Quantum photon emission from a moving mirror in the nonperturbative regime

J. P. F. Mendonça$^{1,3}$, P. A. Maia Neto$^2$ and F. I. Takakura$^1$

$^1$Departamento de Física, ICE, Universidade Federal de Juiz de Fora, 36036-330 Juiz de Fora, Minas Gerais, Brazil
$^2$Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21945-970 Rio de Janeiro, Rio de Janeiro, Brazil
$^3$Department of Physics, Stanford University, Stanford, CA94305, USA

(January 6, 2022)

We consider the coupling of the electromagnetic vacuum field with an oscillating perfectly-reflecting mirror in the nonrelativistic approximation. As a consequence of the frequency modulation associated to the motion of the mirror, low frequency photons are generated. We calculate the photon emission rate by following a nonperturbative approach, in which the coupling between the field sidebands is taken into account. We show that the usual perturbation theory fails to account correctly for the contribution of TM-polarized vacuum fluctuations that propagate along directions nearly parallel to the plane surface of the mirror. As a result of the modification of the field eigenfunctions, the resonance frequency for photon emission is shifted from its unperturbed value.

PACS: 4250, 0365, 4210.

I. INTRODUCTION

When the boundaries of the quantum radiation field are set in motion, low frequency photons may be excited out of the vacuum field state. Such quantum effect is closely related to the radiation pressure exerted on the moving boundaries by the vacuum field. Given its conceptual importance, one has proposed an experiment to observe the effect with the aid of high-Q microwave cavities. Exact relativistic results are known for one-dimensional models, but the analysis of realistic 3D configurations usually demands some perturbation method. Several different perturbation approaches have been successfully employed to compute the radiation effect induced by the nonrelativistic motion of a broad variety of 3D boundaries. In reference, the photon emission rates induced by the oscillation of a single perfectly-reflecting plane mirror were obtained by taking the long wavelength approximation and assuming the motion induced corrections to be small. More recently, this particular perturbation approach was also employed in the problem of two plane mirrors forming a cavity.

The connection between the nonrelativistic limit and the long wavelength approximation may be understood by considering the example of a mirror oscillating at frequency $\Omega_0$ with amplitude $\delta q_0$. Then the nonrelativistic limit translates into $\Omega_0 |\delta q_0| \ll c$, and since the emitted photon frequencies satisfy the inequality $\omega \leq \Omega_0$, they correspond to wavelengths $\lambda$ much larger than $\delta q_0$. In this long wavelength regime, the scattering of a plane wave of frequency $\omega$ usually generates sidebands at frequencies $\omega - m \Omega_0$ with amplitudes of the order of $(\delta q_0/\lambda)^{|m|}$, where $m$ is any integer. Hence the sidebands have amplitudes which are increasingly small as the order $m$ increases. Moreover, the motion induced correction of the elastic scattered component is usually of the order of $(\delta q_0/\lambda)^2$. Therefore, the photon emission effect induced by a nonrelativistic motion may be analyzed by taking a first-order expansion in $\delta q_0/\lambda$, and then the entire motion effect is contained in the sidebands at $\omega \pm \Omega_0$. Accordingly, in this approach the motion induced terms are treated as small perturbations which may be computed directly from the field for a mirror at rest.

However, when the incident wave is polarized so that the magnetic field is perpendicular to the plane of incidence (TM polarization), the first downshifted sideband $(m = -1)$ may be intense if the parameters are such that it corresponds to a grazing wave (peak amplitude value of the order of $\lambda/\delta q_0$). Such enhancement effect, which is reminiscent of the Wood’s anomalies in the scattering by a diffraction grating at rest, first explained by Rayleigh in the beginning of the century, was analyzed in reference from the point-of-view of classical wave optics, always in the long wavelength approximation. It takes place over a very narrow spectral interval, and may be interpreted in terms of resonances associated to the plane symmetry (in the context of diffraction gratings, a discussion along these lines was proposed by Fano). More specifically, the resonances are related to surface waves (plasmons) which are the field eigenfunctions for the unperturbed system (surface at rest). In the perfect-reflection limit considered here, they are expelled from the medium and then degenerate into grazing travelling waves. These unperturbed eigenfunctions are TM polarized (so that the electric field is perpendicular to the surface of the mirror), thus no resonance enhancement occurs for TE scattering in the long wavelength approximation.

As for the motion induced quantum radiation effect, in order to compute the emission rate taking these resonances into account, we follow in this article a nonperturbative scheme, though still assuming a nonrelativistic motion, which...
allows us to employ the long wavelength approximation. We consider in detail the contribution of TM-polarized vacuum fluctuations which propagate along directions nearly parallel to the surface of the mirror. As we show below, the motion induced correction to TM-polarized nearly grazing waves is not a small perturbation in this case, and the dynamical coupling between different sidebands must be taken into account. As a result the elastic field components are strongly modified, leading to a shift of the resonance frequency for generation of photons.

The paper is organized as follows. In section 2, we expand the scattered field as a superposition of sidebands in order to derive a nonperturbative solution of the boundary condition associated to an oscillating mirror, which is then used in section 3 to compute the photon emission rate. Section 4 presents a discussion and some final remarks.

II. BOUNDARY CONDITIONS

We consider a plane perfectly–reflecting mirror oscillating along the direction perpendicular to its surface, which we take along the $x$ axis. The mirror’s position at time $t$ is given by the equation $x = \delta q(t)$. We use MKS units and take $\epsilon_0 = 1$, $c = 1$. As usual we decompose the electromagnetic field into components according to whether the electric field is perpendicular (TE) or parallel (TM) to the plane of incidence. As a consequence of the plane symmetry, the two polarizations are uncoupled in the scattering by the moving mirror, and accordingly we have two separate problems to solve. In the case of TE polarization, the perturbation approach is always valid in the long wavelength regime (as in the problem of reflection by a shallow diffraction grating), so that we only consider in this paper the generation of TM photons out of the vacuum radiation field. We define the vector potential $\mathbf{A}$ as follows:

$$E^{\text{(TM)}} = \nabla \times \mathbf{A},$$  

(1)

$$B^{\text{(TM)}} = \partial_t \mathbf{A}. $$

Moreover, it satisfies the Gauge condition

$$\nabla \cdot \mathbf{A} = 0.$$  

(3)

The key motivation for defining the vector potential as in eqs. (1)–(3) is the fact that $\mathbf{A} \cdot \mathbf{x} = 0$, which entails that $\mathbf{A}$ does not change under a Lorentz boost. For the Lorentz frame in which the mirror is instantaneously at rest the boundary condition for $\mathbf{A}$ simply states that its normal derivative vanishes at the mirror’s position. Then, translating back to the laboratory frame, such condition yields (see [7] for details):

$$[\partial_x + \delta q \partial_t]A(\delta q(t), r_{\parallel}, t) = 0,$$

where $r_{\parallel} = (y, z)$.

The field is written as

$$\mathbf{A} = \mathbf{A}_{\text{in}} + \delta \mathbf{A}_{\text{ret}}$$

(5)

where $\delta \mathbf{A}_{\text{ret}}$ is the retarded field that represents the effect of the motion of the mirror, whereas $\mathbf{A}_{\text{in}}$, representing the input field ($t \to -\infty$), is itself a sum of incident and reflected waves, and hence satisfies the boundary conditions of a mirror at rest. We take periodic boundary conditions on the plane of the mirror over a square of surface $S$, which is to be identified with the surface of the (very large) mirror later. Therefore, the normal mode decomposition of $\mathbf{A}_{\text{in}}$ is given by

$$\mathbf{A}_{\text{in}}(x, r_{\parallel}, t) = \int_0^{\infty} \frac{dk_x}{2\pi} \sum_n \sqrt{\frac{2\hbar}{\epsilon_n s}} \cos(k_x x) e^{ik_{\parallel n} r_{\parallel}} e^{-ik_n t} a_{n,n}(k_x) \hat{e}_n + \text{H.c.},$$

(6)

where H. c. means the Hermitian conjugate, $k_{\parallel n} = (n_y \hat{y} + n_z \hat{z})2\pi/\sqrt{s}$, with $n \equiv (n_y, n_z)$ denoting a pair of integer numbers, $\hat{e}_n = \times \hat{k}_{\parallel n}$ is the polarization vector, and $k_n = \sqrt{k_x^2 + k_{\parallel n}^2}$ is the frequency of a given normal mode. The input bosonic operators satisfy the commutation relations

$$[a_{n,n}(k_x), a_{n,n'}(k'_x)] = 0$$

(7)

and

$$[a_{n,n}(k_x), a_{n,n'}(k'_x)] = 2\pi \delta(k_x - k'_x) \delta_{n,n'}.$$  

(8)
We assume that the oscillatory motion of the mirror is nonrelativistic: $\delta \dot{q} \ll 1$. As discussed in the introduction, this assumption allows us to take the long wavelength approximation in order to solve $\delta \mathbf{A}_{\text{ret}}$ in terms of $\mathbf{A}_{\text{in}}$. Thus, we expand the fields around $x = 0$ keeping terms up to order of $\delta q/\lambda$, and find, from eqs. (10) and (11),

$$\left(\partial_x^2 + \delta q^2 + \delta \dot{q}\partial_t\right)\delta \mathbf{A}_{\text{ret}}(0, r_\parallel, t) = - \left(\delta \dot{q}^2 + \delta \ddot{q}\partial_t\right)\mathbf{A}_{\text{in}}(0, r_\parallel, t).$$

(9)

In the perturbative regime considered in reference [6], $\delta \mathbf{A}_{\text{ret}}$ is already of first–order in $\delta \dot{q}/\lambda$, and then only the first term in the lhs of eq. (10) contributes to the required order of $\delta \dot{q}/\lambda$. In this paper, we consider a regime, involving propagation near the grazing direction, where $\delta \mathbf{A}_{\text{ret}}$ may be of a higher order over a very short spectral range. In fact, eq. (10) itself provides a hint that it is not possible to neglect the last two terms in its lhs when considering waves propagating along the grazing direction, since $\partial_x\delta \mathbf{A}_{\text{ret}}$ vanishes in this case. These terms provide a coupling between the motion induced sidebands, which is to be analyzed later in this section.

It is convenient to define a Fourier representation that takes advantage of the plane symmetry and the corresponding conservation of the parallel component of the wavevector, $\mathbf{k}_\parallel$. Hence we define the Fourier components of a vector field $\mathbf{F}(\mathbf{r}, t)$ as

$$\mathbf{F}_n[x, \omega] = \frac{1}{S} \int dt \int_S d^2 r_\parallel \ e^{i\omega t} e^{-i\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel} \mathbf{F}(x, r_\parallel, t),$$

(10)

and the wave equation reads

$$\partial^2 F_n[x, \omega] = (k^2_\parallel - \omega^2)\mathbf{F}_n[x, \omega].$$

(11)

The normal mode expansion of eq. (3) is written as

$$\mathbf{A}_{\text{in}n}[x, \omega] = \Theta(K(\omega)^2) \sqrt{\frac{2h|\omega|}{K(\omega)^2 S}} \cos(K(\omega)x) \left[\Theta(\omega)\mathbf{a}_{\text{in}n}(K(\omega)) - \Theta(-\omega)\mathbf{a}_{\text{in}n}^\dagger(-K(\omega))\right] \hat{e}_n,$$

(12)

where

$$K(\omega) = \lim_{\varepsilon \to 0^+} [(\omega + i\varepsilon)^2 - k^2_{\parallel n}]^{1/2}$$

(13)

is a function in the complex plane of $\omega$ with a branch cut along the interval in the real axis between $-k_{\parallel}$ and $k_{\parallel}$. It represents the $x$–component of the reflected wave associated to the parameters $\omega$ and $\mathbf{k}_{\parallel n}$. The factor $\Theta(K(\omega)^2)$ in eq. (12) (where $\Theta$ denotes the Heaviside step function) is a consequence of excluding evanescent waves from the mode expansion of eq. (3), whereas the factor $\Theta(\omega)$ establishes the connection between annihilation (creation) operators and positive (negative) frequencies, which is crucial for the understanding of the photon emission process discussed in this paper.

The mirror oscillates around the position $x = 0$:

$$\delta q(t) = \delta q_0 \cos \Omega_0 t,$$

(14)

with $\Omega_0 \delta q_0 \ll 1$, leading to the generation of sidebands at frequencies

$$\omega_m = \omega - m\Omega_0.$$

We take the Fourier transform of eq. (3) as defined by eq. (14) and use eq. (12) to find (from now on we omit explicit reference to $\mathbf{k}_{\parallel n}$):

$$\partial_x\delta \mathbf{A}_{\text{ret}}[0, \omega] + \frac{1}{2}\delta q_0 \sum_{j=-1,1} \left[ (k_{\parallel n}^2 - \omega \omega_j) (\delta \mathbf{A}_{\text{ret}}[0, \omega_j] + \mathbf{A}_{\text{in}}[0, \omega_j]) \right] = 0.$$

(15)

In order to solve eq. (15), the retarded field is written as a superposition of all sidebands generated from the input field. In the half-space corresponding to $x > 0$, we have

$$\delta \mathbf{A}_{\text{ret}}[x, \omega] = \exp (iK(\omega)x) \sum_{m=-\infty}^{\infty} g_m(\omega - \omega_m) \mathbf{A}_{\text{in}}[0, \omega - \omega_m].$$

(16)
According to eqs. (13) and (16) (and to the particular choice of the branch cut on the complex plane associated with the former), $\delta A_{\text{ret}}[x, \omega]$ either corresponds to a wave propagating from the mirror into the half-space $x > 0$ (when $|\omega| > k$) or to an evanescent wave decaying along the positive $x$ direction, since $K$ is of the form $K(\omega) = i|K(\omega)|$ when $|\omega| < k$.

We replace eq. (16) into eq. (15) and use eq. (13) to find, after changing the summation index:

$$
\sum_{m=-\infty}^{\infty} \left\{ iK(\omega)g_m(\omega_m) - \frac{\delta q_0}{2} \sum_{j=-1,1} ([K(\omega_j)]^2 + j\omega_j \Omega_0)(g_{m+j}(\omega_{m}) + \delta_{m-j}) \right\} A_m[0, \omega_m] = 0. \tag{17}
$$

Since eq. (17) applies to any input field, the expression within curly brackets must vanish for all values of $m$ and arbitrary values of $\omega$. When considering a given value of $m$ in eq. (17), we replace $\omega$ by $\omega_m = \omega - m\omega_0$ in order to have the $g$ functions evaluated at the same frequency $\omega$. We obtain an infinite system of coupled linear equations for the functions $g_m(\omega)$ of the form

$$
Mg = Y, \tag{18}
$$

where $M$ is the symmetric tridiagonal infinite matrix given by $M_{m,m} = iK(\omega_m)$,

$$
M_{m+1,m} = M_{m,m+1} = \frac{\delta q_0}{2} H(\omega_m),
$$

and with $M_{m,m'} = 0$ otherwise. We have introduced the auxiliary function

$$
H(\omega) = \omega \Omega_0 - K(\omega)^2,
$$

and $Y$ and $g$ are column vectors, with

$$
Y_m = -\frac{\delta q_0}{2} H(\omega)\delta m_1 - \frac{\delta q_0}{2} H(\omega_{-1})\delta m_{-1},
$$

while the elements of $g$ are the functions $g_m(\omega)$, the integer index $m$ running from $-\infty$ to $\infty$. We may recover the perturbation results by neglecting the nondiagonal elements of the matrix $M$, thus resulting in uncoupled equations whose solution is

$$
g_1(\omega) \approx \frac{i}{2} \frac{\delta q_0 H(\omega)}{K(\omega_1)}; \quad g_{-1}(\omega) \approx \frac{i}{2} \frac{\delta q_0 H(\omega_{-1})}{K(\omega_{-1})}, \tag{19}
$$

and with all other functions, including $g_0(\omega)$, of higher order of $\omega \delta q_0$. Note that these results are in agreement with the analogous expressions of reference [9], and could be more easily obtained directly from eq. (9) by assuming $\delta A_{\text{ret}}$ to be a small perturbation.

We seek a solution that interpolates this perturbative regime, considered in detail in reference [1], with the regime corresponding to the scattering of vacuum fluctuations propagating along the grazing direction, always in the long wavelength approximation. Thus, we assume that the $x$ component of the input field is small: $|K(\omega)| \ll \omega^2 \delta q_0$. In this case, it is no longer possible to neglect the nondiagonal terms in the row corresponding to $m = 0$, so that we cannot neglect the coupling between $g_0$, $g_1$ and $g_{-1}$. We then solve eq. (18) with $m$ running from $-1$ to $1$,

$$
g_1(\omega) = \frac{i}{2} \frac{K(\omega)}{K(\omega_1)} \frac{\delta q_0 H(\omega)}{K(\omega_1)} + \left( \frac{\delta q_0}{2} \right)^2 \left( \frac{H(\omega)^2}{K(\omega_1)} + \frac{H(\omega_{-1})^2}{K(\omega_{-1})} \right), \tag{20}
$$

$$
g_{-1}(\omega) = \frac{K(\omega_1)H(\omega_1)}{K(\omega_{-1})H(\omega)} g_1(\omega), \tag{21}
$$

and

$$
g_0(\omega) + 1 = -2i \frac{K(\omega)}{\delta q_0 H(\omega)} g_1(\omega). \tag{22}
$$

The perturbative results as given by eq. (19) are recovered from eqs. (20) - (22) when $|K(\omega)/\omega| \gg (\omega \delta q_0)^2$. On the other hand, since $g_0$ is of the order of one when $|K(\omega)/\omega| \leq (\omega \delta q_0)^2$, the oscillation of the mirror generates a strong modification of the elastic field components when considering propagation close to a grazing direction. In fact, eq. (22) yields $g_0(\omega) = -1$ when $K(\omega) = 0$. Thus, in this limit, we have $A = A_m + \delta A_{\text{ret}} = 0$, showing that in this case the retarded field $\delta A_{\text{ret}}$ is of course not a small perturbation [12], and that the TM grazing waves, which are the field eigenfunctions in the case of a perfectly-reflecting mirror at rest, are no longer allowed in the motional case. This effect is at the origin of the resonance frequency shift for quantum photon generation out of the vacuum field state, to be discussed in the next two sections.
III. PHOTON EMISSION RATE

In this section, we use the nonperturbative results for the retarded field found in section 2 in order to consider the effect of photon generation induced by the motion of the mirror. As discussed in connection with eq. (12), changing from positive to negative frequencies yields a coupling between annihilation and creation photon operators, which is responsible for the radiation effect considered in this paper.

In order to analyze such coupling, we start by writing the total field, as given by eq. (13), in terms of the advanced solution \( \mathcal{A}_{\text{adv}} \) of the boundary condition. In this case, the homogeneous component of the solution is interpreted as an output field, which represents the limit \( t \rightarrow \infty \):

\[
\mathcal{A}_{\text{in}}[x, \omega] + \mathcal{A}_{\text{ret}}[x, \omega] = \mathcal{A}_{\text{out}}[x, \omega] + \mathcal{A}_{\text{adv}}[x, \omega]. \tag{23}
\]

The output field \( \mathcal{A}_{\text{out}} \) may be expanded in terms of output bosonic operators (also satisfying the commutation relations of eqs. (7) and (8)) and normal modes exactly as in eqs. (6) and (12), because it also satisfies the homogeneous boundary condition associated to a mirror at rest. Moreover, the advanced solution may be expanded as in eq. (16):

\[
\mathcal{A}_{\text{adv}}[x, \omega] = \exp(-i\mathcal{K}(\omega)x) \sum_{m=-\infty}^{+\infty} f_m(\omega-m) \mathcal{A}_{\text{out}}[0, \omega-m], \tag{24}
\]

where the coefficients \( f_m \) are given by \( f_m = g_m^* \).

Eq. (23) establishes a connection between input and output bosonic operators. In what concerns the dependence on the spatial coordinate \( x \), note that the evanescent components contained in both \( \mathcal{A}_{\text{ret}} \) and \( \mathcal{A}_{\text{adv}} \) cancel each other in eq. (23) and do not contribute to the photon emission effect, as expected since the radiation field is associated to travelling waves only. Then, we take \( |\omega| \geq k ||_n \) and assume \( \mathcal{K}(\omega) \) to be real in what follows. As a consequence, the fields may be written entirely in terms of the two linearly independent functions \( \exp(i\mathcal{K}(\omega)x) \) and \( \exp(-i\mathcal{K}(\omega)x) \).

Since eq. (23) applies to any value of \( x \), the coefficient multiplying \( \exp(i\mathcal{K}(\omega)x) \) must vanish, yielding, with the aid of eq. (12):

\[
\sqrt{\frac{\hbar |\omega|}{2\mathcal{K}(\omega)^2 S}} \{ \Theta(\omega) \left[ a_{\text{out}_n}(\mathcal{K}(\omega)) - a_{\text{in}_n}(\mathcal{K}(\omega)) \right] - \Theta(-\omega) \left[ a_{\text{out}_{-n}}(-\mathcal{K}(\omega)) - a_{\text{in}_{-n}}(-\mathcal{K}(\omega)) \right] \} |\epsilon_n\rangle = \mathcal{A}_{\text{ret}}[0, \omega]. \tag{25}
\]

If we take \( \omega < -k ||_n \) we single out the creation operators in the l.h.s of eq. (25). However, the retarded field in its r.h.s evaluated at \( \omega \) may also contain annihilation operators, since \( \omega \) may correspond to a downshifted sideband associated to an initially positive frequency \( \omega + \Omega_0 \). Such mixture between creation and annihilation operators entails that photons are created from vacuum by means of the frequency modulation associated to the motion of the mirror. In Appendix A, we start from eq. (25) and compute the average value of the output number operator associated to given values of \( k ||_n \) and \( k_x \) over the input vacuum field state \( |0\text{in}\rangle \) (we use the shorthand \( \langle \ldots \rangle \) to denote such average):

\[
\langle a_{\text{out}_n}(k_x) a_{\text{out}_{-n}}(k_x) \rangle = \frac{4k^2}{k^2 - \Omega_0} |\theta_1(\Omega_0 - k - k ||_n)\frac{|g_1(\Omega_0 - k)|^2}{\mathcal{K}(\Omega_0 - k)} \Delta, \tag{26}
\]

where \( \Delta \) is a coarse-grained time interval, and \( k = \sqrt{k_x^2 + k ||_n^2} \) is the photon frequency. Eq. (26) explicitly relates the photon creation effect to frequency downshifting, which is dominated by \( g_1 \) in the long wavelength approximation (an equivalent description in terms of frequency upshifting is also possible and leads to the same final results). The step function in its r.-h.-s. expresses the requirement that the input vacuum fluctuations at frequency \( \Omega_0 - k \) correspond to traveling waves. We consider a direction of emission forming an angle \( \theta \) with the \( x \) axis, so that \( k ||_n = k \sin \theta \).

We analyze the behaviour of the photon emission rate at fixed frequency \( k \) and propagation angle \( \theta \) as we tune the mechanical frequency \( \Omega_0 \). It is convenient to define the dimensionless variable

\[
\Delta = \frac{\Omega_0}{k} - 1 - \sin \theta.
\]

According to eq. (26), no photons are generated when \( \Delta < 0 \). At such low values of the mechanical frequency, there are no travelling wave vacuum field modes at frequency \( \Omega_0 - k \) that match the requirement associated to the conservation of the parallel component of the wavevector, \( k ||_n \). As a consequence, regardless of the choice of \( \theta \), the mechanical
frequency must be higher than the analyzed photon frequency $k$. As could be expected, high frequency field modes are unaffected by the motion of the mirror (quasistatic limit). This basic fact is essential to the long wavelength approximation employed in the paper.

At $\Delta = 0$, the vacuum fluctuations that contribute to the photon emission effect correspond to grazing waves. More generally, when $\Delta \ll 1$ the wavevectors of the relevant vacuum fluctuations have their $x$ component given by

$$K(\Omega_0 - k) = k\sqrt{2\sin \theta \Delta} [1 + O(\Delta)]. \tag{27}$$

In the perturbative approximation, the function $g_1$ remains finite (see eq. (13)) and then the photon number as given by eq. (26) diverges as $\Delta \to 0$ since $K(\Omega_0 - k)$ vanishes in this limit. Ref. [13] provides a simple interpretation of the divergence at $\Delta = 0$. In the perturbative theory, photons are emitted in pairs. When the mechanical frequency is set to $\Delta = 0$, the observed photon is such that its ‘twin’ propagates along a grazing direction [13]. TM polarized grazing waves are eigenfunctions for the perfectly-reflecting mirror at rest — in fact they represent the limit of the surface plasmons of a conducting surface as we take the perfectly-reflecting limit. Thus, $\Delta = 0$ is the resonance frequency for the field given the boundary condition of a perfectly-reflecting plane mirror at rest, for in this case the external modulation perfectly matches the conditions for the creation of grazing photons.

On the other hand, according to the nonperturbative result of eq. (20), the coefficient $g_1(\Omega_0 - k)$ is proportional to $K(\Omega_0 - k)$ as $\Delta \to 0$. Therefore, instead of the divergence at $\Delta = 0$ predicted by the perturbative theory, the photon emission rate vanishes in this limit. In Appendix B, we compute the angular distribution rate for $\Delta \geq 0$ from eq. (26):

$$\frac{d^2 R}{dkd\Omega} = \frac{S}{2\pi^3} \frac{k^2 \cos^2 \theta}{\sqrt{\Delta(\Delta + 2 \sin \theta)}} |g_1[\omega = k(\Delta + \sin \theta)]|^2. \tag{28}$$

It follows from eqs. (20), (27) and (28) that the photon emission rate goes as $\sqrt{\Delta}$ in the limit $\Delta \ll (k\delta_0)^4$. On the other hand, according to the discussion following eqs. (20)–(22), the perturbative regime is recovered when the vacuum fluctuations propagate along a direction not close to a grazing direction: $K(\Omega_0 - k)/k \gg (k\delta_0)^2$, which from eq. (27) translates into $\Delta \gg (k\delta_0)^2$. In this limit, we recover the results of Ref. [13].

Since $g_1$ is dimensionless, it depends on $k$ only through $\Delta$ and $k\delta_0$. Thus, apart from the trivial $k^2$ dependence in eq. (28), we may replace $\delta_0$, $\Omega_0$ and $k$ by the two dimensionless parameters $k\delta_0$ and $\Delta$. In figure 1, we plot the photon emission rate as given by eq. (28) as function of $\Delta$ for $\theta = 78^0$ and $k\delta_0 = 0.03$. In the main figure, we also plot the result from perturbation theory (dotted line), which is obtained from eqs. (14) and (28). As discussed above, whereas the latter diverges at $\Delta = 0$, the exact emission rate (solid line) vanishes in this limit. However, the exact result also displays a singularity, which is shifted to a higher value $\Delta_s$. Thus, the resonance frequency for photon emission is shifted from its unperturbed value $\Omega_0 = k(1 + \sin \theta)$ by an amount

$$\delta \Omega = k \Delta_s.$$

The exact value of $\Delta_s$ is best displayed in the insert of figure 1, where we compare the analytical result as given by replacing eq. (20) into eq. (28) (dashed line) with the result obtained from the numerical evaluation (solid line) of the coupled equations [18] for $g_m$ with $m$ running from $-3$ to $3$ (we have checked that truncations at higher dimensions had not changed the results within machine accuracy). The insert shows that the exact frequency shift is slightly smaller than the analytical value for the particular values of $\theta$ and $k\delta_0$ considered in fig. 1. On the other hand, outside the neighbourhood of $\Delta_s$ the analytical result works fairly well, and in fact for the scale employed in the main plot of fig. 1 the two methods provide indistinguishable curves.

From eq. (20), we may derive an analytical approximation for $\Delta_s$ up to lower order of $k\delta_0$:

$$\Delta_s = \frac{(k\delta_0)^4}{32} \sin^3 \theta (1 + \sin \theta)^4 \left( \frac{1}{\cos \theta} - \frac{1}{\sqrt{3\sin^2 \theta + 4\sin \theta + 1}} \right)^2. \tag{29}$$

For $\theta = 78^0$, and $k\delta_0 = 0.03$, eq. (28) yields $\Delta_s = 7.187 \times 10^{-6}$, which is in agreement with the plot of the emission rate as given by the analytical result (dashed line in the insert of fig. 1). In figure 2, we plot $\Delta_s$, as given by eq. (29) as a function of the observation angle $\theta$. The shift vanishes at $\theta = 0$ and increases by several orders of magnitude as $\theta$ increases from zero to $90^0$. In the next section, we discuss such behaviour starting from the analysis of the diagram representing the input vacuum fluctuations that contribute to the radiation emission effect.
IV. DISCUSSION

In the previous section, we discussed the behaviour of the photon emission rate for fixed values of the photon frequency and direction of emission, as we tune the mechanical frequency \( \Omega_0 \). In this section, we consider the complementary situation, in which \( \Omega_0 \) is fixed, in order to understand why the resonance frequency is shifted from its unperturbed value. For the sake of clarity, we analyze the photon emission process as an effect of frequency upshifting (rather than downshifting) in this section.

Each TM-polarized photon have given values of frequency \( \omega \), parallel component of wavevector \( k_\parallel \), and an azimuthal angle defining the direction of \( k_\parallel \). The photons generated from a mechanical modulations at frequency \( \Omega_0 \) correspond to points in the ABC triangle (dark grey) in the \( \omega \times k_\parallel \) plane of figure 3, since they obey the inequalities \( \omega + k_\parallel \leq \Omega_0 \) and \( \omega \geq k_\parallel \geq 0 \). Normal modes with \( \omega < k_\parallel \) do not correspond to travelling waves and hence are not relevant here. They correspond to the light grey region in figure 3.

In the \( \omega \times k_\parallel \) plane the frequency upshifting induced by the motion corresponds to a horizontal shift (since \( k_\parallel \) is conserved) by an amount of \( \Omega_0 \). Hence the vacuum fluctuations that contribute to the emission effect correspond to the A'B'C' triangle in figure 3. Vacuum fluctuations associated to the B'C' line propagate along grazing directions. They give rise to photons associated to points along the BC line, all of them corresponding to \( \Delta = 0 \). In this paper, we have shown that in the neighbourhood of B'C' the perturbative approach fails to provide the amplitudes of the sidebands. Whereas the perturbative theory predicts a divergent emission rate all along the BC line, we have found that the emission rate vanishes in this case, and that the singularity is displaced into the interior neighbourhood of the BC border.

This effect is a consequence of the motion induced modification of the field eigenfunctions. For a metallic surface at rest, the eigenfunctions correspond to surface plasmons. In the perfect-reflecting limit, they are expelled from the interior of the medium and degenerate into TM polarized grazing waves. In the framework of first-order perturbation theory, the dynamical modification of the field eigenfunctions is neglected, and then the resonant enhancement takes place when considering vacuum fluctuations along the B'C' line, which give rise to photons in the field modes corresponding to \( \Delta = 0 \). However, this resonant regime is inconsistent with the underlying assumption that the motion effect is a small perturbation. In this paper, we have employed a nonperturbative approach, and found the correction of the elastic field component, which is represented by the coefficient \( q_0 \) in eq. (16), to be very important in the case of nearly grazing waves (see eq. (22) and the discussion that follows). Thus, TM polarized grazing waves are no longer acceptable solutions in the motional case. Since the field eigenfunctions are strongly modified, the resonant region is displaced away from the B'C' line in figure 3 into the interior of the A'B'C' region. As a consequence, the critical region in the positive-frequency part of the diagram is displaced from the \( \Delta = 0 \) line (BC line), leading to a shift of the resonance frequency.

Vacuum fluctuations associated to the A'B' border give rise to photons propagating along a grazing direction (AB line). In this case, the perturbative approach also fails to provide the correct photon emission rate (see remark [2]). The two critical regions overlap at point B, which corresponds to grazing photons at the subharmonic frequency, generated from grazing vacuum fluctuations. As we approach B along the BC line, the width of the nonperturbative region increases, as well as the frequency shift \( \Delta_s \). In fact, that corresponds to the limit \( \theta \rightarrow 90^\circ \) discussed in connection with eq. (22) (see fig. 2). Note, however, that eq. (22) was derived from eq. (21) by assuming \( \Delta_s \ll 1 \). Moreover, eq. (21) itself was obtained from the coupled equations (18) by neglecting higher order sidebands, which turns out not to be a good approximation in this limit. As shown in the insert of fig. 1, the exact shift is smaller than the value given by eq. (24), and the discrepancy increases as \( \theta \rightarrow 90^\circ \).

The ABC triangle is closed from below by the AC border. The latter corresponds to photons that propagate along the normal direction. Since they are generated from vacuum fluctuations that also propagate along the normal direction (A'C' line), the nonperturbative theory provides no relevant correction in this case. As expected, according to fig. 2 the frequency shift \( \Delta_s \) vanishes as we approach the neighbourhood of point C (\( \theta \rightarrow 0 \)).

In this paper, we have analyzed in detail the resonance effect associated to the excitation of surface plasmons from the vacuum field state when a mirror oscillates in free space. Due to the assumption of perfect reflectiveness, the surface plasmons degenerate into TM-polarized grazing waves, and the resonance appears as a singularity in the photon emission rate. The nonperturbative approach developed in this paper allowed us to calculate the shift of the resonance frequency induced by the motion of the mirror, which is a consequence of the dynamical modification of the field eigenfunctions.

The authors thank Programa Especial de Visitante (PREVI-UFJF), Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for partial support (J. P. R. F. M. and P. A. M. N.) and Programa de Núcleos de Excelência (PRONEX), grant no. 4.1.96.08880.00 – 7035-1 (P. A. M. N.).
APPENDIX A: AVERAGE VALUES OF OUTPUT NUMBER OPERATORS

In this appendix, we derive eq. (26) for the average value of the output number operator. We replace the retarded field in eq. (25) by its expansion in terms of sidebands as given by eq. (16) and use the normal mode decomposition as written in eq. (12). Then, as mentioned in sec. 3, we pick up a frequency \( \omega < -k_{||n} \) to find

\[
\langle a_{out-n}(-\mathcal{K}(\omega))a_{out-n}(-\mathcal{K}(\omega)) \rangle = 4 \left| \frac{\omega - 1}{\omega} \right| \left[ \frac{\mathcal{K}(\omega)}{\mathcal{K}(\omega - 1)} \right]^2 \Theta(\omega - 1 - k_{||n}) |g_1(\omega - 1)|^2 \times \langle a_{in-n}(\mathcal{K}(\omega - 1))a_{in-n}^\dagger(\mathcal{K}(\omega - 1)) \rangle.
\]  

(A1)

Note that it is also possible to compute the photon emission rate by picking up a positive \( \omega \) in eq. (25), but in this case the dominant contribution comes from the function \( g_{-1} \), which describes frequency upshifting. Of course this method leads to the same final result for the average output number operator.

In order to calculate the vacuum correlation function in the rhs of eq. (A1), we introduce a coarse–grained time scale \( \Delta t \), which satisfies \( \Omega_0 \Delta t \gg 1 \), by means of a frequency distribution sharply peaked around the mechanical frequency \( \Omega_0 \):

\[
h(\Omega) = \frac{1}{\pi} \frac{1}{(\Omega - \Omega_0)^2 + (\frac{1}{\Delta t})^2}.
\]  

(A2)

The distribution \( h(\Omega) \) is normalized so as to yield

\[
\int d\Omega \ h(\Omega) = 1.
\]

It represents the spectral profile associated to the mirror’s motion: \( \delta q[\Omega] = \delta q_0 [h(\Omega) + h(-\Omega)]/2 \). Therefore, rather than the sinusoidal motion of eq. (13), we actually have an exponentially damped sinusoidal motion (where \( \Delta t \) is the damping time). In the frequency domain, it amounts to replace the delta function by the distribution \( h(\Omega) \). Since \( h(\Omega) \) is very sharply peaked around \( \Omega_0 \), however, we may replace \( \Omega \) by \( \Omega_0 \) everywhere (and thus keep all the development of sec. 2), except when computing the vacuum correlation function in eq. (A1). For instance, the representation of eq. (16) is replaced by

\[
\delta \mathcal{A}_{ret}[x,\omega] = \exp(i\mathcal{K}(\omega)x) \sum_{m=-\infty}^{\infty} g_m(\omega + m\Omega_0) \int h(\Omega) \mathcal{A}_{in}[0,\omega + m\Omega]d\Omega,
\]

and moreover \( \Omega \) is replaced by \( \Omega_0 \) everywhere in \( \mathcal{A}_{in}[0,\omega + m\Omega] \) except in the argument of the bosonic input operators. The finite mechanical linewidth is taken into account only when computing the vacuum correlation function, for which we take an average over the distribution \( h \):

\[
\langle a_{in-n}(\mathcal{K}-\mathcal{K}_\perp)a_{in-n}^\dagger(\mathcal{K}-\mathcal{K}_\perp) \rangle = \int d\Omega' \int d\Omega'' h(\Omega') h(\Omega'') \langle 0 in | a_{in-n}(\mathcal{K}(\omega + \Omega'))a_{in-n}(\mathcal{K}(\omega + \Omega''))^\dagger | 0 in \rangle.
\]  

(A3)

The unaveraged correlation function is easily calculated from the commutation relations given by eq. (6):

\[
\langle 0 in | a_{in-n}(\mathcal{K})a_{in-n}(\mathcal{K}')^\dagger | 0 in \rangle = 2\pi \delta(\mathcal{K} - \mathcal{K}').
\]  

(A4)

Replacing eq. (A4) into eq. (A3) and using the auxiliary result

\[
\int d\Omega h^2(\Omega) = \frac{\Delta \omega}{2\pi}
\]

yields

\[
\langle a_{in-n}(\mathcal{K}(\omega-1))a_{in-n}^\dagger(\mathcal{K}(\omega-1)) \rangle = \frac{[\mathcal{K}(\omega-1)]}{|\omega-1|} \Delta \omega.
\]  

(A5)

When inserting eq. (A5) into (A1), it is convenient to employ the variables \( k_x \) and \( k_\perp \), defining a given wavevector \( \mathbf{k} \) and the corresponding bosonic operator \( a_{in-n}(k_x) \) [see eq. (1)], rather than the Fourier variable \( \omega \). Thus, we take \( k_x = -\mathcal{K}(\omega) \), and then \( k = \sqrt{k_x^2 + k_\perp^2} = -\omega \). Note that since \( \omega < -k_\perp \), eq. (A5) yields \( k_x > 0 \) as it should. The final result is given by eq. (26).
APPENDIX B: CONNECTION BETWEEN PHOTON EMISSION RATES AND NUMBER OPERATORS

In this appendix, we establish the connection between the averaged number operator, as given by eq. (26) and the photon emission rate. The field normal mode decomposition as given by eq. (6) is such that the lhs of eq. (26) represents, when multiplied by $dk_x/(2\pi)$, the average number of (TM polarized) photons with wavevectors whose $x$ component is between $k_x$ and $k_x + dk_x$ and whose parallel component is $k_{\parallel n}$. Since the number of allowed values of $k_{\parallel n}$ per unit area in the two-dimensional $yz$ reciprocal space is $S/(2\pi)^2$, the average number of photons with frequency between $k$ and $k + dk$ is

$$d^3N = \left\langle a_{\text{out}, n}^\dagger(k_x)a_{\text{out}, n}(k_x) \right\rangle \frac{dk_x}{2\pi} d^2k_{\parallel n} \frac{S}{(2\pi)^2}. \tag{B1}$$

According to eqs. (26) and (B1), $d^3N$ is proportional to the coarse-grained time interval $\Delta t$, which allows us to define the photon production rate $d^2R = d^3N/\Delta t$. Writing the volume element in spherical coordinates,

$$d^3k = dk_x d^2k_{\parallel n} = k^2 dk d\Omega, \tag{B2}$$

where $\Omega$ denotes the solid angle, we obtain from eq. (B1) the connection between the angular distribution rate of photon emission (photon emission rate per frequency and solid angle intervals) and the averaged number operator:

$$\frac{d^2R}{dk d\Omega} = \frac{Sk^2}{(2\pi)^3 \Delta t} \left\langle a_{\text{out}, n}^\dagger(k_x)a_{\text{out}, n}(k_x) \right\rangle. \tag{B3}$$

By replacing eq. (26) into (B3), we obtain the result for the photon emission rate as given by eq. (28).
[1] Ford L H and Vilenkin A 1982 Phys. Rev. D 25 2569; Jaekel M T and Reynaud S 1992 Quantum Optics 4 39; Barton G and Eberlein C 1993 Ann. Phys., N.Y., 227 222; Maia Neto P A 1994 J. Phys. A: Math. Gen. 27 2167; Maia Neto P A and Machado L A S 1995 Brazilian J. Phys. 25 324

[2] Lambrecht A, Jaekel M -T and Reynaud S 1996 Phys. Rev. Lett. 77 615
[3] Moore G T 1970 J. Math. Phys. 11 2679; Dodonov V V and Klimov A B 1992 Phys. Lett. A 167 309
[4] Barton G 1996 Ann. Phys. (NY) 245 361; Eberlein C 1996 Phys. Rev. A 53 2772; Eberlein C 1996 Phys. Rev. Lett. 76 3842

[5] Barton G and North C A 1996 Ann. Phys. (NY) 252 72
[6] Maia Neto P A and Machado L A S 1996 Phys. Rev. A 54 3420
[7] Mundarain D F and Maia Neto P A 1998 Phys. Rev. A 57 1379
[8] Rayleigh J W S 1907 Proc. Roy. Soc. (London) A 79 399
[9] Maia Neto P A 1994 Optics Comm. 105 151
[10] Fano U 1941 J. Opt. Soc. Am. 31 213

[11] We have also compared eqs. (20)-(22) with a numerical solution of eq. (18) taking into account the coupling with higher order $g$ functions. As discussed in the section 3, we have found good agreement as far as the photon emission rate is concerned.

[12] A similar effect takes place when the parameters are such that the first downshifted sideband propagates along the grazing direction. In this regime, considered in detail in ref. 9, we have $\left| \mathcal{M}_{1,1} \right| = \left| \mathcal{K}(\omega_1) \right| \ll \omega^2 \delta q$, and then the best approximation is to consider the coupled equations for $g_0$, $g_1$ and $g_2$.

[13] In fact, the twin photon is generated in the same field mode associated to the vacuum fluctuation giving rise to the emission at $k$ and $\theta$, except for the fact that it propagates in the opposite sense.
Figure Captions

Figure 1. Photon emission rate as function of the normalized mechanical frequency $\Delta$ (see text), with $\theta = 78^\circ$ and $k_0 \delta q = 0.03$. Main plot: as calculated from the perturbation theory (dotted line) and from the nonperturbative approach presented in this paper (solid line). Insert plot: numerical results (solid line) and analytical approximation where high order sidebands are neglected (dashed line). Logarithmic scale is employed in the vertical axis in the insert. Note that for the scale employed in the main plot, it is not possible to distinguish numerical and analytical curves.

Figure 2. Frequency shift $\Delta_s$ as calculated from the analytical nonperturbative theory as a function of observation angle $\theta$. When $\theta$ varies from 0 to $90^\circ$, $\Delta_s$ increases by several orders of magnitude.

Figure 3. $\omega \times k_\parallel$ plane representing the field modes. For a given mechanical frequency $\Omega_0$, the region bounded by the triangle ABC (dark grey) represents the modes that may be excited. The light grey region represents evanescent field modes (not relevant in the present discussion). In this diagram, the photon creation process is associated to frequency upshifting. This is represented by a horizontal displacement (since $k_0$ is conserved) of length $\Omega_0$. Accordingly, photons in field modes along the BC line ($\Delta = 0$) are created from vacuum fluctuations that propagate along grazing directions (B'C' line).