EXTRINSIC AND CONFORMAL UPPER BOUNDS FOR LOWER NEUMMAN AND STEKLOV EIGENVALUES

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Abstract. Let $M$ be an $m$-dimensional compact Riemannian manifold with boundary. We obtain the upper bound of the harmonic mean of the first $m$ nonzero Neumann eigenvalue and Steklov eigenvalues involving the conformal volume and relative conformal volume, respectively. We also give an optimal sharp extrinsic upper bound for closed submanifolds in the space forms. These extend the previous related results for the first nonzero eigenvalue.

1. Introduction

The study of eigenvalues on manifolds is an important topic in Riemannian geometry, and there have been lots of results, including various estimates bounded from below or above in different situations. In this paper, we study the upper bounds of the harmonic mean of the lower Neumann and Steklov eigenvalues.

1.1. Extrinsic upper bounds in space forms. Let $(M, g)$ be an $m$-dimensional compact Riemannian manifold. The eigenvalues of Laplacian on $M$ (with Neumann boundary condition if $\partial M \neq \emptyset$) are discrete and satisfies

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \rightarrow +\infty.$$ 

Let $(\mathbb{R}^n(c), h_c)$ be the $n$-dimensional simply-connected space form of constant curvature $c$ equality with the canonical metric $h_c$, namely, $\mathbb{R}^n(c)$ represents the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n(-1)$ and the unit sphere $\mathbb{S}^n(1)$ for $c = 0, -1$ and 1, respectively. When $M$ is a closed submanifold of $\mathbb{R}^n(c)$, the first nonzero eigenvalue can be bounded by the mean curvature $H$ from above.

**Theorem 1.1** ([11, 32]). Let $M$ be an $m(\geq 2)$-dimensional closed submanifold in $\mathbb{R}^n(c)$. Then the first nonzero eigenvalue of Laplacian satisfies

$$\lambda_1 \leq \frac{m}{V(M)} \int_M (c + |H|^2). \quad (1.1)$$

If we exclude the case that $M$ is minimal in $\mathbb{S}^n(1)$ for $c = 1$, then the equality holds if and only if $M$ is minimally immersed in a geodesic sphere of $\mathbb{R}^n(c)$.

Theorem 1.1 is usually referred “Reilly inequality” as it was first proved by Reilly [32] for $c = 0$ by using the coordinate functions as the test functions after translating the center of mass of $M$ to the origin. By embedding the sphere $\mathbb{S}^n(1) \rightarrow \mathbb{R}^{n+1}$, one can reduce the case $c = 1$ to the case $c = 0$. But this method fails for $c = -1$. The case $c = -1$ was proved by El Soufi and Ilias [11], where they chose the test functions by using a technique of conformal maps.

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There are lots of generalizations for Reilly inequality. For example, one can consider the case that the ambient space is a manifold whose sectional curvature bounded above by a constant, even can have a weighted measure; one can consider other operators, such as so-called $L_T$ operator and $p$-Laplacian, instead of the Laplacian. The readers can refer [4, 17, 25, 30, 33] and the references therein.

Now we are still in the situations of Theorem 1.1 but consider the harmonic mean of the first $m$ nonzero eigenvalues. We write the harmonic mean of $k$ positive numbers $a_1, \cdots, a_k$ as

$$H(a_1, \cdots, a_k) = \left(\frac{a_1^{-1} + \cdots + a_k^{-1}}{k}\right)^{-1}. \quad (1.2)$$

If we assume $a_1 \leq \cdots \leq a_k$, then one can check that

$$a_1 = H(a_1) \leq H(a_1, a_2) \leq \cdots \leq H(a_1, \cdots, a_{k-1}) \leq H(a_1, \cdots, a_k). \quad (1.3)$$

The first main result in this paper is the following sharp extrinsic estimate.

**Theorem 1.2.** Let $M$ be an $m(\geq 2)$-dimensional closed submanifold in $\mathbb{R}^n(c)$. Then the first $m$ nonzero eigenvalues of the Laplacian satisfy:

$$H(\lambda_1, \cdots, \lambda_m) \leq \frac{m}{V(M)} \int_M \left(c + |H|^2\right). \quad (1.4)$$

Moreover, equality in (1.4) holds if and only if either

1. $M$ is a minimal in $\mathbb{S}^n(1)$ (this occurs only when $c = 1$), or

2. $M$ is minimal in a geodesic sphere $\Sigma_c$ of $\mathbb{R}^n(c)$, where the geodesic radius $r_c$ of $\Sigma_c$ is given by

$$r_0 = \left(\frac{m}{\lambda_1}\right)^{1/2}, \quad r_1 = \arcsin r_0, \quad r_1 = \arcsinh r_0.$$

Obviously, Theorem 1.2 is stronger than Theorem 1.1 due to (1.3). It is proved via conformal maps. This technique also allows us to obtain some conformal upper bounds for compact manifolds with boundary.

### 1.2. Conformal upper bounds for Neumann eigenvalues.

In 1982, Li and Yau [24] introduced the concept of the conformal volume in order to study the Willmore conjecture.

**Definition 1.3** (cf. [24]). Let $M$ be an $m$-dimensional compact Riemannian manifold with boundary that admits a conformal map $\phi: M \to \mathbb{S}^n$. Define

$$V_c(M, n, \phi) := \sup_{\gamma \in G} V(M, (\gamma \circ \phi)^* h_1), \quad (1.5)$$

$$V_c(M, n) := \inf_{\phi} V_c(M, n, \phi), \quad (1.6)$$

$$V_c(M) := \lim_{n \to +\infty} V_c(M, n). \quad (1.7)$$

Here $G$ is the group of conformal diffeomorphisms of $\mathbb{S}^n$, and the infimum is over all non-degenerate conformal maps $\phi: M \to \mathbb{S}^n$.

We call $V_c(M, n, \phi), V_c(M, n)$ and $V_c(M)$ the $n$-conformal volume of $\phi$, the $n$-conformal volume of $M$, and the conformal volume $M$, respectively.

We remark that, the conformal map $\phi$ exists for $n$ big enough because of the Nash embedding theorem (via the stereographic projection). And the limit in (1.7) exists since $V_c(M, n) \geq V_c(M, n + 1)$ (cf. [24, Fact 4]).
We set the following condition.

**(C1)** $(M, g)$ admits, up to a homothety, a minimal isometric immersion in $S^n$ given by a subspace of the first eigenfunctions.

Then we have

**Theorem 1.4** ([10, 24]). *Let $(M, g)$ be an $m(\geq 2)$-dimensional compact Riemannian manifold. Then the first nonzero eigenvalue $\lambda_1$ of the Laplacian (with Neumann boundary condition if $\partial M \neq \emptyset$) satisfies

$$\lambda_1 \leq m^2 \left( \frac{V_c(M, n)}{V(M, g)} \right)^{2/m}$$

for all $n$ for which $V_c(M, n)$ is defined.

Equality implies (C1). If $M$ is closed, (C1) is also sufficient for equality holding.*

This estimate was firstly obtained by Li and Yau [24] for $m = 2$, and then El Soufi and Ilias [10] for general $n \geq 3$. Matei [26] generalized it to the $p$-Laplacian. For higher eigenvalues of a closed manifold, Kokarev [20] very recently proved that there is a constant $C(n, m)$ depending only on $m$ and $n$ such that

$$\lambda_k \leq C(n, m) \left( \frac{V_c(M, n)}{V(M, g)} \right)^{2/m} k^{2/m}$$

holds for any $k \geq 1$, which is compatible with the famous Weyl asymptotic formula:

$$\lambda_k V(M, g)^{2/m} \sim \frac{4\pi^2}{\omega_m^2} k^{2/m} \text{ as } k \to \infty,$$

where $\omega_m$ is the volume of the unit ball in $\mathbb{R}^m$. This can be viewed as an improvement of Korevaar’s result [21], which says that

$$\lambda_k \leq C k^{2/m}$$

for some constant $C$ depending on $[g]$ in a rather implicit way. In the same paper [20], the author also proved a version of the Reilly inequality for higher eigenvalues, that is,

$$\lambda_k \leq C(n, m) \left( \frac{1}{V(M)} \int_M (c + |H|^2) \right) k$$

for a closed submanifold $M^m$ of $\mathbb{R}^n(c)$.

We give the following conformal upper bound of the harmonic mean of the first $m$ nonzero eigenvalues.

**Theorem 1.5.** *Let $M$ be an $m(\geq 2)$-dimensional compact Riemannian manifold. Then the first $m$ nonzero eigenvalues of the Laplacian (with Neumann boundary condition if $\partial M \neq \emptyset$) satisfy

$$H(\lambda_1, \ldots, \lambda_m) \leq m^2 \left( \frac{V_c(M, n)}{V(M, g)} \right)^{2/m}$$

for all $n$ for which $V_c(M, n)$ is defined.*

Equality implies (C1). If $M$ is closed, (C1) is also sufficient for equality holding.

**Remark 1.6.** *For some particular manifolds, such as compact symmetric spaces of rank 1 and the minimal Clifford torus with their canonical metric $\tilde{g}$, we have $V_c(M, n) = V(M, \tilde{g}_{can})$ for $n + 1$ greater or equal to the multiplicity of $\lambda_1$ (cf. [10, pp. 266–267]). In these cases the upper bound is explicit.*

For $m = 2$, the upper bound can be controlled by the topology of the surface.
Corollary 1.7. Let \((M, g)\) be a closed surface. Then
\[
\mathcal{H}(\lambda_1, \lambda_2) \leq \frac{8\pi}{V(M, g)} \left[ \frac{\text{genus}(M) + 3}{2} \right],
\]  
where \([\cdot]\) denotes the integer part. In particular, for a 2-sphere we have
\[
\mathcal{H}(\lambda_1, \lambda_2) \leq \frac{8\pi}{V(S^2, g)},
\]
This corollary is directly obtained from Theorem 1.5 since Li and Yau [24] pointed out that
\[
V_c(M, g) \leq 4\pi(\text{genus}(M) + 1),
\]  
and then El Soufi and Ilias [9] remark that the factor \(\text{genus}(M) + 1\) in (1.15) can be improved to \([\frac{\text{genus}(M) + 3}{2}]\).

Let’s review some previous related results for \(m = 2\). Obviously, Corollary 1.7 implies
\[
\lambda_1 \leq \frac{8\pi}{V(M, g)}(\text{genus}(M) + 1),
\]  
which was originally proved by Hersch [18] for \(M = S^2\) (i.e., \(\text{genus}(M) = 0\)) and Yang-Yau [40] for \(\text{genus}(M) \geq 1\). Actually, Yang-Yau [40] proved a stronger result was proved
\[
\mathcal{H}(\lambda_1, \lambda_2, \lambda_3) \leq \frac{8\pi}{V(M, g)}(\text{genus}(M) + 1).
\]  
By (1.15), Li and Yau [24] gave a simpler proof of (1.16); they also proved that ([24, Corollary 2])
\[
\mathcal{H}(\lambda_1, \cdots, \lambda_k) \leq \frac{V_c(M, n - 1)}{V(M)} \text{ for any } n \geq k,
\]  
but the constant is not sharp.

Another remarkable estimate for the higher eigenvalues of a 2-sphere \((S^2, g)\) is
\[
\lambda_k(S^2, g) \leq \frac{8\pi k}{V(S^2, g)}, \text{ for any } k \geq 1.
\]  
The metric \(g\) is smooth outside a finite number of conical singularities. For \(k = 1\), this is just Hersch’s result [18] and the equality in (1.19) is attained if and only if \(g\) is a round metric. For \(k \geq 2\), the inequality in (1.19) is strict, and the equality can be attained in the limit by a sequence of metrics degenerating to a union of \(k\) touching identical round spheres. This was proved by Nadirashvili [28] (and Petrides [31] with a different argument) for \(k = 2\), Nadirashvili and Sire [29] for \(k = 3\), and Karpukhin et al. [19] for any \(k \geq 2\) very recently. From (1.19) we have
\[
\mathcal{H}(\lambda_1, \lambda_2) \leq \frac{4}{3} \frac{8\pi}{V(S^2, g)},
\]
which is weaker than (1.13).

1.3. Conformal upper bounds for Neumann eigenvalues. Beside Neumann eigenvalues, Steklov eigenvalues are intensively studied for a compact Riemannian manifold \((M, g)\) with nonempty boundary \(\partial M\). A real numbers \(\sigma\) is called a Steklov eigenvalue if there exists a non-zero function \(u\) (which is calle the Steklov eigenfunction corresponding to \(\sigma\)) on \(M\) satisfying
\[
\begin{cases}
\Delta u = 0, & \text{on } M; \\
\frac{\partial u}{\partial v} = \sigma u, & \text{on } \partial M,
\end{cases}
\]

\[
\mathcal{H}(\lambda_1, \lambda_2) \leq \frac{8\pi}{V(S^2, g)},
\]
where \( \nu \) is the outward normal on \( \partial M \).

The Steklov problem has a physical background, and it can be traced back to the turn of the 20th century, when Steklov studied liquid sloshing [34, 35]. The readers can refer [22] for the history and a recent survey [16] for researches in this area. There are also the very recent developments and various results (e.g., [7, 8, 14, 15, 37, 39] and the references therein).

The Steklov eigenvalues can be interpreted as the eigenvalues of the Dirichlet-to-Neumann operator \( L \), which is defined as follows:

\[
 L : C^\infty(\partial M) \to C^\infty(\partial M)
\]

\[
 Lu = \frac{\partial \hat{u}}{\partial \nu},
\]

where \( \hat{u} \) is a function on \( \partial M \) and \( \hat{u} \) is the harmonic extension of \( u \), i.e.,

\[
\begin{aligned}
 \Delta \hat{u} &= 0, \quad \text{on } M; \\
 \hat{u} &= u, \quad \text{on } \partial M.
\end{aligned}
\]

\( L \) is a nonnegative self-adjoint operator with discrete spectrum

\[
0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to +\infty.
\]

Fraser and Schoen [13] gave an upper bound involved the relative conformal volume for the first nonzero Steklov eigenvalue, which is analogous to Theorem 1.4 for the first Neumann eigenvalue of the Laplacian.

**Theorem 1.8** ([13, Theorem 6.2]). Let \( (M, g) \) be a compact \( m \)-dimensional Riemannian manifold with nonempty boundary. Then the first nonzero Steklov eigenvalue \( \sigma_1 \) satisfies

\[
\sigma_1 V(\partial M)V(M)^{\frac{2-m}{m}} \leq m V_{rc}(M, n)^{2/m}
\]

for all \( n \) for which \( V_{rc}(M, n) \) is defined.

Equality implies that there exists a conformal harmonic map \( \phi : M \to B^n \) which (after rescaling the metric \( g \)) is an isometry on \( \partial M \), with \( \phi(\partial M) \subset \partial B^n \) and such that \( \phi(M) \) meets \( \partial B^n \) orthogonally along \( \phi(\partial M) \). For \( m > 2 \), this map is an isometric minimal immersion of \( M \) to its image. Moreover, the immersion is given by a subspace of the first eigenspace.

We explain \( V_{rc}(M, n) \) in the above theorem.

**Definition 1.9** (cf. [13]). Let \( M \) be an \( m \)-dimensional compact Riemannian manifold with boundary that admits a conformal map \( \phi : M \to B^n \) with \( \phi(\partial M) \subset \partial B^n \). Define

\[
V_{rc}(M, n, \phi) := \sup_{\gamma \in G} V(f(\phi(M))),
\]

\[
V_{rc}(M, n) := \inf_\phi V_{rc}(M, n, \phi),
\]

\[
V_{rc}(M) := \lim_{n \to +\infty} V_{rc}(M, n).
\]

Here \( G \) is the group of conformal diffeomorphisms of \( B^n \), and the infimum is over all non-degenerate conformal maps \( \phi : M \to B^n \) with \( \phi(\partial M) \subset \partial B^n \).

We call \( V_{rc}(M, n, \phi), V_{rc}(M, n) \) and \( V_{rc}(M) \) the relative \( n \)-conformal volume of \( \phi \), the relative \( n \)-conformal volume of \( M \), and the relative conformal volume of \( M \), respectively. The existence of the limit in (1.27) is from the monotonicity of \( V_{rc}(M, n) \) (cf. [13, Lemma 5.7]).
We extend Theorem 1.8 to the harmonic mean of the first $m$ nonzero eigenvalues.

**Theorem 1.10.** Let $(M, g)$ be a compact $m$-dimensional Riemannian manifold with nonempty boundary. Then the first $m$ nonzero Steklov eigenvalues satisfy

$$\tilde{f}_n(\sigma_1, \cdots, \sigma_m) V(\partial M) V(M) \frac{2-m}{m} \leq m V_{rc}(M, n)^{2/m} \quad (1.28)$$

for all $n$ for which $V_{rc}(M, n)$ is defined.

Equality implies that there exists a conformal harmonic map $\phi : M \to B^n$ with $\phi(\partial M) \subset \partial B^n$. For $m > 2$, this map (after rescaling the metric $g$) is an isometric minimal immersion of $M$ to its image. Moreover, each coordinate functions of this immersion belongs to some eigenspace $E_{\sigma_i} (1 \leq i \leq m)$ (counted with multiplicity).

This paper is organized as follows. In Section 2 we recall some formulae about conformal maps and submanifolds. In Section 3 we prove the Reilly-type inequality Theorem 1.2, and a general result Theorem 3.6 is obtained. We give the proofs of Theorems 1.5 and 1.10 in Sections 4 and 5, respectively. The choice of test functions is analogous in the proofs, but discussions on equalities are different.

2. Preliminaries and notations

In this section, we give the relations between some geometric quantities of $M^m$ as a submanifold of $(N^n, g_N)$ and their corresponding quantities of $M$ as a submanifold in $(N^n, \tilde{g}_N)$, where $\tilde{g}_N$ is conformal to $g_N$. Although the relations are well-known in the literature (cf. [2, 3]), we give a brief proof of these relations for the reader’s convenience. We also recall some basic formulas for submanifolds in space forms.

We use the following convention on the ranges of indices except special declaration:

$$1 \leq i, j, k, \ldots \leq m; \quad m + 1 \leq \alpha, \beta, \gamma, \ldots \leq n; \quad 1 \leq A, B, C, \ldots \leq n.$$ 

2.1. Conformal relations: the ambient manifold. Let $(N, g_N)$ be an $n$-dimensional $n$-dimensional Riemannian manifold.

Let $\{e_A\}_{i=1}^m$ be an orthonormal frame of $(N, g_N)$, where $\{e_i\}_{i=1}^m$ are tangent to $M^1$ and $\{e_i\}_{i=m+1}^n$ are normal to $M$. Let $\{\omega_A\}_{i=A}^n$ be the dual coframe of $\{e_A\}_{i=1}^n$. Then the structure equations of $(N, g_N)$ are (see [6]):

$$\begin{cases} d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \\
\omega_{AB} + \omega_{BA} = 0,
\end{cases} \quad (2.1)$$

where $\{\omega_{AB}\}$ are the connection forms of $(N, g_N)$.

Let $\tilde{g}_N$ be a metric on $N$ conforming to $g_N$. We assume $\tilde{g}_N = e^{2\rho} g_N$ for some $\rho \in C^\infty(N)$. Then $\{\tilde{e}_A = e^{-\rho} e_A\}$ is an orthonormal frame of $(N, \tilde{g}_N)$, and $\{\tilde{\omega}_A = e^\rho \omega_A\}$ is the dual coframe of $\{\tilde{e}_A\}$. The structure equations of $(N, \tilde{g}_N)$ are given by

$$\begin{cases} d\tilde{\omega}_A = \sum_B \tilde{\omega}_{AB} \wedge \tilde{\omega}_B, \\
\tilde{\omega}_{AB} + \tilde{\omega}_{BA} = 0,
\end{cases} \quad (2.2)$$

where $\{\tilde{\omega}_{AB}\}$ are the connection forms of $(N, \tilde{g}_N)$.

\footnote{Here we identify $e_i$ with $\phi_*(e_i)$}
From (2.1) and (2.2), we can derive that
\[ \tilde{\omega}_{AB} = \omega_{AB} + \rho_A \omega_B - \rho_B \omega_A, \quad (2.3) \]
where \( \rho_A \) means the covariant derivative of \( \rho \) with respect to \( e_A \).

Let \( \nabla \) (or \( \tilde{\nabla} \) resp.) denote Levi-Civita connections on \( N \) with respect to metric \( g_N \) (or \( \tilde{g}_N \) resp.). In general, for any \( F \in C^\infty(N) \), we compute the gradient
\[
dF = \sum_{A=1}^{n} F_A \omega_A = \sum_{A=1}^{n} \tilde{\nabla}_A F \omega_A
\]
which gives the relation
\[
\tilde{F}_A = \tilde{\nabla}_A F = e^{-\rho} \nabla_A F = e^{-\rho} F_A, \quad \text{for } A = 1, \cdots, n. \quad (2.4)
\]

2.2. Conformal relations: the submanifold. Let \( M \) be an \( m (\geq 2) \)-dimensional submanifold of \( N \) immersed by \( \phi \).

When consider the metric \( g_N \), \( M \) has an induced metric \( g_M = \phi^* g_N \). Denote \( \phi^* \omega_A = \theta_A, \phi^* \omega_{AB} = \theta_{AB} \), then restricted to \((M, g_M)\), we have (see [6])
\[
\theta_\alpha = 0, \quad \theta_{i\alpha} = \sum_j h^{\alpha}_{ij} \theta_j. \quad (2.5)
\]
and
\[
\begin{aligned}
d\theta_i &= \sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} + \theta_{ji} = 0, \\
d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} &= \frac{1}{2} \sum_{k,l} \tilde{R}_{ijkl} \theta_k \wedge \theta_l,
\end{aligned} \quad (2.6)
\]
where \( \tilde{R}_{ijkl} \) are components of the curvature tensor of \((M, g_M)\) and \( h^{\alpha}_{ij} \) are components of the second fundamental form of \((M, g_M)\) in \((N, g_N)\).

We denote the mean curvature vector of \( M \) (precisely, of \( \phi \)) by
\[
H_{\phi} = \frac{1}{n} \sum_{\alpha} (\sum_i h^{\alpha}_{ii}) e_\alpha = \sum_{\alpha} H^\alpha e_\alpha. \quad (2.7)
\]
We sometimes omit the subscript \( \phi \) and just write \( H \) when the immersion is clear.

Similarly, we consider the induced metric \( \tilde{g}_M = \phi^* \tilde{g}_N \), and denote \( \phi^* \tilde{\omega}_A = \tilde{\theta}_A, \phi^* \tilde{\omega}_{AB} = \tilde{\theta}_{AB} \). Then restricted to \((M, \tilde{g}_M)\), we have
\[
\tilde{\theta}_\alpha = 0, \quad \tilde{\theta}_{i\alpha} = \sum_j \tilde{h}^{\alpha}_{ij} \tilde{\theta}_j. \quad (2.8)
\]
and
\[
\begin{aligned}
d\tilde{\theta}_i &= \sum_j \tilde{\theta}_{ij} \wedge \tilde{\theta}_j, \quad \tilde{\theta}_{ij} + \tilde{\theta}_{ji} = 0, \\
d\tilde{\theta}_{ij} - \sum_k \tilde{\theta}_{ik} \wedge \tilde{\theta}_{kj} &= -\frac{1}{2} \sum_{k,l} \tilde{\tilde{R}}_{ijkl} \tilde{\theta}_k \wedge \tilde{\theta}_l,
\end{aligned} \quad (2.9)
\]
where \( \tilde{\tilde{R}}_{ijkl} \) are components of the curvature tensor of \((M, \tilde{g}_M)\) and \( \tilde{h}^{\alpha}_{ij} \) are components of the second fundamental form of \((M, \tilde{g}_M)\) in \((N, \tilde{g}_N)\).
By pulling back (2.3) to \(M\) by \(\phi\) and using (2.5) and (2.8), we obtain the following relation.
\[
\tilde{h}_{ij}^\alpha = e^{-\rho}(h_{ij}^\alpha - \rho \delta_{ij}), \quad \tilde{H}^\alpha = e^{-\rho}(H^\alpha - \rho \delta).
\] (2.10)
Combining (2.3), (2.4), (2.6), and (2.9), we can obtain the following relation:
\[
e^{2\rho} \tilde{R}_{ijkl} = \tilde{R}_{ijkl} - (\rho_i \rho_j \delta_{ik} - \rho_j \rho_i \delta_{kj} - \rho_j \rho_k \delta_{ij} - \rho_i \rho_k \delta_{ij}) + (\rho_i \rho_k \delta_{ij} - \rho_j \rho_k \delta_{ij} - \rho_j \rho_i \delta_{lj} - \rho_j \rho_i \delta_{kj}) - (\sum m^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),
\] (2.11)
\[
e^{2\rho} \tilde{R}_{ij} = R_{ij} + (n - 2)(\rho_i \rho_j - \rho_{ij} - |\nabla \rho|^2 \delta_{ij}) - (\Delta \rho)\delta_{ij},
\] (2.12)
where \(R_{ij}\) (or \(\tilde{R}_{ij}\) resp.) are components of Ricci curvature with respect to \(g_M\) (or \(\tilde{g}_M\) resp.).

The following proposition gives the relation between the conformal fact and the mean curvature.

**Proposition 2.1** (cf. Proposition 3.3 and Remark 3.4 in [5]). Let \((N, g_N)\) be an \(n\)-dimensional Riemannian manifold (possibly not complete) which admits a conformal immersion \(\gamma\) in the sphere \((\mathbb{S}^n, h_1)\). Assume \(\gamma^* h_1 = e^{2\rho} g_N\).

Let \(M\) be an \((m \geq 2)\)-dimensional submanifold immersed in \((N, g_N)\) by \(\phi\). Then we have
\[
e^{2\rho} = \left( |\mathbf{H}|^2 + \mathcal{R} \right) - \frac{2}{m} \Delta \rho - \frac{m - 2}{m} |\nabla \rho|^2 - |(\nabla \rho)^\perp| - |\mathbf{H}|^2,
\] (2.13)
where
\[
\mathcal{R} = \frac{1}{m(m-1)} \sum_{i,j} K_N(e_i, e_j)
\] (2.14)
for a local orthogonal tangent frame \(\{e_i\}\) on \(M\), and \(K_N\) is the sectional curvature of \((N, g_N)\).

In particular, when \((N, g_N) = (\mathbb{R}^n(c), h_c)\), we have
\[
e^{2\rho} = \left( |\mathbf{H}|^2 + c \right) - \frac{2}{m} \Delta \rho - \frac{m - 2}{m} |\nabla \rho|^2 - |(\nabla \rho)^\perp| - |\mathbf{H}|^2.
\] (2.15)

**Proof.** We denote the immersion of \(M\) to \((N, g_N)\) by \(\phi\). Consider two metric \(g_M = \phi^* g_N\) and \(\tilde{g}_M = \phi^* (\gamma^* h_1) = (\gamma \circ \phi)^* h_1\) on \(M\). The Gauss equations for the immersions \(\phi\) and \(\gamma \circ \phi\) imply:
\[
R_{ij} = \sum_k \tilde{R}_{ikjk} + \sum_{\alpha} mH^\alpha h_{ij}^\alpha - \sum_{k,\alpha} h_{ik}^\alpha h_{kj}^\alpha,
\] (2.16)
\[
\tilde{R}_{ij} = (m - 1)\delta_{ij} + \sum_{\alpha} m\tilde{H}^\alpha h_{ij}^\alpha - \sum_{k,\alpha} \tilde{h}_{ik}^\alpha \tilde{h}_{kj}^\alpha.
\] (2.17)
From (2.10), (2.12), (2.16) and (2.17), we obtain
\[
(m - 2)(\rho_i \rho_j - \rho_{ij} - |\nabla \rho|^2 \delta_{ij}) - (\Delta \rho)\delta_{ij} = e^{2\rho} \tilde{R}_{ij} - R_{ij}
\] (2.18)
\[
\quad = (m - 1)(e^{2\rho} + \sum_{\alpha} \rho_{\alpha}^2) \delta_{ij} - \sum_k \tilde{R}_{ikjk} - (m - 2) \sum_{\alpha} \rho_{\alpha} h_{ij}^\alpha - \sum_{\alpha} m H^\alpha \rho_{\alpha} \delta_{ij}.
\] (2.19)
Contracting (2.18) with \(\delta_{ij}\), we obtain
\[
-2\Delta \rho = (m - 2)|\nabla \rho|^2 + m(e^{2\rho} + \sum_{\alpha} \rho_{\alpha}^2) - 2 \sum_{\alpha} m H^\alpha \rho_{\alpha} - \frac{1}{m-1} \sum_{i \neq k} \tilde{R}_{ikik},
\] (2.19)
which is equivalent to (2.13). \(\square\)
When \((N, g_N) = (\mathbb{R}^n(c), h_c)\), we usually denote the immersion of \(M\) to \(\mathbb{R}^n(c)\) by \(x\) instead of \(\phi\), and \(x\) can be viewed as the position vector. Then we have (cf. [1,6,36]):

\[
dx = \sum_i \theta_i e_i, \quad de = \sum_j \theta_j e_j + \sum_j h^j_\alpha \theta_j e_\alpha - c\theta_i x, \quad de_\alpha = - \sum_{i,j} h^j_\alpha \theta_j e_\beta + \sum_\beta \theta_\alpha e_\beta,
\]

which implies

\[
x_i = e_i, \quad x_{ij} = \sum_\alpha h^i_\alpha e_\alpha - c\delta_{ij} x, \quad \Delta x = mH - mc x.
\]  

(2.20)

### 3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We divide the proof into three parts: the inequality, the sufficient conditions for the equality and the necessary conditions for the equality. To prove the inequality (1.4), a key step is to choose suitable test functions, where we will use a technique of conformal transformation on a sphere which was introduced by Li and Yau [24]. The verification for the sufficient conditions is direct and easier. But the discussion of the necessary conditions is more involved, and we need to write down explicitly the conformal transformation and analyze the equality carefully.

#### 3.1. The inequality in (1.4)

By using the technique in Li-Yau [24], we have the following lemma.

**Lemma 3.1** (see [12,24]). Let \(M\) be an \(m\)-dimensional submanifold in an \(n\)-dimensional space form \(\mathbb{R}^n(c)\). Then there exists a regular conformal map \(\Gamma : \mathbb{R}^n(c) \rightarrow \mathbb{S}^n(1) \subset \mathbb{R}^{n+1}\) such that the immersion \(\Phi = \Gamma \circ x = (\Phi_1, \ldots, \Phi^{n+1})\) satisfies that

\[
\int_M \Phi^A = 0, \quad A = 1, \ldots, n + 1.
\]  

(3.1)

**Remark 3.2.** The immersion \(\Phi\) in Lemma 3.1 can explicitly. For each \(a \in B^{n+1}\), we can define a conformal map \(\gamma_a\) on \(\mathbb{S}^n(1)\) (cf. [27]):

\[
\gamma_a(x) = \frac{x + (\mu f + \lambda)a}{\lambda(1 + f)}, \quad \forall x \in \mathbb{S}^n(1),
\]  

(3.2)

where \(B^{n+1}\) is the open unit ball in \(\mathbb{R}^{n+1}\), \(x\) is the position vector of \(\mathbb{S}^n(1)\), and

\[
\lambda = (1 - |a|^2)^{-1/2}, \quad \mu = (\lambda - 1)|a|^{-2}, \quad f(x) = \langle x, a \rangle.
\]  

(3.3)

When \(a = 0\), we set \(\lambda = 1, \mu = 0, \gamma_0(x) = x\).

When \(c = 1\), \(\Gamma = \gamma_a\) for some \(a \in B^{n+1}\); when \(c = 0\) or \(-1\), \(\Gamma\) is the composition of the stereographic projection and \(\gamma_a\). See Section 3.3 for the details.

Now we set \(N = \mathbb{R}^n(c)\) in Section 2.1, \(g_N = h_c, \tilde{g}_N = \Gamma^* h_1 = e^{2\rho} g_N\), then \(g_M = x^* h_c, \tilde{g}_M = (\Gamma \circ x)^* h_1\). Here \(h_c\) is the standard metric on \(\mathbb{R}^n(c)\).

**Proof of (1.4).** It is well known that the variational characterization of \(\lambda_i(i \geq 1)\)

\[
\lambda_i = \inf_{u \in H^1(M) \setminus \{0\}} \left\{ \frac{\int_M |\nabla u|^2}{\int_M u^2} \mid \int_M uu_j = 0, j = 0, \ldots, i - 1 \right\},
\]  

(3.4)

where \(\{u_i\}_{i \geq 0}\) is an orthonormal set of eigenfunctions satisfying \(\Delta u_i = -\lambda_i u_i\).

Recall (3.1) means that \(\Phi^A\) is \(L^2\)-orthogonal to the first eigenfunction \(u_0\) (a constant).
By using the QR-decomposition via the Gram–Schmidt process, we can assume that \( \{\Phi^1, \ldots, \Phi^{n+1}\} \) satisfy
\[
\int_M \Phi^A u_B = 0, \text{ for } 1 \leq B < A \leq n + 1. \tag{3.5}
\]
Indeed, we denote
\[
d_{AB} = \int_M \Phi^A u_B = 0, \text{ for } 1 \leq A, B \leq n + 1. \tag{3.6}
\]
Then \( D = (d_{AB}) = QR \), where \( Q = (q_{AB}) \in O(n + 1) \) and \( R \) is an upper triangular matrix. This is equivalent to \( R = Q^T D \). Hence, if we replace the old orthonormal basis \( (E_1, \ldots, E_n) \) of \( \mathbb{R}^{n+1} \) by the new one \( (E'_1, \ldots, E'_{n+1}) = (E_1, \ldots, E_n) Q^T \), then these new coordinate functions, still denoted by \( \Phi^A \), satisfy \((3.5)\).

Now by the variational characterization \((3.4)\), for each \( A = 1, \ldots, n + 1 \), we have
\[
\lambda_A \int_M (\Phi^A)^2 \leq \int_M |\nabla \Phi^A|^2 = \int_M e^{2\rho} |\tilde{\nabla} \Phi^A|^2. \tag{3.7}
\]
Denote \( Q_A = |\tilde{\nabla} \Phi^A|^2 \). Since \( 0 \leq Q_A \leq 1 \),
\[
\sum_{A=1}^{n+1} \Phi^2_A = 1, \quad \sum_{A=1}^{n+1} Q_A = m, \tag{3.8}
\]
we obtain
\[
V(M) = \int_M \frac{1}{V(M)} \int_M e^{2\rho} Q_A \leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M e^{2\rho} Q_A + \sum_{A=m+1}^{n+1} \frac{1}{\lambda_A} \int_M e^{2\rho} Q_A \leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M e^{2\rho} Q_A + \frac{1}{\lambda_m} \int_M e^{2\rho} \sum_{A=m+1}^{n+1} Q_A \leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M e^{2\rho} Q_A + \frac{1}{\lambda_m} \int_M e^{2\rho} \sum_{A=1}^{m} (1 - Q_A) \leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M e^{2\rho} Q_A + \frac{1}{\lambda_m} \int_M e^{2\rho} (1 - Q_A) \leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M e^{2\rho}. \tag{3.9}
\]
Then by using \((2.15)\), we have
\[
\frac{1}{\sum_{i=1}^{m} \lambda_i} \leq \frac{1}{V(M)} \int_M e^{2\rho} \leq \frac{1}{V(M)} \int_M \left( c + |H|^2 \right). \tag{3.12}
\]

3.2. The sufficient conditions for the equality in \((1.4)\). Assume that \( M \) is minimal in a geodesic sphere \( \Sigma_c \) of \( \mathbb{R}^n(c) \), where the geodesic radius \( r_c \) of \( \Sigma_c \) is given by
\[
r_0 = \left( \frac{m}{\lambda_1} \right)^{1/2}, \quad r_1 = \arcsin r_0, \quad r_{-1} = \arsinh r_0. \tag{3.13}
\]
We verify the equality holds in (1.4) for such $M$.

We know that $\Sigma_c$ has constant curvature, denoted by $c'$, where $c' \geq 1$ for $c = 1$ and $c' > 0$ for $c = 0, -1$. Moreover, $\Sigma_c$ is totally umbilical in $\mathbb{R}^n(c)$ with the constant principal curvature $k$ satisfying $k = \sqrt{c' - c} \geq 0$. This can be obtained by the Gauss equation, and we can always choose $e_{m+1}$ such that $k \geq 0$, where $e_{m+1}$ is the unit normal vector of the immersion from $\Sigma_c$ to $\mathbb{R}^n(c)$.

Now let $\{e_{m+2}, \ldots, e_n\}$ be the normal frame of the immersion from $M$ to $\Sigma_c$, then $\{e_{m+1}, e_{m+2}, \ldots, e_n\}$ forms a local normal frame of $M$ in $\mathbb{R}^n(c)$ (When $n - m = 1, M = \Sigma_c$ and there is only one normal vector $e_{m+1}$). We choose a local orthonormal tangent frame $\{e_1, \ldots, e_m\}$ on $M$, then we have $h_{ij}^{m+1} = k\delta_{ij}$. Since $M$ is minimal in $\Sigma_c$, we have $\sum_{\beta=m+2}^{n} h_{ij}^{\beta} e_{\beta} = 0$. This implies $H = \frac{1}{m} \sum_i h_{ii}^{m+1} e_{m+1} = k e_{n+1}$. Hence, the right side of (1.4) is $m(c + k^2) = mc'$.

On the other hand, the relation between the geodesic radius $r_c$ and the curvature $c'$ of $\Sigma_c$ is

$$c' = \frac{1}{r_c^2} = \frac{1}{\sin^2 r_1} = \frac{1}{\sinh^2 r_{-1}}.$$

It follows from (3.13) that $\lambda_1 = mc'$. Hence,

$$mc' = \lambda_1 \leq \mathcal{H}(\lambda_1, \ldots, \lambda_m) \leq \frac{m}{V(M)} \int_M (c + |H|^2) = mc'.$$

This means that the inequality must be sharp.

**Remark 3.3.** From the above proof, we see that $\lambda_1 = \cdots = \lambda_m$, which is reasonable. Indeed, since $\Sigma_c$ has constant curvature $c' = \lambda_1/m$ and $M$ is minimal in the sphere $\Sigma_c$, we know that after translating the center of $\Sigma_c$ to the origin of $\mathbb{R}^n$, each coordinate function $y_i (1 \leq i \leq n)$ satisfies $\Delta y_i = -mc'y_i = -\lambda_1 y_i$ (cf. (2.20)). Then we can find at least $m$ coordinate functions which are linear independent since $\dim M = m$, which means the multiplicity of $\lambda_1$ is not less than $m$. This is also true for $M = \Sigma_c$ in the case $m = n - 1$.

When $c = 1$, if $M$ is minimal in $\mathbb{S}^n(1)$, for the same reason as in above remark, we know that

$$\lambda_1 = \cdots = \lambda_m = \mathcal{H}(\lambda_1, \ldots, \lambda_m) = m = \frac{m}{V(M)} \int_M (c + |H|^2).$$

### 3.3. The necessary conditions for the equality in (1.4)

In this section, we discuss the necessary conditions for the equality in (1.4). From now on, we assume that $M$ is not minimal in $\mathbb{S}^n(1)$ when $c = 1$. Firstly, we have

**Lemma 3.4.** If the equality holds in (1.4), then

1. $H = (\nabla \rho)^\perp$ on $M$;
2. $\rho \nabla \rho$ is constant;
3. $\lambda_1 = \cdots = \lambda_m$.

**Proof.** We know that all the inequality are sharp in Sect. 3.1. Assert (1) is directly derived by noticing Proposition 2.1.

When $m \geq 3$, Assert (2) can also be obtained from $|\nabla \rho| = 0$ on $M$ by noticing Proposition 2.1. But this is invalid for $m = 2$. We need more detailed analysis, which works for $m \geq 2$.

The equality in (3.7) gives

$$\Delta \Phi^A = -\lambda_A \Phi^A,$$

(3.15)
then
\begin{equation}
me^{2\rho} = \sum_{A=1}^{n+1} |\nabla \Phi^A|^2 = \sum_{A=1}^{n+1} \left( \frac{1}{2} \Delta (\Phi^A)^2 - \Phi^A \Delta \Phi^A \right) = \sum_{A=1}^{n+1} \lambda_A (\Phi^A)^2. \tag{3.16}
\end{equation}

We notice that \( Q_A \equiv 0 \) implies \( \Phi^A \equiv 0 \). Indeed, since \( Q_A = e^{-2\rho}|\nabla \Phi^A|^2 \), if \( Q_A \equiv 0 \), then \( \Phi^A \) is constant on \( M \), and then \( \Phi^A \equiv 0 \) by using (3.15).

On the other hand, \( Q_A \equiv 1 \) is impossible. Otherwise, \( 1 \equiv Q_A = |\nabla \Phi^A|^2 \leq 1 - (\Phi^A)^2 \), so we have \( \Phi^A \equiv 0 \) and then \( Q_A = e^{-2\rho}|\nabla \Phi^A|^2 = 0 \), a contradiction.

If \( \Phi^s \not\equiv 0 \) for some \( s \geq m+1 \), then \( Q_s \not\equiv 0 \) and the equality in (3.9) gives \( \lambda_m = \lambda_s \). Since \( Q^A \not\equiv 1 \), then the equality in (3.10) gives \( \lambda_t = \lambda_m \) for all \( t \leq m - 1 \). Hence, we conclude from (3.16) that
\begin{equation}
me^{2\rho} = \lambda_m \sum_{1 \leq A \leq n+1} (\Phi^A)^2 = \lambda_m, \tag{3.17}
\end{equation}
which implies the restriction of \( \rho \) to \( M \) is constant.

At last, from the equality in (3.15), we must have \( \frac{1}{m} \sum_{i=1}^{m} \frac{1}{\lambda_i} = \frac{1}{\lambda_m} \), which implies \( \lambda_1 = \cdots = \lambda_m \).

So far, (3) of Lemma 3.4 makes us reduce the equality case in (1.4) to the equality case in (1.1), then one can follow the arguments in [5]. Here we give a little different discussion.

Based on the above lemma, we can further prove that \( M \) is contained in a geodesic.

**Proposition 3.5.** Assume that the equality holds in (1.4). In addition, assume \( H \) is not identically zero when \( c = 1 \). Then \( M \) is contained in a geodesic sphere \( \Sigma_c \) of \( \mathbb{R}^n(c) \). Moreover, \( \rho|_{\Sigma_c} \) is constant and \( \Sigma_c \) has constant curvature \( c' = e^{2\rho}|_{\Sigma_c} \).

**Proof.** Firstly, we prove the first part of the proposition by discussing the three cases \( c = 1, 0 \) or \(-1\), respectively. Noticing the choice of the test functions in Sect. 3.1, we can assume that the conformal map is just as in Lemma 3.1 up to an isometry on \( \mathbb{S}^n \).

**Case 1.** \( c = 1 \). In this case, the conformal map \( \Gamma : \mathbb{S}^n(1) \to \mathbb{S}^n(1) \) in Lemma 3.1 is given by
\begin{equation}
\Gamma = \gamma_a \tag{3.18}
\end{equation}
for certain \( a \in B^{n+1} \), where \( x \) is the position vector of \( \mathbb{S}^n(1) \) (cf. [23, 27]).

We denote \( h_1 \) the standard metric on \( \mathbb{S}^n(1) \), and set \( \Gamma^* h_1 = e^{2\rho} h_1 \), then a direct computation gives
\begin{equation}
e^{2\rho} = \frac{1}{\lambda^2 (1+f)^2}, \quad \rho = -\ln \lambda - \ln (1+f), \quad \rho_A = -\frac{f_A}{1+f} (1 \leq A \leq n). \tag{3.19}
\end{equation}
From the last equation of (3.19), it is obvious that \( \rho \) is constant if and only if \( f \) is constant.

Since \( H \not\equiv 0 \), we claim \( \rho \) is not constant on \( \mathbb{S}^n(1) \), which implies \( a \neq 0 \). Otherwise, \( a = 0 \), then \( \lambda = 1, \mu = 0, \gamma_0(x) = x \), i.e., \( \gamma_0 \) is the identity map, so we have \( \rho \equiv 0 \) on \( \mathbb{S}^n(1) \), which contradicts with (1) of Lemma 3.4.

Now, (2) of Lemma 3.4 implies that \( M \) lies in a level set \( \{ x \in \mathbb{S}^n(1) \mid \rho(x) = b \} \) for some constant \( b \). Recall that \( \rho \) is constant if and only if \( f \) is constant, \( a \neq 0 \), so \( M \) lies in a totally umbilical hypersurface
\[ \Sigma_1 = \{ x \in \mathbb{S}^n(1) \mid f(x) = \langle x, a \rangle = b' \} \]
of \( \mathbb{S}^n(1) \) for some constant \( b' \).
Case 2. $c = 0$. There is a conformal map $\pi_0 : \mathbb{R}^n \to S^n(1) \subset \mathbb{R}^{n+1}$ given by the stereographic projection:

$$
\pi_0(x) = \left( \frac{2x}{1 + |x|^2} \frac{|x|^2 - 1}{1 + |x|^2} \right) \in S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1,
$$

(3.20)

where $x$ is the position vector in $\mathbb{R}^n$. In this case, the conformal map $\Gamma : \mathbb{R}^n \to S^n(1)$ in Lemma 3.1 is given by

$$
\Gamma = \gamma_a \circ \pi_0
$$

(3.21)

for certain $a \in B^{n+1}$.

We denote $h_0$ the standard metric on $\mathbb{R}^n$, and set $\Gamma^* h_1 = \pi_0^* (\gamma_a^* h_1) = e^{2\rho} h_0$. By a direct computation, we have

$$
e^{2\rho} = \frac{4}{(1 + |x|^2)^2} \cdot \frac{1}{\lambda^2 (1 + f \circ \pi_0(x))^2},
$$

(3.22)

where $f : S^n(1) \to \mathbb{R}$ is defined by (3.3). From (3.22), we derive that $\rho$ is constant if and only if $(1 + f \circ \pi_0(x))(1 + |x|^2) = b$ for some constant $b$. Suppose $a = (\bar{a}, a^0) \in \mathbb{R}^N \times \mathbb{R}^1$, then we have $1 + |x|^2 + 2\langle x, \bar{a} \rangle + (|x|^2 - 1)a^0 = b$, which is equivalent to

$$
\left| x + \frac{\bar{a}}{1 + a^0} \right|^2 = \frac{(1 + a^0)b + |\bar{a}|^2 - 1}{(1 + a^0)^2},
$$

(3.23)

which means that $M$ lies in an $(n - 1)$-dimensional geodesic sphere $\Sigma_0$ of $\mathbb{R}^n$.

Case 3. $c = -1$. Let $\mathbb{R}_1^{n+1}$ be the Lorentz space equipped with the Lorentz metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$, and denote the inner product by $\langle , \rangle'$, i.e.,

$$
\langle x, y \rangle' = \langle \tilde{x}, \tilde{y} \rangle - x^0 y^0
$$

for $x = (\tilde{x}, x^0), y = (\tilde{y}, y^0) \in \mathbb{R}_1^{n+1}$. Then

$$
\mathbb{H}^n(-1) = \{ x \in \mathbb{R}_1^{n+1} | \langle x, x \rangle' = -1, x^0 \geq 1 \},
$$

which is equipped with the induced metric from $\mathbb{R}_1^{n+1}$.

There is a conformal map $\pi : \mathbb{H}^n(-1) \to B^n(1) \subset \mathbb{R}^n$ by “stereographic projection”:

$$
\pi(x) = \frac{\tilde{x}}{1 + x^0} =: w(x) \in B^n(1), \quad \pi^{-1}(w) = \left( \frac{2w}{1 - |w|^2}, \frac{1 + |w|^2}{1 - |w|^2} \right) \in \mathbb{R}_1^{n+1},
$$

(3.24)

where $x = (\tilde{x}, x^0)$ is the positive vector in $\mathbb{H}^n(-1)$. Actually, $\left( B^n(1), \frac{4|dw|^2}{(1 - |w|^2)^2} \right)$ is just the Poincaré model of hyperbolic space $\mathbb{H}^n(-1)$ and has constant curvature $-1$.

In this case, the conformal map $\Gamma : \mathbb{H}^n(-1) \to S^n(1)$ in Lemma 3.1 is given by

$$
\Gamma = \gamma_a \circ \pi_0 \circ \pi
$$

(3.25)

for certain $a \in B^{n+1}$.

We denote $h_{-1}$ the standard metric on $\mathbb{H}^n(-1)$, and set $\Gamma^* h_1 = \pi_0^* (\gamma_a^* h_1) = e^{2\rho} h_{-1}$. By a direct computation, we have

$$
e^{2\rho} = \frac{1 - |w(x)|^2)^2}{(1 + |w(x)|^2)^2} \cdot \frac{1}{\lambda^2 (1 + f(|\pi_0(w(x))|))^2},
$$

(3.26)

where $f : S^n(1) \to \mathbb{R}$, $\pi_0 : \mathbb{R}^n \to S^n(1)$ and $w(x)$ are defined by (3.3), (3.20) and (3.24) respectively.
From (3.26), we know that \( \rho \) is constant if and only if 
\[
(1 + f(\pi_0(w(x))))(1 + |w(x)|^2) = b(1 - |w(x)|^2)
\]
for some constant \( b \). Suppose \( a = (\tilde{a}, a^0) \in \mathbb{R}^n \times \mathbb{R}^1 \), then we have
\[
b = \frac{1 + |w|^2 + 2\langle w, \tilde{a} \rangle}{1 - |w|^2} - a^0,
\]
which implies that
\[
1 + b + a^0 = \frac{2(1 + \langle w, \tilde{a} \rangle)}{1 - |w|^2} > 0,
\]
where we used \( |\tilde{a}| < 1, |w| < 1 \). Hence, we obtain that
\[
\left| w(x) + \frac{\tilde{a}}{1 + b + a^0} \right|^2 = 1 - \frac{2}{1 + b + a^0} + \frac{|\tilde{a}|^2}{(1 + b + a^0)^2}.
\]
Note that
\[
1 - \frac{2}{1 + b + a^0} + \frac{|\tilde{a}|^2}{(1 + b + a^0)^2} < 1 - \frac{2|\tilde{a}|}{1 + b + a^0} + \frac{|\tilde{a}|^2}{(1 + b + a^0)^2} = (1 - \frac{|\tilde{a}|}{1 + b + a^0})^2,
\]
thus \( (3.28) \) implies that \( \pi \circ x(M) \) lies in a hypersphere \( S \) of \( B^n(1) \), equivalently, \( M \) lies in a geodesic sphere \( \Sigma_{-1} = \pi^{-1}(S) \) of \( \mathbb{H}^n(-1) \).

Now we prove the second part of the proposition. Since we have proved that \( M \) is contained in \( \Sigma_c \), by denoting \( \Sigma' = \Gamma(\Sigma_c) \), we know that \( \Sigma' \) is totally umbilical in \( S^n(1) \) and \( \Gamma \circ x(M) \) is contained in \( \Sigma' \subset S^n(1) \). Combining (1) of Lemma 3.4 with (2.10), we derive that \( \Gamma \circ x(M) \) is minimal in \( S^n(1) \), so \( \Sigma' \) must be the great sphere of curvature 1. Reviewing the proof of the first part, we obtain \( \rho|_{\Sigma_c} \) is constant. Hence, we obtain that \( \Sigma_c \) has constant curvature \( c' = e^{2\rho}|_{\Sigma_c} \) by using (2.11).

Finally, we only need to prove \( M \) is minimal in \( \Sigma_c \) and determine the radius of \( \Sigma_c \). Let \( \nu \) and \( k \) be the unit normal vector and the principal curvature of \( \Sigma_c \) in \( \mathbb{R}^n(c) \) respectively, then \( H = H' + \nu \nu \), where \( H' \) be the mean curvature of \( M \) in \( \Sigma_c \). The right side of (1.4) is
\[
\frac{m}{V(M)} \int_M (c + |H|^2) = m(c + k^2) + \frac{m}{V(M)} \int_M |H'|^2 = mc' + \frac{m}{V(M)} \int_M |H'|^2
\]
(3.29)

On the other hand, when the equality holds in (1.4), by recalling (3.17) and (3) of the proof of Lemma 3.4, the left side of (1.4) becomes
\[
\lambda_1 = \lambda_m = mc' = mc',
\]
where we used Proposition 3.5 in the last equality. By comparing (3.29) and (3.30), we conclude \( H' \equiv 0 \), i.e., \( M \) is minimal in \( \Sigma_c \). The geodesic radius \( r_c \) can be solved from (3.14) and (3.30). We complete the proof of the necessary condition for the equality in (1.4).

\[\square\]

3.4. A general result. When the ambient is not the space form, we can prove the following theorem, which implies [11, Theorem 2].

**Theorem 3.6.** Let \((N, g_N)\) be a Riemannian manifold of dimension \( n \) (possibly not complete) which admits a conformal immersion in the sphere \((S^n, h_1)\). Then, for any \( m(\geq 2) \)-dimensional closed submanifold \( M \) immersed in \((N, g_N)\) by \( \phi \), we have
\[
\mathcal{S}(\lambda_1, \cdots, \lambda_m) \leq \frac{m}{V(M)} \int_M (|H_\phi|^2 + \bar{\mathcal{R}}_\phi),
\]
where \( K_N \) is the sectional curvature of \((N, g_N)\), and
\[
\bar{\mathcal{R}}_\phi = \frac{1}{m(m - 1)} \sum_{i \neq j} K_N(e_i, e_j)
\]
for a local orthonormal tangent frame \( \{e_i\} \) on \( M \).

Moreover, equality holds if and only if \( \lambda_1 = \cdots = \lambda_m \) and \( |H_\phi|^2 + \bar{\mathcal{R}}_\phi \) equals the constant \( \lambda_1/m \).
Proof. The inequality follows from the same arguments as in Sect. 3.1 (using (2.13) rather than (2.15)). The sufficiency for the equality is obvious. The necessary for the equality is due to Lemma 3.4. □

4. Proof of Theorem 1.5

4.1. The inequality in (1.12). We first prove the inequalities (1.12). Similar with Lemma 3.1, we have

Lemma 4.1 (see [12, 24]). Let $\phi : (M^m, g) \rightarrow (S^n, h_1)$ be a conformal map. Then there exists $\gamma \in$ such that the immersion $\Phi = \gamma \circ \phi = (\Phi_1, \ldots, \Phi^{n+1})$ satisfies that

$$\int_M \Phi^A = 0, \ A = 1, \ldots, n + 1. \quad (4.1)$$

For the Neumann boundary condition, we also have the the variational characterization (3.4), and we can assume (3.5) holds by the same arguments as in Sect. 3.1. Now for each $A = 1, \ldots, n + 1$, we have

$$\lambda_A \int_M (\Phi^A)^2 \leq \int_M |\nabla \Phi^A|^2. \quad (4.2)$$

On the other hand, if we denote $\tilde{g} = \Phi^* h_1 = fg$, then

$$0 \leq |\tilde{\nabla} \Phi^A|^2 \leq 1 - (\Phi^A)^2, \quad \sum_{A=1}^{n+1} |\tilde{\nabla} \Phi^A|^2 = m, \quad (4.3)$$

and

$$mf = \sum_{A=1}^{n+1} |\nabla \Phi^A|^2. \quad (4.4)$$

where $\tilde{\nabla}$ is the gradient operator on $(M, \tilde{g})$.

An analogous computation as in Sect. 3.1 shows that

$$V(M, g) = \int_M dv_g \leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M f |\tilde{\nabla} \Phi^A|^2 + \sum_{A=m+1}^{n+1} \frac{1}{\lambda_A} \int_M f |\nabla \Phi^A|^2 \quad (4.5)$$

$$\leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M f |\tilde{\nabla} \Phi^A|^2 + \frac{1}{\lambda_m} \int_M f \sum_{A=m+1}^{n+1} |\tilde{\nabla} \Phi^A|^2 \quad (4.6)$$

$$= \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M f |\tilde{\nabla} \Phi^A|^2 + \frac{1}{\lambda_m} \int_M f \sum_{A=1}^{m} (1 - |\tilde{\nabla} \Phi^A|^2) \quad (4.7)$$

$$\leq \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M f |\nabla \Phi^A|^2 + \sum_{A=1}^{m} \frac{1}{\lambda_A} \int_M f (1 - |\tilde{\nabla} \Phi^A|^2) \quad (4.8)$$

Hence, by using the Hölder inequalities, we have

$$S(\lambda_1, \ldots, \lambda_m) \leq \frac{m}{V(M, g)} \int_M f dv_g \quad (4.9)$$
\[
\leq \frac{m}{V(M, g)} \left( \int_M f^{m/2} \, dv_g \right)^{2/m} (V(M))^{\frac{m-2}{m}} \tag{4.10}
\]

\[
= m \left( \frac{V(M, \bar{g})}{V(M, g)} \right)^{2/m} \tag{4.11}
\]

\[
\leq m \left( \sup_{\gamma \in G(n)} \frac{V(M, (\gamma \circ \phi)^* h_1)}{V(M, g)} \right)^{2/m}. \tag{4.12}
\]

Taking the infimum over all \( \phi \in I_n(M, [g]) \), we obtain the desired inequality.

### 4.2. Equality in (1.12)

Now we discuss the equality. Choose a sequence of conformal maps \( \phi_j : M \to S^n \) such that

\[
\lim_{j \to \infty} V_c(M, n, \phi_j) = V_c(M, n), \tag{4.13}
\]

and by composing with a conformal transformation of the sphere we may assume

\[
\int_M \phi_j^A = 0, \quad 1 \leq A \leq n + 1. \tag{4.14}
\]

\[
\int_M \phi_j^A u_B = 0, \quad \text{for } 1 \leq B < A \leq n + 1 \tag{4.15}
\]

We denote \( m f_j = \sum_{A=1}^{n+1} |\nabla \phi_j^A|^2 \) and write \( \mathfrak{H}(\lambda_1, \cdots, \lambda_m) \) as \( \mathfrak{H} \) for short.

We reproduce the previous steps from (4.5) to (4.12) for each \( \phi_j \) and then take the limit. It follows that all the inequalities must be sharp. Indeed, noticing (4.5) and (4.8), we have

\[
V(M, g) = \int_M \sum_{A=1}^{n+1} (\phi_j^A)^2 \leq \mathfrak{H}^{-1} \int_M m f_j \tag{4.16}
\]

\[
\leq \mathfrak{H}^{-1} m \left( \int_M f_j^{m/2} \, dv_g \right)^{2/m} (V(M))^{\frac{m-2}{m}} \tag{4.17}
\]

\[
\leq \mathfrak{H}^{-1} m \left( V_c(M, n, \phi_j) \right)^{2/m} (V(M))^{\frac{m-2}{m}}. \tag{4.18}
\]

Letting \( j \to \infty \) and using \( \mathfrak{H} = m \left( \frac{V_c(M, n, \phi_j)}{V(M, g)} \right)^{2/m} \), we obtained

\[
V(M, g) = \lim_{j \to \infty} \int_M \sum_{A=1}^{n+1} (\phi_j^A)^2 = \mathfrak{H}^{-1} \lim_{j \to \infty} \int_M \sum_{A=1}^{n+1} |\nabla \phi_j^A|^2 \tag{4.19}
\]

\[
= \mathfrak{H}^{-1} \lim_{j \to \infty} \left( \int_M \sum_{A=1}^{n+1} |\nabla \phi_j^A|^2 \, dv_g \right)^{2/m} (V(M))^{\frac{m-2}{m}} \tag{4.20}
\]

\[
= V(M, g). \tag{4.21}
\]

Hence, for any fixed \( A \), \( \{\phi_j^A\} \) is a bounded sequence in \( W^{1, m}(M) \). Due to the compact inclusion \( W^{1, m}(M) \hookrightarrow L^2(M) \), by passing to a subsequence we can assume that \( \{\phi_j^A\} \) converges weakly in \( W^{1, m}(M) \), strongly in \( L^2(M) \), and pointwise a.e., to a map \( \psi^A : M \to \mathbb{R} \). We have

\[
\sum_{A=1}^{n+1} (\psi^A)^2 = 1, \text{ a.e. on } M; \tag{4.22}
\]

\[
\lim_{j \to \infty} \int_M (\phi_j^A)^2 = \lim_{j \to \infty} \frac{1}{\lambda_A} \int_M |\nabla \phi_j^A|^2, \tag{4.23}
\]
We give two claims.

Claim 1. For each $A$ fixed, $\psi^A$ belongs to the eigenspace $E_{\lambda_A}$ corresponding to the eigenvalue $\lambda_A$, and then $\psi^A$ is $C^\infty$.

Proof. Let $\pi$ be the orthogonal projection on $E_{\lambda_A}$ and $\pi^\perp$ the projection on $E_{\lambda_A}^\perp$, then

$$\|\phi_j^A\|_{L^2}^2 = \|\pi\phi_j^A\|_{L^2}^2 + \|\pi^\perp\phi_j^A\|_{L^2}^2, \quad (4.24)$$

$$\|\nabla\phi_j^A\|_{L^2}^2 = \|\nabla\pi\phi_j^A\|_{L^2}^2 + \|\nabla\pi^\perp\phi_j^A\|_{L^2}^2. \quad (4.25)$$

Combining with (4.23), we derive that

$$\lim_{j \to \infty} \|\nabla\pi^\perp\phi_j^A\|_{L^2}^2 = \lim_{j \to \infty} \lambda_A\|\nabla\pi\phi_j^A\|_{L^2}^2. \quad (4.26)$$

Noticing (4.15), we have

$$\|\nabla\pi\phi_j^A\|_{L^2}^2 \geq \lambda_B\|\pi^\perp\phi_j^A\|_{L^2}^2, \quad (4.27)$$

where $\lambda_B (> \lambda_A)$ is the next distinct eigenvalue after $\lambda_A$. Hence,

$$0 \geq \lim_{j \to \infty} (\lambda_A - \lambda_B)\|\pi^\perp\phi_j^A\|_{L^2}^2 \geq 0, \quad (4.28)$$

which implies $\lim_{j \to \infty} \|\pi^\perp\phi_j^A\|_{L^2}^2 = 0$ and then $\pi^\perp\psi^A = 0$. This proves Claim 1. $\square$

Claim 2. $\{\phi_j^A\}$ converges to $\psi^A$ strongly in $W^{1,2}(M)$.

Proof. By using Claim 1, we have

$$\|\nabla\phi_j^A - \nabla\psi^A\|_{L^2}^2 = \|\nabla\phi_j^A\|_{L^2}^2 + \|\nabla\psi^A\|_{L^2}^2 - 2\langle\nabla\phi_j^A, \nabla\psi^A\rangle_{L^2} \quad (4.29)$$

$$= \|\nabla\phi_j^A\|_{L^2}^2 + \lambda_A\|\psi^A\|_{L^2}^2 - 2\lambda_A\langle\phi_j^A, \psi^A\rangle_{L^2}. \quad (4.30)$$

By using (4.15), we derive

$$\lim_{j \to \infty} \|\nabla\phi_j^A - \nabla\psi^A\|_{L^2}^2 = 0. \quad (4.31)$$

This proves Claim 2. $\square$

Now we can continue the proof. Denote $mf = \sum_{A=1}^m |\nabla\psi^A|^2$. By taking the limit, all the inequalities from (4.5) to (4.12) for $\phi$ becomes the equalities, that is,

$$V(M, g) = \int_M \sum_{A=1}^m |\psi^A|^2 = \sum_{A=1}^m \frac{1}{\lambda_A} \int_M |\nabla\psi^A|^2 + \sum_{A=m+1}^{n+1} \frac{1}{\lambda_A} \int_M |\nabla\psi^A|^2 \quad (4.32)$$

$$= \sum_{A=1}^m \frac{1}{\lambda_A} \int_M |\nabla\psi^A|^2 + \frac{1}{\lambda_m} \int_M \sum_{A=m+1}^{n+1} |\nabla\psi^A|^2 \quad (4.33)$$

$$= \sum_{A=1}^m \frac{1}{\lambda_A} \int_M |\nabla\psi^A|^2 + \frac{1}{\lambda_m} \int_M f \sum_{A=1}^m \left(1 - \frac{1}{f} |\nabla\psi^A|^2\right) \quad (4.34)$$

$$= \sum_{A=1}^m \frac{1}{\lambda_A} \int_M |\nabla\psi^A|^2 + \sum_{A=1}^m \frac{1}{\lambda_A} \int_M f \left(1 - \frac{1}{f} |\nabla\psi^A|^2\right) \quad (4.35)$$

$$= \sum_{A=1}^m \frac{1}{\lambda_A} \int_M f \quad (4.36)$$

$$=\delta^{-1} m \left(\int_M f^{m/2} \, dv_g\right)^{2/m} (V(M))^{m-2/m} \quad (4.37)$$
\[ = \delta^{-1} m \left( V_c(M,n) \right)^{2/m} \left( V(M) \right)^{m-2/m} = V(M,g). \] (4.37)

The strong convergence in \( W^{1,2}(M) \) proves that
\[ \psi^* h_1 = \lim_{j \to \infty} \phi_j^* h_1 = \lim_{j \to \infty} f_j g = f g, \] (4.38)
which means that \( \psi : M \to S^n \) is a conformal map with the conformal factor \( f \). Similar with the proof of Lemma 3.4, we can prove
\[ \lambda_1 = \cdots = \lambda_m \] (4.39)
and then \( f = \delta/m = \lambda_1 \).

By scaling we may assume \( \lambda_1 = m \). Then \( \psi : M \to S^n \) is an isometric immersion. Since each \( \psi^A \in E_{\lambda_1} \), we know that \( \psi \) is minimal by the Takahashi theorem.

Conversely, the sufficiency for the equality is directly from the following theorem when \( M \) is closed.

**Theorem 4.2** ([11, Theorem 1.1]). Let \((M,g)\) be an \( m(\geq 2) \)-dimensional closed Riemannian manifold. Suppose there exists a minimal isometric immersion \( \phi \) of \((M,g)\) in \( S^n \). Then
\[ V(M,g) = V_c(M,n,\phi) \geq V_c(M,n). \] (4.40)
Moreover, if \((M,g)\) is not isometric to \((S^n,h_1)\), then \( V(M,g) > V(M, (\gamma \circ \phi)^*c) \) for all \( \gamma \in G \setminus O(n+1) \).

**Remark 4.3.** The proof of Theorem 4.2 is used the following fact:

For any \( \gamma \in G \), there exists \( a \in B^{n+1} \) and \( r \in O(n+1) \) such that \( \gamma = r \circ \gamma_a \), where \( \gamma_a \) is defined in Remark 3.2.

This fact allows us to replace \( G \) by the subgroup \( G' = \{ \gamma_a | a \in B^{n+1} \} \) when taking the supremum in the definition of \( V(M,n,\phi) \). (see [10, pp. 259])

### 4.3. A corollary.

**Corollary 4.4.** (1) Let \((M,g)\) be a closed \( m(\geq 2) \)-dimensional Riemannian manifold which can be minimally immersed into \( S^n \) by \( \phi \). If \((M,g)\) is not isometric to \((S^n,h_1)\), then for any metric \( \tilde{g} \in [g] \), we have
\[ \delta(\tilde{g}) V(M,\tilde{g})^{2/m} \leq m V(M,g)^{2/m}, \] (4.41)
where \([g]\) is the conformal class of \( g \), and \( \delta(\tilde{g}) \) is the harmonic mean of the first \( m \) nonzero eigenvalues with respect to \( \tilde{g} \).

Moreover, equality holds if and only if the minimal immersion \( \phi \) is given by a subspace of the first eigenfunctions and \( \tilde{g} = kg \) for some constant \( k > 0 \).

(2) For any metric \( g \) on \( S^n, g \in [h_1] \), we have
\[ \delta(g) V(S^n,g)^{2/m} \leq \delta(h_1) V(S^n,h_1)^{2/m}, \] (4.42)
where equality holds if and only if \( g = kh_1 \) for some constant \( k > 0 \).

**Proof.** (1) The inequality (4.41) is from Theorem 1.5 and Theorem 4.2. Now assume that the equality holds. After rescaling, we can assume that \( \delta(\tilde{g}) = m \). Then \( V(M,\tilde{g}) = V(M,g) \) and there is a minimal immersion \( \psi : (M,\tilde{g}) \to S^n \) with \( \lambda_1(\tilde{g}) = \cdots = \lambda_m(\tilde{g}) = m \) by the case of equality in Theorem 1.5.
By reviewing the proof of Theorem 1.5, we can assume $\psi = \gamma_a \circ \phi$ for some $a \in B^{n+1}$. Since $(M, g)$ is not isometric to $(S^m, h_1)$, $\gamma_a$ must be the identity map by Theorem 4.2 (and Remark 4.3). This follows that $\tilde{g} = \psi^*h_1 = \phi^*h_1 = g$.

(2) This assertion follows from Theorem 1.5, by noting that $V_c(S^m, g) = V_c(S^m, h_1) = V(S^m, h_1)$ and $\lambda_1(h_1) = \lambda_1(h_1) = m$. \hfill $\square$

5. Proof of Theorem 1.10

In this section, we prove Theorem 1.10. Since the main approaches are the same as in Sect. 4, we will omit some same details but emphasize the differences.

5.1. The inequality in (1.28). Recall that the variational characterization of the Steklov eigenvalue $\sigma_i(i \geq 1)$ (cf. [38, 39])

$$\sigma_i = \inf_{u \in H^1(M) \setminus \{0\}} \left\{ \frac{\int_M |\nabla u|^2}{\int_{\partial M} u^2} \mid \int_{\partial M} u v_j = 0, j = 0, \ldots, i - 1 \right\}, \quad (5.1)$$

where $\{v_i\}_{i \geq 0}$ is an orthonormal set of Steklov eigenfunctions.

Let $\phi : (M, g) \to B^n$ be a conformal map with $\phi(\partial M) \subset \partial B^n$. By analogous arguments in previous sections, there exists $\gamma \in G$ such that $\Phi = \gamma \circ \phi = (\Phi_1, \ldots, \Phi_n)$ satisfies that

$$\int_{\partial M} \Phi^A = 0, \quad A = 1, \ldots, n. \quad (5.2)$$

$$\int_{\partial M} \Phi^A v_B = 0, \text{ for } 1 \leq B < A \leq n. \quad (5.3)$$

Let $\tilde{\Phi}^A$ be a harmonic extension of $\Phi^A|_{\partial M}$. Now for each $A = 1, \ldots, n$, we have

$$\sigma_A \int_{\partial M} (\tilde{\Phi}^A)^2 \leq \int_M |\nabla \tilde{\Phi}^A|^2 \leq \int_M |\nabla \Phi^A|^2. \quad (5.4)$$

On the other hand, if we denote $\tilde{g} = \Phi^*g_{B^n} = fg$, then

$$0 \leq |\nabla \Phi^A|^2 \leq 1, \quad \sum_{A=1}^n |\nabla \Phi^A|^2 = m, \quad (5.5)$$

and

$$mf = \sum_{A=1}^n |\nabla \Phi^A|^2. \quad (5.6)$$

where $\tilde{\nabla}$ is the gradient operator on $(M, \tilde{g})$.

An analogous computation as in Sect. 4.1 shows that

$$V(\partial M) = \int_{\partial M} 1 \leq \sum_{A=1}^m \frac{1}{\sigma_A} \int_M f |\nabla \Phi^A|^2 + \sum_{A=m+1}^n \frac{1}{\sigma_A} \int_M f |\nabla \tilde{\Phi}^A|^2$$

$$\leq \sum_{A=1}^m \frac{1}{\sigma_A} \int_M f |\nabla \Phi^A|^2 + \frac{1}{\sigma_m} \int_M f \sum_{A=m+1}^n |\nabla \tilde{\Phi}^A|^2 \quad (5.7)$$

$$= \sum_{A=1}^m \frac{1}{\sigma_A} \int_M f |\nabla \Phi^A|^2 + \frac{1}{\sigma_m} \int_M f (m - \sum_{A=1}^m |\nabla \Phi^A|^2)$$

$$= \sum_{A=1}^m \frac{1}{\sigma_A} \int_M f |\nabla \Phi^A|^2 + \frac{1}{\sigma_m} \int_M f \sum_{A=1}^m (1 - |\nabla \Phi^A|^2)$$
\[
\leq \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M f |\nabla \phi^A|^2 + \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M f (1 - |\nabla \phi^A|^2) \tag{5.8}
\]
\[
= \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M f. \tag{5.9}
\]

Hence, by using the Hölder inequalities, we have

\[
\tilde{S}(\sigma_1, \cdots, \sigma_m)V(\partial M) \leq m \int_M f \, dv_g \tag{5.10}
\]
\[
\leq m \left( \int_M f^{m/2} \, dv_g \right)^{2/m} \left( V(M) \right)^{m-2/m} \tag{5.11}
\]
\[
= m \left( V(M, \tilde{g}) \right)^{2/m} \left( V(M) \right)^{m-2/m} \tag{5.12}
\]
\[
\leq m \left( \sup_{\gamma \in G} V(M, (\gamma \circ \phi)^*g_{B^n}) \right)^{2/m} \left( V(M) \right)^{m-2/m}. \tag{5.13}
\]

Taking the infimum over all non-degenerate conformal maps \( \phi : M \to B^n \) with \( \phi(\partial M) \subset \partial B^n \), we obtain the desired inequality.

### 5.2. Equality in \((1.28)\).

Now, we assume that the equality holds.

Choose a sequence of conformal maps \( \phi_j : M \to B^n \) with \( \phi_j(\partial M) \subset \partial B^n \) such that

\[
\lim_{j \to \infty} V_{rc}(M, n, \phi_j) = V_{rc}(M, n), \tag{5.14}
\]

and by composing with a conformal transformation of the sphere we may assume

\[
\int_{\partial M} \phi_j^A = 0, \quad 1 \leq A \leq n. \tag{5.15}
\]
\[
\int_{\partial M} \phi_j^A v_B = 0, \quad \text{for } 1 \leq B < A \leq n. \tag{5.16}
\]

Denote \( \hat{\phi}_j^A = \phi_j^A|_{\partial M} \). Let \( \hat{\phi}_j^A \) be a harmonic extension of \( \hat{\phi}_j^A \) as before. Denote \( mf_j = \sum_{A=1}^n |\nabla \phi_j^A|^2 \) and write \( \tilde{S}(\sigma_1, \cdots, \sigma_m) \) as \( \tilde{S} \) for short.

Like the arguments in Sect. 4.2, We reproduce the previous steps in Sect. 5.1 for each \( \phi_j \) and then take the limit. It follows that all the inequalities must be sharp.

\[
V(\partial M) = \lim_{j \to \infty} \int_{\partial M} \sum_{A=1}^n (\phi_j^A)^2 = \tilde{S}^{-1} \lim_{j \to \infty} \int_{\partial M} \sum_{A=1}^n |\nabla \phi_j^A|^2 \tag{5.17}
\]
\[
= \tilde{S}^{-1} \lim_{j \to \infty} \left( \int_{\partial M} \left( \sum_{A=1}^n |\nabla \phi_j^A|^2 \right)^{m/2} \, dv_g \right)^{2/m} \left( V(M) \right)^{m-2/m} \tag{5.18}
\]
\[
= V(\partial M). \tag{5.19}
\]

Hence, by passing to a subsequence we can assume that \( \{\phi_j^A\} \) converges weakly in \( W^{1,m}(M) \), strongly in \( L^2(M) \), and pointwise a.e., to a map \( \psi^A : M \to \mathbb{R} \). We have

\[
\sum_{A=1}^n (\psi^A)^2 = 1, \quad \text{a.e. on } \partial M; \tag{5.20}
\]
\[
\sum_{A=1}^{n} (\psi^A)^2 \leq 1, \text{ a.e. on } M; \tag{5.21}
\]
\[
\int_{\partial M} (\phi_j^A)^2 \leq \frac{1}{\sigma_A} \int_M |\nabla \phi_j^A|^2, \text{ for } A = 1, \ldots, n; \tag{5.22}
\]
\[
\lim_{j \to \infty} \sum_{A=1}^{n} \int_{\partial M} (\phi_j^A)^2 = \lim_{j \to \infty} \sum_{A=1}^{n} \frac{1}{\sigma_A} \int_M |\nabla \phi_j^A|^2, \tag{5.23}
\]

**Claim 3.** For each \( A \) fixed, \( \{\phi_j^A\} \) converges to \( \psi^A \) strongly in \( W^{1,2}(M) \); \( \psi^A \) belongs to the eigenspace \( E_{\sigma_A} \) corresponding to the eigenvalue \( \sigma_A \), and \( \psi^A \) is harmonic.

**Proof.** It follows from (5.22) and (5.23) that
\[
\lim_{j \to \infty} \frac{1}{\sigma_A} \int_M |\nabla \phi_j^A|^2 = \lim_{j \to \infty} \int_{\partial M} (\phi_j^A)^2 = \int_{\partial M} (\psi^A)^2 \leq \frac{1}{\sigma_A} \int_M |\nabla \psi^A|^2, \tag{5.24}
\]
On the other hand, \( \phi_j^A \to \psi^A \) weakly in \( W^{1,m}(M) \) implies
\[
\int_M |\nabla \psi^A|^2 \leq \lim_{j \to \infty} \int_M |\nabla \phi_j^A|^2. \tag{5.25}
\]
Therefore, we must have
\[
\lim_{j \to \infty} \int_M |\nabla \phi_j^A|^2 = \int_M |\nabla \psi^A|^2 \tag{5.26}
\]
and
\[
\sigma_A \int_{\partial M} (\psi^A)^2 = \int_M |\nabla \psi^A|^2. \tag{5.27}
\]
These equalities proves Claim 3.

Now we continue the proof. Denote \( mf = \sum_{A=1}^{m} |\nabla \psi^A|^2 \). Claim 3 implies \( \psi : M \to B^n \) is a conformal map with the conformal factor \( f \). Also, \( \psi \) is harmonic with \( \psi(\partial M) \subset \partial B^n \).

By taking the limit, we have
\[
V(\partial M) = \int_{\partial M} \sum_{A=1}^{m} |\psi^A|^2 = \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M |\nabla \psi^A|^2 + \sum_{A=m+1}^{n} \frac{1}{\sigma_A} \int_M |\nabla \psi^A|^2 \tag{5.28}
\]
\[
= \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M |\nabla \psi^A|^2 + \frac{1}{\sigma_m} \int_M \sum_{A=m+1}^{n} |\nabla \psi^A|^2 \tag{5.29}
\]
\[
= \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M |\nabla \psi^A|^2 + \frac{1}{\sigma_m} \int_M f \sum_{A=1}^{m} (1 - \frac{1}{f} |\nabla \psi^A|^2) \tag{5.30}
\]
\[
= \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M |\nabla \psi^A|^2 + \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M f(1 - \frac{1}{f} |\nabla \psi^A|^2) \tag{5.31}
\]
\[
= \sum_{A=1}^{m} \frac{1}{\sigma_A} \int_M f \tag{5.31}
\]
\[
= f^{-1} m \left( \int_M f^{m/2} \, dv_g \right)^{2/m} (V(M))^{\frac{m-2}{m}} \tag{5.32}
\]
\[
= f^{-1} m \left( V_c(M, n) \right)^{2/m} (V(M))^{\frac{m-2}{m}} = V(M, g). \tag{5.33}
\]
The strong convergence in $W^{1,2}(M)$ proves that
\[
\psi^*h_1 = \lim_{j \to \infty} \phi^*_j h_1 = \lim_{j \to \infty} f_j g = fg,
\]
which means that $\psi : M \to S^n$ is a conformal map with the conformal factor $f$.

For $s \geq m + 1$, from (5.29) we know that if $\nabla \psi^s \neq 0$ on $M$, then $\sigma_s = \sigma_m$; if $\nabla \psi^s \equiv 0$ on $M$, then we still have $\sigma_s \int_{\partial M} |\psi|^2 = \int_M |\nabla \psi|^2 = 0$. Therefore, we always have $\sigma_s \in E_{\sigma_m}$ for any $s \geq m + 1$.

When $m > 2$, equality in the Hölder inequality (5.32) implies $f$ is constant. Hence, $\psi$ is an isometric immersion after rescaling ($f \equiv 1$ and $\sum_{A=1}^m \frac{1}{\sigma_A} = m$), and then it is a minimal immersion since each coordinate function is harmonic.

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