Logarithmically complete monotonicity of Catalan-Qi function related to Catalan numbers

Feng Qi\textsuperscript{1,2,*} and Bai-Ni Guo\textsuperscript{4}

Abstract: In the paper, the authors find the logarithmically complete monotonicity of the Catalan–Qi function related to the Catalan numbers.

Subjects: Advanced Mathematics; Analysis - Mathematics; Integral Transforms & Equations; Mathematical Analysis; Mathematics & Statistics; Number Theory; Real Functions; Science; Sequences & Series; Special Functions

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1. Introduction

It is stated in Koshy (2009) that the Catalan numbers $C_n$ for $n \geq 0$ form a sequence of natural numbers that occur in tree enumeration problems such as “In how many ways can a regular $n$-gon be divided into $n-2$ triangles if different orientations are counted separately?” whose solution is the Catalan number $C_{n-2}$. The Catalan numbers $C_n$ can be generated by

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\frac{2}{1 + \sqrt{1 - 4x}} = 1 - \frac{\sqrt{1 - 4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n
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= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \ldots.
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One of explicit formulas of $C_n$ for $n \geq 0$ reads that

$$C_n = \frac{4^n \Gamma(n + 1/2)}{\sqrt{\pi} \Gamma(n + 2)},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0$$

is the classical Euler gamma function. In Graham, Knuth, and Patashnik (1994), Koshy (2009), and Vardi (1991), it was mentioned that there exists an asymptotic expansion

$$C_x \sim \frac{4^x}{\sqrt{\pi}} \left( \frac{1}{x^{3/2}} - \frac{9}{8} \frac{1}{x^{5/2}} + \frac{145}{128} \frac{1}{x^{7/2}} + \cdots \right)$$

(1)

for the Catalan function $C_x$.

A generalization of the Catalan numbers $C_n$ was defined in Hilton and Pedersen (1991), Klarner (1970), and McCarthy (1992) by

$$p d_n = \frac{1}{n} \binom{pn}{n-1} = \frac{1}{(p-1)n+1} \binom{pn}{n}$$

for $n \geq 1$. The usual Catalan numbers $C_n = \frac{1}{2} d_n$ are a special case with $p = 2$.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p, r)$ are defined (Fuss, 1791) as numbers of the form

$$A_n(p, r) = \frac{r}{np + r} \binom{np + r}{n} = \frac{\Gamma(np + r)}{\Gamma(n+1)\Gamma(n(p+1) + r + 1)}.$$

It is easy to see that

$$A_n(2, 1) = C_n, \quad n \geq 0$$

and

$$A_{n-1}(p, p) = p d_n, \quad n \geq 1.$$

There has existed some literature, such as Alexeev, Götze, and Tikhomirov (2010), Aval (2008), Bisch and Jones (1997), Gordon and Griffeth (2012), Lin (2011), Liu, Song, and Wang (2011), Młotkowski (2010), Młotkowski, Pensom, and Życzkowski (2013), Przytycki and Sikora (2000), Stump (2008, 2010), on the investigation of the Fuss–Catalan numbers $A_n(p, r)$.

In Qi, Shi, and Liu (2015a, Remark 1), an alternative and analytical generalization of the Catalan numbers $C_n$ and the Catalan function $C_x$ was introduced by

$$C(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} \left( \frac{b}{a} \right)^x \frac{\Gamma(z + a)}{\Gamma(z + b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \geq 0.$$

For the uniqueness and convenience of referring to the quantity $C(a, b; x)$, we call the quantity $C(a, b; x)$ the Catalan–Qi function and, when taking $x = n \geq 0$, call $C(a, b; n)$ the Catalan–Qi numbers. It is obvious that

$$C \left( \frac{1}{2}, 2; n \right) = C_n, \quad n \geq 0$$
and that

\[ C(a, b; x) = \frac{1}{C(b, a; x)}, \quad C(a, b; x)C(b, c; x) = C(a, c; x) \]

for \( a, b, c > 0 \) and \( x \geq 0 \). In the recent papers of Liu, Shi, and Qi (2015), Mahmoud and Qi (identities), Qi (2015a, 2015c, 2015d), Qi, Mahmoud, Shi, and Liu (2015), Qi et al. (2015a), Qi, Shi, and Liu (2015b, 2015c, 2015d), Shi, Liu, and Qi (2015, among other things, some properties, including the general expression and a generalization of the asymptotic expansion (Equation 1), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers \( C_n \), the Catalan function \( C_x \), the Catalan–Qi function \( C(a, b; x) \), and the Fuss–Catalan numbers \( A_n(p, r) \) were established. Very recently, we discovered in Qi (2015d, Theorem 1.1) a relation between the Fuss–Catalan numbers \( A_n(p, r) \) and the Catalan–Qi numbers \( C(a, b; n) \), which reads that

\[ A_n(p, r) = \frac{r^n}{\prod_{k=1}^{p} C\left(\frac{k+r-1}{p}, 1; n\right)} \prod_{k=1}^{p} C\left(\frac{k+r-1}{p}, 1; 1\right) \]

for integers \( n \geq 0, p > 1, \) and \( r > 0 \).

From the viewpoint of analysis, motivated by the idea in the papers of Qi and Chen (2007), Qi, Zhang, and Li (2014a, 2014b, 2014c) and closely related references cited therein, we will consider in this paper the function

\[ \psi_{a,b;x}(t) = C(a + t, b + t; x), \quad t, x \geq 0, \quad a, b > 0 \]

and study its properties.

Recall from Atanassov and Tsoukrovski (1988), Qi and Chen (2004), Qi and Guo (2004), Schilling, Song, and Vondraček (2012) that an infinitely differentiable and positive function \( f \) is said to be logarithmically completely monotonic on an interval \( I \) if it satisfies

\[ 0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty \]

on \( I \) for all \( k \in \mathbb{N} \).

The main results of this paper are the logarithmically complete monotonicity of the function \( \psi_{a,b;x}(t) \) in \( t \in [0, \infty) \) for \( a, b > 0 \) and \( x \geq 0 \), which can be stated as the following theorem.

**Theorem 1.1**  
For \( x \geq 0 \) and \( a, b > 0 \),

1. the function \( \psi_{a,b;x}(t) \) is logarithmically completely monotonic on \([0, \infty)\) if and only if either \( 0 \leq x \leq 1 \) and \( a \leq b \) or \( x \geq 1 \) and \( a \geq b \),
2. the function \( \frac{1}{\psi_{a,b;x}(t)} \) is logarithmically completely monotonic on \([0, \infty)\) if and only if either \( 0 \leq x \leq 1 \) and \( a \geq b \) or \( x \geq 1 \) and \( a \leq b \).

**2. Proof of Theorem 1.1**

Taking the logarithm of \( \psi_{a,b;x}(t) \) and differentiating with respect to \( t \) gave

\[ [\ln \psi_{a,b;x}(t)]' = \psi(t + b) - \psi(t + a) + x \left( \frac{1}{t + b} - \frac{1}{t + a} \right) \]

\[ + \psi(t + x + a) - \psi(t + x + b). \]
Making use of

\[ \psi(z) = \int_0^\infty \left( \frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}} \right) \, du, \quad \Re(z) > 0 \]

in Abramowitz and Stegun (1972, p. 259, 6.3.21) leads to

\[ \ln \mathcal{G}_{a,b,x}(t)' = \int_0^\infty e^{-au} - e^{-bu} \, du + x \int_0^\infty (e^{-bu} - e^{-au}) e^{-tu} \, du \]

\[ + \int_0^\infty \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} e^{-t+uxu} \, du \]

\[ = \int_0^\infty [e^{-xu} - 1 + x(1 - e^{-u})] \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} e^{-tu} \, du \]

\[ = x \int_0^\infty \left( \frac{1-e^{-u}}{u} - \frac{1-e^{-xu}}{xu} \right) \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} e^{-tu} \, du. \]

It is easy to see that the function \( e^{-u} \) is strictly decreasing on \((0, \infty)\). Hence,

\[ \frac{1-e^{-u}}{u} - \frac{1-e^{-xu}}{xu} \leq 0 \]

for \( u \in (0, \infty) \) if and only if \( x \leq 1 \). It is apparent that

\[ \frac{e^{-bu} - e^{-au}}{1 - e^{-u}} \leq 0 \]

for \( u \in (0, \infty) \) if and only if \( a \leq b \). Recall from Mitrinović, Pečarić, and Fink (1993, Chap. XIII), Schilling et al. (2012, Chap. 1), and Widder (1941, Chapter IV) that an infinitely differentiable function \( f \) is said to be completely monotonic on an interval \( I \) if it satisfies

\[ 0 \leq (-1)^k f^{[k]}(x) < \infty \]

on \( I \) for all \( k \geq 0 \). The famous Bernstein–Widder theorem (Widder, 1941, p. 160, Theorem 12a) states that a necessary and sufficient condition that \( f(x) \) should be completely monotonic in \( 0 \leq x < \infty \) is that

\[ f(x) = \int_0^\infty e^{-xt} \, d\alpha(t), \quad (2) \]

where \( \alpha \) is bounded and non-decreasing and the integral (Equation 2) converges for \( 0 \leq x < \infty \). Consequently,

1. the function \( \ln \mathcal{G}_{a,b,x}(t)' \) is completely monotonic on \([0, \infty)\) if and only if \( x \leq 1 \) and \( a \leq b \),
2. the function \( -[\ln \mathcal{G}_{a,b,x}(t)]' \) is completely monotonic on \([0, \infty)\) if and only if \( x \leq 1 \) and \( a \leq b \).

As a result,

1. the function \( \frac{1}{\mathcal{G}_{a,b,x}(t)} \) is logarithmically completely monotonic on \([0, \infty)\) if and only if \( x \leq 1 \) and \( a \leq b \),
2. the function \( \mathcal{G}_{a,b,x}(t) \) is logarithmically completely monotonic on \([0, \infty)\) if and only if \( x \leq 1 \) and \( a \leq b \).

The proof of Theorem 1.1 is thus complete.

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