On the Non-invasive Measurement of the

Intrinsic Quantum Hall Effect

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Abstract

With a model calculation, we demonstrate that a non-invasive measurement of intrinsic quantum Hall effect defined by the local chemical potential in a ballistic quantum wire can be achieved with the aid of a pair of voltage leads which are separated by potential barriers from the wire. Büttiker’s formula is used to determine the chemical potential being measured and is shown to reduce exactly to the local chemical potential in the limit of strong potential confinement in the voltage leads. Conditions for quantisation of Hall resistance and measuring local chemical potential are given.

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To study the electronic transport properties of a system, it is normal to use at least four leads, attaching two pairs of leads to the system to measure the current passing through and the voltage drop across it. In the macroscopic regime, the scale of system is much larger than the scale of the measurement leads. Consequently, this approach has very little effect on the system being measured and the measurement results can be used to fully characterise the system itself. This desirable situation has changed with the rapid development of semiconductor fabrication techniques which make it possible to investigate two-dimensional electron gas (2DEG) microstructures [1]. In this case both current and voltage leads become an inseparable part of the system being measured. Moreover, the dimensions of the part being measured and the measurement leads are of the same order and can be comparable with the de Broglie wavelength of the electron propagating in the system. Many novel phenomena are observed in this situation. They are attributed to this new partnership between the system being measured and the leads and are explained successfully by using Landauer-Büttiker formulae which reveal the relationship of resistance to transmission coefficient between leads [2, 3]. Büttiker has proposed a general formula to determine the chemical potential measured by a voltage lead through a current-stop procedure [3, 4]:

\[
\mu_l = \frac{T_{ls}\mu_s + T_{ld}\mu_d}{T_{ls} + T_{ld}}
\]  

(1)

where \(T_{ls}(T_{ld})\) is the sum of all the transmission coefficients for a carrier
incident in lead $s(d)$ to be transmitted to lead $l$ and the subscripts $l$, $s$, and $d$ denote voltage, source, and drain leads respectively.

A problem comes when one asks how to determine the intrinsic resistance of such a microstructure system, i.e. its own response to the change of environment. To do this, it is necessary to study the effect of the leads on the resistance measurement in detail. The two kinds of leads (current and voltage) have different interactions with the system. Current leads function as sources and drains which respectively inject electrons into and collect them from the system being measured. Voltage leads do not have any net electron exchange with the system; they determine the potential being measured by a current-stop procedure. Furthermore, different shapes of leads give different results. To be definite we consider ideal current leads, i.e. hard wall ballistic electron waveguides which become an integral part of a system. (They are used as filters to get rid of fluctuations, evanescent modes, etc. coming from the reservoirs and introduce standard propagating modes of electrons into the system.) The injection modes are determined by the character of the ideal leads only. If the shape of the current leads are fixed we just need to concern about the effect of the voltage leads.

There is no confusion as long as we use the one pair of leads to measure the current passing through and the “voltage drop” across a system. The result of such measurements is a conventional longitudinal resistance. When we have separate pairs of leads to measure current and “voltage drop”, which is
essential in studies of the quantum Hall effect (QHE), the resistance measured reflects the behaviour of the original system plus a pair of voltage leads and is voltage-lead dependent. We are not able to isolate the contribution from the system being measured in the total signal. The only way to solve this problem is to reduce the coupling between the system to be measured and the voltage leads. The weaker the coupling, the less the measurement result is affected by the measurement process. However, we know that there is no way to measure a system without some perturbation of the system being measured. What we must do is to make the coupling small enough so that the measured resistance does not change within the accuracy of the measurement instrument when the coupling decreases further. Then, in this sense, the measurement is non-invasive and the measured resistance can be regarded as intrinsic to the system which we measure.

Many papers discuss methods of making a non-invasive voltage leads. Most of the works are carried out at the geometrical edge of the 2DEG microstructure due to technical reason. Li and Thouless suggested using a scanning-tunnelling-microscope tip as a weakly coupled voltage lead to detect the electrostatic potential response of QHE from an etched edge [7]. Field et al. use a separate quantum point contact sited at the side of a gated edge to achieve non-invasive measurement of electrostatic potential [8]. When we work out the resistance of a system, however, we need to know the chemical potential difference rather the electrostatic potential difference between two
points, as is stressed by Engquist and Anderson [9]. Experimental attempts to measure resistance in the weak coupling limit have been made recently by Shepard et al. [10]. It is much more difficult to determine the chemical potential at a certain point of a transport system. The main reason is that the chemical potential in a system is normally not well defined when there is a net current flowing through it. Many suggestions have been made about how to define this quantity locally in a system away from thermal equilibrium [9, 11, 12]. They all lead to the same chemical potential and average electron occupation in an equilibrium system as has been pointed out by Landauer [13, 14]. Different procedures for non-invasive measurement have been suggested, e.g. phase-insensitive [9] and phase-sensitive [15]. They give different results when there is a net current passing through the system with reflections.

To avoid of these problems, a particular formula has been introduced through an assumption of a virtual contact measurement procedure for both single and multi-mode two-terminal cases by Entin-Wohlman et al. [11] and Imry [16] respectively. The advantage of this formula is that it defines a local chemical potential (LCP) in a non-equilibrium system so that we can calculate the resistance between any two points in a system in which net currents are flowing without introducing voltage leads. Büttiker derives a similar expression for a self-consistent electrostatic potential [4]. The same formula for the LCP is obtained in a general multi-mode and multi-terminal case by using only the assumptions inherent in Landauer-Büttiker formulas.
\[ \mu(r) = \frac{\sum_t p_t \mu_t}{\sum_t p_t} \]  

where \( p_t = \sum_m |\psi_{tm}(r)|^2 / v_{tm} \). Here \( t \) labels the leads feeding the microstructure, \( v_{tm} \) is the group velocity of mode \( m \) in lead \( t \) and \( \psi_{tm}(r) \) is the total wave function generated by an incident wave of unit amplitude in mode \( m \) in lead \( t \). We would like to stress that the LCP is phase-sensitive. The phase relation between the incident wave and the reflected wave is fully considered in the calculation of the wave function for the whole system. Moreover, the resistance determined by the LCP is non-local resistance which is not normally additive.

In this paper, we model a non-invasive measurement procedure in a system consisting of a quasi-one-dimensional ballistic quantum wire (BQW) and two voltage leads. The current leads are part of the BQW. Transmission coefficients are calculated and the chemical potential as well as the Hall resistance associated with it are obtained using Büttiker’s formula, Eq. (1). In the strong confinement limit, we prove analytically that Büttiker’s formula is equivalent to the formula for the LCP, Eq. (2), and the Hall resistance approaches the intrinsic Hall resistance defined by the LCP [5]. Numerical results are given to show how the character of the voltage leads affects the Hall resistance and to which every mode therein makes a non-negligible contribution.
The main part of our model system is a non-interacting 2DEG with electron density $n_s$ which is confined in a space of width $W$ in the $x$-$y$ plane by infinite potential barriers at $y = \pm W/2$. The two ends of the BQW are connected to the electron reservoirs with chemical potential $\mu_s$ (at the end where $x < 0$) and $\mu_d$ (at the end where $x > 0$) respectively. When $\mu_s \neq \mu_d$, there is a net current traversing the BQW. To model a four-terminal measurement of the Hall resistance, we use the weak-link model studied by Peeters [17] and later by Akera and Ando [18] to put two voltage leads on the two sides of the wire in the $x$-$y$ plane and parallel to the $y$-axis. The confinement potential in the voltage leads has the form of $m^* \omega_p^2 x^2 / 2$ and is characterised by an equivalent magnetic field $B_p = m^* \omega_p / e$ where $m^*$ is the effective mass of electron. We assume these two types of confinements for the BQW and the voltage leads respectively because they are mathematically simple and are close to the calculated self-consistent potential profiles for the relatively wide and narrow BQWs in which the Fermi energy is the same [19] which is the case when we explore the strong confinement limit in the voltage leads. Moreover, two identical tunnelling barriers with heights $V_b$ and widths $b$ are symmetrically placed between the wire and the ends of the voltage leads. The amount of current leaking into the voltage leads can be made very small by increasing the product $V_b b$ so that we can approach the non-invasive limit defined above. For convenience in the calculations, delta functions of area $V_b b$ are located at $y = \pm W/2$ to describe potential energies of the tunnelling
barriers as $V_y b \delta (y \mp W/2)$.

A magnetic field $B$ is applied in the direction perpendicular to the $x$-$y$ plane and is described in the Landau gauge by writing the vector potential as $A = (-B y, 0, 0)$ for the BQW and as $A = (0, B x, 0)$ for the voltage leads. Taking account of the gauge difference between the two regions, the tunnelling wave function of an electron from the BQW to the voltage lead at $y = W/2 + \epsilon$ ($\epsilon \to 0^+$) is

$$\psi^{(n \pm)}(x, W/2) = C^{(n \pm)} \exp \left[ i \left( \pm k_x^{(n)} + \frac{W}{2l_c^2} \right) x \right]$$

with $C^{(n \pm)} = -\frac{\hbar^2}{2m^* V_y b} \frac{1}{l_c^2} \frac{d\chi^{(n)}(y)}{dy} \bigg|_{y=W/2}$, $l_c^2 = \hbar/e |B|$, and $\chi^{(n)}$ for the $n$-th eigenfunction for an electron in the BQW. The Fermi wave vector $k_y^{(n)}$ is all real and positive and determined with the Fermi energy $E_F$ by a sum constrained to keep $n_s$ fixed \cite{footnote}. The $\pm$ sign refers the mode propagating along $\pm x$ direction.

The eigenfunction of electron in the voltage lead at $y > W/2$ is

$$\phi^{(m)}(x, y) = C^{(m)} e^{-\frac{1}{2} \frac{\eta_m^2}{l_c^2}} H_m(\eta_m)$$

with $C^{(m)} = \frac{(1 + \gamma^2)^{1/8}}{(2m! \pi^{1/2} l_c)^{1/2}} \exp \left[ i k_y^{(m)} \left( y - \frac{W}{2} \right) \right]$, $\eta_m = (1 + \gamma^2)^{1/4} \frac{x}{l_c} + \frac{1}{(1 + \gamma^2)^{3/4}} l_c k_y^{(m)}$, where $m$ is the mode index, $H_m$ is Hermite polynomial, $k_y^{(m)}$ is the Fermi wave vector of the electron, and $\gamma = B_p/B$. The Fermi wave vector $k_y^{(m)}$ is either real or imaginary (corresponding to propagating
and evanescent mode) due to the parabolic potential confinement.

We choose the \( k_y^{(m)} \)'s so that the electron energy in the voltage lead is \( E_F \) when we expand the tunnelling electron wave function in Eq. (3) in the terms of the eigenfunction of electron in the voltage lead at \( y = W/2 \)

\[
\psi^{(n\pm)}(x, \frac{W}{2}) = \sum_m g_m^{(n\pm)} \phi^{(m)}(x, \frac{W}{2}) \tag{5}
\]

The wave functions \( \phi^{(m)}(x, W/2) \) are normalised but they are not orthogonal. Consequently, the \( g_m^{(n\pm)} \) are determined by following equations:

\[
\sum_m f_{jm} g_m^{(n\pm)} = h_j^{(n\pm)} \tag{6}
\]

with \( f_{jm} = \int_{-\infty}^{+\infty} dx \phi^{(j)}(x, W/2) \phi^{(m)}(x, W/2), h_j^{(n\pm)} = \int_{-\infty}^{+\infty} dx \phi^{(j)}(x, W/2) \psi^{(n\pm)}(x, W/2). \)

After solving Eq. (6), we can directly calculate the transmission coefficients from their definitions

\[
T_{ls} = \sum_n T_{ls}^{(n)} = \sum_n \frac{1}{v_n} \sum_m v_m \left| g_m^{(n)} \right|^2, \tag{7}
\]

\[
T_{ld} = \sum_n T_{ld}^{(n)} = \sum_n \frac{1}{v_n} \sum_m v_m \left| g_m^{(n)} \right|^2,
\]

where the summations over \( n \) and \( m \) include all the values for which \( \{n|E_F = E(k_x^{(n)})\} \) and \( \{m|k_y^{(m)} \in \mathbb{R}\} \) respectively. Here, the subscripts have the same meaning as in Eq. (1) and \( v_m \geq 0 \) \((v_n \geq 0)\) is the group velocity of electron of the \( m \)-th \((n\)-th\) propagating mode in the voltage lead (BQW) with its energy equals \( E_F \).

The chemical potential defined by Büttiker’s formula, Eq. (1), can be calculated easily from \( T_{ls} \) and \( T_{ld} \) in Eq. (7). If we have strong potential
confinement in the voltage lead, i.e. $B_p \gg B$, the leading term of the coefficients $f_{jm}$ and $h_j^{(n\pm)}$ are

$$f_{jm} \simeq \delta_{jm}, \quad h_j^{(n\pm)} \simeq \frac{(2\pi)^{1/2} l e}{\gamma^{1/2}} i j C_\phi^{(j)} C_\psi^{(n\pm)} H_j(0).$$

Consequently, we can easily show that

$$\sum_m v_m \left| g_m^{(n\pm)} \right|^2 \simeq \sum_m v_m \left| h_m^{(n\pm)} \right|^2 = \text{const.} \times \gamma^{-1/2} \left| C_\psi^{(n\pm)} \right|^2.$$

Hence, the chemical potential measured by the voltage lead attached to the BQW at the edge $y = W/2$, which is defined by Eq. (1), reduces to

$$\mu_l = \frac{\sum_n \frac{1}{v_n} \left( \left| F^{(n+)} \right|^2 \mu_s + \left| F^{(n-)} \right|^2 \mu_d \right)}{\sum_n \frac{1}{v_n} \left( \left| F^{(n+)} \right|^2 + \left| F^{(n-)} \right|^2 \right)} \quad (8)$$

where $F^{(n\pm)} = \partial \psi^{(n\pm)}(x,y)/\partial y |_{y=W/2}$.

We see by inspection of Eq. (8) that Büttiker’s chemical potential is identical to the LCP at $y = W/2$ defined by Eq. (2) in a BQW with no voltage probes attached. It is important to note that the eigenfunctions of electrons in the voltage lead do not change significantly as we change $B$ when $B_p \gg B$. In this situation, the coupling strength of each mode in the BQW to the voltage lead will depend only on the character of the mode itself and nothing else. As long as these electron modes are undisturbed by the voltage lead, we can make a non-invasive measurement and the LCP defined by
Eq. (2) and the intrinsic Hall resistance associated with it are the quantities being measured.

The Hall resistance \( R_H \) associated with the chemical potential \( \mu_l \) defined by Büttiker’s formula Eq. (1) is obtained by solving Eq. (6) with \( m^* = 0.068 \, m_e \) for GaAs, \( W = 100 \, \text{nm} \), and \( n_s = 1.1 \times 10^{15} \, \text{m}^{-2} \) so that three subbands are populated when \( B = 0 \). We include the necessary number of evanescent modes in the voltage leads, such that no change of the expansion coefficient \( g_m^{(n \pm)} \) (for the \( k_y^{(m)} \in \mathbb{R} \)) occurs when we take more evanescent modes into account. The same zero point of potential is used in both the BQW and the voltage leads.

Fig. 1 shows the changes of the dependence of \( R_H \) on \( B \) from \( B_p \sim B \) to \( B_p \gg B \). The solid line is the result of \( R_H \) associated with the LCP of Eq. (2) as studied in Refs. [5, 20], while the dashed line and the dot-and-dash line are the results of \( R_H \) calculated by Büttiker’s formula Eq. (1) for \( B_p = 1 \, \text{T} \) and \( 11 \, \text{T} \) respectively. The dotted line is for the longitudinal resistance which is perfectly quantised since there are no reflections in the BQW. We verify from Fig. 1 that the \( R_H \) derived from Büttiker’s formula does approach the intrinsic Hall resistance derived from the LCP when we increase \( B_p \) and has it as its limit when \( B_p \gg B \). In the range of \( 0 < B < 0.6 \, \text{T} \), there is a quenching of \( R_H \) for both \( B_p = 1 \, \text{T} \) and \( B_p = 11 \, \text{T} \) as we have found for the intrinsic quantum Hall resistance in the BQW with interacting electrons [20] and the magnitude of \( R_H \) reduces when \( B_p \) increases. The dips of \( R_H \)
are deeper than that displayed in Ref. [20] because only one electron state is used here to calculate $R_H$ rather than the electron states in a small but finite range of energy due to the chemical potential difference between source and drain. We also notice from Fig. 1 is that there is a quantised plateau on the $R_H$ curve around $B \sim 2.2$ T when $B_p = 1$ T instead of the dip found when $B_p = 11$ T. This implies that measurements of $R_H$ made with two weakly confined voltage leads give results which are similar to those found using a Hall bar geometry. On the other hand, we have confirmed both analytically and numerically, that strongly confined voltage leads give the $R_H$ values predicted by the LCP given in Eq. (2).

In Fig. 2, we present results for the single mode form factor $F^{(n)} = (T_{ls}^{(n)} - T_{ld}^{(n)})/(T_{ls}^{(n)} + T_{ld}^{(n)})$ as defined in Ref. [17] for propagating modes. Figs. 2(a) and 2(b) are for $B_p = 1$ T and 11 T respectively. The dotted, dashed, dot-and-dash, and solid lines are for $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, and the total form factor $F = \sum_n(T_{ls}^{(n)} - T_{ld}^{(n)})/\sum_n(T_{ls}^{(n)} + T_{ld}^{(n)})$ respectively. Each mode (not only the one closest to the edge of the BQW) makes a contribution to the total form factor. Quantisation of $R_H$ can be reached when every single mode form factor $F^{(n)} = 1$. Comparing Figs. 2(a) and 2(b), we can see that $F^{(n)}$ is closer to 1 when $B_p \sim B$ than when $B_p \gg B$. In other words, better quantisation plateaus of $R_H$ can be observed by using more loosely confined voltage leads.

In summary, we investigate the possibility of making non-invasive mea-
surement of the LCP and the intrinsic quantum Hall resistance. A model procedure is used for calculation. We proved that the chemical potential described by Büttiker’s formula Eq. (1) has the LCP defined by Eq. (2) as its limit when the potential confinement parameter $B_p$ in the voltage leads increases indefinitely. Numerical calculations are carried out, which confirm the limiting behaviour of the quantum Hall resistance $R_H$. Quenching of $R_H$ is seen in a broad range of $B_p$. Our calculations indicate that it is possible to measure the LCP given in Eq. (2) and the intrinsic QHE non-invasively in some circumstances. It is hoped that further experimental studies of this challenging problem will be made.

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**Figure Caption**

Fig. 1 Hall resistance $R_H$ calculated from Eq. (1) when $B_p = 1$ T (dashed line) and 11 T (dot-and-dash line). The $R_H$ associated with LCP and the longitudinal resistance of BQW are shown by solid line and dotted line respectively.

Fig. 2 The total form factor $F$ (solid line) with three single mode form factor $F^{(1)}$ (dotted line), $F^{(2)}$ (dashed line), and $F^{(3)}$ (dot-and-dash line) for (a) $B_p = 1$ T and (b) $B_p = 11$ T.