Representation stability for the pure cactus group

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Abstract. The fundamental group of the real locus of the Deligne-Mumford compactification of the moduli space of rational curves with $n$ marked points, the pure cactus group, resembles the pure braid group in many ways. As it is the case for several “pure braid like” groups, it is known that its cohomology ring is generated by its first cohomology. In this note we survey what the $FI$-module theory developed by Church, Ellenberg and Farb can tell us about those examples. As a consequence we obtain uniform representation stability for the sequence of cohomology groups of the pure cactus group.

1. Introduction

In this paper we survey the principal notions and consequences of the $FI$-module theory introduced by Church, Ellenberg and Farb in [CEF]. Our main objective is to show how the theory can be readily applied to certain sequences of groups and spaces with cohomology rings that have the structure of a graded $FI$-algebra and are known to be generated by the first cohomology group. We revisit the collection of examples in Section 3 that have in common this behavior (see Table 1).

To illustrate our discussion we work out the details for the sequence of spaces $\{M_n\}$. The space $M_n := \mathcal{M}_{0,n}(\mathbb{R})$ is the real locus of the Deligne-Mumford compactification $\mathcal{M}_{0,n}$ of the moduli space of rational curves with $n$ marked points. We recall the precise definition of $M_n$ in Section 2 below.

As pointed out by Etingof-Henriques-Kamnitzer-Rains in [EHKR10], the space $M_n$ resembles in many ways the configuration space of $n-1$ distinct points in the plane $\mathcal{F}(\mathbb{C}, n-1)$. For instance, the spaces $M_n$ are smooth manifolds and Eilenberg-MacLane spaces ([DJS03]). In particular, it follows that the cohomology of the pure cactus groups $\pi_1(M_n)$ coincides with the cohomology of $M_n$. Another similarity, there exist maps from $M_n$ to $M_{n-1}$ which can be described as “forgetting one marked point” and from $M_n$ to $M_{n+1}$ which can be thought of “inserting a bubble at a marked point”. These maps endow the collection of spaces $\{M_n\}$ with the structure of a simplicial space. Furthermore, as we recall in Theorem 2.2 the rational cohomology of $M_n$ is generated multiplicatively by the 1-dimensional classes coming from the simplest of such spaces, namely, $M_4 = S^1$.

On the other hand, there are also notable differences between the spaces $M_n$ and $\mathcal{F}(\mathbb{C}, n-1)$. For instance, for $n > 5$ the space $M_n$ is not formal [EHKR10].

2000 Mathematics Subject Classification. Primary.
In contrast to the configuration spaces, the forgetful maps $M_n \to M_{n-1}$ are not fibrations. It is not known whether the pure cactus group $\pi_1(M_n)$ is residually nilpotent.

In the present note we show that the spaces $M_n$ share one more property with the configuration spaces and the examples considered in Section 4. Namely, we see in Section 3 that their rational cohomology ring has the structure of a graded $FI$-algebra which is a finitely generated $FI$-module in each degree. As main consequence of this approach we obtain, as in the case of configuration spaces, uniform representation stability in the sense of [CF13].

**Theorem 1.1.** The sequence $\{H^i(M_n, \mathbb{Q})\}_n$ of $S_n$-representations is uniformly representation stable for all $i \geq 0$ and the stability holds for $n \geq 6i$.

Moreover, in [Theorem 3.23] we show that the characters of the $S_n$-representations are eventually given by a unique character polynomial. For instance, for $n \geq 4$ it is known that $H^1(M_n, \mathbb{Q}) = \bigwedge^3 \mathcal{H}_n$, where $\mathcal{H}_n$ is the standard representation of $S_n$. Its character is given by the character polynomial of degree 3:

$$\chi_{H^1(M_n, \mathbb{Q})} = \left( \frac{X_1}{3} \right) + X_3 - X_2X_1 - \left( \frac{X_1}{2} \right) + X_2 + X_1 - 1,$$

where $X_l(\sigma)$ counts the number of $l$-cycles in the cycle decomposition of $\sigma \in S_n$. Notice that for $n \geq 4$, the value of the character in any permutation $\sigma \in S_n$ only depends on cycles of length 1, 2 and 3 in the cycle decomposition of $\sigma$. Furthermore, we recover the following result first proven in [EHKR10, Theorem 6.4].

**Corollary 1.2.** For $n \geq 6i$, the $i$th Betti number of $M_n$ is given by a unique polynomial on $n$ of degree $\leq 3i$.

In particular, for $n \geq 4$ the first Betti number is the cubic polynomial in $n$:

$$\dim_{\mathbb{Q}}(H^1(M_n, \mathbb{Q})) = \chi_{H^1(M_n, \mathbb{Q})}(\text{id}) = \binom{n}{3} - \binom{n}{2} + n - 1 = \frac{(n-1)(n-2)(n-3)}{6}.$$

More is actually known about these representations. In [Rai09, Theorem 1.1] Rains obtained an explicit formula for the graded character of the $S_n$-action on $H^\ast(M_n, \mathbb{Q})$. In particular, his result recovers the product formula [EHKR10, Theorem 6.4] for the Poincaré series of $M_n$.

**Acknowledgments.** The authors would like to thank B. Farb, J. Mostovoy, B. Cisneros and J. Wilson for useful conversations and comments.

## 2. The space $M_n$ and its cohomology ring

The Deligne-Mumford compactification $\overline{M}_{0,n}$ of the moduli space $M_{0,n}$ of rational curves with $n$ marked points is a smooth projective variety that parametrizes stable curves of genus 0 with $n$ labeled points (see [DM69]).

**Definition 2.1.** A stable rational curve with $n$ marked points is a finite union $C$ of projective lines $C_1, C_2, \ldots, C_k$ together with marked points $z_1, \ldots, z_n$ in $C$ such that

- each marked point $z_i$ is in only one $C_j$;
- $C_i \cap C_j$ is either empty or consists only of one point, and in this case the intersection is transversal;
• its associated graph, with a vertex for each \( C_i \) and an edge for each pair of intersecting lines, its a tree;
• each \( C_i \) has at least three special points, where a special point is either an intersection point with another component, or a marked point.

In \( \overline{M}_{0,n} \) two stable curves \( \{ C, (z_1, \ldots, z_n) \} \) and \( \{ C', (z'_1, \ldots, z'_n) \} \) are equivalent if there is an isomorphism of algebraic curves \( f : C \rightarrow C' \) such that \( f(z_i) = z'_i \).

Notice that since each component has at least three special points a stable curve has no non-trivial automorphisms.

Let \( n, m \geq 3 \) and a subset \( S \) of \([n] := \{1, \ldots, n\} \) with \( m \) elements. There is a well-defined “forgetful morphism”
\[
\phi_S : \overline{M}_{0,n} \rightarrow \overline{M}_{0,m}
\]
which first forgets all the marked points whose index are not in \( S \) and then contracts any unstable twig if such appears (see for example [KV07] Chapter 1 and references therein). Furthermore, there is a natural action of the symmetric group \( S_n \) on \( \overline{M}_{0,n} \) by permuting the marked points.

Our space of interest is the real locus of this variety \( M_n := \overline{M}_{0,n}(\mathbb{R}) \). It can be described as the set of equivalence classes of stable curves with \( n \) marked points defined over \( \mathbb{R} \). Since the projective lines are circles over the real numbers, a stable curve is a tree of circles with labeled points on them (as referred in [EHKR10] a “cactus-like” structure). The space \( M_n \) is a compact and connected smooth manifold of dimension \( n - 3 \) (see [DJS03] and [Dev99]). It is clear that \( M_3 \) is a point and \( M_4 \) is a circle. It can also be shown that \( M_5 \) is a connected sum of five real projective planes (see [Dev99]).

The cohomology ring of \( M_n \) was completely determined in [EHKR10]. We recall their description next.

**Theorem 2.2 ([EHKR10] 2.9).** Let \( \Lambda_n \) be the skew-commutative algebra generated by the elements \( w_{ijkl} \), \( 1 \leq i, j, k, l \leq n \), which are antisymmetric in \( i, j, k, l \) with defining relations
\[
w_{ijkl} + w_{jklm} + w_{klmi} + w_{lmij} + w_{mijk} = 0
\]
\[
w_{ijkl}w_{ijkl} = 0.
\]
Then \( \Lambda_n \) is isomorphic to \( H^*(M_n, \mathbb{Q}) \), and the action of \( S_n \) is given by
\[
\pi(w_{ijkl}) = w_{\pi(i)\pi(j)\pi(k)\pi(l)}.
\]

As we mentioned before, for \( n, m \geq 3 \) and each \( m \)-subset \( S \) of \([n]\), there is a well-defined forgetful map \( \phi_S : M_n \rightarrow M_m \) which induces a map in cohomology \( \phi^*_S : H^*(M_m, \mathbb{Q}) \rightarrow H^*(M_n, \mathbb{Q}) \). If \( m = 4 \) and \( S = \{i, j, k, l\} \) with \( 1 \leq i < j < k < l \leq n \), the induced map \( \phi^*_S : H^*(M_4, \mathbb{Q}) \rightarrow H^*(M_n, \mathbb{Q}) \) takes the generator of \( H^*(M_4, \mathbb{Q}) \cong \mathbb{Q} \) to \( w_{ijkl} \).

Observe that since \( w_{ijkl} = w_{1jkl} - w_{1kli} + w_{1lji} - w_{1ijk} \), the set
\[
\{w_{1ijk} | 1 < i < j < k \leq n\}
\]
is a generating set for the algebra \( \Lambda_n \).
3. Representation stability and \( FI \)-modules

In this section we revisit what it means for a sequence of \( S_n \)-representations to be uniformly representation stable in the sense of \[CF13\] and we survey the main notions from the theory of \( FI \)-modules developed by Church, Ellenberg and Farb in \[CEF\] that can be used to prove such phenomenon. Our main purpose is to discuss how these ideas can be readily applied to certain sequences of groups or spaces (see examples in Section 4) with cohomology rings that are known to be generated by the first cohomology group. This approach was first introduced in \[CEF\] Section 4.2 for a more general setting and was applied to another collection of examples in \[CEF\] Section 5. We illustrate the concepts with the example given by the cohomology of the moduli spaces \( M_n \) and obtain the proof of \[Theorem 1.1\].

3.1. Representation stability. Church and Farb introduced in \[CF13\] a notion of stability for certain sequences of \( S_n \)-representations over a field \( k \) of characteristic zero. We recall their definitions here.

**Definition 3.1.** A sequence \( \{V_n, \phi_n\}_n \) of \( S_n \)-representations over \( k \) together with homomorphisms \( \phi_n : V_n \to V_{n+1} \) is said to be consistent if the maps \( \phi_n \) are \( S_n \)-equivariant with respect to the natural inclusion \( S_n \hookrightarrow S_{n+1} \).

**Notation:** Over a field of characteristic zero, irreducible representations of \( S_n \) are defined over \( \mathbb{Q} \) and every \( S_n \)-representation decomposes as a direct sum of irreducible representations. Furthermore, irreducible representations of \( S_n \) are classified by partitions of \( n \). By a partition of \( n \) we mean a collection \( \lambda = (l_1, \ldots, l_r) \) with \( |\lambda| := l_1 + \cdots + l_r = n \) and \( l_i \geq l_{i+1} > 0 \). We will denote such a partition by \( \lambda \vdash n \) and the associated irreducible representation by \( V_\lambda \). Following the notation in \[CEF\], given a partition \( \lambda = (l_1, \ldots, l_j) \vdash m \) of a positive integer \( m \) and an integer \( n \geq m + l_1 \), we define the \( S_n \) representation \( V(\lambda) \) as

\[
V(\lambda) := V_{\lambda[n]},
\]

the irreducible representation that corresponds to the padded partition \( \lambda[n] := (n-m, l_1, \ldots, l_j) \) of \( n \).

**Definition 3.2.** (Uniform representation stability) Let \( \{V_n, \phi_n\}_n \) be a consistent sequence of \( S_n \)-representations. This sequence is uniformly representation stable if there exists a natural number \( N \) such that for \( n \geq N \) the following conditions are satisfied:

1. The map \( \phi_n \) is injective.
2. The span of the \( S_{n+1} \)-orbit of \( \phi_n(V_n) \) is equal to \( V_{n+1} \).
3. If \( V_n \) is decomposed into irreducible representations as \( \bigoplus \lambda c_{\lambda,n}V(\lambda) \), the multiplicities \( c_{\lambda,n} \) for each \( \lambda \) are independent of \( n \).

3.2. The \( FI \)-module structure. It was noticed in \[CEF\] that certain consistent sequences of \( S_n \)-representations could be encoded as a single object and this perspective had strong advantages. Using their notation, we consider the category \( FI \) whose objects are all finite sets and whose morphisms are all injections.

**Definition 3.3.** An \( FI \)-module over a commutative ring \( R \) is a functor from the category \( FI \) to the category of \( R \)-modules. If \( V \) is an \( FI \)-module we denote by \( V_n \) or \( V([n]) \) the \( R \)-module associated to \( n := [n] \) and by \( f_* : V_m \to V_n \) the map corresponding to the inclusion \( f \in \text{Hom}_{FI}(m,n) \).
The category of $FI$-modules over $R$ is closed under covariant functorial constructions on $R$-modules, if we apply functors pointwise. In particular, if $V$ and $W$ are $FI$-modules, then $V \otimes W$ and $V \oplus W$ are $FI$-modules.

Notice that since $\text{End}_{FI}(n) = S_n$, an $FI$-module $V$ encodes the information of the consistent sequence $\{V_n, (I_n)_n\}$ of $S_n$-representations with the maps induced by the natural inclusion $I_n : [n] \to [n + 1]$.

**The $FI$-modules $H^i(M_\bullet, \mathbb{Q})$.** In this paper we are interested in certain sequences of spaces or groups that can be encoded as a contravariant functor $Y_\bullet$ forming the $FI$-module $V$ to the category of spaces, a co-$FI$-space, or to the category of groups, a co-$FI$-group. Therefore, for each $i \geq 0$, we have that $H^i(Y_\bullet, R)$ has the structure of an $FI$-module over $R$ and $H^*(Y_\bullet, R)$ is a functor from $FI$ to the category of graded $R$-modules, what [CEF] calls a graded $FI$-algebra over $R$.

In particular, we can think of the sequence of manifolds $\{M_n\}$ as the objects of the co-$FI$-algebra $M_\bullet$ that takes each $n$ to $M_n$ and each inclusion $f \in \text{Hom}_{FI}(m, n)$ to the corresponding forgetful map $f^* := \phi_S : M_n \to M_m$ defined as before where $S := \text{im}(f)$. In this section, for each $i \geq 0$ we sometimes denote the $FI$-module $H^i(M_\bullet, \mathbb{Q})$ simply by $H^i$ and the graded $FI$-algebra $H^*(M_\bullet, \mathbb{Q})$ by $H^*$.

### 3.3. Finite generation of an $FI$-module

We are not interested in all $FI$-modules, but in those that are finitely generated in the following sense.

**Definition 3.4.** Let $V$ be an $FI$-module over $R$.

1. If $\Sigma$ is a subset of the disjoint union $\bigsqcup_n V_n$ we define span$_V(\Sigma)$, the span of $\Sigma$, to be the minimal sub-$FI$-module of $V$ containing $\Sigma$.
2. We say that $V$ is generated in degree $\leq m$ if $V$ is generated by elements in $V_k$ with $k \leq m$. Thus $V$ is generated in degree $\leq m$ if span$_V(\bigsqcup_{k \leq m} V_k) = V$.
3. We say that $V$ is finitely generated if there is a finite set of elements $\{v_1, \ldots, v_k\}$ with $v_i \in V_n$, which generates $V$, that is, span$_V(\{v_1, \ldots, v_k\}) = V$.

Finite generation of $FI$-modules is a property that is closed under quotients and extensions. It also passes to sub-$FI$-modules when $R$ is a field that contains $\mathbb{Q}$ (see [CEF] Theorem 1.3) and more generally when $R$ is a Noetherian ring (see [CEF] Theorem A). It is key that finite generation is closed under tensor products and we have control of the degree of generation.

**Proposition 3.5.** [CEF] Prop. 2.3.6 If $V$ and $W$ are finitely generated $FI$-modules, so is $V \otimes W$. If $V$ is generated in degree $\leq m_1$ and $W$ is generated in degree $\leq m_2$, then $V \otimes W$ is generated in degree $\leq m_1 + m_2$.

Our examples below are graded $FI$-algebras and we will take advantage of this structure to obtain finite generation.

**Definition 3.6.** Let $A$ be a graded $FI$-algebra over $R$. Given a graded sub-$FI$-module $V$ of $A$, we say that $A$ is generated by $V$ if $V_n$ generates $A_n$ as an $R$-algebra for all $n \geq 0$.

From [Theorem 2.2], the graded $FI$-algebra $H^*(M_\bullet, R)$ is generated by the $FI$-module $H^1(M_\bullet, R)$ (which can be thought as the graded sub-$FI$-module concentrated in grading 1). In all the examples in Section 4, the graded $FI$-algebra of interest $H^*(Y_\bullet, R)$ is a quotient of the free tensor algebra on the generating classes in $H^1(Y_\bullet, R)$. Therefore, Proposition 3.5 reduces the question of finite generation
for the \( FI \)-modules \( H^i(Y_\bullet, R) \) to a question of finite generation for the \( FI \)-module \( H^1(Y_\bullet, R) \).

**Proposition 3.7.** Suppose that the \( FI \)-module \( H^1(Y_\bullet, R) \) is finitely generated in degree \( \leq g \). If the graded \( FI \)-algebra \( H^*(Y_\bullet, R) \) is generated by \( H^1(Y_\bullet, R) \), then for each \( i \geq 1 \) the \( FI \)-module \( H^i(Y_\bullet, R) \) is finitely generated in degree \( \leq g \cdot i \).

**Degree of generation of** \( H^i(M_\bullet, Q) \). Using the explicit description of the cohomology ring in [Theorem 2.2] we can obtain finite generation for the \( FI \)-modules \( H^i \) from Proposition 3.7. Moreover, we can obtain the following upper bound for the degree of generation.

**Lemma 3.8.** The \( FI \)-module \( H^i \) is finitely generated in degree \( \leq 3i + 1 \).

**Proof.** The vector space \( H^i(M_n, Q) \) is generated by elements

\[
x = w_{1k_1l_1m_1} \cdots w_{1k_il_i},
\]

with the indices varying from 2 to \( n \). Hence, if \( n \geq 3i + 1 \), the vector space \( H^i(M_n, Q) \) is nontrivial and a generator element \( x \) has at most \( 3i + 1 \) different indices in the set \( [n] \). Then, after the action of some permutation \( \sigma \) of \( S_n \), \( \sigma \cdot x \in H^i(M_{3i+1}, Q) \). But

\[
x = \sigma^{-1}(\sigma \cdot x) \in \text{span}_{H^i}(H^i_{3i+1}),
\]

therefore \( H^i = \text{span}_{H^i}(H^i_{3i+1}) \). Since \( H^i(M_{3i+1}, Q) \) is of finite dimension, we conclude that the \( FI \)-module \( H^i \) is finitely generated in degree \( \leq 3i + 1 \). \( \square \)

Notice that, in particular, \( H^1 \) is finitely generated in degree \( \leq 4 \) and Proposition 3.7 only implies that \( H^1 \) is finitely generated in degree \( \leq 4i \).

One of the highlights of the theory of \( FI \)-modules is that finite generation of an \( FI \)-module over a field \( k \) of characteristic zero is equivalent to uniform representation stability of the corresponding consistent sequence.

**Theorem 3.9.** [CEF] Theorems 1.14 and 2.58] An \( FI \)-module over \( k \) is finitely generated if and only if the sequence \( \{ V_n, (I_n)_* \} \) of \( S_n \)-representations is uniformly representation stable and each \( V_n \) is finite dimensional. Furthermore, the stable range is \( n \geq s + d \) where \( d \) and \( s \) are the weight and the stability degree of \( V \) respectively.

In Sections 3.4 and 3.5 below we recall the definitions of weight and stability degree of an \( FI \)-module. We focus on \( FI \)-modules over a field \( k \) of characteristic zero.

**Representation stability for the cohomology of** \( M_n \). We just proved finite generation for the \( FI \)-module \( H^1 \) in Lemma 3.8. Therefore, the theorem above implies uniform representation for the sequence \( \{ H^i(M_n, Q) \} \). The specific stable range in [Theorem 1.11] follows from the computations of weight and stability degree of the \( FI \)-module \( H^1 \) in Lemmas 3.15 and 3.21 below.

**Remark 3.10.** Representation stability for the cohomology of the pure cactus groups can also be obtained by the methods in [CFI13] where the notion was first introduced. This approach, however, allows us to prove only the stability of \( H^i(M_n, Q) \) with respect to the \( S_{n-1} \)-action (rather than the action of \( S_n \)) and we obtain a worse estimate for the stable range than the one achieved using \( FI \)-modules.
3.4. Weight of an FI-module. It turns out that finite generation of an FI-module $V$ puts certain constraints on the partitions (shape of the Young diagrams) in the irreducible representations of each representation $V_n$ as we discuss below.

Definition 3.11. Let $V$ be an FI-module over $k$. We say that $V$ has weight $d$ if, for all $n \geq 0$, $d$ is the maximum order $|\lambda|$ over all the irreducible constituents $V(\lambda)_n$ of $V_n$. We write $weight(V) = d$.

By definition, if $W$ is a subquotient of $V$, then $weight(W) \leq weight(V)$. Moreover, the degree of generation of an FI-module gives an upper bound for the weight.

Proposition 3.12. [CEF Prop. 3.2.5] If the FI-module $V$ over $k$ is generated in degree $\leq g$, then $weight(V) \leq g$.

This implies, for example, that for a finitely generated FI-module $V$ the alternating representation cannot appear in the decomposition into irreducibles for $n > 0$. We also have control of the weight under tensor products.

Proposition 3.13. [CEF Prop. 3.2.2.] If $V$ and $W$ are FI-modules over $k$, then $weight(V \otimes W) \leq weight(V) + weight(W)$.

As a consequence we have the following result that can be applied to our examples of interest.

Corollary 3.14. If the graded FI-algebra $H^*(Y, k)$ is generated by the FI-module $H^1(Y, k)$ and $weight(H^1(Y, k)) \leq d$, then for each $i \geq 1$ we have that $weight(H^i(Y, k)) \leq d \cdot i$.

The weight of $H^i(M, \mathbb{Q})$. From Proposition 3.12 and our computation above we have that the weight of $H^i(M, \mathbb{Q})$ is bounded above by $3i + 1$. We can do slightly better.

Let $H_n$ be the standard representation of $S_n$, the subrepresentation of dimension $n - 1$ of the permutation representation $\mathbb{Q}^n$ consisting of vectors with coordinates that add to zero. Recall the description in Theorem 2.2 and notice that there is an isomorphism of $S_n$-representations $H^i(M_n, \mathbb{Q}) \cong \bigwedge^3 H_n$ given by $w_{ijkl} \mapsto (e_i - e_j) \wedge (e_j - e_k) \wedge (e_k - e_l)$. Notice that with our notation above $H^1(M_n, \mathbb{Q}) = V(1, 1, 1)$ for $n \geq 4$ and $weight(H^i(M, \mathbb{Q})) = 3$. From Proposition 3.12 we obtain that weight($H^i(M, \mathbb{Q})$) $\leq 3i$. Hence we have shown.

Lemma 3.15. The FI-module $H^i$ has weight $\leq 3i$.

3.5. Stability degree of an FI-module. In order to define the stability degree we need the notion of a $k[T]$-module.

Definition 3.16. A graded $k[T]$-module $U$ consists of a collection of $k$-modules $U_i$ for each $i \in \mathbb{N}$, endowed with a map $T : U_i \to U_{i+1}$ for each $i \in \mathbb{N}$.

If $V_n$ is an $S_n$-representation, its coinvariants $(V_n)_{S_n}$ is the largest $S_n$-equivariant quotient of $V_n$ on which $S_n$ acts trivially, or equivalently, it is the module $V_n \otimes k_{S_n}$.

If $V$ is an FI-module, we can apply this construction to all the $V_n$ simultaneously, obtaining the $k$-modules $(V_n)_{S_n}$ for each $n \geq 0$. Moreover, all the maps $V_n \to V_{n+1}$ involved in the definition of the FI-module structure induce a single map $(V_n)_{S_n} \to (V_{n+1})_{S_{n+1}}$. Hence, we obtain a graded-$k[T]$-module which we denote by $\Phi_0(V)$.

For each integer $a \geq 0$ fix once and for all some set $\mathfrak{a}$ of cardinality $a$, for instance, take $\mathfrak{a} = \{-1, \ldots, -a\}$.
Definition 3.17. Given an integer $a \geq 0$ and an $FI$-module $V$, we define the graded $k[T]$-module $\Phi_a(V)$ as follows. For $n \in \mathbb{N}$, let $\Phi_a(V)_n$ be the coinvariant quotient
\[ \Phi_a(V)_n := (V \wedge n)S_n. \]

To define $T : \Phi_a(V)_n \to \Phi_a(V)_{n+1}$, take any injection $f : n \hookrightarrow n+1$. This determines an injection $id \sqcup f : \mathbb{A} \sqcup n \to \mathbb{A} \sqcup n + 1$ and thus a map $(id \sqcup f)_* : V_{\mathbb{A} \sqcup n} \to V_{\mathbb{A} \sqcup n+1}$. The map $T$ is defined to be the induced map $(V_{\mathbb{A} \sqcup n})S_n \to (V_{\mathbb{A} \sqcup n+1})S_{n+1}$.

Definition 3.18. The stability degree, $\text{stab-deg}(V)$, of an $FI$-module $V$ is the smallest $s \geq 0$ such that for all $a \geq 0$, the map $T : \Phi_a(V)_n \to \Phi_a(V)_{n+1}$ is an isomorphism for all $n \geq s$.

For $FI$-modules over $\mathbb{Q}$, when taking tensor products, the stability degree is still bounded above.

Proposition 3.19. [KM Prop. 2.32] If the $FI$-modules $V$ and $W$ over $\mathbb{Q}$ have stability degree $\leq s_1, s_2$ and weight $\leq d_1, d_2$ respectively, then $V \otimes W$ has stability degree $\leq \max\{s_1 + d_1, s_2 + d_2, d_1 + d_2\}$.

This implies the following result in the setting of interest for this paper.

Corollary 3.20. If the graded $FI$-algebra $H^*(Y, \mathbb{Q})$ is generated by the $FI$-module $H^1(Y, \mathbb{Q})$ with weight $\leq d$ and stability degree $\leq s$, then for each $i \geq 1$ we have that $\text{stab-deg}(H^i(Y, \mathbb{Q})) \leq \max\{d + s, d \cdot i\}$.

Stability degree of $H^i(M, \mathbb{Q})$. We now find an upper bound for the stability degree for our example of interest.

Lemma 3.21. The $FI$-module $H^i$ has stability degree $\leq 3i$.

Proof. Given $a \geq 0$, the module $\Phi_a(H^1)_n = (H^1(M_{\mathbb{A} \sqcup n}), \mathbb{Q})S_n$ is the module of coinvariants of $H^1(M_{\mathbb{A} \sqcup n}), \mathbb{Q})$ where $S_n$ acts only in the indices in $\{1, \ldots, n\}$. Thus, for $n \geq 1$, this module is generated by the equivalence classes of elements $[x] = [w_{klm}]$ where the indices are now in the set $\mathbb{A} \sqcup n$, and any two $x$ and $y$ are related if there exists some permutation $\sigma$ of $\{1, \ldots, n\}$ such that $\sigma \cdot x = y$. Notice that if at least one of the indices $k, l$ or $m$ is in $[n]$, say $k \in [n]$, then $[w_{klm}] = [w_{klm}] = [-w_{klm}]$, hence it is zero in coinvariants. Then the map $T : \Phi_a(H^i)_n \to \Phi_a(H^i)_{n+1}$, given by $[x] \mapsto [x]$, is always injective and surjective if $n \geq 1$. Therefore $\text{stab-deg}(H^i) = 1$. Since we know that weight($H^1$) = 3, the upper bound for the stability degree of $H^i$ follows from Corollary 3.20.

□

3.6. Polynomiality of characters. We end this section by discussing another way of how finite generation of $FI$-modules imposes strong constraints in the corresponding $S_n$-representations. If $V$ is finitely generated, then the sequence of characters $X_V$ of the $S_n$-representations $V_n$ is eventually polynomial as we now explain. For each $i \geq 1$ and any $n \geq 0$, we consider the class function $X_i : \bigsqcup_n S_n \to \mathbb{Z}$ defined by

$$X_i(\sigma) = \text{number of } l\text{-cycles in the cycle decomposition of } \sigma.$$
Polynomials in the variables $X_l$ are called character polynomials. The degree of the character polynomial $P(X_1, \ldots, X_r)$ is defined by setting $\deg(X_l) = l$.

**Theorem 3.22** ([CEF] Theorem 1.5). Let $V$ be an FI-module over $\mathbb{Q}$. If $V$ is finitely generated then there exist a unique character polynomial $P_V \in \mathbb{Q}[X_1, X_2, \ldots]$ with $\deg P_V \leq d$ such that for all $n \geq s + d$

$$\chi_{V_n}(\sigma) = P_V(\sigma)$$

for all $\sigma \in S_n$, where $d$ and $s$ are the weight and the stability degree of $V$ respectively.

The upper bound on the degree of the character polynomial $P_V$ in Theorem 3.22 implies that $P_V$ only involves the variables $X_1, \ldots, X_d$. This tells us that the character $\chi_{V_n}$ only depends on “short cycles”, i.e., on cycles of length $\leq d$, independently of how large $n$ is. Moreover, we have that the dimension of $V_n$

$$\dim V_n = \chi_{V_n}(\text{id}) = P_V(n, 0, \ldots, 0)$$

which is a polynomial in $n$ of degree $\leq d$ for $n \geq s + d$.

**Characters for the cohomology of $M_n$.** [Theorem 3.22] together with the computations below of upper bounds for the weight and the stability degree for the FI-modules $H^i$ give us the following consequence.

**Theorem 3.23.** For $i \geq 0$ and $n \geq 6i$ the character of the $S_n$-representation $H^i(M_n, \mathbb{Q})$ is given by a unique character polynomial $P_i$ of degree $\leq 3i$.

Corollary 1.2 follows from this theorem and the discussion before.

4. Other examples

We would like to end this note by revisiting the collection of “pure braid like” examples in Table 1 that have been considered using the perspective discussed in Section 3. In general, the bounds in Table 1 are not sharp.

**Configuration space of $n$ points in $\mathbb{C}$ and the pure braid group.** This example was the first studied with this approach in [CEF, Example 5.1.A]. We have the co-FI-space $\mathcal{F}(\mathbb{C}, \bullet)$ given by $n \mapsto \mathcal{F}(\mathbb{C}, n)$. Since these configuration spaces are Eilenberg-MacLane spaces, we could alternatively consider their fundamental groups $P_n$, the pure braid groups. The cohomology ring in this case is the so called Arnol’d algebra which is generated by cohomology classes of degree one. We recall its description in Table 1 and recover the information about the weight and degree of generation from our discussion in Section 3. Uniform representation stability was originally proved in [CF13].

**Moduli space of $n$-pointed curves of genus zero.** The moduli space of $(n+1)$-pointed curves of genus zero $M_{0,n+1}$ has a natural action of $S_{n+1}$ by permuting the marked points. If we focus on the action of the subgroup $S_n$ generated by the transpositions $(2 \ 3), (3 \ 4), \ldots , (n \ n+1)$ in $S_{n+1}$, we can identify its cohomology ring with a subalgebra of $H^*(\mathcal{F}(\mathbb{C}, n), \mathbb{Q})$ as we recall in Table 1. Hence the graded FI-algebra $H^*(M_{0,\bullet+n+1}, \mathbb{Q})$ is generated in degree one classes and our arguments in Section 3 readily apply. As explained in [JR], we can use conclusions on these “shifted” FI-modules to obtain information about $H^*(M_{0,\bullet}, \mathbb{Q})$.
Remark 4.1. The theory of $FI$-modules can also be used to study and obtain representation stability of the cohomology of configuration spaces of $n$ points on connected manifolds and moduli spaces of genus $g \geq 2$ with $n$ marked points (see [Chu12, CEF, JR11 and JR13]). In these cases, little information is known about the corresponding cohomology rings and the arguments needed are more involved than the ones discussed in this note.

**Pure virtual and flat braid groups.** The pure virtual braid groups $P_vB_n$ and the pure flat braid groups $PfB_n$ are generalizations of the braid group $P_n$ that allow virtual crossings of strands. Those virtual crossings were introduced in [Kau99]. The original motivation was to study knots in thickened surfaces of higher genus and the extension of knot theory to the domain of Gauss codes and Gauss diagrams. Explicit definitions and presentations of these groups can be found in [Lee, Section 2.1]. As for the pure braid group $P_n$, there is a well-defined action of $S_n$ on $H^*(P_vB_n; \mathbb{Q})$ and $H^*(PfB_n; \mathbb{Q})$. Moreover, we have well-defined forgetful maps compatible with the $S_n$-action. Hence we can consider the $FI$-modules $H^*(P_vB_n; \mathbb{Q})$ and $H^*(PfB_n; \mathbb{Q})$ in the setting from Section 3.

From the description in Table 1, we see that the corresponding cohomology rings have a graded $FI$-algebra structure and are generated by the first cohomology group. Moreover, from the explicit decompositions of $H^*(P_vB_n, \mathbb{Q})$ and $H^*(PfB_n, \mathbb{Q})$ as a direct sum of induced representations obtained in [Lee, Theorem 3 and 8], it follows that $H^*(P_vB_n, \mathbb{Q})$ and $H^*(PfB_n, \mathbb{Q})$ have the extra-structure of an $FI\#$-module (see [CEF, Def. 4.1.1 and Thm. 4.1.5]). This in particular implies that the corresponding stability degree is bounded above by the weight ([CEF, Prop. 3.1.7]). Theorem 3.9 implies uniform representation stability, which was previously obtained in [Lee] using the approach in [CEF13].

In [Wil14], Wilson extended the work of Church, Ellenberg, Farb, and Nagpal on sequences of representations of $S_n$ to representations of the signed permutation groups $B_n$ and the even-signed permutation groups $D_n$. She introduced the notion of a finitely generated $FI\#$-module, where $\mathcal{W}_n$ is the Weyl group in type $A_{n-1}$, $B_n/C_n$, or $D_n$. Arguments like the ones discussed in Section 3 also apply in this setting ([Wil14 Prop. 5.11]). Uniform representation stability and polynomiality of characters are also consequences of this approach ([Wil14 Thms 4.26 & 4.27] and [Wil Thm 4.16]).

**Group of pure string motions.** The group $\Sigma_n$ of string motions is another generalization of the braid group. It is defined as the group of motions of $n$ smoothly embedded, oriented, unlinked, unknotted circles in $\mathbb{R}^3$ and can be identified with the symmetric automorphism group of the free group $F_n$. We refer the interested reader to [Wil] Section 5.1 and references therein. The subgroup $PS_n$ of pure string motions or pure symmetric automorphisms is the analogue of the pure braid group and has a natural action of the signed permutation group $B_n$. The presentation of the cohomology ring that we recall in Table 1 can be used to show that $H^*(PS_n, \mathbb{Q})$ has the structure of what she calls a graded $FI_{BC}\#$-algebra (see [Wil Theorem 5.3]) which turns out to be finitely generated by $H^1(PS_n, \mathbb{Q})$. By restricting to the $S_n$-action, one recovers finite generation for $H^p(PS_n, \mathbb{Q})$ as $FI$-modules.
Table 1. Examples of “pure braid like” groups and spaces that have cohomology rings with an FI-algebra structure generated by the first cohomology.

| Group or Space $Y_n$ | Cohomology Ring $H^*(Y_n, \mathbb{Q})$ | $H^1(Y_n, \mathbb{Q})$ | $\chi H^1(Y_n, \mathbb{Q})$ | weight of $H^p(Y_n)$ | deg-gen of $H^p(Y_n)$ | stab-deg of $H^p(Y_n)$ |
|----------------------|----------------------------------------|------------------------|-----------------------------|-------------------|-------------------|-------------------|
| $\mathcal{F}(\mathbb{C}, n)$ | Exterior algebra $\mathcal{R}_n$ generated by $w_{i,j}$, $1 \leq i \neq j \leq n$, with relations $w_{i,j} = w_{j,i}$, $w_{i,j}w_{j,k} + w_{j,k}w_{k,i} + w_{k,i}w_{i,j} = 0$ (See [Arn69]) | $V(\cdot) \oplus V(1) \oplus V(2)$ for $n \geq 4$ | $(\chi)^2/2 + X_2$ for $n \geq 4$ | $2p$ | $2p$ | $2p$ |
| $\mathcal{M}_{0,n+1}$ | Subalgebra of $\mathcal{R}_n$ generated by $1$ and $\theta_{i,j} := w_{i,j} - w_{i,2}$ with $\{i, j\} \neq \{1, 2\}$ (See [Ga10] Cor. 3.1 and references therein) | $V(1) \oplus V(2)$ for $n \geq 4$ | $(\chi)^2/2 + X_2 - 1$ for $n \geq 4$ | $2p$ | $4p$ | $2p$ |
| $\mathcal{M}_{0,n}(\mathbb{R})$ | Supercommutative algebra generated by $w_{i,j}$, $1 \leq i, j, k \leq n$, antisymmetric in $i, j, k, l$, with relations $w_{i,j}w_{j,k}w_{k,m} = 0$ and $w_{i,j}w_{j,k}w_{k,l} + w_{k,l}w_{l,m} + w_{m,i} + w_{m,i} = 0$ (See [EHKR10] Thm. 2.9) | $V(1, 1, 1)$ for $n \geq 4$ | $\chi^2/2 + X_3 - X_2X_1$ for $n \geq 4$ | $3p$ | $3p + 1$ | $3p$ |
| $PuB_n$ | Exterior algebra generated by $w_{i,j}$, $1 \leq i \neq j \leq n$, with relations $w_{i,j}w_{j,i} = 0$, $w_{i,j}w_{i,k} = w_{i,j}w_{j,k} - w_{i,k}w_{k,j}$ and $w_{i,k}w_{j,k} = w_{i,j}w_{j,k} - w_{j,1}w_{1,k}$ (See [Lee13] Section 5) and [BEER06] | $V(\cdot) \oplus V(1)^2 \oplus V(1, 1) \oplus V(2)$ for $n \geq 4$ | $2(\chi)^2/2$ for $n \geq 4$ | $2p$ | $2p$ | $2p$ |
| $PfB_n$ | Exterior algebra generated by $w_{i,j}$, $1 \leq i \neq j \leq n$, with relations $w_{i,j} = -w_{j,i}$, $w_{i,j}w_{i,k} = w_{i,j}w_{j,k}$ and $w_{i,k}w_{j,k} = w_{i,j}w_{j,k}$ for $i < j < k$ (See [Lee13] Section 5) and [BEER06] | $V(1) \oplus V(1, 1)$ for $n \geq 3$ | $2(\chi)^2/2$ for $n \geq 3$ | $2p$ | $2p$ | $2p$ |
| $P\Sigma_n$ | Exterior algebra generated by $w_{i,j}$, $1 \leq i \neq j \leq n$, with relations $w_{i,j}w_{j,i} = 0$, and $w_{i,j}w_{j,i} - w_{k,j}w_{k,i} + w_{j,i}w_{k,i} = 0$ (See [JMM06]) | $V(\cdot) \oplus V(1)^2 \oplus V(1, 1) \oplus V(2)$ for $n \geq 4$ | $2(\chi)^2/2$ for $n \geq 4$ | $2p$ | $2p$ | $2p$ |
Hyperplane complements. Another generalization of configuration spaces on the plane is given by hyperplane complements for other Weyl groups. Consider the canonical action of $\mathcal{W}_n$ on $\mathbb{C}^n$ by (signed) permutation matrices and let $A(n)$ be the set of hyperplanes fixed by the (complexified) reflections of $\mathcal{W}_n$. The sequence of hyperplane complements $\mathcal{M}_\mathcal{W}(n) := \mathbb{C}^n - \bigcup_{H \in A(n)} H$ can be encoded as a co-$FI$-space $\mathcal{M}_\mathcal{W}(\bullet)$. In particular $\mathcal{M}_A(\bullet) = \mathcal{F}(\mathbb{C}, \bullet)$. The corresponding fundamental groups are referred as “generalized pure braid groups”. The work in [Bri73] and [OS80] show that the cohomology ring for $\mathcal{M}_\mathcal{W}(n)$ is also generated by cohomology classes of degree one and arguments as before apply. The details are worked out in [Wil], Thm 5.8.

Complex points of the Deligne-Mumford compatification of $\mathcal{M}_{0,n}$. Another related example is the complex locus of the variety $\mathcal{M}_{0,n}$, usually denoted again by $\mathcal{M}_{0,n}$. From our discussion in Section 2 it is clear that $H^*(\mathcal{M}_{0,\bullet}; \mathbb{Q})$ has the structure of a graded $FI$-algebra. Moreover, from the results in [Kee92] it follows that it is generated by the $FI$-module $H^2(\mathcal{M}_{0,\bullet}; \mathbb{Q})$. However, the computations in [Kee92] also imply that the dimension of $H^i(\mathcal{M}_{0,\bullet}; \mathbb{Q})$ is exponential in $n$. Therefore, from Theorem 3.23 it follows that the $FI$-modules $H^i(\mathcal{M}_{0,\bullet}; \mathbb{Q})$ are not finitely generated.

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