On boundary correlators in Liouville theory on AdS$_2$

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Abstract: We consider the Liouville theory in fixed Euclidean AdS$_2$ background. Expanded near the minimum of the potential the elementary field has mass squared 2 and (assuming the standard Dirichlet b.c.) corresponds to a dimension 2 operator at the boundary. We provide strong evidence for the conjecture that the boundary correlators of the Liouville field are the same as the correlators of the holomorphic stress tensor (or the Virasoro generator with the same central charge) on a half-plane or a disc restricted to the boundary. This relation was first observed at the leading semiclassical order (tree-level Witten diagrams in AdS$_2$) in arXiv:1902.10536 and here we demonstrate its validity also at the one-loop level. We also discuss arguments that may lead to its general proof.

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1 Introduction

Study of quantum field theories in AdS$_2$ background is of interest from several points of view (see, e.g., [1–6] and references there). A recent example appeared in the investigation of correlators of operators inserted on a straight or circular Wilson line in the context of AdS$_5$/CFT$_4$ correspondence [7–14]. As discussed in [12], starting with the string action in AdS$_5 \times S^5$ and expanding it near the corresponding minimal surface one gets a 2d field theory action in AdS$_2$ background. The boundary correlators of elementary excitations that should match the strong coupling limit of the Wilson loop "defect" correlators in $N = 4$ super Yang-Mills theory are constrained by the isometry of AdS$_2$ or the 1d conformal group $SO(2, 1) \simeq SL(2, \mathbb{R})$ providing a non-trivial example of a non-gravitational AdS$_2$/CFT$_1$ duality. One technical problem in the approach of [12, 14] is how to systematically compute loop corrections to boundary correlators of 2d quantum fields in AdS$_2$.

Using this as a partial motivation, it is interesting to study loop corrections to boundary correlators in some simple 2d QFTs defined on AdS$_2$ background. Here we shall consider the familiar example of the Liouville theory [15–17]

$$S = \frac{1}{4\pi} \int d^2x \sqrt{g} \left( \partial^a \varphi \partial_a \varphi + \mu e^{2b \varphi} + Q R \varphi \right), \quad Q = b + b^{-1}. \quad (1.1)$$

Defined on a fixed curved 2d background it is a Weyl-covariant quantum theory with the central charge $c = 1 + 6 Q^2$. In the special case of the Euclidean AdS$_2$ background $ds^2 = \frac{1}{27} (dt^2 + dz^2)$ with the curvature $R = -2$ the field $\varphi$ expanded near its constant vacuum value has the classical ($b \ll 1$) mass...
\(m^2 = 2\). Interpreting this model from the point of view of the AdS\(_2\)/CFT\(_1\) duality\(^1\) and assuming the standard (Dirichlet) boundary condition at \(z = 0\) this field should have the boundary asymptotics \(\varphi(t, z)|_{z=0} \sim z^2 \Phi(t) + \ldots\), i.e. should be dual to a 1d boundary CFT operator with the 1d conformal dimension \(\Delta = 2\).\(^2\) One can then compute the corresponding boundary correlators\(^3\)

\[
\langle \Phi(t_1) \cdots \Phi(t_n) \rangle \equiv \lim_{z_1, \ldots, z_n \to 0} z_1^{-2} \cdots z_n^{-2} \langle \varphi(t_1, z_1) \cdots \varphi(t_n, z_n) \rangle
\]

(1.2)

using the standard Witten diagram prescription.

Given the special conformal invariance properties of the Liouville theory one may expect these boundary correlators to have a particularly simple structure. Indeed, it was recently noticed in [19] that at the tree-level \((b < 1 \text{ or } c \gg 1)\) the 2-, 3- and 4-point boundary correlators (1.2) have exactly the same dependence on the boundary points \(t_i\) as the correlators of the holomorphic stress tensor \(T(z)\) (generator of the Virasoro algebra with the same central charge \(c\)) have on the complex coordinates \(z_i\), i.e.

\[
\langle \Phi(t_1) \cdots \Phi(t_n) \rangle = \kappa^n \langle T(z_1) \cdots T(z_n) \rangle |_{t_i \to t_i}.
\]

(1.3)

Here \(\kappa = \kappa(b)\) is a proportionality coefficient in the formal identification \(\Phi(t) \to \kappa T(t)\). Equivalently, one may view the r.h.s. of (1.3) as the correlator of the chiral stress tensor of a CFT on a half-plane restricted to the real boundary \(z_i = t_i + iy_i \to t_i\) (assuming the usual conformal gluing condition \(T(t) = T(t)\)). Explicitly, the Virasoro algebra or the standard OPE, \(T(z') T(z) = \frac{c/2}{(z-z')^4} + \frac{2T(z)}{(z-z')^2} + \frac{\partial T(z)}{z-z'} + \ldots\), fixes the correlators of the stress tensor \(T(z)\) to be

\[
\langle T(z_1) T(z_2) \rangle = \frac{c}{2} \frac{1}{z_1 z_2}, \quad \langle T(z_1) T(z_2) T(z_3) \rangle = \frac{c}{z_1 z_2 z_3 z_4},
\]

(1.4)

\[
\langle T(z_1) T(z_2) T(z_3) T(z_4) \rangle = \frac{c^2}{4} \left( \frac{1}{z_1 z_2 z_3 z_4} + \frac{1}{z_1 z_2 z_4 z_3} + \frac{1}{z_1 z_3 z_2 z_4} + \frac{1}{z_1 z_3 z_4 z_2} + \frac{1}{z_1 z_4 z_2 z_3} + \frac{1}{z_1 z_4 z_3 z_2} \right).
\]

(1.5)

The fact that the boundary operator dual to the Liouville field \(\varphi\) should have the dimension 2 is indeed consistent with the structure of (1.4),(1.5).\(^4\)

Our aim will be to check the conjecture (1.3) beyond the leading tree level approximation discussed in [19] by directly computing the one-loop corrections to the correlators \(\langle \Phi(t_1) \cdots \Phi(t_n) \rangle\) in (1.2) starting with the Liouville action (1.1) in AdS\(_2\). We shall also discuss how one may try to prove the relation (1.3) general and suggest the exact expression for \(\kappa(b)\).

One prediction of the identification (1.3) is that the boundary operator dual to \(\varphi\) should have no anomalous dimension, i.e. the two-point and three-point functions (1.2) should be given by

\[
\langle \Phi(t_1) \Phi(t_2) \rangle = \frac{C_2(b)}{t_1 t_2}, \quad \langle \Phi(t_1) \Phi(t_2) \Phi(t_3) \rangle = \frac{C_3(b)}{t_1^2 t_2 t_3},
\]

(1.6)

i.e. any logarithmic corrections should cancel out. The consistency with (1.3),(1.4) implies

\[
C_2 = \frac{1}{2} c \kappa^2, \quad C_3 = c \kappa^3, \quad \frac{(2C_2)^3}{(C_3)^2} = c, \quad \frac{C_3}{2C_2} = \kappa.
\]

(1.7)

\(^1\)This approach is different in spirit from the earlier investigations in [2] and [5, 18].

\(^2\)This is the special case of the familiar AdS\(_{d+1}\) relation \(m^2 = \Delta(\Delta - d)\).

\(^3\)Here the l.h.s. may be viewed as a symbolic notation for the corresponding boundary CFT correlator.

\(^4\)This has indeed dimension 2 with respect to the \(\text{SL}(2, \mathbb{R})\) subgroup of the 2d conformal group. Note also that \(\langle T \rangle = 0\) by the conformal symmetry in the bulk so one needs to consider only the connected correlators or subtract the expectation value \(\varphi \to \varphi - \langle \varphi \rangle\) in (1.2) to make the identification (1.3) possible.
Since $\kappa(b)$ is the only a priori unknown function (the central charge is assumed to be given by the Liouville theory value $c = 1 + 6(b + b^{-1})^2 = 6b^{-2} + \ldots$) we thus get non-trivial relations between the coefficients in the perturbative expansion of the functions $C_2$ and $C_3$

$$C_2(b) = C_{2,0} + C_{2,1} b^2 + C_{2,2} b^4 + \cdots, \quad C_3(b) = C_{3,0} b + C_{3,1} b^3 + C_{3,2} b^5 + \cdots,$$

in particular,

(i) $\frac{4 (C_{2,0})^3}{(C_{3,0})^2} = 3$, \hspace{1cm} (ii) $\frac{8 (C_{2,0})^2}{(C_{3,0})^3} (3 C_{2,1} C_{3,0} - 2 C_{2,0} C_{3,1}) = 13.$

The tree-level relation (i) in (1.9) was checked in [19] and we will show that the one-loop relation (ii) is also satisfied.

It was also observed in [19] that the 4-point boundary correlator given by the sum of the disconnected and connected tree-level diagrams following from the Liouville action (1.1), i.e. symbolically

$$\langle \Phi \Phi \Phi \rangle = \left[ \begin{array}{c} \text{tree} \\ \text{crossed} \end{array} \right] + b^2 \left[ \begin{array}{c} \text{loop} \\ \text{crossed} \end{array} \right] + O(b^3)$$

has a structure which is consistent with (1.3),(1.5). In general, one should expect to find

$$\langle \Phi(t_1) \cdots \Phi(t_4) \rangle = \langle \Phi(t_1) \cdots \Phi(t_4) \rangle_{\text{disc}} + \langle \Phi(t_1) \cdots \Phi(t_4) \rangle_{\text{conn}},$$

$$\langle \Phi(t_1) \cdots \Phi(t_4) \rangle_{\text{disc}} = [C_2(b)]^2 \left( \frac{1}{t_{12}^2 t_{23}^2} + \frac{1}{t_{13}^2 t_{24}^2} + \frac{1}{t_{14}^2 t_{23}^2} \right),$$

$$\langle \Phi(t_1) \cdots \Phi(t_4) \rangle_{\text{conn}} = C_4(b) \left( \frac{1}{t_{12}^2 t_{23}^2 t_{24}^2 t_{14}^2} + \frac{1}{t_{13}^2 t_{24}^2 t_{14}^2 t_{23}^2} + \frac{1}{t_{12}^2 t_{24}^2 t_{34}^2 t_{13}^2} \right),$$

$$C_4 = c \kappa^4 = C_{4,0} b^2 + C_{4,1} b^4 + C_{4,2} b^6 + \cdots.$$  

Using (1.7),(1.9) to eliminate $\kappa$ we should then have the following relations for the coefficients in the perturbative expansion of $C_4$ in (1.14)

(iii) $C_{4,0} = \frac{(C_{3,0})^2}{2 C_{2,0}}$, \hspace{1cm} (iv) $C_{4,1} = \frac{C_{3,0}}{2 (C_{2,0})^2} \left( 2 C_{2,0} C_{3,1} - C_{2,1} C_{3,0} \right).$

The tree-level relation (iii) was checked in [19] while the non-trivial one-loop relation (iv) will be confirmed below.

Our one-loop results will thus provide a strong evidence for the validity of the conjecture (1.3). One may wonder how it can be proved in general. The conformal symmetry of the Liouville theory in the bulk (on half-plane) viewed as an infinite global symmetry extending the $SL(2, \mathbb{R})$ isometry of the AdS$_2$ implies strong constraints on the boundary correlators which are effectively equivalent to the Virasoro symmetry constraints on the form of the stress tensor correlators (1.4),(1.5). This suggests that here we get an effective CFT$_1$ which is simply a restriction of a CFT$_2$ defined on a half-plane to its real-line boundary so that in this case the AdS$_2$/CFT$_1$ becomes effectively "AdS$_2$/(CFT$_2$)$^{1/2}$ duality".  

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5This may be compared to the discussion of the conjectured duality between a gravitational theory in AdS$_2$ and a chiral half of a 2d CFT in [20]. While here we assume that the AdS$_2$ background is fixed and the Virasoro symmetry is not gauged, in the gravitational context the 2d diffeomorphisms is a gauge symmetry and the Virasoro symmetry (with $c = 0$ due to ghosts, etc.) was assumed to be an asymptotic symmetry (also, the Liouville field in [20] was a gravitational mode). Whether this symmetry is spontaneously broken to $SL(2, \mathbb{R})$ appears to depend on a particular model in question (cf. [21–23]).
In the semiclassical limit of small \( b \) the relation (1.3) may then be argued for as follows. Since the theory (1.1) is Weyl-covariant theory in the bulk, in this limit we may eliminate the conformal factor of the AdS\(_3\) metric by redefining the Liouville field as \( \phi(t, z) \rightarrow \phi(t, z) + b^{-1} \log z \), i.e. transform the action in (1.1) into the flat-space one (cf. [2, 5]). This gives the Liouville CFT for field \( \phi \) defined on a flat upper half-plane (\( w = t + i z, \ z > 0 \)) with the (classical) stress tensor \( T(w) = -(\partial_w \phi)^2 + b^{-1} \partial_w^2 \phi \) (where \( \partial_w = \frac{1}{2} (\partial_t - i \partial_z) \)). The shifted field has then the boundary asymptotics \( \phi(t, z)|_{z \to 0} = z^2 \Phi(t) - b^{-1} \log z + \ldots \) Taking the boundary limit in \( T \ (w \to t) \) gives

\[
T(t) = -\frac{3}{2} b^{-1} \Phi(t) + O(z^2).
\]  

This is precisely the operator relation which is required for the validity of (1.3) with \( \kappa(b) = -\frac{2}{3} b + \ldots \). The same value of the leading term in \( \kappa(b) \) in (1.3),(1.7) will be found by an explicit computation below (see (3.22)).

This argument should admit an extension to the full quantum level, i.e. to all orders in \( b \), allowing to determine an exact expression for \( \kappa(b) \). One important point is the role of the boundary limit: the relation (1.3) between the elementary Liouville field correlators and the stress tensor correlators may hold only at the boundary. A general proof should use the boundary conformal field theory considerations. In particular, the completeness of the OPE in the boundary CFT (cf. Appendix D) implies that it contains only the boundary stress tensor and its descendants; since the boundary operator dual to the \( m^2 = 2 \) Liouville field in AdS\(_3\) should have dimension 2 it can only be (proportional to) the stress tensor itself. Considering the OPE of the stress tensor in the bulk and the Liouville field at the boundary should then fix the value of the proportionality constant \( \kappa(b) \) or \( C_2(b) \) in the two-point function in (1.6).\(^6\)

A possible generalization of the above semiclassical argument is as follows. Starting with the Liouville theory on a complex plane, the (normal ordered) operator \( V_a = e^{2a \phi} \) is a primary with the conformal dimension \( \Delta = a (Q - a) \), i.e. we have

\[
T(w) V_a(w', \bar{w}') = \frac{a(Q - a)}{(w - w')^2} V_a(w', \bar{w}') + \frac{1}{w - w'} \partial_{w'} V_a(w', \bar{w}') + \cdots.
\]  

Expanding this formally in small \( a \) gives \( T(w) \phi(w', \bar{w}') \sim \frac{Q}{2 (w - w')^2} + \frac{\partial_{w'} \phi(w', \bar{w}')} {w - \bar{w}} \). On the upper half-plane (\( w = t + i z, \ z > 0 \)) the boundary conformal Ward identity takes into account also the poles at \( w = \bar{w}' \). Including them in the OPE of \( T \) and \( \phi \) and assuming the exact form of the boundary asymptotics to be\(^7\)

\[
\phi(t, z)|_{z \to 0} = z^2 \Phi(t) - Q \log z + \ldots,
\]  

one finds that the leading \( \frac{1}{(t - t')^2} \) term in \( T(t) \phi(t, 0) \) cancels and we get\(^8\)

\[
T(t) \Phi(t') = -\frac{2Q}{(t - t')^4} + \cdots.
\]  

Taking the expectation value of (1.19) and comparing the result with (1.4),(1.6),(1.7) we get the following prediction for the exact value of the coefficient \( \kappa(b) \) between \( \Phi \) and \( T \):

\[
\Phi = \kappa T, \quad \kappa(b) = -\frac{4Q}{c},
\]  

\(^6\)We are grateful to Simone Giombi and Xi Yin for useful suggestions on this problem.

\(^7\)This generalizes the classical asymptotics where \( Q \to b^{-1} \) in a way which is consistent with the value of the one-loop tadpole (see [5] and (2.16)).

\(^8\)In more details, Eq. (1.19) is the coefficient of \( z^2 \) term in the \( z \to 0 \) expansion of \( \frac{Q}{2 (t - t')^2} + \frac{1}{2} Q \frac{1}{t - \bar{t} - t' - \bar{t}'} + c.c \) where the second term comes from the \(-Q \log z \) term in (1.18) and the complex conjugate contribution comes from poles at \( w = \bar{w}' \).
\[ \kappa(b) = -\frac{4Q}{1 + 6Q^2} = -\frac{4b(1 + b^2)}{(3 + 2b^2)(2 + 3b^2)} = -\frac{2}{3} b + \frac{7}{9} b^3 - \frac{55}{54} b^5 + \cdots . \quad (1.21) \]

The first two terms in this expansion match precisely with the above semiclassical value and the one-loop value that we shall find below in (3.24). Given (1.20) we get from (1.7)

\[ C_2 = \frac{8Q^2}{c} = \frac{4}{3} - \frac{2}{9} b^2 + \ldots , \quad C_3 = -\frac{64Q^3}{c^2} = -\frac{16}{9} b + \frac{64}{27} b^3 + \ldots , \quad (1.22) \]

which are in agreement with (3.25).

A natural generalization of the above duality (1.3) for the Liouville theory is to all conformal Toda theories for finite Lie algebras \cite{24–26} defined on the AdS\(_2\) background. For instance, in the \(A_n\) case, expanding near the minimum of the Toda potential in AdS\(_2\) one finds \(n\) scalar fields \(\varphi_\Delta\) with masses \(m^2 = \Delta(\Delta - 1)\) corresponding to \(\Delta = 2, \ldots, n + 1\) \cite{19}. The duality relation extending (1.3) is then

\[ \langle \Phi_\Delta_1(t_1) \cdots \Phi_\Delta_n(t_n) \rangle = \left( \prod_{i=1}^n \kappa_\Delta_i \right) \langle Q_\Delta_1(z_1) \cdots Q_\Delta_n(z_n) \rangle \bigg|_{z_i \to t_i}, \quad (1.23) \]

where \(Q_\Delta = \{Q_2 \equiv T, Q_3, \ldots, Q_{n+1}\}\) are the generators of the chiral \(W_{n+1}\) algebra generalizing the Virasoro symmetry and having the same central charge as the Toda theory in the bulk. The relation (1.23) was noticed at the tree level in \cite{19} by computing few sample 4-point functions in the Toda theories associated to some rank-2 algebras with two scalar fields (one dual to the stress tensor \(T\) and the other dual to a higher spin current \(Q_s\)). It is natural to expect (1.23) to hold also at the quantum level as should be possible to check by the methods used in the present paper. Another interesting generalization is to the super-Liouville theory on AdS\(_2\), cf. \cite{27}. In this case one will need to compute loop corrections for both the bosonic and fermionic fields.

The structure of the rest of this paper is as follows. In section 2.1 we will discuss two alternative formulations of the Liouville theory on AdS\(_2\). One starts with the flat space action expanded around the classical solution corresponding effectively to the AdS\(_2\) background while the other starts directly with the Liouville theory defined on fixed AdS\(_2\). The two formulations differ in the choice of the regularization of the short distance propagator and also in the coefficient in front of the potential. They turn out to give the equivalent results for the physically relevant connected one-loop correlators.

In section 2.2 we will compute the two-point function of the Liouville field at the one-loop order demonstrating the consistency of its boundary limit with (1.6). The one-loop corrections to the boundary three-point and four-point functions will be computed in sections 3 and 4 respectively finding again the agreement with (1.6) and (1.13). As a result, we will provide a strong support for the validity of the duality (1.3) at the loop level. Appendix A will present some details of the diagram computations on the Poincaré disc. Comments on tadpole diagrams (that lead to different predictions for the 1-point functions in the two formulations) will be made in Appendix B. In Appendix D we will review the structure of the 1d conformal block expansion of the four-point function in (1.5).

2 Liouville theory on Euclidean AdS\(_2\) background

The correlators in the Liouville theory (1.1) in AdS\(_2\) background may be computed using two alternative approaches. In the first (the "ZZ formulation" \cite{5}) one starts with the Liouville action on a flat upper half plane (or flat disc) and expands it near a non-trivial non-constant solution \cite{1, 2} preserving the \(SL(2, \mathbb{R})\) symmetry. In the second (the "AdS formulation") one starts directly with the Liouville
admits a particular solution in either disc or plane parametrization. Taking one point to the boundary we get the corresponding bulk-to-boundary propagator where

\[ C_t (\text{disc (or "pseudosphere") coordinates where } z \in \mathbb{C}, |z|^2 = a^2 + b^2, |a|^2 - |b|^2 = 1) \text{.} \]

An alternative choice is the half-plane parametrization with coordinates \( t \in \mathbb{R}, z \geq 0 \) and the metric \( ds^2 = \frac{1}{z} (dt^2 + dz^2) \). The two are related by the standard conformal map

\[ w \equiv t + iz = \frac{1 + z}{1 - z}, \quad z = \frac{w - i}{w + i}, \quad ds^2 = -\frac{4dwd\bar{w}}{(w - \bar{w})^2} = \frac{4dzd\bar{z}}{(1 - zz')^2}. \]

The analog of the action (2.3) on the Poincaré plane reads

\[ S^{ZZ} = \frac{1}{2\pi} \int dt dz \left( \frac{1}{2} (\partial_t \chi)^2 + \frac{e^{2b \chi} - 2b \chi - 1}{2b^2 (1 - zz')} \right). \]

As follows from (2.3) or (2.5) the field \( \chi \) has the mass \( m^2 = \Delta(\Delta - 1) = 2 \). Assuming the standard Dirichlet boundary condition (corresponding to the \( \Delta = 2 \) choice) the canonically normalized AdS\(_2\) bulk-to-bulk propagator is given by

\[ G_{\Delta=2} = \frac{\xi_2}{16u^2} \, 2F_1 (2, 2, 4; -\frac{1}{u}) = -\frac{1}{4\pi} \left[ (1 + 2u) \log \frac{u}{u + 1} + 2 \right], \]

where \( \xi_2 = \frac{\Gamma(2)}{2\sqrt{\pi} \Gamma(2 + \frac{1}{2})} = \frac{2}{3\pi} \) and \( u \) is the invariant chordal distance. Equivalently (taking into account the factor \( \frac{1}{2\pi} \) in front of the action in (2.3),(2.5)) the free two-point function of the field \( \chi \) is given by

\[ g(z, z') = \langle (\chi(z) \chi(z'))_0 \rangle = 2\pi G_{\Delta=2} = -\frac{1}{2} \left( \frac{1 + \eta}{1 - \eta} \right) \log \eta + 2, \]

\[ \eta = \left| \frac{z - z'}{1 - zz'} \right|^2 = \left| \frac{w - w'}{w - \bar{w}} \right|^2 = \frac{u}{u + 1}, \quad u = \frac{(t - t')^2 + (z - z')^2}{4zz'}. \]

Taking one point to the boundary we get the corresponding bulk-to-boundary propagator \( g_b \) written in either disc or plane parametrization:

\[ z^{-2} g_b (z, z') = g_b(t, u) = \frac{4 \sin^2 (\frac{\theta}{2})}{|e^{-\theta} - z'|^4} = \frac{4}{3} \left( \frac{z'}{z'^2 + (t - t')^2} \right)^2. \]

This is similar to the so-called "geometrical" approach [28] discussed in detail in [18].

We shall use the notation \( z = x_1 + ix_2, \quad dz = dx_1 dx_2 \). As usual, here one also assumes that the stress tensor of the theory (2.1) is \( T = -(\partial \varphi)^2 + Q \partial \varphi^2 \) in order to make the interaction term exactly marginal.

The factor \( \frac{1}{4} \) is the same as \( 2\pi \xi_2 \), where \( \xi_2 \) appeared in (2.6).
where the relation between the Poincaré plane and disc boundary coordinates \( t \) and \( \theta \) is

\[
t(\theta) = i \frac{1 + e^{i \theta}}{1 - e^{i \theta}} = -\cot \frac{\theta}{2}.
\] (2.10)

The starting point of the AdS formulation is the action (1.1) in unit-radius AdS\(_2\) background \( \mathcal{R} = -2 \). Shifting the field by a constant as \( \varphi = \varphi^{(0)} + \chi \) where \( \mu e^{2b\varphi^{(0)}} = \frac{Q}{b} \) we get the following analog of (2.3)

\[
S^{\text{AdS}} = \frac{1}{2\pi} \int d^2z \left[ \frac{1}{2} (\partial_a \chi)^2 + \frac{2Q(e^{2b\chi} - 2b\chi - 1)}{b(1 - z \bar{z})^2} \right].
\] (2.11)

The two coincide in the semiclassical limit of small \( b \) when \( Q = b + b^{-1} \to b^{-1} \) but differ for finite \( b \).

The two formulations will differ also by the required choice of the UV regularization of the propagator (2.7) at the coinciding points but the two differences will happen to compensate each other leading to the equivalent results for the relevant physical correlators (but not for the tadpole values). The ZZ formulation that starts with the flat-space action requires the use of the simple flat-space regularization \( z - z' \to z - z' + \varepsilon \). Omitting the singular \( \log \varepsilon \) term this is equivalent to the subtraction of the \( -[\log(z - z')]_{z \to z'} \) term from (2.7) \[5\]. At the same time, the regularization in the AdS formulation should be the AdS\(_2\) covariant one with \( \eta \to \eta + \varepsilon \) or \( u \to u + \varepsilon \) in (2.7),(2.8). In general, the value of the propagator at the coinciding points may be parametrized as

\[
g(z, z) = q_1 \log(1 - z \bar{z}) - q_2, \quad q_1^{\text{ZZ}} = q_2^{\text{ZZ}} = 1, \quad g^{\text{ZZ}}(z, z) = \lim_{z' \to z} [g(z, z') + \log |z - z'|] = \log(1 - z \bar{z}) - 1, \quad q_1^{\text{AdS}} = q_2^{\text{AdS}} = 0, \quad g^{\text{AdS}}(z, z) = 0.
\] (2.12) (2.13) (2.14)

Here in (2.14) we assumed that the logarithmic UV divergences are cancelled by the mass renormalization with the "minimal" subtraction (more generally, one may keep the constant \( q_2^{\text{AdS}} \) arbitrary).

Starting with (2.3) or (2.11) one may compute the corresponding quantum corrections to bulk correlators in perturbation theory in \( b \). The two actions differ in the coefficient in front of the potential term

\[
V^{\text{AdS}} = \frac{2Q}{b} (e^{2b\chi} - 2b\chi - 1) = (1 + b^2) V^{\text{ZZ}},
\] (2.15)

implying that in the AdS formulation one has extra higher order terms in the \( \chi^2, \chi^3, \ldots \) vertices. In addition, there is a difference in the regularization choices in (2.13),(2.14). While the values of the one-point functions \( \langle \chi^n(z) \rangle \) computed in the ZZ and AdS formulations will differ, the connected correlators at separated points are expected to be same; we shall find evidence for that below.

### 2.2 One-point and two-point correlation functions

Let us consider the leading order corrections to the tadpole or the one-point function \( \langle \chi(z) \rangle \). Computing it for generic choice of the regularization (2.12) one finds (see (A.7),

\[
\langle \chi(z) \rangle = z \bullet \bigcirc + O(b^3) = b \left[ \frac{1}{2} q_1 + q_2 - q_1 \log(1 - z \bar{z}) \right] + O(b^3).
\] (2.16)

The leading-order tadpole thus vanishes in the AdS formulation and is constant in general, as required by the AdS symmetry (under which \( \chi \) transforms as a scalar). Explicitly, one finds

\[
\langle \chi(z) \rangle^{\text{ZZ}} = b \left[ \frac{3}{2} - \log(1 - z \bar{z}) \right] + b^3 \left( \frac{\pi^2}{6} - \frac{13}{12} \right) + O(b^5), \quad \langle \chi(z) \rangle^{\text{AdS}} = O(b^3).
\] (2.17)
Note that combining the one-loop tadpole with the classical solution (2.2) the $z$-dependent part of the tadpole in the $\text{ZZ}$ case gets coefficient $Q = b + b^{-1}$. The $b^3$ term in $(\chi(z))^{\text{ZZ}}$ is in agreement with the results of [5] and [29] (see also Appendix B).\footnote{The tadpole (2.17) is the simplest cumulant that can be extracted from the expectation value of the exponential vertex operator $\langle e^{\alpha \chi(z)} \rangle$ determined in [5]. For a perturbative discussion of the $O(b^3)$ corrections to higher cumulants and a comparison with the bootstrap predictions see [29].}

Next, let us consider the one-loop order $b^2$ correction to the connected part of the two-point function $\langle \chi(z)\chi(z') \rangle$. In the AdS formulation one needs to take into account an extra contribution coming from the $b^2\chi^2$ term in the potential in (2.15). The final result in both formulations can be written as

$$
\langle \chi(z)\chi(z') \rangle_{\text{conn}} = g(z,z') + \Sigma(z,z') + q'_1\left[1 - \eta \log \eta - \log(1-\eta) \left(1 + \frac{(1+\eta) \log \eta}{2(1-\eta)} \right) - \frac{1+\eta}{1-\eta} \text{Li}_2(1-\eta) \right] + O(b^4),
$$

where

$$
\Sigma(z,z') = b^2 \left[3 + \frac{\eta^2 \log^2 \eta}{2(1-\eta)^2} - \frac{1+\eta}{1-\eta} \text{Li}_2(1-\eta) \right],
$$

and the coefficient $q'_1 = q_1 - 1$ in the ZZ case and $q'_1 = q_1$ in the AdS case. As follows from (2.13),(2.14) in both cases

$$
q'_1 = 0,
$$

and so we get the equivalent result for (2.18) given by the sum of the free propagator in (2.7) and the "self-energy" correction in (2.20).

Let us consider the Poincaré plane parametrization and introduce the subtracted and rescaled field

$$
\Phi(t,z) = z^{-2} \left[\chi(t,z) - \langle \chi(t,z) \rangle \right].
$$

Using (2.7),(2.8),(2.20) and taking the limit $z_1, z_2 \to 0$ we get for the two-point boundary correlator in (1.2) (cf. (2.9))

$$
\langle \Phi(t)\Phi(t') \rangle \equiv \lim_{z_1, z_2 \to 0} \langle \chi(t,z)\chi(t',z') \rangle_{\text{conn}} = \frac{C_2(b)}{(t-t')^4}, \quad C_2 = \frac{4}{3} \left(1 - \frac{1}{6} b^2 \right) + O(b^4). \tag{2.23}
$$

Thus there are no $b^2 \log(t-t')$ corrections, i.e. the boundary operator dual to $\chi$ has no anomalous dimension confirming the expected behaviour (1.6). We conclude that the coefficients in the perturbative expansion of $C_2$ introduced in (1.8) are thus

$$
C_{2,0} = \frac{4}{3}, \quad C_{2,1} = -\frac{2}{9}. \tag{2.24}
$$

Let us note that it is only the boundary limit of the 2-point function (2.18) that takes the simple rational form in (2.23) matching the stress tensor two-point function (1.4) according to (1.3). The same will apply to higher-point correlation functions.
3 Three-point boundary correlation function

Next, let us compute the tree-level and one-loop contributions to the connected boundary three-point function. We shall use mainly the ZZ formulation, but will comment on its equivalence with the AdS one. The relevant terms in the expansion of the potential term in (2.3) will be

\[ \int d^2z \, e^{2b \chi - \frac{2b \chi}{b^2}} = \int d^2z \left[ 2\chi^2 + 8b \frac{\chi^3}{b^3} + 16b^2 \frac{\chi^4}{b^4} + 32b^3 \frac{\chi^5}{b^5} + \cdots \right], \quad d^2z \equiv \frac{d^2z}{\pi (1 - |z|^2)^2}. \]  

(3.1)

We will need the tree boundary-to-bulk propagator in the disc parametrization given in (2.9) and also will use the following notation for the conformal prefactor in the three-point function (1.6) (cf. (2.10))

\[ \mathcal{K}_3(\theta) \equiv \mathcal{K}(\theta_1, \theta_2, \theta_3) = \prod_{i<j} |t(\theta_i) - t(\theta_j)|^{-2}. \]  

(3.2)

We shall divide the relevant Witten diagrams in AdS$_2$ by the overall conformal factor $\mathcal{K}_3$, i.e. the resulting expressions will represent the contributions to the coefficients in $C_3(b)$ in (1.6).

The tree level contribution to the three-point function is given by the diagram

\[ C_{3,0} b = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \]  

\[ = \frac{(-8b)}{\mathcal{K}_3(\theta)} \int d^2z \prod_{i=1}^{3} g_{h_i}(\theta_i, z) = (-8b) \left( \frac{4}{3} \right)^3 \left( \frac{1}{4\pi} \right) \times \frac{3\pi}{8} = -\frac{16}{9} b. \]  

(3.3)

The same result in the Poincaré plane coordinates follows from (see, e.g., [30])

\[ \int \frac{dt \, dz}{z^2} \prod_{i=1}^{3} \left[ \frac{z}{z^2 + (t - t_i)^2} \right]^2 = \frac{3\pi}{8} \frac{1}{|t_{12}|^2 |t_{13}|^2 |t_{23}|^2}. \]  

(3.4)

Thus the value of the leading coefficient in $C_3$ in (1.6),(1.8) is

\[ C_{3,0} = -\frac{16}{9}, \]  

(3.5)

which together with $C_{2,0}$ in (2.24) is in agreement with the first relation in (1.9).

The value (3.5) combined with (2.24) and the second relation in (1.9) gives the following prediction for the one-loop coefficient $C_{3,1}$ in $C_3(b)$ in (1.8)

\[ C_{3,1} = \frac{64}{27}. \]  

(3.6)

This coefficient should be given by the sum of the contributions of the four different types of diagrams

\[ C_{3,1} = \sum_{i=1}^{4} C_{3,1}^{(i)}, \]  

(3.7)

that we shall consider below.

\[ ^{13}\text{See (2.9) and note that } \frac{dt \, dz}{z^2} = \frac{4 \pi d^2z}{(1 - |z|^2)^2}. \]
Diagrams with dressed propagators: There are three tree-like diagrams with the one-loop self-energy correction to one of the propagators

\[ C^{(1)}_{3,1} b^3 = \quad \text{diagram 1} + \quad \text{diagram 2} + \quad \text{diagram 3}. \]  

(3.8)

The one-loop corrected propagator is given by (2.18),(2.20) (and is the same in both ZZ and AdS formulations). As a result, like in (2.23), the contribution of each of the diagrams in (3.8) is given by the tree-level expression (3.3) times the \( 1 - \frac{1}{b^2} \) factor, i.e.

\[ C^{(1)}_{3,1} = C_{3,0} \times 3 \times \left( -\frac{1}{b} \right) = \frac{8}{9}. \]  

(3.9)

Diagrams with vertex tadpoles: In the ZZ formulation the contribution of diagrams with tadpoles attached to an internal 3-vertex

\[ \quad = \quad \text{diagram 4} + \quad \text{diagram 5} \]  

(3.10)

turns out to be equivalent to that of the tree diagram (3.3) with the cubic vertex \( 8b^3 \frac{\chi^3}{3!} \) (the same as in (3.1) with \( b \rightarrow b^3 \))^14

\[ C^{(2)}_{3,1} b^3 = C_{3,0} = -\frac{16}{9}. \]  

(3.11)

In the AdS formulation we get the same result (by a mechanism similar to the one in the case of the two-point function (2.18)–(2.21)). Here the tadpole contributions vanish \( (g(z,z) = 0 \text{ in (2.14)}) \) but the cubic vertex \( \sim b \chi^3 \) in (3.1) is rescaled by \( bQ = 1 + b^2 \) (see (2.15)), i.e. \( -8b \chi^3 \rightarrow -8b \chi^3 - 8b^3 \chi^3 \) resulting in an additional one-loop correction. Thus

\[ C^{(2)}_{3,1} = C_{3,0} = -\frac{16}{9}. \]  

(3.12)

Diagrams with cubic and quartic vertices: There are also one-loop diagrams with one cubic and one quartic vertices from (3.1)

\[ C^{(3)}_{3,1} b^3 = \quad \text{diagram 6} + \text{two permutations}. \]  

(3.13)

Explicitly, we get (cf. (3.3))

\[ \quad \frac{1}{2} \left( -8b \right) \left( -16b^2 \right) \frac{X_3(\theta)}{X_3(\theta)} \int d^2 z' d^2 z'' g_0(\theta_1, z') g_0(\theta_2, z'') g_0(\theta_3, z'') [g(z', z'')]^2, \]  

(3.14)

---

^14In the ZZ formulation \( g(z,z) \) is given by (2.13) and taking into account the combinatorics (see (A.9),(A.10)) the contribution is found to be the same as of the tree diagram but with the coupling factor \( -8b \) in (3.3) replaced by \( -8b^3 \). The same expression appears in Eq. (15) of [29].
where $\frac{1}{2}$ is the leg symmetry factor. Part of this diagram (the loop with upper leg and integration over the cubic vertex point) is the same as the function $B(z_2, z_1) = \int d^2 z' \, g(z', z_1) [g(z_2, z')]^2$ in (A.11),(A.14). Taking the point $z_1$ to the boundary then gives, according to (A.14), the boundary-to-bulk propagator (2.9) times the $\frac{1}{8}$ factor

$$\lim_{|z_1| \to 1} \frac{1}{z_1} \quad z_1 \rightarrow z_2 \Rightarrow \frac{1}{8} \theta_1 \rightarrow z_2$$

(3.15)

This should be multiplied by the coupling constants from the two vertices $(-8b)(-16b^2)$, by a symmetry factor $\frac{1}{2}$ and by the diagram multiplicity factor 3. Compared to the tree diagram contribution this gives an extra $-3b^2$ factor in total, i.e.

$$C_{3,1}^{(4)} = -3 C_{3,0} = \frac{16}{3}.$$  

(3.16)

Diagram with one-loop vertex correction: The most complicated diagram is the one with the three $\chi^3$ vertices and thus with the three bulk-to-bulk and the three bulk-to-boundary propagators:

$$C_{3,1}^{(4)} b^3 = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram.png}
\end{array} = \frac{(-8b)^3}{K^3(\theta)} \int d^2 z_1 d^2 z_2 d^2 z_3 \, g(z_1, z_2) \, g(z_1, z_3) \, g(z_2, z_3) \prod_{i=1}^{3} g_b(\theta_i, z_i) .$$

(3.17)

As this is the triple AdS$_2$ integral its analytic computation is non-trivial. According to (3.6),(3.7) and the results in (3.9),(3.12),(3.16) the expected value of $C_{3,1}^{(4)}$ should be

$$C_{3,1}^{(4)} = \frac{64}{27} - \frac{8}{9} + \frac{16}{9} - \frac{16}{3} = -\frac{56}{27} .$$

(3.18)

Since (3.17) should give just the value of the constant $C_{3,1}^{(4)}$ it is sufficient to evaluate it numerically. We did this by using the Suave routine of the Cuba library [31]. One important point is to check that $C_{3,1}^{(4)}$ given by (3.17) is indeed independent of the boundary coordinates $\theta_1, \theta_2, \theta_3$. From a practical perspective, one is interested in the $\theta_1$ dependence of the systematic error in the numerical estimate at fixed computational effort (the total number of integrand evaluations). In Fig. 1 we fixed the two

| $C_{3,1}^{(4)}$ | $\theta_3$ |
|----------------|---------|
| -1.0 | -0.5 | 0.5 | 1.0 |
| -1.95 |
| -2.00 |
| -2.05 |
| -2.10 | -2.15 | -2.20 |

Figure 1. Numerical estimate of $C_{3,1}^{(4)}$ extracted by $10^7$ integrand evaluations for $\theta_1 = \frac{2\pi}{3}$, $\theta_2 = -\frac{2\pi}{3}$ and variable $\theta_3$ with $|\theta_3| < \frac{\pi}{3}$. The horizontal blue line is the expected value $-\frac{56}{27}$. 

- 11 -
points at $\theta_1 = \frac{2\pi}{3}, \theta_2 = -\frac{2\pi}{3}$ and varied $\theta_3$ showing the resulting numerical estimate for $C_{3,1}^{(4)}$. The horizontal line is the expected value $-\frac{56}{27}$ in (3.18). One finds that the integral value is more "flat” when the three external points $\theta_i$ are well separated. This is expected as the integrand has a more singular behaviour when at least two external points get close. Considering the equally spaced points and increasing statistics we found the best estimate to be\footnote{The number in the parenthesis is the statistical uncertainty (the least significant digits of a given numerical value for the average). Here the uncertainty is the standard deviation of the Monte-Carlo evaluation of the loop integral. Explicitly, 2.08(2) \equiv 2.08 \pm 0.02.}

\[
C_{3,1}^{(4)} = -2.08(2) .
\]

This is in good agreement with (3.18) or $-\frac{56}{27} \approx -2.0741$.

The expected value (3.6) and thus (3.18) was found from (1.9) which was based on matching the two-point and three-point correlators (1.2) with the stress tensor ones (1.4) for the Liouville central charge value $c = 1 + 6(b + b^{-1})^2$. Alternatively, we may determine the values of $c$ and $\kappa$ in (1.7) from the explicit tree plus one-loop results for $C_2$ and $C_3$ found above. Omitting the overall conformally covariant factors we may symbolically summarize the results for the two-point and three-point correlators in (2.23) and (3.5),(3.9),(3.10),(3.16),(3.19) as follows

\[
\langle \Phi \Phi \rangle = \frac{4}{3} (1 - \frac{1}{6} b^2) + O(b^4), \quad \langle \Phi \Phi \Phi \rangle = -\frac{16}{9} b + \left( \frac{8}{9} - \frac{16}{3} + C_{3,1}^{(4)} \right) b^3 + O(b^5) .
\]

The consistency of (3.20) with the identification in (1.3),(1.4) or (1.7)

\[
\langle \Phi \Phi \rangle = \frac{1}{2} c \kappa^2 , \quad \langle \Phi \Phi \Phi \rangle = c \kappa^3
\]

implies that

\[
k = -\frac{2}{3} b + \left( \frac{14}{9} + \frac{3}{8} C_{3,1}^{(4)} \right) b^3 + O(b^5), \quad c = 6 b^{-2} + 27 (1 + \frac{1}{4} C_{3,1}^{(4)}) + O(b^2).
\]

Using the numerical estimate for $C_{3,1}^{(4)}$ found in (3.19) gives

\[
c = 6 b^{-2} + 13.0(1) + O(b^2),
\]

which is in good agreement with the value of the Liouville central charge $c = 6 b^{-2} + 13 + 6 b^2$. Note also that for $C_{3,1}^{(4)} = -\frac{56}{27}$ corresponding to the exact value of $c$ we get

\[
k = -\frac{2}{3} b \left( 1 - \frac{7}{6} b^2 \right) + O(b^5) ,
\]

and thus the following expressions for $C_2$ (2.23) and $C_3$ in (1.6),(1.8)

\[
C_2 = \frac{4}{3} (1 - \frac{1}{6} b^2) + O(b^4) , \quad C_3 = -\frac{16}{9} b \left( 1 - \frac{4}{3} b^2 \right) + O(b^5) .
\]

4 Four-point boundary correlation function

The four-point function (1.11) receives contributions from the disconnected and connected diagrams in AdS$_2$. The disconnected part is given by $\langle \Phi(t_1) \Phi(t_2) \Phi(t_3) \Phi(t_4) \rangle$ ( + permutations) corresponding to (1.12). From (3.21),(4.1) the prediction for the non-trivial connected part is

\[
\langle \Phi(t_1) \Phi(t_2) \Phi(t_3) \Phi(t_4) \rangle_{\text{conn}} = \mathcal{C}_4(b) \left( \frac{1}{t_{12}^2 t_{23}^2 t_{34}^2 t_{41}^2} + \frac{1}{t_{13}^2 t_{24}^2 t_{14}^2 t_{23}^2} + \frac{1}{t_{12}^2 t_{24}^2 t_{34}^2 t_{13}^2} \right),
\]
\[ C_4(b) = c \kappa^4 = \frac{32}{27} b^2 - \frac{80}{27} b^4 + \mathcal{O}(b^6), \]  

implying that in (1.14) one should have

\[ C_{4,0} = \frac{32}{27}, \quad C_{4,1} = -\frac{80}{27}. \]  

These values are in agreement with the relations in (1.15) for the coefficients \( C_{2,0} = \frac{4}{3}, \ C_{2,1} = -\frac{2}{9}, \ C_{3,0} = -\frac{16}{9}, \ C_{3,1} = \frac{54}{27} \) found above.

The value of the leading coefficient \( C_{4,0} \) follows from the sum of the tree-level contact diagram and the three exchange diagrams as in (1.10) (see Appendix C)

\[ C_{4,0} b^2 = (C_{4,0}^{\text{cont}} + C_{4,0}^{\text{exch}}) b^2 = \begin{array}{c}
\text{contact} \\
\text{exchange}
\end{array} + \text{crossed} = \frac{32}{27} b^2. \]  

The one-loop coefficient \( C_{4,1} \) is determined from the sum of the contributions of the five classes of diagrams

\[ C_{4,1} = C_{4,1}^{(0)} + \sum_{i=1}^{4} C_{4,1}^{(i)}. \]  

To compute them we shall use the AdS formulation in which there are no tadpoles but the potential has an extra factor \((1 + b^2)\) in (2.15) rescaling the vertices. \( C_{4,1}^{(0)} \) will denote the resulting contribution originating, due to this rescaling, from the contact and exchange tree diagrams in (4.4). Since the contact 4-vertex is rescaled by \(1 + b^2\) while the exchange diagrams with two 3-vertices by \((1 + b^2)^2\) the two are combined now with the relative factor of 2 (cf. (4.4))

\[ C_{4,1}^{(0)} = C_{4,1}^{\text{cont}} + 2 C_{4,1}^{\text{exch}}. \]  

The contributions \( C_{4,1}^{(i)} \) with \( i = 1, 2, 3, 4 \) in (4.5) come from the genuine one-loop diagrams described below.

In writing (4.5) and (4.6) we formally assumed that \( t_i \)-dependence of each individual diagram contribution is given by the same simple conformal factor as in (4.1) (cf. (3.2), (2.10))

\[ \mathcal{K}_4(\theta) = \mathcal{K}_4(\theta_1, \ldots, \theta_4) = \frac{1}{t_{12} t_{23} t_{34} t_{14}} + \frac{1}{t_{13} t_{24} t_{31} t_{12}} + \frac{1}{t_{14} t_{23} t_{32} t_{13}} + \frac{1}{t_{12} t_{24} t_{31} t_{34}}, \quad t_{ij} = t(\theta_i) - t(\theta_j). \]  

However, this need not be true in general. Let us first recall that the tree level four-point boundary correlator (1.10) has the canonical form in (1.12), (1.13) without any logarithmic terms depending on the 1d invariant cross-ratio \( \chi = \frac{t_{12} t_{34}}{t_{13} t_{24}} \). Such terms cancel between the contact and exchange diagram contributions [19] (see Appendix C) but they survive if the contact and exchange contributions are combined with a different relative coefficient like in (4.6) (see (C.2)). These extra terms should still not appear in the total expression (4.1) for the one-loop corrected correlator, i.e. they should cancel against similar "extra" contributions of other diagrams. Checking this directly would require the analytic computation of all the diagrams discussed below which we will not attempt here.

Here we will compute the total value of the coefficient \( C_{4,1} \) numerically by fixing particular values of the boundary coordinates \( t_i \). We will also give the values of the individual diagram contributions \( C_{4,1}^{(k)} \) defined formally as the corresponding integrals divided by the factor \( \mathcal{K}_4 \) in (4.7).
Diagrams with dressed external propagators: These diagrams are analogous to the ones in (3.8) (their contribution is actually proportional to $K_4(\theta)$). As in (3.8),(3.9) the one-loop dressing of the bulk-to-boundary propagators gives the contribution

$$C_{4,1}^{(1)} = C_{4,0} \times 4 \times \left(-\frac{1}{6}\right) = -\frac{64}{81}. \tag{4.8}$$

Diagrams with dressed internal propagator:

$$C_{4,1}^{(2)} b^4 = \begin{array}{c}
\includegraphics{diagram1}
\end{array} + \text{crossed}. \tag{4.9}$$

Here the grey circle stands for the one-loop self energy correction $\Sigma(z, z')$ in (2.20) or explicitly

$$\begin{array}{c}
\includegraphics{diagram2}
\end{array} = \frac{(-8b)^2}{K_4(\theta)} \int d^2 z' d^2 z'' \Sigma(z', z'') g_b(\theta_1, z') g_b(\theta_4, z') g_b(\theta_2, z'') g_b(\theta_3, z''). \tag{4.10}
\end{array}$$

Diagrams with one-loop 1PI cubic vertex correction:

$$C_{4,1}^{(3)} b^4 = \begin{array}{c}
\includegraphics{diagram3}
\end{array} + \begin{array}{c}
\includegraphics{diagram4}
\end{array} + \text{crossed}. \tag{4.11}$$

Here the grey circle stands for the one-loop cubic vertex correction. In the AdS formulation where the tadpoles are absent it is given by the 1PI diagrams

$$\begin{array}{c}
\includegraphics{diagram5}
\end{array} = \begin{array}{c}
\includegraphics{diagram6}
\end{array} + \left[ \begin{array}{c}
\includegraphics{diagram7}
\end{array} + \text{crossed} \right]. \tag{4.12}
\end{array}$$

The expressions for the resulting integrals are similar to the one-loop three-point contributions in (3.14) and (3.17) connected to the remaining tree parts.

Diagrams with 1PI quartic vertex corrections: The final class of diagrams represent irreducible quartic vertex corrections (here we do not indicate explicitly additional diagrams obtained by exchanging the boundary points)

$$C_{4,1}^{(4)} b^4 = \begin{array}{c}
\includegraphics{diagram8}
\end{array} + \begin{array}{c}
\includegraphics{diagram9}
\end{array} + \begin{array}{c}
\includegraphics{diagram10}
\end{array} + \begin{array}{c}
\includegraphics{diagram11}
\end{array} \tag{4.13}
\end{array}$$

The integrals corresponding to these diagrams are

$$\begin{array}{c}
\includegraphics{diagram12}
\end{array} = \frac{(-8b)^4}{K_4(\theta)} \int \prod_{i=1}^{4} d^2 z_i \prod_{i=1}^{4} g_b(\theta_i, z_i) g(z_1, z_2) g(z_2, z_3) g(z_3, z_4) g(z_1, z_4),
\end{array}$$
Below we collect some results about the A AdS$_2$ Theoretical Physics for hospitality and INFN for partial support during the completion of this work.

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\begin{equation}
C_{4,1}^{(0)} = C_{4,1}^{(1)} = C_{4,1}^{(2)} = C_{4,1}^{(3)} = C_{4,1}^{(\text{tot})}
\end{equation}

The resulting total value $C_{4,1}^{(\text{tot})} = -2.99 \pm 0.03$ given by the sum in (4.5) is to be compared with the expected value in (4.2), i.e. $C_{4,1} = -\frac{89}{8} = -2.963...$. The relative error is thus below the 1% level and within the statistical uncertainty of the multidimensional integration. This very good agreement with the prediction (4.3) effectively confirms our assumption that like the tree-level one, the total one-loop correlator has simple dependence on $t_i$ as in (4.1).

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\section{A AdS$_2$ integrals in disc parametrization}

Below we collect some results about the SU(1, 1) covariant integrals over the unit disc that were used in the main text. We shall use the invariant measure

\begin{equation}
\frac{d^2z}{\pi(1-|z|^2)^2},
\end{equation}

\begin{align*}
\int d^2z' d^2z'' g_b(\theta_1, z') g_b(\theta_2, z'') g_b(\theta_3, z''') g_b(\theta_4, z''') \\
\times g(z', z'') g(z', z''') g(z'', z'''),
\end{align*}

\begin{align*}
\int d^2z' d^2z'' g_b(\theta_1, z') g_b(\theta_2, z') g_b(\theta_3, z''') g_b(\theta_4, z''') [g(z', z'')]^2, \\
\int d^2z' d^2z'' g_b(\theta_1, z') g_b(\theta_2, z') g_b(\theta_3, z''') g_b(\theta_4, z''') [g(z', z'')]^2.
\end{align*}

\begin{table}[h]
\begin{tabular}{cccccc}
\hline
\hline
$C_{4,1}^{(0)}$ & $C_{4,1}^{(1)}$ & $C_{4,1}^{(2)}$ & $C_{4,1}^{(3)}$ & $C_{4,1}^{(\text{tot})}$ \\
\hline
2.675(3) & -0.7901... & -0.2191(2) & -5.54(2) & 0.884(6) & -2.99(3) \\
\hline
\end{tabular}
\end{table}
and the propagator \( g(z, z') \) defined in (2.7) with \( g(z, z) \) parametrized as in (2.12). A very useful representation of the propagator is the Fourier expansion derived in [29]. Denoting by \( \vartheta(z, z') \) the angle between the disc points \( z, z' \), one has

\[
g(z, z') = \sum_{n=0}^{\infty} g_n(|z|^2, |z'|^2) \cos(n \vartheta(z, z')) ,
\]

(A.2)

\[
g_n(x, y) = \theta(y - x) a_n(x) b_n(y) + \theta(x - y) a_n(y) b_n(x) ,
\]

(A.3)

\[
a_0(x) = \frac{1 + x}{1 - x} , \quad a_1(x) = \frac{\sqrt{x}}{1 - x} ,
\]

\[
b_0(y) = -\frac{1}{2} \left( \frac{1 + y}{1 - y} \log y + 2 \right) , \quad b_1(y) = \frac{1}{\sqrt{y}} \left( 2y - \log y + 1 + y \right) ,
\]

\[
a_{n \geq 2}(x) = \frac{x^n}{1 - x} \left( 1 - \frac{n - 1}{n + 1} x \right) , \quad b_{n \geq 2}(y) = -\frac{y^{-\frac{n}{2}}}{n(n - 1)} \left( \frac{1 + y}{1 - y} (1 - y^n) - n (1 + y^n) \right) ,
\]

(A.4)

where \( \theta(y - x) \) in (A.3) is the step function.

A.1 Tadpole reduction identity

Let us consider the integral (using (2.12))

\[
I(z) = \int d^2 z' g(z, z') g(z', z) = \int d^2 z' g(z, z') \left[ q_1 \log(1 - |z'|^2) - q_2 \right] .
\]

(A.5)

Substituting (A.2) the only contributing term is \( n = 0 \). After integrating over the angle and setting \( |z'| = r \) we have

\[
I(z) = 2\pi \int_0^1 \frac{r dr}{\pi (1 - r^2)^2} g_0(|z|^2, r^2) \left[ q_1 \log(1 - r^2) - q_2 \right]
\]

\[
= 2 \frac{1 + |z|^2}{1 - |z|^2} \int_0^1 \frac{r dr}{(1 - r^2)^2} b_0(r^2) \left[ q_1 \log(1 - r^2) - q_2 \right]
\]

\[
+ 2 b_0(|z|^2) \int_0^{|z|} \frac{r dr}{(1 - r^2)^2} \left( 1 + r^2 \right) [q_1 \log(1 - r^2) - q_2] .
\]

(A.6)

Integrating over \( x = r^2 \) gives

\[
I(z) = -\frac{1}{8} (q_1 + 2 q_2) + \frac{1}{4} q_1 \log(1 - |z|^2) .
\]

(A.7)

Combining (A.5) and (A.7) we get a simple relation (which follows also from the defining equation for the propagator)

\[
g(z, z) - 4 \int d^2 z' g(z, z') g(z', z') = \frac{q_1}{2} .
\]

(A.8)

It may be used to perform the following reduction of the tadpole diagrams (special cases of which are mentioned in [29])

\[
-16-
\]
Here dots stand for \( N \) external lines and the crossed vertex in the r.h.s. corresponds to the interaction vertex \( \sim \chi^N \). Its precise coefficient may be determined as follows. The explicit form of the l.h.s. of (A.9) is

\[
-\frac{1}{2} \int \frac{1}{ \beta^2} (2b)^{N+1} g(z, z) + \frac{1}{2} \int \frac{1}{ \beta^2} (2b)^{N+1} \frac{1}{ \beta^2} (2b)^3 \int d^2 z' g(z, z') g(z', z') = -q_1 (2b)^N, \tag{A.10}
\]

where we used (A.8) and took into account the \( \frac{1}{2} \) symmetry factor due to the tadpoles. In the ZZ formulation we have \( q_1^{ZZ} = 1 \) (see (2.13)) and thus the r.h.s. corresponds to \( (2b)^N \chi^N \), i.e. \( b^2 \) times the interaction term in the ZZ Lagrangian (2.3). Thus the one-loop tadpoles in the ZZ formulation give the same contribution as the extra couplings in (2.15) in the AdS formulation.

### A.2 One-loop correction to the two-point function

The one-loop correction to the connected two-point function is the sum of the bubble diagram

\[
D(z_1, z_2) = \int d^2 z' g(z_1, z') B(z', z_2) = \int d^2 z' g(z_1, z') B(z', z_2) = \int d^2 z' \left[ g(z_1, z') \right]^2 g(z', z_2), \tag{A.11}
\]

plus the tadpole diagrams contribution (A.9) that, using (A.8), (A.10), is given by \( q_1 \tilde{B}(z_1, z_2) \) with

\[
\tilde{B}(z_1, z_2) = \int d^2 z' g(z_1, z') g(z', z_2). \tag{A.12}
\]

Let us begin with the evaluation of \( B(z_1, z_2) \). The result should be the function of the \( SU(1, 1) \) invariant \( \eta(z, z') = \left| \frac{z-z'}{z-z'\overline{z}} \right|^2 \) in (2.8). We may use the 3 parameters of \( SU(1, 1) \) to first set \( z_1 = 0 \) and \( z_2 = R \in \mathbb{R} \) and then identify \( R^2 \) with \( \eta \) in the final result.\(^{16}\)

Denoting \( g(z, 0) \equiv g(|z|^2) \) and \( x = |z'|^2 \), we get (see (A.3) for the expression for \( g_0 \))

\[
B(0, R) = \int d^2 z' \left[ g(|z'|^2) \right]^2 g(z', R) = \int_0^1 \frac{dx}{(1-x)^2} \left[ g(x) \right]^2 g_0(x, R^2)
\]

\[
= b_0(R^2) \int_0^R \frac{dx}{(1-x)^2} \left[ g(x) \right]^2 a_0(x) + a_0(R^2) \int_0^1 \frac{dx}{(1-x)^2} \left[ g(x) \right]^2 b_0(x)
\]

\[
= \frac{1}{8} \frac{R^2 \log^2 R^2}{8(R^2-1)^2}. \tag{A.13}
\]

The result is thus [5]

\[
B(z_1, z_2) = \frac{1}{8} - \frac{\eta \log^2 \eta}{8(1-\eta)^2}, \quad \eta = \eta(z_1, z_2). \tag{A.14}
\]

It is then straightforward to find the expression for \( D(z_1, z_2) \) in (A.11). We choose again \( z_1 = R > 0 \) and \( z_2 = 0 \) to get

\[
D(z_1, z_2) = D(R, 0) = \int d^2 z' g(R, z') B(z', 0) = \int d^2 z' g(R, z') \left[ \frac{1}{8} - \frac{|z'|^2 \log^2(|z'|^2)}{8(1-|z'|^2)^2} \right]
\]

\(^{16}\)One can always find an \( SU(1, 1) \) transformation sending \( z_2 \to R > 0 \) and \( z_1 \to 0 \): the corresponding parameters are \( b = -a z_1, |a| = 1/(1-|z_1|^2) \). From \( R = (az_2 - az_1)/(\sigma \sigma_1 z_2 + \sigma) = \frac{a}{2} (z_2 - z_1)/(1 - z_2 \sigma_1) \) we find \( R^2 = \eta(z_2, z_1) \) and determine also \( \arg a \).
Using the method of [18], we get to the connected part of the two-point function is (cf. (2.18)).

Taking into account the combinatoric coefficients and the couplings, the total one-loop correction to the connected part of the two-point function is (cf. (2.18))

\[
\begin{align*}
\hat{B}(R, 0) &= \int d^2z' g(R, z') g(z', 0) = \int d^2z' g(R, z') g(|z'|^2) = \int_0^1 \frac{dx}{(1-x)^2} g_0(R^2, x) g(x) \\
&= b_0(R^2) \int_0^1 \frac{dx}{(1-x)^2} a_0(x) g(x) + a_0(R^2) \int_0^1 \frac{dx}{(1-x)^2} b_0(x) g(x) \\
&= -\frac{\eta \log \eta}{6(\eta - 1)} + \log(1-\eta) \left[ \frac{1}{6} - \frac{(\eta+1) \log \eta}{12(\eta - 1)} \right] - \frac{(\eta+1) \text{Li}_2(1-\eta)}{6(\eta - 1)} - \frac{1}{6}. \tag{A.16}
\end{align*}
\]

Taking into account the combinatoric coefficients and the couplings, the total one-loop correction to the connected part of the two-point function is (cf. (2.18))

\[
32 D(z_1, z_2) - 4 q_1 \hat{B}(z_1, z_2) = \frac{3}{2} + \frac{\eta^2 \log^2 \eta}{2(1-\eta)^2} - \frac{1+\eta}{1-\eta} \text{Li}_2(1-\eta) \\
+ \frac{2}{3} (q_1 - 1) \left( 1 - \frac{\eta \log \eta}{1-\eta} - \log(1-\eta) \left[ \frac{1}{2} + \frac{(1+\eta) \log \eta}{2(1-\eta)} \right] - \frac{1+\eta}{1-\eta} \text{Li}_2(1-\eta) \right). \tag{A.17}
\]

For $q_1 = 1$ this gives $\Sigma(z_1, z_2)$ in (2.20) and agrees with the result of [18].

### B On the one-point function in the ZZ and AdS formulations

Let us make few comments on the tadpole contributions in the ZZ and AdS formulations discussed in section 2.1. As follows from (2.17), the tadpole $\langle \chi(z) \rangle^{ZZ}$ has a non-constant $O(b)$ contribution that combined with the classical background (2.2) changes its coefficient from $b^{-1}$ to $Q = b^{-1} + b$. The constant part of the $O(b)$ term in (2.16),(2.17) is regularization dependent.

In general, the two-loop purely tadpole corrections to the one-point function are given by

\[
\mathcal{I}_2(z) = \begin{array}{c}
\begin{tikzpicture}
    
    \node at (0,0) [circle,draw] (z) {};
    \node at (1,0) [circle,draw] (z1) {};
    \node at (2,0) [circle,draw] (z2) {};
    \node at (1,1) [circle,draw] (z3) {};
    \node at (1,-1) [circle,draw] (z4) {};
    
    \draw (z) -- (z1);
    \draw (z1) -- (z2);
    \draw (z2) -- (z3);
    \draw (z3) -- (z4);
    \draw (z4) -- (z) ;
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
    
    \node at (0,0) [circle,draw] (z) {};
    \node at (1,0) [circle,draw] (z1) {};
    \node at (2,0) [circle,draw] (z2) {};
    \node at (1,1) [circle,draw] (z3) {};
    \node at (1,-1) [circle,draw] (z4) {};
    
    \draw (z) -- (z1);
    \draw (z1) -- (z2);
    \draw (z2) -- (z3);
    \draw (z3) -- (z4);
    \draw (z4) -- (z) ;
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
    
    \node at (0,0) [circle,draw] (z) {};
    \node at (1,0) [circle,draw] (z1) {};
    \node at (2,0) [circle,draw] (z2) {};
    \node at (1,1) [circle,draw] (z3) {};
    \node at (1,-1) [circle,draw] (z4) {};
    
    \draw (z) -- (z1);
    \draw (z1) -- (z2);
    \draw (z2) -- (z3);
    \draw (z3) -- (z4);
    \draw (z4) -- (z) ;
\end{tikzpicture}
\end{array}.
\tag{B.1}
\]

These are absent in the AdS formulation (where all tadpoles vanish, see (2.14)) but are non-zero and constant in the ZZ formulation. While separate contributions to (B.1) are not $SU(1, 1)$ covariant, their

---

17 For instance, the diagram associated with $D(z_1, z_2)$ must be multiplied by $(-8)^2$ from the cubic couplings and by the symmetry factor $\frac{3}{2}$ giving $+32$ in (A.17).

18 Note also that $\hat{B}(z_1, z_2)$ coincides with the expression in Eq.(86) of [18] because the r.h.s. of that equation is given by the diagrams associated with $B(z_1, z_2)$. 
This is just a z-independent constant (as one can see, e.g., by using $SU(1,1)$ invariance or comparing (A.5) and (A.7)). In a general regularization scheme in (2.12) one thus finds that $T_2(z) \sim q_1^2$ (cf. (A.8)). This is consistent with the vanishing of $T_2(z)$ in the Hadamard-like regularization (where $q_1 = 0$, $q_2 = \text{const}$ in (2.12)) discussed in [18].

\section*{C Tree level Witten diagram contributions to $\langle \Phi(t_1) \cdots \Phi(t_4) \rangle$}

At the tree level, the connected four-point boundary correlator $C(t_1, \ldots, t_4) \equiv \langle \Phi(t_1) \cdots \Phi(t_4) \rangle_{\text{conn}}$ is given by the sum of the contact and the exchange diagrams (cf. $C_{4,0}^{\text{cont}}$ and $C_{4,0}^{\text{exch}}$ in (4.4)). Their separate contributions is straightforward to compute [19]:

\begin{equation}
C_{4,0}^{\text{cont}}(t_1, \ldots, t_4) b^2 = \begin{align*}
\chi, \\
\end{align*}
\end{equation}

\begin{equation}
C_{4,0}^{\text{exch}}(t_1, \ldots, t_4) b^2 = \begin{align*}
\chi.
\end{align*}
\end{equation}

where the function $D_{n,k,l,m}(t_1, \ldots, t_4)$ are the standard AdS integrals defined in [33–35]. Their explicit evaluation gives

\begin{align*}
C_{4,0}^{\text{cont}} &= \frac{64}{81 t_{12} t_{34}^2} \left[ 2(\chi^2 - \chi + 1)\chi^2 + (2\chi^2 + \chi + 2)\chi \log(1 - \chi) + \frac{(2\chi^2 - 5\chi + 5)\chi^4 \log \chi}{(1 - \chi)^3} \right], \\
C_{4,0}^{\text{exch}} &= \frac{64}{81 t_{12} t_{34}^2} \left[ (\chi^2 - \chi + 1)\chi^2 - (2\chi^2 + \chi + 2)\chi \log(1 - \chi) - \frac{(2\chi^2 - 5\chi + 5)\chi^4 \log \chi}{(1 - \chi)^3} \right],
\end{align*}

where $\chi = \frac{t_{12} t_{34}}{t_{13} t_{24}}$ is the 1d cross-ratio. Note that these two contributions have non-trivial logarithmic dependence on the boundary points $t_1, \ldots, t_4$. However, all the logarithms cancel in their sum which becomes simply proportional to the conformal factor $\mathcal{K}_4$ in (4.7):

\begin{equation}
C_{4,0}^{\text{cont}}(t_1, \ldots, t_4) + C_{4,0}^{\text{exch}}(t_1, \ldots, t_4) = \frac{64}{27 t_{12} t_{34}^2} \chi^2 \frac{(1 - \chi + \chi^2)}{(1 - \chi)^2} = \frac{32}{27} \left( \frac{1}{t_{12} t_{34}^2 t_{14}^2} + \frac{1}{t_{13} t_{23} t_{14}^2 t_{12}^2} + \frac{1}{t_{12} t_{24} t_{34}^2 t_{23}^2} \right),
\end{equation}

reproducing the value for the coefficient $C_{4,0}$ in (4.4).

Note that this cancellation does not happen if the two contributions are combined with different weights like in (4.6) where $C_{4,0}^{\text{cont}}$ and $C_{4,0}^{\text{exch}}$ should then be understood as the functions in (C.2) divided by the factor $\mathcal{K}_4$.

\section*{D Global conformal block decomposition of the four-point correlator}

The four-point correlator in (1.11)–(1.13) has a simple dependence on the boundary points without any logarithmic terms of the 1d cross ratio $\chi$. This means that the 1d conformal operators appearing
in the OPE of the boundary correlator (1.11) will have no anomalous dimensions (cf. [12, 14]). The same structure is fixed by the Virasoro symmetry in the case of the stress tensor correlator (1.5). In view of this connection it is of some interest to discuss the equivalent OPE decomposition of the stress tensor correlator in terms of the conformal blocks of the global part of the 2d conformal symmetry (which is effectively equivalent to 1d conformal block decomposition).

One can represent the correlator in (1.5) as

$$ \langle T(z_1) T(z_2) T(z_3) T(z_4) \rangle = \frac{1}{z_1 z_2 z_3 z_4} G(\chi), \quad \chi \equiv \frac{z_1 z_2 z_3}{z_1 z_3 z_4}, $$

(D.1)

$$ G(\chi) = \frac{c^2}{4} \left[ 1 + \chi^4 + \frac{\chi^4}{(1-\chi)^4} \right] + 2c^2 (1 - \chi + \chi^2)/(1-\chi)^2, $$

(D.2)

where the $c^2$ term is the "generalized free field" part. From the global $SL(2, \mathbb{R})$ invariance we should have

$$ G(\chi) = \sum_{n=0}^{\infty} c_n F_n(\chi), \quad F_n(\chi) = \chi^n 2F_1(n, 2n; \chi), \quad c_n = \sum_{\mathcal{O}_n} C_{2,\mathcal{O}_n}^2, $$

(D.3)

where $\mathcal{O}_n$ stand for the dimension $n$ quasi-primary fields (with $T$ corresponding to $\{\mathcal{O}_2\}$) in the vacuum module and $C_{3,\mathcal{O}}$ and $C_{2,\mathcal{O}}$ are the coefficients in $\langle T T \mathcal{O} \rangle$ and $\langle \mathcal{O} \mathcal{O} \rangle$. One can show by a direct analysis of the algebraic expression in (D.2) that [36]

$$ c_{2p+2} = \left[ \frac{c^2}{144} (2p-1)_6 + 2c [1+2p(2p+3)] \right] \frac{(2p)! (2p+1)!}{(4p+1)!}, \quad c_{2p+1} = 0, \quad p = 0, 1, 2, \ldots. $$

(D.4)

Thus

$$ G(\chi) = c^2/4 + 2c F_2(\chi) + \left( \frac{c^2}{2} + \frac{11c}{5} \right) F_4(\chi) + \left( \frac{10}{9} c^2 + \frac{29}{63} c \right) F_6(\chi) + \mathcal{O}(\chi^8). $$

(D.5)

Here $c_0 = c^2/4$ and $c_2 = 2c$ are associated with $\mathcal{O}_0 = I$ and $\mathcal{O}_2 = T$. At level 4 we have the quasi-primary field (appearing in the regular part of the OPE of $T(z)T(z')$)

$$ \Lambda_4 = \langle TT \rangle - \frac{3}{112} T''', $$

(D.6)

where the brackets denote the normal ordering. Using that $C_{2,\Lambda_4} = C_{3,\Lambda_4} = c(22+5c)/19$, we confirm the value of $c_4$ in (D.5). At level 6 one finds two orthogonal quasi-primaries

$$ \Lambda_6^{(a)} = \langle TT TT \rangle - \frac{9}{8} \langle T'' T' \rangle - \frac{1}{112} \langle T'''' \rangle, $$

$$ \Lambda_6^{(b)} = \langle TT TT \rangle - \frac{28c^2 + 1050c + 2735}{18(42c + 67)} \langle T'' T' \rangle + \frac{35c^2 + 462c + 2062}{18(42c + 67)} \langle T'' T'' \rangle - \frac{5c^2 + 228c + 553}{108(42c + 67)} \langle T'''' \rangle, $$

(D.7)

and a straightforward calculation gives

$$ C_{2,\Lambda_6^{(a)}} = \frac{3}{172} c(28c^2 + 1050c + 2735), \quad C_{3,\Lambda_6^{(a)}} = \frac{3}{25} c(42c + 67), $$

$$ C_{2,\Lambda_6^{(b)}} = \frac{c(2c-1)(5c+22)(7c+68)}{36(42c + 67)^2}, \quad C_{3,\Lambda_6^{(b)}} = -\frac{c(2c-1)(5c+22)(7c+68)}{9(42c + 67)}, $$

(D.8)

As a result, $c_6$ in (D.3) is given by the sum

$$ c_6 = \frac{3c(42c + 67)^2}{7(28c^2 + 1050c + 2735)} + \frac{4c(2c-1)(5c+22)(7c+68)}{9(28c^2 + 1050c + 2735)} = \frac{10}{9} c^2 + \frac{29}{63} c, $$

(D.9)

in agreement with (D.5).
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