Consequences of APSP, triangle detection, and 3SUM hardness for separation between determinism and non-determinism

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Abstract. Let $\text{NTIME}(f(n), g(n))$ denote the class of problems solvable in $O(g(n))$ time by a multi-tape Turing machine using an $f(n)$-bit non-deterministic oracle, and let $\text{TIME}(g(n)) = \text{NTIME}(0, g(n))$. We show that if the all-pairs shortest paths problem (APSP) for directed graphs with $N$ vertices and integer edge weights within a super-exponential range $\{-2^{N^2+k+o(1)}, \ldots, 2^{N^2+k+o(1)}\}$, $k \geq 1$, does not admit a truly subcubic algorithm then for any $\epsilon > 0$, $\text{NTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \not\subseteq \text{TIME}(n^{1+\frac{k}{2^k}-\epsilon})$. If the APSP problem does not admit a truly subcubic algorithm already when the edge weights are of moderate size then we obtain even a stronger implication, namely that for any $\epsilon > 0$, $\text{NTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \not\subseteq \text{TIME}(n^{\omega/2^k-\epsilon})$, where $\omega$ stands for the exponent of fast matrix multiplication. For the more general problem of detecting a minimum weight $\ell$-clique ($\text{MWCL}_\ell$) in a graph with edge weights of moderate size, we show that the non-existence of truly sub-$n^{1/2^k-\epsilon}$-time algorithm yields for any $\epsilon > 0$, $\text{NTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \not\subseteq \text{TIME}(n^{1+\frac{2}{2^k}-\epsilon})$. Finally, we observe that the Exponential Time Hypothesis (ETH) implies $\text{NTIME}(\lceil k \log_2 n \rceil, n) \not\subseteq \text{TIME}(n)$ for some $k > 0$, while the strong ETH (SETH) yields for any $\epsilon > 0$, $\text{NTIME}(\lceil \log_2 n \rceil, n) \not\subseteq \text{TIME}(n^{2-\epsilon})$. For comparison, the strongest known result on separation between non-deterministic and deterministic time only asserts $\text{NTIME}(O(n), n) \not\subseteq \text{TIME}(n)$.

1 Introduction

The complexity status of the all-pairs shortest paths problem (APSP) in directed graphs with arbitrary edge weights is a major open problem in the area of graph algorithms. In spite of several decades of research efforts, no truly subcubic algorithm for this problem is known.

* Research supported in part by VR grant 2017-03750 (Swedish Research Council).
By truly subcubic, Vassilevska Williams and Williams [20] mean \( O(N^{3-\delta} \text{poly} \log M) \) for some \( \delta > 0 \), where \( N \) is the number of vertices in the input graph or the number of rows and columns in the input matrix, and the edge weights or matrix entries respectively are in the range \( \{-M, \ldots, M\} \). This definition assumes that \( M \) is not too large. When \( M \) is very large, e.g., \( M = 2^{N^\phi} \) for some \( \phi > 0 \), then \( \text{poly} \log M \) can become at least polynomial in \( N \). For this reason, we shall adopt a more strict definition of truly subcubic, namely \( O(N^{3-\delta} \log M) \) for some \( \delta > 0 \). This definition is still compatible with the reductions presented in [20], in particular, it allows for multiplication of \( \log M \) bit numbers in \( O(\log M) \) time, and it works for \( M \) of \( 2^{N^k+o(1)} \) size, for any constant \( k \geq 0 \).

Vassilevska Williams and Williams presented a list of eleven problems that they could show to be equivalent to the APSP problem regarding the question of admitting a truly subcubic algorithm [20]. Thus, if any problem on the list could be shown to admit a truly subcubic algorithm then all the remaining problems on the list, in particular APSP, would have truly subcubic algorithms. Besides APSP and the verification of the naturally related distance (i.e., \((\min, +))\) matrix product, the list includes problems of different form ranging from multi-functions to decision problems. Some of the problems on the list immediately reduce to search problems. In particular, the problem of detecting a triangle of negative total edge weight (DNT) belongs to the latter ones. For these reasons, the problems on the list admit quite different upper time bounds in the quantum computational model (cf. Table 2 in Appendix) or in the non-deterministic Turing machine or RAM model. We utilize the presence of the problems on the list that directly reduce to search problems in order to derive among other things the following implication, where \( \text{NDTIME}(f(n), g(n)) \) denotes the class of problems solvable in \( O(g(n)) \) time by a multi-tape Turing machine using an \( f(n) \)-bit non-deterministic oracle, and \( \text{DTIME}(g(n)) = \text{NDTIME}(0, g(n)) \). If the APSP problem for directed graphs with integer edge weights of moderate size does not admit a truly subcubic algorithm then for any \( \epsilon > 0 \), \( \text{NDTIME}([\frac{1}{2} \log_2 n], n) \not\subseteq \text{DTIME}(n^{1.5-\epsilon}) \). By an edge weight of moderate size, we mean a weight requiring \( N^{o(1)} \)-bit representation, where \( N \) is the number of vertices in the input graph or the number of rows and columns in the input matrix, respectively. Observe that the best known and celebrated result separating non-deterministic time from the deterministic one in terms of our notation is just \( \text{NDTIME}(O(n), n) \not\subseteq \text{DTIME}(n) \) [19] (cf. [5]). Furthermore, no result of the form \( \text{NDTIME}(O(n^q), n^q) \not\subseteq \text{DTIME}(n^q) \) for \( q > 1 \) is known. Our more general result is as follows.

Let \( k \geq 0 \), and let \( Q \) stand for the set of integers \( \{-2^{N^k+o(1)}, \ldots, 2^{N^k+o(1)}\} \). If for any \( \epsilon > 0 \), the APSP problem or any of the subcubic-time equivalent problems...
for graphs on \( N \) vertices with edge weights in \( Q \) does not admit an \( \mathcal{O}(N^{3-\epsilon}+o(1)) \)-time algorithm then for any \( \epsilon' > 0 \),

\[
\text{NTIME}(\lceil \frac{1}{2^{k}} \log_2 n \rceil, n) \nsubseteq \text{TIME}(n^{1+\frac{3}{2k} - \epsilon'})
\]

Thus, if it is true that the APSP problem for directed graphs does not admit a truly subcubic algorithm when the edge weights are within a super-exponential range then showing this seems beyond the reach of presently known techniques. Simply, it would imply an enormous breakthrough not only in lower bounds on Boolean circuit size for natural problems but also in separation between deterministic and non-deterministic time.

We also consider the much simpler problem of detecting a triangle (DT) in a (undirected) graph. It immediately reduces to a search problem and in dense graphs it can be solved in \( O(n^\omega) \) time by a well known reduction to fast matrix multiplication [14]. The somewhat informal concept of a truly subcubic algorithm can be naturally generalized to include that of a truly sub-\( N^d \)-time algorithm by replacing \( N^{3-\epsilon} \) with \( N^{d-\epsilon} \), respectively [20]. Similarly, we obtain the following weaker implication: if the DT problem for a graph on \( N \) vertices does not admit a truly sub-\( N^\omega \)-time algorithm then for any \( \epsilon > 0 \),

\[
\text{NTIME}(\lceil \frac{1}{2^{k}} \log_2 n \rceil, n) \nsubseteq \text{TIME}(n^{\omega/2 - \epsilon})
\]

Next, we consider the problem of detecting a triangle of minimum total weight (MWT) in an edge weighted graph. Note that the DNT and DT problems can be easily reduced to the MWT problem. It follows in particular that if the MWT problem admits a truly subcubic algorithm then any problem on the aforementioned list has also this property (cf. Conclusion in [20]). A natural generalization of the MWT problem is that of determining a minimum weight \( \ell \)-clique in an edge weighted graph (MWC-\( \ell \)) conjectured to not admit a truly sub-\( N^\ell \)-time algorithm [1]. We show that if this conjecture holds for graphs with \( N \) vertices and edge weights in \( \{-2^{N^k+o(1)}, \ldots, 2^{N^k+o(1)}\} \) then for any positive \( \epsilon > 0 \),

\[
\text{NTIME}(\lceil \frac{1}{2^{k}} \log_2 n \rceil, n) \nsubseteq \text{TIME}(n^{\omega/2 - \epsilon})
\]

The 3SUM problem, which is to decide if an input set of numbers contains three numbers summing to zero, is widely believed to not admit a truly subquadratic algorithm (weakened 3SUM hypothesis). For this reason, one has shown truly subquadratic reducibility of 3SUM to several other problems believed to have almost quadratic time complexity in order to demonstrate their relative hardness (see, e.g., [8,16,20]). We show that if 3SUM, when the input \( N \) numbers are integers in \( \{-2^{N^k+o(1)}, \ldots, 2^{N^k+o(1)}\} \) for some \( k \geq 0 \), does not admit a truly subquadratic algorithm then for any \( \epsilon > 0 \),

\[
\text{NTIME}(\lceil \frac{1}{2^{k}} \log_2 n \rceil, n) \nsubseteq \text{TIME}(n^{1+\frac{3}{2k} - \epsilon'})
\]

Finally, we observe that the Exponential Time Hypothesis (ETH) [13] implies \( \text{NTIME}(\lceil k \log_2 n \rceil, n) \nsubseteq \text{TIME}(n) \) for some \( k > 0 \), while the
conjecture on implication

|                |                                                                 |
|----------------|-----------------------------------------------------------------|
| APSP and equiv.| for $\epsilon > 0$, $\text{NDTIME}([\frac{1}{2}\log_2 n], n) \not\subseteq \text{DTIME}(n^{1-\epsilon})$ |
| DT             | for $\delta < \omega$, $\text{NDTIME}([\frac{1}{12}\log_2 n], n) \not\subseteq \text{DTIME}(n^{1/2})$ |
| DC-$\ell$ where $\ell \geq 3$ | for $\epsilon > 0$, $\text{NDTIME}((\ell - 2)[\frac{1}{2}\log_2 n], n) \not\subseteq \text{DTIME}(n^{1/2} - \epsilon)$ |
| MWC-$\ell$     | for $\epsilon > 0$, $\text{NDTIME}((\ell - 2)[\frac{1}{2}\log_2 n], n) \not\subseteq \text{DTIME}(n^{1/\ell/2} - \epsilon)$ |
| 3SUM           | for $\epsilon > 0$, $\text{NDTIME}([\log_2 n], n) \not\subseteq \text{DTIME}(n^{0.5} - \epsilon)$ |
| ETH            | for some $k > 0$, $\text{NDTIME}([k\log_2 n], n) \not\subseteq \text{DTIME}(n^{1/k})$ |
| SETH           | for $\epsilon > 0$, $\text{NDTIME}([\log_2 n], n) \not\subseteq \text{DTIME}(n^{1/2} - \epsilon)$ |

Table 1. Implications from the conjectures when the input edge weights or input matrix entries or input numbers are assumed to be integers in $\{-N^{o(1)}, \ldots, N^{o(1)}\}$. For DC-$\ell$, see Section 3.

strong ETH (SETH) \cite{7} yields for any $\epsilon > 0$, \text{NDTIME}([\log_2 n], n) \not\subseteq \text{DTIME}(n^{2-\epsilon})$.

Our implications from the known conjectures are in a form of a negated containment of a linear-time with a logarithmic non-deterministic oracle in a respective deterministic bounded-time class. For interesting or even dramatic consequences of this kind of containment see subsection 1.1.2 in \cite{21}.

Marginally, we present simple $O(N^{1.5})$ time quantum algorithms for the MWT problem and the problem of verifying if an $N \times N$ matrix defines a metric (MDM), occurring on the list in \cite{20}.

Remark 1. All our main results can be viewed as putting straightforward/obvious and/or known implications of the conjectures in a common framework. This in particular shows subtle differences between the implications and exhibits dependency on the range parameter. Thus, the contribution of our paper is mostly a conceptual one formalizing a known intuition.

2 Preliminaries

For a positive integer $r$, $[r]$ will denote $\{1, \ldots, r\}$.

Among the eleven problems on the list of subcubic-time equivalent problems in \cite{20}, we shall refer explicitly to:

- the all-pairs shortest path problem in directed edge weighted graphs (APSP),
- the problem of detecting a triangle of negative total edge weight (DNT), and
- the problem of verifying if a matrix defines a metric (MDM).

Note that an $N \times N$ matrix $K$ defines a metric on $[N]$ if and only if it has non-negative entries, $K[i, j] = 0$ iff $i = j$ and $K[i, j] = K[j, i]$ for all $i$, $j \in [N]$,
and $K[i,j] \leq K[i,k] + K[k,j]$ for all $i, j, k \in [N]$. Of course, the first three conditions can be easily verified in quadratic time.

We shall also consider the problem of finding a triangle of minimum total edge weight in an edge weighted graph (MWT) and the simpler problem of detecting a triangle in a (unweighted) graph (DT), as well as their generalizations where a triangle is replaced by an $\ell$-clique (MWC-$\ell$ and DC-$\ell$, respectively). Note that DNT and DT trivially reduce to MWT.

Furthermore, we shall consider the 3SUM problem which is to decide if a given set of numbers contains three elements whose sum is zero, and the Exponential Time Hypothesis (ETH) [13] as well as its strong version (SETH) [7].

The reductions proving the subcubic-time equivalences between the eleven problems on the list in Theorem 1.1 in [20] do not introduce new very large edge weights or matrix entries. They typically use the edge weights or matrix entities from the reduced problem. The exception is the use of $+\infty$ or $-\infty$ in case of some problems on the list. However, the latter can be easily simulated by multiplying the maximum or minimum of the assumed range by a polynomial in the number of vertices or in the number of matrix rows/columns in the considered problem.

In Definition 3.1 in [20], one requires an $O(m^{3-\delta})$ bound on the time taken by a subcubic reduction, where $m = N \log M$ in our terms. This definition as that of truly subcubic also assumes that the edge weights or matrix entries are not too large. We can replace the required upper time-bound by $O(N^{3-\delta}(\log M)^{1+o(1)})$ to extend the edge weight or matrix entry range to at least $\{2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\}$ for any fixed $k \geq 0$. In fact, the authors show in Section 4.3 of [20] that if random bits are allowed then the polylogarithmic dependence on $M$ can be replaced by a polylogarithmic dependence on $N$ in the aforementioned reductions.

Thus, Theorem 1.1 in [20] holds also when the edge weights or matrix entries in the problems on the list are in the range $\{-2^{-N^{k+o(1)}}, \ldots, 2^{-N^{k+o(1)}}\}$ for some $k \geq 0$ (under the assumption of the more strict definition of “truly subcubic” from the introduction). Hence, we have the following fact.

**Fact 1** Let $k \geq 0$. If the APSP problem or the DNT problem, or the problem of verifying if a matrix defines a metric, or any of the remaining problems on the list in [20], does not admit a truly subcubic algorithm when the edge weights or matrix entries are in $\{-2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\}$ then none of the problems admits a truly subcubic algorithm when the edge weights or matrix entries are in $\{-2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\}$. 
3 Implications of APSP and DT hardness

To start, we observe that the decision version of the MWT problem, in particular of the DNT problem, as well as the complement of the problem of verifying if a matrix defines a metric admit $N^{2+k+o(1)}$-time algorithms with a non-deterministic $\lceil \log_2 N \rceil$-bit oracle, when the edge weights or matrix entries are in the set $\{2^{N^{k+o(1)}}, ..., 2^{N^{k+o(1)}}\}$ for some $k \geq 0$. Recall that $N$ stands for the number of vertices in the input graph or the number of rows/columns in the input matrix, respectively.

**Lemma 1.** Let $k \geq 0$, let $Q$ denote the set of integers $\{-2^{N^{k+o(1)}}, ..., 2^{N^{k+o(1)}}\}$, and let $d \in Q$. The problem of determining if a graph with $N$ vertices and edge weights in $Q$ has a triangle of weight smaller than $d$, as well as the problem of verifying that an $N \times N$ matrix with entries in $Q$ does not define a metric are in $\text{NDTIME}(\lceil \log_2 N \rceil, N^{2+k+o(1)})$.

**Proof.** In order to guess a vertex of a triangle of edge weight smaller than $d$, a non-deterministic $\lceil \log_2 N \rceil$-bit oracle is sufficient. A multi-tape Turing machine can easily verify if the guessed vertex belongs to a triangle of edge weight smaller than $d$, and if so return such a triangle. Simply, for each pair of vertices it can examine if the pair jointly with the guessed vertex forms a triangle of total edge weight smaller than $d$, and if so output the triangle in $N^{2+k+o(1)}$ total time.

A deterministic multi-tape Turing machine can also easily verify if an input $N \times N$ matrix $K$ satisfies the first three condition required by a metric, including the symmetry one, in $N^{2+k+o(1)}$ time. If the three conditions are satisfied it remains to check if $K[i,j] > K[i,k] + K[k,j]$ for some $i$, $j$, $k$. Again, to guess the first index belonging to such a triple of indices a non-deterministic $\lceil \log_2 N \rceil$-bit oracle is sufficient. Similarly, to verify if the guessed first index, say $i$, belongs to a triple of indices violating the triangle inequality condition can be done by checking if $K[i,j] > K[i,k] + K[k,j]$ for all other possible indices $j$, $k$. Again, it can be easily done by multi-tape Turing machine in $N^{2+k+o(1)}$ time.

**Theorem 1.** Let $k \geq 0$, and let $Q$ stand for the set of integers $\{-2^{N^{k+o(1)}}, ..., 2^{N^{k+o(1)}}\}$. If for any $\epsilon > 0$, the APSP problem or any of the subcubic-time equivalent problems for graphs on $N$ vertices with edge weights in $Q$ or $N \times N$ matrices with entries in $Q$ does not admit an $O(N^{3-\epsilon+k+o(1)})$-time algorithm then for any $\epsilon' > 0$, $\text{NDTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \not\subseteq \text{DTIME}(n^{1+\frac{\epsilon'}{2k}} - \epsilon')$ holds.

**Proof.** By the theorem assumptions and Fact[1] we infer that for any $\epsilon > 0$, the problem of detecting a negative triangle (DNT), when the edge weights are in
the range \([-2^{N^k+o(1)}, \ldots, 2^{N^k+o(1)}]\), does not admit an \(O(N^{3-\epsilon+k+o(1)})\)-time RAM algorithm under the logarithmic cost. On the other hand, the so restricted DNT problem is in \(\text{NDTIME}(\lceil \log_2 N \rceil, N^{2+k+o(1)})\) by Lemma [1]. Consequently, if we assume \(N^{-o(1)}\)-bit representation of the edge weights and set \(n = N^{2+k+o(1)}\) then we conclude that the restricted DNT is in \(\text{NDTIME}(\lceil \frac{1}{2+k} \log_2 n \rceil, n)\). Now the proof is by contradiction. Suppose that \(\text{NDTIME}(\lceil \frac{1}{2+k} \log_2 n \rceil, n) \supseteq \text{DTIME}(n^{1+\frac{1}{2+k} \epsilon'})\) holds. Then, the restricted DNT problem admits an \(O((N^{2+k+o(1)})^{1+\frac{1}{2+k} \epsilon'})\)-time algorithm, i.e., an \(O(N^{3+k-(2+k+o(1))\epsilon'+o(1)})\)-time algorithm in the multi-tape (deterministic) Turing machine model. A multi-tape (deterministic) Turing machine of time complexity \(T(n) \geq n\) can be easily simulated by a RAM with logarithmic cost running in \(O(T(n) \log T(n))\) time (see, e.g., section 1.7 in [4]). We infer that the restricted DNT problem admits an \(O(N^{3+k-(2+k+o(1))\epsilon'+o(1)} \log n)\)-time, i.e., an \(O(N^{3+k-(2+k+o(1))\epsilon'+o(1)})\)-time algorithm, in the RAM model. We obtain a contradiction.

In particular, when the edge weights or matrix entries are of moderate size, we obtain the stronger implication of the following form: for any \(\epsilon' > 0\), \(\text{NDTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \not\subseteq \text{DTIME}(n^{1.5-\epsilon'})\). In fact, APSP is assumed to be hard already when the weights are polynomial [3].

For the simpler DT problem, we obtain similarly the following implication.

**Theorem 2.** If for any \(\delta < \omega\), the problem of detecting a triangle in a graph on \(N\) vertices does not admit an \(O(N^{\delta+o(1)})\)-time algorithm then for any positive \(\delta' < \omega\), \(\text{NDTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \not\subseteq \text{DTIME}(n^{\delta'/2})\) holds.

**Proof.** The proof is similar to that of Theorem [1]. First, we observe that the triangle detection problem specified in the theorem is in \(\text{NDTIME}(\lceil \log_2 N \rceil, N^{2 \log N})\) by modifying slightly the proof of Lemma [1]. The difference is that in case of the triangle detection problem we do not have edge weights and we need to operate only on vertex indices of logarithmic size. Also, the verification if three vertices form a triangle is easier than that they form a triangle of total edge weight smaller than \(d\) in an edge weighted graph.

Next, we proceed along the lines of the proof of Theorem [1]. Namely, we set \(n = N^{2 \log_2 N}\) to conclude that the triangle detection problem is in \(\text{NDTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n)\). In order to obtain a contradiction suppose that \(\text{NDTIME}(\lceil \frac{1}{2} \log_2 n \rceil, n) \subseteq \text{DTIME}(n^{\delta'/2})\) holds for some \(\delta' < \omega\). Then, the triangle detection problem admits an \(O((N^2 \log N)^{\delta'/2})\)-time algorithm, i.e., an \(O(N^{\delta'} \text{poly}(\log N))\)-time algorithm in the multi-tape (deterministic) Turing machine model. Hence, we can infer (by the same argument as in the proof of Theorem [1]) that the triangle detection problem admits an
The asymptotically fastest algorithm for the detection of an \( \ell \)-clique in an \( N \) vertex graph (DC-\( \ell \)) is by a straightforward reduction to the triangle problem. In particular, if \( \ell \) is divisible by 3, it runs in \( O(n^{\omega \ell/3}) \) time \cite{18}. If \( \ell \) is not divisible by 3 one uses also fast rectangular multiplication and the expression is more complicated \cite{10}. One can conjecture that the aforementioned asymptotic time cannot be substantially improved. We can easily generalize Theorem 2 to include the consequences of such a conjecture (the main trick is to guess \( \ell - 2 \) vertices of the clique), the details are left to the reader.

**Theorem 3.** Let \( \ell \geq 3 \) be divisible by 3. If for any \( \epsilon > 0 \), the problem of detecting an \( \ell \)-clique in a graph on \( N \) vertices does not admit an \( O(N^{\omega \ell/3-\epsilon + o(1)}) \)-time algorithm, then for any positive \( \epsilon' > 0 \),\n
\[
NDTIME((\ell-2)\left\lceil \frac{1}{2} \log_2 n \right\rceil, n) \not\subseteq DTIME(n^{\omega \ell/6-\epsilon'})
\]

holds.

The MWT problem seems harder than the DT one, it is not clear how fast arithmetic matrix multiplication could be used here. More generally, Abboud et al. conjectured in \cite{1} that the problem of determining a minimum weight \( \ell \)-clique in an edge weighted graph on \( N \) vertices (MWC-\( \ell \)) does not admit a truly sub-\( N^{\ell} \)-time algorithm. The derivation of consequences of this conjecture for the separation between nondeterminism and determinism is quite analogous to the proof of Theorem 3. The differences follow from the fact that now the bound is \( N^{\ell} \) instead of \( N^{\omega \ell/3} \) and the size \( n \) of the input is \( N^{2+k+o(1)} \) (like in Theorem 1) instead of \( N^2 \log_2 N \). The proof details are left to the reader.

**Theorem 4.** Let \( k \geq 0 \), and let \( Q \) stand for the set of integers \( \{-2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\} \). Next, let \( \ell \geq 3 \). If for any \( \epsilon > 0 \), the problem of detecting a minimum weight \( \ell \)-clique in a graph with edge weights in \( Q \) and \( N \) vertices does not admit an \( O(N^{\ell-\epsilon+k+o(1)}) \)-time algorithm then for any positive \( \epsilon' > 0 \),\n
\[
NDTIME((\ell-2)\left\lceil \frac{1}{2+k} \log_2 n \right\rceil, n) \not\subseteq DTIME(n^{1+\frac{\ell-1}{2+k}-\epsilon'})
\]

holds.

### 4 Implication of 3SUM hardness

Gajentaan and Overmars \cite{11} exhibited a large class of geometric problems that were the so called 3SUM hard, i.e., if any of them admitted a substantially sub-quadratic algorithm then 3SUM would also have a substantially subquadratic algorithm. The aforementioned class has been subsequently expanded by not necessarily geometric problems (e.g., \cite{8,16,20}). In this section, we show that if 3SUM, for integers within a bounded (up to super-exponential) range, does
not admit a truly subquadratic algorithm then a strong separation between deterministic and non-deterministic time holds. The proofs in this section are similar to those from Section 3. The key trick in the proof of the following lemma is analogous to that in the footnote on 3SUM on page 2 in [21].

**Lemma 2.** The 3SUM problem, when the input \( N \) numbers are integers in \( \{-2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\} \) for some \( k \geq 0 \), is in \( \text{NTIME}(\lceil \log_2 N \rceil, N^{1+k+o(1)}) \).

**Proof.** In order to guess a number \( q \) that belongs to a triple of numbers whose sum is zero, a non-deterministic \( \lceil \log_2 N \rceil \)-bit oracle is sufficient. A multi-tape Turing machine can verify if the guessed number \( q \) belongs to such a triple as follows. First, it sorts the input numbers in non-decreasing order in \( N^{1+k+o(1)} \) time. Next, it places two copies of the sorted sequence on two tapes and moves its heads over the tapes in opposite directions starting from the opposite ends. When its head over the first tape advances to a number \( r \) then its head over the second tape advances in opposite direction to check if the number \( -q - r \) occurs in the sorted sequence, using auxiliary tapes. The whole process takes \( N^{1+k+o(1)} \) time. In this way, the Turing machine can verify if \( q \) belongs to a triple of numbers summing to zero in \( N^{1+k+o(1)} \) time. \( \square \)

The proof of the following theorem is analogous to that of Theorem 1 in Section 3.

**Theorem 5.** If for any \( \epsilon > 0 \), the 3SUM problem, when the input \( N \) numbers are integers in \( \{-2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\} \), for some \( k \geq 0 \), does not admit an \( O(N^{2-\epsilon+k+o(1)}) \)-time algorithm then for any \( \epsilon' > 0 \), \( \text{NTIME}(\lceil \frac{1}{1+k} \log_2 n \rceil, n) \not\subseteq \text{TIME}(n^{1+\frac{1}{1+k} - \epsilon'}) \) holds.

**Proof.** By the theorem assumptions, we infer that for any \( \epsilon > 0 \), the 3SUM problem, when the input numbers are in the range \( \{-2^{N^{k+o(1)}}, \ldots, 2^{N^{k+o(1)}}\} \), does not admit an \( O(N^{2-\epsilon+k+o(1)}) \)-time RAM algorithm under the logarithmic cost. On the other hand, the so restricted 3SUM problem is in \( \text{NTIME}(\lceil \log_2 N \rceil, N^{1+k+o(1)}) \) by Lemma 2. Consequently, if we assume \( N^{1+k+o(1)}, \) bit representation of the input numbers and set \( n = N^{1+k+o(1)} \) then we can conclude that the restricted 3SUM is in \( \text{NTIME}(\lceil \frac{1}{1+k} \log_2 n \rceil, n) \). Now the proof is by contradiction. Suppose that \( \text{NTIME}(\lceil \frac{1}{1+k} \log_2 n \rceil, n) \subseteq \text{TIME}(n^{1+\frac{1}{1+k} - \epsilon'}) \) holds. Then, the restricted 3SUM problem admits an \( O((N^{1+k+o(1)})^{1+\frac{1}{1+k} - \epsilon'}) \)-time algorithm, i.e., an \( O(N^{2+k-(1+k+o(1))\epsilon' + o(1)}) \)-time algorithm in the multi-tape (deterministic) Turing machine model. A multi-tape (deterministic) Turing machine of time complexity \( T(n) \geq n \) can be easily simulated by a RAM with logarithmic cost running in \( O(T(n) \log T(n)) \) time.
We conclude that the restricted 3SUM problem admits an $O(N^{2+k-(1+k+o(1))\epsilon'+o(1)} \log N)$-time, i.e., an $O(N^{2+k-(1+k+o(1))\epsilon'+o(1)})$-time algorithm, in the RAM model. We obtain a contradiction.

In fact, 3SUM is known to be hard, under randomized reductions, when $k = 0$, more precisely, already when the numbers are in $\{-n^3, \ldots, n^3\}$ [2]. The “strong 3SUM conjecture” even states that it is hard when the numbers are in $\{-n^2, \ldots, n^2\}$.

5 Extension to ETH and SETH

We can even easily derive implications of similar form from the Exponential Time Hypothesis (ETH), involving a logarithmic non-deterministic oracle. Let $kSAT(\ell)$ denote the $k$SAT problem (see [4]) for instances with $\ell$ variables and distinct clauses. Roughly, ETH conjectures that $3SAT(\ell)$ requires $2^{\Omega(\ell)}$ (deterministic) time [13] while its strong version SETH [7] conjectures that when $k$ tends to infinity than the time complexity of $kSAT(\ell)$ tends to at least $2^\ell$.

Lemma 3. $3SAT(\ell)$ is in $NDTIME(\ell,\text{poly}(\ell))$.

Proof. In order to guess an assignment (if any) satisfying the input $3SAT(\ell)$ formula, a non-deterministic $\ell$-bit oracle is enough. Now it is sufficient to observe that w.l.o.g. the number of clauses in the input formula is at most $8\binom{\ell}{3}$.

Theorem 6. If ETH holds then there is $k > 0$ such that $NDTIME([k \log_2 n], n) \not\subseteq DTIME(n)$.

Proof. By ETH, there is a constant $k_0$ such that $3SAT(\ell)$ does not admit any $2^{k_0 \ell} \text{poly}(\ell)$-time algorithm. Let $T(\ell)$ be the time taken by the non-deterministic algorithm for $3SAT(\ell)$ from Lemma 3. We may assume w.l.o.g. that $\ell$ is enough large so the inequality $T(\ell) \leq 2^{k_0 \ell}$ holds. Hence, it follows from Lemma 3 that $3SAT(\ell)$ is in $NDTIME(\ell, 2^{k_0 \ell})$. Let us set $n = 2^{k_0 \ell}$. Then, appropriately padded $3SAT(\ell)$ is in $NDTIME([k_1 \log_2 n], n)$ for $k_1 = \frac{1}{k_0}$ and the containment $NDTIME([k_1 \log_2 n], n) \subseteq DTIME(n)$ cannot hold since it would contradict the non-existence of $2^{k_0 \ell} \text{poly}(\ell)$-time algorithm for $3SAT(\ell)$.

The Orthogonal Vectors problem in dimension $d$ ($OV(d)$) is for two sets $A, B$ of vectors in $\{0, 1\}^d$ to detect a pair of vectors $a \in A$ and $b \in B$ such that $a, b$ are orthogonal in $\mathbb{Z}^d$.

Lemma 4. For any natural number $d$, the $OV(d)$ problem is in $NDTIME([\log N], N d)$. 

Proof. To guess a vector $a \in A$ which is orthogonal to some vector $b \in B$, a $\lceil \log N \rceil$-bit non-deterministic oracle is sufficient. It remains to compute the inner product of $a$ with each vector in $B$. This can be easily accomplished by a multi-tape Turing machine in $O(Nd)$ time. \hfill \Box

The low-dimension OV conjecture asserts that the $OV(d)$ problem does not admit a truly sub-$N^2$-time algorithm when $d = \omega(\log N)$ \cite{12}. It is implied by SETH \cite{22}.

**Theorem 7.** If the low-dimension OV conjecture holds, and hence if SETH holds, then for any $\epsilon > 0$, $\text{NDTIME}(\lceil \log_2 n \rceil, n) \not\subset \text{DTIME}(n^{2-\epsilon})$ holds.

**Proof.** Consider the OV problem, where $d = \omega(\log N)$ and on the other hand, $d = N^{o(1)}$. By the theorem assumption, it does not admit a truly sub-$N^2$-time algorithm. Suppose that for some $\epsilon > 0$, $\text{NDTIME}(\lceil \log_2 n \rceil, n) \subset \text{DTIME}(n^{2-\epsilon})$ holds. Set $n = Nd$. It follows from Lemma 4 that the OV problem is in $\text{NDTIME}(\lceil \log_2 N \rceil, Nd) \subset \text{NDTIME}(\lceil \log_2 n \rceil, n)$. Hence, it can be solved by a Turing machine operating in $O(n^{2-\epsilon})$ time, and consequently by a RAM with logarithmic cost in $O((N^{1+o(1)})^{2-\epsilon}) \log N = O(N^{2-\epsilon+o(1)})$ time. We obtain a contradiction. \hfill \Box

6 Quantum algorithms for MWT and MDM

In this marginal section, we present simple quantum algorithms for finding a triangle of minimum total edge length in an edge weighted graph (MWT) and for verifying if a matrix defines a metric (MDM). Our quantum algorithm for MWT is substantially faster than the fastest known quantum algorithm for APSP in the general case \cite{17}. On the other hand, it is substantially slower than the fastest known quantum algorithm for DT \cite{15}, see Table 2 (Appendix). Note that DT can be regarded as a special case of MWT.

We shall use a specialized variant of Grover’s search due to Dürr and Høyer \cite{6,9} for finding an entry of the minimum value in a table.

**Fact 2** (Dürr and Høyer \cite{9}) Let $T[k]$, $1 \leq k \leq n$, be an unsorted table where all values are distinct. Given an oracle for $T$, the index $k$ for which $T[k]$ is minimum can be found by a quantum algorithm with probability at least $\frac{1}{2}$ in $O(\sqrt{n})$ time.

**Algorithm Q**

**Input:** an oracle $W$ for the weighted adjacency matrix representing an edge weighted graph on $N$ vertices and a positive integer $M$ such that the edge weights are in the range $\{-M, ..., M\}$. 
**Output:** a minimum weight triangle in the graph.

Set an oracle for the virtual table $T[i, j, k]$, $i, j, k \in [N]$, such that $T[i, j, k] = (N + 1)^4(W[i, j] + W[i, k] + W[k, j]) + (N + 1)^2i + (N + 1)j + k$ if $W[i, j]$, $W[i, k]$, $W[k, j]$ are defined and $T[i, j, k] = (N + 1)^4(3M + 1) + (N + 1)^2i + (N + 1)j + k$ otherwise.

Find the indices $i', j', k'$ minimizing $T[i, j, k]$ by using the method from Fact 2 if $T[i', j', k'] < (n + 1)^4(3M + 1)$ then return $(i', j', k')$ else return “No”.

Since the values of the entries of the table $T$ in Algorithm Q are distinct, the method from Fact 2 can be applied to $T$. Hence, by Fact 2 we obtain the following theorem.

**Theorem 8.** Let $G$ be a graph on $N$ vertices with integer edge weights. Given an oracle for the weighted adjacency matrix of $G$, a minimum weight triangle in $G$ (if any) can be detected by a quantum algorithm with high probability in $\tilde{O}(N^{1.5})$ time.

In similar fashion, we obtain a quantum algorithmic solution to the problem of verifying if a matrix defines a metric.

**Theorem 9.** Given an oracle for an $N \times N$ integer matrix, the problem of verifying if the matrix defines a metric admits an $\tilde{O}(N^{1.5})$-time quantum algorithm.

**Proof.** The algorithm is similar to that quantum for MWT, i.e., Algorithm Q. For each of the four properties that the matrix, say $K$, should have, we form a virtual table with distinct values. On the base of the minimum value of an entry in the table, we can decide if the property holds. For searching for the minimum, we use again Fact 2. The largest of the virtual tables is of cubic in $N$ size and it corresponds to the triangle inequality property. For $i, j, k \in [N]$, the virtual table is defined by $T[i, j, k] = (N + 1)^4(K[i, k] + K[k, j] - K[i, j]) + (N + 1)^2i + (N + 1)j + k$. Note that all values of the entries of $T$ are distinct and that the triangle inequality is violated by $K$ if and only if the minimum value of an entry of $T$ is negative. The virtual quadratic table $U$ corresponding to the symmetry property is given by $U[i, j] = (N + 1)^4(U[i, j] - U[j, i])^2 + (N + 1)^2i + (N + 1)j + k$ for $i, j \in [N]$. We leave the rest of details to the reader. The application of the quantum search from Fact 2 to the verification of the triangle inequality property dominates the time complexity. □

**Final remark**

Among the implications derived from the conjectures considered in this paper, the strongest seem to be those from the conjectures on 3SUM, MWC-ℓ, and SETH. The weakest seems the implication from ETH. See Table 1.
Table 2. Known upper bounds on the time complexity of quantum algorithms for problems discussed in this paper. In case of 3SAT $N$ stands for the number of variables in the input formula.

| problem | upper bound | author |
|---------|-------------|--------|
| APSP | $O(N^{2.5})$ | Navebi and Vassilevska Williams [14] |
| MWT | $O(N^{2.5})$ | This paper |
| MDM | $O(N^{1.5})$ | This paper |
| DT | $O(N^{9/7} \times \log N)$ | Lee, Magniez, and Santha [15] |
| 3SUM | $O(N^{1+o(1)})$ | Ambainis [6] |
| 3SAT | $1.153^N \text{poly}(N)$ | Ambainis [6] |

7 Acknowledgments

The author is very grateful to reviewers of an earlier version of this paper for valuable comments and suggestions.

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