Lectures on the Error Analysis of Interpolation on Simplicial Triangulations without the Shape-Regularity Assumption

Part 2: Lagrange Interpolation on Tetrahedrons

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Abstract: This is the second lecture note on the error analysis of interpolation on simplicial elements without the shape regularity assumption 1. In this manuscript, we explain the error analysis of Lagrange interpolation on (possibly anisotropic) tetrahedrons. This topic is hardly explained in standard textbooks of the mathematical theory of finite element methods. The authors hope that this manuscript will be merged into a new textbook in future. Therefore, this manuscript is not intended to be a research paper. Supposed readers are students and researchers who are familiar with the mathematical theory of the finite element methods.

1 Lagrange interpolation on tetrahedrons

This is the second lecture note concerning the error analysis of interpolation on simplicial triangulations without the shape regularity assumption. In this note, we will explain the error analysis of Lagrange interpolation on tetrahedrons. To this end, we summarize the results given in [11, 12, 13, 14]. Readers are referred to the first lecture note [15] for the notation, lemmas, and theorems used in this manuscript.

Throughout this paper, \( T \subset \mathbb{R}^3 \) denotes a tetrahedron with vertices \( x_i, i = 1, \cdots, 4 \), and all tetrahedrons are assumed to be closed sets. Let \( \lambda_i \) be the barycentric coordinates of a tetrahedron with respect to \( x_i \). By definition, \( 0 \leq \lambda_i \leq 1, \sum_{i=1}^{4} \lambda_i = 1 \). Let \( N_0 \) be the set

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1The first one is arXiv:1908.03894 and Memoirs of the Faculty of Science, Ehime University, 24 9–42 (2022).
of nonnegative integers and \( \gamma = (a_1, \ldots, a_4) \in \mathbb{N}_0^4 \) be a multi-index. If \( |\gamma| := \sum_{i=1}^4 a_i = k \), then \( \gamma/k := (a_1/k, \ldots, a_4/k) \) can be regarded as a barycentric coordinate in \( T \). The set \( \Sigma^k(T) \) of points on \( T \) is defined by

\[
\Sigma^k(T) := \left\{ \frac{\gamma}{k} \in T \mid |\gamma| = k, \ \gamma \in \mathbb{N}_0^4 \right\}.
\]

Let \( \mathcal{P}_k(T) \) be the set of polynomials defined on \( T \) whose degree is at most \( k \). For a continuous function \( v \in C^0(T) \), the Lagrange interpolation \( I^k_T v \in \mathcal{P}_k(T) \) of degree \( k \) is defined as

\[
v(x) = (I^k_T v)(x), \quad \forall x \in \Sigma^k(T).
\]

Let \( m \), \( 0 \leq m \leq k \) be an integer, and \( p \), \( 1 \leq p \leq \infty \) be a real. For the mathematical theory of finite element methods, estimating error \( |v - I^k_T v|_{m,p,T} \) of the Lagrange interpolation is an important task. For error analysis, the following condition is usually imposed for the meshes to use in many textbooks [4, 7, 9].

Suppose that \( \mathcal{X} \) is a set of (possibly infinitely many) simplicial elements (triangles or tetrahedrons). For \( T \in \mathcal{X} \), let \( h_T := \text{diam}T \), and \( \rho_T \) be the diameter of its inscribed ball.

**Assumption 1 (Shape regularity)** The set \( \mathcal{X} \) is called shape regular if there exists a constant \( \sigma > 0 \) such that

\[
\frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \mathcal{X}.
\]

The shape regularity assumption requires that any element \( T \in \mathcal{X} \) is not too “flat”, or degenerate. The maximum of the ratio \( h_T/\rho_T \) in \( \mathcal{X} \) is called its chunkiness parameter [4]. The shape regularity condition is sometimes called the inscribed ball condition.

Let \( \tilde{T} \) be a reference element. If we consider about tetrahedrons, the tetrahedron with vertices \((0, 0, 0)^\top\), \((1, 0, 0)^\top\), \((0, 1, 0)^\top\), and \((0, 0, 1)^\top\) is typically taken as the reference element \( \tilde{T} \). Let \( \varphi(x) = Ax + b \) be an affine transformation that maps \( \tilde{T} \) to \( T \), where \( A \) is a \( 3 \times 3 \) regular matrix and \( b \in \mathbb{R}^3 \). Error analysis is first performed on the reference element \( \tilde{T} \). Then, the pull back \( v \circ \varphi \) is used to transfer the result obtained on \( \tilde{T} \) to the “physical element” \( T \). Let \( \|A\| \) denote the matrix norm of \( A \) associated with the Euclidean norm of \( \mathbb{R}^d \) (\( d = 2, 3 \)).

Under the shape regularity assumption, we have the following theorem.

**Theorem 2 ([7], Theorem 3.1.4)** Let \( \sigma > 0 \) be a constant. If \( h_T/\rho_T \leq \sigma \), then there exists a constant \( C = C(\tilde{T}, p, k, m) \) independent of \( T \) such that, for \( v \in W^{k+1,p}(T) \),

\[
|v - I^k_T v|_{m,p,T} \leq C\|A\|^{k+1}\|A^{-1}\|^m|v|_{k+1,p,T} \leq C \frac{h_T^{k+1}}{\rho_T^m} |v|_{k+1,p,T} \leq (C\sigma^m)h_T^{k+1-m}|v|_{k+1,p,T}.
\]
If the chunkiness parameter of $X$ is not small enough (say, $\sigma > 10$), $X$ is called anisotropic. In numerical simulation, we sometimes need to introduce an *adaptive mesh refinement* technique. In a process of mesh refinements, many anisotropic elements may be generated. With such meshes, the standard theory of finite element methods with the shape regularity assumption cannot be applied. The main purpose of this manuscript is to explain the error analysis of Lagrange interpolation on tetrahedrons without the shape regularity assumption.

Figure 1: Two anisotropic triangles; *dagger*: the maximum angle is not close to $\pi$, and the circumradius is not large (left), and *blade*: the maximum angle is close to $\pi$ and the circumradius is large (right).

Let $T$ be a triangle and $R_T$ be its circumradius. Anisotropic triangles can be categorized into only two types as depicted in Figure 1 ([6]). Also, as is explained in [15], the “badness” of an anisotropic triangle can be measured by $R_T$, and the following theorem is known [15].

**Theorem 3 (Circumradius estimates)** Let $T$ be an arbitrary triangle. Then, for the $k$th-order Lagrange interpolation $I^k_T$ on $T$, the estimation

$$|v - I^k_Tv|_{m,p,T} \leq C \left( \frac{R_T}{h_T} \right)^m h_T^{k+1-m} |v|_{k+1,p,T} = CR_T^mh_T^{k+1-2m} |v|_{k+1,p,T}$$

holds for any $v \in W^{k+1,p}(T)$, where the constant $C = C(k, m, p)$ is independent of the geometry of $T$.

Note that by the laws of sines, we have

$$\frac{R_T}{h_T} = \frac{1}{2 \sin \theta_T}, \quad \frac{\pi}{3} \leq \theta_T < \pi$$

where $\theta_T$ is the maximum inner angle of $T$. Hence, if there exists a constant $\theta_{\text{max}} < \pi$ and $\theta_T \leq \theta_{\text{max}}$, we have

$$|v - I^k_Tv|_{m,p,T} \leq C \left( \frac{R_T}{h_T} \right)^m h_T^{k+1-m} |v|_{k+1,p,T} \leq C' h_T^{k+1-2m} |v|_{k+1,p,T}$$

The condition $\theta_T \leq \theta_{\text{max}}$ is called the **maximum angle condition** with $\theta_{\text{max}}$ for triangles.

For the case of tetrahedrons, anisotropic tetrahedrons are usually categorized into nine types as depicted in Figure 2 ([6]). Also, as we will see later, the radius of the
circumsphere does not represent the “badness” of an anisotropic tetrahedron. These facts suggest that the analysis on anisotropic tetrahedrons is much more complicated than the case of anisotropic triangles.

Figure 2: Nine anisotropic tetrahedrons; (top row from left) spire, spear, spindle, spike, splinter, (bottom row from left) wedge, spade, cap, sliver.

Křížek introduced the maximum angle condition for tetrahedrons [16].

**Definition 4 (Maximum angle condition for tetrahedrons)** Let $\theta_{\text{max}}, \pi/2 \leq \theta_{\text{max}} < \pi$ be a constant. Let $T$ be an arbitrary tetrahedron. If all inner angles of the faces of $T$, and all dihedral angles between two faces of $T$ are less than or equal to $\theta_{\text{max}}$, $T$ is said to satisfy the maximum angle condition with $\theta_{\text{max}}$.

For the error analysis of Lagrange interpolation on tetrahedrons without the shape regularity condition, the following theorem is known [16, 8].

**Theorem 5** Let $\theta_{\text{max}}, \pi/2 \leq \theta_{\text{max}} < \pi$ be a constant. Suppose that a tetrahedron $T$ satisfies the maximum angle condition with $\theta_{\text{max}}$. Then, there exists a constant $C = C(\theta_{\text{max}}, p)$ with $p > 2$ such that

$$|v - T^1_Tv|_{1,p,T} \leq Ch_T|v|_{1,p,T},$$

where $C(\theta_{\text{max}}, p) = O((p - 2)^{-1/2})$ as $p \searrow 2$.

By this theorem, we may say that, if a tetrahedron $K$ satisfies the maximum angle condition, the error of the linear Lagrange interpolation is of order $O(h_K)$ in $L^p$-norm with $p > 2$.

To extend the above estimation, a theorem similar to Theorem 3 was desired. For that purpose, an immediate idea is to replace the circumradius of a triangle with the radius of circumsphere of a tetrahedron. However, this idea can be immediately rejected

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2Note that Apel [2] presents a different type of error analysis on anisotropic meshes.
by considering the tetrahedron \( T \) with vertices 
\[
\begin{align*}
\mathbf{x}_1 & := (h, 0, 0) ^ {\top}, \\
\mathbf{x}_2 & := (-h, 0, 0) ^ {\top}, \\
\mathbf{x}_3 & := (0, h, h^\alpha)^ {\top}, \\
\mathbf{x}_4 & := (0, -h, h^\alpha)^ {\top},
\end{align*}
\]
with \( h > 0 \) and \( \alpha > 0 \). This tetrahedron is an example of sliver (see Figure 2). Setting \( v(x, y, z) := x^2 - h^2 + h^2 - \alpha z \), we see that \( I_T^1 v = 0 \), and a simple computation yields that \( |v - I_T^1 v|_{1,\infty,T} = |v|_{1,\infty,T} \geq h^{2-\alpha} \) and \( |v|_{2,\infty,T} = 2 \). Hence, if \( \alpha > 2 \), an inequality such as the one given in Theorem 3 does not hold for the tetrahedron, although the radius of circumshpere of the above \( T \) converges to 0 as \( h \to 0 \).

To express the “badness” of a tetrahedron, the following definition is given [11, 12]. Let \( h_i (i = 1, \ldots, 6) \) be the length of edges of \( T \) with \( h_1 \leq \cdots \leq h_6 = h_T := \text{diam} T \). Then, we define \( R_T \) by
\[
R_T := \frac{h_1 h_2 h_T}{|T|} h_T.
\]

The following is the main theorem of this manuscript.

**Theorem 6 (Main Theorem)** Let \( T \) be an arbitrary tetrahedron and \( R_T \) be defined by (1). Let \( k \) and \( m \) be integers with \( k \geq 1 \) and \( 0 \leq m \leq k \). Let \( p \) be taken as
\[
\begin{align*}
2 < p & \leq \infty \quad \text{if } k - m = 0, \\
\frac{3}{2} < p & \leq \infty \quad \text{if } k = 1, m = 0, \\
1 < p & \leq \infty \quad \text{if } k \geq 2 \text{ and } k - m \geq 1.
\end{align*}
\]

For the Lagrange interpolation \( I_T^k v \) of degree \( k \) on \( T \), the following estimate holds:
\[
B_p^{m,k}(T) := \sup_{u \in \mathcal{I}_T^k(T)} \frac{|u|_{m,p,T}}{|u|_{k+1,p,T}} \leq C_{k,m,p} \left( \frac{R_T}{h_T} \right)^m h_T^{k+1-m},
\]
\[
|v - I_T^k v|_{m,p,T} \leq C_{k,m,p} \left( \frac{R_T}{h_T} \right)^m h_T^{k+1-m} |v|_{k+1,p,T}, \quad \forall v \in W^{k+1,p}(T),
\]
where \( C_{k,m,p} \) is a constant depending on \( k, m, \) and \( p \).

**Remark.** Note that, in (6) and Theorem 6, the restriction \( 2 < p \) for the case \( k = m \) comes from the continuity of the trace operator \( \gamma : W^{1,p}(T) \ni v \mapsto v|_S \in L^1(S) \), where \( S \subset T \) is a non-degenerate segment (see [13, Section 3] and Lemma 19 in Appendix). By the counterexamples given by Shenk [18] and the authors [14], we find that this restriction cannot be improved.

For the maximum angle condition of tetrahedrons, we have the following theorem.

**Theorem 7** Let \( T \) be an arbitrary tetrahedron and \( R_T \) be defined by (1). Then, \( T \) satisfies the maximum angle condition with \( \theta_{\max} \in [\pi/2, \pi) \), if and only if there exists a fixed constant \( D = D(\theta_{\max}) \) such that
\[
\frac{R_T}{h_T} \leq D.
\]
This theorem implies that, with $R_T$ given in (1), the situation for tetrahedrons is very similar to that of triangles. We immediately obtain the following corollary.

**Corollary 8** Let $T$ be an arbitrary tetrahedron that satisfies the maximum angle condition with $\theta_{\text{max}} \in [\pi/2, \pi)$. Let $k$ and $m$ be integers with $k \geq 1$ and $0 \leq m \leq k$. Let $p$ be taken as (6). For the Lagrange interpolation $I_T^k v$ of degree $k$ on $T$, the following estimate holds.

$$|v - I_T^k v|_{m,p,T} \leq C h_T^{k+1-m} |v|_{k+1,p,T}, \quad \forall v \in W^{k+1,p}(T),$$

where $C$ is a constant depending only on $k$, $m$, $p$, and $\theta_{\text{max}}$.

In the sequel of this lecture note, we will explain the proofs of Theorems 6, 7 in detail.

2 Preliminaries

2.1 Notation

A triangle with vertices $x_i$ ($i = 1, 2, 3$) is denoted by $\triangle x_1 x_2 x_3$. The edge connecting $x_i$, $x_j$ and its length are denoted by $x_i x_j$ and $|x_i x_j|$, respectively.

2.2 The Sobolev imbedding theorem

Let $1 < p \leq \infty$. From Sobolev’s imbedding theorem and Morry’s inequality, we have the continuous imbeddings

$$W^{2,p}(T) \subset C^{1,1-3/p}(T), \quad p > 3,$$

$$W^{2,3}(T) \subset W^{1,q}(T) \subset C^{0,1-3/q}(T), \quad \forall q > 3,$$

$$W^{2,p}(T) \subset W^{1,3p/(3-p)}(T) \subset C^{0,2-3/p}(T), \quad \frac{3}{2} < p < 3,$$

$$W^{3,3/2}(T) \subset W^{2,3}(T) \subset W^{1,q}(T) \subset C^{0,1-3/q}(T), \quad \forall q > 3,$$

$$W^{3,p}(T) \subset W^{2,3p/(3-p)}(T) \subset W^{1,3p/(3-2p)}(T) \subset C^{0,3-3/p}(T), \quad 1 < p < \frac{3}{2}.$$ 

For the imbedding theorem, see [1] and [5]. Although Morry’s inequality may not be applied, the continuous imbedding $W^{3,1}(T) \subset C^0(T)$ still holds. For proof of the critical imbedding, see [1, Theorem 4.12] and [4, Lemma 4.3.4]. In the following, we assume that $p$ is taken so that the imbedding $W^{k+1,p}(T) \subset C^0(T)$ holds, that is,

$$1 \leq p \leq \infty, \quad \text{if } k + 1 \geq 3 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \quad \text{if } k + 1 = 2.$$
2.3 Classification of tetrahedrons into two types

As noted in [2, 12, 15], to deal with arbitrary tetrahedrons (including anisotropic ones) uniformly, we need to classify tetrahedrons into two types. Let $T$ be an arbitrary tetrahedron and $x_i$, $i = 1, \cdots, 4$ be its vertices. Let $e_2$ be the shortest edge of $T$ and $e_1$ be the longest edge connected to $e_2$. We assume that $x_1$ and $x_2$ are the endpoints of $e_1$. Let $x_3$ be an endpoint of $e_2$ that is not an endpoint of $e_1$. Then, $e_1$ and $e_2$ are edges of $\triangle x_1x_2x_3$. Note that we still have two cases for assigning $x_1$ and $x_2$ as the endpoints of $e_1$.

Consider the plane that is perpendicular to $e_1$ and intersects $e_1$ at its midpoint. Then, $\mathbb{R}^3$ is divided by this plane into two half-spaces. In this situation, we have two cases, and tetrahedrons are classified as either Type 1 or Type 2 accordingly:

- **Case 1.** If one half-space contains three vertices and the other half-space contains one vertex, then $T$ is classified as Type 1.

- **Case 2.** If the two half-spaces contain two vertices each, then $T$ is classified as Type 2.

If the plane contains a vertex, then $T$ is classified as Type 1.

We now introduce the following assignment of the vertices for each case.

- If $T$ is Type 1, the endpoints of $e_2$ are $x_1$ and $x_3$, and the face $\triangle x_1x_3x_4$ belongs to one half-space. Let $\alpha_2 := |x_1x_3|$.

- If $T$ is Type 2, the endpoints of $e_2$ are $x_2$ and $x_3$, and $e_2$ and $x_1x_4$ belongs to the different half-spaces. Let $\alpha_2 := |x_2x_3|$.

Define $\alpha_1 := |x_1x_2|$ and $\alpha_3 := |x_1x_4|$ for both cases.

![Figure 3: Tetrahedrons of Type 1 (left) and Type 2 (right).](image-url)
2.4 Standard position of tetrahedrons

For considering the geometry of tetrahedrons, it is convenient to assign coordinates of their vertices explicitly. Suppose that an arbitrary tetrahedron $T$ is taken and classified as explained in Section 2.3. Let the parameters $s_1, t_1, s_{21}, s_{22}, t_2$ be such that

$$\begin{cases} s_1^2 + t_1^2 = 1, & s_1 > 0, \quad t_1 > 0, \quad \alpha_2 s_1 \leq \frac{\alpha_1}{2}, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, & t_2 > 0, \quad \alpha_3 s_{21} \leq \frac{\alpha_2}{2}. \end{cases} \tag{4}$$

Suppose that $T$ is Type 1. Then, using translation and rotation, we may move $T$ as $x_1 \mapsto (0,0,0)^\top$, $x_2 \mapsto (\alpha_1,0,0)^\top$, and $x_3 \mapsto (x_3,y_3,0)^\top$ with $y_3 > 0$. Letting $\theta := \angle x_2 x_1 x_3$ and $s_1 := \cos \theta$, $t_1 := \sin \theta > 0$, we have $x_3 = \alpha_2 s_1$, $y_3 = \alpha_2 t_1$. Note that, by the assignment of vertices $x_i$ ($i = 1,2,3$), we have $s_1 > 0$ (otherwise $|x_1 x_2| < |x_3 x_2|$) and $\alpha_2 s_1 \leq \frac{\alpha_1}{2}$. In this situation, $x_4$ might be below $xy$-plain (its $z$-coordinate is negative). If so, we use mirror imaging with respect to $xy$-plain to make it be above $xy$-plain (make its $z$-coordinate positive). Let $(s_{21},s_{22},t_2) := \frac{x_1 x_4}{|x_1 x_4|}$. By these procedure, we may assume without loss of generality that $T$ of Type 1 is transformed to a tetrahedron with vertices

$$x_1 = (0,0,0)^\top, \quad x_2 = (0,0,0)^\top, \quad x_3 = (\alpha_2 s_1, \alpha_2 t_1,0)^\top, \quad x_4 = (\alpha_3 s_{21}, \alpha_3 s_{22}, \alpha_3 t_2)^\top. \tag{5}$$

(Recall that $\alpha_2 = |\overline{x_1 a_3}|$, $\alpha_3 = |\overline{x_1 a_4}|$, and $\alpha_3 s_{21} \leq \alpha_1/2$ by the definition.)

If $T$ is Type 2, we may transform $T$ to a tetrahedron with vertices

$$x_1 = (0,0,0)^\top, \quad x_2 = (0,0,0)^\top, \quad x_3 = (\alpha_1 - \alpha_2 s_1, \alpha_2 t_1,0)^\top, \quad x_4 = (\alpha_3 s_{21}, \alpha_3 s_{22}, \alpha_3 t_2)^\top \tag{6}$$

by a similar manner. We refer to the coordinates in (2.4), (2.4) as the **standard position** of $T$. We always identify $T$ with the tetrahedron with vertices (2.4), (2.4). Note that we have

$$|T| = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 t_1 t_2, \tag{7}$$

where $|T|$ is the volume of $T$.

2.5 Reference tetrahedrons

Because we have two types of tetrahedrons, it is convenient to introduce two reference tetrahedrons to deal with them uniformly. Let $\hat{T}$ and $\overline{T}$ be tetrahedrons that have the following vertices (see Figure 4):

- $\hat{T}$ has the vertices $(0,0,0)^\top, (1,0,0)^\top, (0,1,0)^\top, (0,0,1)^\top$,
- $\overline{T}$ has the vertices $(0,0,0)^\top, (1,0,0)^\top, (1,1,0)^\top, (0,0,1)^\top$. 

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These tetrahedrons are called the **reference tetrahedrons**. In the following, \( \hat{T} \) corresponds to tetrahedrons of Type 1 and \( \tilde{T} \) corresponds to tetrahedrons of Type 2. We denote the reference tetrahedrons by \( T \), that is, \( T \) is either of \( \{ \hat{T}, \tilde{T} \} \).

### 2.6 Linear transformations

For an arbitrary tetrahedron \( T \) written as (2.4) or (2.4) with parameters (2.4), we consider an affine transformation from the reference tetrahedrons. Define the matrices \( \hat{A}, \tilde{A}, D_{\alpha_1\alpha_2\alpha_3} \in GL(3, \mathbb{R}) \) by

\[
\hat{A} := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \tilde{A} := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad D_{\alpha_1\alpha_2\alpha_3} := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}.
\]

We immediately confirm that the following lemma holds.

**Lemma 9** ([14]) *Let \( T \) be an arbitrary tetrahedron in the standard position (2.4) or (2.4) with parameters (2.4). Then, \( T \) is transformed from the reference tetrahedron \( T \) by \( T = \hat{A}D_{\alpha_1\alpha_2\alpha_3}(\hat{T}) \) for Type 1, or \( T = \tilde{A}D_{\alpha_1\alpha_2\alpha_3}(\tilde{T}) \) for Type 2.***

The linear transformation defined by \( D_{\alpha_1\alpha_2\alpha_3} \) is called the **squeezing transformation** [15], and we will show that the squeezing transformation does not reduce approximation property of Lagrange interpolation at all (see Theorem 11).

Note that \( \hat{A} \) and \( \tilde{A} \) are decomposed as \( \hat{A} = XY \hat{Y} \) and \( \tilde{A} = X\tilde{Y} \) with

\[
X := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \hat{Y} := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{Y} := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
respectively. We consider the singular values of \( \hat{A}, \tilde{A}, X, \hat{Y}, \) and \( \tilde{Y} \). A straightforward computation yields

\[
\det (X^\top X - \mu I) = (1 - \mu) \left( \mu^2 - 2\mu + t_1^2 \right),
\]
\[
\det (\hat{Y}^\top \hat{Y} - \mu I) = \det (\tilde{Y}^\top \tilde{Y} - \mu I) = (1 - \mu) \left( \mu^2 - 2\mu + t_1^2 \right).
\]

Thus, we find that, setting \( s_1 := |s_1| \) and \( s_2 := (s_{21}^2 + s_{22}^2)^{1/2} \),

\[
\|X\| = (1 + s_2)^{1/2}, \quad \|X^{-1}\| = (1 - s_2)^{-1/2},
\]
\[
\|Y\| = (1 + s_1)^{1/2}, \quad \|Y^{-1}\| = (1 - s_1)^{-1/2}, \quad Y = \hat{Y} \text{ or } Y = \tilde{Y},
\]
\[
\|A\| \leq \prod_{i=1}^2 (1 + s_i)^{1/2}, \quad \|A^{-1}\| \leq \prod_{i=1}^2 (1 - s_i)^{-1/2}, \quad A = \hat{A} \text{ or } A = \tilde{A}.
\]

Note that

\[
s_i^2 + t_i^2 = 1, \quad i = 1, 2 \quad \text{and} \quad \|A^{-1}\| \leq \prod_{i=1}^2 (1 - s_i)^{-1/2} = \prod_{i=1}^2 \frac{(1 + s_i)^{1/2}}{t_i}.
\]

2.7 Another geometric quantities of tetrahedrons

In (1), a quantity \( R_T \) is defined for a tetrahedron \( T \). Here, we define another quantity \( H_T \) [12], which represents the geometry of \( T \), by

\[ H_T := \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} h_T = \frac{6h_T}{t_1 t_2}, \]

where the last equation is from (2.4). Then, the following lemma holds [12, Lemma 3].

\[ \text{Lemma 10} \quad \text{The two quantities } R_T \text{ and } H_T \text{ are equivalent. That is, for an arbitrary tetrahedron } T, \text{ we have} \]

\[ \frac{1}{2} H_T \leq R_T \leq 2 H_T. \]

\[ \text{Proof. Suppose that we have a triangle with the edge lengths } h_1 \leq h_2 \leq h_3. \text{ Then, } \frac{1}{2} h_3 < h_2 \leq h_3. \text{ Let } T \text{ be an arbitrary tetrahedron } T \text{ in the standard position.} \]

\[ \text{Case 1. Suppose that } T \text{ is of Type 1. Set } \beta := |\mathbf{x}_2 \mathbf{x}_3|, \quad \gamma := |\mathbf{x}_3 \mathbf{x}_4|, \text{ and } \delta := |\mathbf{x}_2 \mathbf{x}_4|. \]

By the definition of the standard position, we have

\[ \alpha_2 \leq \min \{ \alpha_3, \beta, \gamma \} \leq \max \{ \alpha_3, \beta, \gamma \} \leq \alpha_1. \]

Hence, we have either \( h_T = \alpha_1 \) or \( h_T = \delta \). Note that \( \mathbf{x}_1 \mathbf{x}_4 \) is the shortest edge of the triangle \( \triangle \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_4 \) because \( \mathbf{x}_1 \) and \( \mathbf{x}_4 \) belong to the same half-space.
Hence, we have $\alpha_3 \leq \delta$ and

$$\alpha_1 < h_T < 2\alpha_1, \text{ or } \frac{1}{2} h_T < \alpha_1 \leq h_T.$$  

So far, we realize that either $h_2 = \alpha_3$, $h_2 = \beta$, or $h_2 = \gamma$. Recall that $\alpha_2 = h_1$. In the following, we check each case.

- Case of $h_2 = \alpha_3$. In this case, we have $\alpha_1 \alpha_2 \alpha_3 = \alpha_1 h_1 h_2$, and

$$\alpha_1 \alpha_2 \alpha_3 \leq h_1 h_2 h_T < 2\alpha_1 \alpha_2 \alpha_3 \text{ and } H_T \leq R_T < 2H_T.$$  

- Case of $h_2 = \beta$. Note that $h_2 = \beta \leq \alpha_3$, and $\overline{x_1x_2}$ and $\overline{x_1x_3}$ are the longest and shortest edges of $\triangle x_1x_2x_3$, respectively. Therefore, we have

$$\frac{1}{2} \alpha_3 \leq \frac{1}{2} \alpha_1 < \beta = h_2 \leq \alpha_3 \leq \alpha_1.$$  

This means that

$$\frac{1}{2} \alpha_1 \alpha_2 \alpha_3 < h_1 h_2 h_T \leq 2\alpha_1 \alpha_2 \alpha_3 \text{ and } \frac{1}{2} H_T < R_T \leq 2H_T.$$  

- Case of $h_2 = \gamma$. Note that $h_2 = \gamma \leq \alpha_3$, and $\overline{x_1x_4}$ and $\overline{x_1x_3}$ are the longest and shortest edges of $\triangle x_1x_3x_4$, respectively. Therefore, we have

$$\frac{1}{2} \alpha_3 < \gamma = h_2 \leq \alpha_3.$$  

This implies

$$\frac{1}{2} \alpha_1 \alpha_2 \alpha_3 < h_1 h_2 h_T \leq 2\alpha_1 \alpha_2 \alpha_3 \text{ and } \frac{1}{2} H_T < R_T \leq 2H_T.$$  

**Case 2.** Suppose that $T$ is of Type 2. Set $\beta := |x_1x_3|, \gamma := |x_3x_4|$, and $\delta := |x_2x_4|$.

By the definition of the standard position, we have

$$\alpha_2 \leq \min\{\beta, \gamma, \delta\} \leq \max\{\beta, \gamma, \delta\} \leq \alpha_1.$$  

Note that $\overline{x_1x_2}$ is the longest edge of the triangle $\triangle x_1x_2x_4$ because $x_1$ and $x_4$ belong to the same half-space. Hence, we have $\alpha_3 \leq \delta \leq \alpha_1 = h_T$.

Therefore, we realize that either $h_2 = \alpha_3$, $h_2 = \beta$, or $h_2 = \gamma$. In the following, we check each case.

- Case of $h_2 = \alpha_3$. In this case, we have $\alpha_1 \alpha_2 \alpha_3 = h_1 h_2 h_T$ and $H_T = R_T.$

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• Case of $h_2 = \beta$. Note that $h_2 = \beta \leq \alpha_3$, and $x_1x_2$ and $x_2x_3$ are the longest and shortest edges of $\triangle x_1x_2x_3$, respectively. Therefore, we have
\[
\frac{1}{2} \alpha_3 \leq \frac{1}{2} \alpha_1 < \beta = h_2 \leq \alpha_3 \leq \alpha_1.
\]
This implies
\[
\frac{1}{2} \alpha_1 \alpha_2 \alpha_3 < h_1h_2h_T \leq \alpha_1 \alpha_2 \alpha_3 \quad \text{and} \quad \frac{1}{2} H_T < R_T \leq H_T.
\]
• Case of $h_2 = \gamma$. Note that $h_2 = \gamma \leq \alpha_3 \leq \delta$, and $x_2x_4$ and $x_1x_3$ are the longest and shortest edges of $\triangle x_2x_3x_4$, respectively. Therefore, we have
\[
\frac{1}{2} \alpha_3 \leq \frac{1}{2} \delta < \gamma = h_2 \leq \alpha_3 \leq \delta.
\]
This implies
\[
\frac{1}{2} \alpha_1 \alpha_2 \alpha_3 < h_1h_2h_T \leq \alpha_1 \alpha_2 \alpha_3 \quad \text{and} \quad \frac{1}{2} H_T < R_T \leq H_T.
\]
Therefore, all cases are checked and the proof is completed. □

Remark. In [14], the projected circumradius $\tilde{R}_T$ is defined for a tetrahedron $T$ as follows. Take any facet $B$ of $T$, and suppose that $T$ is transformed by translation and rotation so that $B$ is on $xy$-plain. Let $P_{xz}$ be the perpendicular projection of $\mathbb{R}^3$ onto $xz$-plain; $P_{xz}(x,y,z) := (x,0,z)$. Note that the image $P_{xy}(T)$ is a triangle, and let $R_0$ be its circumradius. Now, consider rotating $T$ around the circumcenter of $B$ on $xy$-plain. Let $T_\theta$ be the rotated tetrahedron, where $\theta$ is the angle of the rotation. Let $R_\theta$ be the circumradius of $P_{xz}(T_\theta)$ (see Figure 5). Then, define
\[
R_P := \max_{\theta \in [-\pi/2,\pi/2]} R_\theta, \quad \tilde{R}_T := \min_B \frac{R_P R_B}{h_B},
\]

Figure 5: The image of the projected circumradius of $T$. 

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where $R_B$ is the circumradius of $B$, $h_B := \text{diam}B$, and the minimum is taken over all the facets of $T$. In [14], a theorem similar to Theorem 6 is proved using $\tilde{R}_T$. It is conjectured that $R_T$ defined by (1) and the projected circumradius $\tilde{R}_T$ are equivalent.

While the circumradius of a triangle is a good and simple geometric quantity that represent its “badness” (or “goodness”), it is not so clear what is the best geometric quantity of a tetrahedron that represents its “badness”.

### 2.8 Squeezing theorem

As is explained in Section 2.4, we may assume without loss of generality that an arbitrary tetrahedron $T$ may be in the standard position. Let $T_{\alpha_1\alpha_2\alpha_3} := D T$, where the diagonal matrix $D$ is defined in (2.6). We define the set $T_p^k(T) \subset W^{k+1,p}(T)$ by

$$T_p^k(T) := \left\{ v \in W^{k+1,p}(T) \mid v(x) = 0, \forall x \in \Sigma^k(T) \right\}.$$ 

Then, we have the following squeezing theorem.

**Theorem 11** Let $k$ and $m$ be integers with $k \geq 1$ and $0 \leq m \leq k$. Let $p$ be taken as (6). Then, there exists a constant $C_{k,m,p}$ depending on $k$, $m$, $p$, but independent of $\alpha_i$ ($i = 1, 2, 3$) such that

$$B_p^{m,k}(T_{\alpha_1\alpha_2\alpha_3}) := \sup_{v \in T_p^k(T_{\alpha_1\alpha_2\alpha_3})} \frac{|v|_{m,p,T_{\alpha_1\alpha_2\alpha_3}}}{|v|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}}} \leq \left( \max_{i=1,2,3} \alpha_i \right)^{k+1-m} C_{k,m,p}.$$ 

**Proof.** Because the proof is very similar to that of [15, Theorem 21], we give it in Appendix. □

### 3 Proof of Theorem 6

In this section, we prove Theorem 6 using the setting prepared so far. Suppose that an arbitrary tetrahedron $T$ is in the standard position. Recall that $T = AD(T)$ and $T_{\alpha_1\alpha_2\alpha_3} := DT$, where $(A,T) = (\bar{A},\bar{T})$ or $(A,T) = (\tilde{A},\tilde{T})$ defined by (2.6) according to the type of $T$. Let $v \in W^{k+1,p}(T)$, and $\bar{v} \in W^{k+1,m}(T_{\alpha_1\alpha_2\alpha_3})$ be defined by $\bar{v}(x) = v(Ax)$. Then, it follows from [15, Lemma 12] that

$$|v|_{m,p,T} \leq 3^{m\mu(p)} t^{1/p} \|A^{-1}\|^m |\bar{v}|_{m,p,T_{\alpha_1\alpha_2\alpha_3}},$$

$$3^{-k+1}\mu(p) t^{1/p} \|A\|^{-k} |v|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}} \leq |v|_{k+1,p,T}.$$
Combining the above inequalities and Theorem 11, we obtain
\[
\frac{|v|_{m,p,T}}{|v|_{k+1,p,T}} \leq c_{k,m,p} \left| A \right|^{k+1} \left| A^{-1} \right|^m \frac{|\tilde{v}|_{m,p,T_{\alpha_1\alpha_2\alpha_3}}}{|\tilde{v}|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}}} \leq c_{k,m,p} C_{k,m,p} \left| A \right|^k \left| A^{-1} \right|^m \left( \max_{i=1,2,3} \alpha_i \right)^{k+1-m} \leq c_{k,m,p} C_{k,m,p} \left| A \right|^k \left| A^{-1} \right|^m h_T^{k+1-m}.
\]

where \( c_{k,m,p} := 3^{(k+1+m)\mu(p)} \). Therefore, we obtain the following lemma.

**Lemma 12** For an arbitrary triangle \( T \) in the standard position, we have
\[
B_p^m,k+1(T) := \sup_{v \in T_p^{k}(T)} \frac{|v|_{m,p,T}}{|v|_{k+1,p,T}} \leq c_{k,m,p} C_{k,m,p} \left| A \right|^k \left| A^{-1} \right|^m h_T^{k+1-m}.
\]

Therefore, inserting \( v - I_T^k v \in T_p^k(T) \) into \( v \), we have
\[
|v - I_T^k v|_{m,p,T} \leq c_{k,m,p} C_{k,m,p} \left| A \right|^k \left| A^{-1} \right|^m h_T^{k+1-m} |v|_{k+1,p,T}, \quad \forall v \in W^{k+1,p}(T).
\]

We attempt to obtain upper bounds of \( \left| A \right| \) and \( \left| A^{-1} \right| \). From (2.6), (2.6), (1), and (10), we know that
\[
\left| A \right| \leq 2, \quad \left| A^{-1} \right| \leq \frac{2}{t_1 t_2} = \frac{H_T}{3h_T} \leq \frac{2R_T}{3h_T}.
\]

Hence, redefining the constant \( C_{k,m,p} \) (recall that the Sobolev (semi-)norms may be affected by rotation up to a constant [15, (16)]), Theorem 6 is proved.

### 4 Proof of Theorem 7

In this section, we give a proof of Theorem 7. For the proof, we introduce the following notation convention on \( T \). Let \( F_i \) be the face of \( T \) opposite to \( x_i \). We denote the dihedral angle between the faces \( F_i \) and \( F_j \) by \( \psi^{i,j} \). Note that \( \psi^{i,j} = \psi^{j,i} \). Furthermore, we denote the internal angle at \( x_j \) on \( F_i \) by \( \theta^i_j \), and the angle between \( F_i \) and \( \overline{x_i x_j} \) by \( \phi^i_j \).

| \( x_i \) | the vertices of \( T \). |
| --- | --- |
| \( F_i \) | the face opposite to \( x_i \). |
| \( \psi^{i,j} \) | the dihedral angle between \( F_i \) and \( F_j \). |
| \( \theta^i_j \) | the internal angle of \( F_i \) at \( x_j \). |
| \( \phi^i_j \) | the angle between \( F_i \) and \( \overline{x_i x_j} \). |
Figure 6: Definitions of the angles on $T$.

Let $A$ and $B$ be the feet of perpendicular lines from $x_j$ to $F_j$ and from $x_j$ to $x_nx_k$, respectively (see Figure 6). Then, we have

$$|x_jx_n| \sin \phi_j = |x_jA| = |x_jB| \sin \psi^{j,m} = |x_jx_n| \sin \theta^m \sin \psi^{j,m}.$$  

A similar equation holds for $\phi_n$, $\theta^k$, and $\psi^{k,j}$. Therefore,

$$\sin \phi_j = \sin \theta^k \sin \psi^{k,j} = \sin \theta^m \sin \psi^{m,j}$$

for $j = 1, 2, 3, 4$, $m, n, k \in \{1, 2, 3, 4\} \setminus \{j\}$.  

In the following, we abbreviate “maximum angle condition” as MAC.

**Lemma 13 (Cosine rules on tetrahedrons)** Let $T \subset \mathbb{R}^3$ be a tetrahedron. Let $j = 1, 2, 3, 4$ and $\{k, m, n\} = \{1, 2, 3, 4\} \setminus \{j\}$. Then, we have

$$\cos \theta_j = \cos \theta^m_j \cos \theta^k_j + \sin \theta^m_j \sin \theta^k_j \cos \psi^{m,n},$$

$$\cos \psi^{m,n} = \sin \psi^{m,k} \sin \psi^{n,k} \cos \theta_j - \cos \psi^{m,k} \cos \psi^{n,k}.$$  

**Proof.** See [10, 19]. \(\square\)

**Lemma 14** Let $T \subset \mathbb{R}^2$ be a triangle and let $\theta_i$ ($i = 1, 2, 3$) be the internal angles of $T$ with $\theta_1 \leq \theta_2 \leq \theta_3$. If there exists $\theta_{\text{max}} \in [\pi/3, \pi)$ such that $\theta_3 \leq \theta_{\text{max}}$, then we have

$$\sin \theta_2, \sin \theta_3 \geq \min \left\{ \sin \frac{\pi - \theta_{\text{max}}}{2}, \sin \theta_{\text{max}} \right\}.$$  

**Proof.** Because $\theta_1 + \theta_2 + \theta_3 = \pi$, the assumptions yield

$$2\theta_2 \geq \theta_1 + \theta_2 = \pi - \theta_3 \geq \pi - \theta_{\text{max}}$$

and $\frac{\pi - \theta_{\text{max}}}{2} \leq \theta_2 \leq \theta_3 \leq \theta_{\text{max}},$

which implies (14). \(\square\)
Lemma 15 For \( \gamma \in [\pi/3, \pi) \), we have
\[
0 < \frac{\cos \gamma + 1}{\sin \frac{\gamma}{2} + 1} \leq 1.
\]

Proof. This lemma can be proved immediately from
\[
\frac{\cos \gamma + 1}{\sin \frac{\gamma}{2} + 1} = 2 \left( 1 - \sin \frac{\gamma}{2} \right), \quad \frac{\pi}{6} \leq \frac{\gamma}{2} < \frac{\pi}{2}, \quad \frac{1}{2} \leq \sin \frac{\gamma}{2} < 1. \quad \square
\]

Lemma 16 Let \( T \subset \mathbb{R}^3 \) be a tetrahedron. Suppose that \( T \) satisfies the MAC with \( \theta_{\text{max}} \in [\pi/3, \pi) \). Additionally, assume that \( \theta_n^j \) is not the minimum angle of face \( F_j = \triangle P_m P_n P_k \), and \( \theta_n^j < \pi/2 \), where \( j = 1, 2, 3, 4 \) and \( \{m, n, k\} = \{1, 2, 3, 4\} \setminus \{j\} \). Then, setting \( \delta \) to
\[
\sin \delta = \left( \frac{\cos \theta_{\text{max}} + 1}{\sin \frac{\theta_{\text{max}}}{2} + 1} \right)^{1/2}, \quad 0 < \delta \leq \frac{\pi}{2},
\]
we have either
\[
\psi^{m,j} \geq \delta, \quad \text{or} \quad \psi^{k,j} \geq \delta.
\]

Proof. From Lemma 15, we have
\[
0 < \frac{\cos \theta_{\text{max}} + 1}{\sin \frac{\theta_{\text{max}}}{2} + 1} \leq 1,
\]
and we confirm that \( \delta \) is well-defined.

The proof is by contradiction. Suppose that
\[
0 < \psi^{m,j} < \delta \quad \text{and} \quad 0 < \psi^{k,j} < \delta.
\]
Then, we have \( 0 < \sin \psi^{m,j} \sin \psi^{k,j} < \sin^2 \delta \) and \( 1 > \cos \psi^{m,j} \cos \psi^{k,j} > \cos^2 \delta \). From Lemma 14 and the assumption, we have
\[
\frac{\pi - \theta_{\text{max}}}{2} \leq \theta_n^j < \frac{\pi}{2}, \quad 0 < \cos \theta_n^j \leq \cos \left( \frac{\pi - \theta_{\text{max}}}{2} \right) = \sin \frac{\theta_{\text{max}}}{2}.
\]
Thus, we obtain
\[
\sin \psi^{m,j} \sin \psi^{k,j} \cos \theta_n^j < \sin^2 \delta \sin \frac{\theta_{\text{max}}}{2}.
\]
The cosine rule (13) and the above inequalities yield
\[
\cos \psi^{m,k} = \sin \psi^{m,j} \sin \psi^{k,j} \cos \theta_n^j - \cos \psi^{m,j} \cos \psi^{k,j}
\]
\[
< \sin^2 \delta \sin \frac{\theta_{\text{max}}}{2} - (1 - \sin^2 \delta)
\]
\[
= \frac{\cos \theta_{\text{max}} + 1}{\sin \frac{\theta_{\text{max}}}{2} + 1} \left( \sin \frac{\theta_{\text{max}}}{2} + 1 \right) - 1 = \cos \theta_{\text{max}},
\]
which contradicts the MAC: \( \psi^{m,k} \leq \theta_{\text{max}}. \quad \square \)
Corollary 17 Under the assumptions of Lemma 16, we have

\[ \sin \psi^{m,j} \geq C_0, \quad \text{or} \quad \sin \psi^{k,j} \geq C_0, \quad C_0 := \min \{ \sin \delta, \sin \theta_{\max} \}. \]

Lemma 18 For \( j = 1, 2, 3, 4 \), let \( \{m, n, k\} = \{1, 2, 3, 4\}\setminus\{j\} \). Let \( p \in \{m, n, k\} \), and \( \{q, r\} = \{m, n, k\}\setminus\{p\} \). Suppose that there exists a positive constant \( M \) with \( 0 < M < 1 \) such that \( \sin \phi_p^j \sin \theta_n^j \geq M \). Then, setting \( \gamma(M) := \pi - \sin^{-1} M \left( \frac{\pi}{2} < \gamma(M) < \pi \right) \), the MAC with \( \gamma(M) \) is satisfied on faces \( F_j, F_q, F_r \), and \( \psi_j^{p,q}, \psi_j^{p,r} \leq \gamma(M) \).

Proof. From the assumption, we have

\[ M \leq \sin \phi_p^j \sin \theta_n^j \leq \sin \theta_n^j \quad \text{and} \quad M \leq \phi_p^j. \]

Hence, the definition of \( \gamma(M) \) yields \( \pi - \gamma(M) \leq \theta_n^j \leq \gamma(M) \). Because \( \theta_n^j + \theta_m^j + \theta_k^j = \pi \), we see that \( \theta_n^j, \theta_m^j < \theta_m^j + \theta_k^j \leq \gamma(M) \). That is, the MAC with \( \gamma(M) \) is satisfied on face \( F_j = \triangle P_m P_n P_k \).

Moreover, it follows from (4) that

\[ M \leq \sin \phi_p^j = \sin \theta_p^q \sin \psi_j^{q,j} = \sin \theta_p^r \sin \psi_j^{r,j} \leq \sin \theta_p^q, \sin \theta_p^r, \sin \psi_j^{q,j}, \sin \psi_j^{r,j} \]

By the same reasoning, we find that the MAC with \( \gamma(M) \) is satisfied on faces \( F_q \) and \( F_r \), and \( \psi_j^{q,q}, \psi_j^{r,r} \leq \gamma(M) \). \( \square \)

In the following, we prove Theorem 7 using \( H_T \) instead of \( R_T \). We divide the proof into four cases.

4.1 Type 1: Proof of “MAC implies (7)”

First, we suppose that \( T \) is of Type 1 and satisfies the MAC with \( \theta_{\max}, \pi/3 \leq \theta_{\max} < \pi \). Because \( |T| = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 \sin \theta_1^4 \sin \phi_1^4 \), we have

\[ \frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4}. \]

From the definition of Type 1, we realize that \( \theta_2^4 \leq \theta_1^4 \leq \theta_3^4 \), that is, \( \theta_3^4 \) and \( \theta_2^4 \) are the maximum and minimum angles of face \( F_4 = \triangle P_1 P_2 P_3 \), respectively. Thus, it follows from Lemma 14 that

\[ \frac{\pi - \theta_{\max}}{2} \leq \theta_1^4 \leq \theta_{\max}, \quad \sin \theta_1^4 \geq \min \left\{ \sin \frac{\pi - \theta_{\max}}{2}, \sin \theta_{\max} \right\} =: C_1. \]
Additionally, we may apply Lemma 16 to $\theta_1^4$ and $F_4$, and find that either $\psi^{2,4} \geq \delta$ or $\psi^{3,4} \geq \delta$, where $\delta = \delta(\theta_{\text{max}})$, $0 < \delta \leq \pi/2$ is defined as

$$\sin \delta = \left( \frac{\cos \theta_{\text{max}} + 1}{\sin \frac{\theta_{\text{max}}}{2} + 1} \right)^{1/2}. \tag{15}$$

Suppose that $\psi^{2,4} \geq \delta$. By Corollary 17 and (4), we have

$$\sin \phi_1^4 = \sin \theta_1^2 \sin \psi^{2,4} \geq C_0 \sin \theta_1^2,$$

where $C_0$ is the constant defined in Corollary 17. By the definition of Type 1, $\theta_1^2$ is not the minimum angle of $F_2 = \triangle P_1 P_3 P_4$, and therefore, we have

$$\frac{\pi - \theta_{\text{max}}}{2} \leq \theta_1^2 \leq \theta_{\text{max}}, \quad \sin \theta_1^2 \geq C_1.$$

Thus, we obtain $\sin \phi_1^4 \geq C_0 C_1$.

Next, suppose that $\psi^{3,4} \geq \delta$. Replacing $\psi^{2,4}$, $\theta_1^2$, and $F_2$ with $\psi^{3,4}$, $\theta_1^3$, and $F_3$ in the above argument, we obtain $\sin \phi_1^4 \geq C_0 C_1$ in the same manner.

Gathering the above results, we conclude that

$$\frac{H_T}{h_T} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4} \leq \frac{6}{C_0 C_1^2} =: D$$

in both cases, that is, (7) holds.

### 4.2 Type 1: Proof of “(7) implies MAC”

Now, we suppose that $T$ is of Type 1 and

$$\frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4} \leq D.$$

Because $\theta_1^4 < \pi/2$ and $\sin \theta_1^4 \sin \phi_1^4 < 1$, we have

$$\sin \theta_1^4 \sin \phi_1^4 \geq \frac{6}{D} =: M, \quad 0 < M < 1.$$

By Lemma 18 with $j = 4$ and $p = 1$, setting $\gamma(M) := \pi - \sin^{-1} M$, we have $\frac{\pi}{2} < \gamma(M) < \pi$, and the MAC with $\gamma(M)$ is satisfied on $F_2$, $F_3$, $F_4$, and $\psi^{2,4}$, $\psi^{3,4} \leq \gamma(M)$.

Note that $|T| = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 \sin \theta_1^3 \sin \phi_1^3$, and we have

$$\frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_1^3 \sin \phi_1^3} \leq D.$$

Thus, by Lemma 18 with $j = 3$ and $p = 1$, we find that $\psi^{2,3} \leq \gamma(M)$.
Because $|P_3P_4| < |P_1P_4| + |P_1P_3| \leq 2\alpha_3$ on $F_2 = \Delta P_1P_3P_4$ and $|P_2P_3| \leq \alpha_1$, we note that

$$|T| = \frac{1}{6}\alpha_2|P_2P_3||P_3P_4| \sin \theta_3^4 \sin \phi_3^4 < \frac{1}{3}\alpha_1\alpha_2\alpha_3 \sin \theta_3^4 \sin \phi_3^4.$$

Thus, we have

$$D \geq \frac{H_K}{h_K} > \frac{3}{\sin \theta_3^4 \sin \phi_3^4} \text{ and } \sin \theta_3^4 \sin \phi_3^4 > \frac{3}{D} = \frac{M}{2}.$$

From Lemma 18, setting $\gamma(M/2) := \pi - \sin^{-1}(M/2)$, we have $\frac{\pi}{2} < \gamma(M/2) < \pi$ and MAC with $\gamma(M/2)$ is satisfied on $F_1$, and $\psi^{2,1}, \psi^{4,1} \leq \gamma(M/2)$.

The final thing to prove is the MAC for $\psi^{1,3}$. From the cosine rule (13), we have

$$\cos \psi^{1,3} = \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 - \cos \psi^{3,4} \cos \psi^{4,1}.$$
we have
\[ \frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \leq \frac{6}{\sin \theta_2^4 \sin \phi_2^4}. \] (16)

From the definition of Type 2, we realize that \( \theta_4^1 \leq \theta_4^2 \leq \theta_4^3 \) on \( F_4 \), \( \theta_3^1 \leq \theta_3^2 \leq \theta_3^3 \) on \( F_3 \), and \( \theta_2^1 \) is not the minimum angle of \( F_1 \). Thus, it follows from Lemma 14 that
\[ \frac{\pi - \theta_{\max}}{2} \leq \theta_4^4, \theta_3^4, \theta_2^4 \leq \theta_{\max}, \quad \sin \theta_4^4, \sin \theta_3^4, \sin \theta_2^4 \geq C_1. \]

Additionally, we may apply Lemma 16 to \( \theta_4^4 \) and \( F_4 \), and find that either \( \psi_1,4 \geq \delta \) or \( \psi_3,4 \geq \delta \), where \( \delta = \delta(\theta_{\max}) \) is defined by (4.1).

Suppose that \( \psi_3,4 \geq \delta \). Using the same argument as in Section 4.1, we have
\[ \sin \phi_4^4 = \sin \theta_4^3 \sin \psi_3,4 \geq C_0 \sin \theta_4^3 \geq C_0 C_1. \]
Next, suppose that \( \psi_1,4 \geq \delta \). We have
\[ \sin \phi_2^4 = \sin \theta_2^4 \sin \psi_1,4 \geq C_0 \sin \theta_2^4 \geq C_0 C_1. \]
Combining these results with (4.3), we obtain
\[ \frac{H_T}{h_T} \leq \frac{6}{C_0 C_1^2} =: D, \]
that is, (7) holds.

### 4.4 Type 2: Proof of “(7) implies MAC”

Finally, we suppose that \( T \) is of Type 2 and
\[ \frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \leq D, \quad \sin \theta_2^4 \sin \phi_4^4 \geq \frac{6}{D} =: M. \]

The proof is very similar to that described in Section 4.2. By Lemma 18 with \( j = 4 \) and \( p = 1 \), setting \( \gamma(M) := \pi - \sin^{-1} M \), the MAC with \( \gamma(M) \) is satisfied on \( F_2, F_3, F_4 \), and \( \psi_2^4, \psi_3^4 \leq \gamma(M) \).

Because \( |P_2 P_4| \leq \alpha_1 \), we have
\[ |T| = \frac{1}{6} |P_2 P_3| |P_2 P_4| |P_1 P_4| \sin \theta_2^4 \sin \phi_4^4 \leq \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 \sin \theta_2^4 \sin \phi_4^4. \]
This yields
\[ D \geq \frac{H_T}{h_T} \geq \frac{6}{\sin \theta_2^4 \sin \phi_4^4} \quad \text{and} \quad \sin \theta_2^4 \sin \phi_4^4 \geq \frac{6}{D} = M, \]
and, by Lemma 18 with \( j = 1 \) and \( p = 4 \), we find that the MAC with \( \gamma(M) \) is satisfied on \( F_1 \), and \( \psi^{1,2}, \psi^{1,3} \leq \gamma(M) \).

The final thing to prove is the MAC for \( \psi^{1,4} \) and \( \psi^{2,3} \). By the cosine rule (13) with \( j = 2 \), we have

\[
\cos \psi^{1,4} = \sin \psi^{1,3} \sin \psi^{4,3} \cos \theta_2^3 - \cos \psi^{1,3} \cos \psi^{4,3}, \\
\cos \psi^{2,3} = \sin \psi^{2,4} \sin \psi^{3,4} \cos \theta_1^4 - \cos \psi^{2,4} \cos \psi^{3,4}.
\]

By the definition of Type 2, \( \theta_2^3 \) and \( \theta_1^4 \) are the minimum angles of \( F_3 \) and \( F_4 \), respectively. Therefore, we have \( \cos \theta_2^3, \cos \theta_1^4 \geq \frac{1}{3} \) and thus

\[
\cos \psi^{1,4} > - \cos \psi^{1,3} \cos \psi^{3,4}, \quad \cos \psi^{2,3} > - \cos \psi^{2,4} \cos \psi^{3,4}.
\]

Because \( \sin \psi^{1,3}, \sin \psi^{2,4}, \sin \psi^{3,4} > M \), we find that

\[
\cos \psi^{1,4} > - \cos \psi^{1,3} \cos \psi^{3,4} \geq - \sqrt{1 - \sin^2 \psi^{1,3} \sqrt{1 - \sin^2 \psi^{3,4} > M^2 - 1}}, \\
\cos \psi^{2,3} > M^2 - 1.
\]

Therefore, we conclude that \( \psi^{1,4}, \psi^{2,3} < \cos^{-1}(M^2 - 1) < \pi \), and \( T \) satisfies the MAC with

\[
\theta_{\text{max}} := \max \{ \gamma(M), \cos^{-1}(M^2 - 1) \}.
\]

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Appendix: Proof of Theorem 11

The proof of Theorem 11 is very similar to that of [13, Theorem 13] and [15, Theorem 21]. First, refer to [15, Section 5] for the definition of difference quotients of one and two variable functions. Difference quotients of three variable functions is their simple extension.

For a positive integer \( k \), \( X^k \) is the set of lattice points defined by
\[
X^k := \left\{ \mathbf{x}_\gamma := \gamma/k \in \mathbb{R}^3 \ \middle| \ \gamma \in \mathbb{N}_0^3 \right\},
\]
where \( \gamma/k = (a_1/k, a_2/k, a_3/k) \) is understood as the coordinate of a point in \( \mathbb{R}^3 \). For \( \mathbf{x}_\gamma \in X^k \) and a multi-index \( \delta \in \mathbb{N}_0^3 \), we define the correspondence \( \Delta^\delta \) between nodes by
\[
\Delta^\delta \mathbf{x}_\gamma := \mathbf{x}_\gamma + \delta = (\gamma + \delta)/k.
\]

For two multi-indexes \( \eta = (m_1, m_2, m_3), \delta = (n_1, n_2, n_3), \eta \leq \delta \) means that \( m_i \leq n_i \) \((i = 1, 2, 3)\). Also, \( \delta \cdot \eta \) and \( \delta! \) are defined by \( \delta \cdot \eta := \sum_{i=1}^3 m_i n_i \) and \( \delta! := n_1! n_2! n_3! \), respectively. Suppose that, for \( \gamma, \delta \in \mathbb{N}_0^3 \), both \( \mathbf{x}_\gamma \) and \( \Delta^\delta \mathbf{x}_\gamma \) belong to \( K \). Then, we define the difference quotients for \( f \in C^0(K) \) by
\[
f^{[\delta]}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma] := k^{[\delta]} \sum_{\eta \leq \delta} \frac{(-1)^{[\delta]-[\eta]}}{\eta!(\delta - \eta)!} f(\Delta^\eta \mathbf{x}_\gamma).
\]

For example, we see that
\[
f^4[\mathbf{x}_{(0,0,0)}, \Delta^{(2,1,1)} \mathbf{x}_{(0,0,0)}] = \frac{k^4}{2} \left( f(\mathbf{x}_{(2,1,1)}) - 2f(\mathbf{x}_{(1,1,1)}) + f(\mathbf{x}_{(0,1,1)}) - f(\mathbf{x}_{(2,0,1)}) + 2f(\mathbf{x}_{(1,0,1)}) - f(\mathbf{x}_{(0,0,1)}) - f(\mathbf{x}_{(2,1,0)}) + 2f(\mathbf{x}_{(1,1,0)}) - f(\mathbf{x}_{(0,1,0)}) + f(\mathbf{x}_{(2,0,0)}) - 2f(\mathbf{x}_{(1,0,0)}) + f(\mathbf{x}_{(0,0,0)}) \right).
\]

As explained in [15, Section 5], a differential quotients is expressed concisely by an integral. For that purpose, we introduce the \( s \)-simplex
\[
S_s := \left\{ (x_1, \ldots, x_s)^\top \in \mathbb{R}^s \ \middle| \ x_i \geq 0, \ 0 \leq x_1 + \cdots + x_s \leq 1 \right\},
\]
and the integral of \( g \in L^1(S_s) \) on \( S_s \) is defined by
\[
\int_{S_s} g(w_1, \ldots, w_s) dW_s := \int_0^1 \int_0^{w_1} \cdots \int_0^{w_{s-1}} g(w_1, \ldots, w_s) dw_s \cdots dw_2 dw_1,
\]
where \( dW_s := dw_1 \cdots dw_s \). Then, \( f^s[x_{(t,q)}, \Delta^{(0,s,0)}x_{(t,q)}] \) becomes

\[
f^s[x_{(t,q)}, \Delta^{(0,s,0)}x_{(t,q)}] = \int_{S_s} \partial^{(0,s,0)} f \left( \frac{l}{k}, \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s), \frac{r}{k} \right) dW_s.
\]

For a general multi-index \((t, s, m)\), we can write

\[
f^{t+s+m}[x_{(t,q)}, \Delta^{(t,s,m)}x_{(t,q)}] = \int_{S_s} \int_{S_t} \int_{S_m} \partial^{(t,s,m)} f \left( Z_t, W_s, Y_m \right) dZ_t dW_s dY_m,
\]

\[
Z_t := \frac{l}{k} + \frac{1}{k}(z_1 + \cdots + z_t), \quad dZ_t := dz_1 \cdots dz_t, \quad W_s := \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s), \quad Y_m := \frac{r}{k} + \frac{1}{k}(y_1 + \cdots + y_m), \quad dY_m := dy_1 \cdots dy_m.
\]

Let \( \Box^\delta_\gamma \) be the rectangular parallelepiped defined by \( x_\gamma \) and \( \Delta^\delta x_\gamma \) as the diagonal points. If \( \delta = (t, s, 0) \) or \( (0, s, 0) \), \( \Box^\delta_\gamma \) degenerates to a rectangle or a segment. For \( v \in L^1(\hat{K}) \) and \( \Box^\delta_\gamma \) with \( \gamma = (l, q, r) \), we denote the integral as

\[
\int_{\Box^\delta_\gamma} v := \int_{S_s} \int_{S_t} \int_{S_m} v(Z_t, W_s, Y_m) dZ_t dW_s dY_m.
\]

If \( \Box^\delta_\gamma \) degenerates to a rectangle or a segment, the integral is understood as an integral on the rectangle or on the segment. By this notation, the difference quotient \( f^{[\delta]}[x_\gamma, \Delta^\delta x_\gamma] \) is written as

\[
f^{[\delta]}[x_\gamma, \Delta^\delta x_\gamma] = \int_{\Box^\delta_\gamma} \partial^\delta f.
\]

Therefore, if \( u \in T^k_p(T) \), then we have

\[
0 = u^{[\delta]}[x_\gamma, \Delta^\delta x_\gamma] = \int_{\Box^\delta_\gamma} \partial^\delta u, \quad \forall \Box^\delta_\gamma \subset T.
\]

Let \( S \subset T \) be a segment. In the proof of Theorem 11, the continuity of the trace operator \( t \) defined as \( t : W^{1,p}(T) \ni v \mapsto v|_S \in L^1(S) \) is crucial. For two-dimensional case, the continuity of \( t \) is standard and is mentioned in many textbooks such as [5]. For three dimensional case, the situation becomes a bit more complicated. If the continuous inclusion \( W^{k+1,p}(T) \subset C^0(T) \) holds, the continuity of \( t \) is obvious. Even if this is not the case, we still have the following lemma. For the proof, see [1, Theorem 4.12], [8, Lemma 2.2], and [17, Theorem 2.1].

**Lemma 19** Let \( S \subset T \) be an arbitrary segment. Then, the following trace operators are well-defined and continuous:

\[
t : W^{1,p}(T) \to L^p(S), \quad 2 < p < \infty, \quad t : W^{2,p}(T) \to L^p(S), \quad 1 \leq p < \infty.
\]
Let $p$ be taken as (6). The set $\Xi_{\delta,k}^p \subset W^{k+1-|\delta|,p}(T)$ is then defined by

$$\Xi_{\delta,k}^p := \left\{ v \in W^{k+1-|\delta|,p}(T) \mid \int_{\square_\delta^k} v = 0, \forall \square_\gamma^k \subset T \right\}.$$ 

Note that $u \in T^k_p(T)$ implies $\partial^\delta u \in \Xi_{\delta,k}^p$ by (4.4).

**Lemma 20** We have $\Xi_{\delta,k}^p \cap \mathcal{P}_{k-|\delta|} = \{0\}$. That is, if $q \in \mathcal{P}_{k-|\delta|}$ belongs to $\Xi_{\delta,k}^p$, then $q = 0$.

**Proof.** Note that $\dim \mathcal{P}_{k-|\delta|} = \#\{\square_\delta^k \subset T\}$. For example, if $k = 4$ and $|\delta| = 3$, then $\dim \mathcal{P}_1 = 4$. This corresponds to the fact that, in $T$, there are four cubes of size $1/4$ for $\delta = (1,1,1)$ and there are four rectangles of size $1/2 \times 1/4$ for $\delta = (1,2,0)$. All their vertices (corners) belong to $\Sigma^4(T)$ (see Figure 7). Now, suppose that $q \in \mathcal{P}_{k-|\delta|}$ satisfies $\int_{\square_\delta^k} q = 0$ for all $\square_\gamma^k \subset T$. These conditions are linearly independent and determine $q = 0$ uniquely (see Exercise below).

![Figure 7: The four cubes and four rectangles in T.](image)

**Exercise:** Show that the condition “$\int_{\square_\delta^k} q = 0$ for all $\square_\gamma^k \subset T$” implies $q = 0$ for $q \in \mathcal{P}_{k-|\delta|}$.

(Hint: (1) First, consider the case $d = 1$. For example, show the following: if a polynomial $p \in \mathcal{P}_k$ satisfies $\int_n^{n+1} p(x)dx = 0$, $n = 0, \cdots, k$, then $p = 0$.)

(2) Reduce the proof of the case $d > 1$ to that of the case $d = 1$.

The constant $A_{\delta,k}^p$ is defined by

$$A_{\delta,k}^p := \sup_{v \in \Xi_{\delta,k}^p} \frac{|v|_{0,p,T}}{|v|_{k+1-|\delta|,p,T}}.$$ 

The following lemma is an extension of [3, Lemma 2.1].

**Lemma 21** Let $p$ be such that $2 < p \leq \infty$ if $k+1-|\delta| = 1$ or $1 \leq p \leq \infty$ if $k+1-|\delta| \geq 2$. We then have $A_{\delta,k}^p < \infty$. 

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Proof. The proof is by contradiction. Suppose that \( A_{p}^{δ,k} = \infty \). Then there exists a sequence \( \{w_{k}\}_{k=1}^{\infty} \subset \Xi_{p}^{δ,k} \) such that \( |w_{n}|_{0,p,T} = 1 \) and \( \lim_{n \to \infty} |w_{n}|_{k+1-\delta,p,T} = 0 \). By the Bramble–Hilbert lemma [15, Theorem 14], there exists \( \{q_{n}\} \subset \mathcal{P}_{k-\delta} \) such that

\[
\|w_{n} + q_{n}\|_{k+1-\delta,p,T} \leq \inf_{q \in \mathcal{P}_{k-\delta}} \|w_{n} + q\|_{k+1-\delta,p,T} + \frac{1}{n} \leq C \|w_{n}\|_{k+1-\delta,p,T} + \frac{1}{n}
\]

and \( \lim_{n \to \infty} \|w_{n} + q_{n}\|_{k+1-\delta,p,T} = 0 \). Because \( \{w_{n}\} \subset W^{k+1-\delta,p}(\mathcal{T}) \) is bounded, \( \{q_{n}\} \subset \mathcal{P}_{k-\delta} \) is bounded as well. Hence, there exists a subsequence \( \{q_{n_{i}}\} \) such that \( q_{n_{i}} \) converges to \( \bar{q} \in \mathcal{P}_{k-\delta} \) and \( \lim_{n_{i} \to \infty} \|w_{n_{i}} + \bar{q}\|_{k+1-\delta,p,T} = 0 \). If \( \square_{lp}^{\delta} \) is not degenerate to a rectangle or a segment, we have

\[
\left| \int_{\square_{lp}^{\delta}} (w_{n_{i}} + \bar{q}) \right| \leq \int_{\square_{lp}^{\delta}} |w_{n_{i}} + \bar{q}| \leq C \|w_{n_{i}} + \bar{q}\|_{k+1-\delta,p,T} \to 0 \quad \text{as} \quad h \to 0. \tag{18}
\]

If \( \square_{lp}^{\delta} \) is degenerate to a rectangle or a segment, (4.4) holds as well by Lemma 19. Because \( \int_{\square_{lp}^{\delta}} w_{n_{i}} = 0 \) by the definition, we have

\[
0 = \lim_{n_{i} \to \infty} \int_{\square_{lp}^{\delta}} (w_{n_{i}} + \bar{q}) = \int_{\square_{lp}^{\delta}} \bar{q}, \quad \forall \square_{lp}^{\delta} \subset \mathcal{T}.
\]

Therefore, it follows from Lemma 19 that \( \bar{q} = 0 \). This implies that

\[
0 = \lim_{n_{i} \to \infty} \|w_{n_{i}}\|_{k+1-\delta,p,T} \geq \lim_{n_{i} \to \infty} |w_{n_{i}}|_{0,p,T} = 1,
\]

which is a contradiction. \( \square \)

Define the linear transformation by, for \( (x, y, z)^{\top} \in \mathbb{R}^{3} \),

\[
(x^{*}, y^{*}, z^{*})^{\top} = D_{α_{1}α_{2}α_{3}}(x, y, z)^{\top} = (α_{1}x, α_{2}y, α_{3}z)^{\top}, \quad α_{i} > 0, \ i = 1, 2, 3,
\]

which the diagonal matrix \( D_{α_{1}α_{2}α_{3}} \) is defined by (2.6). This linear transformation squeezes the reference tetrahedron \( \mathcal{T} \) perpendicularly to \( T_{α_{1}α_{2}α_{3}} = D_{α_{1}α_{2}α_{3}} \mathcal{T} \). Take an arbitrary \( v \in \mathcal{T}_{p}^{k}(T_{α_{1}α_{2}α_{3}}) \) and define \( u \in \mathcal{T}_{p}^{k}(\mathcal{T}) \) by \( u(x, y, z) := v(D_{α_{1}α_{2}α_{3}}(x, y, z)^{\top}) \). Let \( p \) be taken as (6) with \( m = |\delta| \). To make formula concise, we introduce the following notation. For a multi-index \( γ = (a, b, c) \in \mathbb{N}_{0}^{3} \) and a real \( t \neq 0 \), and \( (α) := (α_{1}, α_{2}, α_{3}) \), \( (α)^{t} := α_{1}^{a_{1}}α_{2}^{a_{2}}α_{3}^{-a_{3}} \) . Because \( u \in \mathcal{T}_{p}^{k}(\mathcal{T}) \) and \( \partial^{δ}u \in \Xi_{\alpha_{1}α_{2}α_{3}}^{δ,k} \), we may apply Lemma 21 as follows. For \( p, 1 \leq p < \infty \), we have

\[
\frac{|v|_{p,m,p,T_{α_{1}α_{2}α_{3}}}^{p}}{|v|_{p,k+1,p,T_{α_{1}α_{2}α_{3}}}^{p}} = \frac{\sum_{|γ|=m} m! \frac{m!}{|γ|!} (α)^{-|γ|} |\partial^{γ}u|_{0,p,T}^{p}}{\sum_{|δ|=k+1} \frac{(k+1)!}{|δ|!} (α)^{-|δ|} |\partial^{δ}u|_{0,p,T}^{p}} = \frac{\sum_{|γ|=m} m! \frac{m!}{|γ|!} (α)^{-|γ|} |\partial^{γ}u|_{0,p,T}^{p}}{\sum_{|γ|=m} m! \frac{m!}{|γ|!} (α)^{-|γ|} \left( \sum_{|γ|=k+1-m} \frac{(k+1-m)!}{|γ|!} |\partial^{γ}u|_{0,p,T}^{p} \right)}
\]

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Prove Theorem 11 for the case $p$ can be written as

$$\left| C \right| \leq v^2 p = \max_{i=1,2,3} \alpha_i \left( \frac{1}{\alpha_1^2} \right)^{k+1} \frac{m!}{\gamma!} \left( \gamma + \eta \right) \frac{m!}{\gamma!} \left( \gamma + \eta \right) \quad (4.4)$$

where $C_{k,m,p} := \max_{|\gamma|=m} A_p^{\gamma,k}$. Here, we use the equality

$$\frac{(k + 1)!}{\delta!} = \sum_{|\gamma|=m,|\eta|=k+1-m} m! (k + 1 - m)!$$

Hence, Theorem 11 is proved for this case. The proof of the case $p = \infty$ may be done in a similar manner. □

Exercise: (1) Check the above proof in detail. For example, confirm that, if $k = m = 1$, (4.4) can be written as

$$\frac{|u|^p_{1,p,T_{\alpha_1 \alpha_2 \alpha_3}}}{|u|^p_{2,p,T_{\alpha_1 \alpha_2 \alpha_3}}} = \sum_{|\gamma|=1} \frac{1}{\alpha_1^2} \left| \frac{\partial u^p}{\partial x^0} \right| + \frac{1}{\alpha_2^2} \left| \frac{\partial u^p}{\partial y^0} \right| + \frac{1}{\alpha_3^2} \left| \frac{\partial u^p}{\partial z^0} \right|$$

$$= \alpha_1^2 \left| \frac{\partial_x u^p}{\partial x^0} \right| + \alpha_2^2 \left| \frac{\partial_y u^p}{\partial y^0} \right| + \alpha_3^2 \left| \frac{\partial_z u^p}{\partial z^0} \right|$$

$$= \left\{ \begin{array}{ll}
X := \frac{1}{\alpha_1^2} \left| \frac{\partial_x u^p}{\partial x^0} \right| + \frac{1}{\alpha_2^2} \left| \frac{\partial_y u^p}{\partial y^0} \right| + \frac{1}{\alpha_3^2} \left| \frac{\partial_z u^p}{\partial z^0} \right|,
Y := \frac{1}{\alpha_1^2} \left| \frac{\partial_x u^p}{\partial x^0} \right| + \frac{1}{\alpha_2^2} \left| \frac{\partial_y u^p}{\partial y^0} \right| + \frac{1}{\alpha_3^2} \left| \frac{\partial_z u^p}{\partial z^0} \right|,
Z := \frac{1}{\alpha_1^2} \left| \frac{\partial_x u^p}{\partial x^0} \right| + \frac{1}{\alpha_2^2} \left| \frac{\partial_y u^p}{\partial y^0} \right| + \frac{1}{\alpha_3^2} \left| \frac{\partial_z u^p}{\partial z^0} \right|
\end{array} \right\}$$

(2) Prove Theorem 11 for the case $p = \infty$. 

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