QUASI RIGHT-VEERING BRAIDS AND NON-LOOSE LINKS

TETSUYA ITO AND KEIKO KAWAMURO

Abstract. We introduce a notion of “quasi right-veering” for closed braids, which plays an
analogous role to “right-veering” for abstract open books. We show that a transverse link $K$
in a contact 3-manifold $(M, \xi)$ is non-loose if and only if every braid representative of $K$ with
respect to every open book decomposition that supports $(M, \xi)$ is quasi right-veering. We also
show that several definitions for a “right-veering” closed braid are equivalent.

1. Introduction

The dichotomy between tight and overtwisted is fundamental to 3-dimensional contact topology
and detecting tightness of a given contact structure often arises as an important problem.

In the classification of Legendrian and transverse links in contact 3-manifolds, non-loose vs.
loose dichotomy plays a role similar to the tight vs. overtwisted dichotomy in the classification
of contact structures. For example, overtwisted contact structures are classified by homotopy
equivalence \cite{7}, on the other hand loose null-homologous Legendrian (resp. transverse) links are
coarsely classified by classical invariants called the Thurston-Bennequin number and the rotation
number (resp. the self-linking number) \cite{8, 10}. Here, ‘coarse’ means up to contactomorphism,
smoothly isotopic to the identity.

Recall a result of Honda, Kazez and Matić \cite{16}.

**Theorem 1.1.** \cite[Theorem 1.1]{16} A contact 3-manifold $(M, \xi)$ is tight if and only if for every
open book decomposition $(S, \phi)$ of $(M, \xi)$, $\phi$ is right-veering.

In \cite{16}, Honda, Kazez and Matić also define the fractional Dehn twist coefficient (FDTC).
The FDTC is an invariant of an open book decomposition and detects right-veering-ness of the
monodromy. Hence the FDTC can be used to determine tight or overtwisted of the compatible
contact structure \cite{6, 17, 22}.

As a natural counterpart of right-veering mapping classes, right-veering closed braids (with
respect to general open books) have been defined and studied in the literature \cite{2, 3, 26}. As a
counterpart of the FDTC, in \cite{20} we naturally extend it to the FDTC for a closed braid $L$ in an
open book $(S, \phi)$ with respect to a boundary component $C$ of $S$, denoted by $c(\phi, L, C)$ (see also
Definition 2.3 below).

| open book $(S, \phi)$ | closed braid $L$ in $(S, \phi)$ |
|----------------------|-------------------------------|
| right-veering mapping class $\phi \in \text{MCG}(S)$ | (quasi) right-veering closed braid $\phi_L \in \text{MCG}(S, P)$ |
| FDCT of $\phi$ w.r.t. $C$ $c(\phi, C)$ | FDTC of braid $L$ w.r.t. $C$ $c(\phi, L, C)$ |

In \cite{20} we see that various results on open books and the FDTC are translated to results on
closed braids and the FDTC for closed braids. This gives us some hope that open books and
closed braids in open books can be treated in a unified manner.

Date: October 19, 2017.
However, this is too optimistic: Note that any non-right-veering open book supports an over-twisted contact structure [16], but not every non-right-veering closed braid is loose. A simple example of this fact is a non-right-veering closed braid in an open book decomposition of a tight contact 3-manifold.

In this paper, we find a condition on closed braids to be loose. In Definition 3.11 we introduce quasi-right-veering closed braids. After studying basic properties of quasi-right-veering braids we show that it is the quasi-right-veering condition on closed braids that plays the same role as the right-veering condition on open books in Theorem 1.1. Our first main result is the following:

**Theorem 4.1** A transverse link $K$ in a contact 3-manifold $(M, \xi)$ is non-loose if and only if every braid representative of $K$ with respect to every open book decomposition that supports $(M, \xi)$ is quasi right-veering.

In Theorem 4.1 we allow the transverse link $K$ to be empty. Our definition of quasi-right-veering implies that the empty braid with respect to an open book $(S, \phi)$ is quasi-right-veering if and only if $\phi$ is right-veering. Having a loose empty link can be interpreted as having an overtwisted underlying contact structure. Therefore, Theorem 1.1 follows as a corollary of Theorem 4.1.

In Sections 5 and 6 we present more results concerning non-loose links. The invariant depth is a measurement of non-looseness introduced by Baker and Onaran [1]. In Theorem 5.2 we relate depth-one links and non-quasi-right-veering braids.

**Theorem 5.2.** Let $(S, \phi)$ be an open book supporting $(M, \xi)$. Let $B$ denote the binding of $(S, \phi)$ and $L$ be a closed braid in $(S, \phi)$. Let $K := B \cup L$ which is a transverse link in $(M, \xi)$. We have the depth $d(K) = 1$ if and only if the braid $L$ is non-quasi-right-veering.

Theorem 6.1 below a result on braids can be seen as a generalization of [22, Corollary 1.2] a result on open books.

**Theorem 6.1.** Let $L$ be a closed braid with respect to a planar open book $(S, \phi)$. If $c(\phi, L, C) > 1$ for every boundary component $C$ of $S$ then $L$ is non-loose.

Finally in Section 7 we address one subtle but important issue on right-veering closed braids. As mentioned above, a couple of different looking definitions of right-veering closed braids have been existing in the literature (cf. [2, 3, 26]), which we call $\partial$-$(\partial + P)$, $\partial$-$\partial$, and $\partial$-$P$ right-veering (see Definition 7.2). We show that they are essentially equivalent (though there are subtle differences).

**Corollary 7.8.** For $\psi \in \text{MCG}(S, P)$ the following are equivalent.

1. $\psi$ is $\partial$-$(\partial + P)$ right-veering.
2. $\psi$ is $\partial$-$\partial$ right-veering.
3. $\psi$ is $\partial$-$P$ right-veering.

2. The fractional Dehn twist coefficients of closed braids and branched coverings

Let $S \cong S_{g,d}$ be an oriented compact surface with genus $g$ and $d$ boundary components. Throughout the paper we assume $d > 0$. Let $P = \{p_1, \ldots, p_n\}$ be a (possibly empty) finite set of interior points of $S$. Let $\text{MCG}(S, P)$ (denoted by $\text{MCG}(S)$ if $P$ is empty) be the mapping class group of the punctured surface $S \setminus P$, which is the group of isotopy classes of orientation
preserving homeomorphisms of the surface $S$ fixing $\partial S$ pointwise and fixing $P$ set-wise. For the entire section, by abuse of notation, $\phi$ may be often used for a homeomorphism representing the mapping class $\phi \in \text{MCG}(S, P)$. Denote the fractional Dehn twist coefficient (FDTC) \cite{16} with respect a boundary component $C$ by $c(-, C) : \text{MCG}(S, P) \to \mathbb{Q}$.

2.1. The FDTC of a braid. In this subsection, we review the definition of the FDTC for a closed braid that is originally defined in \cite{20}, then prove some well-definedness result in Proposition 2.4.

Let $(S, \phi)$ be an abstract open book supporting an oriented closed contact 3-manifold $(M, \xi) \cong (M(S, \phi), \xi(S, \phi))$. The pages are parametrized by $t \in [0, 1]$ and the page $S_t$ and $S$ are identified under the following diffeomorphism and projection map $p$:

$$\{S_t \mid 0 \leq t \leq 1\} \cong S \times [0, 1] \xrightarrow{p} S$$

$(x, t) \mapsto x$

Suppose that $\phi \in \text{Diff}^+(S, \partial S)$ be a diffeomorphism representing $\phi \in \text{MCG}(S)$. We may choose a collar neighborhood $\nu(\partial S)$ of the boundary $\partial S$ on which $\phi = \text{id}$.

Let $L$ be a possibly empty closed $n$-braid with respect to the open book $(S, \phi)$. Suppose that the $n$ intersection points $L \cap S_0$ are contained in $\nu(\partial S)$ under the projection $p$. Put $P := p(L \cap S_0) \subset \nu(\partial S) \subset S$. Cutting the 3-manifold $M$ along the page $S_0$, the closed braid $L$ gives rise to an $n$-braid denoted by $\beta_L \subset S \times [0, 1]$ joining $P \times \{0\}$ and $P \times \{1\}$. We regard $\beta_L$ as an element of the $n$-stranded surface braid group $B_n(S)$.

Since $\phi|_{\nu(\partial S)} = \text{id}$ and $P \subset \nu(\partial S)$ we have $\phi|_P = \text{id}$ and we may view $\phi$ as an element of $\text{Diff}^+(S, P, \partial S)$. In order to distinguish $\phi$ in $\text{Diff}^+(S, \partial S)$ and $\varphi$ in $\text{Diff}^+(S, P, \partial S)$, we denote the latter by $j(\phi)$.

$$j : \text{Diff}^+(S, \partial S) \longrightarrow \text{Diff}^+(S, P, \partial S)$$

$\varphi \mapsto j(\varphi)$

Note that if $P \not\subset \nu(\partial S)$ then $\varphi(P) \neq P$ in general, and the homomorphism $j$ cannot be defined. The map $j$ induces a homomorphism:

$$j_* : \text{MCG}(S) \longrightarrow \text{MCG}(S, P)$$

$\phi \mapsto [j(\phi)]$

Recall the generalized Birman exact sequence \cite{13} Theorem 9.1

$$1 \rightarrow B_n(S) \xrightarrow{i} \text{MCG}(S, P) \xrightarrow{j} \text{MCG}(S) \rightarrow 1$$

(2.1)

where $i$ is the push map and $f$ is the forgetful map. Since $f \circ j_* = \text{id}_{\text{MCG}(S)}$ the exact sequence splits.

**Definition 2.1.** Let $L$ be a closed braid in $(S, \phi)$ satisfying $P = p(L \cap S_0) \subset \nu(\partial S)$. Let

$$\phi_L := i(\beta_L) \circ j_*(\phi) \in \text{MCG}(S, P)$$

and call it the distinguished monodromy of the closed braid $L$.

We have

$$\text{(M(S, \phi), L) \simeq \langle (S, P) \times [0, 1] \rangle / \sim_{\phi_L}}$$

where the equivalence relation $\sim_{\phi_L}$ satisfies $(x, 1) \sim (\varphi_L(x), 0)$ for $x \in S$ and $(x, 1) \sim (x, t)$ for $x \in \partial S$ and $t \in [0, 1]$ where $\varphi_L \in \text{Diff}^+(S, P, \partial S)$ is a diffeomorphism representing $\phi_L$.

In practice, we tend to identify two braids (say, $L$ and $L'$) if they are braid isotopic, and often call the isotopy class simply a braid, which is sometimes confusing. The distinguished monodromy $\phi_L \in \text{MCG}(S, P)$ is defined for the individual braid representative $L$ satisfying $L \cap S_0 \subset \nu(\partial S)$. The following Proposition 2.2 shows the relation between $\phi_L$ and $\phi_{L'}$. 

**
Proposition 2.2. Let $L$ and $L'$ be closed $n$-braids in an open book $(S, \phi)$ with $P := p(L \cap S_0) \subset \nu(\partial S)$ and $P' := p(L' \cap S_0) \subset \nu(\partial S)$. If $L$ and $L'$ are braid isotopic then there exists an isomorphism

$$\gamma^* : \text{MCG}(S, P) \to \text{MCG}(S, P')$$

such that $\phi_{L'} = \gamma^*(\phi_L)$. When $P' = P$ the isomorphism $\gamma^*$ is an inner automorphism of $\text{MCG}(S, P)$.

Proof. Cutting $M_{(S, \phi)}$ along the page $S_0$, we get $n$-braids $\beta_L$ and $\beta_{L'} \subset S \times [0, 1]$. Since $L$ and $L'$ are braid isotopic we have

$$\beta_{L'} = \gamma^{-1} \circ \beta_L \circ \gamma^* \quad \text{(read from the right to left)}$$

for some $n$-braid

$$\gamma : [0, 1] \to S \times [0, 1]$$

$$t \mapsto \{p_1(t), \ldots, p_n(t)\} \times \{t\}$$

connecting $\gamma(0) = P' \times \{0\}$ and $\gamma(1) = P \times \{1\}$, where $\gamma^{-1}(t) = \gamma(1 - t)$, $\gamma^*$ is an $n$-braid defined by

$$\gamma^*(t) := \{\phi(p_1(t)), \ldots, \phi(p_n(t))\} \times \{t\}$$

and the product in (2.3) is concatenation of braids. Note that $\gamma^*(0) = \gamma(0) = P' \times \{0\}$ and $\gamma^*(1) = \gamma(1) = P \times \{1\}$.

We may regard the $n$-braid $\gamma$ as an isotopy $\{\gamma_t : P' \to S \mid t \in [0, 1]\}$ of ordered $n$ distinct points such that $\gamma_0(P') = P'$ and $\gamma_1(P') = P$. We naturally extend $\{\gamma_t\}$ to an isotopy $\{\tilde{\gamma}_t : S \to S \mid t \in [0, 1]\}$ of the surface so that

- $\tilde{\gamma}_t|_{P'} = \gamma_t$ for all $t \in [0, 1]$,
- $\tilde{\gamma}_0 = \text{id}_{(S, \phi)}$,
- $\tilde{\gamma}_t$ is isotopic to $\text{id}_S$ for all $t \in [0, 1]$ if we forget the marked points in $S$.

Therefore, $\tilde{\gamma}_1 : (S, P') \to (S, P)$ is a homeomorphism and it gives rise to an isomorphism:

$$\gamma^* : \text{MCG}(S, P) \to \text{MCG}(S, P')$$

$$\psi \mapsto [\tilde{\gamma}_1^{-1}] \circ \psi \circ [\tilde{\gamma}_1]$$

Let $j_* : \text{MCG}(S) \to \text{MCG}(S, P)$ denote the homomorphism constructed in the same way as $j_* : \text{MCG}(S) \to \text{MCG}(S, P)$ and replacing $P$ with $P'$. Let $i' : B_n(S) \to \text{MCG}(S, P')$ be the push map in the Birman exact sequence where $P$ is replaced with $P'$.

By (2.3) the we obtain:

$$\phi_{L'} = i'(\beta_{L'}) \circ j_* (\phi)$$

$$= i'(\gamma^{-1} \circ \beta_L \circ \gamma^*) \circ j_* (\phi)$$

$$= [\tilde{\gamma}_1]^{-1} \circ i(\beta_L) \circ (j_*(\phi) \circ [\tilde{\gamma}_1] \circ j_*(\phi)^{-1}) \circ j'_*(\phi)$$

$$= [\tilde{\gamma}_1]^{-1} \circ \phi_L \circ [\tilde{\gamma}_1].$$

$$= \gamma^*(\phi_L)$$

□

Definition 2.3. We define the fractional Dehn twist coefficient (FDTC) of $L$ with respect to $C$ as the FDTC of the distinguished monodromy $\phi_L$ with respect to $C$ and denote it by $c(\phi, L, C)$. Namely,

$$c(\phi, L, C) := c(\phi_L, C).$$

We have $c(\psi \circ \phi_L \circ \psi^{-1}, C) = c(\phi_L, C)$ for any $\psi \in \text{MCG}(S, P)$. In fact a stronger result holds.

Proposition 2.4. Let $L$ and $L'$ be closed $n$-braids in the open book $(S, \phi)$ with $P := p(L \cap S_0) \subset \nu(\partial S)$ and $P' := p(L' \cap S_0) \subset \nu(\partial S)$. If $L$ and $L'$ are braid isotopic then $c(\phi, L, C) = c(\phi, L', C)$ for every boundary component $C$. 

Proof. This is a corollary of Proposition 2.2. By the properties of the isotopy \( \{ \gamma_t \} \) in the proof of Proposition 2.2, the following diagram commutes:

\[
\begin{array}{ccc}
MCG(S, P) & \xrightarrow{\gamma^*} & MCG(S, P') \\
c(-c) & \downarrow & c(-c) \\
\mathbb{Q} & \xrightarrow{c(-c)} & \mathbb{Q}
\end{array}
\]

By (2.4) we have

\[c(\phi_L, C) = c(\gamma^*(\phi_L), C) = c(\phi_L', C).\]

As a remark, when \( P' = P \) the isomorphism \( \gamma^* \) is an inner automorphism of \( MCG(S, P) \) and the commutativity implies invariance of the FDTC under conjugation.

If a braid \( L \) is empty we set \( P = \emptyset \) and define the distinguished monodromy \( \phi_L := \phi \). Hence the FDTC of the empty closed braid is equal to the FDTC of the monodromy of the open book.

In practice, when we consider \( c(\phi, L, C) \) it is often convenient, as done in [20], to take \( P = p(L \cap S_0) \) so that \( P \) is contained in a collar neighborhood of just \( C \) rather than \( \partial S \).

2.2. Branched coverings and the FDTC. In this subsection, we study the behavior of the FDCT \( c(\phi, L, C) \) under a covering of open books branched along a braid, then give applications to contact geometry and geometry.

Given a contact 3-manifold \((M, \xi)\) with a transverse link \( L \subset (M, \xi) \) and a covering \( \pi : \tilde{M} \to M \) branched along \( L \), there exists a contact structure \( \tilde{\xi} \) on \( \tilde{M} \) unique up to isotopy such that \( \pi_*(\tilde{\xi}) \) is isotopic to \( \xi \) through contact structures. See [24, Section 2] for a construction of \( \tilde{\xi} \) and its uniqueness. We call the contact 3-manifold \((\tilde{M}, \tilde{\xi})\) the contact branched covering of \((M, \xi)\) branched along \( L \).

Definition 2.5 (Branched coverings of open books). Let \((S, \phi)\) be an open book supporting the contact manifold \((M, \xi) := (M_{(S, \phi)}, \xi_{(S, \phi)})\). Let \( L \) be a closed \( n \)-braid with respect to \((S, \phi)\) and \( P := p(L \cap S_0) \subset S \). Let \( \pi : \tilde{S} \to S \) be a branched covering of \( S \) branched at the \( n \) points \( P \). Suppose that \( \tilde{S} \) is connected. Put \( \tilde{P} := \pi^{-1}(P) \) and we may denote \( \pi : (\tilde{S}, \tilde{P}) \to (S, P) \) by abusing the notation. We say that an open book \((\tilde{S}, \tilde{\phi})\) is a branched covering of \((S, \phi)\) along the closed braid \( L \) if there exists \( \psi \in MCG(S, \tilde{P}) \) such that \( f(\psi) = \tilde{\phi} \) (the map \( f \) is the forgetful map in the Birman exact sequence (2.1)) and

\[\pi \circ \psi = \phi_L \circ \pi\]

where \( \phi_L \) is the distinguished monodromy introduced in Definition 2.1.

The branched covering \( \pi : (\tilde{S}, \tilde{P}) \to (S, P) \) induces a covering \( \pi : M_{(\tilde{S}, \tilde{\phi})} \to M_{(S, \phi)} \) branched along \( L \). The contact 3-manifold \((M_{(\tilde{S}, \tilde{\phi})}, \xi_{(\tilde{S}, \tilde{\phi})})\) supported by \((\tilde{S}, \tilde{\phi})\) gives a contact branched covering of \((M, \xi)\) branched along \( L \). The preimage \( \tilde{L} := \pi^{-1}(L) \) is a closed braid in the open book \((\tilde{S}, \tilde{\phi})\) and \( \phi_L = \psi \in MCG(S, \tilde{P}) \).

We note that Casey [5, Theorem 3.4.1] has studied branch coverings of a special case where \((S, \phi) = (D^2, id)\) and \((M, \xi) = (S^3, \xi_{std})\). See also Giroux’s work [15, Corollaire 5].
Definition 2.6. Recall that the branched covering \( \pi : (\tilde{S}, \tilde{P}) \to (S, P) \) is fully ramified if for every branch point \( \tilde{p} \in \tilde{P} \) there exists a disk neighborhood \( \tilde{N} \) containing \( \tilde{p} \) such that the restriction \( \pi|_\tilde{N} : \tilde{N} \to \pi(\tilde{N}) \) is a non trivial branched covering with the single branch point.

A typical example of fully ramified branched covering is a quotient map \( \pi : \tilde{S} \to \tilde{S}/G \) for a properly discontinuous action of a group \( G \) on \( \tilde{S} \). The action of \( G \) on \( \tilde{S} \) further induces an action of \( G \) on \( M(\tilde{S}, \tilde{\phi}) \) such that \( M(\tilde{S}, \tilde{\phi})/G = M(S, \phi) \).

The following proposition shows that the FDTC behaves nicely under a fully ramified branched covering map.

Proposition 2.7. Assume that the above branched covering \( \pi : (\tilde{S}, \tilde{P}) \to (S, P) \) is fully ramified and \( \chi(\tilde{S}) < 0 \). For a boundary component \( C \) of \( S \) let \( \tilde{C} \) be a connected component of the preimage \( \pi^{-1}(C) \). Let \( d(\pi, \tilde{C}) \) denote the degree of the covering \( \pi|_C : \tilde{C} \to C \). Then we have

\[
c(\tilde{\phi}, \tilde{L}, \tilde{C}) = c(\tilde{\phi}, \tilde{C})
\]

and

\[
c(\tilde{\phi}, \tilde{L}, \tilde{C}) \cdot d(\pi, \tilde{C}) = c(\phi, L, C).
\]

Proof. We first prove the case where \( \phi_L \in \text{MCG}(S, P) \) is pseudo-Anosov. The definition of the FDTC [16], in terms of stable foliations instead of laminations, states that \( c(\phi, C) \) for a pseudo-Anosov mapping class \( \phi \) counts how much a pseudo-Anosov homeomorphism representing \( \phi \) twists the singular leaves of the stable foliation \( F \) near the boundary component \( C \).

The restriction \( \pi' := \pi|_{\tilde{S}\setminus\tilde{P}} : \tilde{S} \setminus \tilde{P} \to S \setminus P \) gives an honest covering map. Let \( F \subset S \setminus P \) be a stable foliation of \( \phi_L \in \text{MCG}(S, P) \). Then \( F' = \pi'^{-1}(F) \subset \tilde{S} \setminus \tilde{P} \) is a stable foliation of the lift \( \psi \in \text{MCG}(\tilde{S}, \tilde{P}) \) of \( \phi_L \). By the definition of the FTDC it follows that \( c(\phi_L, C) = c(\psi, \tilde{C}) \cdot d(\pi, \tilde{C}). \)

See Figure 1. By Definition 2.3 we get

\[
c(\phi, L, C) = c(\phi_L, C) = c(\psi, \tilde{C}) \cdot d(\pi, \tilde{C}) = c(\tilde{\phi}, \tilde{L}, \tilde{C}) \cdot d(\pi, \tilde{C}).
\]

It remains to show that \( c(\psi, \tilde{C}) = c(\tilde{\phi}, \tilde{C}) \). Every puncture point \( p \in P \) is a singularity of the foliation \( F \). Suppose that \( F \) has \( k \) prongs at \( p \) where \( k \geq 1 \) and \( k \neq 2 \). Then each preimage \( \tilde{p} \in \pi^{-1}(p) \) is a \( kd \)-prong singularity of \( F' \) for some \( d \geq 1 \). The fully ramified assumption on \( \pi \) further imposes that \( d > 1 \). This implies that \( \tilde{p} \) is not a 1-prong singularity of \( F' \) and we can fill the
puncture points \( \tilde{P} \) to get a singular foliation \( \tilde{F} \) on \( \tilde{S} \) which is a stable foliation for \( \tilde{\phi} \in \text{MCG}(\tilde{S}) \). By the definition of the FDTC, \( c(\psi, \tilde{C}) = c(\tilde{\phi}, \tilde{C}) \).

Next, suppose that \( \phi_L \in \text{MCG}(S, P) \) is periodic. There exists \( N \in \mathbb{Z} \) such that \( \phi_L^N \) is freely isotopic to \( id(S, P) \). This means that there exist \( M \in \mathbb{Z} \) and \( T \in \text{MCG}(S, P) \) which is a product of Dehn twists about boundary components of \( \partial S \setminus C \) such that

\[
(2.5) \quad \text{id}(S, P) \approx \phi_L^N = T_C^M \circ T \quad \text{in} \quad \text{MCG}(S, P).
\]

Here, ‘\( \approx \)’ means freely isotopic.

Suppose that the covering \( \pi : (\tilde{S}, \tilde{P}) \to (S, P) \) is \( \delta \)-fold. Let \( D = \delta! \). Then for every boundary component \( X \) of \( \tilde{S} \), \( d(\pi, X) \) divides \( D \). In particular, \( d := d(\pi, \tilde{C}) \) divides \( D \). Taking the \( D \)th power we get

\[
(2.6) \quad \text{id}(\tilde{S}, \tilde{P}) \approx \psi^{ND} = (T_C)^{MD} \circ T^D \quad \text{in} \quad \text{MCG}(\tilde{S}, \tilde{P})
\]

for some product \( T' \) of Dehn twists about boundary components of \( \partial S \setminus \tilde{C} \).

Since \( \chi(\tilde{S}) < 0 \) the space \( \tilde{S} \) is not an annulus; thus, for any two distinct boundary components \( \tilde{C}_1, \tilde{C}_2 \) of \( \tilde{S} \) the images \( f(T_{C_1}) \) and \( f(T_{C_2}) \) under the forgetful map \( f \) in the Birman exact sequence are distinct elements of \( \text{MCG}(\tilde{S}) \). (cf. Remark 2.8 below.) Abusing the notation, we may denote \( f(T_{\tilde{C}}) \) by \( T_{\tilde{C}} \). With this observation, filling the puncture points of \( \tilde{P} \) we obtain

\[
(2.7) \quad \text{id}_{\tilde{S}} \approx \tilde{\phi}^{ND} = (T_{\tilde{C}})^{MD/d} \circ T' \quad \text{in} \quad \text{MCG}(\tilde{S}).
\]

As for the FDTC, equation (2.5) gives \( c(\phi, L, C) = \frac{M}{N} \), and equations (2.6) and (2.7) give \( c(\tilde{\phi}, \tilde{L}, \tilde{C}) = c(\tilde{\phi}, \tilde{C}) = \frac{M}{d} \). Therefore, the statement of the proposition is proved. (For the periodic case, the fully ramified condition on \( \pi \) is not used.)

Finally if \( \phi_L \) is reducible then the statement follows from the pseudo-Anosov and periodic cases.

**Remark 2.8.** If \( \chi(\tilde{S}) = 0 \) (i.e. \( \tilde{S} \) is an annulus) then Proposition 2.7 does not hold. Consider a double branched covering \( \pi : A \to D^2 \) branched at two points \( P = \{p_1, p_2\} \subset D^2 \) and the positive half-twist \( \sigma \in \text{MCG}(D^2, P) \simeq B_2 \). The mapping class \( \sigma \) lifts to the positive Dehn twist \( \tau \in \text{MCG}(A) \) along the core of the annulus \( A \). We have \( c(\tau, \tilde{C}) = 1 \), \( d(\pi, \tilde{C}) = 1 \) and \( c(\sigma, C) = \frac{1}{2} \).

**Corollary 2.9.** Let \((\tilde{S}, \tilde{\phi})\) be a fully ramified branched open book covering of \((S, \phi)\) branched along \( L \). If \( \chi(\tilde{S}) < 0 \) and \( c(\phi, L, C) < 0 \) for some boundary component \( C \) then \((\tilde{S}, \tilde{\phi})\) supports an overtwisted contact structure.

**Proof.** By Proposition 2.7 we have \( c(\tilde{\phi}, \tilde{C}) < 0 \) for every connected component \( \tilde{C} \) of the preimage of \( C \). This means that \( \tilde{\phi} \) is not right-veering. Hence Theorem 1.1 implies that \((\tilde{S}, \tilde{\phi})\) supports an overtwisted contact structure.

In general, taking a branched covering does not preserve the geometric structure. For example, a branched cover of \( S^3 \) along a hyperbolic link is not necessarily hyperbolic. In the following corollary we give a sufficient condition on the FDTC that the geometric structure to be preserved under taking a fully ramified branched cover.

**Corollary 2.10.** Let \((\tilde{S}, \tilde{\phi})\) be a fully ramified branched covering of \((S, \phi)\) branched along a closed braid \( L \) with \( \chi(\tilde{S}) < 0 \). Assume that \( c(\phi, L, C) > 4d(\pi, \tilde{C}) \) for every boundary component \( C \subset \partial S \) and connected component \( C \subset \pi^{-1}(C) \). If \( M_{(S, \phi)} \setminus L \) is Seifert-fibered (resp. toroidal, hyperbolic) then \( M_{(\tilde{S}, \tilde{\phi})} \) is Seifert-fibered (resp. toroidal, hyperbolic).
Proof. We have $|c(\phi, L, C)| > |c(\phi, L, C)/d(\pi, \tilde{C})| > 4$. \cite{20} Theorem 8.4 and $|c(\phi, L, C)| > 4$ yield that $M_{(S, \bar{L})} \setminus L$ is Seifert-fibered (resp. toroidal, hyperbolic) if and only if $\phi_L$ is periodic (resp. reducible, pseudo-Anosov). Since $\tilde{\phi}$ is a lift of $\phi_L$, the branch covering is fully ramified, and $\chi(\tilde{S}) < 0$, the map $\bar{\phi}$ is periodic (resp. reducible, pseudo-Anosov).

By Proposition \ref{prop:2.7} $|c(\tilde{\phi}, \tilde{C})| = |c(\tilde{\phi}, L, \tilde{C})| = |c(\phi, L, C)/d(\pi, \tilde{C})| > 4$. With \cite{20} Theorem 8.3 we conclude that $M_{(S, \bar{L})}$ is Seifert-fibered (resp. toroidal, hyperbolic). \qed

3. Quasi-right-veering maps

In this section we introduce a partial ordering "$\ll_{\text{right}}$" and quasi-right-veering braids, then we compare right-veering and quasi-right-veering. We use the same notations in the previous section.

3.1. Strongly right-veering partial ordering "$\ll_{\text{right}}$". For each boundary component $C$ of $S$, we choose a base point $*_C \in C$. Let $\mathcal{A}_C(S, P)$ be the set of isotopy classes of properly embedded arcs $\gamma : [0, 1] \to S \setminus P$ satisfying $\gamma(0) = *_C$. Here, by isotopy we mean isotopy fixing the end points of the arc. We do not allow $\gamma \in \mathcal{A}_C(S, P)$ to have $\gamma(1) \in P$ but we allow $\gamma(1) \in (C \setminus \{*_C\})$. Abusing the notation, an element $\gamma \in \mathcal{A}_C(S, P)$ often means an actual arc $[0, 1] \to S$ representing $\gamma$ and we may call an element of $\mathcal{A}_C(S, P)$ simply an arc.

We say that two arcs $\alpha$ and $\beta$ intersect efficiently if they attain the minimal geometric intersection number among all the arcs isotopic to them.

Definition 3.1 (Right-veering total ordering $\prec_{\text{right}}$). Let $\alpha, \beta \in \mathcal{A}_C(S, P)$ be arcs intersecting efficiently. We denote $\alpha \prec_{\text{right}} \beta$ and say that $\beta$ lies on the right side of $\alpha$ if the arc $\beta$ lies on the right side of $\alpha$ in a small neighborhood of the base point $*_C$.

In \cite{16} and \cite{20}, where the set $P$ is empty, the symbol "$>$" is used in the place of "$\prec_{\text{right}}$".

The order "$\ll_{\text{right}}$" is a total ordering. For any family of arcs $\{\alpha_i\} \subset \mathcal{A}_C(S, P)$ we can always put them in a position simultaneously so that $\alpha_i$ and $\alpha_j$ intersect efficiently for any pairs $(i, j)$. This can be done, for example, by choosing a hyperbolic metric on $S \setminus P$ and realizing the arcs as geodesics.

Naturally extending the notion of right-veering in \cite{16} we define the following, cf. \cite{3} p.949:

Definition 3.2 (Right-veering).

- We say that $\psi \in \text{MCG}(S, P)$ is right-veering with respect to the boundary component $C$ if $\alpha \prec_{\text{right}} \psi(\alpha)$ or $\alpha = \psi(\alpha)$ for every $\alpha \in \mathcal{A}_C(S, P)$. Since $\prec_{\text{right}}$ is a total ordering on the set $\mathcal{A}_C(S, P)$, $\psi \in \text{MCG}(S, P)$ is right-veering if and only if $\psi(\alpha) \not\prec_{\text{right}} \alpha$ for every $\alpha \in \mathcal{A}_C(S, P)$.
- We say that a closed braid $L$ in an open book $(S, \phi)$ with $P := p(L \cap S_0) \subset \nu(\partial S)$ is right-veering with respect to $C$ if $\phi_L \in \text{MCG}(S, P)$ is right-veering with respect to $C$.

Remark 3.3. By Proposition \ref{prop:2.4} if braids $L$ and $L'$ are braid isotopic then for every boundary component $C$ of $S$ we have $\phi_L \in \text{MCG}(S, P)$ is right-veering with respect to $C$ if and only if $\phi_{L'} \in \text{MCG}(S, P')$ is right-veering with respect to $C$.

Remark 3.4. In \cite{4} \cite{20}, a slightly different definition of “right-veering” is used. See Section \ref{section:7} for the relationship between these two superficially different notions of right-veering.

We will define another ordering "$\ll_{\text{right}}$" which plays a central role in this paper.

Definition 3.5 (Strongly right-veering partial ordering $\ll_{\text{right}}$). For two arcs $\alpha, \beta \in \mathcal{A}_C(S, P)$, we define $\alpha \ll_{\text{right}} \beta$ if there exists a sequence of arcs $\alpha_0, \ldots, \alpha_k \in \mathcal{A}_C(S; P)$ such that

\begin{align*}
(3.1) & \quad \alpha = \alpha_0 \prec_{\text{right}} \alpha_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \alpha_k = \beta \text{ in } \mathcal{A}_C(S; P), \\
(3.2) & \quad \text{Int}(\alpha_i) \cap \text{Int}(\alpha_{i+1}) = \emptyset \text{ for all } i = 0, \ldots, k - 1.
\end{align*}
By the definition it is easy to see that $\preceq_{\text{right}}$ is a partial ordering, i.e., $\alpha \preceq_{\text{right}} \beta$ and $\beta \preceq_{\text{right}} \gamma$ imply $\alpha \preceq_{\text{right}} \gamma$. If the puncture set $P$ is empty, then by \cite{10} Lemma 5.2 the ordering $\preceq_{\text{right}}$ coincides with $\prec_{\text{right}}$. On the other hand, when $P$ is non-empty $\preceq_{\text{right}}$ is not a total ordering and there is difference between $\prec_{\text{right}}$ and $\preceq_{\text{right}}$. To see the difference we use the following notion.

**Definition 3.6** (Boundary right $P$-bigon). Let $\alpha, \beta \in \mathcal{A}_C(S, P)$ with $\alpha \prec_{\text{right}} \beta$. Assume that there exist subarcs $\delta_\alpha \subset \alpha$ and $\delta_\beta \subset \beta$ such that

- $*C \in \delta_\alpha \cap \delta_\beta$
- $\delta_\alpha \cup \delta_\beta$ bounds a (possibly immersed) bigon $D(\subset S)$ which lies on the right side of $\alpha$ (i.e., the orientation of $\delta_\alpha$, as a subarc of $\alpha$, disagrees with the orientation of $\partial D$) and
- $D$ contains some of the marked points of $P$.

We call such a bigon $D$ a **boundary right $P$-bigon** from $\alpha$ to $\beta$.

A boundary right $P$-bigon gives an obstruction for $\alpha \preceq_{\text{right}} \beta$:

**Proposition 3.7.** Let $\alpha, \beta \in \mathcal{A}_C(S, P)$ be arcs with $\alpha \prec_{\text{right}} \beta$. If there is a boundary right $P$-bigon from $\alpha$ to $\beta$ then $\alpha \not\preceq_{\text{right}} \beta$.

**Proof.** If there is a boundary right $P$-bigon $D$ from $\alpha$ to $\beta$ then every arc $\gamma \in \mathcal{A}_C(S, P)$ that satisfies $\alpha \prec_{\text{right}} \gamma \prec_{\text{right}} \beta$ must intersect $D$ and yields either a boundary right $P$-bigon from $\alpha$ to $\gamma$ or from $\gamma$ to $\beta$ (see Figure 2 (a)). Thus for any sequence of arcs $\alpha = \gamma_0 \prec_{\text{right}} \gamma_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \gamma_n = \beta$ for some $i$ there exists a boundary right $P$-bigon from $\gamma_i$ to $\gamma_{i+1}$, which means $\text{Int}(\gamma_i)$ and $\text{Int}(\gamma_{i+1})$ cannot be disjoint. \hfill $\square$

![Figure 2](image.png)

**Figure 2.** (a) The arc $\gamma$ with $\alpha \prec_{\text{right}} \gamma \prec_{\text{right}} \beta$ cuts the boundary right $P$-bigon $D$, yielding a boundary right $P$-bigon from $\beta$ to $\gamma$. (b) $\alpha \prec_{\text{right}} \beta$ and $\beta \preceq_{\text{right}} \gamma$, but $\alpha \not\preceq_{\text{right}} \gamma$. (c) $f(\alpha) \prec_{\text{right}} f(\beta)$ and $\alpha \prec_{\text{right}} \beta$, but $\alpha \not\prec_{\text{right}} \beta$.

As a corollary, we observe that conditions $\alpha \prec_{\text{right}} \beta$ and $\beta \preceq_{\text{right}} \gamma$ may not imply $\alpha \preceq_{\text{right}} \gamma$ in general. (Also $\alpha \preceq_{\text{right}} \beta$ and $\beta \prec_{\text{right}} \gamma$ may not imply $\alpha \preceq_{\text{right}} \gamma$.) For example, the arcs depicted in Figure 2 (b) satisfy $\alpha \prec_{\text{right}} \beta$ and $\beta \preceq_{\text{right}} \gamma$ but by Proposition 3.7 $\alpha \not\prec_{\text{right}} \gamma$.

We conjecture the converse of Proposition 3.7.

**Conjecture 3.8.** $\alpha \prec_{\text{right}} \beta$ if and only if $\alpha \prec_{\text{right}} \beta$ and there exist no boundary right $P$-bigons from $\alpha$ to $\beta$.

The next lemma easily follows from the definition of $\preceq_{\text{right}}$.

**Lemma 3.9.** Let $f : \mathcal{A}_C(S, P) \to \mathcal{A}_C(S)$ be the forgetful map. If $\alpha \preceq_{\text{right}} \beta$ in $\mathcal{A}_C(S; P)$ then we have $f(\alpha) \prec_{\text{right}} f(\beta)$ in $\mathcal{A}_C(S)$.

**Proof.** Since $\alpha \preceq_{\text{right}} \beta$, there is a sequence of arcs $\alpha = \gamma_0 \prec_{\text{right}} \gamma_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \gamma_n = \beta$ in $\mathcal{A}_C(S, P)$ with $\text{Int}(\gamma_i) \cap \text{Int}(\gamma_{i+1}) = \emptyset$ for all $i$. This implies that $\gamma_i$ and $\gamma_{i+1}$ do not even cobound marked bigons. Therefore, $\text{Int}(f(\gamma_i)) \cap \text{Int}(f(\gamma_{i+1})) = \emptyset$ and we can conclude $f(\alpha) = f(\gamma_0) \prec_{\text{right}} f(\gamma_1) \prec_{\text{right}} \cdots \prec_{\text{right}} f(\gamma_n) = f(\beta)$ in $\mathcal{A}_C(S)$; that is, $f(\alpha) \prec_{\text{right}} f(\beta)$ in $\mathcal{A}_C(S)$. \hfill $\square$
Remark. The converse of the lemma does not hold in general, even if we assume $\alpha \prec_{\text{right}} \beta$. See Figure 2 (c).

The next proposition gives a sufficient condition for $\alpha \ll_{\text{right}} \beta$.

**Proposition 3.10.** Let $\alpha, \beta \in A_C(S, P)$ be arcs with $\alpha \prec_{\text{right}} \beta$. If $\alpha$ and $\beta$ do not cobound bigons with marked points (namely, if $\alpha$ and $\beta$, viewed as arcs in the non-punctured surface $S$, intersect efficiently), then $\alpha \ll_{\text{right}} \beta$.

**Proof.** If $\alpha$ and $\beta$ do not cobound bigons with marked points then following the proof of [10, Lemma 5.2] one can construct an arc $\gamma \in A_C(S, P)$ such that $\alpha \prec_{\text{right}} \gamma \prec_{\text{right}} \beta$ with $\#(\alpha, \gamma) < \#(\alpha, \beta)$ and $\#(\gamma, \beta) < \#(\alpha, \beta)$. Here $\#(-, -)$ denotes the geometric intersection number of the interiors of the two arcs. Moreover, the construction of $\gamma$ shows that $\alpha$ and $\gamma$ ($\gamma$ and $\beta$) do not cobound bigons with marked points. Thus iterating this interpolation process, we get a sequence of arcs satisfying the conditions (3.1) and (3.2).

3.2. Quasi-right-veering vs. right-veering. Now we introduce quasi right-veering mapping classes and quasi right-veering closed braids.

**Definition 3.11.** (Quasi-right-veering)

- We say that $\psi \in \text{MCG}(S, P)$ is quasi-right-veering with respect to the boundary component $C$ of $S$ if every arc $\alpha \in A_C(S, P)$ satisfies $\psi(\alpha) \ll_{\text{right}} \alpha$. (Warning: Since “$\ll_{\text{right}}$” is not a total ordering, $\psi(\alpha) \ll_{\text{right}} \alpha$ is not equivalent to $\alpha \ll_{\text{right}} \psi(\alpha)$ or $\alpha = \psi(\alpha)$.)
- We say that a closed braid $L$ in an open book $(S, \phi)$ is quasi-right-veering with respect to a boundary component $C$ if its distinguished monodromy $\phi_L \in \text{MCG}(S, P)$ is quasi-right-veering with respect to $C$.
- We say that $L$ is quasi-right-veering if $L$ is quasi-right-veering with respect to every boundary component of $S$.

**Proposition 3.12.** If braids $L$ and $L'$ in $(S, \phi)$ are braid isotopic then for every boundary component $C$ of $S$ we have $L$ is quasi-right-veering with respect to $C$ if and only if $L'$ is quasi-right-veering with respect to $C$.

**Proof.** Recall the homeomorphism $\tilde{\gamma}_1 : (S, P') \to (S, P)$ and the isomorphism $\gamma^*: \text{MCG}(S, P) \to \text{MCG}(S, P')$ in the proof of Proposition 2.4. Note that $\tilde{\gamma}_1$ naturally induces a bijection $\gamma_* : A_C(S, P') \to A_C(S, P)$.

Using the maps $\gamma_*$ and $\gamma^*$ the statement of the proposition follows.

If $L$ is empty then by identifying $\phi_L = \phi$, the empty closed braid is quasi right-veering if and only if the monodromy $\phi$ is right-veering.

We note that the definitions of “right-veering” and “quasi-right-veering” are independent of a choice of the distinguished point $*_{C}$.

**Proposition 3.13.** A mapping class $\psi \in \text{MCG}(S, P)$ is quasi-right-veering if $\psi$ is right-veering. More generally, $\psi \in \text{MCG}(S, P)$ is quasi-right-veering if $f(\psi) \in \text{MCG}(S)$ is right-veering, where $f : \text{MCG}(S, P) \to \text{MCG}(S)$ is the forgetful map in the generalized Birman exact sequence $\text{(2.1)}$.

As a consequence, every closed braid $L$ in an open book $(S, \phi)$ is quasi-right-veering if $\phi \in \text{MCG}(S)$ is right-veering. In particular, every closed braid in the open book $(D^2, \text{id})$ is quasi-right-veering.

**Proof.** Assume that $\psi \in \text{MCG}(S, P)$ is not quasi-right-veering with respect to some boundary component $C$ of $S$. Then there exists an arc $\alpha \in A_C(S, P)$ such that $\psi(\alpha) \ll_{\text{right}} \alpha$. By Lemma 3.9 we get $f(\psi)(f(\alpha)) = f(\psi(\alpha)) \prec_{\text{right}} f(\alpha)$ in $A_C(S)$; that is, $f(\psi) \in \text{MCG}(S)$ is not right-veering.
It is proved in [10] Section 3 that the right-veeringness of \( \phi \in \mathcal{MCG}(S) \) is almost equivalent to positivity of its FDTC. We say “almost” because the statement is slightly complicated when the FDTC = 0 for non-pseudo Anosov case. If \( \phi \in \mathcal{MCG}(S) \) is pseudo Anosov, \( \phi \) is right-veering with respect to a boundary component \( C \) if and only if \( c(\phi, C) > 0 \). We remark that parallel statements on positivity and right-veering-ness hold for elements \( \psi \in \mathcal{MCG}(S, P) \). Namely if \( \psi \) is right-veering then \( c(\psi, C) \geq 0 \). Moreover, if \( \psi \) is pseudo Anosov then \( \psi \) is right-veering with respect \( C \) if and only if \( c(\psi, C) > 0 \).

The next proposition shows significant difference between quasi right-veering and right-veering. In particular, quasi right-veering is much less related to positivity of the FDTC.

**Proposition 3.14.** Let \((S, \phi)\) be an open book.

1. For a boundary component \( C \) of \( S \) and integers \( N < 0 \) and \( n > 1 \), there exists a closed \( n \)-braid \( L \) in \((S, \phi)\) which is quasi right-veering with respect to \( C \) but not right-veering with \( c(L, \phi, C) \leq N < 0 \).
2. For any negative integer \( N \) there exists a closed braid \( L \) in \((S, \phi)\) which is quasi right-veering and has \( C(L, \phi, C) \leq N < 0 \) for every boundary component \( C \) of \( S \).

**Proof.** Fix a boundary component \( C \) of \( S \). Take a collar neighborhood \( \nu(C) \) of \( C \) so that \( \phi = id \) on \( \nu(C) \). We identify \( \nu(C) \) with the annulus \( A = \{ z \in \mathbb{C} \mid 1 \leq |z| \leq 2 \} \) so that the boundary component \( C \) is identified with \( \{ z \in \mathbb{C} \mid |z| = 1 \} \). We put

\[
P = \left\{ p_1, \ldots, p_n \in \mathbb{C} \mid p_i = 1 + \frac{i}{n+1} (i = 1, \ldots, n) \right\} \subset A \cong \nu(C) \subset S.
\]

For \( k \in \mathbb{N} \) let \( \beta_{C,k} \) be the \( n \)-braid in \( S \times [0, 1] \) whose \( i \)-th strand \( \gamma_{k,i} : [0, 1] \to A \times [0, 1] \subset S \times [0, 1] \) is given by (see Figure 3(1))

\[
\gamma_{k,i}(t) = \begin{cases} ((1 + \frac{1}{n+1}) \exp(2\pi \sqrt{-1} kt), t) & (i = 1) \\ ((1 + \frac{2}{n+1}) \exp(-2\pi \sqrt{-1} kt), t) & (i = 2) \\ (1 + \frac{i}{n+1}, t) & (i = 3, \ldots, n). \end{cases}
\]

Thus, the 1st strand of \( \beta_{C,k} \) winds \( k \) times around \( C \) counterclockwise and the 2nd strand winds \( k \) times clockwise. Let \( L_{C,k} \) be the closed braid in the open book \((S, \phi)\) obtained by closing the braid \( \beta_{C,k} \).

![Figure 3](image)

**Figure 3.** (1) The braid \( L_{C,1} \) is not right-veering but quasi right-veering.
(2) The map \((T_C)^{-1}(T_{C'})^2(T_{C''})^{-1}\) forces to form a boundary right \( P \)-bigon.

With the push map \( i : B_n(S) \to \mathcal{MCG}(S, P) \) in the Birman exact sequence \((2.1)\) we have

\[i(\beta_{C,1}) = (T_C)^{-1}(T_{C'})^2(T_{C''})^{-1},\]

where \( T_C, T_{C'} \) and \( T_{C''} \) are the right-handed Dehn twists along
the curves $C, C' = \{ z \in A \mid |z| = \frac{3}{2\pi + \gamma} \}$ and $C'' = \{ z \in A \mid |z| = \frac{5}{2\pi + \gamma} \}$. The distinguished monodromy of the closed braid $L := L_{C,k}$ is

$$
\phi_L = \phi_{L_{C,k}} = i(\beta C,k) j_*(\phi) = (T_C)^{-k} (T_{C''})^{2k} (T_{C'})^{-k} j_*(\phi) \in \text{MCG}(S, P).
$$

Since $j_*(\phi) = id$ on $\nu(C)$ we have $c(L, \phi, C) = -k$.

For any $\gamma \in A_{C}(S; P)$ the factor $(T_C)^{-k}(T_{C''})^{2k}(T_{C'})^{-k}$ of $\phi_L$ forces to form a boundary right $P$-bigon from $\phi_L(\gamma)$ to $\gamma$. See Figure 3-(2). Thus by Proposition 3.7 $\phi_L(\gamma) \not\rightarrow$ right $\gamma$ for any $\gamma \in A_{C}(S; P)$, which means $L_{C,k}$ is quasi right-veering with respect to $C$. This proves (1).

Next we prove (2). Let $\{C_1, \ldots, C_d\}$ be the set of boundary components of $S$. For each component $C_i$ we take a closed braid $L_{C_i,k}$ given in the proof of (1), and let $L = \bigsqcup_{i=1}^d L_{C_i,k}$ be the disjoint union of $L_{C_i,k}$. By (1) we see that $L$ is quasi right-veering. By Proposition 2.4 we obtain $c(L, \phi, C_i) \leq -k$ for all $i = 1, \ldots, d$.

**Corollary 3.15.** The set of quasi right-veering mapping classes in $\text{MCG}(S, P)$ does not form a monoid.

**Proof.** We use the same notations in Proposition 3.14. Let $\chi = (T_{C'})^{-1} i(\beta_{C,1})^{-1} = T_CT_{C''}T_{C'}^{-1}$ and $\psi = i(\beta_{C,1})$. Both $\chi$ and $\psi$ are quasi right-veering but $\chi \psi = (T_C)^{-1}$ is not quasi right-veering. $\square$

Proposition 6.1 of [16] implies that every contact 3-manifold admits an open book decomposition $(S, \phi)$ with right-veering monodromy. Thus in light of Proposition 3.13 every transverse link in $(M, \xi)$ admits a quasi right-veering closed braid representative with respect to some open book decomposition of $(M, \xi)$. The next proposition shows more is true.

**Proposition 3.16.** Every closed braid $L$ in an open book $(S, \phi)$ can be made right-veering after a sequence of positive stabilizations.

When $(S, \phi) = (D^2, id)$ the same statement is proved in [26, Proposition 3.1].

**Proof.** As usual, we put $L$ so that $P = p(L \cap S_0)$ is contained in a collar neighborhood $\nu(\partial S)$ of $\partial S$. We may assume that $\phi_{|\nu(\partial S)} = id$. Let $C$ be a boundary component of $S$. Let $\nu'(C) \subset \nu(C)$ be a sub-collar neighborhood of $C$ that does not contain (or intersect) $P$. See Figure 4

Choose points $q$ and $q'$ in $\nu'(C)$. Let $\gamma_1 \subset S$ be an arc that connects one of the puncture points in $P$ and the point $q$. Let $\gamma_2 \subset \nu'(C)$ be an arc that connects $q$ and $q' \in \nu'(C)$ and satisfying the following.

1. The interiors of $\gamma_1$ and $\gamma_2$ intersect exactly at one point in $\nu'(C)$. We name it $r$.
2. Let $\gamma_1' \subset \gamma_1$ and $\gamma_2' \subset \gamma_2$ be the sub-arcs connecting $r$ and $q$. Then the simple closed curve $\gamma_1' \cup \gamma_2'$ is homotopic to $C$ in $S \setminus (P \cup \{q\})$.

![Figure 4](image.png)

**Figure 4.** Twice stabilizations about $C$ makes a closed braid right-veering with respect to $C$. 

Let $L'$ be a closed braid obtained from $L$ by positive stabilizations first along $\gamma_1$ and then $\gamma_2$. The distinguished monodromy of $L'$ is $\phi_{L'} = H_{\gamma_2} \circ H_{\gamma_1} \circ \phi_L \in \text{MCG}(S, P \cup \{q, q'\})$, where $H_{\gamma_i}$ is the positive half twist about the arc $\gamma_i$. Since $\phi_L = \text{id}$ on $\nu'(C)$ and every essential arc in $A_C(S, P \cup \{q, q'\})$ intersects either $\gamma_1$ or $\gamma_2$, the monodromy $\phi_{L'}$ is right-veering with respect to $C$.

Applying this operation for every boundary component we get a right-veering closed braid that is transversely isotopic to the original braid $L$. □

4. Proof of Theorem 4.1

We now prove our main theorem:

**Theorem 4.1.** A transverse link $K$ in a contact 3-manifold $(M, \xi)$ is non-loose if and only if every braid representative of $K$ with respect to every open book decomposition that supports $(M, \xi)$ is quasi right-veering.

Our proof of Theorem 4.1 is a generalization of the proof of [19, Theorem 2.4]. We may assume that the readers are familiar with basic definitions and properties of open book foliations that can be found in [18, 20, 21].

**Proof of Theorem 4.1** $(\Rightarrow)$ First we show that non-quasi-right-veering braid is loose. Assume that a transverse link $K$ can be represented by a non-quasi-right-veering closed $L$ with respect to an open book $(S, \phi)$. That is, there exist a boundary component $C \subset \partial S$ and an arc $\alpha \in A_C(S, P)$ such that there is a sequence of arcs $\phi_L(\alpha) = \alpha_0 \prec_{\text{right}} \alpha_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \alpha_k = \alpha$ with $\text{Int}(\alpha_i) \cap \text{Int}(\alpha_{i+1}) = \emptyset$ for all $i = 0, \ldots, k - 1$.

We explicitly construct a transverse overtwisted disk $D_{\text{trans}}$ in $M \setminus L$ by giving its movie presentation. A similar construction can be found in [19]. Here, a *transverse overtwisted disk* (see [18, Definition 4.1] for the precise definition) is a disk admitting a certain type of open book foliation and is bounded by a transverse push-off of a usual overtwisted disk.

For $i = 0, \ldots, k$ denote the endpoint $\alpha_i(1) \in \partial S$ of the arc $\alpha_i$ by $w_i$. Slightly moving $w_i$ along $\partial S$, if necessary, we may assume that all the points $w_0, \ldots, w_{k-1}$ are distinct and still satisfying $\text{Int}(\alpha_i) \cap \text{Int}(\alpha_{i+1}) = \emptyset$. Since $\phi_L(\alpha) = \alpha$ we get $w_0 = w_k$. Fix a sufficiently small $\varepsilon > 0$.

The open book foliation of $D_{\text{trans}}$ contains one negative elliptic point at $*_C$ and $k$ positive elliptic points at $w_0, \ldots, w_{k-1}$.
The movie presentation of $D_{\text{trans}}$ on the page $S_0$ consists of $(k - 1)$ a-arcs emanating from $w_1, \ldots, w_{k-1}$ and a b-arc that is a copy of $\alpha_0$ joining $w_0$ and $*_C$. For $t \in [0, \frac{1}{k+1}]$ the movie presentation on the page $S_1$ is the same as $S_0$.

The movie presentation on the page $S_{\frac{1}{k+1} + \varepsilon}$ contains one hyperbolic point, $h_1$, whose describing arc joining $\alpha_0$ and the a-arc from $w_1$ is a parallel copy of $\alpha_1$ in $S_{\frac{1}{k+1} - \varepsilon}$. Since $\text{Int}(\alpha_0) \cap \text{Int}(\alpha_1) = \emptyset$ the interior of the describing arc is disjoint from all the a-arcs and the b-arc in the page $S_{\frac{1}{k+1} - \varepsilon}$. Since $\alpha_0 \prec_{\text{right}} \alpha_1$ the normal vectors of $D_{\text{trans}}$ point out of the describing arc, thus by Observation 2.5] the sign of the hyperbolic point $h_1$ is positive. The movie presentation on the page $S_{\frac{1}{k+1} + \varepsilon}$ consists of one b-arc which is a copy of $\alpha_1$ connecting $w_1$ and $*_C$ and $(k - 1)$ a-arcs emanating from $w_0, w_2, \ldots, w_{k-1}$.

![Figure 6. Movie for $t \in [\frac{1}{k+1} - \varepsilon, \frac{1}{k+1} + \varepsilon]$: the b-arc $\alpha_0$ disappears and the new b-arc $\alpha_1$ appears at $t = \frac{1}{k+1}$. The black dashed arc is the describing arc and the gray dashed arc is $\alpha_1$. The black solid arrows indicate the orientations of $\partial S$. The gray dashed arrows are normal vectors to $D_{\text{trans}}$.](image)

We inductively apply the above procedure. Let $j = 1, \ldots, k$. The above paragraph describes $j = 1$ case.

On the page $S_{\frac{1}{k+1}}$ ($j > 1$) we put a positive hyperbolic point $h_j$ whose describing arc is a parallel copy of $\alpha_j$. As a consequence the page $S_{\frac{1}{k+1} + \varepsilon}$ has one b-arc which is a copy of $\alpha_j$ connecting $w_j$ and $*_C$ and $(k - 1)$ a-arcs emanating from $w_i$ for $i = 1, \ldots, j - 1, j + 1, \ldots, k - 1$.

On the page $S_1$ the movie presentation consists of one b-arc which is a copy of $\alpha_k = \alpha$ and $(k - 1)$ a-arcs emanating from $w_0, \ldots, w_{k-1}$. Since $\phi_L(\alpha) = \alpha_0$ the slices $D_{\text{trans}} \cap S_1$ and $D_{\text{trans}} \cap S_0$ of $D_{\text{trans}}$ can be identified under the distinguished monodromy $\phi_L$. In other words the movie presentation gives rise to an embedded surface in $M \setminus \{\gamma\}$. The construction tells us that the surface is topologically a disk, and moreover it is a transverse overtwisted disk (see [19]).

$(\Leftarrow)$ Assume that a transverse link $L \subset (M, \xi)$ is loose. By taking a neighborhood of an overtwisted disk $D \subset M \setminus L$, we may regard $(M, \xi)$ as the connected sum $(M', \xi')#(S^3, \xi_{ot})$ such that $L \subset (M', \xi')$. Here $\xi_{ot}$ denotes some overtwisted contact structure on $S^3$. Applying the argument of Honda, Kazez and Matić in [16, p.444] to $(S^3, \xi_{ot})$ we may regard $(M, \xi)$ as $(N, \xi_N)#(S^3, \xi_{ot})$ such that $L \subset (N, \xi_N)$, where $(S^3, \xi_{ot})$ denotes the overtwisted contact structure supported by the annulus open book $(A, T_A^{-1})$ with the left-handed Dehn twist.

Take an open book decomposition $(S_N, \phi_N)$ of $(N, \xi_N)$ and a closed braid representative $L_N$ of $L$. Then the original contact 3-manifold $(M, \xi)$ is supported by the open book $(S, \phi) := (S_N, \phi_N) * (A, T_A^{-1})$ and $L_N$ is in a braid position with respect to $(S, \phi)$. For the co-core $\gamma \subset A$ of the attached 1-handle we have $\phi_{L_N}(\gamma) = \phi(\gamma) \ll_{\text{right}} \gamma$ hence $L_N$ is not quasi right-veering.  

The following statement is a weak version of Theorem 4.1 but still gives a criterion of non-loose links.
**Corollary 4.2.** A transverse link $K$ in a contact 3-manifold $(M, \xi)$ is non-loose if and only if for every closed braid representative $L$ of $K$ with respect to every open book decomposition $(S, \phi)$ that supports $(M, \xi)$ and for every properly embedded arc $\gamma \in S \setminus P$, at least one of the following holds:

1. $\gamma = \phi_L(\gamma)$.
2. $\gamma \prec_{\text{right}} \phi_L(\gamma)$.
3. $\gamma$ and $\phi_L(\gamma)$ cobound bigons that contain points of $P$.

**Proof.** ($\Rightarrow$) If there exists $\gamma$ such that $\phi_L(\gamma) \prec_{\text{right}} \gamma$ and no marked bigons are cobounded by $\phi_L(\gamma)$ and $\gamma$, then Proposition 3.10 shows that $\phi_L(\gamma) \prec R \gamma$. Thus $\phi_L \in \text{MCG}(S, \phi)$ is not quasi-right-veering. Then Theorem 4.1 shows that $K$ is loose.

($\Leftarrow$) This implication holds by exactly the same proof of ($\Leftarrow$ part of) Theorem 4.1. □

## 5. Depth of Transverse Links

Theorem 4.1 can be used to study the depth that measures non-looseness of transverse links. The depth is introduced by Baker and Onaran in [1].

Let $F$ be an oriented surface in an oriented 3-manifold $M$ and $K \subset M$ be an oriented link that transversely intersects $F$. We denote the number of intersection points of $K$ and $F$ by $\#(K \cap F)$, which is not necessarily realizing the geometric intersection number. We also denote the number of positive and negative intersection points of $K$ and $F$ by $\#^+(K \cap F)$ and $\#^-(K \cap F)$, respectively. We have $\#(K \cap F) = \#^+(K \cap F) + \#^-(K \cap F)$.

The depth $d(K)$ of a transverse link $K$ in $(M, \xi)$ is defined by

$$d(K) = \min \{ \#(K \cap D) \mid D \text{ is an overtwisted disk in } (M, \xi) \}.$$ 

Assuming that $(M, \xi)$ is overtwisted, we see that $K$ is loose if and only if $d(K) = 0$.

First we show that the depth of $K$ is equal to the minimal number of the negative intersection points of $K$ with a transverse overtwisted disk ([18 Definition 4.1]). The same result is proved in [23] for the case when $K$ is the binding of an open book.

**Theorem 5.1.** Let $(S, \phi)$ be an open book supporting a contact 3-manifold $(M, \xi)$. Let $K$ be a transverse link in $(M, \xi)$. We have:

$$d(K) = \min \left\{ \#^-(K' \cap D) \mid K' \text{ is a link transversely isotopic to } K, \right.$$  
$$D \text{ is a transverse overtwisted disk in } (S, \phi) \left. \right\}$$

**Proof.** We denote by $d_{\text{trans}}(K)$ the quantity in the right hand side of (5.1). We first show that $d(K) \leq d_{\text{trans}}(K)$.

Let $D_{\text{trans}}$ and $K_0$ be transverse overtwisted disk and transverse link which attain $d_{\text{trans}}(K)$. Therefore, $d_{\text{trans}}(K) = \#^-(K_0 \cap D_{\text{trans}})$. By the structural stability theorem [18 Theorem 2.21], we may assume that

(a) The characteristic foliation $\mathcal{F}_\xi(D_{\text{trans}})$ and the open book foliation $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$ are topologically conjugate.

Let $G_{++}(\mathcal{F}_\xi(D_{\text{trans}}))$ (resp. $G_{--}(\mathcal{F}_\xi(D_{\text{trans}}))$) be the Giroux graph [15 Page 646] consisting of the positive (resp. negative) elliptic points and the stable (resp. unstable) separatrices of positive (resp. negative) hyperbolic points. By the assumption (a), these graphs are identified with the corresponding graph $G_{++}$ and $G_{--}$ in the open book foliation $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$ see [18 Definition 2.17] for the definitions.

Take small neighborhoods $N_+, N_- \subset D_{\text{trans}}$ of the graphs $G_{++}(\mathcal{F}_\xi(D_{\text{trans}}))$ and $G_{--}(\mathcal{F}_\xi(D_{\text{trans}}))$, respectively. By transverse isotopy we move $K_0$ without introducing new intersection points with $D_{\text{trans}}$ so that:
(b) The intersection $K_0 \cap D_{\text{trans}}$ is disjoint from the region $N_+ \cup N_-$. We apply Giroux elimination lemma [14, Lemma 3.3] to remove all the positive elliptic and positive hyperbolic points of $F_\xi(D_{\text{trans}})$ (see Figure 7). Call the resulting disk $D'$. By (a) and the definition of a transverse overtwisted disk, the characteristic foliation $F_\xi(D')$ has a unique negative elliptic point enclosed by a circle leaf. We can find a usual overtwisted disc $D \subset D'$. Since the Giroux elimination is supported on $N_+ \cup N_-$, the condition (b) implies that this process does not produce new intersections, i.e., $K_0 \cap D_{\text{trans}} = K_0 \cap D'$.

![Figure 7. (Left) From a transverse overtwisted disk to a usual overtwisted disk. The graphs $G_{++}$ and $G_{--}$ are depicted by black and gray bold lines, respectively. A dot $\circ$ represents an intersection of $K$ and $D_{\text{trans}}$ which is moved away from the gray regions before applying the Giroux elimination lemma to the gray regions. (Right) Disk $D'$ and an overtwisted disk $D'$ (highlighted in gray).](image)

The proof of [1, Theorem 4.1.4] shows that every positive intersection of a Legendrian link and a (usual) overtwisted disk can be removed by a negative stabilization of the Legendrian link. Also the set of transverse links up to transverse isotopy is naturally identified, through the positive transverse push-off, with the set of Legendrian links up to Legendrian isotopy and negative stabilization [9, 11].

Therefore each positive intersection of $K_0$ and the overtwisted disk $D$ can be removed by a suitable transverse isotopy. That is, there exists a link $K_1$ that is transversely isotopic to $K_0$ such that $\#(K_1 \cap D) = \#(K_0 \cap D) = \#(K_0 \cap D')$. We conclude

$$d(K) \leq \#(K_1 \cap D) = \#(K_0 \cap D) \leq \#(K_0 \cap D') = \#(K_0 \cap D_{\text{trans}}) = d_{\text{trans}}(K).$$

Next we show that $d(K) \geq d_{\text{trans}}(K)$. Let $D$ be an overtwisted disc in $(M, \xi)$ that intersects $K$ at $d(K)$ points.

Take a slightly larger disc, $D'$, which contains $D$ in its interior and is bounded by a positive transverse push-off of the Legendrian unknot $\partial D$ so that $D' \cap K = D \cap K$.

Using transverse isotopy we make $K$ disjoint from the binding of the open book. Following Pavalescu’s proof of Alexander theorem [25, Theorem 3.2] one can find an isotopy of $M$ preserving each page of the open book set-wise and taking the non-braided part of $\partial D' \cup K$ (subsets which are not positively transverse to pages) into a neighborhood of the binding.

Inside the neighborhood of the binding we make $\partial D' \cup K$ braided with respect to the open book using [4]. We call the resulting link and disk $K'$ and $D''$, respectively. It is possible that new positive intersection points of $D''$ and $K'$ may be created if a component of $K$ is transversely isotopic to a binding component. However no new negative intersection points will be introduced. Hence $\#(K' \cap D'') = \#(K \cap D') \leq d(K)$.

Fixing $\partial D''$ and $K'$ and following the proof of [20, Theorem 3.3] we perturb $D''$ so that the resulting disk, $D'''$, admits an essential open book foliation. This process can be done without introducing new intersection points with $K'$ hence $\#(K' \cap D''') = \#(K' \cap D'')$. 

Since the Bennequin-Eliashberg inequality does not hold
\[ \text{sl}(\partial D'', [D'']) = \text{sl}(\partial D'', [D']) = \text{sl}(\partial D', [D]) = \text{tb}(\partial D, [D]) - \text{rot}(\partial D, [D]) = 1 \leq -\chi(D'') \]
we can apply the proof of [18, Theorem 4.3] to \( D'' \) and obtain a transverse overtwisted disc, \( D_{\text{trans}} \).

By the nature of this construction we have
\[
\begin{align*}
\#^- (K' \cap D_{\text{trans}}) &= \#^- (K' \cap D'') \\
\#^+ (K' \cap D_{\text{trans}}) &\geq \#^+ (K' \cap D'')
\end{align*}
\] (5.2)

where a strict inequality ‘\( > \)’ in (5.2) may hold only when a component of \( K' \) is transversely isotopic to a binding component. Summing up, we have
\[ d_{\text{trans}}(K) \leq \#^- (K' \cap D_{\text{trans}}) = \#^- (K' \cap D'') = \#^- (K \cap D') \leq d(K). \]

The following theorem characterizes depth-one links containing the binding.

**Theorem 5.2.** Let \((S, \phi)\) be an open book supporting \((M, \xi)\). Let \(B\) denote the binding of \((S, \phi)\) and \(L\) be a closed braid in \((S, \phi)\). Let \(K := B \cup L\) which is a transverse link in \((M, \xi)\). We have the depth \(d(K) = 1\) if and only if the braid \(L\) is non-quasi-right-veering.

**Proof.** (\(\Rightarrow\)) Suppose that the braid \(L\) is non-quasi-right-veering. As in the proof of Theorem 4.1 we can construct a transverse overtwisted disk with only one negative elliptic point in the complement of \(L\). By Theorem 5.1 we have \(d(K) \leq 1\). On the other hand, since the binding of any open book is non-loose [12] and \(K\) contains the binding \(B\) we have \(d(K) \geq d(B) \geq 1\).

\((\Rightarrow)\) Assume that \(d(K) = 1\). Let \(D\) be a transverse overtwisted disk in \((M, \xi)\) satisfying \(\#(K \cap D) = d(K) = 1\). Since the complement of the binding of a supporting open book decomposition is tight [12], \(\#(D \cap K) = \#(D \cap B) = 1\) and \(\#(D \cap L) = 0\).

Following the proof of Theorem 5.1 (the second half showing \(d(K) \geq d_{\text{trans}}(K)\)) we can construct starting from \(D\) a transverse overtwisted disk \(D_{\text{trans}}\) in the complement of \(L\) such that
\[ \#^- (K \cap D_{\text{trans}}) = \#^- (B \cap D_{\text{trans}}) = 1. \]

Let \(v \in B \cap D_{\text{trans}}\) denote the unique negative intersection point. That is, \(v\) is the unique negative elliptic point in the open book foliation \(F_{ob}(D_{\text{trans}})\) of \(D_{\text{trans}}\). Assume that \(v\) lies on a boundary component \(C\) of \(S\). For a regular page \(S_t\) of the open book let \(b_t \subset S_t\) denote the unique \(b\)-arc in \(F_{ob}(D_{\text{trans}})\) that ends at \(v\). We use \(v\) as the base point \(\ast_C\) of \(C\). Recall the projection map
\[ p : S \times [0, 1] \to S. \]

We view the image \(p(b_t)\) as an element of \(\mathcal{A}_C(S, P)\) where \(P = p(L \cap S_0)\) is a set of punctures given by the intersection of the braid \(L\) and the page \(S_0\).

Let \(S_{t_1}, \ldots, S_{t_k} \ (0 < t_1 < \cdots < t_k < 1)\) be the singular pages of the open book foliation \(F_{ob}(D_{\text{trans}})\) and \(\varepsilon > 0\) be a sufficiently small number such that \(S_{t_i}\) is the only singular page in the interval \((t_i - \varepsilon, t_i + \varepsilon)\). Since \(D_{\text{trans}}\) is a transverse overtwisted disk with one negative elliptic point, by the definition of a transverse overtwisted disk [18, Definition 4.1], all the hyperbolic points of \(F_{ob}(D_{\text{trans}})\) are positive. This shows that \(\pi(b_{t_i - \varepsilon}) \prec_{\text{right}} \pi(b_{t_i + \varepsilon})\) with \(\text{Int}(\pi(b_{t_i - \varepsilon})) \cap \text{Int}(\pi(b_{t_i + \varepsilon})) = \emptyset\) for all \(i = 1, \ldots, k\) (see Figure 8 (ii), or consult Observation 2.5 of [22]). Let us put
\[ \gamma_i := \pi(b_{t_i + \varepsilon}) = \pi(b_{t_{i+1} - \varepsilon}) \in \mathcal{A}_C(S; P). \]

Then the sequence of arcs satisfies
\[ \phi_L(\pi(b_1)) = \pi(b_0) = \gamma_0 \prec_{\text{right}} \gamma_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \gamma_k = \pi(b_1) \]
and \(\text{Int}(\gamma_i) \cap \text{Int}(\gamma_{i+1}) = \emptyset\); hence, \(\phi_L \in \text{MCG}(S, P)\) is not quasi-right-veering. \(\square\)
6. VERY POSITIVE FDTC AND NON-LOOSE LINKS

Proposition 3.14 and Theorem 4.1 show that a negative FDTC $c(\phi, L, C) < 0$ does not always imply looseness of the braid $L$. This makes a sharp contrast to empty braid case, where the negative FDTC $c(\phi, C) < 0$ implies that the contact structure $\xi(S, \phi)$ is overtwisted.

On the other hand, if the FDTC is very positive then there is some similarity between non-empty braid case and empty braid case. In [22, Corollary 1.2] it is proved that a planar open book $(S, \phi)$ with $c(\phi, C) > 1$ for every boundary component $C$ supports a tight contact structure. We may regard this as a special case ($L = \emptyset$) of the following theorem.

**Theorem 6.1.** Let $L$ be a closed braid in a planar open book $(S, \phi)$. If $c(L, \phi, C) > 1$ for every boundary component $C$ of $S$ then $L$ is non-loose.

**Proof.** By (2.2) the distinguished monodromy $\phi_L \in \mathcal{MCG}(S, P)$ gives

$$(S \setminus P) \times [0, 1]) / \sim_{\phi_L} \simeq M \setminus L.$$ 

Recall the forgetful map $f : \mathcal{MCG}(S, P) \to \mathcal{MCG}(S)$ in the Birman exact sequence (2.1). Note that $f(\phi_L) = \phi \in \mathcal{MCG}(S)$. In the following argument, we may use the open book $(S, P, \phi_L)$ instead of $(S, \phi)$.

Assume that $L$ is loose. By Theorem 5.1 there exists a transverse overtwisted disk $D$ in $M \setminus L$. Applying the proof of [22, Theorem 1.1], we can construct a transverse overtwisted disk $D'$ such that every $b$-arc of $\mathcal{F}_{ob}(D')$ ending at a valence $\leq 1$ vertex of the graph $G_{--}(D')$ is an essential arc in the punctured page $S \setminus P$. Using [20, Lemma 5.7] the existence of such a disk $D'$ implies that $c(\phi, L, C) = c(\phi_L, C) \leq 1$ for some boundary component $C$ of $S$. \qed

7. COMPARISON OF PROPOSED DEFINITIONS OF RIGHT-VHEELINGNESS

In this section we discuss a comparison of several proposed definitions of right-veering for the mapping class group of punctured surfaces.
Definition 7.1. We say that an arc \( \gamma : [0, 1] \to S \) is \( \partial P \) (resp. \( \partial \partial \)) arc if the following are all satisfied:

1. \( \gamma(0) \in \partial S \) and \( \gamma \) is transverse to \( \partial S \) at \( \gamma(0) \).
2. \( \gamma(t) \in \text{Int}(S) \setminus P \) for \( t \in (0, 1) \).
3. \( \gamma(1) \in P \) (resp. \( \gamma(1) \in \partial S \) and \( \gamma \) is transverse to \( \partial S \) at \( \gamma(1) \)).
4. \( \text{Int}(\gamma) \) is embedded in \( S \setminus P \) and not boundary-parallel.

For a boundary component \( C \) of \( S \), we say that a \( \partial P \) or \( \partial \partial \) arc is based on \( C \) if \( \gamma(0) \in C \).

As natural generalizations of the right-veering property for \( \text{MCG}(S) \) to \( \text{MCG}(S, P) \) there are three candidates.

Definition 7.2. For a boundary component \( C \) of \( S \) we say that \( \psi \in \text{MCG}(S, P) \) is

1. \( \partial \partial (\partial + P) \) right-veering with respect to \( C \) if \( \gamma \prec_{\text{right}} \psi(\gamma) \) or \( \gamma = \psi(\gamma) \) for all \( \partial \partial \) and \( \partial P \) arcs \( \gamma \) based on \( C \).
2. \( \partial \partial \) right-veering with respect to \( C \) if \( \gamma \prec_{\text{right}} \psi(\gamma) \) or \( \gamma = \psi(\gamma) \) for all \( \partial \partial \) arcs \( \gamma \) based on \( C \).
3. \( \partial P \) right-veering with respect to \( C \) if \( \gamma \prec_{\text{right}} \psi(\gamma) \) or \( \gamma = \psi(\gamma) \) for all \( \partial P \) arcs \( \gamma \) based on \( C \).

We say that \( \psi \in \text{MCG}(S, P) \) is \( \partial \partial (\partial + P) \), \( \partial \partial \), or, \( \partial P \) right-veering, respectively, if \( \psi \) is \( \partial \partial (\partial + P) \), \( \partial \partial \), or, \( \partial P \) right-veering, respectively, with respect to every boundary component of \( S \).

The \( \partial \partial \) right-veering appears in [3]. It is easy to see that our Definition 3.2 of right-veering is equivalent to the \( \partial \partial \) right-veering. Recall that in Definition 3.2 we only consider \( \partial \partial \) arcs starting from the distinguished base point \( *_C \in C \). This restriction is just to define the orderings \( \prec_{\text{right}} \) and \( \ll_{\text{right}} \) on \( \text{MCG}(S; \partial P) \).

On the other hand, in [2] [25] the notion of \( \partial P \) right-veering is used to study the classical braid group \( \text{MCG}(D^2, P) \).

It is asked in [2] Remark 3.3 whether these two superficially different notions of “right-veering” are equivalent or not. One can immediately see that these notions (2) and (3) of “right-veering with respect to \( C \)” are in general not exactly the same.

Example 7.3. Assume that \( S \) has more than one boundary component with marked points \( P \neq \emptyset \). Let \( C \) and \( C' \) be distinct boundary components. Clearly \( T_{C'}^{-1} \in \text{MCG}(S, P) \) is not \( \partial \partial \) right-veering with respect to \( C \). On the other hand \( T_{C'}^{-1} \) preserves all \( \partial P \) arcs based on \( C \). This means that \( T_{C'}^{-1} \) is \( \partial P \) right-veering with respect to \( C \).

More generally we have the following. Let \( \psi \in \text{MCG}(S, P) \) be a \( \partial P \) right-veering map with respect to \( C \). Suppose that \( \psi(\gamma) = \gamma \) for some \( \partial \partial \) arc \( \gamma \) connecting distinct \( C \) and \( C' \). Then \( T_{C'}^{-1} \psi \) is still \( \partial P \) right-veering with respect to \( C \), but is not \( \partial \partial \) right-veering with respect to \( C \) since \( T_{C'}^{-1} \psi(\gamma) = T_{C'}^{-1}(\gamma) \prec_{\text{right}} \gamma \).

It turns out that the difference between \( \partial \partial \) right-veering and \( \partial P \) right-veering only shows up when \( \psi \in \text{MCG}(S, P) \) involves negative Dehn twists along boundary components like in Example 7.3.

Definition 7.4. We say that \( \psi \in \text{MCG}(S, P) \) is special with respect to \( C \) if the following two conditions are satisfied:

- \( \psi \) is not \( \partial \partial \) right-veering with respect to \( C \).
- If a \( \partial \partial \) arc \( \gamma \) that is based on \( C \) and ending at \( C' \) has \( \psi(\gamma) \prec_{\text{right}} \gamma \) then \( C' \neq C \) and \( \psi(\gamma) = T_{C'}^{-n}(\gamma) \) for some \( n > 0 \).
That is, a special map \( \psi \) is not \( \partial \partial \) right-veering with respect to \( C \) only because of negative Dehn twists about some other boundary component \( C' \).

**Theorem 7.5.** Let \( \psi \in \mathcal{MCG}(S, P) \).

1. If \( \psi \) is \( \partial \partial \) right-veering with respect to \( C \), then \( \psi \) is \( \partial P \) right-veering with respect to \( C \).
2. If \( \psi \) is \( \partial P \) right-veering with respect to \( C \), then either
   - \( \psi \) is \( \partial \partial \) right-veering with respect to \( C \), or,
   - \( \psi \) is special with respect to \( C \).

**Proof.** We prove both (1) and (2) by showing the contrapositives.

First we prove (1). Assume that there is a \( \partial P \) arc \( \kappa \) based on \( C \) with \( \psi(\gamma) \prec_{\text{right}} \gamma \). Let \( \kappa \) be a properly embedded arc which is the boundary of a regular neighborhood of \( \gamma \) in \( S \). Then we see that \( \kappa \) is a \( \partial \partial \) arc with \( \psi(\kappa) \prec_{\text{right}} \kappa \).

To see (2), assume that \( \psi \) is not \( \partial \partial \) right-veering with respect to \( C \) and is not special with respect to \( C \). Then there exists a \( \partial \partial \) arc \( \gamma \) based on \( C \) such that \( \psi(\gamma) \prec_{\text{right}} \gamma \). We put \( \psi(\gamma) \) and \( \gamma \) so that they intersect efficiently. Our goal is to show that there exits a \( \partial P \) arc \( \kappa \) based on \( C \) with \( \kappa(0) = \gamma(0) \) and

\[
\psi(\gamma) \prec_{\text{right}} \kappa \prec_{\text{right}} \gamma.
\]

This shows \( \psi(\kappa) \prec_{\text{right}} \psi(\gamma) \prec_{\text{right}} \kappa \); hence, \( \psi \) cannot be \( \partial P \) right-veering with respect to \( C \).

If \( \#(\gamma, \psi(\gamma)) = m > 0 \), we put \( \text{Int}(\gamma) \cap \text{Int}(\psi(\gamma)) = \{p_1, \ldots, p_m\} = \{q_1, \ldots, q_m\} \), where \( p_i = \gamma(t_i) \) with \( 0 < t_1 < t_2 < \cdots < t_m < 1 \) and \( q_i = (\psi(\gamma))(s_i) \) with \( 0 < s_1 < s_2 < \cdots < s_m < 1 \). If \( \#(\gamma, \psi(\gamma)) = m = 0 \) we put \( t_1 = s_1 = 1 \) and \( p_1 = q_1 = \gamma(1) \).

Suppose that \( q_1 = p_k \). Let

\[
\delta := \gamma|_{[0,t_k]} \ast (\psi(\gamma)|_{[0,s_1]})^{-1}
\]

then \( \delta \) is an oriented simple closed curve in \( S \setminus P \). Here \( \ast \) denotes concatenation of paths read from left to right, and \( (-)^{-1} \) means the arc with reversed orientation. If \( \delta \) is separating, we denote by \( R \) the connected component of \( S \setminus (\delta \cup P) \) that lies on the left side of \( \delta \) with respect to the orientation of \( \delta \). If \( \delta \) is non-separating let \( R := S \setminus (\delta \cup P) \).

**Definition 7.6.** We say that the arc \( \gamma \) is bad if the following two properties are satisfied:

- \( R \) is an annulus (possibly a pinched annulus if \( m = 0 \)) with no punctures. (In particular, \( \delta \) is separating.)
- The sign of the intersection of \( \gamma \) and \( \psi(\gamma) \) (in this order) at \( q_1 \) is positive.

Assume that \( \gamma \) is bad. Let \( C' = \partial R \setminus \delta \). Note that \( C' \) is a boundary component of \( S \). Since \( \gamma \) and \( \psi(\gamma) \) intersect efficiently and \( \delta \) is separating, we can see that \( \psi(\gamma) \) cannot exit out of the annulus \( R \) and \( C \neq C' \). See Figure 9. Therefore we have:

**Claim 7.7.** If \( \gamma \) is bad then \( C' \neq C \) and \( \psi(\gamma) = T_{C'}^{-n}(\gamma) \) for some \( n > 0 \).

![Figure 9. A bad arc \( \gamma \) and its image \( \psi(\gamma) \).](image-url)
Since we assume that $\psi$ is not $\partial$-$\partial$ right-veering with respect to $C$ and is not special with respect to $C$, Claim 7.7 implies that $\gamma$ is not bad.

Knowing that $\gamma$ is not bad, we consider two cases to construct $\kappa$:

**Case 1:** $R$ is an annulus with punctures or a non-annulus surface with or without punctures.

The sign of the intersection of $\gamma$ and $\psi(\gamma)$ at $q_1$ can be either positive or negative. Take an arc $\gamma'$ in $S \setminus (P \cup \gamma \cup \delta)$ which connects $q_1$ and some puncture point and efficiently intersects $\psi(\gamma)|_{[s_1,1]}$.

**Case 1A:** There exists such an arc $\gamma'$ which lies on the left side of $\gamma$ near $q_1$.

In this case, define $\kappa := \gamma|_{[0,t_k]} * \gamma'$.

**Case 1B:** No such arc can exist on the left side of $\gamma$ near $q_1$, so $\gamma'$ lies on the right side of $\gamma$ near $q_1$.

If $R$ contains punctures then let $\kappa \subset R$ be an arc connecting $\gamma(0)$ and one of the punctures in $R$ and satisfying $\psi(\gamma) \prec_{\text{right}} \kappa \prec_{\text{right}} \gamma$ and $\text{Int}(\kappa) \cap \delta = \emptyset$. (We do not use $\gamma'$ here.)

Now we may assume that $R$ is a non-annular surface with no punctures. We can take an arc $\gamma''$ in $R \setminus (R \cap \gamma')$ such that:

- $\gamma''(0) = \gamma''(1) = \gamma(0)$.
- $\psi(\gamma) \prec_{\text{right}} \gamma'' \prec_{\text{right}} \gamma$.
- $\text{Int}(\gamma'') \cap \delta = \emptyset$.
- $\gamma''$ is not parallel to $\delta$.
- $\gamma''$ and $\gamma$ efficiently intersect.

Let $q'' := \gamma''(u) = \gamma(t) \in \gamma'' \cap \gamma$ be the intersection point such that $\gamma''|_{[0,u]}$ is disjoint from $\gamma$. If $\text{Int}(\gamma'') \cap \text{Int}(\gamma) = \emptyset$ then we take $q'' := \gamma''(1) = \gamma(0)$. Namely, $u = 1$ and $t = 0$. Define:

$$
\kappa := \begin{cases} 
\gamma''|_{[0,u]} * \gamma|_{[t_k,t]} * \gamma' & \text{if } t < t_k \\
\gamma''|_{[0,1]} * \gamma|_{[t_k,t]} & \text{if } t_k < t 
\end{cases}
$$

![Figure 10. Case 1. A $\partial$-$P$ arc $\kappa$ (dashed arc) is chosen so that it does not intersect $\gamma$ (black bold line) and $\psi(\gamma)|_{[0,s_1]}$ (gray bold arc), possibly with one exceptional point $q_1$.](image)

**Case 2:** $R$ is an annulus with no punctures, and the sign of the intersection of $\gamma$ and $\psi(\gamma)$ at $q_1$ is negative.

Let $k' (\neq k)$ be the number satisfying $q_2 = p_{k'}$.

**Case 2A:** $k' < k$.

Since $\delta$ is separating the sign of the intersection of $\gamma$ and $\psi(\gamma)$ at $q_2$ is positive. Take an arc $\gamma'$ in $S \setminus (P \cup \gamma \cup \psi(\gamma)|_{[0,s_2]})$ which connects $q_2$ and a puncture point and efficiently intersects $\psi(\gamma)|_{[s_2,1]}$. Then put $\kappa := \psi(\gamma)|_{[0,s_1]} * (\gamma|_{[t_k,t_s]})^{-1} * \gamma'$.

**Case 2B:** $k < k'$. 
Let $\gamma'$ be an arc in $S \setminus (P \cup \gamma \cup \psi(\gamma)|_{[0,s_2]})$ that connects $\gamma(0)$ and a puncture point. Put

\[ \kappa := \begin{cases} 
\gamma|_{[0,t_{\kappa}]} \ast (\psi(\gamma)|_{[s_1,s_2]})^{-1} \ast (\gamma|_{[0,s_1]})^{-1} \ast \gamma' & \text{if } \gamma \prec_{\text{right}} \gamma' \\
\gamma|_{[0,t_{\kappa}]} \ast (\psi(\gamma)|_{[s_1,s_2]})^{-1} \ast C \ast (\gamma|_{[0,s_1]})^{-1} \ast \gamma' & \text{if } \gamma' \prec_{\text{right}} \gamma 
\end{cases} \]

In the second case in order to make $\kappa$ embedded it turns along $C$. □

![Figure 11. Case 2. Construction of a $\partial$-$P$ arc $\kappa$ (dashed). $\kappa$ does not intersect $\gamma$ (black bold line) and $\psi(\gamma)|_{[0,s_2]}$ (gray bold arc), possibly with exceptions near $q_1$, $q_2$ and $\gamma(1)$ (if $\gamma(1) \in C$).](image)

As a consequence of Theorem 7.5, the three notions of right-veering with respect to all the boundary components, which is a condition closely related to tight contact structures, are equivalent. In particular, if $S$ has connected boundary then the three notions are equivalent.

**Corollary 7.8.** For $\psi \in \text{MCG}(S, P)$ the following are equivalent.

1. $\psi$ is $\partial$-$\partial + P$ right-veering.
2. $\psi$ is $\partial$-$\partial$ right-veering.
3. $\psi$ is $\partial$-$P$ right-veering.

Therefore in the case of $B_n = \text{MCG}(D^2, \{n \text{ points}\})$ the proposed definitions of right-veering in [3] and [2, 25] are the same. Also, we remark that the subtle difference between $\partial$-$P$ right-veering with respect to $C$ and $\partial$-$\partial$ right-veering with respect to $C$ (existence of a special mapping class $\psi$) only occurs when $c(\psi,C) = 0$.

**Remark 7.9.** One may come up with still different candidates of right-veering. Instead of using embedded arcs, one may use immersed arcs. However, one can check that immersed $\partial$-$\partial + P$ (resp. $\partial$-$\partial$, $\partial$-$P$) right-veering with respect to $C$ is equivalent to the (embedded) $\partial$-$\partial + P$ (resp. $\partial$-$\partial$, $\partial$-$P$) right-veering with respect to $C$.

**Acknowledgements**

The authors would like to thank John Etnyre for many useful comments including one on Proposition 2.7. TI was partially supported by JSPS Grant-in-Aid for Research Activity start-up, Grant Number 25887030. KK was partially supported by NSF grant DMS-1206770 and Simons Foundation Collaboration Grants for Mathematicians.

**References**

[1] K. Baker and S. Onaran, *Nonlooseness of nonloose knots*. Algebr. Geom. Topol. 15 (2015), no. 2, 1031-1066.
[2] J. Baldwin and E. Grigsby, *Categorified invariants and the braid group* Proc. Amer. Math. Soc. 143 (2015), 2801–2814.
[3] J. Baldwin, S. Vela-Vick and V. Vertesi, *On the equivalence of Legendrian and transverse invariants in knot Floer homology* Geom. Topol. 17 (2013), 925–974.
[4] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque, 107-108, (1983) 87-161.
[5] M. Casey, *Branched covers of contact manifolds*, Ph.D. thesis, Georgia Institute of Technology, 2013.
[6] V. Colin and K. Honda, *Reeb vector fields and open book decompositions*, J. Eur. Math. Soc. **15** (2013), 443-507.

[7] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623-637.

[8] Y. Eliashberg and M. Fraser, *Topologically trivial Legendrian knots*, J. Symplectic Geom. **7** (2009), 77-127.

[9] J. Epstein, D. Fuchs, and M. Meyer, *Chekanov-Eliashberg invariants and transverse approximations of Legendrian knots*, Pacific J. Math. **201** (2001), 89–106.

[10] J. Etnyre, *On knots in overtwisted contact structures*, Quantum Topol. **4** (2013), 229–264.

[11] J. Etnyre and K. Honda, *Knots and contact geometry. I. Torus knots and the figure eight knot*, J. Symplectic Geom. **1** (2001), 63–120.

[12] J. Etnyre and S. Vela-Vick, *Torsion and open book decompositions*, Int. Math. Res. Not. IMRN **22** (2010), 4385–4398.

[13] F. Farb and D. Margalit, *A primer on mapping class groups*. Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472 pp.

[14] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66** (1991), no. 4, 637-677.

[15] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematics, vol. II (Beijing, 2002), 405-414.

[16] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. math. **169**, No.2 (2007), 427-449.

[17] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary II*, Geom. Topol. **12** (2008), no. 4, 2057–2094.

[18] T. Ito and K. Kawamuro, *Open book foliations*, Geom. Topol. **18** (2014) 1581-1634.

[19] T. Ito and K. Kawamuro, *Visualizing overtwisted discs in open books*, Publ. Res. Inst. Math. Sci. **50** (2014) 169–180.

[20] T. Ito and K. Kawamuro, *Essential open book foliation and fractional Dehn twist coefficient*, Geom. Dedicata **187** (2017), 17–67.

[21] T. Ito and K. Kawamuro, *Operations on open book foliations*, Algebr. Geom. Topol. **14** (2014), 2983–3020.

[22] T. Ito and K. Kawamuro, *Overtwisted discs in planar open books*, Internat. J. Math. **26** (2015) 1550027, 29 pp.

[23] T. Ito and K. Kawamuro, *Coverings of open books*, Advances in the Mathematical Sciences, Research from the 2015 Association for Women in Mathematics Symposium, Springer, (2016) arXiv:1509.00352

[24] F. Öztürk and K. Niederkrüger, *Brieskorn manifolds as contact branched covers of spheres*, Period. Math. Hungar. **54** (2007), 85–97.

[25] E. Pavelescu, *Braiding knots in contact 3-manifolds*, Pacific J. Math. **253** (2011), 475–487.

[26] O. Plamenevskaya, *Transverse invariants and right-veering*, arXiv. 1509.01732.

[27] R. Winarski, *Symmetry, isotopy, and irregular covers*. Geom. Dedicata, 177:213-227, 2015.

Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama Toyonaka, Osaka 560-0043, JAPAN

E-mail address: tetito@math.sci.osaka-u.ac.jp

URL: http://www.math.sci.osaka-u.ac.jp/~tetito/

Department of Mathematics, The University of Iowa, Iowa City, IA 52242, USA

E-mail address: keiko-kawamuro@uiowa.edu