Bäcklund transforms of the extreme Kerr near-horizon geometry

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Abstract. We apply the method of Bäcklund transformations to generate a new hierarchy of exact solutions to the vacuum Einstein equations starting from the extreme Kerr near-horizon geometry. Solutions with extreme Kerr near-horizon asymptotics containing an arbitrary number of free parameters are included.

1. Introduction

The general relativistic solution describing a uniformly rotating disk of dust [1] admits an interesting parameter limit leading to the extreme Kerr metric outside the horizon (“exterior perspective”). From the “interior perspective”, the same parameter limit can be performed after an appropriate parameter dependent coordinate transformation and then leads to a spacetime that still describes a rotating disk of dust embedded in a non-asymptotically flat vacuum exterior that approaches the extreme Kerr near-horizon geometry at spatial infinity [2]. The same phenomenon was observed numerically for “relativistic Dyson rings” [3]. Moreover, it was shown that a black hole limit of rotating fluid bodies in equilibrium always leads to the extreme Kerr solution [4].

This motivates the systematic investigation of solutions which have the extreme Kerr near-horizon geometry, also called “extreme Kerr throat geometry” [5], as their asymptotics. As a first step, we derive here a new class of solutions by means of Bäcklund transformations [6, 7] using the extreme Kerr near-horizon geometry as the starting point (“seed solution”) and show how the Bäcklund parameters have to be constrained in order to conserve the extreme Kerr near-horizon asymptotics. Of course, the solutions constructed in this way are pure vacuum solutions. Instead of material sources they possess some singularities; note that this guarantees consistency with results published in [8, 9].

The derived solutions might also be of interest within the context of the Kerr/CFT correspondence, see [10].
2. Ernst potential of the extreme Kerr near-horizon geometry

It is well known that the stationary and axially symmetric vacuum Einstein equations are equivalent to the Ernst equation \[^{11, 12}\]
\[
(\Re E) \nabla^2 E = (\nabla E)^2, \tag{1}
\]
where the operator \(\nabla\) has the same meaning as in Euclidean 3-space in which \(r, \theta\) and \(\phi\) are spherical coordinates. The complex Ernst potential \(E\) depends on \(r\) and \(\theta\) only. The spacetime line element reads
\[
ds^2 = f^{-1} \left[ h(dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\phi^2 \right] - f(dt + a d\phi)^2 \tag{2}
\]
with \(f = \Re E\); the other metric functions \(h\) and \(a\) can also be obtained from \(E\). The Ernst potential of the extreme Kerr near-horizon geometry is given by \[^{2}\]
\[
E_{\text{NHG}} = -\Omega^2 r^2 H(\theta), \quad H(\theta) = \frac{2(1 + i \cos \theta)^2}{1 - i \cos \theta} + \sin^2 \theta. \tag{3}
\]
The real constant \(\Omega\) is the angular velocity of the horizon. Note that \(^{3}\) belongs to a family of solutions discovered by Ernst \[^{13}\].

3. Bäcklund transformation

Neugebauer’s general Bäcklund formula \[^{14}\] has been used to investigate huge classes of asymptotically flat solutions to the stationary and axially symmetric vacuum Einstein equations. However, it can be applied to non-asymptotically flat solutions as well. In general, for a given seed solution \(E_0\), the \((2n\text{-fold})\) Bäcklund transform \(E\) reads \[^{14, 15}\]
\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_0 & \alpha_1 \lambda_1 & \alpha_2 \lambda_2 & \cdots & \alpha_{2n} \lambda_{2n} \\
1 & (\lambda_1)^2 & (\lambda_2)^2 & \cdots & (\lambda_{2n})^2 \\
\alpha_0 & \alpha_1 (\lambda_1)^3 & \alpha_2 (\lambda_2)^3 & \cdots & \alpha_{2n} (\lambda_{2n})^3 \\
1 & (\lambda_1)^4 & (\lambda_2)^4 & \cdots & (\lambda_{2n})^4 \\
\alpha_0 & \alpha_1 (\lambda_1)^5 & \alpha_2 (\lambda_2)^5 & \cdots & \alpha_{2n} (\lambda_{2n})^5 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (\lambda_1)^{2n} & (\lambda_2)^{2n} & \cdots & (\lambda_{2n})^{2n}
\end{bmatrix}, \tag{4}
\]
\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \alpha_1 \lambda_1 & \alpha_2 \lambda_2 & \cdots & \alpha_{2n} \lambda_{2n} \\
1 & (\lambda_1)^2 & (\lambda_2)^2 & \cdots & (\lambda_{2n})^2 \\
1 & \alpha_1 (\lambda_1)^3 & \alpha_2 (\lambda_2)^3 & \cdots & \alpha_{2n} (\lambda_{2n})^3 \\
1 & (\lambda_1)^4 & (\lambda_2)^4 & \cdots & (\lambda_{2n})^4 \\
1 & \alpha_1 (\lambda_1)^5 & \alpha_2 (\lambda_2)^5 & \cdots & \alpha_{2n} (\lambda_{2n})^5 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (\lambda_1)^{2n} & (\lambda_2)^{2n} & \cdots & (\lambda_{2n})^{2n}
\end{bmatrix}
\]

\[
\text{det} \quad \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_0 & \alpha_1 \lambda_1 & \alpha_2 \lambda_2 & \cdots & \alpha_{2n} \lambda_{2n} \\
1 & (\lambda_1)^2 & (\lambda_2)^2 & \cdots & (\lambda_{2n})^2 \\
\alpha_0 & \alpha_1 (\lambda_1)^3 & \alpha_2 (\lambda_2)^3 & \cdots & \alpha_{2n} (\lambda_{2n})^3 \\
1 & (\lambda_1)^4 & (\lambda_2)^4 & \cdots & (\lambda_{2n})^4 \\
\alpha_0 & \alpha_1 (\lambda_1)^5 & \alpha_2 (\lambda_2)^5 & \cdots & \alpha_{2n} (\lambda_{2n})^5 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (\lambda_1)^{2n} & (\lambda_2)^{2n} & \cdots & (\lambda_{2n})^{2n}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \alpha_1 \lambda_1 & \alpha_2 \lambda_2 & \cdots & \alpha_{2n} \lambda_{2n} \\
1 & (\lambda_1)^2 & (\lambda_2)^2 & \cdots & (\lambda_{2n})^2 \\
1 & \alpha_1 (\lambda_1)^3 & \alpha_2 (\lambda_2)^3 & \cdots & \alpha_{2n} (\lambda_{2n})^3 \\
1 & (\lambda_1)^4 & (\lambda_2)^4 & \cdots & (\lambda_{2n})^4 \\
1 & \alpha_1 (\lambda_1)^5 & \alpha_2 (\lambda_2)^5 & \cdots & \alpha_{2n} (\lambda_{2n})^5 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (\lambda_1)^{2n} & (\lambda_2)^{2n} & \cdots & (\lambda_{2n})^{2n}
\end{bmatrix}
\]
where the entries in the two $(2n + 1) \times (2n + 1)$ matrices are given by

$$
\alpha_0 = -\frac{E_0^*}{E_0}, \quad \lambda_i = \sqrt{\frac{K_i - re^{i\theta}}{K_i - re^{-i\theta}}} \quad (\lambda_i \to e^{i\theta} \text{ as } r \to \infty)
$$

and solutions $\alpha_i$ to the total Riccati equations

$$
(E_0 + E_0^*) \, d\alpha_i = \left[ \frac{\partial E_0^*}{\partial z} (\alpha_i - \lambda_i) + \frac{\partial E_0}{\partial z} \alpha_i (\alpha_i \lambda_i - 1) \right] \, dz
+ \left[ \frac{\partial E_0^*}{\partial z^*} (\alpha_i - \lambda_i^{-1}) + \frac{\partial E_0}{\partial z^*} \alpha_i (\alpha_i \lambda_i^{-1} - 1) \right] \, dz^*
$$

with the complex coordinates

$$
z = ire^{-i\theta}, \quad z^* = -ire^{i\theta}.
$$

Note that the integrability condition of (6) is equivalent to the Ernst equation for $E_0$.

The (finite) constants $K_i$ must either be real ($K_i = K_i^*$), with the consequence $\lambda_i = 1/\lambda_i^*$, or complex conjugate pairs ($K_j = K_j^*$), to ensure $\lambda_j = 1/\lambda_j^*$. The integration constants of the Riccati equations (6) have to be chosen such that $\alpha_i = 1/\alpha_i^*$ or $\alpha_j = 1/\alpha_j^*$, respectively.

For the particular seed solution (3) we obtain

$$
\alpha_0 = -\frac{H^*}{H}
$$

and

$$
\alpha_i = -\frac{\psi(\lambda_i, \theta) - c_i \psi(-\lambda_i, \theta)}{\chi(\lambda_i, \theta) + c_i \chi(-\lambda_i, \theta)} \quad (i = 1, 2, \ldots, 2n)
$$

with

$$
\psi(\lambda, \theta) = [\chi(1/\lambda^*, \theta)]^* = \frac{A(1 + \lambda^2) + B(1 - \lambda^2) + C\lambda}{e^{-i\theta}(\lambda - e^{i\theta})^2},
$$

$$
A = \frac{\cos \theta + i}{1 + i}, \quad B = \frac{(1 - i) \sin \theta}{\cos \theta - i}
$$

and

$$
C = -(1 + i)(\cos \theta - i).
$$

The constants $c_i$, which can also be chosen infinite [meaning $\alpha_i = \psi(-\lambda_i, \theta)/\chi(-\lambda_i, \theta)$], have to satisfy

$$
c_i = -c_i^* \quad (\text{for real } K_i) \quad \text{or} \quad c_j = -c_j^* \quad (\text{for pairs } K_j = K_j^*).
$$

It is interesting to note that $\psi(1, \theta) = -\chi(1, \theta) = -i$, $\psi(-1, \theta) = \chi(-1, \theta) = 1$ and $\alpha_0$ can be expressed as

$$
\alpha_0 = -\lim_{\lambda \to 1} \frac{\psi(\lambda, \theta) + i\psi(-\lambda, \theta)}{\chi(\lambda, \theta) - i\chi(-\lambda, \theta)}.
$$
4. Solutions with extreme Kerr near-horizon asymptotics

For the discussion of the asymptotic behaviour as $r \to \infty$ a reformulation of (4) in terms of $n \times n$ determinants following [16] is useful, see also [17]. With

$$r_i \equiv -\lambda_i (K_i - re^{-i\theta}) = r \sqrt{\left(1 - \frac{K_i e^{i\theta}}{r}\right) \left(1 - \frac{K_i e^{-i\theta}}{r}\right)}$$  \hspace{1cm} (15)$$

one obtains

$$\mathcal{E} = \mathcal{E}_0 \frac{\det \left(\frac{\alpha_p r_p - \alpha_q r_q}{K_p - K_q} + \alpha_0\right)}{\det \left(\frac{\alpha_p r_p - \alpha_q r_q}{K_p - K_q} + 1\right)}$$  \hspace{1cm} (16)$$

with

$$p = 1, 3, 5, \ldots, 2n - 1; \quad q = 2, 4, 6, \ldots, 2n.$$  \hspace{1cm} (17)$$

(This means: first row $p = 1$, second row $p = 3$, ..., $n$-th row $p = 2n - 1$ and first column $q = 2$, second column $q = 4$, ..., $n$-th column $q = 2n$.) For $n = 1$, Eq. (16) reduces to

$$\mathcal{E} = \mathcal{E}_0 \frac{\alpha_1 r_1 - \alpha_2 r_2 + \alpha_0 (K_1 - K_2)}{\alpha_1 r_1 - \alpha_2 r_2 + K_1 - K_2}.$$  \hspace{1cm} (18)$$

By means of an expansion in powers of $r^{-1}$ it can easily be verified from (5) and (9)–(12) that

$$\alpha_i = -\frac{\psi(\lambda_i, \theta)}{\chi(\lambda_i, \theta)} + \mathcal{O}(r^{-2}) = F(\theta) + \mathcal{O}(r^{-1}) \quad \text{for} \quad c_i \neq \infty$$  \hspace{1cm} (19)$$

and

$$\alpha_i = \frac{\psi(-\lambda_i, \theta)}{\chi(-\lambda_i, \theta)} = G(\theta) + \mathcal{O}(r^{-1}) \quad \text{for} \quad c_i = \infty$$  \hspace{1cm} (20)$$

with certain functions $F(\theta)$ and $G(\theta)$ not depending on $i$. Because of

$$\lim_{r \to \infty} \frac{r_i}{r} = 1,$$  \hspace{1cm} (21)$$

see (15), we find (with $\mathcal{E}_0 = \mathcal{E}_{\text{NHG}}$)

$$\lim_{r \to \infty} \frac{\mathcal{E}}{\mathcal{E}_{\text{NHG}}} = 1$$  \hspace{1cm} (22)$$

for all $\theta$ with $F(\theta) \neq G(\theta)$ if $n$ of the $2n$ constants $c_i$, say $c_p$ ($p = 1, 3, 5, \ldots, 2n - 1$), are chosen finite and the other ones, say $c_q$ ($q = 2, 4, 6, \ldots, 2n$), are chosen infinite. This reduces the number of free real constants contained in the $K_i$'s and $c_i$'s from $4n$ to $3n$. For $n = 1$, $c_1 \neq \infty$ and $c_2 = \infty$ means that $K_1$ and $K_2$ must be real, see (13). For $n > 1$, pairs of complex conjugate $K_i$'s are possible as well. It turns out that the so far excluded special values of $\theta$ defined by $F(\theta) = G(\theta)$ are the same values for which $\alpha_0 \equiv -H^*/H = 1$ holds ($\cos^2 \theta = 2\sqrt{3} - 3$), leading obviously to $\mathcal{E} = \mathcal{E}_{\text{NHG}}$ for all $r$. Hence our solutions have the extreme Kerr near-horizon asymptotics whenever precisely $n$ of the $2n$ constants $c_i$ are chosen infinite.
5. Discussion

Eqs (4, 5) with $E_0 = E_{NHG}$ and (8)–(12) represent a new hierarchy of solutions, the (2n-fold) Bäcklund transforms of the extreme Kerr near-horizon geometry ($n = 1, 2, 3, \ldots$). For given $n$, the solution contains $4n$ arbitrary real parameters. From a mathematical point of view, this result is an example of applying Bäcklund transformations to non-static and non-asymptotically flat seed solutions. From a physical perspective, members of the $3n$-parameter subfamily with extreme Kerr near-horizon asymptotics might be of particular interest.

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