Complexity and scaling in quantum quench in 1 + 1 dimensional fermionic field theories

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ABSTRACT: We consider the scaling behavior of circuit complexity under quantum quench in an a relativistic fermion field theory on a one dimensional spatial lattice. This is done by finding an exactly solvable quench protocol which asymptotes to massive phases at early and late times and crosses a critical point in between. We find a variety of scaling behavior as a function of the quench rate, starting with a saturation for quenches at the lattice scale, a "fast quench scaling" at intermediate rate and a Kibble Zurek scaling at slow rates.
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1 Introduction

Quantum quench at finite quench rates which involve critical points are known to display universal scaling behavior in various regimes. For quench rates which are slow compared to physical mass scales local quantities in many systems obey Kibble Zurek scaling [1–4]. In systems which have relativistic continuum limits there is a different scaling behavior for quench rates which are fast compared to physical mass scales, but slow compared to the UV scale [5–10]. Finally at quench rates at the scale of a UV cutoff one expects that the response saturates as a function of the rate. These scaling behaviors are characteristic of early time response, i.e. for measurements made in the middle of the quench or soon after the quench is over.

While the scalings themselves are quite generic, explicit investigations typically involve solvable models and a lot has been learnt from exactly solvable quench protocols in these models. They have also been studied for models which have holographic descriptions via the AdS/CFT correspondence [11–13]. In fact fast quench scaling was first discovered in holographic studies in [13]. They have been most extensively studied for local quantities like one point functions and correlation functions. For one dimensional harmonic chain scaling has also been found for the entanglement entropy [14] and recently for circuit complexity [15].

Complexity in a field theory quantifies the difficulty in preparing a quantum state starting from some reference state. Study of such measures is motivated by ideas of holographic complexity [16]-[21]. Since this is a quantity which characterizes properties of a quantum state which are not easily captured by correlation functions, it is interesting to study its behavior in non-equilibrium situations. There are several proposals for quantifying complexity in field theories. The proposal we consider in this paper is ”circuit complexity” which relates the length of the optimal circuit of unitary operations relating the reference state and a target state to a geometric quantity in the space of states parametrized in a suitable fashion [22]-[27]. Clearly, because of the dependence on the reference state as well as the unitary gates used, this is not uniquely defined. Nevertheless such a definition is expected to capture the true complexity of a state and seems to agree with holographic expectations. For other approaches to field theoretic complexity, see [28, 29].

In this paper we study scaling of circuit complexity in quantum quench for 1 + 1 dimensional majorana fermions on a spatial lattice with a time (and momentum) dependent mass function - this is the fermionic description of a one dimensional transverse field Ising model with a time dependent transverse field. Following [9] we consider a time dependence for which the dynamics can be solved exactly - this corresponds to a transverse field which asymptotes to constant values at early and late times and passes
through the critical point at some intermediate time which we choose to be \( t = 0 \). The Heisenberg picture state of the system is chosen to be the "in" state, which approaches the ground state of the system at early times. This latter state is also chosen as the reference state. The Heisenberg picture state is then a Bogoliubov transformation of the reference state with time dependent Bogoliubov coefficients. As shown in [25] for such a free fermion theory, circuit complexity (as defined in that paper) can be expressed entirely in terms of these Bogoliubov coefficients. Using the exact expression for this quantity we study the complexity analytically in various regimes.

In the slow regime we use the standard adiabatic-diabatic scenario underlying Kibble Zurek scaling to evaluate the complexity in the middle of the quench. We find a scaling behavior \( \sim \text{constant} + (\delta t)^{-1/2} \) where \( \delta t \) denotes the time scale of the quench. We compare this result with a numerical evaluation of the integral involved in the exact result and find excellent agreement. Interestingly this comes mostly from contribution of modes which remain adiabatic. This is in contrast to what happens for the bosonic theory studied in [15] where the zero momentum modes in fact dominate the result.

In the fast regime, we can perform an expansion of the exact answer in a power series in \( J b \delta t \) where \( J \) denotes the mass scale of the theory and \( b \) is the quench amplitude. In this expansion, the complexity at \( t = 0 \) is proportional to \( \delta t \) for arbitrarily small \( \delta t \). This agrees nicely with a numerical evaluation of the exact answer. The complexity at a slightly later time \( t \ll \delta t \) shows a slightly different behavior: for \( \delta t \) smaller than a \( t \)-dependent threshold value the complexity saturates as a function of \( \delta t \), while for \( \delta t \) larger than this threshold, the above mentioned linear behavior holds.

The content of the paper is as follows: In section 2, we summarize the definition of circuit complexity. In section 3, we introduce 1D Majorana fermion field theory and the derivation of complexity of its quench by considering Bogoliubov transformation. In section 4, we study the scaling of complexity with respect to quench rate, and show some numerical results of the late-time behaviors of complexity.

2 Complexity in Free fermionic theory

We follow the definition of complexity in [22] and [25] which we summarize below: complexity is the minimal number of elementary unitary gates needed to prepare a certain target state \( |\psi_T\rangle \) from a reference state \( |\psi_R\rangle \)

\[
|\psi_T\rangle = U|\psi_R\rangle, \quad U = \prod_{i=1}^{N} V_i.
\]
In continuum limit, \( U \) takes a form of path-ordered exponential of the sum of products of control function \( Y^I(s) \) and a basis of elementary gates \( \mathcal{O}_I \)

\[
U(s) = \mathcal{P} \exp \left[-i \int_0^s ds \sum_I Y^I(s) \mathcal{O}_I \right], \quad U = U(s = 1) \tag{2.2}
\]

And the complexity is defined to be the circuit that minimizes a cost

\[
\mathcal{D}(U(t)) = \int_0^1 ds F(U(s); Y^I(s)). \tag{2.3}
\]

Notice that \( Y^I(s) \) can be interpreted as the \( I^{th} \) component of the tangent vector of trajectory \( U(s) \). The functional \( F \) is a measurement of “distance” from reference state at \( U(0) \) to target state at \( U(1) \); for example, if all classes of gates have equal cost, \( F \) can have a general form \( F_\kappa(U; Y^I) = \sum_I |Y^I|\kappa \). Then minimizing the cost is equivalent to looking for the shortest geodesic on the manifold formed by tangent vector \( \vec{Y}(s) \).

When both the target state \( |\psi_T\rangle \) and the reference state \( |\psi_R\rangle \) are gaussian, there exist two pairs of sets of creation and annihilation operators \( \{a_T\}, \{a_T^\dagger\} \) and \( \{a_T\}, \{a_T^\dagger\} \) s.t. \( a_T|\psi_T\rangle = 0 \) and \( a_R|\psi_R\rangle = 0 \). Then the transformation between the two states can be described by the transformation between the two pairs of creation and annihilation operators. Most of time the transformation is a Bogoliubov transformation\(^1\). Below we give a rudimentary argument about fermions:

For a pair of fermions, the unitary operation \( U \) from reference state \( |\psi_R\rangle \) to target state \( |\psi_T\rangle \) is of the form

\[
\begin{align*}
\tilde{a} &= \alpha a - \beta b^\dagger, \\
\tilde{b}^\dagger &= \alpha^* b^\dagger + \beta^* a,
\end{align*}
\tag{2.4}
\]

where operators \( a, b \) and \( a^\dagger, b^\dagger \) are annihilation and creation operators of reference state, i.e. \( a|\psi_R\rangle = b|\psi_R\rangle = 0 \); Similarly, \( \tilde{a}, \tilde{b} \) and \( \tilde{a}^\dagger, \tilde{b}^\dagger \) are annihilation and creation operators of target state, i.e. \( \tilde{a}|\psi_T\rangle = \tilde{b}|\psi_T\rangle = 0 \). To preserve the anti-commutation relations, \( \alpha \) and \( \beta \) satisfy

\[
|\alpha|^2 + |\beta|^2 = 1. \tag{2.5}
\]

The equation (2.5) implies that all of the possible target states form a unit sphere with the north pole the reference state. This is made explicit by writing \( \alpha \) and \( \beta \) by two angles \( \theta \) and \( \phi \), i.e.

\[
\alpha = \cos \theta, \quad \beta = e^{i\phi} \sin \theta. \tag{2.6}
\]

\(^1\)A single-fermion excited state can be expressed in the form \( a|\psi\rangle = 0 \) as well by letting a special \( a^\dagger \rightarrow a \) while the remain unchanged. However, it cannot be transformed from the corresponding ground state through any Bogoliubov transformation.
Then a definition of the circuit complexity is the length of the geodesic from north pole to the position of target state, i.e. $|\theta|$ gives the minimal cost. This can be generalized to N-pairs of free fermions. Since the Bogoliubov transformation does not mix operators with different momenta, it still takes the form in (2.4) and therefore (2.6) for each pair of fermion with momentum $\vec{k}, -\vec{k}$. On the other hand, to prepare the target state $|\psi_T\rangle$ from reference state $|\psi_R\rangle$, one need to Bogoliubov transform all the independent (momentum) modes. As a result, the circuit complexity is the sum of geodesics $|\theta|(\vec{k})$ of all momenta, i.e.

$$C^{(n)} = \sum_{\vec{k}} |\theta|^n(\vec{k}) \to V \int \frac{d^d k}{(2\pi)^d} |\theta|^n(\vec{k}),$$

where

$$|\theta|(\vec{k}) = \tan^{-1} \frac{|\beta_{\vec{k}}|}{|\alpha_{\vec{k}}|} = \frac{1}{2} \tan^{-1} \frac{2|\alpha_{-\vec{k}}||\beta_{\vec{k}}|}{||\alpha_{\vec{k}}|^2 - ||\beta_{\vec{k}}|^2}.$$ 

(2.8)

### 3 The model and quench dynamics

The model considered in this paper is Majorana fermion field theory of the one dimensional transverse field Ising model with a time dependent transverse field (The model is discussed in details in [9]). The Hamiltonian is given by

$$H = \int \frac{dk}{2\pi} \chi^\dagger(k, t) [-m(k, t)\sigma_3 + G(k)\sigma_1] \chi(k, t).$$

(3.1)

where $\sigma_{1,3}$ are 2D Gamma matrices and $\chi$ denotes the two component spinor field, i.e. $\chi = \begin{pmatrix} \chi_1(k) \\ \chi_2(k) \end{pmatrix}$.

The Heisenberg equation of motion for $\chi(k, t)$ is a superposition of two independent solutions $U(k, t)$ and $V(k, t)$,

$$i\partial_t (U(k, t), V(-k, t)) = [-m(k, t)\sigma_3 + G(k)\sigma_1] (U(k, t), V(-k, t))$$

(3.2)

and

$$\chi(k, t) = a(k)U(k, t) + a^\dagger(-k)V(-k, t).$$

(3.3)

because of Majorana condition $\chi_2(k) = \chi_1^\dagger(-k)$. The operators $a(k)$ and $a^\dagger(k)$ satisfy the usual anti-commutation relations

$$\{a(k), a^\dagger(k')\} = \delta(k - k')$$

$$\{a(k), a(k')\} = \{a^\dagger(k), a^\dagger(k')\} = 0$$

(3.4)
We can relate the spinor to a scalar field $\phi(k, t)$ by letting
\begin{align*}
U(k, t) &= \begin{pmatrix}
-i\partial_t + m(k, t) \\
-G(k)
\end{pmatrix} \phi(k, t), \\
V(-k, t) &= \begin{pmatrix}
G(k) \\
i\partial_t + m(k, t)
\end{pmatrix} \phi^*(k, t),
\end{align*} 
where $\phi(k, t)$ satisfies
\begin{equation}
\partial_t^2 \phi + i\partial_t m \cdot \phi + (m^2 + G^2)\phi = 0.
\end{equation}
according to (3.2), and
\begin{equation}
|\partial_t \phi|^2 + (m^2 + G^2) |\phi|^2 - 2m \cdot \text{Im}(\phi\partial_t \phi^*) = 1
\end{equation}
to preserve anti-commutation relations and the orthonormality of $U(k, t)$ and $V(k, t)$.

An exactly solvable quench dynamics has been found in [9] which we use
\begin{equation}
m(k, t) = A(k) + B\tanh(t/\delta t),
\end{equation}
and the rest of the parameters are
\begin{equation}
A(k) = 2J(a - \cos k), B = 2Jb, G(k) = 2Js \sin k,
\end{equation}
where $J$ is the interaction strength between the nearest-neighbor spins in Ising model. It has dimension of energy. $a$ is the lattice spacing of Ising model. $b$ determines the mass gap.

In the rest of the paper we mainly consider the case $a = 1$, which describes a cross-critical-point (CCP) type-like potential at $k = 0$; another interesting case is when $a = 1 - b$, which corresponds an end-critical-point (ECP) type-like potential at $k = 0$.

The Heisenberg picture state we use is the "in" state. This means that the spinors $U(k, t)$ should asymptote to the positive frequency solution of the equation in the infinite past. This "in" solution is given by (3.6) and (3.7) is
\begin{equation}
\phi_{in}(k, t) = \frac{1}{|G(k)|} \sqrt{\omega_{in} + m_{in}} \exp[-i\omega_+(k) t - i\omega_-(k) \delta t \log(2\cosh(t/\delta t))]
\end{equation}
\begin{equation}
2F_1[1 + i\omega_-(k) \delta t + iB\delta t, i\omega_-(k) \delta t - iB\delta t; 1 - i\omega_{in}(k) \delta t; \frac{1}{2}(1 + \tanh(t/\delta t))],
\end{equation}
where the frequencies $\omega_{in, out, \pm}$ are defined to be
\begin{equation}
\omega_{out, in} = \sqrt{G(k)^2 + (A(k) \pm B)^2}, \omega_\pm = \frac{1}{2}(\omega_{out} \pm \omega_{in}).
\end{equation}
For the reference state we will choose the ground state of the system at infinite past, while the target state is the Heisenberg picture state. For some momentum $k$ the former is annihilated by a set of fermionic oscillators $a_{-\infty}(k)$ and $a_{-\infty}^\dagger(k)$ defined by

$$
\chi(k, t) = a_{-\infty}(k)U_{-\infty}(k, t) + a_{-\infty}^\dagger(-k)V_{-\infty}(-k, t)
$$

(3.12)

where $U_{-\infty}(k, t)$ and $V_{-\infty}(k, t)$ are given by the expressions (3.5) using the asymptotic form $\phi_{-\infty}(k, t)$

$$
\phi_{-\infty}(k, t) = \frac{1}{\sqrt{2\omega_{in}}} e^{-i\omega_{in}t}
$$

(3.13)

The relationship between $a_{-\infty}(k), a_{-\infty}^\dagger(k)$ and $a(k), a(k)$ then become Bogoliubov transformation of the form (2.4)

$$
\begin{align*}
a_{-\infty}(k, t) & = \alpha(k, t)a(k) - \beta(k, t)a^\dagger(-k), \\
a_{-\infty}^\dagger(-k, t) & = \beta^*(k, t)a(k) + \alpha^*(k, t)a^\dagger(-k),
\end{align*}
$$

(3.14)

where $\alpha(k, t) = \alpha(-k, t)$ and $\beta(k, t) = -\beta(-k, t)$; anti-commutation relation requires $|\alpha(k, t)|^2 + |\beta(k, t)|^2 = 1$. Then $\alpha(k, t)$ and $\beta(k, t)$ can be expressed by $U(k, t), V(k, t)$:

$$
\begin{align*}
\alpha(k, t) & = U_{-\infty}^\dagger(k, t)U(k, t) = V^\dagger(-k, t)V_{-\infty}(-k, t), \\
\beta(k, t) & = -U_{-\infty}^\dagger(k, t)V(-k, t) = U^\dagger(k, t)V_{-\infty}(-k, t),
\end{align*}
$$

(3.15)

and therefore $\phi(k, t)$:

$$
\begin{align*}
\alpha(k, t) & = \phi_{-\infty}^*(k, t)\left\{G^2(k) + (-\omega_{in} + m_{in})[-i\partial_t + m(k, t)]\right\}\phi(k, t) \\
\beta(k, t) & = \phi_{-\infty}^*(k, t)G(k)\left\{[i\partial_t + m(k, t)] - (-\omega_{in} + m_{in})\right\}\phi^*(k, t).
\end{align*}
$$

(3.16)

In this paper we consider the measurement closest to the original definition of complexity in discrete case, i.e. $F(U, \tilde{Y}) = \sum_i |Y_i|$, therefore the circuit complexity of the model is

$$
C^{(1)} = V \int \frac{dk}{2\pi} |\theta|(k, t),
$$

(3.17)

where the integrand $\theta(k, t)$ is given by

$$
|\theta|(k, t) \equiv \tan^{-1} \left|\frac{\beta(k, t)}{\alpha(k, t)} \right| = \tan^{-1} \left|\frac{\phi_{-\infty}(k, t)G(k)\left\{[-i\partial_t + m(k, t)] - (-\omega_{in} + m_{in})\right\}\phi(k, t)}{\phi_{-\infty}^*(k, t)\left\{G^2(k) + (-\omega_{in} + m_{in})[-i\partial_t + m(k, t)]\right\}\phi^*(k, t)} \right|
$$

(3.18)

For practical reason, we ignore factor $\frac{V}{2\pi}$.

4 Scaling of Complexity

Now we want to see how circuit complexity scales with respect to quench rate. The behavior of $C^{(1)}(t)$ as a function of $\delta t$ is shown in Fig. 1 and Fig. 2. There are three regimes of the quench rate: slow quench, fast quench and instantaneous quench.
Figure 1. Exact $C^{(1)}(0)-\delta t$ relations in log-log scale. Red and blue dots correspond to $b = 0.01$ and $b = 0.1$ respectively. The orange and yellow fitting curve are $y = ax^c$ and $y = P' + Q'x^{-1/2}$, respectively. The linear fitting coefficient $a = 0.985146$ for $b = 0.1$ and $a = 0.984975$ for $b = 0.01$, which implies the linear relation between $C^{(1)}(0)$ and $\delta t$ in fast quench regime.

Figure 2. Exact $C^{(1)}(t)-\delta t$ relations in log-log scale. Red and blue lines correspond to $b = 0.01$ and $b = 0.1$ respectively. From solid to dashed, the curves correspond to $t = 0.002, 0.001$ and $0.0005$, respectively. We can see the circuit complexity saturates around $\delta t \sim t$ (gridlines), and the saturation value is approximately $8Jbt$ (in yellow dotted lines). As reference, $C^{(1)}(0)-\delta t$ relations are in dotted lines.
4.1 Slow quench

In slow quench region, $J \delta t \gg 1/b$, the asymptotic behavior of circuit complexity with respect to the quench rate is $C^{(1)} \sim P' + Q' \delta t^{-1/2}$, where $P'$, $Q'$ are constants (Fig. 1). This behavior is consistent with Kibble-Zurek scaling. To make it clear, the circuit complexity is plotted mode by mode (See Fig. 3). We find that the momentum-dependence of circuit complexity can be divided two regions by Landau criterion,

\[ \frac{1}{E_g(t)} \frac{dE_g(t)}{dt} \bigg|_{-t_{KZ}} \sim 1, \]

where $E_g$ is the instantaneous energy gap

\[ E_g = \sqrt{m(k,t)^2 + G(k)^2}. \]

In other words, a critical momentum $k_c$ exists, where

\[ \sin k_c = \frac{4}{\sqrt{27}} \sqrt{\frac{b}{J \delta t}}. \]

For $k < k_c$ the adiabatic approximation breaks down and circuit complexity can be evaluated approximately by that at a fixed mass $m = m(-t_{KZ})$; for $k > k_c$, circuit complexity can be evaluated by using the instantaneous mass at $t = 0$,

\[ |\theta| \approx \frac{1}{2} \tan^{-1} \left( \frac{(m_{in} - m)G}{m_{in} \cdot m + G^2} \right). \]
Therefore, an approximation of circuit complexity is
\[
C^{(1)}(0) \approx \frac{1}{2} \int_0^\pi \left| \frac{[m_{\text{in}} - m(t = 0)]G}{m_{\text{in}} \cdot m(t = 0) + G^2} \right| \tan^{-1} k - \frac{1}{2} \int_0^{k_c} \left| \frac{[m_{\text{in}} - m(t = 0)]G}{m_{\text{in}} \cdot m(t = 0) + G^2} \right| + \frac{1}{2} \int_0^{k_c} \tan^{-1} \left| \frac{m_{\text{in}} - m_{\text{KZ}}G}{m_{\text{in}} \cdot m_{\text{KZ}} + G^2} \right|.
\] (4.5)

Simplify it, and we find
\[
C^{(1)}(0) \sim \int_0^{\pi/2} \tan^{-1} \left( \frac{b - b \cot x}{2 - b} \right) + \mathcal{O} \left( \sqrt{J \delta t} \right). \tag{4.6}
\]

Numerically we can find that leading term is 0.207762 when \( b = 0.1 \), which is close to the curve fitting result of exact \( C^{(1)}(0) \), \( P' = 0.207605 \) (yellow fitting in Fig. 1). Therefore, in slow quench part, circuit complexity follows the expectation of KZ.

### 4.2 Fast quench and instataneous quench

In fast quench region, \( J \delta t \ll 1 \), circuit complexity at time \( t \) shows linearity when \( \delta t \gg t \) and saturation when \( \delta t \ll t \); In both cases the circuit complexity is proportional to the gap of mass \( Jb \) (Fig. 2). One can see these features more clearly from the asymptotic behaviors of \( \phi \) and \( \beta \). When \( \delta t \gg t \), \( \tanh(t/\delta t) \rightarrow t/\delta t \). Therefore expand solution \( \phi(k, t) \) and we find
\[
\phi_{\text{in}}(k, t) \approx \frac{1}{|G(k)|} \sqrt{\frac{\omega_{\text{in}} + m_{\text{in}}}{2\omega_{\text{in}}}} e^{-\omega_+ t} \times \left\{ 2^{-\omega_- \delta t} \frac{\omega_+}{\omega_{\text{out}}} \left[ 1 - i \delta t(\omega_- - B) \log \left( \frac{1}{2} (1 + \frac{t}{\delta t}) \right) \right] + 2^{\omega_- \delta t} (1 - \frac{t}{\delta t})^{-i \omega_{\text{out}} \delta t} \frac{\omega_-}{\omega_{\text{out}}} \left[ 1 + i \delta t(\omega_+ + B) \log \left( \frac{1}{2} (1 + \frac{t}{\delta t}) \right) \right] \right\} \tag{4.7}
\]
\[
- \frac{1}{|G(k)|} \sqrt{\frac{\omega_{\text{in}} + m_{\text{in}}}{2\omega_{\text{in}}}} e^{-\omega_+ t} \times t \delta t(\omega_+ + B)(\omega_- - B) \log 2 + (\text{higher order contributions}).
\]

Then the leading term of \( \beta^*(k, t) \) is
\[
\beta^*(k, t) \approx \frac{1}{G} \frac{\omega_{\text{in}} + m_{\text{in}}}{2\omega_{\text{in}}} e^{-\omega_- t - i \omega_{\text{in}} t} \{ i 2 \delta t(\omega_+ + B)(\omega_- - B) \log 2 \}. \tag{4.8}
\]

Since \( J \delta t \ll 1 \), circuit complexity \( C^{(1)}(0) \sim |\beta| \) and as a result,
\[
C^{(1)}(0) \sim b J \delta t \cdot 4 \log 2 \approx 2.77 b J \delta t. \tag{4.9}
\]
This is close to the linear fitting of exact circuit complexity, where the slopes are \( C(0) \approx 0.261913 \) when \( b = 0.1 \) and \( C(0) \approx 0.0261736 \) when \( b = 0.01 \).

When \( \delta t \ll t \), \( \tanh(t/\delta t) \to 1 - e^{-2t/\delta t} \). Expand solutions and we find

\[
\phi_{in}(\vec{k},t) \approx \frac{1}{|G(k)|} \sqrt{\frac{\omega_{in} + m_{in}}{2\omega_{in}}} \exp[-i\omega_{-}\delta te^{-2t/\delta t}] \\
\times \left\{ e^{-i\omega_{out}t} \frac{\omega + B}{\omega_{out}} (1 + i\delta t(\omega - B)e^{-2t/\delta t}) + e^{i\omega_{out}t} \frac{\omega - B}{\omega_{out}} (1 - i\delta t(\omega + B)e^{-2t/\delta t}) \right\}.
\]

Therefore \( \beta(k,t) \)

\[
\beta^{*}(k,t) \approx -i \exp[-i\omega_{-}\delta te^{-2t/\delta t}] \frac{G(m_{out} - m_{in})}{\omega_{in}\omega_{out}} e^{-i\omega_{-}t}\sin\omega_{out}t \\
+ \frac{1}{G} \frac{m_{in} + \omega_{in}}{2\omega_{in}} e^{-i\omega_{-}t} \exp[-i\omega_{-}\delta te^{-2t/\delta t}] 4(\omega - B)e^{-2t/\delta t} e^{i\omega_{out}t},
\]

and then the leading term and subleading term of the circuit complexity is

\[
C^{(1)}(t) \sim 8bJt + \mathcal{O}(te^{-2t/\delta t}).
\]

This gives the saturation when \( \delta t \ll t \).

One can see that relativistic fermionic Ising theory seems to show similar behavior of complexity scaling compared to free bosonic oscillators [15]. However, in the fermionic theory, the zero-mode is unpaired and thus never contributes to complexity (Fig. 4). The single-mode contribution has a peak at some nonzero mode and it moves close to zero mode as quench becomes slower. Most of the contribution to complexity comes from modes that remain adiabatic (Fig. 3). The theory with ECP-type-like potential is slightly different when slow quenched (Fig. 5), complexity saturates much more quickly because the single-mode contribution saturate at large \( J\delta t \) (Fig. 4(a)).
Figure 5. Exact $C^{(1)}(t)-\delta t$ relations in log-log scale when $b = 0.01$. Red and yellow lines correspond to ECP and CCP-type-like potential respectively. From solid to dashed, the curves correspond to $t = 0.002, 0.001$ and $0.0005$, respectively. The plots differ at large $J\delta t$ (Red plots saturate more quickly).

Figure 4. Single-mode contribution to complexity at $t = 0, \theta(k,0)$ in ECP and CCP-like potentials when $b = 0.01$. Purple, red, yellow, green and blue solid lines are $J\delta t = 0.01, 0.1, 1, 10, 100$, respectively.

Finally, we numerically compare the late-time behaviors of circuit complexity ([30, 31] have studied circuit complexity at late time in the bosonic free field with smooth quench and fermionic free field with instantaneous quench, respectively) in ECP-type-like potential ($a = 1 - b$) and CCP-type-like one ($a = 1$), and the results are shown in Fig. 6. It shows that circuit complexity of ECP-type saturates without oscillation, unlike CCP-type potential. This is consistent to quantities such as $\langle \bar{\psi}\psi \rangle$ ($\langle \bar{\chi}\chi \rangle$ in [9]).
(a) ECP-type-like: from solid to dotted lines $J\delta t = 200, 100, 10, 1, 0.1, 0.01$, respectively

(b) CCP-type-like: from solid to dotted lines $J\delta t = 100, 1, 0.01$, respectively

(c) ECP-type-like: from solid to dashed lines $J\delta t = 150, 100$, respectively

(d) CCP-type-like: from solid to dashed lines $J\delta t = 150, 100$, respectively

Figure 6. 6(a)&6(b): Time evolution of complexity $C(t)$ in ECP and CCP-like potentials; 6(c)&6(d): Time evolution of $\langle \bar{\psi} \psi \rangle$ in ECP and CCP-like potentials. From thick solid lines to dotted lines $J\delta t$ decrease. Choose $b = 0.01$.

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