Moduli space of symplectic connections of Ricci type on $T^{2n}$; a formal approach

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Abstract

We consider analytic curves $\nabla^t$ of symplectic connections of Ricci type on the torus $T^{2n}$ with $\nabla^0$ the standard connection. We show, by a recursion argument, that if $\nabla^t$ is a formal curve of such connections then there exists a formal curve of symplectomorphisms $\psi_t$ such that $\psi_t \cdot \nabla^t$ is a formal curve of flat $T^{2n}$-invariant symplectic connections and so $\nabla^t$ is flat for all $t$. Applying this result to the Taylor series of the analytic curve, it means that analytic curves of symplectic connections of Ricci type starting at $\nabla^0$ are also flat.

The group $G$ of symplectomorphisms of the torus $(T^{2n}, \omega)$ acts on the space $\mathcal{E}$ of symplectic connections which are of Ricci type. As a preliminary to studying the moduli space $\mathcal{E}/G$ we study the moduli of formal curves of connections under the action of formal curves of symplectomorphisms.

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Introduction

On any symplectic manifold \((M, \omega)\) the space \(\mathcal{S}\) of symplectic connections is an infinite dimensional affine space whose corresponding vector space is the space of completely symmetric 3-tensors on \(M\). To encode some geometry into a symplectic connection it thus seems reasonable to introduce a selection rule for symplectic connections. A variational principle associated to a Lagrangian density, which is an invariant quadratic polynomial in the curvature, has been considered in \([1]\); the symplectic connections satisfying the Euler–Lagrange equations are said to be preferred. The symplectomorphism group \(G\) of \((M, \omega)\) acts naturally on \(\mathcal{S}\) and stabilises the subspace \(\mathcal{P}\) of preferred symplectic connections. The first question we wanted to address is to give a description of the moduli space \(\mathcal{P}/G\) of preferred connections modulo the action of symplectomorphisms. Such a description was given in \([1]\) when \((M, \omega)\) is a closed surface; but, up to now, very little has been done in the higher dimensional situation.

We have observed that a linear condition on the curvature (the vanishing of one of its irreducible components – the non-Ricci component, \(W\)) implies the Euler–Lagrange equations. Furthermore, this condition seems to imply that many of the properties of the surface situation extend to the higher-dimensional case. We have called symplectic connections satisfying this curvature condition \textit{connections of Ricci type} (all symplectic connections in dimension 2 are of Ricci type). This condition is preserved by symplectomorphisms and so we modify our initial question to the following one: give a description of the space \(\mathcal{E}\) of Ricci type connections and its moduli space \(\mathcal{E}/G\).

This paper is devoted to this modified question in the case where \(M\) is a torus \(T^{2n}\) and \(\omega\) a \(T^{2n}\)-invariant symplectic structure. Although we do not answer this question, we are able, in a formal setting made precise below, to show that the moduli space is infinite dimensional and to give a partial description of it.

If \(\nabla^t\) is a formal curve of symplectic connections, we shall denote by \(W^t\) the \(W\) part of the curvature of \(\nabla^t\). We prove

\textbf{Theorem} Let \(\nabla^t\) be a formal curve of symplectic connections on \((T^{2n}, \omega)\) such that \(\nabla^0\) is the standard flat connection on \(T^{2n}\), and such that \(W^t = 0\). Then the formal curvature \(R^t\) of \(\nabla^t\) vanishes and there exists a formal curve of symplectomorphisms \(\psi_t\) such that \(\tilde{\nabla}^t := \psi_t \cdot \nabla^t\) is a formal curve of flat \(T^{2n}\)-invariant symplectic connections.

This implies

\textbf{Theorem} Let \(\nabla^t\) be an analytic curve of analytic symplectic connections on \((T^{2n}, \omega)\) such that \(\nabla^0\) is the standard flat connection on \(T^{2n}\), and such that \(W^t = 0\). Then the curvature \(R^t\) of \(\nabla^t\) vanishes.

For the moduli space in the formal setting, we show:

\textbf{Proposition} For two curves \(\tilde{\nabla}^t\) and \(\tilde{\nabla}^{t'}\) of invariant flat connections of Ricci-type on \((\mathbb{R}^{2n}, \Omega)\) with \(\tilde{\nabla}^0 = \tilde{\nabla}^{t'} = \tilde{\nabla}\) the trivial connection, there always exists a formal curve of
symplectomorphisms \( \tilde{\psi}_t \) so that \( \tilde{\psi}_t \cdot \tilde{\nabla}^t = \tilde{\nabla}^t \).

**Theorem**  The moduli space of formal curves of Ricci-type symplectic connections starting with the standard flat connection on \((T^{2n}, \omega)\) under the action of formal curves of symplectomorphisms is described by the space of formal curves of linear maps \( A^t : \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R}) \) satisfying \( A^t(X)A^t(Y) = 0 \) and \( A^t(X)Y = A^t(Y)X \), modulo the action of \( \text{Sp}(2n, \mathbb{Z}) \).

The plan of the paper is as follows. In \( \S 1 \) we recall some general properties of symplectic connections having Ricci-type curvature. In \( \S 2 \) we introduce the notion of formal curves of connections and we show that the properties of \( \S 1 \) are still true for a formal curve of symplectic connections with Ricci-type curvature. In \( \S 3 \), we analyse the \( W^t = 0 \) condition at order 1 and order 2 for \( \nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)} \) a formal curve of Ricci-type symplectic connections on \( T^{2n} \) with \( \nabla^0 \) the standard flat connection; in particular, we show that there exists a function \( U^{(1)} \) and a completely symmetric, \( T^{2n} \)-invariant 3-tensor \( Q^{(1)} \) on \( T^{2n} \) such that \( A^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)} \) and we show that \( \nabla'^t = \nabla^0 + tQ^{(1)} \) (with \( \omega(Q^{(1)}(X)Y, Z) = Q^{(1)}(X, Y, Z) \)) defines a curve of invariant flat symplectic connections on \((T^{2n}, \omega)\). This remark can be formulated in a slightly different way: given \( \nabla^t = \nabla^0 + A^t \) a smooth curve of Ricci-type symplectic connections then, up to a symplectomorphism, the tangent vector to this family of connections lies in the finite dimensional space of flat \( T^{2n} \)-invariant symplectic connections. \( \S 4 \) is devoted to a proof of a recurrence lemma which implies the first theorem. In \( \S 5 \) we study the question of when two formal curves of flat invariant connections on \( T^{2n} \) are equivalent by a formal curve of symplectomorphisms.

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## 1 Ricci Type Curvature

A symplectic connection \( \nabla \) on a symplectic manifold \((M, \omega)\) is a linear connection having no torsion and for which \( \omega \) is parallel (\( \nabla \omega = 0 \)). The curvature endomorphism \( R \) of \( \nabla \) is

\[
R(X, Y)Z = \left( \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right) Z
\]

for vector fields \( X, Y, Z \) on \( M \). The symplectic curvature tensor

\[
R(X, Y; Z, T) = \omega(R(X, Y)Z, T)
\]

is antisymmetric in its first two arguments, symmetric in its last two and satisfies the first Bianchi identity

\[
\bigoplus_{X,Y,Z} R(X, Y; Z, T) = 0
\]
where \( \oplus \) denotes the sum over the cyclic permutations of the listed set of elements. The second Bianchi identity takes the form
\[
\oplus_{X,Y,Z} (\nabla_X R)(Y, Z) = 0.
\]
The Ricci tensor \( r \) is the symmetric 2-tensor
\[
r(X, Y) = \text{Trace}[Z \mapsto R(X, Z)Y].
\]
If \( \dim M = 2n \geq 4 \), the curvature \( R \) of such a connection has 2 irreducible components under the action of the symplectic group \( Sp(2n, \mathbb{R}) \). We denote them by \( E \) and \( W \):
\[
R = E + W.
\]
The \( E \) component encodes the information contained in the Ricci tensor of \( \nabla \) and is called the Ricci part of the curvature tensor. It is given by
\[
E(X, Y; Z, T) = \frac{-1}{2(n+1)} \left[ 2\omega(X, Y)r(Z, T) + \omega(X, Z)r(Y, T) + \omega(X, T)r(Y, Z) - \omega(Y, T)r(X, Z) - \omega(Y, Z)r(X, T) - \omega(Y, T)r(X, Z) \right].
\]
The curvature is said to be of Ricci type if the \( W \) component vanishes, i.e. when \( R = E \).

**Lemma 1.1** Let \( (M, \omega) \) be a symplectic manifold of dimension \( 2n \geq 4 \). If the curvature of a symplectic connection \( \nabla \) on \( M \) is of Ricci type then there is a 1-form \( u \) such that
\[
(\nabla_X r)(Y, Z) = \frac{1}{2n+1} (\omega(X, Y)u(Z) + \omega(X, Z)u(Y)).
\]
Conversely, if there is such a 1-form \( u \), the “Weyl” part of the curvature, \( W = R - E \) satisfies
\[
\oplus_{X,Y,Z} (\nabla_X W)(Y, Z; T, U) = 0.
\]
**Proof** The property follows from the second Bianchi’s identity, see [2].

**Corollary 1.2** A symplectic manifold with a symplectic connection whose curvature is of Ricci type is locally symmetric if and only if the 1-form \( u \), defined in the lemma, vanishes.

Denote by \( \rho \) the linear endomorphism such that
\[
r(X, Y) = \omega(X, \rho Y).
\]
The symmetry of \( r \) is equivalent to saying that \( \rho \) is in the Lie algebra of the symplectic group \( Sp(TM, \omega) \). For an integer \( p > 1 \), define
\[
(\rho^p)(X, Y) = \omega(X, \rho^p Y).
\]
It is symmetric when \( p \) is odd and antisymmetric when \( p \) is even.
Lemma 1.3 Let $(M,\omega)$ be a symplectic manifold with a symplectic connection $\nabla$ with Ricci-type curvature. Then, the following identities hold:

(i) There is a function $b$ such that
\[
\nabla u = -\frac{1 + 2n}{2(1 + n)} r^2 + b\omega.
\]

(ii) The differential of the function $b$ is given by
\[
db = \frac{1}{1 + n} i(\overline{u}) r
\]
where $\overline{u}$ is the vector field such that $i(\overline{u}) \omega = u$, so that

(iii)
\[
b + \frac{2n + 1}{4(1 + n)} \text{Trace } \rho^2
\]
is a constant when $M$ is connected.

Proof These identities follow from Lemma 1.1, see [2].

Let the torus $T^{2n}$ be endowed with a $T^{2n}$-invariant symplectic structure $\omega$. Let $\nabla$ be a symplectic connection on $(T^{2n},\omega)$ which is of Ricci-type. The group $G$ of symplectomorphisms of $(T^{2n},\omega)$ acts on the set $\mathcal{E}$ of symplectic connections with $W = 0$. We are interested in the set of orbits of $G$ in $\mathcal{E}$, i.e. in $\mathcal{E}/G$.

We now consider the symplectic vector space $(\mathbb{R}^{2n},\Omega)$ and view $\Omega$ as a translation invariant symplectic structure. A symplectic connection on $\mathbb{R}^{2n}$ will be determined by its values on translation invariant vector fields. If, in addition, the connection $\nabla$ is translation invariant then $B(X)Y := \nabla_X Y$ (for invariant vector fields $X,Y$) defines a linear map $B: \mathbb{R}^{2n} \to \mathfrak{sp}(2n,\mathbb{R})$ which completely determines $\nabla$. The only condition on $B$ is that $\Omega(B(X)Y, Z)$ is completely symmetric.

Proposition 1.4 Let $\nabla$ be a translation invariant symplectic connection on $(\mathbb{R}^{2n},\Omega)$ and let $B(X)Y = \nabla_X Y$ as above. If $\nabla$ is of Ricci type and $2n \geq 4$, then $\nabla$ is flat and $B(X)B(Y) = 0$.

Proof Since $B$ is constant, the curvature endomorphism is given by
\[
R(X,Y) = [B(X), B(Y)]
\]
and so the Ricci tensor is given by
\[
r(X,Y) = \text{Trace}(B(X)B(Y)).
\]
It is easy to see that symplectic curvature tensors $R(X,Y;Z,T)$ are, in fact, determined by the terms of the form $R(X,Y;X,Y)$ so that the equation $W = 0$ is equivalent to

$$R(X,Y;X,Y) = -\frac{2}{n+1}\Omega(X,Y)r(X,Y),$$

and in the present case this has the form

$$(n + 1)\Omega(B(X)X, B(Y)Y) = -2\Omega(X,Y)r(X,Y).$$

Polarising the equation in $X$ we have

$$(n + 1)\Omega(T, B(X)B(Y)Y) = \Omega(X,Y)r(T,Y) + \Omega(T,Y)r(X,Y)$$

so that $W = 0$ is equivalent to

$$(n + 1)B(X)B(Y)Y = \Omega(X,Y)\rho Y + \Omega(X,\rho Y)Y.$$ 

Polarising this in $Y$ we have

$$2(n + 1)B(X)B(Y)Z = \Omega(X,Y)\rho Z + \Omega(X,\rho Y)Z + \Omega(X,Z)\rho Y + \Omega(X,\rho Z)Y \quad (\ast).$$

Now choose dual bases $X^i, X_i$ for $\mathbb{R}^{2n}$ with $\Omega(X^i, X_j) = \delta^i_j$ then an easy calculation shows

$$\rho = \sum_i B(X^i)B(X_i).$$

If we multiply $(\ast)$ by $B(X^i)$, set $X = X_i$ and sum we get

$$(n + 1)\rho B(Y)Z = -B(Y)\rho Z - B(Z)\rho Y.$$

Alternatively we may substitute $B(X_i)Z$ for $Z$ in $(\ast)$, set $Y = X^i$ and sum to give

$$(n + 1)B(X)\rho Z = -\rho B(Z)X + B(Z)\rho X.$$

Adding the two equations after setting $X = Y$ we see that

$$\rho B(X) = -B(X)\rho$$

and hence that

$$(n - 1)\rho B(X) = 0.$$ 

Thus if $2n \geq 4$

$$\rho B(X) = B(X)\rho = 0 \quad \Rightarrow \rho^2 = 0.$$ 

Substituting $\rho Z$ for $Z$ in $(\ast)$ we have

$$0 = r(X,Y)\rho Z + r(X,Z)\rho Y$$

and setting $Z = Y$, applying $\Omega(X,\cdot)$ we get finally

$$0 = r(X,Y)^2.$$ 

Thus the Ricci tensor vanishes, and hence $\nabla$ is flat.

Putting $\rho = 0$ in $(\ast)$ yields $B(X)B(Y) = 0$. 

\[\blacksquare\]
2 Formal curves

Definition 2.1 A formal curve of symplectic connections on a symplectic manifold \((M,\omega)\) is a formal power series

\[ \nabla^t = \nabla + \sum_{k=1}^{\infty} t^k A^{(k)} \]

where \(\nabla\) is a symplectic connection on \(M\), and the \(A^{(k)}\) are \((2,1)\) tensors such that

\[ A^{(k)}(X,Y,Z) = \omega(A^{(k)}(X)Y,Z) \tag{2.1} \]

is totally symmetric.

Definition 2.2 A formal curve of symplectomorphisms is a homomorphism of Poisson algebras

\[ \psi_t: C^\infty(M) \to C^\infty(M)[[t]], \quad \psi_t = \psi^{(0)} + \sum_{k=1}^{\infty} t^k \psi^{(k)} \]

such that \(\psi^{(0)}: C^\infty(M) \to C^\infty(M)\) is an isomorphism.

The leading term \(\psi^{(0)}\) of a formal curve of symplectomorphisms is given by composition with a symplectomorphism \(\psi^{(0)}(f) = f \circ \sigma = \sigma^*(f)\) so that we may take such a term out as a common factor and write \(\psi_t = \sigma^* \circ \phi_t\) and \(\phi_t = \text{id} + \sum_{k \geq 1} t^k \phi^{(k)}\).

If \(\phi_t = \text{id} + \sum_{k \geq 1} t^k \phi^{(k)}\) is a formal curve of symplectomorphisms beginning with the identity then the first order term \(X^{(1)} = \phi^{(1)}\) is a symplectic vector field. Moreover, for any symplectic vector field, \(\exp tX = \text{id} + \sum_{k \geq 1} t^k/k! X^k\) is a formal curve of symplectomorphisms. A straightforward recursion argument then shows that any formal curve of symplectomorphisms beginning with the identity can be written in the form \(\phi_t = \exp X_t\) where \(X_t = \sum_{k \geq 1} t^k X^{(k)}\) is a formal curve of vector fields.

Definition 2.3 A formal 1-parameter group of symplectomorphisms is a formal curve of symplectomorphisms \(\psi_t\) such that \(\psi_{at} \circ \psi_{bt} = \psi_{(a+b)t}\) for all \(a, b \in \mathbb{R}\).

In order for this definition to make sense we first have to extend \(\psi_t\) by linearity over \(\mathbb{R}[t]\) to a morphism of \(\mathbb{R}[t]\)-algebras. The definition then implies that \(\psi^{(0)}\) is the identity and that \(\psi^{(1)}(f) = X(f)\) for some symplectic vector field which we call the infinitesimal generator of \(\psi_t\). It is easy to see that every formal 1-parameter group of symplectomorphisms has the form \(\psi_t = \exp tX\). Moreover, a recursion shows that, if \(X_t\) is a formal curve of symplectic vector fields, we can find a second sequence of symplectic vector fields \(Y^{(k)}\) such that

\[ \exp X_t = \exp tY^{(1)} \circ \exp t^2 Y^{(2)} \circ \cdots \circ \exp t^k Y^{(k)} \circ \cdots \]
and so any formal curve of symplectomorphisms $\psi_t$ can be factorised in two ways

$$\psi_t = \sigma^* \circ \exp X_t = \sigma^* \circ \phi_t^{(1)} \circ \phi_t^{(2)} \circ \cdots \circ \phi_t^{(k)} \circ \cdots$$

where the $\phi_t^{(k)}$ are formal 1-parameter groups of symplectomorphisms.

Remark that a formal curve of symplectomorphisms $\psi_t$ acts on a formal curve of vector fields $X_t$ viewed as a $\mathbb{R}[t]$-linear derivation of $C^\infty(M)[t]$ by

$$(\psi_t \cdot X_t) f = \psi_t(X_t(\psi_t^{-1} f)),$$

and acts on a formal curve of symplectic connections $\nabla^t$ by

$$(\psi_t \cdot \nabla^t)_X Y = \psi_t \cdot \left( \nabla^t_{\psi_t^{-1} X} \psi_t^{-1} \cdot Y \right). \quad (2.2)$$

Let $\nabla^t$ be a formal curve of symplectic connections on a symplectic manifold $(M, \omega)$ of dimension $2n$,

$$\nabla^t = \nabla + \sum_{k=1}^{\infty} t^k A^{(k)}.$$

We denote as in (2.1) by $A^{(k)}$ the corresponding symmetric 3-tensors. The formal curvature endomorphism $R^t$ of $\nabla^t$ is

$$R^t(X, Y) = \nabla^t_X \circ \nabla^t_Y - \nabla^t_Y \circ \nabla^t_X - \nabla^t_{[X,Y]}$$

so that

$$R^t = R^\nabla + \sum_{k=1}^{\infty} t^k R^{(k)}$$

with

$$R^{(k)}(X, Y) = (\nabla_X A^{(k)})(Y) - (\nabla_Y A^{(k)})(X) + \sum_{p+q=k} [A^{(p)}(X), A^{(q)}(Y)]. \quad (2.3)$$

The symplectic curvature tensor $R^t(X, Y; Z, T) = \omega(R^t(X, Y)Z, T)$ is antisymmetric in its first two arguments, symmetric in its last two, satisfies the first Bianchi identity $\nabla^t_{[X,Y,Z]} R^t(X, Y; Z, T) = 0$ and the second Bianchi identity $\nabla^t_X R^t(Y, Z) = 0$.

The formal Ricci tensor is $r^t(X, Y) = \text{Trace}[Z \mapsto R^t(X, Z)Y]$, so that

$$r^t = r^\nabla + \sum_{k=1}^{\infty} t^k r^{(k)}$$

where the $r^{(k)}$ are the symmetric tensors

$$r^{(k)}(X, Y) = \text{Trace}[Z \mapsto (\nabla_Z A^{(k)})(X)Y] + \sum_{p+q=k} \text{Trace} A^{(p)}(X)A^{(q)}(Y). \quad (2.4)$$

The Ricci part $E^t$ of the formal curvature tensor is given by

$$E^t(X, Y; Z, T) = \frac{-1}{2(n+1)} \left[ 2\omega(X, Y)r^t(Z, T) + \omega(X, Z)r^t(Y, T) + \omega(X, T)r^t(Y, Z) \\
- \omega(Y, Z)r^t(X, T) - \omega(Y, T)r^t(X, Z) \right]. \quad (2.5)$$

The formal curvature is said to be of Ricci type when $R^t = E^t$. 
Lemma 2.4 Let \((M, \omega)\) be a symplectic manifold of dimension \(2n \geq 4\). If the formal curvature of a formal curve of symplectic connections \(\nabla^t\) on \(M\) is of Ricci type then there exists a formal curve of 1-forms

\[ u^t = \sum_{k=0}^{\infty} t^k u^{(k)} \]

such that

\[ (\nabla^t_X r^t)(Y, Z) = \frac{1}{2n+1} \left( \omega(X, Y) u^t(Z) + \omega(X, Z) u^t(Y) \right) \]

and there exists a formal curve of functions

\[ b^t = \sum_{k=0}^{\infty} t^k b^{(k)} \]

such that

\[ \nabla^t u^t = -\frac{1+2n}{2(1+n)} r^t + b^t \omega. \]

with \(\omega(X, (\rho^t)Y) = r^t(X, Y) = \omega(X, (\rho^t)^2 Y)\). Also

\[ db^t = \frac{1}{1+n} i(\pi^\alpha) r^t. \]

Lemma 2.5 Let \(\nabla^t\) be a formal curve of translation invariant symplectic connections on \((\mathbb{R}^{2n}, \Omega)\) and let \(B^t(X)Y := \nabla^t_X Y\) (for invariant vector fields \(X, Y\)). If \(\nabla^t\) is of Ricci type and \(2n \geq 4\), then \(\nabla^t\) is flat and \(B^t(X)B^t(Y) = 0\).

Proof We can copy in the formal series setting the proof of Lemma 1.4. Write \(B^t = \sum_{k=0}^{\infty} t^k B^{(k)}\) where the \(B^{(k)}\) are constant maps from \(\mathbb{R}^{2n}\) to \(sp(\mathbb{R}^{2n}, \Omega)\). The formal curvature endomorphism is given by

\[ R^t(X, Y) = [B^t(X), B^t(Y)] \quad \text{i.e.} \quad R^{(k)}(X, Y) = \sum_{p+q=k, p,q \geq 0} [B^p(X), B^q(Y)] \]

and the formal Ricci tensor by

\[ r^t(X, Y) = \text{Trace}(B^t(X)B^t(Y)) \quad \text{i.e.} \quad r^{(k)}(X, Y) = \sum_{p+q=k, p,q \geq 0} \text{Trace} B^p(X)B^q(Y). \]

The equation \(W^t = 0\) is again equivalent to \(2(n+1)B^t(X)B^t(Y)Z = \Omega(X, Y)\rho^t Z + \Omega(X, \rho^t Y)Z + \Omega(X, Z)\rho^t Y + \Omega(X, \rho^t Z)Y\), i.e.

\[ \sum_{p+q=k, p,q \geq 0} 2(n+1)B^{(p)}(X)B^{(q)}(Y)Z = \Omega(X, Y)\rho^{(k)} Z + \Omega(X, \rho^{(k)} Y)Z + \Omega(X, Z)\rho^{(k)} Y + \Omega(X, \rho^{(k)} Z)Y. \]
Choosing dual bases $X^i$, $X_j$ for $\mathbb{R}^{2n}$ with $\Omega(X^i, X_j) = \delta^i_j$ then $\rho^t = \sum_i B'^i(X^i)B^t(X_i)$, i.e. $\rho^{(k)} = \sum_{p+q=k} \sum_i B^p(X^i)B^q(X_i)$. If we multiply (2.3) by $B^{(k')}(X^i)$, set $X = X_i$ and sum over $i$ and over $k, k' \geq 0$ so that $k + k' = K$ we get

$$(n + 1) \sum_{q' + q = K} \sum_{q, q' \geq 0} \rho^{(q')} B^q(Y)Z = -B^k(Y)\rho^{(k)}Z - B^{(k)}(Z)\rho^{(k)}Y.$$  

This can be written in terms of formal series

$$(n + 1)\rho^t B^t(Y)Z = -B^t(Y)\rho^tZ - B^t(Z)\rho^tY.$$  

Alternatively we may substitute $B^{(s)}(X_i)Z$ for $Z$ in (2.9), set $Y = X^i$ and sum to give

$$(n + 1)B^t(X)\rho^tZ = -\rho^t B^t(Z)X + B^t(Z)\rho^tX.$$  

Adding the two equations after setting $X = Y$ as before, we see that $\rho^t B^t(X) = -B^t(X)\rho^t$, so $(n - 1)\rho^tB^t(X) = 0$ and, if $2n \geq 4$, $\rho^tB^t(X) = B^t(X)\rho^t = 0$ thus $(\rho^t)^2 = 0$. This in turn implies $r^t = 0$, hence $R^t = 0$ and $\nabla$ is flat. Putting $\rho^t = 0$ in (2.9) yields $B^t(X)B^t(Y) = 0$.

## 3 Curves of Ricci Type Connections on the Torus

Consider the torus $T^{2n}$ endowed with a $T^{2n}$-invariant symplectic structure $\omega$. Let $\nabla^0$ be the standard flat, $T^{2n}$-invariant symplectic connection on $(T^{2n}, \omega)$. Let

$$\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$$

be a formal curve of symplectic connections such that $W(t) = 0$. We denote as before (2.1) by $A^{(k)}$ the corresponding symmetric 3-tensors $A^{(k)}(X, Y, Z) = \omega(A^{(k)}(X)Y, Z)$.

We consider, as given by Lemma 2.4, the corresponding formal curve of 1-forms $u^t = \sum_{k=0}^{\infty} t^k u^{(k)}$ and the formal curve of functions $b^t = \sum_{k=0}^{\infty} t^k b^{(k)}$; clearly $u^{(0)} = 0$ and $b^{(0)} = 0$ since $r^0 = 0$.

**Lemma 3.1** If $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ is a formal curve of symplectic connections such that $W(t) = 0$, then the formal curvature vanishes at order 1 in $t$ (i.e. one has $b^{(1)} = 0$, $u^{(1)} = 0$, $r^{(1)} = 0$, $R^{(1)} = 0$). Furthermore, there exists a function $U^{(1)}$ and a completely symmetric, $T^{2n}$-invariant 3-tensor $Q^{(1)}$ on $T^{2n}$ such that

$$A^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}.$$  

**Proof**  Denote by $x^a$ ($1 \leq a \leq 2n$) the standard angle variables on $T^{2n}$ and by $\partial_a$ the corresponding $T^{2n}$-invariant vector fields on $T^{2n}$ (the standard flat connection is defined by $\nabla_{\partial_a} \partial_b = 0$).

At order 1, since $b^{(0)} = 0$, $u^{(0)} = 0$, $r^{0} = 0$, we have:
(i) $db^{(1)} = 0$ by (2.8), so $b^{(1)}$ is a constant;

(ii) $du^{(1)} = b^{(1)} \omega$ by (2.7); but $\omega$ is not exact by compactness of $T^{2n}$ so $b^{(1)} = 0$ and $\nabla^0 u^{(1)} = 0$ thus $u^{(1)}(X)$ is a constant for any $T^{2n}$-invariant vector field $X$ on $T^{2n}$;

(iii) the equation (2.6) at order 1 yields $(\nabla^0 r^1)$ as a combination of products of $\omega$ and $u^1$ so that $\partial_a (r^{(1)}(\partial_b, \partial_c))$ is a constant; the periodicity of the angles $x^a$ implies then that $\partial_a (r^{(1)}(\partial_b, \partial_c)) = 0$ so $u^{(1)} = 0$ and $r^{(1)}(\partial_b, \partial_c) = a^{(1)}_{ab}$ is a constant.

The definition of the (formal) Ricci tensor (2.4) at order 1 yields

\[ a^{(1)}_{ab} = 0 \quad \text{so} \quad r^{(1)} = 0 \quad \text{and thus} \quad R^{(1)} = 0. \]

The definition of the (formal) curvature tensor (2.3) at order 1 gives $R^{(1)}_{abcd} = \partial_a A^{(1)}_{bcd} - \partial_b A^{(1)}_{acd}$. Hence, for each value of the indices $c, d$ the 1-form $A^{(1)}_{cd}$ is closed, so there exist functions $k_{cd}$ on $T^{2n}$ and constants $Q^{(1)}_{bcd}$ such that:

\[ A^{(1)}_{bcd} = \partial_b k^{(1)}_{cd} + Q^{(1)}_{bcd}. \]

Since $\nabla^t$ is symplectic, $A^{(1)}_{bcd}$ is totally symmetric; the fact that $A^{(1)}_{bcd} - A^{(1)}_{bdc} = 0$ implies

\[ \partial_b k^{(1)}_{cd} - \partial_c k^{(1)}_{bd} = -Q^{(1)}_{bcd} + Q^{(1)}_{bdc}. \]

When $d$ is fixed, the left-hand side is an exact 2-form. The right-hand side is $T^{2n}$-invariant. Since there are no non-zero exact $T^{2n}$-invariant forms, this implies

\[ Q^{(1)}_{bcd} = Q^{(1)}_{bdc}, \quad \partial_k k^{(1)}_{cd} - \partial_k k^{(1)}_{bd} = 0. \]

Similarly $A^{(1)}_{bcd} - A^{(1)}_{bdc} = 0$ gives

\[ \partial_b k^{(1)}_{cd} - \partial_c k^{(1)}_{bd} = -Q^{(1)}_{bcd} + -Q^{(1)}_{bdc}. \]

In this case, when $c$ and $d$ are fixed, the left-hand side is an exact 1-form, while the right-hand side is $T^{2n}$-invariant. For the same reason as above, we deduce that both members vanish:

\[ Q^{(1)}_{bcd} = Q^{(1)}_{bdc}, \quad k^{(1)}_{cd} - k^{(1)}_{dc} = \text{constant}. \]

Hence $Q^{(1)}_{bcd}$ is completely symmetric. Furthermore, for each fixed index $d$, the 1-form $k^{(1)}_{.d}$ is closed. Hence there exist functions $S^{(1)}_d$ and constants $T_{cd}$ such that

\[ k^{(1)}_{cd} = \partial_c S^{(1)}_d + T^{(1)}_{cd}. \]

The fact that $k^{(1)}_{cd} - k^{(1)}_{dc}$ is a constant implies for the 1-form $S^{(1)}_d$ that $dS^{(1)}$ is $T^{2n}$-invariant, thus $S^{(1)}$ is closed. Hence there exists a function $U^{(1)}$ and constants $V^{(1)}_d$ such that

\[ S^{(1)}_d = \partial_d U^{(1)} + V^{(1)}_d. \]

Substituting, we have:

\[ A^{(1)}_{bcd} = \partial^2_{bcd} U^{(1)} + Q^{(1)}_{bcd}. \]
Lemma 3.2 If $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ is a formal curve of symplectic connections such that $W(t) = 0$, then the curvature vanishes at order 2 in $t$, i.e. $b^{(2)} = 0$, $u^{(2)} = 0$, $r^{(2)} = 0$, $R^{(2)}(0) = 0$.

Writing $A^{(1)}_b = (\nabla^0)^3 U^{(1)} + \omega^{Q(1)}$ as in Lemma 3.1, the formula $\nabla^t = \nabla^0 + t Q^{(1)}$, where $\omega(Q^{(1)}(X, Y, Z) = Q^{(1)}(X, Y, Z)$, defines a curve of invariant flat symplectic connections on $(T^{2n}, \omega)$.

Furthermore, there exist a function $U^{(2)}$ and a $T^{2n}$-invariant, completely symmetric tensor $Q^{(2)}$ such that

$$A^{(2)}_{bcd} = \sum_{abcd} U^{(1)}p_b(Q^{(1)}_{pcm} + \frac{1}{2} U^{(1)}_{pcm}) + \frac{1}{2} U^{(1)}_{pbc} + \delta^3_{bcd} U^{(2)} + Q^{(2)}_{bcd}$$

where

$$U^{(1)}_{p1...pq} = \partial^k_{p1...pq} U^{(1)} \quad U^{(1)}_{q1...qk} = \partial^k_{q1...qk} U^{(1)} \omega^{ap} \quad \omega^{pq} \omega^q = \delta^p_q.$$

Proof At order 2, since $b^{(0)} = b^{(1)} = 0$, $u^{(0)} = u^{(1)} = 0$, $r^{(0)} = r^{(1)} = 0$

(i) $db^{(2)} = 0$ by (2.8), so $b^{(2)}$ is a constant;

(ii) $du^{(2)} = b^{(2)} \omega$ by (2.7); so $b^{(2)} = 0$ and $\nabla^0 u^{(2)} = 0$;

(iii) the equation (2.6) at order 2 yields that $\partial_a (r^{(2)}(\partial_b, \partial_c))$ is a constant; again this implies $u^{(2)} = 0$ and $r^{(2)}(\partial_b, \partial_c) = a^{(2)}_{ab}$ is a constant.

The definition of the (formal) Ricci tensor yields $a^{(2)}_{ab} = -\partial_a A^{(2)}_{ab} + A^{(1)}_{qb} A^{(1)}_{ap}$; Using lemma 3.1 with $Q^{(1)}_{pq} = Q^{(1)}_{qbp} \omega^{kp}$,

$$A^{(1)}_{qb} A^{(1)}_{ap} = Q^{(1)}_{qb} Q^{(1)}_{ap} + \partial_q (Q^{(1)}_{aqp} U^{(1)}_{pb}) + \partial_p (U^{(1)}_{paq} Q^{(1)}_{qbp}) + \partial_q (U^{(1)}_{pab} U^{(1)}_{qap}).$$

Hence:

$$a^{(2)}_{ab} = Q^{(1)}_{qb} Q^{(1)}_{ap} - \partial_q (A^{(2)}_{aq} - U^{(1)}_{pa} Q^{(1)}_{qap} - U^{(1)}_{pqa} Q^{(1)}_{qpb} - U^{(1)}_{pba} U^{(1)}_{qap}).$$

Since there are no exact, non-zero, $T^{2n}$-invariant 2n-form on $T^{2n}$, we have

$$a^{(2)}_{ab} = Q^{(1)}_{qb} Q^{(1)}_{ap} - \partial_q (A^{(2)}_{aq} - U^{(1)}_{pa} Q^{(1)}_{qap} - U^{(1)}_{pqa} Q^{(1)}_{qpb} - U^{(1)}_{pba} U^{(1)}_{qap}) = 0.$$
The fact that there does not exist a non-zero $T^{2n}$-invariant exact 2-form implies on one hand:

$$\partial_a(A_{bcd}^{(2)} - U^{(1)}_{pd}Q^{(1)}_{bc} - U^{(1)}_{pc}Q^{(1)}_{bd} - U^{(1)}_{pd}U^{(1)}_{bc} - U^{(1)}_{pc}U^{(1)}_{bd})$$

and on the other hand:

$$Q^{(1)}_{bc}Q^{(1)}_{apd} - Q^{(1)}_{ac}Q^{(1)}_{bpd} = -\frac{1}{2(n+1)} \left[ 2\omega_{ab}a^{(2)}_{cd} + \omega_{ac}a^{(2)}_{bd} + \omega_{ad}a^{(2)}_{bc} 
- \omega_{bc}a^{(2)}_{ad} - \omega_{bd}a^{(2)}_{ac} \right],$$

where $a^{(2)}_{ab} = Q^{(1)}_{qpb}Q^{(1)}_{q}{ap}.$

This last relation tells us that the $T^{2n}$-invariant connection defined by $\nabla^0 + tQ^{(1)}$ (which is symplectic because of the complete symmetry) has a $W$ tensor which is zero. Lifting everything to $\mathbb{R}^{2n}$ and applying lemma 3 we get that the corresponding curvature vanishes identically. Hence:

$$a^{(2)}_{ab} = 0, \quad Q^{(1)}_{bc}Q^{(1)}_{apd} - Q^{(1)}_{ac}Q^{(1)}_{bpd} = 0.$$  

This in turn implies

$$k^{(2)} = 0, \quad R^{(2)} = 0.$$

The first relation tells us that there exist functions $k_{cd}^{(2)}$ and constants $Q^{(2)}_{bcd}$ such that

$$A_{bcd}^{(2)} - U^{(1)}_{pd}Q^{(1)}_{bc} - U^{(1)}_{pc}Q^{(1)}_{bd} - U^{(1)}_{pd}U^{(1)}_{bc} - U^{(1)}_{pc}U^{(1)}_{bd} = \partial_b k_{cd}^{(2)} + Q^{(2)}_{bcd}.$$  

This can be rewritten as

$$A_{bcd}^{(2)} = \frac{1}{2} U^{(1)}_{pd}b(Q^{(1)}_{pcd} + \frac{1}{2} U^{(1)}_{pcd}) - \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{pcd} = \partial_b k_{cd}^{(2)} + Q^{(2)}_{bcd} \quad (3.10)$$

with

$$k_{cd}^{(2)} = k_{cd}^{(2)} - U^{(1)}_{pd}Q^{(1)}_{pcd} + \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{pcd} - \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{pcd}.$$  

Indeed we have $U^{(1)}_{pd}U^{(1)}_{bc} = \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{bc} + \frac{1}{2} \partial_b(U^{(1)}_{pd}U^{(1)}_{bc}) + \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{bc}$ and also $\frac{1}{2} U^{(1)}_{pd}U^{(1)}_{pcd} = \frac{1}{2} \partial_b(U^{(1)}_{pd}U^{(1)}_{pcd}) - \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{pcd}.$

Now the left hand side of the equation $3.10$ is totally symmetric in its indices $(bcd)$ so the same reasoning as in Lemma $3.1$ shows that $Q^{(2)}$ is totally symmetric and there exists a function $U^{(2)}$ so that $\partial_b k^{(2)} = \partial_{bcd}^{2}U^{(2)}.$ Substituting, we find:

$$A_{bcd}^{(2)} = \frac{1}{2} U^{(1)}_{pd}b(Q^{(1)}_{pcd} + \frac{1}{2} U^{(1)}_{pcd}) + \frac{1}{2} U^{(1)}_{pd}U^{(1)}_{pcd} + \partial_{bcd}^{2}U^{(2)} + Q^{(2)}_{bcd}$$

which ends the proof of the lemma.
4 A Recurrence Lemma

Lemma 4.1 Let $\nabla^t$ be a formal curve of symplectic connections on $(T^{2n}, \omega)$ such that $\nabla^{(0)} = \nabla^0$, and $W^t = 0$. Assume that, for all orders $l < k$, $A^{(l)}$, and thus $r^{(l)}$, $u^{(l)}$, $b^{(l)}$ are $T^{2n}$-invariant. Then, at order $k$, $r^{(k)}$, $u^{(k)}$, $b^{(k)}$ are $T^{2n}$-invariant, and there exist a function $U^{(k)}$ on $T^{2n}$ and a $T^{2n}$-invariant completely symmetric 3 tensor $Q^{(k)}$ such that

$$A^{(k)} = \partial^3 U^{(k)} + Q^{(k)}.$$ 

Proof Assume that, up to order $k - 1$ (included), $A^{(l)}$, $r^{(l)}$, $u^{(l)}$, $b^{(l)}$ are $T^{2n}$-invariant. Then, at order $k$, we have

(i) $R_{abcd} = \partial_a A_{bcd} - \partial_b A_{acd} + \sum_{s + s' = k, s, s' > 0} A^{(s)} p_{bc} A^{(s')}_{apd} - A^{(s)} p_{ac} A^{(s')}_{bpd};$

(ii) $r^{(k)} = -\partial_q A^{(k)} q_{ac} + \sum_{s + s' = k, s, s' > 0} A^{(s)} p_{qc} A^{(s')}_{ap};$

(iii) $\partial_c r^{(k)} = \sum_{s + s' = k, s, s' > 0} A^{(s)} p_{ca} r^{(s')}_{pb} + \Gamma^{(s)} p_{cb} r^{(s')}_{ap} = \frac{1}{2n + 1} (\omega_c b^{(k)}_a + \omega_a u^{(k)}_b);$

(iv) $\partial_b u^{(k)}_a = \sum_{s + s' = k, s, s' > 0} A^{(s)} p_{ba} u^{(s')}_{p} = \frac{1 + 2n}{2(1 + n)} \sum_{s + s' = k, s, s' > 0} r^{(s')}_{bc} r^{(s')}_{ap};$

(v) $\partial_a b^{(k)} = \frac{1}{1 + n} \sum_{s + s' = k, s, s' > 0} \omega^{(s)} c r^{(s')}_{ca}.$

Relation (v) implies that $db^{(k)}$ is $T^{2n}$-invariant. Hence $db^{(k)} = 0$ and $b^{(k)}$ is a constant. Antisymmetrising (iv) we get that $du^{(k)} - b^{(k)} \omega$ is a $T^{2n}$-invariant 2-form, hence $du^{(k)} = 0$ and $b^{(k)} \omega = \frac{1 + 2n}{2(1 + n)} \sum_{s + s' = k, s, s' > 0} r^{(s')}_{bc} r^{(s')}_{ap};$

Also

$$\partial_b u^{(k)}_a = \sum_{s + s' = k, s, s' > 0} A^{(s)} p_{ba} u^{(s')}_{p}.$$

Using periodicity again and the fact that the right hand side is a constant, we see that the $u^{(k)}_a$ are constants. Relation (iii) tells us, for the same reason, that the $r^{(k)}$ are constants. Finally from (i) and the $W^t = 0$ condition, we get that $\partial_a A^{(k)}_{bcd} - \partial_b A^{(k)}_{acd}$ is a constant hence

$$\partial_a A^{(k)}_{bcd} - \partial_b A^{(k)}_{acd} = 0.$$  \hfill (4.11)

The reasoning of Lemma 3.1 applies to equation (4.11) so there exist a function $U^{(k)}$ on $T^{2n}$ and a $T^{2n}$-invariant completely symmetric 3 tensor $Q^{(k)}$ such that

$$A^{(k)} = \partial^3 U^{(k)} + Q^{(k)}.$$
We can now proceed to the proof of the main theorem.

**Theorem 4.2** Let $\nabla^t$ be a formal curve of symplectic connections on $(T^{2n}, \omega)$ with $\nabla^0$ the standard connection, and $W^t = 0$. Then there exists a formal curve of symplectomorphisms $\psi_t$ such that $\tilde{\nabla}^t := \psi_t \cdot \nabla^t$ is a formal curve of symplectic connections which is $T^{2n}$-invariant and has $\tilde{W}^t = 0$, hence is flat. In particular, $\tilde{\nabla}^t$ is flat.

**Proof** If $\nabla^t = \nabla^0 + \sum_{k=0}^{\infty} t^k A^{(p)}$ is any formal curve of symplectic connections, one defines as in 2.2 the action of a formal curve $\psi_t$ of symplectomorphisms on $\nabla^t$:

$$(\psi_t \cdot \nabla^t)_X Y = \psi_t \cdot \left( \nabla^{-1}_Y \psi_t^{-1} \cdot Y \right).$$

Consider a formal one-parameter group $\psi_f(t)$ of symplectomorphisms generated by a Hamiltonian vector field $X_f$ $(i(X_f) \omega = df)$ and consider the formal curve of symplectomorphisms defined by $\psi^{k-1}_f(t) = \psi_f(t^k)$. Write

$$\psi^{k-1}_f(t) \cdot \nabla^t = \nabla^0 + \sum_{p=0}^{\infty} t^p \tilde{A}^{(p)}$$

then $\tilde{A}^{(p)} = A^{(p)}$, $\forall p < k$ and

$$\tilde{A}^{(k)}_X Y = A^{(k)}_X Y + [X_f, \nabla^0_Y Z] - \nabla^0_{[X_f,Y]} Z - \nabla^0_Y [X_f, Z].$$

Observe that $[X_f, \nabla^0_Y Z] - \nabla^0_{[X_f,Y]} Z - \nabla^0_Y [X_f, Z] = R^0_f(Y,Z) + ((\nabla^0)^2 X_f)(Y,Z)$ and

$$\omega((\nabla^0)^3 f)(Y,Z, T) = ((\nabla^0)^3 f)(Y,Z, T).$$

Assume now that the curve $\nabla_t = \nabla^0 + \sum_{k=0}^{\infty} t^k A^{(p)}$ is a curve of symplectic connections on the torus $(T^{2n}, \omega)$ and that $\nabla^0$ is the standard flat connection.

At order 1, we have seen in Lemma 3.1 that $A^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}$ so choosing $f_1 = -U^{(1)}$ and $\psi^{(1)}(t) = \psi_{f_1}(t)$ as defined above we see that

$$\psi^{(1)}(t) \cdot \nabla^t = \nabla^0 + t \tilde{Q}^{(1)} + \sum_{p=2}^{\infty} t^p \tilde{A}^{(p)}$$

with $\omega(\tilde{Q}^{(1)}(X) Y, Z) = Q^{(1)}(X,Y,Z)$. Assume now that one has found a formal curve of symplectomorphisms $\psi^{(k-1)}(t)$ so that

$$\psi^{(k-1)}(t) \cdot \nabla^t = \nabla^0 + \sum_{p=1}^{k-1} t^p \tilde{Q}^{(p)} + \sum_{p=k}^{\infty} t^p \tilde{A}^{(p)}$$

where the $\tilde{Q}^{(p)}$ are $T^{2n}$-invariant.
At order $k$, we have seen in Lemma [1] that $\Delta^{(k)} = (\nabla^0)^3 U^{(k)} + Q^{(k)}$ where $Q^{(k)}$ is $T^{2n}$-invariant, so choosing $f_k = -U^{(k)}$, $\psi^k_f(t)$ as defined above and $\psi^{(k)}(t) = \psi^k_f(t^k) \circ \psi^{(k-1)}(t)$ we see that

$$\psi^{(k)}(t) \cdot \nabla^t = \psi^k_f(t^k) \cdot \psi^{(k-1)}(t) \cdot \nabla^t = \nabla^0 + \sum_{p=1}^{k} t^p Q^{(p)} + \sum_{p=k+1}^{\infty} t^p \tilde{A}^{(p)}$$

with $\omega(Q^{(k)}(X,Y,Z) = Q^{(k)}(X,Y,Z)$. By induction this proves that one can build a formal curve of symplectomorphisms

$$\psi(t) = \ldots \circ \psi^{(k)}_f(t^k) \circ \ldots \circ \psi^{(2)}_f(t^2) \circ \psi^{(1)}_f(t)$$

so that $\tilde{\nabla}(t) := \psi(t) \cdot \nabla(t)$ is a formal curve of symplectic connections which is $T^{2n}$-invariant and has $\tilde{W}(t) = 0$. Lifting the connection to $\mathbb{R}^{2n}$ and using Lemma 2.3 shows that $\tilde{\nabla}(t)$ has vanishing curvature. Since $\nabla(t) = (\psi(t))^{-1} \cdot \tilde{\nabla}(t)$, its curvature is 0 so $\nabla(t)$ is flat.

The above theorem implies:

**Theorem 4.3** Let $\nabla^t$ be an analytic curve of analytic symplectic connections on $(T^{2n}, \omega)$ such that $\nabla^0$ is the standard flat connection on $T^{2n}$, and such that $W^t = 0$. Then the curvature $R^t$ of $\nabla^t$ vanishes.

## 5 Equivalence of formal curves of connections

In this section we study the question of when two formal curves of flat invariant connections on $T^{2n}$ are equivalent by a formal curve of symplectomorphisms. First we consider the question on $(\mathbb{R}^{2n}, \Omega)$. Here it is easy to answer.

The first case to consider is the case of a single flat invariant connection $\nabla^A = \nabla^0 + A$ on $(\mathbb{R}^{2n}, \Omega)$. We have seen that such a connection is given by a linear map $A: \mathbb{R}^{2n} \rightarrow \mathfrak{sp}(2n, \mathbb{R})$ satisfying $A(X)A(Y) = 0$ and $\Omega(A(X)Y, Z)$ completely symmetric. Define $\psi^A: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by

$$\psi^A(x) = x - \frac{1}{2} A(x)x.$$

**Proposition 5.1** $\psi^A$ is a symplectomorphism of $(\mathbb{R}^{2n}, \Omega)$ satisfying $\psi^A \cdot \nabla^0 = \nabla^A$.

**Proof** It is enough to check that $\psi^A$ is a symplectomorphism on constant vector fields. We make extensive use of the fact that $A(X)A(Y) = 0$. If $X$ is a constant vector field then

$$\psi^A_x X = \frac{d}{dt} \psi^A(x + tX) \bigg|_{t=0} = (X - A(x)X)\psi^A(x),$$

thus $\psi^A \cdot X = X - A(\cdot)X$. Hence

$$\Omega(\psi^A \cdot X, \psi^A \cdot Y)(x) = \Omega(X - A(x)X, Y - A(x)Y) = \Omega(X, Y).$$
It is easy to see that \( \psi^{-A} \) is an inverse for \( \psi^A \) so that \( \psi^A \) is a symplectomorphism. Indeed, \( t \mapsto \psi^t \) is a 1-parameter group of symplectomorphisms with generator the symplectic vector field \( (X_A)_x = -\frac{1}{2}A(x)x_x \).

Finally, for constant vector fields \( X, Y \)

\[
(\psi^A \cdot \nabla^0)_X Y = \psi^A \cdot (\nabla^0_{\psi^{-A}X} \psi^{-A} \cdot Y) = \psi^A \cdot ((X + A(\cdot)X)(A(\cdot)Y)).
\]

But

\[
(X + A(\cdot)X)(A(\cdot)Y)_x = \frac{d}{dt}A(x + t(X + A(x)X)) \bigg|_{t=0} = A(X)Y
\]

so

\[
(\psi^A \cdot \nabla^0)_X Y = \psi^A \cdot (A(X)Y) = A(X)Y = \nabla^A_X Y.
\]

If \( \nabla^t = \nabla^0 + A^t \) is a formal curve of invariant flat connections on \((\mathbb{R}^{2n}, \Omega)\) given by a curve of linear maps \( A^t: \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[t] \) satisfying \( A^t(X)A^t(Y) = 0 \) and \( \Omega(A^t(X)Y, Z) \) completely symmetric, we define a formal curve of vector fields \( X_{A^t} \) by

\[
X_{A^t}(f)(x) = -\frac{1}{2}(A_t(x)x)_xf
\]

and set

\[
\psi_{A^t} = \exp X_{A^t}.
\]

**Proposition 5.2** \( \psi_{A^t} \) is a formal curve of symplectomorphisms of \((\mathbb{R}^{2n}, \Omega)\) and \( \psi_{A^t} \cdot \nabla^0 = \nabla^{A^t} \).

**Proof** As the exponential of a derivation, \( \psi_{A^t} \) is invertible with inverse \( \exp -X_{A^t} = \psi_{-A^t} \). Moreover \( \psi_{A^t} \cdot X = \exp \text{ad} X_{A^t}X \) and it is easy to verify that \( \text{ad} X_{A^t}X = A^t(\cdot)X \), \( (\text{ad} X_{A^t})^2X = 0 \) so that \( \psi_{A^t} : X = X - A^t(\cdot)X \) as before. Likewise \( \psi_{-A^t} \cdot X = X + A^t(\cdot)X \) so that

\[
(\psi_{A^t} \cdot \nabla^0)_X Y = \psi_{A^t} \cdot (\nabla^0_{\psi_{-A^t}X} (Y + A^t(\cdot)Y)) = A^t(X)Y.
\]

In particular the above proves

**Theorem 5.3** For two curves \( \nabla^t \) and \( \nabla^{t'} \) of invariant flat connections of Ricci-type on \((\mathbb{R}^{2n}, \Omega)\) with \( \nabla^0 = \nabla^{t'} \) the trivial connection, there always exists a formal curve of symplectomorphisms \( \tilde{\psi}_t \) so that \( \tilde{\psi}_t \cdot \nabla^t = \nabla^{t'} \).

Finally, we need to know what is the general form of a formal curve of symplectomorphisms of \((\mathbb{R}^{2n}, \Omega)\) which fixes the trivial connection \( \nabla^0 \).

**Proposition 5.4** Let \( \psi_t = \sigma^t \circ \exp X_t \) be a formal curve of symplectomorphisms with \( \psi_t \cdot \nabla^0 = \nabla^0 \) then \( \sigma(x) = Cx + d \) and \( (X_t)_x = (C_t(x) + d_t)_x \) where \( C \in Sp(2n, \mathbb{R}) \), \( d \in \mathbb{R}^{2n} \), \( C_t \in t \mathfrak{sp}(2n, \mathbb{R})[t] \) and \( d_t \in t \mathbb{R}^{2n}[t] \).
**Theorem 5.5** Let the moduli space of symplectic connections of Ricci type on symplectomorphism of $E_{\omega}$. Evaluation at $t = 0$ shows that $\sigma \cdot \nabla^0 = \nabla^0$ so that $\sigma(x) = Cx + d$ where $C \in Sp(2n, \mathbb{R})$ and $d \in \mathbb{R}^{2n}$. Hence $\exp X_t \cdot \nabla^0 = \nabla^0$. $\nabla^0$ is the connection for which constant vector fields are parallel, so $(\exp X_t \cdot \nabla^0)_x Y = 0$ for constant vector fields $X, Y$. Hence $\nabla^0_{\exp -X_t \cdot Y} = 0$ and so $\nabla^0_{X_t \cdot Y} = 0$. But the only parallel vector fields for $\nabla^0$ are the constant fields, so $\exp -X_t \cdot Y$ is constant. The leading term is $-t [X^{(1)}, Y]$ and hence $[X^{(1)}, Y]$ is constant. Since $X^{(1)}$ is symplectic, this means $X^{(1)}_x = (C_1 x + d_1)_x$ where $C_1 \in \mathfrak{sp}(2n, \mathbb{R})$. Further $\exp tX^{(1)}$ preserves $\nabla^0$ and $\exp -tX(1) \circ \exp X_t = \exp X'_t$ with $X'_t = O(t^2)$ so we can recurse to conclude that $X_t = (C_t x + d_t)_x$ for formal curves $C_t \in \mathfrak{tsp}(2n, \mathbb{R})[t]$ and $d_t \in t \mathbb{R}^{2n}[t]$.

**Proof** We lift the connections and $\psi_t$ to $\mathbb{R}^{2n}$ and denote the lifts by a tilde. $\tilde{\psi}_t \cdot \tilde{\nabla}^t = \tilde{\nabla}^t$. Then $\tilde{\nabla}^t = \nabla^0 + A^t, \tilde{\nabla}'_t = \nabla^0 + B^t$ where $A^t, B^t: \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[t]$ are linear with the usual properties. Thus

$$(\tilde{\psi}_t \circ \psi_{A^t}) \cdot \nabla^0 = \psi_{B^t} \cdot \nabla^0$$

and hence

$$\tilde{\psi}_t \circ \psi_{A^t} = \psi_{B^t} \circ \sigma^* \circ \exp X_t$$

where $\sigma(x) = Cx + d$ and $(X_t)_x = (C_t x + d_t)_x$.

Now $\psi_{B^t} \circ \sigma^* = \sigma^* \circ \sigma^{-1} \circ \exp X_{B^t} \circ \sigma^* = \sigma^* \circ \exp \sigma \cdot X_{B^t}$ and

$$(\sigma \cdot X_{B^t})_x = (X_{C \cdot B^t})_x + ((C \cdot B^t)(x)d)_x - \frac{1}{2}((C \cdot B^t)(d)d)_x$$

and the last two terms are in the pronilpotent semidirect product $t \mathfrak{sp}(2n, \mathbb{R})[t] + t \mathbb{R}^{2n}[t]$. We can exponentiate this equation in the form

$$\exp \sigma \cdot X_{B^t} = \exp X_{C \cdot B^t} \exp Z_t$$

with $Z_t \in t \mathfrak{sp}(2n, \mathbb{R})[t] + t \mathbb{R}^{2n}[t]$. At order zero we see that $\sigma$ must be the lift of $\psi^0$ and so must preserve the lattice: $C \in Sp(2n, \mathbb{Z})$. Then $\sigma^{-1} \circ \tilde{\psi}_t$ descends to the torus and leads off with the identity, so is of the form $\exp L_t$ where $L_t$ is a formal series of periodic vector fields on $\mathbb{R}^{2n}$. Thus we have, combining the terms in $\exp t \mathfrak{sp}(2n, \mathbb{R})[t] + t \mathbb{R}^{2n}[t]$ and renaming as $Z_t$,

$$\exp L_t = \exp X_{C \cdot B^t} \exp Z_t \exp -X_{A^t}.$$}

Equating the coefficient of $t$ on both sides we see that

$$L^{(1)} = X_{C \cdot B^{(1)}} + Z^{(1)} - X_{A^{(1)}}$$
and since linear and quadratic functions are never periodic we see that $C \cdot B^{(1)} = A^{(1)}$, and $L^{(1)} = Z^{(1)}$ is constant. A simple recursion (moving constant terms past $\exp X_{C \cdot B^t}$) suffices to see that $A^t = C \cdot B^t$.

So we have:

**Theorem 5.6** The moduli space of curves of Ricci-type symplectic connections starting with the standard flat connection on $(T^{2n}, \omega)$ under the action of formal curves of symplectomorphisms is described by the space of formal curves $A^t: \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[t]$ satisfying $A^t(X)A^t(Y) = 0$ and $A^t(X)Y = A^t(Y)X$, modulo the action of $\text{Sp}(2n, \mathbb{Z})$.

It is worth noting that a curve of Ricci type connections on the torus is equivalent to the constant curve at the trivial connection when lifted to $\mathbb{R}^{2n}$.

**References**

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