On the algebra of quantum observables for a certain gauge model

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Abstract
We prove that the algebra of observables of a certain gauge model is generated by unbounded elements in the sense of Woronowicz. The generators are constructed from the classical generators of invariant polynomials by means of geometric quantization.
1 Introduction

One of the fundamental structures in nonperturbative quantum field theory is the algebra of observables and its representations. To construct the observable algebra and to find its irreducible representations for a gauge theory is a complicated task, see [28], [9] and [10] for attempts made in the seventies and eighties. Roughly speaking, one has to start with a model of the field algebra carrying the action of the gauge group by automorphisms, next one has to pass to the algebra of gauge invariant elements and, finally, one has to factorize this algebra by an ideal generated by the Gauss law. Unfortunately, standard charge superselection theory [6, 7, 8] does not apply to genuine local gauge theories, see [3, 4].

In order to separate functional analytical problems related to the mathematical nature of quantum fields on continuous space time from those related to the gauge structure, one is tempted to consider, in a first step, models approximated on a finite lattice. In this context, we have constructed the observable algebras and classified their irreducible representations both for quantum electrodynamics [22, 21] and for quantum chromodynamics [19, 20]. An additional challenge comes from the fact that on the classical level there are nongeneric gauge orbit strata, see [25] for a review, which should have an impact on quantum level as well. In [16] we have shown that one can include these singularities by using the concept of a costratified Hilbert space [13]. In the case of chromodynamics, a full understanding of the observable algebra in terms of generators and defining relations is still lacking, see [17] for preliminary results. Generally speaking, gauge invariant generators are polynomial invariants built from gauge and matter fields, corresponding to classical generators of the algebra of polynomial invariants. Since typical quantum observables are unbounded operators, one cannot hope to incorporate all observables in a naïve sense into the observable algebra. Fortunately, there is a suitable approach developed by Woronowicz in the nineties [30], which makes it possible to say that a given number of unbounded elements generates a certain $C^*$-algebra, with the generators being affiliated with the algebra under consideration in the $C^*$-sense. We remark that recently another construction of a $C^*$-algebra of observables from unbounded physical quantities was invented, see [5]. In [20] we have shown that the field algebra of quantum chromodynamics is a $C^*$-algebra of this type. In the present paper we prove that the algebra of observables of the model studied in [16] is also generated by unbounded operators in the sense of Woronowicz. It is a challenge to extend this result to full chromodynamics on a finite lattice in the future. In the case at hand, the generating operators are the quantum counterparts of the generators of the algebra of real invariant polynomials on the reduced phase space. This is an interesting fact in itself, because in the Woronowicz theory there does not exist a general method to find a set of generators of a given $C^*$-algebra, nor does there exist a general method to find the $C^*$-algebra generated by a given set of unbounded operators.

The paper is organized as follows: In Section 2, we briefly present the underlying classical model. In Section 3, we present the algebra of classical observables and its generators. Section 4 is devoted to quantum observables. First we quantize the classical generators
using geometric quantization. Next, we discuss the spectral properties of the quantized generators and the quantum counterpart of the relation amongst the classical generators. Then, we construct the algebra of quantum observables and discuss the relations between our generators and the generators defined in [20]. Finally, we comment on quantum dynamics and give an outlook.

2 The model

The model was explained in detail in [16]. We recall the main facts. The configuration space is the group manifold \( G = SU(2) \), acted upon by \( G \) itself by inner automorphisms,

\[
g \cdot a = gag^{-1}.
\]

The phase space is given by the cotangent bundle \( T^*G \), acted upon by the lifted action. This action is symplectic and it possesses a natural equivariant momentum mapping \( \mu : T^*G \rightarrow \mathfrak{g}^* \), where \( \mathfrak{g} \) denotes the Lie algebra of \( G \). Thus, the phase space carries the structure of a Hamiltonian \( G \)-manifold. We trivialize \( T^*G \simeq G \times \mathfrak{g} \) by means of an invariant scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \) and left translation. In these coordinates, the lifted action is given by

\[
g \cdot (a, X) = (gag^{-1}, \text{Ad}(g)X), \quad a \in G, X \in \mathfrak{g}, g \in G,
\]

and the natural momentum mapping is given by

\[
\mu(a, X) = aXa^{-1} - X. \tag{1}
\]

W.r.t. the natural decomposition

\[
T_{(a, X)}(G \times \mathfrak{g}) = T_aG \oplus T_X\mathfrak{g}, \tag{2}
\]

tangent vectors at \( (a, X) \in G \times \mathfrak{g} \) can be written in the form

\[
(L'_aA, B), \tag{3}
\]

where \( A, B \in \mathfrak{g} \) and \( L_a \) means left multiplication by \( a \). In this notation, the symplectic potential reads

\[
\theta_{(a, X)}((L'_aA, B)) = \langle X, A \rangle, \quad A, B \in \mathfrak{g}, \tag{4}
\]

and the symplectic form \( \omega = -d\theta \) is given by

\[
\omega_{(a, X)}((L'_aA_1, B_1), (L'_aA_2, B_2)) = \langle A_1, B_2 \rangle - \langle A_2, B_1 \rangle + \langle X, [A_1, A_2] \rangle. \tag{5}
\]

The model can be interpreted as an SU(2)-lattice gauge theory on a single spatial plaquette in the Hamiltonian approach in the tree gauge, or as SU(2)-gauge theory on a space-time cylinder in the temporal gauge and after reduction by the group of based gauge transformations, see [16]. In both cases, the classical Hamiltonian is given by

\[
H(a, X) = -\frac{1}{2}|X|^2 + \frac{\nu}{2}(3 - \Re \text{tr}(a)), \quad a \in G, X \in \mathfrak{g}. \tag{6}
\]
Let $T$ denote the subgroup of $G$ of diagonal matrices and $t$ the subalgebra of $\mathfrak{g}$ of diagonal matrices. Let $W$ denote the Weyl group. It acts on $T$ and $t$ by permutation of entries. The reduced configuration space $\mathcal{X}$ is given by the adjoint quotient

$$\mathcal{X} = G/\text{Ad}(G) \cong T/W.$$ 

For general SU($n$), this is an $(n-1)$-simplex. For SU(2), the parameterization

$$\phi : \mathbb{R} \to T, \quad x \mapsto \text{diag}(e^{ix}, e^{-ix})$$

induces a homeomorphism $[0, \pi] \cong \mathcal{X}$. The reduced phase space is the zero level singular symplectic quotient

$$\mathcal{P} = \mu^{-1}(0)/G.$$ 

Since, according to (1), $\mu(a, X) = 0$ means that $a$ and $X$ commute and hence can be simultaneously diagonalized, $\mathcal{P}$ may be identified with the quotient $(T \times t)/W$. For SU(2), this amounts to the cylinder $U(1) \times \mathbb{R}$, factorized by reflection about the (virtual) line connecting the points $(1, 0)$ and $(-1, 0)$, see Figure 1. The space arising this way is known as the canoe. It coincides with the phase space of a spherical pendulum, reduced at zero angular momentum by the rotations about the vertical axis.

The reduced configuration space and the reduced phase space are stratified by connected components of orbit type subsets,

$$\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_+ \cup \mathcal{X}_-, \quad \mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_+ \cup \mathcal{P}_-,$$

where $\mathcal{X}_\pm$ consists of the class of $\pm 1$ and $\mathcal{P}_\pm$ consists of the class of the zero covector over $\pm 1$, see Figure 1.

### 3 Classical observables

The algebra of classical observables, as provided by standard singular symplectic reduction at level 0, is given by the quotient Poisson algebra

$$\mathcal{O}_c = C^\infty(T^*G)^{G}/V^{G},$$

where $V$ denotes the vanishing ideal of the closed subset $\mu^{-1}(0)$ [1]. This algebra contains as a Poisson subalgebra the quotient $\text{Pol}(T^*G)^{G}/V_{\text{Pol}}^{G}$, where $\text{Pol}(T^*G)$ denotes the algebra of real polynomials on $T^*G$ and $V_{\text{Pol}}$ is the vanishing ideal of $\mu^{-1}(0)$ in this algebra. By definition, a function on $T^*G$ is polynomial if via the diffeomorphism $T^*G \cong G \times G$ it corresponds to a function that is polynomial in the matrix entries.

**Remark 1.** One could also define polynomial functions on $T^*G$ to be functions which via the diffeomorphism $T^*G \cong G^C$ correspond to elements of $\text{Pol}(G^C)$, i.e., to the functions on $G^C$ that are polynomial in the matrix entries. This type of polynomial functions was used in [14]. Since polar decomposition is non-polynomial, the two types of polynomial functions lead to completely different subalgebras of $\mathcal{O}_c$ which intersect only in the constants.
The generators of \( \text{Pol}(T^*G)^G \) are provided by invariant theory. For \( G = \text{SU}(n) \) it is known that, via the diffeomorphism \( T^*G \cong G \times \mathfrak{g} \), a set of generators is provided by the real and imaginary parts of arbitrary trace monomials of order \( 2^n - 1 \) in \( a, a^\dagger \in G \) and \( X \in \mathfrak{t} \). By means of the fundamental trace identity and the Cayley-Hamilton theorem this set of generators can be reduced considerably. For \( \text{SU}(2) \) there remain 3 generators,

\[
\begin{align*}
    f_0(a, X) &= \text{tr}(a), \\
    f_1(a, X) &= \frac{1}{2\beta^2} \text{tr}(aX), \\
    f_2(a, X) &= -\frac{1}{2\beta^2} \text{tr}(X^2).
\end{align*}
\]

Here \( \beta \) is a scaling factor, defined by

\[
\langle X, Y \rangle = -\frac{1}{2\beta^2} \text{tr}(XY), \quad X, Y \in \mathfrak{g}.
\]

The functions \( f_0, f_1, f_2 \) are already real. For convenience, the generators \( f_1 \) and \( f_2 \) have been rescaled by the scaling factor of the invariant scalar product on \( \mathfrak{g} \). This way, \( f_2 \) is twice the kinetic energy. In terms of the generators, the Hamiltonian \((9)\) reads

\[
H = \frac{1}{2} f_2 + \frac{1}{2\beta^2} (3 - f_0).
\]

I.e., up to a shift and up to a coupling parameter, \( f_0 \) is the potential energy of the system.

Remark 2. For \( G = \text{SU}(n) \), \( n \geq 3 \), to cut to size the set of generators one also has to make use of the fact that in the level set \( \mu^{-1}(0) \), \( a \) and \( X \) commute. This is not necessary for \( \text{SU}(2) \) though. I.e., here the set of generators of invariant polynomials for the reduced phase space \( \mathcal{P} \) and for the full quotient \( T^*G/G \) coincide.

The generators \( f_0, f_1, f_2 \) define a map \( \mathcal{P} \to \mathbb{R}^3 \), known as the Hilbert map associated with this set of generators. It is common knowledge, see e.g. [29], that the Hilbert map...
is a homeomorphism onto its image and that the image is a semialgebraic subset, i.e., a subset defined by equalities and inequalities. The defining equalities and inequalities for our case are obtained as follows. Up to diagonal conjugation, an arbitrary element \((a,X) \in G \times g\) can be written

\[
a = \begin{bmatrix} \alpha & 0 \\ 0 & \tau \end{bmatrix}, \quad X = \begin{bmatrix} ix & z \\ -z & -ix \end{bmatrix}, \quad \alpha \in U(1), \ x \in \mathbb{R}, \ z \in \mathbb{C}.
\]

Then

\[
f_0(a,X) = 2\Re(\alpha), \quad f_1(a,X) = -\frac{1}{\beta^2}x \Im(\alpha), \quad f_2(a,X) = \frac{1}{\beta^2}(x^2 + |z|^2).
\]

Eliminating \(x\) and \(\alpha\) we obtain the relations

\[
(\beta^2 f_2 - |z|^2)(4 - f_2^2) - 4\beta^4 f_1^2 = 0, \quad \beta^2 f_2 - |z|^2 \geq 0,
\]

where now \(f_0, f_1\) and \(f_2\) are interpreted as standard coordinates in \(\mathbb{R}^3\). Hence, the image of the full quotient \(T^*G/G\) under the Hilbert map coincides with the set of points \((f_0, f_1, f_2) \in \mathbb{R}^3\) satisfying (10) for some \(z\). Since (10) implies \(4 - f_1^2 \geq 0\) and since \(|z|\) can take any nonnegative value, this subset is given by the two inequalities

\[
f_2(4 - f_0^2) - 4\beta^2 f_1^2 \geq 0, \quad 4 - f_0^2 \geq 0.
\]

If \(a\) and \(X\) commute then, up to diagonal conjugation, \(z = 0\). Hence, the reduced phase space \(\mathcal{P} \subseteq T^*G/G\) corresponds to the subset of (10) defined by \(z = 0\). Thus, this subset is given by the relation

\[
f_2(4 - f_0^2) - 4\beta^2 f_1^2 = 0
\]

and the inequality

\[
f_2 \geq 0.
\]

This subset is shown in Figure 2. It is of course a concrete realization of the canoe, see Figure 1. The image of the full quotient \(T^*G/G\) corresponds to this subset together with the interior.

Remark 3. From Figure 2 it is obvious that, topologically, \(\mathcal{P}\) is just a copy of \(\mathbb{R}^2\). In fact, the Hilbert map defined by the natural generators of the polynomial algebra \(\text{Pol}(G^C)^G\), see Remark 1, identifies \(\mathcal{P}\) with the complex plane. However, as Poisson spaces, \(\mathcal{P}\) and \(\mathbb{C}\) are distinct. This generalizes to \(SU(n)\). See [14] for details.

Next, we compute the Hamiltonian vector fields \(X_{f_i}\) associated with the generators \(f_i\) and the Poisson brackets between the generators. Let \(P : M_2(\mathbb{C}) \to g\) denote the orthogonal projection, i.e.,

\[
P(A) = \frac{1}{2}(A - A^\dagger) - \frac{i}{2}(\text{Im tr } A)1.
\]

The defining equation for \(X_{f_i}\) is \(\omega(X_{f_i}, Y) = -Y(f_i)\) for all vector fields \(Y\) on \(G \times g\). Writing

\[
(X_{f_i})_{(a,X)} = (L'_a A_i, B_i)
\]
and $Y_{(a,X)} = (L'_a C, D)$ and using (5) we obtain

$$\langle A_i, D \rangle - \langle B_i + [A_i, X], C \rangle = - \frac{d}{dt}|_{t=0} f_i (ae^{Ct}, X + tD) \quad \forall \ C, D \in g.$$ 

Evaluating the r.h.s. and solving for $A_i$ and $B_i$ we arrive at

$$A_0 = 0, \quad B_0 = -2\beta^2 P(a),$$

$$A_1 = P(a), \quad B_1 = -P(aX),$$

$$A_2 = -2X, \quad B_2 = 0.$$  

(Calculations are simplified by observing that tr($a$) and tr($aX$) are real, hence the trace term in (13) is absent in both cases.)

**Lemma 1.** The Hamiltonian vector fields $X_{f_0}$, $X_{f_1}$, and $X_{f_2}$ are complete.

**Proof.** The flows of $X_{f_0}$ and $X_{f_2}$ are immediate:

$$\Phi^t_{X_{f_0}}(a, X) = (a, X - 2\beta^2 P(a)t), \quad \Phi^t_{X_{f_2}}(a, X) = (ae^{-2Xt}, X).$$

They are defined for all $t \in \mathbb{R}$. The flow equations for $X_{f_1}$ are

$$\dot{a} = L'_a P(a), \quad \dot{X} = -P(aX).$$

Let $(a(0), X(0))$ be arbitrary but fixed initial values. Since $\dot{a}$ does not depend on $X$ and since $G$ is compact, the solution $a(t)$ exists for all $t \in \mathbb{R}$. Hence, for $X_{f_1}$ to be complete it suffices that $|X(t)|$ be finite for any $t \in \mathbb{R}$. Consider the function $f(t) = |X(t)|^2$. A brief computation using the Cayley-Hamilton theorem for $X$ yields $\frac{d}{dt} f(t) = -\text{tr}(a(t)) f(t)$. Then $\frac{d}{dt} f(t) \leq |\frac{d}{dt} f(t)| \leq 2f(t)$. Thus, $f(t)$ is a nonnegative function whose derivative at $t$ is bounded by $2f(t)$. It follows $f(t) \leq f(0)e^{2t}$, hence the assertion.

Finally, we calculate the Poisson brackets between the generators,

$$\{f_i, f_j \} = X_{f_i} f_j = \frac{d}{dt}|_{t=0} f_j (ae^{A_it}, X + tB_i).$$
Using the Cayley-Hamilton theorem to reduce powers of $a$ and $X$ we obtain
\[
\{f_0, f_1\} = 2 - \frac{1}{2} f_0^2, \quad \{f_0, f_2\} = 4 \beta^2 f_1, \quad \{f_1, f_2\} = -f_0 f_2. \tag{18}
\]
Since restriction of $f_i$ to the singular strata $\mathcal{P}_\pm$ yields $f_0|_{\mathcal{P}_\pm} = \pm 2$ and $f_1|_{\mathcal{P}_\pm} = f_2|_{\mathcal{P}_\pm} = 0,
\{f_i, f_j\}|_{\mathcal{P}_\pm} = 0.
Thus, the Poisson structure of the reduced phase space $\mathcal{P}$, given by (18), reduces consistently to the (necessarily trivial) Poisson structure on the singular strata $\mathcal{P}_\pm$.

4 Quantum observables

4.1 Quantization of classical generators

The algebra of quantum observables will be constructed as follows. We quantize the generators $f_i$ of the algebra of classical observables by means of geometric quantization in the vertical polarization (‘Schrödinger quantization’) on the unreduced phase space and subsequent reduction. As the algebra of quantum observables we will then take the $C^*$-algebra generated by these operators in the sense of Woronowicz. The latter notion will be explained below. We will loosely speak of the quantized generators as quantum observables as well, although they do not belong to the algebra of quantum observables so constructed.

The Hilbert space of Schrödinger quantization on $T^*G$ can be identified canonically with $L^2(G)$ with scalar product
\[
\langle \psi_1 | \psi_2 \rangle = \frac{1}{\text{vol}(G)} \int_G \overline{\psi_1} \psi_2 da.
\]
Here $da$ stands for the volume form associated with the bi-invariant Riemannian metric on $G$ defined by the invariant scalar product on $g$. By virtue of the isomorphism $G \times g \cong TG$, $f_2$ corresponds to the bi-invariant Riemannian metric defined by the invariant scalar product on $g$. Let $\Delta_G$ denote the Laplacian associated with this metric. Since $G$ is closed, $\Delta_G$ is essentially self-adjoint on the domain $C^\infty(G)$. The quantum observable $\hat{f}_2$ associated with $f_2$ is the unique self-adjoint extension of $-\hbar^2 \Delta_G$ see [27, §9.7]. Thus, on the core $C^\infty(G),
\hat{f}_2 \psi = -\hbar^2 \Delta_G \psi, \quad \psi \in C^\infty(G). \tag{19}
In order to determine the quantum observables $\hat{f}_0$ and $\hat{f}_1$ associated with the generators $f_0$ and $f_1$, respectively, we have to recall the main steps in the construction of the Hilbert space and the quantum observables in the Schrödinger quantization [11, 27]. Prequantization renders the complex line bundle $L = T^*G \times \mathbb{C}$ with Hermitian form $h((a, z_1), (a, z_2)) = \overline{z_1} z_2$ and connection $\nabla = d + \theta$, where $\theta$ denotes the symplectic potential of $T^*G$. Let $\pi : T^*G \to G$ denote the canonical projection. The vertical polarization is given by the vertical distribution $D \subseteq T(T^*G)$ induced by the fibres of $T^*G$. 

**Hilbert space:** Consider the tautological complex line bundle \( \kappa := \Lambda^n \text{Ann}(D^2) \), where \( n = \dim(G) \) and \( \text{Ann} \) denotes the annihilator of \( D^2 \) in \( \mathcal{T}(C)^*(T^*G) \). The pull-back \( \pi^* v \) of the volume form \( v \) associated with the Riemannian metric on \( G \) defines a global section in \( \kappa \). Hence \( \kappa \) is trivial and there exists a real line bundle \( \delta \) over \( T^*G \) such that \( \kappa := (\delta \otimes \delta)^C \). The bundle \( \delta \) is called the half-form bundle associated with \( D \). By choosing a square root \( \sqrt{\pi^* v} \) of \( \pi^* v \) one obtains a global nowhere vanishing section in \( \delta \), hence \( \delta \) is trivial, too.

Let \( \Gamma_{\text{pol}}(L \otimes \delta) \) denote the space of polarized sections in \( L \otimes \delta \). By definition, a section \( \varphi \otimes \nu \) in \( L \otimes \delta \) is polarized if so are \( \varphi \) and \( \nu \). A section \( \varphi \in \Gamma(L) \), viewed as a function on \( T^*G \), is polarized if it is constant along the fibres, i.e., if \( \varphi = \pi^* \psi \) for some \( \psi \in \mathcal{C}^\infty(G) \). A section \( \nu \) in \( \delta \) is polarized if \( \nu \otimes \nu = \pi^* \alpha \) for some \( n \)-form \( \alpha \) on \( G \). The Hilbert space \( L^2_{\text{pol}}(L \otimes \delta) \) is defined as the completion of \( \Gamma_{\text{pol}}(L \otimes \delta) \) w.r.t. the norm defined by the following intrinsic scalar product: if \( \varphi_1 \otimes \nu_1 \) and \( \varphi_2 \otimes \nu_2 \) are polarized, \( h(\varphi_1, \varphi_2)\nu_1 \otimes \nu_2 = \pi^* \beta \) for some \( n \)-form \( \beta \) on \( G \). Then

\[
\langle \varphi_1 \otimes \nu_1 | \varphi_2 \otimes \nu_2 \rangle := \frac{1}{\text{vol}(G)} \int_G \beta.
\]

Finally, one can pass from half forms to functions on \( G \) by observing that any element of \( \Gamma_{\text{pol}}(L \otimes \delta) \) can be written in the form \( \varphi \otimes \sqrt{\pi^* v} \) with \( \varphi = \pi^* \psi \) for some \( \psi \in \mathcal{C}^\infty(G) \). By construction,

\[
\langle \pi^* \psi_1 \otimes \sqrt{\pi^* v} | \pi^* \psi_2 \otimes \sqrt{\pi^* v} \rangle = \langle \psi_1 | \psi_2 \rangle.
\]

Hence, the assignment

\[
\psi \mapsto \pi^* \psi \otimes \sqrt{\pi^* v}, \quad \psi \in \mathcal{C}^\infty(G),
\]

defines a unitary isomorphism from \( L^2(G) \) onto \( L^2_{\text{pol}}(L \otimes \delta) \).

**Quantization of polarized classical observables:** A classical observable \( f \in \mathcal{C}^\infty(T^*G) \) is polarized if the Hamiltonian vector field \( X_f \) associated with \( f \) satisfies \( [X_f, \Gamma(D)] \subseteq \Gamma(D) \). The operator \( \hat{f} \) associated with \( f \) is then defined by

\[
\hat{f}(\varphi \otimes \nu) = ((\text{i} \hbar X_f + \theta(X_f) + f)\varphi) \otimes \nu + \varphi \otimes (\text{i} \hbar \mathcal{L}_{X_f} \nu), \quad \varphi \otimes \nu \in \Gamma_{\text{pol}}(L \otimes \delta).
\]

Here, \( \mathcal{L}_X \) denotes the Lie derivative w.r.t. the vector field \( X \) on \( T^*G \), which is defined on sections of \( \delta \) by virtue of the Leibniz rule

\[
(\mathcal{L}_X \nu) \otimes \nu := \frac{1}{2} \mathcal{L}_X (\nu \otimes \nu).
\]

The first term in (21) contains the ordinary quantization formula of Kostant and Souriau, whereas the second term represents the half-form correction. If \( X_f \) is complete, \( \hat{f} \) is essentially self-adjoint [27]. The argument is as follows. For any polarized \( f \), the flow of \( X_f \) lifts to a flow on \( L \otimes \delta \), where the lift to \( \delta \) is natural and the lift to \( L \) is defined by the connection \( \nabla \). If \( X_f \) is complete, the lifted flow induces a strongly continuous 1-parameter group of unitary transformations on \( L^2_{\text{pol}}(L \otimes \delta) \). The self-adjoint generator of this group, which exists due to Stone’s theorem, has the subspace \( \Gamma_{\text{pol}}(L \otimes \delta) \) as a core and on this core it is given by [27]:

\[
\hat{f}(\varphi \otimes \nu) = ((\text{i} \hbar X_f + \theta(X_f))\varphi) \otimes \nu + \varphi \otimes (\text{i} \hbar \mathcal{L}_{X_f} \nu), \quad \varphi \otimes \nu \in \Gamma_{\text{pol}}(L \otimes \delta).
\]
By adding the multiplication operator by the real function $f$ we obtain $\hat{f}$ as an essentially self-adjoint operator. By virtue of the isomorphism between $L^2(G)$ and $L^2_{pol}(L \otimes \delta)$ defined by (20), $\hat{f}$ is mapped to an essentially self-adjoint operator on $L^2(G)$ which will be denoted by $\hat{f}$ as well. According to (21), the defining equation for this operator is

$$\pi^*(\hat{f} \psi) \otimes \sqrt{\pi^* v} = \left((i\hbar X_f + \theta(X_f) + f)\pi^* \psi \right) \otimes \sqrt{\pi^* v} + \pi^* \psi \otimes (i\hbar \mathcal{L}_{X_f} \sqrt{\pi^* v}),$$

where $\psi \in C^\infty(G)$.

**Quantization of $f_0$ and $f_1$:** We check that $f_0$ and $f_1$ are polarized. Since in the decomposition (2), elements of $\Gamma(D)$ are characterized by having zero first component, it suffices to take the commutator of $X_{f_i}$ with the constant vector fields $(0, B)$ where $B \in \mathfrak{g}$. Since, according to (15) and (16), the first components of $X_{f_0}$ and $X_{f_1}$ do not depend on the momentum variable $X$, only their second components can contribute to the commutator $[X_{f_i}, (0, B)]$. Since $\Gamma(D)$ is integrable, then $[X_{f_i}, (0, B)] \in \Gamma(D)$, $i = 0, 1$.

Next, we determine $\hat{f}_0$ and $\hat{f}_1$ using (23). For $f = f_0$, (4) and (15) yield $\theta(X_{f_0}) = 0$ as well as $X_{f_0}\pi^* \psi = 0$ and $\mathcal{L}_{X_{f_0}} \pi^* v = 0$, hence $\mathcal{L}_{X_{f_0}} \sqrt{\pi^* v} = 0$. Thus, (23) yields

$$\hat{f}_0 \psi = f_0 \psi, \quad \psi \in C^\infty(G),$$

where on the r.h.s., $f_0$ is viewed as a function on $G$ rather than on $T^* G$. As $G$ is compact, $f_0$ is bounded, hence $\hat{f}_0$ extends to a bounded self-adjoint operator on $L^2(G)$, which will be denoted by the same symbol.

For $f = f_1$, (4) and (16) yield $\theta(a, X)(X_{f_1}) = \langle X, P(a) \rangle = -\frac{1}{2\pi} tr(Xa) = -f_1(a, X)$. Hence,

$$\theta(X_{f_1}) + f_1 = 0. \quad (25)$$

Furthermore, we observed before that the first component of $X_{f_1}$ in the decomposition (2) does not depend on the momentum variable $X$. Hence, this component defines a vector field $Y_{f_1}$ on $G$. According to (16),

$$(Y_{f_1})_a = L'_a P(a), \quad a \in G. \quad (26)$$

By construction,

$$X_{f_1}\pi^* \psi = \pi^*(Y_{f_1} \psi), \quad (27)$$

$$\mathcal{L}_{X_{f_1}} \pi^* v = \pi^*(\mathcal{L}_{Y_{f_1}} v). \quad (28)$$

A straightforward computation yields

$$\mathcal{L}_{Y_{f_1}} v = \frac{3}{2} f_0 v, \quad (29)$$

see the appendix. Then (22) and (28) yield $\mathcal{L}_{X_{f_1}} \sqrt{\pi^* v} = \frac{3}{4} f_0 \sqrt{\pi^* v}$. Plugging in this as well as (25) and (27) into (23) we arrive at

$$\hat{f}_1 \psi = i\hbar \left(Y_{f_1} + \frac{3}{4} f_0 \right) \psi, \quad \psi \in C^\infty(G). \quad (30)$$
Since according to Lemma 1, the Hamiltonian vector field $X_{\hat{f}_1}$ is complete, $\hat{f}_1$ is essentially self-adjoint. From now on, $\hat{f}_1$ will denote the self-adjoint extension. Thus, all the operators $\hat{f}_0$, $\hat{f}_1$ and $\hat{f}_2$ are self-adjoint and have $C^\infty(G)$ as a common invariant core.

Remark 4. Consider the operator $i\hbar Y_{\hat{f}_1}$ on $C^\infty(G)$. Since $\hat{f}_1$ and $\hat{f}_0$ are symmetric, $i\hbar Y_{\hat{f}_1} = \hat{f}_1 - i\hbar^2 \hat{f}_0$ is not. Thus, the term $i\hbar^2 \hat{f}_0$, playing the role of the half-form correction in the quantization of $f_1$, can be characterized as the unique purely imaginary multiplication operator which has to be added to the 'naïve quantization' $i\hbar Y_{\hat{f}_1}$ of $f_1$ in order to obtain a symmetric operator.

Finally, by reduction after quantization we arrive at the Hilbert space $L^2(G)^G$ of $G$-invariant elements. Since the functions $f_i$ are $G$-invariant, by restriction, the operators $\hat{f}_i$ define self-adjoint operators on $L^2(G)^G$ which will be denoted by the same symbols. The subspace $C^\infty(G)^G$ is a common invariant core for these operators.

An orthonormal basis in $L^2(G)^G$ is provided by the real characters $\chi_n$, where $n = 0, 1, 2, \ldots$ is twice the spin and labels the irreducible representations of $G$. To have the formulae in the following proposition valid for all $n$, let $\chi_{-1} = 0$.

**Proposition 1.** In the basis of characters, the quantum observables $\hat{f}_i$ are given by

$$\hat{f}_0 \chi_n = \chi_{n+1} + \chi_{n-1},$$

$$\hat{f}_1 \chi_n = i\hbar \left( \frac{2n+3}{4} \chi_{n+1} - \frac{2n+1}{4} \chi_{n-1} \right),$$

$$\hat{f}_2 \chi_n = \hbar^2 \beta^2 n(n+2) \chi_n. \quad \text{(33)}$$

Accordingly, their matrix elements are

$$(\hat{f}_0)_{nm} = \delta_{n,m+1} + \delta_{n,m-1},$$

$$(\hat{f}_1)_{nm} = i\hbar \left( \frac{2n+3}{4} \delta_{n,m+1} - \frac{2n+1}{4} \delta_{n,m-1} \right),$$

$$(\hat{f}_2)_{nm} = \hbar^2 \beta^2 n(n+2) \delta_{nm}. \quad \text{(33)}$$

**Proof.** As $\hat{f}_2$ is the negative of the Laplacian on $G$, the formula for $\hat{f}_2$ is standard. As $\hat{f}_0$ is multiplication by $f_0$ and $f_0$ is the character of the fundamental representation, the formula for $\hat{f}_0$ reflects the ordinary reduction formula for tensor products. For $f_1$, it suffices to determine $(\hat{f}_1 \chi_n)(a)$ for $a \in T$. Write $a = \text{diag}(\alpha, \overline{\alpha})$ with $\alpha \in U(1)$. Then

$$\chi_n(a) = \alpha^n + \alpha^{n-2} + \cdots + \alpha^{-n}. \quad \text{(31)}$$

We compute $(Y_{f_1})_a \chi_n = \frac{d}{dt} \big|_{t=0} \chi_n \left( a e^{P(a)t} \right)$. Since $a e^{P(a)t} = \text{diag} \left( \alpha e^{\frac{t}{2}(\alpha - \overline{\alpha})t}, \overline{\alpha} e^{-\frac{t}{2}(\alpha - \overline{\alpha})t} \right)$, we have

$$\chi_n \left( a e^{P(a)t} \right) = \alpha^n e^{\frac{t}{2}(\alpha - \overline{\alpha})t} + \alpha^{n-2} e^{\frac{t}{2}(\alpha - \overline{\alpha})t} + \cdots + \alpha^{-n} e^{-\frac{t}{2}(\alpha - \overline{\alpha})t}. \quad \text{(32)}$$

Taking the derivative and sorting by powers of $\alpha$ we obtain

$$(Y_{f_1})_a \chi_n = \frac{\alpha}{2} \alpha^{n+1} - \alpha^{n-1} - \alpha^{-3} - \cdots - \alpha^{-n+1} + \frac{\alpha}{2} \alpha^{-n-1} = \frac{\alpha}{2} \chi_{n+1}(a) - \frac{\alpha}{2} \chi_{n-1}(a).$$

Combining this with (31) we arrive at (32).
Remark 5. Composition of the trivialization $T^*G \cong G \times g$ with the inverse of the polar decomposition on $G^\mathbb{C}$ yields a natural diffeomorphism $T^*G \cong G^\mathbb{C}$. In our situation, $G^\mathbb{C} = SL(2, \mathbb{C})$. By virtue of this diffeomorphism, the complex structure of $G^\mathbb{C}$ and the symplectic structure of $T^*G$ combine to a Kähler structure. Therefore, in addition to the vertical polarization defined by the fibres, $T^*G$ carries a canonical Kähler polarization defined by the Kähler structure. For quantization in this polarization (Kähler quantization) on a general compact Lie group see [11]. The Hilbert space $H_{\text{Kähler}}$ of Kähler quantization consists of holomorphic function on $G^\mathbb{C}$ which are square-integrable w.r.t. a certain measure. Since the elements of $H_{\text{Kähler}}$ are true functions rather than classes of functions, it is this space on which one constructs the costratified Hilbert space structure that implements the stratification of the reduced phase space on the level of the quantum theory, see [13] and [16]. There exists a natural unitary isomorphism between the Hilbert spaces of the Schrödinger and the Kähler quantization (‘generalized Bargmann-Segal transformation’). For general compact $G$, this isomorphism was first given in [11] in terms of the heat kernel on $G^\mathbb{C}$. Later on, in [15] a Peter-Weyl theorem for the Hilbert space of Kähler quantization was proved and it was used to show that the isomorphism between the two Hilbert spaces can also be obtained by matching irreducible components of the standard $G \times G$-representations on these two Hilbert spaces. For the subspaces of invariants this implies that here the unitary isomorphism is given by mapping each $\chi_n$ to the corresponding character on $G^\mathbb{C}$, normalized w.r.t. the specific scalar product on $H_{\text{Kähler}}$.

4.2 Domains, eigenvalues and spectra of the quantized generators

To investigate the operators $\hat{f}$, we pass from $L^2(G)^G$ to $L^2[0, \pi]$ as follows. Let $C^\infty[0, \pi]$ denote the Whitney smooth functions on the closed interval $[0, \pi]$ (i.e., smooth functions on the open interval $]0, \pi[)$ that can be smoothly extended outside $[0, \pi]$). Take the parameterization $\phi$ of the subgroup $T \subseteq G$ of diagonal matrices, see (7), and define a map $\Gamma : C^\infty(G) \to C^\infty[0, \pi]$ by

$$(\Gamma \psi)(x) = \sqrt{2} \sin(x) \psi(\phi(x)),$$

$x \in [0, \pi]$.

Lemma 2. $\Gamma$ extends to a unitary Hilbert space isomorphism $L^2(G)^G \to L^2[0, \pi]$.

Proof. We have to check that $\Gamma$ is isometric and that its image is dense in $L^2[0, \pi]$. Let $\psi, \varphi \in L^2(G)^G$. From the Weyl integration formula we know that

$$\int_G \overline{\psi} \varphi da = \int_T \overline{\psi} \varphi v dt,$$

where $da$ and $dt$ denote the Haar measures on $G$ and $T$, respectively, and $v$ is a density function that accounts for the volume of the orbits under inner automorphisms of $G$. For $G = SU(2)$,

$$\phi^*(v dt) = \frac{\text{vol}(G)}{\pi} \sin^2(x) dx.$$
Hence,

\[ \langle \psi | \varphi \rangle = \frac{1}{\text{vol}(G)} \int_T \bar{\psi} \varphi v \, dt = \frac{1}{\text{vol}(G)} \int_{-\pi}^{\pi} \phi^* (\bar{\psi} \varphi v) \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi(\phi(x)) \varphi(\phi(x)) \sin^2(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Gamma \psi)(x) (\Gamma \varphi)(x) \, dx. \]

Since \( \psi \) and \( \varphi \) are invariant under inner automorphisms, \( \Gamma \psi \) and \( \Gamma \varphi \) are invariant under reflection \( x \mapsto -x \). Hence, the integral over \([-\pi, \pi]\) gives twice the integral over \([0, \pi]\). This shows that \( \Gamma \) is isometric. Since the image of \( \Gamma \) contains the smooth functions with compact support inside the open interval \([0, \pi]\), it is dense in \( L^2[0, \pi] \). \( \square \)

We will need the image of \( C^\infty(G)^G \) under \( \Gamma \). Let \( C^\infty_{ev}[0, \pi] \) denote the subspace of \( C^\infty[0, \pi] \) of functions whose even order derivatives \( \psi^{(2n)} \), \( n = 0, 1, 2, \ldots \), vanish in 0 and \( \pi \):

\[ C^\infty_{ev}[0, \pi] = \{ \psi \in C^\infty[0, \pi] : \psi^{(2n)}(0) = \psi^{(2n)}(\pi) = 0, \ n = 0, 1, 2, \ldots \} . \]

**Lemma 3.** \( \Gamma(C^\infty(G)^G) = C^\infty_{ev}[0, \pi] \).

**Proof.** First, let \( \varphi \in C^\infty(G)^G \). Define \( \check{\varphi} \in C^\infty(\mathbb{R}) \) by \( \check{\varphi}(x) := \varphi(\phi(x)) \). Then \( \Gamma(\varphi)(x) = \sqrt{2} \sin(x) \check{\varphi}(x) \) and the iterated Leibniz rule yields for the derivative of order \( 2n \)

\[ \Gamma(\varphi)^{(2n)}(x) = \sqrt{2} \left\{ \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{2n}{2k} \right) \sin(x) \check{\varphi}^{(2k)}(x) \right. \]

\[ + \sum_{k=0}^{n-1} (-1)^{n-k-1} \left( \frac{2n}{2k+1} \right) \cos(x) \check{\varphi}^{(2k+1)}(x) \} . \]

By construction, the function \( \check{\varphi} \) is \( 2\pi \)-periodic and has even parity, i.e., \( \check{\varphi}(-x) = \check{\varphi}(x) \). Hence, the derivative \( \varphi^{(k)} \) is \( 2\pi \)-periodic and has even parity for even \( k \) and odd parity for odd \( k \). It follows \( \varphi^{(2n+1)}(0) = \varphi^{(2n+1)}(\pi) = 0 \) and hence \( \Gamma(\varphi)^{(2n)}(0) = \Gamma(\varphi)^{(2n)}(\pi) = 0 \), for any \( n \).

Conversely, let \( \psi \in C^\infty_{ev}[0, \pi] \). Since \( \psi(0) = \psi(\pi) = 0 \), we can extend \( \psi \) to a well-defined function on the whole of \( \mathbb{R} \) by setting \( \psi(-x) = -\psi(x) \) and \( \psi(x + 2\pi m) = \psi(x), x \in [0, \pi], m \) an integer. Then for any \( x \in \mathbb{R} \setminus 2\pi \mathbb{Z} \), any \( k = 0, 1, 2, \ldots \) and any \( m \in \mathbb{Z} \) there holds \( \psi^{(k)}(-x) = -(-1)^k \psi^{(k)}(x) \) and \( \psi^{(k)}(x + 2\pi m) = \psi^{(k)}(x) \). In addition, \( \lim_{x \to 2\pi m} \psi^{(2k)}(x) = 0 \). This implies that derivatives of \( \psi \) of arbitrary order are continuous in \( x = 2\pi m \), hence \( \psi \) is smooth. Now define a function \( \check{\varphi} \) on \( \mathbb{R} \setminus 2\pi \mathbb{Z} \) by \( \check{\varphi}(x) = \frac{1}{\sqrt{2}} \frac{\psi(x)}{\sin(x)} \).

We claim that \( \check{\varphi} \) extends to a smooth function on the whole of \( \mathbb{R} \). To see this, it suffices to show smoothness in \( x = 0 \) and \( x = \pi \). We give the argument for \( x = 0 \) only; the case \( x = \pi \) is analogous. Since the sine function is a local diffeomorphism in a neighbourhood of \( x = 0 \), \( \check{\varphi} \) is smooth in 0 iff so is \( \varphi \circ \arcsin \). Denote \( f(x) = \psi(\arcsin(x)) \). Then \( f \) is smooth in a neighbourhood of \( x = 0 \) and \( \varphi \circ \arcsin(x) = \frac{1}{\sqrt{2}} \frac{f(x)}{x} \). Hence,

\[ (\varphi \circ \arcsin)^{(k)}(x) = \frac{1}{\sqrt{2}} \frac{1}{x^{n+1}} \sum_{l=0}^{k} (-1)^{n-k} \binom{k}{l} (k - l)! x^k f^{(k)}(x) . \]
Since $f(0) = 0$, the r.h.s. yields an indefinite expression for $x \to 0$. The derivative of the enumerator is $x^k f^{(k+1)}(x)$. Hence, the rule of de l’Hospital yields
\[
\lim_{x \to 0} (\tilde{\phi} \circ \arcsin)^{(k)}(x) = \frac{1}{\sqrt{2}} \frac{f^{(k+1)}(0)}{x+1}.
\]
This proves that $\tilde{\phi}$ extends to a smooth function on $\mathbb{R}$. Since it is $2\pi$-periodic and has even parity by construction, there exists $\varphi \in C^\infty(G)^G$ such that $\tilde{\phi} = \varphi \circ \phi$. Then $\psi = \Gamma(\varphi)$. \[\square\]

The operators $\hat{f}_i$ on $L^2(G)^G$ induce operators $\Gamma \hat{f}_i \Gamma^{-1}$ on $L^2[0, \pi]$. These induced operators will also be denoted by $\hat{f}_i$. We derive explicit expressions. Since $\Gamma \hat{f}_0 \Gamma^{-1}$ is multiplication by the function $f_0 \circ \phi$,
\[
\hat{f}_0 \psi(x) = 2\cos(x) \psi(x).
\]
Let $AC[0, \pi]$ denote the space of absolutely continuous functions and let
\[
AC^1[0, \pi] = \{ \psi \in AC[0, \pi] : \psi' \in L^2[0, \pi] \},
\]
\[
AC^2[0, \pi] = \{ \psi \in AC^1[0, \pi] : \psi' \in AC^1[0, \pi] \}.
\]

**Proposition 2.**

The operator $\hat{f}_1$ has domain $D(\hat{f}_1) = \{ \psi \in L^2[0, \pi] : \sin(x) \psi(x) \in AC^1[0, \pi] \}$ and is given by the expression
\[
\hat{f}_1 = \imath \hbar \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right).
\]

The operator $\hat{f}_2$ has domain $D(\hat{f}_2) = \{ \psi \in L^2[0, \pi] : \psi \in AC^2[0, \pi], \psi(0) = \psi(\pi) = 0 \}$ and is given by the expression
\[
\hat{f}_2 = -\hbar^2 \beta^2 \left( \frac{d^2}{dx^2} + 1 \right).
\]

The subspace $C_\text{ev}^\infty[0, \pi]$ is a common invariant core for $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$.

**Remark 6.** For $\psi \in C^\infty[0, \pi]$, one has
\[
\hat{f}_1 \psi = \imath \hbar \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right) \psi(x)
\]
\[
= \imath \hbar \left( \sqrt{\sin(x)} \frac{d}{dx} \sqrt{\sin(x)} \right) \psi(x)
\]
\[
= \imath \hbar \left( \sin(x) \frac{d}{dx} + \frac{1}{2} \cos(x) \right) \psi(x),
\]
whereas it is only the first of these three expressions that extends to the whole of $D(\hat{f}_1)$.

**Remark 7.** For general $\text{SU}(n)$ the Hilbert space $L^2(G)^G$ can be realized as $L^2(\sigma^{n-1}, \text{vdt})$, where $\sigma^{n-1}$ is the $(n - 1)$-simplex (more concretely, a Weyl alcove in the Lie algebra $\mathfrak{g}$) and $\text{vdt}$ is an appropriate measure on $\sigma^{n-1}$. In [31] it is proved that in this realization a core for the group Laplacian $\Delta_G$ is given by Neumann boundary conditions at the boundary of $\sigma^{n-1}$. In our situation, $L^2(\sigma^{n-1}, \text{vdt})$ corresponds to $L^2([0, \pi], \sin^2(x) \text{d}x)$ and the core isolated in [31] amounts to $\{ \psi \in C^\infty[0, \pi] : \psi'(0) = \psi'(\pi) = 0 \}$. By means of the isomorphism $\Gamma$, this core is mapped into $\{ \psi \in C^\infty[0, \pi] : \psi(0) = \psi(\pi) = 0 \}$. Thus, the assertion about the core for $\hat{f}_2$ in Proposition 2 is consistent with the result of [31].
Remark 8. The domain of \( \hat{f}_1 \) is unusually large for a differential operator, it contains e.g. all smooth functions. This is due to the fact that, in \( \hat{f}_1 \), the derivative is combined with the sine function which destroys any information about the boundary values of the function whose derivative is taken. In particular, \( C^\infty[0, \pi] \) may also be taken as a core for \( \hat{f}_1 \).

Remark 9. Occasionally we will have to deal with the operator \( \hat{f}_2^2 \) below. For further use we note that the domain of \( \hat{f}_2^2 \) contains \( AC^2[0, \pi] \) as a proper subspace and that on \( AC^2[0, \pi] \),

\[
\hat{f}_2^2 = -\hbar^2 \left( \sin(x) \frac{d^2}{dx^2} \sin(x) - \frac{1}{4} \cos^2(x) + \frac{1}{2} \right).
\]

**Proof.** The last statement follows from the fact that \( C^\infty(G)^G \) is a common invariant core for \( \hat{f}_0, \hat{f}_1, \hat{f}_2 \) and Lemma 3.

First, consider \( \hat{f}_2 \). According to the general formula for the radial part of the Laplacian on a compact group, see [12, §II.3.4], the restriction of \( \hat{f}_2 \) to \( C^\infty[0, \pi] \) is given by the r.h.s. of (36). The assertion about the domain then follows by standard extension theory for the operator of second derivative. Since the r.h.s. of (36) is well defined on \( AC^2[0, \pi] \), \( \hat{f}_2 \) is given by this expression on the whole of its domain.

Next, consider \( \hat{f}_1 \). According to (30), for \( \psi \in C^\infty[0, \pi] \),

\[
\hat{f}_1 \psi \equiv \Gamma \hat{f}_1 \Gamma^{-1} \psi = i\hbar \Gamma (Y_{\hat{f}_1} + \frac{3}{4} \hat{f}_0) \Gamma^{-1} \psi.
\]

According to (26) and (13),

\[
(Y_{\hat{f}_1}, \psi)(x) = \sqrt{2} \sin(x) \frac{d}{dx} \bigg|_{t=0} (\Gamma^{-1} \psi) \left( \phi(x)e^{\frac{i}{2} (\phi(x) - \phi(x)^*) t} \right).
\]

A brief computation shows \( \phi(x)e^{\frac{i}{2} (\phi(x) - \phi(x)^*) t} = \phi(x + t \sin(x)) \). Hence,

\[
(Y_{\hat{f}_1}, \psi)(x) = \sqrt{2} \sin(x) \frac{d}{dx} \bigg|_{t=0} \frac{\psi(x + t \sin(x))}{\sqrt{2} \sin(x + t \sin(x))} = \psi'(x) \sin(x) - \psi(x) \cos(x).
\]

Together with (34) and (37) this yields \( \hat{f}_1 \psi(x) = i\hbar \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right) \psi(x) \), hence on \( C^\infty[0, \pi] \), \( \hat{f}_1 \) is given by the r.h.s. of (35). Denote \( D = \{ \psi \in L^2[0, \pi] : \sin(x)\psi(x) \in AC^1[0, \pi] \} \). Since the r.h.s. of (35) is well-defined for all \( \psi \in D \), \( \hat{f}_1 \) is given by this expression on the whole of \( D \). It remains to show \( \text{D}(\hat{f}_1) = D \).

Let \( A \) be defined by restriction of \( \hat{f}_1 \) to the core \( C^\infty[0, \pi] \). Since \( \hat{f}_1 \) is self-adjoint,

\[
A^\dagger = A = \hat{f}_1.
\]

Hence, it suffices to show \( \text{D}(A^\dagger) = D \). Let \( \psi \in D \). Then \( \sin(x)\psi(x) \in AC^1[0, \pi] \), hence it has a derivative \( (\sin(x)\psi(x))' \in L^1[0, \pi] \) and \( (\sin(x)\psi(x))' \in L^2[0, \pi] \).

Then \( \psi(x) := i\hbar ((\sin(x)\psi(x))' - \frac{1}{2} \cos(x)\psi(x)) \in L^2[0, \pi] \).

For any \( \varphi \in C^\infty[0, \pi] \), integration by parts yields

\[
\langle \psi | \varphi \rangle = -\frac{i\hbar}{\pi} \int_0^\pi \left( (\sin(x)\psi(x))' - \frac{1}{2} \cos(x)\psi(x) \right) \varphi(x) \, dx
\]

\[
= \frac{i\hbar}{\pi} \int_0^\pi \psi(x) (\sin(x)\varphi' + \frac{1}{2} \cos(x)\varphi(x)) \, dx
\]

\[
= \langle \psi | A \varphi \rangle,
\]
hence \( \psi \in D(A^\dagger) \). Conversely, let \( \psi \in D(A^\dagger) \). Then there exists \( \tilde{\psi} \in L^2[0, \pi] \) such that 
\[
\langle \psi | A \varphi \rangle = \langle \tilde{\psi} | \varphi \rangle
\]
for all \( \varphi \in C^\infty_{ev}[0, \pi] \). Write this equation in the form
\[
\int_0^\pi \sin(x)\psi(x) \ i \frac{d}{dx} \varphi(x) \ dx = \frac{1}{i} \left( \tilde{\psi} + \frac{1}{2} \cos(x)\psi(x) \right) \varphi(x) \ dx, \quad \forall \ \varphi \in C^\infty_{ev}[0, \pi].
\]
We conclude that \( \sin(x)\psi(x) \) belongs to the domain of the adjoint of the restriction of \( i \frac{d}{dx} \) to the subspace \( C^\infty_{ev}[0, \pi] \). Since \( C^\infty_{ev}[0, \pi] \) is a core for \( i \frac{d}{dx} \) and since the domain of the self-adjoint operator \( i \frac{d}{dx} \) is \( AC^1[0, \pi] \) it follows that \( \sin(x)\psi(x) \in AC^1[0, \pi] \), i.e., \( \psi \in D \).

Next, we discuss the eigenvalues and the spectra of the operators \( \hat{f}_1 \). According to \( \text{(33)} \), \( \hat{f}_2 \) has pure point spectrum,
\[
\sigma(\hat{f}_2) = \{ \hbar^2 n(n+2) : n = 0, 1, 2, \ldots \}
\]
and the characters form an orthonormal basis of eigenvectors.

**Proposition 3.** The operators \( \hat{f}_0, \hat{f}_1 \) and \( \hat{f}_2^2 \) do not possess eigenvalues. Their spectra are
\[
\sigma(\hat{f}_0) = [-2, 2], \quad \sigma(\hat{f}_1) = \mathbb{R}, \quad \sigma(\hat{f}_2^2) = [0, \infty[.
\]

**Proof.** First, consider \( \hat{f}_0 \). The eigenvalue equation \( (\hat{f}_0 - \lambda)\psi = 0 \) reads \( (2 \cos(x) - \lambda)\psi(x) = 0 \), hence \( \psi = 0 \) a.e. for any \( \lambda \in \mathbb{R} \). Thus, there are no eigenvalues. The assertion about the spectrum follows from the spectral mapping theorem.

Next, consider \( \hat{f}_1 \). According to \( \text{(33)} \), the eigenvalue equation amounts to the differential equation
\[
(\hat{f}_1 - \lambda)\psi(x) = \left\{ i\hbar \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right) - \lambda \right\} \psi(x) = 0 \tag{39}
\]
which on the open interval \( ]0, \pi[ \) can be written in the form
\[
i\hbar \left\{ \frac{d}{dx} + \left( \frac{\hbar}{\sin(x)} - \frac{1}{2} \cot(x) \right) \right\} (\sin(x)\psi(x)) = 0.
\]
For any \( \lambda \in \mathbb{R} \) the solution is
\[
\psi_\lambda(x) = \frac{1}{\sqrt{2\hbar}} e^{-\frac{\lambda}{\hbar} \ln \tan \left( \frac{x}{\hbar} \right)} \sin(x). \tag{40}
\]
The particular choice of normalization will be justified below. Since neither of the functions \( \psi_\lambda \) is square integrable, \( \hat{f}_1 \) does not have eigenvalues.

To determine the spectrum of \( \hat{f}_1 \), let \( \lambda \in \mathbb{R} \). If \( \hat{f}_1 - \lambda \) had a bounded inverse, there would exist \( C > 0 \) such that \( \| \psi \| = \| (\hat{f}_1 - \lambda)^{-1} (\hat{f}_1 - \lambda) \psi \| \leq C \| (\hat{f}_1 - \lambda) \psi \| \) for any \( \psi \in D(\hat{f}_1) \). Thus, in order to show that \( \lambda \in \sigma(\hat{f}_1) \) it suffices to construct a sequence \( \psi_n \) in \( D(\hat{f}_1) \) such that \( \frac{\| \psi_n \|}{\| (\hat{f}_1 - \lambda) \psi_n \|} \to \infty \). Choose a smooth function \( j \) on \( \mathbb{R} \) with support in the open interval \( ]-1, 1[ \) such that \( 0 \leq j(x) \leq 1 \) and \( \int_{-\infty}^\infty j(x) \ dx = 1 \). Define 
\[
g_n(x) = \int_{-\infty}^{x} \left\{ j(nx' - 2) - j(n(x' - \pi) + 2) \right\} \ dx'
\]
and \( \psi_n := g_n \psi_\lambda \). Since \( g_n \) has support in \( ]0, \pi[ \), \( \psi_n \in L^2[0, \pi] \) and hence \( \psi_n \in D(\hat{f}_1) \). On the open interval \( ]0, \pi[ \) we have 
\[
((\hat{f}_1 - \lambda) \psi_n)(x) = i\hbar \left( \frac{d}{dx} g_n \right)(x) \sin(x) \psi_\lambda(x) + g_n(x) \left( i\hbar \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right) - \lambda \right) \psi_\lambda(x).
\]
The second term vanishes because \( \psi_n \) solves (39) on \( \pi, \pi \). Hence

\[
((\hat{f}_1 - \lambda)\psi_n)(x) = i\sqrt{\frac{n}{2}} (n_j(nx - 2) - nj(n(x - \pi) + 2)) \sqrt{\sin(x)} \ e^{-\frac{1}{\hbar} \lambda \ln \tan(\frac{x}{2})}
\]

and therefore

\[
\|(\hat{f}_1 - \lambda)\psi_n\|^2 = \frac{n}{2\pi} n^2 \left\{ \int_0^\pi j(nx - 2)^2 \sin(x) \, dx + \int_0^\pi j(n(x - \pi) + 2)^2 \sin(x) \, dx \right\}.
\]

The mixed term vanishes for \( n \) large enough because \( j(nx - 2) \) has support in \( \left\{ \frac{1}{n}, \frac{3}{n} \right\} \) and \( j(n(x - \pi) + 2) \) has support in \( \left\{ \pi - \frac{3}{n}, \pi - \frac{1}{n} \right\} \). For the same reason, \( j(nx - 2) \sin(x) \leq \frac{3}{n} \) and \( j(n(x - \pi) + 2) \sin(x) \leq \frac{3}{n} \). Hence

\[
\|(\hat{f}_1 - \lambda)\psi_n\|^2 \leq \frac{3h}{\pi} n \left\{ \int_0^\pi j(nx - 2) \, dx + \int_0^\pi j(n(x - \pi) + 2) \, dx \right\} = \frac{3h}{\pi}.
\]

It follows

\[
\frac{\|\psi_n\|^2}{\|(\hat{f}_1 - \lambda)\psi_n\|^2} \geq \frac{\pi}{3h} \|\psi_n\|^2 = \frac{1}{6\pi} \int_0^\pi \frac{\gamma^2(x)}{\sin(x)} \, dx \to \infty
\]

and hence \( \lambda \in \sigma(\hat{f}_1) \).

Finally, consider \( \hat{f}_2^2 \). According to (38), the eigenvalue equation (\( \hat{f}_2^2 - \lambda^2 \)) \( \psi = 0 \) can be written in the form

\[
-h^2 \sin(x) \left( \frac{d^2}{dx^2} - \frac{4}{3} \cot^2(x) + \left( \frac{1}{2} + \frac{\lambda^2}{h^2} \right) \frac{1}{\sin^2(x)} \right) (\sin(x)\psi(x)) = 0, \quad x \in [0, \pi],
\]

where it is manifest that for any \( \lambda \geq 0 \) the solution space has dimension 2. Hence, in case \( \lambda^2 \neq 0 \), any solution is a linear combination of \( \psi_\lambda \) and \( \psi_{-\lambda} \) and, therefore, is not square-integrable. Thus, \( \lambda^2 \) is not an eigenvalue. In case \( \lambda^2 = 0 \) we observe that, in addition to \( \psi_0(x) = \frac{1}{\sqrt{\sin(x)}} \), a further solution is given by \( \tilde{\psi}_0(x) = \frac{\ln \tan(\frac{x}{2})}{\sqrt{\sin(x)}} \). Since neither \( \psi_0 \) nor \( \tilde{\psi}_0 \) is square-integrable, \( \lambda^2 = 0 \) is not an eigenvalue, too.

To prove the assertion about the spectrum we choose \( \lambda \geq 0 \) and consider the sequence \( \psi_n \) defined above. Obviously, \( \psi_n \in D(\hat{f}_2^2) \) for all \( n \). On \( \pi, \pi \) we find, using (35),

\[
(\hat{f}_2^2 - \lambda^2) \psi_n(x) = -h^2 \left\{ g_n'(x) \sin(x) \left( \frac{1}{2} \cos(x) - i\frac{1}{h} \right) + g_n''(x) \sin^2(x) \right\} \psi_\lambda(x)
\]

\[
+ g_n(x) \left( -h^2 \left( \frac{4}{dx} \sin(x) - \frac{1}{2} \cos(x) \right)^2 - \lambda^2 \right) \psi_\lambda(x).
\]

The last term vanishes. Moreover, \( g_n''(x) = n^2 \left( j'(nx - 2) - j'(n(x - \pi) + 2) \right) \). Hence,

\[
((\hat{f}_2^2 - \lambda^2) \psi_n)(x) = -\sqrt{\frac{n^2}{2}} \left\{ n(j(nx - 2) - j(n(x - \pi) + 2)) \sqrt{\sin(x)} \left( \frac{1}{2} \cos(x) - i\frac{1}{h} \right)
\]

\[
+ n^2 \left( j'(nx - 2) - j'(n(x - \pi) + 2) \right) \sin^3(x) \right\} e^{-\frac{1}{\hbar} \lambda \ln \tan(\frac{x}{2})}.
\]

Consequently, there are two contributions to \( \|(\hat{f}_2^2 - \lambda^2) \psi_n\|^2 \). One is centered near \( x = 0 \) and is given by

\[
\frac{n^2}{2\pi} \int_0^\pi \left\{ \left( \frac{\lambda^2}{h^2} n^2 \right) j(nx - 2)^2 \sin(x) + \left( \frac{1}{2} n j(nx - 2) \cos(x) + n^2 j'(nx - 2) \sin(x) \right)^2 \sin(x) \right\} \, dx, \quad (41)
\]
they define a function \( \varphi \) viewed as a generalized eigenvector of \( \lambda, \mu \). This proves (42). For Proposition 4.

Proof. For any \( L \in \mathbb{R} \), the function \( \psi_\lambda \) defines a linear functional on the subspace \( C^0_0]0, \pi[ \) of \( L^2[0, \pi] \) by

\[
\langle \psi_\lambda | \varphi \rangle := \frac{1}{\pi} \int_0^\pi \psi_\lambda(x) \varphi(x) \, dx, \quad \varphi \in C^0_0]0, \pi[.
\]

Integration by parts yields \( \langle \psi_\lambda | (f_1 - \lambda) \varphi \rangle = 0 \) for all \( \varphi \in C^0_0]0, \pi[ \), so that \( \psi_\lambda \) can be viewed as a generalized eigenvector of \( f_1 \).

**Proposition 4.** The set of generalized eigenvectors \( \{ \psi_\lambda : \lambda \in \mathbb{R} \} \) of \( \hat{f}_1 \) is complete and orthogonal in the distributional sense, i.e.,

\[
\int_{-\infty}^{\infty} \psi_\lambda(x) \psi_\lambda(y) \, d\lambda = \pi \delta(x - y), \quad x, y \in ]0, \pi[ , \tag{42}
\]

\[
\frac{1}{\pi} \int_0^\pi \psi_\lambda(x) \psi_\mu(x) \, dx = \delta(\lambda - \mu), \quad \lambda, \mu \in \mathbb{R}. \tag{43}
\]

The assignment of \( \tilde{\varphi}(\lambda) = \langle \psi_\lambda | \varphi \rangle \) to \( \varphi \in C^0_0]0, \pi[ \) extends to a unitary isomorphism of \( L^2[0, \pi] \) onto \( L^2(\mathbb{R}) \) with inverse \( \varphi(x) = \int_\infty^\infty \psi_\lambda(x) \tilde{\varphi}(\lambda) \, d\lambda \).

Proof. For \( x, y \in ]0, \pi[ \),

\[
\int_{-\infty}^{\infty} \frac{\psi_\lambda(y) \psi_\lambda(y) \, d\lambda}{\sin(x) \sin(y)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{y}{\hbar}(\ln \tan(\frac{x}{2}) - \ln \tan(\frac{y}{2}))} \, d\lambda = \pi \delta(\ln \tan(\frac{x}{2}) - \ln \tan(\frac{y}{2})).
\]

The argument of the \( \delta \)-distribution, viewed as a function of \( x \) with parameter \( y \), has derivative \( \frac{1}{\sin(x)} \) at \( x = y \) and a single zero at \( x = \pi \). Hence, \( \delta(\ln \tan(\frac{x}{2}) - \ln \tan(\frac{y}{2})) = \sin(y) \delta(x - y) \).

This proves (42). For \( \lambda, \mu \in \mathbb{R} \), the substitution \( y = \frac{1}{\hbar} \ln \tan(\frac{x}{2}) \) yields

\[
\frac{1}{\pi} \int_0^\pi \psi_\lambda(x) \psi_\mu(x) \, dx = \frac{1}{2\pi} \int_0^\pi \frac{e^{(\lambda-\mu)\ln \tan(\frac{x}{2})}}{\sin(x)} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\lambda-\mu)y} \, dy = \delta(\lambda - \mu),
\]

hence (43). To prove the last assertion, we observe that (42) implies \( \int_{-\infty}^{\infty} \tilde{\varphi}_1(\lambda) \tilde{\varphi}_2(\lambda) \, d\lambda = \langle \varphi_1 | \varphi_2 \rangle \) for all \( \varphi_1, \varphi_2 \in C^0_0]0, \pi[ \). Hence, \( \tilde{\varphi}(\lambda) \in L^2(\mathbb{R}) \) and the assignment \( \varphi \mapsto \tilde{\varphi} \) extends to an isometric map \( L^2[0, \pi] \to L^2(\mathbb{R}) \). It remains to check that this map has dense image. Let \( \varphi \in C^0_0(\mathbb{R}) \). Since the integrals \( \int_{-\infty}^{\infty} \psi_\lambda(x) \varphi(x) \, d\lambda \) exist for any \( x \in ]0, \pi[ \), they define a function \( \varphi_0 \) on \( ]0, \pi[ \). Due to (43), \( \frac{1}{\pi} \int_0^\pi |\varphi_0(x)|^2 \, dx = \int_{-\infty}^{\infty} |\varphi(x)|^2 \, d\lambda \), hence \( \varphi_0 \in L^2[0, \pi] \). This proves that the extended map is a unitary isomorphism. \( \square \)
4.3 Quantum analogue of the classical relation between generators

Consider the relation (11) and the inequality (12) satisfied by the classical generators $f_i$. Since $\hat{f}_2 \geq 0$, the inequality (12) has an obvious quantum counterpart. Concerning the relation (11), we recall that on the domain of $\hat{f}_2$, $\hat{f}_2^2$ is given by (38). Expressing the r.h.s. of this equation in terms of the $\hat{f}_i$ we obtain

$$\hat{f}_1^2 = \frac{1}{4\beta^2} \sqrt{4 - \hat{f}_0^2} \hat{f}_2 \sqrt{4 - \hat{f}_0^2} - \frac{3\beta^2}{16} \hat{f}_0^2 + \frac{\hbar^2}{2}.$$  

This can be written in the form

$$\sqrt{4 - \hat{f}_0^2} \hat{f}_2 \sqrt{4 - \hat{f}_0^2} - 4\beta^2 \hat{f}_1^2 = \hbar^2 \beta^2 \left( \frac{3}{4} \hat{f}_0^2 - 2 \right).$$  

(44)

We observe that when replacing $\hat{f}_i$ by $f_i$, (44) reproduces the relation (11) in the limit $\hbar \to 0$. We will now derive a relation between the quantum observables $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$ which exactly reproduces the classical relation (11) under the naïve replacement of the operators $\hat{f}_i$ by the phase space functions $f_i$, $i = 0, 1, 2$. The attribute 'naïve' shall remind us that this operation is well-defined on the level of formal expressions in the variables $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$ but not necessarily on the level of the operators defined by these expressions. To begin with, let $A_1$ be the operator defined on $D(\hat{f}_2)$ by the l.h.s. of (44). Since the domain of $\hat{f}_2$ is invariant under $\hat{f}_0$ and contained in the domain of $\hat{f}_1^2$, on $D(\hat{f}_2)$ we can define, additionally, the following operators:

$$A_2 := 4\hat{f}_2 - \hat{f}_0^2 \hat{f}_2 - 4\beta^2 \hat{f}_1^2,$$

$$A_3 := 4\hat{f}_2 - \hat{f}_0 \hat{f}_2 \hat{f}_0 - 4\beta^2 \hat{f}_1^2,$$

$$A_4 := 4\hat{f}_2 - \hat{f}_0 \hat{f}_2 - 4\beta^2 \hat{f}_1^2.$$

Like $A_1$, these operators correspond to the classical phase space function $(4 - f_0^2)f_2 - 4\beta^2 f_1^2$, but contrary to $A_1$ they are polynomial in the $\hat{f}_i$. A straightforward computation using (34)–(36) yields that on $D(\hat{f}_2)$ there holds

$$\frac{1}{2}(A_2 + A_4) = \hbar^2 \beta^2 \left( -\frac{1}{4} \hat{f}_0^2 - 2 \cdot \mathbb{1} \right),$$

$$A_3 = \hbar^2 \beta^2 \left( \frac{3}{4} \hat{f}_0^2 - 6 \cdot \mathbb{1} \right).$$  

(45) (46)

First, we observe that, similar to (44), when replacing $\hat{f}_i$ by $f_i$, both (45) and (46) reproduce the relation (11) in the limit $\hbar \to 0$. In addition, we observe that the three operators $A_1$, $\frac{1}{2}(A_2 + A_4)$ and $A_3$ are contained in the real vector space spanned by $\mathbb{1}$ and $\hat{f}_0^2$. A brief computation reveals that the sum of the coefficients of a vanishing linear combination is nonzero. Hence, these coefficients can be chosen so that they add up to 1. The corresponding linear combination is

$$\frac{3}{4} A_1 + \frac{3}{8}(A_2 + A_4) - \frac{1}{2} A_4 = 0.$$
This yields the relation
\[ \hat{f}_2 - \left( \frac{3}{8} \hat{f}_0^2 \hat{f}_2 - \frac{1}{2} \hat{f}_0 \hat{f}_2 \hat{f}_0 + \frac{3}{8} \hat{f}_2 \hat{f}_0^2 \right) + \frac{3}{4} \sqrt{4 - \hat{f}_0^2} \hat{f}_2 \sqrt{4 - \hat{f}_0^2} - 4\beta^2 \hat{f}_1^2 = 0. \] (47)

It holds exactly on the domain of \( \hat{f}_2 \) and reproduces the classical relation (11) under the naïve replacement of the operators \( \hat{f}_i \) by the phase space functions \( f_i, i = 0, 1, 2 \).

From the above relations we can derive relations, valid on the whole of \( \mathcal{D}(\hat{f}_2) \), expressing the operators \((4 - \hat{f}_0^2)\hat{f}_2, (4 - \hat{f}_0^2)^{\frac{1}{2}} \hat{f}_2 (4 - \hat{f}_0^2)^{\frac{1}{2}} \) or \( \hat{f}_2 (4 - \hat{f}_0^2) \) as polynomials in \( \hat{f}_0, \hat{f}_1 \) and the identity. In any of these expressions, the contribution of the identity is nonzero. Since \((4 - \hat{f}_0^2)\) is given by multiplication by \(4 \sin^2(x)\), any subspace on which such a relation can be resolved for \( \hat{f}_2 \) must be contained in
\[ \{ \psi \in \mathcal{D}(\hat{f}_2) : \frac{\psi(x)}{\sin^2(x)} \in L^2[0, \pi] \}. \] (48)

**Proposition 5.** The subspace (48) is not a core for \( \hat{f}_2 \).

Thus, none of the above relations determines \( \hat{f}_2 \) completely in terms of \( \hat{f}_0 \) and \( \hat{f}_1 \).

**Proof.** Let \( \mathcal{D}_0 \) denote the subspace (48). Let \( \mathcal{A} \) be defined as the restriction of \( \hat{f}_2 \) to the domain \( \mathcal{D}_0 \). First, we show that any \( \psi \in \mathcal{D}_0 \) satisfies \( \psi'(0) = \psi'N(\pi) = 0 \). Indeed, since \( \psi' \in \mathcal{A}^1[0, \pi] \), \( \psi' \) is continuous and can be extended continuously outside \( [0, \pi] \). Hence, \( \psi' \in C^1[0, \pi] \). Consider the function \( \frac{\psi(x)}{\sin(x)} \) on \( [0, \pi] \). Since for \( x \to +0 \), \( \frac{\psi(x)}{\cos(x)} \to \psi'(0) \), by the rule of de l’Hospital, \( \frac{\psi(x)}{\sin(x)} \to \psi'(0) \). Hence there exists \( x_0 > 0 \) such that for any \( x \in [0, x_0] \) there holds \( \frac{\psi(x)}{\sin(x)} \geq \frac{\psi'(0)}{2} \). Since \( \frac{\psi(x)}{\sin^2(x)} \) is square-integrable then \( \psi'(0) = 0 \). A similar argument shows the assertion for \( \psi'(\pi) \). Now, integration by parts yields that for any \( \varphi \in \mathcal{A}^2[0, \pi] \) and any \( \psi \in \mathcal{D}_0 \) there holds \( \langle \varphi, \hat{f}_2 \hat{f}_2 \psi \rangle = \langle -\hbar^2 \beta^2 (\varphi'' + \varphi) \rangle \). It follows that \( \mathcal{A}^2[0, \pi] \subseteq \mathcal{D}(\mathcal{A}^1) \), hence \( \mathcal{D}_0 \) is not a core for \( \hat{f}_2 \), as asserted. \( \square \)

### 4.4 Algebra of quantum observables

As the algebra of quantum observables we would like to take an algebra which is generated, in some natural way, by the quantized generators of the algebra of classical observables (to be precise, the subalgebra of observables polynomial in the position and momentum variables). There is a natural choice for that algebra. It relies on the notion of a \( C^* \)-algebra generated by unbounded operators in the sense of Woronowicz.

**Definition 1.**
Let \( \mathcal{H} \) be a separable Hilbert space. Let \( \mathcal{A} \) be a \( C^* \)-subalgebra of \( B(\mathcal{H}) \) and let \( T_1, \ldots, T_N \) be closed, densely defined operators on \( \mathcal{H} \) affiliated with \( \mathcal{A} \). Then \( \mathcal{A} \) is generated by \( T_1, \ldots, T_N \) in the sense of Woronowicz if for all nondegenerate representations \( \pi \) of \( \mathcal{A} \) on \( \mathcal{H} \) and all nondegenerate \( C^* \)-subalgebras \( \mathcal{B} \subseteq B(\mathcal{H}) \) there holds: if \( \pi(T_1), \ldots, \pi(T_N) \) are affiliated with \( \mathcal{B} \), then \( \pi(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{B}) \) and \( \pi(\mathcal{A}) \mathcal{B} \) is dense in \( \mathcal{B} \).
Here, $M(B) = \{b \in B(\mathcal{H}) : bB, Bb \subseteq B\}$ is the multiplier algebra of $B$. Let us recall the notions entering this definition, see [30]. For a closed, densely defined operator $T$ on $\mathcal{H}$, the $z$-transform is defined by

$$z_T = T(1 + T^*T)^{-\frac{1}{2}}.$$  

This is a bounded operator on $\mathcal{H}$. $T$ can be recovered from $z_T$ by

$$T = z_T(1 - z_T^*z_T)^{-\frac{1}{2}}.$$  \hspace{1cm} (49)

A closed, densely defined operator $T$ on $\mathcal{H}$ is said to be affiliated with $\mathcal{A} \subseteq B(\mathcal{H})$ (in the sense of Baaj and Julg [2]) if $z_T \in M(\mathcal{A})$, $1 - z_T^*z_T \geq 0$ and $(1 - z_T^*z_T)\mathcal{A}$ is dense in $\mathcal{A}$. A nondegenerate representation of $\mathcal{A}$ is a $*$-morphism $\pi: \mathcal{A} \to B(\mathcal{H})$ such that $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$. (Note that the assumption in Definition [1] that $\pi(\mathcal{A})$ is dense in $B$ may not follow from the assumption that $\varphi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$ made here.) The representation $\pi: \mathcal{A} \to B(\mathcal{H})$ can be extended to affiliated operators $T$ by extending $\pi$ to $M(\mathcal{A})$ through

$$\pi(b)\pi(a)\psi = \pi(ba)\psi, \quad b \in M(\mathcal{A}), \ a \in \mathcal{A}, \ \psi \in \mathcal{H},$$

and defining $\pi(T)$ by $z_{\pi(T)} = \pi(z_T)$. This definition makes sense, because $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$.

The fundamental criterion to test whether $\mathcal{A}$ is generated by a given set of affiliated operators is

**Theorem 1.** [30] Thm. 3.3

Let $\mathcal{A}$ be a C*-subalgebra of $B(\mathcal{H})$ and let $T_1, \ldots, T_N$ be closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{A}$. Then $\mathcal{A}$ is generated by $T_1, \ldots, T_N$ if

1. some product built from $(1 + T_i^*T_i)^{-1}$, $(1 + T_iT_i^*)^{-1}$ belongs to $\mathcal{A}$,
2. $T_1, \ldots, T_N$ separate the representations of $\mathcal{A}$ on $\mathcal{H}$.

Separation of representations means that for any two distinct representations $\pi_1, \pi_2 : \mathcal{A} \to B(\mathcal{H})$ there exists an $i \in \{1, \ldots, N\}$ such that $\pi_1(T_i) \neq \pi_2(T_i)$. Prominent examples of C*-algebras generated by unbounded operators are:

1. Let $\mathcal{A}$ be a unital C*-subalgebra of $B(\mathcal{H})$ and let $T_1, \ldots, T_N$ be elements of $\mathcal{A}$ such that the algebra generated by $T_1, \ldots, T_N$ and the identity is dense in $\mathcal{A}$. Then $T_1, \ldots, T_N$ generate $\mathcal{A}$ in the sense of Woronowicz. Thus, the concept is a generalization of the ordinary notion of generation of an algebra by a subset.

2. Let $G$ be a connected Lie group, let $\mathcal{H} = L^2(G)$ and let $\mathcal{A}$ be the group C*-algebra $C^*(G)$. Then $\mathcal{A}$ is generated in the sense of Woronowicz by any basis in the Lie algebra $\mathfrak{g}$, where the basis elements are viewed as first order differential operators on $G$.

3. Let $\mathcal{H} = L^2(\mathbb{R})$, let $\mathcal{A}$ be the C*-algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H}$. Then $\mathcal{A}$ is generated in the sense of Woronowicz by the position operator $T_1 = x$ and the momentum operator $T_2 = \frac{\hbar}{i} \frac{d}{dx}$.

**Remark 10.** At the present stage this theory has the disadvantage of not providing a general method how to construct the algebra for a given set of generators.
Theorem 2. Each of the sets \( \{ \hat{f}_0, \hat{f}_2 \} \) and \( \{ \hat{f}_1, \hat{f}_2 \} \) generates the C*-algebra \( K(L^2(G)^G) \) in the sense of Woronowicz.

Thus, it is natural to define the algebra of quantum observables to be

\[
\mathcal{O}_q := K(L^2(G)^G).
\]

Proof. In the proof, denote \( \mathcal{H} = L^2(G)^G \). The \( \hat{f}_i \) are affiliated with \( K(\mathcal{H}) \), because this holds for any closed and densely defined operator. To prove Theorem 2, we use the criterion given in Theorem 1. The first condition holds, because \( \hat{f}_2 \) has eigenbasis \( \chi_n \) with eigenvalues \( \hbar^4 \beta^4 n^2(n+2)^2 \) and hence has compact resolvent. To check the second condition, let \( k = 0 \) or \( 1 \). Assume that we are given representations \( \pi_1, \pi_2 \) with \( \pi_1(\hat{f}_i) = \pi_2(\hat{f}_i), i = k, 2 \). By definition of the operators \( \pi_j(\hat{f}_i), i = k, 2 \). Hence, \( \pi_1 \) and \( \pi_2 \) coincide on the subalgebra \( \tilde{A} \) of the multiplier algebra of \( K(\mathcal{H}) \) generated by \( 1, z_{\gamma}, z_{\beta} \). To prove \( \pi_1 = \pi_2 \) it suffices to show \( K(\mathcal{H}) \subseteq \tilde{A} \). As the multiplier algebra of \( K(\mathcal{H}) \) is \( B(\mathcal{H}) \), we can apply the following criterion.

Lemma 4. ([18, Prop. 10.4.1])

Let \( \tilde{A} \) be a C*-subalgebra of \( B(\mathcal{H}) \). If
1. \( \tilde{A} \cap K(\mathcal{H}) \neq \emptyset \),
2. \( \tilde{A} \) is irreducibly represented on \( \mathcal{H} \),
then \( K(\mathcal{H}) \subseteq \tilde{A} \).

We check these two conditions. For the first one, we use that for a closed, densely defined operator \( T \) there holds the identity

\[
1 - z_T^4 z_T = (1 + T^4 T)^{-1},
\]
see e.g. [23]. Plugging in \( \hat{f}_2 \) for \( T \) we observe: the l.h.s. belongs to \( \tilde{A} \) and the r.h.s. was shown above to belong to \( K(\mathcal{H}) \). Thus, the first condition holds, indeed. To prove the second condition, we apply the lemma of Schur. Assume that we are given a bounded operator \( S \) on \( \mathcal{H} \) that commutes with all elements of \( \tilde{A} \). Then \( S \) commutes with \( z_{\gamma} \) and \( z_{\beta} \). In particular, \( S \) leaves invariant the eigenspaces of \( z_{\beta} \). According to [19], \( \chi_n \) is a basis of eigenvectors of \( z_{\beta} \) with eigenvalues \( \frac{\hbar^2 \beta^2 n(n+2)}{\sqrt{1 + \hbar^4 \beta^2 n^2(n+2)^2}} \). Since this is a strictly monotonous function of \( n \), the eigenspaces have dimension 1. Hence, \( S \chi_n = \lambda_n \chi_n, n = 0, 1, 2, \ldots \) with \( \lambda_n \in \mathbb{C} \). According to [19], \( [S, z_{\beta}] = 0 \) implies \( S \hat{f}_k \chi_n - \hat{f}_k S \chi_n = 0 \) for all \( n \). This yields, respectively,

\[
(\lambda_{n+1} - \lambda_n) \chi_{n+1} + (\lambda_{n-1} - \lambda_n) \chi_{n-1} = 0 \quad (k = 0)
\]

\[
\frac{2n+3}{4} (\lambda_{n+1} - \lambda_n) \chi_{n+1} - \frac{2n+1}{4} (\lambda_{n-1} - \lambda_n) \chi_{n-1} = 0 \quad (k = 1)
\]

for all \( n \). In both cases, it follows \( \lambda_{n+1} = \lambda_n \) for all \( n \), hence \( S = \lambda I \). Then \( \tilde{A} \) is irreducibly represented on \( \mathcal{H} \) by the lemma of Schur and hence the second condition of Lemma 4 is satisfied. This shows that condition 2 of Theorem 1 holds and, therefore, completes the proof of Theorem 2. \( \square \)
Remark 11. There remains the question whether the set \( \{ \hat{f}_0, \hat{f}_1 \} \) generates \( K(L^2(G)^G) \) as well. The crucial point is Condition 1 of Theorem 1. While according to Proposition 3, the operators \((1 + \hat{f}_0^2)^{-1}\) and \((1 + \hat{f}_1^2)^{-1}\) do not belong to \( K(L^2(G)^G) \), it would be sufficient to show that some product of these operators is compact. We did not succeed to clarify this point.

4.5 Relation with the algebra of bosonic quantum observables in lattice gauge theory

We discuss the relation between the algebra of quantum observables of our model and the bosonic part of the algebra of observables of a quantum lattice gauge theory of [20]. This paper concentrates on the case of lattice quantum chromodynamics, i.e., gauge group \( SU(3) \) rather than \( SU(2) \) as in our model. However, the results are valid for general \( SU(n) \). We recall the construction of the bosonic observable algebra for the case of a single plaquette without external links, after having implemented the tree gauge. The bosonic field algebra is the crossed product algebra \( \mathcal{F} = C(G) \otimes_\alpha G \) associated with the \( C^* \)-dynamical system \( (C(G), G, (L_g - 1)^*) \). For the notions of \( C^* \)-dynamical system and crossed-product algebra, see [24]. \( \mathcal{F} \) carries a natural \( G \)-action. The bosonic observable algebra \( \mathcal{O} \) is defined as the quotient of the subalgebra of \( G \)-invariant elements of \( \mathcal{F} \) by the ideal defined by the generators of the \( G \)-action. This factorization corresponds to imposing the Gauss law, which is the quantum analogue of the restriction of the phase space to the zero level set of the momentum mapping. If there are external links, this definition of \( \mathcal{O} \) yields the subalgebra of internal observables. The natural covariant representation of \( (C(G), G, \alpha) \) on \( L^2(G) \) naturally induces a representation of \( \mathcal{F} \) on \( L^2(G) \), mapping \( \mathcal{F} \) to \( K(L^2(G)) \). It is shown in [20] that this representation is the unique irreducible representation of \( \mathcal{F} \). It is therefore called the generalized Schrödinger representation.

Using this representation, it is then shown that \( \mathcal{O} \) can be identified with the compact operators on the closed subspace \( L^2(G)^G \). Thus, through this identification, the algebra of quantum observables \( \mathcal{O}_q \) of our model coincides with the bosonic observable algebra \( \mathcal{O} \) of [20], specified to the case of a single plaquette without external links.

Next, we compare generators. Let \( U^A_B : G \to \mathbb{C} \) denote the matrix entry functions. Choose a basis \( T_i \) in \( g \) orthonormal w.r.t. the trace form and define vector fields \( E^A_B \) on \( G \) by

\[
E^A_B = \sum_i (T_i)^A_B T_i,
\]

where \( (T_i)^A_B \) are the entries of the basis element \( T_i \) when viewed as a matrix, whereas the second \( T_i \) is viewed as a vector field. It is stated in [20] that \( \mathcal{F} \), when realized as \( K(L^2(G)) \), is generated in the sense of Woronowicz by the multiplication operators \( U^A_B \) and the first order differential operators \( E^A_B \). It was not clarified in [20] whether gauge invariant combinations of \( U^A_B \) and \( E^A_B \) generate the observable algebra.

Our quantum observables \( \hat{f}_0, \hat{f}_1 \) and \( \hat{f}_2 \) can be expressed in terms of the gauge invariant combinations \( U^A_A, U^A_B E^B_A \) and \( E^A_B E^B_A \) as follows. Since \( U^A_A = f_0 \) as functions on
For the multiplication operators we have 
\[ \hat{f}_0 = U^A_A. \]

For the value of the vector field \( U^A_B E^B_A \) at \( a \in G \) we find 
\[ (U^A_B E^B_A)_a = L'_a \sum_i \text{tr}(aT_i)T_i. \]
Since \( T_i \) is orthonormal w.r.t. the trace form, 
\( \sum_i \text{tr}(aT_i)T_i = -P(a) \). According to (26), then \( U^A_B E^B_A = -Y f_1 \) and hence
\[ \hat{f}_1 = i\hbar (-U^A_B E^B_A + \frac{3}{4} U^A_A). \]

Finally, a similar computation shows
\[ \hat{f}_2 = 2\hbar^2 \beta^2 E^A_B E^B_A. \]

Thus, Theorem 2 implies that, in the case of a single plaquette without external links, the algebra of quantum observables of [20] is generated, in the sense of Woronowicz, by \( U^A_A \) and \( E^A_B E^B_A \) or by \( U^A_B E^B_A \) and \( E^A_B E^B_A \). This extends the result of [20] on the generation of the field algebra by unbounded operators to the algebra of observables, at least in the simple case at hand.

In [20] it was argued that on a purely algebraic level the observable algebra is generated by \( U^A_A \) and \( U^A_B E^B_A \) and that all other invariants can be expressed in terms of these generators. This is, however, the pair of operators for which we could not prove that they generate the algebra in the sense of Woronowicz. Moreover, from Proposition 5 we conclude that e.g. the quadratic Casimir operator \( E^A_B E^B_A \) cannot be expressed in terms of \( U^A_A \) and \( U^A_B E^B_A \) on a core. These observations show that any naïve algebraic procedure of reducing the number of independent generators has to be handled with care.

4.6 Towards quantum dynamics

Quantization of the classical Hamiltonian (9) yields the quantum Hamiltonian 
\[ \hat{H} = \frac{1}{2} \hat{f}_2 + \frac{1}{2g^2} (3 - \hat{f}_0) \]
which is a time-independent self-adjoint operator with domain \( D(\hat{H}) = D(\hat{f}_2) \). On the level of pure states (Schrödinger picture), dynamics is given by the 1-parameter group of unitary transformations of \( L^2(G)^G \) generated by \( \hat{H} \),
\[ U_t = e^{-\frac{i}{\hbar} \hat{H} t}. \]

Since the algebra of compact operators is invariant under unitary transformations, \( U_t \) induces a 1-parameter automorphisms group \( \alpha_t \) of the algebra of quantum observables by
\[ \alpha_t(A) = U_t A U_t^\dagger. \]

On the level of observables (Heisenberg picture), dynamics is given by the 1-parameter automorphism group \( \alpha_t \). It is interesting as well to study the dynamics of the generators \( \hat{f}_k \). On the common invariant core \( C^\infty(G)^G \) it is given by
\[ \hat{f}_k(t) = U_t \hat{f}_k U_t^\dagger. \]
The corresponding equation of motion, on this core, reads
\[
\frac{d}{dt} \hat{f}_k(t) = \frac{i}{\hbar} [\hat{H}, \hat{f}_k(t)] , \quad \hat{f}_k(0) = \hat{f}_k , \quad k = 0, 1, 2 .
\] (51)

The automorphism group \( \alpha_t \) and the operators \( \hat{f}_k(t) \) will be studied elsewhere.

We conclude with a discussion of the commutators between the generators \( \hat{f}_k \). These commutators are relevant for the evaluation of the right-hand side of (50) and for the iterative solution of (51), respectively. Since \( \hat{f}_0 \) leaves invariant the domains of \( \hat{f}_1 \) and \( \hat{f}_2 \), the commutators \([\hat{f}_0, \hat{f}_1]\) and \([\hat{f}_0, \hat{f}_2]\) are defined on these domains. A straightforward computation using (34)–(36) yields
\[
[\hat{f}_0, \hat{f}_1] = i\hbar \left( 2 - \frac{1}{2} \hat{f}_0^2 \right) , \quad [\hat{f}_0, \hat{f}_2] = 4\beta^2 i\hbar \hat{f}_1 .
\] (52)

We claim that the commutator of \( \hat{f}_1 \) and \( \hat{f}_2 \) is defined on \( D(\hat{f}_2) \) and is given by
\[
[\hat{f}_1, \hat{f}_2] = -\frac{i}{2} \hbar \left( \hat{f}_0 \hat{f}_2 + \hat{f}_2 \hat{f}_0 + 3\hbar^2 \hat{f}_0 \right) .
\] (53)

To see this, write
\[
\hat{f}_1 \hat{f}_2 - \hat{f}_2 \hat{f}_1 = -i\hbar^3 \beta^2 \left\{ \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right) \left( \frac{d^2}{dx^2} + 1 \right) - \left( \frac{d^2}{dx^2} + 1 \right) \left( \frac{d}{dx} \sin(x) - \frac{1}{2} \cos(x) \right) \right\}
= i\hbar^3 \beta^2 \left\{ \frac{d}{dx} \left( \frac{d^2}{dx^2} \sin(x) - \sin(x) \frac{d^2}{dx^2} \right) + \frac{1}{2} \left( \cos(x) \frac{d^2}{dx^2} - \frac{d^2}{dx^2} \cos(x) \right) \right\}
\]
and observe that for \( \psi \in \text{AC}^2[0, \pi] \),
\[
\left( \frac{d^2}{dx^2} \sin(x) - \sin(x) \frac{d^2}{dx^2} \right) \psi(x) = \left( - \sin(x) + 2 \cos(x) \frac{d}{dx} \right) \psi(x) .
\]

Hence \( \hat{f}_1 \hat{f}_2 - \hat{f}_2 \hat{f}_1 \) contains derivatives up to second order only and is therefore defined on \( D(\hat{f}_2) \subseteq \text{AC}^2[0, \pi] \), indeed. Then a straightforward calculation yields (53). Thus, all the commutators between the quantum observables \( \hat{f}_k \) are defined on \( D(\hat{f}_2) \).

**Remark 12.** Comparing the commutators (52) and (53) with the corresponding Poisson brackets (18) we observe that for the combinations of \( \hat{f}_0 \) with \( \hat{f}_1 \) and \( \hat{f}_2 \) the naïve relation between the commutator and the Poisson bracket (replacing, in the commutator, the operators by their classical counterparts, provided the latter are well-defined) holds exactly. For the combination of \( \hat{f}_1 \) and \( \hat{f}_2 \), this relation holds in the limit \( \hbar \to 0 \).

### 5 Outlook

An obvious future task is to generalize the results of this paper to a general compact Lie group. Another task is to study how the algebra of quantum observables depends on the choice of what phase space functions should be considered polynomial, cf. Remark 1.
E.g., one should carry out a similar construction for the generators of the algebra of real invariant polynomials on $G^C$ and compare the resulting algebra of quantum observables with the one obtained above. The concept used in this paper should be also compared with an alternative approach proposed by Buchholz and Grundling [5], who define the notion of resolvent algebra associated with a symplectic space and propose to take this algebra as the algebra of observables in a bosonic field theory. The resolvent algebra is a unital $C^*$-algebra defined abstractly in terms of generators and relations. Equivalently, it can be viewed as generated by the resolvents $(i\lambda - \phi(f))^{-1}$ of the field operators $\phi(f)$ of some quantum field $\phi$. Following these ideas, in our model one may take the unital $C^*$-algebra generated by the resolvents $(i\lambda - \hat{f}_k)^{-1}$ of the quantized generators $\hat{f}_0, \hat{f}_1, \hat{f}_2$ as the algebra of observables. This algebra will be studied elsewhere. It is definitely distinct from the algebra of quantum observables $\mathcal{O}_q$ constructed above. The trivial reason for that is that this algebra is unital by definition; another reason is that, according to Proposition 3 the resolvents $(i\lambda - \hat{f}_0)^{-1}$ and $(i\lambda - \hat{f}_1)^{-1}$ are not compact.

Furthermore, we address the problem of studying the influence of the stratification of the classical configuration and phase spaces on the quantum theory. For that purpose, one has to find a quantum structure that implements this stratification. On the level of pure states, such a quantum structure is given by a costratification of the Hilbert space [13]. One may think of a costratification as a family of closed subspaces, indexed by the strata, and a family of orthoprojectors, indexed by the inclusion relations between the closures of the strata. For the model under consideration, the costratified Hilbert space was studied in [16]. On the other hand, it is not clear how to implement the stratification on the level of observables. For the concrete algebra of observables at hand, this problem will be studied in detail in a future work.

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Appendix

We prove Formula (29). Choose an orthonormal basis $B_i$ in $\mathfrak{g}$. Let $\beta_i$ denote the elements of the dual basis in $\mathfrak{g}^*$. Then $\beta_i(A) = \langle B_i, A \rangle$ for any $A \in \mathfrak{g}$. We have $v = \beta_1 \wedge \beta_2 \wedge \beta_3$, where the $\beta_i$ are viewed as left-invariant forms. Using the derivation property of the Lie derivative, expanding

$$\mathcal{L}_{Y_{\mathfrak{g}}} \beta_i = \sum_{j=1}^{3} (\mathcal{L}_{Y_{\mathfrak{g}}} \beta_i)(B_j) \beta_j$$

and rewriting $(\mathcal{L}_{Y_{\mathfrak{g}}} \beta_i)(B_i) = -\beta_i(\mathcal{L}_{Y_{\mathfrak{g}}} B_i) = -\langle B_i, \mathcal{L}_{Y_{\mathfrak{g}}} B_i \rangle$ we obtain

$$\mathcal{L}_{Y_{\mathfrak{g}}} v = -\left( \sum_{i=1}^{3} \langle B_i, \mathcal{L}_{Y_{\mathfrak{g}}} B_i \rangle \right) v .$$

(54)
We calculate $\mathcal{L}_{Y_{f_i}}B_i$ by taking derivatives in the ambient vector space $M_2(\mathbb{C})$. According to (26), for $a \in G$,

$$(\mathcal{L}_{Y_{f_i}}B_i)_a = [Y_{f_i}, B_i]_a = \frac{d}{dt}|_{t=0} \frac{d}{ds}|_{s=0} a e^{P(a)t} e^{B_i s} - \frac{d}{dt}|_{t=0} \frac{d}{ds}|_{s=0} a e^{B_i t} e^{P(a e^{B_i t}) s}.$$ 

This yields $aP(a)B_i - aB_i P(a) - aP(aB_i)$, which can be rewritten as $-aP(B_i a)$. Hence,

$$(\mathcal{L}_{Y_{f_i}}B_i)_a = -L_a' P(B_i a).$$

Then

$$\langle B_i, \mathcal{L}_{Y_{f_i}}B_i \rangle(a) = -\langle B_i, P(B_i a) \rangle = \frac{1}{2i\pi} \frac{1}{2} tr(B_i^2(a + a^\dagger)) = \frac{1}{2i\pi} \frac{1}{2} tr(B_i^2) tr(a) = -\frac{1}{2} tr(a),$$

where we have used $B_i^2 = \frac{1}{2} tr(B_i^2) \mathbb{1} = 0$, due to the Cayley-Hamilton theorem. Then (54) yields $\mathcal{L}_{Y_{f_i}}v = \frac{3}{2} f_0 v$, i.e., Formula (29).

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