UPPER BOUNDS FOR BERGMAN KERNELS ASSOCIATED TO POSITIVE HERMITIAN LINE BUNDLES WITH SMOOTH METRICS

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Abstract. Off-diagonal upper bounds are established away from the diagonal for the Bergman kernels associated to high powers $L^\lambda$ of holomorphic line bundles $L$ over compact complex manifolds, asymptotically as the power $\lambda$ tends to infinity. The line bundle is assumed to be equipped with a Hermitian metric with positive curvature form, which is $C^\infty$ but not necessarily real analytic. The bounds are of the form $\exp(-h(\lambda) \sqrt{\lambda \log \lambda})$ where $h \to \infty$ at a non-universal rate. This form is best possible.

1. Introduction

1.1. The setting. Let $X$ be a compact complex manifold, without boundary. Let $X$ be equipped with a $C^\infty$ Hermitian metric $g$, along with the metrics on the bundles $B^{(p,q)}(X)$ of forms of bidegree $(p,q)$ induced by $g$, and the volume form on $X$ associated to the induced Riemannian metric. Denote by $\rho(z,z')$ the Riemannian distance from $z \in X$ to $z' \in X$.

Let $L$ be a positive holomorphic line bundle over $X$. Let $L$ be equipped with a $C^\infty$ Hermitian metric $\phi$ whose curvature is positive at every point. $\phi$ is not assumed to be real analytic.

For each positive integer $\lambda$, let the line bundle $L^\lambda$ be the tensor product of $\lambda$ copies of $L$. $L^\lambda$ inherits from $\phi$ a Hermitian metric in a natural way; if $v \in L_z$ then the $\lambda$-fold tensor product $v \otimes v \otimes \cdots \otimes v$ satisfies $|v \otimes v \otimes \cdots \otimes v| = |v|^\lambda$.

Let $L^2_\lambda = L^2(X, L^\lambda)$ be the Hilbert space of equivalence classes of all square integrable Lebesgue measurable sections of $L^\lambda$. Likewise there are the Hilbert spaces $L^2(X, B^{(0,q)} \otimes L^\lambda)$. Let $H^2_\lambda$ be the closed subspace of $L^2_\lambda$ consisting of all holomorphic sections. The Bergman projection is defined to be the orthogonal projection $B_\lambda$ from $L^2_\lambda$ onto $H^2_\lambda$. The Bergman kernel $B_\lambda(z,z')$ is the associated distribution-kernel; $B_\lambda(z,z')$ is a complex linear endomorphism from the fiber $L^2_{z'}$ to the fiber $L^2_z$.

A great deal is known concerning the nature of these Bergman kernels. In particular, detailed asymptotic expansions are known near the diagonal $z = z'$, that is, when $\rho(z,z')$ is bounded by a constant multiple of $\lambda^{-1/2}$. See for instance [1, 6, 20, 23] as well as the related work [5] of Boutet de Monvel and Sjöstrand on the Bergman and Szegö kernels associated to domains in $\mathbb{C}^{n+1}$. This paper is concerned with upper bounds when $z, z'$ are far apart, that is, behavior for large $\lambda$ when $\rho(z,z')$ is bounded below by a positive quantity independent of $\lambda$. If $\phi$ and $g$ are real analytic, then

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for large $\lambda$, $|B_\lambda(z, z')| \leq C_\delta e^{-c_\delta \lambda}$ whenever $\rho(z, z') \geq \delta > 0$, where $C_\delta < \infty$ and $c_\delta > 0$ are independent of $\lambda$. This is interpreted in the theory of Bleher, Shiffman and Zelditch \cite{3, 16} of random zeroes of sections of $L^\lambda$ as an exponentially small upper bound on the degree of correlation between zeros at distinct points.

1.2. **Subexponential off-diagonal decay.** It was shown in \cite{9} that this exponential decay fails to hold, in general, if $\phi$ is merely infinitely differentiable. More quantitatively, for any function $h$ satisfying $h(t) \to \infty$ as $t \to +\infty$ there exists \cite{9} an example for which

$$\limsup_{\lambda \to \infty} \sup_{\rho(z, z') \geq \delta} e^{h(\lambda) \sqrt{\lambda \log \lambda}} |B_\lambda(z, z')| = \infty$$

for all $\delta > 0$. In this paper we establish an upper bound which dovetails with these lower bounds.

**Theorem 1.** Let $L$ be a positive holomorphic line bundle over a compact complex manifold $X$. Let there be given a $C^\infty$ positive metric on $L$ with strictly positive curvature form, and a $C^\infty$ Hermitian metric on $X$. For any $\delta > 0$ there exist $\Lambda < \infty$ and a function $h$ satisfying $h(\lambda) \to \infty$ as $\lambda \to \infty$ such that for all $z, z' \in X$ satisfying $\rho(z, z') \geq \delta$,

$$|B_\lambda(z, z')| \leq e^{-h(\lambda) \sqrt{\lambda \log \lambda}} \quad \text{for all } \lambda \geq \Lambda.$$  

The analysis below of $B_\lambda$ is based on its connection with the fundamental solution of a partial differential operator, $\Box_\lambda$. Denote by $\bar{\partial}^*_\lambda$ the usual Dolbeault operator, mapping sections of $B^{(0,q)} \otimes L^\lambda$ to sections of $B^{(0,q+1)} \otimes L^\lambda$. Denote by $\bar{\partial}^*_\lambda$ its formal adjoint, with respect to the Hilbert space structures $L^2_\lambda$ defined above. Define

$$\Box_\lambda = \begin{cases} \bar{\partial}^*_\lambda \bar{\partial}^*_\lambda + \bar{\partial}^*_\lambda \partial^*_\lambda & \text{for } n > 1, \\ \bar{\partial}^*_\lambda \partial^*_\lambda & \text{for } n = 1, \end{cases}$$

acting on sections of $B^{(0,1)} \otimes L^\lambda$. For each $\lambda$, $\Box_\lambda$ is an elliptic second-order linear system of partial differential operators with $C^\infty$ coefficients. When it is expressed in local coordinates, its coefficients are $O(\lambda^2)$ in any $C^N$ norm.

Because the metric $\phi$ is positive, there exists a constant $c > 0$ such that for all sufficiently large $\lambda \in \mathbb{N}$,

$$\langle \Box_\lambda u, u \rangle \geq c\lambda \|u\|^2_{L^2}$$

for all twice continuously differentiable sections $u$ of $B^{(0,1)} \otimes L^\lambda$. This bound is deduced from a well-known integration by parts calculation \cite{13}. Because of this lower bound and because $\Box_\lambda$ is formally self-adjoint and elliptic, there exists a unique self-adjoint bounded linear operator $G_\lambda$ on $L^2(X, B^{(0,1)} \otimes L^\lambda)$ satisfying $\Box_\lambda \circ G_\lambda = I$, the identity operator.

The operator $B_\lambda$ is related to $\Box_\lambda$ by

$$B_\lambda = I - \bar{\partial}^*_\lambda \circ G_\lambda \circ \bar{\partial}^*_\lambda.$$  

Thus the Bergman kernel is expressed in terms of certain derivatives of the distribution-kernel for the operator $G_\lambda$. We denote this distribution-kernel by $G_\lambda(z, z')$. Because
G_λ(z, z') is a solution of \( \Box_G \lambda = 0 \) with respect to the variable \( z \) and its complex conjugate is a solution of the same equation with respect to \( z' \), elliptic regularity theory guarantees that \( G_\lambda(z, z') \) is a \( C^\infty \) function of \( (z, z') \) on the complement of the diagonal.

We will show that \( G_\lambda(z, z') = O(e^{-h(\lambda)\sqrt{\lambda\log \lambda}}) \) for \( (z, z') \) at any positive distance from the diagonal. The corresponding bound holds for those partial derivatives that express the distribution-kernel for \( \partial_\lambda \circ G_\lambda \circ \bar{\partial}_\lambda \) at \( (z, z') \) will be an easy consequence.

For real analytic metrics, the Bergman kernel is \( O(e^{-c\lambda}) \) away from the diagonal. Combining the result established here with that of [9], one knows that for \( C^2 \) metrics, decay can in some instances be essentially as slow as \( e^{-h(\lambda)\sqrt{\lambda\log \lambda}} \), but is never slower. Zelditch has raised the question of which, or what, behavior is typical, and whether properties of the metric can be inferred from the off-diagonal decay rate of the associated Bergman kernels. This issue will be examined in [10].

### 1.3. Orientation

A weaker upper bound \(|B_\lambda(z, z')| \leq e^{-c\sqrt{\lambda}}\), valid whenever \( \rho(z, z') \geq \delta \), is a simple consequence of (1.4), and requires only \( C^2 \) or even \( C^{1,1} \) regularity of \( \phi \).

In the context of global analysis on \( \mathbb{C}^1 \), this was shown in [9]. For positive line bundles over complex manifolds, it was noted by Berndtsson [2]. Closely related results are found in works of Delin [11] and Lindholm [15]. The novelty in Theorem 1 is a double improvement of the exponent, from \( c\sqrt{\lambda} \) to \( h(\lambda)\sqrt{\lambda\log \lambda} \).

To establish the weaker bound, consider any real-valued auxiliary weight \( \psi \in C^2(X) \). For any \( \varepsilon > 0 \) and all sufficiently large \( \lambda \),

\[
|B_\lambda(z, z')| \leq e^{-c\sqrt{\lambda}}
\]

for all sections \( u \in C^2(X, B^{(0,1)}) \), where \( C \) depends on the \( C^2 \) norm of \( \psi \). This is \( \geq \|u\|^2 \) for all sufficiently large \( \lambda \), provided that \( \varepsilon \) is chosen to be sufficiently small as a function of \( \|\psi\|_{C^2} \). The inequality (1.6) can alternatively be interpreted as a weighted inequality for the inverse operator \( \Box_\lambda^{-1} \), with weight \( e^{2\varepsilon\sqrt{\lambda\psi}} \). Whenever \( U, U' \) are disjoint sets satisfying distance \( (U, U') \geq \delta > 0 \), by choosing \( \psi \) so that \( \psi \geq 1 \) on \( U' \) and \( \psi \leq 0 \) on \( U \) we conclude that \( \Box_\lambda^{-1} \) maps \( L^2(U) \) to \( L^2(U') \), where these norms are defined without reference to the auxiliary weight \( \psi \), with operator norm \( O(e^{-c\sqrt{\lambda}}) \).

### 2. Variants

Here are two variants of Theorem 1 concerning metrics \( \phi \) with more limited regularity. For simplicity we continue to assume that the underlying Hermitian/Riemannian metric on \( X \) itself is \( C^\infty \).
**Theorem 2.** Let \( n \geq 1 \), let \( X \) be a compact complex manifold, and let \( L \) be a positive holomorphic line bundle over \( X \). Let \( \delta > 0 \). There exist a positive integer \( N_0 \) and \( \eta > 0 \) such that for every \( N \geq N_0 \) and every \( C^N \) metric on \( L \) with positive curvature there exists \( \Lambda \) such that

\[
|B_\lambda(z, z')| \leq e^{-\eta \sqrt{N \lambda \log \lambda}},
\]

whenever \( \rho(z, z') \geq \delta \) and \( \lambda \geq \Lambda \).

A natural question is whether the indicated dependence on \( N \), as \( N \to \infty \), is optimal. The construction in [9] could be used to obtain a bound in the opposite direction; we have not reexamined the details to determine whether it shows that the bound obtained here is optimal. A proof of Theorem 2 is implicit in the proof given below of Theorem 1; it is simply a matter of tracing the dependence on \( N \) of the auxiliary parameter \( A \) introduced in that proof.

The following variant is not proved in this paper, but could be established by augmenting the method used here with a more precise version of Lemma 6, below. Such a refinement can be established by arguments closely related to those in [8]. For \( \alpha \in (0, 1) \) let \( C^{2, \alpha} \) denote the class of all \( C^2 \) functions whose second order partial derivatives are all Hölder continuous of order \( \alpha \).

**Claim.** Let \( n \geq 1 \) and \( \alpha \in (0, 1) \). Let \( L \) be a holomorphic line bundle over a compact complex manifold \( X \) of complex dimension \( n \). Let there be given a positive metric of regularity class \( C^{2, \alpha} \) on \( L \), and a \( C^\infty \) Hermitian metric on \( X \). Then for each \( \delta > 0 \) there exist \( \Lambda < \infty \) and \( \eta > 0 \) such that for all \( \lambda \geq \Lambda \) and for any open sets \( U, U' \subset X \) satisfying \( \rho(U, U') \geq \delta \), for any section \( f \in L^2(X, L^\lambda) \) supported in \( U' \),

\[
\|B_\lambda f\|_{L^2(U)} \leq e^{-\eta \sqrt{\lambda \log \lambda}} \|f\|_{L^2}.
\]

These \( L^2 \) norms are computed with respect to the weight \( \phi \). The proof uses Taylor expansion of degree 2, rather than of high degree.

### 3. Unweighted bounds and twisted operators

It will be convenient to work in an equivalent framework, in a coordinate patch in \( X \), in which \( L \) is trivial and norms are defined by integrals without \( \lambda \)-dependent weights, but the underlying operators \( \bar{\partial}_\lambda, \Box_\lambda \) are twisted. This framework is more natural for discussion of regularity.

Let \( U \) be a small coordinate patch on \( X \), over which \( L \) may be identified with \( \mathbb{C} \). Functions and differential forms may be regarded as scalar–valued. For each \( q \), \( \bar{\partial}_\lambda \), mapping sections of \( B^{(0, q)} \otimes L^\lambda \) over \( U \) to sections of \( B^{(0, q+1)} \otimes L^\lambda \) over \( U \), is naturally identified with the standard Cauchy-Riemann operator \( \bar{\partial} \), which maps sections of \( B^{(0, q)} \) to sections of \( B^{(0, q+1)} \).

\( \phi \in C^\infty \) is \( \mathbb{R} \)-valued, and the positive curvature assumption means precisely that its complex Hessian matrix \( (\partial^2 \phi/\partial z_j \partial \bar{z}_k) \) is strictly positive definite at each point of \( U \). The \( C^\infty \) Hermitian metric \( g \) given for \( X \) is interpreted as a \( C^\infty \) Hermitian metric on \( U \), and gives rise to a volume form, expressed as a measure \( \mu \) on \( U \), which is a smooth nonvanishing multiple of Lebesgue measure on \( \mathbb{C}^n \). It also gives rise, for each \( q \), to a \( C^\infty \) metric on \( B^{(0, q)} \) over \( U \). The \( L^2 \) norm squared of a section of \( B^{(0, q)} \) over \( U \),
regarded as a scalar-valued function $f$, is expressed as $\int_U |f(z)|^2 e^{-2\lambda\phi(z)} \, d\mu(z)$, where $|f(z)|$ is measured according to $g$.

Substituting $fe^{-\lambda\phi} = u$, the norm squared of $f$ with respect to the weight $\phi$ becomes $\|f\|_2^2 = \int_U |u(z)|^2 \, d\mu(z)$; there is no weight in this integral. Moreover

$$e^{-\lambda\phi} \overline{\partial} f = e^{-\lambda\phi} \overline{\partial} (ue^{\lambda\phi}) = \overline{\partial} u + \lambda au$$

where $a = \overline{\partial} \phi \in C^\infty$. For each $\eta$ define

$$\overline{D}_\lambda = e^{-\lambda\phi} \circ \overline{\partial} \circ e^{\lambda\phi} = \overline{\partial} + \lambda a \cdot.$$

This is a first-order linear partial differential operator with smooth coefficients, but with a zero-th order term proportional to the large parameter $\lambda$. The formal adjoint(s) $\overline{D}_\lambda^*$ are defined with respect to the given metric $g$ and associated volume form. These data are assumed to be only $C^\infty$, rather than $C^\omega$, but their potential lack of analyticity is less significant than that of $\phi$ because they are not multiplied by the large parameter $\lambda$.

Define

$$\Delta_\lambda = \begin{cases} \overline{D}_\lambda \overline{D}_\lambda^* + \overline{D}_\lambda^* \overline{D}_\lambda & \text{for } n > 1, \\ \overline{D}_\lambda \overline{D}_\lambda^* & \text{for } n = 1, \end{cases}$$

acting on $(0,1)$ forms over $U$. Under these identifications,

$$\Delta_\lambda = e^{-\lambda\phi} \circ \Box_\lambda \circ e^{\lambda\phi}.$$

The function

$$G_\lambda(z,w) = e^{-\lambda\phi(z) + \lambda\phi(w)} G_\lambda(z,w)$$

represents a fundamental solution for $\Delta_\lambda$ with pole at $w$, in the usual sense. This is a section of the complex endomorphism bundle of $B^{(0,1)}$ over $U \times U$ minus the diagonal; in this local coordinate system, it is a matrix-valued function. Its size $|G_\lambda(z,w)|$ is defined with respect to given smooth metrics which do not depend on $\lambda$, so upper bounds with respect to these metrics are uniformly equivalent to upper bounds with respect to the standard metrics on these bundles.

Theorem 1 is therefore equivalent to an upper bound for all $(z,w)$ in $U \times U$ minus the diagonal of the form

$$|G_\lambda(z,w)| \leq e^{-\frac{A}{\sqrt{\lambda}} \log \lambda} \quad \text{for all } \lambda \geq \Lambda(\delta,A),$$

with corresponding upper bounds for all first and second-order derivatives of $G_\lambda$ with respect to $z,w$ in this same region.

4. A NEAR-DIAGONAL UPPER BOUND

Theorem 1 which is concerned with the nature of $G_\lambda$ far from the diagonal, will be derived from a description of $G_\lambda$ much nearer the diagonal. The main point is the manner in which the bounds depend on $\lambda,A$; these bounds are completely independent of the exponent $A$, provided only that $\lambda$ exceeds a certain threshold, which does depend on $A$. The reasoning below will require bounds for derivatives of $G_\lambda$, as well as for $G_\lambda$ itself. These bounds are more naturally expressed in terms of
the twisted kernels $G_{\lambda}$ introduced above. $\nabla$ will denote the gradient in $\mathbb{C}^n \times \mathbb{C}^n$, with respect to both coordinates $z, z'$.

**Proposition 3.** There exist $c_0, A_0 \in \mathbb{R}^+$ such that for any $A \in [A_0, \infty)$ there exists $\Lambda = \Lambda(A) < \infty$ such that for any $\lambda \geq \Lambda$ and any $z, z' \in U$ satisfying

\begin{align}
A_0 \lambda^{-1/2} \log \lambda \leq |z - z'| \leq A \lambda^{-1/2} \log \lambda, \tag{4.1}
\end{align}

$G_{\lambda}(z, z')$ satisfies

\begin{align}
|G_{\lambda}(z, z')| + |\nabla_{z, z'} G_{\lambda}(z, z')| \leq e^{-c_0 \lambda |z - z'|^2}. \tag{4.2}
\end{align}

Here $\operatorname{op}$ denotes the operator norm on the Hilbert space $L^2(X, B^{(0,1)} \otimes L^\lambda)$.

As is well understood, there is a natural scale $\asymp \lambda^{-1/2}$ inherent in this situation. In the model situation in which $X = \mathbb{C}^n$ and $\phi(z) \equiv \frac{1}{2} |z|^2$, $|G_{\lambda}(z, z')| \asymp e^{-c_0 \lambda |z - z'|^2} |z - z'|^{2 - 2n}$ for $n > 1$, with the power of $|z - z'|$ replaced by $\log(1/|z - z'|)$ for $n = 1$. Proposition 5 asserts essentially that this model upper bound persists up to a distance which is greater by a multiplicative factor of $A \sqrt{\log \lambda}$ than the natural scaled distance, for arbitrarily large $A$. The lower bound $|z - z'| \geq A_0 \lambda^{-1/2} \sqrt{\log \lambda}$ is an inessential technicality introduced in order to simplify the statement and proof of the lemma; otherwise the upper bound would have to be modified in order to take the near-diagonal factor $|z - z'|^{2 - n}$ into account.

In the next section we will show how Theorem 1 is an essentially formal consequence of Proposition 3. We will then review and establish foundational results, none of which involve significant novelty, before finally proving the Proposition.

## 5. The Near-Diagonal Bound Implies the Far-From-Diagonal Bound

$\|T\|_{\operatorname{op}}$ will denote the operator norm of $T$, as an operator on $L^2(X, B^{(0,1)} \otimes L^\lambda)$. Recall that $\rho$ denotes the Riemannian distance function on $X^2$. The following obvious statement is at the heart of the construction.

**Lemma 4.** Let $T_1, T_2$ be bounded linear operators on $L^2(X, B^{(0,q)} \otimes L^\lambda)$. Let $r_i > 0$ and suppose that for $i = 1, 2$, the distribution-kernel associated to $T_i$ is supported in $\{(z, z') \in X^2 : \rho(z, z') \leq r_i\}$. Then the distribution-kernel associated to $T_1 \circ T_2$ is supported in $\{(z, z') \in X^2 : \rho(z, z') \leq r_1 + r_2\}$.

This will be used to prove:

**Lemma 5.** Let $A < \infty$ and $\delta > 0$. There exist $C < \infty$ and $\Lambda < \infty$ such that for every $\lambda \geq \Lambda$ there exists a bounded linear map $T$ from the space of $L^2$ sections of $B^{(0,1)} \otimes L^\lambda$ to itself with these two properties: Firstly, the distribution-kernel for $T$ is supported in $\{(z, z') : \rho(z, z') \leq \delta\}$. Secondly,

\begin{align}
\|T \circ \square_{\lambda} - I\|_{\operatorname{op}} \leq e^{-A \lambda^{1/2} \sqrt{\log \lambda}}. \tag{5.1}
\end{align}

**Proof.** Choose an auxiliary function $\eta \in C^\infty([0, \infty))$ that satisfies $\eta(x) \equiv 1$ for $x \leq \frac{1}{2}$, and $\eta(x) \equiv 0$ for all $x \geq 1$. Let $A < \infty$. Let $P$ be the operator with distribution-kernel

\[ K(z, w) = G_{\lambda}(z, w) \eta(A^{-2} \lambda (\log \lambda)^{-1} \rho^2(z, w)). \]
Thus so that for all sufficiently large \(\lambda\), Proposition 3 holds. In this region, according to Proposition 3, 

\[ |G_\lambda(z, w)| + |\nabla G_\lambda(z, w)| \leq C\lambda^C e^{-c\lambda A^2 \log \lambda} \leq C\lambda^{C-cA^2}. \]

So in all, 

\[ |\Box_\lambda (K(z, w) - G_\lambda(z, w))| \leq \lambda^{C-cA^2} \]

for all sufficiently large \(\lambda\), uniformly for all pairs \((z, w)\) in \(X^2\) minus the diagonal. Since \(\Box_\lambda \circ G_\lambda = I\), this is an upper bound for the operator norm of \(\Box_\lambda \circ P - I\). Since both \(\Box_\lambda\) and \(P\) are formally self-adjoint, the same bound holds for \(P \circ \Box_\lambda - I\).

Given \(\delta > 0\), choose \(N\) to be the largest integer such that \(NA\lambda^{-1/2} \log \lambda^{1/2} \leq \delta\). Thus

\[ N \approx A^{-1} \lambda^{1/2} (\log \lambda)^{-1/2}. \]

Set 

\[ E = I - \Box_\lambda \circ P \quad \text{and} \quad T = P \circ \sum_{j=0}^{N-1} E^j \]

so that 

\[ \Box_\lambda \circ T = I - E^N. \]

Because the distribution-kernel for \(P\) is supported where \(\rho(z, w) \leq A\lambda^{-1/2} \sqrt{\log \lambda}\), the distribution-kernel for \(T\) is supported where 

\[ \rho(z, w) \leq NA\lambda^{-1/2} \sqrt{\log \lambda} \leq \delta, \]

according to Lemma 4.

Since \(\|E\|_{op} = \|\Box_\lambda \circ P - I\|_{op} \leq \lambda^{C-cA^2}\), 

\[ \|E^N\|_{op} \leq \lambda^{(C-cA^2)N} \leq \lambda^{(C-cA^2)A^{-1} \lambda^{1/2} (\log \lambda)^{-1/2} \delta} \leq e^{-c' A \lambda^{1/2} \sqrt{\log \lambda}} \]

for all sufficiently large \(A\). \(\square\)

**Proof of Theorem 1.** Consider any \(z' \neq z'' \in X\). To prove the upper bound for \(B_\lambda(z', z'')\), consider any \(L^2\) section \(f\) of \(B^{(0,1)} \otimes L^\lambda\) that is supported in \(B'' = B(z'', \frac{A}{2} \rho(z', z''))\) and satisfies \(\|f\|_{L^2} \leq 1\). Choose \(T\) as in Lemma 3 with distribution-kernel supported within distance \(\frac{1}{2} \rho(z', z'')\) of the diagonal. Then in \(B' = B(z', \frac{1}{4} \rho(z', z''))\),

\[ G_\lambda f = T \Box_\lambda G_\lambda f + O\left(e^{-A\sqrt{\lambda \log \lambda}} \|G_\lambda f\|\right) = Tf + O\left(e^{-A\sqrt{\lambda \log \lambda}} \|f\|\right). \]

Since \(T\) has distribution-kernel supported within distance \(\frac{1}{2} \rho(z, z')\) of the diagonal, \(Tf \equiv 0\) in \(B'\). Therefore

\[ G_\lambda f = O\left(e^{-A\sqrt{\lambda \log \lambda}} \|f\|\right) \text{ in } L^2(B') \text{ norm}. \]

Thus as an operator from \(L^2(B'')\) to \(L^2(B')\), \(G_\lambda\) has operator norm \(O(e^{-A\sqrt{\lambda \log \lambda}})\). Because \(G_\lambda(z, w)\) is a solution of elliptic linear partial differential equations with
$C^\infty$ coefficients with respect to both variables $z, w$, and because the coefficients of those equations are $O(\lambda^2)$ in every $C^N$ norm, it follows from standard bootstrapping arguments that for any $N$, $G_\lambda \in C^N(B' \times B'')$, with norm $O(e^{-A\sqrt{\log \lambda}})$. Since the Bergman kernel is the distribution-kernel for $I - \bar{\partial}_\lambda G_\lambda \partial_\lambda$, this result with $N = 2$ includes the desired upper bound.\[\square\]

6. Off-the-shelf upper bounds

6.1. Low regularity upper bounds. Thus far the argument has been purely formal. We now state two quantitative estimates on which the proof of Proposition 3 will rely. One concerns metrics with nearly minimal regularity; the other, real analytic metrics. The $C^\infty$ case is intermediate between the two.

**Lemma 6.** For each $n \geq 1$ there exists $N < \infty$ with the following property. Let $L$ be a positive holomorphic line bundle over a compact complex manifold $X$ of dimension $n$, equipped with a Hermitian metric $\phi$ of class $C^N$. Assume that likewise that $X$ is equipped with a Hermitian metric $g$ of class $C^N$. Let $U, G_\lambda$ be as defined above. Then there exists $C < \infty$ such that for all sufficiently large positive integers $\lambda$,

$$|G_\lambda(z, z')| + |\nabla G_\lambda(z, z')| \leq (\lambda + |z - z'|^{-1})^C$$

for all $z \neq z' \in U$.

Here $\nabla$ denotes the gradient with respect to both variables $z, z'$.

Considerably sharper upper bounds can be established, but they will not be needed in the proof of Theorem [8].

**Proof.** The fact that integration with respect to $G_\lambda$ defines a bounded operator on $L^2(U, B^{(0,1)})$, uniformly in $\lambda$ can be interpreted as a weak a priori upper bound for $G_\lambda$, as for instance in [8]. Routine localization and bootstrapping arguments, exploiting the ellipticity of $\Delta_\lambda$ and the $O(\lambda^2)$ bounds for its coefficients, lead directly to (6.1). $\square$

6.2. High regularity upper bounds. We work now in the unweighted twisted framework introduced above. Let $B \subset \mathbb{C}^n$ be any fixed open ball of positive radius, and let $\tilde{B} \subset B$ be an arbitrarily compact subball.

**Lemma 7.** Let the ball $B \subset \mathbb{C}^n$ be equipped with a $C^\omega$ Hermitian metric $g$. Let $L$ be any holomorphic line bundle over $B$, equipped with a positive $C^\omega$ Hermitian metric $\phi$. There exist $\Lambda < \infty$ and $c > 0$ such that for any $\lambda \geq \Lambda$ and any solution $u$ of $\Delta_\lambda u \equiv 0$ on $B$

$$|u(z)| \leq e^{-c\lambda}\|u\|_{L^2(B)}$$

for all $z \in \tilde{B}$.

Moreover, given a family of such metrics $g, \phi, c$ may be taken to be independent of $g, \phi$, provided that $g, \phi$ are uniformly $C^\omega$ and that the metrics $\phi$ are uniformly positive.

Positivity of $\phi$ means that in local coordinates, $\sum_{i,j=1}^n \frac{\partial^2 \phi(z)}{\partial z_i \partial \bar{z}_j} \bar{\zeta}_i \zeta_j \geq a|\zeta|^2$ for all $\zeta \in \mathbb{C}^n$ and all $z$, for some $a > 0$. We say that a family of metrics $\phi$ is uniformly positive if $a$ is bounded below by some positive constant uniformly for all elements.
of the family in question. Likewise we say that such a family is uniformly $C^\omega$ if there exists $C < \infty$ for which

$$
\left| \frac{\partial^\alpha \phi}{\partial (z, \bar{z})^\alpha} \right| \leq C^{1+|\alpha|} |\alpha|!
$$

uniformly on $B$ for every multi-index $\alpha$ and all metrics $\phi$. The same applies to $g$.

**Proof of Lemma 7.** This is a consequence of a fundamental result on analytic hypoellipticity of related subelliptic partial differential equations. Consider first the case $n > 1$. Work in $B \times \mathbb{R}$ with coordinates $(z, t)$, and set $U(z, t) = u(z)e^{i\lambda t}$. Then

$$
e^{i\lambda t} \Delta_\lambda u(z) = \bar{\partial}_b U(z, t),$$

where $\bar{\partial}_b$ is a Cauchy-Riemann operator associated to a strictly pseudoconvex CR structure on $B \times \mathbb{R}$; $\Delta_\lambda$ is related to the Kohn Laplacian $\square_b$ for this CR structure by the corresponding equation

$$
e^{i\lambda t} \Delta_\lambda u(z) = \square_b U(z, t).$$

For $n > 1$, $\square_b$ is analytic hypoelliptic on $(0, 1)$ forms, for any $C^\omega$, strictly pseudoconvex CR structure. Proofs of this and/or closely related results can be found in [7], [12], [18], [19], [22]. Identifying $B \subset \mathbb{C}^n$ with a ball in $\mathbb{R}^{2n}$, we regard $B \times \mathbb{R}$ as a subset of $\mathbb{C}^{2n+1}$, hence as a totally real submanifold of $\mathbb{C}^{2n+1}$. Any real analytic function of $(z, t) \in B \times \mathbb{R}$ thus extends holomorphically to a neighborhood in $\mathbb{C}^{2n+1}$.

Analytic hypoellipticity of $\square_b$ implies such extension, in a quantitative sense: there exist a complex neighborhood $\Omega$ of $\breve{B} \times [-1, 1]$ and a constant $C < \infty$ such that any bounded solution $U$ of $\square_b U = 0$ in $B \times (-2, 2)$ extends to a bounded holomorphic function in $\Omega$, and moreover,

$$
sup_\Omega |U| \leq C \sup_{B \times (-2, 2)} |U|.$$

By analytic continuation, any holomorphic extension of $u(z)e^{i\lambda t}$ must take the product form $\tilde{u}(z)e^{i\lambda t}$. For positive $\lambda$ we then set $t = -i$ to deduce that

$$
sup_B |u(e^{\lambda}) \leq C \sup_B |u|.$$

An examination of any of the proofs [18] [19] [22] of analytic hypoellipticity of $\square_b$ confirms that these provide uniform upper bounds, given uniform upper bounds on the coefficients of $\bar{\partial}_b$ in some fixed coordinate patch, and on the Hermitian metric used to define $\bar{\partial}_b^*$, and given that the hypothesis of strict pseudoconvexity holds in a uniform way. In our setting, the latter amounts to uniform strict positivity of the metric $\phi$.

The case $n = 1$ requires an alternative treatment, because $\square_b = \bar{\partial}_b \bar{\partial}_b^*$ fails to be analytic hypoelliptic for three-dimensional CR manifolds. Instead, a variant of analytic hypoellipticity holds in two alternative (but equivalent) forms. One of these

\[1\] The other alternative asserts that $u$ is $C^\omega$, microlocally outside a conic neighborhood of one of the two ray bundles whose union is the characteristic variety of $\bar{\partial}_b$. This implies holomorphic extendibility to an appropriate wedge, and the above reasoning may then be repeated to gain the factor $\exp(-c\lambda)$. 


asserts that if \( \bar{\partial} \partial^\ast U = 0 \) then
\[
(6.4) \quad \sup_{\Omega} |\bar{\partial}^\ast U| \leq C \sup_{B \times (-2,2)} (|U| + |\bar{\partial}^\ast U|),
\]
with the same type of uniform dependence of the constant \( C \) on the data as for \( n > 1 \). Together with the reasoning above, this yields the conclusion
\[
(6.5) \quad \sup_{\tilde{B}} |D^\lambda u| \leq e^{-c\lambda} \sup_{B} (|u| + |D^\lambda u|).
\]

The bound for \( u \) itself now follows from Lemma 8 below. \( \Box \)

The justification of the above form of analytic hypoellipticity rests on several facts and results, combined according to an outline introduced by Kohn [14] for the analysis of related questions concerning (weakly) pseudoconvex three-dimensional CR manifolds. Denote by \( \Box = \bar{\partial}_b \partial^\ast b \) the Kohn Laplacian over a strictly pseudoconvex three (real) dimensional CR manifold \( M \). Assume that \( \Box u \in C^\omega \) in an open set.

(i) The analytic wave front set of \( u \) is contained in the characteristic variety of \( \Box \).

(ii) This characteristic variety is a real line bundle over \( M \), thus a union of two ray bundles.

(iii) In a conic neighborhood of one of these two ray bundles, \( \bar{\partial}_b \) is of principal type and satisfies (microlocally) the Poisson bracket hypothesis which ensures analytic hypoellipticity [21], and therefore is microlocally analytic hypoelliptic. The microlocal version of this theorem of Treves follows for instance by the techniques in [17]. Consequently since \( \bar{\partial}_b (\bar{\partial}^\ast_b u) \in C^\omega \), the analytic wave front set of \( \bar{\partial}^\ast_b u \) is disjoint from this ray bundle.

(iv) In a conic neighborhood of the complementary ray bundle, \( \Box \) has double characteristics and satisfies the hypotheses of the theorem of Sjöstrand [18]; see also [12] where more degenerate operators are analyzed by the same techniques. Therefore the analytic wave front set of \( u \), and hence also the analytic wave front set of \( \bar{\partial}^\ast_b u \), are disjoint from this ray bundle.

(v) If a distribution has empty analytic wave front set, then it is analytic.

(vi) These steps can be made quantitative, where appropriate, to justify the stated uniformity.

6.3. Exponential localization for a first-order equation.

**Lemma 8.** Let \( n \geq 1 \). Let \( U, U' \) be open subsets of \( X \) with \( U \subseteq U' \). There exists \( c > 0 \) such that for all sufficiently large \( \lambda \geq 0 \), and all \( u \in C^1(U') \),
\[
\|u\|_{L^2(U)} \leq C\|D^\lambda u\|_{L^2(U')} + Ce^{-c\lambda}\|u\|_{L^2(U')}.
\]

**Proof.** It suffices to show that for each \( z_0 \in U \), there exists a neighborhood \( V \) of \( z_0 \) such that \( \|u\|_{L^2(V)} \) satisfies the required upper bound. In a small open set, represent \( D^\lambda = -e^{\lambda\phi}(\partial + a)e^{-\lambda\phi} \) where \( a \in C^\infty \). In a sufficiently small neighborhood it is possible to solve \( \partial \alpha = a \) and thus to write \( D^\lambda = -e^{-\alpha}e^{\lambda\phi}\partial e^{-\lambda\phi}e^\alpha \). Since multiplication by \( e^{\pm\alpha} \) preserves \( L^2 \) norms up to a bounded factor, it suffices to prove the inequality with \( \alpha \equiv 0 \).
It is possible to write, for all $z, w$ in a sufficiently small neighborhood of $z_0$,
\[ \phi(w) = \psi(z, w) + \varphi(z, w) \]
where $\psi, \varphi$ are $C^\infty$ functions, $\varphi(z, w)$ is an antiholomorphic function of $w$ for each $z$, and
\[ \text{Re} \left( \psi(z, w) \right) \geq \text{Re} \left( \psi(z, z) \right) + c|z - w|^2 \]
for a certain constant $c > 0$. Such a decomposition is obtained by exploiting the Taylor series of order 2 for $\phi$ at $z$. Then for each $z$, when acting on functions of $w$,
\[ D^*_\lambda u(w) = -e^{\lambda\psi(z,w)} \left( \partial e^{-\lambda\psi(z, \cdot^*)} \right) u(w). \]

Let $\eta \in C^\infty(X)$ be a function supported in a neighborhood of $z_0$ which is contained in a coordinate patch contained in a relatively compact subset of $U'$, within which the above expression for $\phi$ is valid; and $\eta$ is identically equal to one in a smaller neighborhood. Then $\eta u$ can be regarded as a function defined on $\mathbb{C}^1$. Let $v = D^*_\lambda (\eta u) = \eta D^*_\lambda u - u\partial \eta$.

Since
\[ \partial_w e^{-\lambda\psi(z,w)} (\eta u)(w) = -e^{-\lambda\psi(z,w)} v(w) \]
is a compactly supported continuous function defined on $\mathbb{C}^1$, for each $z$ sufficiently close to $z_0$ one may recover $\eta(z)u(z) = u(z)$ by
\[ u(z) = -c_0 \int_{\mathbb{C}^1} v(w)(\bar{z} - \bar{w})^{-1} e^{\lambda(\psi(z,z) - \psi(z,w))} dm(w) \]
where $m$ denotes Lebesgue measure on $\mathbb{C}^1$ and $c_0$ is a certain constant. Now
\[ |e^{\lambda(\psi(z,z) - \psi(z,w))}| = e^{\lambda|\text{Re} \left( \psi(z,z) - \psi(z,w) \right)|} \leq e^{-c|w - z|^2}. \]
Therefore
\[ |u(z)| \leq C \int_{\mathbb{C}^1} |z - w|^{-1} |v(w)| e^{-c|w - z|^2} dm(w) \]
\[ \leq C \int_{\mathbb{C}^1} \left( |\eta(w)u(w)| + |u(w)\partial \eta(w)| \right) |z - w|^{-1} e^{-c|w - z|^2} dm(w) \]
Since $|z - w|$ is bounded below by a positive quantity uniformly for all $z$ in $U$ and $w$ in the support of $\nabla \eta$, the required bound follows. \(\square\)

7. PROOF OF PROPOSITION 3

7.1. Globalization. We introduce a variant situation in which $X$ is replaced by $\mathbb{C}^n$ and sections of $B^{(0,1)} \otimes L^\lambda$ over $X$ are replaced by sections of $B^{(0,1)}(\mathbb{C}^n)$ over $\mathbb{C}^n$. This variant will facilitate $\lambda$-dependent coordinate changes to be made below.

Let $\varepsilon > 0$ be given. Let $U$ be a relatively compact open subset of a coordinate patch in $X$. Fix a holomorphic coordinate system on that coordinate patch, and express $D^*_\lambda = e^{-\lambda\phi} \partial e^{\lambda\phi}$ where $\phi \in C^\infty$ is $\mathbb{R}$-valued, and satisfies the positivity hypothesis
\[ \left( \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j} \geq c(\delta_{i,j})_{i,j} \]
in the sense of Hermitian forms.

Sections of $L^2$ over $U$ are thus identified with $\mathbb{C}$-valued functions in such a way that the $L^2$ norm squared, over $U$, of such a section can be expressed as $\int_U |f(z)|^2 \alpha(z) \, d\mu(z)$ where $\mu$ is Lebesgue measure on $\mathbb{C}^n$, $\alpha \in C^\infty(\mathbb{C}^n)$ is bounded above in $C^N$ norm for all $N$ by constants independent of $\lambda, z', z$, and $\alpha(z)$ is positive and bounded below by a positive constant independent of $\lambda, z, z'$. Extend $\alpha$ to a strictly positive $C^\infty$ function $\tilde{\alpha}$ on $\mathbb{C}^n$, still with uniform upper and lower bounds. Likewise extend $\gamma$ to a $C^\infty$ Hermitian metric on $\mathbb{C}^n$, independent of $\lambda$. Assign to $(0, k)$ forms $f$ defined on $\mathbb{C}^n$ the $L^2$ norm squared $\int_{\mathbb{C}^n} |f(z)|^2 \tilde{\alpha}(z) \, d\mu(z)$ where $|f(z)|$ is measured using this extension of $\gamma$.

Fix an auxiliary function $\eta \in C_0^\infty(\mathbb{C}^n)$, supported in $\{z : |z| < 4\}$ and satisfying $\eta(z) \equiv 1$ for $|z| \leq 2$. For each $z'$ in a fixed relatively compact subset $U \Subset U'$, make the affine coordinate change

$$B \times U \ni (\zeta, z') \mapsto (z, z') = (z' + \zeta, z') \in U \times U,$$

where $B$ is the ball of radius $\varepsilon_0$ centered at $0 \in \mathbb{C}^n$. In these coordinates, $z'$ is the origin, $\zeta = 0$. We will work in the variable $z \in B$, suppressing $z'$ in the notation; all estimates will be uniform in $z' \in U$, as the proof will show.

Let $Q_2$ be the Taylor polynomial of degree 2 for $\phi$ at $\zeta = 0$. Define

$$\tilde{\phi}(\zeta) = Q_2(\zeta) + \eta(\varepsilon_0^{-1})|\phi(\zeta) - Q_2(\zeta)|.$$

Consider the modified operator $e^{-\lambda \tilde{\phi}} \partial_i e^{\lambda \tilde{\phi}}$, which agrees with $e^{-\lambda \phi} \partial_i e^{\lambda \phi}$ for all sufficiently small $\zeta$, but has the advantage of being defined globally for $\zeta \in \mathbb{C}^n$. For sufficiently large $\lambda$,

$$\nabla^2 \tilde{\phi}(z) - \nabla^2 \phi(0) = O(\varepsilon_0)$$

uniformly for all $z \in \mathbb{C}^n$. Therefore it is possible to choose $\varepsilon_0 > 0$ sufficiently small that for all sufficiently large $\lambda$, the quadratic form defined by $(\partial^2 \tilde{\phi}(z)/\partial z_i \partial z_j)_{i,j=1}^n$ is bounded below by a strictly positive constant, independent of $z$ and $\lambda$. This holds uniformly in $z' \in U$. Choose and fix such a value of $\varepsilon_0$.

Consider the associated operator defined for $n > 1$ by

$$\tilde{\Delta}_\lambda = (e^{-\lambda \tilde{\phi}} \partial_i e^{\lambda \tilde{\phi}})(e^{-\lambda \phi} \partial_i e^{\lambda \phi})^* + (e^{-\lambda \tilde{\phi}} \partial_i e^{\lambda \tilde{\phi}})^*(e^{-\lambda \phi} \partial_i e^{\lambda \phi}),$$

and for $n = 1$ by

$$\tilde{\Delta}_\lambda = (e^{-\lambda \tilde{\phi}} \partial_i e^{\lambda \tilde{\phi}})(e^{-\lambda \phi} \partial_i e^{\lambda \phi})^*,$$

where adjoints are interpreted with respect to the Hilbert space structure on $L^2(\mathbb{C}^n)$ introduced above.

For $n > 1$, for all sufficiently large $\lambda$, a well-known computation based on integration by parts \[13\] gives

$$\langle \tilde{\Delta}_\lambda u, u \rangle \geq c\lambda \|u\|_{L^2}^2$$

for all twice continuously differentiable and compactly supported $(0, 1)$ forms $u$, where $c > 0$ is a positive constant.

For $n = 1$, for all sufficiently large $\lambda$,

$$[e^{-\lambda \tilde{\phi}} \partial_i e^{\lambda \tilde{\phi}}, (e^{-\lambda \phi} \partial_i e^{\lambda \phi})^*] \geq c\lambda I,$$
in the sense of operators on $L^2(\mathbb{C}^n)$ with respect to the same Hilbert space structure. Consequently, (7.2) also holds for $n = 1$.

Since $\tilde{\Delta}_\lambda$ is a formally self-adjoint operator, it follows that there exists a bounded linear operator $\tilde{G}_\lambda$ from $L^2(\mathbb{C}^n, B^{(0,1)})$ to itself such that $\tilde{\Delta}_\lambda \circ \tilde{G}_\lambda$ is the identity operator on $L^2(\mathbb{C}^n, B^{(0,1)})$, and the operator norm of $\tilde{G}_\lambda$ is $O(\lambda^{-1})$ for all sufficiently large $\lambda$.

This inverse is bounded in $L^2$ operator norm, uniformly for all sufficiently large $\lambda$, provided that $\varepsilon_0$ is kept fixed. Lemma 6 also applies to this situation, so the distribution-kernel $\tilde{G}_\lambda(z, 0)$ for $\tilde{G}_\lambda$ with pole at $\zeta = 0$ satisfies

$$\tag{7.4} |\tilde{G}_\lambda(z, 0)| \leq (\lambda + |z|^{-1})^C$$

for all sufficiently large $\lambda$, and the same holds for all of its partial derivatives. These bounds are uniform in $\lambda$ provided that $\lambda$ is sufficiently large.

7.2. Gauge change. Denote by $p$ the harmonic part of the Taylor polynomial of $\tilde{\phi}$ of degree 2 at $w = 0$. That is, expand

$$\tilde{\phi}(z) = \tilde{\phi}(0) + \operatorname{Re} \left( \sum_{k=1}^{n} \alpha_k z_k + \sum_{i,j=1}^{n} \beta_{i,j} z_i z_j \right) + \sum_{i,j=1}^{n} \gamma_{i,j} z_i \bar{z}_j + O(|z|^3),$$

and set

$$p(z) = \tilde{\phi}(0) + \operatorname{Re} \left( \sum_{k=1}^{n} \alpha_k z_k + \sum_{i,j=1}^{n} \beta_{i,j} z_i z_j \right).$$

Denote by $\tilde{p}$ the real-valued harmonic conjugate of $p$, normalized to vanish at 0. Then $[\tilde{\phi}, e^{\lambda p + i \tilde{p}}] = \tilde{\phi}(p + i \tilde{p}) \equiv 0$ and consequently

$$\tag{7.5} e^{-\lambda \tilde{\phi}} \bar{\partial} e^{\lambda \tilde{\phi}} = e^{i \lambda \tilde{p}} e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} e^{-i \lambda \tilde{\phi}}.$$

Likewise

$$\left( e^{-\lambda \tilde{\phi}} \bar{\partial} e^{\lambda \tilde{\phi}} \right)^* = \left( e^{i \lambda \tilde{p}} e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} e^{-i \lambda \tilde{\phi}} \right)^* = e^{i \lambda \tilde{p}} \left( e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} \right)^* e^{-i \lambda \tilde{\phi}}$$

and consequently

$$e^{-\lambda \tilde{\phi}} \bar{\partial} e^{\lambda \tilde{\phi}} \left( e^{-\lambda \tilde{\phi}} \bar{\partial} e^{\lambda \tilde{\phi}} \right)^* + \left( e^{-\lambda \tilde{\phi}} \bar{\partial} e^{\lambda \tilde{\phi}} \right)^* e^{-\lambda \tilde{\phi}} \bar{\partial} e^{\lambda \tilde{\phi}}$$

$$= e^{i \lambda \tilde{p}} \left( e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} \left( e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} \right)^* + \left( e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} \right)^* e^{-\lambda (\tilde{\phi} - p)} \bar{\partial} e^{\lambda (\tilde{\phi} - p)} \right) e^{-i \lambda \tilde{\phi}}.$$

Hence upon replacement of $\tilde{\phi}$ by $\tilde{\phi} - p$ in the definition of $\Box_\lambda$, a unitarily equivalent operator on $L^2(\mathbb{C}^n, B^{(0,1)})$ is obtained. Moreover, the absolute value of the distribution-kernel for the inverse of this unitarily equivalent operator is identically equal to $|\tilde{G}_\lambda|$. In deriving upper bounds for $|G_\lambda(z, w)|$, where $G_\lambda$ is the distribution-kernel for $\Box_\lambda^{-1}$ on $X$, we may therefore assume without loss of generality that the harmonic part of the Taylor polynomial of degree 2 for $\phi$ at $w$ vanishes identically. Likewise, because $\bar{\partial}_\lambda$ and $\bar{\partial}_\lambda^*$ have been conjugated by the unitary multiplicative factor $e^{\tilde{p}}$,
the same assumption can be made when deriving upper bounds for $|\bar{\partial}_x G_\lambda(z, w)|$ and $|\partial_x^* G_\lambda(z, w)|$.

7.3. **Taylor expansion and dilation.** Let $\bar{\phi}$ be as above, and suppose, as we may achieve through a gauge change, that the harmonic portion of the Taylor polynomial of degree 2 for $\bar{\phi}$ at 0 vanishes identically, while the complex Hessian matrix of $\bar{\phi}$ is bounded below by a strictly positive constant, and all partial derivatives of $\bar{\phi}$ are bounded above, uniformly in $\lambda$.

Let $N$ be a large positive integer, independent of $\lambda$, to be chosen below. Define $P_N$ to be the Taylor polynomial of degree $N$ for $\bar{\phi}$, at $\zeta = 0$. For any $r > 0$ satisfying $\lambda^{-1/2} \leq r \leq \lambda^{-1/4}$ define

$$
(7.6) \quad \psi(z) = r^{-2} P_N(r z) + r^{-2} (1 - \eta(z))(P_2(r z) - P_N(r z)).
$$

For all sufficiently large $\lambda$, the complex Hessian of $\psi$ evaluated at an arbitrary point $z \in \mathbb{C}^n$, equals the complex Hessian of $\bar{\phi}$ evaluated at 0, plus $O(r) = O(\lambda^{-1/4})$.

Moreover on $\{z : |z| < 3\}$, where $1 - \eta \equiv 0$, $\psi$ is real analytic, uniformly in $\lambda$ and in $N$ provided that $\lambda \geq \Lambda(N)$ where $\Lambda(N)$ is some appropriately large quantity depending only on $N$ and the data $X, L, \phi, g$. This uniformity, which is crucial to our analysis, is a consequence of the normalizations $\bar{\phi}(0) = 0$, $\nabla \phi(0) = 0$ achieved by subtracting the degree one Taylor polynomial of $\bar{\phi}$; indeed, for $z$ in any bounded set and $N \geq 2$, $P_N(r z) = P_2(r z) + O(r^3 |z|)$ so that

$$
\begin{align*}
& r^{-2} P_N(r z) = P_2(z) + O_{M,N}(r) \\
& \text{in any } C^M \text{ norm on any bounded set}. \\
& \text{Once } M, N \text{ are chosen, the term } O_{M,N}(r) \text{ becomes arbitrarily small as } \lambda \text{ becomes arbitrarily large.}
\end{align*}
$$

Define a globalized locally analytic approximation $g^{\dagger}$ to the Hermitian metric $g$ by

$$
(7.7) \quad g^{\dagger}(z) = P_N(r z) + (1 - \eta(z))(g(0) - P_N(r z))
$$

where now $P_N$ is the Taylor polynomial of degree $N$ for $g$ at 0, in the natural sense. Define

$$
(7.8) \quad \kappa = r^2 \lambda
$$

and

$$
\mathcal{D} = e^{-\kappa \psi} \bar{\partial} e^{\kappa \psi},
$$

that is, $\mathcal{D} u = e^{-\kappa \psi} \bar{\partial}(e^{\kappa \psi} u)$, for $(0, q)$ forms $u$ defined on $\mathbb{C}^n$. Define $\mathcal{D}^*$ to be the adjoint of $\mathcal{D}$ with respect to the Hilbert space structures on $L^2$ sections of $B^{(0, q)}(\mathbb{C}^n)$ specified by $g^{\dagger}(z)$. Define

$$
\Box^{\dagger} = \begin{cases} 
\mathcal{D} \mathcal{D}^* + \mathcal{D}^* \mathcal{D} & \text{for } n > 1, \\
\mathcal{D} & \text{for } n = 1.
\end{cases}
$$

These are differential operators. On the region $|z| < 4$, $\Box^{\dagger}$ is related to $\Box_{\lambda}$ as follows: If $u(z) = v(r z)$ then

$$
(7.9) \quad \Box^{\dagger} u(z) = r^2 \Box_{\lambda} v(r z) + O(\lambda^{-cN}) O(v, \bar{\partial}_{\lambda} v, \bar{\partial}_{\lambda}^* v, \bar{\partial}_{\lambda} (bv), \bar{\partial}_{\lambda}^* (bv))
$$

---

2 Subtraction of the harmonic second degree terms is natural, but inessential here.
where the error term denoted $O(v, \bar{\partial}_A v, \bar{\partial}^*_A v, \bar{\partial}_A (bv), \bar{\partial}^*_A (bv))$ is a linear combination of $v$, $\bar{\partial}_A (v)$, $\bar{\partial}^*_A (v)$, $\bar{\partial}_A (bv)$ and $\bar{\partial}^*_A (bv)$ where all coefficients are bounded uniformly in $\lambda, z$, and $bv$ denotes either the wedge product or the interior product of $v$ with a real analytic $(0, 1)$ form $b$. Moreover, in this region, these forms $b$ are uniformly analytic as $\lambda \to \infty$.

Applying (7.9) with

$$u(z) = G_\lambda(rz, 0),$$

using the upper bounds $|G_\lambda(z, 0)| \leq C \lambda^C$ and $|\bar{\partial}_A G_\lambda(z, 0)| + |\bar{\partial}^*_A G_\lambda(z, 0)| \leq C \lambda^C$ for $|z| \geq \lambda^{-1/2}$, and using the assumption $\lambda^{-1/2} \leq r \leq \lambda^{-1/4}$, we conclude that

$$\|\Box u(z)\| \leq \lambda^{-cN}$$

for $\frac{1}{2} \leq |z| \leq 2$, where $c > 0$ is independent of $\lambda, z$ and of $N$, provided that $\lambda \geq \Lambda(N)$.

Provided that $\kappa = r^2 \lambda$ is sufficiently large, the standard integration by parts calculation together with the uniform lower bound for the complex Hessian of $\psi$ give the lower bound

$$\langle \Box u, u \rangle \geq c \kappa \|u\|_{L^2}^2$$

for all $C^2$ forms $u$ of bidegree $(0, 1)$ with compact support. The effect of the localization and rescaling has been to replace $\lambda$ by $\kappa$.

**7.4. Conclusion of Proof of Proposition 3** Let $N$ be a large positive integer. Suppose that $\lambda$ is large, that $\lambda^{-1/2} \leq r \leq \lambda^{-1/4}$, and that $\kappa = r^2 \lambda$ is large. Consider $u(z) = G_\lambda(rz, 0)$, defined as above using Taylor polynomials of order $N$. In the annular region $\frac{1}{2} < |z| < 2$, $|u| \leq \Lambda^C$ and $\|\Box u\| \leq \lambda^{-cN}$, provided that $\lambda \geq \Lambda(N)$.

Let $\tilde{\eta}$ be a $C^\infty$ function which is identically equal to 1 in $\{z : \frac{1}{3} \leq |z| \leq 3\}$ and supported in $\{z : \frac{1}{2} < |z| < 4\}$. Provided that $\kappa$ is sufficiently large, the global lower bound (7.10) ensures that the equation $\Box u = \tilde{\eta} \Box u$ is solvable in $L^2(\mathbb{C}^n)$, and that there exists a solution satisfying

$$\|v\|_{L^2} \leq C \kappa^{-1} \|\tilde{\eta} \Box u\|_{L^2} \leq \lambda^{-cN},$$

provided that $\lambda \geq \Lambda(N)$.

Now $\Box(u - v) \equiv 0$ where $\frac{1}{2} < |z| < 2$, so Lemma 7 can be applied to conclude that

$$|(u - v)(z)| \leq C e^{-c\kappa} = C e^{-c r^2 \lambda} \quad \text{for} \quad \frac{3}{4} \leq |z| \leq \frac{4}{3}.$$

Therefore in this same region,

$$|G_\lambda(rz, 0)| \leq C e^{-c r^2 \lambda} + C \lambda^{-cN}$$

for all $\lambda \geq \Lambda(N)$.

Equivalently, by choosing $r = |z|^{-1}$, we find that there exists a constant $B < \infty$ such that for all $\lambda \geq \Lambda(N)$ and all $|\zeta| \geq B \lambda^{-1/2}$,

$$|G_\lambda(\zeta, 0)| \leq C e^{-c \lambda |\zeta|^2} + C \lambda^{-cN} = C e^{-c \lambda |\zeta|^2} + C e^{-c N \log \lambda}.$$
If $A_0$ is sufficiently large, if $A < \infty$ is fixed, and if $A_0 \lambda^{-1/2} \sqrt{\log \lambda} \leq |\zeta| \leq A \lambda^{-1/2} \sqrt{\log \lambda}$, choose $N = A^2$ to obtain

\begin{equation}
|G_\lambda(\zeta, 0)| \leq C e^{-c\lambda|\zeta|^2}.
\end{equation}

After reversing the change of variables made above, this is the desired bound $|G_\lambda(z, z')| \leq C e^{-c\lambda \rho(z, z')^2}$.

This analysis cannot be extended to a larger range of $|\zeta|$, because bounds only hold for $\lambda \geq \Lambda(N)$ and a larger range would require that $N$ depend on $|\zeta|$, hence that $N$ depend on $\lambda$, introducing circularity into the reasoning.

Since $G_\lambda(z, z')$ is a solution on the complement of the diagonal $z = z'$ of homogeneous elliptic partial differential equations, separately with respect to each of the two variables $z, z'$, and since the coefficients of these equations are $O(\lambda^2)$ in any $C^M$ norm, it follows from routine bootstrapping arguments that each derivative of $G_\lambda$ satisfies the same upper bound with a possibly smaller value of the constant $c > 0$.

Each of the finitely many steps in the bootstrapping process loses at most a factor of $C\lambda^2$. Since

$$\lambda^C e^{-A \sqrt{\lambda \log \lambda}} \leq e^{-(A-1) \sqrt{\lambda \log \lambda}}$$

for all sufficiently large $\lambda$, the loss of finitely many such factors is of no importance here. \hfill \Box

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