Measure-Expansive Homoclinic Classes for $C^1$ Generic Flows

Manseob Lee

Department of Mathematics, Mokwon University, Daejeon 302-729, Korea; lmsds@mokwon.ac.kr

Received: 6 July 2020; Accepted: 24 July 2020; Published: 27 July 2020

Abstract: In this paper, we prove that for a generically $C^1$ vector field $X$ of a compact smooth manifold $M$, if a homoclinic class $H(\gamma, X)$ which contains a hyperbolic closed orbit $\gamma$ is measure expansive for $X$ then $H(\gamma, X)$ is hyperbolic.

Keywords: expansive; measure-expansive; homoclinic class; generic; hyperbolic

MSC: 37C20; 37C10; 37C29

1. Introduction and Statements

A homoclinic class $H(p, f)$ of a diffeomorphism $f$ is the closure of the transverse of homoclinic points associated with a hyperbolic periodic point $p$. It is an invariant, closed and transitive set. It has a dense orbit and contains a dense set of periodic points which is related to a basic set (see [1]). Many people are paying attention to the study of the homoclinic class for dynamical systems. In fact, the relation between homoclinic classes and expansiveness has been studied by many people. They used various types of $C^1$ perturbations ($C^1$ robustly [2,3], $C^1$ persistently [4], $C^1$ stably [5], R-robustly [6-8] and $C^1$ generic [9-14], etc.). In this paper, we focus on the $C^1$ generic property. Yang and Gan [14] proved that every expansive homoclinic class of a $C^1$ generic diffeomorphism of a compact smooth manifold $M$ is hyperbolic. Morales [15] introduced a general notion of expansiveness which is called measure-expansive. Koo, Lee and Lee [16] proved that if every locally maximal homoclinic class $H(p, f)$ is measure expansive then it is hyperbolic. Lee proved in [6] that if a homoclinic class $H(p, f)$ is R-robustly measure-expansive, then it is hyperbolic. Later, Lee [13] proved that $C^1$ generically, a homoclinic class $H(p, f)$ is expansive if and only if a homoclinic class $H(p, f)$ is measure-expansive.

About the results of diffeomorphisms, we consider the vector fields which is an extended version of diffeomorphisms.

For vector fields, Bautista [17] showed that the geometric Lorenz attractor is a homoclinic class which contains a singular point. Komuro showed in [18] that the geometric Lorenz attractor is $K^*$-expansive. However, the geometric Lorenz attractor is not expansive (see [19]). Nevertheless, in vector fields, the relation between homoclinic classes and expansiveness are still interesting research subjects. Lee and Park [20] proved that every expansive locally maximal homoclinic class $H(\gamma, X)$ of $C^1$ generic vector fields is hyperbolic. Lee and Oh [21] proved that every measure-expansive locally maximal homoclinic class $H(\gamma, X)$ of $C^1$ generic vector fields is hyperbolic. The results used a dynamical condition which is locally maximal. In [10], Lee proved that if a homoclinic class $H(\gamma, X)$ is R-robustly measure-expansive then it is hyperbolic.

We study the hyperbolicity of a measure-expansive homoclinic class for $C^1$ generic vector fields without the locally maximal condition and R-robust property, which is a generalization of previous results.
2. Basic Definitions and Main Theorem

Assume that $M$ is a compact smooth Riemannian manifold. Denote by $X(M)$ the set of a $C^1$ vector fields on $M$, endowed with then $C^1$ topology. Every vector field $X$ generates a $C^1$ flow $X^t : M \to M$, $t \in \mathbb{R}$. Denote by $\text{Sing}(X) = \{x \in M : X^t(x) = x \text{ for all } t \in \mathbb{R}\}$ the set of singularities of $X$, and by $\text{Per}(X) = \{x \in M : \exists \pi(x) > 0 \text{ such that } X^{\pi(x)}(x) = x\}$ the set of periodic points of $X$. A point $x \in M$ is said to be critical point if $x \in \text{Sing}(X) \cup \text{Per}(X)$, and denote by $\text{Crit}(X) = \text{Sing}(X) \cup \text{Per}(X)$.

Let $\mathcal{H}$ be the set of all continuous maps $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$. A vector field $X \in X(M)$ is said to be expansive if for every $\epsilon > 0$ there are $\delta > 0$ and $h \in \mathcal{H}$ such that if for any $x, y \in M$ and $d(X^t(x), X^h(y)) \leq \delta \forall t \in \mathbb{R}$, then $y = X^h(x)$ for some $|t_0| < \epsilon$.

Bowen and Walters [22] proved that if a vector field $X \in X(M)$ is expansive then every singular points is isolated. Oka [23] proved that if a vector field $X \in X(M)$ is expansive, then $\text{Sing}(X) = \emptyset$.

We define the following

$$\Gamma(x, \delta) = \bigcup_{h \in \mathcal{H}} \bigcap_{t \in \mathbb{R}} X^{-h(t)}(B[x, \delta]),$$

where $B[x, \delta]$ is the $\delta$-closed neighborhood of $x$. It is said to be the $\delta$-dynamic ball of $X$.

Then we can rewrite the $\delta$-dynamic ball such as

$$\Gamma(x, \delta) = \{y \in M : d(X^t(x), X^{h(t)}(y)) \leq \delta, \text{ for some } h \in \mathcal{H} \text{ and for all } t \in \mathbb{R}\}.$$

The following concept which is a general notion of expansiveness for flows was defined by Carrasco-Olivera and Morales [24]. Denote by $\mathcal{M}(M)$ the set of all Borel probability measures on $M$ and let $\mathcal{M}^*(M) = \{\mu \in \mathcal{M}(M) : \mu(\text{Orb}(x)) = 0 \text{ for all } x \in M\}$. It is known that $\mathcal{M}^*(M) \subset \mathcal{M}(M)$.

For any $\mu \in \mathcal{M}(M)$, a closed $X^t$-invariant set $\Lambda \subset M$ is $\mu$-expansive of $X$ if there is a constant $\delta > 0$ (which is called an expansive constant of $X$ with respect to $\mu$) such that $\mu(\Gamma(x, \delta)) = 0$ for any $x \in \Lambda$. If $\Lambda = M$ then we say that $X$ is $\mu$-expansive.

**Definition 1.** Let $X \in X(M)$ and let $\Lambda \subset M$ be a closed $X^t$-invariant set. We say that $\Lambda$ is measure-expansive if $\Lambda$ is $\mu$-expansive of $X$, for any $\mu \in \mathcal{M}^*(M)$. If $\Lambda = M$, then $X$ is called measure-expansive.

**Remark 1.** Let $\text{supp}(\mu) = \{x \in M : \mu(U) > 0, \text{ for any open neighborhood } U(\neq \emptyset) \text{ of } x\}$. Corrasco–Olivera and Morales proved in [24] if a vector field $X \in X(M)$ is measure-expansive, then $\text{supp}(\mu) \cap \text{Sing}(X) = \emptyset$.

A closed $X^t$-invariant set $\Lambda \subset M$, we say that $\Lambda$ is a hyperbolic set for $(X^t)_{t \in \mathbb{R}}$ if there exists a $DX^t$-invariant splitting $T_M = E^s \oplus F(x) \oplus E^u$ so that:

(a) $F(x)$ is one dimensional and generated by the vector field $X$,
(b) there are constants $C > 0$ and $0 < \lambda < 1$ so that for every $x \in \Lambda$ and $t \geq 0$,
(i) $\|DX^t|_{E^s}\| \leq Ce^{-\lambda t}$ (uniformly contracting) and
(ii) $\|DX^{-t}|_{E^u}\| \leq Ce^{-\lambda t}$ (uniformly expanding).

Let $\gamma \in \text{Per}(X)$ be hyperbolic. We say that $\eta \in \text{Per}(X)$ is homoclinically related to $\gamma \in \text{Per}(X)$, that is, $\eta \sim \gamma$ if

$$W^u(\eta) \cap W^s(\gamma) \neq \emptyset \text{ and } W^u(\eta) \cap W^s(\gamma) \neq \emptyset.$$

Then we define it as $H(\gamma, X) = \{\eta \in \text{Per}(X) : \eta \sim \gamma\}$.

In the paper, we consider the homoclinic class $H(\gamma, X)$ which contains a hyperbolic saddle type of the periodic orbit $\gamma$. The following is the main theorem of this paper.
Theorem 1. There is a residual set $G_\delta \subset \mathcal{X}(M)$ such that for any $x \in G_\delta$, if a homoclinic class $H(\gamma, X)$ is measure-expansive of $X$, then it is hyperbolic.

3. Proof of Theorem 1

Let $M$ be as before and let $X \in \mathcal{X}(M)$.

Lemma 1. ([21]) For a hyperbolic $\gamma \in \text{Per}(X)$, if the homoclinic class $H(\gamma, X)$ is measure-expansive then $H(\gamma, X)$ does not contain singularities.

The following was proved by [25], where it is called a vector field version of Franks lemma.

Lemma 2. Let $p$ belongs to a periodic orbit for $X$ with period $\pi(p) > 0$, and let $f : N_{p,r_1} \to N_p$ is the Poincaré map of $X$ (for some $r_1 > 0$). Let $0 < r \leq r_1$ be given. Then there are a positive $\delta > 0$ and $0 < \epsilon_0 < r/2$ such that for a linear isomorphism $L : N_p \to N_p$ with $\|L - D_pf\| < \delta$, there exists $Y C^1$ closed to $X$ with the properties:

(a) $Y(x) = X(x)$ if $x \notin U_p(X^i, r, \pi(p))$,
(b) $p$ belongs to a periodic orbit for $Y$,
(c) $g(x) = \begin{cases} 
\exp_p \circ L \circ \exp_p^{-1}(1), & \text{if } x \in B_{\epsilon_0/4}(p) \cap N_{p,r}, \\
\exp_f(x), & \text{if } x \notin B_{\epsilon_0}(p) \cap N_{p,r}.
\end{cases}$

Here, $g : N_{p,r} \to N_p$ is the Poincaré map of $Y$.

A closed orbit $\gamma$ is weakly hyperbolic if for any $p \in \gamma$ and any $q > 0$, $D_pf$ has an eigenvalue $\mu$ such that $(1 - q) < |\mu| < (1 + q)$, where $f$ is the Poincaré map of $X$. Denote by $\mathcal{W}\mathcal{H}(X)$ the set of all weak hyperbolic periodic orbits of $X$.

Lemma 3. Let $\eta \in H(\gamma, X) \cap \text{Per}(X)$ with $\eta \sim \gamma$. If $\eta \in \mathcal{W}\mathcal{H}(X)$ then there is $Y C^1$ close to $X$ such that $g$ has a small arc $J$ with the endpoints are homoclinically related to $p \in \gamma_Y$, where $g$ is the Poincaré map of $Y$ and $\gamma_Y$ is continuation of $\gamma$.

Proof. Let $\eta \in H(\gamma, X) \cap \text{Per}(X)$ with $\eta \sim \gamma$. Suppose that $\eta \in \mathcal{W}\mathcal{H}(X)$. Take $q \in \eta$ and let $f : N_{q,r} \to N_q$ (for some $r > 0$) be the Poincaré map of $X$. Since $\eta$ is a weak hyperbolic periodic orbit of $X$, according to Lemma 2 there exist $\delta(= q) > 0$ and $0 < \epsilon_0 < r/2$ such that for a map $L : N_q \to N_q$ with $\|L - D_qf\| < \delta$, there exists $Y C^1$ closed to $X$ satisfying

(a) $Y(x) = X(x)$ if $x \notin U_q(X^i, r, \pi(q))$,
(b) $q$ belongs to $\eta \in \text{Per}(Y)$,
(c) $\eta \sim \gamma_Y$, and
(d) $g(x) = \begin{cases} 
\exp_q \circ L \circ \exp_q^{-1}(1), & \text{if } x \in B_{\epsilon_0/4}(q) \cap N_{q,r}, \\
\exp_f(x), & \text{if } x \notin B_{\epsilon_0}(q) \cap N_{q,r},
\end{cases}$

where, $g : N_{q,r} \to N_q$ is the Poincaré map of $Y$. Then we have that $\lambda$ is an eigenvalue of $D_qg$ with $|\lambda| = 1$. Take a vector $u(\neq 0)$ which is associated to the eigenvalue $\lambda$ and $\|u\| = \epsilon_0/4$. Then we obtain that $g(\exp_q(u)) = \exp_q \circ L \circ \exp_q^{-1}(\exp_q(u)) = \exp_q(u)$.

Let $J = \{tu : -\epsilon_0/4 \leq t \leq \epsilon_0/4\}$ and let $\gamma = \exp_q(J)$. Then $g^k|_{\gamma}$ is the identity, for some $k > 0$, and $g^i(x)$ goes to $W_{\text{loc}}^i(q)$ for $x \in J$ and $i \geq 0$. Let $q_1$ and $q_2$ be the endpoints of $J$. Then as in the proof of ([4], Proposition 3), there is $h C^1$ close to $g$ (also $C^1$ close to $f$) such that $q_1, q_2$ and $q$ are the only periodic points of $h$, $h^k|_{\gamma_i}$ is the identity and $q_1 \sim p_h \in \gamma_Z$ and $q_2 \sim p_h \in \gamma_Z$, where
\( \gamma_Z \) is the continuation of \( \gamma \) and the Poincaré map \( h \) is associated with a vector field \( Z \) which is \( C^1 \) close to \( Y \). Thus if \( \eta \in WH(X) \) then there is \( Z \in C^1 \) close to \( X \) such that \( h \) has a small arc \( J_1 \) with the endpoints \( q_1 \) and \( q_2 \) are homoclinically related to \( p_h \in \gamma_Z \), where \( h \) is the Poincaré map of \( Z \). This proves Lemma 3. \( \square \)

Let \( f : \mathcal{N}_p \to \mathcal{N}_p \) be the Poincaré map of \( X \in \mathcal{X}(M) \). The following is a vector field version of what introduced by Yanf and Gan in [14]. For any \( \epsilon > 0 \), a closed small arc \( I \) is \( \epsilon \)-periodic curve if
(a) \( f(I) = I \),
(b) the length of \( f(I) \leq \epsilon \) with the endpoints are hyperbolic,
(c) \( I \) is normally hyperbolic.

A subset \( G \subset \mathcal{X}(M) \) is called residual if it contains a countable intersection of open and dense subsets.

**Lemma 4.** There is a residual set \( G_1 \subset \mathcal{X}(M) \) such that for any \( X \in G_1 \), any hyperbolic periodic orbit \( \gamma \) of \( X \), and given \( \epsilon > 0 \), if any \( C^1 \) neighborhood \( U(X) \) of \( X \), there is \( Y \in U(X) \subset \mathcal{X}(M) \) which has an \( \epsilon \)-periodic curve \( \gamma \) for which the two endpoints of \( \gamma \) are homoclinically related to \( \gamma \). Then \( X \) has a 2\( \epsilon \)-periodic curve \( L \) for which the two endpoints of \( L \) are homoclinically related to \( \gamma \) (see [14]).

For any \( x, y \in M \), and \( \delta > 0 \), a sequence \( \{x_0 (= x), x_1, x_2, \ldots, x_n (= y)\} \subset M \) is \( \delta \)-chain from \( x \) to \( y \) if there exist \( t_i \geq 1 \) such that \( d(x_i, x_{i+1}) < \delta \) for \( i = 0, 1, \ldots, n-1 \). We say that \( y \) is chain-attainable from \( x \) if for any \( \delta > 0 \), there is a \( \delta \)-chain from \( x \) to \( y \). A point \( x \in M \) is a chain recurrence point if \( x \) is chain-attainable from itself. Denote by \( CR(X) \) the set of all chain recurrence points of \( X \). Note that chain bi-attainability is closed equivalence relation in \( CR(X) \). For any \( x \in CR(X) \), the equivalent class which has \( x \) is said to be the chain recurrence class of \( x \), and we denote by \( C(x, X) \) or \( C(\text{Orb}(x), X) \).

**Lemma 5.** There is a residual set \( G_2 \subset \mathcal{X}(M) \) such that for any \( X \in G_2 \),
(a) \( X \) is Kupka–Smale, that is, any critical orbit is hyperbolic and \( W^a(\sigma) \) is transverse to \( W^u(\sigma) \), where \( \sigma \) and \( \eta \) are critical orbits of \( X \) (see [26]).
(b) \( H(\gamma, X) = C(\gamma, X), \) for some hyperbolic periodic orbit \( \gamma \) (see [27]).

**Lemma 6.** There is a residual set \( G_3 \subset \mathcal{X}(M) \) such that for any \( X \in G_3 \), if \( H(\gamma, X) \) is measure-expansive then for any \( \eta \in H(\gamma, X) \cap Per(X) \) with \( \eta \sim \gamma \),
\[
H(\gamma, X) \cap WH(X) = \emptyset.
\]

**Proof.** Let \( X \in G_3 = G_1 \cap G_2 \), and let \( H(\gamma, X) \) be measure-expansive. Suppose that there is \( \eta \in H(\gamma, X) \cap Per(X) \) with \( \eta \sim \gamma \) such that \( \eta \in H(\gamma, X) \cap WH(X) \). Take \( p \in \eta \) such that \( p \) is a weak hyperbolic point of \( X \). According to Lemma 3, for any \( \delta > 0 \) there is \( Y \subset C^1 \) close to \( X \) such that \( g \) has a small arc \( J \) with the endpoints are homoclinically related to \( p_Y \in \gamma_Y \) which is a \( \delta \)-periodic curve, and \( g^k \mid J : J \to J \) is the identity map for some \( k > 0 \), where \( g \) is the Poincaré map of \( Y \) and \( \gamma_Y \) is continuation of \( \gamma \). For simplicity, we assume that \( g^k \mid J = g \mid J \). It is known that \( J \subset C(p_Y, g) \subset C(\gamma_Y, Y) \). By Lemma 4, one can see that \( X \) has a 2\( \epsilon \)-periodic curve \( L \) for which the two endpoints of \( L \) are homoclinically related to \( \gamma \). By Lemma 5 (b), \( f \) has a 2\( \epsilon \)-periodic curve \( L \subset H(\gamma, X) = C(\gamma, X) \). Let \( \nu \) be the normalized Lebesgue measure on \( L \). Define a measure \( \chi \) on \( M \) by
\[
\chi(B) = \nu_L(f^{-i}(B \cap f^i(L)))(i \in \mathbb{Z})
\]
for some Borel set \( B \) of \( M \). It is clear that \( \chi(B) \neq 0 \). Take \( \epsilon = \delta \), and let \( x \in L \). Then we define \( \Gamma(x, e) = \{ y \in M : d(f^i(x), f^i(y)) \leq \epsilon \} \) for all \( i \in \mathbb{Z} \). Let \( \Phi(x, e) = \{ z \in L : d(f^i(x), f^i(y)) \leq \epsilon \} \) for all \( i \in \mathbb{Z} \). It is clear that \( \Phi(x, e) \subset \Gamma(x, e) \). Since \( H(\gamma, X) \) is measure-expansive, we know that \( \chi(\Gamma(x, e)) = 0 \). Since \( \Phi(x, e) \subset \Gamma(x, e) \), \( \chi(\Phi(x, e)) \) should be 0. This is a contradiction since \( \chi(\Phi(x, e)) \neq 0 \). \( \square \)
Let \( p \in \gamma \) be a hyperbolic periodic point of \( X \) with the period \( \pi(p) \), and let \( f : N_{p,X} \to N_p \) be the Poincaré map with respect to \( X \). Then if \( \mu_1, \mu_2, \ldots, \mu_{n-1} \) are the eigenvalues of \( D_p f \), then

\[
\xi_i = \frac{1}{\pi(p)} \log |\mu_i|,
\]

for \( i = 1, 2, \ldots, n - 1 \) are called the Lyapunov exponents of \( p \).

Denote by \( H^*(\gamma, X) = \{ x \in M : x \in H(\gamma, X) \text{ and } x \notin \text{Sing} \} \). For vector fields, we assume that \( H^*(\gamma, X) \). Wang proved in [28] that a vector field \( X \) in a dense \( G_\delta \) subset of \( X(M) \), if a homoclinic class \( H^*(\gamma, X) \) is not hyperbolic, then one can find a periodic orbit \( \xi \) of \( X \) that is homoclinically related to \( \gamma \) and has a Lyapunov exponent arbitrarily close to 0.

Hereafter, we say that a property holds for \( C^1 \) generic vector fields if it is satisfied on a dense \( G_\delta \) subset of \( X(M) \). Note that for a \( C^1 \) generic vector field \( X \), if a periodic orbit \( \gamma \) has a Lyapunov exponent arbitrarily close to 0, then one can take a periodic orbit \( \xi \) such that \( \xi \in WH(X) \).

Note that if \( H(\gamma, X) \cap \text{Sing}(X) \neq \emptyset \) then the result of Wang [28] is not true. Indeed, the geometric Lorenz attractor is an example of that. However, we consider \( H^*(\gamma, X) \), then, we can rewrite the result of Wang [28] as follows.

**Lemma 7.** There is a residual set \( G_4 \subset X(M) \) such that for any \( X \in G_4 \), if a homoclinic class \( H^*(\gamma, X) \) is not hyperbolic, then there is a periodic orbit \( \eta \) of \( H^*(\gamma, X) \) such that \( \eta \sim \gamma \) and \( \eta \in WH(X) \).

**End of the proof of Theorem 1.** Let \( X \in G_\delta = G_3 \cap G_4 \) and \( H(\gamma, X) \) be measure expansive. According to Lemma 1, \( H(\gamma, X) \cap \text{Sing}(X) = \emptyset \). Since \( H(\gamma, X) \cap \text{Sing}(X) = \emptyset \), we will use Lemma 7. To prove, we will derive a contradiction. Suppose that \( H(\gamma, X) \) is not hyperbolic. Since \( X \in G_4 \), according to Lemma 7, there is \( \eta \in H(\gamma, X) \cap \text{Per}(X) \) with \( \eta \sim \gamma \) such that \( \eta \) is a weak hyperbolic periodic orbit of \( X \). Since \( H(\gamma, X) \) is measure-expansive, by Lemma 6, \( H(\gamma, X) \cap WH(X) = \emptyset \). This is a contradiction. Thus \( C^1 \) generically, if \( H(\gamma, X) \) is measure-expansive then \( H(\gamma, X) \) is hyperbolic.

Since an expansive flow is a measure-expansive flow, according to Theorem 1, \( C^1 \) generically, if a homoclinic class \( H(\gamma, X) \) which contains a hyperbolic periodic orbit \( \gamma \) is expansive, then \( H(\gamma, X) \) is hyperbolic. Thus we have the following, which is a generalization of the result of [20].

**Corollary 1.** For \( C^1 \) generic \( X \in X(M) \), a homoclinic class \( H(\gamma, X) \) which contains a hyperbolic periodic orbit \( \gamma \) is expansive if and only if \( H(\gamma, X) \) is measure-expansive.

4. **Conclusions**

The paper considers the relationship between measure-expansiveness and homoclinic classes for flows. More in detail, we proved that there is an open and dense \( G_\delta \) in \( X(M) \) for any \( X \in G_\delta \), if a homoclinic class \( H(\gamma, X) \) is measure-expansive then \( H(\gamma, X) \) is hyperbolic for \( X \). The results are an extension and generalization of the previous results (see [6,8,13,14,16,20,21]).

**Funding:** This work is supported by the National Research Foundation of Korea (NRF) of the Korean government (MSIP) (No. NRF-2020R1F1A1A01051370).

**Acknowledgments:** The author would like to thank the referee for valuable help in improving the presentation of this article.

**Conflicts of Interest:** The author declares no conflict of interest.

**References**

1. Smale, S. Differentiable dynamical systems. *Bull. Am. Math. Soc.* **1967**, *73*, 747–817. [CrossRef]
2. Pacifico, M.J.; Pujals, E.R.; Sambarino, M.; Vieitez, J.L. Robustly expansive codimension-one homoclinic classes are hyperbolic. *Ergod. Theory Dynam. Syst.* **2009**, *29*, 179–200. [CrossRef]
3. Pacifico, M.J.; Pujals, E.R.; Vieitez, J.L. Robustly expansive homoclinic classes. Ergod. Theory Dynam. Syst. 2005, 25, 271–300. [CrossRef]
4. Sambarino, M.; Vieitez, J. On $C^1$-persistently expansive homoclinic classes. Discret. Contin. Dyn. Syst. 2006, 14, 465–481. [CrossRef]
5. Lee, K.; Lee, M. Hyperbolicity of $C^1$-stably expansive homoclinic classes. Discret. Contin. Dyn. Syst. 2010, 27, 1133–1145. [CrossRef]
6. Lee, M. Robustly measure expansiveness for $C^1$ Vector fields. Quaest. Math. 2020, 43, 569–582. [CrossRef]
7. Lee, M. Continuum-wise expansive homoclinic classes for robust dynamical systems. Adv. Differ. Equ. 2019, 2019, 333.
8. Lee, M. R-robustly measure expansive homoclinic classes are hyperbolic. J. Math. Compt. Sci. 2018, 18, 146–153. [CrossRef]
9. Lee, M. Continuum-wise expansiveness for generic diffeomorphisms. Nonlinearity 2018, 31, 2982–2988. [CrossRef]
10. Lee, M. Measure expansiveness for $C^1$ generic diffeomorphisms. Dynam. Syst. Appl. 2018, 27, 629–635.
11. Lee, M. General expansiveness for diffeomorphisms from the robust and generic properties. J. Dynam. Cont. Syst. 2016, 22, 459–464. [CrossRef]
12. Lee, M. Continuum-wise expansive homoclinic classes for generic diffeomorphisms. Publ. Math. Debr. 2016, 88, 193–200. [CrossRef]
13. Lee, M. Measure expansive homoclinic classes for generic diffeomorphisms. Appl. Math. Sci. 2015, 73, 3623–3628. [CrossRef]
14. Yang, D.; Gan, S. Expansive homoclinic classes. Nonlinearity 2009, 22, 729–733. [CrossRef]
15. Morales, C.A. Measure expansive systems Preprint IMPA. Unpublished work, 2011.
16. Koo, N.; Lee, K.; Lee, M. Generic diffeomorphisms with measure-expansive homoclinic classes. J. Differ. Equ. Appl. 2014, 20, 228–236. [CrossRef]
17. Bautista, S. The geometric Lorenz attractor is a homoclinic class. Bol. Mat. 2004, 11, 69–78.
18. Komuro, M. Expansive properties of Lorenz attractors. In The Theory of Dynamical Systems and Its Applications to Nonlinear Problems; World Scientific Publishing: Kyoto, Japan, 1984; pp. 4–26.
19. Araújo, V.; Pacifico, M.J. Three Dimensional Flows; Springer: Berlin/Heidelberg, Germany, 2010.
20. Lee, S.; Park, J. Expansive homoclinic classes of generic $C^1$-vector fields. Acta Math. Sin. (Engl. Ser.) 2016, 32, 1451–1458. [CrossRef]
21. Lee, M.; Oh, J. Measure expansive flows for generic view point. J. Differ. Equ. Appl. 2016, 22, 1005–1018. [CrossRef]
22. Bowen, R.; Walters, P. Expansive one-parameter flows. J. Differ. Equ. 1972, 12, 180–193. [CrossRef]
23. Oka, M. Expansiveness of real flows. Tsukuba J. Math. 1990, 14, 1–8. [CrossRef]
24. Carrasco-Olivera, D.; Morales, C.A. Expansive measures for flows. J. Differ. Equ. 2014, 256, 2246–2260.
25. Moriyasu, K.; Sakai, K.; Sumi, N. Vector fields with topological stability. Trans. Am. Math. Soc. 2001, 353, 3391–3408. [CrossRef]
26. Kupka, I. Contribution à la théorie des champs génériques. Contrib. Differ. Equ. 1963, 2, 457–484.
27. Bonatti, C.; Crovisier, S. Récurrence et généricité. Invent. Math. 2004, 158, 180–193. [CrossRef]
28. Wang, X. Hyperbolicity versus weak periodic orbits inside homoclinic classes. Ergod. Theory Dynam. Syst. 2018, 38, 2345–2400. [CrossRef]