A metric-affine version of the Horndeski theory

Thomas Helpin
Institut Denis Poisson, UMR - CNRS 7013,
Université de Tours, Parc de Grandmont, 37200 Tours, France
thomas.helpin@lmpt.univ-tours.fr

Mikhail S. Volkov
Institut Denis Poisson, UMR - CNRS 7013,
Université de Tours, Parc de Grandmont, 37200 Tours, France
and
Department of General Relativity and Gravitation, Institute of Physics,
Kazan Federal University, Kremlevskaya street 18, 420008 Kazan, Russia
michael.volkov@idpoisson.fr

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We study the metric-affine versions of scalar-tensor theories whose connection enters the action only algebraically. We show that the connection can be integrated out in this case, resulting in an equivalent metric theory. Specifically, we consider the metric-affine generalisations of the subset of the Horndeski theory whose action is linear in second derivatives of the scalar field. We determine the connection and find that it can describe a scalar-tensor Weyl geometry without a Riemannian frame. Still, as this connection enters the action algebraically, the theory admits the dynamically equivalent (pseudo)-Riemannian formulation in the form of an effective metric theory with an extra K-essence term. This may have interesting phenomenological applications.

Keywords: Horndeski theory; Palatini approach.

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1. Introduction

Within the framework of scalar-tensor theories of Horndeski\cite{1} and beyond Horndeski\cite{2}\cite{3} and in the Degenerate Higher Order Scalar-Tensor (DHOST) theories\cite{4}\cite{5} one assumes the connection to be Levi-Civita. All of these theories evade the Ostrogradsky ghost\cite{6}. The Horndeski theory is the most general scalar-tensor theory with second order equations of motion. Recently it was rediscovered via the covariantization of the Galileon\cite{7}\cite{8} and was found to be equivalent to the Generalized Galileon theories\cite{9}.

In this text we consider the MAG (Metric-Affine Gravity) generalisations of the Horndeski theory. In these theories the gravitational interaction is encoded in two a priori independent fields: the space-time metric $g_{\mu\nu}$ and the distortion
tensor $C^{\alpha}_{\mu\nu}$. The latter characterizes the deviation of the independent connection $\Gamma^{\alpha}_{\mu\nu} = \{\alpha_{\mu\nu}\}_g + C^{\alpha}_{\mu\nu}$ from the Levi-Civita connection $\{\alpha_{\mu\nu}\}_g$ associated with the spacetime metric. The non-Riemannian part of the connection may have two independent origins: the torsion $T^{\alpha}_{\mu\nu} = 2\Gamma^{\alpha}_{\mu\nu} \{\} + \xi_{\mu} \delta^{\alpha}_{\nu}$. This symmetry acts on the vectorial part of the connection, and for consistency reasons any metric-affine theory containing the Einstein-Hilbert term must be projectively invariant. There is currently a growing interest in the metric-affine scalar-tensor theories. In most of the previously studied cases the Levi-Civita covariant derivatives of the scalar field have been replaced by the covariant derivatives with respect to the independent connection in the “minimal” way: $\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} \phi \rightarrow \Gamma_{\mu}^{\alpha} \Gamma_{\nu}^{\beta} \phi$. We wish to study the effect of relaxing this minimal prescription.

2. General Setting

The Ricci tensor associated to the independent connection can be decomposed as $\hat{R}^{\mu\nu} = \hat{R}_{\mu\nu} + N_{\mu\nu}$ where

$$N_{\mu\nu} = \hat{\nabla}_{\alpha} C^{\alpha}_{\nu\mu} - \hat{\nabla}_{\nu} C^{\alpha}_{\mu\alpha} + C^{\alpha}_{\mu\lambda} C^{\lambda}_{\nu\mu} - C^{\alpha}_{\nu\lambda} C^{\lambda}_{\mu\alpha}$$

encodes the contribution of the non-Riemannian part of the connection. Similarly, the Lagrangian of any standard metric-affine scalar-tensor theory $\mathcal{L}(g, \Gamma, \partial \Gamma, \phi, \nabla \phi, \ldots , (\hat{\nabla} \nabla \phi)^n)$, where $n$ is the maximal power of second derivatives, can be decomposed into the metric part $\mathcal{L}_g$ and the distortion part $\mathcal{L}_C$:

$$\mathcal{L} = \mathcal{L}_g(g, \phi, \partial g, \phi, \nabla \phi, \ldots , (\hat{\nabla} \nabla \phi)^n)$$

$$+ \mathcal{L}_c(g, \phi, \nabla \phi, \ldots , (\hat{\nabla} \nabla \phi)^{n-1}, C, \ldots , C^n, \hat{\nabla} C).$$

When the action contains only terms linear in curvature and when they do not couple to the second derivatives $\hat{\nabla} \nabla \phi$, then the distortion enters the action solely algebraically. In this case it can be treated as an auxiliary field. The on-shell value of the distortion tensor will depend on $(g, \phi, \nabla \phi, \ldots , (\hat{\nabla} \nabla \phi)^{n-1})$ and can be integrated out from the action resulting in a dynamically equivalent metric action. Let us clarify this statement by showing that the equations of motion for the scalar field are indeed

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\(^{a}\)The Riemannian structure of a smooth manifold is related to the metric only, and hence the geometry of the metric-affine theory is always (pseudo-)Riemannian. The “non-Riemann” nature of the theory lies in the assumption that in addition to the Levi-Civita connection there may exist a different gravitational connection which couples to the matter and which should be determined via the least action principle.

\(^{b}\)It is still possible to explicitly break the projective symmetry of metric-affine scalar tensor actions with Lagrange multipliers. [11]
equivalent in the metric-affine and effective metric descriptions, at least if \( n = 1 \).

Let \( S_{\text{eff}}[g, \phi] = S_g[g, \phi] + \tilde{S}_C[g, \phi, C(g, \phi, \nabla \phi)] \) denote the effective metric theory obtained after integrating out the connection. Varying \( S \) and \( S_{\text{eff}} \) with respect to the scalar field yields

\[
E_{\phi} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \phi} = \frac{1}{\sqrt{-g}} \left( \frac{\delta S_g}{\delta \phi} + \frac{\delta \tilde{S}_C}{\delta \phi} \right),
\]

\[
\tilde{E}_{\phi} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{eff}}}{\delta \phi} = \frac{1}{\sqrt{-g}} \left( \frac{\delta S_g}{\delta \phi} + \frac{\delta \tilde{S}_C}{\delta \phi} \right).
\]

(3)

By definition,

\[
\frac{\delta \tilde{S}_C}{\delta \phi} = \sqrt{-g} \frac{\partial \tilde{L}_C}{\partial \phi} - \partial_{\alpha} \left( \frac{\partial (\sqrt{-g} \tilde{L}_C)}{\partial C_{\alpha \mu \nu}} \frac{\partial C_{\beta \mu \nu}}{\partial \phi} \right) - \partial_{\alpha} \left( \frac{\partial (\sqrt{-g} \tilde{L}_C)}{\partial C_{\beta \mu \nu}} \frac{\partial C_{\alpha \mu \nu}}{\partial \phi} \right).
\]

(4)

Let us emphasize that the distortion in \( \tilde{L}_C(g, \phi, \nabla \phi, C) \) is considered as an independent variable, whereas in \( \tilde{L}_C(g, \phi, \nabla \phi, C(g, \phi, \nabla \phi)) \) it is a function of \( \phi \) and \( \nabla \phi \).

Using the chain rule one has,

\[
\frac{\delta \tilde{S}_C}{\delta \phi} = \sqrt{-g} \frac{\partial \tilde{L}_C}{\partial \phi}_{\mid C} - \partial_{\alpha} \left( \frac{\partial (\sqrt{-g} \tilde{L}_C)}{\partial C_{\alpha \mu \nu}} \frac{\partial C_{\beta \mu \nu}}{\partial \phi} - \partial_{\alpha} \left( \frac{\partial (\sqrt{-g} \tilde{L}_C)}{\partial C_{\beta \mu \nu}} \frac{\partial C_{\alpha \mu \nu}}{\partial \phi} \right) \right).
\]

(5)

When the distortion tensor appears in the action algebraically, its field equation is equivalent to \( \frac{\partial \tilde{L}_C}{\partial C_{\alpha \mu \nu}} = 0 \). Therefore we conclude that the equations of motion for the scalar field are the same in both cases: \( E_{\phi} = \tilde{E}_{\phi} \). This proof generalizes to the equations for the metric, and also to any theory with \( n > 1 \) whose action is algebraic in the connection. On the other hand, theories where the second derivatives of the scalar field couple to the curvature terms give rise to first order differential equations for the distortion tensor. For example, any theory whose Lagrangian contains the term

\[
G_5(\phi, X)G^{\alpha \beta} \nabla_{\alpha} \nabla_{\beta} \phi
\]

should support a dynamical connection. Whether or not such theories can be ghost-free is an interesting question, which is however beyond the scope of this paper. We shall now fix the definitions and notation needed in order to solve and analyze the field equations for the distortion tensor.

The Palatini tensor is the variation of the Einstein-Hilbert term with respect to the connection,

\[
P_{\alpha \mu \nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{EH}}}{\delta C_{\alpha \mu \nu}} = \frac{\partial N}{\partial C_{\alpha \mu \nu}},
\]

(7)

with \( N = g^{\mu \nu} N_{\mu \nu} \). If the connection enters the action only algebraically, its equation may be written as

\[
P_{\alpha \mu \nu}(C) = B_{\alpha \mu \nu}(g, \partial \phi, \ldots, (\nabla \nabla \phi)^{n-1}, C, \ldots, C^n).
\]

(8)
From the projective invariance of $\bar{R}$ it follows that \[ P_{\alpha}^{\mu} = 0 \] and therefore the above equations are inconsistent, unless $B_{\lambda}^{\alpha} = 0$. Assuming the latter condition amounts to require the projective invariance of the action $S_C$. Besides, one can show that the Palatini tensor is

\[
P_{\alpha\mu} = Q_{\alpha\mu} - \frac{1}{2} g_{\mu\nu} Q_\alpha + g_{\mu\alpha} \left( \frac{1}{2} Q_\nu - \dot{Q}_\nu \right) + T_{\mu\alpha\nu} - g_{\mu\nu} T_\alpha + g_{\mu\alpha} T_\nu,
\]

where the Weyl vectors are $Q^\alpha \equiv Q^\alpha_{\mu} \equiv Q_{\mu}^{\alpha}$ and $\dot{Q}_\alpha \equiv Q^\mu_{\mu\alpha}$ and the torsion vector is $T_\alpha \equiv T^\alpha_{\alpha\beta}$. Before going further we note that \[ \text{(9)} \] can be inverted to express the distortion tensor $C_{\mu\nu}^{\alpha}$ in term of the Palatini tensor, \[ \text{(9)} \]

\[
C_{\mu\nu}^{\alpha} = \frac{1}{2} \left( P_{\mu}^{\alpha} - P_{\nu}^{\alpha} - P_{\nu}^{\alpha} \right) + \frac{1}{4} g_{\mu\nu} \left( P_{\lambda}^{\alpha} - P_{\alpha}^{\lambda} \right) - \frac{1}{4} \delta_{\mu}^{\alpha} \left( P_{\lambda}^{\beta} - P_{\beta}^{\lambda} \right) + \frac{1}{4} \delta_{\nu}^{\alpha} \left( P_{\lambda}^{\beta} + \frac{1}{4} P_{\mu}^{\lambda} \right) + \frac{1}{3} \delta_{\mu\nu} T_\alpha.
\]

Therefore \[ \text{(8)} \] can also be seen as equations for $P_{\mu}^{\alpha}$, while \[ \text{(9)} \] expresses the distortion tensor in term of $P_{\mu}^{\alpha}$, or equivalently in terms of $B_{\mu\nu}^{\alpha}$, after using \[ \text{(8)} \]. The projective symmetry can be used to impose the gauge condition $T_\mu = 0$.

### 3. Construction of the Theory

In the parametrization of the Generalized Galileon \[ \text{(2)} \] the action of the metric Horndeski theory contains four arbitrary functions $G_i(\phi, X)$ ($i = 2, 3, 4, 5$) depending on $\phi$ and on $X = g_{\alpha\beta} \partial_\beta \phi \partial_\alpha \phi$. The action also contains the second covariant derivatives of the scalar field

\[
^{(0)} \Phi_{\alpha\beta} = \nabla_\alpha \nabla_\beta \phi,
\]

and also their powers up to the cubic order ($n = 3$). The theory is determined by the Lagrangian \[ \mathcal{L}_H = \sqrt{-g} \left( \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right) \]

\[
\mathcal{L}_2 = G_2, \quad \mathcal{L}_3 = G_3^{(0)} \Phi, \quad \mathcal{L}_4 = G_4 R - 2 G_4 X \left( ^{(0)} \Phi^2 - \left( ^{(0)} \Phi_{\alpha\beta} \right)^2 \right), \quad \mathcal{L}_5 = G_5 G_3^{(0)} \Phi_{\alpha\beta} + \frac{1}{3} G_5 X \left( ^{(0)} \Phi^3 - 3 ^{(0)} \Phi ( ^{(0)} \Phi_{\alpha\beta} )^2 + 2 ^{(0)} \Phi_{\alpha\beta} \right). \]

Here $^{(0)} \Phi_{\alpha\beta} = g_{\alpha\beta} \phi$. \[ ^{(0)} \phi \] and \[ ^{(0)} \Phi_{\alpha\beta} = g_{\alpha\beta} \phi \] are total derivatives in a metric theory, while $\mathcal{R} = g_{\mu\nu} R_{\mu\nu} (g)$. Let us stress that a metric theory may have various formulations which are equivalent up to a total derivative, but due to the non-metricity and torsion this equivalence may not hold in its MAG versions. For example the terms $\nabla_\mu \left( G_3 \nabla^\mu \phi \right)$ and $R_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - \left( ^{(0)} \Phi^2 + \left( ^{(0)} \Phi_{\alpha\beta} \right)^2 \right)$ are total derivatives in a metric theory but not in MAG theories. Hence a metric-affine formulation of the Horndeski theory based on a chosen metric parametrization will be just one of many other possible MAG extensions of the theory.

In the metric-affine context the definition of the second derivatives of the scalar field is not unique because of the non-metricity. The possible independent second
order covariant derivative operators which reduce to (11) in the Levi-Civita limit are

\[
\Phi_\alpha^\beta = \Gamma_{\alpha}^{\gamma} \nabla^\gamma \nabla^\beta \phi,
\]
(13)

\[
\Phi_\alpha^\beta = g_{\rho\beta} \Gamma_{\alpha}^{\mu} \nabla^\mu \nabla^\rho \phi,
\]
(2)

\[
\Phi_\alpha^\beta = \frac{1}{4} g_{\mu\nu} \Gamma_{\alpha}^{\mu} \nabla^\nu \nabla^\rho \phi,
\]
(3)

\[
\Phi_\alpha^\beta = \nabla_\rho \left( g^{\rho\alpha} \nabla^\beta \phi \right),
\]
(4)

with the convention \( \nabla^\mu \phi \equiv g^{\mu\nu} \nabla^\nu \phi \). These definitions can be recast in terms of the “minimal” operator

\[
\Phi_\alpha^\beta = (0) \Phi_\alpha^\beta + C_{\mu\alpha} \partial_\mu \phi\] plus terms proportional to the non-metricity:

\[
\Phi_\alpha^\beta = (1) \Phi_\alpha^\beta - Q_{\alpha}^\beta \partial_\mu \phi,
\]
(3)

\[
\Phi_\alpha^\beta = (2) \Phi_\alpha^\beta - \tilde{Q}^\alpha \partial_\beta \phi,
\]
(4)

\[
\Phi_\alpha^\beta = (3) \Phi_\alpha^\beta - \frac{1}{4} Q^\alpha \partial_{\alpha} \phi.
\]
(15)

Note that the \( (i) \Phi_{\alpha\beta} = g_{\alpha\sigma} (i) \Phi^\sigma_\beta \) tensors are not symmetric, hence one should be careful with the order of indices.

It seems natural that the MAG versions of the Horndeski theory should respect the following conditions:

(i) \( \mathcal{L}_g(g, \partial g, \partial \partial g, \phi, \nabla\phi, \ldots, (\nabla \nabla \phi)^n) \) must have the structure of the original metric Horndeski theory.

(ii) \( \mathcal{L}_C(g, \phi, \nabla\phi, \ldots, (\nabla \nabla \phi)^{n-1}, C, \ldots, C^n, \nabla C) \) must originate from generalized curvature tensors related to the independent connection and from a consistent replacement \( \{ (0) \Phi_\alpha^\beta \} \rightarrow \{ (i) \Phi_\alpha^\beta \} \) in the original metric action. The terms \( (i) Y = \partial_\alpha \phi \partial^\beta \phi (i) \Phi_{\alpha\beta} \) may also enter the action.

The most general MAG Lagrangian \( \mathcal{L}^{MA}_3 \) resulting from the metric-affine extension (i)-(ii) of \( \mathcal{L}_3 \) is constructed from \( \{ (1) \Phi, (2) \Phi, (3) \Phi, (1) Y, (2) Y \} \) and can be written after integration by part as

\[
\mathcal{L}^{MA}_3 = G_3(\phi, X) (0) \Phi + C_1 Q^\alpha \partial_\alpha \phi + C_2 \tilde{Q}^\alpha \partial_\alpha \phi + C_3 Q^{\mu\nu} \partial_\alpha \phi \partial^\mu \phi \partial^\nu \phi + C_4 \partial_\alpha \phi T^\alpha,
\]
(16)

where the \( C_i \) are functions of \( \phi \) and \( X \). Within the \( n = 1 \) Horndeski class, the generalized \( \mathcal{L}_4 \) part compatible with our requirements (i)-(iii) is obtained by setting in (12) \( G_4 = G_4(\phi) \), hence

\[
\mathcal{L}^{MA}_4 = G_4(\phi) \, R
\]
(17)

\(^{\text{c}}\text{The traces and contractions with the derivatives of the scalar field of the operators defined in (13-14) are not all independent. One has for example } (2) \Phi = (4) \Phi.\)
with $\bar{R} = g^{\mu\nu} R_{\mu\nu}$. Adding the $G_3$ terms would be incompatible with our assumptions. As a result, including also the $L_2$ term, yields the MAG action

$$S^{\text{MA}} = \int d^4x \sqrt{-g} \left( L_4^{\text{MA}} + L_3 \right).$$

(18)

A subset of this theory build solely upon the scalars $\Phi_1$ and $\Phi_2$ was studied in Ref. [19] for particular models in the context of inflation.

4. Solution for the Connection

Varying (18) with respect to $C_{\alpha}^{\mu\nu}$ yields

$$P_{\alpha\mu\nu} = B_{\alpha\mu\nu}$$

(19)

where

$$B_{\alpha\mu\nu} = \frac{1}{G_4} \left( \Delta_{\alpha\mu\nu} + G_4 \phi (g_{\mu\nu} \nabla_\alpha \phi - g_{\alpha\mu} \nabla_\nu \phi) \right)$$

(20)

with

$$\Delta_{\alpha\mu\nu} = C_2 g_{\mu\nu} \nabla_\alpha \phi + \left( 2C_1 - C_4 \right) g_{\alpha\nu} \nabla_\mu \phi + \left( 2C_1 + C_4 \right) g_{\alpha\mu} \nabla_\nu \phi.$$  

(21)

The projective invariance condition $B_{\alpha}^{\mu\nu\alpha} = 0$ imposes $C_1 = -\frac{1}{4} \left( C_2 + C_3 X - \frac{3}{2} C_4 \right)$ and the solution for the distortion tensor is then obtained by injecting (19) in (10):

$$C_{\alpha}^{\mu\nu} = -\frac{1}{4G_4} \left( 4C_3 \nabla^\alpha \phi \nabla_\nu \phi + g_{\mu\nu} \nabla^\alpha \phi (2G_4 \phi - C_2 - C_3 X - \frac{3}{2} C_4) 
- 2\delta_{\mu}^{\alpha} \nabla_\nu \phi (2G_4 \phi + C_2 + C_3 X - \frac{1}{2} C_4) \right) + \frac{1}{3} \delta_{\mu}^{\alpha} T_\mu.$$  

(22)

A particularly interesting subset of solutions corresponds to the choice $C_3 = 0$ and $C_4 = -2C_2$. In this class of theories, setting $T_\mu = 0$, the solution for the distortion tensor reduces to

$$C_{\alpha}^{\mu\nu} = -g_{\mu\nu} A^\alpha + 2\delta_{\mu}^{\alpha} A_\nu$$  \hspace{1cm} \text{with} \hspace{1cm} A_\mu = \frac{1}{2G_4} (G_4 \phi + C_2) \partial_\mu \phi,$$

(23)

and the non-metricity is

$$Q_{\alpha\mu\nu} = \bar{\nabla}_\alpha g_{\mu\nu} = -2A_\alpha g_{\mu\nu}.$$  

(24)

A geometry defined by a family of conformally related pseudo-Riemannian metrics with a connection respecting the compatibility condition (24) is called Weyl geometry. The connection $\Gamma_{\alpha}^{\mu\nu}$ is invariant under local Weyl transformations

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{2\Lambda(x)} g_{\mu\nu}, \hspace{1cm} \text{and} \hspace{1cm} A_\alpha \rightarrow \tilde{A}_\alpha = A_\alpha - \partial_\alpha \Lambda(x).$$

(25)

There exist other gauge choices such that the connection takes the generalized Weyl form.
The metric-affine theory respecting these conditions is

$$S_{\text{Weyl}} = \int d^4x \sqrt{-g} \left( \mathcal{L}_H + G_4 N + C_2 \left( \bar{Q}^\alpha - Q^\alpha - 2T^\alpha \right) \partial_\alpha \phi \right), \quad (26)$$

where $\mathcal{L}_H$ is the Levi-Civita limit of $\mathcal{L}_4^{\text{MA}} + \mathcal{L}_3^{\text{MA}} + \mathcal{L}_2$ in (18). When $C_2$ vanishes then the 1-form $A_\mu = \frac{1}{2} \partial_\mu \ln G_4(\phi)$ is exact. A Weyl geometry where $A_\mu$ is exact is called *Weyl integrable*. This subset of Weyl geometries is intimately related to the existence of a covariantly constant conformally related metric,

$$\tilde{g}_{\mu\nu} = G_4(\phi) g_{\mu\nu}, \quad \nabla_\alpha \tilde{g}_{\mu\nu} = 0.$$ \quad (27)

In other words, the connection $\Gamma^\alpha_{\mu\nu}$ is totally determined by the metric $\tilde{g}$, in which case the latter is said to define the Riemannian frame. On the contrary, if $C_2 \neq 0$ then $A_\mu$ is not exact and the connection $\Gamma^\alpha_{\mu\nu}$ is not Levi-Civita. Still, as we saw previously, the theory admits a dynamically equivalent effective metric description.

### 5. Dynamically Equivalent Metric Theory

Integrating the connection out from (18) via setting it to the value (22), we obtain the Lagrangian of the dynamically equivalent metric theory,

$$\tilde{L}_H = \sqrt{-\tilde{g}} \left( \mathcal{L}_H + K_{\text{eff}}(\phi, X) \right), \quad (28)$$

where the K-essence term is

$$K_{\text{eff}} = \frac{X}{2G_4} \left( (4C_3)^2 - 9(2C_2 + C_4 + \frac{2}{3}XC_3)^2 + 3(2C_4 - 4G_4\phi)^2 \right). \quad (29)$$

This Lagrangian actually belongs to the Horndeski family, hence the theory is free from the Ostrogradsky ghost. If the connection assumes the Weyl form (23) then one has

$$K_{\text{eff}} = \frac{3X}{2G_4} (C_2 + G_4\phi)^2.$$ \quad (30)

Remarkably, there is a non trivial yet simple limit where $K_{\text{eff}}$ goes to zero:

$$C_2 \rightarrow -XC_3 - G_4\phi, \quad C_4 \rightarrow 2G_4\phi, \quad \Rightarrow \quad C^\alpha_{\mu\nu} \rightarrow -\frac{1}{G_4} C_3 \nabla^\alpha \phi \nabla_\mu \phi \nabla_\nu \phi.$$ \quad (31)

The total equivalence between the effective metric theory and the original metric Horndeski theory in this limit ($\tilde{L}_H = L_H$) results from the fact that for any distortion tensor of the form $C^\alpha_{\mu\nu} = \xi^\alpha \xi_\mu \xi_\nu$, where $\xi$ is an arbitrary vector fields, the non-Riemannian part of the Ricci scalar vanishes: $\tilde{\mathcal{N}} = 0$.

### 6. Conclusion

We analysed the metric-affine scalar-tensor theory linear in second derivatives of the scalar field obtained by relaxing the minimal derivative coupling prescription. The full theory has never been studied before, and it may have interesting applications, for example in cosmology. It could be interesting to see how the introduction of all the $\Phi^{\alpha}_{\beta}$ would impact the results of Refs. 16 and 19.
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