Explicit computation of some families of Hurwitz numbers

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Abstract

We compute the number of (weak) equivalence classes of branched covers from a surface of genus $g$ to the sphere, with 3 branching points, degree $2k$, and local degrees over the branching points of the form $(2, \ldots, 2), (2h+1,1,2,\ldots,2), \pi = (d_i)_{i=1}^h$, for several values of $g$ and $h$. We obtain explicit formulae of arithmetic nature in terms of the local degrees $d_i$. Our proofs employ a combinatorial method based on Grothendieck’s *dessins d’enfant*.

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In this introduction we describe the enumeration problem faced in the present paper and the situations in which we solve it.

Surface branched covers A surface branched cover is a map

$$f : \tilde{\Sigma} \to \Sigma$$

where $\tilde{\Sigma}$ and $\Sigma$ are closed and connected surfaces and $f$ is locally modeled on maps of the form

$$(\mathbb{C}, 0) \ni z \mapsto z^m \in (\mathbb{C}, 0).$$

If $m > 1$ the point 0 in the target $\mathbb{C}$ is called a branching point, and $m$ is called the local degree at the point 0 in the source $\mathbb{C}$. There are finitely many branching points, removing which, together with their pre-images, one gets a genuine cover of some degree $d$. If there are $n$ branching points, the local

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degrees at the points in the pre-image of the $j$-th one form a partition $\pi_j$ of $d$ of some length $\ell_j$, and the following Riemann-Hurwitz relation holds:

$$\chi\left(\tilde{\Sigma}\right) - (\ell_1 + \ldots + \ell_n) = d \left(\chi\left(\Sigma\right) - n\right).$$

Let us now call branch datum a 5-tuple

$$\left(\tilde{\Sigma}, \Sigma, d, n, \pi_1, \ldots, \pi_n\right)$$

and let us say it is compatible if it satisfies the Riemann-Hurwitz relation. (For a non-orientable $\tilde{\Sigma}$ and/or $\Sigma$ this relation should actually be complemented with certain other necessary conditions, but we restrict to an orientable $\Sigma$ in this paper, so we do not spell out these conditions here.)

**The Hurwitz problem** The very old Hurwitz problem asks which compatible branch data are realizable (namely, associated to some existing surface branched cover) and which are exceptional (non-realizable). Several partial solutions to this problem have been obtained over the time, and we quickly mention here the fundamental [3], the survey [16], and the more recent [13, 14, 15, 2, 17]. In particular, for an orientable $\Sigma$ the problem has been shown to have a positive solution whenever $\Sigma$ has positive genus. When $\Sigma$ is the sphere $S$, many realizability and exceptionality results have been obtained (some of experimental nature), but the general pattern of what data are realizable remains elusive. One guiding conjecture in this context is that a compatible branch datum is always realizable if its degree is a prime number. It was actually shown in [3] that proving this conjecture in the special case of 3 branching points would imply the general case. This is why many efforts have been devoted in recent years to investigating the realizability of compatible branch data with base surface $\Sigma$ the sphere $S$ and having $n = 3$ branching points. See in particular [14, 15] for some evidence supporting the conjecture.

**Hurwitz numbers** Two branched covers

$$f_1 : \tilde{\Sigma} \to \Sigma \quad f_2 : \tilde{\Sigma} \to \Sigma$$

are said to be weakly equivalent if there exist homeomorphisms $\tilde{g} : \tilde{\Sigma} \to \tilde{\Sigma}$ and $g : \Sigma \to \Sigma$ such that $f_1 \circ \tilde{g} = g \circ f_2$, and strongly equivalent if the set of branching points in $\Sigma$ is fixed once and forever and one can take $g = \text{id}_\Sigma$. The (weak or strong) Hurwitz number of a compatible branch
datum is the number of (weak or strong) equivalence classes of branched covers realizing it. So the Hurwitz problem can be rephrased as the question whether a Hurwitz number is positive or not (a weak Hurwitz number can be smaller than the corresponding strong one, but they can only vanish simultaneously). Long ago Mednykh in [10, 11] gave some formulae for the computation of the strong Hurwitz numbers, but the actual implementation of these formulae is rather elaborate in general. Several results were also obtained in more recent years in [4, 7, 8, 9, 12].

**Computations**  In this paper we consider branch data of the form

\[(\vartriangledown) \left( \Sigma, \Sigma = S, d = 2k, n = 3, (2, \ldots, 2), (2h + 1, 1, 2, \ldots, 2), \pi = (d_i)_{i=1}^{\ell} \right)\]

for \(h \geq 0\). A direct computation shows that such a datum is compatible for \(h \geq 2g\), where \(g\) is the genus of \(\Sigma\), and \(\ell = h - 2g + 1\). We compute the weak Hurwitz number of the datum for the values of \(g, 2h + 1, \ell\) shown in boldface in Table 1. More values could be obtained, including for instance those within parentheses in the table, using the same techniques as we employ below, but the complication of the topological and combinatorial situation grows very rapidly, and the arithmetic formulae giving the weak Hurwitz numbers are likely to be rather intricate for larger values of \(g\) and \(h\).

For brevity we will henceforth denote by \(\nu\) the number of weakly inequivalent realizations of \((\vartriangledown)\) for any given values of \(g\) and \(h\).

**Theorem 0.1.** For \(g = 0\) and \(0 \leq h \leq 2\) there hold:

- For \(h = 0\), whence \(\ell = 1\) and \(\pi = (2k)\), we always have \(\nu = 1\);
- For \(h = 1\), whence \(\ell = 2\) and \(\pi = (p, 2k - p)\) with \(p \leq k\), we have \(\nu = 1\) for \(p < k\), and \(\nu = 0\) for \(p = k\);

| \(g\) | \(2h + 1 = 1\) | \(\ell = 1\) | \(g = 1\) | \(2h + 1 = 3\) | \(\ell = 2\) | \(g = 2\) | \(2h + 1 = 5\) | \(\ell = 3\) | \(g = 3\) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | — | — | — | — | — | — | — | — | — |
| 1 | — | — | — | — | — | — | — | — | — |
| 2 | — | — | — | — | — | — | — | — | — |
| 3 | — | — | — | — | — | — | — | — | — |

Table 1: Values of \(g, h, \ell\) giving compatible data \((\vartriangledown)\).
• For $h = 2$, whence $\ell = 3$, we have:

(i) $\nu = 0$ if $k = 3m$ and $\pi = (2m, 2m, 2m)$;
(ii) $\nu = 0$ if $k = 2m$ and $\pi = (2m, m, m)$;
(iii) $\nu = 1$ if $\pi = (2t, k - t, k - t)$ with
   
   (a) $1 \leq t < \frac{k}{3}$, or
   (b) $\frac{k}{3} < t < \frac{k}{2}$, or
   (c) $\frac{k}{2} < t < k$;
(iv) $\nu = 1$ if $\pi = (k, k - r, r)$ with $1 \leq r < \frac{k}{3}$;
(v) $\nu = 2$ if $\pi = (2k - q - r, q, r)$ with $1 \leq r < \frac{k}{2}$ and $r < q < k - r$;
(vi) $\nu = 3$ if $\pi = (2k - q - r, q, r)$ with
   
   (a) $1 \leq r < \frac{k}{2}$ and $k - r < q < k - \frac{r}{2}$, or
   (b) $\frac{k}{2} \leq r < \frac{3}{2}k$ and $r < q < k - \frac{r}{2}$.

**Theorem 0.2.** For $g = 1$ and $2 \leq h \leq 3$ there hold:

• For $h = 2$, whence $\ell = 1$, we have $\nu = \left\lfloor \frac{1}{4}(k - 1)^2 \right\rfloor$;
• For $h = 3$, whence $\ell = 2$ and $\pi = (p, 2k - p)$ with $p \leq k$, we have:
   
   – for $p = k$
   $$\nu = \frac{1}{2} \left\lceil \frac{k}{2} \left( \left\lfloor \frac{k}{2} \right\rfloor - 1 \right) + \frac{1}{4} \left\lfloor \frac{k - 1}{2} \right\rfloor \right\rceil;$$
   
   – for $p < k$
   $$\nu = \left\lfloor \frac{1}{4}(p - 1)^2 \right\rfloor + \left\lceil \frac{p}{2} \left( \left\lfloor \frac{p}{2} \right\rfloor - 1 \right) \right\rceil + (k - 3)(k - p - 1)$$
   $$+ \left\lfloor \frac{1}{4} \left( k - \left\lceil \frac{p}{2} \right\rceil - 1 \right)^2 \right\rfloor - \left\lfloor \frac{k - p}{2} \right\rfloor + \left\lceil \frac{1}{4} \left\lceil \frac{p - 1}{2} \right\rceil \right\rceil.$$  

**Theorem 0.3.** For $g = 2$ and $h = 4$, whence $\ell = 1$, we have

$$\nu = \frac{k - 1}{16} \left( 7k^3 - 63k^2 + 197k - 208 \right) + \frac{5}{8} \left( 5 - 2k \right) \cdot \left\lfloor \frac{k}{2} \right\rfloor.$$
1 Computation of weak Hurwitz numbers via dessins d’enfant

Dessins d’enfant were introduced in [5] (see also [1]) and have been already exploited to give partial answers to the Hurwitz problem [6, 16]. Here we show how to employ them to compute weak Hurwitz numbers. Let us fix until further notice a branch datum

\[(\spadesuit) \left( \tilde{\Sigma}, \Sigma = S, d, n = 3, \pi_1 = (d_1)_{i=1}^{\ell_1}, \pi_2 = (d_2)_{i=1}^{\ell_2}, \pi_3 = (d_3)_{i=1}^{\ell_3} \right).\]

A graph $\Gamma$ is bipartite if it has black and white vertices, and each edge joins black to white. If $\Gamma$ is embedded in $\tilde{\Sigma}$ we call region a component $R$ of $\tilde{\Sigma} \setminus \Gamma$, and length of $R$ the number of white (or black) vertices of $\Gamma$ to which $R$ is incident, with multiplicity (\(\partial R\) can be parameterized as a union of locally injective closed possibly non-simple curves, and the multiplicity of a vertex for $R$ is the number of times $\partial R$ goes through it). A pair $(\Gamma, \sigma)$ is called dessin d’enfant representing $(\spadesuit)$ if $\sigma \in S_3$ and $\Gamma \subset \Sigma$ is a bipartite graph such that:

- The black vertices of $\Gamma$ have valence $\pi_{\sigma(1)}$;
- The white vertices of $\Gamma$ have valence $\pi_{\sigma(2)}$;
- The regions of $\Gamma$ are topological discs and have length $\pi_{\sigma(3)}$.

We will also say that $\Gamma$ represents $(\spadesuit)$ through $\sigma$.

Remark 1.1. If $f : \tilde{\Sigma} \to S$ is a branched cover matching $(\spadesuit)$ and $\alpha$ is a segment in $S$ with a black and a white end at the branching points corresponding to $\pi_1$ and $\pi_2$, then $(f^{-1}(\alpha), \text{id})$ represents $(\spadesuit)$, with vertex colours of $f^{-1}(\alpha)$ lifted via $f$.

Proposition 1.2. To a dessin d’enfant $(\Gamma, \sigma)$ representing $(\spadesuit)$ one can associate a branched cover $f : \tilde{\Sigma} \to S$ realizing $(\spadesuit)$, well-defined up to equivalence.

Proof. We choose distinct points $x_1, x_2, x_3 \in S$ and a segment $\alpha$ joining $x_1$ to $x_2$. Then we define $f$ on $\Gamma$ such that the black vertices are mapped to $x_1$ and the white ones to $x_2$, and $f$ restricted to any edge is a homeomorphism onto $\alpha$. For each region $R$ of $\Gamma$, assuming $R$ has length $m$, we fix a point $y$ in $R$ and we extend $f$ so that:

- $f$ maps $y$ to $x_3$ locally as $\mathbb{C}, 0) \ni z \mapsto z^m \in \mathbb{C}, 0)$;
• $f$ is continuous on the closure of $R$;

• $f$ is a genuine $m : 1$ cover from $R \setminus \{y\}$ to $S \setminus (\alpha \cup \{x_3\})$.

This construction is of course possible and gives a realization of $(\bigdiamond)$ with local degrees $\pi_{\sigma(j)}$ over $x_j$. To see that $f$ is well-defined up to equivalence we first note that the choice of $x_1, x_2, x_3, \alpha$ is immaterial up to post-composition with automorphisms of $S$. Now if $f_1, f_2$ are constructed as described for the same $x_1, x_2, x_3, \alpha$ we can first define an automorphism $\tilde{g}$ of $\Gamma$ which is the identity on the vertices and given by $f_1^{-1} \circ f_2$ on each edge. Now suppose that to define $f_j$ on a region $R$ of $\Gamma$, we have chosen the point $y_j \in R$. Then we can define a homeomorphism $\tilde{g} : R \setminus \{y_2\} \rightarrow R \setminus \{y_1\}$ so that $f_1 \circ \tilde{g} = f_2$ on $R \setminus \{y_2\}$. Setting $\tilde{g}(y_2) = y_1$ and patching $\tilde{g}$ with that previously defined on $\Gamma$ we get the desired equality $f_1 \circ \tilde{g} = f_2$ on the whole of $\tilde{\Sigma}$.

We define an equivalence relation $\sim$ on dessins d’enfant generated by:

• $(\Gamma_1, \sigma_1) \sim (\Gamma_2, \sigma_2)$ if $\sigma_1 = \sigma_2$ and there is an automorphism $\tilde{g} : \tilde{\Sigma} \rightarrow \tilde{\Sigma}$ such that $\Gamma_1 = \tilde{g}(\Gamma_2)$ matching colours;

• $(\Gamma_1, \sigma_1) \sim (\Gamma_2, \sigma_2)$ if $\sigma_1 = \sigma_2 \circ (1 2)$ and $\Gamma_1 = \Gamma_2$ as a set but with vertex colours switched;

• $(\Gamma_1, \sigma_1) \sim (\Gamma_2, \sigma_2)$ if $\sigma_1 = \sigma_2 \circ (2 3)$ and $\Gamma_1$ has the same black vertices as $\Gamma_2$ and for each region $R$ of $\Gamma_2$ we have that $R \cap \Gamma_1$ consists of one white vertex and disjoint edges joining this vertex with the black vertices on the boundary of $R$ (see Fig. 1 for an example).

**Theorem 1.3.** The branched covers associated to dessins d’enfant $(\Gamma_1, \sigma_1)$ and $(\Gamma_2, \sigma_2)$ as in Proposition 1.2 are equivalent if and only if $(\Gamma_1, \sigma_1) \sim (\Gamma_2, \sigma_2)$.
Proof. We begin with the “if” part. It is enough to prove that the branched covers \( f_1 \) and \( f_2 \) associated to \((\Gamma_1, \sigma_1)\) and \((\Gamma_2, \sigma_2)\) are equivalent for the three instances of \( \sim \) generating it:

- If \( \sigma_1 = \sigma_2 \) and \( \Gamma_1 = \tilde{g}(\Gamma_2) \) and \( f_1 \) is associated to \((\Gamma_1, \sigma_1)\), then \( f_1 \circ \tilde{g} \) is associated to \((\Gamma_2, \sigma_2)\), so \( f_1 \) is equivalent to \( f_2 \);

- If \( \sigma_1 = \sigma_2 \circ (1 \, 2) \) and \( \Gamma_1 \) is \( \Gamma_2 \) with colours switched, recall that the construction of \( f_1 \) and \( f_2 \) requires the choice of \( x_1, x_2, x_3, \alpha \) in \( S \). Choose an automorphism \( g \) of \( S \) that fixes \( x_3 \), switches \( x_1 \) and \( x_2 \) and leaves \( \alpha \) invariant. Then we see that \( g \circ f_2 \) is associated to \((\Gamma_1, \sigma_1)\), so \( f_1 \) is equivalent to \( f_2 \);

- If \( \sigma_1 = \sigma_2 \circ (2 \, 3) \) and \( \Gamma_1, \Gamma_2 \) are related as described above, choose an automorphism \( g \) of \( S \) that fixes \( x_1 \) and switches \( x_2 \) and \( x_3 \) (so \( g(\alpha) \) is a segment joining \( x_1 \) to \( x_3 \)). Then we see that \( g \circ f_2 \) is associated to \((\Gamma_1, \sigma_1)\), so \( f_1 \) is equivalent to \( f_2 \).

Turning to the “only if” part, suppose that the branched covers \( f_1 \) and \( f_2 \) associated to \((\Gamma_1, \sigma_1)\) and \((\Gamma_2, \sigma_2)\) are equivalent, namely \( f_1 \circ \tilde{g} = g \circ f_2 \) for some automorphisms \( \tilde{g} \) and \( g \) of \( \tilde{\Sigma} \) and \( S \). Using \( \sim \) we can reduce to the case where \( \sigma_1 = \sigma_2 = \text{id} \). Assuming \( f_1, f_2 \) have been constructed using the same \( x_1, x_2, x_3, \alpha \), we may still have that \( g \) permutes \( x_1, x_2, x_3 \) non-trivially if the triple \( \pi_1, \pi_2, \pi_3 \) contains repetitions, but up to replacing \( f_2 \) by a suitable \( h \circ f_2 \) for an automorphism \( h \) of \( S \), we can suppose \( g \) fixes \( x_1, x_2, x_3, \alpha \). Then \( \Gamma_j = f_j^{-1}(\alpha) \) with black vertices over \( x_1 \) and white over \( x_2 \), so \( \Gamma_1 = \tilde{g}(\Gamma_2) \) matching colours.

When the partitions \( \pi_1, \pi_2, \pi_3 \) in the branch datum \( (\spadesuit) \) are pairwise distinct, to compute the corresponding weak Hurwitz number one can stick to dessins d’enfant representing the datum through the identity, namely one can list up to automorphisms of \( \tilde{\Sigma} \) the bipartite graphs with black and white vertices of valence \( \pi_1 \) and \( \pi_2 \) and discal regions of length \( \pi_3 \). When the partitions are not distinct, however, it is essential to take into account the other moves generating \( \sim \), see Fig. 2.

Relevant data and repeated partitions We now specialize again to a branch datum of the form \( (\heartsuit) \). We will compute its weak Hurwitz number \( \nu \) by enumerating up to automorphisms of \( \tilde{\Sigma} \) the dessins d’enfant \( \Gamma \) representing it through the identity, namely the bipartite graphs \( \Gamma \) with black vertices of valence \( (2, \ldots, 2) \), white vertices of valence \( (2h+1, 1, 2, \ldots, 2) \), and discal regions of length \( \pi \). Two remarks are in order:
Call $\Gamma_1, \Gamma_2, \Gamma_3$ the dessins d’enfant appearing in this picture. Then $\Gamma_1$ and $\Gamma_2$ both represent the datum $(S, S, 9, 3, (7, 1, 1), (4, 3, 1, 1), (4, 3, 1, 1))$ through the identity and through (23). They are not even abstractly homeomorphic as uncoloured graphs, but they define equivalent branched coverings, because they are obtained from each other by the last move generating $\sim$. Note also that $\Gamma_3$ is obtained from $\Gamma_1$ by applying the second move generating $\sim$ and then the last move, so $\Gamma_3$ also represents the same datum through (1 3) and through (1 2 3).

- In all the pictures we will only draw the two white vertices of $\Gamma$ of valence $(2h + 1, 1)$, and we will decorate an edge of $\Gamma$ by an integer $a \geq 1$ to understand that the edge contains $a$ black and $a - 1$ white valence-2 vertices;

- Enumerating these dessins d’enfant $\Gamma$ up to automorphisms of $\tilde{\Sigma}$ already gives the right value of $\nu$ except if two of the partitions of $d$ in the datum coincide.

**Proposition 1.4.** In a branch datum of the form (◇) two of the partitions of $d$ coincide precisely in the following cases:

- $g = 0$, $h \geq 0$, $k = h + 1$, with partitions $(2, \ldots, 2), (2h + 1, 1), (2, \ldots, 2)$;

- Any $g$, $h \geq 2g + 1$, $k = 2h - 2g$, with partitions $(2, \ldots, 2), (2h + 1, 1, 2, \ldots, 2), (2h + 1, 1, 2, \ldots, 2)$.

**Proof.** The lengths of the partitions $\pi_1, \pi_2, \pi$ in (◇) are $\ell_1 = k$, $\ell_2 = k - h + 1$ and $\ell = h + 1 - 2g$. We can never have $\pi_1 = \pi_2$. Since $k \geq h + 1$ we can have $\ell_1 = \ell$ only if $g = 0$ and $k = h + 1$, whence the first listed item. We can have $\ell_2 = \ell$ only for $k = 2h - 2g$, whence $h \geq 2g + 1$ and the data in the second listed item.

This result implies that the data (◇) relevant to Theorems 0.1 to 0.3 and containing repetitions are precisely

\[(S, S, 2, 3, (2), (1, 1), (2)) \quad (S, S, 4, 3, (2, 2), (3, 1), (2, 2))\]

\[(S, S, 6, 3, (2, 2, 2), (5, 1), (2, 2, 2)) \quad (S, S, 4, 3, (2, 2), (3, 1), (3, 1))\]
(S, S, 8, 3, (2, 2, 2, 2), (5, 1, 2)) \quad (T, S, 8, 3, (2, 2, 2, 2), (5, 1, 2), (7, 1))

(where T is the torus) for which we easily have $\nu = 1$ in the first case, and $\nu = 0$ in the second and third one by the very even data criterion of [15]. The last three cases will be taken into account in Sections 2 and 3.

2 Genus 0

In this section we prove Theorem 0.1 starting from the very easy cases $h = 0$ and $h = 1$, for which there is only one homeomorphism type of relevant graph and only one embedding in S, as shown in Fig. 3—here and below the notion of $(2h + 1, 1)$ graph abbreviates a graph with vertices of valence $(2h + 1, 1)$. The first graph gives a unique realization of $(2k)$ as $\pi$. The second graph with $a = p$ gives a unique realization of $\pi = (p, 2k - p)$ for $p < k$, while $(k, k)$ is exceptional. Note that a single graph emerges for the realization of the case with repeated partitions $(S, S, 4, 3, (2, 2), (3, 1), (3, 1))$, so its realization is a fortiori unique up to equivalence (and it is also immediate to check that the move of Fig. 1 leads this graph to itself).

For the case $h = 2$, whence $\ell = 3$, we first check that the cases listed in the statement cover all the possibilities for $\pi = (p, q, r)$:

- If $p = q = r$ we have $3p = 2k$, so $k = 3m$ and $p = 2m$, whence case (i);
- If $p \neq q = r$ we have $p = 2k - 2q$, so $p = 2t$ and $q = k - t$, with $k - t \neq 2t$, so $t \neq \frac{k}{3}$, and $0 < t < k$; then either $t = \frac{k}{2}$, so $k = 2m$ and case (ii), or one of the subcases (a), (b) or (c) of (iii);
- If $p > q > r$ we have $p = 2k - q - r$; since $q > r$ we have $2k - 2r > 2k - q - r > q > r$, whence $1 \leq r < \frac{2}{3}k$ and $r < q < k - \frac{r}{2}$; for $r < \frac{k}{2}$ by comparing $q$ with $k - r$ we get case (iv) or (v) or (vi)-(a), while for $r \geq \frac{k}{2}$ we get (vi)-(b).

Even if this is not strictly necessary, we also provide in Tables 2 and 3 two examples of an application of Theorem 0.1 for $h = 2$, proving that each
Let us now get to the actual proof. There are only two inequivalent embeddings in $S$ of the $(5,1)$ graph, shown in Fig. 4 with edges decorated as explained at the end of Section 1, and denoted by $I(a,b,c)$ and $\Pi(a,b,c)$, realizing respectively $\pi = (2a+b+c,b,c)$ and $\pi = (2a+b+b+c,c)$. Note that there there is no automorphism taking $\Pi$ to itself, while there is one taking $I(a,b,c)$ to $I(a,c,b)$.

Given $k$ and $\pi = (p,q,r)$ with $p+q+r = 2k$ we must now count how many realizations we have of $\pi$ as $(2a+b+c,b,c)$ and $\pi = (2a+b+b+c,c)$. Of course if $p = q = r$ there is no realization, whence case (i) of the statement.

If $p = q \neq r$, so $\pi = (2t,k-t,k-t)$ with $t \neq \frac{k}{3}$, we can realize it via
| $\pi$   | Case                  | $\nu$ | Realizations          |
|--------|-----------------------|-------|-----------------------|
| (12,1,1) | (iii)-(c) $\frac{1}{5} = 3.5 < 1 = 6 < k = 7$ | 1     | I(5,1,1)             |
| (11,2,1) | (v) $1 \leq r = 1 < \frac{1}{7} = 3.5$ $r = 1 < q < \frac{2}{7} < k - r = 6$ | 2     | I(4,2,1), II(5,1,1) |
| (10,3,1) | (v) $1 \leq r = 1 < \frac{1}{4} = 3.5$ $r = 1 < q < \frac{3}{4} < k - r = 6$ | 1     | I(3,1,1), II(4,2,1) |
| (10,2,2) | (iii)-(c) $\frac{1}{5} = 3.5 < 1 = 5 < k = 7$ | 1     | I(3,2,2)             |
| (9,4,1)  | (v) $1 \leq r = 1 < \frac{1}{7} = 3.5$ $r = 1 < q < \frac{4}{7} < k - r = 6$ | 2     | I(2,4,1), II(3,3,1) |
| (9,3,2)  | (v) $1 \leq r = 2 < \frac{1}{7} = 3.5$ $r = 2 < q < \frac{3}{7} < k - r = 5$ | 2     | I(2,3,2), II(4,1,2) |
| (8,5,1)  | (v) $1 \leq r = 1 < \frac{1}{5} = 3.5$ $r = 1 < q < \frac{5}{7} < k - r = 6$ | 2     | I(1,5,1), II(2,4,1) |
| (8,4,2)  | (v) $1 \leq r = 2 < \frac{1}{5} = 3.5$ $r = 2 < q < \frac{4}{5} < k - r = 5$ | 2     | I(1,4,2), II(3,2,2) |
| (8,3,3)  | (iii)-(c) $\frac{1}{5} = 3.5 < 1 = 4 < k = 7$ | 1     | I(1,3,3)             |
| (7,6,1)  | (iv) $1 \leq r = 1 < \frac{1}{6} = 3.5$ | 1     | II(1,5,1)           |
| (7,5,2)  | (iv) $1 \leq r = 2 < \frac{1}{6} = 3.5$ | 1     | II(2,3,2)           |
| (7,4,3)  | (iv) $1 \leq r = 3 < \frac{1}{6} = 3.5$ | 1     | II(3,1,3)           |
| (6,6,2)  | (iii)-(a) $1 \leq t = 1 < \frac{1}{6} < \frac{2}{7} - \frac{1}{5}$ | 1     | II(1,4,2)           |
| (6,5,3)  | (vi)-(a) $1 \leq r = 3 < \frac{1}{6} < \frac{3}{5}$ $k - r = 4 < \frac{1}{3} < \frac{4}{5} < k - r = 5$ | 3     | II(1,1,5), II(1,3,3), II(2,2,3) |
| (6,4,4)  | (iii)-(b) $\frac{1}{5} < t = 3 < \frac{1}{4} = 3.5$ | 1     | II(1,2,4)           |
| (5,5,4)  | (iii)-(a) $1 \leq t = 2 < \frac{1}{5} = 2.5$ | 1     | II(2,1,4)           |

Table 3: The case $k = 7$. 

11
I(a, b, c) precisely if

\[
\begin{align*}
2a + b + c &= 2t \\
b &= c = k - t
\end{align*}
\quad \iff
\begin{align*}
a &= 2t - k \\
b &= c = k - t
\end{align*}
\quad \text{for } \frac{k}{2} < t < k
\]

whereas we can realize it via I I(a, b, c) in the following cases:

\[
\begin{align*}
2a + b + c &= k - t \\
c &= 2t
\end{align*}
\quad \iff
\begin{align*}
a &= t \\
b &= k - 3t \\
c &= 2t
\end{align*}
\quad \text{for } 1 \leq t < \frac{k}{3}
\]

\[
\begin{align*}
2a + b + c &= k - t \\
b + c &= 2t
\end{align*}
\quad \iff
\begin{align*}
a &= k - 2t \\
b &= 3t - k \\
c &= k - t
\end{align*}
\quad \text{for } \frac{k}{3} < t < \frac{k}{2}.
\]

We have found the three disjoint instances (a), (b), (c) of case (iii), and we have not found realizations with \( t = \frac{k}{2} \), whence (ii).

Turning to the case \( p > q > r \), so \( p = 2k - q - r \) with \( 1 \leq r < \frac{2}{3}k \) and \( r < q < k - \frac{r}{2} \), the realizations via I(a, b, c) are given by

\[
\begin{align*}
2a + b + c &= 2k - q - r \\
b &= q \\
c &= r
\end{align*}
\quad \iff
\begin{align*}
a &= k - q - r \\
b &= q \\
c &= r
\end{align*}
\quad \text{for } r < q < k - r \quad \text{whence } 1 \leq r < \frac{k}{2}
\]

which gives the first contribution to (v). The realizations via I I(a, b, c) are instead

\[
\begin{align*}
2a + b &= 2k - q - r \\
b + c &= q \\
c &= r
\end{align*}
\quad \iff
\begin{align*}
a &= k - q \\
b &= q - r \\
c &= r
\end{align*}
\]

which gives the only contribution to (iv), the second and last contribution to (v), and the first contribution to (vi)-(a) and (vi)-(b), or

\[
\begin{align*}
b + c &= 2k - q - r \\
2a + b &= q \\
c &= r
\end{align*}
\quad \iff
\begin{align*}
a &= q + r - k \\
b &= 2k - q - 2r \\
c &= r
\end{align*}
\quad \text{for } q > k - r
\]

(\text{note that } q < 2k - 2r \text{ is implied by } q < k - \frac{r}{2} \text{ and } r < \frac{2}{3}k), \text{ or}

\[
\begin{align*}
b + c &= 2k - q - r \\
c &= q \\
2a + b &= r
\end{align*}
\quad \iff
\begin{align*}
a &= q + r - k \\
b &= 2k - 2q - r \\
c &= q
\end{align*}
\quad \text{for } q > k - r
\]
Figure 5: Graphs mapped to themselves by the move of Fig. 1.

(2k − 2q − r > 0 is equivalent to q < k − r); these realizations both give the final contributions to (vi)-(a) and (vi)-(b), which are respectively obtained by comparing r to k − r, since we must have q > max{r, k − r}.

The last issue is to deal with the datum with repeated partitions

(S, S, 8, 3, (2, 2, 2), (5, 1, 2), (5, 1, 2)).

This can be realized as I(1, 2, 1) and as II(2, 1, 1), and we must show that these graphs are not obtained from each other via the move of Fig. 1. This is done in Fig. 5 which proves that the move actually takes each of these graphs to itself.

The proof is complete.

3 Genus 1

Let us prove Theorem 0.2. We begin with the rather easy case h = 2. Up to automorphisms there is only one embedding in T of a (5, 1) graph with a disc as only region, shown in Fig. 6 and subject to the symmetry b ↔ c. So

\[ \nu = \# \{(a, \{b, c\}) : a + b + c = k\} = \sum_{a=1}^{k-2} \left\lfloor \frac{k-a}{2} \right\rfloor = \sum_{j=2}^{k-1} \left\lfloor \frac{j}{2} \right\rfloor. \]

For odd and even k one easily gets the expressions \(\frac{1}{4}(k-1)^2\) and \(\frac{1}{4}k(k-2)\) that one can unify as \(\left\lfloor \frac{1}{4}(k-1)^2 \right\rfloor\).
Turning to $h = 3$, so $\ell = 2$ and $\pi = (p, 2k - p)$ with $1 \leq p \leq k$, we first determine the embeddings in $T$ of the bouquet $B$ of 3 circles with two discs as regions. Of course at least a circle of $B$ is non-trivial on $T$, so its complement is an annulus. Then another circle must joint the boundary components of this annulus, so we can assume two circles of $B$ form a standard meridian-longitude pair on $T$. Then the possibilities for $B$ are as in Fig. 7. Note that these embeddings have respectively a $S_3 \times \mathbb{Z}/2$, a $\mathbb{Z}/2 \times \mathbb{Z}/2$ and a $\mathbb{Z}/2$ symmetry. It easily follows that the relevant embeddings in $T$ of the $(7, 1)$ graph are up to symmetry those shown in Fig. 8.

Moreover $I(a, b, c, d)$ realizes $(a+b+c+2d, a+b+c)$, while $II(a, b, c, d)$ realizes $(a+b+2c, a+b+2d)$, then $III(a, b, c, d)$ realizes $(a+b+2c+2d, a+b)$, next $IV(a, b, c, d)$ realizes $(2a+2b+c, c+2d)$ and finally $V(a, b, c, d)$, $VI(a, b, c, d)$ and $VII(a, b, c, d)$ all realize $(2a+2b+c+2d, c)$. We now count how many different realizations of $\pi = (p, 2k - p)$ with $p \leq k$ exist.

I $\pi$ is realized if $3 \leq p < k$ in

$$\# \{ \{a, b\}, c \} : a + b + c = p = \left[ \frac{1}{4}(p - 1)^2 \right]$$

ways. Note that the expression gives the right value 0 also for $p = 1, 2$. 

---

**Figure 6**: The $(5, 1)$ graph in $T$.

**Figure 7**: A bouquet of 3 circles in $T$ with 2 discs as regions.
Figure 8: Embeddings in $T$ of the $(7,1)$ graph with 2 discs as regions.

II $\pi$ is realized if $4 \leq p \leq k$. For $p < k$ the number of realizations is

$$\#\{(a, b, c) : a + b + 2c = p\} + \#\{(a, b, d) : a + b + 2d = p\}$$

$$= 2\#\{(a, b, c) : a + b + 2c = p\} + 2 \sum_{c=1}^{\lceil p/2 \rceil - 1} \left\lfloor \frac{p - 2c}{2} \right\rfloor$$

$$= 2 \sum_{c=1}^{\lceil p/2 \rceil - 1} \left( \left\lfloor \frac{p}{2} \right\rfloor - c \right) = 2 \left( \left\lfloor \frac{p}{2} \right\rfloor - 1 \right) \left\lceil \frac{p}{2} \right\rceil - \left( \left\lfloor \frac{p}{2} \right\rfloor - 1 \right) \left\lceil \frac{p}{2} \right\rceil$$

$$= \left( \left\lfloor \frac{p}{2} \right\rfloor - 1 \right) \left\lceil \frac{p}{2} \right\rceil$$

(which is correct also for $p = 1, 2, 3$), while for $p = k$ it is $\frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right)$.

III $\pi$ is realized if $1 < p < k - 1$ and the number of realizations is

$$\#\{(a, b) : a + b = p\} \cdot \#\{(c, d) : c + d = k - p\} = (p - 1)(k - p - 1)$$

which works also for $p = 1$ and $p = k - 1$.

IV We first consider the case $p < k$. Then $\pi$ is realized in the following ways:

$$\begin{cases} c + 2d = p \\ 2a + 2b + c = 2k - p \end{cases} \Leftrightarrow \begin{cases} 1 \leq d \leq \left\lfloor \frac{p-1}{2} \right\rfloor \\ c = p - 2d \\ a + b = k - p - d \end{cases} \text{ for } 3 \leq p < k,$$
\[
\begin{align*}
\begin{cases}
2a + 2b + c = p \\
c + 2d = 2k - p
\end{cases} & \iff \\
\begin{cases}
2a + 2b + c = p \\
c + 2d = 2k - p
\end{cases}
\end{align*}
\]

So we get for \(5 \leq p < k\) the number
\[
\left[ \frac{k - 1 - \left\lfloor \frac{p}{2} \right\rfloor}{2} \right] + \left[ \frac{p - 1 - \left\lfloor \frac{p}{2} \right\rfloor}{2} \right] + \left[ \frac{p - 1 - \left\lfloor \frac{p}{2} \right\rfloor}{2} \right].
\]

For \(p = k\) the realizations correspond to
\[
\begin{cases}
2a + 2b + c = k \\
c + 2d = k
\end{cases} \iff \begin{cases}
1 \leq d \leq \left\lfloor \frac{k - 1}{2} \right\rfloor \\
c = k - 2d \\
a + b = d
\end{cases}
\]

so their number is
\[
\left[ \frac{k - 1 - \left\lfloor \frac{p}{2} \right\rfloor}{2} \right] + \left[ \frac{p - 1 - \left\lfloor \frac{p}{2} \right\rfloor}{2} \right] + \left[ \frac{p - 1 - \left\lfloor \frac{p}{2} \right\rfloor}{2} \right].
\]

\(V \pi\) can be realized if \(1 \leq p \leq k - 3\), via
\[
\begin{cases}
2a + 2b + c + 2d = 2k - p \\
c = p
\end{cases} \iff \begin{cases}
1 \leq d \leq k - p - 2 \\
a + b = k - p - d \\
c = p
\end{cases}
\]

so there are \(\left\lfloor \frac{1}{4}(k - p - 1)^2 \right\rfloor\) ways (remember the \(a \leftrightarrow b\) symmetry) and the expression is correct also for \(p = k - 2\) and \(p = k - 1\).

\(VI, VII\) The realizations are exactly the same, but we have no symmetry, so there are \(\binom{k-p-1}{2}\) of them, which again is correct for all \(p < k\).
Figure 9: Moves on dessins d’enfant taking them to themselves.

hence from the dessins d’enfant in the top part of Fig. 9. In the bottom part of the same figure we show the dessins obtained by the move that corresponds to the switch of the partitions (7, 1). Since in each case we obtain the same dessin, the move has no effect on the counting.

To conclude we are left to sum all the contributions. For $p = k$ we have the two summands of the statement. For $p < k$ we have some simplifications: the contributions from III, VI, VII give

$$(p - 1)(k - p - 1) + 2 \cdot \frac{1}{2} (k - p - 1)(k - p - 2) = (k - 3)(k - p - 1)$$

while the second summand from IV and V gives

$$- \left[ \frac{1}{4}(k - p)^2 \right] + \left[ \frac{1}{4}(k - p - 1)^2 \right] = - \left[ \frac{k - p}{2} \right]$$

and the proof is complete.

4 Genus 2

We now prove Theorem 0.3. The topological argument will be more elaborate than that for Theorem 0.2 while the arithmetic part will be much easier. We denote by $2T$ the genus-2 surface and we begin with the following:

Claim. Up to automorphisms, there exist precisely 4 embeddings in $2T$ of the bouquet of 4 circles having a disc as only region. They are given by the
To prove the claim, we note that an embedding as described always gives a realization of $2T$ as $O$ with paired edges, so we must discuss how many of these exist. Note that an edge-pairing gives $2T$ precisely if all the vertices of $O$ are equivalent under it. We cyclically give colours in $\mathbb{Z}/8$ to the edges of $O$, so a pairing consists of 4 pairs $\{\{i_0, i_1\}, \{i_2, i_3\}, \{i_4, i_5\}, \{i_6, i_7\}\}$ with $\{i_0, \ldots, i_7\} = \mathbb{Z}/8$. Up to symmetry we suppose that $i_0 = 0$ and $i_1$ is the minimal distance between two paired edges. Up to symmetry we have $i_1 \leq 4$ and we also have $i_1 \geq 2$ because if $i_1 = 1$ the vertex between 0 and 1 is equivalent to itself only. So we can suppose $i_2 = 1$. The rest of the proof of the claim is pictorially illustrated in Fig. 11.

Suppose that $i_1 = 2$. Up to symmetry we then have $3 \leq i_3 \leq 5$. If $i_3 = 3$ we readily conclude that the other glued pairs are $\{4, 6\}, \{5, 7\}$, so we are in case I of Fig. 10. If $i_3 = 4$ the other glued pairs must be $\{3, 6\}, \{5, 7\}$ and we are in case II. If $i_3 = 5$ the other glued pairs can be $\{3, 6\}, \{4, 7\}$, whence III, or $\{3, 7\}, \{4, 6\}$, but then there are 3 equivalence classes of vertices, so this case must be dismissed.

Now suppose $i_1 = 3$. If $i_3 = 4$ then two edges in $\{5, 6, 7\}$ are paired, absurd. If $i_3 = 5$ the other pairs can only be $\{2, 6\}, \{4, 7\}$ but then there are 3 equivalence classes of vertices, and the same happens with $i_3 = 6$ and other forced pairs $\{2, 5\}, \{4, 7\}$.

For $i_1 = 4$ we of course get only IV and the claim is proved.

Note that II and III have an order-2 symmetry, while I has an order-4 dihedral symmetry and IV has an order-16 dihedral symmetry. This remark easily implies that there are 13 inequivalent embeddings of the $(9, 1)$ graph in $2T$ with a disc as only region, obtained from I, II, III, IV by adding a small leg (with a label e not shown) in one of the positions described in Fig. 12. Moreover these embeddings are subject to the following symmetries (and
Figure 11: Analysis of the edge-pairings of an octagon.

Figure 12: Inequivalent embeddings in $2T$ of the $(9, 1)$ graph with a disc as a only region.
only them):

\[ I.1(a, b, c, d, e) \leftrightarrow I.1(b, a, d, c, e) \quad I.3(a, b, c, d, e) \leftrightarrow I.3(d, c, b, a, e) \]

\[ II.1(a, b, c, d, e) \leftrightarrow II.1(d, c, b, a, e) \quad II.5(a, b, c, d, e) \leftrightarrow II.5(d, c, b, a, e) \]

\[ IV(a, b, c, d, e) \leftrightarrow IV(d, c, b, a, e). \]

Each of the 8 cases without symmetries contributes to \( \nu \) with the number of ordered 5-tuples with sum \( k \), whence an 8\((k-1)\) summand. To compute the contribution of the cases with symmetries, which is the same for all of them, we use the notation of I.1, so we must count the 5-tuples \((a, b, c, d, e)\) with sum \( k \) up to the symmetry \((a, b, c, d, e) \leftrightarrow (b, a, d, c, e)\). If \( b \neq a \) we can take \( a > b \), while for \( b = a \) we can take \( c \leq d \), whence

\[
k - 4 \sum_{e=1}^{k-4} \left( \sum_{a=2}^{k-e-3 \min\{a-1, k-e-a-2\}} \sum_{b=1}^{k-e-a-b-1} (k - e - a - b - 1) + \sum_{a=1}^{[(k-e-2)/2]} \left[ \frac{k - e - 2a}{2} \right] \right).
\]

We concentrate on the first sum and distinguish according to the parity of \( k \). For \( k = 2t \) we split the sum on \( e \) between the even and the odd values of \( e \), so for \( e = 2j \) and \( e = 2j - 1 \). Imposing \( 2j \leq 2t - 4 \) and \( 2j - 1 \leq 2t - 4 \) we get \( j \leq t - 2 \) in both cases, while the inequality \( a - 1 \leq k - e - a - 2 \) is equivalent respectively to \( a \leq t - j - 1 \) and \( a \leq t - j \). So we get

\[
\sum_{j=1}^{t-2} \left( \sum_{a=2}^{t-j-1} \sum_{b=1}^{2t-2j-a-2} (2t - 2j - a - b - 1) + \sum_{a=t-j}^{t-j-1} \sum_{b=1}^{2t-2j-a-2} (2t - 2j - a - b - 1) \right.
\]

\[
+ \sum_{a=2}^{t-j} \sum_{b=1}^{2t-2j-a-1} (2t - 2j - a - b) + \sum_{a=t-j+1}^{t-j} \sum_{b=1}^{2t-2j-a-1} (2t - 2j - a - b) \right).
\]

One can now show that this repeated sum actually equals the expression \( \frac{1}{6}(t-1)(t-2)(2t^2 - 6t + 3) \), which can be done in two ways:

- Since \( \sum_{s=1}^{m} s^p \) is a polynomial of degree \( p + 1 \) in \( m \), the given repeated sum is a polynomial of degree 4 in \( t \); moreover it vanishes at \( t = 1 \) and \( t = 2 \), so it is enough to make sure that the values it attains at \( t = 3 \), \( t = 4 \) and \( t = 5 \) coincide with those of \( \frac{1}{6}(t-1)(t-2)(2t^2 - 6t + 3) \);
• One can substitute in the given repeated sum the explicit expression of $\sum_{s=1}^{m} s^p$ as a polynomial of degree $p + 1$ in $m$, and carry on the computation, which can also be achieved in an automated way.

For $k = 2t + 1$ we similarly have

$$
\sum_{j=1}^{t-2} \left( \sum_{a=2}^{t-j} \sum_{b=1}^{(2t-2j)-2} (2t - 2j - a - b) \right) + \sum_{a=t-j+1}^{t-1} \left( \sum_{b=1}^{(2t-2j)-1} (2t - 2j - a - b) \right)
$$

which is computed to be $\frac{1}{3}t(t-1)^2(t-2)$, in either of the two ways already indicated above. Replacing $t = \frac{k}{2}$ in the first formula and $t = \frac{k-1}{2}$ in the second one gives $\frac{1}{36} \left( k^4 - 12k^3 + 50k^2 - 84k + x \right)$ with $x = 48$ for even $k$ and $x = 45$ for odd $k$. For the second sum we can proceed likewise, getting $\frac{1}{3}t(t-1)(t-2)$ for $k = 2t$, and $\frac{1}{6}t(t-1)(2t-1)$ for $k = 2t + 1$, whence $\frac{1}{27} \left( k^3 - 6k^2 + yk + z \right)$ with $y = 8$, $z = 0$ for even $k$ and $y = 11$, $z = -6$ for odd $k$. Putting the contributions together it is now a routine matter to see that the final formula can be written as stated.

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