On the $f$ Sum Rule and its Extensions

R. Cenni
Istituto Nazionale di Fisica Nucleare – sez. di Genova
Dipartimento di Fisica dell’Università di Genova
Via Dodecaneso 33 – 16146 – Genova – Italy

March 30, 2022

Abstract

The $f$ sum rule is derived in a non-relativistic frame and connected, via Ward Identities, to the two-photon term of the Compton scattering. A generalisation to isospin symmetry in the nuclear case is discussed and linked to the Meson Exchange Currents. The extension to a fully relativistic theory is then discussed and it is shown that the energy-weighted sum rule becomes a relation between the particle-hole and particle-antiparticle emission. Moreover the generalisation to isospin symmetry is derived and provides non-perturbative results.

1 Introduction

In this paper we discuss the energy-weighted sum rule of the scalar-isoscalar nuclear response ($f$) and its extension to a scalar-isovector one ($f'$) in a (as far as possible) non-perturbative frame.

As an introduction, we examine the non-relativistic sum rule. This topic is well known since a long time (see, e.g., [1] for the very beginning and [2] for a comprehensive review of the subject), but still some new facets need to be explored.

As a second step we connect the sum rule with the asymptotic behaviour of the polarisation propagator and, via Ward Identities (WI) with the two-photon term in the electro-magnetic (e.m.) lagrangian (for an electron gas) or with a suitable generalisation of it in nuclear physics.
Next we consider fully relativistic schemes, where the previous derivation breaks down because of the absence of two-photon terms, and we got the seemingly paradoxical result that the sum rule is vanishing. This is not a paradox however, because now the \( f \)-sum rule takes the form of a compensation between the nuclear response in the space-like region and a reduction of the response in the time-like region induced by the nuclear medium (Pauli Blocking). The renormalisation procedure plays here a central role.

Finally this approach opens the way to the extension of the sum rule to the scalar-isovector channel (\( f' \) sum rule): while in a potential theory the sum rule receives an extra contribution from the isospin dependence of the potential (if any: when the potential is isospin-independent then \( f \) and \( f' \) coincide), when the potential is replaced by a dynamical meson exchange, then the above contribution disappears but the sum rule is governed by the averaged squared mesonic field.

### 2 Generalities on the \( f \) Sum Rule

To begin with, the \( f \) sum rule, in a non-relativistic frame, reads

\[
\Xi_1 = \int_0^\infty \omega d\omega R_{T=0}^{S=0}(q,\omega) = \frac{q^2 N}{2m} \tag{1}
\]

where \( R_{T=0}^{S=0} \) is the nuclear response to a scalar-isoscalar probe and \( N \) is the number of nucleons. \( R_{T=0}^{S=0} \) is linked to the corresponding polarisation propagator \( \Pi_{T=0}^{S=0} \) by

\[
R_{T=0}^{S=0}(q,\omega) = -\frac{1}{\pi} \Im \int dx \int dy e^{iq(x-y)}(-i)\left< \Psi_0 | T \{ \rho(x), \rho(y) \} | \Psi_0 > \middle/ < \Psi_0 | \Psi_0 > \right> \tag{2}
\]

and in turn \( \Pi_{T=0}^{S=0} \), in Fourier transform and in Lehmann representation (LR), reads

\[
\Pi_{T=0}^{S=0}(q,\omega) = \sum_n \left( E_n - E_0 \right) \left\{ | < \Psi_0 | \tilde{\rho}(q) | \Psi_n > |^2 + | < \Psi_0 | \tilde{\rho}(-q) | \Psi_n > |^2 \right\} / \omega^2 - (E_n - E_0)^2 + i\eta \tag{3}
\]
(\hat{\rho}(q) being the Fourier transform of \rho(x): note that \hat{\rho}(q)^\dagger = \hat{\rho}(-q); further, \{|\Psi_n>\} is the set of the excited states of the system and \{E_n\} the set of the corresponding energies). The LR also shows that \Pi_{T=0}^{S=0} is even in \omega and, at fixed q, at a first sight seems to behave asymptotically like \omega^{-2}. This statement is by one side crucial, but is, on the other side, not immediately proven and deserves more comments.

In a non-relativistic frame no renormalisation is needed, i.e., no counter-term (that would dominate the asymptotic behaviour of \Pi_{T=0}^{S=0}) are required, but nevertheless our statement, that is equivalent to claim that the series

$$\sum_n (E_n - E_0) \{|<\Psi_0|\hat{\rho}(\pm q)|\Psi_n>|^2\}$$

is finite, can be easily proven only in a perturbative frame.

Assume for sake of simplicity the infinite nuclear matter limit: there \Pi_{T=0}^{S=0}, intended as function of the complex variable \omega at fixed q has a cut along the real positive axis; the cut at the lowest level (Free Fermi Gas, or FFG) ranges between \frac{q^2}{2m} - \frac{ak_F}{m} (or 0) and \frac{q^2}{2m} + \frac{ak_F}{m} (the energy range allowed for a particle-hole pair [p-h; we shall denote particle-antiparticle pairs with p-p]) and thus has a finite upper bound; this in turn amounts to say that the integral over the energy-weighted spectral function of the polarisation propagator is also finite. At higher orders the size of the response region increases, depending on the maximum number of allowed p-h pairs but still is bounded from above, so that the point \omega = \infty is regular at each order and the behaviour of \Pi_{T=0}^{S=0} \sim \omega^{-2} is ensured.

In the case of finite nuclei the proof is laborious, because the response region is extended up to \omega = \infty but, again in a perturbative frame, it falls down sufficiently quickly with \omega and the correct asymptotic behaviour is ensured again.

Since however the perturbative series is (likely) asymptotic, we cannot rule out a priori the existence of non-perturbative phenomena like instantons or so, that could destroy the required behaviour \Pi_{T=0}^{S=0} \sim \omega^{-2}

With this proviso (even more relevant in the relativistic case, because in the QCD vacuum instantons exist indeed) we come back to the sum rule, where a direct calculation provides

$$\Xi_1(q) = \frac{1}{2} <\Psi_0| [\hat{\rho}(-q), [H, \hat{\rho}(q)]] |\Psi_0>$$

(4)
that, if $\rho$ commutes with the potential (i.e., if the potential is local), can be written in the “canonical” form (1).

Let us now exploit the assumed analytical properties of $\Pi_{S=0}^{T=0}$ to link its asymptotic behaviour to the sum rule. From the LR $\Pi_{S=0}^{T=0}$ is analytic on the whole first Riemann sheet but for a cut along the real axis. Thus the sum rule can be rewritten as an integral over the whole $\Pi_{S=0}^{T=0}$ as

$$\Xi_1 = \frac{1}{2\pi i} \int_{C_1} \omega d\omega \Pi_{S=0}^{T=0}(q, \omega) \ ,$$

the integration path $C_1$ being shown in fig. 1. Would the integral be well behaved at the infinity, we could transform $C_1$ into a path along the imaginary axis, namely $C_2$. Actually this is not allowed because the integrand behaves like $1/\omega$ at the infinity: we can however add and subtract its asymptotic value by defining

$$\Pi^\infty(q) = \lim_{\omega \to \infty} \omega^2 \Pi_{S=0}^{T=0}(q, \omega)$$

and then by splitting the integrand as

$$\omega \Pi_{S=0}^{T=0}(q, \omega) = \frac{\omega \Pi^\infty(q)}{\omega^2 - a^2} + \left\{ \omega \Pi_{S=0}^{T=0}(q, \omega) - \frac{\omega \Pi^\infty(q)}{\omega^2 - a^2} \right\} \ ,$$

where $a$ is an irrelevant constant. Now we can transform the integration path in the second term of (7) and since its integrand is odd, its contribution vanishes. The only surviving part comes from the first term of (7) and is

$$\Xi_1 = \frac{1}{2\pi i} \oint d\omega \frac{\omega \Pi^\infty(q)}{\omega^2 - a^2} = \frac{1}{2} \Pi^\infty(q) \ ,$$
the last integral being extended to a circle containing \( a \). This relation is fully general, i.e., it does not depend upon the choice of \( \rho \). Further, it holds in a relativistic context as well, provided the right behaviour of \( \Pi_{S=0}^T(q, \omega) \) for \( \omega \to \infty \) is ensured, and is insensitive of a possible spin and/or isospin dependent interaction.

To exemplify let us consider the FFG. There the response function reads

\[
R_{S=0}^T(q, \omega) = -\frac{4V}{\pi} \Im \Pi^0(q, \omega)
\]

where \( V \) is the box volume and \( \Pi^0 \) denotes the Lindhard function without spin-isospin coefficients, that are factorized in front of it[3]. The simplest way to get the sum rule is to evaluate first the energy integral

\[
\Xi_{\text{FFG}}(q) = 4V \int \omega \, d\omega \int \frac{d^3k}{(2\pi)^3} \theta(|k + q| - k_F)\theta(k_F - k) \delta \left( \omega - \frac{q^2}{2m} - \frac{q \cdot k}{m} \right)
\]

and a trivial calculation shows that the “canonical value” of the sum rule is reproduced.

To compare FFG with eq. (8) we take the real part of \( \Pi^0[3] \), namely

\[
\Re \Pi^0(q, \omega) = 8\omega \int \frac{d^3k}{(2\pi)^3} \theta(|k + q| - k_F)\theta(k_F - k) \left( \frac{q^2}{2m} + \frac{q \cdot k}{m} \right)
\]

and since

\[
\lim_{\omega \to \infty} \omega^2 \Pi^0(q, \omega) = 8 \int \frac{d^3k}{(2\pi)^3} \theta(|k + q| - k_F)\theta(k_F - k) \left( \frac{q^2}{2m} + \frac{q \cdot k}{m} \right)
\]

comparison with (10) shows that the canonical value is reached again.
3 Sum Rules and Ward Identities

Now we consider the case of a conserved charge. We assume that a symmetry group exists, that generates a conserved current \( j^\mu(x) \), i.e., at a classical level, \( \partial_\mu j^\mu(x) = 0 \) and, at a quantum level, no anomalies occur. In the following \( \rho(x) \equiv j^0(x) = j_0(x) \).

The Ward Identities (WI) can be derived for any kind of Green’s function. In particular we need them for the current-current polarisation propagator

\[
i\Pi^{\mu\nu}(x-y) = \frac{\langle \Psi_0 | T \{ j^\mu(x), j^\nu(y) \} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}.
\]

(13)

Our case corresponds to a symmetry group \( U(1) \) exploiting charge or baryon number conservation. The quantity denoted with \( \Pi_{T=S=0}^{00} \) in previous sections turns out to be \( \Pi^{00} = \Pi_{00} \) (neglecting channel indices).

Before going on we need to better specify the meaning of (13), because it will play a central role in our derivation. Eq. (13) is very general indeed and applies to any kind of systems, both in a non-relativistic or relativistic frame. In the latter case however a well known disease arises when the “naive” definition of the \( T \)-product (this terminology could be found for instance in [4]; it could be better defined as Dyson’s \( T \)-product, or simply \( T_D \)) is adopted, namely

\[
T_D \{ j^\mu(x), j^\nu(y) \} = j^\mu(x)j^\nu(y)\theta(x^0 - y^0) + x \leftrightarrow y : \ (14)
\]

it has been discovered by Schwinger long time ago [5] that in such a case \( \Pi^{\mu\nu}(x - y) \) (we shall denote with this symbol the quantity defined in eq. (13) when the \( T \)-product is specified to be the Dyson’s one eq. (14)) is not covariant.

It is customary to introduce another \( T \)-product (also called Wick’s \( T \)-product, \( T_W \)) as

\[
T_W \{ j^\mu(x), j^\nu(y) \} = T_D \{ j^\mu(x), j^\nu(y) \} + Sg^\mu_0 g^\nu_0 \delta(x - y) \ \ (15)
\]

where the covariance is restored. The structure of the extra term in the above, namely \( Sg^\mu_0 g^\nu_0 \delta(x - y) \), is well known (see for instance [6] or [4]): we shall discuss it later, however, after having derived WI. We shall denote with \( \Pi^{\mu\nu}(x - y) \) the current-current polarisation propagator derived from(13) where the \( T \)-product has been replaced by \( T_W \).
The “caveat” we must be aware of is instead that a Schwinger term will alter the asymptotic (in $\omega$) properties of (the Fourier transform of) $\Pi^{\mu\nu}$. In fact the LR eq. (3) applies to $\Pi^{\mu\nu}$ only; thus the Schwinger term in $\tilde{\Pi}^{\mu\nu}$ will destroy the behaviour like $\omega^{-2}$. A final remark is that covariant contact terms could still exist. We shall see later in the relativistic infinite nuclear matter example that a particular renormalisation scheme is able to get rid of them, thus preserving the asymptotic behaviour.

Coming back to our main job, the derivation of the sum rule is a simple extension of the standard way QFT uses to get WI (see for instance textbooks like [7, 4]). The same results have already been obtained by Takahashi, within a less powerful formalism[8] and without linking it to the asymptotic behaviour of $\Pi^{00}$.

We write the classical action of the system as

$$A = \int dx \psi^\dagger(x) \left\{ i \frac{\partial}{\partial x_0} + \frac{\nabla^2}{2m} \right\} \psi(x) \tag{16}$$

$$- \int dx dy \sum_{ij} \psi^\dagger(x) O_i \psi(x) V_{ij}(x-y) \psi^\dagger(y) O_j \psi(y)$$

where the $O_i$ are some spin and isospin (or, if case, the identity) operators. The action is invariant under the global transformation

$$\psi \to e^{-i\Lambda} \psi ; \quad \psi^\dagger \to e^{i\Lambda} \psi^\dagger \tag{17}$$

Assuming $\Lambda \to \Lambda(x)$, up to the second order in $\Lambda$ one has

$$A[\Lambda] = A|_{\Lambda=0} + \int dx j^\mu(x) \partial_\mu \Lambda(x) + \int dx B^{\mu\nu}(x) \partial_\mu \Lambda(x) \partial_\nu \Lambda(x) \tag{18}$$

where

$$\rho(x) = j^0(x) = \psi^\dagger(x) \psi(x) \tag{19}$$

$$j(x) = -\frac{i}{2m} \left\{ \psi^\dagger(x) \nabla \psi(x) - [\nabla \psi^\dagger(x)] \psi(x) \right\} \tag{20}$$

$$B^{0\nu} = B^{\nu0} = 0 \tag{21}$$

$$B^{ij} = -\frac{1}{2m} \delta_{ij} \psi^\dagger(x) \psi(x) \tag{22}$$

and spin-isospin dependence in the potential are irrelevant. We now define a generating functional

$$Z[A_\mu] = \int D[\psi^\dagger, \psi] e^{iA + i \int j^\mu(x) A_\mu(x) dx} \tag{23}$$
where $A_\mu$ is a classical external field, the link with $\Pi^{\mu\nu}$ being

$$i\Pi^{\mu\nu}(x-y) = -\frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} \log Z[A_\rho] \bigg|_{A_\mu = 0}. \quad (24)$$

Note that here the covariant $T$-product (15) comes into play. This is equivalent to say that the WI will take a covariant form (as we shall check explicitly). Observe also that the tie ordering implicit on the definition of the path integral acts not only on the ordering of the currents but on the fields contained in it. Thus a further uncertainty could be added in the covariant part of the contact term. This flaw will not trouble us however, due to the renormalization mechanism we shall adopt in the relativistic case.

Coming to the derivation of the WI, we make the change of variable (17) inside the path integral. By one side the integral remain unchanged, but on the other side $A$ transforms according to (18) and $j^\mu(x)$ becomes

$$j^\mu(x) \rightarrow j^\mu(x) + 2B^{\mu\nu}(x)\partial_\nu\Lambda(x). \quad (25)$$

Since a change of the integration variable cannot alter $Z$ we can equal the generating functional evaluated with $\Lambda = 0$ with the one where $A$ and $j^\mu$ are transformed according to (18) and (25). Next, by expanding in $\Lambda$ up to the first order, we get the identity

$$\int D[\psi^\dagger, \psi] \int dx \left\{ j^\mu(x)\partial_\mu\Lambda(x) + 2B^{\mu\nu}(x)\partial_\mu\Lambda(x)A_\nu(x) \right\} e^{iA+i\int dx j^\mu(x)A_\mu(x)} = 0 \quad (26)$$

Taking its functional derivative with respect to $A_\mu(x)$, putting $A_\mu(x) = 0$ and further deriving with respect to $\Lambda(y)$ we get the WI we are interested in, namely

$$\int D[\psi^\dagger, \psi] \left\{ ( -i\partial^\mu j^\mu(x) ) j^\nu(y) - 2\delta(x-y)\partial^\mu B^{\mu\nu}(x) \right\} e^{iA} = 0, \quad (27)$$

where an extra term proportional to the density arises from the tensor $B^{\mu\nu}$. Using (22) and translating (27) into expectation values of physical quantities we get the wanted W.I.:

$$- i\partial^\mu < \Psi_0 | T \{ j^\mu(x), j^\nu(y) \} | \Psi_0 > = 2\delta(x-y) < \Psi_0 | \partial^\mu B^{\mu\nu}(x) | \Psi_0 > \quad (28)$$
that in Fourier transform becomes

\[ q_\mu \tilde{\Pi}^{\mu\nu} = 2q_\mu < \Psi_0 | B^{\mu\nu} | \Psi_0 > \]  

(29)

namely a quite general result. Note that \( B^{\mu\nu} \) is covariant by construction, so also \( \tilde{\Pi}^{\mu\nu} \) must be.

In the case at hand,

\[ q_\mu \tilde{\Pi}^{\mu0} = 0 \]  

(30)

\[ q_\mu \tilde{\Pi}^{\mu i} = \frac{q_i}{m} \int d^3x < \Psi_0 | \rho(x) | \Psi_0 > = \frac{q_i}{m} N . \]  

(31)

Combining these two relations and putting the z-axis along \( q \) we find

\[ \frac{1}{2} q_0^2 \tilde{\Pi}^{00} - \frac{1}{2} |q|^2 \Pi^{33} = \frac{|q|^2}{2m} N . \]  

(32)

(observe that \( \tilde{\Pi}^{ij} = \Pi^{ij} \)). Finally we need to link \( \tilde{\Pi}^{00} \) with \( \Pi^{00} \): we get

\[ \frac{1}{2} q_0^2 \Pi^{00} + \frac{1}{2} q_0^2 < \Psi_0 | S | \Psi_0 > - \frac{1}{2} |q|^2 \Pi^{33} = \frac{|q|^2}{2m} N . \]  

(33)

We know from [6] that \( S \) is vanishing in this case (more precisely is vanishing as far as no squared time derivatives of the fields are present in the lagrangian), so in the limit \( q_0 \to \infty \) \( (q_0 \equiv \omega) \) the first term (being constructed in such a way to preserve the asymptotic behaviour \( \omega^{-2} \)) provides \( \frac{1}{2} \Pi^\infty \), namely the sum rule, \( |q|^2 \Pi^{33} \) is vanishing because it too behaves like \( \omega^{-2} \) (with the same provisos as for \( \Pi^{00} \)) and the rhs term provides the “canonical” value of the sum rule.

Curiously, the above result does not need, in principle, any knowledge about \( S \) but could be regarded as a way to derive it, because at the leading order in \( q_0^2 \) (33) just reads

\[ \frac{1}{2} q_0^2 < \Psi_0 | S | \Psi_0 >= 0 \implies < \Psi_0 | S | \Psi_0 >= 0 . \]  

(34)

4 A fully relativistic model

4.1 The relativistic sum rule

A careful reader has surely recognised that the definition of \( B^{\mu\nu} \) immediately leads in a relativistic dynamics to get 0 as the result of the sum rule, since
in the Dirac theory the nucleon current and the nucleonic $B^\mu\nu$ term read

$$ j^N_\mu = \bar{\psi}(x)\gamma_\mu\psi(x) \quad B^\mu\nu = 0 . \quad (35) $$

A brief comment about the current is needed. Here we assume nucleons and mesons as structure-less, thus without form factors and anomalous momenta, because both of them arise from the dressing of the elementary particles by means of the perturbative theory: their explicit introduction would get rid of any non-perturbative results. The same holds a fortiori in the case of interacting nucleons and pions.

Consider now a system of nucleons interacting through an isoscalar meson and look to the $f$ sum rule. In such a case no further contributions to $B^\mu\nu = 0$ arise, and the equivalent of eq. (33) is

$$ \frac{1}{2}q^2_0\Pi^{00} + \frac{1}{2}q^2_0 < \Psi_0|S|\Psi_0 > - \frac{1}{2}|q|^2\Pi^{33} = 0 . \quad (36) $$

that entails

$$ \Xi_1 \equiv 0 , \quad < \Psi_0|S|\Psi_0 >= 0 . \quad (37) $$

(again the old result about $S$ is recovered). But, as contradictory as a vanishing sum rule could seem, this results holds true, and stems from the existence of the antiparticles [9]. In fact intuitively the non-relativistic $B^\mu\nu$ can be seen as the low energy limit of the “two-photon” time-ordered diagram of fig. 2.

![Figure 2: The relativistic version of the “two-photon” diagram.](image)

The crucial point is that now the energy-weighted sum rule, as well as the Coulomb sum rule and the longitudinal response function itself are non-vanishing even in the vacuum, provided the time-like region is also considered.
It is in fact always possible to find a pair $q^0, q$ such that the probe can create a $p\bar{p}$ pair. Even more, the sum rule for such a process is divergent.

Thus we are faced with two kind of problems: the first is that in deriving the sum rule it is customary to extend the integration region up to $+\infty$, thus going across the light cone and including nucleon-antinucleon pairs, the second is that the internal structure of the nucleon cannot be simply factorized in front of the sum rules.

To both these question an answer, useful for practical calculation and for a meaningful comparison with experimental data has been given in refs. [10, 11, 12]. In the conclusions we shall further discuss this topic.

### 4.2 The lowest-order case

The main difficulty met here is the separation of the nucleonic world from the nuclear one and this requires an appropriate renormalisation procedure. In order to understand the mechanisms leading to the unexpected result (37) let us follow the FFG example we introduced in sect. 3.

The nucleon Green’s function for a relativistic FFG (RFFG) is usually written as

$$S^m(q) = \frac{\gamma \cdot q + m}{2E_q} \left[ \frac{\theta(|q| - k_F) + \theta(k_F - |q|)}{q_0 - E_q + i\eta} + \frac{1}{q_0 + E_q - i\eta} \right] ,$$

(38)

where $E_q = \sqrt{m^2 + q^2}$. The main difference with respect to FFG is of course the existence of a third term describing the propagation of anti-nucleons. The evaluation of $\Pi^{\mu\nu}$ (remember that in this case $\Pi^{\mu\nu}$ and $\tilde{\Pi}^{\mu\nu}$ coincide) reduces to

$$\Pi^{\mu\nu}(q_0, |q|) = -i \int \frac{d^4k}{(2\pi)^4} j^\mu(q)S^m(q + k)j^\nu(q)S^m(k) .$$

(39)

The explicit evaluation of the frequency integral, assuming $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$
leads to the cumbersome but on the other hand trivial expression

$$\Pi^{00}(q_0, |q|) = -2 \int \frac{d^3 k}{(2\pi)^3} P(q, k) \bigg|_{k_0=-E_k}$$

\begin{align*}
&\left\{ \frac{\theta(|q + k| - k_F)\theta(k_F - |k|)}{q_0 - E_{q+k} + E_k + i\eta} - \frac{\theta(k_F - |q + k|)\theta(|k| - k_F)}{q_0 - E_{q+k} + E_k - i\eta} \right\} \\
&-2 \int \frac{d^3 k}{(2\pi)^3} P(q, k) \bigg|_{k_0=-E_k} \frac{\theta(|q + k| - k_F)}{q_0 - E_{q+k} - E_k - i\eta} \\
&+2 \int \frac{d^3 k}{(2\pi)^3} P(q, k) \bigg|_{k_0=E_k} \frac{\theta(|k| - k_F)}{q_0 + E_{q+k} + E_k - i\eta},
\end{align*}

where we have put, in order to simplify the notation,

$$P(q, k) = \frac{m^2 + 2k_0(k_0 + q_0) - (q \cdot k + k^2)}{E_k E_{q+k}}$$

that is the term coming from the traces. Note that the two last terms in (40) are clearly divergent. Note also that at the non-relativistic limit, $P \to 2$ and the first two terms in (40) are led back to the Lindhard function.

In order to remove the divergences, we introduce the polarisation propagator in the vacuum, namely

$$\Pi^{00}(q)\big|_{\text{vacuum}} = \Pi^{00}(q_0, |q|)|_{k_F=0}$$

\begin{align*}
&= -i \int \frac{d^3 k}{(2\pi)^3} j^\mu(q) S^0(q + k) j^\nu(q) S^0(k) \\
&= -2 \int \frac{d^3 k}{(2\pi)^3} P(q, k) \bigg|_{k_0=-E_k} \frac{1}{q_0 - E_{q+k} - E_k - i\eta} \\
&+ 2 \int \frac{d^3 k}{(2\pi)^3} P(q, k) \bigg|_{k_0=E_k} \frac{1}{q_0 + E_{q+k} + E_k - i\eta},
\end{align*}

where $S^0(q) = 1/(\not{q} - m + i\eta)$. $\Pi^{00}(q)\big|_{\text{vacuum}}$ is of course divergent too and requires a normalisation prescription not defined a priori. It describes the propagation of a $p\overline{p}$ in the vacuum and an obvious way of separating the medium effect from the vacuum properties is that of subtracting (42) from
so getting

\[
\Pi^{00}(q_0, |q|)_{\text{reg}} = -2 \int \frac{d^3k}{(2\pi)^3} P(q, k) \left|_{k_0 = -E_k} \right. \left( k_F^2 \right)
\]

\[
\times \left\{ \frac{\theta(|q + k| - k_F)\theta(k_F - |k|)}{q_0 - E_{q+k} - E_k + i\eta} - \frac{\theta(k_F - |q + k|)\theta(|k| - k_F)}{q_0 - E_{q+k} + E_k - i\eta} \right\}
\]

\[
+ 2 \int \frac{d^3k}{(2\pi)^3} P(q, k) \left|_{k_0 = -E_k} \right. \left( k_F^2 \right)
\]

\[
\times \left\{ \frac{\theta(k_F - |q + k|)}{q_0 - E_{q+k} - E_k - i\eta} - \frac{\theta(k_F - |k|)}{q_0 + E_{q+k} + E_k - i\eta} \right\}
\]

that is now convergent as it should. Before going on let us remark that the prescription of subtracting from a diagram its value taken at \( k_F = 0 \) only works at the lowest order in perturbation theory. In more complicated case a rigorous procedure exists and has been given in ref. [13] and roughly speaking amounts to subtract from any elementarily divergent sub-diagram its value at \( k_F = 0 \) and then to remove the remaining superficial divergence by subtracting the whole diagram again at \( k_F = 0 \). In other words, we can use the Bogoljubov’s recursion formula and replace each counter-term considered there with the corresponding sub-diagram in the vacuum.

Note further that this procedure is quite independent by the regularisation scheme we use (dimensional or Pauli-Villars regularisation or any else) because the counter-terms so introduced pertain to the vacuum and have to subtracted too. So we only need to know that the theory is renormalisable in the vacuum, but the results does not depend (being physically meaningful) upon the chosen regularisation scheme.

\( \Pi^{00} \) at large \( q_0 \) seems to behave like \( q_0^{-1} \). Since however the function must be even, due to its bosonic character, then the correct behaviour as \( q_0^{-2} \) is ensured. Starting from the above equation and remembering that the longitudinal response function for a FFG is connected to \( \Pi^{00} \) by \( R_L = -\frac{1}{\pi} \text{Im} \Pi^{00} \), a cumbersome but conceptually simple calculation provides the response function for a RFFG. Its explicit form can be found in many papers, like, say, [14, 10, 11]. In the last reference the response in the time-like region is also derived.

For the present purpose the best choice is not to use the explicit form of the response but, instead, to carry out first the integral over \( q_0 \) of the imaginary part of (43).
Let us observe first that

$$-\frac{V}{\pi} \Im \Pi^{00}(q_0, |q|) \big|_{k_F=0} = 2V \int \frac{d^3k}{(2\pi)^3} P(q, k) \bigg|_{k_0=-E_k} \delta(q_0 - E_{q+k} - E_k)$$

(44)

and consequently

$$-\frac{V}{\pi} \int q_0 dq_0 \Im \Pi^{00}(q_0, |q|) \big|_{k_F=0} = 2V \int \frac{d^3k}{(2\pi)^3} (E_{q+k}E_k) P(q, k) \bigg|_{k_0=-E_k} = \infty ,$$

(45)

i.e., in the vacuum the $p-\pi$ contribution to the energy-weighted sum rule is infinite. Thus when subtracting the vacuum we also subtract this contribution. The meaning of the various terms in (43) is now clear: the first two terms give a positive response and describe a $p-h$ pair creation (a nucleon is ejected from the nucleus); the two remaining describe the correction to the $p-\pi$ creation due to the existence of the Fermi Gas, that produces a Pauli blocking effect. Thus since the Pauli principle inhibit the response with respect to (44), when the vacuum is subtracted its contribution is negative.

To exemplify, for $q^0 > 0$,

$$\Im \Pi^{00}(q_0, |q|) = 2\pi \int \frac{d^3k}{(2\pi)^3} P(q, k) \bigg|_{k_0=-E_k} \times \theta(|q + k| - k_F)\theta(k_F - |k|)\delta(q_0 - E_{q+k} + E_k)$$

$$- 2\pi \int \frac{d^3k}{(2\pi)^3} P(q, k) \bigg|_{k_0=-E_k} \times \theta(k_F - |q + k|)\delta(q_0 - E_{q+k} - E_k)$$

(46)

and the integral over $q_0$ is trivial. We thus get

$$\Xi^{RFFG}_1 = 2V \int \frac{d^3k}{(2\pi)^3} (E_{q+k} - E_k) P(q, k) \bigg|_{k_0=-E_k,q_0=E_{q+k}-E_k} \times \theta(|q + k| - k_F)\theta(k_F - |k|)$$

$$- 2V \int \frac{d^3k}{(2\pi)^3} (E_{q+k} + E_k) P(q, k) \bigg|_{k_0=-E_k,q_0=E_{q+k}+E_k} \times \theta(k_F - |q + k|) .$$

(47)
By changing variable \((k \rightarrow -k - q)\) in the second integral, with some simple algebra we get, for \(q > 2k_F\),

\[
\Xi_{1}^{RFG} = 2V \int \frac{d^3k}{(2\pi)^3} \frac{1}{E_{q+k}E_k} \theta(k_F - |k|) \frac{(E_{q+k} - E_k)}{E_{q+k}E_k} \left\{ (E_{q+k} - E_k) \left[ m^2 + k^2 + k \cdot q + E_{q+k}E_k \right] \right. \\
\left. + (E_{q+k} + E_k) \left[ m^2 + k^2 + k \cdot q - E_{q+k}E_k \right] \right\} \quad (48)
\]

and the cancellation between \(p-h\) and \(p-\overline{p}\) is complete so that the sum rule vanishes indeed. We can also exactly evaluate the two contributions, getting

\[
\left[ (|q| - k_F)^2 + m^2 \right] \frac{q^2 + 3|q|k_F + k_F^2 + m^2}{20k_F^3|q|} N
\]

for the \(p-h\) term and the opposite for the \(p-\overline{p}\) one. The above expression is cumbersome and not enlightening. Two limits are more interesting: for small \(q\) we get

\[
\frac{q^2}{2\sqrt{k_F^2 + m^2}} N;
\]

if we remember that in the non relativistic limit \(k_F\) is negligible too when compared with \(m\), for small \(k_F\) (a quite reasonable limit, as said before) and without constraint over \(q\) we also get

\[
\frac{q^2}{2\sqrt{q^2 + m^2}} N.
\]

 Needless to say the \(p-\overline{p}\) contributions provide the same result up to a sign.

Now the physics is clear: the sum rule taken by itself (i.e. with no subtraction of the vacuum) is divergent. When the vacuum is subtracted (and hence the theory is renormalised) we also subtract the possibility of creating \(p-\overline{p}\) pairs in the vacuum. What the sum rule tells us is that the integrated energy-weighted strength of the \(p-h\) creation is just equal to the reduction of \(p-\overline{p}\) creation due to the presence of a nuclear medium that imposes, through the Pauli principle, that the emitted particle must have a momentum larger
than $k_F$ (Pauli blocking). Here we have explicitly evaluated $\Xi^{RFFG}_1$ because it clarifies a result that at a first sight could seem paradoxical. Of course at each perturbative order we still get $\Xi_1 = 0$, since this outcome was derived in full generality via WI.

Thus the sum rule maintains its validity in spite of the seemingly contradictory result. The key ingredient is the separation of the vacuum: this is something more than a renormalisation procedure: it also ensures, in fact, that the regularised part of $\Pi^{00}$, as well as of $\Pi^{33}$, conserves a behaviour $\sim q_0^{-2}$ at the infinity.

As a final comment, we want to outline the analogy between the effect described in this subsection and the creation of $e^+ e^-$ pairs in the electric field originated by a medium-heavy nucleus [15] or the so-called Darmstadt effect, where again an $e^+ e^-$ pair is created by heavy ions collision [16].

5 A Sum Rule for the Isovector Channel

The techniques developed so far can also be applied to the derivation of the $f'$ sum rule, i.e., the integrated response to an isospin probe.

Following the path of the previous sections, we introduce an isovector density of the form $\rho^i(x) = \psi^\dagger \tau^i \frac{1}{2} \psi$ and we get

$$\Xi^{T=1}_1(q) = \frac{1}{2} \langle \Psi_0 | [\tilde{\rho}^i(-q), [H, \tilde{\rho}^i(q)]] | \Psi_0 \rangle .$$

Clearly if $H = T + V$ and if the potential is isospin-independent, then $[V, \tilde{\rho}^i(q)] = 0$ and the sum rule comes back to the “canonical” value (1). If instead $V$ depends upon isospin, the sum rule is easily got by directly evaluating the commutator but is now model-dependent.

We wish to consider, instead, a model where the Meson Exchange Currents (MEC) are explicitly embodied in the Lagrangian.

To understand the qualitative changes brought in by the meson dynamics consider the Goldstone diagrams of fig. 3. Let us start from the diagram $a$: it represents the next-to-leading order correction in a potential theory (the horizontal dashed line represents an instantaneous interaction) and clearly it displays two energy denominators carrying the external incoming energy $\omega = q_0$. Thus its limit when multiplied by $\omega^2$ is finite. If we however consider the diagram $b$, where the dashed line is inclined to remind that a meson is exchanged with its delay effects (or its intrinsic energy dependence) then the
corresponding Goldstone diagrams contains three energy denominators, each one carrying one $\omega$ and its behaviour is now like $\omega^{-3}$ and gives no contribution to the sum rule. The same occurs for the energy-weighted sum rule for the spectral function and will be explained in more details in a forthcoming paper.

This simple analysis shows however that a dynamics for the exchanged mesons is required because it intrinsically alter the structure of the $f'$ sum rule. Let us in fact consider the following Lagrangian:

$$L = \psi^\dagger(x) \left\{ i \frac{\partial}{\partial x_0} + \frac{\nabla^2}{2m} \right\} \psi(x)$$

$$+ \frac{1}{2} [\partial_\mu \vec{\phi}(x)]^2 - \frac{\mu^2}{2} \vec{\phi}^2(x) + i \frac{f_\pi}{\mu} \psi^\dagger(x) \sigma \cdot \nabla \left( \vec{\tau} \cdot \vec{\phi} \right) \psi(x)$$

The field $\vec{\phi}$ being an isovector meson, three conserved currents exist, with 0-components

$$j_0^i(x) \equiv \rho^i(x) = \psi^\dagger(x) \frac{\tau^i}{2} \psi(x) + \epsilon^{ijk} \phi^j(x) \partial_0 \phi^k(x) \equiv \rho^i_N(x) + \rho^i_\pi(x)$$

the second term representing a necessary feature of the isospin charge. Physically it corresponds (for the $i = 3$ component) to the pion-in-flight term in the isovector part of the e.m. current. This term is often neglected in nuclear calculations (while usually the 3-vector part is fully accounted for).

Denoting with $H_{\text{int}}$ the $\pi N$ interaction term of the model, an explicit calculation of the double commutator eq. (4) provides the contribution (we
assume for simplicity that $|\Psi_0>$ is isoscalar)

$$-\frac{i}{3\mu} \int d^3x <\Psi_0|\psi^\dagger(x)\sigma \cdot \nabla \vec{\tau} \cdot \vec{\phi}(x)\psi(x)|\Psi_0> \delta_{ij}.$$ \hfill (53)

However, the pion current term also exist and thus other three terms need to be considered, with one or two nucleon currents replaced by the pionic ones. A simple (even if tedious) calculation show that the three remaining terms have the same structure and the same coefficient as (53), up to a sign, and all the four terms cancel out, giving the result

$$\frac{1}{2} <\Psi_0|\tilde{\rho}(-q), [H_{\text{int}}, \tilde{\rho}^j(q)]|\Psi_0> = 0,$$ \hfill (54)

and the sum rule regains the canonical value. This result was expected (at least formally) both because no $B^{\mu\nu}$ term is associated to the interaction Hamiltonians (that contains only one derivative), and thus, following our previous derivation no change was expected in the sum rule, and because the diagrams of fig. 3 already suggested us that the energy dependence of the meson exchange should kill further contributions coming from the interaction.

The term $1/2(\partial \phi^i_\mu)^2$ introduces a new feature in the model, that is absent when MEC are handled statically; namely the arising of a term $B^{\mu\nu}$ coming from the free pion Lagrangian. It corresponds to the tadpole displayed in fig. 4, and clearly will affect, via WI, the sum rule. The present model is however neither covariant nor renormalisable so that the polarisation propagator is meaningless. Thus in order to handle objects whose existence is ensured at any order let us replace the model (51) with a fully covariant one, namely

$$A = \int d^4x \bar{\psi}(i\partial - m)\psi + \frac{1}{2}(\partial_\mu \bar{\phi}) - \frac{\mu^2}{2} \phi^2 + ig\bar{\psi}\gamma_5 \psi \phi,$$ \hfill (55)

where $\psi$ is an isospin doublet.

We follow now the same path of sect. 3, replacing however (with obvious meaning of the symbols) eq. (18) with

$$A[\Lambda] = A|\Lambda=0 + \int dx j^{i\mu(x)}\partial_\mu \Lambda_i(x) + \int dx B^{ij\mu\nu}(x)\partial_\mu \Lambda_i(x)\partial_\nu \Lambda_j(x)$$ \hfill (56)

The associated isospin currents read

$$j^i_\mu = \bar{\psi} \gamma_\mu \gamma^i \psi + \epsilon^{ijk} \phi^j(x)\partial_\mu \phi^k(x)$$ \hfill (57)
and the associated “two-gauge boson term” is given by

$$B^{ij}_{\mu\nu}(x) = \frac{1}{2} g^{\mu\nu} \left\{ \frac{2}{3} \phi^2(x) \delta^{ij} - \left[ \phi^i(x) \phi^j(x) - \frac{1}{3} \phi^2(x) \delta^{ij} \right] \right\},$$

(58)
corresponding to the tadpole of fig. 4, and clearly affects, via WI, the sum rule.

Further, in the non-abelian case eq. (28) does not follow directly from (27) since now, in general, $[j_0^i, j_\nu^j]$ is not vanishing but is instead $\sim \epsilon_{ijk} j_\nu^k$. In the simple case of isosinglet ground state the average value of this contribution vanishes however. In this simplified situation also the isotensor term of $B^{ij}_{\mu\nu}$ is immaterial and we get for the WI the simplified expression

$$q^\mu \tilde{\Pi}^{ij}_{\mu\nu} = \frac{2}{3} q^\nu \delta^{ij} \int d^3x \langle \tilde{\phi}(x) \rangle^2 |\Psi_0\rangle,$$

(59)

where the last line holds provided the ground state is an isosinglet. With the same assumption on the ground state the isospin structure of $\tilde{\Pi}^{ij}_{\mu\nu}$ also simplifies to $\tilde{\Pi}^{ij}_{\mu\nu} \delta^{ij}$. Thus, neglecting isospin indices and following the same procedure as above, i.e., rewriting $\tilde{\Pi}^{00}$ in terms of $\Pi^{00}$ we end up with

$$q_0^2 \Pi_{00} + q_0^2 <\Psi_0|S|\Psi_0> - q^2 \Pi_{33} = (q_0^2 + q^2) \frac{2}{3} \int d^3x <\Psi_0|\tilde{\phi}(x)\rangle^2 |\Psi_0\rangle.$$

(60)
This result does not contain anymore the “canonical value” because of the cancellation described in the previous section but contains $S$. Taking the leading term in $q_0^2$ in (60) we can again derive the expression for it, namely

$$< \Psi_0 | S | \Psi_0 > = \frac{2}{3} \int d^3 x < \Psi_0 | [\phi^i(x)]^2 | \Psi_0 > .$$

(61)

Of course $S$, being a true Schwinger term, could also be written as a commutator (as done for instance in [6] and [4] chapt. 5.1.7). Note also that $S$ is non-vanishing because of the term proportional to $(\dot{\phi})^2$ in the lagrangian. It could be shown that, because of the existence of this term, the translation from the lagrangian to the hamiltonian formalism is not trivial and affects in particular $B^{\mu \nu}$ that is altered just by the amount $S$.

Note further that a subtraction of the vacuum is needed also for the Schwinger term, as well as for the rhs of eq. (61), making its expectation value finite in the medium.

Having determined $< \Psi_0 | S | \Psi_0 >$, we can cancel the highest order terms in $q_0^2$ in (60) and extend the energy-weighted sum rule to the isospin currents, with the nontrivial result

$$\Xi^i = \frac{1}{2} \lim_{q_0 \to \infty} q_0^2 \Pi_{00} = \frac{1}{3} q^2 \int d^4 x < \Psi_0 | [\tilde{\phi}(x)]^2 | \Psi_0 > = \frac{q^2}{2} < \Psi_0 | S | \Psi_0 > .$$

(62)

Of course one still needs to evaluate diagrammatically the last term, but we can also derive a non-perturbative result, namely that

$$\Xi^i(q) = \text{const} \times q^2 .$$

(63)

or, in other word, the constant is model-dependent but the functional dependence of the sum rule upon $q$ is fixed non-perturbatively to be $q^2$.

6 Conclusions

In conclusion we have obtained the following results:

1. We have re-derived the $f$ sum rule in such a way to connect it with the behaviour at infinity (in $\omega$) of the longitudinal polarisation propagator and then, via WI, to the “two-photon” term $B^{\mu \nu}$ of the photon scattering amplitude.
2. The $f$ sum rule, when extended relativistically vanishes due to the cancellation between $p\cdot h$ and $p\cdot \overline{p}$ contributions. The first term is responsible of the non-relativistic sum rule (and in fact the relativistic Free Fermi Gas calculation reproduces the limiting case for small $q$) while the second term completely cancel the former because the $p\cdot \overline{p}$ is inhibited by the Pauli blocking.

3. In the case of $f'$ sum rule a potential theory leads to the violation of the “canonical value”

$$\Xi_1 = \frac{q^2}{2m} A$$

but if we include the mesonic field in the lagrangian a further current is added and when the missing part of the current is accounted for, then we find again

$$< \Psi_0 | \left[ \tilde{\rho}^i(-q), \left[ H_{int}, \tilde{\rho}^i(q) \right] \right] | \Psi_0 > = 0.$$

4. In the $f'$ case however we need a fully relativistic theory. Here the non-perturbative conclusion we can draw is that

$$\Xi^i(q) = Sq^2,$$  \hspace{1cm} (64)

where $S$ is the Schwinger term (suitably renormalised) that is given by

$$S = \frac{2}{3} \int d^3x \,< \Phi_0 | \left[ \tilde{\phi}(x) \right]^2 | \Phi_0 > .$$  \hspace{1cm} (65)

Note that the Schwinger term arises when one requires the Lorentz covariance and thus is well defined only in a fully covariant model. This was not the case for the model (51) because there the lagrangian was not covariant and the definition of the Schwinger term was meaningless.

These conclusions still open new perspectives:

1. First of all it will be interesting, as previously mentioned, to reproduce these results in a Goldstone expansion scheme, that will clarify the rather formal aspect of the present paper.

2. New possibilities are opened, for instance the study of the sum rule in a parity violating response.
3. Finally, it seems to be exciting to analyse a symmetry group like $SU(2) \otimes SU(2)$ where the Adler anomaly breaks the symmetry at the first order in the loop expansion.

Before concluding, a last topic should be mentioned. The present paper makes an attempt to draw non-perturbative conclusions for the $f$ and $f'$ sum rules. In so doing we have discovered that this could be done only extending the integrations beyond the light cone (by the way, the same limitation affected also ref. [17]). Of course in comparing with experimental data the above is a serious disease. One could think maybe to $\vec{p}$ production in heavy ions collisions, in analogy with the Darmstadt effect, but we do not believe that data coming from so different frames could be seriously compared. Thus experimentally one is forcedly limited by the light cone. A series of papers [10, 11, 12] attempted to substantiate a set of sum rules in the space-like region. The algorithm proved to be stable with respect to different nuclear models but still was limited by the initial PWIA assumption. A further disease overcome by those paper was the interconnection, in a relativistic frame, of the nuclear and nucleon dynamics, that manifests itself in a dependence upon $k_F$ of the form factors: the counterpart in the present paper is that the renormalization procedure, and hence the dressing of the vertices, i.e. the introduction of the form factors, must be carried out contextually with the perturbative expansion.

A major point to be understood in this frame is how the cancellations between $p-h$ and $p-\vec{p}$ realize themselves. To reach this goal a perturbative analysis is required: it will show how, when some degrees of freedom are frozen, a relevant contribution to the sum rule comes from the tail: in fact in the above situation we implicitly fix the infinity below the threshold of the frozen degrees of freedom, and this in turn implies an Ansatz on the tail of the integrand in the sum rule. From our analysis this fact further implies an overestimate of the sum rule.

The above discussion seems to be rather irrelevant for the $f$ sum rule. For the $f'$ case instead one can consider an intermediate energy region (say, about 1 GeV for the transferred momentum) and there the kinetic energy term could just provide the “canonical” value, since the threshold for the frozen $\vec{p}$’s is fixed at about two GeV, while the cancellation eq. (54), that has its threshold at quite lower energies, seems to be reasonably ensured.
Thus in the region ranging from 1 to 2 GeV/c one could reasonably expect

\[ \Xi'(q) \simeq \left( \frac{1}{2m} + \sigma \right) Aq^2 \]  

where \( \sigma \) is an \textit{a priori} unknown parameter reasonably stable in a rather wide energy region.

**Acknowledgements**

Profs. A. Polls and A. Ramos are gratefully acknowledged for the many helpful discussions about this topic during my visits at the Dept. of Physics of the University of Barcelona. I wish also to thank Prof. G. Orlandini for her valuable suggestions and advices.
References

[1] P. Nozières and D. Pines. The Theory of Quantum Liquids. W. A. Benjamin, inc., New York, Amsterdam, 1966.

[2] G. Orlandini and M. Traini. Rept. Prog. Phys., 54:257, 1991.

[3] A. L. Fetter and J. D. Walecka. Quantum Theory of Many-Particle Systems. McGraw-Hill, New York, 1971.

[4] C. Itzykson and J. P. Zuber. Quantum Field Theory. McGraw-Hill Book co., Singapore, 1980.

[5] J. Schwinger. Phys. Rev. Letter, 3:296, 1959.

[6] L. S. Brown. Phys. Rew., 150:1338, 1966.

[7] D. J. Amit. Field Theory, the Renormalization Group, and Critical Phenomena. McGraw Hill, New York, 1978.

[8] Y. Takahashi. Quantum Field Theory. Elsevier Science Publishers, 1986.

[9] J. S. Levinger, M. L. Rustgi and K. Okamoto. Phys. Rev., 106:1191, 1957.

[10] R. Cenni, T. W. Donnelly and A. Molinari. Phys. Rev., C56:276, 1997.

[11] P. Amore, R. Cenni, T. W. Donnelly and A. Molinari. Nucl. Phys., A615:353, 1997.

[12] M. B. Barbaro, R. Cenni, A. De Pace, T.W . Donnelly and A. Molinari. Nucl. Phys., A643:137, 1998.

[13] W. M. Alberico, R. Cenni, A. Molinari and P. Saracco. Phys. Rev., C38:2389, 1988.

[14] T. W. Donnelly et al. Nucl. Phys., A541:525, 1992.

[15] E. I. Gol’braikh, A. I. L’vov and V. A. Petrun’kin. Sov. J. Nucl. Phys., 37:868, 1983.

[16] P. Kienle. Ann. Rev. Nucl. Part. Sci., 36:605, 1986.
[17] J. D. Walecka. *Nucl. Phys.*, A399:405, 1983.