Renormalization of crossing probabilities in the dilute Potts model

P. Rigas

Abstract

A recent paper due to Duminil-Copin and Tassion from 2019 introduces a novel argument for obtaining estimates on horizontal crossing probabilities of the random cluster model, in which a range of four possible behaviors is established. To apply the novel renormalization of crossing probabilities that the authors propose can be studied in other models of interest that are not self-dual, we collect results to formulate vertical and horizontal strip, and renormalization, inequalities for the dilute Potts model, whose measure is obtained from the high temperature expansion of the loop $O(n)$ measure supported over the hexagonal lattice in the presence of two external fields $h, h'$. The dilute Potts model was originally introduced in 1991 by Nienhuis and is another model that enjoys the RSW box crossing property in the Continuous Critical phase, which is one of the four possible behaviors that the model is shown to enjoy. Through a combination of the Spatial Markov Property (SMP) and Comparison between Boundary Conditions (CBC) of the high-temperature spin measure, four phases of the dilute Potts model can be analyzed, exhibiting a class of boundary conditions upon which the probability of obtaining a horizontal crossing is significantly dependent. The exponential factor that is inserted into the Loop $O(n)$ model to quantify properties of the high-temperature phase is proportional to the summation over all spins, and the number of monochromatically colored triangles over a finite volume, which is in exact correspondence with the parameter of a Boltzmann weight introduced in Nienhuis’ 1991 paper detailing extensions of the $q$-state Potts model. Asymptotically, in the infinite volume limit we obtain strip and renormalization inequalities that provide conditions on the constants $1 - c$ and $c$ that are known from RSW results that have been classically obtained for Voronoi and Bernoulli percolation. Applications of two phases of the dilute Potts model are provided following arguments for strip and renormalization inequalities.

1 Introduction

1.1 Overview

Russo-Seymour-Welsh (RSW) theory provides estimates regarding the crossing probabilities across rectangles of specified aspect ratios, and was studied by Russo, and then by Seymour and Welsch on the square lattice, with results specifying the finite mean size of percolation clusters [23], in addition to a relationship that critical probabilities satisfy through a formalization of the sponge problem [24]. With such results, other models in statistical physics have been examined, particularly ones exhibiting sharp threshold phenomena [1,7] and continuous phase transitions [13], with RSW type estimates obtained for Voronoi percolation [27], critical site percolation on the square lattice [28], the Kostlan ensemble [2], and the FK Ising model [9], to name a few.

RSW arguments classically rely on model self-duality, which is enjoyed by neither the random cluster nor dilute Potts models. With an adaptation of the 2019 renormalization of crossing probabilities argument due to Duminil-Copin and Tassion [14], crucial modifications for renormalizing crossing probabilities in the dilute Potts model arise not only from spin analogues of random cluster SMP and CBC properties that intrinsically capture the model’s dependence with the hexagonal lattice (introduced in 3.2), but also in arguments for proving that PushPrimal, PushDual, PushPrimal Strip and PushDual Strip conditions hold. The constants provided in the lower bounds of PushPrimal and PushDual conditions additionally impact arguments throughout Sections 7 and 8 surrounding strip and renormalization inequalities, allowing for a classification of four phases of behavior of the dilute Potts model (introduced

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1 **Keywords**: Statistical mechanics, Russo-Seymour-Welsh, crossing probabilities, self-duality, random cluster model, loop $O(n)$ model, symmetric domains, six-vertex model, crossing estimates, high-temperature phase.

2 **MSC Class**: 60K35; 82B02
in 3.3). Although classical RSW arguments are successful for analyzing self-dual models, previous arguments to obtain RSW estimates are not applicable to the dilute Potts model (in correspondence with the high-temperature loop $O(n)$ model in presence of two external fields), that has been studied extensively by Nienhuis [15,19,20] who not only conjectured that the critical point of the model should be $1/\sqrt{2 + \sqrt{2 - n}}$ for $0 \leq n < 2$, but also has provided results for the $O(n)$ model on the honeycomb lattice [22] which has connective constant $\sqrt{2 + \sqrt{2}}$ [12]. It is also known that the loop $O(n)$ model, a model for random collections of loop configurations on the hexagonal lattice, exhibits a phase transition with critical parameter $1/\sqrt{2 + \sqrt{2 - n}}$, in which subcritically the probability of obtaining a macroscopic loop configuration of length $k$ decays exponentially fast in $k$, while at criticality the probability of obtaining infinitely many macroscopic loop configurations, also of length $k$, and centered about the origin is bound below by $c$ and above by $1 - c$ for $c \in (0, 1)$ irrespective of boundary conditions [8]. The existence of macroscopic loops in the loop $O(n)$ model has also been proved in [3] with the XOR trick.

In [14], Duminil-Copin & Tassion proposed alternative arguments to obtain RSW estimates for models that are no self-dual at criticality. The novel quantities of interest in the argument involve renormalization inequalities, which in the case of Bernoulli percolation can be viewed as a coarse graining argument, as well as the introduction of strip densities which are quantities defined as a limit supremum over a real parameter $\alpha$. Ultimately, the paper proves RSW estimates for measures with free or wired boundary conditions in subcritical, supercritical, critical discontinuous & critical continuous cases, with applications of the two theorems relating to the mixing times of the random cluster measure, for systems undergoing discontinuous phase transitions [14,18]. Near the end of the introduction, the authors mention that potential generalizations of their novel renormalization argument can be realized in the dilute Potts model studied by Nienhuis which is equivalent to the loop $O(n)$ model, a model conjectured to exist in the same universality class as the spin $O(n)$ model.

With regards to the loop $O(n)$ model, previous arguments have demonstrated that the model undergoes a phase transition by making use of Smirnov’s parafermionic observable, originally introduced to study conformal invariance of different models in several celebrated works [11,25,26]. As a holomorphic function, the discrete contour integral of the observable vanishes for specific choice of a multiplicative parameter to the winding term in the power of the exponential. Under such assumptions on $\sigma$, Duminil-Copin & coauthors prove exponential decay in the loop $O(n)$ model from arguments relating to the relative weights of paths and a discrete form of the Cauchy Riemann equations [8]. Historically, disorder operators share connections with the parafermionic observable and have been studied to prove the existence of phase transitions through examination of the behavior of expectations of random variables below, and above, a critical point [11,16], while other novel uses of the parafermionic observable have been introduced in [10]. It is of interest to formulate RSW arguments for the six-vertex model which is in preparation for next year in another paper.

### 1.2 Organization of results

We define the models of interest to introduce Spin and Loop configurations, from which modifications to the SMP and CBC conditions (defined in future sections) yields bounds for crossings across symmetric domains. To introduce such arguments in Sections 4 & 5 with the proof of Theorem 1 and Lemma 9*, in Section 3 we define the Loop $O(n)$ measure, from which the dilute Potts measure in the presence of two external fields $h$ and $h'$. In Section 6, we apply the $\mu$ homomorphism to lower bound vertical crossings with horizontal crossings, from which PushPrimal & PushDual conditions are introduced in Section 7 to prove horizontal and vertical strip density formulas. In Sections 8 & 9, we characterize two behaviors of the quadrichotomy, finalizing our characterization of the discontinuous-continuous phases of the quadrichotomy behavior by making use of the parafermionic observable which has already been manipulated to characterize properties of the phase transition for the random cluster model [5,6,13]. In Section 9, we obtain classical results for the dilute Potts measure in each of the four regimes of behavior.

### 2 Background

To execute steps of the renormalization argument in the hexagonal case, we introduce quantities to avoid making use of self duality arguments. For $G = (V, E)$, $n \geq 1$ and the strip $\mathbb{R} \times [-n, 2n] \equiv S_n \subset G$, let $\phi_{S_n}^\xi$, for $\xi \in \{0, 1, 0/1\}$, respectively denote the measures with free, wired and Dobrushin boundary conditions
in which all vertices at the bottom of the strip are wired. From such measures on the square lattice, several planar crossing events are defined in order to obtain RSW estimates for all four parameter regimes \(\text{subcritical, supercritical, discontinuous \& continuous critical}\), including analyses of the intersection of crossing probabilities across a family of non disjoint rectangles \(\mathcal{R}\), each of aspect ratio \([0, \rho n] \times [0, n]\) for \(\rho > 0\), to obtain crossings across long rectangles \(\text{à la FKG inequality}\), three arm events which establish lower bounds of the crossing probabilities across \(\mathcal{R}\) under translation and reflection invariance of \(\phi\), in addition to horizontal rectangular crossings which are used to prove renormalization inequalities through use of PushPrimal \& PushDual relations. To begin, we define the horizontal and vertical crossing strip densities.

**Definition 1 ([14], Theorem 2, Corollary 3):** The strip density corresponding to the measure across a rectangle \(\mathcal{R}\) of aspect ratio \([0, \alpha n] \times [-n, 2n]\) with free boundary conditions is of the form,

\[
p_n = \limsup_{\alpha \to \infty} \left( \phi^{0}_{[0,\alpha n] \times [-n,2n]} \left[ \mathcal{H}_{[0,\alpha n] \times [0,n]} \right] \right)^{\frac{1}{\alpha}},
\]

where \(\mathcal{H}\) denotes the event that \(\mathcal{R}\) is crossed horizontally, whereas for the measure supported over \(\mathcal{R}\) with wired boundary conditions, the crossing density is of the form,

\[
q_n = \limsup_{\alpha \to \infty} \left( \phi^{1}_{[0,\alpha n] \times [-n,2n]} \left[ \mathcal{V}_c^{c}_{[0,\alpha n] \times [0,n]} \right] \right)^{\frac{1}{\alpha}},
\]

where \(\mathcal{V}_c\) denotes the complement of a vertical crossing across \(\mathcal{R}\).

The strip and renormalization inequalities provided in this section are dependent on different quantities for \(\pm\) spin configurations rather than the corresponding inequalities for the random cluster model which only depend on the cluster weight \(q\). Besides the definition of the strip densities \(p_n\) and \(q_n\), another key step in the argument involves inequalities relating \(p_n\) and \(q_n\). The statement of the Lemma below holds under the assumption that the planar random cluster model is neither in the subcritical nor supercritical phase.

**Lemma 1 ([14], Lemma 12)** There exists a constant \(C > 0\) such that for every integer \(\lambda \geq 2\), and for every \(n \in 3\mathbb{N}\),

\[
p_{3n} \geq \frac{1}{\lambda \varepsilon} q_n^{3 + \frac{2}{n}},
\]

while a similar inequality holds between horizontal and the complement of vertical crossing probabilities of the complement \(\mathcal{V}_c\) across \(\mathcal{R}\), which takes the form,

\[
q_{3n} \geq \frac{1}{\lambda \varepsilon} p_n^{3 + \frac{2}{n}}.
\]

Finally, we introduce the renormalization inequalities.

**Lemma 2 ([14], Lemma 15)** There exists \(C > 0\) such that for every integer \(\lambda \geq 2\) and for every \(n \in 3\mathbb{N}\),

\[
p_{3n} \leq \lambda^C p_n^{3 - \frac{2}{n}} \quad \& \quad q_{3n} \leq \lambda^C q_n^{3 - \frac{2}{n}}.
\]

To readily generalize the renormalization argument to the dilute Potts model, we proceed in the spirit of [14] by introducing hexagonal analogues of the crossing events discussed at the beginning of the section.
3 Towards hexagonal analogues of crossing events from the random
cluster renormalization argument

3.1 Loop O(n) measure, hexagonal lattice crossing events

The Gibbs measure on a random configuration $\sigma$ in the loop $O(n)$ model is of the form,

$$P^\xi_{\Lambda,x,n}(\sigma) = \frac{x^{\epsilon(\sigma)}Y^l(\sigma)}{Z^\xi_{\Lambda,x,n}},$$

(Loop measure)

where $\epsilon(\vert\varepsilon\vert)$ denotes the number of edges, $\epsilon(l)$ the number of loops, $\Lambda \subset \mathbf{H}$, $\xi \in \{0,1,0/1\}$ and $Z^\xi_{\Lambda,x,n}$ is the partition function which normalizes $P^\xi_{\Lambda,x,n}$ so that it is a probability measure. In particular, we restrict the parameter regime of $x$ to that of [30], in which the loop $O(n)$ model satisfies the strong FKG lattice condition and monotonicity through a spin representation measure albeit $P^\xi_{\Lambda,x,n}$ not being monotonic. By construction, $P^\xi_{\Lambda,x,n}$ is invariant under $\frac{2\pi}{\pi}$ rotations. Through a particular extension for $n \geq 2$ of the spin representation of $P^\xi_{\Lambda,\sigma(e),\sigma(l)}$, the measure on spin configurations $\sigma' \in \Sigma(G,\tau)$ is of the form,

$$\mu_{G,x,n}(\sigma') = \frac{n^{k(\sigma')x(\sigma')}}{Z^G_{G,x,n}} \exp\left( h^r(\sigma') + h^r(\sigma') \right),$$

(Spin Measure)

where $\tau \in \{-1,+1\}^T$, $\Sigma(G,\tau)$ is the set of spin configurations coinciding with $\sigma'$ outside of $G$, $r(\sigma') = \sum_{u \in G} \sigma'_u$ is the summation of spins inside $G$, $r'(\sigma') = \sum_{\{u,v,w\} \in G} \sigma'_u \sigma'_v \sigma'_w$ is the difference between the spins of monochromatic triangles, and $Z^G_{G,x,n}$ is the partition function which makes $\mu^G_{G,x,n}$ a probability measure. The extension enjoys translation invariance, a weaker form of the spatial/domain Markov property that will be mentioned in Section 5.1, comparison between boundary conditions that is mentioned in Section 3.2, & FKG for $n \geq 1$ and $nx^2 \leq 1$. The dual measure of $\mu^G_{G,x,n}$ is $\mu^{G^*}_{G,x,n}$. Simply put, the superscripts above $\mu$ indicate whether the pushforward of a horizontal or vertical crossing event under the measure is under free, wired, or mixed boundary conditions.

Additionally, the model can be placed into correspondence with the dilute Potts model, originally characterized by occupied, and vacant, faces of $H$. The exponential factor introduced to characterize high temperature behavior of the Loop $O(n)$ model in the Spin Measure equality is in direct correspondence with the dilute Potts model, whose Boltzmann weight is, from [30],

$$W_{ij} \equiv \prod_{i \sim j} \left( 1 - t_it_j + t_it_j \delta_{s_j,s_k} \right) \exp\left( K_1 \sum_i t_i + K_2 \sum_{i \sim j} t_it_j + K_3 \sum_{i \sim j \sim k} t_it_j t_k \right),$$

where the quantities in the nearest-neighbor product above include the occupation number $t_j$, which is either equal to 0 or 1, and the spins $s_j$ and $s_k$ indicated in the nearest neighbor product of $i$ and $j$ can take values between 1, $\cdots$, $q$ corresponding to $q$-state Potts model spins. In the power of the exponential term, we consider occupancy numbers across sites, edges, and faces of triangles, respectively, with $K_1, K_2, K_3$ real constants.

To obtain boundary dependent RSW results on $H$ in all 4 cases, we identify crossing events in the planar renormalization argument in addition to difficulties associated with applying the planar argument to the push forward of similarly defined horizontal and vertical crossing events under $\mu^G_{G,x,n}$ on $(T)^* = H$. In what follows, we describe all planar crossing events in the argument.

First, planar crossing events across translates of horizontal crossings across short rectangles of equal aspect ratio are combined to obtain horizontal crossings across long rectangles, through the introduction of a lower bound to the probability of the intersection that all short rectangles are simultaneously crossed.
horizontally with FKG. On $\mathbf{H}$, the probability of the intersection of horizontal crossing events of first type can be readily generalized to produce longer horizontal crossings from the intersection of shorter ones, through an adaptation of $[14, \text{Lemma 9}]$.

Second, three arm events which determine whether two horizontal crossings to the top of a rectangle of aspect ratio $[0,n] \times [0,\rho n]$ intersect. Planar crossings of second type create symmetric domains over which the conditional probability of horizontal crossings in the symmetric domain can be determined, which for the renormalization argument rely on comparison between random cluster measures with free and wired boundary conditions. For random cluster configurations, comparison between boundary conditions is established in how the number of clusters in a configuration is counted. Comparison between boundary conditions applies to $\mu_{G,x,n}^\tau$ from $[8]$, with hexagonal symmetric domains enjoying $2\pi/3$ symmetry.

Crossing events with wired boundary conditions, of third type induce wired boundary conditions within close proximity of vertical crossings in planar strips. Long horizontal crossings are guaranteed through applications of FKG across dyadic translates of horizontal crossings across shorter rectangles. For hexagonal domains, modifications to planar crossings of first type permit ready generalizations of third type planar crossings.

Fourth, planar horizontal crossing events of fourth type across rectangles establish relations between the strip densities $p_n$ & $q_n$ (Lemma 1). Finally, planar crossing events satisfying PushPrimal & Push-Dual conditions prove Lemma 2.

### 3.2 Comparison of boundary conditions & relaxed spatial markovianity for the $n \geq 2$ extension of the loop $O(n)$ measure

For suitable comparison of boundary conditions in the presence of external fields $h, h'$, the influence of boundary conditions from the fields on the spin representation amount to enumerating configurations differently for wired and free boundary conditions than for the random cluster model in $[14]$. In particular, modifications to comparison between boundary conditions and the spatial Markov property.

The modifications entail that an admissible symmetric domain $\text{Sym}$ inherit boundary conditions from partitions on the outermost layer of hexagons along loop configurations (see Figures 1-3 in later section for a visualization of crossing events from the argument). Through distinct partitions of the +/− assignment on hexagons on the outermost layer to the boundary, appearing in arguments for symmetric domains appearing in 5.1 - 5.4.

**Corollary** ([8, Corollary 10], comparison between boundary conditions for the Spin measure): Consider $G \subset \mathbf{T}$ finite and fix $(n, x, h, h')$ such that $n \geq 1$ and $nx^2 \leq \exp(-|h'|)$. For any increasing event $A$ and any $\tau \leq \tau'$,

$$
\mu_{G,n,x,h,h'}^\tau[A] \leq \mu_{G,n,x,h,h'}^{\tau'}[A] \quad (\mathcal{S} – \text{CBC})
$$

Altogether, modifications to comparison of boundary conditions and the spatial Markov property between measurable spin configurations for $\mu$ is also achieved. We recall the (CBC) inequality for the random cluster model, and for the loop model make use of an "analogy" discussed in [8], in which we associate wired boundary conditions to the + spin, and free boundary conditions to the − spin over $\mathbf{T}$. Specifically, for boundary conditions $\xi, \psi$ distributed under the random cluster measure $\phi$, the measure supported over $G$ satisfies

$$
\phi_{G}^\xi[A] \leq q^{\max\{k_\xi(\omega) - k_\psi(\omega)\omega\} - \min\{k_\xi(\omega) - k_\psi(\omega)\}} \phi_{G}^\psi[A].
$$

A special case of $(\mathcal{S} – \text{CBC})$ property above takes the form,

$$
\mu_{H}^\tau[A'] \leq n^{k_\sigma - k_\sigma'} \exp\left(h(r_{\sigma} - r_{\sigma'}) + \frac{h'}{2}(r_{\sigma'}(\sigma') - r_{\sigma}(\sigma'))\right) \mu_{H}^{\tau'}[A'], \quad (\mathcal{S} – \text{CBC})
$$
Another property that the dilute Potts measure satisfies, for finite volumes \( \mathcal{I} \subset \mathcal{O} \),

\[
\mu^{\tau}_{\mathcal{O}} [ \cdot | \sigma^{\tau}_{\mathcal{O}} \equiv \sigma^{\tau}_{\mathcal{I}} ] \text{ for } \mathcal{I} \cap \mathcal{O} \equiv \mu^{(\tau',\tau)}_{\mathcal{O}} [ \cdot ] . \tag{S – SMP}
\]

where the exponential factor in front of the pushforward in the upper bound results from the difference between the number of monochromatically colored triangles in the configuration distributed under the Spin Measure, the edge weight associated with \( x \), and the number of connected components \( k(\sigma) + 1 \), respectively with boundary conditions \( \tau \) and \( \tau' \). The multiplicative factor arises from comparisons between the Spin Measure and the Random Cluster model measure, particularly by associating the summation over all spins in Spin Measure configurations with the ratio of the number of open edges to the number of closed edges in an FK percolation configuration, the number of connected components in a spin configuration under the loop \( O(n) \) model with the number of clusters in the Random cluster model, and also, the edge weights of \( x \) of spin configurations under the loop \( O(n) \) model with the cluster weights of \( q \) in the Random Cluster Model.

A special case of the inequality above will be implemented several times throughout the renormalization argument, stating,

\[
\mu^{\tau}_{\mathcal{H}} [ A' ] \leq n^{k_{\tau}(\sigma) - k_{\tau'}(\sigma)} \exp(h) \mu^{\tau'}_{\mathcal{H}} [ A' ] ,
\]

which will be introduced when applying \((S – SMP)\), in 5.4.3 and later in 7.1 and 7.2.1. The special case of the multiplicative factor above represents the difference in the number of clusters that are counted under boundary conditions \( \tau, \tau' \), in addition to the corresponding edge weights \( x \) under each boundary condition. We denote the modified properties for spin representations \( S \) with \((S – CBC)\) and \((S – SMP)\). Besides such modifications, \((MON)\) from [14] directly applies, and will be used repeatedly. To probabilistically capture the dependence of the high-temperature measure on the first external field, another special case of \((S – SMP)\) is,

\[
\exp\left( \#\{ \mathcal{H} : 1 \leq i \sim j \leq 6, \sigma_v \in \{\pm 1\}^{\mathcal{H}}, 1_{\sigma_{v_i} = \sigma_{v_j} = 1} \} \right) ,
\]

where \( \mathcal{H} \) is a hexagon. Finally, positive association for increasing events \( A \) and \( B \) is an inequality of the form,

\[
\mu^{\tau}_{\mathcal{H}} [ A \cap B ] \geq \mu^{\tau}_{\mathcal{H}} [ A ] \mu^{\tau}_{\mathcal{H}} [ B ] \tag{FKG}
\]

We refer to free boundary conditions under the Spin measure which represents – boundary conditions, and wired boundary conditions for + boundary conditions. The \( q^k \) analogue from the random cluster model for the high-temperature Spin measure will enter into the novel renormalization argument at several points, stating,

- **Lemma 9**, in which \((S – SMP)\) will be repeatedly used to compare boundary conditions between crossings across the second or third edge of a hexagon, and boundary conditions for crossings across symmetric regions Sym,

- **Corollary 11**, in which \((S – SMP)\) will be used to bound the pushfoward of horizontal crossings under wired boundary conditions, which in light of the homeomorphism \( f \) in 4.1, yields a corresponding bound for the pushforward of a vertical crossing under free boundary conditions,

- **Lemma 1**, in which an application of \((S – SMP)\) and \((MON)\) yield a lower bound for the probability of a horizontal crossing under free boundary conditions with a probability of a horizontal crossing under wired boundary conditions,
• **Lemma 2**, in which a modification to the lower bound obtained in the proof of **Lemma 1** is applied to obtain a lower bound for the probability of a horizontal crossing under wired boundary conditions with the probability of a vertical crossing under free boundary conditions,

• **Quadrichotomy proof**, in which the crossing events from previous results are compared to obtain the standard box crossing estimate that the Gibbs measure on loop configurations satisfies, per conditions of the Continuous Critical phase of the dilute Potts model provided in **Theorem 1**. [8].

### 3.3 Description of results

The result presented for the loop $O(n)$ model mirrors the dichotomy of possible behaviors, in which the standard box crossing estimate reflects the influence of boundary conditions on the nature of the phase transition, namely that the transition is discontinuous, from the discontinuous critical case. To prove the subcritical & supercritical cases, the generalization to the dilute Potts model will make use of planar crossing events of first and second type, while crossing events of third and fourth type proves the remaining continuous & discontinuous critical cases. We denote the vertical strip domain $S_T$ with $T$ hexagons, $S_{T,L}$ the finite domain of $S_T$ of length $L > 0$, and any regular hexagon $H_j \subset S_T$ with side $j$ [12]. The strip densities $p_n^\delta$ and $q_n^\delta$ are defined in 7.

**Theorem 1** *(μ homeomorphism):* For $L \in [0,1]$, there exits an increasing homeomorphism $f_L$ so that for every $n \geq 1$, where $\mathcal{H}_H \equiv \mathcal{H}$ and $\mathcal{V}_H \equiv \mathcal{V}$ denote the horizontal and vertical crossings across a regular hexagon $H$, $\mu(\mathcal{H}) \geq f(\mu(\mathcal{V}))$.

**Theorem 2** *(hexagonal crossing probabilities):* For $x \leq \frac{1}{\sqrt{3}}$, aspect ratio $n$ of a regular hexagon $H \subset S_T$, $c > 0$, and horizontal crossing $\mathcal{H}$ across $H$, estimates on crossing probabilities with free, wired or mixed boundary conditions satisfy the following criterion in the following 4 possible behaviors.

- **Subcritical**: For every $n \geq 1$, under wired boundary conditions, $\mu_{G,x,n}^1[H] \leq \exp(-cn)$,
- **Supercritical**: For every $n \geq 1$, under free boundary conditions, $\mu_{G,x,n}^0[H] \geq 1 - \exp(-cn)$,
- **Continuous Critical** *(Russo-Seymour-Welsh property)*: For every $n \geq 1$, independent of boundary conditions $\tau$, $c \leq \mu_{G,x,n}^1[H] \leq 1 - c$,
- **Discontinuous Critical**: For every $n \geq 1$, $\mu_{G,x,n}^2[H] \geq 1 - \exp(-cn)$ for free boundary conditions, while $\mu_{G,x,n}^0[H] \leq \exp(-cn)$ for wired boundary conditions.

As in the proofs for each set of inequalities located in Section 7 and Section 9.1, we set $p_n^\delta \equiv p_\delta$ and $q_n^\delta \equiv q_\delta$ for simplicity. Each one of the estimates below before letting $\rho \to \infty$ is achieved by concluding the argument with the $q^k$ “analogy” mentioned on the previous page. The $(\mathcal{S}$ CBC) leads to similar estimates for the Spin measure. In the statements below, the factor Stretch appearing in the strip density and renormalization inequalities denotes some nonzero factor that a regular hexagon is “stretched” by along in the vertical degree of freedom.

**Lemma 1** *(7, hexagonal strip density inequalities):* In the **Non(Subcritical)** and **Non(Supercritical)** regimes, for every integer $\lambda \geq 2$, and every $n \in \text{Stretch N}$, there exists a positive constant $C$ satisfying,

$$P \text{ Stretch } n \geq \frac{1}{\lambda^C} \left(q \text{Stretch } n\right)^{\text{Stretch} + \frac{\text{Stretch}}{\lambda}},$$

while a similar upper bound for vertical crossings is of the form,

$$Q \text{ Stretch } n \geq \frac{1}{\lambda^C} \left(p \text{Stretch } n\right)^{\text{Stretch} + \frac{\text{Stretch}}{\lambda}}.$$
With the strip densities for horizontal and vertical crossings, we state closely related renormalization inequalities.

**Lemma 2** (9, hexagonal renormalization inequalities): In the **Non(Subcritical)** and **Non(Supcritical)** regimes, for every integer \( \lambda \geq 2 \), and every \( n \in \text{Stretch } \mathbb{N} \), there exists a positive constant \( C \) satisfying,

\[
p_{\text{Stretch } n} \geq \lambda^C \left( p_{\text{Stretch } n}\right)^{\text{Stretch } - \frac{n}{\lambda}} \quad \& \quad q_{\text{Stretch } n} \geq \lambda^C \left( q_{\text{Stretch } n}\right)^{\text{Stretch } - \frac{n}{\lambda}}.
\]

### 4 Proof of Theorem 1 & Lemma 9* preparation

To prove Theorem 1, we introduce 6-arm crossing events, from which symmetric domains will be crossed with good probability. The arguments hold for the \( n \geq 2 \) extension measure with free, wired or mixed boundary conditions. Previous use of such domains has been implemented to avoid using self duality throughout the renormalization argument [1,13]. Although more algebraica characterizations of fundamental domains on the hexagonal, and other, lattices exist [4], we focus on defining crossing events, from which we compute the probability conditioned on a path \( \Gamma \) crossing the symmetric region.

#### 4.1 Existence of the homomorphism \( \mu \)

The increasing homeomorphism is shown to exist with the following.

**Proposition 8** (homeomorphism existence): For any \( L > 0 \), there exists \( c_0 = c_0(L) > 0 \) so that for \( nL \geq 1 \),

\[
\mu^0[\mathcal{H}] \geq c_0 \mu^1[\mathcal{V}] \frac{1}{c_0} \iff 1 - \mu^1[\mathcal{V}] \geq c_0 \left( 1 - \mu^0[\mathcal{H}] \right) \frac{1}{c_0} \iff \left( 1 - \mu^1[\mathcal{V}] \right) \leq c_0 \left( 1 - \mu^0[\mathcal{H}] \right),
\]

where the final inequality is equivalent to,

\[
\mu^0[\mathcal{H}] \leq 1 - \frac{1}{c_0} \left( 1 - \mu^1[\mathcal{V}] \right),
\]

because by complementarity, \( \mu^0[\mathcal{H}] + \mu^1[\mathcal{V}] = 1 \). The existence of a homeomorphism satisfying \( \mu(\mathcal{H}) \geq f(\mu(\mathcal{V})) \) is equivalent to \( 1 - \mu(\mathcal{V}) \geq f(\mu(\mathcal{V})) \), implying from the upper bound,

\[
1 - \frac{1}{c_0} \left( 1 - \mu^1[\mathcal{V}] \right) = \frac{c_0^\mu - 1 + \mu^1[\mathcal{V}]}{c_0^{-\mu}} = \frac{c_0^\mu - 1}{c_0^{-\mu}} = 1 - c_0^{-\mu} + \frac{c_0^{-\mu}}{c_0^\mu} = 1 - c_0^{-\mu} + c_0^{-\mu} \mu^1[\mathcal{V}].
\]

The homeomorphism can be read off from the inequality, hence establishing its existence. \( \Box \)
4.2 Crossing events for Lemma 9*

For a fixed ordering of all 6 edges that enclose any $H_j \subset S_{T,L}$, \{1$_j$, 2$_j$, 3$_j$, 4$_j$, 5$_j$, 6$_j$\}, crossing events $C$ to obtain hexagonal symmetric domains with rotational and reflection symmetry will be defined. To obtain generalized regions from their symmetric counterparts in the plane from [14], we make use of comparison between boundary condition with the $n \geq 2$ extension measure. For $\mu$, we are capable of readily proving a generalization of the union bound with the following prescription.

First, we define 5-armed crossing events across the box $H_j \in S_{T,L}$, from which families of crossing probabilities across a countable number of domains are introduced.

**Definition 4.2 (crossings events across the hexagonal box)** Fix $x = \{k\}$. From a partition of $x$ into equal $k$ subintervals, each of length $\frac{x}{k}$, we define a countable family of crossing events from the partition $S_j$ of 1$_j$ to the corresponding topmost edge 4$_j$ of $H_j$, as well as crossing events from $S_j$ to all remaining edges of $H_j$. We consider crossing events across finite volumes arranged as follows,

- From our choice of 1$_j$, we horizontally position the line $L \equiv [0, \delta] \times \{0\} \subset H$ for arbitrary $\delta$. We denote the horizontal translate $H_{j+\delta'}$ of $H_j$ along $L$ by $\delta'$ where $\delta' << \delta$.

- From crossing events across a series of any 3 hexagons $\{H_{j-\delta'}, H_j, H_{j+\delta'}\}$, we additionally introduce crossing events across translates by stipulating that the crossing starting from the partition of $L$ into $k$ subintervals to any of the remaining edges $\{2, 3, 4, 5, 6\}$ of $H$ occur in other regions, namely $H_{j-\delta'} \cap H_j$ and $H_j \cap H_{j+\delta'}$ (such a series of finite volumes is provided in Figure 2).

With the properties of the crossings provided above, we conclude by sending $L \to \infty$, generalizing the crossing events on $S_T$ in the weak limit along the infinite hexagonal strip.

Differences emerge in the proofs for the dilute Potts model in comparison to those of the random cluster model, not only in the encoding of boundary conditions for $\mu$ but also in the construction of the family of crossing probabilities, and the cases that must be considered to prove the union bound. We gather these notions below; denote the quantities corresponding to the partition $S_j \subset 1_j$ with the following events,

$$
\begin{align*}
\mathcal{C}_{2_j} &= \{S_j \xleftarrow{H_j+\delta'} 2_{j-\delta'}\}, \quad \mathcal{C}_{3_j} = \{S_j \xleftarrow{H_j+\delta'} 3_{j-\delta'}\}, \quad \mathcal{C}_{4_j} = \{S_j \xleftarrow{H_j} 4_{j}\}, \quad \mathcal{C}_{5_j} = \{S_j \xleftarrow{H_{j-\delta'}} 5_{j+\delta'}\}, \\
\mathcal{C}_{6_j} &= \{S_j \xleftarrow{H_{j-\delta'}} 6_{j+\delta'}\},
\end{align*}
$$

as well as the following crossing events across the left and right translates of $H_j$,

$$
\begin{align*}
\mathcal{C}_{2_j} &= \{S_j \xleftarrow{H_j+\delta'} 2_{j+\delta'}\} \setminus \mathcal{C}_{2_j}, \quad \mathcal{C}_{3_j} = \{S_j \xleftarrow{H_j+\delta'} 3_{j+\delta'}\} \setminus \mathcal{C}_{3_j}, \\
\mathcal{C}_{5_j} &= \{S_j \xleftarrow{H_{j-\delta'}} 5_{j+\delta'}\} \setminus \mathcal{C}_{5_j}, \quad \mathcal{C}_{6_j} = \{S_j \xleftarrow{H_{j-\delta'}} 6_{j+\delta'}\} \setminus \mathcal{C}_{6_j}.
\end{align*}
$$

Along with the right and left translates of $H_j$, we can easily Before proceeding to make use of the 6-arm events to create symmetric domains for Lemma 9* (presented below), we prove 8* below.

**Proof of Proposition 8**. Let $C_j = \{S_j \xleftarrow{H_j \cup H_{j+\delta'}} S_{j+\delta'} \cup S_{j+2\delta'}\}$.
Figure 1: \(0 \leq n < 2\) construction of Sym from macroscopic +/− crossings induced by \(C_2\) and \(C_{2j+2\delta'}\). Loop configurations with distribution \(P\), with corresponding +/− random coloring of faces in \(H\) with distribution \(\mu\), are shown with purple \(\gamma_1\) and red \(\gamma_2\). Each configuration intersects \(2_j\), with crossing events occurring across the box \(H_j\) and its translate \(H_{j+2\delta'}\). Under translation invariance of the spin representation, different classes of Sym domains are produced from the intersection of \(\gamma_1\) and \(\gamma_2\), as well as the connected component of an intersection \(x_I\) incident to \(2_j\), which is shown above the second intersection of the red connected components of \(\gamma_2\). From one such arrangement of \(\gamma_1\) and \(\gamma_2\), a magnification of the symmetric domain is provided, illustrating the contours of Sym which are dependent on the connected components of the outermost \(\gamma_2\) path above \(x_{\gamma_1,\gamma_2}\), while the connected components of \(\gamma_2\) below \(x_{\gamma_1,\gamma_2}\) determine the number of connected components below the intersection. Across \(2_j\), one half of Sym is rotated to obtain the other half about the crossed edge. From paths of the connected components of each configuration, Sym is determined by forming the region from the intersection of the connected components of \(\gamma_1\) and \(\gamma_2\) in the magnified region. We condition on the number of connected components of each path by stipulating that they are equal to form two connected sets along the incident boundary to Sym. At the point of intersection between the red and purple +/− spin configurations, the connected component associated with \(x_I\) determines half of the lowest side of Sym. The region allows for the construction of identical domains under \(C_{3\delta}\) and \(C_{3\delta + 2\delta'}\). Connected components are only shown in the vicinity of \(2_j\) for the identification of boundaries of Sym, running from the intersection of \(\gamma_2\) at the cusp of \(2_j\) and \(3_j\), and from two nearby intersections of \(\gamma_1\) with \(2_j\). The points of intersection of the purple connected components of \(\gamma_1\) with \(2_j\) are labeled \(x_1 \gamma_1, x_2 \gamma_1, x_3 \gamma_1, x_4 \gamma_1, x_I\).
Uniformly in boundary conditions, for $8^*$ horizontal (vertical) crossings $\mathcal{H}$ ($\mathcal{V}$) across $H_j$ can be pushed forwards under $\mu$ to obtain a standard lower bound for the probability of obtaining a longer vertical (horizontal) crossing $\mathcal{V}'$ ($\mathcal{H}'$) through one application of ($\text{FKG}$) to the finite intersection of shorter vertical (horizontal) crossings $\mathcal{H}'_j$ ($\mathcal{V}'_j$),

$$
\mu[\mathcal{H}'] \geq \mu[\bigcap_{j \in J} C_j] \geq \prod_{j \in J} \mu[\mathcal{V}'_j] \geq \left[ \frac{c}{\lambda^5} \mu[\mathcal{V}]^5 \right]^{\left| J \right|},
$$

where the product is taken over admissible $j \in J \equiv \{ j \in \mathbb{R} : \text{there exists a regular hexagon with side length } j \ \& \ H_j \cap S_{T,L} \neq \emptyset \}$, with $c, \lambda > 0$. We denote the sequence of inequalities with ($\text{FKG}$) because the same argument will be applied several times for collections of horizontal and vertical crossings. From a standard lower bound from vertical crossings, the claim follows by setting $\lambda$ equal to the aspect ratio of $H_j$.

\[ \square \]

The lower bound of ($\text{FKG}$) above is raised to the cardinality of $J$. We apply the same sequence of terms from this inequality to several arguments in Corollary 11*, Lemma 1*, Lemma 2*, Lemma 13*, & Lemma 14*. We now turn to a statement of 9*.

**Lemma 9* (6-arm events, existence of c):** For every $\lambda > 0$ there exists a constant $c, \lambda$ such that for every $n \in \mathbb{Z}$,

$$
\mu[C_0] \geq \frac{c}{\lambda^5} \mu[\mathcal{V}]^5.
$$

## 5 Lemma 9* arguments

**Proof of Lemma 9*.** For the 6-arm lower bound, the argument involves manipulation of symmetric domains. In particular, we must examine the crossing event that is the most probable from the union bound, in 3 cases that are determined by the rotational invariance of $\mu$. Under this symmetry, in the union bound it is necessary that we only examine the structure of the crossing events $C$ in the following cases. We include the index $j$ associated with crossing events $C_j$, executing the argument for arbitrary $j$ (in contrast to $j \equiv 0$ in [14]), readily holding for any triplet $j - \delta', j, j + \delta'$ which translates $H_j$ horizontally. Besides exhibiting the relevant symmetric domain in each case, the existence of $c$ will also be justified. Depending on the construction of Sym, we either partition the outermost layer to Sym, called the incident layer to $\partial$ Sym, as well as sides of Sym with $L_{\text{Sym}}$, $R_{\text{Sym}}$, $T_{\text{Sym}}$ and $B_{\text{Sym}}$. Finally, we finish the section with bounds in 5.4.3 to conclude the argument.

### 5.1 $C_j \equiv C_{2j}$

In the first case, crossings across $2_j$ can be analyzed with the events $C_j$ and $C_{j+2\delta'}$. To quantify the conditional probability of obtaining a $2_j + \delta'$ crossing beginning from $S_j + \delta'$, let $\Gamma_2$ and $\Gamma_{2+2\delta'}$ be the set of respective paths from $S_j$ and $S_{j+2\delta'}$ to $2_j$ and $2_j+2\delta'$, and also realizations of the paths as $\gamma_1 \in \Gamma_2$, $\gamma_2 \in \Gamma_{2+2\delta'}$.

To accommodate properties of the dilute Potts model, we also condition that the number of connected components $k_{\gamma_1}$ of $\gamma_1$ equal the number of connected components of $k_{\gamma_2}$ of $\gamma_2$ in the spin configuration sampled under $\mu$ (see Figure 3 for one example, in which the illustration roughly gives one half of the top part of Sym which is above the point of intersection $x_{17^{1/2}}$ of the red and purple connected components, while the remaining purple connected components until $x_{27}$ constitute one half of the lower half of Sym). We denote restrictions of the connected components for $\gamma_1$ and $\gamma_2$ to the magnified region in Figure 3, and with some abuse of notation we still denote $k_{\gamma_1} \equiv k_{\gamma_1} | \gamma_j \cap \gamma_{j+2\delta'}$ and $k_{\gamma_2} \equiv k_{\gamma_1} | \gamma_j \cap \gamma_{j+2\delta'}$ for simplicity.

Finally, assign $\Omega \subset H$ as the points to the left of $\gamma_1$ and to the right of $\gamma_2$, and the symmetric domain as $\text{Sym} \equiv \text{Sym}_{\gamma_j} = \text{Sym}_{\gamma_j}(\Omega)$. To obtain a crossing across Sym, we conditionally pushforward the event
\[ \mu[C_0|\Gamma_2 = \gamma_1, \Gamma_{2+2\delta'} = \gamma_2], \]

which quantifies the probability of obtaining a connected component across \( S_{j+\delta'} \cup S_{j+2\delta'} \). We condition \( C_0 \) through \( \gamma_1 \) and \( \gamma_2 \) because if there exits a spin configuration passing through \( \text{Sym} \) whose boundaries are determined by \( \gamma_1 \) and \( \gamma_2 \), then necessarily the configuration would have a connected component from \( S_j \) to \( S_{j+2} \cup S_{j+4} \) hence confirming that \( C_0 \) occurs. To establish a comparison between this conditional probability and the conditional probability of obtaining a horizontal crossing across \( \text{Sym} \), consider

\[ \mu[\gamma_1 \leftrightarrow \gamma_2|\Gamma_2 = \gamma_1, \Gamma_{2+2\delta'} = \gamma_2], \]

subject to wired boundary conditions on \( R_{\text{Sym}} \) and \( L_{\text{Sym}} \) and free boundary conditions elsewhere. Conditionally this probability is an upper bound for another probability supported over \( \text{Sym} \), as

\[ \mu[\gamma_1 \leftrightarrow \gamma_2|\Gamma_2 = \gamma_1, \Gamma_{2+2\delta'} = \gamma_2, k_{\gamma_1} = k_{\gamma_2}] \geq \mu_{\Omega}^{\{\gamma_1,\gamma_2\}}[\gamma_1 \leftrightarrow \gamma_2], \quad (\text{V-SYMDOM}) \]

with the conditioning on the connected components applying to +/- spin configurations as shown in Figure 3, \( \Omega \) is a region inside the symmetric domain (see Figure 5), and the \( \{\gamma_1, \gamma_2\} \) superscript indicates boundary conditions wired along \( \gamma_1 \) and \( \gamma_2 \). We denote this inequality as (V-SYMDOM), which is short for vertical crossings across the symmetric domain that we introduce and further analyze in 5.2.2. Similarly, conditional on \( \Gamma_2 = \gamma_1 \) & \( \Gamma_{2+2\delta'} = \gamma_2 \), \( \{\gamma_1 \leftrightarrow \gamma_2\} \) occurs.

To quantify the probability of \( \mathcal{S}_2 \setminus (C_0 \cup C_2) \), conditionally that the connect components of the event not intersect those of \( \mathcal{S}_2 \cap \mathcal{S}_{2+2\delta'} \), we introduce modifications through \( (\mathcal{S} - \text{SMP}) \), which impact the boundary conditions of the symmetric domains that will be constructed, while modifications through \( (\mathcal{S} - \text{CBC}) \) impact the number of paths that can be averaged over in \( \Gamma_2 \) and \( \Gamma_{2+2\delta'} \) given the occurrence of \( C_0 \).

5.1.1 Incident layer of hexagons to the symmetric domain boundary

Under \( (\mathcal{S} \text{ SMP}) \), we push boundary conditions away from nonempty boundary \( \partial \text{Sym} \subset \partial H_j \) with the edge of intersection towards \( L_{\text{Sym}} \), to construct \( \text{Sym} \) by reflecting one half of the region enclosed by the realizations \( \{\gamma_1, \gamma_2\} \subset \mathcal{S}_2 \cap \mathcal{S}_{2+2\delta'} \). Because the event \( \mathcal{S}_2 \) necessarily induces the existence of a loop configuration from \( S_j \) to \( 2_j \), under Dobrushin/mixed boundary conditions which stipulate the existence of a wired arc of length \( \frac{n}{6} \) along \( 2_j \), the distribution \( \mu \) over spin configurations satisfying \( \mathcal{S}_2 \) implies that the probability of a crossing occurring across \( \text{Sym} \) supported on \( \mu_{\text{mix}}^3 \).

Formally, boundary conditions are pushed away from the boundary of \( H_j \) onto boundaries of the symmetric domain as follows.

**Definition 9.1** (pushing boundary conditions onto symmetric domains from boundary conditions on \( H_j \))

From boundary conditions along \( 2_j \), before reflecting connected components induced by the crossings event \( \mathcal{S}_2 \) about \( 2_j \), boundary conditions along symmetric domains are obtained with the following procedure:

- To partition vertices in \( \text{Sym} \) for constructing boundary conditions on vertices along the boundaries of symmetric domains, we assign + boundary conditions to a partition of the first layer of hexagons outside of the crossing induced by \( \mathcal{S}_2 \setminus (C_0 \cup C_2) \), conditioned under realizations of paths \( \gamma_1 \) & \( \gamma_2 \).
- To apply \( (\mathcal{S} - \text{SMP}) \), given the crossing \( \mathcal{S}_2 \setminus (C_0 \cup C_2) \), the length of the boundary of the symmetric domain is determined by the number of connected components of the spin configuration, which corresponds to the edges present in the configuration. From the total number of vertices on the boundary, we introduce boundary conditions with \( (\mathcal{S} - \text{CBC}) \). Outside of \( H_j \), the paths \( \gamma_1 \) and \( \gamma_2 \) (see Figure 3 for spin configurations in red and purple yield boundaries of \( \text{Sym} \)).

---

\(^3\)The mix boundary conditions are provided in two separate constructions of \( \text{Sym} \) below.
After having identified the boundaries of the symmetric domain, reflection of one half of Sym is constructed by taking the union $\gamma_1^{x_{11:2}} \cup \gamma_2^{x_{11:2}}$, where the paths in the union denote the restriction of the connected components of $\gamma_1$ and $\gamma_2$ after examining the pushforward of the conditional probability above under spin configurations supported on $\mu$. The stochastic domination above of the conditional probability under no boundary conditions on any side of Sym will be studied for paths $\gamma$. To proceed, we make use of Sym, in addition to the modification of boundary conditions as follows. From an application of (\text{S} – \text{CBC}), the conditional probability introduced at the beginning of the proof, under spin configurations supported on $\mu_{\text{Sym}}$ satisfies, under the conditional measure $\mu_2 \equiv \mu_\Omega[\{\gamma_1 \cap \gamma_2 = \emptyset, \gamma_1 \cap \gamma_3 = \emptyset, k_{\gamma_1} = k_{\gamma_2}\}]$, for measurable events depending on finitely many edges in $\Omega$,

$$\mu[\mathcal{E}_2 \setminus (C_0 \cup C_2) | \Gamma_2] = \gamma_1, \Gamma_{2,j+2\theta'} = \gamma_2] \leq \mu_{\Omega}^{\{\gamma_1, \gamma_2\}^c} [\mathcal{E}_2^{j+2\theta'}] ,$$

after examining the pushforward of the conditional probability above under spin configurations supported in Sym, where the superscript $\{\gamma_1, \gamma_2\}^c$ denotes free boundary conditions along $\gamma_1$ and $\gamma_2$ and wired elsewhere, the complement of $\{\gamma_1, \gamma_2\}$ given in the lower bound of (V-SYMDOM) (provided in 5.1). The stochastic domination above of the conditional probability under no boundary conditions on any side of Sym will be studied for paths $\gamma_3 \in \Gamma_{j+2\theta'}$. The event under $\mu_{\Omega}^{\{\gamma_1, \gamma_2\}^c}$ demands that the connected components of $\gamma_3$ be disjoint for those of $\gamma_1$ and $\gamma_2$ for the entirety of the path.

Particularly, we remove the conditioning from the pushforward in the upper bound because the definition of $\Omega$ implies that connectivity holds in between $\gamma_1$ and $\gamma_2$. Pointwise, the connected components of $\gamma_3$ do not intersect those of $\gamma_1$ and $\gamma_2$. Recalling (V-SYMDOM) in 5.1, we present additional modifications to the renormalization argument through the lower bound of the inequality to exhaust the case for $\mathcal{C}_j \equiv \mathcal{E}_2_j$. Lower bounds for the pushforward under $\mu_\Omega$ can only be obtained for mixed boundary conditions along Sym precisely under partitions of the incident hexagonal layer given in 5.1.1 & 5.1.2.

Under the conditions of (\text{S} – \text{SMP}), crossings in $\Omega$ with boundary conditions $\{\gamma_1, \gamma_2\}$, the lowermost bound for (V-SYMDOM) can only be established when boundary conditions are distributed under 5.1.1 or 5.1.2. For completeness, we first establish the lower bound for 5.1.1, in which the boundary conditions for a crossing distributed under $\mu_{\text{Sym}}^{\{\gamma_1, \gamma_2\}}$ can be compared to a closely related crossing distributed under $\mu_\Omega^{\{\gamma_1, \gamma_2\}^c}$.

To establish the comparison, the edges in Sym $\cap \Omega$, we divide the proof into separate cases depending on whether the boundary conditions for vertices along $\gamma_1$ or $\gamma_2$ are connected together under wired or free boundary conditions. One instance of pushing boundary conditions occurs for $\mathcal{C}_j \equiv \mathcal{E}_2_j$, while another instance of pushing boundary conditions occurs when $\mathcal{C}_j \equiv \mathcal{C}_{\gamma_j}$ in Section 5.4. 4

### 5.2.2 Pushing wired boundary conditions away from $\Omega$ towards Sym

One situation occurs as follows. It is possible that $+ \setminus -$ configurations distributed under $\mu_\Omega$ can be compared to configurations distributed under $\mu_{\text{Sym}}$ by pushing boundary conditions away from the first

\footnote{In contrast to the planar case of [14], considerations through the condition $k_{\gamma_1} = k_{\gamma_2}$ impact the construction of Sym and the rotational symmetry the region enjoys.}
partition of Sym towards $\Omega$; applying $(S - \µ)$ of away to obtain wired boundary conditions along $\gamma$ which holds by virtue of dual boundary conditions of $\µ$. By virtue of (FKG) for the Spin measure, in which we suitably restricted our analysis of $\µ$ for $n \geq 1 \& n2^2 \leq 1$, from which it follows that the event $\{T_\text{Sym} \leftrightarrow B_\text{Sym}\}$ depends on more edges than the conditional event $\{\ getWidth\_str \ get\_str\_end\_str\}\bigl(\gamma_1 \cap \gamma_2 = \emptyset, \gamma_1 \cap \gamma_3 = \emptyset, k_{\gamma_1} = k_{\gamma_2}\bigr)$ under $\µ$ does and is an increasing event. Finally, the simplest comparison, namely the equality

$$\mu^{(T,B)}_{\text{Sym}}[T_\text{Sym} \leftrightarrow B_\text{Sym}] = \mu^{(L,R)}_{\text{Sym}}[L_\text{Sym} \leftrightarrow R_\text{Sym}],$$

holds by virtue of dual boundary conditions of $\µ_{\text{Sym}}$, in which the pushforward of the event $\{T_\text{Sym} \leftrightarrow B_\text{Sym}\}$ under boundary conditions $\{T, B\}$ is equal to the pushfoward of the event $\{L_\text{Sym} \leftrightarrow R_\text{Sym}\}$ under boundary conditions $\{L, R\}$. Hence complementarity implies that the rotation of boundary conditions of Sym gives the following upper bound,

$$\mu[\ getWidth\_str \ get\_str\_end\_str]\bigl(\Gamma_{2j} = \gamma_1, \Gamma_{2j+2^m} = \gamma_2, k_{\gamma_1} = k_{\gamma_2}\bigr) \leq \mu^{(\gamma_1, \gamma_2)}_{\µ}[\ getWidth\_str \ get\_str\_end\_str]$$

which holds by $(S - \text{SMP})$, as wired boundary conditions for $\ getWidth\_str \ get\_str\_end\_str$ in between $\gamma_1$ and $\gamma_2$ can be pushed away to obtain wired boundary conditions along $\gamma_1$ and $\gamma_2$ for $\ getWidth\_str \ get\_str\_end\_str$, in turn transitively yielding,

$$\mu[\ getWidth\_str \ get\_str\_end\_str]\bigl(\Gamma_{2j} = \gamma_1, \Gamma_{2j+2^m} = \gamma_2, k_{\gamma_1} = k_{\gamma_2}\bigr) \leq \mu^{(L,R)}_{\text{Sym}}[L_\text{Sym} \leftrightarrow R_\text{Sym}].$$

Under $\frac{2^m}{3}$ rotational invariance of $\µ$, the argument for this case can be directly applied with $C_j \equiv \ getWidth\_str \ get\_str\_end\_str$. Examining the pushfoward of this crossing event, in addition to $\ getWidth\_str \ get\_str\_end\_str$, which guarantees the existence of a connected component that necessarily crosses $\ getWidth\_str \ get\_str\_end\_str$, leads to the same conclusion with wired boundary conditions from to along Sym. Under duality, the identification between measures under nonempty boundary conditions over Sym readily applies. Hence a combination of $(S - \text{SMP})$, followed by $(S - \text{CBC})$, implies that $\{L_\text{Sym} \leftrightarrow R_\text{Sym}\}$ occurs with substantial probability for $C_j \equiv \ getWidth\_str \ get\_str\_end\_str$ and $\ getWidth\_str \ get\_str\_end\_str$.

5.3 $C_j \equiv \ getWidth\_str \ get\_str\_end\_str$

In the second case, one can apply similar arguments with the following modifications. To identify other possible symmetric regions Sym corresponding to $\ getWidth\_str \ get\_str\_end\_str$ and $\ getWidth\_str \ get\_str\_end\_str$, fix path realizations $\gamma_1 \in \Gamma_{3j}$ and $\gamma_2 \in \Gamma_{3j+2^{m'}}$ (see Figure 2 for the connected components in the Sym construction). From $\gamma_1$ and $\gamma_2$, we construct Sym by reflecting half of the domain across $3_j$ instead of $2_j$. Under $\frac{2^m}{3}$ rotational invariance of $\µ$, Sym constructed in this case correspond to symmetric domains induced by the paths in $\ getWidth\_str \ get\_str\_end\_str$ and $\ getWidth\_str \ get\_str\_end\_str$. Explicitly, the conditional probability is of the familiar form,

$$\mu[\ getWidth\_str \ get\_str\_end\_str]\bigl(\Gamma_{2j} = \gamma_1, \Gamma_{2j+2^m} = \gamma_2\bigr).$$
which by the same argument applied to $\mathcal{C}_3$, is bounded above by

$$\mu_{\operatorname{Sym}}^{\{L,R\}}[L_{\operatorname{Sym}} \leftrightarrow R_{\operatorname{Sym}}],$$

for $\operatorname{Sym}(\Omega) \equiv \operatorname{Sym}$. Applying the same argument to push boundary conditions away from wired boundary conditions on $3_j (\delta_j)$, to $L_{\operatorname{Sym}} (R_{\operatorname{Sym}})$ establishes the same sequence of inequalities, through contributions of $\mu, \mu$ & $\mu_{\operatorname{Sym}}$. $\operatorname{Sym}$ for $\mathcal{C}_3$, corresponds to rotating the crossings of loop configurations, and hence the symmetric region to $5_j$ from the symmetric domain corresponding to $2_j$ in Figure 3.

5.4 $\mathcal{C}_j \equiv \mathcal{C}_{4j}$

In the third case, we denote the events $C_0$ and $C_2$ as bottom to top crossings, respectively across $H_j$ and $H_{j+2\delta}$, with respective path realizations $\Gamma_1$ and $\Gamma_1$ as in the previous two cases. However, the final case for top to bottom crossings stipulates that the construction of $\operatorname{Sym}$ independently of $\Omega$.

We present modifications to the square symmetric region of [14], and partition the region over which connectivity events are quantified through points to the left and right of $\gamma_1$ and $\gamma_2$, respectively. In particular, we denote $\Omega$ as the collection of all points in the hexagonal box $\operatorname{Sym}$, along with the partition $\Omega = \Omega_L \cup \Omega_{(L,R),c} \cup \Omega_R$. In the partition, each set respectively denotes the points to the left of $\gamma_1$, the points in between the left of $\gamma_1$ and the right of $\gamma_2$, and the points to the right of $\gamma_2$. With some abuse of notation we restrict the paths in $\Omega_L$, $\Omega_R$ and $\Omega_{(L,R),c}$ to coincide with crossings in between the top most edge of $H_j$ and $\operatorname{Sym}$, in which $\Omega_L \equiv (\operatorname{Sym} \cap H_j) \cap \Omega_R$, $\Omega_L \equiv (\operatorname{Sym} \cap H_j) \cap \Omega_L$, and $\Omega_{(L,R),c} \equiv (\operatorname{Sym} \cap H_j) \cap \Omega_{(L,R),c}$ (see Figure 5 above for the $\Omega$ partition). We provide such an enumeration to apply $(\Sigma - \text{SMP})$ and then $(\Sigma - \text{CBC})$, when comparing the spin representation measures supported over $\Omega$ and $\operatorname{Sym}$.

Besides the $\Omega$ partition, to apply $(\Sigma - \text{SMP})$ we examine $\mathcal{R}_1 \equiv (\operatorname{Sym} \setminus H_j^c) \cap \Omega_L$ and $\mathcal{R}_2 \equiv (\operatorname{Sym} \setminus H_j^c) \cap \Omega_R$ which denote the collection of points to the left of $\gamma_1$ and to the right of $\gamma_2$ in the region above $H_j$ that is contained in $\operatorname{Sym}$ (see Figure 5 for $H_j$ embedded within the hexagonal symmetric domain). To apply $(\Sigma - \text{CBC})$, it is necessary that we isolate $\mathcal{R}_1$ and $\mathcal{R}_2$ so that $(\Sigma - \text{CBC})$ can be applied to the outermost layer of hexagons incident to $\partial \Omega$ through a partition of the incident layer.

Again, we provide an upper bound for the pushforward of the following conditional probability, for $\mathcal{R} \equiv \{\Gamma_{2j} = \gamma_1 \& \Gamma_{2j+2\delta} = \gamma_2, \gamma_1 \cap \gamma_3 = \emptyset, \gamma_1 \cap \gamma_2 = \emptyset\}$

$$\mu[\mathcal{C}_{4j} \setminus (C_0 \cup C_2) | \mathcal{R}],$$

for the class of hexagonal box symmetric domains $\operatorname{Sym}$, with $\gamma_3 \in \Gamma_{j+\delta}$.

5.4.1 Pushing boundary conditions away from $H_j$ towards $\operatorname{Sym}$

Next, we push boundary conditions away from $H_j$. Under the assumption that the upper half of $\operatorname{Sym}$ is endowed with wired boundary conditions while the lower half is endowed with free boundary conditions. We denote these boundary conditions with Top, and will consider the measures supported over $\operatorname{Sym}$, respectively. From observations in previous cases, to analyze the conditional probability of $C_0$ given $\Gamma_j = \gamma_1$ and $\Gamma_{j+2\delta} = \gamma_2$, we introduce the following lower bound for a connectivity event between $\gamma_1$ and $\gamma_2$ in $\Omega$, with,

$$\mu[C_0 | \mathcal{R}] \geq \mu[\gamma_1 \leftarrow \Omega \rightarrow \gamma_2 | \mathcal{R}],$$

holds from arguments applied when $\mathcal{C}_j \equiv \mathcal{C}_{2j}$. By construction, $H_j \subset \operatorname{Sym}$ implies

$$\mu[\mathcal{C}_{4j} | \mathcal{R}] \leq \mu_{\Omega_{(L,R),c}}^{\{\gamma_1, \gamma_2\}}[\mathcal{C}_{4j} | \mathcal{R}],$$
due to monotonicity in the domain, as the occurrence of $C_3$ conditionally on disjoint connected components of $\gamma_3 \in \Gamma_{j+\delta}$ with those of $\gamma_1$ and $\gamma_2$. In comparison to the conditioning applied through $k_{\gamma_1} = k_{\gamma_2}$ for $C_2$, and $C_3$, the sides of Sym are formed independently of the connected components of $\gamma_1$ and $\gamma_2$; a combination of monotonicity of $\mu$, in addition to $(S -\text{SMP})$ through an equal partition of the incident layer outside of Sym equally into two sets along which +/- spin is constant.

After pushing boundary conditions towards Sym, we make use of rotational symmetry of Sym. In particular, the distribution of boundary conditions from the incident layer partition of $5.4.1$ satisfies the following inequality,

$$
\mu_{\Omega_{(L,R)^c}}[C_0|R] \leq \mu_{\text{(Top Half)}}^{\text{(Top Half)}}[C_0|R] \leq \mu_{\text{Sym}}^{\text{(Top Half)}}[T_{\text{Sym}} \leftarrow B_{\text{Sym}}] = \mu_{\text{Sym}}^{\text{(Top Half)}}[L_{\text{Sym}} \leftarrow R_{\text{Sym}}] ,
$$

where (Top Half) denotes wired boundary conditions along the top half of hexagonal Sym. Within the sequence of inequalities, the leftmost lower bound for $\mu_{\text{Sym}}^{(L,R)}[C_0 | R]$ holds because $\Omega_{(L,R)^c} \subset \text{Sym}$, with $\{L, R\}$ denoting wired boundary conditions along $L_{\text{Sym}}$ and $R_{\text{Sym}}$. The next lower bound for $\mu_{\text{Sym}}^{(L,R)}[T_{\text{Sym}} \leftarrow B_{\text{Sym}}]$ holds because the event $\{T_{\text{Sym}} \leftarrow B_{\text{Sym}}\}$ depends on finitely many more edges in $H$ than $\{C_0 | R\}$ does. Finally, the last inequality holds due to complementarity as in the argument for $C_j \equiv C_3$. $\{L, R\}$ denotes a $\frac{\pi}{3}$ rotation of the boundary conditions supported over Sym. More specifically, rotating the boundary conditions $\{L, R\}$ by $\frac{2\pi}{3}$ to obtain the boundary conditions $\{L, R\}^{\frac{2\pi}{3}}$ amounts to four $\frac{\pi}{3}$ rotations of Sym. With each rotation, the boundary conditions $\{L, R\}^{\frac{\pi}{3}}$ are obtained by rotating the partition of the incident layer along $\partial \text{Sym}$ to its leftmost neighboring edge, in addition to modifications of the connectivity in $C_4$.

Finally, the arguments imply the same result as in other cases, in which

$$
\mu_{\Omega_{(L,R)^c}}[C_0 \cup C_2|R] \leq \mu_{\text{Sym}}^{(\frac{2\pi}{3})}[L_{\text{Sym}} \leftarrow R_{\text{Sym}}] .
$$

We conclude the argument for $9^*$, not only having shown that the same inequality holds for a different classes of symmetric domains in the $C_j \equiv C_4$ case, but also that rotation of boundary conditions wired along the top half of Sym for top to bottom crossings can be used to obtain boundary conditions for left to right crossings.

### 5.4.2 $(S \text{ CBC})$ lower bound for the conditional crossing event $\{C_0|C_0 \cap C_1\}$

We complete the argument by providing the following inequalities for each case. We make use of the special case of $(S - \text{CBC})$ from Section 3.2, in which for $C_j \equiv C_4$, conditionally on top to bottom crossings $C_0$ and $C_2$ from 5.2.2, the pushforward below satisfies,

$$
\mu_{\Omega_{(L,R)^c}}[C_0 \cap C_4] \geq \frac{1}{n^{k_r(\sigma)-k_r(\sigma)x}} \exp(h) \mu_{\text{Sym}}^{(\frac{2\pi}{3})}[C_0 \cap C_4] , \quad ((S-\text{SMP}) - (C_2))
$$

where the normalization to the crossing probability in the lower bound is dependent on the edge weight $x$. One obtains the same bound for crossings $C_j \equiv C_4$. The bound above corresponds to the partition of boundary conditions. Finally, the existence of $c$ such that the inequality in the statement of Lemma $9^*$ holds is of the form. We obtain several factors in the constant for the lower bound, the first of which is proportional to the reciprocal of the number of connected components and positions at which the connected components are located on the lattice.
\[
\frac{1}{n} \left( \sum_{\sigma} \left( e^{r_{\mathcal{H}_0 \cup \mathcal{H}_2} (\sigma)} \prod_{e \in \mathcal{H}_1 \cup \mathcal{H}_1^+} e^{H_1 (\sigma)} \prod_{\tau \in \mathcal{H}_0 \cup \mathcal{H}_2} e^{\tau (\sigma)} \right) \right) \, ,
\]

in addition to the reciprocal of the following factor dependent on the magnitude of the two external fields \( h_1, h_2 \),

\[
e^h \left( \frac{H_0}{H_2} (\sigma_1) - \frac{H_1}{H_2} (\sigma_2) \right) + \frac{1}{2} \left( \frac{H_0}{H_2} (\sigma_1) - \frac{H_1}{H_2} (\sigma_2) \right) \, .
\]

Denoting the product of all the factors above as \( P \), the desired lower bound is of the form,

\[
c = (3I)^{-3} P \, ,
\]

because the superposition of crossing probabilities,

\[
c \mu[C_0] + \mu[C_2] \, ,
\]

where the first crossing probability is magnified with respect to the product of edges, loops, and the exponential of the difference between the external fields that are scaled with respect to the summation of spins, in addition to the number of monochromatically colored hexagons. We denote the spin configurations \( \sigma_1 \) and \( \sigma_2 \) supported over crossings over \( \mathcal{H}_0 \cup \mathcal{H}_2 \) and \( \mathcal{H}_1 \cup \mathcal{H}_1^+ \), respectively. To provide a lower bound for this superposition of crossing probabilities, we bound each term below with,

\[
\mu[C_0 \cap C_1 \cap C_2] + \mu[(C_0 \cup C_2) \cap C_1 \cap C_2] \, ,
\]

where the crossing events \( C_1 \neq C_2 \) are disjoint and can be chosen from the crossing events defined in 4.2. The lower bound holds because the crossing probability event \( C_0 \) depends on more edges than the event \( C_0 \cap C_1 \cap C_2 \) does, with the same observation holding between the crossing events \( C_2 \) and \( (C_0 \cup C_2) \cap C_1 \cap C_2 \). Moreover, the superposition provided in the lower bound itself can be bounded below by the pushforward of the single crossing event,

\[
\mu[C_0 \cap C_2 \cap C_4] \geq (\mu[C_0])^3 \, ,
\]

from which the form of the constant \( c \) defined above follows, due to a previous application of \( (S - \text{SMP}) \) in the lower bound for the conditional crossing event provided in \( (S - \text{SMP}) - (C_2) \). The ultimate inequality follows from the fact that each crossing event in the intersection is independently bounded below by the product of three crossing probabilities. As a result, the union bound,

\[
\max \{ \mu[C_{2j}], \mu[C_{3j}], \mu[C_{4j}] : 0 \leq j < I \} \geq \frac{\mu[\mathcal{V}_H]}{3I} \, ,
\]

holds, which is equivalent to the maximum of the crossing events taken over \( C_{20}, C_{30} \) and \( C_{40} \),

\[
\max \{ \mu[C_{20}], \mu[C_{30}], \mu[C_{40}] \} \geq \frac{\mu[\mathcal{V}_H]}{3I} \, ,
\]
which holds given that the maximum of each one of the crossing probabilities across $2_j$ or $3_j$, across $3I$ events, in addition to the fact that the lower bound for the intersection of crossing events below takes the form,

$$
\mu[C_0] \geq \mu[\mathcal{C}^1 \cap \mathcal{C}^2] \geq \left( \frac{\mu[V_H]}{3I} \right)^2,
$$

where in the upper bound $\mathcal{C}^1$ can be any one of the crossings to a rightmost edge of $H$, $\mathcal{C}_{3j}$ or $\mathcal{C}_{6j}$, while $\mathcal{C}^2$ can be any one of the crossings to a leftmost edge of $H$, $\mathcal{C}_{2j}$ or $\mathcal{C}_{3j}$. Observe that this particular choice of constant is dependent on the magnitude of the $\square$

**Remark** The above constant $c$ for which Lemma 9* is proved, unlike the accompanying constant for the constant provided in the random cluster model case, is dependent on the product of the number of loops, the number of edges, and the difference in the number of monochromatically colored hexagons, as defined with the quantities $r(\sigma)$ and $r'(\sigma)$ in (S-SMP), instead of solely on $q$.

6 Volume of connected components from wired boundary conditions

6.1 Proof of 10* with the $\mu$ homomorphism of Proposition 8*

To study behavior of the dilute Potts model in the Continuous Critical and Discontinuous Critical cases, we turn to studying vertical crossings under $\mu$ under wired boundary conditions. To denote vertical translates of hexagons containing $H_j$, we introduce $H_{j+j+\delta}$ as the hexagonal box whose center coincides with that of $H_j$, and is of side length $j + \delta$. We state the following Lemma and Corollary.

**Lemma 10** (volume of connected components): For $x \in H_j$ and $C \geq 2$, there exists $\epsilon > 0$ such that, given $\mu_{H,Cj}^{1}[H_j \longleftrightarrow \partial H_{j,j+\delta}] < \epsilon$ for some $k$, in $H_j \cap H_j,j+\delta$ there exits a positive $c$ satisfying,

$$
\mu_{H_j}^{1}[\text{Vol}(\text{connected components in the annulus } H_j \cap H_j,j+\delta)] = N \leq e^{-cN},
$$

for every $j, N \geq 2$, taken under wired boundary conditions.

**Proof of Lemma 10**. The arguments require use of hexagonal annuli which for simplicity we denote with $\mathcal{H}_A \equiv H_j \cap H_{j,j+\delta}$, in which one hexagonal box is embedded within another (the same arrangement given in Figure 5 for top to bottom crossings in $C_{j} \equiv \mathcal{C}_j$), and set $P \equiv \{\text{Vol}(\text{connected components in the annulus } H_j \cap H_{j,j+\delta}) = N\}$. The existence of the quantity $\mu^{\mathcal{C}_j}$, where $\mu$ is a finite constant and $\mathcal{C}_j$ is the number of connected components of length $l$ is standard from [29]. To prove the statement, we measure the connected components of length $l$ from the center of $H_j$ in $\mathcal{H}_A$.

From the connected components of $x$ in $H_j$, we can restrict the connected components to the nonempty intersection given by $\mathcal{H}_A$. The argument directly transfers from the planar case to the hexagonal one with little modification, as the restriction of the connected components $\mathcal{C}_j^{l}$ of length $l$ to the annulus implies the existence of a connected set of in $H$, denoted with $S \subset \mathcal{H}_A$ of vertex cardinality $N \setminus |H_j|$ from which a subset of the connected components $\mathcal{S}_C \subset S$ can be obtained. We conclude the proof by analyzing the pushforward of $P$ under wired boundary conditions supported on $H_j$, in which the union bound below over $\mathcal{J}_S$ satisfies,

$$
\mu_{H_j}^{1}[P] \leq \bigcup_{i \in \mathcal{J}_S = \{\text{connected components of size } l \text{ of } \mathcal{H}_A \text{ in } S\}} \mu_{H_j}^{1}[\mathcal{P}_i] \leq \left( \mu_{H_j}[P] \right)^{N \setminus |H_j|} \leq \left( \mu_{\mathcal{S}_C}^{N \setminus |H_j|} \right)^{N \setminus |H_j|} \leq e^{-cN},
$$

where the union is taken over the collection of connected components under the criteria that admissible vertices from $S$ are taken to be of distance $2j$ from one another in $\mathcal{J}_S$, and events $\mathcal{P}_{\mathcal{J}_S}$ denote measurable
events under $\mu_{H_j}$ indexed by the number of admissible vertices from $S_C$. We also apply $(S-\text{SMP})$ and $(S-\text{CBC})$ in the inequality above to push boundary conditions away, with $\epsilon$ arbitrary and small enough.

Next, we turn to the statement of the Corollary below which requires modification to vertical crossings across $H_j$, which can be accommodated with families of boxes $H_j$ with varying height dependent on the usual RSW aspect ratio factor $\rho$. We also make use of $S_{T,L} \equiv S$.

**Corollary 11** (dilate Potts behavior outside of the supercritical and subcritical regimes): For every $\rho > 0$, $L \geq 1$, there exists a positive constant $C$ satisfying the following, in which

- for the **Non(Subcritical)** regime, the crossing probability under wired boundary conditions of a horizontal crossing across $H_j$ supported over the strip, $\mu_S^1[\mathcal{H}_{H_j}] \geq C$,

- for the **Non(Supercritical)** regime, the crossing probability under free boundary conditions of a vertical crossing across $H_j$, $\mu_S^0[\mathcal{V}_{H_j}] \leq 1 - C$, also supported over the strip.

**Proof of Corollary 11**. We present the argument for the first statement in **Non(Subcritical)** from which the second statement in **Non(Supercritical)** follows. For $S$, in the **Non(Subcritical)** phase horizontal crossing probabilities across $S_{T,L} \equiv S$ are bound uniformly away from 0, which for $\mu$ can be demonstrated through examination of crossing events $C_j$ first introduced in **Proof of Proposition 8**. For $S$, the result under which the pushforward with wired boundary conditions takes the form, for any $j \geq 1$,

$$\mu^1_S[C_j] \geq e^{-6c},$$

from an application of $10^*$ to a connected component with unit volume in $\mathcal{H}$ type annuli.

Also, in the following arrangement, we introduce a factor $\rho$ for the aspect length of a regular hexagon in $S_{T,L}$ which mirrors the role of $\rho$ in RSW theory for crossings across rectangles. About the origin, we pushforward vertical crossing events on each side of $H_j = \bigcup_i H_{j+i\delta}$, respectively given by $H_{j+i\delta_k}$ and $H_{j+i\delta_l}$ for $k$ such that $H_{j+i\delta_k}$ and $H_{j+i\delta_l}$ are of equal distance to the left and right of the origin. By construction, in any $H_j$ with the aspect length dependent on $\rho$, intermediate regular hexagons can be embedded within $H_j$ corresponding to the partition of the aspect length $\rho$. Longer horizontal or vertical crossings can be constructed through applications of (FKG) which are exhibited below.

From the lower bound on the volume of a unit connected component, a vertical crossing across a hexagon of aspect height $\delta$, from reasoning as given in (FKG) can be bound below by the product of crossing probabilities of $\delta_i$ translates of vertical crossings across hexagons of aspect height $\delta_i$.

The measure under wired boundary conditions, for a vertical crossing $\mathcal{V}$ across $H_{j+i\delta_k}$, is

$$\mu_{H_j}[\mathcal{V}_{H_{j+i\delta_k}}],$$

where the measure for the vertical crossing event given above is supported over $H_j$.

From the upper bound of (FKG), longer vertical horizontal crossings occur across $2^l$ vertical translates of shorter vertical crossings. The next ingredient includes making use of previous arrangements of horizontal translates of $H_j$, namely the left translate $H_{j-\delta'}$ and the right translate $H_{j+\delta'}$. Under the occurrence of vertical crossings across $H_{j+i\delta_k}$ and $H_{j+i\delta_l}$, this event, to show that some box $H_j$ in between $H_{j+i\delta_k}$ and $H_{j+i\delta_l}$ is crossed vertically, under wired boundary conditions supported over $H_j$, we directly apply previous arguments from (FKG), with the exception that (FKG) is applied to a countable intersection of vertical, instead of horizontal, crossing events $\mathcal{V}$.

Conditionally, if vertical crossings in $H_{j+i\delta_k}$ and $H_{j+i\delta_l}$ occur about arbitrary $H_{j+i\delta_i}$ with $k \leq i \leq l$, then the probability below satisfies, under wired boundary conditions,
\[\mu^1[V_{H_{j+\delta_k}} \cap V_{H_{j+\delta_l}}] \geq \mu^1_{H_j}[V_{H_{j+\delta_k}}]\mu^1_{H_j}[V_{H_{j+\delta_l}}] = \mu^1_{H_j}[V_{H_{j+\delta_k}}]^2 \geq \prod_i \mu^1_{H_j}[V_{H_{j+\delta_i}}] = (\mu^1_{H_j}[V_{H_{j+\delta_l}}])^{2^{1-i}}, \quad (\circ)\]

where \(V_H\) denotes the vertical crossing across hexagons of aspect length which is the same as that of \(H_{j+\delta_k}\), but with and aspect height \(\delta_i\) where \(\delta_i = \cup_i \delta_i\). The union over \(i\) indicates a partition of the aspect height of \(H_{j+\delta_i}\) into \(2^{1-i}\) intervals. Finally,

\[\mu^1_{H_j}[V_{H_{j+\delta_l}}]^{2^{1-i}} \geq e^{-c(2^{1-i})}, \quad (\circ\circ)\]

The lower bound for the inequality above is obtained from an application of \(10^*\) to the volume of a connected component from vertical crossings in \(H_{j+\delta_k}\) and \(H_{j+\delta_l}\). Between the second and third terms in \(\circ\circ\), monotonicity in the domain allows for a comparison between the measure under wired boundary conditions respectively supported over \(H_{j+\delta_i}\) and \(H_j\).

From the partition of \(H_j\), to apply \((S-CBC)\) we consider the region between vertical crossings across \(H_{j+\delta_l}\) and \(H_{j+\delta_k}\). From the previous upper bound, given some \(u\) the vertical event \(\{V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}}\}\) about \(H_{j+\delta_u}\) occurs for some \(k, l < u\). Under wired boundary conditions, the conditional vertical crossing

\[\mu^1_{H_j}[V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}} \mid V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}}],\]

is bounded from below by the lower bound of \((\circ \circ)\). With conditioning on \(\{V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}}\}\), the probability of simultaneous vertical crossings in \(H_{j+\delta_k}\) and \(H_{j+\delta_l}\) and \(j + \delta_k \equiv j + \delta_l\), the pushforward under wired boundary conditions of vertical crossings across two hexagons which entirely overlap with one another gives the upper bound

\[\mu^1_{H_j}[V_{(j+\delta_k \equiv j+\delta_l)}] \geq \mu^1_{H_j}[V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}}] \prod_{i=1}^j \mu^1_{H_j}[V_{H_{j+\delta_k-1}} \cup V_{H_{j+\delta_l-1}} \mid V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}}] \geq e^{-c},\]

where the vertical crossing \(V\) occurs when the indicator is satisfied. As \(\rho \rightarrow +\infty\), the finite volume measure over \(H_j\) under the weak limit of measures yields a similar inequality

\[\mu^1_S[V_{(j+\delta_k \equiv j+\delta_l)}] \geq \mu^1_S[\mathcal{C}_0] \geq e^{-c},\]

with the exception that \(\mu\) under wired boundary conditions is supported along the strip \(S\), and \(\mathcal{C}_0\) denotes the crossing event in which hexagons to the right and left of \(H_0\) are crossed vertically. The exponential bound itself can be bounded below with the desired constant,

\[e^{-c} \geq \mathcal{C},\]

establishing the inequality for the Spin measure under wired boundary conditions. From the union of vertical crossings \(V_{H_{j+\delta_k}} \cup V_{H_{j+\delta_l}}\), applying the \(\mu\) homeomorphism under the conditions on \(c_0\) in \(\text{Theorem 1}^*\),
\[ f(x) = 1 - c_0^{-c_0} + c_0^{-c_0} x , \]

for \( x = \mu^1[\mathcal{V}] \) to the inequality for vertical crossings bounded below by \( C \) implies that the upper bound of \( C \) on can be translated into a corresponding upper bound dependent on \( C \) for horizontal crossings, obtaining a similar upper bound under free boundary conditions,

\[ \mu_{\mathcal{V}}^0[\mathcal{V}_{\mathcal{H}_j}] \leq 1 - C , \]

concluding the argument after having taken the infinite aspect length as \( \rho \rightarrow +\infty \) for a second time. From rotational symmetries in the \( 9^* \) proof, there are six possible rotations from which \( C_j \) can occur, in which \( C \equiv C_{2j}, C \equiv C_{3j} \) or \( C \equiv C_{4j} \). Each upper bound under wired and free boundary conditions has been shown. \( \square \)

7 Vertical and horizontal strip densities

7.1 Towards proving horizontal and vertical crossing densities in Definition 1

In this section, we make use of strip densities similar to those provided for the random cluster model in [14] (defined in 3.3) from which strip density and renormalization inequalities will be presented, in the infinite length aspect ratio limit. In the arguments below, we present boxes \( \mathcal{H}, \mathcal{H}_i \) and \( \mathcal{H}'_i \) across which horizontal and vertical crossings are quantified. For the lower bound of the conditional probability of obtaining no vertical crossings across each \( \mathcal{H}_i \), under the Spin measure with wired boundary conditions, is,

\[ p_n \equiv \lim sup_{\rho \rightarrow \infty} \left( \mu^0_{[0, \rho n] \times \mathcal{H} \times [0, \lambda \text{Stretch}]} \left[ \mathcal{H}_{[0, \rho n] \times \mathcal{H} \times [0, \lambda \text{Stretch}]} \right] \right) \frac{1}{\rho} , \]

while for vertical crossings across \( \mathcal{H} + j \), under the Spin measure with wired boundary conditions, is,

\[ q_n \equiv \lim sup_{\rho \rightarrow \infty} \left( \mu^0_{[0, \rho n] \times \mathcal{H} \times [0, \lambda \text{Stretch}]} \left[ \mathcal{V}_{[0, \rho n] \times \mathcal{H} \times [0, \lambda \text{Stretch}]} \right] \right) \frac{1}{\rho} . \]

We denote \( p_n \equiv p_n^\mu \) and \( q_n \equiv q_n^\mu \). With these quantities, we prove the strip density formulas which describe how boundary conditions induced by vertical crossings under wired boundary conditions across \( H + \delta_j \) relate to horizontal crossings under free boundary conditions.

In the proof below, we make use of arguments from \( 11^* \) to study vertical crossings across hexagons, and through applications of (\( S - \text{SMP} \)) and (\( S - \text{CBC} \)). To prove \( 1^* \), we define additional crossing events as follows. First, the crossing event that three hexagons, with aspect width of \( \mathcal{H}_j \) and aspect length \( \text{Stretch} \) placed on top of each other, is pushed forwards to apply FKG type arguments, with (FKG), over a countable intersection of horizontal crossings across each \( \mathcal{H}_i \), we introduce a slightly larger hexagonal box \( \mathcal{H}' \text{Stretch} \) which has an aspect height ratio \text{insert} times that of \( \mathcal{H}_j \).
in \( \{F | E\} \) are formulated by making use of the monotonicity in the domain assumption, denoted as \( \mathcal{G} \) which is independent of \( \rho \).

Fourth, the intersection of the previous three events is pushed forwards, and by virtue of \((S - \text{SMP})\) and \((S - \text{CBC})\), yields a strip inequality relating \( p_n \) to \( q_n \), and \( q_n \) to \( p_n \). In infinite aspect length as \( \rho \rightarrow \infty \), inequalities corresponding to the horizontal and vertical strip densities are presented.

**Proof of Lemma 1.** The argument consists of six parts; we fix \( \lambda \in \mathbb{N} \), \( n \in 3 \mathbb{N} \). As a matter of notation, below we denote each of the three boxes below as the Cartesian product of the aspect length and height, crossings occur, which are defined as,

\[
\mathcal{H} = [0, \rho n] \times_H [0, (2\lambda) \text{Stretch} + \text{Stretch}],
\]

\[
\mathcal{H}_i = [0, \rho n] \times_H [(2i)\text{Stretch} + \text{Stretch}, (2i)\text{Stretch} + 2\text{Stretch}],
\]

\[
\mathcal{H}'_i = [0, \rho n] \times_H [(2i)\text{Stretch}, (2i)\text{Stretch} + \text{Stretch}],
\]

for every \( 0 \leq i \leq \lambda - 1 \). As indicated above, the notation \( \times_H \), for some nonempty subset \([0, a) \times_H [0, b] \) of \( H \), denotes that the finite volume over the hexagonal lattice of length \( a \) and height \( b \), for \( a \) and \( b \) strictly positive. In the construction, the aspect length is the same as that of \( \mathcal{H} \), while the aspect height of each box is partitioned in \( i \) relative to the scaling of the Stretch factor. Also, a final box with the Stretch scaling itself will be defined,

\[
\mathcal{H}_{\text{Stretch}} = [0, \rho n] \times_H [0, n\lambda \text{Stretch}],
\]

which is supported over which the spin measure with wired bound conditions for a lower bound of \( \mu^1_{\mathcal{H}}[F \mid E] \), and \( n \) is an integer parameter. Second, to apply (FKG) previously used, if \( \mathcal{H}_D \) denotes a horizontal crossing across a finite domain \( D \) of \( S \), we make use of \( \mathcal{H}, \mathcal{H}_i, \mathcal{H}'_i \subset D \) with smaller aspect lengths across which horizontal crossings occur. The lower bound for applying (FKG) across a countable family of horizontal crossings \( \mathcal{H}_i \) is,

\[
\mu^1_{\mathcal{H}}[E] \geq \mu^1_{\mathcal{H}}\left[ \bigcap_{0 \leq i \leq \lambda - 1} \mathcal{H}_i \right] \geq \prod_{0 \leq i \leq \lambda - 1} \mu^1_{\mathcal{H}}[\mathcal{H}_i] \geq \prod_{0 \leq i \leq \lambda - 1} \left( \frac{1}{\lambda C} \right)^\rho \geq \left( \frac{1}{\lambda C} \right)^\lambda \rho,
\]

with the existence of the lower bound guaranteed by Corollary 11*, and \( \lambda \) is the minimum amongst all \( \lambda_i \). Before letting \( \rho \rightarrow +\infty \), pushing forwards the horizontal crossing event across \( \mathcal{H} \subset D \) under wired boundary conditions for vertical crossings across \( \mathcal{H}'_i \) gives,

\[
\mu^1_{\mathcal{H}}[F | E] \geq \mu^1_{\mathcal{H}_i'}\left[ \bigcap_{0 \leq i \leq \lambda} V^1_{\mathcal{H}_i'} \right],
\]

which is bounded below by the product of independent events through repeated applications of (FKG) across \( \lambda - 1 \) crossing events,

\[
\prod_{0 \leq i \leq \lambda - 1} \mu^1_{\mathcal{H}_i'}[V^1_{\mathcal{H}_i'}] \geq \left( \mu^1_{[0, \rho n] \times_H [0, n_1 \lambda \text{Stretch}]} \left[ V^1_{[0, \rho n] \times_H [0, n_1 \lambda \text{Stretch}]} \right] \right)^{\lambda + 1},
\]

for \( n_2 > n_1 \), fromo applications of \((S - \text{SMP})\), monotonicity in the domain, and applying (FKG) to vertical crossing events, instead of horizontal crossing events.
By construction of $\mathcal{E}$, the following lower bound for the conditional event $\{\mathcal{F}|\mathcal{E}\}$,

$$
\mu^1_{\mathcal{F}}[\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}] = \mu^1_{\mathcal{F}}[(\mathcal{E} \cap \mathcal{F}) \cap \mathcal{G}] \geq \mu^1_{\mathcal{F}}[\mathcal{E} \cap \mathcal{F}] \mu^1_{\mathcal{F}}[\mathcal{G}]
$$

$$(\text{FKG})
$$

$$
= \left( n^{k_{\sigma}(\sigma)} \exp \left( \# \{ \mathcal{H}_i \; : \; 1 \leq i \sim j \leq 6, \sigma_v \in \{\pm 1\}^{\mathcal{H}_i}, \mathbf{1}_{\sigma_{n_1} = \cdots = \sigma_{n_6} = 1} \} \right) \right) \mu^1_{\mathcal{F}}[\mathcal{E} \cap \mathcal{F}],
$$

where the edge weight in the lower bound is representative of additional weights in the configuration supported under the wired Spin measure over $\mathcal{F}^\prime$, which we denote as (FKG) $-$ (S $-$ SMP). In the exponential of the first external field, $\mathcal{H} \subset (2\lambda)\text{Stretch} + \text{Stretch}$. In the exponent of the edge weight $x$, by definition the number of connected components in the configuration $\sigma$ sampled under $\mu$ is,

$$
k_{\sigma}(\sigma) = \sum_{k \in \mathcal{C}} \left( \sum_{u \sim v} \mathbf{1}_{\{\sigma_u = \pm \sigma_v\}} + 1 \right),
$$

where the summation is taken over all connected components $k$ so that $\mathcal{G}$ occurs, and all neighboring vertices $u$ and $v$ with the same $\pm$ spin, clearly impacting the number of connected components counted under $\sigma$.

Before completing the next step, we combine the estimates on $\mu^1_{\mathcal{F}}[\mathcal{E}]$ and $\mu^1_{\mathcal{F}}[\mathcal{F} | \mathcal{E}]$ to obtain the strip inequality between horizontal and vertical crossings. The following comparison amounts to making use of (S $-$ SMP) and (MON) to establish the following. First, we know that the measure $\mu^1_{\mathcal{E}}[\cdot]$ can be bound above with,

$$
\mu^1_{\mathcal{E}}[\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}] \leq \mu^1_{\mathcal{E}}[\mathcal{E} | \mathcal{F} \cap \mathcal{G}] \leq \left( \mu^0_{[0,\rho \lambda] \times \mathcal{H}(0, n_2 \lambda \text{Stretch}) \cap \mathcal{G}}[\mathcal{H}(0, \rho \lambda) \times \mathcal{H}(0, n_1 \lambda \text{Stretch})] \right)^{\lambda},
$$

because the event in the upper bound is more likely to occur than the event in the lower bound, in addition to an application of (FKG) for $\lambda$ horizontal crossings, each event of which has equal probability, across $[0, \rho \lambda] \times \mathcal{H}(0, n_2 \lambda \text{Stretch})$, and $n_3 > n_2$. Also, the upper bound to the conditional probability $\mu^1_{\mathcal{E}}[\mathcal{E} | \mathcal{F} \cap \mathcal{G}]$ above is established by making the comparison between measures with free boundary conditions. In comparison to the planar case, modifications to the argument with (S $-$ SMP), while other properties of the random cluster measure $\phi$ directly apply.

In light of the lower bound in (FKG) $-$ (S $-$ SMP) dependent on the edge weight $x$ and Stretch, we consider the horizontal pushforward from the previous upper bound with free boundary conditions,

$$
\mu^0_{[0,\rho \lambda] \times \mathcal{H}(0, n_2 \lambda \text{Stretch}) \cap \mathcal{G}}[\mathcal{H}(0, \rho \lambda) \times \mathcal{H}(0, n_1 \lambda \text{Stretch})],
$$

which can be bound below by establishing comparisons between the measure under wired boundary conditions supported over a smaller hexagonal domain,

$$
\mu^1_{[0,\rho \lambda] \times \mathcal{H}(0, n_1 \lambda \text{Stretch}) \cap \mathcal{G}}[\mathcal{V}(0, \rho \lambda) \times \mathcal{H}(0, n_1 \lambda \text{Stretch})],
$$

as a consequence yielding one estimate for the vertical strip density,

$$
\left( \mu^0_{[0,\rho \lambda] \times \mathcal{H}(0, n_2 \lambda \text{Stretch}) \cap \mathcal{G}}[\mathcal{H}(0, \rho \lambda) \times \mathcal{H}(0, n_1 \lambda \text{Stretch})] \right)^{\lambda} \overset{\rho \rightarrow +\infty}{\approx} \left( p_{\text{Stretch}} \right)^{\lambda}
$$

$$
\geq \left( \mu^1_{[0,\rho \lambda] \times \mathcal{H}(0, n_1 \lambda \text{Stretch}) \cap \mathcal{G}}[\mathcal{V}(0, \rho \lambda) \times \mathcal{H}(0, n_1 \lambda \text{Stretch})] \right)^{\lambda+1} \overset{\rho \rightarrow +\infty}{\approx} \frac{1}{\lambda^{\lambda+1}} \left( q_{\text{Stretch}} \right)^{\lambda+1},
$$

23
due to the fact that,

\[
\left( n^{k_G(\sigma)} x^2 \{ e \in (2\lambda) \text{Stretch} \} \right) \exp\left( \# \{ H : 1 \leq i \sim j \leq 6, \sigma_v \in \{ \pm 1 \}^{\#H}, 1_{\sigma_i = \sigma_j = 1} \} \right)^{\lambda \rho^{-1}} \to 1
\]

as \( \rho \to \infty \), and cancellations of \( \lambda \) crossings in the inequality gives,

\[
p_{\text{Stretch}} n \geq \frac{1}{C} (q_{\text{Stretch}} n)^{\text{Stretch} + \frac{\text{Stretch}}{\lambda}},
\]

resulting from \((S-CBC)\), as in the region below the connected component of the path associated with the crossing \( H \), the induced boundary conditions dominate the measure supported over a smaller domain under wired boundary conditions.

The strip inequality for horizontal and vertical crossings is finally achieved by taking each side of the inequality to the power \( \frac{1}{\lambda \rho} \), which preserves the direction of the inequality as a monotonic decreasing transformation. As \( \rho \to \infty \), we recover the peculiar definition of the horizontal strip density, while the other inequality corresponding to the vertical crossing density can be easily achieved by following the same argument, with the exception of the inequalities leading to the final estimate for vertical crossing events instead of horizontal ones. \( \Box \)

### 7.2 Pushing lemma

We turn to the following estimates. In Lemmas 13* and 14* below, \( \bar{H} \) denotes the box with aspect length \( \rho n \), and variable aspect height defined for each box in the proof. To prove Lemma 13* (see below), we make use of the following property for the Spin measure. With the Pushing Lemma, we provide arguments for the renormalization inequalities in the next section. Under this Lemma, we proceed to obtain results for the PushPrimai and PushDual conditions (listed below), from which are the combined with the accompanying PushPrimai Strip, and PushDual Strip, to probabilistically quantify crossing probabilities across hexagons \( H_i, H_i^+, \) and \( H_i^- \) (introduced in Section 8 to obtain the vertical and horizontal strip density formulas).

**Property (Finite energy for the Spin measure, [8]):** For any \( \tau \in \{ -1, 1 \}^T \) and \( \sigma \in \Sigma(G, \tau), \mu_{G, n, x, h, h'}[\sigma] \geq \epsilon|G| \), for any \( \epsilon > 0 \) depending only on \( (n, x, h, h') \).

#### 7.2.1 Statement of the two Lemmas for strip and planar domains

**Lemma 13* (Pushing Lemma):** There exists positive \( c_1 \equiv c_1(n, k_i, x_i, \text{Stretch}) \), such that for every \( n \geq 1 \), with aspect length \( \rho \), one of the following two inequalities is satisfied,

\[
\mu_{\bar{H}}^{\text{Mixed}}[H_{\bar{H}}] \geq c_1^\rho, \quad \text{(PushPrimai)}
\]

or,

\[
\mu_{\bar{H}}^{\text{Mixed}'}[V_{\bar{H}}] \geq c_1^\rho, \quad \text{(PushDual)}
\]

for every \( \rho \geq 1 \), and the superscript Mixed denotes wired boundary conditions along the left, top and right sides of \( \bar{H} \), and free boundary conditions elsewhere. \( H \) is the same hexagonal box used in previous arguments for Lemma 1* . Under the PushDual condition, the analogous statement holds for the complement of vertical crossings across \( \overline{H} \), under dual boundary conditions (Mixed)' to Mixed.
Lemma 14* (Pushforward of horizontal and vertical crossings under mixed boundary conditions): There exists positive $c_2 = c_2(n, k_i, x_i, \text{Stretch})$ such that for every $n \geq 1$, with aspect length $\rho$, one of the following two inequalities is satisfied,

$$\mu_S^{(\text{Mixed})'}[\mathcal{H}_{\mathcal{F}}] \geq c_2^\rho,$$

(PushPrimal Strip)

or,

$$\mu_S^{(\text{Mixed})'}[\mathcal{V}_{\mathcal{F}}^{c}] \geq c_2^\rho,$$

(PushDual Strip)

for every $\rho \geq 1$. (Mixed)' denote the same boundary conditions from 13*, which manifests in the following.

7.2.2 Proof of Lemma 14* for PushPrimal Strip & PushDual Strip conditions

Proof of Lemma 14*. With some abuse of notation, we denote the hexagonal boxes for this proof as,

$$\mathcal{H}_i = [0, 2n] \times_H \left[ \frac{i}{3} n, \frac{i+1}{3} n \right],$$

for $i = 0, 1, 2$. Furthermore, we introduce the vertical segments along the bottom of each $\mathcal{H}_i$, and hexagons with same aspect length as those of each $\mathcal{H}_i$, in addition to hexagon of the prescribed aspect height below, respectively,

$$\mathcal{I}_i = \left[ \frac{i}{3} n, \frac{i+1}{3} n \right] \times \{0\},$$

$$\mathcal{K}_i = \left[ \frac{i}{3} n, \frac{i+1}{3} n \right] \times_H \left[ -n, n \right],$$

each of which are also indexed by $i$, with the exception that $i$ also runs over $i = 4, 5$. Before presenting more arguments for the connectivity between $\mathcal{I}_1$ and $\mathcal{I}_4$, suppose that either $\mu_S^{(\text{Mixed})'}[\mathcal{V}_{\mathcal{F}}^{c}] \geq \frac{1}{6}$, or $\mu_S^{(\text{Mixed})'}[\mathcal{H}_{\mathcal{F}}] \geq \frac{1}{6}$ for some $i$. In the first case for the pushforward of vertical crossings in $\mathcal{H}_i$, another application of the $\mu$ homeomorphism $f$ from arguments to prove Corollary 11* implies that PushPrimal Strip holds, while in the second case for the pushforward of horizontal crossings in $\mathcal{H}_i$, an application of the same homeomorphism implies that PushDual Strip holds. By complementarity, under $\mu_S^{(\text{Mixed})'}[\cdot]$, the pushforward of the following events respectively satisfy the lower bounds, as $\mu_S^{(\text{Mixed})'}[\mathcal{V}_{\mathcal{F}}^{c}] \geq \frac{5}{6}$, and $\mu_S^{(\text{Mixed})'}[\mathcal{H}_{\mathcal{F}}] \geq \frac{5}{6}$. The same argument that follows applies to lower bounds for crossing probabilities by other constants than $\frac{1}{6}$ or $\frac{5}{6}$ which are provided in [14], modifications to obtaining identical lower bounds in place of different constants are provided with the following.

With such estimates, under the same boundary conditions listed in PushPrimal Strip & PushDual Strip, the Spin measure satisfies

$$\mu_S^{(\text{Mixed})'}[\mathcal{V}_{\mathcal{F}}^{c} \cap \mathcal{H}_{\mathcal{F}} \cap \mathcal{V}_{\mathcal{F}}^{c}] \leq \mu_S^{(\text{Mixed})'}[\mathcal{H}_{\mathcal{F}}] \leq \mu_{\mathcal{F}}^{(\text{Mixed})'}[\mathcal{H}_{\mathcal{F}}],$$

where the upper bound for the probability of the intersection of the three events above only holds under boundary conditions in which the incident layer to the configuration (as given in arguments for the proof of Lemma 9*), the boundary conditions for the measure dominating (Mixed)' boundary conditions holds because every vertex that is wired in the (Mixed)' boundary conditions is also wired in the boundary
conditions for the pushforward in the upper bound. Moreover, the partition of boundary vertices in the boundary conditions for the upper bound is composed of the arc that is wired in the boundary conditions for γ, in addition to a singleton.

Under (Mixed)' boundary conditions introduced for the high-temperature spin measure above, the conditional probability

\[ \mu^{\text{Mixed}}_S [I_1 \leftrightarrow I_4 | \mathcal{H}_1] , \]

can be bound below by conditioning on a horizontal crossing \( \mathcal{H}_1 \) across \( \mathcal{H}_1 \). In particular, conditionally on \( \mathcal{H}_1 \), the connectivity event

\[ \mu^{(\text{Mixed})'}_S [I_1 \leftrightarrow I_4 | \mathcal{H}_1] , \]

can be bounded below as shown above through applications of (S − CBC) and (S − SMP). Each property is applied as follows; for (S SMP), we make use of previous partitions of the incident layer of hexagons to a configuration, in which (S − SMP) can only be applied when the outermost layer of a configuration can be partitioned into two equal sets over which the ± spin is constant.

Concluding, we apply standard arguments for the crossing event below through a lower bound dependent on a conditional probability,

\[ \mu^{\text{Mixed}}_S [I_1 \leftrightarrow I_4] \geq \mu^{\text{Mixed}}_S [I_1 \leftrightarrow I_4 | \mathcal{H}_1] \mu^{\text{Mixed}}_S [\mathcal{H}_1] \geq \mathcal{C} \left( \prod_{i=1}^{\alpha} n^{k_i} x^{e_i} \exp(h_i) \right)^{-1} , \]

from which an application of (FKG), given suitable \( \mathcal{C} > 0 \), for the countable intersection, dependent on \( i \), of horizontal crossings across hexagons of sufficiently small aspect length \( \text{Stretch}_i \). The inverse proportionality in the lower bound is dependent on the product \( \mathcal{T} \), defined in the proof for Lemma 1 on page 21, with \( i \) running over two configurations with respective number of connected components \( k_1 + 1 \) and \( k_2 + 1 \). The lower bound dependent on the edge weight \( x \) arises from multiple applications of (S − SMP) and (MON), in which the modification to (SMP) from the random cluster model argument with (S − SMP) for the Spin Measure results in comparisons between ± configurations and partitions of the incident layer as described in 5.2.

Instead, if we suppose that the lower bounds for \( \mu^{(\text{Mixed})'}_S [V_{\mathcal{H}_1}] \geq c'' \) for real \( c'' \), the lower bound on the second line above takes the form,

\[ \mu^{\text{Mixed}}_S [I_1 \leftrightarrow I_4 | \mathcal{H}_1] \geq \mathcal{C} c'' \left( \prod_{i=1}^{\alpha} n^{k_i} x^{e_i} \exp(h_i) \right)^{-1} . \]

due to the fact that the boundary conditions from the special case of the inequality, where the power to which the product of the edge weight and difference in monochromatically colored triangles is raised to the aspect ratio \( \text{Stretch} \) of \( \mathcal{H}_\text{Stretch} \), and the number of connected components in the exponent of \( n \) is the difference between the number of connected components of a ± configuration respectively sampled under \( \mu^{(\text{Mixed})'}_S \) and \( \mu^{\text{Mixed}}_S \).

Furthermore, the lower bound dependence on the edge weight \( x, n \) and \( e \), emerges from an application of (FKG) to the pushforward below of the connectivity event between \( I_1 \) and \( I_4 \), bounded below above,

\[ \mu^{\text{Mixed}}_S [I_1 \leftrightarrow I_4] \geq \mu^{\text{Mixed}}_S [I_1 \leftrightarrow I_4 | \mathcal{H}_1] \mu^{\text{Mixed}}_S [\mathcal{H}_1] , \]
which can be further bounded below by the product of crossing probabilities,

$$C' c'' \left( \prod_{i=1}^{\alpha} n^{k_i} x^{e_i} \exp(h_i) \right)^{-2},$$

Observe that the horizontal crossing pushed forwards in the inequality above yields the desired pushforwards in the PushDual condition, as the previously mentioned application of (FKG) yields,

$$\mu[H] \geq \mu[H_i]^\alpha \geq \left( C' c'' \left( \prod_{i=1}^{\alpha} n^{k_i} x^{e_i} \exp(h_i) \right)^{-2} \right)^\alpha \equiv (C' c'')^{-\alpha} \left( \prod_{i=1}^{\alpha} n^{k_i} x^{e_i} \exp(h_i) \right)^{-2\alpha},$$

for crossings across each of the hexagons $H = \cup_{i=1}^{\alpha} H_i$, and where the respective powers $k_i, e_i$ and $h_i$ appear in powers of the number of loops, edges, and exponential for the first external field. The form of the constant is provided in the lower bound above, and the same argument can be applied to obtain constant corresponding to the PushDual Strip for vertical hexagonal crossings to obtain the desired constant in the lower bound, concluding the proof. □

7.3 Lemma 13* arguments from Strip conditions

Proof of Lemma 13*. We show that either PushPrimal Strip ⇒ PushPrimal, or that PushDualStrip ⇒ PushDual. Without loss of generality, suppose that PushDual Strip holds; to show that PushDual holds, we introduce the following collection of similarly defined boxes from arguments in 14* on the previous page,

$$\widetilde{H}_i = [0, \rho n] \times H \left[ \frac{i}{3n}, \frac{i + 1}{3n} \right],$$

for $1 \leq i \leq N$, with $N$ sufficiently large. Under (Mixed)' boundary conditions,

$$\mu^{(Mixed)'}[\gamma_{\widetilde{H}_N}] \geq c^\rho,$$

the probability of a complement of the vertical crossing across $\widetilde{H}_N$, and can be bounded below by $c^\rho$ because by assumption PushPrimal Strip holds. Clearly, the probability of obtaining a vertical crossing across the last rectangle over all $i$ can be determined by applying the FKG inequality across each of the $N$ smaller hexagons, yielding an upper bound of $c^{N\rho}$ to the probability of obtaining a longer $N$-hexagon crossing.

Next, with similar conditioning on horizontal crossings in previous arguments, the probability of a horizontal crossing across $\widetilde{H}_i$, given the occurrence of a horizontal crossing across $\widetilde{H}_{i+1}$, satisfies for every $i$,

$$\mu^{(Mixed)'}[\gamma_{\widetilde{H}_i}] \geq c^\rho,$$
with the exception that the pushforward $\overset{\sim}{\mathcal{H}}_{i+1}$, taken under (Mixed)$'$ boundary conditions, in comparison to previous arguments for the wired pushforward

$$\mu_{\mathcal{H}}^{1_H} V_{(j+i+\delta_k=j+i]} \bigg| \overset{\sim}{\mathcal{H}}_{i+1}$$

below by $e^{-c}$ for Corollary 11*, can also be applied to bound the intersection of conditional events, for the event $\{ V^{\mathcal{E}} \big| V^{\mathcal{F} \cap \mathcal{G}} \}$, for all $i$,

$$\prod_{0 \leq i \leq N} \mu_{\mathcal{H}}^{(\text{Mixed})'} V^{\mathcal{E}} \big| V^{\mathcal{F} \cap \mathcal{G}} \mathcal{H}_{i+1} \geq c_N \rho,$$

implying that the identical lower bound from the PushPrimal Strip holds, across the countable intersection of horizontal crossings,

$$\mu_{\mathcal{H}}^{(\text{Mixed})'} V^{\mathcal{E}} \overset{\sim}{\mathcal{H}}_{i+1} \geq c_N \rho.$$

We conclude the argument, having made use of the previous application of FKG across $0 \leq i \leq \lambda - 1$, uniformly in boundary conditions (Mixed)$'$. □

8 Renormalization inequality

We now turn to arguments for the Renormalization inequality. We make use of notation already given in the proof for the vertical and horizontal strip inequalities of Lemma 1*, namely that we make use of a similar partition of the hexagons to the left and right of some $\mathcal{H}$. To restrict the crossings to occur across hexagons of smaller aspect length, we change the assumptions on our choice of $n_i$ and follow the same steps in the argument of Lemma 1* to obtain a lower bound for the pushforward $\mu_{\mathcal{H}}^{1_H} V^{E \cap F \cap G}$, where $E$ denotes the event that each of the three boxes $\overset{\sim}{\mathcal{H}}, \mathcal{H}^{+}, \mathcal{H}^{-}$ which are defined in arguments below. The partition of the aspect length of $\overset{\sim}{\mathcal{H}}, \mathcal{H}^{+}, \mathcal{H}^{-}$ is dependent on $i$. Also, the smaller scale over which we force the horizontal crossings to occur in $E$ is reflected in the partition of the aspect length, which not surprisingly permits for applications of (FKG) with domains that are indexed by an auxiliary parameter for $0 \leq i \leq \lambda - 1$. The partition of $\mathcal{H}$ into the three boxes $\overset{\sim}{\mathcal{H}}, \mathcal{H}^{+}, \mathcal{H}^{-}$ determines corresponding powers, dependent on $\lambda$ to which the horizontal or vertical strip densities are raised before taking $\rho \rightarrow \infty$. As previously mentioned, differences in $(S - \text{SMP})$ emerge in one step of the following argument. We discuss the arguments for the proof when PushPrimal holds, and in the remaining case when PushDual holds, a modification to the argument is provided.

8.1 Arguments for obtaining renormalization inequalities in the thermodynamic limit

Proof of Lemma 2*. Suppose that PushDual holds; the PushPrimal case will be discussed at the end. In light of the brief remark of the argument at the beginning of the section, we introduce the three boxes to partition the middle of $\mathcal{H}$ from the 1* proof,

$$\overset{\sim}{\mathcal{H}}_{i}^{1-} = [0, \rho n] \times H \ (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_1 \text{Stretch}, (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_2 \text{Stretch},$$

$$\overset{\sim}{\mathcal{H}}_{i}^{1+} = [0, \rho n] \times H \ (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_3 \text{Stretch}, (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_4 \text{Stretch},$$

$$\overset{\sim}{\mathcal{H}}_{i}^{-} = [0, \rho n] \times H \ (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_1 \text{Stretch}, (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_2 \text{Stretch},$$

$$\overset{\sim}{\mathcal{H}}_{i}^{+} = [0, \rho n] \times H \ (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_3 \text{Stretch}, (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_4 \text{Stretch},$$

$$\overset{\sim}{\mathcal{H}}_{i}^{+} = [0, \rho n] \times H \ (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_1 \text{Stretch}, (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_2 \text{Stretch},$$

$$\overset{\sim}{\mathcal{H}}_{i}^{-} = [0, \rho n] \times H \ (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_3 \text{Stretch}, (2i)\text{Stretch} + \beta \text{Stretch} + \alpha_4 \text{Stretch},$$
for every \(0 \leq i \leq \lambda - 1\), and will apply steps of the argument from the proof of Lemma \(1^*\), in which we modify all pushforwards under the prescribed boundary conditions for \(\tilde{\mathcal{E}}\). By construction, the boxes \(\mathcal{H}_i, \mathcal{H}_i^+, \text{ and } \mathcal{H}_i^-\), each have the same \(\rho\) aspect ratio, yet differ in the increment of the factors \(\tilde{\alpha}_i \in \beta \mathbb{N}\), given \(\beta\) sufficiently large, is given by,

\[
\tilde{\alpha}_i = 1 + \frac{1}{\beta + i - 1} .
\]

Briefly, we recall the steps with the sequence of inequalities below. Under one simple modification through the lower bound, we analyze the intersection of crossing probabilities as given in (FKG), implying,

\[
\mu^1_{\mathcal{H}}[\tilde{\mathcal{E}}] \geq \prod_{0 \leq i \leq \lambda - 1} \mu^1_{\mathcal{H}}[\mathcal{H}_i] \geq \left( \frac{1}{(\lambda \gamma)^{\rho}} \right) \lambda^\rho ,
\]

from which the conditional probability dependent on \(\tilde{\mathcal{E}}\) can be bound from below as follows,

\[
\mu^1_{\mathcal{H}}[\mathcal{F} | \tilde{\mathcal{E}}] \geq \prod_{0 \leq i \leq \lambda - 1} \mu^1_{\mathcal{H}}[V_{\mathcal{H}_i}^c] \geq \left( \mu^1_{[0, \rho n] \times H [0, n \lambda \text{Stretch}] | \mathcal{H}_1} \right)^{\lambda + 1} .
\]

Further arguments result in the following lower bound for the probability of \(\{\tilde{\mathcal{E}} \cap \mathcal{F} \cap \mathcal{G}\}\),

\[
\mu^1_{\mathcal{H}}[\tilde{\mathcal{E}} \cap \mathcal{F} \cap \mathcal{G}] = \mu^1_{\mathcal{H}}[\tilde{\mathcal{E}} \cap \mathcal{F}] \mu^1_{\mathcal{H}}[\mathcal{F} \cap \mathcal{G}] \geq c^\lambda \text{Stretch}_\mathcal{E} \mu^1_{\mathcal{H}}[\mathcal{E} \cap \mathcal{F}] ,
\]

which is the same lower bound provided in (FKG) – (S – SMP). The exponents of the number of edges \(x\) and the exponential of the first external field are respectively parametrized with respect to the number of edges \(e\), and the number of hexagons \(\mathcal{H}\) that are not monochromatically colored. On the other hand, under the PushDual condition, the conditional pushforward under wired boundary conditions supported over \(\mathcal{H}\) satisfies,

\[
\mu^1_{\mathcal{H}}[\tilde{\mathcal{F}} | \tilde{\mathcal{E}} \cap \mathcal{F} \cap \mathcal{G}] \geq \mu^1_{\mathcal{H}}[\tilde{\mathcal{F}} | \mathcal{F}] \mu^1_{\mathcal{H}}[\mathcal{F} \cap \mathcal{G}] \geq c^{\lambda \alpha \text{Stretch}} ,
\]

which will be used to complete the remaining steps from the \(1^*\) proof. In particular, the intersection \(\{\tilde{\mathcal{E}} \cap \mathcal{F} \cap \mathcal{G}\}\) can be bounded above by the product of \(\lambda\) horizontal crossings below, from (FKG),

\[
\mu^1_{\mathcal{H}}[\tilde{\mathcal{E}} \cap \tilde{\mathcal{F}} \cap \mathcal{G}] = \mu^1_{\mathcal{H}}[\tilde{\mathcal{E}} | \mathcal{F}] \mu^1_{\mathcal{H}}[\mathcal{F} \cap \mathcal{G}] \mu^1_{\mathcal{H}}[\tilde{\mathcal{E}} \cap \mathcal{F} \cap \mathcal{G}] \geq c^{\lambda \alpha \text{Stretch}} \mu^1_{\mathcal{H}}[\mathcal{E} \cap \mathcal{F} \cap \mathcal{G}] ,
\]

through the same application of (S – SMP) and (MON), and where \(\mathcal{F}\) denotes the crossing event that neither of the three hexagonal boxes defined at the beginning of the proof are vertically crossed. As a result, the last application of (FKG) yields, for \(\lambda\) horizontal crossings across thinner hexagons,

\[
\mu^1_{\mathcal{H}}[\tilde{\mathcal{E}} \cap \tilde{\mathcal{F}} \cap \mathcal{G}] \leq \left( \mu^1_{[0, \rho n] \times H [0, n \lambda \text{Stretch}] | \mathcal{H}_1} \right)^{\lambda + 1} ,
\]
under free boundary conditions.

Finally, comparing the pushforward under free boundary conditions to the pushforward under wired boundary conditions yields, after taking the same infinite aspect length limit as in Section 7. From previous applications of \((S \rightarrow \text{SMP})\) and \((\text{MON})\) are used, in order to suitably compare boundary conditions, as a consequence imply a similar estimate as in Section 7,

\[
\left( \mu^0_{[0,\rho n] \times_H [0,n\lambda \text{Stretch}]} [\mathcal{H}_{[0,\rho n] \times_H [0,n\lambda \text{Stretch}]}] \right)^{\lambda^\rho} \xrightarrow{\rho \to +\infty} \left( p_{\text{Stretch} n} \right)^{\lambda},
\]

which, as in previous arguments for Lemma 1\(^a\), is bounded below by the following infinite aspect ratio limit,

\[
\left( \mu^1_{[0,\rho n] \times_H [0,n\lambda \text{Stretch}]} [\mathcal{V}_{c}^{c}_{[0,\rho n] \times_H [0,n\lambda \text{Stretch}]}] \right)^{\lambda+1} \xrightarrow{\rho \to +\infty} \frac{1}{\lambda^c} \left( q_{\text{Stretch} n} \right)^{\lambda+1},
\]

with the exception that the support of the measure with free (-) boundary conditions is over a hexagon with thinner aspect length. The result corresponds to the renormalization inequality for the horizontal crossing probability, concluding the argument under the PushDual assumption. Below, we briefly describe how the same sequence of inequalities applies for the remaining possibility.

Suppose that PushPrimal holds. Under this assumption, denote \(\tilde{F}\) as the crossing event that none of the boxes \(\tilde{H}^\pm_i\) are vertically crossed. From this event, the assumption implies from the definition of the horizontal and vertical strip densities for the Spin Measure that the arguments to bound the conditional probability can be achieved by the same line of argument, possibly with larger \(C\).

9 Quadrichotomy proof

In the final section we classify all possible behaviors of the model. Briefly, we remark that for the continuous critical case, the first part of the argument does not require use of (SMP) and (CBC) for original results in the random cluster model, implying that the entirety of the argument immediately applies. Briefly, we summarize the steps of the argument. We consider horizontal crossing events across a regular hexagon, pushed forwards under free boundary conditions for the Spin measure supported over a slightly larger hexagon. From knowledge of the longest edge in the + path of the horizontal crossing, removing the largest edge from the configuration easily yields a connectivity event along the common set of edges over a subgraph of the triangular lattice that excludes the length of the maximal edge along two points \(x\) and \(y\). These steps demonstrate the ingredients for the Discontinuous Critical case, before obtaining the horizontal strip densities in the infinite aspect length limit as \(\rho \to \infty\). For the discontinuous critical case, the second part of the argument requires use of \((S \rightarrow \text{CBC})\) and \((\text{MON})\). Before proceeding, we cite the following theorem which classifies the probability of obtaining loop configurations of fixed length in the model. In the following application of the inequalities, we choose an aspect ratio of hexagons dependent on \(\rho\), from which horizontal and vertical crossings will be studied.

**Theorem 3\(^a\) (Continuous & Discontinuous critical cases, behaviors of the dilute Potts model quadrichotomy, [8]):** For configurations distributed under the \(\mu\), for \(n \geq 1\) and \(x = \frac{1}{\sqrt{n}},\) one of the two possible behaviors occurs,

\[
\mu^S_{\tilde{\mathcal{A}}} \left[ R \geq k \right] \leq \exp(-ck), \quad \text{(Exponential decay of + paths)}
\]

where \(R\) is the diameter of the largest loop surrounding the origin, demonstrating that + paths are exponentially unlikely for any \(k \geq 1\), or,

\[
c \leq \mu^S_{\tilde{\mathcal{A}}} \left[ \text{there exits a + path which horizontally crosses a rectangle over the triangular lattice} \right] \leq 1 - c, \quad \text{(RSW box-crossing property)}
\]
demonstrating that the RSW box-crossing property is satisfied. Each possibility holds for boundary conditions \( \tau \in \{-1, +1\}^T \) and \( c > 0 \).

Observe that we have slightly rephrased the first condition provided in [8] which is stated instead for the loop measure \( \mathbf{P} \), of Section 3.1. The equivalent condition of obtaining a loop configuration whose largest diameter about the origin is \( k \) is equivalent to obtaining a path of + spins about the origin. From the statement of Theorem 3*, we now study 2*.

### 9.1 Subcritical & Supercritical behaviors

**Proof of Theorem 2* (Discontinuous Critical phase from non Subcritical phase).** As mentioned at the beginning of the section, first suppose that the first possibility holds. To show that this condition implies that the phase transition is discontinuous, consider the following. Define a horizontal crossing across \( H \). From the existence of such an event, the longest edge in the crossing of arbitrary length \( L \) then excluding the length of this longest edge from the crossing implies that another closely related crossing event occurs across a subgraph of the triangular lattice which excludes the maximal edge with length \( L \). Hence there exists vertices in a subgraph of the triangular lattice, such that the vertices \( x \) and \( y \) are connected by a + path in a hexagon of smaller aspect length that is not regular. Collecting these observations implies the following, where the upper and lower bounds of the inequality are taken under − boundary conditions, by the union bound,

\[
\mu^0_{[0, \rho n] \times H[0, 2n]} [H[0, \rho n] \times H[0, n]] \leq c n^2 \mu^0_{[0, \rho n] \times H[0, 2n]} [x [0, \rho n] \times H[0, n]] \leftarrow (x [0, \rho n] \times H[0, n]) \rightarrow y],
\]

where \( x \) and \( y \) are the vertices, with \( c \) an arbitrary positive constant. For the next step, we introduce horizontal translates of \( x \) with \( x_k = x + (4kn, 0) \). Across all horizontal translates of \( x \), yields the following lower bound for the connectivity event between \( x \) and each \( x_k \), by (MON) and (FKG),

\[
\mu^0_S [x \longleftrightarrow x_k] \geq \frac{1}{cn^2} \mu^0_{[0, \rho n] \times H[0, 2n]} [x [0, \rho n] \times H[0, n]] \leftarrow (x [0, \rho n] \times H[0, n]) \rightarrow y],
\]

From previous remarks, the first upper bound given in the proof dependent on \( c \) yields the inequality, as applied in (FKG) several times previously in the argument,

\[
\mu^0_S [x \longleftrightarrow x_k] \geq \frac{1}{cn^2} \mu^0_{[0, \rho n] \times H[0, 2n]} [H[0, \rho n] \times H[0, n]]]^{2k},
\]

from which taking the infinite limit as in previous arguments implies, for \( k \rightarrow +\infty \),

\[
p^2_{2n} \geq \frac{1}{cn^2} \mu^0_{[0, \rho n] \times H[0, 2n]} [H[0, \rho n] \times H[0, n]],
\]

so that the pushforward of the spin measure under free boundary conditions satisfies the strip density estimate from the original definition provided in the beginning of Section 7, from the connected components of + paths from the occurrence of \( \{x \longleftrightarrow \infty\} \). Finally, we observe that the upper bound for the horizontal strip density decays exponentially fast, implying that the pushforward in the lower bound taken under free boundary conditions does as well. As expected, to analyze the other possibility for infinitely long vertical crossings, repeating the same steps of the argument, with the exception that the horizontal crossing event is instead a vertical crossing event, simply yields a similar bound, from an application of 12* for some integer \( \lambda \) satisfying the conditions of the Lemma, that the probability of obtaining an infinitely long vertical crossings is an upper bound of the following inequality,
Figure 2: The configuration of finite volumes across which crossing events are quantified to obtain RSW results in the proof of the remaining case of Theorem 2∗.

\[ q_{2n}^2 \geq \frac{1}{cn^2} \left( 1 - \mu(1; \rho n; [0, 2n] \times \mathbb{H}[0, n]) \right), \]

which nevertheless still exponential decays for the same reason as \( k \to \infty \). This conclude the argument for the model, demonstrating that the horizontal and vertical strip densities hold for infinite aspect ratios. \( \square \)

In the following, we analyze the Continuous critical case to obtain RSW results.

**Proof of Theorem 2∗ (Continuous Critical phase from non Subcritical phase).** To prove RSW results, consider the following four finite volumes, with \( \mathcal{R}_1 \subset \mathcal{R}_2 \), and \( \rho, n > 0 \),

\[
\begin{align*}
\mathcal{R}_1 &\equiv [0, \rho n] \times \mathbb{H}[0, n], \\
\mathcal{R}_2 &\equiv [-n, \rho n + n] \times \mathbb{H}[-n, 2n], \\
\mathcal{R}_3 &\equiv \left[ -\frac{2n}{3}, -\frac{n}{3} \right] \times \mathbb{H}[-n, 2n], \\
\mathcal{R}_4 &\equiv \left[ \rho n + \frac{n}{3}, \rho n + \frac{2n}{3} \right] \times \mathbb{H}[-n, 2n].
\end{align*}
\]

We bound the crossing probability of a the horizontal crossing event below with the product of vertical and horizontal strip densities (from Definition 1∗). To ensure that the appropriate cancellation of crossing events occurs, we bound the probability of a horizontal crossing event with the horizontal strip density \( p_n \), in which,
\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1}] \geq p_n^\alpha \rho ,
\]

for every \( \alpha \geq 1 \) and Mixed boundary conditions (all vertices along the 2 edges of the hexagon on the left and right are wired), due to the fact that,

\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1}] \geq \left( \mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1}] \right)^\alpha \geq \mu_{[0,\rho n] \times H[0,\lambda \text{Stretch}]}^{\alpha}[H_{[0,\rho n] \times H[0,\lambda \text{Stretch}]}] ,
\]

for \( \beta_2 \equiv [-\alpha n, \alpha \rho n + n] \times H \left[ -n, 2n \right] \), and \( \beta_1^{\alpha} \equiv [0, \alpha \rho n] \times H [0, n] \), raising the inequality to \( \frac{1}{\rho} \) as \( \rho \to \infty \). On the other hand, below we bound the probability of the following intersection of crossing events, conditionally on \( H_{\beta_1} \),

\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1} \cap H_{\beta_4} | H_{\beta_1}] ,
\]

which can similarly be lower bounded with the vertical crossing strip density, as,

\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1} \cap H_{\beta_4} | H_{\beta_1}] \geq q_\gamma n ,
\]

for \( \gamma \) sufficiently large, and as the inequality is raised to \( \frac{1}{\rho} \) as \( \rho \to \infty \). The two inequalities can be bounded below by the product of horizontal and vertical strip densities,

\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1} \cap H_{\beta_4}] \geq \mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1} \cap H_{\beta_3} \cap H_{\beta_4}] \geq p_n^\alpha q_\gamma n .
\]

Finally, the ultimate term in the inequality above can be bounded above with the horizontal crossing event of interest, as the conditioning on the other two horizontal crossing events is removed,

\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1}] \geq \mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1} | H_{\beta_3} \cap H_{\beta_4}] .
\]

Altogether, by duality and rotational invariance of \( \mu \), one obtains,

\[
\mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1}] \leq 1 - p_n^\alpha q_\gamma n ,
\]

where Mixed’ is a rotation of Mixed, implying that the RSW inequality for horizontal crossing events is, with respective constants \( c \) and \( 1 - c \) provided in the lower and upper bounds,

\[
p_n^\alpha q_\gamma n \leq \mu_{\beta_2}^{\text{Mixed}}[H_{\beta_1}] \leq 1 - p_n^\alpha q_\gamma n ,
\]

independently of boundary conditions \( \xi \), hence concluding the proof because the Continuous Critical phase occurs. \( \square \)
9.2 Applications of different phases of the quadrichotomy

9.2.1 Subcritical regime

Proposition A (coexistence of wired and free high-temperature measures). The probability of connectivity to distance $n$ under the wired high-temperature Loop $O(n)$ measure is exponentially upper bounded, as

$$
\mu^1_{\mathcal{H}_n}[0 \leftrightarrow \partial \mathcal{H}_n] \leq \exp(-cn),
$$

also implying that $\mu^0 = \mu^1$.

Proof of Proposition A. To show that $\mu^0 = \mu^1$ in Subcritical, we lower bound the horizontal crossing probability in terms of crossings of loops to the left and right sides, respectively $\mathcal{L}_n$ and $\mathcal{R}_n$ of $\mathcal{H}_n$, in which,

$$
\mu^1_{\mathcal{H}_{2n}}[\mathcal{H}_n \leftrightarrow \mathcal{H}_n] \geq \mu^1_{\mathcal{H}_{2n}}[0 \leftrightarrow \mathcal{L}_n] \mu^1_{\mathcal{H}_{2n}}[0 \leftrightarrow \mathcal{R}_n] \geq \frac{1}{32} \mu^1_{\mathcal{H}_{2n}}[0 \leftrightarrow \partial \mathcal{H}_n].
$$

As a consequence, by ($S - \text{SMP}$) the probability of connectivity to distance $n$ decays exponentially fast. The fact that the high-temperature measure under wired and free boundary conditions coincides follows from classical arguments.

9.2.2 Supercritical regime

Proposition B (exponential unlikelihood of obtaining finite connected components). In Supercritical, the probability of an infinite connected component under free boundary conditions of being absent is exponentially unlikely, as

$$
\mu^0_{\mathcal{H}_n}[\mathcal{H}_n \not\leftrightarrow \infty] \leq \exp(-cn).
$$

Proof of Proposition B. By duality, if an infinite connected component does not exist in the primal configuration, then an infinite connected component exists in the dual configuration that is measurable over $T$. With the dual configuration, there does exist a loop whose maximum diameter is $n$, implying that the standard connectivity event at distance $n$ does occur. The inequality follows.

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