Möbius disjointness conjecture for local dendrite maps

El Houcein El Abdalaoui¹, Ghassen Askri²
and Habib Marzougui³ ⁴

¹ Department of Mathematics, E. H. El Abdalaoui, Normandy University of Rouen,
LMRS UMR 6085 CNRS, Avenue de l’Université, BP.12, 76801 Saint Etienne du
Rouvray, France
² Ghassen Askri, University of Carthage, Bizerte Preparatory Engineering Institute,
Jarzouna, 7021, Tunisia
³ Habib Marzougui, University of Carthage, Faculty of Science of Bizerte,
(U.R17ES21), ‘Dynamical systems and their applications’, 7021, Jarzouna, Tunisia

E-mail: elhoucein.elabdalaoui@univ-rouen.fr, askri.ghassen@gmail.com,
elaskri.ghassen@ipeib.u-carthage.tn, hmarzoug@ictp.it and
habib.marzougui@fsb.rnu.tn

Received 16 May 2018, revised 1 October 2018
Accepted for publication 18 October 2018
Published 20 December 2018

Abstract
We prove that the Möbius disjointness conjecture holds for graph maps with
zero topological entropy and for all monotone local dendrite maps. We further
show that this also holds for continuous maps on certain class of dendrites.
Moreover, we see that there is an example of transitive dendrite map with zero
entropy for which Möbius disjointness conjecture holds.

Keywords: dendrite, graph, local dendrite, Möbius function, Möbius
disjointness conjecture, Sarnak conjecture, ω-limit set, minimal set
Mathematics Subject Classification numbers: 37B05, 37B45, 37E99

1. Introduction

Let X be a compact metric space with a metric d and let f : X → X be a continuous map. We
call for short (X, f) a dynamical system. The topological entropy h(f) of such a system is
defined as:

\[ h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \text{sep}(n, f, \varepsilon) \]

⁴ Author to whom any correspondence should be addressed.
where for \( n \) integer and \( \varepsilon > 0 \), \( \text{sep}(n, f, \varepsilon) \) is the maximal possible cardinality of an \((n, f, \varepsilon)\)-separated set in \( X \), this later means that for every two points of it, there exists \( 0 \leq j < n \) with \( d(f^j(x), f^j(y)) > \varepsilon \), where \( f \) denotes the \( j \)th iterate of \( f \). A dynamical system \((X, f)\) is called a null system if its sequence entropy is zero for any sequence; we refer the reader to \([24, 29, 30]\) for the details. The Möbius function \( \mu \) is an ally of the Liouville function \( \lambda \). This later function is defined by \( \lambda(n) = 1 \) if the number of prime factors of \( n \) is even and \(-1\) otherwise. Precisely, the Möbius function is given by

\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
\lambda(n) & \text{if all primes in decomposition of } n \text{ are distinct} \\
0 & \text{otherwise}.
\end{cases}
\]

In 2010, Sarnak [46, 47] initiated the study of the dynamical system generated by the Möbius function, and in the connection with the Möbius randomness law, he stated the following conjecture:

**Sarnak’s conjecture.** Let \((X, f)\) be a dynamical system with \( h(f) = 0 \). Then

\[
S_N(x, \varphi) := \frac{1}{N} \sum_{n=1}^{N} \mu(n) \varphi(f^n(x)) = o(1), \text{as } N \to +\infty
\]  

(1.1)

for each \( x \in X \) and each continuous function \( \varphi : X \to \mathbb{R} \).

We recall that the Möbius randomness law [31] asserts that for any ‘reasonable’ sequence \((a_n)\), we have

\[
\frac{1}{N} \sum_{n=1}^{N} \mu(n)a_n = o(1), \text{as } N \to +\infty.
\]

It turns out that Sarnak’s conjecture (1.1) is connected to the popular Chowla conjecture on the multiple autocorrelations of the Möbius function. This later conjecture asserts that for any \( r \geq 0, 1 \leq a_1 < \cdots < a_r \), for any \( 0 \leq s \leq r \), \( i_s \in \{1, 2\} \) not all equal to 2, we have

\[
\sum_{n \leq N} \mu(i_0(n)) \mu(i_1(n + a_1)) \cdots \mu(i_r(n + a_r)) = o(N).
\]  

(1.2)

The Chowla conjecture implies a weaker conjecture stated by Chowla in [19]. We refer to [19] for the statement of this weaker form of Chowla conjecture. For more details on the connection between Sarnak and Chowla conjectures we refer to the very recent works of the first author [1], Tao [49], Gomilko–Kwietniak–Lemańczyk [22] and Tao–Teräväinen [50].

Note, that in the simplest case, when \( f \equiv \text{const} \), (1.1) is equivalent to the statement

\[
\frac{1}{N} \sum_{n=1}^{N} \mu(n) = o(1), \text{as } N \to +\infty
\]

which is equivalent to the prime number theorem [7]. The conjecture (1.1), also known as the Möbius disjointness conjecture is known to be true for several dynamical systems, see e.g. ([2, 3, 21, 28, 29, 33, 39, 42]) and the references therein. In [33], Karagulyan proved the conjecture for the orientation preserving circle-homeomorphisms and for continuous interval maps of zero entropy. For another proof, see also [32]. In the present paper, we are interested in another natural classes of dynamical systems: the graph, dendrite and local dendrite maps. We thus establish that for the graph maps with zero entropy and for all monotone local dendrite maps, Sarnak’s conjecture holds. We are also able to prove that the Möbius disjointness property holds for a deterministic class of dendrites for which the set of endpoints is closed.
and its derived set is finite. This extends Karagulyan result on the Möbius disjointness of any interval maps with zero entropy and (orientation preserving) circle homeomorphisms.

Recent interests in dynamics on graphs and local dendrites is motivated by the fact that graphs and local dendrites are examples of Peano continua with complex topology structures (e.g. [43], pp 165–87). On the other hand, dendrites often appear as Julia sets in complex dynamics (see [11]). After finishing this version, we learned that Li et al had recently solved the conjecture for graph maps [37]. Notice that our proof of theorem 3.1 and that of [37] are different. Indeed, in their proof, they need a more stronger dynamical property based on the notion of locally mean equicontinuous.

In this paper, we also investigate the Möbius disjointness conjecture for transitive dendrite maps. It turns out that we are able to establish this conjecture for the transitive dendrite map with zero topological entropy introduced by Byszewski et al [18].

According to the recent result of Li et al [38], our investigation can be seen as a deep investigation on Sarnak’s conjecture. Indeed, the authors therein proved that if the Möbius disjointness conjecture holds for any Gehman dendrite map with zero entropy then Sarnak’s conjecture holds.

We further discuss the problem of Möbius disjointness for the transitive dendrite map with positive topological entropy introduced by Špitalky [48]. At this point, let us point out that Sarnak mentioned in his paper [46] that Bourgain constructed a topological dynamical system with positive topological entropy for which the Möbius disjointness holds. Later, Downarowicz and Serafin constructed a class of topological systems with positive topological entropy which satisfy the Möbius randomness law [20].

The plan of the paper is as follows. In section 2, we give some definitions and preliminary properties on graphs, dendrites and local dendrites which are useful for the rest of the paper. Section 3 is devoted to the proof of theorem 3.1 for graph maps of zero entropy. Section 4 is devoted to local dendrite maps of zero entropy. In section 4.1 we will prove theorem 4.1 for monotone local dendrite maps. In section 4.2, we prove the conjecture for continuous map on a certain class of dendrites. Section 4.3, is devoted to the conjecture for an example of transitive dendrite map with zero entropy.

Finally, in section 4.4, we discuss the problem of Möbius disjointness for an example of transitive dendrite map with positive entropy.

2. Preliminaries and some results

Let \( \mathbb{Z}, \mathbb{Z}_+ \) and \( \mathbb{N} \) be the sets of integers, non-negative integers and positive integers, respectively. For \( n \in \mathbb{Z}_+ \), denote by \( f^n \) the \( n \)-th iterate of \( f \); that is, \( f^0 = \text{identity} \) and \( f^n = f \circ f^{n-1} \) if \( n \in \mathbb{N} \). For any \( x \in X \), the subset \( \text{Orb}_f(x) = \{ f^n(x) : n \in \mathbb{Z}_+ \} \) is called the orbit of \( x \) (under \( f \)). A subset \( A \subseteq X \) is called \( f \)-invariant (resp. strongly \( f \)-invariant) if \( f(A) \subseteq A \) (resp., \( f(A) = A \)). It is called a minimal set of \( f \) if it is non-empty, closed, \( f \)-invariant and minimal (in the sense of inclusion) for these properties, this is equivalent to say that it is an orbit closure that contains no smaller one; for example a single finite orbit. When \( X \) itself is a minimal set, then we say that \( f \) is minimal. We denote by \( \overline{A} \) the closure of \( A \) and by \( \text{diam}(A) \) the diameter of \( A \). We define the \( \omega \)-limit set of a point \( x \) to be the set:

\[
\omega_f(x) = \{ y \in X : \exists n_i \in \mathbb{N}, n_i \to \infty, \lim_{i \to +\infty} d(f^{n_i}(x), y) = 0 \}.
\]

A point \( x \in X \) is called:

- periodic of period \( n \in \mathbb{N} \) if \( f^n(x) = x \) and \( f^i(x) \neq x \) for \( 1 \leq i < n - 1 \); if \( n = 1 \), \( x \) is called a fixed point of \( f \) i.e. \( f(x) = x \).

- Almost periodic if for any neighborhood \( U \) of \( x \), there is \( N \in \mathbb{N} \) such that \( \{ f^{i+k}(x) : i = 0, 1, \ldots, N \} \cap U \neq \emptyset \), for all \( k \in \mathbb{N} \). It is well known (see e.g. [12], chapter
We need the following lemmas. In particular, if $C$ of $X$ is a homeomorphism, then it is monotone. Following ([8], corollary 3.6), if $f$ is a dendrite with $E(X)$ is a pair of $X$ and $X$ is said to be a Li–Yorke pair of $f$ if it is proximal but not asymptotic.

In this section, we recall some basic properties of graphs, dendrites and local dendrites. A continuum is a compact connected metric space. An arc is any space homeomorphic to the compact interval $[0, 1]$. A topological space is arcwise connected if any two of its points can be joined by an arc. A circle is any space homeomorphic to the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We use the terminologies from Nadler [43].

By a graph $X$, we mean a continuum which can be written as the union of finitely many arcs such that any two of them are either disjoint or intersect only in one or both of their endpoints. In particular, arcs and circles are graphs. For any point $v$ of $X$, the order of $v$, denoted by $\text{ord}(v)$, is an integer $r \geq 1$ such that $v$ admits a neighborhood $U$ in $X$ homeomorphic to the set $\{z \in \mathbb{C} : z' \in [0, 1]\}$ with the natural topology, with the homeomorphism mapping $v$ to 0. If $r \geq 3$, then $v$ is called a branch point. If $r = 1$, then we call $v$ an endpoint of $X$. If $r = 2$, $v$ is called a regular point of $X$.

Denote by $B(X)$ and $E(X)$ the sets of branch points and endpoints of $X$ respectively. An edge is the closure of some connected component of $X \setminus B(X)$, it is an arc. A sub-graph of $X$ is a subset of $X$ which is a graph itself. Every sub-continuum of a graph is a graph ([43], corollary 9.10.1).

By a dendrite $X$, we mean a locally connected continuum containing no circle. Every sub-continuum of $X$ is a dendrite ([43], theorem 10.10) and every connected subset of $X$ is arcwise connected ([43], proposition 10.9). In addition, any two distinct points $x, y$ of $X$ can be joined by a unique arc with endpoints $x$ and $y$, denoted by $[x, y]$. A point $x \in X$ is called an endpoint if $X \setminus \{x\}$ is connected. It is called a branch point if $X \setminus \{x\}$ has more than two connected components. The number of connected components of $X \setminus \{x\}$ is called the order of $x$ and denoted by $\text{ord}(x)$. Denote by $E(X)$ and $B(X)$ the sets of endpoints, and branch points of $X$, respectively. By ([35], theorem 6, p 304 and theorem 7, p 302), $B(X)$ is at most countable. A point $x \in X \setminus E(X)$ is called a cut point. It is known that the set of cut points of $X$ is dense in $X$ ([35], VI, theorem 8, p 302). Following ([8], corollary 3.6), if $X$ is a dendrite with $E(X)$ closed, then $B(X)$ is discrete. An arc $I$ of $X$ is called free if $I \cap B(X) = \emptyset$.

By a local dendrite $X$, we mean a continuum every point of which has a dendrite neighborhood. A local dendrite is then a locally connected continuum containing only a finite number of circles ([35], theorem 4, p 303). As a consequence, every sub-continuum of a local dendrite is a local dendrite ([35], theorems 1 and 4, p 303). Every graph and every dendrite is a local dendrite. Also $X_\infty$ is a sub-local dendrite of $X$. A continuous map from a local dendrite (resp. graph, resp. dendrite) into itself is called a local dendrite map (resp. graph map, resp. dendrite map). It is well known that every dendrite map has a fixed point (see [43]).

For every topological space $X$, a map $f : X \to X$ is called monotone if $f^{-1}(C)$ is connected for any connected subset $C$ of $X$. In particular, if $f$ is a homeomorphism, then it is monotone. We need the following lemmas.
Lemma 2.1 ([40], lemma 2.3). Let \( X \) be a dendrite, \((C_i)_{i \in \mathbb{N}}\) be a sequence of pairwise disjoint connected subsets of \( X \). Then \( \lim_{n \to +\infty} \text{diam}(C_n) = 0 \).

Lemma 2.2 ([40], lemma 2.1). Let \( X \) be a dendrite with metric \( d \). Then for any \( \varepsilon > 0 \), there is \( 0 < \delta < \varepsilon \) such that if \( d(x, y) < \delta \), then \( \text{diam}([x, y]) < \varepsilon \).

Theorem 3.3 from [8] allows us to deduce the following lemma.

Lemma 2.3. If \( X \) is a dendrite with \( E(X) \) closed, then the order of every branch point is finite.

Lemma 2.4 ([8], corollary 3.5). If \( X \) is a dendrite with \( E(X) \) closed, then \( \overline{B(X)} \setminus B(X) \subset E(X) \).

Applying Dirichlet’s theorem on primes in arithmetical progressions (see [7, p 146]), it easy to see that if \((x_n)_{n \in \mathbb{N}}\) is an eventually periodic sequence of real numbers (i.e. \( x_n = x_{n+m} \) for some fixed number \( m \in \mathbb{N} \) and for any \( n \geq n_0 \), then

\[
\frac{1}{N} \sum_{n=1}^{N} \mu(n)x_n = o(1), \text{ as } N \to +\infty.
\] (2.1)

We also need the following lemma from [33].

Lemma 2.5. Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers such that \( |x_n| \leq 1 \) for any \( n \in \mathbb{N} \). Assume that there is \( n_0, k \in \mathbb{N} \) such that for any \( n, m \geq n_0 \) if \( x_n \neq 0 \), \( x_m \neq 0 \) and \( n \neq m \), then \( |n-m| \geq k \). Then we have

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} x_n \right| \leq \frac{1}{k}.
\]

We shall use the following useful property of \( \omega \)-limit set.

Lemma 2.6 ([10], theorem 3, p 67). Let \((X, f)\) be a dynamical system. Then for each \( x \in X \), there exists an almost periodic point \( y \in \omega_f(x) \) such that \((x, y)\) is a proximal pair.

For the asymptotic pair, we have

Lemma 2.7. Let \((X, f)\) be a dynamical system and let \( x, y \in X \). Let \( \varphi : X \to \mathbb{R} \) be a continuous function. If \( S_N(x, \varphi) = o(1) \), as \( N \to +\infty \) and \((x, y)\) is asymptotic, then

\( S_N(y, \varphi) = o(1) \), as \( N \to +\infty \).

Proof. Fix \( \varepsilon > 0 \). Since \( \varphi \) is uniformly continuous on \( X \), there is \( \alpha > 0 \) such that for any \( u, v \in X \) with \( d(u, v) < \alpha \), we have \( |\varphi(u) - \varphi(v)| < \frac{\varepsilon}{2} \). Since \( \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0 \), there is \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \), \( d(f^n(x), f^n(y)) < \alpha \) and therefore \( |\varphi(f^n(x)) - \varphi(f^n(y))| < \frac{\varepsilon}{2} \) for any \( n \geq n_0 \). Let \( n_1 \geq n_0 \) be such that for any \( N > n_1 \),

\[
\frac{1}{N} \sum_{n=1}^{n_0-1} |\varphi(f^n(x)) - \varphi(f^n(y))| < \frac{\varepsilon}{2}.
\]
Then for any $N > n_1$, we have

$$|S_N(x, \varphi) - S_N(y, \varphi)| = \frac{1}{N} \sum_{n=1}^{N} \mu(n) (\varphi(f^n(x)) - \varphi(f^n(y)))$$

$$\leq \frac{1}{N} \sum_{n=1}^{n_0-1} |\varphi(f^n(x)) - \varphi(f^n(y))|$$

$$+ \frac{1}{N} \sum_{n=n_0}^{N} |\varphi(f^n(x)) - \varphi(f^n(y))|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

Hence $\lim_{N \to +\infty} |S_N(x, \varphi) - S_N(y, \varphi)| = 0$. Since $S_N(x, \varphi) = o(1)$, as $N \to +\infty$, it follows that $S_N(y, \varphi) = o(1)$, as $N \to +\infty$. This completes the proof. 

\section{The case of graph maps}

The aim of this section is to prove the following theorem:

\textbf{Theorem 3.1.} Let $G$ be a graph and $f : G \to G$ be a continuous map with zero topological entropy. Then (1.1) holds.

Let us recall some definitions and results on the structure of $\omega$-limit sets for graph maps which were studied by Blokh in ([13–15]) and Ruelle–Snoha in [45]. Here we follow the notations from [45].

\textbf{Definition 3.2 ([45]).} Let $f : G \to G$ be a graph map. A sub-graph $K$ of $G$ is called periodic of period $k \geq 1$, if $K, f(K), \ldots, f^{k-1}(K)$ are pairwise disjoint and $f^k(K) = K$. The set $\text{Orb}(K) = \bigcup_{i=0}^{k-1} f^i(K)$ is called a cycle of graphs.

For an infinite $\omega$-limit set $\omega_f(x)$, we let

$$C(x) := \left\{ X : X \subseteq G \text{ is a cycle of graphs and } \omega_f(x) \subseteq X \right\}.$$ 

The set $C(x)$ is non-empty since $G_\infty \in C(x)$; indeed, $\Lambda(f) \subseteq G_\infty$ and $G_\infty$ is a 1-periodic cycle of graphs since $f(G_\infty) = G_\infty$.

\textbf{Definition 3.3.} An infinite $\omega$-limit set $\omega_f(x)$ is called a solenoid whenever the periods of the cycles in $C(x)$ are unbounded.

\textbf{Proposition 3.4 ([25]).} Any $\omega$-limit set of a graph map is either finite set, or an infinite closed nowhere dense set or a finite union of non-degenerate subgraphs (which form a cycle of graphs).

Note that if $\omega_f(x)$ is a solenoid, then it is nowhere dense by proposition 3.4.

\textbf{Case 1.} $\omega_f(x)$ is a solenoid.

\textbf{Lemma 3.5 ([45], lemma 11 (see also [13], theorem 1)).} Let $f : G \to G$ be a graph map and let $\omega_f(x)$ be an infinite $\omega$-limit set. If $\omega_f(x)$ is a solenoid, then there exists a sequence of cycles of graphs $(X_n)_{n \geq 1}$ with increasing periods $(k_n)_{n \geq 1}$ such that, for all $n \geq 1$, $X_{n+1} \subseteq X_n$ and $\omega_f(x) \subseteq \bigcap_{n \geq 1} X_n$. Moreover, for all $n \geq 1$, $k_{n+1}$ is a multiple of $k_n$ and every connected
component of $X_n$ contains the same number (equal to $\frac{k+1}{k}$) of components of $X_{n+1}$. Furthermore, $\omega_f(x)$ contains no periodic point.

**Proposition 3.6.** If $\omega_f(x)$ is a solenoid, then (1.1) holds.

**Proof.** Let $\varphi: G \to \mathbb{R}$ be a continuous function.

**Claim.** For any $\varepsilon > 0$, there is a function $\phi: G \to \mathbb{R}$ such that
\[
\|\varphi - \phi\|_{\infty} := \sup_{x \in G} |\varphi(x) - \phi(x)| < \varepsilon,
\]
where $\phi$ is of the form $\phi = \sum_{i=1}^{r} \alpha_i \psi_{U_i}$, with $\alpha_i \in \mathbb{R}$, $U_i$ is an open free arc in $G$ and $\psi_{U_i}$ is given by:
\[
\psi_{U_i}(x) = \begin{cases} 
1 & \text{if } x \in U_i \\
\frac{1}{\text{ord}(x)} & \text{if } x \in U_i \setminus U_i \\
0 & \text{if } x \in G \setminus U_i.
\end{cases}
\]
Note that in the definition of $\psi_{U_i}(x)$, $\text{ord}(x) < +\infty$ by lemma 2.7.

**Proof of the claim.** Since $\varphi$ is uniformly continuous on $G$, there is $\delta > 0$ such that $d(\varphi(x), \varphi(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Then one can write $G = \bigcup_{i=1}^{r} U_i$, where the $U_i$ (for $1 \leq i \leq r$) are pairwise disjoint free open arcs of $G$ such that
- $\text{diam}(U_i) < \delta$ for all $1 \leq i \leq r$,
- $U_i$ and $\overline{U_i}$ are either disjoint or intersect in one of their endpoints, for $i \neq j$.

For each $1 \leq i \leq r$, we let $\phi = \sum_{i=1}^{r} \alpha_i \psi_{U_i}$, where $\alpha_i = \varphi(c_i)$ for some $c_i \in U_i$. Let $x \in G$. If $x \in U_i$ for some $1 \leq i \leq r$, then we have $|\varphi(x) - \phi(x)| = |\varphi(x) - \varphi(c_i)| < \varepsilon$. Now if $x \in \overline{U_i} \setminus U_i$, for $i \in I \subset \{1, 2, \ldots, r\}$, we let $\text{ord}(x) = p$. Then $p = \text{Card}(I)$ and we have
\[
|\varphi(x) - \phi(x)| = |\varphi(x) - \sum_{i \in I} \frac{1}{p} \varphi(c_i)|
\]
\[
= \left| \sum_{i \in I} \frac{1}{p} \varphi(x) - \sum_{i \in I} \frac{1}{p} \varphi(c_i) \right|
\]
\[
\leq \frac{1}{p} \sum_{i \in I} |\varphi(x) - \varphi(c_i)| < \varepsilon.
\]
This proves the claim.

Fix $x \in G$. By lemma 3.5, there is a cycle of graphs $X_r$ with period $k_r > 0$ such that $\omega_f(x) \subset X_r$. Write $X_r = \bigcup_{s=0}^{k_r-1} f^s(K)$, where $K$ is a sub-graph of $G$. There is $s \in \mathbb{N}$ such that $f^s(x) \in K$. Then for any $0 \leq i < k_r$ and $n \geq s$, $f^n(x) \in f^i(K)$ if and only if $n \equiv s + i \mod(k_r)$. Hence, for any $N > s$,
\[
S_N(x, \psi_{U_i}) = \frac{1}{N} \sum_{n=1}^{N} \mu(n)\psi_{U_i}(f^n(x))
\]
\[
= \frac{1}{N} \sum_{s=1}^{N} \mu(n)\psi_{U_i}(f^n(x)) + \frac{1}{N} \sum_{i=0}^{k_r-1} \sum_{s \leq n \leq N} \sum_{f^i(K)} \mu(n)\psi_{U_i}(f^n(x)).
\]
We distinguish two cases.

(1) \( f^t(K) \subset U_j \). In this case, by (2.1):

\[
\frac{1}{N} \sum_{s \leq n \leq N} \mu(n) \psi_{U_j}(f^m(x)) = \frac{1}{N} \sum_{s \leq n \leq N, n+i \equiv \text{mod}(k_r)} \mu(n) = o(1).
\]

(2) \( f^t(K) \not\subset U_j \) and \( f^t(K) \cap U_j \neq \emptyset \). In this case, for any \( n, m \geq s \), \( n \neq m \), if \( f^n(x), f^m(x) \in f^t(K) \), then \( |n - m| \geq k_r \). It follows that by lemma 2.5,

\[
\limsup_{N \to +\infty} \frac{1}{N} \sum_{s \leq n \leq N, f^n(x) \in f^t(K)} \mu(n) \psi_{U_j}(f^m(x)) \leq \frac{1}{k_r}.
\]

The case (2) above can occur at most 2 times and therefore

\[
\limsup_{N \to +\infty} |S_N(x, \psi_{U_j})| \leq \frac{2}{k_r}.
\]

As \( k_r \) is arbitrarily large and \( S_N(x, \phi) = \sum_{i=1}^{r} \alpha_i S_N(x, \psi_{U_j}) \), so \( S_N(x, \phi) = o(1) \), as \( N \to +\infty \).

Since \( \|\phi - \phi\|_\infty \) can be taken arbitrarily small, so \( S_N(x, \phi) = o(1) \), as \( N \to +\infty \) and (1.1) holds.

\[\blacksquare\]

Case 2: \( \omega_f(x) \) is not a solenoid.

Let \( X \) be a finite union of sub-graphs of \( G \) such that \( f(X) \subset X \). We define \( E(X, f) = \{ y \in X : \forall \text{ neighborhood } U \text{ of } y \text{ in } X, \overline{\text{Orb}}_y(U) = X \} \). We call \( E(X, f) \) a basic set if it is infinite and if \( X \) contains a periodic point (see Blokh’s survey [14], see also [45]).

Lemma 3.7 ([45], lemma 13). Let \( f : G \to G \) be a graph map so that \( \omega_f(x) \) is not a solenoid. Then there exists a cycle of graphs \( X \in \mathcal{C}(x) \) such that for any \( Y \in \mathcal{C}(x) \), \( X \subset Y \). The period of \( X \) is maximal among the periods of all cycles in \( \mathcal{C}(x) \).

Lemma 3.8 ([45], lemma 14). Let \( f : G \to G \) be a graph map so that \( \omega_f(x) \) is not a solenoid. Let \( K \) be the minimal cycle of graphs in \( \mathcal{C}(x) \). Then

1. For every \( y \in \omega_f(x) \) and for every relative neighborhood \( U \) of \( y \) in \( K \), \( \overline{\text{Orb}}_y(U) = K \).
2. \( \omega_f(x) \subset E(K, f) \). In particular, \( E(K, f) \) is infinite.

Lemma 3.9 ([45], corollary 21). If a graph map \( f : G \to G \) admits a basic \( \omega \)-limit set, then \( h(f) > 0 \).

Proposition 3.10 ([41], theorem 5.7). Let \( f : G \to G \) be a graph map without periodic points. Then \( (G, f) \) is a null system.

It turns out that the notion of null system is related to the so-called tame system. This later notion was coined by Glasner in [23]. The dynamical system \((X, T)\) is tame if the closure of \( \{ T^n : n \in \mathbb{Z}_+ \} \) in \( X^X \) is Rosenthal compact. We recall that a set \( K \) is Rosenthal compact if and only if there is a Polish space \( P \) such that \( K \) can be embedded into Baire-1(\( P \)), where Baire-1(\( P \)) is the first class of Baire functions i.e. which are pointwise limits of continuous functions.
functions on $P$. By Bourgain–Fremelin–Talagrand’s theorem [16], $K$ is Rosenthal compact if and only if it is a subset of the Borel functions on $P$ with $K = \{ f_n \}, f_n \in C(P, \mathbb{R})$ (the space of continuous functions on $P$).

The precise connection between null systems and tame systems is stated in the following proposition.

**Proposition 3.11** ([23, 26, 34]). Let $(X, f)$ be a dynamical system. If it is a null system, then it is tame.

It is well known that if $(X, T)$ is tame, then the pointwise limit of $T$ along any subsequence is Borel, when it exists. Combining this with Kushnireko’s characterization of the measurable discrete spectrum [36], it can be seen that tame system has a measurable discrete spectrum for any invariant measure. This was observed by Huang in [26]. From this, we see the following:

**Proposition 3.12.** [27, theorem 1.8] Let $(X, f)$ be a tame system. Then (1.1) holds.

**Proposition 3.13.** Let $f : G \to G$ be a graph map without periodic points. Then (1.1) holds.

**Proof.** By proposition 3.10, $(G, f, \varphi)$ is a null system and by proposition 3.11, $(G, f)$ is tame. It follows from proposition 3.12 that (1.1) holds.

**Proof of theorem 3.1.** Let $f : G \to G$ be a graph map with $h(f) = 0$ and let $x \in G$. If $\omega_f(x)$ is finite, then $x$ is asymptotic to some periodic point. By (2.1) and lemma 2.7, (1.1) holds.

Now, suppose that $\omega_f(x)$ is infinite. If $\omega_f(x)$ is a solenoid, then by proposition 3.6, (1.1) holds. Suppose that $\omega_f(x)$ is not a solenoid. Set $X = \bigcup_{i=0}^{\infty} f^i(K)$ be the minimal cycle of $G$ containing $\omega_f(x)$. By lemma 3.8, (2), $E(X, f)$ is finite. Then by lemma 3.9, $f$ does not admit a basic set, that is $X \cap P(f) = \emptyset$. For any $0 \leq i < k$, set $K_i = f^i(K)$ and $g = f^k$. Then $g_i := g_{|K} : K_i \to K_i$ is a graph map without periodic point. By proposition 3.10, $(K_i, g_i)$ is a null system and therefore so is $(X, f^k)$. By propositions 3.11 and 3.12, (1.1) holds for $(X, f^k)$. Let $s \geq 0$ such that $f^s(x) \in X$. Since $f(X) = X$, there is $y \in X$ such that $f^s(y) = f^s(x)$. In particular, $(x, y)$ is asymptotic. Let $\varphi : G \to \mathbb{R}$ be a continuous function. Since $S_N(y, \varphi) = o(1)$, as $N \to +\infty$, so by lemma 2.7, $S_N(x, \varphi) = o(1)$, as $N \to +\infty$. This finishes the proof of theorem 3.1.

4. The case of local dendrite map

4.1. On monotone local dendrite map

The aim of this subsection is to prove the following theorem:

**Theorem 4.1.** Let $f : X \to X$ be a monotone local dendrite map. Then (1.1) holds.

**Corollary 4.2.** If $f : X \to X$ is a homeomorphism on a local dendrite $X$, then (1.1) holds.

We recall the following results.

**Lemma 4.3** ([6], theorem 4.1). Let $f : X \to X$ be a monotone local dendrite map. Then $f$ has no Li–Yorke pair. In particular, $f$ has zero topological entropy.

---

$^6$ A transformation measure-preserving has a measurable discrete spectrum if and only if the orbit of any square integrable function is compact in $L^2(\mu)$, $\mu$ is an invariant measure.
Lemma 4.4 ([5], theorem 1.2). Any ω-limit set of a monotone local dendrite map is a minimal set which is either finite, or a Cantor set, or a circle.

Lemma 4.5 ([44], corollary 3.7). Let \( f : X \to X \) be a monotone dendrite map and \( L \) be an infinite ω-limit set. Then there is a sequence \( \alpha \) of prime numbers such that \( f|_L \) is topologically conjugate to the adding machine \( f_\alpha \).

At this point, let us point out that the adding machine satisfies (1.1) since it has topological discrete spectrum, that is, the eigenfunctions span a dense linear subspace of the \( C(X, \mathbb{R}) \) (the space of continuous functions equipped with the strong topology). This will allow us to prove the following:

Lemma 4.6. Let \( f : X \to X \) be a monotone dendrite map. Then (1.1) holds.

Proof. Let \( x \in X \) and set \( L = \omega_f(x) \). If \( L \) is finite, then \( x \) is asymptotic to some periodic point. By (2.1) and lemma 2.7, (1.1) holds for the point \( x \). Suppose that \( L \) is a Cantor set and so \( f|_L \) acts as the adding machine (lemma 4.5). Hence (1.1) holds for any point of \( L \). But, by lemma 2.6, there exists \( y \in L \) such that \( (x, y) \) is a proximal pair and by lemma 4.3, \( (x, y) \) is asymptotic. As (1.1) holds for the point \( y \), it follows that (1.1) holds for \( x \) by lemma 2.7. This finishes the proof of lemma 4.6.

Let \( X \) be a local dendrite. We define the graph \( G_X \) as the intersection of all graphs in \( X \) containing all the circles. Then \( G_X \) is a sub-graph of \( X \) (with \( G_X = \emptyset \), if \( X \) contains no circle).

Proposition 4.7 ([5], proposition 3.6). Let \( f : X \to X \) be an onto monotone local dendrite map. Then we have the following properties:

(i) \( f(G_X) = G_X \).
(ii) \( f|_{G_X} \) is monotone.

Lemma 4.8. Let \( f : X \to X \) be an onto monotone local dendrite map. Then (1.1) holds.

Proof. Assume that \( X \) is not a dendrite and let \( x \in X \). Set \( L = \omega_f(x) \). By lemma 2.6, there exists \( y \in L \) such that \( (x, y) \) is a proximal pair. By lemma 4.3, \( (x, y) \) is asymptotic. From lemma 4.4, we distinguish the following cases.

Case 1: \( L \) is finite. In this case, (1.1) holds for the point \( x \) similarly as in the proof of lemma 4.6.

Case 2: \( L \) is a circle. In this case \( f|_L \) is a circle map, so by theorem 3.1, (1.1) holds for the point \( y \).

Case 3: \( L \) is a Cantor set. In this case, \( X \) contains only one circle (i.e. \( G_X = C \) a circle). If \( L \) meets \( C \), then \( L \) is included in \( C \) (by minimality of \( L \)). Hence by theorem 3.1, (1.1) holds for the point \( y \). Now if \( L \) is disjoint from \( C \), then it is included in \( X \setminus C \), and so \( f|_L \) acts as the adding machine (lemma 4.5). So (1.1) holds for the point \( y \). It follows that (1.1) holds for \( x \) by lemma 2.7. This finishes the proof of lemma 4.8.

Recall that \( X_{\infty} \) is a sub-local dendrite of \( X \).

Lemma 4.9 ([5], lemma 4.3). The map \( f|_{X_{\infty}} \) is monotone and onto.

Proof of theorem 4.1. First by lemma 4.3, \( f \) has zero topological entropy. Let \( x \in X \) and set \( L = \omega_f(x) \). If \( x \in X_{\infty} \), then (1.1) holds for \( x \) by lemmas 4.9 and 4.8. Assume that
$x \in X \setminus X_{\infty}$. By lemma 2.6, there exists $y \in L$ such that $(x, y)$ is a proximal pair and by lemma 4.3, $(x, y)$ is asymptotic. As $L \subset \Lambda(f) \subset X_{\infty}$, so by lemma 4.8, (1.1) holds for the point $y$ and hence for $x$. The proof is complete.

4.2. On continuous maps on a certain class of dendrites

The aim of this subsection is to prove the following theorem:

**Theorem 4.10.** Let $X$ be a dendrite such that $E(X)$ is closed and its set of accumulation points $E(X)'$ is finite. Let $f : X \to X$ be a continuous map with zero topological entropy. Then (1.1) holds.

We need the following results.

**Lemma 4.11 ([28, 51], theorem 5.16).** If $X$ is at most countable and $f : X \to X$ is a continuous map, then (1.1) holds.

**Lemma 4.12 ([9]).** Let $X$ be a dendrite such that $E(X)$ is closed and $E(X)'$ is finite. Let $f : X \to X$ be a continuous map with zero topological entropy. Let $L$ be an uncountable ω-limit set. Then there exist a sequence of sub-dendrites $(D_k)_{k \geq 1}$ of $X$ and a sequence of integers $n_k \geq 2$ for every $k \geq 1$.

1. $f_{n_k}(D_k) = D_k$, where $\alpha_k = n_1n_2 \ldots n_k$,
2. $\bigcup_{k=0}^{\infty} f_{\alpha_k}(D_k) \subseteq D_j$ for all $j \geq 2$,
3. $L \subseteq \bigcup_{k=0}^{\infty} f_{\alpha_k}(D_k)$,
4. $f(L \cap f^i(D_k)) = L \cap f^{i+1}(D_k)$ for any $0 \leq i \leq \alpha_k - 1$. In particular, $L \cap f^i(D_k) \neq \emptyset$,
5. $f^i(D_k) \cap f^j(D_l)$ has empty interior for any $0 \leq i \neq j \leq \alpha_k$.

**Proof of theorem 4.10.** Let $x \in X$ and set $L = \omega_f(x)$. Let $\varphi : X \to \mathbb{R}$ be a continuous function. We distinguish three cases.

**Case 1** $L$ is finite. In this case, there is a periodic point $b$ such that $(x, b)$ is asymptotic. Then by (2.1) and lemma 2.7, $S_N(x, \varphi) = o(1)$, as $N \to +\infty$.

**Case 2** $L$ is countable. In this case, $Y := \overline{O_f(x)} = O_f(x) \cup L$ is countable and $f$-invariant. So by lemma 4.11 applied to $(Y, f)$, (1.1) holds.

**Case 3** $L$ is uncountable. Let $\varepsilon > 0$ and $k \geq 1$. There is $\delta > 0$ such that $d(\varphi(x), \varphi(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Note that $B(X)$ is discrete and by lemma 2.4, $\overline{B(X)} \setminus B(X) \subset E(X)$. So there exist pairwise disjoint open connected subsets $U_i$ $(1 \leq i \leq r)$ of $X$ such that

- $X = \bigcup_{i=1}^{r} U_i$,
- $\text{diam}(U_i) < \delta$ for all $1 \leq i \leq r$,
- if $U_i \cap E(X)' = \emptyset$, then $U_i$ is an open free arc in $X$.
- if $U_i \cap E(X)' \neq \emptyset$, then $U_i \cap E(X)' = \{e\}$ and $U_i$ is the connected component of $X \setminus \{e\}$ containing $e$, for some $e \in X \setminus E(X)$.
- $U_i$ and $\overline{U_i}$ are either disjoint or intersect in one of their endpoints, for $i \neq j$.

We let $\varphi_0 = \sum_{i=1}^{r} \alpha_i \psi_{U_i}$, where for each $1 \leq i \leq r$, $\alpha_i = \varphi(e_i)$ for some $e_i \in U_i$ and $\psi_{U_i}$ is given by:

$$
\psi_{U_i}(x) = \begin{cases} 
1 & \text{if } x \in U_i \\
\frac{1}{\text{deg}(x)} & \text{if } x \in \overline{U_i} \setminus U_i \\
0 & \text{if } x \in X \setminus \overline{U_i}.
\end{cases}
$$
Similarly as in the proof of proposition 3.6, we get \( \sup_{x \in X} |\varphi(x) - \varphi_0(x)| < \varepsilon \). Now by lemma 4.12, \( L \subseteq \cup_{n=0}^{\infty} f^n(D_k) \). Then there is \( n_0 \geq 0 \) such that \( f^{n_0}(x) \in D_k \). Since \( D := \cup_{0 \leq i < j < \alpha_k} f^i(D_k) \cap f^j(D_k) \) is finite, we may assume that \( f^n(x) \notin D \) for any \( n \geq n_0 \). So for any \( n \geq n_0 \) and \( 0 \leq s < \alpha_k \), \( f^n(x) \in f^s(D_k) \) if and only if \( n \equiv n_0 + s \mod(\alpha_k) \).

Then

\[
S_N(x, \psi_U) = \frac{1}{N} \sum_{n=1}^{N} \mu(n)\psi_U(f^n(x)) = \frac{1}{N} \sum_{n=1}^{n_0-1} \mu(n)\psi_U(f^n(x)) + \frac{1}{N} \sum_{n=n_0}^{N} \mu(n)\psi_U(f^n(x)) = o(1) + \sum_{s=0}^{\alpha_k-1} A^s_N, \text{ as } N \to +\infty,
\]

where \( A^s_N = \frac{1}{N} \sum_{n_0 \leq n < N, f^n(x) \in f^s(D_k)} \mu(n)\psi_U(f^n(x)) \).

For \( 0 \leq s \leq \alpha_k - 1 \), define the sequence

\[
x^s_n = \begin{cases} 0 & \text{if } n < n_0 \\ \psi_U(f^n(x)) \chi_{f^s(D_k)}(f^n(x)) & \text{if } n \geq n_0 \end{cases}
\]

where \( \chi_{f^s(D_k)} \) is the characteristic function of \( f^s(D_k) \). We can rewrite \( A^s_N \) as follows:

\[
A^s_N = \frac{1}{N} \sum_{n=1}^{N} \mu(n)x^s_n, \text{ if } f^s(D_k) \subseteq U_j, \text{ the sequence } (x^s_n)_n \text{ is eventually periodic with period } \alpha_k. \text{ Then by (2.1), } A^s_N = \frac{1}{N} \sum_{n=1}^{N} \mu(n)x^s_n = o(1), \text{ as } N \to +\infty. \text{ Otherwise, there is at most two distinct numbers } 0 \leq s, r \leq \alpha_k - 1 \text{ such that } f^s(D_k) \subseteq U_j, f^r(D_k) \cap U_j \neq \emptyset, f^r(D_k) \subseteq U_j \text{ and } f^s(D_k) \cap U_j \neq \emptyset. \text{ In such case, if } f^n(x), f^p(x) \in f^s(D_k) \text{ and } n \neq p, \text{ then } |n-p| \geq \alpha_k. \text{ Therefore by lemma 2.5,}
\]

\[
\limsup_{N \to +\infty} |A^s_N| = \limsup_{N \to +\infty} |A^s_N| \leq \frac{1}{\alpha_k}.
\]

The integer \( \alpha_k \) can be taken arbitrarily large, then we obtain that \( S_N(x, \psi_U) = o(1) \), as \( N \to +\infty \) and hence \( S_N(x, \varphi_0) = o(1) \), as \( N \to +\infty \). Therefore (1.1) holds for \((X, f)\). \( \blacksquare \)

4.3. On a transitive dendrite map with zero entropy

In [18], Byszewski et al give an example of transitive map \( f \) with zero entropy on the universal dendrite \( D \) with the following properties: (1) \( f \) has a unique fixed point \( o \). (2) \( f \) is uniquely ergodic, with the only \( f \)-invariant Borel probability measure being the Dirac measure \( \delta_o \) concentrated on \( o \). Applying the machinery from [2, p 313], one can see that we have the following. We include the proof for the reader convenience.

**Proposition 4.13.** Let \( f \) be the dendrite map above. Then (1.1) holds.

**Proof.** Let \( \varphi : D \to \mathbb{R} \) be a continuous function and set \( \Phi = \varphi - \varphi(o) \). As \( \delta_o \) is the only \( f \)-invariant Borel probability measure (by (2)), and since \( \int_D |\Phi|d\delta_o = 0 \), so \( \frac{1}{N} \sum_{n=0}^{N-1} |\Phi(f^n(x))| \to 0 \), as \( N \to +\infty \). We have that
We define on $\nu$ with $0 \rightarrow \infty = \rightarrow$ 
so as $\frac{1}{n} \sum_{n=0}^{N-1} \mu(n) \varphi(f^n(x)) \leq \frac{1}{N} \sum_{n=0}^{N-1} |\varphi(f^n(x))| = \frac{1}{N} \sum_{n=0}^{N-1} \mu(n)$ 

Let us notice that the convergence in (1.1) is uniform, since $(D,f)$ satisfy the so-called MOMO property (Möbius Orthogonality on Moving Orbits) (see [4] for the definition). In this direction, it is proved in [4] the following

**Proposition 4.14 ([4]).** If Sarnak’s conjecture (1.1) is true, then for all zero entropy systems $(X,T)$ and for all continuous function $f : X \rightarrow \mathbb{R}$, we have

$$
\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \mu(n) \xrightarrow{N \rightarrow +\infty} 0,
$$

uniformly in $x \in X$.

### 4.4. On a transitive dendrite map with positive entropy

In this subsection, we discuss the problem of Möbius disjointness for the example introduced by Špitálský in [48]. Špitálský constructed his example as a factor of a map $F$ acting on the universal dendrite of order 3. Precisely, let $Q$ be a set of all dyadic rational numbers in $(0,1)$, that is, every $r \in Q$ can be uniquely written as $r = \frac{p}{2^n}$ with $q_r \geq 1$ and $p_r$ is odd in $\{1, \ldots, 2^n\}$. Let us denote by $Q^k, Q^*$ the sets $\{0\}$ and $\cup_{k \geq 0} Q^k$, respectively. The length of an element $\alpha \in Q^*$, denoted by $|\alpha|$ corresponds to the integer $k$ such that $\alpha \in Q^k$. We define on $Q^*$ the concatenation operation as follows:

For $\alpha \in Q^k, \beta \in Q^m$, we put $\gamma = \alpha \beta \in Q^{k+m}$. If $\alpha = r_0 r_1 \cdots r_{k-1} \in Q^k$ with $k \geq 1$, then $\tilde{\alpha}$ denotes $r_0 r_1 \cdots r_{k-2}$. The dendrite of order 3 is given by

$$
X = \bigcup_{m \geq 0} X^{(m)},
$$

where $X^{(0)} = [a_0,b_0]$ is an arc and every $X^{(m)}, m \geq 1$, satisfies

$$
X^{(m)} = X^{(m-1)} \cup \left( \bigcup_{a \in Q^*} \{a, a_\beta\} \right), a_\alpha \in (a_\tilde{\alpha}, b_\tilde{\alpha}], \text{ for } \alpha \in Q^m,
$$

where $(x,y) = [x,y] \setminus \{x\}$.

Špitálský proved that $F$ has positive entropy and for any $x \in X^{(m)}, m \geq 1$, the omega set of $x$ is either $\{a_0\}$ or $\{b_0\}$. Furthermore, if $\nu$ is an $F$-invariant Borel probability measure, then for each $m \geq 1$, $\nu(X^{(m)} \setminus \{a_0, b_0\}) = 0$, and $a_0$ and $b_0$ are the only fixed points of $F$. This yields that the topological entropy of $F_{|x^{(m)}}$ is zero. Therefore, by ([17], proposition 2, (c)), the entropy of $F_{|\cup_{m \geq 1} X^{(m)}}$ is zero. Moreover, by the same arguments as before, we can see easily that for any $x \in \cup_{m \geq 1} X^{(m)}$, for any continuous function $\Phi$, we have

$$
\frac{1}{N} \sum_{n=1}^{N} \Phi(F^n(x)) \mu(n) \xrightarrow{N \rightarrow +\infty} 0.
$$

According to proposition 4.14, if Sarnak’s conjecture is true, then the Möbius disjointness is uniform. But, we cannot apply this result in our situation since the set $\cup_{m \geq 1} X^{(m)}$ is a
Although, the Möbius disjointness holds uniformly on each $X^{(m)}$. We thus asked whether Špitalský’s example satisfy Möbius disjointness or not. This allows us also to ask the following questions.

**Question 4.15.** Let $(X, F)$ be the Špitalský’s example. Do we have that the Möbius disjointness is true for $(X, F)$?

**Question 4.16.** Let $(X, T)$ be a dynamical system with zero topological entropy. Let $Y$ be a dense $T$-invariant subset of $X$. Again by proposition 2, (c) from [17], the topological entropy $T|_Y$ is zero. Assume that the Möbius disjointness for $(Y, T|_Y)$ holds, do we have that Sarnak conjecture is true for $(X, T)$?

**Acknowledgments**

The authors are thankful to the referees for their helpful remarks and suggestions which have improved the presentation of the paper. They also thank Issam Naghmouchi for fruitful discussions on this paper. The third author thanks the University of Rouen Normandy for their hospitality where the revision of this work was done. H Marzougui and G Askri were supported by the research unit: ‘Dynamical systems and their applications’ (UR17ES21), of Higher Education and Scientific Research, Tunisia.

**References**

[1] El Abdalaoui E H 2018 On Veech’s proof of Sarnak’s theorem on the Möbius flow in preparation (arXiv:1711.06326v2 [math.DS])

[2] El Abdalaoui E H, Lemańczyk M and de la Rue T 2014 On spectral disjointness of powers for rank-one transformations and Möbius orthogonality J. Funct. Anal. 266 284–317

[3] El Abdalaoui E H, Kulaga-Przymus J, Lemańczyk M and de la Rue T 2017 The Chowla and the Sarnak conjectures from ergodic theory point of view Discrete Contin. Dyn. Syst. 37 2899–944

[4] El Abdalaoui E H, Kulaga-Przymus J, Lemańczyk M and de la Rue T 2017 Möbius disjointness for models of an ergodic system and beyond in preparation (arXiv:1704.03506)

[5] Abdelli H 2015 $\omega$-Limit sets for monotone local dendrite maps Chaos Solitons Fractals 71 66–72

[6] Abdelli H and Marzougui H 2016 Invariant sets for monotone local dendrite maps Int. J. Bifurcation Chaos Appl. Sci. Eng. 26 1650150

[7] Apostol T M 1976 Introduction to Analytic Number Theory (Undergraduate Texts in Mathematics) (New York: Springer)

[8] Arevalo D, Charatonik W J, Covarrubias P P and Simon L 2001 Dendrites with a closed set of endpoints Topol. Appl. 115 1–17

[9] Askri G 2017 Li–Yorke chaos for dendrite maps with zero topological entropy and $\omega$-limit sets Discrete Contin. Dyn. Syst. 37 2957–76

[10] Auslander J 1988 Minimal Flows and their Extensions (North-Holland Mathematics Studies vol 153) (Amsterdam: North-Holland)

[11] Beardon A F 1991 Iteration of Rational Functions: Complex Analytic Dynamical Systems (Graduate Texts in Mathematics vol 132) (New York: Springer)

[12] Block L S and Coppel W A 1992 Dynamics in One Dimension (Lecture Notes in Mathematics vol 1513) (Berlin: Springer)

[13] Blokh A 1986 Dynamical systems on one-dimensional branched manifolds I (Russian) Teor. Funktsii Funktsional. Anal. i Prilozhen. 46 8–18 
Blokh A 1990 Dynamical systems on one-dimensional branched manifolds I (Russian) J. Sov. Math. 48 500–8 (Engl. transl.)

[14] Blokh A 1995 The Spectral Decomposition for One-Dimensional Maps (Expositions Dynamics Systems (N.S.) vol 4) (Berlin: Springer) p 159
[15] Blokh A 1984 On transitive mappings of one-dimensional branched manifolds Difference Equations and Problems of Mathematical Physics (Akad. Nauk Ukrain. SSR Inst. Mat., Kiev) pp 3–9 (Russian)
[16] Bourgain J, Fremlin D H and Talagrand M 1978 Pointwise compact sets of Baire-measurable functions Am. J. Math. 100 845–86
[17] Bowen R 1973 Topological entropy for noncompact sets Trans. Am. Math. Soc. 184 125–36
[18] Bylszewski J, Falniowski F and Kwietniak D 2017 Transitive dendrite map with zero entropy Ergod. Theor. Dynam. Syst. 37 2077–83
[19] Chowla S 1965 The Riemann Hypothesis and Hilbert’s Tenth Problem (Mathematics and its Applications vol 4) (New York: Gordon and Breach Science Publishers)
[20] Downarowicz T 2017 Almost full entropy subshifts uncorrelated to the Möbius function Int. Math. Res. Not. 1–14 (https://doi.org/10.1093/imrn/rnx192)
[21] Ferenczi S, Kulaga-Przymus J and Lemanczyk M 2018 Sarnak’s conjecture—what’s new Ergodic Theory and Dynamical Systems in their Interactions with Arithmetic and Combinatorics (Lecture Notes in Mathematics vol 2213) (New York: Springer) pp 163–235
[22] Gomilko A, Kwietniak D and Lemanczyk M 2018 Sarnak’s conjecture implies the Chowla conjecture along a subsequence Ergodic Theory and Dynamical Systems in their Interactions with Arithmetic and Combinatorics (Lecture Notes in Mathematics vol 2213) (New York: Springer) pp 237–47
[23] Glasner E 2006 On tame dynamical systems Colloq. Math. 105 283–95
[24] Goodman T N T 1974 Topological sequence entropy Proc. Lond. Math. Soc. 29 331–50
[25] Hric R and Malek M 2017 Omega limit sets and distributional chaos on graphs Topol. Appl. 153 2469–75
[26] Huang W 2006 Tame systems and scrambled pairs under an abelian group action Ergod. Theor. Dynam. Syst. 26 1549–67
[27] Huang W, Wang Z and Ye X 2017 Measure complexity and Möbius disjointness for subshifts of zero entropy Nonlinearity 30 4260–76
[28] Iwaniec H and Kowalski E 2004 Analytic Number Theory (American Mathematical Society Colloquium Publications vol 53) (Providence, RI: American Mathematical Society)
[29] Jiang Y 2018 Zero entropy continuous interval maps and MMLS-MMA for maps of dendrites Int. J. Bifurcation Chaos Appl. Sci. Eng. 28 285–99
[30] Kerr D and Li H 2005 Dynamical entropy in Banach spaces Inventory Math. 162 649–86
[31] Kuratowski K 1968 Topology vol II (New York: Academic)
[32] Kushnirenko A G 1967 On metric invariants of entropy type Russ. Math. Surv. 22 53–61
[33] Li J, Oprocha P, Yang Y and Zeng T 2017 On dynamics of graph maps with zero topological entropy Nonlinearity 30 4260–76
[34] Liu J and Sarnak P 2015 The Möbius function and distal flows Duke Math. J. 164 1353–99
[35] Mai J H and Shi E H 2009 Hyperbolic modular flows Proc. R. Soc. Edinb. Sect. A vol 139 1391–6
[36] Mai J H and Shao S 2007 The structure of graph maps without periodic points Topology Appl. 154 2714–28
[37] Müller C 2017 Automatic sequences fulfill the Sarnak conjecture Duke Math. J. 166 3219–90
[38] Nadler S B ited by F. Takens and B. Weiss Retractions Topology Appl. 158 (New York: Dekker)
[39] Naghmouchi I 2011 Dynamics of monotone graph, dendrite and dendrite maps Int. J. Bifurcation Chaos Appl. Sci. Eng. 21 3205–15
[40] Ruette S and Snoha L 2014 For graph maps, one scrambled pair implies Li–Yorke chaos Proc. Am. Math. Soc. 142 2087–100
[46] Sarnak P 2010 Three lectures on the Möbius function, randomness and dynamics http://publications.ias.edu/sarnak/

[47] Sarnak P 2012 Möbius randomness and dynamics Not. South Afr. Math. Soc. 43 89–97

[48] Špitalský V 2015 Transitive dendrite map with infinite decomposition ideal Discrete Contin. Dyn. Syst. 35 771–92

[49] Tao T 2017 (https://terrytao.wordpress.com/2017/10/20/the-logarithmically-averaged-and-non-logarithmically-averaged-chowla-conjectures/)

[50] Tao T and Teräväinen J 2017 The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures in preparation (arXiv:1708.02610 [math.NT])

[51] Wei F 2016 Entropy of arithmetic functions and Sarnak’s Möbius disjointness conjecture PhD Thesis The University of Chinese Academy of Sciences