Massively Parallel Benders Decomposition for Correlation Clustering

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Abstract

We tackle the problem of graph partitioning for image segmentation using correlation clustering (CC), which we treat as an integer linear program (ILP). We reformulate optimization in the ILP so as to admit efficient optimization via Benders decomposition, a classic technique from operations research. Our Benders decomposition formulation has many subproblems, each associated with a node in the CC instance’s graph, which are solved in parallel. Each Benders subproblem enforces the cycle inequalities corresponding to the negative weight edges attached to its corresponding node in the CC instance. We generate Magnanti-Wong Benders rows in addition to standard Benders rows, to accelerate optimization. Our Benders decomposition approach provides a promising new avenue to accelerate optimization for CC, and allows for massive parallelization.

1 Introduction

Many computer vision tasks involve partitioning (clustering) a set of observations into unique entities. A powerful formulation for such tasks, is that of (weighted) correlation clustering (CC). CC is defined on a sparse graph with real valued edge weights, where nodes correspond to observations, and weighted edges describe the affinity between pairs of nodes.

For example, in image segmentation (on superpixel graphs), nodes correspond to superpixels, and edges indicate adjacency between superpixels. The weight of the edge between a pair of superpixels relates, to the probability, as
defined by a classifier, that the two superpixels belong to the same ground truth entity. This weight is positive if the probability is greater than \( \frac{1}{2} \), and negative if less than \( \frac{1}{2} \). The magnitude of the weight is a function of the confidence of the classifier.

The CC cost function, sums up the weights of the edges separating connected components (referred to as entities) in a proposed partitioning of the graph. Optimization in CC partitions the graph into entities so as to minimize the CC cost. CC is appealing since the optimal number of entities emerges naturally as a function of the edge weights rather, than requiring an additional search over some model order parameter describing the number of clusters (entities) \( [29] \).

Optimization in CC is NP-hard for general graphs \( [5] \). Common approaches for optimization in CC, which are based on linear programming \( [2, 3, 19] \), do not scale easily to large CC problem instances. The goal of this paper is to introduce an efficient mechanism for optimization in CC for domains, where we can exploit massively parallel computation.

In this paper, we apply the classic Benders decomposition from operations research \( [7] \), to CC for computer vision. Benders decomposition is commonly applied in operations research to solve mixed integer linear programs (MILP) that have a special, but common block structure. Benders decomposition receives a partition of the variables in the MILP between a master problem, and a set of subproblems. The block structure requires, that no row of the constraint matrix of the MILP, contains variables from more than one subproblem. Variables explicitly enforced to be integral lie in the master problem.

Optimization in Benders decomposition is achieved using a cutting plane algorithm. Optimization proceeds with the master problem solving optimization over its variables, followed by solving the subproblems in parallel, providing primal/dual solutions over their variables conditioned on the solution to the master problem. The dual solutions to the subproblems, provide constraints to the master problem. Optimization continues until no further constraints are added to the master problem.

Benders decomposition is an exact MILP programming solver, but can be intuitively understood as a coordinate descent procedure, iterating between the master problem, and the subproblems. Here, solving the subproblems not only provides a solution for their variables, but also a lower bound in the form of a hyper-plane over the master problem’s variables. This lower bound is tight at the current solution to the master problem.

Benders decomposition is accelerated using the seminal operations research technique of Magnanti-Wong Benders rows (MWR) \( [17] \). MWR are generated by solving the Benders subproblems with a distinct objective under the hard constraint of optimality regarding the original subproblem objective.

Our contribution in this paper is the use of Benders decomposition with MWR to tackle optimization in CC. In contrast to classic approaches to CC such as \( [2] \), this allows for massive parallelization.
1.1 Outline

We outline this paper as follows. First in Section 2, we review related work in computer vision, and operations research. Next in Section 3, we review the standard cutting plane approach for optimization in CC of [2]. Then in Section 4 we reformulate optimization using Benders decomposition, and derive cutting planes (referred to as Benders rows). Next in Section 5, we consider the use of Magnanti-Wong Benders rows to accelerate Benders decomposition. Then in Section 6, we apply our Benders decomposition approach to image segmentation. Finally in Section 7, we conclude and discuss extensions.

2 Related Work

Correlation clustering has been successfully applied to multiple problems in computer vision including image segmentation, multi-object tracking, instance segmentation, and multi-person pose estimation. In the classical work of [2], the authors model image segmentation as CC, where nodes correspond to superpixels, and optimize CC using an integer linear programming (ILP) branch-and-cut strategy. [15] extends CC to include higher-order cost terms over sets of nodes, which they solve using an approach similar to [2].

[29] attacks CC in the planar graph structured problems commonly found in computer vision. [29] introduces a column generation [13, 6] approach, where the pricing problem corresponds to finding the lowest reduced cost 2-colorable partition of the graph, via a reduction to minimum cost perfect matching [10, 22, 10]. [29] is extended to hierarchical image segmentation in [28], and specific cases of non-planar graphs in [26, 31, 4].

[23] tackles multi-object tracking using a formulation closely related to CC, where nodes correspond to detections of objects, and edges are associated with probabilities of co-association. [14, 20] built off [23], to formulate multi-person pose estimation using CC augmented with node labeling.

Our work is derived from the classical work in operations research on Benders decomposition [7, 8, 12]. Specifically, we are inspired by the fixed charge formulations of [9], which solves a mixed integer linear program over a set of fixed charge variables (opening links), and a larger set of fractional variables (flows of commodities from facilities to customers in a network) associated with constraints. Benders decomposition reformulates optimization so as to use only the integer variables, and converts the fractional variables into constraints. These constraints are referred to as Benders rows. Optimization is then tackled using a cutting plane approach. Optimization is accelerated by the use of MWR [17], which are more binding than the standard Benders rows.

Benders decomposition has recently been introduced to computer vision (though not for CC), for the purpose of multi-person pose estimation [25, 24, 30]. In these works, multi-person pose estimation is modeled so as to admit efficient optimization, using column generation and Benders decomposition jointly. The application of Benders decomposition in our paper is distinct regarding problem
domain, underlying integer program, and the structure of the Benders subproblems.

3 Standard CC Formulation

In this section, we review the standard optimization formulation for CC [2], which corresponds to a graph partitioning problem with respect to the graph $G = (\mathcal{D}, \mathcal{E})$. This problem is defined as follows:

$$
\min_{x_{d_1d_2} \in \{0, 1\}} \sum_{d_1d_2 \in \mathcal{E}^-} -\phi_{d_1d_2} (1 - x_{d_1d_2}) + \sum_{d_1d_2 \in \mathcal{E}^+} \phi_{d_1d_2} x_{d_1d_2} \\
\text{s.t.} \sum_{d_1d_2 \in \mathcal{E}_c^+} x_{d_1d_2} \geq x_{d^c_1d^c_2} \quad \forall c \in \mathcal{C}
$$

where the variables are defined as:

- $d \in \mathcal{D}$: The set of nodes in the graph, which CC is applied on, is denoted $\mathcal{D}$ and indexed by $d$.
- $(d_1, d_2) \in \mathcal{E}$: The set of undirected edges in the graph, which CC is applied on is denoted $\mathcal{E}$, and indexed by nodes $d_1, d_2$. The graph described by $\mathcal{E}$ is very sparse for real problems [29].
- $x_{d_1d_2} \in \{0, 1\}$. We set $x_{d_1d_2} = 1$ to indicate that nodes $d_1, d_2$ are in separate components, and zero otherwise. We refer to an edge $(d_1, d_2)$, where $x_{d_1d_2} = 1$ as a cut edge.
- $\phi_{d_1d_2} \in \mathbb{R}$: We use $\phi_{d_1d_2}$ to denote the weight associated with edge $(d_1, d_2)$. We use $\mathcal{E}^+, \mathcal{E}^-$ to denote the subsets of $\mathcal{E}$, for which $\phi_{d_1d_2}$ is non-negative, and negative respectively.
- $c \in \mathcal{C}$: We use $\mathcal{C}$ to denote the set of (undirected) cycles of edges in $\mathcal{E}$, each of which contains exactly one member of $\mathcal{E}^-$. We index $\mathcal{C}$ with $c$.
- $(d^c_1, d^c_2)$: We use $(d^c_1, d^c_2)$ to denote the only edge in $\mathcal{E}^-$ associated with cycle $c$.
- $\mathcal{E}_c^+$: We use $\mathcal{E}_c^+$ to denote the subset of $\mathcal{E}^+$ associated with the cycle $c$.

The objective in Eq 1 describes the total weight of the cut edges. The constraints described in Eq 2 enforce the standard relaxation of correlation clustering, which requires that transitivity regarding association of nodes with components is respected.

We express Eq 2 with the following example. Consider any cycle of edges $c$ containing exactly one edge in $(d^c_1, d^c_2) \in \mathcal{E}^-$. Eq 2 states that if edge $(d^c_1, d^c_2)$
is cut, then at least one other edge must be cut on the cycle. If this constraint were violated, then it would mean that $d_1$, $d_2$ are in separate components (since $x_{d_1}d_2 = 1$), and that all nodes on the cycle are in the same component (since $x_{d_1}d_2 = 0$ for all $(d_1, d_2) \in \mathcal{E}^+$) creating a contradiction.

We refer to the constraints in Eq 2 as cycle inequalities. Solving Eq 1 is intractable due to the large number of cycle inequalities. To attack such problems, [2] iterates between solving the ILP over a nascent set of constraints $\hat{\mathcal{C}}$ (initialized empty), and adding new constraints from the set of currently violated cycle inequalities. Generating constraints corresponds to iterating over $(d_1, d_2) \in \mathcal{E}^-$, and identifying the shortest path between the $d_1$, $d_2$ in the graph with edges $\mathcal{E}$, and weights equal to the vector $x$. If the corresponding path has total weight less than $x_{d_1}d_2$, then the corresponding constraint is added to $\hat{\mathcal{C}}$. The LP relaxation of Eq 1-2 can be solved instead of the ILP in each iteration until no violated cycle inequalities exist; after which the ILP must be solved in each iteration.

We should note that early work in CC for computer vision did not require that cycle inequalities contain exactly one member of $\mathcal{E}^-$, which is on the right hand side of Eq 2. It is established in [27], that the addition of cycle inequalities, that contain edges in $\mathcal{E}^-\mathcal{E}^+$ on the left hand side, right hand side of Eq 2 respectively do not tighten the ILP in Eq 1-2 or its LP relaxation.

In this section we reviewed the baseline approach for solving CC in the computer vision community. In the subsequent sections we rely on the characterization of CC in Eq 1-2 though not the specific solver of [2].

4 Benders Decomposition for CC

In this section, we introduce a novel approach to CC using Benders decomposition (referred to as BDCC).

Our proposed formulation, is defined by a minimal vertex cover on $\mathcal{E}^-$, with members $N \subset D$ indexed by $n$. Each $n \in N$ is associated with a Benders subproblem, and is referred to as the root of that Benders subproblem. Edges in $\mathcal{E}^-$ are partitioned arbitrarily between the subproblems, such that each $(d_1, d_2) \in \mathcal{E}^-$ is associated with either the subproblem with root $d_1$ or the subproblem with root $d_2$. Here $\mathcal{E}^-_n$ is the subset of $\mathcal{E}^-$ associated with subproblem $n$. The subproblem with root $n$ enforces, the cycle inequalities $\mathcal{C}_n$, where $\mathcal{C}_n$ is the subset of $\mathcal{C}$ containing edges in $\mathcal{E}^-_n$. We use $\mathcal{E}^+_n$ to denote the subset of $\mathcal{E}^+$ adjacent to $n$.

In this section we assume, that we are provided with $N$, which can be produced greedily or using an LP/ILP solver.

Below, we rewrite Eq 1 using an auxiliary function $Q(\phi, n, x)$. Here $Q(\phi, n, x)$ provides the cost to alter $x$ to satisfy all cycle inequalities in $\mathcal{C}_n$, by increasing/decreasing $x_{d_1}d_2$ for $(d_1, d_2)$ in $\mathcal{E}^+/\mathcal{E}^-_n$, respectively. Below we describe the
changes to \( x \) using \( x^n \), which is indexed as \( x \).

\[
\text{Eq 1} = \min_{x_{d_1,d_2} \in \{0,1\}} \sum_{d_1,d_2 \in \mathcal{E}^-} -\phi_{d_1,d_2} (1 - x_{d_1,d_2}) + \sum_{d_1,d_2 \in \mathcal{E}^+} \phi_{d_1,d_2} x_{d_1,d_2} + \sum_{n \in \mathcal{N}} Q(\phi,n,x)
\]

where \( Q(\phi,n,x) \) is defined as follows.

\[
Q(\phi,n,x) = \min_{x^{n}_{d_1,d_2} \in \{0,1\}} \sum_{d_1,d_2 \in \mathcal{E}_n^-} -\phi_{d_1,d_2} (1 - x^{n}_{d_1,d_2}) + \sum_{d_1,d_2 \in \mathcal{E}_n^+} \phi_{d_1,d_2} x^{n}_{d_1,d_2}
\]

\[
s.t. \sum_{d_1,d_2 \in \mathcal{E}_n^+} x_{d_1,d_2} + x^{n}_{d_1,d_2} \geq x_{d_1,d_2} - (1 - x^{n}_{d_1,d_2}) \quad \forall c \in \mathcal{C}_n
\]

Below we map \( \{ x, x^n \forall n \in \mathcal{N} \} \) to a solution \( \{ x^*, x^{n^*} \forall n \in \mathcal{N} \} \), where \( x^* \) satisfies all cycle inequalities by construction, without increasing the cost according to Eq 3. We define \( x^* \) as follows.

\[
x^*_{d_1,d_2} = \min(x_{d_1,d_2}, x^{n}_{d_1,d_2}) \quad \forall (d_1,d_2) \in \mathcal{E}_n^-, n \in \mathcal{N}
\]

\[
x^*_{d_1,d_2} = x_{d_1,d_2} + \max_{n \in \mathcal{N}} x^{n}_{d_1,d_2} \quad \forall (d_1,d_2) \in \mathcal{E}_n^+
\]

Given \( x^* \), the optimizing solution to each Benders subproblem, \( n \) is denoted \( x^{n^*} \), and defined as follows. We set \( x^{n^*}_{d_1,d_2} = 1 \), if and only if \( (d_1,d_2) \in \mathcal{E}_n^- \), and otherwise set it to zero. Observe that cost of \( \{ x^*, x^{n^*} \forall n \in \mathcal{N} \} \), is no greater than that of \( \{ x, x^n \forall n \in \mathcal{N} \} \), with regard to the objective in Eq 3.

Now observe that \( Q(\phi,n,x^*) = 0 \) for all \( n \in \mathcal{N} \). Thus there always exists an optimizing solution to Eq 3, denoted \( x \), such that \( Q(\phi,n,x) = 0 \) for all \( n \in \mathcal{N} \) (See Appendix A for further exposition).

Observe, that there exists an optimal partition \( x^n \), in Eq 1, that is 2-colorable. This is because any partition \( x^n \), can be altered without increasing its cost, by merging adjacent connected components not including the root node \( n \). Note that merging any pair of such components, does not increase the cost, since those components are not separated by negative weight edges.

Given this observation, we rewrite optimization in \( Q(\phi,n,x) \), using the node labeling formulation of min-cut, with notation below.

- \( m_d = 1 \) for \( d \in \mathcal{D} \): indicates that a node \( d \) is not in the component associated with \( n \), and is otherwise zero. To avoid extra notation \( m_n \) is replaced by 0.

- \( f_{d_1,d_2} = 1 \) for \( (d_1,d_2) \in \mathcal{E}^+ \): indicates that the edge between \( d_1, d_2 \) is cut, but is not cut in \( x \). Thus a penalty of \( \phi_{d_1,d_2} \) is added to \( Q(\phi,n,x) \). Observe that \( x^{n}_{d_1,d_2} = f_{d_1,d_2} \) for all \( (d_1,d_2) \in \mathcal{E}^+ \).

- \( f_{d_1,d_2} = 1 \) for \( (d_1,d_2) \in \mathcal{E}_n^- \): indicates that the edge between \( d_1, d_2 \) is not cut, but is cut in \( x \). Thus a penalty of \(-\phi_{d_1,d_2} \) is added to \( Q(\phi,n,x) \). Observe that \( x^{n}_{d_1,d_2} = 1 - f_{d_1,d_2} \) for all \( (d_1,d_2) \in \mathcal{E}_n^- \).
For benefit of readability, we re-orient edges from \((d,n)\) to \((n,d)\).

Below we write \(Q(\phi,n,x)\) as primal/dual LP, with primal constraints associated with dual variables \(\psi, \lambda\), which are noted in the primal. Given binary \(x\), we need only enforce that \(f, m\) are non-negative to ensure, that there is an optimizing solution for \(f, m\) that is binary. This is a consequence of optimization being totally unimodular, given that \(x\) is binary. Total unimodularity is a known property of the min-cut/max flow LP. [11].

\[
Q(\phi,n,x) = \begin{array}{c}
\min_{f_{d1,d2}, \phi_{nd} \geq 0} \sum_{d1,d2 \in E^+} \phi_{d1,d2} f_{d1,d2} - \sum_{nd \in E^-} \phi_{nd} f_{nd} \\
\end{array}
\]

\[
m_{d1} - m_{d2} \leq x_{d1,d2} + f_{d1,d2} \forall (d1,d2) \in (E^+ - E^-), \lambda_{d1,d2}^-
\]

\[
m_{d2} - m_{d1} \leq x_{d1,d2} + f_{d1,d2} \forall (d1,d2) \in (E^+ - E^-), \lambda_{d1,d2}^+
\]

\[
x_{nd} - f_{nd} \leq m_d \forall (n,d) \in E^-_\phi, \psi_d^-
\]

\[
m_d \leq x_{nd} + f_{nd} \forall (n,d) \in E^-_\phi, \psi_d^+
\]

\[
(6)
\]

\[
Q(\phi,n,x) = \begin{array}{c}
\max_{\psi_\geq 0} \sum_{d1,d2 \in (E^+ - E^-)} (\lambda_{d1,d2}^1 + \lambda_{d1,d2}^2) x_{d1,d2} + \sum_{nd \in E^-_\phi} \psi_{d}^+ x_{nd} - \sum_{nd \in E^-_\phi} \psi_{d}^x x_{nd} \\
\end{array}
\]

\[
\text{s.t. } \psi_{d1}^+ [(n,d1) \in E^+_\phi] - \psi_{d1}^- [(n,d1) \in E^-_\phi]
\]

\[
+ \sum_{d2} (\lambda_{d1,d2}^- - \lambda_{d1,d2}^+) + \sum_{d2} (\lambda_{d2,d1}^- - \lambda_{d2,d1}^+) \geq 0 \forall d1 \in D - n
\]

\[
-\phi_{nd} - \psi_{d}^x \geq 0 \forall (n,d) \in E^{-}_\phi
\]

\[
\phi_{nd} - \psi_{d}^+ \geq 0 \forall (n,d) \in E^{+}_\phi
\]

\[
\phi_{d1,d2} - \lambda_{d1,d2}^1 - \lambda_{d1,d2}^2 \geq 0 \forall (d1,d2) \in E - E^+_\phi - E^-_\phi
\]

In Eq[7] and subsequently, we use \(\lfloor \cdot \rfloor\) to denote the binary indicator function, which returns one if the statement is true and zero otherwise. We now consider the constraint that \(Q(\phi,n,x) = 0\). Observe, that any dual feasible solution \(\lambda, \psi\) in Eq[7] describes an affine function of \(x\), that is a lower bound on \(Q(\phi,n,x)\).

We compact the terms \(\lambda, \psi\) into \(\omega\), where \(\omega_{d1,d2}^z\) is associated with the \(x_{d1,d2}\) term.

\[
\omega_{d1,d2}^z = - (\lambda_{d1,d2}^1 + \lambda_{d1,d2}^2) \text{ if } (d1,d2) \in E^+ - E^-_\phi
\]

\[
\omega_{d1,d2}^z = - \psi_{d2}^x \text{ if } (d1,d2) \in E^+_\phi
\]

\[
\omega_{d1,d2}^z = \psi_{d2}^- \text{ if } (d1,d2) \in E^-_\phi
\]

\[
\omega_{d1,d2}^z = 0 \text{ if } (d1,d2) \in E^- - E^-_\phi
\]
We denote the set of all dual feasible solutions across \( n \in \mathbb{N} \) as \( Z \), which we index by \( z \). Observe, that to enforce that \( Q(\phi, n, x) = 0 \), it is sufficient to require that \( 0 \geq \sum_{d_1, d_2 \in E} x_{d_1, d_2} \omega_{d_1, d_2}^z \), for all \( z \in Z \). We formulate CC as optimization using \( Z \) below.

\[
\text{Eq 3} = \begin{array}{cl}
\min_{x_{d_1, d_2} \in \{0, 1\}} & \sum_{d_1, d_2 \in E^+} \phi_{d_1, d_2} x_{d_1, d_2} - \sum_{d_1, d_2 \in E^-} (1 - x_{d_1, d_2}) \phi_{d_1, d_2} \\
\text{s.t.} & 0 \geq \sum_{d_1, d_2 \in E} x_{d_1, d_2} \omega_{d_1, d_2}^z \quad \forall z \in Z 
\end{array}
\]

4.1 Cutting Plane Optimization

Optimization in Eq 8 is intractable, since \( |Z| \) equals the number of dual feasible solutions across subproblems, which is infinite. Since we can not consider the entire set \( Z \), we use a cutting plane approach to construct a set \( \hat{Z} \subset Z \), that is sufficient to solve Eq 8 exactly. We initialize \( \hat{Z} \) as the empty set. We iterate between solving the LP relaxation of Eq 8 over \( \hat{Z} \) (referred to as the master problem), and generating new Benders rows until no violated constraints exist.

This ensures that no violated cycle inequalities, exist but may not ensure that \( x \) in integral. To enforce integrality, we iterate between solving the ILP in Eq 8 over \( \hat{Z} \), and adding Benders rows to \( \hat{Z} \). By solving the LP relaxation first, we avoid unnecessary and expensive calls to the ILP solver.

To generate Benders rows given \( x \), we iterate over \( N \), and generate one Benders row using Eq 6 if \( n \) is associated with a violated cycle inequality. We determine if \( n \) is associated with a violated cycle inequality as follows. Given \( n, x \) we iterate over \((d_1, d_2) \in E^\_n\). We find the shortest path in from \( d_1 \) to \( d_2 \) on graph \( E \), with weights equal to the vector \( x \). If the length of this path, denoted \( Dist(d_1, d_2) \), is strictly less than \( x_{d_1, d_2} \), then we have identified a violated cycle inequality associated with \( n \).

We describe our cutting plane approach below in Alg 1 with line by line description in Appendix B. To accelerate optimization, we add MWR in addition to standard Benders rows, which are described subsequently in Section 5.

Prior to termination of Alg 1, one can produce a feasible integer solution \( x^* \), from any solution \( x \), provided by the master problem, as follows. First for each \((d_1, d_2) \in E \), set \( x^*_{d_1, d_2} = 1 \), if \( x_{d_1, d_2} > \frac{1}{2} \), and otherwise set \( x^*_{d_1, d_2} = 0 \). Second for each \((d_1, d_2) \in E \), set \( x^*_{d_1, d_2} = 1 \), if \( d_1, d_2 \) are in separate connected components of the solution described by \( x^* \), and otherwise set \( x^*_{d_1, d_2} = 0 \). The cost of the feasible integer solution \( x^* \), provides an upper bound on the cost of the optimal solution. In Section C we provide a more sophisticated approach for producing feasible integer solutions.

In this section we characterized CC using Benders decomposition, and provided a cutting plane algorithm to solve the corresponding optimization.
Algorithm 1 Benders Decomposition for CC (BDCC)

1: \( \tilde{Z} \leftarrow \{ \} \)
2: done\_LP \leftarrow \text{False}
3: \textbf{repeat}
4: \( x \leftarrow \) Solve Eq 8 over \( \tilde{Z} \) enforcing integrality if and only if done\_LP=\text{True}
5: \( \text{did\_add} \leftarrow \text{False} \)
6: \textbf{for} \( n \in N \) \textbf{do}
7: \( \text{if } \exists (d_1, d_2) \in E^\_n \text{ s.t. } \text{Dist}(d_1, d_2) < x_{d_1d_2} \text{ then} \)
8: \( z_1 \leftarrow \) Get Benders row via Eq 7
9: \( z_2 \leftarrow \) Get MWR via Section 5.
10: \( \tilde{Z} \leftarrow \tilde{Z} \cup z_1, z_2 \)
11: \( \text{did\_add} \leftarrow \text{True} \)
12: \textbf{end if}
13: \textbf{end for}
14: \textbf{if } \text{did\_add=\text{False} } \text{ then}
15: \( \text{done\_LP} \leftarrow \text{True} \)
16: \textbf{end if}
17: \textbf{until } \text{did\_add=\text{False AND } x_{d_1d_2} \in \{0, 1\}\forall (d_1, d_2) \in E} \)
18: \text{Return } x

5 Magnanti-Wong Benders Rows

In this section we accelerate Benders Decomposition using the classic operation research technique of MWR [17]. The Benders row, given in Eq 7 provides a tight bound at \( x^* \), where \( x^* \) is the master problem solution used to generate the Benders row. However, ideally we want our Benders row to provide good lower bounds for a large set of \( x \) different from \( x^* \), while being tight (or perhaps very active) at \( x^* \). To achieve this, we use a modified version of Eq 7, where we replace the objective, and add one additional constraint.

We follow the tradition of the operations research literature, and use a random negative valued vector (with unit norm) in place of the objective Eq 7. This random vector is unique each time a Benders subproblem is solved. We experimented with using as an objective \( -\frac{1}{0.0001 + |\omega_{d_1d_2}|} \), which encourages the cutting of edges with large positive weight, but it works as well as the random negative objective. Here .0001 is a tiny positive number used to ensure that the terms in the objective do not become infinite.

Below, we enforce that the new Benders row is active at \( x^* \), by requiring that the dual cost is within a tolerance \( \nu \in (0, 1) \), of the optimal with regards
to the objective in Eq. 7:

$$\nu Q(\phi, n, x) \leq - \sum_{d_1, d_2 \in (E^- - E^+_n)} (\lambda^1_{d_1, d_2} + \lambda^2_{d_1, d_2}) x_{d_1, d_2} + \sum_{nd \in E^-} \psi^-_d x_{nd} - \sum_{nd \in E^+_n} \psi^+_d x_{nd}$$  \hspace{1cm} (9)

Here $\nu = 1$ requires optimality with respect to the objective in Eq 7 and $\nu = 0$ ignores optimality. In experiments we found that $\nu = \frac{1}{2}$ provides strong performance.

In this section, we augment our Benders decomposition approach from Section 4 with Benders rows that accelerate optimization called MWR.

6 Experiments: Image Segmentation

In this section, we demonstrate the value of our algorithm BDCC on CC problem instances for image segmentation on the benchmark Berkeley Segmentation Data Set (BSDS) [18]. Our experiments demonstrate the following three findings. (1) BDCC solves CC instances for image segmentation; (2) BDCC successfully exploits parallelization; (3) the use of MWR dramatically accelerates optimization.

To benchmark performance, we employ cost terms provided by the OPENGM2 dataset [2] for BSDS. This allows for a direct comparison of our results to the ones from [2]. We use the random unit norm negative valued objective when generating MWR. We use CPLEX to solve all linear, and integer linear programming problems considered during the course of optimization. We use a maximum total CPU time of 600 seconds, for each problem instance (regardless of parallelization).

We formulate the selection of $N$, as a minimum vertex cover problem, where for every edge $(d_1, d_2) \in E^-$, at least one of $d_1, d_2$ is in $N$. We solve for the minimum vertex cover exactly as an ILP. Given $N$, we assign edges in $E^-$ to a connected selected node in $N$ arbitrarily. We found experimentally, that solving for the minimum vertex cover consumed negligible CPU time, for our data set. We attribute this fact to the structure of our problem domain, since the minimum vertex cover is an NP-hard problem. For problem instances where solving for the minimum vertex cover exactly is difficult, the minimum vertex cover problem can be solved approximately or greedily.

In Fig. 1 (top) we demonstrate the effectiveness of BDCC with various $\nu$ for different problem difficulties. We observe that the presence of MWR dramatically accelerates optimization. However, the exact value of $\nu$ does not effect the speed of optimization dramatically. We show performance with, and without relying on parallel processing. Our parallel processing times assume that we have one CPU for each subproblem. For the problem instances in our application the number of subproblems is under one thousand, each of which are very easy to solve. The parallel and non-parallel time comparisons share only the
\( \epsilon = 0.1 \)

| \( \nu \) | par | 10  | 50  | 100 | 300 |
|----------|-----|-----|-----|-----|-----|
| 0.5      | 0   | 0.149 | 0.372 | 0.585 | 0.894 |
| 0        | 1   | 0.0106 | 0.0532 | 0.0745 | 0.106 |
| 0.5      | 0   | 0.266 | 0.777 | 0.904 | 0.968 |
| 0        | 1   | 0.0426 | 0.0745 | 0.0745 | 0.138 |

\( \epsilon = 1 \)

| \( \nu \) | par | 10  | 50  | 100 | 300 |
|----------|-----|-----|-----|-----|-----|
| 0.5      | 0   | 0.149 | 0.394 | 0.606 | 0.904 |
| 0        | 1   | 0.0106 | 0.0638 | 0.0745 | 0.16 |
| 0.5      | 0   | 0.319 | 0.819 | 0.947 | 0.979 |
| 0        | 1   | 0.0532 | 0.0745 | 0.106 | 0.17 |

\( \epsilon = 10 \)

| \( \nu \) | par | 10  | 50  | 100 | 300 |
|----------|-----|-----|-----|-----|-----|
| 0.5      | 0   | 0.202 | 0.426 | 0.628 | 0.915 |
| 0        | 1   | 0.0532 | 0.0957 | 0.128 | 0.223 |
| 0.5      | 0   | 0.447 | 0.936 | 0.979 | 0.989 |
| 0        | 1   | 0.0638 | 0.128 | 0.181 | 0.287 |

Table 1: We show the percentage of problems solved that have a duality gap of up to tolerance \( \epsilon \), within a certain amount of time (10, 50, 100, 300) seconds, with and without MWR/parallelization. We use \( \text{par} = 1 \) to indicate the use of parallelization and \( \text{par} = 0 \) otherwise. Here \( \nu = 0 \) means that no MWR are generated.

| \( \epsilon \) | \( \nu \) | par | 10  | 50  | 100 | 300 |
|------------|-----|-----|-----|-----|-----|-----|
| 0.1        | 0   | 0.149 | 0.372 | 0.585 | 0.894 |
| 0.5        | 0   | 0.0106 | 0.0532 | 0.0745 | 0.106 |
| 0.5        | 0   | 0.266 | 0.777 | 0.904 | 0.968 |
| 0.5        | 0   | 0.0426 | 0.0745 | 0.0745 | 0.138 |
| 1          | 0   | 0.149 | 0.394 | 0.606 | 0.904 |
| 0          | 1   | 0.0106 | 0.0638 | 0.0745 | 0.16 |
| 0.5        | 0   | 0.319 | 0.819 | 0.947 | 0.979 |
| 0          | 1   | 0.0532 | 0.0745 | 0.106 | 0.17 |
| 10         | 0   | 0.202 | 0.426 | 0.628 | 0.915 |
| 0          | 1   | 0.0532 | 0.0957 | 0.128 | 0.223 |
| 0.5        | 0   | 0.447 | 0.936 | 0.979 | 0.989 |
| 0          | 1   | 0.0638 | 0.128 | 0.181 | 0.287 |

We observe large benefits of parallelization for all settings of \( \nu \). However, when MWR are not used we observe diminished improvement, since the master problem consumes a larger proportion of total CPU time.

In Fig. 1 (bottom), we demonstrate the speed up induced by the use of parallelization. For most problem instances, the total CPU time required when using no MWR was prohibitively large, which is not the case when MWR are employed. Thus most problem instances solved without MWR terminated early.

In Tab. 1, we consider the convergence of the bounds for \( \nu = \frac{1}{2}, 0 \); ( \( \nu = 0 \) means that no MWR are generated). We consider a set of tolerances on convergence regarding the duality gap, which is the difference between the anytime solution (upper bound), and the lower bound on the objective. For each such tolerance \( \epsilon \), we compute the percentage of instances, for which the duality gap is less than \( \epsilon \), after various amounts of time. We observe that the performance of optimization without MWR, but exploiting parallelization performs worse than using MWR, but without parallelization. This demonstrates that, across the data set, MWR is of greater importance than parallelization.

7 Conclusions

We present a novel methodology for finding high quality correlation clustering in arbitrary graphs. Our method exploits the Benders decomposition to avoid the
Figure 1: **Top):** We plot the gap between the upper and lower bounds as a function of time for various values of $\nu$ on selected problem instances. We use red, green, blue for $\nu = [0.5, 0.99, .01]$ respectively, and black for not using Magnanti Wong rows. We show both the computation time with and without exploiting parallelization of subproblems with dotted, and solid lines respectively. We use titles to indicate the approximate difficulty of the problem as ranked by input file size of 100 files.

**Bottom:** We compare the benefits of parallelization, and MWR across our data set. We scatter plot the total running time versus the total running time when solving each subproblem is done on its own CPU across problem instances. We use red to indicate $\nu = 0.5$ and black to indicate that MWR are not used. We draw a line with slope=1 in magenta to better enable appreciation of the red, and black points.

**Note:** The time spent generating Benders rows, in a given iteration of BDCC when using parallel processing, is the maximum time spent to solve any subproblem for that iteration.
enumeration of large numbers of cycle inequalities. This offers a new technique in the toolkit of linear programming relaxations, that we expect will find further use in the application of combinatorial optimization to problems in computer vision.

The exploitation of research from the domain of operations research may lead to improved variants of BDCC. For example, one can intelligently select the subproblems to generate Benders rows over instead of solving all subproblems each iteration. That strategy is referred to as partial pricing in the operations research literature. Similarly one can devote a minimum amount of time in each iteration, to solve the master problem so as to enforce integrality on a subset of the variables of the master problem.

A APPENDIX: $Q(\phi, n, x) = 0$ at Optimality

In this section, we demonstrate that there exists an $x$, that minimizes Eq 3, for which $Q(\phi, n, x) = 0$. We prove this by mapping an arbitrary solution $(x, \{x^n \forall n \in \mathcal{N}\})$ to one denoted $(x, \{x^n^* \forall n \in \mathcal{N}\})$ where $Q(\phi, n, x^*) = 0$, without increasing the objective in Eq 3. We write the updates below in terms of $x^n$.

$$x^n_{d_1 d_2} \leftarrow x_{d_1 d_2} + \max_{n \in \mathcal{N}} x^n_{d_1 d_2} \quad \forall (d_1, d_2) \in \mathcal{E}^+$$  \hspace{1cm} (10)

$$x^n_{d_1 d_2} \leftarrow x_{d_1 d_2} + x^n_{d_1 d_2} - 1 \quad \forall (d_1, d_2) \in \mathcal{E}^+_n, n \in \mathcal{N}$$

$$x^n_{d_1 d_2} \leftarrow 0 \quad \forall (d_1, d_2) \in \mathcal{E}^+$$

$$x^n_{d_1 d_2} \leftarrow 1 \quad \forall (d_1, d_2) \in \mathcal{E}^-_n, n \in \mathcal{N}$$

The updates in Eq 10 are equivalent to the following updates using $f^n, f^n^*$. Here $f^n, f^n^*$ correspond to the optimizing solution for $f$ in subproblem $n$, given $x, x^*$ respectively.

$$x^n_{d_1 d_2} \leftarrow x_{d_1 d_2} + \max_{n \in \mathcal{N}} f^n_{d_1 d_2} \quad \forall (d_1, d_2) \in \mathcal{E}^+$$  \hspace{1cm} (11)

$$x^n_{d_1 d_2} \leftarrow x_{d_1 d_2} - f^n_{d_1 d_2} \quad \forall (d_1, d_2) \in \mathcal{E}^-_n, n \in \mathcal{N}$$

$$f^n_{d_1 d_2} \leftarrow 0 \quad \forall (d_1, d_2) \in \mathcal{E}^+$$

$$f^n_{d_1 d_2} \leftarrow 0 \quad \forall (d_1, d_2) \in \mathcal{E}^-_n$$

Observe that the updates in Eq 10, Eq 11 preserve the feasibility of the primal LP in Eq 6. Also notice, that since $f^n^*$ is a zero valued vector for all $n \in \mathcal{N}$, then $Q(\phi, n, x) = 0$ for all $n \in \mathcal{N}$.

We now consider, the total change in Eq 3 corresponding to edge $(d_1, d_2) \in \mathcal{E}^+$, induced by Eq 10 which is non-positive. The objective of the master problem increases by $\phi_{d_1 d_2} \max_{n \in \mathcal{N}} x^n_{d_1 d_2}$, while the total decrease in the objectives of the subproblems is $\phi_{d_1 d_2} \sum_{n \in \mathcal{N}} x^n_{d_1 d_2}$.

We now consider, the total change in Eq 3 corresponding to edge $(d_1, d_2) \in \mathcal{E}^-$, induced by Eq 10 which is zero. The objective of the master problem, increases by $-\phi_{d_1 d_2} (1 - x^n_{d_1 d_2})$, while objective of subproblem $n$ decreases by $-\phi_{d_1 d_2} (1 - x^n_{d_1 d_2})$. 

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B   Line by Line Description of BDCC

We provide the line by line description of Alg 1.

• Line [1] Initialize the nascent set of Benders rows \( \mathcal{Z} \) to the empty set.
• Line [2] Indicate that, we have not solved the LP relaxation yet.
• Line [3-17] Alternate between solving the master problem, and generating Benders rows, until a feasible integral solution is produced.

1. Line [4] Solve the master problem providing a solution \( x \), which may not satisfy all cycle inequalities. We enforce integrality if we have finished solving the LP relaxation, which is indicated by done\_lp=True.
2. Line [5] Indicate that we have not yet added any Benders rows this iteration.
3. Line [6-13] Add Benders rows by iterating over subproblems, and adding Benders rows, corresponding to subproblems, associated with violated cycle inequalities.
   – Line [7] Check if there exists a violated cycle inequality associated \( \mathcal{E}_n^- \). This is done by iterating over \( (d_1, d_2) \in \mathcal{E}_n^- \), and checking if the shortest path from \( d_1 \) to \( d_2 \) is less than \( x_{d_1d_2} \). This distance is defined on graph \( \mathcal{E} \) with weights equal to \( x \).
   – Lines [8-10] Generate Benders rows associated with subproblem \( n \), and add them to nascent set \( \mathcal{Z} \).
   – Line [11] Indicate that a Benders row was added this iteration.
4. Lines [14-16] If no Benders rows were added this iteration, then we enforce integrality on \( x \), when solving the master problem for the remainder of the algorithm.

• Line [18] Return solution \( x \).

C   Generating Feasible Integer Solutions Prior to Convergence

Prior to the termination of optimization, it is valuable to provide feasible integer solutions on demand. This is so that a practitioner can terminate optimization, when the gap between the objectives of the integral solution, and the relaxation is small. In this section we consider the production of feasible integer solutions, given the current solution \( x^* \) to the master problem, which may neither obey cycle inequalities or be integral. We refer to this procedure as rounding.

Rounding is a coordinate descent approach defined on the graph \( \mathcal{E} \) with weights \( \kappa \), determined using \( x^* \) below.

\[
\begin{align*}
\kappa_{d_1,d_2} &= \phi_{d_1d_2} (1 - x^*_{d_1d_2}) & \forall (d_1, d_2) \in \mathcal{E}^+ \\
\kappa_{d_1,d_2} &= \phi_{d_1d_2} x^*_{d_1d_2} & \forall (d_1, d_2) \in \mathcal{E}^-
\end{align*}
\]
Consider that $x^*$ is integral, and feasible (where feasibility indicates that $x^*$ satisfies all cycle inequalities). Let $x_n^*$ define the boundaries, in partition $x^*$, of the connected component containing $n$. Here $x_n^{d_1d_2} = 1$ if exactly one of $d_1, d_2$ is in the connected component containing $n$ under cut $x^*$. Observe that $Q(\kappa, n, x^n) = 0$, where $x_n^{d_1d_2} = [(d_1, d_2) \in \mathcal{E}_n^-]$, is achieved using $x_n^*$ as the solution to Eq (6). Thus $x_n^*$ is the minimizer of Eq (6). The union of the edges cut in $x_n^*$ across $n \in \mathcal{N}$ is identical to $x^*$. Observe, that when $x^*$ is integral, and feasible then solution produced, below has cost equal to that of $x^*$.

$$x_n^* \leftarrow \minimizer \text{ of } Q(\kappa, n, x^n) \forall n \in \mathcal{N} \tag{13}$$

$$x_{d_1d_2}^+ \leftarrow \max_{n \in \mathcal{N}} x_n^{d_1d_2} \forall (d_1, d_2) \in \mathcal{E}^+$$

$$x_{d_1d_2}^+ \leftarrow x_n^{d_1d_2} \forall (d_1, d_2) \in \mathcal{E}^- \forall n \in \mathcal{N}$$

The procedure of Eq (13) can be used regardless of whether $x^*$ is integral or feasible. Observe that if $x^*$ is close to integral, and close to feasible, then Eq (13) is biased to produce a solution that is similar to $x^*$ by design of $\kappa$.

We now consider a serial version of Eq (13), which may provide improved results. We construct a partition $x^+$ by iterating over $n \in \mathcal{N}$, producing component partitions as in Eq (13). We alter $\kappa$ by allowing for the cutting of edges previously cut with cost zero. We formally describe this serial rounding procedure below in Alg 2.

### Algorithm 2 Generating an Integral and Feasible Solution Given Infeasible and or Non-Integral Input $x^*$

1. $x_{d_1d_2}^+ \leftarrow 0 \forall (d_1, d_2) \in \mathcal{E}$
2. $\kappa_{d_1d_2} \leftarrow \phi_{d_1d_2} x_{d_1d_2}^+ \forall (d_1, d_2) \in \mathcal{E}^-$
3. $\kappa_{d_1d_2} \leftarrow \phi_{d_1d_2} (1 - x_{d_1d_2}^+) \forall (d_1, d_2) \in \mathcal{E}^+$
4. for $n \in \mathcal{N}$ do
   5. $x_n^* \leftarrow \minimizer \text{ for } Q(\kappa, n, x^n) \text{ given fixed } \kappa, n.$
   6. $x_{d_1d_2}^+ \leftarrow \max(x_{d_1d_2}^+, x_{d_1d_2}^n) \forall (d_1, d_2) \in \mathcal{E}$
   7. $\kappa_{d_1d_2} \leftarrow \kappa_{d_1d_2} (1 - x_{d_1d_2}^+) \forall (d_1, d_2) \in \mathcal{E}$
5. end for
6. Return $x^+$

- Line [1] Initialize $x^+$ as the zero vector.
- Line [2][3] Set $\kappa$ according to Eq [12]
- Line [4][5][8] Iterate over $n \in \mathcal{N}$ to construct $x^+$ by cutting edges cut in the subproblem.
  1. Line [5] Produce the lowest cost $x^n$ given altered edge weights $\kappa$ for subproblem $n$.
  2. Line [6] Cut edges in $x^+$ that are cut in $x^n$.  

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3. Line 7: Set $\phi_{d_1,d_2}$ to zero for cut edges in $x^+$.  
• Line 9: Return the solution $x^+$

When solving for the fast minimizer of $Q(\kappa, n, x^{0n})$ we rely on the network flow solver of [21], though we do not exploit its capacity to attack non-submodular problems.

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