Characterizations of finite woven frames

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Abstract

In this article we provide methods of constructing finite woven frames. Several examples are discussed. We also introduce the notion of woven frame sequences and characterize them through the concepts of gaps and angles between spaces.

Keywords: Frames, Woven Frames, Gap, Angle

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1 Introduction

Hilbert space frame was first initiated by D. Gabor [1] in 1946 to reconstruct signals using Fourier coefficients. Later, in 1986, frame theory was reintroduced and popularized by Daubechies, Grossman and Meyer [2]. Since then frame theory has been widely used by mathematicians and engineers in various fields of mathematics and engineering sciences, namely, operator theory [3], harmonic analysis [4], coding theory [5], signal processing [6], sensor network [7], data analysis [8], etc.

Frame theory literature became richer through several generalizations, namely, $G$-frame (generalized frames) [9], $K$-frame (frames for operators (atomic systems)) [10], fusion frame (frames of subspaces) [11, 12, 13], $K$-fusion frame (atomic subspaces) [14], etc. and these generalizations have been proved to be useful in various applications. Also there are some spin-off applications of frame theory by means of robustness to erasures in [15], [16], [17], through which frame theory became much more prosperous in theoretical sciences.

Let us consider a scenario: suppose in a sensor network system, there are sensors $A_1, A_2, \ldots, A_n$ which capture data to produce certain results. These sensors can be characterized by frames. In case one of these sensors, say $A_k$, fails to operate due to some technical reason, then the results obtained from these sensors may contain errors. Now assume that there are another set of sensors $B_1, B_2, \cdots, B_n$ which does play similar role as $A_i$’s. In addition, in the case of $A_k$ fails, $B_k$ can substitute so that obtained results are error free. Such an intertwinedness between two sets of sensors, or in general between two frames, leads to the idea of weaving frames. Weaving frames or woven frames were recently introduced by Bemrose et. al. [18].
This article focuses on study, characterize and explore several properties of woven frames. Keeping in mind the practical application, this article analyzes only finite woven frames. The outline of this article is organized as follows. Section 2 is devoted to the basic definitions and results related to various kinds of frames, angle and gap between subspaces. Moreover, the characterizations of woven frames are analyzed in Section 3. Finally, woven frame sequences are established in Section 4.

Throughout the paper, \( \mathcal{H} \) is a separable Hilbert space. We denote by \( \mathcal{H}^n \) an \( n \)-dimensional Hilbert space, \( \mathcal{L}(\mathcal{H}^n) \) to be a collection of all bounded, linear operators on \( \mathcal{H}^n \), \( R(T) \) is denoted as the range of the operator \( T \), by \( \delta(M,N) \) we denote the gap between two closed subspaces \( M \) and \( N \) of a Hilbert space \( \mathcal{H} \), \( c_0(M,N) \) is denoted as the cosine of the minimal angle between \( M \) and \( N \), \([n] = \{1, 2, \cdots, n\} \) and the index set \( I \) is either finite or countably infinite.

2 Preliminaries

In this section we recall basic definitions and results needed in this paper. For more details we refer the books written by Casazza and Kutyniok [8] and Ole Christensen [19].

2.1 Frame

A collection \( \{f_i\}_{i \in I} \) in \( \mathcal{H} \) is called a frame if there exist constants \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2,
\]

(2.1)

for all \( f \in \mathcal{H} \). The numbers \( A \) and \( B \) are called frame bounds. The supremum over all \( A \)'s and infimum over all \( B \)'s satisfying above inequality are called the optimal frame bounds. If a collection satisfies only the right inequality in (2.1), it is called a Bessel sequence.

Given a frame \( \{f_i\}_{i \in I} \) of \( \mathcal{H} \), the pre-frame operator or synthesis operator is a bounded linear operator \( T : l^2(I) \to \mathcal{H} \) and is defined by \( T\{c_i\} = \sum_{i \in I} c_i f_i \). The adjoint of \( T \), \( T^*: \mathcal{H} \to l^2(I) \), given by \( T^*f = \{\langle f, f_i \rangle\} \), is called the analysis operator. The frame operator, \( S = TT^*: \mathcal{H} \to \mathcal{H} \), is defined by

\[
Sf = TT^*f = \sum_{i \in I} \langle f, f_i \rangle f_i.
\]

It is well-known that the frame operator is bounded, positive, self-adjoint and invertible.

Reconstruction formula: Every element in \( \mathcal{H} \) can be represented using frame elements as follows:

\[
f = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i
\]

(2.2)

Definition 2.1. 1. Let \( \{f_i\}_{i \in I} \) and \( \{g_i\}_{i \in I} \) be two frames for \( \mathcal{H} \). If for all \( f \in \mathcal{H} \), \( f = \sum_{i \in I} \langle f, g_i \rangle f_i \), then \( \{g_i\}_{i \in I} \) is called a dual frame of \( \{f_i\}_{i \in I} \).

2
2. Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \), with the associated frame operator \( S \), then \( \{S^{-1}f_i\}_{i \in I} \) is said to be the canonical dual frame of \( \{f_i\}_{i \in I} \).

**Proposition 2.2.** \([19], [20]\) A finite family \( \{f_i\}_{i \in [m]} \) in \( \mathcal{H}^n \), forms a frame for \( \mathcal{H}^n \) if and only if \( \text{span}\{f_i\}_{i \in [m]} = \mathcal{H}^n \).

### 2.2 Woven and Full Spark Frame

In a Hilbert space \( \mathcal{H} \), a family of frames \( \{f_{ij}\}_{i \in \mathbb{N}, j \in [M]} \) is said to be weakly woven if for any partition \( \{\sigma_j\}_{j \in [M]} \) of \( \mathbb{N} \), \( \{f_{ij}\}_{i \in \sigma_j, j \in [M]} \) forms a frame for \( \mathcal{H} \).

Also, in \( \mathcal{H} \), two frames \( \{f_i\}_{i \in I} \) and \( \{g_i\}_{i \in I} \) are said to be woven if for any \( \sigma \subseteq I \), \( \{f_i\}_{i \in \sigma} \cup \{g_i\}_{i \in \sigma^c} \) also forms a frame for \( \mathcal{H} \).

**Theorem 2.3.** \([18]\) In \( \mathcal{H} \), two frames are weakly woven if and only if they are woven.

Moreover, a frame with \( m \) elements in \( \mathcal{H}^n \), is said to be a full spark frame if every subset of the frame, with cardinality \( n \), is also a frame for \( \mathcal{H}^n \). For example, \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \) is a full spark frame for \( \mathbb{R}^2 \). Furthermore, if every element of a finite frame can be represented as a linear combination of the remaining others, then the frame is called a weak full spark frame.

For example, if \( \{e_1, e_2, e_3\} \) is an orthonormal basis of \( \mathbb{R}^3 \), \( \{e_1, e_2, e_2, e_2, e_3\} \) is a weak full spark frame but not a full spark frame. In this context, it is a fortuitous evident that every nontrivial (other than exact) full spark frame is also a weak full spark frame.

**Theorem 2.4.** \([18]\) In \( \mathcal{H}^n \), two frames \( \{f_i\}_{i \in [m]} \) and \( \{g_i\}_{i \in [m]} \) are woven if and only if for every \( \sigma \subseteq [m] \), \( \{f_i\}_{i \in \sigma} \cup \{g_i\}_{i \in \sigma^c} \) spans \( \mathcal{H}^n \).

### 2.3 Gap and Angle between subspaces

Let \( M \) and \( N \) be two closed subspaces of a Hilbert space \( \mathcal{H} \). Then the gap between \( M \) and \( N \) is given by, \( \delta(M, N) = \max\{\delta(M, N), \delta(N, M)\} \), where \( \delta(M, N) = \sup_{x \in S_M} \text{dist}(x, N), S_M \) is the unit sphere in \( M \) and \( \text{dist}(x, N) \) is the distance from \( x \) to \( N \).

Again the cosine of the angle between two closed subspaces \( M \) and \( N \) of a Hilbert space \( \mathcal{H} \) is given by,

\[
\cos(M, N) = \sup\{\langle x, y \rangle : x \in M \cap (M \cap N)^\perp, \|x\| = 1, y \in N \cap (M \cap N)^\perp, \|y\| = 1\}
\]

and the cosine of the minimal angle of the same is given by,

\[
\cos_0(M, N) = \sup\{\langle x, y \rangle : x \in M, \|x\| = 1, y \in N, \|y\| = 1\}.
\]

For the extensive discussion regarding the gap and the angle between two subspaces, we refer \([21], [22], [23], [24]\).

**Remark 2.5.** \([23]\) Let \( M \) and \( N \) be two closed subspaces of a Hilbert space \( \mathcal{H} \). Then :
1. $\delta(M, N) = 0$ if and only if $M \subset N$.

2. $\delta(M, N) = 0$ if and only if $M = N$.

**Lemma 2.6.** [24] Let $M$ and $N$ be two closed subspaces of a Hilbert space $\mathcal{H}$. Then $c_0(M, N) = 0$ if and only if $M \perp N$.

**Theorem 2.7.** [23] Let $M$ and $N$ be two closed subspaces of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:

1. $c_0(M, N) < 1$.
2. $M \cap N = \{0\}$ and $M + N$ is closed.

## 3 Characterization of Woven Frames

In this section, we characterize woven frames, mainly through constructing frames from a given frame. The proposed constructions are based on the images of a given frame via bounded linear operators. Before presenting these results, we start the discussion with a basic example.

**Lemma 3.1.** Let $\{f_i\}_{i \in [m]}$ be a frame for $\mathcal{H}^n$. Suppose $f_{m+1} = 0$, then $\{(f_i - f_{i+1})\}_{i \in [m]}$ is also a frame for $\mathcal{H}^n$ and these two frames are woven.

**Proof.** Given $\{f_i\}_{i \in [m]}$ is a frame for $\mathcal{H}^n$ and $f_{m+1} = 0$, it is obvious that $\{(f_i - f_{i+1})\}_{i \in [m]}$ is a frame for $\mathcal{H}^n$.

Now we prove $\{f_i\}_{i \in [m]}$ and $\{(f_i - f_{i+1})\}_{i \in [m]}$ are woven. Let $\sigma \subseteq [m]$ such that $|\sigma| = j$, then it is sufficient to prove that $\{f_i\}_{i \in \sigma} \cup \{(f_i - f_{i+1})\}_{i \in \sigma^c}$ spans $\mathcal{H}^n$.

Suppose $f \in \mathcal{H}^n$, then for some scalars $b_i$, $i \in [m]$, $f = \sum_{i \in [m]} b_i (f_i - f_{i+1})$. Now, it may be noted that for $i < m$, $f_i = \sum_{j=1}^{m} (f_j - f_{j+1})$, using this, we can rewrite $f$ as

$$f = b_1 f_1 + (b_2 - b_1) f_2 + \ldots + (b_j - b_{j-1}) f_j + (b_{j+1} - b_j) (f_{j+1} - f_{j+2}) + \ldots + (b_{m-1} - b_j) (f_{m-1} - f_m) + (b_m - b_j) f_m.$$

Therefore $\{f_i\}_{i \in \sigma} \cup \{(f_i - f_{i+1})\}_{i \in \sigma^c}$ spans $\mathcal{H}^n$ for all $\sigma$ with $|\sigma| = j$ and hence using Theorem 2.4 it forms a frame for $\mathcal{H}^n$. \hfill $\blacksquare$

**Remark 3.2.** In the above Lemma instead of $f_{m+1} = 0$, if $f_{m+1} = f_1$, then $\{(f_i - f_{i+1})\}_{i \in [m]}$ may not be a frame for $\mathcal{H}^n$. For example, let us consider $\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ as a frame for $\mathbb{R}^3$. But clearly, $\{(f_i - f_{i+1})\}_{i \in [4]} = \{\begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\}$ is not a frame for $\mathbb{R}^3$.\hfill $\square$
Remark 3.5. It is to be noted that, if one of the conditions of $f = \sum_{i=1}^{m} b_i (f_i - f_{i+1})$ can be written as $f = (b_1 - b_j)f_1 + (b_2 - b_1)f_2 + ...((b_j - b_{j-1})f_j + (b_{j+1} - b_j)(f_{j+1} - f_{j+2}) + ...((b_{m-1} - b_j)(f_{m-1} - f_m) + (b_m - b_j)(f_m - f_{m+1})$.

Remark 3.3. If $\{f_i\}_{i \in [m]}$ is a frame for $\mathcal{H}^n$ and suppose $f_{m+1} = 0$, then $\{(\alpha f_i + \beta f_{i+1})\}_{i \in [m]}$, $\alpha, \beta \neq 0$, is also a frame for $\mathcal{H}^n$ and they are woven.

In the following proposition, we present conditions under which image of a given frame under an idempotent operator forms woven frame.

**Proposition 3.4.** Let $F \in \mathcal{L}(\mathcal{H}^n)$ be an idempotent operator with $R(F) = R(F^*)$. Suppose $\{f_i\}_{i \in [m]}$ is a frame for $R(F^*)$, then $\{Ff_i\}_{i \in [m]}$ also forms a frame for $R(F^*)$ and these two frames are woven.

**Proof.** Since $\{f_i\}_{i \in [m]}$ is a frame for $R(F^*)$, for every $f \in R(F^*)$ we have,

$$f = \sum_{i \in [m]} a_i f_i,$$

for some scalars $a_i$.

Therefore, $Ff = F \sum_{i \in [m]} a_i (Ff_i)$ and since $R(F) = R(F^*)$, using Moore-Penrose pseudo inverse, we obtain $f = \sum_{i \in [m]} a_i (Ff_i)$. Hence $\{Ff_i\}_{i \in [m]}$ is a frame for $R(F^*)$.

Again, to show that they are woven, it is sufficient to prove that for a $\sigma = \{1, 2, ..., k\} \subset [m]$, $\{f_i\}_{i \in \sigma} \cup \{Ff_i\}_{i \notin \sigma}$ spans $R(F^*)$.

Since $\{Ff_i\}_{i \in [m]}$ is a frame for $R(F^*)$, for every $f \in R(F^*)$ we have,

$$f = \sum_{i \in [m]} b_i Ff_i,$$

for some scalars $b_i$.

Therefore, $Ff = F(\sum_{i \in \sigma} b_i f_i + \sum_{j \notin \sigma} b_j Ff_j)$ and since $R(F) = R(F^*)$, we get $f = \sum_{i \in \sigma} b_i f_i + \sum_{j \notin \sigma} b_j Ff_j$. Consequently, these two frames are woven. \qed

**Remark 3.5.** It is to be noted that, if one of the conditions of $F^2 = F$ and $R(F) = R(F^*)$ fails, then the conclusion of the above proposition may not hold. This is evident from the following two examples.

**Example 3.6.** Consider an idempotent operator on $\mathbb{R}^2$, whose matrix representation is $F = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. Then

$$R(F) = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \neq \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = R(F^*).$$

Now for the frame $\mathcal{F} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$, $F(\mathcal{F}) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is not a frame for $R(F^*)$.  


Example 3.7. Consider an operator on \( \mathbb{R}^3 \), whose matrix representation is
\[
F = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then \( F^2 \neq F \) but \( R(F) = R(F^*) \). Now let us choose a frame \( \{f_i\}_{i=3} \) \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\} \) for \( R(F^*) \). Then \( \{Ff_i\}_{i=3} \) is also a frame for \( R(F^*) \), but they are not woven, which can be verified for \( \sigma = \{1,3\} \).

Every frame in finite dimensional spaces, naturally, gives rise to a woven frame through duality.

Theorem 3.8. In \( \mathcal{H}^n \), every frame is woven with its associated dual frame.

Proof. Let \( \mathcal{G} = \{g_i\}_{i=m} \) be a dual frame for \( \mathcal{F} = \{f_i\}_{i=m} \) in \( \mathcal{H}^n \). Then for all \( f \in \mathcal{H}^n \) we have,
\[
f = \sum_{i=m} \langle f, g_i \rangle f_i.
\]

To show \( \mathcal{F} \) and \( \mathcal{G} \) are woven, it is sufficient to prove that for a \( \sigma = \{1,2,...,k\} \subset [m] \), \( \{f_i\}_{i=\sigma} \cup \{g_i\}_{i=\sigma^c} \) spans \( \mathcal{H}^n \). This is evident from the following,
\[
\text{span} (\{f_i\}_{i=\sigma} \cup \{g_i\}_{i=\sigma^c}) = \text{span} \left\{ f_1, f_2, \sum_{i=m} \langle g_{k+1}, g_i \rangle f_i, ..., \sum_{i=m} \langle g_m, g_i \rangle f_i \right\} = \text{span} \{f_i\}_{i=m} = \mathcal{H}^n.
\]

Corollary 3.9. Every frame for \( \mathcal{H}^n \) is woven with its canonical dual frame.

Remaining results of this section are devoted to study woven-ness of images of frames under invertible operators.

Lemma 3.10. If \( \{f_i\}_{i=m} \) is a frame for \( \mathcal{H}^n \) with the associated frame operator \( S \), then \( \{f_i\}_{i=m} \) and \( \{Sf_i\}_{i=m} \) are woven.

Proof. Since \( S \) is the frame operator of \( \{f_i\}_{i=m} \), for all \( f \in \mathcal{H}^n \) we have,
\[
Sf = \sum_{j=m} \langle f, f_j \rangle f_j.
\]
Let $\sigma = \{1, 2, ..., k\} \subset [m]$. Therefore we obtain,

\[
\text{span} \ (\{f_i\}_{i \in \sigma} \cup \{Sf_i\}_{i \in \sigma^c}) = \text{span} \ \left(\{f_i\}_{i \in \sigma} \cup \left\{\sum_{j \in [m]} \langle f_i, f_j \rangle f_j \right\}_{i \in \sigma^c}\right) = \text{span} \ \{f_i\}_{i \in [m]} = \mathcal{H}^n.
\]

Hence our goal is executed. 

\[
\square
\]

**Proposition 3.11.** Finite dimensional full spark frames are invariant under invertible operators.

**Proof.** The result follows from the fact that invertible bounded linear operators preserve linear independency. 

\[
\square
\]

**Remark 3.12.** From the above two results we can conclude that if $\{f_i\}_{i \in [m]}$ is a full spark frame for $\mathcal{H}^n$ with associated frame operator $S$, then $\{Sf_i\}_{i \in [m]}$ is a full spark frame and it is woven with $\{f_i\}_{i \in [m]}$.

**Remark 3.13.** In general, the images of full spark frame under invertible operators may not be woven with the full spark frame. To see this, consider an invertible operator on $\mathbb{R}^3$, whose matrix representation is

\[
T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}
\]

and consider a full spark frame for $\mathbb{R}^3$ as $\mathcal{F} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Then

\[
T\mathcal{F} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}. \text{ Let } \sigma = \{1, 2\}, \text{ then the corresponding weaving is}
\]

\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\},
\]

which is not a frame for $\mathbb{R}^3$ and hence they are not woven.

**Remark 3.14.** In general, the images of a frame under invertible operators may not be woven with the frame.
4 Woven frame sequence

A family \( \{f_i\}_{i \in I} \) in \( \mathcal{H} \) is said to be a frame sequence if it forms a frame for its closed, linear span. It may be noted that \( \{f_i\}_{i \in I} \) may not be a frame for \( \mathcal{H} \). In this section we explore the possibilities of two finite frame sequences together, through the concept of woven frames, form a frame for \( \mathcal{H}^n \).

**Definition 4.1.** Two frame sequences \( \mathcal{F} = \{f_i\}_{i \in [m]} \) and \( \mathcal{G} = \{g_i\}_{i \in [m]} \) in \( \mathcal{H}^n \), are said to be woven frame sequences, if for every non-trivial subset \( \sigma \) of \([m] \), \( \{f_i\}_{i \in \sigma} \cup \{g_i\}_{i \in \sigma^c} \) forms a frame for \( \mathcal{H}^n \).

**Example 4.2.** For example, In \( \mathbb{R}^3 \), the frame sequences
\[
\begin{bmatrix}
1 & 1 & -1 \\
0 & 0 & 0 \\
2 & -1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 1 & -1 \\
-1 & 2 & 3 \\
0 & 0 & -2
\end{bmatrix}
\]
are woven frame sequences, whereas
\[
\begin{bmatrix}
1 & 1 & -1 \\
0 & 0 & 0 \\
2 & -1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 1 & -1 \\
0 & 0 & 0 \\
2 & 2 & 3
\end{bmatrix}
\]
are not.

The notion of woven frame sequences is beneficial for its practical importance, because instead of two given frames, if we consider two frame sequences, then less restriction is there in the primary assumption and due to this fact, it is cost-effective.

**Lemma 4.3.** Let \( \mathcal{F} = \{f_i\}_{i \in [m]} \) and \( \mathcal{G} = \{g_i\}_{i \in [m]} \) be two frame sequences in \( \mathcal{H}^n \). Then the following statements are satisfied:

1. \( \mathcal{F} \) and \( \mathcal{G} \) are not woven if there exists a non-trivial \( \sigma \subset [m] \) so that \( c_0(\text{span}(\mathcal{F}_\sigma \cup \mathcal{G}_{\sigma^c}), \text{span}(\mathcal{F}_{\sigma^c} \cup \mathcal{G}_\sigma)) < 1 \).

2. If for every non-trivial \( \sigma \subset [m] \), \( \hat{\delta}(\text{span}(\mathcal{F}_\sigma \cup \mathcal{G}_{\sigma^c}), \text{span}(\mathcal{F}_{\sigma^c} \cup \mathcal{G}_\sigma)) = 0 \) and \( c_0(\text{span}(\mathcal{F}_\sigma \cup \mathcal{G}_{\sigma^c}), \text{span}(\mathcal{F}_{\sigma^c} \cup \mathcal{G}_\sigma)) = 0 \) then \( \mathcal{F} \) and \( \mathcal{G} \) are woven.

**Proof.** Using the Lemma 2.6 and the Theorem 2.7, our assertions are quickly plausible.

**Proposition 4.4.** In \( \mathcal{H}^n \), if \( \mathcal{F} = \{f_i\}_{i \in [m]} \) and \( \mathcal{G} = \{g_i\}_{i \in [m]} \) are two woven frame sequences, then for every non-trivial \( \sigma \subset [m] \),
\[
\hat{\delta}(\text{span}(\mathcal{F}_\sigma \cup \mathcal{G}_{\sigma^c}), \text{span}(\mathcal{F}_{\sigma^c} \cup \mathcal{G}_\sigma)) = 0 .
\]

**Proof.** If \( \mathcal{F} \) and \( \mathcal{G} \) are woven, then for every non-trivial \( \sigma \subset [m] \), both \( \mathcal{F}_\sigma \cup \mathcal{G}_{\sigma^c} \) and \( \mathcal{F}_{\sigma^c} \cup \mathcal{G}_\sigma \) constitute frames for \( \mathcal{H}^n \). Hence the conclusion directly follows from the Remark 2.5.

**Remark 4.5.** It is to be noted that, the two foregoing outcomes also hold for characterizing woven frames.
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