Stability of multi-dimensional nonlinear piezoelectric beam with viscoelastic infinite memory

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Abstract. The longtime behavior of a kind of fully magnetic effected nonlinear multi-dimensional piezoelectric beam with viscoelastic infinite memory is considered. The well-posedness of this nonlinear coupled PDEs’ system is showed by means of the semigroup theories and Banach fixed-point theorem. Based on frequency-domain analysis, it is proved that the corresponding coupled linear system can be indirectly stabilized exponentially by only one viscoelastic infinite memory term, which is located on one equation of these strongly coupled PDEs. Then, the exponential decay of the solution to the nonlinear coupled PDEs’ system is established by the energy estimation method under certain conditions.

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1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary \( \partial \Omega \) satisfying \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \), and \( \mathbf{n} \) be the unit outward normal vector of \( \Gamma_1 \).

In the present work, we consider the fully dynamic magnetic effect on the model of piezoelectric beam with only one viscoelastic memory involved, whose dynamic behavior is described by elasticity equation and charge equation coupling via the piezoelectric constants, which is given as follows:

\[
\begin{align*}
\rho v_{tt}(x, t) &= \alpha \Delta v(x, t) - \gamma \beta \Delta p(x, t) + f_1(v, p) + \int_0^{\infty} g(s) \Delta v(x, t - s) \, ds, & x \in \Omega, t > 0, \\
\mu p_{tt}(x, t) &= \beta \Delta p(x, t) - \gamma \beta \Delta v(x, t) + f_2(v, p), & x \in \Omega, t > 0, \\
v(x, t) &= p(x, t) = 0, & x \in \Gamma_0, t > 0, \\
\frac{\partial v}{\partial n}(x, t) - \gamma \beta \frac{\partial p}{\partial n}(x, t) &= \beta \frac{\partial p}{\partial m}(x, t) - \gamma \beta \frac{\partial v}{\partial m}(x, t) = 0, & x \in \Gamma_1, t > 0, \\
v(x, 0) &= v_0(x), v_t(x, 0) = v_1(x), p(x, 0) = p_0(x), p_t(x, 0) = p_1(x), & x \in \Omega, \\
v(x, -s) &= h(x, s), & x \in \Omega, s > 0,
\end{align*}
\]  

where \( v(x, t) \) and \( p(x, t) \) are denoted by the transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point \( x \in \Omega \), respectively. \( v_0, v_1, p_0, p_1 \) are the given initial data. The coefficients \( \rho, \alpha, \gamma, \mu, \beta > 0 \) are the mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, and impermeability coefficient of the beam, respectively, and satisfy \( \alpha > \gamma^2 \beta \), and thus, there always exists a positive constant \( \alpha_1 \) such that \( \alpha_1 = \alpha - \gamma^2 \beta \). The functions \( f_i(v, p) \), \( i = 1, 2 \) and \( h(x, s) \) are nonlinear source terms and memory history function, respectively. \( g(s) \) is the memory kernel function (also named as “relaxation function”), and the following assumptions on \( g(s) \) are imposed:

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(A1) \( g \in L^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+) \) satisfies \( 0 < \zeta := \int_0^\infty g(s)ds < \alpha_1 \) and \( g(s) > 0 \) for \( s \in \mathbb{R}^+ \);

(A2) For any \( s \in \mathbb{R}^+ \), \( g'(s) < 0 \) and there exist two constants \( k_0 > 0 \) and \( k_1 > 0 \) such that \( -k_0 g(s) \leq g'(s) \leq -k_1 g(s) \).

The piezoelectric material, as a kind of multi-functional smart material, is tremendously applied in industrial fields in recent years, such as ultrasonic welders, micro-sensors, inchworm robots, wearable human–machine interfaces, and so on. For more details, we refer to the typical works [5, 13, 14, 22, 23, 25] and the references therein. It is worth mentioning that based on Hamilton’s principle, Morris and Özer in [13] first proposed the theory of piezoelectric beam with fully magnetic effect and derived a kind of PDEs’ dynamical model:

\[
\begin{align*}
\rho v_{tt}(x, t) &= \alpha v_{xx}(x, t) - \gamma \beta p_{xx}(x, t), & x \in (0, L), & t > 0, \\
\mu p_{tt}(x, t) &= \beta p_{xx}(x, t) - \gamma \beta v_{xx}(x, t), & x \in (0, L), & t > 0, \\
v(0, t) &= p(0, t) = 0, & t > 0, \\
\alpha v_x(L, t) - \gamma \beta p_x(L, t) &= 0, & t > 0, \\
\beta p_x(L, t) - \gamma \beta v_x(L, t) &= -\frac{V(t)}{k}, & t > 0,
\end{align*}
\]

where \( V(t) \) denotes the voltage applied at the electrodes. This proposed model aims to compensate for the deficiency of the traditional piezoelectric beam theory dominated by Maxwell’s equation which ignores the dynamic interactions of electro-magnetism, and the electromagnetic effect must be taken into account in some situations, as mentioned in reference [28]. Recently, based on its abundant applications in industrial fields, the longtime behavior of the piezoelectric beam system with various damping mechanisms has been attracted by many scholars. In [14, 15], Morris and Özer proved that the system can be strongly stabilized, but not exponentially stable if only one frictional damping is actuated on either of these two boundaries. However, it was proved in [13] that the system can achieve exponential stability if the frictional damping is actuated on both boundaries simultaneously. Compared to boundary damping, [21] showed that only one internal frictional damping acting on either of these two equations is sufficient to stabilize this coupled system exponentially. Different from the above-mentioned literature, we replace the frictional damping by a viscoelastic infinite memory term \( \int_0^\infty g(s)\Delta v(x, t - s)ds \), which is presented only in the first equation of the strongly coupled system (1.1). This type of memory term may also exist in thermoelastic materials in low temperature, the stability of which was discussed by [27]. The objective of this work is to identify to which extent, this multi-dimensional coupled PDEs system can be “indirectly” stabilized through a viscoelastic memory term acting only in one of these two equations.

Besides, it is a known fact that the nonlinear models can describe the natural phenomena more accurately. As a great number of extra uncertainties often occur in practical engineering, many nonlinear elastic systems (vibrating strings, beams, or others) are proposed and aroused mathematicians and engineers’ interest. For instance, Rivera et al. in [24] studied a nonlinear system of Timoshenko type in a one-dimensional bounded domain and investigated the exponential stability of the system. Mustafa et al. in [16] obtained an explicit and general decay result of a class of nonlinear Timoshenko beam system. [11] studied the decay rates of a one-dimensional porous-elastic beam with infinite memory under a nonlinear damping mechanism. [7] studied the longtime dynamics of a kind of nonlinear piezoelectric beam with fractional damping and thermal effects, and [8] considered a nonlinear piezoelectric system with delay effect, in which the global attractor and exponential attractor are studied. For other kinds of PDEs on this issue, we refer to [19] for a semilinear wave equation with viscoelastic damping and delay feedback and [4] for two nonlinear systems including Korteweg–de Vries–Burgers and Kuramoto–Sivashinsky equations with memory, and the references therein.

In this paper, we consider the stability of a nonlinear piezoelectric beam system (1.1) with only one viscoelastic memory involved. The novelties and contributions are mainly in twofold:
1. Through the in-depth research, we found that there is no work on the longtime behavior of the multi-dimensional nonlinear piezoelectric beam system with infinite memory terms. Different from the existing literature on damped piezoelectric beam systems, we consider the damping mechanism by a viscoelastic memory term \( \int_0^\infty g(s)\Delta v(x, t - s)ds \). Particularly, it actuates only on one equation of the coupled PDEs (1.1) and the other equation in this system is “indirectly” stabilized through the coupling. We first show that the corresponding linear coupled PDEs system can be indirectly stabilized exponentially by the viscoelastic infinite memory. Particularly, this decay rate is irrelevant to the relationships of wave speeds \( \frac{c}{a} \) and \( \frac{c}{3} \), which is totally different from the well-known Timoshenko beam (see [1,2]). Moreover, the exponential decay of the nonlinear system (1.1) is further proved by a careful energy estimate.

2. For the previous works on the stability analysis of the piezoelectric beams, the energy multiplier method is used exclusively. However, the infinite memory term living in such a model leads to the difficulty in the construction of energy multipliers. In order to solve this problem, the frequency-domain method is adopted in this work to prove the exponential decay of the piezoelectric beam system. To the best of the authors’ knowledge, this could be the first work that the frequency-domain method is employed to discuss the stability of the multi-dimensional piezoelectric beam system with viscoelastic memory term. Some novel frequency multipliers are developed to help us overcome the technical difficulties caused by the strong coupling characteristics and the memory term.

The rest of this paper is organized as follows. In Sect. 2, the preliminary assumptions and spaces are presented. In Sect. 3, the generation of semigroup \( \{e^{At}\}_{t \geq 0} \) of the linear part of the system (1.1) is discussed by the semigroup theories and the well-posedness of the nonlinear system is further dealt with by the Banach fixed-point theory. Section 4 is devoted to discussing the exponential stability of system (1.1). Finally, a concluding remark is given in Sect. 5.

2. Preliminaries and problem setting

This section is devoted to considering system (1.1) in an appropriate Hilbert space setting. According to the approach of Dafermos [6], let us first define a new variable \( \eta(x, t, s) \) for system (1.1)

\[
\eta(x, t, s) = v(x, t) - v(x, t - s), \quad x \in \Omega, \quad t, s > 0.
\]

It is easy to verify that \( \eta_t(x, t, s) = v_t(x, t) - \eta_s(x, t, s) \) and

\[
\eta(x, t, 0) = 0, \quad x \in \Omega, \quad t > 0.
\]

Thus, system (1.1) can be rewritten as follows:

\[
\begin{align*}
\rho v_{tt}(x, t) &= (\alpha - \zeta)\Delta v(x, t) - \gamma \beta \Delta p(x, t) + f_1(v, p) + \int_0^\infty g(s)\Delta \eta(x, t, s)ds, \quad x \in \Omega, \quad t > 0, \\
\mu p_{tt}(x, t) &= \beta \Delta p(x, t) - \gamma \beta \Delta v(x, t) + f_2(v, p), \quad x \in \Omega, \quad t > 0, \\
\eta_t(x, t, s) &= v_t(x, t) - \eta_s(x, t, s), \quad x \in \Omega, \quad t, s > 0, \\
v(x, t) &= p(x, t) = 0, \quad x \in \Gamma_0, \quad t > 0, \\
\alpha \frac{\partial \eta}{\partial n}(x, t) - \gamma \beta \frac{\partial p}{\partial n}(x, t) &= \beta \frac{\partial \eta}{\partial n}(x, t) - \gamma \beta \frac{\partial v}{\partial n}(x, t) = 0, \quad x \in \Gamma_1, \quad t > 0, \\
v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x), \quad p(x, 0) = p_0(x), \quad p_t(x, 0) = p_1(x), \quad x \in \Omega, \quad t > 0, \\
v(x, -s) &= h(x, s), \quad x \in \Omega, \quad s > 0.
\end{align*}
\]

Define \( H^k_{\Gamma_0}(\Omega) = \{ f \in H^k(\Omega) \mid f = 0 \text{ on } \Gamma_0 \} \), where \( H^k(\Omega) \) is \( k \)-order Sobolev space. We define the space \( \Xi \) by

\[
\Xi := \left\{ \eta(x, s) \mid \begin{array}{l}
\eta(x, s) \in H^1_{\Gamma_0}(\Omega), \eta_s(x, s) \in H^1_{\Gamma_0}(\Omega), \\
\int_0^\infty \int_\Omega g(s)|\nabla \eta(x, s)|^2 dx ds < \infty
\end{array} \right\}.
\]
equipped with the inner product
\[(\eta, \tilde{\eta})_\Xi = \int_0^\infty \int_\Omega g(s)\nabla \eta(x, s) \cdot \nabla \tilde{\eta}(x, s) dx ds.\]

Choose the state space
\[\mathcal{H} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times \Xi \quad (2.4)\]

with
\[(v, u, p, q, \eta) \sim (\tilde{v}, \tilde{u}, \tilde{p}, \tilde{q}, \tilde{\eta}) \quad (2.5)\]

Under the assumptions (A1) and (A2), it is easy to check that \((\mathcal{H}, (\cdot, \cdot)_\mathcal{H})\) is a Hilbert space.

Define the system operator \(A\) in \(\mathcal{H}\) by
\[A \begin{pmatrix} v(x) \\ u(x) \\ p(x) \\ q(x) \\ \eta(x, s) \end{pmatrix} = \begin{pmatrix} u(x) \\ \frac{1}{\rho}[(\alpha - \zeta)\Delta v(x) - \gamma\beta\Delta p(x) + \int_0^\infty g(s)\Delta \eta(x, s) ds] \\ \frac{1}{\rho} (\beta\Delta p(x) - \gamma\beta\Delta v(x)) \\ u(x) - \eta_s(x, s) \end{pmatrix} \quad (2.6)\]

with domain
\[D(A) = \left\{(v, u, p, q, \eta)^T \in ((H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega))^2 \times \Xi \mid \alpha \frac{\partial u}{\partial n} - \beta \frac{\partial p}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 \right\}. \quad (2.7)\]

The source terms are described by a nonlinear function \(F: \mathcal{H} \to \mathcal{H}\) defined as
\[F(v, u, p, q, \eta) = \left(0, \frac{1}{\rho} f_1(v, p), 0, \frac{1}{\mu} f_2(v, p), 0\right)^T. \quad (2.8)\]

Set \(X(t) = (v(x, t), v_t(x, t), p(x, t), p_t(x, t), \eta(x, s, t))^T\) and \(X_0 = (v_0, v_1, p_0, p_1, \eta_0)^T\). With the above notations, problem (2.3) can be reformulated into the following abstract evolution equation:
\[
\begin{cases}
\frac{dX(t)}{dt} = AX(t) + F(X(t)), & t > 0, \\
X(0) = X_0.
\end{cases}
\quad (2.9)
\]

3. Well-posedness

3.1. The generation of \(C_0\)-semigroup \(e^{At}\)

**Theorem 3.1.** Assume that the assumptions (A1)–(A2) hold. Then, the operator \(A\) generates a \(C_0\) semigroup of contractions \(e^{At}\) on \(\mathcal{H}\).
Proof. First, we show that $A$ is dissipative in $\mathcal{H}$. In fact, for any $W = (v, u, p, q, \eta)^T \in D(A)$, a direct calculation yields that

$$
\Re \langle AW, W \rangle_{\mathcal{H}} = \Re \left( \int_\Omega (\alpha_1 - \zeta) \nabla u \cdot \nabla \bar{v} dx + \int_\Omega \rho \left[ \frac{1}{\rho} \left( (\alpha - \zeta) \Delta v - \gamma \Delta p + \int_0^\infty g(s) \Delta \eta(x, s) ds \right) \right] u dx 
+ \int_\Omega \beta (\gamma \nabla u - \nabla q) \cdot (\gamma \nabla \bar{v} - \nabla \bar{p}) dx + \int_\Omega \mu \left( \frac{1}{\mu} (\beta \Delta p - \gamma \beta \Delta v) \right) \bar{q} dx 
+ \int_0^\infty \int_\Omega g(s) \nabla (u - \eta_s(x, s)) \cdot \nabla \bar{\eta}(x, s) dx ds \right)
= \frac{1}{2} \int_0^\infty \int_\Omega g'(s)|\nabla \eta|^2 dx ds \leq 0, \tag{3.1}
$$

which implies that $A$ is dissipative.

Second, we show $0 \in \varrho(A)$ (where $\varrho(A)$ denotes the resolvent point set of $A$). Indeed, for any $(\xi_1, \xi_2, z_1, z_2, \nu) \in \mathcal{H}$, let us discuss the solvability of the equation

$$
A(v, u, p, q, \eta)^T = (\xi_1, \xi_2, z_1, z_2, \nu)^T, \quad \text{for } (v, u, p, q, \eta)^T \in D(A), \tag{3.2}
$$

that is,

$$
u = \xi_1, \tag{3.3}
$$
$$
\frac{1}{\rho} \left[ (\alpha - \zeta) \Delta v - \gamma \Delta p + \int_0^\infty g(s) \Delta \eta(x, s) ds \right] = \xi_2, \tag{3.4}
$$
$$
q = z_1, \tag{3.5}
$$
$$
\frac{1}{\mu} (\beta \Delta p - \gamma \beta \Delta v) = z_2, \tag{3.6}
$$
$$
u - \eta_s = \nu. \tag{3.7}
$$

Solving (3.7), along with (2.2) and (3.3), we have

$$
\eta = \int_0^s [\xi_1 - \nu(r)] dr. \tag{3.8}
$$

Substituting $\alpha_1 = \alpha - \gamma^2 \beta$ into (3.4), together with the assumption (A2) and (3.6), we transform (3.3–3.7) into the following ones:

$$
\Delta v = \frac{1}{\alpha_1 - \zeta} \left[ \gamma \mu z_2 + \rho \xi_2 - \int_0^\infty g(s) \Delta \eta(x, s) ds \right], \tag{3.9}
$$
$$
\Delta p = \frac{1}{\alpha_1 - \zeta} \left[ \frac{(\alpha - \zeta) \mu}{\beta} z_2 + \rho \gamma \xi_2 - \gamma \int_0^\infty g(s) \Delta \eta(x, s) ds \right]. \tag{3.10}
$$
Let \((\varphi, \psi) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)\). Multiplying (3.9) and (3.10) by \(\varphi\) and \(\psi\) and taking \(L^2\)-inner product, respectively, then integrating by parts yields that

\[
\int_{\Omega} \nabla v \cdot \nabla \varphi dx + \int_{\Omega} \nabla p \cdot \nabla \psi dx = -\int_{\Omega} \frac{1}{\alpha_1 - \zeta} \left[ \gamma \mu z_2 + \rho \xi_2 - \int_0^\infty g(s) \Delta \eta(x, s) ds \right] \varphi dx \\
- \int_{\Omega} \frac{1}{\alpha_1 - \zeta} \left[ \frac{(\alpha - \zeta) \mu}{\beta} z_2 + \rho \gamma \xi_2 - \gamma \int_0^\infty g(s) \Delta \eta(x, s) ds \right] \psi dx.
\] (3.11)

In order to study the variation Eq. (3.11), we introduce the space \(V^1(\Omega) := H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)\) equipped with the norm

\[
\| (f, f') \|_{V^1} = \left( \int_{\Omega} (|\nabla f|^2 + |\nabla f'|^2) dx \right)^{\frac{1}{2}}.
\]

Define the bilinear functional \(B\) by

\[
B((v, p), (\varphi, \psi)) = \int_{\Omega} \nabla v \cdot \nabla \varphi dx + \int_{\Omega} \nabla p \cdot \nabla \psi dx.
\] (3.12)

It is obvious that \(B\) is bounded and coercive in \(V^1(\Omega) \times V^1(\Omega)\).

Introduce the functional \(F\) defined on \(V^1(\Omega)\) as

\[
F(\varphi, \psi) = -\int_{\Omega} \frac{1}{\alpha_1 - \zeta} \left[ \gamma \mu z_2 + \rho \xi_2 - \int_0^\infty g(s) \Delta \eta(x, s) ds \right] \varphi dx \\
- \int_{\Omega} \frac{1}{\alpha_1 - \zeta} \left[ \frac{(\alpha - \zeta) \mu}{\beta} z_2 + \rho \gamma \xi_2 - \gamma \int_0^\infty g(s) \Delta \eta(x, s) ds \right] \psi dx.
\] (3.13)

By Hölder inequality and Poincaré inequality, simple calculation shows that \(F\) is bounded in \(V^1(\Omega)\). Then, it follows by Lax–Millgram theorem that there is a unique solution \((v, p) \in V^1(\Omega)\) of (3.11). According to the result in [26, Chapter 3, p.92], Eq. (3.2) admits a unique solution \((v, u, p, q, \eta)^T \in D(A)\) and hence \(0 \in g(A)\). Therefore, the well-known Lumer–Phillips theorem [3,18] shows that the desired result follows. \(\square\)

Note that the generation of \(C_0\)-semigroup implies the homogeneous problem (the linear part of (2.9)) is well-posed. In the following subsection, we discuss the well-posedness of the nonlinear problem (2.9).

3.2. Well-posedness of nonlinear system

We now prove a global existence result for problem (2.9) with sufficiently small initial conditions. In order to ensure the solvability of the nonlinear problem (1.1), we assume the following hypotheses on the nonlinear source terms \(f_i(v, p), i = 1, 2\):

(H1) \(f_i(v, p)\) are locally Lipschitz continuous;
(H2) \(f_i(v, p)\) fulfill the growth condition: there exist \(r \geq 0\) and \(c > 0\) such that
\[ |f_1(v,p)| \sqrt{|f_2(v,p)|} \leq c(|v| + |p|)(|v|^r + |p|^r + 1), \]  
(3.14)

where \( |f_1| \sqrt{|f_2|} \) denotes the maximum of \(|f_1|\) and \(|f_2|\).

Inspired by the work [7], we obtain the following lemma.

**Lemma 3.1.** Suppose that the assumption (H1) holds, then \( \mathcal{F} : \mathcal{H} \to \mathcal{H} \) is locally Lipschitz continuous.

Thanks to Theorem 3.1, we know that \( \mathcal{A} \) generates a \( C_0 \) semigroup \( e^{\mathcal{A}t} \) on \( X \), and the problem (2.9) can be expressed in the form of an equivalent functional integral equation

\[
X(t) = e^{\mathcal{A}t}X_0 + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{F}(X(s))ds, \quad t \geq 0. 
\]  
(3.15)

Define \( C(I,\mathcal{H}) \) be the space consisting of the \( \mathcal{H} \)-value continuous functions on the compact set \( I \subset \mathbb{R}_+ \) and equipped with the classical norm \( \|x\|_\infty = \max_{t \in I} \|x(t)\|_\mathcal{H}, \) then \( C(I,\mathcal{H}) \) is a Banach space. By means of Banach fixed-point theorem, we will show the existence and uniqueness of the solution of the integral equation (3.15).

**Theorem 3.2.** Let \( X_0 \) and \( \mathcal{F} \) be defined as in (2.9). Assume that the hypotheses (H1) and (H2) hold. Then, for any initial value \( X_0 \in \mathcal{H} \) with \( \|X_0\|_\mathcal{H} \) sufficiently small, the integral equation (3.15) has a unique global solution on \( C([0,a],\mathcal{H}) \).

**Proof.** It can be easily verified that \( X(\cdot) \) is a \( \mathcal{H} \)-value continuous function with respect to variable \( t \). For \( a > 0 \) fixed, we define an operator \( \mathcal{J} \) on \( C([0,a],\mathcal{H}) \) by

\[
\mathcal{J}X(t) = e^{\mathcal{A}t}X_0 + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{F}(X(s))ds, \quad \forall t \in [0,a].
\]

Note that the solvability of (3.15) is equivalent to the fixed point of the operator equation \( \mathcal{J}X = X \).

Since \( e^{\mathcal{A}t} \) is a \( C_0 \)-semigroup, there exist \( M > 0 \) and \( \omega > 0 \) such that \( \|e^{\mathcal{A}t}\| \leq Me^{\omega t}, \quad t \geq \mathbb{R}_+ \). In view of the hypothesis (H1), we know that Lemma 3.1 holds. Denote \( B_K = \{ \varphi \in C([0,a],\mathcal{H}) \ | \|\varphi\|_\mathcal{H} \leq K \} \). For any given \( X, \tilde{X} \in B_K \), it follows from Lemma 3.1 that

\[
\|\mathcal{F}(X) - \mathcal{F}(\tilde{X})\|_\mathcal{H} \leq L(K)\|X - \tilde{X}\|_\mathcal{H},
\]

where \( L(\cdot) \) is the Lipschitz coefficient. Then, \( \mathcal{F} \) is uniformly bound on \( B_K \), and hence we denote \( N = \sup_{X \in B_K} \|\mathcal{F}(X)\|_\mathcal{H}. \) Choose a sufficiently small constant \( b > 0 \) such that \( b \leq a \), combining with the assumption for the small initial value, it deduces that \( (ML(K))^2M_0b < 1 \) and \( Me^{\omega b}(\|X_0\|_\mathcal{H} + Nb/\omega) \leq K \), where \( M_0 = \frac{1}{2\omega}(e^{2\omega} - 1) \). Let \( D = C([0,b],B_K) \), which is a closed sphere of radius \( K \) in \( C([0,b],\mathcal{H}) \). It is easy to check that \( \mathcal{J} : D \to D \). For \( X,Y \in D \), making use of Hölder inequality, we have

\[
\|\mathcal{J}X(s) - \mathcal{J}Y(s)\|^2_\mathcal{H} \leq \left( \int_0^t \|e^{\mathcal{A}(t-s)}[\mathcal{F}(X(s)) - \mathcal{F}(Y(s))]\|_\mathcal{H}ds \right)^2 \leq M^2 \left( \int_0^t e^{\omega(t-s)}\|\mathcal{F}(X(s)) - \mathcal{F}(Y(s))\|_\mathcal{H}ds \right)^2 \\
\leq M^2M_0(L(K))^2 \int_0^t \|X(s) - Y(s)\|^2_\mathcal{H}ds \\
\leq (ML(K))^2M_0b\|X - Y\|^2_\infty.
\]
which shows that $J$ is a contraction operator. By Banach fixed-point theorem, there is a unique solution $X(t) \in C([0,b],\mathcal{H})$. This proves the local existence and uniqueness of the solution to system (2.9).

Next, we claim the global existence of the solution. Similar to [17], in order to prove the global existence, we shall derive a priori estimate of this solution. Let $X(t)$ be a solution of (3.15) on the interval $[0,T]$, $T > 0$, we first estimate the nonlinear term. Using the hypothesis (H2) and Poincaré inequality, it follows that

$$
\|F(X)\|_{\mathcal{H}}^2 = \frac{1}{\rho} \int_{\Omega} |f_1(v,p)|^2 dx + \frac{1}{\mu} \int_{\Omega} |f_2(v,p)|^2 dx
$$

$$
\leq \left( \frac{1}{\rho} + \frac{1}{\mu} \right) c^2 \int_{\Omega} (|v| + |p|)^2 (|v|^r + |p|^r + 1)^2 dx
$$

$$
\leq 2 \left( \frac{1}{\rho} + \frac{1}{\mu} \right) c^2 l_0 \int_{\Omega} (|\nabla v|^2 + |\nabla p|^2) dx
$$

$$
\leq 2 \left( \frac{1}{\rho} + \frac{1}{\mu} \right) c^2 l_0 \delta \int_{\Omega} (\alpha_1 - \zeta)|\nabla v|^2 + \beta|\gamma \nabla v - \nabla p|^2 dx
$$

$$
\leq \vartheta \|X\|_{\mathcal{H}}^2,
$$

where $\delta = \max\{\frac{2^r+1}{\alpha_1 - \zeta}, \frac{1}{\beta}\}$ and $\vartheta = 2\left( \frac{1}{\rho} + \frac{1}{\mu} \right) c^2 l_0 \delta$, in which $c$ is the same as in (3.14) and $l_0 > 0$ is related to $r$ and Poincaré constant. Thus,

$$
\|X(t)\|_{\mathcal{H}}^2 \leq M^2 e^{2\omega T} \|X_0\|_{\mathcal{H}}^2 + M^2 M_T \vartheta \int_0^t \|X(s)\|_{\mathcal{H}}^2 ds,
$$

(3.16)

where $M_T = \frac{1}{2\xi}(e^{2\omega T} - 1)$. Applying Gronwall’s inequality to (3.16), it follows that

$$
\|X(t)\|_{\mathcal{H}} \leq e^{(M\sqrt{M_T \vartheta})^2 T} M e^{\omega T} \|X_0\|_{\mathcal{H}}, \ t \in [0,T].
$$

Since $T > 0$ is arbitrary chosen and can be large enough, the global existence of the solution is verified. \[\square\]

4. Stability analysis

This section is devoted to discussing the longtime behavior of the solution to the nonlinear system (1.1). To do this, let us first consider the stability of the $C_0$-semigroup associated with the linear part of this system.

4.1. Exponential stability of $C_0$-semigroup $e^{At}$

Based on the frequency-domain method, we obtain the following result.

**Theorem 4.1.** Assume that the assumptions (A1)–(A2) hold. Then, the semigroup $e^{At}$ associated with the linear part of system (2.3) is exponentially stable on $\mathcal{H}$, that is, there are two constants $\tilde{M} > 0$ and $\tilde{r} > 0$ such that the semigroup $e^{At}$ satisfies

$$
\|e^{At}(v_0, v_1, p_0, p_1, \eta_0)\|_{\mathcal{H}} \leq \tilde{M} e^{-\tilde{r} t} \|e^{At}(v_0, v_1, p_0, p_1, \eta_0)\|_{\mathcal{H}}.
$$

(4.1)

In order to show the above theorem, let us recall the following frequency characteristics on the exponential stability of $C_0$-semigroups of contractions on Hilbert spaces (see [9,10,20]).
Lemma 4.1. Let $A$ be the infinitesimal generator of a bounded $C_0$ semigroup $e^{At}$ on a Hilbert space $\mathcal{H}$ and satisfy $i\mathbb{R} \subset \sigma(A)$. Then, $e^{At}$ is exponentially stable if and only if the following condition holds:

$$
\sup_{\lambda} \{ \| (i\lambda - A)^{-1} \|_{\mathcal{H}} \mid \lambda \in \mathbb{R} \} < \infty. \tag{4.2}
$$

Proof of Theorem 4.1. By Lemma 4.1, it is sufficient to verify that the conditions $i\mathbb{R} \subset \sigma(A)$ and (4.2) hold. For clarity, the proof is divided by two parts.

**Part I. We show (4.2) holds.**

By contradiction, if it is not true, then thanks to Banach–Steinhaus theorem, there exist sequences $X_n = (v_n, u_n, p_n, q_n, w_n) \in \mathcal{D}(A)$ with $\|X_n\|_{\mathcal{H}} = 1$ and $\lambda_n \to \infty$ such that

$$(i\lambda_n - A)X_n \equiv (\xi^1_n, \xi^2_n, z^1_n, z^2_n, v_n) \to 0 \text{ in } \mathcal{H}, \tag{4.3}$$

specifically,

$$
i\lambda_n v_n - u_n = \xi^1_n \to 0, \text{ in } H^1_{\Gamma_0}(\Omega), \tag{4.4}$$

$$
i\lambda_n u_n - \frac{1}{\rho} \left[ (\alpha - \zeta) \Delta v_n - \gamma \beta \Delta p_n + \int_0^\infty g(s) \Delta \eta_n(x,s)ds \right] = \xi^2_n \to 0, \text{ in } L^2(\Omega), \tag{4.5}$$

$$
i\lambda_n p_n - q_n = z^1_n \to 0, \text{ in } H^1_{\Gamma_0}(\Omega), \tag{4.6}$$

$$
i\lambda_n q_n - \frac{1}{\mu} (\beta \Delta p_n - \gamma \beta \Delta v_n) = z^2_n \to 0, \text{ in } L^2(\Omega), \tag{4.7}$$

$$
i\lambda_n \eta_n - (u_n - \eta_{n,s}) = v_n \to 0, \text{ in } \Xi \tag{4.8}$$

with the boundary conditions

$$v_n = p_n = 0, \text{ on } \Gamma_0, \quad (4.9)$$

$$\alpha \frac{\partial v_n}{\partial n} - \gamma \beta \frac{\partial p_n}{\partial n} = \beta \frac{\partial p_n}{\partial n} - \gamma \beta \frac{\partial v_n}{\partial n} = 0, \text{ on } \Gamma_1. \tag{4.10}$$

It is easy to check that (4.9)–(4.10) can be simplified as Dirichlet–Neumann type:

$$v_n|_{\Gamma_0} = p_n|_{\Gamma_0} = \frac{\partial v_n}{\partial n}|_{\Gamma_1} = \frac{\partial p_n}{\partial n}|_{\Gamma_1} = 0. \tag{4.11}$$

Note that $\eta_n, \eta_{n,s}, u_n = 0$ on $\Gamma_0$, then (4.8) implies that

$$i\lambda_n \eta_n - (u_n - \eta_{n,s}) \to 0, \text{ in } L^2_g(\mathbb{R}_+, L^2(\Omega)), \tag{4.12}$$

where the norm in $L^2_g(\mathbb{R}_+, L^2(\Omega))$ is defined as the conventional one, that is,

$$\|\varphi\|_g = \int_0^\infty \int_\Omega g(s)|\varphi(x,s)|^2dxds.$$  

In the sequel of this part, we aim to show

$$|\eta_n|_{\Xi}, \|u_n\|_{L^2(\Omega)}, \|\nabla v_n\|_{L^2(\Omega)}, \|\gamma\nabla v_n - \nabla p_n\|_{L^2(\Omega)}, \|q_n\|_{L^2(\Omega)} = o(1). \tag{4.13}$$

As long as (4.13) is proved, we can get directly that $\|X_n\|_{\mathcal{H}} = o(1)$ due to (2.5), which contradicts the fact that $\|X_n\|_{\mathcal{H}} = 1$, and thus, the proof of Part I can be finished.

For this aim, we divide it by five steps.

**Step 1.** $|\eta_n|_{\Xi} \to 0, \quad n \to \infty.$
By virtue of the dissipativeness of $A$ (see (3.1)) and (4.3), we get
\[
\int_0^\infty \int_\Omega g'(s)|\nabla \eta_n|^2 \, dx \, ds \to 0, \text{ in } C.
\]

Due to the assumption (A2), we obtain that
\[
0 \leftarrow - \int_0^\infty \int_\Omega g'(s)|\nabla \eta_n|^2 \, dx \, ds \geq k_1 \int_0^\infty \int_\Omega g(s)|\nabla \eta_n|^2 \, dx \, ds \geq 0,
\]
which implies that
\[
\|\eta_n\|_\Xi \to 0. \quad (4.14)
\]

**Step 2.** $\|u_n\|_{L^2(\Omega)} \to 0, \quad n \to \infty$.

Taking the $L^2$-inner product of (4.12) with $g(s)u_n(x)$ and then integrating it with respect to $s$, we have
\[
\int_0^\infty \int_\Omega i\lambda_n g(s)\overline{\eta_n} u_n \, dx \, ds + \int_0^\infty \int_\Omega g(s)\overline{\eta_n} u_n \, dx \, ds - \int_0^\infty \int_\Omega g(s)|u_n|^2 \, dx \, ds \to 0. \quad (4.15)
\]

We now estimate each term in (4.15). Recall $\zeta = \int_0^\infty g(s) \, ds > 0$.

**Observation I.** In view of (4.5), using Hölder inequality, we have
\[
\left| \int_0^\infty \int_\Omega i\lambda_n g(s)\overline{\eta_n} u_n \, dx \, ds \right|^2 \\
= \frac{1}{\rho^2} \left| \int_0^\infty \int_\Omega g(s)\overline{\eta_n}(x, s)((\alpha - \zeta)\Delta v_n - \gamma\beta \Delta p_n + \int_0^\infty g(r)\Delta \eta_n(x, r) \, dr) \, dx \, ds \right|^2 \\
= \frac{1}{\rho^2} \left| - (\alpha - \zeta) \int_0^\infty \int_\Omega g(s)\nabla \overline{\eta_n} \cdot \nabla v_n \, dx \, ds + \gamma \int_0^\infty \int_\Omega g(s)\nabla \overline{\eta_n} \cdot \nabla p_n \, dx \, ds \\
- \int_0^\infty \int_\Omega g(s)\nabla \overline{\eta_n}(x, s) \cdot \left( \int_0^\infty g(r)\nabla \eta_n(x, r) \, dr \right) \, dx \, ds \right|^2 \\
\leq \frac{3}{\rho^2} (\alpha - \zeta)^2 \zeta \|\nabla v_n\|_{L^2(\Omega)}^2 \int_0^\infty \int_\Omega g(s)|\nabla \eta_n|^2 \, dx \, ds \\
+ \frac{3\gamma^2\beta^2}{\rho^2} \zeta \|\nabla p_n\|_{L^2(\Omega)}^2 \int_0^\infty \int_\Omega g(s)|\nabla \eta_n|^2 \, dx \, ds \\
+ \frac{3\zeta^2}{\rho^2} \left( \int_0^\infty \int_\Omega g(s)|\nabla \eta_n(x, s)|^2 \, dx \, ds \right)^2,
\]
which along with $\|\eta_n\|_{\Xi} \to 0$ yields that the first term in (4.15) satisfies

$$\int_0^\infty \int_\Omega i\lambda_n g(s)\eta_n \varpi_n \eta_n \, dx \, ds \to 0.$$  

**Observation II.** Thanks to (A2) and (4.14), integrating by part and applying Cauchy–Schwarz inequality induce that

$$\left| \int_0^\infty \int_\Omega g(s)\eta_{n, s} \varpi_n \eta_n \, dx \, ds \right|^2 \leq k_0^2 \left| \int_0^\infty \int_\Omega g(s)\eta_n \varpi_n \eta_n \, dx \, ds \right|^2$$

$$\leq k_0^2 \left( \int_\Omega \left| \int_0^\infty \int_\Omega g(s)\eta_n \varpi_n \eta_n \, dx \, ds \right| \right)^{1/2} \left( \int_\Omega \left| \int_0^\infty \int_\Omega g(s)\eta_n \varpi_n \eta_n \, dx \, ds \right| \right)^{1/2} ds^2$$

$$\leq k_0^2 \|u_n\|_{L^2(\Omega)}^2 \int_0^\infty \int_\Omega g(s)ds \int_\Omega \int_\Omega g(s)|\eta_n(x, s)|^2ds \, dx \, ds$$

$$\leq k_0^2 \|u_n\|_{L^2(\Omega)}^2 \int_0^\infty \int_\Omega g(s)|\nabla \eta_n(x, s)|^2ds \, dx \, ds \to 0.$$  

Substituting **Observation I** and **Observation II** into (4.15) yields that $\int_0^\infty \int_\Omega g(s)u_n\varpi_n \, dx \, ds \to 0$. Note that

$$0 \leftarrow \int_0^\infty \int_\Omega g(s)u_n\varpi_n \, dx \, ds = \int_0^\infty \int_\Omega g(s)ds \int_\Omega |u_n|^2 \, dx,$$

which together with the assumption (A1) shows that

$$\|u_n\|_{L^2(\Omega)} \to 0.$$  

(4.16)

**Step 3.** $\|\nabla v_n\|_{L^2(\Omega)} \to 0, \quad n \to \infty$.

Substituting (4.4) and (4.6) into (4.5) and (4.7), respectively, we get

$$-\lambda_n^2 \rho v_n - [(\alpha - \zeta)\Delta v_n - \gamma \beta \Delta p_n + \int_0^\infty g(s)\Delta \eta_n(x, s)ds] \to 0, \text{ in } L^2(\Omega),$$

(4.17)

$$-\lambda_n^2 \mu p_n - (\beta \Delta p_n - \gamma \beta \Delta v_n) \to 0, \text{ in } L^2(\Omega).$$

(4.18)

Note that thanks to $\alpha_1 = \alpha - \gamma^2 \beta > 0$, (4.17) and (4.18) can be transformed into the following form, respectively.

$$-\lambda_n^2 \rho v_n - (\alpha_1 - \zeta)\Delta v_n - \gamma \beta (\gamma \Delta v_n - \Delta p_n) - \int_0^\infty g(s)\Delta \eta_n(x, s)ds \to 0, \text{ in } L^2(\Omega),$$

(4.19)

$$-\lambda_n^2 \gamma \mu p_n + \gamma \beta (\gamma \Delta v_n - \Delta p_n) \to 0, \text{ in } L^2(\Omega).$$

(4.20)
Thus, adding (4.19) and (4.20) so as to eliminate the common item $\gamma\beta(\gamma\Delta v_n - \Delta p_n)$, we obtain
\[
\lambda_n^2 \rho v_n + \lambda_n^2 \gamma \mu p_n + (\alpha_1 - \zeta) \Delta v_n + \int_0^\infty g(s) \Delta \eta_n(x, s) ds \to 0, \text{ in } L^2(\Omega).
\] (4.21)

Then, taking the $L^2$-inner product of (4.21) with $v_n$, along with the boundary conditions (4.11), we have
\[
\int_\Omega \lambda_n^2 \rho |v_n|^2 dx + \int_\Omega \lambda_n^2 \gamma \mu p_n v_n dx - (\alpha_1 - \zeta) |\nabla v_n|^2 dx - \int_0^\infty \int_\Omega g(s) \nabla \eta_n \cdot \nabla v_n dx ds \to 0. \tag{4.22}
\]

By (4.4), (4.6) and Cauchy–Schwartz inequality, we get that the second term in (4.22) satisfies
\[
\left| \int_\Omega \lambda_n^2 \gamma \mu p_n v_n dx \right|^2 \sim \left| \gamma \mu \int_\Omega q_n v_n dx \right|^2 \leq (\gamma \mu)^2 \int_\Omega |v_n|^2 dx \int_\Omega |q_n|^2 dx \to 0. \tag{4.23}
\]

Here, we have used (4.16) and the boundedness of $\|q_n\|_{L^2(\Omega)}$.

By Cauchy–Schwartz inequality, we have
\[
\left| - \int_0^\infty \int_\Omega g(s) \nabla \eta_n \cdot \nabla v_n dx ds \right|^2 \leq \left( \int_0^\infty \int_\Omega |g(s) \nabla \eta_n|^2 dx \right)^2 \left( \int_\Omega |\nabla v_n|^2 dx \right)^2 \leq \|\nabla v_n\|_{L^2(\Omega)}^2 \int_0^\infty \int_\Omega |g(s)| |\nabla \eta_n|^2 dx ds \to 0. \tag{4.24}
\]

This together with (4.14), (4.22) and (4.23) shows that
\[
\int_\Omega \lambda_n^2 \rho |v_n|^2 dx - (\alpha_1 - \zeta) |\nabla v_n|^2 dx \to 0,
\]
which along with (4.4) and (4.16) leads to
\[
\|\nabla v_n\|_{L^2(\Omega)} \to 0. \tag{4.25}
\]

**Step 4.** $\|q_n\|_{L^2(\Omega)} \to 0$, $n \to \infty$.

Taking the $L^2$-inner product of (4.21) with $p_n$ and integrating it by parts, along with the boundary conditions (4.11), we obtain
\[
\int_\Omega \lambda_n^2 \rho v_n p_n dx + \int_\Omega \lambda_n^2 \gamma \mu |p_n|^2 dx - (\alpha_1 - \zeta) |\nabla v_n| \cdot |\nabla p_n| dx - \int_0^\infty \int_\Omega g(s) \nabla \eta_n \cdot \nabla p_n dx ds \to 0. \tag{4.26}
\]

On the one hand, we claim that
\[
\int_\Omega \lambda_n^2 \rho v_n p_n dx \to 0. \tag{4.27}
\]

Indeed, by (4.4) and (4.16), we know that $\|\lambda_n v_n\|_{L^2(\Omega)} \to 0$, which together with Hölder inequality, (4.6) and the boundedness of $\|q_n\|_{L^2(\Omega)}$, yields (4.27).
On the other hand, using Hölder inequality again, along with (4.14), (4.25) and the boundedness of \( \| \nabla p_n \|_{L^2(\Omega)} \), we obtain that the last two terms in (4.26) satisfy \( \int_\Omega (\alpha_1 - \zeta) \nabla v_n \cdot \nabla p_n dx \to 0 \), and 
\[
\int_0^\infty \int_\Omega g(s) \nabla \eta_n \cdot \nabla p_n dx ds \to 0.
\]

Hence, by (4.26), we get
\[
\int_\Omega \lambda^2 \gamma \mu |p_n|^2 dx \to 0,
\]
which along with (4.6) leads to
\[
\| q_n \|_{L^2(\Omega)} \to 0. \tag{4.28}
\]

**Step 5.** \( \| \gamma \nabla v_n - \nabla p_n \|_{L^2(\Omega)} \), \( n \to \infty \).

Taking the \( L^2 \)-inner product of (4.17) and (4.18) with \( v_n \) and \( p_n \), respectively, and using (4.11), then integrating by parts yields that
\[
- \int_\Omega \lambda_n^2 \rho |v_n|^2 dx + \int_\Omega (\alpha - \zeta) |\nabla v_n|^2 dx - \int_\Omega \gamma \beta \nabla \eta_n \cdot \nabla p_n dx + \int_0^\infty \int_\Omega g(s) \nabla \eta_n \cdot \nabla \eta_n dx ds \to 0, \text{ in } L^2(\Omega),
\]
\[
- \int_\Omega \lambda_n^2 \mu |p_n|^2 dx + \int_\Omega \beta |\nabla p_n|^2 dx - \int_\Omega \beta \gamma \nabla v_n \cdot \nabla p_n dx \to 0, \text{ in } L^2(\Omega). \tag{4.29}
\]

Adding (4.29) and (4.30), and by virtue of (4.4), (4.6), (4.24) and \( \alpha_1 = \alpha - \gamma^2 \beta > 0 \), we obtain
\[
\int_\Omega (\alpha_1 - \zeta) |\nabla v_n|^2 dx - \int_\Omega \rho |u_n|^2 dx + \int_\Omega \beta |\nabla v_n - \nabla p_n|^2 dx - \int_\Omega \mu |q_n|^2 dx \to 0. \tag{4.31}
\]

Substituting (4.16), (4.25) and (4.28) into (4.31) gives that
\[
\int_\Omega \beta |\gamma \nabla v_n - \nabla p_n|^2 dx \to 0,
\]
that is, \( \| \gamma \nabla v_n - \nabla p_n \|_{L^2(\Omega)} \to 0 \).

Summing up the above five steps, we have obtained that \( \| X_n \|_H \to 0 \), which contradicts the fact \( \| X_n \|_H = 1 \). Thus, the condition (4.2) in Lemma 4.1 has been verified.

**Part II. We show that** \( \mathbb{i} \mathbb{R} \subset \varrho(A) \).

The proof by contradiction is still employed. Suppose that \( \mathbb{i} \mathbb{R} \notin \varrho(A) \). Thus, due to \( 0 \in \varrho(A) \) and \( \varrho(A) \) is an open set, we have
\[
0 < \hat{\lambda} < \infty,
\]
in which
\[
\hat{\lambda} := \sup \{ R > 0 : [\mathbb{R} \setminus \mathbb{R}] \subset \varrho(A) \}.
\]
Thanks to Banach–Steinhaus theorem, there exist sequences \( X_n = (v_n, u_n, p_n, q_n, \eta_n) \in \mathcal{D}(A) \) with \( \|X_n\|_\mathcal{H} = 1 \) and \( \lambda_n \to \hat{\lambda} \) such that

\[
(i\lambda_n - A)X_n \to 0 \text{ in } \mathcal{H},
\]

that is, (4.4)–(4.8) hold for \( \lambda_n \to \hat{\lambda} \), with the boundary conditions (4.11).

Note that Steps 1–5 in Part I still holds for \( \lambda_n \to \hat{\lambda} \). Thus, following these five steps, we can also achieve the contradiction \( \|X_n\|_\mathcal{H} = o(1) \), and hence \( \mathbb{R} \ni \rho(A) \) holds.

Therefore, by Part I, II along with Lemma 4.1, the result in Theorem 4.1 holds. The proof is completed.

\[\Box\]

### 4.2. Stability analysis of nonlinear system (2.3) or (2.9)

In this section, we prove the exponential stability of the nonlinear system (2.3) with small initial data. Inspired by [7,12], we assume that

(H3) There exists a Gâteaux differentiable functional \( F : (H^1_0(\Omega))^2 \to \mathbb{C}^2 \) satisfying \( F(0,0) = 0 \) such that

\[
\nabla F(v, p) = (f_1(v, p), f_2(v, p)),
\]

where \( f_1, f_2 \) are the nonlinear source terms in system (1.1). It should be mentioned that \( \nabla F \) denotes the unique vector representing the Gâteaux derivative \( DF \) in the Riesz isomorphism.

Note that Theorem 4.1 indicates that the semigroup \( e^{At} \) associated with the linear part of system (2.3) is exponentially stable on \( \mathcal{H} \), that is, there exist \( \bar{M} > 0 \) and \( \bar{\omega} > 0 \) such that

\[
\|e^{At}\|_{\mathcal{H}} \leq \bar{M}e^{-\bar{\omega}t}, \quad t \geq 0.
\] \[
(4.32)
\]

Based on (4.32), we have the following result on the stability of the nonlinear system (2.3) with initial value satisfying \( \|X_0\|_{\mathcal{H}} \leq \rho_0 \), where \( \rho_0 > 0 \) is any given constant.

**Theorem 4.2.** Assume that (H3) holds, and \( f_i, \ i = 1,2 \) satisfy (H1) and (H2) such that for any \( \|X\|_{\mathcal{H}}, \|\tilde{X}\|_{\mathcal{H}} \leq C\rho_0 \), there exists a constant \( L(C\rho_0) < \frac{\bar{\omega}}{\bar{M}} \) satisfying

\[
\|\mathcal{F}(X) - \mathcal{F}(\tilde{X})\|_{\mathcal{H}} \leq L(C\rho_0)\|X - \tilde{X}\|_{\mathcal{H}}.
\] \[
(4.33)
\]

Then, the solution to system (2.3) decays to zero exponentially for every initial value \( X_0 \in \mathcal{H} \) satisfying \( \|X_0\|_{\mathcal{H}} \leq \rho_0 \).

**Proof.** We now define the total energy functional \( E(t) \) of the system (2.3) by

\[
E(t) = \frac{1}{2} \left[ \int_{\Omega} (\alpha_1 - \zeta)|\nabla v|^2dx + \int_{\Omega} \rho|u|^2dx + \int_{\Omega} \beta|\gamma\nabla v - \nabla p|^2dx + \int_{\Omega} \mu|q|^2dx 
\right.
\]

\[
+ \int_0^\infty \int_{\Omega} g(s)|\nabla \eta(x,s)|^2dxds \right] - \int_{\Omega} F(v,p)dx.
\] \[
(4.34)
\]

Note that, apart from the last term in (4.34), \( E(t) \) is the natural energy for the linear part of (2.3). By taking the \( L^2 \)-inner product of (2.3)1 and (2.3)2 with \( v_t \) and \( p_t \), respectively, then integrating by part and using (4.11) yield that

\[
E'(t) = \int_0^\infty \int_{\Omega} g'(s)|\nabla \eta|^2dxds \leq 0,
\]

which implies that \( E(t) \leq E(0) \).
We mainly estimate the last term of $E(t)$. In view of the differential mean value inequality, along with the assumption (H3), we have

$$\left|\int_{\Omega} F(v,p) dx\right| \leq \int_{\Omega} |\nabla F(\theta(v,p))||v,p|| dx$$

$$\leq c \int_{\Omega} (|f_1(\theta(v,p))| + |f_2(\theta(v,p))|)|v| + |p| dx, \ \theta \in (0,1). \quad (4.35)$$

Thanks to the assumption (3.14) in (H2), combining (4.35) with Hölder and Poincaré inequality, we have

$$\left|\int_{\Omega} F(v,p) dx\right| \leq 4c \int_{\Omega} (\frac{1}{\alpha_1 - \zeta})|\nabla v|^2 + \beta |\nabla v - \nabla p|^2 dx,$$

Where $\delta = \max\{\frac{1}{\alpha_1 - \zeta}, \frac{1}{\beta}\}$. Therefore, substituting (4.36) into (4.34) shows that

$$E(t) \geq \frac{1}{2} \left[ \int_{\Omega} (\alpha_1 - \zeta)|\nabla v|^2 dx + \int_{\Omega} \rho |u|^2 dx + \int_{\Omega} \beta |\nabla v - \nabla p|^2 dx + \int_{\Omega} \mu |q|^2 dx + \int_{\Omega} g(s)|\nabla \eta(x,s)|^2 dx ds \right] - 4c\delta \int_{\Omega} (\alpha_1 - \zeta)|\nabla v|^2 + \beta |\nabla v - \nabla p|^2 dx. \quad (4.37)$$

Choosing $c < \frac{1}{8c\delta}$ and denoting $k_0 = 1 - 8c\delta > 0$, then it yields that

$$E(t) \geq \frac{k_0}{2} \parallel X \parallel_{\mathcal{H}^t}, \ t \in \mathbb{R}_+.$$

(4.38)

Note that by (4.36) and (4.37), choosing $t = 0$, we obtain that

$$E(0) \leq \frac{\tilde{k}_0}{2} \parallel X_0 \parallel_{\mathcal{H}^t}, \ \text{where} \ \tilde{k}_0 = 1 + 8c\delta. \quad (4.39)$$

Thus, by (4.38) and (4.39), we have

$$\parallel X \parallel_{\mathcal{H}} \leq \sqrt{\frac{\tilde{k}_0}{k_0}} \parallel X_0 \parallel_{\mathcal{H}} \leq \sqrt{\frac{k_0}{\tilde{k}_0}} \rho_0 =: C_{\rho_0}, \ t \in \mathbb{R}_+.$$

Let us recall (3.15), direct calculation yields that

$$\parallel X(t) \parallel_{\mathcal{H}} \leq \hat{M} e^{-\hat{\omega}t} \parallel X_0 \parallel_{\mathcal{H}} + \hat{M}L(C_{\rho_0}) e^{-\hat{\omega}t} \int_{0}^{t} e^{\hat{\omega}s} \parallel X(s) \parallel_{\mathcal{H}} ds,$$
and hence,

$$e^{\hat{\omega}t}\|X(t)\|_{\mathcal{H}} \leq \hat{M}\|X_0\|_{\mathcal{H}} + \hat{M}L(C_{\rho_0}) \int_0^t e^{\hat{\omega}s}\|X(s)\|_{\mathcal{H}}ds. \quad (4.40)$$

Applying Gronwall’s inequality to (4.40), we obtain

$$\|X(t)\|_{\mathcal{H}} \leq \hat{M}e^{-[\hat{\omega}-\hat{M}L(C_{\rho_0})]t}\|X_0\|_{\mathcal{H}}, \quad t \in \mathbb{R}_+. \quad (4.41)$$

\[\square\]

**Remark 4.1.** Based on the proof of Theorem 4.2, the decay rate of the energy $E(t)$ (given as in (4.34)) associated with the nonlinear system (2.3) can also be estimated. In fact, thanks to the estimation (4.36), there exists a constant $\hat{C}>0$ such that $E(t) \leq \hat{C}\|X\|_{\mathcal{H}}^2$, which together with (4.41) implies the exponential decay of $E(t)$.

## 5. Conclusions

In this paper, we considered a kind of multi-dimensional nonlinear piezoelectric beam system subject to only one viscoelastic infinite memory in the elasticity equation. Under appropriate Hilbert settings and the suitable conditions on nonlinear source terms, the existence and uniqueness of global solution of the nonlinear system were shown by the semigroup theories and fixed-point theorem. We further proved that its solution decays exponentially for small initial value based on frequency-domain analysis and energy estimates. Particularly, this decay rate is irrelevant to the relationships of the wave speeds $\frac{\rho}{\alpha}$ and $\frac{\mu}{\beta}$, which is totally different from the well-known Timoshenko beam.

One promising research problem is to investigate the large time behavior of the following abstract coupled system:

$$\begin{cases}
v_{tt}(t) + a_1Av(t) - \kappa A^{\beta}p(t) - \int_0^\infty g(s)A^\alpha v(t-s)ds = 0, \\
p_{tt}(t) + a_2Ap(t) - \kappa A^{\beta}v(t) = 0, \quad \alpha, \beta \in [0,1),
\end{cases} \quad (5.1)$$

where $A$ is a positive self-adjoint operator. When $\alpha = \beta = 1$ and $A = -\Delta$, the system becomes a concrete one, that is, the linear part of system (1.1). From this view, system (5.1) is a more generalized case compared to system (1.1). Note that the choices of $\alpha$ and $\beta$ are related to the memory properties and coupling properties, respectively. Based on the previous results on weakly coupled PDEs systems and single second-order systems with memories, it is reasonable to predict that the coupling and memory properties can simultaneously affect the stability of the system (5.1). Thus, in a future work, we shall identify how the choices of $\alpha$ and $\beta$ determine the decay rates of the solutions to system (5.1).

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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