Thermal effective potential of the linear sigma model

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We have attempted an approach to the chiral phase transition of QCD using the linear sigma model as an effective theory. In order to get some insight into how the phase transition could proceed, we have calculated the finite temperature effective potential of this model in the Hartree and large N approximations using the Cornwall-Jackiw-Tomboulis formalism of composite operators.

I. INTRODUCTION

It is well known the chiral symmetry of QCD is broken due to the small quark masses. A large amount of the current research activity is devoted to the chiral phase transition of QCD by using effective models or lattice techniques. An account on the physics of thermal QCD and further references is given in [1]. In studies of symmetry restoring phase transitions the finite temperature effective potential is an important and popular theoretical tool [2]. The finite temperature effective potential $V(\phi, T)$ is defined through an effective action $\Gamma(\phi)$ which is the generating functional of the one particle irreducible vacuum graphs and it has the meaning of the free energy density of the system under consideration.

A generalised version is the effective potential $V(\phi, G)$ for composite operators introduced by Cornwall, Jackiw and Tomboulis (CJT) [3] and originally was written at zero temperature but it has been extended to finite temperature by Amelino-Camelia and Pi [4]. In this case the effective action $\Gamma(\phi, G)$ is the generating functional of the two particle irreducible vacuum graphs and the effective potential depends on $\phi(x)$ – a possible expectation value of a quantum field $\Phi(x)$ – and on $G(x, y)$ – a possible expectation value of the time ordered product of the field operator $T\Phi(x)\Phi(y)$.

Physical solutions demand that the effective potential should satisfy the stationarity requirements

$$\frac{dV(\phi, G)}{d\phi} = 0 \quad \text{and} \quad \frac{dV(\phi, G)}{dG} = 0. \quad (1)$$

The conventional effective potential results as $V(\phi) = V(\phi; G_0)$ at the solution $G(\phi) = G_0(\phi)$ of the second equation.

There is an advantage in using the CJT method to calculate the effective potential in Hartree approximation. As it was shown in a recent investigation of the $\lambda\phi^4$ theory [5], we need to evaluate only one type of graphs – those in fig. 1a – using an ansatz for a “dressed propagator”, instead of summing the infinite class of “daisy” and “super daisy” graphs – figs. 1b, 1c respectively – using the usual tree level propagators.

FIG. 1. The double bubble and examples of daisy and superdaisy diagrams.

As in many other cases in physics, in order to deal with the chiral phase transition of QCD we can use effective models to describe the physical situation. The linear sigma model [6] is such a theory since it exhibits the correct chiral properties and has attracted much attention recently, especially in studies of Disoriented Chiral Condensates [7,8,14]. The generalised version of the linear sigma model – without fermions – is called the $O(N)$ or vector model and is based on a set of $N$ real scalar fields. The $O(N)$ model Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}m^2\Phi^2 - \frac{1}{6N}\lambda\Phi^4 - \varepsilon\sigma, \quad (2)$$

and in order our notation to be consistent with applications on pion phenomenology we can identify the $\Phi_1$ with the $\sigma$ field and the remaining $N - 1$ components as the pion fields, so $\Phi = (\sigma, \pi_1, \ldots, \pi_{N-1})$. The last term $-\varepsilon\sigma$ has been introduced in order to generate the observed masses of the pions.

The contact with phenomenology is obtained by considering the case $N = 4$. Then the model consists of four scalar fields, the sigma $\sigma$ field and the usual three pion fields $\pi^0, \pi^\pm$ which form a four vector $(\sigma, \pi_i)$ $i = 1, 2, 3$. The $\sigma$ field can be used to represent the quark condensate, the order parameter for the chiral phase transition, since they exhibit the same behaviour under chiral transformations [4]. The pions are very light particles and can be considered approximately as massless Goldstone bosons. In order to be consistent with the observed pion mass of $m_\pi \approx 138$ MeV and the usually adopted sigma mass of $m_\sigma \approx 600$ MeV we choose $\varepsilon = f_\pi m_\pi^2$, where $f_\pi = 93$ MeV is the pion decay constant. The coupling constant $\lambda$ and the negative mass parameter $m^2$ of the model are defined as $\lambda = 3(m_\sigma^2 - m_\pi^2)/f_\pi^2$ and $-m^2 = (m_\sigma^2 - 3m_\pi^2)/2 > 0$. 

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II. THE HARTREE APPROXIMATION

We examine the $O(4)$ model first by setting $N=4$ in the Lagrangian (2). In the chiral limit when the pions are considered as massless the last term is ignored. The four types of bubble diagrams relevant for Hartree approximation are given in fig. 2.

\[
\begin{array}{cccc}
\sigma & \pi_i & \pi_j\neq i & \pi_i \\
3 & 9 & 6 & 6
\end{array}
\]

FIG. 2. Graphs which contribute to the effective potential for the $O(4)$ linear sigma model in the Hartree approximation. The numbers show the weight of each type of bubble in the expression for the effective potential.

Following the procedure described in [10] the thermal effective potential reads as

\[
\begin{align*}
V(\phi, M) &= \frac{1}{2} m^2 \phi^2 + \frac{1}{24} \lambda \phi^4 + \frac{1}{2} \int \ln G_{\sigma}^{-1}(k) \\
&+ \frac{3}{2} \int \beta \ln G_{\pi}^{-1}(k) + \frac{1}{2} \int [D_{\sigma}^{-1}(k) G_{\sigma}(k) - 1] \\
&+ \frac{3}{2} \int [D_{\pi}^{-1}(k) G_{\pi}(k) - 3] \\
&+ \frac{3}{24} \left[ \int G_{\sigma}(k) \right]^2 + 15 \frac{\lambda}{2} \left[ \int G_{\pi}(k) \right]^2 \\
&+ \frac{3}{24} \left[ \int G_{\pi}(k) \right] \left[ \int G_{\pi}(k) \right].
\end{align*}
\]

(3)

In our calculations we use the imaginary time formalism and for simplicity we use the following notation

\[
\int \frac{d^3k}{(2\pi)^3} \rightarrow \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \equiv \int_\beta.
\]

(4)

The tree propagators appearing above are given by

\[
\begin{align*}
D_{\sigma}^{-1}(\phi; k) &= k^2 + m^2 + \frac{1}{2} \lambda \phi^2 \\
D_{\pi}^{-1}(\phi; k) &= k^2 + m^2 + \frac{1}{6} \lambda \phi^2,
\end{align*}
\]

(5)

and as in [8] we adopt the following form for the dressed propagator

\[
G_{\sigma/\pi}^{-1}(\phi; k) = k^2 + M_{\sigma/\pi}^2.
\]

(6)

Introducing a thermal effective mass $M_{\sigma/\pi}$. Minimizing the potential with respect to $G_{\sigma/\pi}$ we obtain a system of gap equations for the thermal effective masses, which after performing the Matsubara frequency sums appears as

\[
\begin{align*}
M_{\sigma}^2 &= m^2 + \frac{1}{2} \lambda \phi^2 + \frac{1}{2} F_\beta(M_{\sigma}) + \frac{1}{2} F_\beta(M_{\pi}) \\
M_{\pi}^2 &= m^2 + \frac{1}{6} \lambda \phi^2 + \frac{1}{6} F_\beta(M_{\sigma}) + \frac{5}{6} F_\beta(M_{\pi}),
\end{align*}
\]

(7)

and at the level of our approximation we keep only the finite temperature part of the resulting integrals, so

\[
F_\beta(M_{\pi/\sigma}) = \frac{T^2}{2\pi^2} \int_0^{\infty} \frac{x^2 dx}{\alpha^{1/2} \exp(a^{1/2}) - 1},
\]

(8)

with $a = x^2 + y^2$ and $y = M_{\pi/\sigma}/T$. Minimizing with respect to the order parameter $\phi$ we find one more equation

\[
0 = m^2 + \frac{1}{6} \lambda \phi^2 + \frac{1}{2} F_\beta(M_{\sigma}) + \frac{1}{2} F_\beta(M_{\pi}).
\]

(9)

This system is solved numerically and the solution is given in fig. 3.

\[\text{FIG. 3. (a) The sigma and pion effective masses as functions of temperature. (b) Order parameter as a function of temperature.}\]

The finite temperature effective potential $V(\phi, T)$ written in compact form is given by

\[
\begin{align*}
V(\phi, M) &= \frac{1}{2} m^2 \phi^2 + \frac{1}{24} \lambda \phi^4 + Q_\beta(M_{\sigma}) \\
&+ 3 Q_\beta(M_{\pi}) - \frac{\lambda}{8} [F_\beta(M_{\sigma})]^2 \\
&- \frac{5\lambda}{8} [F_\beta(M_{\pi})]^2 - \frac{\lambda}{4} F_\beta(M_{\sigma}) F_\beta(M_{\pi}).
\end{align*}
\]

(10)

where $F_\beta(M)$ is given by the equation (8) and

\[
Q_\beta(M) = \frac{T^4}{2\pi^2} \int_0^{\infty} dx x^2 \ln \left[ 1 - \exp(-a^{1/2}) \right]
\]

(10)

with $a = x^2 + y^2$ and $y = M_{\pi/\sigma}/T$. The evolution of the potential with temperature is given in fig. 4 and we easily observe indication of a first order phase transition.
FIG. 4. Evolution of the effective potential $V(\phi, T)$ as a function of the order parameter $\phi$ for several temperatures in steps of 2 MeV. The two minima appear as degenerate at $T_c \approx 182$ MeV. The second minimum at $\phi \neq 0$ dissapears at a temperature $T_{c2} \approx 187$ MeV.

When the chiral symmetry is broken minimizing the thermal effective potential with respect to dressed propagators we find the same set of gap equations (7) for the effective masses as before, but minimizing with respect to the order parameter we obtain the following equation

$$m^2 + \frac{1}{6} \lambda \phi^2 + \frac{\lambda}{2} F_{\beta}(M_\sigma) + \frac{\lambda}{2} F_{\beta}(M_\pi) \phi - \varepsilon = 0.$$

(11)

We have solved the system of equations (7) and (11) numerically and the solution is given in fig. 5.

When we consider $N - 1$ pion fields the Lagrangian of the model is given in equation (2) and in the chiral limit we ignore the last term. Then the thermal effective potential looks like \cite{10,11}

$$V(\phi, M) = \frac{1}{2} m^2 \phi^2 + \frac{1}{6N} \lambda \phi^4 + \frac{N - 1}{2} \int \ln G_\pi^{-1}(k)$$

$$+ \frac{1}{2} \int \ln G_\sigma^{-1}(k) + \frac{1}{2} \left[ \int D_\pi^{-1}(k) G_\pi(k) - 1 \right]$$

$$+ \frac{N - 1}{2} \left[ \int D_\pi^{-1}(k) G_\sigma(k) - (N - 1) \right]$$

$$+ \frac{\lambda}{24} \left[ \int G_\pi(k) \right]^2 + \frac{\lambda(N^2 - 1)}{6N} \left[ \int G_\sigma(k) \right]^2$$

$$+ \frac{\lambda(N - 1)}{6N} \left[ \int G_\sigma(k) \right] \left[ \int G_\pi(k) \right].$$

(12)

Minimizing the potential with respect to the dressed propagators we find a set of nonlinear gap equations which in the large $N$ approximation reduces to

$$M_{\sigma}^2 = m^2 + \frac{1}{2} \lambda \phi^2 + \frac{2\lambda}{3} F_{\beta}(M_\pi)$$

$$M_{\pi}^2 = m^2 + \frac{1}{6} \lambda \phi^2 + \frac{2\lambda}{3} F_{\beta}(M_\pi).$$

(13)

Minimizing the potential with respect to $\phi$ we obtain

$$\frac{dV(\phi, M)}{d\phi} = \phi M_\sigma^2 = 0$$

(14)

which means that in this approximation the Goldstone theorem is satisfied. The solution of the system is given in fig. 6.

There is no longer any phase transition. We rather observe a crossover phenomenon from the low temperature phase to the high temperature phase.

III. LARGE $N$ APPROXIMATION

When we consider $N - 1$ pion fields the Lagrangian of the model is given in equation (2) and in the chiral limit we ignore the last term. Then the thermal effective potential looks like \cite{10,11}

$$V(\phi, M) = \frac{1}{2} m^2 \phi^2 + \frac{1}{6N} \lambda \phi^4 + \frac{N - 1}{2} \int \ln G_\pi^{-1}(k)$$

$$+ \frac{1}{2} \int \ln G_\sigma^{-1}(k) + \frac{1}{2} \left[ \int D_\pi^{-1}(k) G_\sigma(k) - 1 \right]$$

$$+ \frac{N - 1}{2} \left[ \int D_\pi^{-1}(k) G_\sigma(k) - (N - 1) \right]$$

$$+ \frac{\lambda}{24} \left[ \int G_\pi(k) \right]^2 + \frac{\lambda(N^2 - 1)}{6N} \left[ \int G_\sigma(k) \right]^2$$

$$+ \frac{\lambda(N - 1)}{6N} \left[ \int G_\sigma(k) \right] \left[ \int G_\pi(k) \right].$$

(12)

Minimizing the potential with respect to the dressed propagators we find a set of nonlinear gap equations which in the large $N$ approximation reduces to

$$M_{\sigma}^2 = m^2 + \frac{1}{2} \lambda \phi^2 + \frac{2\lambda}{3} F_{\beta}(M_\pi)$$

$$M_{\pi}^2 = m^2 + \frac{1}{6} \lambda \phi^2 + \frac{2\lambda}{3} F_{\beta}(M_\pi).$$

(13)

Minimizing the potential with respect to $\phi$ we obtain

$$\frac{dV(\phi, M)}{d\phi} = \phi M_\sigma^2 = 0$$

(14)

which means that in this approximation the Goldstone theorem is satisfied. The solution of the system is given in fig. 6.

FIG. 5. (a) Solution of the system of gap equations in the case when $\varepsilon \neq 0$. At low temp the pions appear with the observed masses. (b) Evolution of the order parameter as a function of temperature.

FIG. 6. (a) Solution of the system of gap equations in the large $N$ approximation in the chiral limit. At low temperatures the pions appear as massless. (b) Evolution of the order parameter with temperature in the large $N$ approximation.
In the high temperature phase when $\phi = 0$, the two equations become degenerate and can be used to calculate the critical temperature $T_c = \sqrt{3}f_\pi \approx 161\text{MeV}$.

The presence of the symmetry breaking term in the Lagrangian results – after minimizing the potential – the same set of equations as before, but minimizing with respect to $\phi$ we obtain the equation

$$m^2 + \frac{1}{6}\lambda\phi^2 + \frac{2\lambda}{3}F_\beta(M_\pi)\phi - \varepsilon = 0.$$ 

We have solved the system of equations numerically and the solution is given in fig 7.

![Graph showing solution of gap equations](image)

**FIG. 7.** (a) Solution of the system of gap equations in the large $N$ approximation in the case of broken chiral symmetry $\varepsilon \neq 0$. At low temperatures the pions appear with the observed masses. (b) Evolution of the order parameter with temperature.

As is the Hartree case we do not observe a phase transition. We encounter the crossover phenomenon again; the difference being now the smoother behaviour of the order parameter.

### IV. CONCLUSIONS

The calculation of the thermal effective potential of the linear sigma model in the Hartree and large $N$ approximations using the CJT formalism of composite operators proved to be very handy since we actually need to calculate only one type of diagram. In both cases we have solved numerically the system of the resulting gap equations and found the evolution with temperature of the thermal effective masses. In the chiral limit, when the pion mass is ignored, the Hartree approximation predicts a first order phase transition, but in contrast to that, in the large $N$ approximation we find a second order phase transition. This last observation seems to be in agreement with different approaches to the chiral phase transition based on the argument that the linear sigma model belongs to the same universality class as other models which are known to exhibit second order phase transitions. However in the large $N$ approximation the sigma contribution is ignored and this of course introduces errors when we calculate the critical temperature. In the $N = 4$ case which is closer to phenomenology we could probably obtain a better approximation in calculating the transition temperature if we had considered all the interactions described by the Lagrangian of the model. We are planning some investigations in order to include these effects in the calculation of the effective potential.

When we introduce a symmetry breaking term, $\varepsilon\sigma$, which generates the observed pion masses in the model, both in the Hartree and large $N$ approximations we found that there is no longer any phase transition. We rather observe a crossover phenomenon which confirms other results reported recently using the linear sigma model. In this analysis there is also an indication of a first order phase transition in the chiral limit. First order phase transition is also reported in [14,15].

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