Stability and Relevance in Incomplete Argumentation Frameworks

Technical Appendix

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Abstract. This document contains the proofs of the paper Stability and Relevance in Incomplete Argumentation Frameworks, submitted to COMMA'22, that were not included into the paper itself, due to space restrictions. If accepted, the proofs will be made available online.

The document is structured as follows: we provide proofs for each of the three decision problems stated in the paper: JUSTIFICATION (Section 1), STABILITY (Section 2) and RELEVANCE (Section 3). The results are summarised in Table 1. New results are marked with a lemma or proposition number.

1. Justification status

This section provides proofs for complexity results related to JUSTIFICATION problems. We start with a lemma for justification similar to Lemma 1 for STABILITY:

Lemma 3 (Justification status IN and OUT). For any given $\sigma \in \{\text{GR, CP, PR, ST}\}$ and $c \in \{\text{sceptical, credulous}\}$, the complexity of $\sigma$-c-OUT-JUSTIFICATION equals the complexity of $\sigma$-c-IN-JUSTIFICATION.

Proof. Let $AF = (A, R)$ be an AF and $A \in A$ an argument. Now construct $A'$ as $A \cup \{B\}$ (where $B \notin A$) and $R' = R \cup \{(A, B)\}$; let $AF' = (A', R')$. Then $A$ is $\sigma$-c-OUT in $AF$ iff $B$ is $\sigma$-c-IN in $AF'$; in addition, $A$ is $\sigma$-c-IN in $AF$ iff $B$ is $\sigma$-c-OUT in $AF'$. $\square$

In the remainder of this section, we give the complexity results of UNDEC-JUSTIFICATION for various semantics.

Lemma 4. ST-credulous-UNDEC-JUSTIFICATION and ST-sceptical-existent-UNDEC-JUSTIFICATION are trivial.

Proof. In ST semantics, each extension is defined in such a way that each argument is either in the extension or attacked by some argument in the extension. Consequently,
Table 1. Overview of all complexity results related to this paper. If a reference is specified, this complexity result is trivial from an earlier result in the literature. New results are printed bold; their proofs are presented in this appendix. Results for ST-sceptical-existence justification and stability (recall Definition 8 from the paper) are given in parentheses. The complexities for IN- and OUT- JUSTIFICATION are the same; this follows from Lemma 3. Similarly, complexities for IN- equal those for OUT- STABILITY by Lemma 1.

| σ | c/s | status | JUSTIFICATION          | STABILITY            | RELEVANCE         |
|---|-----|--------|------------------------|----------------------|-------------------|
| ST | c   | IN/OUT | NP-c [1,2]             | Π₁²-c [3]            |                   |
| ST | c   | UNDEC  | Trivial (no) L4        | Trivial (no) P6      |                   |
| ST | s   | IN/OUT | CoNP-c [1]             | CoNP-c (Π₁²-c) [3]   |                   |
| ST | s   | UNDEC  | CoNP-c [1], L5 (Trivial (no) L4) | CoNP-c P5 (Trivial (no) P6) |                   |
| CP | c   | IN/OUT | NP-c [1,2]             | Π₁²-c [3]            |                   |
| CP | c   | UNDEC  | P-c L7                 | CoNP-c P2            |                   |
| CP | s   | IN/OUT | P-c [4]                | CoNP-c [3]           | NP-c P8           |
| CP | s   | UNDEC  | CoNP-c L8              | CoNP-c P3            |                   |
| GR | c   | IN/OUT | P-c [4]                | CoNP-c [3]           | NP-c P8           |
| GR | c   | UNDEC  | P-c L7                 | CoNP-c P2            |                   |
| GR | s   | IN/OUT | P-c [4]                | CoNP-c [3]           |                   |
| GR | s   | UNDEC  | P-c L7                 | CoNP-c P2            |                   |
| PR | c   | IN/OUT | NP-c [1,2]             | Π₁²-c [3]            |                   |
| PR | c   | UNDEC  | Σ²₀-c L9               | Π₁²-c P4            |                   |
| PR | s   | IN/OUT | Σᵡ²₀-c [5]            | Π₁²-c [3]           |                   |
| PR | s   | UNDEC  | CoNP-c L8              | CoNP-c P3            |                   |

Table 1. Overview of all complexity results related to this paper. If a reference is specified, this complexity result is trivial from an earlier result in the literature. New results are printed bold; their proofs are presented in this appendix. Results for ST-sceptical-existence justification and stability (recall Definition 8 from the paper) are given in parentheses. The complexities for IN- and OUT- JUSTIFICATION are the same; this follows from Lemma 3. Similarly, complexities for IN- equal those for OUT- STABILITY by Lemma 1.

each instance of ST-credulous-UNDEC-JUSTIFICATION or ST-sceptical-existent-UNDEC-JUSTIFICATION is False.

In addition, an argument can only be ST-sceptical-UNDEC in a given argumentation framework AF if AF does not have any ST extension.

**Lemma 5.** ST-sceptical-UNDEC-JUSTIFICATION is CoNP-complete.

**Proof.** For each AF $⟨A, R⟩$ and argument $A ∈ A$, $A$ is ST-sceptical-UNDEC in $⟨A, R⟩$ iff no stable extension exists for $⟨A, R⟩$. The problem of deciding if a given AF has a stable extension is NP-complete [1], so the complementary problem of deciding if an AF has no stable extension is CoNP-complete.

The “simplest” nontrivial UNDEC-JUSTIFICATION problems are P-complete, as we will show in Lemma 7. Before we do so, we prove that these problems are equivalent in the following lemma:

**Lemma 6.** Given an argumentation framework $AF = ⟨A, R⟩$ and an argument $A ∈ A$, $A$ is CP-credulous-UNDEC in $AF$ iff $A$ is GR-credulous-UNDEC in $AF$.

**Proof.** We prove this lemma in both directions:

$→$ Suppose that $A$ is CP-credulous-UNDEC in $AF$. Then there is some CP extension $S$ that does not contain $A$, which implies that the GR extension does not contain $A$ (since the GR extension is part of each complete extension). Furthermore, $A$ is not attacked by any argument in $S$, hence it is not attacked by any argument in the GR extension. Consequently, $A$ is GR-credulous-UNDEC in $AF$. 

$←$
If $A$ is GR-credulous-UNDEC in $AF$ then $A \not\in S$ and $A$ is not attacked by any argument in $S$, where $S$ is the GR extension. Since $S$ is also a complete extension, $A$ is CP-credulous-UNDEC in $AF$.

In the following, we use the previous result to show P-completeness for three variants of UNDEC-JUSTIFICATION.

**Lemma 7.** CP-credulous-UNDEC-JUSTIFICATION, GR-sceptical-UNDEC-JUSTIFICATION and GR-credulous-UNDEC-JUSTIFICATION are P-complete.

**Proof.** These problems and their complexities coincide, by Lemma 6 and because the GR semantics is a single-status semantics. We consider GR-sceptical-UNDEC-JUSTIFICATION. This problem is in P, as it can be decided in polynomial time by constructing the grounded extension $G$ and verifying that neither the argument itself, nor any argument attacking this argument is in $G$. In addition, the problem is P-hard, which can be shown by a reduction from GR-sceptical-IN-JUSTIFICATION: consider an instance of GR-sceptical-IN-JUSTIFICATION with argumentation framework $AF = \langle A, R \rangle$ and argument $A \in A$. W.l.o.g. let $B$ be an argument not in $A$ and construct $AF' = \langle A', R' \rangle$ where $A' = A \cup \{B\}$ and $R' = R \cup \{(A, B), (B, B)\}$. Then $A$ is GR-sceptical-IN in $AF$ iff $B$ is not GR-sceptical-UNDEC in $AF'$.

Finally, for some semantics, UNDEC-JUSTIFICATION is more complicated, especially for PR semantics. We prove this in Lemmas 8 and 9. In order to do so, we use translations from (co-)QBF instances to argumentation frameworks as specified in Definitions 12 and 13 and illustrated in Figures 1 and 2. These are extensions from the translations in [6, Reductions 3.6 and 3.7].

**Definition 12 (Translation $T_1$).** Let $(\phi, X)$ be an instance of co-QBF-1 (UNSAT) and let $\phi = \lor_j \alpha_j$ and $c_i = \lor_j \alpha_j$ for each clause $c_i$ in $\phi$, where $\alpha_j$ are the literals over $X$ that occur in clause $c_i$. We define the corresponding AF for this instance as $T_1(\phi, X) = (\langle A, R \rangle, \phi)$, where:

- $A = \{x_i, \overline{x}_i \mid x_i \in X\} \cup \{c_i \mid c_i \in \phi\} \cup \{\phi, \overline{\phi}\};$
- $R = \{\langle \overline{x}_i, x_i \rangle, \{x_i, \overline{x}_i\} \mid x_i \in X\} \cup \{\langle x_k, \overline{c}_i \rangle, \{\overline{x}_k, \overline{c}_i\} \mid \neg x_k \in c_i\} \cup \{\langle c_i, \phi \rangle \mid c_i \in \phi\} \cup \{\langle \phi, \overline{\phi} \rangle, \langle \overline{\phi}, \phi \rangle\}.$

![Figure 1. Visualisation of the IAF created for the clauses $c_1 = x_1 \lor \neg x_2$ and $c_2 = x_2 \lor \neg x_3$ using the reduction of Definition 12. We use this reduction for CP and PR sceptical-UNDEC-JUSTIFICATION.](image)
Lemma 8. CP-sceptical-UNDEC-JUSTIFICATION and PR-sceptical-UNDEC-JUSTIFICATION are CoNP-complete.

Proof. First, we show that these problems are in CoNP. For a negative instance \( (\langle A, R \rangle, A) \), a suitable certificate would be some set \( S \subseteq A \) such that \( S \) is an admissible set (which can be verified in polynomial time) containing either \( A \) or some argument attacking \( A \). If such a set exists, then there is some PR extension \( S' \supseteq S \) which is a PR extension and contains \( A \) itself or an argument attacking \( A \) - note that \( S' \) is also complete. So this proves that \( (\langle A, R \rangle, A) \) is a negative instance, which implies that CP-sceptical-

extension and contains \( A \) itself and an argument attacking \( A \) - note that \( S' \) is also complete. So this proves that \( (\langle A, R \rangle, A) \) is a negative instance, which implies that CP-sceptical-

UNDEC-JUSTIFICATION and PR-sceptical-UNDEC-JUSTIFICATION are in CoNP.

For CoNP-completeness, we use the translation \( T_1 \) specified in Definition 12 and show that for each co-QBF-1 instance \( (\phi, X) \): \( (\phi, X) \) = True iff \( \overline{\phi} \) is PR-sceptical-UNDEC and CP-sceptical-UNDEC in AF where \( (AF, A) = T_1(\phi, X) \).

- Let \( (\phi, X) \) be an arbitrary positive co-QBF-1 instance and let \( (AF, A) = T_1(\phi, X) \).
  Since there exists no assignment to \( X \) that makes \( \phi \) True, there exists no CP (and therefore no PR) extension \( S \) such that \( \phi \in S \). Then for each CP (and therefore for each PR) extension \( S \) of \( AF \), the argument \( \overline{\phi} \) is attacked by itself and an argument not in \( S \); it must therefore be CP-sceptical-UNDEC and PR-sceptical-UNDEC.

- Now let \( (\phi, X) \) be an arbitrary negative co-QBF-1 instance and let \( (AF, A) = T_1(\phi, X) \).
  Then there exists an assignment to \( X \) that makes \( \phi \) True; let \( S = \{ x_i \mid x_i \in X, x_i \text{ is assigned True} \} \cup \{ \overline{x_i} \mid x_i \in X, x_i \text{ is assigned False} \} \). Then the arguments \( S \cup \{ \phi \} \) constitute an extension that is CP and PR: each argument in \( S \) defends itself against its only attacker and the argument \( \phi \) is in the extension since each \( c_j \)-argument attacking it is attacked by some argument in \( S \). Note that \( S \) is a PR extension since adding any more arguments would violate conflict-freeness. As there exists some CP and PR extension (\( S \)) containing \( \phi \), the argument \( \overline{\phi} \) cannot be CP-sceptical-UNDEC or PR-sceptical-UNDEC. \( \square \)

The following translation is from [6, Reduction 3.7], where it is used to show that sceptical acceptance under PR semantics is \( \Sigma_2^p \)-complete. We will use the same translation in Lemma 9 to prove that PR-credulous-UNDEC-JUSTIFICATION is \( \Sigma_2^p \)-complete.

Definition 13 (Translation \( T_2 \)). Let \( (\phi, X, Y) \) be an instance of QBF-2 and let \( \phi = \land_i c_i \) and \( c_i = \lor_j \alpha_j \), for each clause \( c_i \) in \( \phi \), where \( \alpha_j \) are the literals over \( X \cup Y \) that occur in clause \( c_i \). We define the corresponding AF for this instance as \( T_2(\phi, X, Y) = (\langle A, R \rangle, \phi) \), where:

\[
A = \{ x_i, \overline{x_i} \mid x_i \in X \} \cup \{ y_i, \overline{y_i} \mid y_i \in Y \} \cup \{ c_i \mid c_i \in \phi \} \cup \{ \phi, \overline{\phi} \};
\]

\[
R = \{ (\overline{c_i}, x_i), (x_i, \overline{c_i}) \mid x_i \in X \} \cup \{ (\overline{c_i}, y_i), (y_i, \overline{c_i}) \mid y_i \in Y \} \cup \{ (\overline{c_i}, \overline{X}), (\overline{X}, \overline{c_i}) \mid \neg y_k \in c_i \} \cup \{ (y_k, \overline{c_i}), (\overline{c_i}, \overline{y_k}) \mid y_k \in c_i \} \cup \{ (\overline{c_i}, \phi), (\phi, \overline{c_i}) \} \cup \{ (\overline{\phi}, y_k), (\phi, \overline{y_k}) \mid y_k \in Y \}.
\]

Lemma 9. PR-credulous-UNDEC-JUSTIFICATION is \( \Sigma_2^p \)-complete.

Proof. First we show that PR-credulous-UNDEC-JUSTIFICATION is in \( \Sigma_2^p \). Let \( (\langle A, R \rangle, A) \) be a positive instance. Then a suitable certificate would be some set \( S \subseteq A \) such that \( A \notin S \) and no argument in \( S \) attacks \( A \). Using a CoNP oracle, one can check that \( S \) is a PR extension [1].
In this section, we give complexity results for variants of the STABILITY problem. We start with relating IN- and OUT-STABILITY.

**Lemma 1.** For any given \( \sigma \in \{ \text{GR, CP, PR, ST} \} \) and \( c \in \{ \text{sceptical, credulous} \} \), the complexity of \( \sigma\cdot c\)-OUT-STABILITY equals the complexity of \( \sigma\cdot c\)-IN-STABILITY.

**Proof.** Let \( IAF = \langle A, A', R, R' \rangle \) be an IAF and \( A \in A \) an argument. W.l.o.g. suppose that \( B \notin A \cup A' \) and construct \( IAF' = \langle A \cup \{ B \}, A', R \cup \{ (A, B) \}, R' \rangle \). We claim that \( A \) is stable-\( \sigma\cdot c\)-IN w.r.t. \( IAF \) iff \( B \) is stable-\( \sigma\cdot c\)-OUT w.r.t. \( IAF' \).

→ First suppose towards a contradiction that \( A \) is stable-\( \sigma\cdot c\)-IN w.r.t. \( IAF \) while \( B \) is not stable-\( \sigma\cdot c\)-OUT w.r.t. \( IAF' \). Then there is some element in \( F(IAF') \) with the certain specification \( AF' = \langle A' \cup \{ B \}, R' \cup \{ (A, B) \} \rangle \) such that \( B \) is not \( \sigma\cdot c\)-OUT in \( AF' \).

- If \( c = \text{sceptical} \) then for some \( \sigma\)-extension \( S \) of \( AF' \), \( B \) is not attacked by any argument in \( S \). Given that \( B \) is only attacked by \( A \), it must be that \( A \notin S \), so \( A \) is not \( \sigma\)-sceptical-IN in \( AF' \).

- Alternatively, \( c = \text{credulous} \); then for each \( \sigma\)-extension \( S \) of \( AF' \), \( B \) is not attacked by any argument in \( S \), hence \( A \notin S \). Thus \( A \) is not \( \sigma\)-credulous-IN in \( AF' \).

Given that \( A \) is not \( \sigma\cdot c\)-IN in \( AF' \), it was not \( \sigma\cdot c\)-IN in \( \langle A', R' \rangle \) either while \( \langle A', \emptyset, R', \emptyset \rangle \in F(IAF) \). This contradicts with \( \sigma\cdot c\)-IN-stability of \( A \) in \( IAF \).

← Now suppose that \( B \) is stable-\( \sigma\cdot c\)-OUT w.r.t. \( IAF' \) while \( A \) is not stable-\( \sigma\cdot c\)-IN w.r.t. \( IAF \). Then some element in \( F(IAF) \) has the certain specification \( AF = \langle A, R \rangle \) in which \( A \) is not \( \sigma\cdot c\)-IN. Now consider \( AF'' = \langle A' \cup \{ B \}, R' \cup \{ (A, B) \} \rangle \); since \( A \) is attacked by the same arguments in \( AF' \) as in \( AF'' \), \( A \) is not \( \sigma\cdot c\)-IN in \( AF'' \).

Being attacked by only \( A \), \( B \) cannot be \( \sigma\cdot c\)-OUT in \( AF'' \). This contradicts with the assumption that \( B \) is stable-\( \sigma\cdot c\)-OUT w.r.t. \( IAF' \) and since \( AF'' \) is a certain specification of some IAF in \( F(IAF) \).

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**Figure 2.** Visualisation of the IAF created for the clauses \( c_1 = x_1 \lor \neg y_1 \) and \( c_2 = y_1 \lor y_2 \) using the reduction of Definition 13. We use this reduction for PR-credulous-UNDEC/JUSTIFICATION.
Similarly, \( A \) is stable-\( \sigma \text{-c-out} \) w.r.t. \( IAF \) iff \( B \) is stable-\( \sigma \text{-c-in} \) w.r.t. \( IAF \).

For the lower bound, we give a reduction from the UNSAT problem.

**Proposition 2.** GR-credulous-UNDEC-STABILITY, GR-credulous-UNDEC-STABILITY and CP-credulous-UNDEC-STABILITY are CoNP-complete.

**Proof.** Let \( (\phi, X) \) be a co-l-QBF (UNSAT) instance and let \( IAF = (A, A^1, R^1) \) be the IAF according to Definition 7, where \( Y = \emptyset \). As GR semantics results in a single extension, which is the intersection of all CP extensions, the problems GR-credulous-UNDEC-STABILITY, GR-credulous-UNDEC-STABILITY and CP-credulous-UNDEC-STABILITY coincide. We discuss GR-credulous-UNDEC-STABILITY here.

We claim that the argument \( \overline{\sigma} \) is stable-GR-credulous-UNDEC in \( IAF \) iff \( (\phi, X) \) is a positive UNSAT instance:

\[ \leftarrow \text{First assume that } (\phi, X) \text{ is a positive UNSAT instance. Then there is no interpretation to the variables in } X \text{ such that } \phi \text{ is True. Let } IAF' \text{ be an arbitrary specification of } IAF \text{ and let } AF' = (A', R') \text{ be its certain projection. Given that there is no interpretation to } X \text{ such that } \phi \text{ is True, it must be that in each interpretation to } X \text{ there is some clause } c_i \text{ of } \phi \text{ that is False. This can only be the case if all variables corresponding to positive literals in } c_i \text{ are False and all variables corresponding to negative literals in } c_i \text{ are True. So in } AF', \text{ the argument for } \overline{\sigma} \text{ is GR-credulous-IN. This implies that the argument for } \phi \text{ is GR-credulous-OUT, so the argument for } \overline{\sigma} \text{ must be GR-credulous-UNDEC. Since we chose } IAF' \text{ as an arbitrary specification of } IAF, \text{ this applies for all specifications of } IAF. \text{ Consequently, } \overline{\sigma} \text{ is stable-GR-credulous-UNDEC w.r.t. } IAF. \]

\[ \rightarrow \text{Now assume that } (\phi, X) \text{ is a negative UNSAT instance. Then there exists some assignment to variables in } X \text{ such that } \phi \text{ is True. Let } IAF' = (A', R) \text{ be the specification of } IAF \text{ corresponding to this assignment (having } A' = A \cup \{ g_i | x_i \text{ is assigned False} \}) \text{ and let } AF = (A', R). \text{ Then each argument corresponding to a clause is GR-credulous-OUT in } AF, \text{ since at least one of the arguments attacking it is GR-credulous-IN. Consequently, the argument for } \phi \text{ must be GR-credulous-IN, which implies that the argument for } \overline{\sigma} \text{ cannot be GR-credulous-UNDEC in } AF'. \text{ Given that } IAF' \in F(IAF), \text{ we have that the argument for } \overline{\sigma} \text{ cannot be stable-GR-credulous-UNDEC w.r.t. } IAF. \]

In view of the above we have that the following items are equivalent:

1. \( (\phi, X) \) is a positive UNSAT instance;
2. the argument for \( \overline{\sigma} \) is stable-GR-credulous-UNDEC in \( IAF \);
3. the argument for \( \overline{\sigma} \) is stable-GR-credulous-UNDEC in \( IAF \);
4. the argument for \( \overline{\sigma} \) is stable-CP-credulous-UNDEC in \( IAF \).

CoNP-hardness from UNDEC-STABILITY follows from CoNP-hardness of UNSAT. From Proposition 1 and the fact that the corresponding JUSTIFICATION problems are in \( P \), we conclude CoNP-completeness.

**Proposition 3.** CP-credulous-UNDEC-STABILITY and PR-credulous-UNDEC-STABILITY are CoNP-complete.
Proof. First, we show that CP-sceptical-UNDEC-STABILITY and PR-sceptical-UNDEC-STABILITY are in CoNP. Given a negative instance \(((A,A',R,R'),\phi)\) of any of these problems, a suitable certificate would be a tuple \(((A',R'),S)\). We can verify in polynomial time that:

1. \((A',A'',R',R'')\in F((A,A',R,R'))\);
2. \(A\in A\);
3. \(S\) is an admissible set of \((A',R')\); and
4. either \(A\in S\) or \(S\) contains some argument attacking \(A\).

If each of these requirements are fulfilled, then there is an admissible set \(S\) either containing \(A\) or an argument attacking \(A\). This implies that there exists a PR and CP extension (which is a non-strict superset of \(S\)) either containing \(A\) or an argument attacking \(A\). This proves that \(A\) cannot be stable-CP-sceptical-UNDEC or stable-PR-sceptical-UNDEC w.r.t. \((A,A',R,R')\).

For hardness, note that these STABILITY problems can be reduced to corresponding (CoNP-hard) JUSTIFICATION problems, leaving the uncertain part empty. \(\square\)

The following translation is from [3, Definition 26], where it is used to prove a different problem. We use the same translation to prove Proposition 4.

Definition 14 (Translation T3). Let \((\phi,X,Y,Z)\) be an instance of QBF-3 and let \(\phi = \bigwedge_i c_i\) and \(c_i = \bigvee_j \alpha_j\) for each clause \(c_i\) in \(\phi\), where \(\alpha_j\) are the literals over \(X\cup Y\cup Z\) that occur in clause \(c_i\). We define the corresponding IAF for this instance as \(T_3(\phi,X,Y,Z) = ((A,A',R,R,\phi),())\), where:

- \(A = \{x_i,x_i^\overline{} | x_i \in X\} \cup \{y_i,y_i^\overline{} | y_i \in Y\} \cup \{z_i,z_i^\overline{} | z_i \in Z\} \cup \{\overline{\tau_i} | c_i \in \phi\} \cup \{\phi,\overline\phi\};
- \(A' = \{g_i | x_i \in X\};
- \(R = \{(\overline{\tau_i},x_i),(x_i,\overline{\tau_i}) | x_i \in X\} \cup \{(\overline{\tau_i},y_i),(y_i,\overline{\tau_i}) | y_i \in Y\} \cup \{(\overline{\tau_i},z_i),(z_i,\overline{\tau_i}) | z_i \in Z\} \cup \{(\overline{\tau_i},z_i^\overline{}),(z_i^\overline{},\overline{\tau_i}) | z_i \in Z\} \cup \{(\overline{\tau_i},y_i^\overline{}),(y_i^\overline{},\overline{\tau_i}) | y_i \in Y\} \cup \{(\overline{\tau_i},g_i),(g_i,\overline{\tau_i}) | g_i \in X\} \cup \{(\overline{\tau_i},c_i),(c_i,\overline{\tau_i}) | c_i \in \phi\} \cup \{(\overline{\tau_i},\overline{\phi}),(\overline{\phi},\overline{\tau_i}) | \overline{\phi} \in \phi\};
- \(R' = \{(\overline{\tau_i},x_i),(x_i,\overline{\tau_i}) | x_i \in X\} \cup \{(\overline{\tau_i},y_i),(y_i,\overline{\tau_i}) | y_i \in Y\} \cup \{(\overline{\tau_i},z_i),(z_i,\overline{\tau_i}) | z_i \in Z\} \cup \{(\overline{\tau_i},c_i),(c_i,\overline{\tau_i}) | c_i \in \phi\} \cup \{(\overline{\tau_i},\overline{\phi}),(\overline{\phi},\overline{\tau_i}) | \overline{\phi} \in \phi\} \cup \{(\overline{\tau_i},\overline{\phi}),(\overline{\phi},\overline{\tau_i}) | \overline{\phi} \in \phi\};

Proposition 4. PR-credulous-UNDEC-STABILITY is \(\Pi^0_3\)-complete.

Proof. First, we prove that the PR-credulous-UNDEC-STABILITY problem is in \(\Pi^0_3\); given an arbitrary negative instance \(((A,A',R,R'),\phi)\), a suitable certificate would be some specification \(IAF'\). It can be verified in polynomial time that \(IAF' \in F((A,A',R,R'))\). Using a \(\Sigma^0_2\) oracle, one can verify that for each PR extension \(S\) of \(IAF'\), either \(\phi \in S\) or \(\phi\) is attacked by an argument in \(S\).

For \(\Pi^0_3\)-hardness, we use the earlier result from [3] (expressed in the notation used in our paper) that \((\phi,X,Y,Z)\) is a positive instance of QBF-3 iff there is some specification of \(T_3(\phi,X,Y,Z)\) such that each PR extension contains \(\phi\). Then \((\phi,X,Y,Z)\) is a positive instance of co-QBF-3 iff in each specification of \(T_3(\phi,X,Y,Z)\), some PR extension does not contain \(\phi\). Since \(\overline{\phi}\) attacks itself, it can never be IN; it can be OUT only if \(\phi\) is IN and otherwise it is UNDECIDED. So \((\phi,X,Y,Z)\) is a positive instance of co-QBF-3 iff \(\overline{\phi}\) is stable-PR-credulous-UNDEC in \(T_3(\phi,X,Y,Z)\). \(\square\)

Finally, we turn to ST semantics.

Proposition 5. ST-sceptical-UNDEC-STABILITY is CoNP-complete.
Proof. The problem is in CoNP, as a negative instance \( (\langle A, A', R', R'\rangle, A) \) can be verified in polynomial time given a certificate \( (\langle A', R'\rangle, S) \) such that \( \langle A', A'', R'', R''\rangle \in F(\langle A, A', R', R'\rangle) \), \( A \in A \) and \( S \) is a \( \text{ST} \) extension of \( \langle A', R'\rangle \). If \( S \) is a \( \text{ST} \) extension then each argument \( A \in A \) is either in \( S \) or attacked by \( S \); therefore no argument can be stable-\( \text{ST} \)-sceptical-\text{UNDEC} w.r.t. \( \langle A, A', R', R'\rangle \). For hardness, we can reduce from the CoNP-complete problem \( \text{ST} \)-sceptical-\text{UNDEC}-\text{JUSTIFICATION}. \qed

**Proposition 6.** \( \text{ST} \)-credulous-\text{UNDEC}-\text{STABILITY} and \( \text{ST} \)-sceptical-existent-\text{UNDEC}-\text{STABILITY} are trivial.

Proof. For each argumentation framework \( AF = \langle A, R \rangle \) such that a \( \text{ST} \) extension \( S \) exists, each argument in \( A \) is either in \( S \) or attacked by an argument in \( S \). Therefore, \( A \) cannot be \( \text{ST} \)-credulous-\text{UNDEC} or \( \text{ST} \)-sceptical-existent-\text{UNDEC} in \( AF \). This applies for each \( AF \), including all certain projections of all specifications of each possible \( IAF \), so each instance of \( \text{ST} \)-credulous-\text{UNDEC}-\text{STABILITY} and \( \text{ST} \)-sceptical-existent-\text{UNDEC}-\text{STABILITY} must be negative. \qed

### 3. Relevance

The following lemma shows that the relevance of adding an uncertain argument can be validated by checking the justification status of the certain projections of two particular future specifications. This property will be useful for proving an upper bound on \( j \)-minimal-relevance.

**Lemma 2.** Given an incomplete argumentation framework \( IAF = \langle A, A', R', R'\rangle \), a certain argument \( A \in A \) and a justification status \( j \):

1. For each \( U \in A' \), addition of \( U \) is \( j \)-minimal-relevant for \( A \) w.r.t. \( IAF \) iff there exists some \( IAF' = \langle A',\{U\}, R', \emptyset \rangle \in F(IAF) \) such that \( A \) is not \( j \) in \( \langle A', R' \rangle \) and \( A \) is \( j \) in \( \langle A' \cup \{U\}, R' \rangle \).

2. For each \( U \in R' \), addition of \( U \) is \( j \)-minimal-relevant for \( A \) w.r.t. \( IAF \) iff there exists some \( IAF' = \langle A', \emptyset, R', \{U\} \rangle \in F(IAF) \) such that \( A \) is not \( j \) in \( \langle A', R' \rangle \) and \( A \) is \( j \) in \( \langle A' \cup \{U\}, R' \rangle \).

3. For each \( U \in A' \), removal of \( U \) is \( j \)-minimal-relevant for \( A \) w.r.t. \( IAF \) iff there exists some \( IAF' = \langle A',\{U\}, R', \emptyset \rangle \in F(IAF) \) such that \( A \) is \( j \) in \( \langle A', R' \rangle \) and \( A \) is not \( j \) in \( \langle A' \cup \{U\}, R' \rangle \).

4. For each \( U \in R' \), removal of \( U \) is \( j \)-minimal-relevant for \( A \) w.r.t. \( IAF \) iff there exists some \( IAF' = \langle A',\{U\}, R', \emptyset \rangle \in F(IAF) \) such that \( A \) is \( j \) in \( \langle A', R' \rangle \) and \( A \) is not \( j \) in \( \langle A' \cup \{U\}, R' \rangle \).

Proof. We prove both directions of the first item; proofs for the other items are analogous.

Let \( IAF = \langle A, A', R, R'\rangle \) be an incomplete argumentation framework, \( A \in A \) a certain argument and \( j \) a justification status.

\( \rightarrow \) If addition of \( U \) is \( j \)-minimal-relevant for \( A \) w.r.t. \( IAF \) then \( IAF' = \langle A' \cup \{U\}, \emptyset, R', \emptyset \rangle \) is a minimal stable-\( j \) specification for \( A \) w.r.t. \( IAF \). This implies that \( A \) is stable-\( j \) w.r.t. \( IAF \) but not w.r.t. \( \langle A', \{U\}, R', \emptyset \rangle \), hence \( A \) cannot be \( j \) in \( \langle A', R' \rangle \).
️ Suppose that there exists some $IAF = \langle A', \{U\}, R', 0, \emptyset \rangle \in F(IAF)$ such that $A$ is not $j$ in $(A', R')$ and $A$ is $j$ in $(A' \cup \{U\}, R')$. Then $A$ is stable-$j$ w.r.t. $(A' \cup \{U\}, 0, R', \emptyset)$. Consequently, there must be some minimal stable-$j$ specification $IAF' = \langle A'', A''', R'', R''' \rangle$ for $A$ w.r.t. $IAF$ such that $A'' \subseteq A' \cup \{U\}$ and $R'' \subseteq R'$. Note that $U \in A''$, otherwise $(A', 0, R', \emptyset)$ would be in $F(IAF'')$, which contradicts the assumption that $A$ is not $j$ in $(A', R')$. To conclude, addition of $U$ is $j$-minimal-relevant for $A$ w.r.t. $IAF$. 

In the following proposition, we use the results from Lemma 2 to prove an upper bound on the complexity of $j$-minimal-relevance.

**Proposition 7** (Upper bound $j$-minimal relevance). Given an incomplete argumentation framework $IAF = \langle A, A', R, R' \rangle$, a certain argument $A \in A$, an uncertain argument or rule $U \in A' \cup R'$ and a justification status $j$, if the complexity of deciding $j$'s justification status in $(A, R)$ is $C$, then an upper bound on the problem of deciding if addition and/or removal of $U$ is $j$-relevant for $A$ w.r.t. $IAF$ is $NP^C$.

**Proof.** In order to validate that a given $U \in A' \cup R'$ is $j$-minimal-relevant for a given $A \in A$, a suitable polynomial-sized certificate would be some $IAF = \langle A', A'', R', R'' \rangle$ as specified in Lemma 2 (so $A'' \cup R'' = \{U\}$). The following procedure can be used to validate that $U$ is $j$-minimal-relevant for $A$ w.r.t. $IAF$:

1. Check in polynomial time if $IAF' \in F(IAF)$ and store the result in $r_1$;
2. Call the $C$ oracle to check if $A$ is $j$ w.r.t. $(A', R')$ and store the result in $r_2$;
3. Let $AF' = \langle A' \cup \{U\}, R' \rangle$ if $U \in A''$ and $AF' = \langle A', R' \cup \{U\} \rangle$ otherwise. Then call the $C$ oracle to check if $A$ is $j$ w.r.t. $AF'$ and store the result in $r_3$.

Then by Lemma 2, addition of $U$ is $j$-minimal relevant for $A$ w.r.t. $IAF$ iff $r_1 \land \neg r_2 \land r_3$. Removal of $U$ is $j$-minimal relevant for $A$ w.r.t. $IAF$ iff $r_1 \land r_2 \land \neg r_3$. These checks can be performed in polynomial time. 

**Proposition 8** (Lower bound grounded minimal relevance). GR-sceptical-IN-RELEVANCE, GR-credulous-IN-RELEVANCE and CP-sceptical-IN-RELEVANCE are $NP$-complete.

**Proof.** Given an instance of 1-QBF ($SAT$) $(\phi, X)$, let $IAF$ be the IAF constructed according to Definition 11. We claim that addition of $\chi$ is GR-sceptical-IN-relevant for $A$ w.r.t. $IAF$ iff the SAT instance is positive:

$\rightarrow$ Suppose that addition of $\chi$ is GR-sceptical-IN-relevant for $A$ w.r.t. $IAF$. Then there is some $IAF' = \langle A', A'', R', R'' \rangle \in F(IAF)$ such that $\phi$ is GR-sceptical-IN in $(A', R')$ and $U \in A'$. This implies that each of the clause arguments in $IAF'$ is GR-sceptical-OUT in $(A', R')$, which means that at least one of their literals must have been GR-sceptical-IN. In the corresponding assignment to $X$ (where all literals for which there is an argument in the GR extension are assigned True and all others are False), each of the clauses are True, so the formula $\phi$ is satisfiable.

$\leftarrow$ Now suppose that the SAT instance is positive. Then there exists some assignment to $X$ such that $\phi$ is True. Let $IAF' = \langle A', 0, R, 0 \rangle$ be the IAF corresponding to this assignment (having $A' = A \cup \{g_i \mid x_i$ is assigned positive$\} \cup \{\chi\}$). Then the
argument for $\phi$ is in the GR extension of $\langle A', \mathcal{R} \rangle$. However, it would not be in the GR extension of $\langle A' \setminus \{\chi\}, \mathcal{R} \rangle$, because in that AF, it would be attacked by the unattacked argument $\Psi$. Then by Lemma 2, addition of $\chi$ is GR-sceptical-IN-relevant for $A$ w.r.t. IAF.

Note that the GR extension is the intersection of all CP extensions, so addition of $\chi$ is GR-sceptical-IN-relevant for $A$ w.r.t. IAF iff addition of $\chi$ is CP-sceptical-IN-relevant for $A$ w.r.t. IAF.

NP-completeness follows from Proposition 8 and the fact that the corresponding JUSTIFICATION problems are in P.

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