On removable singularities for solutions of Neumann problem for elliptic equations involving variable exponent

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Abstract
We study the removability of a singular set in the boundary of Neumann problem for elliptic equations with variable exponent. We consider the case where the singular set is compact, and give sufficient conditions for removability of this singularity for equations in the variable exponent Sobolev space $W^{1, p(\cdot)}(\Omega)$.

Keywords  Variable exponent · Singular set · Removable singularity · Neumann problem

Mathematics Subject Classification  35A21 · 35D30 · 35J57 · 35J60

1 Introduction

This paper is devoted to the study of conditions guaranteeing the removability of singular set for solutions of nonlinear elliptic equations with Neumann boundary conditions of the form:

\[
\begin{align*}
-\text{div}A(x, u, \nabla u) + a(x, u) + g(x, u) &= 0 \quad \text{in } \Omega, \\
A(x, u, \nabla u) \cdot \nu + b(x, u) + h(x, u) &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Throughout the whole article $\Omega$ is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial \Omega$, and $\Gamma \subset \partial \Omega$ is a compact set. We always equip $\partial \Omega$ with the $(n - 1)$-dimensional Hausdorff measure.

We assume that $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $b, h : \partial \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable functions, $g$ and $h$ are locally bounded. Furthermore, there exists $\mu > 0$ such that the following conditions are satisfied almost everywhere:
\[ \langle A(x, u, \eta), \eta \rangle \geq \mu |\eta|^{p_i(x)}, \tag{2} \]

\[ |A(x, u, \eta)| \leq \mu^{-1} (|\eta|^{p_i(x)-1} + |u|^{p_i(x)-1} + 1), \tag{3} \]

\[ |a(x, u)| \leq \mu^{-1} (|u|^{p_i(x)-1} + 1), \tag{4} \]

\[ g(x, u)\text{sign}u \geq \mu |u|^{p_i(x)} - \mu^{-1}, \tag{5} \]

\[ |b(x, u)| \leq \mu^{-1} (|u|^{q_i(x)-1} + 1), \tag{6} \]

\[ h(x, u)\text{sign}u \geq \mu |u|^{q_i(x)} - \mu^{-1}, \tag{7} \]

\[ A(x, u, -\eta) = -A(x, u, \eta), \tag{8} \]

where \( p_1, p_2 : \Omega \to \mathbb{R}, q_1, q_2 : \partial\Omega \to \mathbb{R} \) are measurable functions satisfying

\[ p_i^-, p_i^+ \in (1, n), \quad q_i > 1, \quad \min \{p_i^-, q_i^-\} \geq \max \{p_i^+, q_i^+\} + 1 > 0. \tag{9} \]

Also, for some \( d \geq 0, \)

\[ \text{ess sup}_{\Omega} \frac{p_1 p_2}{p_2 - p_1 + 1} < n - d, \tag{10} \]

where \( p_i^- = \text{ess inf}_{\Omega} p_i, \quad p_i^+ = \text{ess sup}_{\Omega} p_i, \quad q_i^- = \text{ess inf}_{\partial\Omega} q_i \) and \( q_i^+ = \text{ess sup}_{\partial\Omega} q_i; \) \( i = 1, 2. \)

Concerning the singular set \( \Gamma \subset \partial\Omega, \) we suppose that \( |\Gamma| = 0, \) and there is a small enough \( r_0 \in (0, 1) \) so that \( \mathcal{U} = \{ x \in \Omega \mid \text{dist}(x, \Gamma) < 2r_0 \} \) is an open set with Lipschitz boundary \( \partial\mathcal{U}, \) \( \{ x \in \Omega \mid \text{dist}(x, \Gamma) = 2r_0 \} \neq \emptyset, \) and for a suitable positive constant \( C > 0: \)

\[ \frac{|\{ \text{dist}(\cdot, \Gamma) - \epsilon < r \}|}{|\{ \text{dist}(\cdot, \Gamma) = r \}|} \leq C r^{n-d}, \frac{|\{ \text{dist}(\cdot, \Gamma) = r \}|}{|\partial\mathcal{U} \cap \{ \text{dist}(\cdot, \Gamma) - \epsilon < r \}|} \leq C r^{n-d-1}, \tag{11} \]

where \( 0 < r \leq \epsilon \leq r_0. \)

The main result of this paper is the following theorem.

**Theorem 1** Suppose that the conditions (2)–(11) are satisfied. Let \( u \in W^{1, p_i(x)}_{\text{loc}} (\overline{\Omega} \setminus \Gamma) \cap L^\infty_{\text{loc}} (\overline{\Omega} \setminus \Gamma) \) be a solution of (1) in \( \overline{\Omega} \setminus \Gamma. \) Then, the singularity of \( u \) at \( \Gamma \) is removable.

We follow the same lines as in [5] and [7]; which handle the case when the singular set is an interior point of \( \Omega, \) and the solution of an elliptic equation \( u \) has no boundary conditions.
2 Preliminaries

We first recall some facts on spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. Denote by $\mathbf{P}(\Omega)$ the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For the details see [1, 4, 6, 8]. Let $p \in \mathbf{P}(\Omega)$, we define the functional

$$\rho_p(u) = \int_{\Omega \setminus \Omega_\infty} |u|^p \, dx + \operatorname{ess sup}_{\Omega_\infty} |u|,$$

where $\Omega_\infty = \{ x \in \Omega \mid p(x) = \infty \}$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is the class of all functions $u$ such that $\rho_p(tu) < \infty$, for some $t > 0$. $L^{p(\cdot)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 \mid \rho_p \left( \frac{u}{\lambda} \right) \leq 1 \right\};$$

see [6, Theorem 2.5].

**Proposition 2** (see [6, Theorem 2.1]) Let $p \in \mathbf{P}(\Omega)$. If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, then

$$\int_{\Omega} |uv| \, dx \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$p'(x) = \begin{cases} \infty & \text{if } p(x) = 1, \\ 1 & \text{if } p(x) = \infty, \\ \frac{p(x)}{p(x) - 1} & \text{if } p(x) \neq 1 \text{ and } p(x) \neq \infty. \end{cases}$$

**Proposition 3** (see [4, Theorem 1.3].) Let $p \in \mathbf{P}(\Omega)$ with $p^* < \infty$. For any $u \in L^{p(\cdot)}(\Omega)$, we have

1. If $\|u\|_{L^{p(\cdot)}(\Omega)} \geq 1$, then $\|u\|_{L^{p(\cdot)}(\Omega)}^{p^*} \leq \int_{\Omega} |u|^p \, dx \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p^*}$
2. If $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, then $\|u\|_{L^{p(\cdot)}(\Omega)}^{p^*} \leq \int_{\Omega} |u|^p \, dx \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p^*}$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is the class of all functions $u \in L^{p(\cdot)}(\Omega)$ which have the property $|\nabla u| \in L^{p(\cdot)}(\Omega)$. The space $W^{1,p(\cdot)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

More precisely, we have
Proposition 4 (see [6, Theorem 3.1].) Let \( p \in P(\Omega) \). The space \( W^{1,p(\cdot)}(\Omega) \) is a Banach space, which is separable if \( p \in L^\infty(\Omega) \) and reflexive if \( 1 < p^- \leq p^+ < \infty \).

Next we will see the definitions that we use in this work. Firstly we will make some observations regarding to the trace. Let \( p \in P(\Omega) \). Obviously \( W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega) \) because \( p^- \geq 1 \). From \( W^{1,1}(\Omega) \rightarrow L^1(\partial \Omega) \) we know that for all \( u \in W^{1,p(\cdot)}(\Omega) \) there already holds \( u|_{\partial \Omega} \in L^1(\partial \Omega) \). Thus for \( W^{1,p(\cdot)}(\Omega) \), the trace \( u|_{\partial \Omega} \) has definite meaning; see [3, p. 1398].

Define

\[
\begin{align*}
L^{p(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) := \{ u : \Omega \to \mathbb{R} | u \in L^{p(\cdot)}(U) \text{ for all open subset } U \subset \Omega \text{ with } \overline{U} \cap \Gamma = \emptyset \} \\
W^{1,p(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) := \{ u : \Omega \to \mathbb{R} | u \in W^{1,p(\cdot)}(U) \text{ for all open subset } U \subset \Omega \text{ with } \overline{U} \cap \Gamma = \emptyset \}.
\end{align*}
\]

Similarly we define \( L^{\infty}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \).

Let \( v : \Omega \to \mathbb{R} \) be a function, we call \( \text{supp} v = \{ x \in \Omega | v(x) \neq 0 \} \) the support of \( v \). For \( E \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), we denote by \( d(x,E) \) the Euclidean distance from \( x \) to \( E \).

**Definition 1** We will say that \( u \in W^{1,p(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^{\infty}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \) is a (weak) solution of (1) in \( \overline{\Omega}\setminus\Gamma \) if

\[
\int_{\Omega} \langle A(\cdot, u, \nabla u), \nabla \varphi \rangle + a(\cdot, u)\varphi + g(\cdot, u)\varphi dx + \int_{\partial \Omega} b(\cdot, u)\varphi + h(\cdot, u)\varphi d\sigma = 0
\]

for all \( \varphi \in W^{1,p(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^{\infty}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \), with \( \text{supp} \varphi \subset \overline{\Omega}\setminus\Gamma \).

Let us observe that the trace of \( u \in W^{1,p(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^{\infty}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \) possibly is not defined on \( \partial \Omega \), however it is defined on \( \{ x \in \partial \Omega | d(x, \Gamma) > r \} \), and is essentially bounded, for small enough values \( r > 0 \).

**Definition 2** We will say that the solution \( u \) of (1) in \( \overline{\Omega}\setminus\Gamma \) has a removable singularity at \( \Gamma \): if \( u \in W^{1,p(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^{\infty}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \) implies \( u \in W^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega) \) and the equality (12) is fulfilled for all \( \varphi \in W^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega) \).

### 3 The behavior of solutions near the singular set

The main result of this section is Theorem 8, which will be used in the proof of Theorem 1. We begin with the following results.
Lemma 5 (see [2, p. 1004]). Let $0 < \theta < 1$, $\sigma > 0$, $\xi(h)$ be a nonnegative function on the interval $[1/2, 1]$, and let

$$
\xi(k) \leq C_0(h - k)^{-\sigma}(\xi(h))^\theta, \quad 1/2 \leq k < h \leq 1,
$$

for some positive constant $C_0$. Then, there exists $C_1(\sigma, \theta) > 0$ such that

$$
\xi(1/2) \leq C_1 C_0^{1/\theta}.
$$

Lemma 6 If $p \in (1, n)$, $u \in W^{1,p}(\mathcal{U})$ and $u = 0$ on $\{x \in \tilde{\Omega} \mid d(x, \Gamma) = 2r_0\}$, then

$$
\left(\int_{\mathcal{U}} |u|^q dx\right)^{\frac{1}{q}} \leq C \left(\int_{\mathcal{U}} |\nabla u|^p dx\right)^{\frac{1}{p}}
$$

(13)

for each $q \in \left[p, \frac{np}{n-p}\right)$, where $C = C(n, p, q, \mathcal{U})$ is a positive constant.

**Proof** The proof is by contradiction, considering that $W^{1,p}(\mathcal{U})$ is compactly embedded in $L^q(\mathcal{U})$.

We define,

$$
V_{\xi, r} = \{x \in \Omega \mid |d(x, \Gamma) - \xi| < r\}.
$$

Proposition 7 Assume that the conditions (2)–(7), (9) and (11) are satisfied. Suppose that $u \in W^{1,p_{1}(\cdot)}_{\text{loc}} (\tilde{\Omega} \setminus \Gamma) \cap L^\infty_{\text{loc}} (\tilde{\Omega} \setminus \Gamma)$ satisfies

$$
\int_\Omega \langle A(\cdot, u, \nabla u), \nabla \varphi \rangle + a(\cdot, u)\varphi + g(\cdot, u)\varphi dx + \int_{\partial \Omega} b(\cdot, u)\varphi + h(\cdot, u)\varphi d\sigma \leq 0,
$$

for all $\varphi \in W^{1,p_{1}(\cdot)}_{\text{loc}} (\tilde{\Omega} \setminus \Gamma) \cap L^\infty_{\text{loc}} (\tilde{\Omega} \setminus \Gamma)$, $\varphi \geq 0$, with $\text{supp} \varphi \subset \tilde{\Omega} \setminus \Gamma$. Then, if $0 < r < \xi < r_0$ we have the estimate

$$
\|\max\{u, 0\}\|_{L^\infty(V_{\xi, r_0})} \leq Cr^{-\tau},
$$

(15)

where $C = C(n, \mu, p_1, p_2, q_1, q_2, \mathcal{U}) > 0$ and $\tau = \tau(n, \mu, p_1, p_2, q_1, q_2, \mathcal{U}) > 0$

$\max\{p_1^+, q_1^\oplus\} - \min\{p_2^+, q_2^\oplus\} + 1$.

**Proof** 1. Let $u = C_{w} w$, where $C_{w} > 1$ is a number that will be determined below. We assume that $\{x \in V_{\xi, r_0/2} \mid w(x) > 0\} \neq \emptyset$, otherwise, (15) is immediate. Set $\Omega' = \{x \in V_{\xi, r} \mid w(x) > 0\}$. Take
\[ m_t = \text{ess sup} \{ w(x) \mid x \in V_{t} \cap \Omega' \}, \quad 1/2 \leq t \leq 1. \]

Let \( 1/2 \leq s < t \leq 1 \). Define the functions \( z : \Omega \to \mathbb{R}, z_k : \Omega \to \mathbb{R} \) by

\[
z(x) = w(x) - m_t \xi((d(x, \Gamma) - \ell'),
\]

\[
z_k(x) = \begin{cases} 
\max \{ w(x) - m_t \xi((d(x, \Gamma) - \ell')) - k, 0 \} & \text{if } x \in V_{t}, \\
0 & \text{if } x \in \Omega \setminus V_{t},
\end{cases}
\]

where \( 0 \leq k \leq \text{ess sup}_\Omega z \), and \( \xi : \mathbb{R} \to \mathbb{R} \) is a smooth function satisfying: \( \xi \equiv 0 \) on \((-\infty, sr] \), \( \xi \equiv 1 \) on \([s/t, r, \infty) \),

\[
0 \leq \xi \leq 1 \quad \text{and} \quad |\xi'| \leq \frac{C_1}{r(t - s)} \quad \text{on } \mathbb{R},
\]

where \( C_1 \) is a suitable positive constant. Observe that \( z_k \in W^{1,p_1(\cdot)}_{\text{loc}}(\Omega \setminus \Gamma) \cap L^\infty(\bar{\Omega} \setminus \Gamma) \) and \( \text{supp} z_k \subset \Omega \cap \{(d(\cdot, \Gamma) - \ell') \leq t r\} \subset \bar{\Omega} \setminus \Gamma \). It is assumed that \( m_{1/2} > 1 \). The conclusion is obviously right for the case of \( 0 < m_{1/2} \leq 1 \). For simplicity we write \( \xi((d(x, \Gamma) - \ell')) = \xi(x) \) and \( \xi'((d(x, \Gamma) - \ell')) = \xi'(x) \).

Take \( k \in [0, K] \), where \( K = \sup \{ k \in [0, \text{ess sup}_\Omega z] \mid \{ x \in V_{t} \mid z_k(x) > 0 \} \neq \emptyset \} \). Observe that \( K \geq m_s \geq m_{1/2} > 1 \). Substituting \( \varphi = z_k \) into (14), we obtain

\[
\int_{\Omega} \langle A(\cdot, u, \nabla u), \nabla z_k \rangle + a(\cdot, u)z_k + g(\cdot, u)z_k \, dx + \int_{\partial \Omega} b(\cdot, u)z_k + h(\cdot, u)z_k \, d\sigma \leq 0.
\]

Denote \( \Omega_k = \{ x \in V_{t} \mid z_k(x) > 0 \} \). Then,

\[
\int_{\Omega_k} \left( A(\cdot, C_* w, C_* \nabla w), \nabla w - m_t \xi' \frac{d(d(\cdot, \Gamma) - \ell')}{d(\cdot, \Gamma) - \ell} \nabla d(\cdot, \Gamma) \right) \, dx + \int_{\Omega_k} g(\cdot, u)z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} h(\cdot, u)z_k \, d\sigma \leq \int_{\Omega_k} |a(\cdot, u)|z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} |b(\cdot, u)|z_k \, d\sigma.
\]

By (2)–(7), we have

\[
\int_{\Omega_k} \mu C_*^{p_1 - 1}|\nabla w|^{p_1} \, dx - \int_{\Omega_k} \frac{\mu^{-1} C_* m_t}{r(t - s)} (|C_* \nabla w|^{p_1 - 1} + |w|^{p_1 - 1} + 1) \, dx + \int_{\Omega_k} \left( \mu |C_* w|^{p_2} - \mu^{-1} \right) z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} \left( \mu |C_* w|^{q_2} - \mu^{-1} \right) z_k \, d\sigma \leq \int_{\Omega_k} \mu^{-1} (|C_* w|^{p_1 - 1} + 1) z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} \mu^{-1} (|C_* w|^{q_1 - 1} + 1) z_k \, d\sigma.
\]

Since \( m_t > 1, C_* > 1 \), and observing that \( w \geq k \) and \( m_t \geq w \geq z_k \) on \( \Omega_k \), we have
\[
\int_{\Omega_k} C_1^{p_1} |\nabla w|^{p_1} \, dx + \int_{\Omega_k} C_2^{p_2} k^{p_2} \nabla z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} C_3^{q_2} k^{q_2} z_k \, d\sigma \\
\leq C_2(\mu) \int_{\Omega_k} \frac{m_t C_1^{p_1}}{r(t-s)} |\nabla w|^{p_1} \, dx \\
+ C_2 C_\ast \max \{p_1^{-1}, q_1^{-1}\}^{-1} \left[ \frac{m_t}{r(t-s)} \right]^{\max \{p_1^{-1}, q_1^{-1}\}} (|\Omega_k| + |\partial \Omega_k \cap \partial \Omega|).
\]

On the other hand, using Young’s inequality,
\[
\int_{\Omega_k} \frac{m_t C_1^{p_1}}{r(t-s)} |\nabla w|^{p_1} \, dx \leq \int_{\Omega_k} C_1^{p_1} \left\{ C_3(\varepsilon_1, p_1) \left[ \frac{m_t}{r(t-s)} \right]^{p_1} + \varepsilon_1 |\nabla w|^{p_1} \right\} \, dx.
\]

Take \( \varepsilon_1 = \frac{1}{2C_2} \). Then, by (16) and (17),
\[
\frac{1}{2} \int_{\Omega_k} C_1^{p_1} \left( \frac{1}{2p_1-1} |\nabla z_k|^{p_1} - m_t |\xi'|^{p_1} \right) \, dx + \int_{\Omega_k} C_2^{p_2} k^{p_2} \nabla z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} C_3^{q_2} k^{q_2} z_k \, d\sigma \\
\leq C_4(\mu, p_1) C_\ast \max \{p_1^{-1}, q_1^{-1}\}^{-1} \left[ \frac{m_t}{r(t-s)} \right]^{\max \{p_1^{-1}, q_1^{-1}\}} (|\Omega_k| + |\partial \Omega_k \cap \partial \Omega|).
\]

Note that \( \nabla z_k = \nabla w - m_t \xi' d(\cdot, \Gamma) - \ell \nabla d(\cdot, \Gamma) \) in \( \Omega_k \), therefore
\[
\frac{1}{2} \int_{\Omega_k} C_1^{p_1} \left( \frac{1}{2p_1-1} |\nabla z_k|^{p_1} - m_t |\xi'|^{p_1} \right) \, dx + \int_{\Omega_k} C_2^{p_2} k^{p_2} \nabla z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} C_3^{q_2} k^{q_2} z_k \, d\sigma \\
\leq C_4 C_\ast \max \{p_1^{-1}, q_1^{-1}\}^{-1} \left[ \frac{m_t}{r(t-s)} \right]^{\max \{p_1^{-1}, q_1^{-1}\}} (|\Omega_k| + |\partial \Omega_k \cap \partial \Omega|).
\]

We have
\[
|\nabla z_k|^{p_1} \leq 1 + |\nabla z_k|^{p_1}.
\]

Then, from (18) and (19),
\[
\frac{1}{2p_1} \int_{\Omega_k} C_1^{p_1} \left( |\nabla z_k|^{p_1} - 1 \right) \, dx + \int_{\Omega_k} C_2^{p_2} k^{p_2} \nabla z_k \, dx + \int_{\partial \Omega_k \cap \partial \Omega} C_3^{q_2} k^{q_2} z_k \, d\sigma \\
\leq C_5(\mu, p_1) C_\ast \max \{p_1^{-1}, q_1^{-1}\}^{-1} \left[ \frac{m_t}{r(t-s)} \right]^{\max \{p_1^{-1}, q_1^{-1}\}} (|\Omega_k| + |\partial \Omega_k \cap \partial \Omega|).
\]

On the other hand, let \( \gamma \in \left( p_1^{-1}, \frac{(n-1)p_1}{n-p_1} \right) \) and \( \alpha \in \left( p_1^{-1}, \gamma \right) \) be arbitrary. Observe that \( \text{supp} z_k \cap \Omega \subset \Omega \cap \{ |d(\cdot, \Gamma) - \ell | \leq tr \} \subset U \). By (13), and since the trace \( W^{1,p_1}(U) \to L^\gamma(\partial U) \) is continuous,
where $C_6 = C_\delta(n, \mu, p_1, \gamma, \mathcal{U}) > 0$. Using Hölder’s inequality, we get

$$
\int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx \leq \left( \int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx \right)^{\frac{1}{\alpha}} |\Omega_k|^{-\frac{1}{\alpha}}, \quad \text{(22)}
$$

$$
\int_{\partial V_{\ell,p} \cap \partial \Omega} z_k \, d\sigma \leq \left( \int_{\partial V_{\ell,p} \cap \partial \Omega} \frac{z_k}{z_k^{\alpha}} \, d\sigma \right)^{\frac{1}{\alpha}} |\partial \Omega_k \cap \partial \Omega|^{-\frac{1}{\alpha}}. \quad \text{(23)}
$$

From (20)–(23), we have

$$
\frac{C_\delta^{p_i-1}}{2p_i} \left[ C_\delta^{-1} \left( |\Omega_k| + |\partial \Omega_k \cap \partial \Omega| \right)^{-\frac{1}{\alpha}} \frac{1}{r} \left( \int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx + \int_{\partial V_{\ell,p} \cap \partial \Omega} \frac{z_k}{z_k^{\alpha}} \, d\sigma \right) \right]^{p_i-1} + C_\delta \min \left\{ p_\gamma^-, q_\gamma^+ \right\} \min \left\{ k^{p_\gamma^-}, k^{p_\gamma^+}, k^{q_\gamma^-}, k^{q_\gamma^+} \right\} \left( \int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx + \int_{\partial V_{\ell,p} \cap \partial \Omega} \frac{z_k}{z_k^{\alpha}} \, d\sigma \right) \leq C_5 C_\delta \max \left\{ p_1^+, q_1^+ \right\}^{-1} \left[ \frac{m_t}{r(t-s)} \right]^{\max \left\{ p_1^+, q_1^+ \right\}} \left( |\Omega_k| + |\partial \Omega_k \cap \partial \Omega| \right) + C_\delta^{p_i-1} \frac{1}{2p_i} |\Omega_k|. \quad \text{(24)}
$$

2. Take $\varepsilon \in (0, 1)$. Then, (24) implies

$$
\varepsilon C_\delta^{p_i-1} \left( \int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx + \int_{\partial V_{\ell,p} \cap \partial \Omega} \frac{z_k}{z_k^{\alpha}} \, d\sigma \right)^{p_i-1} \left( \int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx + \int_{\partial V_{\ell,p} \cap \partial \Omega} \frac{z_k}{z_k^{\alpha}} \, d\sigma \right)^{p_i-1} \frac{1}{r} \left( \int_{V_{\ell,p}} \frac{z_k}{z_k^{\alpha}} \, dx + \int_{\partial V_{\ell,p} \cap \partial \Omega} \frac{z_k}{z_k^{\alpha}} \, d\sigma \right) \leq C_7 C_\delta \max \left\{ p_1^+, q_1^+ \right\}^{-1} \left( \frac{m_t}{r(t-s)} \right)^{\max \left\{ p_1^+, q_1^+ \right\}} + \left( |\Omega_k| + |\partial \Omega_k \cap \partial \Omega| \right)^{1+p_1^+} \frac{1}{\alpha}, \quad \text{(25)}
$$

where $C_7 = C_7(C_5, C_\delta, \mu, p_1) > 0$. Applying Young’s inequality $a^\varepsilon b^{1-\varepsilon} \leq \varepsilon a + (1-\varepsilon)b$ in the left-hand side of (25), we obtain
\[
\text{min} \left\{ k \rho_1^\alpha (1 - \epsilon), k \rho_2^\alpha (1 - \epsilon), k \rho_3^\alpha (1 - \epsilon), k \rho_4^\alpha (1 - \epsilon) \right\} C_\ast \left( \rho_1^{\alpha - \epsilon} + \rho_2^{\alpha - \epsilon} + \rho_3^{\alpha - \epsilon} + \rho_4^{\alpha - \epsilon} \right)
\]
\[
\leq C_\gamma C_\ast \max \left\{ \rho_1^{\alpha - \epsilon}, \rho_2^{\alpha - \epsilon}, \rho_3^{\alpha - \epsilon}, \rho_4^{\alpha - \epsilon} \right\} \left\{ \frac{m_t}{r(t - s)} \right\} \max \left\{ \rho_1^{\alpha - \epsilon}, \rho_2^{\alpha - \epsilon}, \rho_3^{\alpha - \epsilon}, \rho_4^{\alpha - \epsilon} \right\} \left( |\Omega_k| + |\partial \Omega_k \cap \partial \Omega| \right)^{1 - \rho_1^{\alpha - \epsilon}}.
\]

Therefore,
\[
\text{min} \left\{ k \rho_1^\alpha (1 - \epsilon), k \rho_2^\alpha (1 - \epsilon), k \rho_3^\alpha (1 - \epsilon), k \rho_4^\alpha (1 - \epsilon) \right\} C_\ast \leq (2C_\gamma)^2 \int_{\Omega_k} \left( z_d \text{dx} + \int_{\partial \Omega_k \cap \partial \Omega} z_d \text{d}\sigma \right) \rho_1^{\alpha - \epsilon} \rho_2^{\alpha - \epsilon} \rho_3^{\alpha - \epsilon} \rho_4^{\alpha - \epsilon} \left( \frac{m_t}{r(t - s)} \right) \left( |\Omega_k| + |\partial \Omega_k \cap \partial \Omega| \right) \text{d}k.
\]

(26)

where \( \beta = 1 + \rho_1^{\alpha - \epsilon} \).

3. Integrating (26) with respect to \( k \),
\[
\text{min} \left\{ k \rho_1^\alpha (1 - \epsilon), k \rho_2^\alpha (1 - \epsilon), k \rho_3^\alpha (1 - \epsilon), k \rho_4^\alpha (1 - \epsilon) \right\} C_\ast \leq (2C_\gamma)^2 \int_{0}^{\mathcal{K}} \left( \int_{\Omega_k} \left( z_d \text{dx} + \int_{\partial \Omega_k \cap \partial \Omega} z_d \text{d}\sigma \right) \rho_1^{\alpha - \epsilon} \rho_2^{\alpha - \epsilon} \rho_3^{\alpha - \epsilon} \rho_4^{\alpha - \epsilon} \left( \frac{m_t}{r(t - s)} \right) \left( |\Omega_k| + |\partial \Omega_k \cap \partial \Omega| \right) \text{d}k \right) \text{d}k.
\]

Let us consider the equalities
\[
\frac{d}{dk} \left( \int_{\Omega_k} z_d \text{dx} \right) = -|\Omega_k| \quad \text{and} \quad \frac{d}{dk} \left( \int_{\partial \Omega_k \cap \partial \Omega} z_d \text{d}\sigma \right) = -|\partial \Omega_k \cap \partial \Omega|.
\]

Since \( 1 - \frac{\rho_1^{\alpha - \epsilon}}{\rho_1^{\alpha - \epsilon}} > 0 \) and \( \mathcal{K} > 1 \),
\[
\mathcal{K}^{1 + \min \left\{ k \rho_1^\alpha (1 - \epsilon), k \rho_2^\alpha (1 - \epsilon), k \rho_3^\alpha (1 - \epsilon), k \rho_4^\alpha (1 - \epsilon) \right\} C_\ast} \leq \varepsilon^{-1} C_8 \rho_1^{\alpha - \epsilon} \rho_2^{\alpha - \epsilon} \rho_3^{\alpha - \epsilon} \rho_4^{\alpha - \epsilon} \left( \frac{m_t}{r(t - s)} \right) \left( \int_{\Omega_0} z_0 \text{dx} + \int_{\partial \Omega_0 \cap \partial \Omega} z_0 \text{d}\sigma \right) \left( \frac{m_t}{r(t - s)} \right) \left( |\Omega_0| + |\partial \Omega_0 \cap \partial \Omega| \right) \left( \frac{m_t}{r(t - s)} \right) \left( \frac{m_t}{r(t - s)} \right),
\]

where \( C_8 = C_8 (n, \mu, \alpha, \gamma, p_1, p_2, q_2, t, \mathcal{U}) > 0 \). We note \( \mathcal{K} \geq m_t \). Apply the estimates
\[
\int_{\Omega_0} z_0 \text{dx} \leq m_t |V_{\mathcal{E}, r}| \quad \text{and} \quad \int_{\partial \Omega_0 \cap \partial \Omega} z_0 \text{d}\sigma \leq m_t |\partial V_{\mathcal{E}, r} \cap \partial \Omega|,
\]
we have
where $C_9 = C_9(n, \mu, \epsilon, \alpha, \gamma, p_1, p_2, q_2, \mathcal{U}) > 0$.

Now we take $C_* > 0$ such that
\[
C_* = r^{-\tau}, \quad \text{where}
\tau = -\left[ \frac{p_1 (n-d-1)(\gamma - \alpha) \epsilon}{\gamma \alpha \beta} \left( \frac{\max \{p_1^+, q_1^+\}}{\beta} - (n-d-1) \left( 1 - \frac{p_1^- \epsilon + 1 - \epsilon}{\beta} \right) \right) \right. \\
\left. \left[ \min \{p_2^-, q_2^-\} - \max \{p_1^+, q_1^+\} + 1 - \left( \frac{\min \{p_2^-, q_2^+\} - p_1^- + 1}{\beta} \right) \right]^{-1} \right. \\
\left. > \frac{\max \{p_1^+, q_1^+\}}{\min \{p_2^-, q_2^-\} - \max \{p_1^+, q_1^+\} + 1} > 0, \right.
\]
if $1 > (n-d-1) \left( 1 - \frac{p_1^-}{\gamma} \right)$ and $\epsilon \in \left( 0, \frac{\min \{p_2^-, q_2^-\} - \max \{p_1^+, q_1^+\} + 1}{\min \{p_2^-, q_2^-\} - p_1^- + 1} \right)$.

On the other hand,
\[
m_s \leq C_9 \frac{\max \{p_1^+, q_1^+\}^\beta m_t^{\beta}}{(t-s)^\sigma},
\]
where
\[
\theta = \left[ \frac{\max \{p_1^+, q_1^+\}}{\beta} + 1 - \frac{p_1^- \epsilon + 1 - \epsilon}{\beta} \right] \left[ 1 + \frac{\min \{p_2^-, q_2^-\} (1 - \epsilon)}{\beta} \right]^{-1} < 1,
\]
\[
\sigma = \left[ \frac{\max \{p_1^+, q_1^+\}}{\beta} + 1 - \frac{\min \{p_2^-, q_2^-\} (1 - \epsilon)}{\beta} \right]^{-1}.
\]
By virtue of Lemma 5, we derive
\[
m_{1/2} \leq C_{10}(n, \mu, p_1, p_2, q_1, q_2, \mathcal{U}).
\]
From the substitution $u = C_* w$ we obtain
\[
\text{ess sup}\{u(x) \mid x \in V_{\epsilon, r/2} \cap \Omega'\} = C_* m_{1/2} \leq C_{10} C_* = C_{10} r^{-\tau}.
\]
Therefore, we conclude the proof of the Proposition 7. \qed
The next theorem follows easily from the Proposition 7.

**Theorem 8** Suppose that the conditions (2)–(9) and (11) are satisfied. Let \( u \in W^{1,p_1(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^\infty_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \) be a solution of equation (1) in \( \overline{\Omega}\setminus\Gamma \). Then, in \( \{x \in \Omega \mid 0 < d(x,\Gamma) < r_0\} \), the following inequality holds almost everywhere:

\[
|u(x)| \leq Cd(x,\Gamma)^{-\tau},
\]

where \( C = C(n,\mu,p_1,p_2,q_1,q_2,\mathcal{U}) > 0 \) and \( \tau = \tau(n,\mu,p_1,p_2,q_1,q_2,\mathcal{U}) > \frac{\max\{p_1^+,q_2^+\}}{\min\{p_2^+,q_1^+\} - \max\{p_1^+,q_2^+\} + 1} \).

Proceeding in the same way as in the Proposition 7 and Theorem 8, we have the following results.

**Proposition 9** Suppose that the conditions (2)–(5) and (11) are satisfied, additionally

\[
p_2^- - p_1^+ + 1 > 0.
\]

Assume that \( u \in W^{1,p_1(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^\infty_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \) satisfies

\[
\int_{\Omega} \langle A(\cdot,u,\nabla u),\nabla \varphi \rangle + a(\cdot,u)\varphi + g(\cdot,u)\varphi \,dx \leq 0,
\]

for all \( \varphi \in W^{1,p_1(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^\infty_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \), \( \varphi \geq 0 \), with \( \text{supp} \varphi \subset \overline{\Omega}\setminus\Gamma \). Then, if \( 0 < r < \ell' < r_0 \) we have the estimate

\[
\|\max\{u,0\}\|_{L^\infty(V_{\tau',r/\ell})} \leq C r^{-\tau},
\]

where \( C = C(n,\mu,p_1,p_2,\mathcal{U}) > 0 \) and \( \tau = \tau(n,\mu,p_1,p_2,\mathcal{U}) > \frac{p_1^+}{p_2^- - p_1^+ + 1} \).

**Theorem 10** Suppose that the conditions (2)–(5), (8), (11) and (28) are satisfied. Let \( u \in W^{1,p_1(\cdot)}_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \cap L^\infty_{\text{loc}}(\overline{\Omega}\setminus\Gamma) \) be a solution of equation (1) in \( \overline{\Omega}\setminus\Gamma \), with \( b \equiv h \equiv 0 \). Then, in \( \{x \in \Omega \mid 0 < d(x,\Gamma) < r_0\} \), the following inequality holds almost everywhere:

\[
|u(x)| \leq Cd(x,\Gamma)^{-\tau},
\]

where \( C = C(n,\mu,p_1,p_2,\mathcal{U}) > 0 \) and \( \tau = \tau(n,\mu,p_1,p_2,\mathcal{U}) > \frac{p_1^+}{p_2^- - p_1^+ + 1} \).

### 4 The removability of singular set

Next, we prove the main theorem of this paper. Before, we start with the following
Lemma 11 Suppose that the conditions (2)–(7), (9)–(11) are satisfied. If \( u \in W^{1,p_1} _{\text{loc}} (\bar{\Omega} \setminus \Gamma) \cap L^\infty _{\text{loc}} (\bar{\Omega} \setminus \Gamma) \) satisfies (14), then
\[
\max \{ u, 0 \} \in L^\infty (\Omega).
\]

Proof We proceed by contradiction. For \( r \in (0, r_0^2) \), we denote
\[
\Lambda(r) = \text{ess sup}\{ \max \{ u(x), 0 \} | r \leq d(x, \Gamma) \leq r_0^2, x \in \Omega \}.
\]
We have \( \lim_{r \to 0} \Lambda(r) = \infty \). For sufficiently small values \( r \) we define the function \( \psi_r : \mathbb{R} \to \mathbb{R} \) as follows:
\[
\psi_r(t) = \begin{cases} 
0 & \text{if } t < r, \\
1 & \text{if } t > \sqrt{r}, \\
\frac{2}{\ln \frac{1}{r}} \ln \frac{t}{r} & \text{if } r \leq t \leq \sqrt{r}.
\end{cases}
\]
Choosing \( \delta > 0 \) such that \( \Lambda(\delta) > 1 \), set
\[
\varphi = \left( \ln \max \left\{ \frac{u}{\Lambda(\delta)}, 1 \right\} \right) \psi_r' \circ d(\cdot, \Gamma),
\]
where \( \gamma = \text{ess sup}_\Omega \psi_{r_0}^{\frac{p_1 p_2}{p_2 - p_1 + 1}} \). We have \( \varphi \in W^{1,p_1}_{\text{loc}} (\bar{\Omega} \setminus \Gamma) \cap L^\infty _{\text{loc}} (\bar{\Omega} \setminus \Gamma) \), with \( \text{supp} \varphi \subset \bar{\Omega} \cap \{ d(\cdot, \Gamma) \geq r \} \subset \bar{\Omega} \setminus \Gamma \). For simplicity we write \( \psi_r \circ d(\cdot, \Gamma) = \psi_r \) and \( \psi_r' \circ d(\cdot, \Gamma) = \psi_r' \).
Denote \( \Omega_{\delta} = \{ x \in \Omega | u(x) > \Lambda(\delta) \} \). Substituting \( \varphi \) into (14), we obtain
\[
\int_{\Omega_{\delta}} \left[ \frac{\psi_r'}{u} \langle A(\cdot, u, \nabla u), \nabla u \rangle + \gamma \psi_r' \psi_r'^{-1} \left( \ln \frac{u}{\Lambda(\delta)} \right) \langle A(\cdot, u, \nabla u), \nabla d(\cdot, \Gamma) \rangle \\
\right. 
+ a(\cdot, u) \psi_r' \left( \ln \frac{u}{\Lambda(\delta)} \right) + g(\cdot, u) \psi_r' \left( \ln \frac{u}{\Lambda(\delta)} \right) \right] \, dx
\]
\[
+ \int_{\partial \Omega_{\delta} \cap \partial \Omega} \left[ b(\cdot, u) \psi_r' \left( \ln \frac{u}{\Lambda(\delta)} \right) + h(\cdot, u) \psi_r' \left( \ln \frac{u}{\Lambda(\delta)} \right) \right] \, d\sigma \leq 0.
\]
Since \( u > \Lambda(\delta) > 1 \) in \( \Omega_{\delta} \), and by virtue of the conditions (2)–(7), we have
\[
\int_{\Omega_\delta} \mu \frac{\psi'}{u} |\nabla u|^{p_1} \, dx + \int_{\Omega_\delta} \left( \mu - \frac{3\mu^{-1}}{\Lambda(\delta)^{\beta_{q_1}^{-1}} + 1} \right) \psi' \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, dx \\
+ \int_{\partial \Omega_\delta \cap \Omega} \left( \mu - \frac{3\mu^{-1}}{\Lambda(\delta)^{\beta_{q_1}^{-1}} + 1} \right) \psi' \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, d\sigma \\
\leq \int_{\Omega_\delta} \mu^{-1} \gamma \psi_r \psi_r^{p_1} \left( \ln \frac{u}{\Lambda(\delta)} \right) (|\nabla u|^{p_1} + u^{p_1} + 1) \, dx \\
\leq \int_{\Omega_\delta} \mu^{-1} \gamma \psi_r \psi_r^{p_1} \left\{ C_1 (\epsilon_1, p_1) \left[ \psi' \psi_r^{p_1} \left( \ln \frac{u}{\Lambda(\delta)} \right) \right]^{p_1} + \epsilon_1 |\nabla u|^{p_1} \right\} \, dx \\
+ \int_{\Omega_\delta} \mu^{-1} \gamma \left( \ln \frac{u}{\Lambda(\delta)} \right) \left[ C_2 (\epsilon_2, p_1, p_2) + 1 \right] (\psi')^{p_2} \psi_r^{p_1} \psi_r^{p_1} + \epsilon_2 \psi_r^{p_1} \psi_r^{p_1} \, dx.
\]

We can assume that

\[
\epsilon_3 = \mu - \frac{\mu^{-1}(3 + \gamma)}{\Lambda(\delta)^{\min\{p_2, q_1^*\} - \max\{p_2, q_1^*\} + 1}} > 0,
\]

because \( \lim_{r \to 0^+} \Lambda(r) = \infty \). Take \( \epsilon_1 = \frac{\mu}{2r} \), \( \epsilon_2 = \frac{\mu \epsilon_3}{2r} \). Since \( \frac{(\gamma - 1)p_2}{p_1 - 1} \geq \gamma \) and \( \psi_r \leq 1 \),

\[
\int_{\Omega_\delta} \frac{\psi'}{u} |\nabla u|^{p_1} \, dx + \int_{\Omega_\delta} \psi' \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, dx + \int_{\partial \Omega_\delta \cap \Omega} \psi' \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, d\sigma \\
\leq C_3 \int_{\Omega_\delta} \left( \psi_r^{p_1} \psi_r^{p_1 - p_1} \left( \ln \frac{u}{\Lambda(\delta)} \right)^{p_1} + \left( \ln \frac{u}{\Lambda(\delta)} \right)^{p_2} \psi_r^{p_2} \right) \, dx,
\]

where \( C_3 = C_3 (\mu, \gamma, \epsilon_3, p_1, p_2) > 0 \).

Additionally, let us consider

\[
(\psi')^{p_1} \psi_r^{p_1 - p_1} \left( \ln \frac{u}{\Lambda(\delta)} \right)^{p_1} \\
\leq \left( \ln \frac{u}{\Lambda(\delta)} \right) \left\{ C_4 (\epsilon_4, p_1, p_2) \left[ (\psi')^{p_1} \left( \ln \frac{u}{\Lambda(\delta)} \right)^{p_1 - p_1} \right]^{p_1} + \epsilon_4 \psi_r^{p_1} \psi_r^{p_1} \right\}^{p_2} \psi_r^{p_2 - p_1 + 1} \psi_r^{p_2 - p_1 + 1} \, dx,
\]

\[
\frac{(\gamma - 1)p_2}{p_1 - 1} \geq \gamma \quad \text{and} \quad \psi'(t) = \frac{2}{t \ln \frac{1}{r}} > 1 \text{ for } t \in \left( r, \sqrt{r} \right).
\]

Choose \( \epsilon_4 = \frac{1}{2C_3} \). By (30),

\[
\int_{\Omega_\delta} \frac{\psi'}{u} |\nabla u|^{p_1} \, dx + \int_{\Omega_\delta} \psi' \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, dx + \int_{\partial \Omega_\delta \cap \Omega} \psi' \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, d\sigma \\
\leq C_5 (\mu, \gamma, \epsilon_3, p_1, p_2) \int_{\Omega_\delta \cap \{ r \leq d(\cdot, \Gamma) \leq \sqrt{r} \}} \left[ \ln \frac{u}{\Lambda(\delta)} \right] \left[ \ln \frac{u}{\Lambda(\delta)} \right]^{p_1 - p_1} + 1 \left( \frac{2}{d(\cdot, \Gamma) \ln \frac{1}{r}} \right)^{p_1} \, dx.
\]

We can assume that \( 1 < \ln \frac{1}{d(x, \Gamma)} \) if \( 0 < d(x, \Gamma) \leq \sqrt{r} \). Using (27), we see
\[
\int_{\Omega_\delta} \frac{\psi_r^T}{u} |\nabla u|^p_1 \, dx + \int_{\Omega_\delta} \psi_r^T \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, dx \\
\leq C_6 \left( \ln \frac{1}{r} \right)^{-\frac{p_1^+ p_2^- (n-1)}{p_2^- (n-1)+1}} \int_{\Omega_\delta \cap \{ r \leq d(\cdot, \Gamma) \leq \sqrt{r} \}} \left( \ln \frac{1}{d(\cdot, \Gamma)} \right)^{(1+\frac{p_1^+ p_2^-}{p_2^- (n-1)+1})} \left( \frac{1}{d(\cdot, \Gamma)} \right)^{\frac{p_1^+ p_2^-}{p_2^- (n-1)+1}} \, dx \\
\leq C_7 \left( \ln \frac{1}{r} \right)^{-\frac{p_1^+ p_2^- (n-1)}{p_2^- (n-1)+1}} \int_{r}^{\sqrt{r}} \left( \ln \frac{1}{r} \right)^{1+\frac{p_1^+ p_2^-}{p_2^- (n-1)+1}} \left( \frac{1}{t} \right)^{p_2^- (n-1)+1} \, dt \\
= C_7 \left( \ln \frac{1}{r} \right)^{-\frac{p_1^+ p_2^- (n-1)}{p_2^- (n-1)+1}} \left( \ln \frac{1}{r} \right)^{1+\frac{p_1^+ p_2^-}{p_2^- (n-1)+1}} \left( \frac{1}{r} \right)^{\frac{n-d-\gamma}{2}} \left( \frac{r}{n-d-\gamma} \right)^{1-\frac{n-d-\gamma}{2}},
\]
where \( C_i = C_i(n, \tau, \mu, \gamma, \varepsilon_3, p_1, p_2, q_1, q_2, \mathcal{U}) > 0, \ i = 6, 7. \) Since \( \lim_{r \to 0^+} 1/r = \infty, \)
we can assume that \( (\ln 1/r)^{1+\frac{p_1^+ p_2^-}{p_2^- (n-1)+1}} \leq r^{-\frac{n-d-\gamma}{4}}. \) Therefore, if \( r \to 0^+, \)
\[
\int_{\Omega_\delta} \frac{|\nabla u|^p_1}{u} \, dx + \int_{\Omega_\delta} \left( \ln \frac{u}{\Lambda(\delta)} \right) u^{p_2} \, dx = 0.
\]
Hence, \( u(x) = \Lambda(\delta) \) almost every in \( \Omega_\delta. \) Thus, we have a contradiction, and proves
the Lemma 11. \( \square \)

Now we are ready to prove the main Theorem 1.

**Proof of Theorem 1** 1. First we prove \( u \in W^{1,p_1^+} \cap L^\infty. \) As consequence of a
Lemma 11, \( u \in L^\infty. \) Next, for \( r < 2r_0/5, \) let \( \psi_r : \mathbb{R} \to \mathbb{R} \) be a smooth function
such that
\[
\psi_r(t) = \begin{cases} 
0 & \text{if } t < r/2 \lor t > 5r/2, \\
1 & \text{if } r < t < 2r,
\end{cases}
\]
\( 0 \leq \psi_r \leq 1 \) and \( |\psi'_r| \leq C/r, \) where \( C \) is a suitable positive constant. Set
\[
\phi = \left( \psi_r^{p_1^+} \circ d(\cdot, \Gamma) \right) u.
\]
We have \( \phi \in W^{1,p_1^+} \cap L^\infty \), with \( \text{supp} \phi \subset \Omega \cap \{ r/2 \leq d(\cdot, \Gamma) \leq 5r/2 \}. \)
For simplicity we write \( \psi_r = \psi_r \circ d(\cdot, \Gamma) \) and \( \psi'_r = \psi'_r \circ d(\cdot, \Gamma). \) Substituting \( \phi \) into
(12),
\[
\int_{\Omega} \left( A(\cdot, u, \nabla u), p_1^+ \psi'_r u \nabla \phi + \psi_r u \nabla \phi \right) + a(\cdot, u) \psi_r^{p_1^+} u + g(\cdot, u) \psi_r^{p_1^+} u \, dx + \int_{\partial \Omega} b(\cdot, u) \psi_r^{p_1^+} u + h(\cdot, u) \psi_r^{p_1^+} u \, d\sigma \leq 0
\]
By conditions (2)–(7), we have
\[
\int_\Omega |\nabla u|^{p_i} \psi_i^{p_i} \, dx + \int_\Omega \mu_1 |\psi_i^{p_i} - u|^{p_i+1} \, dx \\
+ \int_{\partial \Omega} \mu_1 \psi_i^{p_i} |u|^{q_i+1} \, d\sigma \leq \int_\Omega \mu_1^{-1} p_i |\psi_i^{p_i-1} - u|^{p_i-1} |u|^{p_i} \, dx \\
+ \int_{\partial \Omega} \mu_1^{-1} (|u|^{q_i-1} + 1) \psi_i^{p_i} |u| + \mu_1 |\psi_i^{p_i} - u| \, dx + \int_{\partial \Omega} \mu_1^{-1} (|u|^{q_i-1} + 1) \psi_i^{p_i} |u| + \mu_1 |\psi_i^{p_i} - u| |u| \, d\sigma.
\]

Since \( |u|^{p_i} \leq \max \left\{ \|u\|^{p_i}_{L^\infty(\Omega)}, \|u\|^{p_i}_{L^p(\Omega)} \right\} \) and \( |u|^{q_i} \leq \max \left\{ \|u\|^{q_i}_{L^\infty(\partial \Omega)}, \|u\|^{q_i}_{L^\infty(\Omega)} \right\} \),

\[
\int_\Omega |\nabla u|^{p_i} \psi_i^{p_i} \, dx \\
\leq \mu_1^{-1} p_i \int_\Omega \left[ \psi_i^{p_i-1} \frac{|u|^{p_i-1} - |\psi_i^{p_i-1}|}{\psi_i^{p_i-1}} \right]^{p_i} + \epsilon_1 \left[ |\nabla u|^{p_i-1} \psi_i^{p_i-1} \right] \, dx + C_2 r^{n-d-1},
\]

where \( C_2 \) is a positive constant independent of \( r \). Choosing \( \epsilon_1 = \frac{\mu_1^2}{2p_i} \),

\[
\int_{\Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} |\nabla u|^{p_i} \, dx \leq C_3 r^{n-d-p_i},
\]

where \( C_3 \) is a positive constant independent of \( r \). Therefore,

\[
\int_{\Omega \cap \{d(\cdot, \Gamma) \leq 2r\}} |\nabla u|^{p_i} \, dx \\
= \sum_{i=0}^{\infty} \int_{\Omega \cap \{2^{-i}r \leq d(\cdot, \Gamma) \leq 2^{-i+1}r\}} |\nabla u|^{p_i} \, dx \leq C_3 \sum_{i=0}^{\infty} \left( \frac{r}{2^{i}} \right)^{n-d-p_i} 2^{i} < \infty.
\]

So \( |\nabla u| \in L^{p_i}(\Omega) \), and thus we have proved that \( u \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega) \).

2. Now, we will show that \( u \) is a solution of equation (1) in the domain \( \Omega \). For \( r \in (0, r_0) \), let \( \xi_r : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function such that

\[
\xi_r(t) = \begin{cases} 
1 & \text{if } |t| \leq r, \\
0 & \text{if } 2r \leq |t|,
\end{cases}
\]

\( 0 \leq \xi \leq 1 \) and \( |\xi_r'| \leq C/r \), where \( C \) is a suitable positive constant. For simplicity we write \( \xi = \xi_r \odot d(\cdot, \Gamma) \) and \( \xi' = \xi_r' \odot d(\cdot, \Gamma) \). Let \( \varphi \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega) \). We have \((1 - \xi \odot d(\cdot, \Gamma))\varphi \in W^{1,p_i}_{\text{loc}}(\bar\Omega \setminus \Gamma) \cap L^\infty_{\text{loc}}(\bar\Omega \setminus \Gamma)\), with \( \text{supp} \varphi \subset \bar\Omega \setminus \Gamma \). Then, (12) yields

\[
\int_\Omega \left\langle A(\cdot, u, \nabla u), (1 - \xi) \nabla \varphi - \varphi \xi' \Delta d(\cdot, \Gamma) \right\rangle \\
+ a(\cdot, u)(1 - \xi) \varphi + g(\cdot, u)(1 - \xi) \varphi \, dx \\
+ \int_{\partial \Omega} b(\cdot, u)(1 - \xi) \varphi + h(\cdot, u)(1 - \xi) \varphi \, d\sigma = 0. \tag{31}
\]

When \( r \to 0^+ \), for all \( \varphi \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega) \), the equality (31) implies (12). Indeed, we have
\[
\lim_{r \to 0} \int_{\Omega} \left\langle A(\cdot, u, \nabla u), (1 - \xi_r) \nabla \varphi \right\rangle + a(\cdot, u)(1 - \xi_r) \varphi \\
+ g(\cdot, u)(1 - \xi_r) \varphi \, dx + \int_{\partial \Omega} b(\cdot, u)(1 - \xi_r) \varphi + h(\cdot, u)(1 - \xi_r) \varphi \, d\sigma \\
= \int_{\Omega} \left\langle A(\cdot, u, \nabla u), \nabla \varphi \right\rangle + a(\cdot, u) \varphi + g(\cdot, u) \varphi \, dx + \int_{\partial \Omega} b(\cdot, u) \varphi + h(\cdot, u) \varphi \, d\sigma.
\]

Additionally, by (3), Proposition 2 and 3:
\[
\left| \int_{\Omega} \left\langle A(\cdot, u, \nabla u), \varphi \xi_r \nabla \varphi(\cdot, \Gamma) \right\rangle \, dx \right| \\
\leq C_4 \frac{r}{r} \int_{\Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} (|\nabla u|^{p_1-1} + |u|^{p_1-1} + 1) \varphi \, dx \\
\leq C_5 \frac{r}{r} \int_{\Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} |\nabla u|^{p_1-1} + |u|^{p_1-1} + 1 \, dx \\
\leq C_6 \frac{r}{r} \left\| |\nabla u|^{p_1-1} \right\|_{L^{p_1} \Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} \left\| 1 \right\|_{L^{p_1} \Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} + C_6 r^{p_1-1} \\
\leq C_7 \frac{r}{r} \max \left\{ \left( \int_{\Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} |\nabla u|^{p_1} \, dx \right)^{\frac{p_1-1}{p_1}}, \left( \int_{\Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}} |\nabla u|^{p_1} \, dx \right)^{\frac{p_1-1}{p_1}} \right\} \\
\cdot |\Omega \cap \{r \leq d(\cdot, \Gamma) \leq 2r\}|^{\frac{1}{\Gamma_1}} + C_6 r^{p_1-1} \\
\leq C_8 r^{\frac{n-d+p_1}{\Gamma_1}} \to 0 \quad \text{as} \quad r \to 0,
\]

where \(C_i, i = 4, \ldots, 8,\) are positive constants independents of \(r.\) So, we have obtained that equality (12) is fulfilled for all \(\varphi \in W^{1,p_1(\cdot)}(\Omega) \cap L^\infty(\Omega).\) Therefore, the singular set \(\Gamma\) is removable for solutions of equation (1).

Similarly as in the Lemma 11 and in the Theorem 1 we have the following results.

**Lemma 12** Suppose that the conditions (2)–(5), (9)–(11) and (28) are satisfied. If \(u \in W^{1,p_1(\cdot)}_{loc}(\Omega \setminus \Gamma) \cap L^\infty_{loc}(\Omega \setminus \Gamma)\) satisfies (29), then
\[
\max \{u, 0\} \in L^\infty(\Omega).
\]

**Theorem 13** Suppose that the conditions (2)–(5), (8)–(11) and (28) are satisfied. Let \(u \in W^{1,p_1(\cdot)}_{loc}(\Omega \setminus \Gamma) \cap L^\infty_{loc}(\Omega \setminus \Gamma)\) be a solution of equation (1) in \(\Omega \setminus \Gamma,\) with \(b \equiv h \equiv 0.\) Then, the singularity of \(u\) at \(\Gamma\) is removable.

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