Is the Statistical Interpretation of Quantum Mechanics
Implied by the Correspondence Principle?

Kurt Gottfried
Laboratory for Nuclear Studies, Cornell University, Ithaca NY 14853
November 29, 1998

Our impresari, Anton Zeilinger, ignored my pleas of ignorance and prevailed on me
to talk about my discussions with John Bell about the foundations of quantum mechanics.
This ‘debate’ was aborted by John’s tragic death shortly after we last met at a wonderful
workshop in Amherst attended by several people in this audience. At that time, John’s
last paper “Against Measurement” was about to be published. It featured a wonderfully
barbed attack on the treatment of measurement in my 1966 textbook. I was delighted
that the most profound student of quantum mechanics since the Founding Fathers, and an
old friend from CERN, had paid close attention to what I had written, because with but
one exception, no one publishing in the field had ever mentioned my work even when
espousing a position that I had taken long before.

John would not have changed his views had he been able to hear my response at the
CERN memorial symposium. Furthermore, I was far from satisfied with what I published
back then. On the other hand, I continue to find John’s critique of orthodoxy to be rather
less overwhelming than his superb rhetoric. Nevertheless, what I will say was stimulated
by reflecting on the views John espoused in our last conversations and in his last paper.

In large part, the interpretation of quantum mechanics is so controversial because the
basic equations of the theory seem not to say what they mean in terms of pre-existing
concepts. In retrospect, the successful developments of physics followed a clear path from
the Principia until it disappeared into a fog in 1925. Whatever Thomas Kuhn and his
disciples may say, before the advent of quantum mechanics physics had ultimately proven
to be a cumulative pursuit, with new knowledge built on prior concepts.

Newton’s equations of motion defined all its new concepts in terms of then accepted
conceptions of space and time. The next giant steps concerned heat and electromagnetism.
Statistical mechanics, while full of subtleties, nevertheless demonstrated how thermody-
namics could be related to an underlying Newtonian description. In electrodynamics, the
meaning of the new concept, the electromagnetic field, is defined by Maxwell’s equations
and the Lorentz force law by means of Newton’s equations. Admittedly, special relativity
introduced a new conception of space and time, but their meaning was defined by a more
penetrating examination of pre-existing concepts using familiar tools: measuring rods and
clocks. General relativity, while profoundly new, is, if you pardon the expression, trivial,
provided you are freely falling and short-sighted. Furthermore, Einstein’s equations tell you
that – you do not need him whispering in your ear.

1 Presented at the conference “Epistemological and Experimental Perspectives on Quantum Physics,”
Vienna, September 3-6, 1998; to appear in 7th. Yearbook, Institute Vienna Circle, D. Greenberger, W.L.
Reiter and A. Zeilinger (eds.), Kluwer: Dordrecht 1999.
Quantum mechanics does not fit this pattern, it would appear. On its own, the formalism
does not seem to disclose what the wave function means, and gurus such as Born, Bohr and
Heisenberg were needed to translate the equations into our language. I acknowledged this
in my response to John:

If one were to hand the Schrödinger equation to Maxwell, and tell him that it
describes the structure of matter in terms of various point particles whose masses
and charges are to be seen in the equation, this knowledge would not, by itself,
enable Maxwell to figure out what is meant by the wave function. Eventually
he would need help: “Oh, I forgot to tell you that according to Rabbi Born, a
great thinker in the yeshiva that flourished in Göttingen in the early part of the
20th century, $|\Psi(r_1, \ldots, t)|^2$ is . . . ”

My aim is to explore whether the situation is really this bleak – to ask what extent, if
any, the statistical interpretation of quantum mechanics is revealed by the theory’s formal-
ism. I make no claim to having a fully satisfactory answer to this question. What I have
to say is really the reading of an old book from the back towards the front.

I now put myself into the shoes of this fictional Maxwell. The task of fathoming the
physical content of the Schrödinger equation leads me to adopt the following assumptions
and ground rules:

- The mathematical formalism of orthodox quantum mechanics provides a complete and
  consistent description of Nature as it stands. The implications of the formalism, such
  as the superposition principle, are not to be tampered with.

- It is not permissible to invoke the statistical interpretation, the collapse of the wave
  function, or the influence of some dynamical environment.

- If quantum mechanics is indeed complete, there must exist conditions under which
  classical mechanics provides an essentially exact approximation to quantum mechanics
  for systems having properties that are defined by this very limit.

The goal is to examine the classical limit of the quantum mechanical formalism to learn
the extent to which this limit compels the statistical interpretation. If such a link could be
established it would continue the tradition of connecting new developments in physics to
the conceptual roots of the discipline.

I will present an argument which claims that the Schrödinger equation, when examined
in the classical limit, leads to the statistical interpretation for degrees of freedom described
by finite-dimensional Hilbert spaces having no classical counterpart. In contrast, the ar-
gument, as it stands here, does not lead to the statistical interpretation for the degrees of
freedom that have a classical counterpart.

Incidentally, my title is purposely close to that of a remarkable paper by Van Vleck
written 70 years ago, “The Correspondence Principle in the Statistical Interpretation of
Quantum Mechanics,” which had a somewhat similar goal. So I am singing for my supper
from an old though unfinished score.
II

Schrödinger’s very first equation in his first paper on wave mechanics is the classical Hamilton-Jacobi equation; then he made the great leap to his wave equation. Is there a return path?

The first step is to note that the Schrödinger equation implies that in the naive classical limit \( \hbar \to 0 \), the wave function \( \Psi \) has an essential singularity \[\Psi = e^{i\Theta(q,t)/\hbar}, \quad \Theta = S + \frac{\hbar}{i} U + O(\hbar^2),\] where \( q \equiv (q_1, \ldots, q_N) \) is a point in the configuration space \( \mathbb{C} \) of an \( N \) particle system. At this first stage, I ignore “internal” variables, such as spin, inhabiting finite dimensional Hilbert spaces with no classical counterpart.

The Schrödinger’s equation then produces the following familiar facts:

1. Ansatz (1) is only legitimate – only produces a respectable asymptotic series, if
   \[\frac{|\partial \Theta / \partial q_k|^2}{|\partial^2 \Theta / \partial q^2_l|} \ll \hbar;\]
   when this condition is violated the approximation (1) is not valid, and the Schrödinger equation itself must be used.

2. The leading term \( S \) satisfies the classical Hamilton-Jacobi equation in the region accessible to classical motion. Recall that the classical trajectories are the curves in \( \mathbb{C} \) everywhere normal to the surfaces of constant \( S \). Thus \( \Psi \) is related not to one classical trajectory, but to a family or set of such trajectories, \( \{q(t)\} \).

3. The next order term \( U \) is related to \( S \) by
   \[\frac{\partial U}{\partial t} + \sum_{k=1}^{N} \frac{1}{2m_k} \left( \frac{\partial^2 S}{\partial q_k^2} + 2 \frac{\partial S}{\partial q_k} \cdot \frac{\partial U}{\partial q_k} \right) = 0.\]
   Thus \( U \), which is the \( \hbar \)-independent factor in \( \Psi \), is real in the classically accessible regions, and therefore
   \[w(q,t) \equiv \exp[2U(q,t)] = \lim_{\hbar \to 0} |\Psi|^2.\]
   When rephrased in terms of \( w \), (3) becomes a classical continuity equation in \( \mathbb{C} \):
   \[\frac{\partial w}{\partial t} + \sum_k \frac{\partial}{\partial q_k} (w \dot{q}_k) = 0,\]
   where \( \dot{q}_k \) is the velocity at time \( t \) of the \( k^{th} \) particle as determined by the classical equations of motion. Thus \( w \) plays the role of a density and \( w \dot{q} \) that of a 3\( N \) dimensional current vector in \( \mathbb{C} \); i.e., Eq. (5) is the \( \hbar \to 0 \) limit of the Schrödinger continuity equation.
The quantity $w(q, t)$, apart from an overall arbitrary constant, is Van Vleck’s determinant $D$, a purely classical quantity (involving derivatives of $S$). For a given $|\Psi|^2$, $D$ determines how the trajectories that form the set $\{q(t)\}$ are “populated.”

In short, in the classical limit the Schrödinger equation does not describe a single system, but a population of replicas of such a system moving along the trajectories $\{q(t)\}$. As $\Psi$ depends only on the degrees of freedom of a single system, and because experiments can be done on individual systems, this population is to be be visualized as a set of identical specimens which, one at a time, follow one or another of the allowed classical trajectories $\{q(t)\}$, and which, in retrospect, produced a population of these trajectories as specified by $w(q, t)$.

For any specific solution $\Psi$ of a specific Schrödinger equation, that is all that can be said about the the trajectories and their population. To have better knowledge of which trajectories are being followed given this $\Psi$, one must intervene by changing the Hamiltonian in the Schrödinger equation, e.g., with a potential that does not allow trajectories to proceed unless they pass through an aperture of dimension $a$ with edges smooth enough to leave the WKB approximation valid. Thereafter, the trajectories and their population will change; i.e., $\Psi$ will change.

The preceding discussion does not purport to be a derivation of the Born interpretation of $\Psi(q, t)$. To avert such a misperception, I have used the word “population” instead of probability. Furthermore, the classical description of what can be extracted from $\Psi$ is only valid as long as the inequality (2) is satisfied, i.e., if the system is sufficiently heavy, the forces sufficiently smooth, and the time interval over which the semiclassical description is needed is sufficiently short. The Born interpretation has no such restrictions, quite aside from the profound difference between quantum mechanical probabilities and probability distributions for a population of identical systems obeying the laws of classical mechanics.

The restrictions imposed by the semiclassical approximation do not emasculate the argument being made here. For the purpose at hand, it suffices that systems exist that, on the one hand, have properties which allow some of their degrees of freedom to be described semiclassically under appropriate circumstances, and on the other, have inherently nonclassical degrees of freedom that must be treated in strict accordance with the laws of quantum mechanics.

III

Consider, then, such a system. The degrees of freedom with a classical counterpart will be the position $q$ in a configuration space $C$, while the degrees of freedom with no such counterpart live in a finite dimensional “internal” Hilbert space $H$.

I should explain that the rules of my game allowed the fictional Newton to know the physical phenomena of electrodynamics when he was asked to unravel the meaning of the new concepts defined by Maxwell’s equations. In the same spirit, the fictional Maxwell is to know the phenomena that quantum mechanics supposedly accounts for. For example, he would be aware of the Stern-Gerlach experiment, and thus know that if an atomic species has a magnetic moment, this moment only displays a discrete set of orientations, while from the Schrödinger equation he would know that angular momentum is quantized.

In this scenario, therefore, the internal space of the systems of interest, $H$, is spanned by a finite set of states, which without loss of anything essential can be taken to be the spinors
\( \chi_{\mu}(m) \), where \( m \) is the direction of the quantization axis and \( \mu = j, j-1, \ldots, -j \). As in the paradigmatic example of the Stern-Gerlach experiment, the Hamiltonian is assumed to contain a time-independent perturbation \( H_{\text{ext}}(q) \) which is an operator both in \( \mathbb{C} \) and \( \mathbb{H} \), is nonzero only in some bounded region \( \mathbb{C}_{\text{ext}} \) of configuration space, and is brought to diagonal form by the basis whose quantization axis is \( n \).

Let \( \psi_{\text{in}}(q,t) \) be a wave function that is accurately approximated by the WKB Ansatz, and which, for \( t < t_1 \), describes a set of classical trajectories \( \{q_{\text{in}}(t)\} \) that form a beam moving towards \( \mathbb{C}_{\text{ext}} \), which they enter at \( t = t_1 \). Furthermore, let \( \chi_{\mu}(m) \) be the internal state of this beam, so that the incident state is represented by

\[
\Psi_{\text{in}}(q,t) = \psi_{\text{in}}(q,t) \chi_{\mu}(m) . \tag{6}
\]

For \( t_1 < t < t_2 \) this beam traverses the region \( \mathbb{C}_{\text{ext}} \) where the perturbation \( H_{\text{ext}}(q) \) exists, and provided this is sufficiently smooth, the WKB approximation will continue to be valid. The state (6) is then expanded in the diagonal basis, and for \( t > t_1 \) evolves into

\[
\Psi(q,t) = \sum_{\sigma=-j}^{j} c_{\sigma} \psi_{\sigma}(q,t) \chi_{\sigma}(n) , \quad c_{\sigma} = (\chi_{\sigma}^{*}(n), \chi_{\mu}(m)) . \tag{7}
\]

Each \( \psi_{\sigma} \) has a phase which is a solution of a separate Hamilton-Jacobi equation with a potential that depends on the quantum number \( \sigma \), and describes a set of trajectories \( \{q_{\sigma}(t)\} \) that move in distinct directions. Each such set has a density \( w_{\sigma}(q,t) \) satisfying a separate continuity equation, which means that

\[
\int dq \ w_{\sigma}(q,t) = C_{\sigma} , \quad t_1 < t < t_2 , \tag{8}
\]

where the constants \( C_{\sigma} \) could depend on \( \sigma \). But \( \psi_{\sigma}(q,t) \to \psi_{\text{in}}(q,t_1) \) as \( t \to t_1 \) from above, and therefore all the \( C_{\sigma} \) are equal:

\[
\int dq \ w_{\sigma}(q,t) = \int dq \ w_{\text{in}}(q,t) = 1 , \tag{9}
\]

where the last equality is just a convention.

By design, the region in which the perturbation \( H_{\text{ext}} \) acts is large enough to produce beams that are well separated for \( t > t_2 \), so that for such times the density is

\[
w(q,t) = \sum_{\sigma} |c_{\sigma}|^2 w_{\sigma}(q,t) . \tag{10}
\]

Because of (10), the fraction of the incident population that ends up in the beam bearing the label \( \sigma \) is \( |c_{\sigma}|^2 \). All but one of these beams, with \( \sigma = \rho \), can then be eliminated by another interaction term in \( H \). This filtered beam can be sent through a second arrangement of the Stern-Gerlach variety with any other orientation \( k \), which will produce a set of beams with population fractions

\[
|\langle \chi_{\rho}^{*}(n), \chi_{\sigma}(k) \rangle|^2 . \tag{11}
\]

All the familiar statements about the complex coefficients \( c_{\sigma} \) as probability amplitudes emerge, therefore, though thus far only expressed as fractions of various populations that pass through some combination of filters and fields.
This then leads to the question of whether a label “σ along n” can be assigned to a specimen following a trajectory in the initial set \( \{q_{\text{in}}(t)\} \), before it enters the force field that produces the separation into the distinct sets \( \{q_\sigma(t)\} \). If this could be done, then each member of the population would have an inherent property called \( \sigma \), which is revealed by the subsequent segregation into the distinct sets \( \{q(t)_\sigma\} \). But this cannot be done, because the initial internal state \( \chi_\sigma(n) \) in (6) can be expanded in any of the infinity of bases in \( \mathbb{H} \), and the appropriate choice is only revealed after the beam has entered the separating field.

Thus an intrinsic property “σ along n” cannot be assigned to individual specimens; all that can be said is that if a specimen passed through the field oriented along \( n \), the probability that it will emerge in the population “σ” is \( |c_\sigma|^2 \).

This is just the orthodox meaning of probabilities in quantum mechanics: the probability of a specific outcome as revealed by measurement, John Bell notwithstanding.

The same combination of semiclassical and quantum mechanical descriptions can be given for experiments of the Bohm-EPR type – for example, a system at rest that disintegrates in two fragments which then follow opposed classical trajectories \( \{q_1(t)\} \) and \( \{q_2(t)\} \), and a suitable correlated state in the joint internal Hilbert space \( \mathbb{H}_1 \otimes \mathbb{H}_2 \). When the widely separated fragments are passed through fields that produce distinct trajectories for the various eigenvalues \( (\sigma_1, \sigma_2) \) along the directions \( (n_1, n_2) \), they will produce correlations that violate the Bell inequalities, with all the familiar implications that follow therefrom.

I thank David Mermin for asking several pointed questions.

References

[1] J.S. Bell, Physics World, August 1990, pp. 33-40.

[2] M. Cini, Nuovo Cimento 73B, 27 (1983), who states “that the right answer to the question about the origin of the so-called wave function collapse has been outlined already” in my 1966 book, and that this “has remained . . . unnoticed for more than fifteen years.” I might add that it continued to go unnoticed in Quantum Theory Without Reduction, M. Cini and J-M. Lévy-Leblond (eds.), Adam Hilger: Bristol 1990.

[3] K. Gottfried, Physics World, October 1991, pp. 34-40.

[4] Here I am not engaging in ‘Whig’ history, or any other sort of history, but considering how we view the accomplishments of the past in the light of present knowledge.

[5] ‘Measurement’ was not the only word that Bell sought to banish from a discussion of fundamentals; environment was also forbidden. By “dynamical” environment I mean systems having their own separate degrees of freedom not contained in the Hamiltonian of the system under discussion, and which then enter at a critical juncture to save the day.

[6] Bell would have reworded my sentence to read “. . . under which classical mechanics provides an essentially exact approximation, FAPP . . . ”, where FAPP is the denigrating
acronym “for all practical purposes.” For a discussion of the compatibility of classical and quantum mechanics, see M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer: New York 1990; §12.1.

[7] Because classical dynamical systems tend to have chaotic regimes, the classical limit is now known to be far more subtle and complex than was recognized several decades ago; see M.C. Gutzwiller, *Am. J. Phys.* 66, 304 (1998). On the other hand, for the purpose of inferring the statistical interpretation, I believe it suffices to deal with a restricted class of systems in regimes that do not lead to chaotic classical motions.

[8] J.H. Van Vleck, *Proc. Nat. Acad. Sc.* 14, 178 (1928).

[9] That this is not just an educated guess was shown by G.D. Birkhoff, *Am. Math. Soc. Bull.* 39, 681 (1933).

[10] In the limit $a \to 0$, the Van Vleck determinant (or $w$) describes a population of trajectories with a uniform momentum distribution emanating from the aperture, which can be said to be required by the uncertainty principle. (Cf. e.g., M.V. Berry and K.E. Mount, *Rep. Prog. Phys.* 35, 315 (1972); Gutzwiller, §12.5.) While it is true that the uncertainty principle is implicit in the Schrödinger equation, I find this result to be quite remarkable: the semiclassical approximation is at best questionable when the aperture is shrunk to a pinhole, and Van Vleck’s determinant is a purely classical expression.

[11] One might worry about interference terms. As soon as the beams cease to overlap in space, there is no interference. In any event, there never is any interference because the internal wave functions are orthogonal. The fictional Maxwell would surely realize this as he would have known long before about the analogous phenomenon of light propagation in a medium immersed in a magnetic field.