RIGOROUS DERIVATION OF A LINEAR SIXTH-ORDER THIN FILM EQUATION AS A REDUCED MODEL FOR THIN FLUID - THIN STRUCTURE INTERACTION PROBLEMS

MARIO BUKAL AND BORIS MUHA

Abstract. We analyze a linear 3D/3D fluid-structure interaction problem between a thin layer of a viscous fluid and a thin elastic plate with the aim of deriving a simplified reduced model. Based on suitable energy-energy dissipation inequalities quantified in terms of two small parameters, thickness of the fluid layer and thickness of the elastic structure, we identify the right time scale and relation between small parameters which eventually, on the vanishing limit of small parameters, provide a reduced model. The reduced model is given in terms of a linear sixth-order thin-film equation describing the out-of-plane displacement of the plate. The linear thin-film equation is rigorously justified in terms of weak convergence results relating its solution to the solutions of the original problem. Furthermore, error estimates for approximate solutions constructed from the reduced model are obtained, which provide even strong convergence results.

1. Introduction

Physical models involving fluids lubricating underneath elastic structures are common phenomena in nature with ever-increasing application area in technology. In nature, such examples range from geophysics, like the growth of magma intrusions [32, 34], the fluid-driven opening of fractures in the Earth’s crust [6, 24], and subglacial floods [14, 46], to biology, for instance the passage of air flow in the lungs [25], and the operation of vocal cords [45]. They have also become inevitable mechanism in industry, for example in manufacturing of silicon wafers [27, 28] and suppression of viscous fingering [40, 41]. In the last two decades we witness an emergence of a huge area of microfluidics [29, 26, 44] with particular applications to so called lab-on-a-chip technologies [42, 15], which revolutionized experimentations in biochemistry and biomedicine. All those examples belong to a wider
class of physical models, called the fluid–structure interaction (FSI) models, which have recently gained a huge attention in applied mathematics community due to their important and increasing applications in medicine [5, 7], aero-elasticity [16], marine engineering [48], etc.

Mathematical models describing the above listed examples are coupled systems of partial differential equations, where fluids are typically described by the Stokes or Navier-Stokes equations, while structures are either described by linear elasticity equations or some lower-dimensional (nonlinear) model if the structure is relatively thin and has a plate-like geometry. If fluids are also considered to be relatively thin, like in our case, the lubrication approximation is formally employed giving rise to the Reynolds equation for the pressure (see e.g. [4, 37]). Coupling the Reynolds equation with the structure equation yields in further, after appropriate time scaling, to a reduced (simplified) model given in terms of a higher-order (fourth or sixth) evolution equation. Such models are common and favorable in engineering literature [26, 41, 24, 32]. They are typically derived based on some physical assumptions, heuristic arguments, and asymptotic expansion techniques. Despite the numerous application and abundance of the literature on reduced FSI models, they often lack rigorous mathematical derivation in the sense that there is no convergence of solutions (not even in a weak sense) of the original problem to solutions of the reduced problem, i.e. the literature on the topic of rigorous derivation of reduced models, which we outline below, is very scarce.

In the last 20 years there has been a lot of progress in well-posedness theory for the FSI problems (see e.g. [1, 5, 10, 12, 23, 36] and references within). Starting from various FSI problems, Čanić, Mikelić and others [11, 35, 43] studied the flow through a long elastic axially symmetric channel and using asymptotic techniques obtained several reduced (effective) models of Biot-type. In [11] they provided a rigorous justification of the reduced model through a weak convergence result and the corresponding error estimates. In [38] Panasenko and Stavre analyzed a periodic flow in thin channel with visco-elastic walls. The problem was initially described by a linear 2D (fluid)/1D (structure) FSI model and under the special ratio of the channel height and rigidity of the wall, a linear sixth-order thin-film equation describing the wall displacement emanated as a reduced model. A similar problem has been also considered in [13], resulting again in the reduced model described by a linear sixth-order equation. In both papers, reduced models have been rigorously justified by the appropriate convergence results. Finally, in [39] starting from a linear, again simplified, 2D/2D FSI model Panasenko and Stavre justified the simplified 2D/1D FSI model, which was the starting point in [38].

To the best of our knowledge, rigorous derivation of a reduced 2D model starting from a simple linear 3D/3D FSI model, where thicknesses of both parts (structure and fluid layer) vanish simultaneously, is lacking in the literature. Our aim in this manuscript is not only to fill this gap, but good understanding of the linear model is a natural first step towards rigorous derivation of nonlinear thin-film equations, for instance [26], which originate form more realistic FSI problems.
1.1. Problem formulation. We consider a physical model in which a three dimensional channel of relative height \( \varepsilon > 0 \) is filled with incompressible viscous fluid described by the Stokes equations, and the channel is covered by an elastic plate of relative height \( h > 0 \) which is described by the linear elasticity equations. Upon nondimensionalization of the model (domain and equations), we denote (non-dimensional) material configuration domain by \( \Omega_{\varepsilon,h} = \Omega_{\varepsilon} \cup \omega \cup \Omega_{h} \), where \( \Omega_{\varepsilon} = (0,1)^2 \times (-\varepsilon,0) \) denotes the fluid domain, \( \omega = (0,1)^2 \times \{0\} \) is the interface between the two phases, which we often identify with \( \omega \equiv (0,1)^2 \), and \( \Omega_{h} = (0,1)^2 \times (0,h) \) denotes the structure domain. The problem is then described by the system of partial differential equations:

\[
\begin{align*}
\rho_f \partial_t v - \text{div} \sigma_f(v,p) &= f, \quad \Omega_{\varepsilon} \times (0,T_\varepsilon), \quad (1) \\
\text{div} v &= 0, \quad \Omega_{\varepsilon} \times (0,T_\varepsilon), \quad (2) \\
\rho_s \partial_t u - \text{div} \sigma_s(u) &= 0, \quad \Omega_{h} \times (0,T_\varepsilon), \quad (3)
\end{align*}
\]

where fluid and structure stress tensors are given respectively by

\[
\begin{align*}
\sigma_f(v,p) &= 2\eta \text{sym} \nabla v - pI_3, \quad (4) \\
\sigma_s(u) &= 2\mu \text{sym} \nabla u + \lambda(\text{div} u)I_3,
\end{align*}
\]

\( \text{sym}(\cdot) \) denotes symmetric part of the matrix, \( f \) denotes the density of an external fluid force, and \( T_\varepsilon > 0 \) is a given time horizon. Unknowns in the above system are nondimensional quantities: the fluid velocity \( v \), the fluid pressure \( p \), and the plate displacement \( u \). Constitutive laws \( [4] \) are given in terms of non-dimensional numbers, which are in place of physical parameters: the fluid viscosity \( \eta \) and Lamé constants \( \mu \) and \( \lambda \), while \( \rho_f \) and \( \rho_s \) denote non-dimensional numbers in place of the density of the fluid and the structure, respectively.

The two subsystems (fluid and structure) are coupled through the interface conditions on the fixed interface \( \omega \):

\[
\begin{align*}
\partial_t u &= v, \quad \omega \times (0,T_\varepsilon), \quad (\text{kinematic – continuity of velocities}), \\
(\sigma_f(v,p) - \sigma_s(u))e_3 &= 0, \quad \omega \times (0,T_\varepsilon), \quad (\text{dynamic – stress balance}).
\end{align*}
\]

Remark 1.1. Contrary to the intuition of the moving interface in FSI problems, system \([1]–[6]\) is posed on the fixed domain with fixed interface. This simplification can be seen as a linearization of a truely nonlinear dynamics under the assumption of small displacements \([17]\). In particular, such models are relevant for describing the high frequency, small displacement oscillations of elastic structures immersed in low Reynolds number viscous fluids \([17]\).

Boundary and initial conditions. For simplicity of exposition we assume periodic boundary conditions in horizontal variables for all unknowns. On the bottom of the channel we assume no-slip condition \( v = 0 \), and the plate is free on the top boundary, i.e. \( \sigma_s(u)e_3 = 0 \). The system is for simplicity supplemented by trivial initial conditions:

\[
\begin{align*}
v(0) &= 0, \quad u(0) = 0, \quad \partial_t u(0) = 0.
\end{align*}
\]
Remark 1.2. All obtained results will also hold for nontrivial initial conditions under some additional assumptions, which we discuss in Appendix 3. We could also involve a nontrivial volume force on the structure (nontrivial right hand side in (3)) under certain scaling assumptions, similar to (A1) and (A2) below for the fluid volume forces. However, again for simplicity we take the trivial one, which is in fact a common choice for applications in microfluidics [42].

Remark 1.3. The above settled framework also incorporates physically more relevant problem which involves prescribed pressure drop between inlet and outlet of the channel, instead of the periodic boundary conditions. As described in [38], this is a matter of the right choice of the fluid volume force $f$.

Scaling ansatz and assumptions on data. In our analysis we will assume that small parameters $\varepsilon$ and $h$ are related through a power law

(S1) $\varepsilon = h^\gamma$ for some $\gamma > 0$ independent of $h$.

Lamé constants and structure density are also assumed to depend on $h$ as

(S2) $\mu_h = \hat{\mu} h^{-\kappa}$, $\lambda_h = \hat{\lambda} h^{-\kappa}$ and $\rho_s h = \hat{\rho}_s h^{-\kappa}$ for some $\kappa > 0$,

and $\hat{\mu}$, $\hat{\lambda}$ and $\hat{\rho}_s$ independent of $h$. Finally, the time scale of the system will be set as

(S3) $T = h^\tau$ for some $\tau \in \mathbb{R}$.

Scaling ansatz of the structure data is motivated by the fact that Lamé constants and density are indeed large for solid materials, and parameter $\kappa$ may be interpreted as a measure of stiffness of the structure material [9].

For the fluid volume force $f$ we assume:

(A1) $\|f\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\partial^2_\alpha f\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C\sqrt{\varepsilon}$ for $\alpha = 1, 2$,

(A2) $\|\partial_t f\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C\sqrt{\varepsilon T\varepsilon}$,

where $C > 0$ is independent of $f$ and $\varepsilon$.

Remark 1.4. (A1) is relatively weak assumption necessary for the derivation of the energy estimate [8], and consequently derivation of the reduced model (cf. Sec. 2), while (A2) is mainly needed for the error estimate analysis (cf. Sec. 4). Notice also that these assumptions are not “small data” assumptions, since the small factor $\sqrt{\varepsilon}$ comes from the size of the domain. For instance, any bounded load satisfies (A1), while additional boundedness of the time derivative asserts (A2).

Let us emphasize at this point that unknowns of the system are ansatz free, and our first aim is to determine the right scaling of unknowns, which will eventually lead to a nontrivial reduced model as $h, \varepsilon \downarrow 0$. The appropriate scaling of unknowns will be determined solely from a priori estimates, which are quantified in terms of small parameters $\varepsilon$ and $h$. 
1.2. Main results. Key ingredient of our convergence results, which provides all necessary a priori estimates, is the following energy estimate. Let \((v^\varepsilon, u^h)\) be a solution to (1)-(7), precisely defined in Section 2, and assume (A1), then

\[
\frac{\partial}{\partial t} \int_{\Omega} |v^\varepsilon(t)|^2 \, dx + \frac{\eta}{2} \int_0^t \int_{\Omega} |\nabla v^\varepsilon|^2 \, dx \, ds + \frac{\eta_s}{2} \int_{\Omega_h} |\partial_t u^h(t)|^2 \, dx \\
+ \int_{\Omega_h} \left( \mu \left| \text{sym} \nabla u^h(t) \right|^2 + \frac{\lambda}{2} |\text{div} u^h(t)|^2 \right) \, dx \leq C \varepsilon^3
\]

for a.e. \(t \in [0, T]\), where \(C > 0\) is independent of \((v^\varepsilon, u^h), \varepsilon, \) and of time variable \(t\). The proof of (8) is given in Section 2.4 (Proposition 2.3).

Rescaling the thin domain \(\Omega_{\varepsilon,h}\) to the reference domain \(\Omega = \Omega_- \cup \omega \cup \Omega_+\), as detailed described in Section 2 in detail, and rescaling time and data according to the above scaling ansatz, the rescaled energy estimate (8) together with the weak formulation suggest to take

\[
\tau = \kappa - 3 \gamma - 3 \quad \text{and} \quad \tau \leq -1
\]

in order to obtain a nontrivial limit model as \(h \downarrow 0\). Employing (9) in the rescaled problem (1)-(7) we obtain weak convergence results and identify a reduced FSI model. The following theorem summarizes our first main result.

**Theorem 1.1.** Let \((v(\varepsilon), p(\varepsilon), u(h))\) be a solution to the rescaled problem of (1)-(7), then the following convergence results hold. For the fluid part we have

\[
\varepsilon^{-2} v(\varepsilon) \rightharpoonup (v_1, v_2, 0) \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_-; \mathbb{R}^3)), \\
\varepsilon^{-2} \partial_3 v(\varepsilon) \rightharpoonup (\partial_3 v_1, \partial_3 v_2, 0) \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_-; \mathbb{R}^3)), \\
p(\varepsilon) \rightharpoonup p \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_-))
\]

on a subsequence as \(\varepsilon \downarrow 0\). The limit velocities are explicitly given in terms of the pressure in the sense of distributions

\[
v_\alpha(y, t) = \frac{1}{2\eta} y_3 (y_3 + 1) \partial_\alpha p(y', t) + F_\alpha(y, t) + (1 + y_3) \partial_t a_\alpha(t), \quad (y, t) \in \Omega_- \times (0, T),
\]

where \(F_\alpha(\cdot, y_3, \cdot) = \frac{y_3 + 1}{\eta} \int_{-1}^{y_3} \zeta_3 f_\alpha(\cdot, \zeta_3, \cdot) \, d\zeta_3 + \frac{1}{\eta} \int_{-1}^{y_3} (y_3 - \zeta_3) f_\alpha(\cdot, \zeta_3, \cdot) \, d\zeta_3\) and \(\partial_t a_\alpha \in L^\infty(0, T)\) denote limit of translational structure velocities (cf. Section 2.5).

For the structure part we find the linear bending plate model

\[
h^{2-\kappa} \begin{pmatrix} u_1(h) - a_1(h) \\ u_2(h) - a_2(h) \\ hu_3(h) \end{pmatrix} \rightharpoonup \begin{pmatrix} - (z_3 - \frac{1}{2}) \partial_1 w_3 \\ - (z_3 - \frac{1}{2}) \partial_2 w_3 \\ w_3 \end{pmatrix} \quad \text{weakly} \ast \quad \text{in} \quad L^\infty(0, T; H^1(\Omega_+; \mathbb{R}^3)),
\]
where \( w_3 \in L^\infty(0, T; H^2_\text{per}(\omega)) \) and \( a_\alpha(h) \subset L^\infty(0, T) \) denote horizontal translations of the structure. Furthermore, the vertical limit displacement \( w_3 \) is related to the limit pressure \( p \) in the sense of distributions as

\[
 p = \chi_\tau \hat{\rho}_s \partial_{tt}w_3 + \frac{8\hat{\mu}(\hat{\mu} + \hat{\lambda})}{3(2\hat{\mu} + \hat{\lambda})}(\Delta')^2 w_3,
\]

where \( \chi_\tau = 0 \) for \( \tau < -1 \) and \( \chi_\tau = 1 \) for \( \tau = -1 \), \( \hat{\lambda}, \hat{\mu} \) and \( \hat{\rho}_s \) are rescaled Lamé constants and material density according to (S2), while \( (\Delta')^2 \) denotes the bi-Laplace operator in horizontal variables. Finally, the system (10)–(11) is closed with a sixth order evolution equation for \( w_3 \)

\[
\partial_t w_3 - \chi_\tau \frac{\hat{\rho}_s}{12\eta} \Delta' \partial_{tt}w_3 - \frac{2\hat{\mu}(\hat{\mu} + \hat{\lambda})}{9\eta(2\hat{\mu} + \hat{\lambda})}(\Delta')^3 w_3 = F
\]

with periodic boundary conditions and trivial initial datum. The right hand side \( F \) is given by \( F(y', t) = -\int_{-1}^0 (\partial_1 F_1 \, dy_3 + \partial_2 F_2) \, dy_3 \).

We refer to equation (12) as the linear sixth-order thin-film equation. Complete proof of Theorem 1.1 with detailed discussions is given in Section 3.

Evolution equation (12) now serves as a reduced FSI model of the original problem (1)–(7). Namely, by solving (12), we can approximately reconstruct solutions of the original FSI problem in accordance with the convergence results of the previous theorem. Let \( w_3 \) be a solution of equation (12). The approximate pressure \( p^\varepsilon \) is defined by

\[
p^\varepsilon(x, t) = p(x', t), \quad (x, t) \in \Omega_\varepsilon \times (0, T),
\]

where \( p \) is given by (11) and the approximate fluid velocity \( v^\varepsilon \) is defined by

\[
v^\varepsilon(x, t) = \varepsilon^2 \left( v_1(x', \frac{x_3}{\varepsilon}, t), v_2(x', \frac{x_3}{\varepsilon}, t), 0 \right), \quad (x, t) \in \Omega_\varepsilon \times (0, T),
\]

with \( v_\alpha \) given by (10). Accordingly, we also define the approximate displacement \( u^h \) as

\[
u^h(x, t) = h^{\kappa-3} \left( h^{-\gamma} a_1 - \left( x_3 - \frac{h}{2} \right) \partial_1 w_3(x', t), h^{-\gamma} a_2 - \left( x_3 - \frac{h}{2} \right) \partial_2 w_3(x', t), w_3(x', t) \right)
\]

for all \( (x, t) \in \Omega_h \times (0, T) \), where \( a_\alpha(t) = \int_0^t \partial_t a_\alpha \, ds, \alpha = 1, 2, \) and \( \partial_t a_\alpha \) are given by (70). Observe that approximate solutions are defined on the original thin domain \( \Omega_{\varepsilon,h} \), but in rescaled time.

Our second main result provides error estimates for approximate solutions, and thus strong convergence results in respective norms.
Theorem 1.2. Let \((v^\varepsilon, p^\varepsilon, u^h)\) be a solution to the original FSI problem (1)–(7) in rescaled time and let \((v^\varepsilon, p^\varepsilon, u^h)\) be approximate solution constructed from the reduced model as above. Let us additionally assume that max\{2\gamma + 1, \frac{7}{4} \gamma + \frac{3}{2}\} \leq \kappa < 2 + 2\gamma, then
\[
\|v^\varepsilon - v^\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon^{5/2}h^{\min\{\gamma/2, 2\gamma - \kappa + 2\}},
\]
\[
\|p^\varepsilon - p^\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon^{1/2}h^{\min\{2\gamma - \kappa + 2\}},
\]
\[
\|u^h_\alpha - u^h_\alpha\|_{L^\infty(0,T;L^2(\Omega_h))} \leq Ch^{k-3/2}h^{\min\{1, \gamma/2, 2\gamma + 2 - \kappa\}} + C\sqrt{h}\|a^h_\alpha - h^{\kappa-3-\gamma}a_\alpha\|_{L^\infty(0,T)},
\]
\[
\|u^h_3 - u^h_3\|_{L^\infty(0,T;L^2(\Omega_h))} \leq Ch^{\kappa-5/2}h^{\min\{1/2, 2\gamma + 2 - \kappa\}},
\]
where \(C > 0\) denote generic positive constants independent of \(\varepsilon\) and \(h\).

Remark 1.5. Note that the error estimate of horizontal fluid velocities relative to the norm of velocities as well as the relative error estimate of the pressure is \(O(h^{\min\{\gamma/2, 2\gamma - \kappa + 2\}})\). Hence, for \(\kappa \leq \frac{3}{2}\gamma + 2\), this convergence rate is \(O(\sqrt{\varepsilon})\). Since \(v^\varepsilon\) is of lower order, we would need to construct better (higher-order) corrector for establishing error estimates in the vertical component of the fluid velocity. Such construction would require additional tool and would thus exceed the scope of this paper. In the leading order of the structure displacement, the vertical component, we have relative convergence rate \(O(h^{\min\{1/2, \gamma/2, 2\gamma - \kappa + 2\}})\), which for \(\kappa \leq \frac{3}{2}\gamma + 2\) means \(O(h^{\min\{1/2, \gamma/2\}})\), i.e. \(O(\sqrt{\varepsilon})\) for \(\gamma \leq 1\) and \(O(\sqrt{h})\) for \(\gamma > 1\). In horizontal components, in-plane displacements, dominant part of the error estimates are errors in horizontal translations, which are artefact of periodic boundary conditions (cf. Section 2.5). Neglecting these errors which cannot be controlled in a better way, the relative error estimate of horizontal displacements is \(O(h^{\min\{1/2, \gamma/2, (2\gamma + 2 - \kappa)\}})\). For \(\kappa \leq \frac{3}{2}\gamma + 2\), this estimate is \(O(h^{\min\{1, \gamma/2\}})\), which in addition means \(O(\sqrt{\varepsilon})\) for \(\gamma \leq 2\) and \(O(h)\) for \(\gamma > 2\).

Let us point out that one cannot expect better convergence rates for such first-order approximation without dealing with boundary layers, which arise around interface \(\omega\) due to mismatch of the interface conditions for approximate solutions. For example, in [33] the obtained convergence rate for the Poiseuille flow in the case of rigid walls of the fluid channel is \(O(\sqrt{\varepsilon})\). On the other hand, convergence rate for the clamped Kirchhoff-Love plate is found to be \(O(\sqrt{h})\) [22]. Additional conditions on parameters \(\kappa\) and \(\gamma\) which appear in the theorem are mainly due to technical difficulties of dealing with structure translations in horizontal directions. If these translations were not present in the model, the error estimates of Theorem 1.2 would improve.

The proof of Theorem 1.2 is demonstrated in Section 4.

2. Energy estimates and weak solutions

2.1. Notation and definitions. Let \(x = (x', x_3) = (x_1, x_2, x_3) \in \Omega_{\varepsilon,h}\) denotes the spatial variables in original thin domain and let \(y = (y', y_3) = (x', x_3/\varepsilon) \in (0,1)^2 \times (-1,0) =: \Omega_-\) and \(z = (z', z_3) = (x', x_3/h) \in (0,1)^2 \times (0,1) =: \Omega_+\) denote respectively, the fluid and the
Figure 1. Sketch of the original thin domain $\Omega_{\varepsilon,h}$.

structure variables in the reference domain. Sketch of the original thin domain is depicted in Figure 1. Solutions in the original domain $\Omega_{\varepsilon,h}$ will be denoted by $\varepsilon$ or $h$ in superscripts, i.e. $v^\varepsilon$, $p^\varepsilon$ and $u^h$. On the reference domain, solutions will be denoted by $\varepsilon$ or $h$ in parentheses and they are defined according to

\[
(13) \quad v(\varepsilon)(y,t) := v^\varepsilon(x,t), \quad p(\varepsilon)(y,t) := p^\varepsilon(x,t), \quad u(h)(z,t) := u^h(x,t)
\]

for all $(x,t) \in \Omega_{\varepsilon,h} \times (0,T)$. Next, we denote scaled gradients by $\nabla \varepsilon = (\partial_{y_1}, \partial_{y_2}, \frac{1}{\varepsilon} \partial_{y_3})$ and $\nabla h = (\partial_{z_1}, \partial_{z_2}, \frac{1}{h} \partial_{z_3})$, and they satisfy the following identities

\[
(14) \quad \nabla v^\varepsilon(x,t) = \nabla \varepsilon v(\varepsilon)(y,t), \quad \nabla u^h(x,t) = \nabla h u(h)(z,t), \quad (x,t) \in \Omega_{\varepsilon,h} \times (0,T).
\]

When the domain of a function is obvious, partial derivatives $\partial_x$, $\partial_y$ or $\partial_z$ will be simply denoted by $\partial_i$ for $i = 1, 2, 3$. Greek letters $\alpha, \beta$ in indices will indicate only horizontal variables, i.e. $\alpha, \beta = 1, 2$.

The basic energy estimate for the original FSI problem (1)–(7), given in Section 2.2 below, suggest the following functions spaces to be appropriate for definition of weak solutions and test functions. For fluid velocity, the appropriate function space appears to be

\[
\mathcal{V}_F(0,T;\Omega_\varepsilon) = L^\infty(0,T;L^2(\Omega_\varepsilon;\mathbb{R}^3)) \cap L^2(0,T;V_F(\Omega_\varepsilon)),
\]

where $V_F(\Omega_\varepsilon) = \{v \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : \text{div } v = 0, \ v|_{x_3=-\varepsilon} = 0, \ v \text{ is } \omega\text{-periodic}\}$, and $T > 0$ is a given time horizon. Similarly, the structure function space will be

\[
\mathcal{V}_S(0,T;\Omega_h) = W^{1,\infty}(0,T;L^2(\Omega_h;\mathbb{R}^3)) \cap L^\infty(0,T;V_S(\Omega_h)),
\]

where $V_S(\Omega_h) = \{u \in H^1(\Omega_h;\mathbb{R}^3) : u \text{ is } \omega\text{-periodic}\}$. Finally, the solution space of the coupled problem (1)–(7) on the thin domain will be compound of previous spaces involving
the kinematic interface condition \((5)\) as a constraint:

\[
\mathcal{V}(0, T_\varepsilon; \Omega_{\varepsilon, h}) = \{(v, u) \in \mathcal{V}_F(0, T_\varepsilon; \Omega_\varepsilon) \times \mathcal{V}_S(0, T_\varepsilon; \Omega_h) : v(t) = \partial_t u(t) \text{ on } \omega \text{ for a.e. } t \in (0, T_\varepsilon)\}.
\]

Now we can state the definition of weak solutions to our problem in the sense of Leray and Hopf.

**Definition 2.1.** We say that a pair \((v^\varepsilon, u^h)\) is a weak solution to the linear FSI problem (1)–(7), if the following variational equation holds in \(D'(0, T_\varepsilon)\):

\[
\begin{align*}
\varrho_f & \frac{d}{dt} \int_{\Omega_\varepsilon} v^\varepsilon \cdot \phi \, dx - \varrho_f \int_{\Omega_\varepsilon} v^\varepsilon \cdot \partial_t \phi \, dx + 2\eta \int_{\Omega_\varepsilon} \text{sym} \nabla v^\varepsilon : \text{sym} \nabla \phi \, dx \\
\varrho_s & \frac{d}{dt} \int_{\Omega_h} u^h \cdot \partial_t \psi \, dx - \varrho_s \int_{\Omega_h} \partial_t u^h \cdot \partial_t \psi \, dx + \int_{\Omega_h} (2\mu \text{sym} \nabla u^h : \text{sym} \nabla \psi + \lambda \text{div} u^h \text{div} \psi) \, dx = \int_{\Omega_\varepsilon} f^\varepsilon \cdot \phi \, dx
\end{align*}
\]

for all \((\phi, \psi) \in \mathcal{W}(0, T_\varepsilon; \Omega_{\varepsilon, h})\), where

\[
\mathcal{W}(0, T_\varepsilon; \Omega_{\varepsilon, h}) = \{(\phi, \psi) \in C^1([0, T_\varepsilon]; V_F(\Omega_\varepsilon) \times V_S(\Omega_h)) : \phi(t) = \psi(t) \text{ on } \omega \text{ for all } t \in [0, T_\varepsilon]\}
\]

denotes the space of test functions. Moreover, \((v^\varepsilon, u^h)\) verify energy dissipation inequality \((16)\) given below.

### 2.2. Basic energy estimate.

First we derive a basic energy estimate, quantified only in terms of the relative fluid thickness \(\varepsilon\), while the full geometry of the system will be utilized later.

**Proposition 2.1.** Let us assume (A1) and let \((v^\varepsilon, u^h) \in \mathcal{V}(0, T_\varepsilon; \Omega_{\varepsilon, h})\) be a solution to \((16)\). There exists a constant \(C > 0\), independent of \(\varepsilon\) and \(T_\varepsilon\), such that the following energy estimate holds:

\[
\begin{align*}
\frac{\varrho_f}{2} \int_{\Omega_\varepsilon} |v^\varepsilon(t)|^2 \, dx + \eta \int_0^t \int_{\Omega_\varepsilon} |\text{sym} \nabla v^\varepsilon(s)|^2 \, dx \, ds + \frac{\varrho_s}{2} \int_{\Omega_h} |\partial_t u^h(t)|^2 \, dx \\
+ \mu \int_{\Omega_h} |\text{sym} \nabla u^h(t)|^2 \, dx + \frac{\lambda}{2} \int_{\Omega_h} |\text{div} u^h(t)|^2 \, dx \leq Ct_\varepsilon,
\end{align*}
\]

for a.e. \(t \in [0, T_\varepsilon]\).
\begin{proof}
Here we present just formal argument for the basic energy which can be made rigorous in the standard way, see e.g. \cite{19}. Let us take solution \((v^\varepsilon, \partial_t u^h)\) as test functions in \((16)\).
\[
\frac{\partial f}{2} \int_{\Omega_\varepsilon} |v^\varepsilon(t)|^2 \, dx + 2\eta \int_0^t \int_{\Omega_\varepsilon} |\nabla v^\varepsilon(s)|^2 \, dx \, ds + \frac{\eta}{2} \int_{\Omega_h} |\partial_t u^h(t)|^2 \, dx
\]
\[
+ \mu \int_{\Omega_h} |\nabla u^h(t)|^2 \, dx + \frac{\lambda}{2} \int_{\Omega_h} |\text{div} \, u^h(t)|^2 \, dx = \int_0^t \int_{\Omega_\varepsilon} f^\varepsilon \cdot v^\varepsilon \, dx \, ds.
\]
Now let us estimate the right hand side. First, apllying the Cauchy-Schwarz inequality, then employing the assumption (A1) on the volume force \(f^\varepsilon\), and utilizing Poincaré and Korn inequalities from Proposition A.2 we obtain respectively,
\[
\left| \int_0^t \int_{\Omega_\varepsilon} f^\varepsilon \cdot v^\varepsilon \, dx \, ds \right| \leq \int_0^t \| f^\varepsilon \|_{L^2(\Omega_\varepsilon)} \| v^\varepsilon \|_{L^2(\Omega_\varepsilon)} \, ds \leq C \int_0^t \sqrt{\varepsilon \varepsilon} \| \partial_3 v^\varepsilon \|_{L^2(\Omega_\varepsilon)} \, ds
\]
\[
\leq C t \varepsilon + \eta \int_0^t \| \nabla v^\varepsilon \|_{L^2(\Omega_\varepsilon)}^2 \, ds.
\]
The latter inequality is obtained by choosing a suitable constant in the application of the Young inequality such that the last term can be absorbed in the left-hand side of \((18)\), which finishes the proof.
\end{proof}

Remark 2.1. In a special case, when the fluid volume force is vertical, i.e. \(f^\varepsilon = f_3^\varepsilon e_3\), one can obtain energy estimate of order \(O(\varepsilon^3)\). This is due to the fact that we don’t need to employ the Korn inequality. Namely, the right hand side of \((18)\) can be estimated as
\[
\left| \int_0^t \int_{\Omega_\varepsilon} f^\varepsilon \cdot v^\varepsilon \, dx \, ds \right| \leq C \int_0^t \varepsilon^{3/2} \| \partial_3 v^\varepsilon \|_{L^2(\Omega_\varepsilon)} \, ds \leq C t \varepsilon^3 + \eta \int_0^t \| \nabla v^\varepsilon \|_{L^2(\Omega_\varepsilon)}^2 \, ds.
\]

2.3. Existence and regularity of weak solutions. Although \((1)–(7)\) is a linear problem, the existence analysis is not trivial. Well-posedness for related (but geometrically different) problem has been first established in \cite{17} using a Galerkin approximation scheme, and later in \cite{2, 3} using the semigroup approach. The existence analysis for a completely analogous problem to \((1)–(7)\), but in 2D case, has been performed in \cite{39} using the Galerkin approximation scheme. The following proposition, which establishes the existence of a unique solution to \((16)\), is a compound of analogous results from \cite{39} and \cite{1}.

Proposition 2.2. Let \(T_\varepsilon > 0\) be given and let assumption (A1) holds. There exists a unique solution \((v^\varepsilon, u^h)\) \(\in \mathcal{V}(0,T_\varepsilon; \Omega_{\varepsilon,h})\) to \((16)\), which additionally satisfies:
(a) (time regularity)
\[
\partial_t v^\varepsilon \in L^\infty(0,T_\varepsilon; L^2(\Omega_\varepsilon; \mathbb{R}^3)) \quad \text{and} \quad \partial_t u^h \in L^\infty(0,T_\varepsilon; L^2(\Omega_h; \mathbb{R}^3)),
\]
(b) (space regularity)
\[
v^\varepsilon \in L^\infty(0,T_\varepsilon; H^2(\Omega_\varepsilon; \mathbb{R}^3)) \quad \text{and} \quad u^h \in L^\infty(0,T_\varepsilon; H^2(\Omega_h; \mathbb{R}^3)).
\]
Moreover, there exists a unique pressure $p^\varepsilon \in L^2(0,T; H^1(\Omega_\varepsilon))$ such that $(v^\varepsilon, p^\varepsilon, u^h)$ solves the original problem (1)-(7) in a classical sense.

**Proof.** The existence of a unique weak solution, together with its time regularity, follows by the straightforward application of the proof of Theorem 3.1 from [39] in the case of three spatial dimensions. Spatial regularity of the fluid velocity and the existence of the pressure of the corresponding regularity can also be recovered from [39, Theorem 3.2]. However, for the structure displacement, only $\text{div} \sigma(u^h) \in L^\infty(0,T; L^2(\Omega_h; \mathbb{R}^3))$ has been established. To complete the proof we invoke [1, Theorem 2.1], where the required spatial regularity of the structure displacement has been performed for a related FSI problem. \hfill \square

**Remark 2.2.** We emphasize at this point that in the subsequent analysis we will not need the full spatial regularity of solutions provided in the previous proposition, but only the corresponding regularity with respect to the horizontal variables. Such regularity can be shown as below within the derivation of improved energy estimate.

**2.4. Improved energy estimates.** Next, we aim to quantitatively improve, in terms of the small parameter $\varepsilon$, the basic energy estimate (17).

**Proposition 2.3.** Assume that the volume force $f^\varepsilon$ satisfies assumption (A1) and let $(v^\varepsilon, u^h) \in V(0,T; \Omega_{\varepsilon,h})$ be the solution to (16). There exists a constant $C > 0$, independent of $\varepsilon$ and $T_\varepsilon$, such that the following energy estimate holds

$$
\frac{\theta_f}{2} \int_{\Omega_\varepsilon} |v^\varepsilon(t)|^2 \, dx + \frac{\eta}{2} \int_0^t \int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx \, ds + \frac{\theta_s}{2} \int_{\Omega_h} |\partial_t u^h(t)|^2 \, dx \\
+ \int_{\Omega_h} \left( \mu |\text{sym} \nabla u^h(t)|^2 + \frac{\lambda}{2} |\text{div} u^h(t)|^2 \right) \, dx \leq Ct\varepsilon^3
$$

for a.e. $t \in [0,T_\varepsilon)$.

Observe that in (20), unlike in (17), we control the full gradient of the fluid velocity.

**Proof.** Take formally $(-\partial_\alpha^2 v^\varepsilon, -\partial_\alpha^2 \partial_t u^h)$ for $\alpha = 1, 2$, as test functions in (16). After integrating by parts in horizontal variables, for a.e. $t \in [0,T_\varepsilon)$ we obtain

$$
\frac{\theta_f}{2} \int_{\Omega_\varepsilon} |\partial_\alpha v^\varepsilon(t)|^2 \, dx + 2\eta \int_0^t \int_{\Omega_\varepsilon} |\nabla \partial_\alpha v^\varepsilon| \, dx \, ds + \frac{\theta_s}{2} \int_{\Omega_h} |\partial_t \partial_\alpha u^h(t)|^2 \, dx \\
+ \mu \int_{\Omega_h} |\text{sym} \partial_\alpha u^h(t)|^2 \, dx + \frac{\lambda}{2} \int_{\Omega_h} |\text{div} \partial_\alpha u^h(t)|^2 \, dx = -\int_0^t \int_{\Omega_\varepsilon} \partial_\alpha^2 f^\varepsilon \cdot v^\varepsilon \, dx \, ds,
$$

for $\alpha = 1, 2$. The above formalism for weak solutions can be justified by standard arguments using finite differences instead of partial derivatives. Under assumption (A1), the right-hand side can be estimated using the Poincaré and Korn inequalities on thin domains from
Proposition A.2 as well as the basic energy inequality (17)
\[
\left| \int_0^t \int_{\Omega_\varepsilon} \partial_\alpha^2 f^\varepsilon \cdot v^\varepsilon \, dx \, ds \right| \leq C \left| \int_0^t \varepsilon^{3/2} \| \partial_3 v^\varepsilon \|_{L^2(\Omega_\varepsilon)} \, ds \right| \leq C \left( \int_0^t \varepsilon^{1/2} \| \text{sym} \nabla v^\varepsilon \|_{L^2(\Omega_\varepsilon)} \right) \leq C t \varepsilon .
\]

Therefore, we have
\[
\frac{\partial f}{2} \int_{\Omega_\varepsilon} |\partial_\alpha v^\varepsilon(t)|^2 \, dx + 2 \eta \int_{\Omega_\varepsilon} |\text{sym} \nabla \partial_\alpha v^\varepsilon(s)|^2 \, dx \, ds + \frac{\partial f}{2} \int_{\Omega_h} |\partial_\alpha \partial_\alpha u^h(t)|^2 \, dx
\]
\[
+ \mu \int_{\Omega_h} |\text{sym} \nabla \partial_\alpha u^h(t)|^2 \, dx + \frac{\lambda}{2} \int_{\Omega_h} |\text{div} \partial_\alpha u^h(t)|^2 \, dx \leq C t \varepsilon , \quad \alpha = 1, 2 .
\]

Next, observe that for every \( t \in (0, T_\varepsilon) \) we have
\[
\| \text{sym} \nabla v^\varepsilon \|_{L^2(0,t;L^2(\Omega_\varepsilon))}^2 = \frac{1}{2} \| \nabla v^\varepsilon \|_{L^2(0,t;L^2(\Omega_\varepsilon))}^2 + \frac{1}{2} \int_0^t \int_{\Omega_\varepsilon} \nabla v^\varepsilon : \nabla^T v^\varepsilon \, dx \, ds .
\]

Using the Einstein’s summation convention and integrating by parts with respect to space variables, the last term on the right hand side equals
\[
\int_0^t \int_{\Omega_\varepsilon} \nabla v^\varepsilon : \nabla^T v^\varepsilon \, dx \, ds = \int_0^t \int_{\Omega_\varepsilon} \partial_i v_i^\varepsilon \partial_j v_j^\varepsilon \, dx \, ds
\]
\[
= - \int_0^t \int_{\Omega_\varepsilon} v_j^\varepsilon \partial_i \partial_j v_i^\varepsilon \, dx \, ds + \int_0^t \int_{\partial \Omega_\varepsilon} v_j^\varepsilon \partial_j \partial_i v_i^\varepsilon n_i \, ds ,
\]
where \( n = (n_1, n_2, n_3) \) denotes the outward unit normal on \( \Omega_\varepsilon \). The divergence free condition together with imposed boundary conditions finally provide
\[
\int_{\Omega_\varepsilon} \nabla v^\varepsilon : \nabla^T v^\varepsilon = \int_\omega v_j^\varepsilon \partial_i v_i^\varepsilon = 2 \int_\omega (v_1^\varepsilon \partial_1 v_1^\varepsilon + v_2^\varepsilon \partial_2 v_2^\varepsilon + v_3^\varepsilon \partial_3 v_3^\varepsilon ) .
\]

Taking \((v^\varepsilon, \partial_\alpha u^h)\) as a test function in (16) and using the identity (23), we find
\[
\frac{\partial f}{2} \int_{\Omega_\varepsilon} |v^\varepsilon(t)|^2 \, dx + \eta \int_{\Omega_\varepsilon} \| \nabla v^\varepsilon \|^2 \, dx \, ds
\]
\[
+ \frac{\partial f}{2} \int_{\Omega_h} |\partial_\alpha u^h(t)|^2 \, dx + \int_{\Omega_h} \left( \mu |\text{sym} \nabla u^h(t)|^2 + \frac{\lambda}{2} |\text{div} u^h(t)|^2 \right) \, dx
\]
\[
= \int_0^t \int_{\Omega_\varepsilon} f^\varepsilon \cdot v^\varepsilon \, dx \, ds - 2 \eta \int_0^t \int_{\omega} (v_1^\varepsilon \partial_1 v_1^\varepsilon + v_2^\varepsilon \partial_2 v_2^\varepsilon ) \, ds ,
\]
for a.e. \( t \in [0, T_\varepsilon) \). Now we again estimate terms on the right hand side. The force term is estimated in a similar fashion like in (19):
\[
\left| \int_0^t \int_{\Omega_\varepsilon} f^\varepsilon \cdot v^\varepsilon \, dx \, ds \right| \leq \int_0^t \varepsilon^{3/2} \| \partial_3 v^\varepsilon \|_{L^2(\Omega_\varepsilon)} \, ds \leq C t \varepsilon^3 + \frac{\eta}{4} \int_0^t \| \partial_3 v^\varepsilon \|^2_{L^2(\Omega_\varepsilon)} \, ds ,
\]
but now controlling the full gradient of $v^\varepsilon$, which provides the better estimate in terms of $\varepsilon$.

The interface terms in (24) are estimated in the following way, separately for every $\alpha = 1, 2$. First, using the Cauchy-Schwarz inequality and the trace inequality from Proposition A.2, we obtain

$$
\left| \int_0^t \int_\omega v^\varepsilon_\alpha \partial_\alpha v^\varepsilon_3 \, dx \, ds \right| \leq \int_0^t \| v^\varepsilon_\alpha \|_{L^2(\omega)} \| \partial_\alpha v^\varepsilon_3 \|_{L^2(\omega)} \, ds \leq C\varepsilon \int_0^t \| \partial_3 v^\varepsilon \|_{L^2(\Omega_s)} \| \partial_\alpha \partial_\alpha v^\varepsilon_3 \|_{L^2(\Omega_s)} \, ds.
$$

Observe that the term $\partial_3 \partial_\alpha v^\varepsilon_3$ is a diagonal element of $\text{sym} \nabla \partial_\alpha v^\varepsilon$. Hence, according to (22), we further estimate

$$
(26) \quad \left| \int_0^t \int_\omega v^\varepsilon_\alpha \partial_\alpha v^\varepsilon_3 \, dx \, ds \right| \leq C t^{1/2} \varepsilon^{3/2} \| \partial_3 v^\varepsilon \|_{L^2(0,t;L^2(\Omega_s))} \leq C t \varepsilon^3 + \frac{\eta}{4} \int_0^t \| \partial_3 v^\varepsilon \|_{L^2(\Omega_s)}^2 \, ds.
$$

Going back to (24) we conclude the improved energy estimate (20). □

Assuming additional regularity of solutions and repeating formally the above arguments, one obtains improved higher-order energy estimates, which will be used in estimating the pressure and later in the error analysis in Section 4.

**Corollary 2.4.** Let us assume that (A2) holds and let $(v^\varepsilon, u^h) \in \mathcal{V}(0, T_\varepsilon; \Omega, h)$ be the solution to (16). There exists a constant $C > 0$, independent of $\varepsilon$ and $T_\varepsilon$, such that the following a priori estimates hold:

$$
(27) \quad \frac{\theta_1}{2} \int_{\Omega_\varepsilon} |\partial_3 v^\varepsilon(t)|^2 \, dx + \frac{\eta}{2} \int_0^t \int_{\Omega_\varepsilon} |\nabla \partial_\alpha v^\varepsilon|^2 \, dx \, ds + \frac{\theta_2}{2} \int_{\Omega_h} |\partial_\alpha u^h(t)|^2 \, dx
$$

$$
+ \int_{\Omega_h} \left( \mu |\text{sym} \nabla \partial_\alpha u^h(t)|^2 + \frac{\lambda}{2} |\text{div} \partial_\alpha u^h(t)|^2 \right) \, dx \leq C T^{-1} \varepsilon^3
$$

$$
(28) \quad \frac{\theta_1}{2} \int_{\Omega_\varepsilon} |\partial_3 v^\varepsilon(t)|^2 \, dx + \frac{\eta}{2} \int_0^t \int_{\Omega_\varepsilon} |\nabla \partial_\alpha v^\varepsilon|^2 \, dx \, ds + \frac{\theta_2}{2} \int_{\Omega_h} |\partial_\alpha u^h(t)|^2 \, dx
$$

$$
+ \int_{\Omega_h} \left( \mu |\text{sym} \nabla \partial_\alpha u^h(t)|^2 + \frac{\lambda}{2} |\text{div} \partial_\alpha u^h(t)|^2 \right) \, dx \leq C t \varepsilon^3
$$

for a.e. $t \in [0, T_\varepsilon)$ and $\alpha = 1, 2$.

2.5. **Rigid body displacements.** Since the boundary conditions for the structure equations are periodic on the lateral boundaries and only stress is prescribed on the interface and upper boundary, the structure is not anchored and nontrivial rigid body displacements arise as part of solutions. However, the periodic boundary conditions prevent rotations and due to the coupling with the fluid, translations can also be controlled. First, the kinematic coupling in the vertical direction together with the incompressibility of the fluid imply

$$
\frac{d}{dt} \int_\omega u^h_3(t) \, dx = \int_\omega \partial_3 u^h_3(t) \, dx = \int_\omega v^\varepsilon_3(t) \, dx = \int_{\Omega_\varepsilon} \text{div} v^\varepsilon(t) \, dx = 0.
$$
Therefore, due to the trivial initial conditions we have \( \int_\omega u_3^h(t) \, dx' = 0 \) for every \( t \in (0, T_\epsilon) \), which implies that there are no translations in the vertical direction. Let us now estimate translations in the tangential directions. Using the trace inequality from Proposition A.2 we have

\[
\left| \frac{d}{dt} \int_\omega u_3^h(t) \, dx' \right| = \left| \int_\omega v_3^\alpha(t) \, dx' \right| \leq C \| v_3^\alpha(t) \|_{L^2(\omega)} \leq C \sqrt{\epsilon} \| \nabla v_3^\alpha(t) \|_{L^2(\Omega_\epsilon)}.
\]

Employing the the last inequality together with improved energy estimate (20) we obtain

\[
(29) \quad \left| \int_\omega u_3^h(t) \, dx' \right| = \left| \int_0^t \frac{d}{dt} \int_\omega u_3^h(s) \, dx' \, ds \right| \leq C \sqrt{\epsilon} \int_0^t \| \nabla v_3^\alpha(s) \|_{L^2(\Omega_\epsilon)} \, ds \leq C \sqrt{\epsilon} \sqrt{t} \| \nabla v_3^\alpha(s) \|_{L^2(0,t;L^2(\Omega_\epsilon))} \leq C \epsilon^2 t
\]

for every \( t \in (0, T_\epsilon) \). Estimate (29) shows that for large time scales, which are of particular interest in the lubrication approximation regime, the tangential translations can be of order \( O(1) \) or bigger. On the other hand, these translations are actually artefact of the periodic boundary conditions, which we consider in order to avoid unnecessary technical complications that would arise with other type of boundary conditions. For example, if the structure is clamped or anchored in some other way, these translations would not be present. Moreover, we will see in the subsequent section that these translations do not play the role in the derivation of reduced FSI model (cf. Section 3), but they do play a role in construction of approximate solutions and error analysis.

3. Derivation of reduced FSI model — proof of Theorem 1.1

In this section we prove our first main result, Theorem 1.1. The proof is divided into several steps throughout the following subsections. First we employ the scaling ansatz (S1)–(S3), rescale the energy estimate and obtain uniform estimates on the reference domain. Based on these estimates we further rescale the unknowns and finally identify the reduced model by means of weak convergence results.

3.1. Uniform estimates on the reference domain. The key source of uniform estimates is the energy estimate (20). In order to obtain a nontrivial reduced model we need to rescale the time and structure data. Let us denote the new time variable with hat and define it according to \( t = \hat{T} \hat{t} \), where \( \hat{T} > 0 \) denotes the time scale of the system satisfying (S3). Functions depending on the new time are then defined by \( \hat{w}(\hat{t}) = w(t) \), and its time derivative equals \( \partial_\hat{t} \hat{w} = T^{-1} \partial_t w \).
3.1.1. Rescaled energy estimates. Recalling the scaling ansatz (S1)–(S2) and neglecting hats in further notation, the rescaled energy estimate (20) now reads

$$\frac{\eta \tau}{2} \int_{\Omega_\varepsilon} |v^\varepsilon(t)|^2 \, dx + \frac{\eta \tau}{2} \int_0^t \int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx \, ds + \frac{\eta}{2} h^{-\kappa} \int_{\Omega_\varepsilon} |\partial_t u^h(t)|^2 \, dx$$

$$+ h^{-\kappa} \int_{\Omega_\varepsilon} \left( \mu \text{sym} \nabla u^h(t)^2 + \frac{\lambda}{2} |\text{div} u^h(t)|^2 \right) \, dx$$

for a.e. $t \in (0, T)$, where $T = \tau T_\varepsilon$ denotes the rescaled time horizon.

Furthermore, the rescaled higher-order energy estimate (27) reads

$$\frac{\eta \tau}{2} \int_{\Omega_\varepsilon} |\partial_t v^\varepsilon(t)|^2 \, dx + \frac{\eta \tau}{2} \int_0^t \int_{\Omega_\varepsilon} |\nabla \partial_t v^\varepsilon|^2 \, dx \, ds + \frac{\eta}{2} h^{-\kappa} \int_{\Omega_\varepsilon} |\partial_t u^h(t)|^2 \, dx$$

$$+ h^{-\kappa} \int_{\Omega_\varepsilon} \left( \mu |\text{sym} \partial_t u^h(t)|^2 + \frac{\lambda}{2} |\text{div} \partial_t u^h(t)|^2 \right) \, dx \leq C \tau \varepsilon^3$$

for a.e. $t \in (0, T)$. Observe that the last inequality is quantitatively (in terms of small parameters) of the same type as (30).

In order to perform the dimension reduction (i.e. obtain limits as $\varepsilon, h \downarrow 0$) we also need to move to the reference domain $\Omega = \Omega_- \cup \omega \cup \Omega_+$ by the standard change of variables introduced in (13)–(14) and obtain the uniform energy estimates there. After the change of variables, the energy estimate (30) on the reference domain reads: for a.e. $t \in (0, T)$ it holds

$$\frac{\eta \tau}{2} \int_{\Omega_-} |v(\varepsilon)(t)|^2 \, dy + \frac{\eta \tau}{2} \int_0^t \int_{\Omega_-} |\nabla v(\varepsilon)|^2 \, dy \, ds + \frac{\eta}{2} h^{-\kappa-2\tau+1} \int_{\Omega_+} |\partial_t u(h)(t)|^2 \, dz$$

$$+ h^{-\kappa+1} \int_{\Omega_+} \left( \mu |\text{sym} \nabla_h u(h)(t)|^2 + \frac{\lambda}{2} |\text{div}_h u(h)(t)|^2 \right) \, dz \leq C \tau \varepsilon^3.$$

3.1.2. Uniform estimates for the fluid velocity. The rescaled energy estimate (32) gives us uniform bound

$$\frac{\eta \tau}{2} \int_0^T \int_{\Omega_-} |\nabla v(\varepsilon)|^2 \, dy \, ds \leq C \tau \varepsilon^3,$$

which directly implies

$$\int_0^T \int_{\Omega_-} |\partial_3 v(\varepsilon)|^2 \, dy \, ds \leq C \varepsilon^4.$$

Using the boundary condition $v(\varepsilon)|_{y_3=-1} = 0$, we have the identity

$$v(\varepsilon)(y', y_3, t) = \int_{-1}^{y_3} \partial_3 v(\varepsilon)(y', \zeta, t) \, d\zeta,$$
which together with (33) then provides

\[ \| \mathbf{v}(\varepsilon) \|_{L^2(0,T; L^2(\Omega \times \mathbb{R}^3))} \leq C\varepsilon^2. \]

This uniform estimate motivates rescaling of the fluid velocity according to

\[ \tilde{\mathbf{v}}(\varepsilon) = \varepsilon^{-2} \mathbf{v}(\varepsilon). \]

Neglecting the bar notation, uniform a priori estimates imply the following convergence results of the rescaled fluid velocity (on a subsequence as \( \varepsilon \downarrow 0 \)):

\[ (34) \quad \mathbf{v}(\varepsilon) \rightharpoonup \mathbf{v} \quad \text{and} \quad \partial_t \mathbf{v}(\varepsilon) \rightharpoonup \partial_t \mathbf{v} \quad \text{weakly in} \quad L^2(0,T; L^2(\Omega \times \mathbb{R}^3)). \]

### 3.1.3. Uniform estimates for the pressure

According to Proposition 2.2 there exists a unique pressure \( p^\varepsilon \in L^2(0,T; H^1(\Omega_\varepsilon)) \) such that the triplet \( (\mathbf{v}^\varepsilon, p^\varepsilon, u^h) \) satisfies the system (1)–(3) in the \( L^2 \)-sense. Regularity results of Proposition 2.2 allow us to weaken the regularity of test functions. Thus, we multiply (1) and (3) by test functions \( \phi \) and \( \psi \), respectively, where \( (\phi, \psi) \in C_c((0,T); \tilde{V}(\Omega_\varepsilon) \times V_S(\Omega_h)) \) such that \( \phi(t) = \psi(t) \) on \( \omega \) for every \( t \in [0,T) \), and \( \tilde{V}(\Omega_\varepsilon) = \left\{ \mathbf{v} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{v}|_{\{x_3=\varepsilon\}} = 0, \mathbf{v} \text{ is } \omega \text{-periodic} \right\} \). Integrating with respect to the new (rescaled) time and original space variables we find

\[
\begin{align*}
\phi(f &- \frac{T}{\varepsilon^2}) \int_0^T \int_{\Omega_\varepsilon} \partial_t \mathbf{v}^\varepsilon \cdot \phi \, d\mathbf{x} \, dt + 2\eta \int_0^T \int_{\Omega_\varepsilon} \text{sym} \nabla \mathbf{v}^\varepsilon : \text{sym} \nabla \phi \, d\mathbf{x} \, dt - \int_0^T \int_{\Omega_\varepsilon} p^\varepsilon \text{div} \phi \, d\mathbf{x} \, dt \\
+ &\frac{T}{\varepsilon^2} \int_0^T \int_{\Omega_h} \partial_t u^h \cdot \psi \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega_h} (2\mu^h \text{sym} \nabla u^h : \text{sym} \nabla \psi + \lambda^h \text{div} u^h \text{div} \psi) \, d\mathbf{x} \, dt \\
= &\int_0^T \int_{\Omega_\varepsilon} \mathbf{f}^\varepsilon \cdot \phi \, d\mathbf{x} \, dt.
\end{align*}
\]

Unlike in the Stokes equations solely, where the pressure is determined up to a function of time, in the case of the FSI problem the pressure is unique. This is a consequence of the fact that in the Stokes system the boundary (wall) is assumed to be rigid and therefore cannot feel the pressure, while in the present case elastic wall feels the pressure. Therefore, we define \( \pi^\varepsilon(t) = \int_{\Omega_\varepsilon} p^\varepsilon(x,t) \, d\mathbf{x} \) to be the mean value of the pressure at time \( t \in (0,T) \).

Let us first estimate the zero mean value part of the pressure \( p^\varepsilon - \pi^\varepsilon \) in a classical way. For an arbitrary \( q \in C_c((0,T); L_0^2(\Omega_\varepsilon)) \), where \( L_0^2(\Omega_\varepsilon) = \left\{ q \in L^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} q \, d\mathbf{x} = 0 \right\} \), there exists \( \phi_q \in C_c((0,T); V(\Omega_\varepsilon)) \), such that \( \text{div} \phi_q(t) = q(t) \), \( \phi_q(t)|_\omega = 0 \) for all \( t \in [0,T) \) and \( \| \phi_q \|_{L^2(0,T; H^1(\Omega_\varepsilon))} \leq C\varepsilon^{-1} \| q \|_{L^2(0,T; L^2(\Omega_\varepsilon))} \) (cf. [33] Lemma 9). Taking \( (\phi, \psi) = (\phi_q, 0) \) as test
functions in (35) we have
\[ \int_0^T \int_{\Omega_\varepsilon} \rho_f T^{-1} \int_{\Omega_\varepsilon} \partial_t v^\varepsilon \cdot \phi_q \, dx \, dt + 2\eta \int_0^T \int_{\Omega_\varepsilon} \text{sym} \, \nabla v^\varepsilon : \text{sym} \, \nabla \phi_q \, dx \, dt \]
\[ - \int_0^T \int_{\Omega_\varepsilon} f^\varepsilon \cdot \phi_q \, dx \, dt. \]
Using energy estimates (30) and (31), assumption (A1) for the fluid volume force and the Poincaré inequality from Proposition A.2 we conclude
\[ \left| \int_0^T \int_{\Omega_\varepsilon} \rho_f T^{-1} \int_{\Omega_\varepsilon} \partial_t v^\varepsilon \cdot \phi_q \, dx \, dt + 2\eta \int_0^T \int_{\Omega_\varepsilon} \text{sym} \, \nabla v^\varepsilon : \text{sym} \, \nabla \phi_q \, dx \, dt \right| \leq C \sqrt{\varepsilon} \| q \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \]
for all \( q \in C_c((0,T); L^2_0(\Omega_\varepsilon)) \). Employing a density argument, the latter inequality implies
\[ (36) \quad \| p^\varepsilon - \pi^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \sqrt{\varepsilon}. \]
In order to conclude the pressure estimate we still need to estimate the mean value \( \pi^\varepsilon \). Let us define test functions by \((\phi, \psi) = (0, 0, x_3 + \varepsilon, 0, 0, \varepsilon)\) for an arbitrary \( \zeta \in C_c((0,T)) \). Notice that \( \text{div} \phi(t) = \zeta(t) \) for every \( t \in (0,T) \). Taking \((\phi, \psi)\) as test functions in (35) we obtain
\[ (37) \quad \int_0^T \pi^\varepsilon \zeta \, dt = \int_0^T \zeta \int_{\Omega_\varepsilon} (x_3 + \varepsilon) \partial_t v_3^\varepsilon \, dx \, dt + 2\eta \int_0^T \int_{\Omega_\varepsilon} \zeta \int_{\Omega_\varepsilon} \partial_3 v_3^\varepsilon \, dx \, dt \]
\[ - \int_0^T \int_{\Omega_\varepsilon} f_3^\varepsilon \cdot dx \, dt + \rho_s^h T^{-2} \int_0^T \int_{\Omega_\varepsilon} \varepsilon \partial_t u_3^h \, dx \, dt. \]
Let us estimate the right hand side of (37) using energy estimates (30) and (31):
\[ \left| \int_0^T \pi^\varepsilon \zeta \, dt \right| \leq C T^{-1/2} \varepsilon^3 \int_0^T |\zeta| \, dt + C \varepsilon^2 \| \zeta \|_{L^2(0,T)} + C T^{-1/2} \varepsilon^{5/2} h^{-\frac{k}{2}+1} \int_0^T |\zeta| \, dt. \]
Under assumption of scaling ansatz (S1) and (S3), and assuming that \( \tau = k - 3 \gamma - 3 \leq -1 \), the worst term above, \( T^{-1/2} \varepsilon^{5/2} h^{-\frac{k}{2}+1} / 2 \) is of order less or equal to \( O(\varepsilon) \) (cf. Section 3.2 for justification of this assumption). Therefore, we have
\[ \left| \int_0^T \pi^\varepsilon \zeta \, dt \right| \leq C \varepsilon \| \zeta \|_{L^2(0,T)}, \]
which implies
\[ (38) \quad \| \pi^\varepsilon \|_{L^2(0,T)} \leq C \varepsilon. \]
Combining (38) with (36) we find the pressure estimate
\[ \| p^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \sqrt{\varepsilon}, \]
which in further yields the uniform estimate for the pressure \(p(\varepsilon)(y) := p^\varepsilon(x)\) defined on the reference domain

\[
\|p(\varepsilon)\|_{L^2(0,T;L^2(\Omega_+))} \leq C.
\]

Finally, we conclude that there exists \(p \in L^2(0, T; L^2(\Omega_+))\) such that (on a subsequence as \(\varepsilon \downarrow 0\)) we have

\[
p(\varepsilon) \rightharpoonup p \quad \text{weakly in } L^2(0, T; L^2(\Omega_-)).
\]

### 3.1.4. Uniform estimates for the structure displacement.

The energy estimate (32) provides an \(L^\infty\)-\(L^2\) estimate of the symmetrized scaled gradient,

\[
\text{ess sup}_{t \in (0,T)} \int_{\Omega_+} |\text{sym } \nabla_h u(h)|^2 \leq C h^{3\gamma - 1 + \tau + \kappa}.
\]

This motivates the following rescaling of displacements

\[
\bar{u}(h) = h^{-(3\gamma - 1 + \tau + \kappa)/2} u(h).
\]

Neglecting bars in further notation we have the uniform bound on the reference domain

\[
\text{ess sup}_{t \in (0,T)} \int_{\Omega_+} |\text{sym } \nabla_h u(h)|^2 \leq C.
\]

In the analysis of structure displacements we rely on the Griso decomposition \cite{[21]}. For every \(h > 0\), scaled structure displacement \(u(h)\) is, at almost every time instance \(t \in (0, T)\), decomposed into a sum of so called elementary plate displacement and warping as follows (cf. (110) in Appendix A)

\[
u(h)(z) = w(h)(z') + r(h)(z') \times (z_3 - \frac{1}{2}) e_3 + \bar{u}(h),
\]

where

\[
w(h)(t, z') = \int_0^1 u(h)(t, z)dz_3, \quad r(h)(t, z') = \frac{3}{h} \int_0^1 (z_3 - \frac{1}{2}) e_3 \times u(h)(t, z)dz_3,
\]

\(\bar{u}(h) \in L^\infty(0, T; H^1(\Omega_+))\) is the warping term, and \(\times\) denotes the cross product in \(\mathbb{R}^3\).

Moreover, the following uniform estimate holds (cf. (111) in Appendix A)

\[
\|\text{sym } \nabla_h (w(h)(z') + r(h)(z') \times (z_3 - 1/2) e_3)\|_{L^\infty(0,T;L^2(\Omega_+))}^2 + \|\nabla_h \bar{u}(h)\|_{L^\infty(0,T;L^2(\Omega_+))}^2 + \frac{1}{h^2} \|\bar{u}(h)\|_{L^\infty(0,T;L^2(\Omega_+))}^2 \leq C,
\]

with \(C > 0\) independent of \(h\) and \(u(h)\).
According to [21, Theorem 2.6], the above uniform estimate implies the existence of a sequence of in-plane translations \( a(h) = (a_1(h), a_2(h)) \subset (L^\infty(0, T))^2 \), as well as limit displacements \( w_1, w_2 \in L^\infty(0, T; H^1_{\text{per}}(\omega)) \), \( w_3 \in L^\infty(0, T; H^2_{\text{per}}(\omega)) \) and \( \bar{u} \in L^2(\omega; H^1((0, 1); \mathbb{R}^3)) \) such that the following weak-* convergence results hold:

\[
\begin{align*}
(45) & \quad w_\alpha(h) - a_\alpha(h) \xrightarrow{*} w_\alpha \quad \text{in} \quad L^\infty(0, T; H^1_{\text{per}}(\omega)), \quad \alpha = 1, 2, \\
(46) & \quad hw_3(h) \xrightarrow{*} w_3 \quad \text{in} \quad L^\infty(0, T; H^1_{\text{per}}(\omega)), \\
(47) & \quad u_\alpha(h) - a_\alpha(h) \xrightarrow{*} w_\alpha - (z_3 - \frac{1}{2})\partial_3 w_\alpha \quad \text{in} \quad L^\infty(0, T; H^1(\Omega_+)), \quad \alpha = 1, 2, \\
(48) & \quad hu_3(h) \xrightarrow{*} w_3 \quad \text{in} \quad L^\infty(0, T; H^1(\Omega_+)), \\
(49) & \quad \text{sym} \nabla_h u(h) \xrightarrow{*} \frac{1}{2} \left( \text{sym} \nabla'(w_1, w_2) - (z_3 - \frac{1}{2})\nabla^2 w_3 \right) + \text{sym} \left( e_3 \otimes (\partial_3 \bar{u}) \right).
\end{align*}
\]

To estimate in-plane translations \( a_\alpha(h) \) we first use [29] to get:

\[
(50) \quad \left| \int_\omega u_\alpha(h)(t)dx' \right| \leq C\varepsilon^2 h^{\bar{r} - (3\bar{r} - 1 + \tau + \kappa)/2} = Ch^{(\tau + \kappa + 1 - \bar{r})/2}, \quad t \in [0, T), \quad \alpha = 1, 2.
\] Combining (50) with (47) we get

\[
(51) \quad \|a_\alpha(h)\|_{L^\infty(0, T)} \leq C h^{(\gamma + \bar{r} - 1)/2},
\]
and therefore

\[
h^{(\kappa - \gamma - \bar{r} - 1)/2}a_\alpha(h) \xrightarrow{*} a_\alpha \quad \text{in} \quad L^\infty(0, T).
\]

Employing the higher order energy estimate (31) on the reference domain, we find analogous convergence results to (45)–(49) for the respective time derivatives. Moreover, it holds

\[
(52) \quad \|\partial_t a_\alpha(h)\|_{L^\infty(0, T)} \leq C h^{(\tau - \kappa + \gamma + 1)/2},
\]
which implies

\[
h^{(\kappa - \gamma - \bar{r} - 1)/2}\partial_t a_\alpha(h) \xrightarrow{*} \partial_t a_\alpha \quad \text{in} \quad L^\infty(0, T).
\]

### 3.2. Identification of reduced model.

Taking all the above rescalings into account, the rescaled variational equation (35), which now includes the pressure, on the reference domain reads

\[
-\varrho f h^{-\gamma - \bar{r}} \int_0^T \int_{\Omega_+} \mathbf{v}(\varepsilon) \cdot \partial_t \phi \, d\mathbf{y} dt + 2\eta \varepsilon^3 \int_0^T \int_{\Omega_+} \text{sym} \nabla_\varepsilon \mathbf{v}(\varepsilon) : \text{sym} \nabla_\varepsilon \phi \, d\mathbf{y} dt
\]

\[
-\varepsilon \int_0^T \int_{\Omega_+} p(\varepsilon) \text{div}_h \phi \, d\mathbf{y} dt + \varrho_s h^{2\mu} \int_0^T \int_{\Omega_+} \mathbf{u}(h) \cdot \partial_h \psi \, dz dt + h^{\bar{r}} \int_0^T \int_{\Omega_+} (2\mu \text{sym} \nabla_h \mathbf{u}(h) : \text{sym} \nabla_h \psi + \lambda \text{div}_h \mathbf{u}(h) \text{div}_h \psi) \, dz dt = \varepsilon \int_0^T \int_{\Omega_-} \mathbf{f}(\varepsilon) \cdot \phi \, d\mathbf{y} dt,
\]
for all \((\phi, \psi) \in C^2(0, T); V(\Omega) \times V_s(\Omega^+))\) such that \(\phi(t) = \psi(t)\) on \(\omega\) for all \(t \in [0, T]\), and where \(\delta = (3\gamma - 1 + \tau + \kappa)/2 + 1 - \kappa\). In order to obtain a nontrivial coupled reduced model on a limit as \(h \downarrow 0\) we need to adjust \(\delta = -1\). This is due to the linear theory of plates (cf. [8, Section 1.10]). Namely, the fluid pressure which is here \(O(1)\) is acting as a normal force on the structure and therefore has to balance the structure stress terms in the right way. This condition then yields the choice of the right time scale \(T = h^\tau\) with

\[
\tau = \kappa - 3\gamma - 3.
\]

The above weak formulation then becomes

\[
-\bar{\rho} h^{6\gamma - \kappa + 3} \int_0^T \int_{\Omega^-} v(\varepsilon) \cdot \partial_t \phi \, dy \, dt + 2\eta \varepsilon^3 \int_0^T \int_{\Omega^-} \text{sym} \nabla v(\varepsilon) : \text{sym} \nabla \phi \, dy \, dt
\]

\[
-\varepsilon \int_0^T \int_{\Omega^-} \sigma(\varepsilon) \text{div}_x \phi \, dy \, dt + \bar{\rho} h^{6\gamma - 2\kappa + 5} \int_0^T \int_{\Omega^+} u(h) \cdot \partial_x \psi \, dz \, dt
\]

\[
+ h^{-1} \int_0^T \int_{\Omega^+} (2\mu \text{sym} \nabla_h u(h) : \text{sym} \nabla_h \psi + \lambda \text{div}_h u(h) \text{div}_h \psi) \, dz \, dt = \varepsilon \int_0^T \int_{\Omega^-} f(\varepsilon) \cdot \phi \, dy \, dt.
\]

Expanding [54] with test functions of the form \(\phi = (\phi_1, \phi_2, 0)\) and \(\psi = (\psi_1, \psi_2, 0)\), multiplying the equation with \(h^2\) and employing the weak*-convergence results for the structure [47]–[49], we find

\[
\mu \int_0^T \int_{\Omega^+} (\partial_3 \bar{u}_1 \partial_3 \psi_1 + \partial_3 \bar{u}_2 \partial_3 \psi_2) \, dz \, dt = 0.
\]

Taking sequences \((\psi_{1,n})\) and \((\psi_{2,n})\) which approximate \(\bar{u}_1\) and \(\bar{u}_2\), respectively, in the sense of \(L^2\)-convergence, we conclude \(\partial_3 \bar{u}_1 = \partial_3 \bar{u}_2 = 0\). Similarly, taking test functions \(\phi = (0, 0, \phi_3)\) and \(\psi = (0, 0, \psi_3)\), we obtain

\[
\mu \int_0^T \int_{\Omega^+} \left(\left(2\mu + \lambda\right)\partial_3 \bar{u}_3 + \lambda(\partial_1 w_1 + \partial_2 w_2) - \left(z_3 - \frac{1}{2}\right) \Delta' w_3\right) \partial_3 \psi_3 \, dz \, dt = 0,
\]

from which we conclude

\[
\partial_3 \bar{u}_3 = -\frac{\lambda}{2\mu + \lambda} \left(\partial_1 w_1 + \partial_2 w_2 - \left(z_3 - \frac{1}{2}\right) \Delta' w_3\right).
\]

Previous calculations are equivalent to those from [8, Theorem 1.4-1]. Now we have complete information on the limit of the scaled strain [49] given in terms of limit displacements \((w_1, w_2, w_3)\).

Next, we will take test functions to imitate the shape of the limit of scaled displacements [47]–[48], i.e. we take \(\psi = (h \psi_1, h \psi_2, \psi_3)\) satisfying \(\partial_1 \psi_3 + \partial_3 \psi_1 = \partial_2 \psi_3 + \partial_3 \psi_2 = \partial_3 \psi_3 = 0\).
(cf. [8 Theorem 1.4-1]), while for the fluid part we accordingly take (in order to satisfy the interface conditions) \( \hat{\phi} = (h\phi_1, h\phi_2, \phi_3) \). With this choice of test functions, under assumption \( \tau \leq -1 \) (i.e. \( \kappa \leq 3\gamma + 2 \)), the weak limit form of (54) (on a subsequence as \( h \downarrow 0 \)) reads

\[
- \int_0^T \int_{\Omega_-} p\partial_3 \phi_3 \, dy \, dt + \chi_\tau \varrho s \int_0^T \int_\omega w_3 \partial t \psi_3 \, dz' \, dt
\]

(55) \[+ \int_0^T \int_{\Omega_+} \left( 2\mu (\text{sym} \nabla'(w_1, w_2) - (z_3 - \frac{1}{2})\nabla'^2 w_3) : \text{sym} \nabla'(\psi_1, \psi_2) + \frac{2\mu \lambda}{2\mu + \lambda} \text{div} ((w_1, w_2) - (z_3 - \frac{1}{2})\nabla' w_3) \text{div}(\psi_1, \psi_2) \right) \, dz \, dt = 0 , \]

where \( \chi_\tau = 1 \) for \( \tau = -1 \) and \( \chi_\tau = 0 \) for \( \tau < -1 \). Notice that for \( \tau = -1 \), i.e. \( \kappa = 3\gamma + 2 \), we have \( \delta - 2\tau = 1 \), and the inertial term of the vertical displacement of the structure survives in the limit.

The obtained limit model (55) is a linear plate model (cf. [8]) coupled with the limit pressure from the fluid part, which acts as a normal force on the interface \( \omega \) of the structure (cf. equation (56) below). Let us consider the pressure term more in detail. Taking test function \( \phi \in C^1_c([0, T); C^\infty_c(\Omega_-; \mathbb{R}^3)) \), i.e. smooth and with compact support in space, and \( \psi = 0 \) in (54) we find

\[
\int_0^T \int_{\Omega_-} p \partial_3 \phi_3 \, dy \, dt = 0 ,
\]

which implies \( \partial_3 p = 0 \) in the sense of distributions. As a consequence of this we have that \( p \) is independent of the vertical variable \( z_3 \), and therefore \( p \) (although \( L^2 \)-function) has the trace on \( \omega \). Since \( \phi_3 = \psi_3 \) on \( \omega \times (0, T) \), after integrating by parts in the pressure term, the limit form (55) then becomes

\[
- \int_0^T \int_\omega p \psi_3 \, dz' \, dt + \chi_\tau \varrho s \int_0^T \int_\omega w_3 \partial t \psi_3 \, dz' \, dt
\]

(56) \[+ \int_0^T \int_{\Omega_+} \left( 2\mu (\text{sym} \nabla'(w_1, w_2) - (z_3 - \frac{1}{2})\nabla'^2 w_3) : \text{sym} \nabla'(\psi_1, \psi_2) + \frac{2\mu \lambda}{2\mu + \lambda} \text{div} ((w_1, w_2) - (z_3 - \frac{1}{2})\nabla' w_3) \text{div}(\psi_1, \psi_2) \right) \, dz \, dt = 0 . \]

Recall that structure test functions in (56) satisfy \( \partial_1 \psi_3 + \partial_3 \psi_1 = \partial_2 \psi_3 + \partial_3 \psi_2 = \partial_3 \psi_3 = 0 \). Following Ciarlet [8 Theorem 1.4-1 (c)], this is equivalent with

\[ \psi_\alpha = \zeta_\alpha - (z_3 - \frac{1}{2})\partial_3 \zeta_3, \quad \text{and} \quad \psi_3 = \zeta_3 , \]

for some \( \zeta_\alpha \in C^1_c([0, T); H^1_{\text{per}}(\omega)) \), \( \alpha = 1, 2 \), and \( \zeta_3 \in C^1_c([0, T); H^2_{\text{per}}(\omega)) \). Next we resolve (56) into equivalent formulation, which decouples horizontal and vertical displacements.
First, choosing the test function \( \psi = (- (z_3 - \frac{1}{2}) \partial_1 \zeta_3, -(z_3 - \frac{1}{2}) \partial_2 \zeta_3, \zeta_3) \), for arbitrary \( \zeta_3 \in C^2_c([0, T); H^1_{\text{per}}(\omega)) \), after explicit calculations of integrals we find

\[
\int_0^T \int_\omega \eta \int_\Omega_- (\partial_3 v_1 \partial_3 \phi_1 + \partial_3 v_2 \partial_3 \phi_2) \, dy \, dt - \int_0^T \int_\omega \eta \int_\Omega_- (p \partial_1 \phi_1 + p \partial_2 \phi_2) \, dy \, dt = \int_0^T \int_\Omega_- (f_1 \phi_1 + f_2 \phi_2) \, dy \, dt, \tag{57}
\]

Second, taking the test function \( \psi = (\zeta_1, \zeta_2, 0) \), for arbitrary \( \zeta_\alpha \in C^1_c([0, T); H^1_{\text{per}}(\omega)) \), we obtain the variational equation for horizontal displacements only,

\[
\int_0^T \int_\omega \left( \frac{4 \mu}{3} \nabla^2 w_3 : \nabla^2 \zeta_3 + \frac{4 \mu \lambda}{3(2 \mu + \lambda)} \Delta' w_3 \Delta' \zeta_3 \right) \, dz'dt = 0. \tag{58}
\]

Equation (58) implies that horizontal displacements \( (w_1, w_2) \) are spatially constant functions, and as such they will not affect the reduced model. Moreover, they are dominated by potentially large horizontal translations, hence we omit them in further analysis. Thus, the limit system (56) is now essentially described with (57), which relates the limit fluid pressure \( p \) with the limit vertical displacement of the structure \( w_3 \).

In order to close the limit model, we need to further explore on the fluid part. First, we analyze the divergence free condition on the reference domain. Multiplying \( \text{div}_\varepsilon \mathbf{v}(\varepsilon) = 0 \) by a test function \( \varphi \in C^1_c([0, T); H^1_{\text{per}}(\omega)) \), integrating over space and time, integrating by parts and employing the rescaled kinematic condition \( \varepsilon^2 \mathbf{v}(\varepsilon) = \eta(h^{3\gamma - 1 + \tau + \kappa})/2 - \tau \partial_t \mathbf{u}(h) \), which with relation (53) and (S1) becomes \( \varepsilon^{-1} \mathbf{v}(\varepsilon) = \eta \partial_t \mathbf{u}(h) \) a.e. on \( \omega \times (0, T) \), we have

\[
- \int_0^T \int_\Omega_- \left( v_1(\varepsilon) \partial_1 \varphi + v_2(\varepsilon) \partial_2 \varphi \right) \, dy \, dt - \int_0^T \int_\omega h u_3(h) \partial_t \varphi \, dy \, dt = 0. \tag{59}
\]

Utilizing convergence results (34) and (48) in (59), we find (on a subsequence as \( \varepsilon \downarrow 0 \))

\[
- \int_0^T \int_\Omega_- \left( v_1 \partial_1 \varphi + v_2 \partial_2 \varphi \right) \, dy \, dt - \int_0^T \int_\omega w_3 \partial_t \varphi \, dy \, dt = 0, \tag{60}
\]

which relates the limit vertical displacement of the structure with limit horizontal fluid velocities.

Relation between horizontal fluid velocities \( (v_1, v_2) \) and pressure \( p \) is obtained from (54) as follows. Take test functions \( \phi = (\phi_1/\varepsilon, \phi_2/\varepsilon, 0) \) with \( \phi_\alpha \in C^1_c([0, T); C^\infty(\Omega_-)) \) and \( \psi = 0 \), then convergence results (34) and (40) yield

\[
\eta \int_0^T \int_\Omega_- (\partial_3 v_1 \partial_3 \phi_1 + \partial_3 v_2 \partial_3 \phi_2) \, dy \, dt - \int_0^T \int_\Omega_- (p \partial_1 \phi_1 + p \partial_2 \phi_2) \, dy \, dt = \int_0^T \int_\Omega_- (f_1 \phi_1 + f_2 \phi_2) \, dy \, dt, \tag{61}
\]
and the reduced (limit) model composed of (57), (60) and (61) is now closed.

Before exploring the limit model more in detail, let us conclude that \( v_3 = 0 \). Namely, for an arbitrary \( \varphi \in C^1((0,T); H^1(\Omega_-)) \), integrating the divergence-free condition we calculate

\[
\lim_{\varepsilon \downarrow 0} \int_0^T \int_{\Omega_-} \partial_3 v_3(\varepsilon) \varphi \, dy \, dt = \lim_{\varepsilon \downarrow 0} \left( \varepsilon \int_0^T \int_{\Omega_-} \left( v_1(\varepsilon) \partial_1 \varphi + v_2(\varepsilon) \partial_2 \varphi \right) \, dy \, dt \right) = 0,
\]

which implies \( \partial_3 v_3 = 0 \), and therefore \( v_3 = 0 \), due to the no-slip boundary condition.

### 3.3. A single equation.

Since \( p \) is independent of the vertical variable \( y_3 \), equation (61) can be solved for \( v_\alpha \) explicitly in terms of \( y_3 \) and \( p \). Let us first resolve the boundary conditions for \( v_\alpha \) in the vertical direction. The bottom condition is inherited from the original no-slip condition, i.e. \( v_\alpha(\cdot,-1,\cdot) = 0 \), while for the interface condition we derive \( v_\alpha(\cdot,0,\cdot) = \partial_1 a_\alpha \), \( \alpha = 1,2 \), where \( \partial_1 a_\alpha \) are translational limit velocities of the structure defined by (52). Recall the rescaled kinematic condition \( v_\alpha(\varepsilon) = \varepsilon h \partial_1 a_\alpha(h) \) on \( \omega \times (0,T) \). Multiplying this with a test function \( \varphi \in C^1((0,T); H^1(\omega)) \) and using convergence results (47) and (51) we have

\[
\lim_{\varepsilon \downarrow 0} \int_0^T \int_{\omega} v_\alpha(\varepsilon) \varphi \, dy' \, dt = - \lim_{h \downarrow 0} h^{\gamma+1} \int_0^T \int_{\omega} u_\alpha(h) \partial_1 \varphi \, dz' \, dt = \int_0^T \int_{\omega} \partial_1 a_\alpha \varphi \, dz' \, dt.
\]

Since (34) implies \( v_\alpha(\varepsilon) \to v_\alpha \) weakly in \( L^2(0,T;L^2(\omega)) \), we conclude that \( v_\alpha = \partial_1 a_\alpha \) a.e. on \( \omega \times (0,T) \). Explicit solution of \( v_\alpha (\alpha = 1,2) \) from (61) is then given by

\[
v_\alpha(y,t) = \frac{1}{2\eta} y_3(y_3+1) \partial_3 p(y',t) + F_\alpha(y,t) + (1 + y_3) \partial_1 a_\alpha, \quad (y,t) \in \Omega_- \times (0,T),
\]

where \( F_\alpha(\cdot, y_3, \cdot) = \frac{y_3+1}{\eta} \int_{-1}^{y_3} \zeta_3 f_\alpha(\cdot, \zeta_3, \cdot) \, d\zeta_3 + \frac{y_3 - y_3}{{\eta}} \int_{-1}^{y_3} (y_3 - \zeta_3) f_\alpha(\cdot, \zeta_3, \cdot) \, d\zeta_3.\)

From equation (60) we have

\[
\int_0^T \int_{\omega} \left( \partial_1 \int_{-1}^{y_3} v_1 \, dy_3 + \partial_2 \int_{-1}^{y_3} v_2 \, dy_3 + \partial_3 w_3 \right) \varphi \, dy' \, dt = 0.
\]

Replacing \( v_\alpha \) with (62) it follows a Reynolds type equation

\[
\int_0^T \int_{\omega} \left( - \frac{1}{12\eta} \Delta' p - F + \partial_3 w_3 \right) \varphi \, dy' \, dt = 0,
\]

where \( F(y',t) = - \int_{-1}^{y_3} (\partial_1 F_1 \, dy_3 + \partial_2 F_2) \, dy_3. \) Considering equation (57) in the sense of distributions, i.e.

\[
p = \chi \rho_\varepsilon \partial_3 w_3 + \frac{8\mu(\mu + \lambda)}{3(2\mu + \lambda)} (\Delta')^2 w_3,
\]
where \((\Delta')^2\) denotes the bi-Laplacian in horizontal variables, we finally obtain the reduced model in terms of the vertical displacement only

\begin{equation}
\frac{\partial}{\partial t} w_3 - \chi \frac{g_s}{12\eta} \Delta' \partial_{tt} w_3 - \frac{2\mu(\mu + \lambda)}{9\eta(2\mu + \lambda)} (\Delta')^3 w_3 = F.
\end{equation}

This is an evolution equation for \(w_3\) of order six in spatial derivatives. Mixed space and time derivatives are present only in the case of \(\tau = -1\). Equation (66) is accompanied by trivial initial data \(w_3(0) = \partial_t w_3(0) = 0\) and periodic boundary conditions. Knowing \(w_3\), the pressure and horizontal velocities of the fluid are then calculated according to (65) and (62), respectively. This finishes the proof of Theorem 1.1.

4. Error estimates — proof of Theorem 1.2

This section is devoted to the proof of our second main result — Theorem 1.2, which provides the error estimates for approximation of solutions to the original FSI problem (1)–(7) by approximate solutions constructed from the reduced model (66). In the subsequent analysis we will assume additional regularity of solutions \((\mathbf{v}^\varepsilon, p^\varepsilon, \mathbf{u}^h)\) to the original problem, together with sufficient regularity of solutions \(w_3\) to the reduced problem (66), as well as regularity of external forces. In the sequel we work on the original thin domain \(\Omega_{\varepsilon,h}\), but in the rescaled time variable with scaling parameter \(\tau < -1\).

4.1. Construction of approximate solutions and error equation. Recall the limit model (66) in terms of the scaled vertical displacement \(w_3\) (for \(\tau < -1\)):

\begin{equation}
\frac{\partial}{\partial t} w_3 - \frac{2\mu(\mu + \lambda)}{9\eta(2\mu + \lambda)} (\Delta')^3 w_3 = F \quad \text{in} \quad \omega \times (0, T),
\end{equation}

\(w_3(0) = 0\).

This is a linear partial differential equation with periodic boundary conditions. The classical theory of linear parabolic equations provides the well-posedness and smoothness of the solution. Based on (67) we reconstruct the limit fluid pressure and horizontal velocities according to:

\begin{equation}
p = \frac{8\mu(\mu + \lambda)}{3(2\mu + \lambda)} (\Delta')^2 w_3,
\end{equation}

\begin{equation}
v_\alpha = \frac{1}{2\eta} y_3(y_3 + 1) \partial_{\alpha} p + F_\alpha + (y_3 + 1) \partial_t a_\alpha, \quad \alpha = 1, 2,
\end{equation}

where \(F_\alpha\) is defined like in (62). Limit structure velocities \(\partial_t a_\alpha\) will be specified by an additional interface condition \(\int_\omega \partial_3 v_\alpha \, dz' = 0\) for a.e. \(t \in (0, T)\), which can be formally seen as a weakened limit stress balance condition and it will be justified by the convergence
result of Theorem 1.2. Using the periodic boundary conditions of the pressure, the interface condition implies

\[ \partial_t a_\alpha(t) = - \int_\omega \partial_3 F_\alpha(y', 0, t) dy', \quad \alpha = 1, 2. \]

Let us first construct an approximate fluid velocity

\[ \tilde{\mathbf{v}}^\varepsilon(x, t) = \varepsilon^2 \left( \mathbf{v}_1(x', \frac{x_3}{\varepsilon}, t), \mathbf{v}_2(x', \frac{x_3}{\varepsilon}, t), 0 \right), \quad (x, t) \in \Omega_\varepsilon \times (0, T), \]

and the approximate pressure by

\[ p^\varepsilon(x, t) = p(x', t), \quad (x, t) \in \Omega_\varepsilon \times (0, T), \]

where \( p \) and \( v_\alpha \) are given by (68) and (69), respectively. Notice that \( \tilde{\mathbf{v}}^\varepsilon \) is not a divergence free vector field. Therefore, following [18], we define a divergence corrector

\[ \tilde{\mathbf{v}}_3^\varepsilon(x, t) = - \varepsilon^3 \int_{-1}^{x_3/\varepsilon} (\partial_1 v_1 + \partial_2 v_2)(x', \xi, t) d\xi. \]

Denoting \( \mathbf{v}^\varepsilon = \tilde{\mathbf{v}}^\varepsilon + \tilde{\mathbf{v}}_3^\varepsilon \mathbf{e}_3 \), we now have \( \text{div} \mathbf{v}^\varepsilon = 0 \) and therefore \( \mathbf{v}^\varepsilon \in L^2(0, T; V_F(\Omega_\varepsilon)) \). Furthermore, \( \mathbf{v}^\varepsilon \) then solves the modified Stokes system

\[ \rho_f T^{-1} \partial_t \mathbf{v}^\varepsilon - \text{div} \sigma_f(\mathbf{v}^\varepsilon, p^\varepsilon) = f^\varepsilon - f_3^\varepsilon \mathbf{e}_3 + \mathbf{r}_f^\varepsilon, \]

where the residual term \( \mathbf{r}_f^\varepsilon \) is given by

\[ \mathbf{r}_f^\varepsilon = \rho_f T^{-1} \partial_t \mathbf{v}^\varepsilon - \eta \Delta \mathbf{v}^\varepsilon - \eta \partial_3 \mathbf{v}^\varepsilon e_3. \]

From the definition of the fluid residual term \( \mathbf{r}_f^\varepsilon \) we immediately have

\[ \| \mathbf{r}_f^\varepsilon \|_{L^2(0, T; L^2(\Omega_\varepsilon))} \leq C \varepsilon^{3/2}. \]

Multiplying equation (72) by a test function \( \phi \in C^1_\text{c}([0, T); V_F(\Omega_\varepsilon)) \), and then integrating over \( \Omega_\varepsilon \times (0, T) \), we find

\[ -\rho_f \int_0^T \int_{\Omega_\varepsilon} \mathbf{v}^\varepsilon \cdot \partial_t \phi \, dx \, dt + 2\eta T \int_0^T \int_{\Omega_\varepsilon} \text{sym} \nabla \mathbf{v}^\varepsilon : \text{sym} \nabla \phi \, dx \, dt \]

\[ -T \int_0^T \int_{\omega} \sigma_f(\mathbf{v}^\varepsilon, p^\varepsilon) \phi \cdot \mathbf{e}_3 \, dx \, dt = T \int_0^T \int_{\Omega_\varepsilon} (f_1^\varepsilon \phi_1 + f_2^\varepsilon \phi_2) \, dx \, dt + T \int_0^T \int_{\Omega_\varepsilon} \mathbf{r}_f^\varepsilon \cdot \phi \, dx \, dt. \]
Expanding the boundary term we get
\[
\int_0^T \int_\omega \sigma_f(v^e, p^e) \phi \cdot e_3 \, dx \, dt
\]
\[
= \int_0^T \int_\omega (\eta(\partial_1 v_3^e + \partial_3 v_1^e)\phi_1 + \eta(\partial_2 v_3^e + \partial_3 v_2^e)\phi_2 + (2\eta\partial_3 v_3^e - p^e)\phi_3) \, dx \, dt
\]
\[
= \int_0^T \int_\omega \left( -\varepsilon^2 \eta \int_0^1 (\partial_{11} v_1 + \partial_{12} v_2)\phi_1 - \varepsilon^2 \eta \int_{-1}^0 (\partial_{21} v_1 + \partial_{22} v_2)\phi_2 
+ \varepsilon\eta\partial_3 v_1\phi_1 + \varepsilon\eta\partial_3 v_2\phi_2 - 2\varepsilon^2 \eta(\partial_1 v_1 + \partial_2 v_2)\phi_3 - p\phi_3 \right) \, dx \, dt.
\]
According to the limit form \([56]\), the pressure term can be evaluated as
\[
-\int_0^T \int_\omega p\phi_3 \, dz' \, dt = \int_0^T \int_{\Omega^+} \left( 2\mu (z_3 - \frac{1}{2})\nabla^2 w_3 : \text{sym} \nabla' (\psi_1, \psi_2) + \frac{2\mu\lambda}{2\mu + \lambda} \text{div}(z_3 - \frac{1}{2})\nabla' w_3 \text{div}(\psi_1, \psi_2) \right) \, dz \, dt,
\]
where the test function \(\psi\) satisfies \(\partial_1 \psi_3 + \partial_3 \psi_1 = \partial_2 \psi_3 + \partial_3 \psi_2 = \partial_3 \psi_3 = 0\) and \(\psi_3 = \phi_3\) on \(\omega\).

Next, we define the approximate displacement by
\[
\mathbf{u}^h(x, t) = h^{\kappa-3} \left( h^{-\gamma} a_1 - \left( x_3 - \frac{h}{2} \right) \partial_1 w_3(x', t), h^{-\gamma} a_2 - \left( x_3 - \frac{h}{2} \right) \partial_2 w_3(x', t), w_3(x', t) \right),
\]
for all \((x, t) \in \Omega_h \times (0, T)\), where \(w_3\) is the solution of \([67]\), and \(a_\alpha\) are horizontal time-dependent translations calculated by \(a_\alpha(t) = \int_0^t \partial_\alpha a_\alpha \, ds, \alpha = 1, 2\), with \(\partial_\alpha a_\alpha\) given by \([70]\).

Straightforward calculations reveal that \(\mathbf{u}^h\) satisfies
\[
-\int_0^T \int_\omega p\phi_3 \, dx' \, dt = \int_0^T \int_{\Omega_h} \left( 2\mu^h \text{sym} \nabla \mathbf{u}^h : \text{sym} \nabla (h\psi_1, h\psi_2) + \frac{2\mu^h\lambda^h}{2\mu^h + \lambda^h} \text{div} \mathbf{u}^h \text{div}(h\psi_1, h\psi_2) \right) \, dx \, dt
\]
for all test functions \(\psi\) satisfying \(\partial_1 \psi_3 + \partial_3 \psi_1 = \partial_2 \psi_3 + \partial_3 \psi_2 = \partial_3 \psi_3 = 0\) and \(\psi_3 = \phi_3\) on \(\omega\).

Furthermore, since \(\text{sym} \nabla \mathbf{u}^h\) has only \(2 \times 2\) nontrivial submatrix, the latter identity can be written as
\[
\int_0^T \int_\omega p\phi_3 \, dx' \, dt = \int_0^T \int_{\Omega_h} \left( 2\mu^h \text{sym} \nabla \mathbf{u}^h : \text{sym} \nabla \psi + \lambda^h \text{div} \mathbf{u}^h \text{div} \psi \right) \, dx \, dt
\]
\[
-\int_0^T \int_{\Omega_h} \left( \frac{(\lambda^h)^2}{2\mu^h + \lambda^h} \text{div} \mathbf{u}^h \text{div}(h\psi_1, h\psi_2) + \lambda^h \text{div} \mathbf{u}^h \partial_3 \psi_3 \right) \, dx \, dt,
\]
with a test function $\psi = (h\psi_1, h\psi_2, \psi_3)$ which now satisfies only $\psi_3 = \phi_3$ on $\omega$. Going back to (75) and taking $\psi = (\psi_1, \psi_3, \psi_3)$ in further calculations, we find the weak form of approximate solutions to be of the same type as the original weak formulation (16):

$$-\varrho_f \int_0^T \int_{\Omega_e} \mathbf{v}^\varepsilon \cdot \partial_t \phi \, d\mathbf{x} \, dt + 2\eta T \int_0^T \int_{\Omega_e} \text{sym} \nabla \mathbf{v}^\varepsilon : \text{sym} \nabla \phi \, d\mathbf{x} \, dt$$

(77)

$$-\varrho_s T^{-1} \int_0^T \int_{\Omega_h} \partial_t \mathbf{u}^h \cdot \partial_t \psi \, d\mathbf{x} \, dt + T \int_0^T \int_{\Omega_h} (2\mu^h \text{sym} \nabla \mathbf{u}^h : \text{sym} \nabla \psi + \lambda^h \text{div} \mathbf{u}^h \text{div} \psi) \, d\mathbf{x} \, dt$$

$$= T \int_0^T \int_{\Omega_e} (f_1^\varepsilon \psi_1 + f_2^\varepsilon \psi_2) \, d\mathbf{x} \, dt + T \int_0^T \int_{\Omega_e} r^\varepsilon \cdot \phi \, d\mathbf{x} \, dt + T \int_0^T \int_{\omega} r^h_s \cdot \phi \, dx' \, dt + \langle r^h_s, \psi \rangle.$$

Above, $r^h_s$ denotes the boundary residual term given by

$$r^h_s = \eta \left(\varepsilon \partial_3 v_1 - \varepsilon^3 \int_{-1}^0 (\partial_{11} v_1 + \partial_{12} v_2), \varepsilon \partial_3 v_2 - \varepsilon^3 \int_{-1}^0 (\partial_{21} v_1 + \partial_{22} v_2), -2\varepsilon^2 (\partial_1 v_1 + \partial_2 v_2) \right)$$

and $\langle r^h_s, \psi \rangle$ denotes the structure residual term $r^h_s$ acting on a test function $\psi$ as

$$\langle r^h_s, \psi \rangle = -\varrho_s T^{-1} \int_0^T \int_{\Omega_h} \partial_t \mathbf{u}^h \cdot \partial_t \psi \, d\mathbf{x} \, dt$$

$$+ T \int_0^T \int_{\Omega_h} \left(\frac{(\lambda^h)^2}{2}\mu^h + \lambda^h \text{div} \mathbf{u}^h \text{div}(\psi_1, \psi_2) + \lambda^h \text{div} \mathbf{u}^h \partial_3 \psi_3 \right) \, d\mathbf{x} \, dt.$$

Let us define the fluid error $\mathbf{e}^\varepsilon_f := \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon$ and the structure error $\mathbf{e}^h_s := \mathbf{u}^h - \mathbf{u}^h$. Subtracting (77) from the original problem (16), in the rescaled time, we find the variational equation for the errors:

$$-\varrho_f \int_0^T \int_{\Omega_e} \mathbf{e}^\varepsilon_f \cdot \partial_t \phi \, d\mathbf{x} \, dt + 2\eta T \int_0^T \int_{\Omega_e} \text{sym} \nabla \mathbf{e}^\varepsilon_f : \text{sym} \nabla \phi \, d\mathbf{x} \, dt$$

(78)

$$-\varrho_s T^{-1} \int_0^T \int_{\Omega_h} \partial_t \mathbf{e}^h_s \cdot \partial_t \psi \, d\mathbf{x} \, dt + T \int_0^T \int_{\Omega_h} (2\mu^h \text{sym} \nabla \mathbf{e}^h_s : \text{sym} \nabla \psi + \lambda^h \text{div} \mathbf{e}^h_s \text{div} \psi) \, d\mathbf{x} \, dt$$

$$= T \int_0^T \int_{\Omega_e} f_3^\varepsilon \phi_3 \, d\mathbf{x} \, dt - T \int_0^T \int_{\Omega_e} r^\varepsilon \cdot \phi \, d\mathbf{x} \, dt - T \int_0^T \int_{\omega} r^h_s \cdot \phi \, dx' \, dt - \langle r^h_s, \psi \rangle.$$

for all test functions $(\phi, \psi) \in \mathcal{W}(0, T; \Omega_{\varepsilon, h}).$

4.2. **Basic error estimate.** Let us first introduce some notation. For an $L^2$-function $\psi \in L^2(0, h)$ we introduce orthogonal decomposition (w.r.t. $L^2(0, h)$-inner product) denoted
by \( \psi = \psi^e + \psi^o \), where \( \psi^e \) and \( \psi^o \) denote even and odd part of \( \psi \), respectively. Furthermore, functions \( \psi \in L^2(\Omega_h) \) will be considered as \( \psi \in L^2(\omega; L^2(0,h)) \) and the orthogonal decomposition \( \psi = \psi^e + \psi^o \) will be performed in a.e. point of \( \omega \).

Our key result for proving Theorem 1.2 is an energy type estimate for errors, which we derive from equation (78) based on a careful selection of test functions.

**Proposition 4.1.** Let us assume that the fluid volume force verifies assumption (A2) then for a.e. \( t \in (0,T) \) we have

\[
\frac{\partial f}{4} \int_{\Omega_\varepsilon} |e^\varepsilon_j(t)|^2 \, dx + \frac{\eta T}{2} \int_0^t \int_{\Omega_\varepsilon} |\nabla e^\varepsilon_j|^2 \, dx \, ds + \frac{\theta^h T^{-2}}{4} \int_{\Omega_h} \left( (\partial_t e^e_{s,\alpha}(t))^2 + (\partial_t e^o_{s,3}(t))^2 \right) \, dx
\]

\[
+ \int_{\Omega_h} \left( \mu^h |\text{sym} \nabla (e^e_{s,1}, e^e_{s,2}, e^o_{s,3})(t)|^2 + \frac{\lambda h}{2} \left| \text{div}(e^e_{s,1}, e^e_{s,2}, e^o_{s,3})(t) \right|^2 \right) \, dx 
\leq C T \varepsilon^3 (h^\gamma + h^{4\gamma-2\kappa+4}).
\]

**Proof of Proposition 4.1.** Since the elasticity equations appear to be more delicate for the analysis, we first choose

\[
\psi = T^{-1}(\partial_t e^e_{s,1}, \partial_t e^e_{s,2}, \partial_t e^o_{s,3}),
\]

where superscripts denote even and odd components of the orthogonal decomposition with respect to the variable \((x_3 - h/2)\). Observe from (76) that components of the approximate displacement \( u^h \) are respectively odd, odd and even with respect to \((x_3 - h/2)\). The idea of using this particular test function comes from the fact that such \( \psi \) annihilates large part of the structure residual term \( r^h_s \) on the right hand side in (78) and the rest can be controlled (cf. estimate (99) below).

In construction of the test function for the fluid part, first observe that approximate solutions do not satisfy the kinematic interface condition in the horizontal components, i.e. \( v^\varepsilon_3 \neq \partial_t u^h_3 \) on \( \omega \times (0,T) \) and therefore \((v^\varepsilon, u^h)\) does not belong to the space \( V(0,T; \Omega_{\varepsilon,h}) \). For the third component however, the interface condition is satisfied. Namely, using (63) and the definition of \( v^\varepsilon_3 \) we find:

\[
T^{-1} \partial_t u^h_3 = h^{-\tau+\kappa-3} \partial_t w_3 = -h^3 \gamma \int_{-1}^0 (\partial_1 v_1 + \partial_2 v_2) \, dy_3 = v^\varepsilon_3 \quad \text{a.e. on } \omega \times (0,T).
\]

In order to match interface values of \( \psi \), the fluid test function \( \phi \) will be accordingly corrected fluid error, i.e. we take

\[
\phi = e^\varepsilon + \varphi.
\]
where the correction $\varphi$ satisfies

\begin{align}
\text{div } \varphi &= 0 \quad \text{on } \Omega_\varepsilon \times (0, T), \\
\varphi_\alpha|_{\omega \times (0, T)} &= -T^{-1} \partial_t u_\alpha^0|_{\omega \times (0, T)}, \\
\phi_3|_{\omega \times (0, T)} &= -T^{-1} \partial_t e_{s,3}^e|_{\omega \times (0, T)}, \\
\varphi|_{\{x_3 = -\varepsilon\} \times (0, T)} &= 0,
\end{align}

and $\varphi(\cdot, t)$ is $\omega$-periodic for every $t \in (0, T)$. This choice of $\varphi$ ensures the kinematic boundary condition $\phi = \psi$ a.e. on $\omega \times (0, T)$. Moreover, the corrector $\varphi$ satisfies the uniform bound

Lemma 4.2.

\begin{align}
\|\nabla \varphi\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} &\leq C \varepsilon^{5/2},
\end{align}

where $C > 0$ is independent of $\varphi$ and $\varepsilon$.

\textbf{Proof.} Following \cite[Lemma 9]{33}, solution $\varphi$ of the problem (82)–(85) can be estimated as

\begin{align}
\|\nabla \varphi\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} &\leq C T^{-1} \left( \sum_{\alpha=1}^{\ell} \frac{1}{\sqrt{\varepsilon}} \|\partial_t u_\alpha^0\|_{L^\infty(0, T; L^2(\omega))} + \|\partial_t e_{s,3}^e\|_{L^\infty(0, T; L^2(\omega))} \right),
\end{align}

where $C > 0$ is independent of $\varepsilon$ and $t$. Let us now estimate the right hand side of (87).

First, employing inequalities on thin domains: the trace inequality from \cite{30}, the Poincaré and the Korn inequality (cf. Proposition A.2), respectively, we find

\begin{align}
\sum_{\alpha=1}^{\ell} \|\partial_t u_\alpha^0\|_{L^2(0, T; L^2(\Omega_\varepsilon))} &\leq C \left( \frac{1}{h} \|\partial_t u^0\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} + h \|\nabla \partial_t u^0\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \right) \\
&\leq C \left( \frac{1}{h} \|\nabla \partial_t u^0\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} + \frac{1}{h} \|\text{sym } \nabla \partial_t u^0\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \right) \\
&\leq C \frac{1}{h^3} \|\text{sym } \nabla \partial_t u^0\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \leq C \frac{1}{h^3} \|\text{sym } \nabla \partial_t u\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \\
&\leq C T \varepsilon^{3} h^{\kappa-3}.
\end{align}

In the latter inequality we used the higher-order energy estimate (31).

Utilizing the Griso decomposition for the third component

$$
\partial_t e_{s,3}^e = \partial_t (w_3^h + \bar{u}_3^o) - h^{\kappa-3} \partial_t w_3 = \partial_t w_3^h - \partial_t \bar{u}_3^o - h^{\kappa-3} \partial_t w_3
$$
and estimating the second term by using the trace inequality \[30\], we have
\[
\left\| \partial_t \epsilon_{s,3}^\ell \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq \frac{C}{h^2} \left\| \partial_t (u_3^h - \tilde{u}_3^0) \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + Ch \left\| \nabla \partial_t (u_3^h - \tilde{u}_3^0) \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + h^{2\kappa-6} \left\| \partial_t \omega \right\|_{L^\infty(0,T;L^2(\Omega))}^2
\]
\[
\leq \frac{C}{h} \left\| \partial_t u_3^h \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + Ch \left\| \nabla \partial_t u_3^h \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \frac{C}{h} \left\| \partial_t \tilde{u}_3^0 \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + Ch \left\| \nabla \partial_t \tilde{u}_3^0 \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + h^{2\kappa-6} \left\| \partial_t \omega \right\|_{L^\infty(0,T;L^2(\Omega))}
\]

Performing the Griso decomposition of the structure velocity \( \partial_t \mathbf{u}^h \) and employing the Griso estimates (cf. \[11\]), the higher-order energy inequality \([31]\) implies
\[
\frac{1}{h^2} \left\| \partial_t \mathbf{u}^h \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \left\| \nabla \partial_t \mathbf{u}^h \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 \leq CT \varepsilon^3 h^\kappa
\]
and \( \left\| \partial_t \omega \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C \). Using the latter together with the Poincaré inequality we further estimate
\[
(89) \quad \left\| \partial_t \epsilon_{s,3}^\ell \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq \frac{C}{h} \left\| \nabla \partial_t u_3^h \right\|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \frac{C}{h^2} \text{ess sup}_{t \in (0,T)} \left| \int_{\Omega_h} \partial_t u_3^h \, dx \right|^2 + C T \varepsilon^3 h^{\kappa+1} + Ch^{2\kappa-6}.
\]
In order to conclude the estimate we need the following lemma.

**Lemma 4.3.** The mean values of the vertical structure displacement and velocity satisfy
\[
\text{ess sup}_{t \in (0,T)} \left| \int_{\Omega_h} u_3^h(t) \, dx \right| + \text{ess sup}_{t \in (0,T)} \left| \int_{\Omega_h} \partial_t u_3^h \, dx \right| \leq C h^\kappa.
\]

**Proof.** Let us define
\[
U_3^h(x_3, t) := \int_\omega u_3^h(x', x_3, t) \, dx', \quad x_3 \in (0, h), \ t \in (0, T).
\]
Then for a.e. \( t \in (0, T) \), using the Cauchy-Schwarz inequality we have
\[
\left| \int_{\Omega_h} u_3^h(t) \, dx \right| = \int_0^h U_3^h(x_3, t) \, dx_3 \leq \sqrt{h} \left\| U_3^h(t) \right\|_{L^2(0,h)}.
\]
Since \( U_3^h(0, t) = \int_\omega u_3^h(x', 0, t) \, dx' = 0 \), the Poincaré inequality on \((0, h)\) gives
\[
\left| \int_{\Omega_h} u_3^h(t) \, dx \right| \leq Ch^{3/2} \left\| \partial_t U_3^h(t) \right\|_{L^2(0,h)}.
\]
Next, using the Jensen’s inequality and energy estimate (30) we find
\[
\|\partial_3 u_3^h(t)\|_{L^2(0,h)}^2 = \int_0^h (\partial_3 u_3^h(t))^2 \, dx = \int_0^h \left( \int_\omega (\partial_3 u_3^h(x', x_3, t)) \, dx' \right)^2 \, dx_3 \\
\leq \int_{\Omega_h} (\partial_3 u_3^h(x', x_3, t))^2 \, dx \leq C T \varepsilon^3 h^\kappa.
\]
Therefore, employing the latter inequality together with relation (53), we conclude
\[
\text{ess sup}_{t \in (0, T)} \left| \int_{\Omega_h} u_3^h \, dx \right| \leq C h^{3/2} \|\partial_3 u_3^h\|_{L^\infty(0, T; L^2(0, h))} \leq C h^\kappa.
\]
Due to the higher-order energy estimate (31), which is of the same type as (30), the analogous conclusion can be performed also for \(\partial_t u_3^h\). □

Going back to (89) and applying the previous lemma with the Korn inequality and energy estimate (31), we obtain
\[
\|\partial_t e_{s,3}^h\|_{L^\infty(0, T; L^2(\Omega_h))} \leq C \frac{\varepsilon}{h^3} \|\nabla \partial_t u_3^h\|_{L^\infty(0, T; L^2(\Omega_h))} + C h^{2\kappa-2} + C h^{2\kappa-6} \\
\leq C T \varepsilon^3 h^\kappa - 3 + C h^{2\kappa-6}.
\]
Combining (88) and (90), from estimate (87) and relation (53) we conclude
\[
\|\nabla \varphi\|_{L^\infty(0, T; L^2(\Omega_h))} \leq C \left( T^{-1} \varepsilon^2 h^{\kappa-3} + T^{-1} \varepsilon^3 h^{\kappa+1} + T^{-2} h^{2\kappa-6} \right) \leq C \varepsilon^5,
\]
which finishes the proof of Lemma 4.2 □

Now we continue with the proof of Proposition 4.1. Utilizing the above constructed test functions \((\phi, \psi)\) in the variational equation (78) and using the orthogonality property of the decomposition to even and odd functions with respect to the variable \((x_3 - h/2)\), then for a.e. \(t \in (0, T)\) we have
\[
\frac{\eta f}{2} \int_{\Omega_c} |e_f^\varepsilon(t)|^2 \, dx + 2\eta T \int_0^t \int_{\Omega_c} \text{sym} \nabla e_f^\varepsilon \, dx \, ds + \frac{\eta h T^{-2}}{2} \int_{\Omega_h} \left( (\partial_t e_{s,\alpha}^\varepsilon)^2 + (\partial_t e_{s,3}^\varepsilon)^2 \right) \, dx \\
+ \int_{\Omega_h} \left( \mu^h \left| \text{sym} \nabla (e_{s,1}^\varepsilon, e_{s,2}^\varepsilon, e_{s,3}^\varepsilon) \right|^2 + \frac{\lambda^h}{2} \left| \text{div}(e_{s,1}^\varepsilon, e_{s,2}^\varepsilon, e_{s,3}^\varepsilon) \right|^2 \right) (t) \, dx = -\eta f \int_{\Omega_c} e_f^\varepsilon(t) \cdot \varphi \, dx \\
- 2\eta T \int_0^t \int_{\Omega_c} \text{sym} \nabla e_f^\varepsilon : \text{sym} \nabla \varphi \, dx \, ds + T \int_0^t \int_{\Omega_c} f_3 e_f^\varepsilon + \varphi_3 \, dx \, ds \\
- T \int_0^t \int_{\Omega_c} r_f^h \cdot (e_f^\varepsilon + \varphi) \, dx \, ds - T \int_0^t \int_\omega r_f^h \cdot (e_f^\varepsilon + \varphi) \, dx' \, ds - (r_s^h, \psi).
Using the Cauchy-Schwarz and the Young inequality together with inequalities from Proposition A.2, we estimate the right hand side of the latter equation as follows:

$$\frac{\partial f}{4} \int_{\Omega_\varepsilon} |e_\varepsilon (t)|^2 \, d\mathbf{x} + \eta T \int_0^t \int_{\Omega_\varepsilon} |\text{sym} \nabla e_\varepsilon|^2 \, d\mathbf{x} ds + \frac{\rho_\varepsilon T^2}{2} \int_{\Omega_\varepsilon} \left( (\partial_t e_{s,0}^e) + (\partial_t e_{s,3}^o)^2 \right) (t) \, d\mathbf{x}$$

$$+ \int_{\Omega_\varepsilon} \left( \mu^h |\text{sym} \nabla (e_{s,1}^e, e_{s,2}^e, e_{s,3}^o)|^2 + \frac{\lambda^h}{2} |\text{div} (e_{s,1}^e, e_{s,2}^e, e_{s,3}^o)|^2 \right) (t) \, d\mathbf{x} \leq \frac{\partial f}{4} \int_{\Omega_\varepsilon} |\varphi (t)|^2 \, d\mathbf{x}$$

(92)

$$\quad + \eta T \int_0^t \int_{\Omega_\varepsilon} |\text{sym} \nabla \varphi|^2 \, d\mathbf{x} ds + T \int_0^t \| f_{3}^\varepsilon \|_{L^2(\Omega_\varepsilon)} \left( \| e_{f,3}^\varepsilon \|_{L^2(\Omega_\varepsilon)} + \varepsilon \| \nabla \varphi_3 \|_{L^2(\Omega_\varepsilon)} \right) ds$$

$$\quad + CT \int_0^t \varepsilon \| r_{f}^\varepsilon \|_{L^2(\Omega_\varepsilon)} \left( \| \nabla e_{f}^\varepsilon \|_{L^2(\Omega_\varepsilon)} + \| \nabla \varphi \|_{L^2(\Omega_\varepsilon)} \right) ds + T \int_0^t \left| \int_\Omega \int_{\Omega_\varepsilon} \varphi_{\text{div}} e_{f}^\varepsilon \, d\mathbf{x} \right|$$

The right hand side in (92) is further estimated term by term as follows. The first two terms are bounded by

$$\frac{\partial f}{4} \int_{\Omega_\varepsilon} |\varphi (t)|^2 \, d\mathbf{x} + \eta T \int_0^t \int_{\Omega_\varepsilon} |\text{sym} \nabla \varphi|^2 \, d\mathbf{x} ds \leq C \varepsilon^7 + CT \varepsilon^5 \leq CT \varepsilon^5.$$  

(93)

Higher-order energy estimate (28) directly provides

$$\| \nabla \partial_e \mathbf{u}^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \varepsilon^3.$$  

(94)

Then the Poincaré inequality on thin domains implies

$$\| \partial_e \mathbf{u}^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \varepsilon^{5/2}.$$  

Another application of the Poincaré inequality in a combination with the divergence free condition yields

$$\| e_{f,3}^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \varepsilon^{7/2}.$$  

(95)

The latter trivially implies

$$\| e_{f,3}^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} = \| e_{f,3}^\varepsilon - v_3^\varepsilon \|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \varepsilon^{7/2}.$$  

Therefore, the force term can be bounded as

$$T \int_0^t \| f_{3}^\varepsilon \|_{L^2(\Omega_\varepsilon)} \left( \| e_{f,3}^\varepsilon \|_{L^2(\Omega_\varepsilon)} + \varepsilon \| \nabla \varphi_3 \|_{L^2(\Omega_\varepsilon)} \right) ds \leq CT \varepsilon^4.$$  

(96)

For the fluid residual term we employ the apriori estimates to conclude

$$CT \int_0^t \varepsilon \| r_{f}^\varepsilon \|_{L^2(\Omega_\varepsilon)} \left( \| \nabla e_{f}^\varepsilon \|_{L^2(\Omega_\varepsilon)} + \| \nabla \varphi \|_{L^2(\Omega_\varepsilon)} \right) ds \leq CT \varepsilon^4.$$  

(97)
For the boundary residual term, which is only $O(\varepsilon)$ in the leading order, we invoke the Griso decomposition to conclude that

$$e_j^\varepsilon \big|_\omega = T^{-1} \partial_t u^h \big|_\omega - \varepsilon^2 \partial_t a_\alpha$$

is dominantly constant on $\omega$. Due to the interface condition $\int_\omega \partial_3 v_0 \, dz' = 0$ the leading order term vanishes and the rest can be controlled as

$$\left| T \int_0^t \int_\omega \mathbf{r}_s^\varepsilon \cdot \mathbf{e}_j^\varepsilon \, ds \right| + T \int_0^t \varepsilon^{1/2} \| \mathbf{r}_s^\varepsilon \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} \, ds \leq CT \varepsilon^4 .$$

(98)

Finally, due to the orthogonality properties and integrating by parts in time, for the structure residual term we have

$$\langle \mathbf{r}_s^h, \psi \rangle = \int_0^t \int_{\Omega_h} h^{k-3-\gamma} (\partial_1 a_1 \partial_1 e_{s,1}^e + \partial_1 a_2 \partial_1 e_{s,2}^e) \, dx \, ds$$

$$= \int_0^t \int_{\Omega_h} h^{k-3-\gamma} (\partial_1 a_1 \partial_1 e_{s,1}^e + \partial_1 a_2 \partial_1 e_{s,2}^e) (t) \, dx$$

$$- \int_0^t \int_{\Omega_h} h^{k-3-\gamma} (\partial_1 a_1 \partial_1 e_{s,1}^e + \partial_1 a_2 \partial_1 e_{s,2}^e) \, dx \, ds .$$

The latter can be estimated as

$$\langle \mathbf{r}_s^h, \psi \rangle \leq \frac{C h^2 - h^{2k-6-2\gamma+1}}{4} \int_{\Omega_h} (|\partial_1 e_{s,1}^e|^2 + |\partial_1 e_{s,2}^e|^2) (t) \, dx$$

$$+ \frac{h^2 - h^{2k-6-2\gamma+1}}{4} \int_{\Omega_h} (|\partial_1 e_{s,1}^e|^2 + |\partial_1 e_{s,2}^e|^2) \, dx \, ds .$$

(99)

Going back to [92] and employing previously established bounds together with the Gronwall inequality, we find

$$\int_0^t \int_{\Omega_\varepsilon} \left| e_j^\varepsilon(t) \right|^2 \, dx + \eta T \int_0^t \int_{\Omega_\varepsilon} \left| \text{sym} \, \nabla e_j^\varepsilon \right|^2 \, dx \, ds + \frac{h^2 - h^{2k-6-2\gamma+1}}{4} \int_{\Omega_h} (|\partial_1 e_{s,1}^e|^2 + |\partial_1 e_{s,2}^e|^2) \, dx$$

$$+ \int_{\Omega_h} \left( \mu h \left| \text{sym} \, \nabla (e_{s,1}^e, e_{s,2}^e, e_{s,3}^o) \right|^2 + \frac{h^2}{2} \left| \text{div} (e_{s,1}^e, e_{s,2}^e, e_{s,3}^o) \right|^2 \right) \, dx$$

$$\leq CT (\varepsilon^4 + h^{2k-2\gamma+1}) = CT \varepsilon^3 (h^\gamma + h^{4\gamma-2k+1}) .$$

(100)

In order to finish with the proof of Proposition 4.1, we still need to estimate tangential fluid errors on the interface. Namely,

$$\int_0^t \int_{\Omega_\varepsilon} \left| \text{sym} \, \nabla e_j^\varepsilon \right|^2 \, dx \, ds = \frac{1}{2} \int_0^t \int_{\Omega_\varepsilon} \left| \nabla e_j^\varepsilon \right|^2 \, dx \, ds + \int_0^t \int_{\omega} \left( e_{j,1}^\varepsilon \partial_1 e_{f,3}^\varepsilon + e_{j,2}^\varepsilon \partial_2 e_{f,3}^\varepsilon \right) \, dx' \, ds .$$
The interface terms are then estimated as
\[
\left| \int_0^t \int_{\omega} e^\varepsilon_{f, \alpha} \partial_\alpha e^\varepsilon_{f, 3} \, dx \, dt \right| \leq \| e^\varepsilon_{f, \alpha} \|_{L^2(0, t; L^2(\omega))} \| \partial_\alpha e^\varepsilon_{f, 3} \|_{L^2(0, t; L^2(\omega))} \\
\leq \varepsilon \| \partial_3 e^\varepsilon_{f, \alpha} \|_{L^2(0, t; L^2(\omega_\alpha))} \| \partial_3 \alpha e^\varepsilon_{f, 3} \|_{L^2(0, t; L^2(\omega_\varepsilon))} \\
\leq \frac{\eta}{4} \| \partial_3 e^\varepsilon_{f, \alpha} \|_{L^2(0, t; L^2(\omega_\alpha))}^2 + C \varepsilon^2 \| \partial_3 \alpha e^\varepsilon_{f, 3} \|_{L^2(0, t; L^2(\omega_\varepsilon))}^2 \\
\leq \frac{\eta}{4} \| \partial_3 e^\varepsilon_{f, \alpha} \|_{L^2(0, t; L^2(\omega_\alpha))}^2 + C \varepsilon^5 ,
\]
where the last bound follows from \([94]\). Employing this estimate into \((100)\) we arrive to
\[
\frac{\partial f}{4} \int_{\Omega_\varepsilon} |e^\varepsilon_f(t)|^2 \, dx + \frac{\eta T}{4} \int_0^t \int_{\Omega_\varepsilon} |\nabla e^\varepsilon_f|^2 \, dx \, ds + \frac{\eta T}{4} \int_{\Omega_\varepsilon} \left( (\partial_\varepsilon e^\varepsilon_{\alpha, \varepsilon})^2 + (\partial_\alpha e^\varepsilon_{\alpha, 3})^2 \right) \, dx \\
+ \int_{\Omega_\varepsilon} \left( \mu^h \left| \text{sym} \left( e^\varepsilon_{s, 1}, e^\varepsilon_{s, 2}, e^\varepsilon_{s, 3} \right) \right|^2 + \frac{\lambda^h}{2} \left| \text{div} \left( e^\varepsilon_{s, 1}, e^\varepsilon_{s, 2}, e^\varepsilon_{s, 3} \right) \right|^2 \right) \, dx \leq C T \varepsilon^3 (h^\gamma + h^{4\gamma - 2\kappa + 4}) ,
\]
which finishes the proof of Proposition 4.1.

4.3. Error estimates for fluid velocities. Let us now start with the proof of Theorem 1.2. Proposition 4.1 directly implies
\[
\| \nabla e^\varepsilon_f \|_{L^2(0, T; L^2(\omega_\varepsilon))} \leq C \varepsilon^3 (h^\gamma + h^{4\gamma - 2\kappa + 4}) ,
\]
while the Poincaré inequality then gives
\[
\| e^\varepsilon_{f, \alpha} \|_{L^2(0, T; L^2(\omega_\varepsilon))} \leq C \varepsilon^5 (h^\gamma + h^{4\gamma - 2\kappa + 4}) .
\]
The latter implies the desired error estimate
\[
(101) \quad \| e^\varepsilon_{f, \alpha} \|_{L^2(0, T; L^2(\omega_\varepsilon))} \leq C \varepsilon^{5/2} h^{\min\{\gamma/2, (2\gamma - \kappa + 2)\} + } , \quad \alpha = 1, 2 ,
\]
where \((2\gamma - \kappa + 2)\) = \(\max\{2\gamma - \kappa + 2, 0\}\).

4.4. Error estimates for structure displacements. For the structure displacement error, Proposition 4.1 provides only
\[
\| \text{sym} \left( e^\varepsilon_{s, 1}, e^\varepsilon_{s, 2}, e^\varepsilon_{s, 3} \right) \|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \leq C \left( h^{2\kappa + \gamma - 3} + h^{4\gamma + 1} \right) .
\]
According to the Griso decomposition for \(u^h_\alpha\), we have
\[
e^\varepsilon_{s, \alpha} = w^h_\alpha + \tilde{u}^\varepsilon_\alpha - h^{c - 3 - \gamma} a_\alpha , \quad \alpha = 1, 2 ,
\]
where \(\tilde{u}^\varepsilon_\alpha\) denotes the even part of the warping \(\tilde{u}_\alpha\). Employing now the Griso decomposition for the error \(e^\varepsilon_{s, \alpha}\) together with the Griso estimate for the corresponding elementary plate
displacement, we have
\[
\| \text{sym} \nabla'(w_1^h + \frac{1}{h} \int_0^h \tilde{u}_1^e dx_3, w_2^h + \frac{1}{h} \int_0^h \tilde{u}_2^e dx_3) \|_{L^\infty(0,T;L^2(\Omega_h))}^2 \leq \| \text{sym} \nabla' (e_{s,1}^e, e_{s,2}^e) \|_{L^\infty(0,T;L^2(\Omega_h))}^2 \leq C \left(h^{2\kappa+\gamma-3} + h^{4\gamma + 1}\right),
\]
i.e.
\[
\| \text{sym} \nabla'(w_1^h + \frac{1}{h} \int_0^h \tilde{u}_1^e dx_3, w_2^h + \frac{1}{h} \int_0^h \tilde{u}_2^e dx_3) \|_{L^\infty(0,T;L^2(\omega))}^2 \leq C \left(h^{2\kappa+\gamma-4} + h^{4\gamma}\right).
\]
According to the Korn inequality and the Griso estimate for the warping terms \(\tilde{u}_e\), for the sequence of spatially constant functions \((a_h^\alpha) \subset L^\infty(0,T)\) we have
\[
(102) \quad \| (w_1^h - a_1^h, w_2^h - a_2^h) \|_{L^\infty(0,T;L^2(\omega))}^2 \leq C \left(h^{2\kappa+\gamma-4} + h^{4\gamma} + h^{2\kappa-2}\right).
\]
Again the Griso decomposition and a priori estimate of the warping terms \(\tilde{u}_e\) imply
\[
\| e_{s,1}^h - w_1^h + h^{\kappa-3-\gamma} a_1 - (x_3 - \frac{h}{2}) (h^{\kappa-3} \partial_1 w_3 + r_2^h) \|_{L^\infty(0,T;L^2(\Omega_h))} \leq C h^{2\kappa-1},
\]
\[
\| e_{s,2}^h - w_2^h + h^{\kappa-3-\gamma} a_2 - (x_3 - \frac{h}{2}) (h^{\kappa-3} \partial_2 w_3 - r_1^h) \|_{L^\infty(0,T;L^2(\Omega_h))} \leq C h^{2\kappa-1}.
\]
Using the triangle inequality and estimate \((102)\) we have
\[
(103) \quad \| e_{\alpha,s}^h \|_{L^\infty(0,T;L^2(\Omega_h))}^2 \leq C h^{2\kappa-3} \left(h^2 + h^{\gamma} + h^{4\gamma+4-2\kappa}\right) + h \| a_\alpha^h - h^{\kappa-3-\gamma} a_\alpha \|_{L^\infty(0,T)}^2.
\]
provided the following lemma holds true.

**Lemma 4.4.**
\[
(104) \quad \| h^{\kappa-3} \partial_1 w_3 + r_2^h \|_{L^\infty(0,T;L^2(\Omega_h))}^2 + \| h^{\kappa-3} \partial_2 w_3 - r_1^h \|_{L^\infty(0,T;L^2(\Omega_h))}^2 \leq C h^{2\kappa-5} \left(h^2 + h^{\gamma} + h^{4\gamma+4-2\kappa}\right).
\]

**Proof.** In order to prove \((104)\) we first employ the test function on the structure part \(\psi = T^{-1} \partial_3 e_3^e\) in \((78)\). For the fluid part we take the test function
\[
\phi = e_3^e + \varphi,
\]
where the correction \(\varphi\) satisfies
\[
\text{div} \varphi = 0 \quad \text{on} \quad \Omega_\varepsilon \times (0,T),
\]
\[
\varphi_{\alpha}|_{\omega} = T^{-1} \left(\frac{h^{\kappa-2}}{2} \partial_\alpha w_3 - \partial_3 e_3^e_{\alpha}\right)|_{\omega}, \quad \varphi_3|_{\omega} = -T^{-1} \partial_3 e_3^e_{3,\alpha}|_{\omega},
\]
\[
\varphi|_{\{x_3 = -\varepsilon\}} = 0,
\]
and $\varphi$ is periodic on the lateral boundaries. The estimate on $\varphi$ now reads
\begin{equation}
\|
abla \varphi \|_{L^\infty(0,T;L^2(\Omega_t))} \leq CT^{-1} \left( \frac{1}{\sqrt{\varepsilon}} \left\| \frac{h^{n-2}}{2} \partial_\alpha w_3 - \partial_t e_{s,\alpha}^e \right\|_{L^\infty(0,T;L^2(\omega))} + \| \partial_t e_{s,3}^e \|_{L^\infty(0,T;L^2(\omega))} \right),
\end{equation}
The last term is already estimated above with $O(\varepsilon^3)$. Therefore, using the trace and Korn inequalities on thin domains together with estimates (100) and (27) we have
\begin{equation}
\| \nabla \varphi \|_{L^\infty(0,T;L^2(\Omega_t))} \leq CT^{-2} \left( h^{2n-4-\gamma} + \frac{1}{\varepsilon h} \| \partial_t e_{s,\alpha}^e \|_{L^\infty(0,T;L^2(\Omega_h))}^2 \right)
\end{equation}
\begin{equation}
+ \frac{h}{\varepsilon} \| \nabla \partial_t e_{s,\alpha}^e \|_{L^\infty(0,T;L^2(\Omega_h))}^2 \right) + C \varepsilon^6
\end{equation}
\begin{equation}
\leq C \left( h^{5\gamma+2} + \frac{T \varepsilon^4}{h^2} + \frac{T^{-2}}{\varepsilon h} \| \text{sym} \nabla \partial_t u^h \|_{L^\infty(0,T;L^2(\Omega_h))} \right)
\end{equation}
\begin{equation}
\leq C \left( h^{5\gamma+2} + h^{2\kappa-4} + h^{6\gamma} \right) \leq C(\varepsilon^5 + h^{2\kappa-4}) \right).
\end{equation}

Using the above test functions in the weak form for the errors (78) and estimating like in (92) we find
\begin{equation}
\frac{\partial_t}{4} \int_{\Omega_t} |e_f^{\varepsilon}(t)|^2 \, dx + \frac{\eta T}{2} \int_0^t \int_{\Omega_t} |\nabla e_f^{\varepsilon}(t)|^2 \, dx \, ds + \frac{\eta T}{4} \int_0^t \int_{\Omega_h} |\partial_t e_s^{\alpha}(t)|^2 \, dx \, ds
\end{equation}
\begin{equation}
+ \frac{\lambda h}{2} \int_0^t \int_{\Omega_h} |\partial_t e_s^{\alpha}(t)|^2 \, dx \, ds}
\end{equation}
\begin{equation}
\leq C T \left( \varepsilon^4 + h^{2\kappa-4} + \varepsilon^{3/2}(\varepsilon^5 + h^{2\kappa-4})^{1/2} \right)
\end{equation}
\begin{equation}
+ C T \varepsilon^3 + \frac{\lambda h}{4} \int_{\Omega_h} |\text{div} e_s^{\alpha}(t)|^2 \, dx + \frac{\lambda h}{2} \int_0^t \int_{\Omega_h} |\text{div} e_s^{\alpha}(t)|^2 \, dx \, ds.
\end{equation}

The last line of the above inequality arises from estimating the structure residual term $T^{-1}(r_s^h, \partial_t e_s^{\alpha})$. Therefore, employing the Gronwall lemma we close the estimate (106) with
\begin{equation}
\frac{\partial_t}{4} \int_{\Omega_t} |e_f^{\varepsilon}(t)|^2 \, dx + \frac{\eta T}{2} \int_0^t \int_{\Omega_t} |\nabla e_f^{\varepsilon}(t)|^2 \, dx \, ds + \frac{\eta T}{4} \int_0^t \int_{\Omega_h} |\partial_t e_s^{\alpha}(t)|^2 \, dx \, ds
\end{equation}
\begin{equation}
+ \frac{\lambda h}{2} \int_0^t \int_{\Omega_h} |\partial_t e_s^{\alpha}(t)|^2 \, dx \, ds \leq C T \left( \varepsilon^3 + h^{2\kappa-4} \right).
\end{equation}

Having this at hand, we conclude
\begin{equation}
\int_{\Omega_h} (\partial_\alpha e_{s,3}^{\alpha} + \partial_3 e_{s,\alpha}^{\alpha})^2 \, dx \leq Ch^{2\kappa-5} \left( h^2 + h^{2\kappa-3\gamma-2} \right), \quad \alpha = 1, 2.
\end{equation}
Thus, using the triangle inequality we conclude the desired estimate which is equivalent to

\[ \int_{\Omega_h} \left( (r_2^h + h^{\kappa-3}\partial_1 w_3 + \partial_3 \tilde{u}_1^o + \partial_1 \tilde{u}_3^o)^2 + (-r_1^h + h^{\kappa-3}\partial_2 w_3 + \partial_3 \tilde{u}_2^o + \partial_2 \tilde{u}_3^o)^2 \right) \, dx \leq C h^{2\kappa-5} (h^2 + h^{2\kappa-3\gamma-2}). \]

From the basic Griso inequality and a priori estimate we have

\[ \int_{\Omega_h} ((\partial_3 \tilde{u}_1^o + \partial_1 \tilde{u}_3^o)^2 + (\partial_3 \tilde{u}_2^o + \partial_2 \tilde{u}_3^o)^2) \, dx \leq \| \nabla \tilde{u}_h \|^2_{L^\infty(0; T; L^2(\Omega_h))} \]

\[ \leq \| \text{sym} \nabla \tilde{u}_h \|^2_{L^\infty(0; T; L^2(\Omega_h))} \leq C h^{2\kappa-3}. \]

One immediately sees that for \( \kappa \geq \max\{2\gamma + 1, \frac{7}{4}\gamma + \frac{3}{2}\} \) it holds \( h^{2\kappa-3\gamma-2} \leq h^\gamma + h^{4\gamma+4-2\kappa} \). Thus, using the triangle inequality we conclude the desired estimate

\[ \| h^{\kappa-3}\partial_1 w_3 + r_2^h \|^2_{L^\infty(0; T; L^2(\Omega_h))} + \| h^{\kappa-3}\partial_2 w_3 - r_1^h \|^2_{L^\infty(0; T; L^2(\Omega_h))} \leq C h^{2\kappa-5} (h^2 + h^\gamma + h^{4\gamma+4-2\kappa}). \]

\[ \square \]

Recall again the Griso estimate for the elementary plate displacement, we have

\[ \int_{\Omega_h} ((\partial_1 w_3^h + r_2^h)^2 + (\partial_2 w_3^h - r_1^h)^2) \, dx \leq C h^{2\kappa-3}, \]

which combined with \([104]\) implies

\[ \| \partial_1 w_3^h - h^{\kappa-3}\partial_1 w_3 \|^2_{L^\infty(0; T; L^2(\Omega_h))} + \| \partial_2 w_3^h - h^{\kappa-3}\partial_2 w_3 \|^2_{L^\infty(0; T; L^2(\Omega_h))} \leq C h^{2\kappa-5} (h^2 + h^\gamma + h^{4\gamma+4-2\kappa}). \]

The Poincaré inequality gives

\[ \| w_3^h - h^{\kappa-3}w_3 \|^2_{L^\infty(0; T; L^2(\Omega_h))} \leq C h^{2\kappa-4} + C h^{2\kappa-5} (h^2 + h^\gamma + h^{4\gamma+4-2\kappa}), \]

which eventually provides

\[ (108) \quad \| \epsilon_{s,3}^h \|^2_{L^\infty(0; T; L^2(\Omega_h))} \leq C h^{2\kappa-5} (h^1 + h^\gamma + h^{4\gamma+4-2\kappa}). \]

### 4.5. Error estimate for the pressure.

Finally we prove the error estimate for the pressure. We define \( e_p^h = p^f - p^s \). Similarly as in the a priori pressure estimate, the error estimate will be performed in two steps. In the first step we estimate zero mean value part of the error \( e_p^h \). This is classical and follows directly from the error estimates for the fluid velocity.

The second step is specific for our problem and is related to the fact that the pressure is unique due to the fact that elastic wall can be deformed by the pressure force. Let us denote by \( \pi_e(t) = \int_{\Omega_e} e_p^h(t, \cdot) \) the mean value of the pressure error. The test function \( \phi \in V_F(0, T; \Omega_e) \) is constructed such that \( \text{div} \phi(t, \cdot) = e_p^h(t, \cdot) - \pi_e(t) \) and \( \phi \) vanishes on the interface. This
can be done in a standard way by using the Bogovskij construction, see e.g. \cite[Section 3.3]{20}. Moreover, the following estimates hold (see e.g. \cite[Lemma 9]{33}):\[
\|\phi\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq \frac{C}{\varepsilon} \|e_\varepsilon^p - \pi_\varepsilon e\|_{L^2(0,T;L^2(\Omega_\varepsilon))}, \\
\|\phi_3\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \|e_\varepsilon^p - \pi_\varepsilon e\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.
\]
By construction \((\phi, 0)\) is an admissible test function for error formulation and therefore we get the following estimate:

\[
\int_0^T \int_{\Omega_\varepsilon} (e_\varepsilon^p - \pi_\varepsilon e)^2 dx \, ds = \left| \int_0^T \int_{\Omega_\varepsilon} e_\varepsilon^p \text{div} \phi \, dx \, ds \right|
\]
\[
= \left| \int_0^T \int_{\Omega_\varepsilon} \left( T^{-1} \varrho_\varepsilon \partial_t e_\varepsilon^f - r_\varepsilon^f + f_3^\varepsilon e_3 \right) \cdot \phi \, dx \, ds \right|
\]
\[
- 2\eta \int_0^T \int_{\Omega_\varepsilon} \text{sym} \nabla e_\varepsilon^f : \text{sym} \nabla \phi \, dx \, ds \right|
\]
\[
\leq C \varepsilon^{1/2} \left( h^{3\gamma/2+3/2-\kappa/2} + \varepsilon + h^{\min\{\gamma/2,(2\gamma-\kappa+2)+\}} \right) \|e_\varepsilon^p - \pi_\varepsilon e\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.
\]

Here we used the higher-order energy estimate of Corollary \cite[2.4]{24} to control the time derivatives, definition of the fluid residual term \(r_\varepsilon^f\) and the estimate of Proposition \cite[4.1]{41}.

To estimate the mean value term \(\pi_\varepsilon e\) we follow the same steps as in the proof of estimate \cite[39]{39} with \(\zeta = \pi_\varepsilon e\). However, we do not gain anything in comparison to the a priori estimates because we have not derived higher-order estimates for \(\partial_\varepsilon^2 e_\varepsilon^h\). Therefore, for \(\kappa \geq 2\gamma + 1\) we proved the following error estimates for the pressure:

\[
(109) \quad \|p_\varepsilon - p\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \varepsilon^{1/2} h^{\min\{\gamma/2,(2\gamma-\kappa+2)+\}} + C \varepsilon h^{-1-\tau} \leq C \varepsilon^{1/2} h^{\min\{\gamma/2,(2\gamma-\kappa+2)+\}}.
\]

This finishes the proof of Theorem \cite[1.2]{1.2}.

5. Conclusions

Let us summarize our main achievements and key steps. We analyzed a linear FSI problem between a thin layer of a viscous fluid and a thin elastic plate with the aim of deriving a reduced model. The FSI problem is a coupled system, where the fluid is described by the Stokes equations and the structure by the linear elasticity equations, both in three spatial dimensions. We first derived an energy estimate quantified in terms of the relative fluid thickness \(\varepsilon\) and relative structure thickness \(h\), which are both assumed to be small parameters and related through \(\varepsilon = h^\gamma\), for some \(\gamma > 0\). Lamé constants and material density of the elastic structure are assumed to be large and behave like \(h^{-\kappa}\).

Based on the Griso decomposition of the structure displacement and energy estimates we derive key a priori estimates and identify that the right time scale at which nontrivial coupled reduced model appears is \(T = h^{\kappa-3\gamma-3}\). The reduced model is given by a linear sixth-order thin-film equation for the out-of-plane displacement of the linear bending plate.
It is a consequence of simultaneous dimension reduction in the structure and in the fluid. In the structure, the linear elasticity equations reduce to the linear bending plate model with the limit fluid pressure acting as a normal force, and in the fluid, the so called lubrication approximation is performed. Finally, utilizing the reduced model we construct approximate solutions to the original FSI problem and derive basic error estimates which provide even strong convergence results. In this procedure we pushed forward a novel approach in the error analysis of the structure part, the application of the Griso decomposition \cite{21}, which is of its own interest.

Since relation $\tau = \kappa - 3\gamma - 3$ has two degrees of freedom, one might come up with many different physical configurations of the thin FSI problem, which are all well described by the limit model. However, from the applicational point of view it seems most likely to prescribe parameters $\kappa$ – rigidity of the structure material and $\gamma$ – describing geometry of the system. Then if these two parameters assert $\kappa - 3\gamma \leq 2$, which is equivalent to $\tau \leq -1$, the corresponding time scaling will eventually provide (66) as the correct reduced model.

Appendix A. Griso decomposition and inequalities on thin domains

The following result is directly from \cite{21, Theorem 2.3}, tailored to the specific boundary conditions and geometry considered in this paper.

Theorem A.1. Let $h > 0$, then every $u^h \in V_S(\Omega_h)$ can be decomposed as

$$u^h(x) = w^h(x') + (x_3 - h/2)e_3 \times r^h(x') + \tilde{u}^h(x), \quad (x', x_3) \in \Omega_h,$$

or written componentwise

$$u_1^h(x) = w_1^h(x') + (x_3 - h/2)r_2^h(x') + \tilde{u}_1^h(x),$$
$$u_2^h(x) = w_2^h(x') - (x_3 - h/2)r_1^h(x') + \tilde{u}_2^h(x),$$
$$u_3^h(x) = w_3^h(x') + \tilde{u}_3^h(x),$$

where

$$w^h(x') = \frac{1}{h} \int_0^h u^h(x', x_3)dx_3, \quad r^h(x') = \frac{3}{h^3} \int_0^h (x_3 - h/2)e_3 \times u^h(x', x_3)dx_3,$$

and $\tilde{u}^h \in V_S(\Omega_h)$ is so called warping or residual term. The main part of the decomposition, denoted by $u^h_E = w^h(x') + (x_3 - h/2)e_3 \times r^h(x')$, is called the elementary plate displacement. Moreover, the following estimate holds

$$\|\text{sym} \nabla u^h_E\|_{L^2(\Omega_h)}^2 + \|\nabla \tilde{u}^h\|_{L^2(\Omega_h)}^2 + \frac{1}{h^2} \|\tilde{u}^h\|_{L^2(\Omega_h)}^2 \leq C \|\text{sym} \nabla u^h\|_{L^2(\Omega_h)}^2,$$

where $C > 0$ is independent of $u^h$ and $h$.

In the next proposition we collect a few simple, but important facts, which are used in the analysis of our system.
Proposition A.2. Let $0 < \varepsilon \ll 1$ and $\mathbf{v}^\varepsilon \in V_F(\Omega_\varepsilon)$, then the following uniform inequalities hold:

\begin{align}
\| \mathbf{v}^\varepsilon \|_{L^2(\Omega_\varepsilon)} & \leq C\varepsilon \| \partial_3 \mathbf{v}^\varepsilon \|_{L^2(\Omega_\varepsilon)}, \quad \text{(Poincaré inequality),} \\
\| \mathbf{v}^\varepsilon \|_{L^2(\omega)} & \leq C\sqrt{\varepsilon} \| \partial_3 \mathbf{v}^\varepsilon \|_{L^2(\Omega_\varepsilon)}, \quad \text{(Trace inequality),} \\
\| \partial_\alpha \mathbf{v}^\varepsilon_3 \|_{L^2(\Omega_\varepsilon)} & \leq \frac{C}{\varepsilon} \| \text{sym } \nabla \mathbf{v}^\varepsilon \|_{L^2(\Omega_\varepsilon)}, \quad \alpha = 1, 2, \quad \text{(Korn inequality).}
\end{align}

All above constants $C$ are positive and independent of $\varepsilon$.

Proof. [112] Utilizing the Cauchy-Schwarz inequality, we calculate

\[
\| \mathbf{v}^\varepsilon \|_{L^2(\Omega_\varepsilon)}^2 = \int_\omega \int_{-\varepsilon}^0 \mathbf{v}^\varepsilon(x', x_3)^2 dx'dx_3 = \int_\omega \int_{-\varepsilon}^0 \left( \int_{x_3}^{x_3+\varepsilon} \partial_3 \mathbf{v}^\varepsilon(x', s) ds \right)^2 dx'dx_3 \\
\leq \int_\omega \int_{-\varepsilon}^0 (x_3 + \varepsilon) \int_{-\varepsilon}^{x_3+\varepsilon} (\partial_3 \mathbf{v}^\varepsilon)^2(x', s) ds \leq \int_\omega \int_{-\varepsilon}^0 (x_3 + \varepsilon) \int_{-\varepsilon}^0 (\partial_3 \mathbf{v}^\varepsilon)^2(x', s) ds \\
\leq \| \partial_3 \mathbf{v}^\varepsilon \|_{L^2(\omega)}^2 \int_{-\varepsilon}^0 (x_3 + \varepsilon) dx_3 = \frac{3}{2} \varepsilon^2 \| \partial_3 \mathbf{v}^\varepsilon \|_{L^2(\omega)}^2.
\]

[113] Similar calculations with application of the Jensen’s inequality give:

\[
\| \mathbf{v}^\varepsilon \|_{L^2(\omega)}^2 = \int_\omega |\mathbf{v}^\varepsilon(x', 0)|^2 dx' = \int_\omega \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \partial_3 \mathbf{v}^\varepsilon(x) dx_3 \frac{1}{\varepsilon} dx' \leq \varepsilon \int_{\Omega} |\partial_3 \mathbf{v}^\varepsilon(x)|^2 dx.
\]

[114] Finally, the Korn inequality follows directly from Theorem A.3 and boundary condition $\mathbf{v}^\varepsilon = 0$ on bottom part of the fluid domain $x_3 = -\varepsilon$. \hfill \Box

Theorem A.3 (Korn inequality on thin domains). Let $\omega \subset \mathbb{R}^2$ be Lipschitz domain and $\gamma \subset \partial \omega$ part of its boundary of positive measure, then there exists a constant $C_K > 0$ and $h_0 > 0$ such that for every $0 < h < h_0$

\[
\| (\psi_1, \psi_2, h\psi_3) \|_{H^1(\Omega; \mathbb{R}^3)}^2 \leq C_K \left( \| (\psi_1, \psi_2, h\psi_3) \|_{L^2(\Omega; \mathbb{R}^3)}^2 + \| \text{sym } \nabla_h \psi \|_{L^2(\Omega; \mathbb{R}^3)}^2 \right), \quad \forall \psi \in H^1(\Omega; \mathbb{R}^3),
\]

where $\Omega = \omega \times (0, 1)$. The Korn constant $C_K$ depends only on $\omega$ and $\gamma$.

Proof. The proof follows by the Griso’s decomposition of $\psi \in H^1(\Omega; \mathbb{R}^3)$ (see [21]) and application of the Korn inequality for functions defined on $\omega$. \hfill \Box

Appendix B. Discussion on nontrivial initial conditions

The existence analysis follows the same paths like in the trivial case. Here we mainly discuss on how nontrivial initial conditions reflect the energy estimates, which are crucial in the derivation of the reduced model.
Let us denote the energy of initial datum \((v_0^h, u_0^h, u_1^h)\) by
\[
\mathcal{E}_0^\varepsilon = \frac{1}{2} \left( \varrho_f \|v_0^h\|^2_{L^2(\Omega)} + 2\mu^h \|\nabla u_0^h\|^2_{L^2(\Omega)} + \lambda^h \|\text{div } u_0^h\|^2_{L^2(\Omega)} + \varrho_s^h \|u_1^h\|^2_{L^2(\Omega)} \right).
\]
Furthermore, higher-order energies are denoted by
\[
\mathcal{E}_{0,\alpha}^\varepsilon = \frac{1}{2} \left( \varrho_f \|\partial_\alpha v_0^h\|^2_{L^2(\Omega)} + 2\mu^h \|\nabla \partial_\alpha u_0^h\|^2_{L^2(\Omega)} + \lambda^h \|\text{div } \partial_\alpha u_0^h\|^2_{L^2(\Omega)} + \varrho_s^h \|\partial_\alpha u_1^h\|^2_{L^2(\Omega)} \right),
\]
for \(\alpha = 1, 2\). With this notation in hand energy estimates (17) and (22) become
\[
\varrho_f \int_{\Omega_e} |v^\varepsilon(t)|^2 \, dx + \eta \int_0^t \int_{\Omega_e} \|\nabla v^\varepsilon(s)\|^2 \, dx \, ds + \varrho_s^h \int_{\Omega_h} |\partial_\alpha u^h(t)|^2 \, dx
\]
(115)
\[
\quad + \mu^h \int_{\Omega_h} |\nabla u^h(t)|^2 \, dx + \frac{\lambda^h}{2} \int_{\Omega_h} |\text{div } u^h(t)|^2 \, dx \leq C t \varepsilon + \mathcal{E}_0^\varepsilon
\]
and
\[
\varrho_f \int_{\Omega_e} |\partial_\alpha v^\varepsilon(t)|^2 \, dx + 2\eta \int_0^t \int_{\Omega_e} \|\nabla \partial_\alpha v^\varepsilon(s)\|^2 \, dx \, ds + \varrho_s^h \int_{\Omega_h} |\partial_\alpha \partial_\alpha u^h(t)|^2 \, dx
\]
(116)
\[
\quad + \mu^h \int_{\Omega_h} \|\nabla \partial_\alpha u^h(t)\|^2 \, dx + \frac{\lambda^h}{2} \int_{\Omega_h} |\text{div } \partial_\alpha u^h(t)|^2 \, dx \leq C t \varepsilon + \mathcal{E}_{0,\alpha}^\varepsilon, \quad \alpha = 1, 2,
\]
respectively. In order to derive improved energy estimate in the case of nontrivial initial conditions, it remains to analyze how initial conditions influence the estimate of interface terms. Applying (116) instead of (22) in (26) we find
\[
\left| \int_0^t v_\beta^\varepsilon \partial_\alpha v_\beta^\varepsilon \, dx \right| \leq C \varepsilon \sqrt{C t \varepsilon + \mathcal{E}_{0,\alpha}^\varepsilon} \|\partial_\beta v^\varepsilon\|_{L^2(0,T;L^2(\Omega_e))}.
\]
Finally, the improved energy estimate (analogue of (30)) with nontrivial initial conditions, and rescaled time and data reads
\[
\varrho_f \int_{\Omega_e} |v^\varepsilon(t)|^2 \, dx + \frac{\eta T}{2} \int_0^t \int_{\Omega_e} \|\nabla v^\varepsilon\|^2 \, dx \, ds + \varrho_s^h T^{-2} \int_{\Omega_h} |\partial_\alpha u^h(t)|^2 \, dx
\]
(117)
\[
\quad + h^{-\kappa} \int_{\Omega_h} \left( \mu \|\nabla u^h(t)\|^2 + \frac{\lambda}{2} |\text{div } u^h(t)|^2 \right) \, dx \leq C T \varepsilon^3 + \mathcal{E}_0^\varepsilon + C \varepsilon^2 \mathcal{E}_{0,\alpha}^\varepsilon.
\]
Clearly, we are interested in initial conditions whose energies are dominated by \(C T \varepsilon^3\). Therefore, we require
\[
\mathcal{E}_0^\varepsilon \lesssim T \varepsilon^3 \quad \text{and} \quad \mathcal{E}_{0,\alpha}^\varepsilon \lesssim T \varepsilon.
\]
The latter means that initial fluid velocity hast to satisfy \(\|v_0^h\|^2_{L^2(\Omega_e)} \lesssim T \varepsilon^3\), while the initial displacement and structure velocity have to satisfy
\[
\|u_0^h\|^2_{H^1(\Omega_h)} + \|u_1^h\|^2_{L^2(\Omega_h)} \lesssim T \varepsilon^3 h^\kappa.
\]
Additional conditions on initial data verifying $E_{0,\alpha}^\varepsilon \lesssim \varepsilon$ are in order.

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**University of Zagreb, Faculty of Electrical Engineering and Computing**

Unska 3, 10000 Zagreb, Croatia

*E-mail address: mario.bukal@fer.hr*

**University of Zagreb, Faculty of Science, Department of Mathematics, Bijenčka cesta 30, 10000 Zagreb, Croatia**

*E-mail address: borism@math.hr*