Wigner’s quantum phase space flow in weakly anharmonic weakly excited two-state systems

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Wigner flow, quantum mechanics’ phase space flow, reveals fine details of quantum dynamics – finer than is ordinarily thought possible according to quantum folklore invoking Heisenberg’s uncertainty principle. Here, we study the features of Wigner flow of bound states of time-independent conservative, weakly anharmonic one-dimensional quantum-mechanical systems which are weakly excited. Only energy eigenstates and their pure two-state superpositions are considered. We focus on the simplest, most intuitive and analytically accessible aspects of Wigner flow. We stress connections between anharmonic cases and the well-studied harmonic case. We show that Wigner flow of weakly anharmonic potentials can be grouped into three distinct classes, associated with hard, soft, and odd potentials. It is shown that their Wigner flow patterns can be characterised by the flow’s discrete stagnation points, how these arise and how a quantum system’s dynamics is constrained by their topological charge conservation.

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I. INTRODUCTION

The phase space dynamics of a time-independent, conservative, continuous system tends to be much harder to characterise in the quantum case than in the classical case. The main reason for this difference lies in the fact that classical systems’ dynamics generates sharply defined phase space trajectories which can be analysed each on their own. In the quantum case such trajectories do not exist [1]. Moreover, the set of all trajectories, the classical system’s phase portrait, is ordered by the fact that the dynamics is Liouvillian and can be analysed with powerful mathematical tools [2–4].

Classical phase space trajectories can be of appealing simplicity allowing the viewer to characterise the system’s dynamics at a glimpse. They also reveal rich structures such as the ‘strange’ attractors of ‘chaotic’ systems full of intricacies and beauty [2].

The power of classical phase space techniques does not readily translate to the quantum realm. At the core of this incompatibility lies Werner Heisenberg’s uncertainty principle of quantum mechanics [1]: it establishes that quantum indeterminacy does not allow us to define a sharply defined quantum phase space-based trajectory representing the evolution of a state of a system. Since no trajectories can be studied, the power of classical phase space methods is severely curtailed in the quantum case.

Recently, it became clear that these detrimental effects of the uncertainty principle do not apply so stringently to Wigner flow, the quantum phase space flow of Wigner’s function, and its streamlines [5].

Wigner flow was introduced by Eugene Wigner in his 1932 paper [6] on quantum phase space dynamics. It arises when Schrödinger’s equation is recast as a quantum phase space-based evolution equation for Wigner’s function giving it the form of the continuity equation (7) [6–8] (also known as the quantum Liouville equation [9]). Its flow, the system’s Wigner flow, has, so far, been little studied [8, 10–13] and essential features were missed until recently [5]: Wigner flow reveals detail of quantum dynamics’ finest features, much like classical phase portraits.

In the study of dynamic systems’ flows, flow stagnation points, the locations in phase space where the dynamics stops, are also referred to as equilibrium, stationary, fixed, critical, invariant and rest points [4]. This multitude of terms testifies to their central importance in classical mechanics as well as wave theory, such as the field of ‘singular’ optics [14] (where they are called singular points).

Analogously, Wigner flow’s stagnation points, the locations in phase space where Wigner flow vanishes, are the most important points in quantum phase space: firstly, because of their topological nature, Wigner flow’s stagnation points order the flow in surrounding sectors of phase space. Secondly, they carry a conserved topological charge of the surrounding flow’s orientation winding number [5]. This makes their appearance robust to small perturbations.

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Here, we study Wigner flow of conservative, time-independent, one-dimensional quantum-mechanical systems featuring nearly harmonic potentials. We only consider the bound energy-eigenstates of weakly excited systems in pure two-state superpositions. This renders the dynamics strictly periodic.

Generalisations to pure state superpositions of more than two states and mixed states will be discussed elsewhere. Since Wigner flow patterns in phase space, the quantum analog of classical phase portraits, are at present still a little-studied concept [5], we stress aspects which can be understood intuitively. We particularly emphasise the connection of Wigner flow of weakly anharmonic systems with that of the well-studied harmonic case of quantum optics [9].

In section II we introduce Wigner functions and Wigner flow, and some of their mathematical features. Section III discusses Wigner flow for harmonic oscillators. Section IV establishes that, from the perspective of Wigner flow, there are three classes of weakly anharmonic potentials: hard, soft, or odd potentials. They distort Wigner flow’s stagnation features in three characteristic fashions. A perturbation analysis of these distortions is performed in section V for eigenstates and in section VI for two-state superpositions. Subsections VA and VB particularly stress an intuitive understanding of how the Wigner flow patterns emerge due to the presence of a potential’s anharmonicity. The features these flow patterns have in common are referred to in sections V and VI. In subsection VI B we investigate a Rabi-oscillation scenario generalising and combining our results of sections V and VI. The Rabi scenario is chosen because it allows us to study the dynamics of a time-dependent but reversible two-state system. We conclude in section VII.

II. WIGNER FUNCTIONS AND WIGNER FLOW

We parameterize pure quantum two-state superpositions of energy eigenstates $\psi_m$ (with eigenenergies $E_m$) by the mixing angle $\theta$

$$\Psi_{m,n}(x,t;\theta) = \cos(\theta)e^{-\frac{i}{\hbar}E_m t}\psi_m(x) + \sin(\theta)e^{-\frac{i}{\hbar}E_n t}\psi_n(x),$$

(1)

here, $x$ and $t$ denote position and time, and $\hbar = \hbar/(2\pi)$ is Planck’s constant (rescaled). Their density matrix has the pure state form

$$\rho(x+y,x-y,t) = \Psi^\ast(x+y,t) \cdot \Psi(x-y,t).$$

The associated Wigner function $W(x,p,t)$ [6, 15] is defined as the Fourier transform to momentum $p$, of the density matrix’ off-diagonal coherences, parameterized by the shift $y$,

$$W_\rho(x,p,t) \equiv \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dy \rho(x+y,x-y,t) \cdot e^{2i\hbar py}.$$  

(2)

By construction $W$ is real-valued, non-local (through $y$), and normalized

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \ W(x,p,t) = 1.$$  

(3)

Wigner’s function $W(x,p,t)$ is set apart from other quantum phase space distributions [15] by the fact that its marginals are the respective probability distributions of its state $\Psi(x,t)$ (or $\Psi$’s momentum representation $\Phi(p,t)$), namely

$$\int_{-\infty}^{\infty} dp \ W(x,p,t) = |\Psi(x,t)^2|$$  

(4)

and

$$\int_{-\infty}^{\infty} dx \ W(x,p,t) = |\Phi(p,t)^2|.$$  

(5)

To describe $W$’s dynamics in phase space we use the shortened notation $\frac{\partial^2}{\partial x^2} = \partial_x^2$, etc., for partial derivatives. Using it, Schrödinger’s equation

$$i\hbar\partial_t \Psi(x,t) = -\frac{\hbar^2}{2M}\partial_x^2 \Psi(x,t) + V(x)\Psi(x,t),$$  

(6)

can be cast into the form of a continuity equation [6]

$$\partial_t W + \partial_x J_x + \partial_p J_p = 0,$$  

(7)

where $J(x,p,t)$ denotes the Wigner flow [5], and explicit reference to functional dependence on $x$, $p$, and $t$ has been dropped. In general $J$ has an integral representation, just like $W$ itself. In the case of potentials $V(x)$ that can be
expanded into a Taylor series in $x$, $J$ assumes the form [6, 9, 11]

$$J = \left( \frac{J_x}{J_p} \right) = \begin{pmatrix} \frac{d}{d^2} W \\ -\sum_{l=0}^{\infty} \frac{(ih/2)^{2l}}{(2l+1)!} \frac{\partial^{2l} W}{\partial x^{2l+1}} \end{pmatrix},$$

where $M$ is the mass of the particle. For numerical reasons, we frequently truncate this expression after $L$ terms yielding the truncated flow with cutoff $L$

$$J_{[L]} = \left( \frac{J_x}{J_{p,[L]}} \right) = \begin{pmatrix} \frac{d}{d^2} W \\ \sum_{l=0}^{L} J_{p,l} \end{pmatrix},$$

where

$$J_{p,l} = -\frac{(ih/2)^{2l}}{(2l+1)!} \frac{\partial^{2l} W}{\partial x^{2l+1}}.$$ (10)

Eigenstates of a conservative real potential are real functions; their Wigner functions $W_{n,n} = W_{\psi_n^*\psi_n}$ are time-independent and therefore obey the symmetry relation

$$W_{n,n}(x,p) = W_{n,n}(x,-p),$$ (11)

which, for pure eigenstates, implies the following symmetries in momentum

$$J_x(x,p) = -J_x(x,-p)$$ (12)

and $J_p(x,p) = J_p(x,-p)$. (13)

The Wigner flow’s orientation winding number [5]

$$\omega(L,t) = \frac{1}{2\pi} \oint_L d\varphi$$ (14)

tracks the orientation angle $\varphi$ of the flow vectors $J$ along continuous, closed, self-avoiding loops $L$ in phase space. Because the components of the flow are continuous functions, $\omega$ is zero except for the case when the loop contains stagnation points, such as those sketched in Fig. 1. In such a case a non-zero value of $\omega$ can occur and this value is conserved unless the system’s dynamics transports a stagnation point across $L$ [5].

![Diagram showing Wigner flow stagnation points](image)

**FIG. 1.** **Wigner flow stagnation points are characterised** by their integer orientation winding number (14) or topological charge [14], $\omega$, of the flow around them. We use a red plus sign for stagnation points with charge $\omega = +1$, a yellow minus sign for $\omega = -1$, and a white circle for $\omega = 0$. Notably, the flow near stagnation points can be skewed in phase space, can feature skewed separatrices, and saddle flow with a saddle oriented in the $p$-direction. The topological charges [14] can be combined or split through the system’s time evolution while their sum remains conserved [5].

### III. WIGNER FLOW OF HARMONIC OSCILLATORS

We rescale weakly anharmonic potentials to match their minimum’s curvature to our choice

$$V^{\odot}(x) = \frac{k}{2} x^2 = \frac{M\Omega^2}{2} x^2 = \frac{x^2}{2}$$ (15)
FIG. 2. Top row: Wigner flow and Wigner functions for the harmonic oscillator potential (15). Bottom row: Streamlines of the integrated Wigner flow \(J(x,p,t)\) while time \(t\) is held fixed. Shown are the cases of the first excited state \(\psi_1\) (left column) and the superposition \(|\Psi_{1,0}(\frac{T}{4},\frac{3}{4})\rangle\) (right column). Dashed black lines show the locations of \(W = 0\) which implies that there \(J^x_0 = 0\) and \(J^p_0 = 0\). In the top row the background colouring refers to the respective Wigner functions’ values (compare insets). The normalised Wigner flow \(J/|J|\) is depicted with red arrows if the Wigner function is positive. For negative Wigner function green arrows are used, demonstrating flow reversal [5]. White arrows on top of the normalised flow show the unnormalized Wigner flow \(J\). The dashed black line forms a circular flow stagnation line with constant radius \(R^\circ_1\) and centre the further displaced from the origin (18) the smaller the mixing angle \(\theta\) in (1).

Bottom row: The depicted Wigner flow streamlines run through randomly picked points, the small black dots in the figures. At these points the local speed of Wigner flow is determined and the streamline it lies on gets colored accordingly: we chose the colors of the rainbow, red represents relatively fastest flow (“hot”) over yellow, green to blue representing slow flow (“cold”). The stagnation point at the origin carries a flow orientation winding number \(\omega = +1\) and is labelled as in Fig. 1.

for a harmonic reference potential with circular trajectories (rather than ellipses [16]), see Fig. 2. Having such circular trajectories is the main motivation for this particular choice. This constitutes a choice of units of mass \(M = 1\), spring constant \(k = 1\), and, setting \(\hbar = 1\), leads to an angular frequency of \(\Omega^\circ = 1\) and an oscillation period of

\[ T^\circ = \frac{2\pi}{\Omega^\circ} = 2\pi. \tag{16} \]

Wigner flow for \(V^\circ\), according to Eq. (8), has the ‘classical’ form

\[ J^\circ = \begin{pmatrix} J^x_0 & J^p_0 \\ J^p_0 & J^x_0 \end{pmatrix} = W^\circ(x,p,t) \cdot \begin{pmatrix} p \\ -x \end{pmatrix}. \tag{17} \]

A. Degenerate Wigner flow

Wigner functions are continuous and have negativities [6], they therefore feature lines of zero in phase space which, in the case of the harmonic oscillator, because of the form (17) of \(J\), become lines of zero for both components, \(J^x_0\) and \(J^p_0\), giving rise to lines of stagnation of the flow, see Fig. 2.

To make the existence of such lines of stagnation of the flow intuitively plausible, it is helpful to remember that Wigner’s function is normalized to unity while featuring positive and negative regions. Its dynamics can therefore resemble a flow of two charged liquids of opposing sign with an overall conserved positive unit charge. In this analogy,
the lines of the stagnation of the flow form just like areas in a charged plasma where local flow separation creates (or conversely local flow merger annihilates) spatial charge densities.

It is well known from quantum optics that the Wigner functions $W_{n,n}^\odot(x,p)$ of the harmonic oscillator Fock states $\psi_n^\odot$ resemble Mexican hats centred on the origin, with concentric fringes of alternating polarity. Their zero lines thus form concentric circles [9].

In the case of superposition states, we primarily investigate superpositions of ground and first excited state $\Psi_{0,1}(t;\theta)$ (1), in which case the circular zero line remains a circle with constant radius $R_1^\odot = 1/\sqrt{2}$ but shifted centre position $C_{01}$. The centre is the further displaced from the origin the larger the ground state contribution ($\theta \to 0$) and rotates around the origin, with frequency $\Omega^\odot = 1$, according to

$$C_{01}(t;\theta) = (-R_1^\odot \cos(t) \cot(\theta), R_1^\odot \sin(t) \cot(\theta)),$$

compare Fig. 2.

IV. THE THREE CLASSES OF WEAKLY ANHARMONIC POTENTIALS

![FIG. 3. Top row: Representatives of the three classes of weakly anharmonic potentials displayed side by side: Eckart potentials ($V^E$, left, in blue) are hard, Rosen-Morse potentials ($V^R$, centre, in green) are soft and Morse potentials ($V^M$, right, in red) are odd. The potentials feature differing amounts of anharmonicity, $\alpha_\nu$ (19), while all are rescaled to have the same minimum curvature as the harmonic reference potential $V^\odot$ (15) (displayed behind each class as a thick grey line). Bottom row: Cartoons of associated classical phase portraits (for one fixed potential strength in each column) superimposed on harmonic oscillator’s phase portraits (thick grey lines).](image)

Weakly anharmonic potentials $V(x)$ that admit a Taylor expansion in $x$ are characterized by their leading anharmonic term $\alpha_\nu$, in what we will refer to as their truncation $V_\nu^V$ of order $\nu$ and representative $V$, namely,

$$V(x) \approx V_\nu^V(x) = \frac{x^2}{2} + \alpha_\nu x^\nu = V^\odot(x) + \alpha_\nu x^\nu.$$  \hspace{1cm} (19)

The precise order of the truncations’ leading terms is quite unimportant, it is the qualitative class of the potential that determines its qualitative dynamic features.

With respect to qualitative features of Wigner flow for weakly excited bound state systems, just as for the associated phase portraits in the classical case, only three classes of anharmonic potentials exist: hard, soft, and odd potentials. We checked this numerically for several potentials and it is plausible from our discussion below.

All potentials with an even order leading term that is positive have qualitatively similar classical phase space profiles. They correspond to springs harder than their Hookian reference (15), compare left column of Fig. 3. Soft potentials have a negative leading term of even order. For potentials of a leading term of odd order we always set the leading term $\alpha_\nu < 0$, making the odd potentials soft for $x > 0$ and hard for $x < 0$.

For each class a representative exists for which all bound state eigenfunctions and eigenenergies are known in simple closed form. As such representatives we choose the Eckart, Rosen-Morse, and Morse potentials [17], denoted by superscripts $V = E, R, M$; they are hard, soft, and odd, respectively; see Fig. 3.

Potentials $V_\nu^V$ which, based on their truncation $V_\nu^V$, are classed as even or odd can contain higher order Taylor terms which are not necessarily only even or odd. The influence of such higher terms can be neglected since we limit
our investigation to weakly excited systems. If we were to regard the truncated right hand side of Eq. (19) as the full potential, soft and odd potentials would obviously be open, without any bound eigenstates, we exclude such cases. With these provisions, studying one representative of each class allows us to cover qualitative features of Wigner flow of the bound states of all weakly excited weakly anharmonic potentials.

Since we only consider two-state superpositions $\Psi_{m,n}^V$ (1) the associated revolution time describing the periodic movement in phase space is

$$T_{m,n}^V = \frac{2\pi\hbar}{|E_n^V - E_m^V|}. \quad (20)$$

In the harmonic case, evidently, $T_{m,n}^\odot = T^\odot/|n-m|$.

A. The Hard Eckart Potential

The Eckart potential

$$V^E(x) = D \tan^2 \left( \frac{x}{\sqrt{2D}} \right), \quad (21)$$

parametrized by its reciprocal hardness parameter $D$, has an even fourth-order truncation of the form

$$V_4^E = V^\odot(x) + \alpha_4^E x^4 = \frac{x^2}{2} + \frac{x^4}{6D} \quad (22)$$

confirming it is a hard potential. Its eigenstates $\psi_n^E$ are given in [17] and have energy eigenvalues [17]

$$E_n^E = \frac{\sqrt{1 + 16D^2 + 1}}{8D} (2n + 1) + \frac{2n^2}{8D}. \quad (23)$$

B. The Soft Rosen-Morse Potential

The Rosen-Morse potential

$$V^R(x) = D \tanh^2 \left( \frac{x}{\sqrt{2D}} \right), \quad (24)$$

parameterized by its depth $D$, has an even fourth-order truncation of the form

$$V_4^R = V^\odot(x) + \alpha_4^R x^4 = \frac{x^2}{2} - \frac{x^4}{6D} \quad (25)$$

confirming it is a soft potential. This truncation is not bounded from below and thus, by itself, does not hold any bound states.

The full potential’s bound states $\psi_n^R$ are given in reference [17] and have the respective energy eigenvalues

$$E_n^R = \frac{(\sqrt{1 + 16D^2} - 1) (2n + 1) - 2n^2}{8D}. \quad (26)$$

The full potential $V^R$ holds

$$B^R = \left\lfloor \frac{1}{2} \sqrt{1 + 16D^2} + \frac{1}{2} \right\rfloor \quad (27)$$

bound states, where ‘...’ denotes floor rounding.
C. The Odd Morse Potential

The Morse potential

\[ V^M(x) = D \left( 1 - e^{-x/\sqrt{2D}} \right)^2, \]  

(28)

with depth \( D \), has a third-order truncation of the form

\[ V_3^M = V \circ \left( x \right) + \alpha_3^M x^3 = \frac{x^2}{2} - \frac{x^3}{2\sqrt{2D}}, \]  

(29)

confirming it is an odd potential. The truncation is open, i.e., does not have bound states. The full potential \( V^M \)'s bound eigenstates are given in [17], they have energy eigenvalues

\[ E_n^M = D \left( 1 - \left( 1 - \frac{2n+1}{4D} \right)^2 \right). \]  

(30)

and the potential holds

\[ B^M = \left\lfloor \frac{2D}{D} + \frac{1}{2} \right\rfloor \]  

(31)

bound states. For large values of \( D \), the Rosen-Morse and Morse potentials hold roughly the same number of bound states. Thus over the energy range of their depth \( D \), their average density of states is roughly twice that of the harmonic oscillator.

For the Morse case all Wigner functions (for all diagonal and off-diagonal density matrix entries) are known [16]. Since they contain modified Bessel function of the second kind with complex order [18], they make analytical calculations not much easier than numerical calculations [16] and are, here, primarily used to check some of the numerical results.

V. WIGNER FLOW PATTERNS FOR EIGENSTATES OF ANHARMONIC POTENTIALS

A. Formation of distinct Stagnation Points of Wigner Flow

In the limit of vanishing anharmonicity, the Wigner functions of anharmonic potentials converge pointwise towards those of the harmonic oscillator.

Pointwise convergence does not occur for the associated Wigner flow patterns. The degeneracy of Eq. (17) leads to formation of lines of stagnation [5] in the harmonic case, see section III A.

The presence of higher order terms in \( J_p \) in Eq. (8) lifts this degeneracy for anharmonic potentials; it shifts the lines of stagnation of the individual components of the flow (\( J_x = 0 \)) and (\( J_p = 0 \)) by differing amounts. They therefore cross each other at discrete points in phase space forming separate stagnation points at their intersections. We colour zero lines of \( J \)'s \( x \)-component in green and use blue for lines of \( J_p = 0 \). Wherever blue and green lines cross a Wigner flow stagnation point exists, see Figs. 4 and 5.

How these respective stagnation lines of the flow’s components get shifted when we increase the anharmonicity and how this gives rise to an intuitive understanding of the ensuing Wigner flow patterns is presented next.

B. Qualitative Effects of Anharmonicities: emergence of eigenstates’ flow stagnation points

Heisenberg’s uncertainty principle \( \Delta x \cdot \Delta p_x \geq \hbar/2 \) implies constancy of the size of an uncertainty domain in phase space [19] (note that this argument must not be taken too far [20]). Hard potentials squash phase space trajectories in position thus elliptically expanding them in momentum, see bottom row of Fig. 3. This observation can be applied to the shape of zero circles (III A) as well: compare the green lines in Fig. 5 and in the top row of Fig. 4. Soft potentials invert this scenario, expansion in \( x \) leads to an elliptical squeeze in \( p \), see middle row of Fig. 4. Odd potentials are effectively hard on the left and soft on the right side. This leads to a growth in position spread and reduction in momentum spread, similar to the case of soft potentials; but, additionally, phase space features tend to be moved to the right, see bottom row of Fig. 4.
FIG. 4. **Wigner flow patterns of the first excited state** for the three classes of weakly anharmonic potentials: top, Eckart (hard; left to right: $J_{[10]}$, $J_{[50]}$, $J_{[50]}$ (9)), middle, Rosen-Morse (soft; left to right: $J_{[10]}$, $J_{[10]}$, $J_{[5]}$), and, bottom row, Morse (odd; left to right: $J_{[10]}$, $J_{[10]}$, $J_{[10]}$). Note that the flow patterns of odd potentials feature shapes of the hard (for $x < 0$) and the soft case (for $x > 0$). The anharmonicity increases from left to right with the respective values for $\alpha$ (19) quoted in each plot.

Symbols and their coloring have been adopted from Fig. 2. Thick blue and green lines refer to lines of zero of $J_p$ and $J_x$, respectively. The stagnation points at their crossings are labelled as in Fig. 1.

The $x$-axis is coloured green to mark the vanishing of the component $J_x$, yielding two stagnation points for all (blue) $J_p$-zero circles intersecting it.

Similarly, the $p$-axis is a blue line of zero of $J_p$ in the harmonic case, and, for symmetry reasons, also for even potentials. For odd potentials these $J_p$ zeros do not lie on the $p$-axis but are displaced to the right. One might wonder whether an alternative to the break-up of the $J_p$ lines in the bottom row of Fig. 4 can exist for odd truncations. The answer is –it cannot; to the left of the $p$-axis an odd potential is hard and therefore has to yield the characteristic pattern displayed in the top row, to the right it is soft, yielding the middle row pattern. Near the $p$-axis both patterns meet but cannot be connected due to the continuity of $J_x$ and $J_p$ as functions of $x$ and $p$. The only option, respecting continuity, is the cut-and-reconnect pattern we see realised in the bottom row of Fig. 4.

Per zero circle four stagnation points form near the diagonals $|x| = |p|$, i.e., in the limit of vanishing anharmonicity they lie at odd multiples of 45 degrees (counted from the $x$-axis). These 45-degree positions can be understood from the above observations. The elliptic squashing and expansion of the zero circles ($J_x = 0$) and ($J_p = 0$) is weak. We therefore have to consider the deformation of a zero circle into two ellipses with small, slightly different eccentricities, common centres and equal area which are aligned with the coordinate axes of phase space. In the limit of vanishing eccentricities these intersect at odd multiples of 45 degrees forming the **diagonal stagnation points** we observe in Fig. 4 or Fig. 5.

To summarize this qualitative discussion, we have shown that for even potentials $8n + 1$ stagnation points are to be expected for all low lying eigenstates $\psi_n$: one near the origin, and, per zero circle each, 2 on $x$- and $p$-axis, respectively,
FIG. 5. Qualitative discussion of emergence of type and positioning of Wigner flow stagnation points for first excited state of a hard potential. White arrows represent normalized Wigner flow as observed in the harmonic case; clockwise, far from the origin and, anti-clockwise (inverted flow), very close to the origin (compare left panels of Fig. 2). Green and blue delineated ellipses represent the zero circles, see section III A, deformed by the anharmonicity of the potential. They are coloured green when $J_x$ changes sign and blue when $J_p$ does, this also applies to $x$ and $p$-axis. To ‘fill the plane’ we track the orientation of the flow as we move its location across phase space. The corresponding sequence features arrows with ever darker shades of grey which eventually wrap around the rightmost ‘$+1$’-stagnation point. Whenever this sequence crosses a zero-line, when $J_x = 0$ or $J_p = 0$, the arrows are framed green or blue, respectively. Since $J_x$ and $J_p$ vary continuously across phase space we can similarly track the flow’s orientation around the boundaries of the deformed zero circles and along $x$ and $p$-axis. Green arrows with blue fringe are orientated horizontally ($J_p = 0$) and invert direction whenever the blue line they are pinned to crosses a green line. Blue arrows with green fringe are tied to green lines, are vertically aligned, and behave analogously. At every crossing of a green with a blue line a stagnation point exists, but nowhere else. Having ‘filled the plane’ we can work out the topological charge of the stagnation points, labelled as the symbols of Fig. 1. The quantitative plots in the top row of Fig. 4 confirm this qualitative analysis.

There are 2 diagonal stagnation points. For odd potentials there are $6n + 1$ stagnation points per eigenstate, since the $p$-axis stagnation points are all avoided by the cut-and-reconnect mechanism.

C. Displacement of the minimum vortex

FIG. 6. Quantum displacement $\delta x_{J_p}(t)$ of the Morse potential minimum’s stagnation point along the $x$-axis versus time for superposition state $\Psi_{0,2}(x,t; \pi)$. Anharmonicity parameter $\alpha_{M}^{3} = -0.0884$. Black curve: position of minimum of potential. Red curve: numerically determined displacement using $J_{[10]}$ of Eq. (9). Grey curve: analytical approximation of Eq. (34), namely: $\delta x_{J_p} = \sqrt{2} \cos \left( 2t - \frac{\pi}{T} \right) + 0.1$. Here, with $\Psi(x,t)$, $\delta x_{J_p}(t)$ depends on $t$.

A full analysis of the displacement of the vortex near the minimum entails determination of the Wigner functions of the anharmonic system. To determine the qualitative behaviour we investigate the shift to first order in $\alpha$ only.
this, use of the Wigner functions of the harmonic case suffices.

At the position \( (x = 0) \) of the potential’s minimum \( \partial_x V = 0 \), \( \partial^2_x V \approx 1 \) and \( \partial_p W \approx \partial_x W^\circ = 0 \), therefore, approximations of \( J_p \) and \( \partial_x J_p \), up to first order \( \mathcal{O}(\alpha^1) \) (evaluated at phase space position \( (x = X, p = P) \), and referred to as \( \partial_x J_p|_{(X,P)} \), etc.), are

\[
J_{p,[1]}(0,P) \approx \frac{\hbar^2}{24} \partial_x^2 V \partial_p^2 W^\circ \left|_{(0,P)} \right. = \frac{\hbar^2 \nu (\nu - 1)(\nu - 2)}{24} \cdot \alpha_p x^{\nu - 3} \cdot \partial_p^2 W^\circ \left|_{(0,P)} \right. + \mathcal{O}(\alpha^2),
\]

and

\[
\partial_x J_{p,[1]}(0,P) \approx \left( -V^\circ \partial_x^2 V + \frac{\hbar^2}{24} \partial^2_x W^\circ \partial^2_p V \right) \left|_{(0,P)} \right. \approx -V^\circ (0,0) + \mathcal{O}(\alpha^1).
\]

With a Newton gradient approximation for the \( x \)-shift of the zero of \( J_p \) at the origin \( \delta x_{J_p}(0,0) = -J_p(0,0)/\partial_x J_p(0,0) \) we find that the minimum vortex’ shift is

\[
\delta x_{J_p}(0,0) \approx \frac{\hbar^2 \nu (\nu - 1)(\nu - 2)}{24} \cdot \alpha_p x^{\nu - 3} \cdot \partial_p^2 W^\circ \frac{\partial_x W}{W} \left|_{(0,0)} \right. + \mathcal{O}(\alpha^2).
\]

The stagnation point of \( J \) near the minimum of even potentials does not shift since \( \partial^2_x V(0) = 0 \), which implies \( \delta x_{J_p}(0,0) = 0 \). This result conforms with our expectation (VB) that, for symmetry reasons, the vortex at the origin of eigenstates of even potentials does not shift. This can easily be confirmed, to all orders in \( \alpha \), using (10).

The stagnation point of \( J \) near the minimum of the potential only shifts for odd potentials. If the potential is anharmonic in higher than third order a higher order expansion has to be performed. We assume that \( \alpha_3 < 0 \), the Mexican hat profiles of the harmonic oscillator’s Wigner functions imply that \( \partial_p^2 W^\circ/W^\circ|_{(0,0)} < 0 \). Therefore, according to Eq. (34) \( \delta x_{J_p}(0,0) > 0 \). This confirms the shift to the right, in the direction of the potential’s opening, as predicted in the qualitative discussion in section VB and visible in the bottom row of Fig. 4; for depiction of a time-dependent displacement, for a superposition state, see Fig. 6 below.

Note that in the classical case the minimum does not shift at all, compare Fig. 3, the shift of the minimum vortex is a pure quantum effect.

VI. WIGNER FLOW PATTERNS FOR TWO-STATE SUPERPOSITIONS

A. The Ferris Wheel Effect – alignment with \( x \) and \( p \)-axes

According to the discussion in section VB, four diagonal stagnation points form per zero circle of every eigenstate. If we ‘perturb’ an eigenstate by, say, mixing in a little bit of groundstate \( \Psi_{m,0}(\theta) \) with \( \theta \ll 1 \), the zero circles get displaced from the origin (compare discussion leading up to Eq. (18)). Yet, for small values of \( \theta \) the four diagonal stagnation points remain pinned to the zero circle while it rotates around the origin as time progresses. They do this is such a way that they keep their relative orientation with respect to the axes of phase space as seen from the zero circle’s centre. In other words, while they travel through phase space they behave somewhat like markers on a Ferris wheel cabin, where the zero line, \( J_x = 0 \), denotes the cabin’s outline, see top row of Figs. 7, 8 and 9.

The remaining stagnation points are pinned to the intersections of the zero circles with \( x \)- or \( p \)-axes. When the off-centre displacement becomes larger the diagonal stagnation points start to interact with stagnation points on the axes leading to repulsion or coalescence. Such interactions are constrained by the topological charge associated with the stagnation points’ flow winding number. If the magnitude of the sum of their charges is not greater than one, they can merge, otherwise they are topologically protected and repel each other, this is illustrated in Figs. 7, 8, 9, and 10.

B. Rabi scenario: modified two-state dynamics

To investigate reversible but time-dependent two-state system dynamics, we study a resonantly driven Rabi system described by

\[
\Psi_{0,1}^R(x,t;\theta(t)) = \Psi_{0,1}(x,t; \frac{\Omega R}{2} t + \frac{\pi}{2}),
\]

(35)
where $\Omega_R$ is the Rabi frequency \[19\].

It displays Wigner flow patterns associated with the systems’ (fast) intrinsic dynamics while (slowly) shifting the weighting of the superposition of the state: for the ratio of these two system frequencies we choose $\Omega_R/\Omega^\otimes = 1/8$ in Fig. 11.

Fig. 11 shows plots with zero circles \(18\) tied to a spiral centred on $t = 0$ (since $\Psi_{0,1}^R (t = 0) = \psi_1$) which expands outward as more of the groundstate gets mixed in with increasing values of $|t|$. We notice that the Ferris wheel-effect tends to keep the orientation of the stagnation points on the zero circle aligned with $x$- and $p$-axes. With our choice of $\Omega_R/\Omega^\otimes = 1/8$, around $|t| = 2T$ the mixing angle is roughly $|\theta| = \pi/4$ which is why those sections of Fig. 11 resemble the plots in Fig. 10. At this stage the zero circle gets displaced by its radius and stagnation points on the circle interact with those on $x$- and $p$-axes giving rise to repulsion, attraction, coalescence and splitting of stagnation points – all constrained by conservation of the local flow orientation number $\omega$.\[^{11}\]
FIG. 8. Wigner flow patterns $J_{[5]}$ (9), as in Fig. 7, for the Rosen-Morse potential (25) with $\alpha^R_4 = -0.042$.

FIG. 9. Wigner flow patterns $J_{[10]}$ (9), as in Figs. 7, and 8, for the Morse potential (29) with $\alpha^M_3 = -0.088$. Note that the Morse potential is odd, i.e. it is hard on the left ($x < 0$) and soft on the right side ($x > 0$). Accordingly, the flow patterns on the left resemble those of the (hard) Eckhart potential depicted in Fig. 7 and those on the right resemble those for the (soft) Rosen-Morse potential depicted in Fig. 8.
FIG. 10. Time evolution of the stagnation points of Wigner flow $J_{\Omega}$ for a superposition state $|1\rangle$, $\Psi_{0,1}(\pi/4, 0.95)$, with probability 0.54 for ground and 0.46 for first excited state. The panels feature, from top to bottom, a soft, hard and odd potential (19) with anharmonicities $\alpha^E_4 = 0.0017$, $\alpha^R_4 = -0.0017$ and $\alpha^M_3 = -0.0035$, respectively. The helical tube depicting the position of the zero circle (18) nearly touches the origin. To guide the eye, every tenth of the full period is denoted by a dashed black zero circle painted onto the tube which is rainbow-colored to display the flow of time, one period (20) per rainbow. The position of stagnation points is depicted by red lines if they carry charge $\omega = +1$ and yellow if $\omega = -1$, compare Fig. 1. The stagnation point positions are additionally projected along the $x$-axis onto the blue wall in the back and along the $p$-axis downward onto the green floor.

Winding number conservation implies that positively and negatively charged stagnation points originate and annihilate together, this is seen as red and yellow lines forming loops which are reminiscent of the formation of the torus reported in Fig. 4 of reference [5]. As mentioned in the caption of Fig. 4 above, plots for the odd Morse potential (bottom panel), contain features of the soft Rosen-Morse potential (middle panel) for $x > 0$ and of the hard Eckart Potential (top panel) for $x < 0$.

C. Other Superpositions

Instead of changing the weighting angle dynamically, as in the Rabi-scenario depicted in Fig. 11, we can also monitor the changes that arise when we change the weighting angle ‘by hand’. The topological nature of the stagnation points conserves the flow winding number in this case as well, see Fig. 12.

Other superposition states, such as $\Psi_{1,2}$, can show symmetric flower petal arrangements, see insets in Fig. 13, which have recently been observed experimentally [21]. Fig. 13 shows how the three different types of weakly anharmonic potentials give rise to flow patterns which generalise our previous discussions in sections V B and VI A.

VII. CONCLUSIONS

We have reminded the reader in section II of the mathematical basis of Wigner flow and in III of its degenerate character in the harmonic case. In section IV we showed that there are three types of weakly anharmonic weakly excited one-dimensional quantum mechanical systems which feature, respectively, soft, hard and odd potentials. The overall distortions of phase space in the case of soft potentials (expansion in the $x$-direction together with compression in the $p$-direction) and the reversed distortion in the case of hard potentials (compressed in $x$, expanded in $p$) allowed
FIG. 11. Rabi scenario for state (35) with frequency ratio $\Omega_R/\Omega^\odot = 1/8$. The panels use the same symbols as Fig. 10 and feature the same potentials with the same anharmonicities. At time $t = 0$ the system is in the first excited state; at other times the groundstate contribution displaces the zero circle from the origin such that, over time, it sweeps out a helix with varying width. Every full period is denoted by a dashed black zero circle. As mentioned in Figs. 4 and 10 above, the bottom panel, for the odd Morse potential, contains features of the soft Rosen-Morse potential (middle panel) for $x > 0$ and those of the hard Eckart Potential (top panel) for $x < 0$.

us to qualitatively consider the emergence of stagnation points of Wigner flow and the associated flow patterns for eigenstates of such potentials, in section V. Odd potentials turn out to conform with the established patterns for soft and hard potentials in that they are a hybrid of these and behave as such. In section VI we investigate superposition states and find that the position of stagnation points, like in the case of eigenstates, tend to align with $x$- and $p$-axis in phase space. This Ferris wheel-effect (VI A) tends to apply to various superposition states. We again see that flow patterns for odd potentials are hybrids of those of hard and soft potentials.

Our investigations show that subtle flow patterns on the sub-Planck scale are common in quantum phase space and that these can be understood to some extent from phase space distortions and alignment with $x$- and $p$-axes. We believe that their monitoring might provide new insights into the dynamic behaviour of systems commonly investigated using quantum phase space techniques, such as quantum optical systems, quantum chemical systems and quantum-
FIG. 12. Demonstration of conservation of flow winding number for fixed time $t = T/8$ but varying weighting angle $\theta$ of superposition state $\Psi_{0,1}(x, \frac{T}{8}; \theta)$. We observe movements of stagnation points leading to their merger and splitting. The top row refers to the hard Eckart and bottom row to the soft Rosen-Morse potential with identical parameters as in Figs. 7 and 8, respectively. Note the non-Liouvillian nature of the flow featuring regions of pronounced flow expansion and compression.

FIG. 13. Wigner flow pattern for state $\Psi_{1,2}(x, \frac{3T}{4}; \pi)$ for, from left to right, Eckart, Rosen-Morse and Morse potentials, with anharmonicities $\alpha^E_4 = 0.042$, $\alpha^R_4 = 0.033$, and $\alpha^M_3 = 0.125$, respectively. The same symbols as in Fig. 4 are used. Zeros of the Wigner function (thick green lines) intersect with zeros of the momentum component of Wigner flow (thick blue line) yielding fairly intricate arrangements of stagnation points of Wigner flow. Similarly to Figs. 4, 10 and 11, the Morse potential’s case inherits features of a stiff potential for $x < 0$ and of a soft potential for $x > 0$.

mechanical oscillators. Wigner flow should be useful for benchmarking of quantum phase space based numerical propagator codes [7, 11, 22] and its patterns might even make quantum dynamics appear less alien when introduced to students versed in classical phase space techniques [3, 4].

It would be interesting to see whether investigations of Wigner flow might lead to new insights on the nature of ‘chaologic’ systems [23] and quantum classical scenarios.
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