The Rayleigh-Taylor Instability in a Bose-Einstein Condensate

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We show how a two dimensional Bose-Einstein Condensate trapped in a nonequilibrium state can be driven into the Rayleigh-Taylor instability if an outward in-plane force is exerted on it. If the condensate is inside a semiconductor, the above force could arise from an inhomogeneous strain applied on the host semiconductor crystal. On the other hand, for a BEC of alkali atoms, the force could be exerted by an inhomogeneous magnetic field turned on after the BEC has been released from its magnetic trap. During its expansion, the condensate will break into droplets each a separate BEC. Therefore, one can create BEC droplets out of a single large BEC and study the coherence properties of the droplets with respect to each other. The discussion on the linear onset of the Rayleigh-Taylor instability in a BEC is generalized to three dimensions.

I. INTRODUCTION

The Rayleigh-Taylor instability (RT) [1,2] is observed in fingers that develop on industrial smoke spreading from factory chimneys [3], blast remnants of supernovae [4], superfluid droplets from charged superfluid jets [5], and in exciton droplets in bulk Ge [6]. It occurs in inhomogeneous fluids with density gradients when a force acts in the direction of less density. Layers of constant density become unstable towards the formation of undulations or wiggles. Since a wiggle displaces mass in the direction of the force, a constant density layer which has undulations has a lower potential energy than one which is straight [4,7]. The wavelength of the wiggles is bounded from below by the surface tension or the viscosity, since there is a greater cost in surface energy for a wiggle of shorter wavelength and therefore, greater surface area, while viscosity does not permit adjacent fluid elements to have very different velocities as is needed for short wavelength undulations. A necessary condition for the RT instability to evolve in a fluid is that it should take a shorter time for sound to propagate through the medium than the time in which the instability grows. Therefore, \( s \geq \lambda/\tau \) must hold, where \( s \) is the sound speed in the fluid, while \( \lambda \) and \( \tau \) are the wavelength and growth time of the instability.

Ever since BECs were first created in the laboratory, experiments in BECs, particularly on their collapse and revival [8], the quantum interference of two BECs [9] and the superfluid to Mott-Insulator transition in optical lattices [10] have justified the hope that the laws of quantum mechanics can be studied in such systems. BECs made out of atoms are trapped in magnetic traps and are physically isolated from the walls of the vacuum chamber. This limits decoherence processes to interactions with uncondensed particles which remain in the vicinity of the condensate. On the other hand, BECs in semiconductor systems usually have a shorter decoherence time, \( \tau_d \), compared to their atomic counterparts because of greater interaction with the host, since generally \( \tau_d \sim \tau_e \), where \( \tau_e \) is the interaction time [11]. BECs in semiconductors trapped in more than one dimension are truly extended systems with the host environment making quantum mechanical measurements of the density of the condensate wavefunction at different parts of the condensate, hence quickly destroying the long-range order. The onset of decoherence through the destruction of long range order as above is related to the Kibble-Zurek mechanism [12], which is important in cosmology. The presence of additional channels of decoherence within a solid due to a greater interaction with the host makes it more difficult to make BECs in solids that can survive long enough to be observed. Magnon condensation has been reported in the magnetic compounds, \( \text{Cs}_2\text{CuCl}_4 \) and \( \text{TlCuCl}_3 \) [13]. A solid state system, which is an interesting but not yet proven as a candidate system for Bose-Einstein Condensation is a cold excitonic cloud trapped in a coupled quantum well (CQW) [14] [15] [16].

When it comes to BECs, hydrodynamic phenomena play a very important role. For example, vortices in rotating BECs are essentially hydrodynamic in nature. To make safe inferences about the quantum mechanics, it is necessary to have an understanding of hydrodynamic instabilities, such as the RT instability, that may develop in them in certain experimental conditions. Furthermore, hydrodynamic instabilities may give rise to new situations in which quantum mechanics can be further tested. In this paper, we show how in certain conditions a Rayleigh-Taylor instability may develop in a BEC giving rise to the formation of droplets. These droplets, each of which is a BEC, can have interesting properties that need future study. One interesting possibility is to make clones of a parent BEC and then make quantum mechanical operations on each of them separately.

The Rayleigh-Taylor instability in BECs has not yet been investigated. This paper aims to address that issue. In Sec. II, we consider a two dimensional BEC that can be unstable towards RT instability. Later, we shall see that our results can easily be generalized to three dimensions. In Sec. III, we describe the linear onset of the RT instability. In Sec. IV, we remark on the future evolution of the instability. In Sec. V, we discuss two possible BEC systems in which we might see the formation of Rayleigh-Taylor droplets.
II. THE RAYLEIGH-TAYLOR INSTABILITY IN A BEC

A. Initial State of the 2D BEC at T=0

We consider a two dimensional BEC of excitons at rest in a harmonic trap of frequency, \( w \). The initial potential is \( V(r) \sim w^2 r^2/2 \). The Thomas-Fermi (TF) density of the exciton cloud, \( \rho_0 \sim (A - r^2)^{1/2} \) at \( r < \sqrt{\hbar/mw} \), with \( A = (2\mu/mw^2) \) and the chemical potential, \( \mu \).

We now consider the release of the trap and in its place, the immediate application of a radial force outward from the former trap’s center. We call this as time, \( t = 0 \). The condensate is in non-equilibrium at \( y > \delta \). We also assume that the density varies slowly over time, so that the perturbation of the density is given by

\[ \rho(r, t + \delta t) = \rho_0(r - \delta \vec{r}(t), t) \]

\[ = \rho_0(r, t) - \nabla \rho_0 \cdot \delta \vec{r}(t), \]

so that the perturbation of the density is given by

\[ \frac{\partial \delta \rho}{\partial t} = -\delta \vec{u} \cdot \nabla \rho_0. \]

The continuity equation,

\[ \frac{\partial P}{\partial t} + \nabla \cdot (P \delta \vec{u}) = 0. \]

when linearized becomes,

\[ \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_0 \delta \vec{u}) = 0; \]

leading to

\[ \nabla \cdot \delta \vec{u} = 0. \]

The linearized equation of motion for the condensate with only the lowest order terms in the perturbation is,

\[ m \rho_0 \frac{\partial \delta \vec{u}}{\partial t} = -\nabla \delta P + m \rho_0 \vec{g} - \frac{\hbar^2}{4m} \nabla (\nabla^2 \rho). \]

The line of discontinuity in density, \( \rho_0 \), will become slightly deformed in the perturbed state. Following [2], let the line be then defined by \( y_s + \delta y_s \), with

\[ \delta y_s(x, t) = \delta y_s(0)e^{i(kx - wt)} \]

. The discontinuity in normal stress required by equilibrium is \( T_s \delta y_s/dx^2 \), where \( T_s \) is the ‘effective tension’ of the condensate, which we define as the kinetic energy per length along the locus of the density drop, similar to the surface tension defined in Ref. [18].

\[ T_s = \frac{\hbar^2}{2m} \int_{-\delta/2}^{\delta/2} dy |\nabla \psi|^2 \sim \frac{\hbar^2}{4m}(\rho_1 - \rho_2)^2 \]

assuming a linear fall in the density at the defect over a surface width, \( \delta \). Therefore, we should add a term
corresponding to the effective tension to the right side of Eq. (5), which leads to,
\[
mρ \frac{∂δu}{∂t} = - \nabla δP + m g δρ - \frac{ℏ^2}{4m} (\nabla^2 δρ) - y k^2 τ T_s δu_y (y),
\]
where we have used \( dδy_i/dt = δu_y (y = 0) \), and therefore, \( δu_y = τ δu_y (y = 0) \).
Following Chandrashekhar [2], we look for an instability of transverse wavelength \( k \) such that any perturbed quantity, \( Q \), grows in time as,
\[
δQ(x, y, t) = Q'(y) e^{i k x + t/τ}.
\]
Taking the x-component of Eq. (11),
\[
\frac{mρ δu_x}{τ} = -ik δP - \frac{ℏ^2 k}{4m} \nabla^2 δρ.
\]
Since, from Eq. (4),
\[
δρ = -\frac{dρ_0}{dy} δu_y,
\]
is nonzero only at \( y = 0 \), solving Eq. (13) for \( y \neq 0 \) gives
\[
\frac{mρ δu_x}{τ} = -ik δP.
\]
Similarly the y-component of Eq. (11), for \( y \neq 0 \), is,
\[
\frac{mρ δu_y}{τ} = \frac{dδP}{dy}.
\]
Differentiating Eq. (13) with respect to \( x \) leads to
\[
\frac{mρ δu_x}{τ} = k^2 δP.
\]
Eqs. (4) and (14) lead to,
\[
- \frac{mρ δu_y}{τ} = k^2 δP.
\]
Differentiating with respect to \( y \), and using Eq. (16), we find,
\[
- \frac{mρ δ^2 u_y}{τ dy^2} = k^2 \frac{dδP}{dy} = - \frac{mρ k^2}{τ} δu_y.
\]
Therefore,
\[
\left( \frac{d^2}{dy^2} - k^2 \right) δu_y = 0.
\]
with \( k > 0 \), since \( δu_y \) must be continuous across the interface. It should be noted that \( δu_x \) need not be since the condensate is inviscid.

To determine the solution and thus the growth of the instability, we require the boundary conditions across the interface. Taking the y-component of Eq. (11), we find,
\[
\frac{dδP}{dy} = - \frac{mρ δu_y}{τ} - \frac{τ m g δu_y}{dy} - k^2 τ T_s δu_y (y) + \frac{ℏ^2 k}{4m} \left( \nabla^2 \left( \frac{dρ_0}{dy} δu_y \right) \right).
\]
Integrating Eq. (22) from \( y = 0 \) to \( 0+ \) gives,
\[
\Delta [δP] = -Δ \left[ m g δu_y δρ_0 - k^2 τ T_s δu_y (y = 0) \right] + \frac{ℏ^2 k}{4m} \Delta \left( \nabla^2 \left( \frac{dρ_0}{dy} δu_y \right) \right),
\]
where \( Δ \) denotes the difference across the interface. On the other hand, Eqs. (7), (13) and (14) lead to,
\[
\Delta [δP] = - \frac{m}{k^2 τ} \Delta \left[ \frac{du_y}{dy} \right],
\]
since \( dρ_0/dy = 0 \) for \( y \neq 0 \), and therefore, \( Δ[\nabla^2 (u_y dρ_0/dy)] = 0 \). Eliminating \( Δ[δP] \) from Eqs. (23) and (24), we find the boundary condition,
\[
\Delta \left[ \frac{m}{k^2 τ} \frac{du_y}{dy} \right] = Δ \left[ m g τ δu_y (y = 0) δρ_0 + k^2 τ T_s δu_y (y = 0) \right],
\]
which, after substitution for \( δu_y \) using Eq. (21), and Eq. (10), gives the relation between the growth time of the instability and the wavevector,
\[
τ = \frac{1}{k \sqrt{2}} \left[ g \left( \frac{ρ_1 - ρ_2}{ρ_1 + ρ_2} \right) - \frac{ℏ^2 k^2}{4 m^2 δ^2} \left( \frac{ρ_1 - ρ_2}{ρ_1 + ρ_2} \right)^2 \right]^{-1/2}.
\]
For \( ρ_1 > ρ_2 \), the solution is unstable (\( τ \) is real and positive) for \( 0 < k < k_c \) where \( k_c = \left\{ m(ρ_1 - ρ_2)g/T_s \right\}^{1/2} \).
Using \( δτ/dk = 0 \) in Eq. (26) gives us the fastest growing mode, which has a wavevector and growth time,
\[
κ = k_c/√3; \quad τ = \frac{3^{3/2} ℏ}{4m(gρδ)^{3/2}} \left( \frac{ρ_1 + ρ_2}{ρ_1 - ρ_2} \right)^{1/4}.
\]

C. Future Evolution of the RT Instability

We expect the instability to progress into the nonlinear regime through the development of spikes and bubbles \([6, 7]\). Spikes longer than their diameters are unstable to varicose perturbations leading to their breakup into droplets. Spike breakup can also occur due to the Kelvin-Helmholtz Instability at the edges of the spikes \([7]\) in both 2D and 3D cases. Spike breakup leads to droplet formation in both cases instead of a mixing zone when
surface tension is significant, since droplets minimize surface energy. The diameter of the droplets is then half the wavelength of the fastest growing mode, $\lambda/2$.

For low non-zero temperatures, the motion of the condensate and the excitations are decoupled. Since the excitations are dilute and of long wavelength, the exciton fluid continues to be RT unstable with the droplet size determined by the effective tension of the condensate and the applied stress or electric field gradient. For weak interactions and low temperatures, $\rho_0 U_0 << T << T_c$, the condensate moves inviscidly in a mean field due to the excitations in the Hartree-Fock approximation. If the external potential is modified to include the mean field due to excitations as a correction, the discussion in Sec. III can be used to show that the BEC is RT unstable with the applied force and the effective tension again determining the droplet size. At $T > T_c$, however, when the fluid is normal and has viscosity, the character of the flow changes. Though the fluid can still be RT unstable, the droplet size is now determined by the viscosity and applied force. Therefore, one can observe in experiments the variation in droplet size as T is increased beyond $T_c$.

III. APPLICATION TO SPECIAL CASES

Below we investigate the effects of the Rayleigh-Taylor instability in two systems of special interest.

A. Atomic BECs

Consider a typical experiment in which we have a BEC of alkali atoms in a three dimensional magnetic trap. When the trap potential is switched off, the condensate is in a non-equilibrium state. Repulsive interactions make the condensate expand slowly. After a time of flight of several nanoseconds, a snapshot of the freely expanding BEC is usually taken. We propose one modification of the experiment which can lead to the formation of droplets of BEC at the surface of the condensate while the whole gas expands. We suggest that after turning off the magnetic trap, we turn on an “inverted magnetic trap” with a potential profile, $V(\vec{r}) \sim -1/2 w^2 r^2$, resulting in a radial outward force acting on the expanding condensate. This outward force will result in a Rayleigh-Taylor instability at the surface of the expanding condensate.

It is simple to generalize to three dimensions the results of Sec. III. We continue to choose $y$ as the linear axis perpendicular to the surface. In 3D, the Eqs. (11), (20) and (21) hold exactly for the growth time and wavelength, Eqs. (26) and (27) hold exactly with $k$ now being given by $k^2 = k_x^2 + k_y^2 + k_z^2$, and every perturbed quantity, $Q$, growing in time as,

$$\delta Q(x, y, t) = Q'(y)e^{ik_x x + ik_y y + ik_z z + i\omega t}.$$ (28)

For $^{87}$Rb atoms, a magnetic trap of frequency, $24 - 240$ Hz can exert a restoring force equivalent to an acceleration of $g \sim 4.55 - 455 \text{ cm/s}^2$, at a distance of 2 $\mu$m away from the trap center. A BEC contained in a trap of frequency, 24 Hz will have a radius $\sim 2 \mu$m. After the BEC is released from this trap, an inhomogeneous magnetic field can be turned on so that the force exerted points outwards from the center instead of inwards. The BEC, which is now in a nonequilibrium state will then begin to expand. If the force of expansion is of the appropriate magnitude, there will be an RT instability driven by the force of expansion which will lead to the formation of RT droplets. A strong outward force corresponding to an acceleration, $g \sim 5 \times 10^4 \text{ cm/s}^2$, leads to a fastest growing wavelength of 0.1 $\mu$m (Eq. (27)), if we assume that the surface width, $\delta$, is almost the same in magnitude as the wavelength. This leads to droplets of size, $> 0.05 \mu$m, which can be observed. The growth time for such a droplet is given by Eq. (27) to be $\tau \sim 3 \mu$s. Weaker forces leads to larger wavelengths for the instability. Since for an observable instability, we need the wavelength, $\lambda < \pi D$, where $D$ is the diameter of the cloud, we find that the maximum value of acceleration at the surface is, $g_{\text{max}} \sim 0.8 \text{ cm/s}^2$, which leads to a minimum growth time for observable instability, $\tau_{\text{min}} \sim 0.02 \text{ sec}$ for $\lambda_{\text{max}} \sim \lambda / \delta \sim 4 \mu$m (Eq. (27)).

A suitably designed experiment can therefore be used to create Rayleigh-Taylor droplets in an atomic BEC. In an actual experiment, the condensate will be allowed to expand to several times its initial size before measurements are made. Therefore, the droplets seen at measurement will be scaled up.

B. Excitonic BEC in Layered Semiconductors?

The possibility exists that an indirect exciton cloud in a layered semiconductor if cooled below $T_c$ may form two dimensional BECs trapped in impurity driven in-plane shallow traps \[14,19\]. This type of BEC has not yet been conclusively observed. If, however, such a BEC can be created in the future, it may be possible to overturn the trap by applying an outward radial force through an inhomogeneous stress. This would lead to a two-dimensional BEC in a non-equilibrium state, which is driven to expand by the radial force resulting in an RT instability. We note here that inhomogeneous stress has previously been used to create a harmonic trap in a semiconductor bilayer \[20\]. It is reasonable to assume that it may be possible to use the same means to create an outward, instead of inward, radial force field.

For indirect excitons, $m = 0.1 m_e$, $\rho \sim 10^{16} \text{ cm}^{-3}$ in the wells, and $m^* \sim m_0 \sim 0.1 - 1 \text{ meV}$ \[14\]. If $g = 10^{15} \text{ cm s}^{-2}$ as achieved in Ref. \[20\] through inhomogeneous stress and electric fields, and making the reasonable assumption that $\delta \sim \bar{\lambda}$, under which conditions the RT instability still exists, we find $\lambda \sim 5.4 \mu$m and $\tau \sim 0.4 \text{ ns}$. Moreover, it can be verified that $s \gtrsim \bar{\lambda}/\tau$. Therefore, we would expect BEC droplets, 5-10 microns in diameter, but as the exciton cloud starts to expand radially,
the size of the droplets will scale accordingly.

IV. CONCLUSIONS

While different statistical distributions are used to correlate data on the drop size in normal fluids, such as the Nukiyama-Tanasawa law \[7\] \[21\], which is arrived at through assumptions on droplet formation, a theory for the distribution in the droplet size in a quantum fluid such as a BEC needs to be developed. We also need new experiments to study BEC droplets to further our understanding of the quantum mechanics.

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