Mass dependence of Wightman function and light front singularity

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Abstract. The Wightman function for a massive free scalar field is studied within the light front formulation, while a special attention is paid to its mass dependence. The long lasting inconsistency is successfully solved by means of the novel Fourier representation for scalar fields. The new interpretation of the light front singularities as the high momentum phenomena is presented and adequate regularizations are implemented.

1. Introduction

The light front (LF) formulation of the relativistic quantum field theory, which starts from the pioneering paper by Dirac on forms of the relativistic dynamics[1], has been extensively studied during last twenty years (for some reviews see [2], [3], [4]). In this approach one introduces the LF coordinates usually as \(x^\pm = (x^0 \pm x^1)/\sqrt{2}\) and then picks \(x^+\) as the LF temporal evolution parameter, while \(x^-\) is treated as the LF surface coordinate. The standard procedure, where \(x^0 = t\) is the evolution parameter, referred to as the equal-time (ET) approach, differs substantially from the LF procedure. The triviality of LF physical vacuum usually is given as the main advantage of the LF approach. However when one considers a free field theory then literally no difference should appear for all physical quantities defined in either procedure. Especially, the Wightman functions should be the same but, as it has been observed already in 1977 by Nakanishi and Yabuki [5], it is not the case. They have argued that the ET Wightman function

\[
\Delta^+_{ET}(x) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^3p}{2\omega(p)} e^{-i[\omega(p)(t-x^0)]}\]

(1)

where \(\omega(p) = \sqrt{m^2 + p^2}\) and \(x^0 = t\), is a smooth mass-dependent function at the LF surface \(x^+ = 0\)

\[
\lim_{x^+ \to 0} \Delta^+_{ET}(x) = \frac{m}{4\pi^2 \sqrt{x^2_\perp}} K_1(m\sqrt{x^2_\perp}).
\]

(2)

They have opposed the LF Wightman function

\[
\Delta^+_{LF}(x) = \frac{1}{(2\pi)^3} \int_0^\infty dp_- \int_{-\infty}^{\infty} d^2p_\perp e^{-i[p_.(-x^- + \frac{m^2 + p^2_\perp}{2p_-} - p_\perp \cdot x_\perp)]},
\]

(3)

which as long as \(x^+ \neq 0\) coincides with (1) - this can be shown by performing explicitly all momentum integrations. Then taking the limit \(x^+ \to 0\) gives the expected result (2). The inconcistency arises when one takes \(x^+ \to 0\) before doing the momentum integrations, since then one obtains

\[
\Delta^+_{LF}(x^+ = 0, x^-, x_\perp) = \frac{1}{(2\pi)^3} \int_0^\infty dp_- \int_{-\infty}^{\infty} d^2p_\perp e^{-i[p_.(-x^- - p_\perp \cdot x_\perp)]}.
\]

(4)

The above expression is explicitly mass independent and the integral over \(p_-\) variable is ill defined. This situation is unexceptable since the value of the Wightman function
at the LF surface \((x^+ = 0)\) is crucial for the LF quantization procedure. Further one may believe that the mass dependence and the LF singularity are quite close related phenomena.

The singularities at \(p_- = 0\) are permanent difficulties within the LF formulation and different regularizations are used for keeping them under control. However neither the \(\nu\) theory of [3] or the DLCQ approach of [7] can fix the above mentioned mass-dependence inconsistency. For the free scalar fields the mass appears only via \(p_+ = m^2 + p_{\perp}^2\), then one may force that \(x^+\) coordinates of fields should not coincide for the two point Wightman function. Since this would stay in a conflict with the LF quantization at the fixed \(x^+\) surface, one may allow for the imaginary part of \(x^+ \to x^+ - i\epsilon\) in (3), which effectively leads to the formula [5]

\[
\Delta_{LF}^+(x^+ = 0, \bar{x}) = \frac{1}{(2\pi)^3} \int_0^\infty \frac{dp_-}{2p_-} \int_{-\infty}^{\infty} d^2p_{\perp} e^{-i[p_--(m^2 + p_{\perp}^2)\frac{\mu_k}{2p_-} - p_{\perp} \cdot x_{\perp}]}.
\]

(5)

Thus, as long as \(\epsilon > 0\), one has both the regularization for small values of \(p_-\) and the mass dependence. Another solution is even more drastic - one should forget about the LF quantization [8]. Quite recently there was an attempt to solve this mass dependence inconsistency within the distribution theory [9], but the conclusion that in the sense of distributions one can define only objects which are mass-independent misses the point.

The partial success of all these approaches can be connected with the LF dogma that one has to solve the singularity at \(p_- \to 0\) first, then all other problems will be properly solved. We think that one may reverse the logic and start with a different mass-dependent Fourier representation of scalar massive field at the LF surface, at least for the free field case.

Our paper is organized as follows. In section 2 we present the standard LF formulation the Wightman function for the free scalar massive field. In section 3 we present the modified LF formulation. In section 4 we discuss the mass dependence of modified LF Wightman function. In section 5 we present different UV regularizations. At last, the conclusions and possible further research are presented. The notation and technical details of calculations are given in Appendix A. In Appendix B we present LF canonical formalism for higher derivative Lagrangian. In Appendix C we show how the LF singularity may arise within the ET formulation.

2. Standard light front formulation

We start our concise presentation of the standard formulation of the light front field theory with the free real massive field \(\phi(x)\), which when quantized on the LF surface \((x^+ = 0)\) is represented as the Fourier integral

\[
\phi(x) = \int \frac{d^2k_{\perp}}{(2\pi)^3} \int_0^\infty \frac{dk_-}{2k_-} \left[ e^{-i k_{\perp} \cdot x_{\perp}} e^{-i k_- x^-} e^{-i \frac{\mu_k}{2p_-} x^+} a(k_{\perp}, k_-) + h.c. \right],
\]

(6)

where h.c. stands for the Hermitian conjugate and

\[
\mu_k = \sqrt{(k_{\perp}^2 + m^2)/2}.
\]

(7)
The anihilation and creation operators \( a(\mathbf{k}_\perp, k_-), a^\dagger(\mathbf{p}_\perp, p_-) \) have a nonvanishing commutator
\[
[a(\mathbf{k}_\perp, k_-), a^\dagger(\mathbf{p}_\perp, p_-)] = (2\pi)^3 2k_- \delta(k_- - p_-) \delta^2(k_\perp - p_\perp)
\] (8)
The Wightman function \( \Delta_+(x) \) is defined as the vacuum expectation value
\[
\Delta_+(x) = \langle 0 | \phi(x) \phi(0) | 0 \rangle
\] (9)
which due to (6) and (8) has the Fourier representation
\[
\Delta_+(x) = \int \frac{d^3k_\perp}{(2\pi)^3} \int_0^\infty \frac{dk_-}{2k_-} e^{-ik_-x^-} e^{-ik_\perp \cdot x_\perp} e^{-i\frac{\mu^2}{m^2}x^+}.
\] (10)
The integral over the longitudinal momentum \( k_- \) must be treated with a high caution since for \( x^+ = 0 \) \( x^- = 0 \) it diverges logarithmically at the upper (lower) limit of integration. Therefore we see that the LF Wightman function is ill defined for the space-like separation of points \( x^2 = -x_{\perp}^2 < 0 \), contrary to the ET result (2).

Thus in order to perform consistently all integrations in (10) we need to keep \( x^\pm \neq 0 \).

Since for the free field case the Wightman function is known for arbitrary separation of points, one may accept this as a kind of technical details which are specific for the LF formulation. However from the more general perspective the problem is much more serious, since from (6) one usually infers the Fourier representation for an interacting theory at the LF quantization surface \( x^+ = 0 \)
\[
\phi(x) = \int \frac{d^3k_\perp}{(2\pi)^3} \int_0^\infty \frac{dk_-}{2k_-} \left[ e^{-ik_-x^-} e^{-ik_\perp \cdot x_\perp} a(\mathbf{k}_\perp, k_-) + h.c. \right].
\] (11)
Thus we see that any solution of the LF Wightman problem for the free scalar field may have a serious consequences for other LF models with interacting scalar fields. Finally we would like to present the 4-dimensional integral representation of the LF Wightman function
\[
\Delta_+(x) = \int \frac{d^3k_\perp}{(2\pi)^3} \int dk_- e^{-ik_\perp \cdot x_\perp} \Theta(k_-) \delta(2k_\perp k_- - k_{\perp}^2 - m^2),
\] (12)
which evidently is equivalent to (10) and it explicitly shows that there is no symmetry between \( k_{\pm} \) momenta.

3. Modified light front formulation

We propose to take another Fourier representation for free real scalar field
\[
\phi(x) = \int \frac{dk_\perp}{(2\pi)^3} \int_0^\infty \frac{dk_-}{2k_-} \left[ e^{-ik_-x^-} e^{-ik_\perp \cdot x_\perp} e^{-i\frac{\mu^2}{m^2}x^+} a(\mathbf{k}_\perp, k_-) + h.c. \right]
\] (13)
\[
+ \int \frac{dk_\perp}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_+}{2k_+} \left[ e^{-ik_-x^-} e^{-ik_\perp \cdot x_\perp} e^{-i\frac{\mu^2}{m^2}x^+} b(\mathbf{k}_\perp, k_+) + h.c. \right],
\] (14)
where the nonvanishing commutators are
\[
[a(\mathbf{k}_\perp, k_-), a^\dagger(\mathbf{p}_\perp, p_-)] = (2\pi)^3 2k_- \delta(k_- - p_-) \delta^2(k_\perp - p_\perp),
\] (15)
\[
[b(\mathbf{k}_\perp, k_+), b^\dagger(\mathbf{p}_\perp, p_+)] = (2\pi)^3 2k_- \delta(k_+ - p_+) \delta^2(k_\perp - p_\perp).
\] (16)
This leads to the modified LF Wightman function

$$\Delta_+(x) = \int \frac{d^2k_+}{(2\pi)^2} \int_{\mu(k_+)}^{\infty} \frac{dk_-}{2k_-} e^{-ik_-x^-} e^{-ik_+\cdot x^+} e^{\frac{i{k_+}^2}{2}x^+} +$$

$$+ \int \frac{d^2k_+}{(2\pi)^2} \int_{\mu(k_-)}^{\infty} \frac{dk_-}{2k_-} e^{-ik_-\cdot x^-} e^{-ik_+x^+} e^{\frac{i{k_-}^2}{2}x^-},$$

(17)

which can be equivalently rewritten as

$$\Delta_+(x) = \int \frac{d^2k_+}{(2\pi)^3} \int dk_- e^{-ik\cdot x} \Theta(k_- - \mu_k) \delta(2k_+ k_- - k_-^2 - m^2) +$$

$$+ \int \frac{d^2k_+}{(2\pi)^3} \int dk_- e^{-ik\cdot x} \Theta(k_+ - \mu_k) \delta(2k_+ k_- - k_+^2 - m^2),$$

(18)

thus our modification restores the symmetry $k_+ \leftrightarrow k_-$. 

Both integrals in (17) diverge logarithmically in their upper limits when either $x^-=0$ or $x^+=0$, respectively. This leads us to the conclusion that there are possible two UV divergences, contrary to the standard formulation where there is one UV and one IR divergency. This new interpretation seems to be more physical since when fields are massive no IR divergence should appear.

When $x^+ \neq 0$, we may change the integration variables

$$k_+ = \frac{\mu_k^2}{k_-},$$

(19)

in the second integral and return to the standard result (10).

The free field representation (13) can be taken as a basis for the new canonical representation (at $x^+=0$)

$$\phi(\bar{x}) = \int \frac{dk_-}{(2\pi)^3} \int_{\mu(k_-)}^{\infty} \frac{dk_-}{2k_-} \left[ e^{-ik_-\cdot x^-} a(k_-, k_-) + h.c. \right] +$$

$$+ \int \frac{dk_-}{(2\pi)^3} \int_{\mu(k_+)}^{\infty} \frac{dk_+}{2k_+} \left[ e^{-ik_+\cdot x^+} b(k_+, k_+) + h.c. \right]$$

(20)

for an interacting scalar field. Thus we may compare two LF integral representations (11) and (20) for their interpretations of modes, which do not depend on $x^-$. In the former one, which is currently commonly accepted, these modes are low momentum $k_- = 0$ and should be treated as other IR contributions (e.g. by putting the system into a finite box in $x^-$ coordinate). In the latter one, which is novel, these modes are high momentum ($k_+ \to \infty$), thus one should treat them in an adequate way (e.g. by the Pauli-Villars regularization or by the higher derivative terms).

We may check the consistency of our modified formula (17) by inspecting the LF commutator function which is defined as

$$\Delta(\bar{x}) = \Delta_+(\bar{x}) - \Delta_+(-\bar{x}) =$$

$$= \int \frac{d^2k_+}{(2\pi)^2} \int_{\mu(k_+)}^{\infty} \frac{dk_-}{2k_-} \left( e^{-ik_-x^-} - e^{-ik_-x^-} \right) e^{-ik_+\cdot x^+} +$$

$$+ \int \frac{d^2k_+}{(2\pi)^2} \int_{\mu(k_-)}^{\infty} \frac{dk_-}{2k_-} \left( e^{-ik_+\cdot x^+} - e^{-ik_+\cdot x^+} \right) e^{-ik_-\cdot x^-}. $$

(21)
Now we notice that the $k_\pm$ integrals are convergent, therefore we may easily perform these integrations

$$\Delta(x) = -i \int \frac{d^2k_-}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_-}{2k_-} \sin(k_-x^-) e^{-ik_-x_-} +$$

$$-i \int \frac{d^2k_+}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_+}{2k_+} \sin\left(\frac{\mu^2(k_+)}{k_+} x^-\right) e^{-ik_+x_+} =$$

$$-i \frac{1}{2} \text{sgn}(x^-) \int \frac{d^2k_+}{(2\pi)^3} \left[ \pi - 2Si(\mu(k_+)|x^-|) \right] e^{-ik_+x_+} -$$

$$i \text{sgn}(x^-) \int \frac{d^2k_+}{(2\pi)^3} Si(\mu(k_+)|x^-|) e^{-ik_+x_+} = -i \frac{1}{4} \text{sgn}(x^-) \delta^2(x_+), \quad (22)$$

where we have used relations (1.1b)-(1.1c) and we conclude that the commutator function has the proper form.

4. Mass dependence of modified LF Wightman function

Since the mass dependence (or rather mass independence) of the LF Wightman function has lead to serious problems concerning the consistency of LF field theory [8], thus here we will concentrate on the LF surface $x^+ = 0$ for the modified LF Wightman function

$$\Delta_+(\bar{x}) = \int \frac{d^2k_+}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_+}{2k_+} e^{-ik_+x^-} e^{-ik_+x_+} +$$

$$+ \int \frac{d^2k_+}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_+}{2k_+} e^{-i\frac{x^-}{k_+}x^-} e^{-ik_+x_+}. \quad (23)$$

The simplest criterion, whether our modified LF Wightman effectively depends on $m$ or not, is to calculate its derivative with respect to $m$

$$\frac{d}{dm} \Delta_+(\bar{x}) = -\frac{m}{2} \int \frac{d^2k_+}{(2\pi)^3} \frac{1}{\mu_k^2} e^{-i\mu_kx^-} e^{-ik_+x_+} -$$

$$-\frac{i}{2} x^- m \int \frac{d^2k_+}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_+}{k_+^2} e^{-\frac{x^-}{k_+}x^-} e^{-ik_+x_+}, \quad (24)$$

where we use the simple relation

$$\frac{d}{dm} \mu_k = \frac{m}{2\mu_k}. \quad (25)$$

Though our integral (23) is logarithmically divergent, then its parametric derivative is already convergent and the integration over $k_+$ is elementary

$$\frac{d}{dm} \Delta_+(\bar{x}) = -\frac{m}{2} \int \frac{d^2k_+}{(2\pi)^3} \frac{1}{\mu_k^2} e^{-i\mu_kx^-} e^{-ik_+x_+} +$$

$$+ \frac{m}{2} \int \frac{d^2k_+}{(2\pi)^3} \frac{1}{\mu_k^2} e^{-ik_+x_+} \left( e^{-i\mu_kx^-} - 1 \right) =$$

$$= -m \int \frac{d^2k_+}{(2\pi)^3} \frac{1}{m^2 + k_+^2} e^{-ik_+x_+}. \quad (26)$$

Thus quite unexpectedly we find that the final integral does not depend on the LF coordinate $x^-$ but only on the transverse coordinates $x_\perp$. More we see that the mass
dependence of the modified LF Wightman function \( \Delta_+ = \frac{\partial^2}{\partial x^2} \) is by no means trivial. The 2-dimensional integral over \( k_\perp \) can be performed in the cylinder coordinates

\[
\begin{align*}
k_2 &= \rho \sin \phi, & k_3 &= \rho \cos \phi, \\
x_2 &= \rho \sin \theta, & x_3 &= \rho \cos \theta,
\end{align*}
\]

where the angle integration gives

\[
\int_0^{2\pi} d\phi e^{i\rho \cos(\phi - \theta)} = 2\pi J_0(\rho),
\]

with \( J_0(\rho) \) being the Bessel function. Then the last radial integral gives

\[
\frac{d}{dm} \Delta_+(\bar{x}) = \frac{m}{(2\pi)^2} \int_0^\infty d\rho \frac{J_0(\rho)}{m^2 + \rho^2} = -\frac{m}{4\pi^2} K_0(m\rho),
\]

where \( K_0(x) \) is the modified Bessel function and we have found a well defined mass differential equation for the Wightman function. As a quick consistency check for our calculations we may take the Wightman function \( \Delta_+ = \frac{\partial^2}{\partial x^2} \) and find that it satisfies the above mass differential equations, due to the property of the modified Bessel functions

\[
\frac{d}{dx} [xK_1(x)] = -xK_0(x).
\]

However one may raise the objection, that if one changes the integration variables

\[
k_+ = \frac{\mu^2}{k_-},
\]

in the second integral of (23), then one gets the usual LF representation

\[
\Delta_+(x) = \int \frac{d^2k_\perp}{(2\pi)^3} \int_0^\infty \frac{dk_-}{2k_-} e^{-ik_-x^-} e^{-ik_\perp \cdot x^\perp},
\]

which is evidently mass independent. However, since here this second integral is divergent for \( k_+ \to \infty \), thus such a change of the integration variables is not legitimate. Instead, one should first regularize this evident UV divergence, by means of some UV regularization and only then, one may change the integration variables.

### 5. UV regularizations

In this section we will define different regularized Wightman function and our starting point is the cuttoff expression

\[
\Delta_+^\Lambda(\bar{x}) = \int \frac{d^2k_\perp}{(2\pi)^3} \int_0^\infty \frac{dk_-}{2k_-} e^{-ik_-x^-} e^{-ik_\perp \cdot x^\perp} + \int \frac{d^2k_\perp}{(2\pi)^3} \int_{\mu_k}^\infty \frac{dk_+}{2k_+} \left( e^{-\frac{\mu_k^2}{4}x^-} - 1 \right) e^{-ik_\perp \cdot x^\perp} + \int \frac{d^2k_\perp}{(2\pi)^3} \int_{\mu_k}^\Lambda \frac{dk_+}{2k_+} e^{-ik_\perp \cdot x^\perp}.
\]

We have subtracted the divergent contribution in a way, which allows to perform easily integrations over \( k_\perp \)

\[
\Delta_+^\Lambda(\bar{x}) = -\frac{1}{2} \int \frac{d^2k_\perp}{(2\pi)^3} e^{-ik_\perp \cdot x^\perp} \left( \gamma + \log(\mu_k |x^-|) + \frac{\pi}{2} \text{sgn}(x^-) + \log \frac{\mu_k}{\Lambda} \right),
\]
where we have used the relations (1.1a)-(1.1c). Further calculations will depend on the regularization method that we will choose to get rid of the cutoff parameter $\Lambda$.

5.1. Pauli-Villars regularization

The Pauli-Villars regularization is the method of doing perturbative calculations in both gauge and Lorentz-invariant way, where one uses the regularized propagators. For instance, the massive scalar propagator is

$$\Delta_{F}^{\text{reg}}(x) = \int \frac{d^{4}k}{(2\pi)^{4}} \left( \frac{1}{m^{2} - p^{2} - i\epsilon} + \sum_{a=1}^{N} \frac{C_{a}}{M_{a}^{2} - p^{2} - i\epsilon} \right) e^{-ip\cdot x},$$

(36)

with the following conditions upon the coefficients $C_{a}$:

$$1 + \sum_{a=1}^{N} C_{a} = 0, \quad m^{2} + \sum_{a=1}^{N} C_{a}M_{a}^{2} = 0, \quad \text{etc.}$$

(37)

However this regularization is by no means restricted to the perturbation theory, but on contrary it can be applied to almost all problems where high momenta divergencies appear. For the free scalar field, this method has been applied in the original paper [10], where Pauli and Villars start with the regularization of singular functions $\Delta(x)$ and $\Delta^{(1)}(x)$, which are closely related to the Wightman function $\Delta^{+}(x)$

$$\Delta^{+}(x) = \frac{1}{2} \left( \Delta(x) + i\Delta^{(1)}(x) \right).$$

(38)

So even in the ET formulation the Pauli-Villars regularization can be applied for the free massive scalar field.

Since the LF Wightman function depends on the cutoff $\Lambda$ logarithmically, then the Pauli-Villars regularization of this function should have only one auxiliary field (with $M_{1}, C_{1}$). Accordingly we define the Pauli-Villars regularization of the LF Wightman function as

$$\Delta_{\text{F}}^{\text{reg}}(x) = \int \frac{d^{2}k_{\perp}}{(2\pi)^{3}} \int_{\mu_{k}}^{\infty} \frac{dk_{-}}{2k_{-}} e^{-ik_{-}x^{-}} e^{-ik_{\perp} \cdot x_{\perp}} +$$

$$+ \int \frac{d^{2}k_{\perp}}{(2\pi)^{3}} \int_{\mu_{k}}^{\infty} \frac{dk_{+}}{2k_{+}} e^{i\mu_{k}^{2}k_{\perp}^{2}x^{-}} e^{-ik_{\perp} \cdot x_{\perp}} +$$

$$+ C_{1} \int \frac{d^{2}k_{\perp}}{(2\pi)^{3}} \int_{\mathcal{M}_{k}}^{\infty} \frac{dk_{-}}{2k_{-}} e^{-ik_{-}x^{-}} e^{-ik_{\perp} \cdot x_{\perp}} +$$

$$+ C_{1} \int \frac{d^{2}k_{\perp}}{(2\pi)^{3}} \int_{\mathcal{M}_{k}}^{\infty} \frac{dk_{+}}{2k_{+}} e^{-i\mu_{k}^{2}k_{\perp}^{2}x^{-}} e^{-ik_{\perp} \cdot x_{\perp}},$$

(39)

where $\mathcal{M}_{k} = \sqrt{(k_{\perp}^{2} + M_{1}^{2})/2}$. In the current case, only the first condition should be taken from (37)

$$1 + C_{1} = 0,$$

(40)

and further we may use our former results for the integrations over $k_{\pm}$ given in (35)

$$\Delta_{\text{F}}^{\text{reg}}(x) = -\frac{1}{2} \int \frac{d^{2}k_{\perp}}{(2\pi)^{3}} e^{-ik_{\perp} \cdot x_{\perp}} \log \left( \frac{\mu_{k}^{2}}{\mathcal{M}_{k}^{2}} \right).$$

(41)
Mass dependence of Wightman function

Since we may introduce the integral representation

$$\log \left( \frac{\mu_k^2}{M_k^2} \right) = - \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha \mu_k^2} - e^{-\alpha M_k^2} \right),$$

into the integrand, then integrals over the transverse momenta $k_\perp$ become Gaussians

$$\Delta_+^{reg}(x) = \frac{1}{2} \int_0^\infty \frac{d\alpha}{\alpha} \int \frac{d^2k_\perp}{(2\pi)^3} e^{-i k_\perp \cdot x_\perp} e^{-\frac{\alpha}{2} k_\perp^2} \left( e^{-\frac{\alpha}{2} m^2} - e^{-\frac{\alpha}{2} M_1^2} \right),$$

which lead to

$$\Delta_+^{reg}(x) = \frac{1}{8\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\frac{x^2}{2\alpha}} \left( e^{-\frac{\alpha}{2} m^2} - e^{-\frac{\alpha}{2} M_1^2} \right) =$$

$$= \frac{1}{4\pi^2} \frac{m}{x_\perp^2} K_1(m \sqrt{x_\perp^2}) - \frac{1}{4\pi^2} \frac{M_1}{x_\perp^2} K_1(M_1 \sqrt{x_\perp^2})$$

As the last step we may push the auxiliary mass to infinity $M_1 \to \infty$ and this leads to

$$\lim_{M_1 \to \infty} \Delta_+^{reg}(x) = \frac{1}{4\pi^2} \frac{m}{x_\perp^2} K_1(m \sqrt{x_\perp^2}),$$

which is the expected result (2).

5.2. Higher derivative regularization

Another popular regularization is based on the modified Lagrangians, where one adds the higher derivative (HD) terms which effectively change the high momenta behaviour of the perturbative propagators. However here, we will use this method for a free field theory. The LF canonical procedure for this model is presented in Appendix B and here we will just quote the crucial points. The Wightman function within the HD regularization is defined as

$$D_{+}^{HD}(x) = \langle 0 | \phi(x) \phi(0) | 0 \rangle = \frac{1}{(c_+ - c_-)^2} \left( \langle 0 | \Phi_+ (x) \Phi_+(0) | 0 \rangle + \langle 0 | \Phi_- (x) \Phi_-(0) | 0 \rangle \right),$$

where we have used (2.16a)-(2.16b), (2.24a). Next we may take (2.25) and write

$$D_{+}^{HD}(\bar{x}) = \frac{1}{\sqrt{1 + 4\alpha m^2}} \int \frac{d^2k_\perp}{(2\pi)^3} e^{-i k_\perp \cdot x_\perp} \left( \int_{\mu_k^+}^{\infty} \frac{dk_-}{2k_-} e^{-i k_- x_-} - \int_{\mu_k^-}^{\infty} \frac{dk_-}{2k_-} e^{-i k_- x_-} \right) +$$

$$+ \frac{1}{\sqrt{1 + 4\alpha m^2}} \int \frac{d^2k_\perp}{(2\pi)^3} e^{-i k_\perp \cdot x_\perp} \left( \int_{\mu_k^+}^{\infty} \frac{dk_+}{2k_+} e^{-i k_+ x_+} - \int_{\mu_k^-}^{\infty} \frac{dk_+}{2k_+} e^{-i k_+ x_+} \right),$$

which is very similar to the Pauli-Villars regularization; thus omitting here the intermediatry steps, we write down the solution

$$D_{+}^{HD}(\bar{x}) = \frac{1}{\sqrt{1 + 4\alpha m^2}} \frac{1}{4\pi^2 \sqrt{x_\perp^2}} \left( M_+ K_1(M_+ \sqrt{x_\perp^2}) - M_- K_1(M_- \sqrt{x_\perp^2}) \right),$$

\[1\] One should not be surprised that the HD results are very similar to those from the Pauli-Villars regularization, since one may prove that, for some models, these methods are equivalent \[11\].
where we write after (2.22a)-(2.22b) and (2.23)
\[ M_+^2 = \frac{1}{2\alpha}(1 - \sqrt{1 + 4\alpha m^2}) \xrightarrow{\alpha \to 0} m^2, \]  
\[ M_-^2 = \frac{1}{2\alpha}(1 + \sqrt{1 + 4\alpha m^2}) \xrightarrow{\alpha \to 0} \infty. \]  
Therefore we may take the limit \( \alpha \to 0 \), removing the HD regularization,
\[ \lim_{\alpha \to 0} D_{\pm}^{HD}(\bar{x}) = \frac{m}{4\pi^2 \sqrt{x_+^2}} K_1(m\sqrt{x_+^2}), \]  
which again coincides with (2).

6. Conclusions and further prospects

In this paper we have proposed the modification of the LF Wightman function, which properly solves the problem of mass dependence. Our modification starts with the Fourier representation for the free massive scalar field operator and perfectly agrees with the canonical commutation relations. Further we find that there is no longer any IR singularity problem in this LF field theory, but rather a new UV divergency arises. We point out that the distribution \( \text{sgn}(x^-) \) in the commutation function (22) is generated by the high momentum behaviour of the LF Wightman function. We may further speculate that for an interacting theory one should take another Fourier representation (20), which may lead to new description of the LF systems.

The analogous analysis for the higher spin fields, specially fermions and gauge fields, will be given elsewhere, since these cases are technically more complicated and here we have decided to present our conjecture for the simplest case of quantum field theory.

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Appendix A. Notation and useful integrals

We use the natural units \( c = \hbar = 1 \). Our LF notation starts with the definitions of null components for the coordinates \( x^\pm = (x^0 \pm x^1)/\sqrt{2} \), while the transverse components are \( x^i = (x^2, x^3) \). The similar definitions are taken for any 4-vectors. The LF surface coordinates are denoted as \( \bar{x} = (x^-, x^i) \). The partial derivatives are taken with respect to contravariant coordinates, thus we have \( \partial_- = \partial/\partial x^+ \), \( \partial_+ = \partial/\partial x^- \), \( \partial_i = \partial/\partial x^i \). The metric tensor has non vanishing components \( g_{++} = g_{--} = 1, g_{ij} = -\delta_{ij} \). The scalar product of 4-vectors is \( a \cdot b = a_+ b_+ + a_- b_- - a_i b_i \), while for the LF surface components we have \( \bar{a} \cdot \bar{b} = a^- b_- - a_i b_i \).
In the main text, the following integrals are helpful

\[ \int_0^\infty \frac{dk}{k} \cos k + \int_0^\mu \frac{dk}{k} (\cos k - 1) = -\gamma - \log \mu \]  

(1.1a)

and

\[ \int_\mu^\infty \frac{dk}{k} \sin kx = \frac{1}{2} \text{sgn}(x) \left[ \pi - 2 \text{Si}(\mu|x|) \right], \]  

(1.1b)

\[ \int_\mu^\infty \frac{dk}{k} \sin \frac{\mu^2}{k}x = \text{sgn}(x) \text{Si}(\mu|x|), \]  

(1.1c)

where \( \text{Si}(x) \) is the integral sine function. In Appendix C we use the following integral

\[ \int_0^\infty \frac{d\rho J_0(\rho)}{\rho} \frac{1}{(m^2 + \rho^2)^\delta} = \frac{1}{\Gamma(\delta)} \left( \frac{2m}{\rho} \right)^{1-\delta} K_{1-\delta}(m\rho), \]  

(1.2)

which is valid for \( \delta > 1/4 \). Also we use another type of integral [12]

\[ \int_0^\infty \frac{d\alpha}{\alpha^{1-\delta}} e^{iu\alpha} = \frac{\Gamma(\delta)}{|u|^\delta} e^{i\pi\delta/2\text{sgn}(u)}, \]  

(1.3)

which is valid for \( 0 < \text{Re} \delta < 1 \).

**Appendix B. Simplified LF canonical formalism for higher derivative Lagrangian**

Let us describe the free scalar field with the higher order derivative and start with the Lagrangian density

\[ \mathcal{L}_\alpha = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{\alpha}{2} \left( \partial^2 \phi \right)^2 \]  

(2.1)

which leads to the Euler-Lagrange equation of motion

\[ \left[ \partial^2 + m^2 - \alpha \left( \partial^2 \right)^2 \right] \phi = 0, \]  

(2.2)

where we use \( \partial^2 = \partial_\mu \partial^\mu \) and \( \alpha > 0 \) is the regularization parameter.

For the sake of canonical quantization procedure, we prefer to extend the field content of our system by adding another scalar field \( \chi \) which leads to the equivalent Lagrangian density

\[ \mathcal{L}'_\alpha = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\chi^2}{2\alpha} - 2 \partial_\mu \chi \partial^\mu \phi \]  

(2.3)

with the system of Euler-Lagrange equations of motion

\[ \left[ \partial^2 + m^2 \right] \phi - \partial^2 \chi = 0, \]  

(2.4)

\[ \frac{1}{\alpha} \chi - \partial^2 \phi = 0. \]  

(2.5)

In the LF formulation, we explicitly we write

\[ \mathcal{L}'_\alpha = \partial_+ \phi \partial_- \phi - \frac{1}{2} \left( \partial_+ \phi \right)^2 - \partial_+ \phi \partial_- \chi - \partial_- \phi \partial_+ \chi - \partial_i \phi \partial_i \chi - \frac{m^2}{2} \phi^2 - \frac{\chi^2}{2\alpha}. \]  

(2.6)
The $T^{++}$ component of the energy-momentum tensor is defined as

$$T^{++} = \frac{\delta L_\alpha'}{\delta \partial_x \phi} \partial^+ \phi + \frac{\delta L_\alpha'}{\delta \partial_x \chi} \partial^+ \chi = (\partial_- \phi)^2 - 2 \partial_- \phi \partial_- \chi. \quad (2.7)$$

where $L_\alpha' = \int d^3x' L_\alpha'$. Here we will use the recently proposed simplified LF canonical quantization procedure [13], which is based on the trivial equations

$$i \partial_\phi(x) = \left[\phi(x), P^+(x^+)\right], \quad (2.8)$$

$$i \partial_\chi(x) = \left[\chi(x), P^+(x^+)\right], \quad (2.9)$$

with the translation generator

$$P^+(x^+) = \int d^3\bar{x} T^{++}(\bar{x}) = \int d^3\bar{x} \left[(\partial_- \phi(\bar{x}) - \partial_- \chi(\bar{x}))^2 - (\partial_- \chi(\bar{x}))^2\right]. \quad (2.10)$$

From the diagonal structure of $P^+$, we immediately recognize fields $\phi - \chi$ and $\chi$ as the independent LF canonical fields, with nonvanishing LF commutators

$$2 \left[\phi(\bar{x}) - \chi(\bar{x}), \partial_- \phi(y) - \partial_- \chi(y)\right]_{x^+ = y^+} = i \delta^3(\bar{x} - y), \quad (2.11)$$

$$2 \left[\chi(\bar{x}), \partial_- \chi(y)\right]_{x^+ = y^+} = - i \delta^3(\bar{x} - y), \quad (2.12)$$

or equivalently

$$2 \left[\phi(\bar{x}), \partial_- \chi(y)\right]_{x^+ = y^+} = - i \delta^3(\bar{x} - y), \quad (2.13)$$

$$2 \left[\chi(\bar{x}), \partial_- \phi(y)\right]_{x^+ = y^+} = - i \delta^3(\bar{x} - y), \quad (2.14)$$

$$[\phi(\bar{x}), \partial_- \phi(y)]_{x^+ = y^+} = 0. \quad (2.15)$$

Now we will look for the independent modes for our system and suppose that they are given as the linear combinations

$$\Phi_+ = c_+ \phi + \chi, \quad (2.16a)$$

$$\Phi_- = c_- \phi + \chi, \quad (2.16b)$$

while their equations of motions are

$$\left(2 \partial_+ \partial_- - \Delta_\perp + M^2_\pm\right) \Phi_\pm = 0. \quad (2.17)$$

These expressions are compatible with the equations of motion [2.4] and [2.6]

$$(2 \partial_+ \partial_- - \Delta_\perp) \phi = \frac{\chi}{\alpha}, \quad (2.18a)$$

$$(2 \partial_+ \partial_- - \Delta_\perp) \chi = \frac{\chi}{\alpha} + m^2 \phi, \quad (2.18b)$$

provided the following relations are satisfied

$$M^2_\pm c_\pm = - m^2, \quad (2.19a)$$

$$M^2_\pm = - \frac{1 + c_\pm}{\alpha}, \quad (2.19b)$$

with the consistency condition

$$c_\pm^2 + c_\pm - \alpha m^2 = 0. \quad (2.20)$$

If we choose the regularization parameter to satisfy the following inequalities

$$- \frac{1}{4m^2} \leq \alpha \leq 0, \quad (2.21)$$
then
\[
c_+ = \frac{-1 + \sqrt{1 + 4\alpha m^2}}{2} < 0, \quad (2.22a)
\]
\[
c_- = \frac{-1 - \sqrt{1 + 4\alpha m^2}}{2} < 0, \quad (2.22b)
\]
and we end up with two nontachyonic modes, since we have
\[
M^2_{\pm} = \frac{m^2}{c_\pm} > 0. \quad (2.23)
\]

Now we may find the LF commutators for the independent modes
\[
\begin{align*}
\left[ \Phi_+(x^+, \bar{x}), \Phi_-(x^+, \bar{y}) \right] &= -i (c_+ + c_- + 1) \delta^3(\bar{x} - \bar{y}) = 0, \quad (2.24a) \\
\left[ \Phi_+(x^+, \bar{x}), \Phi_+(x^+, \bar{y}) \right] &= -i (2c_+ + 1) \delta^3(\bar{x} - \bar{y}) = -i \sqrt{1 + 4\alpha m^2} \delta^3(\bar{x} - \bar{y}), \quad (2.24b) \\
\left[ \Phi_-(x^+, \bar{x}), \Phi_-(x^+, \bar{x}) \right] &= -i (2c_- + 1) \delta^3(\bar{x} - \bar{y}) = \sqrt{1 + 4\alpha m^2} \delta^3(\bar{x} - \bar{y}), \quad (2.24c)
\end{align*}
\]
which indicate that \( \Phi_- \) and \( \Phi_+ \) are the positive and negative metric fields, respectively. Thus the nonvanishing LF Wightman functions for the independent modes can be defined, accordingly to (17), as
\[
\Delta_+(x, a) = \langle 0 | \Phi_a(x) \Phi_a(0) | 0 \rangle = \\
= a \sqrt{1 + 4\alpha m^2} \int \frac{d^2k_\perp}{(2\pi)^3} \int_{\mu_k}^{\infty} \frac{dk_-}{2k_-} e^{-ik_-x^+} e^{ik_-x^+} e^{\frac{\alpha m^2}{2} x_+} + \\
+ a \sqrt{1 + 4\alpha m^2} \int \frac{d^2k_\perp}{(2\pi)^3} \int_{\mu_k}^{\infty} \frac{dk_+}{2k_+} e^{ik_+x^+} e^{-ik_+x^+} e^{\frac{\alpha m^2}{2} x^-}, \quad (2.25)
\]
with \((a = \pm)\) and \(\mu_{k_\pm} = \sqrt{(M^2_{\pm} + k^2_\perp)/2}\).

Appendix C. ET momentum integrals in cylinder coordinates

The LF singularities that we have discussed in the main part of this paper, can be found also within the ET quantum field theory. In order to encounter them one has to perform the momentum integrations in (1) in the cylinder coordinates and in \( d \) space dimensions \([14]\). Thus we take coordinates \((p_1, \varrho = \sqrt{p_2^2 + \ldots + p_d^2}, \Omega_{d-1})\) and the volume element is
\[
d^d\rho = \varrho^{d-2}d\varrho \, dp_1 \, d\Omega_{d-1}, \quad (3.1)
\]
where \(\Omega_{d-1}\) is the solid angle in \( d-1 \) transverse directions. Then the angular integral over transverse directions is
\[
\int d\Omega_{d-1} e^{i p_1 \cdot x_\perp} = 2\pi \left( \frac{2\pi}{\varrho} \right)^{(d-3)/2} J_{(d-3)/2}(\varrho \rho), \quad (3.2)
\]
where \(\rho = \sqrt{x_2^2 + \ldots + x_d^2}\).

Now we may concentrate on the integration over variable \( p_1 \),
\[
I(t, x^1) = \int_{-\infty}^{\infty} \frac{dp_1}{2\omega} e^{-i(\omega t - p_1^2)} = \frac{1}{2} \int_{-\infty}^{\infty} d\eta e^{-i\sqrt{m^2 + \varrho^2}(t \cosh \eta - x^1 \sinh \eta), \quad (3.3)}
\]
where we have introduced $\eta$ variable through the parameterization
\[ p_1 = \sqrt{m^2 + \rho^2 \sinh \eta}. \] (3.4)

The result depends on the value of $s^2$ [12]

\[ I(t, x^1) = \begin{cases} \dfrac{1}{2} \int_{-\infty}^{\infty} d\eta e^{i\sqrt{m^2 + \rho^2 \sinh \eta}} = K_0(\sqrt{m^2 + \rho^2 \sinh \eta}) & \text{for } s^2 < 0, \\ \dfrac{1}{2} \int_{-\infty}^{\infty} d\eta e^{-i\sqrt{m^2 + \rho^2 \cosh \eta}} = (-1)^{\sigma} \dfrac{i\pi}{2} H_0^{(\sigma)}(\sqrt{m^2 + \rho^2 \cosh \eta}) & \text{for } s^2 > 0, \\ \dfrac{1}{2} \int_{-\infty}^{\infty} d\eta e^{-i\sqrt{m^2 + \rho^2 \cosh \eta}} & \text{for } t = x^1, \\ \dfrac{1}{2} \int_{-\infty}^{\infty} d\eta e^{-i\sqrt{m^2 + \rho^2 \sinh \eta}} & \text{for } t = -x^1, \end{cases} \] (3.5)

where $\sigma = 1$ for $t > 0$, and $\sigma = 2$ for $t < 0$ and evidently at the LF surfaces $x^\pm = 0$ this integral is ill-defined.

Let us consider the singularity at $x^0 = x^1$ and we introduce the modified integral

\[ I^\delta(t, x^1) = \frac{1}{2} \int_{-\infty}^{\infty} d\eta e^{i\sqrt{m^2 + \rho^2 (t \cosh \eta - x^1 \sinh \eta)}} e^{-\eta} e^{-\delta \eta} \] (3.6)

with $0 < \delta < 1$. Thus we find

\[ I^\delta(t, t) = \frac{1}{2} \int_{-\infty}^{\infty} d\eta e^{i\sqrt{m^2 + \rho^2 \eta}} e^{-\delta \eta} = \frac{1}{2} \int_{0}^{\infty} \frac{da}{\alpha^{1-\delta}} e^{a t \sqrt{m^2 + \rho^2 \alpha}} \] (3.7)

The remaining integration over $\rho$ is simple

\[ \int_{0}^{\infty} d\rho \rho^{s+1} J_s(\rho t) = \frac{1}{(m^2 + \rho^2)^{\delta/2}} \left( \frac{m}{\rho} \right)^{1-\delta/2} K_{\delta/2-1-\epsilon}(m \rho), \] (3.8)

for $\delta > 1/2$, where $\epsilon = (d - 3)/2$.

Now we would like to remove our regularization parameter $\delta$ by taking the limit $\delta \to 0$, but this is allowed only if $\epsilon < -1/2$ and this is the reason why we have started with the dimensional regularization $d < 3$. Thus in $d$ dimensions we have the ET Wightman function

\[ \Delta_d^+(t, \rho; m^2) = \lim_{\delta \to 0} \frac{(2\pi)^{1+\epsilon}}{(2\pi)^d} \left( \frac{m}{\rho} \right)^{1+\epsilon-\delta/2} \frac{\Gamma(\delta)}{\Gamma(\delta/2)} e^{\text{sgn}(t) \delta \pi/2} K_{\delta/2-1-\epsilon}(m \rho) \] (3.9)

At last, we may go to physical three space dimensions and obtain for any space-like distances $x^2 < 0$

\[ \Delta^+(x^2, m^2) = \frac{m}{4\pi^2 \rho} K_1(m \rho). \] (3.10)
Though our final result has the commonly known form (2), we stress that we have encountered in our ET calculations similar problems to those, which appear within the LF formulation. The cylinder coordinates clearly single out one space direction, thus two light fronts $x^\pm = 0$ appear naturally. In contrast, when one uses the spherical coordinates all space direction are equivalent and no LF singularity arises.
References

[1] Dirac P A M 1949 Rev. Mod. Phys. 21 392.
[2] Burkardt M 1996 Adv. Nucl. Phys. 23 1
[3] Brodsky S J, Pauli H-C and Pinsky S, 1998 Phys.Rept. 301 299-486.
[4] Heinzl T 2001 Lect. Notes Phys 572 55
[5] Nakanishi N, Yabuki H, 1977 Lett. Math. Phys. 1 371
[6] Nakanishi N and Yamawaki K 1977 Nucl. Phys. B 122 15.
[7] Pauli H-C and Brodsky S J 1985 Phys. Rev. D32 1993, 2001.
[8] Tsujimaru S and Yamawaki K 1998 Phys. Rev. D 57 4942.
[9] Ullrich P and Werner E 2005 arXiv:hep-th/0503176.
[10] Pauli W and Villars F 1949 Rev. Mod. Phys. 21 434.
[11] Stoilov M N 1997 arXiv:hep-th/9706106.
[12] Bateman H 1953 Higher Transcendental Functions vol 2 (McGraw-Hill Book Company, Inc.)
[13] Przeszowski J A 2005 J. Phys. A: Math. Gen. (in press), arXiv:hep-th/0505221.
[14] Nakanishi N 1976 Comm. Math. Phys. 48 97.