ON NILPOTENT ORBITS OF $\text{SL}_n$ AND $\text{Sp}_{2n}$ OVER A LOCAL NON-ARCHIMEDEAN FIELD

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Abstract. We relate the partition-type parametrization of rational (arithmetic) nilpotent adjoint orbits of the classical groups $\text{SL}_n$ and $\text{Sp}_{2n}$ over local non-Archimedean fields with a parametrization, introduced by DeBacker in 2002, which uses the associated Bruhat-Tits building to relate the question to one over the residue field.

1. Introduction

Let $k$ be a local non-Archimedean field and let $G$ be a reductive linear algebraic group defined over $k$. In [3], DeBacker parametrizes the set of $k$-rational (that is, arithmetic) nilpotent adjoint orbits of $G$ by equivalence classes of objects coming from the Bruhat-Tits building of the group. This parametrization forms a key step in DeBacker’s proof of the range of validity of the Harish-Chandra-Howe character expansion in [4].

As a special case of the Dynkin-Kostant classification, one can parametrize the algebraic (that is, geometric) nilpotent adjoint orbits of a classical algebraic group $G$ explicitly by way of the action of $\mathfrak{sl}_2$-triples on the standard representation $V$, and this classification can be conveniently interpreted via partitions of $n = \dim(V)$. The $k$-rational points of each algebraic orbit decompose into one or more rational orbits under the action of $G(k)$. The parametrization of these rational orbits thus additionally involves terms dependent on the field $k$, such as equivalence classes of nondegenerate quadratic forms.

In this paper, we give such a partition-type classification of rational nilpotent orbits of $\text{SL}_n$ in Proposition 4.1 and of $\text{Sp}_{2n}$ in Propositions 5.1 and 5.5 based on the argument for real groups given in [2]. We then interpret DeBacker’s parameter set explicitly for the groups $\text{SL}_n$ and $\text{Sp}_{2n}$, and define a map from this partition-type parametrization to the DeBacker one for rational nilpotent orbits of these groups. This is the content of Theorem 4.2 and Theorem 5.6.

One also may classify algebraic nilpotent adjoint orbits of reductive linear algebraic groups via the Bala-Carter classification, and DeBacker describes his parametrization as “an affine analogue of Bala-Carter theory” [3]. A real analogue of the Bala-Carter classification was provided in [13] by Noël. Though originally proven over algebraically closed fields of characteristic either zero or sufficiently large, Bala-Carter theory has been extended to algebraically closed fields of good characteristic.

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characteristic by Pommerening. Recent work of McNinch, including \cite{10, 11}, has also advanced the theory for rational orbits over fields of low characteristic. More generally and classically, given an algebraic orbit one may apply Galois cohomology towards understanding the corresponding rational orbits; this was used previously by the author in studying admissible nilpotent orbits of exceptional $p$-adic groups in \cite{13}. Waldspurger gave a parametrization of rational nilpotent orbits of symplectic, orthogonal and unitary groups over the $p$-adic numbers in \cite[1.6]{15}; his symplectic case is equivalent to our Proposition 5.1 for such $k$.

These various parametrization schemes have different ranges of validity and of applicability. The Bala-Carter parametrization of algebraic orbits, as well as the DeBacker parametrization of rational orbits, applies to all reductive linear algebraic groups, whereas partition-type classifications are restricted to the classical groups. The Bala-Carter classification avoids the problematic (in low characteristic) use of Lie triples upon which the DeBacker parametrization in \cite{3} and partition-type classifications rely. Partition-type classifications yield explicit representatives and encode much information about the corresponding algebraic orbit, including dimension, the closure ordering on orbits, and whether the orbit is special, even or distinguished. In contrast, we see here that these properties are not easily read from the DeBacker parametrization.

There are a number of interesting questions to pursue with regards to DeBacker’s parametrization. The first is that the parametrization varies with the choice of a real number $r$. For different choices of $r$, the number of classes of objects in the building ($r$-facets) to which orbits are associated can increase, and so correspondingly the number of orbits associated to each $r$-facet will decrease. In particular, two orbits associated to the same class of $r$-facet may be associated to different classes of $s$-facets, for $r \neq s$. That said, given the $r$-facet to which an orbit is associated, plus additional information about the orbit (a Lie triple and adapted one-parameter subgroup) it is easy to find the $s$-facet to which it is associated, for any $s$ (Corollary 3.7). Choosing $r$ to be irrational maximizes the number of classes of $r$-facets, a feature we exploit in the proofs of Theorems 4.2 and 5.6.

Secondly, the classification of the classes of $r$-facets seems quite difficult in general. We determine some equivalence classes for the case of $\text{SL}_n$ in Corollary 4.3. The problem can be reduced to a finite computation, since it suffices to consider all $r$-facets meeting a fundamental chamber, but an elegant solution seems elusive.

Thirdly, the dimension of the $r$-facet to which a given rational orbit is associated is not in general an invariant of the algebraic orbit. This feature may perhaps offer more tools to distinguish between the various rational orbits in one algebraic class. We prove the dimension is an invariant for $\text{SL}_n$ in Corollary 4.3 and is not for $\text{Sp}_{2n}$ in Corollary 5.8.

Finally, the DeBacker parametrization is proven under hypotheses which require large residual characteristic, although DeBacker conjectures \cite{3} that the correspondence should hold more generally. We explore this question through some examples in Section 6 where we see that one sticking point is that the intrinsic definition of the classes arising in the correspondence (distinguished pairs, Definition 5.8) doesn’t extend to small residual characteristic. We can also show that (not surprisingly) the correspondence cannot hold if the residual characteristic is not good for $\mathbb{G}$.
This paper is organized as follows. In Section 2 we set our notation and recall several well-known results about Lie \((\mathfrak{sl}_2)\) triples, the partition-type classification of algebraic nilpotent orbits, Bruhat-Tits buildings and Moy-Prasad filtrations. In Section 3 we summarize the key results needed here about DeBacker’s parametrization of rational nilpotent orbits.

In Section 4 we turn our attention to the group \(\text{SL}_n\). We first give the parametrization of rational nilpotent orbits of \(\text{SL}_n(k)\) in Proposition 4.1; this is presumably well-known. Our purpose is to produce preferred representatives of the orbits, which we use to deduce the corresponding DeBacker parameters in Theorem 4.2. We conclude the section by stating several Corollaries of the main theorem.

In Section 5, we consider the group \(\text{Sp}_{2n}\). We begin by describing the partition-type classification of rational nilpotent orbits in Section 5.1. To give explicit representatives of these orbits, we briefly recall some facts about quadratic forms over local fields, and then choose preferred representatives for equivalence classes of non-degenerate quadratic forms, in Section 5.2. The corresponding orbit representatives are given in Section 5.3 and our main theorem is presented in Section 5.4.

We conclude in Section 6 with some illustrative examples and discussion about issues arising in small residual characteristic.

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## 2. Preliminaries

Let \(k\) be a non-Archimedean local field of characteristic \(p\) with finite residue field \(\mathfrak{p}\) of characteristic \(p\). Then either \(k\) is a \(p\)-adic field of characteristic 0, or \(k\) is a field of Laurent series over a finite field \(\mathfrak{p}\), and \(p = p_\ell > 0\). For the DeBacker correspondence we will require \(p_\ell\) to be sufficiently large, but for the partition-type classification it will suffice to ask that \(p\) is either zero or sufficiently large.

Let \(\mathcal{O}\) denote the integer ring of \(k\) and \(\mathfrak{p}\) the maximal ideal of \(\mathcal{O}\). Let \(\varpi\) denote a uniformizer of \(k\) and let the discrete valuation on \(k\) be normalized so that \(\text{val}(\varpi) = 1\). Let \(K\) be an algebraic closure of \(k\).

Let \(G\) be a classical linear algebraic group defined over \(k\), and identify \(G = G(k)\). We also write \(V\) in place of \(V(k)\), where this will not cause confusion.

In this paper, we will consider \(G = \text{SL}_n\) or \(G = \text{Sp}_{2n}\), for some \(n \geq 2\). The group \(\text{SL}_n\) consists of unimodular matrices. Set \(J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\), where \(I\) is the \(n \times n\) identity matrix. Then, writing \(A^\dagger\) for the transpose of the matrix \(A\), we embed \(\text{Sp}_{2n}(K)\) into the general linear group \(\text{GL}_{2n}(K)\) as \(\{ g \in \text{GL}_{2n}(K) : g^\dagger J g = J \}\). We denote the symplectic form on \(V\) corresponding to \(J\) by \(\langle x, y \rangle = x^\dagger J y\).

A *partition* \(\lambda\) of a positive integer \(n\) is a sequence \((\lambda_1, \ldots, \lambda_t)\) of positive integers in weakly decreasing order with the property that \(\sum_{i=1}^t \lambda_i = n\). The \(\lambda_i\) are called the *parts* of the partition \(\lambda\) and for any \(j\), the multiplicity of \(j\) in \(\lambda\), denoted \(m_j^\lambda\) or \(m_j\), is the number of parts \(\lambda_i\) such that \(\lambda_i = j\). Write \(\gcd(\lambda)\) for the greatest common divisor of the parts of \(\lambda\).
2.1. Some $\mathfrak{sl}_2(k)$-modules. Consider the basis $\{e, h, f\}$ of $\mathfrak{sl}_2(k)$ for which $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. For each positive integer $j$, we can construct a $j$-dimensional module $W_j$ of $\mathfrak{sl}_2(k)$ with basis $\{e^{j-1}w, e^{j-2}w, \cdots, ew, w\}$, subject to $h(e^jw) = (2i+1-j)e^jw$, $e^jw = 0$, and for $i > 0$, $f(e^jw) = i(j-i)e^{i-1}w$. The $k$-subspace spanned by $w$ is called the lowest weight space of this module; in general we refer to the eigenspaces of $h$ as weight spaces. Write $\pi_j$ for the representation afforded by this module; then with respect to this basis we have $\pi_j(e) = J_j$, the upper triangular matrix in Jordan normal form corresponding to a single Jordan block. We also have

\begin{equation}
\begin{aligned}
\pi_j(h) &= H_j = \text{diag}(j-1, j-3, \cdots, -j+1), \\
\pi_j(f) &= Y_j = J_j^\dagger \text{diag}(j-1, 2(j-2), \cdots, j-1, 0),
\end{aligned}
\end{equation}

which are diagonal and lower triangular matrices, respectively.

Now let $W$ be a finite-dimensional $\mathfrak{sl}_2(k)$-module affording a representation $\pi$ such that there exists a positive integer $m$ with $\pi(e)^m = \pi(f)^m = 0$. Then [1 Thm 5.4.8] if $k$ has characteristic zero or $p > m + 1$, $W$ decomposes as a direct sum of irreducible submodules, each one of which is isomorphic to $(\pi_j, W_j)$ for some $1 \leq j \leq m$. The dimensions of these irreducibles, counted with multiplicity, define a partition $\lambda$ of $\dim(W)$. Conversely, any partition of a positive integer $n$ (such that if $p > 0$ then $n + 1 < p$) defines a unique isomorphism class of $\mathfrak{sl}_2(k)$-module.

In particular, choosing bases for the irreducibles as above, the matrices for $\pi(e)$, $\pi(h)$ and $\pi(f)$ take block-diagonal form, with blocks of size equal to the parts of $\lambda$, in decreasing order. Write $J_\lambda$, $H_\lambda$ and $Y_\lambda$ for these $n \times n$ matrices; so $J_\lambda$ is the upper triangular matrix in Jordan normal form corresponding to the partition $\lambda$.

2.2. Lie triples. Suppose $k$ is a field of characteristic zero, or of characteristic $p > 3(h-1)$ where $h$ is the Coxeter number of $\mathfrak{g}$. (Recall that $h = n$ for $\mathbb{G} = \text{SL}_n$ and $h = 2n$ for $\mathbb{G} = \text{Sp}_{2n}$.) Let $X$ be a nonzero nilpotent element in $\mathfrak{g}$. Then by the Jacobson-Morozov Theorem (see, for instance, [1 §5.4]) there exists a Lie algebra homomorphism $\phi: \mathfrak{sl}_2 \to \mathfrak{g}_K$ defined over $k$ such that $\phi(e) = X$; we call $(\phi(f), \phi(h), \phi(e))$ (or, by abuse of notation, the map $\phi$) a Lie triple corresponding to $X$.

Furthermore, there exists a homomorphism of algebraic groups $\varphi: \text{SL}_2 \to \mathbb{G}$, defined over $k$, such that $d_\varphi = \phi$. Define a one-parameter subgroup $\lambda$ of $\mathbb{G}$ via $\lambda(t) = \varphi \left( \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \right)$ for $t \in k^\times$. Then $d\lambda(1) = \phi(h)$ and $\lambda$ is said to be adapted to the Lie triple $(\phi(f), \phi(h), \phi(e))$ [3 Def 4.5.6].

2.3. Classification of algebraic nilpotent orbits. Let $k, K$ be as above. The material in this section is adapted from [2] (where the hypothesis $p = 0$ was assumed).

Let $X$ be a nonzero nilpotent element in $\mathfrak{g}_K$ and $\phi$ a corresponding Lie triple. Through $\phi$, $V$ is a completely reducible $\mathfrak{sl}_2(K)$-module. Let $V(j)$ denote the isotypic component of $V$ of all $j$-dimensional irreducible submodules; then there is an isomorphism

$$\phi_j: L(j) \otimes W_j \to V(j) \quad \quad \quad v \otimes e^jw \mapsto X^jv,$$
where \( L(j) \) denotes the subspace of lowest weight vectors of \( V(j) \). Then we may write
\[
\tag{2.2} V = \bigoplus_{j \in \mathbb{N}} V(j) \cong \bigoplus_{j \in \mathbb{N}} L(j) \otimes W_j.
\]

If \( X' \) is another nilpotent element in the same \( G \)-orbit, with associated \( \text{Lie} \) triple \( \phi' \), then by the Jacobson-Morozov Theorem there exists an element \( g \in \mathfrak{g} \) intertwining \( \phi \) and \( \phi' \) as representations of \( \mathfrak{sl}_2(K) \) (see, for example, \cite[Proposition 5.6.4]{[1]}). It follows that the modules \( (2.2) \) arising from \( X \) and \( X' \) are isomorphic, and thus give rise to the same partition of \( \dim(V) \). In fact, we have the following well-known result about nilpotent orbits over \( K \); see, for example, \cite[§3.1, 5.1]{[2]}.

**Proposition 2.1.** When \( G = \text{SL}_n \), the set of nilpotent adjoint orbits is in one-to-one correspondence with the set of partitions of \( n \). When \( G = \text{Sp}_{2n} \), the set of nilpotent adjoint orbits is in one-to-one correspondence with the set of partitions of \( 2n \) in which all odd parts occur with even multiplicity.

When \( G \) is quasi-split one deduces from \cite[Theorem 4.2]{[3]} or \cite{[4]} that every nilpotent \( G \)-orbit which is defined over \( k \) contains a \( k \)-rational element. Thus each algebraic nilpotent orbit of \( \text{SL}_n \) and \( \text{Sp}_{2n} \) decomposes into one or more rational orbits.

More precisely, we may apply the same reasoning as above to deduce that nilpotent \( G \)-orbits are in one-to-one correspondence with \( G \)-orbits of \( k \)-rational \( \text{Lie} \) triples in \( \mathfrak{g} \); the proofs of the Jacobson-Morozov theorem and Kostant’s theorem in \cite[Chap 3]{[2]} are unchanged over \( k \). However, two non-conjugate \( \text{Lie} \) triples \( \phi \) and \( \phi' \) may give rise to the same partition \( \lambda \); this happens exactly when none of the endomorphisms from \( V \) to \( V \) which intertwine the two representations \( \phi \) and \( \phi' \) of \( \mathfrak{sl}_2(k) \) lie in the group \( G \). Distinguishing these cases is the subject of the first parts of each of Sections \cite{[4]} and \cite{[5]}

### 2.4. Buildings and Moy-Prasad filtrations

Let \( k \) be a field of zero or odd characteristic. We consult \cite{[12]} for the theory of Moy-Prasad filtrations. Let \( G = \text{SL}_n(k) \) or \( \text{Sp}_{2n}(k) \) so that \( G \) is connected and split over \( k \). Let \( \mathcal{B}(G) = \mathcal{B}(G, k) \) denote the Bruhat-Tits building of \( G \). The building is endowed with a \( G \)-action. For any \( x \in \mathcal{B}(G) \), set \( G_x = \{ g \in G : g \cdot x = x \} \). Since \( G \) is simple and simply-connected, this is a parahoric subgroup of \( G \). Now let \( T \) be a maximal torus of \( G \) which is \( k \)-split and set \( T = T(k) \). Let \( \mathcal{A} = \mathcal{A}(T) \subset \mathcal{B}(G) \) be the corresponding apartment. Then \( \mathcal{A} \) is the affine space underlying \( X_*(T) \otimes \mathbb{Z} \mathbb{R} \) where \( X_*(T) \) is the group of \( k \)-rational cocharacters (one-parameter subgroups) of \( T \).

Let \( X^*(T) \) be the group of \( k \)-rational characters of \( T \). Recall there is a natural pairing \( X^*(T) \times X_*(T) \to \mathbb{Z} \), which we denote by \( (\alpha, \mu) \mapsto m(\alpha, \mu) \).

Let \( \Phi = \Phi(G, T) \subset X^*(T) \) be the set of roots of \( T \) in \( G \), and let \( \Psi = \Psi(G, T) \) denote the set of affine roots relative to \( \Phi \) and the valuation on \( k \). If we fix an origin, as above, then we may write
\[
\Psi = \{ \phi + n : \phi \in \Phi, n \in \mathbb{Z} \}.
\]

We say \( \phi = \psi \) is the gradient of \( \psi \). Each element of \( \Psi \) defines a map \( \psi : \mathcal{A} \to \mathbb{R} \) by
\[
\psi(\lambda \otimes s) = (\phi + n, \lambda \otimes s) = sm(\phi, \lambda) + n.
\]

For \( \alpha \in \Phi \), let \( \mathfrak{g}_\alpha \) denote the \( \alpha \)-root subspace of \( \mathfrak{g} \). To each \( \psi \in \Psi \), we associate an \( \mathcal{R} \)-submodule \( \mathfrak{g}_\psi \) of the corresponding root space \( \mathfrak{g}_\psi \) by choosing an isomorphism
\(\gamma: k \to g_\psi\) such that \(\gamma(R) = g_\psi \cap g(R)\) and setting

\[g_\psi = \{X \in g_\psi: \text{val}(\gamma^{-1}(X)) \geq \psi - \psi\}\]

To each pair \((x,r) \in B(G) \times R\), Moy and Prasad have associated an \(R\)-subalgebra of \(g\), denote \(g_{x,r}\), defined as follows. Choose an apartment \(A = A(T)\) such that \(x \in A\). Set \(t\) to be the Lie algebra of \(T\); define

\[t_r = \{H \in t: \text{val}(d\chi(H)) \geq r \ \forall \chi \in X^*(T)\}.

Then

\[g_{x,r} = t_r \oplus \sum_{\psi \in \Psi: \psi(x) \geq r} g_\psi.

Similarly, we define \(t_{r+}\) and in turn \(g_{x,r+}\) by replacing each inequality above with a strict inequality. Then \(g_{x,r+} = g_{x,s}\) for some \(s > r\) depending on \(x\).

Explicitly, for \(G = SL_n\), consider the apartment \(A\) corresponding to the diagonal torus \(T\); then identifying \(A = X^*(T) \otimes \mathbb{R}\) we have

\[\Phi = \{e_i - e_j: 1 \leq i \neq j \leq n\}\]

where \(e_i(\text{diag}(t^{x_1}, t^{x_2}, \ldots, t^{x_n}) \otimes s) = sx_i\). In particular, for any \(a \in A\), \(\sum_{i=1}^n e_i(a) = 0\). Our choice of simple system is the set of roots \(\{\alpha_i = e_i - e_{i+1}: 1 \leq i \leq n\}\). Each root space is one-dimensional. An element \(X \in g_{e_i - e_j}\), given in matrix form, has \(X_{kl} = 0\) unless \((k,l) = (i,j)\).

Similarly for \(G = Sp_{2n}\) consider the apartment \(A\) corresponding to the diagonal torus \(T\). By our embedding of \(Sp_{2n}\) the elements of \(T\) are diagonal matrices of the form \(\tau = \text{diag}(t_1, t_2, \ldots, t_n, t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1})\). Then working as above, we have

\[\Phi = \{e_i - e_j, \pm(e_i + e_j), \pm 2e_i: 1 \leq i \neq j \leq n\},

where here \(e_i(\text{diag}(t^{x_1}, \ldots, t^{x_n}, t^{-x_1}, \ldots, t^{-x_n}) \otimes s) = sx_i\). The one-dimensional root spaces may be identified in matrix form as follows. We have

\[g_{e_i - e_j} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^\dagger \end{bmatrix} : A \in M_{n \times n}(k), A_{kl} = 0 \ \text{unless} \ (k,l) = (i,j) \right\} ,\]

\[g_{e_i + e_j} = \left\{ \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} : A \in M_{n \times n}(k), A = A^\dagger, A_{kl} = 0 \ \text{unless} \ (k,l) = (i,j) \right\} ,\]

\[g_{2e_i} = \left\{ \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} : A \in M_{n \times n}(k), A_{kl} = 0 \ \text{unless} \ k = l = i \right\} ,\]

corresponding to the positive roots, and for each \(\alpha \in \Phi^+, X \in g_{-\alpha}\) if and only if \(X^\dagger \in g_\alpha\).

3. DeBacker’s parametrization of nilpotent orbits

All material in this section is summarized from [3]. For the case of split simple groups, the hypotheses of [3] §4.2] (under which Lemmas 3.1 and 3.2 and Theorem 3.3 recalled below, are proven) are satisfied by taking the residual characteristic \(p_f > 3(h-1)\).

Let \(r \in R\). Fix \(A, \Phi, \Psi\) as above. For each \(\psi \in \Psi\) define

\[H_{\psi,r} = \{x \in A: \psi(x) = r\} .\]
Given a finite subset $S \subseteq \Psi$, define $H_S = \bigcap_{\psi \in S} H_{\psi - r}$. Then any connected component $F$ in $H_S$ of the complement

$$H_S \setminus \bigcup_{\psi \in \Psi \setminus S} H_{\psi - r},$$

for some $S$, is called an $r$-facet of $A$. Denote the smallest affine subspace of $A$ containing $F$ by $A(F, A)$ and define $\dim(F) = \dim A(F, A)$.

An $r$-facet has the property that for each $x, y \in F$,

$$g_{x, r} = g_{y, r} \quad \text{and} \quad g_{x, r^+} = g_{y, r^+}.$$

More generally, for each $x \in B(G)$, the generalized $r$-facet $F^B_x$ containing $x$ is the set of all $y \in B(G)$ satisfying (3.1). Then $F^B_x$ is an open convex subset of $B(G)$ whose intersection with any apartment, when nonempty, is an $r$-facet; moreover each such nonempty intersection has the same dimension. Call two generalized $r$-facets $F^B_x$ and $F^B_y$ $r$-associate if there exists a $g \in G$ and an apartment $A$ such that $A(F^B_x \cap A, A) = A(gF^B_y \cap A, A) \neq \emptyset$.

Now for each $x \in F \subset F^B_x$, consider the quotient space

$$V_{F^B_x} = V_F \odot V_{x, r} \odot g_{x, r}/g_{x, r^+};$$

this is naturally a vector space over the residue field $\mathfrak{f}$ with a $\mathfrak{f}$-linear action by the parahoric subgroup $G_x$. Note that $V_{x, r}$ is entirely defined by the roots $\psi \in \Psi$ such that $\psi(x) = r$, which in turn only depends only on $A(F, A)$. In fact there is a natural identification of $V_{F^B_1}$ with $V_{F^B_2}$ whenever $F_1$ and $F_2$ are such that $A(F^B_1, A) = A(F^B_2, A)$.

Call an element $v \in V_{x, r}$ degenerate if there exists a nilpotent element $X \in g_{x, r}$ mapping to $v$ under the quotient map. We consider the set

$$I^n_r \equiv \{(F, v) : v \in \text{a degenerate element of } V_F\}$$

(or use generalized $r$-facets in place of $r$-facets, as in [3 Def 5.3.1]). Define an equivalence relation $\sim$ on $I^n_r$ via $(F_1, v_1) \sim (F_2, v_2)$ if there exists a $g \in G$ such that $A(F^B_1, A) = A(gF^B_2, A)$ and such that, under the natural identification of $V_{F^B_1}$ with $Ad(g)V_{F^B_2}$, the elements $v_1$ and $Ad(g)v_2$ lie in the same orbit under $G_x$ for any $x \in F_1$.

**Lemma 3.1.** [3 Corollary 5.2.4] Let $(F, v) \in I^n_r$, $v \neq 0$. Extend $v$ to a Lie triple

$$(w, h, v) \in V_{2, -r} \times V_{2, 0} \times V_{2, r}.$$ 

Let

$$(Y, H, X) \in g_{2, -r} \times g_{2, 0} \times g_{2, r}$$

be any lift of this triple to a Lie triple over $k$. Then we have that $O(F, v) \doteq Ad(G)X$ is the unique nilpotent orbit of minimal dimension whose intersection with the coset $v = X + g_{2, r^+}$ is nontrivial.

If $v = 0$, define $\mathcal{O}(F, v) = \{0\}$. Let $\text{Nil}(k)$ denote the set of rational nilpotent orbits in $g$.

**Lemma 3.2.** [3 Lemma 5.3.3] The map

$$\gamma : I^n_r / \sim \to \text{Nil}(k)$$

$$(F, v) \mapsto O(F, v),$$

is well defined and surjective.
The map $\gamma$ is not bijective; see Example 3.8 for an interesting example.

Given a Lie triple $(Y, H, X)$ over $k$, set
\[ B(Y, H, X) = B_r(Y, H, X) = \{ x \in B(G) : Y \in g_{x,-r}, X \in g_{x,r} \}. \]

This is a nonempty closed convex subset of $B(G)$ which is the union of generalized $r$-facets, such that any two generalized $r$-facets of maximal dimension in $B(Y, H, X)$ are $r$-associate. Moreover, given $(F, v)$ and an associated triple $(Y, H, X)$ as in Lemma 3.1, we have that $F^B \subset B(Y, H, X)$.

**Definition 3.3.** The degenerate pair $(F, v)$ is distinguished if $F^B$ is a generalized $r$-facet of maximal dimension in $B(Y, H, X)$, where $(F, v)$ and $(Y, H, X)$ are related as in Lemma 3.1.

Denote the set of distinguished pairs by $I^d_r$, and the restriction of $\gamma$ to $I^d_r/\sim$ by $\gamma_d$.

**Theorem 3.4.** [3, Theorem 5.6.1] The map $\gamma_d : (I^d_r/\sim) \to \text{Nil}(k)$ is a bijection.

The following results, which are implicit in [3], are key to establishing our correspondence.

**Proposition 3.5.** The equivalence class of $(F, v)$ is distinguished if and only if $(F, v)$ is maximal, in terms of the dimension of $F$, among all degenerate pairs occurring in $\gamma^{-1}(O(F, v))$.

At issue is that the maximality of $F$ in $B(Y, H, X) \cap A$ does not imply maximality of $F^B$ in $B(Y, H, X)$; see Example 3.8.

**Proof of Proposition 3.5.** Let $O \in \text{Nil}(k)$ and let $(F, v)$ represent an element of $\gamma^{-1}(O)$. Let $(F_0, v_0)$ be a representative of $\gamma_d^{-1}(O)$. Let $(Y, H, X)$ and $(Y_0, H_0, X_0)$ be Lie triples corresponding to $(F, v)$ and $(F_0, v_0)$, respectively. Since they represent the same orbit, these Lie triples are conjugate under $G$, so there exists a $g \in G$ such that
\[ B(Y, H, X) = B(Ad(g)Y_0, Ad(g)H_0, Ad(g)X_0) = gB(Y_0, H_0, X_0). \]

Consequently the maximal generalized $r$-facets in $B(Y, H, X)$ and $B(Y_0, H_0, X_0)$ have the same dimension, which is equal to $\dim(F_0)$ by hypothesis. It follows that $\dim(F) \leq \dim(F_0)$. If equality holds then $Ad(g^{-1})F$ and $F_0$ are $r$-associate by their maximality in $B(Y_0, H_0, X_0)$; thus $(F, v) \sim (F_0, v_0)$ since they represent also the same orbit.

\[ \square \]

**Proposition 3.6.** Let $(Y, H, X)$ be a Lie triple in $g$, and let $\lambda$ be a one-parameter subgroup adapted to $(Y, H, X)$. Then we have
\[ B_r(Y, H, X) = B_0(Y, H, X) + \frac{r}{2} \lambda. \]

In particular the sum is well-defined.

**Proof.** This is the content of [3, Remark 5.1.5]. Set $M = C_G(\lambda)$ and let $M$ be the Levi subgroup of $G$, defined over $k$, such that $M(k) = M$. By [3, Corollary 4.4.2], $B(M) = B(G)^\lambda_k$ and by [3, Corollary 4.5.9], for any $r \in \mathbb{R}$ and $x \in B(G)$, one has $B_r(Y, H, X) \subseteq B(M)$. Hence for any $x \in B_0(Y, H, X)$ there exists a $k$-split torus $T \subset M$ so that $x \in A(T)$ and $\lambda \in X_*(T)$, so the sum is well-defined in $A(T)$. 
To see the equality, note first that for any such $A$ and $x \in A$, $x \in B_r(Y, H, X)$ is equivalent to
\[ X \in \sum_{\psi: \psi(x) \geq r} g_{\psi} \text{ and } Y \in \sum_{\psi: \psi(x) \geq r} g_{-\psi}. \]

Hence the result follows by noting that
\[ \{ \psi: \psi(x) \geq 0, \psi(\lambda) = 2 \} = \{ \psi: \psi(x + \frac{r}{2}\lambda) \geq r, \psi(\lambda) = 2 \}. \]

\[ \square \]

**Corollary 3.7.** Let $(Y, H, X)$, $\lambda$ be as above. If $F_r \subset A$ corresponds to a maximal generalized $r$-facet of $B_r(Y, H, X)$, then there exists $x \in F_r$ and an $s$-facet $F_s$ such that $x + \frac{s-r}{2}\lambda \in F_s'$ and $F_s'$ corresponds to a maximal generalized $s$-facet in $B_s(Y, H, X)$.

**Proof.** By Proposition 3.6 we have that $F_r + \frac{s-r}{2}\lambda \subset B_s(Y, H, X) \cap A$ and by dimension we have that this subset must meet a maximal $s$-facet in an open nonempty set. More precisely, given $x \in F_r$, maximality implies that the only $\psi \in \Psi$ for which $\psi(x) = r$ are among those for which $\psi(\lambda) = 2$; in which case we have $\psi(x + \frac{s-r}{2}\lambda) = s$. Therefore $x + \frac{s-r}{2}\lambda$ will lie in a maximal $s$-facet if and only if for all other $\psi$, $\psi(x) \neq s - \frac{s-r}{2}\psi(\lambda)$. In particular, the set of such $x$ is dense in $F_r$.

\[ \square \]

Corollary 3.7 shows that given a Lie triple $(Y, H, X)$, identifying a maximal $r$-facet in $B_r(Y, H, X)$ for just one value of $r$ determines the $s$-associativity class of $s$-facets which are maximal in $B_s(Y, H, X)$, for any $s$. Thus, in Sections 3 and 5 it suffices to establish our correspondence for values of $r \in \mathbb{R} \setminus \mathbb{Q}$, for example.

The following example illustrates the difficulties one needs to address to explicitly realize the correspondence.

**Example 3.8.** Consider $G = \text{Sp}_4$ and choose $r \in \mathbb{R} \setminus \mathbb{Q}$. Consider the rational nilpotent orbit represented by
\[ X_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

Then the orbit $G \cdot X_1$ contains also a representative of the form
\[ X_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

where $-t \in \mathbb{R}^\times$. Complete these to Lie triples $(X_1, H, Y_1)$ and $(X_0, H, Y_0)$ with $H = \text{diag}(1, 1, -1, -1)$. Then we have
\[ B(Y_1, H, X_1) \cap A = H_{e_1 + e_2 - r} \]
\[ B(Y_0, H, X_0) \cap A = H_{2e_1 - r} \cap H_{2e_2 - r}. \]

Choose $F_1 \in B(Y_1, H, X_1) \cap A$ and $F_0 \in B(Y_0, H, X_0) \cap A$ with $\dim(F_i) = i$, so that each $gF_i$ is maximal in $B(Y_i, H, X_i) \cap A$, and set
\[ v_1 = X_1 + gF_1+ \in V_{F_1} \]
\[ v_0 = X_0 + gF_0+ \in V_{F_0}. \]

By Lemma 3.1 we conclude that
\[ O \cong O(F_1, v_1) = \text{Ad}(G)X_1 = \text{Ad}(G)X_0 = O(F_0, v_0). \]
It follows that \((\mathcal{F}_0, v_0) \in \gamma^{-1}(\mathcal{O})\) and that \(\mathcal{F}_0\) is a maximal \(r\)-facet in \(\mathcal{B}(Y_0, H, X_0) \cap \mathcal{A}\) but that it does NOT correspond to a maximal generalized \(r\)-facet in \(\mathcal{B}(Y_0, H, X_0)\). Hence \((\mathcal{F}_0, v_0) \notin I^d_r\).

On the other hand we may conclude that \((\mathcal{F}_1, v_1) \in I^d_r\), as follows. Since every \(r\)-associativity class necessarily meets the apartment \(\mathcal{A}\), then if \((\mathcal{F}_1, v_1)\) were not distinguished, then we would have

\[
2 \geq \dim(\mathcal{B}(Y_1, H, X_1)) > \dim(\mathcal{B}(Y_1, H, X_1) \cap \mathcal{A}) = 1.
\]

Now if \(\dim(\mathcal{B}(Y_1, H, X_1)) = 2 = \dim(\mathcal{A})\) it would follow that there exists another representative \(X' \in \mathcal{O}\) such that for a corresponding Lie triple \((X', H', Y')\), we have \(\mathcal{B}(Y', H', X') \cap \mathcal{A} \supseteq \mathcal{F}'\) with \(\mathcal{F}'\) an \(r\)-alcove. However, for \(r\) non-integral \(V_{\mathcal{F}} = \{0\}\); hence the unique corresponding orbit under the DeBacker correspondence (for any \(r\)) is the trivial one. This contradicts the choice of \(X'\). Hence \(\dim(\mathcal{B}(Y_1, H, X_1)) = 1\) and \((\mathcal{F}_1, v_1)\) is distinguished.

**Remark 3.9.** As we’ll see in Sections 4 and 5 Lemma 3.1 implies that given a good choice of representative of a nilpotent orbit \(\mathcal{O}\) it is relatively easy to identify elements of \(\gamma^{-1}(\mathcal{O})\). The difficult step in identifying \(\gamma^{-1}(\mathcal{O})\) is the necessity of verifying that one has indeed chosen the \(r\)-associate class of maximal dimension. As Example 3.8 illustrates, to identify maximal generalized \(r\)-facets in \(\mathcal{B}(Y, H, X)\) (and hence distinguished pairs) it does not suffice to consider maximal \(r\)-facets of \(\mathcal{B}(Y, H, X) \cap \mathcal{A}\) for an given apartment \(\mathcal{A}\).

Conversely, suppose we begin with a list of representatives of all the classes of pairs \((\mathcal{F}, v)\) with \(\mathcal{F} \in \mathcal{A}\). To identify the distinguished pairs, apply Proposition 3.5 and work inductively downwards on the dimensions of the facets: for each pair \((\mathcal{F}, v)\), use Lemma 3.1 to determine \(\mathcal{O} = \mathcal{O}(\mathcal{F}, v)\); if \(\mathcal{O}\) has not been obtained from any previous pair, then the pair \((\mathcal{F}, v)\) is distinguished.

One difficulty in this latter approach seems to be to generate a list of representatives \(I^d_r\). The number of classes of facets varies with \(r\) and, as DeBacker notes, identifying orbits of distinguished elements in \(V_{\mathcal{F}}\) is not easy in general; choosing \(r\) irrational is helpful here, as it increases the number of equivalence classes of facets while decreasing the number of orbits associated to each class of facets.

4. The rational nilpotent orbits of \(\text{SL}_n(k)\)

4.1. Partition-type parametrization of rational nilpotent orbits of \(\text{SL}_n\).

Let \(k\) be as in Section 2.2 \(\mathbb{G} = \text{SL}_n\) and \(\lambda\) be a partition of \(n\) with corresponding nilpotent \(\mathbb{G}\)-orbit \(\mathcal{O}_\lambda\). One representative of \(\mathcal{O}_\lambda\) is \(J_\lambda\); this representative is clearly \(k\)-rational.

Suppose that \(X, X' \in \mathcal{O}_\lambda(k)\) and let \(\phi\) and \(\phi'\) be corresponding \(k\)-rational Lie triples. Then there exists \(g \in \text{GL}_n(k)\) which conjugates \(\phi\) to \(\phi'\); so \(g\) preserves the direct sum decomposition \((2.2)\), as well as each of the weight spaces \(V(j)\). Write \(g_j\) for the restriction of \(g\) to \(L(j)\). On each nonzero \(V(j) \simeq L(j) \otimes W_j\), we have \(g(v \otimes u) = (g_j v) \otimes u\) for all \(u \in W_j\), so \(\det(g)|_{V(j)} = \det(g_j)^j\). Taking \(g_j\) to be the identity if \(L(j) = \{0\}\), we have \(\det(g) = \prod_j (\det(g_j))^{j_1}\). This product takes values only in \((k^\times)^m\), where \(m = \gcd(j : L(j) \neq \{0\}) = \gcd(\lambda)\), whence the following result.

**Proposition 4.1.** Let \(\lambda\) be a partition of \(n\) and set \(m = \gcd(\lambda)\). For any \(d \in k^\times\), define \(D(d) = \text{diag}(1, 1, \ldots, 1, d)\).
(1) For each \( d \in k^x \), the matrix \( J_\lambda D(d) \) represents a \( k \)-rational orbit in \( O_\lambda(k) \), and conversely every orbit has a representative of this form.

(2) The \( SL_n(k) \)-orbits represented by \( J_\lambda D(d) \) and \( J_\lambda D(d') \) coincide if and only if \( \lambda = \lambda' \) and \( d \equiv d' \) in \( k^x/(k^x)^m \).

Thus there are exactly \( |k^x/(k^x)^m| \) rational orbits of \( SL_n(k) \) in \( O_\lambda(k) \).

Proof. Let \( X \in O_\lambda(k) \) and set \( X' = J_\lambda \). Then there exists \( g \in GL_n(k) \) such that \( gXg^{-1} = J_\lambda \). Set \( d = \det(g) \); then \( D(d^{-1})g \in SL_n(k) \) and

\[
(D(d^{-1})g)^{-1} = J_\lambda D(d).
\]

Now let \( X = J_\lambda D(d) \) and \( X' = J_\lambda D(d') \); these are nilpotent elements in \( g \). Choose corresponding Lie triples \( \phi \) and \( \phi' \) respectively. That \( \lambda = \lambda' \) follows from Proposition 2.1 and rest from the discussion preceding Proposition 4.1. \( \square \)

It follows immediately, for instance, that if the smallest part of the partition \( \lambda \) defining the algebraic orbit of \( X \) is 1, then the algebraic orbit contains a unique rational orbit. More generally, we note that \( \gcd(\lambda) \) divides \( n \) and all divisors of \( n \) occur for some partition \( \lambda \).

4.2. Correspondence with the DeBacker parametrization. We now determine the \( r \)-associativity class to which each nilpotent orbit is to be associated via the DeBacker correspondence. Assume now additionally that if \( p = 0 \) then the residual characteristic \( p_1 \) is greater than \( 3(h - 1) \).

Given a partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \) of \( n \) and \( D = \text{diag}(d_1, d_2, \cdots, d_n) \in T \), set \( m = \gcd(\lambda) \) and define the set

\[
I_\lambda = \{1, 2, 3, \cdots, n\} \setminus \{\lambda_1, \lambda_1 + \lambda_2, \cdots, \sum \lambda_i = n\}.
\]

This set identifies the nonzero entries of the matrix of \( X = J_\lambda D \); the value \( d_{i+1} \) is the \((i, i+1)\)st entry, for each \( i \in I_\lambda \); all others are zero. Recall that we denote the simple roots of \( G \) by \( \alpha_i = e_i - e_{i+1} \).

**Theorem 4.2.** Let \( \lambda \) and \( D \) be as above. Define

\[
H_{\lambda, D} = \bigcap_{i \in I_\lambda} H_{\alpha_i + \text{val}(d_{i+1}) - r} \subset A
\]

and let \( \mathcal{F} \) be any maximal \( r \)-facet in \( H_{\lambda, D} \). We have \( X = J_\lambda D \in g_{\mathcal{F}} \); set \( v \) to be its image in \( V_\mathcal{F} \). Then \( (\mathcal{F}, v) \in I^d_{\lambda} \) and \( \mathcal{O}(\mathcal{F}, v) = \text{Ad}(G)X \).

Proof. As noted previously, the zero orbit corresponds to the \( r \)-associate class of any \( r \)-alcove, so we may assume from now on that \( X \neq 0 \) and so \( I_\lambda \neq \emptyset \). Set \( N = |I_\lambda| \); then \( N = \text{rank } X \).

Since the set \( \{\alpha_i : i \in I_\lambda\} \subseteq \Phi \) is linearly independent, \( H_{\lambda, D} \) is nonempty and has dimension \( n - 1 - |I_\lambda| \). It is a union of \( r \)-facets; moreover, all maximal \( r \)-facets \( \mathcal{F} \) contained in \( H_{\lambda, D} \) satisfy \( A(\mathcal{F}, A) = H_{\lambda, D} \), and so are strongly \( r \)-associate.

Let \( \mathcal{F} \) be such a facet and \( x \in \mathcal{F} \). Then recall

\[
g_{x, r} = t_r \oplus \sum_{\psi : \psi(x) \geq r} g_\psi.
\]

By construction, \( X = J_\lambda D \in \bigoplus_{i \in I_\lambda} g_{\alpha_i + \text{val}(d_{i+1})} \), and \( \alpha_i(x) + \text{val}(d_{i+1}) = r \), so \( X \in g_{x, r} \). Let \( v \) be the image of \( X \) in \( V_{x, r} = g_{x, r}/g_{x, r+} \).
A Lie triple corresponding to $X$ is $(Y, H, X)$ with $H = H_N$ and $Y = D^{-1}Y_N$. By
the hypotheses on $p_1$, the equalities of the characteristic function $H_N$ and $Y_N$.
Thus $H_N \in t_0 \subset \mathfrak{g}_{x,0}$. Moreover, since val$(d^{-1}) = -\text{val}(d)$ we deduce
$$Y = D^{-1}Y_N \in \bigoplus_{i \in I_N} \mathfrak{g}_{-n_i - \text{val}(d_{i+1})},$$
which is in turn a subset of $\mathfrak{g}_{x,-r}$. Let $(w, h, v) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ be the image
of $(Y, H, X) \in \mathfrak{g}_{x,-r} \times \mathfrak{g}_{x,0} \times \mathfrak{g}_{x,r}$; then these two Lie triples correspond as in
Lemma 5.1 and $O(F, v) = \text{Ad}(G)X$.

Suppose now $(F_0, v_0) \in I_0^n$ is another pair with $F_0 \subset A$ such that $O(F_0, v_0) = \text{Ad}(G)X$. We claim that $\text{dim}(F_0) \leq n - 1 - N$, from which we may conclude that
$(F, v) \in I^r_0$ by maximality of dimension. By Corollary 5.7 it suffices to show this
for just one choice of $r$; let $r \in \mathbb{R} \setminus \mathbb{Q}$.

Let $x \in F_0$. Let $(v_0, h_0, v_0) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ be a Lie triple corresponding
to $v_0$. By [3 Proposition 4.3.2], we have that any lift of $v_0$ to an element $X_0$ in $\mathfrak{g}_{x,r}$
(and lying in the 2-weight space defined by an adapted one-parameter subgroup)
extends to a lift $(Y_0, H_0, X_0)$ of the Lie triple $(v_0, h_0, v_0)$. Furthermore, since $r \notin \mathbb{Z}$
we have $t_r/t_{r+} = \{0\}$. Thus we may choose a lift $X_0$ such that
\begin{equation}
X_0 \in \bigoplus_{\psi : \psi(x) = r} \mathfrak{g}_\psi.
\end{equation}
Since rank $X_0 = \text{rank } X = N$, there exist at least $N$ rows of the matrix of $X_0$ with
nonzero, non-diagonal entries. Permuting the indices as necessary, we may assume
these nonzero entries correspond to affine roots
$$\psi_i = e_i - e_{j_i} + n_i, \quad 1 \leq i \leq N < n, \quad 1 \leq j_i \leq n, \quad i \neq j_i$$
for some $n_i \in \mathbb{Z}$. By (4.2) we have that $\psi_i(x) = r$ for each $i$, for each $x \in F_0$.

Thus any $x \in F_0$ satisfies the inhomogeneous system of $N + 1$ linear equations in
$n$ unknowns
\begin{equation}
(e_i - e_{j_i}) (x) = r - n_i, \quad 1 \leq i \leq N,
\end{equation}
$$\sum_{i=1}^n e_i(x) = 0.$$ To conclude the desired result, it suffices to show that this system is not overdeter-
determined.

For suppose it was. Then there would exist $c_1, c_2, \ldots, c_N, c$, not all zero, such that
\begin{equation}
\sum_{i=1}^N c_i (e_i - e_{j_i}) + c \sum_{i=1}^n e_i = 0.
\end{equation}
Form a directed graph $\Gamma$ with vertices equal to the set of indices $\{1, 2, \ldots, n\}$ and
an edge from $i$ to $j_i$ for each $i \in \{1, 2, \ldots, N\}$. It is an exercise in linear algebra to see that the system (4.4) has rank $N + 1 - k$ where $k$ is the number of distinct closed
cycles of $\Gamma$, and the characteristic functions $f$ of the closed cycles parametrize a
basis for the space of solutions to (4.4) by setting $c_i = f(i)$. Let $B$ denote the
set of vertices in one cycle. Then the corresponding basic solution of (4.4) is
\[ \sum_{i \in B} (e_i - e_j)(x) = 0. \] Together with (4.3) this implies

\[ 0 = \sum_{i \in B} (e_i - e_j)(x) \equiv |B|r \mod \mathbb{Z}, \]

which contradicts the choice of \( r \notin \mathbb{Q} \). Hence \( k = 0 \) and we conclude that (4.3) has a solution space of dimension \( n - 1 - N \). Thus \( \dim(\mathcal{F}_0) = n - 1 - N \), as required. \[\square\]

One immediate corollary of the proof is the following result.

**Corollary 4.3.** If \((\mathcal{F},v) \in I_r^d\) satisfies \(\mathcal{O}(\mathcal{F},v) \subset \mathcal{O}_\lambda(k)\) then

\[ \dim \mathcal{F} = |\lambda| - 1, \]

where \( |\lambda| \) denotes the number of parts, counted with multiplicity, in the partition \( \lambda \).

**Proof.** We saw in the proof of Theorem 4.2 that if \( X \in \mathcal{O}(\mathcal{F},v) \), then \( \dim(\mathcal{F}) = n - 1 - \text{rank}(X) \). When \( \mathcal{O}(\mathcal{F},v) \) is a rational orbit of \( \mathcal{O}_\lambda \), we have \( \text{rank} X = \text{rank} J_\lambda = n - |\lambda| \).

\[\square\]

**Remark 4.4.** Recall from [2] that \( \dim_K(\mathcal{O}_\lambda) = n^2 - |\lambda|^2 - \sum_{i=2}^n \left( \sum_{j \geq i} m_i \right)^2 \),

where \( m_i \) denotes the multiplicity of \( i \) in \( \lambda \). Thus there is no direct relationship between \( \dim(\mathcal{O}(\mathcal{F},v)) \) and \( \dim(\mathcal{F}) \). (As we shall see in Corollary 5.8, \( \dim(\mathcal{F}) \) is not even an invariant of the algebraic orbit.)

In general it is difficult to identify \( r \)-associate classes of facets in \( \mathcal{A} \). However, for the special case that these facets are defined by intersections of hyperplanes corresponding to simple roots, we have a complete answer, as follows.

**Corollary 4.5.** Let \( S \subseteq \{1,2,\ldots,n-1\} \) and for each \( i \in S \) let \( k_i \in \mathbb{Z} \). Suppose \( \mathcal{F} \) is an \( r \)-facet of maximal dimension in

\[ \bigcap_{i \in S} H_{a_i+k_i-r} \subset \mathcal{A}. \]

Then \( \mathcal{F} \) is \( r \)-associate to any \( r \)-facet of maximal dimension in

\[ H_S = \begin{cases} \bigcap_{i \in S} H_{a_i-r} & \text{if } n - 1 \notin S; \\ \bigcap_{i \in S, j \notin n-1} H_{a_i-r} \cap H_{a_{n-1}+K-r} & \text{if } n - 1 \in S, \end{cases} \]

where \( K \) is defined as follows. Let \( \lambda \) be the unordered partition of \( n \) determined by the maximally consecutive subsets of \( S \) and let \( x_l = \sum_{j \leq l} \lambda_j \). Then \( K \) is any integer in the equivalence class modulo \( \gcd(\lambda) \) of the sum

\[ -\sum_{l=1}^{|\lambda|} \lambda_l x_l - 1. \]

**Proof.** Given \( \mathcal{F} \) as above, we can construct a nilpotent \( X \) and its image in \( VX \) such that \((\mathcal{F},v) \in I_r^d\) and \( \text{Ad}(G)X = \mathcal{O}(\mathcal{F},v) \) using the methods of the proof of Theorem 4.2. Namely, if for each \( i \in S \) we choose \( X_i \in g_{a_i+k_i} \setminus g_{a_i+k_i+1} \) then \( X = \sum_{i \in S} X_i \) is such an element. We may write \( X = J_\lambda D \) for some diagonal matrix \( D \) whose diagonal entries \( d_i \) satisfy \( \text{val}(d_{i+1}) = k_i \) for each \( i \in S \).

Now let \( \mathcal{F}_0 \) be a maximal \( r \)-facet in \( H_S \). By DeBacker’s theorem, if we can show that there exists \( X_0 \) so that with \( v_0 = X_0 + V_{\mathcal{F}_0}^+ \), \((\mathcal{F}_0,v_0)\) is another distinguished pair such that \( \mathcal{O}(\mathcal{F}_0,v_0) = \mathcal{O}(\mathcal{F},v) \), then we may deduce that \( \mathcal{F} \) and \( \mathcal{F}_0 \) are \( r \)-associate.
By Proposition 4.1 if we construct a \( g \in GL_n \) such that \( gXg^{-1} = J_\lambda \), then \( X \) lies in the same \( SL_n \) orbit as \( X_0 = J_\lambda D(d) \), for any \( d \equiv \det(g) \mod \gcd(\lambda) \). It is clear that \( X_0 \) is the lift of an element of \( V_{\mathbb{F}_0} \), and hence that we are done, if \( K \equiv \det(d) \mod \gcd(\lambda) \).

Note that the unordered partition \( \lambda \) is defined by first choosing \( 1 = x_1 < x_2 < \cdots < x_t \leq n \) so that for each \( x_i \), either \( x_i \notin S \cup \{n\} \), or \( x_i \in S \cup \{n\} \) and \( x_i - 1 \notin S \). Then we set \( \lambda_l = x_{l+1} - x_l \) and \( \lambda_t = n + 1 - x_t \); \( |\lambda| = t \) is the number of parts in \( \lambda \).

For each \( l \) with \( 1 \leq l \leq t \) consider the matrix \( J_{\lambda_l} D_l \) with
\[
D_l = \text{diag}(d_{x_1}, d_{x_1+1}, \ldots, d_{x_l+\lambda_l-1}).
\]
Then the diagonal matrix
\[
g_l = \text{diag}(\prod_{j=1}^{\lambda_1-1} d_{x_1+j}^{-1}, \prod_{j=2}^{\lambda_2-1} d_{x_1+j}^{-1}, \ldots, d_{x_l+\lambda_l-1})
\]
satisfies \( g_l(J_{\lambda_l} D_l)g_l^{-1} = J_{\lambda_l} \) (as does \( ag_l \) for any nonzero scalar \( a \)) and \( \det(g_l) = \prod_{j=1}^{\lambda_l-1} (d_{x_1+j}^{-1})^j \). Since \( X \) is the direct sum of the \( J_{\lambda_l} D_l \), and \( \det(d_j) = k_{j-1} \), we have
\[
\text{val}(\det(g)) = \sum_{l=1}^{t} \sum_{j=1}^{\lambda_l-1} -jk_{x_1+j-1},
\]
as we wished to show. \( \square \)

5. The rational nilpotent orbits of \( \text{Sp}_{2n}(k) \)

5.1. Nilpotent orbits. Let \( k \) be a local non-Archimedean field of characteristic zero or of characteristic \( p > 3(h-1) \) and assume additionally that the residual characteristic \( p_H \) is odd.

Let \( \lambda \) be a partition of \( 2n \) in which odd parts occur with even multiplicity. Then by Proposition 2.1 there is an algebraic nilpotent adjoint orbit \( O_\lambda \) of \( G = \text{Sp}_{2n} \), the \( k \)-points of which form one or more rational orbits under the action of \( G = \text{Sp}_{2n}(k) \). These rational orbits are parametrized by isometry classes of quadratic forms, as described in Proposition 5.1 below. This theorem is derived using the notation and approach of [2 Chap 9.3]; the result is functionally equivalent to [15 I.6].

**Proposition 5.1.** Let \( \lambda \) be a partition of \( 2n \) and write \( m_j \) for the multiplicity of \( j \) in \( \lambda \). Suppose \( m_j \) is even whenever \( j \) is odd. The \( G \)-orbits in \( O_\lambda(k) \) are parametrized by \( n \)-tuples
\[
\mathcal{Q} = (Q_2, Q_4, \cdots, Q_{2n})
\]
where \( Q_j \) represents the isometry class of a nondegenerate quadratic form over \( k \) of dimension \( m_j \) (taking \( Q_j = 0 \) if \( m_j = 0 \)).

**Proof.** Let \( X \in O_\lambda(k) \) and let \( \phi \subset g \) be a corresponding Lie triple. Then under \( \phi \) the symplectic vector space \( V \) decomposes as
\[
V = \bigoplus_{j \in \lambda} V(j) = \bigoplus_{j \in \lambda} L(j) \otimes W_j,
\]
with \( \dim(L(j)) = m_j \), and each \( V(j) \) a symplectic vector space. The restriction of \( \langle , \rangle \) to \( V(j) \) naturally induces a nondegenerate form \( \langle , \rangle_j \) on the lowest weight space \( L(j) \) via the formula
\[
(v, w)_j = \langle v, X^{j-1} w \rangle \quad \forall v, w \in L(j).
\]
Note that this form is symplectic if $j$ is odd; such a form exists only if $\dim L(j) = m_j$ is even, and then it is unique up to equivalence. If $j$ is even, the form (5.1) is symmetric and nondegenerate; such forms are not unique (cf. the first column of Table 5.4). Given $X' \in O_\lambda(k)$ and a corresponding Lie triple $\phi'$, suppose there exists a $g \in G$ satisfying $\text{Ad}(g)\phi = \phi'$. Then the restriction $g_j$ of $g$ to each $V(j)$ induces an isometry between $(L(j), (\cdot, \cdot)_j)$ and $(L(j), (\cdot, \cdot)_j)$. Conversely, any collection of such isometries lifts to an element of $G = \text{Sp}_{2n}(k)$.

Finally, given any choice of nondegenerate symmetric form on $L(j)$, for each even $j$, one can use (5.1) and (2.2) to define a symplectic form on $V(j)$, and hence build a symplectic form $(\cdot, \cdot)'$ on $V$, with the property that $\phi \subset \text{sp}(V, (\cdot, \cdot))$. Since $\text{sp}(V, (\cdot, \cdot)) \cong g$, it follows that all equivalences classes of nondegenerate symmetric forms (equivalently, of nondegenerate quadratic forms) on $L(j)$ arise for some choice of $X \in O_\lambda(k)$. □

Example 5.2. Consider the groups $\text{Sp}_4$ and $\text{Sp}_6$, and suppose $p = 0$ or $p > 3(h - 1)$ so that Proposition 2.1 can be applied. In Tables 5.1 and 5.2 respectively, we enumerate the algebraic orbits by partition $\lambda$, and deduce the number of rational orbits in $O_\lambda(k)$ using Proposition 5.1 and Table 5.4. The dimension of each orbit can be determined from the partition; see [2 Chap 6]. We include the dimension of the $r$-facet $\mathcal{F}$ associated to $O$ as a preview of Corollary 5.8.

To achieve our orbit correspondence, we need “best” representatives of the different quadratic forms, which is the goal of the next subsection.

5.2. Classification of quadratic forms. In this subsection, $k$ may be any local field of characteristic 0 or odd. Let $\varepsilon \in k^\times$ be an non-square such that $\text{val}(\varepsilon) = 0$.

Let $\Omega$ be a quadratic form on an $m$-dimensional vector space over $k$; write $Q$ for a matrix representing $\Omega$. Define $\dim(\Omega) = m$ and $\text{Det}(\Omega)$ as the class of $\det(Q)$ in $k^\times / (k^\times)^2$. For $a, b \in k^\times$, the Hilbert symbol $(a, b)_k$ takes values in $\pm 1$. It equals 1 if and only if $ax^2 + by^2 = 1$ has a solution $(x, y) \in k^2$; see Table 5.3. Given a
diagonal representative \( Q = \text{diag}(a_1, a_2, \ldots, a_m) \) of \( \Omega \), the Hasse invariant of \( \Omega \) is defined as \( \text{Hasse}(\Omega) = \prod_{1 \leq i < j} (a_i, a_j)_k \).

The following theorem is well-known; see for example [9, Theorem VI.2.12].

**Theorem 5.3.** Let \( k \) be a local field of characteristic 0 or odd. Two nondegenerate quadratic forms \( \Omega \) and \( \Omega' \) over \( k \) are isometric if and only if

\[
\text{dim}(\Omega) = \text{dim}(\Omega'), \quad \text{Det}(\Omega) = \text{Det}(\Omega'), \quad \text{and} \quad \text{Hasse}(\Omega) = \text{Hasse}(\Omega').
\]

A quadratic form \( \Omega \) is called anisotropic if there is no nonzero \( x \) such that \( \Omega(x) = 0 \), and isotropic otherwise. Following [9 VI.2], we list the number of nondegenerate quadratic forms, together with the number of those which are anisotropic, in Table 5.4. A key example of a nondegenerate isotropic quadratic form is the hyperbolic plane, which can be represented by the matrix

\[
q_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Thus \( \text{dim}(q_0) = 2 \), \( \text{Det}(q_0) = -1 \) and \( \text{Hasse}(q_0) = 1 \). Any orthogonal direct sum of hyperbolic planes is called a hyperbolic space. Every nondegenerate quadratic form \( \Omega \) may be uniquely decomposed as a direct sum of hyperbolic and anisotropic forms, which we may write as

\[
\Omega = q_0^m \oplus Q_{\text{aniso}}
\]

where \( q_0^m \) denotes the direct sum of \( m \) copies of \( q_0 \), for some \( m \leq \frac{1}{2} \text{dim}(\Omega) \).

**Lemma 5.4.** Suppose \( p_1 \neq 2 \). Given a quadratic form \( \Omega \), its anisotropic part \( Q_{\text{aniso}} \) is either the zero subspace, or one of the 15 anisotropic quadratic forms in Table 5.5.
Proof. The values for dimensions 1 and 4 are from [8, Theorem VI.2.2]. For the rest, one produces representatives of all equivalence classes of quadratic forms from among diagonal matrices with entries in the set \{1, \varepsilon, \varpi, \varepsilon \varpi\}, and then calculates their invariants. Note that any isotropic form of dimension 3 must be of the form $q_0 \oplus q_1$ for some $q_1 \in \{1, \varepsilon, \varpi, \varepsilon \varpi\}$, and so are easy to eliminate from the list. The final result in Table 5.5 is condensed by setting $\alpha = \varepsilon$ if $-1 \in k^\times$, but $\alpha = 1$ and $\varepsilon = -1$ if $-1 \not\in k^\times$. \hfill \square

Given a quadratic form $Q$, a matrix representative of the form $Q = q_0^m \oplus Q_{\text{aniso}}$, where $Q_{\text{aniso}}$ is one of the diagonal matrices given in Table 5.5 will henceforth be called a minimal matrix representative of $Q$.

5.3. Explicit parametrization of nilpotent orbits. We return to the hypotheses on $k$ from Section 5.1. In [2, Ch 5.2], Collingwood and McGovern construct explicit Lie triples representing each of the algebraic orbits $O_\lambda$. In this subsection, we construct triples for each of the rational orbits in $O_\lambda(k)$ using similar ideas, though we must group the indices in a slightly different way.

Let $\lambda$ be a partition of $2n$ such that each odd part occurs with even multiplicity, and let $m_j$ denote the multiplicity of $j$ in $\lambda$. Let $\underline{Q} = (Q_2, Q_4, \cdots, Q_{2n})$ be an $n$-tuple of quadratic forms corresponding to $\lambda$ and choose a minimal matrix representative $Q_i$ for each $Q_i$. We construct a representative $X$ of the corresponding nilpotent orbit in $\mathfrak{g}$ by first producing a decomposition of $\mathfrak{g}$ of the form (2.2), which corresponds to the partition $\lambda$, and then defining $X$ by its action on each symplectic subspace $V(j)$.

Denote by $\{p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n\}$ an ordered symplectic basis for $V$. By this we mean that $\langle p_i, q_j \rangle = \delta_{ij}$, $\langle q_i, p_j \rangle = -\delta_{ij}$, and all other pairings are zero. For each $i \in \{1, 2, \ldots, 2n\}$, define a set of indices

$$s_i = \sum_{j<i} \frac{1}{2} jm_j \in \mathbb{Z};$$
Then we have $V = \bigoplus_{j \neq 0} V(j)$, so to define $X$ it suffices to give its action on each such nonzero $V(j)$.

If $j$ is odd, let $\mu$ be the partition of $\frac{1}{2}j m_j$ given by $\frac{1}{2}m_j$ copies of $j$, and define the restriction of $X$ to $V(j)$ with respect to the basis $(5.4)$ by

$$X|_{V(j)} = \begin{bmatrix} J_\mu & 0 \\ 0 & -J_\mu^\dagger \end{bmatrix}.$$ (5.5)

If $j = 2N$ is even, let $Z$ denote the $m_j(N - 1) \times m_j(N - 1)$ zero matrix. Then define the restriction of $X$ to $V(j)$ with respect to the basis $(5.4)$ by

$$X|_{V(j)} = \begin{bmatrix} J_{N m_j}^{m_j} & Z \oplus (-1)^N Q_j \\ 0 & -(J_{N m_j}^{m_j})^\dagger \end{bmatrix}.$$ (5.6)

Note that $J_{N m_j}^{m_j} \dagger (J_{N m_j}^{m_j})^\dagger = J_N \otimes I_{m_j}$, where $I_{m_j}$ the $m_j \times m_j$ identity matrix.

**Proposition 5.5.** Let $\lambda$ be as above. The matrix $X \in g$ defined by $(5.5)$ and $(5.6)$ is a representative of the G-orbit in $O_{\lambda}(k)$ corresponding to the n-tuple of quadratic forms $Q$.

**Proof.** We complete $X$ to a Lie triple $(Y, H, X)$ in $g$, and then verify that $V$ decomposes according to $\lambda$ and that the resulting form $(5.1)$ coincides with $Q$ for each even $j$. As done for $X$, we define $H$ and $Y$ by their action on the subspaces $V(j)$.

Following the notation of $(2.1)$, for each odd $j$, let $H_j^\oplus \frac{1}{2}m_j$ denote the direct sum of $\frac{1}{2}m_j$ copies of the diagonal matrix $H_j$ and let $H|_{V(j)} = H_j^\oplus \frac{1}{2}m_j \oplus -H_j^\oplus \frac{1}{2}m_j$.

Similarly, we may set $Y|_{V(j)} = Y_j^\oplus \frac{1}{2}m_j \oplus -\left(Y_j^\oplus \frac{1}{2}m_j\right)^\dagger$.

For each even $j = 2N$, let $H_{N m_j}$ be the $N m_j \times N m_j$ matrix given by $H_{N m_j} = (j - 1)I_{m_j} \oplus (j - 3)I_{m_j} \oplus \cdots \oplus I_{m_j}$, and set

$$H|_{V(j)} = H_{N m_j} \oplus -H_{N m_j}.$$ (5.7)

Similarly, set

$$Y_{N m_j} = \left(J_N \text{diag}(j - 1, 2(j - 2), \ldots, (N - 1)(N + 1), 0) \right) \otimes I_{m_j}$$

and let

$$Y|_{V(j)} = \begin{bmatrix} Y_{N m_j} & 0 \\ Z \oplus (-1)^N N^2 Q_j^{-1} & -Y_{N m_j}^\dagger \end{bmatrix}.$$ (5.8)

One checks directly that each $(Y|_{V(j)}, H|_{V(j)}, X|_{V(j)})$ is a Lie triple in the Lie subalgebra $\mathfrak{sp}(V(j))$ and hence by orthogonality $(Y, H, X)$ is a Lie triple in $g$. By construction the corresponding decomposition of $V$ into isotypic subspaces under this Lie triple corresponds to the partition $\lambda$ so $X \in O_{\lambda}(k)$.

It remains to verify that when $j$ is even, the quadratic form $(5.1)$ is isometric to $Q$. First note that by $(5.7)$, with respect to the basis $(5.4)$ of $V(j)$, the lowest weight space of this isotypic component is

$$L(j) = \text{span}\{q_{s_j+1}, q_{s_j+2}, \ldots, q_{s_j+M_j}\}.$$ (5.9)

Given $v, w \in L(j)$, write $\tilde{v}, \tilde{w}$ for their coordinate vectors with respect to this basis.
By construction, we have that

\[ X^{j-1}|_{V(j)} = \begin{bmatrix} 0 & -Q_j \oplus Z \\ 0 & 0 \end{bmatrix}, \]

which allows us to deduce directly that \( \langle v, X^{j-1}w \rangle = \bar{\pi}^t Q_j \bar{\pi}, \) as required. \( \square \)

5.4. **Correspondence with the DeBacker parametrization.** Suppose now that \( p_f > 3(h-1). \) As in Section 4.2, we assert that our choice of representative defines the equivalence class of the corresponding pair in \( I_f^d. \) To make this precise, we need to identify the affine roots which occur in the expression of \( X \) as a sum of root vectors; Theorem 5.6 below says that the orbit of \( X \) is associated to the corresponding intersection of affine \( r \)-hyperplanes. Unlike the case of \( SL_n, \) however, the roots which arise are not generally simple, so the precise statement is more cumbersome.

Set our notation as in Section 5.3 and suppose that \( X \in O_\lambda(k) \) is as defined in Proposition 5.5. For each odd \( j, \) let

\[ I_j = \{1, 2, \ldots, \frac{1}{2} jm_j\} \setminus \{j, 2j, \ldots, \frac{1}{2} jm_j\} \]

and let \( S_j = S_j^1 \) denote the set of simple roots

\[ S_j^1 = \{e_{s_j+k} - e_{s_j+k+1}: k \in I_j\}. \]

For each even \( j, \) suppose \( Q_j = q_0^m \oplus Q_{aniso}, \) with \( 0 \leq 2m \leq m_j \) and set \( M_j = (\frac{1}{2} j - 1)m_j \) for simplicity. Then we take instead \( S_j = S_j^1 \cup S_j^2, \) where these are the sets of positive roots defined by

\[ S_j^1 = \{e_{s_j+k} - e_{s_j+k+m_j}: 1 \leq k \leq M_j\} \]

\[ \cup \{e_{s_j+M_j+2i-1} + e_{s_j+M_j+2i}: i \in \{1, 2, \ldots, m\}\} \]

and

\[ S_j^2 = \{2e_{s_j+M_j+i}: 2m < i \leq m_j\}. \]

Finally, if \( Q_{aniso} = \text{diag}(a_{2m+1}, a_{2m+2}, \ldots, a_{m_j}), \) define for \( \alpha_i = 2e_{s_j+M_j+i} \) the integer \( v_{\alpha_i} = \text{val}(a_i) \) for each \( i \in \{2m+1, 2m+2, \ldots, m_j\}. \)

Now let \( S = \bigcup_{j: m_j \neq 0} S_j \) and define \( H_{\lambda, \bar{\pi}} \) to be the common intersection of all the hyperplanes \( H_{\alpha, -r} \) for \( \alpha \in S_1 \) and \( H_{\alpha+v_{\alpha}, -r} \) for \( \alpha = S_2. \)

**Theorem 5.6.** The affine subspace \( H_{\lambda, \bar{\pi}} \subset A \) is a nonempty union of \( r \)-facets. Let \( F \) be any maximal \( r \)-facet in \( H_{\lambda, \bar{\pi}}, \) and let \( v \) denote the projection of \( X \) in \( V_F. \) Then \( (F, v) \in I_f^d \) and \( O(F, v) = \text{Ad}(G)X. \)

**Proof.** The first assertion follows from the construction of \( S \) as a linearly independent subset of \( \Phi, \) and of \( H_{\lambda, \bar{\pi}} \) as an intersection of hyperplanes of the form \( H_{\alpha, -r}. \)

Given \( F \subset H_{\lambda, \bar{\pi}} \) of maximal dimension, and \( x \in F, \) we deduce from the proof of Proposition 5.5 that \( X \in g_{x,r}, H \in g_{x,0} \) and \( Y \in g_{x,-r}. \) Furthermore, by the hypotheses on the residual characteristic, this Lie triple projects onto a Lie triple \( (w, h, v) \in V_{x,-r} \times V_{x,0} \times V_{x,r}. \) Thus we deduce that \( O(F, v) = \text{Ad}(G)X, \) as in the proof of Theorem 4.2.

It remains to show that \( (F, v) \) is distinguished, which we do by demonstrating that \( \dim(F) \geq \dim(F_0) \) for any other pair \( (F_0, v_0) \) such that \( O(F_0, v_0) = \text{Ad}(G)X. \)

Start with \( (F_0, v_0) \) and \( x_0 \in F_0. \) Complete \( v_0 \) to a Lie triple \( (w_0, h_0, v_0) \) with adapted one-parameter subgroup \( \mu \) over \( \mathfrak{f}. \) Using the argument in [3 §4.3], we may
conjugate the triple and $\mu$ by an element which fixes $x \in A$ to obtain a new Lie
 triple such that the adapted one-parameter subgroup lies in $X_*(T)$. Lift this to a
Lie triple $(Y_0, H_0, X_0) \in g_{x,-r} \times g_{x,0} \times g_{x,r}$.

There exists a choice of $g$ normalizing $T$ so that $\text{Ad}(g)H_0$ is a dominant toral
element. Since $g$ preserves $A$, we may without loss of generality replace $F$ with $gF$
and $(Y_0, H_0, X_0)$ with its $\text{Ad}(g)$-conjugate. (Note that in general this is not equal
to the Lie triple $(Y, H, X)$, since our $H$ is not generally dominant.)

We wish to show that if $\text{Ad}(G)X = \text{Ad}(G)X_0$, then the dimension of $F_0$ is
at most equal to $\dim(F)$. As in the proof of Theorem 1.2 we assume that $r$ is
irrational and begin by deducing that

$$X_0 \in \bigoplus_{\psi : \psi(x) = r} g_{\psi}.$$ 

Now let $\Phi(X_0) = \{\psi \in \Phi : \psi(x) = r\}$; then we have

$$X_0 \in \bigoplus_{\alpha \in \Phi(X_0)} g_{\alpha}.$$ 

Let $s = \dim(\text{span}(\Phi(X_0)))$. Then $\dim(F_0) = n - s$ so our goal is to minimize $s$. As
was the case for $\text{SL}_n$, this is closely related to the goal of minimizing the number of
nonzero entries in the matrix representing $X_0$; however, as some root spaces contain
matrices with two nonzero entries, there is more to say.

Let $V[i]$ denote the $i$-weight space of the toral element $H_0$. Since $H_0$ is dominant,
these weight spaces are strictly ordered with respect to the basis (5.4) of $V$. More
precisely, we can define a decreasing list of indices $k_i$ by

$$k_i = \sum_{j > i} \dim V[j], \quad 0 \leq i \leq 2n.$$

Setting $k_{-i} = k_i$ and noting than $k_0 \leq n$, we have

$$V[i] = \begin{cases} 
\text{span}\{p_{k_i+1}, p_{k_i+2}, \ldots, p_{k_i+\dim V[i]} = p_{k_i-1}\} & \text{if } i > 0; \\
\text{span}\{q_{k_i+1}, q_{k_i+2}, \ldots, q_{k_i+\dim V[i]} = q_{k_i-1}\} & \text{if } i < 0; \\
\text{span}\{p_{k_0+1}, \ldots, p_n, q_{k_0+1}, \ldots, q_n\} & \text{if } i = 0.
\end{cases}$$

In particular, $V[0]$, as well as $V[-i] \oplus V[i]$ for each $i \geq 1$, are symplectic subspaces
of $V$.

For each $i$, the restriction of $X_0$ to the $i$-weight space gives a map $X_0 : V[i] \to V[i+2]$; when $i \geq -1$, this map is surjective. Let us decompose $X_0$ into a sum of
simpler elements in $g$ via these restricted maps.

Since $V[-1] \oplus V[1]$ is a symplectic subspace of $V$, the map $X_0^{(-1)}$ defined by
extension by zero of the restriction $X_0 : V[-1] \to V[1]$ is a well-defined element
of $g$. Similarly, for $i \geq 0$, define $X_0^{(i)}$ to be the restriction of $X_0$ to the domain
$V[i] \oplus V[-i-2]$, and zero on all other weight spaces; then $X_0^{(i)} \in g$. Altogether,
we have a decomposition of $X_0$ as

$$X_0 = \sum_{i \geq -1} X_0^{(i)}.$$ 

By construction, these components are supported on disjoint subsets of the basis
(5.4) of $V$ associated to the root system $\Phi$. It follows that we may decompose the
This form is given by \((v, w)\) as a disjoint union
\[
\Phi(X_0) = \Phi(-1) \cup \Phi(0) \cup \cdots \cup \Phi(2n-2)
\]
such that for each \(i \geq -1,\)
\[
X_0^{(i)} \in \bigoplus_{\alpha \in \Phi(i)} g_\alpha.
\]  
(5.10)

We can say more about these sets \(\Phi(i)\).

Suppose first that \(i \geq 1\). Then we deduce from (5.9) that
\[
\Phi(i) \subseteq \{e_i - e_j : l \in \{k_{i+2} + 1, \ldots, k_{i+1}\}, j \in \{k_{i+1}, \ldots, k_{i-1}\}\}.
\]  
(5.11)

Furthermore, since the restriction \(X_0 : V[i] \to V[i + 2]\) is surjective, the matrix \(A\) representing this map has at least one nonzero entry in each row. In terms of the roots which occur in \(\Phi(i)\), one may deduce that for each \(l \in \{k_{i+2} + 1, \ldots, k_{i+1}\}\) there exists a \(j \in \{k_{i+1}, \ldots, k_{i-1}\}\) such that \(e_l - e_j \in \Phi(i)\). Hence we conclude that \(|\Phi(i)| \geq \dim(\text{span}\Phi(i)) \geq \dim V[i + 2]\).

Next consider \(i = 0\). Since \(V[0]\) meets both \(\text{span}\{p_1, \ldots, p_n\}\) and \(\text{span}\{q_1, \ldots, q_n\}\), it follows that additionally \(\Phi(0)\) may contain roots of the form \(e_l + e_j\), for indices \(l\) and \(j\) corresponding to \(V[0]\) and \(V[2]\), respectively. Consequently,
\[
\Phi(0) \subseteq \{e_l + e_j : l \in \{k_2 + 1, \ldots, k_1\}, j \in \{k_0 + 1, \ldots, n\}\}.
\]

Since \(X_0^{(0)}\) has full rank, we deduce as above that \(|\Phi(0)| \geq \dim(\text{span}\Phi(0)) \geq \dim V[2]\).

Finally, suppose that \(i = -1\). The map \(X_0^{(-1)}\) sends the span of \(\{q_{k_{i+1}}, \ldots, g_{k_0}\}\) onto the span of \(\{p_1, \ldots, p_{k_0}\}\), so we have that
\[
\Phi(-1) \subseteq \{e_l + e_j : l \in \{k_{1+1}, \ldots, k_1\}, j \in \{k_{1+1}, \ldots, k_0\}\}.
\]  
(5.12)

This time, however, there is little connection between the number of roots in \(\Phi(-1)\) and the rank \(\dim V[1]\) of \(X_0^{(-1)}\). Namely, when \(l = j\), a root vector corresponding to the root \(e_l + e_j\) has a single nonzero entry, whereas when \(l \neq j\), it has two nonzero entries, in two distinct rows.

To better understand the constraints on \(\Phi(-1)\) we consider the symmetric matrix \(B\) representing \(X_0 : V[-1] \to V[1]\) with respect to the bases (5.9). We have the following Lemma.

**Lemma 5.7.** If \(X_0\) represents the \(G\)-orbit defined by the pair \((\lambda, \underline{\nu})\) then the matrix \(B\) represents the quadratic form
\[
-\mathcal{Q}_2 \oplus \mathcal{Q}_4 \oplus \cdots \oplus (\lambda)[\underline{\nu}] \mathcal{Q}_{2n}.
\]  
(5.13)

**Proof.** Given the Lie triple \((Y_0, H_0, X_0)\), consider the decomposition \(\mathcal{Q}_2\) of \(V\) it determines. For each \(t \geq 1\), the lowest weight space \(L(2t)\) of the isotypic component \(V(2t)\) (if nonzero) carries the quadratic form \(\mathcal{Q}_{2t}\) of dimension \(m_{2t} = \dim L(2t)\). This form is given by \(\langle v, w \rangle_{2t} = \langle v, X_0^{2t-1}w \rangle\). Note that \(L(2t) \subseteq V[-2t + 1]\). There is a natural map from \(L(2t)\) to \(V[-1]\) given by \(v \mapsto X_0^{t}v\); denote its image \(E(2t)\). Then \(V[-1]\) decomposes as direct sum
\[
V[-1] = \bigoplus_{t \geq 1} E(2t).
\]  
(5.14)
The form $(,)_{2t}$ on $L(2t)$ induces a form $(,)_{t}$ on $E(2t)$, for each $t$, and hence by orthogonal direct sum a quadratic form on $V[-1]$. It is given on each $E(2t)$ by the formula

$$(X_0^tv, X_0^tw) = \langle X_0^tv, X_0X_0^tw \rangle = (-1)^t \langle v, X_0^{2t-1}w \rangle = (-1)^t \langle v, w \rangle_{2t}.$$ 

Since $V[-1]$ and $V[1]$ have complementary bases under $\langle, \rangle$ the Lemma follows. □

Given all the above, we wish to bound the value of $|\Phi(X_0)|$. We consider two possible cases.

In the first case, suppose that each subspace $E(2t)$ of the decomposition \ref{5.14} is spanned by a subset of $\{q_{k_1+1}, \ldots, q_{k_0}\}$. Then by orthogonality $X_0 E(2t)$ has as basis the complementary subset of $\{p_{k_1+1}, \ldots, p_{k_0}\}$, and so the matrix of $B$ is block-diagonal, corresponding to the decomposition \ref{5.13}. Hence the problem of minimizing $|\Phi(-1)|$ is reduced to minimizing the roots required to represent each of the quadratic forms $Q_{2t}$, which by Section 5.2 implies one should choose for each $Q_{2t}$ the minimal matrix representative $q_0^t \oplus Q_{aniso}$ (up to order of the summands).

Consequently, in this first case, we deduce that the minimal possible value of $|\Phi(X_0)|$ is exactly $|\Phi(X)|$, and thus that $\dim(F_0) \leq \dim(F)$, as required.

In the second case, suppose that some of the subspaces $E(2t)$ are not aligned with the symplectic basis. The only way to reduce the size of $|\Phi(-1)|$ would be to choose a matrix representative $B$ of the quadratic form \ref{5.13} which implies fewer roots than our preferred choice, above. This in turn can only happen if we form a hyperbolic plane from vectors coming from different isotypic components. We claim that any such reduction in $|\Phi(-1)|$ would be offset by a corresponding increase in some $|\Phi(j)|$ with $i \geq 1$, as follows.

Suppose that $B$ contains a summand $q_0$ at indices $t_1$ and $t_2$. So we have $q_{t_1}, q_{t_2} \in V[-1]$ such that

$$X_0q_{t_1} = p_{t_2} \quad \text{and} \quad X_0q_{t_2} = p_{t_1}.$$ 

If $q_{t_1}$ and $q_{t_2}$ lie in the same isotypic component of $V$, then this hyperbolic plane falls under the analysis of the first case so assume this is not the case.

For each $i = 1, 2$, consider the sequences $(p_i, X_0p_i, X_0^2p_i, \cdots)$. If there exists an $i \in \{1, 2\}$ and a least $w > 0$ such that $X_0^{w-1}p_i$ is not again a scalar multiple of some $p_j$, then write $X_0^{w-1}p_i = c p_j$ and $X_0^{w-1}p_i = X_0(cp_j) = \sum_{i=1}^n c_i p_i$ with at least two coefficients $c_k, c_l$ different from zero. Now $p_j \in V[2s+1]$ for some $s \geq 1$, so by \ref{5.10}, it follows that $e_k - e_j, e_l - e_j \in \Phi(2s+1)$. Using rank arguments as before (this time counting columns with nonzero entries), we conclude that $|\Phi(2s+1)| \geq 1 + \dim V[2s+3]$. Thus the decrease by one in the size of $\Phi(-1)$ is offset by an increase of at least one in the size of $\Phi(2s+1)$.

So we may assume from now on that both sequences consist of multiples of vectors from our symplectic basis. That is, for each $i$ we have indices $s(i, l)$ and an integer $w_i$ so that up to nonzero scalar multiples

$$(p_i, X_0p_i, X_0^2p_i, \cdots) = (p_i, p_{s(i,1)}, \cdots, p_{s(i,w_i)}, 0, \cdots).$$ 

Now note that for any $k \in \{t_1, s(i,1), \cdots, s(i, w_i) : i \in \{1, 2\}\}$, if $X_0q_k = cp_j \neq 0$ then $(X_0q_j, p_k) = -\langle q_j, X_0p_k \rangle = -\langle q_j, cp_j \rangle = c$, so $X_0q_j = cq_k + \text{other terms}$. If the other terms were nonzero, then the same analysis as above would yield an $s \geq 1$ such that $|\Phi(2s+1)|$ is not minimal. Hence we may assume $X_0q_j = cq_k$.

From the sequence above we have that $p_{s(i,w_i)}$ is a vector of highest weight $2w_i + 1$; hence $q_{s(i,w_i)}$ has weight $-2w_i - 1$ since they are complementary in the
symplectic basis. By the preceding paragraph, however, we have that the smallest invariant subspace containing $q_{i + (w_1)}$ contains $p_{i_j}$ for $j \neq i \in \{1, 2\}$, as a highest weight vector; so necessarily $w_j \geq w_i$. We deduce that $w_1 = w_2 = w$, which implies $q_1$ and $q_2$ both lie in $E(2w + 2)$, contrary to assumption.

Finally, we remark that given any number of distinct hyperbolic pairs arising from $V[-1]$, the roots added to $\Phi(X_0)$ through the above argument will be distinct. We conclude that one cannot do better than $\Phi(X)$, so $\dim(\mathcal{F}) \geq \dim(F_0)$. □

**Corollary 5.8.** Let $(\lambda, \overline{\Sigma})$ represent a rational nilpotent orbit $O$. Suppose that $(\mathcal{F}, v) \in I_d^i$ such that $O = \mathcal{O}(\mathcal{F}, v)$. For each $i$, let $a_i$ denote the dimension of the largest anisotropic subspace of $\mathcal{Q}_{2i}$. Then

$$\dim(\mathcal{F}) = \frac{1}{2} \left( |\lambda| - \sum_{i=1}^n a_i \right).$$

**Proof.** Recall that $|\lambda|$ denotes the number of parts in the partition $\lambda$, counting with multiplicity.

We have that $\dim \mathcal{F} = \dim(H_{\lambda, \overline{\Sigma}}) = n - |S|$. When $j$ is odd, $|S_j| = |I_j| = \frac{1}{2}jm_j - \frac{1}{2}m_j = \frac{1}{2}m_j(j - 1)$. When $j = 2i$ is even, we have $2m + a_i = m_i$, so $|S_j| = m_j(\frac{j}{2}) - 1 + \frac{1}{2}(m_j - a_i)$ and $|S_j^2| = a_i$, so $|S_j| = \frac{1}{2}m_j(j - 1) + \frac{1}{2}a_i$. Hence since $\sum jm_j = 2n$, we have $|S| = \sum_{j=1}^{2n} \frac{1}{2}m_j(j - 1) + \frac{1}{2} \sum_{i=1}^n a_i = \frac{1}{2}(2n - \sum_{j=1}^{2n} m_j + \sum_{i=1}^n a_i)$, and the result follows.

See Tables 5.1 and 5.2 for examples.

**Remark 5.9.** Recall the notion of distinguished orbits, key to the Bala-Carter classification. By [7, Lemma 4.2], these may be characterized for $Sp_{2n}$ as the orbits with partitions consisting of distinct even parts. So if $O$ is distinguished, $a_i = 1$ for each $i$ and it follows from Corollary 5.8 that $\dim(\mathcal{F}) = 0$; the distinguished orbits correspond to vertices. (This also holds for $SL_n$, where only the principal orbit (corresponding to the partition $(n)$) is distinguished.) As we can see already from $Sp_4$ and $Sp_6$, the converse is false in general.

Note also that the $r$-associativity classes of vertices are easy to determine for any $r$; there is one representative of each in the fundamental domain.

6. SOME REMARKS ON THE CASE OF SMALL RESIDUAL CHARACTERISTIC

The parametrization of nilpotent orbits via conjugacy classes of Lie triples fails to hold in small characteristic. Since this parametrization, over finite fields, forms the backbone of DeBacker’s results in [3], it is not clear to what extent the DeBacker parametrization will hold over $p$-adic fields with small residual characteristic. We explore this question in this section.

So let $k$ be a $p$-adic field with residual characteristic $p_l$. Then the partition-type parametrization of nilpotent orbits discussed in Section 4 is valid for all $p_l$ and that in Section 5 is valid for all $p_l > 2$. In Section 6 we have assumed that $p_l > 3(h - 1)$. Let us consider now values of $p_l$ less than this bound.

Consider the following essential characteristics of the correspondence $(\mathcal{F}, v) \mapsto O$:

1. That $O$ is the unique nilpotent orbit of minimal dimension meeting the coset $v = X + g_{\mathcal{F}, +}$.
2. That $(\mathcal{F}, v)$ corresponds to a maximal generalized $r$-facet in $\mathcal{B}(Y, H, X)$. 
That $\gamma_d$ is bijective.

We illustrate how (2) can fail when $p_f$ is small with an example.

Example 6.1. Consider the group $G = SL_n$. Suppose $n = 4$ and $p_f = 3$; let $\lambda = (4)$ be the full partition, corresponding to the principal nilpotent orbit. The rational orbits contained in $O_\lambda(k)$ are parametrized by

$$X_d = J_\lambda D(d) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $d$ running over a list of representatives of the distinct quartic classes in $k^\times$. One has $H = H_\lambda = \text{diag}(3, 1, -1, -3)$ and

$$Y = D(d^{-1})Y_\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3d^{-1} & 0 \end{bmatrix}.$$

Let us suppose $\text{val}(d) = 0$ for definiteness.

Define $H_{\lambda, D(d)}$ as in Theorem 4.2. Let $F$ be a maximal $r$-facet in $H_{\lambda, D(d)}$; this is a single vertex. Then one readily verifies that the subgroups $g_F$ and $g_{F+}$ are of the following form (with the usual abuse of notation):

$$g_F = \begin{bmatrix} \mathcal{P} & \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \mathcal{P} & \mathcal{P} & \mathcal{R} & \mathcal{R} \\ \mathcal{P} & \mathcal{P} & \mathcal{R} & \mathcal{R} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{P} \end{bmatrix}$$

and

$$g_{F+} = \begin{bmatrix} \mathcal{P} & \mathcal{P} & \mathcal{R} & \mathcal{R} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{R} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{R} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{P} \end{bmatrix}.$$

It follows that any element in the coset $X_d + g_{F+}$ has rank at least 3, and so any nilpotent orbit meeting this coset is regular, and hence of dimension at least (in fact equal to) $\dim(\text{Ad}(G)X_d)$. We note directly that for all $g$ in the upper triangular Borel subgroup $B$, $\text{Ad}(g)X_d \notin X_{d'} + g_{F+}$ if $d \neq d'$ mod $(\mathbb{R}^\times)^4$, and that if $g$ is in any other Bruhat cell, then $\text{Ad}(g)X_d$ does not meet any such cosets. Hence $\text{Ad}(G)X_d$ is the unique nilpotent orbit of minimum dimension meeting the coset $X_d + g_{F+}$, proving characterization (1).

Now set $X = X_d$ for simplicity and let us determine $B(Y, H, X) \cap A$. From our explicit matrix choices for $Y$ and $X$ we have that $Y \in \mathfrak{g}_{x, -r}$ for some $x \in A$ if and only if

$$-\alpha_1(x) + \text{val}(3) \geq -r$$

$$-\alpha_2(x) + \text{val}(4) \geq -r$$

$$-\alpha_3(x) + \text{val}(3) \geq -r,$$

whereas $X \in \mathfrak{g}_{x, r}$ if and only if $\alpha_i(x) \geq r$ for $i \in \{1, 2, 3\}$.

Note that $\text{val}(3) \geq 1$ and $\text{val}(4) = 0$ in this case. Thus $B(Y, H, X) \cap A = \{ x \in A : Y \in \mathfrak{g}_{x, -r}, X \in \mathfrak{g}_{x, r} \}$ is the set of all points $x \in A$ such that

$$r \leq \alpha_1(x) \leq r + \text{val}(3)$$

$$r \leq \alpha_2(x) \leq r$$

$$r \leq \alpha_3(x) \leq r + \text{val}(3).$$
This region contains some 2-dimensional $r$-facets, and thus $F$ is not maximal, so the characterization (2) fails for this choice of $(F, v)$.

Let us now further show that a choice of $(F, v)$ satisfying (2) cannot satisfy (1).

To fix our ideas, let us suppose $r \in (0, 1)$ and let $F'$ be the $r$-facet containing the point $x \in \mathcal{A}$ such that $\alpha_1(x) = \alpha_2(x) = \frac{1}{4}$ and $\alpha_2(x) = r$. Then $F'$ is a maximal $r$-facet in $B(Y, H, X) \cap \mathcal{A}$. Furthermore, for any $x \in F'$, we have that $r \leq \phi(x) < 1 + r$ for all positive roots $\phi \in \Phi^+$ and $-1 + r < \phi(x) < r$ for all negative roots $\phi$, with exactly one equality (for $\phi = \alpha_2$). So the elements of $\mathfrak{g}_{x,r}$ have their strictly upper triangular entries in $R$, and all lower triangular entries in $P$. The strictness of all but one inequality above implies that the elements of $V_{x,r} = \mathfrak{g}_{x,r} / \mathfrak{g}_{x,r+}$ may be represented by matrices of the form

$$v_a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $a \in F^\times$. Similarly, we deduce that $\mathfrak{g}_{x,-r}$ and $\mathfrak{g}_{x,0}$ have the form

$$\mathfrak{g}_{x,-r} = \begin{bmatrix} R & R & R & R \\ P & R & R & R \\ P & R & R & R \\ P & P & P & R \end{bmatrix}, \quad \mathfrak{g}_{x,0} = \begin{bmatrix} R & R & R & R \\ P & R & R & R \\ P & P & R & R \\ P & P & P & R \end{bmatrix}.$$ 

It is true that $(Y, H, X) \in \mathfrak{g}_{x,-r} \times \mathfrak{g}_{x,0} \times \mathfrak{g}_{x,r}$. However, now consider instead the Lie triple

$$Y' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

This triple certainly satisfies the condition that $(Y', H', X') \in \mathfrak{g}_{x,-r} \times \mathfrak{g}_{x,0} \times \mathfrak{g}_{x,r}$, but now $\dim(\text{Ad}(G)X') < \dim(\text{Ad}(G)X)$. Thus $\text{Ad}(G)X$ is not the orbit of minimal dimension meeting the coset $X + \mathfrak{g}_{x,r+}$. Hence the distinguished pair $(F', v')$ (condition (2)) does not satisfy the minimality requirement (condition (1)).

This example illustrates that the set $B(Y, H, X)$ is too large to uniquely identify a facet which is associated to the nilpotent orbit $\text{Ad}(G)X$.

Nonetheless, through Theorem 4.2 and Theorem 5.6 there exist obvious candidates for the “distinguished pairs” $(F, v)$, namely, those satisfying Proposition 3.3. Unfortunately, Proposition 3.3 is a less satisfying definition because it is not intrinsic. In any case what still needs to be proven is that property (3) continues to hold for these in small residual characteristic.

For our final remark, we note that even this approach will encounter some obstacles for the case of $G = \text{SL}_n$, as the Example 6.3 below, illustrates. We first recall a well-known result; one can prove it by adapting the proof of [9, Thm VI.2.22], which is the special case $m = 2$.

**Proposition 6.2.** Let $m \geq 2$ and let $\mu_m(k)$ denote the group of $m$th roots of unity in $k$. Then the number of distinct cosets of $(k^\times)^m$ in $k^\times$ is

$$|k^\times / (k^\times)^m| = m|\mathbb{R}^\times / (\mathbb{R}^\times)^m|$$

and $|\mathbb{R}^\times / (\mathbb{R}^\times)^m| = |\mu_m(k)| q^{\text{val}(m)}$. 

Applying now Proposition 1, we have a more precise result about the number of rational nilpotent orbits of $SL_n$. We may now present an example to illustrate how (3) may fail when $p$ is not very good for $G$.

**Example 6.3.** Suppose $G = SL_n$ and that $p_1$ is a prime which is not very good for $G$, meaning that $p_1$ divides $n$. Let $\lambda$ be a partition of $n$ such that $(\gcd(\lambda), p_1) = m > 1$. Then $\text{val}(m) \geq 1$ and so by Proposition 6.2, there are $q^{\text{val}(m)} | \mu_m(k) |$ distinct representatives of $k^\times/(k^\times)^m$ of any given valuation. In particular, there exist distinct cosets represented by elements $d,d'$ of the same valuation, such that their difference $d - d'$ has strictly larger valuation. Thus the corresponding representatives of the nilpotent orbit will descend to the same element in $g_F/g_{F+}$.

In this example, the number of rational orbits in $O_\lambda(k)$ was too large to be parametrized by $G_z$ orbits of $v$ in $V_F$, for any collection of pairs $(\mathcal{F}, v)$. Thus the DeBacker correspondence cannot hold. We note that this issue is also a problem for the Bala-Carter classification over algebraically closed fields with $p$ not very good.

In contrast, for the group $Sp_{2n}$, the classical parametrization requires only information about the square classes in $k$, which is in turn completely answered by the same question in $f$ when $p > 2$. It follows that one could apply the arguments in this paper to identify equivalence classes of pairs $(\mathcal{F}, v)$ in $\gamma_d^{-1}(O)$ for any nilpotent orbit $\mathcal{O}$ of $Sp_{2n}(k)$, for $p > 2$ (which is the set of very good primes for $Sp_{2n}$).

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