SURGERY ON HERMAN RINGS OF THE STANDARD
BLASCHKE FAMILY

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ABSTRACT. Let $B_{\alpha,a}$ be the Blaschke product of the following form:

$$B_{\alpha,a}(z) = e^{2\pi i \alpha} z^{d+1} \left( \frac{z-a}{1-az} \right)^d.$$  

If $B_{\alpha,a}|_{S^1}$ is analytically linearizable, then there is a Herman ring admitting the unit circle as an invariant curve in the dynamical plane of $B_{\alpha,a}$. Given an irrational number $\theta$, the parameters $(\alpha, a)$ such that $B_{\alpha,a}|_{S^1}$ has rotation number $\theta$ form a curve $T_d(\theta)$ in the parameter plane. Using quasiconformal surgery, we prove that if $\theta$ is of Brjuno type, the curve can be parameterized real analytically by the modulus of the Herman ring, from $a = M(\theta)$ up to $\infty$ with $M(\theta) \geq 2d + 1$, for which the Herman ring vanishes. Moreover, we can show that for a certain set of irrational numbers $\theta \in B \setminus H$, the number $M(\theta)$ is strictly greater than $2d + 1$ and the boundary of the Herman rings consist of two quasicircles not containing any critical point.

1. Introduction. For any given $d \in \mathbb{N}$, the standard Blaschke family $\mathcal{F}$ is a two-parameter family defined by

$$\mathcal{F} = \left\{ B_{\alpha,a} \left| B_{\alpha,a}(z) = e^{2\pi i \alpha} z^{d+1} \left( \frac{z-a}{1-az} \right)^d, \; 0 \leq \alpha \leq 1, \; a > 2d + 1 \right. \right\}.$$

These maps can be regarded as simple perturbations of some rigid rotation when $a$ is large enough. So this family is one of the simplest families a map of which may have a Herman ring. It is well studied how their dynamics vary in terms of the parameters $\alpha$ and $a$.

Since a Blaschke product preserves the unit circle $S^1$, if we restrict the map $B_{\alpha,a}$ on the unit circle, we can regard $\mathcal{F}$ as a two-parameter $(\alpha, a)$ family of circle maps, the bifurcation diagram of this family (the parameter $a$ is exchanged by $1/a$) shows a similar diagram to that of Arnold family. As we can write

$$B_{\alpha,a}(e^{1z}) = e^{i(z + 2\pi \alpha + H_a(z))},$$

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we consider the lifting map $\hat{B}_{\alpha,a}(x) = x + 2\pi \alpha + H_\alpha(x)$. The map $H_\alpha(x)$ will appear later. Then we can define the rotation number $\rho(B_{\alpha,a}|_{S^1})$ of $B_{\alpha,a}|_{S^1}$ as follows:

$$
\rho(B_{\alpha,a}|_{S^1}) = \lim_{n \to \infty} \frac{\hat{B}_{\alpha,a}^n(x) - x}{n} \pmod{\mathbb{Z}}.
$$

For any given $d \in \mathbb{N}$, the level set $T_d(\theta)$ of the rotation number $\theta$ in the parameter plane is given by:

$$
T_d(\theta) = \{(\alpha, a) \in [0,1] \times (2d+1, \infty) \mid \rho(B_{\alpha,a}|_{S^1}) = \theta\}.
$$

We are mainly interested in the case when the rotation number is irrational. For the Arnold family $f_{\alpha,\beta}(x) = x + \alpha + \beta \sin x \pmod{2\pi}$, the corresponding level set has been successfully studied. By using the complexification

$$
F_{\alpha,\beta}(z) = ze^{i\alpha} \exp\left\{\frac{1}{2} \beta \left(z - \frac{1}{z}\right)\right\},
$$

$\alpha \in [0,2\pi), \beta \in (0,1)$

of the Arnold family, Fagella and Geyer [4] showed that if $\theta$ is of Brjuno type, the corresponding level set $T(\theta)$ can be parameterized real analytically by the modulus of the Herman ring, from $\beta = 0$ up to a point $(\alpha_0, \beta_0)$ with $\beta_0 \leq 1$, for which the Herman ring collapses. Moreover they treated the problem whether $\beta_0 < 1$ holds or not for some irrational numbers. Using a result of Herman [8], they showed that there exists a set of irrational rotation numbers such that $\beta_0$ is strictly less than 1. In this paper we use their ideas to give a complete description for the parameters where the rational maps in $F$ have Herman rings.

Maps in the standard Blaschke family $F$ are real analytic on the unit circle. As a consequence, if the rotation number $\theta$ of a given map $B_{\alpha,a}|_{S^1}$ is irrational, $B_{\alpha,a}|_{S^1}$ is topologically conjugate to the rigid rotation $R_\theta(z) = e^{2\pi i \theta}z$, i.e. there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $B_{\alpha,a}|_{S^1} \circ \varphi = \varphi \circ R_\theta$. If $\varphi$ is analytic, we say that $B_{\alpha,a}|_{S^1}$ is analytically linearizable (this is equivalent to say that $B_{\alpha,a}$ has a Herman ring around the unit circle).

For the analytical linearization problem, Yoccoz [14] gave the sharpest results which were named as Global Conjugacy Theorem and Local Conjugacy Theorem. His Global Conjugacy Theorem says that if $\theta \notin \mathcal{H}$, then any analytic circle diffeomorphism with rotation number $\theta$ is analytically linearizable. The set $\mathcal{H}$ of irrational numbers has full Lebesgue measure and includes all Diophantine numbers $D$. It is a proper subset of Brjuno numbers $B$ (for precise definitions of these sets the reader can refer to [14].) The Global Conjugacy Theorem is sharp in the following sense: if $\theta \notin \mathcal{H}$, there exists an analytic circle diffeomorphism $f$ with rotation number $\theta$, such that $f$ is not analytically linearizable. Yoccoz’s Local Conjugacy Theorem states that if $f$ has rotation number $\theta \in B$ and is close to the rigid rotation, then it is analytically linearizable. For the standard Blaschke family, closeness to the rigid rotation means having $a > M(\theta)$, where $M(\theta) > 0$ is possibly very large. For the standard Blaschke family, Local Conjugacy Theorem is sharp. Indeed, it was shown by Henriksen [6] and Chu [3] that: if $B_{\alpha,a}|_{S^1}$ is analytically linearizable, then the rotation number $\rho(B_{\alpha,a}|_{S^1}) \in B$.

Hence, based on the argument above, the set $\mathbb{R} \setminus \mathbb{Q}$ can be divided into the following three subsets corresponding three different types of irrational curves in the parameter space of the standard Blaschke family $F$:

1. if $\theta \in \mathcal{H}$, all maps in $T_d(\theta)$ are analytically linearizable;
2. if \( \theta \in \mathcal{B} \setminus \mathcal{H} \) and \((\alpha, a) \in T_d(\theta)\) then there exists \( M(\theta) \geq 2d + 1 \) such that \( B_{\alpha,a} \) is analytically linearizable for all \( a > M(\theta) \);

3. if \( \theta \notin \mathcal{B} \), no map in \( T_d(\theta) \) is analytically linearizable.

For the second case, no information is given about those maps in \( T_d(\theta) \) with \( a \leq M(\theta) \). In this paper, we aim to study irrational curves of type (1) and (2) in detail. More precisely, given an irrational curve \( T_d(\theta) \) such that \( \theta \in \mathcal{B} \), we shall study the set of parameters \((\alpha, a) \in T_d(\theta)\) such that \( B_{\alpha,a} \) is analytically linearizable. Obviously in the case \( \theta \in \mathcal{H} \), this set is exactly \( T_d(\theta) \); but if \( \theta \in \mathcal{B} \setminus \mathcal{H} \), it might, a priori, be a disconnected set.

We are also interested in the regularity of the curves \( T_d(\theta) \). And there are some partial result about the regularity: Arnold [2] and Herman [7] showed that \( T_d(\theta) \) is real analytic if \( \theta \) satisfies some Diophantine condition; using different techniques, Risler [12] proved that \( T_d(\theta) \) is real analytic for \( a > N(\theta) \) for some constant \( N(\theta) > 0 \) if \( \theta \in \mathcal{B} \).

We shall use the method of complex dynamics in order to study these questions. The standard Blaschke family can be regard as rational maps, and these are holomorphic self maps of \( \hat{C} \). Observe that \( B_{\alpha,a} \) is symmetric with respect to the unit circle \( S^1 \).

The dynamical plane \( \hat{C} \) of the rational map \( B_{\alpha,a} \) consists of two completely invariant subsets: the Fatou set \( \mathcal{F}(B_{\alpha,a}) \), where the dynamics is stable and the Julia set \( \mathcal{J}(B_{\alpha,a}) = \hat{C} \setminus \mathcal{F}(B_{\alpha,a}) \) where chaotic dynamics occurs. The periodic Fatou components are completely classified for rational maps by Sullivan [11]. Among these components we are interested in Herman rings. A component \( A \) of the Fatou set \( \mathcal{F}(f) \) is called a fixed Herman ring if \( A \) is conformally isomorphic to some standard annulus \( A_R = \{ z \in \mathbb{C} \mid 1/R < |z| < R \} \), and \( f \) is conjugate to some rigid irrational rotation \( R_\theta(z) = e^{2\pi i \theta} z \) on the standard annulus \( A_R \). The number \( R \) is unique and the modulus of \( A \) is \( \text{mod}(A_R) = \pi^{-1} \log R \).

If \( \theta \in \mathcal{B} \setminus \mathcal{H} \), maps in \( T_d(\theta) \) with a large enough can have Herman rings. As we decrease \( a \) along the curve \( T_d(\theta) \), the ring vanishes at some point \( M(\theta) \) before arriving at \( a = 2d + 1 \) and never appears again.

The main result in this paper is the following theorem.

**Theorem 1.1.** For any given \( d \in \mathbb{N} \) and \( \theta \in \mathcal{B} \), let \((\alpha, a) \in T_d(\theta)\) be such that \( B_{\alpha,a} \) has a Herman ring \( A \) admitting \( S^1 \) as an invariant curve. Let \( R \in (1, \infty) \) be such that \( \text{mod}(A) = \pi^{-1} \log R \). Then there exists a real analytic map

\[
\gamma : (0, \infty) \to T_d(\theta)
\]

\[
s \mapsto (\alpha(s), a(s))
\]

such that:

1. for each \( s \in (0, \infty) \), the map \( B_{\alpha(s),a(s)} \) has a Herman ring \( A_s \) of rotation number \( \theta \) and modulus \( \pi^{-1} s \log R \);

2. the map \( s \mapsto a(s) \) is strictly increasing;

3. \( \lim_{s \to \infty} a(s) = \infty \);

4. \( \lim_{s \to 0} a(s) = M(\theta) \geq 2d + 1 \); and

5. for any \((\alpha, a) \in T(\theta)\) such that \( a \leq M(\theta) \), the map \( B_{\alpha,a} \) has no Herman ring.

From Theorem 1.1 we know that the irrational curves \( T_d(\theta) \) are analytically parameterized by the modulus of the Herman ring down to the point \( M(\theta) \), where the ring vanishes. And this does not be implied by the general regularity results
mentioned above. More important, after the point $M(\theta)$, the ring never appears again. By Yoccoz’s Global Conjugacy Theorem: $M(\theta) = 2d + 1$ when $\theta \in \mathcal{H}$. The important question at this point is that whether the case $M(\theta) > 2d + 1$ ever occurs and if so, for which rotation numbers $\theta$.

Combining a result of Herman [8] with some ideas from [5], we give an affirmative answer to this question. The precise statement is as follows.

**Theorem 1.2.** For any given $a_0 \in (2d + 1, \infty)$, there exists $\alpha_0 \in (0, 1)$ such that $(\alpha_0, a_0) \in T_d(\theta)$ for some $\theta \in \mathcal{B} \setminus \mathcal{H}$ satisfying $M(\theta) = a_0$ and $B_{\alpha_0, a_0}$ has no Herman ring. Furthermore, if $(\alpha, a) \in T_d(\theta)$ and $a > a_0$, the boundary of the Herman ring $\mathcal{A}$ of $B_{\alpha, a}$ consists of two quasicircles not containing the critical points of $B_{\alpha, a}$.

An interesting open question is whether the standard Blaschke family is a ‘prototype’ for the Global Conjugacy Theorem, i.e. whether the $M(\theta) > 2d + 1$ for all rotation numbers $\theta \in \mathcal{B} \setminus \mathcal{H}$. To our knowledge, there is no explicit prototype family which is known to have this property.

2. Preliminaries.

2.1. Quasiconformal surgery. Quasiconformal surgery is based on the quasiconformal mapping theory. The standard references for quasiconformal mappings are [1] and [9]. Quasiconformal surgery is a way to construct new rational maps with certain dynamical properties from existing holomorphic maps. In order to do this, we first construct a map that is not holomorphic (at least in some part of the domain) but is still quasiregular. Then we appeal to the measurable Riemann mapping theorem to recover the analyticity. By passing through quasiregular mappings, we gain a flexibility in the construction.

Let $U$ be an open subset of the complex plane $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a map.

**Definition 2.1.** We say that $f$ is $K$–quasiregular ($K \geq 1$) if

- $f \in C(U) \cap W^{1,2}_{\text{loc}}(U)$, that is to say: $f$ is continuous and belongs to the Sobolev class $W^{1,2}_{\text{loc}}(U)$, i.e. with distributional first derivatives which are locally square-integrable;
- $|\overline{\partial f}| \leq \frac{K-1}{K+1}|\partial f|$ almost everywhere on $U$.

Now we will give a key quasiconformal extension lemma which is very important in constructing new dynamics.

**Lemma 2.2.** For $r > 1$, let $\text{Int}\partial A_r$ and $\text{Ext}\partial A_r$ denote the interior and exterior boundary of the standard annulus $A_r$. Let

$$\varphi^{(1/r)} : \text{Int}\partial A_r \to \text{Int}\partial A_r$$

$$\varphi^{(r)} : \text{Ext}\partial A_r \to \text{Ext}\partial A_r$$

be two quasisymmetric maps symmetric with respect to $S^1$. Then there exists a quasiconformal mapping $\Phi : A_r \to A_r$ such that

1. $\Phi|_{\text{Int}\partial A_r} = \varphi^{(1/r)}$ and $\Phi|_{\text{Ext}\partial A_r} = \varphi^{(r)}$;
2. $\Phi$ is symmetric with respect to $S^1$.

The basic idea of the proof is to lift by universal covering to parallel strips, where one can just use linear interpolation. The whole construction can be made in a symmetric way, so the resulting map will be symmetric with respect to the unit circle. For more details of the proof the reader can refer to [3] and [4].
We end this subsection with a powerful tool which we will use later. The following QC-lemma due to Shishikura [13] when he estimated the number of non-repelling periodic cycles for a nonlinear rational map with fixed degree.

**Lemma 2.3.** Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a quasiregular mapping. Suppose that there are disjoint open sets \( E_i \) of \( \hat{\mathbb{C}} \), quasiconformal mappings \( \Phi_i : E_i \to E'_i \) and integer \( N \geq 0 \), satisfying the following conditions:

1. \( f(E) \subset E \), where \( E = E_1 \cup \cdots \cup E_m \)
2. \( \Phi \circ f \circ \Phi_i^{-1} \) is holomorphic in \( E'_i = \Phi(E_i) \), where \( \Phi : E \to \hat{\mathbb{C}} \) is defined by \( \Phi|_{E_i} = \Phi_i \)
3. \( \frac{\partial \Phi}{\partial z} = 0 \ (a.e) \) on \( \mathbb{C} \setminus f^{-N}(E) \)

Then there exists a quasiconformal mapping \( \psi \) of \( \hat{\mathbb{C}} \) such that \( \psi \circ f \circ \psi^{-1} \) is a rational map. Moreover, \( \psi \circ \Phi_i^{-1} \) is conformal in \( E'_i \) and \( \frac{\partial \psi}{\partial z} = 0 \ (a.e) \) on \( \hat{\mathbb{C}} \setminus \bigcup_{n \geq 0} f^{-n}(E) \)

## 2.2. Dynamics of Blaschke product \( B_{\alpha,a} \)

\[
B_{\alpha,a}(z) = e^{2\pi i \alpha} z^{d+1} \left( \frac{z - a}{1 - \bar{a} z} \right)^d
\]

\[
B'_{\alpha,a}(z) = -e^{2\pi i \alpha} z^d \left( \frac{z - a}{1 - \bar{a} z} \right)^{d-1} \left( \bar{a}(d+1)z^2 - (2d+1+|a|^2)z + a(d+1) \right).
\]

Since the rational maps \( B_{\alpha,a} \) have degree \( 2d + 1 \), they have \( 4d \) critical points counted with multiplicity. The fixed points \( z = 0 \) and \( z = \infty \) are critical points of multiplicity \( d \) and hence superattracting fixed points of local degree \( d - 1 \). The zeros \( z_0 = a \) and poles \( z_\infty = 1/\bar{a} \) are critical points of multiplicity \( d + 1 \). The other two critical points, denoted by \( c_\pm \), are given by

\[
c_\pm = a \cdot \left( 2d + 1 + |a|^2 \right) \pm \sqrt{(2d + 1 - |a|^2)(2d + 1 + |a|^2 - 1)}
\]

\[
\frac{2|a|^2(d+1)}{2|a|^2(d+1)}
\]

If \( |a| < 1 \) or \( |a| > 2d + 1 \) the critical points \( c_+ \) and \( c_- \) are free and satisfy \( |c_+| > 1, |c_-| < 1 \) and \( c_+ = 1/c_-^* \). Consequently, their orbits are symmetric with respect to \( S^1 \). On the other hand, if \( 1 < |a| < 2d + 1 \), then the critical points satisfy \( c_+ = a \cdot k \) and \( c_- = a \cdot k \), where

\[
k = \left( \frac{2d + 1 + |a|^2}{2|a|^2(d+1)} \right) + \frac{i \sqrt{(2d + 1 - |a|^2)(2d + 1 + |a|^2 - 1)}}{2|a|^2(d+1)} \in \mathbb{C}.
\]

In this case \( c_+ \) and \( c_- \) are not symmetric with respect to the unit circle and it follows that \( |c_+| = |c_-| = 1 \) since otherwise its symmetric points would lead to two extra critical points.

When \( |a| < 1 \) we have that both critical points \( c_\pm \) lie on the half ray containing \( a \). Moreover, \( |c_-| < 1 \) and \( |c_+| > 1 \). The only pole \( z_\infty = 1/\bar{a} \) has modulus greater than one. Hence \( B_{\alpha,a} : \mathbb{D} \to \mathbb{D} \) is a holomorphic self map of \( \mathbb{D} \) having \( z = 0 \) as a superattracting fixed point. Since there no preimage of the unit disk outside the unit circle, \( B_{\alpha,a}|_\mathbb{D} \) is a degree \( 2d + 1 \) branched covering. By Schwarz lemma we have that \( z = 0 \) is the only attracting point of \( B_{\alpha,a} \) in \( \mathbb{D} \) and attracts all orbits in \( \mathbb{D} \).

Summarizing above, we have:

**Lemma 2.4.** If \( |a| < 1 \), the immediate basin of attraction \( A(0) = \mathbb{D} \) and by symmetry \( A(\infty) = \mathbb{C} \setminus \mathbb{D} \). Hence, the Julia set \( J(B_{\alpha,a}) = S^1 \).
When \(|a| = 1\) both critical points and the preimages of 0 and \(\infty\) collapse at the point \(z = a\), where the function is not formally defined. Everywhere else we have the equality

\[
B_{\alpha,a} = e^{2\pi i \alpha} z^{d+1} \left( \frac{z - a}{1 - z/a} \right)^d = e^{2\pi i \alpha} (-a)^d z^{d+1}.
\]

When \(1 < |a| < 2d + 1\) the two critical points lie on the unit circle, i.e., \(|c_{\pm}| = 1\). Consequently, the critical orbits lie on \(S^1\) and are not related to each other by symmetry.

When \(|a| = 2d + 1\) there is a unique critical point \(a/(2d + 1)\) of multiplicity 2 on the unit circle.

**Lemma 2.5.** For \(|a| \leq 2d + 1\), \(B_{\alpha,a}\) has no Herman rings.

**Proof.** Shishikura [13] proved that if a rational map has a Herman ring, then it has two different critical points whose orbits accumulate on the two different components of its boundary. It follows that \(B_{\alpha,a}\) can have at most one cycle of Herman rings.

If \(|a| \leq 1\), the Julia set \(J(B_{\alpha,a}) = S^1\), so \(B_{\alpha,a}\) cannot have Herman rings. If \(1 < |a| \leq 2d + 1\), the two critical points lie on \(S^1\) and, hence, there can be no Herman rings.

When \(|a| > 2d + 1\) both critical points \(c_{\pm}\) lie on the half ray containing \(a\) and are symmetric with respect to \(S^1\). And in this case \(B_{\alpha,a}\) may have Herman rings.

### 2.3. Lift of the Blaschke product in \(\mathcal{F}\).

**Lemma 2.6.** Let \(B_{\alpha,a} \in \mathcal{F}\). If we consider \(B_{\alpha,a}\) as a mapping from the unit circle onto itself, we have

\[
B_{\alpha,a}(e^{ix}) = e^{i(x + 2\pi \alpha + H_a(x))}
\]

where

\[
H_a(x) = -2d \arcsin \left( \frac{\sin x}{\sqrt{a^2 - 2a \cos x + 1}} \right).
\]

We consider \(\tilde{B}_{\alpha,a}(x) = x + 2\pi \alpha + H_a(x) \pmod{2\pi}\).

**Proof.** The proof is an easy computation.

\[
B_{\alpha,a}(e^{ix}) = e^{2\pi i \alpha} e^{i(d+1)x} \left( \frac{e^{ix} - a}{1 - ae^{ix}} \right)^d
\]

\[
= e^{i(x + 2\pi \alpha)} \left( \frac{e^{ix} - a}{e^{-ix} - a} \right)^d.
\]

Since

\[
\left| \frac{e^{ix} - a}{e^{-ix} - a} \right| = 1,
\]

let

\[
\frac{e^{ix} - a}{e^{-ix} - a} = e^{2i\phi}.
\]

From this one, we can get

\[
\sin \phi = \frac{-\sin x}{\sqrt{a^2 - 2a \cos x + 1}}.
\]
Define another two families of Blaschke products of degree $2d + 1$:
$$E = \{B_{\alpha,a} \mid 0 \leq \alpha \leq 1, |a| > 2d + 1\}$$
and
$$\mathcal{E} = \{B_{\alpha,a} \mid 0 \leq \alpha \leq 1, |a| > 2d + 1\}.$$
Obviously, $\mathcal{F} \subset \mathcal{E} \subset \mathcal{E}$. Here the following characterization holds.

**Lemma 2.7.** A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $2d + 1$ belongs to $\mathcal{E}$ if and only if
1. $f$ maps $S^1$ diffeomorphically onto $S^1$;
2. $f$ has two zeros: one zero at 0 with multiplicity $d$, the other at $a$ with multiplicity $(d - 1)$;
3. $f$ has two poles: one pole at $\infty$ with multiplicity $(d + 1)$, the other at $1/a$ with multiplicity $d$.

Moreover, every map $f \in \mathcal{E}$ commutes with the reflection $J(z) = \frac{1}{z}$.

We use the following lemma [11] to prove Lemma 2.7.

**Lemma 2.8.** A rational map $f$ of degree $d$ carries the unit circle into itself if and only if it can be written as a Blaschke product
$$f(z) = e^{2\pi i \theta} \beta_{a_1}(z) \beta_{a_2}(z) \cdots \beta_{a_d}(z) \quad \text{with} \quad \beta_{a_i}(z) = \frac{z - a_j}{1 - \bar{a}_j z}$$
for some constants $e^{2\pi i \theta} \in \partial \mathbb{D}$ and $a_1, a_2, \ldots, a_d \in \mathbb{C} \setminus \partial \mathbb{D}$.

**Proof.** According to the Lemma 2.8, $f$ can be written as
$$f(z) = e^{2\pi i \theta} \beta_{a_1}(z) \beta_{a_2}(z) \cdots \beta_{a_{2d+1}}(z) \quad \text{with} \quad \beta_{a_j}(z) = \frac{z - a_j}{1 - \bar{a}_j z}$$
for some constants $e^{2\pi i \theta} \in \partial \mathbb{D}$ and $a_1, a_2, \ldots, a_{2d+1} \in \mathbb{C} \setminus \partial \mathbb{D}$. By the zeros and poles of $f$, we can get $f$ has the following form:
$$f(z) = e^{2\pi i \theta} z^{d+1} \left( \frac{z - a}{1 - \bar{a} z} \right)^d.$$

Since $f$ maps $S^1$ diffeomorphically onto $S^1$, so there no critical points on the unit circle. By the argument in section 2.2, we must have $|a| < 1$ or $|a| > 2d + 1$. 

2.4. **Herman’s theorem.** We cite the following theorem [8], see [4]:

**Theorem 2.9.** Let $f$ be an orientation-preserving $C^\infty$–diffeomorphism of $\mathbb{R}/2\pi \mathbb{Z}$ such that no iterate of $f + \alpha$ lifts to the identity map, for any $\alpha \in \mathbb{R}$. Then, there exists $\alpha_0 \in \mathbb{R}$ and irrational number $\theta$ such that:
1. $f + \alpha_0$ has an irrational rotational number $\theta$;
2. $f + \alpha_0 = \varphi \circ r_\theta \circ \varphi^{-1}$ where $\varphi$ is a quasisymmetric map of $\mathbb{R}/2\pi \mathbb{Z}$ and $\varphi(1) = 1$;
3. $\varphi$ is not $C^2$.

Here $r_\theta(x) = x + 2\pi \theta$ (mod $2\pi$). Applying this theorem to the Blaschke product $B_{\alpha,a}$, we have

**Corollary 1.** Let $\tilde{f}(x) = x + H_\alpha(x) \pmod{2\pi}$ and $\tilde{B}_{\alpha,a}$ as in Lemma 2.6. Then, for any $\alpha_0 \in (2d + 1, \infty)$, there exists $\alpha_0 \in \mathbb{R}$ and irrational number $\theta$ such that
1. $\tilde{B}_{\alpha_0,a}$ has an irrational rotational number $\theta$;
2. $\tilde{B}_{\alpha_0,a} = \varphi \circ r_{\theta} \circ \varphi^{-1}$ where $\varphi$ is a quasisymmetric map of $\mathbb{R}/2\pi \mathbb{Z}$ and $\varphi(1) = 1$;
3. \( \varphi \) is not \( C^2 \).

In particular, \( B_{\alpha,a_0} \) has no Herman ring admitting \( S^1 \) as an invariant curve.

**Proof.** By the Lemma 4.3 in [10], no iterate of \( \tilde{f} + \alpha \) lifts to the identity map. Thus, it only needs to be shown that \( B_{\alpha,a_0} \) has no Herman ring. Suppose that \( B_{\alpha,a_0} \) has a Herman \( \mathcal{A} \) ring admitting \( S^1 \) as an invariant curve. Because \( B_{\alpha,a_0} \) commutes with \( J(z) = \frac{1}{z} \). Thus there exists a conformal mapping \( \psi : A_r \rightarrow \mathcal{A} \) such that

\[
B_{\alpha,a_0} = \psi \circ R_\theta \circ \psi^{-1}
\]

where \( R_\theta(z) = e^{2\pi i \theta} z \) is the rigid rotation. This contradicts that \( \varphi \) is not analytic. \( \square \)

3. **Proof of Theorem 1.1.** In this section, we fix \( \theta \in B \). Let \( (\alpha,a) \in T_d(\theta) \) be such that \( B_{\alpha,a} \) has a Herman ring \( \mathcal{A} \) admitting \( S^1 \) as an invariant curve with rotation number \( \theta \). We know that this ring must be fixed. Since \( B_{\alpha,a} \) commutes with \( J \), it follows that \( \mathcal{A} \) is symmetric with respect to \( S^1 \). Let \( R \in (1,\infty) \) be such that \( \text{mod}(\mathcal{A}) = \pi^{-1} \log R \).

Let \( \varphi : A_R \rightarrow \mathcal{A} \) be the unique conformal map such that \( \varphi^{-1}(1) > 0 \) which maps the exterior boundary component of \( A_R \) onto the exterior boundary component of \( \mathcal{A} \). Then \( \varphi \) conjugates \( B_{\alpha,a} \) on \( \mathcal{A} \) to the rigid rotation \( R_\theta \) on \( A_R \), i.e.

\[
B_{\alpha,a} \circ \varphi = \varphi \circ R_\theta.
\]

It is easy to see that \( J \circ \varphi \circ J \) is also a conjugate map satisfying the same conditions. Therefore, \( \varphi = J \circ \varphi \circ J \). Thus \( \varphi \) is symmetric with respect to \( S^1 \).

For \( s \in (0,\infty) \), let \( \phi_s : A_{R^s} \rightarrow A_R \) be the quasiconformal mapping \( \phi_s(z) = z|z|^{-s-1} \). This map commutes with \( R_\theta \) and satisfies \( \phi_s(1) = 1 \). Clearly, \( S^1 \) is invariant under \( \phi_s \) and \( J \circ \phi_s = \phi_s \circ J \). Let

\[
\Phi_s = \varphi \circ \phi_s : A_{R^s} \rightarrow \mathcal{A},
\]

\( \Phi_s \) also is symmetric with respect to \( S^1 \). We define the map \( g_s : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) as follows:

\[
g_s(z) = \begin{cases} f & \text{on } \hat{\mathbb{C}} \setminus \mathcal{A} \\ \Phi \circ R_\theta \circ \Phi^{-1} & \text{on } \mathcal{A} \end{cases}.
\]

Then \( g_s \) commutes with \( J \). Since \( f \) has zeros of order \( d+1 \) at the origin and of order \( d \) at \( 1/a \), the definition shows that \( g_s \) also has zeros of order \( d+1 \) at the origin and of order \( d \) at some other point.

Here we need the QC-Lemma 2.3, take \( g_s \) constructed above as \( f \), \( \mathcal{A} \) as \( E \) and \( \Phi_s \) as \( \Phi \), it is easy to check that the conditions of QC-lemma hold. So there exists a quasiconformal mapping \( \psi_s \) such that

\[
G_s = \psi_s \circ g_s \circ \psi_s
\]

is a rational map and \( \psi \) commutes with \( J \). From the construction, \( G_s \) has Herman ring \( \psi_s(A) \) of rotation number \( \theta \) and modulus \( \pi^{-1} s \log R \).

It is necessary to check whether \( G_s \) belongs to the family \( \mathcal{E} \). According to the Lemma 2.7, \( G_s \) belongs to the family \( \mathcal{E} \), i.e. there exists a real number \( \alpha(s) \), \( 0 \leq \alpha(s) \leq 1 \) and a complex number \( a(s) \) satisfying \( |a(s)| > 2d+1 \) or \( |a(s)| < 1 \) such that

\[
G_s(z) = e^{2\pi \imath \alpha(s)} |z|^{d+1} \left( \frac{z - a(s)}{1 - \bar{a(s)}z} \right)^d.
\]
The case of \(|a(s)| < 1\) does not occur by Lemma 2.5, because \(G_s\) has a Herman ring. Thus \(G_s \in \mathcal{E}\). Without loss any generality, we can assume \(a(s) \in \mathbb{R}\), otherwise we can make a suitable rotation conjugation. So \(G_s = B_{a(s), a(s)} \in \mathcal{F}\).

In the following we show that \(a(s)\) and \(a(s)\) are real analytic functions. As \(a(s)\) is the critical point of \(G_s\), according to the dependence of the measurable Riemann mapping theorem \(a(s) = \psi_s(a)\) is analytic with respect to \(s\). Since \(G_s(1) = e^{2\pi i \alpha(s)}\) analytically depends on \(s\), \(a(s)\) depends analytically on \(s\).

Next we go on to show that the results of (2), (3), (4) and (5) of Theorem 1.1 are satisfied. We need the following lemma in [9].

**Lemma 3.1.** If the ring domain \(A\) separates the pair of points \(z_1, w_1\) from the pair \(z_2, w_2\) and if the spherical distant satisfies \(d(z_i, w_i) \geq \delta > 0\), \(i = 1, 2\), then the modulus \(M(A)\) of \(A\) satisfies

\[ M(A) \leq \frac{\pi^2}{2\delta^2}. \]

Let \(z_1 = 0, w_1 = 1/a(s)\) and \(z_2 = \infty, w_2 = a(s)\) and \(A = \psi_s(A)\). By Lemma 3.1 if \(M(A) > \frac{\pi^2}{2\delta^2}\), then \(d(z_i, w_i) < \delta\). This implies that the modulus \(\pi^{-1}s \log R\) increases then the critical point tends to \(\infty\).

Suppose that \(a(s_1) = a(s_2)\). Since \(B_{a(s_1), a(s_1)}\) and \(B_{a(s_2), a(s_2)}\) have Herman rings with the same rotation number \(\theta\), we have that both pairs of parameters belong to the irrational curve \(T_d(\theta)\). But it well know that the irrational curves \(T_d(\theta)\) are graphs in the \(\alpha - a\) plane, i.e. there are no two different points in any irrational curve with the same parameter \(a\) [10]. Hence \(a(s_1) = a(s_2)\). As the Herman rings of \(B_{a(s_1), a(s_1)}\) and \(B_{a(s_2), a(s_2)}\) have modulus \(\pi^{-1}s \log R\) and \(\pi^{-1}s_2 \log R\), respectively, we get \(s_1 = s_2\). This shows that \(a(s)\) is injective. This completes the proof of (2) and (3). It remains only to see what happens when \(s\) tends to 0. Since no Herman ring may exists for any \(a \leq 2d + 1\), and \(a(s)\) is continuous, we must have

\[ \lim_{s \to 0} a(s) = M(\theta) \geq 2d + 1. \]

This implies that (4) is correct.

Finally, if there exists \(a < M(\theta)\) such that \(B_{a,a}\) has a Herman ring for some \(a\) satisfying \((\alpha, a) \in T_d(\theta)\), then the previous procedure applied to \(B_{a,a}\) shows that there is a non-zero lower bound on the modulus of the Herman ring of \(B_{a(s), a(s)}\) for any \(s\), contradicting the fact that these moduli tend to zero as \(s\) tends to 0.

4. **Proof of Theorem 1.2.** As we mention in the introduction, according to Yoccoz’s theorem [14] if \(\theta \in \mathcal{H}\), every map in \(T_d(\theta)\) is analytically linearizable, i.e. there exists an analytic circle diffeomorphism \(\varphi\) such that

\[ B_{\alpha, \alpha}|_{S^1} \circ \varphi = \varphi \circ R_\theta. \]

Therefore, every rational map \(B_{\alpha, \alpha}\) has Herman ring for \((\alpha, a) \in T_d(\theta)\).

On the other hand, if \(\theta \notin \mathcal{B}\), then no map for \((\alpha, a) \in T_d(\theta)\) has Herman ring [3]. Therefore the most interesting and uncertain case is \(\theta \in \mathcal{B} \setminus \mathcal{H}\).

For any fixed \(a_0 \in (2d + 1, \infty)\), by Theorem 2.9, there exists \(\alpha_0 \in [0, 1]\) and irrational number \(\theta\) such that:

1. \(B_{\alpha_0, \alpha_0}|_{S^1}\) has rotation number \(\rho(B_{\alpha_0, \alpha_0}|_{S^1}) = \theta\);
2. \(B_{\alpha_0, \alpha_0}|_{S^1} \circ \varphi = \varphi \circ R_\theta\) where \(\varphi\) is a quasisymmetric map of \(S^1\);
3. \(\varphi\) is not \(C^2\).
Then by Corollary 1, $B_{α_0,a_0}$ has no Herman ring and hence $θ \notin \mathcal{H}$. Since $θ$ is an irrational number, $α_0$ is unique such that $(α_0,a_0) ∈ T_d(θ)$ according to Lemma 4.1 in [10].

In what follows we will construct a one-parameter curve of maps of the Blaschke product $B_{α(r),a(r)}$ for $r ∈ (1, ∞)$, each of them has a Herman ring $A_r$ of rotation number $θ$ and modulus $π^{-1} \log r$. If so, it will imply that $θ ∈ \mathcal{B}$. Observe that, by Theorem 1.1, $r → (α(r), a(r))$ is a reparametrization of the curve $T_d(θ)$ from $∞$ down to $M(θ)$ and hence it covers all maps in $T_d(θ)$ with Herman ring. The boundaries of the Herman rings $A_r$ we construct will be quasicircles not containing the critical points of $B_{α(r),a(r)}$.

The idea of the construction is as follows. We start in the dynamical plane of $B_{α_0,a_0}$. We make some space around $S^1$ in order to glue a standard annulus $A_r$ there. Then we define a new self map $f_r$ of $\hat{\mathbb{C}}$, which equals the rigid rotation inside $A_r$ and equals $B_{α_0,a_0}$ (after a change of scale) outside this annulus. This map is not holomorphic, but satisfies Shishikura’s QC-Lemma. Applying QC-lemma we can obtain a rational map with desired dynamical properties.

We now proceed to make this construction precise. For simplicity we take $B = B_{α_0,a_0}$. Let $r > 1$ be arbitrarily fixed. Consider the standard annulus $A_r = \{z ∈ \mathbb{C} \mid 1/r < |z| < r\}$. Then we cut the complex plane along the unit circle dividing into two parts and make the interior part shrink and also enlarge the exterior part in order to make the annulus $A_r$ around the unit circle. As $B$ maps $S^1$ onto itself. We define the map on $\hat{\mathbb{C}} \setminus A_r$ as follows:

$$f_r(z) = \begin{cases} rB(z/r) & \text{if } |z| ≥ r \\ 1/rB(rz) & \text{if } |z| ≤ 1/r \end{cases}.$$

We define two circle maps

$$φ_r^{(1/r)} : Int∂A_r → Int∂A_r$$

by $(1/r)φ(rz)$ and

$$φ_r^{(r)} : Ext∂A_r → Ext∂A_r$$

by $rφ(z/r)$. According to Corollary 1, $φ_r^{(1/r)}$ and $φ_r^{(r)}$ are quasisymmetrically conjugate to the rigid rotation $R_θ$ on $Int∂A_r$ and $Ext∂A_r$, respectively.

By the quasiconformal extension Lemma 2.2, we obtain a quasiconformal mapping $φ_r : A_r → A_r$ such that

$$φ_r(z) = \begin{cases} φ_r^{(1/r)} & \text{if } z ∈ Int∂A_r \\ φ_r^{(r)} & \text{if } z ∈ Ext∂A_r \end{cases}$$

and $φ_r$ commutes with reflection in the unit circle.

Then combining the map defined on $\hat{\mathbb{C}} \setminus A_r$ and $A_r$. We define the map $f_r : \hat{\mathbb{C}} → \hat{\mathbb{C}}$ as follows:

$$f_r = \begin{cases} f_r & \text{on } \hat{\mathbb{C}} \setminus A_r \\ φ_r^{-1} ◦ R_θ ◦ φ_r & \text{on } A_r \end{cases}.$$

It is easily seen that the map coincides on the boundary and finally $f_r$ is continuous on $\hat{\mathbb{C}}$. Actually, it is quasiconformal mapping inside the annulus and a holomorphic
map outside the annulus. Therefore, $f_r$ is quasiregular mapping in $\hat{\mathbb{C}}$ and commutes with $J$.

Applying the QC-Lemma 2.3, take $f_r$ constructed above as $f$, $A_r$ as $E = E'$ and $\varphi_r$ as $\Phi$. Then, it is easy to check that all the conditions are satisfied. So there exists a quasiconformal mapping $\psi$ such that

$$F_r = \psi \circ f_r \circ \psi^{-1}$$

is a rational map.

From the construction, $F_r$ has a Herman ring $A_r = \psi(A_r)$ of rotation number $\theta$ and modulus $\pi^{-1} \log r$.

It is necessary to check whether $F_r$ belongs to the family $\mathcal{E}$. By construction, $F_r$ is symmetric with respect to the unit circle. Moreover, it is clear that $F_r$ has only two free critical points which are bounded away from the boundary of the Herman ring, since the critical points of the original map $B$ were bounded away from the unit circle. According to the Lemma 2.7, $F_r$ belongs to the family $\mathcal{E}$, i.e. there exists a real number $\alpha(r)$, $0 \leq \alpha(r) \leq 1$ and a complex number $a(r)$ satisfying $|a(r)| > 2d + 1$ or $|a(r)| < 1$ such that

$$F_r(z) = e^{2\pi i \alpha(r)} z^{d+1} \left( \frac{z - a(r)}{1 - \overline{a(r)} z} \right)^d$$

The case of $|a(r)| < 1$ does not occur by Lemma 2.5, because $F_r$ has the Herman ring $A_r$. Thus $F_r \in \mathcal{E}$. Without loss any generality, we can assume $a(r) \in \mathbb{R}$, otherwise we can make a suitable rotation conjugation. So $F_r = B_{\alpha(r), a(r)} \in \mathcal{F}$. The set $A_r = \psi(A_r)$ is a Herman ring of rotation number $\theta$ and modulus $\pi^{-1} \log r$, as required.

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