Monopole currents and Dirac sheets in U(1) lattice gauge theory

Werner Kerler\textsuperscript{a}, Claudio Rebbi\textsuperscript{b}, Andreas Weber\textsuperscript{a}

\textsuperscript{a} Fachbereich Physik, Universität Marburg, D-35032 Marburg, Germany
\textsuperscript{b} Department of Physics, Boston University, Boston, MA 02215, USA

Abstract

We show that the phases of the 4-dimensional compact U(1) lattice gauge theory are unambiguously characterized by the topological properties of minimal Dirac sheets as well as of monopole currents lines. We obtain the minimal sheets by a simulated-annealing procedure. Our results indicate that the equivalence classes of sheet structures are the physical relevant quantities and that intersections are not important. In conclusion we get a percolation-type view of the phases which holds beyond the particular boundary conditions used.
1. Introduction

We investigate the 4-dimensional compact U(1) lattice gauge theory with Wilson action supplemented by a monopole term [1]:

\[ S = \beta \sum_{\mu > \nu, x} (1 - \cos \Theta_{\mu \nu, x}) + \lambda \sum_{\rho, x} |M_{\rho, x}|, \]

where \( M_{\rho, x} = \epsilon_{\rho \sigma \mu \nu} (\bar{\Theta}_{\mu \nu, x + \sigma} - \Theta_{\mu \nu, x}) / 4\pi \) and the physical flux \( \bar{\Theta}_{\mu \nu, x} \in [-\pi, \pi) \) is given by \( \Theta_{\mu \nu, x} = \bar{\Theta}_{\mu \nu, x} + 2\pi n_{\mu \nu, x} \) [2]. We consider periodic boundary conditions except for the discussion in Section 5 where we extend our study to a system with open boundary conditions.

As is well known, the system has two phases, separated for \( \lambda = 0 \) by a first order phase transition. The strength of this transition decreases with increasing \( \lambda \) until the transition ultimately becomes of second order [3]. We have used this property to set up a very efficient algorithm [3, 4] for Monte Carlo simulations in which \( \lambda \) becomes a dynamical variable. The \( \lambda \) dependence has also allowed us to study the dynamics of the transition in detail [3].

We argue that the analysis of the topological structure of the configurations is a valuable tool for studying the phase structure, a tool which can provide much more detailed information than just the measurement of global observables. In the present work we use it by analyzing monopole currents and Dirac sheets.

For individual loops the topological characterization is straightforward, but this becomes less trivial when loops are entangled in networks of monopole currents. Recently we have been able to produce a mathematically sound characterization of the latter [3]. This is not only necessary for the unambiguous identification of their physical features but also for the analysis of huge networks by computer.

Here the analysis of configurations is extended to Dirac sheets and the fact that appropriately specified topological structures signal the phases is confirmed in more detail. In conclusion we point out a general principle for characterizing the phases, which does not rely on a particular choice of boundary conditions.

2. Dual-lattice structures

On the dual lattice the current \( J_{\rho, x} = M_{\rho, x+\rho} \), defined over the links, obeys the conservation law \( \sum_{\rho} (J_{\rho, x} - J_{\rho, x-\rho}) = 0 \). Current lines are defined in terms of the current as follows: for \( J_{\rho, x} = 0 \) there is no line on the link, for \( J_{\rho, x} = \pm 1 \) there is
one line, and for $J_{\rho,x} = \pm 2$ there are two lines, in positive or negative direction, respectively. Networks of currents are connected sets of current lines. For a network $N$ disconnected from the rest one can define a net current flow $\vec{f}$, with components $f_{\mu3} = \sum_{x_{\mu0}x_{\mu1}x_{\mu2}} J_{\mu3,x}, J_{\mu,x} \in N$.

The Dirac string content of the plaquettes on the dual lattice is described in terms of a variable $p_{\rho\sigma,x} = -\frac{1}{2} \epsilon_{\rho\sigma\mu\nu} n_{\mu\nu,x+\rho+\sigma}$, which satisfies the field equation $\sum_{\sigma}(p_{\rho\sigma,x} - p_{\rho\sigma,x-\sigma}) = J_{\rho,x}$. To every plaquette in the dual lattice we will associate no Dirac plaquette if $p_{\rho\sigma,x} = 0$, one Dirac plaquette if $p_{\rho\sigma,x} = \pm 1$, with similar or opposite orientation according to the sign of $p$, and two Dirac plaquettes if $p_{\rho\sigma,x} = \pm 2$.

Dirac sheets are formed by connecting Dirac plaquettes with a common edge and appropriate orientation, so that $J_{\rho,x} = 0$ except at the boundaries of the sheets. Dirac sheet structures are not gauge invariant. However, they belong to equivalence classes which cannot be deformed into each other by gauge transformations. For given boundaries there are topologically distinct possibilities for the sheet structures. This is illustrated in Fig. 1 by a simple example in 2 dimensions, where a topologically nontrivial current network may be accompanied by a trivial or a nontrivial Dirac sheet. The equivalence classes of Dirac sheets thus carry more information than the related current networks.

We represent the equivalence classes of the sheet structures by their members with minimal area, which are obtained by minimizing the number of Dirac plaquettes by a gauge transformation. This permits a unique specification and, in view of the large initial number of Dirac plaquettes, is necessary for a computational analysis.

To address the connectedness of sheet structures, we construct minimal Dirac sheets by making first the connections where only two Dirac plaquettes meet at an edge with appropriate orientation. Then we consider the places where more than two Dirac plaquettes meet at an edge. If the sheets can be deformed (by a gauge transformation) in such a way that no more than two plaquettes meet we connect the respective pairs. Otherwise all must to be connected.

In Fig. 2 we use some simple 3-dimensional examples to illustrate typical possibilities for the intersection of Dirac sheets or for the meeting of more than two plaquettes. In Fig. 2a, deforming one of the sheets by a gauge transformation, the sheets can be separated. Consequently there is no connection. The same holds for Fig. 2b, where one must realize that common sites are not considered as a connection of sheets. In Fig. 2c, although one can identify two sheets, the structure cannot be separated by a gauge transformation and, therefore, is to be considered as connected. In Fig. 2d separation is obviously not possible.
Figure 1: Example of current network with two possibilities for a Dirac sheet on a 2-dimensional lattice with periodic boundary conditions. The shaded (unshaded) area represents a topologically non-trivial (trivial) sheet with given boundary. (b) shifted against (a) by $L/2$ in both directions.
Figure 2: Typical possibilities for intersections of Dirac sheets.
3. Topological analysis

The elements of the fundamental homotopy group $\pi_1(X, b)$ of a space $X$ with base point $b$ are equivalence classes of paths starting and ending at $b$ which can be deformed continuously into each other. Its generators may be obtained embedding a sufficiently dense network $N$ into $X$ and performing suitable transformations which preserve homotopy. We observe that, if a given network $N$ does not wrap around in all directions, then only the generators of a subgroup are produced. This provides an unambiguous characterization of networks.

Choosing one vertex point of $N$ to be the base point $b$ and considering all paths which start and end at $b$, we note that a mapping which shrinks one edge to zero length preserves the homotopy of all of these paths. By a sequence of such mappings we can then shift all other vertices to $b$ without changing the group content until we finally obtain a bouquet of paths which all start and end at $b$.

Describing a path by a vector which is the sum of oriented steps along the path, for $N$ with $K_0$ vertices and $K_1$ edges the bouquet has $K = K_1 - K_0 + 1$ loops, represented by vectors $\mathbf{s}_i$, with $i = 1, \ldots, K$ and $j$-th components $s_{ij} = w_{ij} L_j$, where $L_j$ is the lattice size. The bouquet matrix $w_{ij}$ must then be analyzed for the content of generators of $\pi_1(T^4, b) = \mathbb{Z}^4$.

Current networks have the additional properties of path orientation and current conservation. The maps reducing the bouquet matrix $w_{ij}$, therefore, in addition to homotopy have to respect current conservation. This leads to a modified Gauss elimination procedure, whereby the addition of a row to another requires its simultaneous subtraction from a further row. In this way one arrives at the minimal form, with rows $\mathbf{a}_1, \ldots, \mathbf{a}_r, \mathbf{t}, \mathbf{0}, \ldots, \mathbf{0}$ where $r \leq 4$. The significance of this minimal matrix becomes obvious if one switches to the pair form with rows $\mathbf{a}_i, -\mathbf{a}_i, \ldots, \mathbf{a}_r, -\mathbf{a}_r, \mathbf{f}, \mathbf{0}, \ldots, \mathbf{0}$, which exhibits the relation to the net current flow $\mathbf{f}$ explicitly. The number of independent pairs determines the number of nontrivial directions.

For the topological analysis of minimal Dirac sheets we use the network of plaquettes as a sufficiently dense auxiliary network. The bouquet matrix then is obtained as described before. The reduction of the bouquet matrix is simpler here because the usual Gauss elimination applies, which gives the minimal form with rows $\mathbf{a}_1, \ldots, \mathbf{a}_r, \mathbf{0}, \ldots, \mathbf{0}$. Now the number of independent vectors corresponds to the number of nontrivial directions.
4. Numerical results

Fig. 3 shows our results for the probability $P_{\text{net}}$ to find a current network which is nontrivial in four directions as function of $\lambda$, with $\beta$ chosen at the corresponding phase transition points [3, 4]. For larger $L$ or negative $\lambda$, where the peaks of the energy distribution related to the phases are well separated [3, 4], $P_{\text{net}}$ is seen to be very close to 1 for the hot (confining) phase, and very close to 0 for the cold (Coulomb) phase. We thus see that the two phases have an unambiguous topological characterization, provided by the existence of a nontrivial network in the hot phase and its absence in the cold phase.

For $\lambda = 0$ the events with $\vec{f} \neq 0$ are rare; for lattice sizes $L = 8$ and $L = 16$ their percentage is less than 2% and 0.5%, respectively. For positive $\lambda$ they get slightly more frequent [3]. As is obvious from the pair representation of the bouquet matrix in Section 3, the net current flow $\vec{f}$ is not central for the topological characterization. In this context we should note that the quantity introduced in [3] to classify the networks, because of current conservation, is equal to $\vec{f}$.

For minimizing the number of Dirac plaquettes via a sequence of gauge transformations we have used an annealing technique [3]. Our procedure uses Metropolis sweeps based on a probability distribution $P(\eta) \sim \exp(-\alpha \eta)$, where $\eta = \sum_{x,\mu>\nu} |n_{\mu\nu,x}|$. 

Figure 3: Probability $P_{\text{net}}$ for the occurrence of a nontrivial network in the hot and cold phase at the transition point as function of $\lambda$ for lattice sizes $8^4$ (circles) and $16^4$ (crosses).
implemented through random local gauge transformations with angles maintained within $[-\pi, \pi)$. In subsequent simulations we increase the parameter $\alpha$ in appropriately chosen steps. First we determine $\eta$ as function of $\alpha$ (which changes little with $L$ and $\lambda$) using tiny step sizes and large sweep numbers. Then we continue the procedure with 45 values of $\alpha$ (chosen in the interval $[0.03,100]$ in such a way as to get approximately equal changes in $\eta$), performing up to 100 sweeps for each $\alpha$. To be sure about the results, for each configuration we have generally performed three such runs as well as some additional runs with many more sweeps. Repeated applications of the method, with different random numbers, lead to consistent values for minimal numbers of Dirac plaquettes, with deviations smaller than 2%. Thus our procedure appears to produce a rather reliable determination of the absolute minimum.

We found that the careful annealing described above is absolutely necessary. A minimization of the number of Dirac plaquettes based only on iterative gauge transformations gives strongly fluctuating results, which on the average are roughly factors 2 and 1.3 larger in the cold and hot phase, respectively, than the minimal numbers obtained by our procedure. This is not only inappropriate in principle but also not sufficient for a proper computational identification of the topological structures.

Before applying the annealing procedure the numbers of Dirac plaquettes are similar in both phases, but the numbers become markedly different after annealing. For example, for $L = 8$ and $\lambda = 0$, in typical configurations with 8703 and 8793 plaquettes in the hot and cold phase, respectively, our procedure reduced the numbers to 306 and 846. The stronger reduction in the cold phase is mainly due to the disappearance of closed trivial structures. We also note that the procedure not only affects sheets separately, but involves the full set of them. For example, even pairs of oppositely oriented nontrivial sheets, first observed in [8], disappear with our annealing procedure.

Our analysis of minimal Dirac sheet structures leads to results which agree with those derived from the study of current networks. The hot phase is characterized by the existence of a Dirac sheet which is nontrivial in all directions and the cold phase by its absence. To understand the significance of this result it is important to remember that, given a nontrivial current network, the sheets could be trivial or nontrivial (as illustrated by the example of Fig. 1). Thus our results show that in the hot phase the placement of the largest sheet is one which makes it topologically nontrivial. This points to the remarkable fact that the (equivalence classes of) Dirac sheet structures are the physically relevant objects.
In this context it should be remembered that some time ago Grösch et al. [7] were able to identify nontrivial Dirac sheets without boundaries as the structures responsible for the metastable states occurring under certain conditions in the cold phase. Here we have shown that nontrivial sheet structures with boundaries characterize the hot phase.

To study the role of connectedness, we construct minimal Dirac sheets by making first the connections where only two Dirac plaquettes meet at an edge with appropriate orientation. We note then that making either all or none of the connections where more than two plaquettes meet (with correct orientation) changes the numbers of sheets and networks, and the sizes of the largest sheets, only by 10% or less and does not affect the topology. Looking into more detail we find that, for the most part, these changes are due to intersections of the type of Figs. 2a and 2b, where the sheets must actually be kept separated. Taking this into account the change reduces to 3% or less. We thus obtain the further remarkable result that intersections of minimal sheets are not important.

In Fig. 4 we present the sizes of the largest current network and of the largest minimal Dirac sheet. Obviously the sheets give a more sensitive signal for the different phases. This may be explained by the fact that area instead of length enters. It is to be noted that the definition we have used for the connectedness of
sheets plays an important role in this result. If one considers as connected sheets which have just sites in common, then the signal provided by the size of the largest minimal sheets becomes comparable to the one given by the networks.

Fig. 5 indicates that the probability $P_{\text{net}}$ of the occurrence of a non-trivial network, taking values very close 1 or 0 for hot and cold phase respectively, may be a more sensitive order parameter than $n_{\text{max}}/n_{\text{tot}}$, the relative size of the largest network, advocated in [9].

Figure 5: Order parameters $P_{\text{net}}$ and $n_{\text{max}}/n_{\text{tot}}$ for $\lambda = 0$ as functions of $\beta$ on $8^4$ lattice.
5. Discussion

The success of our topological order parameter confirms that the phase transition of the 4-dimensional compact U(1) gauge field theory coincides with a percolation-type transition of the topological structures. Examples of theories with different behaviors have recently been discussed in [9].

In order to see the general meaning of this results we note that a topologically nontrivial structure on a finite lattice with periodic boundary conditions, i.e. on the torus $T^4$, corresponds to an infinite structure on an infinite lattice. Thus the hot phase is more generally characterized by the existence of an infinite current network or Dirac sheet and the cold phase by its absence, where on finite lattices “infinite” is to be defined in accordance with the particular boundary conditions which are used.

To test this characterization we also performed simulations with open boundary conditions [10]. In the corresponding analysis the prescription “nontrivial in all directions” is replaced by “touching the boundary in all directions”. In this way we again get an unambiguous signal for the phases. The comparison of the order parameters looks very similar to Fig. 5, the advantage of our order parameter being even more pronounced. Because of finite-size effects, however, the width of transition region becomes larger, for $L = 16$, by a factor of approximately 100.

In order to obtain information about the order of the phase transition the most immediate thing to do is to look whether there is an energy gap for sufficiently large $L$. With periodic boundaries $L = 8$ is sufficient for seeing a gap whereas $L = 4$ is not. With open boundaries a gap is not yet seen for $L = 16$. The width of the peak is relatively narrow (comparable to the width of the peaks with $L = 8$ in the periodic-boundary case) [10]. It appears to us that lattices well larger than $L = 16$ will be needed in order to see a possible gap in systems with open boundary conditions.

In [11] the simulations have been performed on the surface of a 5 dimensional cube which is homeomorphic to the sphere $S^4$. Considering lattices up to $L = 10$ the authors observe only one peak. Its width for $L = 10$ is comparable to those of the peaks on $T^4$ for $L = 8$. The extension of the transition region, which could also provide information about finite-size effects, has not been reported.

In [12] simulations for lattices up to $L = 16$ have been done with fixed boundaries (i.e. setting the group elements to $1$ at the boundaries), which makes the lattice again homeomorphic to $S^4$. With these boundary conditions the transition region was observed to be wider by a factor of about $10^3$, which reflects the huge finite-size
effects caused by the strong inhomogeneity of the system. Because of low statistics the results in this work are of semiqualitative nature and the energy distribution is not given.

It seems to us that in the two geometries homeomorphic to $S^4$ mentioned above larger lattices would also be necessary for deciding about the existence of a gap. Further, in these geometries the analysis of configurations of monopole currents and Dirac sheets would be of great interest, too. Appropriate definitions of “infinite” on finite lattices should of course be given. It should be realized that these geometries are different from a physical point of view. Fixed boundary conditions can be expected to lead to the same (flat-space) limit as periodic or open boundary conditions, while for the surface of a 5-dimensional hypercube some curvature (depending on the details of the limit) could persist.

A further type of boundary condition, proposed in [13], consists in the suppression of (all) monopoles at the boundaries. This is done by using an action leading to the suppression of monopoles [14] at the boundaries and the usual Wilson action elsewhere. Though the definition of “infinite” gets somewhat dilute in that case we expect it to fit into our picture as well. The existence of a gap with this type of boundary condition remains to be confirmed.

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