Quantization In Finite Volumes
Using Symplectic Quantization Programm

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Abstract

We use the ideas of symplectic quantization for quantizing fields in finite volumes. We consider, as examples, the Klein-Gordon and electromagnetic fields in three different boxes. As a second idea we consider the given boundary conditions as primary constrains. Consistency of primary constrains leads to infinite chains of constraints at the boundaries. Without solving the equation of motion, we impose the set of constrains on suitable expansions of the fields. We show that if the new set of variables, such as Fourier modes, are chosen appropriately, imposing the constraints omits a few number of canonical pairs. Hence, the reduced phase space, with canonical pairs as coordinates, is achieved.

Keywords: Symplectic Quantization, Constraints, Boundary conditions

1 Introduction

Every student of quantum field theory may have encountered the question "what is the reason the people consider the coefficients of Fourier expansions of the fields as creation and annihilation operators and why this assumption works?". A possible answer is: "one can retain the original canonical brackets among the fields by using the assumed creation-annihilation algebra".

Such a statement does not show the essential meaning inherent in the quantum modes and the way the assumed algebra has been emerged. Moreover, it should be explained how the constraint equations as well as the gauge conditions are remained valid as relations among the operators. There exist also difficulties with boundary conditions. The problem is the canonical algebra among the fields are violated on the boundaries. The reason is the boundary conditions and their consistency in time constitute a set of second class constraints which change the Poisson brackets to Dirac brackets.

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3As an example, the contradiction between the canonical Poisson brackets and the mixed boundary conditions in the model of bosonic string in the background B-field leaded people to new ideas such as non-commutative coordinates of the string at the boundaries[2 3]
Hence we come to conclusion that the canonical quantization program should be managed in a more convincing way within the framework of Dirac constraint theory\cite{1,4,5}. This program is a systematic and universal quantization method based on the canonical structure of the classical system. This framework is inevitably necessary for singular Lagrangians such as gauge systems. The boundary conditions viewed as essential identities among the coordinates and (velocities or) momenta should also be considered as Dirac constraints where their consistency in time may give more constraints.

The whole procedure leads us to omitting redundant variables considered initially in the model due to the symmetry structure of the theory. Hence, one finds the ”pure” space of physical variables which is called the ”reduced phase space”. These variables are the only meaningful things to be considered as quantum operators. This procedure is often escaped in the textbooks on quantum field theory. Hence, some physical concepts may be sacrificed or remain unclear in such ”operational methods”, since they use ”ad-hoc quantization assumptions” emphasizing only on the results.

One approach to canonical quantization is based on Fadeev-Jackie\cite{8} method of analyzing constrained systems and is recognized in the related literature\cite{6} as ”symplectic quantization”. In ref.\cite{7} the basic concepts and proofs of the ”symplectic quantization program” are proposed and applied to the model of a massive bosonic string in a background B-field. The essential aspects of the symplectic quantization can be summarized in the following steps:

1) Investigating the constraint structure of the system, including the boundary conditions and their consistency in time;

2) fixing the gauges by imposing appropriate gauge fixing conditions;

3) choosing suitable coordinates for the phase space to impose the constraints, finding the physical modes as the coordinates of the reduced phase space and writing the expansions of the fields and momentum fields in terms of them;

4) inserting the above expansions in the symplectic two-form;

5) inverting the symplectic matrix of the reduced phase space to find the Dirac brackets of the physical modes and converting them to quantum commutators;

6) writing the Hamiltonian in terms of the physical modes, finding their equations of motion and then writing the physical modes in terms of their initial values (Schrodinger modes);

7) writing the expansions of the fields in terms of Schrodinger modes, and finding their time dependent commutation relations by using the commutator algebra of Schrodinger modes.

According to symplectic approach (steps 4 and 5) for a theory with dynamical fields $\phi_s(x,t)$ and momentum fields $\pi_s(x,t)$ in $d+1$ dimensions the symplectic two-form is defined as

$$\Omega = 2 \sum_s \int d^d x d\pi_s(x,t) \wedge d\phi_s(x,t).$$  \hspace{1cm} (1)

Under imposing the constraints (and gauge fixing conditions) the original fields can be written in terms of physical modes $a^n(t)$ which are the coordinates of the reduced phase space. These are the main physical quantities of the theory. Inserting the expansions of the original fields and momentum fields in the symplectic two-form, it can be written as

$$\Omega = \sum_{n,m} \omega_{nm} da^n \wedge da^m.$$  \hspace{1cm} (2)
It has been shown [7] that the antisymmetric tensor $\omega^{mn}$ which is the inverse of the symplectic tensor $\omega_{nm}$ defines the classical brackets (in fact the Dirac brackets) among the physical modes as
$$\{a^n, a^m\} = \omega^{nm}.$$  

(3)

In this paper we want to show that "the symplectic quantization program" gives the desired answer for some familiar field theories in finite volumes. Our emphasize is on the "method" and not on the "results" (which we expect anyhow to be consistent with the literature). Our main goal is to show that the standard quantization idea of Dirac (converting the Dirac brackets to commutators) is enough for each model and it is not needed to "assume" alternative "quantization assumptions" in different cases. We use this program for the following models: Klein-Gordon field in a rectangular box, a long cylinder, and a sphere; and the electromagnetic field inside a rectangular box, a cylindrical box and a sphere. In the following section we explain our method in more details by investigating the Klein-Gordon field in a rectangular box. Then other examples are considered more briefly in the subsequent sections. In the last section we will give our concluding remarks.

2 Klein-Gordon field in rectangular box

The classical action of the Klein-Gordon field is
$$S = \frac{1}{2} \int \left[ \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right] d^4x.$$  

(4)

Suppose our system is confined inside a rectangular box with boundaries at $x = 0, a; y = 0, b; z = 0, c$. Using the Lagrangian
$$L = \frac{1}{2} \int d^3x \left[ \dot{\phi}^2 - \nabla \phi^2 - m^2 \phi^2 \right],$$  

(5)

the momentum field is $\pi(x) = \dot{\phi}(x)$ and the canonical Hamiltonian reads
$$H_c = \frac{1}{2} \int d^3x \left[ \pi^2 + \nabla \phi^2 + m^2 \phi^2 \right].$$  

(6)

Suppose we are given Dirichlet boundary condition on the boundaries which can be considered as the following primary constraints
$$\phi(0, y, z) \approx 0, \quad \phi(x, 0, z) \approx 0, \quad \phi(x, y, 0) \approx 0,$$
$$\phi(a, y, z) \approx 0, \quad \phi(x, b, z) \approx 0, \quad \phi(x, y, c) \approx 0;$$  

(7)

where the symbol $\approx$ means weak equality. The total Hamiltonian is symbolically
$$H_T = H_c + \sum_{i=1}^{6} \int \lambda_i \psi_i^{(0)},$$  

(8)

where $\psi_i^{(0)}$ are the six constraints given in Eqs. [7], and the integration is over the remaining coordinates on each boundary. For example the $i = 1$ term is $\int dy dz \lambda_1(y, z) \phi(0, y, z)$, etc.
As in any constrained systems, we should investigate the consistency of primary constraints in the course of the time. Consider, for instance, the consistency of the constraint \( \psi_1^{(0)}(y, z) = \varphi(0, y, z) \) which using Eq. (5) and the fundamental Poisson brackets,

\[
\{ \varphi(x), \pi(x') \} = \delta^3(x - x'),
\{ \varphi(x), \varphi(x') \} = 0,
\{ \pi(x), \pi(x') \} = 0,
\]

gives

\[
\chi_1^{(0)}(y, z) = \{ \psi_1^{(0)}(y, z), H_T \} = \pi(0, y, z) \approx 0. \tag{10}
\]

In this way we find second level constraints \( \chi_1^{(0)}, \cdots, \chi_6^{(0)} \) for momentum field \( \pi(x, y, z) \) at the boundaries, similar to \( \varphi_1^{(0)}, \cdots, \varphi_6^{(0)} \) in Eqs. (7). At this step we have found 6 constraints \( \chi_1^{(0)}, \cdots, \chi_6^{(0)} \) that are conjugate to 6 primary constraints \( \psi_1^{(0)}, \cdots, \psi_6^{(0)} \). In the context of the theory of constrained system, the process of consistency of constraints should end here since the Lagrange multipliers would be determined by consistency of secondary constraints \( \chi_i^{(0)} \). However, the situation is different for the constraints emerged from boundary conditions. Here, the consistency of \( \chi_i^{(0)}(y, z) \), for instance, gives

\[
\dot{\chi}_1^{(0)}(y, z) = \{ \pi(0, y, z), H_C \} + \int dy' dz' \lambda_1(y', z')\{ \pi(0, y, z), \varphi(0, y', z') \} \approx 0 \tag{11}
\]

The first term at the right hand side of Eq. (11), after straightforward calculations using Eqs. (6) and (9), is equal to \( \nabla^2 \varphi(x, y, z)|_{x=0} \). The second term, however, needs a little care, due to Poisson brackets at the sharp boundary on \( x = 0 \). To this end, we use delta functions to convert the fields within the bracket to their usual form. In this way the corresponding term reads

\[
\int dx dx' dy' dz' \lambda_1(y', z')\delta(x)\delta(x')\{ \pi(x, y, z), \varphi(x', y', z') \} =

- \int dx dx' \lambda_1(y', z')\delta(x)\delta(x')\delta^3(x - x') \tag{12}
\]

If we consider Eq. (11) as an equation to determine the Lagrange multiplier \( \lambda_1(y, z) \), then Eq. (12) shows that the coefficient of the unknown is one order more singular than the first term in Eq. (11), (see more details in [2] by regularizing the delta functions in Eq. (12)).

The only way to satisfy the consistency condition (11) is \( \lambda_1(y, z) \) vanish for all \( (y, z) \) and at the same time the first term in Eq. (11) is considered as a new constraint. Hence, we find the third level constraints \( \psi_1^{(1)}, \cdots, \psi_6^{(1)} \) as \( \nabla^2 \psi_1, \cdots, \nabla^2 \psi_6 \) at the corresponding boundaries while the lagrange multiplier \( \lambda_i \) has been determined to be zero. In this way the consistency procedure is not terminated, although the lagrange multipliers are determined. Next we should impose the consistency of third level constraints \( \psi_1^{(1)}, \cdots, \psi_6^{(1)} \). Direct calculation using Eqs. (6) and (9) shows that the forth level constraints are \( \chi_1^{(1)}, \cdots, \chi_6^{(1)} \equiv \nabla^2 \pi|_{\text{boundaries}} \).

The consistency procedure continues unlimitedly leading to the following infinite chains of constraints at the boundaries.

\[
\psi_i^{(n)} = (\nabla^2)^n \varphi|_{\text{boundaries}}
\chi_i^{(n)} = (\nabla^2)^n \pi|_{\text{boundaries}} \tag{13}
\]

where \( n \) is any integer and the index \( i \) refers to six boundaries involved.
Next we want to see how the constraints \([13]\) restricts the physical degrees of freedom. In its original form, the phase space of the theory consists of the field variable \(\varphi(x,y,z)\) and \(\pi(x,y,z)\) at all points within the corresponding cubic box, up to the constraints given in Eqs. \([13]\). As is well-known, the Dirac procedure of quantization of such a theory requires transforming the Dirac brackets of the fields to quantum commutators. This procedure involves inverting the matrix of second class constraints. A little look at Eqs. \([13]\) shows that for our case we have a system of second class constraints with a complicated infinite matrix of Poisson brackets, whose elements are derivatives of different orders of delta functions at the boundaries. It is practically impossible to compute such a matrix and invert it. However, it is possible to quantize the theory in a simpler way by changing the dynamical variables.

As explained in details in [9], in our case for instance, one prefers to do a canonical transformation from the original variables \(\varphi(x)\) and \(\pi(x)\) to a new set of suitable canonical coordinates in which the constraints \([13]\) lead to omitting a number of canonical coordinate-momentum pairs. This should finally leave us with a reduced phase space in which the remaining canonical pairs act as the physical degrees of freedom. The Fourier transformation sometimes dose this job for models with Dirichlet or Neuman boundary conditions. In fact, this is the mysterious behind the fact that Fourier transformation is, most of the time, the first step toward the aim of quantization of the fields.

Let us consider the following Fourier transformations of the original fields.

\[
\varphi(x,t) = \frac{1}{(2\pi)^{3/2}} \int a(k,t) e^{ikx} d^3k, \tag{14}
\]

\[
\pi(x,t) = \frac{1}{(2\pi)^{3/2}} \int b(k,t) e^{-ikx} d^3k. \tag{15}
\]

Using the inverse Fourier transformations and the fundamental Poisson brackets \([9]\) we can find the Poisson brackets of the new variables \(a(k,t)\) and \(b(k,t)\) as

\[
\{a(k,t), b(k',t)\} = \delta^3(k - k'),
\]

\[
\{a(k,t), a(k',t)\} = 0,
\]

\[
\{b(k,t), b(k',t)\} = 0. \tag{16}
\]

This shows that the Fourier transformation given in Eqs. \([14]\) and \([15]\) is a canonical transformation. Now let us impose the constraints \([13]\) on the expansions \([14]\) and \([15]\) to omit nonphysical modes. Imposing, for instance, the constraints \(\psi_1^{(n)} = (\nabla^2)^n \varphi|_{x=0}\) on \(\varphi(x,t)\) gives

\[
\int a(k,t) (k^2)^n e^{i(k_yy + k_zz)} d^3k = 0. \tag{17}
\]

This condition can be satisfied for arbitrary integer \(n\) and all points \((y,z)\) if \(a(k_x, k_y, k_z, t)\) is an odd function of \(k_x\). Imposing this restriction on Eq.\([14]\) shows that the term \(\cos k_x x\) in \(e^{ik_x x}\) should be absent. The same thing happens for \(\pi(x,t)\) and also for the terms \(\cos k_y y\) and \(\cos k_z z\) in \(e^{ik_y y}\) and \(e^{ik_z z}\). In refs. \([2]\) and \([3]\) the authors have tried to do this for a bosonic string in a background B-field. But it seems practically impossible to find the answer without using the results known from other approaches.
and cos $k_\zeta z$. We are left finally just with the sin terms in the expansions (14) and (15) as

$$\varphi(x,t) = \frac{1}{(2\pi)^{3/2}} \int a(k,t) \sin k_x x \sin k_y y \sin k_z z d^3k$$

$$\pi(x,t) = \frac{1}{(2\pi)^{3/2}} \int b(k,t) \sin k_x x \sin k_y y \sin k_z z d^3k$$

(18) (19)

where $a(k,t)$ and $b(k,t)$ are odd functions of $k_x$, $k_y$ and $k_z$. Imposing the constraints $\psi_4^{(n)} \equiv \left(\nabla^2\right)^n \varphi(x)|_{x=a}$ on the expansion of $\varphi(x,t)$ in Eq. (14) gives

$$\int a(k,t) k^{2n} \sin k_x a \sin k_y y \sin k_z z d^3k = 0.$$ 

(20)

This equation should be satisfied for arbitrary integer $n$ and for all points $(y,z)$. Since the integrand is even with respect to $k_x$, the only possibility is that $a(k,t) = 0$ when $\sin k_x a \neq 0$. In other words, we are left with modes of the form $k_x = p\pi x / a$, where $p$ is some integer. The same argument for boundary conditions on the surfaces $y = b$ and $z = c$, and for the momentum field as well, gives the following expansion for the fields;

$$\varphi(x,t) = \sum_{p,r,s} A_{prs}(t) \sin \frac{p\pi x}{a} \sin \frac{r\pi y}{b} \sin \frac{s\pi z}{c},$$

(21)

$$\pi(x,t) = \sum_{p,r,s} B_{prs}(t) \sin \frac{p\pi x}{a} \sin \frac{r\pi y}{b} \sin \frac{s\pi z}{c}.$$ 

(22)

Since we were left previously with odd functions $a(k,t) = 0$ and $b(k,t) = 0$, the summations in Eqs. (21) and (22) are only on positive integers.

As we see the reduced phase space is much smaller than the original one. We began with the fields $\varphi(x,t)$ and $\pi(x,t)$ which introduce innumerably infinite number of degrees of freedom. Going to the Fourier modes $a(k,t)$ and $b(k,t)$, half of degrees of freedom are killed because of the constraints at $x = 0$, $y = 0$ and $z = 0$ surfaces, and within the remaining ones most of them are omitted due to the constraints at $x = a$, $y = b$ and $z = c$ surfaces. Finally we are left with infinitely numerable degrees of freedom labeled by the positive integers $(p,r,s)$.

We emphasize that in order to recognize the space of physical variables we need not to solve the full dynamics of the system. In other words, we have considered so far just the dynamics of the constraints. If we force the fields in Eqs. (21) and (22) to satisfy the equations of motion, then $A_{prs}(t)$ and $B_{prs}(t)$ should have definite time dependence. Upon quantization we can find the commutation relations among the physical degrees of freedom $A_{prs}(t)$ and $B_{prs}(t)$, while their evolution during time depends on the specific form from of the Hamiltonian.

Now let us find the brackets of the physical modes $A_{prs}(t)$ and $B_{prs}(t)$ using the symplectic approach. The symplectic two-form $\Omega = \int d^3x d\pi(x,t) \wedge d\varphi(x,t)$ in terms of the physical modes $A_{prs}(t)$ and $B_{prs}(t)$ is

$$\Omega = \sum_{prs} \frac{abc}{4} dB_{prs} \wedge dA_{prs}.$$ 

(23)

Comparing this with Eq. (2) shows that $\omega = (abc/8)J$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

(24)
$J$ is the standard symplectic matrix with entries written in $AA$, $AB$, $BA$ and $BB$ blocks respectively. Since $J^{-1} = -J$ the (Dirac) brackets of physical modes are as follows;

$$\{A_{prs}(t), B_{p'r's'}(t)\} = \frac{8}{abc} \delta_{pp'} \delta_{rr'} \delta_{ss'}.$$  \(25\)

Using the above brackets and the final expansions in Eqs. (21) the equal time brackets of the original fields are

$$\{\varphi(x,t), \pi(x',t)\}_{DB} = \frac{8}{abc} \sum_{prs} \sin \frac{p \pi x}{a} \sin \frac{p \pi x'}{a} \sin \frac{r \pi y}{b} \sin \frac{r \pi y'}{b} \sin \frac{s \pi z}{c} \sin \frac{s \pi z'}{c}$$

$$\{\varphi(x,t), \varphi(x',t)\}_{DB} = 0$$
$$\{\pi(x,t), \pi(x',t)\}_{DB} = 0$$  \(26\)

It is easily seen that the non vanishing Dirac bracket can be written as follows

$$\{\varphi(x,t), \pi(x',t)\}_{DB} = \begin{cases} \delta^3(x - x') & \text{inside the box} \\ 0 & \text{on the boundaries.} \end{cases}$$  \(27\)

It is important to note that the boundary conditions have changed the original Poisson brackets given in Eqs. (9) to Dirac brackets of Eqs. (27). Now everything is ready to take our final step and quantize the system. Upon quantization the physical modes $A_{prs}(t)$ and $B_{prs}(t)$ should be considered as quantum operators and the Eqs. (25)-(27) with right hand sides multiplied by $(i\hbar)$ as commutation relations.

We can compute the canonical Hamilton in the reduced phase space by inserting expansions (21) in Eq. (6). The answer is

$$H_c = \frac{abc}{16} \sum_{prs} \left[ B_{prs}^2 + \omega_{prs}^2 A_{prs}^2 \right],$$  \(28\)

where

$$\omega_{prs}^2 = m^2 + \left( \frac{p^2 \pi^2}{a^2} + \frac{r^2 \pi^2}{b^2} + \frac{s^2 \pi^2}{c^2} \right).$$  \(29\)

Eq.(25) shows that the Hamiltonian (28) is a superposition of infinite number of simple harmonic oscillators. Hence, the equations of motion for the physical modes read:

$$\dot{A}_{prs} = B_{prs}$$
$$\dot{B}_{prs} = -\omega_{prs}^2 A_{prs},$$

which acquire the following solutions;

$$A_{prs}(t) = A_{prs}(0) \cos(\omega_{prs} t) + \frac{B_{prs}(0)}{\omega_{prs}} \sin(\omega_{prs} t)$$  \(30\)
$$B_{prs}(t) = -A_{prs}(0) \omega_{prs} \sin(\omega_{prs} t) + B_{prs}(0) \cos(\omega_{prs} t)$$  \(31\)

Inserting Eqs.(30) and (31) into Eqs. (21) and (22) gives original fields in terms of the Schrodinger modes $A_{prs}(0)$ and $B_{prs}(0)$. Since the algebra of the modes given in Eq.(25) is independent of time \(7\) the Poisson brackets of the Schrodinger modes is the same as time dependent modes. Note that the Schrodinger modes are in fact the coefficients of the expansions of the fields in terms of the solutions of the equations of motion, i.e. the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0.$$  \(32\)
As explained in the introduction, the quantization may be achieved by "assuming" suitable algebra among the Schrodinger modes (as is almost done in the literature on quantum field theory). However, instead of an ad-hoc assumption which works well, we have derived the algebra of Schrodinger modes in a systematic way. Note also that the Klein-Gordon field $\varphi(x,t)$ is non commutative at different points of space, as expected.

3 Klein-Gordon field inside a cylinder

Consider the Klein-Gordon field inside an infinite cylinder of radius $a$, along the z-axis. The Lagrangian and canonical Hamiltonian is as before (Eqs.5 and 6), while the primary constraint is given by $\phi(a, \varphi, z) \approx 0$. Consistency of primary constraint, using the total Hamiltonian

$$H_T = H_c + \int d\varphi dz \lambda(\varphi, z) \phi(a, \varphi, z)$$

(33)

and the fundamental Poisson brackets in cylindrical coordinates gives the second level constraint $\pi(a, \varphi, z)$. Consistency of $\pi(a, \varphi, z)$ results to vanishing of $\lambda(\varphi, z)$ and at the same time emerging the third level constraint $\nabla^2 \phi |_{\rho=a}= 0$. This happens due to the same reason as stated after Eqs.(12). Fourth level constraint is also derived as $\nabla^2 \pi |_{\rho=a}= 0$. The process continues to give two sets of constraints $(\nabla^2)^n \phi |_{\rho=a}= 0$ and $(\nabla^2)^n \pi |_{\rho=a}= 0$.

Let us expand our canonical fields, in the most general form, as the following Bessel-Fourier integrals;

$$\phi(\rho, \varphi, z, t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dk A_m(\lambda, k, t) e^{i\lambda z} e^{im\varphi} J_m(k\rho),$$

(34)

$$\pi(\rho, \varphi, z, t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dk B_m(\lambda, k, t) e^{-i\lambda z} e^{-im\varphi} J_m(k\rho).$$

(35)

These expansions can be considered as a suitable canonical transformation from the original variables $\phi(\rho, \varphi, z)$ and $\pi(\rho, \varphi, z)$ to $A_m(\lambda, k)$ and $B_m(\lambda, k)$ which are more compatible with the symmetry of the system.

Now imposing the set of constraints $\nabla^2 j \phi|_{\rho=a}= 0$ on $\phi(\rho, \phi, z)$ gives

$$\sum_{m} \int d\lambda dk A_m(\lambda, k, t) e^{i\lambda z} e^{im\varphi} (-k^2 - \lambda^2)^j J_m(k\rho)|_{\rho=a}= 0,$$

(36)

by using the Bessel equation, $(\frac{d^2}{dx^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{m^2}{x^2}) J_m(x) = -k^2 J_m(x).$ . This equation for arbitrary $j$ shows that $A_m(\lambda, k, t)$ should vanishes except for values of $k$ that

$$J_m(ka) = 0 \Rightarrow k = k_{mn} \equiv \frac{x_{mn}}{a},$$

(37)

where $x_{mn}$ are the roots of the Bessel function $J_m(x)$. The same thing happens for $\pi(\rho, \varphi, z, t)$ and we can write the following expansion for the fields;

$$\phi(\rho, \varphi, z, t) = \sum_{mn} \int d\lambda A_{mn}(\lambda, t) e^{i\lambda z} e^{im\varphi} J_m(\frac{x_{mn} \rho}{a}),$$

(38)

$$\pi(\rho, \varphi, z, t) = \sum_{mn} \int d\lambda B_{mn}(\lambda, t) e^{-i\lambda z} e^{-im\varphi} J_m(\frac{x_{mn} \rho}{a}).$$

(39)
Inserting the above expansions in the symplectic two-form of Eq. (1) and using orthogonality of the functions involved, gives

$$\Omega = 2 \sum_{mn} 2\pi^2 a^2 K_{mn}^2 dB_{mn}(\lambda, t) \wedge dA_{mn}(\lambda, t),$$

(40)

where $K_{mn} = J_{m+1}(x_{mn})$. The non-vanishing brackets among physical modes can be found by inverting the symplectic matrix of Eq. (40) as

$$\{A_{mn}(\lambda, t), B_{m'n'}(\lambda', t')\} = \frac{1}{2\pi^2 a^2 K_{mn}^2} \delta_{mn} \delta_{m'n'} \delta(\lambda - \lambda').$$

(41)

Using the above brackets and the expansions (38) and (39) the equal time brackets of the original fields is as follows;

$$\{\phi(\rho, \varphi, z, t), \pi(\rho', \varphi', z', t)\} = \sum_{mn} \frac{1}{2\pi^2 a^2 K_{mn}^2} \int d\lambda e^{i\lambda(z-z')} e^{im(\varphi - \varphi')} J_m(x_{mn}\rho) J_m(x_{mn}\rho')$$

(42)

which is in fact equivalent to a delta function inside the tube and zero on the boundary, similar to Eq.(27).

4 Klein-Gordon field inside a sphere

Consider the Klein-Gordon field inside a sphere of radius $R$. The canonical Hamiltonian is again as in Eq.(6), while the primary constraint is given by $\phi(R, \theta, \varphi) \approx 0$. Using the total Hamiltonian

$$H_T = H_c + \int d\theta d\varphi \lambda(\theta, \varphi) \phi(R, \theta, \varphi),$$

(43)

the consistency procedure gives, similar to the cylindrical coordinates, two infinite sets of constraints as

$$(\nabla^2)^n \phi|_{r=R} \approx 0, \quad (\nabla^2)^n \pi|_{r=R} \approx 0.$$  

(44)

According to the spherical symmetry of the problem, we expand our fields inside the sphere, in their most general form, as the following Bessel-Fourier expansions

$$\phi(r, \theta, \varphi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} dk A_{lm}(k, t) j_l(kr) Y_{lm}(\theta, \varphi),$$

(45)

$$\pi(r, \theta, \varphi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} dk B_{lm}(k, t) j_l(kr) Y_{lm}(\theta, \varphi).$$

(46)

where $j_l(x)$’s are spherical Bessel functions and $Y_{lm}$’s are spherical harmonics. We can find the reduced phase space of the system by imposing the set of constraints (44) on the expansion of the fields. Using the spherical Bessel equation we find, for $\phi$ in Eq.(45),

$$\sum_{lm} \int dk A_{lm}(k, t)(k^2)^l [j_l(kr)]_{r=R} Y_{lm}(\theta, \varphi) = 0.$$  

(47)
This equation for arbitrary $j$ shows that $A_{lm}(k,t)$ should vanish except for
\[ j_i(kr)|_{r=R} = 0 \Rightarrow k_{ln} \equiv \frac{x_{ln}}{R} \] (48)
where $x_{ln}$ for positive integers $n$ are roots of the spherical Bessel function $j_i(x)$. The same thing happen for $\pi(r,\theta,\varphi,t)$, and we can write the following expansion for fields in the reduced phase space
\[ \phi(r,\theta,\varphi,t) = \sum_{lmn} A_{lmn}(t) j_i(x_{ln}r/R) Y_{lm}(\theta,\varphi), \] (49)
\[ \pi(r,\theta,\varphi,t) = \sum_{lmn} B_{lmn}(t) j_i(x_{ln}r/R) Y_{lm}(\theta,\varphi). \] (50)

Again we quantize the theory by using the symplectic approach. Using the orthogonality of spherical harmonics and spherical Bessel functions, the symplectic two-form (1) turns out to be
\[ \Omega = \sum_{lmn} R^3[j_{l+1}(x_{ln})]^2 dB_{lmn} \wedge dA_{lmn}. \] (51)
Inverting the symplectic matrix, the non-vanishing brackets among physical modes are
\[ \{A_{lmn}, B_{l'm'n'}\} = \frac{2}{R^3[j_{l+1}(x_{ln})]^2} \delta_{ll'} \delta_{mm'} \delta_{nn'}. \] (52)
Using the above bracket and Eqs.(49) and (50), one can show the Dirac brackets of the original coordinate and momentum field are as follows
\[ \{\phi(r,\theta,\varphi,t), \pi(r',\theta',\varphi',t)\} = \sum_{lmn} \frac{2}{R^3[j_{l+1}(x_{ln})]^2} j_i(x_{ln}r/R) j_i(x_{ln}r') \frac{R}{R} Y_{lm}(\theta,\varphi) Y_{lm}(\theta',\varphi'), \] (53)
which is equal to delta function inside the sphere and vanishes on the boundary similar to Eq. (27).

5 Electromagnetic field in a rectangular box

In the subsequent sections we find the reduced phase space of the Electromagnetic field in a finite volume. We begin with a rectangular box, however, before that we consider some general aspects of the electromagnetic field in Hamiltonian formalism.

The classical Lagrangian of the free electromagnetic field is given as
\[ L = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^4x, \] (54)
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The canonical momenta are $\pi_\mu(x) = \partial L/\partial \dot{A}_\mu(x) = -F^0_\mu$, giving $\phi_1 \equiv \pi_0$ as the primary constraint. The basic Poisson bracket among the canonical variables are
\[ \{A^\mu(x), \pi_\mu(x')\} = \delta^\mu_\nu \delta^3(x - x'), \]
\[ \{A^\mu(x), A^\nu(x')\} = 0, \]
\[ \{\pi_\mu(x), \pi_\nu(x')\} = 0. \] (55)
The canonical Hamiltonian reads
\[ H_c = \int \left( \frac{1}{2} \pi_i \pi_i + \pi_i \partial_i A_0 + \frac{1}{4} F_{ij} F_{ij} \right) d^3x. \] (56)

The total Hamiltonian is \( H_T = H_c + \int d^3x u(x,t) \phi_1(x,t), \) where \( u(x,t) \) is the Lagrange multiplier. Consistency condition of primary constraint gives the secondary constraint \( \phi_2 \equiv \{ \phi_1, H_T \} = -\partial_t \pi_i. \) The constraints \( \phi_1 \) and \( \phi_2 \) are first class, i.e. \( \{ \phi_1, \phi_2 \} = 0. \) Consistency of the constraint \( \phi_2 \) is satisfied identically, hence no more constraint would emerge.

We fix the gauge generated by \( \phi_2 \) by imposing the gauge fixing conditions \( \Omega_2 \equiv \partial_i A_i \approx 0. \) Consistency of the gauge fixing condition \( \Omega_2 \) then gives the next gauge fixing condition as \( \Omega_1 \equiv A_0 \approx 0 \) which in turn fixes the gauge generated by the constraint \( \phi_1. \) We denote the four constraints \( \phi_1, \phi_2, \Omega_1 \) and \( \Omega_2, \) which emerge due to the singular structure of the Lagrangian and are valid throughout the volume of the system, as the "bulk constraints". Besides the bulk constraints there are also the "boundary constraints" emerging from the geometry and physical properties of the boundaries.

Now consider the electromagnetic field inside a rectangular cubic box with boundaries at \( x = 0, a, \) \( y = 0, b, \) \( z = 0, c. \) Suppose the walls are ideal conductors with infinite magnetic permeability, so the constraints due to boundary conditions are as follows
\[ A_y(x,y,z)\big|_{x=0,x=a} = A_z(x,y,z)\big|_{x=0,x=a} = 0, \quad \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\big|_{x=0,x=a} = 0, \]
\[ A_x(x,y,z)\big|_{y=0,y=b} = A_z(x,y,z)\big|_{y=0,y=b} = 0, \quad \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\big|_{y=0,y=b} = 0, \]
\[ A_x(x,y,z)\big|_{z=0,z=c} = A_y(x,y,z)\big|_{z=0,z=c} = 0, \quad \frac{\partial A_y}{\partial y} - \frac{\partial A_x}{\partial x}\big|_{z=0,z=c} = 0. \] (57)

We denote the above constraints as \( \psi_i, i = 1, \ldots 18. \) We should then investigate the consistency of the boundary constraints, together with applying the bulk constraints. Mathematically we are free to do this in any order we desire. We use this possibility to follow the simplest way to investigate the consistency of the constraints and avoid a large amount of useless algebra. For this reason we omit first the fields \( \pi_0 \) and \( A_0 \) from the very beginning to simplify the problem. Hence, the new total Hamiltonian is
\[ H_T = \int d^3x \left( \frac{1}{2} \pi_i \pi_i + \frac{1}{4} F_{ij} F_{ij} \right) + \sum_{i=1}^{18} \int_{\text{boundaries}} \lambda_i \psi_i. \] (58)

In the last term of the Eq. (58) the \( i = 1 \) term, for instance, is understood as \( \int dydz \lambda_1(y,z) A_y(0,y,z) \) and so on. Using the Poisson brackets (55), consistency of the constraint \( \psi_1(y,z) = A_y(0,y,z) \) gives the new constraint \( \chi_1(y,z) \equiv \{ A_y(0,y,z), H_T \} = \pi_y(0,y,z) \approx 0. \) Hence we find eighteen second level constraints \( \chi_i \) similar to primary constraints of Eqs. (57) for momentum fields \( \pi(x,y,z). \) Consistency of \( \pi_0(y,0,z) \) then gives \( (\partial_y \partial_z) A_i - \nabla^2 A_x \) (while determining the corresponding Lagrange multiplier \( \lambda_1 \) to be zero, as discussed for the Klein-Gordon field after the relation (12)). Since \( \partial_y A_i \) is the bulk constraint \( \Omega_2, \) consistency of \( \chi_1 \) gives \( \nabla^2 \psi_1 \) at the same boundary.

The consistency procedure leads in this way to infinite chains of constraints at the boundaries which are obtained by acting \((\nabla^2)^n\) on the constraints \( \psi_i \) in (57) as well as similar constraints \( \chi_i \) for momentum fields.
In order to impose the constraints we use the usual Fourier transformations of the real fields as combinations of sine and cosine terms. Imposing the boundary constraints $(\nabla^2)^n \psi_i$ for arbitrary $n$ as well as the bulk constraint $\partial_i A_i$ leads to remaining special products of sine and cosine terms as the following combinations of discrete modes

$$A_x(x, t) = \sum_k a_x(k, t) \cos(k_x x) \sin(k_y y) \sin(k_z z) ,$$
$$A_y(x, t) = \sum_k a_y(k, t) \sin(k_x x) \cos(k_y y) \sin(k_z z) ,$$
$$A_z(x, t) = \sum_k a_z(k, t) \sin(k_x x) \sin(k_y y) \cos(k_z z) ,$$

(59)

where $k = (m\pi/a, n\pi/b, l\pi/c)$ for integers $m$, $n$, and $l$ and $a.k = 0$ due to the bulk constraints $\partial_i A_i = 0$. Defining the orthonormal set of basis vectors $\hat{\epsilon}_1(k), \hat{\epsilon}_2(k)$ and $\hat{k}$, the components of the vector potential $A$ can be written in terms of the physical modes $a_{mnl}^1(t)$ and $a_{mnl}^2(t)$ as follows

$$A_x(x, t) = \sum_{mnl}^{2} \sum_{i=1}^{2} a_{mnl}^i(t) \epsilon_{i x}(k) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{l\pi z}{c} ,$$
$$A_y(x, t) = \sum_{mnl}^{2} \sum_{i=1}^{2} a_{mnl}^i(t) \epsilon_{i y}(k) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \frac{l\pi z}{c} ,$$
$$A_z(x, t) = \sum_{mnl}^{2} \sum_{i=1}^{2} a_{mnl}^i(t) \epsilon_{i z}(k) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{l\pi z}{c} .$$

(60)

The same story should be repeated for the momentum fields to get

$$\pi_x(x, t) = \sum_{mnl}^{2} \sum_{i=1}^{2} b_{mnl}^i(t) \epsilon_{i x}(k) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{l\pi z}{c} ,$$
$$\pi_y(x, t) = \sum_{mnl}^{2} \sum_{i=1}^{2} b_{mnl}^i(t) \epsilon_{i y}(k) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \frac{l\pi z}{c} ,$$
$$\pi_z(x, t) = \sum_{mnl}^{2} \sum_{i=1}^{2} b_{mnl}^i(t) \epsilon_{i z}(k) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{l\pi z}{c} .$$

(61)

Now we can quantize the system be calculating the symplectic two-form of the system. Using the general formula (11) and orthogonality of physical modes given in Eqs.(59) we find

$$\Omega = \sum_{i=1}^{2} \sum_{mnl}^{2} \frac{abc}{8} db_{mnl}^i(t) \wedge da_{mnl}^i(t) .$$

(62)

Hence, non-vanishing bracket among the physical modes are as follows

$$[a_{mnl}^i(t), b_{m'n'l'}^j(t)] = \frac{16}{abc} \delta^{ij} \delta_{mnl} \delta_{m'n'l'} \delta_{l'l'} .$$

(63)
As is seen the reduced phase space is described by the canonical conjugate pairs \((a_{mnl}^i, b_{mnl}^i)\). Once again we can compute the Dirac brackets of the original fields as follows

\[
\begin{align*}
[A^x(x, t), \pi_x(x', t)] &= \sum_{mnl} \frac{32}{abc} \cos \frac{m\pi x}{a} \cos \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b} \sin \frac{l\pi z}{c} \sin \frac{l\pi z'}{c}, \\
[A^y(y, t), \pi_y(y', t)] &= \sum_{mnl} \frac{32}{abc} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \cos \frac{n\pi y}{b} \cos \frac{n\pi y'}{b} \sin \frac{l\pi z}{c} \sin \frac{l\pi z'}{c}, \\
[A^z(z, t), \pi_z(z', t)] &= \sum_{mnl} \frac{32}{abc} \sin \frac{m\pi x}{a} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b} \cos \frac{l\pi z}{c} \cos \frac{l\pi z'}{c}.
\end{align*}
\]

which differs from the primary Poisson brackets given in Eqs.\((55)\) in the sense that the constraints \((57)\) (and their partners for the momentum fields) are satisfied strongly on the corresponding walls. For example \(A^z(0, y, z)\) as well as \([\partial A_x/\partial y - \partial A_y/\partial z](0, y, z)\) have vanishing Dirac bracket with everything. Moreover \(\nabla \cdot A\) has vanishing Dirac bracket with all of the fields everywhere inside the box.

The canonical Hamiltonian of the system can be written in terms of the physical modes as

\[
H_c = \frac{abc}{16} \sum_{mnl} \sum_{i=1}^2 \left[ (b_{mnl}^i)^2 + \omega_{mnl}^2 (a_{mnl}^i)^2 \right],
\]

where

\[
\omega_{mnl}^2 = \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} + \frac{l^2 \pi^2}{c^2}.
\]

In this way the problem in the reduced phase space reduces to a summation over simple harmonic oscillators obeying the following dynamics

\[
\begin{align*}
a_{mnl}^i(t) &= a_{mnl}^i(0) \cos(\omega_{mnl} t) + b_{mnl}^i(0) \sin(\omega_{mnl} t), \\
b_{mnl}^i(t) &= -a_{mnl}^i(0) \omega_{mnl} \sin(\omega_{mnl} t) + b_{mnl}^i(0) \cos(\omega_{mnl} t).
\end{align*}
\]

Inserting the time dependent modes of \((67)\) and \((68)\) into the expansions \((60)\) gives the vector potential components in terms of Schrodinger modes \(a_{mnl}^i(0)\) and \(b_{mnl}^i(0)\) which obey the same algebra as given in Eq.\((69)\). Using \(B = \nabla \times A\) and \(E = -\partial A/\partial t\) in the Coulomb gauge considered above, we can write the physical fields \(E(x, t)\) and \(B(x, t)\) as well as every physical quantity of our interest in terms of Schrodinger modes. These are also the basic quantum objects upon quantization. Quantum commutators of the Schrodinger modes come from the fundamental Dirac prescription as

\[
[a_{mnl}^i(0), b_{mn'l'}^j(0)] = i\hbar \frac{16}{abc} \delta^i_j \delta_{mn} \delta_{m'n'} \delta_{l'l}.
\]

### 6 Electromagnetic field in a cylindrical box

Consider the Electromagnetic field inside a cylindrical box of radius \(R\), and length \(d\) made of an ideal conductor. The canonical Hamiltonian is given by Eq.\((56)\). The boundary conditions due to vanishing of the tangential components of electric field are

\[
A^\rho|_{z=0} = A^\rho|_{z=d} = 0, \quad A^\phi|_{z=0} = A^\phi|_{z=d} = 0, \quad A^z|_{r=R} = A^\phi|_{\rho=R} = 0,
\]
while vanishing of normal component of the magnetic field gives
\[ \frac{1}{\rho} \left( \frac{\partial A^z}{\partial \phi} - \frac{\partial A^\phi}{\partial z} \right) |_{\rho=R} = 0, \quad \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho A^\phi) - \frac{\partial A^\phi}{\partial \varphi} \right) |_{z=0, z=d} = 0. \] (71)

The fundamental Poisson brackets among field components can be written in cylindrical coordinates as
\[ \{ A^i(\rho, \varphi, z), \pi_j(\rho', \varphi', z') \} = \frac{1}{\rho} \delta^i_j \delta(\rho - \rho') \delta(\varphi - \varphi') \delta(z - z'). \] (72)

Imposing first the bulk constraints \( A^0 \approx 0 \) and \( \pi_0 \approx 0 \), the total Hamiltonian read
\[ H_T = H_c + \sum_{i=1}^{9} \lambda_i \psi_i^0. \] (73)

where the summation is over the nine constraints \( \psi_i^0 \) given in Eqs.(70) and (71). Consistency of the constraints \( \psi_i^0 \) gives a copy of them in terms of momentum fields which we call them \( \chi_i^0 \). Consistency of the new constraints should be achieved by writing the Hamiltonian in terms of cylindrical coordinate and using the Poisson bracket (72). The procedure is as before and straightforward; however, care is needed to considering the spacial derivatives of unit vectors. Fortunately, the additional terms are proportional to the previous constraints which vanish weekly. Similar to previous cases we find two sets of infinite constraints at the boundaries as
\[ \psi_i^n = \nabla^{2n} \psi_i^0 \approx 0, \quad \chi_i^n = \nabla^{2n} \chi_i^0 \approx 0 \] (74)

Noting the cylindrical geometry of the problem, we expand our fields in the most general form as
\[ A^\alpha(\rho, \varphi, z, t) = \sum_m \int d\lambda d\gamma A^\alpha_m(\lambda, \gamma, t) e^{i\lambda z} e^{im\varphi} J_m(\gamma \rho) \] (75)
\[ \pi^\alpha(\rho, \varphi, z, t) = \sum_m \int d\lambda d\gamma B^\alpha_m(\lambda, \gamma, t) e^{-i\lambda z} e^{-im\varphi} J_m(\gamma \rho) \] (76)

where \( \alpha \) runs over \( \rho, \phi, z \) and \( m \) is an integer, since the fields should be single valued at each point of space. Imposing the constraints \( \nabla^{2n} A^\alpha |_{\rho=R} \) for \( \alpha = \varphi \) and \( z \) gives
\[ \sum_m \int d\lambda d\gamma A^\alpha_m(\lambda, \gamma, t) e^{i\lambda z} e^{im\varphi} \left[ (-\gamma^2 - \lambda^2)^n J_m(\gamma \rho) \right] |_{\rho=R} = 0 \] (77)
for arbitrary \( n \). Hence, \( A^\varphi_m \) and \( A^z_m \) should vanish except for \( \gamma_m R = x_mn \) where \( x_mn \) is the \( n \)th root of the Bessel function \( J_m(x) \). Imposing the constraints \( \nabla^{2n} A^\rho |_{z=0, d} = 0 \) and \( \nabla^{2n} A^\varphi |_{z=0, d} = 0 \) gives the \( z \) dependence of the corresponding fields as \( \sin(lz \pi/d) \) for integer \( l \). We should also impose the gauge fixing condition
\[ \partial_i A_i \equiv \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A^\rho) + \frac{1}{\rho} \frac{\partial A^\varphi}{\partial \varphi} + \frac{\partial A^z}{\partial z} = 0. \] (78)

This can be satisfied at any arbitrary point in space only if the \( \gamma \) and \( \lambda \) acquire the same quantized values in all of the expansions. From Eq.(78) it is clear that the \( z \)-dependence
of $A^z$ should be of the form $\cos(l\pi z/d)$ and $\rho$-dependence of $A^\rho$ should be combinations of $J_m(\gamma_{mn}\rho)$ for suitable indices $m$. Functions $J'_m(x)$ and $J_m(x)/x$ for $x = \gamma_{mn}\rho$ can be written in terms of $J_{m\pm 1}(x)$ and may turn out to be useful.

Hence, imposing all of the requirements of the constraints (74) and (78) leads to determining $A^\rho$, $A^\varphi$ and $A^z$ in terms of TM modes,

$$Q^\rho_{mnl}(\rho, \varphi, z, t) = -\frac{l\pi}{d\gamma_{mn}} \sin \frac{l\pi z}{d} e^{im\varphi} J'_m(\gamma_{mn}\rho),$$

$$Q^\varphi_{mnl}(\rho, \varphi, z, t) = -\frac{l\pi}{d^2\gamma_{mn}} \frac{im}{\rho} \sin \frac{l\pi z}{d} e^{im\varphi} J_m(\gamma_{mn}\rho),$$

$$Q^z_{mnl}(\rho, \varphi, z, t) = \cos \frac{l\pi z}{d} e^{im\varphi} J_m(\gamma_{mn}\rho),$$

and TE modes,

$$R^\rho_{mnl}(\rho, \varphi, z, t) = -\frac{im}{\gamma_{mn}\rho} \sin \frac{l\pi z}{d} e^{im\varphi} J_m(\gamma_{mn}\rho),$$

$$R^\varphi_{mnl}(\rho, \varphi, z, t) = \sin \frac{l\pi z}{d} e^{im\varphi} J'_m(\gamma_{mn}\rho),$$

$$R^z_{mnl}(\rho, \varphi, z, t) = 0.$$

The general form of dynamical fields $A(\rho, \varphi, z, t)$ in terms of physical modes is

$$A(\rho, \varphi, z, t) = \sum_{m,n,l} A^1_{mnl}(t)Q_{mnl}(\rho, \varphi, z) + A^2_{mnl}(t)R_{mnl}(\rho, \varphi, z).$$

Similar results should be considered for the momentum field $\pi$ as

$$\pi(\rho, \varphi, z, t) = \sum_{m,n,l} B^1_{mnl}(t)Q^*_{mnl}(\rho, \varphi, z) + B^2_{mnl}(t)R^*_{mnl}(\rho, \varphi, z)$$

Hence our reduced phase space is described by variables $A^i_{mnl}(t)$ and $B^i_{mnl}$ for $i = 1, 2$. Using the equations $E = -\partial A/\partial t$ and $B = \nabla \times A$ in the coulomb gauge (which is used here) one can show that the above modes are consistent with the standard results given in the text books.

Now we can quantize the system by using the symplectic two-form given in formula (11). Fortunately the physical modes $Q_{mnl}(\rho, \varphi, z)$ and $R_{mnl}(\rho, \varphi, z)$ constitute an orthogonal set of vector functions inside the cylindrical box. Using orthogonality conditions

$$\int d^3x Q^*_{mnl}(x).Q_{m'n'l'}(x) = \frac{V}{2}K^2_{mn} \frac{\omega^2_{mnl}}{12\gamma_{mn}^2} \delta_{mn'} \delta_{nn'} \delta_{ll'}$$

$$\int d^3x R^*_{mnl}(x).R_{m'n'l'}(x) = \frac{V}{2}K^2_{mn} \delta_{mn'} \delta_{nn'} \delta_{ll'}$$

we find

$$\Omega = \sum_{mnl} V K^2_{mn} \left[ \frac{\omega^2_{mnl}}{12\gamma_{mn}^2} dB^1_{mnl} \wedge dA^1_{mnl} + dB^2_{mnl} \wedge dA^2_{mnl} \right],$$

where $\omega^2_{mnl} = \gamma^2_{mn} + l^2\pi^2/d^2$, $K_{mn} = J_{m+1}(x_{mn})$ and $V$ is the volume of the cylinder. Since the symplectic matrix is diagonal in the basis of TM and TE modes, it is an easy task to
invert it and read the non-vanishing Dirac brackets among physical modes as follows

\[
[A_{mn}^1(t), B_{m'n'}^1(t)] = \frac{2}{VK_{mn}^2} \frac{\gamma_{mn}^2}{\omega_{mn}^2} \delta_{mm'} \delta_{nn'} \delta_{ll'}
\]

\[
[A_{mn}^2(t), B_{m'n'}^2(t)] = \frac{2}{VK_{mn}^2} \delta_{mm'} \delta_{nn'} \delta_{ll'}
\]  

(86)

Note once again that we have not solved the equations of motion completely; so that the time dependence of the physical modes \(A_{mn}^i(t)\) and \(B_{m'n'}^j(t)\) are not determined yet. This, however, can be achieved by writing the canonical Hamiltonian in terms of physical modes as follows

\[
H = \sum_{mnl} \frac{V}{2} K_{mn}^2 \left[ \frac{\omega_{mn}^2}{\gamma_{mn}^2} \left( \omega_{mn}^2 A_{mn}^1 \omega_{mn}^2 + B_{mn}^1 \right)^2 + \left( \omega_{mn}^2 A_{mn}^2 + B_{mn}^2 \right)^2 \right]
\]  

(87)

Solving the equations of motion for physical modes gives, as in the previous cases

\[
\dot{A}^i = [A_{mn}^i, H] = B_{mn}^i
\]

\[
\dot{B}^i = [B_{mn}^i, H] = -\omega_{mn}^2 A_{mn}^i
\]  

(88)

for \(i = 1, 2\). By straightforward calculation using the Dirac brackets of Eqs. (86) and the following completeness relations (for divergence-less vector functions satisfying our boundary conditions) of physical modes

\[
\sum_{mnl} \frac{2}{VK_{mn}^2} \left[ \frac{\gamma_{mn}^2}{\omega_{mn}^2} Q_{mn}^\alpha(x) Q_{mn}^\beta(x') + R_{mn}^\alpha(x) R_{mn}^\beta(x') \right] = \delta^{\alpha\beta} \delta(x-x')
\]  

(89)

where \(\alpha\) and \(\beta\) refer to different components in cylindrical coordinates, we can find the Dirac brackets of the fields inside the cylindrical box as

\[
\{ A^\alpha(x, t), \pi^\beta(x', t) \} = \delta^{\alpha\beta} \delta(x-x').
\]  

(90)

However, the result is not the same as the Poisson brackets which we begin with. In fact special construction (81) and (82) of fields in terms of physical modes (79) and (80) shows that the Dirac brackets on the walls are consistent with the constraints (70) and (71). For example \(\{ A^\phi(x, t), \pi^\beta(x', t) \}, \{ A^\phi(x, t), \pi^\beta(x', t) \}\) and \(\{ \frac{1}{\rho} (\frac{\partial}{\partial \rho} (\rho A^\phi) - \frac{\partial A^\phi}{\partial \rho})(x, t), \pi^\beta(x', t) \}\) vanish on the end faces \(z = 0\) and \(z = d\) while \(\{ \nabla.A(x, t), \pi^\beta(x', t) \}\) vanishes everywhere inside the box and on the walls.

### 7 Electromagnetic field inside a sphere

Consider the Electromagnetic field inside a sphere of radius \(R\). Assuming the interior wall of the sphere is an ideal conductor, the boundary conditions are

\[
A^\phi|_{r=R} = 0, \quad A^\theta|_{r=R} = 0, \quad \frac{1}{r \sin \theta} \left[ \left( \frac{\partial}{\partial \theta} (\sin \theta A^\phi) - \frac{\partial A^\phi}{\partial \varphi} \right) \right]_{r=R} = 0
\]  

(91)
The total Hamiltonian in the gauge $A^0 = 0$ is $H_T = H_c + \sum_{i=1}^{3} \lambda_i \psi_i^0$, where the summation is over the three boundary constraints given by Eq. (91). The fundamental Poisson brackets among the field components read

$$\{ A^\alpha(r, \theta, \varphi), \pi_\beta(\rho', \theta', \varphi') \} = \frac{\delta^\alpha_\beta}{r^2 \sin^2 \theta} \delta(r - r') \delta(\theta - \theta') \delta(\varphi - \varphi'), \quad (92)$$

where $\alpha$ and $\beta$ run over $r, \theta$ and $\varphi$. The procedure of consistency of the constraints leads to two infinite sets of constraints as follows

$$\nabla^{2n} \psi_i^0 = 0, \nabla^{2n} \chi_i^0 = 0 \quad (93)$$

where $\chi_i^0$ are similar to $\psi_i^0$ of Eq. (91) in terms of momentum fields $\pi_\alpha$. Let impose the constraints on the most general form of Bessel-Fourier expansions as

$$A^\alpha_r(r, \theta, \varphi, t) = \sum_{lmn} \int dk A^\alpha_{lm}(k, t) j_l(kr) Y_{lm}(\theta, \varphi). \quad (94)$$

Using the differential equation of spherical Bessel functions as

$$\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2 \right) j_l(kr) = 0, \quad (95)$$

the constraints $\nabla^{2n} A^\theta = 0$ and $\nabla^{2n} A^\varphi = 0$ at $r = R$ give

$$\sum_{lmn} \int dk A^\alpha_{lm}(k, t) \left[ (-k^2)^n j_l(kr) \right]_{r=R} Y_{lm}(\theta, \varphi) = 0. \quad (96)$$

Hence $A_{lm}(k, t)$ should vanish except for

$$j_l(kr)|_{r=R} = 0 \Rightarrow k_{ln} \equiv \frac{x_{ln}}{R} \quad (97)$$

where $x_{ln}$ are roots of the equation $j_l(x) = 0$. Up to this stage, we can write the following expansion for the first two components of the $A$-field

$$A^\theta(r, \theta, \varphi) = \sum_{lmn} A^\theta_{lmn}(t) j_l(k_{ln}r) Y_{lm}(\theta, \varphi)$$

$$A^\varphi(r, \theta, \varphi) = \sum_{lmn} A^\varphi_{lmn}(t) j_l(k_{ln}r) Y_{lm}(\theta, \varphi) \quad (98)$$

The gauge fixing condition $\nabla.A = 0$ in spherical coordinate, i.e.

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A^r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A^\theta) + \frac{1}{r \sin \theta} \frac{\partial A^\varphi}{\partial \varphi} = 0 \quad (99)$$

can be satisfied at any point, if the variable $k$ in the expansion of $A^r$ is of the form $x_{ln}/R$, i.e.

$$A^r(r, \theta, \varphi) = \sum_{lmn} A^r_{lmn}(t) j_l(k_{ln}r) Y_{lm}(\theta, \varphi) \quad (100)$$

Hence, the final answer for each $l$ is the product of $j_l(x_{ln}r/R)$ with some suitable vector combination of $Y_{lm}$’s which satisfy the gauge $\nabla.A = 0$. This leads us naturally to the notion
of "vector spherical harmonics". As is well-known from the standard text books on special
functions, for each \( l, m \) there exist three orthogonal vector combinations \( X_{lm}, U_{lm}, \) and \( V_{lm}, \) which are vector eigenfunctions of the angular part of Laplacian operator; among them only
\( X_{lm} \) is the suitable one such that the combination \( A_{lm}(r, \theta, \varphi) \equiv \sum_m a_{lmj}(kr)Y_{lm}(\theta, \varphi) \)
satisfies \( \nabla A_{lm} = 0. \) This function is defined as

\[
X_{lm} = \hat{\theta} \left\{ \frac{-m}{[l(l+1)]^{1/2}} \sin \theta \right\} Y_{lm} + \hat{\varphi} \left\{ -i \left[ \frac{-i}{[l(l+1)]^{1/2}} \partial Y_{lm} \right] \right\}.
\]

This is the only reasonable answer. In other words, no other mode can be found that satisfies all of our physical conditions. This is specular for spherical coordinates which the gauge condition \( \nabla A = 0 \) leaves us just with one mode (in each pair \( lm \)) as follows

\[
A(r, \theta, \varphi) = \sum_{lmnn} A_{lmnnj}(x_{ln}r/R)X_{lm}(\theta, \varphi).
\]

Similar results should be written for the momentum fields in terms of the physical modes \( B_{lnm}. \) Using the orthogonality relations

\[
\int X_{lm}(\theta, \varphi) X_{lm'}(\theta, \varphi) d\Omega = \delta_{ll'} \delta_{mm'}.
\]

\[
\int j_l(x_{ln}r/R) j_l(x_{lm'}r/R) r^2 dr = [j_{l+1}(x_{ln}r/R)]^2
\]

and the general formula (11) for the symplectic two-form, we find

\[
\Omega = R^3 j_{l+1}(x_{ln}) dB_{mnl}^\theta(t) \wedge dA_{mnl}^\theta(t)
\]

Fortunately the symplectic two-form is in the diagonal form between conjugate pairs \( (A_{lnn}, B_{lnn}). \) Hence it can be inverted easily to give the following brackets among physical modes,

\[
[A_{mnl}(t), B_{mn'l'}(t)] = \frac{2}{R^3 j_{l+1}(x_{ln})} \delta_{ll'} \delta_{mm'} \delta_{nn'}.
\]

Finally for Dirac brackets of the original fields we have \( [A^\alpha(r, \theta, \varphi, t), \pi_\beta(r', \theta', \varphi', t')] = \delta^\alpha_\beta \delta^3(x - x'), \) inside the box; while the Dirac brackets of the constraints vanish strongly on the surface of the sphere. Moreover, \( \nabla A, \nabla \pi, \) as well as \( A^0 \) and \( \pi_0 \) have vanishing Dirac brackets with all of the fields.

8 Conclusions

Our aim in this paper was presenting a systematic method for quantization fields with given
boundary conditions on the walls of a box which the fields live in it. We avoid proposing
different ad-hoc quantization assumptions for individual models. Instead, we think the
brilliant prescription of Dirac (i.e. converting the Dirac brackets to commutators) is the
only needed tool for this reason. In other words, it is not just a criterion to be satisfied
by a quantization assumption; it is, on the other hand, a road-map for quantizing every
desired model.
Traditionally quantization of the fields is achieved by assuming certain commutation relations among the coefficients of the expansions of the fields in terms of solutions of the equations of motion and then showing that the assumed commutation relation gives the standard canonical commutation relations among the original fields. We have three objections against this way of thinking. First, there is not enough logic in this point of view, by its own. In fact, the main stress is on the standard canonical commutation relations coming from the classical Poisson brackets. Second, there is no guaranty about the uniqueness of the method, since the classical brackets are not the beginning point of the quantization procedure. Why no other quantization assumption can be proposed which are consistent with the same set of canonical brackets?

Finally, the third and most important objection is inherent in this question: "which set of classical brackets should be resulted from the assumed commutation relations among the coefficients, Poisson brackets or Dirac brackets?". Text books often refer to Poisson brackets in the bulk of the medium. However, in the presence of boundary conditions this is no more correct; since the Poisson brackets in the bulk are not consistent with the boundary conditions. For example, when you are given a Dirichlet boundary condition, the vanishing field can no more give delta function (unity) in its bracket with the corresponding momentum field. As we know in the context of constrained systems, in the presence of second class constraints (which is the case for boundary conditions as Dirac constraints) the Poisson brackets should be replaced with Dirac brackets before quantization.

We show in this paper, in almost all of the examples, that the Dirac brackets are different with the Poisson brackets which we begin with (see for instance Eqs. 26 and 27 and 64). The consistency procedure of Dirac, together with omitting the modes which are associated with the second class constraints are essential steps to obtain classical brackets which are consistent with the given boundary conditions. The second step is equivalent to considering the Dirac brackets instead of Poisson brackets. Hence, the quantization postulates should be based on the Dirac brackets of the fields. This simple point is not as clear and well-known as it may seem. In fact concentrating on the contradiction between the canonical commutators and the mixed boundary conditions in string theory observed by Seiberg and Witten [11] led to a stream of papers on non-commutativity on the brain coordinates linking by bosonic strings.

We used here a special approach to Dirac method based on finding the symplectic matrix in the basis of "physical modes" and inverting it to find the (Dirac) brackets among them. By physical modes we mean degrees of freedom which are compatible with the constraints of the system including first class constraints (as generators of gauge transformations), gauge fixing conditions, intrinsic second class constraints, and finally primary boundary conditions as well as secondary conditions on the boundaries emerging from their consistency in time. This does not mean "the solutions of the equations of motion", since the latter depends on the particular form of the Hamiltonian of the system.

We considered six examples. In each case we were able to find "suitable basis" for the space of physical variables, so that imposing the constraints leaded to omitting a number of conjugate pairs as second class constraints. Hence, we were left with a reduced phase space with a prescribed canonical basis, which converted to canonical operators upon quantization. One important point is that the Fourier modes are not necessarily the canonical operators of a quantum theory; this is only the case for a geometry consistent with cartesian coordinates. Note that in the general case there is no guaranty to find a "suitable basis". Orthogonality plays an important role in this regard. If the physical modes are orthogonal
the symplectic matrix, in p-q and q-p blocks, would be diagonal. Hence, the Dirac brackets coming from inverting the symplectic two-form (see Eq. 3) will be canonical. This means that we have been lucky enough to find the canonical basis of the reduced phase space. However, on the basis of the famous Darboux theorem [6], we are only sure about the existence of a canonical basis in the reduced phase space.

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