Gauge-Yukawa-Finite Unified Theories
and their Predictions

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Abstract

Gauge-Yukawa unified theories, and in particular those which are finite beyond the unification scale, have been extended to include a soft supersymmetry breaking (SSB) sector. In the case of the Finite Unified Theories a new solution to the condition of two-loop finiteness of the SSB parameters is found, which requires a sum rule for the relevant scalar masses. The sum rule permits violations of the universality of the scalar masses which is found to lead to phenomenological problems. The minimal supersymmetric $SU(5)$ Gauge-Yukawa model and two Finite-Gauge-Yukawa models have been examined using the sum rule. The characteristic features of these models are: a) the old agreement of the top quark mass prediction persists using the new data, b) the lightest Higgs boson is predicted to be around 120 GeV, c) the s-spectrum starts above 200 GeV.

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1 Introduction

One of the most challenging problems of the Standard Model (SM) is the reduction of its free parameters. In a series of papers it has been shown that such a reduction can be achieved by embedding first the SM in a supersymmetric GUT, and subsequently searching for renormalization group invariant relations (RGI) among gauge and Yukawa couplings, which could have been established at the Planck scale in the framework of a more fundamental theory. Of particular interest are the Finite-Gauge-Yukawa unified theories which apparently are touching one of the most fundamental problems of Elementary Particle Physics. They are based on the fact that in certain theories with vanishing one-loop $\beta$-functions there exist RGI relations among couplings that guarantee finiteness even to all-orders in perturbation theory. Given the searches of strings, non-commutative geometry, and quantum groups with similar aims, it is very interesting to note that finiteness does not require gravity.

2 Reduction of Couplings and Finiteness in $N = 1$ SUSY Gauge Theories

A RGI relation among couplings, $\Phi(g_1, \ldots, g_N) = 0$, has to satisfy the partial differential equation (PDE) $\mu \frac{d\Phi}{d\mu} = \sum_{i=1}^{N} \beta_i \frac{d\Phi}{dg_i} = 0$, where $\beta_i$ is the $\beta$-function of $g_i$. There exist $(N-1)$ independent $\Phi$'s, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs), $\beta_g (dg_i/dg) = \beta_i$, $i = 1, \ldots, N$, where $g$ and $\beta_g$ are the primary coupling and its $\beta$-function. Using all the $(N-1)$ $\Phi$'s to impose RGI relations, one can in principle express all the couplings in terms of a single coupling $g$. The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, where the uniqueness of such a power series solution can be investigated at the one-loop level. The completely reduced theory contains only one independent coupling with the corresponding $\beta$-function. This possibility of coupling unification is attractive, but it can be too restrictive and hence unrealistic. To overcome this problem, one may use fewer $\Phi$'s as RGI constraints.

It is clear by examining specific examples, that the various couplings in supersymmetric theories have easily the same asymptotic behaviour. Therefore searching for a power series solution to the REs is justified. This is not the case in non-supersymmetric theories.

Let us then consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group $G$ with gauge coupling constant $g$. The superpotential of the theory is given by

$$W = \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k ,$$

(1)
where \( m^{ij} \) and \( C^{ijk} \) are gauge invariant tensors and the matter field \( \Phi_i \) transforms according to the irreducible representation \( R_i \) of the gauge group \( G \).

The one-loop \( \beta \)-function of the gauge coupling \( g \) is given by

\[
\beta^{(1)}_g = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[ \sum_i l(R_i) - 3 C_2(G) \right],
\]

where \( l(R_i) \) is the Dynkin index of \( R_i \) and \( C_2(G) \) is the quadratic Casimir of the adjoint representation of the gauge group \( G \). The \( \beta \)-functions of \( C^{ijk} \), by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix \( \gamma^j_i \) of the matter fields \( \Phi_i \) as:

\[
\beta^{ijk}_{C} = \frac{d}{dt} C^{ijk} = C^{ijp} \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \gamma^{kp(n)}_p + (k \leftrightarrow i) + (k \leftrightarrow j).
\]

At one-loop level \( \gamma^j_i \) is

\[
\gamma^{(1)}_{ij} = \frac{1}{2} C_{ipq} C^{jpq} - 2 g^2 C_2(R_i) \delta^j_i,
\]

where \( C_2(R_i) \) is the quadratic Casimir of the representation \( R_i \), and \( C^{ijk} = C^{*ijk} \).

As one can see from Eqs. (2) and (4) all the one-loop \( \beta \)-functions of the theory vanish if \( \beta^{(1)}_g \) and \( \gamma^{(1)}_{ij} \) vanish, i.e.

\[
\sum_i l(R_i) = 3 C_2(G), \quad \frac{1}{2} C_{ipq} C^{jpq} = 2 \delta^j_i g^2 C_2(R_i).
\]

A very interesting result is that the conditions (5) are necessary and sufficient for finiteness at the two-loop level.

The one- and two-loop finiteness conditions (5) restrict considerably the possible choices of the irreps. \( R_i \) for a given group \( G \) as well as the Yukawa couplings in the superpotential (1). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a \( U(1) \) gauge group is incompatible with the condition (5), due to \( C_2[U(1)] = 0 \). This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.

A natural question to ask is what happens at higher loop orders. There exists a very interesting theorem [3] which guarantees the vanishing of the \( \beta \)-functions to all orders in perturbation theory, if we demand reduction of couplings, and that all the one-loop anomalous dimensions of the matter field in the completely and uniquely reduced theory vanish identically.
3 Supersymmetry breaking terms

The above described method of reducing the dimensionless couplings has been extended\[6\] in the soft supersymmetry breaking (SSB) dimensionfull parameters of $N = 1$ supersymmetric theories. In addition it was found\[7\] that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule. Here we would like to describe how the use of the available two-loop RG functions and the requirement of finiteness of the SSB parameters up to this order leads to the soft scalar-mass sum rule\[8\].

Consider the superpotential given by (1) along with the Lagrangian for SSB terms,

$$- L_{\text{SB}} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} h^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)_{ij} \phi^c_i \phi_j + \frac{1}{2} M \lambda \lambda + \text{H.c.} \quad (6)$$

where the $\phi_i$ are the scalar parts of the chiral superfields $\Phi_i$, $\lambda$ are the gauginos and $M$ their unified mass. Since we would like to discuss only finite theories here, we assume that the gauge group is a simple group and the one-loop $\beta$ function of the gauge coupling $g$ vanishes. We also assume that the reduction equations admit power series solutions of the form

$$C_{ijk} = \sum_{n=0}^{\infty} \rho_{(n)}^{ijk} g^{2n}, \quad (7)$$

According to the finiteness theorem of ref.\[3\], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{(1)}$ vanish. The one- and two-loop finiteness for $h^{ijk}$ can be achieved by

$$h^{ijk} = - MC^{ijk} + \ldots = - M \rho_{(0)}^{ijk} g + O(g^5). \quad (8)$$

Now, to obtain the two-loop sum rule for soft scalar masses, we assume that the lowest order coefficients $\rho_{(0)}^{ijk}$ and also $(m^2)_j$ satisfy the diagonality relations

$$\rho_{ipq(0)}^{jpq} \propto \delta^i_j \text{ for all } p \text{ and } q \text{ and } (m^2)_{ij} = m^2 \delta^i_j, \quad (9)$$

respectively. Then we find the following soft scalar-mass sum rule

$$\frac{(m_i^2 + m_j^2 + m_k^2) / M M^1}{1 + \frac{g^2}{16\pi^2} \Delta^{(1)} + O(g^4)} \quad (10)$$

for $i$, $j$, $k$ with $\rho_{(0)}^{ijk} \neq 0$, where $\Delta^{(1)}$ is the two-loop correction, which vanishes for the universal choice in accordance with the previous findings of ref.\[4\].
4 Gauge-Yukawa-Unified Theories

4.1 Finite Unified Models

A predictive Gauge-Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma^{(1)}_{ij} \propto \delta_{ij}$, according to the assumption (9).

2. Three fermion generations, $\mathbf{5}_i$ ($i = 1, 2, 3$), obviously should not couple to $\mathbf{24}$. This can be achieved for instance by imposing $B - L$ conservation.

3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model.

A: The model of ref. [1].

B: A slight variation of the model A.

The superpotential which describe the two models takes the form [1, 8]

$$W = \sum_{i=1}^{3} \left[ \frac{1}{2} g_i^u \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \mathbf{10}_i \mathbf{5}_i \overline{H}_i \right] + g_{23}^u \mathbf{10}_2 \mathbf{10}_3 H_4$$

$$+ g_{23}^d \mathbf{10}_2 \overline{\mathbf{5}}_3 \overline{H}_4 + g_{32}^d \mathbf{10}_3 \overline{\mathbf{5}}_2 \overline{H}_4 + \sum_{a=1}^{4} g_a^f H_a \mathbf{24} \overline{H}_a + \frac{g^\lambda}{3} (\mathbf{24})^3,$$

where $H_a$ and $\overline{H}_a$ ($a = 1, \ldots, 4$) stand for the Higgs quintets and anti-quintets.

The non-degenerate and isolated solutions to $\gamma^{(1)}_{ij} = 0$ for the models $\{A, B\}$ are:

$$(g_1^u)^2 = \left(\frac{8}{5}, \frac{8}{5}\right) g^2, \quad (g_1^d)^2 = \left(\frac{6}{5}, \frac{6}{5}\right) g^2, \quad (g_2^u)^2 = (g_3^u)^2 = \left(\frac{8}{5}, \frac{4}{5}\right) g^2,$$

$$(g_2^d)^2 = (g_3^d)^2 = \left(\frac{6}{5}, \frac{3}{5}\right) g^2, \quad (g_{23}^d)^2 = \left(0, \frac{4}{5}\right) g^2, \quad (g_{32}^d)^2 = \left(0, \frac{3}{5}\right) g^2,$$

$$(g^\lambda)^2 = \frac{15}{7} g^2, \quad (g_{4}^f)^2 = (g_{4}^f)^2 = \left(0, \frac{1}{2}\right) g^2, \quad (g_{4}^f)^2 = \left(0, 1\right) g^2.$$  }

According to the theorem of ref. [3] these models are finite to all orders. After the reduction of couplings the symmetry of $W$ is enhanced [1, 8].

The main difference of the models $A$ and $B$ is that three pairs of Higgs quintets and anti-quintets couple to the $\mathbf{24}$ for $B$ so that it is not necessary to mix them with $H_4$ and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of $SU(5)$.
4.2 The minimal supersymmetric $SU(5)$ model

Next let us consider the minimal supersymmetric $SU(5)$ model. The field content is minimal. Neglecting the CKM mixing, one starts with six Yukawa and two Higgs couplings. We then require GYU to occur among the Yukawa couplings of the third generation and the gauge coupling. We also require the theory to be completely asymptotically free. In the one-loop approximation, the GYU yields

$$g_{t,b}^2 = \sum_{m,n=1}^{\infty} \kappa_{t,b}^{(m,n)} h^m f^n g^2 \quad (h \text{ and } f \text{ are related to the Higgs couplings}).$$

Where $h$ is allowed to vary from 0 to $15/7$, while $f$ may vary from 0 to a maximum which depends on $h$ and vanishes at $h = 15/7$. As a result, it was obtained [2]:

$$0.97 g^2 \lesssim g_{t,b}^2 \lesssim 1.37 g^2, \quad 0.57 g^2 \lesssim g_{b}^2 = g_{\tau}^2 \lesssim 0.97 g^2.$$

It was found [2, 5] that consistency with proton decay requires $g_{t,b}^2$, $g_{b}^2$ to be very close to the left hand side values in the inequalities.

5 Predictions of Low Energy Parameters

Since the gauge symmetry is spontaneously broken below $M_{GUT}$, the finiteness conditions do not restrict the renormalization property at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings [12] and the $h = -MC$ relation [8] and the soft scalar-mass sum rule [10] at $M_{GUT}$. So to find the predictions of these models we examine the evolution of these parameters according to their renormalization group equations at two-loop for dimensionless parameters and at one-loop for dimensional ones with these boundary conditions. Below $M_{GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale $M_s$ so that below $M_s$ the SM is the correct effective theory.

The predictions for the top quark mass $M_t$ are $\sim 183$ and $\sim 174$ GeV in models A and B respectively, and $\sim 181$ GeV for the minimal $SU(5)$ model. Comparing these predictions with the most recent experimental value $M_t = (175.6 \pm 5.5)$ GeV, and recalling that the theoretical values for $M_t$ may suffer from a correction of less than $\sim 4\%$ [8], we see that they are consistent with the experimental data.

Turning now to the SSB sector of these models we first note that preliminary results on the minimal $SU(5)$ model show that the low energy SSB sector contains a single arbitrary parameter, the gaugino mass $M$, which characterizes the scale of supersymmetry breaking. The lightest supersymmetric particle is found to be a neutralino of $\sim 220$ GeV for $M(M_{GUT}) \sim 0.5$ TeV. Concerning the SSB sector of the finite theories A, B we look for the parameter space in which the lighter s-tau mass squared $m_{\tilde{\tau}}^2$ is larger than the lightest neutralino mass squared $m_{\chi}^2$ (which is the LSP). For the case where all the soft scalar masses are universal at the unification scale, there is no region of $M_s = M$ below $O$(few) TeV in which $m_{\tilde{\tau}}^2 > m_{\chi}^2$ is satisfied. But once the universality condition is relaxed this problem can be solved naturally (provided the sum rule). More
specifically, using the sum rule \((10)\) and imposing the conditions a) successful radiative electroweak symmetry breaking b) \(m_{F2} > 0\) and c) \(m_{F2} > m_{\chi2}\), we find a comfortable parameter space for both models (although model B requires large \(M \sim 1\) TeV). The particle spectrum of models A and B in turn is calculated in terms of 3 and 2 parameters respectively.

In Fig. 1 we present the \(m_{10}\) dependence of \(m_h\) for for \(M = 0.8\) (dashed) 1.0 (solid) TeV for the finite Model B, which shows that the value of \(m_h\) is stable. Similar results hold also for the minimal supersymmetric \(SU(5)\) model. In particular, in Fig.2 it is shown the dependence of the lightest Higgs mass \(m_h\) in terms of the single free parameter \(M\).

6 Conclusions

The search for realistic Finite Unified theories started a few years ago \([1, 2, 5]\) with the successful prediction of the top quark mass, and it has now been complemented with a new important ingredient concerning the finiteness of the SSB sector of the theory. Specifically, a sum rule for the soft scalar masses has been obtained which guarantees the finiteness of the SSB parameters up to two-loops \([8]\), avoiding at the same time serious phenomenological problems related to the previously known “universal” solution. It is found that this sum rule coincides with that of a certain class of string models in which the massive string modes are organized into \(N = 4\) supermultiplets. Using the sum rule we can now determine the spectrum of realistic models in terms of just a few parameters. In addition to the successful prediction of the top quark mass the characteristic features of the spectrum are that 1) the lightest Higgs mass is

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**Figure 1:** \(m_h\) as function of \(m_{10}\) for \(M = 0.8\) (dashed) 1.0 (solid) TeV.
predicted $\sim 120$ GeV and 2) the $s$-spectrum starts above 200 GeV. Therefore, the next important test of these Finite Unified theories will be given with the measurement of the Higgs mass, for which the models show an appreciable stability in their prediction. The minimal supersymmetric $SU(5)$ gives similar results.

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