Learning nonlinear dynamics in synchronization of knowledge-based leader-following networks

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Abstract

Knowledge-based leader-following synchronization of heterogeneous nonlinear multi-agent systems is a challenging problem, since the leader’s dynamic information is unknown to any follower node. This paper proposes a learning-based fully distributed observer for a class of nonlinear leader systems, which can simultaneously learn the leader’s dynamics and states. This class of leader dynamics is rather general and does not require a bounded Jacobian matrix. Based on this learning-based distributed observer, we further synthesize an adaptive distributed control law for solving the leader-following synchronization problem of multiple Euler-Lagrange systems subject to an uncertain nonlinear leader system. The results are illustrated by a simulation example.

Keywords: Distributed observer, Euler-Lagrange system, multi-agent system, parameter estimation, synchronization.

I. Introduction

Synchronization of networked multi-agent systems has found its applications in various scenarios, such as flocking (Bullo, Cortés, & Martinez, 2009), coupled distributed estimation (Olfati-Saber & Jalaianmali, 2012), and formation of multiple robots (Döfler & Francis, 2010; Feng, Hu, & Candras, 2019; Roy, Baldi, & Ioannou, 2021), and has been extensively studied for the past several decades. A central class of synchronization problems are the so-called leader-following synchronization problems (also called cooperative tracking problems (Zhang and Lewis, 2012)), which aim to drive each follower system to behave in the same motion or oscillate concurrently with the leader system (Chen, 2014; Isidori, Marconi, & Casadei, 2014; Radenković & Krstić, 2018).

According to Wang and Slotine (2006), leader-following synchronization problems can be divided into two categories: power-based leader-following synchronization problems and knowledge-based leader-following synchronization problems. For the former one, both followers and the leader node share the same system dynamics. For the latter one, the leader’s dynamic knowledge (i.e., parameters) is inaccessible to any follower and each follower adaptively learns the leader’s dynamic knowledge through neighborhood interactions. In this sense, the leader is also called an uncertain leader in the literature (Wang & Huang, 2018; Baldi, Azzollini, & Ioannou, 2021).

Most recent works (Wu, Lu, Shi, Su, & Wu, 2017; Klotz, Kan, Shea, Pasiliao, & Dixon, 2014) solved the leader-following synchronization problem by assuming that some/all followers know the leader’s parameters. In contrast, solving knowledge-based leader-following synchronization problem is much challenging, since no followers can get access to the leader’s knowledge (i.e., parameters) and only part of them can sense the leader’s state or output signals. There are generally two approaches to handle the uncertain leader. One approach is the so-called distributed adaptive internal model design approach (Su & Huang, 2013). Then, under the assumption that the leader has a unitary relative degree and strongly minimum phase properties, the synchronization problem can be converted to a global adaptive robust stabilization problem. Another is the adaptive distributed observer design approach, which was studied recently in Wang and Huang (2018) and Wang and Meng (2021), where each follower node maintains an observer, estimating the state and/or learning the parameters of the leader node in a distributed manner, and then a decentralized controller is designed for each follower node to drive it to track the output of its observer. Some early tries of the observer design can also be found in Moraes, Nageshrao, Lopes, Babuska, and Lewis (2016), where the convergence analysis of the estimated parameters is not provided. Very recently, Baldi, Azzollini, and Ioannou (2021) also studied the knowledge-based leader-
following synchronization problem and proposed an adaptive distributed observer to simultaneously estimate the state and learn the knowledge of a linear leader. They pointed out that it is a starting point to extend this approach to more general cases. Although much attention has been paid to uncertain linear leader dynamics, insufficient investigation has been reported for knowledge-based leader-following networks with nonlinear dynamics, to the best of the authors’ knowledge.

However, in practical applications, dynamics of the leader are often described by nonlinear systems (Wang & Slotine, 2005, 2006; Zhang & Lewis, 2012), and the nonlinear leader system may contain some parameters unknown to any of the followers. It is noted that the techniques developed in Wang and Huang (2018) and Baldi, Azzollini, and Ioannou (2021) for linear leaders with unknown parameters cannot be applied to nonlinear systems. In this paper, we consider an uncertain nonlinear leader. Compared with its linear counterpart, the knowledge-based leader-following synchronization problem with a nonlinear leader is recognized to be much challenging, and is worthy of further investigation.

Note that by considering the multi-agent system consisting only of the leader node and the learning-based observers, the distributed observer design problem is essentially the same with the knowledge-based leader-following synchronization problem (Wang & Slotine, 2006) and has the well-known frequency estimation problem (Hsu, Ortega, & Damm, 1999) as a special case. In the literature, synchronization of coupled nonlinear systems can be achieved under either of the following three common conditions: the global Lipschitz condition (Wang & Slotine, 2005; Zhou, Lu, & Lü, 2006; Yu, Chen, & Lü, 2009), the quadratic condition (Lu & Chen, 2004; DeLellis, Di Bernardo, Gorochowski, & Russo, 2010), and the contracting condition (Wang & Slotine, 2006). The relationship among these conditions was reported in DeLellis, di Bernardo, and Russo (2011). Essentially, the aforementioned works try to use a sufficiently large constant to stabilize the coupled nonlinear dynamic networks. For more general scenarios, when the nonlinear dynamics of the leader do not satisfy the above conditions, a practical approach for synchronization of networked nonlinear systems is to use neural networks to approximate the unknown nonlinearities (Zhang & Lewis, 2012), which only guarantees boundedness of the synchronization error, without estimating the leader’s dynamics.

It is worth noting that, in the literature of multi-agent systems, many works require the knowledge of the whole communication graph, such as its Laplacian matrix, which is a certain global information. This destroys the scalability of the design in some extent. For example, to handle a nonlinear leader with known leader’s parameters, Dong and Chen (2021) assume that the term related to the nonlinearity and the network topology can be bounded by some smooth function. This approach relies on the global knowledge of the communication graph, which is not fully distributed according to Li, Wen, Duan, and Ren (2014).

Motivated by the above-mentioned statements, in this paper, we consider a more practical scenario of knowledge-based leader-following synchronization problems, where the leader node has a nonlinear dynamical model which is not known by any followers. We aim at designing a fully distributed control law, which does not require any global knowledge of the communication graph. Therefore, the design is not affected by the topology change of the communication graph. In other words, it is totally scalable. We first establish a learning-based fully distributed observer for a class of nonlinear leader systems whose nonlinearities can be written as the product between a weight vector and a regressor matrix. Under some standard assumptions, this learning-based distributed observer can estimate and pass the leader’s state to each follower through local interactions without knowing the leader’s parameters. If the leader’s regressor matrix is persistently exciting, this distributed observer can also asymptotically adaptively learn the leader’s parameters. Since the nonlinear dynamics of the leader and information of the communication network gets involved in the analysis and design of learning-based fully distributed observer, the design of learning-based fully distributed observer proposed in Wang and Huang (2018) and Baldi, Azzollini, and Ioannou (2021) cannot be used to estimate and learn the state and knowledge of a nonlinear leader. Our observer design removes the bounded Jacobian matrices assumption or the quadratic condition on the coupled nonlinear systems (Wang & Slotine, 2005, 2006; Zhou, Lu, & Lü, 2006). Moreover, the global information of the communication graph has been completely removed by using an adaptive technique. It is different from the one proposed in Li, Wen, Duan, and Ren (2014) and Dong and Chen (2021), which is not applicable in our case of nonlinear leader dynamics. As an application, based on this learning-based distributed observer, we further synthesize an adaptive distributed control law for solving the leader-following synchronization problem of multiple Euler - Lagrange systems subject to an uncertain nonlinear leader system.

The contributions of this article are: (1) relaxing the bounded Jacobian matrices assumption or the quadratic condition on the coupled nonlinear systems (Wang and Slotine, 2005, 2006; Zhou, Lu, and Lü, 2006) and (2) using an adaptive technique to obtain fully distributed design without relying on any global information of the communication graph.

The rest of this paper is organized as follows: Section II formulates the problem. In Section III, we establish a learning-based distributed observer for a nonlinear leader system whose parameters are unknown to any follower nodes. As an application of the learning-based distributed observer, we apply the learning-based distributed observer to synthesize an adaptive distributed control law for solving the leader-following synchronization problem of multiple Euler - Lagrange systems subject to an uncertain nonlinear leader system in Section IV. A simulation ex-
ample is given in Section V, and Section VI concludes the paper.

Notation: Let \( \otimes \) denote the Kronecker product of matrices. A function with \( k \) continuous derivatives is called a \( C^k \) function. \( \mathbb{I}_N = \text{col}(1, \ldots, 1) \), and \( I_n \) denotes the identity matrix with dimension \( n \). \( \mathbb{R}^+ \) denotes all the positive real numbers. For \( X_1, \ldots, X_k \in \mathbb{R}^{n \times m} \), \( \text{col}(X_1, \ldots, X_k) = [X_1^T, \ldots, X_k^T]^T \) and

\[
\text{blkdiag}(X_1, \ldots, X_k) = \begin{bmatrix} X_1 & \cdots & X_k \\ \end{bmatrix}.
\]

For \( x \in \mathbb{R}^m \), unless indicated otherwise, \( x_i \) denotes the \( i \)-th component of \( x \), \( \|x\| \) and \( \|x\|_p \) denote the 2-norm and the \( p \)-norm of \( x \), respectively.

II. Preliminaries and problem formulation

II.1. Graph theory

The communication/interaction network of a multi-agent system composed of \( N \) followers and one leader is described by a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) with \( \mathcal{V} = \{1, \ldots, N, N + 1\} \) and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) being the node set and the edge set, respectively. Here node \( N + 1 \) is associated with the leader and nodes \( i, i = 1, \ldots, N \), are associated with the followers. For \( i = 1, \ldots, N, N + 1 \) and \( j = 1, \ldots, N, (i, j) \in \mathcal{E} \) if and only if node \( j \) can get the information of node \( i \) for control purpose. Let \( \mathcal{N}_i = \{ j | (j, i) \in \mathcal{E} \} \) denote the neighbor set of agent \( i \). Let \( \tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}) \) denote the induced subgraph of \( \mathcal{G} \) with \( \tilde{\mathcal{V}} = \{1, \ldots, N\} \), which captures the interaction among follower nodes. Assume that \( \tilde{\mathcal{G}} \) contains a spanning tree with node \( N + 1 \) being the root, and \( \tilde{\mathcal{G}} \) is an undirected graph. Let \( \tilde{\mathcal{L}} \) be the Laplacian matrix of the graph \( \tilde{\mathcal{G}} \), and \( \tilde{H} \) is obtained by deleting the last row and column of \( \tilde{\mathcal{L}} \). Then, \( \tilde{H} \) is a symmetric positive definite matrix (Zhang & Lewis, 2012). More details of the graph theory can be found in Godsil and Royle (2001).

II.2. Problem formulation

Let the follower nodes be described by general nonlinear systems

\[
\begin{align*}
\dot{x}_i &= f_i(x_i, \tau_i), \\
y_i &= h_i(x_i, \tau_i), & i = 1, \ldots, N,
\end{align*}
\]

where \( x_i \in \mathbb{R}^{n_i}, y_i \in \mathbb{R}^{n} \) and \( \tau_i \in \mathbb{R}^{m_i} \) are the state, measurement output and control input of the \( i \)-th follower, and \( f_i(\cdot) \) and \( h_i(\cdot) \) are globally defined and sufficiently smooth functions vanishing at the origin.

The leader’s output signal \( q_{N+1} \in \mathbb{R}^{n} \), is generated by the following nonlinear system

\[
\begin{align*}
\dot{v} &= p(v, \omega), \\
q_{N+1} &= Ev,
\end{align*}
\]

where \( v \in \mathbb{R}^{m}, p(\cdot, \omega) \) is a globally defined and sufficiently smooth function vanishing at the origin, \( \omega \in \mathbb{R}^l \) is a constant vector consisting of the unknown parameters of the leader node, \( E \in \mathbb{R}^{n \times m} \) is a known constant matrix. The uncertain nonlinear system (2) is assumed to satisfy

\[
p(v, \omega) = \phi(v) \omega, \tag{3}
\]

where the regressor matrix \( \phi(\cdot) \in \mathbb{R}^{m \times l} \) is known. It is assumed that given any compact set \( \mathcal{V}_0 \), there exists a compact set \( \mathcal{V} \) such that, for any \( v(0) \in \mathcal{V}_0, v(t) \in \mathcal{V} \) for all \( t \geq 0 \). The system (2) encompasses a class of nonlinear systems that can generate stable limit cycles, such as the well-known Van der Pol system, which can describe lots of periodic behavior arising from physics, biology, chemistry and engineering (Buchli, Righetti, & Ijspeert, 2006).

The design schematic is shown in Figure 1. We consider the following class of distributed control laws

\[
\begin{align*}
\tau_i &= k_i(x_i, \hat{v}_i, \hat{\omega}_i), \\
\dot{\hat{v}}_i &= g_i \left( \hat{\omega}_i, \hat{v}_i, \sum_{j \in \mathcal{N}_i} (\hat{v}_j - \hat{v}_i) \right), \\
\dot{\hat{\omega}}_i &= p_i \left( \hat{v}_i, \sum_{j \in \mathcal{N}_i} (\hat{v}_j - \hat{v}_i) \right),
\end{align*}
\]

where \( \hat{v}_{N+1} = v \), and for \( i = 1, \ldots, N, \hat{v}_i \in \mathbb{R}^{m_i} \) and \( \hat{\omega}_i \in \mathbb{R}^{l_i} \) are the estimation of \( v \) and \( \omega \), respectively; \( k_i(\cdot), g_i(\cdot) \) and \( p_i(\cdot) \) are sufficiently smooth functions vanishing at the origin.

Now we are ready to formulate the knowledge-based leader-following synchronisation (KLFS) problem for a class of nonlinear multi-agent systems.

Problem 1 (KLFS Problem). Consider the multi-agent system (1) and (2), and the corresponding communication graph \( \mathcal{G} \). Given any compact set \( \mathcal{V}_0 \subset \mathbb{R}^m \) containing the origin, design a distributed control law in the form of (4), such that for any initial condition \( x_i(0) \in \mathbb{R}^{m_i}, \hat{v}_i(0) \in \mathbb{R}^{m_i}, \hat{\omega}_i(0) \in \mathbb{R}^{l_i} \), and \( v(0) \in \mathcal{V}_0, x_i(t) \) is bounded for all \( t \geq 0 \), and

\[
\lim_{t \to \infty} (y_i(t) - q_{N+1}(t)) = 0, \quad i = 1, 2, \ldots, N.
\]

Note that for the special case that \( v \) and \( \omega \) are known to all followers, the control law (4) will reduce to the following form

\[
\tau_i = k_i(x_i, v, \omega). \tag{5}
\]

Then the KLFS problem will reduce to the traditional adaptive nonlinear output regulation problem, which has been well studied in Huang (2004) and Isidori, Marconi, and Casadei (2014).

In practice, the leader’s dynamic knowledge (i.e., parameters) may be unknown to all followers, and only a few followers who have direct links to the leader can sense the leader’s state or output information. Then, in order to solve the KLFS problem, we need to design a learning-based distributed observer for each follower node to estimate the state and simultaneously learn the parameters of
the leader node. The learning-based distributed observer, which is the main kernel in solving Problem 1, is defined as follows.

**Definition 1 (Learning-based distributed observer).** A dynamic of the form (4b) and (4c) is called a learning-based distributed observer for the leader (2) if, given a digraph $\mathcal{G}$ and any compact set $\mathcal{V}_0 \subset \mathbb{R}^m$ containing the origin, there exist global defined and sufficiently smooth functions $g_i(\cdot)$ and $p_i(\cdot)$, such that for any initial condition $\hat{\omega}_i(0) \in \mathbb{R}^s$, $\hat{v}_i(0) \in \mathbb{R}^m$ and $v(0) \in \mathcal{V}_0$, $\hat{v}_i(t)$ is bounded for all $t \geq 0$, and

$$\lim_{t \to \infty} (\hat{v}_i(t) - v(t)) = 0.$$  

Further, under certain conditions, the observer (4c) can adaptively learn the actual values of the knowledge-based leader in the sense that

$$\lim_{t \to \infty} (\hat{\omega}_i(t) - \omega) = 0.$$  

Before proceeding, we recall the definition of persistent excitation and two useful lemmas, which will be used in the sequel.

**Definition 2.** (Anderson, 1977) A bounded piecewise continuous function $f : [0, +\infty) \to \mathbb{R}^{n \times m}$ is said to be persistently exciting if there exist positive constants $\epsilon, t_0$, and $T_0$ such that,

$$\frac{1}{T_0} \int_{t_0}^{t+T_0} f(s) f^T(s) ds \geq \epsilon I_n, \quad \forall t \geq t_0.$$  

**Lemma 1.** (Chen & Huang, 2015) (i) For any continuous function $f(x, d) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$, there exist smooth functions $a(x), b(d) \geq 0$, such that $|f(x, d)| \leq a(x) b(d)$.

(ii) For any continuous function $f(x, d) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ satisfying $f(0, d) = 0$, there exist smooth functions $m(x, d) \geq 0$, such that $|f(x, d)| \leq m(x, d)|x|$.

**Lemma 2.** (Huang, 2004) Let $f : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ be a $C^1$ function satisfying $f(0, 0, d) = 0$ for all $d \in \mathbb{D}$, with $\mathbb{D}$ being a compact set of $\mathbb{R}^p$. Then, there exist smooth functions $F_1 : \mathbb{R}^m \to \mathbb{R}$ and $F_2 : \mathbb{R}^n \to \mathbb{R}$ satisfying $F_1(0) = 0$ and $F_2(0) = 0$ such that

$$|f(x, y, d)| \leq F_1(x) + F_2(y), \quad \forall x \in \mathbb{R}^m, y \in \mathbb{R}^n, d \in \mathbb{D}.$$  

**III. Learning-based distributed observer design**

This section devotes to the design and analysis of the learning-based distributed observers. Then in Section IV, we will apply this observer to solve a leader-following synchronisation problem of multiple Euler-Lagrange systems.

**III.1. Learning-based distributed observers design**

Note that the parameter vector $\omega$ in the leader system (2) is constant, and thus bounded. To estimate both states and parameters of the leader node, we design the following learning-based distributed observer for node $i$,

$$\dot{\hat{v}}_i = p(\hat{v}_i, \hat{\omega}_i) + \hat{\kappa}_i \rho_i(z_i) z_i,$$  

$$\dot{\hat{\omega}}_i = \mu \phi^T(\hat{v}_i) z_i,$$  

$$\hat{\kappa}_i = \rho_i(z_i) z_i^T z_i,$$  

$$z_i = \sum_{j \in \mathcal{N}_i} (\hat{\omega}_j - \hat{\omega}_i), \quad i = 1, \ldots, N,$$  

where $\hat{v}_{N+1} = v$, $\hat{v}_i \in \mathbb{R}^m$ is the estimation of $v$, $\hat{\omega}_i \in \mathbb{R}^s$ is the estimation of $\omega$, $\hat{\kappa}_i \in \mathbb{R}$ is a dynamic gain to relax the dependence on an unknown bound $\kappa_i$ induced by the matrix $H$, uniform bound of state $v$ and unknown parameter $\omega$. Here

$$p(\hat{v}_i, \hat{\omega}_i) = \phi(\hat{v}_i) \hat{\omega}_i,$$

where $\phi(\cdot) \in \mathbb{R}^{m \times l}$ is defined in (3), $\rho_i(\cdot) \geq 1$ is a smooth positive function to be designed and $\mu$ can be selected as any positive scalar.

**Remark 1.** A salient feature of the observer (6) is its fully distributed nature, and only those follower nodes who have direct links from the leader need to get access to the leader’s state information, instead of its parameters $\omega$. Closely related results can be found in Wang and Slotine (2005, 2006) and Zhou, Lu, and Lü (2006). However, they use a fixed linear gain to stabilize the coupled nonlinear dynamic networks under assumptions that each coupled nonlinear agent must satisfy the global bounded Jacobian matrix assumption or the quadratic condition proposed in Lu and Chen (2003).

Moreover, the observer design (6) is fully distributed in the sense that the coupled global information $\hat{\kappa}$ induced by the communication graph and knowledge-based leader system has been completely removed by using an adaptive technique, i.e., equation (6c). Note that this technique is
different from methods proposed in Li, Wen, Duan, and Ren (2014) and Sun, Rantzer, Li, and Robertsson (2021), in which the existence of time-varying coupling weights is based on the assumption of the global Lipschitz condition and the quadratic condition, respectively. However, such conditions are not imposed in this paper. Therefore, the methods in Li, Wen, Duan, and Ren (2014) and Sun, Rantzer, Li, and Robertsson (2021) are not applicable in our case.

**Remark 2.** The choice of $p_i(\cdot)$ depends on the structure of $p(\cdot,\omega)$, where $\omega$ is unknown. For a class of polynomial differential systems (2), with $p(v,\omega)$ being real polynomials of degree $m_0 \geq 1$, $\|p(v,\omega)\|^2 \leq \sum_{k=0}^{2m_0} c_k(\omega)\|v\|^k$

where $c_k(\omega)$ is some unknown constant induced by the uncertain parameter $\omega$. Then $p_i(z_i)$ can be chosen as $p_i(z_i) = a_i \sum_{k=0}^{2m_0-2} \|z_i\|^k + b_i,$

where $a_i$ and $b_i$ are any positive numbers.

### III.2. Convergence analysis

Define the estimation errors as $\tilde{v}_i = \hat{v}_i - v_i$, $\tilde{\omega}_i = \hat{\omega}_i - \omega$, and $\tilde{z}_i = z_i - \omega$, respectively. Their time derivatives along the trajectories (2) and (6) are

\[
\begin{align*}
\dot{\tilde{v}}_i &= \tilde{\kappa}_i p_i(z_i) z_i + \phi(\tilde{v}_i) \tilde{\omega}_i \\
&\quad + \tilde{\kappa}_i p_i(z_i) z_i + (\phi(\hat{v}_i) - \phi(v)) \omega, \\
\dot{\tilde{\omega}}_i &= \mu \phi^T(\hat{v}_i) z_i, \\
\dot{\tilde{z}}_i &= p_i(z_i) z_i, i = 1, \ldots, N.
\end{align*}
\]

(7a)–(7c)

Define $\tilde{v} = \text{col}(\tilde{v}_1, \ldots, \tilde{v}_N)$, $\tilde{\omega} = \text{col}(\tilde{\omega}_1, \ldots, \tilde{\omega}_N)$, $\tilde{\kappa} = \text{blkdiag}(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_N)$, $\Phi(\tilde{v}) = \text{blkdiag}(\phi(\tilde{v}_1), \ldots, \phi(\tilde{v}_N))$, $\Phi(\hat{v}) = \text{blkdiag}(\phi(\hat{v}_1), \ldots, \phi(\hat{v}_N))$, and $z = \text{col}(z_1, \ldots, z_N)$. Then, we have

\[
\begin{align*}
\dot{z} &= -(H \otimes I_m) \tilde{v},
\end{align*}
\]

(8)

where $H$ is defined in Section II.1. Equations (7a) and (7b) can be rewritten in the following compact form

\[
\begin{align*}
\dot{\tilde{v}} &= (\tilde{\kappa} \Phi d \otimes I_m) z + \Phi d (\tilde{\omega}) + (\tilde{\kappa} \Phi d \otimes I_m) z \\
&\quad + [\phi_d(\tilde{v}) - I_N \otimes \phi(v)] (I_N \otimes \omega), \\
\dot{\tilde{\omega}} &= \mu \Phi_d^T(\hat{v}) z.
\end{align*}
\]

(9a)–(9b)

where $I_N \in 2^{RN}$ and $I_m \in \mathbb{R}^{m \times m}$ denote the vector of all ones and the identity matrix, respectively.

Before analyzing the convergence of the proposed learning-based distributed observer, we first establish the following result.

**Lemma 3.** Consider systems (2) and (6).

1. There exist sufficiently smooth positive function $\gamma_i(z_i)$ and positive constants $\beta_M$ and $\lambda_M$ such that

\[
\sum_{i=1}^{N} \|p(\tilde{v}_i, \omega) - p(v,\omega)\|^2 \leq \beta_M \lambda_M \sum_{i=1}^{N} \gamma_i(z_i) \|z_i\|^2.
\]

(10)

2. Moreover, if system (2) is a class of polynomial differential system with degree $m_0$, then

\[
\gamma_i(z_i) = \sum_{k=0}^{2m_0-2} \|z_i\|^k.
\]

**Proof:**

Define $f_i(\tilde{v}_i, v,\omega) = \|p(\tilde{v}_i + v,\omega) - p(v,\omega)\|,$

(11)

which is a continuously differentiable function satisfying $f_i(0,v,\omega) = 0$, for $i = 1, \ldots, N$. By the second part of Lemma 1, there exists a smooth function $m_i(\tilde{v}_i(t), v(t),\omega) \geq 0$ such that

\[
f_i(\tilde{v}_i(t), v(t),\omega) \leq m_i(\tilde{v}_i(t), v(t),\omega) \|\tilde{v}_i(t)\|, \quad \forall t \geq 0.
\]

(12)

For the continuous function $m_i(\tilde{v}_i(t), v(t),\omega)$, by the first part of Lemma 1, there exist smooth functions $\alpha_i(\tilde{v}_i(t)) \geq 0$ and $\beta_i(v(t),\omega) \geq 0$, such that

\[
m_i(\tilde{v}_i(t), v(t),\omega) \leq \alpha_i(\tilde{v}_i(t)) \times \beta_i(v(t),\omega), \quad \forall t \geq 0.
\]

(13)

Since $v(t)$ is uniformly bounded in $t$ and $\omega$ is an unknown constant vector, there exists a positive constant $\beta_M = \sup_{t \geq 0} \beta_i(v(t),\omega)$ such that $\beta_i(v(t),\omega) \leq \beta_M$, for all $t \geq 0$. Define

\[
F(\tilde{v}, v,\omega) \equiv \sum_{i=1}^{N} f_i^2(\tilde{v}_i, v,\omega)
\]

(14)

Then, from equations (11), (12), (13) and (14), we have

\[
F(\tilde{v}, v,\omega) = \sum_{i=1}^{N} f_i^2(\tilde{v}_i, v,\omega)
\]

\[
\leq \sum_{i=1}^{N} m_i^2(\tilde{v}_i(t), v(t),\omega) \|\tilde{v}_i(t)\|^2
\]

\[
\leq \sum_{i=1}^{N} \alpha_i^2(\tilde{v}_i(t)) \times \beta_M^2(v(t),\omega) \|\tilde{v}_i(t)\|^2
\]

\[
\leq \sum_{i=1}^{N} \beta_M^2 \alpha_i^2(\tilde{v}_i(t)) \|\tilde{v}_i(t)\|^2 \equiv \tilde{g}(\tilde{v}).
\]

As $\tilde{v}(z)$ is a function of $z$ via the relationship

\[
\tilde{v} = -(H^{-1} \otimes I_m) z.
\]

Therefore,

\[
\tilde{g}(\tilde{v}) = \tilde{g}(-(H^{-1} \otimes I_m) z) \equiv g(z)
\]

Since the smooth positive definite function $g(\cdot)$ and its first derivative vanishes at the origin, by Lemma 2, there exist
smooth functions $\gamma_i(z_i)$ and some positive constant $\lambda_M$, such that

$$F(\hat{v}, v, \omega) \leq g(z) \leq \beta_M^2 \lambda_M \sum_{i=1}^N \gamma_i(z_i) \|z_i\|^2.$$ 

**2:** For $l = 1, \ldots, m$, $p_l(\cdot, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ are continuously differentiable functions and $p_l(\cdot, \cdot)$ is a class of polynomial differential systems with the largest degree $m_0$. It is noted that $\tilde{v}_i = \hat{v}_i - v_i$, for $i = 1, \ldots, N$. Then, for any $\tilde{v}_i$ and $v$, we have

$$p_l(\tilde{v}_i, v) - p_l(v, v) = \left[ \int_0^1 \frac{\partial p_l(x, v)}{\partial x} |_{x = v + \theta \tilde{v}_i} d\theta \right] \tilde{v}_i.$$ 

Then, for $l = 1, \ldots, m$, we will have

$$\|p_l(\tilde{v}_i, v) - p_l(v, v)\| \leq \left\| \int_0^1 \frac{\partial p_l(x, v)}{\partial x} |_{x = v + \theta \tilde{v}_i} d\theta \right\| \|\tilde{v}_i\|.$$ 

In addition, $p_l(x, v)$ is in real polynomials form with the largest degree $m_0$. Thus

$$\sum_{l=1}^m \left\| \int_0^1 \frac{\partial p_l(x, v)}{\partial x} |_{x = v + \theta \tilde{v}_i} d\theta \right\| \leq \sum_{k=0}^{m_0-1} \beta_k(\omega, v) \|\tilde{v}_i\|^k,$$

where $\beta_k(\omega, v(t))$ are some unknown positive smooth functions induced by the uncertain parameter $\omega$ and unknown state $v(t)$. Since $v(t)$ is uniform bounded in $t$, and $\omega$ is an unknown constant vector, there exists a positive constant $\beta_M = \sup_{t>0} \beta_k(v(t), \omega)$ such that $\beta_k(v(t), \omega) \leq \beta_M$, for all $t \geq 0$. Hence, we have

$$\|p(v, \omega) - p(\hat{v}, v)\|^2 \leq \beta_M^2 \sum_{k=0}^{2m_0-2} \|\tilde{v}_i\|^{k+2}.$$ 

It is noted that

$$\|\tilde{v}\| = \|(H^{-1} \otimes I_m)z\| \leq \bar{h} \|z\|,$$

where $\bar{h} = \|H^{-1}\|$, and

$$\|x\|_p \leq \|x\| \leq n^{\frac{p-1}{p}} \|x\|_p,$$

for any $p \geq 2$ and $x \in \mathbb{R}^n$, which further implies

$$\|x\|_p^p \leq \|x\|^p \leq n^\frac{p}{2-1} \|x\|_p^p.$$ 

Then, for any $k \geq 2$, we have

$$\sum_{i=1}^N \|\tilde{v}_i\|^k \leq \left(\sum_{i=1}^N \|\tilde{v}_i\|^2 \right)^{k/2} \leq \bar{h}^k \left(\sum_{i=1}^N \|z_i\|^2 \right)^{k/2} \leq \bar{h}^k \left(\|z\|^2 \right)^{k/2} \leq \bar{h}^k (\|z\|)^k \leq N \bar{h}^{-1} \bar{h}^k \sum_{i=1}^N \|z_i\|^k.$$ 

Thus

$$\sum_{i=1}^N \|p(v, \omega) - p(\hat{v}, v)\|^2 \leq \beta_M^2 \sum_{i=1}^N \sum_{k=2}^{2m_0} N^{k-1} \bar{h}^{k+1} \sum_{i=1}^N \|z_i\|^k.$$ 

Then, by letting $\gamma_i(z_i) = \sum_{k=0}^{2m_0-2} \|z_i\|^k$, we have

$$\sum_{i=1}^N \|p(v, \omega) - p(\hat{v}, v)\|^2 \leq \beta_M^2 \lambda_M \sum_{i=1}^N \gamma_i(z_i) \|z_i\|^2$$

for some positive constant $\lambda_M = N^{k-1} \bar{h}^{k+2}$.

**Theorem 1.** Consider systems (2) and (6). There exist some sufficiently smooth positive functions $\rho_i(\cdot) \geq 1$ for all $i = 1, 2, \ldots, N$, such that, for all $x > 0$, $\hat{v}_i(0) \in \mathbb{V}_0$ and $\tilde{w}_i(0) \in \mathbb{R}$, the signals $\hat{v}_i(t)$, $\tilde{w}_i(t)$ and $\tilde{w}_i(t)$ exist and are bounded for all $t \geq 0$ and satisfy

$$\lim_{t \to \infty} \hat{v}_i(t) = 0,$$

$$\lim_{t \to \infty} \tilde{w}_i(t) = 0,$$

$$\lim_{t \to \infty} \tilde{w}_i(t) = 0.$$ 

Moreover, if $\tilde{w}(\cdot)$ is persistently exciting, then

$$\lim_{t \to \infty} \tilde{w}_i(t) = 0.$$ 

**Proof:** Consider the Lyapunov function candidate

$$V = \frac{1}{2} \left( \tilde{w}_o^T (H \otimes I_m) \tilde{w}_o + \mu^{-1} \tilde{w}_o^T \tilde{w}_o + \sum_{i=1}^N \tilde{k}_i^T \tilde{k}_i \right),$$

where $H$ is defined in Section II.1 and is symmetric positive definite. Differentiating (19) along the trajectory of (7) gives

$$\dot{V} = \tilde{w}_o^T (H \otimes I_m) \dot{v}_o + \mu^{-1} \tilde{w}_o^T \dot{\tilde{w}}_o + \sum_{i=1}^N \tilde{k}_i^T \dot{\tilde{k}}_i$$

$$= - \tilde{w}_o^T (\tilde{k}_o \dot{p}_d \otimes I_m) z - \tilde{w}_o^T (\tilde{k}_o \dot{p}_d \otimes I_m) z$$

$$- \tilde{w}_o^T \phi_v(\tilde{v}) - I_N \otimes \phi_v(v) \| \tilde{w}_o \|^2$$

$$- \tilde{w}_o^T \phi_v(\tilde{v}) \tilde{w}_o + \mu^{-1} \tilde{w}_o^T \tilde{w}_o + \sum_{i=1}^N \tilde{k}_i^T \dot{\tilde{k}}_i$$

$$= - \sum_{i=1}^N \tilde{k}_i \rho_i(z_i) \|z_i\|^2 - t_i^2 \left[ \phi(\tilde{v}_i) - \phi(v) \right]$$

$$- \sum_{i=1}^N \tilde{k}_i \rho_i(z_i) \tilde{w}_o^T \tilde{w}_o - \tilde{w}_o^T \left[ p(\tilde{v}_i, v) - p(\hat{v}_i, v) \right].$$

By using the $S$ procedure and the inequality (10) in Lemma 3, the equation in (20) implies $\dot{V} \leq 0$ if and only if there exists a $\lambda \geq 0$ such that

$$\sum_{i=1}^N \left[ -\tilde{k}_i \rho_i(z_i) \right] \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \leq \lambda \lambda_M \beta_M^2 \sum_{i=1}^N \left[ -\gamma_i(z_i) \right].$$

A sufficient condition to make the above inequality hold is

$$\left[ -\tilde{k}_i \rho_i(z_i) + \lambda \lambda_M \beta_M^2 \gamma_i(z_i) \right] \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} < 0.$$ 

Therefore, for any sufficiently smooth positive function $\rho_i(\cdot)$ with the form

$$\rho_i(z_i) = a_i \gamma_i(z_i) + b_i \geq \lambda \lambda_M \beta_M^2 \gamma_i(z_i)/\tilde{k}_i$$

(22)
where $a_i$ and $b_i$ are any positive numbers, there exist $\bar{\kappa}_i$ and $\lambda$ to make (21) hold and
\[
\dot{V} \leq -\varepsilon \sum_{i=1}^{N} z_i^2 + \sum_{i=1}^{N} \kappa_i \rho_i(z_i) \| z_i \|^2 - z_i^2 \left[ p(v_i, \omega) - p(v, \omega) \right],
\]
where $a_i > 0$. Then, $V(t)$ is bounded, which means $\bar{v}(t)$, $\bar{\omega}(t)$, and $\bar{\kappa}(t)$ are bounded, for all $t \geq 0$, and $\lim_{t \to \infty} V(t)$ exists and is finite. Since $\bar{v}(t)$ is bounded, $z(t)$ is bounded from (8), and $\bar{v}(t)$ is bounded, for all $t \geq 0$. From (10) and the smoothness of $\gamma_i$, $i = 1, \ldots, N$, we know that $\phi_d(t)$ is bounded, for all $t \geq 0$. Again, using the smoothness of $\gamma_i(z_i)$, $i = 1, \ldots, N$, $\rho_d(z(t))$ and $\rho_d(z(t))$ are all bounded from (22) and the fact that $z(t)$ is bounded, for all $t \geq 0$. From (9), $\dot{v}(t)$ is bounded because $\rho_d(t)$, $z(t)$, $\phi_d(\bar{v}(t))$, $\bar{\omega}(t)$, $\phi(v(t))$, and $\bar{\kappa}(t)$ are all bounded for all $t \geq 0$. The time derivative of $\dot{V}$ is
\[
\dot{V} = -\sum_{i=1}^{N} \left[ \kappa_i \rho_i(z_i) \| z_i \|^2 - z_i^2 \left[ p(v_i, \omega) - p(v, \omega) \right] \right] + 2\kappa_i \rho_i(z_i) z_i^T z_i - \sum_{i=1}^{N} \kappa_i \rho_i(z_i) z_i^T \dot{z}_i - \dot{z}_i^T \left[ \rho(v_i, \omega) - \rho(v, \omega) \right].
\]
From Lemma 3 and the fact that $z$ is bounded for all $t \geq 0$, we have that $\| p(v_i, \omega) - p(v, \omega) \|$ is bounded for all $t \geq 0$. Furthermore, by part (i) of Lemma 1, we can easily conclude that $\| p(v_i, \omega) \|$ and $\| \phi(v) \|$ are bounded for all $t \geq 0$ by sufficiently smooth positive functions of $v_i$ and $v$, respectively. In addition, we have proven that all the items $z_i$, $\dot{z}_i$, $\rho_i(z_i)$, $\dot{v}_i$ and $v$ in (24) are bounded for all $t \geq 0$. which further implies $\dot{V}(t)$ is bounded from (24), for all $t \geq 0$. By the Lyapunov-Like Lemma in Slotine and Li (1991), we have $\lim_{t \to \infty} \dot{V}(t) = 0$. From (23), we have
\[
-\dot{V} \geq \varepsilon z^T z \geq 0
\]
which implies that $\lim_{t \to \infty} z(t) = 0$. From (8) and (9b), we can further have (15) and (16), respectively.

To show (17), differentiating $\bar{v}$ gives
\[
\dot{v} = (\kappa_d \rho_d \otimes I_m) z + (\kappa_d \rho_d \otimes I_m) \dot{z} + \left[ \phi_d(\bar{v}) - I_N \otimes \phi(\bar{v}) \right] (I_N \otimes \omega) \]
\[
+ \phi_d(\bar{v}) \bar{\omega} + \phi_d(\bar{v}) \bar{\omega} + (\kappa_d \rho_d \otimes I_m) \dot{z} + (\kappa_d \rho_d \otimes I_m) z + (\kappa_d \rho_d \otimes I_m) \dot{z}.
\]
We have shown that $\rho(t)$ and $\phi(t)$ are smooth, and $z(t)$, $\dot{z}(t)$, $\bar{\omega}(t)$, $\bar{\omega}(t)$, and $\bar{\kappa}(t)$ are all bounded, for all $t \geq 0$. We can also show that $\bar{\kappa}(t)$ is bounded from (7c), for all $t \geq 0$. Thus, $\dot{v}(t)$ is bounded from (25), for all $t \geq 0$. By the Barbalat’s lemma, we have $\lim_{t \to \infty} \dot{v}(t) = 0$, which further implies
\[
\lim_{t \to \infty} \left[ \phi(\bar{v}(t)) \bar{\omega}(t) + (\phi(\bar{v}(t)) - \phi(v(t))) \omega \right] = 0.
\]
Besides, for $i = 1, \ldots, N$,
\[
\lim_{t \to \infty} \left[ \phi(\bar{v}(t)) - \phi(v(t)) \right] \omega = 0,
\]
for all $t \geq 0$. From (15), then,
\[
\lim_{t \to \infty} \phi(\bar{v}(t)) \bar{\omega}(t) = 0.
\]
Thus, (17) holds. Eqs. (15) and (17) imply
\[
\lim_{t \to \infty} \phi(\bar{v}(t)) \bar{\omega}(t) = \lim_{t \to \infty} \phi(v(t)) \bar{\omega}(t) = 0.
\]
If $\phi^T(v)$ is persistently exciting, by Lemma 2.4 of Chen and Huang (2015), we have (18) from (16).

**Remark 3.** As a result of Theorem 1, it is trivial to show that
\[
\lim_{t \to \infty} (\bar{E} \bar{v}(t) - q_{N+1}(t)) = 0,
\]
\[
\lim_{t \to \infty} (\bar{E} \bar{v}(t) - q_{N+1}(t)) = 0, \quad i = 1, \ldots, N.
\]

**Remark 4.** The learning-based distributed observer (6) features itself in two aspects. Unlike the nonlinear distributed observers proposed in many literatures, such as Liu and Huang (2019) and Dong and Chen (2021), the observer (6) is independent of the leader’s dynamic knowledge, i.e., parameters $\omega$. Moreover, the observer (6) does not depend on the eigenvalues of either the Laplacian matrix or the adjacency matrix of the communication graph, which is normally required in the literature of distributed observer design. Therefore, our proposed observer is fully distributed. In particular, the observer (6) can adaptively learn the true values of the leader’s parameters.

**Example 1.** This example shows that the well-known Van der Pol system (Slotine & Li, 1991) satisfies the requirements of the leader node. Suppose the leader’s states are generated by a Van der Pol oscillator of the following form
\[
\dot{v} = p(v, \omega) = \left[ \begin{array}{c} 0v_2 \\ -bv_1 + c(1 - v_1^2) v_2 \end{array} \right],
\]
where $v = \text{col}(v_1, v_2)$ and $\omega = \text{col}(a, b, c)$. When $a$, $b$, and $c$ are positive constants, the Van der Pol system will have a stable limit cycle with periodic $T$ for any nonzero initial condition, which implies there exists to such that $v_1(t) = v_1(t + T), \forall t \geq 0, \forall \epsilon = 1, 2$ (Buchli, Righetti, & Ijspeert, 2006). Then, we have
\[
\int_{t}^{t+T} v_2(s) ds = \int_{t}^{t+T} 1 - \frac{1}{v_1(s)} ds = 0,
\]
\[
\int_{t}^{t+T} v_1(s) v_2(s) (1 - v_1^2(s)) ds = 0.
\]
For any $t \geq t_0$, both $v_1(t)$ and $v_2(t)$ are periodic functions with periodic $T$,
\[
\int_{t}^{t+T} v_1^2(s) ds > 0,
\]
\[
\int_{t}^{t+T} v_2^2(s) ds > 0.
\]
\[ \int_t^{t+T} v_1^2(s)v_2^2(s) (1-v_1^2(s))^2 \, ds > 0. \]

From (27) we have
\[ \frac{\partial p(v, \omega)}{\partial \omega} = \begin{bmatrix} v_2 & 0 & 0 \\ 0 & -v_1 & (1-v_1^2) v_2 \end{bmatrix} = \phi(v). \]

Then, it can be verified that
\[ \phi^T(v) \phi(v) = \begin{bmatrix} v_2^2 & 0 & 0 \\ 0 & v_1^2 & -v_1 v_2 (1-v_1^2) \\ 0 & -v_1 v_2 (1-v_1^2) & v_2^2 (1-v_1^2) \end{bmatrix}. \]

Hence, for any \( t \geq t_0 \), we will have
\[ \int_t^{t+T} \phi^T(v(s)) \phi(v(s)) \, ds = \begin{bmatrix} \int_t^{t+T} v_2^2 \, ds & 0 & 0 \\ 0 & \int_t^{t+T} v_1^2 \, ds & 0 \\ 0 & 0 & \int_t^{t+T} v_2^2 (1-v_1^2)^2 \, ds \end{bmatrix} > 0, \]

which implies that \( \phi^T(v(t)) \) is persistently exciting.

IV. An application to leader-following synchronization of multiple Euler-Lagrange systems

The Euler-Lagrange equations are often used to describe the evolution of a mechanical system subject to holonomic constraints, such as the dynamics of robot manipulators (Lewis, Dawson, & Abdallah, 2003; Roy, Baldi, & Ioannou, 2021). In this section, we apply the proposed learning-based distributed observer to solve the leader-following synchronization problem of multiple Euler-Lagrange systems subject to an uncertain nonlinear leader.

IV.1. Problem formulation

Consider a group of \( N \) follower systems described by the following Euler-Lagrange system
\[ M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + G_i(q_i) = \tau_i, \]
where, for \( i = 1, \ldots, N \), \( q_i \in \mathbb{R}^n \) is the vector of generalized position vector, \( M_i(q_i) \in \mathbb{R}^{n \times n} \) is the symmetric positive definite inertia matrix, \( C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n \times n} \) is the coriolis and centripetal matrix, and \( \tau_i \in \mathbb{R}^n \) is the control torque. For \( i = 1, \ldots, N \), system (30) has the following three properties (Lewis, Dawson, & Abdallah, 2003):

1. The inertia matrix \( M_i(q_i) \) is symmetric and uniformly positive definite, i.e. that there exists positive constants \( \alpha \) and \( \beta \) such that, for all \( q_i \),
\[ \beta I \geq M_i(q_i) \geq \alpha I, \]
where \( I \) is the identity matrix with appropriate dimensions.

2. For all \( a, \dot{a} \in \mathbb{R}^n \),
\[ M_i(q_i) \ddot{a} + C_i(q_i, \dot{q}_i) \dot{a} + G_i(q_i) = Y_i(q_i, \dot{q}_i, a, \dot{a}) \theta_i, \]
where \( Y_i(q_i, \dot{q}_i, a, \dot{a}) \in \mathbb{R}^{n \times p} \) is a known regressor matrix and \( \theta_i \in \mathbb{R}^p \) is a constant vector consisting of the uncertain parameters of (30).

3. \( (M_i(q_i) - 2C_i(q_i, \dot{q}_i)) \) is skew symmetric, \( \forall q_i, \dot{q}_i. \)

**Problem 2.** Consider the multi-agent system consisting of (2) and (30). Let \( V_0 \subset \mathbb{R}^m \) be a compact set containing the origin. Design a distributed control law \( \tau_i \), such that for any initial condition \( q_i(0) \in \mathbb{R}^n \), \( \dot{q}_i(0) \in \mathbb{R}^n \), and \( v(0) \in V_0, q_i(t), \) and \( \dot{q}_i(t) \) are bounded for all \( t \geq 0 \), and
\[ \lim_{t \to \infty} (q_i(t) - q_{N+1}(t)) = 0. \]

IV.2. Distributed observer based control law

Define \( \ddot{q}_i = q_i - E \ddot{v}_i \) as the “leader output tracking error”, where \( E \ddot{v}_i \) can be regarded as an estimate of leader’s output \( q_{N+1} \) in (2). The estimated “leader output derivative”
\[ \dot{\hat{q}}_i = E \phi(\hat{v}_i) \omega_i - \alpha \ddot{q}_i \]
for \( i = 1, \ldots, N \), can be interpreted as a “leader output derivative error”. Take the adaptive control law to be
\[ \tau_i = -K_i s_i + Y_i \dot{\theta}_i, \]
\[ \dot{\hat{q}}_i = -\Gamma Y_i^T s_i, \]
where \( \Gamma_i \) is a diagonal matrix with positive diagonal entries and \( K_i \) is a positive definite matrix, \( \dot{\theta}_i \) is the parameter estimates, for \( i = 1, \ldots, N \). The control law (34) includes a “feedforward” term \( Y_i \dot{\theta}_i \).

IV.3. Stability analysis

**Theorem 2.** Consider systems (2), (30), and a digraph \( G \). For any initial condition \( q_i(0) \in \mathbb{R}^n \), \( \dot{q}_i(0) \in \mathbb{R}^n \), and \( v(0) \in V_0 \), Problem 2 is solvable by the control law (34).

**Proof:** Differentiating (32) and (33), and considering \( \ddot{q}_i = q_i - E \ddot{v}_i \) give,
\[ \dddot{q}_i = E \phi(\dot{v}_i) \omega_i + E \phi(\dot{v}_i) \omega_i - \alpha (q_i - E \ddot{v}_i), \]
\[ \dddot{\hat{q}}_i = \dddot{q}_i - \dddot{\hat{q}}_i. \]
By equation (31), there exists a known matrix
\[ Y_i = Y_i(q_i, \dot{q}_i, \ddot{q}_i, \dddot{q}_i) \]
and an unknown constant vector \( \theta_i \) such that,
\[
Y_i \theta_i = M_i (q_i) \ddot{q}_i + C_i (q_i, \dot{q}_i) \dot{q}_i + G_i (q_i).
\]  
(36)
Substituting \( Y_i \theta_i \) into (30) gives
\[
M_i (q_i) (\ddot{q}_i - \ddot{\bar{q}}_i) + C_i (q_i, \dot{q}_i) (\dot{q}_i - \dot{\bar{q}}_i) \\
+ G_i (q_i) - G_i (\bar{q}_i) + Y_i \theta_i = \tau_i.
\]  
(37)
Considering (33), (37) can be written as
\[
M_i (q_i) \dot{s}_i = -C_i (q_i, \dot{q}_i) s_i + \tau_i - Y_i \theta_i.
\]  
(38)
Substituting (34) into (38) gives, for \( i = 1, \ldots, N \),
\[
M_i (q_i) \dot{s}_i = -C_i (q_i, \dot{q}_i) s_i - K_i s_i + Y_i \dot{\bar{q}}_i
\]  
(39)
where \( \bar{q}_i = \hat{q}_i - q_i \). For \( i = 1, \ldots, N \), consider the following Lyapunov function candidate
\[
V_i = \frac{1}{2} \left[ s_i^T M_i (q_i) s_i + \hat{\theta}_i^T \Gamma_i^{-1} \hat{\theta}_i \right],
\]  
(40)
The time derivative of (40) along the trajectory (39) yields
\[
\dot{V}_i = s_i^T M_i (q_i) \dot{s}_i + \frac{1}{2} s_i^T M_i (q_i) s_i + \hat{\theta}_i^T \Gamma_i^{-1} \hat{\theta}_i \\
= -s_i^T K_i s_i \leq 0, \ i = 1, \ldots, N.
\]  
(41)
Now since \( V_i (t) \geq 0 \) and \( \dot{V}_i (t) \leq 0 \), \( V_i (t) \) remains bounded, for all \( t \geq 0 \). Equation (40) further implies that both \( s_i (t) \) and \( \hat{\theta}_i (t) \) are bounded, for all \( t \geq 0 \). Furthermore, this in turn implies that \( \ddot{q}_i (t), \dot{q}_i (t), \) and \( \theta_i (t) \) are all bounded, for all \( t \geq 0 \). Noticing the closed-loop dynamics in (39), this shows that \( \dot{s}_i (t) \) is also bounded, for all \( t \geq 0 \). Since, given (41), one has \( \dot{V}_i = -2s_i^T K_i \dot{s}_i \), which shows that \( V_i (t) \) is bounded, for all \( t \geq 0 \). Furthermore, Barbalat’s lemma indicates that \( V_i (t) \) tends to zero, which implies that \( s_i (t) \to 0 \) as \( t \to \infty \).

Using (32), (33) and (6a) gives
\[
\ddot{\bar{q}}_i - E \ddot{v} \bar{v}_i + \alpha (q_i - E \bar{v}_i) = s_i - \kappa_i \rho_i (z_i) E z_i.
\]  
(42)
With \( \dddot{q}_i = (q_i - E \bar{v}_i) \), equation (42) can be rewritten as
\[
\dddot{q}_i + \alpha \dddot{q}_i = s_i - \kappa_i \rho_i (z_i) E z_i.
\]  
(43)
Since, by Lemma 1, \( \lim_{t \to \infty} z_i (t) = 0 \), for \( i = 1, \ldots, N \), equation (43) can be viewed as a stable differential equation in \( \dddot{q}_i \) with \( (s_i (t) - \kappa_i \rho_i (z_i (t)) E z_i (t)) \) as the input, which are bounded over \( t \geq 0 \) and tend to zero as \( t \to \infty \). Thus, we conclude that both \( \dddot{q}_i (t) = (q_i (t) - E \bar{v}_i (t)) \) and \( \dddot{\bar{q}}_i (t) = (\bar{q}_i (t) - E \bar{v}_i (t)) \) are bounded over \( t \geq 0 \) and will tend to zero as \( t \to \infty \). These facts together with (26a) and (26b) complete the proof. \( \Box \)

V. A simulation example

Consider a group of six Euler-Lagrange systems connected according to the communication topology shown in Fig. 2. It is clear that the communication graph contains a spanning tree with the leader (i.e., node 7) being the root. Each follower (i.e., nodes 1-6) is a two-link planar elbow arm whose dynamics are described by (30), where \( q_i = \text{col} (q_{i1}, q_{i2}) \) is the joint variable representing the angles of both joints of the two-link robot arm,
\[
M_i (q_i) = \begin{bmatrix}
\theta_{i1} + \theta_{i2} + 2\theta_{i3} \cos q_{i2} & \theta_{i2} + \theta_{i3} \cos q_{i2} \\
\theta_{i3} \sin q_{i2} & \theta_{i3} \sin q_{i2}
\end{bmatrix},
\]
\[
C_i (q_i, \dot{q}_i) = \begin{bmatrix}
-\theta_{i3} q_{i2} \sin q_{i2} & -\theta_{i3} (\dot{q}_{i1} + \dot{q}_{i2}) \sin q_{i2} \\
\theta_{i3} \dot{q}_{i1} \sin q_{i2} & 0
\end{bmatrix},
\]
\[
G_i (q_i) = \begin{bmatrix}
\theta_{i4} \cos q_{i1} + \theta_{i5} \cos (q_{i1} + q_{i2}) \\
\theta_{i5} \cos (q_{i1} + q_{i2})
\end{bmatrix},
\]
and the unknown vector \( \hat{\theta}_i = \text{col} (\theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4}, \theta_{i5}) \) is induced by the unknown link masses and link offsets (Lewis, Dawson, & Abdallah, 2003, p. 117).

The leader’s signal is generated by a Van Der Pol system in the form (27) with
\[
\frac{\partial p (v, \omega)}{\partial \omega} = \begin{bmatrix}
v_2 & 0 & 0 \\
0 & -v_1 & (1 - v_1^2) v_2
\end{bmatrix},
\]
\[
\omega = \text{col} (a, b, c),
\]
where \( a, b \) and \( c \) are unknown positive constant scalars. The states of the Van Der Pol system will converge to a stable limit cycle (Slotine & Li, 1991). Therefore, \( v (t) \) is bounded for all \( t \geq 0 \) and for any initial condition \( v (0) \).

It is noticed that \( p (x, \omega) \) is in the polynomial differential form of degree 3. Then, for any \( x \in \mathbb{R}^2 \), we have
\[
\frac{\partial p (x, \omega)}{\partial x} = \begin{bmatrix}
\frac{\partial p_1 (x, \omega)}{\partial x_1} & \frac{\partial p_1 (x, \omega)}{\partial x_2} \\
\frac{\partial p_2 (x, \omega)}{\partial x_1} & \frac{\partial p_2 (x, \omega)}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
0 & \alpha \\
-b - 2c a x_1 x_2 & c (1 - x_1^2)
\end{bmatrix}.
\]
Hence, for some unknown sufficiently large positive constant \( \omega \), we have
\[
\sum_{i=1}^{2} \left\| \frac{\partial p_i (x, \omega)}{\partial x} \right\| \leq a + \|b + 2c x_1 x_2\| + \|c (1 - x_1^2)\|
\]
1The picture of robot arm is adopted from www.vector-4free.com
\[ \leq a + b + c + 2c\|x_1\|\|x_2\| + c\|x_1\|^2 \leq 2\bar{\omega} + 2\bar{\omega}\|x\|^2. \]

Then, we have
\[
\sum_{l=1}^{\infty} \int_0^1 \left\| \frac{\partial p_l(x, \omega)}{\partial x} \right\|_{x=v+\delta v_l} d\theta_l \leq 2\bar{\omega}[1 + (\|v\| + \|\tilde{\omega}_l\|)^2] \leq \sum_{k=0}^{\infty} c_k(\omega, v)\|\tilde{\omega}_l\|^k.
\]

where \( c_0(\omega, v) = 2\bar{\omega}[1 + \|v\|^2] \), \( c_1(\omega, v) = 4\bar{\omega}\|v\| \) and \( c_2(\omega, v) = 1 \). Then, by Lemma 3, we can choose
\[
\rho_i(z_i) = 2 + 6(\|z_i\| + \|\tilde{z}_i\|^2 + \|z_i\|^3 + \|\tilde{z}_i\|^4).
\] (44)

Please be noted that \( b_i = 2 \) and \( a_i = 6 \) in \( \rho_i(z_i) \) are randomly chosen in \( \mathbb{R}^+ \). By Lemma 3, for \( i = 1, \ldots, 6 \), we can design a learning-based distributed observer for (2) in the form (34a) with the following parameters:

\[ K_i = 20I_2, \quad \alpha = 2, \quad \Gamma_i = 10. \]

For \( i = 1, \ldots, N \), the actual values of \( \theta_i \) are given as follows:

\[
\begin{align*}
\theta_1 & = \text{col}(0.64, 1.10, 0.08, 0.64, 0.32), \\
\theta_2 & = \text{col}(0.76, 1.17, 0.14, 0.93, 0.44), \\
\theta_3 & = \text{col}(0.91, 1.26, 0.22, 1.27, 0.58), \\
\theta_4 & = \text{col}(1.10, 1.36, 0.32, 1.67, 0.73), \\
\theta_5 & = \text{col}(1.21, 1.16, 0.12, 1.45, 1.03), \\
\theta_6 & = \text{col}(1.31, 1.56, 0.22, 1.65, 1.33).
\end{align*}
\]

Simulation is conducted with the following initial conditions:

\[ q_1(0) = \text{col}(-1, 2), \quad q_2(0) = \text{col}(-2, -1), \quad q_3(0) = \text{col}(1, -1), \quad q_4(0) = \text{col}(2, -1), \quad q_5(0) = \text{col}(-3, 2), \quad q_6(0) = \text{col}(-1, 1), \quad \hat{q}_1 = 0, \quad \hat{q}_2 = 0, \quad \hat{q}_3 = 3.3689, \quad \hat{q}_4 = 3.4607, \quad \hat{q}_5 = 3.9816, \quad \hat{q}_6 = 3.1564, \quad \kappa_1(0) = 3.8555, \quad \kappa_2(0) = 3.6448, \quad \kappa_3(0) = 0, \quad \kappa_4(0) = 0, \quad \kappa_5(0) = 0. \]

The leader’s initial
condition $v(0) = \text{col}(2, 2)$ is chosen in $V_0 = \{v(0) \mid \|v(0)\| \leq 8\}$, and the actual unknown constants are $a = 1, b = 1$ and $c = 1$. Figure 3 shows that both joint angles $q_i$ = $\text{col}(q_{i1}, q_{i2})$ of all six robot arms asymptotically achieve synchronization with the output $q_{N+1}$ of the leader system. The synchronization errors $e_i = \dot{q}_i - \dot{q}_N$ and $\dot{e}_i = \dot{q}_i - \dot{q}_N$ of both joint angles and joint angular velocities vanishes as $t \to \infty$, i.e., $\|e_i(t)\| \to 0$ and $\|\dot{e}_i(t)\| \to 0$ (see Fig. 4). Moreover, the adaptive parameters $\dot{\hat{\omega}}_i$ eventually converge to some constants (see Fig. 5). By Remark 1, $\phi^T(v)$ is persistently exciting for any $v(0) \neq 0$. Thus, the distributed observer (6) will guarantee $\hat{\omega}_i(t) \to 0$ as $t \to \infty$, for $i = 1, \ldots, 6$ (see Fig. 6).

Remark 5. In this example, each follower agent does not know any knowledge of the leader. It is hard to find a time-varying bound coupled by the network and nonlinear leader, and thus one cannot find a sufficiently large smooth function to design the nonlinear distributed observer as in Dong and Chen (2021). Furthermore, since the leader is described by a nonlinear dynamics, those designs proposed in Wang and Huang (2018) and Baldi, Azzollini, and Ioannou (2021) for linear leaders are not applicable in our case. More importantly, the solution of the closed-loop system may be unbounded in our case, therefore, the global Lipschitz condition (Wang & Slotine, 2005; Zhou, Lu, & Liu, 2006; Yu, Chen, & Liu, 2009), the quadratic condition (Lu & Chen, 2004; DeLellis, Di Bernardo, Gorochowski, & Russo, 2010), and the contracting condition (Wang & Slotine, 2006) cannot be used to analyze the convergence of our learning-based distributed observer.

VI. Conclusion

This paper studied the leader-following synchronization problem of heterogeneous multi-agent systems subject to a class of uncertain nonlinear leader systems. For this purpose, we first established a learning-based distributed observer for the nonlinear leader system whose parameters are unknown. This observer can globally estimate and pass the leader’s state to each follower through the communication network without knowing the leader’s parameters. Further, under certain persistent excitation condition, this observer can also asymptotically learn the unknown parameters of the leader’ system. Some common assumptions in the analysis of the synchronization problem, such as the bounded Jacobian matrix assumption, the quadratic condition, and the known parameters assumption, are removed. Based on this distributed observer, finally, we synthesized an adaptive distributed control law for solving the leader-following synchronization problem of multiple heterogeneous Euler-Lagrange systems via the certainty equivalence principle. The Lyapunov function candidate (19) relies heavily on the symmetry property of the Laplacian matrix of the communication graph of the undirected graph. It turns out to be very challenging in extending Theorem 1 to the case of directed graphs. In the future, we will consider developing a learning-based distributed observer for multi-agent systems with an uncertain leader over general directed graphs.

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