Flux bias-controlled chaos and extreme multistability in SQUID oscillators

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The radio frequency (rf) Superconducting QUantum Interference Device (SQUID) is a highly nonlinear oscillator exhibiting rich dynamical behavior. It has been studied for many years and it has found numerous applications in magnetic field sensors, in biomagnetism, in non-destructive evaluation, and gradiometers, among others. Despite its theoretical and practical importance, there is relatively very little work on its multistability, chaotic properties, and bifurcation structure. In the present work, the dynamical properties of the SQUID in the strongly nonlinear regime are demonstrated using a well-established model whose parameters lie in the experimentally accessible range of values. When driven by a time-periodic (ac) flux either with or without a constant (dc) bias, the SQUID exhibits extreme multistability at frequencies around the (geometric) resonance. This effect is manifested by a “snake-like” form of the resonance curve. In the presence of both ac and dc flux, multiple bifurcation sequences and secondary resonance branches appear at frequencies above and below the geometric resonance. In the latter case, the SQUID exhibits chaotic behavior in large regions of the parameter space; it is also found that the state of the SQUID can be switched from chaotic to periodic or vice versa by a slight variation of the dc flux.

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In this work we study the dynamics of a nonlinear driven oscillator which serves as a model for a radio-frequency (rf) Superconducting QUantum Interference Device (SQUID), hereafter referred to as “SQUID”. When arranged in periodic structures, SQUIDs form metamaterials with extraordinary electromagnetic properties and important quantum technology applications. Besides their appeal as superconducting devices, SQUID metamaterials provide a unique testbed for exploring complex spatio-temporal dynamics. Their properties are affected to a large extent by those of their constitutive elements. Thus, an exploration of the dynamical properties of the single SQUID is not only theoretically interesting but it may also help for the construction of SQUID-based devices and metamaterials with improved capabilities. Here we revisit the single SQUID oscillator and undertake a systematic study of the flux bias effects on the system’s dynamics. By using experimentally relevant parameters, we show that the single SQUID system can exhibit highly multistable responses to an externally applied driving force. This is reflected in the resonance curve which exhibits a “snake-like” structure around the geometric resonance frequency, while at lower frequencies a very delicate sequence of bifurcations leading to chaos is revealed, which we have quantified using suitable measures. For the first time, we address in detail the role of an additional constant term in the external force driving the SQUID and show that it can induce new rich dynamics and complex bifurcation scenarios which are absent otherwise. The flux bias, which serves as yet another control parameter of the SQUID dynamics, has been largely ignored in prior works, and this is the focus of the interesting results presented in this manuscript.

I. INTRODUCTION

Superconducting metamaterials are artificially structured media of weakly coupled discrete elements that exhibit extraordinary properties\cite{1,2,3}. Besides their main appeal as superconducting devices, these metamaterials provide a unique testbed for exploring complex spatio-temporal dynamics. The Superconducting Quantum Interference Device (SQUID) as a meta-atom (building block of a superconducting metamaterial) consists of a supercon-
ducting ring interrupted by a Josephson junction (JJ) as shown schematically in Fig. 1. When driven by an alternating magnetic field, the induced supercurrents around the ring are determined through the celebrated Josephson relations. Metamaterials comprising SQUIDs, i.e., SQUID metamaterials, are very attractive for both theoretical and experimental research, holding a great promise for novel applications. A prominent feature of SQUID metamaterials is negative diamagnetic permeability that has been predicted both for the quantum and the classical regime. Other properties of SQUID metamaterials include tunability of the resonance frequency with a flux bias, dynamic multistability, and self-induced broadband transparency. Recently, the collective behavior of coupled SQUID oscillators was also studied and counter-intuitive dynamic states referred to as chimera states were found in a wide parameter regime.

Importantly, it has been shown that many of the properties of SQUID metamaterials emerge, to a large extent, from those of the individual SQUIDs, if all elements in the metamaterial are biased homogeneously and the coupling between them is weak. In particular, the key ingredient to the formation of the chimera states, i.e., states of coexisting synchronized and unsynchronized clusters, is the multistability of the single SQUID. From the viewpoint of nonlinear dynamics, the SQUID is a strongly nonlinear oscillator with a resonant response to an externally applied alternating magnetic field; it exhibits rich dynamical behavior which we revisit in this article, revealing new interesting effects in a systematic way, which can be controlled by a flux bias term.

The magnetic flux \( \Phi \) threading the loop of the SQUID is given by:

\[
\Phi = \Phi_{ext} + LI, \tag{1}
\]

where \( \Phi_{ext} \) is the external flux applied to the SQUID, \( L \) is the self-inductance of the SQUID ring, and

\[
I = -C \frac{d^2 \Phi}{dt^2} - \frac{1}{R} \frac{d\Phi}{dt} - I_c \sin \left( \frac{2\pi \Phi}{\Phi_0} \right), \tag{2}
\]

is the current in the SQUID as provided by the resistively and capacitively shunted junction (RCSJ) model of the JJ. In Eq. (2), within the RCSJ framework, \( C \) is the capacitance of the JJ of the SQUID, \( R \) is the resistance, \( I_c \) is the critical current which characterizes the JJ, \( \Phi_0 \) is the flux quantum, and \( t \) is the temporal variable. Combining Eqs. (1) and (2),
gives the equation:

\[ C \frac{d^2 \Phi}{dt^2} + \frac{1}{R} \frac{d\Phi}{dt} + I_c \sin \left( 2\pi \frac{\Phi}{\Phi_0} \right) + \frac{\Phi - \Phi_{\text{ext}}}{L} = 0, \quad (3) \]

which can be obtained from direct applications of Kirchhoff’s laws to the equivalent electrical circuit for the SQUID (shown in Fig. 1). The external flux takes the form:

\[ \Phi_{\text{ext}} = \Phi_{\text{dc}} + \Phi_{\text{ac}} \cos(\omega t), \quad (4) \]

i. e., it contains both constant (dc) flux bias \( \Phi_{\text{dc}} \) and an alternating (ac) flux of amplitude \( \Phi_{\text{ac}} \) and frequency \( \omega \). The normalized equation for the flux through the loop of the SQUID can be obtained by combining Eqs. (3) and (4) and transforming the resulting equation using the relations:

\[ \phi = \frac{\Phi}{\Phi_0}, \quad \phi_{\text{ac,dc}} = \frac{\Phi_{\text{ac,dc}}}{\Phi_0}, \quad \tau = \frac{t}{\omega_L}, \quad \Omega = \frac{\omega}{\omega_{LC}}, \quad (5) \]

where \( \omega_{LC} = 1/\sqrt{LC} \) is the inductive-capacitive SQUID frequency, and the definitions

\[ \beta = \frac{I_c L}{\Phi_0} = \frac{\beta_L}{2\pi}, \quad \gamma = \frac{1}{R} \sqrt{\frac{L}{C}}, \quad (6) \]

for the rescaled SQUID parameter and the loss coefficient, respectively. Thus we get:

\[ \ddot{\phi} + \gamma \dot{\phi} + \phi + \beta \sin(2\pi \phi) = \phi_{\text{dc}} + \phi_{\text{ac}} \cos(\Omega \tau), \quad (7) \]
which can be also written as
\[
\ddot{\phi} + \gamma \dot{\phi} = -\frac{d u_{SQ}}{d \phi},
\]  
(8)

where the normalized SQUID potential reads:
\[
u_{SQ} = -\phi_{\text{ext}}(\tau)\phi + \frac{1}{2} \left( \phi^2 - \frac{\beta}{\pi} \cos(2\pi \phi) \right),
\]  
(9)

and the normalized external flux is given by:
\[
\phi_{\text{ext}}(\tau) = \phi_{dc} + \phi_{ac} \cos(\Omega \tau).
\]  
(10)

\[\text{Figure 2. The SQUID potential } u_{SQ} \text{ from Eq. (9) for } \beta = 0.1369 \ (\beta_L \simeq 0.86). \ (a) } u_{SQ}(\phi) \text{ for } \phi_{ac} = 0 \text{ and } \phi_{dc} = 0 \text{ (black-solid), 0.18 (red-dashed), 0.36 (green-long-dashed), 0.54 (blue-dotted).} \ 
(b) \text{ Temporal evolution of } u_{SQ}(\phi) \text{ during half of the driving period } T = 2\pi/\Omega, \text{ for } \phi_{ac} = 0.16, \beta = 0.1369 \ (\beta_L \simeq 0.86), \Omega = 0.345, \text{ and } \phi_{dc} = 0. \ (c) \text{ Similar to (b) for } \phi_{dc} = 0.36. \text{ The arrows point the direction of the time increasing from } \tau = 0 \text{ to } \tau = T/2.\]

The SQUID potential \( u_{SQ} \), given by Eq. (9), becomes time-dependent for \( \phi_{ac} \neq 0 \). For \( \phi_{ac} = 0 \), although the potential is constant in time, its shape changes with \( \phi_{dc} \); more specifically, while it is symmetric for \( \phi_{dc} = 0 \), it becomes more and more asymmetric with increasing \( \phi_{dc} \), as can be observed in Fig. 2(a). For \( \phi_{ac} \neq 0 \), several snapshots of the time-dependent \( u_{SQ} \) are shown in Figs. 2(b) and (c), for \( \phi_{dc} = 0 \) and \( \phi_{dc} = 0.36 \), respectively, during the first half of the driving period \( T = 2\pi/\Omega \).

The SQUID model has been studied in the past in the hysteretic regime (\( \beta_L > 1 \)), where low dimensional chaos was reported for varying ac flux\(^{18,21}\). The effect of noise has also been studied in the SQUID system, in particular with respect to stochastic resonance\(^{22}\).
Moreover, in Refs.\textsuperscript{23,24} the Melnikov method was applied to the SQUID in the case of small ac field and the threshold conditions for the onset of homoclinic behavior leading to chaos were found. In all of these works, the dc component of the external magnetic flux was set to zero. The role of $\phi_{dc}$ on the SQUID dynamics has been investigated in\textsuperscript{26,27} but only through an analytical approximation of the SQUID equation. The effect of a dc flux bias on the dynamics of a hysteretic SQUID has been also discussed recently in Ref.\textsuperscript{17}. In particular, it was demonstrated theoretically that the state of the system can be shifted from one fixed point to another via the dc flux. In the present work, we revisit the dynamics of the full model for a SQUID oscillator in the non-hysteretic regime ($\beta_L < 1$) and reveal the complex behavior induced by all parameters of the flux bias, with emphasis on the effect of the dc term. Our work contributes significantly to recent experimental findings, where a number of dynamic properties of single, non-hysteretic SQUIDs were demonstrated, such as multistability, switching\textsuperscript{11}, and broad-band tunability of the resonance frequency by magnetic fields and temperature\textsuperscript{8–10}.

The design parameters of a SQUID are its self-inductance $L$, the capacitance $C$ of the JJ, the critical current $I_c$ of the JJ, and the subgap resistance $R$. Typical values of these parameters are $L = 120$ pH, $C = 1.1$ pF, $I_c = 2.35$ µA, and $R = 500$ Ω\textsuperscript{9,10}. These parameters provide the values of the dimensionless coefficients $\beta \simeq 0.1369$ ($\beta_L \simeq 0.86$) and $\gamma \simeq 0.024$ which appear in the normalized Eq. (7) for the flux $\phi = \Phi/\Phi_0$ through the loop of the SQUID. They also provide $f_{LC} = \omega_{LC}/(2\pi) \simeq 13.9$ GHz ($\Omega \simeq 1$) and $f_{SQ} = \omega_{SQ}/(2\pi) \simeq 18.9$ GHz ($\Omega = \Omega_{SQ} \simeq 1.364$) for the geometric and the linear resonance frequency of the SQUID, respectively, which are also typical in single-SQUID experiments\textsuperscript{8–10}. The values of the externally controlled parameters $\phi_{dc}$, $\phi_{ac}$, and $\Omega$ used here, are within the range of the experimentally accessible values, i. e., $\phi_{dc}$ in the interval $[-1, 2]$\textsuperscript{10}, $\phi_{ac}$ in the interval $[0.001, 0.18]$\textsuperscript{10}, and $\Omega$ in the interval $\frac{2\pi}{\omega_{LC}}[10, 22.5]$ GHz\textsuperscript{8,10}.

\section{Snake-like resonance curve and subresonances}

As mentioned previously, in the present work we focus on the non-hysteretic regime of our system. In the context of SQUID dynamics, the terms hysteretic and non-hysteretic refer to its static properties. In practice, the potential $u_{SQ}$ of a non-hysteretic SQUID has only one minimum (as shown in Fig. \textsuperscript{2}), while the corresponding potential $u_{SQ}$ of a
Figure 3. The “snake-like” resonance curve of the SQUID for $\beta = 0.1369$ ($\beta_L = 0.86$), $\gamma = 0.024$, external ac flux amplitude $\phi_{ac} = 0.16$ and dc flux $\phi_{dc} = 0.36$. Blue and red lines correspond to branches of stable and unstable periodic solutions, respectively. The black dashed curve corresponds to $\phi_{dc} = 0$. Insets: Enlargement around the maximum multistability frequency (left) and around lower frequencies (right). The symbols “SN” in the left inset denote saddle-node bifurcations of limit cycles.

Hysteretic SQUID has more than one, which gives rise to multistability. In the case of a single-well $u_{SQ}$ (which is the case here), the multistability is not related to the minima of the potential, but instead it is a purely dynamical effect. In the linear regime, i.e., for $|\phi_{ext}(\tau)| \ll 1$, the flux amplitude-frequency (resonance) curve of the SQUID resembles that of a harmonic oscillator with eigenfrequency $\Omega_{SQ}$. However, with increasing $\phi_{ac}$ and/or $\phi_{dc}$, nonlinear effects become more and more significant. The resonance curve bends more and more; in the strongly nonlinear regime, it acquires a “snake-like” form with several stable and unstable branches. By setting $\gamma = 0$ and $\phi_{ext} = 0$ into Eq. 7, linearizing, and using $\beta \sin(2\pi\phi) \simeq \beta_L \phi$, the linear resonance frequency of the SQUID can be obtained as $\Omega_{SQ} = \sqrt{1 + \beta_L}$. Note that the geometric resonance frequency is $\Omega = 1$ (or $\omega = \omega_{LC}$ in
natural units), which is always lower than $\Omega_{SQ}$. The resonance frequency of the SQUID can be tuned by varying $\phi_{ac}$ and/or $\phi_{dc}$; thus that frequency can be shifted from $\Omega \simeq \Omega_{SQ}$ in the linear regime, to $\Omega \simeq 1$ in the strongly nonlinear regime. It is the latter regime that is investigated here. When the SQUID is driven by an ac flux, the flux through the loop of the SQUID oscillates with a particular amplitude; its frequency or equivalently its period of oscillation is that of the driving flux (although oscillations with periods several times that of the driving flux or even chaotic oscillations can be also observed in the strongly nonlinear regime, as we shall see below).

In this Section, we present in detail the delicate structure of the SQUID resonance curve in a wide range of values for the driving frequency $\Omega$. Figure 3 shows the amplitude of the flux oscillations calculated from Eq. (7) over the frequency of the ac flux field $\Omega$ for finite ac and dc flux. This curve has been obtained using a very powerful software tool that executes a root-finding algorithm for continuation of steady state solutions and bifurcation problems. This tool also allows us to determine the stability of the system’s periodic solutions through the calculation of the corresponding Floquet multipliers. In relation to stability, in Fig. 3 the blue and red branches mark the stable and unstable periodic (period-1) solutions, respectively. This “snake-like” form of the resonance curve has also been observed in the Duffing equation.

Notably, a snaking resonance curve for a nonlinear superconducting quantum oscillator which is very similar to the one shown in Fig. 3 has been reported in figure 3 of Ref. The curve depicts the number of photons absorbed for the occurrence of quantum phase slips (which are the cause of nonlinearity) as a function of the driving frequency. It could be argued that this curve is the dual quantum analogue of the snaking resonance curves presented here. In our model the “snake-like” resonance curve was first reported in the context of chimera states, where its crucial role for the emergence of these patterns was discussed. In Ref. the dc flux $\phi_{dc}$ was set to zero, which in Fig. 3 corresponds to the black dashed line. Both curves exhibit a winding behavior around the geometric resonance frequency $\omega_{LC}$, where multiple saddle-node bifurcations of limit cycles take place when stable and unstable branches merge (marked by “SN” in the left inset).

However, the inclusion of a dc component in the external magnetic flux creates some new effects at lower and secondarily at higher frequencies. As shown by the right inset, at lower frequencies, $\phi_{dc}$ induces new stable and unstable periodic solutions which create a complex structure of branches which are absent for zero dc flux. We can distinguish multiple
secondary maxima one of which appears around Ω = 0.45 like in the zero dc flux curve. This value is about one third of the SQUID linear resonance frequency.

The other subresonances are shifted to lower values compared to other fractions of Ω_{SQ} and this is due to the nonlinearity which is more prominent for finite φ_{dc}. This is related to the fact that the SQUID potential becomes more and more asymmetric as φ_{dc} increases (see Fig. 2) enhancing, thus, the nonlinear effects in our system. The bifurcation structure in the low frequency regime is very complex and one example is shown in the enlargement in Fig. 4(a). We can see that in this small interval of Ω, four bifurcations take place: One saddle-node (“SN”) and three period-doubling bifurcations denoted by the letter “P”. In Fig. 4(b) we plot the phase portraits of the periodic solutions corresponding to points A – D of Fig. 4(a). The orbit corresponding to A is a stable period-2 solution which becomes
unstable at the second period-doubling bifurcation. This is evident by the corresponding Floquet multipliers (Fig. 4(c)) whose real part cross the complex unit circle through $-1$. Simultaneously, a new stable period-4 orbit is created belonging to the stable branch which emerges at the “P” bifurcation point after point $A$. Its phase portrait for $\Omega = 0.6167$ corresponds to point $C$ and is shown in Fig. 4(b) along with its Floquet multipliers in Fig. 4(c) which, as expected, lie in the complex unit circle. As $\Omega$ increases further, the real part of the Floquet multipliers exit the complex unit circle through $+1$ and a saddle-node bifurcation (“SN”) takes place, giving birth to an unstable orbit (point $D$).

III. BIFURCATION DIAGRAMS AND ATTRACTORS

The sequence of bifurcations described above, takes place on the respective branches of periodic solutions. These coexist with multiple other branches which can be found by continuation from different initial conditions. Moreover, the period-doubling bifurcations discussed in Fig. 4 may lead the system to chaotic dynamics. Therefore, the full bifurcation diagram with $\Omega$ as the control parameter is much more complex, and parts of it can be seen in the upper panels of Fig. 5. In Fig. 5(a) we observe a typical period-doubling bifurcation cascade which leads to chaos at $\Omega \approx 0.263$. In Fig. 5(b) we observe a period-doubling bifurcation cascade of a period-8 solution which is followed by a reverse such cascade. In the upper panel of Fig. 5(c), besides the period-doubling cascade at low frequencies $\Omega$, we observe the formation of a stable period-4 “bubble”$^{32}$. The latter is created (destroyed) through a period-doubling (reverse period-doubling) bifurcation at $\Omega \approx 0.54$ ($\Omega \approx 0.62$). The appearance of bubbles here is a manifestation of “antimonotonicity”, i.e., the concurrent creation and destruction of periodic orbits, which has been observed in several physical systems such as driven nonlinear $RLC$ circuits$^{33}$. The above described bifurcation diagrams have been produced via direct numerical integration of our model equation (Eq. 7) and continuation in $\Omega$, for different initial condition realizations.

Next, we proceed to the quantification of the chaotic orbits through suitable measures. There are several ways to define the fractal dimension of a chaotic attractor, which is a measure of its geometric scaling properties or its “complexity” and it has been considered its most basic property. These methods fall into two categories, those derived from the topology, and those derived from the dynamics. Perhaps the most common of the former metrics is the
correlation dimension and the most common of the latter type is the Lyapunov dimension $d_L$, proposed by Kaplan and Yorke. According to the definition of Kaplan and Yorke,

$$d_L = k + \frac{1}{|\lambda_{k+1}|} \sum_{i=1}^{k} \lambda_i,$$

where the Lyapunov exponents $\lambda_i$ are calculated from the system’s equations of motion using the method described in and $k$ is defined by the condition that

$$\sum_{i=1}^{k} \lambda_i \geq 0 \text{ and } \sum_{i=1}^{k+1} \lambda_i < 0.$$  

For example, the Lyapunov dimension of the chaotic attractor for $\Omega = 0.5$ shown in the upper-right panel of Fig. 5(a) is $d_L \approx 2.53$. In Figs. 5(a)-(c), we see a clear one-to-one correspondence between the bifurcation plots and the curves of the Lyapunov exponents. That is, chaotic dynamics emerges when the largest Lyapunov exponent becomes positive, while the bifurcation points correspond to its zeroing. In the lower panel of each subfigure, the calculated fractal (Lyapunov or Kaplan-Yorke) dimension of the phase space attractors of the SQUID are also shown. Note that the Lyapunov dimension $d_L$ of chaotic attractors is always between 2 and 3 as it should be, i.e., the chaotic phase space attractor is topologically complicated more than a limit cycle and less than a three-dimensional object. The geometric complexity of phase space attractors can be compared through their Lyapunov dimension.

In order to observe the complexity of these phase-space attractors, we have also obtained the stroboscopic maps (as the complete phase space attractors are not very easy to visualize). In Fig. 6(a), four such attractors are shown for four different values of the driving frequency $\Omega$ (indicated on each subfigure).

Chaotic behavior arises by varying all three parameters of the external magnetic flux. This is depicted in Figs. 6(b)-(d), where the largest Lyapunov exponent $\lambda_1$ is calculated in the relevant parameter spaces. In general, chaos is observed for moderate to higher values of $\phi_{ac}$ (Figs. 6(b), (c)) and for $\Omega$ values lower than 0.7. Note that the chaotic regimes show approximately periodic structure in dependence of the dc flux $\phi_{dc}$. In order to explore this further, we take a cross section of Figs. 6(b) and (c) at $\phi_{ac} = 0.16$ and follow a single branch of periodic solutions up to $\phi_{dc} = 1.0$. These are depicted in the left panels of Fig. 7(a) and (b) where again, blue and red denote stable and unstable periodic solutions and in Fig. 7(b) we have also noted the bifurcations that occur along the branch: There is a cascade of
Figure 5. Dynamical behavior of a SQUID subjected to an external flux having both ac and dc components (Eq. (10)) as a function of the driving frequency $\Omega$ for three different intervals. For all three subfigures: Upper panel: bifurcation diagram. Middle panel: all three Lyapunov exponents. Lower panel: the corresponding fractal (Kaplan-Yorke or Lyapunov) dimension $d_L$. Other parameters as in Fig. 3.

eight period doubling bifurcations preceded and followed by a sequence of four saddle-node bifurcations, which is repeated periodically for $\phi_{dc} > 1$. The periodicity with the dc flux was discussed in reference 9, where an analytical expression of the solution of the linearized version of Eq. 7 was found as a function of $\phi_{dc}$. Note that there is also a symmetry in the bifurcation points, around $\phi_{dc} = 0$, which is also reflected in Fig. 7(c) where the phase portraits for various $\phi_{dc}$ values are shown for $\Omega = 0.51$: The orbit for $\phi_{dc} = 0.5$ lies in the center and around it we have two symmetrical period-1 orbits of low amplitude for $\phi_{dc} = 0.0, 1.0$ and two symmetrical period-2 orbits of higher amplitude for $\phi_{dc} = 0.25, 0.75$.

The external flux bias $\phi_{dc}$ certainly affects the “snake-like” resonance curve of the SQUID as well. First of all, the frequency $\Omega$ at which the first saddle-node bifurcation emerges fluctuates periodically with $\phi_{dc}$ (not shown here). Moreover, around the resonance, the saddle-node bifurcations change from sub- to supercritical. Most importantly, for sufficiently high values of $\phi_{ac} > 0.12$, there are particular values of $\phi_{dc}$ at which those saddle-node bifurcations disappear! This is evident in Fig. 7(d), in which the resonance curve is shown for two values of $\phi_{dc}$. We can see that for $\phi_{dc} = 0.25$ the saddle-node bifurcations vanish and the real part of the corresponding Floquet multipliers never crosses the value +1 (Fig. 7(e)). The
Figure 6. (a) Stroboscopic maps of chaotic attractors on the $\phi - \dot{\phi}$ phase space for four different frequencies $\Omega$ indicated in each subfigure, for $\phi_{ac} = 0.16$, and $\phi_{dc} = 0.36$. (b) The maximum Lyapunov exponent $\lambda_1$ mapped onto the $(\phi_{dc}, \phi_{ac})$ plane for $\Omega = 0.345$, and (c) $\Omega = 0.51$. (d) The maximum Lyapunov exponent $\lambda_1$ mapped onto the $(\phi_{dc}, \Omega)$ plane for $\phi_{ac} = 0.16$. Other parameters as in Fig. 3.

same holds periodically for $\phi_{dc} = 0.75, 1.25, \ldots$ (not shown here). Note that for relatively low values of $\phi_{ac}$ the saddle-node bifurcations do not disappear for any value of $\phi_{dc}$.

In addition to the effects described above, $\phi_{dc}$ can create multiple periodic solution branches which coexist and may lead to chaos through complex bifurcation scenarios. This is depicted in Fig. 8 where the bifurcation diagrams for two different values of $\Omega$ are plotted. In the enlargements of the right panel of (a), we can see two period-doubling cascades coexisting and leading to a common chaotic regime which ends in eight branches of periodic solutions. Another interesting feature is the formation of four period-2 “bubbles” for higher values of $\phi_{dc}$. In the enlargement of (b) we see a similar period doubling cascade as in (a),
Figure 7. Amplitude of the magnetic flux $\phi$ in dependence of the dc flux for two different values of the external frequency (a) $\Omega = 0.345$ and (b) $\Omega = 0.51$. Blue and red lines correspond to branches of stable and unstable periodic solutions, while “SN” and “P” in (b) denote saddle-node and period doubling bifurcations, respectively. (c) Phase portraits of the magnetic flux for five different dc fluxes and $\Omega = 0.51$. (d) Resonance curve for $\phi_{dc} = 0.10$ (black) and $\phi_{dc} = 0.25$ (red) and (e) real part of the corresponding leading Floquet multiplier. Other parameters as in Fig. 3.

but here the two branches lie within one-another.

IV. CONCLUSIONS

The dynamic properties of an rf SQUID, which is a highly nonlinear oscillator, were explored numerically using the well-established model Eq. (7) governing the temporal evolution of the flux $\phi(\tau)$ through the loop of the SQUID. The coefficients entering the model equation have been obtained from relevant experimental parameters, and thus our results can be in principle experimentally testable. The SQUID was subjected to the sum of a constant and a sinusoidal flux. It is demonstrated that a non-zero flux bias crucially affects
the dynamics of the SQUID in a wide range of driving flux amplitudes and frequencies.

At frequencies around the geometrical resonance $\omega_{LC}$, in particular, the flux bias $\phi_{dc}$ changes the snaking resonance curve by varying the location of the saddle-node bifurcations while it transforms them from subcritical to supercritical or vice versa. At frequencies higher than $\omega_{LC}$, it strongly enhances secondary resonances, which may also exhibit multistability similar to that of the primary resonance curve. At frequencies lower than $\omega_{LC}$, wide-band chaotic behavior has been observed for reasonable and experimentally accessible values of the flux bias $\phi_{dc}$ and the driving amplitude $\phi_{ac}$. A wealth of nonlinear dynamics effects such as period-doubling and reverse period-doubling, multi-periodic solutions, saddle-node bifurcations, bubbling and multistability, has been observed in this region. It should be noted that no period-doubling bifurcation cascades and subsequent transitions to chaos have been observed for $\phi_{dc} = 0$. This is probably due to the symmetry of time-independent part of the SQUID potential in the absence of a constant flux bias, which renders the SQUID equation (Eq. [7]) symmetric according to the considerations in Ref. [38]. That symmetry suppresses
period-doubling bifurcations in a large class of systems, including the sinusoidally driven damped pendulum. The bifurcation diagrams with varying $\phi_{dc}$ at frequencies lower than $\omega_{LC}$ clearly reveal multistability, where periodic/multi-periodic and chaotic solutions may coexist. This allows for switching the SQUID dynamics from periodic to multi-periodic to chaotic with a slight variation of $\phi_{dc}$.

Extensive calculations of the maximal Lyapunov exponent in two external parameter spaces indicates that wide-band chaotic behavior appears at frequencies lower than $\omega_{LC}$ and relatively high $\phi_{dc}$ and $\phi_{ac}$. However, some regions of $\phi_{dc}$ and $\phi_{ac}$ are still experimentally accessible. Note that for relatively high $\phi_{ac}$, the resonance curve may lose its “snake-like” form for a certain value of $\phi_{dc} = 0.25$. The vanishing of the resonance curve “snake-like” form is repeated periodically in the dc fluc and marks the switching from chaotic to regular behavior. This study can prove very useful for the deeper understanding of the collective dynamics of coupled SQUIDs which form metamaterials that find important technological applications. Moreover, it may initiate further experimental work on the dynamics of single SQUIDs to confirm the predictions above.

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