Classification of terminal simplicial reflexive $d$-polytopes with $3d - 1$ vertices.

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July 6, 2017

Abstract

We classify terminal simplicial reflexive $d$-polytopes with $3d - 1$ vertices. They turn out to be smooth Fano $d$-polytopes. When $d$ is even there is 1 such polytope up to isomorphism, while there are 2 when $d$ is uneven.

1 Introduction

Let $N \cong \mathbb{Z}^d$ be a $d$-dimensional lattice, and let $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R} \cong \mathbb{R}^d$. Let $M$ be the dual lattice of $N$ and $M_\mathbb{R}$ the dual of $N_\mathbb{R}$. A reflexive $d$-polytope $P$ in $N_\mathbb{R}$ is a fully-dimensional convex lattice polytope, such that the origin is contained in the interior and such that the dual polytope $P^* := \{ x \in M_\mathbb{R} | \langle x, y \rangle \leq 1 \ \forall y \in P \}$ is also a lattice polytope. The notion of a reflexive polytope was introduced in [3]. Two reflexive polytopes $P$ and $Q$ are called isomorphic if there exists a bijective linear map $\varphi : N_\mathbb{R} \to N_\mathbb{R}$, such that $\varphi(N) = N$ and $\varphi(P) = Q$. For every $d \geq 1$ there are finitely many isomorphism classes of reflexive $d$-polytopes, and for $d \leq 4$ they have been completely classified using computer algorithms ([10],[11]).

Simplicial reflexive $d$-polytopes have at most $3d$ vertices ([6] theorem 1). This upper bound is attained if and only if $d$ is even and $P$ splits into $d/2$ copies of del Pezzo 2-polytopes $V_2 = \text{conv}\{\pm e_1, \pm e_2, \pm (e_1 - e_2)\}$, where $\{e_1, e_2\}$ is a basis of a 2-dimensional lattice.

A reflexive polytope $P$ is called terminal, if $N \cap P = 0 \cup V(P)$. An important subclass of terminal simplicial reflexive polytopes is the class of smooth reflexive polytopes, also known as smooth Fano polytopes: A reflexive polytope $P$ is called smooth if the vertices of every face $F$ of $P$ is a part of a basis of the lattice $N$. Smooth Fano $d$-polytopes have been intensively studied and completely classified up to dimension 4 ([1],[4],[14],[16]). In higher dimensions not much is known. There are classification results valid in any dimension, when the polytopes have few vertices ([2],[9]) or if one assumes some extra symmetries ([5],[8],[15]). Some of these results have been generalized to simplicial reflexive polytopes ([13]).
In this paper we classify terminal simplicial reflexive $d$-polytopes with $3d-1$ vertices for arbitrary $d$. It turns out that these are in fact smooth Fano $d$-polytopes.

**Theorem 1.** Let $P \subset \mathbb{N}^{\mathbb{R}}$ be a terminal simplicial reflexive $d$-polytope with $3d-1$ vertices. Let $e_1, \ldots, e_d$ be a basis of the lattice $\mathbb{N}$.

If $d$ is even, then $P$ is isomorphic to the convex hull of the points

$$e_1, \pm e_2, \ldots, \pm e_d, \pm(e_1-e_2), \ldots, \pm(e_{d-1}-e_d).$$

(1)

If $d$ is uneven, then $P$ is isomorphic to either the convex hull of the points

$$\pm e_1, \ldots, \pm e_{d-1}, e_d, \pm(e_1-e_2), \ldots, \pm(e_{d-2}-e_{d-1}), e_1-e_d.$$  

(2)

or the convex hull of the points

$$\pm e_1, \ldots, \pm e_d, \pm(e_2-e_3), \ldots, \pm(e_{d-1}-e_d).$$

(3)

In particular, $P$ is a smooth Fano $d$-polytope.

A key concept in this paper is the notion of a special facet: A facet $F$ of a simplicial reflexive $d$-polytope $P$ is called special, if the sum of the vertices $\mathcal{V}(P)$ of $P$ is a non-negative linear combination of vertices of $F$. In particular, $\langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle \geq 0$, where $u_F \in \mathbb{M}_{\mathbb{R}}$ is the unique element determined by $\langle u_F, F \rangle = \{1\}$. The polytope $P$ is reflexive, so $\langle u_F, v \rangle$ is an integer for every $v \in \mathcal{V}(P)$. As $\langle u_F, v \rangle \leq 1$ with equality if and only if $v \in F$, there are at most $d$ vertices $v$ such that $\langle u_F, v \rangle \leq -1$. For simplicity, let $H(F, i) := \{ x \in \mathbb{N} | \langle u_F, x \rangle = i \}, i \in \mathbb{Z}$. It is well-known that at most $d$ vertices of $P$ are situated in $H(F, 0)$ for any facet $F$ of $P$ ([7] section 2.3 remarks 5(2)). If $P$ has $3d-1$ vertices and $F$ is a special facet of $P$, then

$$d - 1 \leq |\mathcal{V}(P) \cap H(F, 0)| \leq d,$$

and there are only three possibilities for the placement of the $3d-1$ vertices of $P$ in the hyperplanes $H(F, i)$ as shown in the table below.

| $|\mathcal{V}(P) \cap H(F, 1)|$ | Case 1 | Case 2 | Case 3 |
|--------------------------|--------|--------|--------|
| $|\mathcal{V}(P) \cap H(F, 0)|$ | $d$    | $d$    | $d$    |
| $|\mathcal{V}(P) \cap H(F, -1)|$ | $d - 1$ | $d - 2$ | $d$    |
| $|\mathcal{V}(P) \cap H(F, -2)|$ | $0$    | $1$    | $0$    |
| $|\mathcal{V}(P)|$ | $3d - 1$ | $3d - 1$ | $3d - 1$ |
We prove theorem 1 by considering these three cases separately for terminal simplicial reflexive $d$-polytopes.

The paper is organised as follows: In section 2 we define some notation and prove some well-known basic facts about simplicial reflexive polytopes. In section 3 we define the notion of special facets. In section 4 we prove some lemmas needed in section 5 for the proof of theorem 1.

Acknowledgments. The author would like to thank his advisor Johan P. Hansen for advice and encouragement.

2 Notation and basic results

In this section we fix the notation and prove some basic facts about simplicial reflexive $d$-polytopes.

From now on $N$ denotes a $d$-dimensional lattice, $N \cong \mathbb{Z}^d$, $d \geq 1$, and $M$ denotes the dual lattice. Let $N_R := N \otimes \mathbb{Z} R$ and let $M_R$ denote the dual of $N_R$.

By $\text{conv} K$ we denote the convex hull of a set $K$. A polytope is the convex hull of finitely many points, and a $k$-polytope is a polytope of dimension $k$. Recall that faces of a polytope of dimension 0 are called vertices, while codimension 1 and 2 faces are called facets and ridges, respectively. The set of vertices of any polytope $P$ is denoted by $V(P)$.

2.1 Simplicial polytopes containing the origin in the interior

A $d$-polytope $P$ in $N_R$ is called simplicial if every face of $P$ is a simplex. In this section $P$ will be a simplicial $d$-polytope in $N_R$ with $0 \in \text{int} P$.

For any facet $F$ of $P$, we define $u_F$ to be the unique element in $M_R$ where $\langle u_F, x \rangle = 1$ for every point $x \in F$. Certainly for any vertex $v$ and any facet $F$ of $P$, $\langle u_F, v \rangle \leq 1$ with equality if and only if $v$ is a vertex of $F$.

We also define some points $u^v_F \in M_R$ for any facet $F$ of $P$ and any vertex $v \in \mathcal{V}(F)$: $u^v_F$ is the unique element where $\langle u^v_F, v \rangle = 1$ and $\langle u^v_F, w \rangle = 0$ for every $w \in \mathcal{V}(F) \setminus \{v\}$. In other words, $\{u^v_F | v \in \mathcal{V}(F)\}$ is the basis of $M_R$ dual to the basis $\mathcal{V}(F)$ of $N_R$.

When $F$ is a facet of $P$ and $v \in \mathcal{V}(F)$, there is a unique ridge $R = \text{conv}(\mathcal{V}(F) \setminus \{v\})$ of $P$ and a unique facet $F'$ of $P$, such that $F \cap F' = R$.

We denote this facet by $N(F, v)$ and call it a neighboring facet of $F$. The set $\mathcal{V}(N(F, v))$ consists of the vertices $\mathcal{V}(R)$ of the ridge $R$ and a unique vertex $v'$, which we call a neighboring vertex of $F$ and denote it by $n(F, v)$. See figure 1.

The next lemma shows how $u_F$ and $u_{F'}$ are related, when $F'$ is a neighboring facet of the facet $F$. 

Lemma 2. Let $P \subset N_R$ be a simplicial $d$-polytope containing the origin in the interior. Let $F$ be a facet of $P$ and $v \in V(F)$. Let $F'$ be the neighboring facet $N(F, v)$ and $v'$ the neighboring vertex $n(F, v)$.

Then for any point $x \in N_R$,

$$\langle u_{F'}, x \rangle = \langle u_F, x \rangle + ((u_{F'}, v) - 1)\langle u_v, x \rangle.$$ 

In particular,

- $\langle u_{F'}, x \rangle < 0$ iff $\langle u_{F'}, x \rangle > \langle u_F, x \rangle$.
- $\langle u_{F'}, x \rangle > 0$ iff $\langle u_{F'}, x \rangle < \langle u_F, x \rangle$.
- $\langle u_{F'}, x \rangle = 0$ iff $\langle u_{F'}, x \rangle = \langle u_F, x \rangle$.

Proof. The vertices of $F$ span $N_R$, and

$$x = \sum_{w \in V(F)} \langle u_w, x \rangle w \quad \text{and} \quad \langle u_F, x \rangle = \sum_{w \in V(F)} \langle u_w, x \rangle.$$ 

The vertices of the neighboring facet $F' = N(F, v)$ are $\{v'\} \cup V(F) \setminus \{v\}$. So

$$\langle u_{F'}, x \rangle = \langle u_{F'}, x \rangle \langle u_{F'}, v \rangle + \langle u_F, x \rangle - \langle u_v, x \rangle$$

$$= \langle u_F, x \rangle + ((\langle u_{F'}, v \rangle - 1) \langle u_v, x \rangle.$$ 

The vertex $v$ is not on the facet $F'$, and then the term $(\langle u_{F'}, v \rangle - 1)$ is negative. From this the equivalences follow. \qed
2.2 Simplicial reflexive polytopes

A polytope \( P \subset \mathbb{N} \subset \mathbb{R} \) is called a lattice polytope if \( V(P) \subset N \). A lattice polytope is called reflexive, if \( 0 \in \text{int}(P) \) and \( V(P^*) \subset M \), where 
\[
P^* := \{ x \in M \mid \langle x, y \rangle \leq 1 \ \forall y \in P \}
\]
is the dual of \( P \).

From now on \( P \) denotes a simplicial reflexive \( d \)-polytope.

Reflexivity guarantees that \( u_F \in M \) for every facet \( F \) of \( P \), and every vertex of \( P \) lies in one of the lattice hyperplanes 
\[
H(F, i) := \{ x \in N \mid \langle u_F, x \rangle = i \} \quad i \in \{1, 0, -1, -2, \ldots \}
\]
In particular, for every facet \( F \) and every vertex \( v \) of \( P \): \( v \not\in F \) iff \( \langle u_F, v \rangle \leq 0 \).

This can put some restrictions on the points of \( P \).

**Lemma 3.** Let \( P \) be a simplicial reflexive polytope. For every facet \( F \) of \( P \) and every vertex \( v \) of \( \mathcal{V}(F) \) we have
\[
\langle u_F, x \rangle - 1 \leq \langle u_F, v \rangle
\]
for any \( x \in P \). In case of equality, \( x \) is on the facet \( N(F, v) \).

**Proof.** The inequality is obvious, when \( \langle u_F, x \rangle > 0 \). So assume \( \langle u_F, x \rangle \leq 0 \).

Let \( F' \) be the neighboring facet \( N(F, v) \).

Since \( x \in P \), \( \langle u_F, x \rangle \leq 1 \) with equality iff \( x \in F' \). From lemma 2 we then have
\[
\langle u_F, x \rangle - 1 \leq (1 - \langle u_F, v \rangle)\langle u_F, x \rangle \leq \langle u_F, x \rangle
\]
as \( \langle u_F, v \rangle \leq 0 \). \( \square \)

The next lemma concerns an important property of simplicial reflexive polytopes.

**Lemma 4 (\cite{7} section 2.3 remarks 5(2), \cite{12} lemma 5.5).** Let \( F \) be a facet and \( x \in H(F, 0) \) be vertex of a simplicial reflexive polytope \( P \). Then \( x \) is a neighboring vertex of \( F \).

More precisely, for every \( w \in \mathcal{V}(F) \) where \( \langle w_F, x \rangle < 0 \), \( x \) is equal to \( n(F, w) \).

In particular, for every \( w \in \mathcal{V}(F) \) there is at most one vertex \( x \in H(F, 0) \cap \mathcal{V}(P) \), with \( \langle w_F, x \rangle < 0 \).

As a consequence, there are at most \( d \) vertices of \( P \) in \( H(F, 0) \).

**Proof.** Since \( \langle u_F, x \rangle = \sum_{w \in \mathcal{V}(F)} \langle w_F, x \rangle = 0 \) and \( x \neq 0 \), there is at least one \( w \in \mathcal{V}(F) \) for which \( \langle w_F, x \rangle < 0 \). Choose such a \( w \) and consider the neighboring facet \( F' = N(F, w) \). By lemma 2 we get that \( 0 < \langle u_{F'}, x \rangle \leq 1 \).

As \( P \) is reflexive, \( \langle u_{F'}, x \rangle = 1 \) and then \( x = n(F, w) \).

The remaining statements follow immediately. \( \square \)
3 Special facets

Now we define the notion of special facets, which will be of great use to us in the proof of theorem 1.

**P is a simplicial reflexive d-polytope in this section.**

Consider the sum of all the vertices of \( P \),

\[
\nu_P := \sum_{v \in \mathcal{V}(P)} v.
\]

There exists at least one facet \( F \) of \( P \) such that \( \nu_P \) is a non-negative linear combination of vertices of \( F \), i.e. \( \langle u_F, \nu_P \rangle \geq 0 \) for every \( w \in \mathcal{V}(F) \). We call facets with this property **special**.

Let \( F \) be a special facet of \( P \). In particular we have that

\[
0 \leq \langle u_F, \nu_P \rangle,
\]

which implies that

\[
0 \leq \sum_{v \in \mathcal{V}(P)} \langle u_F, v \rangle = \sum_{i \leq 1} i|H(F, i) \cap \mathcal{V}(P)| = d + \sum_{i \leq -1} i|H(F, i) \cap \mathcal{V}(P)|. \tag{4}
\]

As there are at most \( d \) vertices in \( H(F,0) \) we can easily see that \(|\mathcal{V}(P)| \leq 3d\), which was first proved by Casagrande using a similar argument ([6] theorem 1). Notice that \( \langle u_F, v \rangle \geq -d \) for every vertex \( v \) of \( P \). Notice also, that when \(|\mathcal{V}(P)|\) is close to \( 3d \), the vertices of \( P \) tend to be in the hyperplanes \( H(F, i) \) for \( i \in \{1,0,-1\} \).

4 Many vertices in \( H(F,0) \)

We now study some cases of many vertices in \( H(F,0) \), where \( F \) is a facet of a simplicial reflexive d-polytope. The lemmas proven here are ingredients in the proof of theorem 1.

**Lemma 5.** Let \( F \) be a facet of a simplicial reflexive d-polytope \( P \). Suppose there are at least \( d - 1 \) vertices \( v_1, \ldots, v_{d-1} \) in \( \mathcal{V}(F) \), such that \( n(F, v_i) \in H(F,0) \) and \( \langle u_F^v, n(F, v_i) \rangle = -1 \) for every \( 1 \leq i \leq d - 1 \).

Then \( \mathcal{V}(F) \) is a basis of the lattice \( N \).

**Proof.** Follows from statement 3 in [12] lemma 5.5. \( \square \)

**Lemma 6.** Let \( P \) be a simplicial reflexive d-polytope, such that

\[
|\mathcal{V}(P) \cap H(F,0)| \geq d - 1
\]

for every facet \( F \) of \( P \). Then there exists a facet \( G \) of \( P \), such that \( \mathcal{V}(G) \) is a basis of \( N \).
Proof. By lemma 5 we are done, if there exists a facet $G$ such that the set

$$\{ v \in V(G) \mid n(G, v) \in H(G, 0) \text{ and } \langle u_G^n, n(G, v) \rangle = -1 \}$$

is of size at least $d - 1$. So we suppose that no such facet exists.

For every facet $F$ we denote the volume of the $d$-simplex $\text{conv}(\{0\} \cup V(F))$ by $\text{vol}(F)$. When $v_1, \ldots, v_d$ are the vertices of $F$, the volume $\text{vol}(F)$ is equal to $\frac{1}{d!} |\det A_F|$, where $A_F$ is the matrix

$$A_F := \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}.$$

The volume $\text{vol}(N(F, v_i))$ of the neighboring facet $N(F, v_i)$ is then

$$\text{vol}(N(F, v_i)) = \frac{1}{d!} |\det A_{F'}| = \frac{|\langle u_{F'}^{v_i}, n(F, v_i) \rangle|}{d!} |\det A_F|.$$

Now, let $F_0$ be an arbitrary facet of $P$. There must be at least one vertex $v$ of $F_0$, such that $v' = n(F_0, v) \in H(F_0, 0)$, but $\langle u_{F_0}^{v'}, v' \rangle \neq -1$. Then $0 > \langle u_{F'}^{v'}, v' \rangle > -1$ by lemma 3. Let $F_1$ denote the neighboring facet $N(F_0, v)$. Then $\text{vol}(F_0) > \text{vol}(F_1)$.

We can proceed in this way to produce an infinite sequence of facets

$$F_0, F_1, F_2, \ldots \quad \text{where} \quad \text{vol}(F_0) > \text{vol}(F_1) > \text{vol}(F_2) > \ldots.$$

But there are only finitely many facets of $P$. A contradiction.

Lemma 7. Let $F$ be a facet of a simplicial reflexive polytope $P \in \mathbb{N}_R$. Let $v_1, v_2 \in V(F)$, $v_1 \neq v_2$, and set $y_1 = n(F, v_1)$ and $y_2 = n(F, v_2)$. Suppose $y_1 \neq y_2$, $y_1, y_2 \in H(F, 0)$ and $\langle u_F^{y_1}, y_1 \rangle = \langle u_F^{y_2}, y_2 \rangle = -1$. Then there are no vertex $x \in \mathcal{V}(P)$ in $H(F, -1)$ with $\langle u_F^x, x \rangle = \langle u_F^{y_1}, x \rangle = -1$.

Proof. Suppose the statement is not true.

For simplicity, let $G = \text{conv}(V(F) \setminus \{v_1, v_2\})$. The vertex $x$ written as a linear combination of $V(F)$ is then

$$x = -v_1 - v_2 + \sum_{w \in V(G)} \langle u_F^w, x \rangle w.$$

The vertices of the facet $F_1 = N(F, v_1)$ are $\{y_1\} \cup (V(F) \setminus \{v_1\})$, where

$$y_1 = -v_1 + \langle u_F^{v_2}, y_1 \rangle v_2 + \sum_{w \in V(G)} \langle u_F^w, y_1 \rangle w.$$
In the basis (of $N_R$) $F_1$ provides we have
\[ x = y_1 + (-1 - \langle u_F^{v_2}, y_1 \rangle)v_2 + \sum_{w \in \mathcal{V}(G)} \langle u_F^{v_2}, x - y_1 \rangle w \]
The vertex $x$ is in $H(F,0)$ by lemma 2. Certainly, $\langle u_F^{v_2}, y_1 \rangle \leq 0$, otherwise we would have a contradiction to lemma 3. On the other hand, $\langle u_F^{v_2}, y_1 \rangle \geq 0$, as $n(F, v_2) \neq y_1$. So $\langle u_F^{v_2}, y_1 \rangle = 0$ and $x = n(F, v_2)$.
Similarly, $\langle u_F^{v_2}, y_2 \rangle = 0$.
\[ y_2 = -v_2 + \sum_{w \in \mathcal{V}(G)} \langle u_F^{v_2}, y_2 \rangle w. \]
But then $y_2$ and $x$ are both in $H(F,0)$ and both have negative $v_2$-coordinate. This is a contradiction to lemma 4. \hfill \Box

4.1 The terminal case

If we assume that the simplicial reflexive $d$-polytope $P$ is terminal, we can sharpen our results in case of $d$ vertices in $H(F,0)$ for some facet $F$ of $P$. Recall, that a reflexive polytope is called terminal if $\mathcal{V}(P) \cup \{0\} = P \cap N$.

**Lemma 8.** Let $P$ be a terminal simplicial reflexive $d$-polytope. If there are $d$ vertices of $P$ in $H(F,0)$ for some facet $F$ of $P$, then
\[ \mathcal{V}(P) \cap H(F,0) = \{ -y + z_y \mid y \in \mathcal{V}(F) \} \]
where $z_y \in \mathcal{V}(F)$ for every $y$.
In particular $\mathcal{V}(F)$ is a basis of the lattice $N$.

**Proof.** Let $y \in \mathcal{V}(F)$. By lemma 4 there exists exactly one vertex $x \in H(F,0)$, such that $x = n(F,y)$. Conversely, there are no vertex $y' \neq y$ of $F$, such that $x = n(F,y')$. So $x$ is on the form
\[ x = -by + a_1w_1 + \ldots + a_kw_k \quad , \quad 0 < b \leq 1 , \quad 0 < a_i \text{ and } w_i \in \mathcal{V}(F) \setminus \{y\} \forall i , \]
where $b = \sum_{i=1}^{k} a_i$.
Suppose there exists a facet $G$ containing both $x$ and $y$. Then
\[ 1 + b = \langle u_G, x + by \rangle = \langle u_G, a_1w_1 + \ldots + a_kw_k \rangle \leq \sum_{i=1}^{k} a_i = b. \]
Which is a contradiction. So there are no such facets. Consider the lattice point $z_y = x + y$. For any facet $G$ of $P$, $\langle u_G, z_y \rangle \leq 1$ as both $\langle u_G, x \rangle, \langle u_G, y \rangle \leq 1$ and both cannot be equal to 1. So $z_y$ is a lattice point in $P$. Since $P$ is terminal, $z_y$ is either a vertex of $P$ or the origin. As $1 = \langle u_F, x + y \rangle = \langle u_F, z_y \rangle$, $z_y$ must be a vertex of $F$ and $y \neq z_y$. And then we’re done.
The vertex set $\mathcal{V}(F)$ is a basis of $N$ by lemma 5. \hfill \Box
Figure 2: Terminality is important in lemma 8. This is a simplicial reflexive (self-dual) 2-polytope with 5 vertices. Consider the facet \( F \) containing 3 lattice points. The two vertices in \( H(F,0) \) is not on the form \(-y + z \) for vertices \( y, z \in V(F) \).

The proof of lemma 8 is inspired by proposition 3.1 in [12].

**Lemma 9.** Let \( F \) be a facet of a terminal simplicial reflexive \( d \)-polytope \( P \subset \mathbb{N}_\mathbb{R} \), such that \( |H(F,0) \cap V(P)| = d \). If \( x \in H(F,1) \cap P \), then \(-x \in V(F)\).

**Proof.** The vertex set \( V(F) \) is a basis of the lattice \( N \), and every vertex in \( H(F,0) \) is of the form \(-y + z \) for some \( y, z \in V(F) \) (lemma 8). Let \( x \) be vertex of \( P \) in \( H(F,1) \).

\[
x = \sum_{w \in V(F)} \langle u^w_F, x \rangle w,
\]

where \( \langle u^w_F, x \rangle \in \mathbb{Z} \) for every \( w \in V(F) \). If \( \langle u^w_F, x \rangle \leq -2 \) for some \( w \in V(F) \), then \( x = n(F,w) \) (lemma 3), which is not the case. So \( \langle u^w_F, x \rangle \geq -1 \) for every \( w \in V(F) \). Furthermore, by lemma 7 \( x \) is only allowed one negative coordinate with respect to the basis \( V(F) \). The only possibility is then \( x = -w \), where \( w \in V(F) \). \( \square \)

## 5 Proof of main result

In this section we will prove theorem 1.

*Throughout the section \( P \) is a terminal simplicial reflexive \( d \)-polytope in \( \mathbb{N}_\mathbb{R} \) with \( 3d - 1 \) vertices, whose sum is \( \nu_P \), and \( \{e_1, \ldots, e_d\} \) is a basis of the lattice \( N \).*

\[
\nu_P := \sum_{v \in V(P)} v.
\]

By the existing classification we can check that theorem 1 holds for \( d \leq 2 \) ([12] proposition 4.1). So we may assume that \( d \geq 3 \).

Let \( F \) be a special facet of \( P \), i.e. \( \langle u^w_F, \nu_P \rangle \geq 0 \) for every \( w \in V(F) \). Of course, there are \( d \) vertices of \( P \) in \( H(F,1) \). The remaining \( 2d - 1 \) vertices
are in the hyperplanes \( H(F, i) \) for \( i \in \{0, -1, -2, \ldots, -d\} \), such that

\[
0 \leq \langle u_F, \nu_P \rangle = d + \sum_{i \leq -1} i \cdot |V(P) \cap H(F, i)|.
\]

So there are three cases to consider.

| Case 1 | Case 2 | Case 3 |
|--------|--------|--------|
| \( |V(P) \cap H(F, 1)| \) | \( d \) | \( d \) | \( d \) |
| \( |V(P) \cap H(F, 0)| \) | \( d \) | \( d \) | \( d - 1 \) |
| \( |V(P) \cap H(F, -1)| \) | \( d - 1 \) | \( d - 2 \) | \( d \) |
| \( |V(P) \cap H(F, -2)| \) | 0 | 1 | 0 |
| \( |V(P)| \) | \( 3d - 1 \) | \( 3d - 1 \) | \( 3d - 1 \) |

We will consider these cases separately.

**Case 1.** There are \( d \) vertices in \( H(F, 0) \), so by lemma 8 \( V(F) \) is a basis of \( N \). We may then assume that \( V(F) = \{ e_1, \ldots, e_d \} \).

The sum of the vertices is a lattice point on \( F \), since \( \langle u_F, \nu_P \rangle = 1 \). As \( P \) is terminal, this must be a vertex \( e_i \) of \( F \), say \( \nu_P = e_1 \). Then a facet \( F' \) of \( P \) is a special facet iff \( e_1 \in V(P) \).

There are \( d - 1 \) vertices in \( H(F, -1) \), so by lemma 9 we get

\[
V(P) \cap H(F, -1) = \{-e_1, \ldots, -e_{j-1}, -e_{j+1}, \ldots, -e_d\},
\]

for some \( 1 \leq j \leq d \). Now, there are two possibilities: \( j = 1 \) or \( j \neq 1 \), that is \(-e_1 \notin V(P) \) or \( e_1 \in V(P) \).

\(-e_1 \notin V(P) \). Then \(-e_i \in V(P) \) for every \( 2 \leq i \leq d \). There are \( d \) vertices in \( H(F, 0) \), so by lemma 8 there is a vertex \(-e_1 + e_{a_1} \), which we can assume to be \(-e_1 + e_2 \).

Consider the facet \( F' = N(F, e_2) \). This is a special facet, so we can show that

\[
V(P) \cap H(F', -1) = V(-F') \setminus \{-e_1\}.
\]

The vertex \(-e_1 + e_2 \) is in the hyperplane \( H(F', -1) \). So \( e_1 - e_2 \) is a vertex of \( F' \) (lemma 9), and then of \( P \).

For every \( 3 \leq i \leq d \) we use the same procedure to show that \(-e_i + e_{a_i} \) and \(-e_{a_i} + e_i \) are vertices of \( P \). This shows that \( d \) is even and that \( P \) is isomorphic to the convex hull of the points in \( \{e_i\} \).

\(-e_1 \in V(P) \). We may assume \(-e_d \notin V(P) \). The sum of the vertices \( V(P) \) is \( e_1 \), so there are exactly two vertices in \( H(F, 0) \) of the form \(-e_k + e_1 \) and \(-e_l + e_1 \), \( k \neq l \). We wish to show that \( k = d \) or...
l = d. This is obvious for d = 3. So suppose d ≥ 4 and k, l ≠ d, that is \(-e_k, -e_l \in \mathcal{V}(P)\).
Consider the facet \(F' = N(F, e_k)\), which is a special facet. So by the arguments above we get that
\[
\mathcal{V}(P) \cap H(F', -1) = \mathcal{V}(-F') \setminus \{-e_d\}.
\]
As \(\mathcal{V}(F') = \{e_1, \ldots, e_{k-1}, e_{k+1}, e_d, -e_k + e_1\}\), we have that \(-e_1 + e_k\) must be a vertex of \(P\). In a similar way we get that \(-e_1 + e_l\) is a vertex of \(P\). But this is a contradiction. So \(k\) or \(l\) is equal to \(d\), and without loss of generality, we can assume that \(k = 2\) and \(l = d\).
For \(3 ≤ i ≤ d - 1\) we proceed in a similar way to get that both \(-e_i + e_{a_i} = -e_{a_i} + e_i\) are vertices of \(P\).
And so we have showed that \(d\) must be uneven and that \(P\) is isomorphic to the convex hull of the points in \(\{e_1, \ldots, e_d\}\).

**Case 2.** Since \((u_F, v_P) = 0\), the sum of the vertices is the origin, so every facet of \(P\) is special. There are \(d\) vertices in \(H(F, 0)\), so \(\mathcal{V}(F)\) is a basis of \(N\) (lemma \(\xi\)). Without loss of generality, we can assume \(\mathcal{V}(F) = \{e_1, \ldots, e_d\}\). By lemma \(\nu\)
\[
x ∈ \mathcal{V}(P) ∩ H(F, -1) \implies x = -e_i\text{ for some } 1 ≤ i ≤ d.
\]
Consider the single vertex \(v\) in the hyperplane \(H(F, -2)\). If \(\langle u_F^j, v \rangle > 0\) for some \(j\) then \(\langle u_F^j, v \rangle < -2\) for the facet \(F' = N(F, e_j)\) (lemma \(\eta\)), which is not the case as \(F'\) is special. So \(\langle u_F^j, v \rangle ≤ 0\) for every \(1 ≤ j ≤ d\). As \(v\) is a primitive lattice point we can without loss of generality assume \(v = -e_1 - e_2\).
There are \(d\) vertices in \(H(F, 0)\), so there is a vertex of the form \(-e_1 + e_j\) for some \(j ≠ 1\). If \(j = 2\), then \(-e_1 \in \text{conv}\{-e_1 + e_2, -e_1 - e_2\}\) which is not the case as \(P\) is terminal. So we may assume \(j = 3\). In \(H(F, 0)\) we also find the vertex \(-e_2 + e_i\) for some \(i\). A similar argument yields \(i ≠ 1\).
Let \(G = N(F, e_1)\). Then \(\mathcal{V}(G)\) is a basis of the lattice \(N\). Write \(v\) in this basis.
\[
v = (-e_1 + e_3) - e_3 - e_2.
\]
As \(i ≠ 1\), \(-e_2 + e_i\) is in \(H(G, 0)\) and is equal to \(n(G, e_2)\) (lemma \(\zeta\)). If \(v ≠ n(G, e_3)\) we have a contradiction to lemma \(\zeta\). Therefore \(v = n(G, e_3)\), and \(\text{conv}\{v, -e_1 + e_3, e_2\}\) is a face of \(P\).
As \(e_3\) and \(-e_1 + e_3\) are vertices of \(P\), there are at least two vertices of \(P\) with positive \(e_3\)-coordinate (with respect to the basis \(F\) provides). There is exactly one vertex in \(H(F, 0)\) with negative \(e_3\)-coordinate,
namely $-e_3 + e_k$ for some $k$. Any other has to be in $H(F, -1)$. The vertices of $P$ add to 0, so the point $-e_3$ must be a vertex of $P$.

But $-e_3 = -(e_1 + e_3) + v + e_2$, which cannot be the case as $P$ is simplicial.

We conclude that case 2 is not possible.

Case 3. In this case we also have $\langle u_F, \nu_P \rangle = 0$, so every facet is special. Case 2 was not possible, so $-1 \leq u_G(v) \leq 1$ for any facet $G$ and any vertex $v$ of $P$. By lemma 6 we may assume that $V(F)$ is a basis of $N$, say $V(F) = \{e_1, \ldots, e_d\}$. As case 2 was not possible,

$$V(P) \cap H(F, -1) = \{-e_1, \ldots, -e_d\}.$$  

Let $x$ be any vertex in $H(F, 0)$. There exists at least one $1 \leq i \leq d$, such that $\langle u_F^i, x \rangle = -1$. Consider the facet $F' = N(F, e_i)$: $V(F') = (V(F) \setminus \{e_i\}) \cup \{x\}$ is a basis of $N$. Case 2 was not possible, so $V(P) \cap H(F', -1) = -V(F')$, which implies $-x \in V(P)$.

This shows that $P$ is centrally symmetric and $d$ must be uneven. By [12] theorem 5.9 $P$ is isomorphic to the convex hull of the points in (3).

This ends the proof of theorem 1.

References

[1] V. V. Batyrev, Toroidal Fano 3-folds, Math. USSR-Izv. 19 (1982), 13–25.

[2] V. V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. 43 (1991), 569–585.

[3] V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Algebraic Geom. 3 (1994), no. 3, 493–535.

[4] V. V. Batyrev, On the classification of toric Fano 4-folds, J. Math. Sci. (New York) 94 (1999), 1021–1050.

[5] C. Casagrande, Centrally symmetric generators in toric Fano varieties, Manuscr. Math. 111 (2003), 471–485.

[6] C. Casagrande, The number of vertices of a Fano polytope, Ann. Inst. Fourier 56 (2006), 121–130.

[7] O. Debarre, Toric Fano varieties in Higher dimensional varieties and rational points, lectures of the summer school and conference, Budapest 2001, Bolyai Society Mathematical Studies 12, Springer, 2001.
REFERENCES

[8] G. Ewald, *On the classification of toric Fano varieties*, Discrete Comput. Geom. 3 (1988), 49–54.

[9] P. Kleinschmidt, *A classification of toric varieties with few generators*, Aequationes Math 35 (1988), no.2-3, 254–266.

[10] M. Kreuzer & H. Skarke, *Classification of reflexive polyhedra in three dimensions*, Adv. Theor. Math. Phys. 2 (1998), 853–871.

[11] M. Kreuzer & H. Skarke, *Complete classification of reflexive polyhedra in four dimensions*, Adv. Theor. Math. Phys. 4 (2000), 1209–1230.

[12] B. Nill, *Gorenstein toric Fano varieties*, Manuscr. Math. 116 (2005), 183–210.

[13] B. Nill. *Classification of pseudo-symmetric simplicial reflexive polytopes*, Preprint, math.AG/0511294, 2005.

[14] H. Sato, *Toward the classification of higher-dimensional Toric Fano varieties*, Tohoku Math. J. 52 (2000), 383–413.

[15] V.E. Voskresenskij & A. Klyachko, *Toric Fano varieties and systems of roots*. Math. USSR-Izv. 24 (1985), 221–244.

[16] K. Watanabe & M. Watanabe, *The classification of Fano 3-folds with torus embeddings*, Tokyo Math. J. 5 (1982), 37–48.

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