Abstract

The aim of this paper is to design a theoretical framework that allows us to perform the computation of regular expression derivatives through a space of generic structures. Thanks to this formalism, the main properties of regular expression derivation, such as the finiteness of the set of derivatives, need only be stated and proved one time, at the top level. Moreover, it is shown how to construct an alternating automaton associated with the derivation of a regular expression in this general framework. Finally, Brzozowski’s derivation and Antimirov’s derivation turn out to be a particular case of this general scheme and it is shown how to construct a DFA, a NFA and an AFA for both of these derivations.

1 Introduction

The (left) quotient of a language $L$ over an alphabet $\Sigma$ with respect to a word $w$ in $\Sigma^*$ is the language obtained by stripping the leading $w$ from the words in $L$ that are prefixed by $w$. The quotient operation plays a fundamental role in language theory and is especially involved in two main issues. First, checking whether a word $w$ belongs to a language $L$ turns out to be equivalent to checking whether the empty word belongs to the quotient of $L$ w.r.t. $w$. Secondly it was proved by Myhill [17] and Nerode [18] that a language is regular if and only if the set of its quotients w.r.t. all the words in $\Sigma^*$ is finite. In the case of a regular language $L$, the different quotients are the states of the so-called quotient automaton of $L$ that is isomorphic to its minimal deterministic automaton.

Since the equality of two languages amounts to the isomorphism of their minimal deterministic automata, the construction of the quotient automaton via the computation of the left quotients is intractable. The seminal work of Brzozowski [4] that introduced the notion of a word derivative of a regular expression and the construction of the associated DFA, gave rise to a long series...
of studies (see for example [3, 1, 9, 13, 7]) that are all based on the simulation of the computation of a language quotient by the one of an expression derivative. In all these research works, the first rule is that if the expression \( E \) denotes the language \( L \), then the derivative of \( E \) w.r.t. \( w \), for any \( w \), denotes the quotient of \( L \) w.r.t. the word \( w \). Thus, checking whether a word \( w \) belongs to the language denoted by the expression \( E \) is equivalent to checking if the empty word belongs to the language denoted by the derivative of \( E \) w.r.t. \( w \). The second rule is that as far as the set \( D \) of all the derivatives of \( E \) is finite, a finite automaton recognizing the language denoted by \( E \) can be constructed, admitting \( D \) as a set of states.

Let us notice that Brzozowski derivatives [4] handle unrestricted regular expressions and provide a deterministic automaton; Antimirov derivatives [1] only address simple regular expressions and provide both a deterministic automaton and a non-deterministic one; Antimirov derivatives have been recently extended to regular expressions [7] and this extension provides a deterministic automaton, a non-deterministic one and, as shown in this paper, an alternating automaton. Berry and Sethi continuations [3] are based on the linearization of the (simple) input expression and allow the construction of its Glushkov (non-deterministic) automaton. Champarnaud and Ziadi c-continuations [9] and Ilie and Yu derivatives [13] allow both the construction of the Glushkov automaton and of the Antimirov non-deterministic automaton. Let us mention that derivation has been extended to expressions with multiplicity [13] [8].

As mentioned by Antimirov [1], derivatives of regular expressions have proved to be a productive concept to investigate theoretical topics such as the algebra of regular expressions [11] or of \( K \)-regular expressions [13], the systems of language equations [6], the equivalence of simple regular expressions [12] or of regular expressions [2]. More recently, Brzozowski introduced a new approach for finding upper bounds for the state complexity of regular languages, based on the counting of their quotients (or of their derivatives) [5].

Moreover, derivatives provide a useful tool to implement regular matching algorithms: Brzozowski’s DFA and Antimirov’s NFA turn out to be competitive matching automata [20], compared for instance with Thompson’s \( \varepsilon \)-automaton [21]. The derivative-based techniques are well-suited to functional languages, that are characterized by a good support for symbolic term manipulation. As an example, two derivative-based scanner generators have been recently developed, one for PLT Scheme and one for Standard ML, as reported in [19]. Similarly, Brzozowski’s derivatives are used in the implementation of the XML schema language RELAX NG [10]. Finally, let us notice that derivatives can be extended to context-free grammars, seen as recursive regular expressions, yielding a system for parsing context-free grammars [16].

The aim of this paper is to design a general framework where the computation of the set of derivatives of a regular expression, called derivation, is performed over a space of generic structures. Of course Brzozowski’s derivation and Antimirov’s one appear as particular cases of this general scheme. A first benefit of this formalism is that the properties inherent to the mechanism of derivation, such as the equality between the language denoted by a derivative
and the corresponding quotient, the finiteness of the set of derivatives and the way for constructing the associated automata, need only be stated and proved one time, at the top level. A second benefit is that the general framework allows us to design the construction of an AFA from the set of derivatives. As a consequence, we show how to construct a DFA, a NFA and an AFA for any finite derivation, including both Brzozowski’s one and Antimirov’s one.

The next section is a preliminary section; it gathers classical notions concerning regular languages, regular expressions and finite automata, as well as boolean formulas and alternating automata. The notion of a regular expression derivation via a support is defined in Section 3 and the properties of the corresponding derivatives are investigated. Section 4 is devoted to the construction of the alternating automaton associated with the derivation of a regular expression via a support. This construction is illustrated in Section 5, where the support is based on the set of clausal forms over the alphabet of the regular expressions.

2 Preliminaries

Let \( B = \{0, 1\} \). A boolean formula \( \phi \) over a set \( X \) is inductively defined by \( \phi = x \) where \( x \in X \), or \( \phi = f_B(\phi_1, \ldots, \phi_k) \), where \( f_B \) is the operator associated to the \( k \)-ary boolean function \( f \) from \( B^k \) to \( B \) and \( \phi_1, \ldots, \phi_k \) are boolean formulas over \( X \). The set of the boolean formulas over \( X \) is denoted by \( \text{BoolForm}(X) \). Let \( v \) be a function from \( X \) to \( B \). The evaluation of \( \phi \) with respect to \( v \) is the boolean \( \text{eval}_v(\phi) \) inductively defined by: \( \text{eval}_v(x) = v(x) \), \( \text{eval}_v(f_B(\phi_1, \ldots, \phi_k)) = f(\text{eval}_v(\phi_1), \ldots, \text{eval}_v(\phi_k)) \). The set \( \text{Atom}(\phi) \) is the subset of \( X \) inductively defined by \( \text{Atom}(x) = \{x\} \) and \( \text{Atom}(f_B(\phi_1, \ldots, \phi_k)) = \bigcup_{1 \leq j \leq k} \text{Atom}(\phi_j) \).

Let \( \Sigma \) be an alphabet, \( w \) be a word in \( \Sigma^* \) and \( L \) be a language over \( \Sigma \). Deciding whether \( w \) belongs to \( L \) is called the membership problem for the language \( L \). We denote by \( r_w(L) \) the boolean equal to 1 if \( w \in L \), 0 otherwise. Let \( L_1, \ldots, L_k \) be \( k \) languages. We denote by \( \cdot, \ast \) and \( f_L \) for any \( k \)-ary boolean function \( f \) the operators defined as follows: \( L_1 \cdot L_2 = \{w_1 \cdot w_2 \in \Sigma^* \mid r_w(L_1) \land r_w(L_2) = 1\} \), \( L_1^* = \{\varepsilon\} \cup \{w_1 \cdots w_n \in \Sigma^* \mid n \in \mathbb{N} \land \forall k \in \{1, \ldots, n\}, r_{w_k}(L_1) = 1\} \), \( f_L(L_1, L_2) = \{ w \in \Sigma^* \mid f(r_w(L_1), \ldots, r_w(L_k)) = 1\} \). The quotient of a regular language \( L \) with respect to a word \( w \), that is defined as the set \( w^{-1}(L) = \{w' \in \Sigma^* \mid r_{w'w}(L) = 1\} \) can be inductively computed as follows: \( \varepsilon^{-1}(L) = L \), and for \( a \in \Sigma \),

\[
a^{-1}(L) = \begin{cases} a^{-1}(L_1) \cdot L_2 \cup a^{-1}(L_2) & \text{if } L = L_1 \cdot L_2 \land \varepsilon \in L_1, \\ a^{-1}(L_1) \cdot L_2 & \text{if } L = L_1 \cdot L_2 \land \varepsilon \notin L_1, \\ a^{-1}(L_1) \cdot L_1 & \text{if } L = L_1, \\ f_L(a^{-1}(L_1), \ldots, a^{-1}(L_n)) & \text{if } L = f_L(L_1, \ldots, L_n), \\ \{\varepsilon\} & \text{if } L = \{a\}, \\ \emptyset & \text{otherwise}. \end{cases}
\]

\((a \cdot w')^{-1}(L) = w'^{-1}(a^{-1}(L))\) for \( w' \in \Sigma^* \).

An Alternating Automaton (AFA) is a 5-tuple \( A = (\Sigma, Q, I, F, \delta) \) where \( \Sigma \) is an alphabet, \( Q \) is a set of states, \( I \) is a boolean formula over \( Q \), \( F \) is a function
from \(Q\) to \(\mathbb{B}\) and \(\delta\) is a function from \(Q \times \Sigma\) to \(\text{BoolForm}(Q)\). The function \(\delta\) is extended from \(\text{BoolForm}(Q) \times \Sigma^*\) to \(\text{BoolForm}(Q)\) as follows: \(\delta(\phi, \varepsilon) = \phi, \delta(\phi, aw) = \delta(\delta(\phi, a), w), \delta(f_\Sigma(\phi_1, \ldots, \phi_k), a) = f_\Sigma(\delta(\phi_1, a), \ldots, \delta(\phi_k, a))\) where \(a\) is any symbol in \(\Sigma\), \(w\) is any word in \(\Sigma^*\) and \(\phi_1, \ldots, \phi_k\) are any \(k\) boolean formulas over \(Q\). The accessible part of \(A\) is the alternating automaton \((\Sigma, Q', I, F', \delta')\) defined by: \(Q' = \{q \in Q \mid \exists w \in \Sigma^*, q \in \text{Atom}(\delta(I, w))\}; \forall q \in Q', F'(q) = F(q); \forall a \in \Sigma, \forall q \in Q', \delta'(q, a) = \delta(q, a)\). The language recognized by the alternating automaton \(A\) is the subset \(L(A)\) of \(\Sigma^*\) defined by \(L(A) = \{w \in \Sigma^* \mid \text{eval}_F(\delta(I, w)) = 1\}\). Whenever \(Q\) is a finite set, \(A\) is said to be an Alternating Finite state Automaton (AFA).

A regular expression \(E\) over an alphabet \(\Sigma\) is inductively defined by: \(E = a, E = 1, E = 0, E = E_1 \cdot E_2, E = E_1^*\) or \(E = f_k(E_1, \ldots, E_n)\), where \(\forall k \in \mathbb{N}, E_k\) is a regular expression, \(a \in \Sigma\) and \(f_k\) is the operator associated to the \(k\)-ary boolean function \(f\), e.g. \(+\) is the operator associated to \(\lor\). In the following, we assume that the regular expression operators as well as the boolean formula operators have no specific algebraic properties, unlike the boolean functions. For instance, the operator \(\lor\) is not associative, not commutative, nor idempotent. A regular expression is said to be simple if the only boolean operator used is the sum.

The set of the regular expressions over an alphabet \(\Sigma\) is denoted by \(\text{Exp}(\Sigma)\). The language denoted by \(E\) is the subset \(L(E)\) of \(\Sigma^*\) inductively defined as follows: \(L(E \cdot F) = L(E) \cdot L(F), L(E^*) = (L(E))^*, L(f_k(E_1, \ldots, E_n)) = f_k(L(E_1), \ldots, L(E_n)), L(a) = \{a\}, L(0) = \emptyset,\) and \(L(1) = \{\varepsilon\}\) with \(E\) and \(F\) any two regular expressions, \(a\) any symbol in \(\Sigma\) and \(f_k\) the operator associated with \(f\) (\(\lor\) is associated to \(\lor\)). Whenever two expressions \(E_1\) and \(E_2\) denote the same language, \(E_1\) and \(E_2\) are said to be equivalent (denoted by \(E_1 \sim E_2\)). In the following, we denote by \(r_w(E)\) the boolean \(r_w(L(E))\). Notice that the boolean \(r_w(E)\) is straightforwardly computed as follows:

\[
\forall \alpha \in \Sigma \cup \{1, 0\}, r_w(\alpha) = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{otherwise,} \end{cases}
\]

\[
r_w(f_k(E_1, \ldots, E_k)) = f_k(r_w(E_1), \ldots, r_w(E_k)),
\]

\[
r_w(E_1 \cdot E_2) = r_w(E_1) \land r_w(E_2),
\]

\[
r_w(E_1^*) = 1.
\]

Given a regular expression \(E\) over an alphabet \(\Sigma\) and a symbol \(a\) in \(\Sigma\), the Brzozowski derivative of \(E\) w.r.t. \(a\) is the expression \(\frac{d}{da}(E)\) inductively defined by:

\[
\frac{d}{da}(E(a)) = 1, \frac{d}{da}(b) = \frac{d}{da}(1) = \frac{d}{da}(0) = 0,
\]

\[
\frac{d}{da}(f_k(E_1, \ldots, E_k)) = f_k(\frac{d}{da}(E_1), \ldots, \frac{d}{da}(E_k)), \frac{d}{da}(E_1^*) = \frac{d}{da}(E_1) \cdot E_1^*,
\]

\[
\frac{d}{da}(E_1 \cdot E_2) = \begin{cases} \frac{d}{da}(E_1) \cdot E_2 + \frac{d}{da}(E_2) & \text{if } r_w(E_1) = 1, \\ \frac{d}{da}(E_1) \cdot E_2 & \text{otherwise,} \end{cases}
\]

where \(b\) is any symbol in \(\Sigma \setminus \{a\}\), \(E_1, \ldots, E_k\) are any \(k\) regular expressions over \(\Sigma\), and \(f_k\) is the operator associated with the \(k\)-ary boolean function \(f\). Notice that Brzozowski defines the dissimilar derivative of \(E\) as the expression \(\frac{d}{da}(E)\) inductively computed by substituting the operator \(+_{ACI}\) to the \(+\) operator in the derivative formulas, where \(+_{ACI}\) is the associative, commutative and idempotent version of \(+\).
Given a simple regular expression \( E \) over an alphabet \( \Sigma \) and a symbol \( a \) in \( \Sigma \), the Antimirov partial derivative of \( E \) w.r.t. \( a \) is the set of expression \( \frac{\partial}{\partial a}(E) \) inductively defined by:

\[
\frac{\partial}{\partial a}(1) = \{1\}, \quad \frac{\partial}{\partial a}(b) = \frac{\partial}{\partial a}(1) = \frac{\partial}{\partial a}(0) = \emptyset,
\]

\[
\frac{\partial}{\partial a}(E_1 + E_2) = \frac{\partial}{\partial a}(E_1) \cup \frac{\partial}{\partial a}(E_2), \quad \frac{\partial}{\partial a}(E_1^*) = \frac{\partial}{\partial a}(E_1) \cdot E_1^*,
\]

\[
\frac{\partial}{\partial a}(E_1 \cdot E_2) = \begin{cases} \frac{\partial}{\partial a}(E_1) \cdot E_2, & \text{if } r_x(E_1) = 1, \\ \frac{\partial}{\partial a}(E_1) \cdot E_2, & \text{otherwise}, \end{cases}
\]

where \( b \) is any symbol in \( \Sigma \setminus \{a\} \), \( E_1, \ldots, E_k \) are any \( k \) regular expressions over \( \Sigma \), and for any set \( \mathcal{E} \) of regular expression, for any regular expression \( F \), \( \mathcal{E} \cdot F = \bigcup_{E \in \mathcal{E}} \{E \cdot F\} \).

## 3 Derivation via a Support

We now introduce the notion of a derivation via a support. Recall that a Brzozowski derivative is an expression, whereas an Antimirov derivative is a set of expressions. In our framework, a derivative is more generally an element of an arbitrary set, called the structure set. For example, a structure can be an expression, a set of expressions or a set of set of expressions. A support is essentially made of a structure set equipped with operators, and of a mapping that transforms a structure into an expression.

**Definition 1.** Let \( \Sigma \) be an alphabet. Let \( \mathcal{E} \) be a set and \( h \) be a mapping from \( \mathcal{E} \) to \( \text{Exp}(\Sigma) \). Let \( O \) be a set containing:

- for any \( k \)-ary boolean function \( f \), an operator \( f_\mathcal{E} \) from \( \mathcal{E}^k \) to \( \mathcal{E} \),
- an operator \( \cdot_\mathcal{E} \) from \( \mathcal{E} \times \text{Exp}(\Sigma) \) to \( \mathcal{E} \).

Let \( 1_\mathcal{E} \) and \( 0_\mathcal{E} \) be two elements in \( \mathcal{E} \). The 6-tuple \( S = (\Sigma, \mathcal{E}, h, O, 1_\mathcal{E}, 0_\mathcal{E}) \) is said to be a support if the three following conditions are satisfied:

1. for any \( k \) elements \( \mathcal{E}_1, \ldots, \mathcal{E}_k \) in \( \mathcal{E} \):
   \[
   h(f_\mathcal{E}(\mathcal{E}_1, \ldots, \mathcal{E}_k)) \sim f_c(h(\mathcal{E}_1), \ldots, h(\mathcal{E}_k)),
   \]
2. for any element \( \mathcal{E} \) in \( \mathcal{E} \), for any expression \( E \) in \( \text{Exp}(\Sigma) \):
   \[
   h(\mathcal{E} \cdot_\mathcal{E} E) \sim h(\mathcal{E}) \cdot E,
   \]
3. \( h(1_\mathcal{E}) \sim 1 \) and \( h(0_\mathcal{E}) \sim 0 \).

Notice that the expressions \( h(f_\mathcal{E}(\mathcal{E}_1, \ldots, \mathcal{E}_k)) \) and \( f_c(h(\mathcal{E}_1), \ldots, h(\mathcal{E}_k)) \) need not to be identical. They are only required to define the same language. A support is based on a set of generic structures that can be used to handle regular expressions. We now define the notion of regular expression derivation via a support.

**Definition 2.** Let \( S = (\Sigma, \mathcal{E}, h, O, 1_\mathcal{E}, 0_\mathcal{E}) \) be a support. The derivation via \( S \) is the mapping \( D \) from \( \Sigma^+ \times \text{Exp}(\Sigma) \) to \( \mathcal{E} \) inductively defined for any \( a \in \Sigma \), for any word \( w \) in \( \Sigma^+ \) and for any regular expression \( E \) in \( \text{Exp}(\Sigma) \) by:
Proposition 1. The regular expression via 3.1 Classical Derivations are Derivations via Zwozowski’s one are derivations via This subsection illustrates the fact that both Antimirov’s derivation and Brzo-Definition 3. We denote by is finite, the derivation D

\[
D(a, E) = \begin{cases} 
\text{f}_g(D(a, E_1), \ldots, D(a, E_k)) & E = \text{f}_e(E_1, \ldots, E_k), \\
(D(a, E_1) \lor E_2) \lor \varepsilon D(a, E_2) & E = E_1 \cdot E_2 \land \varepsilon \in L(E_1), \\
D(a, E_1) \lor E_2 & E = E_1 \land \varepsilon \notin L(E_1), \\
1_\varepsilon & E = a, \\
0_\varepsilon & E \in \Sigma \setminus \{a\} \cup \{1, 0\}.
\end{cases}
\]

\[D(w, E) = D(a, h(D(a, E))) \text{ if } w = au \land u \in \Sigma^+\]

Furthermore, if for all expression E in Exp(\Sigma), the set \(D(w, E) \mid w \in \Sigma^+\) is finite, the derivation D is said to be a finite derivation.

3.1 Classical Derivations are Derivations via a Support

This subsection illustrates the fact that both Antimirov’s one are derivations via a support.

Definition 3. We denote by \(S_A = (\Sigma, E = 2^{\text{Exp}(\Sigma)}, h_A, O_A, \{1\}, \emptyset)\) the 6-tuple defined by:

- for any \(E \in 2^{\text{Exp}(\Sigma)}\), \(h_A(E) = \sum_{E \in E} E\),
- \(O_A = \{f_\varepsilon \mid f \text{ is a k-ary boolean function}\} \cup \{\varepsilon\}\) where for any elements \(E_1, \ldots, E_k\) in \(2^{\text{Exp}(\Sigma)}\),
  - \(E \not\varepsilon F = \bigcup_{E \in E} \{E \cdot F\}\),
  - \(f_\varepsilon(E_1, \ldots, E_k) = E_1 \cup E_2\) if \(k = 2\) and \(f = \lor\),
  - \(f_\varepsilon(E_1, \ldots, E_k) = \{f_e(h_A(E_1), \ldots, h_A(E_k))\}\) otherwise.

Proposition 1. The 6-tuple \(S_A\) is a support. Furthermore, for any simple regular expression \(E\) over \(\Sigma\), for any symbol \(a\), it holds \(D_A(a, E) = \frac{\partial}{\partial a}(E)\), where \(D_A\) is the derivation via \(S_A\).

Proof. Let \(E_1\) and \(E_2\) be two sets of simple regular expressions and \(E\) be a simple regular expression. The condition (1) of Definition \(D\) is satisfied since:

- **(a)** if \(k = 2\) and \(f = \lor\),

\[
L(h_A(E_1 \cup E_2)) = L(\sum_{E \in E_1 \cup E_2} E) = \bigcup_{E \in E_1 \cup E_2} L(E) = \bigcup_{E \in E_1} L(E_1) \cup \bigcup_{E \in E_2} L(E_2) = L(\sum_{E \in E_1} E_1 + \sum_{E \in E_2} E_2) = L(h_A(E_1) + h_A(E_2))
\]

- **(b)** and otherwise,

\[
L(h_A(E_1, \ldots, E_k)) = L(h_A(\{f_e(h_A(E_1), \ldots, h_A(E_k))\})) = L(h_A(E_1, \ldots, h_A(E_k)))
\]

The condition (2) of Definition \(D\) is satisfied since:

\[
L(h_A(E_1 \cdot E)) = L(h_A(\bigcup_{E \in E_1} \{E_1 \cdot E\})) = L(\sum_{E \in E_1} E_1 \cdot E) = L(h_A(E_1) \cdot L(E)) = L(h_A(E_1)) \cdot L(E)
\]
The condition (3) of Definition 1 is satisfied since:
\[ L(h_A(\{1\})) = L(1) \]
\[ L(h_A(\emptyset)) = \emptyset = L(0) \]

Consequently, \( S_A \) is a support. Moreover, since the operators \( \cup \) and \( \cdot \) in \( O_A \) are the operations used in partial derivation, it can be shown by induction that for any simple regular expression \( E \) over \( \Sigma \) and for any symbol \( a \), \( D_A(a, E) = \frac{\partial}{\partial a}(E) \).

**Definition 4.** We denote by \( S_B = (\Sigma, \mathcal{E}', \mathcal{E}, h_B, O_B, 1_E, 0_E) \) the 6-tuple defined by:

- for any \( E \in \mathcal{E}(\Sigma) \), \( h_B(E) = E \),
- \( O_B = \{ f_E \mid f \text{ is a } k\text{-ary boolean function}\} \cup \{ \cdot_E \}, \) where for any \( E_1, \ldots, E_k \) elements in \( \mathcal{E}(\Sigma) \),
  - \( E \cdot_E F = E \cdot F \),
  - \( f_E(E_1, \ldots, E_k) = E_1 + E_2 \) if \( k = 2 \) and \( f = \lor \),
  - \( f_E(E_1, \ldots, E_k) = f(E_1, \ldots, E_k) \) otherwise.

**Proposition 2.** The 6-tuple \( S_B \) is a support. Furthermore, for any regular expression \( E \) over \( \Sigma \), for any symbol \( a \), it holds \( D_B(a, E) = \frac{\partial}{\partial a}(E) \), where \( D_B \) is the derivation via \( S_B \).

**Proof.** Since \( h_B \) is the identity and \( \mathcal{E}' = \mathcal{E}(\Sigma) \), it is obvious that \( S_B \) is a support. Moreover, since the operators in \( O_B \) are the operators of regular expressions, it can be shown by induction that for any regular expression \( E \) over \( \Sigma \) and for any symbol \( a \), \( D_B(a, E) = \frac{\partial}{\partial a}(E) \). \( \square \)

Let us notice that, by definition, the derivation \( D_A \) more generally addresses unrestricted expressions; therefore it provides a natural extension for Antimirov derivation. See [7] and Section 5 for alternative extensions.

### 3.2 Main Properties of Supports

We now show that the language denoted by the expression associated with any derivative \( D(w, E) \) is equal to the corresponding quotient.

**Proposition 3.** Let \( D \) be the derivation via a support \( S = (\Sigma, \mathcal{E}, h, O, 1_E, 0_E) \). Then for any word \( w \) in \( \Sigma^+ \), for any expression \( E \) in \( \mathcal{E}(\Sigma) \), it holds:
\[ L(h(D(w, E))) = w^{-1}(L(E)) \]

**Proof.** By recurrence over the length of \( w \).

1. Let \( w = a \in \Sigma \). By induction over the structure of \( E \).
\[ L(h(D(a, a))) = L(h(1_E)) = \{ \epsilon \} = a^{-1}(L(a)) \]
\[ L(h(D(a, b))) = L(h(D(a, 1))) = L(h(D(a, 0))) \]
\[ = L(h(0_E)) = \emptyset = a^{-1}(L(b)) = a^{-1}(L(1)) = a^{-1}(L(0)) \]
Let us consider that \( \varepsilon \in L(E_1) \):
\[
L(h(D(a,E_1 \cdot E_2))) = L(h(D(a,E_1) \cdot \varepsilon E_2))
\]
\[
= L(h(D(a,E_1))) \cup L(h(D(a,E_2)))
\]
\[
= a^{-1}(L(E_1)) \cdot L(E_2) \cup a^{-1}(L(E_2))
\]
\[
= a^{-1}(L(E_1)) \cdot L(E_2) = a^{-1}(L(E_1 \cdot E_2))
\]

Let us consider that \( \varepsilon \notin L(E_1) \):
\[
L(h(D(a,E_1 \cdot E_2))) = L(h(D(a,E_1) \cdot \varepsilon E_2))
\]
\[
= L(h(D(a,E_1))) \cdot L(E_2)
\]
\[
= a^{-1}(L(E_1)) \cdot L(E_2)
\]
\[
= a^{-1}(L(E_1 \cdot E_2))
\]

2. Let \( w = au \) with \( a \in \Sigma \) and \( u \in \Sigma^+ \). According to the recurrence hypothesis,
\[
L(h(D(w,E))) = L(h(D(u,h(D(a,E)))))
\]
\[
= u^{-1}(L(h(D(a,E)))) = u^{-1}(a^{-1}(L(E))) = (au)^{-1}(L(E))
\]

From Proposition 3 we deduce that \( r_\varepsilon(h(D(w,E))) = r_\varepsilon(w^{-1}L(E)) \). This property does not depend whether the derivation is finite or not and since the boolean \( r_\varepsilon(E) \) can be inductively computed for any regular expression \( E \), any support defines a syntactical solution of the membership problem of the language \( L(E) \).

**Corollary 1.** For a given regular expression \( E \), any derivation via a support can be used to solve the membership problem for \( L(E) \).

As an example, the support \( S_B \) of Definition 4 can be used to solve the membership test, even if the associated derivation is not finite.

Given an expression, the finiteness of the set of its derivatives is a necessary condition for the construction of an associated finite automaton. It is well-known that the set of Brzozowski’s derivatives is not necessarily finite whereas the set of dissimilar derivatives and the set of Antimirov’s derived terms are finite sets. We now give two sufficient conditions of finiteness in the general case. The first one has already been stated in the case of Brzozowski derivatives 4. The second one is related to the mapping \( h \) of the support, that needs to satisfy specific properties.
Proposition 4. Let $D$ be the derivation via a support $S = (\Sigma, E, h, \mathcal{O}, f_\varepsilon, 0_\varepsilon)$. The following set of conditions is sufficient for the mapping $D$ to be a finite derivation:

1. $\forall \varepsilon$ is associative, commutative and idempotent ($\textbf{H1}$);
2. for any $k$-ary boolean function $f$, for any $k$ elements $E_1, \ldots, E_k$ in $\mathcal{E}$, $D(a, h(f_\varepsilon(E_1, \ldots, E_k))) = f_\varepsilon(D(a, h(E_1)), \ldots, D(a, h(E_k)))$ ($\textbf{H2}$).

Proof. Let us show by induction over the structure of regular expressions that for any expression $E$ in $\text{Exp}(\Sigma)$, the set $\{D(w, E) \mid w \in \Sigma^+\}$ is finite.

If $E \in \Sigma \cup \{1, 0\}$: According to the definition of derivation, the proposition holds.

If $E = E_1^*$: Let $w$ be a word in $\Sigma^+$. Let us show by recurrence over the length of $w$ that $D(w, E_1^*)$ is a finite $\forall \varepsilon$-combination of elements in the set $\{D(w', E_1) \mid w' \neq \varepsilon \text{ is a suffix of } w\}$.

1. Let $w = a \in \Sigma$. Since $D(a, E_1^*) = D(a, E_1) \cdot E_1^*$, the property holds.
2. Let $w = ua$ with $a \in \Sigma$ and $u \in \Sigma^+$. By definition, $D(ua, E_1^*) = D(a, h(D(u, E_1^*)))$. By the recurrence hypothesis, $D(u, E_1^*)$ is a finite $\forall \varepsilon$-combination of elements in the set $\{D(w', E_1^*) \mid w' \neq \varepsilon \text{ is a suffix of } u\}$.

As a consequence, since the set $\{D(w, E_1) \mid w \in \Sigma^+\}$ is a finite set by induction hypothesis, since $\text{Card}(\bigcup_{w \in \Sigma^+} \{D(w', E_1) \mid w' \neq \varepsilon \text{ is a suffix of } w\}) = \text{Card}(\bigcup_{w \in \Sigma^+} \{D(w, E_1)\})$ and since $\forall \varepsilon$ is associative, commutative and idempotent, we get:

$$\text{Card}(\{D(w, E_1^*) \mid w \in \Sigma^+\}) \leq 2^{\text{Card}(\{D(w, E_1) \mid w \in \Sigma^+\})}.$$ 

If $E = f_\varepsilon(E_1, \ldots, E_k)$: Let $w$ be a word in $\Sigma^+$. Let us show by recurrence over the length of $w$ that $D(w, f_\varepsilon(E_1, \ldots, E_k)) = f_\varepsilon(D(w, E_1), \ldots, D(w, E_k))$.

1. Let $w = a \in \Sigma$. According to the definition of $D$, the property holds.
2. Let $w = ua$ with $a \in \Sigma$ and $u \in \Sigma^+$. By definition, $D(ua, f_\varepsilon(E_1, \ldots, E_k)) = D(a, h(D(u, f_\varepsilon(E_1, \ldots, E_k))))$. By the recurrence hypothesis, $D(u, f_\varepsilon(E_1, \ldots, E_k)) = f_\varepsilon(D(u, E_1), \ldots, D(u, E_k))$. According to hypothesis $\textbf{H2}$, $D(a, h(D(u, E_1), \ldots, D(u, E_k))) = f_\varepsilon(D(a, h(D(u, E_1))), \ldots, D(a, h(D(u, E_k)))) = f_\varepsilon(D(ua, E_1), \ldots, D(ua, E_k))$.

As a consequence, since for all integer $j$ in $\{1, \ldots, k\}$ the set $\{D(w, E_j) \mid w \in \Sigma^+\}$ is finite by induction hypothesis, we get:
it is possible to compute a finite set of dissimilar derivatives using a quotient
function $D$ which is associative, commutative and idempotent, and since for any
word $w$, the length of $w$ is finite by induction hypothesis and since
$\cup$ is associative, commutative and idempotent, we get:
\[
\{ w \} = \{ w \} \times \cdots \times \{ w \}.
\]

If $E = E_1 \cdot E_2$: Let $w$ be a word in $\Sigma^+$. Let us show by recurrence over
the length of $w$ that either $D(w, E_1 \cdot E_2) = D(w, E_1) \cdot E_2$ or $D(w, E_1 \cdot E_2) =
(D(w, E_1) \cdot E_2) \vee \emptyset$ where $\emptyset$ is a finite $\vee$-combination of elements in the set
$\{ D(w', E_2) \mid w' \neq \varepsilon \text{ is a suffix of } w \}$.

1. Let $w = a \in \Sigma$. According to the definition of $D$, the property holds.

2. Let $w = ua$ with $a \in \Sigma$ and $u \in \Sigma^+$.

By definition, $D(ua, E_1 \cdot E_2) = D(a, h(D(u, E_1 \cdot E_2)))$.

Two cases have to be considered:

(a) $D(u, E_1 \cdot E_2) = D(u, E_1) \cdot E_2$. Either $D(u, E_1 \cdot E_2) = D(a, h(D(u, E_1) \cdot E_2))$, or $D(u, E_1 \cdot E_2) = D(a, h(D(u, E_1) \cdot E_2)) \vee \emptyset D(a, h(E))$.

According to the recurrence hypothesis, both of these cases satisfy
the proposition.

(b) $D(u, E_1 \cdot E_2) = (D(u, E_1) \cdot E_2) \vee \emptyset E$ where $E$ is a finite $\vee$-combination
of elements in the set $\{ D(w', E_2) \mid w' \neq \varepsilon \text{ is a suffix of } u \}$. Either
$D(u, E_1 \cdot E_2) = D(a, h(D(u, E_1) \cdot E_2)) \vee \emptyset D(a, h(E))$ or $D(u, E_1 \cdot
E_2) = D(a, h(D(u, E_1) \cdot E_2)) \vee \emptyset D(a, h(E))$. According to the recurrence hypothesis $E^2 \vee E' = D(a, h(E))$ is a finite $\vee$-combination
of elements in the set $\{ D(a, h(D(w', E_2))) \mid w' \neq \varepsilon \text{ is a suffix of } u \}$, set
that equals $\{ D(w', E_2) \mid w' \neq \varepsilon \text{ is a suffix of } ua \}$.

Consequently, $D(u, E_1 \cdot E_2) = (D(u, E_1) \cdot E_2) \vee \emptyset E$ where $E$ is a finite $\vee$-combination
of elements in the set $\{ D(w', E_2) \mid w' \neq \varepsilon \text{ is a suffix of } u \}$.

As a consequence, since the sets $\{ D(w, E_1) \mid w \in \Sigma^+ \}$ and $\{ D(w, E_2) \mid w \in \Sigma^+ \}$
are finite by induction hypothesis and since $\vee$ is associative, commutative
and idempotent, we get:

\[
\text{Card}(\{ D(w, E_1 \cdot E_2) \mid w \in \Sigma^+ \}) \leq \text{Card}(\{ D(w, E_1) \mid w \in \Sigma^+ \}) \times \text{Card}(\{ D(w, E_2) \mid w \in \Sigma^+ \}).
\]

The derivation $D_A$ of Definition 3 is an example of finite derivation since
$\cup$ is associative, commutative and idempotent, and since for any $k$-ary boolean
function $f$ and for any $k$ elements $\mathcal{E}_1, \ldots, \mathcal{E}_k$ in $2^{\Exp(\Sigma)}$, we have:
\[
D_A(a, h_A(f_A(\mathcal{E}_1, \ldots, \mathcal{E}_k))) = f_2(D_A(a, h_A(\mathcal{E}_1)), \ldots, D_A(a, h_A(\mathcal{E}_k))).
\]

On the opposite, since $+$ is not an ACI law, Proposition 4 does not allow us
to conclude for the derivation $D_B$ of Definition 4. Brzozowski showed in 4 that
it is possible to compute a finite set of dissimilar derivatives using a quotient
of the expressions w.r.t. an ACI sum. It can be achieved by considering the
support $S_B'$ defined as follows:
Definition 5. We denote by $S_B' = (\Sigma, \mathbb{E}'' = 2^{\exp(\Sigma)}, h_A, O_B', \{1\}, \{0\})$ the 6-tuple defined by:

- for any $E \in 2^{\exp(\Sigma)}$, $h_A(E) = \sum_{E \in \mathcal{E}} E$ (Definition 3);
- $O_B' = \{f_{E'} | f$ is a $k$-ary boolean function$\} \cup \{\cdot_{E'}\}$, where for any $E_1, \ldots, E_k$ elements in $\mathbb{E}''$,
  - $\cdot_{E_1 E_2} F = (\sum_{E \in E_1} E) \cdot F$,
  - $f_{E_1, \ldots, E_k}(E_1) = \bigcup_{E_2} E_2$ if $k = 2$ and $f = \lor$,
  - $f_{E_1, \ldots, E_k}(E_1, \ldots, E_k) = \{f_e(h_A(E_1), \ldots, h_A(E_k))\}$ otherwise.

Proposition 5. The 6-tuple $S_B'$ is a support. Furthermore, for any regular expression $E$ and for any symbol $a$, it holds that $h_A(D_B'(a, E))$ is the dissimilar derivative of $E$ w.r.t. $a$, where $D_B'$ is the derivation via the support $S_B'$.

Proof. According to Definition 3, Definition 5, and Proposition 1, the conditions (1) and (3) of Definition 1 are satisfied by $S_B'$.

The condition (2) of Definition 1 is satisfied since:

$$
L(h_A(\cdot_{E'} E)) = L(h_A((\sum_{F \in E_1} F) \cdot E)) = L((\sum_{F \in E_1} F) \cdot E) = L(\sum_{F \in E_1} F) \cdot L(E) = L(h_A(E_1)) \cdot L(E)
$$

Consequently, $S_B'$ is a support.

Moreover, since the operator $\cup$ is an ACI law and since the catenation product $\cdot_{E'}$ returns a singleton, it can be shown by induction that for any regular expression $E$ and for any symbol $a$, it holds that $h_A(D_B'(a, E))$ is the dissimilar derivative of $E$ w.r.t. $a$. $\square$

4 From Derivation via a Support to Automata

Computing the set of derivatives of a regular expression $E$ w.r.t. a derivation $D$ is similar as computing the transition function $\delta$ of an automaton, where $\delta(E, w) = h(D(w, E))$. As far as alternating automata are concerned, the resulting expression $h(D(w, E))$ needs to be transformed into a boolean formula. This computation is performed through a base function defined as follows.

Definition 6. Let $\Sigma$ be an alphabet. A base function $B$ is a mapping from $\exp(\Sigma)$ to $\text{BoolForm}(\exp(\Sigma))$ such that for any expression $E$ and for any word $w$ in $\Sigma^*$:

$$
w \in L(E) \iff \text{eval}_{r_w}(B(E)) = 1.
$$

Definition 7. Let $B$ be a base function and $D$ be the derivation via the support $S = (\Sigma, \mathbb{E}, h, O, 1_{\mathbb{E}}, 0_{\mathbb{E}})$. Let $E$ be an expression in $\exp(\Sigma)$. Let $A = (\Sigma, Q, I, F, \delta)$ be the automaton defined by:

- $Q = \exp(\Sigma)$,
$I = E,$

$\forall q \in Q, F(q) = \begin{cases} 1 & \text{if } \varepsilon \in L(q), \\ 0 & \text{otherwise}, \end{cases}$

$\forall a \in \Sigma, \forall q \in Q, \delta(q, a) = B(h(D(a, q))).$

The accessible part of $A$ is said to be the $(D, B)$-alternating automaton of $E$. Notice that there may exist an infinite number of states.

**Theorem 1.** The $(D, B)$-alternating automaton of a regular expression $E$ recognizes $L(E)$.

**Proof.** Let $A = (\Sigma, Q, I, F, \delta)$ be the $(D, B)$-alternating automaton of $E$. Let $D$ be the derivation via the support $S = (\Sigma, E, h, O, 1_\Sigma, 0_\Sigma)$. Let $w$ be a word in $\Sigma^+$. Let us show by recurrence over the length of $w$ that for any boolean formula $\phi = f_\Sigma(q_1, \ldots, q_k)$ in FormBool($Q$), the following (P) proposition holds:

$\text{eval}_F(\delta(\phi, w)) = \text{eval}_r(f_\Sigma(B(h(D(w, q_1))), \ldots, B(h(D(w, q_k)))).$  

1. If $w = a \in \Sigma$, by definition of the transition function $\delta$, $\delta(\phi, a) = f_\Sigma(\delta(q_1, a), \ldots, \delta(q_k, a))$. By construction, for any integer $j$ in $\{1, \ldots, k\}$, $\delta(q_j, a) = B(h(D(a, q_j)))$.

Since for any state $q \in Q$, $F(q) = 1 \Leftrightarrow \varepsilon \in L(q)$, the following proposition holds: $\text{eval}_F(\delta(\phi, a)) = \text{eval}_r(f_\Sigma(B(h(D(a, q_1))), \ldots, B(h(D(a, q_k))))$.

2. Let $w = au$ with $a \in \Sigma$ and $u \in \Sigma^+$. Then it holds:

$\text{eval}_F(\delta(\phi, au)) = \text{eval}_F(\delta(\phi, a), u))$ (Definition of $\delta$)

$= \text{eval}_F(\delta(f_\Sigma(\delta(q_1, a), \ldots, \delta(q_k, a))), u))$ (Definition of $\delta(\phi, a)$)

$= \text{eval}_F(f_\Sigma(\delta(\delta(q_1, a), u), \ldots, \delta(\delta(q_k, a), u)))$ (Definition of $\delta$)

$= \text{eval}_F(f_\Sigma(\delta(B(h(D(a, q_1))), u), \ldots, \delta(B(h(D(a, q_k))))))$ (Construction of $\delta$)

$= f(\text{eval}_F(\delta(B(h(D(a, q_1))), u)), \ldots, \text{eval}_F(\delta(B(h(D(a, q_k))))))$ (Definition of $\text{eval}_F$)

$= f(\text{eval}_r(B(h(D(u, h(D(a, q_1)))))), \ldots, \text{eval}_r(B(h(D(u, h(D(a, q_k)))))$ (Induction hypothesis)

$= \text{eval}_r(f_\Sigma(B(h(D(u, h(D(a, q_1)))))), \ldots, B(h(D(u, h(D(a, q_k)))))$ (Definition of $\text{eval}_F$)

$= \text{eval}_r(f_\Sigma(B(h(D(au, q_1)))), \ldots, B(h(D(au, q_k))))$ (Definition of $D$)

Finally, it holds:

$w \in L(E) \Leftrightarrow \varepsilon \in L(h(D(w, E)))$ (Proposition 3)

$\Leftrightarrow \text{eval}_r(B((h(D(w, E))))) = 1$ (Definition 6)

$\Leftrightarrow \text{eval}_F(\delta(E, w)) = 1$ (Proposition (P))

$\Leftrightarrow w \in L(A)$ (Definition of the language of an AA)
Previous definitions and properties address non necessarily finite automata. We now give sufficient conditions for the finiteness of the automaton. The basic idea is that if it is equivalent to derive an expression \( E \) or to derive its atoms \( \text{Atom}(B(E)) \), then the set of atoms obtained by a finite derivation is finite.

**Definition 8.** Let \( S = (\Sigma, E, h, O, 1_{E}, 0_{E}) \) be a support, let \( D \) be the derivation via \( S \) and \( B \) be a base function. The couple \((D, B)\) satisfies the atom-derivability property if for any expression \( E \) in \( \text{Exp}(\Sigma) \) and for any symbol \( a \) in \( \Sigma \):

\[
\text{Atom}(B(h(D(a, E)))) = \bigcup_{E' \in \text{Atom}(B(E))} \text{Atom}(B(h(D(a, E')))).
\]

**Theorem 2.** Let \( A \) be the \((D, B)\)-alternating automaton of a regular expression \( E \). If \( D \) is finite and if \((D, B)\) satisfies the atom-derivability property, then:

\( A \) is an AFA.

**Proof.** Let \( D \) be the derivation via the support \( S = (\Sigma, E, h, O, 1_{E}, 0_{E}) \). Let \( w \) be a word in \( \Sigma^+ \).

1. Let us show by recurrence over the length of \( w \) that for any expression \( q \) in \( Q \), \( \text{Atom}(B(h(D(w, q)))) = \text{Atom}(\delta(q, w)) \).

   \( (a) \) If \( w = a \in \Sigma \), \( \delta(q, a) = B(h(D(a, q))) \). Then, \( \text{Atom}(B(h(D(a, q)))) = \text{Atom}(\delta(q, a)) \).

   \( (b) \) If \( w = ua \) with \( a \in \Sigma \) and \( u \in \Sigma^+ \), by the recurrence hypothesis, \( \text{Atom}(\delta(q, u)) = \text{Atom}(B(D(u, q))) \).

\[
\text{Atom}(\delta(q, ua))
= \text{Atom}(\delta(\delta(q, u), a)) \tag{Definition of \( \delta \)}
= \text{Atom}(\delta(f_q(q_1', \ldots, q_j'), a)) \tag{Definition of \( \delta \)}
= \bigcup_{q' \in \{q_1', \ldots, q_j'\}} \text{Atom}(\delta(q', a)) \tag{Definition of Atom}
= \bigcup_{q' \in \text{ Atom}(\delta(q, u))} \text{Atom}(\delta(q', a)) \tag{Induction hypothesis and construction of \( \delta \)}
= \bigcup_{q' \in \text{Atom}(\delta(q, u))} \text{Atom}(B(h(D(u, q')))) \tag{Atom-derivability property}
= \text{Atom}(B(h(D(a, h(D(u, q))))))
\]

2. As a direct consequence of the previous point, since the set \( \{D(w, E) \mid w \in \Sigma^+ \} \) is finite, so are the sets \( \{B(h(D(w, E))) \mid w \in \Sigma^+ \} \) and \( \bigcup_{w \in \Sigma^+} \text{Atom}(B(h(D(w, q)))) \). Finally, the set \( Q \), that is equal to \( \bigcup_{w \in \Sigma^+} \text{Atom}(\delta(q, w)) \), is a finite set.

\[\square\]

**Example 1.** Let \( D_A \) and \( D_B' \) be the derivations of Definition 9 and Definition 8. Let \( B_A \) be the base inductively defined for any expression by \( B_A(E + F) = B_A(E) \lor B_A(F), B_A(E) = E \) otherwise. Let \( B_B \) the base defined for any expression by \( B_B(E) = E \). It can be shown that any couple in \( \{D_A, D_B'\} \times \{B_A, B_B\} \) satisfies the atom-derivability property.
Furthermore, the \((D_A, B_A)\)-AFA of \(E\) can be straightforwardly transformed into the derived term NFA of \(E\) defined by Antimirov in \([2]\); the \((D_A, B_B)\)-AFA of \(E\) can be transformed into the partial derivative DFA of \(E\) defined by Antimirov in \([1]\); the \((D_B, B_B)\)-AFA of \(E\) can be transformed into the dissimilar derivative DFA of \(E\) defined by Brzozowski in \([3]\). Notice that the \((D_B', B_B)\)-AFA of \(E\) can be transformed into a NFA that is different from the NFA of Antimirov.

Finally, the base \(B_C\) inductively defined by \(B_C(f)(E_1, \ldots, E_k) = f_B(B_C(E_1), \ldots, B_C(E_k))\) for any operator \(f\), \(B_C(E) = E\) otherwise, provides an AFA construction both from \(D_A\) and \(D_B'\).

5 Derivation via the Set of Clausal Forms

In this section, we show that the set of clausal forms over the set of regular expressions and equipped with the right operators is a derivation support and that the associated \(D_C\) derivation is finite. Furthermore we prove that the atom-derivability property is satisfied whenever the \(D_C\) derivation is associated with a base function in the set \(\{B_A, B_B, B_C\}\). Finally we illustrate these results by the construction of the \((D_C, B_C)\)-AFA of the expression \(E = ((ab)^a) \text{XOR} ((abab)^a)\).

Let us first recall some definitions about clausal forms and their operators. A clausal form over a set \(X\) is an element in \(C(X) = 2^{2^X} \cup X\) where \(X = \{\overline{x} \mid x \in X\}\). Let \(\oplus\) and \(\otimes\) be the two mappings from \(C(X) \times C(X)\) to \(C(X)\) and \(\ominus\) be the function from \(C(X)\) to \(C(X)\) defined for any \(C_1, C_2\) in \(C(X)\) by:

- \(C_1 \oplus C_2 = C_1 \cup C_2\),
- \(C_1 \otimes C_2 = \bigcup_{(C_1, C_2) \in C_1 \times C_2} (C_1 \cup C_2)\),
- \(\ominus(C_1) = \begin{cases} \{\emptyset\} & \text{if } C_1 = \emptyset, \\ \emptyset & \text{if } C_1 = \{\emptyset\}, \\ \ominus C_{\mathcal{E} \in C_1} \oplus_{c \in C} \{\mathcal{n}(c)\} & \text{otherwise}, \end{cases}\)

where \(\mathcal{n}(c) = \begin{cases} c' & \text{if } c = \overline{c'}, \\ \overline{c} & \text{otherwise}. \end{cases}\)

It can be shown that for any element \(x \in X \cup \overline{X}\), for any clause \(C_1, C_2, C_3\) in \(C(X)\), the following conditions are satisfied:

- \(C_1 \otimes (C_2 \oplus C_3) = (C_1 \otimes C_2) \oplus (C_1 \otimes C_3)\),
- \(\ominus \ominus \{x\} = \{x\}\),
- \(\ominus(C_1 \otimes C_2) = \ominus(C_1) \otimes \ominus(C_2)\),
- \(\ominus(C_1 \otimes C_2) = \ominus(C_1) \oplus \ominus(C_2)\).
Furthermore, let us notice that there exist clauses $C_1, C_2, C_3$ such that $\ominus \ominus C_1 \neq C_1$ or $C_1 \ominus (C_2 \ominus C_3) = (C_1 \ominus C_2) \ominus (C_1 \ominus C_3)$.

From now on, we will consider the set $\mathbb{C} = \mathbb{C}(\mathbb{E}(\Sigma))$ of the clausal forms over the set of regular expressions. We now explain how a clausal form over the set of regular expressions is transformed into a regular expression. Let us consider the function $h_C$ defined from $\mathbb{C}$ to $\mathbb{E}(\Sigma)$ for any element $C$ in $\mathbb{C}$ by:

$$h_C(C) = \begin{cases} 0 & \text{if } C = \emptyset, \\ \sum_{C \in \mathbb{C}} \neg \left( \bigwedge_{E \in C} c(E) \right) & \text{if } C \neq \emptyset, \end{cases}$$

where $c(E) = \begin{cases} \neg e(E) & \text{if } E = E', \\ E & \text{otherwise.} \end{cases}$

Let us now give the definition of the support operators. For any $k$-ary boolean function $f$, let $f_C$ be the operator from $(\mathbb{C})^k$ to $\mathbb{C}$ associated with $f$ defined by:

- $f_C(C_1, \ldots, C_k) = \bigoplus_{b=(b_1, \ldots, b_k) \in \mathbb{B}^k} \bigvee_{1 \leq j \leq k} g(b_j, C_j)$,

where $g(b_j, C_j) = \begin{cases} C_j & \text{if } b_j = 1, \\ \ominus C_j & \text{otherwise.} \end{cases}$

The operator $\cdot_C$ from $\mathbb{C} \times \mathbb{E}(\Sigma)$ to $\mathbb{C}$ is defined, for any clause $C$ in $\mathbb{C}$ and for any expression $F$ in $\mathbb{E}(\Sigma)$, by:

$$C \cdot_C F = \bigcup_{C \in \mathbb{C}} \{ h_C\{C\} \cdot F \}.$$  

Finally, we consider the operation set $\mathbb{O}_C$ defined by

$$\mathbb{O}_C = \{ \ominus, \cdot, f_C \} \cup \{ f_C \mid f \text{ is a } k\text{-ary boolean function different from } \vee \}.$$  

Let us notice that, by definition, the operator $\oplus$ (resp. $\ominus$) is equal to the operator $\ominus$ (resp. $\land$) whereas the operator $\ominus$ is different from $\lor$, since:

$$C_1 \lor C_2 = (\ominus(\ominus C_1) \ominus C_2) \ominus (\ominus(\ominus C_1) \ominus C_2)) \ominus (\ominus C_1 \ominus C_2).$$

There exist several expressions (combinations of $\oplus$, $\ominus$ and $\ominus$) for a given $f_C$ operator. As an example, with the ternary $\lor^3(b_1, b_2, b_3) = b_1 \lor b_2 \lor b_3$ is associated an operator $\lor^3_C$ that can be expressed as the combination of two $\ominus$ operators:

$$\lor^3_C(C_1, C_2, C_3) = (C_1 \ominus C_2) \ominus C_3.$$  

Reduced expressions can be found using Karnaugh maps for instance.  

We now consider the 6-tuple $\mathbb{S}_C = (\Sigma, \mathbb{C}, \mathbb{O}_C, h_C, \{\{1\}\}, \emptyset)$ and show that it is a support.

**Proposition 6.** The 6-tuple $\mathbb{S}_C$ is a support.

**Proof.** Properties of support are trivially checked for the clauses $\{\{1\}\}$, $\emptyset$ and $\{\emptyset\}$. Let us consider that $C, C_1, \ldots, C_k$ are elements in $\mathbb{C} \setminus \{\emptyset, \{\emptyset\}\}$. According to the definitions of $\ominus$ and $\ominus$:
Hence, according to definition of $L(h_C(\oplus(C_1, C_2)))$
\begin{align*}
L(h_C(C_1 \oplus C_2)) &= L(h_C(C_1 \cup C_2)) \\
&= L(\bigcup_{c \in C_1 \cup C_2} \wedge_{E \in E_C} c(E)) \\
&= \bigcup_{c \in C_1 \cup C_2} L(\wedge_{E \in E_C} c(E)) \\
&= \bigcup_{c \in C_1} L(\bigwedge_{E \in E_C} c(E)) \cup \bigcup_{c \in C_2} L(\bigwedge_{E \in E_C} c(E)) \\
&= L(h_C(C_1)) \cup L(h_C(C_2)) \\
&= L(h_C(C_1)) \cup h_C(C_2)) \\
L(h_C(C_1 \oplus C_2)) &= L(h_C(\bigcup_{(c_1, c_2) \in C_1 \times C_2} \{C_1 \cup C_2\})) \\
&= \bigcup_{(c_1, c_2) \in C_1 \times C_2} L(h_C(\{C_1 \cup C_2\})) \\
&= \bigcup_{(c_1, c_2) \in C_1 \times C_2} L(\bigwedge_{E \in E_C} c(E)) \\
&= \bigcup_{(c_1, c_2) \in C_1 \times C_2} L(h_C(\{C_1\}) \cap L(h_C(\{C_2\})) \\
&= \bigcup_{c \in C} L(h_C(\{C\}) \cap L(h_C(\{C\})) \\
&= L(h_C(C_1) \wedge h_C(C_2)) \\

Moreover, 
\begin{align*}
L(h_C(\oplus(\{c\}))) &= L(h_C(\{n(c)\})) \\
&= \begin{cases} 
L(h_C(\{E\})) & \text{if } c = E \\
L(h_C(\{E\})) & \text{if } c = \overline{E} \\
L(\lnot E) & \text{if } c = E \\
L(E) & \text{if } c = \overline{E} \\
\lnot L(E) & \text{if } c = E \\
\lnot L(L(\lnot E)) & \text{if } c = \overline{E} \\
\lnot L(L(h_C(\{c\}))) & \text{if } c = E \\
\lnot L(L(h_C(\{c\}))) & \text{if } c = \overline{E}
\end{cases}
\end{align*}

Consequently :
\begin{align*}
L(h_C(\oplus(\{C_1\}))) &= L(h_C(\bigoplus_{C_1 \in C_1} \bigoplus_{c \in C} \{n(c)\})) \\
&= \bigcap_{c \in C_1} L(h_C(\bigoplus_{c \in C} \{n(c)\})) \\
&= \bigcap_{c_1 \in C_1} L(h_C(\bigoplus_{c \in C} \{c\})) \\
&= \bigcap_{c_1 \in C_1} L(h_C(\bigoplus_{c \in C} \{n(c)\})) \\
&= \bigcap_{c_1 \in C_1} L(h_C(\bigoplus_{c \in C} \{c\})) \\
&= \bigcap_{c_1 \in C_1} L(h_C(\bigoplus_{c \in C} \{c\})) \\
&= \bigcap_{c_1 \in C_1} \lnot L(h_C(\{c\})) \\
&= \bigcap_{c_1 \in C_1} \lnot L(h_C(\{c\})) \\
&= \lnot L(h_C(\{c\})) \\
&= \lnot L(h_C(\{c\}))) \\
&= L(h_C(C_1)) \\
&= L(\lnot L(h_C(\{c\}))) \\
&= L(\lnot L(h_C(\{c\})))
\end{align*}

Hence, according to definition of $f_C$:
\begin{align*}
L(h_C(f_C(\subseteq(C_1, \ldots, C_k)))) &= L(h_C(\bigoplus_{b \in (b_1, \ldots, b_k)} \{b_1 \leq j \leq k \bigwedge g(b_j, C_j)\}) \\
&= \bigcup_{b \in (b_1, \ldots, b_k)} \bigcap_{1 \leq j \leq k} \bigwedge_{1 \leq j \leq k} L(h_C(g(b_j, C_j))) \\
&= \bigcup_{b \in (b_1, \ldots, b_k)} \bigcap_{1 \leq j \leq k} L(h_C(g(b_j, C_j))) \\
&= \bigcup_{b \in (b_1, \ldots, b_k)} \bigcap_{1 \leq j \leq k} L(h_C(g(b_j, C_j))) \\
&= f_L(L(h_C(C_1)), \ldots, L(h_C(C_k))) \\
&= L(f, L(h_C(C_1)), \ldots, L(h_C(C_k)))
\end{align*}
Furthermore,
\[ L(h_C(C \cdot F)) = L(h_C(\bigcup_{C \in C}((\bigwedge_{E \in C} c(E)) \cdot F))) \]
\[ = \bigcup_{C \in C} L((\bigwedge_{E \in C} c(E)) \cdot F) \]
\[ = \bigcup_{C \in C} L((\bigwedge_{E \in C} c(E))) \cdot L(F) \]
\[ = (\bigcup_{C \in C} L(\bigwedge_{E \in C} c(E))) \cdot L(F) \]
\[ = L(\bigcup_{C \in C} \bigwedge_{E \in C} c(E)) \cdot L(F) \]
\[ = L(h_C(C)) \cdot L(F) \]
\[ = L(h_C(C) \cdot F) \]
\[ \square \]

We now study the properties of the derivation \( D_C \) associated with the support \( S_C \).

**Theorem 3.** Let \( E \) be an expression in \( \text{Exp}(\Sigma) \). Let \( (D, B) \) be a couple in \( \{D_C\} \times \{B_A, B_B, B_C\} \). Then:

The \( (D, B) \)-automaton of \( E \) is an AFA recognizing \( L(E) \).

**Proof.** (I) Let us show that the derivation \( D_C \) satisfies the sufficient conditions for finiteness of Proposition \[\text{H1} \] (a) Since \( \oplus = \cup \), the function \( \oplus \) is associative, commutative and idempotent.  (b) According to the definition of the operators \( \oplus, \otimes \) and \( \oplus \):

\[ D_C(a, h_C(C_1 \oplus C_2)) = D_C(a, h_C(C_1 \cup C_2)) \]
\[ = D_C(a, \sum_{C \in C_1 \cup C_2} \left\{ (\neg c_0 \text{ if } C = \emptyset, \bigwedge_{E \in C} c(E) \text{ otherwise, } \right\}) \]
\[ = D_C(a, \sum_{C \in C_1 \cup C_2} \left\{ D_C(a, \neg c_0) \text{ if } C = \emptyset, D_C(a, \bigwedge_{E \in C} c(E)) \text{ otherwise, } \right\}) \]
\[ = \bigcup_{C \in C_1} \left\{ D_C(a, \neg c_0) \text{ if } C = \emptyset, D_C(a, \bigwedge_{E \in C} c(E)) \text{ otherwise, } \right\} \]
\[ \cup \bigcup_{C \in C_2} \left\{ D_C(a, \neg c_0) \text{ if } C = \emptyset, D_C(a, \bigwedge_{E \in C} c(E)) \text{ otherwise, } \right\} \]
\[ = D_C(a, \sum_{C \in C_1} \left\{ \bigwedge_{E \in C} c(E) \text{ otherwise, } \right\}) \]
\[ \cup D_C(a, \sum_{C \in C_2} \left\{ \bigwedge_{E \in C} c(E) \text{ otherwise, } \right\}) \]
\[ = D_C(a, h_C(C_1)) \oplus D_C(a, h_C(C_2)). \]

\[ D_C(a, h_C(C_1 \otimes C_2)) = D_C(a, h_C(\bigcup_{(C_1, C_2) \in C_1 \times C_2} \{C_1 \cup C_2\})) \]
\[ = \bigcup_{(C_1, C_2) \in C_1 \times C_2} D_C(a, h_C(\{C_1 \cup C_2\})) \]
\[ = \bigcup_{(C_1, C_2) \in C_1 \times C_2} D_C(a, h_C(\{C_1\} \cup \{C_2\})) \]
\[ = \bigcup_{(C_1, C_2) \in C_1 \times C_2} D_C(a, h_C(\{C_1\})) \cup D_C(a, h_C(\{C_2\})) \]
\[ = D_C(a, h_C(C_1)) \otimes D_C(a, h_C(C_2)). \]
Let us show that any couple in \((II)\) satisfies the atom-derivability property. (a) By definition of \(B_B\), the atom-derivability property is satisfied by \((D_C,B_B)\). (b) By induction over the structure of \(E\). Let \(a\) be a symbol in \(\Sigma\). (i) If \(E \in \Sigma \cup \{1,0\}\) or if \(E = F \cdot G\) or if \(E = F^*\), since \(B_A(E) = B_C(E) = E\), \(\text{Atom}(B_A(E)) = \text{Atom}(B_C(E)) = \{E\}\). (ii) Let \(f\) be a \(k\)-ary boolean function. Let us first prove that the equation of the atom-derivability property is satisfied for the operators \(\oplus\), \(\otimes\) and \(\ominus\). If \(C_1\) or \(C_2\) equal \(\emptyset\) or \(\{\emptyset\}\), equation is trivially satisfied. Let \(C_1\) and \(C_2\) be two clauses different from \(\emptyset\) and \(\{\emptyset\}\).

\[
\begin{align*}
\text{Atom}(B_C(h_C(\cup C_2)))) &= \text{Atom}(B_C(\sum_{c \in C_1 \cup C_2} \land c \in E \in C_2(E))) \\
&= \text{Atom}(\bigvee_{C \in C_1 \cup C_2} B_C(\land c \in E \in C_2(E))) \\
&= \bigcup_{C \in C_1 \cup C_2} \text{Atom}(B_C(\land c \in E \in C_2(E))) \\
&= \bigcup_{C \in C_1} \text{Atom}(B_C(\land c \in E \in C_2(E))) \\
&\quad \cup \bigcup_{C \in C_2} \text{Atom}(B_C(\land c \in E \in C_2(E))) \\
&= \text{Atom}(\bigvee_{C \in C_1 \cup C_2} B_C(\land c \in E \in C_2(E))) \\
&\quad \cup \text{Atom}(B_C(\sum_{C \in C_1 \cup C_2} \land c \in E \in C_2(E))) \\
&\quad \cup \text{Atom}(B_C(\sum_{C \in C_1 \cup C_2} \land c \in E \in C_2(E))) \\
&= \text{Atom}(B_C(h_C(C_1))) \cup \text{Atom}(B_C(h_C(C_2))) \\
\end{align*}
\]
According to Theorem 2, from (I) and (II), the theorem holds. □

The following example illustrates the computation of an AFA from a regular expression. In order to improve readability, regular expressions are simplified according to the following rules:

\[
\begin{align*}
    E + 0 &\equiv 0 + E \equiv E \\
    E \cdot 0 &\equiv 0 \cdot E \equiv 0 \\
    E \cdot 1 &\equiv 1 \cdot E \equiv E
\end{align*}
\]
Example 2. Let $E = ((ab)^*a)\ XOR((abab)^*a)$. We now construct the $(D_C, B_C)$-automaton of $E$, that is the AFA $(\Sigma, Q, E, F, \delta)$. We first compute the derivatives $C$ of $E$ w.r.t. any symbol in $\Sigma$, according to the derivation $D_C$ and then compute the derivatives of the expressions that are atoms of the base of $h_C(C)$.

This scheme is repeated until no more expression is produced.

$$D_C(a, E) = \bigoplus(D_C(a, (ab)^*a)) \otimes D_C(a, (abab)^*a) \bigoplus(D_C(a, (ab)^*a) \otimes D_C(a, (abab)^*a))$$

$$= \bigoplus\{(\{b(ab)^*a\}, \{1\}) \otimes \{\{bab(abab)^*a\}, \{1\}\}\}$$

$$= \{(\{b(ab)^*a, 1\}) \otimes \{\{bab(abab)^*a, 1\}\}\}$$

$$= \{(b(ab)^*a, 1), (bab(abab)^*a, 1), (b(ab)^*a, 1, bab(abab)^*a, 1)\}$$

From the computation of the derivatives, we deduce:

- the set $Q = \{q_1, \ldots, q_8\}$ of states:
  
  $q_1 = E$, $q_2 = b(ab)^*a$, $q_3 = 1$, $q_4 = bab(abab)^*a$,
  $q_5 = ab(abab)^*a$, $q_6 = b(abab)^*a$, $q_7 = (ab)^*a$, $q_8 = (abab)^*a$,

- the function $F$ from $Q$ to $\mathbb{B}$:
  
  $F(q_3) = F(q_7) = F(q_8) = 1$,
  $F(q_1) = F(q_2) = F(q_4) = F(q_5) = F(q_6) = 0$,

- and the function $\delta$ from $Q \times \Sigma$ to $\text{BoolForm}(Q)$:

|   | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q_7$ | $q_8$ |
|---|------|------|------|------|------|------|------|------|
| $a$ | $\neg q_2 \land \neg q_3 \land \neg q_4$ | $\lor$ | $\neg q_2 \land \neg q_3 \land \neg q_3$ | $\lor$ | $q_2 \land \neg q_4 \land \neg q_3$ | $\lor$ | $q_3 \land \neg q_4 \land \neg q_3$ | $0$ | $0$ | $0$ | $0$ | $q_2 \lor q_3$ | $q_4 \lor q_3$ |
| $b$ | | $q_7$ | $0$ | $q_5$ | $0$ | $q_8$ | $0$ | $0$ | $0$ |

Let us notice that in this example, substituting $B_C(E)$ to $E$ in $I$ would produce a smaller automaton.
6 Conclusion

This paper provides two main results. First, the theoretical scheme of derivations via a support allows us to formalize intrinsic properties of (unrestricted) regular expression derivations. As a by-product we obtain a kind of unification of the classical derivations that compute word derivatives, partial derivatives or extended partial derivatives. Secondly, the notion of base function that associates a boolean formula with a regular expression allows us to show how to deduce an alternating automaton equivalent to a given regular expression from the set of its derivatives via a given support. We are now investigating new features: it is possible, for example, to define morphisms from one support to another one in order to study the relations between the associated automata. An other perspective is to replace the derivation mapping by an other mapping (right derivation or left-and-right derivation for example, or any transformation with good properties). There also exist well-know algorithms to reduce boolean formulas (Karnaugh, Quine-McCluskey); we intend to investigate reduction techniques based on derivation. Finally we intend to extend the theoretical derivation scheme in order to handle the derivation of regular expression with multiplicities.

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