Roberge-Weiss endpoint at the physical point of $N_f = 2 + 1$ QCD

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We study the phase diagram of $N_f = 2 + 1$ QCD in the $T - \mu_B$ plane and investigate the critical point corresponding to the onset of the Roberge-Weiss transition, which is found for imaginary values of $\mu_B$. We make use of stout improved staggered fermions and of the tree level Symanzik gauge action, and explore four different sets of lattice spacings, corresponding to $N_t = 4, 6, 8, 10$, and different spatial sizes, in order to assess the universality class of the critical point. The continuum extrapolated value of the endpoint temperature is found to be $T_{RW} = 208(5)$ MeV, i.e. $T_{RW}/T_c \sim 1.34(7)$, where $T_c$ is the chiral pseudocritical temperature at zero chemical potential, while our finite size scaling analysis, performed on $N_t = 4$ and $N_t = 6$ lattices, provides evidence for a critical point in the 3d Ising universality class.

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I. INTRODUCTION

The remarkable changes expected for the properties of strongly interacting matter when it is put under extreme conditions are the subject of intense ongoing theoretical and experimental research. Various parameters of phenomenological interest enter the description of such extreme conditions, like temperature, chemical potentials or external background fields. Part of this research consists in the study of the QCD phase diagram, i.e. in mapping the various phases of strongly interacting matter in equilibrium conditions, and the associated phase transitions and critical points, as a function of those parameters.

At high temperature confinement and chiral symmetry breaking are expected to disappear, and QCD is expected to be described in terms of quark and gluon effective degrees of freedom (Quark-Gluon Plasma). Lattice QCD simulations show that, indeed, a rapid change of properties takes place around a well defined temperature $T_c$. There is no compelling reason for expecting a true phase transition, since no exact symmetry of QCD, which could possibly change its realization at $T_c$, is known: chiral symmetry is exact only for vanishing quark masses, while the $Z_3$ center symmetry is exact only in the pure gauge theory, where its spontaneous breaking is associated to deconfinement. In fact, lattice simulations have shown that only a smooth crossover is present in the case of physical quark masses, at a temperature $T_c \sim 155$ MeV [1, 2].

The situation could be different in the presence of other external parameters. In particular, the crossover could turn into a real transition for large enough baryon chemical potential $\mu_B$, starting from a critical endpoint in the $T - \mu_B$ plane. Such a critical point, and the associated critical behavior around it, could have a huge impact on strong interactions phenomenology, so that large theoretical and experimental efforts are being dedicated to prove its existence and locate it. Unfortunately, numerical progress by lattice QCD simulations is strongly hindered by the sign problem affecting the path integral formulation at non-zero baryon chemical potential.

There are, however, well defined locations, in an extended QCD phase diagram, where exact symmetries are known for any value of the quark masses. Critical points associated with their spontaneous symmetry breaking are predicted to exist and can be investigated by standard lattice simulations. This is the case of QCD with a purely imaginary baryon chemical potential $\mu_B$ [3, 4], the partition function of which is

$$Z(T, \theta_B) = \text{Tr} \left( e^{-\frac{T}{\theta_B} e^{i\theta_B}} \right) \quad (1)$$

where $H$ is the QCD Hamiltonian, $B$ is the baryon charge and $\theta_B = \text{Im}(\mu_B)/T$. All physical states of the theory, over which the trace is taken, are globally color neutral and carry an integer valued baryon charge $B$, hence $Z$ is $2\pi$-periodic in $\theta_B$, or alternatively $2\pi/N_c$-periodic in $\theta_B = \text{Im}(\mu_B)/T$, where $\mu_B = \mu_B/N_c$ is the quark chemical potential and $N_c$ is the number of colors. That can also be proven by making use of center transformations in the path-integral formulation of the partition function, as we review in Section III.

On the other hand, in the high-$T$ phase, quarks, which carry a baryon charge $1/N_c$, become the effective degrees of freedom propagating through the thermal medium: modes which are $2\pi$-periodic in $\theta_B$, hence $2\pi N_c$ periodic modes...
in $\theta_B$, appear in the functional dependence of the partition function. As a consequence, the $2\pi$ periodicity in $\theta_B$ is possible only through the appearance of a non-analytic behavior in $Z(T, \theta_B)$, associated with first order phase transition lines present for $\theta_B = \pi$ or odd multiples of it, which are known as Roberge-Weiss (RW) transitions \[10\] and have been widely studied by lattice QCD simulations \[8, 9, 11–25\].

In correspondence with such points, analogously to what happens when $\theta_B$ is a multiple of $2\pi$, the theory is invariant under charge conjugation, but contrary to that case charge conjugation is spontaneously broken at high $T$, where the system develops a non-zero expectation value for the imaginary part of the baryon number density: the temperature $T_{\text{RW}}$ where the spontaneous breaking takes place is precisely the endpoint of the Roberge-Weiss first order transition lines. An alternative point of view about the same transition is to look at it as a quantum (i.e. zero temperature) transition, with an associated spontaneous breaking of charge conjugation, driven by the compactification of one of the spatial directions beyond a critical size $L_C = 1/T_{\text{RW}}$ (finite size transition \[20, 27\]). Since charge conjugation is a $Z_2$ symmetry, one expects a 3$d$-Ising universality class if the transition is second order, or alternatively a first order transition with the development of a latent heat.

The temperature $T_{\text{RW}}$ and the critical behavior to which it is related represent universal properties of strong interactions, directly related to the change in the effective degrees of freedom propagating in the thermal medium, hence to deconfinement. They can be carefully studied by lattice QCD simulations, since the path integral measure is real and positive for imaginary chemical potentials. Despite being related to a critical point located in an unphysical region of the QCD phase diagram, their importance and relevance to a full understanding of strong interactions stems from various considerations:

\begin{itemize}
  \item[i)] The RW endpoint may influence physics in a critical region around it. Moreover, if at the RW endpoint a first order transition is present, the endpoint is actually a triple point, with further departing first order lines, the endpoints of which may be even closer to the $\mu_B = 0$ axis, with more interesting consequences.
  \item[ii)] Early studies have shown that the RW endpoint transition is first order for small quark masses, second order for intermediate masses, and again first order for large masses; the three regions are separated by two tricritical points \[13–15\]. The emergence of this interesting structure has induced many further studies in effective models \[28–40\] which try to reproduce the essential features of QCD. Moreover, interesting proposals have been made on the connection of this phase structure with that present at $\mu_B = 0$ (the so-called Columbia plot) and on the possibility to exploit the whole phase structure at imaginary chemical potential in order to clarify currently open issues on the phase structure at $\mu_B = 0$, like the order of the chiral transition for $N_f = 2$ \[21, 24\].
  \item[iii)] Once the RW endpoint has been precisely located, it can be taken as a test ground to compare the lattice techniques presently used to locate the critical point at real $\mu_B$, so as to assess their reliability and guide future research on the subject.
  \item[iv)] The relation of the RW endpoint to the other symmetries of QCD, which are present at least in well defined limits of strong interactions, is an interesting issue by itself, which can help elucidate some fundamental non-perturbative properties of the theory.
\end{itemize}

In this paper we study the properties of the RW endpoint by lattice simulations of QCD with physical quark masses. Its location $T_{\text{RW}}$ is determined for various lattice spacings, corresponding to temporal extensions $N_t =$ 4, 6, 8, 10, and then extrapolated to the continuum limit. Moreover we are able to determine its universality class, through a finite size scaling analysis, at two different lattice spacings, namely $N_t = 4, 6$. Finally, in order to approach the issue of the interconnection between chiral symmetry and the RW endpoint, we consider the relation of the endpoint location to the analytic continuation of the pseudocritical chiral transition temperature $T_c(\mu_B)$ to imaginary chemical potentials.

The paper is organized as follows. In Section II we review the general framework regarding the RW endpoint in a path-integral approach and present details about our numerical setup and the observables used to investigate the critical behavior. In Section III we report on our numerical results regarding the universality class of the endpoint, the continuum extrapolated value of $T_{\text{RW}}$ and its relation with $T_c(\mu_B)$. Finally, in Section IV we draw our conclusions.

\section{General Framework and Numerical Setup}

We consider a staggered discretization of the $N_f = 2+1$ QCD partition function in the presence of imaginary quark chemical potentials:

\begin{equation}
Z = \int DU e^{-S_{\text{YM}}} \prod_{f = u, d, s} \det \left( M_{\text{st}}(U, \mu_f, i) \right)^{1/4},
\end{equation}

\begin{equation}
S_{\text{YM}} = -\frac{\beta}{3} \sum_{i, \mu \neq \nu} \left[ \frac{5}{6} W_{i, \mu \nu}^{1 \times 1} - \frac{1}{12} W_{i, \mu \nu}^{1 \times 2} \right],
\end{equation}

\begin{equation}
(M_{\text{st}})_{i, j} = a m_f \delta_{i, j} + \sum_{\nu = 1}^{4} \frac{\eta_{f, \nu}}{2} \left[ e^{i \mu_f, i \delta_{\nu, \nu}} U_{i, \nu}^{(2)} \delta_{i, j - \nu} - e^{-i \mu_f, i \delta_{\nu, \nu}} U_{i, \nu}^{(2)\dagger} \delta_{i, j + \nu} \right].
\end{equation}

The gauge link variables $U$ are used to construct the tree level improved Symanzik pure gauge action \[11, 12\], $S_{\text{YM}}$, where $W_{i, \mu \nu}^{1 \times 2}$ is the trace of the $n \times m$ rectangular loop constructed along the directions $\mu, \nu$ departing from the $i$ site. The staggered Dirac operator $(M_{\text{st}})$, instead, is built up in terms of the two times stout-smearred \[42\] links $U_{i, \nu}^{(2)}$, in order to reduce taste symmetry violations,
with an isotropic smearing parameter $\rho = 0.15$. As usual, the rooting procedure is adopted to remove the residual degeneracy of the staggered Dirac operator.

When thermal boundary conditions (periodic/anti-periodic for boson/fermion fields) are taken in the temporal direction, the temperature of the system is given by $T = 1/(N_t a)$, where $N_t$ is the number of temporal lattice sites and $a$ is the lattice spacing, related to the bare parameters of the theory. For a given number of lattice sites in the temporal direction, we can choose the simulated temperature by tuning the value of the bare coupling constant $\beta$ and the quark masses $m_s$ and $m_u = m_d = m_l$, in order to change the lattice spacing while remaining on a line of constant physics, where $m_\pi \simeq 135$ MeV and $m_s/m_l = 28.15$; this line has been determined by a spline interpolation of the results reported in Refs. [44–46].

Let us now sketch the structure of the phase diagram at imaginary $\mu_B$. This has already been done in the introduction, by considering the effective degrees of freedom at work in the different regimes: now we will proceed through an analysis of the properties of the path integral. In the presence of a purely baryonic chemical potential (i.e. $\mu_Q = 0$ and $\mu_S = 0$), one has $\mu_u = \mu_d = \mu_s = \mu_q = \mu_B/3$. When $\mu_q$ is purely imaginary, its introduction is equivalent to a global rotation of fermionic boundary conditions in the temporal direction by an angle $\theta_q = \text{Im}(\mu_q)/T$, therefore one expects at least a $2\pi$-periodicity in $\theta_q \Sim 2\pi N_c$ in $\theta_B$. However, the actual periodicity is $2\pi/N_c$, since a rotation of the fermionic boundary conditions by that angle is equivalent to a center transformation on the gauge fields, hence it can be reabsorbed without modifying the path integral [10].

Numerical simulations show that such a periodicity is smoothly realized at low temperatures [8–10]. At high $T$, instead, since the Polyakov loop $L$ (trace of the temporal Wilson line normalized by $N_c$) enters the fermionic determinant expansion multiplied by $\exp(i\theta_q)$, the value of $\theta_q$ selects the true vacuum among the three different minima of the Polyakov loop effective potential, which are related to each other by center transformations. Hence, phase transitions occur as $\theta_q$ crosses the boundary between two different center sectors, i.e. for $\theta_q = (2k + 1)\pi/N_c$ and $k$ integer (in which case $\theta_B$ is an odd multiple of $\pi$), where $\langle L \rangle$ jumps from one center sector to the other [10]; the phase of $L$ can serve as a possible order parameter in this case. The $T-\theta_q$ phase diagram then consists of a periodic repetition of first order lines (RW lines) in the high-$T$ regime, which disappear at low $T$. Therefore they have an endpoint at some temperature $T_{RW}$, where an exact $Z_2$ symmetry breaks spontaneously. A schematic view of the diagram is reported in Fig. 1.

An alternative order parameter is represented by any of the quark number densities (where $q = u, d, s$)

$$\langle n_q \rangle \equiv \frac{1}{V_4} \frac{\partial \log Z}{\partial \mu_q}$$

where $V_4$ is the four dimensional lattice volume. Since $Z$ is an even function of $\mu_B$, each $\langle n_q \rangle$ is odd and, for purely imaginary $\mu_B$, it is purely imaginary as well. Invariance under charge conjugation, or alternatively oddness and the required $2\pi$ periodicity in $\theta_B$, implies that $\langle n_q \rangle$ vanishes for $\theta_B = \pi$ or integer multiples of it, unless a discontinuity takes place at such points, in correspondence of a spontaneous breaking of charge conjugation invariance. This is exactly what happens at the RW lines, so that a non-zero $\langle n_q \rangle$ signals the onset of the RW transition.

In the following, it will be convenient to consider one particular RW line, corresponding to $\theta_q = \pi$, for which the imaginary part of the Polyakov loop, together with the imaginary part of the quark number density, can be taken as an order parameter. The order parameter susceptibility is then defined as

$$\chi_L \equiv N_s N_s^3 \left( \langle \text{Im}(L) \rangle^2 - \langle |\text{Im}(L)| \rangle^2 \right),$$

where $N_s$ ($N_t$) is the spatial (temporal) size in lattice units. The susceptibility $\chi_L$ is expected to scale, moving around the endpoint at fixed $N_t$ and $\theta_q$, as

$$\chi_L = N_s^{\gamma/\nu} \phi(tN_4^{1/\nu}),$$

where $t = (T - T_{RW})/T_{RW}$ is the reduced temperature, which is proportional to $(\beta - \beta_{RW})$ close enough to the critical point. That means that the quantity $\chi_L/N_s^{\gamma/\nu}$, measured on different spatial sizes, should lie on the same curve when plotted against $(\beta - \beta_{RW})N_s^{1/\nu}$. Alternatively, we will consider also the susceptibility of the imaginary part of the quark number density, which is defined, for every flavor $q$, by

$$\chi_q \equiv N_t N_s^3 \left( \langle \text{Im}(n_q) \rangle^2 - \langle |\text{Im}(n_q)| \rangle^2 \right),$$

and is expected to show a scaling behavior as in Eq. (7).
III. NUMERICAL RESULTS

In this Section we present our numerical results, starting from an analysis of the critical behavior around the RW endpoint transition, in order to assess its order and universality class on lattices with $N_t = 4, 6$. Then we will consider also lattices with $N_t = 8, 10$ in order to provide a continuum extrapolated value for $T_{RW}$.

Since we are interested in studying the behavior near the phase transition, long histories are required, to cope with the critical slowing down (see Fig. 2); for the couplings around the critical value, we used $\sim 40 - 50K$ trajectories for each run when performing the finite size analysis.

A. Finite size scaling and universality class of the transition

The effective theory associated with the spontaneous breaking of the charge conjugation at finite temperature is a three dimensional theory with $Z_2$ symmetry, so the transition can be either first order or second order in the three-dimensional Ising universality class. A tricritical scaling is in principle possible as well; however the tricritical point is just a single point at the boundary of first and second order regions. As a consequence (apart from the unlikely case of being exactly on it) tricritical indices can be observed only as scaling corrections, the ultimate large volume behavior being either first order or Ising $3d$. The critical indices that will be used in the following are reported for convenience in Table I.

We will now present the finite size scaling analysis performed to identify the nature of the transition on lattices with temporal extent $N_t = 4$ and 6. As previously discussed, we adopt two different order parameters, namely the imaginary part of the average Polyakov loop and the quark number density; the former turned out to have a smaller correction to scaling, so we will start our analysis from the study of the susceptibility $\chi_L$ defined in Eq. (6).

Fig. 3 shows $\chi_L$ obtained on $N_t = 4$ lattices and rescaled according to Eq. (7), using alternatively the critical indices of the 3d Ising universality class or those corresponding to a first order transition (the values used for the critical coupling are the ones reported in Table I). Using 3d Ising indices the results on different volumes collapse on top of each other, whereas this is not the case using first order indices, which strongly indicates that the transition is second order for $N_t = 4$. Note that, since we are performing simulations on a line of constant physics, the mass parameters change with $\beta$; it is thus not possible to use standard reweighting methods. In Fig. 4 we repeat the same analysis using the Polyakov loop measured on lattices with temporal extent $N_t = 6$. Again, the 3d-Ising universality class appears to describe the scaling of the susceptibility of the Polyakov loop significantly better than a first order, although larger corrections to scaling are present with respect to the $N_t = 4$ case.

A confirmation of the previous analysis comes from the study of the fourth-order Binder ratio, which in our case is defined as

$$B_4 = \frac{\langle |\text{Im}L|^4 \rangle}{\langle |\text{Im}L|^2 \rangle^2}. \quad (9)$$

It is easy to show that, in the thermodynamical limit, $B_4 \to 3$ in the absence of a phase transition, while $B_4 \to 1$ if a first order transition is present. At second order transitions $B_4$ assumes non-trivial values, which are characteristic of the universal critical behavior associated with the transition $\nu, \gamma, \gamma/\nu, 1/\nu$. For the particular case of the three-dimensional Ising universality class the critical value is $B_4 = 1.604(1)$, see Ref. [51]. From these general properties the following simple procedure follows to locate the critical endpoint of a line of first order transition: study the behavior of $B_4$ as a function of the coupling for different values of the lattice size; the endpoint coupling value will correspond (up to scaling corrections) to the crossing point of these curves.

In Fig. 5 we show the values of $B_4$ in a neighborhood of the critical coupling at three different volumes both on $N_t = 4$ and $N_t = 6$ temporal extent. The behavior of the Binder ratio as a function of $\beta$ is clearly the one expected at a critical endpoint and the value at the crossing point is in reasonable agreement with that expected for a transition of the 3d Ising universality class, while a first order is clearly excluded.

| $3D$ Ising | $\nu$ | $\gamma$ | $\gamma/\nu$ | $1/\nu$ |
|-----------|-------|---------|-------------|--------|
| $1^\text{st}$ Order | 1/3 | 1 | 3 | 3 |

TABLE I: The critical exponents relevant for this study (see e.g. [50, 51]).

FIG. 2: Monte Carlo histories of $|\text{Im}L|$ for $N_t = 4$ and the $\beta$ values closest to the peak of $\chi_L$, showing the peculiar features expected near a second order transition: the increase of the autocorrelation time and the absence of a double peak structure in the histogram.
The same conclusions are obtained by studying the susceptibility of the $u$ quark number density defined in Eq. (8), although in this case the scaling corrections appear to be larger. As an example in Fig. 6 we show the behavior of $\chi_u$ on $N_t = 4$ lattices, rescaled according to Eq. (7): again, the 3d-Ising critical indices are favored.

B. Critical temperature: continuum extrapolated value

Having established that the RW-transition is second order for lattices with temporal extent $N_t = 4$ and 6, we now proceed to estimate the continuum value of $T_{RW}$. To this purpose, simulations have been performed also on lattices with $N_t = 8$ and 10, considering a limited number of spatial volumes (one or two) per simulation setup.

The pseudocritical value of the coupling has been determined for each lattice size by estimating the position of the maximum of $\chi_L$ and $\chi_u$. To this purpose, we have fitted the peak with a Lorentzian function:

$$f(\beta) = \frac{a}{1 + (\beta - \beta_{pc})^2 / c^2}. \tag{10}$$

The results for the large volume limit of $\beta_{pc}$, denoted by $\beta_c$, are reported in Table II; the error also takes into account the systematics related to the choice of the fit range. The volume dependence of the pseudocritical coupling is very mild for lattice with aspect ratio 4 or larger, with variations at the level of 0.1 % in terms of $\beta$ (which become 0.5 % in terms of temperature), as can be seen in Fig. 7 for the case of the $N_t = 4$ lattices. The pseudocritical couplings determined by using $\chi_L$ or $\chi_u$ have a priori to coincide only in the thermodynamical limit, however in all the cases the differences between the two determinations are well below 0.1% and, with the exception of the lattice $4 \times 16^3$, they are compatible with each other at one standard deviation.

In order to convert the critical temperatures to physical units we used the lattice spacings values reported in Tab. III which are obtained by a spline interpolation of the results presented in [44–46]. The systematic uncertainty on these lattice spacings is $2 - 3\%$ [44–46] and this is by far the largest source of error in the final tempera-
3.445 3.4475 3.45 3.4525 3.455 3.4575 3.46

β

1

1.5

2

2.5

3

B

4

N

s = 16

N

s = 24

N

s = 32

3.615 3.62 3.625 3.63 3.635 3.64

β

1

1.5

2

2.5

3

3.5

4

B

4

N

s = 24

N

s = 32

N

s = 40

FIG. 5: Binder fourth order ratio of the Polyakov loop imaginary part computed on \( N_t = 4 \) lattices (top) and \( N_t = 6 \) lattices (bottom). The horizontal line denotes the value expected for a second order transition of the 3d Ising universality class.

Table II: Critical values of the coupling for different \( N_t \) values (estimated by using lattices of spatial extent \( N_s \)) and corresponding values for the lattice spacing. Only the statistical error of the lattice spacing is reported in the table, the systematic error is about 2 – 3\% [44–46].

| \( N_t \) | \( \beta_c \) | \( N_s \) | \( a \) (fm) |
|----------|----------|----------|----------|
| 4        | 3.4498(7)| 16, 24, 32 | 0.2424(6) |
| 6        | 3.6310(15)| 21, 32, 40 | 0.1714(3) |
| 8        | 3.7540(25)| 32, 40   | 0.1233(3) |
| 10       | 3.8600(25)| 40       | 0.0968(2) |

FIG. 6: Disconnected susceptibility of the light baryon number computed on \( N_t = 4 \) lattices and rescaled with the critical exponents of the 3d Ising universality class (top) or corresponding to a first order transition (bottom).

C. Relation with the pseudocritical chiral transition line

An interesting issue that remains to be investigated is the relation between the RW endpoint and the chiral transition.
FIG. 8: Continuum extrapolation of the critical temperature.

FIG. 9: Phase diagram of QCD in the presence of an imaginary baryon chemical potential obtained from numerical simulations on $N_t = 8$ lattices alone. Bands denote fits to polynomials in $\mu_B^4$; the orange (longer) band is obtained using terms up to order $\mu_B^6$, the violet (shorter) one using up to $\mu_B^8$ terms.

FIG. 10: Renormalized light chiral susceptibility on $N_t = 6$ and 8 lattices. The vertical bands denote the position of the RW endpoint on lattices of the corresponding temporal extent.

IV. CONCLUSIONS

We have investigated the properties of the RW endpoint by lattice simulations of $N_f = 2 + 1$ QCD with physical quark masses and making use of two different order parameters for the transition, namely the imaginary part of the Polyakov line and the imaginary part of terms up to the sixth order in $\theta_B^6$, which includes the RW endpoint as a part of the pseudocritical line. The result is the dashed line reported in Fig. 9, which has been continued also to the other center sectors, so as to reproduce a realistic version (i.e. for $N_f = 2 + 1$ QCD with physical quark masses, even if just for $N_t = 8$) of the phase diagram sketched in Fig. 11.

transition. In particular, the question can be posed in the following way: does the pseudocritical line really get to the RW endpoint, as assumed in Fig. 1 and as suggested by early studies on the subject?

A number of investigations appeared recently, reproducing the pseudocritical line for imaginary chemical potentials at or close to the physical point and with the setup of chemical potentials relevant to the RW endpoint, i.e. $\mu_s = \mu_t = \mu_B/3$, see Refs. [54–59]. A possible way to approach the question is to try extrapolating the location of the pseudocritical line up to $\theta_B = \pi$ on the basis of those determinations. To this aim we considered results for $T_c(\theta_B)$ obtained in Ref. [58] on $N_t = 8$ lattices and adopting the same discretization used in the present study. In Fig. 10 we present two different extrapolations of such data, corresponding to the fit ansatz

$$T_c(\theta_B) = T_c(1 + \kappa\theta_B^2 + b\theta_B^4 + c\theta_B^6)$$  \hspace{1cm} (11)

with or without the sixth order term included (a simple linear dependence on $\theta_B^2$ was excluded in Ref. [58]). In both cases one gets reasonably close, within errors, to the RW endpoint.

Of course, the issue can be checked also directly, by determining the location of the pseudocritical line exactly at $\theta_B = \pi$. To that aim, in Fig. 11 we plot the renormalized light chiral susceptibility (as defined, e.g., in Ref. [58]) for lattices with temporal extent $N_t = 6, 8$, together with the positions of the RW endpoint as previously determined on the same lattices. It is clearly seen that the location of the maxima of the chiral susceptibility is compatible with the position of the RW endpoints. For instance for $N_t = 8$ and $N_s = 32$ we obtain, by fitting the chiral susceptibility to a Lorentzian peak, $\beta_c = 3.749(3)$, which is at just one standard deviation from the RW endpoint coupling reported in Table [11].

We can thus confirm, within present errors, evidence that the RW endpoint is located at a point where the analytic continuation of the pseudocritical line and the RW first order line meet each other. To conclude, based on this evidence, we have performed a final fit, including...
the quark number density, which have led to consistent results.

The temperature of the endpoint, $T_{\text{RW}}$, has been determined at four different values of the lattice temporal extent, $N_t = 4, 6, 8, 10$, from which we have obtained a continuum extrapolated value $T_{\text{RW}} = 208(5)$ MeV, where the error includes both statistical and systematic contributions, stemming mostly from the determination of the physical scale. That leads to the estimate $T_{\text{RW}}/T_c = 1.34(7)$, where the error also takes into account the systematics involved in the determination of $T_c$, originating both from the scale setting and from the difficulties in defining a critical temperature when no real transition is present. This ratio is significantly larger than the ones obtained in previous studies; indeed, with unimproved actions, unphysical quark masses and no extrapolation to the continuum limit, $T_{\text{RW}}$ was typically found to be only about 10% larger than $T_c$. The larger value is partially due to the larger curvature $\kappa$, and partially to the more significant contribution from non-linear terms in $\mu_B^2$ (see Eq. (11)), which are present in the case $\mu_u = \mu_d = \mu_s$ (see Ref. [28]).

Regarding the order of the transition, our finite size scaling analysis provides evidence that a second order transition of the 3d-Ising universality class takes place, rather than a first order one, at least for $N_t = 4$ and $N_t = 6$ lattices. Our investigation has been performed at a fixed value of the pion mass, corresponding to its physical value $m_\pi \simeq 135$ MeV.

Previous studies on the subject, performed in the $N_f = 2$ theory with both staggered and Wilson fermions, have shown that the order of the transition changes as a function of $m_\pi$: in particular, there are two tricritical pion masses, $m_{\pi,\text{tric.light}}$ and $m_{\pi,\text{tric.heavy}}$, and the transition is second order for $m_{\pi,\text{tric.light}} < m_\pi < m_{\pi,\text{tric.heavy}}$ and first order for lighter or heavier pion masses. The value of the heavy tricritical mass is typically well above the GeV scale. The lighter critical pion mass has been found to be $m_{\pi,\text{light}} \sim 400$ MeV for standard staggered fermions on $N_t = 4$ lattices [15], and around 930 and 680 MeV for standard Wilson fermions on respectively $N_t = 4$ [19] and $N_t = 6$ [23] lattices. Given these results, even if we have studied just the physical value of the pion mass, we can conclude the following: for stout improved staggered fermions, one has $m_{\pi,\text{light}} < 135$ MeV on both the $N_t = 4$ and $N_t = 6$ lattices. When compared to previous results, that demonstrates the presence of significant cut-off effects on the values of this tricritical mass, even when working at fixed $N_t$ but with a different action. Moreover, based on the observed tendency of the tricritical mass to decrease with the increase of $N_t$, we suggest that $m_{\pi,\text{light}}$ should be smaller than $m_{\pi,\text{phys}} = 135$ MeV in the continuum limit, so that the RW endpoint should be a second order transition in the continuum limit at the physical pion mass.

We must however remark that the mechanism driving the change of nature of RW endpoint transition, from second to first order as the pion mass decreases, is still unknown. If such a mechanism is related to the chiral properties of quarks, unexpected behaviors could occur as the continuum chiral symmetry group is fully recovered. This is known to happen, at least for staggered fermions, for lattice spacings well below those explored in the present study (see Ref. [60] for a recent investigation about this issue).

Let us spend a few words about what, in our opinion, future studies should clarify. First of all, one would like to check the second order nature of the RW endpoint at the physical point on finer lattices, i.e. for $N_t > 6$. Then, our study with stout improved staggered fermions should be extended to different values of the pion mass, in order to locate the values of the tricritical masses $m_{\pi,\text{tric.light}}$ and $m_{\pi,\text{tric.heavy}}$ and possibly extrapolate them to the continuum limit. Such a program, which goes beyond our present computational capabilities, would clarify the universal properties of the only critical point of strong interactions (in the presence of finite quark masses) that one can predict a priori, based on the known symmetries of QCD.

Finally, another open issue regards the relation of the RW critical point to those predicted in well defined limits of QCD. The relation to the deconfinement transition present in the quenched case is obvious, since the two transitions trivially coincide in this case and are both related to center symmetry. The relation to the chiral transition in the limit of massless quarks is far less trivial. Suppose to move (varying the temperature) along the line $\theta_B = \pi$ in the presence of massless quarks; in principle one expects two different critical temperatures, one at which chiral symmetry is restored, $T_\chi$, and one at which the $Z_2$ charge conjugation symmetry spontaneously breaks, $T_{\text{RW}}$. What is the relation between $T_\chi$ and $T_{\text{RW}}$? Our present results at finite quark masses prove that the location of the peak of the renormalized chiral susceptibility coincides, within errors, with $T_{\text{RW}}$, see Fig. 10, so that the analytic continuation of the pseudo-critical line meets the RW line at its endpoint. However, in order to obtain a definite answer, the issue should be explored while approaching the chiral limit: this is something which goes beyond the purpose of the present study and is left to future investigations.

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