SKEW CONSTACYCLIC CODES OVER THE LOCAL FROBENIUS NON-CHAIN RINGS OF ORDER 16

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ABSTRACT. We introduce skew constacyclic codes over the local Frobenius non-chain rings of order 16 by defining non-trivial automorphisms on these rings. We study the Gray images of these codes, obtaining a number of binary and quaternary codes with good parameters as images of skew cyclic codes over some of these rings.

1. Introduction

Cyclic codes are one of the most widely studied classes of codes. They were first studied by Prange in 1957 in [26] and then by several researchers, some of which are given in [5, 6, 18, 20, 27]. This is, in large part, due to the fact that they have a canonical algebraic representation. Numerous generalizations of cyclic codes have been introduced including quasi-cyclic (QC), quasi-twisted (QT) ([3, 4, 10, 21, 22, 24, 23]), skew cyclic and skew constacyclic codes [7, 8, 9]. One of the benefits of these generalizations of cyclic codes is that they have been used to produce codes with optimal or best known parameters.

Since the Chinese Remainder Theorem, as applied to codes, shows that codes over arbitrary rings, can be decomposed as codes over local rings, the most important family of rings for coding theory is the family of local rings. Specifically, any ring can

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be viewed as the direct product of local rings via the Chinese Remainder Theorem (see [11] for a complete description of this). Additionally, alphabets of size \( 2^k \) for some \( k \) have been increasingly important due to the fact that many have a natural Gray map to the binary Hamming space, see [13] for a complete description. Codes over the rings of order 4 and order 8 as well as codes over chain rings have been well studied. Hence, the next case to consider is non-chain local rings of order 16.

In [25], the local Frobenius non-chain rings of order 16 were classified. Codes over the local Frobenius non-chain rings were studied in [15] and [16] and a Gray map to the binary Hamming space was defined which applies to each of the rings in this family. Cyclic and constacyclic codes over these rings were studied in [12], and in [14] respectively, and relationship between cyclic and constacyclic codes was given. Numerous optimal binary codes were also obtained via the Gray map.

In this paper, we shall extend this work by studying skew constacyclic codes over these rings. Central to this study, we define non-trivial automorphisms on each of the rings. We use the Gray map, established in [13], to study the binary images of codes over these rings and we find numerous optimal binary codes. Additionally, we use a Gray map to the quaternary space to find a number of new linear codes over \( \mathbb{Z}_4 \).

2. Preliminares

2.1. Codes. We denote the finite field of order \( q \) by \( \mathbb{F}_q \) and integers modulo \( n \) by \( \mathbb{Z}_n \). The alphabets for the codes that we shall use in this paper are all commutative Frobenius rings. They can be characterized as follows. Let \( R \) be a finite ring and assume that all rings contain a multiplicative identity.

Let \( G \) be a finite abelian group. The set of all characters of \( G \) is the set \( \hat{G} = Hom_{\mathbb{Z}}(G, C^*) \), where \( C^* \) is the multiplicative group of nonzero complex numbers. The set \( \hat{G} \) is a group and is isomorphic to \( G \) but not canonically. If a finite abelian group \( G \) is the additive group of a module \( M \) over a ring \( R \), then the character group \( \hat{M} \) inherits an \( R \)-module structure (see [28] for details). See [11], for a complete description of the use of the character group in coding theory.

Let \( \hat{R} \) be the character module of the ring \( R \). For a finite ring \( R \) the following are equivalent.

- \( R \) is a Frobenius ring.
- As a left module, \( \hat{R} \cong \hat{R}R \).
- As a right module, \( \hat{R} \cong R\hat{R} \).

A nonzero module over \( R \) is simple if it has no non-trivial submodules. Given any left \( R \)-module \( M \), the socle, \( soc(M) \), is the sum of all the simple submodules of \( M \), see [28] for details.

This family of rings is the canonical choice for alphabets since it is the largest class of rings for which the MacWilliams relations apply (see [11, 28, 29], for a complete explanation).

A code \( C \) of length \( n \) is a subset of \( R^n \) where \( R \) is a finite commutative Frobenius ring. If the code is an \( R \)-submodule of \( R^n \) then we say that it is linear. The usual inner-product is attached to the ambient space, namely \( \langle v, w \rangle = \sum v_i w_i \) and the orthogonal with respect to this inner-product is denoted by \( C^\perp \). Since the ring is Frobenius we know that for all linear codes \( C \), we have that \( |C||C^\perp| = |R|^n \) (see [11] for a proof of this fact).

A linear code \( C \) over a ring \( R \) is constacyclic if \( \mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in C \) implies
that \((\lambda c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C\) for some unit \(\lambda \in R\). If \(\lambda = 1\), then the code is said to be cyclic and if \(\lambda = -1\), then the code is said to be negacyclic.

**Definition 2.1.** Let \(c \in \mathbb{F}_2^{4n}\) with \(c = (c_0, c_1, \ldots, c_{4n-1}) = (c^{(0)}|c^{(1)}|c^{(2)}|c^{(3)})\), \(c^{(i)} \in \mathbb{F}_2^n\) for \(i = 0, 1, 2, 3\). Let \(\sigma \otimes 4\) be a map from \(\mathbb{F}_2^{4n}\) to \(\mathbb{F}_2^{4n}\) given by

\[
\sigma \otimes 4(c) = (\sigma (c^{(0)}), \sigma (c^{(1)}), \sigma (c^{(2)}), \sigma (c^{(3)}))
\]

where \(\sigma\) is a cyclic shift from \(\mathbb{F}_2^n\) to \(\mathbb{F}_2^n\) given by

\[
\sigma (c^{(i)}) = ((c^{(i,n-1)}), (c^{(i,0)}), (c^{(i,1)}), (c^{(i,2)}), \ldots, (c^{(i,n-2)}))
\]

for every \(c^{(i)} = (c^{(i,0)}, \ldots, c^{(i,n-1)})\) where \(c^{(i,j)} \in \mathbb{F}_2, 0 \leq j \leq n - 1\). Then a code of length \(4n\) over \(\mathbb{F}_2\) is said to be a quasi-cyclic (QC) code of index 4 if \(\sigma \otimes 4(C) = C\).

Note that one may define quasi-cyclic in a different manner, but this definition is permutation equivalent to the definition which states that a quasi-cyclic code of index 4 is a code held invariant by 4 applications of the cyclic shift. We use this notation to aid in the comprehension of later results.

For a complete description of codes over commutative Frobenius rings and any undefined terms see [11].

### 2.2. Rings

In [25], the local Frobenius non-chain rings of order 16 are given as follows

1. \(R_1 = \mathbb{F}_2[u, v]/\langle u^2, v^2 \rangle\).
2. \(R_2 = \mathbb{F}_2[u, v]/\langle u^2 + u, uv \rangle\).
3. \(R_3 = \mathbb{Z}_4[x]/\langle x^2 \rangle\).
4. \(R_4 = \mathbb{Z}_4[x]/\langle x^2 - 2x \rangle\).
5. \(R_5 = \mathbb{Z}_4[x, y]/\langle x^2, xy - 2, y^2, 2x, 2y \rangle\).
6. \(R_6 = \mathbb{Z}_4[x, y]/\langle x^2 - 2, xy - 2, y^2, 2x, 2y \rangle\).
7. \(R_7 = \mathbb{Z}_8[x]/\langle x^2 - 4, 2x \rangle\).

We shall maintain this notation throughout the paper, that is when referring to the ring \(R_i\) we mean the ring \(R_i\) in this list.

In [25], the structure of these codes is described as follows. Let \(R\) be a local Frobenius non-chain ring of order 16. Let \(m\) be the maximal ideal of \(R\), with \(|m| = 8\). Then the ring has two elements \(u\) and \(v\) such that \(m = \langle u, v \rangle\). Moreover, there is an element \(w\) in these rings with \(\text{soc}(R) = \langle w \rangle = \{0, w\}\). The following diagram shows the ideal structure for these rings.

```
(1)  
|    |
|--|--|
|  |  |
(0) (u, v) (u) (v) (u+v) (w)
```

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If \( a \) is an ideal in a ring, then we denote the dual of the ideal by \( a^\perp \). See [11] for a description of ideals in terms of coding theory.

We have that \( \text{soc}(R) = m^\perp \) and we note that each ring has 5 non-trivial ideals. The maximal ideal \( m = \langle u, v \rangle \) is of size 8, the socle is of size 2 and the remaining 3 ideals are of size 4.

Given the ideal structure of these rings it is possible to write every element of the ring uniquely in the form \( a + bu + cv + dw \) where \( a, b, c, d \in \mathbb{F}_2 \). Of course, this does not imply that the additive structure of the ring is \( \mathbb{Z}_2 \). In general, it is not. The possibilities for the additive structure are \( \mathbb{Z}_2^4, \mathbb{Z}_4 \times \mathbb{Z}_2^2, \mathbb{Z}_8 \times \mathbb{Z}_2 \) and \( \mathbb{Z}_{16} \) (see [15, 16, 25], for a complete explanation).

The group of units of the ring \( R \) consists of precisely half the elements and is \( U(R) = \{1, 1 + u + v, 1 + u + v, 1 + w, 1 + u + w, 1 + v + w, 1 + u + v + w\} \).

The Gray map used in [15] and [16] follows from the general Gray map given in [13] and is defined as follows: \( \phi : R \rightarrow \mathbb{F}_2^{4n} \),

\[
\phi(a + bu + cv + dw) = (d, c + d, b + d, a + b + c + d).
\]

This map is extended component-wise to \( \phi : R^n \rightarrow \mathbb{F}_2^{4n} \), by

\[
\phi(a_0, \ldots, a_{n-1}) = (d_0, \ldots, d_{n-1}, c_0 + d_0, \ldots, c_{n-1} + d_{n-1}, b_0 + d_0, \ldots, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1})
\]

where \( a_i = a_i + ub_i + vc_i + wd_i \) for \( i = 0, \ldots, n - 1 \).

The Hamming weight of a vector \( c \) is denoted by \( w_H(c) \) and is defined as the number of non-zero coordinates of the vector \( c \). The Lee weight of a vector \( c \) is denoted by \( w_L(c) \) and is defined as the Hamming weight of the Gray image of the vector \( c \), that is, \( w_L(c) = w_H(\phi(c)) \). The minimum weight of a code is the smallest weight of all non-zero vectors in the code.

It is shown in [15] and [16] that if \( C \) is a linear code over a Frobenius local ring of order 16 of length \( n \), size \( 2^k \) and minimum Lee weight \( d \), then \( \phi(C) \) is a binary code with length \( 4n \), minimum Hamming distance \( d \) and size \( 2^k \).

3. Skew Constacyclic Codes over \( R \)

We shall now describe the setting for skew constacyclic codes. Let \( \theta \) be a non-trivial automorphism on \( R \). The ring

\[
R[x, \theta] = \{a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} : a_i \in R, n \in \mathbb{N}\}
\]

is called the skew polynomial ring. The addition in the ring \( R[x, \theta] \) is the usual polynomial addition and multiplication is defined as follows:

\[
(ax^i)(bx^j) = a\theta^i(b)x^{i+j}.
\]

It follows that this ring is, in general, a non-commutative ring.

**Definition 3.1.** A subset \( C \) of \( R^n \) is called a skew cyclic code of length \( n \) if \( C \) satisfies the following conditions:

1. The code \( C \) is an \( R \)-submodule of \( R^n \).
2. If \( c = (c_0, c_1, \ldots, c_{n-1}) \in C \), then \( \sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C \),
   where \( \sigma_\theta \) is the skew cyclic shift operator of \( C \).

If \( \theta \) is the identity then the code is said to be a cyclic code.

Define the following quotient space \( R_{\theta, n} = R[x, \theta]/\langle x^n - 1 \rangle \).
Let $f(x) + \langle x^n - 1 \rangle$ be an element in $R_{\theta,n}$ and let $r(x) \in R[x, \theta]$. Define multiplication on the left as follows,

$$r(x)(f(x) + \langle x^n - 1 \rangle) = r(x)f(x) + \langle x^n - 1 \rangle$$

for any $r(x) \in R[x, \theta]$. Using this definition, it is straightforward to prove the following. See [7] for details.

**Theorem 3.2.** Let $R$ be a finite commutative ring, then $R_{\theta,n}$ is a left $R[x, \theta]$-module.

We make the standard correspondence between vectors and polynomials, namely the vector $(c_0, c_1, \ldots, c_{n-1})$ is associated with the polynomial $\sum_{i=0}^{n-1} c_i x^i$. This leads to the following theorem.

**Theorem 3.3.** A code $C$ over $R$ of length $n$ is a skew cyclic code if and only if $C$ is a left $R[x, \theta]$-submodule of $R_{\theta,n}$.

**Proof.** Since the code is linear over $R$, it is an $R$-submodule of $R^n$ giving that the sum of any two elements and the scalar multiple of a vector in the submodule of $R_{\theta,n}$ is in the submodule. Given a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ in the submodule, multiplication on the left by $x$ gives $\theta(a_{n-1}) + \theta(a_0)x + \theta(a_1)x^2 + \cdots + \theta(a_{n-2})x^{n-1}$. This gives the second condition of being a skew cyclic code. The proof of the other direction is identical. \qed

This theorem allows us to study skew cyclic codes as left $R[x, \theta]$-submodules of the non-commutative ring $R_{\theta,n}$ and to use the theory of non-commutative rings when studying this ambient ring (see [29], for a detailed explanation).

The next theorem follows in the usual study of skew cyclic codes. (See [7] for similar results). It is not intended to give a complete description of all skew cyclic codes, but rather describe skew cyclic codes in manner that aids in their computation.

**Theorem 3.4.** Let $C$ be a skew cyclic code over $R$ of length $n$ and let $f(x)$ be a polynomial in $C$ of minimal degree. If $f(x)$ is a monic polynomial, then $C = \langle f(x) \rangle$, where $f(x)$ is a right divisor of $x^n - 1$.

**Proof.** The standard proof applies noting that we require $f(x)$ to be a right divisor of $x^n - 1$ so that the code is a left submodule. \qed

From this theorem, it is then of the utmost importance to factor the polynomial $x^n - 1$ in the skew polynomial ring $R[x, \theta]$.

We can naturally extend the definition of a skew cyclic code to a skew constacyclic code in the following manner.

**Definition 3.5.** Let $\lambda$ be a unit in $R$ and $\theta$ be a non-trivial automorphism of $R$. A linear code $C$ of length $n$ over $R$ is said to be a skew $\lambda$-constacyclic code if and only if $C$ is invariant under the skew $\lambda$-constacyclic shift operator $\sigma_{\theta, \lambda}$, defined by

$$\sigma_{\theta, \lambda}(c_0, c_1, \ldots, c_{n-1}) = (\lambda \theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})).$$

This leads naturally to the following characterization.

**Theorem 3.6.** Let $C$ be a skew $\lambda$-constacyclic code over $R$ of length $n$ and let $f(x)$ be a polynomial in $C$ of minimal degree. If $f(x)$ is a monic polynomial, then $C = \langle f(x) \rangle$, where $f(x)$ is a right divisor of $x^n - \lambda$.

**Proof.** The proof is similar to the the proof of Theorem 3.4, except that we are using the polynomial $x^n - \lambda$ rather than $x^n - 1$. \qed
3.1. Non-trivial automorphisms. To produce skew cyclic codes over a ring $R$, we need a non-trivial automorphism $\theta$ of the ring $R$. For each of the local non-chain Frobenius rings of order 16, we shall give at least one non-trivial automorphism.

3.1.1. Non-trivial automorphisms on the finite rings $R_1$, $R_2$ and $R_5$. We begin with the rings $R_1$, $R_2$ and $R_5$.

1. The ring $\mathbb{F}_2[u,v]/\langle u^2, v^2 \rangle$, where $w = uv$, was given as $R_1$ in [12]. Notice that this ring is a member of the family of rings $R_k$ studied in [17], [18], and [19]. This family of rings has been shown to be very useful in finding good binary self-dual codes as well as good binary QC codes. A non-trivial automorphism of $R_1$ is given as follows:

$$\theta_1(a + bu + cv + d uv) = a + cu + bv + d uv.$$ 

Then we have that

$$w_H(\phi(\theta_1((a + bu + cv + d uv)))) = w_H(\phi(a + cu + bv + d uv)) = w_H(d, b + d, c + d, a + b + c + d)$$

and

$$w_H(\phi((1 + u)\theta_1(a + bu + cv + d uv))) = w_H(\phi((1 + u)(a + cu + bv + d uv))) = w_H(\phi(a + (a + c)u + bv + (d + b)uv)) = w_H(b + d, d, a + b + c + d, c + d).$$

Similarly, we have that

$$w_H(\phi((1 + v)\theta_1(a + bu + cv + d uv))) = w_H(c + d, a + b + c + d, b + d).$$

2. The ring $\mathbb{F}_2[u,v]/\langle u^2 + v^2, uv \rangle$, where $w = u^2$, was given as $R_2$ in [12]. A non-trivial automorphism of $R_2$ is given as follows:

$$\theta_2(a + bu + cv + d v^2) = a + cu + bv + d v^2.$$ 

Then we have that

$$w_H(\phi(\theta_2((a + bu + cv + d v^2)))) = w_H(\phi(a + cu + bv + d v^2)) = w_H(d, b + d, c + d, a + b + c + d)$$

and

$$w_H(\phi((1 + u + v + v^2)\theta_2(a + bu + cv + d v^2))) = w_H(a + b + c + d, c + d, b + d, d).$$

3. The ring $\mathbb{Z}_4[x,y]/\langle x^2, xy - 2, y^2, 2x, 2y \rangle$, where $w = xy = 2$, was given as $R_5$ in [12]. A non-trivial automorphism on $R_5$ is given as follows:

$$\theta_5(a + bx + cy + d 2) = a + cx + by + d 2.$$ 

Then we have that

$$w_H(\phi(\theta_5((a + bx + cy + d 2)))) = w_H(\phi(a + cx + by + d 2)) = w_H(d, b + d, c + d, a + b + c + d)$$

and

$$w_H(\phi((1 + x)\theta_5(a + bx + cy + d 2))) = w_H(b + d, d, a + b + c + d, c + d).$$

Similarly we have

$$w_H(\phi((1 + y)\theta_5(a + bx + cy + d 2))) = w_H(c + d, a + b + c + d, d, b + d).$$
Proposition 1. Let $R_1 = \mathbb{F}_2[u,v]/\langle u^2, v^2 \rangle$. Let $\phi$ be the map given in Equation 2.2, $\sigma_{\theta_1}$ be the map defined in Definition 3.1 and $\sigma^{\otimes 4}$ be as defined in Definition 2.1. Then

$$\phi \sigma_{\theta_1} = \Pi \sigma^{\otimes 4} \phi$$

where $\Pi$ is a permutation defined by $\Pi(x,y,z,t) = (x,z,y,t)$ for $x, y, z, t \in F_2^n$.

Proof. Let $r_i = a_i + b_i u + c_i v + d_i uv$ be the elements of $R_1$ for $i = 0, 1, ..., n-1$. Then

$$\sigma_{\theta_1}(r_0, ..., r_{n-1}) = (\theta_1(r_{n-1}), \theta_1(r_0), ..., \theta_1(r_{n-2})) = (a_{n-1} + c_{n-1} u + b_{n-1} v + d_{n-1} uv, a_0 + c_0 u + b_0 v + d_0 uv, ..., a_{n-2} + c_{n-2} u + b_{n-2} v + d_{n-2} uv).$$

By applying $\phi$, we have

$$\phi(\sigma_{\theta_1}(r_0, ..., r_{n-1})) = \begin{pmatrix} d_{n-1}, ..., d_{n-2}, b_{n-1} + d_{n-1}, ..., b_{n-2} + d_{n-1}, c_{n-1} + d_{n-1}, \ldots, c_{n-2} + d_{n-1}, a_{n-1} + c_{n-1} + d_{n-1}, \ldots, a_{n-2} + c_{n-2} + d_{n-1} \end{pmatrix}.$$

On the other hand

$$\phi(r_0, ..., r_{n-1}) = \begin{pmatrix} d_0, ..., d_{n-1}, c_0 + d_0, ..., c_{n-1} + d_0, b_0 + d_0, ..., b_{n-1} + d_0, a_0 + b_0 + c_0 + d_0, ..., a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} \end{pmatrix}.$$

If we apply $\sigma^{\otimes 4}$, we have

$$\sigma^{\otimes 4} \phi(r_0, ..., r_{n-1}) = \begin{pmatrix} d_{n-1}, ..., d_{n-2}, c_{n-1} + d_{n-1}, ..., c_{n-2} + d_{n-1}, b_{n-1} + d_{n-1}, \ldots, b_{n-2} + d_{n-1}, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2} \end{pmatrix}.$$

By applying $\Pi$ we have the expected result. \qed

Theorem 3.7. The Gray image of a skew cyclic code over $R_1$ of length $n$ is permutation equivalent to a QC code of index 4 over $\mathbb{Z}_2$ with length $4n$.

Proof. Let $C$ be a skew cyclic code over $R_1$ of length $n$. That is $\sigma_{\theta_1}(C) = C$. If we apply $\phi$, we have $\phi(\sigma_{\theta_1}(C)) = \phi(C)$. From Proposition 1, we get $\phi(\sigma_{\theta_1}(C)) = \Pi \sigma^{\otimes 4}(\phi(C))$. So $\phi(C)$ is permutation equivalent to a QC code of index 4 over $\mathbb{Z}_2$ with length $4n$. \qed

Proposition 2. Let $R_1 = \mathbb{F}_2[u,v]/\langle u^2, v^2 \rangle$ and $\lambda_1 = 1 + u$. Let $\phi$ be the map given in Equation 2.2, $\sigma_{\theta_1}$ be the map defined in Definition 3.1 and $\sigma^{\otimes 4}$ be as defined in Definition 2.1. Then

$$\phi \sigma_{\theta_1, \lambda_1} = \Gamma_1 \Pi \sigma^{\otimes 4} \phi$$

where $\sigma_{\theta_1, \lambda_1}$ is the skew $\lambda_1$-cyclic shift, $\Pi$ is a permutation defined by $\Pi(x,y,z,t) = (x,z,y,t)$ for $x, y, z, t \in F_2^n$ and $\Gamma_1$ is a permutation of $F_2^{4n}$ defined by $\Gamma_1(a_0, a_1, ..., a_{4n-1}) = (a_{\tau(0)}, a_{\tau(1)}, ..., a_{\tau(4n-1)})$ where $\tau = (0,n)(2n, 3n)$ is a permutation of $\{0, 1, ..., 4n-1\}$. 
Proof. Let $r_i = a_i + b_i u + c_i v + d_i u v$ be the elements of $R_1$ for $i = 0, 1, \ldots, n-1$. Then
\[
\sigma_{\theta_1, 1+u}(r_0, \ldots, r_{n-1}) = (1 + u) \theta_1(r_{n-1}), \theta_1(r_0), \ldots, \theta_1(r_{n-2})
\]
\[
= \left( a_{n-1} + (a_{n-1} + c_{n-1}) u + b_{n-1} v + (b_{n-1} + d_{n-1}) u v, \right.
\]
\[
a_0 + c_0 u + b_0 v + d_0 u v, \ldots, a_{n-2} + c_{n-2} u + b_{n-2} v + d_{n-2} u v \right).
\]
By applying $\phi$, we have
\[
\phi(\sigma_{\theta_1, 1+u}(r_0, \ldots, r_{n-1}))
\]
\[
= \left( b_{n-1} + d_{n-1} + d_0, \ldots, d_{n-2} + d_{n-1}, b_0 + d_0, \ldots, b_{n-2} + d_{n-2},
\right.
\]
\[
a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, c_0 + d_0, \ldots, c_{n-2} + d_{n-2},
\]
\[
c_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, \ldots, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} + d_{n-1}.
\]
On the other hand,
\[
\phi(r_0, \ldots, r_{n-1}) = \left( d_0, \ldots, d_{n-1}, c_0 + d_0, \ldots, c_{n-1} + d_{n-1}, b_0 + d_0, \ldots, b_{n-1} + d_{n-1},
\right.
\]
\[
a_0 + b_0 + c_0 + d_0, \ldots, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}.
\]
If we apply $\sigma^{\otimes 4}$, we have
\[
\sigma^{\otimes 4} \phi(r_0, \ldots, r_{n-1})
\]
\[
= \left( d_{n-1} + d_{n-2} + c_{n-1} + d_{n-1}, \ldots, c_{n-2} + d_{n-2} + b_{n-1} + d_{n-1},
\right.
\]
\[
\ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2}.
\]
By applying $\Pi$ and $\Gamma_1$, respectively, we have $\phi \sigma_{\theta_1, 1+u} = \Gamma_1 \Pi \sigma^{\otimes 4} \phi$. \hfill \square

Theorem 3.8. The Gray image of a skew $\lambda_1$-cyclic code over $R_1$ of length $n$ is permutation equivalent to a QC code of index 4 over $\mathbb{Z}_2$ with length $4n$.

Proof. Let $C$ be a skew $\lambda_1$-cyclic code over $R_1$ of length $n$. That is $\sigma_{\theta_1, 1+u}(C) = C$. If we apply $\phi$, we have $\phi(\sigma_{\theta_1, 1+u}(C)) = \phi(C)$. From Proposition 2, we get $\phi(\sigma_{\theta_1, 1+u}(C)) = \phi(C) = \Gamma_1 \Pi \sigma^{\otimes 4} \phi(C)$. Therefore, $\phi(C)$ is permutation equivalent to a QC code of index 4 over $\mathbb{Z}_2$ with length $4n$. \hfill \square

Proposition 3. Let $R_2 = \mathbb{F}_2[u, v]/(u^2 + v^2, uv)$. Let $\phi$ be the map given in Equation 2.2, $\sigma_{\theta_1}$ be the map defined in Definition 3.1 and $\sigma^{\otimes 4}$ as defined in Definition 2.1. Then
\[
\phi \sigma_{\theta_2} = \Pi \sigma^{\otimes 4} \phi
\]
where $\Pi$ is a permutation defined by $\Pi(x, y, z, t) = (x, z, y, t)$ for $x, y, z, t \in \mathbb{F}_2^2$.

Proof. Let $r_i = a_i + b_i u + c_i v + d_i u v^2$ be the elements of $R_2$ for $i = 0, 1, \ldots, n-1$. Then
\[
\sigma_{\theta_2}(r_0, \ldots, r_{n-1}) = (\theta_2(r_{n-1}), \theta_2(r_0), \ldots, \theta_2(r_{n-2})
\]
\[
= \left( a_{n-1} + c_{n-1} u + b_{n-1} v + d_{n-1} u v^2, a_0 + c_0 u + b_0 v + d_0 u v^2, \right.
\]
\[
\ldots, a_{n-2} + c_{n-2} u + b_{n-2} v + d_{n-2} u v^2 \right).
\]
By applying $\phi$, we have
\[
\phi(\sigma_{\theta_2}(r_0, \ldots, r_{n-1}))
\]
\[
= \left( d_{n-1} + d_{n-2} + b_{n-1} + d_{n-1}, \ldots, b_{n-2} + d_{n-2} + c_{n-1} + d_{n-1},
\right.
\]
\[
\ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2} \right).\]
On the other hand
\[ \phi(r_0, ..., r_{n-1}) = \left( d_0, ..., d_{n-1}, c_0 + d_0, ..., c_{n-1} + d_{n-1}, b_0 + d_0, ..., b_{n-1} + d_{n-1} \right). \]

If we apply \( \sigma_4^{\otimes 4} \), we have
\[ \sigma_4^{\otimes 4} \phi(r_0, ..., r_{n-1}) = \left( d_{n-1}, ..., d_{n-2}, c_{n-1} + d_{n-1}, ..., c_{n-2} + d_{n-2}, b_{n-1} + d_{n-1}, ..., b_{n-2} + d_{n-2}, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, ..., a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2} \right). \]

By applying \( \Pi \) we have the expected result.

**Theorem 3.9.** The Gray image of a skew cyclic code over \( R_2 \) of length \( n \) is permutation equivalent to a QC code of index 4 over \( \mathbb{Z}_2 \) with length \( 4n \).

**Proof.** Let \( C \) be a skew cyclic code over \( R_2 \) of length \( n \). That is \( \sigma_2(C) = C \). If we apply \( \phi \), we have \( \phi(\sigma_2(C)) = \phi(C) \). From Proposition 3, we get \( \phi(\sigma_2(C)) = \Pi \sigma_4^{\otimes 4}(\phi(C)) \). So \( \phi(C) \) is permutation equivalent to a QC code of index 4 over \( \mathbb{Z}_2 \) with length \( 4n \).

**Proposition 4.** Let \( R_2 = \mathbb{F}_2[u, v]/(u^2 + v^2, uv) \) and \( \lambda_2 = 1 + u + v + v^2 \). Let \( \phi \) be the map given in Equation 2.2, \( \sigma_2 \) be the map defined in Definition 3.1 and \( \sigma_4^{\otimes 4} \) be as defined in Definition 2.1. Then
\[ \phi \sigma_2, \lambda_2 = \Gamma_2 \Pi \sigma_4^{\otimes 4} \phi \]
where \( \sigma_2, \lambda_2 \) is the skew \( \lambda_2 \)-cyclic shift, \( \Pi \) is a permutation defined by \( \Pi(x, y, z, t) = (x, z, y, t) \) for \( x, y, z, t \in F_2^n \) and \( \Gamma_2 \) is a permutation of \( F_2^{4n} \) defined by
\[ \Gamma_2(a_0, a_1, ..., a_{4n-1}) = (a_{r(0)}, a_{r(1)}, ..., a_{r(4n-1)}) \]
where \( r = (0, 3n)(n, 2n) \) is a permutation of \( \{0, 1, ..., 4n - 1\} \).

**Proof.** Let \( r_i = a_i + b_i u + c_i v + d_i u^2 \) be the elements of \( R_2 \) for \( i = 0, 1, ..., n-1 \). Then
\[ \sigma_{2,1++1++...}(r_0, ..., r_{n-1}) = ((1 + u + v + v^2)\theta_2(r_{n-1}), \theta_2(r_0), ..., \theta_2(r_{n-2})) \]
\[ = \left( a_{n-1} + (a_{n-1} + c_{n-1})u + (a_{n-1} + b_{n-1})v + (a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1})v^2, a_0 + c_0u + b_0v + d_0v^2, ..., a_{n-2} + c_{n-2}u + b_{n-2}v + d_{n-2}v^2 \right). \]

By applying \( \phi \), we have
\[ \phi(\sigma_{2,1++1++...}(r_0, ..., r_{n-1})) = \left( a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, ..., a_{n-2} + c_{n-2} + d_{n-2} + d_{n-1}, a_0 + b_0 + c_0 + d_0, ..., a_{n-2} + c_{n-2} + d_{n-2} + d_{n-1} \right). \]

On the other hand
\[ \phi(r_0, ..., r_{n-1}) = \left( d_0, ..., d_{n-1}, c_0 + d_0, ..., c_{n-1} + d_{n-1}, b_0 + d_0, ..., b_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, ..., a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} \right). \]

If we apply \( \sigma_4^{\otimes 4} \), we have
\[ \sigma_4^{\otimes 4} \phi(r_0, ..., r_{n-1}) = \left( d_{n-1}, ..., d_{n-2}, c_{n-1} + d_{n-1}, ..., c_{n-2} + d_{n-2}, b_{n-1} + d_{n-1}, ..., b_{n-2} + d_{n-2}, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, ..., a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2} \right). \]
By applying \( \Pi \) and \( \Gamma_2 \), respectively, we have \( \phi \sigma_{\theta_2,1+n+v+s+2} = \Gamma_2 \Pi \sigma^{\otimes 4} \phi \). \( \square \)

**Theorem 3.10.** The Gray image of a skew \( \lambda_2 \)-cyclic code over \( R_2 \) of length \( n \) is permutation equivalent to a QC code of index 4 over \( \mathbb{Z}_2 \) with length \( 4n \).

**Proof.** Let \( C \) be a skew \( \lambda_2 \)-cyclic code over \( R_2 \) of length \( n \). That is \( \sigma_{\theta_2,1+n+v+s+2}(C) = C \). If we apply \( \phi \), we have \( \phi(\sigma_{\theta_2,1+n+v+s+2}(C)) = \phi(C) \). From the Proposition 4, we get \( \phi(\sigma_{\theta_2,1+n+v+s+2}(C)) = \phi(C) = \Gamma_2 \Pi \sigma^{\otimes 4}(\phi(C)) \). Therefore, \( \phi(C) \) is permutation equivalent to a QC code of index 4 over \( \mathbb{Z}_2 \) with length \( 4n \). \( \square \)

The proofs for the following four results are similar to the proofs just given.

**Proposition 5.** Let \( R_5 = \mathbb{Z}_4[x, y]/\langle x^2, xy - 2, y^2, 2y \rangle \). Let \( \phi \) be the map given in Equation 2.2, \( \sigma_{\theta_5} \) be the map defined in Definition 3.1 and \( \sigma^{\otimes 4} \) be as defined in Definition 2.1. Then

\[
\phi \sigma_{\theta_5} = \Pi \sigma^{\otimes 4} \phi
\]

where \( \Pi \) is a permutation defined by \( \Pi(x, y, z, t) = (x, z, y, t) \) for \( x, y, z, t \in F_4^n \).

**Theorem 3.11.** The Gray image of a skew cyclic code over \( R_5 \) of length \( n \) is permutation equivalent to a QC code of index 4 over \( \mathbb{Z}_2 \) with length \( 4n \).

**Proposition 6.** Let \( R_5 = \mathbb{Z}_4[x, y]/\langle x^2, xy - 2, y^2, 2x, 2y \rangle \) and \( \lambda_5 = 1 + y \) (or \( 1 + x \)). Let \( \phi \) be the map given in Equation 2.2, \( \sigma_{\theta_5} \) be the map defined in Definition 3.1 and \( \sigma^{\otimes 4} \) be as defined in Definition 2.1. Then

\[
\phi \sigma_{\theta_5, \lambda_5} = \Gamma_5 \Pi \sigma^{\otimes 4} \phi
\]

where \( \sigma_{\theta_5, \lambda_5} \) is the skew \( \lambda_5 \)-cyclic shift, \( \Pi \) is a permutation defined by \( \Pi(x, y, z, t) = (x, z, y, t) \) for \( x, y, z, t \in F_4^n \) and \( \Gamma_5 \) is a permutation of \( F_4^{4n} \) defined by \( \Gamma_5(a_0, a_1, \ldots, a_{4n-1}) = (a_{\tau(0)}, a_{\tau(1)}, \ldots, a_{\tau(4n-1)}) \) where \( \tau = (0, 2n)(n, 3n) \) (or \( \tau = (0, n)(2n, 3n) \)) is a permutation of \( \{0, 1, \ldots, 4n - 1\} \).

**Theorem 3.12.** The Gray image of a skew \( \lambda_5 \)-cyclic code over \( R_5 \) of length \( n \) is permutation equivalent to a QC code of index 4 over \( \mathbb{Z}_2 \) with length \( 4n \).

3.1.2. Non-trivial automorphisms of the finite rings \( R_3, R_4, R_6 \) and \( R_7 \). We now proceed to the rings \( R_3, R_4, R_6 \) and \( R_7 \). It is possible, of course, to discuss skew constacyclic codes over these rings, however their images under the canonical Gray map are not necessarily QC codes.

4. The ring \( \mathbb{Z}_4[x]/\langle x^2 \rangle \), where \( w = 2x \) was given as \( R_3 \) in [4]. Two non-trivial automorphisms on \( R_3 \) are given as follows, with \( \Theta_{3,i} : R_3 \to R_3 \):

(1) \[ \Theta_{3,1}(a + bx) = a + 3bx \]

and

(2) \[ \Theta_{3,2}(a + bx) = (a + 2b) + 3bx \]

where \( a, b \in \mathbb{Z}_4 \).

5. The ring \( \mathbb{Z}_4[x]/\langle x^2 - 2x \rangle \), where \( w = 2x \) was given as \( R_4 \) in [4]. Three non-trivial automorphisms on \( R_4 \) are given as follows, with \( \Theta_{4,i} : R_4 \to R_4 \):

(3) \[ \Theta_{4,1}(a + bx) = a + 3bx \]

(4) \[ \Theta_{4,2}(a + bx) = (a + 2b) + 3bx \]
and

\[ \Theta_{4,3}(a + bx) = (a + 2b) + bx \]

where \( a, b \in \mathbb{Z}_4 \).

6. The ring \( \mathbb{Z}_4[x, y]/(x^2 - 2, xy - 2, y^2, 2x, 2y) \), where \( w = xy = 2 \) was given as \( R_6 \) in [4]. Three non-trivial automorphisms on \( R_6 \) are given as follows with \( \Theta_{6,i} : R_6 \rightarrow R_6 \):

\[ \Theta_{6,1}(a + bx + cy + d2) = a + bx + cy + (b + d)2 \]

\[ \Theta_{6,2}(a + bx + cy + d2) = a + bx + cy + (c + d)2 \]

and

\[ \Theta_{6,3}(a + bx + cy + d2) = a + bx + cy + (b + c + d)2 \]

where \( a, b, c, d \in \mathbb{Z}_2 \).

7. The ring \( \mathbb{Z}_8[u]/(u^2 + 4, 2u) \), where \( w = 4 \) was given as \( R_7 \) in [4]. A non-trivial automorphism on \( R_7 \) is given as follows, with \( \Theta_7 : R_7 \rightarrow R_7 \):

\[ \Theta_7(a + bu) = (a + 4b) + bu \]

where \( a, b \in \mathbb{Z}_8 \).

Note that for the rings \( R_3, R_4, R_6, R_7 \), we do not necessarily get QC codes as their images, since we do not have the equality \( \phi \sigma_{\theta_{i,j}} = \Pi_{i,j} \sigma^{\otimes 4} \phi \) when we are using our Gray map where \( \Pi_{i,j} \) is as follows:

- \( \Pi_{i,j}(x, y, z, t) = (x, y, 0, y) \) for \( i = 3, j = 1 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, y, y, y) \) for \( i = 3, j = 2 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, y, 0, 0) \) for \( i = 4, j = 1 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, y, y, 0) \) for \( i = 4, j = 2 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, y, z, y + t) \) for \( i = 6, j = 1 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, y, z, z + t) \) for \( i = 6, j = 2 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, z, y + z + t) \) for \( i = 6, j = 3 \),
- \( \Pi_{i,j}(x, y, z, t) = (x, 0, y, z) \) for \( i = 7, j = 0 \), with \( \theta_{7,0} = \theta_7 \)

and \( x, y, z, t \in F_2^2 \).

For example, let \( R_3 = \mathbb{Z}_4[x]/(x^2) \). Let \( C \) be a code of length 4 over \( R_3 \) and let \( c = (1, 2 + x, 2x, x) \) be a codeword in \( C \). Then \( \sigma_{\theta_{3,1}}(c) = \sigma_{\theta_{3,1}}((1, 2 + x, 2x, x)) = (\theta_{3,1}(1), \theta_{3,1}(2 + x), \theta_{3,1}(2x)) = (3x, 1, 2 + 3x, 0) \). Applying \( \phi \), \( \phi(\sigma_{\theta_{3,1}}(c)) = \phi(\sigma_{\theta_{3,1}}(0x, 1, 2 + 3x, 0)) = \phi((0, 1, 0, 0) + (1, 0, 1, 0)x + (0, 0, 1, 0)2x + (1, 0, 1, 0)2x) = (1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1). \) Then we get

\[ \sigma^{\otimes 4}(c) = (0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1) \].

Applying \( \Pi_{3,1} \), we get

\[ \Pi_{3,1} \sigma^{\otimes 4}(c) = (0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1) \]

which is not equal to \( \phi \sigma_{\theta_{3,1}} \).
4. Computational results

In this section, we present codes with good parameters obtained as images of skew cyclic codes over some of the local Frobenius non-chain rings of order 16. Our computational results, which are carried out using Magma software [30], suggest that these rings are promising in terms of obtaining binary and quaternary codes (codes over $\mathbb{Z}_4$) with good parameters.

4.1. Optimal and new binary codes. As shown in [16], the Gray images of linear codes over the first two rings in our list are binary linear codes. From skew cyclic codes over some of the local Frobenius non-chain rings of order 16, our computational results, which are carried out using Magma software [30], suggest that these rings are promising in terms of obtaining binary and quaternary codes (codes over $\mathbb{Z}_4$) with good parameters.

The skew cyclic code generated by $g(x) = x^5 + ux + 1$ over $R_1$ has length 8 and rank 6 over $R_1$. Each such divisor generates a free skew cyclic code of length $n$ and rank $k$ over $R_1$, where a free code is isomorphic to $R^*_n$ as a module. Its Gray image will be a binary linear code of length $4n$ and dimension $4k$. As an example, let $g(x) = x^2 + ux + 1$. Then $g(x)$ is a right divisor of $x^8 - 1$ in the skew polynomial ring over $R_1$ with $x^8 - 1 = g(x)h(x)$ where $h(x) = x^8 + ux^5 + (uv + 1)x^4 + (uv + 1)x^3 + ux + 1$. The skew cyclic code generated by $g(x)$ has length 8 and rank 6 over $R_1$. Its Gray image has parameters [32, 24, 4] which turns out to be an optimal code according to [32]. In the tables below, we will give either $g(x)$ or $h(x)$, whichever one has lower degree. Table 1 lists the optimal binary linear codes obtained from skew cyclic codes $R_1$.

| $n$ | $g(x)$ or $h(x)$ | Binary Parameters |
|-----|------------------|-------------------|
| 8   | $g = x^4 + ux + 1$ | [32, 24, 4]        |
| 8   | $g = x^4 + (uv + u)x^3 + (v + u)x^2 + ux + 1$ | [32, 16, 8] |
| 6   | $h = x^2 + (u + v + 1)x + 1$ | [24, 8, 8] |
| 6   | $h = x + uv + 1$ | [24, 4, 12] |

The codes in Table 2 have the parameters of best known linear codes in [32].

| $n$ | $g(x)$ or $h(x)$ | Binary Parameters |
|-----|------------------|-------------------|
| 12  | $h = x^5 + (uv + 1)x^4 + x^3 + x^2 + (uv + v + 1)x + v + 1$ | [48, 20, 12] |
| 12  | $g = x^4 + (uv + u)x^3 + x + 1$ | [48, 32, 6] |
| 12  | $g = x^2 + ux^4 + x^3 + (u + v + 1)x^2 + v + 1$ | [48, 28, 8] |
| 16  | $h = x^4 + (uv + v)x^3 + ux^2 + 1$ | [64, 24, 16] |

In Table 3, we give information about the new binary QC codes obtained from skew cyclic codes $R_1$ according to [31].

4.2. New quaternary codes. Of the seven rings we consider in this paper, 4 of them are extensions of $\mathbb{Z}_4$: $R_3 = \mathbb{Z}_4[x]/(x^2)$, $R_4 = \mathbb{Z}_4[x]/(x^2 - 2x)$, $R_5 = \mathbb{Z}_4[x, y]/(x^2, xy - 2, y^2, 2x, 2y)$, $R_6 = \mathbb{Z}_4[x, y]/(x^2 - 2, xy - 2, y^2, 2x, 2y)$. It is possible to obtain either binary codes (not necessarily linear) or quaternary linear codes (by...
a “quaternary code” in this paper, we mean a code over \( \mathbb{Z}_4 \) from codes over any of these 4 rings. Quaternary codes have a special place in codes over rings, hence there is an online database of quaternary codes [33]. Using the third ring in our list, that is \( R_3 = \mathbb{Z}_4[x]/(x^2) \), and the automorphism \( \theta(a + bx) = a + 3bx = a - bx \) we obtained some quaternary linear codes that were not known before. Hence we contribute new codes to the database [1, 33]. There is more than one possible linear map from an extension \( R \) of \( \mathbb{Z}_4 \) to obtain codes over \( \mathbb{Z}_4 \). The following two maps are used in [2], where \( \phi_1 : R \to \mathbb{Z}_4^2 \):

\[
\phi_1(a + ub) = (b, a + b)
\]

and

\[
\phi_2(a + ub) = (a, a + 3b, 3a + 3b).
\]

These injective linear maps can be naturally extended to maps from \( R^n \) to \( \mathbb{Z}_4^{2n} \) and \( \mathbb{Z}_4^{3n} \) respectively. In the final table below, we present parameters of the new quaternary linear codes obtained from skew cyclic codes over \( R_3 \) using either \( \phi_1 \) or \( \phi_2 \). Whether \( \phi_1 \) or \( \phi_2 \) is used to obtain the quaternary code is not explicitly stated but it can be deduced by comparing the value of \( n \) in the first column, and the length of the quaternary code in the last column. If the length of the quaternary code is \( 2n \) then \( \phi_1 \) is used, and if it is \( 3n \) then \( \phi_2 \) is used. As in the case of \( R_1 \), we start with a skew cyclic code \( C \) over \( R_3 \) generated by a monic polynomial \( g(x) \) where \( g(x) \) divides \( x^n - 1 \) in the skew polynomial ring over \( R_3 \). Therefore, \( C \) is a free code of length \( n \) and rank \( k \). Its image \( \phi_1(C) \) (or \( \phi_2(C) \)) has parameters \( [2n, 4^k 2^0] \) \((3n, 4^k 2^0)\)). Because the \( \mathbb{Z}_4 \) type of the image is always of the form \( 4^k 2^0 \) for these codes, we will simply write the \( \mathbb{Z}_4 \) parameters as \( [N, 4^k, d] \) where \( d \) is the Lee weight over \( \mathbb{Z}_4 \).

**Table 3. New binary QC codes**

| \( n \) | \( g(x) \) or \( h(x) \) | Binary Parameters |
|---|---|---|
| 8 | \( g = x^3 + (uv + 1)x^2 + x + 1 \) | 32, 20, 4 |
| 10 | \( g = x^4 + (u + v + 1)x^3 + x^2 + x + 1 \) | 40, 24, 4 |
| 12 | \( h = x^3 + ux^2 + 1 \) | 48, 36, 4 |
| 14 | \( h = x^3 + (uv + 1)x^2 + x^2 + x + 1 \) | 56, 20, 7 |
| 14 | \( h = x^4 + (u + v + 1)x^3 + x^2 + 1 \) | 56, 16, 12 |
| 14 | \( h = x^3 + (u + v + 1)x^2 + 1 \) | 56, 44, 3 |
| 14 | \( h = x^3 + (u + v + 1)x^2 + 1 \) | 56, 12, 16 |
| 16 | \( h = x^5 + (uv + v + 1)x^4 + u x^3 + x^2 + vx + u + v + 1 \) | 64, 24, 16 |
| 18 | \( h = x^4 + x^2 + x^2 + x + u + v + 1 \) | 80, 16, 28 |
| 20 | \( h = x^4 + x^2 + x^2 + x + u + v + 1 \) | 72, 16, 9 |
| 28 | \( h = x + u + v + 1 \) | 112, 4, 56 |
| 28 | \( h = x^4 + x^2 + (u + 1)x^2 + (u + 1)v + 1 \) | 112, 16, 40 |
| 30 | \( h = x + uv + 1 \) | 120, 4, 60 |
| 32 | \( h = x + u + v + 1 \) | 128, 4, 64 |
| 32 | \( h = x^4 + (u + 1)x^2 + x + 1 \) | 128, 12, 32 |
Table 4. New quaternary codes

| n  | g(x) or h(x)                                            | Z₄ Parameters |
|----|---------------------------------------------------------|----------------|
| 8  | g = x³ + (u + 1)x² + 3x + u + 1                         | 24, 10, 9      |
| 8  | g = x³ + (3u + 2)x + 3u + 3                             | 24, 12, 7      |
| 8  | g = x + u + 1                                           | 16, 14, 2      |
| 12 | g = x⁶ + (u + 3)x + 1                                   | 36, 20, 8      |
| 12 | h = x⁶ + (3u + 1)x² + (u + 2)x₃ + (3u + 3)x + 3u + 3    | 24, 8, 12      |
| 12 | h = x⁶ + (3u + 3)x³ + (u + 2)x₄ + (u + 3)x + u + 1      | 36, 8, 20      |
| 12 | h = x⁶ + (2u + 1)x⁴ + 3x₃ + (2u + 1)x + 3x + 1          | 24, 10, 8      |
| 12 | h = x⁶ + 2ux⁴ + (3u + 3)x³ + (2u + 3)x₂ + (3u + 2)x + 3 | 36, 10, 17     |
| 12 | g = x⁶ + 3ux⁴ + x³ + (3u + 1)x₃ + 1                     | 36, 14, 13     |
| 14 | h = x⁶ + (3u + 3)x³ + (u + 3)x² + ux + u + 1            | 28, 8, 18      |
| 14 | h = x⁶ + (3u + 3)x³ + (u + 3)x + ux + u + 1             | 42, 8, 26      |
| 14 | h = x⁶ + (u + 3)x₃ + (3u + 2)x + u + 1                  | 28, 6, 18      |
| 14 | h = x⁶ + (u + 3)x₃ + (3u + 2)x + u + 1                  | 42, 6, 29      |
| 14 | g = x⁶ + (u + 3)x³ + x² + (3u + 2)x + 2u + 1            | 28, 20, 6      |
| 14 | g = x⁶ + (u + 3)x³ + x² + (3u + 2)x + 2u + 1            | 42, 20, 13     |
| 16 | g = x⁶ + (u + 2)x⁴ + ux + 2x + 1                        | 48, 24, 13     |
| 16 | g = x⁶ + (u + 3)x₃ + (3u + 1)x + 3u + 3                 | 32, 26, 4      |
| 16 | g = x⁶ + (u + 3)x₃ + (3u + 1)x + 3u + 3                 | 48, 26, 11     |
| 16 | g = x⁶ + (u + 2)x + 1                                   | 32, 28, 2      |
| 16 | g = x⁶ + (u + 2)x + 1                                   | 48, 28, 9      |
| 18 | g = x⁶ + (2u + 3)x₄ + ux + (2u + 1)x + 3 + 3            | 54, 28, 13     |
| 18 | g = x⁶ + ux + 2x + 1                                    | 54, 30, 10     |
| 18 | h = x⁶ + (3u + 2)x₃ + 3ux + 2u + 3                      | 54, 6, 34      |
| 20 | g = x⁶ + 3x⁵ + (3u + 1)x + x + 1                        | 60, 32, 13     |
| 20 | g = x⁶ + (u + 3)x₃ + x³ + x + 1                          | 40, 32, 4      |
| 20 | g = x⁶ + (3u + 3)x³ + (2u + 1)x + u + 3                 | 60, 34, 12     |
| 24 | g = x⁶ + 3ux⁴ + (u + 2)x + 1                            | 48, 42, 4      |
| 24 | g = x⁶ + 3ux⁴ + (u + 2)x + 1                            | 72, 42, 10     |
| 30 | g = x⁶ + (u + 1)x³ + 1                                  | 60, 52, 3      |
| 30 | g = x⁶ + (u + 1)x³ + 1                                  | 90, 52, 8      |
| 32 | g = x + u + 1                                           | 64, 62, 2      |
| 32 | g = x + u + 1                                           | 96, 62, 5      |
| 32 | h = x⁶ + ux + 2x + 3u + 1                               | 96, 8, 60      |

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