Generalizing the Converse to Pascal’s Theorem via Hyperplane Arrangements and the Cayley-Bacharach Theorem

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Abstract

Using a new point of view inspired by hyperplane arrangements, we generalize the converse to Pascal’s Theorem, sometimes called the Braikenridge-Maclaurin Theorem. In particular, we show that if \(2k\) lines meet a given line, colored green, in \(k\) triple points and if we color the remaining lines so that each triple point lies on a red and blue line then the points of intersection of the red and blue lines lying off the green line lie on a unique curve of degree \(k - 1\). We also use these ideas to extend a second generalization of the Braikenridge-Maclaurin Theorem, due to Möbius. Finally we use Terracini’s Lemma and secant varieties to show that this process constructs a dense set of curves in the space of plane curves of degree \(d\), for degrees \(d \leq 5\). The process cannot produce a dense set of curves in higher degrees. The exposition is embellished with several exercises designed to amuse the reader.

Dedicated to H.S.M. Coxeter, who demonstrated a heavenly syzygy: the sun and moon aligned with the Earth, through a pinhole.
(Toronto, May 10, 1994, 12:24:14)

1 Introduction

In Astronomy the word syzygy refers to three celestial bodies that lie on a common line. More generally, it sometimes is used to describe interesting geometric patterns.

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For example, in a triangle the three median lines that join vertices to the midpoints of opposite sides meet in a common point, the centroid, as illustrated in Figure 1. Choosing coordinates, this fact can be viewed as saying that three objects lie on a line: there is a linear dependence among the equations defining the three median lines. In Commutative Algebra and Algebraic Geometry, a syzygy refers to any relation among the generators of a module.

Figure 1: The three medians of a triangle intersect at the centroid.

Pappus’s Theorem, which dates from the fourth century A.D., describes another syzygy. It is one of the inspirations of modern projective geometry.

**Theorem 1** (Pappus). If 3 points $A, B, C$ lie on one line, and three points $a, b, c$ lie on another, then the lines $Aa, Bb, Cc$ meet the lines $aB, bC, cA$ in three new points and these new points are collinear, as illustrated in the left diagram of Figure 2.

Pappus’s Theorem appears in his text, *Synagoge* [14][15], a collection of classical Greek geometry with insightful commentary. David Hilbert observed that Pappus’s Theorem is equivalent to the claim that the multiplication of lengths is commutative (see e.g. Coxeter [5] p. 152). Thomas Heath believed that Pappus’s intention was to revive the geometry of the Hellenic period [12] p. 355, but it wasn’t until 1639 that the sixteen year-old Blaise Pascal generalized Pappus’s theorem [6] Section 3.8, replacing the two lines with a more general conic section.

**Theorem 2** (Pascal). If 6 points $A, B, C, a, b, c$ lie on a conic section, then the lines $Aa, Bb, Cc$ meet the lines $aB, bC, cA$ in three new points and these new points are collinear, as illustrated in the right diagram of Figure 2.
Pascal’s theorem is sometimes formulated as the Mystic Hexagon Theorem: if a hexagon is inscribed in a conic then the 3 points lying on lines extending from pairs of opposite edges of the hexagon are collinear, as in Figure 3.

Figure 3: The Mystic Hexagon Theorem.

It is not clear why the theorem deserves the adjective mystic. Perhaps it refers
to the case where a regular hexagon is inscribed in a circle. In that case, the three pairs of opposite edges are parallel and the theorem then predicts that the parallel lines should meet (at infinity), and all three points of intersection should be collinear. Thus, a full understanding of Pascal’s theorem requires knowledge of the projective plane, a geometric object described in some detail in Section 2.

Pascal’s Theorem has an interesting converse, sometimes called the Braikenridge–Maclaurin theorem after the two British mathematicians William Braikenridge and Colin Maclaurin.

**Theorem 3 (Braikenridge–Maclaurin).** If three lines meet three other lines in nine points and if three of these points lie on a line then the remaining six points lie on a conic.

Braikenridge and Maclaurin seem to have arrived at the result independently, though they knew each other and their correspondence includes a dispute over priority.

In 1848 the astronomer and mathematician August Ferdinand Möbius generalized the Braikenridge–Maclaurin Theorem. Suppose a polygon with \(4n + 2\) sides is inscribed in a nondegenerate conic and we determine \(2n + 1\) points by extending opposite edges until they meet. If \(2n\) of these \(2n + 1\) points of intersection lie on a line then the last point also lies on the line. Möbius’s had developed a system of coordinates for projective figures, but surprisingly his proof relies on solid geometry. In Section 3 we prove an extension of Möbius’s result, using the properties of projective plane curves – in particular, the Cayley-Bacharach Theorem.

The Cayley–Bacharach Theorem is a wonderful result in projective geometry. In its most basic form (sometimes called the 8 implies 9 Theorem) it says that if two cubic curves meet in 9 points then any cubic through 8 of the nine points must also go through the ninth point. For the history and many equivalent versions of the Cayley-Bacharach Theorem, see Eisenbud, Green and Harris’s elegant paper [8]. A strong version of the Cayley–Bacharach Theorem, described in Section 3, can be used to establish another generalization of the Braikenridge–Maclaurin Theorem. The following existence theorem is well-known (see Kirwan’s book on complex algebraic curves [17, Theorem 3.14]) but we also claim a uniqueness result. The statement of Theorem 4 is inspired by the study of hyperplane arrangements – in this case, by collections of colored lines in the plane.

**Theorem 4.** Suppose that \(2k\) lines in the projective plane meet another line in \(k\) triple points. Color the lines so that the line containing all the triple points is green and each of the \(k\) collinear triple points has a red and a blue line passing through it. Then there is a unique curve of degree \(k - 1\) passing through the points where the red lines meet the blue lines off the green line.

When the red and blue lines have generic slopes, they meet in \(k^2 - k\) points off the green line. Since \(\binom{k+1}{2} - 1 = \frac{k^2 + k - 2}{2}\) points in general position determine a unique
curve of degree \( k - 1 \) passing through the points, it is quite remarkable that the curve passes through all \( k^2 - k \) points of intersection off the green line. The Braikenridge–Maclaurin Theorem is just the instance \( k = 3 \) of Theorem 4. The case where \( k = 4 \) is illustrated in Figure 4.

![Figure 4: An illustration of Theorem 4 when \( k = 4 \).](image)

We use the Cayley-Bacharach Theorem to prove Theorem 4 in Section 3. In Section 4 we consider the kinds of curves produced by the construction in Theorem 4. For instance, we use the group law on an elliptic curve to give a constructive argument that, in a way that will be made precise, almost every degree-3 curve arises in this manner. More generally, almost every degree-4 and degree-5 curve arises in this manner. A simple dimension argument is given to show that most curves of degree 6 or higher do not arise in this manner. The proofs for degree 4 and 5 involve secant varieties – special geometric objects that have been quite popular recently because of their applications to algorithmic complexity, algebraic statistics, mathematical biology and quantum computing (see, for example, Landsberg [18, 19]).

The last section contains some suggestions for further reading. As well, Sections 2 and 5 contain amusing exercises that expand on the topic of the paper.
A paper generalizing a classical result in geometry cannot reference all the relevant literature. One recent paper by Katz [16] is closely related to this work. His Mystic 2d-Gram [16, Theorem 3.3] gives a nice generalization of Pascal’s Theorem; see Exercise 16.6. He also raises an interesting constructibility question: which curves can be described as the unique curve passing through the $d^2 - 2d$ points of intersection of $d$ red lines and $d$ blue lines that lie off a conic through $2d$ intersection points?

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2 Projective Geometry

The general statement of Pascal’s Theorem suggests that parallel lines should meet in a point and that as we vary the pairs of parallel lines the collection of such intersection points should lie on a line. This is manifestly false in the usual Cartesian plane, but the plane can be augmented by adding points at infinity, after which Pascal’s Theorem holds. The resulting projective plane $\mathbb{P}^2$ is a fascinating object with many nice properties.

One powerful model of the projective plane identifies points in $\mathbb{P}^2$ with lines through the origin in 3-dimensional space. To see how this relates to our usual plane, consider the plane $z = 1$ in 3-dimensional space as a model for $\mathbb{R}^2$ and note that most lines through the origin meet this plane. The line passing through $(x, y, 1)$ is identified with the point $(x, y) \in \mathbb{R}^2$. But what about the lines that don’t meet this plane? These are parallel to $z = 1$ and pass through $(0, 0, 0)$ so they are lines in the $xy$-plane. Each of these lines can be viewed as a different point at infinity since they’ve been attached to our copy of $\mathbb{R}^2$.

In 1827 Möbius developed a useful system of coordinates for points in projective space [23], later extended by Grassmann. If we consider the punctured 3-space $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and the equivalence relation

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \Leftrightarrow \lambda \neq 0,$$

then each equivalence class corresponds to a line in $\mathbb{R}^3$ through the origin. We denote the equivalence class of points on the line through $(x, y, z)$ by $[x : y : z]$. This is a sensible notation since the ratios between the coordinates determine the direction of the line. Returning to our earlier model of $\mathbb{P}^2$, the points with $z \neq 0$ correspond to
points in our usual copy of $\mathbb{R}^2$, while the points with $z = 0$ correspond to points at infinity.

If points in $\mathbb{P}^2$ correspond to lines through the origin, then what do lines in $\mathbb{P}^2$ look like? Considering a line in $\mathbb{R}^2$ as sitting in the plane $z = 1$ we see that the points making up this line correspond to lines through the origin that, together, form a plane. Any line in $\mathbb{R}^2$ can be described by an equation of the form $ax + by + c = 0$; the reader should check that this determines the plane $ax + by + cz = 0$. Thus, lines in $\mathbb{P}^2$ correspond to dimension-2 subspaces of $\mathbb{R}^3$. In particular, the line in $\mathbb{P}^2$ whose equation is $z = 0$ is the line at infinity.

We can also add points at infinity to $\mathbb{R}^n$ to create $n$-dimensional projective space $\mathbb{P}^n$. Again, points in $\mathbb{P}^n$ can be identified with 1-dimensional subspaces of $\mathbb{R}^{n+1}$ and each point is denoted using homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$. Similarly, we can construct the complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$, the points of which can be identified with 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$.

**Exercise 5.** If this is the first time you’ve met projective space, you might try these, increasingly complicated, exercises.

1. Show that if $ax + by + cz = 0$ and $dx + ey + fz = 0$ are two lines in $\mathbb{P}^2$ then they meet in a point $P = [g : h : i]$ given by the cross product,
   \[ \langle g, h, i \rangle = \langle a, b, c \rangle \times \langle d, e, f \rangle. \]

2. Show that the line $ax + by + cz = 0$ in $\mathbb{P}^2$ consists of all the points of the form $[x : y : 1]$ such that $ax + by + c = 0$, together with a single point at infinity (the point $[b : -a : 0]$). We say that the line $ax + by + cz = 0$ is the **projectivization** of the line $ax + by + c = 0$.

3. Show that the projectivizations of two parallel lines $ax + by + d = 0$ and $ax + by + d = 0$ in $\mathbb{R}^2$ meet at a point at infinity.

4. The projectivization of the hyperbola $xy = 1$ in $\mathbb{R}^2$ is the set of points in $\mathbb{P}^2$ that satisfy $xy - z^2 = 0$. Show that whether a point $[x : y : z]$ lies on the projectivization of the hyperbola or not is a well-defined property (i.e. the answer doesn’t depend on which representative of the equivalence class $[x : y : z]$ we use). Where does the projectivization meet the line at infinity?

5. (a) By picking representatives of each equivalence class carefully, show that $\mathbb{P}^2$ can be put into 1-1 correspondence with the points on a sphere $S^2 \subset \mathbb{R}^3$, as long as we identify antipodal points, $(x, y, z) \sim (-x, -y, -z)$.
   (b) Considering only the top half of the sphere, show that the points in $\mathbb{P}^2$ can be identified with points in a disk where antipodal points on the boundary circle are identified.
   (c) Considering a thin band about the equator of the sphere from part (a), show
that \( \mathbb{P}^2 \) can be constructed by sewing a Möbius band onto the boundary of a disk. How many times does the band twist around as we go along the boundary of the disk?

(d) **Blowing up** is a common process in algebraic geometry. When we blow up a surface at a point we replace the point with a projectivization of its tangent space (that is, the space of lines through the base point in the tangent space). Show that if we blow up a point on the sphere \( S^2 \subset \mathbb{R}^3 \) we get \( \mathbb{P}^2 \). Show that if we blow up a second point we get a Klein bottle, the surface obtained by sewing two Möbius bands together along their edges.

6. (a) Viewing \( \mathbb{P}^2 \) as the set of lines in \( \mathbb{R}^3 \) through the origin, check that the map \( \iota : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^4 \) given by

\[
\iota(x,y,z) = \left( \frac{x^2}{x^2 + y^2 + z^2}, \frac{xy}{x^2 + y^2 + z^2}, \frac{z^2}{x^2 + y^2 + z^2}, \frac{(x+y)z}{x^2 + y^2 + z^2} \right)
\]

induces a well-defined embedding of \( \mathbb{P}^2 \) into \( \mathbb{R}^4 \).

(b) Samuelson [28] gives an elegant argument to show that \( \mathbb{P}^2 \) cannot be embedded in \( \mathbb{R}^3 \). Maehara [20], exploiting work of Conway and Gordon [4] and Sachs [27], gives another simple argument for this fact based on the observation that any embedding of the complete graph \( K_6 \) in \( \mathbb{R}^3 \) contains a pair of linked triangles. Complete the argument by drawing \( K_6 \) on \( \mathbb{P}^2 \) in such a way that no triangles are homotopically linked.

(c) Amiya Mukherjee [24] showed that the complex projective plane \( \mathbb{P}^2_{\mathbb{C}} \) can be smoothly embedded in \( \mathbb{R}^7 \). Research Question: Can \( \mathbb{P}^2_{\mathbb{C}} \) be holomorphically embedded in \( \mathbb{C}^3 \)? Can \( \mathbb{P}^2_{\mathbb{C}} \) be smoothly embedded in \( \mathbb{R}^n \) with \( n < 7 \)?

### 2.1 The Power of Projective Space

Projective space \( \mathbb{P}^2 \) enjoys many nice properties that Euclidean space \( \mathbb{R}^2 \) lacks. Many theorems are much easier to state in projective space than in Euclidean space. For instance, in Euclidean space any two lines meet in either one point or in no points (in the case where the two lines are parallel). By adding points at infinity to Euclidean space, we’ve ensured that any two distinct lines meet in a point. This is just the first of a whole sequence of results encapsulated in Bézout’s Theorem.

Each curve \( C \) in the projective plane can be described as the zero set of a homogeneous polynomial \( F(x,y,z) \):

\[
C = \{ [x : y : z] : F(x,y,z) = 0 \}.
\]

The polynomial needs to be homogeneous (all terms in the polynomial have the same degree) in order for the curve to be well-defined (see Exercise 5.4). It is traditional
to call degree-\(d\) homogeneous polynomials \textit{degree-\(d\) forms}. The curve \(C\) is said to be a degree-\(d\) curve when \(F(x, y, z)\) is a degree-\(d\) polynomial. We say that \(C\) is an irreducible curve when \(F(x, y, z)\) is an irreducible polynomial. When \(F(x, y, z)\) factors then the set \(C\) is actually the union of several component curves each determined by the vanishing of one of the irreducible factors of \(F(x, y, z)\). If \(F(x, y, z) = G(x, y, z)H(x, y, z)\) then removing the component \(G(x, y, z) = 0\) leaves the \textit{residual curve} \(H(x, y, z) = 0\).

**Theorem 6** (Bézout’s Theorem). If \(C_1\) and \(C_2\) are curves of degrees \(d_1\) and \(d_2\) in the complex projective plane \(\mathbb{P}^2_{\mathbb{C}}\) sharing no common components then they meet in \(d_1d_2\) points, counted appropriately.

Bézout’s Theorem requires that we work in \textit{complex} projective space: in \(\mathbb{P}^2_{\mathbb{R}}\) two curves may not meet at all. For instance, the line \(y - 2z = 0\) misses the circle \(x^2 + y^2 - z^2 = 0\) in \(\mathbb{P}^2_{\mathbb{R}}\); the points of intersection have complex coordinates.

To say what it means to count appropriately requires a discussion of intersection multiplicity. This can be defined in terms of the length of certain modules \([10]\), but an intuitive description will be sufficient for our purposes. When two curves meet transversally at a point \(P\) (there is no containment relation between their tangent spaces) then \(P\) counts as 1 point in Bézout’s Theorem. If the curves are tangent at \(P\) or if one curve has several branches passing through \(P\) then \(P\) counts as multiple points. One way to determine how much the point \(P\) should count is to look at well-chosen families of curves \(C_1(t)\) and \(C_2(t)\) so that \(C_1(0) = C_1\) and \(C_2(0) = C_2\) and to count how many points in \(C_1(t) \cap C_2(t)\) approach \(P\) as \(t\) goes to 0. For instance, the line \(y = 0\) meets the parabola \(yz = x^2\) in one point \(P = [0 : 0 : 1]\). Letting \(C_1(t)\) be the family of curves \(y - t^2z = 0\) and letting \(C_2(t)\) be the family consisting only of the parabola, we find that \(C_1(t) \cap C_2(t) = \{(t : t^2 : 1), [-t : t^2 : 1]\}\) and so two points converge to \(P\) as \(t\) goes to 0. In this case, \(P\) counts with multiplicity two. The reader interesting in testing their understanding could check that the two concentric circles \(x^2 + y^2 - z^2 = 0\) and \(x^2 + y^2 - 4z^2\) meet in two points, each of multiplicity two.

More details can be found in Fulton’s lecture notes \([9\text{ Chapter 1}]\).

It is traditional to call this result Bézout’s Theorem because it appeared in a widely-circulated and highly-praised book\(^1\). Indeed, in his position as Examiner of the Guards of the Navy in France, Étienne Bézout was responsible for creating new textbooks for teaching mathematics to the students at the Naval Academy. However, Issac Newton proved the result over 80 years before Bézout’s book appeared! Kirwan \([17]\) gives a nice proof of Bézout’s Theorem.

\(^1\)Both the MathSciNet and Zentralblatt reviews of the English translation \([2]\) are entertaining and well-worth reading. The assessment in the MathSciNet review is atypically colorful: “This is not a book to be taken to the office, but to be left at home, and to be read on weekends, as a romance”, while the review in Zentralblatt Math calls it “an immortal evergreen of astonishing actual relevance”. 

9
Higher projective spaces arise naturally when considering moduli spaces of curves in the projective plane. For instance, consider a degree-2 curve \( C \) given by the formula
\[
a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 = 0. 
\]
Multiplying the formula by a nonzero constant gives the same curve, so the curve \( C \) can be identified with the point \([a_0 : a_1 : a_2 : a_3 : a_4 : a_5]\) in \( \mathbb{P}^5 \). More generally, letting \( S = \mathbb{C}[x, y, z] = \oplus_{d \geq 0} S_d \) be the polynomial ring in three variables, the degree-\( d \) curves in \( \mathbb{P}^2 \) are identified with points in the projective space \( \mathbb{P}(S_d) \), where we identify polynomials if they are nonzero scalar multiples of one another. A basis of the vector space \( S_d \) is given by the \( D = \binom{d+2}{2} \) monomials of degree \( d \) in three variables, so the degree-\( d \) curves in \( \mathbb{P}^2 \) are identified with points in the projective space \( \mathbb{P}(S_d) \cong \mathbb{P}^{D-1} \). Returning to the case of degree-2 curves in \( \mathbb{P}^2 \), if we require \( C \) to pass through a given point, then the coefficients \( a_0, \ldots, a_5 \) of \( C \) must satisfy the linear equation produced by substituting the coordinates of the point into (1). Now if we require \( C \) to pass through 5 points in \( \mathbb{P}^2 \) the coefficients must satisfy a homogeneous system of 5 linear equations. If the points are in general position (so that the resulting system has full rank), then the system has a one-dimensional solution space and so there is just one curve passing through all 5 points. In general, we expect a unique curve of degree \( d \) to pass through \( D-1 \) points in general position and we expect no curves of degree \( d \) to pass through \( D \) points in general position.

3 A Generalization of the Braikenridge–Maclaurin Theorem

The following Theorem is a version of the Cayley-Bacharach Theorem that was first proven by Michael Chasles. He used it to prove Pascal’s Mystic Hexagon Theorem, Theorem \(^2\). Because of its content, the theorem is often called the 8 ⇒ 9 Theorem.

**Theorem 7** (8 ⇒ 9 Theorem). Let \( C_1 \) and \( C_2 \) be two plane cubic curves meeting in 9 distinct points. Then any other cubic passing through any 8 of the nine points must pass through the ninth point too.

Inspired by Husemöller’s book on Elliptic Curves \(^[13]\), Terry Tao recently gave a simple proof of the 8 ⇒ 9 Theorem in his blog\(^[7]\).

**Proof.** (After Tao) The proof exploits the special position of the points in \( C_1 \cap C_2 = \{P_1, \ldots, P_9\} \). Let \( F_1 = 0 \) and \( F_2 = 0 \) be the homogeneous equations of the curves \( C_1 \) and \( C_2 \). We will show that if \( F_3 \) is a cubic polynomial and \( F_3(P_1) = \cdots = F_3(P_8) = 0 \), then \( F_3(P_9) = \).

\(^2\)See Tao’s July 15, 2011 post at terrytao.wordpress.com.
0 then $F_3(P_0) = 0$. To do this it is enough to show that there are constants $a_1$ and $a_2$ so that $F_3 = a_1 F_1 + a_2 F_2$ because then $F_3(P_0) = a_1 F_1(P_0) + a_2 F_2(P_0) = 0$. Aiming for a contradiction, suppose that $F_1$, $F_2$ and $F_3$ are linearly independent elements of $S_3$.

To start, no four of the points $P_1, \ldots, P_8$ can be collinear otherwise $C_1$ intersects the line in $4 > (3)(1)$ points so the line must be a component of $C_1$ by Bézout’s Theorem. Similarly, the line must be a component of $C_2$. But $C_1$ and $C_2$ only intersect in 9 points so this cannot be the case.

Now we show that there is a unique conic through any 5 of the points $P_1, \ldots, P_8$. If two conics $Q_1$ and $Q_2$ were to pass through 5 of the points then by Bézout’s Theorem they must share a component. So either $Q_1 = Q_2$ or both $Q_1$ and $Q_2$ are reducible and share a common line. Since 4 of the points cannot lie on a line, the common component must pass through no more than three of the five points. The remaining two points determine the residual line precisely so $Q_1 = Q_2$.

Now we argue that in fact no three of the points $P_1, \ldots, P_8$ can be collinear. Aiming for a contradiction suppose that three of the points lie on a line $L$ given by $H = 0$ and the remaining 5 points lie on a conic $C$. Since no 4 of the points $P_1, \ldots, P_8$ lie on a line, we know that the 5 points on $C$ do not lie on $L$. Pick constants $b_1$, $b_2$ and $b_3$ so that the polynomial $F = b_1 F_1 + b_2 F_2 + b_3 F_3$ vanishes on a fourth point on $L$ and at another point $P \notin L \cup C$. Since the cubic $F = 0$ meets the line $L$ in 4 points, $L$ must be a component of the curve $F = 0$. But the residual curve given by $F/H = 0$ is a conic going through 5 of the 8 points so it must be $C$ itself. So $F = 0$ is the curve $L \cup C$. But $F(P) = 0$ by construction and $P$ does not lie on $L \cup C$, producing the contradiction.

Now note that no conic can go through more than 6 of the points $P_1, \ldots, P_8$. Bézout’s Theorem shows that the conic can not go through 7 of the points, else $C_1$ and $C_2$ would share a common component. So suppose that a conic $C$ given by $G = 0$ goes through 6 of the points. Then there is a line $L$ going through the remaining 2 points. The polynomial $G$ does not vanish at either of these 2 remaining points because no 7 of the points lie on a conic. Pick constants $b_1$, $b_2$ and $b_3$ so that the cubic $F = b_1 F_1 + b_2 F_2 + b_3 F_3$ vanishes on a seventh point on the conic $C$ and another point $P \notin L \cup C$. Now $F/G$ is a linear form that vanishes on the two remaining points so $F/G = 0$ must determine the line $L$. It follows that $F = 0$ is the curve $L \cup C$. Again, $F(P) = 0$ by construction and $P$ does not lie on $L \cup C$, producing the contradiction.

Now let $L$ be the line through $P_1$ and $P_2$ and let $C$ be the conic through $P_3, \ldots, P_7$. From what we’ve proven above, $P_8$ does not lie on $L \cup C$. Pick constants $c_1$, $c_2$ and $c_3$ so that the cubic $F = c_1 F_1 + c_2 F_2 + c_3 F_3$ vanishes on two more points $P$ and $P'$, both on $L$ but neither on $C$. Since $F = 0$ meets $L$ in four points, $F = 0$ contains $L$ as a component. Since $L$ cannot go through any of $P_3, \ldots, P_7$, the residual curve must be $C$, so $F = 0$ is the curve $L \cup C$. But then $F(P_8) = 0$ by construction and $P_8$ does not
lie on $L \cup C$, producing the final contradiction.

Note that we proved slightly more: any cubic curve passing through 8 of the nine points must be a linear combination of the two cubics $C_1$ and $C_2$.

Pascal’s Mystic Hexagon Theorem, Theorem 2, follows from an easy application of the $8 \Rightarrow 9$ Theorem. Let $C_1$ be the cubic consisting of the 3 lines formed by extending an edge of the hexagon and its two adjacent neighbors. Let $C_2$ be the cubic consisting of the 3 lines formed by extending the remaining, opposite, edges. $C_1$ and $C_2$ meet in 6 points on the conic $Q$ and in three points off the conic. Let $L$ be the line through two of the three points of intersection not on $Q$. Then $Q \cup L$ is a cubic curve through 8 of the 9 points of $C_1 \cap C_2$. By the $8 \Rightarrow 9$ Theorem, $Q \cup L$ must contain the ninth point too. The point cannot lie on $Q$ so it must lie on $L$. That is, the three points of intersection not on $Q$ are collinear.

To prove the Braikenridge–Maclaurin Theorem, Theorem 3, using the Cayley-Bacharach Theorem, just observe that each collection of three lines is a cubic curve (it is determined by the vanishing of a degree-3 polynomial) and if three of the points lie on a line $L$ and five of the remaining six points lie on a conic $C$ then $L \cup C$ is a cubic curve passing through 8 of the nine points and so it must pass through all nine points. However, the ninth point cannot lie on the line $L$ if the original cubics meet only in points, otherwise $L$ would meet each of the original cubics in more than three points. So the ninth point must be on the conic $C$.

A more powerful version of the Cayley-Bacharach Theorem can be found in the last exercise of the Eisenbrick [7, p. 554] (also see Eisenbud, Green and Harris [8, Theorem CB5]). Before stating this result, we introduce some notation. Requiring a degree-$d$ curve in $\mathbb{P}^2$ to go through a point $p \in \mathbb{P}^2$ imposes a non-trivial linear condition on the coefficients of the defining equation of the curve. If a set $\Gamma$ of $\gamma$ points imposes only $\lambda$ independent linear conditions on the coefficients of a curve of degree $d$, then we say that $\Gamma$ fails to impose $\gamma - \lambda$ independent linear conditions on forms of degree $d$. For example, 9 collinear points fail to impose 5 independent linear conditions on forms of degree 3 – any cubic that passes through 4 of the points must pass through them all. More generally, any set of $k$ collinear points fails to impose $k - (d + 1)$ conditions on forms of degree $d \leq k - 1$.

**Theorem 8** (Cayley-Bacharach). Suppose that two curves of degrees $d_1$ and $d_2$ meet in a finite collection of points $\Gamma \subset \mathbb{P}^2$. Partition $\Gamma$ into disjoint subsets: $\Gamma = \Gamma' \cup \Gamma''$ and set $s = d_1 + d_2 - 3$. If $d \leq s$ is a non-negative integer then the dimension of the space of forms of degree $d$ vanishing on $\Gamma'$, modulo those vanishing on $\Gamma$, is equal to the failure of $\Gamma''$ to impose independent conditions on forms of degree $s - d$.

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3 An affectionate name for David Eisenbud’s excellent (and mammoth) tome on Commutative Algebra.
We restate our generalization of the Braikenridge-Maclaurin Theorem and give a proof using the Cayley-Bacharach Theorem. Kirwan [17, Theorem 3.14] gives a simple proof for the existence part of the Theorem that is well-worth examining.

**Theorem 4.** Suppose that $2k$ lines in the projective plane meet another line in $k$ triple points. Color the lines so that the line containing all the triple points is green and each of the $k$ collinear triple points has a red and a blue line passing through it. Then there is a unique curve of degree $k - 1$ passing through the points where the red lines meet the blue lines (off the green line).

**Proof.** The cases $k = 1$ and $k = 2$ are trivial so assume $k \geq 3$. Suppose that the red lines are cut out by the forms $L_1, \ldots, L_k$, the blue lines are cut out by the forms $M_1, \ldots, M_k$ and the green line is cut out by the form $G$. Let $\Gamma$ be the points of intersection of the two degree $k$ forms $L_1 \cdots L_k$ and $M_1 \cdots M_k$. Note that there are no degree-$(k - 1)$ curves that pass through all the points of $\Gamma$: any such curve meets the line $M_i = 0$ in $k$ points so each $M_i$ divides the equation of the curve, leading to a contradiction on the degree of the defining equation. Now let $\Gamma'$ be the points of $\Gamma$ that lie off the green line and $\Gamma''$ the points of $\Gamma$ on the green line. The $k$ collinear points in $\Gamma''$ impose $k - 1$ independent conditions on forms of degree $k - 2$. So $\Gamma''$ fails to impose $k - (k - 1) = 1$ condition on forms of degree $k - 2$. Because there are no curves of degree $k - 1$ going through all of $\Gamma$, the Cayley-Bacharach Theorem says that the dimension of the space of forms of degree $k - 1$ vanishing on $\Gamma'$ is equal to the failure of $\Gamma''$ to impose independent conditions on forms of degree $k + k - 3 - (k - 1) = k - 2$, which is one. So up to scaling, there is a unique equation of degree $k - 1$ passing through the points of $\Gamma$ off the green line.

In the generic case, just one red and one blue line pass through each point of intersection, and the curve $S$ passes through all $k^2 - k$ points of intersection between the red lines and the blue lines that do not lie on the green line. If we are not in the generic case the uniqueness claim needs further interpretation. For those that know about intersection multiplicity, the curve $S$ is the unique curve whose intersection multiplicity with the union of the red lines at the points of intersection off the green line equals the intersection multiplicity of the union of the blue lines with the union of the red lines at those same points (and we can replace red with blue in this statement).

Möbius [22] also generalized the Braikenridge-Maclaurin Theorem, but in a different direction. Suppose a polygon with $4n + 2$ sides is inscribed in an irreducible conic and we determine $2n + 1$ points by extending opposite edges until they meet. If $2n$ of these $2n + 1$ points of intersection lie on a line then the last point also lies on the line. Using the Cayley-Bacharach Theorem allows us to extend Möbius's result, relaxing the constraint on the number of sides of the polygon.
Theorem 9. Suppose that a polygon with $2k$ sides is inscribed in an irreducible conic. Working around the perimeter of the polygon, color the edges alternately red and blue. Extending the edges to lines, consider the $k^2 - 2k$ points of intersection of the red and blue lines, omitting the original $2k$ vertices of the polygon. If $k - 1$ of these points lie on a green line, then in fact another of these points lies on the green line as well.

The theorem is illustrated when $k = 4$ in Figure 5: both the green and purple lines contain 3 of the 8 points off the conic, so they must each contain a fourth such point too.

Figure 5: An illustration of Theorem 9 when $k = 4$.

Proof of Theorem 9. First compute the dimension of the degree-$(k - 2)$ curves that go through all the points off the conic. Since there are no degree-$(k - 2)$ curves through all the $k^2$ points of intersection of the extended edges, the Cayley-Bacharach Theorem gives that this dimension equals the failure of $2k$ points that lie on a conic
to impose independent conditions on curves of degree $k + k - 3 - (k - 2) = k - 1$. Because the conic is irreducible, these degree-$(k - 1)$ curves must contain the conic as a component. So the failure is $2k$ minus the difference between the dimension of the space of degree $k - 1$ curves in $\mathbb{P}^2$ and the dimension of the space of degree $k - 3$ curves in $\mathbb{P}^2$. The failure is thus

$$2k - \left[\binom{k+1}{2} - \binom{k-1}{2}\right] = 1.$$ 

So, up to scaling, there is a unique curve of degree $k - 2$ through all the points off the conic. Since this curve meets the green line in at least $k - 1$ points, Bézout’s Theorem shows that it must contain the line as a component. Taking the union of the residual curve (of degree $k - 3$) with the conic gives a curve of degree $k - 1$ through all the points on both the red and blue lines that do not lie on the green line. Now the Cayley-Bacharach Theorem says that the dimension of all degree-$(k - 1)$ curves through all the points off the green line equals the failure of the points on the green line to impose independent conditions on forms of degree $k - 2$. There are at least $k - 1$ points on the green line, so the failure equals

$$(\# \text{ points on the line}) - \left[\binom{k}{2} - \binom{k-1}{2}\right] = (\# \text{ points on the line}) - (k - 1).$$

But there is such a curve so this number must be at least one, in which case the number of points on the green line must be at least $k$. Of course, the number of points on the green line is bounded by the number of points on the intersection of the green line with the $k$ red lines so there are precisely $k$ intersection points on the green line. This shows that the last point must also lie on the green line and establishes Möbius’s result.

4 Constructions

Let’s take a constructive view of Theorem 4, our extension of the Braikenridge-Maclaurin Theorem. Say that a curve $X$ of degree $d$ is constructible if there exist $d + 1$ red lines $\ell_1, \ldots, \ell_{d+1}$ and $d + 1$ blue lines $L_1, \ldots, L_{d+1}$ so that the $d + 1$ points $\{\ell_i \cap L_i : 1 \leq i \leq d + 1\}$ are collinear and the other $d(d + 1)$ points $\{\ell_i \cap L_j : i \neq j\}$ lie on $X$.

We turn to the question of which curves are constructible. In particular, we aim to show that for a certain range of degrees $d$ almost all curves of degree $d$ are constructible and for degrees outside of this range, almost no curves of degree $d$ are constructible. One way to make such statements precise is to introduce the Zariski topology on projective space.

The Zariski topology is the coarsest topology that makes polynomial maps from $\mathbb{P}^m$ to $\mathbb{P}^n$ continuous. More concretely, every homogeneous polynomial $F$ in $n + 1$
variables determines a closed set in $\mathbb{P}^n$

$$\mathbb{V}(F) = \{ P \in \mathbb{P}^n : F(P) = 0 \},$$

and every closed set is built up by taking finite unions and arbitrary intersections of such sets. Closed sets in the Zariski topology are called varieties. The nonempty open sets in this topology are dense: their complement is contained in a set of the form $\mathbb{V}(F)$.

We’ll say that the construction is dense for degree-$d$ curves if there is a nonempty Zariski-open set of degree-$d$ curves $U$ such that each $X \in U$ is constructible.

**Question 10.** For which degrees is the construction dense?

The construction is clearly dense for degree-1 curves (lines). Pascal’s Theorem, Theorem 2 shows that the construction is dense for degree-2 curves.

We give a simple argument to show that the construction cannot be dense if $d \geq 6$. Consider the number of parameters that can be used to define an arrangement of $2d+3$ lines so that there are $d+1$ triple points on one of the lines. Two parameters are needed to define the green line and then we need $d+1$ parameters to determine the triple points and $2(d+1)$ parameters to choose the slopes of pairs of lines through these points. So a $(3d+5)$-dimensional space parameterizes the line arrangements. The space of degree-$d$ curves is parameterized by a $\left( \binom{d+2}{2} - 1 \right)$ projective space. Since

$$\binom{d+2}{2} - 1 = \frac{d^2 + 3d}{2} > 3d + 5,$$

when $d \geq 6$, it is impossible for the line arrangements to parameterize a nonempty Zariski-open set of dimension $\binom{d+2}{2} - 1$ when $d \geq 6$.

Using the group law on elliptic curves allows us to show that the construction is dense for degree-3 curves.

**Theorem 11.** The construction is dense for degree-3 curves.

**Proof.** The set of smooth plane curves of degree 3 is a nonempty Zariski-open set in the space $\mathbb{P}^9$ parameterizing all degree-3 curves [29, Theorem 2 in Section II.6.2]. Such curves are called elliptic curves and their points form a group: three distinct points add to the identity element in the elliptic curve group if and only if they are collinear. Given an elliptic curve $X$ we pick 5 points, $p_1, \ldots, p_5$ on the curve, no three of which are collinear. We will construct the red, blue and green lines; the reader may wish to refer to the schematic diagram in Figure 6 as the construction proceeds.

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\[\text{If two of the points are the same then the line must also be tangent to } X \text{ at this point, while if all three points are the same then the tangent line to } X \text{ at the point must intersect } X \text{ with multiplicity 3.}\]
We draw a red line connecting points $p_1$ and $p_2$, meeting $X$ in the third point $-(p_1 + p_2)$. A blue line joining $p_1$ and $p_4$ meets $X$ at $-(p_1 + p_4)$ and a blue line joining $p_2$ and $p_3$ meets $X$ at $-(p_2 + p_3)$. A red line joining $p_4$ and $p_5$ meets $X$ at $-(p_4 + p_5)$. A red line joins $-(p_1 + p_4)$ and $-(p_2 + p_3)$ and meets $X$ in the point $p_1 + p_2 + p_3 + p_4$. A blue line joins $p_1 + p_2 + p_3 + p_4$ to $p_5$, meeting $X$ in $-(p_1 + p_2 + p_3 + p_4 + p_5)$. A red line joins $-(p_1 + p_2 + p_3 + p_4 + p_5)$ to $p_3$, meeting $X$ in the point $p_1 + p_2 + p_4 + p_5$. A blue line through $-(p_1 + p_2)$ and $-(p_1 + p_4)$ also hits $X$ at $p_1 + p_2 + p_4 + p_5$. The four red lines meet the four blue lines in 16 points, 12 of which lie on the elliptic curve $X$. We will prove that the other 4 points, circled in the schematic Figure 6, are collinear (lying on the green line) using the Cayley-Bacharach Theorem. Indeed, let $\Gamma$ be the 16 points where the red lines meet the blue lines and let $\Gamma''$ be the 12 points lying on the cubic $X$. Let $\Gamma' = \Gamma \setminus \Gamma''$ be the residual set of the four circled points. Since there are no degree-1 curves vanishing on the 16 points of $\Gamma$, the Cayley-Bacharach Theorem says that the dimension of the space of degree-1 curves vanishing on all four points of $\Gamma'$ equals the failure of $\Gamma''$ to impose independent conditions on curves of degree 4 + 4 − 3 − 1 = 4. The failure equals 12 minus the codimension of the degree-4 forms vanishing on $\Gamma''$ in the space of all degree-4 forms. This is equal to three less than the dimension of the vector space of degree-4 forms.
vanishing on $\Gamma''$:

$$12 - \left[ \binom{6}{2} \right] = \dim \text{ degree-4 forms vanishing on } \Gamma'' - 3.$$

Now any linear form times the equation of the cubic $X$ gives a degree-4 form vanishing on $\Gamma''$ so the degree-4 forms vanishing on $\Gamma''$ is a vector space of dimension at least 3. However, the defining ideal of the four red lines also vanishes on $\Gamma''$, so in fact the dimension is at least 4. It follows that the failure is at least 1 so the four points in $\Gamma'$ are collinear. So the construction produces all elliptic curves and is dense in degree 3.

To establish that the construction is dense in degrees 4 and 5, we use Terracini’s beautiful lemma about secant varieties [3, Lemma 3.1] (stated below in a restricted form, though it holds for higher secant varieties too). If $X$ is a subvariety of $\mathbb{P}^n$ and $p_1 \neq p_2$ are two points on $X$ then the line joining $p_1$ to $p_2$ is a secant line to $X$. The secant line variety $\text{Sec}(X)$ is the Zariski-closure of the variety of points $q \in \mathbb{P}^n$ that lie on a secant line to $X$.

**Lemma 12** (Terracini’s Lemma). Let $p$ be a generic point on $\text{Sec}(X) \subset \mathbb{P}^n$, lying on the secant line joining the two points $p_1 \neq p_2$ of $X$. Then $T_p(\text{Sec}(X))$, the (projectivized) tangent space to $\text{Sec}(X)$ at $p$, is $\langle T_{p_1}(X), T_{p_2}(X) \rangle$, the projectivization of the linear span of the two vector spaces $T_{p_1}(X)$ and $T_{p_2}(X)$. In particular,

$$\dim \text{Sec}(X) = \dim \langle T_{p_1}(X), T_{p_2}(X) \rangle.$$

We apply this in the setting where $X = \mathbb{X}_{15}$, the variety of completely reducible forms of degree 5 on $\mathbb{P}^2$. Letting $S = \mathbb{C}[x, y, z] = \oplus_{d \geq 0} S_d$, $\mathbb{X}_{15}$ is a subvariety of the parameter space $\mathbb{P}(S_5)$ of all degree-5 curves:

$$\mathbb{X}_{15} = \{ [F_1 \cdots F_5] : \text{ each } F_i \in S_1 \}.$$ 

Fortunately, Carlini, Chiantini and Geramita recently described the tangent space to $\mathbb{X}_{15}$.

**Lemma 13** ([3, Proposition 3.2]). The tangent space to a point $p = [F_1 \cdots F_d]$ in $\mathbb{X}_{1d} \subset \mathbb{P}(S_d)$ is the projectivization of the degree-$d$ part of the ideal

$$I_p = \langle G_1, \ldots, G_d \rangle,$$

where $G_i = (F_1 F_2 \cdots F_d)/F_i$. 

18
If the forms $F_1, \ldots, F_5$ are distinct then $(I_p)_5$ has dimension 11. To see this, note that $\dim (I_p)_5 = (\dim S_1)(\dim (J_p)_4) - \dim (\text{Syz}(G_1, \ldots, G_5)_1) = 3(5) - \dim (\text{Syz}(G_1, \ldots, G_5)_1)$, where

$$\text{Syz}(G_1, \ldots, G_5)_1 = \{(L_1, \ldots, L_5) \in (S_1)^5 : L_1G_1 + \cdots + L_5G_5 = 0\}$$

is the degree-1 part of the syzygy module. However, if $L_1G_1 + \cdots + L_5G_5 = 0$ then $L_1G_1 = -(L_2G_2 + \cdots + L_5G_5)$. Since $F_1$ divides each of the terms on the right-hand side of this equality, $F_1$ must divide $L_1G_1$. But $F_1$ does not divide $G_1$ so $F_1 | L_1$. Similarly, $F_i | L_i$ for $i = 1, \ldots, 5$. It follows that the only degree-1 syzygies are generated by the 4 linearly independent syzygies $F_1e_1 - F_ie_i$ ($i = 2, \ldots, 5$). As a result, $T_p(X_{15})$ is the projectivization of an 11-dimensional vector space.

Terracini’s Lemma shows that if $p$ is a generic point on the line between $p_1$ and $p_2$, $T_p(\text{Sec}X_{15})$ is the projectivization of the linear span of the vector spaces $(I_{p_1})_5$ and $(I_{p_2})_5$. This span has dimension $2(11) - \dim (I_{p_1} \cap I_{p_2})_5$. If $I$ is an ideal in $S$, let $\mathcal{V}(I)$ denote the set of points $P \in \mathbb{P}^2$ such that $F(P) = 0$ for all $F \in I$. Now if $p_1 = [F_1 \cdots F_{15}]$ and $p_2 = [F_{21} \cdots F_{25}]$ and if all the lines $\mathcal{V}(F_{ij})$ are distinct, then $\mathcal{V}(I_{p_1} \cap I_{p_2}) = \mathcal{V}(I_{p_1}) \cup \mathcal{V}(I_{p_2})$ is a collection of 20 points (counted with multiplicities): 10 of the points are given by the intersections of the $\binom{5}{2}$ pairs of lines $F_{1i}(x, y, z) = 0$ and 10 of the points are given by the intersections of the $\binom{5}{3}$ pairs of lines $F_{2i}(x, y, z) = 0$. A polynomial in $(I_{p_1} \cap I_{p_2})_5$ is a curve that goes through these 20 points. If the 20 points were in general position, we would expect only 1 curve to go through all 20 points and so $\dim (I_{p_1} \cap I_{p_2})_5 = 1$. However, the 20 points are in special position – for example, many collections of four of the points are collinear – so we cannot trust our intuition blindly. Note that an element $H$ of $(I_{p_1} \cap I_{p_2})_5$ corresponds to a solution to a system of equations $Av = 0$ where $A$ is a $20 \times 21$ matrix whose columns correspond to the monomials of $S_5$ and whose rows correspond to the 20 points in $\mathcal{V}(I_{p_1}) \cup \mathcal{V}(I_{p_2})$. The 21 entries of a solution $v$ are the coefficients of $H(x, y, z)$. The entries of $A$ in the column corresponding to a given monomial are obtained by plugging in the coordinates of a point into the monomial. The entries of the point at the intersection of $F_{ij}(x, y, z) = a_{ij}x + b_{ij}y + c_{ij}z = 0$ and $F_{ik}(x, y, z) = a_{ik}x + b_{ik}y + c_{ik}z = 0$ are given by the cross product $(a_{ij}, b_{ij}, c_{ij}) \times (a_{ik}, b_{ik}, c_{ik})$. So the entries in the matrix $A$ are degree-10 polynomials in the coefficients of the $F_{ij}$. This matrix will have full rank unless all the $20 \times 20$ minors are zero. This means that the matrix has full rank off the closed set where all the maximal minors vanish. So the matrix has full rank (and $\dim (I_{p_1} \cap I_{p_2})_5 = 1$) generically and so for a generic point $p$ of $\text{Sec}(X_{15})$, we see that the tangent space at $p$ is the projectivization of a 21-dimensional space; that is,
Sec($X_{15}$) ⊂ $\mathbb{P}^{20}$ is a 20-dimensional projective variety and so Sec($X_{15}$) = $\mathbb{P}^{20}$. We’re now ready to tackle the constructibility question for curves of degrees 4 and 5.

**Theorem 14.** The construction is dense for curves of degree 4.

**Proof.** We’ll show that there is a dense open subset of constructible irreducible curves of degree 4. The irreducible curves of degree 4 are themselves dense and open in the set of all curves of degree 4; see Shafarevich’s book [29, Section 5.2] for details. First we note that the set of degree-5 forms $Z ∈ \mathbb{P}(S_5)$ such that there exist $p_1 ∈ X_{15}$ and $p_2 ∈ X_{15}$ so that $Z$ is a linear combination of $p_1$ and $p_2$ and the 10 lines in $\mathcal{V}(p_1) ∪ \mathcal{V}(p_2)$ are not distinct is contained in a 19-dimensional subvariety $W$ of $\mathbb{P}(S_5)$. The dimension count is easy: there are two parameters for each of the 9 (possibly) distinct lines determining $p_1$ and $p_2$ and 1 parameter to reflect where $Z$ lies on the line joining $p_1$ and $p_2$. Now fix a linear form $L$ and consider the map $φ_L : \mathbb{P}(S_4) → \mathbb{P}(S_5)$ given by multiplication by $L$. The inverse image $φ_L^{-1}(W^c)$ of the complement of $W$ is open in $\mathbb{P}(S_4)$. Given an irreducible degree-4 form $F$ in this open set, there exist $p_1$ and $p_2$ in $X_{15}$ so that $FL$ is a linear combination of $p_1$ and $p_2$ and the 10 lines in $p_1$ and $p_2$ are distinct. Then $\mathcal{V}(FL)$ contains the 25 points of intersection between the lines in $p_1$ and the lines in $p_2$. We claim that 5 of the 25 points in $\mathcal{V}(p_1) ∩ \mathcal{V}(p_2)$ lie on $\mathcal{V}(L)$ and the remaining 20 points lie on $\mathcal{V}(F)$. If more than 5 points lie on $\mathcal{V}(L)$ then Bézout’s Theorem shows that $L$ must divide $p_1$. Similarly, $L$ must divide $p_2$. This is impossible because the 10 lines in $\mathcal{V}(p_1)$ and $\mathcal{V}(p_2)$ are distinct. Similarly, if $\mathcal{V}(F)$ goes through more than 20 points of $\mathcal{V}(p_1) ∩ \mathcal{V}(p_2)$ then $F$ and $p_1$ must have a nontrivial common divisor. But $F$ is irreducible so this cannot occur. It follows that $\mathcal{V}(F)$ is constructible (the red lines are the lines in $\mathcal{V}(p_1)$, the blue lines are the lines in $\mathcal{V}(p_2)$ and the green line is the line $\mathcal{V}(L)$).

It follows that an open set of irreducible degree-4 curves is constructible. We give an example to show that this open set is nonempty. Take the green line to be $y = 0$, the red lines to be $x + 2z = 0$, $x + z = 0$, $x = 0$, $x - z = 0$, and $x - 2z = 0$, and the blue lines to be $x - y + 2z = 0$, $x - y + z = 0$, $x - y = 0$, $x - y - z = 0$, and $x - y - 2z = 0$. The red lines intersect the blue lines in 20 distinct points off the green line and the polynomial

$$5x^4 - 10x^3y + 10x^2y^2 - 5xy^3 + y^4 - 15x^2 + 15xy - 5y^2 + 4$$

vanishes on each of the 20 points. You can, for example, dehomogenize the polynomial (set $z = 1$) and use Maple’s evala(FACTOR(·)) command to check that the polynomial is irreducible.

**Theorem 15.** The construction is dense for curves of degree 5.

**Proof.** First we note that $X_{1,1}$, the subvariety of $\mathbb{P}(S_6)$ consisting of degree 6 forms that factor into a linear form times an irreducible degree-5 form, is in fact a subvariety.
of Sec($X_{1\ell}$). If $L$ is a linear form and $Q$ is an irreducible degree-5 form then $Q \in P(S_5) = Sec(X_{1\ell})$ so there are completely reducible forms $p_1$ and $p_2$ of degree 5 so that $Q$ is a linear combination of $p_1$ and $p_2$. It follows that the form $LQ \in X_{1,V}$ is a linear combination of $Lp_1$ and $Lp_2$ so $X_{1,V} \subseteq Sec(X_{1\ell})$. Moreover, $X_{1,V}$ is a closed set in $P(S_6)$ since it is the image of the regular map $P(S_1) \times V \rightarrow P(S_6)$, where the map is given by multiplication and $V$ is projectivization of the irreducible forms of degree 5. This shows that $X_{1,V}$ is a subvariety of $Sec(X_{1\ell})$.

Now pick $Z \in V$ constructible so that $L$ is the defining equation of the green line (for some set of distinct red and blue lines). For example, we can take the green line to be $y = 0$, the red lines to be $x + 3z = 0$, $x + 2z = 0$, $x + z = 0$, $x = 0$, $x - z = 0$ and $x - 2z = 0$, and the blue lines to be $x - y + 3z = 0$, $x - y + 2z = 0$, $x - y + z = 0$, $5x - y = 0$, $5x - y - 5z = 0$ and $5x - y - 10z = 0$. The red lines intersect the blue lines in 30 distinct points off the green line and the irreducible polynomial

$$\begin{align*}
450x^5 - 615x^4y + 396x^3y^2 - 123x^2y^3 + 18xy^4 - y^5 &+ 675x^4z - 150x^3yz \\
-234x^2y^2z + 93xy^3z - 9y^4z &- 2400x^3z^2 + 2250x^2yz^2 - 504xy^2z^2 \\
+ 29y^3z^2 - 2025x^2z^3 - 375xyz^3 + 141y^2z^3 &+ 2400xz^4 - 460yz^4 + 300z^5
\end{align*}$$

vanishes on each of the 30 points. Fixing $L$, consider the map $\phi_L : V \rightarrow Sec(X_{1\ell})$ given by sending an irreducible degree-5 form $F$ to $FL \in X_{1,V} \subseteq Sec(X_{1\ell})$. The set of points in $Sec(X_{1\ell})$ that lie on the line connecting completely reducible forms $p_1$ and $p_2$ where the 12 lines forming $\mathcal{V}(p_1) \cup \mathcal{V}(p_2)$ are not distinct is a closed set $W$ of dimension no larger than 23. We leave it to the reader to check that $Sec(X_{1\ell})$ has dimension 25; the proof is similar to the argument given above that $Sec(X_{1\ell})$ has dimension 20. It follows that the inverse image $\phi_L^{-1}(W^c)$ of the complement of $W$ is open in $V$. Since $\phi_L(Z) \notin W$, the open set is nonempty. Now if $F$ is a degree-5 irreducible form with $FL \notin W$, there exist $p_1$ and $p_2$ in $X_{1\ell}$ so that $FL$ is a linear combination of $p_1$ and $p_2$ and the 12 lines in $p_1$ and $p_2$ are distinct. Then $\mathcal{V}(FL)$ contains the 36 points of intersection between the lines in $p_1$ and the lines in $p_2$. Now, as in the proof of Theorem 14, Bézout’s Theorem shows that 6 of the 36 points in $\mathcal{V}(p_1) \cap \mathcal{V}(p_2)$ lie on $\mathcal{V}(L)$ and the remaining 30 points lie on $\mathcal{V}(F)$. This allows us to use the lines in $\mathcal{V}(p_1)$ as our red lines, the lines in $\mathcal{V}(p_2)$ as the blue lines and the line $\mathcal{V}(L)$ as our green line to construct the curve $\mathcal{V}(F)$.

We’ve shown that a nonempty open subset of the irreducible degree-5 curves consists of constructible curves. The result follows because the collection of irreducible degree-5 curves form an open set in the parameter space $P(S_6)$. \hfill \Box

We have not provided an example of a curve of degree less than 6 that is not constructible. It may be that the set of constructible curves is Zariski-closed. In this case, every curve of degree less than 6 would be constructible because projective spaces are
connected in the Zariski-topology: the only sets in projective space that are both open and closed are the empty set and the whole space.

5  Further Reading and Exercises

Pappus’s Theorem inspired a lot of amazing mathematics. The first chapter of a fascinating new book by Richter-Gebert [26] describes the connections between Pappus’s Theorem and many areas of mathematics, including cross-ratios and the Grassmann-Plücker relations among determinants.

The history and implications of the Cayley-Bacharach Theorem is carefully considered in Eisenbud, Green and Harris’s amazing survey paper [8]. They connect the result to a host of interesting mathematics, including the Riemann-Roch Theorem, residues and homological algebra. Their exposition culminates in the assertion that the theorem is equivalent to the statement that polynomial rings are Gorenstein.

My approach to the Braikenridge-Maclaurin Theorem was inspired by thinking about hyperplane arrangements. A good introduction to these objects from an algebraic and topological viewpoint is the book by Orlik and Terao [25]. For a more combinatorial viewpoint, see Stanley’s lecture notes [30].

One way to view what we’ve done is to note that if $\Gamma$ is a complete intersection – a codimension $d$ variety (or, more generally, scheme) defined by the vanishing of $d$ polynomials – and $\Gamma$ is made up of two subvarieties, then special properties of one subvariety are reflected in special properties of the other subvariety. This point of view leads to the beautiful subject of liaison theory. The last chapter of Eisenbud [7] introduces this advanced topic in Commutative Algebra; more details can be found in Migliore and Nagel’s notes [21].

Exercise 16. The following exercises are roughly in order of increasing difficulty.

1. Pascal’s Theorem says that if a regular hexagon is inscribed in a circle then the 3 pairs of opposite edges lie on lines that intersect in 3 collinear points. Which line do the three points lie on? Is it surprising that it doesn’t matter where in the plane the circle is centered?

2. When working with lines in $\mathbb{P}^2$ it is desirable to have a quick way to compute their intersection points. Show that the lines $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$ meet in the point $[a_3 : b_3 : c_3]$ where

$$\langle a_3, b_3, c_3 \rangle = \langle a_1, b_1, c_1 \rangle \times \langle a_2, b_2, c_2 \rangle.$$

Interpret the result in terms of the geometry of 3-dimensional space. Also describe how to use this result to compute the intersection of two lines in $\mathbb{R}^2$. 22
3. There is an interesting duality between points and lines in $\mathbb{P}^2$. Fixing a nondegenerate inner product on 3-dimensional space, we define the dual line $\overline{P}$ to a point $P \in \mathbb{P}^2$ to be the projectivization of the 2-dimensional subspace orthogonal to the 1-dimensional subspace corresponding to $P$. Similarly, if $L$ is a line in $\mathbb{P}^2$, it corresponds to a 2-dimensional subspace in 3-dimensional space and we define the dual point $\overline{L}$ to be the projectivization of the 1-dimensional subspace orthogonal to this subspace.

(a) Show that a line $L$ in $\mathbb{P}^2$ goes through two points $P_1 \neq P_2$ if and only if the dual point $\overline{L}$ lies on the intersection of the two dual lines $\overline{P}_1$ and $\overline{P}_2$.

(b) Use part (a) and Exercise 16.2 to develop a cross product formula for the line through 2 points in $\mathbb{P}^2$. Extend the formula to compute the equation for a line through 2 points in $\mathbb{R}^2$.

(c) It turns out that the duals of all the tangent lines to an irreducible conic $C$ form a collection of points lying on a dual irreducible conic $\hat{C}$, and vice-versa (see Bachelor, Ksir and Traves [1] for details). Show that dualizing Pascal’s Theorem gives Brianchon’s Theorem: If an irreducible conic is inscribed in a hexagon, then the three lines joining pairs of opposite vertices intersect at a single point.

4. Provide a proof for one of the assertions in the paper: any set of $k$ collinear points fails to impose $k - (d + 1)$ conditions on forms of degree $d \leq k - 1$.

5. Establish the following result due to Möbius [22] using the Cayley-Bacharach Theorem. Consider two polygons $P_1$ and $P_2$, each with $m$ edges, inscribed in a conic, and associate one edge from $P_1$ with one edge from $P_2$. Working counterclockwise in each polygon, associate the other edges of $P_1$ with the edges of $P_2$. Extending these edges to lines, Möbius proved that if $m - 1$ of the intersections of pairs of corresponding edges lie on a line then the last pair of corresponding edges also meets in a point on this line.

6. Establish the following result due to Katz [16, Theorem 3.3], his Mystic 2d-Gram Theorem. If $d$ red lines and $d$ blue lines intersect in $d^2$ points and if $2d$ of these points lie on an irreducible conic then there is a unique curve of degree $k - 2$ through the other $d^2 - 2d$ intersection points. Katz’s interesting paper [16] contains several open problems.

7. Use the Cayley-Bacharach Theorem to show that if two degree-5 curves meet in 25 points, 10 of which lie on an irreducible degree-3 curve, then there is a unique degree-4 curve through the other 15 points. Also convince yourself that the hypotheses of this exercise can actually occur.

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5 Like Bézout, Charles Julien Brianchon (1783-1864) was a professor at a French military academy. The French military of the 19th century seems to have played an interesting role in supporting the development and teaching of mathematics.
8. If a degree-8 curve meets a degree-9 curve in 72 points and if 17 of these points lie on an irreducible degree-3 curve, then what is the dimension of the family of degree-9 curves through the remaining 55 points? Convince yourself that the hypotheses of this exercise can actually occur.

9. Use the $8 \Rightarrow 9$ Theorem to show that the group law on an elliptic curve is associative.

10. In general you might expect that if $X \subset \mathbb{P}^n$ then $\dim \text{Sec}(X) = \min(2\dim(X), n)$. Varieties $X$ where this inequality fails to hold are called defective. Check that $\text{Sec}(X_{16})$ is not defective: it has dimension 25.

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