A new approach to analysis of 2D higher order quantum superintegrable systems

Bjorn K. Berntson¹, Ian Marquette², and Willard Miller, Jr.³

¹ Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden
bbernts@kth.se

² School of Mathematics and Physics, The University of Queensland, Brisbane, Australia
i.marquette@uq.edu.au, WWW home page: https://smp.uq.edu.au/profile/211/ian-marquette

³ School of Mathematics, University of Minnesota, Minneapolis, Minnesota, U.S.A.
mille003@math.umn.edu, WWW home page: http://www-users.math.umn.edu/mille003

Abstract. We revise a method by Kalnins, Kress and Miller (2010) for constructing a canonical form for symmetry operators of arbitrary order for the Schrödinger eigenvalue equation $H\Psi \equiv (\Delta_2 + V)\Psi = E\Psi$ on any 2D Riemannian manifold, real or complex, that admits a separation of variables in some orthogonal coordinate system. Most of this paper is devoted to describing the method. Details will be provided elsewhere. As examples we revisit the Tremblay and Winternitz derivation of the Painlevé VI potential for a 3rd order superintegrable flat space system that separates in polar coordinates and, as new results, we show that the Painlevé VI potential also appears for a 3rd order superintegrable system on the 2-sphere that separates in spherical coordinates, as well as a 3rd order superintegrable system on the 2-hyperboloid that separates in spherical coordinates and one that separates in horocyclic coordinates. The purpose of this project is to develop tools for analysis and classification of higher order superintegrable systems on any 2D Riemannian space, not just Euclidean space.

Keywords: quantum superintegrable systems, Painlevé VI equation, Weierstrass equation

1 Introduction

In the paper [1] the authors constructed a canonical form for symmetry operators of any order in 2D and used it to give the first proof of the superintegrability of the quantum Tremblay, Turbiner, and Winternitz (TTW) system [2] in polar coordinates, for all rational values of the parameter $k$. In the original method the various potentials were given and the problem was the construction of higher order symmetry operators that would verify superintegrability. The method was
highly algebraic and required the solution of systems of difference equations on a lattice. Here, we consider an arbitrary space admitting a separation in some orthogonal coordinate system (hence admitting a 2nd order symmetry operator), and search for all potentials $V$ for which the Schrödinger equation admits an additional independent symmetry operator of order higher than 2. Now the problem reduces to solving a system of partial differential equations.

We give a brief introduction to the method and then specialize it to 3rd order superintegrable systems where we treat a few examples. We revisit the Tremblay and Winternitz derivation of the Painlevé VI potential for a 3rd order superintegrable flat space system that separates in polar coordinates, and we show among other new results that the Painlevé VI potential also appears for a 3rd order superintegrable system on the 2-sphere that separates in spherical coordinates, as well as a 3rd order superintegrable system on the 2-hyperboloid that separates in spherical coordinates.

2 The canonical form for a symmetry operator

We consider a Schrödinger equation on a 2D real or complex Riemannian manifold with Laplace-Beltrami operator $\Delta_2$ and potential $V$:

$$H\Psi \equiv \left( -\frac{\hbar^2}{2} \Delta_2 + V \right) \Psi = E\Psi$$ (1)

that also admits an orthogonal separation of variables. If $\{u_1, u_2\}$ is the orthogonal separable coordinate system the corresponding Schrödinger operator can always be put in the form

$$H = -\frac{\hbar^2}{2} \Delta_2 + V(u_1, u_2) = \frac{1}{f_1(u_1) + f_2(u_2)} \left( -\frac{\hbar^2}{2} \partial^2_{u_1} - \frac{\hbar^2}{2} \partial^2_{u_2} + v_1(u_1) + v_2(u_2) \right)$$ (2)

and, due to the separability, there is the second-order symmetry operator

$$L_2 = \frac{f_2(u_2)}{f_1(u_1) + f_2(u_2)} \left( -\frac{\hbar^2}{2} \partial^2_{u_1} + v_1(u_1) \right) - \frac{f_1(u_1)}{f_1(u_1) + f_2(u_2)} \left( -\frac{\hbar^2}{2} \partial^2_{u_2} + v_2(u_2) \right),$$

i.e., $[H, L_2] = 0$. We look for a partial differential symmetry operator of arbitrary order $\tilde{L}(H, L_2, u_1, u_2)$ that satisfies

$$[H, \tilde{L}] = 0.$$ (3)

We require that the symmetry operator take the standard form

$$\tilde{L} = \sum_{j,k} \left( A^{j,k}(u_1, u_2) \partial_{u_1 u_2} - B^{j,k}(u_1, u_2) \partial_{u_1} - C^{j,k}(u_1, u_2) \partial_{u_2} + D^{j,k}(u_1, u_2) \right) H^j L_2^k.$$ (4)

This can always be done. Note that if the formal operator $\tilde{L}$ contained partial derivatives in $u_1$ and $u_2$ of orders $\geq 2$ we could rearrange terms to achieve the unique standard form (4).
Details of the derivation can be found in [1].

Note that condition (4) makes sense, at least formally, for infinite order differential equations. Indeed, one can consider \( H, L_2 \) as parameters in these equations. Then once \( \hat{L} \) is expanded as a power series in these parameters, the terms are reordered so that the powers of the parameters are on the right, before they are replaced by explicit differential operators. Of course (4) is defined rigorously for finite order symmetry operators.

In this view we can write

\[
\hat{L}(H, L_2, u_1, u_2) = A(u_1, u_2)\partial_{u_1}u_2 - B(u_1, u_2)\partial_{u_1} - C(u_1, u_2)\partial_{u_2} + D(u_1, u_2),
\]

and consider \( \hat{L} \) as an at most second-order order differential operator in \( u_1, u_2 \) that is analytic in the parameters \( H, L_2 \). Then the integrability condition for (7), (8) is (with the shorthand \( \hat{L} \))

\[
\frac{\hbar^2}{2} (\partial_{u_1}^2 B - \partial_{u_2}^2 B) - 2\partial_{u_2} A v_2 - \hbar^2 \partial_{u_2} D - Av_2' + (2\partial_{u_2} A f_2 + Af_2') H - 2\partial_{u_2} A L_2 = 0, \tag{7}
\]

\[
\frac{\hbar^2}{2} (\partial_{u_1}^2 C + \partial_{u_2}^2 C) - 2\partial_{u_2} Av_1 - \hbar^2 \partial_{u_2} D - Av_1' + (2\partial_{u_2} A f_1 + Af_1') H + 2\partial_{u_2} A L_2 = 0, \tag{8}
\]

\[
-\frac{\hbar^2}{2} (\partial_{u_1}^2 D + \partial_{u_2}^2 D) + 2\partial_{u_1} B v_1 + 2\partial_{u_2} C v_2 + Bv_1' + Cv_2'
\]

\[= (2\partial_{u_1} B f_1 + 2\partial_{u_2} C f_2 + Bf_1' + Cf_2') H + (-2\partial_{u_1} B + 2\partial_{u_2} C) L_2 = 0. \tag{9}\]

We can view (6) as an equation for \( A, B, C \) and (7), (8) as the defining equations for \( \partial_{u_1} D, \partial_{u_2} D \). Then \( \hat{L} \) is \( L \) with the terms in \( H \) and \( L_2 \) interpreted as (4) and considered as partial differential operators.

We can simplify this system by noting that there are two functions \( F(u_1, u_2, H, L_2) \), \( G(u_1, u_2, H, L_2) \) such that (6) is satisfied by

\[
A = F, \quad B = \frac{1}{2} \partial_{u_2} F + \partial_{u_1} G, \quad C = \frac{1}{2} \partial_{u_1} F - \partial_{u_2} G, \quad \tag{10}
\]

Then the integrability condition for (7), (8) is (with the shorthand \( \partial_{u_j} F = F_j, \partial_{u_j} \partial_{u_j} F = F_{jj}, \) etc., for \( F \) and \( G \)),

\[
-\hbar^2 G_{1222} = \frac{1}{4} \hbar^2 F_{2222} + 2F_{22}(v_2 - f_2 H + L_2) + 3F_2(v_2' - f_2' H) + F(v_2'' - f_2'' H) = \frac{1}{4} \hbar^2 F_{1111} + 2F_{11}(v_1 - f_1 H - L_2) + 3F_1(v_1' - f_1' H) + F(v_1'' - f_1'' H), \tag{11}
\]

and equation (9) becomes

\[
\frac{1}{4} \hbar^2 F_{1112} - 2F_{12}(v_1 - f_1 H) - F_1(v_2' - f_2' H) + \frac{1}{4} \hbar^2 G_{1111} - 2G_{11}(v_1 - f_1 H - L_2)
- G_1(v_1' - f_1' H) = -\frac{1}{4} \hbar^2 F_{2222} + 2F_{22}(v_2 - f_2 H) + F_2(v_1' - f_1' H) + \frac{1}{4} \hbar^2 G_{2222} - 2G_{22}(v_2 - f_2 H + L_2) - G_2(v_2' - f_2' H). \tag{12}\]

We remark that any solution of (11), (12) with \( A, B, C \) not identically 0 corresponds to a symmetry operator that does not commute with \( L_2 \), hence is algebraically independent of the symmetries \( H, L_2 \).
3 3rd order superintegrability

To illustrate how equations (11) and (12) can be used to find potentials for superintegrable systems, we provide detailed derivations of the determining equations for 3rd order superintegrability. First, we note that the most general 3rd order operator must be of the form (11) with

\[ A = A^0(x, y), \quad B = B^0(x, y) + B^H(x, y)H + B^L(x, y)L, \]
\[ C = C^0(x, y) + C^H(x, y)H + C^L(x, y)L, \quad D = D^0(x, y) + D^H(x, y)H + D^L(x, y)L, \]

or, in view of (10),

\[ F(x, y) = F^0(x, y), \quad G(x, y) = G^0(x, y) + G^H(x, y)H + G^L(x, y)L. \]  

Substituting (13) into (11), (12) and noting that the coefficients of independent powers of \( H \) and \( L \) in these expressions must vanish, we obtain 9 equations, (the first 3 from (11) and the next 6 from (12)):

\[
0 = -6v_1F_1^0 + 6v_2F_2^0 - 4v_1F_{11}^0 + 4v_2F_{22}^0 - 2h^2G_{1112}^0 - 2h^2G_{1222}^0 + 2F^0v_2^0 - 2F^0v_1^0, \\
0 = F_1^0 + F_2^0, \\
0 = -h^2G_{1112}^H - h^2G_{1222}^H + 3f_1F_1^0 - 3f_1F_2^0 + 2f_1F_{11}^0 - 2f_2F_{22}^0 - F_0f_1^0 + F_0f_2^0, \\
0 = v_1F_1^0 + v_1F_2^0 + v_1G_1^0 - v_2G_2^0 + 2F_{12}v_2 + 2F_{12}v_1 + 2v_1G_{11}^0 - 2v_2G_{22}^0 - \frac{1}{4}h^2G_{1111}^0 + \frac{1}{4}h^2G_{2222}^0, \\
0 = v_1G_1^L - v_2G_2^L + 2v_1G_{11}^L - 2G_{11}^0 - 2v_2G_{22}^L - 2G_{22}^0, \\
0 = G_{11}^L + G_{22}^L, \\
0 = -f_1F_1^0 - f_2F_2^0 + v_1G_1^H - f_1G_1^0 - v_2G_2^H + f_2G_2^0 - 2F_{12}f_2 - 2F_{12}f_1 + 2v_1G_{11}^H - 2f_2G_{22}^H + 2f_2G_{22}^0 - \frac{1}{4}h^2G_{1111}^H + \frac{1}{4}h^2G_{2222}^H, \\
0 = -f_1G_1^L + f_2G_2^L + 2f_2G_{22}^L - 2f_1G_{11}^L - 2G_{11}^H - 2G_{22}^H, \\
0 = -f_1G_1^L + f_2G_2^L + 2f_2G_{22}^L - 2f_1G_{11}^L.
\]

4 Some examples (mostly new)

We are particularly interested in potentials with nonlinear defining equations. First, we show that we get the result of Tremblay and Winternitz [3] that the quantum system separating in polar coordinates in 2D Euclidean space admits potentials that are expressed in terms of the sixth Painlevé transcendent or in terms of the Weierstrass elliptic function. To do this we must put the system in the canonical form (2). The separable polar coordinates are \((r \cos(\theta), r \sin(\theta))\). For the canonical form we use the coordinates \(\{u_1, u_2\}\), where
Quantum superintegrable systems

\[ r = \exp(u_1), \quad \theta = u_2. \] Thus, \( f_1(u_1) = \exp(2u_1) \) and \( f_2(u_2) = 0 \). We know that these extreme potentials can appear only if the potential depends on the angular variable alone, so we set \( v_1(u_1) = 0 \). Since we want only systems that satisfy nonlinear equations alone, whenever an explicit linear equation for the potential appears, we require that it vanish identically.

We obtain a solution

\[
F^0 = 4h^2 \exp(-u_1) \sin(u_2), \quad G^L = -8 \exp(-u_1) \cos(u_2) + a_4 u_2 + a_3, \\
G^0 = -U_1(u_2) \exp(-u_1) + U_2(u_2), \quad G^H = a_5,
\]

subject to the conditions

\[
0 = a_4 \frac{d^2}{du_2^2} + 2 \frac{d^3}{du_2^3}, \tag{14}
\]
\[
0 = h^2 \frac{d^4}{du_2^4} + 4a_4 \frac{d^3}{du_2^3} - 4 \frac{d^2}{du_2^2} - U_1, \tag{15}
\]
\[
0 = 8v_2 \cos(u_2) + 4 \frac{d^2}{du_2^2} \sin(u_2) - \frac{d^2}{du_2^2}, \tag{16}
\]
\[
0 = \frac{d^2}{du_2^2} - 4h^2 \frac{d^3}{du_2^3} \sin(u_2) - 4h^2 \frac{d^2}{du_2^2} \cos(u_2), \tag{17}
\]
\[
+ 2 \sin(u_2) \left( h^2 + 4v_2 \frac{d}{du_2} - 2v_2 \left( 2h^2 \cos(u_2) - 8v_2 \cos(u_2) + U_1 \right) \right).
\]

There are basically two cases to consider:

1. \( a_4 = 0 \).

Then condition (14) says that \( U_2 \) is linear in \( u_2 \). Thus condition (15) is a linear equation for \( u_2 \) which must vanish. Then condition (16) can be solved for \( U_1(u_2) \) and the result substituted into condition (17) to obtain an equation for \( v_2(u_2) \). After some manipulation (using the fact that \( V_2 \) is unchanged under transformations \( W \to W + c \), where \( c \) is a constant), we obtain an equation characterizing Painlevé VI, in agreement with [3], equation (4.27):

\[
\hbar^2 \left( \sin(u_2) \frac{d^4}{du_2^4} + 4 \cos(u_2) \frac{d^3}{du_2^3} - 6 \sin(u_2) \frac{d^2}{du_2^2} - 4 \cos(u_2) \frac{d}{du_2} \right), \tag{18}
\]

\[
-12 \sin(u_2) \frac{d^2}{du_2^2} - 4 \cos(u_2) W \frac{d^2}{du_2^2} - 4(\beta_1 \sin(u_2) - \beta_2 \cos(u_2)) \frac{d^2}{du_2^2},
\]

\[
-16 \cos(u_2) \left( \frac{d}{du_2} \right)^2 + 8 \sin(u_2) W \frac{d}{du_2} - 8(\beta_1 \cos(u_2) + \beta_2 \sin(u_2)) \frac{d}{du_2} = 0.
\]

Here \( v_2(u_2) = \frac{dW}{du_2} \).
2. $a_4 \neq 0$.

Solving condition (14) for $v_2(u_2)$ and substituting the result and (14) into (15) we obtain the equation that characterizes the Weierstrass $\wp$-function (in fact it is a translated and rescaled version):

$$h^2 d^3 v_2 \frac{dv_2}{du_2} - 12 \frac{dv_2}{du_2} v_2 + 12a_1 \frac{dv_2}{du_2} = 0.$$  \hspace{1cm} (19)

Thus $v_2(u_2) = \bar{h}^2 \wp(u_2 - u_2; 0; g_2, g_3) + a_1$, where $u_2$, $g_2$, and $g_3$ are arbitrary constants. As shown in [3] this solution is subject to the compatibility conditions (16) and (17), which leads to a complicated nonlinear differential equation for $v_2(u_2)$.

With this verification out of the way, we consider the analogous system on the 2-sphere, separable in spherical coordinates. Here $s_1 = \sin(\theta) \cos(\phi)$, $s_2 = \sin(\theta) \sin(\phi)$, $s_3 = \cos(\theta)$ with $s_1^2 + s_2^2 + s_3^2 = 1$. This system is in canonical form with coordinates $u_1, u_2$ where

$$\sin(\theta) = (\cosh(u_1))^{-1}, \quad \phi = u_2, \quad f_1(u_1) = (\cosh(u_1))^{-2}, \quad f_2(u_2) = 0.$$  \hspace{1cm} (20)

As before we look for solutions such that $v_1(u_1) = 0$ and $v_2$ satisfies a nonlinear equation only.

The computation is very similar to that for the Euclidean space example. We obtain the solution

$$F^0 = 4h^2 \cosh(u_1) \sin(u_2), \quad G^L = 8 \sinh(u_1) \cos(u_2) + a_4 u_2 + a_3,$$

subject to the conditions (14-17), exactly the same as for Euclidean space. Thus the system on the 2-sphere also admits Painlevé VI and special Weierstrass potentials for 3rd order superintegrability. It is clear from these results that these systems in Euclidean space can be obtained as Böcher contractions, [4], chapter 15, of the corresponding systems on the 2-sphere.

Next we consider spherical coordinates on the hyperboloid $s_1^2 - s_2^2 - s_3^2 = 1$,

$$s_1 = \cosh(x), \quad s_2 = \sinh(x) \cos(\phi), \quad s_3 = \sinh(x) \sin(\phi).$$

For the canonical form we find

$$\tanh \left( \frac{u_1}{2} \right) = \exp(x), \quad u_2 = \phi, \quad f_1(u_1) = (\sinh(u_1))^{-2}, \quad f_2(u_2) = 0,$$

and we look for solutions such that $v_1(u_1) = 0$ and $v_2(u_2)$ satisfies only a nonlinear equation. We obtain the solution

$$F^0 = 4h^2 \sinh(u_1) \sin(u_2), \quad G^L = 8 \cosh(u_1) \cos(u_2) + a_4 u_2 + a_3.$$
subject to the conditions (14), (17), again exactly the same as for flat space. Thus
the system on the 2-hyperboloid admits Painlevé VI and special Weierstrass

horocyclic coordinates \{u_1, u_2\} on the hyperboloid \(s_1^2 - s_2^2 - s_3^2 = 1\), e.g. [5], section 7.7:

\[
s_1 = \frac{1}{2} \left( u_1 + \frac{u_2^2 + 1}{u_1} \right), \quad s_2 = \frac{1}{2} \left( u_1 + \frac{u_2^2 - 1}{u_1} \right), \quad s_3 = \frac{u_2}{u_1}.
\]  

There are again two basic cases here:

We obtain the solution

\[
F^0 = -\frac{1}{2} a_8 \hbar^2 u_1, \quad G^L = \frac{u_2^2 (a_8 u_2 + a_9)}{2} - \frac{a_9 u_2^3}{6} - \frac{a_9 u_2^2}{2} + a_{10} u_2,
\]

\[
G^0 = \frac{u_2^2}{2} U_1(u_2) + U_2(u_2), \quad G^H = a_7,
\]

subject to the conditions

\[
0 = a_8 \hbar^2 u_2 + 2 \frac{d^2 U_1}{du_2^2}, \tag{22}
\]

\[
0 = \frac{1}{\hbar^2} a_8 \frac{d^3 v_2}{du_2^3} - 4a_8 \frac{dv_2}{du_2} v_2 + 4 \frac{dv_2}{du_2} \frac{dU_1}{du_2}, \tag{23}
\]

\[
0 = (2a_{10} - 2a_9 u_2 - a_8 u_2^2) \frac{dv_2}{du_2} - 4(a_9 + a_8 u_2) v_2 + 4 U_1 + 4 \frac{d^2 U_1}{du_2^2}, \tag{24}
\]

\[
0 = -2\hbar^2 a_8 u_2 \frac{dv_2}{du_2} + 16(a_9 + a_8 u_2) v_2^2 - 4(2a_{10} - 2a_9 u_2 + a_8 u_2^2) \frac{dv_2}{du_2} v_2 \tag{25}
\]

\[
+ \frac{\hbar^2}{2} (2a_{10} - 2a_9 u_2 - a_8 u_2^2) \frac{d^3 v_2}{du_2^3} - 4\hbar^2 (a_9 + a_8 u_2)
\]

\[
- 16 V_2 U_1 + 8 \frac{dv_2}{du_2} \frac{dU_2}{du_2}.
\]

There are again two basic cases here:

1. \(a_8 = 0\).

Then conditions (22) and (23) say that \(U_1\) is a constant: \(U_1(u_2) = d_1\). Then
condition (24) can be solved for \(U_2(u_2)\) and the result substituted into condition (25) to obtain an equation for \(v_2(u_2)\):

\[
-4a_9 \frac{dW}{du_2} \left( -3a_9 u_2 + 3a_{10} \frac{d^2 W}{du_2^2} + 4d_1 \right) \frac{dW}{du_2} + (-a_9 W + 2d_1 u_2 - 2d_3) \frac{d^2 W}{du_2^2} \tag{26}
\]

\[
+ \hbar^2 a_9 \frac{d^3 W}{du_2^3} - \frac{1}{4} \hbar^2 (-a_9 u_2 + a_{10}) \frac{d^4 W}{du_2^4} = 0, \quad \text{where} \quad v_2(u_2) = \frac{dW(u_2)}{du_2}.
\]
2. \( a_3 \neq 0 \).
Here we can solve (22) for \( v_2(u_2) \) and substitute the result into (23) to obtain
the equation
\[
\hbar^2 \frac{d^3 v_2}{du_2^3} - 12v_2 \frac{dv_2}{du_2} + 12a_1 \frac{dv_2}{du_2} = 0.
\]  
(27)
and it follows that \( v_2(u_2) = \hat{\hbar}^2 \hat{\varphi}(u_2 - u_{2,0}; g_2, g_3) + a_1 \), where \( u_{2,0}, g_2, \) and \( g_3 \) are arbitrary constants.

Acknowledgments

We thank Pavel Winternitz for helpful discussions and Adrian Escobar for pointing out the relevance of the paper [1] to classification of 3rd order superintegrable systems. W.M. was partially supported by a grant from the Simons Foundation (# 412351 to Willard Miller, Jr.). I.M. was supported by the Australian Research Council Discovery Grant DP160101376 and Future Fellowship FT180100099.

References

1. Kalnins, E.G., Kress, J.M. and Miller, W., Jr., Superintegrability and higher order integrals for quantum systems, *J. Phys. A: Math. Theor.* 43 (2010) 265205.
2. Tremblay, F., Turbiner, V.A. and Winternitz, P., An infinite family of solvable and integrable quantum systems on a plane. *J. Phys. A: Math. Theor.* 42 (2009) 242001.
3. Tremblay, F. and Winternitz, P., Third order superintegrable systems separating in polar coordinates, *J. Phys., A43*, 175206 (2010).
4. Kalnins, E.G., Kress, J.M. and Miller, W. Jr, Separation of variables and Superintegrability: The symmetry of solvable systems, *Institute of Physics, UK*, 2018, ISBN: 978-0-7503-1314-8, e-book.
5. Kalnins, E.G., Kress, J.M. and Miller, W. Jr, Separation of variables and Superintegrability: The symmetry of solvable systems, Institute of Physics, UK, 2018, ISBN: 978-0-7503-1314-8, http://iopscience.iop.org/book/978-0-7503-1314-8