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Carleman estimates and null controllability of a class of singular parabolic equations

https://doi.org/10.1515/anona-2016-0266
Received December 12, 2016; revised January 18, 2018; accepted February 2, 2018

Abstract: In this paper, we consider control systems governed by a class of semilinear parabolic equations, which are singular at the boundary and possess singular convection and reaction terms. The systems are shown to be null controllable by establishing Carleman estimates, observability inequalities and energy estimates for solutions to linearized equations.

Keywords: Carleman estimate, null controllability, singular equation

MSC 2010: 93B05, 93C20, 35K67

1 Introduction

This paper concerns the null controllability of the system governed by the following first initial-boundary value problem:

\[ x^\alpha u_t - u_{xx} + c(x, t)u = h(x, t)\chi_\omega, \quad (x, t) \in Q_T = (0, 1) \times (0, T), \]
\[ u(0, t) = u(1, t) = 0, \quad t \in (0, T), \]
\[ u(x, 0) = u_0(x), \quad x \in (0, 1), \]

where \( \alpha > 0, T > 0, c \in L^\infty(Q_T), \omega = (x_0, x_1) (0 < x_0 < x_1 < 1) \) is the control region and \( \chi_\omega \) is the characteristic function of \( \omega, h \in L^2(Q_T) \) is the control function, and \( u_0 \in \mathcal{S}_a \) with

\[ \mathcal{S}_a = \{ \zeta \text{ is a measurable function in} (0, 1) : x^{\alpha/2} \zeta \in L^2(0, 1) \}. \]

System (1.1)–(1.3) is said to be null controllable if for each \( u_0 \in \mathcal{S}_a \) there exists \( h \in L^2(Q_T) \) such that its solution \( u \) satisfies \( u(\cdot, T)|_{(0,1)} = 0 \). It is noted that (1.1) is singular at the boundary \( x = 0 \). Equations with such a boundary singularity arise in some physical problems such as the propagation of a thermal wave in an inhomogeneous medium [32, 36] and the Ockendon model for the flow in a channel of a fluid whose viscosity is temperature-dependent [21, 35]. By the coordinate transformation

\[ y = x^{\alpha+1}, \quad 0 < x < 1, \]
problem (1.1)–(1.3) can be transformed into
\begin{align}
\dot{u}_t - (\alpha + 1)^2 (y^{\alpha/(\alpha + 1)} \dot{u}_y)_y + y^{-\alpha/(\alpha + 1)} \dot{c}(y, t) \dot{u} = y^{-\alpha/(\alpha + 1)} \dot{h}(y, t) \chi_{\dot{w}}, & \quad (y, t) \in Q_T, \\
\dot{u}(0, t) = \dot{u}(1, t) = 0, & \quad t \in (0, T), \\
\dot{u}(y, 0) = u_0(y^{1/(\alpha + 1)}), & \quad y \in (0, 1),
\end{align}
where \( \dot{w} = (x_0^{\alpha + 1}, x_1^{\alpha + 1}) \) and
\begin{align}
\dot{u}(y, t) = u(y^{1/(\alpha + 1)}, t), & \quad \dot{c}(y, t) = c(y^{1/(\alpha + 1)}, t), \\
\dot{h}(y, t) = h(y^{1/(\alpha + 1)}, t), & \quad (y, t) \in Q_T.
\end{align}

Here, (1.4) is degenerate at the boundary \( y = 0 \), and its reaction term is singular.

Controllability theory, containing approximate controllability and exact controllability, has been widely investigated for semilinear uniformly parabolic equations over the last forty years and there have been a great number of results (see for instance [3, 18–20, 31] and the references therein for a detailed account). The study of the controllability for semilinear degenerate or singular parabolic equations just began about ten years ago and it is far from being completely solved although many results have been known. Parabolic equations may be not exact controllable for a general terminal datum since there is a smoothing effect for their solutions. As usual, one considers null controllability. That is to say, the zero function is chosen as the terminal datum.

There are some studies on the null controllability of systems governed by equations with boundary degeneracy, whose simple example is
\[ u_t - a_0(x^\lambda u_x)_x + c(x, t) u = h(x, t) \chi_{\dot{w}}, \quad (x, t) \in Q_T, \]
with \( a_0 > 0, \lambda > 0 \) and \( c \in L^\infty(Q_T) \). The degeneracy of (1.7) is classified as a weak one \((0 < \lambda < 1)\) and a strong one \((\lambda \geq 1)\), and different boundary conditions are prescribed for the two cases (see [2, 12, 13, 34, 39, 43]). Precisely, for each \( \lambda > 0 \) and each \( u_0 \in L^2(0, 1) \), the problem of (1.7) subject to the following boundary and initial conditions is well-posed
\begin{align}
\dot{u}(0, t) = 0 & \quad \text{if } 0 < \lambda < 1, \; \quad (x^\lambda u_x)(0, t) = 0 & \quad \text{if } \lambda \geq 1, \\
\dot{u}(1, t) = 0 & \quad t \in (0, T), \\
\dot{u}(x, 0) = u_0(x) & \quad x \in (0, 1).
\end{align}

System (1.7)–(1.10) was proved to be null controllable if \( 0 < \lambda < 2 \) (see [2, 12, 13, 34]), while not if \( \lambda \geq 2 \) (cf. [11]). Besides, Wang [39] and Cannarsa et al. [7, 10, 11] proved that it is approximately controllable in \( L^2(0, 1) \) and regional null controllable, respectively, for each \( \lambda > 0 \). Here, the approximate controllability in \( L^2(0, 1) \) means that for each \( u_0, u_d \in L^2(0, 1) \) and \( \epsilon > 0 \) there exists \( h \in L^2(Q_T) \) such that the solution \( u \) satisfies \( \|u(\cdot, T) - u_d(\cdot)\|_{L^2(0, 1)} < \epsilon \), while the regional null controllability means that for each \( u_0 \in L^2(0, 1) \) and \( \delta \in (0, 1 - x_0) \) there exists \( h \in L^2(Q_T) \) such that the solution \( u \) satisfies \( u(\cdot, T)|_{(x_0 + \delta, 1)} = 0 \). In [23, 24, 37], it was proved that the system of
\[ u_t - a_0(x^\lambda u_x)_x + \frac{c_0}{x^\beta} u = h(x, t) \chi_{\dot{w}}, \quad (x, t) \in Q_T, \]
subject to (1.8)–(1.10) is null controllable, where \( a_0 > 0, \lambda \in [0, 2), 0 < \beta < 2 - \lambda \) and \( c_0 \in \mathbb{R} \), or \( a_0 = 1, \lambda \in [0, 1) \cup (1, 2), \beta = 2 - \lambda \) and \( c_0 \geq - (1 - \lambda)^2/4 \). Since systems (1.1)–(1.3) and (1.4)–(1.6) are equivalent, choosing \( a_0 = (\alpha + 1)^2 \) and \( \lambda = \alpha/(\alpha + 1) \) in (1.7) and (1.11) shows that (1.1)–(1.3) is null controllable if \( x^{-\alpha} c \in L^\infty(Q_T) \) or \( x^{-\beta} c \) is a constant in \( Q_T \) with some \( \beta \in (-\alpha, 2) \). However, the general case that \( c \in L^\infty(Q_T) \) is unknown yet. There are also other studies on the controllability for semilinear degenerate or singular parabolic equations such as [16, 22, 40, 41] for equations with first-order terms, [8, 9, 25, 26] for equations in nondivergence form, [26–29] for equations with interior degeneracy, [4, 14, 17, 33, 38] for multi-dimensional equations and [1, 15] for coupled systems. Moreover, [5, 6] studied null controllability for heat equations with singular potentials.
The null controllability of system (1.1)–(1.3) is based on the Carleman estimate for solutions to the conjugate problem

\[
\begin{align*}
    x^a v_t + v_{xx} - c(x, t)v &= 0, \quad (x, t) \in Q_T, & (1.12) \\
    v(0, t) &= v(1, t) = 0, & t \in (0, T), \quad (1.13) \\
    v(x, T) &= v_T(x), & x \in (0, 1), \quad (1.14)
\end{align*}
\]

where \(v_T \in \mathcal{S}_d\). A Carleman estimate is a weighted inequality that relates a global (weighted) energy with a weighted local norm of the solution. Since (1.12) is singular, the reaction term cannot be controlled by the diffusion term generally in establishing the Carleman estimate for solutions to problem (1.12)–(1.14). Therefore, there should be some further restrictions on \(c_t\) or \(c_{xx}\). In the present paper, we prescribe the restriction on \(c_t\) as \(x^\alpha c_t \in L^{\infty}(Q_T)\). More generally, \(c \in L^{\infty}(Q_T)\) can be relaxed by \(x^\beta c \in L^{\infty}(Q_T)\) with some \(\beta \in [0, 2)\). Moreover, the Carleman estimate still holds if there are convection and reaction terms with suitable weights in (1.12). Precisely, we can establish the Carleman estimate of solutions to the problem of

\[
\begin{align*}
    x^a v_t + v_{xx} + \beta(x, t)v_x - c(x, t)v - x^{a/2-1}y(x, t)v &= 0, \quad (x, t) \in Q_T, \quad (1.15)
\end{align*}
\]

subject to (1.13) and (1.14), where \(b, x^\beta c, x^2 c_t, y \in L^{\infty}(Q_T)\) with some \(\beta \in [0, 2)\). In establishing the Carleman estimate, the convection term and the second reaction term can be controlled by the diffusion term thanks to their weights. Since (1.15) is singular, solutions to problem (1.15), (1.13), (1.14) are weak and it is not convenient to estimate them. Thus we consider the regularized problem

\[
\begin{align*}
    (x + \eta)^a v_t + v_{xx} + (x + \eta)^{a/2}b(x, t)v_x - c(x, t)v - (x + \eta)^{a/2-1}y(x, t)v &= 0, \quad (x, t) \in Q_T, \quad (1.16) \\
    v(0, t) &= v(1, t) = 0, & t \in (0, T), \quad (1.17) \\
    v(x, T) &= v_T(x), & x \in (0, 1), \quad (1.18)
\end{align*}
\]

and estimate its solutions uniformly with respect to \(\eta\), where \(0 < \eta < 1\). To show the idea on the choice of the weights, we use the method of undetermined functions to determine the suitable weights. By complicated and detailed estimates, we establish the local Carleman estimate, uniformly with respect to \(\eta \in (0, 1)\), near the singular point \(x = 0\). Combining this local Carleman estimate and the classical one for uniformly parabolic equations, we get the uniform Carleman estimate and thus the uniform observability inequality for solutions to problem (1.16)–(1.18), which imply the ones for solutions to problem (1.15), (1.13), (1.14) by a limit process as \(\eta \to 0^+\). Owing to the uniform observability inequality, we can prove that the linear system

\[
\begin{align*}
    (x + \eta)^a u_t - u_{xx} + (x + \eta)^{a/2}b(x, t)u_x + c(x, t)u + (x + \eta)^{a/2-1}y(x, t)u &= h(x, t)\chi_\omega, \quad (x, t) \in Q_T, \quad (1.19) \\
    u(0, t) &= u(1, t) = 0, & t \in (0, T), \quad (1.20) \\
    u(x, 0) &= u_0(x), & x \in (0, 1), \quad (1.21)
\end{align*}
\]

is null controllable and the control function is uniformly bounded by considering a family of functional minimum problems, where \(b, c, c_t, y \in L^{\infty}(Q_T)\) and \(v_T \in L^2(0, 1)\). It is noted that there is no other restriction on \(b\) and \(y\) except for \(b, y \in L^{\infty}(Q_T)\). By a fixed point argument, we can show that the semilinear system of

\[
\begin{align*}
    (x + \eta)^a u_t - u_{xx} + \bar{p}(x, t, u)x + \bar{q}(x, t, u) = h(x, t)\chi_\omega, \quad (x, t) \in Q_T, \quad (1.22)
\end{align*}
\]

subject to (1.20) and (1.21) is null controllable and the control function is uniformly bounded, where \(\bar{p}\) and \(\bar{q}\) are two measurable functions in \(Q_T \times \mathbb{R}\) such that for \((x, t) \in Q_T\) and \(u, v \in \mathbb{R}\),

\[
\begin{align*}
    \bar{p}(x, t, 0) &= 0, & |\bar{p}(x, t, u) - \bar{p}(x, t, v)| & \leq K(x + \eta)^{a/2}|u - v|, \quad (1.23) \\
    \bar{q}(x, t, 0) &= 0, & |\bar{q}(x, t, u) - \bar{q}(x, t, v) - c(x, t)(u - v)| & \leq K(x + \eta)^{a/2-1}|u - v| \quad (1.24)
\end{align*}
\]

with some \(K > 0\). Then, by the uniform estimates on the control functions and solutions of system (1.22), (1.20), (1.21) and a limit process as \(\eta \to 0^+\), we prove that the semilinear system of the singular parabolic equation

\[
\begin{align*}
    x^a u_t - u_{xx} + (p(x, t, u)x + q(x, t, u) = h(x, t)\chi_\omega, \quad (x, t) \in Q_T = (0, 1) \times (0, T), \quad (1.25)
\end{align*}
\]

subject to (1.2) and (1.3) is null controllable, where \(p\) and \(q\) are two measurable functions in \(Q_T \times \mathbb{R}\) such
that for \((x, t) \in Q_T\) and \(u, v \in \mathbb{R}\),
\[
  p(x, t, 0) = 0, \quad |p(x, t, u) - p(x, t, v)| \leq Kx^{\alpha/2}|u - v|, \quad (1.26)
\]
\[
  q(x, t, 0) = 0, \quad |q(x, t, u) - q(x, t, v) - c(x, t)(u - v)| \leq Kx^{\alpha/2-1}|u - v|. \quad (1.27)
\]
In particular, in the case \(0 < \alpha \leq 2\), the system of
\[
x^\alpha u_t - u_{xx} + g(x, t, u) = h(x, t)\eta, \quad (x, t) \in Q_T,
\]
subject to (1.2) and (1.3) is null controllable, where \(g\) is a measurable function in \(Q_T \times \mathbb{R}\)

The paper is organized as follows: In Section 2, we prove the well-posedness of the singular problems
(1.25), (1.2), (1.3) and (1.15), (1.13), (1.14) by doing energy estimates for solutions to the regularized
problems (1.19)–(1.21) and (1.16)–(1.18), respectively. Carleman estimates and observability inequalities
are proved in Section 3, where we first establish the uniform ones for solutions to the regularized problem
(1.16)–(1.18) by complicated and detailed estimates and then get the ones for solutions to the singular
problem (1.15), (1.13), (1.14) by a limit process. Subsequently, the null controllability is studied in Section 4.
Owing to the uniform observability inequality, we show that the regularized system (1.19)–(1.21) is null
controllable and the control function is uniformly bounded by considering a family of functional minimum
problems. Then it follows from a fixed point argument that the semilinear system (1.22), (1.20), (1.21) is
null controllable and the control function is uniformly bounded, which yields the null controllability of the
semilinear singular system (1.25), (1.20), (1.21) by the uniform estimates and a limit process.

## 2 Energy estimates and well-posedness

In this section, we prove the well-posedness of the singular problems (1.25), (1.2), (1.3) and (1.15), (1.13),
(1.14), which is based on uniform energy estimates for solutions to the regularized problems (1.19)–(1.21)
and (1.16)–(1.18), respectively. More generally, instead of (1.15), we consider its semilinear case
\[
x^\alpha v_t + v_{xx} + \hat{g}(x, t, v, \eta) = 0, \quad (x, t) \in Q_T, \quad (2.1)
\]
where \(\hat{g}\) is a measurable function in \(Q_T \times \mathbb{R}^2\) such that for \((x, t) \in Q_T\) and \(u, v, \eta \in \mathbb{R}\),
\[
  \hat{g}(x, t, 0, 0) = 0, \quad |\hat{g}(x, t, u, \eta) - \hat{g}(x, t, v, \eta) - c(x, t)(u - v)| \leq K(x^{\alpha/2-1}|u - v| + x^{\alpha/2}|\eta|). \quad (2.2)
\]

**Definition 2.1.** A function \(u \in L^2(0, T; H^1_0(0, 1))\) with \(x^{\alpha/2}u \in L^\infty(0, T; L^2(0, 1))\) is said to be a solution to problem (1.25), (1.2), (1.3) if
\[
  \iint_{Q_T} \left( -x^\alpha u\zeta_t + u_x \zeta_x - p(x, t, u)\zeta_x + q(x, t, u)\zeta \right) dx dt = \int_0^1 h_x\zeta dx dt + \int_0^1 x^\alpha u_0(x)\zeta(x, 0) dx
\]
for each \(\zeta \in C^1(\overline{Q_T})\) with \(\zeta(\cdot, T)|_{(0, 1)} = 0\) and \(\zeta(0, \cdot)|_{(0, T)} = \zeta(1, \cdot)|_{(0, T)} = 0\).

**Definition 2.2.** A function \(v \in L^2(0, T; H^1_0(0, 1)) \cap H^1_{loc}(Q_T)\) with \(x^{\alpha/2}v \in L^\infty(0, T; L^2(0, 1))\) is said to be a solution to problem (2.1), (1.13), (1.14) if
\[
  \iint_{Q_T} \left( x^\alpha v\zeta_t + v_x \zeta_x - \hat{g}(x, t, v, \eta)\zeta \right) dx dt = \int_0^1 x^\alpha v_T(x)\zeta(x, T) dx
\]
for each \(\zeta \in C^1(\overline{Q_T})\) with \(\zeta(\cdot, 0)|_{(0, 1)} = 0\) and \(\zeta(0, \cdot)|_{(0, T)} = \zeta(1, \cdot)|_{(0, T)} = 0\).

**Remark 2.3.** Like the heat equation where \(\alpha = 0\), both \(x^\alpha u\) and \(x^\alpha v\) are continuous from \([0, T]\) to \(L^2(0, 1)\) with respect to the weak topology of \(L^2(0, 1)\), where \(u\) and \(v\) are solutions to problems (1.25), (1.2), (1.3)
and (1.15), (1.13), (1.14), respectively. That is to say, \(u\) and \(v\) have some continuity with respect to the time
variable so that they make sense at a time.
2.1 Energy estimates for solutions to linear problems

Since (1.25) and (1.15) are singular at the boundary \( x = 0 \), the existence of solutions to problems (1.25), (1.2), (1.3) and (1.15), (1.13), (1.14) are based on energy estimates for solutions to the regularized problems (1.19)–(1.21) and (1.16)–(1.18), respectively. To do so, we first prove the following lemma similar to the Poincaré inequality.

**Lemma 2.4.** Assume that \( 0 < \eta < 1 \) and \( u \in H^1(0, 1) \) with \( u(0) = 0 \). Then for each \( \delta \in (0, 1] \) and \( \lambda \in (0, 2) \) it holds that

\[
\int_0^\delta (x + \eta)^2 v^2(x) \, dx \leq 4 \int_0^\delta v^2(x) \, dx, \quad \int_0^\delta (x + \eta)^{-\lambda} v^2(x) \, dx \leq \frac{1}{2 - \lambda} \delta^{2-\lambda} \int_0^\delta v^2(x) \, dx.
\]

**Proof.** It follows from the Hölder inequality and the Fubini theorem that

\[
\int_0^\delta (x + \eta)^2 v^2(x) \, dx = \int_0^\delta (x + \eta)^2 \left( \int_0^x v'(y) \, dy \right)^2 \, dx
\]

\[
\leq \int_0^\delta (x + \eta)^2 \left( \int_0^x (y + \eta)^{-1/2} \, dy \right) \left( \int_0^x (y + \eta)^{1/2} v^2(y) \, dy \right) \, dx
\]

\[
\leq 2 \int_0^\delta (x + \eta)^{-3/2} \left( \int_0^x (y + \eta)^{1/2} v^2(y) \, dy \right) \, dx
\]

\[
= 2 \int_0^\delta (y + \eta)^{1/2} v^2(y) \left( \int_y^\delta (x + \eta)^{-3/2} \, dx \right) \, dy
\]

\[
\leq 4 \int_0^\delta v^2(y) \, dy.
\]

The Hölder inequality gives

\[
\int_0^\delta (x + \eta)^{-\lambda} v^2(x) \, dx = \int_0^\delta (x + \eta)^{-\lambda} \left( \int_0^x v'(y) \, dy \right)^2 \, dx
\]

\[
\leq \int_0^\delta (x + \eta)^{-\lambda} \left( \int_0^x v^2(y) \, dy \right) \, dx
\]

\[
\leq \int_0^\delta x^{1-\lambda} \, dx \int_0^\delta v^2(x) \, dx
\]

\[
= \frac{1}{2 - \lambda} \delta^{2-\lambda} \int_0^\delta v^2(y) \, dy.
\]

Consider problem (1.19)–(1.21).

**Proposition 2.5.** Assume that \( a > 0, 0 \leq \beta < 2 \) and \( \| b \|_{L^\infty(Q_T)}, \| (x + \eta)^{\beta} c \|_{L^\infty(Q_T)}, \| y \|_{L^\infty(Q_T)} \leq N \) for some \( N > 0 \).

Then for each \( h \in L^2(Q_T) \) and \( u_0 \in L^2(0, 1) \) problem (1.19)–(1.21) admits a unique solution

\[
u \in L^2(0, T; H^1_0(0, 1)) \cap L^\infty(0, T; L^2(0, 1)).
\]

Furthermore, the solution satisfies

\[
\| (x + \eta)^{\alpha/2} u \|_{L^\infty(0, T; L^2(0, 1))} + \| u \|_{L^2(Q_T)} + \| u_x \|_{L^2(Q_T)} \leq M(\| (x + \eta)^{\alpha/2} u_0 \|_{L^2(0, 1)} + \| h \|_{L^2(Q_T)})
\]  

(2.3)
and for each \( \varepsilon \in (0, T) \),
\[
\int_0^T \int_0^1 (u(x, \tau + \varepsilon) - u(x, \tau))^2 \, dx \, d\tau \leq M \varepsilon^{1/(2+\alpha)} \|\eta\|^\alpha_{L^\infty(0, 1)} \|u_0\|^2_{L^2(Q_0)} + \|h\|^2_{L^2(Q_T)},
\]
where \( M > 0 \) depends only on \( N, T, a \) and \( \beta \).

**Proof.** The classical theory for parabolic equations (cf. [30, 42]) shows that problem (1.19)–(1.21) admits a unique solution \( u \in L^2(0, T; H^1_0(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \). It suffices to prove (2.3) and (2.4). Without loss of generality, we can assume that \( u \) is a classical solution. Otherwise, one can consider the approximating problem with mollified \( b, c, y, h_\varepsilon \) and \( u_0 \) (cf. [42]). For convenience, we use \( C_i (1 \leq i \leq 7) \) to denote generic constants depending only on \( N, T, a \) and \( \beta \) in the proof.

For each \( \tau \in (0, T) \), multiplying (1.19) by \( u \) and integrating over \( Q_\tau = (0, 1) \times (0, \tau) \) by parts with (1.20), we get
\[
\int_0^1 \left( (x + \eta)^a u_\varepsilon^2(x, \tau) \right) \, dx + \int_0^\tau \int_{Q_\tau} u_\varepsilon^2 \, dx \, dt
\]
\[
= \int_0^1 \left( (x + \eta)^a u_\varepsilon^2(x) \right) \, dx + \int_0^\tau \int_{Q_\tau} h x_\varepsilon u \, dx \, dt + \int_0^\tau \int_{Q_\tau} (x + \eta)^{a/2} b u u_\varepsilon \, dx \, dt - \int_0^\tau \int_{Q_\tau} c u^2 \, dx \, dt
\]
\[
- \int_0^\tau \int_{Q_\tau} (x + \eta)^{a/2 - 1} y u^2 \, dx \, dt
\]
\[
\leq \int_0^1 \left( (x + \eta)^a u_0^2(x) \right) \, dx + \frac{1}{2} \left( 1 + 2^a b_{L^\infty(Q_T)}^2 \right) \int_0^\tau \int_{Q_\tau} u^2 \, dx \, dt + \frac{1}{2} \int_0^\tau \int_{Q_\tau} h^2 \, dx \, dt + \frac{1}{2} \int_0^\tau \int_{Q_\tau} u_\varepsilon^2 \, dx \, dt
\]
\[
+ \| (x + \eta)^\beta c_{L^\infty(Q_T)} \|_{L^\infty(Q_T)} \int_0^\tau \int_{Q_\tau} (x + \eta)^{-\beta} u^2 \, dx \, dt + 2^a \| b_{L^\infty(Q_T)} \|_{L^\infty(Q_T)} \int_0^\tau \int_{Q_\tau} (x + \eta)^{-1} u^2 \, dx \, dt,
\]
which, together with Lemma 2.4, yields
\[
\int_0^1 \left( (x + \eta)^a u_\varepsilon^2(x, \tau) \right) \, dx + \int_0^\tau \int_{Q_\tau} u_\varepsilon^2 \, dx \, dt
\]
\[
\leq \int_0^1 \left( (x + \eta)^a u_0^2(x) \right) \, dx + \int_0^\tau \int_{Q_\tau} h^2 \, dx \, dt + C_1 \int_0^\tau \int_{Q_\tau} (1 + (x + \eta)^{-\beta} + (x + \eta)^{-1}) u^2 \, dx \, dt
\]
\[
+ C_1 \int_0^\tau \int_{Q_\tau} (1 + (x + \eta)^{-\beta} + (x + \eta)^{-1}) u^2 \, dx \, dt
\]
\[
\leq \int_0^1 \left( (x + \eta)^a u_0^2(x) \right) \, dx + \int_0^\tau \int_{Q_\tau} h^2 \, dx \, dt + C_1 \left( \frac{1}{2} \delta^2 + \frac{1}{2} \delta^{2 - \beta} + \delta \right) \int_0^\tau \int_{Q_\tau} u_\varepsilon^2 \, dx \, dt
\]
\[
+ C_1 (\delta^{-a} + \delta^{-a-\beta} + \delta^{-a-1}) \int_0^\tau \int_{Q_\tau} (x + \eta)^a u_\varepsilon^2 \, dx \, dt
\]
for each \( \delta \in (0, 1) \). Choosing \( \delta \in (0, 1) \) such that
\[
\frac{1}{2} \delta^2 + \frac{1}{2} \delta^{2 - \beta} + \delta = \min \left\{ \frac{1}{2}, \frac{1}{2C_1} \right\},
\]
we obtain
\[
\int_0^1 \left( (x + \eta)^a u_\varepsilon^2(x, \tau) \right) \, dx + \frac{1}{2} \int_0^\tau \int_{Q_\tau} u_\varepsilon^2 \, dx \, dt \leq \int_0^1 \left( (x + \eta)^a u_0^2(x) \right) \, dx + \int_0^\tau \int_{Q_\tau} h^2 \, dx \, dt + C_2 \int_0^\tau \int_{Q_\tau} (x + \eta)^a u_\varepsilon^2 \, dx \, dt.
\]
(2.5)
Using the Gronwall inequality in (2.5), one gets that for each $\tau \in (0, T]$,

$$\int_0^1 (x + \eta)^\alpha u^2(x, \tau) \, dx \leq C_3 \left( \int_0^1 (x + \eta)^\alpha u_0^2(x) \, dx + \iint_{Q_T} h^2 \, dx \, dt \right).$$

(2.6)

Choosing $\tau = T$ in (2.5), together with (2.6), leads to

$$\iint_{Q_T} u_0^2 \, dx \, dt \leq C_4 \left( \int_0^1 (x + \eta)^\alpha u_0^2(x) \, dx + \iint_{Q_T} h^2 \, dx \, dt \right).$$

(2.7)

Then (2.3) follows from (2.6), (2.7) and Lemma 2.4.

Now we turn to proving (2.4). For each $\varepsilon \in (0, T)$ and $(x, \tau) \in (0, 1) \times (0, T - \varepsilon)$, multiplying (1.19) by $u(x, \tau + \varepsilon) - u(x, \tau)$ and then integrating over $(\tau, \tau + \varepsilon)$ with respect to $t$, we get

$$(x + \eta)^\alpha (u(x, \tau + \varepsilon) - u(x, \tau))^2 = \left( u_{xx}(x, t) - ((x + \eta)^{\alpha/2} b(x, t) u(x, t))_x \right) (u(x, \tau + \varepsilon) - u(x, \tau)) \, dt$$

$$+ \int_{\tau}^{\tau + \varepsilon} (h(x, t) \chi_\omega - (x + \eta)^{\alpha/2-1} \gamma(x, t) u(x, t)) (u(x, \tau + \varepsilon) - u(x, \tau)) \, dt$$

$$- \int_{\tau}^{\tau + \varepsilon} c(x, t) u(x, t) (u(x, \tau + \varepsilon) - u(x, \tau)) \, dt.$$ (2.8)

Integrating (2.8) over $(0, 1) \times (0, T - \varepsilon)$ by parts, together with (1.20), Lemma 2.4 and (2.3), leads to

$$\int_{0}^{T-\varepsilon} \int_{0}^{1} (x + \eta)^\alpha (u(x, \tau + \varepsilon) - u(x, \tau))^2 \, dx \, d\tau$$

$$= -\int_{0}^{T-\varepsilon} \int_{0}^{1} \int_{0}^{\tau} (u_x(x, t) - (x + \eta)^{\alpha/2} b(x, t) u(x, t)) (u_x(x, \tau + \varepsilon) - u_x(x, \tau)) \, dt \, dx \, d\tau$$

$$+ \int_{0}^{T-\varepsilon} \int_{0}^{1} \int_{\tau}^{\tau + \varepsilon} (h(x, t) \chi_\omega - (x + \eta)^{\alpha/2-1} \gamma(x, t) u(x, t)) (u(x, \tau + \varepsilon) - u(x, \tau)) \, dt \, dx \, d\tau$$

$$- \int_{0}^{T-\varepsilon} \int_{0}^{1} \int_{\tau}^{\tau + \varepsilon} c(x, t) u(x, t) (u(x, \tau + \varepsilon) - u(x, \tau)) \, dt \, dx \, d\tau$$

$$\leq \left( T - \varepsilon \right) \int_{0}^{1} \int_{\tau}^{\tau + \varepsilon} (u_x - (x + \eta)^{\alpha/2} b u)^2 \, dx \, dt \right)^{1/2} \left( \varepsilon \int_{0}^{T-\varepsilon} \int_{0}^{1} (u_x(x, \tau + \varepsilon) - u_x(x, \tau))^2 \, dx \, d\tau \right)^{1/2}$$

$$+ \left( T - \varepsilon \right) \int_{0}^{1} \int_{\tau}^{\tau + \varepsilon} (h \chi_\omega - (x + \eta)^{\alpha/2-1} y u)^2 \, dx \, dt \right)^{1/2} \left( \varepsilon \int_{0}^{T-\varepsilon} \int_{0}^{1} (u(x, \tau + \varepsilon) - u(x, \tau))^2 \, dx \, d\tau \right)^{1/2}$$

$$+ \| (x + \eta)^{\beta} c \|_{L^\infty(Q_T)} \left( T - \varepsilon \right) \int_{0}^{1} (x + \eta)^{-\beta} u^2 \, dx \, dt \right)^{1/2}$$

$$\cdot \left( \varepsilon \int_{0}^{T-\varepsilon} \int_{0}^{1} (x + \eta)^{-\beta} (u(x, \tau + \varepsilon) - u(x, \tau))^2 \, dx \, d\tau \right)^{1/2}$$

$$\leq 2T^{1/2} \varepsilon^{1/2} \| u_t \|_{L^2(Q_T)} (2) \| u_x \|_{L^2(Q_T)} + 2T^{1/2} \varepsilon^{1/2} \| u \|_{L^2(Q_T)} (2) \| h \|_{L^2(Q_T)} + 2T^{1/2} \varepsilon^{1/2} \| u \|_{L^2(Q_T)} (2) \| \gamma \|_{L^2(Q_T)} (x + \eta)^{-1} \| u \|_{L^2(Q_T)} (2)$$

$$+ 2T^{1/2} \varepsilon^{1/2} \| (x + \eta)^{\beta} c \|_{L^\infty(Q_T)} \| (x + \eta)^{-\beta/2} u \|_{L^2(Q_T)}^2$$
Here consider the approximating problem with mollified
The classical theory for parabolic equations (cf. [30, 42]) gives the well-posedness. It suffices to prove
to denote generic constants depending only on
\[ \int_0^T (x + \eta)^{a_0} u_0^2(x) \, dx + \int_{Q_T} h^2 \, dx \, dt, \]  
(2.9)

It follows from Lemma 2.4, (2.3) and (2.9) that
\[
\int_0^T \int_0^{\tau - \epsilon} (u(x, \tau + \epsilon) - u(x, \tau))^2 \, dx \, d\tau \leq \int_0^{\tau - \epsilon} \int_0^{\tau - \epsilon} (u(x, \tau + \epsilon) - u(x, \tau))^2 \, dx \, d\tau + \int_0^{\tau - \epsilon} \int_0^{\tau - \epsilon} (u(x, \tau + \epsilon) - u(x, \tau))^2 \, dx \, d\tau
\]
\[ \leq 2 \int_0^{T} \int_0^{\delta} u^2 \, dx \, dt + \delta^{-a} \int_0^{T} \int_0^{\delta} (x + \eta)^a (u(x, \tau + \epsilon) - u(x, \tau))^2 \, dx \, d\tau \]
\[ \leq \delta^2 \int_0^{T} \int_0^{\delta} u^2 \, dx \, dt + C_6 \delta^{-a} \epsilon^{1/2} \int_0^{T} \int_{Q_T} (x + \eta)^a u_0^2(x) \, dx + \int_{Q_T} h^2 \, dx \, dt \]
\[ \leq C_7 \epsilon^{1/(2 + a)} \left( \int_0^{T} \int_{Q_T} (x + \eta)^a u_0^2(x) \, dx + \int_{Q_T} h^2 \, dx \, dt \right), \]

where \( \delta = \min \{1, \epsilon^{1/(4 + 2a)} \}. \]

Turn to problem (1.16)–(1.18).

**Proposition 2.6.** Assume that \( \alpha > 0 \), \( 0 \leq \beta < 2 \) and \( b, c, \gamma, \sigma \in L^\infty(Q_T) \). Then for each \( v_T \in L^2(0, 1) \), problem
(1.16)–(1.18) admits a unique solution \( v \in L^2(0, T; H^1_0(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \cap H^1_{\text{loc}}(Q_T) \). Furthermore, the solution satisfies
\[ \| (x + \eta)^{\alpha/2} v \|_{L^\infty(0, \tau; L^2(0, 1))} + \| v \|_{L^2(Q_T)} + \| v_T \|_{L^2(Q_T)} \leq M_1 \| (x + \eta)^{\alpha/2} v_T \|_{L^2(0, 1)} \]  
(2.10)

and for each \( \epsilon \in (0, T) \),
\[ \| v \|_{L^\infty(0, \tau; H^1_0(1))} + \| (x + \eta)^{\alpha/2} v \|_{L^2((0, 1) \times (0, \tau - \epsilon))} \leq M_1 \epsilon^{-1/2} \| (x + \eta)^{\alpha/2} v_T \|_{L^2(0, 1)} \]  
(2.11)

If \( v_T \in H^1_0(0, 1) \), then \( v \in L^\infty(0, T; H^1_0(0, 1)) \cap H^1(Q_T) \cap L^2(0, T; H^2(0, 1)) \) and
\[ \| v \|_{L^\infty(0, T; H^1_0(1))} + \| (x + \eta)^{\alpha/2} v \|_{L^2(Q_T)} \leq M_2 \| (x + \eta)^{\alpha/2} v_T \|_{L^2(0, 1)} + \| v_T \|_{L^2(Q_T)} \]  
(2.12)

Here \( M_1, M_2 > 0 \) depend only on \( \| b \|_{L^\infty(Q_T)}, \| (x + \eta)^{\beta} c \|_{L^\infty(Q_T)}, \| (x + \eta)^{a/2} c_i \|_{L^\infty(Q_T)}, \| y \|_{L^\infty(Q_T)}, T, \alpha \) and \( \beta \).

**Proof.** The classical theory for parabolic equations (cf. [30, 42]) gives the well-posedness. It suffices to prove
(2.10)–(2.12). Without loss of generality, we can assume that \( v \) is a classical solution. Otherwise, one can consider
the approximating problem with mollified \( b, c, \gamma \) and \( v_T \) (cf. [42]). For convenience, we use \( C_i (1 \leq i \leq 6) \)
to denote generic constants depending only on \( \| b \|_{L^\infty(Q_T)}, \| (x + \eta)^{\beta} c \|_{L^\infty(Q_T)}, \| (x + \eta)^{a/2} c_i \|_{L^\infty(Q_T)}, \| y \|_{L^\infty(Q_T)}, T, \alpha \)
and \( \beta \) in the proof.

Similarly to the proof of (2.3), one can prove (2.10). Below we prove (2.12). For each \( \tau \in [0, T) \), multiplying
(1.16) by \( v_T \) and integrating over \( \bar{Q}_\tau = (0, 1) \times (\tau, T) \) by parts, we get
\[
\int_{\bar{Q}_\tau} (x + \eta)^{a} v_T^2 \, dx \, dt + \frac{1}{2} \int_0^{1} v_T^2(x, \tau) \, dx
\]
\[ = \frac{1}{2} \int_0^{1} (v_T'(x))^2 \, dx \quad - \quad \int_{\bar{Q}_\tau} (x + \eta)^{\alpha/2} b v_T v_T \, dx \, dt + \frac{1}{2} \int_0^{1} c(x, T) v_T^2(x) \, dx
\]
\[ - \frac{1}{2} \int_0^{1} c(x, \tau) v^2(x, \tau) \, dx \quad - \quad \int_{\bar{Q}_\tau} (x + \eta)^{a/2 - 1} y v_T v_T \, dx \, dt
\]
Then (2.12) follows from (2.14), (2.15) and Lemma 2.4.

Choosing $\tau$ such that

$$
\tau \leq \left(1 + \frac{C_1}{2 - \beta}\right) \int (\int_Q (x + \eta)^{a} v_2^2 dx dt + \frac{1}{\alpha} \int (x + \eta)^{a} v_2^2 dx dt)
$$

for each $\delta \in (0, 1]$. Choosing $\delta \in (0, 1)$ such that $C_1 \delta^{2 - \beta} = \min\{1 - \frac{\beta}{2}, \frac{C_1}{2}\}$ and using (2.10), we obtain

$$
\int (x + \eta)^{a} v_2^2 dx dt + \frac{1}{\alpha} \int (x + \eta)^{a} v_2^2 dx dt
$$

Using the Gronwall inequality in (2.13), we get that for each $\tau \in [0, T)$,

Finally, let us prove (2.11). For each $\varepsilon \in (0, T)$, inequality (2.10) implies that there exists $\tilde{\varepsilon} \in (0, \varepsilon)$ such that $v(\cdot, T - \tilde{\varepsilon}) \in H^1_0(0, 1)$ and

$$
\|v_4(\cdot, T - \tilde{\varepsilon})\|_{L^2(0, 1)} \leq \left(\frac{1}{\varepsilon} \int_0^1 v_2^2 dx dt\right)^{1/2} \leq C_6 \varepsilon^{-1/2} \|(x + \eta)^{a/2} v_2\|_{L^2(0, 1)}.
$$
Regarding \( v \) as a solution to problem (1.16)–(1.18) in \((0, 1) \times (0, T - \varepsilon)\), we get (2.11) from (2.12), (2.10) and (2.16).

**Remark 2.7.** Propositions 2.5 and 2.6 still hold if \( \|x + \eta\|^{2}c_{\ast}^{(Q_{T})} \leq c_{0} < \frac{1}{4} \) \((M, M_{1}, M_{2} \text{ depend also on } c_{0} \text{ in this case})\). The restriction \( c_{0} < \frac{1}{4} \) is owing to the first estimate in Lemma 2.4.

### 2.2 Well-posedness for semilinear singular problems

**Theorem 2.8.** Assume that \( a > 0, 0 < \beta < 2, x^{\beta}c, x^{2}c_{t} \in L^{\infty}(Q_{T}) \) and \( p, q \) satisfy (1.23), (1.24). Then for each \( h \in L^{2}(Q_{T}) \) and \( u_{0} \in \mathcal{H}_{a}, \) there exists uniquely a solution \( u \in L^{2}(0, T; H^{1}_{0}(0, 1)) \) with \( x^{\beta}u \in L^{\infty}(0, T; L^{2}(2(0, 1))) \) to problem (1.15), (1.13), (1.14).

**Proof.** Assume that \( b, \gamma \in L^{\infty}(Q_{T}) \) and \( \|b\|_{L^{\infty}(Q_{T})}, \|\gamma\|_{L^{\infty}(Q_{T})} \leq K \). Using the uniform estimates in Propositions 2.5 and 2.6, we can show the well-posedness of the problem of the linear equation

\[
x^{\beta}u_{t} - u_{xx} + (x^{\beta}b(x, t)u)_{x} + c(x, t)u + x^{\beta}u_{x} = h(x, t)x_{w}, \quad (x, t) \in Q_{T},
\]

subject to (1.2) and (1.3). Furthermore, \( u \) satisfies

\[
\|x^{\beta}u\|_{L^{\infty}(0, T; L^{2}(1, 1))} + \|u\|_{L^{2}(Q_{T})} + \|u_{x}\|_{L^{2}(Q_{T})} \leq M\left(\|x^{\beta}u_{0}\|_{L^{2}(0, 1)} + \|h\|_{L^{2}(Q_{T})}\right)
\]

and for each \( \varepsilon \in (0, T) \),

\[
\int_{0}^{T-\varepsilon} \int_{0}^{T} \left(u(x, \tau + \varepsilon) - u(x, \tau)\right)^{2} dx d\tau \leq M\varepsilon^{1/(2+\alpha)}\left(\|x^{\beta}u_{0}\|_{L^{2}(0, 1)} + \|h\|_{L^{2}(Q_{T})}\right),
\]

where \( M > 0 \) depends only on \( K, \|x^{\beta}c\|_{L^{\infty}(Q_{T})}, T, a \) and \( \beta \). Using the uniform estimates (2.17) and (2.18), we can get the existence of solutions to problem (1.25), (1.2), (1.3) by a fixed point argument. And the uniqueness can be proved by the Holmgren method. The proof is so standard (see, for example, [39, Theorem 3.1]) that we omit the details here. \( \square \)

Similarly, one can prove the well-posedness of problem (2.1), (1.13), (1.14).

**Theorem 2.9.** Assume that \( a > 0, 0 < \beta < 2, x^{\beta}c, x^{2}c_{t} \in L^{\infty}(Q_{T}) \) and \( g \) satisfies (2.2). Then for each \( v_{T} \in \mathcal{H}_{a} \), there exists uniquely a solution \( v \in L^{2}(0, T; H^{1}_{0}(0, 1)) \cap H^{1}_{\text{loc}}(Q_{T}) \) with \( x^{\beta}v \in L^{\infty}(0, T; L^{2}(0, 1)) \) to problem (2.1), (1.13), (1.14).

**Remark 2.10.** Due to Remark 2.7, Theorems 2.8 and 2.9 still hold if \( x^{\beta}c \in L^{\infty}(Q_{T}) \) \((\beta \in [0, 2))\) is relaxed by \( x^{2}c \in L^{\infty}(Q_{T}) \) with \( \|x^{2}c\|_{L^{\infty}(Q_{T})} \leq c_{0} < \frac{1}{4} \).

### 3 Carleman estimates and observability inequalities

In this section, we establish Carleman estimates and observability inequalities of solutions to the regularized problem (1.16)–(1.18) and the singular problem (1.15), (1.13), (1.14).

#### 3.1 Estimates near the singular point

In this subsection, we consider the regularized problem (1.16)–(1.18) and we always assume that \( a > 0, 0 \leq \beta < 2, 0 < x_{0} < x_{1} < 1, 0 < \eta < 1, v_{T} \in H_{0}^{1}(0, 1), \) and \( b, c, c_{t}, y \in L^{\infty}(Q_{T}) \) satisfying

\[
\|x + \eta\|^{\beta}c \leq N, \quad \|x + \eta\|^{2}c_{\ast}^{(Q_{T})} \leq N, \quad \|b\|_{L^{\infty}(Q_{T})} \leq K, \quad \|y\|_{L^{\infty}(Q_{T})} \leq K
\]

with some \( N, K > 0 \). As shown in Proposition 2.6, problem (1.16)–(1.18) admits a unique solution

\[
v \in L^{\infty}(0, T; H_{0}^{1}(0, 1)) \cap H^{1}(Q_{T}) \cap L^{2}(0, T; H^{2}(0, 1)).
\]
We first cut off v in the following way: Set
\[ \hat{w} = \left( \frac{2x_0 + x_1}{3}, \frac{x_0 + 2x_1}{3} \right). \] (3.2)
Define
\[ w(x, t) = \xi(x)v(x, t), \quad (x, t) \in Q_T, \] (3.3)
where \( \xi \in C^\infty([0, 1]) \) satisfies
\[ \left\{ \begin{array}{ll}
\xi = 1, & x \in \left[ 0, \frac{2x_0 + x_1}{3} \right], \\
\in [0, 1], & x \in \hat{w}, \\
= 0, & x \in \left[ \frac{x_0 + 2x_1}{3}, 1 \right].
\end{array} \right. \] (3.4)

It follows from (1.16) and (3.3) that \( w \) satisfies
\[ (x + \eta)^2w_t + w_{xx} - c(x, t)w = f(x, t), \quad (x, t) \in Q_T, \] (3.5)
where
\[ f(x, t) = -(x + \eta)^{\alpha/2}b(x, t)\xi(x)v(x, t) + 2\xi(x)v(x, t) + \xi''(x)v(x, t) \]
\[ + (x + \eta)^{\alpha/2-1}y(x, t, \xi(x)v(x, t), (x, t) \in Q_T. \] (3.6)

**Remark 3.1.** The convection term and the second reaction term of (1.16) are regarded as a known function in (3.5) since they can be controlled by the diffusion term for the Carleman estimate. However, the first reaction term of (1.16) has to be treated as a reaction term in (3.5).

Reformulate (3.5) in a similar way as for the classical Carleman estimate. For \( s > 0 \), set
\[ z(x, t) = e^{\psi(x, t)}w(x, t), \quad \psi(x, t) = \theta(t)\psi(x), \quad (x, t) \in Q_T. \] (3.7)
Here, \( \theta \) takes the form of
\[ \theta(t) = \frac{1}{(t(T - t))^s}, \quad t \in (0, T), \] (3.8)
as usual, which satisfies
\[ |\theta'(t)| \leq C_0 \theta^2(t), \quad |\theta''(t)| \leq C_0 \theta^2(t), \quad t \in (0, T), \] (3.9)
with \( C_0 > 0 \) depending only on \( T \), while \( \psi \in C^2([0, 1]) \) is a negative function and will be determined below (see (3.15)). It follows from (3.5) and (3.7) that \( z \) satisfies
\[ L_s z = e^{\psi(x + \eta)^\alpha(e^{-s\psi}z), (x + \eta)^\alpha(x, \psi)(e^{-s\psi}z)} = f \psi, \quad (x, t) \in Q_T. \] (3.10)

Decompose \( L_s z \) into
\[ L_s z = L^+ s + L^- s, \quad (x, t) \in Q_T, \]
where
\[ L^+ s = z_{xx} - s(x + \eta)^\alpha \psi_\psi(x, t)z + s^2 \psi_\psi^2(x, t)z - c(x, t)z, \quad (x, t) \in Q_T, \]
\[ L^- s = (x + \eta)^\alpha z_t - 2s \psi_\psi(x, t)z - s \psi_\psi(x, t)z, \quad (x, t) \in Q_T. \]

It follows from (3.10) that
\[ \int_{Q_T} (x + \eta)^{-\alpha} L^+ s z L^- s z \ dx \ dt \leq \int_{Q_T} (x + \eta)^{-\alpha} f^2 e^{2s\psi} \ dx \ dt. \] (3.11)
The following lemma gives the formula of the left-hand side of (3.11).

**Lemma 3.2.** For each $s > 0$,

$$\left\langle (x + \eta)^{-a}L^*_{x} z L_{x} z \mathbf{d}x \mathbf{d}t \right\rangle = \int_{\Omega} \left( \int_{0}^{T} (x + \eta)^{-a} \varphi_{xx} - a (x + \eta)^{-a-1} \varphi_{x} \right) z_{x}^{2} \mathbf{d}x \mathbf{d}t$$

$$+ s^{3} \int_{\Omega} \left( \int_{0}^{T} (x + \eta)^{-a} \varphi_{xx} - a (x + \eta)^{-a-1} \varphi_{x} \right) \varphi_{x}^{2} z^{2} \mathbf{d}x \mathbf{d}t - 2s^{2} \int_{\Omega} \varphi_{x} \varphi_{xt} z^{2} \mathbf{d}x \mathbf{d}t$$

$$+ \frac{s}{2} \int_{\Omega} \left( (x + \eta)^{a} \varphi_{tt} - (x + \eta)^{-a} \varphi_{xx} \right)_{xx} + 2(x + \eta)^{-a} \varphi_{xx} z^{2} \mathbf{d}x \mathbf{d}t$$

$$+ \frac{1}{2} \int_{\Omega} c_{z} z^{2} \mathbf{d}x \mathbf{d}t + 2s \int_{\Omega} (x + \eta)^{-a} c \varphi_{x} z_{x} z \mathbf{d}x \mathbf{d}t + s \int_{0}^{T} (x + \eta)^{-a} \varphi_{x} z_{x}^{2} \mathbf{d}t \bigg|_{x = 0}. \quad (3.12)$$

**Proof.** Note that

$$\left\langle (x + \eta)^{-a}L^*_{x} z L_{x} z \mathbf{d}x \mathbf{d}t \right\rangle = \int_{\Omega} \left( \int_{0}^{T} (x + \eta)^{-a} \varphi_{xx} - a (x + \eta)^{-a-1} \varphi_{x} \right) z_{x}^{2} \mathbf{d}x \mathbf{d}t$$

$$+ s^{2} \int_{\Omega} \left( (x + \eta)^{-a} \varphi_{xx} - a (x + \eta)^{-a-1} \varphi_{x} \right) \varphi_{x}^{2} z^{2} \mathbf{d}x \mathbf{d}t - \int_{\Omega} (x + \eta)^{-a} c \varphi_{x} z_{x}^{2} \mathbf{d}x \mathbf{d}t. \quad (3.13)$$

We compute the four integrals on the right-hand side of (3.13), respectively. Integrating by parts and using

$$z(x, 0) = z(x, T) = 0 \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad z(0, t) = z(1, t) = z_{x}(1, t) = 0 \quad \text{for} \quad 0 < t < T,$$

we get

$$\int_{\Omega} (x + \eta)^{-a} z_{xx} L_{x} z \mathbf{d}x \mathbf{d}t = \int_{0}^{T} z_{x} z_{t} \mathbf{d}t \bigg|_{x = 0}^{x = 1} - \frac{1}{2} \int_{0}^{\infty} z_{x}^{2} \mathbf{d}x \mathbf{d}t \bigg|_{t = 0}^{t = T} - s \int_{0}^{T} (x + \eta)^{-a} \varphi_{x} z_{x}^{2} \mathbf{d}t \bigg|_{x = 0}^{x = 1}$$

$$+ s \int_{\Omega} \left( (x + \eta)^{-a} \varphi_{x} \right) z_{x}^{2} \mathbf{d}x \mathbf{d}t - s \int_{0}^{T} (x + \eta)^{-a} \varphi_{xx} z_{x} z_{t} \mathbf{d}t \bigg|_{x = 0}^{x = 1}$$

$$+ \frac{s}{2} \int_{0}^{T} ((x + \eta)^{-a} \varphi_{xx})_{x} z^{2} \mathbf{d}t \bigg|_{x = 0}^{x = 1} - \frac{s}{2} \int_{\Omega} ((x + \eta)^{-a} \varphi_{xx})_{xx} z^{2} \mathbf{d}x \mathbf{d}t$$

$$+ s \int_{\Omega} (x + \eta)^{-a} \varphi_{xx} z_{x}^{2} \mathbf{d}x \mathbf{d}t$$

$$= s \int_{0}^{T} (x + \eta)^{-a} \varphi_{x} z_{x}^{2} \mathbf{d}t \bigg|_{x = 0}^{x = 1} + s \int_{\Omega} \left( (x + \eta)^{-a} \varphi_{xx} - a (x + \eta)^{-a-1} \varphi_{x} \right) z_{x}^{2} \mathbf{d}x \mathbf{d}t$$

$$- \frac{s}{2} \int_{\Omega} ((x + \eta)^{-a} \varphi_{xx})_{xx} z^{2} \mathbf{d}x \mathbf{d}t,$$

$$- s \int_{\Omega} \varphi_{x} z_{x} z \mathbf{d}x \mathbf{d}t = -\frac{s}{2} \int_{0}^{T} (x + \eta)^{a} \varphi_{tt} z^{2} \mathbf{d}x \mathbf{d}t \bigg|_{t = 0}^{t = T} + \frac{s}{2} \int_{\Omega} (x + \eta)^{a} \varphi_{tt} z^{2} \mathbf{d}x \mathbf{d}t + s^{2} \int_{0}^{T} \varphi_{x} z_{x} z^{2} \mathbf{d}t \bigg|_{x = 0}^{x = 1}$$

$$- s^{2} \int_{\Omega} \varphi_{xt} \varphi_{x} z_{x} z \mathbf{d}x \mathbf{d}t$$

$$= \frac{s}{2} \int_{\Omega} (x + \eta)^{a} \varphi_{tt} z^{2} \mathbf{d}x \mathbf{d}t - s^{2} \int_{\Omega} \varphi_{x} z_{x} z^{2} \mathbf{d}x \mathbf{d}t.$$
Then (3.12) follows by substituting the above four identities into (3.13).

**Remark 3.3.** The penultimate term in (3.12) is \(2s \iint_{Q_T} (x + \eta)^{-a} c \varphi_x z z_x dx dt\), which, integrated by parts, is equal to \(-s \iint_{Q_T} ((x + \eta)^{-a} c \varphi_x)_x z^2 dx dt\) if \(c_x \in L^{\infty}(Q_T)\). Note that the assumption on \(c\) is only \(c, c_t \in L^{\infty}(Q_T)\). So this term is treated as an integral not of \(z^2\) but of \(z z_x\).

Owing to (3.12), in order to get a global weighted energy, one should choose \(\psi\) such that

\[
2(x + \eta)^{-a} \psi''(x) - a(x + \eta)^{-a-1} \psi'(x) > 0, \quad \psi'(x) \neq 0, \quad 0 < x < 1. \tag{3.14}
\]

A choice of \(\psi \in C^2([0, 1])\) being a negative function and satisfying (3.14) is

\[
\psi(x) = \frac{1}{2 + a} (x + \eta)^{2+a} - 2^{2+a}, \quad x \in (0, 1), \tag{3.15}
\]

which satisfies

\[
2(x + \eta)^{-a} \psi''(x) - a(x + \eta)^{-a-1} \psi'(x) = 2 + a, \quad \psi'(x) = (x + \eta)^{1+a} > 0, \quad 0 < x < 1. \tag{3.16}
\]

For such \(\psi\), one gets the following lemma.

**Lemma 3.4.** There exist two positive constants \(s_0\) and \(M_0\), depending only on \(N, T, a\) and \(\beta\), such that for each \(s \geq s_0\),

\[
\iint_{Q_T} (x + \eta)^{-a} L^+_x z L^-_x z dx dt \geq M_0 \iint_{Q_T} (s \theta z^2 + s^3 (x + \eta)^{2+2a} \theta^3 z^2) dx dt. \tag{3.17}
\]

**Proof.** Substituting the definition of \(\varphi\) into (3.12) and using (3.16), we get after a direct calculation that

\[
\iint_{Q_T} (x + \eta)^{-a} L^+_x z L^-_x z dx dt
\]

\[
\geq (2 + a)s \iint_{Q_T} \theta z^2 dx dt + (2 + a)s^3 \iint_{Q_T} (x + \eta)^{2+2a} \theta^3 z^2 dx dt - 2s^2 \iint_{Q_T} (x + \eta)^{2+2a} \theta \theta' z^2 dx dt
\]

\[
+ \frac{5}{2} \iint_{Q_T} (x + \eta)^{a} \psi'' z^2 dx dt + (1 + a)s \iint_{Q_T} c \theta z^2 dx dt
\]

\[
+ \frac{1}{2} \iint_{Q_T} c_z z^2 dx dt + 2s \iint_{Q_T} (x + \eta) c \theta z z_x dx dt. \tag{3.18}
\]
We estimate the last five terms on the right-hand side of (3.18). On one hand, one obtains from (3.9), the Hölder inequality and Lemma 2.4 that

\[
-2s^2 \int_{Q_T} (x + \eta)^{2s-1} \theta^2 z^2 \, dx \, dt + \frac{\alpha}{2} \int_{Q_T} (x + \eta)^{s-1} \theta^2 z^2 \, dx \, dt \leq 2C_0s^2 \int_{Q_T} (x + \eta)^{2s-1} \theta^2 z^2 \, dx \, dt + \frac{C_0}{2}s \int_{Q_T} (x + \eta)^{s-1} \theta^2 z^2 \, dx \, dt \\
\leq (2C_0s^2 + C_0^2s^2) \int_{Q_T} (x + \eta)^{2s-1} \theta^2 z^2 \, dx \, dt + \frac{1}{16} \int_{Q_T} (x + \eta)^{-2} \theta z^2 \, dx \, dt \\
\leq (2C_0 + C_0^2s^2) \int_{Q_T} (x + \eta)^{2s-1} \theta^2 z^2 \, dx \, dt + \frac{1}{4} \int_{Q_T} \theta z_x^2 \, dx \, dt. \tag{3.19}
\]

On the other hand, a direct calculation and Lemma 2.4 show

\[
\left| (1 + \alpha)s \int_{Q_T} c \theta z^2 \, dx \, dt + \frac{\alpha}{2} \int_{Q_T} c_1 z^2 \, dx \, dt + 2s \int_{Q_T} c(x + \eta) \theta z_z \, dx \, dt \right| \\
\leq (1 + \alpha)\| (x + \eta)^\beta c \|_{L^\infty(Q_T)} s \int_{Q_T} (x + \eta)^{-\beta} \theta z^2 \, dx \, dt + \left( \frac{T}{2} \right)^{\beta} \| (x + \eta)^2 c \|_{L^\infty(Q_T)} \int_{Q_T} (x + \eta)^{-2} \theta z^2 \, dx \, dt \\
+ 2\| (x + \eta)^{2\beta} c^2 \|_{L^2(Q_T)} s \int_{Q_T} (x + \eta)^{-\beta} \theta z^2 \, dx \, dt + \frac{1}{2} s \int_{Q_T} \theta z_x^2 \, dx \, dt \\
\leq \frac{1}{2} s \int_{Q_T} \theta z_x^2 \, dx \, dt + C_1 s \int_{Q_T} \int (x + \eta)^{-\beta} \theta z^2 \, dx \, dt + C_1 s \int_{Q_T} \int (x + \eta)^{-\beta} \theta z^2 \, dx \, dt \\
+ C_1 \int_{Q_T} \int (x + \eta)^{-2} \theta z^2 \, dx \, dt \tag{3.20}
\]

\[
\leq \frac{1}{2} s \int_{Q_T} \theta z_x^2 \, dx \, dt + \frac{1}{2 - \beta} \delta^{2-\beta} C_1 s \int_{Q_T} \theta z_x^2 \, dx \, dt \\
+ C_1 \delta^{-\beta-2s} s \int_{Q_T} \int (x + \eta)^{2s-1} \theta^3 z_x^2 \, dx \, dt + 4C_1 \int_{Q_T} \theta z_x^2 \, dx \, dt \\
\leq \left( \frac{1}{2} + \frac{1}{2 - \beta} \delta^{2-\beta} C_1 s \right) \int_{Q_T} \theta z_x^2 \, dx \, dt + C_1 \delta^{-\beta-2s} s \int_{Q_T} \int (x + \eta)^{2s-1} \theta^3 z_x^2 \, dx \, dt + 4C_1 \int_{Q_T} \theta z_x^2 \, dx \, dt \tag{3.21}
\]

for each $\delta \in (0, 1)$, where $C_1 > 0$ depends only on $N, T, \alpha$ and $\beta$. Choose $\delta \in (0, 1)$ so small that

$$\delta^{2-\beta} C_1 = \min \left\{ 1 - \frac{\beta}{2}, \frac{C_1}{2} \right\}.$$

Then it follows from (3.21) that

\[
\left| (1 + \alpha)s \int_{Q_T} c \theta z^2 \, dx \, dt + \frac{\alpha}{2} \int_{Q_T} c_1 z^2 \, dx \, dt + 2s \int_{Q_T} (x + \eta) c \theta z_z \, dx \, dt \right| \\
\leq (s + 4C_1) \int_{Q_T} \theta z_x^2 \, dx \, dt + C_2 s \int_{Q_T} \int (x + \eta)^{2s-1} \theta^3 z^2 \, dx \, dt \tag{3.22}
\]

with $C_2 > 0$ depending only on $N, T, \alpha$ and $\beta$. By substituting (3.19) and (3.22) into (3.18) and choosing $s_0 = 2C_0 + C_0^2 + 4C_1 + C_2 + \frac{1}{4}$, one gets (3.17).

\[\square\]

**Remark 3.5.** In Lemma 3.4, $M_0$ depends on $c_1$, which is caused because we distribute $cz$ into $L^2_q z$. If $cz$ is distributed into $L^2_{q^*} z$, then $M_0$ will depend on $c_x$ and $c_{xx}$. 

Remark 3.6. Lemma 3.4 still holds if \( \| (x + \eta)^{\beta}c \|_{L^\infty(Q_T)} \leq N (\beta \in [0, 2]) \) is relaxed by

\[
\| (x + \eta)^{2}c \|_{L^\infty(Q_T)} \leq C_0 < \frac{1}{4}.
\]

But \( s_0 \) and \( M_0 \) depend also on \( c_0 \).

Below we prove the following Caccioppoli inequality.

**Lemma 3.7.** There exists \( C > 0 \) depending only on \( N, K, x_0, x_1, T, \alpha \) and \( \beta \) such that for each \( s > 0 \),

\[
\int_0^T \int_\omega v^2 e^{2sp} dx dt \leq C \int_0^T \int_\omega v^2 dx dt, \tag{3.23}
\]

where \( \tilde{\omega} \) is defined in (3.2).

**Proof.** Let \( \zeta \in C^0(0, 1) \) satisfy

\[
\zeta(0) = 1, \quad \zeta(T) = 0, \quad \zeta \in [0, 1], \quad \zeta \in \omega \setminus \tilde{\omega}, \quad \zeta \in [0, 1] \setminus \omega.
\]

For \( s > 0 \), the definition of \( \varphi \) implies

\[
0 = \int_0^T \frac{d}{dt} \int_0^1 (x + \eta)^{\alpha} \zeta^2 v^2 e^{2sp} \, dx \, dt dt = 2s \int_0^T \int_\omega (x + \eta)^{\alpha} \varphi_t \zeta^2 v^2 e^{2sp} \, dx \, dt + 2 \int_0^T \int_\omega (x + \eta)^{\alpha} \zeta^2 v v_t e^{2sp} \, dx \, dt. \tag{3.24}
\]

Substituting (1.16) into (3.24) leads to

\[
0 = 2s \int_0^T \int_\omega (x + \eta)^{\alpha} \varphi_t \zeta^2 v^2 e^{2sp} \, dx \, dt - 2 \int_0^T \int_\omega \zeta^2 v \varphi_x e^{2sp} \, dx \, dt - 2 \int_0^T \int_\omega (x + \eta)^{\alpha/2} b \zeta^2 v v_x e^{2sp} \, dx \, dt
\]

\[
+ 2 \int_0^T \int_\omega c \zeta^2 v^2 e^{2sp} \, dx \, dt + 2 \int_0^T \int_\omega (x + \eta)^{\alpha/2 - 1} b \zeta^2 v^2 e^{2sp} \, dx \, dt, \quad s > 0. \tag{3.25}
\]

Integrating by parts gives

\[
-2 \int_0^T \int_\omega \zeta^2 v \varphi_x e^{2sp} \, dx \, dt
\]

\[
= 2 \int_0^T \int_\omega \zeta^2 v_x^2 e^{2sp} \, dx \, dt + 4 \int_0^T \int_\omega \zeta \varphi_x v v_x e^{2sp} \, dx \, dt + 4s \int_0^T \int_\omega \varphi_x \zeta^2 v v_x e^{2sp} \, dx \, dt, \quad s > 0. \tag{3.26}
\]

Owing to (3.25), (3.26) and the choice of \( \zeta \), one gets from the Hölder inequality that

\[
\int_0^T \zeta^2 v_x^2 e^{2sp} \, dx \, dt = -s \int_0^T \int_\omega (x + \eta)^{\alpha} \varphi \zeta v^2 e^{2sp} \, dx \, dt - 2 \int_0^T \int_\omega \zeta \varphi_x v v_x e^{2sp} \, dx \, dt + 2s \int_0^T \int_\omega \varphi_x \zeta^2 v v_x e^{2sp} \, dx \, dt
\]

\[
+ \int_0^T \int_\omega (x + \eta)^{\alpha/2} b \zeta^2 v v_x e^{2sp} \, dx \, dt - \int_0^T \int_\omega \zeta^2 v^2 e^{2sp} \, dx \, dt
\]

\[
- \int_0^T \int_\omega (x + \eta)^{\alpha/2 - 1} b \zeta^2 v^2 e^{2sp} \, dx \, dt
\]

\[
\leq \frac{1}{2} \int_0^T \int_\omega \zeta^2 v_x^2 e^{2sp} \, dx \, dt + \tilde{C}(s^2 + 1) \int_0^T \int_\omega \theta^2 v^2 e^{2sp} \, dx \, dt, \quad s > 0,
\]

with \( \tilde{C} > 0 \) depending only on \( N, K, x_0, x_1, T, \alpha \) and \( \beta \). Therefore,

\[
\int_0^T \int_\omega v^2 e^{2sp} \, dx \, dt \leq \int_0^T \int_\omega \zeta^2 v_x^2 e^{2sp} \, dx \, dt \leq 2 \tilde{C}(s^2 + 1) \int_0^T \int_\omega \theta^2 v^2 e^{2sp} \, dx \, dt, \quad s > 0. \tag{3.27}
\]

Then (3.23) follows from (3.27) with \( C = 2\tilde{C} \sup\{(s^2 + 1)\theta^2 e^{2sp} : s > 0, (x, t) \in Q_T\} \).
3.2 Carleman estimates

We are ready to establish the local Carleman estimate for solutions to problem (1.16)–(1.18), uniformly with respect to \( \eta \in (0, 1) \), near the singular point \( x_0 \).

**Proposition 3.8.** Assume that \( a > 0 \), \( 0 \leq \beta < 2 \), \( 0 < x_0 < x_1 < 1 \), \( 0 < \eta < 1 \) and \( b, c, c_1, \gamma \in L^\infty(Q_T) \) satisfying (3.1). Then there exist two positive constants \( s_1 > 0 \) and \( M_1 > 0 \), depending only on \( N, K, x_0, x_1, T, a \) and \( \beta \), such that for each \( \nu_T \in L^2(0, 1) \) and \( s \geq s_1 \) the solution \( \nu \) to problem (1.16)–(1.18) satisfies

\[
\int_0^T \int_0^{T(2x_0+\xi_1)/3} (s\theta v_\xi^2 + s^3(x + \eta)^{2+2a} \theta^3 v^2) e^{2s\phi} \, dx \, dt \leq M \int_0^T \nu^2 \, dx \, dt.
\]

**Proof.** Without loss of generality, we can assume \( \nu_T \in H^1_0(0, 1) \). Otherwise, (3.28) can be proved by a limit process. Due to (3.11) and (3.17), one gets that for each \( s > s_0 \),

\[
\int_Q (s\theta z^2 + s^3(x + \eta)^{2+2a} \theta^3 z^2) \, dx \, dt \leq \frac{1}{M_0} \int_Q (x + \eta)^{-a} f^2 e^{2s\phi} \, dx \, dt,
\]

where \( z \) is given in (3.7). For \( s \geq s_0 \), substituting (3.7), (3.3) and (3.6) into (3.29) yields

\[
\int_0^T \int_0^{T(2x_0+\xi_1)/3} (s\theta v_\xi^2 + s^3(x + \eta)^{2+2a} \theta^3 v^2) e^{2s\phi} \, dx \, dt
\]

\[
\leq s \int_Q \theta w^2 e^{2s\phi} \, dx \, dt + s^3 \int_Q (x + \eta)^{2+2a} \theta^3 w^2 e^{2s\phi} \, dx \, dt
\]

\[
\leq 2s \int_Q \theta z^2 \, dx \, dt + 3s^3 \int_Q (x + \eta)^{2+2a} \theta^3 z^2 \, dx \, dt
\]

\[
\leq \frac{3}{M_0} \int_Q (x + \eta)^{-a} \left( (x + \eta)^{a/2} b \xi v_x + 2 \xi' v_x + \xi'' v + (x + \eta)^{a/2-1} \gamma \xi v \right)^2 e^{2s\phi} \, dx \, dt
\]

\[
\leq M \int_0^T (v^2 + v_\nu^2) e^{2s\phi} \, dx \, dt + M \int_0^T \theta(v_\xi^2 + (x + \eta)^{-2} \nu^2) e^{2s\phi} \, dx \, dt,
\]

where \( M > 0 \) depends only on \( N, K, x_0, x_1, T, a \) and \( \beta \). Lemma 2.4 shows

\[
\int_0^T \int_0^{T(2x_0+\xi_1)/3} (x + \eta)^{-2} \theta v^2 e^{2s\phi} \, dx \, dt
\]

\[
\leq 4 \int_0^T \int_0^{T(2x_0+\xi_1)/3} \theta(v_x + 2s \nu_x v)^2 e^{2s\phi} \, dx \, dt
\]

\[
\leq 8 \int_0^T \int_0^{T(2x_0+\xi_1)/3} \theta v_\xi^2 e^{2s\phi} \, dx \, dt + 32s^2 \int_0^T \int_0^{T(2x_0+\xi_1)/3} (x + \eta)^{2+2a} \theta^3 v^2 e^{2s\phi} \, dx \, dt.
\]

Choose \( s_1 = s_0 + 64M \). Then it follows from (3.30) and (3.31) that

\[
\int_0^T \int_0^{T(2x_0+\xi_1)/3} (s\theta v_\xi^2 + s^3(x + \eta)^{2+2a} \theta^3 v^2) e^{2s\phi} \, dx \, dt \leq 2M \int_0^T (v^2 + v_\nu^2) e^{2s\phi} \, dx \, dt, \quad s \geq s_1,
\]

which, together with (3.23), yields (3.28).

**Remark 3.9.** The assumption on \( c_i \), the factors \( (x + \eta)^{a/2} \) and \( (x + \eta)^{a/2-1} \) in the convection term and the second reaction term of (1.16) are necessary when one establishes the Carleman estimate in such a way.

Define

\[
\Phi(x, t) = \xi(x) \phi(x, t) + (1 - \xi(x)) \phi(x, t), \quad (x, t) \in Q_T,
\]
where $\xi$, $\varphi$ are given by (3.4), (3.7), (3.8) and (3.15), while
\[
\phi(x, t) = (e^{r(1-x)} - e^{2r})\theta(t), \quad (x, t) \in Q_T,
\]
with $r$ being a positive constant. Then one can prove the following Carleman estimate.

**Theorem 3.10 (Uniform Carleman estimate).** Assume that $\alpha > 0$, $0 \leq \beta < 2$, $0 < x_0 < x_1 < 1$, $0 < \eta < 1$ and $b, c, c_1, \gamma \in L^\infty(Q_T)$ satisfying (3.1). Then there exist $r > 0$, $s_2 > 0$ and $M_2 > 0$, depending only on $N, K, x_0, x_1, T, \alpha$ and $\beta$, such that for each $v_T \in L^2(0, 1)$ and $s \geq s_2$ the solution $v$ to problem (1.16)–(1.18) satisfies
\[
\iint_{Q_T} (s\theta_0 v_{xx}^2 + s^3(x + \eta)^{2+2\alpha}\theta^3 v^2) e^{s2\phi} dx dt \leq M_2 \int_0^T \iint_{\Omega} v^2 dx dt.
\]

**Proof.** For convenience, we use $C_i$ $(1 \leq i \leq 4)$ to denote generic constants depending only on $N, K, x_0, x_1, T, \alpha$ and $\beta$. Let
\[
\tilde{v}(x, t) = (1 - \xi(x))v(x, t) - w(x, t), \quad (x, t) \in Q_T.
\]
Then $\tilde{v}$ solves the following problem:
\[
\begin{cases}
(x + \eta)^{n/2}b(x, t)(1 - \xi(x))v_x(x, t) - 2\xi'(x)v_x(x, t) - \xi''(x)v(x, t) + (x + \eta)^{n/2-1}v(x, t)(1 - \xi(x))v(x, t), & (x, t) \in (x_0, 1) \times (0, T), \\
\tilde{v}(x, t) = \tilde{v}(1, t) = 0, & t \in (0, T), \\
\tilde{v}(x, T) = (1 - \xi(x))v(T), & x \in (x_0, 1),
\end{cases}
\]
where
\[
\tilde{f}(x, t) = -(x + \eta)^{n/2}b(x, t)(1 - \xi(x))v_x(x, t) - 2\xi'(x)v_x(x, t) - \xi''(x)v(x, t) + (x + \eta)^{n/2-1}f(x, t)(1 - \xi(x))v(x, t), \quad (x, t) \in (x_0, 1) \times (0, T).
\]
Due to the classical Carleman estimate (see [3]), there exist $r > 0$, $\hat{s}_1 > 0$ and $\hat{M}_1 > 0$, depending only on $N, K, x_0, x_1, T, \alpha$ and $\beta$, such that for each $s \geq \hat{s}_1$,
\[
\iint_{Q_D} (s\theta_0 v_{xx}^2 + s^3\theta^3 v^2) e^{s2\phi} dx dt \leq \hat{M}_1 \left( \int_0^T \int_{x_0}^1 f^2 e^{2\phi} dx dt + \int_0^T \int_{\Omega} \tilde{v}^2 dx dt \right).
\]
Therefore, for each $s \geq \hat{s}_1$,
\[
\int_0^T \int_0^{(x_0 + x_1)/3} \sum_{i=1}^3 \sum_{j=1}^3 (s\theta_0 v_{xx}^2 + s^3(x + \eta)^{2+2\alpha}\theta^3 v^2) e^{s2\phi} dx dt \leq C_1 \left( \int_0^T \int_{\Omega} v_{xx}^2 e^{2\phi} dx dt + \int_0^T \int_{\Omega} v^2 dx dt \right). \tag{3.32}
\]
Combine (3.28) and (3.32) to get that for each $s \geq \max[\hat{s}_1, s_1]$,
\[
\int_0^T \int_0^{(x_0 + x_1)/3} (s\theta_0 v_{xx}^2 + s^3(x + \eta)^{2+2\alpha}\theta^3 v^2) e^{s2\phi} dx dt \leq (M_1 + C_1) \left( \int_0^T \int_{\Omega} v_{xx}^2 dx dt + \int_0^T \int_{\Omega} (s\theta_0 v_{xx}^2 + s^3(x + \eta)^{2+2\alpha}\theta^3 v^2) e^{s2\phi} dx dt \right), \tag{3.33}
\]
Similar to the proof of (3.23), one can prove the following Caccioppoli inequalities:
\[
s \int_0^T \int_{\Omega} v_{xx}^2 e^{2\phi} dx dt \leq C_2 \int_0^T \int_{\Omega} v^2 dx dt, \quad \int_0^T \int_{\Omega} v_{xx}^2 e^{2\phi} dx dt \leq C_3 \int_0^T \int_{\Omega} v^2 dx dt, \quad s > 0. \tag{3.34}
\]
It is clear that
\[
\int_0^T \int_0^\infty (x + \eta)^{2+2\alpha} v^2 e^{2s\Phi} \, dx \, dt \leq C_\delta \int_0^\infty v^2 \, dx \, dt, \quad s > 0.
\] (3.35)

Then the theorem is proved by combining (3.33)–(3.35).

\section{3.3 Observability inequalities}

\textbf{Theorem 3.11} (Uniform observability inequality). Assume that \( \alpha > 0, 0 \leq \beta < 2, 0 < x_0 < x_1 < 1, 0 < \eta < 1 \) and \( b, c, c_I, y \in L^{\infty}(Q_T) \) satisfying (3.1). Then there exists \( M_3 > 0 \), depending only on \( N, K, x_0, x_1, T, \alpha \) and \( \beta \), such that for each \( \nu_T \in L^2(0, 1) \) the solution \( \nu \) to problem (1.16)–(1.18) satisfies

\[
\int_0^1 (x + \eta)^\alpha v^2(x, 0) \, dx \leq M_3 \int_0^T v^2 \, dx \, dt.
\] (3.36)

\textbf{Proof.} Without loss of generality, we can assume \( \nu_T \in H^1_0(0, 1) \). Otherwise, the theorem can be proved by a limit process. For convenience, we use \( C_i (1 \leq i \leq 3) \) to denote generic constants depending only on \( N, K, x_0, x_1, T, \alpha \) and \( \beta \). Set

\[
m(s) = \min_{0 < x < 1, 0 < \eta < \frac{1}{3T/4}} \{ \phi(t)e^{2s\Phi(x, \eta)} \, s^3 \theta^3(t)e^{2s\Phi(x, \eta)} \}, \quad s > 0.
\]

Then for each \( s > 0 \),

\[
\int_0^{3T/4} \int_0^1 ((x + \eta)^{2+2\alpha} v^2 + v_x^2) \, dx \, dt \leq m(s) \int_0^T \int_0^1 ((x + \eta)^{2+2\alpha} v^2 + v_x^2) \, dx \, dt,
\]

which, together with Theorem 3.10, leads to

\[
\int_0^1 (x + \eta)^\alpha v^2(x, \tau) \, dx \leq C_1 \int_0^T v^2 \, dx \, dt.
\] (3.36)

For each \( \tau \in (0, T) \), multiplying (1.16) by \( e^{Mr} \nu \) with \( M > 0 \) to be determined, and then integrating over \((0, 1) \times (0, \tau)\) by parts, we get that

\[
\frac{1}{2} \left[ (x + \eta)^\alpha e^{Mr} \nu^2(x, \tau) \right]_0^1 \geq \frac{1}{2} \left[ (x + \eta)^\alpha \nu^2(x, 0) \right]_0^1
\]

\[
\begin{align*}
&= \frac{M}{2} \int \left( \int (x + \eta)^\alpha e^{Mr} \nu^2 \, dx \, dt + \int \left( \int e^{Mr} \nu^2 \, dx \, dt - \int (x + \eta)^{\alpha/2} b e^{Mr} v v_x \, dx \, dt \right) \\
&\quad + \int \left( \int c e^{Mr} \nu^2 \, dx \, dt + \int \left( \int (x + \eta)^{\alpha/2} c e^{Mr} \nu^2 \, dx \, dt \right) \\
&\quad \geq \frac{M}{2} \int \left( \int (x + \eta)^\alpha e^{Mr} \nu^2 \, dx \, dt + \frac{1}{2} \left( \int \left( \int e^{Mr} \nu^2 \, dx \, dt - 2^{a-1} \| b \|_{L^\infty(Q_T)} \right) \int e^{Mr} \nu^2 \, dx \, dt \\
&- \| (x + \eta)^\beta \|_{L^\infty(Q_T)} \left( \int (x + \eta)^{-\beta} e^{Mr} \nu^2 \, dx \, dt - 2^{a/2} \| b \|_{L^\infty(Q_T)} \right) \int (x + \eta)^{-1} e^{Mr} \nu^2 \, dx \, dt \\
&\quad - \int \left( \int e^{Mr} \nu^2 \, dx \, dt - C_2 \int e^{Mr} \nu^2 \, dx \, dt \\
&\quad - C_2 \int \left( \int (x + \eta)^{-\beta} e^{Mr} \nu^2 \, dx \, dt - C_2 \int (x + \eta)^{-1} e^{Mr} \nu^2 \, dx \, dt \right).
\end{align*}
\] (3.37)
Lemma 2.4 gives
\[
\int_0^b e^{Mt}v^2 \, dx \, dt \leq \frac{1}{2} \delta^2 \int_0^\delta e^{Mt}v_x^2 \, dx \, dt + \int_{\delta}^1 e^{Mt}v^2 \, dx \, dt, \tag{3.38}
\]

\[
\int_0^b (x+\eta)^{-\beta} e^{Mt}v^2 \, dx \, dt \leq \frac{1}{2-\beta} \delta^2 \int_0^\delta e^{Mt}v_x^2 \, dx \, dt + \int_{\delta}^1 (x+\eta)^{-\beta} e^{Mt}v^2 \, dx \, dt, \tag{3.39}
\]

\[
\int_0^b (x+\eta)^{-1} e^{Mt}v^2 \, dx \, dt \leq \delta \int_0^\delta e^{Mt}v_x^2 \, dx \, dt + \int_{\delta}^1 (x+\eta)^{-1} e^{Mt}v^2 \, dx \, dt \tag{3.40}
\]

for each \(\delta \in (0, 1]\). Choose \(\delta \in (0, 1]\) such that
\[
\frac{1}{2} \delta^2 + \frac{1}{2-\beta} \delta^2 - \beta = \min \left\{ \frac{1}{2C_2}, 1 \right\}.
\]

It follows from (3.37)–(3.40) that
\[
\int_0^1 (x+\eta)^a e^{Mt}v^2(x, \tau) \, dx - \int_0^1 (x+\eta)^a v^2(x, 0) \, dx \geq M \int_0^1 (x+\eta)^a e^{Mt}v^2 \, dx \, dt - C_3 \int_0^1 (x+\eta)^a e^{Mt}v^2 \, dx \, dt. \tag{3.41}
\]

Choose \(M = C_3\) in (3.41) to obtain
\[
\int_0^1 (x+\eta)^a v^2(x, 0) \, dx \leq e^{C_3T} \int_0^1 (x+\eta)^a v^2(x, t) \, dx, \quad 0 < t < T. \tag{3.42}
\]

Integrating (3.42) over \((\frac{T}{2}, \frac{3T}{4})\) and using Lemma 2.4, we get
\[
\int_0^{\frac{T}{4}} (x+\eta)^a v^2(x, 0) \, dx \leq \frac{2}{T} e^{C_1T} \int_{\frac{T}{4}}^{\frac{3T}{4}} (x+\eta)^a v^2 \, dx \, dt \\
\leq \frac{21+a}{T} e^{C_1T} \int_{\frac{T}{4}}^{\frac{3T}{4}} v^2 \, dx \, dt \\
\leq \frac{2a}{T} e^{C_1T} \int_{\frac{T}{4}}^{\frac{3T}{4}} v_x^2 \, dx \, dt. \tag{3.43}
\]

Then the theorem follows from (3.36) and (3.43).

\[\square\]

**Remark 3.12.** Proposition 3.8 and Theorems 3.10 and 3.11 still hold if \(\|x+\eta\|_{L^\infty(Q_T)} \leq N (\beta \in [0, 2))\) is relaxed by \(\|x+\eta\|^2 \|\|_{L^\infty(Q_T)} \leq C_0 < \frac{1}{2}\). But \(s_1, s_2, M_1, M_2, M_3\) depend also on \(c_0\).

As a corollary of Theorems 3.10 and 3.11, one can get the following Carleman estimate and observability inequality for the singular problem (1.15), (1.13), (1.16).

**Theorem 3.13** (Carleman estimate, observability inequality). Assume that \(a > 0, 0 \leq \beta < 2, 0 < x_0 < x_1 < 1\) and \(b, x^\beta c, x^2 \gamma \in L^\infty(Q_T)\). Then for each \(v_T \in \mathcal{F}_a\) the solution \(v\) to problem (1.15), (1.13), (1.14) satisfies
\[
\int_{Q_T} (s \theta v_x^2 + x^2 \theta^3 v^2) e^{2c\Phi} \, dx \, dt \leq M_2 \int_0^T v^2 \, dx \, dt, \quad s \geq s_2,
\]

and
\[
\int_0^T x^2 v^2(x, 0) \, dx \leq M_3 \int_0^T v^2 \, dx \, dt,
\]

where \(M_2, s_2\) and \(M_3\) are given in Theorems 3.10 and 3.11, which depend only on \(\|b\|_{L^\infty(Q_T)}\), \(\|x^\beta c\|_{L^\infty(Q_T)}\), \(\|x^2 \gamma \|_{L^\infty(Q_T)}\), \(x_0, x_1, T, a\) and \(\beta\).
Proof. Choose \( |v^n| \in C^1(\mathbb{Q}_T) \) such that
\[
\|(x + \eta)^{\alpha/2} v^n_T\|_{L^2(0,1)} + \|(x + \eta)^{\beta} c^n\|_{L^\infty(\mathbb{Q}_T)} + \|(x + \eta)^{\beta} c^n T\|_{L^\infty(\mathbb{Q}_T)} \\
\leq |x^{\alpha/2} v^n_T|_{L^2(0,1)} + \|x^\beta c^n|_{L^\infty(\mathbb{Q}_T)} + \|x^2 c^n|_{L^\infty(\mathbb{Q}_T)}, \quad 0 < \eta < 1,
\]
(3.44)
and
\[
\lim_{\eta \to 0} \frac{1}{2} \int_0^1 ((x + \eta)^{\alpha/2} v^n_T(x) - x^{\alpha/2} v_T(x))^2 \, dx = 0, \quad \lim_{\eta \to 0} \int_{\mathbb{Q}_T} ((x + \eta)^{\beta} c^n(x) - x^\beta c(x))^2 \, dx \, dt = 0.
\]
(3.45)

Denote by
\[
v^n \in L^2(0, T; H^1(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \cap H^1_{\text{loc}}(\mathbb{Q}_T)
\]
the solution to problem (1.16)–(1.18) with \( \nu = \nu^n \) and \( c = c^n \). That is to say, for each \( \varsigma \in C^1(\mathbb{Q}_T) \) with \( \varsigma(\cdot, 0)_{|\{0,1\}} = 0 \) and \( \varsigma(0, \cdot)_{|\{0,\eta\}} = \varsigma(1, \cdot)_{|\{0,\eta\}} = 0 \) it holds that
\[
\int_{\mathbb{Q}_T} \left( (x + \eta)^\beta v^n_T \varsigma_t + v^n_T \varsigma_x - (x + \eta)^{\alpha/2-2} v^n_T \varsigma + c^n v^n \varsigma + (x + \eta)^{\alpha/2-2} v^n_\nu \varsigma \right) \, dx \, dt
= \int_0^1 (x + \eta)^\beta v^n_T(x, T) \varsigma(x, T) \, dx.
\]
(3.46)

Proposition 2.6 shows that
\[
\|(x + \eta)^{\alpha/2} v^n_T\|_{L^\infty(0, T; L^2(0,1))} + \|v^n_T\|_{L^\infty(\mathbb{Q}_T)} + \|v^n_T\|_{L^2(\mathbb{Q}_T)} \leq M \|(x + \eta)^{\alpha/2} v_T^n\|_{L^2(0,1)}, \quad (3.47)
\]
\[
\|v^n_T\|_{L^\infty(0, T-c; H^1(0,1))} + \|(x + \eta)^{\alpha/2} v^n_T\|_{L^2(0,1) \times (0, T-c)} \leq M e^{-cT/2} \|(x + \eta)^{\alpha/2} v_T^n\|_{L^2(0,1)}, \quad (3.48)
\]
for each \( \epsilon \in (0, T) \), where \( M > 0 \) depends on \( \|\beta\|_{L^\infty(\mathbb{Q}_T)}, \|x^\beta c^n\|_{L^\infty(\mathbb{Q}_T)}, \|x^2 c^n|_{L^\infty(\mathbb{Q}_T)}, \|v^n_T\|_{L^\infty(\mathbb{Q}_T)}, \|v^n_T\|_{L^2(\mathbb{Q}_T)}, |\eta\|_{L^\infty(\mathbb{Q}_T)}, T, \alpha \) and \( \beta \). Owing to Theorems 3.10 and 3.11, \( v^n \) satisfies
\[
\int_{\mathbb{Q}_T} (s \theta(v^n_x)^2 + s^3 (x + \eta)^{2+2\alpha} \theta^3 (v^n)^2) e^{2s \theta} \, dx \, dt \leq M_2 \int_0^T \int_0^\omega (v^n)^2 \, dx \, dt, \quad s \geq s_2, \quad (3.49)
\]
\[
\int_0^1 (x + \eta)^\beta (v^n(x, 0))^2 \, dx \leq M_3 \int_0^\omega (v^n)^2 \, dt, \quad (3.50)
\]
where \( M_2, s_2 \) and \( M_3 \) are given in Theorems 3.10 and 3.11, and which depend only on \( \|\beta\|_{L^\infty(\mathbb{Q}_T)}, \|x^\beta c^n\|_{L^\infty(\mathbb{Q}_T)}, \|x^2 c^n|_{L^\infty(\mathbb{Q}_T)}, \|v^n_T\|_{L^\infty(\mathbb{Q}_T)}, \|v^n_T\|_{L^2(\mathbb{Q}_T)}, \forall x \in (0,1), T, \alpha \) and \( \beta \). Due to (3.44), (3.47) and (3.48), there exist \( |\eta|_{L^\infty(\mathbb{Q}_T)} < (0, 1) \) with \( \lim_{n \to \infty} \eta_n = 0 \), and \( v \in L^2(0, T; H^1(0,1)) \) with \( x^{\alpha/2} v \in L^\infty(0, T; L^2(0,1)) \), such that
\[
v^{n_0} \rightharpoonup v \text{ in } L^2(\mathbb{Q}_T), \quad (3.51)
\]
\[
v^{n_0} \rightharpoonup v \text{ in } L^2(\omega \times (0, T)) \text{ as } n \to \infty, \quad (3.51)
\]
\[
(x + \eta)^{\alpha/2} v^{n_0}(x, 0) \rightharpoonup x^{\alpha/2} v(x, 0) \text{ weakly in } L^2(0,1), \quad (3.52)
\]

For each \( \varsigma \in C(\mathbb{Q}_T) \) with \( \varsigma(0, \cdot)_{|\{0,1\}} = \varsigma(1, \cdot)_{|\{0,\eta\}} = 0 \), choosing \( \eta = \eta_n \) in (3.46) and then letting \( n \to \infty \), together with (3.45) and (3.51), we get that \( v \) is the solution to problem (1.15), (1.13), (1.14). Thanks to (3.51) and (3.52), the proof of the theorem is complete by letting \( n \to \infty \) in (3.49) and (3.50).

\textbf{Remark 3.14.} Owing to Remark 3.12, Theorem 3.13 still holds if \( x^\beta c \in L^\infty(\mathbb{Q}_T) \) (\( \beta \in [0, 2] \)) is relaxed by \( x^2 c \in L^\infty(\mathbb{Q}_T) \) with \( \|x^2 c\|_{L^\infty(\mathbb{Q}_T)} \leq c_0 < 1/4 \).

\section{4 Null controllability}

In this section, we prove the null controllability of system (1.25), (1.2), (1.3). It can be proved by using Theorem 3.13. However, (1.25) is singular and its solutions are weak. It is more convenient that one first shows
the null controllability of the regularized system (1.22), (1.20), (1.21) by using the uniform observability inequality (Theorem 3.11) and then taking a limit by means of uniform energy estimates (Propositions 2.5 and 2.6).

### 4.1 Linear case

**Lemma 4.1.** Assume \( a > 0, 0 < \beta < 2, 0 < x_0 < x_1 < 1, 0 < \eta < 1, \| (x + \eta)^\beta c \|_{L^\infty(Q_T)}, \| (x + \eta)^2 c \|_{L^\infty(Q_T)} \leq N \) and \( \| b \|_{L^\infty(Q_T)}, \| b \|_{L^\infty(Q_T)} \leq K \) with some \( N, K > 0 \). Then for each \( u_0 \in L^2(0, 1) \) there exists \( h \in L^2(Q_T) \) such that the solution \( u \) to problem (1.19)–(1.21) satisfies

\[
u(x, t) = 0, \quad x \in (0, 1).
\]

Furthermore, there exists \( M > 0 \) depending only on \( N, K, x_0, x_1, T, \alpha \) and \( \beta \) such that

\[
\| h \|_{L^2(Q_T)} \leq M\| (x + \eta)^{\alpha/2} u_0 \|_{L^2(0, 1)}.
\]

**Proof.** For each \( \epsilon \in (0, 1) \), choose \( b^\epsilon \in C^1(\bar{Q}_T) \) to satisfy

\[
\| b^\epsilon \|_{L^\infty(Q_T)} \leq \| b \|_{L^\infty(Q_T)} \leq K, \quad \| b^\epsilon - b \|_{L^\infty(Q_T)} < \epsilon.
\]

Consider the following minimum problem:

\[
\min_{h \in L^2(Q_T)} \left( \iint_{Q_T} h^2 \, dx \, dt + \frac{1}{\epsilon} \int_0^1 u^2(x, T) \, dx \right),
\]

where \( u \) is the solution to problem (1.19)–(1.21) with \( b = b^\epsilon \). By a standard argument (cf. [3]), one can prove that the minimum problem (4.4) admits a unique solution \((u^\epsilon, h^\epsilon)\) with

\[
h^\epsilon = \chi_\omega v^\epsilon \quad \text{in } Q_T,
\]

where

\[
u^\epsilon \in L^2(0, T; H^1_0(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \cap L^\infty(\tau, T; H^1(0, 1)) \cap H^1((0, 1) \times (\tau, T))
\]

(for each \( \tau \in (0, T) \)) solves the problem

\[
\begin{aligned}
(x + \eta)^\alpha u_{\epsilon t}^\epsilon - u_{\epsilon xx}^\epsilon + ((x + \eta)^{\alpha/2} b^\epsilon(x, t)u^\epsilon) \chi_x + c(x, t)u^\epsilon + (x + \eta)^{\alpha/2-1} \gamma(x, t)u^\epsilon = v^\epsilon(x, t)\chi_\omega, & \quad (x, t) \in Q_T, \\
u^\epsilon(0, t) = u^\epsilon(1, t) = 0, & \quad t \in (0, T),
\end{aligned}
\]

\[
u^\epsilon(x, 0) = u^\epsilon(x), & \quad x \in (0, 1),
\]

while \( v^\epsilon \in L^\infty(0, T; H^1_0(0, 1)) \cap H^1(Q_T) \) solves the conjugate problem

\[
\begin{aligned}
(x + \eta)^\alpha v_{\epsilon t}^\epsilon + v_{\epsilon xx}^\epsilon + ((x + \eta)^{\alpha/2} b^\epsilon(x, t) v^\epsilon) \chi_x - c(x, t)v^\epsilon - (x + \eta)^{\alpha/2-1} \gamma(x, t)v^\epsilon = 0, & \quad (x, t) \in Q_T, \\
v^\epsilon(x, 0) = v^\epsilon(1, t) = 0, & \quad t \in (0, T),
\end{aligned}
\]

\[
v^\epsilon(x, T) = -\frac{1}{\epsilon} u^\epsilon(x, T), & \quad x \in (0, 1).
\]

That is to say, for each \( \zeta \in L^2(0, T; H^1_0(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \cap H^1(Q_T) \) and \( g \in L^2(0, T; H^1_0(0, 1)) \) it holds that

\[
\begin{aligned}
\iint_{Q_T} (-(x + \eta)^\alpha u_{\epsilon t}^\epsilon \zeta_t + u_{\epsilon xx}^\epsilon \zeta_x - (x + \eta)^{\alpha/2} b^\epsilon u^\epsilon \zeta_x + cu^\epsilon \zeta + (x + \eta)^{\alpha/2-1} \gamma u^\epsilon \zeta) \, dx \, dt \\
= \iint_{Q_T} v^\epsilon(0, \tau) \zeta(0, \tau) \, dx \, dt + \frac{1}{\epsilon} \int_0^1 (x + \eta)^\alpha u_0(x) \zeta(x, 0) \, dx - \frac{1}{\epsilon} \int_0^1 (x + \eta)^\alpha u^\epsilon(x, T) \zeta(x, T) \, dx
\end{aligned}
\]

and

\[
\begin{aligned}
\iint_{Q_T} ((x + \eta)^\alpha v_{\epsilon t}^\epsilon \bar{g} - v_{\epsilon xx}^\epsilon \bar{g}_x + (x + \eta)^{\alpha/2} b^\epsilon v^\epsilon \bar{g} - c v^\epsilon \bar{g} - (x + \eta)^{\alpha/2-1} \gamma v^\epsilon \bar{g}) \, dx \, dt = 0.
\end{aligned}
\]
Choosing $\zeta = v^\epsilon$ in (4.5) and $\varrho = u^\epsilon$ in (4.6) yields

$$
\int_0^T (v^\epsilon)^2 \, dx \, dt + \frac{1}{\epsilon} \int_0^T (x + \eta)^{\alpha}(u^\epsilon)^2(x, T) \, dx = - \int_0^T (x + \eta)^{\alpha} v^\epsilon(x, 0) u_0(x) \, dx. 
$$

(4.7)

It follows from the Hölder inequality and Theorem 3.11 together with the first inequality in (4.3) that

$$
\left| \int_0^T (x + \eta)^{\alpha} v^\epsilon(x, 0) u_0(x) \, dx \right| \leq \left( \int_0^T (x + \eta)^{\alpha}(v^\epsilon)^2(x, 0) \, dx \right)^{1/2} \left( \int_0^T (x + \eta)^{\alpha} u_0^2(x) \, dx \right)^{1/2}
$$

$$
\leq \tilde{M} \left( \int_0^T (v^\epsilon)^2 \, dx \, dt \right)^{1/2} \left( \int_0^T (x + \eta)^{\alpha} u_0^2(x) \, dx \right)^{1/2}
$$

$$
\leq \frac{1}{2} \int_0^T (v^\epsilon)^2 \, dx \, dt + \frac{1}{\epsilon} \int_0^T (x + \eta)^{\alpha} u_0^2(x) \, dx,
$$

where $\tilde{M} > 0$ depends only on $N$, $K$, $x_0$, $x_1$, $T$, $\alpha$ and $\beta$. Combining this estimate with (4.7) leads to

$$
\frac{1}{2} \int_{Q_T} (h^\epsilon)^2 \, dx \, dt + \frac{1}{\epsilon} \int_0^T (x + \eta)^{\alpha}(u^\epsilon)^2(x, T) \, dx \leq \frac{1}{2} \tilde{M} \int_0^T (x + \eta)^{\alpha} u_0^2(x) \, dx. 
$$

(4.8)

Due to the first inequality in (4.3), (4.8) and Proposition 2.5, there exist a positive sequence $\{\epsilon_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\lim_{n \to \infty} \epsilon_n = 0$, $h \in L^2(Q_T)$ and $u \in L^2(Q_T; H_0^1(0, 1)) \cap L^\infty(0, 1)$ such that

$$
h^\epsilon_n \to h, \quad u^\epsilon_n \to u, \quad u_x^\epsilon_n \to u_x \quad \text{weakly in } L^2(Q_T) \text{ as } n \to \infty, 
$$

(4.9)

$$
\lim_{n \to \infty} \int_0^1 (x + \eta)^{\alpha}(u^\epsilon_n)^2(x, T) \, dx = 0. 
$$

(4.10)

For each $\zeta \in C^1(Q_T)$ with $\zeta(0, \cdot)|_{(0, \eta)} = \zeta(1, \cdot)|_{(0, \eta)} = 0$, choosing $\epsilon = \epsilon_n$ in (4.5) and then letting $n \to \infty$, together with the second inequality in (4.3), (4.9) and (4.10), we get that

$$
\int_{Q_T} \left( -(x + \eta)^{\alpha} u_{\zeta} + u_x \zeta - (x + \eta)^{\alpha/2} bu_\zeta + cu_\zeta + (x + \eta)^{\alpha/2 - 1} yu_\zeta \right) \, dx \, dt
$$

$$
= \int_{Q_T} h \zeta \omega \, dx \, dt + \int_0^1 (x + \eta)^{\alpha} u_0(x) \zeta(x, 0) \, dx.
$$

Therefore, $u$ is the solution to problem (1.19)–(1.21) and satisfies (4.1). Finally, (4.2) follows from (4.8) and (4.9).

\end{proof}

### 4.2 Semilinear case

Owing to Lemma 4.1 and Proposition 2.5, one can prove by a fixed point argument (see, for example, [7, 10]) that the semilinear system (1.22), (1.20), (1.21) is null controllable. The proof is standard and we omit it here.

**Lemma 4.2.** Assume that $\alpha > 0$, $0 \leq \beta < 2$, $0 < x_0 < x_1 < 1$, $\|x^\beta c\|_{L^\infty(Q_T)}$, $\|x^2 c\|_{L^\infty(Q_T)} \leq N$ for some $N > 0$, and $\tilde{p}$, $\tilde{q}$ satisfy (1.23), (1.24). Then for each $u_0 \in L^2(0, 1)$ there exists $h \in L^2(Q_T)$ such that the solution $u$ to problem (1.22), (1.20), (1.21) satisfies

$$
u(x, T) = 0, \quad x \in (0, 1).$$

Furthermore, there exists $M > 0$ depending only on $N$, $K$, $x_0$, $x_1$, $T$, $\alpha$ and $\beta$ such that

$$
\|h\|_{L^2(Q_T)} \leq M\|(x + \eta)^{\alpha/2} u_0\|_{L^2(0, 1)}.
$$
Remark 4.3. Lemmas 4.1 and 4.2 still hold if \( \| (x + \eta)^{\beta} c \|_{L^{\infty}(Q_T)} \leq N (\beta \in [0, 2)) \) is relaxed by
\[
\| (x + \eta)^{2} c \|_{L^{\infty}(Q_T)} \leq C_0 < \frac{1}{4}.
\]
But \( M \) depends also on \( C_0 \).

Turn to the null controllability of system (1.25), (1.2), (1.3).

**Theorem 4.4.** Assume \( \alpha > 0, 0 \leq \beta < 2, 0 < x_0 < x_1 < 1, x^\beta c, x^2 c_t \in L^{\infty}(Q_T) \) and \( p, q \) satisfy (1.26), (1.27). Then system (1.25), (1.2), (1.3) is null controllable. More precisely, for each \( u_0 \in \mathcal{F}_a \) there exists \( h \in L^2(Q_T) \) such that the solution \( u \) to problem (1.25), (1.2), (1.3) satisfies
\[
u(x, T) = 0, \quad x \in (0, 1).
\]
Furthermore, there exists \( M > 0 \), depending only on \( \| x^\beta c \|_{L^{\infty}(Q_T)}, \| x^2 c_t \|_{L^{\infty}(Q_T)}, K, x_0, x_1, T, \alpha \) and \( \beta \), such that
\[
\| h \|_{L^1(Q_T)} \leq M \| x^{\alpha/2} u_0 \|_{L^2(0, 1)}.
\]

**Proof.** Set
\[
\hat{p}(x, t, u) = x^{-\alpha/2} p(x, t, u), \quad \hat{q}(x, t, u) = x^{1-\alpha/2} (q(x, t, u) - c(x, t) u), \quad (x, t, u) \in Q_T \times \mathbb{R}.
\]
Then (1.25) is equivalent to
\[
x^\alpha u_t - u_{xx} + (x^{\alpha/2} \hat{p}(x, t, u))_x + c(x, t) u + x^{\alpha/2-1} \hat{q}(x, t, u) = h(x, t) \chi_\omega, \quad (x, t) \in Q_T.
\]
Choose \( \{u_0^\eta\}_{0 < \eta < 1} \subset L^2(0, 1) \) and \( c_t^\eta \in C^1(\overline{Q_T}) \) such that
\[
\| (x + \eta)^{\alpha/2} u_0^\eta \|_{L^2(0, 1)} + \| (x + \eta)^{\beta} c_t^\eta \|_{L^{\infty}(Q_T)} + \| (x + \eta)^{\alpha} c_t^\eta \|_{L^{\infty}(Q_T)} \leq \| x^{\alpha/2} u_0 \|_{L^2(0, 1)} + \| x^\beta c_t \|_{L^{\infty}(Q_T)} + \| x^2 c_t \|_{L^{\infty}(Q_T)}, \quad 0 < \eta < 1,
\]
and
\[
\lim_{\eta \to 0} \frac{1}{0} \int ((x + \eta)^{\alpha/2} u_0^\eta(x) - x^{\alpha/2} u_0(x))^2 \, dx = 0, \quad \lim_{\eta \to 0} \int_{Q_T} ((x + \eta)^{\alpha} c_t^\eta(x) - x^\beta c(x))^2 \, dx \, dt = 0.
\]
Owing to Lemma 4.2, for each \( \eta \in (0, 1) \) there exists \( h^\eta \in L^2(Q_T) \) with
\[
\| h^\eta \|_{L^2(Q_T)} \leq M \| (x + \eta)^{\alpha/2} u_0^\eta \|_{L^2(0, 1)}
\]
such that the solution \( u^\eta \in L^2(0, T; H^1_0(0, 1)) \cap L^{\infty}(0, T; L^2(0, 1)) \) to the problem
\[
\begin{cases}
(x + \eta)^{\alpha/2} u^\eta_t - u^\eta_{xx} + ((x + \eta)^{\alpha/2} \hat{p}(x, t, u^\eta))_x + c(x, t) u^\eta + (x + \eta)^{\alpha/2-1} \hat{q}(x, t, u^\eta) = h^\eta(x, t) \chi_\omega, & (x, t) \in Q_T, \\
u^\eta(0, t) = \nu^\eta(1, t) = 0, \quad & t \in (0, T), \\
u^\eta(0, 0) = u_0^\eta(x), \quad & x \in (0, 1)
\end{cases}
\]
satisfies
\[
u^\eta(x, T) = 0, \quad x \in (0, 1),
\]
where \( M > 0 \) depends only on \( \| x^\beta c \|_{L^{\infty}(Q_T)}, \| x^2 c_t \|_{L^{\infty}(Q_T)}, K, x_0, x_1, T, \alpha \) and \( \beta \). That is to say,
\[
\int_{Q_T} (- (x + \eta)^{\alpha} u^\eta \zeta_t + u^\eta \zeta_x) - (x + \eta)^{\alpha/2} \hat{p}(x, t, u^\eta) \zeta_x + c_t^\eta u^\eta \zeta + (x + \eta)^{\alpha/2-1} \hat{q}(x, t, u^\eta) \zeta) \, dx \, dt
\]
\[
= \int_{Q_T} h^\eta \chi_\omega \zeta \, dx \, dt + \frac{1}{0} (x + \eta)^{\alpha} u_0^\eta(x) \zeta(x, 0) \, dx
\]
for each $\zeta \in C^1(\overline{Q}_T)$ with $\zeta(0, \cdot)|_{(0, T)} = \zeta(1, \cdot)|_{(0, T)} = 0$. Rewrite the equation of $u^0$ as the following linear equation:

$$(x + \eta)^{\alpha/2} u^0_t - u^0_{xx} + ((x + \eta)^{\alpha/2} b^0(x, t) u^0_t + c^0(x, t) u^0 + (x + \eta)^{\alpha/2 - 1} y^0(x, t) u^0 = h^0(x, t) x_\omega, \quad (x, t) \in Q_T,$$

where

$$b^0(x, t) = \begin{cases} \frac{\hat{p}(x, t, u^0(x, t))}{u^0(x, t)}, & u^0(x, t) \neq 0, \\ 0, & u^0(x, t) = 0, \end{cases} \quad (x, t) \in Q_T,$$

$$y^0(x, t) = \begin{cases} \frac{\tilde{q}(x, t, u^0(x, t))}{u^0(x, t)}, & u^0(x, t) \neq 0, \\ 0, & u^0(x, t) = 0, \end{cases} \quad (x, t) \in Q_T.$$

Then Proposition 2.5 shows

$$\| (x + \eta)^{\alpha/2} u^0_t \|_{L^\infty(0, T; L^2(0, 1))} + \| u^0_t \|_{L^2(Q_T)} + \| u^0_r \|_{L^2(Q_T)} \leq \tilde{M} \left( \| (x + \eta)^{\alpha/2} u^0_0 \|_{L^2(0, 1)} + \| h^0 \|_{L^2(Q_T)}^2 \right) \quad (4.17)$$

and

$$\int_0^{T-\varepsilon} \int_0^1 (u^0_t(x, t + \varepsilon) - u^0_t(x, t))^2 \, dx \, dt \leq \tilde{M} \varepsilon \left( \| (x + \eta)^{\alpha/2} u^0_0 \|_{L^2(0, 1)}^2 + \| h^0 \|_{L^2(Q_T)}^2 \right) \quad (4.18)$$

for each $\varepsilon \in (0, T)$, where $\tilde{M} > 0$ depends only on $\| x^0 c^0 \|_{L^\infty(Q_T)}$, $K$, $T$, $\alpha$ and $\beta$. Owing to (4.13), (4.15), (4.17) and (4.18), there exist $\eta_n \in (0, 1)$ ($n \geq 1$) with $\lim_{n \to \infty} \eta_n = 0$, $E_m \subset Q_T$ ($m \geq 1$) with $\lim_{m \to \infty} \text{meas}(E_m) = 0$, $h \in L^2(Q_T)$ and $u \in L^2(0, T; H^1_0(0, 1))$ with $x^{\alpha/2} u \in L^\infty(0, T; L^2(0, 1))$, such that

$$h^{\eta_n} \to h, \quad u^{\eta_n} \to u, \quad u^{\eta_n}_x \to u_x \quad \text{weakly in } L^2(Q_T) \text{ as } n \to \infty,$$

$$u^{\eta_n} \to u \quad \text{uniformly in } Q_T \setminus E_m \text{ as } n \to \infty \text{ for each positive integer } m. \quad (4.19)$$

For each $\zeta \in L^2(Q_T)$, it holds that

$$\left| \int_{Q_T} ((x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u))^2 \zeta \, dx \, dt \right|$$

$$\leq \left| \int_{Q_T} ((x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u))^2 \zeta \, dx \, dt \right|$$

$$+ \left| \int_{B_\delta \setminus E_m} ((x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u))^2 \zeta \, dx \, dt \right|$$

$$+ \left| \int_{B_\delta \setminus E_m} ((x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u))^2 \zeta \, dx \, dt \right|$$

$$\leq \| (x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u) \|_{L^2(Q_T)} (\| \zeta \|_{L^2(A_\delta)} + \| \zeta \|_{L^2(B_\delta \setminus E_m)})$$

for each $\zeta \in L^2(Q_T)$, it holds that

$$\lim_{n \to \infty} \int_{Q_T} ((x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u))^2 \zeta \, dx \, dt = 0,$$

which shows

$$\frac{(x + \eta_n)^{\alpha/2} \hat{p}(x, t, u^{\eta_n}) - x^{\alpha/2} \hat{p}(x, t, u)}{u^{\eta_n}} \to \frac{x^{\alpha/2} \hat{p}(x, t, u)}{u} \quad \text{weakly in } L^2(Q_T) \text{ as } n \to \infty.$$

A similar discussion gives

$$\frac{(x + \eta_n)^{\alpha/2 - 1} \tilde{q}(x, t, u^{\eta_n}) - x^{\alpha/2 - 1} \tilde{q}(x, t, u)}{u^{\eta_n}} \to \frac{x^{\alpha/2 - 1} \tilde{q}(x, t, u)}{u} \quad \text{weakly in } L^2(Q_T) \text{ as } n \to \infty.$$

(4.21)
For each $\zeta \in C^1 (\overline{Q}_T)$ with $\zeta (0, \cdot )|_{(0, T)} = \zeta (1, \cdot )|_{(0, T)} = 0$, letting $n \to \infty$ in (4.16), together with (4.19), (4.21), (4.22) and (4.14), leads to

$$\iint_{Q_T} (-x^n u_{\zeta T} + u_x \zeta_x - x^{n/2} p(x, t, u) \zeta_x + cu \zeta + x^{n/2-1} q(x, t, u) \zeta) \, dx \, dt = \iint_{Q_T} h^\omega \zeta \, dx \, dt + \int_0^1 x^n u_0(x) \zeta(x, 0) \, dx,$$

or equivalently to

$$\iint_{Q_T} (-x^n u_{\zeta T} + u_x \zeta_x - p(x, t, u) \zeta_x + q(x, t, u) \zeta) \, dx \, dt = \iint_{Q_T} h^\omega \zeta \, dx \, dt + \int_0^1 x^n u_0(x) \zeta(x, 0) \, dx.$$

Hence, $u$ is the solution to problem (1.25), (1.2), (1.3) and satisfies (4.11). Finally, (4.12) follows from (4.15) and (4.19).

**Remark 4.5.** Owing to Remarks 4.3 and 2.7, Theorem 4.4 still holds if $x^\beta c \in L^\infty (Q_T)$ ($\beta \in [0, 2)$) is relaxed by $x^\beta c \in L^\infty (Q_T)$ with $\|x^\beta c\|_{L^\infty (Q_T)} \leq c_0 < \frac{1}{2} t$.

**Remark 4.6.** Theorem 4.4 still holds if $\omega \subset (0, 1)$ is a nonempty open set.

**Corollary 4.7.** Assume that $0 < \alpha \leq 2$, $0 < x_0 < x_1 < 1$ and $g$ satisfies (1.29). Then system (1.28), (1.2), (1.3) is null controllable.

**Acknowledgment:** The authors would like to express their sincerely thanks to the referees and the editor for their helpful comments on the original version of the paper.

**Funding:** Supported by the German–Chinese research project on PDEs and the National Natural Science Foundation of China (nos. 11222106 and 11571137).

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