TOPOLOGICAL COMPUTATION OF THE STOKES MATRICES OF
THE WEIGHTED PROJECTIVE LINE \( \mathbb{P}(1,3) \)

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Abstract. The localized Fourier–Laplace transform of the Gauß–Manin system
of \( f : \mathbb{G}_m \to \mathbb{A}^1, x \mapsto x + x^{-3} \) is a \( \mathcal{D}_{\mathbb{G}_m} \)-module, having a regular singularity
at 0 and an irregular one at \( \infty \). By mirror symmetry it is closely related to
the quantum connection of the weighted projective line \( \mathbb{P}(1,3) \). Following [6] we
compute its Stokes multipliers at \( \infty \) by purely topological methods. We compare
it to the Gram matrix of the Euler–Poincaré pairing on \( D^b(\text{Coh}(\mathbb{P}(1,3))) \).

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Introduction

In [6], A. D’Agnolo, M. Hien, G. Morando and C. Sabbah describe how to compute
the Stokes multipliers of the enhanced Fourier–Sato transform of a perverse sheaf on
the affine line by purely topological methods. To a regular holonomic \( \mathcal{D} \)-module
\( M \in \text{Mod}_{rh}(\mathcal{D}_{\mathbb{A}^1}) \) on the affine line one associates a perverse sheaf via the regular
Riemann–Hilbert correspondence
\[
\text{RHom}(\bullet, \mathcal{O}^{an})[1] : \text{Mod}_{rh}(\mathcal{D}_{\mathbb{A}^1}) \xrightarrow{\sim} \text{Perv}(\mathcal{C}_{\mathbb{A}^1}).
\]
Let \( \Sigma \subset \mathbb{A}^1 \) denote the set of singularities of \( M \). Following [6, Section 4.2], after
suitably choosing a total order on \( \Sigma \), the resulting perverse sheaf \( F \in \text{Perv}_\Sigma(\mathcal{C}_{\mathbb{A}^1}) \)
can be described by linear algebra data, namely the quiver
\[
(\Psi(F), \Phi_\sigma(F), u_\sigma, v_\sigma)_{\sigma \in \Sigma},
\]
where \( \Psi(F) \) and \( \Phi_\sigma(F) \) are finite dimensional \( \mathbb{C} \)-vector spaces and \( v_\sigma : \Psi(F) \to \Phi_\sigma(F) \)
and \( v_\sigma : \Phi_\sigma(F) \to \Psi(F) \) are linear maps s.t. \( 1 - u_\sigma v_\sigma \) is invertible for any \( \sigma \). The main
result in [6] is a determination of the Stokes multipliers of the enhanced Fourier–Sato
transform of \( F \) and therefore of the Fourier–Laplace transform of \( M \) in terms of the
quiver of \( F \).

Mirror symmetry connects the weighted projective line \( \mathbb{P}(1,3) \) with the Landau–
Ginzburg model
\[
\left( \mathbb{G}_m, f = x + \frac{1}{x^3} \right).
\]
The quantum connection of $\mathbb{P}(1,3)$ is closely related to the Fourier–Laplace transform of the Gauß–Manin system $H^0(\mathcal{f}_f \mathcal{O})$ of $f$. We compute that

$$F = Rf_* \mathbb{C}[1] \in \text{Perv}_\Sigma(\mathbb{C}_h),$$

where $\Sigma$ denotes the set of singular values of $f$, is the perverse sheaf associated to $H^0(\mathcal{f}_f \mathcal{O})$ by the Riemann–Hilbert correspondence. In section 1 we compute the localized Fourier–Laplace transform of $f$. In section 2, analogous to the examples in [6, Section 7], we carry out the topological computation of the Stokes multipliers of the Fourier–Laplace transform of $H^0(\mathcal{f}_f \mathcal{O})$. In section 3 we compare the Stokes matrix $S_\mathcal{O}$ that we obtained from our topological computations to the Gram matrix of the Euler–Poincaré pairing on $D^b(\text{Coh}(\mathbb{P}(1,3)))$ w.r.t. a suitable full exceptional collection. Following Dubrovin’s conjecture about the Stokes matrix of the quantum connection, proven for the weighted projective space $\mathbb{P}(\omega_0, \ldots, \omega_n)$ in [16] by S. Tanabé and K. Ueda and in [5] by J. A. Cruz Morales and M. van der Put, they are known to be equivalent after appropriate modifications. We give the explicit braid of the braid group $B_4$ that deforms the Gram matrix into the Stokes matrix $S_\mathcal{O}$.

1. Gauß–Manin system and its Fourier–Laplace transform

Let $X$ be affine and $f$ a regular function $f : X \to \mathbb{A}^1$ on $X$. Denote by $\mathcal{f}_f$ the direct image in the category of $\mathcal{D}$-modules and by $M := H^0(\mathcal{f}_f \mathcal{O}_X) \in \text{Mod}_{\text{rh}}(\mathcal{D}_{\mathbb{A}^1})$ the zeroth cohomology of the Gauß–Manin system of $f$. Following [8, Section 2.c] it is given by

$$M = \mathcal{O}^n(X)[\partial_\theta]/(d - \partial_\theta d f \wedge) \mathcal{O}^{n-1}(X)[\partial_\theta].$$

Denote by $G := \mathcal{M}[\tau^{-1}]$ the Fourier–Laplace transform of $M$ localized at $\tau = 0$. It is given by

$$G = \mathcal{O}^n(X)[\tau, \tau^{-1}]/(d - \tau d f \wedge) \mathcal{O}^{n-1}(X)[\tau, \tau^{-1}].$$

If $f$ fulfills some tameness condition, $G$ is a free $\mathcal{C}[\tau, \tau^{-1}]$-module of finite rank. Rewriting in the variable $\theta = \tau^{-1}$ gives the $\mathcal{C}[\theta, \theta^{-1}]$-module

$$G = \mathcal{O}^n(X)[\theta, \theta^{-1}]/(\theta d - df \wedge) \mathcal{O}^{n-1}(X)[\theta, \theta^{-1}].$$

$G$ is endowed with a flat connection given as follows. For $\gamma = \left[\sum_{k \in \mathbb{Z}} \omega_k \theta^k\right] \in G$, where $\mathcal{O}^n(X) \ni \omega_k = 0$ for a.a. $k$, the connection is given by (cf. [11, Def. 2.3.1]):

$$\theta^2 \nabla \frac{\partial}{\partial \theta}(\gamma) = \left[\sum_k f \omega_k \theta^k + \sum_k k \omega_k \theta^{k+1}\right].$$

It is known that $(G, \nabla)$ has a regular singularity at $\theta = \infty$ and possibly an irregular singularity at $\theta = 0$. Rewriting in $\tau = \theta^{-1}$ yields the irregular singularity at $\tau = \infty$.

We now consider the Laurent polynomial $f = x + x^{-3} \in \mathbb{C}[x, x^{-1}]$, being a regular function on the multiplicative group $\mathbb{G}_m$. Since $f$ is convenient and nondegenerate w.r.t. its Newton polytope $\Delta_\infty$ at $\infty$, $f$ is cohomologically tame (cf. [15, Part 2]). The localized Fourier–Laplace transform of the Gauß–Manin module $H^0(\mathcal{f}_f \mathcal{O})$ is therefore free of finite rank over $\mathcal{C}[\tau, \tau^{-1}]$. Since $f$ is of relative dimension 0, $\mathcal{f}_f \mathcal{O}$ is concentrated in degree 0, i.e., $\mathcal{f}_f \mathcal{O} \simeq H^0(\mathcal{f}_f \mathcal{O}) \in \text{Mod}_{\text{rh}}(\mathcal{D}_{\mathbb{A}^1})$. For our computations we pass to the variable $\theta = \tau^{-1}$. We compute that for the given $f$, $G$ is given by the free $\mathcal{C}[\theta, \theta^{-1}]$-module

$$G = \mathcal{C}[x, x^{-1}]dx[\theta, \theta^{-1}]/\left(\theta d - \left(dx - \frac{3}{\tau^3}dx\right) \wedge\right) \mathcal{C}[x, x^{-1}][\theta, \theta^{-1}].$$
with basis over \( \mathbb{C}[\theta, \theta^{-1}] \) given by \( \frac{dx}{x}, \frac{dx}{x^2}, \frac{dx}{x^3}, \frac{dx}{x^4} \). In this basis, the connection is given by

\[
\theta \nabla_{\bar{\theta}} = \theta \partial_{\theta} + \begin{pmatrix}
0 & 4 & 0 & 0 \\
0 & \frac{1}{4} & \frac{4}{3} & 0 \\
0 & 0 & \frac{1}{3} & \frac{4}{9} \\
\frac{1}{4} & 0 & 0 & 1
\end{pmatrix}.
\]

(1)

Via the cyclic vector \( m = (1, 0, 0, 0) \) we compute the relation

\[
\nabla_{\theta}^4 \theta \partial_{\theta} m + 4 \nabla_{\theta}^3 \theta \partial_{\theta} m + \frac{32}{9} \nabla_{\theta}^2 \theta \partial_{\theta} m - \frac{256}{27 \theta^2} m = 0
\]

and therefore associate the differential operator

\[
P = (\theta \partial_{\theta})^4 + 4(\theta \partial_{\theta})^3 + \frac{32}{9} (\theta \partial_{\theta})^2 - \frac{256}{27 \theta^2} \in \mathbb{C}[\theta, \theta^{-1}] \langle \partial_{\theta} \rangle = \mathcal{D}_{\mathbb{C}_m}.
\]

The Newton polygon in figure 1 confirms that \( P \) – and therefore system (1) – has the nonzero slope 1 and therefore is irregular singular at \( \theta = 0 \) and regular singular at \( \theta = \infty \).

2. Topological computation of the Stokes matrices

We consider the Laurent polynomial \( f = x + \frac{1}{x} : \mathbb{C}_m \to \mathbb{A}^1 \). Its critical points are given by \( \{ \pm \sqrt{3}, \pm i \sqrt{3} \} \). The critical values of \( f \) are given by

\[
\Sigma = \left\{ \pm \frac{4}{\sqrt{27}}, \pm \frac{4i}{\sqrt{27}} \right\} \subset \mathbb{A}^1.
\]

The preimages of

- \( \frac{4}{\sqrt{27}} \) are \( \sqrt{3} \) (double), \( \frac{1-\sqrt{3}i}{\sqrt{27}} \) and \( \frac{1+i\sqrt{3}i}{\sqrt{27}} \),
- \( -\frac{4}{\sqrt{27}} \) are \( -\sqrt{3} \) (double), \( \frac{1+i\sqrt{3}i}{\sqrt{27}} \) and \( \frac{1-\sqrt{3}i}{\sqrt{27}} \),
- \( \frac{i}{\sqrt{27}} \) are \( i \sqrt{3} \) (double), \( \frac{-i-\sqrt{3}i}{\sqrt{27}} \) and \( \frac{-i+i\sqrt{3}i}{\sqrt{27}} \),
- \( -\frac{i}{\sqrt{27}} \) are \( -i \sqrt{3} \) (double), \( \frac{i+i\sqrt{3}i}{\sqrt{27}} \) and \( \frac{-i-\sqrt{3}i}{\sqrt{27}} \).

Since \( f \) is proper we compute by the adjunction formula that

\[
\text{RHom}_{\mathcal{D}^{an}} \left( \left( \int_f \mathcal{O} \right)^{an}, \mathcal{O}^{an} \right) \simeq Rf^{an}_* \text{RHom}_{\mathcal{D}^{an}}(\mathcal{O}^{an}, f^! \mathcal{O}^{an}) \simeq Rf^{an}_* \mathbb{C}.
\]
Since $f$ is semismall, $Rf_!\mathbb{C}[1] \in \text{Perv}(\mathbb{C}A_1)$ is a perverse sheaf (cf. [7]). Outside of $\Sigma$, $f$ is a covering of degree 4, therefore $Rf_!\mathbb{C}[1] \in \text{Perv}_\Sigma(\mathbb{C}A_1)$. By the regular Riemann–Hilbert correspondence

$$\text{Sol}(\bullet)[\dim X] := \text{RHom}_{\mathcal{D}_X^{an}}((\bullet)^{an}, \mathcal{O}_X^{an})[\dim X]: \text{Mod}_{\text{rh}}(\mathcal{D}_X) \xrightarrow{\simeq} \text{Perv}(\mathbb{C}_X),$$

we associate to $\mathcal{H}^0(f_!\mathcal{O})$ the perverse sheaf $F := Rf_!\mathbb{C}[1]$.

We fix $\alpha = e^{\frac{2\pi i}{8}} \in \mathbb{A}^1, \beta = e^{\frac{3\pi i}{8}} \in (\mathbb{A}^1)^\vee$, s.t. $\Re(\langle \alpha, \beta \rangle) = 0, \Im(\langle \alpha, \beta \rangle) = 1$. This induces the following ordering on $\Sigma$ (cf. [6, Section 4]):

$$\sigma_1 = \frac{4i}{\sqrt{27}} <_\beta \sigma_2 = -\frac{4}{\sqrt{27}} <_\beta \sigma_3 = \frac{4}{\sqrt{27}} <_\beta \sigma_4 = -\frac{4i}{\sqrt{27}}.$$

In figure 5 the $\sigma_i$ are depicted in the following colors:

- $\sigma_1$: green,
- $\sigma_2$: red,
- $\sigma_3$: purple,
- $\sigma_4$: orange.

**Figure 2.** LHS: $\{x | \Re(f(x)) \geq 0\}$, RHS: $\{x | \Im(f(x)) \geq 0\}$

**Figure 3.** Preimages of imaginary (blue) and real (red) axis
The blue area in figure 2 depicts where $f$ has real resp. imaginary part greater than or equal to 0. In figure 3 the preimages of the imaginary (blue) and real (red) axis under $f$ are plotted. We consider lines passing through the singular values with phase $\frac{\pi}{8}$, as depicted in figure 5. The preimages of these lines are plotted in figure 4. We fix a base point $e$ with $\Re(e) > \Re(\sigma_i)$ and denote its preimages by $e_1, e_2, e_3, e_4$ as depicted in figure 6. In the following we adopt the notation of [6, Section 4]. The nearby and global nearby cycles of $F$ are given by

$$\Psi_{\sigma_i}(F) := R\Gamma_c(\mathbb{A}^1; \mathcal{E}_{\sigma_i} \otimes F) \simeq H^0 R\Gamma_c(\mathcal{E}_{\sigma_i}^*; F) \cong \bigoplus_{e_j \in f^{-1}(e)} \mathbb{C}_{e_j} \cong \mathbb{C}^4,$$

$$\Psi(F) := R\Gamma_c(\mathbb{A}^1; \mathcal{C}_{\mathbb{A}^1 \setminus \ell_5} \otimes F)[1] \simeq \Psi_{\sigma_i}(F) \cong \mathbb{C}^4.$$

Furthermore we fix an isomorphism $i_{\sigma_i}^{-1} F[-1] \cong \bigoplus_{e_j \in f^{-1}(\sigma_i)} \mathbb{C}_{e_j} \cong \mathbb{C}^3$. 

Figure 4. Preimages under $f$

Figure 5. Lines passing through $\sigma_i$ with phase $\frac{\pi}{8}$
The exponential components at \( \infty \) of the Fourier–Laplace transform of \( H^0(f_O) \) are known to be of linear type with coefficients given by the \( \sigma_i \). The Stokes rays are therefore given by
\[
\left\{ 0, \pm \frac{\pi}{4}, \pm \frac{\pi}{2}, \pm \frac{3\pi}{4}, \pi \right\}.
\]

We consider loops \( \gamma_{i}, i=1,2,3,4 \), starting at \( e_j \) and running around the singular value \( \sigma_i \) in counterclockwise orientation\(^1\) as depicted in figure 6. We denote by \( \gamma^j_{i} \) the preimage of \( \gamma_{i} \), starting at \( e_j \), \( j=1,2,3,4 \).

From figure 6 we read, in the basis \( e_1, e_2, e_3, e_4 \), the monodromies \( b_{\sigma_i} \), encode which lift of \( \ell_{\sigma_i} \) starts at which preimage of \( \sigma_i \), induced from the corresponding boundary map in homology. More explicitly from figure 7 we read the following:

\( \sigma_1: \ell^1_{\sigma_1} \mapsto \sigma^1_1, \ell^2_{\sigma_1} \mapsto \sigma^1_1, \ell^3_{\sigma_1} \mapsto \sigma^1_1, \ell^4_{\sigma_1} \mapsto \sigma^3_1. \)

Therefore \( b_{\sigma_1} \) is the transpose of \( \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \).

\( \sigma_2: \ell^1_{\sigma_2} \mapsto \sigma^3_2, \ell^2_{\sigma_2} \mapsto \sigma^1_2, \ell^3_{\sigma_2} \mapsto \sigma^2_2, \ell^4_{\sigma_2} \mapsto \sigma^2_2. \)

Therefore \( b_{\sigma_2} \) is the transpose of \( \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

\( \sigma_3: \ell^1_{\sigma_3} \mapsto \sigma^1_3, \ell^2_{\sigma_3} \mapsto \sigma^2_3, \ell^3_{\sigma_3} \mapsto \sigma^3_3, \ell^4_{\sigma_3} \mapsto \sigma^3_3. \)

Therefore \( b_{\sigma_3} \) is the transpose of \( \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \).

\( \sigma_4: \ell^1_{\sigma_4} \mapsto \sigma^1_4, \ell^2_{\sigma_4} \mapsto \sigma^3_4, \ell^3_{\sigma_4} \mapsto \sigma^4_4, \ell^4_{\sigma_4} \mapsto \sigma^2_4. \)

Therefore \( b_{\sigma_4} \) is the transpose of \( \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \).

We obtain, in the ordered bases \( \sigma_i^1, \sigma_i^2, \sigma_i^3 \) and \( \ell^1_{\sigma_i}, \ell^2_{\sigma_i}, \ell^3_{\sigma_i}, \ell^4_{\sigma_i} \) each:

\( b_{\sigma_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_{\sigma_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \)

\( b_{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b_{\sigma_4} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \)

\(^1\) counterclockwise orientation since the imaginary part of \( \langle \alpha, \beta \rangle \) is positive
Denote by $u_i := u_{v_i}$, $v_i := v_{t_i}$, $T_i := T_{v_i}$ and $\Phi_i := \Phi_{v_i}$. We obtain $\Phi_i(F) \xrightarrow{v_i} \Psi(F)$ as the cokernels of the diagrams

$$i_{\sigma_i}^{-1}F[-1] \xrightarrow{b_{\sigma_i}} \Psi(F)$$

We identify the cokernels of $b_{\sigma_i}$ in the following way:

- $\text{coker } b_{\sigma_1} \simeq \mathbb{C}$ via $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} v_1 - v_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

- $\text{coker } b_{\sigma_2} \simeq \mathbb{C}$ via $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} v_2 - v_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

- $\text{coker } b_{\sigma_3} \simeq \mathbb{C}$ via $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} v_1 - v_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

- $\text{coker } b_{\sigma_4} \simeq \mathbb{C}$ via $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} v_1 - v_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We obtain that $(\Phi_i(F) \xrightarrow{v_i} \Psi(F)) \simeq (\mathbb{C} \xrightarrow{v_i} \mathbb{C}^4)$, where

$$u_1 = (1, -1, 0, 0), \quad u_2 = (0, 1, -1, 0), \quad u_3 = (1, 0, 0, -1), \quad u_4 = (1, 0, -1, 0),$$

and $v_i = u_i^*$.

Remembering carefully all the choices, by [6, Theorem 5.2.2] we obtain the following Stokes multipliers of the Fourier–Laplace transform of $H^0(\mathcal{F})$ at $\infty$:

$$S_\beta = \begin{pmatrix} 1 & u_1 v_2 & u_1 v_3 & u_1 v_4 \\ 0 & 1 & u_2 v_3 & u_2 v_4 \\ 0 & 0 & 1 & u_3 v_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_{-\beta} = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ -u_2 v_1 & T_2 & 0 & 0 \\ -u_3 v_1 & -u_3 v_2 & T_3 & 0 \\ -u_4 v_1 & -u_4 v_2 & -u_4 v_3 & T_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix} = -S_\beta^T,$$

where $T_i := 1 - u_i v_i$. $S_{-\beta}$ describes crossing $h_{\pm \beta}$ from $H_\alpha$ to $H_{-\alpha}$, where

$$H_\alpha = \left\{ r e^{i \varphi} \in (\mathbb{A}^1)^\vee \mid r > 0, \varphi \in \left[\frac{-5 \pi}{8}, \frac{3 \pi}{8}\right] \right\},$$

$$H_{-\alpha} = \left\{ r e^{i \varphi} \in (\mathbb{A}^1)^\vee \mid r > 0, \varphi \in \left[\frac{3 \pi}{8}, \frac{11 \pi}{8}\right] \right\},$$

denote the closed sectors at $\infty$ and $h_{\pm \beta} = \pm \mathbb{R}_{>0} \beta \subset (\mathbb{A}^1)^\vee$, s.t. $H_\alpha \cap H_{-\alpha} = h_\beta \cup h_{-\beta}$. 

Figure 6. $\gamma_{\sigma_1}$ and its preimages under $f$
Figure 7. $\ell_{\sigma_i}$ and its preimages under $f$
3. Quantum connection and Dubrovin’s conjecture

3.1. Quantum connection. The quantum connection of a Fano variety (resp. an orbifold) $X$ is a connection on the trivial vector bundle over $\mathbb{P}^1$ with fiber $H^*(X, \mathbb{C})$ (resp. $H^*_\text{orb}(X, \mathbb{C})$), the standard inhomogenous coordinate on $\mathbb{P}^1$ being denoted by $z$.

By [10, (2.2.1)] the quantum connection is the connection given by

$$\nabla_{z\partial_z} = z \frac{\partial}{\partial z} - \frac{1}{z} (-K_X^{\circ}) + \mu,$$

where the first term on the right hand side is ordinary differentiation, the second is pointwise quantum multiplication by $(-K_X)$, and the third term is the grading operator

$$\mu(a) := \left( \frac{i}{2} - \frac{\dim X}{2} \right) a \text{ for } a \in H^1(X, \mathbb{C}).$$

The quantum connection is regular singular at $z = 0$ and irregular singular at $z = \infty$. For the weighted projective line $\mathbb{P}(a, b)$, the orbifold cohomology ring is given by (cf. [13, Example 3.20])

$$H^*_\text{orb}(\mathbb{P}(a, b), \mathbb{C}) = \mathbb{C}[x, y, \xi]/(xy, ax^{\frac{a}{2}} - by^{\frac{b}{2}}\xi^{a-m}, \xi^d - 1),$$

where $d = \gcd(a, b)$ and $m, n \in \mathbb{Z}$ s.t. $am + bn = d$. The grading is given as follows (cf. [2, Section 9]): $\deg x = \frac{1}{a}$, $\deg y = \frac{1}{b}$, $\deg \xi = 0$, where $A = \frac{a}{2}$, $B = \frac{b}{2}$. Quantum multiplication$^2$ is computed in

$$QH^*_\text{orb} = \mathbb{C}[x, y, \xi]/(xy - 1, ax^{\frac{a}{2}} - by^{\frac{b}{2}}\xi^{a-m}, \xi^d - 1).$$

For $\gcd(a, b) = 1$, $-K_{\mathbb{P}(a, b)}$ is given by the element $[x^a + y^b] \in H^1$. Here the grading is scaled by 2, s.t. the grading operator is defined by $\mu(a) = (i - \dim X)a$ for $a \in H^1$.

We obtain the quantum connection of $\mathbb{P}(1, 3)$ as follows.

$$H^*_\text{orb}(\mathbb{P}(1, 3), \mathbb{C}) = \mathbb{C}[x, y]/(xy, x - 3y^3)$$

with grading given by $\deg x = 1, \deg y = \frac{1}{3}$. A basis over $\mathbb{C}$ is given by $1, y, y^2, y^3$. Quantum multiplication by $-K_{\mathbb{P}(1, 3)} = [x + y^3] = [4y^3]$ in this basis is given by the matrix

$$\begin{pmatrix}
0 & \frac{4}{3} & 0 & 0 \\
0 & 0 & \frac{4}{3} & 0 \\
0 & 0 & 0 & \frac{4}{3} \\
4 & 0 & 0 & 0
\end{pmatrix}.$$

The grading $\mu$ is given by the matrix

$$\begin{pmatrix}
-\frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{6} & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}.$$

Therefore the quantum connection of $\mathbb{P}(1, 3)$ is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z} \begin{pmatrix}
0 & \frac{4}{3} & 0 & 0 \\
0 & 0 & \frac{4}{3} & 0 \\
0 & 0 & 0 & \frac{4}{3} \\
4 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
-\frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{6} & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}.$$

It is irregular singular at $z = 0$ and regular singular at $z = \infty$. Rewriting in $z^{-1}$ yields the irregular singularity at $\infty$. 

$^2$for $q = 1$
**Observation.** By the gauge transformation $h = \text{diag}(\theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}})$ which substracts $\frac{1}{2}$ on the diagonal entries and passing to $-\theta$, connection (1) coming from the Landau–Ginzburg model is exactly the quantum connection (2) of $\mathbb{P}(1,3)$, as predicted by mirror symmetry.

### 3.2. Dubrovin’s conjecture.

Let $X$ be a Fano variety (or an orbifold), s.t. the bounded derived category $D^b(\text{Coh}(X))$ of coherent sheaves on $X$ admits a full exceptional collection $\{E_1, \ldots, E_n\}$, where the collection $\{E_1, \ldots, E_n\}$ is called

- exceptional if $\text{RHom}(E_i, E_i) = \mathbb{C}$ for all $i$ and $\text{RHom}(E_i, E_j) = 0$ for $i > j$,
- full if $D^b(\text{Coh}(X))$ is the smallest full triangulated subcategory of $D^b(\text{Coh}(X))$ containing $E_1, \ldots, E_n$.

In [9], B. Dubrovin conjectured that, under appropriate choices, the Stokes matrix of the quantum connection of $X$ equals the Gram matrix of the Euler–Poincaré pairing w.r.t. some f.e.c. – modulo some action of the braid group, sign changes and permutations (cf. [4, Section 2.3]). The second Stokes matrix then is the transpose of the first one. The Euler–Poincaré pairing is given by the bilinear form

$$\chi(E, F) := \sum_k (-1)^k \dim \mathbb{C} \text{Ext}^k(E, F), \quad E, F \in D^b(\text{Coh}(X)).$$

The Gram matrix of $\chi$ w.r.t. to a f.e.c. is upper triangular with ones on the diagonal.

For $\mathbb{P}(a, b)$, $\langle O, O(1), \ldots, O(a + b - 1) \rangle$ is a f.e.c. of $D^b(\text{Coh}(\mathbb{P}(a, b)))$ (cf. [1, Theorem 2.12]). Following [3, Theorem 4.1], the cohomology of the twisting sheaves for $k \in \mathbb{Z}$ is given by

- $H^0(\mathbb{P}(a, b), O(k)) = \bigoplus_{(m, n) \in I_0} \mathbb{C} x^m y^n$, where $I_0 = \{(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} | am + bn = k\}$,
- $H^1(\mathbb{P}(a, b), O(k)) = \bigoplus_{(m, n) \in I_1} \mathbb{C} x^m y^n$, where $I_1 = \{(m, n) \in \mathbb{Z}_{< 0} \times \mathbb{Z}_{< 0} | am + bn = k\}$,
- $H^i(\mathbb{P}(a, b), O(k)) = 0$ for all $i \geq 2$.

We only need to compute $\text{Ext}^i(O(i), O(j))$ for $i < j$ which is given by $H^i(O(j - i))$ (cf. [14, Lemma 4.5]). Therefore the zeroth cohomologies of the twisting sheaves $O(j - i)$ are the only ones that contribute to the Gram matrix of $\chi$. For $\mathbb{P}(1, 3)$ we obtain the cohomology groups

$$H^0(O(1)) \cong \mathbb{C}, \quad H^0(O(2)) \cong \mathbb{C}, \quad H^0(O(3)) \cong \mathbb{C}^2$$

and therefore the Gram matrix of the Euler–Poincaré pairing w.r.t. the f.e.c.

$$E : = \langle O, O(1), O(2), O(3) \rangle$$

is given by

$$S_{\text{Gram}} = \begin{pmatrix}
1 & 1 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

\[3\text{from now on abbreviated to f.e.c.} \]
3.3. Comparison of the Gram and Stokes matrix. Mirror symmetry relates the Laurent polynomial \( f = x + \frac{1}{x^3} \) to the weighted projective line \( \mathbb{P}(1,3) \). The pair \((\mathbb{G}_m, f = x + \frac{1}{x^3})\) is a Landau–Ginzburg model of the weighted projective line \( \mathbb{P}(1,3) \). According to Dubrovin’s conjecture, the Stokes matrix of the quantum connection of \( \mathbb{P}(1,3) \) is given by the Gram matrix of the Euler–Poincaré pairing w.r.t. some f.e.c. of \( D^b(\text{Coh}(\mathbb{P}(1,3))) \). Note that there is a natural action of the braid group on the Stokes matrix reflecting variations in the choices involved to determine the Stokes matrix (cf. [12]). In our case we have to consider the braid group

\[
B_4 = \langle \beta_1, \beta_2, \beta_3 \mid \beta_1 \beta_2 \beta_1 = \beta_2 \beta_1 \beta_2, \beta_2 \beta_3 \beta_2 = \beta_3 \beta_2 \beta_3 \rangle.
\]

We computed that the Gram matrix of \( \chi \) w.r.t. the f.e.c. \( E \) is given by (3). Via the action of the braid \( \beta_1 \in B_4 \), we find that it is equivalent to \( S_\beta \). Following [12, Section 6] the braid \( \beta_1 \) acts on the Gram matrix as

\[
S_{\text{Gram}} \mapsto S_{\text{Gram}}^{\beta_1} := A^{\beta_1}(S_{\text{Gram}}) \cdot S_{\text{Gram}} \cdot (A^{\beta_1}(S_{\text{Gram}}))^t,
\]

where \( A^{\beta_1}(S_{\text{Gram}}) \) is given by

\[
A^{\beta_1}(S_{\text{Gram}}) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We obtain that

\[
S_{\text{Gram}}^{\beta_1} = \begin{pmatrix}
1 & -1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix} = S_\beta.
\]

Remark. \( S_{\text{Gram}}^{\beta_1} = S_\beta \) is the Gram matrix of the Euler–Poincaré pairing w.r.t. the right mutation \( \mathbb{R}_1E \) of the f.e.c. \( E \) (cf. [4, Proposition 13.1]). The action of the braid \( \beta_1 \in B_4 \) should correspond to a counterclockwise rotation of \( \beta \). Therefore, we could expect to have the braid \( \beta_1 \) acting on our Stokes matrix a priori.

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