The $p$-Faber-Krahn Inequality Noted

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Abstract When revisiting the Faber-Krahn inequality for the principal $p$-Laplacian eigenvalue of a bounded open set in $\mathbb{R}^n$ with smooth boundary, we simply rename it as the $p$-Faber-Krahn inequality and interestingly find that this inequality may be improved but also characterized through Maz’ya’s capacity method, the Euclidean volume, the Sobolev type inequality and Moser-Trudinger’s inequality.

1 The $p$-Faber-Krahn Inequality Introduced

Throughout this article, we always assume that $\Omega$ is a bounded open set with smooth boundary $\partial \Omega$ in the $2 \leq n$-dimensional Euclidean space $\mathbb{R}^n$ equipped with the scalar product $\langle \cdot, \cdot \rangle$, but also $dV$ and $dA$ stand respectively for the $n$ and $n-1$ dimensional Hausdorff measure elements on $\mathbb{R}^n$. For $1 \leq p < \infty$, the $p$-Laplacian of a function $f$ on $\Omega$ is defined by

$$\Delta_p f = - \text{div} (|\nabla f|^{p-2} \nabla f).$$

As usual, $\nabla$ and $\text{div} (|\nabla|^{p-2}\nabla)$ mean the gradient and $p$-harmonic operators respectively (cf. [8]). If $W^{1,p}_0(\Omega)$ denotes the $p$-Sobolev space on $\Omega$ – the closure of all smooth functions $f$ with compact support in $\Omega$ (written as $f \in C^\infty_0(\Omega)$) under the norm

$$\left( \int_\Omega |f|^p dV \right)^{1/p} + \left( \int_\Omega |\nabla f|^p dV \right)^{1/p},$$

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then the principal $p$-Laplacian eigenvalue of $\Omega$ is defined by

$$\lambda_p(\Omega) := \inf \left\{ \frac{\int_{\Omega} |\nabla f|^p dV}{\int_{\Omega} |f|^p dV} : 0 \neq f \in W^{1,p}_0(\Omega) \right\}.$$ 

This definition is justified by the well-known fact that $\lambda_2(\Omega)$ is the principal eigenvalue of the positive Laplace operator $\Delta_2$ on $\Omega$ but also two kinds of observation that are made below. One is the normal setting: If $p \in (1, \infty)$, then according to [26] there exists a nonnegative function $u \in W^{1,p}_0(\Omega)$ such that the Euler-Lagrange equation

$$\Delta_p u - \lambda_p(\Omega)|u|^{p-2}u = 0 \text{ in } \Omega$$

holds in the weak sense of

$$\int_{\Omega} (|\nabla u|^{p-2} \nabla u \cdot \nabla \phi) dV = \lambda_p(\Omega) \int_{\Omega} |u|^{p-2} u \phi dV \quad \forall \phi \in C^\infty_0(\Omega).$$

The other is the endpoint setting: If $p = 1$, then since $\lambda_1(\Omega)$ may be also evaluated by

$$\inf \left\{ \frac{\int_{\Omega} |\nabla f|^2 dV + \int_{\partial \Omega} |f|^2 dA}{\int_{\Omega} |f|^2 dV} : 0 \neq f \in BV(\Omega) \right\},$$

where $BV(\Omega)$, containing $W^{1,1}_0(\Omega)$, stands for the space of functions with bounded variation on $\Omega$ (cf. [9, Chapter 5]), according to [7, Theorem 4] (cf. [16]) there is a nonnegative function $u \in BV(\Omega)$ such that

$$\Delta_1 u - \lambda_1(\Omega)|u|^{-1}u = 0 \text{ in } \Omega$$

in the sense that there exists a vector-valued function $\sigma : \Omega \mapsto \mathbb{R}^n$ with

$$||\sigma||_{L^\infty(\Omega)} = \inf \{c : |\sigma| \leq c \text{ a.e. in } \Omega \} < \infty$$

and

$$\text{div}(\sigma) = \lambda_1(\Omega), \quad \langle \sigma, \nabla u \rangle = |\nabla u| \text{ in } \Omega, \quad \langle \sigma, n \rangle u = -|u| \text{ on } \partial \Omega,$$

where $n$ represents the unit outer normal vector along $\partial \Omega$. Moreover, it is worth pointing out that

$$\lambda_1(\Omega) = \lim_{p \to \infty} \lambda_p(\Omega), \quad (1.1)$$

and so that $\Delta_1 u = \lambda_1(\Omega)|u|^{-1}u$ has no classical nonnegative solution in $\Omega$: In fact, if not, referring to [18, Remark 7] we have that for $p > 1$ and $|\nabla u(x)| > 0$