On the algebraicity of polyquadratic plectic points

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Abstract

We establish direct evidence of the arithmetic significance of plectic Stark–Heegner points for elliptic curves of arbitrarily large rank. The main contribution is a proof of the algebraicity of plectic points associated to polyquadratic CM extensions of totally real number fields. Moreover, we relate the non-vanishing of plectic points to analytic and algebraic ranks of elliptic curves.

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1. Introduction

In previous work, plectic Stark–Heegner points were associated to quadratic extensions $E/F$ of number fields and modular elliptic curves $A/F$ under some technical assumptions. The construction generalizes the description of classical Heegner points on Shimura curves admitting a $p$-adic uniformization, and combines Nekovář–Scholl's plectic insights ([NS16], [Nek16]) with Darmon's pioneering work on Stark-Heegner points [Dar01]. Moreover, Conjectures 1.3, 1.5 of [FG21] predict the algebraicity of plectic Stark–Heegner points and their significance for elliptic curves of higher rank, for which some numerical evidence was provided in [FGM21]. The aim of this paper is to establish direct evidence for the aforementioned conjectures in the polyquadratic CM case.
1.1 Conjectures on plectic Stark–Heegner points

Even though the formulation of the conjectures does not require any restriction on the possible signatures of the fields $E$ and $F$, we only consider CM extensions in this article; that is, $F$ is a totally real number field and $E/F$ a totally complex quadratic extension. All prime divisors of the conductor $f$ of the elliptic curve $A$ are assumed to be unramified in $E/F$. We fix a rational prime $p$ and a set $S = \{p_1, \ldots, p_r\}$ of $p$-adic primes of $F$ such that

- $A/F$ has multiplicative reduction at every $p \in S$,
- the primes in $S$ are all inert in $E$.

Given $p \in S$ we also denote by $p$ the unique prime of $E$ above $p$. Furthermore, we let $a_p \in \{\pm 1\}$ equal $+1$ (resp. $-1$) when $A$ has split (resp. non-split) multiplicative reduction at $p$. The quantity $a_p$ is closely related to the local root number $\varepsilon_p(A/F)$ of $A$:

$$a_p = -\varepsilon_p(A/F).$$

By setting $p_S := p_1 \cdots p_r$, we may write

$$f = p_S \cdot n^S \cdot n^b,$$

for coprime ideals $n^S, n^b \in \mathcal{O}_F$ such that $n^S$ is divisible by every prime divisor of $f$ split in $E/F$.

Assumption 1.1 (Plectic Heegner hypothesis for $(A, E, S)$). We require that

- $n^b$ is square-free,
- the number of prime factors of $n^b$ is congruent to $[F : \mathbb{Q}]$ modulo 2.

Since $E/F$ is a CM extension, Assumption 1.1 implies that the sign of the functional equation of the $L$-function $L(A/E, s)$ of the base-change of $A/F$ to $E$ is equal to $\varepsilon(A/E) = (-1)^r$.

We introduce the following useful notations. First, if $H$ is any commutative prodiscrete group, we denote by $\widehat{H}$ the torsion-free part of its pro-$p$ completion. Second, if $M$ is an abelian group and $\Omega$ a field of characteristic zero, we use $M_{\Omega}$ and $\bigwedge^r(M_{\Omega})$ to respectively denote the tensor product $M \otimes_{\mathbb{Z}} \Omega$, and the $r$-th exterior power of the $\Omega$-vector space $M_{\Omega}$.

Consider the tensor product

$$\widehat{A}(E_S) := \widehat{A}(E_{p_1}) \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} \widehat{A}(E_{p_r}).$$

Under the running assumptions, the plectic Stark–Heegner point (for the trivial character) is

$$P_{A,S} \in \widehat{A}(E_S)_{\mathbb{Q}}$$

as constructed in (FG21, Section 4.4), where it was denoted $P^1_A$. The notion of algebraicity for plectic Stark–Heegner points is formulated in terms of a determinant map. Writing $i_p : E \hookrightarrow E_p$ for the canonical embedding for every $p \in S$, we can consider the homomorphism

$$\det : \bigwedge^r(A(E)_{\mathbb{Q}}) \longrightarrow \widehat{A}(E_S)_{\mathbb{Q}}$$

given by

$$\det(P_1 \wedge \cdots \wedge P_r) = \det \begin{pmatrix} i_{p_1}(P_1) & \cdots & i_{p_r}(P_1) \\ \vdots & \ddots & \vdots \\ i_{p_1}(P_r) & \cdots & i_{p_r}(P_r) \end{pmatrix}.$$

As usual, $r_{\text{alg}}(A/E)$ denotes the rank of the finitely generated abelian group $A(E)$, and $r_{\text{an}}(A/E)$ the order of vanishing of the $L$-function $L(A/E, s)$ at $s = 1$. 
Conjecture 1.2. If \( r_{\text{alg}}(A/E) \geq r \), then there exists an element \( w_{A,S} \in \mathcal{N}(A(E)_{\mathbb{Q}}) \) such that
\[
P_{A,S} = \det(w_{A,S}).
\]
Moreover, if \( P_{A,S} \neq 0 \), then \( r_{\text{alg}}(A/E) = r \).

Remark 1.3. When \( S \) consists of a single prime, Conjecture 1.2 follows from the generalization of the Gross–Zagier–Kolyvagin theorem for totally real number fields by Nekovář and Zhang. Indeed, Čerednik–Drinfeld’s uniformization of Shimura curves implies that the plectic point \( P_{A,S} \) is a Heegner point when \( |S| = 1 \) (see Section 3.2 for more details).

There are also plectic Stark–Heegner points associated to non-trivial anticyclotomic characters of \( E/F \). Aiming for clarity in the introduction, we discuss them only in the body of the paper.

1.1.1 Eigenspaces for partial Frobenii. Let \( \sigma_p \) be the generator of the Galois group of \( E_p/F_p \). It naturally acts on \( \widehat{A}(E_p) \), and thus also on \( \widehat{A}(E_S) \) via its action on the \( p \)-th factor. We set
\[
\widehat{A}(E_p)^\pm := \widehat{A}(E_p)^{\sigma_p=\pm a_p}_{/\mathbb{Q}} \quad \text{and} \quad \widehat{A}(E_S)^\pm := \otimes_{p \in S} \widehat{A}(E_p)^\pm.
\]
There are two eigenspace projections
\[
pr_S^\pm: \widehat{A}(E_S)_{\mathbb{Q}} \rightarrow \widehat{A}(E_S)^\pm, \quad pr_S^\pm = \prod_{p \in S} (1 \pm a_p \cdot \sigma_p).
\]
The main results of this article (Theorems A & B) establish the first cases of the minus part of the following conjecture, a direct consequence of Conjecture 1.2.

Conjecture 1.4. If \( r_{\text{alg}}(A/E) \geq r \), then there exists an element \( w_{A,S} \in \mathcal{N}(A(E)_{\mathbb{Q}}) \) such that
\[
pr_S^\pm(P_{A,S}) = pr_S^\pm(\det(w_{A,S})).
\]
Moreover, if \( pr_S^\pm(P_{A,S}) \neq 0 \), then \( r_{\text{alg}}(A/E) = r \).

The special cases that we treat in Theorems A, B and C are singled out precisely to leverage the known properties of classical Heegner points. The key idea is to further suppose that \( F \) is a polyquadratic extension of another totally real number field \( F_0 \), the elliptic curve is the base change of an elliptic curve \( A_\circ \) defined over \( F_0 \), and \( E \) is the compositum of \( F \) with a quadratic CM extension \( E_\circ/F_0 \). Then, under an appropriate Heegner hypothesis, we use Heegner points for \( A_{\circ/E_\circ} \) and its twists by characters of \( \text{Gal}(F/F_0) \), plus a factorization of anticyclotomic \( p \)-adic \( L \)-functions to establish our results.

1.1.2 Plectic \( p \)-adic invariants. Anticyclotomic \( p \)-adic \( L \)-functions come in to play in the proofs of our theorems because of the \( p \)-adic Gross-Zagier formula ([FG21], Theorem A) relating higher order derivatives to plectic \( p \)-adic invariants. These invariants, denoted \( Q_{A,S} \), are canonical lifts of the points \( pr_S^\pm(P_{A,S}) \) with respect to a “plectic” Tate parametrization: as the elliptic curve \( A/F \) has multiplicative reduction at every \( p \in S \), Tate’s \( p \)-adic uniformization results provides surjections \( \phi_p^{\text{Tate}}: E_p^x \rightarrow A(E_p) \) whose kernels are generated by Tate periods \( q_p \in F_p^x \setminus \mathcal{O}_{F_p}^x \). If we denote by \( E_p^{-} := (E_p^x)^{\sigma_p=-1} \) the subgroup of \( E_p^x \) on which \( \sigma_p \) acts via inversion, and we set
\[
\widehat{E}_{S,\otimes} := \widehat{E}_p^{-} \otimes_{p \in S} \mathbb{Z}_p, \quad \widehat{E}_p^{-},
\]
then the plectic \( p \)-adic invariant \( Q_{A,S} \) is the unique element of \( \widehat{E}_{S,\otimes} \) satisfying
\[
\phi_S^{\text{Tate}}(Q_{A,S}) = pr_S^\pm(P_{A,S}).
\]
Here $\phi^\text{Tate}_S : \widehat{E}_S^{\otimes} \to \widehat{A}(E_S)$ denotes the tensor product of Tate’s local uniformizations. Since the restriction of $\phi^\text{Tate}_S$ to $\widehat{E}_S^{\otimes}$ is injective, we have that $Q_{A,S} \neq 0$ if and only if $\text{pr}^-_{S}(P_{A,S}) \neq 0$.

1.2 The polyquadratic setup

For the rest of the introduction we suppose that the totally real number field $F$ is a polyquadratic extension of degree $2^t$ of a number field $F_0$, i.e., $F/F_0$ is a Galois extension with Galois group $G := \text{Gal}(F/F_0) \cong (\mathbb{Z}/2\mathbb{Z})^t$. Further, we assume that $E$ is the compositum of $F$ with a quadratic CM extension $E_o/F_0$ and that the following technical assumptions are satisfied.

**Assumption 1.5.** We require that

- every non-trivial subextension of $F/F_0$ is ramified,
- all primes of $F_0$ that ramify in $F$ split in $E_o$,
- the elliptic curve $A/F$ is the base change of a modular elliptic curve $A_o$ defined over $F_0$, whose conductor $f_o$ is unramified in $F/F_0$,
- the set $S$ consists of all primes of $F$ lying above a single prime $\wp$ of $F_0$, totally split in $F$.

**Remark 1.6.** The need for the first assumption is explained in Remark 1.2. The splitting in $E_o$ of the primes ramified in $F/F_0$ is necessary to construct Heegner points on twists of $A/F$ by characters of $G$, while the total splitting of the prime $\wp$ in $F/F_0$ is just a simplifying hypothesis for the proof of Proposition 1.5.

Assumption 1.5 implies that the elliptic curve $A/F$ is modular by quadratic base change for Hilbert modular forms. Moreover, we deduce that the cardinality $r = |S|$ equals $2^t$, and that the prime $\wp$ is inert in $E_o/F_0$. By a small abuse of notation, we denote by $\wp$ the unique prime of $E_0$ lying above $\wp$. Since $\wp$ is completely split in $F/F_0$, the elliptic curve $A_o/F_0$ has multiplicative reduction at $\wp$. Furthermore, if we set $a_\wp = 1$ (resp. $a_\wp = -1$) in case $A_o/F_0$ has split (resp. non-split) multiplicative reduction, we have

$$a_p = a_\wp \quad \forall \ p \in S. \quad (1)$$

Now, write the conductor $f_o$ of $A_o/F_0$ as

$$f_o = \wp \cdot n_o^s \cdot n_o^b$$

where $n_o^s$ is divisible by every prime divisor of $f_o$ split in $E_o/F_0$.

**Assumption 1.7 (Generalized Heegner hypothesis for $(A_o, E_0, \wp)$).** We require that

- $n_o^b$ is square-free,
- the number of prime factors of $n_o^b$ is congruent to $|F_0 : \mathbb{Q}|$ modulo 2.

Under Assumption 1.7, the sign of the functional equation for $A_o/E_0$ equals $\varepsilon(A_o/E_0) = -1$. Hence, the BSD-conjecture predicts that $A_o(E_0)$ is non-torsion. Moreover, for any character $\eta : G \to \{\pm 1\}$, the twist $A_o^\eta/F_0$ also fulfils the generalized Heegner hypothesis and we have

$$\varepsilon(A_o^\eta/E_0) = -1$$

because every prime ramified in $F/F_0$ splits in $E_o/F_0$ by Assumption 1.5. Thus, we expect that

$$r_\text{alg}(A/E) \geq [F : F_0] = r,$$

with equality if and only if $r_\text{alg}(A_o^\eta/E_0) = 1$ for every character $\eta : G \to \{\pm 1\}$.
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Remark 1.8. Under the running assumptions \([\text{L3}]\) and \([\text{L7}]\), the plectic Heegner hypothesis for \((A, E, S)\) holds if and only the number of prime divisors of \(n^b\) is even. For example, this is the case when the conductor \(f_\circ\) of \(A_\circ/F_\circ\) is completely split in \(F/F_\circ\).

Remark 1.9. Simple examples satisfying all our hypotheses can be found by considering \(F_\circ = \mathbb{Q}\), \(E_\circ/\mathbb{Q}\) an imaginary quadratic field, \(F/\mathbb{Q}\) a real quadratic field, and \(A_\circ\) a rational elliptic curve of conductor \(f_\circ = p \cdot q\) for two rational primes both inert in \(E_\circ/\mathbb{Q}\), and with \(p\) split in \(F/\mathbb{Q}\).

1.3 Main results

As \(\varphi\) is completely split in \(F\), we have for every \(p \in S\) canonical identifications \(F_\circ,\varphi = F_p\) and \(E_\circ,\varphi = E_p\). The resulting identifications \(\hat{A}_\circ(E_\circ,\varphi) = \hat{A}(E_p)\) are used to define the norm map

\[
N_{S/\varphi} : \hat{A}(E_S) \sim \hat{A}_\circ(E_\circ,\varphi)^{\otimes r} \longrightarrow \text{Sym}^r_{\mathbb{Z}_p} (\hat{A}_\circ(E_\circ,\varphi)),
\]

where the second arrow is the canonical projection.

Remark 1.10. If \(\varphi\) is of degree one, the restriction of the norm map \(N_{S/\varphi}\) to \(\hat{A}(E_S)^\pm\) is injective.

Under our running assumptions \([\text{L1}]\), \([\text{L3}]\), \([\text{L5}]\), \([\text{L7}]\), we deduce the following theorems about plectic points from the known properties of Heegner points, the \(p\)-adic uniformization of Shimura curves, and a factorization of anticyclotomic \(p\)-adic \(L\)-functions (Corollary \([\text{L4}]\)).

**Theorem A** (Arithmetic significance). The following implication holds:

\[
N_{S/\varphi} \left( \text{pr}_S^\varphi(P_{A,S}) \right) \neq 0 \implies r_{\text{alg}}(A/E) = r \quad & \quad r_{\text{an}}(A/E) = r.
\]

**Theorem B** (Algebraicity). There is a quadratic extension \(\Omega/\mathbb{Q}\) and \(w_{A,S} \in \mathcal{N}(A(E)_{\Omega})\) s.t.

\[
N_{S/\varphi} \left( \text{pr}_S^\varphi(P_{A,S}) \right) = N_{S/\varphi} \left( \text{pr}_S^\varphi(\text{det}(w_{A,S})) \right).
\]

Remark 1.11. Aside from the quadratic extension \(\Omega/\mathbb{Q}\), Theorems \([\text{A}]\) and \([\text{B}]\) provide a proof of the minus part of Conjecture \([\text{L4}]\) in the polyquadratic setup when \(\varphi\) is of degree one. Using the main theorem of \([\text{HM22}]\) one can apply the same strategy to prove the plus part of the conjecture. This will be explained in more detail in future work.

Remark 1.12. The quadratic extension \(\Omega/\mathbb{Q}\) is generated by the square-root of a rational number that is the product of various explicit terms: Petersson norms, discriminants, Euler factors and special values of Dedekind zeta functions. It would be interesting to know whether that rational number is in fact a square. Similar questions were raised and shown to be implied by the Birch–Swinnerton-Dyer conjecture in \([\text{Mok10}]\).

Now, set \(A_{/F}^+ = A_{/F}\) and denote by \(A_{/F}^-\) the quadratic twist of \(A_{/F}\) with respect to the extension \(E/F\). We partition \(S = S^+ \cup S^-\) by declaring that the subset \(S^+ \subseteq S\) contains all the primes in \(S\) of split multiplicative reduction for \(A_{/F}^+\), and define \(q_A(S) := \max \left\{ r_{\text{alg}}(A^+/F) + |S^+| \right\} \).

**Theorem C**. We also have

\[
N_{S/\varphi} \left( \text{pr}_S^\varphi(P_{A,S}) \right) \neq 0 \iff r_{\text{an}}(A/E) = r \quad & \quad q_A(S) = r.
\]

Remark 1.13. If \(\varphi\) is a prime of degree one, Theorem \([\text{C}]\) establishes \([\text{FGM21}]\), Conjecture 1.5) in the polyquadratic CM case.

We note that in the main body of this article, we prove generalizations of Theorems \([\text{A}]\) \([\text{B}]\) for plectic Stark–Heegner points associated to anticyclotomic characters of \(E/F\) that are restrictions of anticyclotomic characters of \(E_\circ/F_\circ\).
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2. Preliminaries

We gather some basic results on symmetric powers and completed group algebras.

2.1 Symmetric powers

Let us fix a commutative ring $R$. Given an $R$-module $M$ and an integer $n \geq 0$ we write

$$M^\otimes n := \underbrace{M \otimes_R \ldots \otimes_R M}_{n \text{ times}}.$$

Recall that the symmetric algebra $\text{Sym}_R(M)$ of $M$ is the quotient of the tensor algebra

$$T_R(M) := \bigoplus_{n \geq 0} M^\otimes n$$

by the ideal generated by $x \otimes y - y \otimes x$ for $x, y \in M$. As this ideal is graded, the natural grading on $T_R(M)$ induces a grading on $\text{Sym}_R(M)$:

$$\text{Sym}_R(M) = \bigoplus_{n \geq 0} \text{Sym}^n_R(M).$$

We denote the image of an element $m \in M^\otimes n$ in $\text{Sym}^n_R(M)$ by $[m]$. Given a homomorphism $f: M \rightarrow N$ of $R$-modules we write

$$\text{Sym}^n_R(f): \text{Sym}^n_R(M) \rightarrow \text{Sym}^n_R(N)$$

for the induced homomorphism. If $M$ and $N$ are $R$-modules, there is a canonical isomorphism

$$\text{Sym}^n_R(N \oplus M) = \text{Sym}^n_R(M) \otimes_R \text{Sym}^n_R(N)$$

of $R$-algebras. Now, suppose $M$ is a finitely generated free $R$-module with generators $m_1, \ldots, m_\ell$, then there is an isomorphism of graded $R$-algebras

$$R[x_1, \ldots, x_\ell] \overset{\sim}{\rightarrow} \text{Sym}_R(M), \quad x_i \mapsto m_i. \quad (3)$$

**Lemma 2.1.** Let $R$ be an integrally closed domain, $M$ a finitely generated free $R$-module, and

$$(-)^2: \text{Sym}^n_R(M) \rightarrow \text{Sym}^{2n}_R(M)$$

the squaring map. If $x, y$ are elements of $\text{Sym}^n_R(M)$ and $C \in R \setminus \{0\}$ is a non-zero constant satisfying $x^2 = C \cdot y^2$, then there exists a square-root $\sqrt{C} \in R$ such that $x = \sqrt{C} \cdot y$.

**Proof.** As $R$ is an integrally closed domain, equation $(3)$ implies that the $R$-algebra $\text{Sym}_R(M)$ is one as well. Thus, the equality $C = (x/y)^2$ in the fraction field of $\text{Sym}_R(M)$ implies that $\sqrt{C} := x/y$ is an element of $\text{Sym}_R(M)$. Moreover, $\sqrt{C} \in R$ because its square belongs to $R$. $\Box$

The following lemma can be easily deduced from $(2)$ and $(3)$.

**Lemma 2.2.** Let $M_1, \ldots, M_n$ and $M$ be finitely generated free $R$-modules.

(a) The canonical map

$$\mu: M_1 \otimes_R \ldots \otimes_R M_n \rightarrow \text{Sym}^n_R(M_1 \oplus \ldots \oplus M_n), \quad m_1 \otimes \ldots \otimes m_n \mapsto [m_1 \otimes \ldots \otimes m_n]$$
is injective.

(b) The following diagram is commutative

\[
\begin{array}{ccc}
M^\otimes n & \xrightarrow{\mu} & \text{Sym}^n_R(M^\otimes n) \\
\downarrow & & \downarrow \\
\text{Sym}^n_R(id^\otimes n) & \xrightarrow{} & \text{Sym}^n_R(M).
\end{array}
\]

2.2 Completed group algebras

Let \( G = \varprojlim G_i \) be a topologically finitely generated commutative profinite group. Recall that the **completed group algebra** of \( G \) with coefficients in a commutative ring \( R \) is defined as

\[
R[G] = \varprojlim R[G_i].
\]

We denote by \((-)^\vee : R[G] \to R[G] \) the involution induced by inversion on \( G \). In the rest of this section we always consider the coefficient ring \( R = \mathbb{Z}_p \). Let \( I(G) \subseteq \mathbb{Z}_p[G] \) be the **augmentation ideal**, i.e. the kernel of the natural map \( \mathbb{Z}_p[G] \to \mathbb{Z}_p \). More generally, if \( Q \) is a quotient of \( G \) by an open subgroup, the **relative augmentation ideal** is defined as

\[
I_Q(G) := \ker(\mathbb{Z}_p[G] \to \mathbb{Z}_p[Q]).
\]

Note that the quotients \( I_Q(G)^n/I_Q(G)^{n+1} \) are modules over the group ring \( \mathbb{Z}_p[G] \). If \( \Theta \) is an element of \( I_Q(G)^n \), we write \( \partial^n_Q(\Theta) \) for its image in \( I_Q(G)^n/I_Q(G)^{n+1} \). The map \( G \to I(G) \), \( g \mapsto [g] - 1 \) induces an isomorphism of \( \mathbb{Z}_p \)-modules

\[
G \otimes_\mathbb{Z} \mathbb{Z}_p \xrightarrow{\sim} I(G)/I(G)^2,
\]

and for every integer \( n \geq 1 \) a surjection of \( \mathbb{Z}_p \)-modules

\[
\text{Sym}^n_{\mathbb{Z}_p}(G \otimes_\mathbb{Z} \mathbb{Z}_p) \twoheadrightarrow I(G)^n/I(G)^{n+1}.
\]

When \( G \) is a finitely generated free \( \mathbb{Z}_p \)-module, a choice of topological generators \( \{g_1, \ldots, g_s\} \) determines an isomorphism

\[
\mathbb{Z}_p[G] \xrightarrow{\sim} \mathbb{Z}_p[t_1, \ldots, t_s], \quad [g_i] \mapsto t_i + 1,
\]

mapping the augmentation ideal to the ideal \( (t_1, \ldots, t_s) \). It follows that the surjective maps

\[
\text{Sym}^n_{\mathbb{Z}_p}(G) \xrightarrow{\sim} I(G)^n/I(G)^{n+1}
\]

are isomorphisms for all \( n \geq 1 \). Furthermore, when \( G \) is a product \( G = H \times Q \) with \( Q \) finite, it is easy to see that the canonical map

\[
I(H)^n/I(H)^{n+1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to I(G)^n/I(G)^{n+1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
\]

is an isomorphism for all \( n \geq 1 \). The following lemma gives a slight generalization of this fact.

**Lemma 2.3.** Let \( G \) be a finitely generated commutative profinite group, \( H \leq G \) an open subgroup that is a finitely generated \( \mathbb{Z}_p \)-module, and \( Q \) a finite quotient of \( G \) such that \( H \subseteq \ker(G \to Q) \). Then, the canonical \( \mathbb{Q}_p[Q] \)-linear map

\[
I(H)^n/I(H)^{n+1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p[Q] \to I_Q(G)^n/I_Q(G)^{n+1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
\]

is injective for all \( n \geq 1 \).

Now, both the augmentation ideal \( I(G) \) and the relative versions \( I_Q(G) \) are clearly stable under \((-)^\vee \). Equation (4) implies that \((-)^\vee \) induces multiplication with \(-1\) on \( I(G)/I(G)^2 \) and,
thus, it induces multiplication with $(-1)^n$ on $I(G)^n/I(G)^{n+1}$. This observation readily implies the following relative statement: under the assumptions of Lemma 2.3 the following diagram commutes

$$
\begin{array}{ccc}
I(H)^n/I(H)^{n+1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[Q] & \longrightarrow & I_Q(G)^n/I_Q(G)^{n+1} \\
(-1)^n \mathrm{id} \otimes (-)^\vee & & (-)^\vee \\
I(H)^n/I(H)^{n+1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[Q] & \longrightarrow & I_Q(G)^n/I_Q(G)^{n+1}.
\end{array}
$$

3. Plectic points and $p$-adic $L$-functions

We begin by explaining how the construction of plectic Stark–Heegner points recovers the $p$-adic uniformization of classical Heegner points as a special case. Then, we recall the $p$-adic Gross–Zagier formula ([FG21], Theorem A) relating plectic points to derivatives of certain anticyclotomic $p$-adic $L$-functions, for which we also state precise interpolation formulas. Note that in this section we work in the setup of Subsection 1.1, that is, $E/F$ is an arbitrary quadratic CM extension and Assumption 1.1 is supposed to hold. In particular, we never assume that we are in a polyquadratic situation.

3.1 Plectic Stark–Heegner points

We fix an $\mathcal{O}_F$-ideal $c$ coprime to the conductor of $A/F$. Let $E_{c}/E$ denote the anticyclotomic extension of conductor $c$ defined in ([FG21], Section 4.2.1) with Galois group $\mathcal{G}_c = \operatorname{Gal}(E_{c}/E)$. Recall that for any character $\chi: \mathcal{G}_c \rightarrow \mathbb{C}^\times$ its conductor is the maximal divisor $c_{\chi}$ of $c$ such that $\chi$ factors through $\mathcal{G}_c \rightarrow \mathcal{G}_{c_{\chi}}$. We write $\mathbb{Q}_{\chi}$ for the extension of $\mathbb{Q}$ generated by the values of $\chi$. With a small change of notation compared to ([FG21], Section 4.4), we denote the plectic Stark–Heegner point associated to an anticyclotomic character $\chi$ by

$$
P_{A,S}^{\chi} \in \widehat{A}(E_{c})_{\mathbb{Q}_{\chi}}.
$$

It follows easily from the construction of plectic Stark–Heegner points that there is an element

$$
P_{A,S}^{\mathcal{G}_{c_{\chi}}} \in \widehat{A}(E_{c}) \otimes_{\mathbb{Z}} \mathbb{Q}[\mathcal{G}_c]
$$

such that the equality $\chi(P_{A,S}^{\mathcal{G}_{c_{\chi}}}) = P_{A,S}^{\chi}$ holds for any character $\chi: \mathcal{G}_c \rightarrow \mathbb{C}^\times$ of conductor $c$. A similar statement also holds for plectic $p$-adic invariants ([FG21], Section 4.2): there is an element

$$
Q_{A,S}^{\mathcal{G}_{c_{\chi}}} \in \widehat{E}_{S,\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{G}_c]
$$

such that the equality $\chi(Q_{A,S}^{\mathcal{G}_{c_{\chi}}}) = Q_{A,S}^{\chi}$ holds for any character $\chi: \mathcal{G}_c \rightarrow \mathbb{C}^\times$ of conductor $c$. Moreover, the two elements are related by the following equation

$$
\phi_S^{\text{Tate}}(Q_{A,S}^{\mathcal{G}_{c_{\chi}}}) = p_{\mathbb{Q}}(P_{A,S}^{\mathcal{G}_{c_{\chi}}}).
$$

3.2 Plectic points are Heegner points when $|S| = 1$

The construction of plectic Stark–Heegner points generalizes the $p$-adic description of classical Heegner points given in ([BD98], [Mok11]). In this subsection, we recall the precise relation between the two constructions when the set $S$ consists of a single prime $\{p\}$ and $\chi: \mathcal{G}_c \rightarrow \mathbb{C}^\times$ is a character of conductor $c$.

We fix embeddings $\iota_p: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\iota_{\nu}: \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ respectively inducing the $p$-adic prime $p$ and an Archimedean place $\nu$ of $F$. Since $p$ is inert in $E/F$, it splits completely in the anticyclotomic...
extension $E_c/E$. Thus, the embedding $\iota_p$ restricts to an embedding $\iota_p: E_c \hookrightarrow E_p$ inducing an injective homomorphism

$$\iota_{A,p}: A(E_c)_{Q_\chi} \rightarrow \hat{A}(E_p)_{Q_\chi}.$$  

We write $r_{\text{alg}}(A/E, \chi)$ for the $Q_\chi$-dimension of the $\chi$-component $A(E_c)_{Q_\chi}^\chi$ and define $r_{\text{an}}(A/E, \chi)$ to be the order of vanishing of the $L$-function $L(A/E, \chi, s)$ at $s = 1$. As explained in ([GMM20], Appendix A.1.2), there exists a Heegner point $P_{A,\chi} \in A(E_c)$ of conductor $\mathfrak{c}$, arising from a Shimura curve associated to the $F$-quaternion algebra ramified exactly at $p\mathfrak{m}$ and all Archimedean places different from $\nu$, such that the image of

$$P_{A,\chi}^\chi := \sum_{\sigma \in G_\chi} \chi^{-1}(\sigma) \cdot \sigma(P_{A,\chi}) \in A(E_c)_{Q_\chi}^\chi$$  

under $\iota_{A,p}$ is a non-zero rational multiple of the plectic Stark–Heegner point associated to the triple $(A, E, \{p\})$, i.e., there exists $k_{A,p}^\chi \in Q_\chi$ such that:

$$k_{A,p}^\chi \cdot P_{A,\iota(p)}^\chi = \iota_{A,p}(P_{A,\chi}^\chi).$$  

**Proposition 3.1.** Let $\chi: G_\chi \rightarrow \mathbb{C}^\times$ be a character of conductor $\mathfrak{c}$. We have

$$P_{A,\iota(p)}^\chi \neq 0 \iff r_{\text{an}}(A/E, \chi) = 1,$$

and both statements imply $r_{\text{alg}}(A/E, \chi) = 1$. Moreover, if $\chi$ is not quadratic,

$$r_{\text{an}}(A/E, \chi) = 1 \implies Q_{A,\iota(p)}^\chi \neq 0.$$  

**Proof.** By equation (10), the equivalence between the non-triviality of $P_{A,\iota(p)}^\chi$ and the analytic rank one statement follows from ([Zha01], Theorem 1.2.1), while the relation with the algebraic rank is a consequence of the main theorem in [Nek07]. For the second claim, we begin by noting that the involution $A(E_c) \rightarrow A(E_c)$, $P \mapsto \overline{P}$, induced by the complex conjugation associated to the Archimedean place $\nu$, yields an isomorphism

$$A(E_c)_{Q_\chi}^\chi \cong A(E_c)_{Q_\chi}^{\chi^{-1}}.$$  

Then, we observe that equations (8) and (10) imply the equality

$$k_{A,p}^\chi \cdot \phi_{\iota(p)}^{\text{Tate}}(Q_{A,\iota(p)}^\chi) = \iota_{A,p}(P_{A,\chi}^\chi - a_\iota \cdot \overline{P_{A,\chi}^\chi})$$  

because $\sigma_{\iota} \circ \iota_{A,p}(P_{A,\chi}^\chi) = \iota_{A,p}(P_{A,\iota(p)}^\chi)$ by ([BD98], Theorem 4.7). Now, our assumption is that $r_{\text{an}}(A/E, \chi) = 1$, and ([Zha01], Theorem 1.2.1) implies that $P_{A,\chi}^\chi \neq 0$. When $\chi$ is not quadratic, the intersection $A(E_c)_{Q_\chi}^\chi \cap A(E_c)_{Q_\chi}^{\chi^{-1}}$ is trivial and the claim follows.

**Remark 3.2.** Let $\chi = 1$ be the trivial character and assume that $r_{\text{an}}(A/E) = 1$. Using equation (11) and ([Mok11], Corollary 4.2), we deduce that $\text{pr}_{\iota}(P_{A,\iota(p)}^\chi) \neq 0$ is equivalent to either

$$\left(a_\iota = +1 \land r_{\text{alg}}(A/F) = 0\right) \text{ or } \left(a_\iota = -1 \land r_{\text{alg}}(A/F) = 1\right).$$  

**3.3 The anticyclotomic Gross–Zagier formula**

Let $E_c, S$ be the union of the anticyclotomic extensions of $E$ of conductor $\mathfrak{c} \cdot \prod_{p \in S} p^n$ for $n \geq 0$. We put $G_{c,S} = \text{Gal}(E_{c,S}/E)$ and denote by

$$I_c \subseteq \mathbb{Z}_p[G_{c,S}]$$
the relative augmentation ideal with respect to the quotient map $G_{\epsilon,S} \to G_{\epsilon}$. Restriction of the global Artin homomorphism to the local components at $p \in S$ induces the homomorphism

$$\text{rec}_S : \bigoplus_{p \in S} E_p^- \longrightarrow G_{\epsilon,S}.$$

Since $E/F$ is a quadratic CM extensions, the kernel of $\text{rec}_S$ is torsion and its image is an open subgroup of $\ker(G_{\epsilon,S} \to G_{\epsilon})$. Moreover, Lemma 2.2 and Lemma 2.3 imply that the map

$$d \text{rec}_S : \hat{E}_{S,\mathcal{O}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_{\epsilon}] \longrightarrow \text{Sym}_p^r((\hat{E}_{p_1}^- \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} \hat{E}_{p_r}^-) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_{\epsilon}]) \longrightarrow (I_\epsilon^r/I_\epsilon^{r+1})_\mathbb{Q}$$

is injective. Associated to the quadruple $(A, S, E, \epsilon)$ there is the square-root $p$-adic $L$-functions

$$L_S(A/E)_{\epsilon} \in \mathbb{Z}[G_{\epsilon,S}]$$

constructed in ([BG18], Definition 5.2) and also in ([FG21], Definition 5.7). On the one hand, the values of this $p$-adic $L$-function at finite order characters not satisfying certain ramification conditions are always equal to zero (see [FG21], Theorem 5.10), while the (squares of the) non-trivial values are explicitly calculated in ([BG18], Theorem 5.8). We recall the interpolation formula in Theorem 3.8 below. For the rest of this subsection we consider $L_S(A/E)_{\epsilon}$ as an element of $\mathbb{Z}_p[G_{\epsilon,S}]$. Recall that ([BG18], Theorem 5.5) shows that

$$L_S(A/E)_{\epsilon} \in I_\epsilon^r.$$

The following is a reformulation of the main theorem of [FG21].

**Theorem 3.3 (Anticyclotomic Gross-Zagier formula).** The equality

$$2^r \cdot \partial_{G_{\epsilon}}^r(L_S(A/E)_{\epsilon}) = d \text{rec}_S ((Q_{A,S}^{G_{\epsilon}})^\vee)$$

holds in $(I_\epsilon^r/I_\epsilon^{r+1})_\mathbb{Q}$.

Interestingly, ([BG18], Proposition 5.6) allows us to describe the behaviour of $L_S(A/E)_{\epsilon}$ under the involution $(-)^\vee$ in terms of the global root number $\varepsilon(A/F)$ of $A_F$ and a product of local root numbers $\varepsilon_S(A/F) := \prod_{p \in S} \varepsilon_p(A/F)$.

**Proposition 3.4.** The equality

$$(L_S(A/E)_{\epsilon})^\vee = \varepsilon(A/F) \cdot \varepsilon_S(A/F) \cdot L_S(A/E)_{\epsilon}$$

holds up to multiplication with an element in $G_{\epsilon,S}$.

**Corollary 3.5.** There exists an element $g \in G_{\epsilon}$ such that the equality

$$Q_{A,S}^{-1} = \chi(g) \cdot \varepsilon(A/F) \cdot \varepsilon_S(A/F) \cdot (-1)^r \cdot Q_{A,S}^r$$

holds. In particular, if $\chi = 1$ is the trivial character, we have

$$Q_{A,S}^r \neq 0 \quad \Longrightarrow \quad (-1)^r = \varepsilon(A/F) \cdot \varepsilon_S(A/F).$$

**Proof.** Thanks to the commutative diagram (7) and Theorem 3.3 we have

$$\chi^{-1}(d \text{rec}_S(Q_{A,S}^{G_{\epsilon}})) = (-1)^r \chi(d \text{rec}_S(Q_{A,S}^{G_{\epsilon}})^\vee)$$

$$= (-2)^r \chi(\partial_{G_{\epsilon}}^r(L_S(A/E)_{\epsilon})).$$

Proposition 3.4 implies that there is $g \in G_{\epsilon}$ such that

$$\chi(\partial_{G_{\epsilon}}^r(L_S(A/E)_{\epsilon})) = \chi(g) \cdot \varepsilon(A/F) \cdot \varepsilon_S(A/F) \cdot \chi(\partial_{G_{\epsilon}}^rL_S(A/E)_{\epsilon}),$$

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and applying Theorem 3.3 once more we obtain the equality
\[ \chi^{-1}(\text{drec}_S(Q_{A,S}^\text{rec})) = \chi(g) \cdot \varepsilon(A/F) \cdot \varepsilon_S(A/F) \cdot (-1)^r \cdot \chi(\text{drec}_S(Q_{A,S}^\text{rec})). \]
The claim now follows from the injectivity of drec \(_S\).

**Remark 3.6.** Corollary 3.5 proves one implication of ([FGM21], Conjecture 1.6) in the CM case.

### 3.4 Interpolation formula

The square-root \( p \)-adic \( L \)-function \( \mathcal{L}_S(A/E)_\chi \) does not interpolate values of complex \( L \)-functions but – as the name suggests – a choice of their square-roots.

**Definition 3.7.** The anticyclotomic \( p \)-adic \( L \)-function is defined as the product
\[ \mathcal{L}_S(A/E)_\chi := \mathcal{L}_S(A/E)_\chi \cdot \mathcal{L}_{S}(A/E)^\gamma_\chi \in \mathbb{Z}[\mathcal{G}_{c,S}]. \]

It is this \( p \)-adic \( L \)-function that interpolates special values of complex \( L \)-functions. We introduce some notation to state the interpolation property: the normalized special value of the complex \( L \)-function \( L(A/E, \chi, s) \) at \( s = 1 \) is given by
\[ L^\text{nm}(A/E, \chi) := {L(A/E, \chi, 1) \over L(\pi_A, \text{Ad}, 1)} \cdot \prod_{\nu \mid \infty} C_{\nu}(E, \pi_A, \chi) \]
where \( L(\pi_A, \text{Ad}, s) \) is the adjoint \( L \)-function of the \( \text{GL}_2 \)-automorphic representation \( \pi_A \) attached to \( A/F \), and \( C_{\nu}(E, \pi_A, \chi) \) is the Archimedean factor defined in ([FMP17], Section 7B). Since the quadratic extension \( E/F \) is CM, the factors \( C_{\nu}(E, \pi_A, \chi) \) are all equal to a fixed constant \( C_{\infty} \) independent of \( \nu, A/F, E \) and \( \chi \). Note that we always include the Archimedean Euler factors in the definition of the \( L \)-functions and \( \zeta \)-functions that occur.

By our assumptions, the automorphic representation \( \pi_A \) admits a Jacquet–Langlands lift to group of units of the totally definite quaternion algebra \( B/F \) that is ramified exactly at those finite primes that divide \( \mathfrak{n}^\flat \). As \( B/F \) is totally definite, we may normalize the newform of the Jacquet-Langlands lift such that it takes rational values. Let \( f_A^\text{R} \) be the rational newform involved in the construction of the \( p \)-adic \( L \)-function in ([BG18]).

**Theorem 3.8.** For all locally constant characters \( \chi : \mathcal{G}_{c,S} \rightarrow \mathbb{C}^\times \) we have
\[ \chi(\mathcal{L}_S(A/E)_\chi) = {1 \over 2} \cdot \langle f_A^\text{R}, f_A^B \rangle_B \cdot L_{\Sigma_A}(1, \omega_{E/F}) \cdot L_{\Sigma_A^{\text{add}}(\pi_A, \text{Ad}, 1)} \cdot \prod_{q \in \Sigma_A \setminus S} e_q(E/F) \]
\[ \times \zeta_{E}(2) \cdot \sqrt{{\Delta_F \over \Delta_E}} \cdot L_{\text{nm}}^{}(A/E, \chi, 1) \cdot \prod_{q \mid p \in \mathcal{C}} C_{\text{ord}_p(\epsilon), \chi_p, A_p} \]
where
- \( \langle \cdot, \cdot \rangle_B \) denotes the Petersson inner product on automorphic forms of the unit group of \( B/F \),
- \( \Sigma_A \) denotes the set of primes of \( F \) at which \( A/F \) has bad reduction, and \( \Sigma_A^{\text{add}} \subset \Sigma_A \) the subset of primes of additive reduction,
- \( \omega_{E/F} \) is the quadratic character associated to \( E/F \),
- \( e_q(E/F) \) denotes the ramification degree of \( q \) in \( E/F \),
- \( \Delta_F \) are \( \Delta_E \) the absolute values of the discriminants of \( F \) and \( E \) respectively,
- \( C_{\text{ord}_p(\epsilon), \chi_p, A_p} \in \mathbb{Q}_\chi \) are constants that only depend on the the \( p \)-adic valuation of \( \epsilon \), the restriction of \( \chi_p \) to a decomposition group at \( p \), and the base change \( A/F_p \),

\[ \text{for all } p \text{-adic } \varepsilon \text{ and } \chi \text{ on } \mathbb{G}_{c,S}. \]
Proof. In case the conductor of $\chi$ is exactly $c$ this follows from ([BG18], Theorem 5.8), and the explicit Waldspurger formula of [FMP17]. The general case follows from the norm relations of ([BG18], Theorem 5.8).

4. Artin formalism for plectic points

We keep the same notation as in Subsection 3.3. In addition, we assume that we are in the polyquadratic setup of Subsection 1.2, and that $c = c \cdot \mathcal{O}_F$ for some ideal $c$ of $F_0$. Let $E_{o,c}$ denote the ring class field of conductor $c$ and $E_{o,c,\{p\}}$ the union of the ring class fields of $E_o$ of conductor $c_p^n$ for $n \geq 0$. We put $G_c = \text{Gal}(E_{o,c}/E_o)$ and $G_{c,\{p\}} = \text{Gal}(E_{o,c,\{p\}}/E_o)$, so that there are natural maps $G_c \to G_c$ and $G_{c,\{p\}} \to G_{c,\{p\}}$.

**Assumption 4.1.** We require that

- every prime divisor of $c$ is completely split in $F/F_0$ and unramified in $E/F_0$,
- the character $\chi: G_c \to \mathbb{C}^\times$ is the restriction of an anticyclotomic character $\xi: G_c \to \mathbb{C}^\times$.

**Remark 4.2.** Since every non-trivial subextension of $F/F_0$ is ramified by Assumption 1.3 requiring the ideal $c$ to be unramified in $E/F_0$ implies that the fields $E$ and $E_{o,c}$ are linearly disjoint over $E_0$. We impose the total splitting in $F/F_0$ of the prime divisors of $c$ just as a simplifying hypothesis for the proof of Proposition 4.5.

4.1 Factorization of complex $L$-functions

We write $G^* := \text{Hom}_{gr}(G, \{\pm 1\})$ for the Pontryagin dual of $G = \text{Gal}(F/F_0) \cong \text{Gal}(E/E_0)$. For any number field $\Omega$ we write $G_\Omega := \text{Gal}(\mathbb{Q}/\Omega)$ for its absolute Galois group.

**Lemma 4.3.** There exists an isomorphism of $G_{F_0}$-representations:

$$\text{Ind}^{G_{F_0}}_{G_E} (\chi) \cong \bigoplus_{\eta \in G^*} \text{Ind}^{G_{F_0}}_{G_{E_0}} (\xi \cdot \eta).$$

**Proof.** First, we claim that

$$\text{Ind}^{G_{F_0}}_{G_E} (\chi) \cong \bigoplus_{\eta \in G^*} \xi \cdot \eta.$$

To prove the claim note that all the characters $\xi \cdot \eta$, for $\eta \in G^*$, have the same restriction to $G_E$, namely the character $\chi$. By Frobenius reciprocity we have

$$\text{Hom}_{G_{F_0}} (\xi \cdot \eta, \text{Ind}^{G_{F_0}}_{G_E} (\chi)) \cong \text{Hom}_{G_E} (\chi, \chi) \neq 0$$

and, therefore, the claim follows from semi-simplicity of representations of finite groups in characteristic zero. The statement of the lemma follows from transitivity of induction.

**Corollary 4.4.** The following equality of complex $L$-functions holds:

$$L^{nm}(A/E, \chi, s) = \prod_{\eta \in G^*} L^{nm}(A_0^*/E_0, \xi, s).$$

**Proof.** This is a direct consequence of Lemma 4.3 and Artin formalism for complex $L$-functions.
4.2 Factorization of $p$-adic $L$-functions

For each $\eta \in \mathcal{G}^*$, the quadruple $(A_0^\eta, E_0, \varphi, c)$ fulfills the conditions of Subsection 3.3 and thus, we can define the $p$-adic $L$-functions

$$L_{(\varphi)}(A_0^\eta/E_0)c, L_{(\varphi)}(A_0^\eta/E_0)c \in \mathbb{Z}[G_{c,p}].$$

Let $R$ be a commutative ring. The homomorphism $G_{c,S} \to G_{c,p}$ induces the restriction map

$$\text{res}_{S,p}: R[G_{c,S}] \to R[G_{c,p}]$$

between completed group algebras.

**Proposition 4.5.** There is a constant $C \in \mathbb{Q}^\times$ such that the equality

$$\text{res}_{S,p}(L_S(A/E)c) = C \cdot \prod_{\eta \in \mathcal{G}^*} L_{(\varphi)}(A_0^\eta/E_0)c,$$

holds in $\mathbb{Z}[G_{c,p}]$.

**Proof.** To prove the statement it is enough to use Theorem 3.8 to show equality of both sides after evaluation at every finite order character of $G_{c,p}$. We note that the first line of the interpolation formula is a non-zero rational number and, thus, we may neglect it. By Corollary 4.4, the normalized special values on both sides cancel out. In addition, the local constants $C_{\text{ord}_p(c)}$, $\chi_p$, $A_p$ cancel out as well because we assumed that all the primes of $F_0$ dividing $c\varphi$ are totally split in $F/F_0$. As we are taking a product over $r = 2^t$ factors on the right hand side, we see that it suffices to show that

$$\sqrt{\frac{\Delta_F}{\Delta_E} \cdot \frac{\zeta_F(2)}{\zeta_{F_0}(2)^r}}$$

is a rational number. Using the functional equation of the Dedekind zeta function, we may rewrite this term as

$$\sqrt{\frac{\Delta_F}{\Delta_E} \cdot \frac{\zeta_F(-1)\Delta_F^{-3/2}}{\zeta_{F_0}(-1)^r \Delta_{F_0}^{-3r/2}}}.$$

By the Klingen–Siegel Theorem ([Sie37], [Kli62]), the special values of the Dedekind zeta function (excluding the Archimedean Euler factors) at negative integers are rational. The Archimedean factors of $\zeta_F(s)$ and $\zeta_{F_0}(s)^r$ agree, so we are left to show that $\Delta_E \in \mathbb{Q}^\times$ is a square. Let $\Delta_{E/F_0}$ denote the relative discriminant of $E/F_0$. The formula for relative discriminants in towers gives

$$\Delta_E = N_{F_0/\mathbb{Q}}(\Delta_{E/F_0}) \cdot \Delta_{E/F_0}^{[E:F_0]}.$$

As $[E : F_0]$ is a power of 2, it suffices to show that $\Delta_{E/F_0}$ is a square. At this point we conclude the argument because ([Kha19], Theorem 1.2) implies

$$\Delta_{E/F_0} = \Delta_{E_0/F_0}^{r} \cdot \Delta_{F/F_0}^{2},$$

since the relative discriminants $\Delta_{E_0/F_0}$ and $\Delta_{F/F_0}$ are coprime by Assumption 1.5.

4.3 Factorization of plectic invariants

Recall that by Assumption 4.4 the character $\chi$ is the restriction to $G_c$ of an anticyclotomic character $\xi: G_c \to \mathbb{C}^\times$. For every $\eta \in \mathcal{G}^*$ we can consider the plectic $p$-adic invariant

$$Q^\xi_{A_0^\eta(\varphi)} \in \hat{E}_0^{-} \otimes_{\mathbb{Z}} \mathbb{Q}_\xi$$

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holds up to multiplication by an element in \( G \) which implies \( r \) commutes. Thus, we may conclude by invoking Theorem 3.3.

5.1 Theorem

We keep the same notation as in Section 4. In particular, we are in the polyquadratic CM setting and Assumptions 1.1, 1.5, 1.7 and 4.1 hold. Moreover, we suppose that the conductor of the character \( \xi \) is \( c \). One can always achieve this by shrinking \( c \). We have the equalities

\[
r_\bullet(A/E, \chi) = \sum_{\eta \in \mathfrak{G}^*} r_\bullet(A^\eta_{0}/E_{0}, \xi), \quad \bullet \in \{\text{an, alg}\},
\]

which imply \( r_{\text{an}}(A/E, \chi) \geq r \) because \( r_{\text{an}}(A^\eta_{0}/E_{0}, \xi) \) is odd for every character \( \eta \in \mathfrak{G}^* \).

**Theorem 5.1.** The following implication holds:

\[
N_{S/p} \left( \text{pr}_{S}^{*}(P_{A,S}^{\chi}) \right) \neq 0 \implies r_{\text{alg}}(A/E, \chi) = r \quad \text{and} \quad r_{\text{an}}(A/E, \chi) = r.
\]

Suppose \( p \) does not split in \( \mathbb{Q}_\chi \) and that the character \( \chi \) is not quadratic, then

\[
r_{\text{an}}(A/E, \chi) = r \implies N_{S/p} \left( \text{pr}_{S}^{*}(P_{A,S}^{\chi}) \right) \neq 0 \quad \text{and} \quad r_{\text{alg}}(A/E, \chi) = r.
\]
The first claim follows from (12). The second claim is proved similarly using Corollary 4.7. Hence, the point $P_{A,\eta}$ is non-zero for every $\eta \in G^\ast$ and, thus, Proposition 3.1 gives

$$r_{\text{alg}}(A_0^0/E_0, \xi) = r_{\text{an}}(A_0^0/E_0, \xi) = 1 \quad \forall \eta \in G^\ast.$$ 

The first claim follows from (12). The second claim is proved similarly using Corollary 4.7. ◻

Set $A_F^+ = A/F$ and denote by $A_{F}^-$ the quadratic twist of $A/F$ with respect to the extension $E/F$. We partition $S = S^+ \cup S^-$ by declaring that the subset $S^+ \subseteq S$ contains all the primes in $S$ of split multiplicative reduction for $A_F^+$, and define $g(A) := \max\{r_{\text{alg}}(A^\pm/F) + |S^\pm|\}$.

**Theorem 5.2.** We also have

$$N_{S}^S(p_{S}^-(P_{A,\eta}^1)) \neq 0 \iff r_{\text{an}}(A/E) = r \quad \& \quad g_A(S) = r.$$ 

**Proof.** Under our hypotheses we either have $S = S^+$ or $S = S^-$. For simplicity we assume that $S = S^+$, or equivalently that $a_F = +1$, since the other case is proved by similar arguments.

Suppose $N_{S}^S(p_{S}^-(P_{A,\eta}^1))$ is non-zero. By Theorem 5.1, it is enough to show that $g_S(A) = r$. Now, Corollary 4.7 implies that $p_{S}^-(P_{A,\eta}^1) = 0$ for every $\eta \in G^\ast$, and since $a_F = +1$, Remark 3.2 tells us that $r_{\text{alg}}(A_0^0/E_0) = 0$ for every $\eta \in G^\ast$. We deduce $r_{\text{alg}}(A^+/F) = 0$ and $r_{\text{alg}}(A^-/F) = r$ as required. For the converse implication, note that the assumptions imply that $r_{\text{alg}}(A_0^0/E_0) = 1$ and $r_{\text{alg}}(A_0^0/E_0) = 0$ for every $\eta \in G^\ast$. Then, Remark 3.2 shows that $p_{S}^-(P_{A,\eta}^1) = 0$ for every $\eta \in G^\ast$ and the claim follows from Corollary 4.7. ◻

As in Subsection 3.2 we fix an embedding $\iota_{\varphi} : E_{0,c} \hookrightarrow E_{0,p}$ extending the canonical embedding $E_0 \hookrightarrow E_{0,p}$. Furthermore, we choose an embedding $\iota_{\varphi} : E_0 \hookrightarrow E_{0,p}$ that restricts to the chosen embedding $\iota_{\varphi}$ as well as to the canonical embedding $E \hookrightarrow E_{0,p}$. Under our assumptions, the fields $E$ and $E_{0,c}$ are linearly disjoint over $E_0$, thus for every $p \in S$ we can choose an element $\tau_p \in \text{Gal}(E/E_{0,c})$ whose restriction to $\text{Gal}(E/E_0) \cong G$ sends $p$ to $p_1$, and such that $\iota_{\varphi} := \iota_{p_1} \circ \tau_p$ corresponds to the prime $p$. Then, we define the determinant map $\det : N(A(E))_{\mathbb{Q}} \to \widehat{A}(E)_{\mathbb{Q}}$ by setting

$$\det(P_1 \wedge \cdots \wedge P_r) = \det \begin{pmatrix} \iota_{A,p_1}(P_1) & \cdots & \iota_{A,p_r}(P_1) \\ \vdots & \ddots & \vdots \\ \iota_{A,p_1}(P_r) & \cdots & \iota_{A,p_r}(P_r) \end{pmatrix}.$$ 

For every $\eta \in G^\ast$ we may view $A_0^0(E_{0,c})$ as a subgroup of $A(E)$, and by construction we have

$$\iota_{A,p}(P) = \eta(\tau_p) \cdot \iota_{A,p_1}(P) \quad \forall \quad P \in A_0^0(E_{0,c}). \quad (13)$$

**Theorem 5.3.** Suppose $p$ does not split in $\mathbb{Q}_X$. There exists a quadratic extension $\Omega_X/\mathbb{Q}_X$, in which $p$ splits, and an element $w_{A,S}^{X} \in N(A(E_c)_{\Omega_X})$ such that

$$N_{S}^S(p_{S}^-(P_{A,\eta}^X)) = N_{S}^S(p_{S}^-(\det(w_{A,S}^{X})))$$

in $\text{Sym}^\ast_{\mathbb{Q}_p}(\widehat{A}(E_{0,p}))_{\mathbb{Q}_X}$.

**Proof.** We choose an ordering $\eta\{1\}, \ldots, \eta\{r\}$ of the elements of the character group $G^\ast$. For $\eta \in G^\ast$ we consider the Heegner point $P_{A,\eta}^X$ of equation (10), and set

$$w_{A,S}^{X} := P_{A,\eta(1)}^X \wedge \cdots \wedge P_{A,\eta(r)}^X.$$
Using (13), the formula for the determinant gives
\[
N_{\mathcal{S}/\wp} \left( \det(\tilde{w}_{A,S}^\chi) \right) = N_{\mathcal{S}/\wp} \left( \sum_{\sigma \in S_r} \text{sgn}(\sigma) \cdot \iota_{p_1}(P_{A_{\wp}^0}^{\xi}(\sigma(1))) \otimes \ldots \otimes \iota_{p_r}(P_{A_{\wp}^0}^{\xi}(\sigma(r))) \right)
\]
\[
= C_{\wp} \cdot \prod_{\eta \in \wp^*} \iota_{A_{\wp}^0,\wp}(P_{A_{\wp}^0}^{\xi}),
\]
where
\[
C_{\wp} := \sum_{\sigma \in S_r} \text{sgn}(\sigma) \cdot \prod_{i=1}^r \eta \{ \sigma(i) \} (\tau_{p_i})
\]
is the determinant of the character table of the group \( \wp \). By orthogonality of characters, the determinant is non-zero. In fact, it is equal to \( \pm \sqrt{r} \). Let \( k_{A_{\wp}^0,\wp}^{\xi} \in \mathbb{Q}^\times \) be the constants appearing in (10), and \( \sqrt{C_\chi} \in \mathbb{Q}^\times_{A_{\wp}^0} \) the constant appearing in Corollary 4.7. By setting
\[
w_{A,S}^\chi := \frac{\sqrt{C_\chi} \cdot \prod_{\eta \in \wp^*} k_{A_{\wp}^0,\wp}^{\xi}}{C_{\wp} \cdot \prod_{\eta \in \wp^*} k_{A_{\wp}^0,\wp}^{\xi}} \cdot \tilde{w}_{A,S}^\chi
\]
we get that
\[
N_{\mathcal{S}/\wp} \left( \det(w_{A,S}^\chi) \right) = \sqrt{C_\chi} \cdot \prod_{\eta \in \wp^*} P_{A_{\wp}^0}^{\xi},\wp.
\]
The claim follows from Corollary 4.7 after applying the minus projector on both sides. \( \square \)

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