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On the $q$-Bessel Fourier transform

Lazhar Dhaouadi *

Abstract

In this work, we are interested by the $q$-Bessel Fourier transform with a new approach. Many important results of this $q$-integral transform are proved with a new constructive demonstrations and we establish in particular the associated $q$-Fourier-Neumen expansion which involves the $q$-little Jacobi polynomials.

1 Introduction

In the recent mathematical literature one finds many articles which deal with the theory of $q$-Fourier analysis associated with the $q$-Hankel transform. This theory was elaborated first by Koornwinder and R.F. Swarttouw \[12\] and then by Fitouhi and Al \[5, 8\].

It should be noticed that in \[5\] we provided the mains results of $q$-Fourier analysis in particular that the $q$-Hankel transform is extended to the $L_{q,2,\nu}$ space like an isometric operator. Often we use the crucial properties namely the positivity of the $q$-Bessel translation operator to prove some results but these last property is not ensured for any $q$ in the interval $[0, 1]$. Thus, we will prove some main results of $q$-Fourier analysis without the positivity argument especially the following statements:
- Inversion Formula in the $L_{q,p,\nu}$ spaces with $p \geq 1$.
- Plancherel Formula in the $L_{q,p,\nu} \cap L_{q,1,\nu}$ spaces with $p > 2$.
- Plancherel Formula in the $L_{q,2,\nu}$ spaces.

Note that in the paper \[7\] we have proved that the positivity of the $q$-Bessel translation operator is ensured in all points of the interval $]0, 1[$ when $\nu \geq 0$. In this article we will try to show in a clear way the part in which the positivity of the $q$-Bessel translation operator plays a role in $q$-Bessel Fourier analysis. In particular, when we try to prove a $q$-version of the Young’s inequality for the associated convolution.

Many interesting result about the uncertainty principle for the $q$-Bessel transform was proved in the last years. We cite for examples \[2, 3, 4, 9\]. There are some differences of the results cited above and our result:
- In this paper the Heisenberg uncertainty inequality is established for functions in $L_{q,2,\nu}$ space. The Hardy’s inequality discuss here is a quantitative uncertainty principles which give an information about how a function and its $q$-Bessel Fourier transform are linked.

In the end of this paper we use the remarkable work in \[1\] to establish a new result about the $q$-Fourier-Neumen expansion involving the $q$-little Jacobi polynomials.

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2 The $q$-Bessel transform

The reader can see the references [10, 11, 16] about $q$-series theory. The references [5, 8, 12] are devoted to the $q$-Bessel Fourier analysis. Throughout this paper, we consider $0 < q < 1$ and $\nu > -1$. We denote by

$$\mathbb{R}_q^+ = \{ q^n, \ n \in \mathbb{Z} \}.$$ 

The $q$-Bessel operator is defined as follows [5]

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - (1 + q^{2\nu}) f(x) + q^{2\nu} f(qx) \right].$$

The eigenfunction of $\Delta_{q,\nu}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto \mathcal{J}_\nu(\lambda x, q^2)$, where $\mathcal{J}_\nu(., q^2)$ is the normalized $q$-Bessel function defined by [5, 8, 10, 14, 16]

$$\mathcal{J}_\nu(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\nu+2}, q^2)n(q^2, q^2)_n} x^{2n}.$$ 

The $q$-Jackson integral of a function $f$ defined on $\mathbb{R}_q^+$ is

$$\int_0^{\infty} f(t) d_q t = (1 - q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

We denote by $\mathcal{L}_{q,p,\nu}$ the space of functions $f$ defined on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,\nu} = \left( \int_0^{\infty} |f(x)|^p x^{2\nu+1} d_q x \right)^{1/p}$$

exist.

We denote by $\mathcal{C}_{q,0}$ the space of functions defined on $\mathbb{R}_q^+$ tending to 0 as $x \to \infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)|.$$ 

The normalized $q$-Bessel function $\mathcal{J}_\nu(., q^2)$ satisfies the orthogonality relation

$$c_{q,\nu}^2 \int_0^{\infty} \mathcal{J}_\nu(xt, q^2) \mathcal{J}_\nu(yt, q^2) t^{2\nu+1} d_q t = \delta_q(x, y), \quad \forall x, y \in \mathbb{R}_q^+$$

where

$$\delta_q(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{(1-q)x^{2(\nu+1)}} & \text{if } x = y \end{cases}.$$

and

$$c_{q,\nu} = \frac{1}{1-q} \frac{(q^{2\nu+2}, q^2)_{\infty}}{(q^2, q^2)_{\infty}}.$$ 

Let $f$ be a function defined on $\mathbb{R}_q^+$ then

$$\int_0^{\infty} f(y) \delta_q(x, y) y^{2\nu+1} d_q y = f(x).$$

The normalized $q$-Bessel function $\mathcal{J}_\nu(., q^2)$ satisfies

$$|\mathcal{J}_\nu(q^n, q^2)| \leq \frac{(-q^2; q^2)_{\infty}(-q^{2\nu+2}; q^2)_{\infty}}{(q^{2\nu+2}, q^2)_{\infty}} \begin{cases} 1 & \text{if } n \geq 0 \\ q^{-n(2\nu+1)} & \text{if } n < 0 \end{cases}.$$
The q-Bessel Fourier transform $F_{q,\nu}$ is defined by [5] [8] [12]

$$F_{q,\nu}f(x) = c_{q,\nu} \int_0^\infty f(t) j_\nu(xt, q^2 t^{2\nu+1} dt, \quad \forall x \in \mathbb{R}_q^+.$$  

**Proposition 1** Let $f \in \mathcal{L}_{q,1,\nu}$ then $F_{q,\nu} f \in C_{q,0}$ and we have

$$\|F_{q,\nu}(f)\|_{q,\infty} \leq B_{q,\nu}\|f\|_{q,1,\nu}$$

where

$$B_{q,\nu} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2
u+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$  

**Theorem 1** Let $f$ be a function in the $\mathcal{L}_{q,p,\nu}$ space where $p \geq 1$ then

$$F_{q,\nu}^2 f = f.$$  

**Proof.** If $f \in \mathcal{L}_{q,p,\nu}$ then $F_{q,\nu} f$ exist, and we have

$$F_{q,\nu}^2 f(x) = c_{q,\nu} \int_0^\infty F_{q,\nu} f(t) j_\nu(xt, q^2 t^{2\nu+1} dt \right. \left. t^{2\nu+1} dt \right. \frac{y^{2\nu+1} d_q y}{y^{2\nu+1} d_q t} = \int_0^\infty f(y) \delta_q(x, y) y^{2\nu+1} d_q y = f(x).$$

The computations are justified by the Fubuni’s theorem: If $p > 1$ then we use the Hölder’s inequality

$$\int_0^\infty |f(y)| \left[ \int_0^\infty |j_\nu(xt, q^2 t^{2\nu+1} dt \right. \left. t^{2\nu+1} dt \right. \frac{y^{2\nu+1} d_q y}{y^{2\nu+1} d_q t} = \left[ \int_0^\infty |f(y)|^p y^{2\nu+1} d_q y \right]^{1/p} \times \left[ \int_0^\infty \sigma(y)^p y^{2\nu+1} d_q y \right]^{1/p}.\right.$$  

The numbers $p$ and $\bar{p}$ above are conjugates and

$$\sigma(y) = \int_0^\infty |j_\nu(xt, q^2 t^{2\nu+1} dt, \quad \text{then}$$

$$\int_0^\infty \sigma(y)^p y^{2\nu+1} d_q y$$

$$= \int_1^\infty \sigma(y)^p y^{2\nu+1} d_q y + \int_1^\infty \sigma(y)^p y^{2\nu+1} d_q y.$$  

Note that

$$\int_0^1 \sigma(y)^p y^{2\nu+1} d_q y$$

$$\leq \|j_\nu(\cdot, q^2)\|_{q,\infty} \int_0^1 \left[ \int_0^\infty |j_\nu(xt, q^2 t^{2\nu+1} dt \right. \left. t^{2\nu+1} dt \right. \frac{y^{2\nu+1} d_q y}{y^{2\nu+1} d_q y}$$

$$\leq \|j_\nu(\cdot, q^2)\|_{q,\infty} \|j_\nu(\cdot, q^2)\|_{q,1,\nu}^{-2(\nu+1)} \int_0^1 y^{2\nu+1} d_q y < \infty,$$
and
\[ \int_1^\infty \sigma(y) y^{2\nu+1} dy \]
\[ \leq \|j_\nu(., q^2)\|_{q,1,\nu} \|j_\nu(., q^2)\|_{q,1,\nu} \int_1^\infty \frac{y^{2\nu+1}}{y^{2(\nu+1)p}} dy \]
\[ \leq \|j_\nu(., q^2)\|_{q,1,\nu} \|j_\nu(., q^2)\|_{q,1,\nu} \int_1^\infty \frac{1}{y^{2(\nu+1)(p-1)+1}} dy < \infty. \]

If \( p = 1 \) then
\[ \int_0^\infty \|f(y)\| \left[ \int_0^\infty |j_\nu(xt, q^2)j_\nu(yt, q^2)| t^{2\nu+1} d_q t \right] y^{2\nu+1} d_q y \]
\[ \leq \|f\|_{q,1,\nu} \|j_\nu(., q^2)\|_{q,\infty} \|j_\nu(., q^2)\|_{q,1,\nu} \frac{1}{x^{2(\nu+1)}}. \]

**Theorem 2** Let \( f \) be a function in the \( L_{q,1,\nu} \cap L_{q,p,\nu} \) space, where \( p > 2 \) then
\[ \|F_{q,\nu}f\|_{q,2,\nu} = \|f\|_{q,2,\nu}. \]

**Proof.** Let \( f \in L_{q,1,\nu} \cap L_{q,p,\nu} \) then by Theorem 1 we see that
\[ F_{q,\nu}^2 f = f. \]

This implies
\[ \int_0^\infty F_{q,\nu}^2 f(x)^2 x^{2\nu+1} d_q x = \int_0^\infty F_{q,\nu} f(x) \left[ c_{q,\nu} \int_0^\infty f(t) j_\nu(xt, q^2) t^{2\nu+1} d_q t \right] x^{2\nu+1} d_q x \]
\[ = \int_0^\infty f(t) \left[ c_{q,\nu} \int_0^\infty F_{q,\nu} f(x) j_\nu(xt, q^2) x^{2\nu+1} d_q x \right] t^{2\nu+1} d_q t \]
\[ = \int_0^\infty f(t)^2 t^{2\nu+1} d_q t. \]

The computations are justified by the Fubuni’s theorem
\[ \int_0^\infty |f(t)| \left[ c_{q,\nu} \int_0^\infty |F_{q,\nu} f(x)||j_\nu(xt, q^2)| x^{2\nu+1} d_q x \right] t^{2\nu+1} d_q t \]
\[ \leq \left[ \int_0^\infty |f(t)|^p t^{2\nu+1} d_q t \right]^{1/p} \times \left[ \int_0^\infty |\phi(t)|^{p/\nu} t^{2\nu+1} d_q t \right]^{1/p}, \]
where
\[ \phi(t) = c_{q,\nu} \int_0^\infty |F_{q,\nu} f(x)||j_\nu(xt, q^2)| x^{2\nu+1} d_q x, \]
then
\[ \|F_{q,\nu} f(x)\| \leq c_{q,\nu} \int_0^\infty |f(y)||j_\nu(xy, q^2)| y^{2\nu+1} d_q y \]
\[ \leq c_{q,\nu} \left[ \int_0^\infty |f(y)|^p y^{2\nu+1} d_q y \right]^{1/p} \times \left[ \int_0^\infty |j_\nu(xy, q^2)|^{p/\nu} y^{2\nu+1} d_q y \right]^{1/p} \]
\[ \leq c_{q,\nu} \left[ \int_0^\infty |f(y)|^p y^{2\nu+1} d_q y \right]^{1/p} \times \left[ \int_0^\infty |j_\nu(y, q^2)|^{p/\nu} y^{2\nu+1} d_q y \right]^{1/p} x^{-2(\nu+1)/p} \]
\[ \leq c_{q,\nu} \|f\|_{q,p,\nu} \|j_\nu(., q^2)\|_{q,\nu} x^{-2(\nu+1)/p}. \]
This gives
\[
\phi(t) \leq c_{q,\nu} \left\| f \right\|_{q,\nu} \left\| j_{\nu} \left( \cdot, q^2 \right) \right\|_{q,\nu} \int_0^\infty \left| j_{\nu}(xt, q^2) \right| t^{(2\nu+1) - 2(\nu+1)/p} dq
\]
\[
\leq c_{q,\nu} \left\| f \right\|_{q,\nu} \left\| j_{\nu} \left( \cdot, q^2 \right) \right\|_{q,\nu} \left[ \int_0^\infty \left| j_{\nu}(x, q^2) \right| t^{-2(\nu+1)/p} dq \right] t^{2(\nu+1)/p - 2(\nu+1)}
\]
\[
\leq C_1 t^{-2(\nu+1)/p},
\]
and
\[
\phi(t) = c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x)| \left\| j_{\nu}(xt, q^2) \right\|_{q,\nu} x^{2(\nu+1)/p} dx
\]
\[
= \left[ c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(x/t)| \left\| j_{\nu}(x, q^2) \right\|_{q,\nu} x^{2(\nu+1)/p} dx \right] t^{-2(\nu+1)}
\]
\[
\leq c_{q,\nu} \left\| \mathcal{F}_{q,\nu} f \right\|_{q,\nu} \times \left\| j_{\nu} \left( \cdot, q^2 \right) \right\|_{q,\nu} \times t^{-2(\nu+1)}
\]
\[
\leq C_2 t^{-2(\nu+1)}.
\]

Note that
\[
\{ -1 < -2(\nu+1)\frac{p}{\nu} + 2\nu + 1 \} \Leftrightarrow \{ 0 < -2(\nu+1)(\frac{p}{\nu} - 2) \} \Leftrightarrow \{ -2(\nu+1)(\frac{\nu}{\nu} - 1) < 0 \} \Leftrightarrow 1 < \frac{\nu}{\nu} < 2 \Leftrightarrow p > 2.
\]

Hence
\[
\int_0^\infty |\phi(t)| \frac{t^{2\nu+1}}{\nu} dq \leq \int_0^1 |\phi(t)| \frac{t^{2\nu+1}}{\nu} dq + \int_1^\infty |\phi(t)| \frac{t^{2\nu+1}}{\nu} dq \leq C_1 \int_0^1 t^{-2(\nu+1)/\nu} \frac{t^{2\nu+1}}{\nu} dq + C_2 \int_1^\infty t^{-2(\nu+1)/\nu} \frac{t^{2\nu+1}}{\nu} dq < \infty,
\]
which prove the result.

**Theorem 3** Let \( f \) be a function in the \( L_{q,\nu} \) space then
\[
\left\| \mathcal{F}_{q,\nu} f \right\|_{q,\nu} = \left\| f \right\|_{q,\nu}.
\]

**Proof.** We introduce the function \( \psi_x \) as follows
\[
\psi_x(t) = c_{q,\nu} j_{\nu}(tx, q^2).
\]

The inner product \( \langle \cdot, \cdot \rangle \) in the Hilbert space \( L_{q,\nu} \) is defined by
\[
f, g \in L_{q,\nu} \Rightarrow \langle f, g \rangle = \int_0^\infty f(t)g(t)t^{2\nu+1} dq.
\]

Using (1) we write
\[
x \neq y \Rightarrow \langle \psi_x, \psi_y \rangle = 0
\]
\[
\left\| \psi_x \right\|^2_{q,\nu} = \frac{1}{1-q} x^{-2(\nu+1)}.
\]

We have
\[
\mathcal{F}_{q,\nu} f(x) = \langle f, \psi_x \rangle,
\]
which prove the result.

\[\boxed\]
and by Theorem 1
\[ f \in L_{q,2,\nu} \Rightarrow F_{q,\nu}^2 f = f, \]
then
\[ \langle f, \psi_x \rangle = 0, \forall x \in \mathbb{R}_q^+ \Rightarrow F_{q,\nu} f(x) = 0, \forall x \in \mathbb{R}_q^+ \Rightarrow f = 0. \]
Hence, \{\psi_x, x \in \mathbb{R}_q^+\} form an orthogonal basis of the Hilbert space \( L_{q,2,\nu} \) and we have
\[ \{\psi_x, \forall x \in \mathbb{R}_q^+\} = L_{q,2,\nu}. \]
Now
\[ f \in L_{q,2,\nu} \Rightarrow f = \sum_{x \in \mathbb{R}_q^+} \frac{1}{\|\psi_x\|^2_{q,2,\nu}} \langle f, \psi_x \rangle \psi_x, \]
and then
\[ \|f\|^2_{q,2,\nu} = \sum_{x \in \mathbb{R}_q^+} \frac{1}{\|\psi_x\|^2_{q,2,\nu}} \langle f, \psi_x \rangle^2 = (1 - q) \sum_{x \in \mathbb{R}_q^+} x^{2(\nu+1)} F_{q,\nu} f(x)^2 = \|F_{q,\nu} f\|^2_{q,2,\nu}, \]
which achieve the proof. ■

**Proposition 2** Let \( f \in L_{q,p,\nu} \) where \( p \geq 1 \) then \( F_{q,\nu} f \in L_{q,p,\nu} \). If \( 1 \leq p \leq 2 \) then
\[ \|F_{q,\nu} f\|_{q,p,\nu} \leq B_{q,\nu}^{\frac{2}{p}-1} \|f\|_{q,p,\nu}. \]  \( (4) \)
**Proof.** This is an immediate consequence of Proposition 3, Theorem 3, the Riesz-Thorin theorem and the inversion formula (2). ■

The \( q \)-translation operator is given as follow
\[ T_{q,x}^\nu f(y) = c_{q,\nu} \int_0^\infty F_{q,\nu} f(t) j_\nu(yt, q^2) j_\nu(xt, q^2) t^{2\nu+1} dt. \]
Let us now introduce
\[ Q_\nu = \{ q \in [0,1[, \ T_{q,x}^\nu \text{ is positive for all } x \in \mathbb{R}_q^+ \} \]
the set of the positivity of \( T_{q,x}^\nu \). We recall that \( T_{q,x}^\nu \) is called positive if \( T_{q,x}^\nu f \geq 0 \) for \( f \geq 0 \).
In a recent paper [6] it was proved that if \(-1 < \nu < \nu' \) then \( Q_\nu \subset Q_{\nu'} \). As a consequence :
- If \( 0 \leq \nu \) then \( Q_\nu = [0,1[. \)
- If \( -\frac{1}{2} \leq \nu < 0 \) then \([0,q_0] \subset Q_{-\frac{1}{2}} \subset Q_\nu \subset [0,1[, \ q_0 \simeq 0.43. \)
- If \( -1 < \nu \leq -\frac{1}{2} \) then \( Q_\nu \subset Q_{-\frac{1}{2}}. \)
Theorem 4 Let \( f \in \mathcal{L}_{q,p,\nu} \) then \( T_{q,x}^\nu f \) exists and we have
\[
\int_0^\infty T_{q,x}^\nu f(y)y^{2\nu+1} d_q y = \int_0^\infty f(y)y^{2\nu+1} d_q y.
\]
and
\[
T_{q,x}^\nu f(y) = \int_0^\infty f(z)D_\nu(x,y,z)z^{2\nu+1} d_q z,
\]
where
\[
D_\nu(x,y,z) = \frac{c_{q,\nu}^2}{c_{q,\nu}} \int_0^\infty j_\nu(xs,q^2 j_\nu(ys,q^2 j_\nu(zs,q^2 s)^{2\nu+1} d_q s.
\]
If we suppose that \( T_{q,x}^\nu \) is a positive operator then for all \( p \geq 1 \) we have
\[
\|T_{q,x}^\nu f\|_{q,p,\nu} \leq \|f\|_{q,p,\nu}.
\]
Proof. We write the operator \( T_{q,x}^\nu \) in the following form
\[
T_{q,x}^\nu f(y) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(z)j_\nu(xz,q^2)j_\nu(yz,q^2)z^{2\nu+1} d_q z
\]
\[
= \mathcal{F}_{q,\nu} \left[ \mathcal{F}_{q,\nu} f(z)j_\nu(xz,q^2) \right](y).
\]
So we have
\[
\int_0^\infty T_{q,x}^\nu f(y)y^{2\nu+1} d_q y = \int_0^\infty \mathcal{F}_{q,\nu} \left[ \mathcal{F}_{q,\nu} f(z)j_\nu(xz,q^2) \right](y)y^{2\nu+1} d_q y
\]
\[
= \frac{1}{c_{q,\nu}} c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} \left[ \mathcal{F}_{q,\nu} f(z)j_\nu(xz,q^2) \right](y)j_\nu(0,q^2)y^{2\nu+1} d_q y
\]
\[
= \frac{1}{c_{q,\nu}} \int_0^\infty \mathcal{F}_{q,\nu} f(0)
\]
\[
= \int_0^\infty f(y)y^{2\nu+1} d_q y.
\]
On the other hand
\[
T_{q,x}^\nu f(y) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(z)j_\nu(xz,q^2)j_\nu(yz,q^2)z^{2\nu+1} d_q z
\]
\[
= c_{q,\nu} \int_0^\infty \left[ c_{q,\nu} \int_0^\infty f(t)j_\nu(tz,q^2)t^{2\nu+1} dt \right] j_\nu(xz,q^2)j_\nu(yz,q^2)z^{2\nu+1} d_q z
\]
\[
= \int_0^\infty \left[ c_{q,\nu}^2 \int_0^\infty j_\nu(xz,q^2)j_\nu(yz,q^2)j_\nu(tz,q^2)z^{2\nu+1} d_q z \right] f(t)t^{2\nu+1} dt
\]
\[
= \int_0^\infty D_{q,\nu}(x,y,t)f(t)t^{2\nu+1} dt.
\]
The computations are justified by the Fubuni’s theorem
\[
\int_0^\infty \left[ \int_0^\infty |f(t)| |j_\nu(tz,q^2)| t^{2\nu+1} dt \right] |j_\nu(xz,q^2)| |j_\nu(yz,q^2)| z^{2\nu+1} d_q z
\]
\[
\leq \|f\|_{q,p,\nu}\int_0^\infty \left[ \int_0^\infty |j_\nu(tz,q^2)| t^{2\nu+1} dt \right] \|j_\nu(xz,q^2)| |j_\nu(yz,q^2)| z^{2\nu+1} d_q z
\]
\[
\leq \|f\|_{q,p,\nu}\|j_\nu(.,q^2)\|_{q,p,\nu}\int_0^\infty |j_\nu(xz,q^2)| |j_\nu(yz,q^2)| z^{2(\nu+1)}(1-\nu)^{-1} d_q z.
\]
Now suppose that $T_{q,x}^\nu$ is positive. Given a function $f \in C_{q,0}$ we obtains
\[
|T_{q,x}^\nu f(y)| = \left| \int_0^\infty D_{q,\nu}(x, y, t) f(t) t^{2\nu+1} d_q t \right|
\leq \int_0^\infty |D_{q,\nu}(x, y, t)| |f(t)| t^{2\nu+1} d_q t
\leq \left[ \int_0^\infty D_{q,\nu}(x, y, t) t^{2\nu+1} d_q t \right] \|f\|_{q,\infty} = \|f\|_{q,\infty}
\]
which implies
\[
\|T_{q,x}^\nu f\|_{q,\infty} \leq \|f\|_{q,\infty}.
\]
If the function $f \in L_{q,1,\nu}$ then we obtains
\[
\|T_{q,x}^\nu f\|_{q,1,\nu} = \int_0^\infty \left| T_{q,x}^\nu f(y) \right| y^{2\nu+1} d_q y
\leq \int_0^\infty \left[ \int_0^\infty |D_{q,\nu}(x, y, t)| |f(t)| t^{2\nu+1} d_q t \right] y^{2\nu+1} d_q y
\leq \int_0^\infty \left[ \int_0^\infty D_{q,\nu}(x, y, t) y^{2\nu+1} d_q y \right] |f(t)| t^{2\nu+1} d_q t
\leq \int_0^\infty |f(t)| t^{2\nu+1} d_q t = \|f\|_{q,1,\nu}.
\]
The result is a consequence of the Riesz-Thorin theorem.

Notice that the kernel $D_{q,\nu}(x, y, t)$ can be written as follows
\[
D_{q,\nu}(x, y, t) = c_{q,\nu}^2 \int_0^\infty j_\nu(xz, q^2) j_\nu(yz, q^2) j_\nu(tz, q^2) z^{2\nu+1} d_q z
= c_{q,\nu} F_{q,\nu} \left[ j_\nu(xz, q^2) j_\nu(yz, q^2) \right] (t),
\]
which implies
\[
\int_0^\infty D_{q,\nu}(x, y, t) t^{2\nu+1} d_q t = c_{q,\nu} F_{q,\nu} \left[ j_\nu(xz, q^2) j_\nu(yz, q^2) \right] (t) t^{2\nu+1} d_q t
= F_{q,\nu}^2 \left[ j_\nu(xz, q^2) j_\nu(yz, q^2) \right] (0) = 1.
\]

The $q$-convolution product is defined by
\[
f *_q g = F_{q,\nu} [F_{q,\nu} f \times F_{q,\nu} g].
\]

**Theorem 5** Let $1 \leq p, r, s$ such that
\[
\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}
\]
Given two functions $f \in L_{q,p,\nu}$ and $g \in L_{q,r,\nu}$ then $f *_q g$ exists and we have
\[
f *_q g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(y) g(y) y^{2\nu+1} d_q y.
\]
and

\[ f \ast_q g \in \mathcal{L}_{q,s,\nu}. \]

\[ \mathcal{F}_{q,\nu}(f \ast_q g) = \mathcal{F}_{q,\nu}(f) \times \mathcal{F}_{q,\nu}(g). \]

If \( s \geq 2 \) then

\[ \| f \ast_q g \|_{q,s,\nu} \leq B_{q,\nu} \| f \|_{q,p,\nu} \| g \|_{q,r,\nu}. \]  

(6)

If we suppose that \( T_{q,\nu}^\nu \) is a positive operator then

\[ \| f \ast_q g \|_{q,s,\nu} \leq c_{q,\nu} \| f \|_{q,p,\nu} \| g \|_{q,r,\nu}. \]  

(7)

**Proof.** We have

\[
 f \ast_q g(x) = \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g](x)
\]
\[
 = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu}f(y) \times \mathcal{F}_{q,\nu}g(y) j_{\nu}(xy, q^2) y^{2\nu+1} d_y
\]
\[
 = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu}f(y) \times \left[ c_{q,\nu} \int_0^\infty g(z) j_{\nu}(zy, q^2) z^{2\nu+1} d_z \right] j_{\nu}(xy, q^2) y^{2\nu+1} d_y
\]
\[
 = c_{q,\nu} \int_0^\infty \left[ c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu}f(y) j_{\nu}(zy, q^2) j_{\nu}(xy, q^2) y^{2\nu+1} d_y \right] g(z) z^{2\nu+1} d_z
\]
\[
 = c_{q,\nu} \int_0^\infty T_{q,\nu}^\nu f(z) g(z) z^{2\nu+1} d_z.
\]

The computations are justified by the Fubini’s theorem

\[
 \int_0^\infty |\mathcal{F}_{q,\nu}f(y)| \times \left[ \int_0^\infty |g(z)| \times |j_{\nu}(zy, q^2)| z^{2\nu+1} d_z \right] |j_{\nu}(xy, q^2)| y^{2\nu+1} d_y \leq \|g\|_{q,r,\nu} \int_0^\infty |\mathcal{F}_{q,\nu}f(y)| \times \left[ \int_0^\infty |j_{\nu}(zy, q^2)| z^{2\nu+1} d_z \right] \frac{1}{y^{2\nu+2}} y^{2\nu+1} d_y
\]
\[
 \leq \|g\|_{q,r,\nu} \|j_{\nu}(., q^2)\|_{q,\pi,\nu} \int_0^\infty \mathcal{F}_{q,\nu}f(y) \times \left[ |j_{\nu}(xy, q^2)| y^{\frac{2\nu+2}{p}} \right] y^{2\nu+1} d_y
\]
\[
 \leq \|g\|_{q,r,\nu} \|j_{\nu}(., q^2)\|_{q,\pi,\nu} \|\mathcal{F}_{q,\nu}f\|_{q,\pi,\nu} \left( \int_0^\infty \left[ |j_{\nu}(xy, q^2)| y^{\frac{2\nu+2}{p}} \right] y^{2\nu+1} d_y \right)^\frac{1}{p}
\]
\[
 \leq \|g\|_{q,r,\nu} \|j_{\nu}(., q^2)\|_{q,\pi,\nu} \|\mathcal{F}_{q,\nu}f\|_{q,\pi,\nu} \left( \int_0^\infty |j_{\nu}(xy, q^2)| y^{2(\nu+1)(1-\frac{1}{p})-1} d_y \right)^\frac{1}{p}.
\]

From Proposition 2 we deduce that

\[ \mathcal{F}_{q,\nu}f \in \mathcal{L}_{q,\pi,\nu} \text{ and } \mathcal{F}_{q,\nu}g \in \mathcal{L}_{q,\pi,\nu}. \]

Then, using the H"older inequality and the fact that

\[ \frac{1}{p} + \frac{1}{r} = \frac{1}{s} \]

to conclude that

\[ \mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g \in \mathcal{L}_{q,\pi,\nu}. \]

Which implies that

\[ f \ast_q g = \mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g] \in \mathcal{L}_{q,s,\nu}. \]
and by the inversion formula (2) we obtain

\[ \mathcal{F}_{q,\nu}(f \ast_q g) = \mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g. \]

Suppose that \( s \geq 2 \), so \( 1 \leq \frac{s}{q} \leq 2 \) and we can write

\[
\|f \ast_q g\|_{q,s,\nu} = \|\mathcal{F}_{q,\nu}[\mathcal{F}_{q,\nu}f \times \mathcal{F}_{q,\nu}g]\|_{q,s,\nu}
\leq B_{q,\nu}^{-1} \|\mathcal{F}_{q,\nu}f\|_{q,\nu} \|\mathcal{F}_{q,\nu}g\|_{q,\nu}
\leq B_{q,\nu}^{-1} B_{q,\nu}^{-1} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}
\leq B_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}.
\]

Now suppose that \( T_{q,x}^\nu \) is a positive operator.
We introduce the operator \( K_f \) as follows

\[
K_f g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(z)g(z)z^{2\nu+1}d_qz.
\]

By the Hölder inequality and (5) we get

\[
\|K_f g\|_{q,\infty} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,\nu}.
\]

The Minkowski inequality leads to

\[
\|K_f g\|_{q,p,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,1,\nu}.
\]

Hence we have

\[
K_f : \mathcal{L}_{q,\nu} \rightarrow \mathcal{C}_{q,0}, \quad \tilde{K}_f : \mathcal{L}_{q,1,\nu} \rightarrow \mathcal{L}_{q,p,\nu}.
\]

Then the operator \( K_f \) satisfies

\[
K_f : \mathcal{L}_{q,r,\nu} \rightarrow \mathcal{L}_{q,s,\nu}
\]

and

\[
\|f \ast_q g\|_{q,s,\nu} = \|K_f g\|_{q,s,\nu} \leq c_{q,\nu} \|f\|_{q,p,\nu} \|g\|_{q,r,\nu}.
\]

\[ \blacksquare \]

**Remark 1** We discuss here the sharp results for the Hausdorff-Young inequality provided above. An inequality already sharper than (6) is given in formula (7). In fact we have \( c_{q,\nu} < B_{q,\nu} \).

To obtained (7) without the positivity argument, we can do by using which is a \( q \)-Riemann-Liouville fractional integral generalizing the \( q \)-Mehler integral representation for the \( q \)-Bessel function \( j_\nu(.,q^2) \) which can be proved in a straightforward way \[8\]

\[
j_\nu(\lambda, q^2) = [2\nu]_q \int_0^1 (q^2t^2, q^2)_{\infty} j_0(\lambda t, q^2)t d_qt
\]

together with the inequalities for the \( q \)-Bessel function which is given as formula (24) in the paper \[4\]

\[
|j_0(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+.
\]

Combine this formulas we arrive at

\[
|j_\nu(x; q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+, \quad \nu \geq 0.
\]
Then the inequalities (4) can be written as follows

\[ \|F_{q,\nu}f\|_{q,\nu} \leq c_{q,\nu}^2 \|f\|_{q,\nu}. \]

This should give the sharpest version of (6) in the cases \( \nu \geq 0 \). Unfortunately the positivity of the operator \( T_{q,x}^\nu \) is satisfied in this case. In fact we can prove that if we are in the positivity cases then

\[ \|j_\nu(.,q^2)\|_{q,\nu} \leq 1. \]

To prove this recalling that

\[ T_{q,x}^\nu j_\nu(y,q^2) = j_\nu(x,q^2)j_\nu(y,q^2). \]

So we have

\[ \int_0^\infty D_\nu(x,y,t)j_\nu(t,q^2)t^{2\nu+1}d_qt = j_\nu(x,q^2)j_\nu(y,q^2). \]

We obtain for all \( x,y \in \mathbb{R}_q^+ \)

\[ |j_\nu(x,q^2)| \times |j_\nu(y,q^2)| \leq \int_0^\infty D_\nu(x,y,t) |j_\nu(t,q^2)| t^{2\nu+1}d_qt \leq \left[ \int_0^\infty D_\nu(x,y,t)t^{2\nu+1}d_qt \right] \|j_\nu(.,q^2)\|_{q,\nu}. \]

The fact that

\[ \int_0^\infty D_\nu(x,y,t)t^{2\nu+1}d_qt = 1 \]

implies

\[ \|j_\nu(.,q^2)\|^2_{q,\nu} \leq \|j_\nu(.,q^2)\|_{q,\nu}. \]

which gives the result.

3 Uncertainty principle

We introduce two \( q \)-difference operators

\[ \partial_q f(x) = \frac{f(q^{-1}x) - f(x)}{x} \]

and

\[ \partial_q^* f(x) = \frac{f(x) - q^{2\nu+1}f(qx)}{x}. \]

Then we have

\[ \partial_q \partial_q^* f(x) = \partial_q \partial_q f(x) = \Delta_{q,\nu} f(x). \]

**Proposition 3** If \( \langle \partial_q f, g \rangle \) exist and \( \lim_{a \to \infty} |a^{2\nu+1} f(q^{-1}a)g(a)| = 0 \) then

\[ \langle \partial_q f, g \rangle = -\langle f, \partial_q^* g \rangle. \]
**Proof.** The following computation

\[
\int_0^a \partial_q f(x) g(x) x^{2\nu+1} d_q x = \int_0^a \frac{f(q^{-1}x) - f(x)}{x} g(x) x^{2\nu+1} d_q x
\]

\[
= \int_0^a \frac{f(q^{-1}x)}{x} g(x) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} g(x) x^{2\nu+1} d_q x
\]

\[
= q^{2\nu+1} \int_0^a \frac{f(q^{-1}x)}{x} g(x) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} \partial_q g(x) x^{2\nu+1} d_q x
\]

\[
= q^{2\nu+1} \int_0^a \frac{f(x)}{x} \partial_q g(qx) x^{2\nu+1} d_q x - \int_0^a \frac{f(x)}{x} g(x) x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a)
\]

\[
= - \int_0^a \frac{f(x)}{x} g(x) - q^{2\nu+1} g(qx) x^{2\nu+1} d_q x + a^{2\nu+1} f(q^{-1}a) g(a)
\]

leads to the result. ■

**Corollary 1** If \( f \in L_{q,2,\nu} \) such that \( x F_{q,\nu} f \in L_{q,2,\nu} \) then

\[\|\partial_q f\|_2 = \|x F_{q,\nu} f\|_2.\]

**Proof.** In fact we have

\[\|\partial_q f\|_2^2 = \langle \partial_q f, \partial_q f \rangle = - \langle f, \partial_q^* \partial_q f \rangle = - \langle f, \Delta_{q,\nu} f \rangle = - \langle F_{q,\nu} f, F_{q,\nu} \Delta_{q,\nu} f \rangle = \langle F_{q,\nu} f, x^2 F_{q,\nu} f \rangle = \|x F_{q,\nu} f\|_2^2,\]

which prove the result. ■

**Theorem 6** Assume that \( f \) belongs to the space \( L_{q,2,\nu} \). Then the \( q \)-Bessel transform satisfies the following uncertainty principal

\[\|f\|_2^2 \leq k_{q,\nu} \|xf\|_2 \|xF_{q,\nu} f\|_2\]

where

\[k_{q,\nu} = \frac{1 + \sqrt{q} \times q^{\nu+1}}{1 - q^{2(\nu+1)}}.\]

**Proof.** In fact

\[\partial_q^* x f = f(x) - q^{2\nu+2} f(qx)\]

\[x \partial_q f = f(q^{-1}x) - f(x)\].
We introduce the following operator
\[ \Lambda_q f(x) = f(qx), \]
then
\[ \langle \Lambda_q f, g \rangle = q^{-2(\nu+1)} \langle f, \Lambda_q^{-1} g \rangle. \]
So
\[ \frac{1}{1 - q^{2(\nu+1)}} \left[ \partial_q^n xf(x) - q^{2\nu+2} \Lambda_q x \partial_q f(x) \right] = f(x) \]
Assume that \( xf \) and \( xF_{q,\nu} f \) belongs to the space \( L_{q,2,\nu} \). Then we have
\[ \langle f, f \rangle = -\frac{1}{1 - q^{2(\nu+1)}} \| x f \|_2 \| \partial_q f \|_2 + \frac{1}{1 - q^{2(\nu+1)}} \| \partial_q f \|_2 \| x \Lambda_q^{-1} f \|_2. \]
By Cauchy-Schwartz inequality we get
\[ \langle f, f \rangle \leq \frac{1}{1 - q^{2(\nu+1)}} \| x f \|_2 \| \partial_q f \|_2 + \frac{1}{1 - q^{2(\nu+1)}} \| \partial_q f \|_2 \| x \Lambda_q^{-1} f \|_2^2. \]
On the other hand
\[ \| x \Lambda_q^{-1} f \|_2 = \sqrt{q} \times q^{\nu+1} \| x f \|_2, \]
Corollary \( \blacksquare \) leads to the result.

4 Hardy’s theorem

The following Lemma from complex analysis is crucial for the proof of our main theorem.

**Lemma 1** For every \( p \in \mathbb{N} \), there exist \( \sigma_p > 0 \) for which
\[ |z|^{2p} |j_\nu(z, q^2)| < \sigma_p e^{|z|}, \quad \forall z \in \mathbb{C}. \]

**Proof.** In fact
\[ |z|^{2p} |j_\nu(z, q^2)| \leq \frac{1}{(q^2, q^2)^{\infty}(q^{2\nu+2}, q^2)^{\infty}} \sum_{n=0}^{\infty} q^{n(n-1)} |z|^{2n+2p} \]
\[ \leq \frac{q^{p(p+1)}}{(q^2, q^2)^{\infty}(q^{2\nu+2}, q^2)^{\infty}} \sum_{n=p}^{\infty} q^{n(n-2p-1)} |z|^{2n}. \]
Now using the Stirling’s formula
\[ n! \sim \sqrt{2\pi n} n^n e^{-n}, \]
we see that there exist an entire \( n_0 \geq p \) such that
\[ q^{n(n-2p-1)} < \frac{1}{(2n)!}, \quad \forall n \geq n_0, \]
which implies
\[ \sum_{n=n_0}^{\infty} q^{n(n-2p-1)} |z|^{2n} < \sum_{n=n_0}^{\infty} \frac{1}{(2n)!} |z|^{2n} < e^{|z|}. \]
Finally there exist $\sigma_p > 0$ such that

$$\frac{|z|^{2p}|j_\nu(z, q^2)|}{e^{|z|}} < \sigma_p, \quad \forall z \in \mathbb{C}$$

This complete the proof. ■

**Lemma 2** Let $h$ be an entire function on $\mathbb{C}$ such that

$$|h(z)| \leq Ce^{a|z|^2}, \quad z \in \mathbb{C},$$

$$|h(x)| \leq Ce^{-ax^2}, \quad x \in \mathbb{R},$$

for some positive constants $a$ and $C$. Then there exist $C^* \in \mathbb{R}$ such

$$h(x) = C^* e^{-ax^2}.$$

The reader can see the reference [17] for the proof.

Now we are in a position to state and prove the $q$-analogue of the Hardy’s theorem

**Theorem 7** Suppose $f \in L_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R}_q^+, \quad (8)$$

$$|\mathcal{F}_{q,\nu} f(x)| \leq Ce^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R},$$

where $C$ is a positive constant. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu} \left( e^{-\frac{1}{2}x^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

**Proof.** We claim that $\mathcal{F}_{q,\nu} f$ is an analytic function and there exist $C' > 0$ such that

$$|\mathcal{F}_{q,\nu} f(z)| \leq C'e^{\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{C}.$$ 

We have

$$|\mathcal{F}_{q,\nu} f(z)| \leq c_{q,\nu} \int_0^\infty |f(x)||j_\nu(zx, q^2)|x^{2\nu+1}d_qx.$$ 

From the Lemma [11] if $|z| > 1$ then there exist $\sigma_1 > 0$ such that

$$x^{2\nu+1}|j_\nu(zx, q^2)| = \frac{1}{|z|^{2\nu+1}} \left( |z| x \right)^{2\nu+1} |j_\nu(zx, q^2)| < \frac{\sigma_1}{1 + |z|^2x^2} e^{|z|^2}, \quad \forall x \in \mathbb{R}_q^+.$$

Then we obtain

$$|\mathcal{F}_{q,\nu} f(z)| \leq C\sigma_1 c_{q,\nu} \left[ \int_0^\infty \frac{1}{1 + |z|^2x^2} e^{\frac{1}{2}|z|^2} \right] e^{|z|^2} < C\sigma_1 c_{q,\nu} \left[ \int_0^\infty \frac{1}{1 + x^2} d_qx \right] e^{|z|^2}.$$ 

Now, if $|z| \leq 1$ then there exist $\sigma_2 > 0$ such that

$$x^{2\nu+1}|j_\nu(zx, q^2)| \leq \sigma_2 e^x, \quad \forall x \in \mathbb{R}_q^+.$$

14
Therefore
\[ |\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_2c_{q,\nu}\left[ \int_0^\infty e^{-\frac{1}{2}x^2+x}d_qx \right] \leq C\sigma_2c_{q,\nu}\left[ \int_0^\infty e^{-\frac{1}{2}x^2+x}d_qx \right] e^{\frac{1}{2}|z|^2}, \]
which leads to the estimate (8). Using Lemma 2, we obtain
\[ \mathcal{F}_{q,\nu}f(z) = \text{const}.e^{-\frac{1}{2}x^2}, \quad \forall z \in \mathbb{C}, \]
and by Theorem 1, we conclude that
\[ f(z) = \text{const.}\mathcal{F}_{q,\nu}\left( e^{-\frac{1}{2}t^2} \right)(z), \quad \forall z \in \mathbb{C}. \]

Corollary 2 Suppose \( f \in \mathcal{L}_{q,1,\nu} \) satisfying the following estimates
\[ |f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+, \]
\[ |\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}, \]
where \( C, p, \sigma \) are a positive constant and \( p\sigma = \frac{1}{4} \). We suppose that there exist \( a \in \mathbb{R}_q^+ \) such that \( a^2 p = \frac{1}{2} \). Then there exist \( A \in \mathbb{R} \) such that
\[ f(z) = Ac_{q,\nu}\mathcal{F}_{q,\nu}\left( e^{-\sigma t^2} \right)(z), \quad \forall z \in \mathbb{C}. \]

Proof. Let \( a \in \mathbb{R}_q^+ \), and put
\[ f_a(x) = f(ax), \]
then
\[ \mathcal{F}_{q,\nu}f_a(x) = \frac{1}{a^{2v+2}}\mathcal{F}_{q,\nu}f(x/a). \]
In the end, applying Theorem 7 to the function \( f_a \). ■

Corollary 3 Suppose \( f \in \mathcal{L}_{q,1,\nu} \) satisfying the following estimates
\[ |f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+, \]
\[ |\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}, \]
where \( C, p, \sigma \) are a positive constant and \( p\sigma > \frac{1}{4} \). We suppose that there exist \( a \in \mathbb{R}_q^+ \) such that \( a^2 p = \frac{1}{2} \). Then \( f \equiv 0 \).

Proof. In fact there exists \( \sigma' < \sigma \) such that \( p\sigma' = \frac{1}{4} \). Then the function \( f \) satisfying the estimates of Corollary 2 if we replacing \( \sigma \) by \( \sigma' \). Which implies
\[ \mathcal{F}_{q,\nu}f(x) = \text{const}.e^{-\sigma'x^2}, \quad \forall x \in \mathbb{R}. \]
On the other hand, \( f \) satisfying the estimates (9), then
\[ \left| \text{const}.e^{-\sigma'x^2} \right| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}. \]
This implies \( \mathcal{F}_{q,\nu}f \equiv 0 \), and by Theorem 1 we conclude that \( f \equiv 0 \). ■
5 The $q$-Fourier-Neumann Expansions

The little $q$-Jacobi polynomials are defined for $\nu, \beta > -1$ by \[ p_n(x; q^\nu, q^\beta; q) = 2\phi_1 \left( q^{n+\nu+\beta+1}/q^{\nu+1}; q; qx \right). \]

We define the functions
\[ P_{\nu,n}(x; q^2) = \sigma_{q,\nu}(n) q^{-n(n+1)} (q^{2+2n}; q^2)_\infty (q^{2+2n}; q^2)_\infty p_n(x^2; q^{2\nu}, 1; q^2) \]
and
\[ J_{\nu,n}(x; q^2) = \sigma_{q,\nu}(n) J_{\nu+2n+1}(q^n x; q^2) \]
where
\[ \sigma_{q,\nu}(n) = \sqrt{1 - q^{2n+4\nu+2} / (1 - q)}. \]

Consider $L^\nu_{q,2}$ as an Hilbert space with the inner product
\[ \langle f | g \rangle = \int_0^1 f(x) g(x) x^{2\nu+1} d_qx. \]

The $q$-Paley-Wiener space is defined by
\[ PW^\nu_q = \left\{ f \in L_{q,2,\nu} : f(x) = c_{q,\nu} \int_0^1 u(t) j_\nu(xt, q^2) t^{2\nu+1} d_q t, \quad u \in L^\nu_{q,2} \right\}. \]

**Proposition 4** $PW^\nu_q$ is a closed subspace of $L_{q,2,\nu}$ and with the inner product given in (3) is an Hilbert space.

**Proof.** In fact, given $f \in L_{q,2,\nu}$ and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of element of $PW^\nu_q$ which converge to $f$ in $L^2$-norm. For $n \in \mathbb{N}$, there exist $u_n \in L^\nu_{q,2}$ such that
\[ f_n(x) = c_{q,\nu} \int_0^1 u_n(t) j_\nu(xt, q^2) t^{2\nu+1} d_q t. \]
Moreover
\[ \lim_{n \to \infty} \| f_n - f \|_{q,2,\nu} = 0. \]
This give
\[ \lim_{n \to \infty} \| F_{q,\nu} f_n - F_{q,\nu} f \|_{q,2,\nu} = 0, \]
and then
\[ \lim_{n \to \infty} \left[ \int_0^1 |F_{q,\nu} f_n - F_{q,\nu} f|^2 x^{2\nu+1} d_q x + \int_1^\infty |F_{q,\nu} f|^2 x^{2\nu+1} d_q x \right] = 0, \]
which implies
\[ \int_1^\infty |F_{q,\nu} f(x)|^2 x^{2\nu+1} d_q x = 0 \Rightarrow F_{q,\nu} f(x) = 0, \quad \forall x \in \mathbb{R}^+_q \cap [1, +\infty[. \]

Then $f \in PW^\nu_q$. \[ \square \]
Proposition 5 We have
\[ F_{q,\nu}(J_{\nu,n})(x) = P_{\nu,n}(x; q^2)\chi_{[0,1]}(x), \quad \forall x \in \mathbb{R}^+_q. \]

As a consequence
\[ \int_0^1 P_{\nu,n}(x; q^2)P_{\nu,m}(x; q^2)x^{2\nu+1} \, dq \, x = \delta_{n,m}. \]

Proof. The following proof is identical to the proof of Lemma 1 in [1]. Using an identity established in [12, 13],
\[ \int_0^\infty t^{-\lambda} J_\mu(q^m t; q^2) J_\theta(q^n t; q^2) \, dq \, t = \left(1 - q\right) q^{(\lambda-1)+(m-n)\mu} \frac{(q^{1+\lambda+\theta-\mu}, q^{2\mu+2}; q^2)_\infty}{(q^{1-\lambda+\theta+\mu}, q^{2\mu+2}; q^2)_\infty} \times 2 \phi_1 \left( q^{1-\lambda+\mu+\theta}, q^{1-\lambda+\mu-\theta}; q^{2\mu+2} \bigg| q^2, q^{2n-2m+1+\lambda+\theta-\mu} \right), \quad \text{(10)} \]
where \( n, m \in \mathbb{Z} \) and \( \theta, \mu, \lambda \in \mathbb{C} \) such that \( \text{Re}(1 - \lambda + \theta + \mu) > 0, \theta, \mu \) are not equal to a negative integer and
\[ (\lambda + \theta + 1 - \mu)/2, \quad m - n + (\lambda + \theta + 1 - \mu)/2 \]
are not a non-positive integer [13].

To evaluate \( F_{q,\nu}(J_{\nu,n})(x) \) when \( x = q^m \leq 1 \), we take in (10)
\[ q^m = x, \mu = \nu, \theta = \nu + 2n + 1, \lambda = 0 \]
then
\[ \mathcal{F}_{q,\nu}(J_{\nu,n})(x) = \sigma_{q,\nu}(n) \frac{x}{1 - q} \int_0^\infty J_\mu(x t; q^2) J_{\nu+2n+1}(q^n t; q^2) \, dq \, t = \sigma_{q,\nu}(n) q^{-n(\nu+1)} \frac{(q^{2+2n}, q^{2n+2}; q^2)_\infty}{(q^{2+2n+2\nu}, q^{2n}; q^2)_\infty} 2 \phi_1 \left( q^{2+2n+2\nu}, q^{-2n}; q^2, q^2 x^2 \right) = P_{\nu,n}(x; q^2). \]

To evaluate \( F_{q,\nu}(J_{\nu,m})(x) \) when \( x = q^n > 1 \), we consider in (10)
\[ q^n = x, \mu = \nu + 2m + 1, \theta = \nu, \lambda = 0 \]
In this way, \( 1 + \lambda + \theta - \mu = -2m \). This gives for \( m \in \mathbb{N} \) a factor
\[ (q^{-2m}; q^2)_\infty = 0 \]
on the numerator and then
\[ \mathcal{F}_{q,\nu}(J_{\nu,m})(x) = 0, \quad x > 1 \]
By setting \( \lambda = 1, \theta = \nu + 2n + 1, \) and \( \mu = \nu + 2m + 1 \) in , it is clear that, for \( n, m = 0,1,2, \ldots \),
\[ \int_0^\infty J_{\nu+2n+1}(q^n x; q^2) J_{\nu+2m+1}(q^m x; q^2) \frac{dq}{x} = \frac{1}{\sigma_{q,\nu}(n)^2} \delta_{n,m}. \]
and then
\[ \int_0^\infty J_{\nu,n}(x; q^2) J_{\nu,m}(x; q^2) x^{2\nu+1} \, dq \, x = \delta_{n,m}. \]

Now we use the arguments of $q$-Bessel Fourier analysis provided in this paper to show that
\[ \langle P_{\nu,n}x_{[0,1]}, P_{\nu,m}x_{[0,1]} \rangle = \langle F_{q,\nu}(J_{\nu,n}), F_{q,\nu}(J_{\nu,m}) \rangle = \langle J_{\nu,n}, J_{\nu,m} \rangle = \delta_{n,m}. \tag{11} \]

Another proof of the orthogonality of the little $q$-Jacobi polynomials can be found in [15].

**Proposition 6** The systems
\[ \{J_{\nu,n}\}_{n=0}^\infty, \quad \{P_{\nu,n}\}_{n=0}^\infty \]
form two orthonormal basis respectively of the Hilbert spaces $PW^q_\nu$ and $L^\nu_{q,2}$.

**Proof.** From (11) we derive the orthonormality. To prove that the system \( \{J_{\nu,n}\}_{n=0}^\infty \) is complete in $PW^q_\nu$, given a function $f \in PW^q_\nu$ such that
\[ \langle f, J_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}. \]

Then
\[ \langle F_{q,\nu}(f), F_{q,\nu}(J_{\nu,n}) \rangle = 0, \quad \forall n \in \mathbb{N}, \]
which implies
\[ \langle F_{q,\nu}(f), P_{\nu,n}x_{[0,1]} \rangle = \langle F_{q,\nu}(f)x_{[0,1]}, P_{\nu,n} \rangle = \langle F_{q,\nu}(f), P_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N}. \]

From the definition of the polynomial $P_{\nu,n}$ we conclude that
\[ \langle F_{q,\nu}(f), t^{2n} \rangle = 0, \quad \forall n \in \mathbb{N}. \]

Then
\[ c_{q,\nu} \sum_{n=0}^\infty (-1)^n \frac{q^{n(n+1)}}{(q^2, q^2)_{n}(q^{2\nu+2}, q^2)_n} \langle F_{q,\nu}(f), t^{2n} \rangle x^{2n} = 0, \quad \forall x \in \mathbb{R}^+_q, \]
which can be written as
\[ F^2_{q,\nu}(f)(x) = 0, \quad \forall x \in \mathbb{R}^+_q. \]

By the inversion formula (2) we conclude that $f = 0$. From (11) we derive the orthonormality. To prove that the system \( \{P_{\nu,n}\}_{n=0}^\infty \) is complete in $L^\nu_{q,2}$, given a function $f \in L^\nu_{q,2}$ such that
\[ \langle f| P_{\nu,n} \rangle = 0, \quad \forall n \in \mathbb{N} \]

Then
\[ \langle f| t^{2n} \rangle = 0, \quad \forall n \in \mathbb{N}. \]

Which leads to the result. 

\[ 18 \]
Proposition 7 Let $\lambda \in \mathbb{R}^+_q$ then
\[
c_{q,\nu}J_\nu(\lambda x; q^2) = \sum_{n=0}^{\infty} J_n(\lambda; q^2) P_{n,\nu}(x), \quad \forall x \in [0, 1] \cap \mathbb{R}^+_q.
\]
As a consequence we have
\[
\sum_{n=0}^{\infty} [P_{n,\nu}(x; q^2)]^2 = \frac{x^{-2(\nu+1)}}{1 - q}, \quad \forall x \in [0, 1] \cap \mathbb{R}^+_q
\]
and for all $\lambda \in \mathbb{R}^+_q$
\[
\sum_{n=0}^{\infty} [J_n(\lambda; q^2)]^2 = -\frac{q^\nu}{2(1 - q)\lambda^{1 + 2\nu}}
\times \left[ \frac{\lambda}{q} J_{\nu+1}(\lambda; q^2)J_\nu'(\lambda/q; q^2) - J_{\nu+1}(\lambda; q^2)J_\nu(\lambda/q; q^2) - J_{\nu+1}'(\lambda; q^2)J_\nu(\lambda/q; q^2) \right].
\]

Proof. Let $\lambda \in \mathbb{R}$ and consider the function
\[
\psi_\lambda : [0, 1] \cap \mathbb{R}^+_q \to \mathbb{R}, \quad x \mapsto c_{q,\nu}J_\nu(\lambda x; q^2).
\]
Then $\psi_\lambda \in L^2_{q,\nu}$ and we can write
\[
\psi_\lambda(x) = \sum_{n=0}^{\infty} \langle \psi_\lambda | P_{n,\nu} \rangle P_{n,\nu}(x), \quad \forall x \in [0, 1] \cap \mathbb{R}^+_q. \tag{12}
\]
Note that
\[
\langle \psi_\lambda | P_{n,\nu} \rangle = \langle \psi_\lambda, P_{n,\nu} \chi[0,1] \rangle = \langle \psi_\lambda, F_{q,\nu}(J_{n,\nu}) \rangle = F_{q,\nu}^2(J_{n,\nu})(\lambda) = J_{n,\nu}(\lambda; q^2).
\]
Then we deduce the result. Using the Parseval’s theorem and (12) we obtain
\[
\sum_{n=0}^{\infty} [P_{n,\nu}(x; q^2)]^2 = \| \psi_x \|_{q,2,\nu}^2 = \frac{x^{-2(\nu+1)}}{1 - q}.
\]
The second identity is deduced also from the Parseval’s theorem
\[
\sum_{n=0}^{\infty} [J_n(\lambda; q^2)]^2 = N_{q,2,\nu}^2(\psi_\lambda),
\]
and the following relation proved in [14]
\[
\int_0^1 [J_\nu(aq t; q^2)]^2 t dq t = -\frac{(1 - q)q^{\nu-1}}{2a}
\times \left[ aJ_{\nu+1}(aq; q^2)J_\nu(a; q^2) - J_{\nu+1}(aq; q^2)J_\nu(a; q^2) - J_{\nu+1}'(aq; q^2)J_\nu(a; q^2) \right].
\]
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