ON THE KINETIC PROPERTIES OF SOLITONS IN NONLINEAR SCHRÖDINGER EQUATION

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Abstract

The Boltzmann type kinetic equation for solitons in Nonlinear Schrödinger equation has been constructed on the base of analysis of two soliton collision. Possible applications for Langmuir solitons in plasma and solitons in optic fibers are discussed.
1. Nonlinear Schrödinger equation (NSE) is one of the most popular equations for describing nonlinear phenomena in condensed matter and plasma. Such effects as small amplitude localized waves in magnets, localized excitations in quasi-one-dimensional biologic systems and optic fibers are described by NSE (see for example [1-3]). In spite of wide field of applications, kinetic properties of solitons in NSE as well as the another models were investigated much less than kinematic and dynamic properties of solitons. Some aspects of kinetic behavior of solitons in Korteveg-de-Vries, $\phi^4$ and Sine-Gordon equations have been investigated in [4-8]. Fokker-Plank equation for the distribution function of soliton in NSE damping and fluctuations were considered in [9]. In present paper the kinetic equation for solitons in NSE following the approach suggested in [10] is proposed. Possible applications for solitons in plasma and optic fibers are discussed.

2. NSE in dimensionless form for slow varying nonlinear envelope is written as:

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2 = 0.$$  \hspace{1cm} (1)

Construction of kinetic equation for solitons needs analysis of the process of soliton-soliton interaction. It is well known that solitons in exactly integrable models do not change there forms and velocities and additional shifts of there coordinates $\Delta x$ and phases $\Delta \phi$ appear as the result of their interaction[1-3]. For NSE the shifts $\Delta x$ and $\Delta \phi$ can be written as [1]:

$$\Delta x_i = \zeta_i^{-1} \ln |Z_{1,2}| \quad \Delta \phi_i = 2\text{arg}Z_{1,2}, \quad i = 1, 2$$

$$Z_{1,2} = \frac{2i(\zeta_1 + \zeta_2) - (v_1 - v_2)}{2i(\zeta_1 - \zeta_2) - (v_1 - v_2)}.$$  \hspace{1cm} (2)

Here $\zeta_i$ - is dimensionless parameter characterizing width and amplitude of soliton, $v_i$ -is soliton velocity.

To simplify the problem we will analyse the case of low density of soliton, therefore we will consider the case when shift of soliton coordinate much more then soliton width. In the region of parameters:

$$4(\zeta_1 - \zeta_2)^2 < (v_1 - v_2)^2 \ll 4(\zeta_1 + \zeta_2)^2,$$

one can obtain that

$$\Delta x_1 \gg \zeta_1^{-1}, \Delta x_2 \gg \zeta_2^{-1}$$

and

$$|\Delta \phi_1| \approx |\Delta \phi_2| \approx 1,$$  \hspace{1cm} (3) \hspace{1cm} (4) \hspace{1cm} (5)

that gives us the possibility to follow only for the evolution of soliton positions.

Let us introduce the distribution function of solitons $f(x, p, t)$. To construct collision integral we follow [10] and apply Boltzmann approach taking into account that in contrast to usual particles solitons change there positions. Also it is necessary to consider that the interaction between solitons is absent when the distance become much more then soliton width. Let us analyse the number of solitons arriving to the point $(x, p_1)$ and leaving this point due to two soliton collisions for example in the case $p_2 > p_1 > 0$. 

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Due to following process:

\[(x, p_1) \rightarrow (x + \Delta x_1, p_1)\]
\[(x, p_2) \rightarrow (x - \Delta x_2, p_2)\] (6)

the number of solitons leaving the point \(x, p_1\) can be written as:

\[N(-) = dx \int_{v_1}^{\infty} |v_1 - v_2| f(x, p_1, t) f(x, p_2, t) dp_2.\] (7)

The number of solitons arriving at the point \((x, p_1)\) and corresponding to following process

\[(x - \Delta x_1, p_1) \rightarrow (x, p_1)\]
\[(x - \Delta x_1, p_2) \rightarrow (x - \Delta x_1 - \Delta x_2, p_2)\] (8)

can be presented as:

\[N(+) = dx \int_{v_1}^{\infty} |v_1 - v_2| f(x - \Delta x_1, p_1, t) f(x - \Delta x_1, p_2, t) dp_2.\] (9)

From formulae (7) and (9) one can write soliton-soliton collision integral in the case \(p_2 > p_1 > 0\). Considering all possible ratio for \(p_1\) and \(p_2\) soliton-soliton collision integral can be written as:

\[\mathcal{L} = \int_{-\infty}^{\infty} |v_1 - v_2| \{f(x - \Delta x_1, p_1, t) f(x - \Delta x_1, p_2, t) - f(x, p_1, t) f(x, p_2, t)\} dp_2.\] (10)

In the approximation

\[|\Delta x_1| \approx |\Delta x_2| \approx |\Delta x|, \quad \Delta x = \zeta^{-1} \text{sign}(v_1 - v_2) \ln \left( \frac{4\zeta}{v_1 - v_2} \right)^2\] (11)
collision integral coincide with one obtained for Sine-Gordon equation in [10]. Assuming that \(f(x, p, t)\) is slowly varying in scales comparable to \(\Delta x\) and expanding \(f(x, p, t)\) in powers of \(\Delta x\) and keeping the leading terms we can rewrite the expression for \(\mathcal{L}\) in the form:

\[\mathcal{L} = -\frac{\partial}{\partial x} [f(x, p_1, t)u(x, p_1, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [f(x, p_1, t)D(x, p_1, t)],\] (12)

where

\[u(x, p_1, t) = \int_{-\infty}^{\infty} |v_1 - v_2| \Delta x f(x, p_2, t) dp_2,\] (13)
describes the renormalization of soliton velocity and

\[D(x, p_1, t) = \int_{-\infty}^{\infty} |v_1 - v_2| (\Delta x)^2 f(x, p_2, t) dp_2,\] (14)
describes the solitons diffusion process.
The kinetic equation can be written as:
\[ \frac{\partial f(x, p_1, t)}{\partial t} + \frac{\partial}{\partial x} [v_1 + u(x, p_1, t)] f(x, p_1, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} D(x, p_1, t) f(x, p_1, t). \] (15)

From the kinetic equation (15) it is easy to show, that soliton-soliton collisions lead to the entropy production in the soliton gas in the case of nonuniform distribution function of soliton in coordinate space (see [10]). Applying standard methods from (10) it is easy to derive transport equations and calculate the relaxation time (see [10,11]).

3. Let us consider some applications that result from kinetic equation (15). NSE appears in analysis of nonlinear electron plasma waves [12-15]. The conditions of solitons creation [16] and collapse [17,18] and interpretation of Langmuir turbulence in soliton terms have been analysed in the frame of NSE (see [13-15]).

On the base of proposed approach it is possible to estimate the relaxation time \( \tau_{ss} \) for soliton gas. Indeed this estimation can be written as:
\[ \frac{1}{\tau_{ss}} \sim (q x_0)^2 n v_T, \] (16)
where \( x_0 \) - is soliton size, \( q \) is characteristic scale of inhomogeneity in soliton gas, \( n \)-is soliton density, \( v_T = \sqrt{T/m} \) - is thermal velocity of solitons, \( T \) -is temperature of solitons and \( m \) is its mass.

Here
\[ x_0 = \zeta^{-1} \sqrt{\beta \alpha}, \quad \beta = \frac{1}{2} \frac{dv_g}{dk}, \quad \alpha = - \frac{\partial^2 \omega}{\partial |\psi|^2}, \] (17)
where \( v_g \) - is group velocity of the high frequency wave, \( \omega \) and \( k \) - its frequency and wave vector, \( \psi \) -is slow varying nonlinear envelope.

To come nearer from proposed scheme to real situation it is necessary to take into account two facts: the interactions which destroy the integrability of the system are always existed and one dimensional soliton is unstable under the influence of 2D and 3D perturbations. In the nonintegrable system solitons collide with momentum changing, but in the case close to completely integrable model it is not difficult to estimate the relaxation time \( \frac{1}{\tau_{ss}^*} \sim \Delta E/E \), where \( \Delta E/E \) is relative energy change due to collision. Therefore we have two steps of the relaxation: first one deal with shifts of the soliton positions and second deal with momentum exchange [10,19]. The characteristic time \( \tau_{un} \) of soliton instability in 2D space is finite and \( \tau_{un} \sim L^2 \) (see [14]), where \( L \) is characteristic size in perpendicular direction. Comparing these estimations with formula (16) it is possible to conclude that above considered kinetic behaviour of solitons deal with shift of its positions can be realized as a intermediate regime before solitons collapse and can be interpreted as soliton turbulence. Soliton turbulence phenomena have been considered in [15] were kinetic equation for many particle distribution function of soliton like waves has been proposed. In fact in [15] processes with momentum exchange only were considered.

4. Another application of NSE deal with soliton propagation in optic fibers [20,21]. For single mode case NSE can be written as:
\[ i \frac{\partial \chi}{\partial \xi} + \frac{\partial^2 \chi}{\partial \tau^2} + \sigma \chi |\chi|^2 = 0. \] (18)
Here $\chi$ is amplitude envelope of the pulse, $\xi = z/L_\omega$ - is the coordinate along the fiber, $L_\omega$ - is characteristic disperse length, $\tau = (z - v_0 t)/v_0 T_0$, where $v_0$ - is the group velocity, $T_0$ is the initial duration of the pulse, the value $\sigma$ is defined by dispersion of group velocity and refraction coefficient.

From applied point of view the problem of relaxation of solitons interacting with defects is one of the most actual among kinetic effects. The possible results of soliton - defect interaction are transmission, reflection or capturing of soliton by defect (see [22-25]). Dynamic properties of solitons in optic fiber interacting with dispersion-spectrum inhomogeneities have been studied in [26]. Obviously that the most natural way to analyze relaxation of solitons is to formulate soliton-defect collision integral. In general form it can be written as:

$$\mathcal{L}_{sd} = \int_{-\infty}^{\infty} W(p_1, p_2) \{ f(x, p_1, t) - f(x, p_2, t) \} dp_2. \quad (19)$$

Concrete expression for $W(p_1, p_2)$ in Sine-Gordon model was calculate in [11] for the case of elastic scattering of soliton by impurity. Here we estimate the relaxation time $\tau_{sd}$ in the case of low density of solitons comparing with defects concentration $C_i$. Really from dimensionally consideration:

$$\frac{1}{\tau_{sd}} \sim C_i \frac{\Delta E_{sd}}{E}, \quad (20)$$

where $\Delta E_{sd}$ is relative energy loss due to interaction with defect.

The defects appearance can deal with fiber irradiation. As a result the characteristics of light and sound propagation have changed under the influence of the ionizing radiation. These effects have been applied for measuring of the radiation dose [27,28]. Obviously that the relaxation time and kinetic coefficients of solitons depend from the intensity of fiber irradiation.

Let us consider the expression for the concentration of defects in the case of $\gamma$-rays. The $\gamma$-rays can be generated in the processes of electron beam braking on heavy elements targets. The breaking radiation has continuous spectrum with the maximum energy of photons equal to the energy of electrons (see for example [29]). In this case [30]:

$$C_i = t J_e K, \quad (21)$$

where $t$ - is the time of irradiation, $J_e$ - is the current of electrons, coefficient $K=2.1$ for Si [30] and energy of $\gamma$-rays $E_{\gamma} \geq 20$Mev.

Therefore:

$$J_e \sim \frac{1}{\tau_{sd} t (\Delta E_{sd}/E)}. \quad (22)$$

Consequently due to stability of soliton and low damping in optical fiber this effect can be applied for measurement of irradiation dose.

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