Equivalence classes of non-local unitary operations

W. Dür\textsuperscript{1} and J. I. Cirac\textsuperscript{2}

\textsuperscript{1}Sektion Physik, Ludwig-Maximilians-Universität München, Theresienstr. 37, D-80333 München, Germany
\textsuperscript{2}Max Planck Institut für Quantenoptik, D-85748 München, Germany

(March 31, 2022)

We study when a multipartite non–local unitary operation can deterministically or probabilistically simulate another one when local operations of a certain kind—in some cases including also classical communication—are allowed. In the case of probabilistic simulation and allowing for arbitrary local operations, we provide necessary and sufficient conditions for the simulation to be possible. Deterministic and probabilistic interconversion under certain kinds of local operations are used to define equivalence relations between gates. In the probabilistic, bipartite case this induces a finite number of classes. In multiqubit systems, however, two unitary operations typically cannot simulate each other with non-zero probability of success. We also show which kind of entanglement can be created by a given non–local unitary operation and generalize our results to arbitrary operators.

03.67.-a, 03.65.Bz, 03.65.Ca, 03.67.Hk

I. INTRODUCTION

In the last decade, there has been big effort to characterize qualitatively and quantitatively entanglement properties of pure and mixed states. This relies in part on the fact that entanglement is thought to be the key ingredient for many applications in Quantum Information Theory (QIT). A proper understanding of entanglement is expected to lead not only to possible new applications in quantum computation and quantum communication, but also to a more satisfactory understanding of the basic principles of quantum mechanics and especially of QIT.

Only quite recently, it was realized that also entanglement properties of physical operations are of relevance, as after all we deal with interactions in experiments and the interactions allow us to create entangled states. In recent years, first steps have been taken in this direction. In particular, the possibility to implement non–local operations consuming an entangled state [1–3], the capability to create entanglement in an optimal way given an interaction Hamiltonian [3–4], the simulation of an interaction Hamiltonian by some other one [3–4] as well as a connection of the entanglement properties of operations to the entanglement properties of states [4–5] have been established. Many application of this last relation, including the storage, tomography, teleportation, cloning and purification of operations, as well as the possibility to decide whether a given operation can create entanglement or not, have been found [5–6].

However—compared to the extensive knowledge on the structure of entangled quantum states—, only few is known for quantum operations. The most successful approach to characterize the entanglement properties of bipartite and multipartite pure states is concerned with the study of equivalence relations under certain classes of allowed operations, e.g. local unitaries (LU), local operations (LO), local operations and classical communication (LOCC) or stochastic local operations and classical communication (SLOCC), possible applied to many copies of a system [6]. That is, two states are identified if they can be obtained from each other by means of a certain class of operations, e.g. LU. In this case, we say that two states belong to the same (equivalence) class under LU. For bipartite systems, when considering equivalence classes under LOCC, this leads to the well known Schmidt decomposition, while the Schmidt number, i.e. the number of non–zero Schmidt coefficients, turns out to be the relevant quantity when considering equivalence classes under SLOCC [17]. For three qubit systems, this approach allowed to identify two inequivalent kinds of tripartite entanglement under SLOCC represented by the states $(GHZ)$ and $|W\rangle$, defined in Eq. (2a) [17]. When applied to many copies of a bipartite system [18], considering equivalence classes under LOCC operations and allowing for small imperfections, this criterion leads to identify all kinds of bipartite pure state entanglement with that of the EPR–Bohm state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ [19].

In this paper, we introduce a similar notion of equivalence relations under certain classes of operations for non–local unitary operations. First steps in this direction have been recently reported in [20], where equivalence classes for single bipartite unitary operators $U$ under SLOCC were considered. That is, whether two non–local unitary operations $U$ and $\tilde{U}$ can simulate each other with non–zero probability of success. In this paper, we generalize this approach and provide a general framework which covers also single multipartite unitary operations, multiple copies of unitary operations as well as other classes of allowed local operations such as LU, LO, and LOCC. Our classification allows us to put a partial order in the set of multipartite nonlocal unitary operations. In the bipartite case of probabilistic simulation, we obtain a complete hierarchic classification [20].

Note that such classification is not only of theoretical interest, but might also be of some practical relevance. For example, it allows to decide whether a given unitary operations—e.g. produced by a weak interaction—is already sufficient to implement a relevant task in QIT, e.g.
entanglement purification, one of the basic primitives for long range quantum communication using quantum repeaters \[21\]. Standard entanglement purification schemes \[22,23\] require the possibility to implement CNOT gates \[23\]. In certain physical systems — e.g. the polarization modes of singe photons —, one might however not be able to implement a CNOT gate but rather only a weak interaction between two particles, e.g. a phase gate with a small phase. It is a relevant question in this context whether such gates are already sufficient to implement a CNOT gate — and thus entanglement purification — with certain probability of success. We derive necessary and sufficient conditions in terms of SLOCC equivalence of pure states. This allows e.g. to answer the problem mentioned above in a positive way, i.e. any phase gate with an arbitrary small phase allows to realize entanglement purification. Note that in the context of entanglement purification, it is sufficient to consider only probabilistic simulation of the gates, as entanglement purification itself is already a probabilistic process.

In the context of quantum computation, however, such probabilistic simulation may change the complexity class of a given algorithm and might thus not be suitable. In this case, deterministic simulation plays a more important role, which corresponds to equivalence classes of unitary operations under deterministic LOCC.

The paper is organized as follows: We start in Sec. \[III\] by reviewing some relevant results on equivalence classes under LU, LOCC and SLOCC for pure states and a previously introduced isomorphism which relates non–local physical operations and states. In Sec. \[IV\], we fix some notation and define in a similar way gate simulation and equivalence classes under LU, LOCC and SLOCC for unitary operations. We also briefly discuss gate simulation under LU and LOCC in Sec. \[IV\]. The isomorphism of Sec. \[III\] turns out to be the main tool to establish necessary and sufficient conditions for probabilistic gate simulation, as it provides a connection of this problem to the well studied problem of SLOCC conversion of pure states. In Sec. \[V\], this connection is established and the implications for bipartite and multipartite unitary operations are discussed in detail. Sec. \[VI\] is concerned with the question, which kind of entanglement a given unitary operation can create, and necessary and sufficient conditions in terms of LU, LO, LOCC and SLOCC conversion of pure states are given. In Sec. \[VII\], we generalize our results to arbitrary operators and summarize and conclude in Sec. \[VIII\].

II. RELEVANT RESULTS FOR PURE STATES AND CONNECTION BETWEEN STATES AND OPERATIONS

In this section, we review some relevant results on equivalence classes under certain classes of operations for multipartite pure states as well as the isomorphism between physical operations and states.

A. Equivalence classes under LU, LOCC and SLOCC for pure states

A widely studied subject in QIT is concerned with the entanglement properties of multiparticle pure states. A situation of particular interest consists of several spatially separated parties, each of them holding one of the systems of a multiparticle pure state they share. In this setting, the parties are restricted to apply some kind of local operations and eventually to communicate classically. In this scenario, it turned out to be a very fruitful approach to identify pure states which can be converted into each other using a certain kind X of local operations, where X \( \in \{ \text{LU, LU+ancilla, LO, LOCC, SLOCC} \} \). This allows to identify in some sense the entanglement properties of two states. In the case of LU, LO+ancilla, LO and LOCC, both states allow to perform exactly the same QIT tasks, while in case of SLOCC operations, the probability of a successful performance of the task may differ.

We say that a multipartite entangled pure state \( |\Psi\rangle \) can be converted to some other pure state \( |\Phi\rangle \) under a certain class of operations \( X \), \( |\Psi\rangle \xrightarrow{X} |\Phi\rangle \), if there exists a sequence of local operations of the kind \( X \) which transform the state \( |\Psi\rangle \) exactly into the state \( |\Phi\rangle \). In case of SLOCC, only a probabilistic conversion is required, i.e. the state \( |\Phi\rangle \) has to be obtained only with some non–vanishing probability of success.

Note that above relation induces an equivalence relation in the set of pure states, namely two states \( |\Psi\rangle \) and \( |\Phi\rangle \) are equivalent under operations of the kind \( X \), \( |\Psi\rangle \equiv_X |\Phi\rangle \iff |\Psi\rangle \xrightarrow{X} |\Phi\rangle \) and \( |\Psi\rangle \xleftarrow{X} |\Phi\rangle \). That is, the states can be converted into each other and are said to belong to the same equivalence class.

1. Bipartite systems

The definition of equivalence classes under LU allows to write any pure state \( |\Psi\rangle \in \mathcal{A}^d \otimes \mathcal{A}^d \) in its Schmidt decomposition, i.e. to identify \( |\Psi\rangle \) with a state of the normal form

\[
\sum_{i=1}^{n_{\Psi}} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle = U_A \otimes U_B |\Psi\rangle; \quad n_{\Psi} \leq d, \quad (1)
\]

where \( S = \{|i\rangle\}_{i=1}^{d} \) is an orthonormal basis. The real, positive coefficients \( \lambda_i \neq 0 \) sum up to unity and we have that \( n_{\Psi} \) is the so called Schmidt number, i.e. the number of nonzero Schmidt coefficients. The Schmidt decomposition turned out to be a very useful tool in many applications in QIT.

\[^1\]One may also include in this list catalytic assisted operations of each kind.
It is known that considering equivalence classes under LOCC does not allow for a further reduction of the relevant parameters and thus for no further simplification. This is due to the fact that two pure states $|\Psi\rangle$ and $|\Phi\rangle$ can be obtained with certainty from each other by means of LOCC if and only if they are related by LU $|25|^{[1]}$. Thus one has to deal with infinitely many classes (i.e. kinds of entanglement) even in the simplest scenario of bipartite two level systems $|26|^{[2]}$.

Considering equivalence classes under SLOCC, however, allows for a further simplification and to identify a finite number of $d$ inequivalent classes. As shown e.g. in Ref. $|17|$, two entangled pure states $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent under SLOCC, $|\Psi\rangle \sim_{\text{SLOCC}} |\Phi\rangle$ $\Leftrightarrow n_{\Psi} = n_{\Phi}$, i.e. they have the same Schmidt number. Conversion of $|\Psi\rangle$ to $|\Phi\rangle$ under SLOCC, $|\Psi\rangle \rightarrow_{\text{SLOCC}} |\Phi\rangle$ is possible if and only if $n_{\Psi} \geq n_{\Phi}$. This provides a complete, hierarchical classification for bipartite pure states of arbitrary dimension.

The concept of equivalence classes can be applied also to many copies of a bipartite system. In this case, it turns out $|28|$ that equivalence under deterministic LOCC — when allowing for small imperfections — leads to identifying all bipartite pure-state entanglement with that of the EPR-Bohm state $1/\sqrt{2}(|00\rangle + |11\rangle)$ $|29|^{[1]}$. That is, the entanglement of any pure state $|\psi\rangle_{AB}$ is asymptotically equivalent, under deterministic LOCC, to that of the EPR-Bohm state, the entropy of entanglement $E(\psi_{AB})$ — the entropy of the reduced density matrix of either system $A$ or $B$ — quantifying the amount of EPR-Bohm entanglement contained asymptotically in $|\psi\rangle_{AB}$.

### 2. Multiparticle systems

In multiparticle systems, equivalence classes under LU have also been studied, however no such particularly simple form as the Schmidt decomposition for bipartite systems could be obtained (see however Ref. $|28|^{[30]}$). This lack of Schmidt decomposition is one of the reasons for our still restricted knowledge on the entanglement properties of multiparticle pure states.

When considering equivalence classes under SLOCC in three qubit systems, this classification allowed to identify two inequivalent kinds of true tripartite entanglement $|17|$, represented by the states

$$|W\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle),$$

$$|GHZ\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle),$$

respectively. The corresponding equivalence classes under SLOCC are called $W$-class and GHZ-class. In multiparticle systems or tripartite systems of higher dimension, infinitely many classes under SLOCC exist, so typically two multiparticle entangled states cannot be transformed into each other with non-zero probability of success $|17|$. Remarkably, all equivalence classes for four qubit systems have been identified recently $|31|$.

### B. Isomorphism between physical operations and states

In $|28|$, an isomorphism which relates non-local physical operations [equivalently completely positive maps (CPM) $E$] acting on two systems and (unnormalized) states (positive operators $E$) was introduced and generalized to $N$-partite systems in $|17|$. When applied to unitary operations $U$, it turns out that the corresponding state is pure. We review this isomorphism —specialized to unitary operations— in detail, as it provides the proper tool to connect the problem of classification of operations under SLOCC to the well studied problem of classification of pure states under SLOCC.

We consider several spatially separated systems $A, B, \ldots, Z$, each possessing several $d$-level systems. Let

$$|\Phi\rangle_{A_1,2} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_{A_1} \otimes |i\rangle_{A_2},$$

be a maximally entangled state (MES), $|\Phi\rangle \in \mathcal{G}^d$. We denote by $P_{\Phi}$ the projector on this state.

We consider a $N$-partite unitary operation $U$ acting on several $d$-level systems, one located in each site $A, B, \ldots, Z$. Let $|\Psi_U\rangle_{A_1,2 \ldots Z_{1,2}} \in \mathcal{G}^{d^N}$ be a $N$-partite pure state and $P_{U \Phi} \equiv |\Psi_U\rangle \langle \Psi_U|$ the corresponding projector on this state. As shown in $|28|$, one obtains the following relations between the unitary operation $U$ and a pure state $|\Psi_U\rangle$:

$$|\Psi_U\rangle_{A_1,2 \ldots Z_{1,2}} = U_{A_1 \ldots Z_1} (|\Phi\rangle_{A_1,2} \otimes \ldots \otimes |\Phi\rangle_{Z_{1,2}}),$$

(4a)

$$U \rho_{A_1 \ldots Z_1} U^\dagger = d^{2N} \text{tr}_{A_2 \ldots Z_2} \left( P_{\Psi_U}^{A_1, \ldots Z_{1,2}} \rho_{A_3 \ldots Z_3}^{A_2, \ldots Z_{2,3}} \ldots P_{\Phi}^{Z_{2,3}} \right).$$

(4b)

These equations have a very simple interpretation: On one hand, $|28|$ states that $|\Psi_U\rangle$ can be created from a $N$-party product state deterministically, given a single application of the unitary operation $U$, where each party prepares locally a MES. On the other hand, $|17|$ tells us that given $|\Psi_U\rangle$ (particles $A_{1,2}B_{1,2} \ldots Z_{1,2}$), one can implement the multi—particle operation $U$ on an arbitrary state $\rho$ of $N$ $d$—level systems (particles $A_2B_3 \ldots Z_3$) probabilistically, by measuring locally the projector $P_{\Phi}$ on particles $A_2, B_2, Z_2, \ldots, Z_3$ in each of the locations. Note that the probability of success is given by $p = 1/d^{2N}$.

Exactly the same relations hold when considering an arbitrary operator $O$ instead of the unitary operation $U$. Also in this case, the corresponding state, $|\Psi_O\rangle$, is pure. We have that to each $N$—partite operator $O \in \mathcal{G}$ corresponds one unique pure state in $\mathcal{G}^d$, and to any such pure state corresponds an operator $O$. However, not any such pure state corresponds to a unitary operation $U$. 


III. DEFINITION OF EQUIVALENCE CLASSES FOR NON–LOCAL UNITARIES

Motivated by the various insights following from the definition of equivalence classes under certain classes of operations for pure states (see Sec. [1A for details), we define a similar notion for unitary operations. We consider $N$ spatially separated, $d$-dimensional systems and non–local unitary operations $U, \tilde{U} \in SU(d^N)$ acting on those systems. We allow for a certain restricted class of local operations, e.g. LU, and are interested in the simulation of non–local unitary operations under these conditions.

**Definition 1:** A unitary operation $U$ can simulate $\tilde{U}$ under a specific class of operations $X$, $U \rightarrow_X \tilde{U}$, where $X \in \{LU, LU+ancilla, LO, LOCC, SLOCC\}$ if the action of $\tilde{U}$ on any input state $\rho$ can be obtained using a sequence of operations of the kind $X$ applied before and after a single application of $U$.

Note that in the case of LU, LU+ancilla, LO and LOCC operations, the simulation has to be deterministic, while for SLOCC operations, only a certain non-zero probability of success is required. In the case of LU+ancilla, LO, LOCC and SLOCC operations, additional auxiliary systems are allowed and it is not required that the operation $U$ has to be performed on the input state $\rho$ directly. We only demand that the total action of the sequence of operations we perform —after tracing out auxiliary systems— is given by $\tilde{U} \rho \tilde{U}^\dagger$ for arbitrary input states $\rho$.

**Definition 2:** Two unitary operations $U$ and $\tilde{U}$ are equivalent under operations of the class $X$, $U \equiv_X \tilde{U}$ if $U \rightarrow_X \tilde{U}$ and $U \leftarrow_X \tilde{U}$, i.e. the two operations can simulate each other under the class of operations $X$.

Definition 2 defines an equivalence relation in the set of non–local unitary operations and thus allows to identify equivalence classes. Two non–local operations $U, \tilde{U}$ belong to the same equivalence class under operations of the kind $X$ if $U \equiv_X \tilde{U}$. Together with definition 1, this allows to obtain a partial order in the set of unitary operations. The aim of this paper is to identify equivalence classes. In what follows, we will mainly focus on equivalence classes under SLOCC.

Note that the same definitions also makes sense when applied to multiple copies of operations, i.e. $U = V \otimes^N$ and $\tilde{U} = V \otimes^M$, where $N$ and $M$ may be different. In this case, one is concerned with the question whether $N$ simultaneous applications of a given operation $V$ can simulate $M$ simultaneous applications of an operation $\tilde{V}$.

We would also like to point out that the problem of simulation of unitary operations is not equivalent to the problem of Hamiltonian simulation. In the latter case, intermediate local operations can be applied, while in the former case, one considers a fixed, non–local unitary operation $U$ —e.g. given by some black box—, and local operations can only be applied before and after the application of $U$. That is, the process of interaction is inaccessible for some reason, e.g. because it is taking place at a very short timescale.

IV. EQUIVALENCE CLASSES UNDER LU AND LOCC FOR NON–LOCAL UNITARY OPERATIONS

In this section, we review some results on equivalence classes under LU and LOCC for unitary operations. We show that for unitary operations —in contrast to pure state conversion—, equivalence under LU is not the same as equivalence under LOCC.

A. Equivalence classes under LU

The only well studied example of equivalence classes under LU for unitary operations is the case of two qubits, i.e. $U \in SU(2^2)$. Kraus et. al showed in [3] (see also [2]) that any bipartite unitary operations $U$ acting on two qubits can uniquely be written as

$$U_{AB} = V_A \otimes W_B e^{-iH} \tilde{V}_A \otimes \tilde{W}_B,$$  

(5a)

$$H = \sum_{i=1}^{3} H_i, \quad H_i = \mu_i \sigma^A_i \otimes \sigma^B_i,$$  

(5b)

$$\pi/4 \geq \mu_1 \geq \mu_2 \geq \mu_3 \geq |\mu_3| \geq 0.$$  

(5c)

That is, any unitary operation $U$ can up to local unitaries be written in the normal form $e^{-iH}$, which might be compared to the Schmidt decomposition for bipartite pure states. Two unitary operations $U$ and $\tilde{U}$ belong to the same equivalence class under LU if and only if $\mu_i(U) = \mu_i(\tilde{U}) \forall i$. This normal form already turned out to be useful in a number of applications [4]. No similar result is known for unitary operations acting on higher dimensional systems or multipartite operations.

B. Equivalence classes under LOCC

Only few is known also on deterministic simulation of unitary operations under LOCC. Very recently, deterministic simulation of two–qubit unitary operations given a CNOT [SWAP] operation [22] were studied in Ref. [23]. In Ref. [14], it was shown that catalytic equivalence under LOCC —i.e. allowing to use in addition to LOCC some entangled state, which has to be given back undisturbed at the end of the process— is not the same as equivalence under LOCC. In fact, deterministic gate simulation under catalytic LOCC turns out to be possible in some cases where it is impossible under LOCC alone. Also equivalence under LU and equivalence under LU+ancilla turned out to be different [13].

We show that in contrast to what happens for pure state conversion, equivalence under LU(+ancilla) and
equivalence under LOCC are different when considering unitary operations. To see this, we consider two copies of the CNOT operation, $U_{\text{CNOT},2}$, and the SWAP operation, $U_{\text{SWAP},2}$. It is easy to show that $U_{\text{CNOT},2} \equiv_{\text{LOCC}} U_{\text{SWAP},2}$, while $U_{\text{CNOT},2} \not\equiv_{\text{LU}} U_{\text{SWAP},2}$. The first relation can be checked by noting that the CNOT operation can create 1 ebit of entanglement, while the SWAP operation can create 2 ebits out of a product state. Given the facts that (i) using classical communication and one ebit of entanglement, one can implement a CNOT gate, and (ii) given 2 ebits of entanglement plus classical communication, one can implement a SWAP operation (see e.g. [16]), it readily follows that the two operations in question can simulate each other under LOCC.

On the other hand, one can show that $U_{\text{CNOT},2} \not\equiv_{\text{LU}} U_{\text{SWAP},2}$. The impossibility of this process is based on the fact that the classical communication capacity of two CNOT operations is given by 2 bits, while the classical communication capacity of the SWAP operation is just one bit. This last property follows from the fact that whenever one applies a SWAP operation to an arbitrary (possibly locally manipulated) product input state, a two-dimensional subspace coming from Alice side appears at Bob’s (and vice versa). However, a two-dimensional subspace cannot contain more than one bit of classical information. Since for complete simulation of operations under LU, it is required that exactly the same tasks can be performed (including also classical information transmission, which has to be considered as a resource in this scenario) and due to the fact that LU do not change the classical capacity, this implies that the two operations in question are not equivalent under LU.

V. EQUIVALENCE CLASSES UNDER SLOCC FOR NON-LOCAL UNITARIES

In this section, we establish a connection between equivalence classes under SLOCC for unitary operations as stated in Sec. 11 and equivalence classes under SLOCC for entangled pure states (see Sec. 11A). The isomorphism (11), discussed in detail in Sec. 11B, turns out to be the central tool. This relation is expressed in the following.

Result 1: $U$ can simulate $\tilde{U}$ under SLOCC ($U \equiv_{\text{SLOCC}} \tilde{U}$) $\iff |\Psi_U\rangle$ can be converted to $|\Psi_{\tilde{U}}\rangle$ by means of SLOCC ($|\Psi_U\rangle \equiv_{\text{SLOCC}} |\Psi_{\tilde{U}}\rangle$).

Proof: ($\Rightarrow$): Given that $U$ can simulate $\tilde{U}$, it is easy to show that $|\Psi_{\tilde{U}}\rangle$ can be converted to $|\Psi_U\rangle$ by means of SLOCC. The conversion takes place as follows: According to (11), $|\Psi_{\tilde{U}}\rangle$ can be used to implement $U$ with certain probability of success. Now a single application of $U$ allows to simulate $\tilde{U}$ probabilistically. According to (12), $\tilde{U}$ can be used to create $|\Psi_U\rangle$ out of a product state, which finishes the proof in one direction.

($\Leftarrow$): Given that $|\Psi_U\rangle$ can be converted to $|\Psi_{\tilde{U}}\rangle$ by means of SLOCC, we have to show that $U$ can simulate $\tilde{U}$ probabilistically. The proof goes as follows: $U$ is used to create the state $|\Psi_{\tilde{U}}\rangle$ using (14), which can be converted by means of SLOCC to $|\Psi_{\tilde{U}}\rangle$. Now $|\Psi_{\tilde{U}}\rangle$ can be used to implement $\tilde{U}$ with certain probability of success according to (15), which finishes the proof of the statement.

Note that from Result 1 follows that two unitary operations $U$ and $\tilde{U}$ belong to the same equivalence class under SLOCC if and only if the corresponding pure states $|\Psi_U\rangle$ and $|\Psi_{\tilde{U}}\rangle$ are equivalent under SLOCC, i.e. they can be converted into each other by means of SLOCC [7].

$$U \equiv_{\text{SLOCC}} \tilde{U} \iff |\Psi_U\rangle \equiv_{\text{SLOCC}} |\Psi_{\tilde{U}}\rangle.$$ (6)

As LOCC are included in SLOCC, it also follows that equivalence of unitary operations under SLOCC is a necessary (but not sufficient) condition for their equivalence under LOCC.

In the following, we are going to illustrate Result 1 and Eq. (6) and apply it to bipartite and multipartite unitary operations.

A. Bipartite unitary operations

In this section, we apply Result 1 to bipartite unitary operations. The results we derive here are already obtained in Ref. [20], however we review the derivation in detail in order to illustrate Result 1. We are going to show that there always exists a finite set of inequivalent classes under SLOCC of bipartite unitary operations which are hierarchically ordered. For unitary operations $U \in SU(d^2)$ acting on two $d$-level systems, at most $d^2$ inequivalent classes exist. In the case of two qubit unitary operations, i.e. $d = 2$, only three classes remain. The operations CNOT and SWAP [22] appear as natural representatives.

From Sec. 11A and Result 1 follows that the relevant quantity that determines the equivalence class under SLOCC for a unitary operation $U$ is the Schmidt number $n_{\Psi_U}$ of the corresponding pure state $|\Psi_U\rangle$. This follows from the fact that two bipartite pure states are equivalent under SLOCC if and only if they have the same Schmidt number (see Sec. 11A). That is, two unitary operations $U$ and $\tilde{U}$ are equivalent under SLOCC if and only if the corresponding states $|\Psi_U\rangle, |\Psi_{\tilde{U}}\rangle$ (see Eq. (14)) have the same Schmidt number, i.e. $n_{\Psi_U} = n_{\Psi_{\tilde{U}}}$. A unitary operation $U$ can simulate another operation $\tilde{U}$ under SLOCC $\iff n_{\Psi_U} \geq n_{\Psi_{\tilde{U}}}$, which provides the announced hierarchical classification. Since $n_{\Psi_U} \leq d^2$ for $U \in SU(d^2)$, one obtains the announced upper bound for the number of classes, $d^2$.

When applied to unitary operations $U \in SU(2^2)$ acting on two qubits, one finds that at most four inequivalent classes exist, corresponding to Schmidt numbers $n_{\Psi_U} \in \{1, 2, 3, 4\}$. We will show, however, that the case $n_{\Psi_U} = 3$ does not exist. That is, the pure state corresponding to any bipartite unitary operation $U$ has either Schmidt
number $n_{\Phi_U}$, 1 or 2. Recall that although to any unitary operation corresponds a unique pure state via Eq. (4), not to any pure state corresponds a unitary operation. In fact, it turns out that all pure states with Schmidt number 3 do not correspond to a unitary operation (but to some non-unitary operator $O$).

To see this, recall that any bipartite unitary operation acting on two qubits can be written as indicated in Eq. (5). Using that $|1\rangle$ acting on two qubits can be written as indicated in Eq. (6) and $d = 2$, we can write the state $|\Psi_U\rangle$ corresponding to $U$ via Eq. (7) as

$$|\Psi_U\rangle = V_{A_1} \otimes W_{B_1} \otimes \tilde{V}_{A_2}^T \otimes \tilde{W}_{B_2}^T e^{-iH_{A_1}b_1} |\Phi\rangle_{A_1,2} |\Phi\rangle_{B_1,2}. \quad (7)$$

The local unitary operations $V, W, \tilde{V}, \tilde{W}$ do not change the Schmidt number of $|\Psi_U\rangle$, so we may set them to 1 without loss of generality. We denote an orthogonal basis of maximally entangled states by $\{\Phi_i\}_{i=0,1,2,3}$ with

$$|\Phi_i\rangle = \sigma_i \otimes |\Phi\rangle,$$

and introduce the shorthand notation $c_{\mu_i} \equiv \cos(\mu_i)$, $s_{\mu_i} \equiv \sin(\mu_i)$. Using that $e^{-iH_1 + H_2 + H_3} = e^{-iH_1} e^{-iH_2} e^{-iH_3}$, were $H_i$ are defined in Eq. (5), we find that

$$|\Psi_U\rangle = \sum_{k=0}^{3} a_k |\Phi_i\rangle_{A_1,2} |\Phi_i\rangle_{B_1,2}. \quad (9)$$

which already corresponds —up to some irrelevant phase factors— to the Schmidt decomposition. The coefficients $a_k$ are given by

$$a_0 = c_{\mu_1} c_{\mu_2} c_{\mu_3} - i s_{\mu_1} s_{\mu_2} s_{\mu_3}, \quad (10a)$$

$$a_1 = c_{\mu_1} c_{\mu_2} s_{\mu_3} - i s_{\mu_1} c_{\mu_2} c_{\mu_3}, \quad (10b)$$

$$a_2 = s_{\mu_1} c_{\mu_2} s_{\mu_3} - i c_{\mu_1} s_{\mu_2} c_{\mu_3}, \quad (10c)$$

$$a_3 = s_{\mu_1} c_{\mu_2} c_{\mu_3} - i c_{\mu_1} s_{\mu_2} s_{\mu_3}. \quad (10d)$$

It is now straightforward to check that whenever one demands that one of the coefficients $a_k$ should be zero (which corresponds to having a Schmidt number 3 or less), automatically also a second coefficient (or even two others) vanishes. E.g., $a_0 = 0$ implies that $c_{\mu_1} c_{\mu_2} c_{\mu_3} = s_{\mu_1} s_{\mu_2} s_{\mu_3} = 0$. Assuming e.g. that $c_{\mu_1} = s_{\mu_2} = 0$, one finds that also $a_3 = 0$ etc.. This implies that $|\Psi_U\rangle$ cannot have $n_{\Phi_U} = 3$ and thus no bipartite unitary operation with this property exists.

Thus there exist three classes of $SU(4)$ unitary operations under SLOCC:

- **Class 1**: $n_{\Phi_U} = 1$: This are local unitary operations, with $\mu_1 = \mu_2 = \mu_3 = 0$ in Eq. (8).

- **Class 2**: $n_{\Phi_U} = 2$: This are nonlocal unitary operations with $\mu_1 \neq 0$ and $\mu_2 = \mu_3 = 0$ in Eq. (9).

It is natural to choose the corresponding state to be a maximally entangled state with Schmidt number $n_{\Phi_U} = 2$ as a representative in this case, which leads to the CNOT operation (3) as a natural representative of this class. Note that the CNOT is up to non-local unitaries equivalent to an operation of the form (3) with $\mu_1 = \pi/4, \mu_2 = \mu_3 = 0$. This can be seen by noting that

$$|\Psi_{\text{CNOT}}\rangle = \frac{1}{\sqrt{2}} (|00\rangle_{A_1,2} |\Phi_0\rangle_{B_1,2} + |11\rangle_{A_1,2} |\Phi_1\rangle_{B_1,2}).$$

- **Class 3**: $n_{\Phi_U} = 4$: This are nonlocal unitary operations with $\mu_1, \mu_2 \neq 0$ and $\mu_3$ arbitrary. One may choose as a representation an operation which corresponding state is a maximally entangled state with Schmidt number $n_{\Phi_U} = 4$. This leads to the SWAP operation (2) as a natural representative of this class. An operation of the form (3) with $\mu_1 = \mu_2 = \mu_3 = \pi/4$ is up to local unitaries equivalent to the SWAP operation. This can be seen by noting that

$$|\Psi_{\text{SWAP}}\rangle = |\Phi_0\rangle_{A_1,2} |\Phi_0\rangle_{B_1,2}. \quad (12)$$

Recall that any operations of class 3 can simulate operations of class 2 under SLOCC, however the reverse process is not possible. This implies on the one hand that any non-local unitary operation can be used to simulate a CNOT operation probabilistically (and thus to implement entanglement purification), while the CNOT operation can e.g. not be used to simulate $e^{-iH (\sigma_1^x \otimes \sigma_2^y + \sigma_3^z \otimes \sigma_4^y)}$ with non-zero probability of success even for $t \ll 1$. The procedure sketched in the proof of Result 1 also provides a practical protocol to achieve this task.

**B. Multiparticle unitary operations**

For multiparticle unitary operations $U \in SU(d^N)$ acting on $N$ $d$-level systems, $N \geq 3$, we have that the corresponding pure state $|\Psi_U\rangle \in \mathbb{C}^{d^N}$. As even for $d = 2$, typically two entangled pure states of this kind are not equivalent under SLOCC (This follows from the fact that in multipartite systems —except the case of three qubits— infinitely many equivalence classes exist (17)), Result 1 leads us to expect that typically two multipartite unitary operations $U$ and $\tilde{U}$ will be inequivalent under SLOCC. We can not offer a formal proof of this statement, because equivalence classes under SLOCC for multipartite pure states have not been completely identified yet (important exceptions are all bipartite systems and systems of three and four qubits). However, we illustrate with the help of a simple example that in fact infinitely many inequivalent classes under SLOCC of unitary operations exist in the case of fourpartite unitary operations acting on qubits.
To this aim, we consider a one parameter family of unitary operations \( U(t) \in SU(2^4) \), generated by the interaction Hamiltonian \( H \) applied for some time \( t \), \( 0 < t < \pi/4 \), i.e. \( U(t) = e^{-iHt} \), with
\[
H = \sigma_x^A \otimes \sigma_y^B \otimes \sigma_z^C \otimes \sigma_z^D + 1^A \otimes 1^B \otimes \sigma_z^C \otimes \sigma_z^D
+ \sigma_x^A \otimes \sigma_x^B \otimes 1^C \otimes 1^D. \tag{13}
\]
We show that \( U(t) \) and \( U(\bar{t}) \) are inequivalent under SLOCC if \( t \neq \bar{t} \). This might be surprising at the first sight, because this means that a unitary operation generated by a certain interaction switched on for a certain time \( t \) cannot be used to simulate — not even probabilistically — a unitary evolution generated by the same interaction, switched on e.g. for some smaller time \( \bar{t} \). This is in contrast to what happens for bipartite unitary operations.

It is easy to show that \( U(t) \) is of the form \( U(t) = \alpha(t)1 + \beta(t)H \) with complex coefficients \( \alpha(t), \beta(t) \). This implies that the state \( |\Psi_{U(t)}\rangle \) corresponding to \( U(t) \) via \( |1\rangle \) — after a local basis change \( |\Phi_0\rangle \rightarrow |0\rangle \) and \( |\Phi_1\rangle \rightarrow |1\rangle \) in all four locations — can be written as
\[
|\Psi_{U(t)}\rangle = \alpha(t)(0000) + \beta(t)(1111) + \beta(t)(0011) + (1100) \tag{14}.
\]
It is now straightforward to check — applying the results of Verstraete et al. \([31]\) — that if \( t \neq \bar{t} \), then \( |\Psi_{U(t)}\rangle \neq _{\text{SLOCC}} |\Psi_{U(\bar{t})}\rangle \) (by calculation the corresponding normal form for different \( t \)). This implies that \( U(t) \neq U(\bar{t}) \) and simulation is impossible in both directions.

Note that one may also use \([32]\) the results of Ref. \([30]\) to identify equivalence classes under SLOCC for multiparticle unitary operations. There, it was shown that the problem reduces to establish whether two tensors are equivalent under LU.

C. Many copy case:

One may also apply Result 1 to the case where \( N \) copies of the same bipartite unitary operations \( U \) should be performed simultaneously and used to implement \( M \) copies of some other unitary operation \( \bar{U} \), i.e. we investigate whether \( U^{\otimes N} \approx _{\text{SLOCC}} U^{\otimes M} \) is possible. For example, one may want to know whether a single copy of a (strong entangling) unitary operation, e.g. a CNOT gate, can be used to implement \( M \) (weakly entangling) operations, e.g. \( \bar{U} = e^{-i\sigma_y^A \otimes \sigma_z^B} \) with \( t < 1 \). Given that \( n_{\Psi_{\text{CNOT}}} = 2 \) and \( n_{\Psi_{U^{\otimes M}}} = 2^M \), it follows from Result 1 that such a simulation is impossible, i.e \( U_{\text{CNOT}} \not\approx _{\text{SLOCC}} U^{\otimes M} \), even with an arbitrary small probability of success. This implies that \( U_{\text{CNOT}} \not\approx _{\text{SLOCC}} U^{\otimes M} \). The reverse process, \( U^{\otimes M} \approx _{\text{SLOCC}} U_{\text{CNOT}} \), is however possible.

This also implies that when demanding \textit{exact} simulation, bipartite unitary operations can even in the asymptotic case not be reduced to a single one which serves as a representative for all kinds of bipartite operations. One should — as in the case of pure state transformations in the asymptotic limit, where all kinds of bipartite entanglement turn out to be equivalent to the one of the EPR-Bohm state — allow for small imperfections. This still leaves open the possibility that an equivalence relation under LOCC, allowing for some small imperfections — quantified e.g. by the fidelity for unitary gates as defined in \([21, 25]\), — such as
\[
U^{\otimes N} \approx _{\text{LOCC}} U^{\otimes M}_{\text{CNOT}}, \tag{15}
\]
could exist. In this case, \( M/N \) would be a measure for the non–locality of \( U \) and the CNOT operation could be used as an universal resource to store without losses arbitrary bipartite unitary operations. To proof or disproof such a relation would be of great interest.

VI. GENERATION OF ENSTATE\(S \) GIVEN \( U \)

Not only a classification based on (probabilistic) simulation of operations might be of interest, sometimes a more practical approach might be desirable. For example, one might want to know how powerful a unitary operation is and which kind of entanglement such an operation can create. In this section, we will investigate such questions and show which kind of entanglement a non–local unitary operation \( U \) can create probabilistically. The set–up we consider is similar to the one of the previous section, i.e. we consider several spatially separated parties, each possessing several \( d \)-level systems. The initial state of the whole system is product. We may use, before and after the application of \( U \), an arbitrary sequence of local operations and classical communication as well as arbitrary local resources including auxiliary systems. We are interested in the kind of states which can be \textit{probabilistically} created in this way, that is we are interested in which kind of entanglement a given unitary operation can produce under SLOCC. Note that we demand that the state is created only with some nonzero probability of success. Again, it turns out that this problem is closely related to the problem of SLOCC conversion of pure states, which is expressed in the following

\textbf{Result 2:} \( U \) can generate a state \( |\Psi\rangle \) with non-zero probability of success \( \Leftrightarrow |\Psi_{U}\rangle \) can be converted to \( |\Psi\rangle \) by means of SLOCC.

\textit{Proof:} (\( \Rightarrow \)): We need to show that \( |\Psi_{U}\rangle \) can be converted to \( |\Psi\rangle \) by means of SLOCC, given that \( U \) can generate the state \( |\Psi\rangle \). This easily follows from the fact that \( |\Psi_{U}\rangle \) can be used to implement \( U \) with certain probability of success (see \([33]\)), while \( U \) can be used to create \( |\Psi\rangle \) out of a product state.

(\( \Leftarrow \)): We have to show that \( U \) can generate a state \( |\Psi\rangle \), given that \( |\Psi\rangle \) can be obtained from \( |\Psi_{U}\rangle \) by means of SLOCC. Since \( U \) can create the state \( |\Psi_{U}\rangle \) out of a product state (see \([34]\)), which can by assumption be trans-
formed to the state $|\Psi\rangle$ by means of SLOCC, the claim follows.

Let us illustrate Result 2 with help of some examples.

A. Bipartite systems

We consider a bipartite system of two $d$-level systems and an arbitrary non-local unitary operation $U$. The Schmidt number $n_{\Psi_U}$ of the corresponding state $|\Psi_U\rangle$ — that is the number of nonzero coefficients in the Schmidt decomposition — completely determines which kind of bipartite entanglement can be created by the unitary operation $U$. By means of SLOCC, all bipartite states with lower or equal Schmidt number can be obtained from $|\Psi_U\rangle$, while all other states cannot be created (see Sec. [IA]). Thus it follows from Result 2 that $U$ can generate all entangled pure states $|\Psi\rangle$ for which $n_{\Psi} \leq n_{\Psi_U}$. For example, the unitary operation $U = e^{-it\sigma^A_x \otimes \sigma^B_x}$, $0 < t < \pi/4$, can generate all entangled pure states of Schmidt number 2, but cannot generate Schmidt number 3 or 4 states. This follows from the fact that $|\Psi_U\rangle = \cos(t)|\Phi_0\rangle_A|\Phi_0\rangle_B - i\sin(t)|\Phi_1\rangle_A|\Phi_1\rangle_B$ has $n_{\Psi_U} = 2$.

Note that even if an operation is capable of creating Schmidt number 3 states, such states cannot be created directly. One first has to create a Schmidt number 4 state which is then reduced to a Schmidt number 3 state by local measurements. This follows from the fact that the corresponding pure state of any unitary operation has either Schmidt number 1, 2 or 4.

B. Multipartite systems

Consider as a second example a tripartite system of three qubits and a non-local unitary operation of the form

$$U = e^{-it\sigma^A_x \otimes \sigma^B_x \otimes \sigma^C_x}, \quad 0 < t < \pi/4 \quad (16)$$

We have that the corresponding state $|\Psi_U\rangle$ is given by

$$|\Psi_U\rangle = \cos(t)|\Phi_0\rangle_A|\Phi_0\rangle_B|\Phi_0\rangle_C - i\sin(t)|\Phi_1\rangle_A|\Phi_1\rangle_B|\Phi_1\rangle_C, \quad (17)$$

where the states $|\Phi_i\rangle$ are defined in Eq. (3). This is — after a change of local basis $|\Phi_0\rangle \rightarrow |0\rangle$, $|\Phi_1\rangle \rightarrow |1\rangle$ — a state in $(\mathbb{C}^2)^\otimes 3$ and thus effectively a state of three qubits. Note that $|\Psi_U\rangle$ belongs to the GHZ-class and can thus not be converted into the state $|W\rangle$ (see Sec. [IA]). Using Result 2, this implies that $U$ cannot create the state $|W\rangle$ — not even with a very small probability of success. However, $U$ can be used to create all states within the GHZ-class.

Note that it also happens that three qubit unitary operation can generate both kinds of tripartite qubit entanglement. One such example is the unitary operation $U_W = e^{-itH_W}$, $0 < t < \pi/4$, where $H_W = |W\rangle\langle W|$ and the state $|W\rangle$ is defined in Eq. (2a). Using that $H_W^2 = H_W$, one readily observes that $U_W = 1 + \gamma(t)H_W$, where $\gamma(t) = \sum_{k=1}^{\infty}(-it)^k/k!$. It follows that

$$U_W|001\rangle = |001\rangle + \gamma(t)/\sqrt{3}|W\rangle, \quad (18a)$$

$$U_W\frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |101\rangle) = \frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + \gamma(t)/\sqrt{3}|W\rangle), \quad (18b)$$

where the state (18a) is a state in the $W$-class and the state (18b) is a state in the GHZ-class. Thus both states, $|W\rangle$ and $|GHZ\rangle$, can be created probabilistically out of a product state.

This can be understood as follows. The state $|\Psi_U\rangle$ corresponding to a tripartite unitary operation $U$ acting on three qubits is in general a state acting on $\mathcal{H} = (\mathbb{C}^4)^\otimes 3$. This implies that $U$ can create not only qubit–type entanglement such as $|GHZ\rangle$ or $|W\rangle$, but also certain higher dimensional entangled states, belonging in principle to different classes under SLOCC. Recall that in tripartite four level systems, there exist infinitely many inequivalent classes under invertible SLOCC [15]. Although $|GHZ\rangle$ or $|W\rangle$ belong to inequivalent classes under SLOCC, both of them may be created from a higher dimensional state by means of non–invertible SLOCC, which happens e.g. in the example discussed above. Similarly, maximally entangled pure states shared between two parties, $A - B$, $A - C$ and $B - C$ can be created from $|GHZ\rangle$ and $|W\rangle$ by means of non–invertible SLOCC, although they belong to different equivalence classes under invertible SLOCC.

VII. EQUIVALENCE CLASSES FOR ARBITRARY OPERATORS

It is straightforward to generalize our results obtained for unitary operations to the more general case of arbitrary operators. In particular, since the isomorphism [11] also holds for operators $O$, Results 1 and 2 also hold in this case and the proof is exactly the same. Due to the fact that any pure state $|\Psi_O\rangle \in \mathcal{G}^{\otimes d^2}$ corresponds to an operator $O$ via Eq. (11), we have that in contrast to what happens for unitary operations $U$, exactly $d^2$ equivalence classes under SLOCC for bipartite operators exist.

VIII. SUMMARY AND CONCLUSIONS

In this paper, we have provided a general framework to identify equivalence classes of non–local unitary operations and arbitrary operators under certain classes of local operations. For stochastic local operations, assisted by classical communication, we provided necessary and sufficient conditions for gate simulation in terms of SLOCC conversion for pure states. This allowed us to obtain a complete, hierarchic classification of bipartite unitary operations as well as to obtain a number of results
—including a partial order—for multipartite unitary operations. While for bipartite operations always a finite number of inequivalent classes under SLOCC exists, we showed that for multipartite operations one obtains infinitely many classes. The important case of bipartite unitary operations acting on qubits was studied in detail, and we identified three different kinds of bipartite unitary operations under SLOCC, represented by product operations, the CNOT operation and the SWAP operation respectively.

We also showed which kind of entanglement a unitary operation can create. Again, this was done by obtaining a connection to the problem of state conversion under SLOCC. We provided a complete solution in the single and multi-copy case.

In particular, it would be interesting to obtain normal forms for high dimensional bipartite unitary operations under LU, i.e. to identify the corresponding equivalence classes, and to identify equivalence classes under LOCC in the single and multi-copy case.

ACKNOWLEDGEMENTS

We would like to thank G. Vidal for many useful discussions and comments. This work was supported by European Community under project EQUIP (contract IST-1999-11053) and through grant HPMF-CT-2001-01209 (W.D., Marie Curie fellowship), the ESF and the Institute for Quantum Information GmbH.

[1] D. Gottesman, quant-ph/9807006
[2] A. Chefles, C. R. Gilson and S. M. Barnett, quant-ph/0003062 and quant-ph/0006104.
[3] J. Eisert, K. Jacobs, P. Papadopoulos and M. B. Plenio, Phys. Rev. A 62 052317 (2000).
[4] D. Collins, N. Linden and S. Popescu, quant-ph/0005102
[5] J. I. Cirac, W. Dür, B. Kraus and M. Lewenstein, Phys. Rev. Lett. 86, 544 (2001)
[6] P. Zanardi, C. Zalka and L. Faoro, Phys. Rev. A 62, 30301R (2000)
[7] W. Dür, G. Vidal, J. I. Cirac, N. Linden and S. Popescu, Phys. Rev. Lett. 87, 137901 (2001)
[8] B. Kraus and J. I. Cirac, quant-ph/0011050
[9] J. L. Dodd, M. A. Nielsen, M. J. Bremner, and R. T. Thew, quant-ph/0106064; M. A. Nielsen, M. J. Bremner, J. L. Dodd, A. M. Childs, and C. M. Dawson quant-ph/0109067
[10] P. Wocjan, D. Janzing, Th. Beth, quant-ph/0109063, P. Wocjan, M. Roetteler, D. Janzing, Th. Beth, quant-ph/0109088.
[11] C. H. Bennett, J. I. Cirac, M. S. Leifer, D. W. Leung, N. Linden, S. Popescu, and G. Vidal, quant-ph/0107035
[12] D. W. Leung, quant-ph/0107041
[13] G. Vidal and J. I. Cirac, quant-ph/0108076
[14] G. Vidal and J. I. Cirac, quant-ph/0108076, quant-ph/0108077
[15] W. Dür and J. I. Cirac, Phys. Rev. A 64, 012317 (2001).
[16] C. H. Bennett, S. Popescu, D. Rohrlich, J.A. Smolin and A.V. Thapliyal, quant-ph/9908073.
[17] W. Dür, G. Vidal and J. I. Cirac, Phys. Rev. A 62, 062314 (2000)
[18] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A 53, 2046(1996)
[19] Einstein, Podolsky & Rosen, Phys. Rev. 47, 777-780 (1935); D. Bohm, Quantum Theorie, Prentice-Hall, Englewood Cliffs, New Jersey (1951)
[20] W. Dür, G. Vidal and J. I. Cirac, quant-ph/0112124
[21] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 81, 5932 (1998); W. Dür, H.-J. Briegel, J. I. Cirac, and P. Zoller, Phys. Rev. A 59, 169-181 (1999).
[22] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996); C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[23] The CNOT operation is defined by the following mapping of states, written in the standard basis: $|i\rangle_A |j\rangle_B \rightarrow |i\rangle_A |i \oplus j\rangle_B$, where $\oplus$ denotes addition modulo 2. The SWAP operation is given by the mapping $|i\rangle_A |j\rangle_B \rightarrow |j\rangle_A |i\rangle_B$
[24] Throughout this paper, we will use gates and unitary operations synonymously.
[25] G. Vidal, Journ. of Mod. Opt. 47, 355 (2000);
[26] N. Linden and S. Popescu, Fortsch. Phys. 46, 567 (1998);
[27] A. Sudbery, quant-ph/0001116;
[28] H. A. Carteret and A. Sudbery, quant-ph/0001093.
[29] J. Kempe, Phys. Rev. A60, 910-916 (1999);
[30] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000); A. Acín, A. Andrianov, E. Jané, and R. Tarrach, J. Phys. A: Math. Gen. 34, 6725 (2001).
[31] F. Verstraete, J. Dehaene, B. De Moor, H. Verschelde, quant-ph/0109033
[32] N. Khaneja, R. Brockett and S. J. Glaser, Phys. Rev. A 63, 032308 (2001)
[33] The uniqueness of this decomposition can easily derived using the results of Ref. [8].
[34] G. Vidal, private communication.
[35] F. Verstraete, private communication.
[36] F. Verstraete, J. Dehaene and B. De Moor, quant-ph/0105094.
[37] A. Acín, E. Jane, and G. Vidal, Phys. Rev. A 64, 050302(R) (2001).
[38] A. Acín, Phys. Rev. Lett. 87, 177901 (2001).