Theorem for solutions of this problem which follows by the same arguments as in the case \(k = 1\) above. Finally, we describe some results for the singular \(v_k\)-Yamabe problem, where \(v_k\) denotes the \(k\)th Poincaré-Einstein renormalized volume coefficient, with proofs deferred to a future paper. It holds that \(v_k = \sigma_k(g^{-1} P)\) when \(k = 2\) or \(g\) is locally conformally flat, and it was pointed out in [CF] that in several regards, \(\sigma_k(g^{-1} P)\) should be replaced by \(v_k\) for \(k > 2\) and general metrics. The results are: a generalization of Theorem 1.8 to higher dimensions, a generalization to higher \(k\) of the result of [CS], [GoW] that the Euler-Lagrange equation for the energy \(E\) is a multiple of \(L\), and a higher-dimensional version of the Chern-Gauss-Bonnet Theorem for solutions of the singular \(v_k\)-Yamabe problem with \(2k = n + 1\), generalizing a theorem of [CQY] for Poincaré-Einstein metrics. These results indicate that the singular \(v_k\)-Yamabe problem is perhaps the natural setting for these questions. However, existence of solutions of the singular \(v_k\)-Yamabe problem has not been studied for general metrics when \(k > 2\). It would be interesting to investigate the possibility of extending to these equations the existence and uniqueness results of [GLN] for viscosity solutions.

2. **Proofs of Proposition 1.2 and Theorem 1.1**

We are interested in singular Yamabe metrics \(g\) in dimension \(n + 1\), \(n \geq 2\), admitting a smooth compactification \(\overline{g} = u^2g\). We use the normal exponential map \(\exp : [0, \delta)_r \times \)
\( \partial M \to M \) with respect to \( \overline{\gamma} \) to identify a neighborhood of \( \partial M \) with \([0, \delta)_r \times \partial M \). In this identification, \( \overline{\gamma} \) takes the form
\begin{equation}
\overline{\gamma} = dr^2 + h_r
\end{equation}
for a smooth one-parameter family of metrics \( h_r \) on \( \partial M \). In particular, \( r \) is the \( \overline{\gamma} \)-distance to \( \partial M \). We denote by \( h = h_0 \) the induced metric on \( \partial M \). It follows from [M], [ACF] that the defining function \( u \) has an asymptotic expansion of the form \((1.1)\), where \( \mathcal{L} \) and the indicated \( u^{(j)} \) are smooth, locally determined functions on \( \partial M \).

We first consider the asymptotics of the Einstein tensor \( E = tf(\text{Ric}(g)) = \text{Ric}(g) - \frac{R}{n+1}g \). The conformal transformation law for Ricci applied to \( g = u^{-2}\overline{\gamma} \) gives
\begin{equation}
E = tf \left( \text{Ric}(\overline{\gamma}) + (n-1)u^{-1}\nabla^2 u \right)
\end{equation}
(see [2.5] below). In particular, it is clear that \(|E|_{\overline{\gamma}} = O(r^{-1})\). In order to prove Proposition 1.2 we need to see that the leading term vanishes if \( g \) is singular Yamabe and \( \partial M \) is umbilic.

**Proof of Proposition 1.2.** We argue by putting \( g \) into asymptotically hyperbolic normal form. Upon choosing a representative for its conformal infinity, we can write \( g \) as \( g = s^{-2}(ds^2 + k_s) \) relative to an identification \( M \cong [0, \delta)_s \times \partial M \) near \( \partial M \) (typically this differs from the identification induced by \( \overline{\gamma} \) discussed above). Here \( k_s \) is a 1-parameter family of metrics on \( \partial M \). Because of the log terms in the expansion \((1.1)\), the compactification of \( g \) with respect to a smooth defining function (for example, \( r \)) is not typically \( C^\infty \). So \( k_s \) and the diffeomorphism putting \( g \) into normal form typically have log terms in their expansions. It is not hard to verify that the log terms occur far enough out that they do not affect the subsequent argument.

It is straightforward to calculate the leading term of \( \text{Ric}(g) \) for \( g = s^{-2}(ds^2 + k_s) \), for instance by applying the conformal transformation law for Ricc with conformal factor \( s^{-2} \). See, for example, (2.4)-(2.6) of [GH]. The result is that \( \tilde{E} := \text{Ric}(g) + ng \) is given by:
\begin{equation}
\begin{aligned}
\tilde{E}_{ij} &= \frac{1}{2s}[(n-1)k'_i + k'^jq'_p k_{pj}] + O(1) \\
\tilde{E}_{i0} &= O(1) \\
\tilde{E}_{00} &= \frac{1}{2s}k'^q q'_p + O(1),
\end{aligned}
\end{equation}
where \( \cdot' = \partial_s \). Taking the trace gives
\[
R + n(n+1) = \text{tr}_g \tilde{E} = s^2(k^{ij} \tilde{E}_{ij} + \tilde{E}_{00}) = nsk^{ij}k'_{ij} + O(s^2).
\]
Hence \( R + n(n+1) = O(s^2) \) if and only if \( k^{ij}k'_{ij} = 0 \) at \( s = 0 \). And \( \partial M \) is umbilic for \( g \) if and only if \( tf_k k' = 0 \) at \( s = 0 \). So if \( g \) is singular Yamabe and \( \partial M \) is umbilic, then \( k'|_{s=0} = 0 \). In this case \((2.3)\) shows that all components of \( \tilde{E} \) are \( O(1) \), so also all components of \( E = tf \tilde{E} \) are \( O(1) \), so \(|E|_{\overline{\gamma}} \in L^\infty(M) \).

We remark that in \((4.6)\) below, we identify the leading \( r^{-1} \) term of \( E \) for a singular Yamabe metric written in the form \( g = u^{-2}\overline{\gamma} \). This gives an alternate proof of Proposition 1.2.

Recall that the Chern-Gauss-Bonnet formula \((1.4)\) implies that \( \int_M \sigma_2(\overline{\gamma}^{-1}P) \, dv_{\overline{\gamma}} \) is conformally invariant on a compact 4-dimensional manifold without boundary. Thus under a conformal change \( g = u^{-2}\overline{\gamma} \), the quantity \( \sigma_2(\overline{\gamma}^{-1}P) - u^{-4}\sigma_2(g^{-1}P) \) must be expressible as a divergence with respect to \( \overline{\gamma} \). The next lemma identifies this divergence.
Lemma 2.1. For \( n = 3 \), one has

\[
4\sigma_2(\overline{g}^{-1} P) = 4u^{-4}\sigma_2(g^{-1} P) + 2\nabla^\alpha \left( u^{-3}|du|^2_g u_\alpha - u^{-2}(\nabla u)u_\alpha + u^{-2}u_{\alpha\beta}u^\beta + u^{-1}\overline{R}_{\alpha\beta}u^\beta - \frac{1}{2}u^{-1}\overline{R}u_\alpha \right)
\]

On the right-hand side, indices are raised using \( \overline{g} \) and all covariant derivatives (as in \( u_{\alpha\beta} \)) are with respect to \( \nabla \). Our sign convention is \( \overline{\Delta} = \nabla^\alpha\nabla_\alpha \).

Proof. This is a reformulation of the transformation law of \( \sigma_2(g^{-1} P) = \frac{1}{2}(J^2 - |P|^2) \) under conformal change. Recall that under the change \( g = e^{2\omega}\overline{g} \), the Schouten tensor transforms by \( P_{\alpha\beta} = \overline{P}_{\alpha\beta} - \omega_{\alpha\beta} + \omega_{\alpha}\omega_{\beta} - \frac{1}{2}|\omega|^2\overline{g}_{\alpha\beta} \). Setting \( \omega = -\log u \) gives

\[
P_{\alpha\beta} = \overline{P}_{\alpha\beta} + u^{-1}u_{\alpha\beta} - \frac{1}{2}u^{-2}|du|^2\overline{g}_{\alpha\beta},
\]

and taking the trace gives \( u^{-2}J = \overline{J} + u^{-1}\overline{\Delta}u - 2u^{-2}|du|^2 \). Now substitute into \( 4\sigma_2(g^{-1} P) = 2(J^2 - |P|^2) \) and simplify to obtain

\[
4u^{-4}\sigma_2(g^{-1} P) = 4\sigma_2(\overline{g}^{-1} P) + 6u^{-4}|du|^4 - 6u^{-3}\overline{\Delta}u|du|^2
+ 2u^{-2}[|\overline{\Delta}u|^2 - |\nabla u|^2 - 3\overline{J}|du|^2] - 4u^{-1}[\overline{P}_{\alpha\beta}u^\alpha u^\beta - \overline{J}\overline{\Delta}u].
\]

Equation (2.4) reduces to this same relation upon expanding the divergence. \[\square\]

Proof of Theorem 1.1. First we give the proof modulo identification of the explicit form of \( E_\overline{g} \). Then we calculate \( \overline{E}_\overline{g} \).

First apply the Chern-Gauss-Bonnet Theorem for smooth manifolds-with-boundary to the metric \( \overline{g} \) on \( \{ r \geq \epsilon \} \) with \( \epsilon > 0 \) small. It states

\[
8\pi^2 \chi(M) = \int_{r > \epsilon} \left( \frac{1}{4}|\overline{W}|^2_{\overline{g}} + 4\sigma_2(\overline{g}^{-1} P) \right) dv_{\overline{g}} + \int_{r = \epsilon} S dv_{h_\epsilon},
\]

where the boundary integrand \( S \) can be written in the form

\[
S = \overline{R}H - 2\overline{R}_{00}H - 2\overline{R}^k_{ijk}L^{ij} + \frac{2}{3}H^3 - 2H|L|^2 + \frac{4}{3}\text{tr}(L^3)
\]

(see, for example, [C]). In this formula for \( S, L \) and \( H \) refer to the fundamental form and mean curvature of \( \{ r = \epsilon \} \) for the metric \( \overline{g} \) with respect to the inward pointing unit normal. Use \( |\overline{W}|^2_{\overline{g}}dv_{\overline{g}} = |W|^2_g dv_g \) and substitute (2.4) and then (1.5) with \( R = -12 \), and integrate the divergence by parts to obtain

\[
8\pi^2 \chi(M) = \frac{1}{4}\int_{r > \epsilon} |W|^2_g dv_g - \frac{1}{2}\int_{r > \epsilon} |E|^2_g dv_g + 6\text{Vol}_g(\{ r > \epsilon \})
- 2\int_{r = \epsilon} \left( u^{-3}|du|^2_{\overline{g}}u_0 - u^{-2}(\nabla u)u_0 + u^{-2}u_{0\beta}u^\beta + u^{-1}\overline{R}_{0\beta}u^\beta - \frac{1}{2}u^{-1}\overline{R}u_0 \right) dv_{h_\epsilon}
+ \int_{r = \epsilon} S dv_{h_\epsilon}.
\]

As \( \epsilon \to 0 \), the first term on the right-hand side converges to \( \frac{1}{4}\int_M |W|^2_g dv_g \) and the last term converges to \( \int_{\partial M} S dv_h \). Thus the sum of the other three terms on the right-hand side
converges as $\epsilon \to 0$. However, typically each of them diverges individually. The expansion of $\text{Vol}_g(\{r > \epsilon\})$ is given by (1.2). It follows from (2.2) and (1.1) that

$$|E|^2_g \, dv_g = |E|^2_g \, dv_\pi = (F_2 r^{-2} + F_3 r^{-1} + O(1)) \, dv_\pi$$

for smooth functions $F_2, F_3$ on $\partial M$. Therefore integration shows that

$$\int_{r > \epsilon} |E|^2_g \, dv_g = a \epsilon^{-1} + F \log \frac{1}{\epsilon} + \text{fp} \int_{r > \epsilon} |E|^2_g \, dv_g + o(1), \quad a, F \in \mathbb{R},$$

where by definition $\text{fp} \int_{r > \epsilon} |E|^2_g \, dv_g$ denotes the constant term in the expansion. As for the boundary integral, consideration of the form which results upon substituting (1.1) into each term individually shows that

$$\left( u^{-3} |du|^2_g u_0 - u^{-2} (\nabla u) u_0 + u^{-2} u_{0\beta} u^\beta + u^{-1} T_{0\beta} u^\beta - \frac{1}{2} u^{-1} R u_0 \right) |_{r = \epsilon} \, dv_h = (B_0 \epsilon^{-3} + B_1 \epsilon^{-2} + B_2 \epsilon^{-1} + B_3 + o(1)) \, dv_h$$

for smooth locally determined functions $B_0, B_1, B_2, B_3$ on $\partial M$. The log term in $u$ does not affect the expansions to this order: it generates an $\epsilon \log \epsilon$ term in this expansion.

Combining the terms, we deduce first that the divergent terms must cancel:

$$2 \int_{\partial M} B_0 \, dv_h = 6c_0$$

$$2 \int_{\partial M} B_1 \, dv_h = 6c_1$$

$$2 \int_{\partial M} B_2 \, dv_h = 6c_2 - \frac{1}{2} a$$

$$0 = 6E - \frac{1}{2} F.$$

Then taking the limit gives

$$8\pi^2 \chi(M) = \frac{1}{4} \int_M |W|^2_g \, dv_g - \frac{1}{2} \text{fp} \int_{r > \epsilon} |E|^2_g \, dv_g + 6V + \int_{\partial M} (-2B_3 + S) \, dv_h.$$

This proves Theorem 1.1 once we carry out the calculation that $S - 2B_3 = B_\pi$ modulo divergence terms.

In order to calculate $B_3$, we need to expand all the ingredients appearing in (2.9), namely $u, \pi, \text{and} \, dv_h$, to high enough order to evaluate the constant term. The expansions were calculated in [G3] to one order lower than required here.

Begin with $\pi = dr^2 + h_r$. Denoting $\partial_r$ by $'$, the derivatives of $h_r$ at $r = 0$ are given by:

$$h_{ij}' = -2L_{ij}, \quad h_{ij}'' = -2T_{000j} + 2L_{ik}L_{kj}, \quad h_{ij}''' = -2T_{000j,0} + 8L_{(i}R_{j)0k0}.$$

The latter two equations can be derived by writing out the expressions for the curvature components $\nabla \pi$ and $\nabla \pi_{000j}$ in local coordinates. Taking the trace with respect to the metric $h = h_0$ gives

$$\text{tr} \, h' = -2H, \quad \text{tr} \, h'' = -2T_{00} + 2|L|^2 \quad \text{tr} \, h''' = -2T_{00,0} + 8L_{ij}T_{000j}.$$

Composing the expansion of $\sqrt{1 + x}$ with that of the determinant shows that for any 1-parameter family of metrics $h_r$, one has

$$\sqrt{\frac{\det h_r}{\det h_0}} = 1 + D_1 r + D_2 r^2 + D_3 r^3 + \cdots.$$
with

\[ D_1 = \frac{1}{2} \text{tr} h' \]
\[ D_2 = \frac{1}{4} \left[ \text{tr} h'' - |h'|^2 + \frac{1}{2} (\text{tr} h')^2 \right] \]
\[ D_3 = \frac{1}{2} \left[ \text{tr} h''' - 3(h', h'') + 2 \text{tr}(h'^3) + \frac{3}{2} (\text{tr} h')(\text{tr} h'') - \frac{3}{2} (\text{tr} h')(h')^2 + \frac{1}{4} (\text{tr} h')^3 \right]. \]

Substituting (2.11) and (2.12) gives

\[ D_1 = -H \]
\[ D_2 = \frac{1}{2} \left[ -\overline{R}_{00} - |L|^2 + H^2 \right] \]
\[ D_3 = \frac{1}{6} \left[ -\overline{R}_{00,0} - 2L^i\overline{R}_{0i0j} - 2 \text{tr}(L^3) + 3H \overline{R}_{00} + 3H |L|^2 - H^3 \right]. \]

The above formulas hold in general dimension.

The expansion of \( u \) is determined by the condition \( R_g = -n(n + 1) \) with \( g = u^{-2}\overline{g} \). Necessarily \( g \) is asymptotically hyperbolic (i.e., \( |du|^2 = 1 \) on \( \partial M \)), since the scalar curvature of the conformally compact metric \( u^{-2}\overline{g} \) is asymptotic to \( -n(n + 1)|du|^2 \). Thus we write \( u = r + r^2 \varphi \). Now write the equation \( R_g = -n(n + 1) \) in terms of \( u \) via conformal transformation, and then write the resulting equation in terms of \( \varphi \). The result (see the derivation of (2.5) of [G3]) is that \( \varphi \) satisfies

\[
(1 + r \varphi) \left[ r^2 \varphi_{rr} + 4r \varphi_r + 2 \varphi + \frac{1}{2} h^{ij} h_{ij} (1 + 2r \varphi + r^2 \varphi_r) + r^2 \Delta h_r \varphi \right] \\
- n + 1 \left[ 2(r \varphi_r + 2 \varphi) + r(r \varphi_r + 2 \varphi)^2 + r^3 h^{ij} \partial_i \varphi \partial_j \varphi \right] \\
+ \frac{1}{2n} r(1 + r \varphi)^2 R_{\overline{g}} = 0. 
\]

We need to determine the Taylor expansion of \( \varphi \) through order 2 by successive differentiation of (2.15) at \( r = 0 \). The evaluation of \( \varphi |_{r=0} \) and \( \partial_r \varphi |_{r=0} \) was given in [G3], although there the Gauss equation was used to rewrite some expressions in terms of intrinsic curvature of \( h \). Here we leave everything in terms of \( L \) and curvature of \( \overline{g} \).

Setting \( r = 0 \) in (2.15) and solving for \( \varphi \) give

\[ \varphi |_{r=0} = -\frac{1}{2n} H. \]

Differentiating once, setting \( r = 0 \), and solving for \( \varphi_r \) give

\[ 3(n - 1) \varphi_r |_{r=0} = \frac{1}{2} \text{tr} h'' - \frac{1}{2} |h'|^2 + \frac{3}{2} \varphi \text{tr} h' - 2n \varphi^2 + \frac{1}{2n} \overline{R}. \]

Upon substituting from (2.11), (2.12) and (2.16), and decomposing \( |L|^2 = |\overline{L}|^2 + \frac{1}{n} H^2 \), this simplifies to

\[ 3(n - 1) \varphi_r |_{r=0} = -\overline{R}_{00} - |\overline{L}|^2 + \frac{1}{n} \overline{R}. \]

Differentiating (2.15) twice, setting \( r = 0 \), and solving for \( \varphi_{rr} \) give

\[ 4(n - 2) \varphi_{rr} |_{r=0} = \frac{1}{2} \text{tr} h''' - \frac{3}{2} (h', h'') + \text{tr}(h'^3) + 3\varphi \text{tr} h'' - 3\varphi |h'|^2 + 4\varphi_r \text{tr} h' \\
+ 2 \varphi_r^2 \text{tr} h' + 4(1 - 3n) \varphi \varphi_r + 2 \Delta h_r \varphi + \frac{1}{n} \overline{R}_{00} + \frac{2}{3} \varphi \overline{R}. \]

Henceforth we take \( n = 3 \). Substituting from (2.11), (2.12), (2.16) and (2.17), and simplifying, this can be written

\[ 12 \varphi_{rr} |_{r=0} = -3 \overline{R}_{00,0} + \overline{R}_{00} - \Delta h H - 6L^i \overline{R}_{0i0j} - 6 \text{tr}(L^3) \\
+ \frac{13}{3} H |\overline{L}|^2 + \frac{13}{3} H \overline{R}_{00} - \frac{5}{3} H \overline{R} + \frac{2}{3} H^3. \]
Now we can determine the $B_j$, $0 \leq j \leq 3$, by expanding the left-hand side of (2.19). Recalling $u = r + r^2 \varphi$, we write $\varphi = f_0 + f_1 r + f_2 r^2 + o(r^2)$ with each $f_j \in C^\infty(\partial M)$, so
\[
u = r + f_0 r^2 + f_1 r^3 + f_2 r^4 + o(r^4) = r(1 + f_0 r + f_1 r^2 + f_2 r^3 + o(r^3))\]
and
\[f_0 = \varphi|_{r=0}, \quad f_1 = \varphi_r|_{r=0}, \quad f_2 = \frac{1}{2} \varphi_{rr}|_{r=0}\]
are determined above.

First we evaluate the expansions of the ingredients to the relevant orders. Details of the verifications of these expansions are left to the reader.

\[u_0 = 1 + 2 f_0 r + 3 f_1 r^2 + 4 f_2 r^3 + o(r^3)\]
(2.19)
\[u_i = O(r^2)\]
so
\[|du|^2 = u_0^2 + O(r^4) = 1 + 4 f_0 r + (6 f_1 + 4 f_0^2) r^2 + (8 f_2 + 12 f_0 f_1) r^3 + o(r^3).\]

The inverse powers are given by
\[
u^{-3} = r^{-3}[1 - 3 f_0 r + (-3 f_1 + 6 f_0^2) r^2 + (-3 f_2 + 12 f_0 f_1 - 10 f_0^3) r^3 + o(r^3)]\]
\[
u^{-2} = r^{-2}[1 - 2 f_0 r + (-2 f_1 + 3 f_0^2) r^2 + o(r^2)]\]
\[
u^{-1} = r^{-1}[1 - f_0 r + o(r)].\]

The Christoffel symbols of $\varphi$ are:
\[\Gamma^0_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} h'_{ij} \end{pmatrix}, \quad \Gamma^k_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} h^{kl} h'_{ij} \\ \frac{1}{2} h'_{ij} \end{pmatrix},\]
where here unadorned $h'$s and $\Gamma$'s refer to $h_r$. Using (2.21), one calculates for the second covariant derivatives $u_{\alpha\beta} = \nabla_{\alpha} \nabla_{\beta} u$:
\[u_{00} = \partial_{00}^2 u = 2 f_0 + 6 f_1 r + 12 f_2 r^2 + o(r^2)\]
(2.22)
\[u_{0i} = O(r).\]

The tangential second covariant derivatives are $u_{ij} = \partial_{ij}^2 u - \Gamma^0_{ij} u_{\alpha} = \frac{1}{2} h'_{ij} u_0 + \nabla_{ij}^2 f_0 r^2 + o(r^2)$, where $\nabla_{ij}^2$ refers to the covariant derivatives for the metric $h_0$. Taylor expanding $h'_{ij}$ and multiplying by $u_0$ from (2.19) give
\[u_{ij} = \frac{1}{2} h'_{ij}(0) + \left[ \frac{1}{2} h''_{ij}(0) + h'_{ij}(0) f_0 \right] r + \left[ \nabla_{ij}^2 f_0 + \frac{1}{4} h''_{ij}(0) + h'_{ij}(0) f_0 + \frac{3}{2} h'_{ij}(0) f_1 \right] r^2 + o(r^2).\]

Since
\[h''_{ij} = h''_{ij}(0) - (h'_{ij}(0)) r + \frac{1}{2} \left[ - (h''_{ij}(0)) r + 2(h'_{ij}(0)) r^2 + o(r^2) \right],\]
one obtains upon substituting and expanding:
\[\Delta u = u_{00} + h'_{ij} u_{ij}\]
(2.24)
\[= \left[ 2 f_0 + \frac{1}{2} h''_{ij} h'_{ij} \right] + \left[ 6 f_1 + \frac{1}{2} \text{tr} h'' + \text{tr} h' f_0 - \frac{1}{2} |h'|^2 \right] r\]
\[+ \left[ 12 f_2 + \Delta f_0 + \frac{1}{2} \text{tr} h'' + \text{tr} h' f_0 + \frac{3}{2} \text{tr} h' f_1 - \frac{3}{2} \langle h', h'' \rangle - |h'|^2 f_0 + \frac{1}{2} \text{tr}(h^3) \right] r^2\]
+ $o(r^2)$.
Here all \( h \) and derivatives are evaluated at \( r = 0 \), and \( \Delta = \Delta_{h_0} \). The remaining expansions that we need are
\[
\begin{align*}
\mathcal{R}_{00} &= \mathcal{R}_{00}(0) + \mathcal{R}_{00,0}(0)r + o(r) \\
\mathcal{R} &= \mathcal{R}(0) + \mathcal{R}_0(0)r + o(r).
\end{align*}
\]

Now multiply out the expansions for the terms in the left-hand side of (2.9) term-by-term to obtain:
\[
\begin{align*}
\mathcal{I} &= u^{-3} |du|^2 u_0 - u^{-2} (\Delta u) u_0 + u^{-2} u_{0\beta} u^\beta + u^{-1} \mathcal{R}_{0\beta} u^\beta - \frac{1}{2} u^{-1} \mathcal{R}_0 u_0.
\end{align*}
\]
Combining terms and then substituting (2.11), (2.12), (2.16), (2.17), (2.18) and setting \( r = \epsilon \) result in
\[
\begin{align*}
\mathcal{I}|_{r=\epsilon} &= \epsilon^{-3} + \mathcal{I}_1 \epsilon^{-2} + \mathcal{I}_2 \epsilon^{-1} + \mathcal{I}_3 + o(1),
\end{align*}
\]
with
\[
\begin{align*}
\mathcal{I}_1 &= \frac{1}{7} H \\
\mathcal{I}_2 &= \mathcal{R}_0 - \frac{1}{3} \mathcal{R} \\
24 \mathcal{I}_3 &= 9 \mathcal{R}_{00} - 3 \mathcal{R}_0 - 5 \Delta H - 30 L^{ij} \mathcal{R}_{00 ij} - 30 \text{tr}(L^3) \\
&\quad + 17 H |\hat{L}|^2 + 13 H \mathcal{R}_{00} - \frac{2}{3} H \mathcal{R} + \frac{26}{3} \mathcal{R}^2.
\end{align*}
\]
Now (2.9) gives
\[
B_0 = 1 \\
B_1 = \mathcal{I}_1 + D_1 \\
B_2 = \mathcal{I}_2 + \mathcal{I}_1 D_1 + D_2 \\
B_3 = \mathcal{I}_3 + \mathcal{I}_2 D_1 + \mathcal{I}_1 D_2 + D_3.
\]
Substituting (2.14) and (2.28) and simplifying give finally
\[
\begin{align*}
B_0 &= 1 \\
B_1 &= -\frac{1}{7} H \\
B_2 &= \frac{7}{3} \mathcal{R}_0 - \frac{1}{3} \mathcal{R} - \frac{1}{7} |\hat{L}|^2 \\
24 B_3 &= 5 \mathcal{R}_{00} - 3 \mathcal{R}_0 - 5 \Delta H - 38 L^{ij} \mathcal{R}_{00 ij} - 38 \text{tr}(L^3) \\
&\quad + 23 \mathcal{R} |\hat{L}|^2 - 7 \mathcal{R}_0 + \frac{22}{3} H \mathcal{R} + \frac{42}{3} \mathcal{R}^2.\end{align*}
\]
Now substitute (2.32) for the expansions (1.1) and (1.2) for solutions of the singular \(S\)-Yamabe problem on \(M^{n+1}\). As described in the Introduction, we choose a metric \(\tilde{g} = \frac{1}{2} \tilde{g} + \frac{1}{3} \tilde{H} \tilde{g}\) and apply

\[
\text{tr}(L^3) = \text{tr}(\tilde{L}^3) + H|\tilde{L}|^2 + \frac{1}{3} H^3
\]

\[
L^{ij} \tilde{R}_{ij} = \tilde{L}^{ij} \tilde{R}_{ij} + \frac{1}{3} H \tilde{R} - \frac{1}{3} H \tilde{R}_{00}
\]

\[
L^{ij} \tilde{R}_{00ij} = \tilde{L}^{ij} \tilde{W}_{00ij} + \frac{1}{2} \tilde{L}^{ij} \tilde{R}_{ij} + \frac{1}{3} H \tilde{R}_{00}
\]

to write in terms of trace-free parts. Collecting terms gives

\[
12(S - 2B_3) = \frac{1}{2} \tilde{R}_{00} - 5L^{ij} \tilde{R}_{ij} + \frac{25}{3} \Delta H + 62L^{ij} \tilde{R}_{00ij} + 54 \text{tr}(\tilde{L}^3)
\]

\[
+ 12 \tilde{L}^{ij} \tilde{R}_{ij} + 7H|\tilde{L}|^2 + 3H \tilde{R}_{00} - \frac{5}{3} H \tilde{R} - \frac{8}{3} H^3.
\]

Upon comparison with (1.8), one concludes that \(S - 2B_3 = \mathcal{B}_\tilde{g}\) modulo divergence terms, as claimed.

### 3. The singular \(\sigma_2\)-YAMABE PROBLEM

The main goal of this section is to prove Theorems 1.6 and 1.8. We begin by discussing the expansions (1.1) and (1.2) for solutions of the singular \(\sigma_k\)-Yamabe problem on \(M^{n+1}\). As described in the Introduction, we choose a metric \(\tilde{g} \in C^\infty(M)\) and we assume that we have a polyhomogeneous defining function \(u\) so that (1.16) holds, where \(g = u^{-2}\tilde{g}\). The first task is to identify the relevant indicial root of the equation, which determines the form of the asymptotic expansion of \(u\).

Let \(u_0\) be a smooth defining function for \(M\) such that \(g_0 = u_0^{-2}\tilde{g}\) is asymptotically hyperbolic. Thus \(u_0 = r + O(r^2)\), where \(r\) is the \(\tilde{g}\)-distance to \(\partial M\). Consider a perturbation \(u = u_0 + vr^\gamma\) for some \(\gamma > 1\) and \(v \in C^\infty(M)\), and set \(g = u^{-2}\tilde{g}\). The uniformly degenerate structure of the equation (1.16) (see [MP]) implies that

\[
\sigma_k(-g^{-1}P_g) = \sigma_k(-g_0^{-1}P_{g_0}) + I(\gamma)vr^\gamma + o(r^\gamma)
\]
for a quadratic polynomial $I(\gamma)$ called the indicial polynomial, whose roots are called the indicial roots. In \[MP\], the unknown was taken to be $u_{MP}$, where $u = e^{-u_{MP}}$. In particular, perturbing $u_{MP}$ at order $\gamma$ corresponds to perturbing $u$ at order $\gamma + 1$, so the indicial polynomial which arose in \[MP\] was $I_{MP}(\gamma) = I(\gamma + 1)$. The polynomial $I_{MP}(\gamma)$ was identified near the bottom of p. 179 of \[MP\]: $I_{MP}(\gamma) = c_{k,n}(\gamma^2 - n\gamma) - 2k\beta_k$, with roots

$$\gamma_\pm = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{2k\beta_k}{c_{k,n}}}.$$ 

Here $\beta_k^0 = 2^{-k}(\frac{n+1}{k})$ is the constant on the right-hand side of (1.16) and $c_{k,n}$ is the constant such that

$$T_{k-1}(\frac{1}{2}I) = c_{k,n}I,$$

where $T_{k-1}$ is the $(k-1)$-st Newton transform and $I$ is the $(n+1) \times (n+1)$ identity matrix. It was stated in \[MP\] that $c_{k,n} = 2^{-k}(\frac{n}{k})$, but in fact the correct value is $c_{k,n} = 2^{1-k}(\frac{n}{k-1})$. Making this correction, one obtains $\gamma_+ = n+1$, $\gamma_- = -1$. So for $u$ the indicial roots are $n+2$, 0. The lower value of 0 is an indicial root of the formal linearization of the problem, but it is not relevant for us since we require $u$ to vanish at $\partial M$. So the only relevant indicial root for $u$ is $n+2$. The usual inductive order-by-order derivation then shows that the expansion of a polyhomogeneous solution $u$ necessarily is of the form (1.1). Since the volume form is given by $du = u^{-(n+1)}dv_g$, it then follows that $\text{Vol}(\{r > \epsilon\})$ has an expansion of the form (1.2) by exactly the same reasoning as in the case $k = 1$ in \[G3\] (or see \[GoW\]).

**Proof of Theorems 1.6 and 1.8.** As in the proof of Theorem 1.1 we begin by applying the Chern-Gauss-Bonnet Theorem for smooth manifolds-with-boundary for the metric $\overline{g}$ on $\{r \geq \epsilon\}$ with $\epsilon > 0$ small:

$$8\pi^2 \chi(M) = \int_{r > \epsilon} \left( \frac{1}{4} |W_g|^2 + 4\sigma_2(\overline{g}^{-1}P) \right) dv_g + \int_{r = \epsilon} S dv_{h_\epsilon},$$

where $S$ is given by (2.6), and $L, H$ refer to the second fundamental form and mean curvature of $\{r = \epsilon\}$ for the metric $\overline{g}$ with respect to the inward pointing unit normal. Using the conformal invariance of the Weyl tensor, applying Lemma 2.1 with $\sigma_2(g^{-1}P) = 3/2$, then integrating by parts give

$$8\pi^2 \chi(M) = \frac{1}{4} \int_{r > \epsilon} |W_g|^2 dv_g + 6 \text{Vol}_g(\{r > \epsilon\})$$

$$-2 \int_{r = \epsilon} \left( u^{-3} |du|^2_{\overline{g}} u_0 - u^{-2}(\Delta u) u_0 + u^{-2} u_{0\beta} u^\beta + u^{-1} R_{0\beta} u^\beta - \frac{1}{2} u^{-1} R u_0 \right) dv_{h_\epsilon}$$

$$+ \int_{r = \epsilon} S dv_{h_\epsilon}$$

As before, as $\epsilon \to 0$ the first term on the right-hand side converges to $\frac{1}{4} \int_M |W_g|^2 dv_g$ and the last term converges to $\int_{\partial M} S dv_h$. Thus the sum of the other two terms on the right-hand side converges as $\epsilon \to 0$ (though, as in the case of singular Yamabe metrics, each of them diverges individually). The expansion of $\text{Vol}_g(\{r > \epsilon\})$ is given by (1.1). For the boundary integral, we will show that the expansion (1.1) implies that

$$\left( u^{-3} |du|^2_{\overline{g}} u_0 - u^{-2}(\Delta u) u_0 + u^{-2} u_{0\beta} u^\beta + u^{-1} R_{0\beta} u^\beta - \frac{1}{2} u^{-1} R u_0 \right) \bigg|_{r = \epsilon} dv_{h_\epsilon} = (A_0 \epsilon^{-3} + A_1 \epsilon^{-2} + A_2 \epsilon^{-1} + A_3 + o(1)) dv_h$$

(3.2)
for smooth locally determined functions $A_0, A_1, A_2, A_3$ on $\partial M$. As before, the log term in the expansion of the solution $u$ does not enter into the expansion (3.2) to this order; it generates an $\epsilon \log \epsilon$ term in the expansion.

We can immediately deduce that $E^2 = 0$: since the divergent terms must cancel and there is no log term in (3.2), there cannot be one in (1.2) either. This proves Theorem 1.8.

The coefficients of the negative powers of $\epsilon$ must cancel as well, so it follows that

$$2 \int_{\partial M} A_0 \, dv_h = 6c_0,$$

(3.3)

$$2 \int_{\partial M} A_1 \, dv_h = 6c_1,$$

$$2 \int_{\partial M} A_2 \, dv_h = 6c_2.$$

Then taking the limit in (3.1) gives

$$8\pi^2 \chi(M) = \frac{1}{4} \int_M |W|^2 \, dv_g + 6V(g, \overline{g}) + \int_{\partial M} (-2A_3 + S) \, dv_h.$$

This will complete the proof of Theorem 1.6 once we show that $S - 2A_3 = R^2_{\overline{g}}$ modulo divergence terms.

In order to calculate $A_3$, we need to expand all the ingredients appearing in (3.2). The expansion of $u$ is determined by the condition

$$\sigma_2(g^{-1}P_g) = \frac{3}{2},$$

(3.4)

with $g = u^{-2} \overline{g}$. We rewrite this equation using (2.4). Expand out the divergence on the right-hand side of (2.4) term-by-term using the Leibniz rule and multiply the whole equation by $\frac{1}{2} u^4$. For the second and third terms inside the divergence, note that the Ricci identity implies

$$\nabla^\alpha (u_{\alpha \beta} u^\beta) - \nabla (\overline{u}) u_\alpha = |\nabla^2 u|^2 - |\overline{u}|^2 + R_{\alpha \beta} u^\alpha u^\beta.$$

Upon collecting terms, one concludes that the equation (3.4) can be written as

$$2u^4 \sigma_2(g^{-1}P) = 3(1 - |du|^2) + 3u|du|^2 \overline{\Delta u} + u^2 |\nabla^2 u|^2 - u^2 (\overline{\Delta u})^2$$

$$+ \frac{1}{2} u^2 R |du|^2 + u^3 R_{\overline{\alpha \beta}} u^{\alpha \beta} - \frac{1}{2} u^3 R \Delta u.$$

(3.5)

As in Section 2 we expand $u$ as

$$u = r + f_0 r^2 + f_1 r^3 + f_2 r^4 + O(r^5 \log r),$$

where $f_j \in C^\infty(\partial M)$. Our goal is to substitute this into (3.5) and then calculate modulo $o(r^3)$. The derivatives of $u$ are given by (2.19), (2.22), (2.23), and $|du|^2, \overline{\Delta u}$ by (2.20), (2.24). Upon multiplying the expansions and substituting (2.11) and (2.12) for the derivatives of $h$, we obtain the following expressions for each of the terms appearing in (3.5), all modulo
\( o(r^3) : \)

\[
|du|^4 = 1 + 8f_0r + \left[ 12f_1 + 24f_0^2 \right] r^2 + \left[ 16f_2 + 72f_0f_1 + 32f_0^3 \right] r^3
\]

\[
u|du|^2\nabla^2 u = [2f_0 - H] r + \left[ 6f_1 + 10f_0^2 - 7Hf_0 - |L|^2 \right] r^2
\]

\[
+ \left[ 12f_2 + 44f_0f_1 + \Delta f_0 + 16f_0^3 - 10Hf_1 - 18Hf_0^2 \right]
\]

\[
- 7|L|^2 r^2 - \frac{7}{2} \mathcal{R}_{000} r_0 - \frac{1}{2} \mathcal{R}_{000,0} - L^{ij} \mathcal{R}_{000j} - \text{tr}(L^3) \right] r^3
\]

\[
u^2 |\nabla^2 u|^2 = \left[ 4f_0^2 + |L|^2 \right] r^2
\]

\[
+ \left[ 24f_0f_1 + 8f_0^3 + 6|L|^2 f_0 + 2 \text{tr}(L^3) + 2L^{ij} \mathcal{R}_{000j} \right] r^3
\]

\[
u^2 (\nabla u)^2 = (2f_0 - H)^2 r^2 + \left[ 24f_0f_1 + 8f_0^3 - 12Hf_1 - 16f_0^2 + 6H^2 \right] r^3
\]

\[- 4|L|^2 f_0 - 4\mathcal{R}_{000} f_0 + 2H \mathcal{R}_{000} + 2H |L|^2 \right] r^3
\]

\[
u^2 \mathcal{R}_0|du|^2 = \mathcal{R}_0 r^2 + \left[ \mathcal{R}_0 - 6 \mathcal{R}_0 f_0 \right] r^3
\]

\[
u^2 \mathcal{R}_0 \alpha \beta u^{\alpha \beta} = 2\mathcal{R}_0 f_0 - L^{ij} \mathcal{R}_{ij} r^3
\]

\[
u^2 \mathcal{R}_0 \alpha \beta u^{\alpha \beta} = (2f_0 - H) \mathcal{R}_{00} r^3
\]

Combining the terms, (3.3) becomes

\[
o(r^3) = \left[ -18f_0 - 3H \right] r + \left[ -18f_1 - 42f_0^2 - 17Hf_0 - 2|L|^2 - H^2 - 3\mathcal{R}_{000} + \frac{1}{2} \mathcal{R} \right] r^2
\]

\[
+ \left[ -12f_2 - 84f_0f_1 + 3\Delta f_0 - 48f_0^3 - 18Hf_1 - 38Hf_0^2 - 11|L|^2 f_0 - 6H^2 f_0
\]

\[
- 15\mathcal{R}_{000} f_0 + 2\mathcal{R}_0 f_0 - \frac{3}{2} \mathcal{R}_{000,0} + \frac{1}{2} \mathcal{R}_0 - L^{ij} \mathcal{R}_{000j} - \text{tr}(L^3) - 2H \mathcal{R}_{000} - 2H |L|^2
\]

\[- L^{ij} \mathcal{R}_{ij} + \frac{1}{2} H \mathcal{R}_0 \right] r^3.
\]

Setting the coefficients successively to zero and solving, it follows that

\[
f_0 = - \frac{1}{6} H,
\]

\[
18f_1 = - 2|L|^2 - 3\mathcal{R}_{000} + \frac{1}{2} \mathcal{R},
\]

\[
24f_2 = - 3\mathcal{R}_{000,0} + \mathcal{R}_0 - \Delta H - 2L^{ij} \mathcal{R}_{000j} - 2L^{ij} \mathcal{R}_{ij} - 2 \text{tr}(L^3)
\]

\[
+ \frac{7}{3} H \mathcal{R}_{000} + \frac{1}{3} H \mathcal{R} + \frac{5}{3} H |L|^2 + \frac{8}{3} H^3.
\]

**Remark 3.1.** The fact that the two solutions \( u_1 \) (for the singular Yamabe problem) and \( u_2 \) (for the singular \( \sigma \)-Yamabe problem) transform the same way under conformal change implies that the first nonzero coefficient in the expansion of \( u_1 - u_2 \) must be a conformal invariant up to scale. This is confirmed upon comparing (3.6) with the \( n = 3 \) case of (2.16) and (2.17): the coefficients of \( r \) and \( r^2 \) agree in the expansions of \( u_1 \) and \( u_2 \), and the coefficients of \( r^3 \) differ by a multiple of \( |L|^2 \).

As before, we define \( J \) by (2.25), the integrand of the first boundary term in (3.1). Using the expansions in (2.25), the formulas in (3.6), and the expansion of the metric in (2.11) and (2.12), we obtain (2.27), this time with

\[
J_1 = \frac{1}{2} H
\]

\[
J_2 = \frac{1}{3} |L|^2 + \mathcal{R}_{000} - \frac{1}{3} \mathcal{R}
\]

\[
24J_3 = 9\mathcal{R}_{000,0} - 3\mathcal{R}_{00} - 5\Delta H + 6L^{ij} \mathcal{R}_{000j} - 18L^{ij} \mathcal{R}_{ij} + 6 \text{tr}(L^3)
\]

\[
- 5H \mathcal{R}_{000} + \frac{16}{3} H \mathcal{R} - \frac{37}{3} H |L|^2 - \frac{10}{9} H^3.
\]
Using the expansion of the volume form \((2.13)\) together with \((3.2)\) gives
\[
A_0 = 1 \\
A_1 = \mathcal{I}_1 + \mathcal{D}_1 \\
A_2 = \mathcal{I}_2 + \mathcal{I}_1 \mathcal{D}_1 + \mathcal{D}_2 \\
A_3 = \mathcal{I}_3 + \mathcal{I}_2 \mathcal{D}_1 + \mathcal{I}_1 \mathcal{D}_2 + \mathcal{D}_3.
\]
Substituting the formulas from \((2.14)\) and \((3.7)\) and simplifying give
\[
A_0 = 1 \\
A_1 = -\frac{1}{2} H \\
A_2 = \frac{1}{2} \overline{R}_{00} - \frac{1}{3} \overline{R} - \frac{1}{3} |\overline{\mathcal{L}}|^2 - \frac{1}{3} H^2 \\
24A_3 = 5\overline{R}_{00,0} - 3\overline{R}_{0,0} - 5\Delta H - 2L^j_\mathcal{R}_{0i0j} - 18L^j_\mathcal{R}_{ij} - 2 \text{tr}(L^3) \\
- 23H\overline{R}_{00} + 40\frac{H}{3} \overline{R} - \frac{23}{3} H |\overline{\mathcal{L}}|^2 + \frac{24}{9} H^3.
\]
Consequently,
\[
12(S - 2A_3) = -5\overline{R}_{00,0} + 3\overline{R}_{0,0} + 5\Delta H + 26L^j_\mathcal{R}_{0i0j} - 6L^j_\mathcal{R}_{ij} + 18 \text{tr}(L^3) \\
- H\overline{R}_{00} - \frac{4}{3} H \overline{R} - \frac{23}{3} H |\overline{\mathcal{L}}|^2 - \frac{26}{9} H^3.
\]
Using the contracted second Bianchi identity \((2.32)\) and rewriting in terms of the trace-free components of the intrinsic and extrinsic curvatures give
\[
12(S - 2A_3) = \frac{1}{2} \overline{R}_{i,0} - 5\dot{L}_\mathcal{R}^{ji} + \frac{25}{3} \Delta H + 26L^j_\mathcal{R}_{0i0j} + 18 \text{tr}(\dot{L}^3) \\
+ 12\dot{L}^j_\mathcal{R}_{ij} + \frac{26}{3} H |\dot{\mathcal{L}}|^2 + 3H\overline{R}_{00} - \frac{3}{2} H \overline{R} - \frac{3}{2} H^3.
\]
Upon comparison with \((1.17)\), one concludes that \(S - 2A_3 = \mathcal{B}_g^2\) modulo divergence terms, as claimed. \(\square\)

4. Renormalized Volume Coefficients and Anomalies

In this section we discuss renormalized volume coefficients and identify them explicitly for the singular Yamabe problem and the singular \(\sigma_2\)-Yamabe problem when \(n = 3\). We prove a general result to the effect that the infinitesimal anomaly for the renormalized volume can be written explicitly in terms of renormalized volume coefficients. We also indicate how these identifications can be used to give alternate direct proofs of some of the results of the previous two sections.

Let \(\overline{g}\) be a smooth metric on \(M\), let \(u\) be a defining function for \(\partial M\) with an asymptotic expansion of the form \((1.1)\), and set \(g = u^{-2}\overline{g}\). If \(\overline{g}\) is written in the geodesic normal form \((2.1)\), then
\[
dv_g = u^{-n-1}dv_{\overline{g}} = u^{-n-1} \sqrt{\frac{\text{det} h}{\text{det} h_0}} dr dv_{h_0}.
\]
It follows that
\[
dv_g = r^{-n-1}[v^{(0)} + v^{(1)}r + v^{(2)}r^2 + \cdots + v^{(n)}r^n + O(r^{n+1} \log r)] dr dv_{h_0}
\]
for some functions \(v^{(j)} \in C^{\infty}(\partial M)\) called the renormalized volume coefficients for \(g\) relative to \(\overline{g}\). The \(c_j\) and \(E\) in \((1.2)\) are then given by
\[
c_j = \frac{1}{n-j} \int_{\partial M} v^{(j)} dv_{h_0}, \quad 0 \leq j \leq n - 1, \quad E = \int_{\partial M} v^{(n)} dv_{h_0}.
\]
In the volume expansion (1.2) to the integrals of the divergences in the Chern-Gauss-Bonnet formula. Since the case, in §2 we wrote $u = r(1 + r\varphi)$. From (4.1), it follows that the renormalized volume coefficients are determined by:

$$(1 + r\varphi)^{-n-1} \sqrt{\frac{\det h_r}{\det h_0}} = v^{(0)} + v^{(1)} r + v^{(2)} r^2 + \cdots + v^{(n)} r^n + O(r^{n+1} \log r).$$

Now $v^{(0)} = 1$ and $v^{(1)}$, $v^{(2)}$ are derived for general $n$ in (4.5) of [G3]. Setting $n = 3$ and rewriting $v^{(2)}$ using the Gauss equation ((4.3) of [G3]), these become

$$
\begin{align*}
\quad v^{(0)} &= 1 \\
\quad v^{(1)} &= -\frac{1}{3} H \\
\quad v^{(2)} &= -\frac{1}{9} R + \frac{1}{6} R_{00} - \frac{1}{3} H^2 + \frac{1}{6} |\hat{L}|^2.
\end{align*}
$$

Using the expansions of $\varphi$ and $h_r$ derived in [2] we calculated

$$
v^{(3)} = \frac{2}{3}(\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}) + \frac{1}{2} L_{ij} - \frac{1}{6} \Delta H, \quad n = 3.
$$

In particular, the log term coefficient in (1.2) is given by

$$
\mathcal{E} = 2 \int_{\partial M} (\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}) \, dv_h, \quad n = 3.
$$

Equation (1.2) expresses the expansion of the volume using the defining function $r$ for the exhaustion. One may choose to use other defining functions. The coefficients in the expansion of $\text{Vol}_g\{u > \epsilon\}$ were calculated in [G3]. It is a general fact (see e.g. [G3]) that the log term coefficient is independent of the defining function chosen for the exhaustion. Indeed, (4.5) agrees with the coefficient of the log term in the corresponding expansion (4.15) of [G3].

In the proof of Theorem 1.1 in [2] the identities (2.10) relating the coefficients $c_j$, $\mathcal{E}$ in the volume expansion (1.2) to the integrals of the $B_j$ were deduced via cancellation of divergences in the Chern-Gauss-Bonnet formula. Since the $c_j$ and $\mathcal{E}$ are given in terms of the renormalized volume coefficients by (1.2), our identifications (1.3), (1.4) of these coefficients can be used to give an alternate proof of (2.10) by direct calculation.

One further piece of information is needed to carry out such a direct proof. The last two equations of (2.10) involve the coefficients $a$, $\mathcal{F}$, which are determined by (2.8). These can also be calculated directly. Starting with the conformal transformation law (2.2) of Ricci, we derived for general $n$:

$$
\begin{align*}
E_{ij} &= -(n-1) \hat{L}_{ij} r^{-1} + (n-1) \left(\frac{1}{2n} H \hat{L}_{ij} - \hat{W}_{0ij} + \hat{L}_{ik} \hat{L}_{kj} + \frac{1}{n-1} |\hat{L}|^2 h_{ij} \right) + O(r) \\
E_{0i} &= O(1) \\
E_{00} &= O(1).
\end{align*}
$$

Upon setting $n = 3$ and expanding, one obtains

$$
|E|^2 dv_g = |E|^2 dv_g = \left(4|\hat{L}|^2 r^{-2} + 8 \left(\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}\right) r^{-1} + O(1)\right) dr dv_{h_0}.
$$

Thus

$$
\begin{align*}
a &= 4 \int_{\partial M} |\hat{L}|^2 dv_h \\
\mathcal{F} &= 8 \int_{\partial M} (\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}) \, dv_h.
\end{align*}
$$

See [G3, GoW] for further discussion.

Consider first the case where $g = u^{-2}g$ is a solution of the singular Yamabe problem. In this case, in §2 we wrote $u = r(1 + r\varphi)$. From (4.1), it follows that the renormalized volume coefficients are determined by:

$$
\begin{align*}
(1 + r\varphi)^{-n-1} \sqrt{\frac{\det h_r}{\det h_0}} = v^{(0)} + v^{(1)} r + v^{(2)} r^2 + \cdots + v^{(n)} r^n + O(r^{n+1} \log r).
\end{align*}
$$

Now $v^{(0)} = 1$ and $v^{(1)}$, $v^{(2)}$ are derived for general $n$ in (4.5) of [G3]. Setting $n = 3$ and rewriting $v^{(2)}$ using the Gauss equation ((4.3) of [G3]), these become

$$
\begin{align*}
\quad v^{(0)} &= 1 \\
\quad v^{(1)} &= -\frac{1}{3} H \\
\quad v^{(2)} &= -\frac{1}{9} R + \frac{1}{6} R_{00} - \frac{1}{3} H^2 + \frac{1}{6} |\hat{L}|^2.
\end{align*}
$$

Using the expansions of $\varphi$ and $h_r$ derived in [2] we calculated

$$
v^{(3)} = \frac{2}{3}(\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}) + \frac{1}{2} L_{ij} - \frac{1}{6} \Delta H, \quad n = 3.
$$

In particular, the log term coefficient in (1.2) is given by

$$
\mathcal{E} = 2 \int_{\partial M} (\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}) \, dv_h, \quad n = 3.
$$

Equation (1.2) expresses the expansion of the volume using the defining function $r$ for the exhaustion. One may choose to use other defining functions. The coefficients in the expansion of $\text{Vol}_g\{u > \epsilon\}$ were calculated in [G3]. It is a general fact (see e.g. [G3]) that the log term coefficient is independent of the defining function chosen for the exhaustion. Indeed, (4.5) agrees with the coefficient of the log term in the corresponding expansion (4.15) of [G3].

In the proof of Theorem 1.1 in [2] the identities (2.10) relating the coefficients $c_j$, $\mathcal{E}$ in the volume expansion (1.2) to the integrals of the $B_j$ were deduced via cancellation of divergences in the Chern-Gauss-Bonnet formula. Since the $c_j$ and $\mathcal{E}$ are given in terms of the renormalized volume coefficients by (1.2), our identifications (1.3), (1.4) of these coefficients can be used to give an alternate proof of (2.10) by direct calculation.

One further piece of information is needed to carry out such a direct proof. The last two equations of (2.10) involve the coefficients $a$, $\mathcal{F}$, which are determined by (2.8). These can also be calculated directly. Starting with the conformal transformation law (2.2) of Ricci, we derived for general $n$:

$$
\begin{align*}
E_{ij} &= -(n-1) \hat{L}_{ij} r^{-1} + (n-1) \left(\frac{1}{2n} H \hat{L}_{ij} - \hat{W}_{0ij} + \hat{L}_{ik} \hat{L}_{kj} + \frac{1}{n-1} |\hat{L}|^2 h_{ij} \right) + O(r) \\
E_{0i} &= O(1) \\
E_{00} &= O(1).
\end{align*}
$$

Upon setting $n = 3$ and expanding, one obtains

$$
|E|^2 dv_g = |E|^2 dv_g = \left(4|\hat{L}|^2 r^{-2} + 8 \left(\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}\right) r^{-1} + O(1)\right) dr dv_{h_0}.
$$

Thus

$$
\begin{align*}
a &= 4 \int_{\partial M} |\hat{L}|^2 dv_h \\
\mathcal{F} &= 8 \int_{\partial M} (\text{tr}(\hat{L}^3) + \hat{L}^i w_{0i0j}) \, dv_h.
\end{align*}
$$

See [G3, GoW] for further discussion.
Equations (2.10) can now be verified upon comparing the above formulas with (2.29) for the $B_j$. Namely, substitute (2.29) into the integrals in (2.10), substitute (4.2) for the $c_j$ and $E$, with the $v^{(j)}$ given by (4.3), (4.4), substitute (4.8) for $a$ and $F$, and compare.

Next let $g = u^{-2}g$ be a solution of the singular $\sigma_2$-Yamabe problem. The renormalized volume coefficients are again defined by (4.1) and the coefficients in the volume expansion (1.2) are again given by (4.2). We calculated the $v^{(j)}$ using the expansions of $u$ and $h_r$ derived in [3] analogously to the case $k = 1$. The results are:

$$
\begin{align*}
v^{(0)} &= 1 \\
v^{(1)} &= -\frac{1}{3}H \\
v^{(2)} &= -\frac{1}{9}R + \frac{1}{6}T_{00} - \frac{1}{18}H^2 - \frac{1}{18}|\mathcal{L}|^2 \\
v^{(3)} &= -\frac{1}{9}L_{ij} - \frac{1}{6}\Delta H.
\end{align*}
$$

Since $v^{(3)}$ is a divergence, this provides an alternate proof of Theorem 1.8 by direct calculation. Comparing with (3.8) gives an independent verification of (3.3).

We next discuss how the anomaly of the renormalized volume can be expressed in terms of renormalized volume coefficients. The anomaly of the renormalized volume $V(g, \overline{g})$ of a solution of the singular $\sigma_k$-Yamabe problem is defined to be the left-hand side of (1.3). An anomaly is determined by its linearization (infinitesimal anomaly): $\partial_t|_{t=0}V(g, e^{2\omega} \overline{g})$. For Poincaré-Einstein metrics with $n$ even, the infinitesimal anomaly is $\int_{\partial M} v^{(n)} \omega dv_h$; see §3 of [G3]. Since for the singular $\sigma_k$-Yamabe problem the rescaling occurs on a manifold-with-boundary, in this case normal derivatives of $\omega$ also appear in the infinitesimal anomaly. The following result shows that in this setting, the infinitesimal anomaly can be identified with the full set $(v^{(0)}, \cdots, v^{(n)})$ of renormalized volume coefficients.

**Proposition 4.1.** Let $g$ be an asymptotically hyperbolic metric with renormalized volume coefficients $v^{(j)}$ determined by (4.1) and renormalized volume $V(g, \overline{g})$ determined by (1.2) relative to a compactification $\overline{g}$. (It is not assumed that $g = r^{-2} \overline{g}$.) Let $\omega \in C^\infty(M)$. Then

$$
\partial_t|_{t=0}V(g, e^{2\omega} \overline{g}) = \int_{\partial M} \left( \sum_{j=0}^{n} \frac{v^{(n-j)}}{(j+1)!} \partial_t^j \omega \right) dv_h.
$$

**Proof.** The proof follows the same outline as in [GrW], [G3]. Set $\overline{g} = e^{2\omega} \overline{g}$ and let $r_t$ denote the distance to the boundary with respect to $\overline{g}_t$. Then $r_t = e^{\overline{\gamma}_t}r$ for a smooth function $\overline{\gamma}_t$ on $M$. Use the normal exponential map of $\overline{g}$ to identify $M$ near $\partial M$ with $[0, \delta]_r \times \partial M$ as above. For fixed $x \in \partial M$ and $t > 0$, we can solve the relation $s = e^{\overline{\gamma}_t(x, r)}$ for $r$ as a function of $s$: $r = sb_t(x, s)$, where $b_t(x, s)$ is a smooth nonvanishing function. Set $\epsilon_t(x, \epsilon) = eb_t(x, \epsilon)$. Then $r_t > \epsilon$ is equivalent to $r > \epsilon_t(x, \epsilon)$. From (4.1) it follows that

$$
\begin{align*}
\text{Vol}_g(\{r_t > \epsilon\}) - \text{Vol}_g(\{r > \epsilon\}) &= \int_{r_t > \epsilon} dv_g - \int_{r > \epsilon} dv_g \\
&= \int_{\partial M} \int_0^\epsilon \sum_{0 \leq j \leq n} v^{(j)}(x)r^{-n-1+j} dr dv_h + o(1) \\
&= \sum_{0 \leq j \leq n-1} \int_{\partial M} v^{(j)}(x) \frac{b_t(x, \epsilon)^{n-j} - 1}{n-j} dv_h \\
&- \int_{\partial M} v^{(n)}(x) \log b_t(x, \epsilon) dv_h + o(1).
\end{align*}
$$
Now \( V(g, \overline{g}_t) - V(g, \overline{g}) \) is the constant term in the expansion in \( \epsilon \) of this expression, so

\[
V(g, \overline{g}_t) - V(g, \overline{g}) = \sum_{0 \leq j \leq n-1} \int_{\partial M} \frac{\nu^{(j)}(x)}{(n-j)(n-j)!} \mathcal{D}^{n-j}(b_t(x, \epsilon)^{-n+j}) |_{\epsilon=0} dv_{h_0} \\
- \int_{\partial M} \nu^{(n)}(x) \log b_t(x, 0) dv_{h_0}.
\]

(4.11)

Thus to evaluate \( \partial_t|_{\epsilon=0} V(g, \overline{g}_t) \), we need to identify the Taylor expansion in \( \epsilon \) of \( b_t(x, \epsilon) \) to first order in \( t \).

First consider \( \Upsilon_t \). Since \( \Upsilon_0 = 0 \), we have \( \Upsilon_t = O(t) \). (Here and in the sequel, \( O(t^k) \) means \( t^k \) times a smooth function of \( t \) and the other variables.) Now \( \Upsilon_t \) is determined by the equation \( |dr_t|^2_{\overline{g}_t} = 1 \), which can be written

\[
2r \partial_r \Upsilon_t + r^2 |d\Upsilon_t|^2_{\overline{g}_t} = e^{2(\omega - \Upsilon_t)} - 1.
\]

(4.12)

So

\[
r \partial_r \Upsilon_t = (t \omega - \Upsilon_t) + O(t^2).
\]

(4.13)

Equation (4.12) implies \( \Upsilon_t = t \omega \) at \( r = 0 \). Differentiating (4.13) successively at \( r = 0 \) gives for the Taylor expansion of \( \Upsilon_t \) in \( r \):

\[
(j + 1) \partial^j_r \Upsilon_t|_{r=0} = t \partial^j_r \omega|_{r=0} + O(t^2), \quad j \geq 1.
\]

Next consider \( b_t \). The defining relations of \( \Upsilon_t, b_t \) and \( \epsilon_t \) show that

\[
\partial_t b_t(x, \epsilon) \epsilon_t(x, \epsilon) = 1.
\]

(4.14)

Evaluating at \( \epsilon = 0 \) gives \( \log b_t(x, 0) = - \Upsilon_t(x, 0) = -t \omega(x, 0) \). Equation (4.14) implies \( b_t(x, \epsilon) = 1 - \Upsilon_t(x, \epsilon) + O(t^2) \). Since \( \partial_t \epsilon_t = 1 + O(t) \), differentiating and applying the chain rule successively give \( \partial^j_t b_t(x, \epsilon) = -(\partial^j_t \Upsilon_t)(x, \epsilon) + O(t^2) \). Thus

\[
\partial^j_t b_t|_{\epsilon=0} = -\frac{t}{j+1} \partial^j_r \omega|_{r=0} + O(t^2), \quad j \geq 1.
\]

Now \( \partial_t (b_t^{-1}) = -b_t^{-l-1} \partial_t b_t = -l \partial_t b_t + O(t^2) \). Differentiating further gives \( \partial^j_t (b_t^{-l}) = -l \partial^j_t b_t + O(t^2) \), so in particular

\[
\partial^j_t (b_t^{-n+j})|_{\epsilon=0} = -(n-j) \partial^j_t b_t|_{\epsilon=0} + O(t^2) = \frac{t}{n-j} \partial^j_t \omega|_{r=0} + O(t^2).
\]

Substituting into (4.11) and applying \( \partial_t|_{\epsilon=0} \) give (4.10).

One way to view Corollary 1.4 is that it asserts that in the umbilic case for \( k = 1 \) and \( n = 3 \), the functional \( V(g, \overline{g}) \) has the property that its anomaly agrees with the anomaly of a functional given by integration of a local curvature expression. Likewise, Theorem 1.6 implies the corresponding statement for the general case when \( k = 2 \) and \( n = 3 \). This is an unusual property of a functional arising from a global construction. Since the functionals arising as integrals of local expressions are invariant under constant rescalings of \( \overline{g}_t \), the vanishing of the anomaly for \( \omega = constant \) is a necessary condition for a global functional to have anomaly equal to the anomaly of the integral of a local expression. Via this condition, one can easily see, for instance, that the anomaly for the renormalized volume of Poincaré-Einstein metrics with \( n \) even and the anomaly for functional determinants of natural differential operators do not agree with the anomaly of the integral of a local expression. (In these examples, the conformal rescaling occurs on a closed manifold, rather than on a manifold-with-boundary as in the singular \( \sigma_k \)-Yamabe problem.)
Under a rescaling $\hat{g} = e^{2\omega} g$ with constant $\omega$, the distance function transforms by $\hat{r} = e^{\omega} r$, so the renormalized volume for a solution of the singular $\sigma_k$-Yamabe problem transforms by $V(g, \hat{g}) = V(g, \overline{g}) + \mathcal{E} \omega$. Consequently, $\mathcal{E} = 0$ is a necessary condition for the anomaly to equal the anomaly of the integral of a local expression. That $\mathcal{E}^{\sigma_2} = 0$ for $n = 3$ is Theorem 1.8, and for $k = 1, n = 3$ and $\overline{g}$ umbilic, $\mathcal{E} = 0$ follows from (4.5).

Proposition 4.1 can be used to give alternate proofs by direct calculation of Corollary 1.4 and of the conformal invariance of $\tilde{V}^{\sigma_2}(g, \overline{g})$ in Theorem 1.6. For $k = 1$ it can be used more generally to give a direct proof of the infinitesimal conformal invariance of (1.13) in Theorem 1.6, from which its full conformal invariance is a consequence. Among other things, this verifies the numerical coefficients in (1.8), (1.17) (although this computation does not test the coefficients of $\text{tr}(\hat{\mathcal{L}}^3)$ or $\hat{\mathcal{L}}^{ij} \hat{\mathcal{W}}_{00ij}$, since these are pointwise conformally invariant).

To calculate the invariance of (1.13), one also needs to know the infinitesimal anomaly of $\frac{1}{2} \int_{r > \epsilon} |E|^2_g \, dv_g$. This can be derived in the same way as in Proposition 4.1. The argument is the same except that one uses (4.7) instead of (4.1), and it gives the same result with the $v(n-j)$ in (4.10) replaced by the coefficients in (4.7):

\[
(4.15) \quad \partial_t|_{t=0} \left( \frac{1}{2} \int_{r > \epsilon} |E|^2_g \, dv_g \right) = \int_{\partial M} \left[ 2|\hat{\mathcal{L}}|^2 \partial_r \omega + 8 \left( \text{tr}(\hat{\mathcal{L}}^3) + \hat{\mathcal{L}}^{ij} \hat{\mathcal{W}}_{00ij} \right) \right] \, dv_h.
\]

The infinitesimal conformal change of $B^g_\overline{g}$ and $B^{\sigma_2}_\overline{g}$ can be calculated term-by-term via usual conformal transformation laws of local curvature quantities. Combining with (4.10), (4.15) for the infinitesimal anomalies and substituting (4.3), (4.4), (4.9) for the renormalized volume coefficients which appear, one verifies that the infinitesimal anomalies vanish for (1.13) for $k = 1$, and for $\tilde{V}^{\sigma_2}(g) = V(g, \overline{g}) + \frac{1}{6} \int_{\partial M} B^{\sigma_2}_\overline{g} \, dv_h$ for $k = 2$. Consequently these expressions are conformally invariant. The computations are straightforward but tedious, so are omitted.

5. RELATED PROBLEMS

In [GSW], Gursky-Streets-Warren studied the singular $\sigma_k(\text{Ric})$-problem: given a Riemannian manifold-with-boundary $(M^{n+1}, \partial M, \overline{g})$, find a defining function $u$ so that $g = u^{-2} \overline{g}$ satisfies

\[
(5.1) \quad \sigma_k(-g^{-1} \text{Ric}_g) = n^k \binom{n+1}{k}.
\]

The constant is again the value on hyperbolic space. They showed that, just as for the singular Yamabe problem (and unlike the singular $\sigma_k$-Yamabe problem), there is always a unique solution. In [W], Wang studied the asymptotics of $u$ for domains in Euclidean space, and, among other things, showed that the indicial roots are once again 0 and $n+2$. These are the indicial roots for the equation (5.1) on a general manifold-with-boundary as well. Thus the formal asymptotic expansion of $u$ again takes the form (1.1) and the volume expansion has the same form (1.2).

When $n = 3$, exactly the same arguments as in the case $k = 1$ above can be used to prove a Chern-Gauss-Bonnet Theorem for solutions to the singular $\sigma_2(\text{Ric})$-problem. The relevant identity replacing (1.5) is

\[
4\sigma_2(g^{-1} P_g) = \frac{1}{3} \sigma_2(g^{-1} \text{Ric}_g) - \frac{4}{9} |E_g|^2.
\]
The same proof shows that there is a boundary term \( \mathcal{R}_{g}^{\sigma_2(\text{Ric})} \) so that if \( g = u^{-2}\overline{g} \) solves \( \text{5.1} \), then

\[
8\pi^2 \chi(M) = \frac{1}{4} \int_{M} |W|^2 \, dv_g - \frac{4}{9} \int_{r > \epsilon} |E|^2 \, dv_g + 6V(g, \overline{g}) + \int_{\partial M} \mathcal{R}_{g}^{\sigma_2(\text{Ric})} \, dv_h.
\]

Since the constant multiplying \( \int |E|^2 \) is again negative, the subsequent conclusions about the specialization to the umbilic case also hold, including the variational characterization of Poincaré-Einstein metrics based on Proposition \( \text{1.3} \).

We conclude with a discussion of another variant of the singular \( \sigma_k \)-Yamabe problem, this one motivated by Theorem \( \text{1.8} \). Recall that in \( \text{[G3]} \) and \( \text{[GoW]} \) it was proved that for the singular Yamabe problem, the variation of \( \mathcal{E} \) is a nonzero multiple of \( \mathcal{L} \), where \( \mathcal{E} \) is viewed as a conformally invariant energy of the varying hypersurface \( \Sigma = \partial M \) in the conformal manifold \( (M, \overline{g}) \). Otherwise stated, \( \mathcal{L} = 0 \) is the Euler-Lagrange equation for the functional \( \mathcal{E} \), thought of as a function of \( \Sigma \). If this variational relation were true also for \( k = 2 \) and \( n = 3 \), we could conclude from Theorem \( \text{1.8} \) that \( \mathcal{E}^{2} = 0 \), i.e. the expansion of the solution \( u \) has no log terms. This also raises the question of whether this variational relation between \( \mathcal{E} \) and \( \mathcal{L} \) holds more generally.

A crucial ingredient in our analysis of the Chern-Gauss-Bonnet formula was the divergence identity \( \text{2.4} \). Divergence structure also plays an important role in the proof for the singular Yamabe problem of the variational relation between \( \mathcal{E} \) and \( \mathcal{L} \). It has been known for some time that such divergence structure is lacking in general for \( \sigma_k(g^{-1}P) \) for \( k > 2 \), and Branson-Gover showed in \( \text{[BG]} \) that the equation \( \sigma_k(g^{-1}P) = \text{const} \) for \( k > 2 \) is an Euler-Lagrange equation if and only if \( g \) is locally conformally flat. However, Chang-Fang realized in \( \text{[CF]} \) that a modification of \( \sigma_k(g^{-1}P) \), which we denote \( v_k(g) \), has a variational/divergence structure, leading to the conclusion that at least for some purposes, \( v_k(g) \) is the correct replacement for \( \sigma_k(g^{-1}P) \) for \( k > 2 \). If \( k = 1 \) or \( 2 \), or if \( 3 \leq k \leq n \) and \( g \) is locally conformally flat, then \( v_k(g) = \sigma_k(g^{-1}P) \). These observations motivate us to consider the "singular \( v_k \)-Yamabe problem": given \( (M^{n+1}, \partial M, \overline{g}) \), find a defining function \( u \) so that \( g = u^{-2}\overline{g} \) satisfies

\[
v_k(g) = (-2)^{-k} \binom{n+1}{k}.
\]

We briefly recall the definition of \( v_k(g) \) and refer to \( \text{[CF]}, \text{[G2]} \) for details and elaboration. Consider the formal asymptotics of Poincaré-Einstein metrics: if \( g \) is a metric on a manifold \( M^n \), one searches for a metric \( g_+ = r^{-2}(dr^2 + g_r) \) which satisfies \( \text{Ric}(g_+) = -ng_+ \) to high order, where \( g_r \) is a one-parameter family of metrics on \( M \) with \( g_0 = g \). For \( n \) even, this determines the Taylor expansion of \( g_r \) to order \( n \), and for \( n \) odd, this, together with the condition that \( g_r \) be even in \( r \), determines the Taylor expansion of \( g_r \) to infinite order. Then one considers the expansion of the volume form \( dv_{g_r} = (1 + v^{2k}) r^2 + \ldots \) \( dv_g \). The curvature quantities \( v_k \) are a multiple of the renormalized volume coefficients appearing in this expansion: \( v_k = (-2)^{-k} v^{2k} \). So \( v_k \) is defined for all \( k \geq 0 \) for \( n \) odd, but only for \( k \leq n/2 \) for \( n \) even for general metrics. However, if \( g \) is Einstein or locally conformally flat, it is possible to continue the expansion of \( g_r \) to infinite order, uniquely upon imposing an appropriate auxiliary condition, so in these cases \( v_k(g) \) is defined for all \( k \) also for \( n \) even. It turns out that \( v_k(g) = \sigma_k(g^{-1}P_g) \) if \( k = 1 \) or \( 2 \), and also for \( 3 \leq k \leq n \) if \( g \) is locally conformally flat or Einstein.

For the singular \( v_k \)-Yamabe problem, we begin with \( (M^{n+1}, \partial M, \overline{g}) \), and \( g = u^{-2}\overline{g} \) is supposed to satisfy \( \text{5.2} \). So we must replace \( n \) by \( n + 1 \) in the above definition of \( v_k(g) \).
We always assume $k \leq n + 1$, and also require $2k \leq n + 1$ if $n$ is odd and $g$ is not locally conformally flat. The indicial roots of the equation (5.2), viewed as an equation for $u$, are again 0 and $n + 2$. So once again, the formal expansion of $u$ has the form (1.1) and the volume expansion has the form (1.2). Just as for the problems discussed above, the coefficients $\mathcal{E}^v_k$ and $\mathcal{L}^v_k$ are determined by formal calculations alone, so are well-defined independently of existence theory for the equation. And both of them satisfy the same conformal invariance relations as before: under conformal change $\overline{g} = \Omega^2 g$, one has $\overline{\mathcal{E}}^v_k = \mathcal{E}^v_k$ and $\overline{\mathcal{L}}^v_k = (\Omega|_\Sigma)^{-n-1} \mathcal{L}^v_k$.

The first result is a generalization of Theorem 1.8 to the singular $v_k$-Yamabe problem:

**Theorem 5.1.** If $n \geq 3$ is odd and $2k = n + 1$, then $\mathcal{E}^v_k = 0$.

The second result is a generalization to $k > 1$ of the variational relationship between $\mathcal{E}$ and $\mathcal{L}$. To formulate this result, note that $\mathcal{E}^v_k$ and $\mathcal{L}^v_k$ are determined just by the local geometry of $\partial M$ in $(M, \overline{g})$, so they can be defined for a general hypersurface $\Sigma$ with chosen normal direction in a Riemannian manifold $(M^{n+1}, \overline{g})$ (it must be assumed for $\mathcal{E}^v_k$ that $\Sigma$ is compact to carry out the integration). Suppose that $F_t : \Sigma \to M$, $0 \leq t \leq \delta$, is a variation of $\Sigma$, i.e. a smoothly varying one-parameter family of embeddings with $F_0 = \text{Id}$. Set $\Sigma_t = F_t(\Sigma)$ and denote by $\mathcal{E}^v_{k,t}$ the corresponding quantity for $\Sigma_t$. Write $\bar{F} = \partial_t F|_{t=0} \in \Gamma(TM|_\Sigma)$ and $\mathcal{E}^v_k = \partial_t \mathcal{E}^v_{k,t}|_{t=0}$. Let $\bar{\nu}$ denote the inward pointing $\overline{g}$-unit normal to $\Sigma$ in $M$.

**Theorem 5.2.** Suppose $n \geq 2$ and $1 \leq k \leq n + 1$. Suppose also that $2k \leq n + 1$ if $n$ is odd and $\overline{g}$ is not locally conformally flat. Then

$$\mathcal{E}^v_k = (n + 2)(n - 2k + 1) \int_\Sigma (\bar{F}, \bar{\nu}) \mathcal{L}^v_k \, dv_\Sigma.$$  

Thus the coefficient relating $\mathcal{E}^v_k$ and $\mathcal{L}^v_k$ vanishes when $2k = n + 1$, and in this case one can make no conclusions about $\mathcal{L}^v_k$ from the fact that $\mathcal{E}^v_k = 0$. In particular, when $k = 2$, $n = 3$, there is no conclusion about $\mathcal{L}^{c2}$ from the fact that $\mathcal{E}^{c2} = 0$.

Finally, we state a version of the Chern-Gauss-Bonnet Theorem in higher dimensions for solutions of the singular $v_k$-Yamabe problem, $2k = n + 1$. This is motivated by Theorem 1.6 above and by Theorem 3.3 in [CQY], which generalizes Anderson’s formula to Poincaré-Einstein metrics in higher even dimensions. For this result, we assume that our solution of the $v_k$-Yamabe problem is smooth in $M$ with a polyhomogeneous expansion at $\partial M$. Its renormalized volume $V(g, \overline{g})$ is defined as usual by (1.2).

**Theorem 5.3.** Let $n \geq 3$ be odd and $2k = n + 1$. There is a scalar pointwise conformal invariant $J$ of weight $-(n+1)$ and a boundary term $\mathcal{B}^v_k$ so that if $g = u^{-2} \overline{g}$ is a solution of the singular $v_k$-Yamabe problem which is smooth in $M$ and polyhomogeneous at $\partial M$, then

$$c_n \chi(M) = \int_M J_g \, dv_g + \bar{V}(g), \quad c_n = \frac{(-1)^{n+1} \Gamma(n+2)}{\Gamma(n+1)},$$

where

$$\bar{V}(g) = V(g, \overline{g}) + \int_{\partial M} \mathcal{B}^v_k \, dv_\overline{g}$$

is independent of the choice of compactification $\overline{g}$.

Note that the conformal invariance of $J$ implies that $J_g dv_g = J_{\overline{g}} dv_{\overline{g}}$, so that $\int_M J_g dv_g$ converges. Proofs of Theorems 5.1, 5.2, and 5.3 will be given elsewhere.
We remark that in dimension 2, constant Gauss curvature metrics play the role of both Einstein metrics and metrics of constant sectional curvature. So for $n = 1$, every singular Yamabe metric should be regarded as Poincaré-Einstein. The quantities discussed here for singular Yamabe metrics have the same properties when $n = 1$ as for Poincaré-Einstein metrics with $n \geq 3$ odd. Namely, $E$ and $L$ both vanish (see [G3]), and the renormalized volume defined using a geodesic defining function for $g$ is conformally invariant, i.e. independent of the geodesic defining function. In this case, the analogue of Theorem 5.3 is the result (Ep) that the renormalized volume defined using a geodesic defining function equals $-2\pi \chi(M)$. One can also consider the renormalized volume defined using an arbitrary defining function (corresponding to choosing an arbitrary compactification $\overline{g}$), in which case the analogue of (5.3) takes the form

$$-2\pi \chi(M) = V(g, \overline{g}) + \frac{1}{2} \int_{\partial M} H_{\overline{g}} d\sigma_{\overline{g}}.$$

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