Semi-algebraic description of the closure of the image of a semi-algebraic set under a polynomial

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Abstract

Given a polynomial \( f \) and a semi-algebraic set \( S \), we provide a symbolic algorithm to find the equations and inequalities defining a semi-algebraic set \( Q \) which is identical to the closure of the image of \( S \) under \( f \), i.e.,

\[
Q = \overline{f(S)}.
\]

Consequently, every polynomial optimization problem whose optimum value is finite has an equivalent form with attained optimum value, i.e.,

\[
\min_{t \in Q} t = \inf_{x \in S} f(x)
\]

whenever the right-hand side is finite. Given \( d \) being the upper bound on the degrees of \( f \) and polynomials defining \( S \), we prove that our method requires \( O(d^{O(n)}) \) arithmetic operations to produce polynomials of degrees at most \( d^{O(n)} \) that define \( \overline{f(S)} \).

Keywords: semi-algebraic set; Tarski–Seidenberg’s theorem; quantifier elimination; sum of squares; Nichtnegativstellensatz; polynomial optimization; semidefinite programming

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1 Introduction

Let \( \mathbb{R}[x] \) stand for the ring of polynomials of real coefficients in vector of variables \( x = (x_1, \ldots, x_n) \). The set \( W \subseteq \mathbb{R}^n \) is called elementary semi-algebraic, if it can be written in the form

\[
W = \{ x \in \mathbb{R}^n : p_1(x) > 0, \ldots, p_m(x) > 0, q_1(x) = \cdots = q_l(x) = 0 \},
\]

where \( p_i, q_j \in \mathbb{R}[x] \). We call \( p_i \) (resp. \( q_j \)) inequality (resp. equality) polynomial defining \( W \). A set is called semi-algebraic, if it is the union of a finite number of elementary semi-algebraic sets. Denote by \( \overline{A} \) the closure of a set \( A \subseteq \mathbb{R}^n \) in the usual topology on \( \mathbb{R}^n \).

Known as a central result of real algebraic geometry, Tarski–Seidenberg’s theorem \([21,22]\) says that the projection of a semi-algebraic subset of \( \mathbb{R}^n \) onto the space spanned by the first \( n-1 \) coordinates is semi-algebraic. Using a finite number of arithmetic operations, we can write down explicitly the equations and inequalities defining this projection. Therefore, algorithms that allow us to do so are fundamental and widely used in various applications of real algebraic geometry.

Contribution. In this paper, we provide an algorithm to symbolically compute polynomials defining the closure of the image of a semi-algebraic set \( S \) under a polynomial \( f \). Moreover, we obtain explicitly \( \overline{f(S)} \) as a finite union of closed semi-algebraic sets of the form

\[
\{ t \in \mathbb{R} : g_1(t) \geq 0, \ldots, g_v(t) \geq 0, h_1(t) = \cdots = h_k(t) = 0 \},
\]

where \( g_i, h_j \) are univariate polynomials. Our method consists of the following three steps:

1. By Tarski–Seidenberg’s theorem, the image \( f(S) \) is a semi-algebraic set. Moreover, we obtain its (in)equality polynomials by using Basu’s algorithm in \([1\text{ Section 3]}\).

2. We then convert these polynomial descriptions to new ones defining elementary semi-algebraic sets \( B_1, \ldots, B_r \) whose union is \( f(S) \) such that the following conditions hold:

   (a) Each inequality polynomial defining \( B_j \) has only simple roots.

   (b) Any inequality polynomial defining \( B_j \) has no root in the variety determined by the equality polynomials defining \( B_j \).

   (c) Any couple of inequality polynomials determining each \( B_j \) has no common root.

3. Replacing strict inequalities defining \( B_j \) with non-strict ones, we obtain a new semi-algebraic set \( \overline{B}_j \). We guarantee that the closure \( \overline{f(S)} \) is identical to the union of \( \overline{B}_j \).

The main tools utilized in the last two steps are the fundamental theorem of algebra and the greatest common divisor.

Let \( d \) be an upper bound on the degrees of \( f \) and polynomials defining \( S \). Our method has a complexity of \( O(d^{O(n)}) \) to return polynomials of degrees at most \( d^O(n) \) that define \( \overline{f(S)} \). This complexity follows from the complexity of Tarsi-Seidenberg’s theorem analyzed by Basu in \([1\text{ Theorem 1}]\).

(Another possible way is to rely on Basu’s method in \([1\text{ Section 2.1, Example}]) for computing the closure of a semi-algebraic set to find the semi-algebraic description of \( \overline{f(S)} \). However, we are also not sure if this method allows us to obtain \( \overline{f(S)} \) as a finite union of closed semi-algebraic sets of the form \([1]\), on which we can build some Nichtnegativstellensätze for univariate polynomials.)
Motivation. One of the applications of our method is to address the attainability issue in polynomial optimization. More explicitly, consider

\[ f^* = \inf_{x \in S} f(x), \]

where \( f \) is a polynomial in \( \mathbb{R}[x] \) and \( S \) is a semi-algebraic subset of \( \mathbb{R}^n \). Assume that the optimum value \( f^* \) is finite, i.e., \( f^* \in \mathbb{R} \).

It is well-known that when \( S \) is a compact semi-algebraic set of the form

\[ \{ x \in \mathbb{R}^n : p_1(x) \geq 0, \ldots, p_m(x) \geq 0, q_l(x) = \cdots = q_k(x) = 0 \}, \]

the optimal value \( f^* \) is approximated from below as closely as desired by the sum-of-squares strengthenings introduced by Lasserre in \([11]\). If \( f \) attains \( f^* \) on non-compact \( S \) of form \([0]\), we can approximately compute \( f^* \) using Mai–Lasserre–Magron’s in \([15]\). Moreover, tools that allow us to compute \( f^* \) exactly in Demmel–Nie–Powers’ and Mai’s work \([6, 14]\) require the attainability of \( f^* \).

However, the problem \([5]\) possibly has no optimal solution (i.e., \( f^* \) is not attained). For instance, if \( f = x_1 \) and \( S = \{ x \in \mathbb{R}^2 : x_1 x_2^2 = 1 \} \), then \( f^* = 0 \) is not attained by \( f \) on \( S \).

We now formulate \([5]\) as \( f^* = \inf f(S) \). A simple idea to make \( f^* \) attained is replacing the image \( f(S) \) with its closure. This gives

\[ f^* = \min \{ f(S) \} \]

since \( f^* \in \overline{f(S)} \). The challenge is to transform \([\overline{f(S)}]\) back to a polynomial optimization problem with attained optimum value \( f^* \). In other words, we find a polynomial \( g \) and a semi-algebraic set \( Q \) such that

\[ f^* = \min_{y \in Q} g(y). \]

Our method allows us to obtain \( Q = \overline{f(S)} \) and \( g \) as the identity polynomial in a single variable, i.e., \( g(t) = t \). In principle, \( f(S) \) is a semi-algebraic subset of \( \mathbb{R} \) and therefore is the union of finitely many intervals \( I_1, \ldots, I_r \) in \( \mathbb{R} \). Here \( I_j \) are of the following forms: \( (a, b), [a, b), (a, b], [a, b], (a, \infty), [a, \infty), (-\infty, b], (-\infty, b), [a, a] = \{ a \} \) with \(-\infty < a < b < \infty\).

The closure \( \overline{f(S)} \) is then the union of \( \overline{T_1}, \ldots, \overline{T_r} \), where each \( \overline{T_j} \) contains \( I_j \) and its endpoints. Note that the endpoints of \( I_j \) are the real roots of (in)equality polynomials defining \( f(S) \). It leads to using numerical methods to find the real roots of a system of univariate polynomials. In contrast, our method relies only on symbolic computation to obtain the equations and inequalities determining \( \overline{f(S)} \).

It remains to solve the problem \(([5]\), where \( Q = \overline{f(S)} \) is a semi-algebraic subset of \( \mathbb{R} \) and \( g \) is the identity polynomial. In this case, we can exactly compute the optimum value \( f^* \) by using the sum-of-squares strengthenings under the boundedness assumption of \( f(S) \).

Previous works. In \([13]\), the author provides a symbolic algorithm to compute polynomials defining the image \( f(S) \) when \( f(S) \) is finitely many points in \( \mathbb{R} \) and \( S \) is of form \([5]\). In this case, \( f(S) = \overline{f(S)} \), and \( f(S) \) is also identical to its Zariski closure, the smallest variety containing \( f(S) \). Here a variety is an elementary semi-algebraic set defined only by equations. The method in \([13]\) depends on the computations of real radical generators and Groebner bases.

Regarding the attainability issue for the polynomial optimization problem \([11]\), we refer the readers to \([17, 18, 20]\). In \([20]\), Schweighofer deals with the case where \( S = \mathbb{R}^n \) and \( f^* \) is not attained by \( f \). He replaces the gradient ideals used in Nie–Demmel–Sturmfels’ method in \([17]\) with the gradient tentacles to obtain appropriate sum-of-squares strengthenings for the problem \([9]\). Hà and Pham handle the constraint case.
of S of form (6) in [7]. They propose the truncated tangency variety Λ for
f and S with property saying that f has infimum value f⋆ on Λ under regularity assumption even if f does not attain on S. It also allows them to obtain sum-of-squares strengthenings for the problem (5). Dinh, Ha, and Pham provide in [6] a Frank–Wolfe type theorem for f⋆ bounded from below on S = {x ∈ Rn : p1(x) ≥ 0, ..., pm(x) ≥ 0}. It says that if the polynomial map x ↦ (f(x), p1(x), ..., pm(x)) is convenient and non-degenerate at infinity, then f attains its infimum f⋆ on S. In [18], Pham derives versions at infinity of the Fitz John and Karush–Kuhn–Tucker conditions to deal with the case where f⋆ is not attained by f on S of form (6). He also indicates the existence of polynomial g and semi-algebraic set Q such that (8) holds under the non-degenerate assumption at infinity.

For comparison purposes, our method in this paper does not require any assumption on f and S except the finiteness of f⋆. Besides, we convert the original problem (6) in n-dimensional space to equivalent form (8) in one-dimensional space. The methods in [6, 7, 18, 20] preserve the dimension in obtaining equivalent forms. Moreover, their transformations to equivalent forms are much simpler than ours as we use a lot of arithmetic operations on f and polynomials defining S.

Organization. We organize the paper as follows: Section 2 presents some preliminaries and lemmas needed to prove our main results. Section 3 is to provide an algorithm that allows us to obtain the product of linear factors of a polynomial occurring odd times. Section 4 is to prove that replacing strict inequalities defining a “nice” elementary semi-algebraic set with non-strict ones allows us to obtain the closure of this set. Section 5 is to present the main algorithm that enables us to obtain semi-algebraic descriptions of the closures of the images of semi-algebraic sets under polynomials.

2 Preliminaries

2.1 Projections of semi-algebraic sets and quantifier elimination

We restate Tarski–Seidenberg’s theorem in the following lemma:

Lemma 1. Let S be a semi-algebraic subset of Rn and π : Rn → Rn−1 be the projector onto the space spanned by the first n−1 coordinates. Then the projection π(S) is semi-algebraic.

A first-order formula in the language of real closed fields with free variables x = (x1, ..., xn) is obtained as follows recursively:

1. If f ∈ R[x], n ≥ 1, then f = 0 and f > 0 are first-order formulas and \{x ∈ Rn : f(x) = 0\} and \{x ∈ Rn : f(x) > 0\} are respectively the subsets of Rn such that the formulas f = 0 and f > 0 hold.
2. If ϕ and ψ are first-order formulas, then ϕ ∧ ψ (conjunction), ϕ ∨ ψ (disjunction) and ¬ϕ (negation) are also first-order formulas.
3. If ϕ is a first-order formula and x is a variable ranging over R, then ∃xϕ and ∀xϕ are first-order formulas.

The formulas obtained by using only rules 1 and 2 are called quantifier-free formulas.

It follows from definitions that a subset S ⊆ Rn is semi-algebraic if and only if there exists a quantifier-free formula ϕ such that

S = \{x ∈ Rn : ϕ(x)\}. (9)
Eliminating the quantifier from $\exists x_n \varphi$, is the same as computing a quantifier-free description of the image

$$\pi(S) = \{x' \in \mathbb{R}^{n-1} : \exists x_n \varphi(x', x_n)\},$$

(10)

where $x' = (x_1, \ldots, x_{n-1})$ and $\pi$ is the projector defined as in Lemma 1. Tarski–Seidenberg’s theorem (Lemma 1) allows us to replace $\exists x_n \varphi(x', x_n)$ by some quantifier-free formula with vector of variables $x'$.

The following example is to illustrate the equivalence between eliminating a quantifier and computing semi-algebraic description of the projection of a semi-algebraic set:

**Example 1.** Since $\exists t(at^2 + bt + c = 0)$ is equivalent to the quantifier-free formula $(a \neq 0 \land b^2 - 4ac \geq 0) \lor (a = 0 \land b \neq 0) \lor (a = 0 \land b = 0 \land c = 0)$, the projection of semi-algebraic set

$$\{(a, b, c, t) \in \mathbb{R}^4 : at^2 + bt + c = 0\}$$

onto the space spanned by the first three coordinates is semi-algebraic set

$$\{(a, b, c) \in \mathbb{R}^3 : a \neq 0, b^2 - 4ac \geq 0\} \cup \{(a, b, c) \in \mathbb{R}^3 : a = 0, b \neq 0\} \cup \{(a, b, c) \in \mathbb{R}^3 : a = b = c = 0\}.$$  

(12)

Basu suggests his quantifier elimination algorithm in [1] Section 3. We use his method in the first step of our main algorithm stated later.

**Remark 1.** Heintz, Roy, and Solerno analyze the complexity of Tarski–Seidenberg’s theorem in [9]. An improvement is given by Basu in [1] using terms of quantifier elimination. Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set determined by $s$ polynomials of degree at most $d$. Basu’s method requires $s^2d^{O(1)}$ arithmetic operations to find polynomials of degrees upper bounded by $d^{O(1)}$ that define the projection of $S$ onto the space spanned by the first $n - 1$ coordinates.

### 2.2 Polynomial images of semi-algebraic sets

A mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ is called semi-algebraic if its graph $\{(x, \varphi(x)) : x \in \mathbb{R}^n\}$ is a semi-algebraic subset of $\mathbb{R}^{n+d}$. The following lemma follows from Tarski–Seidenberg’s theorem (Lemma 4):

**Lemma 2.** Let $S$ be a semi-algebraic subset of $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be a semi-algebraic mapping. Then the image $\varphi(S)$ is semi-algebraic.

**Remark 2.** Korkina and Kushnirenko provide in [9] an algorithm to find the equations and inequalities defining the image $\varphi(S)$ (see also Coste’s book [4, Sections 1 and 2]).

The following lemma is a trivial consequence of Lemma 2 and Remark 2.

**Lemma 3.** Let $f$ be a polynomial and $S$ be a semi-algebraic set. Then the image $f(S)$ is a semi-algebraic subset of $\mathbb{R}$. Moreover, there is a symbolic algorithm that produces the equations and inequalities defining $f(S)$.

Next, we analyze the complexity of the algorithm mentioned in Lemma 3.

**Remark 3.** Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set determined by $s$ polynomials of degree at most $d$. Let $f$ be a polynomial of degree at most $d$. It is not hard to prove that $f(S)$ is the projection of semi-algebraic set

$$\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = f(x), x \in S\}$$

onto the space spanned by the last coordinate. It is equivalent to eliminating a block of quantifiers of length $n$. Thanks to the complexity given in [1] Theorem 1, it requires $s^{n+1}d^{O(n)}$ arithmetic operations to find the polynomials of degrees upper bounded by $d^{O(n)}$ defining $f(S)$.
2.3 Closures of semi-algebraic sets in high-dimensional spaces

Set $\|x\|_2^2 := x_1^2 + \cdots + x_n^2$. We recall some properties of semi-algebraic sets in the following lemma:

**Lemma 4.** The following statements hold:

1. With $A$ and $B$ being semi-algebraic subsets of $\mathbb{R}^n$, the sets $A \cup B$, $A \cap B$, and $\mathbb{R}^n \setminus A$ are also semi-algebraic.
2. The closure of a semi-algebraic set is semi-algebraic.
3. A semi-algebraic subset of $\mathbb{R}$ is a finite union of intervals and points in $\mathbb{R}$.

**Proof.** The first and third statements are given in [19, Proposition 1.1.1] and [19, Example 1.1.1], respectively. The second one is proved by Coste in [4, Corollary 2.5] as follows: Let $S$ be a semi-algebraic set defined as in (9). The closure of $S$ is

$$S = \{ x \in \mathbb{R}^n : (\forall \varepsilon > 0) (\exists y) (\|x - y\|_2^2 < \varepsilon^2) \land \varphi(y) \}.$$  

(14)

and can be written as

$$S = \mathbb{R}^n \setminus (\pi_1(\{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} : \varepsilon > 0\} \setminus \pi_2(B))).$$  

(15)

where

$$B = \{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : (\|x - y\|_2^2 < \varepsilon^2) \land \varphi(y)\},$$  

(16)

$\pi_1(x, \varepsilon) = x$ and $\pi_2(x, \varepsilon, y) = (x, \varepsilon)$. Then observe that $B$ is semi-algebraic. Hence $S$ is semi-algebraic thanks to Lemma 1 and the first statement.

**Remark 4.** In [12], Magron, Henrion, and Lasserre approximate the closure $S$ in Lemma 4 using semidefinite programming. Basu develops in [1, Section 2.1, Example] a symbolic method to compute the closure of a semi-algebraic set defined as in (9), where $\varphi(x)$ is a quantifier-free first order formula involving $s$ polynomials with degrees bounded by $d$. It is based on quantifier elimination and has complexity $s^{2(n+1)}d^{O(n^2)}$. His idea is to eliminate two blocks of quantifiers corresponding to $\varepsilon$ and $y$ for the description of $S$ in (14).

**Remark 5.** With $\varphi$ and $S$ being as in Lemma 4, the closure $\overline{\varphi(S)}$ is semi-algebraic thanks to the second statement of Lemma 4. By the third statement of Lemma 4, the image $f(S)$ in Lemma 3 is a finite union of points and intervals of $\mathbb{R}$. It implies that the closure $\overline{f(S)}$ is semi-algebraic and therefore is a finite union of closed intervals and points of $\mathbb{R}$.

2.4 Fundamental theorem of algebra and greatest common divisor

Let $\mathbb{R}[t]$ be the ring of polynomials in single-variable $t$. Let $\deg(p)$ stand for the degree of a polynomial $p \in \mathbb{R}[t]$.

We restate the fundamental theorem of algebra in the following lemma:

**Lemma 5.** Every polynomial $p \in \mathbb{R}[t]$ of positive degree can be decomposed uniquely as

$$p = c \times \prod_{i=1}^s (t - a_i)^{m_i},$$  

(17)

where $c \in \mathbb{R}$, $a_i \in \mathbb{C}$, and $s, m_i \in \mathbb{N}$ such that $c \neq 0$, $m_i > 0$ and $a_1, \ldots, a_s$ are disjoint. Moreover, for every $i \in \{1, \ldots, s\}$ satisfying $a_i \in \mathbb{C} \setminus \mathbb{R}$, there is $j \in \{1, \ldots, s\} \setminus \{i\}$ such that $m_j = m_i$, and $a_j$ is the conjugate of $a_i$. 

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We call $a_i$ in Lemma 6 a (complex) root of $p$ with multiplicity $m_i$. If $m_i > 1$ (resp. $m_i = 1$), $a_i$ is called multiple (resp. simple) root of $p$.

Let $\gcd(p_1, \ldots, p_m)$ stand for the greatest common divisor of polynomials $p_1, \ldots, p_m$ in $\mathbb{R}[t]$. To find $\gcd(p_1, \ldots, p_m)$, we use the well-known polynomial Euclidean algorithm (see, e.g., Basu–Pollack–Roy’s book [2]).

**Remark 6.** Belhaj–Kahla’s algorithm in [2] to compute $\gcd(p_1, p_2)$ via Hankel matrices has a cost of $O(d^3)$ arithmetic operations, where $d = \max\{\deg(p_1), \deg(p_2)\}$. Since $\gcd(p_1, p_2, \ldots, p_m) = \gcd(p_1, \gcd(p_2, \ldots, p_m))$, we obtain the number of arithmetic operations to compute $\gcd(p_2, \ldots, p_m)$ as $(m - 1) \times O(d^2)$, where $d = \max_{j=1,\ldots,m} \deg(p_j)$.

Denote by $p'$ the derivative of a polynomial $p \in \mathbb{R}[t]$. The following two lemmas present the basic properties of the roots of a polynomial and its derivative:

**Lemma 6.** Let $a$ be a root of a polynomial $p \in \mathbb{R}[t]$. Then $a$ is a multiple root of $p$ if and only if $a$ is also a root of $p'$.

**Proof.** Since $a$ is a root of $p$, there exists a polynomial $q$ such that $p = (t - a)q$. Using the product rule of derivatives, we know that $p' = q + (t - a)q'$. But then $(t - a)$ only divides $p'$ if and only if it also divides $q$.

**Lemma 7.** If a polynomial $p \in \mathbb{R}[t]$ has only simple roots. Then $p, p'$ are relatively prime, i.e., $\gcd(p, p') = 1$.

**Proof.** Assume that $p, p'$ are not relatively prime. Then, they have a common irreducible factor of degree 1. Since by definition, two polynomials are relatively prime if their only common factor is of degree 0. Then there exists a polynomial of the form $t - a$ that divides both $p$ and $p'$. Then from Lemma 6 $a$ is a multiple root of $p$. But this is impossible.

We recall the radical of a polynomial in the following lemma:

**Lemma 8.** Let $p$ be polynomial in $\mathbb{R}[t]$ with decomposition (17). Then

$$
\gcd(p, p') = \prod_{i=1}^{s} (t - a_i)^{m_i - 1},
$$

(18)

$$
\frac{p}{\gcd(p, p')} = \prod_{i=1}^{s} (t - a_i)
$$

(19)

are polynomials in $\mathbb{R}[t]$.

**Proof.** Set

$$
h = \prod_{i=1}^{s} (t - a_i) \quad \text{and} \quad w = \prod_{i=1}^{s} (t - a_i)^{m_i - 1}
$$

(20)

Then $p = wh$ and

$$
h' = \sum_{j=1}^{s} \prod_{i=1 \atop i \neq j}^{s} (t - a_i). \quad (21)
$$

Lemma 6 says that $\gcd(h, h') = 1$ since $h$ has only simple roots. Simple computation gives

$$
p' = \sum_{j=1}^{s} m_j (t - a_j)^{m_j - 1} \prod_{i=1 \atop i \neq j}^{s} (t - a_i)^{m_i} = wh'.
$$

(22)

From this and $\gcd(h, h') = 1$, we obtain $\gcd(p, p') = w\gcd(h, h') = w$, which yields (18) and (19). In addition, $w$ and $h$ are polynomials thanks to the second statement of Lemma 6. Hence the result follows.

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Lemma 8 says that we can symbolically compute the product of linear factors of a polynomial. The following lemma shows how to get the product of linear factors of a polynomial occurring once:

**Lemma 9.** Let $p$ be polynomial in $\mathbb{R}[t]$ with decomposition (17). Set $q := \gcd(p, p')$. Then

$$
\frac{p \times \gcd(q, q')}{q^2} = \prod_{i=1}^{s} (t-a_i)
$$

is a polynomial in $\mathbb{R}[t]$.

**Proof.** Applying Lemma 8, we get

$$
\frac{q}{\gcd(q, q')} = \prod_{i=1}^{s} (t-a_i). \quad (23)
$$

It implies that

$$
\frac{p}{\gcd(p, p')} \times \frac{\gcd(q, q')}{q} = \prod_{i=1}^{s} (t-a_i) \div \prod_{j=1}^{s} (t-a_j) = \prod_{i=1}^{s} (t-a_i). \quad (24)
$$

Hence the result follows.

**2.5 Closures of semi-algebraic sets in one-dimensional spaces**

The following lemma describes the properties of the closures of the intersection and union of several sets:

**Lemma 10.** Let $A_1, \ldots, A_m$ be a finite collection of subsets of $\mathbb{R}^n$. Then the following two statements hold:

1. $\bigcap_{i=1}^{m} A_i \subset \bigcap_{i=1}^{m} \overline{A_i}$.
2. $\bigcup_{i=1}^{m} A_i = \bigcup_{i=1}^{m} \overline{A_i}$.

**Proof.** It is not hard to prove the second statement. Let us prove the first one. For $j = 1, \ldots, m$, we get $\bigcap_{i=1}^{m} A_i \subset A_j$, which gives $\bigcap_{i=1}^{m} A_i \subset \overline{A_j}$. Hence the result follows.

We say that a polynomial $p \in \mathbb{R}[t]$ changes sign at its real root $a$, if

$$
\lim_{t \to a^+} \frac{1}{p(t)} \neq \lim_{t \to a^-} \frac{1}{p(t)}.
$$

(26)

Note that these limits have only values $\pm \infty$. The following lemma shows that the sign change of a polynomial depends on the multiplicities of real roots:

**Lemma 11.** Let $p$ be a polynomial in $\mathbb{R}[t]$ of positive degree. Then $p$ changes (resp. does not change) sign at its real roots with odd (resp. even) multiplicities.

**Proof.** Let $a$ be a real root of $p$ with multiplicity $m$. Then $p = q(t-a)^m$, where $q$ is some polynomial in $\mathbb{R}[t]$ that is not zero at $a$. Consider the following two cases:

- Case 1: $m$ is odd. Suppose that $q(a) > 0$. By continuity $q$ is positive on an open interval containing $a$ implying that $\lim_{t \to a^+} \frac{1}{p(t)} = \infty \neq -\infty = \lim_{t \to a^-} \frac{1}{p(t)}$. Thus $p(t)$ changes sign at $a$. The same will be true if $q(a) < 0$. 

Then we claim that the following conditions hold:

Lemma 13. A semi-algebraic description of an open semi-algebraic set defined only by a single inequality is present in the following lemma:

Lemma 12. Let $p$ be a polynomial in $\mathbb{R}[t]$ of positive degree. Suppose that each real root of $p$ has odd multiplicity. Then $\{p \geq 0\} = \{p > 0\}$.

Proof. Let $a_1, \ldots, a_s$ be the real roots of $p$ such that $a_1 < \cdots < a_s$. By assumption, these roots have odd multiplicities. Since $p$ changes sign at each $a_i$ (by Lemma 11), one of the following two cases occurs:

- Case 1: The leading coefficient of $p$ is negative, i.e., $\text{lc}(p) < 0$. Then
  \[
  \{p > 0\} = \left\{(-\infty, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_s, \infty) \right\} \text{ if } s \text{ is even,} \]
  \[
  \left\{(-\infty, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_{s-1}, a_s) \right\} \text{ otherwise.} \tag{27}
  \]

- Case 2: $\text{lc}(p) > 0$. Then
  \[
  \{p > 0\} = \left\{(a_1, a_2) \cup (a_3, a_4) \cup \cdots \cup (a_s, \infty) \right\} \text{ if } s \text{ is odd,} \]
  \[
  \left\{(-\infty, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_s, \infty) \right\} \text{ otherwise.} \tag{28}
  \]

Thus $\{p \geq 0\} = \{p > 0\} \cup \{a_1, \ldots, a_s\} = \{p > 0\}$ yields the result.

Denote by $\text{int}(A)$ the interior of a set $A \subset \mathbb{R}^n$. Let $\partial(A)$ stand for the boundary of $A \in \mathbb{R}^n$, i.e., $\partial(A) = \overline{A} \setminus \text{int}(A)$. The following lemma shows a semi-algebraic description of the closure of an open semi-algebraic set:

Lemma 13. Let $p_1, \ldots, p_m$ be polynomials in $\mathbb{R}[t]$ of positive degrees. Suppose that the following conditions hold:

1. For $i = 1, \ldots, m$, each real root of $p_i$ has odd multiplicity.
2. Any couple of $p_1, \ldots, p_m$ has no common real root.

Then

\[
\bigcap_{i=1}^m \{p_i > 0\} = \bigcap_{i=1}^m \{p_i \geq 0\}. \tag{29}
\]

Proof. Set

\[
A = \bigcap_{i=1}^m \{p_i \geq 0\} \quad \text{and} \quad B = \bigcap_{i=1}^m \{p_i > 0\}. \tag{30}
\]

We claim that $\overline{B} \subset A$. By assumption, Lemma 12 yields $\{p_i > 0\} = \{p_i \geq 0\}$. Form this and the first statement of Lemma 10 we obtain

\[
\overline{B} \subset \bigcap_{i=1}^m \{p_i > 0\} = \bigcap_{i=1}^m \{p_i \geq 0\} = A. \tag{31}
\]

It is sufficient to prove that $A \subset \overline{B}$. Let $a \in A$. If $a \in \text{int}(A)$, then $a \in B$ implies that $a \in \overline{B}$. Assume that $a \in \partial(A)$. Then $a$ is a root of $p_j$ for some $j \in \{1, \ldots, m\}$. By assumption, $a$ is a root of $p_j$ with odd multiplicity. By Lemma 11, $p_j$ changes sign at $a$. Thus there is a sequence $(a_k)_{k \in \mathbb{N}} \subset \{p_j > 0\}$ such that $\lim_{k \to \infty} a_k = a$. By
assumption, \( a \) is not a root of \( p_i \) for any \( i \in \{1, \ldots, m\} \setminus \{j\} \). Since \( a \in A \), it implies that \( a \) is in the open set
\[
U = \bigcap_{i=1}^{m} \{p_i > 0\}.
\]
Since the sequence \((a_k)_{k \in \mathbb{N}} \subset \{p_j > 0\}\) converges to \( a \), there exists a subsequence \((a_k')_{k' \in \mathbb{N}} \subset B = U \cap \{p_j > 0\}\) that converges to \( a \). It implies that \( a \in \mathbb{R} \). Hence the result follows.

In the following lemma, we present the property of the intersection of the closure of an open semi-algebraic set with a variety:

**Lemma 14.** Let \( p_1, \ldots, p_m \) be polynomials in \( \mathbb{R}[t] \) of positive degrees. Let \( V \) be a finite subset of \( \mathbb{R} \) such that any \( p_i \) has no root in \( V \). Then
\[
\bigcap_{i=1}^{m} \{p_i > 0\} \cap \overline{V} = \bigcap_{i=1}^{m} \{p_i > 0\} \cap V.
\]

**Proof.** Set \( A \) as in (30). The first statement of Lemma 14 yields \( \overline{A} \cap V \subseteq \overline{A} \cap V \). It is sufficient to prove that \( \overline{A} \cap V \subseteq \overline{A} \cap V \). Let \( a \in \overline{A} \cap V \). Since \( a \in \overline{A} \), there is a sequence \((a_k)_{k \in \mathbb{N}} \subset A\) such that \( \lim_{k \to \infty} a_k = a \). Since \( a \in \overline{V} \), \( a \) is not a root of any \( p_i \). By Lemma 11 each \( p_i(a) \) is positive or negative. By the continuity of \( p_i \), \( p_i(a) = \lim_{k \to \infty} p_i(a_k) \geq 0 \) implies \( a \in \{p_i > 0\} \). Then \( a \) is in \( A \), and therefore, is in \( A \cap V \). Thus \( a \in A \cap V \) yields the result.

### 2.6 Elimination of common roots for polynomial (in)equalities

Let \( B \) be an elementary semi-algebraic subset of \( \mathbb{R} \) defined by
\[
B = \{ t \in \mathbb{R} : p_1(t) > 0, \ldots, p_m(t) > 0, q_1(t) = \cdots = q_l(t) = 0 \},
\]
where \( p_i, q_j \) are polynomials in \( \mathbb{R}[t] \). For simplicity, we write \( B \) as
\[
B = \{ p_1 > 0, \ldots, p_m > 0, q_1 = \cdots = q_l = 0 \}.
\]
The set \( \{ \hat{q}_1 = \cdots = \hat{q}_l = 0 \} \) is a variety defined by polynomials \( q_1, \ldots, q_l \).

The following two lemmas allow us to obtain a new semi-algebraic description of an elementary semi-algebraic set \( B \) such that the new inequality polynomials have no root in the variety defined by the new equality polynomials determining \( B \):

**Lemma 15.** Let \( B \subset \mathbb{R} \) be an elementary semi-algebraic set defined as in (35) such that each \( q_j \) has only simple roots. Let \( i \in \{1, \ldots, m\} \) be fixed. Set \( w = \gcd(p_i, q_1, \ldots, q_l) \). For \( j = 1, \ldots, l \), set \( \hat{q}_j = q_j/w \). Then the following two statements hold:
1. \( B = \{ p_1 > 0, \ldots, p_m > 0, \hat{q}_1 = \cdots = \hat{q}_l = 0 \} \).
2. \( p_i \) has no root in \( \{ \hat{q}_1 = \cdots = \hat{q}_l = 0 \} \).

**Proof.** Set \( A = \{ p_1 > 0, \ldots, p_m > 0, \hat{q}_1 = \cdots = \hat{q}_l = 0 \} \). Let \( z \in B \). If \( w(z) = 0 \), then \( p_i(z) = 0 \) by assumption. It is impossible since \( p_i(z) > 0 \). Thus \( w(z) \neq 0 \). Since \( w(z)\hat{q}_j(z) = q_j(z) = 0 \), \( \hat{q}_j(z) = 0 \) gives \( z \in A \). It implies \( B \subset A \). It is clear that \( A \subset B \) since \( \hat{q}_j(z) = 0 \) implies \( q_j(z) = 0 \). Thus \( A = B \) yields the first statement. Assume by contradiction that \( p_i \) has root \( a \in \{ \hat{q}_1 = \cdots = \hat{q}_l = 0 \} \). Then \( q_j(a) = w(a)\hat{q}_j(a) = 0 = p_i(a) \) leads to \( t - a \) divides \( w = \gcd(p_i, q_1, \ldots, q_l) \). Note that \( t - a \) also divides \( \hat{q}_j \). It implies that \( (t - a)^2 \) divides \( q_j = w\hat{q}_j \). It is impossible since \( q_j \) has only simple roots. Hence the second statement follows.
Remark 7. Obtaining \( \tilde{q}_i \)s in Lemma 16 requires \((l+1) \times O(d^2)\) arithmetic operations, where \(d\) is the upper bound on the degrees of \(p_i\)s and \(q_j\)s.

In the following lemma, we generalize the result in Lemma 16 by removing the assumption that each inequality polynomial has only simple roots:

**Lemma 16.** Let \( B \subset \mathbb{R} \) be an elementary semi-algebraic set defined as in \((35)\). Then there is a symbolic algorithm which produces \( \tilde{q}_1, \ldots, \tilde{q}_s \) such that the following two statements hold:

1. \( B = \{ p_1 > 0, \ldots, p_m > 0, \tilde{q}_1 = \cdots = \tilde{q}_s = 0 \} \).
2. For \( i = 1, \ldots, m, p_i \) has no root in \( \{ \tilde{q}_1 = \cdots = \tilde{q}_s = 0 \} \).

**Proof.** For \( j = 1, \ldots, l \), set \( \tilde{q}_j = q_j / \gcd(q_j, q_j') \). Lemma 8 says that \( \tilde{q}_j \) has only simple roots and \( \{ \tilde{q}_j = 0 \} = \{ q_j = 0 \} \). It implies that

\[
B = \{ p_1 > 0, \ldots, p_m > 0, \tilde{q}_1 = \cdots = \tilde{q}_s = 0 \}. \tag{36}
\]

Applying Lemma 16 several times with \( i = 1, \ldots, m \), we obtain \( \tilde{q}_1, \ldots, \tilde{q}_s \) as in the conclusion.

**Remark 8.** Thanks to Remark 7, obtaining \( \tilde{q}_j \)s in Lemma 16 requires \( l + (m + 1)(l + 1) \times O(d^2) \) arithmetic operations, where \( d \) is the upper bound on the degrees of \( p_i\)s and \( q_j\)s.

The following lemma is an induction step in the proof of later Lemma 26:

**Lemma 17.** Let \( B \subset \mathbb{R} \) be an elementary semi-algebraic set defined as in \((35)\). Let \( s < r < m \) and assume that the following conditions hold:

1. For \( i = 1, \ldots, m \), polynomial \( p_i \) has only simple roots.
2. For \( i = 1, \ldots, m \), polynomial \( p_i \) has no root in \( \{ q_1 = \cdots = q_r = 0 \} \).
3. For every \((i, j) \in \{1, \ldots, r\}^2\) satisfying \( i \neq j \), polynomials \( p_i \) and \( p_j \) have no common root.
4. For \( i = 1, \ldots, s \), polynomials \( p_{r+1} \) and \( p_i \) have no common root.

Set \( u_0 := \gcd(p_{r+1}, p_{r+1}) \). Let \( u_{s+1} = p_{r+1}/u_0 \) and \( u_{r+1} = p_{r+1}/u_0 \). Then \( B = B_+ \cup B_- \), where

\[
B_+ = \{ u_0 > 0, p_1 > 0, \ldots, p_s > 0, u_{s+1} > 0, p_{r+2} > 0, \ldots, p_r > 0, u_{r+1} > 0, p_{r+2} > 0, \ldots, u_m > 0, q_1 = \cdots = q_r = 0 \} \quad \text{and} \quad B_- = \{ -u_0 > 0, p_1 > 0, \ldots, p_s > 0, -u_{s+1} > 0, p_{r+2} > 0, \ldots, p_r > 0, -u_{r+1} > 0, p_{r+2} > 0, \ldots, p_m > 0, q_1 = \cdots = q_r = 0 \} \tag{37}
\]

are elementary semi-algebraic sets such that the following statements hold:

1. Polynomials \( u_0, u_{s+1}, u_{r+1} \in \mathbb{R}[t] \) have only simple roots.
2. Polynomials \( u_0, u_{s+1}, u_{r+1} \in \mathbb{R}[t] \) have no root in \( \{ q_1 = \cdots = q_r = 0 \} \).
3. For \( i \in \{1, \ldots, s\} \cup \{ s+2, \ldots, r\} \), polynomials \( u_0 \) and \( p_i \) have no common root.
4. For \( i \in \{1, \ldots, s\} \cup \{ s+2, \ldots, r\} \), polynomials \( u_{s+1} \) and \( p_i \) have no common root.
5. Polynomials \( u_0 \) and \( u_{s+1} \) have no common root;
6. For \( i = 1, \ldots, s \), polynomials \( u_{s+1} \) and \( p_i \) have no common root;
7. Polynomials \( u_{r+1} \) and \( u_0 \) have no common root;
8. Polynomials \( u_{r+1} \) and \( u_{s+1} \) have no common root.
Remark 9. We interpret the result of Lemma \[\text{17}\] more simply as follows: The elementary semi-algebraic sets \( B_+ \) and \( B_- \) in Lemma \[\text{17}\] are of the form

\[
\{ \hat{p}_0 > 0, \ldots, \hat{p}_m > 0, q_1 = \cdots = q_r = 0 \},
\]

which satisfies the following conditions:

1. For \( i = 0, 1, \ldots, m \), polynomial \( \hat{p}_i \) has only simple roots.
2. For \( i = 0, 1, \ldots, m \), polynomial \( \hat{p}_i \) has no root in \( \{ q_1 = \cdots = q_r = 0 \} \).
3. For every \( (i, j) \in \{ 0, 1, \ldots, r \}^2 \) satisfying \( i \neq j \), polynomials \( \hat{p}_i \) and \( \hat{p}_j \) have no common root.
4. For \( i = 0, 1, \ldots, s + 1 \), polynomials \( \hat{p}_{r+1} \) and \( \hat{p}_i \) have no common root.

Remark 10. Obtaining \( u_0, u_{s+1}, u_{r+1} \) in Lemma \[\text{17}\] requires \( O(d^2) + 2 \) arithmetic operations, where \( d \) is the upper bound on the degrees of \( p_i \)s and \( q_j \)s.

2.7 Nichtnegativstellensätze for univariate polynomials

Let \( \Sigma[t] \) stand for the cone of sums of squares of polynomials in \( \mathbb{R}[t] \). For given \( g \in \mathbb{R}[t], p = \{ p_1, \ldots, p_m \} \subset \mathbb{R}[t] \) and \( q = \{ q_1, \ldots, q_r \} \subset \mathbb{R}[t] \), define:

\[
S(p, q) := \{ t \in \mathbb{R} : p_1(t) \geq 0, \ldots, p_m(t) \geq 0, q_1(t) = \cdots = q_r(t) = 0 \},
\]

and for all \( k \in \mathbb{N} \),

\[
T_k(p, q)[t] := \left\{ \sum_{\alpha \in \{0,1\}^m} \sigma_\alpha p^\alpha + \sum_{j=1}^r \psi_j q_j \middle| \begin{array}{c}
\sigma_i \in \Sigma[t], \psi_j \in \mathbb{R}[t], \\
\deg(\sigma_i p^\alpha) \leq 2k, \deg(\psi_j q_j) \leq 2k
\end{array} \right\},
\]

\[
\rho_k(g, p, q) := \sup \{ \lambda \in \mathbb{R} : g - \lambda \in T_k(p, q)[t] \},
\]

where \( p^\alpha := p_1^{\alpha_1} \cdots p_m^{\alpha_m} \).

Originally developed by Lasserre in \[\text{11}\], problem \[\text{12}\] is called the sums-of-squares strengthening of order \( k \) for polynomial optimization problem

\[
\rho^*(g, p, q) := \inf_{t \in S(p, q)} g(t).
\]

Moreover, we can formulate problem \[\text{12}\] as a semidefinite program, the optimization of a linear function over the intersection of an affine subspace with the cone of positive semidefinite matrices.

Lemma 18. Let \( p = \{ p_1, \ldots, p_m \} \) and \( q = \{ q_1, \ldots, q_r \} \) be subsets of \( \mathbb{R}[t] \) such that the following conditions hold:

---

Proof. Since \( p_{s+1} = u_{s+1}u_0 \) and \( p_{r+1} = u_{r+1}u_0 \) have only simple roots and have no root in \( \{ q_1 = \cdots = q_r = 0 \} \), the first two statement hold. In addition, we get

\[
\{ p_{s+1} > 0, p_{r+1} > 0 \} = \{ u_0 > 0, u_{s+1} > 0, u_{r+1} > 0 \} \cup \{ -u_0 > 0, -u_{s+1} > 0, -u_{r+1} > 0 \}. \tag{38}
\]

It implies that \( B = B_+ \cup B_- \). The next two statements follow since \( p_{s+1} = u_{s+1}u_0 \) and the couple \( p_{s+1}, p_i \) has no common root, for \( i \in \{ 1, \ldots, s \} \cup \{ s + 2, \ldots, r \} \). The fifth statement is due to the fact that \( p_{s+1} = u_{s+1}u_0 \) has only simple roots. The sixth statement is because for every \( i = 1, \ldots, s \), \( p_{r+1} = u_{r+1}u_0 \) and \( p_i \) have no common root. The seventh statement is due to the fact that \( p_{r+1} = u_{r+1}u_0 \) has only simple roots. If \( w = \gcd(u_{s+1}, u_{r+1}) \) is not constant, \( wu_0 \) divides both \( p_{s+1} \) and \( p_{r+1} \). This is impossible since \( u_0 = \gcd(p_{s+1}, p_{r+1}) \). Thus the eighth statement holds. 

\[ \square \]
1. For $i = 1, \ldots, m$, polynomial $p_i$ has only simple roots.

2. For every $(i, j) \in \{1, \ldots, m\}^2$ satisfying $i \neq j$, polynomials $p_i$ and $p_j$ have no common root.

Then one of the following two cases occurs:

1. The set $S(p, q)$ is a finite union of isolated points.

2. The set $S(p, q)$ is a finite union of closed intervals and has no isolated point. In this case, for each endpoint $a$ of $S(p, q)$, there exists $i \in \{1, \ldots, m\}$ such that $(t - a)$ divides $p_i$, but $(t - a)^2$ does not.

**Proof.** Consider the following two cases:

- **Case 1:** $\{q_1 = \cdots = q_l = 0\} \neq \emptyset$. Then $S(p, q) \subset \{q_1 = \cdots = q_l = 0\}$ is a finite union of isolated points.

- **Case 2:** $\{q_1 = \cdots = q_l = 0\} = \emptyset$. Then we get $S(p, q) = S(p, \emptyset)$. By the third statement of Lemma 4, $S(p, \emptyset)$ is a finite union of closed intervals and points. Assume by contradiction that $S(p, \emptyset)$ has an isolated point $a$. Then $a$ is a root of $p_j$ for some $j \in \{1, \ldots, m\}$. By assumption, $a$ is a root of $p_j$ with odd multiplicity. Lemma 11 yields that $p_j$ changes sign at $a$. Then $[a - \epsilon, a] \cup [a, a + \epsilon]$ or $[a - \epsilon, a]$ is a subset of $\bigcup_{j=1}^m \{p_j \geq 0\}$ for sufficiently small $\epsilon > 0$. By assumption, $a$ is not a root of $p_i$ for any $i \in \{1, \ldots, m\} \setminus \{j\}$. It implies that $p_i(a) > 0$, for $i \in \{1, \ldots, m\} \setminus \{j\}$. Then we take $\epsilon$ small enough such that

$$[a - \epsilon, a + \epsilon] \subset \bigcap_{i \neq j} \{p_i \geq 0\}. \quad (44)$$

Thus $[a, a + \epsilon]$ or $[a - \epsilon, a]$ is a subset of $S(p, \emptyset) = \bigcap_{i=1}^m \{p_i \geq 0\}$. It is impossible since $a$ is an isolated point of $S(p, \emptyset)$. Thus $S(p, q)$ has no isolated point. Assume that $a$ is an endpoint of $S(p, q)$. Then there exists $i \in \{1, \ldots, m\}$ such that $p_i(a) = 0$. It implies that $(t - a)$ divides $p_i$. By the first condition, $(t - a)^2$ does not divide $p_i$.

Hence the result follows. \qed

The following lemma follows from Nie’s Nichtnegativstellensatz in [16, Theorem 4.1]:

**Lemma 19.** Let $h \in \mathbb{R}[t]$, $p = \{p_1, \ldots, p_m\} \subset \mathbb{R}[t]$, $q = \{q_1, \ldots, q_l\} \subset \mathbb{R}[t]$. Assume that $S(p, q)$ is a finite union of isolated points and $h$ is nonnegative on $S(p, q)$. Then there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $h \in T_k(p, q)[t]$.

The following lemma restates Kuhlmann–Marshall–Schwartz’s Nichtnegativstellensatz in [10, Corollary 3.3]:

**Lemma 20.** Let $p = \{p_1, \ldots, p_m\}$ be a subset of $\mathbb{R}[t]$ and $q = \emptyset$ such that $S(p, q) = \{p_1 > 0, \ldots, p_m > 0\}$ is compact and has no isolated points. Then

$$\{h \in \mathbb{R}[t] : h \geq 0 \text{ on } S(p, q)\} = \bigcup_{k=1}^\infty T_k(p, q)[t], \quad (45)$$

if and only if, for each endpoint $a$ of $S(p, q)$, there exists $i \in \{1, \ldots, m\}$ such that $(t - a)$ divides $p_i$, but $(t - a)^2$ does not.

The following lemma is a consequence of Lemmas 19 and 20.
Lemma 21. Let \( q \in \mathbb{R}[t] \), \( p = \{p_1, \ldots, p_m\} \subset \mathbb{R}[t] \), \( q = \{q_1, \ldots, q_t\} \subset \mathbb{R}[t] \), and \( \rho^*(g, p, q) \) be as in (4). Let the assumption of Lemma 18 hold. Assume that \( \rho^*(g, p, q) \) is compact. Then there exists \( K \in \mathbb{N} \) such that for all \( k \geq K \), \( g - \rho^*(g, p, q) \in T_k(p, q)[t] \) and therefore \( \rho_p(g, p, q) = \rho^*(g, p, q) \).

Proof. By assumption, \( g - \rho^*(g, p, q) \) is nonnegative on \( \rho_p^*(g, p, q) \). Due to Lemma 18, one of the two cases mentioned in this lemma occurs. In the first case, we apply Lemma 19 to get \( K \in \mathbb{N} \) satisfying that for all \( k \geq K \), \( g - \rho^*(g, p, q) \in T_k(p, q)[t] \). In the second case, Lemma 20 allows us to obtain such nonnegative integer \( K \). Hence the conclusion follows. \( \square \)

3 Products of linear factors occurring odd times

The following algorithm allows us to obtain the product of linear factors of a polynomial occurring odd times:

Algorithm 1. Products of linear factors occurring odd times.

- Input: Polynomial \( p \) in \( \mathbb{R}[t] \).
- Output: Rational function \( u_{i^*} \) in single-variable \( t \).

1. Set \( u_0 := 1 \), \( h_0 := p \), and \( i = 0 \).
2. Set \( q_i := \gcd(h_i, h'_i) \) and \( w_i := \gcd(q_i, q'_i) \).
3. Update \( u_{i+1} := u_i \times \frac{h_i w_i}{q_i} \).
4. If \( w_i \) is not constant, update \( h_{i+1} := w_i \), \( i := i + 1 \) and run again Steps 2, 3, 4. Otherwise, set \( i^* = i + 1 \) and stop.

Lemma 22. Let \( u_{i^*} \) be the output of Algorithm 1 with input \( p \) as in (17). Then there exists a constant \( C \in \mathbb{R} \) such that

\[
 u_{i^*} = C \times \prod_{\substack{i=1 \atop m_i \in \text{odd}}}^s (t - a_i) \tag{46}
\]

is a polynomial in \( \mathbb{R}[t] \).

Proof. Let us prove that Algorithm 1 terminates after a finite number of steps. By Lemma 8, the first two steps produce \( h_0 w_0 \) as the right-hand side of (23). It is the product of linear factors of \( p \) occurring once and thus is a polynomial. In addition, Lemma 8 yields

\[
 w_0 = \prod_{\substack{i=1 \atop m_i > 1}}^s (t - a_i)^{m_i - 2} \tag{47}
\]

It implies \( \deg(w_0) \leq \deg(h_0) - 2s \). The third step gives \( u_1 \) as the right-hand side of (23). In the fourth step, we obtain \( h_1 = w_0 \), which gives \( \deg(h_1) = \deg(w_0) \leq \deg(h_0) - 2s \). Repeating this process, we get \( \frac{h_i w_i}{q_i} \) as the product of linear factors of \( p \) with multiplicity \( 2i + 1 \) and \( \deg(h_{i+1}) < \deg(h_i) < \deg(p) \). Then \( u_{i+1} \) is the product of linear factors of \( p \) occurring \( 1, 3, \ldots, 2i + 1 \) times. Moreover, Algorithm 1 cannot run forever since \( \deg(h_{i+1}) < \deg(h_i) \) and \( \deg(h_i) < \deg(p) \). Hence the result follows. \( \square \)

Remark 11. Thanks to the complexity given in Remark 8. Algorithm 1 with input \( p \) of degree \( d \) has complexity \( O(d^2) \).

Given a polynomial \( p \in \mathbb{R}[t] \), denote by \( \text{lc}(p) \) the leading coefficient of \( p \).
Definition 1. For a given polynomial $p$ in $\mathbb{R}[t]$ with decomposition (17), we define polynomial $\tilde{p} \in \mathbb{R}[t]$ by

$$
\tilde{p} := \begin{cases} 
\text{lcm}(p) \times u_{i^*} & \text{if } u_{i^*} \neq 0, \\
0 & \text{otherwise}
\end{cases} \quad (48)
$$

where $u_{i^*}$ is the output of Algorithm 1 with input $p$.

By definition, $\tilde{p}$ has the same leading coefficient as $p$. Roughly speaking, polynomial $\tilde{p}$ enables us to remove all linear factors of a polynomial $p$ occurring even times.

The following lemma states some properties of polynomial $\tilde{p}$ needed for the later results:

Lemma 23. Let $p$ be polynomial in $\mathbb{R}[t]$ with decomposition (17). The following statements hold:

1. The polynomial $\tilde{p}$ is the right-hand side of (46). Consequently, $\tilde{p}$ has only simple roots.
2. The polynomial $p$ can be decomposed as $p = \tilde{p} \times w$, where

$$
w = \prod_{j=1}^{s} (t - a_j)^{m_j} \times \prod_{i=1}^{s} (t - a_i)^{m_i - 1} \quad (49)
$$

is a nonnegative polynomial in $\mathbb{R}[t]$. Consequently, $p(t) > 0$ if and only if $\tilde{p}(t) > 0$.

Proof. The first statement is due to Lemma 22. The second statement relies on the first statement and the decomposition (17). It is clear that $w \in \mathbb{R}[t]$ thanks to the second statement of Lemma 5. The non-negativity of $w$ is because $w$ is the product of the squares of linear factors of $p$. \(\square\)

4 Replacement of strict inequalities by non-strict ones

In this section, we demonstrate that replacing strict inequalities defining a “nice” elementary semi-algebraic set with non-strict ones allows us to obtain the closure of this set.

Definition 2. For a given elementary semi-algebraic set $B \subset \mathbb{R}$ defined as in (34), we define elementary semi-algebraic set $\tilde{B} \subset \mathbb{R}$ by replacing strict inequalities $p_i(t) > 0$ determining $B$ with non-strict inequalities $\tilde{p}_i(t) \geq 0$, i.e.,

$$
\tilde{B} = \{t \in \mathbb{R} : \tilde{p}_1(t) \geq 0, \ldots, \tilde{p}_m(t) \geq 0, q_1(t) = \cdots = q_l(t) = 0\}. \quad (50)
$$

Remark 12. The set $\tilde{B}$ in (50) is semi-algebraic since it can be written as

$$
\tilde{B} = \bigcup_{* \in \{>,=\}^m} \{t \in \mathbb{R} : \tilde{p}_1(t) *_1 0, \ldots, \tilde{p}_m(t) *_m 0, q_1(t) = \cdots = q_l(t) = 0\}. \quad (51)
$$

Lemma 24. Let $B \subset \mathbb{R}$ be an elementary semi-algebraic set defined as in (34). Then

$$
B = \{\tilde{p}_1 > 0, \ldots, \tilde{p}_m > 0, q_1 = \cdots = q_l = 0\}. \quad (52)
$$

Proof. By using the second statement of Lemma 23 we get the equivalence: $p_i(t) > 0$ if and only if $\tilde{p}_i(t) > 0$. Hence the result follows. \(\square\)
The following example given in [1, Section 2.1] shows that replacing strict inequalities in elementary semi-algebraic set $B$ without removing the factors of the inequality polynomials occurring even times might not allow for the closure of $B$ to be obtained:

**Example 2.** Let $B = \{ t \in \mathbb{R} : t^2(t-1) > 0 \}$. Then $B = (1, \infty)$. Replacing the strict inequality in $B$ by a non-strict inequality, we obtain $B = \{ t \in \mathbb{R} : t^2(t-1) \geq 0 \} = \{ 0 \} \cup [1, \infty)$, which is not the closure of $B$. However, by definition, $\overline{B} = \{ t \in \mathbb{R} : t - 1 \geq 0 \} = [1, \infty)$ is exactly the closure of $B$.

The following example indicates the case where $B$ is not the closure of an elementary semi-algebraic set $B$:

**Example 3.** Let $B = \{ p_1 > 0, p_2 > 0 \}$ with $p_1 = -t(t-1)$ and $p_2 = -t(t+1)$. Then $\{ p_1 > 0 \} = (0,1)$ and $\{ p_2 \geq 0 \} = (-1,0)$ yields $B = \{ p_1 > 0 \} \cap \{ p_2 \geq 0 \} = \emptyset$. However, since $\tilde{p}_1 = p_1$, $\{ \tilde{p}_1 > 0 \} = [0,1]$ and $\{ \tilde{p}_2 \geq 0 \} = [-1,0]$ yields $\overline{B} = \{ \tilde{p}_1 > 0 \} \cap \{ \tilde{p}_2 \geq 0 \} = \{ 0 \}$. Thus $\overline{B} = \emptyset \neq \{ 0 \} = \overline{B}$.

To address the issues in Examples 2 and 3, we need the following lemma:

**Lemma 25.** Let $B \subset \mathbb{R}$ be an elementary semi-algebraic set of the form $[39]$. Suppose that the following conditions hold:

1. Any couple of inequality polynomials defining $B$ have no common real root with odd multiplicity.
2. Any inequality polynomial defining $B$ has no root in $\{ q_1 = \cdots = q_s = 0 \}$ with odd multiplicity.

Then $\overline{B}$ is the closure of $B$, i.e., $\overline{B} = \overline{B}$.

**Proof.** By Lemma 24, equality $[39]$ holds. For $i = 1, \ldots, m$, let $a_i, r_{ik}$ be the real roots of $p_i$, with odd multiplicities. Lemma 23 yields that $a_{i1}, \ldots, a_{ir_i}$ are all simple real roots of $\tilde{p}_i$. By assumption, for every $(i,r) \in \{ 1, \ldots, m \}$ satisfying $i \neq j$, $\tilde{p}_i$ and $\tilde{p}_j$ have no common root. From this, Lemma 14 yields

$$\bigcap_{i=1}^{m} \{ \tilde{p}_i \geq 0 \} = \bigcap_{i=1}^{m} \{ \tilde{p}_i \geq 0 \}.$$  (53)

Set $V = \{ q_1 = \cdots = q_s = 0 \}$. By assumption, each $\tilde{p}_i$ has no root in $V$. From this, Lemma 14 implies

$$\bigcap_{i=1}^{m} \{ \tilde{p}_i \geq 0 \} \cap V = \bigcap_{i=1}^{m} \{ \tilde{p}_i \geq 0 \} \cap V.$$  (54)

Combining this with $[39]$, $[39]$, and $[39]$, we obtain the result.

Although not all elementary semi-algebraic sets have properties as in Lemma 25 we can always decompose each elementary semi-algebraic set as the union of a finite number of elementary semi-algebraic sets with these properties:

**Lemma 26.** Let $B \subset \mathbb{R}$ be an elementary semi-algebraic set. Then there is a symbolic algorithm which produces elementary semi-algebraic subsets $B_1, \ldots, B_s$ of $\mathbb{R}$ such that $B = \bigcup_{j=1}^{s} B_j$ and the following two statements hold:

1. For $j = 1, \ldots, s$, each inequality polynomial defining $B_j$ has only simple roots.
2. For $j = 1, \ldots, s$, any inequality polynomial defining $B_j$ has no root in the variety determined by the equality polynomials defining $B_j$.
3. For $j = 1, \ldots, s$, any couple of inequality polynomials defining $B_j$ has no common root.
Proof. Suppose that $B$ is of the form (54). By Lemma 24 equality (52) holds. Using Lemma 16 we get $\tilde{q}_i \in \mathbb{R}[t]$ such that

$$\tilde{B} = \{ t \in \mathbb{R} : \tilde{p}_1(t) \geq 0, \ldots, \tilde{p}_m(t) \geq 0, \tilde{q}_1(t) = \cdots = \tilde{q}_s(t) = 0 \}.$$  

(55)

and each $\tilde{p}_i$ has no root in $\{q_1(t) = \cdots = \tilde{q}_s(t) = 0\}$. Note that the first statement of Lemma 24 implies that each $\tilde{p}_i$ has only simple roots. Applying Lemma 17 several times with $s = 1, \ldots, r - 1$ and $r = 1, \ldots, m - 1$, we obtain $B_1, \ldots, B_s$ as in the conclusion.

Remark 13. Thanks to Remarks 11, 8, and 10, the algorithm mentioned in Lemma 26 has complexity

$$l + (m + 1)(l + 1) \times O(d^3) + m \times O(d^3) + m! \times (O(d^3) + 2),$$

(56)

where $B$ is of the form (54) and $d$ is the upper bound on the degrees of $p_i, s$ and $q_j, s$.

5 Main algorithm and application to polynomial optimization

The following algorithm enables us to obtain semidefinite descriptions of the closures of the images of semi-algebraic sets under polynomials:

Algorithm 2. Closures of the images of semi-algebraic sets under polynomials.

- **Input:** Semi-algebraic set $S \subset \mathbb{R}^n$ and polynomial $f \in \mathbb{R}[x]$.
- **Output:** Semi-algebraic description of $Q \subset \mathbb{R}$.

1. Compute $p_{i,j}, q_{k,j} \in \mathbb{R}[t]$ such that $f(S) = \bigcup_{j=1}^s B_j$ with

$$B_j = \{ p_{i,j} > 0, \ldots, p_{m,j} > 0, q_{1,j} = \cdots = q_{s,j} = 0 \}.$$  

(57)

2. For $j = 1, \ldots, s$, do:

   a. Compute $p_{r,j}, q_{r,j} \in \mathbb{R}[t]$ such that $B_j = \bigcup_{k=1}^{r,j} B_{r,j}$ with

$$B_{r,j} = \{ p_{r,j} > 0, \ldots, p_{m,j,r-j} > 0, q_{1,j} = \cdots = q_{r,j,s,j} = 0 \}.$$  

(58)

and the following three conditions hold:

i. Each of $p_{1,j}, \ldots, p_{m,j,r-j}$ has only simple root.
ii. Any of $p_{1,j}, \ldots, p_{m,j,r-j}$ has no root in $\{ q_{1,j} = \cdots = q_{r,j,s,j} = 0 \}$.
iii. Any couple of $p_{1,j}, \ldots, p_{m,j,r-j}$ has no common root.

b. Compute $\tilde{p}_{r,j}$ as in Definition 1 by using Algorithm 1.

3. Set $Q := \bigcup_{j=1}^s \bigcup_{r=1}^{r,j} \tilde{B}_{r,j}$ with $\tilde{B}_{r,j}$ as in Definition 2, i.e.,

$$\tilde{B}_{r,j} = \{ \tilde{p}_{r,j} \geq 0, \ldots, \tilde{p}_{m,j,r-j} \geq 0, q_{1,j} = \cdots = q_{r,j,s,j} = 0 \}.$$  

(59)

The first step of Algorithm 2 relies on Basu’s method in [1, Section 3]. We use the algorithm mentioned in Lemma 26 to do Step 2 (a) of Algorithm 2.

We guarantee that the output $Q$ of Algorithm 2 is identical to the closure $f(S)$ in the following theorem:

Theorem 1. Let $f$ be a polynomial and $S$ be a semi-algebraic set. Then Algorithm 2 with input $f, S$ produces polynomials defining $A_1, \ldots, A_w$ of the form (1) such that $Q = \bigcup_{j=1}^w A_j$ is exactly the closure of $f(S)$ and the following conditions hold:
1. Each inequality polynomial defining \( A_j \) has only simple roots.

2. Any couple of inequality polynomials determining each \( A_j \) has no common root.

Proof. Lemma 23 implies that \( \tilde{B}_{r,j} = B_{r,j} \). From this and the second statement of Lemma 10 we get

\[
Q = \bigcup_{j=1}^{s} \bigcup_{r=1}^{s_j} \tilde{B}_{r,j} = \bigcup_{j=1}^{s} \bigcup_{r=1}^{s_j} B_{r,j} = \bigcup_{j=1}^{s} B_j = f(S). \quad (60)
\]

Setting \( A_1, \ldots, A_w \) as \( \tilde{B}_{r,j}s \), we obtain the result. \( \square \)

Remark 14. Assume that \( S \subset \mathbb{R}^n \) is determined by \( s \) polynomials of degree at most \( d \) and \( f \) has degree at most \( d \). By Remark 3, the first step of Algorithm 2 requires \( s^{n+1}dO(n) \) arithmetic operations and the degrees of \( p_{i,j}, q_{i,j} \) are upper bounded by \( dO(n) \). Remark 13 yields that Step 2 (a) of Algorithm 2 has a complexity of \( O \left( \rho_j \right) \). By Remark 11, the complexity of Step 2 (b) of Algorithm 2 is of \( O(d^{3s}O(n)) \times \sum_{j=1}^{n} m_{r,j} \). Note that the degrees of the polynomials do not increase in Steps 2 and 3 of Algorithm 2.

Thus Algorithm 2 has the number of arithmetic operations as \( O(d^O(n)) \) to produce polynomials of degrees at most \( d^{O(n)} \) that define the closure \( f(S) \).

The following corollary makes use of Algorithm 2 in polynomial optimization, especially in the case where the objective polynomials do not attain minimum value on the semi-algebraic sets:

Corollary 1. Let \( f \) be a polynomial and \( S \) be a semi-algebraic set. Consider polynomial optimization problem:

\[
f^* = \inf_{x \in S} f(x). \quad (61)
\]

Assume that the image \( f(S) \) is bounded. Let \( Q \) be as the output of Algorithm 2 with input \( f, S \). Then there exist \( p^{(1)} \subset \mathbb{R}[t] \) and \( p^{(2)} \subset \mathbb{R}[t] \) such that

\[
f^* = \min_{t \in Q} g(t), \quad (62)
\]

where \( g(t) = t \) and \( Q = \bigcup_{j=1}^{w} S(p^{(j)}, q^{(j)}) \) satisfying the following conditions:

1. Each polynomial in \( p^{(j)} \) has only simple roots.

2. Any couple of polynomials in \( p^{(j)} \) has no common root.

Moreover, there exists \( K \in \mathbb{N} \) such that for all \( k \geq K \),

\[
f^* = \min_{j=1, \ldots, w} \rho_k(g, p^{(j)}, q^{(j)}). \quad (63)
\]

Proof. Theorem 14 yields \( Q = f(S) \), which implies the first statement. Let us prove the second one. Since \( f(S) \) is bounded, so is \( S(p^{(j)}, q^{(j)}) \). From \( Q = \bigcup_{j=1}^{w} S(p^{(j)}, q^{(j)}) \), we get

\[
f^* = \min_{j=1, \ldots, w} \rho^*(g, p^{(j)}, q^{(j)}). \quad (64)
\]

Lemma 24 says that there exists \( K_j \in \mathbb{N} \) such that for all \( k \geq K_j, \rho_k(g, p^{(j)}, q^{(j)}) = \rho^*(g, p^{(j)}, q^{(j)}) \). Set \( K = \max_{j=1, \ldots, w} K_j \). Hence the result follows thanks to (64). \( \square \)

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References

[1] S. Basu. New results on quantifier elimination over real closed fields and applications to constraint databases. *Journal of the ACM (JACM)*, 46(4):537–555, 1999.

[2] S. Basu, R. Pollack, and M. Roy. *Algorithms in Real Algebraic Geometry*. Algorithms and computation in mathematics. Springer, 2003.

[3] S. Belhaj and H. B. Kahla. On the complexity of computing the GCD of two polynomials via Hankel matrices. *ACM Communications in Computer Algebra*, 46(3/4):74–75, 2013.

[4] M. Coste. An introduction to semialgebraic geometry, 2000.

[5] J. Demmel, J. Nie, and V. Powers. Representations of positive polynomials on noncompact semialgebraic sets via KKT ideals. *Journal of pure and applied algebra*, 209(1):189–200, 2007.

[6] S. T. Dinh, H. V. Ha, and T. S. Pham. A Frank–Wolfe type theorem for non-degenerate polynomial programs. *Mathematical Programming*, 147(1):519–538, 2014.

[7] H. V. Hà and T. S. Pham. Solving polynomial optimization problems via the truncated tangency variety and sums of squares. *Journal of Pure and Applied Algebra*, 213(11):2167–2176, 2009.

[8] J. Heintz, M.-F. Roy, and P. Solernó. Sur la complexité du principe de Tarski-Seidenberg. *Bulletin de la Société mathématique de France*, 118(1):101–126, 1990.

[9] E. Korkina and A. G. Kushnirenko. Another proof of the Rarski-Seidenberg theorem. *Siberian Mathematical Journal*, 26(5):703–707, 1985.

[10] S. Kuhlmann, M. Marshall, and N. Schwartz. Positivity, sums of squares and the multi-dimensional moment problem II. 2005.

[11] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[12] V. Magron, D. Henrion, and J.-B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. *SIAM Journal on Optimization*, 25(4):2143–2164, 2015.

[13] N. H. A. Mai. A symbolic algorithm for exact polynomial optimization strengthened with Fritz John conditions. *arXiv preprint arXiv:2206.02643*, 2022.

[14] N. H. A. Mai. Exact polynomial optimization strengthened with Fritz John conditions. *arXiv preprint arXiv:2205.04254*, 2022.

[15] N. H. A. Mai, J.-B. Lasserre, and V. Magron. Positivity certificates and polynomial optimization on non-compact semialgebraic sets. *Mathematical Programming*, pages 1–43, 2021.

[16] J. Nie. Polynomial optimization with real varieties. *SIAM Journal On Optimization*, 23(3):1634–1646, 2013.

[17] J. Nie, J. Demmel, and B. Sturmfels. Minimizing polynomials via sum of squares over the gradient ideal. *Mathematical programming*, 106(3):587–606, 2006.

[18] T.-S. Pham. Optimality conditions for minimizers at infinity in polynomial programming. *Mathematics of Operations Research*, 44(4):1381–1395, 2019.

[19] T. S. Pham and H. H. Vui. *Genericity in polynomial optimization*, volume 3. World Scientific, 2016.

[20] M. Schweighofer. Global optimization of polynomials using gradient tentacles and sums of squares. *SIAM Journal on Optimization*, 17(3):920–942, 2006.
[21] A. Seidenberg. A new decision method for elementary algebra. *Annals of Mathematics*, pages 365–374, 1954.

[22] A. Tarski. A decision method for elementary algebra and geometry, revised. *Berkeley and Los Angeles*, 1951.