IRREDUCIBLE CHARACTERS OF FINITE ALGEBRA GROUPS

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1. Introduction

Let $p$ be a prime number, let $q = p^e$ ($e \geq 1$) be a power of $p$ and let $\mathbb{F}_q$ denote the finite field with $q$ elements. Let $A$ be a finite dimensional $\mathbb{F}_q$-algebra. (Throughout the paper, all algebras are supposed to have an identity element). Let $J = J(A)$ be the Jacobson radical of $A$ and let

$$G = 1 + J = \{1 + a : a \in A\}.$$ 

Then $G$ is a $p$-subgroup of the group of units of $A$. Following [7], we refer to a group arising in this way as an $\mathbb{F}_q$-algebra group. As an example, let $J = u_n(q)$ be the $\mathbb{F}_q$-space consisting of all nilpotent uppertriangular $n \times n$ matrices over $\mathbb{F}_q$. Then $J$ is the Jacobson radical of the $\mathbb{F}_q$-algebra $A = \mathbb{F}_q \cdot 1 + J$ and the $p$-group $G = 1 + J$ is the group $U_n(q)$ consisting of all unipotent uppertriangular $n \times n$ matrices over $\mathbb{F}_q$.

A subgroup $H$ of an $\mathbb{F}_q$-algebra group $G$ is said to be an algebra subgroup of $G$ if $H = 1 + U$ for some multiplicatively closed $\mathbb{F}_q$-subspace $U$ of $J$. It is clear that an algebra subgroup of $G$ is itself an $\mathbb{F}_q$-algebra group and that it has $q$-power index in $G$.

The main purpose of this paper is to prove the following result. (Throughout this paper, all characters are taken over the complex field.)

Theorem 1.1. Let $G$ be an $\mathbb{F}_q$-algebra group and let $\chi$ be an irreducible character of $G$. Then there exist an algebra subgroup $H$ of $G$ and a linear character $\lambda$ of $H$ such that $\chi = \lambda^G$.

As a consequence, we obtain Theorem A of [7] (see also [8, Theorem[26.7]]) which asserts that all irreducible characters of an (arbitrary) $\mathbb{F}_q$-algebra group have $q$-power degree. (However, this result will used in the proof of Theorem 1.1.) Following the terminology of [7], we say that a finite group $G$ is a $q$-power-degree group if every irreducible character of $G$ has $q$-power degree. Hence, [7, Theorem A] asserts that every $\mathbb{F}_q$-algebra group is a $q$-power-degree group. In particular, the unitriangular group $U_n(q)$ is a $q$-power-degree group (which is precisely the statement of [7, Corollary B]). On the other hand, our Theorem 1.1 generalizes Theorem C of [7] and answers the question made by I. M. Isaacs immediately before that theorem. We note, moreover, that the statement of our Theorem 1.1 is precisely the

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assertion made by E. A. Gutkin in [5]. The argument used by Gutkin to prove this assertion was defective and a counterexample was given by Isaacs to illustrate its flaw (see Section 10 of [7]).

A result similar to our Theorem 1.1 was proved by D. Kazhdan for the group $G = U_n(q)$ in the case where $p \geq n$. Kazhdan’s result appears in the paper [9] (see also [12, Theorem 7.7]) and applies to other finite unipotent algebraic groups. However, Kazhdan imposes a restriction on the prime $p$ in order to use the exponential map. In this paper, we replace the exponential map by the bijection $J \rightarrow 1 + J$ defined by the (natural) correspondence $a \mapsto 1 + a$. Then we follow Kazhdan’s idea and we use Kirillov’s method of coadjoint orbits (see, for example, [10]) to parametrize the irreducible characters of the $\mathbb{F}_q$-algebra group $G = 1 + J$.

2. Class functions associated with coadjoint orbits

Let $J = J(A)$, where $A$ is a finite dimensional $\mathbb{F}_q$-algebra, and let $G = 1 + J$. Let $J^* = \text{Hom}_{\mathbb{F}_q}(J, \mathbb{F}_q)$ be the dual space of $J$ and let $\psi$ be an arbitrary non-trivial linear character of the additive group $\mathbb{F}_q^+$ of the field $\mathbb{F}_q$. For each $f \in J^*$, let $\psi_f : J \rightarrow \mathbb{C}$ be the map defined by

\begin{equation}
\psi_f(a) = \psi(f(a))
\end{equation}

for all $a \in J$. Then $\psi_f$ is a linear character of the additive group $J^+$ of $J$ and, in fact,

\begin{equation}
\text{Irr}(J^+) = \{\psi_f : f \in J^*\}.
\end{equation}

(For any finite group $X$, we denote by Irr($X$) the set of all irreducible characters of $X$.)

The group $G$ acts on $J^*$ by $(x \cdot f)(a) = f(x^{-1}ax)$ for all $x \in G$, all $f \in J^*$ and all $a \in J$. (Usually, we refer to this action as the coadjoint action of $G$.) Let $\Omega(G)$ denote the set of all $G$-orbits on $J^*$. We claim that the cardinality $|\Omega|$ of any $G$-orbit $\Omega \in \Omega(G)$ is a $q^2$-power. To see this, let $f \in J^*$ be arbitrary and define $B_f : J \times J \rightarrow \mathbb{F}_q$ by

\begin{equation}
B_f(a, b) = f([ab])
\end{equation}

for all $a, b \in J$ (here $[ab] = ab - ba$ is the usual Lie product of $a, b \in J$). Then $B_f$ is a skew-symmetric $\mathbb{F}_q$-bilinear form. Let $n = \dim J$, let $(e_1, \ldots, e_n)$ be an $\mathbb{F}_q$-basis of $J$ and let $M(f)$ be the skew-symmetric matrix which represents $B_f$ with respect to this basis. Then $M(f)$ has even rank (see, for example, [2, Theorem 8.6.1]). Let

$$\text{Rad}(f) = \{a \in J : f([ab]) = 0 \text{ for all } b \in J\}$$

be the radical of $B_f$. Then $\text{Rad}(f)$ is an $\mathbb{F}_q$-subspace of $J$ and

\begin{equation}
\dim \text{Rad}(f) = \dim J - \text{rank } M(f).
\end{equation}

We have the following result.
Proposition 2.1. Let \( f \in J^* \) be arbitrary. Then \( \text{Rad}(f) \) is a multiplicatively closed \( \mathbb{F}_q \)-subspace of \( J \). Moreover, the centralizer \( C_G(f) \) of \( f \) in \( G \) is the algebra subgroup \( 1 + \text{Rad}(f) \) of \( G \). In particular, if \( O \in \Omega(G) \) is the \( G \)-orbit which contains \( f \), then \( |O| = q^{\text{rank}(M(f))} \) is a \( q^2 \)-power.

Proof. Since \( [ab,c] = [a, bc] + [b, ca] \), we clearly have \( f([ab,c]) = 0 \) for all \( a, b \in \text{Rad}(f) \) and all \( c \in J \). Thus \( \text{Rad}(f) \) is multiplicatively closed.

On the other hand, let \( x \in G \) be arbitrary. Then \( x \in C_G(f) \) if and only if \( f(x^{-1}bx) = f(b) \) for all \( b \in J \). Hence \( x \in C_G(f) \) if and only if \( f(bx) = f(xb) \) for all \( b \in J \). Now, let \( a = x - 1 \in J \). Then \( f(bx) = f(b) + f(ba) \) and \( f(xb) = f(b) + f(ab) \), and so \( f([ab]) = f(xb) - f(bx) \) for all \( b \in J \). It follows that \( x \in C_G(f) \) if and only if \( a \in \text{Rad}(f) \).

For the last assertion, we note that \( |G| = |C_G(f)| \cdot |O| \), that \( |G| = q^{\dim J} \) and (as we have just proved) that \( |C_G(f)| = q^{\dim \text{Rad}(f)} \). Therefore, by (2.4), we deduce that \( |O| = q^{\dim J - \dim \text{Rad}(f)} = q^{\text{rank}(M(f))} \).

For each \( O \in \Omega(G) \), we define the function \( \phi_O : G \to \mathbb{C} \) by the rule

\[
\phi_O(1 + a) = \frac{1}{\sqrt{|O|}} \sum_{f \in O} \psi_f(a)
\]

for all \( a \in J \). It is clear that \( \phi_O \) is a class function of \( G \) of degree

\[
\phi_O(1) = \sqrt{|O|} = q^{\text{rank}(M(f))}
\]

where \( M(f) \) is as before. Moreover, we have the following result.

Proposition 2.2. The set \( \{ \phi_O : O \in \Omega(G) \} \) is an orthonormal basis for the \( \mathbb{C} \)-space \( \text{cf}(G) \) consisting of all class functions on \( G \). In particular, we have

\[
\frac{1}{|G|} \sum_{x \in G} \phi_O(x) \overline{\phi_{O'}(x)} = \delta_{O, O'}
\]

for all \( O, O' \in \Omega(G) \). (Here, \( \delta \) denotes the usual Kronecker symbol.)

Proof. Let \( \langle \cdot, \cdot \rangle \) denote the Frobenius scalar product on \( \text{cf}(G) \). Let \( O, O' \in \Omega(G) \) be arbitrary. Then

\[
\langle \phi_O, \phi_{O'} \rangle_G = \frac{1}{|G|} \sum_{x \in G} \phi_O(x) \overline{\phi_{O'}(x)}
\]

\[
= \frac{1}{|J|} \sum_{a \in J} \frac{1}{\sqrt{|O|}} \frac{1}{\sqrt{|O'|}} \sum_{f \in O} \sum_{f' \in O'} \psi_f(a) \overline{\psi_{f'}(a)}
\]

\[
= \frac{1}{\sqrt{|O|}} \frac{1}{\sqrt{|O'|}} \sum_{f \in O} \sum_{f' \in O'} \left( \frac{1}{|J|} \sum_{a \in J} \psi_f(a) \overline{\psi_{f'}(a)} \right)
\]
\begin{align*}
&= \frac{1}{\sqrt{|O|}} \frac{1}{\sqrt{|O'|}} \sum_{f \in O} \sum_{f' \in O'} \langle \psi_f, \psi_{f'} \rangle_{J^+} \\
&= \frac{1}{\sqrt{|O|}} \frac{1}{\sqrt{|O'|}} \sum_{f \in O} \sum_{f' \in O'} \delta_{f,f'}
\end{align*}

(\text{using (2.2)}) and so

\[ \langle \phi_O, \phi_{O'} \rangle_G = \delta_{O,O'} \]

To conclude the proof, we claim that \(|\Omega(G)|\) equals the class number \(k_G\) of \(G\); we recall that \(k_G = \dim \chi \operatorname{cf}(G)\) (see, for example, [8, Corollary 2.7 and Theorem 2.8]). Firstly, we observe that \(k_G\) is the number of \(G\)-orbits on \(J\) for the \textit{adjoint action}: \(x \cdot a = xax^{-1}\) for all \(x \in G\) and all \(a \in J\). Let \(\theta\) be the permutation character of \(G\) on \(J\) (see [8] for the definition). Then, by [8, Corollary 5.15],

\[ k_G = \langle \theta, 1_G \rangle_G. \]

Moreover, by definition, we have

\[ \theta(x) = |\{a \in J : x \cdot a = a\}| \]

for all \(x \in G\).

On the other hand, consider the action of \(G\) on \(\operatorname{Irr}(J^+)\) given by

\[ x \cdot \psi_f = \psi_{x \cdot f} \]

for all \(x \in G\) and all \(f \in J^*\). We clearly have

\[ (x \cdot \psi_f)(x \cdot a) = \psi_f(a) \]

for all \(x \in G\), all \(f \in J^*\) and all \(a \in J\). It follows from Brauer's Theorem (see [8, Theorem 6.32]) that

\[ \theta(x) = |\{f \in J^* : x \cdot \psi_f = \psi_f\}| \]

for all \(x \in G\). Therefore, \(\theta\) is also the permutation character of \(G\) on \(\operatorname{Irr}(J^+)\) and so

\[ \langle \theta, 1_G \rangle_G = |\Omega(G)|. \]

The claim follows and the proof is complete.

We will prove (see Theorem 4.1 in Section 4) that

\[ \operatorname{Irr}(G) = \{\phi_O : O \in \Omega(G)\}. \]

(This is the key for the proof of Theorem 1.1.) Therefore, the next result will of course be a consequence of that theorem. However, we give below a very easy proof (independent of Theorem 1.1) of the second orthogonality relations for the functions \(\phi_O\) for \(O \in \Omega(G)\).

\textbf{Proposition 2.3.} Let \(x, y \in G\) be arbitrary. Then

\[ \sum_{O \in \Omega(G)} \phi_O(x) \overline{\phi_O(y)} = \begin{cases} |C_G(x)|, & \text{if } x \text{ and } y \text{ are } G\text{-conjugate}, \\ 0, & \text{otherwise}. \end{cases} \]
Proof. Let $a = x - 1$ and $b = y - 1$. Then
\[
\sum_{O \in \Omega(G)} \phi_O(x) \overline{\phi_O(y)} = \sum_{O \in \Omega(G)} \frac{1}{|O|} \sum_{f \in O} \sum_{g \in O} \psi_f(a) \overline{\psi_g(b)}
\]
\[
= \sum_{O \in \Omega(G)} \frac{1}{|O|} \sum_{f \in O} \frac{1}{|C_G(f)|} \left( \sum_{z \in G} \psi_f(a) \overline{\psi_f(z^{-1}bz)} \right)
\]
\[
= \sum_{O \in \Omega(G)} \sum_{f \in O} \frac{1}{|G|} \sum_{z \in G} \psi_f(a - z^{-1}bz)
\]
\[
= \frac{1}{|G|} \sum_{z \in G} \sum_{f \in J^+} \psi_f(a - z^{-1}bz)
\]
where $\rho_{J^+}$ denotes the regular character of $J^+$. It follows that
\[
\sum_{O \in \Omega(G)} \phi_O(x) \overline{\phi_O(y)} = \sum_{z \in G} \delta_{a,z^{-1}bz} = \{ z \in G : a = z^{-1}bz \}
\]
and this clearly completes the proof. 

As a consequence we obtain the following additive decomposition of the regular character $\rho_G$ of $G$ (which is also a consequence of Theorem 4.1).

**Corollary 2.1.** $\rho_G = \sum_{O \in \Omega(G)} \phi_O(1) \phi_O$.

**Proof.** Let $x \in G$ be arbitrary. By the previous proposition,
\[
\sum_{O \in \Omega(G)} \phi_O(1) \phi_O(x) = \delta_{x,1} |G| = \rho_G(x).
\]
The result follows (by definition of $\rho_G$). 

3. **Maximal algebra subgroups**

In this section, we consider restriction and induction of the class functions defined in the previous section. We follow Kirillov’s theory on nilpotent Lie groups (see, for example, [3]). As before, let $A$ be a finite dimensional $F_q$-algebra, let $J = J(A)$ and let $G = 1 + J$.

Let $U$ be a maximal multiplicatively closed $F_q$-subspace of $J$. Then $J^2 \subseteq U$; otherwise, we must have $U + J^2 = J$ and this implies that $U = J$ (see [6, Lemma 3.1]). It follows that $U$ is an ideal of $A$ and so $H = 1 + U$ is a normal subgroup of $G$ (in the terminology of [4], we say that $H$ is an ideal subgroup of $G$). Moreover, we have $\dim U = \dim J - 1$ and so $|G : H| = q$.

Let $\pi : J^* \rightarrow U^*$ be the natural projection (by definition, for any $f \in J^*$, $\pi(f) \in U^*$ is the restriction of $f$ to $U$). Then the kernel of $\pi$ is the $F_q$-subspace
\[
U^\perp = \{ f \in J^* : f(a) = 0 \text{ for all } a \in U \} \]
of $U$. On the other hand, for any $f \in U^*$, the fibre $\pi^{-1}(\pi(f))$ of $\pi(f) \in U^*$ is the subset

$$\mathcal{L}(f) = \{g \in U^*: g(a) = f(a) \text{ for all } a \in U\}$$

of $J^*$. It is clear that

$$\mathcal{L}(f) = f + U^\perp = \{f + g: g \in U^\perp\}$$

for all $f \in J^*$.

Let $f \in J^*$ be arbitrary and let $f_0$ denote the projection $\pi(f) \in U^*$. Let $O \in \Omega(G)$ be the $G$-orbit which contains $f$ and let $O_0 \in \Omega(H)$ be the $H$-orbit which contains $f_0$. Since $\pi(x \cdot f) = x \cdot \pi(f)$ for all $x \in G$ (because $H$ is normal in $G$, hence $U$ is invariant under the adjoint $G$-action), the projection $\pi(O) \subseteq U^*$ of $O$ is $G$-invariant. Thus $O_0 \subseteq \pi(O)$; in fact, $\pi(O)$ is a disjoint union of $H$-orbits. It follows

$$|O_0| \leq |\pi(O)| \leq |O|.$$ 

Let $\pi_O: O \to \pi(O)$ denote the restriction of $\pi$ to $O$. Since $\pi_O$ is surjective, $O$ is the disjoint union

$$O = \bigcup_{g_0 \in \pi(O)} \pi^{-1}(g_0).$$

Since $\pi^{-1}(\pi(g)) = \mathcal{L}(g) \cap O$ for all $g \in O$, we conclude that

$$|O| = \sum_{g \in \Gamma_O} |\mathcal{L}(g) \cap O|,$$

where $\Gamma_O \subseteq O$ is a set of representatives of the fibres $\pi^{-1}(g_0)$ for $g_0 \in \pi(O)$. Now, we clearly have $\mathcal{L}(x \cdot g) = x \cdot \mathcal{L}(g)$ for all $x \in G$ and all $g \in J^*$. It follows that

$$x \cdot (\mathcal{L}(g) \cap O) = \mathcal{L}(x \cdot g) \cap O$$

for all $x \in G$ and all $g \in J^*$. In particular, we have $|\mathcal{L}(g) \cap O| = |\mathcal{L}(f) \cap O|$ for all $g \in O$. Since $|\Gamma_O| = |O|$, we conclude that

$$|O| = |\pi(O)| \cdot |\mathcal{L}(f) \cap O|.$$ 

In particular, we deduce that

$$|O| \leq q \cdot |\pi(O)|$$

(because $\mathcal{L}(f) = f + U^\perp$, hence $|\mathcal{L}(f)| = q$). We claim that $|\pi(O)|$ is a power of $q$. In fact, we have the following.

**Lemma 3.1.** Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi: J^* \to U^*$ be the natural projection. Let $f \in J^*$ be arbitrary and let $O \in \Omega(G)$ be the $G$-orbit which contains $f$. Then:

(a) The centralizer $C_G(f_0)$ of $f_0 = \pi(f)$ in $G$ is an algebra subgroup of $G$; in fact, $C_G(f_0) = 1 + R$ where $R = \{a \in J: f((ab)) = 0 \text{ for all } b \in U\}$.

(b) $|\pi(O)|$ is a power of $q$; in fact, either $|O| = |\pi(O)|$, or $|O| = q \cdot |\pi(O)|$. 
Proof. The proof of (a) is analogous to the proof of Proposition 2.1. On
the other hand, since $G$ acts transitively on $\pi(O)$, we have
\[ |\pi(O)| = |G| \cdot |C_G(f_0)|^{-1} = q^{\dim J - \dim R}. \]
The last assertion is clear because $|\pi(O)| \leq |O| \leq q |\pi(O)|$. \hfill $\square$

Following [11], we say a $G$-orbit $O \in \Omega(G)$ is of type I (with respect to $H$) if $|O| = |\pi(O)|$; otherwise, if $|O| = q |\pi(O)|$, we say that $O$ is of type II (with respect to $H$). The following result asserts that our definition is, in fact, equivalent to Kirillov’s definition.

**Proposition 3.1.** Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi: J^* \to U^*$ be the natural projection. Let $O \in \Omega(G)$ be arbitrary. Then:

(a) $O$ is of type I (with respect to $H$) if and only if $\mathcal{L}(f) \cap O = \{f\}$ for all $f \in O$;

(b) $O$ is of type II (with respect to $H$) if and only if $\mathcal{L}(f) \subseteq O$ for all $f \in O$.

**Proof.** Suppose that $|O| = |\pi(O)|$ (i.e., $O$ is of type I). Then (3.2) implies that $|\mathcal{L}(f) \cap O| = 1$ and so $\mathcal{L}(f) \cap O = \{f\}$. On the other hand, if $|O| = q |\pi(O)|$ (i.e., if $O$ is of type II), we must have $|\mathcal{L}(f) \cap O| = q$ and so $\mathcal{L}(f) \subseteq O$ (because $|\mathcal{L}(f)| = q$). The result follows by (3.1). \hfill $\square$

Now, let $n = \dim J$ and let $(e_1, \ldots, e_n)$ be an $F_q$-basis of $J$ such that $e_i \in U$ for all $1 \leq i \leq n - 1$. Moreover, let $M = M(f)$ be the $n \times n$ skew-symmetric matrix which represents the bilinear form $B_f$ with respect to the basis $(e_1, \ldots, e_n)$. By Proposition 2.1, $|O| = q^{\rank M}$. Moreover, the matrix $M$ has the form
\[
M = \begin{bmatrix}
M_0 & -\nu^T \\
\nu & 0
\end{bmatrix}
\]
where $M_0 = M(f_0)$ is the $(n - 1) \times (n - 1)$ skew-symmetric matrix which represents the bilinear form $B_{f_0}: U \times U \to F_q$ with respect to the $F_q$-basis $(e_1, \ldots, e_{n-1})$ of $U$, and $\nu$ is the row vector $\nu = [f(e_1 e_1) \cdots f(e_n e_{n-1})]$. Since $O_0$ is the $H$-orbit of the element $f_0 \in U^*$, we have $|O_0| = q^{\rank M_0}$ (by Proposition 2.1). Since $M$ and $M_0$ are skew-symmetric matrices, they have even ranks and so, either $\rank M = \rank M_0$, or $\rank M = \rank M_0 + 2$. This concludes the proof of the following.

**Lemma 3.2.** Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi: J^* \to U^*$ be the natural projection. Let $O \in \Omega(G)$ be arbitrary and let $O_0 \in \Omega(H)$ be such that $O_0 \subseteq \pi(O)$. Then, either $|O| = |O_0|$, or $|O| = q^2 |O_0|$.

We note that, since $|O_0| \leq |\pi(O)| \leq |O|$, the equality $|O| = |O_0|$ implies that $O$ is of type I (with respect to $H$); hence, $|O| = q^2 |O_0|$ whenever $O$ is of type II (with respect to $H$). Our next result shows that the dichotomy of the preceding lemma characterizes the $G$-orbit $O$ with respect to the subgroup $H$. 

Proposition 3.2. Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi : J^* \to U^*$ be the natural projection. Let $O \in \Omega(G)$ be arbitrary and let $O_0 \in \Omega(H)$ be such that $O_0 \subseteq \pi(O)$. Moreover, let $f \in O$ be such that the projection $f_0 = \pi(f)$ lies in $O_0$. Then:

(a) The following are equivalent:
   
   (i) $|O|$ is of type I (with respect to $H$);
   (ii) $|O| = |O_0|$;
   (iii) $\dim \text{Rad}(f) = \dim \text{Rad}(f_0) + 1$;
   (iv) $|C_G(f)| = q |C_H(f_0)|$.

(b) The following are equivalent:

   (i) $|O|$ is of type II (with respect to $H$);
   (ii) $|O| = q^2 |O_0|$;
   (iii) $\dim \text{Rad}(f) = \dim \text{Rad}(f_0) - 1$;
   (iv) $|C_G(f)| = q^{-1} |C_H(f_0)|$.

Proof. The equivalence (iii) $\Leftrightarrow$ (iv) (in both (a) and (b)) follows from Proposition 2.2. On the other hand, the equivalence (ii) $\Leftrightarrow$ (iii) (in both (a) and (b)) follows from 2.4 (using also Proposition 2.1). We have already proved that (ii) $\Rightarrow$ (i) (which is equivalent to (i) $\Rightarrow$ (ii) in (b)). Conversely, suppose that $|O|$ is of the type I (with respect to $H$). Then $|\pi(O)| = |O|$. Since $G$ acts transitively on $\pi(O)$, we deduce that

$$|G| = |C_G(f_0)| \cdot |\pi(O)| = |C_G(f_0)| \cdot |O|.$$

It follows that $|C_G(f_0)| = |C_G(f)|$. Since $C_G(f) \subseteq C_G(f_0)$, we conclude that $C_G(f_0) = C_G(f)$. On the other hand, $C_H(f_0) \subseteq C_G(f_0)$ and so $C_H(f_0) \subseteq C_G(f)$. By the equivalence (ii) $\Leftrightarrow$ (iv) (in both (a) and (b)), we conclude that $|O| = |O_0|$. This completes the proof of (i) $\Rightarrow$ (ii) in (a). Hence, the implication (ii) $\Rightarrow$ (i) in (b) is also true. The proof is complete.

We note that, since $O_0 \subseteq \pi(O)$, the equality $|O| = |O_0|$ implies that $\pi(O) = O_0$ (hence, the projection $\pi(O)$ of the $G$-orbit $O$ is an $H$-orbit). This concludes the proof of part (a) of the following result.

Proposition 3.3. Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi : J^* \to U^*$ be the natural projection. Let $O \in \Omega(G)$ be arbitrary and let $O_0 \in \Omega(H)$ be such that $O_0 \subseteq \pi(O)$. Moreover, let $f \in O$ be such that the projection $f_0 = \pi(f)$ lies in $O_0$. Then the following statements hold:

(a) If $O$ is of type I (with respect to $H$), then $\pi(O) = O_0$ is a single $H$-orbit on $U^*$.

(b) Suppose that $O$ is of type II (with respect to $H$). Let $e \in J$ be such that $J = U \oplus \mathbb{F}_e$ and, for each $\alpha \in \mathbb{F}_q$, let $x_{\alpha}$ denote the element $1 + \alpha e \in G$. Then $\pi(O)$ is the disjoint union

$$\pi(O) = \bigcup_{\alpha \in \mathbb{F}_q} O_{\alpha}.$$
where, for each \( \alpha \in \mathbb{F}_q \), \( O_\alpha \subseteq U^* \) is the \( H \)-orbit which contains the element \( x_\alpha \cdot f_0 \in U^* \). We have
\[
O_\alpha = x_\alpha \cdot O_0 = \{ x_\alpha \cdot g : g \in O_0 \}
\]
for all \( \alpha \in \mathbb{F}_q \). Moreover, the set \( \{ x_\alpha : \alpha \in \mathbb{F}_q \} \) can be replaced by any set of representatives of the cosets of \( H \) in \( G \).

**Proof.** It remains to prove part (b).

Let \( \alpha \in \mathbb{F}_q \) be arbitrary. Since \( \pi \) is \( G \)-invariant, we have \( x_\alpha \cdot \pi(f) = \pi(x_\alpha \cdot f) \) and so \( x_\alpha \cdot f_0 \in \pi(O) \) (we recall that \( f_0 = \pi(f) \)). It follows that \( O_\alpha \subseteq \pi(O) \).

Next, we prove that \( O_\alpha = x_\alpha \cdot O_0 \). To see this, let \( x \in H \) be arbitrary. Then \( x \cdot (x_\alpha \cdot f_0) = (xx_\alpha) \cdot f_0 = (x_\alpha \cdot f_0) = x_\alpha \cdot ((x_\alpha^{-1}xx_\alpha) \cdot f_0) \). Since \( H \) is normal in \( G \), we have \( x_\alpha^{-1}xx_\alpha \in H \) and so \((x_\alpha^{-1}xx_\alpha) \cdot f_0 \in O_0 \). Since \( x \in H \) is arbitrary, we conclude that
\[
(3.3) \quad O_\alpha \subseteq x_\alpha \cdot O_0.
\]

It follows that \( |O_\alpha| \leq |O_0| = q^{-2} |O| \) (by Proposition 3.2) and so \( |O_\alpha| < |O| \).

By Lemma 3.3, we conclude that \( |O_\alpha| = q^{-2} |O| \) and so \( |O_\alpha| = |O_0| = |x_\alpha \cdot O_0| \). By (3.3), we obtain \( O_\alpha = x_\alpha \cdot O_0 \) as required.

Now, suppose that \( \beta \in \mathbb{F}_q \) is such that \( x_\beta \cdot f_0 \in O_\alpha \). Then, there exists \( x \in H \) such that \( x_\beta \cdot f_0 = x \cdot (x_\alpha \cdot f_0) = (xx_\alpha) \cdot f_0 \). It follows that \( x_\beta^{-1}xx_\alpha \in C_H(f_0) \). In particular, \( x_\beta^{-1}xx_\alpha x_\beta^{-1}x_\alpha \in H \) and so \( x_\beta^{-1}x_\alpha \in H \) (because \( H \) is normal in \( G \) and \( x \in H \)). Since \( x_\beta^{-1} = 1 - \beta e + a \) for some \( a \in J^2 \), we have
\[
x_\beta^{-1}x_\alpha = 1 + (\alpha - \beta)e - (\alpha\beta)e^2 + \alpha e a.
\]

Since \( J^2 \subseteq U \), we conclude that \((\alpha - \beta)e \in U \) and this implies that \( \alpha = \beta \). It follows that the \( H \)-orbits \( O_\alpha \subseteq \pi(O) \), for \( \alpha \in \mathbb{F}_q \), are all distinct. Hence, the union \( \bigcup_{\alpha \in \mathbb{F}_q} O_\alpha \) is disjoint and so
\[
\left| \bigcup_{\alpha \in \mathbb{F}_q} O_\alpha \right| = \sum_{\alpha \in \mathbb{F}_q} |O_\alpha| = q \cdot |O_0| = q^{-1} \cdot |O| = |\pi(O)|
\]
(because \( O \) is of type \( I \), hence \( |O| = q |\pi(O)| = q^2 |O_0| \)). Since \( O_\alpha \subseteq O \) for all \( \alpha \in \mathbb{F}_q \), we conclude that
\[
\pi(O) = \bigcup_{\alpha \in \mathbb{F}_q} O_\alpha
\]
as required.

Finally, let \( \Gamma \subseteq G \) be a set of representatives of the cosets of \( H \) in \( G \). Then \( G \) is the disjoint union
\[
G = \bigcup_{x \in \Gamma} xH.
\]
Since \(|G:H|=q\), we have \(|\Gamma|=q\). Moreover, for each \(x \in \Gamma\), there exists a unique \(\alpha \in \mathbb{F}_q\) such that \(x \in x_\alpha H\). It follows that \(x \cdot \mathcal{O}_0 \subseteq x_\alpha \cdot \mathcal{O}_0\) and, by order considerations, the equality must occur.

The proof is complete. \(\square\)

Next, given an arbitrary \(G\)-orbit \(\mathcal{O} \in \Omega(G)\), we consider the restriction \((\phi_\mathcal{O})_H\) of the class function \(\phi_\mathcal{O}\) to the maximal algebra subgroup \(H\). For simplicity, we shall write \(\phi = \phi_\mathcal{O}\). We recall that, by definition (see (2.5)), we have

\[
\phi(1 + a) = \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{f \in \mathcal{O}} \psi_f(a)
\]

for all \(a \in J\).

Suppose that \(\mathcal{O}\) is of type I. Then, by Proposition 3.3, we have \(\pi(\mathcal{O}) = \mathcal{O}_0\). Let \(a \in U\) be arbitrary and consider the class function \(\phi_{\mathcal{O}_0} \in \text{cf}(H)\); for simplicity, we shall write \(\phi_0 = \phi_{\mathcal{O}_0}\). We have

\[
\phi_0(1 + a) = \frac{1}{|\mathcal{O}_0|} \sum_{g \in \mathcal{O}_0} \psi_g(a) = \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{g \in \pi(\mathcal{O})} \psi_g(a).
\]

On the other hand, since \(L(f) \cap \mathcal{O} = \{f\}\) for all \(f \in \mathcal{O}\) (by Proposition 3.1), the map \(\pi\) determines naturally a bijection between the \(G\)-orbit \(\mathcal{O} \subseteq J^*\) and the \(H\)-orbit \(\mathcal{O}_0 = \pi(\mathcal{O}) \subseteq U^*\). Therefore

\[
\sum_{g \in \pi(\mathcal{O})} \psi_g(a) = \sum_{f \in \mathcal{O}_0} \psi_f(a).
\]

Since \(|\mathcal{O}| = |\mathcal{O}_0|\) (by Proposition 3.3), we conclude that

\[
\phi(a) = \phi_0(a)
\]

for all \(a \in U\). It follows that

\[
\phi_H = \phi_0.
\]

Now, suppose that \(\mathcal{O}\) is of type II. Then, by Proposition 3.3, \(\pi(\mathcal{O})\) is the disjoint union

\[
\pi(\mathcal{O}) = \bigcup_{\alpha \in \mathbb{F}_q} \mathcal{O}_\alpha
\]

of \(H\)-orbits \(\mathcal{O}_\alpha\) for \(\alpha \in \mathbb{F}_q\). Let \(a \in U\) be arbitrary. In this case, we have

\[
\sum_{g \in \pi(\mathcal{O})} \psi_g(a) = \sum_{\alpha \in \mathbb{F}_q} \sum_{g \in \mathcal{O}_\alpha} \psi_g(a) = \sum_{\alpha \in \mathbb{F}_q} \sqrt{|\mathcal{O}_\alpha|} \cdot \phi_\alpha(1 + a)
\]

where, for any \(\alpha \in \mathbb{F}_q\), \(\phi_\alpha\) denotes the class function \(\phi_{\mathcal{O}_\alpha} \in \text{cf}(H)\). Since \(|\mathcal{O}_\alpha| = q^{-2} |\mathcal{O}|\) for all \(\alpha \in \mathbb{F}_q\) (see the proof of Proposition 3.3), we conclude that

\[
\sum_{\alpha \in \mathbb{F}_q} \phi_\alpha(1 + a) = \frac{q}{|\mathcal{O}|} \sum_{g \in \pi(\mathcal{O})} \psi_g(a).
\]

On the other hand, we have \(L(f) \subseteq \mathcal{O}\) for all \(f \in \mathcal{O}\) (by Proposition 3.1). Hence, there exist elements \(f_1, \ldots, f_r \in \mathcal{O}\) such that \(\mathcal{O}\) is the disjoint union

\[
\mathcal{O} = L(f_1) \cup \ldots \cup L(f_r).
\]
It follows that
\[ \phi(1 + a) = \frac{1}{\sqrt{|O|}} \sum_{f \in O} \psi_f(a) = \frac{1}{\sqrt{|O|}} \sum_{i=1}^{r} \sum_{f \in \mathcal{L}(f_i)} \psi_f(a) = \frac{q}{\sqrt{|O|}} \sum_{i=1}^{r} \psi_{f_i}(a) \]
(because \( f(a) = f_i(a) \) for all \( f \in \mathcal{L}(f_i) \)). Finally, we clearly have \( r = |\pi(O)| \) and \( \pi(O) = \{ \pi(f_1), \ldots, \pi(f_r) \} \). Therefore,
\[ \sum_{i=1}^{r} \psi_{f_i}(a) = \sum_{g \in \pi(O)} \psi_g(a). \]
It follows that
\[ \phi_H = \sum_{\alpha \in F_q} \phi_\alpha. \]
Finally, we note that, by Proposition 3.3, for each \( \alpha \in F_q \), the \( H \)-orbit \( O_\alpha \) considered above may be chosen to be \( x_\alpha \cdot O_0 \). Now, let \( \alpha \in F_q \) and \( a \in U \) be arbitrary. Then,
\[ \phi_\alpha(1 + a) = \frac{1}{\sqrt{|O_\alpha|}} \sum_{g \in O_\alpha} \psi_g(a) = \frac{1}{\sqrt{|O_0|}} \sum_{f \in O_0} \psi_{x_\alpha f}(a). \]
Let \( f \in O_0 \) be arbitrary. Then, by definition,
\[ \psi_{x_\alpha f}(a) = \psi((x_\alpha \cdot f)(a)) = \psi(f(x_\alpha^{-1}ax_\alpha)) = \psi(f(x_\alpha^{-1}ax_\alpha)). \]
This concludes the proof of the following.

**Proposition 3.4.** Let \( H \) be a maximal algebra subgroup of \( G \), let \( U \subseteq J \) be such that \( H = 1 + U \) and let \( \pi: J^* \rightarrow U^* \) be the natural projection. Let \( O \in \Omega(G) \) be arbitrary and let \( \phi \) denote the class function \( \phi_O \in \text{cf}(G) \). Then, the following statements hold:

(a) If \( O \) is of type I (with respect to \( H \)), then the restriction \( \phi_H \) of \( \phi \) to \( H \) is the class function \( \phi_0 = \phi_{O_0} \in \text{cf}(H) \) corresponding to the \( H \)-orbit \( O_0 = \pi(O) \subseteq U^* \).

(b) Suppose that \( O \) is of type II (with respect to \( H \)). Let \( \{ x_\alpha : \alpha \in F_q \} \) be a set of representatives of the cosets of \( H \) in \( G \), let \( O_0 \in \Omega(H) \) be an (arbitrary) \( H \)-orbit satisfying \( O_0 \subseteq \pi(O) \) and, for each \( \alpha \in F_q \), let \( O_\alpha \in \Omega(H) \) be the \( H \)-orbit \( O_\alpha = x_\alpha \cdot O_0 \). Then, the restriction \( \phi_H \) of \( \phi \) to \( H \) is the sum
\[ \phi_H = \sum_{\alpha \in F_q} \phi_\alpha \]
of the (linearly independent) class functions \( \phi_\alpha \in \text{cf}(H), \alpha \in F_q \), which correspond to the \( H \)-orbits \( O_\alpha, \alpha \in F_q \). Moreover, for each \( \alpha \in F_q \), \( \phi_\alpha \) is the class function defined by \( \phi_\alpha(x) = \phi_0(x_\alpha^{-1}xx_\alpha) \) for all \( x \in H \).

In the next result, we use Frobenius reciprocity to obtain the decomposition of the class function \( \phi_{O_0}^G \in \text{cf}(G) \) induced from the class function \( \phi_{O_0} \in \text{cf}(H) \) which each associated with an arbitrary \( H \)-orbit \( O_0 \in \Omega(H) \).
Proposition 3.5. Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi: J^* \to U^*$ be the natural projection. Let $O_0 \in \Omega(H)$ be arbitrary and let $O \in \Omega(G)$ be an arbitrary $G$-orbit satisfying $O_0 \subseteq \pi(O)$. Moreover, let $\phi_0$ denote the class function $\phi_{O_0} \in cf(H)$. Then, the following statements hold:

(a) Suppose that $O$ is of type I (with respect to $H$). Let $e \in J$ be such that $J = U \oplus \mathbb{F}_q e$ and let $e^* \in U^\perp$ be such that $e^*(e) = 1$. Let $f \in O$ be arbitrary and, for each $\alpha \in \mathbb{F}_q$, let $O(\alpha) \in \Omega(G)$ denote the $G$-orbit which contains the element $f + \alpha e^* \in J^*$. Then, the $G$-orbits $O(\alpha)$, for $\alpha \in \mathbb{F}_q$, are all distinct and the induced class function $\phi_0^G$ is the sum

$$\phi_0^G = \sum_{\alpha \in \mathbb{F}_q} \phi_{O(\alpha)}$$

of the (linearly independent) class functions $\phi_{O(\alpha)}$, for $\alpha \in \mathbb{F}_q$.

(b) If $O$ is of type II (with respect to $H$), then $\phi_0^G$ is the class function $\phi = \phi_O$ which corresponds to the $G$-orbit $O \in \Omega(G)$.

Proof. By Proposition 2.2, we have

$$\phi_0^G = \sum_{\mathcal{O}' \in \Omega(G)} \mu_{\mathcal{O}'} \phi_{\mathcal{O}'}$$

where $\mu_{\mathcal{O}'} = \langle \phi_0^G, \phi_{\mathcal{O}'} \rangle_G$ for all $\mathcal{O}' \in \Omega(G)$. Let $\mathcal{O}' \in \Omega(G)$ be arbitrary. By Frobenius reciprocity, we have

$$\langle \phi_0^G, \phi_{\mathcal{O}'} \rangle_G = \langle \phi_0, (\phi_{\mathcal{O}'})_H \rangle_H.$$ 

Therefore, by Proposition 3.4, $\mu_{\mathcal{O}'} \neq 0$ if and only if $O_0 \subseteq \pi(\mathcal{O}')$; moreover, if this is the case, we have $\mu_{\mathcal{O}'} = 1$. On the other hand, let $f \in O$ be such that $\pi(f) \in O_0$; we note that, in the case where $O$ is of type I (with respect to $H$), we have $\pi(O) = O_0$ (by Proposition 3.3) and so $\pi(f) \in O_0$ for all $f \in O$. Let $\mathcal{O}' \in \Omega(G)$ be such that $O_0 \subseteq \pi(\mathcal{O}')$. Then $\pi(f) = \pi(f')$ for some $f' \in \mathcal{O}'$, hence $f' \in \mathcal{L}(f)$. Since $U^\perp = \mathbb{F}_q e^*$, we have $\mathcal{L}(f) = f + \mathbb{F}_q e^*$ and so there exists $\alpha \in \mathbb{F}_q$ such that $f' = f + \alpha e^*$. Therefore, $\mathcal{O}'$ is the $G$-orbit which contains the element $f + \alpha e^*$; we denote this $G$-orbit by $O(\alpha)$. It follows that

$$\phi_0^G = \sum_{\alpha \in \Gamma} \phi_{O(\alpha)}$$

where $\Gamma \subseteq \mathbb{F}_q$ is such that the $G$-orbits $O(\alpha)$, for $\alpha \in \Gamma$, are all distinct.

Suppose that $O$ is of type II. Then $\mathcal{L}(f) \subseteq O$ (by Proposition 3.1) and so $O(\alpha) = O$ for all $\alpha \in \mathbb{F}_q$. It follows that, in this case, $|\Gamma| = 1$ and so

$$\phi_0^G = \phi$$

as required (in part (b)).

On the other hand, suppose that $O$ is of type I. Let $\alpha \in \mathbb{F}_q$ be arbitrary. Then, by Proposition 3.1, the $G$-orbit $O(\alpha)$ is also of type I; otherwise,
Proposition 3.3. \[ \pi \] that suppose that \( O \) and so
\[
q \sqrt{|O_0|} = \phi_0^G(1) = \sum_{\alpha \in \Gamma} \phi_{O(\alpha)}(1) = \sum_{\alpha \in \Gamma} \sqrt{|O(\alpha)|} = |\Gamma| \sqrt{|O_0|}.
\]
It follows that \( |\Gamma| = q \) and so \( \Gamma = \mathbb{F}_q \). In particular, we conclude that the \( G \)-orbits \( O(\alpha) \), for \( \alpha \in \mathbb{F}_q \), are all distinct. Moreover, we obtain
\[
\phi_0^G = \sum_{\alpha \in \mathbb{F}_q} \phi_{O(\alpha)}
\]
as required (in part (a)).

The proof is complete. \( \square \)

As a consequence (of the proof) we deduce the following result.

Proposition 3.6. Let \( H \) be a maximal algebra subgroup of \( G \), let \( U \subseteq J \) be such that \( H = 1 + U \) and let \( \pi : J^* \to U^* \) be the natural projection. Let \( O_0 \in \Omega(H) \) be arbitrary and let \( O \in \Omega(G) \) be an arbitrary \( G \)-orbit satisfying \( O_0 \subseteq \pi(O) \). Then, the following statements hold:

(a) Suppose that \( O \) is of type I (with respect to \( H \)). Let \( e \in J \) be such that \( J = U \oplus \mathbb{F}_qe \) and let \( e^* \in U^\perp \) be such that \( e^*(e) = 1 \). Let \( f \in O \) be arbitrary and, for each \( \alpha \in \mathbb{F}_q \), let \( O(\alpha) \in \Omega(G) \) denote the \( G \)-orbit which contains the element \( f + \alpha e^* \in J^* \). Then, the \( G \)-orbits \( O(\alpha) \), for \( \alpha \in \mathbb{F}_q \), are all distinct and the inverse image \( \pi^{-1}(O_0) \) decomposes as the disjoint union
\[
\pi^{-1}(O_0) = \bigcup_{\alpha \in \mathbb{F}_q} O(\alpha)
\]
of the \( G \)-orbits \( O(\alpha) \in \Omega(G) \), for \( \alpha \in \mathbb{F}_q \).

(b) If \( O \) is of type II (with respect to \( H \)), then \( \pi^{-1}(O_0) \subseteq O \). Moreover, this inclusion is proper.

Proof. Suppose that \( O \) is of type I. Let \( \alpha \in \mathbb{F}_q \) be arbitrary. Then, as we have seen in the proof of Proposition 3.3, \( O(\alpha) \) is also of type I and so \( \pi(O(\alpha)) = O_0 \) (by Proposition 3.3 because \( O_0 \subseteq \pi(O(\alpha)) \)). It follows that \( O(\alpha) \subseteq \pi^{-1}(O_0) \). Conversely, suppose that \( g \in J^* \) is such that \( \pi(g) \in O_0 \). Since \( O_0 = \pi(O) \), there exists \( x \in G \) such that \( \pi(g) = \pi(x \cdot f) \). It follows that \( g \in \mathcal{L}(x \cdot f) \). Since \( \mathcal{L}(x \cdot f) = x \cdot \mathcal{L}(f) \) and since \( \mathcal{L}(f) = f + U^\perp \), we conclude that \( g = x \cdot (f + \alpha e^*) \) (hence, \( g \in O(\alpha) \)) for some \( \alpha \in \mathbb{F}_q \). This completes the proof of part (a).

Now, suppose that \( O \) is of type II. Let \( g \in J^* \) be such that \( \pi(g) \in O_0 \). Since \( O_0 \subseteq \pi(O) \), there exists \( f \in O \) such that \( \pi(g) = \pi(f) \). Therefore, \( g \in \mathcal{L}(f) \). Since \( O \) is of type II, \( \mathcal{L}(f) \subseteq O \) (by Proposition 3.4) and so \( g \in O \). Since \( g \in \pi^{-1}(O_0) \) is arbitrary, we conclude that \( \pi^{-1}(O_0) \subseteq O \).

To see that this inclusion is proper, it is enough to choose \( g \in O \) such that \( \pi(g) \in x_\alpha \cdot O_0 \) for some \( \alpha \in \mathbb{F}_q \), \( \alpha \neq 0 \), where \( \{x_\alpha : \alpha \in \mathbb{F}_q\} \) is as in Proposition 3.3. \( \square \)
4. Irreducible characters

The purpose of this section is the proof of the following result.

**Theorem 4.1.** Let $G$ be an arbitrary $\mathbb{F}_q$-algebra group. Then, for each $O \in \Omega(G)$, the class function $\phi_O$ is an irreducible character of $G$. Moreover, we have $\text{Irr}(G) = \{ \phi_O : O \in \Omega(G) \}$.

By Proposition 2.2, it is enough to show that the class functions $\phi_O \in \text{cf}(G)$, $O \in \Omega(G)$, are, in fact, characters. And, to see this, we proceed by induction on $|G|$ (the result being clear if $|G| = 1$). The key step is solved by the following lemma. As before, $G = 1 + J$ where $J = J(A)$ is the Jacobson radical of a finite dimensional $\mathbb{F}_q$-algebra $A$.

**Lemma 4.1.** Let $H$ be a maximal algebra subgroup of $G$, let $U \subseteq J$ be such that $H = 1 + U$ and let $\pi : J^* \to U^*$ be the natural projection. Let $O \in \Omega(G)$ and let $O_0 \in \Omega(H)$ be any $H$-orbit with $O_0 \subseteq \pi(O)$. Assume that the class function $\phi_{O_0} \in \text{cf}(H)$ is a character of $H$. Then, the class function $\phi_O \in \text{cf}(G)$ is a character of $G$.

The proof of Lemma 4.1 relies on the following result (and on its corollary).

**Lemma 4.2.** Let $H$ be a maximal algebra subgroup of $G$, let $\chi \in \text{Irr}(G)$ and let $\theta \in \text{Irr}(H)$ be an irreducible constituent of $\chi_H$. Then one (and only one) of the following two possibilities occurs:

(a) $\chi_H = \theta$ is an irreducible character of $H$ and $\theta^G$ has $q$ distinct irreducible constituents each one occurring with multiplicity one. The irreducible constituents of $\theta^G$ are the characters $\lambda\chi$ for $\lambda \in \text{Irr}(G/H)$ (as usual, $\text{Irr}(G/H)$ is naturally identified with a subset of $\text{Irr}(G)$).

(b) $\theta^G = \chi$ is an irreducible character of $G$ and $\chi_H$ has $q$ distinct irreducible constituents each one occurring with multiplicity one.

**Proof.** By [8, Corollary 11.29], we know that $\chi(1)/\theta(1)$ divides $|G : H|$ (we recall that $H$ is a normal subgroup of $G$). Since $G$ and $H$ are $\mathbb{F}_q$-algebra subgroups, [8, Theorem A] asserts that $\chi(1)$ and $\theta(1)$ are powers of $q$. Moreover, being a maximal algebra subgroup of $G$, we know that $H$ has index $q$ in $G$. It follows that, either $\chi(1) = \theta(1)$, or $\chi(1) = q\theta(1)$.

Suppose that $\chi(1) = \theta(1)$. Then, we must have $\chi_H = \theta$ and this is the situation of (a). The assertion concerning the induced character is an easy application of a result of Gallagher (see [8, Corollary 6.17]).

On the other hand, suppose that $\chi(1) = q\theta(1)$. Then $\chi = \theta^G$ (because $\theta^G(1) = q\theta(1)$ and because, by Frobenius reciprocity, $\chi$ is an irreducible constituent of $\theta^G$). By Clifford’s Theorem (see, for example, [8, Theorem 6.2]), we have

$$\chi_H = e \sum_{i=1}^t \theta_i$$

where $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are all the distinct conjugates of $\theta$ in $G$ and where $e = \langle \chi_H, \theta \rangle_H$. (The $G$-action on $\text{Irr}(H)$ is defined as usual by $\theta^g(y) =$
Proof of Lemma 4.1. For simplicity, we write \( \chi \) of (4.2) reducible, then \( \langle \chi, \theta \rangle = 1 \) (because \( \chi = \theta^G \)).

It follows that \( \chi(1) = \sum_{i=1}^t \theta_i(1) = t\theta(1) \) (because \( \theta_i(1) = \theta(1) \) for all \( 1 \leq i \leq t \)) and so \( t = q \).

The proof is complete.

The following easy consequence will also be very useful.

Corollary 4.1. Let \( H \) be a maximal algebra subgroup of \( G \) and let \( \chi, \chi' \in \text{Irr}(G) \). Then, the following hold:

(a) If \( \chi \) is irreducible, then \( \langle \chi, \chi' \rangle_H \neq 0 \) if and only if \( \chi' = \lambda \chi \) for some \( \lambda \in \text{Irr}(G/H) \).

(b) If \( \chi \) is reducible, then \( \langle \chi, \chi' \rangle_H \neq 0 \) if and only if \( \chi' = \chi \). Moreover, we have \( \langle \chi, \chi \rangle_H = q \).

Proof. Suppose that \( \chi \) is irreducible. By Frobenius reciprocity, we have \( \langle \theta, \chi \rangle_H = \langle \theta^G, \chi' \rangle \) and so \( \langle \chi, \chi' \rangle_H \neq 0 \) if and only if \( \chi' = \lambda \chi \) for some \( \lambda \in \text{Irr}(G/H) \) (by part (a) of Lemma 1.2).

On the other hand, suppose that \( \chi \) is reducible and let \( \chi' \in \text{Irr}(G) \) be such that \( \langle \chi, \chi' \rangle_H \neq 0 \). Then, there exists \( \theta \in \text{Irr}(H) \) such that \( \langle \theta, \chi \rangle_H \neq 0 \) and \( \langle \theta, \chi' \rangle_H \neq 0 \). By part (b) of Lemma 1.2, we have \( \chi = \theta^G \) and so, using Frobenius reciprocity, we deduce that \( \langle \chi, \chi' \rangle = \langle \theta, \chi' \rangle = \langle \chi, \chi' \rangle_H \neq 0 \). Since \( \chi, \chi' \in \text{Irr}(G) \), we conclude that \( \chi = \chi' \). Finally, by part (b) of Lemma 1.2, it is clear that \( \langle \chi, \chi \rangle_H = q \).

The proof is complete.

We are now able to prove Lemma 1.1.

Proof of Lemma 1.1. For simplicity, we write \( \phi = \phi_O \) and \( \phi_0 = \phi_{O_g} \). If \( O \) is of type II with respect to \( H \), then \( \phi = \langle \phi_0 \rangle^G \) (by Proposition 3.5) and the result is clear. On the other hand, suppose that \( O \) is of type I with respect to \( H \). Then, by Proposition 1.4, \( \phi_0 = \phi_H \) and \( \langle \phi_0, \phi_0 \rangle = q \). By Proposition 2.2, we have

\[
\phi = \sum_{\chi \in \text{Irr}(G)} \mu_\chi \chi
\]

where \( \mu_\chi \in \mathbb{C} \) for all \( \chi \in \text{Irr}(G) \). Let \( I = \{ \chi \in \text{Irr}(G) : \mu_\chi \neq 0 \} \) be the support of \( \phi \). Since \( \langle \phi, \phi \rangle = 1 \), we have

\[
\sum_{\chi \in I} |\mu_\chi|^2 = 1.
\]

Now, we consider the restriction

\[
\phi_0 = \phi_H = \sum_{\chi \in I} \mu_\chi \chi_H
\]

of \( \phi \) to \( H \).

We claim that \( \chi_H \in \text{Irr}(H) \) for all \( \chi \in I \). To see this, suppose that \( \chi_H \) is reducible for some \( \chi \in I \). Let \( \theta \in \text{Irr}(H) \) be an irreducible constituent of \( \chi_H \).
Then, $\theta$ occurs in the sum of the right hand side of (4.2) with coefficient $\mu_\chi$; in fact, $\langle \theta, \chi_H \rangle_H = 1$ (by Lemma 4.2) and $\langle \theta, \chi_H' \rangle_H = 0$ for all $\chi' \in \text{Irr}(G)$ with $\chi' \neq \chi$ (otherwise, $\langle \chi_H, \chi_H' \rangle_H \neq 0$ and this is in contradiction with Corollary 4.1). It follows that (4.2) has the form

$$\phi_0 = \mu_\chi \theta_1 + \cdots + \mu_\chi \theta_q + \xi$$

(4.3)

where $\theta_1 = \theta, \theta_2, \ldots, \theta_q$ are the $q$ distinct irreducible constituents of $\chi_H$ (see Lemma 4.2) and where $\xi \in \text{cf}(H)$ is a $C$-linear combination of $\text{Irr}(H) \setminus \{\theta_1, \ldots, \theta_q\}$. Now, since $\phi_0$ is a character of $H$ (by assumption) and since $\langle \phi_0, \phi_0 \rangle_H = 1$ (by Proposition 2.2), we have $\phi_0 \in \text{Irr}(H)$. Since $\text{Irr}(H)$ is a $C$-basis of $\text{cf}(H)$, the equality (4.3) implies that $\mu_\chi = 0$ and this is in contradiction with $\chi \in \mathcal{I}$. This completes the proof of our claim, i.e., $\chi_H \in \text{Irr}(H)$ for all $\chi \in \mathcal{I}$.

Now, for each $\theta \in \text{Irr}(H)$, let $\mathcal{I}_\theta = \{\chi \in \mathcal{I} : \chi_H = \theta\}$ and let

$$\mu_\theta = \sum_{\chi \in \mathcal{I}_\theta} \mu_\chi.$$ 

Then $\mathcal{I}$ is the disjoint union

$$\mathcal{I} = \bigcup_{\theta \in \text{Irr}(H)} \mathcal{I}_\theta$$

and so

$$\phi_0 = \sum_{\theta \in \text{Irr}(H)} \mu_\theta \theta.$$ 

Since $\phi_0 \in \text{Irr}(H)$, we conclude that $\mu_\theta = \delta_{\theta, \phi_0}$ for all $\theta \in \text{Irr}(H)$. Hence,

$$\sum_{\chi \in \mathcal{I}} \mu_\chi = \sum_{\theta \in \text{Irr}(H)} \mu_\theta = 1.$$ 

Using (4.1), we easily deduce that there exists a unique $\chi \in \text{Irr}(G)$ with $\mu_\chi \neq 0$ and, in fact, $\mu_\chi = 1$.

The proof is complete. \hspace{1cm} \Box

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. As before, let $A$ be a finite dimensional $F_q$-algebra, let $J = J(A)$ be the Jacobson radical of $A$ and let $G = 1 + J$ be the $F_q$-algebra group defined by $J$.

We consider the chain $J \supseteq J^2 \supseteq J^3 \supseteq \ldots$ of ideals of $A$. Since $J$ is nilpotent, there exists the smallest integer $m$ with $J^m \neq \{0\}$. Moreover, we may refine the chain

$$J \supseteq J^2 \supseteq \ldots \supseteq J^m \supseteq \{0\}$$

to obtain a (maximal) chain

$$\{0\} = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n = J$$

of ideals of $A$ satisfying

$$\dim U_{i+1} = \dim U_i + 1$$
for all $0 \leq i < n$. Let $f \in J^*$ be arbitrary and, for each $0 \leq i \leq n$, let
\[ R_i = \{ a \in U_i : f([ab]) = 0 \text{ for all } b \in U_i \} . \]
Finally, let
\[ U = R_1 + \cdots + R_n. \]

It is clear that $U$ is an $\mathbb{F}_q$-subspace of $J$. Now, let $a, b \in U$ and suppose that $a \in R_i$ and $b \in R_j$ for some $1 \leq i, j \leq n$ with $i \leq j$. We claim that $ab \in R_i$. To see this, let $c \in U_i$ be arbitrary. Then,
\[ f([ab, c]) = f([a, bc]) + f([b, ca]). \]
Since $U_i$ is an ideal of $A$, we have $bc \in U_i$, hence $f([a, bc]) = 0$ (because $a \in R_i$). On the other hand, we have $ca \in U_i$ since $U_i \subseteq U_j$ (because $i \leq j$) and since $b \in R_j$, we conclude that $f([b, ca]) = 0$. Thus
\[ f([ab, c]) = f([a, bc]) + f([b, ca]) = 0 \]
and so $ab \in R_i$ (because $c \in U_i$ is arbitrary). It follows that $U$ is a multiplicatively closed $\mathbb{F}_q$-subspace of $J$. Hence, $H = 1 + U$ is an algebra subgroup of $G$. Moreover, a similar argument shows that
\[ f([ab]) = 0 \]
for all $a, b \in U$. This means that $U$ is an $f$-isotropic $\mathbb{F}_q$-subspace of $J$ (i.e., $U$ is isotropic with respect to the skew-symmetric bilinear form $B_f$ which was defined in Section 3). Next, we claim that $U$ is a maximal $f$-isotropic $\mathbb{F}_q$-subspace of $J$. By Witt’s Theorem (see, for example, Theorems 3.10 and 3.11), it is enough to prove that
\[ \dim U = \frac{1}{2} (\dim J + \dim \text{Rad}(f)) . \]
To see this, we proceed by induction on $\dim J$. If $\dim J = 1$, then $U = J = \text{Rad}(f)$ and the claim is trivial. Now, suppose that $\dim J > 1$ and consider the ideal $U_{n-1}$ of $J$. Let
\[ U' = R_1 + \cdots + R_{n-1}. \]
By induction, we have
\[ \dim U' = \frac{1}{2} \left( \dim U_{n-1} + \dim \text{Rad}(f') \right) \]
where $f'$ is the restriction of $f$ to $U_{n-1}$. Using Proposition 2.1 and Proposition 3.2, we conclude that, either $\dim \text{Rad}(f) = \dim \text{Rad}(f') - 1$, or $\dim \text{Rad}(f) = \dim \text{Rad}(f') + 1$. In the first case, we deduce that
\[ \dim U' = \frac{1}{2} \left( \dim J - 1 + \dim \text{Rad}(f) + 1 \right) = \frac{1}{2} \left( \dim J + \dim \text{Rad}(f) \right) . \]
Therefore, $U'$ is a maximal $f$-isotropic $\mathbb{F}_q$-subspace of $J$. Since $U' \subseteq U$, we conclude that $U' = U$ and (5.1) follows in this case. On the other hand, suppose that $\dim \text{Rad}(f) = \dim \text{Rad}(f') + 1$. Then,
\[ \dim U' = \frac{1}{2} \left( \dim J - 1 + \dim \text{Rad}(f) - 1 \right) = \frac{1}{2} \left( \dim J + \dim \text{Rad}(f) \right) - 1. \]
Since $U' \subseteq U$, we have $\dim U' \leq \dim U$. If $\dim U' = \dim U$, then $U' = U$ and so $\text{Rad}(f) \subseteq U' \subseteq U_{n-1}$. If this were the case, we should have $\text{Rad}(f) \subseteq \text{Rad}(f')$ and so $\dim \text{Rad}(f) \leq \dim \text{Rad}(f')$, a contradiction. It follows that $\dim U' < \dim U$.

Since $U$ is an $f$-isotropic $\mathbb{F}_q$-subspace of $J$, we deduce that

$$\dim U' \leq \dim U - 1 \leq \frac{1}{2}(\dim J + \dim \text{Rad}(f)) - 1 = \dim U'. $$

The proof of (5.1) is complete.

Given an arbitrary element $f \in J^*$, we will say that a multiplicatively closed $\mathbb{F}_q$-subspace $U$ of $J$ is an $f$-polarization if $U$ is a maximal $f$-isotropic $\mathbb{F}_q$-subspace of $J$. Hence, we have finished the proof of the following.

**Proposition 5.1.** Let $f \in J^*$ be arbitrary. Then, there exists an $f$-polarization $U \subseteq J$.

Now, it is easy to conclude the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $\chi$ be an (arbitrary) irreducible character of $G$. Then, by Theorem 4.1, $\chi = \phi_{\mathcal{O}}$ for some $G$-orbit $\mathcal{O} \in \Omega(G)$. Let $f \in \mathcal{O}$ be arbitrary and let $U \subseteq J$ be an $f$-polarization. Then, $H = 1 + U$ is an algebra subgroup of $G$. Let $f_0 \in U^*$ be the restriction of $f$ to $U$. Since $U$ is $f$-isotropic, we have $\text{Rad}(f_0) = U$, hence $C_H(f_0) = H$ (by Proposition 2.1).

It follows that $\mathcal{O}_0 = \{f_0\}$ is a single $H$-orbit on $U^*$ (i.e., an element of $\Omega(H)$). We denote by $\lambda_f$ the class function $\phi_{\mathcal{O}_0}$ of $H$; by definition, $\lambda_f: H \to \mathbb{C}$ is defined by

$$\lambda_f(1 + a) = \psi_f(a) = \psi(f(a))$$

for all $a \in U$. By Theorem 4.1, we know that $\lambda_f$ is an irreducible character of $H$. Moreover, $\lambda_f(1) = \sqrt{|\mathcal{O}|}$, i.e., $\lambda_f$ is a linear character of $H$. To conclude the proof, we claim that

$$\phi_{\mathcal{O}} = \lambda_f^G. \tag{5.2}$$

To see this, we evaluate the Frobenius product $\langle \phi_{\mathcal{O}}, \lambda_f^G \rangle_G$. Using Frobenius reciprocity (and the definition of $\phi_{\mathcal{O}}$), we deduce that

$$\langle \phi_{\mathcal{O}}, \lambda_f^G \rangle_G = \langle (\phi_{\mathcal{O}})_H, \lambda_f \rangle_H = \frac{1}{|H|} \sum_{x \in H} \phi_{\mathcal{O}}(x) \overline{\lambda_f(x)} = \frac{1}{|U|} \sum_{a \in U} \left( \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{g \in \mathcal{O}} \psi_g(a) \right) \overline{\psi_f(a)}$$

$$= \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{g \in \mathcal{O}} \left( \frac{1}{|U|} \sum_{a \in U} \psi_g(a) \psi_f(a) \right)$$

$$= \frac{1}{\sqrt{|\mathcal{O}|}} \langle \psi_g, \psi_f \rangle_{U^+}. $$
By (2.2), given any \( g \in J^* \), we have \( \langle \psi_g, \psi_f \rangle_{U^+} \neq 0 \) if and only if \( f(a) = g(a) \) for all \( a \in U \); in other words, \( \langle \psi_g, \psi_f \rangle_{U^+} \neq 0 \) if and only if \( g \in f + U^\perp \). Since \( \psi_g \) is linear for all \( g \in J^* \), we conclude that

\[
\langle \phi_O, \lambda_f^G \rangle_G = \frac{|(f + U^\perp) \cap O|}{\sqrt{|O|}}.
\]

(5.3)

It follows that \( \langle \phi_O, \lambda_f^G \rangle_G \neq 0 \) (because \( f \in (f + U^\perp) \cap O \)), hence \( \phi_O \) is an irreducible constituent of \( \lambda_f^G \). On the other hand, we have \( \phi_O(1) = \sqrt{|O|} \) and

\[
\lambda_f^G(1) = |G : H| \lambda_f(1) = q^{\dim J - \dim U}
\]

\[
= \sqrt{q^{\dim J - \dim \text{Rad}(f)}} = \sqrt{|G : C^G(f)|} = \sqrt{|O|}
\]

(using (5.1) and Proposition 2.1). The claim (5.2) follows and the proof of Theorem 1.1 is complete. \( \square \)

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