Characterization of Branched Covers with Simplicial Branch Sets

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Joint work with Rami Luisto
Definition

A branded cover is a continuous map $f: \Omega \rightarrow \mathbb{R}^n$, where $\Omega$ is a domain in $\mathbb{R}^n$, that is discrete and open.
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At most points $f$ is a local homeomorphism. The \textit{branch set} of $f$, denoted $B_f$, is the set of points where $f$ fails to be a local homeomorphism.
Definition

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- At most points $f$ is a local homeomorphism. The branch set of $f$, denoted $B_f$, is the set of points where $f$ fails to be a local homeomorphism.
- Branched covers are topological generalization of quasiregular maps.
In two dimensions the typical example of a branched cover is a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

- The branch set is the finite set of critical points of $f$.
- Near the branch points, $f$ behaves like the map $z^d$, where $d$ is the degree of the map.
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- The branch set is the finite set of critical points of $f$.
- Near the branch points, $f$ behaves like the map $z^d$, where $d$ is the degree of the map.
- Topologically, this map is equivalent to a winding map: $(r, \theta) \mapsto (r, d\theta)$.
Define \( f : \mathbb{C}/\mathbb{Z}^2 \to \hat{\mathbb{C}} \) as:

\[
\begin{array}{|c|c|}
\hline
A & B \\
\hline
B & A \\
\hline
\end{array}
\]

This is topologically the same as the Weierstrass \( p \)-function. Locally near the branch point the maps behaves like a winding map. Can extend this to a PL-map \( F : \mathbb{R}^2 \to S^2 \times \cdots \times S^2 \).
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- Locally near the branch point the maps behaves like a winding map.
- Can extend this to a PL-map $F : \mathbb{R}^{2n} \to S^2 \times \cdots \times S^2$. 
Up to homeomorphism, rational maps characterize every branched cover.

**Theorem (Stoïlow)**

Let \( f : S^2 \to \hat{\mathbb{C}} \) be a branched cover. Then there exists a homeomorphism \( h : \hat{\mathbb{C}} \to S^2 \) so that \( f \circ h \) is a rational map.
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**Corollary**

Every branched cover from \( S^2 \to S^2 \) is equivalent up to a homeomorphism to a piecewise linear (PL) map.
A map \( f : \Omega \rightarrow \mathbb{R}^n \) is \( K \)-quasiregular if \( f \in W^{1,n}_{\text{loc}}(\Omega) \) and for almost every \( x \in \Omega \),

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\|Df\|^n \leq KJ_f,
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where \( Df \) is the derivative of \( f \) and \( J_f = \det(Df) \).
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- By a theorem due to Reshetnyak, quasiregular maps are branched covers.
- The converse is false in general and it is difficult to construct quasiregular maps.
- PL maps are typically quasiregular.
An \( n \)-dimensional manifold \( M \) is \textit{Quasiregularly Elliptic} if there exists a quasiregular map \( f : \mathbb{R}^n \to M \).

In dimension 2, \( M \) is homeomorphic to \( \mathbb{C}, \mathbb{C} \hat{\mathbb{C}}, S^1 \times \mathbb{R} \) or \( S^1 \times S^1 \).

In dimension 3, closed quasiregularly elliptic manifolds are quotients of either \( S^3, S^1 \times S^1 \times S^1 \) or \( S^2 \times S^1 \).
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**Theorem (P., ’19)**

If $M$ is a closed, orientable Riemannian manifold of dimension $d$ that admits a quasiregular map from $\mathbb{R}^d$, then $\dim H^\ell(M) \leq \binom{d}{\ell}$.

If $\ell = d/2$, then $b^+_{d/2}(M), b^-_{d/2}(M) \leq \frac{1}{2}\binom{d}{d/2}$. 

**Eden Prywes**

**Branched Covers**
The geometry of the branch set can give information on the behavior of the map.

**Theorem (Church and Hemmingsen, '60)**

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover, where $\Omega$ is a domain in $\mathbb{R}^n$. If $f(B_f)$ can be embedded into a codimension 2 subspace, then $f$ is topologically equivalent to a winding map.
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- The $k$-winding map for $k \in \mathbb{N}$ is $w_k(r, \theta, x_2, \ldots, x_n) := (r, k\theta, x_2, \ldots, x_n)$.
- By a theorem due to Černavskiǐ and Väisälä, $B_f$ and $f(B_f)$ have topological dimension less than or equal to $n - 2$.
- In dimension 2 this hypothesis is always satisfied, but it is not always satisfied in higher dimensions.
Counterexample to Church and Hemmingsen

Let \( P \) be the Poincaré homology sphere.
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- $S^3$ is the universal covering space of $P$. 

$\Sigma P$ is not a topological manifold, but $\Sigma \Sigma P \simeq S^5$. So $\Sigma \pi : S^5 \to S^5$ is a branched cover with branch set $B \Sigma \pi$, $\Sigma \pi (B \Sigma \pi) \simeq S^1$.

Note that $\pi_1 (S^5 \setminus \Sigma \pi (B \Sigma \pi))$ has order 120.
Let $P$ be the Poincaré homology sphere.

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- If $\pi: S^3 \to P$ is the covering map, then we can take the suspension of both sides to get a map

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Let \( P \) be the Poincaré homology sphere.

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- \( \Sigma P \) is not a topological manifold, but \( \Sigma \Sigma P \cong S^5 \). So
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  is a branched cover with branch set \( B_{\Sigma \Sigma \pi}, \Sigma \Sigma \pi(B_{\Sigma \Sigma \pi}) \cong S^1 \).
- Note that \( \pi_1(S^5 \setminus \Sigma \Sigma \pi(B_{\Sigma \Sigma \pi})) \) has order 120.

The natural choice for an open neighborhood of a point in \( \Sigma \Sigma \pi(B_{\Sigma \Sigma \pi}) \) has boundary that is homeomorphic to \( \Sigma P \) not \( S^4 \).
Generalizing Church and Hemmingsen

Theorem (Martio and Srebro, '79)

Let \( f : \Omega \rightarrow \mathbb{R}^3 \) be a branched cover and \( x_0 \in B_f \). If there exists an open neighborhood \( V \) of \( x_0 \) so that the image of the branch set \( f(B_f \cap V) \) can be embedded into a union of finitely many line segments originating from \( f(x_0) \), then \( f \) is topologically equivalent on \( V \) to a cone of a rational map \( g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \).

- A cone of a map \( g \) is the map

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g \times \text{id} : \text{cone}(\hat{\mathbb{C}}) \rightarrow \text{cone}(\hat{\mathbb{C}}),
\]

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\text{cone}(\hat{\mathbb{C}}) = \frac{\hat{\mathbb{C}} \times [0, 1]}{\{(z, 0) \sim (w, 0)\}}
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(\( \hat{\mathbb{C}} \times [0, 1] \) with this identification is homeomorphic to \( B^3 \)).
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($\hat{\mathbb{C}} \times [0, 1]$ with this identification is homeomorphic to $B^3$).
- This implies that $f$ is topologically equivalent to a PL map.
Theorem (Luisto and P., '19)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood $V$ of $x_0$ so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n - 2)$-simplicial complex, then $f$ is topologically equivalent on $V$ to a cone of a PL map $g : S^{n-1} \to S^{n-1}$. This implies that $f$ is topologically equivalent to a PL map. This theorem also extends to a global result for $f : S^n \to S^n$. 
Main Result

Theorem (Luisto and P., ’19)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood $V$ of $x_0$ so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n - 2)$-simplicial complex, then $f$ is topologically equivalent on $V$ to a cone of a PL map $g : S^{n-1} \to S^{n-1}$.

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- This implies that \( f \) is topologically equivalent to a PL map.
- This theorem also extends to a global result for \( f : S^n \to S^n \).
We can use this result to construct quasiregular maps.

**Corollary**

For each $n \in \mathbb{N}$ there exists a quasiregular map $f : \mathbb{R}^{2n} \to \mathbb{C}P^n$. 
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- There exists a quasiregular map from $\mathbb{R}^{2n} \to (\mathbb{C}P^1)^n$.
- The map

\[
([z_1 : w_1], \ldots, [z_n : w_n]) \mapsto \left[ z_1 \cdots z_n : \sum_{i=1}^n z_1 \cdots \hat{z_i} \cdots z_n w_i : \cdots : w_1 \cdots w_n \right]
\]

is a branched cover from $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 \to \mathbb{C}P^n$. 
The map can be thought of as the coefficients of the polynomial

\[ p(u, v) = (z_1 u + w_1 v) \cdots (z_n u + w_n v). \]

So the branch set is

\[ \{ ([z_1 : w_1], \ldots, [z_n : w_n]) \in (\mathbb{CP}^1)^n : [z_i : w_i] = [z_j : w_j] \text{ for some } i \ne j \}. \]

The image of this can be given a simplicial structure and so there is a PL version of the map.
For dimension 4:

**Theorem (Piergallini and Zuddas, ’18)**

If $M$ is of the form $\#_m \mathbb{C}P^2 \#_n \overline{\mathbb{C}P^2}$ or $\#_n (S^2 \times S^2)$, then $N$ admits a PL (and quasiregular) map from $\mathbb{R}^4$ when $b_2^+(M), b_2^-(M) \leq 3$.

If $M$ is quasiregularly elliptic, then $b_2^+(M), b_2^-(M) \leq \frac{1}{2} (\binom{n}{n/2})$. 
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- Is there a manifold that admits a quasiregular (PL) map from $\mathbb{R}^d$, but does not admit a quasiregular (PL) map from $T^d$?

- All the above examples factor through the torus.
Theorem (Luisto and P., '19)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood $V$ of $x_0$ so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n-2)$-simplicial complex, then $f$ is topologically equivalent on $V$ to a cone of a PL map $g : S^{n-1} \to S^{n-1}$.

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in \Omega$ be a point. There exists a radius $r_0 > 0$ and a family of neighborhoods, denoted $U(x_0, r)$, such that for $0 < r \leq r_0$

- $x_0 \in U(x_0, r)$
- $f(U(x_0, r)) = B(f(x_0), r)$
- $f(\partial U(x_0, r)) = \partial B(f(x_0), r)$
- $f^{-1}\{f(x_0)\} \cap U(x_0, r) = \{x_0\}$
Suppose that near $x_0$, $\partial U(x_0, r)$ is homeomorphic to $S^{n-1}$.

It is a fact that restricted to $\partial U(x_0, r)$, $f$ is still a branched cover. So if we induct on the dimension, $f: \partial U(x_0, r) \to S^{n-1}$ is equivalent to a PL map.
Outline of Proof

- Suppose that near $x_0$, $\partial U(x_0, r)$ is homeomorphic to $S^{n-1}$.
- It is a fact that restricted to $\partial U(x_0, r)$, $f$ is still a branched cover. So if we induct on the dimension, $f : \partial U(x_0, r) \to S^{n-1}$ is equivalent to a PL map.
- By a path lifting argument we show that $f$ behaves the same way topologically on the boundaries of $U(x_0, r)$ for all sufficiently small $r$. 
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So $f$ is equivalent to a cone of a PL map.
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By a path lifting argument we show that $f$ behaves the same way topologically on the boundaries of $U(x_0, r)$ for all sufficiently small $r$.

So $f$ is equivalent to a cone of a PL map.

It is not clear that $\partial U(x_0, r) \simeq S^{n-1}$, in fact it may not even be a manifold.
Path Lifting

\[ U(z_0, f, R) \]

\[ f(x_0) \]

\[ \beta \]

\[ f(z_0) \]

\[ z_0 \]

\[ x_0 \]

\[ r \]

\[ \alpha_1 \]

\[ \alpha_2 \]

\[ U(x_0, f, r) \]

\[ \alpha_1(s_0) \]

\[ \alpha_2(s_0) \]

\[ f \]

\[ f \circ \gamma_3 \]

\[ f(z_0) \]

\[ S \]

\[ U_1 \]

\[ U_2 \]

\[ f(x_0) \]

\[ f(z_0) \]

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In dimensions two and three the proof simplifies.

- In dimension two, $f$ is locally injective on $\partial U(x_0, r)$ and so $\partial U(x_0, r)$ is a manifold and is homeomorphic to $S^1$. 

- In dimension three, Martio and Srebro first show that $\partial U(x_0, r)$ is a manifold. Like in dimension 2, $\partial U(x_0, r)$ is a manifold away from the branch set of $f$. The image of the branch set is "ray-like" by assumption and so intersects $B(f(x_0), r)$ at a discrete set of points. Topologically $f$ behaves like the power map $z \mapsto z^d$ on $\partial U(x_0, r)$ near the intersection by Church and Hemmingsen's theorem and so still can be used to define a chart for $\partial U(x_0, r)$. $\partial U(x_0, r)$ is homeomorphic to $S^2$ if it is simply connected. This follows because $U(x_0, r)$ is contractible.
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- $\partial U(x_0, r)$ is homeomorphic to $S^2$ if it is simply connected. This follows because $U(x_0, r)$ is contractible.
\( \partial U(x_0, r) \) is a Manifold

In the general case when \( f : \Omega \to \mathbb{R}^n \),

- \( f \) restricted to \( \partial U(x_0, r) \) is a branched cover and away from the branch set is a covering map. So \( \partial U(x_0, r) \setminus B_f \) is a manifold.

- As in dimension 3, if \( x \in \partial U(x_0, r) \cap B_f \), then we consider the map \( f \) restricted to a normal neighborhood of \( x \) in \( \partial U(x_0, r) \).

- We continue to go down in dimension considering more and more nested normal neighborhoods.
We show a stronger fact that for any point $x$, if we consider normal neighborhoods of dimension $k + 1$, then their boundaries will be homeomorphic to $S^k$ when taken sufficiently close to $x$.

- By the path lifting argument, normal neighborhoods will have a cone structure. So if $U$ is a normal neighborhood of a point $x$, then $U \simeq \text{cone}(\partial U)$.
\( \partial U(x_0, r) \) is homeomorphic to a sphere

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- By the path lifting argument, normal neighborhoods will have a cone structure. So if \( U \) is a normal neighborhood of a point \( x \), then \( U \cong \text{cone}(\partial U) \).
- Let \( \gamma : S^\ell \to \partial U \cong U \setminus \{x_0\} \). There is a homotopy sending \( \gamma \) to a point in \( U \times (0, 1)^{n-k-1} \). It can be chosen to avoid \( \{x_0\} \times (0, 1)^{n-k-1} \) when \( 1 \leq \ell < k \). So \( \pi_\ell(V) = 0 \).
There is a partial converse to the Martio-Srebro result.

**Theorem (Martio and Srebro, '79)**

Let \( f : \Omega \rightarrow \mathbb{R}^3 \) be a branched cover so that at \( x \in \Omega \) there exists an \( r_0 > 0 \) with the property that for all \( r \leq r_0 \), \( \partial U(x_0, r) \) is a manifold. Then at \( x_0 \), \( f \) is equivalent to a path of rational maps.
There is a partial converse to the Martio-Srebro result.

**Theorem (Martio and Srebro, '79)**

Let $f : \Omega \rightarrow \mathbb{R}^3$ be a branched cover so that at $x \in \Omega$ there exists an $r_0 > 0$ with the property that for all $r \leq r_0$, $\partial U(x_0, r)$ is a manifold. Then at $x_0$, $f$ is equivalent to a path of rational maps.

We show a corresponding result:

**Theorem (Luisto and P, '19)**

Let $f : \Omega \rightarrow \mathbb{R}^n$ be a branched cover so that at $x \in \Omega$ there exists an $r_0 > 0$ with the property that for all $r \leq r_0$, $U(x_0, r)$ is a manifold with boundary. Then at $x_0$, $f$ is equivalent to a path of branched covers.
We can iterate the previous result to get a lower bound on the topological dimension of $B_f$.

**Corollary (Luisto and P, ’19)**

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover so that for some $k$, $2 \leq k \leq n - 2$, all the normal domains of dimension less than $k$ are manifolds with boundary, then $\dim_{\text{top}}(B_f) \geq n - k$. 
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It is not possible to show that if all the normal domains are manifolds, then $f$ is equivalent to a PL-map. Let $w : \mathbb{R}^3 \to \mathbb{R}^3$ be a winding map and let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism that takes the set $B = \{(0, t^2 \cos(1/t), t), t \in \mathbb{R}\}$ to the $z$-axis near 0. Define $f := w \circ h \circ w$. The branch set is a union of the $z$-axis and $B$. 
Thank you!