Tame Class Field Theory for Singular Varieties over Algebraically Closed Fields

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1 Introduction

Let $X$ be a (possibly singular) separated scheme of finite type over an algebraically closed field $k$ and let $m$ be a natural number. We construct a pairing between the first mod $m$ algebraic singular homology $H_{S}^{1}(X, \mathbb{Z}/m\mathbb{Z})$ and the first mod $m$ tame étale cohomology group $H_{1}^{t}(X, \mathbb{Z}/m\mathbb{Z})$, which classifies finite abelian étale coverings of $X$ whose pull-back to every regular curve extends to an at most tamely ramified covering of its regular compactification (cf. [KS]). For $\pi_{1,\text{ab}}^{X}(X, \mathbb{Q}/\mathbb{Z})$ we prove the following analogue of Hurewicz’s theorem in algebraic topology:

**Theorem 1.1** (=Theorem 7.1). The induced homomorphism

$$\text{rec}_{X}: H_{S}^{1}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \pi_{1,\text{ab}}^{X}(X)/m$$

is surjective. It is an isomorphism of finite abelian groups if $(m, \text{char}(k)) = 1$ and for general $m$ if resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over $k$.

The motivation for constructing our pairing comes from topology: For a locally contractible Hausdorff space $X$ and a natural number $m$, the canonical duality pairing

$$\langle \cdot, \cdot \rangle: H_{1}^{\text{sing}}(X, \mathbb{Z}/m\mathbb{Z}) \times H_{1}^{\text{sing}}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z},$$

between singular homology and sheaf cohomology with mod $m$ coefficients can be given explicitly in the following way: represent $b \in H_{1}^{\text{sing}}(X, \mathbb{Z}/m\mathbb{Z})$ by a $\mathbb{Z}/m\mathbb{Z}$-torsor $T \rightarrow X$ and $a \in H_{1}^{\text{sing}}(X, \mathbb{Z}/m\mathbb{Z})$ by a 1-cycle $\alpha$ in the singular complex of $X$. Then

$$\langle a, b \rangle = \Phi_{\text{par}}^{-1} \circ \Phi_{\text{taut}} \in \mathbb{Z}/m\mathbb{Z}, \quad \text{where} \quad \Phi_{\text{taut}}, \Phi_{\text{par}}: \alpha^{*}(T)|_{0} \sim \alpha^{*}(T)|_{1},$$

are the isomorphisms between the fibres over 0 and 1 of the pull-back torsor $\alpha^{*}(T) \rightarrow \Delta^{1} = [0, 1]$ given tautologically $(0^{*}\alpha = 1^{*}\alpha)$ and by parallel transport (every $\mathbb{Z}/m\mathbb{Z}$-torsor on $[0, 1]$ is trivial).

For a variety $X$, the pairing between $H_{S}^{1}(X, \mathbb{Z}/m\mathbb{Z})$ and $H_{1}^{t}(X, \mathbb{Z}/m\mathbb{Z})$ inducing the homomorphism $\text{rec}_{X}$ of our Main Theorem 1.1 will be constructed in the same way. However, 1-cycles in the algebraic singular complex are not linear combinations of morphisms but finite correspondences from $\Delta^{1}$ to $X$. In order to mimic the above construction, we thus have to define the pull-back of a torsor along a finite correspondence, which requires the construction of the push-forward torsor along a finite surjective morphism.
To prove the theorem, we first consider the case of a smooth curve $C$. If $A$ is the Albanese variety of $C$, then we have isomorphisms

$$H^1_S(C, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\delta} mH^0_S(C, \mathbb{Z}) \cong mA(k).$$  

(1)

The first isomorphism follows from the coefficient sequence together with the divisibility of $H^1_S(C, \mathbb{Z})$, and the second from the Abel-Jacobi theorem. On the other hand,

$$\text{Hom}(mA(k), \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\tau} H^1_t(C, \mathbb{Z}/m\mathbb{Z}).$$  

(2)

This follows because the maximal étale subcovering $\tilde{A} \to A$ of the $m$-multiplication map $A \to A$ is the quotient of $A$ by the connected component of the finite group scheme $mA$, and the maximal abelian tame étale covering of $C$ with Galois group annihilated by $m$ is $\tilde{C} := C \times_A \tilde{A}$. The heart of the proof of Theorem 1.1 for smooth curves is to show that under the above identifications, our pairing agrees with the evaluation map.

We then show surjectivity of $\text{rec}_X$ for general $X$ by reducing to the case of smooth curves. Finally, we use duality theorems to show that both sides of $\text{rec}_X$ have the same order: For the $p$-primary part, we use resolution of singularities to reduce to the smooth projective case considered in [Ge3]. For $(m, \text{char}(k)) = 1$, Suslin and Voevodsky [SV1] construct an isomorphism

$$\alpha_X : H^1_\text{et}(X, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} H^1_\text{et}(X, \mathbb{Z}/m\mathbb{Z}).$$

Hence the source and the target of $\text{rec}_X$ have the same order and therefore $\text{rec}_X$ is an isomorphism. Moreover, we show that $\text{rec}_X$ is the dual map to $\alpha_X$. Thus, for $(m, \text{char}(k)) = 1$, our construction gives an explicit description of the Suslin-Voevodsky isomorphism $\alpha_X$, which zig-zags through Ext-groups in various categories and is difficult to understand.

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## 2 Torsors and finite correspondences

All occurring schemes in this section are separated schemes of finite type over a field $k$. For any abelian group $A$ and a finite surjective morphism $\pi : Z \to X$ with $Z$ integral and $X$ normal, connected, we have transfer maps

$$\pi_* : H^i_{\text{et}}(Z, A) \to H^i_{\text{et}}(X, A)$$

for all $i \geq 0$ (see [MVW], 6.11, 6.21). The group $H^1_{\text{et}}(Z, A)$ classifies isomorphism classes of étale $A$-torsors (i.e. principal homogeneous spaces) over the scheme $Z$. We are going to construct a functor

$$\pi_* : \mathcal{PHS}(Z, A) \to \mathcal{PHS}(X, A)$$

from the category of étale $A$-torsors on $Z$ to the category of étale $A$-torsors on $X$, which coincides with $\pi_* : H^1_{\text{et}}(Z, A) \to H^1_{\text{et}}(X, A)$ after passing to isomorphism classes.

We recall how to add and subtract torsors. For an abelian group $A$ and $A$-torsors $\mathcal{T}_1, \mathcal{T}_2$ on a scheme $Y$, define

$$\mathcal{T}_1 + \mathcal{T}_2$$
to be the quotient scheme of $\mathcal{T}_1 \times_{Y} \mathcal{T}_2$ by the action of $A$ given by $(t_1, t_2) + a = (t_1 + a, t_2 - a)$. It carries the structure of an $A$-torsor by setting

$$(t_1, t_2) + a := (t_1 + a, t_2) = (t_1, t_2 + a).$$

The functor

$$+ : \mathcal{PHS}(Y, A) \times \mathcal{PHS}(Y, A) \longrightarrow \mathcal{PHS}(Y, A), \quad (\mathcal{T}_1, \mathcal{T}_2) \longmapsto \mathcal{T}_1 + \mathcal{T}_2,$$

lifts the addition in $H^1_{et}(Y, A)$ to torsors (cf. [Mi], III, Rem. 4.8 (b)). Note that “+” is associative and commutative up to natural functor isomorphisms. In particular, we can multiply a torsor by any natural number $m$, putting $m \cdot \mathcal{T} = \mathcal{T} + \cdots + \mathcal{T}$ ($m$ times). If $mA = 0$, then we have a natural isomorphism of torsors

$$m \cdot \mathcal{T} \sim \mathcal{T} \times A, \quad (t_1, \ldots, t_m) \mapsto (t_1 - t_1) + \cdots + (t_m - t_1) \in A, \quad (1)$$

where $Y \times A$ is the trivial $A$-torsor on $Y$ representing the constant sheaf $A$ over $Y$. Here $t_i - t_j$ denotes the unique element $a \in A$ with $t_i = t_j + a$.

Furthermore, given a torsor $\mathcal{T}$, define $(-\mathcal{T})$ to be the torsor which is isomorphic to $\mathcal{T}$ as a scheme and on which $a \in A$ acts as $-a$. This yields a functor

$$(-1) : \mathcal{PHS}(Y, A) \longrightarrow \mathcal{PHS}(Y, A), \quad \mathcal{T} \longmapsto (-\mathcal{T}),$$

which lifts multiplication by $(-1)$ from $H^1_{et}(Y, A)$ to an endofunctor of $\mathcal{PHS}(Y, A)$. We have a natural isomorphism of torsors

$$\mathcal{T} + (-\mathcal{T}) \sim \mathcal{T} \times A, \quad (t_1, t_2) \mapsto t_1 - t_2 \in A. \quad (2)$$

Now let $\pi : Z \rightarrow X$ be finite and surjective, $Z$ integral, $X$ normal, connected, and let $\mathcal{T}$ be an $A$-torsor on $Z$. For every point $x \in X$, the base change $Z \times_X X^x$ is a product of strictly henselian local schemes. Therefore we find an étale cover $(U_i \rightarrow X)_{i \in I}$ of $X$ such that $\mathcal{T}$ trivializes over the pull-back étale cover $(\pi^{-1}(U_i) \rightarrow Z)_{i \in I}$ of $Z$.

Next choose a pseudo-Galois covering $\tilde{\pi} : \tilde{Z} \rightarrow X$ dominating $Z \rightarrow X$. Recall that this means that $k(\tilde{Z})/k(X)$ is a normal field extension and that the natural map $\text{Aut}_X(\tilde{Z}) \rightarrow \text{Aut}_X(k(\tilde{Z}))$ is bijective (cf. [SV1], Lemma 5.6). Let $\pi_{in} : X_{in} \rightarrow X$ be the quotient scheme $\tilde{Z}/G$, where $G = \text{Aut}_X(\tilde{Z})$. Then $X_{in}$ is the normalization of $X$ in the maximal purely inseparable subextension $k(X)^{in}/k(X)$ of $k(\tilde{Z})/k(X)$. Consider the object

$$\tilde{\mathcal{T}} := \sum_{\varphi \in \text{Mor}_X(\tilde{Z}, Z)} \varphi^*(\mathcal{T}) \in \mathcal{PHS}(\tilde{Z}, A),$$

which is defined up to unique isomorphism. Starting from any trivialization of $\mathcal{T}$ over $(\pi^{-1}(U_i) \rightarrow Z)_{i \in I}$, we obtain a trivialization of the restriction of $\tilde{\mathcal{T}}$ to $(\tilde{\pi}^{-1}(U_i) \rightarrow \tilde{Z})_{i \in I}$ of the form

$$\tilde{U}_{i \in I} \cong \tilde{\pi}^{-1}(U_i) \times A,$$

where $G = \text{Aut}_X(\tilde{Z})$ acts on the right hand side in the canonical way on $\tilde{\pi}^{-1}(U_i)$ and trivially on $A$. Therefore the quotient scheme $\tilde{\mathcal{T}}/G$ is an $A$-torsor on $\tilde{Z}/G = X_{in}$ in a natural way. Since $X_{in} \rightarrow X$ is a topological isomorphism, $\tilde{\mathcal{T}}/G$ comes by base change from a unique $A$-torsor $\mathcal{T}'$ on $X$. 

3
Definition 2.1. The push-forward $A$-torsor $\pi_*(T)$ on $X$ is defined by

$$\pi_*(T) = [k(Z) : k(X)] \cdot T'.$$

The assignment $T \mapsto \pi_*(T)$ defines a functor

$$\pi_* : \mathcal{PHS}(Z, A) \to \mathcal{PHS}(X, A).$$

The functor $\pi_*$ is additive in the sense that it commutes with the functors "+" and "(-1)" up to a canonical functor isomorphism.

Let $T \in \mathcal{PHS}(Z, A)$ and assume that there exists a section $s : Z \to T$ to the projection $T \to Z$ (so $T$ is trivial and $s$ gives a trivialization). Let again be $\pi : Z \to X$ be finite and surjective, $Z$ integral, $X$ normal, connected. Then

$$\tilde{T} := \sum_{\varphi \in \text{Mor}_X(\tilde{Z}, Z)} \varphi^*(T) \in \mathcal{PHS}(\tilde{Z}, A)$$

has the canonical section $\sum_{\varphi \in \text{Mor}_X(\tilde{Z}, Z)} \varphi^*(s)$ over $\tilde{Z}$. It descents to a section of $T/G$ over $\tilde{Z}/G = X_{in}$. Descending to $X$ and multiplying by $[k(X_{in}) : k(X)]$, we obtain a section

$$\pi_*(s) : X \to \pi_*(T).$$

In other words, we obtain a map

$$\pi_* : \Gamma(Z, T) \to \Gamma(X, \pi_*(T));$$

hence every trivialization of $T$ gives a trivialization of $\pi_*(T)$ in a natural way.

In order to see that $\pi_*$ gives back the transfer homomorphism $\pi_* : H^1_{et}(Z, A) \to H^1_{et}(X, A)$ after passing to isomorphism classes, we formulate the construction of $\pi_*$ on the level of Čech 1-cocycles. As explained above, we find an étale cover $(U_i \to X)_{i \in I}$ such that $T$ trivializes over the étale cover $(\pi^{-1}(U_i) \to Z)_{i \in I}$ of $Z$.

We fix a trivialization and obtain a Čech 1-cocycle $a = (a_{ij} \in \Gamma(\pi^{-1}(U_i \times_X U_j), A))$ over $(\pi^{-1}(U_i) \to Z)_{i \in I}$ which defines $T$. As before choose a pseudo-Galois covering $\tilde{\pi} : \tilde{Z} \to X$ dominating $Z \to X$. Now for all $i, j$ consider the element

$$\sum_{\varphi \in \text{Mor}_X(\tilde{Z}, Z)} \varphi^*(a_{ij}) \in \Gamma(\tilde{\pi}^{-1}(U_i \times_X U_j), A)$$

which, by Galois invariance lies in

$$\Gamma(\pi^{-1}_{in}(U_i \times_X U_j), A) = \Gamma(U_i \times_X U_j, A).$$

The Čech 1-cocycle given by

$$[k(Z) : k(X)]_{in} \cdot \left( \sum_{\varphi \in \text{Mor}_X(\tilde{Z}, Z)} \varphi^*(a_{ij}) \right) \in \Gamma(U_i \times_X U_j, A).$$

now defines a trivialization of $\pi_*(T)$ over $(U_i \to X)_{i \in I}$. Since the transfer map on étale cohomology is defined on Čech cocycles in exactly this way (see [MVW], 6.11, 6.21), we obtain
Lemma 2.2. Passing to isomorphism classes, the functor $\pi_* : \mathcal{PHS}(Z, A) \to \mathcal{PHS}(X, A)$ constructed above induces the transfer homomorphism
\[
\pi_* : H^1_{\text{et}}(Z, A) \to H^1_{\text{et}}(X, A)
\]
on the first étale cohomology groups.

Assume now that $X$ is regular and $Y$ arbitrary. The group of finite correspondences $\text{Cor}(X, Y)$ is defined as the free abelian group on the set of integral subschemes $Z \subset X \times Y$ which project finitely and surjectively to a connected component of $X$. For such a $Z$, we define $p_{Z \to X}^* : \mathcal{PHS}(Z, A) \to \mathcal{PHS}(X, A)$ by extending (if $X$ is not connected) the push-forward torsor defined in the last section in a trivial way to those connected components of $X$ which are not dominated by $Z$. We consider the functor $[Z]^* = p_{Z \to X}^* \circ p_{Z \to Y}^* : \mathcal{PHS}(Y, A) \to \mathcal{PHS}(X, A)$.

Using the operations “+” and “(-1)” we extend this construction to arbitrary finite correspondences.

Definition 2.3. Let $X$ be regular, $Y$ arbitrary and $\alpha = \sum n_i Z_i \in \text{Cor}(X, Y)$ a finite correspondence. Then
\[
\alpha^* : \mathcal{PHS}(Y, A) \to \mathcal{PHS}(X, A)
\]
is defined by setting
\[
\alpha^*(\mathcal{T}) := \sum n_i [Z_i]^*(\mathcal{T}).
\]

Using the isomorphism (2) above, we immediately obtain

Lemma 2.4. For $\alpha_1, \alpha_2 \in \text{Cor}(X, Y)$ and $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{PHS}(Y, A)$, $n_1, n_2 \in \mathbb{Z}$, we have a natural isomorphism
\[
(\alpha_1 + \alpha_2)^*(n_1 \mathcal{T}_1 + n_2 \mathcal{T}_2) \cong n_1 \alpha_1^*(\mathcal{T}_1) + n_1 \alpha_2^*(\mathcal{T}_1) + n_2 \alpha_1^*(\mathcal{T}_2) + n_2 \alpha_2^*(\mathcal{T}_2).
\]

If $X$ and $Y$ are regular and $Z$ is arbitrary, we have a natural composition law
\[
\text{Cor}(X, Y) \times \text{Cor}(Y, Z) \to \text{Cor}(X, Z), \quad (\alpha, \beta) \mapsto \beta \circ \alpha,
\]
(see [MVW], Lecture 1).

Proposition 2.5. Let $X$ and $Y$ be regular and $Z$ arbitrary. Let $\alpha \in \text{Cor}(X, Y)$ and $\beta \in \text{Cor}(Y, Z)$. Then, for any $\mathcal{T} \in \mathcal{PHS}(Z, A)$, we have a canonical isomorphism
\[
\alpha^*(\beta^*(\mathcal{T})) \cong (\beta \circ \alpha)^*(\mathcal{T}).
\]

Proof. This is a straightforward but lengthy computation unfolding all occurring definitions. We leave it to the reader. \qed
Finally, assume that \( mA = 0 \) for some natural number \( m \). Then (using the isomorphism (1) above), we have for any \( \alpha, \beta \in \text{Cor}(X, Y) \), \( T \in \mathcal{PHS}(Y, A) \), a natural isomorphism
\[
(\alpha + m\beta)^* (T) \cong \alpha^* (T).
\]
Therefore, we have an \( A \)-torsor
\[
\tilde{\alpha}^* (T) \in \mathcal{PHS}(X, A)
\]
given up to unique isomorphism for any \( \tilde{\alpha} \in \text{Cor}(X, Y) \otimes \mathbb{Z}/m\mathbb{Z} \). In other words, we obtain the

**Lemma 2.6.** Assume that \( mA = 0 \), and let \( \alpha, \beta \in \text{Cor}(X, Y) \) have the same image in \( \text{Cor}(X, Y) \otimes \mathbb{Z}/m\mathbb{Z} \). Then there is a natural isomorphism of functors
\[
\alpha^* \cong \beta^* : \mathcal{PHS}(Y, A) \rightarrow \mathcal{PHS}(X, A).
\]

We are mainly interested in the case \( A = \mathbb{Z}/m\mathbb{Z} \). Let \( p \geq 0 \) be the characteristic of the base field \( k \).

**Definition 2.7.** We call an étale \( \mathbb{Z}/m\mathbb{Z} \)-torsor \( T \) on \( X \) tame if the irreducible components of \( T \) are curve-tame étale coverings of \( X \) in the sense of [KS], §4.

Recall that an étale morphism is curve-tame if its base change to any regular curve is an étale covering which extends to an at most tamely ramified covering of the regular compactification of the curve. If \( p \nmid m \), then every \( \mathbb{Z}/m\mathbb{Z} \) torsor is tame.

In general, we have the subgroup
\[
H^1_t(X, \mathbb{Z}/m\mathbb{Z}) \subset H^1_{et}(X, \mathbb{Z}/m\mathbb{Z})
\]
of isomorphism classes of tame \( \mathbb{Z}/m\mathbb{Z} \)-torsors. Both groups coincide if \( X \) is proper over \( k \).

The next lemma says that the pull-back of a tame torsor under a finite correspondence is again tame.

**Lemma 2.8.** Let \( Z \) be integral, \( X \) normal, connected, \( \pi : Z \rightarrow X \) finite, surjective and \( f : Z \rightarrow Y \) any morphism. Let \( T \) be a tame torsor on \( Y \). Then \( \pi_*(f^*(T)) \) is a tame torsor on \( X \).

**Proof.** By definition, \( f^* \) preserves curve-tameness. So we may assume \( Z = Y \), \( f = \text{id} \). Again by the definition of curve-tameness and using Proposition 2.5, we may reduce to the case that \( X \) is a regular curve. Furthermore, we may assume that \( \text{char}(k) = p > 0 \) and \( m = p^r \), \( r \geq 1 \).

Let \( \bar{Z} \) be the canonical compactification of \( Z \), i.e. the unique proper curve over \( k \) which contains \( Z \) as a dense open subscheme and such that all points of \( \bar{Z} \setminus Z \) are regular points of \( \bar{Z} \). By the definition of tame coverings of curves, \( T \) extends to a \( \mathbb{Z}/p^r \mathbb{Z} \)-torsor on \( \bar{Z} \). Hence also \( \pi_*(T) \) extends to the canonical compactification \( \bar{X} \) of \( X \) and so is tame. \( \square \)
If \( p = \mathrm{char}(k) > 0 \), any tame \( \mathbb{Z}/p^n\mathbb{Z} \)-torsor on a regular scheme \( X \) uniquely extends to any regular compactification:

**Proposition 2.9.** Let \( \bar{X} \) be a proper and regular scheme over \( k \) and let \( X \subset \bar{X} \) be a dense open subscheme. Let \( p = \mathrm{char}(k) > 0 \). Then for any \( r \geq 1 \) the natural inclusion
\[
H^1_{et}(\bar{X}, \mathbb{Z}/p^r\mathbb{Z}) \hookrightarrow H^1_{et}(X, \mathbb{Z}/p^r\mathbb{Z})
\]
induces an isomorphism
\[
H^1_{et}(X, \mathbb{Z}/p^r\mathbb{Z}) \cong H^1_{et}(X, \mathbb{Z}/p^r\mathbb{Z}) \subset H^1_{et}(X, \mathbb{Z}/p^r\mathbb{Z}).
\]

**Proof.** Let \( \mathcal{T}_0 \) be any connected component of a tame \( \mathbb{Z}/p^r\mathbb{Z} \)-torsor \( \mathcal{T} \) on \( X \). Then \( \mathcal{T}_0 \) is the normalization of \( X \) in the abelian field extension \( k(\mathcal{T}_0)/k(X) \), which has a \( p \)-power degree. Since \( \mathcal{T}_0 \to X \) is curve tame, it is numerically tamely ramified along \( X \setminus X \) by [KS], Thm. 5.4. (b). That means that the inertia groups in \( k(\mathcal{T}_0)/k(X) \) of all points \( x \in X \subset X \) are of order prime to \( p \), hence trivial. Therefore \( \mathcal{T}_0 \), and thus \( \mathcal{T} \) extends to \( X \).

**Corollary 2.10.** Let \( \Delta^n = \text{Spec}(k[T_0, \ldots, T_n]/\sum T_i = 1) \) be the \( n \)-dimensional standard simplex over \( k \). If \( k \) is separably closed, then
\[
H^1_{et}(\Delta^n, \mathbb{Z}/m\mathbb{Z}) = 0
\]
for all \( m \geq 1 \).

**Proof.** Note that \( \Delta^n \cong \mathbb{A}^n \). If \( p \nmid m \), we obtain:
\[
H^1_{et}(\Delta^n, \mathbb{Z}/m\mathbb{Z}) = H^1_{et}(\mathbb{A}^n, \mathbb{Z}/m\mathbb{Z}) = 0.
\]
If \( p = \mathrm{char}(k) > 0 \) and \( m = p^r \), \( r \geq 1 \), Proposition 2.9 yields
\[
H^1_{et}(\Delta^n, \mathbb{Z}/p^r\mathbb{Z}) = H^1_{et}(\mathbb{A}^n, \mathbb{Z}/p^r\mathbb{Z}) \cong H^1_{et}(\mathbb{P}^n, \mathbb{Z}/p^r\mathbb{Z}) = H^1_{et}(\mathbb{P}^n, \mathbb{Z}/p^r\mathbb{Z}) = 0.
\]

In the following, let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \) and let \( X \) be a separated scheme of finite type over \( k \). Let \( H^i_{S}(X, \mathbb{Z}/m\mathbb{Z}) \) denote the mod-\( m \) Suslin homology, i.e. the \( i \)-th homology group of the complex
\[
\text{Cor}(\Delta^*, X) \otimes \mathbb{Z}/m\mathbb{Z}.
\]

We are going to construct a pairing
\[
H^i_{S}(X, \mathbb{Z}/m\mathbb{Z}) \times H^1_{et}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}
\]
as follows: let \( \mathcal{T} 
\to X \) be a tame \( \mathbb{Z}/m\mathbb{Z} \)-torsor representing a class in \( H^1_{et}(X, \mathbb{Z}/m\mathbb{Z}) \) and let \( \alpha \in \text{Cor}(\Delta^*, X) \) be a finite correspondence representing a 1-cocycle in the mod-\( m \) Suslin complex. Then
\[
\alpha^*(\mathcal{T})
\]
is a torsor over \( \Delta^1 \). Since \( \alpha \) is a cocycle modulo \( m \), \( (0^* - 1^*)(\alpha) \) is of the form \( m \cdot z \) for some \( z \in \text{Cor}(\Delta^0, X) = \mathbb{Z}/(X(k)) \). We therefore obtain a canonical identification
\[
\Phi_{tous} : 0^*(\alpha^*(\mathcal{T})) \sim 1^*(\alpha^*(\mathcal{T}))
\]
of \( \mathbb{Z}/m\mathbb{Z} \)-torsors over \( \Delta^0 = \text{Spec}(k) \). Furthermore, by Corollary 2.10, the tame torsor \( \alpha^*(T) \) on \( \Delta^1 \) is trivial, hence a disjoint union of \( m \) copies of \( \Delta^1 \). By parallel transport, we obtain another identification

\[
\Phi_{par} : 0^*(\alpha^*(T)) \sim 1^*(\alpha^*(T)).
\]

By \( \mathbb{Z}/m\mathbb{Z} \)-equivariance, there is a unique \( \gamma(\alpha, T) \in \mathbb{Z}/m\mathbb{Z} \) such that

\[
\Phi_{par} = (\text{translation by } \gamma(\alpha, T)) \circ \Phi_{taut}.
\]

**Proposition 2.11.** The element \( \gamma(\alpha, T) \in \mathbb{Z}/m\mathbb{Z} \) only depends on the class of \( T \) in \( H^1_t(X, \mathbb{Z}/m\mathbb{Z}) \) and on the class of \( \alpha \) in \( H^1_{S1}(X, \mathbb{Z}/m\mathbb{Z}) \). We therefore obtain a bilinear pairing

\[
\langle \cdot, \cdot \rangle : H^1_{S1}(X, \mathbb{Z}/m\mathbb{Z}) \times H^1_t(X, \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}.
\]

**Proof.** Replacing \( T \) by another torsor isomorphic to \( T \) does not change anything. The nontrivial statement is that \( \langle \alpha, T \rangle \) only depends on the class of \( \alpha \) in \( H^1_{S1}(X, \mathbb{Z}/m\mathbb{Z}) \). For \( \beta \in \text{Cor}(\Delta^1, X) \), we have

\[
\langle \alpha, T + m\beta \rangle = \langle \alpha, T \rangle + m\langle \beta, T \rangle = \langle \alpha, T \rangle.
\]

It therefore remains to show that

\[
\langle \partial^*(\Phi), T \rangle = 0,
\]

for all \( \Phi \in \text{Cor}(\Delta^2, X) \), where \( \partial_i : \Delta^1 \to \Delta^2, i = 0, 1, 2 \), are the face maps and \( \partial^*(\Phi) = \Phi \circ \partial_1 - \Phi \circ \partial_0 + \Phi \circ \partial_2 \). Using Proposition 2.5, we can replace \( T \) by \( \Phi^*(T) \) and move the problem from \( X \) to \( \Delta^2 \), i.e. it remains to show that for any tame torsor \( T \) on \( \Delta^2 \)

\[
\langle \partial_0 - \partial_1 + \partial_2, T \rangle = 0.
\]

By Corollary 2.10, \( T \) is trivial on \( \Delta^2 \). Choosing a trivialization, i.e. a section \( \Delta^2 \to T \), we obtain compatible trivializations for all pull-backs and the parallel transport coincides with the identity. Hence the result.

**Definition 2.12.** We define

\[
\text{rec}_X : H^1_{S1}(X, \mathbb{Z}/m\mathbb{Z}) \to \pi^1_{1, ab}(X)/m
\]

as the homomorphism induced by the pairing of Proposition 2.11 combined with the isomorphism \( H^1_t(X, \mathbb{Z}/m\mathbb{Z}) \cong \pi^1_{1, ab}(X)/m \).

The statement of the next lemma immediately follows from the definition of \( \text{rec} \).

**Lemma 2.13.** Let \( f : X' \to X \) be a morphism of separated schemes of finite type over \( k \). Then the induced diagram

\[
\begin{array}{ccc}
H^1_{S1}(X', \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\text{rec}_{X'}} & \pi^1_{1, ab}(X')/m \\
\downarrow f_* & & \downarrow f_* \\
H^1_{S1}(X, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\text{rec}_X} & \pi^1_{1, ab}(X)/m
\end{array}
\]

commutes.
3 Rigid Čech complexes

We consider étale sheaves $F$ on the category $\text{Sch}/k$ of separated schemes of finite type over a field $k$. By a result of M. Artin, Čech cohomology $\check{H}^*(X,F)$ and sheaf cohomology $H^*(X,F)$ coincide in degree $\leq 1$ and in arbitrary degree if $X$ is quasiprojective (cf. [Mi], III Thm. 2.17). Comparing the Čech complex for a covering $U$ and that for a finer covering $V$, the refinement homomorphism

$$\check{C}^*(U,F) \rightarrow \check{C}^*(V,F)$$

is canonical only up to chain homotopy and hence only the induced map $\check{H}^*(U,F) \rightarrow \check{H}^*(V,F)$ is well-defined. We can remedy this problem in the spirit of Friedlander [Fr], chap.4, by using rigid coverings:

We fix an algebraic closure $\overline{k}/k$. A rigid étale covering $U$ of $X$ is a family of pointed separated étale morphisms $(U_x,x_x) \rightarrow (X,x)$, $x \in X(\overline{k})$, with $U_x$ connected and $u_x \in U_x(\overline{k})$ maps to $x$. For an étale sheaf $F$ the rigid Čech complex is defined by

$$\check{C}^*(U,F) : \check{C}^n(U,F) = \prod_{(x_0,...,x_n) \in X(k)^{n+1}} \Gamma(U_{x_0} \times_X \cdots \times_X U_{x_n}, F)$$

with the usual differentials. One defines in the obvious manner what it means for a rigid covering $V$ to be a refinement of $U$. Because the marked points map to each other, there is exactly one refinement morphism. So we obtain a canonical refinement morphism on the level of complexes

$$\check{C}^*(U,F) \rightarrow \check{C}^*(V,F).$$

The set of rigid coverings is cofiltered (form the fibre product for each $x \in X(\overline{k})$ and restrict to the connected components of the marked points). Therefore we can define the rigid Čech complex of $X$ with values in $F$ as the filtered direct limit

$$\check{C}^*(X,F) := \lim_{\rightarrow} \check{C}^*(U,F),$$

where $U$ runs through all rigid coverings of $X$. Forgetting the marking, we can view a rigid covering as a usual covering. Every covering can be refined by a covering which arises by forgetting the marking of a rigid covering. Hence the cohomology of the rigid Čech complex coincides with the usual Čech cohomology of $X$ with values in $F$.

For a morphism $f : Y \rightarrow X$ and a rigid Čech covering $U/X$, we obtain by taking base extension to $Y$ and restricting to the connected components of the marked points a rigid Čech covering $f^*U/Y$, and in the limit a homomorphism

$$f^* : \check{C}^*(X,F) \rightarrow \check{C}^*(Y,F).$$

Lemma 3.1. If $\pi : Y \rightarrow X$ is quasi-finite, then the rigid coverings of the form $\pi^*U$ are cofinal among the rigid coverings of $Y$.

Proof. This is an immediate consequence of the fact that a quasi-finite and separated scheme $Y$ over the spectrum $X$ of a henselian ring is of the form $Y = Y_0 \sqcup Y_1 \sqcup \ldots Y_r$ with $Y_0 \rightarrow X$ not surjective and $Y_i \rightarrow X$ finite surjective with $Y_i$ the spectrum of a henselian ring, $i = 1, \ldots, r$, cf. [Mi], I, Thm. 4.2. □
Lemma 3.2. If $F$ is qfh-sheaf on $\text{Sch}/k$, then for any $n \geq 0$ the presheaf $\mathcal{C}^n(-,F)$ given by

$$X \mapsto \mathcal{C}^n(X,F)$$

is a qfh-sheaf. The obvious sequence

$$0 \to F \to \mathcal{C}^0(-,F) \to \mathcal{C}^1(-,F) \to \mathcal{C}^2(-,F) \to \cdots$$

is exact as a sequence of étale (and hence also of qfh) sheaves.

Proof. We show that each $\mathcal{C}^n(-,F)$ is a qfh-sheaf. For this, let $\pi : Y \to X$ be a qfh-covering, i.e. a quasi-finite universal topological epimorphism. We denote the projection by $\pi : Y \times_X Y \to X$. By Lemma 3.1, we have to show that the sequence

$$\lim_{\mathcal{U}} \mathcal{C}^n(\mathcal{U},F) \to \lim_{\mathcal{U}} \mathcal{C}^n(\pi^*\mathcal{U},F) \Rightarrow \lim_{\mathcal{U}} \mathcal{C}^n(\pi_*\mathcal{U},F)$$

is exact, where $\mathcal{U}$ runs through the rigid coverings of $X$. Since filtered colimits commute with finite limits, it suffices to show the exactness for a single sufficiently small $\mathcal{U}$. This, however, follows from the assumption that $F$ is a qfh-sheaf.

Finally, the exactness of $0 \to F \to \mathcal{C}^0(-,F) \to \mathcal{C}^1(-,F) \to \cdots$ as a sequence of étale sheaves easily follows by considering stalks.

Being qfh-sheaves, the sheaves $F$ and $\mathcal{C}^n(-,F)$ admit transfer maps, see [SV1], §5. For later use, we make the relation between the transfers of $F$ and of $\mathcal{C}^n(-,F)$ explicit: Let $Z$ be integral, $X$ regular and $\pi : Z \to X$ finite and surjective. Let $F$ be a qfh-sheaf on $\text{Sch}/k$. For $x \in X(k)$ we have

$$X^{sh}_x \times_X Z = \coprod_{z \in \pi^{-1}(x)} Z^h_z,$$

where $\pi^{-1}(x)$ denotes the set of morphisms $z : \text{Spec}(k) \to Z$ with $\pi \circ z = x$. For sufficiently small chosen étale $(U_x, u_x) \to (X,x)$ the set of connected components of $U_x \times_X Z$ is in 1-1-correspondence to the set $\pi^{-1}(x)$ and to each family of étale morphisms

$$(V_z, v_z) \to (Z,z), \quad z \in \pi^{-1}(x),$$

there is (after possibly making $U_x$ smaller) a uniquely defined morphism

$$(U_x \times_X Z \to \coprod_{z \in \pi^{-1}(x)} V_z,$$

over $Z$, which sends the connected component associated with $z$ of $U_x \times_X Z$ to $V_z$, and the point $(u_z, z)$ to $u_z$.

In this way we obtain for finitely many points $(x_0, \ldots, x_n)$, $n \geq 0$, to every family

$$(V_{z_i, u_{z_i}}) \to (Z, z_i), \quad z_i \in \pi^{-1}(x_i),$$

and sufficiently small chosen

$$(U_{x_i, u_{x_i}}) \to (X, x_i), \quad i = 0, \ldots, n,$$

a homomorphism

$$\prod_{i=0}^n \Gamma(V_{z_i}^{x_i} \times_Z \cdots \times_Z V_{z_n}^{x_n}, F) \to \Gamma(U_{x_0} \times_X \cdots \times_X U_{x_n} \times_X Z, F).$$
Since $F$ is a qfh-sheaf, we can compose this with the transfer map associated with the finite morphism

$$U_{x_0} \times_X \cdots \times_X U_{x_n} \times X \to U_{x_0} \times_X \cdots \times_X U_{x_n}.$$  

Forming for fixed $n$ the product over all $(x_0, \ldots, x_n) \in X(k)^{n+1}$ and passing to the limit over all rigid coverings, we obtain the transfer homomorphism

$$\pi_* : \check{C}^\bullet (Z, F) \to \check{C}^\bullet (X, F).$$

Passing to cohomology (in degree 0 or 1, or in any degree if the schemes are quasiprojective), we obtain the usual transfer on étale cohomology.

Next we give the pairing constructed in Proposition 2.11.

constructed in Proposition 2.11 for $k$ algebraically closed the following interpretation in terms of the rigid Čech complex:

Let $a \in H^2_\text{ét}(X, \mathbb{Z}/m\mathbb{Z})$ and $b \in H^1_\text{ét}(X, \mathbb{Z}/m\mathbb{Z})$ be given and let $\alpha \in \text{Cor}_k(\Delta^1, X)$ and $\beta \in \ker(C^1(X, \mathbb{Z}/m\mathbb{Z}) \to C^2(X, \mathbb{Z}/m\mathbb{Z}))$ be representing elements. Note that $(0^* - 1^*)(\alpha) \in m\text{Cor}(\Delta^0, X)$ by assumption. Consider the diagram

$$
\begin{array}{ccc}
\check{C}^0(X, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{d} & \check{C}^1(X, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{d} & \check{C}^2(X, \mathbb{Z}/m\mathbb{Z}) \\
\downarrow{\alpha^*} & & \downarrow{\alpha^*} & & \downarrow{\alpha^*} \\
\check{C}^0(\Delta^1, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{d} & \check{C}^1(\Delta^1, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{d} & \check{C}^2(\Delta^1, \mathbb{Z}/m\mathbb{Z}) \\
\downarrow{0^*-1^*} & & \downarrow{0^*-1^*} & & \downarrow{0^*-1^*} \\
\mathbb{Z}/m\mathbb{Z} & \xrightarrow{\cdot} & \check{C}^0(\Delta^0, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{d} & \check{C}^1(\Delta^0, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{d} & \check{C}^2(\Delta^0, \mathbb{Z}/m\mathbb{Z})
\end{array}
$$

Since $\beta$ represents a tame torsor $T$ on $X$, $\alpha^*(\beta)$ represents the torsor $\alpha^*(T)$, which is tame by Lemma 2.8. By Corollary 2.10, there exists $\gamma \in \check{C}^0(\Delta^1, \mathbb{Z}/m\mathbb{Z})$ with $d\gamma = \alpha^*(\beta)$. Since $d(0^* - 1^*)(\gamma) = (0^* - 1^*)\alpha^*(\beta) = 0$,

we conclude that $(0^* - 1^*)(\gamma)$ lies in

$$\mathbb{Z}/m\mathbb{Z} = H^0(\Delta^0, \mathbb{Z}/m\mathbb{Z}) = \ker(\check{C}^0(\Delta^0, \mathbb{Z}/m\mathbb{Z}) \to \check{C}^1(\Delta^0, \mathbb{Z}/m\mathbb{Z}))$$

Its is a standard verification to show that the assignment

$$\langle \cdot, \cdot \rangle : (a, b) \mapsto (0^* - 1^*)(\gamma) \in \mathbb{Z}/m\mathbb{Z}$$

does not depend on the choices made. By the explicit geometric relation between Čech 1-cocycles and torsors, and since our construction of finite push-forwards of torsors is compatible with the construction of transfers for qfh-sheaves given in [SV1], §5, we see that the pairing constructed above coincides with the one constructed in Proposition 2.11.

Finally, let

$$\mathbb{Z}/m\mathbb{Z} \hookrightarrow I^0 \to I^1 \to I^2$$

(*)
be a (partial) injective resolution of the constant sheaf \( \mathbb{Z}/m\mathbb{Z} \) in the category of \( \mathbb{Z}/m\mathbb{Z} \)-module sheaves on \((\text{Sch}/k)_{qfh}\). Let \( \phi : (\text{Sch}/k)_{qfh} \to (\text{Sch}/k)_{et} \) denote the natural map of sites. Since \( \phi^* \) is exact, \( \phi_* \) sends injective sheaves to injective sheaves. By [SV1], Thm. 10.2, we have \( R^i\phi_*(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \) and \( R^i\phi_*(\mathbb{Z}/m\mathbb{Z}) = 0 \) for \( i \geq 1 \). Hence \((*)\) considered as a sequence of \( \text{étale} \) \( \mathbb{Z}/m\mathbb{Z} \)-module sheaves is also a partial resolution of \( \mathbb{Z}/m\mathbb{Z} \) by injectives. We choose a quasi-isomorphism

\[
[0 \to C^0(-, \mathbb{Z}/m\mathbb{Z}) \to C^1(-, \mathbb{Z}/m\mathbb{Z}) \to C^2(-, \mathbb{Z}/m\mathbb{Z})] \longrightarrow [0 \to I^0 \to I^1 \to I^2]
\]

of complexes of \( qfh \)-sheaves. Since Čech- and étale cohomology agree in dimension \( \leq 1 \), the induced map on global sections is a quasi-isomorphism, too. Hence the pairing of Proposition 2.11 can also be obtained by the same procedure as above but using the diagram

\[
\begin{array}{ccc}
I^0(X) & \xrightarrow{d} & I^1(X) \\
\downarrow{\alpha^*} & & \downarrow{\alpha^*} \\
I^0(\Delta^1) & \xrightarrow{d} & I^1(\Delta^1) \\
\downarrow{0^*-1^*} & & \downarrow{0^*-1^*} \\
\mathbb{Z}/m\mathbb{Z} & \longrightarrow & I^0(\Delta^0) \\
\end{array}
\]

The same argument applies with a partial injective resolution of the constant sheaf \( \mathbb{Z}/m\mathbb{Z} \) in the category of \( \mathbb{Z}/m\mathbb{Z} \)-module sheaves on \((\text{Sch}/k)_h\).

4 Connected components

Let \( K \) be a field, \( L_1|K, \ldots, L_n|K \) finite separable extensions and \( K^{sep} \) a fixed separable closure of \( K \). Put \( G_K = \text{Gal}(K^{sep}|K) \). We consider the homomorphism

\[
\Phi : L_1 \otimes_K \cdots \otimes_K L_n \longrightarrow \prod_{(\sigma_1, \ldots, \sigma_n) \in \prod_{i=1}^n \text{Hom}_K(L_i, K^{sep})} \sigma_1(L_1) \cdots \sigma_n(L_n),
\]

\[
a_1 \otimes \cdots \otimes a_n \longmapsto (\sigma_1(a_1) \cdots \sigma_n(a_n))_{(\sigma_1, \ldots, \sigma_n)}.
\]

The composite fields on the right are formed in \( K^{sep} \). There is a natural left \( G_K \)-action on the right hand side by setting

\[
(g \cdot x)_{(\sigma_1, \ldots, \sigma_n)} = g \cdot x_{(g^{-1} \sigma_1, \ldots, g^{-1} \sigma_n)},
\]

and the image of \( \Phi \) is obviously contained in the subgroup of \( G_K \)-invariants.

**Lemma 4.1.** \( \Phi \) induces an isomorphism

\[
\Phi : L_1 \otimes_K \cdots \otimes_K L_n \longrightarrow \left( \prod_{(\sigma_1, \ldots, \sigma_n) \in \prod_{i=1}^n \text{Hom}_K(L_i, K^{sep})} \sigma_1(L_1) \cdots \sigma_n(L_n) \right)^{G_K}.
\]

**Proof.** Since all \( L_i|K \) are separable, \( L_1 \otimes_K \cdots \otimes_K L_n \) is a product of finite separable extension fields of \( K \). Hence the homomorphism

\[
L_1 \otimes_K \cdots \otimes_K L_n \longrightarrow \prod_{\sigma \in \text{Hom}_K(L_1 \otimes_K \cdots \otimes_K L_n, K^{sep})} K^{sep}
\]

is an isomorphism.
is injective. This shows the injectivity of $\Phi$. In order to show that $\Phi$ is an isomorphism, it suffices to show that both sides have the same dimension as $K$-vector spaces. After fixing one $K$-embedding $L_i \hookrightarrow K^{\text{sep}}$ for each $i = 1, \ldots, n$, this equality of dimensions follows from the orbit decomposition formula for the $G_K$-left action on $(G_K/G_{L_1} \times \cdots \times G_K/G_{L_n})$.

**Corollary 4.2.** Let $K$ be a field, $L_1|K, \ldots, L_n|K$ finite extensions and $\bar{K}$ an algebraic closure of $K$. Then there is a natural isomorphism

$$\pi_0\left(\text{Spec}(L_1 \otimes_K \cdots \otimes_K L_n)\right) \cong \left(\prod_{i=1}^{n} \text{Hom}_K(L_i, \bar{K})\right)/\text{Aut}_K(\bar{K}).$$

**Proof.** Purely inseparable extensions induce universal homeomorphisms on $\text{Spec}$. Hence both sides do not change when we replace $\bar{K}$ by its maximal separable subextension for $i = 1, \ldots, n$. The result follows from Lemma 4.1.

**Proposition 4.3.** Let $X$ be a connected normal scheme and let

$$Y_i \to X, \quad i = 1, \ldots, n,$$

be quasi-finite dominant morphisms of finite type. We assume that all $Y_i$ are normal, connected and that all but possibly one of the morphisms $Y_i \to X$ are étale. Let $\bar{K}$ be an algebraic closure of the function field $K = k(X)$ of $X$. Then there is a natural isomorphism

$$\pi_0(Y_1 \times_X \cdots \times_X Y_n) \cong \left(Y_1(\bar{K}) \times \cdots \times Y_n(\bar{K})\right)/\text{Aut}_K(\bar{K}).$$

**Proof.** By the given assumptions, the scheme $Y_1 \times_X \cdots \times_X Y_n$ is étale over a normal scheme, hence normal by [Mi], I.3.17. Therefore we can calculate the set of connected components after passage to generic points. The result follows from Corollary 4.2.

For a subfield $L \subset \bar{K}$ we write $G_L = G(\bar{K}|L) = \text{Aut}_L(\bar{K})$. If a $K$-point of $Y_i$, i.e. a $K$-embedding $k(Y_i) \hookrightarrow \bar{K}$ is given for all $i$, we can reformulate the statement of Proposition 4.3 in terms of (pseudo) Galois groups as an isomorphism

$$\pi_0(Y_1 \times \cdots \times_X Y_n) \cong G_K \backslash (G_K/G_{k(Y_1)} \times \cdots \times G_K/G_{k(Y_n)}).$$

From now on we assume that $\pi : Z \to X$ is a finite and surjective morphism between connected curves over an algebraically closed field $k$. For $x \in X(k)$ we have

$$X_x^{sh} \times_X Z = \prod_{z \in \pi^{-1}(x)} Z_x^{sh}.$$

For a sufficiently small chosen étale neighbourhood $(U_x, u_x) \to (X, x)$ of $x$ we therefore have a bijection

$$\pi_0(U_x \times_X Z) \cong \pi^{-1}(x).$$

Our aim is to understand the map

$$z_x : U_x \times_X Z \longrightarrow \pi^{-1}(x), \quad (u, z) \longmapsto z_x(u, z), \quad (*)$$
where \( z_x(u,z) \in \pi^{-1}(x) \) is the unique point in \( \pi^{-1}(x) \) such that \((u_x, z_x)\) lies in the same connected component of \( U_x \times_X Z \) as \((u, z)\).

We identify closed points on a smooth projective curve \( C \) over \( k \) with their local rings (which are discrete valuation rings) in the function field \( L = k(C) \) and we loosely write \(+ z \in L^{'}/(z \text{ is a prime of } L')\) if \( z \) is a closed point of \( C \). For a finite field extension \( L'/L \) and a closed point \( x' \in L' \) we write \( x'|_L \in L \) for the image \( x \) of \( x' \) in \( L \). Along with these identifications we adopt the convention that Galois groups act from the left on points.

**Definition 4.4.** We call a point \( z \in L \) pseudo-unramified in a finite extension \( L'|L \) if it is unramified in the maximal separable subextension.

If \( L'|L \) is normal, this means that the inertia group \( T_{x'}(L'|L) \subset G(L'|L) \) of one (hence every) extension \( x' \) of \( z \) to \( L' \) is trivial.

We fix \( K \)-embeddings \( k(U_x) \hookrightarrow \bar{K} \) and \( k(Z) \hookrightarrow \bar{K} \). Then Proposition 4.5 yields a bijection

\[
G_K \setminus \left( G_K / G_{k(U_x)} \times G_K / G_{k(Z)} \right) \cong \pi_0(U_x \times_X Z).
\]

By Lemma 4.1, the function field of the connected component corresponding to the class of \((g_1,g_2)\) is naturally isomorphic to the composite field of \( g_1(k(U_x)) \) and \( g_2(k(Z)) \) in \( \bar{K} \). Pulling back a point \( \bar{y} \in \bar{K} \) along the embeddings

\[
g_1 : k(U_x) \subset \bar{K} \overset{g_1}{\rightarrow} \bar{K}, \quad g_2 : k(Z) \subset \bar{K} \overset{g_2}{\rightarrow} \bar{K}
\]

yields the point \((g_1^{-1}\bar{y}|_{k(U_x)},g_2^{-1}\bar{y}|_{k(Z)})\) which lies on the compactification of the connected component of \( U_x \times_X Z \) associated with \((g_1,g_2)\).

Let \( \bar{x} \in \bar{K} \) be any point lying over \( u_x \in k(U_x) \), in other words, \( \bar{x} \) is an extension of \( x \) to \( \bar{K} \) with \( \bar{x}|_{k(U_x)} = u_x \). Pulling back \( \bar{y} := g_1\bar{x} \) as above, yields the point

\[
(u_x,g_2^{-1}g_1\bar{x}|_{k(Z)}) \in U_x \times_X Z
\]

which lies on the connected component parameterized by \((g_1,g_2)\).

Starting from any point \((u,z) \in U_x \times_X Z \), we want to identify the element of \( G_K \setminus \left( G_K / G_{k(U_x)} \times G_K / G_{k(Z)} \right) \) which parameterizes its connected component. We denote the common image of \( u \) and \( z \) in \( X \) by \( y \) and choose an extension \( \bar{y} \in \bar{K} \) of \( y \). If \((u,z) \) lies on the \((g_1,g_2)\)-component, there exists an \( h \in G_K \) with

\[
u = g_1^{-1}h\bar{y}|_{k(U_x)} \quad \text{and} \quad z = g_2^{-1}h\bar{y}|_{k(Z)}.
\]

Now assume that \( y \) is pseudo-unramified in \( k(U_x)k(Z)/K \) (this excludes only finitely many \( y \)). Then, \( T_{yZ} = hT_yh^{-1} \subset G_{k(U_x)} \cap G_{k(Z)} \) for all \( h \in G_K \). Hence \( u = g_1^{-1}h\bar{y}|_{k(U_x)} \), which a priori detects the double coset of \( g_1^{-1} \) in \( G_{k(U_x)} \backslash G_K / T_y \), detects the class of \( g_1 \) in \( G_K / G_{k(U_x)} \). By the same argument, \( z \) detects the class of \( g_2 \) in \( G_K / G_{k(Z)} \). We conclude

**Proposition 4.5.** Consider the map

\[
z_x : U_x \times_X Z \longrightarrow \pi^{-1}(x), \quad (u,z) \mapsto z_x(u,z)
\]

defined above in (\(*\)). Let \((u,z) \in U_x \times_X Z \) and let \( y \) be the common image of \( u \) and \( z \) in \( X \). Assume that \( y \) is unramified in the field extension \( k(U_x)K \) and pseudo-unramified in \( k(Z)K \). Choose a point \( \bar{x} \in \bar{K} \) over \( x \) restricting to \( u_x \) in \( k(U_x) \) and a point \( \bar{y} \in \bar{K} \) over \( y \). Let \( u = g_1^{-1}\bar{y}|_{k(U_x)} \) and \( z = g_2^{-1}\bar{y}|_{k(Z)} \) for \( g_1, g_2 \in G_K \). Then

\[
z_x(u,z) = g_2^{-1}g_1\bar{x}|_{k(Z)} \in \pi^{-1}(x).
\]
5 The case of smooth curves

In this section we prove Theorem 1.1 in the case that $X = C$ is a smooth curve.

Let $k$ be an algebraically closed field of characteristic $p \geq 0$, $C$ a smooth not necessary projective curve over $k$ and $A$ its generalized Jacobian with respect to the modulus given by the sum of the points on the boundary of the regular compactification $\bar{C}$ of $C$ (cf. [Se], Ch.5). $A$ is a semi-abelian variety. The group $A(k)$ is the subgroup of degree zero elements of the relative Picard group $\text{Pic}(\bar{C}, C \setminus C)$. By [SV1], Thm. 3.1 (see [Li], for the case $C = \bar{C}$), there is an isomorphism

$$H^S_0(C, \mathbb{Z})^0 := \ker(H^S_0(C, \mathbb{Z}) \xrightarrow{\text{deg}} \mathbb{Z}) \cong A(k),$$

in particular, $A(k)$ is a quotient of the group of zero cycles of degree zero on $C$. From the coefficient sequence together with the divisibility of $H^S_1(C, \mathbb{Z})$ (which is isomorphic to $k^\times$ if $C$ is proper and zero otherwise), we obtain an isomorphism

$$H^S_1(C, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\delta} mH^S_0(C, \mathbb{Z}) \cong mA(k). \quad (1)$$

After fixing a closed point $P_0$ of $C$, the morphism $C \to A, P \mapsto P - P_0$, is universal for morphisms of $C$ to semi-abelian varieties, i.e. $A$ is the generalized Albanese variety of $C$ ([Se], V, Th. 2).

Consider the $m$-multiplication map $A \xrightarrow{m} A$. Its maximal étale subcovering $\tilde{A} \to A$ is the quotient of $A$ by the connected component of the finite group scheme $mA$ (if $(p, m) = 1$, the connected component is trivial). By [Se], Ch. IV, $\tilde{C} := C \times_A \tilde{A}$ is the maximal abelian tame étale covering of $C$ with Galois group annihilated by $m$. We therefore obtain an isomorphism

$$\text{Hom}(mA(k), \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\tau} H^1_t(C, \mathbb{Z}/m\mathbb{Z}). \quad (2)$$

**Theorem 5.1.** The diagram

$$\begin{array}{ccc}
H^S_1(C, \mathbb{Z}/m\mathbb{Z}) & \times & H^1_t(C, \mathbb{Z}/m\mathbb{Z}) \\
\delta \downarrow & & \tau \downarrow \\
mA(k) & \times & \text{Hom}(mA(k), \mathbb{Z}/m\mathbb{Z}) \\
\end{array}$$

where $\langle \cdot, \cdot \rangle$ is the pairing from Proposition 2.11 and $\text{eval}$ is the evaluation map, commutes. In particular, the upper pairing is perfect and the induced homomorphism $H^S_1(C, \mathbb{Z}/m\mathbb{Z}) \to \pi_1^{t, ab}(C)/m$ is an isomorphism.

Before proving Theorem 5.1, we need some preparations. The projection $A \to \tilde{A}$ is a universal homeomorphism, in particular, it induces a bijection on closed points, so we can identify $\tilde{A}(k)$ with $A(k)$ and with respect to this identification the projection $\tilde{A}(k) \to A(k)$ is the $m$-multiplication map. Furthermore, every element in $G(\tilde{A}, A)$ is translation by an element of $mA(k)$ and vise versa. Therefore we obtain an isomorphism

$$\tilde{A} \times_A \tilde{A} \xrightarrow{\sim} \coprod_{mA(k)} \tilde{A},$$

which sends a pair $(a, b)$ to $a$ in the $a - b \in mA(k)$-component.
On the level of closed points, $\tilde{C} = C \times_A \tilde{A}$ can be identified with the set of $a \in \tilde{A}(k) = A(k)$ such that $ma = P - P_0$ for some point $P \in C$ ($a$ projects to $P$ in $C$). We see that the induced morphism

$$\tilde{C} \times_C \tilde{C} \to \tilde{A} \times_A \tilde{A} \cong \coprod_{t \in mA(k)} \tilde{A},$$

induces a surjective morphisms

$$\tilde{C} \times_C \tilde{C} \to \coprod_{mA(k)} \text{Spec}(k), \quad (P, Q) \mapsto P - Q.$$

The induced map on connected components is thus surjective, hence bijective as both sets have the same cardinality. We obtain

**Lemma 5.2.** The morphism

$$\tilde{C} \times_C \tilde{C} \to \tilde{A}, \quad (P, Q) \mapsto P - Q,$$

induces a bijection between $\pi_0(\tilde{C} \times_C \tilde{C})$ and $mA(k)$.

Let $\pi : C \to \Delta^1$ be a finite surjective morphism and let $\hat{C}$ be the normalization of $\tilde{C}$ in an extension field $k(\hat{C})$ of $k(\tilde{C})$ which is normal over $k(\Delta^1)$. We have finite morphisms

$$\tilde{C} \to \hat{C} \to C \to \Delta^1$$

and the composition $\hat{C} \to \Delta^1$ is pseudo-Galois.

**Lemma 5.3.** Let $\phi, \psi \in \text{Mor}_C(\hat{C}, \tilde{C})$ and let $\hat{x}, \hat{y}$ be closed points of $\hat{C}$. Then

$$\phi(\hat{x}) - \phi(\hat{y}) = \psi(\hat{x}) - \psi(\hat{y}) \quad \in A(k).$$

**Proof.** By Galois theory, there exists an $h \in G(\hat{C}|C)$ with $\phi = h \circ \psi$. The $h$-action on $\hat{C}$ is translation (in $\hat{A}$) by an element $a_h \in mA(k)$. Hence

$$\phi(\hat{x}) - \phi(\hat{y}) = (a_h + \psi(\hat{x})) - (a_h + \psi(\hat{y})) = \psi(\hat{x}) - \psi(\hat{y}).$$

Let $\tilde{x}, \tilde{y}$ be closed points of $\tilde{C}$ and $\tilde{x}|_{\tilde{C}}, \tilde{y}|_{\tilde{C}}$ their images in $\tilde{C}$. As in the last section, we adopt the convention that (pseudo) Galois groups act from the left on points, which are identified with their local rings inside the function fields. For $g \in G(\hat{C}|\Delta^1)$ we consider

$$g\tilde{x}|_{\tilde{C}} - g\tilde{y}|_{\tilde{C}} \in \tilde{A}(k) \cong A(k).$$

A priori, this difference depends on the class of $g$ in $G(\hat{C}|\tilde{C}) \setminus G(\hat{C}|\Delta^1)$. However, applying Lemma 5.3 with $\phi : \hat{C} \to \tilde{C}, \hat{C} \mapsto \hat{C}|_{\hat{C}},$ and $\psi = \phi \circ h, h \in G(\hat{C}|C),$ to $g\tilde{x}$ and $g\tilde{y}$, we see that the difference in (1) only depends on the class of $g$ in $G(\hat{C}|C) \setminus G(\hat{C}|\Delta^1)$.
Proposition 5.4. We have
\[ \sum_{g \in G(\tilde{C}) \setminus G(\tilde{C}|\Delta^1)} g\tilde{x}|_C - g\tilde{y}|_C = 0 \in A(k). \]

Proof. Nothing changes if we replace $\Delta^1$ by its normalization in the maximal purely inseparable subextension of $k(C)|k(\Delta^1)$, which is again isomorphic to $\Delta^1$. Therefore we may assume that the morphism $C \to \Delta^1$ is separable. Replacing $\tilde{C}$ by the normal closure of $\tilde{C} \to \Delta^1$, we may assume that $\tilde{C} \to \Delta^1$ is separable.

For a $k$-variety $X$ we denote the group of zero cycles on $X$ by $Z_0(X)$ and its subgroup of degree zero cycles by $Z_0(X)^0$. We write $\pi_{\tilde{C}} : \tilde{C} \to \tilde{C}, \pi_{\tilde{C}} : \tilde{C} \to C, \pi_{\Delta^1} : \tilde{C} \to \Delta^1$ for the respective projections and denote the induced push-forward and pull-back maps on $Z_0$ by $\pi_C^*$ and $\pi_{\Delta^1,*}$, respectively. For $\tilde{x} \in \tilde{C}$ we have
\[ \pi_{\tilde{C}}^* \pi_{\tilde{C},*}(\tilde{x}) = \sum_{\sigma \in G(\tilde{C}|C)} \sigma \tilde{x} \in Z_0(\tilde{C}), \]
hence
\[ \pi_{\tilde{C}}^* \left( \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} \pi_{\tilde{C},*}(g\tilde{x}) = \sum_{g, \sigma} g\tilde{x} = \sum_{s \in G(\tilde{C}|\Delta^1)} s\tilde{x} = \pi_{\Delta^1}^* \pi_{\Delta^1,*}(\tilde{x}). \]
Since $\pi_{\tilde{C}}^*$ is injective, we obtain
\[ \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} \pi_{\tilde{C},*}(g\tilde{x}) = \pi_{\tilde{C},\Delta^1,*}(\tilde{x}). \]
The diagram
\[ \begin{array}{ccc} Z_0(C)^0 & \longrightarrow & H^0_S(C, Z)^0 = A(k) \\
\pi_{\tilde{C},\Delta^1} & & \pi_{\tilde{C},\Delta^1} \\
Z_0(\Delta^1)^0 & \longrightarrow & H^0_S(\Delta^1, Z)^0 = 0 \end{array} \]
commutes. Therefore
\[ \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} g\tilde{x}|_C - g\tilde{y}|_C = \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} \pi_{\tilde{C},*}(g\tilde{x} - g\tilde{y}) = 0 \in A(k), \]
since it lies in the image of
\[ \pi_{\tilde{C},\Delta^1}^* : H^0_S(\Delta^1, Z)^0 \to H^0_S(C, Z)^0 = A(k). \]
We have $m \cdot \tilde{x}|_C = \tilde{x}|_C - P_0$ as elements of $A(k)$. Hence
\[ m \cdot \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} g\tilde{x}|_C - g\tilde{y}|_C = \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} g\tilde{x}|_C - g\tilde{y}|_C = 0 \in A(k). \]
Therefore the sum considered in the Proposition lies in $mA(k)$.

Now we choose representatives $g_1, \ldots, g_r$ of $G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)$ and consider the morphism
\[ \Phi : \tilde{C} \times_k \tilde{C} \longrightarrow \tilde{A}, \quad (\tilde{x}, \tilde{y}) \mapsto \sum_{i=1}^r \pi_{\tilde{C}} \circ g_i(\tilde{x}) - \sum_{i=1}^r \pi_{\tilde{C}} \circ g_i(\tilde{y}). \]
As seen above, the composition of $\Phi$ with the projection $\tilde{A} \to A$ is zero. Therefore $\Phi$ maps to the kernel of $\tilde{A} \to A$, which is a finite étale group scheme. Hence $\Phi$ is locally constant. The diagonal of the connected surface $\tilde{C} \times C$ maps to zero, hence $\Phi$ is zero. This proves the Proposition.

**Proof of Theorem 5.1.** Let $\zeta \in H^1(C, \mathbb{Z}/m\mathbb{Z})$. By [SV1], Thm. 3.1, $\delta(\zeta) \in mH^0(C, C) = mA(k)$ is given by a zero-cycle $z$ on $C$ such that $mz = g^*(0) - g^*(1)$ for some finite morphism $g : C \to \Delta^1$. Interpreting $g$ as a correspondence from $\Delta^1$ to $C$, its class in $H^2_{et}(C, \mathbb{Z}/m\mathbb{Z})$ is a preimage of $\delta(\zeta)$ under the isomorphism $H^2_{et}(C, \mathbb{Z}/m\mathbb{Z}) \cong mH^0(C, C)$, hence $\zeta$ is represented by $g$. We thus have to show that 

$$\phi(\delta(g)) = (g, \tau(\phi))$$

for all finite morphisms $g : C \to \Delta^1$ with $g^*(0), g^*(1) \in mZ_0(C)$ and every $\phi \in \text{Hom}(m, A(k), \mathbb{Z}/m\mathbb{Z})$.

In order to calculate $(g, \tau(\phi))$, we have to find a rigid Čech cocycle representing $\tau(\phi)$. A non-rigid cocycle is given for the covering $\tilde{C} \to C$ as a locally constant function on $\tilde{C} \times C$ by the composition

$$(*) \quad \pi_0(\tilde{C} \times C) \xrightarrow{(5.2)} \pi_0(C \times C) \xrightarrow{\phi} \mathbb{Z}/m\mathbb{Z}.$$ 

Since $\tilde{C}$ is connected, we can obtain a rigid cocycle easily as follows: For every point $c \in C$ we fix once and for all a point $\tilde{c} \in \tilde{C}$ above $c$. We denote $\tilde{C}$ with marked point $\tilde{c}$ by $\tilde{C}_c$. Then we consider the Čech-1-cocycle $a$ on the rigid covering

$$\mathcal{U} = (\tilde{C}_c)_{c \in C(k)} = \{(\tilde{C}, \tilde{c}) \to (C, c)\}_{c \in C(k)},$$

which is given by $(*)$ on $\tilde{C}_c \times C_c'$ in the same way for any pair $(c, c') \in C(k)^2$. This rigid cocycle $a$ represents $\tau(\phi) \in H^1_{et}(C, \mathbb{Z}/m\mathbb{Z})$. The direct image $g_*\langle a \rangle \in \mathcal{C}^1(\Delta^1, \mathbb{Z}/m\mathbb{Z})$ is a coboundary by Lemma 2.8 and Corollary 2.10. Hence there exists $b \in \mathcal{C}^0(\Delta^1, \mathbb{Z}/m\mathbb{Z})$ with $db = g_*\langle a \rangle$. The element $(0^* - 1^*)(b)$ lies in

$$\ker(d : \mathcal{C}^0(\Delta^0, \mathbb{Z}/m\mathbb{Z}) \to \mathcal{C}^1(\Delta^0, \mathbb{Z}/m\mathbb{Z})) = H^0_{et}(\Delta^0, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$$

and equals $\langle \zeta, \tau(\phi) \rangle$ by the construction of this pairing. It therefore remains to show that

$$(0^* - 1^*)(b) = \phi(z) \in \mathbb{Z}/m\mathbb{Z},$$

where $z$ is the unique zero cycle (of degree zero) on $C$ with $mz = g^*(0) - g^*(1)$ and $\phi(z)$ is the value of $\phi$ on the image of $z$ in $mA(k)$. Let $U_0$ and $U_1$ be sufficiently small connected étale neighbourhoods of $0, 1 \in \Delta^1$ such that $b(U_0), b(U_1) \in \mathbb{Z}/m\mathbb{Z}$ are defined. Then $b$ takes the value $0^*(b)$ on $U_0$ and the value $1^*(b)$ on $U_1$. Hence $db = g_*\langle a \rangle$ takes the value $(0^* - 1^*)(b)$ on every connected component of $U_0 \times \Delta^1, U_1$. In order to calculate $(0^* - 1^*)(b)$ it therefore suffices to calculate $a$ on the intersection (= fibre product over $C$) of sufficiently small neighbourhoods of the fibres $g^*0(0)$ and $g^*1(1)$.

Let $(U_0, u_0)$ and $(U_1, u_1)$ sufficiently small connected étale neighbourhoods of $0, 1 \in \Delta^1$. Sufficiently small means that we have bijections

$$\pi_0(U_0 \times \Delta^1 C) \cong g^{-1}(0), \quad \pi_0(U_1 \times \Delta^1 C) \cong g^{-1}(1)$$

and that $U_i \times \Delta^1 C$ refines the partial rigid covering $(\tilde{C}_i)_{\zeta \in g^{-1}(i)}$ for $i = 0, 1$, i.e. the function field of any connected component of $U_i \times \Delta^1 C$ contains the function field of $\tilde{C}$. After removing finitely many points from $U_i, i = 0, 1$, we may assume that any point different from $i \in \text{im}(U_i \to \Delta^1)$ is pseudo-unramified (see Definition 4.4).
For a point \( y \in U_0 \times \Delta^1 \) we denote by \( c_0(y_0, c) \) the uniquely defined element \( c_0 \in g^{-1}(0) \) such that \( (y_0, c) \) lies in the same connected component as \( (u_0, c_0) \). We use the analogous notation \( c_1(y_1, c) \in g^{-1}(1) \) for \( (y_1, c) \in U_1 \times \Delta^1 \). The refinement morphisms

\[
U_0 \times \Delta^1, C \rightarrow \coprod_{c_0} \tilde{C}_{c_0}
\]

sends the connected component parameterized by \( c_0 \in g^{-1}(0) \) to the \( c_0 \)-component on the right hand side and is uniquely determined by the requirement that the point \( (u_0, c_0) \) maps to \( \tilde{c}_0 \in \tilde{C}_{c_0} \). The refinement morphism \( U_1 \times \Delta^1, C \rightarrow \coprod_{c_1} \tilde{C}_{c_1} \) is defined similarly. Combining these, we obtain the morphism

\[
U_0 \times \Delta^1, U_1 \times \Delta^1, C = (U_0 \times \Delta^1, C) \times_C (U_1 \times \Delta^1, C) \xrightarrow{\Phi} \coprod \tilde{C}_{c_0} \times \tilde{C}_{c_1}
\]

which, together with the transfer for the finite morphism \( U_0 \times \Delta^1, U_1 \times \Delta^1, C \rightarrow U_0 \times \Delta^1, U_1 \) gives the transfer for rigid 1-cocycles on the \( U_0 \times \Delta^1, U_1 \)-component.

Let \( (y_0, y_1) \in U_0 \times \Delta^1, U_1 \) be any point, \( y \in \Delta^1 \) the common image of \( y_0 \) and \( y_1 \), and \( \tilde{y} \in k(\tilde{C}) \) be a point with \( \tilde{y}|_{\Delta^1} = y \). Then, by definition of the transfer,

\[
(\delta^0 - \delta^1)(b) = [k(C) : k(\Delta^1)]_{\text{in}}, \sum_{g \in G(\tilde{C}) \setminus G(\tilde{C}|_{\Delta^1})} a(\Psi(y_0, y_1, g\tilde{y}|_{C})).
\]

It remains to show that the right hand side equals \( \phi(z) \). By definition of the 1-cocycle \( a \), it remains to show that

\[
[k(C) : k(\Delta^1)]_{\text{in}}, \sum_{g \in G(\tilde{C}) \setminus G(\tilde{C}|_{\Delta^1})} \Psi(y_0, y_1, g\tilde{y}|_{C}) = z \in mA(k),
\]

where \( \Psi : U_0 \times \Delta^1, U_1 \times \Delta^1, C \rightarrow mA(k) \) is the map obtained by composing \( \Phi \) with the map

\[
\coprod \tilde{C}_{c_0} \times_C \tilde{C}_{c_1} \rightarrow mA(k),
\]
The normalization of a curve to the diagram of fields above, the point \( x \) be an abstract blow-up square, i.e. a cartesian diagram of schemes such that of a perfect field \( k \). All schemes in this section are separated schemes of finite type over the spectrum which is given by \( U \times \Delta \). We have to identify \( \Phi(y_0, y_1, \tilde{g}|_C) \) considered as a point of \( \tilde{C} \times_C \tilde{C} \). With respect to the diagram of fields above, the point \( \tilde{y} \) defines the point \( (\tilde{g}|_{U_0}, \tilde{g}|_{U_1}, \tilde{g}|_C) \) on \( U_0 \times_{\Delta} U_1 \times_{\Delta} C \). We consider the first component \( \Phi_0 \) of \( \Phi \) (the description of the second component is similar).

By definition of \( c_0(y_0, \tilde{g}|_C) \in g^{-1}(0) \), the point \( (y_0, \tilde{g}|_C) \) lies in the \( c_0(y_0, \tilde{g}|_C) \)-component of \( U_0 \times_{\Delta} C \). Put \( 0 = \tilde{0}|_C \) and let \( g_0 \in G(\tilde{C}|\Delta^1) \) be given by \( y_0 = g_0^{-1} \tilde{y}|_{U_0} \). By Proposition 4.5, we have

\[
c_0(y_0, \tilde{g}|_C) = gg_0 \tilde{0}|_C.
\]

For \( g \in G(\tilde{C}|C) \), we compare the morphism \( \tilde{C} \rightarrow \tilde{C}, \tilde{x} \mapsto g \tilde{x}|_C \) with the morphism \( \tilde{C} \rightarrow \tilde{C} \) which sends \( \tilde{x} \) to \( \Phi_0(g_0^{-1} \tilde{x}|_{U_0}, g \tilde{x}|_C) \). A priori, the second morphism is defined on the open subscheme of points \( \tilde{x} \) with \( g_0^{-1} \tilde{x}|_{k(U_0)} \in U_0 \), but since \( \tilde{C} \) is a regular curve, it extends to the whole of \( \tilde{C} \) at hand. By Lemma 5.3, applied to \( \tilde{y} \) and \( g_0 \tilde{0} \), we obtain

\[
\Phi_0(y_0, \tilde{g}|_C) = \Phi_0(u_0, gg_0 \tilde{0}|_C) + \tilde{g}|_C - gg_0 \tilde{0}|_C.
\]

By definition, \( \Phi_0(u_0, gg_0 \tilde{0}|_C) = gg_0 \tilde{0}|_C \). Using similar notation for the second component, we obtain

\[
[k(C) : k(\Delta^1)]_{g_0} \cdot \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} \Psi(y_0, y_1, \tilde{g}|_C) = [k(C) : k(\Delta^1)]_{g_0} \cdot \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} gg_0 \tilde{0}|_C + \tilde{g}|_C - gg_0 \tilde{0}|_C - gg_1 \tilde{1}|_C + gg_1 \tilde{1}|_C
\]

\[\overset{(5.4)}{=} [k(C) : k(\Delta^1)]_{g_0} \cdot \sum_{g \in G(\tilde{C}|C) \setminus G(\tilde{C}|\Delta^1)} \tilde{g}|_C - g \tilde{1}|_C.
\]

To conclude the proof, it suffices to show that the last term equals \( z \) as an element in \( Z_0(\tilde{C})^0 \) or, by definition of \( z \), that \( m \) times the last term equals \( g^* (0) - g^* (1) \). This follows from \( m \tilde{c} = c - P_0 \) for all \( c \in C \) and from the definition of the cycle theoretic pull-back.

\[\square\]

6 The blow-up sequences

All schemes in this section are separated schemes of finite type over the spectrum of a perfect field \( k \). A curve on a scheme \( X \) is a closed one-dimensional subscheme. The normalization of a curve \( C \) is denoted by \( \tilde{C} \).

Now let

\[
\begin{array}{ccc}
Z' & \longrightarrow & X' \\
\downarrow & & \downarrow \pi \\
Z & \xrightarrow{i} & X
\end{array}
\]

be an abstract blow-up square, i.e. a cartesian diagram of schemes such that \( \pi : X' \rightarrow X \) is proper, \( i : Z \rightarrow X \) is a closed embedding and \( \pi \) induces an isomorphism: \( X' \setminus Z' \cong X \setminus Z \).
**Proposition 6.1.** Given an abstract blow-up square and an abelian torsion group $A$, there is a natural exact sequence

$$0 \rightarrow H^0_{et}(X, A) \rightarrow H^0_{et}(X', A) \oplus H^0_{et}(Z, A) \rightarrow H^0_{et}(Z', A)$$

$$\delta \rightarrow H^1_{et}(X, A) \rightarrow H^1_{et}(X', A) \oplus H^1_{et}(Z, A) \rightarrow H^1_{et}(Z', A).$$

**Proof.** We omit the coefficients $A$ and put $H^0_{et}(X) = H^0_{et}(X)$. We call an abstract blow-up square trivial, if $i$ or $\pi$ is an isomorphism. For a trivial abstract blow-up square we have short exact sequences

$$0 \rightarrow H^i(X) \rightarrow H^i(X') \oplus H^i(Z) \rightarrow H^i(Z') \rightarrow 0$$

for $i = 0, 1$. Every abstract blow-up square with $X$ a regular curve is trivial.

Now let an arbitrary abstract blow-up square be given. The proper base change theorem implies (cf. [Ge2], 3.2 and 3.6) that we have the exact sequence claimed in the proposition for $H^*_et$ instead of $H^*_et$. We first show, that the image of the boundary map $\delta : H^0_{et}(Z') \rightarrow H^1_{et}(X)$ has image in $H^1_{et}(X)$, thus showing the existence of $H^0_{et}(Z') \rightarrow H^1_{et}(X)$ and, at the same time, the exactness of the sequence at $H^1_{et}(X)$. Let $\tilde{C} \rightarrow X$ be the normalization of a curve in $X$. The base change

$$Z_{\tilde{C}} \rightarrow X_{\tilde{C}}$$

$$\downarrow \quad \pi$$

$$Z_{\tilde{C}} \rightarrow \tilde{C}$$

of our abstract blow-up square to $\tilde{C}$ is a trivial abstract blow-up square. Therefore, for any $\alpha \in H^0_{et}(Z')$ the pull-back of $\alpha$ to $H^0_{et}(Z'_{\tilde{C}})$ lies in the image of $H^0_{et}(X'_{\tilde{C}}) \oplus H^0_{et}(Z'_{\tilde{C}}) \rightarrow H^0_{et}(Z'_{\tilde{C}})$ and has therefore trivial image under $\delta : H^0_{et}(Z'_{\tilde{C}}) \rightarrow H^1_{et}(\tilde{C})$. Therefore, $\delta(\alpha) \in H^1_{et}(X)$ has trivial image in $H^1_{et}(\tilde{C})$ for every curve $C \subset X$, in particular, it lies in $H^1_{et}(X)$.

It remains to show exactness at $H^1_{et}(X') \oplus H^1_{et}(Z)$. Let $\alpha$ be in this group with trivial image in $H^1_{et}(Z')$. Then there exists $\beta \in H^1_{et}(X)$ mapping to $\alpha$ and it remains to show that $\beta$ lies in the subgroup $H^1_{et}(X)$. But this is clear, because for every curve $C \subset X$ we have $H^1_{et}(\tilde{C}) = \ker(H^1_{et}(X'_{\tilde{C}}) \oplus H^1_{et}(Z_{\tilde{C}}) \rightarrow H^1_{et}(Z'_{\tilde{C}}))$. \qed

**Remark 6.2.** If $\pi : X' \rightarrow X$ is finite, then the same proof (using the exactness of $\pi_*$ instead of the proper base change theorem) shows the exactness of the sequence of Proposition 6.1 for the cohomology with values in an arbitrary abelian group $A$.

**Proposition 6.3.** Given an abstract blow-up square

$$Z' \rightarrow X'$$

$$\downarrow \quad \pi$$

$$Z \rightarrow X$$

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there is a natural exact sequence of Suslin homology groups with values in an abelian group $A$
\[
H^S_1(Z', A) \to H^S_1(X', A) \oplus H^S_1(Z, A) \to H^S_1(X, A)
\]
\[
\delta : H^S_0(Z', A) \to H^S_0(X', A) \oplus H^S_0(Z, A) \to H^S_0(X, A) \to 0.
\]

**Proof.** Consider the exact sequences
\[
C_\bullet(Z', A) \hookrightarrow C_\bullet(X', A) \oplus C_\bullet(Z, A) \rightarrow C_\bullet(X, A) \rightarrow K^A\]
and
\[
C_\bullet(Z') \hookrightarrow C_\bullet(X') \oplus C_\bullet(Z) \rightarrow C_\bullet(X) \rightarrow K_\bullet,
\]
where $K^A_\bullet$ and $K_\bullet$ are defined to make the sequences exact. Since the complexes $C_\bullet(\_)$ consist of free abelian groups, in order the show the statement of the proposition, it suffices to show that $H_i(K^A_\bullet) = 0$ for $i \leq 2$. Let $\text{Sm}/k$ be the full subcategory of Sch$/k$ consisting of smooth schemes. For $Y \in \text{Sch}/k$ we consider the presheaf $c(Y)$ on $\text{Sm}/k$ given by $c(Y)(U) = \text{Cor}(U, Y)$. Then, by [SV2], Thm. 5.2, 4.7 and its proof, the sequence
\[
0 \rightarrow c(Z') \rightarrow c(X') \oplus c(Z) \xrightarrow{(\pi, \iota)} c(X)
\]
is exact and $F := \text{coker}(\pi_*, \iota_*)$ has the property that, for any $U \in \text{Sm}/k$ of dimension $\leq 2$ and any $x \in F(U)$, there exists a proper birational morphism $\phi : V \to U$ with $V$ smooth such that $\phi^*(x) = 0$. Let $F_\bullet$ be the complex of presheaves given by $F_n(U) = F(U \times \Delta^n)$ with the obvious differentials and let $(F_\bullet)_{\text{Nis}}$ be the associated complex of sheaves on $(\text{Sm}/k)_{\text{Nis}}$. Then by [SS], Thm. 2.4, the Nisnevich sheaves
\[
\mathcal{H}_i((F_\bullet)_{\text{Nis}})
\]
vanish for $i \leq 2$. Evaluating at $U = \text{Spec}(k)$ yields the result.

Now assume that $k$ is algebraically closed. Let
\[
\text{rec}_1 : H^S_1(X, \mathbb{Z}/m\mathbb{Z}) \to H^1_1(X, \mathbb{Z}/m\mathbb{Z})^*
\]
be the reciprocity map constructed in section 2 and let
\[
\text{rec}_0 : H^S_0(X, \mathbb{Z}/m\mathbb{Z}) \to H^0_0(X, \mathbb{Z}/m\mathbb{Z})^*
\]
be the homomorphism induced by the pairing
\[
\langle \cdot, \cdot \rangle : H^0_0(X, m\mathbb{Z}) \times H^0_0(X, \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}
\]
defined as follows: Given $a \in H^0_0(X, \mathbb{Z}/m\mathbb{Z})$ and $b \in H^0_0(X, \mathbb{Z}/m\mathbb{Z})$, we represent $a$ by a correspondence $\alpha \in \text{Cor}(\Delta^0, X)$ and put $\langle a, b \rangle = \alpha^*(b) \in H^0_0(\Delta^0, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$. This is well-defined since the homomorphisms $0^*, 1^* : H^0_0(\Delta^1, \mathbb{Z}/m\mathbb{Z}) \to H^0_0(\Delta^0, \mathbb{Z}/m\mathbb{Z})$ agree.

**Lemma 6.4.** $\text{rec}_{0, X}$ is an isomorphism.

**Proof.** For connected $X$, the structure morphism $\phi : X \to \text{Spec}(k)$ induces the identity $\mathbb{Z}/m\mathbb{Z} = H^0_0(k, \mathbb{Z}/m\mathbb{Z}) \to H^0_0(X, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. Hence, for connected $X$, $\text{rec}_{0, X} : H^S_0(X, \mathbb{Z}/m\mathbb{Z}) \to H^0_0(X, \mathbb{Z}/m\mathbb{Z})^* = \mathbb{Z}/m\mathbb{Z}$ coincides with the mod $m$ degree map $H^S_0(X, \mathbb{Z}/m\mathbb{Z}) \to H^0_0(k, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$.
In particular, $\text{rec}_{0,X}$ is surjective for arbitrary $X$ and is an isomorphism if $\dim X = 0$. If $X$ is a smooth connected curve, then $H^0_S(X, \mathbb{Z}) = \text{Pic}(\bar{X}, \bar{X} \setminus X)$, where $\bar{X}$ is the smooth compactification of $X$ (cf. [SV1], Thm. 3.1). The subgroup Pic$^0(\bar{X}, \bar{X} \setminus X)$ of degree zero elements is the group of $k$-rational points of the Albanese of $X$ and hence divisible. Therefore $\text{rec}_{0,X}$ is an isomorphism for connected, and hence for all smooth curves. Considering the normalization morphism of a arbitrary scheme of dimension 1 and the exact sequences of Propositions 6.1 and 6.3, the five-lemma shows that $\text{rec}_{0,X}$ is an isomorphism for connected, and hence for all smooth curves. Considering the normalization morphism of an arbitrary scheme of dimension 1 and the exact sequences of Propositions 6.1 and 6.3, the five-lemma shows that $\text{rec}_{0,X}$ is an isomorphism for connected, and hence for all smooth curves. Considering the normalization morphism of a arbitrary scheme of dimension 1 and the exact sequences of Propositions 6.1 and 6.3, the five-lemma shows that $\text{rec}_{0,X}$ is an isomorphism for connected, and hence for all smooth curves. Considering the normalization morphism of a arbitrary scheme of dimension 1 and the exact sequences of Propositions 6.1 and 6.3, the five-lemma shows that $\text{rec}_{0,X}$ is an isomorphism for connected, and hence for all smooth curves.

It remains to show that $\text{rec}_{0,X}$ is injective for arbitrary $X$. We may assume $X$ to be connected. Let $a \in \ker(\text{rec}_{0,X})$ and let $\alpha \in Z_0(X)$ be a representing 0-cycle. Since $\text{supp}(\alpha)$ is finite, we find a connected 1-dimensional closed subscheme $Z \subset X$ containing $\text{supp}(\alpha)$ (use, e.g., [Mu], II §6 Lemma). Since $\text{rec}_{0,Z}$ is injective and $a$ is in the image of $H^S_0(Z, \mathbb{Z}/m\mathbb{Z}) \to H^S_0(X, \mathbb{Z}/m\mathbb{Z})$, we conclude that $a = 0$. 

**Corollary 6.5.** Let $k$ be an algebraically closed field and let $X \in \text{Sch}/k$ be connected. Then the kernel of the degree map $$\deg : H^S_0(X, \mathbb{Z}) \longrightarrow H^S_0(k, \mathbb{Z}) \cong \mathbb{Z}$$ is divisible.

**Proposition 6.6.** Let $k$ be algebraically closed and let $$
abla' \xrightarrow{i'} X' \\
abla' \xrightarrow{\pi} Z \xrightarrow{i} X$

be an abstract blow-up square. Then for any integer $m \geq 1$ the diagram $$H^S_0(X, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\delta} H^S_0(Z', \mathbb{Z}/m\mathbb{Z}) \\
\downarrow \text{rec}_1 \hspace{2cm} \downarrow \text{rec}_0 \\
H^1_0(X, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\delta^*} H^0_\text{et}(Z', \mathbb{Z}/m\mathbb{Z})^*,$$

commutes. Here $\delta$ is the boundary map of Proposition 6.3 and $\delta^*$ is the dual of the boundary map of Proposition 6.1.

**Proof.** We have to show that the diagram

$$
\begin{array}{ccc}
H^S_0(X, \mathbb{Z}/m\mathbb{Z}) & \times & H^1_0(X, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{(\cdot, \cdot)} \mathbb{Z}/m\mathbb{Z} \\
\downarrow \delta & & \downarrow \delta \\
H^S_0(Z', \mathbb{Z}/m\mathbb{Z}) & \times & H^0_\text{et}(Z', \mathbb{Z}/m\mathbb{Z}) \xrightarrow{(\cdot, \cdot)} \mathbb{Z}/m\mathbb{Z}.
\end{array}
$$

commutes. Given $a \in H^S_0(X, \mathbb{Z}/m\mathbb{Z})$ and $b \in H^0_\text{et}(Z', \mathbb{Z}/m\mathbb{Z})$, we choose a representing correspondence $\alpha \in C_1(X, \mathbb{Z}/m\mathbb{Z}) = \text{Cot}(\Delta^1, X) \otimes \mathbb{Z}/m\mathbb{Z}$ in such a
way that it has a pre-image $\hat{\alpha} \in C_1(X', \mathbb{Z}/m\mathbb{Z}) \oplus C_1(Z, \mathbb{Z}/m\mathbb{Z})$. By definition, $\delta a \in H^0_\delta(Z', \mathbb{Z}/m\mathbb{Z})$ is represented by a correspondence $\gamma \in C_0(Z', \mathbb{Z}/m\mathbb{Z})$ such that the diagram

$$
\begin{array}{c}
\Delta^0 \xrightarrow{0 -1} \Delta^1 \\
\gamma \downarrow \quad \downarrow \hat{\delta} \\
Z^i \overset{i - \pi'}{\longrightarrow} X' \sqcup Z
\end{array}
$$

of correspondences commutes modulo $m$. Next choose an injective resolution $0 \to \mathbb{Z}/m\mathbb{Z} \to I^\bullet$ of $\mathbb{Z}/m\mathbb{Z}$ in the category of sheaves of $\mathbb{Z}/m\mathbb{Z}$-modules on $(\text{Sch}/k)_h$ in order to compute the pairings (cf. the end of section 3). Consider the following diagram

$$
\begin{array}{c}
I^0(X') \oplus I^0(Z) \xrightarrow{\pi^*, \pi'} I^0(Z') \\
\downarrow d \quad \quad \downarrow d \\
I^1(X) \xrightarrow{(\pi^*, \pi')} I^1(X') \oplus I^1(Z) \xrightarrow{\pi^* - \pi'} I^1(Z') \\
\downarrow \alpha^* \quad \downarrow \hat{\alpha}^* \\
I^0(\Delta^1) \xrightarrow{d} I^1(\Delta^1) \xrightarrow{\alpha^*} I^1(\Delta^1) \xrightarrow{d} I^2(\Delta^1) \\
\downarrow \alpha^* - \epsilon^* \quad \downarrow \alpha^* - \epsilon^* \\
I^0(\Delta^0) \xrightarrow{d} I^1(\Delta^0) \xrightarrow{d} I^2(\Delta^0)
\end{array}
$$

By the argument of [MVW] Lemma 12.7, the sequence

$$0 \to F(X) \to F(X') \oplus F(Z) \to F(Z')$$

is exact for every $h$-sheaf $F$. Therefore the second line in the diagram is exact. The proper base change theorem implies (cf. [Ge2], 3.2 and 3.6) that

$$I^\bullet(X) \xrightarrow{\cdot^*} I^\bullet(X') \oplus I^\bullet(Z) \xrightarrow{\cdot^*} I^\bullet(Z')$$

is an exact triangle in $D(Ab)$. For the exact sequence of complexes

$$0 \to I^\bullet(X) \to I^\bullet(X') \oplus I^\bullet(Z) \to I^\bullet(Z') \to coker^\bullet \to 0,$$

this implies that the complex $coker^\bullet$ is exact. Therefore $b \in \ker(I^0(Z') \to I^1(Z'))$ has a pre-image $\hat{\beta} \in I^0(X') \oplus I^0(Z)$. Then

$$d\hat{\beta} \in \ker(I^1(X') \oplus I^1(Z) \to I^1(Z'))$$

and there exists a unique $\epsilon \in I^1(X)$ with $(\pi^*, \epsilon_*)^*(\epsilon) = d\hat{\beta}$ representing $d b \in H^1_\delta(X)$. We see that $\hat{\alpha}^*(d\hat{\beta}) = \alpha^*(\epsilon)$. It follows that

$$d(\hat{\alpha}^*(\hat{\beta})) = \hat{\alpha}^*(d\hat{\beta}) = \alpha^*\epsilon \in \ker(I^1(\Delta^1) \xrightarrow{\epsilon^* - \epsilon^*} I^1(\Delta^0)).$$

By definition of $\langle , \rangle$, we obtain

$$\langle a, \delta(b) \rangle = (0^* - 1^*)\hat{\alpha}^*\hat{\beta} \in \ker(I^0(\Delta^0) \to I^1(\Delta^0)) = \mathbb{Z}/m\mathbb{Z}.$$
On the other hand, \( \langle \delta u, b \rangle = \gamma^*(b) \in H^0_\acute{\text{e}}t(\Delta^0) \) is represented by \( \gamma^* \beta \in I^0(\Delta^0) \) and the commutative diagram of correspondences above implies

\[
\gamma^* \beta = \gamma^*(i'^* - \pi'^*)(\tilde{\beta}) = (0^* - 1^*)\tilde{\alpha}^* \tilde{\beta}.
\]

This finishes the proof.

\[\square\]

**Proposition 6.7.** Let \( X \) be a normal, generically smooth, connected scheme of finite type over a field \( k \) and let \( A \subset H^0_\acute{\text{e}}t(X, \mathbb{Z}/m\mathbb{Z}) \) be a finite subgroup. Then there exists a regular curve \( C \) over \( k \) and a finite morphism \( \phi : C \to X \) such that \( A \) has trivial intersection with the kernel of \( \phi^* : H^0_\acute{\text{e}}t(X, \mathbb{Z}/m\mathbb{Z}) \to H^0_\acute{\text{e}}t(C, \mathbb{Z}/m\mathbb{Z}) \).

**Proof.** For a normal scheme \( Z \) and a dense open subscheme \( Z' \subset Z \), the induced map \( H^0_\acute{\text{e}}t(Z, \mathbb{Z}/m\mathbb{Z}) \to H^0_\acute{\text{e}}t(Z', \mathbb{Z}/m\mathbb{Z}) \) is injective. Hence we may replace \( X \) by an open subscheme and assume that \( X \) is smooth. Let \( Y \to X \) be the finite abelian étale covering corresponding to the kernel of \( \pi^\text{ab}_1(X) \to A^* \). We have to find a regular curve \( C \) and a finite morphism \( C \to X \) such that \( C \times_X Y \) is connected.

Choose a separating transcendence basis \( t_1, \ldots, t_d \) of \( k(X) \) over \( k \). This yields a rational map \( X \to \mathbb{P}^d_R \). Let \( t \) be another indeterminate and let \( X_t \) (resp. \( Y_t \)) be the base change of \( X \) (resp. \( Y \)) to the rational function field \( k(t) \). Consider the composition \( \phi : Y_t \to X_t \to \mathbb{P}^d_{k(t)} \). Since \( k(t) \) is Hilbertian [FJ], Thm. 12.10, we find a rational point \( P \in \mathbb{P}^d_{k(t)} \) over which \( \phi \) is defined and such that \( P \) has exactly one preimage \( y_i \) in \( Y_t \). The image \( x_i \in X_t \) of \( y_i \) has exactly one preimage in \( Y_t \). Let \( x \) be the image of \( x_i \) in \( X \). If \( \text{trdeg}_k k(x) = 1 \) put \( x' = x \), if \( \text{trdeg}_k k(x) = 0 \) (i.e. \( x \) is a closed point in \( X \)) choose any \( x' \in X \) with \( \text{trdeg}_k k(x') = 1 \) such that \( x \) is a regular point of the closure of \( x' \). In both cases the normalization \( C \) of the closure of \( x' \) in \( X \) is a regular curve with the desired property. \[\square\]

7 **Proof of the main theorem**

In this section we prove our main result. We say that “resolution of singularities holds for schemes of dimension \( \leq d \) over \( k \)” if the following two conditions are satisfied.

1. For any integral separated scheme of finite type \( X \) of dimension \( \leq d \) over \( k \) there exists a projective birational morphism \( Y \to X \) with \( Y \) smooth over \( k \) which is an isomorphism over the regular locus of \( X \).
2. For any integral smooth scheme \( X \) of dimension \( \leq d \) over \( k \) and any birational proper morphism \( Y \to X \) there exists a tower of morphisms \( X_n \to X_{n-1} \to \cdots \to X_0 = X \), such that \( X_n \to X_{n-1} \) is a blow-up with a smooth center for \( i = 1, \ldots, n \), and such that the composite morphism \( X_n \to X \) factors through \( Y \to X \).

**Theorem 7.1.** Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \), \( X \) a separated scheme of finite type over \( k \) and \( m \) a natural number. Then

\[
\text{rec}_X : H^1_\acute{\text{e}}t(X, \mathbb{Z}/m\mathbb{Z}) \to \pi^1,\text{ab}(X)/m
\]

is surjective. It is an isomorphism of finite abelian groups if \( (m, p) = 1 \), and for general \( m \) if resolution of singularities holds for schemes of dimension \( \leq \dim X + 1 \) over \( k \).
The proof will occupy the rest of this section. Following the notation of section 6, we write $H^0_{\text{et}} = H^0_{\text{ct}}$ and consider the maps

$$\text{rec}_{1,X} : H^i_S(X, \mathbb{Z}/m\mathbb{Z}) \to H^i_S(X, \mathbb{Z}/m\mathbb{Z})^*$$

for $i = 0, 1$ (i.e., $\text{rec}_X = \text{rec}_{1,X}$). Given a morphism $X' \to X$ we have a commutative diagram of pairings defining $\text{rec}_i$ for $i = 0, 1$.

$$
\begin{array}{ccc}
H^i_S(X', \mathbb{Z}/m\mathbb{Z}) & \times & H^i_S(X, \mathbb{Z}/m\mathbb{Z}) \\
\phi_* & & \phi^* \\
H^i_S(X, \mathbb{Z}/m\mathbb{Z}) & \times & H^i_S(X, \mathbb{Z}/m\mathbb{Z})
\end{array}
$$

**Step 1:** $\text{rec}_{1,X}$ is surjective for arbitrary $X$.

We may assume that $X$ is reduced and proceed by induction on $d = \dim X$. The case $\dim X = 0$ is trivial. Consider the normalization morphism $X' \to X$, which is an isomorphism outside a closed subscheme $Z \subset X$ of dimension $\leq d - 1$. Using the exact sequences of Propositions 6.1 and 6.3, we obtain an isomorphism of finite abelian groups and by Proposition 2.9, we have an isomorphism $H^i_S(X, \mathbb{Z}/m\mathbb{Z})$. Given a morphism $X' \to X$ we have a commutative diagram of pairings defining $\text{rec}_i$ for $i = 0, 1$.

**Step 2:** Theorem 7.1 holds if $(m, p) = 1$.

If $(m, p) = 1$, $H^i_S(X, \mathbb{Z}/m\mathbb{Z})$ and $H^i_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z})^*$ are isomorphic finite abelian groups by [SV1]. In particular, they have the same order. Hence the surjective homomorphism $\text{rec}_{1,X}$ is an isomorphism.

**Step 3:** Theorem 7.1 holds for arbitrary $X$ if $m = p^r$ and resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over $k$.

We may assume that $X$ is reduced. Using resolution of singularities and Chow’s Lemma, we obtain a morphism $X' \to X$ with $X'$ smooth and quasi-projective, which is an isomorphism over a dense open subscheme of $X$. Using the exact sequences of Propositions 6.1 and 6.3, Lemma 6.4, **Step 1**, induction by dimension and the five-lemma, it suffices to show the result for smooth, quasiprojective schemes.

Let $X$ be smooth, quasiprojective and let $\bar{X}$ be a smooth, projective variety containing $X$ as a dense open subscheme. Then we have isomorphisms

$$H^i_{\text{et}}(\bar{X}, \mathbb{Z}/p^r\mathbb{Z})^* \cong \text{CH}_0(\bar{X}, 1, \mathbb{Z}/p^r\mathbb{Z}) \quad [\text{Ge}3], \S 5, \text{Duality},$$

$$\text{CH}_0(\bar{X}, 1, \mathbb{Z}/p^r\mathbb{Z}) \cong H^i_S(\bar{X}, \mathbb{Z}/p^r\mathbb{Z}) \quad [\text{SS}], \text{Thm. 2.7}.$$

By Proposition 7.2 below the natural homomorphism

$$H^i_S(X, \mathbb{Z}/p^r\mathbb{Z}) \to H^i_S(\bar{X}, \mathbb{Z}/p^r\mathbb{Z})$$

is an isomorphism of finite abelian groups and by Proposition 2.9, we have an isomorphism

$$H^i_{\text{et}}(\bar{X}, \mathbb{Z}/p^r\mathbb{Z}) \cong H^i_S(\bar{X}, \mathbb{Z}/p^r\mathbb{Z}).$$
Hence the groups $H_1^i(X, \mathbb{Z}/p^r\mathbb{Z})^*$ and $H_1^S(X, \mathbb{Z}/p^r\mathbb{Z})$ are isomorphic finite abelian groups, in particular, they have the same order. Since $rec_1X$ is surjective, it is an isomorphism.

In order to conclude the proof of Theorem 7.1 it remains to show

**Proposition 7.2.** Let $k$ be a perfect field, $X \in \text{Sch}/k$ smooth, $U \subset X$ a dense open subscheme and $n \geq 0$ an integer. Assume that resolution of singularities holds for schemes of dimension $\leq \dim X + n$ over $k$. Then for any $r \geq 1$ the natural map

$$H_1^S(U, \mathbb{Z}/p^r\mathbb{Z}) \to H_1^S(X, \mathbb{Z}/p^r\mathbb{Z})$$

is an isomorphism of finite abelian groups for $i = 0, \ldots, n$.

**Remark 7.3.** A proof of Proposition 7.2 for $n = 1$ and $k$ algebraically closed independent of the assumption on resolution of singularities would relax the condition in Theorem 7.1 to:

There exists a smooth, projective scheme $X' \in \text{Sch}/k$ and a dense open subscheme $X' \subset X'$ together with a birational morphism $X' \to X$.

In particular, Theorem 7.1 would hold for $\dim X \leq 3$ without any assumption on resolution of singularities [CV].

**Proof of Proposition 7.2.** We set $R = \mathbb{Z}/p^r\mathbb{Z}$. By [MVW], Lecture 14, we have

$$H_1^S(X, R) = \text{Hom}_{DM^{eff}_{Nis}}((k,R))(R[i], R(X, R)).$$

Let $d = \dim X$. Choose a series of open subschemes $U = X_d \subset \cdots \subset X_1 \subset X_0 = X$ such that $Z_j := X_j \setminus X_{j+1}$ is smooth of dimension $j$ for $j = 0, \ldots, d - 1$. Using the exact Gysin triangles [MVW, 15.15]

$$M(X_{j+1}, R) \to M(X_j, R) \to M(Z_j)(d-j)[2d-2j][1] \to M(X_{j+1}, R)[1]$$

and induction, it suffices to show that

$$\text{Hom}_{DM^{eff}_{Nis}}((k,R))(R[i], M(Z_j, R)(s)[2s]) = 0$$

for $j = 0, \ldots, d - 1$, $i = 0, \ldots, n + 1$ and $s \geq 1$. Using smooth compactifications of the $Z_j$ and induction again, it suffices to show

$$\text{Hom}_{DM^{eff}_{Nis}}((k,R))(R[i], M(Z, R)(s)[2s]) = 0$$

for $Z$ connected, smooth, projective, $i = 0, \ldots, d - d_Z + n$ and $s \geq 1$.

By the comparison of higher Chow groups and motivic cohomology [V] and by [GL], Thm. 8.5, the restriction of $R(s)$ to the small Nisnevich site of a smooth scheme $Y$ is isomorphic to $\nu_r^*[s]$, where $\nu_r^*$ is the logarithmic de Rham Witt sheaf of Milne and Illusie. In particular, $R(s)|_Y$ is trivial for $s > \dim Y$.

For an étale $k$-scheme $Z$ we obtain

$$\text{Hom}_{DM^{eff}_{Nis}}((k,R))(R[i], M(Z, R)(s)[2s]) = H_{Nis}^{2s-i}(Z, R(s)) = 0.$$  

for $s \geq 1$ and all $i \geq 0$. Now assume $\dim Z \geq 1$. Using resolution of singularities for schemes of dimension $\leq d + n$, the same method as in the proof of [SS], Thm. 2.7 yields isomorphisms

$$\text{Hom}_{DM^{eff}_{Nis}}((k,R))(R[i], M(Z, R)) \cong \text{CH}^{dZ}(Z, i, R).$$
for \( i = 0, \ldots, d - 1 + n \). Applying this to \( Z \times \mathbf{P}^s \) and using the decompositions given by the projective bundle theorem on both sides implies isomorphisms

\[
\text{Hom}_{\text{DM}_{\text{Nis}}^e(k,R)}(R[i], M(Z, R)(s)[2s]) \cong \text{CH}^{d_2 + s}(Z, i, R)
\]

for \( i = 0, \ldots, d - 1 + n \). By \([V]\), the latter group is isomorphic to

\[
\text{Hom}_{\text{DM}_{\text{Nis}}^e(k,R)}(M(Z, R)[2d + 2s - i], R(d + s)) \cong H^{2d + 2s - i}(Z, R(d + s)),
\]

which vanishes for \( s \geq 1 \). This finishes the proof. \( \square \)

**Remark 7.4.** The assertion of Proposition 7.2 remains true for non-smooth \( X \) if \( U \) contains the singular locus of \( X \) (see \([Ge4]\), Prop. 3.3).

### 8 Comparison with the isomorphism of Suslin-Voevodsky

**Theorem 8.1.** Let \( k \) be an algebraically closed field, \( X \in \text{Sch}/k \) and \( m \) an integer prime to \( \text{char}(k) \). Then the reciprocity isomorphism

\[
\text{rec}_X : H^{2d_2}(X, \mathbf{Z}/m\mathbf{Z}) \longrightarrow \pi_1^\text{ab}(X)/m
\]

is the dual of the isomorphism

\[
\alpha_X : H^1_\text{et}(X, \mathbf{Z}/m\mathbf{Z}) \longrightarrow H^1_\text{S}(X, \mathbf{Z}/m\mathbf{Z})
\]

of \([SV1]\), Cor. 7.8.

The proof will occupy the rest of this section. Let \( i : \mathbf{Z}/m\mathbf{Z} \hookrightarrow I^0 \) be an injection into an injective sheaf in the category of \( \mathbf{Z}/m\mathbf{Z} \)-module sheaves on \( \text{Sch}/k \). Then \( J^1 = \text{coker}(i) \). Then (see the end of section 3) the pairing between \( H^2(Y, \mathbf{Z}/m\mathbf{Z}) \) and \( H^1_\text{et}(X, \mathbf{Z}/m\mathbf{Z}) \) constructed in Proposition 2.11 can be given as follows: For \( a \in H^2(Y, \mathbf{Z}/m\mathbf{Z}) \) choose a representing correspondence \( \alpha \in \text{Cor}(\Delta^1, X) \) and for \( b \in H^1_\text{et}(X, \mathbf{Z}/m\mathbf{Z}) \) a preimage \( \beta \in J^1(X) \). Consider the diagram

\[
\begin{array}{ccc}
I^0(X) & \longrightarrow & J^1(X) \\
\downarrow \alpha^* & & \downarrow \alpha^* \\
I^0(\Delta^1) & \longrightarrow & J^1(\Delta^1)
\end{array}
\]

\[
\begin{array}{ccc}
0^* - 1^* & \longrightarrow & 0^* - 1^*
\end{array}
\]

Then \( \alpha^*(\beta) \) is the image of some element \( \gamma \in I^0(\Delta^1) \) and \((0^* - 1^*)(\gamma) \in \mathbf{Z}/m\mathbf{Z} = \ker(I^0(\Delta^0) \rightarrow J^1(\Delta^0)) \) equals \( \langle a, b \rangle \).

For \( Y \in \text{Sch}/k \) let \( \mathbf{Z}_{\text{qfh}}^\text{qfh} \) be the free qfh-sheaf generated by \( Y \). We set \( A = \mathbf{Z}[1/\text{char}(k)] \) and \( L_Y = \mathbf{Z}_{\text{qfh}}^\text{qfh} \otimes A \). For smooth \( U \) the homomorphism

\[
\text{Cor}(U, X) \otimes A \rightarrow \text{Hom}_{\text{qfh}}(L_U, L_X)
\]
is an isomorphism by [SV1], Thm. 6.7. We have
\[ H^1_{et}(X, \mathbb{Z}/m\mathbb{Z}) = H^1_{qfh}(X, \mathbb{Z}/m\mathbb{Z}) = \text{Ext}^1_{qfh}(L_X, \mathbb{Z}/m\mathbb{Z}) = \text{coker}(\text{Hom}_{qfh}(L_X, J^0) \to \text{Hom}_{qfh}(L_X, J^1)) \]
The diagram can be rewritten in terms Ext-groups as follows:

\[
\begin{array}{ccc}
\text{Hom}_{qfh}(L_X, I^0) & \longrightarrow & \text{Hom}_{qfh}(L_X, J^1) \\
\alpha^* & & \alpha^* \\
\text{Hom}_{qfh}(L_{\Delta^1}, I^0) & \longrightarrow & \text{Hom}_{qfh}(L_{\Delta^1}, J^1) \\
0^* - 1^* & & 0^* - 1^* \\
\mathbb{Z}/m\mathbb{Z} & \longrightarrow & \text{Hom}_{qfh}(L_{\Delta^0}, I^0) \longrightarrow \text{Hom}_{qfh}(L_{\Delta^0}, J^1).
\end{array}
\]

We denote the morphism \( L_X \to J^1 \) corresponding to the element \( \beta \in J^1(X) \cong \text{Hom}_{qfh}(L_X, J^1) \) by the same letter \( \beta \). Putting \( E := I^0 \times_{J^1, \beta} L_X \), the extension
\[ 0 \to \mathbb{Z}/m\mathbb{Z} \to E \to L_X \to 0 \]
represents \( b \in \text{Ext}^1_{qfh}(L_X, \mathbb{Z}/m\mathbb{Z}) \). Consider the diagram

\[
\begin{array}{ccc}
\text{Hom}_{qfh}(L_X, E) & \longrightarrow & \text{Hom}_{qfh}(L_X, L_X) \\
\alpha^* & & \alpha^* \\
\text{Hom}_{qfh}(L_{\Delta^1}, E) & \longrightarrow & \text{Hom}_{qfh}(L_{\Delta^1}, L_X) \\
0^* - 1^* & & 0^* - 1^* \\
\mathbb{Z}/m\mathbb{Z} & \longrightarrow & \text{Hom}_{qfh}(L_{\Delta^0}, E) \longrightarrow \text{Hom}_{qfh}(L_{\Delta^0}, L_X).
\end{array}
\]

Because diagram (3) maps to diagram (2) via \( \beta^* \) and \( \text{id} \in \text{Hom}_{qfh}(L_X, L_X) \) maps under \( \beta^* \) to \( \beta \in \text{Hom}_{qfh}(L_X, J^1) \), we can calculate the pairing using diagram (3) after replacing \( \beta \) by \( \text{id} \). Since \( \text{id} \) maps to \( \alpha \in \text{Hom}_{qfh}(L_{\Delta^1}, L_X) \) under \( \alpha^* \), we see, writing the lower part of diagram (3) in the form

\[
\begin{array}{ccc}
\mathbb{Z}/m\mathbb{Z} & \longrightarrow & E(\Delta^1) \longrightarrow L_X(\Delta^1) \\
0 & \downarrow & 0^* - 1^* \\
\mathbb{Z}/m\mathbb{Z} & \longrightarrow & E(\Delta^0) \longrightarrow L_X(\Delta^0).
\end{array}
\]

that
\[ \langle a, b \rangle = h(\alpha) \mod m \in \ker(E(\Delta^0)/m \to L_X(\Delta^0)/m) = \mathbb{Z}/m\mathbb{Z}, \]
where \( h \) is the unique homomorphism making diagram (4) commutative. We consider the complex \( C_*(X) = \text{Cor}(\Delta^*, X) \otimes A = L_X(\Delta^*) \) with the obvious differentials. By the above considerations, the homomorphism induced by the pairing of Proposition 2.11
\[ H^1_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z}) = H^1_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z}) \]

\[ H^0_{\text{qfh}}(X, \mathbb{Z}/m\mathbb{Z})^* = \text{Ext}^1_A(C_\bullet(X), \mathbb{Z}/m\mathbb{Z}) = \text{Hom}_{D(A)}(C_\bullet(X), \mathbb{Z}/m\mathbb{Z}[1]), \]

is given by sending an extension class \([\mathbb{Z}/m\mathbb{Z} \hookrightarrow E \twoheadrightarrow L_X]\) to the morphism \(C_\bullet(X) \to \mathbb{Z}/m\mathbb{Z}[1]\) in the derived category of \(A\)-modules represented by the morphism

\[ C_\bullet(X) \to [0 \to E(\Delta^0) \to L_X(\Delta^0) \to 0] \]

which is given by \(\text{id} : L_X(\Delta^0) \to L_X(\Delta^0)\) in degree zero and by \(h : L_X(\Delta^1) \to E(\Delta^0)\) in degree one.

The same construction works for any qfh-sheaf of \(A\)-modules \(F\) instead of \(L_X\), i.e. setting \(C_\bullet(F) = F(\Delta^\bullet)\) and starting from an element

\[ [\mathbb{Z}/m\mathbb{Z} \hookrightarrow E \twoheadrightarrow F] \in \text{Ext}^1_{\text{qfh}}(F, \mathbb{Z}/m\mathbb{Z}), \]

we get a map \(C_\bullet(F) \to \mathbb{Z}/m\mathbb{Z}[1]\) in the derived category of \(A\)-modules. We thus constructed a homomorphism

\[ \text{Ext}^1_{\text{qfh}}(F, \mathbb{Z}/m\mathbb{Z}) \to \text{Ext}^1_A(C_\bullet(F), \mathbb{Z}/m\mathbb{Z}), \]

which for \(F = L_X\) and under the canonical identifications coincides with the map

\[ H^1_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z}) \to H^0_{\text{qfh}}(X, \mathbb{Z}/m\mathbb{Z})^* \]

induced by the pairing constructed in Proposition 2.11.

Now we compare the map \((*)\) with the map

\[ \alpha_X : \text{Ext}^1_{\text{qfh}}(F, \mathbb{Z}/m\mathbb{Z}) \to \text{Ext}^1_A(C_\bullet(F), \mathbb{Z}/m\mathbb{Z}) \]

constructed by Suslin-Voevodsky [SV1] (cf. [Ge1] for the case of positive characteristic). Let \(F_\sim^\bullet\) be the complex of qfh-sheaves associated with the complex of presheaves \(F_\bullet(U) = F(U \times \Delta^\bullet)\). By [SV1], the inclusion \(F \to F_\sim^\bullet\) induces an isomorphism

\[ \text{Ext}^1_{\text{qfh}}(F_\sim^\bullet, \mathbb{Z}/m\mathbb{Z}) \sim \to \text{Ext}^1_{\text{qfh}}(F, \mathbb{Z}/m\mathbb{Z}), \]

and evaluation at \(\text{Spec}(k)\) induces an isomorphism

\[ \text{Ext}^1_{\text{qfh}}(F_\sim^\bullet, \mathbb{Z}/m\mathbb{Z}) \sim \to \text{Ext}^1_A(C_\bullet(F), \mathbb{Z}/m\mathbb{Z}). \]

The map \((**\*)\) of Suslin-Voevodsky is the composite of the inverse of \((5)\) with \((6)\).

We construct the inverse of \((5)\). Let a class \([\mathbb{Z}/m\mathbb{Z} \hookrightarrow E \twoheadrightarrow F] \in \text{Ext}^1_{\text{qfh}}(F, \mathbb{Z}/m\mathbb{Z})\) be given. As a morphism in the derived category this class is given by the homomorphism

\[
\begin{array}{c c c c}
0 & 0 & F & 0 \\
\| & | & | & id \\
0 & E & F & 0
\end{array}
\]

We therefore have to construct a homomorphism \(F_1 \to E\) making the diagram

\[
\begin{array}{c c c c}
F_2 & F_1 & F_0 & 0 \\
\| & | & | & id \\
0 & E & F & 0
\end{array}
\]

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commutative. The construction is a sheafified version of what we did before. Let \( U \in \text{Sch}/k \) be arbitrary. Consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}(U) & \longrightarrow & E(U \times \Delta^2) & \longrightarrow & F(U \times \Delta^2) & \longrightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow \delta^0 - \delta^1 + \delta^2 & & \downarrow \delta^0 - \delta^1 + \delta^2 & & \\
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}(U) & \longrightarrow & E(U \times \Delta^1) & \longrightarrow & F(U \times \Delta^1) & \longrightarrow & 0 \\
& & \downarrow 0 & & \downarrow 0^* - 1^* & & \downarrow 0^* - 1^* & & \\
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}(U) & \longrightarrow & E(U) & \longrightarrow & F(U) & \longrightarrow & 0
\end{array}
\]

Let \( \alpha_1 \in F(U \times \Delta^1) \) be given. By the smooth base change theorem and since \( H^1_{et}(\Delta^1, \mathbb{Z}/m\mathbb{Z}) = 0 \), we can lift \( \alpha_1 \) to \( E(U \times \Delta^1) \) after replacing \( U \) by a sufficiently fine étale cover. Applying \( 0^* - 1^* \) to this lift, we get an element in \( E(U) \). This gives the homomorphism \( F_1 \to E \). Now let \( \alpha_2 \in F(U \times \Delta^2) \) arbitrary. After replacing \( U \) by a sufficiently fine étale cover, we can lift \( \alpha_2 \) to \( E(U \times \Delta^2) \). Since \( (0^* - 1^*)(\delta^0 - \delta^1 + \delta^2) = 0 \) this shows that \( (\delta^0 - \delta^1 + \delta^2)(\alpha_2) \) maps to zero in \( E(U) \).

This describes the inverse isomorphism to (5). Evaluating at \( U = \text{Spec}(k) \) gives back our original construction, hence (*) and (**) are the same maps. This finished the proof.

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