The Jones Polynomial and Khovanov Homology of Weaving Knots $W(3, n)$

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Abstract

In this paper we compute the signature for a family of knots $W(k, n)$, the weaving knots of type $(k, n)$. By work of E. S. Lee the signature calculation implies a vanishing theorem for the Khovanov homology of weaving knots. Specializing to knots $W(3, n)$, we develop recursion relations that enable us to compute the Jones polynomial of $W(3, n)$. Using additional results of Lee, we compute the ranks of the Khovanov Homology of these knots. At the end we provide evidence for our conjecture that, asymptotically, the ranks of Khovanov Homology of $W(3, n)$ are normally distributed.
1 Introduction

Weaving knots originally attracted interest, because it was conjectured that their complements would have the largest hyperbolic volume for a fixed crossing number. Here is the weaving knot \( W(3, 4) \).

Enumerating strands 1, \ldots, \( p \) from the outside inward, our example is the closure of the braid \((\sigma_1\sigma_2^{-1})^4\) on three strands. Thus, \( \sigma_1 \) is a righthand twist involving strands 1 and 2, and \( \sigma_2 \) is a righthand twist involving strands 2 and 3, and so on.

In words, the weaving knot \( W(p, q) \) is obtained from the torus knot \( T(p, q) \) by making the standard diagram of the torus knot alternating. Symbolically, \( T(p, q) \) is the closure of the braid \((\sigma_1\sigma_2\cdots\sigma_{p-1})^q\), and \( W(p, q) \) is the closure of the braid \((\sigma_1\sigma_2^{-1}\cdots\sigma_{p-1}^\pm)^q\). Obviously, the parity of \( p \) is important. If the greatest common divisor \( \text{gcd}(p, q) > 1 \), then \( T(p, q) \) and \( W(p, q) \) are both links with \( \text{gcd}(p, q) \) components. In general we are interested only in the cases when \( W(p, q) \) is an actual knot.

In [1] the main result is the following theorem.

\textbf{Theorem 1.1} (Theorem 1.1, [1]). If \( p \geq 3 \) and \( q \geq 7 \), then

\[ v_{\text{oct}}(p-2)q \left( 1 - \frac{(2\pi)^2}{q^2} \right)^{3/2} \leq \text{vol}(W(p, q)) < (v_{\text{oct}}(p-3) + 4v_{\text{tet}})q. \]

Champanerkar, Kofman, and Purcell call these bounds asymptotically sharp because their ratio approaches 1, as \( p \) and \( q \) tend to infinity. Since the crossing number of \( W(p, q) \) is known to be \((p-1)q\), the volume bounds in the theorem imply

\[ \lim_{p,q \to \infty} \frac{\text{vol}(W(p, q))}{c(W(p, q))} = v_{\text{oct}} \approx 3.66 \]

Their study raised the question of examining the asymptotic behavior of other invariants of weaving knots. In this paper we start a study of the asymptotic behavior of Khovanov homology of weaving knots.

Briefly, since weaving knots are alternating knots by definition, we may specialize certain properties of the Khovanov homology of alternating knots to get started. This is accomplished in section 2 where we explain how to calculate the signature of weaving knots. The second main ingredient in our analysis is the fact that for alternating knots knowing
the Jones polynomial is equivalent to knowing the Khovanov homology. How this works explicitly in our examples is explained in section 5. In section 3 we prepare to follow the development of the Jones polynomial in [2], starting from representations of braid groups into Hecke algebras. For weaving knots $W(3, n)$, which are naturally represented as the closures of braids on three strands, we develop recursive formulas for their representations in the Hecke algebras. These formulas are used in computer calculations of the Jones polynomials we need. Section 4 builds on the recursion formulas to develop information about the Jones polynomials $V_{W(3, n)}(t)$. After we explain how to obtain the two-variable Poincaré polynomial for Khovanov homology in section 6 we present the results of calculations in a few relatively small examples. Our observation is that the distributions of dimensions in Khovanov homology resemble normal distributions. We explore this further in section 7 where we present tables displaying summaries of calculations for weaving knots $W(3, n)$ for selected values of $n$ satisfying gcd$(3, n) = 1$ and ranging up to $n = 326$. The standard deviation $\sigma$ of the normal distribution we attach to the Khovanov homology of a weaving knot is a significant parameter. The geometric significance of this number is an open question.

2 Generalities on Weaving knots

We have already mentioned that weaving knots are alternating by definition. Various facts about alternating knots facilitate our calculations of the Khovanov homology of weaving knots $W(3, n)$. For example, we appeal first to the following theorem of Lee.

Theorem 2.1 (Theorem 1.2, [4]). For any alternating knot $L$ the Khovanov invariants $H^i_j(L)$ are supported in two lines

$$j = 2i - \sigma(L) \pm 1.$$  \hfill $\Box$

We will see that this result also has several practical implications. For example, to obtain a vanishing result for a particular alternating knot, it suffices to compute the signature. Indeed, it turns out that there is a combinatorial formula for the signature of oriented non-split alternating links. To state the formula requires the following terminology.

Definition 2.2. For a link diagram $D$ let $c(D)$ be the number of crossings of $D$, let $x(D)$ be number of negative crossings, and let $y(D)$ be the number of positive crossings. For an oriented link diagram, let $o(D)$ be the number of components of $D(\emptyset)$, the diagram obtained by $A$-smoothing every crossing.

\begin{center}
\includegraphics[width=0.5\textwidth]{positive_negative_crossings.png}
\end{center}

Figure 1: Positive and negative crossings

\begin{center}
\includegraphics[width=0.3\textwidth]{asmoothing.png}
\end{center}

Figure 2: $A$-smoothing a positive, resp., negative, crossing

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In words, \(A\)-regions in a neighborhood of a crossing are the regions swept out as the upper strand sweeps counter-clockwise toward the lower strand. An \(A\)-smoothing removes the crossing to connect these regions. With these definitions, we may cite the following proposition.

**Proposition 2.3** (Proposition 3.11, [4]). For an oriented non-split alternating link \(L\) and a reduced alternating diagram \(D\) of \(L\), \(\sigma(L) = o(D) - y(D) - 1\).

We now use this result to compute the signatures of weaving knots. For a knot or link \(W(m, n)\) drawn in the usual way, the number of crossings \(c(D) = (m - 1)n\). In particular, for \(W(2k+1, n)\), \(c(W(2k+1, n)) = 2kn\); for \(W(2k, n)\), \(c(W(2k, n)) = (2k-1)n\). Evaluating the other quantities in definition 2.2, we calculate the signatures of weaving knots.

**Proposition 2.4.** For a weaving knot \(W(2k+1, n)\), \(\sigma(W(2k+1, n)) = 0\), and for \(W(2k, n)\), \(\sigma(W(2k, n)) = -n+1\).

*Proof.* Consider first the example \(W(3, n)\), illustrated by figures 3 and 4 for \(W(3, 4)\) drawn below. After \(A\)-smoothing the diagram, the outer ring of crossings produces a circle bounding the rest of the smoothed diagram. On the inner ring of crossings the \(A\)-smoothings produce \(n\) circles in a cyclic arrangement. Therefore \(o(W(3, n)) = 1 + n\). The outer ring of crossings consists of positive crossings and the inner ring of crossings consists of negative crossings, so \(x(D) = y(D) = n\). Applying the formula of theorem 2.3, we obtain the result \(\sigma(W(3, n)) = 0\).

For the general case \(W(2k+1, n)\), we have the following considerations. The crossings are organized into \(2k\) rings. Reading from the outside toward the center, we have first a ring of positive crossings, then a ring of negative crossings, and so on, alternating positive and negative. Thus \(y(D) = kn\). Considering the \(A\)-smoothing of the diagram of \(W(2k+1, n)\), as in the special case, a bounding circle appears from the smoothing of the outer ring. A chain of \(n\) disjoint smaller circles appears inside the second ring. No circles appear in the third ring, nor in any odd-numbered ring thereafter. On the other hand, chains of \(n\) disjoint smaller circles appear in each even-numbered ring. Since there are \(k\) even-numbered rings, we have \(o(D) = 1 + kn\). Applying the formula of proposition 2.3

\[
\sigma(W(2k+1, n)) = o(D) - y(D) - 1 = (1 + kn) - kn - 1 = 0.
\]

These figures illustrate the main points of the \(W(3, n)\)-cases, and, as explained above, the main points of the \(W(2k+1, n)\)-cases.

For the case \(W(2k, n)\), we show \(W(4, 5)\) below in figures 5 and 6 as an example. Our standard diagram may be organized into \(2k-1\) rings of crossings. In each ring there are \(n\) crossings, so the total number of crossings is \(c(D) = (2k-1)n\). In our standard representation, there is an outer ring of \(n\) positive crossings, next a ring of \(n\) negative crossings, alternating until we end with an innermost ring of \(n\) positive crossings. There are thus \(k\) rings of \(n\) positive crossings and \(k-1\) rings of \(n\) negative crossings. Therefore, \(y(D) = kn\) and \(x(D) = (k-1)n\). Considering the \(A\)-smoothing of the diagram, a bounding circle appears

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from the smoothing of the outer ring. As before, a chain of $n$ disjoint smaller circles appears inside the second ring and in each successive even-numbered ring. As previously noted, there are $k-1$ of these rings. No circles appear in odd-numbered rings, until we reach the last ring, where an inner bounding circle appears. Thus, $o(D) = 1 + (k-1)n + 1 = (k-1)n + 2$. Consequently,

$$\sigma(W(2k,n)) = o(D) - y(D) - 1 = ((k-1)n + 2) - kn - 1 = -n + 1.$$ 

\[\square\]

**Theorem 2.5.** For a weaving knot $W(2k+1,n)$ the non-vanishing Khovanov homology $\mathcal{H}^{i,j}(W(2k+1,n))$ lies on the lines

$$j = 2i \pm 1.$$
For a weaving knot $W(2k, n)$ the non-vanishing Khovanov homology $H^{i,j}(W(2k, n))$ lies on the lines

$$j = 2i + n - 1 \pm 1$$

Proof. Substitute the calculations made in lemma 2.4 into the formula of theorem 2.1. \hfill \Box

3 Recursion in the Hecke algebra

We review briefly the definition of the Hecke algebra $H_{N+1}$ on generators $T_1$ through $T_N$, and we define the representation of the braid group $B_3$ on three strands in $H_3$. Theorem 3.2 sets up recursion relations for the coefficients in the expansion of the image in $H_3$ of the braid $(\sigma_1\sigma_2^{-1})^n$, whose closure is the weaving knot $W(3, n)$. The recursion relations are essential for automating the calculation of the Jones polynomial for the knots $W(3, n)$. Proposition 3.4 uses these relations developed in theorem 3.2 to prove a vanishing result for one of the coefficients. Being able to ignore one of the coefficients speeds up the computations slightly.

Definition 3.1. Working over the ground field $K$ containing an element $q \neq 0$, the Hecke algebra $H_{N+1}$ is the associative algebra with 1 on generators $T_1, \ldots, T_N$ satisfying these relations.

$$T_i T_j = T_j T_i, \quad \text{whenever } |i - j| \geq 2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq N-1,$$

and, finally,

$$T_i^2 = (q-1)T_i + q, \quad \text{for all } i.$$  \hfill (3.3)

It is well-known \cite{2} that $(N+1)!$ is the dimension of $H_{N+1}$ over $K$.

Recasting the relation $T_i^2 = (q-1)T_i + q$ in the form $q^{-1}(T_i - (q-1)) \cdot T_i = 1$ shows that $T_i$ is invertible in $H_{N+1}$ with $T_i^{-1} = q^{-1}(T_i - (q-1))$. Consequently, the specification $\rho(\sigma_i) = T_i$, \hfill \hfill (3.4)
combined with relations (3.1) and (3.2), defines a homomorphism \( \rho: B_{N+1} \to H_{N+1} \) from \( B_{N+1} \), the group of braids on \( N+1 \) strands, into the multiplicative monoid of \( H_{N+1} \).

For work in \( H_3 \), choose the ordered basis \( \{ 1, T_1, T_2, T_1T_2, T_2T_1, T_1T_2T_1 \} \). The word in the Hecke algebra corresponding to the knot \( W(3,n) \) is formally

\[
\rho((T_1T_2^{-1})^n) = q^{-n}(C_{n,0} + C_{n,1} \cdot T_1 + C_{n,2} \cdot T_2 + C_{n,12} \cdot T_1T_2 + C_{n,21} \cdot T_2T_1 + C_{n,121} \cdot T_1T_2T_1),
\]

where the coefficients \( C_{n,*} = C_{n,*}(q) \) of the monomials in \( T_1 \) and \( T_2 \) are polynomials in \( q \). For \( n = 1 \),

\[
\rho(\sigma_1\sigma_2^{-1}) = T_1T_2^{-1} = q^{-1} \cdot (T_1(-(q-1) + T_2)) = q^{-1}(-(q-1) \cdot T_1 + T_1T_2),
\]

so we have initial values

\[
C_{1,0}(q) = 0, \ C_{1,1}(q) = -(q-1), \ C_{1,2}(q) = 0, \ C_{1,12}(q) = 1, \ C_{1,21}(q) = 0, \ \text{and} \ C_{1,121}(q) = 0.
\]

**Theorem 3.2.** These polynomials satisfy the following recursion relations.

\[
C_{n,0}(q) = q^2 \cdot C_{n-1,21}(q) - q(q-1) \cdot C_{n-1,1}(q) \tag{3.6}
\]

\[
C_{n,1}(q) = -(q-1)^2 \cdot C_{n-1,1}(q) - (q-1) \cdot C_{n-1,0}(q) + q^2 \cdot C_{n-1,121}(q) \tag{3.7}
\]

\[
C_{n,2}(q) = q \cdot C_{n-1,1}(q) \tag{3.8}
\]

\[
C_{n,12}(q) = (q-1) \cdot C_{n-1,1}(q) + C_{n-1,0}(q) \tag{3.9}
\]

\[
C_{n,21}(q) = -(q-1) \cdot C_{n-1,2}(q) + q \cdot C_{n-1,12}(q)
\]

\[
\quad - (q-1)^2 \cdot C_{n-1,21}(q) + q(q-1) \cdot C_{n-1,121}(q) \tag{3.10}
\]

\[
C_{n,121}(q) = C_{n-1,2}(q) + (q-1) \cdot C_{n-1,21}(q) \tag{3.11}
\]

**Proof.** We have

\[
\rho(T_1T_2^{-1})^n = \rho(T_1T_2^{-1})^{n-1} \cdot \rho(T_1T_2^{-1})
\]

\[
= q^{-n} \left( C_{n-1,0} + C_{n-1,1} \cdot T_1 + C_{n-1,2} \cdot T_2 + C_{n-1,12} \cdot T_1T_2 + C_{n-1,21} \cdot T_2T_1 + C_{n-1,121} \cdot T_1T_2T_1 \right)
\]

\[
\quad \cdot \left( -(q-1) \cdot T_1 + T_1T_2 \right)
\]

\[
= q^{-n} \left( -(q-1)C_{n-1,0} \cdot T_1 - (q-1)C_{n-1,1} \cdot T_1^2 - (q-1)C_{n-1,2} \cdot T_2T_1
\]

\[
\quad - (q-1)C_{n-1,12} \cdot T_1T_2T_1 - (q-1)C_{n-1,21} \cdot T_2T_1^2 - (q-1)C_{n-1,121} \cdot T_1T_2T_1^2
\]

\[
\quad + C_{n-1,0} \cdot T_1T_2 + C_{n-1,1} \cdot T_1^2T_2 + C_{n-1,2} \cdot T_2T_1T_2
\]

\[
\quad + C_{n-1,12} \cdot T_1T_2T_1T_2 + C_{n-1,21} \cdot T_2T_1^2T_2 + C_{n-1,121} \cdot T_1T_2T_1^2T_2 \right)
\]

\[
= q^{-n} \left( \left( -(q-1)C_{n-1,0} \cdot T_1 - (q-1)C_{n-1,1} \cdot T_2T_1 - (q-1)C_{n-1,12} \cdot T_1T_2T_1 + C_{n-1,0} \cdot T_1T_2 \right)
\]

\[
\quad + \left\{ -(q-1)C_{n-1,1} \cdot T_1^2 - (q-1)C_{n-1,2} \cdot T_2T_1^2 - (q-1)C_{n-1,12} \cdot T_1T_2T_1^2 + C_{n-1,1} \cdot T_1^2T_2
\]

\[
\quad + C_{n-1,2} \cdot T_2T_1T_2 + C_{n-1,12} \cdot T_1T_2T_1T_2 + C_{n-1,21} \cdot T_2T_1^2T_2 + C_{n-1,121} \cdot T_1T_2T_1^2T_2 \right\} \right) \tag{3.12}
\]
after collecting powers of $q$ and expanding. In the last grouping, the first four terms inside the parentheses $\left(\right)$ involve only elements of the preferred basis; the second eight terms in the pair of braces $\left\{\right\}$ all require further expansion, as follows.

\begin{align*}
-(q-1)C_{n-1,1} \cdot T_1^2 &= -(q-1)C_{n-1,1} \cdot ((q-1)T_1 + q) \\
&= -(q-1)^2C_{n-1,1} \cdot T_1 - q(q-1)C_{n-1,1} \\
-(q-1)C_{n-1,21} \cdot T_2T_1 &= -(q-1)C_{n-1,21} \cdot T_2((q-1)T_1 + q) \\
&= -(q-1)^2C_{n-1,21} \cdot T_2T_1 - q(q-1)C_{n-1,21} \cdot T_2 \\
-(q-1)C_{n-1,121} \cdot T_1T_2T_1 &= -(q-1)C_{n-1,121} \cdot T_1T_2((q-1)T_1 + q) \\
&= -(q-1)^2C_{n-1,121} \cdot T_1T_2T_1 - q(q-1)C_{n-1,121} \cdot T_1T_2 \\
C_{n-1,1} \cdot T_1^2T_2 &= C_{n-1,1} \cdot ((q-1)T_1 + q)T_2 \\
&= (q-1)C_{n-1,1} \cdot T_1T_2 + qC_{n-1,1} \cdot T_2 \\
C_{n-1,12} \cdot T_1T_2T_1T_2 &= C_{n-1,12} \cdot T_1^2T_2T_1 = C_{n-1,12}((q-1)T_1 + q)T_2T_1 \\
&= (q-1)C_{n-1,12} \cdot T_1T_2T_1 + qC_{n-1,12} \cdot T_2T_1 \\
C_{n-1,21} \cdot T_1^2T_2 &= C_{n-1,21} \cdot T_1((q-1)T_1 + q)T_2 = (q-1)C_{n-1,21} \cdot T_1T_2 + qC_{n-1,21} \cdot T_2^2 \\
&= (q-1)C_{n-1,21} \cdot T_1T_2 + qC_{n-1,21} \cdot ((q-1)T_2 + q) \\
&= (q-1)^2C_{n-1,21} \cdot T_1T_2T_1 + q(q-1)C_{n-1,21} \cdot T_2T_1 + q^2C_{n-1,21} \cdot T_2 \\
C_{n-1,121} \cdot T_1T_2T_1 &= C_{n-1,121} \cdot T_1T_2((q-1)T_1 + q)T_2 \\
&= (q-1)C_{n-1,121} \cdot T_1T_2T_1T_2 + qC_{n-1,121} \cdot T_1T_2^2 \\
&= (q-1)C_{n-1,121} \cdot T_1T_2T_1 + qC_{n-1,121} \cdot ((q-1)T_2 + q) \\
&= (q-1)^2C_{n-1,121} \cdot T_1T_2T_1 + q(q-1)C_{n-1,121} \cdot T_2T_1 + q^2C_{n-1,121} \cdot T_1 \\
&\quad+ q(q-1)C_{n-1,121} \cdot T_1T_2 + q^2C_{n-1,121} \cdot T_2
\end{align*}

Collecting the constant terms from (3.13) and (3.19), we get

\[ C_{n,0} = -q(q-1)C_{n-1,1} + q^2C_{n-1,21}. \]

Collecting coefficients of $T_1$ from (3.12), (3.13), (3.20), we get

\[ C_{n,1} = -(q-1)C_{n-1,0} - (q - 1)^2C_{n-1,1} + q^2C_{n-1,121}. \]

Collecting coefficients of $T_2$ from (3.14), (3.16), and (3.19), we get

\[ C_{n,2} = -q(q-1)C_{n-1,21} + qC_{n-1,1} + q(q-1)C_{n-1,21} = qC_{n-1,1}. \]
Collecting coefficients of $T_1T_2$ from (3.12), (3.15), (3.16), and (3.20), we get

$$C_{n,12} = C_{n-1,0} - q(q-1)C_{n-1,121} + (q-1)C_{n-1,1} + q(q-1)C_{n-1,121} = C_{n-1,0} + (q-1)C_{n-1,1}$$

Collecting coefficients of $T_2T_1$ from (3.12), (3.14), (3.18), and (3.20), we get

$$C_{n,21} = -(q-1)C_{n-1,2} - (q-1)^2C_{n-1,21} + qC_{n-1,12} + q(q-1)C_{n-1,121}.$$

Collecting coefficients of $T_1T_2T_1$ from (3.12), (3.15), (3.17), (3.18), (3.19), and (3.20), we get

$$C_{n,121} = -(q-1)C_{n-1,12} - (q-1)^2C_{n-1,121} + C_{n-1,2}
+ (q-1)C_{n-1,12} + (q-1)C_{n-1,21} + (q-1)^2C_{n-1,121}
= C_{n-1,2} + (q-1)C_{n-1,21}$$

Up to simple rearrangements and expansion of notation, these are formulas (3.6) through (3.11).

**Example 3.3.** Applying the recursion formulas just proved to the table of initial polynomials, or by computing $\rho ((\sigma_1\sigma_2^{-1})^2)$ directly from the definitions, we find

$$C_{2,0}(q) = q^2 \cdot C_{1,21}(q) - q(q-1) \cdot C_{1,1}(q) = q(q-1)^2, \quad (3.21)$$
$$C_{2,1}(q) = -(q-1)^2 \cdot C_{1,1}(q) - (q-1) \cdot C_{1,0}(q) = (q-1)^3, \quad (3.22)$$
$$C_{2,2}(q) = q \cdot C_{1,1}(q) = -q(q-1), \quad (3.23)$$
$$C_{2,12}(q) = (q-1) \cdot C_{1,1}(q) + C_{1,0}(q) = -(q-1)^2, \quad (3.24)$$
$$C_{2,21}(q) = -(q-1) \cdot C_{1,2}(q) + q \cdot C_{1,12}(q) - (q-1)^2 \cdot C_{1,21}(q) = q, \quad (3.25)$$
$$C_{2,121}(q) = 0. \quad (3.26)$$

As a first application, we have the following vanishing result.

**Proposition 3.4.** For all $n$, $C_{n,121}(q) = 0$.

**Proof.** For $n \geq 1$, we claim $C_{n+1,121}(q) = 0$. Make the inductive assumption that $C_{k,121}(q) = 0$ for $1 \leq k \leq n$. Apply (3.11), (3.10), and the inductive hypothesis to write

$$C_{n+1,121}(q) = C_{n,2}(q) + (q-1) \cdot C_{n,21}(q)
= C_{n,2}(q)
+ (q-1) \left( (q-1) \cdot C_{n-1,2}(q) + q \cdot C_{n-1,12}(q) - (q-1)^2 \cdot C_{n-1,21}(q) + q(q-1) \cdot C_{n-1,121} \right)
= C_{n,2}(q) + (q-1) \left( (q-1) \cdot C_{n-1,2}(q) + q \cdot C_{n-1,12}(q) - (q-1)^2 \cdot C_{n-1,21}(q) \right).$$
Using (3.8) to replace the first term $C_{n,2}(q)$ and (3.9) to replace the third term factor $C_{n-1,12}(q)$ on the right,

$$C_{n+1,121}(q) = q \cdot C_{n-1,1}(q) - (q-1)^2 C_{n-1,2}(q) + q(q-1)((q-1)C_{n-2,1}(q) + C_{n-2,0}(q))$$

$$= q \cdot C_{n-1,1}(q) - (q-1)^2 C_{n-1,2}(q) + (q-1)^2(qC_{n-2,1}(q)) + q(q-1)C_{n-2,0}(q)$$

$$= q \cdot C_{n-1,1}(q) - (q-1)^2 C_{n-1,2}(q) + (q-1)^2 C_{n-1,2}(q) + q(q-1)C_{n-2,0}(q)$$

$$= q \cdot C_{n-1,1}(q)$$

and using (3.8) in reverse to rewrite the term $qC_{n-2,1}(q)$. Making the obvious cancellation,

$$C_{n+1,121} = q \cdot C_{n-1,1}(q) + q(q-1) \cdot C_{n-2,0}(q) - (q-1)^3 \cdot C_{n,21}(q)$$

$$= q\left(C_{n-1,1} + (q-1)C_{n-2,0}\right) - (q-1)^3 \cdot C_{n-1,21}$$

$$= q\left(-((q-1)^2 \cdot C_{n-2,1} - (q-1) \cdot C_{n-2,0}) + (q-1) \cdot C_{n-2,0}\right) - (q-1)^3 \cdot C_{n-1,21},$$

since

$$C_{n-1,1}(q) = -(q-1)^2 \cdot C_{n-2,1}(q) - (q-1) \cdot C_{n-2,0}(q) + q^2 \cdot C_{n-2,121}(q)$$

$$= -(q-1)^2 \cdot C_{n-2,1}(q) - (q-1) \cdot C_{n-2,0}(q)$$

by (3.7) and the inductive hypothesis. Therefore,

$$C_{n+1,121}(q) = -q(q-1)^2 \cdot C_{n-2,1}(q) - (q-1)^3 \cdot C_{n-2,121}(q)$$

$$= -(q-1)^2 \cdot C_{n-1,2}(q) - (q-1)^3 \cdot C_{n-1,21}(q),$$

using (3.8) in the form $C_{n-1,2}(q) = q \cdot C_{n-2,1}(q)$,

$$= -(q-1)^2\left(C_{n-1,2}(q) - (q-1) \cdot C_{n-1,21}(q)\right)$$

$$= -(q-1)^2 \cdot C_{n,121}(q) = 0,$$

using (3.11) and the inductive hypothesis.

\[\square\]

## 4 Obtaining the Jones Polynomial

Following the construction given in [2] p.288 we work over the function field $K = \mathbb{C}(q, z)$, and we put $w = 1 - q + z$. Let $H_{N+1}$ be the Hecke algebra over $K$ corresponding to $q$ with $N$ generators as in definition 3.1. The starting point is the following theorem.

**Theorem 4.1.** For $N \geq 1$ there is a family of trace functions $\text{Tr} : H_{N+1} \to K$ compatible with the inclusions $H_N \to H_{N+1}$ satisfying
1. $\text{Tr}(1) = 1$,

2. Tr is $K$-linear and $\text{Tr}(ab) = \text{Tr}(ba)$,

3. If $a, b \in H_N$, then $\text{Tr}(ab) = z \text{Tr}(ab)$.

Property 3 enables the calculation of Tr on basis elements of $H_{N+1}$ through use of the defining relations and induction. For $H_3$, note that

$$\text{Tr}(T_1) = \text{Tr}(T_2) = z, \quad \text{Tr}(T_1T_2) = \text{Tr}(T_2T_1) = z^2, \quad \text{Tr}(T_1T_2T_1) = z \text{Tr}(T_1^2) = z((q-1)z+q).$$

The next step toward the Jones polynomial of the knot that is the closure of the braid $e$ where

$$W = \sigma_1\sigma_2^{-1},$$

leads to the one-variable Jones polynomial

$$V_{\alpha}(q, z) = \left(\frac{1}{z}\right)^{(N+e(\alpha))/2} \cdot \left(\frac{q}{w}\right)^{(N-e(\alpha))/2} \cdot \text{Tr}(\rho(\alpha)),$$

where $e(\alpha)$ is the exponent sum of the word $\alpha$. The expression defines an element in the quadratic extension $K(\sqrt{q/w})$. For the weaving knot $W(3, n)$, viewed as the closure of $(\sigma_1\sigma_2^{-1})^n$, we have the exponent sum $e = 0$, and $N = 2$, and

$$\rho((\sigma_1\sigma_2^{-1})^n) = (T_1T_2^{-1})^n = q^{-n}(C_{n,0}(q)+C_{n,1}(q)\cdot T_1+C_{n,2}(q)\cdot T_2+C_{n,12}(q)\cdot T_1T_2+C_{n,21}(q)\cdot T_2T_1),$$

thanks to proposition 3.4, which says the expression for $(T_1T_2^{-1})^n$ requires only the use of the basis elements $1, T_1, T_2, T_1T_2$ and $T_2T_1$. Then we have

$$V_{(\sigma_1\sigma_2^{-1})^n}(q, z)$$

$$= \left(\frac{1}{z}\right) \cdot \left(\frac{q}{w}\right) \cdot q^{-n} \cdot \text{Tr}(C_{n,0}(q)+C_{n,1}(q)\cdot T_1+C_{n,2}(q)\cdot T_2+C_{n,12}(q)\cdot T_1T_2+C_{n,21}(q)\cdot T_2T_1)$$

$$= \left(\frac{q}{zw}\right) \cdot q^{-n} \cdot \left(C_{n,0}(q)+C_{n,1}(q)\cdot z+C_{n,2}(q)\cdot z+C_{n,12}(q)\cdot z^2+C_{n,21}(q)\cdot z^3\right),$$

using the facts that $\text{Tr} T_1 = \text{Tr} T_2 = z$ and $\text{Tr} T_1T_2 = \text{Tr} T_2T_1 = z^2$. Consequently, the sums $C_{n,1} + C_{n,2}$ and $C_{n,12} + C_{n,21}$ are essential for understanding the two-variable Jones polynomial of $W(3, n)$, the closure of $\alpha = (\sigma_1\sigma_2^{-1})^n$. Making the substitutions

$$q = t, \quad z = \frac{t^2}{1+t}, \quad w = \frac{1}{1+t}$$

leads to the one-variable Jones polynomial

$$V_{W(3,n)}(t) = \frac{t(1+t)^2}{t^2} \cdot t^{-n} \cdot \left((C_{n,0}(t)+(C_{n,1}(t)+C_{n,2}(t))\cdot \frac{t^2}{1+t}+(C_{n,12}(t)+C_{n,21}(t))\cdot \frac{t^4}{(1+t)^2}\right)$$

$$= t^{-n-1} \cdot \left((1+t)^2 \cdot C_{n,0}(t)+(1+t) \cdot (C_{n,1}(t)+C_{n,2}(t)) \cdot t^2+(C_{n,12}(t)+C_{n,21}(t)) \cdot t^4\right).$$
Example 4.2. For $W(3,1)$, which is the unknot, we have

$$V_{W(3,1)}(t) = t^{-2} \cdot ((t+1)^2 \cdot C_{1,0}(t) + (1+t) \cdot (C_{1,1}(t) + C_{1,2}(t)) \cdot t^2 + (C_{1,12}(t) + C_{1,21}(t)) \cdot t^4)$$

$$= t^{-2} \cdot ((t+1)^2 \cdot 0 + (1+t) \cdot (-t + 1 + 0) \cdot t^2 + (1 + 0) \cdot t^4)$$

$$= t^{-2} \cdot ((1-t^2)t^2 + t^4) = 1.$$

Example 4.3. For $W(3,2)$, which is the figure-8 knot, we have

$$V_{W(3,2)}(t) = t^{-3} \cdot ((t+1)^2 \cdot C_{2,0}(t) + (1+t) \cdot (C_{2,1}(t) + C_{2,2}(t)) \cdot t^2 + (C_{2,12}(t) + C_{2,21}(t)) \cdot t^4)$$

$$= t^{-3} \cdot ((t+1)^2 \cdot (t-1)^2 + (1+t) \cdot ((t-1)^3 - t(t-1)) \cdot t^2 + (-t^2 + t) \cdot t^4)$$

$$= t^{-3} \cdot (t^5 - t^4 + t^3 - t^2 + t) = t^2 - t + 1 - t^{-1} + t^{-2}$$

Now we take a closer look at the formal expression

$$V_{W(3,n)}(t) =$$

$$t^{-n-1} \cdot ((t+1)^2 \cdot C_{n,0}(t) + (1+t) \cdot (C_{n,1}(t) + C_{n,2}(t)) \cdot t^2 + (C_{n,12}(t) + C_{n,21}(t)) \cdot t^4)$$

for the Jones polynomial of the weaving knot $W(3,n)$.

Proposition 4.4. We have a uniform bound on the degrees of the polynomials $C_{n,*}$. Namely,

$$\deg(C_{n,*}) \leq 2n - 1,$$

and the sharper bounds

$$\deg(C_{n,2}) \leq 2n - 2, \quad \deg(C_{n,12}) \leq 2n - 2, \quad \text{and} \quad \deg(C_{n,21}) \leq 2n - 3. \quad (4.1)$$

Proof. These are easy arguments by induction, using either the formulas $(3.5)$ or the formulas in example $3.3$ to start the inductions. Use the recursion formulas $(3.6)$ through $(3.10)$ along with the fact that $C_{n,121} = 0$, proved in proposition $3.4$ for the inductive step. We have

$$\deg(C_{n,0}) \leq \max\{\deg(C_{n+1,21}) + 2, \deg(C_{n+1,1}) + 2\} \leq \max\{(2n-5) + 2, (2n-3) + 2\} = 2n - 1; \quad (4.2)$$

$$\deg(C_{n,1}) \leq \max\{\deg(C_{n+1,1}) + 2, \deg(C_{n+1,0}) + 1\} \leq \max\{(2n-3) + 2, (2n-1) + 1\} = 2n - 1; \quad (4.3)$$

$$\deg(C_{n,2}) = \deg(C_{n+1,1}) + 1 \leq (2n-3) + 1 = 2n - 2; \quad (4.4)$$

$$\deg(C_{n,12}) \leq \max\{\deg(C_{n+1,1}) + 1, \deg(C_{n+1,0})\} \leq \max\{(2n-3) + 1, 2n - 3\} = 2n - 2; \quad (4.5)$$

$$\deg(C_{n,21}) \leq \max\{\deg(C_{n+1,2}) + 1, \deg(C_{n+1,12}) + 1, \deg(C_{n+1,21}) + 2\} \leq \max\{(2n-4) + 1, (2n-4) + 1, (2n-5) + 2\} = 2n - 3. \quad (4.6)$$
Accordingly, set

\[ C_{n,0}(q) = \sum_{i=0}^{2n-1} c_{n,0,i} q^i; \quad C_{n,1}(q) = \sum_{i=0}^{2n-1} c_{n,1,i} q^i; \quad C_{n,2}(q) = \sum_{i=0}^{2n-2} c_{n,2,i} q^i; \]

\[ C_{n,12}(q) = \sum_{i=0}^{2n-2} c_{n,12,i} q^i, \quad \text{and} \quad C_{n,21}(q) = \sum_{i=0}^{2n-3} c_{n,21,i} q^i. \]

**Lemma 4.5.** In the polynomial \( C_{n,0}(q) \), the constant term \( c_{n,0,0} = 0 \) for all \( n \geq 1 \), and the degree one coefficient \( c_{n,0,1} = (-1)^{n-2} \) for \( n \geq 2 \).

**Proof.** The first polynomial \( C_{1,0}(q) = 0 \), and setting \( q = 0 \) in the recurrence relation (3.6) immediately yields

\[ c_{n,0,0} = C_{n,0}(0) = 0. \]

Differentiate the recursion relation (3.6) with respect to \( q \), obtaining

\[ C'_{n,0}(q) = (2q \cdot C_{n-1,21}(q) + q^2 \cdot C'_{n-1,21}(q)) - ((2q-1) \cdot C_{n-1,1}(q) + q(q-1) \cdot C'_{n-1,1}(q)). \]

Substituting \( q = 0 \) yields immediately \( c_{n,0,1} = C'_{n,0}(0) = C_{n-1,1}(0) = c_{n-1,1,0} \). We have

\[ C_{n,1}(q) = -(q-1)^2 \cdot C_{n-1,1}(q) - (q-1) \cdot C_{n-1,0}(q), \]

simplifying relation (3.7) using proposition 3.4 to get \( C_{n-1,121}(q) = 0 \). Now we prove \( c_{n,1,0} = (-1)^{n-1} \) for all \( n \geq 1 \). We have \( C_{1,1}(q) = -(q-1) \), so \( c_{1,1,0} = 1 \) as claimed. Substituting \( q = 0 \) and using \( c_{n,0,0} = 0 \) we get

\[ c_{n,1,0} = C_{n,1}(0) = -(1)^2 \cdot C_{n-1,1}(0) - (1) \cdot C_{n-1,0}(0) = -c_{n-1,1,0} + 0 = -(1)^{n-2} = (-1)^{n-1} \]

Thus \( c_{n,0,1} = c_{n-1,1,0} = (-1)^{n-2} \), as claimed.

Thus, we improve the expression for \( C_{n,0}(q) \) slightly, obtaining \( C_{n,0}(q) = \sum_{i=1}^{2n-1} c_{n,0,i} q^i \). Now we examine

\[ V_{W(3n)}(t) = t^{n-1} \cdot \left( (1+t)^2 \cdot C_{n,0}(t) + (t^2+t^3) \cdot (C_{n,1}(t) + C_{n,2}(t)) + t^4 \cdot (C_{n,12}(t) + C_{n,21}(t)) \right) \]

\[ = t^{n-1} \cdot \left( (1+t)^2 \cdot \left( \sum_{i=1}^{2n-1} c_{n,0,i} t^i \right) \right. \]

\[ + \left. (t^2+t^3) \cdot \left( \sum_{i=0}^{2n-1} c_{n,1,i} t^i + \sum_{i=0}^{2n-2} c_{n,2,i} t^i \right) \right. \]

\[ + \left. t^4 \cdot \left( \sum_{i=0}^{2n-2} c_{n,12,i} t^i + \sum_{i=0}^{2n-3} c_{n,21,i} t^i \right) \right). \]

In the expression for \( V_{W(3n)}(t) \) the highest degree term is apparently

\[ t^{n-1} \cdot (c_{n,1,2n-1} \cdot t^{2n+2} + c_{n,12,2n-2} \cdot t^{2n+2}), \]

but we claim this term is actually zero.
Proof. Using $\deg(C_{n-1,1}) \leq 2n - 3$, $\deg(C_{n-1,0}) \leq 2n - 3$, the fact that $C_{n,121} = 0$, and simplifying the recursion formulas (3.7) and (3.9), respectively, to

$$C_{n,1}(q) = -(q-1)^2 \cdot C_{n-1,1}(q) - (q-1) \cdot C_{n-1,0}(q)$$

and

$$C_{n,12}(q) = (q-1) \cdot C_{n-1,1}(q) + C_{n-1,0}(q),$$

we compute

$$C_{n,1,2n-1} + C_{n,12,2n-2} = -C_{n-1,1,2n-3} + C_{n-1,1,2n-3} = 0.$$  

So the degree of the highest term in $V_W(3,n)$ is no more than the degree of $t^{-n-1} \cdot t^{2n+1}$ which is $n$. Lemma 4.5 shows that the term of lowest degree is $\pm t^{-n-1} \cdot t = \pm t^{-n}$, so the span of the Jones polynomial $V_W(3,n)$ is no more than $2n$. Of course, Kauffman [3, Theorem 2.10] has proved that the span of the polynomial is, in fact, precisely $2n$.

5 From the Jones Polynomial to Khovanov homology

In this section we amplify Theorem 2.5 at least the first part of it.

Theorem 2.5. For a weaving knot $W(2k+1,n)$ the non-vanishing Khovanov homology $\mathcal{H}^{i,j}(W(2k+1,n))$ lies on the lines

$$j = 2i \pm 1.$$  

For a weaving knot $W(2k,n)$ the non-vanishing Khovanov homology $\mathcal{H}^{i,j}(W(2k,n))$ lies on the lines

$$j = 2i + n - 1 \pm 1.$$  

We have the following definition of the bi-graded Euler characteristic associated to Khovanov homology.

$$Kh(L)(t, Q) \overset{\text{def}}{=} \sum i Q^i \dim \mathcal{H}^{i,j}(L)$$

Theorem 5.1 (Theorem 1.1, [4]). For an oriented link $L$, the graded Euler characteristic

$$\sum_{i,j \in \mathbb{Z}} (-1)^i Q^j \dim \mathcal{H}^{i,j}(L)$$

of the Khovanov invariant $\mathcal{H}(L)$ is equal to $(Q^{-1} + Q)$ times the Jones polynomial $V_L(Q^2)$ of $L$.

In terms of the associated polynomial $Kh(L)$,

$$Kh(L)(-1, Q) = (Q^{-1} + Q)V_L(Q^2). \quad (5.1)$$

Theorem 5.2 (Compare Theorem 1.4 and subsequent remarks, [4]). For an alternating knot $L$, its Khovanov invariants $\mathcal{H}^{i,j}(L)$ of degree difference $(1,4)$ are paired except in the 0th cohomology group.
This fact may be expressed in terms of the polynomial $Kh(L)$, as follows. There is another polynomial $Kh'(L)$ in one variable and an equality

$$Kh(L)(t, Q) = Q^{-\sigma(L)} \{ (Q^{-1} + Q) + (Q^{-1} + tQ^2 \cdot Q) \cdot Kh'(L)(tQ^2) \} \tag{5.2}$$

When we combine theorems \ref{thm:5.1} and \ref{thm:5.2} we find that the bi-graded Euler characteristic and the Jones polynomial of an alternating link determine one another. Obviously, the equality \ref{eq:5.1} shows that one knows $V_L$ if one knows $Kh(t, Q)$.

To obtain $Kh(t, Q)$ from $V_L(Q^2)$ requires a certain amount of manipulation. Implementing these manipulations in Maple and Mathematica is an important step in our experiments. Setting $t = -1$ in \ref{eq:5.2} and combining with equation \ref{eq:5.1}, one has

$$ (Q^{-1} + Q) \cdot V_L(Q^2) = Q^{-\sigma(L)} \{ (Q^{-1} + Q) + (Q^{-1} - Q^3) \cdot Kh'(L)(-Q^2) \}. $$

Consequently,

$$ V_L(Q^2) = Q^{-\sigma(L)} \left\{ 1 + \frac{(Q^{-1} - Q^3)}{(Q^{-1} + Q)} \cdot Kh'(L)(-Q^2) \right\} $$

$$ = Q^{-\sigma(L)} \left\{ 1 + (1 - Q^2) \cdot Kh'(L)(-Q^2) \right\}. $$

Furthermore,

$$ Q^{\sigma(L)} \cdot V_L(Q^2) - 1 = (1 - Q^2) \cdot Kh'(L)(-Q^2), $$

or

$$ Kh'(L)(-Q^2) = (1 - Q^2)^{-1} \cdot (Q^{\sigma(L)} \cdot V_L(Q^2) - 1). $$

Replacing $Q^2$ in the last equation by $-tQ^2$ is the last step to obtain $Kh'(L)$ from the Jones polynomial. Within a computer algebra system, one must first replace $Q^2$ by $-X$ and then replace $X$ by $tQ^2$. Once one has $Kh'(L)(tQ^2)$, one obtains $Kh(t, Q)$ directly from equation \ref{eq:5.2}.

**Example 5.3.** We have computed $V_{W(3,2)}(t) = t^{-2} - t^{-1} + 1 - t + t^2$ in example \ref{ex:4.3} so

$$Kh'(W(3, 2))(-Q^2) = (1 - Q^2)^{-1} \cdot (Q^0 \cdot (Q^{-4} - Q^{-2} - Q^2 + Q^4))$$

$$ = (1 - Q^2)^{-1} \cdot ((1 - Q^2) \cdot (Q^{-4} - Q^2))$$

$$ = Q^{-4} - Q^2.$$ 

It follows that $Kh'(W(3, 2))(tQ^2) = t^{-2}Q^{-4} + tQ^2$, and

$$Kh(W(3, 2))(t, Q) = (Q + Q^{-1}) + (Q^{-1} + tQ^2)(t^{-2}Q^{-4} + tQ^2)$$

$$ = t^{-2}Q^{-5} + t^{-1}Q^{-1} + Q^{-1} + Q + tQ + t^2Q^5.$$
6 Khovanov homology examples

Once one has the Khovanov polynomial one can make a plot of the Khovanov homology in an \((i, j)\)-plane as in this example. The Betti number \(\dim KH^{i,j}(W(3,11))\) is plotted at the point with coordinates \((i, j)\). Clearly, as \(n\) gets larger, it is going to be harder to make sense of such plots. Notice that the \((1, 4)\)-periodicity of the Khovanov homology for these knots makes the information on one of the lines \(j - 2i = \pm 1\) redundant.

Taking advantage of this feature, we simplify by recording the Betti numbers along the line \(j - 2i = 1\). In order to study the asymptotic behavior of Khovanov homology we have to normalize the data. This is done by computing the total rank of the Khovanov homology along the line and dividing each Betti number by the total rank. We obtain normalized Betti numbers that sum to one.

This raises the possibility of approximating the distribution of normalized Betti numbers by a probability distribution. For our baseline experiments we choose to use the normal \(N(\mu, \sigma^2)\) probability density function

\[
f_{\mu,\sigma^2}(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]

Fit a quadratic function \(q_n(x) = -(\alpha x^2 - \beta x + \delta)\) to the logarithms of the normalized Khovanov dimensions along the line \(j = 2i + 1\) and exponentiate the quadratic function. Since the total of the normalized dimensions is 1, we normalize the exponential, obtaining

\[
\rho_n(x) = A_n e^{q_n(x)} \quad \text{satisfying} \quad \int_{-\infty}^{\infty} \rho_n(x) \, dx = 1.
\]

To obtain a formula for \(A_n\), complete the square

\[
q_n(x) = -\alpha \cdot (x - (\beta/2\alpha))^2 + ((\beta^2/4\alpha) - \delta).
\]
Then consider

\[ 1 = A_n \int_{-\infty}^{\infty} \exp q_n(x) \, dx \]
\[ = A_n \cdot \int_{-\infty}^{\infty} \exp((\beta^2/4\alpha) - \delta) \cdot \exp(-\alpha \cdot (x - (\beta/2\alpha))^2) \, dx \]
\[ = A_n \cdot ((\beta^2/(4\alpha) - \delta) \cdot \int_{-\infty}^{\infty} \exp(-\alpha \cdot (x - (\beta/2\alpha))^2) \, dx \]
\[ = A_n \cdot ((\beta^2/(4\alpha) - \delta) \cdot \sqrt{\pi/\alpha} \]

Thus, the expression for \( A_n \) is

\[ A_n = \exp -((\beta^2/4\alpha) - \delta) \cdot \sqrt{\alpha/\pi}. \]

Equating the expressions

\[ \rho_n(x) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_n)^2}{2\sigma_n^2}\right) \quad \text{and} \quad \rho_n(x) = A_n \exp(q_n(x)) \]

\( \mu_n = \beta/2\alpha \) and the efficient way to the parameter \( \sigma_n \) is by solving the equation

\[ \frac{1}{\sigma_n \sqrt{2\pi}} = \rho_n(\frac{\beta}{2\alpha}) = A_n \exp(q_n(\beta/2\alpha)) = \exp\left(-\left(\frac{\beta^2}{4\alpha} - \delta\right) \cdot \sqrt{\frac{\alpha}{\pi}} \exp((\beta^2/4\alpha) - \delta) \right) \]

obtaining \( \sigma_n = 1/\sqrt{2\alpha} \).

Working this out for \( W(3, 10) \), and carrying only 3 decimal places, the raw dimensions are

| \( i \) | -9 | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
| dim | 1 | 9 | 36 | 94 | 196 | 346 | 529 | 721 | 879 | 970 |

| \( i \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| dim | 971 | 879 | 721 | 529 | 346 | 196 | 94 | 36 | 9 | 1 |

and, to three significant digits, the logarithms of the normalized dimensions are

| \( i \) | -9 | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
| -17.9 | -15.7 | -14.3 | -13.3 | -12.6 | -12.0 | -11.6 | -11.3 | -11.1 | -11.0 |
| \( i \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| -11.0 | -11.1 | -11.3 | -12.0 | -12.6 | -13.3 | -14.3 | -15.7 | -17.9 |

Fitting a quadratic to this information, we get

\[ q_{10}(x) = -10.7 + 0.0720 \, x - 0.0720 \, x^2, \quad \alpha = \beta = 0.0720, \quad \delta = 10.7. \]

To three significant digits \( \mu_{10} = 0.500 \) and \( \sigma_{10} = 2.64 \).
By the symmetry of Khovanov homology, the mean $\mu_n$ approaches $1/2$ rapidly, so this parameter is of little interest. On the other hand, relating the parameter $\sigma_n$ to some geometric quantity, say, some hyperbolic invariant of the complement of the link, is a very interesting problem.

For $W(3, 10)$, the density function is

$$\rho_{10}(x) = 11686.8431618280538 \sqrt{\pi} \left( e^{-10.7018780565714309 + 0.0716848579220777243 x - 0.0716848579220778631 x^2} \right)$$

When placed into standard form, $\mu_{10} = 0.5000054030$ and $\sigma_{10} = 2.640882970$. Here is the comparison plot.

![Figure 8: normalized homology of $W(3, 10)$ compared with density function](image)

For the knot $W(3, 11)$ the expression for the density function is

$$\rho_{11}(x) = 29676.8676257830375 \sqrt{\pi} \left( e^{-11.6724860231789886 + 0.0661625395821569817 x - 0.0661623073574252735 x^2} \right)$$

When placed into standard form, $\mu_{11} = 0.5000017550$ and $\sigma_{11} = 2.749031276$. Here is the comparison plot.

![Figure 9: normalized homology of $W(3, 11)$ compared with density function](image)
For $W(3, 22)$, the density function is

$$\rho_{22}(x) = 833596689.149608016 \sqrt{\pi} e^{-22.2219365040983057 + 0.0353061029354434300 x - 0.0353061029347388616 x^2}$$

When placed into standard form, $\mu_{22} = 0.5000000000$ and $\sigma_{22} = 3.763224354$. Here is the comparison plot.

![Figure 10: normalized homology of $W(3, 22)$ compared with density function](image)

For $W(3, 23)$, the density function is

$$\rho_{23}(x) = 2113964949.23002362 \sqrt{\pi^{-1}} e^{-23.1731352596503442 + 0.0338545815354610105 x - 0.0338545815348441914 x^2}$$

When placed into standard form, $\mu_{23} = 0.5000000000$ and $\sigma_{23} = 3.843052143$. Here is the comparison plot.

![Figure 11: normalized homology of $W(3, 23)$ compared with density function](image)

Maple worksheets and, later, Mathematica notebooks will be available at URL prepared by the second-named author.
7 Data Tables

This section contains tables of data generated using *Maple* to implement some of the results of earlier sections. The first table collects data for weaving knots $W(3, n)$ for $n \equiv 1 \mod 3$. The first column lists the dimension; the second column lists the total dimension of the Khovanov homology lying along the line $j = 2i + 1$; and the third column lists the dimension of the vector space $\mathcal{H}^{0,1}(W(3, n))$. In section 6 we approximate the distribution of normalized Khovanov dimensions by a standard normal distribution, and we have displayed graphics comparing the actual distribution with the approximation.

To quantify those visual impressions, we compute two total deviations. Let

$$\text{Total dimension} = \sum_{i=-n}^{2n+1} \dim \mathcal{H}^{i,2i+1}(W(3, n)).$$

For the $L^2$-comparison, we compute

$$\left( \sum_{i=-n}^{2n+1} \left( \rho_n(i) - \frac{\dim \mathcal{H}^{i,2i+1}(W(3, n))}{\text{Total dimension}} \right)^2 \right)^{1/2}$$

For the $L^1$-comparison, we compute

$$\sum_{i=-n}^{2n+1} \left| \rho_n(i) - \frac{\dim \mathcal{H}^{i,2i+1}(W(3, n))}{\text{Total dimension}} \right|$$

The $L^2$ comparisons appear to tend to 0, whereas the $L^1$ comparisons appear to be growing slowly.
| $n$ | Total dimension | $\dim \mathcal{H}^{0,1}$ | $\sigma$ | $L^2$-comparison | $L^1$ comparison |
|-----|----------------|-----------------|---------|------------------|------------------|
| 10  | 7563           | 970             | 2.64088 | 0.040509         | 0.134828         |
| 13  | 135721         | 15418           | 2.95616 | 0.041329         | 0.150599         |
| 16  | 2435423        | 250828          | 3.24564 | 0.040792         | 0.155995         |
| 19  | 43701901       | 4146351         | 3.51395 | 0.040145         | 0.161336         |
| 22  | 784198803      | 69337015        | 3.76322 | 0.039413         | 0.165763         |
| 25  | 14071876561    | 1169613435      | 3.99810 | 0.038678         | 0.167576         |
| 28  | 252509579303   | 19864129051     | 4.22032 | 0.037971         | 0.167790         |
| 31  | 4531100550901  | 339205938364    | 4.43167 | 0.037306         | 0.170736         |
| 34  | 81307300336923 | 5818326037345   | 4.63358 | 0.036675         | 0.172391         |
| 37  | 1459000305513721 | 100173472277125 | 4.82378 | 0.036089         | 0.173119         |
| 40  | 26180698198910063 | 173013573194046 | 5.01342 | 0.035541         | 0.173178         |
| 43  | 46979356724867421 | 2996302681609060 | 5.19305 | 0.035027         | 0.173811         |
| 46  | 84301031248703523 | 520131503664409798 | 5.36671 | 0.034547         | 0.175059         |
| 49  | 1.51272 \cdot 10^{20} | 9.04765 \cdot 10^{18} | 5.53502 | 0.0340935       | 0.175779         |
| 52  | 2.71447 \cdot 10^{21} | 1.57670 \cdot 10^{20} | 5.69838 | 0.033667         | 0.176100         |
| 55  | 4.87091 \cdot 10^{22} | 2.75210 \cdot 10^{21} | 5.85721 | 0.033265         | 0.176098         |
| 58  | 8.74050 \cdot 10^{23} | 4.81071 \cdot 10^{22} | 6.01187 | 0.032885         | 0.175898         |
| 61  | 1.56842 \cdot 10^{25} | 8.42017 \cdot 10^{23} | 6.16267 | 0.032524         | 0.176778         |
| 64  | 2.81441 \cdot 10^{26} | 1.47552 \cdot 10^{25} | 6.30989 | 0.032182         | 0.177369         |
| 67  | 5.05026 \cdot 10^{27} | 2.58843 \cdot 10^{26} | 6.45376 | 0.031857         | 0.177716         |
| 70  | 9.06233 \cdot 10^{28} | 4.54520 \cdot 10^{27} | 6.59451 | 0.031547         | 0.177859         |
| 73  | 1.62617 \cdot 10^{30} | 7.98842 \cdot 10^{28} | 6.73233 | 0.031251         | 0.177831         |
| 76  | 2.91804 \cdot 10^{31} | 1.40517 \cdot 10^{30} | 6.86740 | 0.030968         | 0.176657         |
| 79  | 5.23621 \cdot 10^{32} | 2.47359 \cdot 10^{31} | 6.99986 | 0.030697         | 0.177995         |
| 82  | 9.39600 \cdot 10^{33} | 4.35747 \cdot 10^{32} | 7.12988 | 0.030437         | 0.178445         |
| 85  | 1.68604 \cdot 10^{35} | 7.68116 \cdot 10^{33} | 7.25757 | 0.030188         | 0.178746         |
| 88  | 3.02548 \cdot 10^{36} | 1.35483 \cdot 10^{35} | 7.38305 | 0.029948         | 0.178918         |
| 91  | 5.42901 \cdot 10^{37} | 2.39106 \cdot 10^{36} | 7.50645 | 0.029718         | 0.178976         |
| 94  | 9.74196 \cdot 10^{38} | 4.22211 \cdot 10^{37} | 7.62786 | 0.029496         | 0.178935         |
| 97  | 1.74812 \cdot 10^{40} | 7.45910 \cdot 10^{38} | 7.74736 | 0.029282         | 0.178807         |
| 100 | 3.13688 \cdot 10^{41} | 1.31840 \cdot 10^{40} | 7.86506 | 0.029075         | 0.178890         |
| 121 | 1.87923 \cdot 10^{50} | 7.18477 \cdot 10^{48} | 8.64424 | 0.027805         | 0.179577         |
| 142 | 1.12580 \cdot 10^{59} | 3.97500 \cdot 10^{57} | 9.35886 | 0.026769         | 0.180247         |
| 163 | 6.74436 \cdot 10^{67} | 2.22337 \cdot 10^{66} | 10.0227 | 0.025900         | 0.180596         |
| 184 | 4.04037 \cdot 10^{76} | 1.25398 \cdot 10^{75} | 10.6453 | 0.025156         | 0.180629         |
| 205 | 2.42049 \cdot 10^{85} | 7.11854 \cdot 10^{83} | 11.2334 | 0.024508         | 0.180907         |
| 247 | 8.68689 \cdot 10^{102} | 2.32816 \cdot 10^{101} | 12.3258 | 0.023423         | 0.181027         |
| 289 | 3.11764 \cdot 10^{120} | 7.72623 \cdot 10^{118} | 13.3289 | 0.022542         | 0.181268         |
Table 2: Data for $W(3, n)$ with $n \equiv 2 \mod 3$

| $n$ | Total dimension | $\dim \mathcal{A}^{0,1}$ | $\sigma$ | $L^2$-comparison | $L^1$ comparison |
|-----|-----------------|--------------------------|---------|---------------------|------------------|
| 11  | 19801           | 2431                     | 2.74903 | 0.040906            | 0.141925         |
| 14  | 355323          | 38983                    | 3.05533 | 0.041079            | 0.153170         |
| 17  | 6376021         | 637993                   | 3.33710 | 0.040595            | 0.156595         |
| 20  | 114413063       | 10591254                 | 3.59850 | 0.039905            | 0.163190         |
| 23  | 2053059121      | 177671734                | 3.84035 | 0.039166            | 0.166596         |
| 26  | 36840651123     | 3004390818               | 4.07348 | 0.038438            | 0.167789         |
| 29  | 661078661101    | 51124396786              | 4.29190 | 0.037744            | 0.168941         |
| 32  | 11862575248703  | 874400336044             | 4.49997 | 0.037089            | 0.171411         |
| 35  | 21286527815561  | 15018149469823           | 4.69899 | 0.036476            | 0.172723         |
| 38  | 3819712389431403| 258853011125599          | 4.89004 | 0.035903            | 0.173203         |
| 41  | 6854195773949701| 4474997964407374         | 5.07400 | 0.035366            | 0.173083         |
| 44  | 1229935526821663223| 77563025486587315      | 5.25158 | 0.034864            | 0.174290         |
| 47  | 22070297525055988321| 1347390412214087833    | 5.42341 | 0.034392            | 0.175346         |
| 50  | 3.96035 \cdot 10^{20} | 2.34525 \cdot 10^{19}      | 5.59000 | 0.033949            | 0.175926         |
| 53  | 7.10657 \cdot 10^{21} | 4.08927 \cdot 10^{20}      | 5.75181 | 0.033531            | 0.176131         |
| 56  | 1.27522 \cdot 10^{23} | 7.14133 \cdot 10^{21}       | 5.90921 | 0.033136            | 0.176037         |
| 59  | 2.28829 \cdot 10^{24} | 1.24888 \cdot 10^{23}       | 6.06255 | 0.032763            | 0.176227         |
| 62  | 4.10617 \cdot 10^{25} | 2.18679 \cdot 10^{24}       | 6.21213 | 0.032408            | 0.177005         |
| 65  | 7.36823 \cdot 10^{26} | 3.83347 \cdot 10^{25}       | 6.35821 | 0.032072            | 0.177510         |
| 68  | 1.32218 \cdot 10^{28} | 6.72713 \cdot 10^{26}       | 6.50102 | 0.031752            | 0.177785         |
| 71  | 2.37255 \cdot 10^{29} | 1.18163 \cdot 10^{28}       | 6.64077 | 0.031446            | 0.177867         |
| 74  | 4.25736 \cdot 10^{30} | 2.07736 \cdot 10^{29}       | 6.77765 | 0.031555            | 0.177787         |
| 77  | 7.63953 \cdot 10^{31} | 3.65504 \cdot 10^{30}       | 6.91183 | 0.030876            | 0.177602         |
| 80  | 1.37086 \cdot 10^{33} | 6.43571 \cdot 10^{31}       | 7.04347 | 0.030609            | 0.178163         |
| 83  | 2.45990 \cdot 10^{34} | 1.13397 \cdot 10^{33}       | 7.17269 | 0.030353            | 0.178561         |
| 86  | 4.41412 \cdot 10^{35} | 1.99933 \cdot 10^{34}       | 7.29963 | 0.030107            | 0.178817         |
| 89  | 7.92082 \cdot 10^{36} | 3.52717 \cdot 10^{35}       | 7.42441 | 0.029871            | 0.178949         |
| 92  | 1.42133 \cdot 10^{38} | 6.22605 \cdot 10^{37}       | 7.54714 | 0.029643            | 0.178972         |
| 95  | 2.55048 \cdot 10^{39} | 1.09958 \cdot 10^{38}       | 7.66790 | 0.029424            | 0.178901         |
| 98  | 4.57665 \cdot 10^{40} | 1.94290 \cdot 10^{39}       | 7.78679 | 0.029212            | 0.178747         |
| 119 | 2.74175 \cdot 10^{49} | 1.05696 \cdot 10^{48}       | 8.57308 | 0.027914            | 0.175650         |
| 140 | 1.64251 \cdot 10^{58} | 5.84051 \cdot 10^{56}       | 9.29316 | 0.026859            | 0.180257         |
| 161 | 9.83989 \cdot 10^{66} | 3.26385 \cdot 10^{65}       | 9.96138 | 0.025977            | 0.180552         |
| 182 | 5.89483 \cdot 10^{75} | 1.83951 \cdot 10^{74}       | 10.5875 | 0.025223            | 0.180539         |
| 203 | 3.53144 \cdot 10^{84} | 1.04367 \cdot 10^{83}       | 11.1787 | 0.024566            | 0.180926         |
| 245 | 1.26740 \cdot 10^{102} | 3.41053 \cdot 10^{100}      | 12.2759 | 0.023469            | 0.181064         |
| 287 | 4.54858 \cdot 10^{119} | 1.13115 \cdot 10^{118}      | 13.2829 | 0.022580            | 0.181221         |
| 329 | 1.63244 \cdot 10^{137} | 3.79224 \cdot 10^{135}      | 14.2187 | 0.021838            | 0.181399         |
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