Understanding Boolean Function Learnability on Deep Neural Networks

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Abstract

Computational learning theory states that many classes of boolean formulas are learnable in polynomial time. This paper addresses the understood subject of how, in practice, such formulas can be learned by deep neural networks. Specifically, we analyse boolean formulas associated with the decision version of combinatorial optimisation problems, model sampling benchmarks, and random 3-CNFs with varying degrees of constrainedness. Our extensive experiments indicate that: (i) regardless of the combinatorial optimisation problem, relatively small and shallow neural networks are very good approximators of the associated formulas; (ii) smaller formulas seem harder to learn, possibly due to the fewer positive (satisfying) examples available; and (iii) interestingly, underconstrained 3-CNF formulas are more challenging to learn than overconstrained ones. Source code and relevant datasets are publicly available.

1 Introduction

The construction of Artificial Intelligence systems that integrate the fundamental cognitive abilities of reasoning and learning has been pointed out by Turing Award winner and machine learning pioneer Leslie Valiant as a key challenge for computer science (Valiant 2003, 2013). There remain, however, several challenges with respect to closing the gap between theoretical and practical advances in machine learning that would allow for such effective integration. In the recent Montreal AI Debate between Yoshua Bengio and Gary Marcus and at the recent AAAI2020 conference in New York, leading researchers including deep learning pioneers Bengio, Hinton and LeCun have singled out the effective development of integrated reasoning mechanisms as a key challenge to machine learning (Kahneman et al. 2020). Further, machine learning in general and deep learning in particular have achieved noticeable technological advances in a wide range of applications. These include natural language processing, machine translation, computer vision and image understanding, to name a few (LeCun, Bengio, and Hinton 2015; Schmidhuber 2015).

The integration of machine learning and symbolic reasoning has been the subject of recent debates in AI and as a methodology that can lead to the answer to challenging AI problems (Galassi et al. 2020; van Steenkiste et al. 2018; Kahneman et al. 2020). To respond to these challenges, neural-symbolic methods have recently been the subject of intense investigation and great interest of both academic and industry researchers (d’Avila Garcez et al. 2019; Mao et al. 2019; Raedt et al. 2020; Marcus 2020; Raghavan 2019).

Nobel prize winner Daniel Kahneman also referred explicitly to the need for symbolic reasoning systems being integrated to deep learning. At the AAAI2020 conference, when Kahneman compared his own classification of the two modes of thinking described in (Kahneman 2011), i.e. System 1: that operates automatically and quickly with little or no effort and no sense of voluntary control and System 2: that allocates attention to the effortful mental activities that demand it, including complex computations, with the so-called AI systems 1 (deep learning) and 2 (reasoning layer), Kahneman clearly emphasized the need for a symbolic layer as necessary to achieve richer AI models: “…so far as I’m concerned, System 1 certainly knows language... System 2... does involve certain manipulation of symbols.”. Moreover, at the AAAI2020 Robert S. Engelmore Memorial Lecture, Henry Kautz proposed several challenges for AI in his address titled The Third AI Summer. These challenges include the development of effective systems along the lines of Valiant by stating that “The next steps in AI are tighter symbolic-neuro integration.”

In order to achieve such integration in AI, one has to consider the challenges and questions still open in machine learning and computational learning theory, such as effective algorithms for reasoning and learning over classes of boolean formulas, learnable in polynomial time. Yet there remains outstanding questions to be addressed, particularly referring to effective experimentation on classes of boolean functions (Kearns, Li, and Valiant 1994). Further, the development of efficient learning algorithms for learning boolean formulas remains a challenge in AI (Valiant 2013). Learning unrestricted Disjunctive Normal Formulas (DNFs) still remains a difficult problem as proven by (Klivans and Servedio 2004).

¹Please see (Kahneman 2011) for a full account of the meaning of the terms and the origins of this terminology from psychology.
²Henry Kautz AAAI2020 Robert S. Engelmore memorial lecture is available at https://vimeo.com/389560858. Slides are available at https://www.cs.rochester.edu/u/kautz/talks/index.html

https://github.com/machine-reasoning-ufrgs/mlbf
A k-CNF is a CNF where each clause has no more than k of one or more clauses, which are disjunctions of literals.

Given a Boolean Formula, the Boolean Satisfiability problem (SAT) is to find an assignment of the variables where the BF evaluates to true, or to provide a proof that no satisfying assignment exists. Usually, SAT solvers take as input a BF in the CNF format. The SAT problem is important because it can be found in different areas in science and also in numerous practical problems. Besides that, it is general enough that several other problems that, at first, are not related to SAT, can be converted into a SAT problem - for example, some problems in graph theory, such as the clique and coloring problems (Garey and Johnson 1979).

Related work
Computational learning theory presents hardness results on the learnability of boolean functions related to certain classes of problems, such as cryptography (Kivest 1991), robust learning (Gourdeau et al. 2019) and distribution learning (Kearns et al. 1994). It also presents many positive results on polynomial-time learnability of boolean formulas, which are of our interest. Here we describe results concerning conjunctive normal formulas (CNFs) and neural networks.

Within the probably approximately correct (PAC) learning framework, (Valiant 1984) shows that conjunctive normal formulas with a bounded number of literals per clause (k-CNFs) are learnable in general, although not mentioning neural networks specifically.

Artificial Neural networks (ANNs) are universal learners of boolean formulas (Blum 1989; Steinbach and Kohut 2002), since classical perceptrons can be arranged to implement any logical gate and such gates can be arranged to implement any boolean formula, also with the possibility of extracting boolean formulas from trained neural networks (Tsukimoto 1997). Moreover, even single-hidden-layer networks are universal boolean function learners (Anthony 2010), although the worst-case number of neurons in the hidden layer is exponential on the number of inputs. Anthony (2010) further provides lower bounds on the sample complexity of boolean functions, relating the VC-dimension (Vapnik and Chervonenkis 1971), with the tolerance (expected error margin) and confidence of the PAC learning framework. The width of the neural network can be traded-off by depth to alleviate the worst-case requirement for the number of neurons (Anthony 2010). Other neural networks have also been proven to universally implement boolean formulas, such as the binary pi-sigma network (Shin and Ghosh 1991), the binary product-unit network (Zhang, Yang, and Wu 2011), since classical perceptrons can be arranged to implement any logical gate and such gates can be arranged to implement any boolean formula, also with the possibility of extracting boolean formulas from trained neural networks (Tsukimoto 1997). Moreover, even single-hidden-layer networks are universal boolean function learners (Anthony 2010), although the worst-case number of neurons in the hidden layer is exponential on the number of inputs. Anthony further provides lower bounds on the sample complexity of boolean functions, relating the VC-dimension (Vapnik and Chervonenkis 1971), with the tolerance (expected error margin) and confidence of the PAC learning framework. The width of the neural network can be traded-off by depth to alleviate the worst-case requirement for the number of neurons (Anthony 2010). Other neural networks have also been proven to universally implement boolean formulas, such as the binary pi-sigma network (Shin and Ghosh 1991), the binary product-unit network (Zhang, Yang, and Wu 2011).

A body of empirical work followed the positive theoretical results on the learnability of boolean functions: (Miller 1999) shows that parity and multiplier functions are efficiently learnable, (Franco and Anthony 2004; Franco 2006) study complexity metrics that related to the generalisation abilities of boolean functions implemented via neural networks, (Subirats et al. 2006; Subirats, Jerez, and Franco 2008) and (Zhang, Ma, and Yang 2002). All these results concern a different perspective on the learnability of boolean functions and are not directly related to our setting.
show algorithms for learning boolean circuits with thresholding neural networks, while (Prasad and Beg 2009) studies pre-processing techniques for using ANNs to learn boolean circuits and in (Beg, Prasad, and Beg 2008) they study approximating a boolean function’s complexity using an ANN. (Pan and Srikumar 2016) showcases how neural networks with ReLU activation implement boolean functions much more compactly than with threshold linear units.

Moreover, Unigen2 handles large boolean for-

universal hashing to partition the model space into roughly distributed over the model space. This is done by the use of

Our generation of negative samples

Dataset generation Our generation of negative samples from a formula \( f \) is trivial: each variable is assigned a either truth-value with 50% probability and we add the resulting assignment to the dataset if it falsifies \( f \). Our boolean formulas of interest have much more negative than positive ex-

amples, hence this procedure is expected to take linear time on the desired number of negative samples.

To generate positive samples for \( f \) means to solve \( f \) multiple times to retrieve different satisfying assignments (models). A trivial way to do this is to employ any traditional SAT solver such as MiniSAT (Sorensson and Een 2003) or Glucose (Audemard and Simon 2018) to sequentially enumerate solutions. This is done by augmenting \( f \) with a clause containing the negated model recently found, and calling the solver in the augmented formula. However, the enumerated solutions will be poorly distributed over the model space. For example, when any model with free variables is found, the solver will retrieve subsequent models by sequentially assigning truth-values to the free variables. Any satisfying assignment with \( t \) free variables yields \( 2^t \) such models and this number might already be greater than the total number of desired positive samples (in this paper, for example, we aim at only 500 positive examples).

The issue of poor model sampling of traditional SAT solvers is tackled by Unigen2 (Chakraborty et al. 2013), which provably generates positive samples quasi-uniformly distributed over the model space. This is done by the use of universal hashing to partition the model space into roughly equal “cells”. Moreover, Unigen2 handles large boolean formulas by leveraging their minimal independent support if it

is known\(^4\). We thus generate positive examples with Unigen2, falling back to the “trivial” procedure with Glucose if Unigen2 fails. Unigen2 relies on a previous model counting procedure (exactly with sharpSAT (Thurley 2006) or approxi-

mately with ApproxMC (Chakraborty, Meel, and Vardi 2013)) and it fails with low model counts or if the model counter itself times out. The retrieved positive and negative samples of \( f \) are then shuffled to generate its dataset.

**Performance evaluation** Theoretical studies on learnability are usually concerned with the hardness, in terms of computational complexity classes, of learning certain families of boolean formulas or worst-case sample complexity to achieve certain tolerance and confidence thresholds within the PAC learning framework. The focus of this study is conjunctive normal formulas with bounded number of literals per clause (k-CNFs). In particular, we study 3-CNFs encoding graph 3- and 5-coloring problems and random 3-CNFs with various constrainedness (clause-to-variable ratios).

(Valiant 1984) demonstrated that k-CNFs are learnable in polynomial time, without mention of neural networks in particular. Hence our interest in showing empirical evidence of neural network learning capabilities on such formulas. To do this, given a CNF \( f \), we generate a dataset with the aforementioned procedure and, without loss of generality, assess the learning capabilities of a deep neural network on \( f \) via the \( k \)-fold cross-validation accuracy in the resulting dataset. This gives an individual (per-formula) measure of performance.

To evaluate the learnability over a set \( S \) of formulas belonging to a specific class, we use the average and minimum accuracy over the formulas in \( S \). We also use the ratio of perfectly-learned formulas in \( S \). That is, we count in how many formulas the \( k \)-fold cross-validation accuracy is 100% and divide by \( |S| \). This is a more strict measure of performance that we use in all evaluations.

**Experimental setup** We use scikit-learn’s (Pedregosa et al. 2011) implementation of fully-connected multi-layer perceptrons (MLPs) as our deep neural network. We use Adam optimisation with recommended parameters (learning rate \( = 10^{-3} \), \( \beta_1 = 0.9, \beta_2 = 0.999 \) (Kingma and Ba 2014)), 200 training epochs and L2 regularisation term \( = 10^{-4} \). Experiments in Section 4 are performed with ReLU activation and a neural network with two hidden layers containing 200 and 100 neurons, respectively. Experiments in Section 5 are performed with variations on activation functions and number of neurons in a single hidden layer.

Our experiments are performed on a computer with a 6-core (12 threads) Intel Core i7-8700 CPU @ 3.20GHz and 32 GB DDR4 RAM.

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\(^4\)An independent support is a set containing variables that are sufficient to determine the value of the remaining ones (the dependent support) on satisfying assignments. We made a change in Unigen2 so that it returns values for variables in the dependent support as well.
4 Learnability on Combinatorial Optimisation Problems

We perform learnability experiments on CNF boolean formulas encoding the decision version of two NP-complete combinatorial optimisation problems: graph colouring (is there a way to colour the graph vertices with k colours such that adjacent vertices have different colours?) and clique (is there a complete subgraph with k vertices?) (Arora and Barak 2009). We refer to these problems as k-GCP and k-clique hereafter. The number of satisfying assignments of formulas encoding instances of these problems is the number of k-colourings and k-cliques, respectively.

GCP instances are on flat and morphed graphs, retrieved from SATLIB. GCP flat instances are 3-colourable quasi-random graphs. Different graphs are generated with the same number of nodes and edges, whose connectivity is arranged to make them difficult to solve by the Brelaz heuristic (Hogg 1996). GCP morphed instances are 5-colourable graphs, constructed by merging regular ring lattices, whose vertices are ordered cyclically and each vertex is connected to its 5 closest in this ordering, with random graphs from the Erdös and Rényi (1959) model. An r-morph of two graphs \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) is a graph \( G = (V, E) \) where \( E \) contains all edges in \( E_1 \cap E_2 \), a fraction \( r \) of edges in \( E_1 - E_2 \), and a fraction \( 1 - r \) of edges in \( E_2 - E_1 \).

Our k-clique instances are generated on random \( G(n, p) \) graphs (Erdős and Rényi 1959) with CNFgen (Lauria et al. 2019), where we control the number of nodes \( n \) and the probability of each edge \( p \). We do this by calculating \( p \) according to Equation (1), derived from (Bollobás and Erdös 1976), where \( E(Y_k) \) is the expected number of k-cliques in the graph:

\[
p = \frac{\binom{k}{2} E(Y_k)}{\binom{n}{k}}
\]  

In particular, we aim for 500 3-cliques on average, hence \( k = 3 \) and \( E(Y_3) = 500 \). We generate graphs with \( n = 50, 100 \) and 150 nodes, giving \( p = 0.2944, 0.1457 \) and 0.0908, respectively, according to Eq. (1).

Our deep neural network in this test is a MLP with two hidden layers containing 200 and 100 neurons, respectively. For each problem size (i.e. \#nodes and \#edges), \(|S|\) denotes the number of datasets generated, i.e., for how many formulas we were able to generate examples. We also show the size of the resulting formulas (\#variables and \#clauses), the mean and minimum accuracies of the MLP on the 5-fold cross-validation across all formulas and the ratio of formulas with perfect (i.e. 100%) accuracy.

In general, the mean and minimum accuracies are very close to, but not 100% throughout formulas on all sizes and problems, which means that the MLP is a good approximator of these formulas. The metric that varies most across sets of formulas is the ratio of perfectly learned formulas (% perfect). Thus we use this metric as a proxy for learnability.

Formulas encoding 3-GCP on flat graphs were more challenging on the smallest and largest graphs (30 and 200 nodes, respectively), with %perfect rates around 65%, whereas it remained close to 90% on the other graph sizes. Regarding the number of variables, these are respectively the smallest and largest sets of formulas in this experiment. We further investigate the difficulty imposed by large and small formulas in the sequel.

In 3-clique problems, the ratio of perfectly-learned formulas is very similar and close to 100% across all graph sizes. This suggests that formulas encoding problems over random \( G(n, p) \) graphs are easier to learn. 5-GCPs on morphed graphs showed an interesting behaviour of the ratio of perfectly-learned formulas on different morph ratios \( r \): it is 100% on the largest 4 morph ratios, falling up to 26% on intermediate morph ratios and rising again to 100% on \( r=0 \). Higher \( r \) yields graphs with more edges from a random graph, and these are easier to learn, as per the 3-clique experiments. On the other hand, small \( r \) yields graphs with more edges from the ring lattice, whose regular structure results in easier GCP-encoding formulas. Mixing the structure of random and regular graphs with intermediate morph ratios yields the most challenging GCP-encoding formulas.

Further investigation on large formulas

Experiments so far showed that formulas encoding combinatorial optimisation problems are, in general, easily learnable by MLPs. Nevertheless, the largest set of formulas encoding 3-GCP problems was challenging. In this section, we investigate the performance of a deep neural network on even larger boolean formulas, containing up to hundreds of thousands of variables. The formulas are from the model sampling benchmark of Unigen2 (Chakraborty et al. 2015).

In this test, our deep neural network is also a MLP with two hidden layers containing 200 and 100 neurons, respectively. Table 2 shows the average accuracy of a 5-fold cross validation for each formula. We also report the performance of a decision tree (DT), executed with default scikit-learn parameters (Pedregosa et al. 2011), for comparison.

The MLP perfectly learned all formulas, whereas the decision tree had a high accuracy overall, but did not fully learn any formula.

5 Satisfiability versus Learnability on 3-CNF Formulas

A large body of work has investigated how to generate hard-to-satisfy boolean formulas (Cheeseman, Kanefsky, and Taylor 1991, Crawford and Auton 1996, Selman, Mitchell, and Levesque 1995). They have shown that, random 3-CNF formulas have a phase transition region associated with the clause-to-variable ratio. Formulas are easy to prove satisfiable when the clause-to-variable ratio is below the phase transition region and are easy to prove unsatisfiable when it is above the phase transition region. We refer to the clause-to-variable ratio as constrainedness, denoting the region below the phase transition as underconstrained and the region above it as overconstrained. We denote the constrainedness of the sweet spot between the under- and overconstrained, as “on phase”. Formulas “on phase” are the hardest to solve.
In this section we investigate how hardness in satisfiability relates to hardness in learnability on random 3-CNF formulas. We investigate formulas with 10 from 100 variables in increments of 10. Note that, these formulas are relatively small so that state-of-the-art SAT solvers routinely solve. This is on purpose because large formulas were easily learned (see Table 2) and the set with the smallest 3-GCP formulas of Table 1 was the hardest among sets with varying number of variables.

The clause-to-variable ($c/v$) ratio of phase transition regions for each set of formulas is depicted on Table 3. For each number of variables, we generate sets of formulas with 11 values of constrainedness: 5 sets with underconstrained formulas (with phase $c/v$ subtracted by 0.1, 0.2, ..., 0.5), 1 set on phase and 5 sets with overconstrained formulas (with phase $c/v$ added by 0.1, 0.2, ..., 0.5). Each set contains 1000 formulas. In total, we generate $10 \times 11 \times 1000 = 110,000$ boolean formulas in this study.

We assess the learnability of each generated formula as follows: train a single-hidden-layer neural network on the dataset generated for the formula, varying the number of hidden-layer neurons from 1 to 256 in powers of 2, and check how many neurons are sufficient to “learn” the formula perfectly (with 100% accuracy on the 5-fold cross validation). If perfect accuracy is not achieved with 256 neurons, we say that the formula has not been (perfectly) learned.

We test neural networks with ReLU and logistic (sigmoid) activation functions. Figure 1 shows, for each activation function, the percent of perfectly-learned formulas according to the number of variables and constrainedness, which we depict as ranging from -5 denoting the least constrained, 0 being on phase transition (hardest solubility) and +5 denoting the most constrained.

Interestingly, the smaller the formulas, the harder it is to learn them. In fact, no 10-variable formula was perfectly learned at all, regardless of the constrainedness (Fig. 1a). The ReLu neural network learned more formulas overall compared to logistic, specially with fewer variables. This is aligned with experimental (Nair and Hinton 2010; Dahl, Sainath, and Hinton 2013) and theoretical (Fiat, Malach, and Shalev-Shwartz 2019) evidence favouring ReLU over other activation functions in deep learning.

Figure 1 also shows that as constrainedness increases up to a point, more formulas are learned. Beyond such point, learnability slightly drops for all formulas, except those with 40 variables, where the drop is sharp on both activations. The point of highest learnability never coincides with the point of hardest solubility (the phase transition point, highlighted by the vertical dashed line in Fig. 1a).

Figure 1B also highlights the similarity on the behaviour of onphase and overconstrained formulas as the number of variables increase, in contrast with underconstrained formu-

### Table 1: Learnability results on combinatorial optimisation problems.

| Problem           | #nodes | #edges | $|S|$ | #vars | #clauses | c/v | Mean acc (%) | Min acc (%) | % Perfect |
|-------------------|--------|--------|-----|-------|----------|-----|--------------|-------------|-----------|
| 3-fGCP            | 30     | 60     | 100 | 90    | 300      | 3.33| 99.92        | 99.46       | 65.00     |
| 3-fGCP            | 50     | 15     | 998 | 150   | 545      | 3.63| 99.97        | 99.17       | 89.00     |
| 3-fGCP            | 75     | 80     | 100 | 225   | 840      | 3.73| 99.98        | 99.40       | 94.00     |
| 3-fGCP            | 100    | 239    | 100 | 300   | 1117     | 3.72| 99.97        | 99.21       | 88.00     |
| 3-fGCP            | 125    | 301    | 100 | 375   | 1403     | 3.74| 99.98        | 99.21       | 90.00     |
| 3-fGCP            | 150    | 360    | 99  | 450   | 1680     | 3.73| 99.98        | 99.60       | 91.90     |
| 3-fGCP            | 175    | 417    | 82  | 525   | 1951     | 3.72| 99.98        | 98.44       | 73.20     |
| 3-fGCP            | 200    | 479    | 64  | 600   | 2237     | 3.73| 99.98        | 98.24       | 64.10     |
| 5-mGCP, r=1       | 100    | 400    | 94  | 500   | 3100     | 6.20| 100.00       | 100.00      | 100.00    |
| 5-mGCP, r=0.5     | 100    | 400    | 100 | 500   | 3100     | 6.20| 100.00       | 100.00      | 100.00    |
| 5-mGCP, r=0.25    | 100    | 400    | 100 | 500   | 3100     | 6.20| 100.00       | 100.00      | 100.00    |
| 5-mGCP, r=0.125   | 100    | 400    | 100 | 500   | 3100     | 6.20| 100.00       | 100.00      | 100.00    |
| 5-mGCP, r=2^{-4}  | 100    | 400    | 100 | 500   | 3100     | 6.20| 99.95        | 99.47       | 75.00     |
| 5-mGCP, r=2^{-5}  | 100    | 400    | 100 | 500   | 3100     | 6.20| 99.79        | 99.20       | 26.00     |
| 5-mGCP, r=2^{-6}  | 100    | 400    | 100 | 500   | 3100     | 6.20| 99.93        | 99.47       | 66.00     |
| 5-mGCP, r=2^{-7}  | 100    | 400    | 100 | 500   | 3100     | 6.20| 100.00       | 99.84       | 98.00     |
| 5-mGCP, r=2^{-8}  | 100    | 400    | 100 | 500   | 3100     | 6.20| 99.99        | 99.99       | 94.60     |
| 5-mGCP, r=0       | 100    | 400    | 1    | 500   | 3100     | 6.20| 100.00       | 100.00      | 100.00    |
| 3-clique          | 50     | 360.64*| 100  | 150   | 10091.16*| 67.27| 100.00       | 99.88       | 97.00     |
| 3-clique          | 100    | 721.22*| 100  | 300   | 42695.94*| 142.32| 100.00       | 99.76       | 98.00     |
| 3-clique          | 150    | 1081.74*| 100  | 450   | 97790.55*| 217.31| 100.00       | 99.88       | 96.00     |

In this section we investigate how hardness in satisfiability relates to hardness in learnability on random 3-CNF formulas. We investigate formulas with 10 from 100 variables in increments of 10. Note that, these formulas are relatively small so that state-of-the-art SAT solvers routinely solve. This is on purpose because large formulas were easily learned (see Table 2) and the set with the smallest 3-GCP formulas of Table 1 was the hardest among sets with varying number of variables.
Figure 1: Percent of (perfectly) learned formulas, i.e. with 100% accuracy, according to the number of variables and constrainedness. In (a), the constrainedness appears in the x axis: underconstrained from -5 to -1, the vertical line at 0 marks the solubility phase transition point and overconstrained from 1 to 5. In (b) the number of variables is in the x axis and each line shows the average % of learned formulas for constrainedness grouped into under, on phase and overconstrained.

| Instance       | #vars | #clauses | MLP | DT   |
|----------------|-------|----------|-----|------|
| s1238a_3_2     | 686   | 1850     | 100 | 97.12|
| s1196a_3_2     | 690   | 1850     | 100 | 98.11|
| s832a_15_7     | 693   | 2017     | 100 | 98.21|
| case_1_b12_2   | 827   | 2725     | 100 | 98.21|
| squaring16     | 1627  | 5835     | 100 | 98.01|
| squaring7      | 1628  | 5837     | 100 | 98.41|
| LoginService2  | 11511 | 41411    | 100 | 98.01|
| sort.sk_8_52   | 12125 | 49611    | 100 | 97.52|
| 20             | 15475 | 60994    | 100 | 97.61|
| enqueue        | 16466 | 58515    | 100 | 97.51|
| karatsuba      | 19594 | 82417    | 100 | 98.31|
| llreverse      | 63797 | 257657   | 100 | 97.13|
| tutorial3.sk_4_31 | 486193 | 2598178   | 100 | 92.56|

Table 2: Results (5-fold cross validation accuracy) of multilayer perceptron (MLP) and decision tree (DT) on large boolean formulas from a subset of Unigen2 model sampling benchmark (Chakraborty et al. 2015). The MLP perfectly learns all formulas and the decision tree (DT) approximates them well, but does not learn to perfection.

Smaller formulas require more neurons to be learned. With logistic activation, the required number of neurons varies only slightly with few variables, regardless of the constrainedness. From 80 or more variables, the required number of neurons seem to relate with the difficulty in satisfiability: formulas with low and high constrainedness, which are easy to satisfy, require more neurons to learn, whereas formulas on phase transition, which are hard to satisfy, require less neurons. The stability of the number of neurons required to learn smaller formulas regardless of their constrainedness also occurs with ReLU. Moreover, the number of neurons required to learn formulas with 30 to 100 variables is very similar for the smallest constrainedness. Differently from logistic activation, where the average number of neurons is the highest for under and overconstrained formulas, with ReLU

| #variables | Phase c/v | Source |
|------------|-----------|--------|
| 10         | 5.500     | CA*    |
| 20         | 4.550     | CA     |
| 30         | 4.433     | CA*    |
| 40         | 4.375     | CA*    |
| 50         | 4.360     | SML    |
| 60         | 4.317     | CA*    |
| 70         | 4.300     | CA*    |
| 80         | 4.287     | CA*    |
| 90         | 4.289     | CA*    |
| 100        | 4.310     | SML    |

Table 3: Satisfiability phase transition (c/v stands for the clause-to-variable ratio) for random 3-CNFs for each the number of variables. Source denotes the paper where c/v was taken or calculated: CA stands for (Crawford and Anton 1996) (an * marks values calculated with the formula: $c = 4.258v + 58.26 + v^{(-2/3)}$ derived on the paper instead of directly measured on their experiments) and SML for (Selman, Mitchell, and Levesque 1996).
Figure 2: Average number of neurons required to learn the formulas of each number of variables and constrainedness. No line with 10 variables appears because no such formulas were perfectly learned at all (see Fig. 1). In (a), the constrainedness appears in the x axis: underconstrained from -5 to -1, the vertical line at 0 marks the solubility phase transition point and overconstrained from 1 to 5. The discontinuity with 20 variables and logistic activation happened because no formula with constrainedness=3 was learned. In (b), the number of variables is in the x axis and each line shows the average number of neurons for constrainedness grouped into under, on phase and overconstrained.

activation, the overall trend is a monotonic decrease on the number of neurons as the constrainedness increases.

Figure 2 further shows that for formulas grouped by their constrainedness behave similarly for networks with logistic activation (requiring less neurons for larger formulas). On ReLU networks, however, the required number of neurons for underconstrained formulas decreases much slower with more than 50 variables compared to on phase and overconstrained. Fig. 2b also highlights that ReLU neural networks are more efficient in the sense of requiring less neurons to learn formulas on phase or overconstrained. On larger underconstrained formulas, logistic neural networks are more efficient. We remark that these results apply to the formulas that have been learned, as ReLU neural nets were able to effectively learn more formulas (Fig. 1), regardless of constrainedness.

6 Conclusions

The recent success of deep learning techniques over various domains and the positive results from computational learning theory regarding the learnability of boolean formulas prompt the question on how effective deep learning approaches are on learning boolean formulas, which can encode relevant problems in symbolic reasoning, via propositional logic, and combinatorial optimisation. We have tackled this subject by assessing the learning capabilities of multi-layer perceptrons, which are the basic components of deep learning systems, over boolean formulas encoding combinatorial optimisation problems, large model sampling benchmarks, and random 3-CNFs with various constrainedness (clause-to-variable ratios). Our methodology consists of generating positive (satisfying) and negative (falsifying) examples for each formula, and verifying the cross-validation accuracy of a MLP as a means to assess its capability to learn the formula.

The MLPs were very good approximators of all studied formulas, with no cross-validation accuracy below 99%. We thus use the more challenging measure of how many formulas were perfectly learned, i.e. with 100% accuracy, as a proxy for learnability. Our extensive experiments have three main findings: (i) relatively small and shallow neural networks are very good approximators of all studied formulas; (ii) very large formulas (with up to hundreds of thousands of variables) are easy to learn, whereas smaller formulas are harder, possibly due to the fewer positive (satisfying) examples available; and (iii) underconstrained 3-CNF formulas are more challenging to learn than overconstrained ones.

Our results can provide ground for more empirical or new theoretical studies on the learning capabilities of deep learning models. For example, the relationship of constrainedness and learnability of random 3-CNF formulas can be further studied. Moreover, it would be interesting to investigate whether the harder learnability of small formulas is related with the structure of the associated factor graphs (as in [Yolcu and Póczos 2019], where edges connect variables to clauses). Another line of future work could be on the deduction procedure already mentioned by (Valiant 1984): extract the formula learned by the MLP (using knowledge extraction methods [Garcez, Broda, and Gabbay 2001; Tran and Garcez 2018]) and check to which extent it matches the original one.
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