RIGID LEVI DEGENERATE HYPERSURFACES
WITH VANISHING CR-CURVATURE

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Abstract. We continue our study, initiated in article [I3], of a class of rigid hypersurfaces in \( \mathbb{C}^3 \) that are 2-nondegenerate and uniformly Levi degenerate of rank 1, having zero CR-curvature. We drop the restrictive assumptions of [I3] and give a complete description of the class. Surprisingly, the answer is expressed in terms of solutions of several well-known differential equations, in particular, the equation characterizing conformal metrics with constant negative curvature and a nonlinear \( \bar{\partial} \)-equation.

1. Introduction

This paper is a continuation of articles [I2, I3], and we will extensively refer the reader to these papers in what follows. In particular, a brief review of CR-geometric concepts is given in [I2, Section 2], and we will make use of those concepts without further reference.

We consider connected \( C^\infty \)-smooth real hypersurfaces in \( \mathbb{C}^n \), with \( n \geq 2 \). Specifically, we look at rigid hypersurfaces, i.e., hypersurfaces given by an equation of the form

\[
\text{Re} \, z_n = F(z_1, \overline{z}_1, \ldots, z_{n-1}, \overline{z}_{n-1}),
\]

where \( F \) is a smooth real-valued function defined on a domain in \( \mathbb{C}^{n-1} \). Rigid hypersurfaces have rigid CR-structures in the sense of [BRT] and are invariant under the 1-parameter family of holomorphic transformations

\[
(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, z_n + it), \quad t \in \mathbb{R}.
\]

We only consider rigid hypersurfaces passing through the origin and will be interested in their germs at 0. Thus, we assume that \( F \) in (1.1) is defined near the origin and \( F(0) = 0 \), with the domain of \( F \) being allowed to shrink if necessary.

We will utilize a natural notion of equivalence for germs of rigid hypersurfaces as introduced in [I3]. Namely, two germs of rigid hypersurfaces at 0 are called rigidly equivalent if there exists a map of the form

\[
(z_1, \ldots, z_n) \mapsto \left( f_1(z_1, \ldots, z_{n-1}), \ldots, f_{n-1}(z_1, \ldots, z_{n-1}), \right.
\]

\[
ax_n + f_n(z_1, \ldots, z_{n-1}), \quad a \in \mathbb{R}^*,
\]

nondegenerate at the origin, that transforms one hypersurface germ into the other, where \( f_j \) is holomorphic near 0 with \( f_j(0) = 0 \) for \( j = 1, \ldots, n \).

Our ultimate goal is

(*) to classify, up to rigid equivalence, the germs of rigid hypersurfaces that are CR-flat, that is, have identically vanishing CR-curvature as explained below.

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The problem of describing CR-flat structures that possess certain symmetries (e.g., as in (1.2)) is a natural one but so far has only been addressed for Levi nondegenerate CR-hypersurfaces. In particular, under the assumption of CR-flatness, homogeneous strongly pseudoconvex CR-hypersurfaces have been studied (see [BS]), and tube hypersurfaces in complex space have been extensively investigated and even fully classified for certain signatures of the (nondegenerate) Levi form (see [II] for a detailed exposition). Compared to the tube case, the case of rigid hypersurfaces is the next situation up in terms of complexity. Although not mentioned explicitly, problem (*) was first looked at in article [S] for real-analytic strongly pseudoconvex hypersurfaces in $\mathbb{C}^2$. Even in this simplified setup, determining all rigid CR-flat hypersurfaces turned out to be highly nontrivial, with only a number of examples found in [S]. A complete list (not entirely explicit but presented in an acceptable form) of the germs of real-analytic strongly pseudoconvex rigid CR-flat hypersurfaces in $\mathbb{C}^2$ was only recently obtained in [ES].

Our task is much more ambitious as we attempt to relax the Levi nondegeneracy assumption and do not assume real-analyticity. Specifically, in article [I3] we initiated an investigation of problem (*) for a class of Levi degenerate 2-non-degenerate rigid hypersurfaces in $\mathbb{C}^3$ and obtained a partial description up to rigid equivalence. As part of our considerations, we analyzed CR-curvature for this class.

We will now briefly discuss the concept of CR-curvature. Generally, CR-curvature is defined in a situation when the CR-structures in question are reducible to absolute parallelisms with values in a Lie algebra $\mathfrak{g}$. Indeed, let $\mathfrak{c}$ be a class of CR-manifolds. Then the CR-structures in $\mathfrak{c}$ are said to reduce to $\mathfrak{g}$-valued absolute parallelisms if to every $M \in \mathfrak{c}$ one can assign a fiber bundle $\mathcal{P}_M \to M$ and an absolute parallelism $\omega_M$ on $\mathcal{P}_M$ such that for every $p \in \mathcal{P}_M$ the parallelism establishes an isomorphism between $T_p(\mathcal{P}_M)$ and $\mathfrak{g}$, and for all $M_1, M_2 \in \mathfrak{c}$ the following holds:

(i) every CR-isomorphism $f : M_1 \to M_2$ can be lifted to a diffeomorphism $F : \mathcal{P}_{M_1} \to \mathcal{P}_{M_2}$ satisfying

$$F^* \omega_{M_2} = \omega_{M_1},$$

and

(ii) any diffeomorphism $F : \mathcal{P}_{M_1} \to \mathcal{P}_{M_2}$ satisfying (1.4) is a bundle isomorphism that is a lift of a CR-isomorphism $f : M_1 \to M_2$.

In this situation one considers the $\mathfrak{g}$-valued CR-curvature form

$$\Omega_M := d\omega_M - \frac{1}{2} [\omega_M, \omega_M],$$

and the CR-flatness of $M$ is the condition of the identical vanishing of $\Omega_M$ on the bundle $\mathcal{P}_M$.

Reduction of CR-structures to absolute parallelisms was initiated by É. Cartan who considered the case of 3-dimensional Levi nondegenerate CR-hypersurfaces (see [C]). Since then there have been numerous developments under the Levi nondegeneracy assumption (see [I2, Section 1] for references). On the other hand, the first result for a reasonably large class of Levi degenerate manifolds is fairly recent. Namely, in article [IZ] we looked at the class $\mathfrak{c}_{2,1}$ of connected 5-dimensional CR-hypersurfaces that are 2-non-degenerate and uniformly Levi degenerate of rank 1 and showed that the CR-structures in $\mathfrak{c}_{2,1}$ reduce to $so(3,2)$-valued parallelisms. Alternative constructions for $\mathfrak{c}_{2,1}$ were proposed in [MS], [MP], [Poc], [FM] (see also [Por], [PZ] for reduction in higher-dimensional cases).

Everywhere in this paper we understand CR-curvature and CR-flatness for the class $\mathfrak{c}_{2,1}$ in the sense of article [IZ]. One of the results of [IZ] states that a manifold $M \in \mathfrak{c}_{2,1}$ is CR-flat if and only if in a neighborhood of its every point $M$ is CR-equivalent to an open subset of the tube hypersurface over the future light cone in
\[ M_0 : \quad (\text{Re } z_1)^2 + (\text{Re } z_2)^2 - (\text{Re } z_3)^2 = 0, \quad \text{Re } z_3 > 0. \]

In fact, one can show that the germ of \( M_0 \) at its every point is CR-equivalent to the germ at the origin of the following rigid hypersurface:

\[ \tilde{M}_0 : \quad \text{Re } z_3 = \frac{|z_1|^2}{1 - |z_2|^2} + \frac{z_2}{2(1 - |z_2|^2)} z_1^2 + \frac{z_2}{2(1 - |z_2|^2)} z_3^2 \]

(see [GM] and [FK, Proposition 4.16]). We thus deduce that for the class \( C_{2,1} \) problem (\( \ast \)) reduces to the determination, up to rigid equivalence, of all germs of rigid hypersurfaces in \( C_{2,1} \) that are CR-equivalent to the germ of \( \tilde{M}_0 \).

For real hypersurfaces in \( \mathbb{C}^3 \) of the class \( C_{2,1} \), the condition of local CR-equivalence to \( \tilde{M}_0 \) near the origin (i.e., the CR-flatness condition) can be expressed as the simultaneous vanishing of two CR-invariants, called \( J \) and \( W \), introduced by S. Pocchiola in [Poc] (cf. [FM]). These invariants are given explicitly in terms of a graphing function of the hypersurface. The general formulas in [Poc] for \( J \) and \( W \) are rather lengthy and hard to work with. Luckily, they substantially simplify in the rigid case, and our arguments are based on those shorter formulas.

In article [I3] we initiated the study of a class of solutions of the system

\[
\begin{aligned}
J &= 0, \\
W &= 0
\end{aligned}
\tag{1.5}
\]

for rigid hypersurfaces in \( \mathbb{C}^3 \) lying in \( C_{2,1} \). The class is given by conditions (2.6) and (2.11) stated in the next section. Quite unexpectedly, the study of even this special class of solutions turns out to be rather nontrivial and leads to interesting mathematics. In [I3], we only determined the solutions in the class satisfying certain additional assumptions as specified in Remark 2.4. In the present paper, we drop those assumptions and give a complete description, up to rigid equivalence, of all germs of rigid hypersurfaces in \( \mathbb{C}^3 \) lying in \( C_{2,1} \) whose graphing functions satisfy (2.6) and (2.11). This is our main result; it is stated in Theorem 2.5. One consequence of Theorem 2.5 is the analyticity of solutions (see Corollary 2.6). It would be curious to see whether analyticity holds true without any further constraints as discussed in Remark 2.7.

It is interesting to note that in the course of our analysis various classical differential equations kept appearing. Indeed, first of all, the complex homogeneous Monge-Ampère equation (2.1) arose simply because the Levi form of a hypersurface in \( C_{2,1} \) is everywhere degenerate. Secondly, the condition \( J = 0 \) mysteriously led to an analogue of the Monge equation (see (2.8)). Thirdly, the equation describing conformal metrics with constant negative curvature unexpectedly came up (see (2.23)). Fourthly, we came across a nonlinear \( \partial \bar{\partial} \)-equation, which is a special case of the much-studied equation \( \partial u/\partial \zbar = Au + B\zbar + f \) (see (2.27)). The description given in Theorem 2.5 is expressed in terms of solutions of (2.23), (2.27), as well as those of more elementary equations. We thus see that the geometry of CR-flat manifolds in \( C_{2,1} \) is rather rich, has surprising connections with classical differential-geometric structures, and so deserves further investigation.

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2. Results

From now on, we will look at the germs of rigid hypersurfaces at the origin in \( \mathbb{C}^3 \). They are given by equations of the form

\[ \text{Re } z_3 = F(z_1, \zbar_1, z_2, \zbar_2), \]
where $F$ is a smooth real-valued function defined near the origin with $F(0) = 0$.

Consider the germ $M$ of a rigid hypersurface that is uniformly Levi degenerate of rank 1. Then the complex Hessian matrix of $F$ has rank 1 at every point, hence $F$ is a solution of the complex homogeneous Monge-Ampère equation

\( F_{11} F_{22} - |F_{12}|^2 = 0 \)  

(2.1)

(here and below subscripts 1, 2, ̅ indicate partial derivatives with respect to $z_1$, $z_1$, $z_2$, $z_2$, respectively). Clearly, we have either $F_{11}(0) \neq 0$ or $F_{22}(0) \neq 0$, so, up to rigid equivalence, we may additionally assume

\( F_{11} > 0 \) everywhere.

Set

\( S := \left( \begin{array}{c} F_{12} \\ F_{11} \end{array} \right) \).  

(2.2)

The condition of 2-nondegeneracy of $M$ is then expressed as the nonvanishing of $S$ (see [MP, Poc]). Thus, supposing that $M$ is 2-nondegenerate, we have

\( S \neq 0 \) everywhere.

Further, for real hypersurfaces in $\mathbb{C}^3$ of the class $\mathcal{C}_{2,1}$ the CR-flatness condition is equivalent to the simultaneous vanishing of two CR-invariants, called $J$ and $W$, introduced in [Poc]. These invariants are given by explicit expressions in terms of a graphing function of the hypersurface. The formulas in [Poc] for $J$ and $W$ are quite lengthy and hard to handle in general. However, these complicated formulas significantly simplify in the rigid case as the following proposition shows:

**Proposition 2.1.** For the germ $M$ of a rigid hypersurface in the class $\mathcal{C}_{2,1}$ with $F$ satisfying (2.2), we have

\[
J = \frac{5(S_1)^2 F_{111}}{18S^2 F_{11}} + \frac{1}{3} \frac{F_{111}}{F_{11}} \left( \frac{F_{111}}{F_{11}} \right)_{11} + \frac{S_1}{9S} \frac{(F_{111})^2}{(F_{11})^2} + \frac{20(S_1)^3}{27S^3} - \frac{5S_1 S_{11}}{6S^2} + \frac{S_1}{6S} \left( \frac{F_{111}}{F_{11}} \right)_{11} - \frac{S_{11}}{6S^2} \frac{F_{111}}{F_{11}} - \frac{2}{27} \left( \frac{F_{111}}{F_{11}} \right)^3 - \frac{1}{6} \left( \frac{F_{111}}{F_{11}} \right)_{11} + \frac{S_{11}}{S_1},
\]

(2.3)

\[
W = \frac{2S_1}{3S} + \frac{S_1}{3S} \left( \frac{F_{21} S_1 - S_2}{F_{11}} \right) - \frac{1}{3S^2} \left( \frac{F_{21} S_{11} - S_{21}}{F_{11}} \right).
\]

The proof of Proposition 2.1 goes by straightforward manipulation of formulas in [Poc], and we omit it.

Finding the germs of the graphs of all solutions of system (1.5) up to rigid equivalence is apparently very hard. In article [I3] we made some initial steps towards this goal. Specifically, we discussed solutions having the property

\( S_1 = 0, \quad S_1 = 0. \)

(2.6)

Our motivation for introducing conditions (2.6) comes from the tube case, where these conditions are equivalent to the equation $W = 0$ (see [I3, Lemma 3.1]). At this stage, we do not know whether the same holds true in the rigid case as well, but it is clear from (2.5) that (2.6) implies $W = 0$.

Furthermore, as can be easily seen from (2.5), conditions (2.6) lead to the following simplified expression for $J$:

\[
J = \frac{1}{3} \frac{F_{111}}{F_{11}} \left( \frac{F_{111}}{F_{11}} \right)_{11} - \frac{2}{27} \left( \frac{F_{111}}{F_{11}} \right)^3 - \frac{1}{6} \left( \frac{F_{111}}{F_{11}} \right)_{11}.
\]

(2.7)

Formula (2.7) yields that under assumption (2.6) the equation $J = 0$ is equivalent to

\[
9F_{11111}(F_{111})^2 - 45F_{1111} F_{111} F_{11} + 40(F_{111})^3 = 0,
\]

(2.8)
which looks remarkably similar to the well-known Monge equation. Recall that the classical single-variable Monge equation is

$$9f^{(V)}(f')^2 - 45f^{(IV)}f''f'' + 40(f'')^3 = 0$$

and that it encodes all planar conics (see, e.g., [Lan, pp. 51–52], [Las]). In analogy with (2.9), we call (2.8) the complex Monge equation with respect to $z_1$.

Thus, we arrive at a natural class of CR-flat rigid hypersurfaces in $\mathbb{C}_{2,1}$ described by the system of partial differential equations

$$\begin{align*}
\text{the complex Monge equation w.r.t. } z_1 \text{ (2.8),} \\
\text{the complex Monge-Ampère equation (2.1),} \\
\text{equations (2.6),}
\end{align*}$$

where conditions (2.2) and (2.4) are satisfied.

We will now recall [I3, Proposition 5.4], which states that (2.8) can be integrated three times with respect to $z_1$. Note that an analogous fact holds for the classical Monge equation (see, e.g., [12]).

**Proposition 2.2.** A function $F$ satisfying (2.2) is a solution of (2.8) if and only if

$$\frac{1}{(F_1)^2} = f(z_2, \bar{z}_2)|z_1|^4 + g(z_2, \bar{z}_2)z_1^2\bar{z}_1 + g(\bar{z}_2, z_2)\bar{z}_1^2z_1^2 + h(z_2, \bar{z}_2)|z_1|^{2p} + q(z_2, \bar{z}_2)z_1^2 + q(\bar{z}_2, z_2)\bar{z}_1^2z_1^2 + x(z_2, \bar{z}_2),$$

where $f, g, h, p, q, x$ are smooth functions, with $f, h, x$ being real-valued.

In [I3] we began investigating the simplest possible situation arising from Proposition 2.2 by assuming that in formula (2.10) one has

$$f = g = h = p = q = 0.$$  

This means that $F_1 = r(z_2, \bar{z}_2)$ or, equivalently,

$$F = r(z_2, \bar{z}_2)|z_1|^2 + s(z_1, z_2, \bar{z}_2) + s(z_1, z_2, \bar{z}_2)$$

for some functions $r$ and $s$ smooth near the origin, with $r > 0$ everywhere and $\text{Re } s(0) = 0$. As explained in [I3] and can be seen from what follows, exploring even this very special case is far from being trivial and leads to some interesting analysis. In [I3] we introduced additional assumptions (shown in Remark 2.4 below) and obtained a partial classification of the corresponding rigid hypersurfaces germs. In the present paper, we focus on functions of the form (2.12) without any further constraints. Our goal is to determine $r$ and $s$ as explicitly as possible.

We will utilize the complex Monge-Ampère equation (2.1). Indeed, plugging (2.12) in (2.1) leads to

$$r(r_2|z_1|^2 + s_2r_2 + s_2r_2) - |r_2|^2|z_1|^2 - |s_1|^2 - r_2s_1z_1 - r_2s_1z_1 = 0.$$  

We will now differentiate (2.13). First, applying the operator $\partial^2/\partial z_1 \partial \bar{z}_1$ to (2.13) yields

$$rr_{22} - |r_2|^2 - |s_{112}|^2 = 0.$$  

Further, differentiating (2.14) with respect to $z_1$, we obtain

$$s_{1112}s_{112} = 0,$$

which implies that $s_{1112} = 0$, and therefore we have

$$s_2 = t_0(z_2, \bar{z}_2)z_1^2 + u_0(z_2, \bar{z}_2)z_1 + v_0(z_2, \bar{z}_2)$$

for some smooth functions $t_0, u_0, v_0$ near the origin. Solving the $\bar{z}$-equations

$$t_2(z_2, \bar{z}_2) = t_0(z_2, \bar{z}_2), \quad u_2(z_2, \bar{z}_2) = u_0(z_2, \bar{z}_2), \quad v_2(z_2, \bar{z}_2) = v_0(z_2, \bar{z}_2)$$
on a neighborhood of the origin, we obtain
\[ s = t(z_2, \bar{z}_2)z_1^2 + u(z_2, \bar{z}_2)z_1 + v(z_2, \bar{z}_2), \]
with \( t, u, v, w \) being smooth functions defined near the origin and \( \text{Re}(v(0) + w(0)) = 0 \). Since \( w \) is in fact holomorphic and we study rigid hypersurfaces up to rigid equivalence, by absorbing \( w \) into \( z_3 \) we may assume that \( w = 0 \), so we have
\[ s = t(z_2, \bar{z}_2)z_1^2 + u(z_2, \bar{z}_2)z_1 + v(z_2, \bar{z}_2), \]
with \( \text{Re} v(0) = 0 \). Note that by condition (2.4) we have \( t_2 \neq 0 \) everywhere.

Next, applying the operator \( \partial^2/\partial z_2^2 \) to (2.13), we obtain
\[ rs_{1122} - s_{1112}^2 - r_{2112} - 2r_{2112} = 0, \]
which, upon taking into account expression (2.15), leads to
\[ rt_{22} - 2r_t = 0, \]
and hence to
\[ \frac{t_{22}}{t_2} = 2 \frac{r_2}{r}. \]
Passing to logarithms and integrating (2.17) we arrive at the equation
\[ t_2 = w(z_2)r^2, \]
where \( w(z_2) \) is everywhere nonvanishing.

**Lemma 2.3.** Up to rigid equivalence, one can assume that in (2.18) we have \( w = 1/4 \).

**Proof.** Let us perform the transformation
\[ (z_1, z_2, z_3) \mapsto \left( \frac{z_1}{2\sqrt{w(z_2)}}, z_2, z_3 \right). \]
Clearly, (2.19) is a map of the form (1.3) for \( n = 3 \) and preserves condition (2.2).

Under this transformation the germ of the graph of \( F \) transforms into the germ of the graph of the function
\[ \tilde{F} = \tilde{r}(z_2, \bar{z}_2)|z_1|^2 + \tilde{s}(z_1, z_2, \bar{z}_2) + \tilde{s}(z_1, z_2, \bar{z}_2), \]
with
\[ \tilde{s} = \tilde{t}(z_2, \bar{z}_2)z_1^2 + \tilde{u}(z_2, \bar{z}_2)z_1 + v(z_2, \bar{z}_2), \]
where
\[ \tilde{t}(z_2, \bar{z}_2) := 4|w(z_2)|r(z_2, \bar{z}_2), \]
\[ \tilde{u}(z_2, \bar{z}_2) := 4w(z_2)\tilde{f}(z_2, \bar{z}_2), \]
\[ \tilde{v}(z_2, \bar{z}_2) := 2\sqrt{w(z_2)}u(z_2, \bar{z}_2). \]
Now, from (2.18) and (2.20) we see
\[ \tilde{t}_2 = \frac{1}{4}r^2 \]
as required. \( \square \)

By Lemma 2.3, we may assume
\[ t_2 = \frac{1}{4}r^2. \]
Plugging (2.15) and (2.21) in (2.14), we obtain an equation for \( r \):
\[ rr_{22} - |r_2|^2 - \frac{1}{4}r^4 = 0. \]
It then follows that the function $R := \ln r$ satisfies the equation

$$\Delta R = e^{2R}. \tag{2.23}$$

This shows that $r$ is a conformal metric of constant curvature -1 on a disk $U$ around the origin (see, e.g., [KR]). Since $U$ is simply connected, by Liouville’s theorem we have

$$r(z_2, \bar{z}_2) = \frac{2|\rho' (z_2)|}{1 - |\rho(z_2)|^2}, \tag{2.24}$$

where $\rho$ is a holomorphic function on $U$ with nowhere vanishing derivative and values in the unit disk. In particular, $r$ is real-analytic.

**Remark 2.4.** Formula (2.24) is the most explicit expression for $r$ that one can hope to obtain without making further assumptions. We note that for $r$ depending only on either $\Re z_2$ or $|z_2|^2$, it is possible to derive more precise formulas as shown in [I3, Theorems 5.5 and 5.9]. The dependence of $r$ on either $\Re z_2$ or $|z_2|^2$, as well as the conditions $u = 0$, $v = 0$, were the additional assumptions imposed in [I3], and it is under these assumptions the partial classification of the corresponding rigid hypersurfaces was produced.

Now, for $r$ found in (2.24), we may solve $\tilde{\partial}$-equation (2.21) as follows:

$$t = \int \frac{r^2}{4} d\bar{z}_2 + w(z_2),$$

where the integral in the right-hand side stands for the term-by-term integration, with respect to $\bar{z}_2$, of the power series in $z_2$, $\bar{z}_2$ representing $r^2/4$ near the origin. As $w$ is holomorphic, by absorbing $\bar{z}_2^2 w(z_2)$ into $z_3$ we may assume that $w = 0$, so we have

$$t = \int \frac{r^2}{4} d\bar{z}_2. \tag{2.25}$$

In particular, $t$ is real-analytic.

Now that we have found $r$ and $t$, in order to determine the function $F$ it remains to compute $u$ and $\Re v$ in formula (2.15). Plugging (2.15) into (2.13), collecting the terms linear in $z_1$, and utilizing (2.21), we obtain

$$ru_{z_2} - 2t\bar{z}_2 - r_2 u_2 = 0, \tag{2.26}$$

which yields

$$\left( \frac{u_2}{r} \right)_2 = \frac{1}{2} \bar{u}_2.$$  

Integrating we get

$$u_2 = r \left( \frac{1}{2} u + w(z_2) \right).$$

By adding to $z_3$ the term $4z_1 w(z_2)$ we may assume that $w = 0$, thus $u$ satisfies

$$u_2 = \frac{r}{2} \bar{u}. \tag{2.27}$$

Every solution of (2.27) is real-analytic (see [V, pp. 143–144, Section 3.4] and references therein, as well as [M, Section 6.6]).

Note that by the Cauchy-Pompeiu formula, on any disk $\tilde{U}$ around the origin relatively compact in $U$, the function $u$ solves the integral equation

$$u(z_2, \bar{z}_2) = -\frac{1}{2\pi i} \int_{\tilde{U}} \frac{r(\zeta, \bar{\zeta})u(\zeta, \bar{\zeta})}{2(\zeta - z_2)} d\zeta \wedge d\bar{\zeta} + w(z_2), \tag{2.28}$$
where the holomorphic function $w$ is given by
\[ w(z_2) = \frac{1}{2\pi i} \int_{\gamma(z_2)} \frac{u(\zeta, \overline{\zeta})}{\zeta - z_2} d\zeta. \]

Thus, $u$ is a solution of an integral equation of the form (2.28) for a suitable function $w$. Regarding the existence, regularity and uniqueness of solutions of such integral equations we refer the reader to [V, Chapter III] and, in several variables, to [NW, pp. 436–438]. Other representations of $u$ can be found in the references provided in [V, pp. 143–144, Section 3.4].

Now, we plug (2.15) into (2.13) and collect the terms independent of $z_1$. Taking into account equation (2.27), we obtain
\[
(2.29) \quad \Re v = \frac{r}{8}|u|^2.
\]

As $r$ and $u$ are real-analytic, we see
\[
(\Re v)_{22} = \frac{r}{8}|u|^2.
\]

In particular, $\Re v$ is real-analytic.

We arrive at the following result:

**Theorem 2.5.** The germ of a rigid hypersurface in $\mathbb{C}^3$ of the class $\mathcal{C}_{2,1}$ with graphing function satisfying conditions (2.6) and (2.11) is rigidly equivalent to the germ of a rigid hypersurface with graphing function of the form (2.12), where $s$ is given by (2.15) and the functions $r, t, u, \Re v$ are determined from (2.24), (2.25), (2.27), (2.30), respectively.

Conversely, the germ of a rigid hypersurface with graphing function of the form (2.12), where $s$ is given by (2.15) and the functions $r, t, u, \Re v$ are determined from (2.24), (2.25), (2.27), (2.30), respectively, is of the class $\mathcal{C}_{2,1}$ and the graphing function satisfies conditions (2.6), (2.11).

**Proof.** We only need to prove the converse implication. First, an easy calculation shows that every function that arises in the right-hand side of (2.24) is a solution of (2.22). Next, by plugging (2.15) into (2.13) we see that conditions (2.22), (2.25), (2.27), (2.30) guarantee that equation (2.13) is satisfied. The latter equation is the complex Monge-Ampère equation in the case at hand. Furthermore, $F_{11} = r$, which is positive by (2.24), and therefore (2.2) holds. We have thus shown that the graph of $F$ has Levi form of rank 1 everywhere.

Next, from (2.3), (2.15) we have
\[
(2.31) \quad S = \frac{2t_2}{r},
\]
and (2.25) yields
\[
t_2 = \frac{r^2}{4} \neq 0 \text{ everywhere.}
\]
Therefore, by (2.4) the graph of $F$ is 2-nondegenerate, hence lies in the class $\mathcal{E}_{2,1}$. It also follows from (2.31) that conditions (2.6) are satisfied. Finally, (2.11) trivially holds, which concludes the proof.

We have an immediate consequence:

**Corollary 2.6.** The germ of a rigid hypersurface in $\mathbb{C}^3$ of the class $\mathcal{E}_{2,1}$ satisfying conditions (2.6) and (2.11) is real-analytic.

**Remark 2.7.** It would be interesting to see whether the real-analyticity result of Corollary 2.6 holds true for all CR-flat rigid hypersurfaces in $\mathbb{C}^3$ of the class $\mathcal{E}_{2,1}$, regardless of assumptions (2.6), (2.11). Note that in general a CR-flat manifold does not have to be real-analytic. For example, it is easy to construct, for any $n \geq 2$, an example of a hypersurface in $\mathbb{C}^n$ that is CR-equivalent to an open subset of $S^{2n-1} \subset \mathbb{C}^n$ and $C^2$-smooth but not real-analytic (see, e.g., [I1, Remark 3.3]). In the Levi nondegenerate case the tubularity condition forces real-analyticity (see [I1, Proposition 3.1]) but it is unknown whether the rigidity condition is powerful enough for that. We stress that the work [S, ES] for rigid Levi nondegenerate hypersurfaces in $\mathbb{C}^2$ assumes real-analyticity as the techniques of the proofs rely on power series representations.

**References**

[BRT] Baouendi, M. S., Rothschild, L. P. and Treves, F., CR structures with group action and extendability of CR functions, *Invent. Math.* 82 (1985), 359–396.

[BS] Burns, D. and Shnider, S., Spherical hypersurfaces in complex manifolds, *Invent. Math.* 33 (1976), 223–246.

[C] Cartan, É., Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes: I, *Ann. Math. Pura Appl.* 11 (1933), 17–90; II, *Ann. Scuola Norm. Sup. Pisa* 1 (1932), 333–354.

[ES] Ezhov, V. and Schmalz, G., Explicit description of spherical rigid hypersurfaces in $\mathbb{C}^2$, *Complex Anal. Syner.* 1:2 (2015), DOI: 10.1186/2197-120X-1-2.

[FK] Fels, G. and Kaup, W., CR-manifolds of dimension 5: A Lie algebra approach, *J. reine angew. Math.* 604 (2007), 47–71.

[FM] Foo, W. G. and Merker, J., Differential $\mathcal{E}$-structures for equivalences of 2-nondegenerate Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$, preprint, available from https://arxiv.org/abs/1901.02028.

[GM] Gaussier, H. and Merker, J., A new differential of a uniformly Levi degenerate hypersurface in $\mathbb{C}^3$, *Ark. Mat.* 41 (2003), 85–94; erratum *Ark. Mat.* 45 (2007), 269–271.

[I1] Isaev, A. V., Spherical Tube Hypersurfaces, *Lecture Notes in Mathematics* 2020, Springer, New York, 2011.

[I2] Isaev, A. V., Affine rigidity of Levi degenerate tube hypersurfaces, *J. Differential Geom.* 104 (2016), 111–141.

[I3] Isaev, A. V., Zero CR-curvature equations for Levi degenerate hypersurfaces via Pocchiola's invariants, preprint, available from https://arxiv.org/abs/1809.03029.

[IZ] Isaev, A. and Zaitsev, D., Reduction of five-dimensional uniformly Levi degenerate CR structures to absolute parallelisms, *J. Geom. Anal.* 23 (2013), 1571–1605.

[KR] Kraus, D. and Roth, O., Conformal metrics, in: *Topics in Modern Function Theory*, Ramanujan Math. Soc. Lect. Notes Ser. 19, Ramanujan Math. Soc., Mysore, 2013, pp. 41–83.

[Lan] Landsberg, J. M., Differential-geometric characterizations of complete intersections, *J. Differential Geom.* 44 (1996), 32–73.

[Las] Lasley, J. W., Jr., On Monge's differential equation, *Amer. Math. Monthly* 43 (1936), 284–286.

[MS] Medori, C. and Spiro, A., The equivalence problem for five-dimensional Levi degenerate CR manifolds, *Int. Math. Res. Not. (IMRN)* (2014), 5602–5647.

[MP] Merker, J. and Pocchiola, S., Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1, to appear in *J. Geom. Analysis*, published online, DOI: 10.1007/s12220-018-9988-3.

[M] Morrey, C. B., *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, 2008.
[NW] Nijenhuis, A. and Woolf, W. B., Some integration problems in almost-complex and complex manifolds, *Ann. Math.* **77** (1963), 424–489.

[Poc] Pocchiola, S., Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank 1, preprint, available from https://arxiv.org/abs/1312.6400.

[Por] Porter, C., The local equivalence problem for 7-dimensional 2-nondegenerate CR manifolds whose cubic form is of conformal unitary type, preprint, available from http://arxiv.org/abs/1511.04019.

[PZ] Porter, C. and Zelenko, I., Absolute parallelism for 2-nondegenerate CR structures via bigraded Tanaka prolongation, preprint, available from https://arxiv.org/abs/1704.03999.

[S] Stanton, N., A normal form for rigid hypersurfaces in $\mathbb{C}^2$, *Amer. J. Math.* **113** (1991), 877–910.

[V] Vekua, I. N., *Generalized Analytic Functions*, Pergamon Press, New York, 1962.

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