Hermite-Halphen-Bloch solution of two-gap Lamé equation

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We construct the Hermite–Halphen–Bloch solution for the two-gap \((n = 2)\) Lamé equation of Jacobian form and give closed formulae to calculate the energy band dispersion relation.

The Jacobian form of the Lamé equation,[1]  
\[
\frac{d^2}{dx^2} + n(n+1)\kappa^2 \sin^2 x \psi(x) = \varepsilon \psi(x). \tag{1}
\]

has wide application in physics, where the Jacobi elliptic functions \(\sinh = \sin(x, \kappa), \cosh = \cosh(x, \kappa), \) and \(\text{dn} = \text{dn}(x, \kappa)\) are doubly-periodic functions in the complex plane, with modulus \(\kappa\) \((0 \leq \kappa \leq 1)\). The Lamé equation appears in a wide range of physics[2–9] . For example, the Gaussian (one-loop) fluctuations in the one-dimensional sine-Gordon model[10] and the \(\varphi^4\)-model[11] obey respectively 1-gap and 2-gap Lamé equations.

The solutions of Eq. (1) for positive integer \(n\) are given by  
\[
\psi(x) = \prod_{j=1}^{n} \left[ \frac{H(x + \alpha_j)}{\Theta(x)} e^{-\varepsilon \int \alpha_j} \right], \tag{2}
\]

which is referred to as a Hermite–Halphen solution[11,12]. Here \(\Theta, H \) and \(Z\) are Jacobi’s Theta, Eta and Zeta functions, with periods \(4K, 2K, 2K\), respectively, where \(K = K(\kappa)\) is the complete elliptic integral of the first kind. The complex parameters \(\alpha_1, \alpha_2, \ldots, \alpha_n\) are determined by the constraints equations,[1]  
\[
\varepsilon = \frac{1}{\sum_{j=1}^{n} \frac{(n\alpha_j)}{\Theta(x)^2}} = \left[ \sum_{j=1}^{n} \frac{\cosh \alpha_j \sinh \alpha_j}{\sinh \alpha_j} \right]^2, \tag{3}
\]

\[
\sum_{j=1}^{n} \frac{\sinh \alpha_j \cosh \alpha_j \sinh \alpha_p \sinh \alpha_p}{\sinh^2 \alpha_j - \sinh^2 \alpha_p} = 0 \quad (j \neq p). \tag{4}
\]

When \(x\) is restricted to real axis, the equation (1) is regarded as a Schrödinger equation for a particle moving in a one-dimensional potential, \(V(x) = \kappa^2 n(n+1) \sin^2 x\), which is bounded and periodic with its period being \(2K\). Therefore, Eq. (1) is a kind of Hill’s equation[13]. According to standard results on Hill’s equation, imposing a quasi-periodic boundary condition  
\[
\psi(x + 2K) = e^{-i(2K)k} \psi(x) = \xi \psi(x), \tag{5}
\]

with a real parameter \(k\) being fixed and a Floquet multiplier satisfying \(|\xi| = 1\), defines a self-adjoint boundary value problem and there exists the Bloch-wave solution with crystal momentum \(k\). For a positive integer \(n, V(x)\) is called the \(n\)-gap Lamé potential, because the Bloch spectrum consists of the \(n+1\) bands \(\varepsilon_1 \leq \varepsilon \leq \varepsilon_2, \varepsilon_3 \leq \varepsilon \leq \varepsilon_4, \ldots, \varepsilon_{2n+1} \leq \varepsilon \leq \infty\), i.e., there are \(n\) finite bands followed by a continuum band, separated by \(n\) forbidden lacuna. The Bloch-wave functions at the \(2n+1\) band edges \(\varepsilon_1, \ldots, \varepsilon_{2n+1}\) are periodic and anti-periodic depending of the Floquet multiplier being \(\xi = +1\) or \(\xi = -1\) and are represented in a form of polynomials (so called Lamé polynomials) in \(\sin, \cosh, \) and \(\sinh\) functions. Comparing the general solution (2) and the Bloch form (5), we find  
\[
k = -i \sum_{j=1}^{n} \frac{Z(\alpha_j)}{2K}. \tag{6}
\]

The expression for the momentum in the Weierstrass form of the Lamé equation is given in the Appendix A. The allowed energy bands correspond to a real value of the wavenumber \(k\), i.e., to the condition  
\[
\text{Re} \left[ \sum_{j=1}^{n} Z(\alpha_j, k) \right] = 0. \tag{7}
\]

Now, a full set of equations (2), (3), (4), (6), and (7) give the Bloch band dispersion relations, i.e., \(\varepsilon = \varepsilon(k)\).
The physical origin of the \( n \)-gap band structure is understood by observing,

\[
\kappa^2 \text{sn}^2 x = - \left( \frac{\pi}{2K} \right)^2 \sum_{\ell=-\infty}^{\infty} \text{sech}^2 \left[ \frac{\pi}{2K} (x - 2K\ell) \right] + \frac{E}{K},
\]

[ derivation of this formula is given in Appendix B]. This relation indicates that the Lamé potential consists of a periodic array of the modified Pöschl-Teller potentials form a lattice, the bound state overlaps and the energy band may be formed. Even after the band formation, the potentials form a lattice, the bound state overlaps and the energy band may be formed. Even after the band formation, the n gaps between the bound states and the scattering continuum retain. Therefore, the resulting band is split into the lower valence bands and the upper conduction band. The scenario for the case \( n = 1 \) has been discussed in the seminal work by Sutherland [10].

To obtain a closed analytic form of the Bloch wave solution in a Hermite–Halphen form (we call this Hermite–Halphen–Bloch solution), we need to specify the paths of the complex parameters \( \alpha_1, \alpha_2, \ldots, \alpha_\ell \) on a complex plane which satisfies the conditions \( 2), (3), (4), (6), and (7). The case of \( n = 1 \) has been well known [10] for an arbitrary \( 0 < \kappa \leq 1 \), but the extension to \( n > 1 \) is even numerically nontrivial. Fortunately, a recent paper by Maier [12] offers a method alternative to the Hermite–Krishever Ansatz. Based on this ansatz, Maier succeeded in obtaining the band dispersion relations for any integer \( n \) in terms of the \( n = 1 \) relations. However, in viewpoint of physical applications it may be still useful to seek a closed form of the Hermite–Halphen–Bloch solution for \( n > 1 \), which is still absent as far as the authors know. In this paper, we report on how to construct the Hermite–Halphen–Bloch solution for \( n = 2 \).

Based on the Hermite–Krishever Ansatz, which expresses a solution of the Lamé equation in terms of the known \( n = 1 \) solution, Maier [12] derived the spectral polynomial,

\[
L_2(\varepsilon|\kappa) = [\varepsilon^2 - 4(\kappa^2 + 1)\varepsilon + 12\kappa^2](\varepsilon - \kappa^2 - 1) \times (\varepsilon - 4\kappa^2 - 1)(\varepsilon - \kappa^2 - 4) = 0,
\]

whose roots

\[
\begin{align*}
\varepsilon_1 &= 2(\kappa^2 + 1) - 2\sqrt{\kappa^4 - \kappa^2 + 1}, \\
\varepsilon_2 &= \kappa^2 + 1, \\
\varepsilon_3 &= 4\kappa^2 + 1, \\
\varepsilon_4 &= \kappa^2 + 4, \\
\varepsilon_5 &= 2(\kappa^2 + 1) + 2\sqrt{\kappa^4 - \kappa^2 + 1},
\end{align*}
\]

give the energy eigenvalues at the band edges, i.e., three allowed bands consist of the first band \( \varepsilon_1 \leq \varepsilon \leq \varepsilon_2 \) \(|k| \leq \pi/2K\), the second band \( \varepsilon_3 \leq \varepsilon \leq \varepsilon_4 \) \((\pi/2K \leq |k| \leq \pi/K)\) and the third one \( \varepsilon_5 \leq \varepsilon \) \((\pi/K \leq |k|)\). The construction of the bands and the band edges values for the \( n = 2 \) Lamé equation in the Weierstrass form is given in the Appendix C.

Our goal is to determine the pathways of \( (\alpha_1, \alpha_2) \) on a complex plane which parametrize these three bands. Noting

\[
\text{Re}[Z(iy)] = 0, \quad \text{Re}[Z(K + iy)] = 0, \quad \text{Re}[Z(-x + iy) + Z(x + iy)] = 0,
\]

where \( x, y \in \mathbb{R} \) and taking heed that the zeta function \( Z(x) \) is a singly periodic function with the period \( 2K \), we can locate the following trajectories.

(1) For the first band \( \varepsilon_1 \leq \varepsilon \leq \varepsilon_2 \), we take

\[
\alpha_1 = K + iy_1, \quad \alpha_2 = K + iy_2,
\]

where \( (y_1, y_2) \) lie in a fundamental region \((-3K/2 < y_1 \leq -K, \ K < y_2 \leq 2K)\). The condition \( 4 \) becomes

\[
\frac{\text{sn}y_1}{y_1} + \frac{\text{sn}y_2}{y_2} = 0,
\]

where we used the notation \( \text{sn}y_1 = \text{sn}(y_1, \kappa) \) and alike. In Fig.(a), we explicitly show the pathways for \((\alpha_1 = K + iy_1, \alpha_2 = K + iy_2)\), and in Fig.(b) we show the corresponding trajectory of \((y_1, y_2)\). On this segment, the energy becomes

\[
\varepsilon = 2\kappa^2 + \kappa^2 \frac{\text{cn}^2 y_1}{y_1} + \kappa^2 \frac{\text{cn}^2 y_2}{y_2} + 2\kappa^4 \frac{\text{sn}y_1 \text{cn}y_1 \text{sn}y_2 \text{cn}y_2}{y_1 y_2} \frac{\text{dn}y_1}{y_1} \frac{\text{dn}y_2}{y_2},
\]

(21)
where \((y_1, y_2)\) lie in the fundamental region \(0 \leq y_1 \leq \bar{K}, 0 \leq y_2 \leq K/2\). The condition (11) becomes
\[
\kappa^2 \sin y_1 \sin y_2 = \frac{\sin y_2 \sin y_1}{\sin^2 y_2} \frac{\sin y_1}{\sin y_2},
\]

On this segment, the energy becomes
\[
\varepsilon = 2\kappa^2 + \kappa^2 \frac{\sin^2 y_1}{\sin^2 y_2} + \frac{\sin^2 y_2}{\sin^2 y_1} + 2\kappa^2 \frac{\sin y_1 \sin y_2 \sin y_1 \sin y_2}{\sin^2 y_1 \sin^2 y_2},
\]

Noting (28), we have
\[
\varepsilon = 2\kappa^2 + \kappa^2 \frac{\sin^2 y_1}{\sin^2 y_2} + \frac{\sin^2 y_2}{\sin^2 y_1} + 2\kappa^2 \frac{\sin y_1 \sin y_2 \sin y_1 \sin y_2}{\sin^2 y_1 \sin^2 y_2}.
\]

In Fig.2(a), we show the pathways for \((\alpha_1 = K + iy_1, \alpha_2 = iy_2)\), and in Fig.2(b) we show the corresponding trajectory of \((y_1, y_2)\). The band bottom \((k = \pi/2K)\) corresponds to \(y_1 = K\), \(y_2 = 0\) while the band top \((k = \pi/K)\) does to \(y_1 = 0, y_2 = 0\).

(3) For the third band \((\varepsilon_3 \leq \varepsilon \leq \varepsilon_4)\), we take
\[
\alpha_1 = -x + iy, \quad \alpha_2 = x + iy,
\]

where \((y_1, y_2)\) lie in the fundamental region \(0 \leq x \leq K/2, 0 \leq y \leq \bar{K})\). The condition (11) becomes
\[
\sin^2 y = \frac{\kappa^2 \sin^2 y_1 \sin^2 y_2 + \sin^2 x \sin^2 y - \cos^2 x \sin^2 y}{(\kappa^2 \sin^2 y_1 - \cos^2 x - \kappa^2 \sin^2 x) \sin^2 x}.
\]

FIG. 1: (a) The pathways for \(\alpha = K + iy_1\) and \(\alpha_2 = K + iy_2\). We choose \(\kappa^2 = 1/2\) to obtain these plot. (b) Trajectory of the point \((y_1, y_2)\).

where we used the condition (20).

The band bottom corresponds to
\[
\alpha_1 = K - i\alpha_0, \quad \alpha_2 = K + i\alpha_0,
\]

which gives
\[
k = -iZ(K - i\alpha_0) - iZ(K + i\alpha_0) + \frac{\pi}{K}
\]

\[= \frac{\pi}{K} = 0, \quad \text{(mod } \pi/K\text{)}\]

where we used
\[
Z(K + iy)
\]

\[= i \left( \frac{\sin y}{\sin^2 y} - \bar{Z}(y) - \frac{\pi}{2K} y - \kappa^2 \frac{\sin y}{\cos y} \frac{\sin y}{\sin^2 y} \right).\]

Here, \(\bar{Z} = Z(\bar{k})\) and \(\alpha_0\) is determined by (11) and (21), i.e.,
\[
\varepsilon_1 = 2\kappa^2 + 2\kappa^2 \frac{\sin^2 c_0}{\cos c_0} - 2\kappa^2 \frac{\sin^2 c_0 \sin^2 c_0}{\sin^2 c_0} = 2\kappa^2 + 2\kappa^2 \sin^2 c_0,
\]

that gives the location
\[
\bar{k}^2 \sin^2 c_0 = 1 - \sqrt{\kappa^4 - \kappa^2 + 2}.
\]

The band top \(\varepsilon_2\) is given by \(\alpha_1 = K - i\bar{k}, \alpha_2 = K + 2i\bar{k}\), which actually gives \(k = \pi/2K, \varepsilon = \varepsilon_2\).

(2) For the second band \((\varepsilon_3 \leq \varepsilon \leq \varepsilon_4)\), we take
\[
\alpha_1 = K + iy_1, \quad \alpha_2 = iy_2,
\]
In Fig. 3 we show the pathways for \((\alpha_1 = -x + iy, \alpha_2 = x + iy)\). On this segment,

\[
\varepsilon = 2\kappa^2 + \text{dn}^2\alpha_1 + \text{dn}^2\alpha_2 - 2\frac{\text{cn}\alpha_1 \text{dn}\alpha_1 \text{cn}\alpha_2 \text{dn}\alpha_2}{\text{sn}\alpha_1 \text{sn}\alpha_2}
\]

(33)

FIG. 3: Relevant branches of \(\alpha_1 = -x + iy\) and \(\alpha_2 = x + iy\) which properly reproduces the third band. The case \(\kappa^2 = 1/2\) is presented.

The band bottom \((k = \pi/K)\) corresponds to

\[
\alpha_1 = -x_0, \; \alpha_2 = x_0,
\]

(34)

where \(x_0\) is determined by (33) and (15), i.e.,

\[
\text{dn}^2x_0 + \frac{\text{cn}^2x_0 \text{dn}^2x_0}{\text{sn}^2x_0} = 1 + \sqrt{\kappa^4 - \kappa^2} + 1
\]

(35)

and the band top \((k \to \infty)\) is given by \(\alpha_1 = \alpha_2 = iK\), because \(Z(\alpha)\) and \(\text{dn}(\alpha)\) have a pole at \(\alpha = iK\).

Using the obtained results, we enable to compute the Bloch band dispersion for an arbitrary \(0 \leq \kappa \leq 1\). In Fig. 4 we show the results. It is clearly seen that as the \(\kappa\) approaches unity the lower two bands become flatter and finally utterly flat at the limit of \(\kappa \to 1\). This phenomena is easily understood as follows. As \(\kappa\) increases from 0 to 1, the period of the Lamé potential \(2K\) increases from \(\pi\) to \(\infty\). Therefore, the overlap between the modified Pöschl-Teller potentials [see Eq. (9)] becomes smaller. Consequently, two bound states originating from an independent modified Pöschl-Teller potential with \(n = 2\) become more localized. On the other hand, for smaller \(\kappa\), the overlap becomes large and the bound states form energy bands with a larger band width. The connection between these results and that of obtained in Ref. [10] in terms of Weierstrass elliptic functions is discussed in Appendix D.

In summary, in this paper, we succeeded in constructing the Hermite–Halphen–Bloch solution for \(n = 2\) Lamé equation and obtained closed formulae which give the band dispersion relation. From a physical viewpoint, these dispersions give fluctuation spectra around the soliton lattice solution of the classical \(\phi^4\) -field theory. We hope our results may be useful to promote physical analysis related with this model.

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\text{dn}^2x_0 + \frac{\text{cn}^2x_0 \text{dn}^2x_0}{\text{sn}^2x_0} = 1 + \sqrt{\kappa^4 - \kappa^2} + 1
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FIG. 4: The Bloch band dispersions for various values of \(\kappa\).

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Appendix A: The momentum in the Weierstrass form of the Lamé equation

Henceforth, we use the notations of the paper [16], where the potential $U(x) = -u(n+1)\wp(ix + \omega)$, determined via the Weierstrass elliptic function $\wp$, has the period $2|\omega'|$, the real and the imaginary half periods are $\omega$ and $\omega'$, respectively.

The fundamental solution of the Lamé equation with $n = 2$ in the Weierstrass form is given by (see Ref. [11])

$$
\Lambda_1(ix + \omega) = \prod_{r=1}^{2} \left\{ \frac{\sigma(t_r + ix + \omega)}{\sigma(ix + \omega)\sigma(t_r)} \right\} \times \exp\left\{ -i\omega \sum_{r=1}^{2} \zeta(t_r) \right\}, \quad (A1)
$$

where $\sigma$ and $\zeta$ are the Weierstrass’s sigma and zeta functions, respectively.

The factor $\exp\left\{ -i\omega \sum_{r=1}^{2} \zeta(t_r) \right\}$ is the constant and it can be dropped that gives

$$
\Lambda_1(ix + \omega) = \prod_{r=1}^{2} \left\{ \frac{\sigma(t_r + ix + \omega)}{\sigma(ix + \omega)\sigma(t_r)} \right\} \times \exp\left\{ -ix \sum_{r=1}^{2} \zeta(t_r) \right\}. \quad (A2)
$$

To convert the solution into the Bloch form $\Lambda_1(ix + \omega) = u(x)e^{-ikx}$, where the function $u(x)$ has the period of the potential $u(x + 2|\omega'|) = u(x - 2i\omega') = u(x)$, the constants $\alpha_r$ are introduced

$$
\Lambda_1(ix + \omega) = \prod_{r=1}^{2} \left\{ \frac{\sigma(t_r + ix + \omega)}{\sigma(ix + \omega)\sigma(t_r)} \right\} \times \exp\left\{ -ix \sum_{r=1}^{2} (\zeta(t_r) - \alpha_r) \right\} = u_r(x)\exp\{ -ikx \}. \quad (A3)
$$

By using the property

$$
\sigma(x + 2\omega') = -e^{2\eta_2(x + \omega')}\sigma(x), \quad (A4)
$$

where the constant $\eta_2 = \zeta(\omega')$, we obtain the periodicity $u_r(x - 2i\omega') = u_r(x)$ provided

$$
\alpha_r = \frac{\eta_2}{2}\omega't_r. \quad (A5)
$$
Thereby, the momentum is given by

\[ k = \sum_{r=1}^{2} (\zeta(t_r) - \alpha_r) = \sum_{r=1}^{2} \left( \zeta(t_r) - \frac{n_r}{\omega_t} t_r \right). \]  

(A6)

**Appendix B: Derivation of Eq. (8)**

We start with the Fourier series for the Zeta function

\[ Z(x) = \frac{\pi}{K} \sum_{n=1}^{\infty} \frac{\sin(n \pi x / K)}{\sinh(n \pi K / K)}. \]  

(B1)

and obtain

\[ \kappa^2 \sin^2 x = 1 - \frac{E}{K} - (\pi / K)^2 \sum_{n=1}^{\infty} f(x, n), \]  

(B2)

where

\[ f(x, n) = \frac{n \cos(n \pi x / K)}{\sinh(n \pi K / K)}. \]  

(B3)

Noting \( \lim_{n \to 0} \left[ n \cos(n \pi x / K) / \sinh(n \pi K / K) \right] = K / (\pi K) \) and the Legendre’s identity \( K E + KE - K K = \pi / 2 \), we have

\[ \kappa^2 \sin^2 x = \frac{E}{K} - \frac{\pi^2}{2K^2} \sum_{n=0}^{\infty} f(x, n). \]  

(B4)

The 2nd term on the r.h.s. is computed by using the Poisson summation formula,

\[ S(x) = \sum_{n=-\infty}^{\infty} f(x, n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, \zeta) e^{-2\pi i m \zeta} d\zeta. \]  

(B5)

The integral is evaluated as

\[ I(x) = \int_{-\infty}^{\infty} f(x, \zeta) e^{-2\pi i m \zeta} d\zeta = \oint_{C} f(x, z) e^{-2\pi i m \zeta} d\zeta, \]  

(B6)

where \( C \) is a upper and lower semicircle on the complex \( z \)-plane for \( m < 0 \) and \( m > 0 \), respectively [see Fig. 5].

Picking up residues at the poles of \( f(x, z) \), \( z \ell = (iK/K) \ell \), we obtain

\[ \oint_{C} f(x, z) e^{-2\pi i m z} dz = \frac{1}{2} \left( \frac{K}{X} \right)^2 \sum_{\ell=-\infty}^{\infty} \text{sech}^2 \left( \frac{\pi}{2K} (x - 2K \ell) \right). \]  

(B7)

Plugging this result into Eq. (B4), we arrive at Eq. (8).

**Appendix C: The spectrum of the Lamé equation in the Weierstrass form**

The Lamé equation in the Weierstrass form

\[ \frac{d^2 \Lambda}{du^2} = \{ n(n + 1) \hat{\vartheta}(u) + B \} \Lambda \]  

(C1)

has two independent solutions

\[ \Lambda_{1,2} = \sqrt{X} \exp \left\{ Q \int \frac{du}{X} \right\}, \]  

(C2)

where \( Q \) is the Wronskian, and \( X \) is the product of the pair, \( X = \Lambda_1 \Lambda_2 \), obeying the equation

\[ \frac{d^3 X}{du^3} - 4 \{ n(n + 1) \hat{\vartheta}(u) + B \} \frac{dX}{du} - 2n(n + 1) \hat{\vartheta}'(u) X = 0. \]  

(C3)
For the case $n = 2$ the form of $X$ can be taken as
\[ X(u) = C_0 \varphi^2(u) + C_1 \varphi(u) + C_2, \]  
(C4)
where $C_i$ are the constants, that corresponds to the solution in descending powers of $\varphi(u) - e_2$ (see Ref. [1]).

By finding the coefficients of the same powers of $\varphi$ and excluding the higher-order derivatives through the identities 
\[ \Phi'' = 6\Phi' - \frac{1}{2}g_2, \quad \Phi''' = 12\Phi'' + 9g_2, \tag{C8} \]
and by calculating with the aid of Eq. (C4)
\[ \partial_x X = 2C_0 \varphi \partial_x \varphi + C_1 \partial_x \varphi, \tag{C6} \]
where $g_2$ is the invariant, we obtain from Eq. (C5)
\[
-\partial_x^2 X + 4(6\varphi(ix + \omega) + E) \partial_x X + 12\partial_x \varphi (ix + \omega) X = 0,
\]
(C5)

where we use the identities $\varphi + E = -E$ and $\varphi (u) = -\varphi (ix + \omega)$, where $E$ is the energy.

By finding the coefficients of the same powers of $\varphi$ and $\varphi'$ we get the relations $C_1 = -(E/3) C_0$ and $C_2 = -(E/3) C_1 - (C_0/4) g_2$.

The choice $C_0 = 18$ of the Ref. [10] results in
\[ X = 18\varphi^2(ix + \omega) - 6E \varphi (ix + \omega) + 2E^2 - \frac{9}{2}g_2. \tag{C11} \]
The polynomial can be presented as
\[ X = 18 \left( \varphi (ix + \omega) - \varphi (t_1) \right) \left( \varphi (ix + \omega) - \varphi (t_2) \right), \tag{C12} \]
where
\[ \varphi (t_1) = \frac{E}{6} + \frac{1}{2} \sqrt{3g_2 - E^2}, \tag{C13} \]
\[ \varphi (t_2) = \frac{E}{6} - \frac{1}{2} \sqrt{3g_2 - E^2}. \tag{C14} \]
give the parametric form for the spectrum of the Lamé equation with $n = 2$ (see Ref. [10]).

To find the band edges we note that the phase factor in the solution (C2) turns into zero at the band edges, what is equivalent to the requirement $Q = 0$.

Given $X$, the Wronskian can be found through the relation
\[ n(n + 1) \varphi (u) + B = \frac{1}{2X du^2} - \frac{1}{4X^2} \left( \frac{dX}{du} \right)^2 + \frac{Q^2}{X^2}, \tag{C15} \]
which takes the form
\[ Q^2 = -4 \{ 6\varphi(ix + \omega) + E \} X^2 + 2X \partial_x^2 X - (\partial_x X)^2 \tag{C16} \]
in the notions of Ref. [10].

By using the result (C11), we obtain after simplification
\[ Q^2 = - (E^2 - 3g_2) (16E^3 - 36Eg_2 + 108g_3), \tag{C17} \]
where we use the identities $e_1 e_2 + e_2 e_3 + e_1 e_3 = -g_2/4$ and $e_1 e_2 e_3 = g_3/4$. Here, $g_3$ is the invariant.

This yields the band edges
\[ E_1 = - \sqrt{3g_2}, \quad E_2 = -3e_1, \]
\[ E_3 = -3e_2, \quad E_4 = -3e_3, \quad E_5 = \sqrt{3g_2} \tag{C19} \]
as given in Ref. [10].

Appendix D: The connection between Weierstrass's and Jacobi's forms of the solutions

The relationship between the results (11, 12, 13, 14, 15) and (19) is reached via the formula
\[ \varphi (u) = e_3 + (e_1 - e_3) \operatorname{ns}^2 \left( u \sqrt{e_1 - e_3} \right). \tag{D1} \]
The Jacobi’s elliptic function having its modulus given by the equation
\[ \kappa^2 = \frac{e_2 - e_3}{e_1 - e_3}. \tag{D2} \]
The semiperiods $\omega_{1,2}$ of the Weierstrass functions are related with $K$ and $\bar{K}$ by
\[ \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_2 = i \frac{\bar{K}}{\sqrt{e_1 - e_3}}. \tag{D3} \]
Let us compare, for instance, the results for the spectrum with $\kappa^2 = 1/2$, when $K = \bar{K}$.

Given $\omega_1 = K$, we get from Eq. (D3) $e_1 - e_3 = 1$.

Then, as follow from Eq. (D2), $e_2 - e_3 = \kappa^2 = 1/2$.

By using the identity $e_1 + e_2 + e_3 = 0$, we find

$$e_1 = \frac{1}{3}(2 - \kappa^2) = \frac{1}{2}, \quad (D4)$$

$$e_2 = -\frac{1}{3}(1 - 2\kappa^2) = 0, \quad (D5)$$

$$e_3 = -\frac{1}{3}(1 + \kappa^2) = -\frac{1}{2}. \quad (D6)$$

Therefore, the invariant $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 1$.

According to Ref. [16], the width of the first band is [see Eq. (C19) in Appendix B]

$$\sqrt{3g_2 - 3e_1} \approx 0.23 \quad (D7)$$

that coincides with the result followed from Eqs. (11,12)

$$2\sqrt{\kappa^4 - \kappa^2 + 1 - (\kappa^2 + 1)} \approx 0.23. \quad (D8)$$

The value of the first gap in the Weierstrass form is given by $3(e_1 - e_2) = 1.5$ that agrees with the Jacobian’s result $3\kappa^2 = 1.5$, See Eqs. (12,13).

By similar way, we determine the width of the second band in the Weierstrass form, $3(e_2 - e_3) = 1.5$, and get the same result in the Jacobi’s form, $3(1 - \kappa^2) = 1.5$, as predicted by Eqs. (13,14).

At last, in the Weierstrass form the second gap equals,

$$\sqrt{3g_2 + 3e_3} \approx 0.23, \quad \text{that is in an utter accordance with the result of Eqs. (14,15)}, \quad \kappa^2 - 2 + 2\sqrt{\kappa^4 - \kappa^2 + 1} \approx 0.23.$$