GROUP INVARIANT SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $M$ be a complete Riemannian manifold and $G$ a Lie subgroup of the isometry group of $M$ acting freely and properly on $M$. We study the Dirichlet Problem

$$\begin{cases}
\text{div} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) = 0 \text{ in } \Omega \\
u|_{\partial\Omega} = \varphi
\end{cases}$$

where $\Omega$ is a $G$–invariant domain of $C^{2,\alpha}$ class in $M$ and $\varphi \in C^{2,\alpha}(\partial\Omega)$ a $G$–invariant function. Two classical PDE’s are included in this family: the $p$–Laplacian ($a(s) = s^{p-1}$, $p > 1$) and the minimal surface equation ($a(s) = s/\sqrt{1+s^2}$). Our motivation, by using the concept of Riemannian submersion, is to present a method in studying $G$-invariant solutions for noncompact Lie groups which allows the reduction of the Dirichlet problem on unbounded domains to one on bounded domains.

1. INTRODUCTION

Let $M$ be a complete Riemannian manifold and $G$ a Lie subgroup of the isometry group of $M$ acting freely and properly on $M$. In this paper we study the Dirichlet Problem (DP)

$$\begin{cases}
\text{div} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) = 0 \text{ in } \Omega \\
u|_{\partial\Omega} = \varphi
\end{cases}$$

where $\Omega$ is a $G$–invariant domain of $C^{2,\alpha}$ class in $M$ with non empty boundary and $\varphi \in C^{2,\alpha}(\partial\Omega)$ a $G$–invariant function. We require, as minimal conditions, that

$$a \in C^0([0, \infty]) \cap C^1([0, \infty]), \quad a(s) > 0, \quad a'(s) > 0$$

for $s > 0$. Two classical PDE’s are included in this family: the $p$–Laplacian ($a(s) = s^{p-1}$, $p > 1$) and the minimal surface equation ($a(s) = s/\sqrt{1+s^2}$). The main motivation in studying problem (Ω) is to reduce, by considering non compact Lie groups, the DP in unbounded domains of $M$ to bounded domains in the quotient space $M/G$. One should mention that the existence and uniqueness of solutions to the Dirichlet problem for the minimal surface equation on unbounded domains has a history which goes back to the investigations of J. C. C. Nitsche on the so called exterior Dirichlet problem.

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Several authors continued Nitsche’s study in the Euclidean space [7], [8], [9], [12] and in Riemannian spaces in [3]. Existence and uniqueness for more general unbounded domains were studied in [4], [1], [14] in the Euclidean space and in [15] in the Riemannian setting.

To state our main results we need to remark some facts and to introduce some terminology. We first observe that the assumptions on the action of $G$ on $M$ guarantee that the orbit space $M/G = \{ G(p) \mid p \in M \}$, where $G(p) = \{ g(p) \mid g \in G \}$, $p \in M$, is a differentiable manifold with respect to the quotient topology, and the projection $\pi : M \to M/G$ is a submersion. Since the action of $G$ on $M$ is by isometries we may, and we will, consider in $M/G$ the Riemannian metric such that $\pi$ becomes a Riemannian submersion.

Regarding the PDE in (1), the $p$–Laplace and the minimal surface equation are representatives of two classes of PDE’s which are distinguished as follows (see also [11]). The PDE in (1) may be written in the equivalent form

$$\|\nabla u\|^2 \Delta u + \left( \frac{\|\nabla u\| a'(\|\nabla u\|)}{a(\|\nabla u\|)} - 1 \right) \nabla^2 u (\nabla u, \nabla u) = 0$$

where $\Delta$ and $\nabla^2$ denote the Laplacian and the Hessian. The quadratic form

$$q(\xi, \xi) = \|\nabla u\|^2 |\xi|^2 + b(\|\nabla u\|) \langle \xi, \nabla u \rangle^2$$

associated with (2), where

$$b(s) = \frac{sa'(s)}{a(s)} - 1,$$

has the eigenvalue

$$\beta = \|\nabla u\|^2 (1 + b(\|\nabla u\|))$$

in the direction of $\nabla u$ and the maximal eigenvalue

$$\Lambda = \|\nabla u\|^2 \max \{1, 1 + b(\|\nabla u\|)\}.$$

We may easily see that

$$\frac{\beta}{\Lambda} = 1 + b^-$$

where $b^- = \min \{b, 0\}$. We consider the following two possibilities, which include, respectively, the $p$–Laplacian and the minimal surface equation:

- **Condition I** *Mild decay of the eigenvalue ratio:*
  
  $$(1 + b^-(s)) s^2 \geq g(s), \ s \geq s_0 > 0$$

  where $g$ is non decreasing and
  
  $$\int_{s_0}^{\infty} \frac{g(s)}{s^2} ds = +\infty$$

- **Condition II** *Strong decay of the eigenvalue ratio:*
  
  $$(1 + b^-(s)) s^2 \geq g(s), \ s \geq s_0 > 0$$
where \( g \) is non increasing and
\[
\int_{s_0}^\infty \frac{g(s)}{s} \, ds = +\infty.
\]

The MDER case was introduced by James Serrin in [13] as regularly elliptic equations. We observe that in both MDER and SDER classes the equations can be singular or degenerated. A typical example, which occurs in the MDER class, is the \( p \)-Laplacian PDE (it is singular if \( 1 < p < 2 \) and degenerated if \( p > 2 \)).

In this paper we consider the same class of equations as in [11], namely:

Writing
\[
a(s) = s^{p-1}A(s), \quad s \geq 0,
\]
for some \( p > 1 \) we require that
\[
(5) \quad A \in C^2([0, \infty[), \quad A(s) > 0 \text{ for } s \geq 0
\]
and that
\[
(6) \quad \min\{1, p-1\} + \frac{sA'(s)}{A(s)} > 0
\]
for all \( s \geq 0 \). Note that (6) implies that \( a'(s) > 0 \) for \( s > 0 \).

We call \( p = 2 \) the regular case to which the classical theory of elliptic differential equations can be applied. The study of the DP in the nonregular case \( p \neq 2 \) can be reduced to the regular one by a perturbation technique [11]. The minimal surface equation is regular but not the \( p \)-Laplacian if \( p \neq 2 \), as it is easy to see.

Depending on the case which is being investigated extra conditions have to be required on the PDE. They are:

- **Condition III** There are \( \beta > 0 \) and a function \( h : [0, \infty[ \to \mathbb{R} \) with \( h(s) \to \infty \) (\( s \to \infty \)) such that
  \[
  (b(s) + 1 - \beta b'(s) s^2) s^2 \geq h(s), \quad s \in [0, \infty[.
  \]

- **Condition IV** there are positive numbers \( \alpha \) and \( s_0 \) such that
  \[
  (-b'(s)s - (b(s) + 1)) s^2 \geq \alpha, \quad s \geq s_0
  \]

To state our first result in the MDER case, for smooth boundary data, we recall that \( u \in C^1(\Omega) \cap C^0(\overline{\Omega}) \) is a weak solution of (1) if \( u|_{\partial\Omega} = \varphi \) and
\[
\int_{\Omega} a(\|\nabla u\|) (\nabla u, \nabla g) \omega = 0
\]
for all \( g \in C^\infty(\overline{\Omega}) \) with compact support in \( \Omega \), where \( \omega \) is the volume form of \( M \). If the PDE is regular then regularity theory implies that a weak solution \( u \) is in \( C^{2,\alpha}(\overline{\Omega}) \) and satisfies (1) in the classical sense.
**Theorem 1** (the MDER case for smooth boundary data). Let $M$ be a complete Riemannian manifold and $G$ a Lie subgroup of the isometry group of $M$ acting freely and properly on $M$. Let $\Omega$ be a $G$–invariant domain of $C^{2,\alpha}$ class. Assume that $\pi(\Omega)$ is bounded in $M/G$ and that Conditions I and III are satisfied. Then the Dirichlet problem (1) has an unique $G$–invariant weak solution $u \in C^1(\overline{\Omega})$ for any $G$–invariant boundary data $\varphi$ of $C^{2,\alpha}$ class. If $p = 2$ the solution is of $C^{2,\alpha}$ class in $\Omega$ and hence a classical solution.

Notice that Conditions I and III are fulfilled if $b(s) + 1 = cs^m$ for some $c > 0$, $m \geq 0$. In particular, Theorem 1 holds for the $p$–Laplace equation where $b(s) = p - 2$ ($p > 1$).

Differently of the mild decay eigenvalue ratio, the solvability of the Dirichlet problem in the strong decay case requires the usual mean convexity of the domain.

**Theorem 2** (the SDER case for smooth boundary data). Let $M$ be a complete Riemannian manifold and let $G$ be a Lie subgroup of the isometry group of $M$ acting freely and properly on $M$. Let $\Omega \subset M$ be a $G$–invariant $C^{2,\alpha}$ domain with bounded projection on $M/G$. Assume that Conditions II and IV are satisfied and that the mean curvature of $\partial \Omega$ with respect to the interior normal vector of $\partial \Omega$ as well as of the inner parallel hypersurfaces of $\partial \Omega$ in some neighborhood of $\partial \Omega$ is nonnegative. Then the Dirichlet problem (1) has an unique weak $G$–invariant solution for any $G$–invariant boundary data $\varphi \in C^{2,\alpha}(\partial \Omega)$. If $p = 2$ the solution is of $C^{2,\alpha}$ class in $\overline{\Omega}$ and hence a classical solution.

Since $\Omega$ is invariant by $G$ and has compact projection on $M/G$, the mean convexity of the parallel hypersurfaces of $\partial \Omega$ holds in a neighborhood of $\partial \Omega$ if one requires $\partial \Omega$ to be strictly mean convex. It also holds if $\partial \Omega$ is only mean convex and $M$ has nonnegative Ricci curvature in an uniform neighborhood of $\partial \Omega$.

We conclude the paper with two examples to illustrate the power of Theorem 2. In the first one we consider the Dirichlet problem for the minimal surface equation on $\mathbb{R}^3$ which invariance under helicoidal motions, in the second one the asymptotic Dirichlet problem on $\mathbb{H}^n$ with invariance under transvections.

2. **A new PDE**

Let $M$ be a complete Riemannian manifold and let $G$ be a Lie subgroup of the isometry group of $M$. Assume that $G$ acts freely and properly on $M$. Given $p \in M$ let $G(p) = \{g(p) \mid g \in G\}$ be the orbit of $G$ through $p$. Then the orbit space

$$M/G := \{G(p) \mid p \in M\}$$
is a differentiable manifold with the quotient topology and the projection \( \pi : M \to M/G \) is a submersion. We consider in \( M/G \) the Riemannian metric such that \( \pi \) becomes a Riemannian submersion.

We denote by \( \vec{H}_G \) the mean curvature vector of the orbits of \( G \) that is, 
\[
\vec{H}_G = \sum_{i=1}^{k} (\nabla_{E_i} E_i) ^\perp,
\]
where \( \{ E_i \} \) is a local orthonormal frame tangent to a orbit of \( G \). Note that \( \vec{H}_G \) is \( G \)-invariant, \( g_* \vec{H}_G = \vec{H}_G \circ g \) for all \( g \in G \). Then it projects into a vector field in \( M/G \) which we denote by \( J \).

We denote by \( \nabla \) the gradient and also the Riemannian connections in \( M \) and \( M/G \). The meaning of the notation will be clear from the context.

If \( X \) is a vector field in \( M/G \) we denote by \( X \) the vector field in \( M \) determined by the horizontal lift of \( X \) to \( M \) namely, \( \pi^* X = X \circ \pi \) and \( X(p) \in T_p G(p) ^\perp \) for all \( p \in M \).

**Proposition 3.** Let \( \Omega \) be a \( G \)-invariant domain of \( C^2 \) class in \( M \), \( u \in C^2 (\Omega) \cap C^0 (\overline{\Omega}) \) and \( \varphi \in C^0 (\partial \Omega) \) \( G \)-invariant functions, \( \Lambda = \pi(\Omega) \subset M/G \), \( v \in C^2 (\Lambda) \cap C^0 (\overline{\Lambda}) \) and \( \psi \in C^0 (\partial \Lambda) \) such that \( u = v \circ \pi \), \( \varphi = \psi \circ \pi \). Then
\[
a (\| \nabla u \|) / \| \nabla u \| = a (\| \nabla v \|) / \| \nabla v \| \circ \pi
\]
and \( u \) is a solution of the DP
\[
\begin{align*}
\text{div}_M \left( a (\| \nabla u \|) \nabla u \right) = 0 \quad & \text{in } \Omega \\
u |_{\partial \Omega} = \varphi
\end{align*}
\]
if and only if \( v \) is a solution of the DP
\[
\begin{align*}
\text{div}_{M/G} \left( a (\| \nabla v \|) \nabla v \right) - a (\| \nabla v \|) \langle \nabla v, J \rangle = 0 \quad & \text{in } \Lambda \\
v |_{\partial \Lambda} = \psi.
\end{align*}
\]

**Proof.** Assume that \( n = \dim M \) and \( k = \dim G \). Given \( p \in M \) let \( E_1, \ldots, E_m \) be a local orthonormal frame in a neighborhood of \( p \) which is orthogonal to the orbits of \( G \), \( m = n - k \), and let \( X_1, \ldots, X_k \) be Killing fields determined by \( G \) which are orthonormal at \( p \).

Using that \( E_i = F_i \) where \( F_i, i = 1, \ldots, n - 1 \), is an local orthonormal frame around \( \pi(p) \) in \( M/G \) we obtain \( \nabla u = \nabla v \). It follows that the function \( a (\| \nabla u \|) / \| \nabla u \| \) is invariant by \( G \) and then it is well defined in \( M/G \) and we have
\[
a (\| \nabla u \|) / \| \nabla u \| = a (\| \nabla v \|) / \| \nabla v \| \circ \pi.
\]

Moreover, at \( p \),
\[
\vec{H}_G(p) = \sum_{i=1}^{k} (\nabla_{X_i} X_i) ^\perp (p)
\]
and
\[
\text{div}_M \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) = \sum_{i=1}^{m} \left\langle \nabla_{E_i} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) , E_i \right\rangle + \sum_{i=1}^{k} \left\langle \nabla_{X_i} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) , X_i \right\rangle.
\]

Also,
\[
\left\langle \nabla_{E_i} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) , E_i \right\rangle = E_i \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \right) \langle \nabla u, E_i \rangle + \frac{a(\|\nabla u\|)}{\|\nabla u\|} (\nabla_{E_i} \nabla u, E_i) = F_i \left( \frac{a(\|\nabla v\|)}{\|\nabla v\|} \right) (\nabla v, F_i) \circ \pi + \frac{a(\|\nabla v\|)}{\|\nabla v\|} \circ \pi \left\langle \nabla_{F_i} \nabla u, F_i \right\rangle.
\]

Using O'Neal's formula for Riemannian submersions ([2], exercises of Ch. 8) namely,
\[(8) \quad \nabla_X Y = \nabla_X Y + \frac{1}{2} [X, Y]^V,\]
where \(X, Y\) are vector fields on \(M/G\), \(\overline{X}\) and \(\overline{Y}\) their horizontal lift in \(M\) and \(V\) the orthogonal projection on \(TG\), we obtain
\[
\left\langle \nabla_{F_i} \nabla u, F_i \right\rangle = \langle \nabla_{F_i} \nabla v, F_i \rangle = \langle \nabla_{F_i} \nabla v, F_i \rangle = \langle \nabla_{F_i} \nabla v, F_i \rangle \circ \pi
\]
and hence
\[
\sum_{i=1}^{m} \left\langle \nabla_{E_i} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) , E_i \right\rangle = \sum_{i=1}^{m} \left\{ F_i \left( \frac{a(\|\nabla v\|)}{\|\nabla v\|} \right) \langle \nabla v, F_i \rangle \circ \pi + \left( \frac{a(\|\nabla v\|)}{\|\nabla v\|} \sum_{i=1}^{m} \langle \nabla_{F_i} \nabla v, F_i \rangle \right) \circ \pi \right\} = \text{div}_{M/G} \left( \frac{a(\|\nabla v\|)}{\|\nabla v\|} \nabla v \right) \circ \pi.
\]

Since \(\langle \nabla u, X_i \rangle = 0\), \(1 \leq i \leq k\), it follows that, at \(p\),
\[
\sum_{i=1}^{k} \left\langle \nabla_{X_i} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) , X_i \right\rangle = \frac{a(\|\nabla u\|)}{\|\nabla u\|} \sum_{i=1}^{k} \langle \nabla_{X_i} \nabla u, X_i \rangle = \frac{a(\|\nabla u\|)}{\|\nabla u\|} \sum_{i=1}^{k} \langle \nabla_{X_i} \nabla u, X_i \rangle = \frac{a(\|\nabla v\|)}{\|\nabla v\|} \left\langle \nabla v, J \right\rangle \circ \pi
\]
which concludes the proof of the proposition. □

We use the notations of the proposition above to prove a lemma to be used in the proof of Theorem 2:

**Lemma 4.** Under the same hypothesis of Proposition 3, denote by $H_{\partial \Omega}$ and $H_{\partial \Lambda}$ the non normalized mean curvature of $\partial \Omega$ and $\partial \Lambda$ with respect to the unit normal vector fields pointing to the interior of the domains. Then

$$H_{\partial \Omega} = H_{\partial \Lambda} \circ \pi + \left\langle \vec{H}_G, \nu \right\rangle$$

where $\nu$ is the unit normal vector field along $\partial \Omega$ pointing to $\Omega$.

**Proof.** Let $p \in \partial \Omega$ be given. Let $F_1, \ldots, F_{m-1}$ be a local orthonormal frame in a neighborhood of $\pi(p)$, tangent to $\partial \Lambda$. Let $E_i = \mathcal{F}_i$ be the horizontal lift of $F_i$ to $M$. Let $X_1, \ldots, X_k$ be Killing fields determined by $G$ which are orthonormal at $p$. Let $\nu$ be the unit normal vector field orthogonal to $\partial \Omega$ pointing to $\Omega$. Since clearly $\nu$ is invariant by $G$ and horizontal, it projects into the unit normal vector field $\eta$ orthogonal to $\partial \Lambda$ pointing to $\Lambda$. We then have, at $p$, using (8),

$$H_{\partial \Omega} = \sum_{i=1}^{m-1} \left\langle \nabla E_i, E_i, \nu \right\rangle + \sum_{i=1}^k \left\langle \nabla X_i, X_i, \nu \right\rangle$$

$$= \sum_{i=1}^{m-1} \left\langle \nabla \mathcal{F}_i, \mathcal{F}_i, \eta \right\rangle + \sum_{i=1}^k \left\langle (\nabla X_i)^\perp, \nu \right\rangle$$

$$= \sum_{i=1}^{m-1} \left\langle \nabla \mathcal{F}_i, \mathcal{F}_i, \eta \right\rangle \circ \pi + \sum_{i=1}^k \left\langle \vec{H}_G, \nu \right\rangle$$

$$= H_{\partial \Lambda} \circ \pi + \sum_{i=1}^k \left\langle \vec{H}_G, \nu \right\rangle$$

proving the lemma. □

2.1. **Remark.** It is easy to see that if $\Omega$ is a bounded $C^1$ domain then the PDE in (1) is the Euler-Lagrange equation of the functional

$$F(u) = \int_{\Omega} \phi(\|\nabla u\|), \quad u \in C^1(\overline{\Omega}),$$

where $\phi' = a$.

Let us now assume that $G$ is compact, $\Omega$ a $G$–invariant domain and $u \in C^1(\overline{\Omega})$ a $G$–invariant function. Then, setting $\Lambda = \pi(\Omega)$, defining $v \in C^1(\overline{\Lambda})$ such that $u = v \circ \pi$ and denoting by $\mathcal{H}^s$ the $s$–dimensional
Hausdorff measure of $M$ we have, by the coarea formula,

$$F(u) = \int_{x \in \Lambda} \left( \int_{\pi^{-1}(x)} \frac{1}{\|\text{Jac}(\pi)\|} \phi(\|\nabla u\|) d\mathcal{H}^k \right) d\mathcal{H}^m$$

$$= \int_{x \in \Lambda} \left( \int_{\pi^{-1}(x)} \phi(\|\nabla v(x)\|) d\mathcal{H}^k \right) d\mathcal{H}^m$$

$$= \int_{x \in \Lambda} \phi(\|\nabla v(x)\|) \left( \int_{\pi^{-1}(x)} d\mathcal{H}^k \right) d\mathcal{H}^m$$

$$= \int_{x \in \Lambda} \phi(\|\nabla v(x)\|) \text{Vol}(\pi^{-1}(x)) d\mathcal{H}^m.$$  

where we have used that $\|\text{Jac}(\pi)\| = 1$ as it is easy to see. The last integral defines a functional on $M/G$ namely

$$F_G(v) = \int_{\Lambda} V \phi(\|\nabla v\|) d\mathcal{H}^m, \ v \in C^1(\overline{\Lambda}),$$

where $V \in C^1(\overline{\Lambda})$ is defined by $V(x) = \text{Vol}(\pi^{-1}(x))$, $x \in \overline{\Lambda}$, and it follows that the new PDE (the PDE in (7)) is the Euler-Lagrange equation of $F_G$.

### 3. Proofs of Theorems 1 and 2

In this section we prove that the conditions for the solvability of the DP

$$\begin{cases}
\text{div}_N \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) - \frac{a(\|\nabla u\|)}{\|\nabla u\|} \langle \nabla u, J \rangle = 0 \text{ in } \Lambda \\
u|_{\partial \Lambda} = \psi,
\end{cases}$$

where $N$ is a complete Riemannian manifold, $\Lambda$ a bounded domain of $C^{2,\alpha}$ class in $N$ and $J$ is a smooth vector field on $N$ are the same as in the case that $J = 0$ with exception to the boundary condition as we explain later. We deal only with the regular case ($p = 2$). The case of nonregular equations equations are dealt with by the same perturbation technique used in [11]. For proving Theorems 1 and 2 we then take $N = M/G$, $\Lambda = \pi(\Omega)$, $\psi$ such that $\phi \circ \pi = \psi$ and apply Proposition 3.

We begin observing that the terminologies regular, MDER (Condition I) and SDER (Condition II) as well as Conditions III and IV introduced previously apply to the in PDE (9) since they depend only on the behavior of the function $a$.

Since the equation (9) is not of divergence form the comparison principle for weak solutions as in [11] is not immediately applicable. Instead, we may use the classical maximum and minimum principles for sub and supersolutions of $C^2$ class, respectively ([5], Chapter 3). On account of our assumption that $A$ is in $C^2([0, \infty[)$ one easily sees that if $v, w \in C^2(\Lambda)$ are sub and supersolutions of (9) then $w - v$ satisfies an elliptic linear differential
inequality with locally bounded coefficients to which the maximum principle applies. Hence, if
\[
\liminf_{x \to \partial \Lambda} (w - v) (x) \geq 0
\]
then \( w - v \geq 0 \) in \( \Lambda \) that is, the PDE (9) satisfies the comparison principle.
In particular, since the constant functions are solutions of (9), it follows that
if \( u \in C^2 (\Lambda) \cap C^0 (\overline{\Lambda}) \) and \( u|\partial \Lambda = \psi \) then
\[
\sup_{\Lambda} |u| \leq \sup_{\partial \Lambda} |\psi|.
\]

3.1. **Boundary gradient estimates. Barriers.** In this section we obtain estimates of the norm of the gradient of a solution at the boundary of the domain. We use the technique and several calculations done in [11].

We first observe that the PDE in (9) is equivalent to the PDE
\[
Q_J [w] := \Delta u + b (\|\nabla u\|) \nabla^2 u \left( \frac{\nabla u}{\|\nabla u\|}, \frac{\nabla u}{\|\nabla u\|} \right) - \langle \nabla u, J \rangle = 0.
\]

Assume that \( \psi \in C^{2,\alpha} (\overline{\Lambda}) \) and let \( \delta_0 > 0 \) be such that \( d (x) := d (x, \partial \Lambda) \), \( x \in \overline{\Lambda} \), is \( C^2 \) in the strip
\[
\overline{\Lambda}_{\delta_0} := \{ x \in \overline{\Lambda} \mid d (x) \leq \delta_0 \}.
\]
The barrier is of the form \( w = \psi + f (d) \) with \( f (0) = 0 \), \( f \in C^2 ([0, \infty[) \). A computation gives
\[
Q_J [w] = L_w g + f' L_w d + f'' \left( 1 + b \left( \frac{\nabla d}{\|\nabla w\|} \right)^2 \right)
\]
where \( L \) is the linear differential operator
\[
L_w v = \Delta v + b (\|\nabla w\|) \nabla^2 v \left( \frac{\nabla w}{\|\nabla w\|}, \frac{\nabla w}{\|\nabla w\|} \right) - \langle \nabla v, J \rangle.
\]

We have
\[
|L_w v| \leq \sqrt{mB} \|\nabla^2 v\| + \|J\| \|\nabla v\|
\]
\[
\leq B \left( \sqrt{m} + \|J\| \right) (\|\nabla^2 v\| + \|\nabla v\|)
\]
where \( B = \max \{1, 1 + b\} \), \( m = \dim N \).

Setting
\[
c_1 = \max_{\overline{\Lambda}_{\delta_0}} \|\nabla \psi\|
\]
we have
\[
f' - c_1 \leq \|\nabla w\| \leq f' + c_1
\]
and hence
\[
\frac{2}{3} f' \leq \|\nabla w\| \leq \frac{4}{3} f'
\]
provided that
\[(15) \quad f' \geq \alpha \geq \max \{1, 3c_1\},\]
where the number \(\alpha\) will be appropriately chosen later on. We assume \(f'' \leq 0\) and construct a supersolution.

In the mild decay case we have
\[
1 + b \left\langle \nabla d, \frac{\nabla w}{\|\nabla w\|} \right\rangle^2 \geq (1 + b) \left\langle \nabla d, \frac{\nabla w}{\|\nabla w\|} \right\rangle^2
\]
\[
= \frac{1 + b}{f'^2} \left\langle \nabla w - \nabla \psi, \frac{\nabla w}{\|\nabla w\|} \right\rangle^2 \geq \frac{1 + b}{f'^2} (\|\nabla w\| - c_1)^2
\]
\[
\geq \frac{1 + b}{4} \|\nabla w\|^2.
\]
This implies
\[
\frac{4}{f' B} Q_J [w] \leq C + \frac{f'' b + 1}{f'^2} B \|\nabla w\|^2
\]
with
\[
C = 4 \max \Omega \delta ((\sqrt{m} + \|J\|) (1 + \|\nabla \psi\| + |\nabla^2 \psi| + |\nabla^2 d|))
\]
Now, from Condition I we obtain
\[
\frac{4}{f' B} Q [w] \leq C + \frac{f''}{f'^3} \left( \frac{2}{3} f' \right).
\]
We may now apply exactly the same calculation done in (III, pp 16 - 17) to obtain a supersolution.

In the strong decay eigenvalue ratio case we require the condition
\[
\Delta d - \langle \nabla d, J \rangle \leq 0 \text{ in } \Lambda_{b_0},
\]
With this condition we obtain
\[
f' L_w d \leq \frac{f' b}{\|w\|^2} \nabla^2 d \left( \nabla \psi + f' \nabla d, \nabla \psi + f' \nabla d \right)
\]
\[
= \frac{f'^2 b}{\|w\|^2} \left( 2\nabla^2 d (\nabla \psi, \nabla d) + \frac{1}{f'} \nabla^2 d (\nabla \psi, \nabla \psi) \right)
\]
\[
\leq \frac{9}{4} B \|\nabla^2 d\| (2c_1 + c_1^2) \leq B c_0
\]
where \(c_1\) is given by (12) and (14) is supposed to hold. It follows that
\[
\frac{4}{B} Q_J [w] \leq 4 \left( \sqrt{m} + \|J\| \right) \left( \|\nabla \psi\| + |\nabla^2 \psi| \right) + 4c_0 + \frac{f''}{f'^2} \frac{1 + b}{B} \|\nabla w\|^2.
\]
With Condition II from the strong decay we finally get
\[
\frac{4}{B} Q_J [w] \leq C + \frac{f''}{f'^2} \varphi \left( \frac{4f'}{3} \right).
\]
where the constant $C$ depends only on $m$ and
\[
\sup_{\lambda_{\delta_0}} (\|\nabla \psi\| + |\nabla^2 \psi| + \|J\| + |\nabla^2 d|).
\]

This leads to a supersolution as in ([11] pp 18 - 19). We then have

**Proposition 5.** Let $\Lambda$ be a bounded domain of class $C^2$ in $N$ and let $\delta_0 > 0$ be such that the distance $d(x, \partial \Lambda)$, $x \in \Lambda$, is $C^2$ in the strip $\bar{\Lambda}_{\delta_0} := \{ x \in \bar{\Lambda} \mid d(x) \leq \delta_0 \}$.

Let $u \in C^1(\bar{\Lambda}) \cap C^2(\Lambda)$ be solution of (10) with $\psi \in C^2(\bar{\Lambda})$. We assume that either Condition I or II of Section 1 are satisfied and in case that Condition II holds we require furthermore the existence of $0 < \delta \leq \delta_0$ such that the mean curvature $H_{\partial \Lambda_d}$ of $\partial \Lambda_d$ with respect to the interior normal vector field $\eta_d$ of $\partial \Lambda_d$, $0 < d \leq \delta$ satisfies
\[
H_{\partial \Lambda_d} \geq - \langle J, \eta_d \rangle.
\]

Then the normal derivative of $u$ on $\partial \Lambda$ can be estimated by a constant depending only on $|\psi|_{C^2(\Omega)}$, $|\nabla^2 d|$ and $\|J\|$.

### 3.2. Local and global gradient estimates.

In this section we obtain local gradient estimates of solutions of (10) and use Proposition 5 to obtain global gradient estimates of solutions of (9) with smooth boundary data.

Let $u$ be a solution of (10) of $C^3$ class. We obtain an equation for $\|\nabla u\|$ by differentiating (10) in direction $\nabla u$.

**Lemma 6.** If $u \in C^3(\Lambda)$ solves (10) then, in an orthonormal frame $E_1, \ldots, E_m$ with $E_1 = \|\nabla u\|^{-1} \nabla u$ on a neighborhood of $\Lambda$ where $\nabla u$ is non zero, $m = \dim N$, the following equality holds
\[
(b + 1) \|\nabla u\| \nabla^2 \|\nabla u\| (E_1, E_1) + \|\nabla u\| \sum_{i=2}^{m} \nabla^2 \|\nabla u\| (E_i, E_i)
\]
\[
+ b' \|\nabla u\| \nabla^2 u (E_1, E_1)^2 + b \sum_{i=2}^{m} \nabla^2 u (E_1, E_i)^2 - \sum_{i=1, j=2}^{m} \nabla^2 u (E_i, E_j)^2
\]
\[
- \text{Ric} (\nabla u, \nabla u) - \|\nabla u\| \|\nabla^2 u (E_1, J) - \|\nabla u\| (E_1, \nabla E_i, J) = 0
\]

where Ric is the Ricci tensor of $N$.

**Proof.** We write (10) in the equivalent form
\[
\|\nabla u\|^2 \Delta u + b (\|\nabla u\|) \nabla^2 u (\nabla u, \nabla u) - \|\nabla u\|^2 (\nabla u, J) = 0,
\]
differentiate this equation in direction $\nabla u$ and afterwards divide the result by $\|\nabla u\|^2$. Also using the relation
\[
\nabla^2 u (\nabla u, \nabla u) = \frac{1}{2} \langle \nabla \|\nabla u\|^2, \nabla u \rangle
\]
and Bochner’s formula
\[
\langle \nabla \Delta u, \nabla u \rangle = \frac{1}{2} \Delta \|\nabla u\|^2 - |\nabla^2 u|^2 - \text{Ric} (\nabla u, \nabla u)
\]
we thus get
\[
\frac{1}{2} \Delta \|\nabla u\|^2 - \|\nabla^2 u\|^2 - \text{Ric} (\nabla u, \nabla u) + (b' \|\nabla u\| - 2b) \|\nabla u\|^{-4} \nabla^2 u (\nabla u, \nabla u)^2 \\
+ 2 \|\nabla u\|^{-2} \nabla^2 u (\nabla u, \nabla u)^2 (\nabla u, J) + \frac{1}{2} b \|\nabla u\|^{-2} \nabla u \left\langle \nabla \|\nabla u\|^2, \nabla u \right\rangle \\
- \|\nabla u\|^{-2} \nabla u \left( \|\nabla u\|^2 (\nabla u, J) \right) = 0.
\]

As in [11], Lemma 3.5, we get for the last two terms
\[
\nabla u \left\langle \nabla \|\nabla u\|^2, \nabla u \right\rangle = \|\nabla u\|^2 \left[ \nabla^2 \|\nabla u\|^2 (E_1, E_1) + 2 \sum_{i=1}^{m} \nabla^2 u (E_1, E_i)^2 \right]
\]
and
\[
\nabla u \left( \|\nabla u\|^2 (\nabla u, J) \right) = \left\langle \nabla \|\nabla u\|, \nabla u \right\rangle (\nabla u, J) + \|\nabla u\|^2 \nabla u \left\langle \nabla u, J \right\rangle \\
= 2\nabla^2 u (\nabla u, \nabla u) \left\langle \nabla u, J \right\rangle \\
+ \|\nabla u\|^2 \left( \left\langle \nabla \nabla u, \nabla u \right\rangle (\nabla u, J) + \left\langle \nabla \nabla u, J \right\rangle \nabla u \right) \\
= \|\nabla u\|^2 \left[ 2\nabla^2 u (E_1, E_1) \left\langle \nabla u, J \right\rangle + \nabla^2 u (\nabla u, J) \\
+ \|\nabla u\|^2 (E_i, E_1) \right].
\]

This results in
\[
\frac{1}{2} \left( b + 1 \right) \nabla^2 \|\nabla u\|^2 (E_1, E_1) + \frac{1}{2} \sum_{i=2}^{m} \nabla^2 \|\nabla u\|^2 (E_i, E_i) \\
+ (b' \|\nabla u\| - b) \nabla^2 u (E_1, E_1)^2 + b \sum_{i=2}^{m} \nabla^2 u (E_1, E_i)^2 - \sum_{i,j=1}^{m} \nabla^2 u (E_i, E_j)^2 \\
- \text{Ric} (\nabla u, \nabla u) - \|\nabla u\| \nabla^2 u (E_1, J) - \|\nabla u\|^2 (E_1, \nabla E_1J) = 0.
\]

Using finally the relation
\[
\frac{1}{2} \nabla^2 \|\nabla u\|^2 (E_i, E_i) = \|\nabla u\| \nabla^2 \|\nabla u\| (E_i, E_i) + (\nabla^2 u (E_1, E_i))^2, \quad 1 \leq i \leq m,
\]
to convert the last equation into one for \(\|\nabla u\|\) instead of \(\|\nabla u\|^2\) we arrive at the equation in Lemma 6. \hfill \square

**Lemma 7.** If \(u \in C^3 (\Lambda)\) solves (10) and the function \(G(x) = g(x)f(u)F(\|\nabla u\|)\) attains a local maximum in an interior point \(y_0\) of \(\Lambda\) with \(\nabla u (y_0) \neq 0\) then, in terms of a local orthonormal basis \(E_1 := \|\nabla u\|^{-1} \nabla u, E_2, \ldots, E_m\) of \(T_{y_0}N\) we obtain, at \(y_0\), the relations

\[
(17) \quad \frac{F'}{F} \nabla^2 u (E_1, E_1) = - \frac{1}{g} \langle \nabla g, E_i \rangle - \frac{f'}{f} \langle \nabla u, E_i \rangle
\]
and

\[
0 \geq \left[ -\frac{F''}{F} + (b+1) \left( \frac{F''}{F} - \frac{F'^2}{F^2} \right) \right] \nabla^2 u(E_1, E_1)^2 + \frac{F'}{F} \| \nabla u \| \sum_{i,j \geq 2} \nabla^2 u(E_i, E_j)^2
\]

\[
+ \left[ -\frac{F' b}{F | \nabla u |} + \frac{F''}{F} - \frac{F'^2}{F^2} \right] \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2 + (b+1) \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) \| \nabla u \|^2
\]

\[
+ \frac{\| \nabla u \| F'}{F} \left( \text{Ric}(E_1, E_1) + \langle \nabla E_1, J, E_1 \rangle \right)
\]

\[
+ \frac{1}{g} \left( (b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right)
\]

\[
- \frac{1}{g} \left[ \sum_{i=1}^{n} \langle \nabla g, E_i \rangle \langle J, E_i \rangle \right] - \frac{1}{g^2} \left[ (b+1) \langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right].
\]

**Proof.** The vanishing of

\[
\nabla \ln G = \frac{1}{g} \nabla g + \frac{f'}{f} \nabla u + \frac{F'}{F} \nabla \| \nabla u \|
\]

at \( y_0 \) gives (17), see [11], Lemma 3.6 for details. Since (10) is elliptic and the matrix \( \nabla^2 \ln G (E_i, E_j) \) is nonpositive at \( y_0 \) the expression

\[
\theta := (b+1) \nabla^2 \ln G (E_1, E_1) + \sum_{i \geq 2} \nabla^2 \ln G (E_i, E_i)
\]

will be non positive at \( y_0 \). The evaluation of \( \theta \) at \( y_0 \) is now essentially the same as in [11], Lemma 3.6, the only difference being the additional term involving \( J \). This gives

\[
\theta = \frac{F'}{F \| \nabla u \|} \left\{ \| \nabla u \| (b+1) \nabla^2 \| \nabla u \| (E_1, E_1) + \| \nabla u \| \sum_{i \geq 2} \nabla^2 \| \nabla u \| (E_i, E_i) \right\}
\]

\[
+ (b+1) \left( \frac{F''}{F} - \frac{F'^2}{F^2} \right) \nabla^2 u(E_1, E_1)^2 + \left( \frac{F''}{F} - \frac{F'^2}{F^2} \right) \sum_{i \geq 2} \nabla^2 u(E_1, E_i)^2
\]

\[
+ (b+1) \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) \| \nabla u \|^2 + \frac{1}{g} \left[ (b+1) \nabla^2 g(E_1, E_1) + \sum_{i \geq 2} \nabla^2 g(E_i, E_i) \right]
\]

\[
- \frac{1}{g^2} \left[ (b+1) \langle \nabla g, E_1 \rangle^2 + \sum_{i \geq 2} \langle \nabla g, E_i \rangle^2 \right] + \frac{f'}{f} \| \nabla u \| \langle E_1, J \rangle.
\]
We now employ Lemma 6 to eliminate the second derivatives of \( \| \nabla u \| \).
Moreover, the relation
\[
\nabla^2 u (J, E_1) = -\frac{F}{F'} \frac{f'}{f} \| \nabla u \| \langle E_1, J \rangle - \frac{F}{F'} \frac{1}{g} \sum_{i=1}^{n} \langle \nabla g, E_i \rangle \langle J, E_i \rangle,
\]
which follows from (17), leads to a cancellation of the term
\[
\frac{f'}{f} \| \nabla u \| \langle E_1, J \rangle
\]
in the above expression for \( \theta \). Rearranging terms immediately gives the statement of the lemma.

\[ \square \]

We now realize that Lemma 7 coincides with Lemma 3.6 in [11] with only the following modifications: Ric \((E_1, E_1)\) has to be replaced by Ric \((E_1, E_1) + \langle \nabla E_1 J, E_1 \rangle\) and the additional term
\[
\frac{1}{g} \sum_{i=1}^{m} \langle \nabla g, E_i \rangle \langle J, E_i \rangle
\]
appears. These terms do not depend on \( u \) and have the same structure as the other terms involving \( g \). Hence, the complete analysis following Lemma 3.6 in [11] applies to the present situation and the global and local gradient estimates for the solutions of (10) hold under the same conditions as in [11] (see Theorems 3.8, 3.10, 3.12 and 3.14 in [11]). Of course, these gradient bounds now also depend on \( \| J \| \) and \( \| \nabla J \| \). We will need, in particular, local gradient estimates for the minimal surface equation in the second example of Section 4.

3.3. Proofs of Theorems 1 and 2: conclusion. Theorems 1 is consequence of Proposition 3, and the local and global gradient estimates of Section 3.2. Theorem 2 follows from Proposition 5. As to this last proposition we note that the mean convexity in \( M \) of a \( G \)-invariant domain \( \Omega \) is equivalent to the condition
\[
H_{\partial \Lambda} \geq -\langle J, \eta \rangle
\]
where \( \Lambda = \pi (\Omega) \). Indeed: at \( \partial \Omega \) we have
\[
\langle \nabla d, J \rangle \circ \pi = \langle \eta, J \rangle \circ \pi = \langle \eta, \mathcal{J} \rangle = \langle \nu, \overline{H}_G \rangle
\]
and the claim then follows from Lemma 4. As in this lemma, \( \eta \) denotes the inner unit normal of \( \Lambda \).

As already remarked, Theorem 1 and Theorem 2 are initially proved for regular equations \((p = 2)\) and then carried over to the nonregular case by the perturbation method presented in [11].
4. Two examples with the minimal surface PDE

1) The DP for the minimal surface equation on unbounded helicoidal domains of $\mathbb{R}^3$

Given $\lambda \geq 0$ let $G_\lambda = \{h_t\}_{t \in \mathbb{R}}$ be the helicoidal group of isometries of $\mathbb{R}^3$ acting as

$$h_t(x, y, z) = \left(\begin{array}{cc} \cos(\lambda t) & \sin(\lambda t) \\ -\sin(\lambda t) & \cos(\lambda t) \end{array}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + z, \quad (x, y, z) \in \mathbb{R}^3, \quad t \in \mathbb{R}.$$

Let $\gamma : I \rightarrow \mathbb{R}^2 = \{z = 0\}$ be an arc length curve and let $S$ be the surface generated by $\gamma$ under the action of the helicoidal group. A parametrization of $S$ is given by

$$\varphi(s, t) = h_t(\gamma(s)) = (x(s) \cos(\lambda t) + y(s) \sin(\lambda t), -x(s) \sin(\lambda t) + y(s) \cos(\lambda t), t)$$

$t \in \mathbb{R}$, $s \in I$. We have that $S$ is mean convex if and only if

$$\kappa(\lambda^2 t^2 + 1) + \lambda^2 (yx' - xy') \geq 0 \quad \text{(19)}$$

where $\kappa$ is the curvature of $\gamma$ and $r = \sqrt{x^2 + y^2}$. A sufficient condition for (19) is

$$\kappa \geq \frac{\lambda^2 t}{\lambda^2 t^2 + 1}.$$

Corollary 8. Consider a bounded convex $C^{2,\alpha}$ domain $\Lambda$ in $\mathbb{R}^2 = \{z = 0\} \subset \mathbb{R}^3$ and let $\psi \in C^0(\partial \Lambda)$ be given. Set $\Omega = G_\lambda(\Lambda)$ and let $\varphi \in C^0(\partial \Omega)$ be defined by $\varphi = \psi \circ \pi$. Assume that $\Omega$ is mean convex. Then the Dirichlet problem

$$\begin{cases} \text{div} \frac{\nabla u}{\sqrt{1 + ||\nabla u||^2}} = 0 \text{ in } \Omega \\ u|_{\partial \Omega} = \varphi \end{cases}$$

has a unique $G_\lambda$-invariant solution. A sufficient condition for the mean convexity of $\Omega$ is

$$\kappa \geq \frac{\lambda^2 t}{\lambda^2 t^2 + 1}$$

where $\kappa$ is the curvature of $\partial \Lambda$ and $r$ the distance function to $(0, 0, 0)$ restricted to $\partial \Lambda$.

2) The asymptotic DP in the hyperbolic space for the minimal surface equation with singularities at infinity

In the half space model

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}, \quad g_{ij} = \delta_{ij}/x_n^2,$$

for the hyperbolic space $\mathbb{H}^n$ consider the one parameter subgroup of isometries

$$\varphi_t(x) = e^t x, \quad t \in \mathbb{R}, \quad x \in \mathbb{H}^n.$$
In order to compute the mean curvature vector $\mathbf{H}$ of the orbits we choose the arc length parametrization

$$\alpha(s) = e^{x_n s} x, \ s \in \mathbb{R}, \ |x| = 1,$$

where $|\ |$ denotes the Euclidean norm. The Christoffel symbols with respect to the standard basis of $\mathbb{R}^n$ are given by

$$\Gamma^k_{ij} = 0 \text{ for } i < n, \ j < n, \ k < n,$$

$$\Gamma^k_{in} = -\frac{1}{x_n} \delta_{ik} \text{ for } k < n,$$

$$\Gamma^n_{ij} = \frac{1}{x_n} \delta_{ij} \text{ for } (i, j) \neq (n, n), \ \Gamma^n_{nn} = -\frac{1}{x_n}.$$

The mean curvature vector

$$\mathbf{H} = \nabla_{\frac{d}{ds}} \alpha'(s) = (H_1, \ldots, H_n)$$

with

$$H_k = \alpha''_k + \sum_{i,j} \Gamma^k_{ij} \alpha'_i \alpha'_j$$

is then computed as

$$H_k = -x_n^2 e^{x_n s} x_k, \ 1 \leq k \leq n - 1$$

$$H_n = x_n e^{x_n s} \sum_{i=1}^{n-1} x_i^2.$$

The map

$$x \mapsto |x|^{-1} x, \ x \in \mathbb{H}^n,$$

is seen to be a Riemannian submersion from $\mathbb{H}^n$ onto the $(n - 1)$-dimensional hyperbolic space

$$\mathcal{S} = \{ x \in \mathbb{H}^n \mid |x| = 1 \}$$

with its induced metric. Hence $\mathcal{S}$ is an isometric model for $\mathbb{H}^n/G$, $G = \{ \varphi_t \}_{t \in \mathbb{R}}$, and, since the orbits are orthogonal to $\mathcal{S}$ it follows that $\mathbf{H}$ is tangential to $\mathcal{S}$ and hence $J = \mathbf{H}|\mathcal{S}$.

One immediately sees from (20) that $\mathbf{H}$ is orthogonal to the geodesic spheres of $\mathcal{S}$ centered at $o = (0, \ldots, 0, 1) \in \mathcal{S}$ and points towards $o$.

Note that since equivalence classes of geodesic rays (classes of directions) are preserved by isometries, $G$ acts continuously on $\partial_\infty \mathbb{H}^n$. Moreover, it has two fixed points, $z_n = +\infty$ and $(0, \ldots, 0)$. Let $P : \mathbb{H}^n \to \mathcal{S}$ be the projection

$$P(p) = G(p) \cap \mathcal{S}$$

where $G(p)$ is the orbit through $p$. Note that $P$ extends continuously to a map

$$P : \partial_\infty \mathbb{H}^n \setminus \{ z_n = +\infty, (0, \ldots, 0) \} \to \partial_\infty \mathcal{S}.$$

Given $\phi \in C^0(\partial_\infty \mathcal{S})$ define $\psi \in C^0(\partial_\infty \mathbb{H}^n \setminus \{ z_n = +\infty, (0, \ldots, 0) \})$ by $\psi = \varphi \circ P$. With these notations and remarks, we have:
Corollary 9. There is one and only one $G$–invariant solution

$$u \in C^\infty (\mathbb{H}^n) \cap C^0 (\partial_\infty \mathbb{H}^n \setminus \{z_n = +\infty, (0, \ldots, 0)\})$$

of the minimal surface equation on $\mathbb{H}^n$ such that

$$u|_{\partial_\infty \mathbb{H}^n \setminus \{z_n = +\infty, (0, \ldots, 0)\}} = \psi.$$  

Proof. Denote also by $\varphi$ a $C^0$ extension of $\varphi \in C^0 (\partial_\infty \mathcal{S})$ to $\overline{\mathcal{S}}$. Given $k > 0$, from the results of Section 3 used to prove Theorem 2, we see that there is a solution $u_k \in C^\infty (B_k) \cap C^0 (\overline{B_k})$ of the minimal surface equation on the geodesic ball $B_k$ of $\mathcal{S}$ centered at $o$ and with radius $k$ such that $u_k|_{\partial B_k} = \varphi|_{\partial B_k}$ if inequality (18) is satisfied. But this is the case since $H_{\partial B_k} > 0$ and, from what we have seen above

$$-\langle \eta, J \rangle = \langle \nabla r, \overline{H} \rangle < 0.$$

From the diagonal method and the local gradient estimates (as remarked at the end of Section 3.2), it follows that $u_k$ contains a subsequence converging uniformly on compact subsets of $\mathcal{S}$ to a solution $u \in C^2 (\mathcal{S})$ of the PDE (10). Regularity theory gives $u \in C^\infty (\mathcal{S})$. To prove that $u \in C^0 (\overline{\mathcal{S}})$ and $u|_{\partial_\infty \mathcal{S}} = \varphi|_{\partial_\infty \mathcal{S}}$ it is enough to prove that the PDE (9) is regular at infinity (see Section 6.1 of [11]).

Given $p \in \partial_\infty \mathcal{S}$ and a neighborhood $W \subset \partial_\infty \mathcal{S}$ of $p$, let $T$ be a totally geodesic hypersurface of $\mathbb{H}^n$ such that $\partial_\infty T \subset W$. Let $\Omega$ be the connected component of $\mathbb{H}^n \setminus T$ such that $W \subset \partial_\infty \Omega$. We may assume wlg that $o \notin \Omega$. We shall construct a barrier in $\Omega$ of the forma $w = g(s)$ where $s : \Omega \to \mathbb{R}$ is the distance to $T = \partial \Omega$. Below we shall show that condition

$$\langle \nabla s, J \rangle < 0 \tag{21}$$

is satisfied and so it turns out that $w$ will be a special case of Lemma 6.2 of [11]. For the convenience of the reader we repeat the computation in our special case, also leading to an explicit formula.

We may compute $\Delta s = (n - 2) \tanh s$. Then, for $w = g(s)$ with $g'(s) < 0$ we have

$$\text{div} \left( a \frac{\|\nabla w\|}{\|\nabla w\|} \nabla w \right) = -a \left( -g'(s) \right) \Delta s + a \left( -g'(s) \right) g''(s)$$

$$= - (n - 2) a \left( -g'(s) \right) \tanh s + a \left( -g'(s) \right) g''(s)$$

and, because of (21), $w$ will be a supersolution of our equation if $g$ solves the equation

$$0 = \tanh s - \frac{d'}{a} \left( -g'(s) \right) g''(s)$$

$$= \frac{d}{ds} \left( (n - 2) \ln \cosh s + \ln a \left( -g'(s) \right) \right)$$

leading to

$$(\cosh s)^{n-2} a \left( -g'(s) \right) = c = \text{constant}.$$
With 
\[ a(v) = \frac{v}{\sqrt{1 + v^2}} \]
we thus get
\[ g(s) = c \int_s^\infty \frac{(\cosh t)^{n-2}}{\sqrt{1 - c^2 (\cosh t)^4 - 2n}} dt \]
where \(0 < c < 1\) and obviously \(g(0) \to +\infty\) for \(c \to 1\). We show next that (21) is satisfied.

Let \(p \in \Omega\) and let \(\Delta\) be the totally geodesic hypersurface of \(\mathbb{H}^n\) through \(p\) orthogonal to \(\nabla s\) at \(p\). Let \(\alpha\) be the level hypersurface \(s^{-1}(s(p))\). Since \(\alpha\) is convex towards the connected component of \(S\setminus\alpha\) that contains \(T\) it follows that \(\Delta\) is contained in the closure of the connected component of \(S\setminus\alpha\) which does not contain \(T\). The unit normal vector \(\eta\) along \(\Delta\) such that \(\eta(p) = \nabla s(p)\) points to the connected component of \(S\setminus\Delta\) which does not contain \(T\). Let \(p_0 \in \Delta\) be such that \(d(o, p_0) = d(o, \Delta)\). Then the geodesic sphere centered at \(o\) and passing through \(p_0\) is tangent to \(\Delta\) at \(p_0\). It follows that \(\langle \eta(p_0), J(p_0) \rangle = -\|J(p_0)\| < 0\), because \(J(p_0) = \tilde{H}(p_0)\) and, as we have seen above, \(\tilde{H}(p_0)\) is orthogonal to this sphere and points to its center. We then have that \(\langle \eta, J \rangle\) is everywhere negative otherwise it would exist a point where \(J\) and \(\Delta\) would be tangent. By uniqueness of the geodesics, a geodesic of \(\Delta\) would coincide with a geodesic issuing from \(o\), contradiction! Therefore the PDE is regular at infinity and thus \(u \in C^\infty(\mathbb{H}^n) \cap C^0(\partial_\infty \mathbb{H}^n \setminus \{z_n = +\infty, (0, \ldots, 0)\})\) and 
\[ u|_{\partial_\infty \mathbb{H}^n \setminus \{z_n = +\infty, (0, \ldots, 0)\}} = \psi, \]
proving the corollary. \(\square\)

**Remarks**

a) It is clear that these examples hold for the family of PDE’s (1) under the conditions of Theorems 1 and 2. For simplicity we consider only the more interesting case of the minimal surface equation.

b) The one parameter subgroup of isometries of the hyperbolic space considered in the second example above is a particular case of transvections along a geodesic, which is defined in any symmetric space (see [6]). Thus, it makes sense to investigate a possible extension of Corollary 8 to non compact symmetric spaces, specially on rank 1 symmetric spaces since these have strictly negative curvature.

c) A well known problem which is being investigated in the last decades is the existence or not of non constant bounded harmonic functions, and more recently bounded non constant solutions of the \(p\)-Laplace PDE and the minimal surface equation on a Hadamard manifold. A way of proving existence is by solving the asymptotic Dirichlet problem for non constant continuous boundary data at infinity. However, in a Hadamard manifold which is a Riemannian product \(N = M \times \mathbb{R}\), since the sectional curvature in vertical planes are zero, it is likely true that any solution which
extends continuously to $\partial_\infty N$ is necessarily constant. It is a trivial remark that bounded nonconstant $\mathbb{R}$–invariant solutions of the the minimal surface equation which are continuous on $\overline{N}$ except for two points in $\partial_\infty N$ exist when $M$ is a 2–dimensional Hadamard manifold with curvature bounded by above by a negative constant.

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