Canonical approach to 2D induced gravity

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Abstract

Using canonical method the Liouville theory has been obtained as a gravitational Wess-Zumino action of the Polyakov string. From this approach it is clear that the form of the Liouville action is the consequence of the bosonic representation of the Virasoro algebra, and that the coefficient in front of the action is proportional to the central charge and measures the quantum braking of the classical symmetry.

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I. INTRODUCTION

In the paper [1] we introduced new canonical method to investigate anomalous gauge theories. By definition these are the theories where symmetries of the classical action are broken by the quantum effects. In the hamiltonian language it means that the classical theory has first class constraints (FCC) $t_m$, satisfying closed Poisson bracket (PB) algebra, and that after quantization the corresponding commutator algebra of the operators $\hat{t}_m$ obtains the central charge and consequently constraints become the second class (SCC). The basic idea of the new approach is to introduce the variables $T_m$ and postulate that their classical PB algebra is isomorphic to the quantum commutator algebra of $\hat{t}_m$. This defines the effective theory. To find the effective action of this theory we first parameterize the constraints $T_m$ by some phase-space variables and "solve" the PB constraints obtaining the expressions for $T_m$ in terms of coordinates and momenta. Then we use the general canonical formalism [2,3,1] connecting the action $I$ with the expressions for the hamiltonian $H_0$ and the primary constraints $T_m$ in terms of the phase-space coordinates. It states that the action

$$I = \int d^2 \xi (p_i \dot{q}^i - H_0 - h^m T_m), \quad (1.1)$$

is invariant under gauge transformations

$$\delta R = \varepsilon^m \{R, T_m\}, \quad \delta h^m = \varepsilon^m - h^n \varepsilon^k U^m_{kn} - \varepsilon^n V^m_{kn}, \quad (1.2)$$

of any quantity $R(p,q)$ and lagrange multipliers $h^m$ if the constraints $T_m$ are FCC satisfying PB algebra

$$\{T_m, T_n\} = U^k_{mn} T_k, \quad \{H_0, T_m\} = V^m_{nT} T_n. \quad (1.3)$$

If $T_m$ are SCC, then instead of the first eq. (1.3) we have

$$\{T_m, T_n\} = U^k_{mn} T_k + \Delta_{mn}, \quad (1.4)$$

and the variation of the action is proportional to the Schwinger term $\Delta_{mn}$.
\[ \delta I = - \int d\tau h^m \Delta_{mn} \xi^n. \] 

(1.5)

We will use this statement in both directions, to find the symmetry transformations from the known action, and the more important to us, to construct the effective action from the constraints in terms of the coordinates and momenta.

In the paper [1] we started with the non-Abelian symmetry of the classical action whose generators satisfy the Kac-Moody algebra. The symmetry has an axial anomaly and we obtained the gauged WZNW model as an effective theory.

In this paper we are going to replace the non-Abelian symmetry with the diffeomorphisms. We expect to obtain Virasoro algebra as an algebra of constraints, and reparameterization or trace anomalies, depending on the regularization procedure. As an effective action we will get a Liouville action. We choose the Polyakov string theory [4] as an example. The quantization of this theory has been obtained by path integral approach, which in the case of string becomes a sum over random surfaces [4,5]. The group of authors [6] also develops a canonical treatment of 2D gravity using Batalin, Fradkin and Vilkovisky method. We believe that our approach is not just one more in the series but the simplest one, and besides works naturally in the hamiltonian formalism.

In Sec. II we introduce the Polyakov string. We fix the reparameterization symmetry and make a canonical analysis. All constraints are of the FCC and as a consequence of the reparameterization symmetry the generators of the scalar and Faddeev-Popov (FP) parts satisfy the same Virasoro algebra, but they have different representations.

In Sec. III we quantize the theory, introducing operators instead of the fields, and presence of the central terms changes the nature of the constraints from the FCC to the SCC. The only difference between scalar and FP part appears in the values of the central charges \( c^x \) and \( c^{gh} \), which is the only remnant of the original system.

Sec. IV is a central part of the paper. We find an expression for the constraints, in terms of phase-space coordinates, from the requirement that its PB algebra is isomorphic to the commutator algebra of the Sec. III. With the help of such bosonic representation of the
Virasoro algebra we construct the effective action. Elimination of the momentum variables on their equations of motion yields the Liouville action. This is the gravitational Wess-Zumino (WZ) action because it measures the breaking of the reparameterization invariance. By adding some finite local counterterm we obtain the action which is invariant under diffeomorphisms but not under Weyl rescaling or in the other words we shift the anomaly from the Virasoro to the trace one.

Sec. V is devoted to concluding remarks.

II. CANONICAL ANALYSIS OF THE POLYAKOV STRING

A. Constraints in the scalar theory

Let us consider the action for Polyakov string $S_0$

$$S_0(x^M, g_{\mu\nu}) = \frac{1}{2} \int d^2 \xi \sqrt{-g} g^{\mu\nu} \partial_\mu x^M \partial_\nu x_M,$$  \hspace{1cm} (2.1)

where $M = 0, 1, ..., D - 1$, and $\mu, \nu = 0, 1$. One can rewrite the action as

$$S_0(x^M, h) = \int d^2 \xi \frac{1}{h^- - h^+} (\partial_0 + h^- \partial_1)x^M (\partial_0 + h^+ \partial_1)x^M,$$  \hspace{1cm} (2.2)

in terms of the light-cone variables $h^-, h^+$ and $F$, defined by the expression

$$g_{\mu\nu} = e^{2F} \tilde{g}_{\mu\nu} = \frac{1}{2} e^{2F} \begin{vmatrix} -2h^- h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{vmatrix}.$$  \hspace{1cm} (2.3)

As a consequence of the conformal symmetry the action (2.2) does not depend on $F$.

The canonical momenta are

$$p_M = \frac{\delta S_0}{\delta \dot{x}^M} = \frac{1}{h^- - h^+} [2\dot{x}^M + (h^- + h^+)x^M],$$  \hspace{1cm} (2.4)

$$p_\pm = \frac{\delta S_0}{\delta h^\pm} = 0,$$  \hspace{1cm} (2.5)

where $\dot{X} = \partial_0 X$ and $X' = \partial_1 X$ for any variable $X$. The canonical hamiltonian density $\mathcal{H}_c$ can be expressed in terms of the currents

4
\[ j^M_{\pm} = \frac{p^M \pm x^{M'}}{\sqrt{2}} , \quad (2.6) \]

as

\[ \mathcal{H}_c = h^+ t^x_+ + h^- t^x_- , \quad t^x_{\pm} = \mp \frac{1}{2} j^M_{\pm} j^{M'}_{\pm} . \quad (2.7) \]

The index \( x \) denotes that this energy-momentum tensor belongs to the matter fields \( x^M \).

Starting with the basic PB \( \{ x^M(\sigma), p_N(\bar{\sigma}) \} = \delta^M_N \delta(\sigma - \bar{\sigma}) \) we have

\[ \{ j^M_{\pm}, j^M_{\pm} \} = \pm \delta^M_N \delta' \quad (2.8) \]

which implies

\[ \{ t^x_{\pm}, j^M_{\pm} \} = - j^M_{\pm} \delta' , \quad \{ t^x_{\pm}, j^M_{\mp} \} = 0 \quad (2.9) \]

and

\[ \{ t^x_{\pm}(\sigma), t^x_{\pm}(\bar{\sigma}) \} = - [ t^x_{\pm}(\sigma) + t^x_{\mp}(\bar{\sigma}) ] \delta'(\sigma - \bar{\sigma}) , \quad \{ t^x_{\pm}, t^x_{\mp} \} = 0 \quad (2.10) \]

In relations (2.8-2.10) we recognize the semidirect product of \( D \) independent abelian Kac-Moody (KM) algebras (2.8) and Virasoro algebras (2.10).

We introduce the total hamiltonian

\[ H_T = \int d\sigma [ \mathcal{H}_c + \lambda^+ p^+ + \lambda^- p^- ] , \quad (2.11) \]

and from the consistency condition for the primary constraints \( p^\pm \)

\[ \dot{p}^\pm = \{ p^\pm, H_T \} = - t^x_{\pm} \quad (2.12) \]

we conclude that \( t^x_{\pm} \) are secondary constraints.

With the help of PB (2.10) it is clear that all constraints \( p^\pm \) and \( t^x_{\pm} \) are FCC, and that there are no more constraints. The canonical hamiltonian \( \mathcal{H}_c \) is weakly equal to zero.

We can write action (2.2) in the hamiltonian form

\[ S_0(x^M, p_M, h) = \int d^2 \xi (p_M \dot{x}^M - h^+ t^x_+ - h^- t^x_- ) , \quad (2.13) \]
which is convenient for obtaining the local symmetries. It is easy to check that on the
equation of motion for the momenta $p_M$ the action (2.13) yields (2.2).

Comparing (2.13) with (1.1) and (2.10) with first equation (1.3) we can conclude from
second equation (1.2) that

$$\delta h^\pm = (\partial_0 + h^\pm \partial_1 - h^{\pm'})\varepsilon_\mp = \sqrt{2} \sqrt{-\hat{g}} \hat{\nabla}_\mp c_\mp, \quad (2.14)$$

where the covariant derivative on tensor $V_n \ (n \in \mathbb{Z}$ stands for the sum of the indices taking
1 for plus and -1 for minus) is $\hat{\nabla}_n = (\hat{\partial}_n + n\hat{\omega}_n)V_n$ with $\hat{\partial}_n = \sqrt{2 n - n^2} (\partial_0 + h^\pm \partial_1)$ and
$\hat{\omega}_n = \sqrt{n^2 - n^2} h^\mp'$.  

With the help of (2.3) we recognize (2.14) as the diffeomorphism transformation of the
metric density $\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$, after introducing new parameters $\varepsilon^\pm = \varepsilon^1 - h^\pm \varepsilon^0$, see [3] for
more details.

**B. Gauge fixing and constraints in Faddeev-Popov action**

It is well known that classically we can gauge away all components of the metric tensor.
Due to the presence of the anomaly at the quantum level, we are not allowed to fix all these
symmetries at the classical level. Let us fix the diffeomorphism using BRST method.

To obtain the BRST transformations we replace gauge parameters with ghost fields

$$\varepsilon_\pm \rightarrow c_\pm, \quad (2.15)$$

so that instead of (2.14) we have

$$s h^\pm = \sqrt{2} \sqrt{-g} \hat{\nabla}_\mp c_\mp. \quad (2.16)$$

We introduce antighosts $\bar{c}_\pm$ and auxiliary fields $b_{\pm\pm}$ with the BRST transformations

$$s \bar{c}_{\pm\pm} = b_{\pm\pm}, \quad s b_{\pm\pm} = 0, \quad (2.17)$$

and choose the gauge fermion $\Psi$ in the form
\[ \Psi = \bar{c}_{++}(h^+ - h_b^+) + \bar{c}_{--}(h^- - h_b^-), \] (2.18)

to fix the gauge fields \( h^\pm \) to some background fields \( h_b^\pm \).

Starting from the expression

\[ -s\Psi = \mathcal{L}_{GF} + \mathcal{L}_{FP}, \] (2.19)

one can integrate over auxiliary fields \( b_{\pm\pm} \), and then over \( h^\pm \) obtaining \( h^\pm = h_b^\pm \), and the FP action

\[
S_{FP} = \int d^2\xi \mathcal{L}_{FP} = \sqrt{2} \int d^2\xi \sqrt{-\hat{g}}(\bar{c}_{++} \hat{\nabla}_- c_- + \bar{c}_{--} \hat{\nabla}_+ c_+) \\
= \int d^2\xi [\bar{c}_{++}(\partial_0 + h^+ \partial_1 - h^+)c_- + \bar{c}_{--}(\partial_0 + h^- \partial_1 - h^-)c_+]. \quad (2.20)
\]

From now on only the background fields \( h_b^\pm \) exist, and for simplicity we will omit index \( b \) and write simply \( h^\pm \).

This action is already in the Hamiltonian form. The coordinates are ghosts \( c_\pm \), the conjugate momenta

\[
\pi_{\pm\pm} = \frac{\delta L_{SP}}{\delta \dot{c}_{\mp}} = -\bar{c}_{\pm\pm},
\] (2.21)

are antighosts and the energy-momentum tensor

\[
t^{\phi h}_{\pm\pm} = \frac{\delta S_{FP}}{\delta h^\pm} = \bar{c}_{\pm\pm}c'_{\mp} + (\bar{c}_{\pm\pm}c_{\mp})' = 2\bar{c}_{\pm\pm}c'_{\mp} + \bar{c}'_{\pm\pm}c_{\mp}, \quad (2.22)
\]

plays the role of the primary constraints corresponding to the Lagrange multipliers \( h^\pm \). In terms of the momenta we obtain

\[
t^{\phi h}_{\pm\pm} = -2\pi_{\pm\pm}c'_{\mp} - \pi'_{\pm\pm}c_{\mp}. \quad (2.23)
\]

Starting with the basic PB for \( c \) and \( \pi \)

\[
\{c_{\mp}(\sigma), \pi_{\pm\pm}(\bar{\sigma})\} = \delta(\sigma - \bar{\sigma}), \quad (2.24)
\]

one can find that PB of \( t' \)'s satisfies two independent copies of Virasoro algebras without central charges,
\{t_\pm^{gh}(\sigma), t_\pm^{gh}(\bar{\sigma})\} = -[t_\pm^{gh}(\sigma) + t_\pm^{gh}(\bar{\sigma})]\delta'(\sigma - \bar{\sigma}), \quad \{t_\pm^{gh}, t_\pm^{gh}\} = 0. \quad (2.25)

Note that the Virasoro algebras for matter fields (2.10) and for the ghosts (2.25) are classically identical. The different expressions for the energy momentum tensors \(t_x^{\pm} (2.7)\) and \(t_{gh}^{\pm} (2.23)\) will cause the different quantum algebras.

III. CANONICAL QUANTIZATION OF THE POLYAKOV STRING

In this section we are going to perform quantization of the matter fields \((x^M)\) and the ghost fields \((c_\pm, \bar{c}_{\pm})\) in the gauge fixed Polyakov string theory, with the action

\[S = S_0(x, h) + S_{FP}(\bar{c}, c, h). \quad (3.1)\]

The method developed in [1] for non-Abelian gauge symmetry will be applied here for reparameterization symmetry.

Transition from the classical to the quantum theory is achieved by introducing the operators \(\hat{\Omega}_\pm = \{\hat{j}_\pm, \hat{c}_\mp, \hat{\pi}_{\pm\mp}\}\) instead of the fields \(\Omega_\pm = \{j_\pm, c_\mp, \pi_{\pm\mp}\}\), replacing the PB (2.8) and (2.24) by the (anti)commutators

\[
[\hat{j}_\pm, \hat{j}_N] = \pm i\hbar \delta_N^M \delta', \quad [\hat{\pi}_\pm, \hat{\pi}_{\pm\pm}] = i\hbar \delta, \quad (3.2)
\]

and defining the composite operators using normal ordering prescription

\[
\hat{t}_\pm = \pm \frac{1}{2} : \hat{j}_\pm^M \hat{j}_{\mp M} : , \quad \hat{t}_{gh} = -2 : \hat{\pi}_{\pm \pm} \hat{c}_\mp : - : \hat{\pi}'_{\pm \pm} \hat{c}_\mp : . \quad (3.3)
\]

In order to obtain commutator algebra for \(t'\)’s we decompose operators in positive and negative frequencies in the position space [7,1]

\[
\hat{\Omega}^{(\pm)}(\tau, \sigma) = \int_{-\infty}^{+\infty} d\bar{\sigma} \delta^{(\pm)}(\sigma - \bar{\sigma}) \hat{\Omega}(\tau, \bar{\sigma}), \quad (3.4)
\]

where delta function is given by

\[
\delta^{(\pm)}(\sigma) = \delta^{(+)}(\sigma) + \delta^{(-)}(\sigma), \quad (3.5)
\]

\[
\delta^{(\pm)}(\sigma) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \theta(\mp k)e^{ik(\sigma \mp i\varepsilon)} = \frac{\pm i}{2\pi(\sigma \pm i\varepsilon)}, \quad (\varepsilon > 0). \quad (3.6)
\]
We define $\hat{\Omega}_\pm^{(\mp)}$ as creation operators and $\hat{\Omega}_\pm^{(\pm)}$ as annihilation operators

$$
< 0 | \hat{\Omega}_\pm^{(\mp)} = 0, \quad \hat{\Omega}_\pm^{(\pm)} | 0 > = 0.
$$

(3.7)

With this choice we preserve the symmetry under parity transformations on the quantum level. It corresponds to the left-right symmetric regularization scheme and consequently we will obtain anomaly for both Virasoro algebras. Converting the Virasoro anomaly to the trace anomaly is possible due to finite local counterterm (Sec. IV C).

Up to central terms $\Delta_\pm$, the commutator algebra for both matter and ghost fields should be the same,

$$
[\hat{t}_\pm, \hat{t}_\pm] = -i\hbar [\hat{t}_\pm(\sigma) + \hat{t}_\pm(\bar{\sigma})] \delta'(\sigma - \bar{\sigma}) + \Delta_\pm(\sigma, \bar{\sigma}),
$$

(3.8)

$$
[\hat{t}_+, \hat{t}_-] = 0.
$$

(3.9)

To find the central term we take the vacuum expectation value of $[\hat{t}_\pm, \hat{t}_\pm]$. Note that under parity transformation $P : \hat{\Omega}_+(\tau, \sigma) \rightarrow \hat{\Omega}_-(\tau, -\sigma)$ so that $P \hat{t}_\pm(\tau, \sigma) P = -\hat{t}_\mp(\tau, -\sigma)$. Consequently we have $\Delta_+ = -\Delta_+$ and it is enough to calculate only $\Delta_+$. From the basic commutation relations (3.2) we find

$$
\left[ \hat{\gamma}_+^{(\pm)M}, \hat{\gamma}_N^{(\mp)} \right] = i\hbar \delta^M_N \delta^{(\pm)'},
$$

(3.10)

and

$$
\left[ \hat{c}_-^{(\pm)}, \hat{\pi}_+^{(\mp)} \right] = i\hbar \delta^{(\pm)}, \quad \left[ \hat{c}_-^{(\pm)'}, \hat{\pi}_+^{(\mp)} \right] = i\hbar \delta^{(\pm)'}, \quad \left[ \hat{c}_-^{(\pm)}, \hat{\pi}_+^{(\mp)'} \right] = -i\hbar \delta^{(\pm)'}.
$$

(3.11)

After straightforward calculation we obtain

$$
\Delta_\pm(\sigma, \bar{\sigma}) = \pm i\hbar \frac{\hbar}{24\pi} c^{\prime\prime\prime} \delta''(\sigma - \bar{\sigma}),
$$

(3.12)

where the central charges for the matter and ghost fields are respectively

$$
c^x = D, \quad c^{gh} = -26.
$$

(3.13)
IV. 2D INDUCED GRAVITY AS AN EFFECTIVE ACTION

We want to find effective theory for the matter and ghost part of the Polyakov action. Instead of the FCC PB algebras (2.10) and (2.25) of diffeomorphism generators, we obtained the SCC commutator algebras (3.8, 3.9). These algebras have the central terms and therefore break the reparameterization symmetry and become the algebras of dynamical variables.

A. Bosonic representation of the Virasoro algebra

The algebras for matter and ghost fields have the same structure up to the central charges, and so we will investigate both cases together. Following [1] we introduce new variables $T_\pm$ instead of $\hat{t}_\pm$ and postulate their classical PB

$$\{T_\pm, T_\pm\} = -[T_\pm(\sigma) + T_\pm(\bar{\sigma})]\delta' \pm \kappa_0 c\delta''', \quad (\kappa_0 \equiv \frac{h}{24\pi}) \quad (4.1)$$

$$\{T_+, T_-\} = 0, \quad (4.2)$$

to be isomorphic to the commutator algebras (3.8, 3.9) of the operators $\hat{t}_\pm$. To find the solution of (4.1, 4.2) for $T_\pm$ as a function of the canonical bosonic variables $\varphi$ and $\pi$ with the PB

$$\{\varphi, \pi\} = \delta, \quad (4.3)$$

we use the ansatz

$$T_\pm = a_\pm K^2_\pm + b_\pm K'_\pm + V_\pm, \quad (4.4)$$

where $K_\pm = (\pi + \alpha_\pm \varphi')$ are currents, $V_\pm = V_\pm(\varphi, \pi)$ is momentum dependent potential and $a_\pm, b_\pm$ and $\alpha_\pm$ are constants.

The currents of opposite chirality should commute

$$\{K_+, K_-\} = 0, \quad (4.5)$$

so we get $\alpha_+ = -\alpha_- \equiv \alpha$ and consequently
\[ K_\pm = (\pi \pm \alpha \varphi') . \] (4.6)

The currents of the same chirality satisfy abelian KM algebras

\[ \{ K_\pm, K_\pm \} = \pm 2\alpha \delta' . \] (4.7)

After some calculation we find that

\[ \{ T_\pm, T_\pm \} = \pm 4\alpha a_\pm \left\{ [T_\pm(\sigma) + T_\pm(\bar{\sigma})] \delta' - \frac{b_\pm^2}{2a_\pm} \delta'' \right\} , \] (4.8)

if the potential is the solution of the equation

\[ V_\pm = \frac{1}{2} K_\pm \partial_\pi V_\pm + \frac{1}{2q_\pm} \left[ \partial_\varphi V_\pm + \alpha (\partial_\pi V_\pm)' \right] , \] (4.9)

with

\[ q_\pm \equiv \frac{2\alpha a_\pm}{b_\pm} . \] (4.10)

Comparing (4.8) with (4.1) we conclude that the conditions

\[ \pm 4\alpha a_\pm = -1, \quad 2\alpha b_\pm^2 = -\kappa_0 c , \] (4.11)

should be satisfied.

Let us solve the condition (4.9). Instead of \( (\partial_\pi V_\pm)' \) we can write \( \partial^2_\pi V_\pm \pi' + \partial_\varphi \partial_\pi V_\pm \varphi' \) and because \( V_\pm \) does not depend on \( \pi' \) and \( \varphi' \), respective coefficients should be equal to zero. The first one

\[ \partial^2_\pi V_\pm = 0 , \] (4.12)

implies that \( V_\pm \) is linear in the momenta

\[ V_\pm = \pi u_\pm(\varphi) + v_\pm(\varphi) . \] (4.13)

The second condition

\[ \partial_\pi \left( V_\pm \pm \frac{1}{q_\pm} \partial_\varphi V_\pm \right) = 0 , \] (4.14)
and the form of \( V_\pm \) we have just obtained in (4.13) give us the equation for \( u \)

\[
u_\pm \pm \frac{1}{q_\pm} \partial_\varphi u_\pm = 0.
\] (4.15)

From remaining part of \( V_\pm \) in (4.9) we obtain again (4.15) and the new equation

\[
v_\pm = \pm \frac{1}{2q_\pm} \partial_\varphi v_\pm.
\] (4.16)

Now \( u_\pm \) and \( v_\pm \) are given as

\[
\begin{align*}
u_\pm &= u_0 \pm e^{\mp q_\pm \varphi}, & v_\pm &= v_0 \pm e^{\mp 2q_\pm \varphi},
\end{align*}
\] (4.17)

where \( u_0 \pm \) and \( v_0 \pm \) are constants and the solution for \( V_\pm \) is

\[
V_\pm = u_0 \pm e^{\mp q_\pm \varphi} \pi + v_0 \pm e^{\mp 2q_\pm \varphi}.
\] (4.18)

We can proceed in finding consequences of the relation \( \{T_+, T_-\} = 0 \). All conditions connecting expressions of \( V_+ \) with \( V_- \) should be valid for any \( \varphi \). So, we have \(-q_+ = q_- \equiv q\) and with the help of (4.10)

\[
\frac{a_+}{b_+} = -\frac{a_-}{b_-}.
\] (4.19)

After some calculation three groups of conditions are obtained, because coefficients in front of \( \delta'' \), \( \delta' \) and \( \delta \) must be zero. Any of these groups gives few expressions because coefficients in front of \( \pi, \pi^2, \varphi', \varphi'', \varphi \pi', \varphi \pi'' \) vanish separately. But some conditions are giving relations already obtained, so we get only four new relations

\[
\begin{align*}
a_+ u_0_- - a_- u_0_+ &= 0, & a_+ v_0_- - a_- v_0_+ &= 0, \quad (4.20) \\
b_+ u_0_- + b_- u_0_+ &= 0, & b_+ v_0_- + b_- v_0_+ &= 0. \quad (4.22)
\end{align*}
\]

Now from (4.11), (4.19) and (4.10) it is easy to conclude that \( a_+ = -a_- = -\frac{1}{4a}, \quad b_+ = b_- \equiv b, \quad q = \frac{1}{2b}, \quad u_0_+ = -u_0_- \equiv \frac{\varphi}{2q}, \quad v_0_+ = -v_0_- \equiv \mu \) and
\[ \frac{\alpha}{2q^2} = -\kappa_0 c. \]  

(4.24)

The expressions for \( V_\pm \) and \( T_\pm \) obtain the following form

\[ V_\pm = \pm \left( \frac{\theta}{2q} e^{2q\varphi} \mp 5\mu e^{2q\varphi} \right), \]  

(4.25)

and

\[ T_\pm = \pm \frac{1}{8q^2\kappa_0 c} K_\pm^2 + \frac{1}{2q} K_\pm' \pm \mu e^{2q\varphi} \pm \frac{\theta}{2q} e^{2q\varphi} \pi, \]  

(4.26)

where

\[ K_\pm = \pi \mp 2q^2\kappa_0 c \phi'. \]  

(4.27)

We can eliminate the parameter \( q \) rescaling the canonical variables \( 2q\varphi = \phi, \frac{1}{2q} \pi = p \), and the currents \( K_\pm = 2qJ_\pm \) so that

\[ \{ \phi, p \} = \delta, \]  

(4.28)

and

\[ J_\pm = p \mp \frac{\kappa_0 c}{2} \phi'. \]  

(4.29)

Finally we rewrite the solution of the equations (4.1) and (4.2) in the form

\[ T_\pm = \pm \frac{1}{2\kappa_0 c} J_\pm^2 + J_\pm' \pm \mu e^{2q\varphi} \pm \theta pe^{\phi/2}. \]  

(4.30)

Let us stress that besides the well known terms \( J^2, J' \) and \( e^\phi \), we have obtained the new one \( \theta pe^{\phi/2} \), linear in the momenta.

**B. Effective action**

Using the general canonical formalism described in the introduction we are ready to derive the effective action

\[ W(\phi, p, h^+, h^-) = \int d^2 \xi (p\dot{\phi} - h^+T_+ - h^-T_-), \]  

(4.31)
knowing the expressions for the constraints $T_{\pm}$ (4.29-4.30).

To eliminate the momentum variable $p$, in order to obtain the second-order form of the action, we consider the equation of motion to be fulfilled

$$\hat{\phi} - \frac{1}{\kappa_0 c}(h^+ J_+ - h^- J_-) + (h^+ + h^-)' + \theta(h^- - h^+) e^{\phi/2} = 0. \tag{4.32}$$

With the help of (4.29) it yields

$$J_{\pm} = -\frac{\kappa_0 c}{\sqrt{2}}[\hat{J}_{\pm} \phi + \hat{\omega}_- - \hat{\omega}_+ + \sqrt{2} \theta e^{\phi/2}]. \tag{4.33}$$

Substituting this back in (4.31) we obtain

$$W(\phi, h^+, h^-) = -\frac{\kappa_0 c}{2}[W_L(\phi, h^+, h^-) + W_\omega(h^+, h^-) + \theta W_\Omega(\phi, h^+, h^-)], \tag{4.34}$$

where

$$W_L = \int d^2 \xi \sqrt{-\hat{g}} \{\hat{\partial}_+ \hat{\phi} \hat{\partial}_- \phi + \hat{R} \phi + M^2 e^{\phi}\}, \quad \left( M^2 = 2\theta^2 - \frac{4\mu}{\kappa_0 c} \right) \tag{4.35}$$

$$W_\omega = \int d^2 \xi \sqrt{-\hat{g}} (\hat{\omega}_- - \hat{\omega}_+)^2 = 2 \int d^2 \xi \sqrt{-\hat{g}} \hat{g}^{00} \left( \left( \hat{g}_{01} \right)^2 \right), \tag{4.36}$$

and

$$W_\Omega = 2\sqrt{2} \int d^2 \xi \sqrt{-\hat{g}} (\hat{\phi}_+ - \hat{\omega}_+ + \hat{\omega}_- + \hat{\phi}_-) e^{\phi/2} = \int d^2 \xi \hat{\phi} \hat{\phi}, \tag{4.37}$$

with

$$\hat{R} = 2\hat{\nabla}_- \hat{\omega}_+ - 2\hat{\nabla}_+ \hat{\omega}_-, \quad \Omega^0 = 4e^{\phi/2}, \quad \Omega^1 = 2(h^- + h^+) e^{\phi/2}. \tag{4.38}$$

The first term, $W_L$, is well known Liouville action. The cosmological term has two independent contributions, usual one proportional to $\mu$ and a new one proportional to $\theta^2$. The second term $W_\omega$ depends only on the two components of the gravitational fields $h^+$ and $h^-$ and is Weyl invariant. It is exactly the same as in the ref. [6]. The third term $W_\Omega$ is the new one. It stems from the term linear in the momenta in the hamiltonian expression for the energy-momentum tensor. In the lagrangian formulation it is a total derivative and for
manifolds without boundary, as we have supposed here, it vanishes. In that case, the only contribution of the hamiltonian $\theta$-term to the lagrangian is the cosmological term. So, for manifolds without boundary the complete effective action is

$$W = -\frac{\kappa_0 c}{2}(W_L + W_\omega).$$  \hspace{1cm} (4.39)

C. From the Virasoro to the trace anomaly

The central term in the Virasoro algebras changes the constraints from the FCC to the SCC and breaks the diffeomorphism invariance. Using the extension of the general canonical method to the case when SCC are present [1], with the help of (1.5) and (3.12) we can easily find the variation of the effective action

$$\delta W = -\int d\tau d\sigma \int d\bar{\sigma} [h^+(\sigma)\Delta_+(\sigma, \bar{\sigma})\varepsilon_- (\bar{\sigma}) + h^-(\sigma)\Delta_-(\sigma, \bar{\sigma})\varepsilon_+ (\bar{\sigma})]$$

$$= \kappa_0 c \int d^2\xi [\varepsilon_-(h^+)^m - \varepsilon_+(h^-)^m].$$  \hspace{1cm} (4.40)

It is possible to add finite local counterterm, depending only on the gravitational fields $\Delta W(h^+, h^-, F)$, and to shift anomaly. Here we want to obtain the reparameterization invariance, so we require

$$\delta \Delta W = -\delta W.$$  \hspace{1cm} (4.41)

It is easy to find expression for $\Delta W$ from our previous derivations. In (4.30) we just put $c \rightarrow -c, \theta \rightarrow 0$, and the most important $\phi \rightarrow 2F$ in order to obtain the energy-momentum tensor for $\Delta W$. We change the sign of $c$ to ensure (4.41) and expunge $\theta$ term for simplicity. There are two reasons for substituting auxiliary field $\phi$ with the conformal part of the metric $2F$. First, the local counterterm $\Delta W$ must depend only on the metric components and second the field $\frac{\phi}{2}$ has desirable transformation properties. From the first transformation (1.2), for $R \rightarrow \frac{\phi}{2}$, we obtain

$$\delta \frac{\phi}{2} = -\frac{1}{2} (\varepsilon_+ + \varepsilon_-)' + (\varepsilon_+ - \varepsilon_-) \frac{(h^- + h^+)'}{2(h^- - h^)} - \frac{1}{\sqrt{2}} (\varepsilon_- \partial_+ - \varepsilon_+ \partial_-) \frac{\phi}{2},$$  \hspace{1cm} (4.42)
which for \( \varepsilon^\pm = \varepsilon^1 - h^\pm \varepsilon^0 \) and \( \phi \to 2F \) gives

\[
\delta F = -\partial_1 \varepsilon^1 + \frac{1}{2} (h^- + h^+) \partial_1 \varepsilon^0 - \varepsilon^\mu \partial_\mu F .
\]

(4.43)

This equation together with (2.14) completes the transformation of the metric tensor (2.3) (see last ref. [3]), yielding the well known expression

\[
\delta g^{\mu \nu} = g^{\mu \rho} \partial_\rho \varepsilon^\nu + g^{\nu \rho} \partial_\rho \varepsilon^\mu - \varepsilon^\rho \partial_\rho g^{\mu \nu} .
\]

(4.44)

Then, in analogy with (4.34) we obtain

\[
\Delta W = \frac{\kappa_0 c}{2} [ W_L(2F, h^+, h^-) + W_\omega(h^+, h^-) ] ,
\]

(4.45)

and the complete effective action is

\[
W + \Delta W = -\frac{\kappa_0 c}{2} W_L(g_{\mu \nu}, \varphi) ,
\]

(4.46)

with

\[
W_L(g_{\mu \nu}, \varphi) = W_L(\phi, h^+, h^-) - W_L(2F, h^+, h^-)
\]

\[
= \int d^2 \xi \sqrt{-g} [ \partial_+ \varphi \partial_- \varphi + R \varphi + M^2 (e^\varphi - 1) ] .
\]

(4.47)

Here we have used (2.3) as well as its consequence \( \sqrt{-g} R = \sqrt{-\hat{g}} (\hat{R} - 4\hat{\nabla}_+ \hat{\nabla}_- F) \) and introduced new field

\[
\varphi = \phi - 2F .
\]

(4.48)

From (4.42) and (4.43) we can find that \( \varphi \) is a scalar field \( \delta \varphi = -\varepsilon^\mu \partial_\mu \varphi \), because the \( \phi \) and \( F \) independent terms are canceled. Note that from the same reason \( W_\omega \) term disappeared.

The new expression for the Liouville action (4.47) is manifestly reparameterization invariant, according to our construction. To achieve this, we have introduced the field \( F \), conformal part of the metric. Under Weyl rescaling \( g_{\mu \nu} \to e^{2\sigma} g_{\mu \nu} \) we have \( F \to F + \sigma \) while \( h^\pm \) are invariant. So, from (4.47) we can find

\[
\delta W_L = 2\sqrt{-g} (R + M^2) \sigma ,
\]

(4.49)
which is known expression for the trace anomaly.

Now, we are ready to come back to the equation (3.1) and find the effective action for our model. Both parts, the scalar and the FP one, have the same form of the effective action (4.46) proportional to the Liouville action (4.47). Only the central charges are different. From (3.13) we have $c^x = D$ and $c^{gh} = -26$ so that

$$W_{eff} = -\frac{\hbar}{48\pi}(D - 26)W_L(g_{\mu\nu}, \varphi), \quad (4.50)$$

where we put already obtained expression for $\kappa_0 = \frac{\hbar}{24\pi}$.

The coefficient in front of the Liouville action is just $\frac{1}{2}$ of the central term for total energy-momentum tensor

$$T_\pm = T^x_\pm + T^{gh}_\pm. \quad (4.51)$$

In the critical dimension $D = 26$, the constraints $T_\pm$ become FCC and the anomaly disappear. Originally, the expression for the induced gravity has been obtained from trace anomaly. Here, we calculated reparameterization anomaly, and only after shifting it with the counterterm $\Delta W$ we obtain the result of ref. [4].

V. CONCLUSION

In this paper we derived the Liouville theory as gravitational Wess-Zumino term using hamiltonian method. This approach explicitly shows why the result, up to coefficient in front of the effective action, does not depend on the starting theory. It only depend on a sort of the symmetry, and here as consequence of the reparameterization invariance we got the Virasoro algebras of the constraints and Liouville action as an effective action. The trace of the original theory is in the central charge, which measures the numbers of the new degrees of freedom obtained after quantization, and it appears as a coefficient in front of Liouville action in the effective action. Here we have two examples: the scalar and the ghost fields. They have different structure of the energy-momentum tensors: the square of the
currents bilinear in the momenta for scalar fields (2.7) and linear in the momenta for the ghosts (2.23).

In Sec. II we fixed the reparameterization invariance in the Polyakov string theory and made the canonical analysis for both scalar and ghost part. Then we performed the quantization of the matter and ghost fields in sec. III, obtaining the central charges in the algebra of constraints, which change the nature of constraints from the FCC to the SCC.

Sec. IV contains the main results of the paper. Here we proved that the bosonic representation of Virasoro algebra, with arbitrary central charge, must have a constraints of the form (4.30). They differ from the standard expressions by a new term linear in the momenta. We constructed the new bosonic theory which is equivalent to the quantum theory of the original action. For the ghost part it is bosonization, but for the matter field it is a non trivial procedure because of the presence of anomaly. The hamiltonian $\theta$ term yielded two terms in the lagrangian formulation, one was a total derivative and the other contributed to the mass term. The first one could be important for the manifolds with boundary, which we will investigate in the separate article. Note that even in the case when the $\mu$-term is absent in the hamiltonian approach the lagrangian will contain the mass term through the $\theta$-part.

According to our normal ordering prescription, which has the role of the regularization scheme we obtained the reparameterization anomaly. In order to recover the diffeomorphism invariance we added the finite local counterterm, which effectively changed the regularization scheme and shifted anomaly to the trace one.

Our final results (4.50) is a complete and independent derivation of the known Polyakov result, using hamiltonian method.

The canonical method, introduced in [1] for non-Abelian gauge symmetry and developed here for reparameterization symmetry could be applied to the supersymmetry and to the case where gauge fields and metric are dynamical variables. We will publish it separately.
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