Killing Spinor Identities

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ABSTRACT

We have found generic Killing spinor identities which bosonic equations of motion have to satisfy in supersymmetric theories if the solutions admit Killing spinors. Those identities constrain possible quantum corrections to bosonic solutions with unbroken supersymmetries.

As an application we show that purely electric static extreme dilaton black holes may acquire specific quantum corrections, but the purely magnetic ones cannot.
In recent years there was a strong interest in finding exact classical solutions in theories including gravity. Some of these solutions have very interesting properties, which suggest that they remain exact even after quantum corrections are taken into account. Non-renormalization theorem were used to show that some solutions do not acquire quantum corrections. In this article we will show that in addition to the methods used before there exists a new method, which can be used in order to find and investigate classical solutions, which are not affected by quantum corrections. In some special cases, where the solutions possess Killing spinors, there exist identities, which serve as a powerful tool which help to control quantum corrections.

In order to explain the origin of the new identities, let us consider a classical Lagrange equation derived from some action principle.

\[ \frac{\delta S}{\delta \phi^b} \equiv S_{\phi^b} = 0. \] (1)

Quantum corrections modify those equations by introducing a right-hand side, which is absent in the classical approximation:

\[ S_{\phi^b} = J_{\phi^b}. \] (2)

Generically, no information is available about the quantum corrections to the equations of motion in 4-dimensional gravitational problems. We will exhibit here some particular situations when such information is available due to symmetry properties of \( J_{\phi^b} \), which follow from the symmetry properties of \( S_{\phi^b} \). A well known example is given by the Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}. \] (3)

The stress-energy tensor \( T_{\mu\nu} \) appears in the right-hand side of Einstein equations and may describe both classical sources as well as quantum corrections to classical equations. The left-hand side of Einstein equations satisfies the identities

\[ \nabla^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0. \] (4)

The right-hand side of Einstein equations also must satisfy the covariant conservation law

\[ \nabla^\mu T_{\mu\nu} = 0. \] (5)

This law was used in numerous problems of general relativity. Even when \( T_{\mu\nu} \) is not known, some of its properties can be deduced from its covariant conservation law, supplemented by specific assumptions like time independence, etc.

Historically it was the other way around. In his first attempt \(^{[1]}\) to find the field equations for gravity Einstein proposed

\[ R_{\mu\nu} = T_{\mu\nu}. \] (6)

He realized very soon that in general this equation contradicts the covariant conservation of the energy momentum tensor. The first paper was submitted on November 11. On November 18 Einstein submitted an Addendum \(^{[2]}\) where he required that \( T_{\mu\nu} = 0 \) for consistency with conservation of energy and momentum. Finally, on December 2 he submitted the second paper

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where the crucial final step was made to introduce eq. (3). It was also realized that when Einstein equations are derived from the Einstein-Hilbert Lagrangian, the left hand side of eq. (3) appears automatically, and the identity (4) is the consequence of the general covariance of the action. Thus, either one knows some identities which the right-hand side of the equations satisfies due to some symmetry principle (Einstein started with $\nabla^\mu T_{\mu\nu} = 0$) and requires the same identities to be satisfied by the left-hand-side (which enforces general covariance in Einstein’s theory) or vice versa.

The purpose of this paper is to show that for bosonic configurations of supersymmetric theories with unbroken supersymmetries, the equations of motion satisfy some generic identities. These identities are due to the symmetries of the theory and the solutions. This forces the right-hand side of these equations, coming both from classical sources and from quantum corrections, to satisfy the same identities. An investigation of these identities for specific configurations may give a substantial information about the properties of quantum corrections which may differ from one configuration to another.

We will first rewrite the information discussed above in a language which will be easy generalizable to our problem.

The fact that the action $S = \int d^4x \sqrt{-g} R$ is general covariant means that

$$\delta_\xi S \equiv \frac{\delta S}{\delta g^{\mu\nu}} \delta_\xi g^{\mu\nu} = 0 ,$$

(7)

where $\delta_\xi g^{\mu\nu} \equiv 2\nabla^{(\mu}\xi^{\nu)}$ is the general covariance transformation of the metric. Let us show that eq. (7) does imply the identity (4). Indeed, in detailed form eq. (7) is

$$\int d^4x \frac{\delta S}{\delta g^{\mu\nu}} \nabla^\mu \xi^\nu = \int d^4x \sqrt{-g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \nabla^\mu \xi^\nu = 0 .$$

(8)

Since $\xi^\nu$ is an arbitrary function of $x$, we may vary equation (7) over $\xi^\nu$. The result is

$$\frac{\delta S}{\delta \xi^\nu(x)} = -2\sqrt{-g} \nabla^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 ,$$

(9)

which is equivalent to eq. (4). Thus it is the symmetry of the action, which is responsible for the fact that some mix of the variational derivatives of the action over the metric, as given in eq. (7), has to be zero. Therefore if one wants to consider the equation

$$\frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{-g} T_{\mu\nu} ,$$

(10)

the right-hand side of this equation also has to satisfy the identity, following from replacing $\frac{\delta S}{\delta g^{\mu\nu}}$ in eq. (8) by $\sqrt{-g} T_{\mu\nu}$. The covariant conservation of the energy-momentum tensor (5) follows.

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3Actually he first introduced it in the form $R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$. 
Let us start with some action $S$ which has local supersymmetry. This means that there exist some supersymmetric transformations $\delta_\epsilon \phi^b$, $\delta_\epsilon \phi^f$ with parameter $\epsilon(x)$ of all bosonic fields $\phi^b$ and of all fermionic fields $\phi^f$ such that

$$\delta_\epsilon S \equiv \sum_b S_{,b} \delta_\epsilon \phi^b + \sum_f S_{,f} \delta_\epsilon \phi^f = 0 .$$

(11)

This identity relates all variational derivatives of the action over bosons ($\frac{\delta S}{\delta \phi^b} \equiv S_{,b}$) to those over fermions ($\frac{\delta S}{\delta \phi^f} \equiv S_{,f}$).

Identity (11) is satisfied for any (on- or off-shell) values of bosonic and fermionic fields and arbitrary local fermionic parameter $\epsilon(x)$. We would like to vary this identity over the fermions once, and after the variation to set all fermions to zero. It is useful to remember that $S_{,b}$ and $\delta_\epsilon \phi^b$ are even functions of fermions, and $S_{,f}$ and $\delta_\epsilon \phi^f$ are odd functions of fermions. Therefore we get

$$\left( \delta_\epsilon S \right)_{,f_2} \equiv \left. \left( \sum_b S_{,b} \left( \delta_\epsilon \phi^b \right)_{,f_2} + \sum_f S_{,f_1} \delta_\epsilon \phi^f \right) \right|_{\phi^f=0} = 0 .$$

(12)

In this form this identity is still not very useful for the purely bosonic part of the system, in which we are interested. One may expect that quantum corrections will change the bosonic equations from $S_{,b} = 0$ to $S_{,b} = J_b$. We want to extract some information about the left-hand side of this equation from symmetries of the theory and use them as the constraints on the possible form of $J_b$.

Equation (12), which follows only from the symmetries of the theory, is purely bosonic, but still in addition to bosonic variational derivatives $S_{,b}$ of the action the second variational derivatives of the action over fermions $S_{,f_1 f_2} \equiv \frac{\delta^2 S}{\delta \phi_{,b}^f \delta \phi^f_{,f_2}}$ enters in the second term. Therefore we have to impose some additional assumption about the configuration for which we are going to study the proposed equation (2): if we consider only bosonic configurations admitting Killing spinors (KS), i. e. supersymmetry parameters $\epsilon(x)$ satisfying

$$\delta_\epsilon K\text{illing } \phi^f \bigg|_{\phi^f=0} = 0 ,$$

(13)

then we are left with the following identities

$$\left( \delta_\epsilon K\text{illing } S \right)_{,f} \equiv \sum_b S_{,b} \left( \delta_\epsilon K\text{illing } \phi^b \right)_{,f} \bigg|_{\phi^f=0} = 0 .$$

(14)

Our proposed equation (2) with quantum corrections taken into account is consistent only if

$$\sum_b S_{,b} \left( \delta_\epsilon K\text{illing } \phi^b \right)_{,f} \bigg|_{\phi^f=0} = \sum_b J_b \left( \delta_\epsilon K\text{illing } \phi^b \right)_{,f} \bigg|_{\phi^f=0} = 0 .$$

(15)

We will call these identities Killing Spinor Identities (KSI). Although they were derived from supersymmetry, they are identities for bosonic fields only. In words, equations (15) tell us that
one has to check the consistency of eqs. (2) as follows: For every bosonic field of the theory one has to multiply the \( J_b \)-term by the corresponding bosonic supersymmetry transformation with the KS as a parameter, and take a sum over all bosons. The sum, varied over each fermionic field, should vanish!

Equation (13) is our main result. It is interesting that in numerous studies of bosonic solutions of equations of motions which have unbroken supersymmetries, i.e. admit KS’s, only fermionic transformations were used to establish the configuration for which eq. (13) is satisfied. Our KSI may help to find the most general solution of KS equations (13), which either permits the non-trivial \( J_b \) terms in the right-hand side of eq. (2) or forbids them. In the first case quantum corrections to supersymmetric bosonic solutions of classical field equations are possible, in the second case there are no such corrections. The additional input comes from the knowledge of the supersymmetric transformation rules for bosonic fields, which was not used before in the context of bosonic configurations with unbroken supersymmetries.

We would like to apply the KSI (14) to the configuration of \( U(1) \) dilaton and axion-dilaton 4d black holes. We do not want to use the already known extreme solutions of classical field equations \[3\], \[4\]. Those solutions have some unbroken supersymmetries \[3\], but our purpose here is to find whether the KS equations (13) have more general solutions than those which solve exactly the classical field equations. The action with \( N = 4 \) local supersymmetry is the \( SU(4) \) version of \( N = 4 \) supergravity \[6\]. Its bosonic part, which will be considered here, depends on the vierbein \( g_{\mu\nu} \), on the abelian vector field \( A_\mu \) and on the dilaton-axion field \( \lambda = a + ie^{-2\phi} \). Our notation is defined in \[5\]. The equations of motion, which may acquire a non-trivial right-hand side due to quantum corrections are

\[
\frac{\delta S_{cl}}{\delta g_{\mu\nu}} = J^{\mu\nu}, \quad \frac{\delta S_{cl}}{\delta \lambda} = J, \\
\frac{\delta S_{cl}}{\delta A_\mu} = J^\mu, \quad \frac{\delta S_{cl}}{\delta \bar{\lambda}} = \bar{J}.
\]

(16)

The axion-dilaton extreme black hole solutions \[3\], \[4\] are solutions of equations (16) with

\[
J^{\mu\nu} = J^\mu = J = \bar{J} = 0.
\]

(17)

The extreme solutions are supersymmetric \[3\], i.e. they admit KS’s (13). The four gravitinos \( \Psi_{\mu I} \) and four dilatinos \( \Lambda_I \) have vanishing local supersymmetry transformations in presence of gravity, vector field and dilaton.

\[
\delta_{\text{Killing}} \Psi_{\mu I} = 0, \quad \delta_{\text{Killing}} \Lambda_I = 0, \quad I = 1, 2, 3, 4.
\]

(18)

\textit{A priori} it is not known whether there exist solutions of Killing equations (18) which simultaneously solve the dynamical equations (19) with non-trivial right-hand sides. Using the KSI

\footnote{We will consider only solutions with one vector field for simplicity and take, as in \[3\], \( A_\mu = A_\mu^3 \).}
derived above one can address this problem. One has to take into account the following supersymmetry transformations of our bosonic fields:

$$\delta \epsilon g_{\mu \nu} = \epsilon^I \gamma_\mu \Psi^I_\nu + \bar{\epsilon} J_\mu \gamma^\nu \Psi^I_\nu + \epsilon^I \gamma_\nu \Psi^I_\mu ,$$

$$\delta \epsilon A_\mu = -\frac{1}{\sqrt{2}} e^\phi (\epsilon^I \alpha \bar{\epsilon} J^I_\mu \Psi^J_\mu + \bar{\epsilon} J^I_\mu \gamma^\mu \Lambda^J_\mu - \epsilon^I \alpha \bar{\epsilon} J^I_\mu \Lambda^J_\mu ) ,$$

$$\delta \epsilon \lambda = -4 i e^{-2 \phi} \epsilon J^I_\lambda , \quad \delta \epsilon \bar{\lambda} = 4 i e^{-2 \phi} \bar{\epsilon} J^I_\bar{\lambda} .$$

(19)

The next step is to form a product of each “source-term” $J_b$ from the right-hand side of eq. (16) with the proper $\delta \epsilon \phi^b$, i.e. to write the function

$$\Omega \equiv \sum_b J_b \delta \epsilon \phi^b = J^{\mu \nu} \delta \epsilon g_{\mu \nu} + J^\mu \delta \epsilon A_\mu + \frac{1}{2} (J \delta \epsilon \lambda + c.c.) .$$

(20)

This function is linear in fermions and we have to differentiate it over all types of fermions and the result has to vanish. The KSI take the form $\Omega_{J_f} = 0$. Let us list these identities for our example.

$$\Omega_{\lambda J} = -J^\mu \frac{1}{\sqrt{2}} e^\phi \epsilon J^I_\lambda \gamma_\mu + 2 i J e^{-2 \phi} \epsilon J^I = 0 ,$$

$$\Omega_{\lambda J} = -J^\mu \frac{1}{\sqrt{2}} e^\phi \epsilon J^I_\lambda \gamma_\mu - 2 i J e^{-2 \phi} \bar{\epsilon} J^I = 0 ,$$

$$\Omega_{\psi J^I} = -2 J^{\mu \nu} \epsilon J^I_\mu + J^\mu \frac{1}{\sqrt{2}} e^\phi \epsilon J^I_\lambda \gamma_\nu = 0 ,$$

$$\Omega_{\psi J^I} = -2 J^{\mu \nu} \epsilon J^I_\mu + J^\mu \frac{1}{\sqrt{2}} e^\phi \epsilon J^I_\lambda \gamma_\nu = 0 .$$

(21)

We assume that the KS equations (18) for our metric, vector field and axion-dilaton are satisfied (i.e. $\epsilon = \epsilon_{\text{Killing}}$) and that simultaneously these configurations solve the equations of motion (16). The first pair of equations (21) has solutions only if

$$J_\mu J^\mu = 8 e^{-6 \phi} |J|^2 .$$

(22)

The second pair of equations implies

$$J^\mu J^\nu = 8 e^{-2 \phi} J^\mu_\eta J^\nu_\eta .$$

(23)

These conditions can be more or less restrictive for different configurations. We will study the simplest supersymmetric black hole configurations: $U(1)$ static dilaton black holes. Without

\(^5\)In this particular example we do not consider $\alpha'$ corrections to the $N = 4$ supersymmetry transformation rules. However, the general Killing Spinor Identities given in eq. (15) are correct also when one includes them.
axion field \((a = 0, J = -\bar{J} = \frac{1}{2}e^{2\phi}J_0)\) the static black hole solutions are either electric \((F_{ij} = 0\) and \(J_i = 0)\) or magnetic \((F_{0i} = 0\) and \(J_0 = 0)\). In the first case condition \((22)\) is satisfied and

\[ J_0^2 = 2e^{-2\phi}J_0^2 . \tag{24} \]

In the magnetic case equation \((22)\) cannot be satisfied, since the right-hand side is positive, and the left-hand side is negative. This proves that the only purely magnetic dilaton black hole with unbroken supersymmetries is the well known extreme solution of classical equations \([4], [5]\). For electric case the solution exists under the condition that \(J_{ab}\) has only time components \(J_{00}\), and that there exists the relation

\[ |J_\phi| = |J_0| \frac{1}{\sqrt{2}e^{\phi}} = 2|J_{00}| . \tag{25} \]

After having established that our KSI are satisfied in the electric case with non-vanishing corrections to the classical equations and in the magnetic case with vanishing corrections, we can confirm this by looking on the KS equations \((18)\). The analysis proceeds as the one performed in \([5]\), however this time we solve in addition to \((18)\) the dynamical equations with some unknown quantum corrections \(J^\mu, J_\phi, J_{\mu\nu}\):

\[ 4\sqrt{-g}\nabla_\mu(e^{-2\phi}F^{\mu\nu}) = J^\mu , \]

\[ -4\sqrt{-g}(\nabla^2\phi - \frac{1}{2}e^{-2\phi}F^2) = J_\phi , \]

\[ -\sqrt{-g}\{R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R - 2[\nabla_\mu\phi \cdot \nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2] - 2e^{-2\phi}[F_{\mu\lambda}F_{\nu\delta}g^{\lambda\delta} - \frac{1}{4}g_{\mu\nu}F^2]\} = J_{\mu\nu} . \tag{26} \]

The static electric solution of eqs. \((26)\) and \((18)\) with unbroken \(N = 2\) supersymmetry exists and is given by

\[ ds^2 = e^{2U}dt^2 - e^{-2U}dx^2 , \]

\[ A = \psi dt , \quad F = d\psi \wedge dt , \]

\[ \sqrt{2}\psi = \pm e^{+2U}e^{\phi_0} , \quad \phi = U + \phi_0 , \]

\[ \partial_i\partial_i e^{-2U} = -\frac{1}{2}e^{-2U}J_\phi , \tag{27} \]

where the right-hand side of the last equation is an arbitrary function \(J_\phi(\vec{x})\), which can come either from external classical sources or from quantum corrections. For this solution the terms in the right-hand side of equation \((27)\) defining corrections to classical equations are subject to the constraint \(J_\phi = \pm J_0 \frac{1}{\sqrt{2}}e^\phi = 2J_{00}\), in agreement with the requirement from the KSI \((25)\).
We have checked that the most general static solution of the purely magnetic type, which admits KS’s of $N = 4$ supersymmetry (18) and solves the dynamical equations (26), exists only if $J^\mu = J_\phi = J_{\mu\nu} = 0$, i.e. the extreme magnetic dilaton black hole cannot acquire quantum corrections, in a complete agreement with the prediction from the KSI (15). Equations (18) ensure the existence of a dual magnetic potential [5], and the solution is

$\tilde{A} = \tilde{\psi} dt$, $\tilde{F} = i d\tilde{\psi} \wedge dt$,

$\sqrt{2} \tilde{\psi} = \pm e^{+2U-\phi_0}$, $\phi = -U + \phi_0$,

$\partial_i \partial_i e^{-2U} = 0$.  

(28)

Thus we have found a rather unexpected difference in the properties of electric and magnetic supersymmetric dilaton black hole solutions at the quantum level. This difference follows from eq. (22), which requires $J_\mu J^{\mu} \geq 0$. In absence of $J^0$ term in the right-hand side of the equation, defining the metric, these two solutions are related by $SL(2, R)$-symmetry [6]. But in presence of quantum corrections the magnetic solution remains the same, the metric $g^{00} = e^{-2U}$ is still a harmonic function and for the electric solution a new unknown function may appear in the right-hand side, and both types of solutions have unbroken supersymmetries.

Our first experience of using KSI for the purpose of investigations of the non-perturbative solutions in quantum gravity looks promising. More general black hole solutions, like the ones with axion field may be tested for quantum corrections. Any other bosonic configurations with unbroken supersymmetries may be tested in the same way. The KSI presented in eq. (13) of this paper are valid for any locally supersymmetric theory in any dimension and with any number of supersymmetries with or without $\alpha'$ corrections. Thus, each time equations $\delta_\chi \Psi = 0$ have been used to find a bosonic supersymmetric configuration, an additional new information about how those configurations are affected by quantum corrections may be available from Killing spinor identities (15), applied to each of those configurations. They involve the specific use of supersymmetry transformation rules of bosonic fields of the theory. As we have seen, in the black hole case the result is the following: static purely magnetic supersymmetric extreme dilaton black hole is not affected by quantum corrections, however, the electric one may be affected. In particular, we have found specific relations between quantum corrections to different equations. We believe that the new method of using Killing spinor identities in the context of supersymmetric bosonic configurations improves essentially the quality and the power of the arguments, which are used in proving the so-called “supersymmetric non-renormalization theorems”.

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