NORMAL SPLIT SUBMANIFOLDS OF RATIONAL HOMOGENEOUS SPACES

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Abstract. Let $M \subset X$ be a submanifold of a rational homogeneous space $X$ such that the normal sequence splits. We prove that $M$ is also rational homogeneous.

1. Introduction

Let $M \subset \mathbb{P}^n$ be a projective manifold such that the normal sequence

$$0 \to T_M \to T_{\mathbb{P}^n} \otimes \mathcal{O}_M \to N_{M/\mathbb{P}^n} \to 0$$

splits. Since the tangent bundle of the projective space is ample, the existence of a splitting map $T_{\mathbb{P}^n} \otimes \mathcal{O}_M \to T_M$ yields that $T_M$ is also ample, hence $M \simeq \mathbb{P}^{\dim M}$ by Mori’s theorem [Mor79]. In fact a theorem of van de Ven [vdV59] states that $M \subset \mathbb{P}^n$ is a linear subspace. It is natural to expect that similar statements exist for submanifolds of arbitrary homogeneous spaces. The case of tori has been solved by Jahnke [Jah05, Thm.3.2], so we consider the following

1.1. Problem. Let $X \simeq G/P$ be a rational homogeneous space. Let $i : M \hookrightarrow X$ be a submanifold such that the normal sequence

$$(1) \qquad 0 \to T_M \xrightarrow{\tau_i} T_X \otimes \mathcal{O}_M \to N_{M/X} \to 0$$

splits, i.e. there exists a morphism $s_M : T_X \otimes \mathcal{O}_M \to T_M$ such that $s_M \circ \tau_i = \text{id}_{T_M}$.

- Describe $M$ up to isomorphism.
- Describe the embedding $i : M \hookrightarrow X$.

Both problems have been solved for special classes of rational homogeneous spaces like hyperquadrics, Grassmannians [Jah05, Thm.4.7, Prop.5.2], irreducible compact Hermitian symmetric spaces [Din22, Thm.2.3] or certain blow-ups of the projective space [Jah05, Li22]. In this paper we focus on the first problem and prove a structure result that holds without further assumptions on $X$, thereby improving [Jah05, Prop.2.2] and [Din22, Thm.2.1]:

1.2. Theorem. Let $X \simeq G/P$ be a rational homogeneous space, and let $M \subset X$ be a submanifold that is normal split (cf. Definition 2.1). Then $M$ is rational homogeneous.

Since the vector bundle $T_X \otimes \mathcal{O}_M$ is globally generated and $M \subset X$ is normal split, the tangent bundle of $T_M$ is globally generated. Thus $M$ is homogeneous and by [BR62, Satz 1] we have

$$M \simeq A \times Y$$

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where $A$ is an abelian variety and $Y$ is rational homogeneous. Thus in order to prove Theorem 1.2 we can assume without loss of generality that $M \simeq A$ is abelian (cf. Lemma 2.2). Since the tangent bundle of the rational homogeneous space $T_X$ is big [Ric74] and the tangent bundle of an abelian variety is not, one is tempted to argue as in the case of the projective space. Note however that the quotient of a big vector bundle is in general not big, so a priori neither $T_X \otimes O_A$ nor its quotient $T_A$ are big. Therefore our argument proceeds in a different fashion: denote by

$$\pi : \mathbb{P}(T_X) \to X$$

the Grothendieck projectivisation of the tangent bundle. The global sections of $T_X$ define a morphism

$$(2) \quad \varphi_X : \mathbb{P}(T_X) \to \mathbb{P}(H^0(X, T_X))$$

that is generically finite onto its image. Since $A$ is abelian, the splitting map allows to define a lifting $A \to \mathbb{P}(T_X)$ such that the image $\tilde{A}$ is contracted by $\varphi_X$. If the fibres of $\varphi_X$ are smooth and rationally connected the existence of $\tilde{A}$ is often enough to obtain a contradiction, cp. [Jah05]. For an arbitrary rational homogeneous space $X$ the fibres of $\varphi_X$ are not necessarily smooth (cf. also Remark 3.1), but we show in Lemma 3.2 that manifolds contracted by $\varphi_X$ are always integral submanifolds with respect to the contact structure on $\mathbb{P}(T_X)$. Theorem 1.2 then follows without too much difficulty.

Our result can be extended to normal split submanifolds of projective manifolds such that the tangent bundle $T_X$ is nef and big, cf. Remark 3.6. By a well-known conjecture of Campana and Peternell [CP91], these are exactly the rational homogeneous spaces.

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### 2. Notation and basic facts

We work over the complex numbers, for general definitions we refer to [Har77]. Varieties will always be supposed to be irreducible and reduced. We use the terminology of [Deb01] [KM98] for birational geometry and notions from the minimal model program. We follow [Laz04] for algebraic notions of positivity.

Given a morphism $\varphi : A \to B$ between complex manifolds we denote by

$$\tau_\varphi : T_A \to \varphi^*T_B$$

the tangent map. For a submanifold $M \hookrightarrow X$ of a complex manifold $X$, we will simply denote the tangent map by

$$\tau_M : T_M \to T_X \otimes O_M.$$  

#### 2.1. Definition. Let $X$ be a complex manifold, and let $i : M \to X$ be a submanifold. We say that the $M$ is normal split in $X$ if the inclusion

$$0 \to T_M \xrightarrow{\tau_M} T_X \otimes O_M$$

admits a splitting morphism $s_M : T_X \otimes O_M \to T_M$ such that $s_M \circ \tau_i = \text{id}_{T_M}$. 

Remark. While $\tau_i$ is determined by the embedding, the splitting morphism $s_M$ is not unique. Our statements will not depend on the choice of $s_M$.

2.2. Lemma. Let $X \simeq G/P$ be a rational homogeneous space, and let $M \subset X$ be a submanifold that is normal split. If $M$ is not rational homogeneous, there exists an abelian variety of positive dimension $A \subset X$ that is normal split.

Proof. Since $M$ is homogeneous, we have by [BR62, Satz 1] that $M \simeq A \times Y$ with $A$ an abelian variety and $Y$ rational homogeneous. Since $M$ is not rational homogeneous, we have $\dim A > 0$. Yet $M$ is a product, so the abelian variety $A$ is normal split in $M$. Thus by [Jah05, 1.2.2] the abelian variety $A$ is normal split in $X$. □

3. Geometry of the projectivised tangent bundle

Let $X$ be a complex manifold, denote by $\pi : P(T_X) \to X$ the Grothendieck projectivisation of its tangent bundle. Let

$$0 \to \mathcal{O}_{P(T_X)} \to \pi^* \Omega_X \otimes \mathcal{O}_{P(T_X)}(1) \to T_{P(T_X)}/X \to 0$$

be the relative Euler sequence. Dualising and tensoring by $\mathcal{O}_{P(T_X)}(1)$ we obtain the canonical quotient map

$$q : \pi^* T_X \to \mathcal{O}_{P(T_X)}(1).$$

The composition of the tangent map $\tau_{\pi} : T_{P(T_X)} \to \pi^* T_X$ with the canonical quotient map $q$ defines an exact sequence

$$0 \to F \to T_{P(T_X)} \xrightarrow{q \circ \tau_{\pi}} \mathcal{O}_{P(T_X)}(1) \to 0.$$

The corank one distribution $F \subset T_{P(T_X)}$ is a contact distribution [Bla10, Sect.13.2], i.e. the map

$$2 \bigwedge \mathcal{F} \to T_{P(T_X)}/\mathcal{F} \simeq \mathcal{O}_{P(T_X)}(1)$$

induced by the Lie bracket on $P(T_X)$ is surjective.

Assume now that $X \simeq G/P$ is rational homogeneous, so the tautological bundle $\mathcal{O}_{P(T_X)}(1)$ is nef and big [Ric74].

Since $T_X$ is globally generated, the tautological line bundle $\mathcal{O}_{P(T_X)}(1)$ is globally generated and defines a morphism

$$\varphi|_{\mathcal{O}_{P(T_X)}(1)} : P(T_X) \to P(H^0(X,T_X))$$

such that $\mathcal{O}_{P(T_X)}(1) \simeq \varphi^*|_{\mathcal{O}_{P(T_X)}(1)} \mathcal{O}_{P(H^0(X,T_X))}(1)$. We denote by

$$\varphi_X : P(T_X) \to Y$$

the Stein factorisation of $\varphi|_{\mathcal{O}_{P(T_X)}(1)}$ and by $L$ the pull-back of $\mathcal{O}_{P(H^0(X,T_X))}(1)$ to $Y$. By construction $L$ is ample and $\mathcal{O}_{P(T_X)}(1) \simeq \varphi_X^* L$.

3.1. Remark. Since $\mathcal{O}_{P(T_X)}(1)$ is nef and big, the morphism $\varphi_X$ is birational. The canonical bundle of $P(T_X)$ is isomorphic to $\mathcal{O}_{P(T_X)}(-n)$, so it is trivial on the fibres of $\varphi_X$. Thus we have

$$K_Y^* \simeq L^\otimes n$$
and $Y$ is a Fano variety with canonical Gorenstein singularities. In fact it is not difficult to see that $Y$ has an induced singular contact structure \cite{CF02} given by a global section of $H^0(Y, \Omega_Y^1 \otimes L)$.

We can apply \cite[Thm.2]{Kaw91} to see that the irreducible components of $\varphi_X$-fibres are uniruled. We also know by \cite[Cor.1.5]{HM07} that the fibres are rationally chain-connected. Note however that this does not imply that the irreducible components are rationally chain-connected \cite[p.119]{HM07}.

The following statement, inspired by \cite[Prop.5.9]{MOSC+15} is certainly well-known to experts, we give a detailed proof for the convenience of the reader:

\textbf{3.2. Lemma.} Let $X \simeq G/P$ be a rational homogeneous space, and let $\pi : \mathbb{P}(T_X) \to X$ be its projectivised cotangent bundle. Let $F \subset \mathbb{P}(T_X)$ be a smooth quasi-projective subvariety that is contracted by the birational map \eqref{5} onto a point. Then $F$ is an integral variety with respect to the contact distribution $\mathcal{F} \subset T_{\mathbb{P}(T_X)}$, i.e. one has $T_F \subset \mathcal{F} \otimes \mathcal{O}_F$.

We will see that this lemma is a translation of the fact that fibres of a symplectic resolution are isotropic with respect to the symplectic form \cite[Thm.1.2]{Wie03}, \cite[Lemma 4.1]{Nam01}, a strategy that appears in the literature at several places, e.g. \cite{Bea00, SCW04}.

Let us recall the relation between the two setups: let $Y \to Y$ be the total space of $L^*$ with the zero section removed, and set $X := \mathbb{P}(T_X) \times_Y Y$. Denote the natural maps by $\tilde{\varphi}_X : X \to Y$, $\tau : X \to \mathbb{P}(T_X)$. Then $X \to \mathbb{P}(T_X)$ is a $\mathbb{C}^*$-bundle and in fact the total space of $\mathcal{O}_{\mathbb{P}(T_X)}(-1)$ with the zero section removed \cite[Lemma 4.1]{Fu06}. The spaces $X$ and $Y$ are symplectic and $\tilde{\varphi}_X$ is a symplectic resolution \cite[Lemma 4.2]{Fu06}. More precisely the contact form on $\mathbb{P}(T_X)$ is obtained from the symplectic form $\tilde{\omega}$ on $X$ by contracting with a vector field generated by the $\mathbb{C}^*$-action (cf. proof of \cite[Lemma 4.2]{Fu06}).

\textit{Proof of Lemma 3.2.} By \cite{1} the contact distribution $\mathcal{F}$ is the kernel of the contact map $T_{\mathbb{P}(T_X)} \to \mathcal{O}_{\mathbb{P}(T_X)}(1) \to 0$.

We identify the contact map with a section

$$\theta \in H^0(\mathbb{P}(T_X), \Omega_{\mathbb{P}(T_X)} \otimes \mathcal{O}_{\mathbb{P}(T_X)}(1)).$$

Since $\mathcal{O}_{\mathbb{P}(T_X)}(1) \simeq \varphi_X^*L$ we can find an analytic neighbourhood $U$ of $\varphi^{-1}_X(F)$ such that $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ is trivial on $U$. Hence

$$\theta|_U \in H^0(U, \Omega_U)$$

and we are done if we show that

$$\theta|_F \in H^0(F, N^*_F/\mathbb{P}(T_X)),$$

where $N^*_F/\mathbb{P}(T_X) \subset \Omega_U \otimes \mathcal{O}_F$ is the conormal bundle of $F$.

\textit{Proof of the claim.} We will reduce the claim to the corresponding statement for the symplectic forms: let $T \simeq \mathbb{C}^*$ be the fibre of $Y \to Y$ over $\varphi_X(F)$. Then $F \times_Y T \simeq F \times T \subset X$ and we set

$$\sigma : F \times T \to T,$$
where $\sigma$ is the restriction of $\tilde{\varphi}_X$ to $F \times T$. Then by [Kal06, Lemma 2.9] there exists a dense open subset $T_0 \subset T$ and a holomorphic two-form $\omega_T$ on $T_0$ such that

$$\sigma^* \omega_T = \eta|_{F \times T_0}$$

where $\eta$ is the image of $\tilde{\omega}$ under the restriction map

$$H^0(X, \Omega^2_X) \to H^0(F \times T, \Omega^2_X \otimes \mathcal{O}_{F \times T}) \to H^0(F \times T, \Omega^2_{F \times T})$$

Since $T_0$ is a curve, the holomorphic two-form $\omega_T$ is zero, hence $\eta = 0$. By [Har77, II, Ex. 5.16] the conormal sequence

$$0 \to N^*_{F \times T/X} \to \Omega_X \otimes \mathcal{O}_{F \times T} \to \Omega_{F \times T} \to 0$$

induces a filtration of the kernel $\mathcal{K}$ of the surjection $\Omega^2_X \otimes \mathcal{O}_{F \times T} \to \Omega^2_{F \times T}$:

$$0 \to \bigwedge^2 N^*_{F \times T/X} \to \mathcal{K} \to N^*_{F \times T/X} \otimes \Omega_{F \times T} \to 0.$$  

By what precedes we know that

$$\tilde{\omega}|_{F \times T} \in H^0(F \times T, \mathcal{K}),$$

we will now deduce $\theta|_F \in H^0(F, N^*_F/\mathcal{P}(T_X))$: by the discussion before the proof $\theta|_F$ is obtained from $\tilde{\omega}|_{F \times T}$ by contracting with a vector field $v$ generated by the $\mathbb{C}^*$-action. Since this vector field is mapped onto zero by $\tau$, the contraction with a 2-form that is a pull-back from $F$ is equal to zero. Since

$$N^*_F = \tau^* N^*_F/\mathcal{P}(T_X)$$

we obtain from (3) that the contraction map $\mathcal{K} \xrightarrow{\iota^*} \Omega_{\mathcal{P}(T_X)/F}$ factors through a morphism

$$N^*_F \otimes \Omega_{F \times T} \xrightarrow{\iota^*} \Omega_{\mathcal{P}(T_X)/F}.$$  

Yet

$$\Omega_{F \times T} \cong \tau^* \Omega_F \oplus \varphi^*_X \Omega_T,$$

so if we decompose $\tilde{\omega}|_{F \times T} = \tilde{\omega}_1 + \tilde{\omega}_2$ according to the direct sum

$$N^*_F \otimes \Omega_{F \times T} \cong \left( \tau^* N^*_F/\mathcal{P}(T_X) \otimes \tau^* \Omega_F \right) \oplus \left( \tau^* N^*_F/\mathcal{P}(T_X) \otimes \varphi^*_X \Omega_T \right)$$

we see that $\tilde{\omega}_1 \cdot v = 0$ while

$$\tilde{\omega}_2 \cdot v \in H^0(F, N^*_F/\mathcal{P}(T_X)).$$

Since $\theta|_F = (\tilde{\omega}|_{F \times T})|F = \tilde{\omega}_2|F$ this shows the claim. 

The following example shows that the crucial point in Lemma 3.2 is that the contact form on $\mathbb{P}(T_X)$ is a reflexive pull-back from the singular space $Y$, i.e. we use that $T_X$ is nef and big.

### 3.3. Example.

Let $X = \mathbb{C}^2/\Lambda$ be an abelian surface, so $X$ is homogeneous and the natural map $\tilde{\varphi}_X$ is given by the projection

$$\varphi_X : \mathbb{P}(T_X) \cong X \times \mathbb{P}^1 \to \mathbb{P}^1.$$  

We will now follow the notation of [Bla10, Sect. 13.2] for the local computation of the contact form $\theta$: for linear coordinates $z_1, z_2$ on $\mathbb{C}^2$ the contact form on $\mathbb{P}(T_X)$ is

$$\theta = \sum_{i=1}^2 \frac{dz_i \otimes \zeta_i}{5}$$
where \( \sum_{i=1}^{2} \zeta_i dz_i \) are fibrewise coordinates on \( \mathbb{P}(T_{X,x}) \simeq \mathbb{P}(\Omega_{X,x}) \) (where \( \mathbb{P}(\Omega_{X,x}) \) is the space of lines in \( \Omega_{X,x} \)).

Assume now that \( A \subset X \) is an elliptic curve corresponding to the linear subspace \( \mathbb{C}z_1 \subset \mathbb{C}^2 \), then

\[
\frac{\partial}{\partial z_1} \mapsto \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial z_2} \mapsto 0
\]

defines a splitting \( T_X \otimes O_A \to T_A \) of the tangent map \( \tau_A \). The curve \( \tilde{A} \subset \mathbb{P}(T_X) \) is contained in \( X \times \mathbb{P}(dz_1) \subset X \times \mathbb{P}(\Omega_X) \), so it is contracted by \( \varphi_X \). The restriction of \( \theta \) to \( X \times \mathbb{P}(dz_1) \) is simply the form \( dz_1 \), in particular the composition

\[
T_{\tilde{A}} \hookrightarrow T_{\mathbb{P}(T_X)} \otimes O_{\tilde{A}} \xrightarrow{\nu \mapsto dz_1(\nu)} O_{\tilde{A}}
\]

is surjective.

We make a basic observation:

3.4. Lemma. Let \( X \) be a projective manifold, and let \( \pi : \mathbb{P}(T_X) \to X \) be its projectivised cotangent bundle. Let \( A \subset X \) be an abelian variety that is normal split with splitting map \( s_A : T_X \otimes O_A \to T_A \) and fix a quotient \( q_A : T_A \to O_A \). Let \( \sigma_A : A \to \mathbb{P}(T_X) \) be the lifting determined by the quotient line bundle

\[
q_A \circ s_A : T_X \otimes O_A \to O_A,
\]

and denote by \( \tilde{A} \subset \mathbb{P}(T_X) \) its image. Since \( \tilde{A} \) maps isomorphically onto its image in \( X \), we can consider the tangent morphism

\[
\tau_A : T_{\tilde{A}} \simeq T_A \to T_X \otimes O_A \simeq \pi^*(T_X) \otimes O_{\tilde{A}}.
\]

Then the composition with the canonical quotient map gives a surjective map

\[
q \circ \tau_A : T_{\tilde{A}} \simeq T_A \to \pi^*(T_X) \otimes O_{\tilde{A}} \to O_{\mathbb{P}(T_X)}(1) \otimes O_{\tilde{A}}.
\]

Proof of Lemma 3.4. The lifting \( \sigma_A \) is determined by the quotient line bundle \( q_A \circ s_A : T_X \otimes O_A \to O_A \), so by the universal property of the projectivisation [Laz04] App.A| the pull-back of the canonical quotient map \( q : \pi^*T_X \to O_{\mathbb{P}(T_X)}(1) \) via \( \sigma_A \) identifies to \( q_A \circ s_A \). Since \( s_A \) is a splitting map for \( \tau_A : T_A \to T_X \otimes O_A \), the composition \( q_A \circ s_A \circ \tau_A = q_A \) is surjective. \( \Box \)

3.5. Remark. It is instructive to compare the situation with liftings of rational curves: let \( X \) be a smooth quadric surface, and let \( l \subset X \) be a line of a ruling. Then \( l \subset X \) is normal split: we have

\[
T_X \otimes O_l \simeq T_l \oplus O_l,
\]

so the trivial quotient \( T_X \otimes O_l \to O_l \) determines a lifting of \( l \) to \( \mathbb{P}(T_X) \) such that the image \( \tilde{l} \) is contained in a \( \varphi_X \)-fibre. However the morphism

\[
T_l \to \pi^*(T_X) \otimes O_l \to O_l
\]

is not surjective: we have \( T_l \simeq O_{\mathbb{P}^1}(2) \), so any morphism to a trivial bundle must vanish. The difference to Lemma 3.4 is that the trivial quotient \( T_X \otimes O_l \to O_l \) does not factor through a morphism \( T_X \otimes O_l \to T_l \).
Proof of Theorem 1.2. We argue by contradiction and assume that the submanifold
\( M \subset X \) is not rational homogeneous. By Lemma 2.2 we can assume without loss of
generality that \( M \cong A \) is an abelian variety. We fix a splitting map \( s_A : T_X \otimes O_A \to T_A \) and a trivial quotient \( q_A : T_A \to O_A \). Denote by \( \tilde{A} \subset \mathbb{P}(T_X) \) the lifting of \( A \) to \( \mathbb{P}(T_X) \) determined by the quotient \( q_A \circ s_A \). By Lemma 3.4 the map
\[
q \circ \tau_A : T_{\tilde{A}} \to O_{\mathbb{P}(T_X)}(1) \otimes O_{\tilde{A}}
\]
is surjective. Since \( \tilde{A} \subset \mathbb{P}(T_X) \) the tangent map \( \tau_A \) factors through the
tangent map
\[
\tau_{\tilde{A}} : T_{\tilde{A}} \to T_{\mathbb{P}(T_X)} \otimes O_{\tilde{A}}
\]
and we have a commutative diagram
\[
\begin{array}{ccc}
T_{\tilde{A}} & \xrightarrow{\tau_A} & T_{\mathbb{P}(T_X)} \otimes O_{\tilde{A}} \\
\downarrow{=} & & \downarrow{\tau_\pi} \\
T_A & \xrightarrow{\tau_A} & \pi^* T_X \otimes O_A \xrightarrow{q} O_{\mathbb{P}(T_X)}(1) \otimes O_{\tilde{A}}
\end{array}
\]
By construction the contact map (4) is the composition \( q \circ \tau_\pi \). Since \( q \circ \tau_A \) is
surjective, we obtain that \( q \circ \tau_\pi \circ \tau_{\tilde{A}} \) is surjective, in particular \( \tilde{A} \subset \mathbb{P}(T_X) \) is not
integral with respect to the contact structure. Yet the lifting \( A \to \mathbb{P}(T_X) \) is given
by a trivial line bundle, so
\[
\varphi_X^* L \otimes O_A \cong O_{\mathbb{P}(T_X)}(1) \otimes O_{\tilde{A}} \cong O_{\tilde{A}}.
\]
Since \( L \) is an ample line bundle on \( Y \), we see that \( \tilde{A} \) is contracted by \( \varphi_X \) onto a
point. Thus we have a contradiction to Lemma 3.2. \( \square \)

3.6. Remark. Let us conclude by indicating a variant of Theorem 1.2 under the
weaker assumption that \( T_X \) is nef and big. We claim that in this case a normal
split submanifold \( M \subset X \) is a Fano manifold with semiample tangent bundle.

Proof. The basepoint-free theorem implies that \( T_X \) is semiample in the sense of
\cite{Fuj}, cf. \cite[Prop.5.5]{MOS} for a proof. In particular we can define the birational
morphism \( 5 \) using the global sections of some positive multiple of \( O_{\mathbb{P}(T_X)}(1) \), and
Lemma \ref{lem:3.2} holds for this morphism.

Since \( M \subset X \) is normal split, its tangent bundle \( T_M \) is also semi-ample. Moreover,
by \cite[Main Thm.]{DPS} there exists a finite étale cover \( \eta : M' \to M \) such that
\( M' \) admits a smooth fibration \( f : M' \to A \) onto an abelian variety such that the
general fibre is Fano. Arguing by contradiction we assume that \( A \) is not a point.
Since \( \eta \) is étale, the splitting map \( s_M : T_X \otimes O_M \to T_M \) lifts to splitting map
\[
\bar{s}_M : \eta^*(T_X \otimes O_M) \to \eta^* T_M \cong T_{M'}.
\]
Since \( T_{M'} \) is semiample \cite[Lemma 1]{Fuj} the tangent map \( \tau_f : T_{M'} \to T_A \) splits by
\cite[Cor.4]{Fuj}. Thus we have \( T_{M'} \cong T_{M'/A} \otimes O_{M'}^{\dim A} \) and we can use a quotient line
bundle \( T_{M'} \to O_{M'} \) to define a lifting \( M' \to \mathbb{P}(T_X) \) such that the image is contracted
by \( \varphi_X \) onto a point. Now the proof of Theorem 1.2 yields a contradiction. \( \square \)
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