Infinitesimal Torelli theorem for surfaces with \( c_1^2 = 3 \), \( \chi = 2 \), and the torsion group \( \mathbb{Z}/3 \)

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Abstract

We prove the infinitesimal Torelli theorem for general minimal complex surfaces \( X \)'s with the first Chern number 3, the geometric genus 1, and the irregularity 0 which have non-trivial 3-torsion divisors. We also show that the coarse moduli space for surfaces with the invariants as above is a 14-dimensional unirational variety.

0 Introduction

In the present paper, we will prove the infinitesimal Torelli theorem for general minimal complex surfaces \( X \)'s with \( c_1^2 = 3 \), \( \chi(\mathcal{O}) = 2 \), and \( \text{Tors}(X) \cong \mathbb{Z}/3 \), where \( c_1 \), \( \chi(\mathcal{O}) \), and \( \text{Tors}(X) \) are the first Chern class, the Euler characteristic of the structure sheaf, and the torsion part of the Picard group of \( X \), respectively. We will also show that all surfaces with the invariants as above are deformation equivalent to each other, and that their coarse moduli space \( \mathcal{M} \) is a 14-dimensional unirational variety. Here, let us remark that the condition \( \text{Tors}(X) \cong \mathbb{Z}/3 \) is a topological one; minimal surfaces with \( c_1^2 = 3 \) and \( \chi(\mathcal{O}) = 2 \) have the geometric genus \( p_g = 1 \) and the irregularity \( q = 0 \), hence the torsion group \( \text{Tors}(X) \) isomorphic to the first homology group \( H_1(X, \mathbb{Z}) \).

As is well known today, Torelli type theorems do not necessarily hold for surfaces. One of the most famous counter examples are surfaces of general type with \( p_g = q = 0 \). Although Torelli type theorems have been proved for many classes of surfaces, finding what conditions we should impose still remains as a challenging problem. So it makes sense to study period maps for concrete classes of surfaces.

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Let us recall some results on period maps for surfaces of general type with \( p_g = 1 \) and \( q = 0 \). In [1], Catanese proved the infinitesimal Torelli theorem for general minimal surfaces with \( c_1^2 = 1, \) \( p_g = 1, \) and \( q = 0, \) while in [2], that the global period mapping has degree at least 2. He first showed that any such surface is essentially a weighted complete intersection of type \((6,6)\) in the weighted projective space \( \mathbb{P}(1,2,2,3,3) \), and used this complete description to study the period map for these surfaces. Meanwhile for the case \( c_1^2 = 2, \) \( p_g = 1, \) and \( q = 0, \) the torsion group is either 0 or \( \mathbb{Z}/2. \) Using a complete description for the case of \( \mathbb{Z}/2 \) by Catanese and Debarre [3], Oliverio studied in [8] the infinitesimal period maps for the case of non-trivial 2-torsion divisors by the same method as in [1].

Consider the case \( c_1^2 = 3. \) In this case, the order \( \#\text{Tors}(X) \) is at most 3 by a result in [6]. Moreover, in [7], the author showed that any surface \( X \) of this class with \( \text{Tors}(X) \cong \mathbb{Z}/3 \) is essentially a quotient of a \((3,3)\)-complete intersection in \( \mathbb{P}^4 \) by a certain free action by \( \mathbb{Z}/3. \) Using this complete description, we will show in the present paper the infinitesimal Torelli theorem for general \( X \)'s by the same method as in [1] and [8]. Here, general \( X \) means any surface corresponding to a point in a certain Zariski open subset of the coarse moduli space \( \mathcal{M}. \)

In Section 1 we state our main theorems of the present paper and recall our previous results given in [7]. In Section 2 we show the unirationality of the coarse moduli space \( \mathcal{M}. \) Finally in Section 3 we prove the infinitesimal Torelli theorem for our surfaces \( X \)'s. Throughout this paper, we work over the complex number field \( \mathbb{C}. \)

**Notation**

Let \( S \) be a compact complex manifold of dimension 2. We denote by \( p_g(S), \) \( q(S), \) and \( K_S, \) the geometric genus, the irregularity and a canonical divisor of \( S, \) respectively. We denote by \( \text{Tors}(S) \) the torsion part of the Picard group, and call it the torsion group of \( S. \) For a coherent sheaf \( \mathcal{F} \) on \( S, \) we denote by \( h^i(\mathcal{F}) \) the dimension of the \( i \)-th cohomology group \( H^i(S, \mathcal{F}). \) The sheaf \( \mathcal{O}_S, \mathcal{O}^p_S, \) and \( \Theta_S \) are the structure sheaf, the sheaf of germs of holomorphic \( p \)-forms, and that of germs of holomorphic vector fields on \( S, \) respectively. As usual, \( \mathbb{P}^n \) is the projective space of dimension \( n. \) We denote by \( \varepsilon = \exp(2\pi \sqrt{-1/3}) \) a third root of unity.

## 1 Statement of main results

In a previous paper [7], the author gave a complete description for minimal algebraic surfaces \( X \)'s with \( c_1^2 = 3, \) \( \chi(O) = 2, \) and \( \text{Tors}(X) \cong \mathbb{Z}/3, \) where \( c_1, \)
\( \chi(\mathcal{O}) \), and \( \text{Tors}(X) \) are the first Chern class, the Euler characteristic of the structure sheaf, and the torsion part of the Picard group of \( X \), respectively.

In the present paper, we will give proofs for the following two theorems:

**Theorem 1.** All minimal algebraic surfaces \( X \)'s with \( c_1^2 = 3 \), \( \chi(\mathcal{O}) = 2 \), and \( \text{Tors}(X) \cong \mathbb{Z}/3 \) are deformation equivalent to each other. Their coarse moduli space \( \mathcal{M} \) is a 14-dimensional unirational variety.

**Theorem 2.** Let \( X \) be any general surface as in Theorem 1. Then the infinitesimal period map \( \mu : H^1(\Theta_X) \to \text{Hom}(H^0(\Omega_X^2), H^1(\Omega_X^1)) \) is injective.

**Remark 1.** The surfaces \( X \)'s as in Theorem 1 have the geometric genus \( p_g = 1 \) and the irregularity \( q = 0 \). We refer the readers to [4] for the existence of the coarse moduli space \( \mathcal{M} \). See also [3] for the infinitesimal period map.

In order to give proofs for the theorems above, let us first recall the main results given in [7]. See [7] for proofs of the following two theorems:

**Theorem 3 ([7]).** Let \( X \) be a minimal algebraic surface with \( c_1^2 = 3 \), \( \chi(\mathcal{O}) = 2 \), and \( \mathbb{Z}/3 \subset \text{Tors}(X) \). Let \( \pi : Y \to X \) be the unramified Galois triple cover corresponding to a non-trivial 3-torsion divisor. Then both the fundamental group \( \pi_1(X) \) and the torsion group \( \text{Tors}(X) \) are isomorphic to the cyclic group \( \mathbb{Z}/3 \). Further, the canonical model \( Z \) of \( Y \) is a complete intersection in the 4-dimensional projective space \( \mathbb{P}^4 \) defined by two homogeneous polynomials \( \tilde{F}_1 \) and \( \tilde{F}_2 \) of degree 3 satisfying

\[
\tilde{F}_i(W_0, \varepsilon X_1, \varepsilon X_2, \varepsilon^{-1} Y_3, \varepsilon^{-1} Y_4) = \tilde{F}_i(W_0, X_1, X_2, Y_3, Y_4) \quad (i = 1, 2).
\]

Here, \( (W_0, X_1, \cdots, Y_4) \) is a homogeneous coordinate of \( \mathbb{P}^4 \), and the constant \( \varepsilon = \exp(2\pi \sqrt{-1}/3) \) is a third root of unity.

**Theorem 4 ([7]).** Let \( X \) be a surface as in Theorem 1. If \( X \) has an ample canonical divisor \( K_X \), then \( h^1(\Theta_X) = 14 \) and \( h^2(\Theta_X) = 0 \), hence the Kuranishi space of \( X \) is smooth and of dimension 14.

**Remark 2.** Explicit forms of the two polynomials in Theorem 3 are given by

\[
\tilde{F}_i = a_0^{(i)} W_0^3 + W_0 \tilde{g}_i(X_1, X_2, Y_3, Y_4) + \tilde{\alpha}_i(X_1, X_2) + \tilde{\beta}_i(Y_3, Y_4) \quad (1)
\]

for \( i = 1, 2 \), where

\[
\tilde{g}_i = a_1^{(i)} X_1 Y_3 + a_2^{(i)} X_1 Y_4 + a_3^{(i)} X_2 Y_3 + a_4^{(i)} X_2 Y_4,
\]

\[
\tilde{\alpha}_i = a_5^{(i)} X_1^3 + a_6^{(i)} X_1^2 X_2 + a_7^{(i)} X_1 X_2^2 + a_8^{(i)} X_2^3,
\]

\[
\tilde{\beta}_i = a_9^{(i)} Y_3^3 + a_{10}^{(i)} Y_3^2 Y_4 + a_{11}^{(i)} Y_3 Y_4^2 + a_{12}^{(i)} Y_4^3,
\]

are homogeneous polynomials of \( X_1, \cdots, Y_4 \) with coefficients \( a_j^{(i)} \in \mathbb{C} \).
Remark 3. The complete intersection $Z$ is the image of the canonical map $\Phi_{K_Y} : Y \to \mathbb{P}^4$. We have a natural action on $Z$ by the Galois group $G = \text{Gal}(Y/X) \simeq \mathbb{Z}/3$ of $Y$ over $X$. This action is given by

$$\tau_0 : (W_0 : X_1 : X_2 : Y_3 : Y_4) \mapsto (W_0 : \varepsilon X_1 : \varepsilon X_2 : \varepsilon^{-1} Y_3 : \varepsilon^{-1} Y_4),$$

where $\tau_0$ is a generator of the group $G$. Since this action on $Z$ has no fixed points, the coefficients $a_i$'s satisfy the following three conditions:

i) at least one out of $a_0^{(1)}$ and $a_0^{(2)}$ are not equal to zero,
ii) two polynomials $\alpha_1$ and $\alpha_2$ have no common zeroes on $\mathbb{P}^1 = \{(X_1 : X_2)\}$,
iii) two polynomials $\beta_1$ and $\beta_2$ have no common zeroes on $\mathbb{P}^1 = \{(Y_3 : Y_4)\}.

For each integer $n \geq 0$, we have a natural isomorphism

$$H^0(\mathcal{O}_Y(nK_Y)) \simeq \bigoplus_{m=0,1,-1} H^0(\mathcal{O}_X(nK_X - mT_0))$$

(3)
corresponding to the action by $G$, where $T_0$ is a generator of the torsion group Tors($X$). Note that this is a decomposition into homogeneous eigen spaces, and that, in Theorem 3, the sets $\{W_0\}$, $\{X_1, X_2\}$, and $\{Y_3, Y_4\}$ correspond to a base of $H^0(\mathcal{O}_X(K_X))$, that of $H^0(\mathcal{O}_X(K_X - T_0))$, and that of $H^0(\mathcal{O}_X(K_X + T_0))$, respectively. The polynomials $\tilde{F}_1$ and $\tilde{F}_2$ generate the linear space consisting of all the elements in $H^0(\mathcal{O}_{\mathbb{P}^1}(3H))$ vanishing along $Z$, where $H$ is a hyperplane in $\mathbb{P}^4$.

2 Unirationality of the moduli space

In this section, we will give a proof for Theorem 1. We denote by $W = \mathbb{P}^4$ and $(W_0 : X_1 : X_2 : Y_3 : Y_4)$, the 4-dimensional complex projective space and its homogeneous coordinate, respectively.

Let $\tilde{B}$ be the set of all $(a_j^{(i)})_{0 \leq j \leq 12} \in \mathbb{C}^{26}$ satisfying the conditions i), ii) and iii) in Remark 3 such that two polynomials $\tilde{F}_1$ and $\tilde{F}_2$ given by (2) define in $W = \mathbb{P}^4$ a complete intersection with at most rational double points as its singularities. We denote by $\tilde{B}_0$ the set of points in $\tilde{B}$ corresponding to non-singular complete intersections. Note by [7] Remark 1, we have $\tilde{B}_0 \neq \emptyset$, hence the spaces $\tilde{B}$ and $\tilde{B}_0$ are dense Zariski open subsets of $\mathbb{C}^{26}$. We have a flat family $\tilde{Y} \to \tilde{B}$ whose fiber on each $(a_j^{(i)}) \in \tilde{B}$ is a complete intersection defined by $\tilde{F}_1$ and $\tilde{F}_2$ with $a_j^{(i)}$'s as their coefficients. This $\tilde{Y}$ is a subvariety of $\tilde{B} \times W$ stable under the action by $G \simeq \langle \text{id}_{\tilde{B}} \times \tau_0 \rangle \simeq \mathbb{Z}/3$, where $\tau_0$ is an automorphism of $W$ given by (2). Taking a quotient of $\tilde{Y}$ by this action, we
obtain a family $\tilde{X} \to \tilde{B}$ whose fibers are the canonical models of surfaces $X$’s as in Theorem 3. Note that both restrictions $\tilde{Y}|_{\tilde{B}_0} \to \tilde{B}_0$ and $\tilde{X}|_{\tilde{B}_0} \to \tilde{B}_0$ are analytic families.

**Lemma 2.1.** Let $X$ be an algebraic surface as in Theorem 3. Then there exist bases of $H^0(\mathcal{O}_X(K_X))$, $H^0(\mathcal{O}_X(K_X - T_0))$, and $H^0(\mathcal{O}_X(K_X + T_0))$ such that the polynomials $\tilde{F}_1$ and $\tilde{F}_2$ satisfy $a_0^{(1)} = 1$, $a_5^{(1)} = a_9^{(1)} = a_7^{(1)} = a_8^{(1)} = a_{12}^{(1)} = 0$, $a_8^{(2)} = a_{12}^{(2)} = 1$, and $a_5^{(2)} = a_9^{(2)} = a_2^{(2)} = 0$.

Proof. Take those bases and $\tilde{F}_i$’s in Theorem 3 in such a way that each $\tilde{\alpha}_i$ for $i = 1, 2$ has a zero of order at least 2 at $(X_1 : X_2) = (i - 1 : 2 - i)$ and that each $\tilde{\beta}_i$ for $i = 1, 2$ has a zero at $(Y_3 : Y_4) = (i - 1 : 2 - i)$. This is possible, since $(X_1 : X_2) \mapsto (\tilde{\alpha}_1(X_1, X_2) : \tilde{\alpha}_2(X_1, X_2))$ is a morphism of degree 3. Then, by the conditions ii) and iii) in Remark 3, we have $a_5^{(1)} \neq 0$, $a_8^{(2)} \neq 0$, $a_9^{(1)} \neq 0$ and $a_{12}^{(2)} \neq 0$. Now, by replacing the elements in these bases by their multiples by non-zero constants, and changing indices if necessary, we easily obtain the assertion.

Consider the case of $X$ for which $\tilde{F}_i$’s as in Lemma 2.1 satisfy $a_0^{(2)} \neq 0$. In this case, we replace $X_2$ and $Y_4$ by their multiples by a non-zero constant such that the equality $a_0^{(2)} = 1$, as much as the equalities in the lemma above, holds. Then the defining polynomials $F_i = \tilde{F}_i$’s of $Z$ in $W$ are given by

$$F_i = W_0^3 + W_0\gamma_i(X_1, X_2, Y_3, Y_4) + \alpha_i(X_1, X_2) + \beta_i(Y_3, Y_4), \quad (4)$$

for $i = 1, 2$, where

$$\gamma_1 = a^{(1)}X_1Y_3 + b^{(1)}X_1Y_4 + c^{(1)}X_2Y_3 + d^{(1)}X_2Y_4,$$
$$\gamma_2 = a^{(2)}X_1Y_3 + b^{(2)}X_1Y_4 + c^{(2)}X_2Y_3 + d^{(2)}X_2Y_4,$$
$$\alpha_1 = X_1^3 + e^{(1)}X_1^2X_2,$$
$$\alpha_2 = g^{(2)}X_1X_2^2 + X_3^2,$$
$$\beta_1 = Y_3^3 + h^{(1)}Y_3^2Y_4 + l^{(1)}Y_4^2X_4,$$
$$\beta_2 = h^{(2)}Y_3^2Y_4 + l^{(2)}Y_3Y_4^2 + Y_3^3$$

are homogeneous polynomials with coefficients in $\mathbb{C}$. We have a natural inclusion $\mathbb{C}^{14} = \{(a^{(1)}, b^{(1)}, \cdots, l^{(2)})\} \hookrightarrow \mathbb{C}^{26} = \{(a_j^{(i)})\}$, since the $F_i$’s above are special cases of $\tilde{F}_i$’s. We put $B_0 = \mathbb{C}^{14} \cap \tilde{B}_0$, and denote by $\psi : \mathcal{Y}_0 \to B_0$ and $\varphi : \mathcal{X}_0 \to B_0$, the pull-back of $\tilde{Y} \to \tilde{B}$ and that of $\tilde{X} \to \tilde{B}$, respectively.

Now we are ready to prove Theorem 1. We use the same method as in Theorem 2, 11] and [2, Theorem 2.3].

**Proof of Theorem 1** Let $\mathcal{M}$ be the coarse moduli space for surfaces $X$’s as in Theorem 3. By Theorem 3 any surface $X$ as in Theorem 3 corresponds
3 The infinitesimal period map

In this section, we will give a proof for Theorem 2. Let $\psi : \mathcal{Y}_0 \to B_0$ and $\varphi : \mathcal{X}_0 \to B_0$ be the two analytic families given in Section 2. For each $t = (a^{(1)}, b^{(1)}, \ldots, t^{(2)}) \in B_0$, the fibers $X = \varphi^{-1}(t)$ and $Y = \psi^{-1}(t)$ are a surface with invariants as in Theorem [4] and its universal cover, respectively. Note that $Y$ is a complete intersection in $W = \mathbb{P}^4$ defined by $F_1$ and $F_2$ as in (4). We denote by $\pi : Y \to X$ and $\iota : Y \to W$ the natural projection and the natural inclusion, respectively.

Let $T_{B_0,t}$ be the holomorphic tangent space at $t \in B_0$. We denote by $\rho : T_{B_0,t} \to H^1(\Theta_X)$ and $\rho' : T_{B_0,t} \to H^1(\Theta_Y)$ the Kodaira-Spencer map of $\varphi$ and that of $\psi$, respectively. In order to prove Theorem 2, it only suffices, by Theorem [4] and the equality $\dim T_{B_0,t} = 14$, to show the injectivity of $\mu \circ \rho$ for general $t \in B_0$, where $\mu$ is the morphism given in Theorem 2. Note that the composite $\mu \circ \rho$ corresponds to the infinitesimal period map of $\varphi$. Let $\omega \in H^0(\Omega_X^2)$ be a non-zero holomorphic 2-form on $X$ such that $\pi^* \omega$ corresponds to the section $W_0 \in H^0(\Omega_Y^2)$ in Remark 3. Since $p_g(X) = 1$, the kernel of $\mu \circ \rho$ is equal to that of the morphism $T_{B_0,t} \ni \xi \mapsto (\mu \circ \rho)(\xi)(\omega) \in H^1(\Omega_X^2)$.

Meanwhile we have the following commutative diagram:

$$
\begin{array}{ccc}
T_{B_0,t} & \xrightarrow{\rho} & H^1(\Theta_X) \\
\| \downarrow \id & & \downarrow \\
T_{B_0,t} & \xrightarrow{\rho'} & H^1(\Theta_Y) \\
\end{array}
$$

$$
\begin{array}{ccc}
H^1(\Theta_X) & \xrightarrow{\omega \times} & H^1(\Omega_X^2 \otimes \Theta_X) \simeq H^1(\Omega_X^2) \\
\downarrow & & \downarrow \\
H^1(\Theta_Y) & \xrightarrow{W_0 \times} & H^1(\Omega_Y^2 \otimes \Theta_Y) \simeq H^1(\Omega_Y^2),
\end{array}
$$

where the vertical morphisms $H^1(\Theta_X) \to H^1(\Theta_Y)$ and $H^1(\Omega_X^2 \otimes \Theta_X) \to H^1(\Omega_Y^2 \otimes \Theta_Y)$ are natural inclusions induced by the decomposition of $\pi_* \Theta_Y$ and $\pi_* (\Omega_X^2 \otimes \Theta_Y) \simeq \pi_* \Omega_Y^2$ associated with the action of the Galois group $G = \text{Gal}(Y/X)$. Note that $((\mu \circ \rho)(\xi))(\omega) = ((\omega \times) \circ \rho)(\xi)$ for any $\xi \in T_{B_0,t}$.

Thus in order to prove the injectivity of $\mu \circ \rho$, we only need to show that of the morphism $(W_0 \times) \circ \rho' : T_{B_0,t} \to H^1(\Omega_Y^2 \otimes \Theta_Y)$. 

6
Let us prove the injectivity of $(W_0 \times) \circ \rho'$ for general $t \in B_0$. We denote by $R \simeq \oplus_{m=0}^{\infty} R_n$ and $R_n$ the graded ring $\mathbb{C}[W_0, X_1, X_2, Y_3, Y_4]/\langle F_1, F_2 \rangle$ and its homogeneous part of degree $n$, respectively. This graded ring $R$ is naturally isomorphic to the canonical ring of $Y$. For each $m = 0, 1, -1$, we denote by $R_n^{(m)}$ the set of all $F \in R_n$ satisfying

$$F(W_0, \varepsilon X_1, \varepsilon X_2, \varepsilon^{-1} Y_3, \varepsilon^{-1} Y_4) = \varepsilon^m F(W_0, X_1, X_2, Y_3, Y_4).$$

This space $R_n^{(m)}$ corresponds to the eigenspace $H^0(\mathcal{O}_X(nK_X - mT_0))$ via the isomorphism (3).

We have a natural exact sequence $0 \to \Theta_Y \to \iota^* \Theta_W \to \mathcal{O}_Y(3)_{\oplus 2} \to 0$ of $\mathcal{O}_Y$-modules. By the similar argument as in Catanese [1] and Oliverio [8], we obtain, from this short exact sequence, the following commutative diagram:

$$
\begin{align*}
R_1^{\oplus 5} & \longrightarrow R_3^{\oplus 2} \longrightarrow H^1(\Theta_Y) \longrightarrow 0 \\
\downarrow W_0 \times & \downarrow W_0 \times \downarrow W_0 \times & \downarrow (5) \\
R_2^{\oplus 5} & \delta \longrightarrow R_4^{\oplus 2} \longrightarrow H^1(\mathcal{O}_Y^2 \otimes \Theta_Y) \longrightarrow \mathbb{C} \longrightarrow 0,
\end{align*}
$$

where both of the horizontal sequences are exact, and the morphisms $R_1^{\oplus 5} \to R_3^{\oplus 2}$ and $\delta : R_2^{\oplus 5} \to R_4^{\oplus 2}$ are given by the matrix

$$
\begin{pmatrix}
\frac{\partial F_1}{\partial W_0} & \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \frac{\partial F_1}{\partial Y_3} & \frac{\partial F_1}{\partial Y_4} \\
\frac{\partial F_1}{\partial W_0} & \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \frac{\partial F_1}{\partial Y_3} & \frac{\partial F_1}{\partial Y_4}
\end{pmatrix}.
$$

Let $A' : T_{B_0,t} \to R_3^{\oplus 2}$ be the morphism given by $\frac{\partial}{\partial t} \mapsto (\frac{\partial}{\partial t} F_1, \frac{\partial}{\partial t} F_2)$, that is, the morphism giving the infinitesimal displacement of the deformation $\psi : Y_0 \to B_0$ of the submanifold $Y \subset W$. Since the composite $(W_0 \times) \circ A'$ maps $T_{B_0,t}$ into the subspace $R_4^{(0)\oplus 2} \subset R_4^{\oplus 2}$, we obtain a restriction $A : T_{B_0,t} \mapsto R_4^{(0)\oplus 2}$ of $(W_0 \times) \circ A'$. We put $V = R_2^{\oplus 5} \oplus R_2^{(1)} \oplus R_2^{(1)} \oplus R_2^{(-1)} \oplus R_2^{(-1)} \subset R_2^{\oplus 5}$, and denote by $C : V \to R_4^{(0)\oplus 2}$ the restriction of $\delta : R_2^{\oplus 5} \to R_4^{\oplus 2}$ to this subspace. Then from the commutative diagram (3), we infer the equality $\ker((W_0 \times) \circ \rho') = A^{-1}(C(V))$, where $C(V)$ is the image of the morphism $C$.

Let $M'$ be a 26-dimensional subspace of $R_4^{(0)\oplus 2}$ spanned by the following linearly independent elements:

\begin{align*}
(W_0^3, 0), & \quad (W_0 X_1 X_2^2, 0), \quad (W_0 Y_2^3, 0), \quad (W_0 Y_4^2, 0), \quad (X_1^2 Y_3^2, 0), \\
(X_1^2 Y_3 Y_4, 0), & \quad (X_1^2 Y_4^2, 0), \quad (X_1 X_2 Y_2^2, 0), \quad (X_1 X_2 Y_3 Y_4, 0), \\
(X_1 X_2 Y_4^2, 0), & \quad (X_1^2 Y_2^2, 0), \quad (X_1^2 Y_3 Y_4, 0), \quad (X_1^2 Y_4^2, 0), \\
(0, W_0^3), & \quad (0, W_0 X_1 X_2 Y_3), \quad (0, W_0 Y_2^3), \quad (0, W_0 Y_2 Y_4), \quad (0, X_1^2 Y_2^2), \\
(0, X_1^2 Y_3 Y_4), & \quad (0, X_1^2 Y_2^2), \quad (0, X_1 X_2 Y_3), \quad (0, X_1 X_2 Y_3 Y_4), \\
(0, X_1 X_2 Y_4^2), & \quad (0, X_1^2 Y_2^2), \quad (0, X_1^2 Y_3 Y_4), \quad (0, X_1^2 Y_2^2).
\end{align*}
Then, denoting the image of $A : T_{B_0, t} \to R_1^{(0)\oplus 2}$ by $M$, we have $R_1^{(0)\oplus 2} = M \oplus M'$. Thus there exist two morphisms $D : V \to M$ and $D' : V \to M'$ such that $C = D + D'$. Note that $C(V) \cap M \simeq D(\ker D')$. By this together with the injectivity of $A$, we obtain

$$\ker((W_0 \times) \circ \rho') = A^{-1}(C(V)) \simeq D(\ker D').$$

Meanwhile we have $\dim V = 25$ and

$$(W_0^2, W_0X_1, W_0X_2, W_0Y_3, W_0Y_4) \in \ker C = \ker D \cap \ker D'.$$

Thus in order to prove the injectivity of $(W_0 \times) \circ \rho' : T_{B_0, t} \to H^1(\Omega_Y^1 \otimes \mathcal{O}_Y)$, we only need to show the equality $\text{rank} D' = 24$.

So, in what follows, we will show $\text{rank} D' = 24$ for general $t \in B_0$. We employ the following base of $V$:

$$(W_0^2)_{1}, \ (X_1 Y_3)_{1}, \ (X_1 Y_4)_{1}, \ (X_2 Y_3)_{1}, \ (X_2 Y_4)_{1},$$

$$(W_0X_1)_{2}, \ (W_0X_2)_{2}, \ (Y_3^2)_{2}, \ (Y_3 Y_4)_{2}, \ (Y_4^2)_{2},$$

$$(W_0X_1)_{3}, \ (W_0X_2)_{3}, \ (Y_3 Y_4)_{3},$$

$$(W_0X_3)_{4}, \ (W_0X_4)_{4}, \ (X_1 X_2)_{4}, \ (X_2^2)_{4},$$

$$(W_0Y_3)_{5}, \ (W_0Y_4)_{5}, \ (X_1 X_2)_{5}, \ (X_2^2)_{5},$$

(7)

where, for each $u \in R_2$ and $1 \leq i \leq 5$, we denote by $(u)_i$ the element $(v_1, v_2, v_3, v_4, v_5) \in R_2^{\otimes 5}$ given by $v_i = u, v_j = 0 (j \neq i)$. Let $L_1$ be the $26 \times 25$ matrix of $D'$ corresponding to the bases $[7]$ of $V$ and $[6]$ of $M'$: i.e.

$$L_1 = \begin{pmatrix}
L_{1,1}^{(1)} & L_{1,2}^{(1)} & L_{1,3}^{(1)} \\
L_{2,1}^{(1)} & L_{2,2}^{(1)} & L_{2,3}^{(1)}
\end{pmatrix},$$

where $13 \times 5$ matrices $L_{1,1}^{(1)}, L_{2,1}^{(1)}$, and $13 \times 10$ matrices $L_{1,2}^{(1)}, L_{2,2}^{(1)}, L_{1,3}^{(1)}, L_{2,3}^{(1)}$ are given by

$$L_{1,1}^{(1)} = \begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
a^{(1)} \\
b^{(1)} \\
a^{(1)} \\
b^{(1)} \\
c^{(1)} \\
d^{(1)} \\
a^{(1)} \\
b^{(1)} \\
c^{(1)} \\
d^{(1)}
\end{bmatrix}, \quad L_{2,1}^{(1)} = \begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
a^{(2)} \\
b^{(2)} \\
a^{(2)} \\
b^{(2)} \\
c^{(2)} \\
d^{(2)} \\
a^{(2)} \\
b^{(2)} \\
c^{(2)} \\
d^{(2)}
\end{bmatrix}.$$
\[
L_{1,2}^{(1)} = \begin{bmatrix}
-3 & 2e^{(1)} & -e^{(1)} \\
-3 & a^{(1)} & b^{(1)} & -e^{(1)} & 0 & c^{(1)} & d^{(1)} \\
3 & 3 & e^{(1)} & e^{(1)} & e^{(1)} \\
2e^{(1)} & 2e^{(1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
L_{2,2}^{(1)} = \begin{bmatrix}
-g^{(2)} & 2g^{(2)} & -3 \\
0 & a^{(2)} & 2g^{(2)} & c^{(2)} & d^{(2)} \\
0 & b^{(2)} & -3 & 0 & 0 \\
g^{(2)} & g^{(2)} & 0 & 2g^{(2)} & 2g^{(2)} \\
g^{(2)} & g^{(2)} & 3 & 3 & 3
\end{bmatrix},
\]

\[
L_{1,3}^{(1)} = \begin{bmatrix}
-a^{(1)} & -c^{(1)} & -b^{(1)} & -d^{(1)} \\
3 & -a^{(1)} & * & -b^{(1)} & * & 0 & -b^{(1)} & d^{(1)} & d^{(1)} \\
-3 & -a^{(1)} & h^{(1)} & h^{(1)} & 0 & -b^{(1)} & -d^{(1)} \\
2h^{(1)} & 2h^{(1)} & 0 & 0 & 0 & 0 & 0 \\
2h^{(1)} & 2h^{(1)} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

9
Here empty entries are zero. For general \( t \in B_0 \), we strike off the rows and the columns of \( L_1 \) meeting the following entries by doing the operations in this order: \((3, 16),(1, 6),(17, 22),(14, 12),(2, 7),(4, 17),(15, 11),(16, 21)\). Then we see \( \text{rank} \ L_1 = \text{rank} \ L_2 + 8 \), where \( L_2 \) is the \( 18 \times 16 \) matrix obtained by removing from \( L_1 \) the following: i) the rows and columns meeting the 8 entries given above, and ii) the first column. Thus we only need to show \( \text{rank} \ L_2 = 16 \) for general \( t \in B_0 \).

Let \( L_3 \) be the \( 18 \times 16 \) matrix obtained by specializing \( L_2 \) by \( e^{(1)} = g^{(2)} = 0 \). We can strike off the rows and the columns of \( L_3 \) meeting the following entries: \((1, 5),(2, 6),(3, 7),(16, 8),(17, 9),(18, 10)\). Thus we see \( \text{rank} \ L_2 \geq \text{rank} \ L_3 = \text{rank} \ L_4 + 6 \) for general \( t \in B_0 \), where \( L_4 \) is the \( 12 \times 10 \) matrix obtained by removing from \( L_3 \) the rows and columns meeting the 6 entries above. Hence we only need to show \( \text{rank} \ L_4 = 10 \) for general \( t \in B_0 \).

It now suffices to show \( \det \ L_5 \neq 0 \) for general \( t \in B_0 \), where the \( 10 \times 10 \) matrix

\[
L_5 = \begin{bmatrix}
  c^{(1)} & a^{(1)} & 3 & h^{(1)} \\
  d^{(1)} & b^{(1)} & 2h^{(2)} & 2l^{(1)} \\
  b^{(2)} & a^{(2)} & 2h^{(2)} & 2l^{(2)} \\
  c^{(2)} & b^{(2)} & 2h^{(2)} & 3 \\
  d^{(2)} & b^{(2)} & 3 & h^{(2)} \\
  d^{(2)} & b^{(2)} & 2l^{(2)} & 3 \\
\end{bmatrix}
\]

is the one obtained by removing from \( L_4 \) its 6-th and 7-th rows. But,
when we compute $\det L_5$ by the definition of the determinant, the monomial $(d^{(2)})^2(a^{(1)})^2(l^{(2)})^2(h^{(2)})^2(t^{(1)})^2$ appears only once, i.e., from the term passing the entries $(9,1), (10,2), (1,3), (2,4), (7,5), (3,6), (5,7), (6,8), (8,9)$, and $(4,10)$ of $L_5$. Thus, for general $t \in B_0$, we have $\det L_5 \neq 0$, and hence $\text{rank} D' = 24$, which completes the proof of Theorem 2.

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