Neutrino conversion in a neutrino flux: towards an effective theory of collective oscillations

Rasmus S.L. Hansen and Alexei Yu. Smirnov

Max-Planck-Institut für Kernphysik,
Saupfercheckweg 1, 69117 Heidelberg, Germany

E-mail: rasmus@mpi-hd.mpg.de, smirnov@mpi-hd.mpg.de

Received February 5, 2018
Revised March 29, 2018
Accepted April 11, 2018
Published April 26, 2018

Abstract. Collective oscillations of supernova neutrinos above the neutrino sphere can be completely described by the propagation of individual neutrinos in external potentials and are in this sense a linear phenomenon. An effective theory of collective oscillations can be developed based on certain assumptions about time dependence of these potentials. General conditions for strong flavor transformations are formulated and these transformations can be interpreted as parametric resonance effects induced by periodic modulations of the potentials. We study a simplified and solvable example, where a probe neutrino is propagating in a flux of collinear neutrinos, such that $\nu\nu$—interactions in the flux are absent. Still, this example retains the main feature — the coherent flavor exchange. Properties of the parametric resonance are studied, and it is shown that integrations over energies and emission points of the flux neutrinos suppress modulations of the potentials and therefore strong transformations. The transformations are also suppressed by changes in densities of background neutrinos and electrons.

Keywords: supernova neutrinos, core-collapse supernovas, neutrino astronomy, neutrino theory

ArXiv ePrint: 1801.09751
1 Introduction

Neutrino-neutrino scattering results in flavor exchange between the interacting neutrinos [1]. When a given neutrino propagates in a background containing other neutrinos, the flavor exchange can be coherent producing both diagonal and off-diagonal potentials. In central regions of supernovae with a large density of neutrinos, this coherent flavor exchange may lead to various effects of collective oscillations [2–25].

Finding an exact solution of the evolution equations is an extremely difficult problem and has not been solved in realistic conditions of collapsing stars. With some approximations (stationary situation, symmetries, effective $\nu\nu$ scattering description, elimination of usual matter potential, etc.) effects of bi-polar oscillations [2–5], spectral splits/swaps [3, 4, 6–10], and fast flavor transformations in the early evolution [11–18] have been found. In more than one dimension, multi-angle effects (angles of neutrino propagations) can suppress the
flavor conversion [19, 20]. Still strong transitions have been obtained, e.g., in the case of two intersecting fluxes [21, 22].

The main question is whether the flavor transformations that have been found still exist under realistic conditions or they are artefacts of approximations and simplifications. There are some indications that the collective transformations are either very strongly suppressed or lead to flavor equilibration in realistic situations (see e.g. [19, 20, 22–24]). Indeed, strong transitions imply extremely strong correlations between the flavor evolution of neutrinos produced with different energies in different space-time points and at different directions.

In this paper consider the flavor evolution of individual neutrinos rather than the neutrino field. The problem is linear in a sense that will be described in section 2, and the non-linearity discussed in the literature is a consequence of certain simplifications and approximations which allow the identification of the probe neutrino and background neutrinos. Consequently, the evolution of an individual neutrino can be completely described as propagation in external potentials. These potentials have flavor diagonal as well as flavor off-diagonal terms with non-trivial time (distance) dependence. Using a general parameterisation of the Hamiltonian of evolution, we formulate conditions for strong flavor transitions. We show that in the presence of a large matter potential, strong transformations can only be due to a parametric resonance. On this basis one can develop the effective theory of collective oscillations which is based on certain conjectures about the time dependence of the potentials.

In this connection we consider here a simplified model of the background neutrinos, which still retains the main feature of the coherent flavor exchange. In this model all the background (flux) neutrinos propagate with the same angle, so that $\nu\nu$ interactions in the flux are absent. The latter allows us to explicitly compute the time dependence of the potentials for the probe neutrino. This, in turn, allows us to find an explicit solution to the evolution equation for the probe neutrino. The main feature of the potentials is their periodic (quasi-periodic) dependence on time (distance) which, under certain conditions, leads to the parametric resonance and parametric enhancement of the flavor transition for the probe neutrino.

The simple background model allows us to find an analytic expression for the conversion probability and study the details of the parametric resonance. Furthermore, it allows us to explicitly study the effects of different integrations, in particular, integration over the production point along a given trajectory and averaging over energy. Finally, the effect of a varying matter and neutrino density is explored. In particular, we find that our example reproduces the effect of a spectral split.

The main question left is to which extent our results for the simplified background can be applied to a realistic case with $\nu\nu$ interactions in the flux.

The paper is organised as follows. In section 2, after a discussion of the linearity, we construct the Hamiltonian which describes the evolution of individual neutrinos. This allows us to formulate the conditions for strong flavor transformations. In section 3 we consider a solvable model for the background: namely the flux of neutrinos propagating with the same angle. We compute the neutrino potentials explicitly and consider approximations for the Hamiltonian which reproduce very well the exact numerical solution. In section 4 we provide an analytical solution of the problem. In section 5 we perform integration (averaging) over energies of the flux neutrinos and their production points, while we consider a varying background density in section 6. Our discussions and conclusions are in section 7.
2 Towards an effective theory of collective effects

2.1 On linearity

The evolution equation is linear in the sense that a given neutrino does not affect its own flavor evolution. It does not affect the evolution immediately, since the wave function of this neutrino does not appear in the Hamiltonian that describes its flavor evolution. This is related to the fact that $\nu\nu$ interactions are given by the vector product of the corresponding polarisation vectors. Furthermore it does not affect the evolution indirectly: a given neutrino does not affect the evolution of other neutrinos with which it interacts before the interaction point. This is true for SN neutrinos propagating outwards along a straight trajectory without bending.

Linearity in this sense follows from a simple geometric consideration (see figure 1). The probe neutrino $\nu_p$ emitted from the point $(x_0, t_0)$ interacts in a given space-time point $(x, t)$ with neutrinos $\nu_B$ which move along the trajectory with angle $\Delta\beta$ with respect to the trajectory of $\nu_p$. The previous evolution of $\nu_B$ (before collision with $\nu_p$) was not affected by $\nu_p$. The evolution of $\nu_B$ can be affected by another background neutrino $\nu'_B$ which crosses both the trajectory of $\nu_B$ in the point $(x_B, t_B)$ and the trajectory of $\nu_p$ in a point $(x', t')$, but it did this before $\nu_p$ arrived at $x'$, i.e. $t' < t'_0$. The points $x'$, $x_B$ and $x$ form a triangle: $\nu_p$ propagates along the side $x' - x$, whereas the background neutrinos should propagate along the two other sides: $x' - x_B$ and $x_B - x$. The latter trajectory is longer than the former one and therefore the same $\nu_p$ can not interact with $\nu'_B$ in the point $x'_B$ and with $\nu_B$ in the point $x$. Another probe neutrino $\nu'_p$ emitted before $\nu_p$ can interact with $\nu'_B$ in $x'$; then $\nu'_B$ interacts with $\nu_B$ in $x_B$, and in turn, $\nu_B$ can interact with $\nu_p$ in $x$. But $\nu'_p$ and $\nu_p$ are different neutrinos emitted in different moments of time.

In the stationary situation, $\nu'_p$ can be formally identified with $\nu_p$ since they have identical flavor evolutions and arrive at $x'$ in the same flavor state. This is one of the cases where a symmetry leads to effective non-linear equations.

Another effective non-linearity appears when $\nu'_p$, propagating with the same angle as $\nu_p$, is emitted from a space point different from that of $\nu_p$. Then $\nu'_p$ can influence the evolution of $\nu_B$ and the latter can influence the evolution of $\nu_p$. Again the evolution of $\nu'_p$ and $\nu_p$ are related since they have the same flavor at the same distance from the production point. This corresponds to a translational symmetry.
Here we have neglected the finite size of the wave packet. If the size of the wave packet is long enough, the first part of the wave packet can in principle influence the last part of the wave packet of a given probe neutrino. However, this effect has not been considered elsewhere in the literature and will be left for future work.

Due to the absence of non-linearity, we can consider the flavor evolution of individual neutrinos propagating in an external background described by a potential with non-trivial dependence on distance along the trajectory. This description is complete in the sense that all possible effects obtained by solving the equations for neutrino polarisation vectors or density matrices must, if they are real, be reproduced in this description. Inversely, effects which are shown not to exist in our approach should not appear in the usual consideration.

### 2.2 Evolution equations

We study a probe neutrino with momentum $\mathbf{p}$ which propagates in a medium composed of usual matter including electrons with density $n_e$ and background neutrinos. We consider a 2ν system ($\nu_e, \nu_\tau$) with vacuum mixing angle $\theta$ and mass squared splitting $\Delta m^2$. The eigenfrequency of the probe neutrino is $\omega_p \equiv \Delta m^2 / 2E$. In numerical computations, we use the value of mixing angle $\theta = 0.1485$ rad ($\sin^2 2\theta = 0.087$), $\omega_p > 0$ for normal mass ordering (NO) and $\omega_p < 0$ for inverted mass ordering (IO). In what follows, we present results for the NO unless IO is explicitly indicated.

The background neutrinos arriving at the space-time point $(x, t)$ can be characterised by:

1. The flavor at their production; $a = e, \tau$.
2. The 3-momentum $k$. Notice that in general, neutrinos are produced with wide energy spectrum which depends on the flavor $a$.
3. The length of the trajectory $l$ from the production point to the interaction point $(x, t)$.

The length $l$, the momentum $k$ and $(x, t)$ determine the production point $(x_0, t_0)$. $l$ varies in the interval determined by the width of the neutrino sphere for a given $a$ and momentum $k$.

All neutrinos with the same set $(a, k, l)$ have the same evolution. $n_a^\nu(k, l)$ denotes the number density of neutrinos emitted from $(x_0, t_0)$ in the point $(x, t)$.

The evolution equation for the flavor of the probe neutrino

$$i\partial_t \psi = H^{(p)} \psi$$  \hspace{1cm} (2.1)

has the Hamiltonian

$$H^{(p)} = \frac{1}{2} \left( -c_{2\theta}\omega_p + V_e + V_\nu s_{2\theta}\omega_p + 2\bar{V}_\nu e^{i\phi B} - V_e - V_\nu \right),$$  \hspace{1cm} (2.2)

where $V_e \equiv \sqrt{2}G_F n_e$ is the usual matter potential, $c_x \equiv \cos x$, $s_x \equiv \sin x$, while $V_\nu(t)$ and $\bar{V}_\nu(t)e^{i\phi_B(t)}$ ($V_\nu > 0$) are the neutrino potentials that describe the neutrino-neutrino interactions.

The diagonal potential $V_\nu$ is real and can be written as

$$V_\nu = \sum_a \int \! dk \int \! dl V_\nu^a(k, l) \left[ \psi^\dagger_a(k, l) \psi^a_a(k, l) - \psi^\dagger_a(k, l) \psi^a_\tau(k, l) \right] ,$$  \hspace{1cm} (2.3)
where

\[ V_{\nu}^a(k, l) \equiv \sqrt{2} G_F n_0^a(k, l) \left[ 1 - \frac{p \cdot k}{|p| \cdot |k|} \right]. \]

The contribution from scattering of the probe neutrino on antineutrinos \( V_{\bar{\nu}} \) can be obtained from the previous expressions for \( V_{\nu} \) by the substitutions

\[ V_{\nu}^a \rightarrow -V_{\bar{\nu}}^a, \quad \psi_a^a \psi_a^{*} \rightarrow \psi_a^{\bar{a}} \psi_{\bar{y}}^{\bar{a}} \]

with \( \{ x, y \} = \{ e, \tau \} \).

We can also introduce the total potential

\[ V_{\nu}^0 \equiv \sum_a \int dk dl \ V_{\nu}^a(k, l) \]

and the ratio of the potentials

\[ \xi \equiv \frac{V_{\nu}^0}{V_{e}}, \]

which will play a crucial role in our considerations.

The potential (2.3) can be rewritten as

\[ V_{\nu} = \int dk \int dl [V_{\nu}^e(k, l)(1 - 2P_{e\tau}(k, l)) - V_{\nu}^\tau(k, l)(1 - 2P_{\tau e}(k, l))], \]

where

\[ P_{ab}(k, l) = \psi_b^a(k, l)\psi_b^{a*}(k, l) \]

is the probability of the transition \( \nu_a \rightarrow \nu_b \), and we used the unitarity relation \( P_{ae}(k, l) = \psi_e^a(k, l)\psi_e^{a*}(k, l) = 1 - P_{a\tau}(k, l) \). In the case of T-invariance \( P_{\tau e}(k, l) = P_{e\tau}(k, l), \) eq. (2.6) can be rewritten as

\[ V_{\nu} = \int dk \int dl [V_{\nu}^e(k, l) - V_{\nu}^\tau(k, l)](1 - 2P_{e\tau}(k, l)). \]

The off-diagonal \( \nu^2 \) potential in the Hamiltonian (2.2) equals

\[ \tilde{V}_{e} \ e^{i \phi_B} = \sum_a \int dk \int dl \ V_{\nu}^a(k, l) A_{ae}(k, l) A_{e}^{*a}(k, l), \]

and we can obtain the potential for forward scattering on antineutrinos with the substitutions in eq. (2.4) as before. Since \( \psi_e^a(k, l) = A_{ae}(k, l), \) \( \psi_{\bar{e}}^a(k, l) = A_{a\tau}(k, l), \) where the latter is the amplitude of probability of the \( \nu_a \rightarrow \nu_e \) transition, etc., we can rewrite the integral in (2.8) as

\[ \int dk \int dl \sum_a V_{\nu}^a(k, l) A_{ae}(k, l) A_{e}^{*a}(k, l). \]

Then using the unitarity of the evolution matrix (matrix of amplitudes)

\[ A_{ae} A_{e}^{*a} + A_{ce} A_{c}^{*e} = 0, \]

we have

\[ \tilde{V}_{\nu} \ e^{i \phi_B} = \int dk \int dl [V_{e}^e(k, l) - V_{\tau}^\tau(k, l)] A_{ce}(k, l) A_{e\tau}^{*a}(k, l). \]
It can be represented in terms of the oscillation probability as

\[
\bar{V}_\nu e^{i\phi_B} = \int dk \int dl \left[ V^e_\nu(k, l) - V^\tau_\nu(k, l) \right] e^{i\phi_B(k, l)} \sqrt{P_{ee}(k, l)(1 - P_{\tau\tau}(k, l))},
\]

where

\[
\phi_b(k, l) = \text{Arg}[A_{\tau e}(k, l)A^*_{e\tau}(k, l)].
\]

Then the moduli of the potential, \(\bar{V}_\nu\), and the phase \(\phi_B\) equal

\[
\bar{V}_\nu = \sqrt{\bar{V}_R^2 + \bar{V}_I^2}, \quad \tan \phi_B = \frac{\bar{V}_I}{\bar{V}_R},
\]

(2.10)

where

\[
\bar{V}_R = \int dk \int dl \left[ V^e_\nu - V^\tau_\nu \right] \cos \phi_b \sqrt{P_{ee}(1 - P_{\tau\tau})},
\]

\[
\bar{V}_I = \int dk \int dl \left[ V^e_\nu - V^\tau_\nu \right] \sin \phi_b \sqrt{P_{ee}(1 - P_{\tau\tau})},
\]

and \(V^e_\nu, \phi_b\) and \(P_{\tau\tau}\) are functions of \((k, l)\). Generalisation to the case of three generations is straightforward. The neutrino potentials disappear if \(V^e_\nu(k, l) = V^\tau_\nu(k, l)\), and in general, eq. (2.7) and (2.9) show that the probe neutrino is only affected by the difference \(V^e_\nu(k, l) - V^\tau_\nu(k, l)\).

The Hamiltonian in eq. (2.2) is similar to that for a normal medium, like the Earth, but with non-standard interactions (NSI). The difference from the usual NSI is that we now deal with a strong and non-trivial dependence of these potentials on distance (or time of propagation of the probe neutrino).

Let us remove the complex phases from the Hamiltonian. The off-diagonal element of the Hamiltonian (2.2) can be rewritten as

\[
V' e^{i\phi'},
\]

where

\[
V' = \sqrt{4\bar{V}_R^2 + 4s_{2\theta}\omega_p \cos \phi_B \bar{V}_\nu + s_{2\theta}\omega_p^2},
\]

(2.11)

and the phase \(\phi'\) is determined by

\[
\tan \phi' = \frac{2\bar{V}_\nu \sin \phi_B}{2\bar{V}_\nu \cos \phi_B + s_{2\theta}\omega_p} = \frac{\sin \phi_B}{\cos \phi_B + R}.
\]

(2.12)

Here

\[
R \equiv \frac{s_{2\theta}\omega_p}{2\bar{V}_\nu}
\]

is the ratio of the vacuum to the neutrino contributions to the off-diagonal elements of \(H^{(p)}\), and \(\bar{V}_\nu\) is determined by eq. (2.10). If the \(\nu\nu\) contribution dominates, \(\phi' \approx \phi_B\).

The complex phase can be eliminated from the off-diagonal elements, and consequently from the Hamiltonian by performing the transformation

\[
\hat{\psi} = U \psi', \quad U = \text{diag} \left( e^{i\frac{1}{2}\phi'}, e^{-i\frac{1}{2}\phi'} \right).
\]

(2.13)
The Hamiltonian of the evolution equation for $\psi'$ is then

$$H^{(p)} = \frac{1}{2} \begin{pmatrix} V^r & V' \\ V' & -V^r \end{pmatrix},$$

(2.14)

where

$$V^r \equiv V_{\nu} + V_{\nu} - c_2 \omega_p + \dot{\phi'},$$

(2.15)

and $V'$ is determined in (2.11). From (2.12) we find

$$\dot{\phi'} = \frac{(1 + R \cos \phi_B) \dot{\phi}_B + R \sin \phi_B \hat{V}_\nu / \hat{V}_\nu}{1 + 2R \cos \phi_B + R^2}.$$ 

The elimination of the phase from the off-diagonal elements in the Hamiltonian leads to the appearance of $\dot{\phi'}$ in the diagonal elements. It is easy to show that for the neutrino polarisation vector, the transformation (2.13) goes to a reference frame rotating around the flavor axis $z$. Therefore, the transformation in eq. (2.13) does not change the flavor oscillation probabilities, and the probability for the $\psi'$ state coincide with the probabilities for $\psi$.

The Hamiltonian in (2.14) determines the instantaneous mixing angle in the medium $\theta_{pm}$ for the probe particle via

$$\tan 2\theta_{pm} = -\frac{V'}{V^r} = -\frac{\sqrt{4V_{\nu}^2 + 4\omega_p \cos \phi_B \delta_{2g} V_{\nu} + \omega_p^2 \delta_{2g}^2}}{-c_2 \omega_p + V_{\nu} + V_{\nu} + \phi'},$$

(2.16)

and difference of the eigenvalues

$$\Delta_{pm}^p = \sqrt{V^r^2 + V'^2}.$$ 

The phase $\phi_B$ is defined via (2.8). Here and below we use a super-script $p$ for the probe neutrino and no superscript for the background neutrinos when naming oscillation parameters.

### 2.3 Conditions for strong flavor transformations

A number of results can be obtained from the general form of the Hamiltonian (2.14). The key feature is that $V^r$ and $V'$ have an oscillatory dependence on distance (time), which originates from their dependence on the oscillation probabilities $P_{\alpha\tau}(k, l)$ and the phase $\phi_B$.

Strong flavor transformations can proceed under the following circumstances:

1. **Resonance oscillations.** Oscillations with nearly maximal depth occur if $V^r \ll V'$.

   Explicitly, the resonance condition reads

   $$V_{\nu} + V_{\nu} + \dot{\phi'} - \cos 2\theta_{pm} \approx 0.$$ 

   In the central regions of a star (near the neutrino sphere), $V_{\nu} \gg V_{\nu} \gg c_2 \omega_p$, so $V_{\nu}$ determines the highest frequency in the system. It may happen that

   $$\dot{\phi'} \approx -V_{\nu} - V_{\nu}.$$ 

   Under this condition, the system oscillates with nearly maximal depth at a frequency given by $V'$. Furthermore, neutrinos and antineutrinos will oscillate in the same way which resembles the regime of bi-polar oscillations [2–5].
2. Adiabatic conversion. Performing a series of field transformations, one can exclude fast time variations in $V^r$ and $V'$. Then for the rest of the Hamiltonian, the adiabaticity condition may be satisfied and a strong transition occurs if $V^r \gg V'$ to $V^r \ll V'$ (level crossing).

3. Parametric resonance. If $V^r \gg V'$ during the whole evolution, the only possibility for a strong transition is to build up a large transition probability over many periods of oscillations, that is, due to parametric enhancement. The condition for a parametric resonance is that the oscillation period of the probe neutrino $T_p$ coincide with the period of change for the mixing angle $T_\theta$. Since $V^r$ and $V'$ depend on time, the period of oscillations $T_p$ (precession in the polarisation vector picture) is determined from

$$\int_0^{T_p} dt \Delta^p_m = \int_0^{T_p} dt \sqrt{V^r(t)^2 + V'(t)^2} = 2\pi.$$  \hspace{1cm} (2.17)

The mixing angle $\theta^p_m$ defined through (2.16) also has an oscillatory dependence. Denoting the period of this dependence by $T_\theta$, we can write the parametric resonance condition as

$$T_p = T_\theta.$$ \hspace{1cm} (2.18)

Using this general consideration one can develop an effective theory of collective oscillations making various assumptions (conjectures) about the form of potentials which could lead to strong flavor transformations.

2.4 On effective theory

Let us summarise the main points of the effective theory approach.

- Collective oscillation effects can be completely described by following the evolution of individual neutrinos in external potentials produced by usual matter and other neutrinos. Both flavor diagonal and flavor off-diagonal potentials are generated by the background neutrinos. The potentials have non-trivial time dependence which can lead to complicated flavor transformations of the individual neutrinos. Thus the problem of describing collective effects is reduced to the determination of the potentials and their time dependence.

- The main idea of the approach is to obtain some results using the general form of the evolution equation and to determine the potentials without solving the evolution equations for many neutrinos simultaneously.

- Using the general form of the evolution equation, one can formulate conditions for the potentials and their time dependence which can lead to strong flavor transformations as it was done in section 2.3.

- In the case of two neutrino mixing, the problem is reduced to determining or restricting the potential $V_r(t)$ and $V'(t)$ as functions of time. Using the general expressions (2.11) and (2.15), one can explore properties of these functions. According to (2.7) and (2.9), the potentials are integrals of the oscillation amplitudes which have an oscillatory dependence on time. Therefore the potentials are also expected to be oscillatory functions determined by the intrinsic frequencies of the system: $V_e$, $V_\nu$ and $\omega$. 
• Some results of integrations can be obtained in general. Also flavor averaging or suppression of flavor transitions can be found from the general form. Integration over energy and especially production point affects the potentials substantially.

• One can look for rules or principles for constructing the potentials using various limits, etc.

Known numerical solutions for collective oscillation effects such as for two intersecting fluxes can be used to reconstruct the corresponding potentials. Exploring the dependence of these reconstructed potential on external parameters, $V_e, V^0_\nu, \omega$ may reveal rules for reconstructing the potentials.

• One can use some solvable simplified examples to find rules for reconstructing the potentials.

In what follows we will proceed with the last item and comment on other points.

3 Neutrino conversion in a neutrino flux

To get some idea about the time dependence of the potentials, we will consider here a simple model of the background which allows us to explicitly compute the neutrino potentials. This solvable model retains the main feature - the coherent flavor exchange. This example, however, misses another main feature - $\nu \nu$ interactions in the background. Nevertheless, as we will see, the example allows to reproduce some effects which show up in previous studies, such as strong transitions in high density matter, bi-polar oscillations, and spectral splits. Results obtained with this example can be used as a tool for further explorations.

We assume that a probe electron neutrino is emitted from the surface at an angle $\beta_p$ with respect to the surface, and that the frequency of the probe neutrino is $\omega_p$.

3.1 Background model

Let us consider a flux of neutrinos with a wide energy spectrum produced in a layer with width $r$. We assume that all the flux neutrinos are collinear and propagate in the same direction. Consequently, there are two key features of the background:

• There is no $\nu \nu$ interactions in the flux since the forward scattering potential is proportional to $(1 - v \cdot v')$.

• There is no feedback of the probe neutrinos onto the flux neutrinos. The effect of a single probe neutrino on the neutrino flux can be neglected. In fact, for a single probe neutrino there is no such interaction even in principle.

Under these conditions, the background neutrinos evolve in the usual way with forward scattering on background electrons which generates a potential $V_e$.

We consider an original flux of electron neutrinos, while the inclusion of a $\nu_\tau$ flux can be accounted for by substituting $V^0_e \rightarrow (V^0_e - V^0_\tau)$.

Let us consider first the flux of flux neutrinos with fixed momentum $k$ and the corresponding frequency

$$\omega_k \equiv \frac{\Delta m^2}{2E_k}.$$

The flux is emitted from the same surface as $\nu_p$ at an angle $\pi - \beta_h$. 

- 9 -
The evolution equation for the flux neutrino wave function $\psi^T \equiv (\psi_e, \psi_\tau)$ is
\begin{equation}
    i \partial_t \psi = H \psi, \tag{3.1}
\end{equation}
where the Hamiltonian has the standard form
\begin{equation}
    H = \frac{1}{2} \begin{pmatrix}
    -c_2 \omega_k + V_e & s_2 \omega_k \\
    s_2 \omega_k & c_2 \omega_k - V_e
    \end{pmatrix}. \tag{3.2}
\end{equation}
It determines the mixing angle in matter $\theta_m$ and the level splitting $\Delta_m$ for the flux neutrinos:
\begin{align*}
    \tan 2\theta_m &= \frac{s_2 \omega_k}{c_2 \omega_k - V_e}, \\
    \Delta_m &= \sqrt{(V_e - c_2 \omega_k)^2 + s_2^2 \omega_k^2}. 	ag{3.3}
\end{align*}
If the initial state is $\nu_e$, the wave functions of $\nu_e$ and $\nu_\tau$ in the moment of time $t$ equal
\begin{align*}
    \psi_e(t) &= \cos \frac{1}{2} \Delta_m t + i \cos 2\theta_m \sin \frac{1}{2} \Delta_m t, \\
    \psi_\tau(t) &= -i \sin 2\theta_m \sin \frac{1}{2} \Delta_m t. 	ag{3.4}
\end{align*}
They give the transition amplitudes for $\nu_e \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_\tau$.

Therefore the transition probability $\nu_e \rightarrow \nu_\tau$ is given by
\begin{align*}
    P_{e\tau} = |\psi_\tau(t)|^2 &= \sin^2 2\theta_m \sin^2 \frac{1}{2} \Delta_m t \approx \left(\frac{\omega_k}{V_e}\right)^2 \sin^2 2\theta \sin^2 \frac{1}{2} \Delta_m t, 	ag{3.5}
\end{align*}
where in the second equality we used that for $\omega_k \ll V_e$, so that
\begin{equation}
    \sin 2\theta_m \approx \frac{\omega_k}{V_e} \sin 2\theta. \tag{3.6}
\end{equation}
The probability is strongly suppressed by the electron density. Its maximal value is given by
\begin{equation}
    \sin^2 2\theta_m \approx \left(\frac{\omega_k}{V_e}\right)^2 \sin^2 2\theta \approx 10^{-7}, \tag{3.7}
\end{equation}
and the period equals $2\pi V_e^{-1} \approx 6 \cdot 10^{-3} \omega_p^{-1}$ for $V_e = 1000 \omega_p$ which we will use as a benchmark value. The depth and length of oscillations are constant.

Using eq. (3.3) we find the off-diagonal neutrino potential in eq. (2.8):
\begin{equation}
    \psi_e \psi_\tau^* = \frac{1}{2} \sin 2\theta_m [-\cos 2\theta_m (1 - \cos \Delta_m t) + i \sin \Delta_m t]. \tag{3.8}
\end{equation}

\subsection*{3.2 Evolution of the probe particle. Neutrino potentials}

The solution for the flux neutrinos in eq. (3.4) and eq. (3.6) allows to explicitly compute the neutrino potentials in the equation for the probe particle. The integrals in (2.3), (2.6) and (2.8) are absent, and we have
\begin{align*}
    \dot{V}_\nu &= V_\nu \psi_e \psi_\tau^* = V_\nu \sqrt{P_{e\tau}(1 - P_{e\tau})}, \tag{3.9}
    \\
    \phi_B &= \text{Arg} \psi_e \psi_\tau^*, \\
    V_\nu &= V_\nu (1 - 2P_{e\tau}), \tag{3.10}
\end{align*}
and

\[ V_\nu^0 = \sqrt{2} G_F n_\nu \left[ 1 - \cos(\pi - \beta_p - \beta_b) \right]. \]

\( P_{e\tau} \) is given in eq. (3.4). For a given moment of time \( t \), the probe neutrino interacts with a flux neutrino which has travelled the distance

\[ l = A_\beta t, \]

from the production point, where

\[ A_\beta \equiv \frac{\sin \beta_p}{\sin \beta_b}. \]

(We assume \( v \approx c \)). Therefore, the phase acquired by the flux neutrino when it encounters the probe neutrino equals

\[ \phi_m = A_\beta \Delta m t. \]

This phase should be used in \( P_{e\tau} \) and the expressions for the potentials.

For the diagonal potential (2.15) we have

\[ V_r = V_e + V_\nu^0 (1 - 2P_{e\tau}) - c_2 \theta \omega_p + \dot{\phi}'. \]

Using eqs. (3.8) and (3.6) we obtain the phase of the neutrino potentials \( \phi_B \) in terms of phase of the flux neutrinos \( \phi_m \):

\[ \cos \phi_B = \frac{-\cos 2\theta_m (1 - \cos \phi_m)}{\sqrt{\cos^2 2\theta_m (1 - \cos \phi_m)^2 + \sin^2 \phi_m}} \]

\[ = \frac{-\cos 2\theta_m \sqrt{\cos^2 2\theta_m + \cot^2 \frac{1}{2} \phi_m}}{-\cos 2\theta_m \sin \frac{1}{2} \phi_m} \]

\[ = \frac{-\cos 2\theta_m |\sin \frac{1}{2} \phi_m|}{\sqrt{1 - P_{e\tau}}}. \] (3.10)

Eq. (3.10) can also be found from the geometric picture in figure 2.

Notice that for \( A_\beta = 1 \), the flavor evolution of the probe neutrino and the flux neutrinos are identical. Since flavor exchange in this case does not produce any physical effect, neither flux nor probe neutrinos change (see appendix A).

The phase of the off-diagonal element of \( H^{(p)} \) (2.12) can be found explicitly in terms of \( \phi_m \) using (3.10):

\[ \tan \phi' = \frac{V_\nu^0 \sin \phi_m}{-V_\nu^0 \cos 2\theta_m (1 - \cos \phi_m) + \Delta_m (\omega_p/\omega_k)}. \]

Then the derivative \( \dot{\phi}' \) equals

\[ \dot{\phi}' = \Delta_m A_\beta \frac{-\cos 2\theta_m (\cos \phi_m - 1) + \Delta_m \frac{\omega_p}{\omega_k} \cos \phi_m}{\left[ -\cos 2\theta_m (1 - \cos \phi_m) + \Delta_m \frac{\omega_p}{\omega_k} \right]^2 + \sin^2 \phi_m}. \] (3.11)

The potentials \( V_r \) and \( V' \) as functions of the neutrino propagation time are shown in figure 3 for \( A_\beta = 1.1001 \), and different values of \( \xi \) which was defined in eq. (2.5). The potentials have periodic dependencies on time. Furthermore, for \( \xi = 0.1 \), the time dependence can be described by a cosine. For larger \( \xi \) there is the deviation from the cosine dependence.
The neutrino polarisation vector is given by $\vec{P} = \psi \vec{\sigma} \psi^{\dagger}$, where $\vec{\sigma}$ is a 3-vector of Pauli matrices. The upper circle traces out the flavor evolution of $\psi$, while the lower circle gives the projection of the circle on the $xy$-plane relevant for $\phi_B$.

Figure 2. Relationship between $\phi_m$, $\theta_m$, and $\phi_B$. Eq. (3.10) is demonstrated in polarisation space.

Figure 3. The potentials $V_r$ and $V'$ from eq. (2.15) and eq. (2.11) as functions of the time. We use $\omega_k = \omega_p$ and $A_\beta = 1.1001$.

Although the sizes of $V_r$ and $V'$ are different, the relative amplitude of the time dependence is of order $\xi$ for both. That will be explained when we derive approximate expressions for $V_r$ and $V'$.

The exact numerical solution of eq. (2.1) with the Hamiltonian from eq. (2.14) with $V_r$ given in (3.9) and (3.11), and $V'$ in (2.11) is shown as the red, solid line in figure 4. We use the same value for $V_e$ as before and for all $\xi$, $A_\beta$ is fixed 1% below the value $A_\beta^{R} = (V_e + V_{\nu} - \cos 2\theta \omega_p) / \Delta m$ which we will show is the resonant value in section 4. According to figure 4,
Figure 4. Conversion probabilities as function of time for different $\xi$ close to the parametric resonance. The full solution of eq. (2.1) (solid red) uses eqs. (3.9) and (2.11), the approximation (dashed blue) uses eqs. (3.15) and (3.16), while the analytical solution (dash-dotted magenta) is given in eq. (4.11). The frequencies are related by $\omega_k = \omega_p$, while $A_B = 0.99A_B^F$.

The interaction with the neutrino flux leads to a moderate (factor of 100) enhancement of the conversion probability for the probe neutrino in the case of $\xi = 0.1$. The period of the fast oscillations is determined by $\Delta m \approx V_e$, while the long period is given by $V_\nu$. We give a detailed interpretation of these results using an approximate analytical study in section 4.
3.3 Approximations for the potentials and Hamiltonian

In the following, we present simplified Hamiltonians which reproduce the results of the full calculation to a good approximation and will allow us to find an analytic solution for the oscillation probability of the probe neutrino.

The first approximation is based on the smallness of the oscillation depth of the background neutrino (3.5). The same quantity gives deviation of $|\cos 2\theta_m|$ from 1. Indeed, according to (3.5) $P_{e\tau} \leq 10^{-7}$. Therefore we can neglect $P_{e\tau}$ in comparison to 1 and take $\cos 2\theta_m \approx -1$.

In this approximation, $\bar{V}_\nu$ (3.7) can be rewritten using (3.4) as

$$\bar{V}_\nu \approx V_0\sqrt{P_{e\tau}} = s_{2\theta} \xi \omega |\sin \frac{1}{2} \phi_m|,$$  \hspace{1cm} (3.12)

and from eq. (3.10) we obtain

$$\cos \phi_B \approx |\sin \frac{1}{2} \phi_m|.$$  \hspace{1cm}

The diagonal element of the Hamiltonian (3.9) becomes

$$V^r \approx V_e \left[ 1 + \xi - \frac{\omega_p c_{2\theta}}{V_e} + \xi A \frac{\omega_p}{\omega_k} \cos \phi_m - \xi \frac{(1 - \cos \phi_m)}{2} \right].$$  \hspace{1cm} (3.13)

The off-diagonal element $V'$ (2.11) can be written as

$$V' = s_{2\theta} \omega_p \sqrt{1 + 4 \xi \omega_k \sin^2 \frac{1}{2} \phi_m} \left[ 1 + \xi \omega_k \omega_p \xi \cos \phi_m \right].$$  \hspace{1cm} (3.14)

We find that the solution of the evolution equation with (3.14) and (3.13) practically coincides with the full solution in figure 4.

Notice that the approximation $P_{e\tau} \ll 1$ may not work if the neutrinos in the background are also subjected to the $\nu\nu$ interaction and their oscillations therefore are parametrically enhanced.

Another simplification can be obtained if the density of neutrinos is much smaller than density of electrons $\xi \ll 1$. In this case we have according to (3.12)

$$s_{2\theta} \omega_k \gg \bar{V}_\nu,$$

which means that the off-diagonal neutrino potential is suppressed by $\xi$ with respect to the vacuum term. Using eq. (3.14), we find

$$V' \approx s_{2\theta} \omega_p \left[ 1 + 2 \xi \frac{\omega_k}{\omega_p} \sin^2 \frac{1}{2} \phi_m \right].$$

It can be rewritten as

$$V' \approx s_{2\theta} \omega_p \left( 1 + \xi \frac{\omega_k}{\omega_p} - \frac{\omega_k}{\omega_p} \xi \cos \phi_m \right).$$  \hspace{1cm} (3.15)

For $V^r$ in eq. (3.13), we obtain

$$V^r \approx V_e \left( 1 + \xi - \frac{c_{2\theta} \omega_p}{V_e} + A \frac{\omega_k}{\omega_p} \xi \cos \phi_m \right).$$  \hspace{1cm} (3.16)
by neglecting the highest powers of $\xi$. Let us underline that the periodic term in $V^r$ is mainly due to $\dot{\phi}'$ (after we have neglected $P_{e\tau}$). Periodic contributions to the potentials are due to the $\nu\nu$ scattering and therefore they are proportional to $\xi$. In every place, $\cos \phi_m$ and $\xi$ enter together in the combination

$$B + \xi \cos \phi_m, \quad B = \mathcal{O}(1).$$

For small $\xi$ the periodic dependence in the potentials can be expanded in series of $\xi \cos \phi_m$, and to lowest order $V'$ and $V^r$ have linear dependencies on $\xi \cos \phi_m$. In units of $t\omega_p$, it equals $2\pi 10^{-3}/A_3$.

1. With an increase of $\xi$, the approximation breaks down: the lines deviate from the simple dependence on $\cos \phi_m$. At the same time, the period of oscillations determined by $A_3 \Delta_m \approx A_3 V_e$ does not change with $\xi$. In units of $t\omega_p$, it equals $2\pi 10^{-3}/A_3$. The Hamiltonian with $V^r$ in (3.16) and $V'$ in (3.15) can be presented as

$$H(p) = \frac{1}{2}\begin{pmatrix} d + h \cos \phi_m & g + f \cos \phi_m \\ g + f \cos \phi_m & -d - h \cos \phi_m \end{pmatrix}, \quad (3.17)$$

where

$$d \equiv V_e + V_\nu^0 - c_{2\theta} \omega_p, \quad h \equiv A_3 \frac{\omega_k}{\omega_p} V_\nu^0, \quad g \equiv s_{2\theta} (\omega_p + \xi \omega_k), \quad f \equiv -s_{2\theta} \omega_k \xi. \quad (3.18)$$

These quantities have the following hierarchy:

$$d \approx V_e, \quad h = \mathcal{O}(\xi V_e), \quad g = \mathcal{O}(s_{2\theta} \omega_p), \quad f = \mathcal{O}(s_{2\theta} \omega_p \xi), \quad d \gg h \gg g \gg f.$$  

Notice that the off-diagonal elements of $H(p)$ are suppressed with respect to the diagonal ones as

$$\frac{g}{d} \approx \frac{f}{h} \approx \frac{s_{2\theta} \omega_p}{V_e}.$$  

Furthermore, in each element of the Hamiltonian, the periodic terms are suppressed by $\xi$. In the limit $\xi \to 0$, $V^r$ and $V'$ are reduced to the standard expressions in matter. If higher orders of $\xi$ are included, higher powers of $\cos \phi_m$ appear, and we will obtain a series expansion in $(\xi \cos \phi_m)^k$.

4 Analytic solution of the equation

4.1 Solution for the resonant mode

The Hamiltonian in eq. (3.17) is of the type that can give rise to parametric resonances [26–30]. Close to the resonance, the corresponding evolution equation can be solved analytically. Performing a rotation of the fields $\psi = U'(\theta') \psi'$ by the angle

$$\sin 2\theta' = -\frac{g}{\sqrt{d^2 + g^2}} \approx -\frac{g}{d}, \quad (4.1)$$

we can diagonalise the constant part of the Hamiltonian (3.17), so that in the basis $\psi'$ it becomes

$$H' = \frac{\sqrt{d^2 + g^2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2\sqrt{d^2 + g^2}} \cos \phi_m \begin{pmatrix} -hd + fg & -hg + fd \\ -hg + fd & -hd - fg \end{pmatrix}.$$
Here we can neglect $g^2$ in comparison to $d^2$ and $(fg)$ with respect to $(hd)$ which are of the same order approximations as neglecting $P_{e\tau}$ in comparison to $1$. Then the Hamiltonian equals

$$H' \approx \frac{1}{2} \begin{pmatrix} d + h \cos \phi_m & f' \cos \phi_m \\ f' \cos \phi_m & -d - h \cos \phi_m \end{pmatrix},$$  \hspace{1cm} (4.2)

where

$$f' \equiv f - \frac{gh}{d}. \hspace{1cm} (4.3)$$

Both terms in $f'$ are of the same order: $f \sim gh/d \sim s^2 \theta \omega_p \xi$. It can be written as

$$f' = -s^2 \theta \omega_k \xi \left(1 + A_\beta(\xi) \frac{1}{1 + \xi}\right) \approx -2s^2 \theta \omega_k \xi, \hspace{1cm} (4.4)$$

where we used $A_\beta = A^{R}_\beta$ in the second equality.

Next, we will average the periodic dependence in the diagonal elements. Indeed, the effect of the potentials’ variation on the oscillation probability is related to variations of the mixing angle. According to the Hamiltonian (4.2)

$$\tan 2\theta^p_m = \frac{f' \cos \phi_m}{d + h \cos \phi_m} \approx \frac{f'}{d} \left(1 - \frac{h}{d} \cos \phi_m\right) \cos \phi_m.$$  

The effect of the periodic term in the diagonal elements given by the second term in the last expression is suppressed by $h/d \sim \xi$. Thus, the variations of the diagonal elements produce small depth modulations of the main mode with higher frequencies.

After averaging over the phase in the diagonal elements, we obtain

$$H' \approx \frac{1}{2} \begin{pmatrix} d & f' \cos \phi_m \\ f' \cos \phi_m & -d \end{pmatrix},$$

Using $\cos \phi_m = \frac{1}{2}(e^{i\phi_m} + e^{-i\phi_m})$, we can split the Hamiltonian in two parts:

$$H = H_0 + \Delta H,$$

where

$$H_0 = \frac{1}{4} \begin{pmatrix} 2d & f'e^{-i\phi_m} \\ f'e^{i\phi_m} & -2d \end{pmatrix},$$  \hspace{1cm} (4.5)

and

$$\Delta H = \frac{f'}{4} \begin{pmatrix} 0 & e^{i\phi_m} \\ e^{-i\phi_m} & 0 \end{pmatrix}.$$  

(Similarly one can consider another splitting when the phases in $H_0$ and $\Delta H$ switch signs.) This splitting makes sense since only one frequency mode is enhanced, either $A_\beta \Delta_m$ or $-A_\beta \Delta_m$. If resonance takes place for $A_\beta \Delta_m$, then the part of the Hamiltonian with $-A_\beta \Delta_m$ can be considered as a correction.

Let us make the transformation of the fields

$$\psi' = U_\alpha \psi_\alpha, \hspace{1cm} U_\alpha = \text{diag} \left(e^{-i\frac{1}{2} \phi_m}, \ e^{i\frac{1}{2} \phi_m}\right)$$

that removes the phases in $H_0$ (4.5). Then for the transformed fields $\psi_\alpha$, the Hamiltonian can be written as

$$H^5 = H^{a}_0 + \Delta H^5,$$  \hspace{1cm} (4.6)
where
\[ H_0^{\alpha} = \frac{1}{2} \left( \frac{d - A_\beta \Delta_m}{2} f' - d + A_\beta \Delta_m \right) \]
(4.7)

(here we included the terms with \( A_\beta \Delta_m \) which follow from differentiation of \( U_\alpha \)) and
\[ \Delta H^{\alpha} = \frac{f'}{4} \begin{pmatrix} 0 & e^{i2\phi_m} \\ e^{-i2\phi_m} & 0 \end{pmatrix} \]
(4.8)

Notice that the phase is doubled in \( \Delta H^{\alpha} \).

Let us first find a solution, \( S_0 \), of the evolution equation with the Hamiltonian \( H_0^{\alpha} \):
\[ i \dot{S}_0 = H_0^{\alpha} S_0 \]
(4.9)

thus neglecting \( \Delta H^{\alpha} \). All the parameters in \( H_0^{\alpha} \) (4.7) are constants. Therefore the solution to eq. (4.9) is the usual oscillation solution with a mixing angle given by
\[ \sin 2\theta^{\alpha p} m = \frac{f'}{2\gamma}, \quad \cos 2\theta^{\alpha p} m = -\frac{d - A_\beta \Delta_m}{\gamma} \]
where
\[ \gamma \equiv \sqrt{(d - A_\beta \Delta_m)^2 + f'^2/4} \]
is the level splitting. The \( S_0 \) matrix can be written as
\[ S_0 = U^{p}_m S^{\text{diag}} U^{p}_{m} \]
\[ S^{\text{diag}} = \text{diag}(e^{i\phi^{p}_m}, e^{-i\phi^{p}_m}) \]
with the phase
\[ \phi^{p}_m = \frac{1}{2} \gamma t, \]
and \( U^{p}_m \) being a rotation by the angle \( \theta^{\alpha p} m \).

Explicitly,
\[ S_0 = \cos \phi^{p}_m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin \phi^{p}_m \begin{pmatrix} \cos 2\theta^{\alpha p} m & -\sin 2\theta^{\alpha p} m \\ -\sin 2\theta^{\alpha p} m & \cos 2\theta^{\alpha p} m \end{pmatrix}. \]
(4.10)

The solution in the flavor basis is then
\[ S = U' U_\alpha S_0 U_\alpha^t U'^t. \]
The matrix \( U' \) of rotation by the angle \( \theta' \) (4.1) can be approximated as
\[ U' \approx \begin{pmatrix} 1 & -\frac{g}{2d} \\ \frac{g}{2d} & 1 \end{pmatrix} \]
resulting in
\[ S = \cos \phi^{p}_m I + i \sin \phi^{p}_m \begin{pmatrix} \cos 2\theta^{\alpha p} m + \frac{g}{2} \cos \phi_m \sin 2\theta^{\alpha p} m & -e^{-i\phi_m} \sin 2\theta^{\alpha p} m + \frac{g}{2} \cos 2\theta^{\alpha p} m \\ -e^{i\phi_m} \sin 2\theta^{\alpha p} m + \frac{g}{2} \cos 2\theta^{\alpha p} m & \cos 2\theta^{\alpha p} m + \frac{g}{2} \cos \phi_m \sin 2\theta^{\alpha p} m \end{pmatrix}. \]

Consequently, the transition probability is
\[ P^p_{e\tau} = |S_{12}|^2 = \frac{1}{\gamma^2} \left[ \frac{f'^2}{4} + \frac{g^2}{4d^2} (d - A_\beta \Delta_m)^2 + \frac{g}{d} f'(d - A_\beta \Delta_m) \cos \phi_m \right] \sin^2 \frac{1}{2} \gamma t. \]
(4.11)
The probability averaged over the fast modulations equals

\[ P_{ee}^p = |S_{12}|^2 = \frac{1}{\gamma^2} \left[ \frac{f'^2}{4} + \frac{g^2}{d^2} (d - A_\beta \Delta m)^2 \right] \sin^2 \frac{1}{2} \gamma t. \]

The expression in eq. (4.11) is used for obtaining the dash-dotted magenta curve in figure 4. So, eq. (4.11) provides a good approximation when \( \xi \ll 1 \), and in order to improve it further, we need to include \( \Delta H^\alpha \) of eq. (4.8) (see appendix B).

According to eq. (4.11), the parametric resonance condition is

\[ d = A_\beta \Delta m. \] (4.12)

Recall that this condition is obtained after averaging of the diagonal elements of the Hamiltonian in the linear approximation: \( d = \langle V \rangle \). Under this condition the oscillations in eq. (4.11) proceed with maximal depth independently of \( f' \), while the oscillation length is determined by \( f'/2 \):

\[ l_m = \frac{4\pi}{f'} \approx \frac{2\pi}{s_{2\theta}\omega p} = \frac{1}{s_{2\theta}\xi} l_\nu. \]

Here \( l_\nu \) is the vacuum oscillation length.

The resonance condition (4.12) does not depend on \( t \), and modulations are absent. It can be written explicitly as

\[ V_e + V_\nu^0 - c_2 \theta \omega_\nu = A_\beta (V_e - c_2 \theta \omega_k). \]

The difference between the left and right hand sides of this equation is the factor \( A_\beta \) and the absence of \( V_\nu^0 \) on the right hand side. The latter is only the case since in our model, background neutrinos have no \( \nu \nu \) interactions. If \( V_\nu^0 \) would appear on the right hand side, the resonance condition would be reduced to \( A_\beta = 1 \) for any density and neutrino energy. At this condition, however, the neutrino background effect disappears, as we discussed in section 3. For \( V_\nu^0 = 0 \) it is reduced to the MSW resonance condition \( V_e - c_2 \theta \omega_\nu = 0 \).

With an analytical description of the parametric resonance, the results in figure 4 can now be analysed. The value

\[ A^R_\beta = \frac{d}{\Delta m} = \frac{V_e(1 + \xi)}{\Delta m} \approx (1 + \xi), \] (4.13)

satisfies the resonance condition in eq. (4.12). Changing \( \xi \) means further departure from the resonance condition which would produce the main effect. Therefore, for the computations in figure 4, we change \( A_\beta \) simultaneously with \( \xi \) in such a way that the departure from resonance remain 1%. That is, \( A_\beta \) increases with \( \xi \) according to (4.13).

Let us introduce the deviation from resonance \( D \)

\[ D \equiv 1 - A_\beta \frac{\Delta m}{d}, \]

so that

\[ d - A_\beta \Delta m = dD \]

(in our computations \( D = 0.01 \)). From this equation we have

\[ A_\beta = (1 - D) \frac{d}{\Delta m} \approx (1 - D)(1 + \xi) \]

since \( \Delta m \approx V_e \).
For $\xi = 0.1$ there is a good agreement between the results of computations with the exact Hamiltonian and the approximation in eq. (3.15) and eq. (3.16): the depth and period of the fast modulations are the same. The average value of the probability computed with approximate Hamiltonian is about 10% larger (as expected). With the increase of $\xi$, the approximation breaks down: for $\xi = 0.5$, and $\xi = 1$ the average probability is 45%, 80% larger correspondingly. So, the deviations approximately increase linearly with $\xi$.

In terms of $D$, the frequency of the parametric oscillations squared equals

$$\gamma^2 = d^2 D^2 + \frac{f'^2}{4}, \quad (4.14)$$

and the depth of parametric oscillations (prefactor in (4.11)) averaged over fast modulations can be written as

$$P_{e\tau}^\text{max} = \frac{f'^2 + 4g^2 D^2}{f'^2 + 4d^2 D^2}. \quad (4.15)$$

Using expressions for the parameters in eqs. (4.3) and (3.18) and taking for simplicity $\omega_k = \omega_p$, we obtain from (4.15)

$$P_{e\tau}^\text{max} = \frac{\xi^2 (1 + A_\beta)^2 + 4D^2 (1 + \xi)^2}{\xi^2 (1 + A_\beta)^2 + 4(V_e/s_{2\theta}\omega_p)^2 D^2 (1 + \xi)^2}. \quad (4.16)$$

The second term in the nominator and the first term in the denominator can be neglected, so that

$$P_{e\tau}^\text{max} \approx \frac{s_{2\theta} \omega_p^2 \xi^2 [2 - D + (1 - D)\xi]^2}{4D^2 V_e^2 (1 + \xi)^2}. \quad (4.17)$$

For selected values of parameters, we have

$$P_{e\tau}^\text{max} \approx 2.2 \times 10^{-4} \frac{\xi^2 (2.01 + \xi)^2}{(1 + \xi)^2}.$$

Thus, the depth increases with $\xi$ (almost as $\xi^2$ for small $\xi$).

In the expression for the frequency (4.14), the first term dominates

$$\gamma \approx dD = DV_e (1 + \xi). \quad (4.18)$$

Correspondingly, the period of parametric oscillations

$$T = \frac{2\pi}{\gamma} = \frac{2\pi}{1 + \xi} \frac{1}{DV_e}$$

decreases with the increase of $\xi$ for fixed $D$.

For the results of the approximate computations (blue, dashed lines) there is a perfect agreement with the results of the formulas for the depth (4.17) and frequency (4.18). These approximations are in a good agreement with the exact computations for $\xi = 0.1$. However, for larger $\xi = 0.5$ and $\xi = 1.0$, the approximate results differ substantially from the exact result.

The relative depth of the high frequency modulations is given by

$$\frac{4g d - A_\beta \Delta_m}{f'} \approx 4D \frac{1 + \xi}{\xi (2 + \xi)}.$$
according to (4.11), and it decreases with the increase of $\xi$. These fast modulations have a larger depth in the approximate analytic expression than in the exact solution. The non-resonance contribution to the Hamiltonian due to $\Delta H$ also leads to high frequency modulations, and the two contributions can cancel each other. Notice that this cancellation can not be inferred immediately from the results in appendix B which are only valid exactly in the resonance. In resonance, the modulations of the approximate probability (4.11) disappear.

### 4.2 The parametric resonance

Let us study the parametric resonance in more details. In figure 5 the full solution and the approximate solutions are shown together at resonance and for values of $\omega_k/\omega_p$ close to resonance. The depth of oscillations and the period satisfy the standard relation: $P_{\text{er}}^{\text{max}}/l_m^2 \approx \text{constant}$.

In general, the resonance condition can be formulated using physical variables: the period of rotation of the neutrino polarisation vector $T_p$, and the period of change of mixing angle (which determines the axis of precession), $T_\theta$. The former is determined by the condition in eq. (2.17), and the latter (the period of the axis motion) is given by

$$T_\theta = \frac{2\pi}{A_\beta \Delta m}$$

in our example. Therefore the exact resonance condition (2.18) becomes

$$A_\beta \Delta m = \sqrt{V^2 + V'^2} \approx V^r,$$

assuming that $V_\nu \gg V_0, \omega_p$, which coincides with eq. (4.12).

The dependence of the parametric oscillation depth on $\omega_k$ has a resonance character. In figure 6 we show $P_{\text{er}}^{\text{max}}$ as a function of $\omega_k/\omega_p$ for all other parameters being fixed in such a way that the resonance satisfies is at $\omega_k/\omega_p = 1$.

Let us consider the depth of parametric oscillations averaged over fast modulations. Then the shape of the resonance can be obtained from our analytical results in eq. (4.17). In resonance, $D = 0$, the second term in both nominator and denominator of (4.16) vanish, and the conversion probability equals 1. Close to resonance, the second term in the nominator of eq. (4.17) is suppressed with respect to the first term by $(\omega_p/V_\nu)^2$ and can be safely neglected. Consequently, the depth of oscillations can be written as

$$P_{\text{er}}^{\text{max}} = \left[ 1 + 4 \frac{d^2}{f^2} \left( 1 - A_\beta \frac{\Delta m}{d} \right)^2 \right]^{-1}.$$  \hspace{1cm} (4.19)

According to this expression, the resonance width at half the height (the expression above equals 1/2) is determined by the condition

$$2 \frac{d}{f^2} \left| 1 - A_\beta \frac{\Delta m}{d} \right| = 1.$$  \hspace{1cm} (4.20)

The resonance value of $\Delta_m$ equals

$$\Delta_m^R = \frac{d}{A_\beta},$$

and in turn, $\Delta_m^R$ determines the resonant frequency $\omega_k^R$. In terms of $\Delta_m^R$ we have

$$D = 1 - A_\beta \frac{\Delta m}{d} = 1 - \frac{\Delta_m}{\Delta_m^R}.$$  \hspace{1cm} (4.21)
Figure 5. Parametric oscillations: the dependence of the probability $P_{\tau\tau}$ on time of evolution for different values of $\omega_k/\omega_p$. The upper panel is at resonance, while the middle and lower panels are detuned as indicated. The other parameters are $\xi = 0.1$, and $\Delta_\beta = 1.1001$. The lines are calculated using the same equations as in figure 4.

Using the approximate expression

$$\Delta_m \approx V_e \left(1 - c_{2\phi} \frac{\omega_k}{V_e}\right),$$

we obtain from eq. (4.21)

$$1 - \frac{\Delta_m}{\Delta_m^R} = c_{2\phi} \frac{\omega_k - \omega_k^R}{V_e}.$$

(4.22)
Figure 6. Parametric resonance: dependence of the depth of parametric oscillations on frequency of the background neutrinos for the exact numerical solution (solid line) and the analytic solution (dashed line). We use $V_\nu^0 = 100\omega_p$ and $A_\beta = 1.1001$.

Insertion of (4.22) in (4.20) gives

$$2 \frac{d}{f^{c_{28}}} \frac{[\omega_k - \omega_k^R]}{V_e} = 1.$$ 

Using the expressions for $d$ and $f'$ in the lowest order in $\xi$, we obtain

$$|\omega_k - \omega_k^R| \approx \frac{1}{2} f',$$

i.e., the width is determined by the off-diagonal element of the Hamiltonian. The relative width of the resonance defined as

$$\frac{\Gamma_\omega}{\omega_k^R} \equiv 2 \frac{|\omega_k - \omega_k^R|}{\omega_k^R} = s_{28}\xi(1 + A_\beta). \quad (4.23)$$

For the values of parameters we use, this gives $\Gamma_\omega/\omega_k^R = 0.065$ in very good agreement with the result of figure 6. The width increases with $\xi$, so that for large $\xi$ one expects strong transformations in a wider energy range.

### 4.3 Inverted mass ordering and a probe antineutrino

A change of mass ordering from the normal to inverted one means $\omega \rightarrow -\omega$. For the normal ordering, the difference of the eigenstates is $H_{2m} - H_{1m} = \Delta_m (|\omega|)$, where $\Delta_m > 0$ is defined in (3.2).\footnote{Recall that the levels are enumerated by according to the flavor content in vacuum.} Therefore $\phi_m = (H_{2m} - H_{1m})t = \Delta_m t$, $\cos 2\theta_m = (\cos 2\theta - V_e)/\Delta_m$, and $\sin 2\theta_m = \sin 2\theta\omega/\Delta_m$. For $V_e \gg \omega$, the mixing parameters are $\cos 2\theta \approx -1$ and $\sin 2\theta_m \approx 0$ ($\sin 2\theta_m > 0$). In the case of inverted ordering, $H_{2m} - H_{1m} = -\Delta_m (-|\omega|)$,
so that \( \phi_m = -A_\beta \Delta_m t \). Now \( \cos 2\theta_m = (\cos 2\theta |\omega| + V_e)/\Delta_m \), and \( \sin 2\theta_m = \sin 2\theta |\omega|/\Delta_m \). Consequently, the resonance condition for the inverted ordering is satisfied for \( e^{i\phi_m} \) and not \( e^{-i\phi_m} \) in the decomposition of \( \cos \phi_m \) in eq. (4.5).

Let us consider a probe antineutrino. The corresponding Hamiltonian, \( \bar{H} \), can be obtained from eq. (2.2) by taking the complex conjugate of the phase factors and changing the sign in front of \( V_e, V_0^0 \), and \( V_\nu \). Consequently, the resonance condition becomes \( A_\beta \Delta_m = -d \).

To satisfy this equality, the opposite sign of \( \phi_m \) must be chosen in \( H_0 \) in eq. (4.5), like in the case of IO. The transition probability for a probe antineutrino is very similar to the transition probability for a probe neutrino for the simple model that has been considered here. Therefore, the analytical approximation in eq. (4.11) is also valid for \( \bar{\nu} \) as a probe particle if \( d = -V_e - V_0^0 + c_{2\theta} \), \( f = s_{2\theta}\omega k \xi \), and \( g = s_{2\theta}(\omega_p - \xi \omega_k) \) are used, and \( (d - A_\beta \Delta_m) \) is replaced by \( (d + A_\beta \Delta_m) \). This means that in the same background, a probe \( \nu \) and a probe \( \bar{\nu} \) will evolve in almost the same way. They will both have a parametric resonance and almost the same transition probabilities. This can potentially reproduce the bi-polar oscillations.

5 Integrations

The results obtained in sections 3 and 4 correspond to a neutrino flux with fixed energy, angle and production point. Let us perform the integration of potentials over energies and production points (see general formulas (2.3) and (2.8)). This integration does not change the dynamics of propagation. In the present model, the strong transitions steam entirely from the periodicity of the neutrino background flavor, and therefore the integration which leads to averaging of phases will suppress the flavor transformations.

5.1 Integration over production point

Integration over the production point is due to the finite width of the neutrino sphere. Notice that usually the width of the neutrino sphere is ignored when discussing collective oscillations in supernovae, although the effects that can arise due to different neutrino spheres for \( \nu_e, \bar{\nu}_e \) and \( \nu_x \) have recently been considered (see e.g. [11–14, 31]). The reason for ignoring the width is that the large density of electrons keeps the neutrino flavor frozen well beyond the neutrino sphere.

The Hamiltonian in eq. (3.17) depends on the production point through the phase \( \phi_m \) and the distribution of neutrino sources, \( n_\nu(l) \). Of these two, the dependence on \( \phi_m \) is most important since strong flavor conversion mainly arises from a parametric resonance where the frequency \( \dot{\phi}_m \) is in resonance with the oscillation frequency of the probe neutrino. If sources of neutrinos are distributed in the interval \( r \) with density \( dn_\nu/dl \), averaging of oscillatory terms is given by the integral

\[
\langle n_\nu \cos \phi_m \rangle = \int_{l-r}^{l} dl' \frac{dn_\nu(l')}{dl'} \cos(\Delta_m A_\beta l').
\]

If the neutrino sources are uniformly distributed in \( r \), so that \( dn_\nu/dl = n_\nu/r \), the integral in eq. (5.1) can be computed:

\[
\langle n_\nu \cos \phi_m \rangle = \frac{n_\nu}{r} \int_{l-r}^{l} dl' \cos \Delta_m A_\beta l' = \frac{n_\nu}{r} \frac{1}{\Delta_m A_\beta} [\sin \Delta_m A_\beta l - \sin \Delta_m A_\beta (l - r)].
\]
This shows that the oscillatory term is suppressed by a factor
\[
2 \frac{1}{r \Delta m A_\beta} \approx \frac{1}{V e A_\beta r}. \quad (5.3)
\]
A typical neutrino sphere has a radius \( \sim 10 \text{ km} \), and for a simple model of the density and temperature profile during the accretion phase [32], the width can be estimated to be \( \sim 10\% \) or \( \sim 1 \text{ km} \). On the other hand, for densities close to the neutrino sphere
\[
\frac{1}{\Delta m A_\beta} \approx \frac{1}{V e A_\beta} \sim (10^{-3} - 10^{-2}) \text{ cm}. 
\]
Therefore the suppression factor (5.3) equals \( \sim (10^{-8} - 10^{-6}) \), which means that the integration washes out the oscillatory terms of the Hamiltonian. Actually, the parametric resonance is not removed by the integration in eq. (5.2), but the conversion scale increases by at least a factor of \( 10^6 \). Without the suppression the conversion scale equals \( \gamma \sim 10^{-2} \text{ km}^{-1} \). With the suppression, the conversion scale becomes \( \sim 10^4 \text{ km} \), i.e., it extends far beyond the dense part of a supernova.

If the periodic terms vanish, the rest of the Hamiltonian in (3.17), given by \( d \) and \( g \), coincides with the standard Hamiltonian in matter. The only difference is a small constant correction due to \( V_0^0 \). This leads to oscillations with very small depth suppressed by \( s_{2\theta} \phi / (V e + V_\nu) \).

5.2 Integration over neutrino energy

The integration over energy (frequency) produces a smaller suppression effect since the energy enters \( \Delta m \) which is relevant for the phase averaging with a suppression. Indeed, expanding the phase we have
\[
\phi_m \approx \phi_k = (V e - c_{2\theta} \omega_k) t, \quad (5.4)
\]
and the second term in brackets is three orders of magnitude smaller than the first one for the parameters that we consider. The effect of integration over \( \omega \) on the time dependence of \( P e \) is shown in figure 7. For fixed \( \omega_k / \omega_p = 1 \), the maximal \( P e \) is 1, while it is reduced down to \( 2 \times 10^{-3} \) for the box-type spectrum.

To understand the effect let us consider the averaging of the off-diagonal element in the Hamiltonian (3.17) in the interval \( (\omega_k - \Delta \omega) - (\omega_k + \Delta \omega) \):
\[
\langle \frac{1}{2} f \cos \phi_m \rangle = \frac{s_{2\theta} \xi}{4 \Delta \omega} \int_{\omega_k - \Delta \omega}^{\omega_k + \Delta \omega} d\omega_k' \omega_k' \cos \phi_k(\omega_k'), \quad (5.5)
\]
where \( \phi_k \) is defined in eq. (5.4). The integration in (5.5) gives
\[
\langle \frac{1}{2} f \cos \phi_m \rangle = \frac{s_{2\theta} \xi}{2 \Delta \omega (c_{2\theta} t)^2} [c_{2\theta} t \omega_k \sin \Delta \phi \cos \phi_k + (\sin \Delta \phi - \Delta \phi \cos \Delta \phi) \sin \phi_k], \quad (5.6)
\]
where \( \Delta \phi \equiv c_{2\theta} t \Delta \omega \) is the difference of phases due to the difference of frequencies.

The integration over frequency (energy) leads to additional time dependence of the oscillatory terms. At the condition \( \Delta \phi \ll 1 \) (early times), which can be written as
\[
t \ll t_c \equiv \frac{1}{c_{2\theta} \Delta \omega}, \quad (5.7)
\]
the expression (5.6) is reduced to the original one: \( \frac{1}{2} s_{2\theta} \xi \cos \phi_k \) with a single central frequency.
Figure 7. The effect of integration over the energy spectrum on $P_{e\tau}$ as function of evolution time. The conversion probability $P_{e\tau}$ is shown for different energies, $\omega_k$. The frequency $\omega_k = \omega_p$ corresponds to resonance, so that $P_{e\tau}$ eventually reaches 1, while $\omega_k = 0.5\omega_p$ and $1.5\omega_p$ are off resonance. The box-like neutrino spectrum extends from $0.5\omega_p$ to $1.5\omega_p$. The analytic approximation for $\omega_k = \omega_p$ from eq. (4.11) is also shown. The other parameters we use are $V_\nu^0 = 100\omega_p$ and $A_\beta = 1.1001$.

For later times, $t > t_c$, the oscillatory terms are suppressed as

$$\langle \frac{1}{2} f \cos \phi_m \rangle = \frac{1}{2} s_{2\theta} \xi \omega_k \cos \phi_k \frac{\sin \Delta \phi}{c_{2\theta} \Delta \omega t_c}.$$

Thus, the transition probability is parametrically enhanced at $t < t_c$ in the way we discussed above, whereas at $t > t_c$ enhancement is terminated due to suppression of the oscillatory terms, and the probability converges to a constant.

One can estimate this asymptotic probability in the following way. According to (4.11) the transition probability in the moment of time $t_c$ at the resonance frequency, $\omega_k^R$, equals:

$$P^p_{e\tau} = \sin^2 \frac{1}{2} \gamma(\omega_k^R)t_c \approx \left[ \frac{1}{2} \gamma(\omega_k^R)t_c \right]^2 \approx \left[ \frac{s_{2\theta} \xi \omega_k \gamma}{2c_{2\theta} \Delta \omega} \right]^2,$$

(5.8)

where we used that at resonance $\gamma \approx f'/2 \approx 2s_{2\theta} \xi \omega_k^R$. The probability increases with $\xi$ and with the decrease of the integration interval $\Delta \omega$.

Integration over energy also leads to a shift of the effective resonance frequency to larger values due to the presence of $\omega_k^R$ under the integral in (5.5). In our computations $\omega_k^R/\Delta \omega \sim 2$. Therefore, the probability in eq. (5.8) becomes

$$P^p_{e\tau} = \left[ \frac{s_{2\theta} \xi}{2c_{2\theta}} \right]^2 \sim 10^{-3}$$

in good agreement with the upper panel in figure 7. This consideration means that one should consider another regime when $\xi$ is not small: $\xi \geq 1$ in order to have strong transitions in the present model.
Notice that the integration over energy is equivalent to the effect of loss of coherence due to the spatial separation of neutrino eigenstates wave packets. Indeed, complete separation occurs during the time
\[ t_{\text{coh}} = \frac{l_\nu}{2\pi} \frac{E}{\Delta E}. \]

On the other hand using
\[ \Delta \omega = \frac{\Delta m^2}{2E^2} \Delta E = \frac{2\pi}{l_\nu} \frac{\Delta E}{E}, \]
we find from (5.7) that \( t_c = t_{\text{coh}} \).

6 Varying densities

6.1 Adiabaticity and asymptotic values of \( P_{\tau \tau} \)

Let us first consider the change of the density of electrons \( V_e(r) \) on the neutrino oscillations. The typical scale of density change in a SN \( r_d \) is much larger than the oscillation length:
\[ r_d = V_e \left( \frac{dV_e}{dr} \right)^{-1} \sim r \gg \frac{1}{V_e}. \]

Therefore the change of density is adiabatic, and we can use the adiabatic approximation result for \( P_{\tau \tau} \) in the formulas of the previous sections:
\[ P_{\tau \tau} = \frac{1}{2} \left[ 1 - \cos 2\theta_m(t_0) \cos 2\theta_m(t) - \sin 2\theta_m(t_0) \sin 2\theta_m(t) \cos \phi_m \right], \]
where \( \theta_m(t_0) \) and \( \theta_m(t) \) are the values of the mixing angles in the production moment \( t_0 \) and in a given moment \( t \), and
\[ \phi_m = \int_{t_0}^{t} dt' \Delta m(t'). \]

In the lowest order approximation for \( \sin 2\theta_m \), \( \sin 2\theta_m \approx \sin 2\theta \omega/V_e \), we find approximately:
\[ P_{\tau \tau} = \frac{1}{4} \sin^2 2\theta \omega^2 \left[ \frac{1}{V_e^2(t_0)} + \frac{1}{V_e^2(t)} - \frac{2}{V_e(t_0)V_e(t)} \cos \phi_m(t) \right]. \tag{6.1} \]

The background neutrinos are only subject to an electron background, so they will follow eq. (6.1). The resonance value of the potential is \( V_e^R = c_2g\omega_k \) and strong transformations occur when the final value of the potential \( V_e(t) \ll V_e^R \), i.e. at very large distances. In contrast, strong transformations of the probe neutrinos due to the parametric oscillations can occur at much smaller distances, determined by \( V_\nu \gg c_2g\omega_k \). In the case of varying density, however, the conditions for parametric enhancement can be destroyed unless the density changes slowly enough.

For the probe neutrinos, the parametric resonance condition (4.12) can be satisfied in a certain layer of a medium with varying density. It can be rewritten as
\[ \xi \approx A_\beta - 1. \tag{6.2} \]
So, it depends on the ratio of the potentials and essentially does not depend on the vacuum term. For a fixed \( A_\beta \), eq. (6.2) determines the resonance value \( \xi_R \). The position of resonance is determined by \( V_e \approx V_\nu/(A_\beta - 1) \) and it is much earlier (at higher densities) than the MSW
resonance which occur at $\nu_e \approx c^2 \theta \omega$. If the densities (potentials) change slowly enough, crossing the resonance can lead to strong flavor transformations.

We can generalise the usual MSW adiabatic condition to the case of parametric oscillations since the Hamiltonian (4.7) essentially coincides with the usual Hamiltonian. The difference is that the off-diagonal elements $\frac{1}{2} f^I \approx s_{2\theta} \omega \xi$ (4.4) depend on the potential and that the diagonal elements depend on the densities giving a resonance at (6.2).

Let us compute the adiabaticity parameter in resonance which is given by the ratio of the spatial (evolution time) width of the resonance layer, $\Delta t_R$, and the oscillation length in the resonance $l_m^R$ [33]:

$$\kappa_R = \frac{\Delta t_R}{l_m^R}. \quad (6.3)$$

Then, adiabaticity is satisfied if $\kappa_R > 1$.

Using eq. (4.20), we obtain the width of resonance in the $\xi$, $\Delta \xi$:

$$\Delta \xi = \frac{f^I}{V_e^R} = \frac{2 s_{2\theta} \omega \xi}{V_e^R}. \quad (6.4)$$

Notice that in contrast to the width in energy, here the relative width is very small: $\Delta \xi/\xi \approx 10^{-3}$ for our benchmark values of parameters.

We assume that $\xi$ has a power-law dependence on distance (evolution time)

$$\xi \propto t^\eta, \quad \dot{\xi} = \eta \xi / t_R,$$

where $t_R$ determines the position of the resonance layer. Using this and eq. (6.4) we find the spatial width of the resonance layer:

$$\Delta t_R = \Delta \xi \frac{\dot{\xi}}{\xi} = \frac{2 s_{2\theta} \omega k_{t_R}}{\eta V_e^R}. \quad (6.5)$$

Taking the position of the resonance layer to be $t_R \sim 0.5 l_m^R$, we obtain

$$\kappa_R = \frac{s_{2\theta} \omega k}{\eta V_e^R} \approx 3 \cdot 10^{-4}$$

by plugging eq. (6.5) into eq. (6.3). Thus, for the parameter values that we use and $\eta = O(1)$, the adiabaticity is strongly broken. Adiabatic conversion would imply $\eta \approx 3 \cdot 10^{-4}$.

In this case one expects the following behaviour of the transition probability: far from the parametric resonance layer, the adiabaticity is satisfied or weakly broken, so that $P_{\nu_e}^\tau$ is parametrically enhanced in the way we discussed before. As the resonance (which is very narrow) is approached, the adiabaticity is broken and as it happens in the usual MSW case, $P_{\nu_e}^\tau$ stops to increase and approaches some asymptotic value. In fact, $\kappa_R$ gives an idea about the size of the asymptotic probability: $P_{\nu_e}^\tau \sim \kappa_R$.

The asymptotic value, $P_{\nu_e}^{\tau_{\text{ass}}}$, can be estimated from the condition that in the spatial region $\Delta t$, where $P_{\nu_e}^{\text{max}}$ (around the resonance) is bigger than $P_{\nu_e}^{\tau_{\text{ass}}}$, one oscillation length is obtained:

$$\Delta t = l_m. \quad (6.6)$$

Here $l_m$ is some effective value of the oscillation in the interval $\Delta x$. Recall that $l_m$ is small at the edge of the interval but quickly increases towards the centre of the resonance. Using the expression for $P_{\nu_e}^{\text{max}}$ in eq. (4.19), we find that the region of $\xi$ where $P_{\nu_e}^{\text{max}} > P_{\nu_e}^{\tau_{\text{ass}}}$ is given by

$$\Delta \xi_P = 2 |\xi - \xi_R| = \frac{|f^I|}{V_e^R \sqrt{P_{\nu_e}^{\tau_{\text{ass}}}}}, \quad (6.7)$$
under the assumption that $P_{\epsilon\tau} \ll 1$. Then the corresponding spatial region equals

$$
\Delta t = \frac{\Delta \xi_P}{\xi} = \frac{\Delta \xi_P t}{\eta \xi}, \quad (6.8)
$$

where again we assumed the power law for $\xi(t)$. The oscillation length at the border of the interval $\Delta \xi(t)$ is given by $2\pi/\gamma$. Since the interval is much bigger than the width of resonance, we can neglect $f'$ in $\gamma$: $\gamma \approx |d - A_{\beta} \Delta m| \approx V_e |\xi - \xi_R| = \frac{1}{2} V_e \Delta \xi_P$. Thus, the effective oscillation length equals

$$
l_m = \frac{2\pi}{\gamma}. \quad (6.9)
$$

Inserting (6.9) and (6.8) into our condition (6.6), we obtain

$$
\frac{(\Delta \xi_P)^2 t}{\eta \xi} = \frac{4\pi}{V_e}. \quad (6.10)
$$

Inserting here eq. (6.7) for $\Delta \xi_P$ we obtain

$$
P_{\epsilon\tau}^{\text{ass}} = \frac{f'^2 t}{4\pi \eta V_e},
$$

and finally, inserting $f' \approx 2s_{2\beta} \omega_p \xi$, we have

$$
P_{\epsilon\tau}^{\text{ass}} = \frac{s_{2\beta}^2 \omega_p^2 \xi t_R}{\pi \eta V_e} = \frac{s_{2\beta}^2 \omega_p^2}{\pi \eta} \frac{\omega_p t_R}{V_e}. \quad (6.10)
$$

For our benchmarks parameters and $\eta = 1$, we have from eq. (6.10)

$$
P_{\epsilon\tau}^{\text{ass}} = 3 \cdot 10^{-4}, \quad (6.11)
$$

and the effective oscillation length in eq. (6.9) is expected to be

$$
l_m \omega_p \approx 4. \quad (6.12)
$$

Furthermore, outside the resonance not only the gradient of $\xi$, but also the gradients of potentials separately determine the adiabaticity. Since the potentials have larger gradients, this suppresses the transition further.

### 6.2 Numerical solution

For illustration we take the background neutrino potential in the form

$$
V^0_\nu = 6.5 \cdot 10^{10} \omega_p \frac{(t \omega_p + 10)^4}{(t \omega_p + 10)^4}, \quad (6.13)
$$

in our numerical computations, and for the potential due to scattering on electrons, we use two different profiles with high $V_e^h$ and low $V_e^l$ gradients:

$$
V_e^h = \frac{10^{14} \omega_p}{(t \omega_p + 10)^5}, \quad V_e^l = \frac{1.8 \cdot 10^{12} \omega_p}{(t \omega_p + 10)^{1/2}}. \quad (6.14)
$$

These profiles are typical for a supernova during the accretion phase. The parameters in eqs. (6.13) and (6.14) have been fixed in such a way that the benchmark values of $\xi \approx 0.1$, $V_e \approx 10^5 \omega_p$, and $V^0_\nu \approx 10^2 \omega_p$ are achieved at $t \omega_p = 140$. 

---

[Page 28]
Using $V_e$, we first solved the evolution equation for the background neutrinos eq. (3.1). Then, the obtained amplitudes $\psi_e$, $\psi_\tau$ and the probability $P_{e\tau}$ were used to solve the evolution equation for the probe neutrino. The results are shown in figure 8. The evolution starts at $t\omega_p = 100$.

The lower black lines (areas) show the probabilities for the background neutrinos. Their average values follow the adiabatic solution in eq. (6.1). For them the level crossing occur at $t\omega_p = 630$ (high gradient) and $t\omega_p = 830$ (low gradient) and the adiabaticity in the resonance is fulfilled in both cases. Notice that fast oscillations due to $\Delta m$ are not resolved in the figures.

**Figure 8.** Conversion probability for changing densities. $\omega_b = \omega_p$ and $A_{\beta} = 1.0976$ such that the resonant condition is fulfilled at $t\omega_p = 140$. Upper panel: $V^h_e$ (with high gradient), lower panel: $V^h_e$ (with lower gradient).
The upper (red) lines show the probabilities for the probe neutrinos. The oscillatory pattern of the these lines is due to parametric oscillations. At $t\omega_p = 100$ the conversion probability of the probe neutrino is larger than that for the background neutrino due to parametric enhancement (although $\xi \ll \xi_R$). A zoom would reveal oscillations somewhat similar to the upper panel of figure 4. Close to the parametric resonance at $t\omega_p = 140$, the conversion probability stops to increase and then approaches a constant value which is much smaller than 1. This is due to the breaking of adiabaticity as it was discussed in the previous section. The rapid rise takes place in $\Delta t\omega_p \sim 4$ in good agreement with eq. (6.12) for $\eta = 1$, but the asymptotic value of $P$ is smaller than the result in eq. (6.11) by a factor of $\sim 5$. This is due to the rapidly decreasing $V_e$ and $V_\nu$ which has not been taken into account in eq. (6.11).

With the decrease of the gradient $\eta$, the asymptotic value increases as $1/\eta$. For $\eta = 0.2$ that is a factor of $\sim 5$ which agrees well with the lower panel in figure 8.

When the neutrino density $V_\nu$ decreases and becomes comparable or smaller than the vacuum frequency $\omega$, spectral splits occur [3, 4, 6–10]. For frequencies larger than the split frequency $\omega_C$, $\omega_p > \omega_C$, neutrinos change flavor, whereas for $\omega_p < \omega_C$ they return back to the initial flavor state. The width of the transition region is determined by the degree of adiabaticity — the gradient of density change.

In the limit of $\xi \gg 1$, our example reproduces the spectral splits: we find [34] that for a background flux with neutrino frequencies $\omega_k$ the split frequency is $\omega_C = \omega_k$. So, if the frequency of the probe neutrino is larger than $\omega_k$ a large flavor conversion occurs. In the opposite case $\omega_p < \omega_k$, conversion is suppressed. Hence the spectral split itself is not a unique feature of $\nu\nu$ interactions in the background flux, but can also be seen in simpler models such as the one just analysed. However, the spectral split is not reproduced in the simple background model when $\xi \ll 1$ because the transition through the parametric resonance becomes non-adiabatic. One can expect that $\nu\nu$ interactions in the background flux will improve adiabaticity and restore the spectral split even in the presence of a large matter background.

7 Conclusions

1. We consider an approach to collective neutrino oscillations in supernovae based on the flavor evolution of individual neutrinos in external potentials. The flavor evolution of supernovae neutrinos in the presence of the $\nu\nu$ interactions is linear in the sense that a given probe neutrino does not affect its own evolution as well as the evolution of other neutrinos which, in turn, could affect the probe neutrino. Therefore, the evolution is described by the usual Schrödinger-like equation with potentials generated by $\nu\nu$ scattering both in the diagonal and off-diagonal elements of the Hamiltonian. These potentials have non-trivial dependence on evolution time (distance) of the probe neutrino. Consequently, an effective theory of collective oscillations based on certain assumptions on time dependence of the potentials can be developed. In particular, conditions for strong flavor transformations can be formulated. Strong transitions occur when the diagonal elements of the Hamiltonian vanish or when elements of the Hamiltonian have periodic time (distance) modulations. In the latter case, the parametric enhancement of flavor transitions, which we considered in this paper, can be realised.

There are new features of the parametric effects in the case of a neutrino in a neutrino flux in comparison with known realisations of the parametric effects: usually periodicity
of the potential is due to the density modulations of electrons or usual matter. In our case neither electron nor total neutrino densities are modulated and the modulations originate from the flavor change of the flux neutrinos. Furthermore, the potential has flavor off-diagonal components and contains a complex phase which plays a crucial role. So, we deal here with a rather non-trivial realisation of the parametric resonance.

2. To get an idea about the possible time dependence of the potentials, we considered a simplified solvable model of the background neutrinos which still retains the main feature of coherent flavor exchange. In this model a probe neutrino propagates in a flux of background neutrinos moving in the same direction, so that $\nu\nu$ interactions in the flux are absent.

We have computed the potentials explicitly, and their main feature is the quasi-periodic dependence on time which is related to the flavor oscillations of the background neutrinos. The modulation frequency depends on $V_e$, $V_\nu$ and $\omega$.

At certain conditions, periodic modulations of the potentials lead to parametric enhancement of the probe neutrino’s oscillations. At the parametric resonance, the oscillations proceed with maximal depth. This indicates that, in the realistic case with $\nu\nu$ interactions in the flux, strong transitions can be interpreted as being due to a parametric resonance.

The relative width of the parametric resonance in the energy scale is proportional to the vacuum mixing and relative density of the background neutrinos $s_2\theta\xi$. The length of parametric oscillations in resonance is also given by $2\pi/(s_2\theta\omega_p) = l_\nu/(s_2\theta\xi)$. So, with an increase of the neutrino density, the resonance becomes wider and the oscillation length smaller.

3. Integrations over the energy spectrum as well as over the production point of background neutrinos lead to strong suppression of the flavor transitions. Moreover, the latter is more important.

In the case of varying densities, and consequently potentials, strong transitions are possible if the adiabaticity condition is satisfied in the parametric resonance. We find that the adiabaticity is strongly broken for typical parameters of supernovae, thus leading to suppression of flavor transformations.

Integration over the angles in the background would imply $\nu\nu$ interactions and so can not be described without changing the dynamics of the model. One exception is when the background neutrino flux is emitted from a small sphere, such that there is no $\nu\nu$ interactions in the flux, but a probe neutrino will see background neutrinos with changing angle or $A_\beta$. The effect is similar to the case of a varying neutrino density.

4. The main question is to which extend our conclusions for the simplified model of the background can be extended to the realistic case with $\nu\nu$ interactions in the flux. Our results imply that extremely strong correlations (tuning) between the evolution of the probe and background neutrinos is required in order to get strong transitions. Turning on the $\nu\nu$ interactions in the background can in general lead to an enhancement of transitions in the background (instead of constant $P_{\rm max}^{\nu\nu}$), and consequently to faster transitions of the probe neutrino. At this point one can use iteration: take the solution found for the probe neutrino due to parametric enhancement and use it as the background for the flux neutrinos.
In more details, we used the solution of the evolution equation without $\nu\nu$ interactions for the flux neutrinos to reconstruct the potentials as the first step. The solution is just for oscillations in constant density matter with constant depth and average probability. As the next step one can use the solution obtained for the probe neutrino for the flux neutrinos. Since the average oscillation probability increases for this solution one expects faster transition for the probe particle [34]. In other words, one can use the solution for the probe neutrino for the flux neutrino to reconstruct the potentials.

The first iteration will bring another frequency to the potentials determined by the neutrino density. The $\nu\nu$ interactions in the background can lead to a stronger correlation between the background and probe neutrinos. An increase of $\xi$ expands the region of strong effects in energy scale, and to some extent, mitigate the averaging over energy. It also improves adiabaticity. A detailed study with iterations will be presented elsewhere [34].

5. The simplest example of a background with $\nu\nu$ interactions is two intersecting fluxes with collinear neutrinos and constant densities in each. The solution for this case has been found numerically [35]. Using such a solution in the homogeneous case, we have reconstructed the corresponding potentials. To a good approximation, the off-diagonal potential can be parametrised by the product of two periodic factors:

$$\bar{V}_{\nu}e^{i\alpha B} = ae^{-i\nu e^{-w\cos^2(\pi t/p)},}$$

(7.1)

where $a$, $w$, and $p$ are free parameters. While $w$ only show weak dependence on $\nu$, $\omega$ and $\nu$, $a$ and $p$ depend strongly on $\nu$ and $\omega$. The diagonal potential has little effect on the evolution, and $\nu \approx 0$ is a good approximation.

The first exponential in $\bar{V}_{\nu}e^{i\alpha B}$ has frequency $\nu$ and coincides precisely with the periodic factor we have found in our example. This factor leads to the parametric resonance, and any large conversion is absent when it is not present. The second exponential with period $p$ can not be obtained in our example of a neutrino flux without $\nu\nu$ interactions. This type of enhancement is expected to appear in the iteration procedure mentioned before.

Acknowledgments

R.S.L.H. is funded by the Alexander von Humboldt Foundation. The work of A.S. is supported by Max-Planck senior fellow grant M.FW.A.KERN0001.

A No neutrino background effect for $A_\beta = 1$

For $A_\beta = 1$ the flavor evolutions of the probe neutrino and the neutrino from the flux are identical. This can be understood considering collisions of the probe neutrino with individual neutrinos from the flux. In each collision (starting from the first one) flavor exchange does not produce any change. Both neutrinos (probe and flux) arrive at the collision point (point of crossing of trajectories) in the same state. Therefore the flavor exchange does not produce any change. As a result, the effect of neutrino-neutrino interactions drops out. In our formalism this is obtained since the neutrino state becomes the eigenstate of the part of the
Hamiltonian which depends on the neutrino densities. Indeed, according to (2.2)

\[ H_\nu \psi = V_\nu^0 \begin{pmatrix} \psi_e \psi_e^* & \psi_e \psi_\tau^* \\ \psi_\tau^* \psi_\tau & \psi_\tau \psi_\tau^* \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_\tau \end{pmatrix} = V_\nu^0 \begin{pmatrix} \psi_e \\ \psi_\tau \end{pmatrix}, \]

where we added the matrix \(0.5(\psi_e \psi_e^* + \psi_\tau \psi_\tau^*)I\) proportional to the unit matrix (which does not affect flavor evolution), and we have taken into account that \(\psi_e \psi_e^* + \psi_\tau \psi_\tau^* = 1\). Thus \(H_\nu \psi = V_\nu^0 I \psi\), that is the contribution to the Hamiltonian from neutrino - neutrino interactions is proportional to the unit matrix and therefore can be omitted.

**B Correction due to the non-resonant mode**

Let us search for a solution of the evolution equation with the complete Hamiltonian (4.6) in the form

\[ S_\alpha = S_0 S_1, \quad (B.1) \]

where \(S_0\) is given in (4.10). Inserting \(S_\alpha\) in the evolution equation with the total Hamiltonian and taking into account (4.9), we obtain the equation for \(S_1\):

\[ i \dot{S}_1 = \Delta H_1 S_1, \quad \Delta H_1 \equiv S_0^\dagger \Delta H^\alpha S_0. \]

Let us find the solution of this equation in the resonance: \(A_\beta \Delta_m = \delta, (4.12)\) when \(\theta_m^p = \pi/4\). In this case the matrix (4.10) simplifies

\[ S_0 = \begin{pmatrix} \cos \phi_m^p & -i \sin \phi_m^p \\ -i \sin \phi_m^p & \cos \phi_m^p \end{pmatrix}. \quad (B.2) \]

Then the Hamiltonian \(\Delta H_1\) in (4.10) becomes

\[ \Delta H_1 = \frac{f'}{4} \begin{pmatrix} \sin 2\phi_m^p \sin 2\phi_m & \cos 2\phi_m + i \cos 2\phi_m \sin 2\phi_m \\ \cos 2\phi_m - i \cos 2\phi_m \sin 2\phi_m & -\sin 2\phi_m \sin 2\phi_m \end{pmatrix}. \]

Here \(2\phi_m^p = \gamma t = \frac{1}{2} f't\). In eq. (B.2) we have two frequencies \(\frac{1}{2} f' \ll 2 A_\beta \Delta_m \propto 2 V_\nu\). It is this large frequency \(A_\beta \Delta_m\) that modulates the oscillations generated by (B.2).

The off-diagonal element of (B.2) can be written as

\[ (\Delta H_1)_{12} = ze^{i\chi}, \]

where

\[ z \equiv \sqrt{1 - \eta^2}, \quad \sin \chi = \frac{1}{z} \sin 2\phi_m \cos 2\phi_m^p \quad (B.3) \]

and

\[ \eta \equiv \sin 2\phi_m \sin 2\phi_m^p. \]

In terms of \(\eta\) and \(\chi\) the Hamiltonian can be rewritten as

\[ \Delta H_1 = \frac{f'}{4} \begin{pmatrix} \eta \sqrt{1 - \eta^2} e^{i\chi} \\ \eta \sqrt{1 - \eta^2} e^{-i\chi} \end{pmatrix}. \quad (B.4) \]

Let us make a transformation of the fields

\[ \psi_\alpha = U_\chi \psi_\chi, \quad U_\chi = \text{diag}(e^{i\frac{\chi}{2}}, e^{-i\frac{\chi}{2}}) \]
that eliminates the phase from the off-diagonal elements of \((B.4)\). Then the evolution equation of the transformed fields, \(\psi_\chi\), has the Hamiltonian
\[
\Delta H_\chi = -\frac{1}{2} \left( \frac{1}{2} f' \eta + \dot{\chi} \right) \left( \frac{1}{2} f' \sqrt{1 - \eta^2} - \frac{1}{2} f' \eta - \dot{\chi} \right).
\]
\((B.5)\)

Since \(A_\beta \Delta_m \gg f\) at early times of evolution, we can take \(\sin 2\phi^p_m \approx 0\), and consequently, \(\eta \approx 0\). In this case the Hamiltonian \((B.5)\) becomes
\[
\Delta H_\chi = \frac{1}{2} \left( \dot{\chi} \left( \frac{f'}{2} \sqrt{1 - \eta^2} \right) \right).
\]

According to \((B.3)\) \(\chi \approx 2\phi_m\), so that \(\dot{\chi} \approx 2A_\beta \Delta_m\). Therefore \(\Delta H_\chi\) describes oscillations with large frequency (total level split):
\[
\sqrt{\dot{\chi}^2 + (f'/2)^2} \approx \dot{\chi}
\]
and with the depths
\[
\sin^2 2\theta_\chi = \frac{f'^2}{4\dot{\chi}^2 + f'^2} \approx \frac{f'^2}{4\dot{\chi}^2} \approx \frac{f'^2}{16A_\beta^2 \Delta_m}.
\]

The corresponding \(S_\chi\) matrix has the same form as in \((4.10)\) with \(\theta^0_m \to \theta_\chi\) and \(\phi^0_m \to \phi_\chi\) = \(\frac{1}{2} \chi\):
\[
\phi^p_m \to \phi_\chi = \frac{1}{2} \sqrt{\dot{\chi}^2 + (f^p/2)^2} \approx \frac{1}{2} \dot{\chi} = \frac{1}{2} \chi.
\]

In the first approximation in small mixing \(\theta_\chi\), we have
\[
S_\chi \approx \begin{pmatrix} e^{i\phi_\chi} & -i \sin 2\theta_\chi \sin \phi_\chi \\ -i \sin 2\theta_\chi \sin \phi_\chi & e^{-i\phi_\chi} \end{pmatrix}.
\]

Returning back to the \(\alpha\) basis gives
\[
S_1 = U_{\chi} S_\chi U_{\chi}^\dagger \approx \begin{pmatrix} e^{i\phi_\chi} & -i \epsilon e^{i\chi} \\ -i \epsilon e^{-i\chi} & e^{-i\phi_\chi} \end{pmatrix},
\]
\((B.6)\)

where \(\epsilon \equiv \sin 2\theta_\chi \sin \phi_\chi\). The total \(S\) matrix equals the product \((B.1)\) of \(S_0\) in \((B.2)\) and \(S_1\) \((B.6)\):
\[
S_0 S_1 = \begin{pmatrix} \cos \phi^p_m e^{i\phi_\chi} & -\sin \phi^p_m e^{-i\chi} \\ -i(\sin \phi^p_m e^{i\phi_\chi} + \epsilon \cos \phi^p_m e^{i\chi}) & \cos \phi^p_m e^{-i\phi_\chi} - \epsilon \sin \phi^p_m e^{i\chi} \end{pmatrix}.
\]

Rotating back to the flavor basis
\[
S = U \alpha S_0 S_1 U^\dagger_{\alpha},
\]
we obtain for the 12 element:
\[
S_{12} = \sin \phi^p_m e^{-i(\phi_\chi + \phi_m)} + \cos \phi^p_m \sin \phi_\chi \left( \sin 2\theta_\chi e^{i(\chi - \phi_m)} - \frac{g}{d} \right),
\]
where a common factor \(-i\) is omitted. Then keeping the first correction to the main (first) term, we have for the probability
\[
P_{e\tau} \approx \sin^2 \phi^p_m + 2\phi^p_m \sin \phi_\chi \left[ \sin 2\theta_\chi \cos 3\phi_m - \frac{g}{d} \cos 2\phi_m \right],
\]
\((B.7)\)

where we used that \(\phi_\chi \approx \frac{1}{2} \chi \approx \frac{1}{2} \phi_m\).
Thus, the zero order solution given by the first term $\sin^2 \phi_m$ (oscillations with maximal depth and large period given by $f'$ in our example) is modulated by fast oscillations ($2A_\beta \Delta m$ frequency) with small depth proportional to $\sin \theta \sim f' \dot{\chi} \sim f' A_\beta \Delta m \sim s^2 \omega_k \xi V_e$ and $g \sim s^2 \omega_k V_e$.

Let us compare the probability in (B.7) which takes into account corrections due to the non-resonant mode ($S_1(\Delta H_\alpha)$ and corresponds to the exact resonance, and the lowest order ($S_0$) probability in eq. (4.11) which can be used also outside the resonance. We can rewrite eq. (B.7) as

$$P_{pe}^e \approx \left[ 1 - \frac{g}{d} \tan \frac{1}{2} \gamma t \sin \phi_m \left( \cos 2\phi_m - \frac{f'}{4g} \cos 3\phi_m \right) \right] \sin^2 \frac{1}{2} \gamma t. \quad \text{(B.8)}$$

In turn, the probability (4.11) equals approximately

$$P_{pe}^e = \frac{1}{\gamma^2} \left[ \frac{f'^2}{4} + \frac{g}{d} (d - A_\beta \Delta m) f' \cos \phi_m \right] \sin^2 \frac{1}{2} \gamma t.$$

In resonance $P_{pe}^e = \sin^2 \frac{1}{2} \gamma t$. Close to resonance, $|d - A_\beta \Delta m| \sim |f'|$, we have

$$P_{pe}^e \approx \frac{f'^2}{4\gamma^2} \left( 1 + \frac{4g}{d} \cos \phi_m \right) \sin^2 \frac{1}{2} \gamma t.$$

The coefficient in front of $\cos \phi_m$ has smallness $\sim \frac{g}{d}$ with respect to the first term. Comparing this with (B.8), we conclude that the oscillating term that arises from the transformation from $S_\alpha$ to $S$ is comparable to the correction that arises from $S_1$ (B.7). In resonance, modulations due to transition to the $\alpha$ basis are absent and the modulations are due to the $S_1$ correction only. According to (B.8) corrections to the probability due to $S_1$ can be enhanced due to $\tan \frac{1}{2} \gamma t$ in the early evolution when the phase is small.

Hence eq. (4.11) is not correct to order $\frac{g}{d}$. However, for the hierarchy $V_e \gg V_\nu \gg \omega$, the correction $\frac{g}{d} \ll \xi$ and the analytic approximation is still valid.

References

[1] J.T. Pantaleone, Neutrino oscillations at high densities, Phys. Lett. B 287 (1992) 128 [INSPIRE].

[2] H. Duan, G.M. Fuller and Y.-Z. Qian, Collective neutrino flavor transformation in supernovae, Phys. Rev. D 74 (2006) 123004 [astro-ph/0511293] [INSPIRE].

[3] H. Duan, G.M. Fuller, J. Carlson and Y.-Z. Qian, Simulation of Coherent Non-Linear Neutrino Flavor Transformation in the Supernova Environment. I. Correlated Neutrino Trajectories, Phys. Rev. D 74 (2006) 105014 [astro-ph/0606161] [INSPIRE].

[4] H. Duan, G.M. Fuller, J. Carlson and Y.-Z. Qian, Coherent Development of Neutrino Flavor in the Supernova Environment, Phys. Rev. Lett. 97 (2006) 241101 [astro-ph/0608050] [INSPIRE].

[5] S. Hannestad, G.G. Raffelt, G. Sigl and Y.Y.Y. Wong, Self-induced conversion in dense neutrino gases: Pendulum in flavour space, Phys. Rev. D 76 (2007) 029901 [astro-ph/0608695] [INSPIRE].
[6] G.L. Fogli, E. Lisi, A. Marrone and A. Mirizzi, *Collective neutrino flavor transitions in supernovae and the role of trajectory averaging*, JCAP 12 (2007) 010 [arXiv:0707.1998] [inSPIRE].

[7] G.G. Raffelt and A. Yu. Smirnov, *Self-induced spectral splits in supernova neutrino fluxes*, Phys. Rev. D 76 (2007) 081301 [Erratum ibid. D 77 (2008) 029903] [arXiv:0705.1830] [inSPIRE].

[8] G.G. Raffelt and A. Yu. Smirnov, *Adiabaticity and spectral splits in collective neutrino transformations*, Phys. Rev. D 76 (2007) 125008 [arXiv:0709.4641] [inSPIRE].

[9] B. Dasgupta, A. Dighe, G.G. Raffelt and A. Yu. Smirnov, *Multiple Spectral Splits of Supernova Neutrinos*, Phys. Rev. Lett. 103 (2009) 051105 [arXiv:0904.3542] [inSPIRE].

[10] R.F. Sawyer, *Speed-up of neutrino transformations in a supernova environment*, Phys. Rev. D 76 (2007) 125008 [arXiv:0709.4641] [inSPIRE].

[11] I. Izaguirre, G. Raffelt and I. Tamborra, *Fast Neutrino Flavor Conversion as Oscillations in a Quartic Potential*, Phys. Rev. D 97 (2018) 023017 [arXiv:1709.08671] [inSPIRE].

[12] F. Capozzi, B. Dasgupta, E. Lisi, A. Marrone and A. Mirizzi, *Fast flavor conversions of supernova neutrinos: Classifying instabilities via dispersion relations*, Phys. Rev. D 96 (2017) 043016 [arXiv:1706.03360] [inSPIRE].

[13] G.L. Fogli, E. Lisi, A. Marrone and A. Mirizzi, *Collective neutrino flavor transitions in supernovae and the role of trajectory averaging*, JCAP 01 (2016) 028 [arXiv:1507.05769] [inSPIRE].

[14] B. Dasgupta and A. Mirizzi, *Temporal Instability Enables Neutrino Flavor Conversions Deep Inside Supernovae*, Phys. Rev. D 92 (2015) 125030 [arXiv:1509.03171] [inSPIRE].

[15] F. Capozzi, B. Dasgupta and A. Mirizzi, *Self-induced temporal instability from a neutrino antenna*, JCAP 04 (2016) 043 [arXiv:1603.03288] [inSPIRE].

[16] S. Abbar and H. Duan, *Fast neutrino flavor conversion: roles of dense matter and spectrum crossing*, arXiv:1712.07013 [inSPIRE].

[17] R.S. Hansen and S. Hannestad, *Chaotic flavor evolution in an interacting neutrino gas*, Phys. Rev. D 90 (2014) 025009 [arXiv:1404.3833] [inSPIRE].

[18] G. Mangano, A. Mirizzi and N. Saviano, *Damping the neutrino flavor pendulum by breaking homogeneity*, Phys. Rev. D 89 (2014) 073017 [arXiv:1403.1892] [inSPIRE].

[19] A. Esteban-Pretel et al., *Role of dense matter in collective supernova neutrino transformations*, Phys. Rev. D 78 (2008) 085012 [arXiv:0807.0659] [inSPIRE].

[20] S. Chakraborty, R.S. Hansen, I. Izaguirre and G. Raffelt, *Self-induced flavor conversion of supernova neutrinos on small scales*, JCAP 01 (2016) 028 [arXiv:1507.05769] [inSPIRE].

[21] B. Dasgupta and A. Mirizzi, *Temporal Instability Enables Neutrino Flavor Conversions Deep Inside Supernovae*, Phys. Rev. D 92 (2015) 125030 [arXiv:1509.03171] [inSPIRE].

[22] F. Capozzi, B. Dasgupta and A. Mirizzi, *Self-induced temporal instability from a neutrino antenna*, JCAP 04 (2016) 043 [arXiv:1603.03288] [inSPIRE].

[23] R.S. Hansen and S. Hannestad, *Chaotic flavor evolution in an interacting neutrino gas*, Phys. Rev. D 90 (2014) 025009 [arXiv:1404.3833] [inSPIRE].

[24] G. Mangano, A. Mirizzi and N. Saviano, *Damping the neutrino flavor pendulum by breaking homogeneity*, Phys. Rev. D 89 (2014) 073017 [arXiv:1403.1892] [inSPIRE].

[25] G. Raffelt, S. Sarikas and D. de Sousa Seixas, *Axial Symmetry Breaking in Self-Induced Flavor Conversion of Supernova Neutrino Fluxes*, Phys. Rev. Lett. 111 (2013) 091101 [Erratum ibid. 113 (2014) 239903] [arXiv:1305.7140] [inSPIRE].
[26] G.D. Pusch, *Neutron Oscillations In A Periodically Varying Magnetic Field*, Nuovo Cim. A 74 (1983) 149 [inSPIRE].

[27] V.K. Ermilova, V.A. Tsarev and V.A. Chechin, *Parametric enhancement of neutrino oscillations in matter*, Krak. Soobshch. Fiz. 5 (1986) 26.

[28] E.K. Akhmedov, *Neutrino oscillations in inhomogeneous matter* (in Russian), Sov. J. Nucl. Phys. 47 (1988) 301 [Yad. Fiz. 47 (1988) 475] [inSPIRE].

[29] P.I. Krastev and A. Yu. Smirnov, *Parametric Effects in Neutrino Oscillations*, Phys. Lett. B 226 (1989) 341 [inSPIRE].

[30] E.K. Akhmedov, *Parametric resonance of neutrino oscillations and passage of solar and atmospheric neutrinos through the earth*, Nucl. Phys. B 538 (1999) 25 [hep-ph/9805272] [inSPIRE].

[31] I. Tamborra, L. Huedepohl, G. Raffelt and H.-T. Janka, *Flavor-dependent neutrino angular distribution in core-collapse supernovae*, Astrophys. J. 839 (2017) 132 [arXiv:1702.00060] [inSPIRE].

[32] M.T. Keil, G.G. Raffelt and H.-T. Janka, *Monte Carlo study of supernova neutrino spectra formation*, Astrophys. J. 590 (2003) 971 [astro-ph/0208035] [inSPIRE].

[33] S.P. Mikheev and A. Yu. Smirnov, *Resonant amplification of neutrino oscillations in matter and solar neutrino spectroscopy*, Nuovo Cim. C 9 (1986) 17 [inSPIRE].

[34] R.S.L. Hansen, A.Y. Smirnov, work in progress.

[35] A. Mirizzi, G. Mangano and N. Saviano, *Self-induced flavor instabilities of a dense neutrino stream in a two-dimensional model*, Phys. Rev. D 92 (2015) 021702 [arXiv:1503.03485] [inSPIRE].