Optimal Vehicle Dispatching Schemes via Dynamic Pricing

Mengjing Chen
IIS, Tsinghua University
cchmj@qq.com

Weiran Shen
IIS, Tsinghua University
emersongswr@gmail.com

Pingzhong Tang
IIS, Tsinghua University
kenshinping@gmail.com

Song Zuo
IIS, Tsinghua University
songzuo.z@gmail.com

Abstract

Recently, shared mobility has been proven to be an effective way to relieve urban traffic congestion and reduce energy consumption. Despite the emergence of several nationwide platforms, the pricing schemes and the vehicle dispatching problem of such platforms are optimized in an ad hoc manner. In this paper, we introduce a general framework that incorporates geographic information and time-sensitive dynamic environment parameters (such as the dynamically changing demand) and models the pricing and dispatching problem as a Markov Decision Process with continuous state and action spaces. Despite of the PSPACE-hardness of general MDPs, we provide efficient algorithms finding the exact revenue (or welfare) optimal (potentially randomized) pricing schemes. We also characterize the optimal solution via primal-dual analysis of a convex program. Finally, we also discuss generalizing our model by showing how to reduce a wide range of general settings in practice to our model.

1 Introduction

The emerging applications of shared mobility, such as ride-sharing (e.g., Didi, Uber, and Lyft), bike-sharing (e.g., Ofo and Mobike), Citi Bike in NYC), and car-sharing (e.g., car2go and zipcar), have been proven to be an efficient way of resource usage and social welfare improvement [9] and intensive researches have been done on related topics [10, 17, 23]. Despite that many researches have been done around these applications, the optimal pricing schemes for the resource dispatching problem are still not well-understood. Traditional dispatching/pricing schemes for taxi [18, 13, 14] and airplane [12, 24, 20] do not capture the dynamic nature of the new system: (i) the dispatching system for these shared vehicles can make use of dynamic pricing to further boost revenue, while taxi fees are calculated from a fixed function of traveling distance and time; (ii) the prices of light tickets are sold via long booking periods, while the customers of shared vehicles make their decisions rather immediately.

In this paper, we first model the vehicle dispatching in shared mobility as a pricing problem on a strongly connected digraph. The entire graph refers to a city and each node refers to a region in this city. Each directed edge refers to a pair of potential origin and destination that customers would travel along and an exogenous cost of a ride is associated with each edge. The action of the platform is to set a price for the rides along each directed edge at each time slot and the dispatching along such edges is induced by the customers who take the prices for their rides. The platform is given the demand curves along each edge at each time slot and aims at maximizing the objective, which could be revenue, social welfare, throughput, or even any convex combination of them.

Unlike some previous studies that consider drivers as selfish agents [3, 11], we consider the drivers to be not strategic meaning that they will not reject the rides with low prices (so they are more like employees of the platform). We remark that this is the nature of the bike-sharing and the car-sharing applications and even for the ride-sharing,
the mechanism design problem can be decomposed into the pricing problem for the rides (between the platform and customers) and the dynamic revenue sharing problem (between the platform and drivers) [2]. Therefore, the payout to the driver for each ride is almost independent of the price paid by the customer for this ride and hence the drivers are not sensitive to the prices.

The model we proposed then naturally induces a Markovian Decision Process (MDP) with the driver populations on each node as states, the throughputs along each edge as actions, and the objective (e.g., revenue, welfare, or throughput) as the rewards. While it is in general PSPACE-hard to solve MDPs, we come up with an efficient (polytime) algorithm that finds the exact optimal solution: We first translate the demand curves along each edge into throughput-reward tradeoffs (i.e., the maximum reward can be achieved along this edge with fixed throughput), and then formulate the problem as an optimization with throughputs (along each edge at edge time slot) as variables.

In our algorithm, we resolve two major difficulties: (i) we first reduce the throughput-reward tradeoff, which in general is not concave, to its convex hull by allowing for randomized pricing schemes (commonly adopted in mechanism design and easy to implement in practice), and (ii) then we solve the MDP with continuous state by formulating it as a convex program. In particular, all the constraints in the convex program are linear.

Dynamic Environment  As the environment varies over time in practice, we allow the traveling cost along each edge, the valuation and demand of the customers along each source and destination pair, and even the total driver population (supply) over the graph to be different at each time period. Despite of the high complexity of the dynamics in such an environment, we are able to reduce any general instance to a restricted simply instance and the optimal solution of the general instance can be easily recovered from the optimal solution of the simplified instance (see Section 7).

Our Contribution  In this paper, we introduce a simple model for the dispatching problem that can be used for a wide range of general settings (see Section 7 for detailed discussions on the general settings and how they can be reduced to our simple model). To the best of our knowledge, we are the first to model the dynamic environment where the prior information of both the demand (e.g., frequent and values of requests), the supply (e.g., the driver population in ride-shares), and even the traveling costs may be different for each time period. Note that such dynamics commonly exist in practice [8] and add new challenges to the pricing and dispatching problem.

Within such a model, we then formalize the problem as a mathematical program with linear constraints by introducing the throughput-reward tradeoffs (see Section 4). Although in general the program is not convex, we convexify it by introducing randomized prices (see Section 3) and hence the optimal solution can be computed efficiently. One major merit of this approach is that one can easily introduce additional constraints to the problem with few change to the algorithm. For example, in practice, the supply might be asymmetric and correlated over time and the platform may want the distribution of the supply to be stationary in long run.

Finally, we also provide a characterization of the optimal solution via primal-dual analysis. In particular, a dynamic pricing scheme is optimal if and only if the marginal contribution of the throughput along each edge equals to the system-wise marginal contribution of additional supply minus the difference of the long term contributions of unit supply at the origin and the destination (see Section 6).

1.1 Related Work

Driven by real-life applications, a large number of researches have been done regarding the ride-share markets. Some of them employ queuing networks to model the market [15][3][25]. Iglesias et al. [15] describe the market as a closed, multi-class BCMP queuing network which captures the randomness of customer arrivals. They assume that the number of customers is fixed, since customers only change their locations but don’t leave the network. In contrast, the number of customer are dynamic in our model and we only consider the one who asks for a ride (or sends a request to the platform). Banerjee et al. [3] also use a queuing theoretic approach to analyze the ride-share market and mainly focus on the behaviors of drivers and customers. They assume that drivers enter or leave the market with certain possibilities. Bimpikis et al. [6] take account for the spatial dimension of pricing schemes in the ride-share market. They price for each region and their goal is to rebalance the supply and demand of the whole market, but we price for each routing and aim to maximize the total revenue or social welfare of the platform. We also refer the

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3One major operation cost of bike-sharing companies (Ofo and Mobike) is to redistribute the bikes in the city.
readers to the line of researches initiated by [19] for the problems about the car-pooling in the ride-sharing systems [1][26][7].

Many works on ride-sharing consider both the customers and the drivers to be strategic, where the drivers may reject the requests or leave the system if the prices are too low [3][11]. As we mentioned, if the revenue sharing ratios between the platform and the drivers can be dynamic, then the pricing problem and the revenue sharing problem could be independent and hence the drivers are non-strategic in the pricing problem. In addition, the platform can also increase the profit by adopting dynamic revenue sharing schemes [2].

Another work closely related to ours is by Banerjee et al. [4]. Their work is concurrent and has been developed independently from ours. In particular, the customers arrive according to a queuing model and their pricing policy is state-independent and depends on the transition volume. Both their and our models are built upon the underlying Markovian transitions between the states (the distribution of drivers over the graph). The major differences are: (i) our model is built for the dynamic environments with a very large number of customers (each of them is non-atomic) to meet the practical situations, while theirs adopts discrete agent settings; (ii) they overcome the non-convexity of the problem by relaxation and focus only on concave objectives, while we solve the problem via randomized pricing and transform the problem to a convex program; (iii) they prove approximation bounds of the relaxation problem, while we give exact optimal solutions of the problem by efficiently solving the convex program.

2 Model

We propose a discrete time model. Our model incorporates geographical information as well as requests and driver distributions. Two settings are considered: dynamic environments with a finite time horizon and static environments with an infinite time horizon.

2.1 Geographical information

We use a strongly connected digraph $G = (V, E)$ to model regions of a city, where a node $v \in V$ refers to a region. In particular, we will not distinguish the locations within the same region (i.e., at the same node). Each directed edge $e = (s, t)$ refers to a pair of valid origin and destination and hence the graph $G$ is not a planar graph in general. Each edge $e \in E$ is associated with a cost $c_e(\tau) \geq 0$ for completing the request from $s$ to $t$, where $\tau$ denotes the time step. This cost may be the compensation for the driver’s fuel cost or the emission pollutants from vehicles in reality. It is without loss of generality to assume that the travel time of each edge is exactly in a unit time (see Section 7 for discussions on the generalization to non-uniform traveling time settings). With this assumption, the drivers are always on some node of the graph (no driver is on the way between two nodes) at the end of each time step.

2.2 Drivers

For any $v \in V$, let $w_v(\tau)$ be the number of drivers at node $v$ at time step $\tau$. We assume that the total number of drivers is very large, which is often the case in practice, and consider each of the driver to be non-atomic. Moreover, we normalize the total amount of drivers to be 1 and thus $w_v(\tau)$ is a real number in $[0, 1]$.

2.3 Requests

Each request is a tuple $(e, x)$, where $e = (s, t)$ represents that a passenger sends a request for a ride from node $s$ to $t$, and $x \geq 0$ is the passenger’s value for the ride, i.e. the passenger is willing to pay at most $x \geq 0$ for the ride. At the end of each time step, the platform determines a price $p$ for each request, and the request is served if $x \geq p$ and rejected otherwise. Here we adopt the common setting of short live passengers, that is, the passenger only lives in the system during this ride (the ride ends immediately if being rejected).

We also assume that the number of requests in a time step is large and consider non-atomic requests. We normalize the number of requests on each edge with the total number of drivers on the graph. Note that the amount of requests on an edge $e$ can be more than 1, if there are more requests on $e$ than the total drivers on the graph. We use the demand function $D_e(p|\tau)$ to describe the effect of the price $p$. In particular, $D_e(p|\tau)$ is the amount of served requests on $e$ at time step $\tau$, if the price is $p$. We also assume that $D_e(p|\tau)$ is continuous and monotone decreasing with respect to the price $p$. 

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2.4 Prices and objectives

Aligned with the common practice, the platform is only allowed to set prices for the requests. In particular, the prices for requests along the same edge should be the same (while randomized prices drawn from the same distribution is allowed), but the prices for requests along different edges could be different. Thus the price is a function of $e$.

Once the price $p(e)$ is set, a request $(e, x)$ with $x < p(e)$ will be rejected and the corresponding passenger leaves the system. A straightforward constraint for the pricing problem is that the amount of served requests that start from a node $s$ must be more than the amount of drivers currently on $s$.

In this paper, we are interested in maximizing either the platform’s revenue (sum of collected payments minus the costs) or the social welfare of the entire system (sum of the values of accepted requests minus the costs). While in general, our techniques work for any general objectives that is irrelevant of the driver distribution.

2.5 Dynamic and static environments

In the dynamic environment setting, the costs $c_t(e)$ and the demand function $D_t(p_e)$ change dynamically over time. We consider the finite horizon case where there are $T$ discrete time steps and at each time step $\tau \in [T]$, the graph $G$ is unchanged while $c_t(e)$ and $D_t(p_e)$ on each edge $e$ might be different. We assume that $c_t(e)$ and $D_t(p_e)$ are known to the platform ahead of time.

In the static environment setting, we assume that $c_t(e)$ and $D_t(p_e)$ do not change over time ($c_t(e) \equiv c(e)$ and $D_t(p_e) \equiv D(p_e)$). In this setting, we consider the infinite horizon case, and focus on stable policies that preserve a certain system state.

3 Problem Formulation

Since we normalize the total amount of drivers to be 1, we have:

$$\sum_{v \in V} w_{\tau}(v) = 1, \forall \tau \in [T]$$

Let $q_{\tau}(e) = D_{\tau}(p_{\tau}(e)|e)$ be the flow (the amount of accepted requests) along edge $e$. Then $p_{\tau}(e) = D_{\tau}^{-1}(q_{\tau}(e)|e)$ can be uniquely determined since the demand function is monotone decreasing. It is clear that

$$w_{\tau+1}(v) = w_{\tau}(v) - \sum_{t \in V, e=(v, t) \in E} q_{\tau}(e) + \sum_{s \in V, e=(s, v) \in E} q_{\tau}(e)$$

The above equation indicates that once the driver distribution $w_{\tau}(v)$ and the price $p_{\tau}(e)$ are given, the driver distribution for the next time step $w_{\tau+1}(v)$ is uniquely determined. Therefore, the dispatching problem can be formulated as an MDP (Markov decision process): driver distribution $w_{\tau}(v)$ is the state of the graph, and the Equation[3.1] is the transition function which is implied by the prices and the states. Our goal is to dynamically set prices $p_{\tau}(e)$ for each edge to maximize a certain objective function.

We consider a class of state-irrelevant objective functions in our model. A function is state-irrelevant if its value only depends on the request flows on each edge $q_{\tau}(e)$ but not the state of the system $w_{\tau}(v)$. Let $g_{\tau}(q|e)$ be any such function. We will only use this function as our objective from now on and do not consider any specific objective.

Remark 3.1. Note that a wide range of objectives, such as the revenue and the welfare, are included in our class of objectives:

- **Revenue**: $g_{\tau}(q_{\tau}(e)|e) = (p_{\tau}(e) - c_{\tau}(e)) \cdot q_{\tau}(e)$;
- **Welfare**: $g_{\tau}(q_{\tau}(e)|e) = \int_0^{q_{\tau}(e)} (p_{\tau}(e) - c_{\tau}(e)) dq$.

3.1 Randomized pricing

We allow randomized decisions where the price $p_{\tau}(e)$ is drawn from a distribution $Q_{\tau}(p|e)$. Note that the randomization is performed per-request, thus the total flow along edge $e$ at time step $\tau$ is $E_{p_{\tau}(e)}[q_{\tau}(e)]$. Let $\hat{g}_{\tau}(q|e)$ (ironed objective function) be the smallest concave function that upper-bounds $g_{\tau}(q|e)$ (see Figure[1]). The following theorem enables us to drastically simplify our model and use a deterministic flow $q_{\tau}(e)$ at each edge as our decisions.
Theorem 3.2. Given any \( \hat{q} \in [0,1] \), the optimal expected objective at time \( \tau \) that a randomized pricing scheme \( Q_\tau(p|e) \) with \( \mathbb{E}_{p_\tau(e)}[q_\tau(e)] = \hat{q} \) can achieve is \( \hat{g}_\tau(\hat{q}|e) \).

Proof. By definition of the function \( \hat{g}_\tau(q|e) \), we have:

\[
\mathbb{E}_{p_\tau(e)}[g_\tau(q|e)] \leq \mathbb{E}_{p_\tau(e)}[\hat{g}_\tau(q|e)]
\]

Since \( \hat{g}_\tau(q|e) \) is a concave function, applying Jensen’s inequality yields:

\[
\mathbb{E}_{p_\tau(e)}[\hat{g}_\tau(q|e)] \leq \hat{g}_\tau(\mathbb{E}_{p_\tau(e)}[q_\tau(e)]|e) = \hat{g}_\tau(\hat{q}|e)
\]

Now it suffices to show that the upper bound \( \hat{g}_\tau(\hat{q}|e) \) is attainable.

If \( \hat{g}_\tau(\hat{q}|e) = g_\tau(\hat{q}|e) \), then setting a deterministic price \( p_\tau(e) \) such that \( \hat{q} = D_\tau(p_\tau(e)|e) \) achieves the optimal value. Otherwise, let \( I = (a, b) \) be the ironed interval (where \( \hat{g}_\tau(q|e) \neq g_\tau(q|e), \forall q \in I \) but \( \hat{g}_\tau(a|e) = g_\tau(a|e) \) and \( \hat{g}_\tau(b|e) = g_\tau(b|e) \)) containing \( \hat{q} \). Thus \( \hat{q} \) can be written as a convex combination of the two end points \( a \) and \( b \): \( \hat{q} = \lambda a + (1 - \lambda) b \). It is clear that the function \( \hat{g}_\tau \) is linear in the interval \( I \). Therefore

\[
\lambda g_\tau(a|e) + (1 - \lambda) g_\tau(a|e) = \lambda \hat{g}_\tau(a|e) + (1 - \lambda) \hat{g}_\tau(a|e) = \hat{g}_\tau(\lambda a + (1 - \lambda) b|e) = \hat{g}_\tau(\hat{q}|e)
\]

Let \( p_a \) and \( p_b \) be the prices such that \( q_a = D_\tau(p_a|e) \) and \( q_b = D_\tau(p_b|e) \). The above equation indicates that if we set price \( p_a \) with probability \( \lambda \) and \( p_b \) with probability \( 1 - \lambda \), we can achieve the optimal objective while at the same time ensure that the total flow in this time step is still \( \hat{q} \). \hfill \Box

3.2 Markov decision process

The driver distribution for the next time step \( w_{\tau+1}(v) \) depends on the expected flows of the edges. Thus replacing the randomized decision \( Q_\tau(p|e) \) by a deterministic one \( \hat{q} \) does not affect the state transition. And according to Theorem 3.2, randomized pricing can increase objective in some cases. In fact, the theorem implies that any optimal strategy with expected flow \( \hat{q} \) achieves an objective value of \( \hat{g}_\tau(\hat{q}|e) \). And given an expected flow \( \hat{q} \), we can quickly construct a randomized pricing scheme that achieves objective \( \hat{g}_\tau(\hat{q}|e) \). Therefore, we use the expected flow for each edge as our decisions from now on, and use the ironed function \( \hat{g}_\tau(\hat{q}|e) \) to compute the objective. We also abuse notation and use \( q_\tau \) to denote the expected flow \( \hat{q}_\tau \).

Now we give the formal definition of the dispatching problem.

Definition 3.3. The vehicle dispatching problem is a Markov decision process, denoted by a tuple \( (G, C, F, S, A, W) \) where:

- \( G = (V, E) \) is the given graph;
- \( C = \{c_\tau(e) | \forall \tau \in [T], \forall e \in E \} \) is the set of all cost functions for all edges;
- \( F = \{F_\tau(x|e) | \forall \tau \in [T], \forall e \in E \} \) is the set of all request distributions;
- \( S = \Delta(V) \) is the state space, where \( \Delta(V) \) is the set of all possible probability distributions over the set \( V \);
- \( A = \{a_\tau(e) | \forall \tau \in [T], \forall e \in E \} \) is the set of all actions;
• $A = [0, 1]^{|E|}$ is the action space;

• $W$ is Equation 3.2 which describes the state transition rule.

For the finite horizon setting, the objective function is:

$$OBJ = \sum_{\tau \in [T]} \sum_{e \in E} \hat{g}_\tau(q|e)$$

For the infinite horizon setting, we consider the time-average objective:

$$OBJ = \lim_{T \to \infty} \frac{1}{T} \sum_{\tau \in [T]} \sum_{e \in E} \hat{g}_\tau(q|e)$$

4 Dynamic Environment

In this setting, we consider a more general case where the objective function $\hat{g}_\tau(q|e)$ is time-dependent and we focus our attention on the finite time horizon case. Intuitively, the optimization problem here is to find a "valid" time-dependent transition flow on the graph, $q_1(e), \ldots, q_T(e)$, that maximizes the accumulated objective:

$$OBJ = \sum_{\tau \in [T]} \sum_{e \in E} \hat{g}_\tau(q|e).$$

Since for each $e$, the function $\hat{g}_\tau(q|e)$ is a concave function, the objective above is also a concave function with respect to the flows.

4.1 Convex program formulation

We show that all the "valid" flows can be characterized by a set of linear constraints and hence the optimization program is a convex program with linear constraints.

Variables  There are two types of decision variables in our program:

• $q_\tau(e)$: the amount of drivers moving along edge $e$ during period $\tau$;

• $w_\tau(v)$: the amount of drivers located at node $v$ at the beginning of period $\tau$.

Constraints  We need the following three types of constraints:

• The total number of drivers on the graph must be 1 at any time.

$$\sum_{v \in V} w_\tau(v) = 1 \quad \forall \tau \in [T].$$

• The state transition: the amount of drivers at a node is equal to that of the previous time step, plus the arriving driver flow and minus the leaving driver flow.

$$w_\tau(v) - \sum_{t \in V, e=(v,t) \in E} q_\tau(e) + \sum_{s \in V, e=(s,v) \in E} q_\tau(e) = w_{\tau+1}(v) \quad \forall v \in V, \forall \tau \in [T]$$

• The amount of drivers leaving any node $v$ in an time step cannot exceed the amount of drivers that is currently at the node.

$$\sum_{t \in V, e=(v,t) \in E} q_\tau(e) \leq w_\tau(v).$$
It is easy to see that any “valid” transition flow must follow the constraints above. On the other hand, any set of variables satisfying all the constraints above must correspond to a “valid” transition flow. To summarize, we have the following convex program:

$$\begin{align*}
\text{max} & \quad \sum_{\tau=1}^{T} \hat{g}_\tau(q|e) \\
\text{subject to} & \quad \sum_{e \in T} w_\tau(v) = 1 \quad \forall \tau \in [T] \\
& \quad \sum_{e \in \tau} q_\tau(e) \leq w_\tau(v) \quad \forall v \in V, \forall \tau \in [T] \\
& \quad w_\tau(v) - \sum_{e \in \tau} q_\tau(e) + \sum_{e \in \tau} q_\tau(e) = w_{\tau+1}(v) \quad \forall v \in V, \forall \tau \in [T]
\end{align*}$$

**Alternating the constraints** In practice, there could be some alternative or additional constraints (see Section 7 for further discussions). As long as the new constraints are linear, the optimal solution can still be computed efficiently. For example, one can add constraints to specify a minimum/maximum amount of drivers at some place in certain period, or fixed initial and final distributions of drivers on the graph. Both are quite important in practice.

## 5 Static Environment

In this setting, we restrict our attention to the case where the objective function does not change over time, i.e. \(\forall \tau \in [T], \hat{g}_\tau(q|e) \equiv \hat{g}(q|e)\). We aim to find optimal stationary policies that maximize the objective function, i.e., the decisions \(q_\tau\) depends only on the current state \(w_\tau\).

The state space of our MDP problem is the set of all driver distributions over the nodes \(V\). Traditional solutions such as value iteration and policy iteration [5] can not handle uncountable state spaces such as ours. A natural idea is to discretize the state space and find an approximately optimal policy. However, even with discretizations, such algorithms suffer from unacceptable computational costs. Recently, deep reinforcement learning grows rapidly and succeeds in many application. However, training a deep neural network, such as deep Q-learning network [21] or asynchronous advantage actor-critic (A3C) [22] is also time consuming. However, with the introduction of the ironed objective function \(\hat{g}_\tau\), we can restrict our attentions on stable dispatching schemes and we show that for any discretization scheme, the optimal stationary policy of the induced discretized MDP is dominated by a stable dispatching scheme.

**Definition 5.1.** A stable dispatching scheme is a pair of state and policy \((w_\tau, \pi)\), such that if policy \(\pi\) is applied, the following conditions are satisfied:

- **stability:** the state \(w_\tau\) is a stable state, i.e. for any \(v \in V\), \(w_{\tau+1}(v) = w_\tau(v)\)

  or equivalently,

  $$\sum_{e \in \tau} q_\tau(e) = \sum_{e \in \tau} q_\tau(e)$$

  where \(q_\tau = \pi(w_\tau)\).

- **feasibility:** for any \(v \in V\),

  $$\sum_{e \in \tau} q_\tau(e) \leq w_\tau(v)$$

**Definition 5.2.** Let \(M = (G, C, F, S, A, W)\) be the original MDP problem. A discretized MDP \(\mathcal{D}M\) with respect to \(M\) is a tuple \((G_d, C_d, F_d, S_d, A_d, W_d)\), where \(G_d = G\), \(C_d = C\), \(F_d = F\), \(W_d = W\), \(S_d\) is a finite subset of \(S\), and \(A_d\) is a finite subset of \(A\) that contains all feasible transition flows between every two states in \(S_d\).
Lemma 5.3. Let $\mathcal{DM}$ and $M$ be a discretized MDP and the corresponding original MDP. Let $\pi_d : S_d \to A_d$ be a optimal stationary policy of $\mathcal{DM}$. Then there exists a stable dispatching scheme $(w, \pi)$, such that the time-average objective of $\pi$ in $M$ is no less than that of $\pi_d$ in $\mathcal{DM}$.

Proof. Consider policy $\pi_d$ in $\mathcal{DM}$. Starting from any state in $S_d$ with policy $\pi_d$, let $\{w_k\}_0^\infty$ be the subsequent state sequence. Since $\mathcal{DM}$ has finitely many states and policy $\pi_d$ is a stationary policy, there must be an integer $n$, such that $w_m = w_m$ for some $m < n$ and from time step $m$ on, the state sequence become a periodic sequence. Define

$$\hat{w} = \frac{1}{n-m} \sum_{k=m}^{n-1} w_k$$

and

$$\hat{q} = \frac{1}{n-m} \sum_{k=m}^{n-1} \pi_d(w_k)$$

Denote by $\pi_d(w_k|e)$ or $q_d(e)$ the flow at edge $e$ of the decision $\pi_d(w_k)$. Sum the transition equations for all the time steps $m \leq k < n$, and we get:

$$\sum_{k=m}^{n-1} w_{k+1}(v) = \sum_{k=m}^{n-1} w_k(v) - \sum_{k=m}^{n-1} \left( \sum_{e \in E} \pi_d(w_k|e) \right) + \sum_{k=m}^{n-1} \left( \sum_{e \in E} \pi_d(w_k|e) \right)$$

$$\hat{w}(v) = \hat{w}(v) - \left( \sum_{e \in E} \hat{q}(e) \right) + \left( \sum_{e \in E} \hat{q}(e) \right)$$

Also, policy $\pi_d$ is a valid policy, so $\forall v \in V$ and $\forall m \leq k < n$:

$$\sum_{e \in E} q_k(e) \leq w_k(v)$$

Summing over $k$, we have:

$$\sum_{e \in E} \hat{q}(e) \leq \hat{w}(v)$$

Now consider the original problem $M$. Let $w = \hat{w}$ and $\pi$ be any stationary policy such that:

- $\pi(w) = \hat{q}$;
- starting from any state $w' \neq w$, policy $\pi$ leads to state $w$ within finitely many steps.

Note that the second condition can be easily satisfied since the graph $G$ is strongly connected.

With the above definitions, we know that $(w, \pi)$ is a stable dispatching scheme. Now we compare the objectives of the two policies $\pi_d$ and $\pi$. The time-average objective function is not sensitive about the first finitely many immediate objectives. And since the state sequences of both policies $\pi_d$ and $\pi$ are periodic, Their time-average objectives can be written as:

$$OBJ(\pi_d) = \frac{1}{n-m} \sum_{k=m}^{n-1} \sum_{e \in E} \hat{g}(q_d(e)|e) \quad \text{and} \quad OBJ(\pi) = \sum_{e \in E} \hat{g}(\hat{q}(e)|e)$$

Since the function $\hat{g}(x|e)$ is concave, we have:

$$OBJ(\pi_d) = \frac{1}{n-m} \sum_{k=m}^{n-1} \sum_{e \in E} \hat{g}(q_d(e)|e) \leq \sum_{e \in E} \hat{g} \left( \frac{1}{n-m} \sum_{k=m}^{n} q_d(e) \right) |e|$$

$$= \sum_{e \in E} \hat{g}(\hat{q}(e)|e)$$

$$= OBJ(\pi)$$

\qed
With Lemma 5.3, we now only focus on stable dispatching schemes. We formulate the problem of finding an optimal stable dispatching scheme as a convex program with linear constraints:

$$\max \sum_{e \in E} \hat{g}(q|e)$$

subject to

$$\sum_{v \in V} w(v) = 1$$

$$\sum_{e \in E, v \in \pi(v,e) \in E} q(e) \leq w(v) \quad \forall v \in V$$

$$\sum_{e \in E, v \in \pi(v,e) \in E} q(e) = \sum_{v \in V, e \in \pi(s,v) \in E} q(e) \quad \forall v \in V$$

Because \(\hat{g}(q|e)\) is concave, the program is convex. Since all convex programs can be solved in polynomial time, our algorithm is polynomial.

6 Characterization of Optimality

In this section, we characterize the optimal solution via dual analysis. For ease of presentation, we focus on the dual program in the static environment with infinite horizon while remark that the characterization directly extends to the optimal solution in dynamic environments.

**Primal:**

$$\max \sum_{e \in E} \hat{g}(q|e)$$

subject to

$$\sum_{e \in E} q(e) \leq 1$$

$$\sum_{e \in \out(v)} q(e) - \sum_{e \in \in(v)} q(e) = 0 \quad \forall v \in V$$

**Lagrange:**

$$L(q, \lambda, \mu) = -\sum_{e \in E} \hat{g}(q|e) + \lambda \left( \sum_{e \in E} q(e) - 1 \right) + \sum_{v \in V} \mu_v \left( \sum_{e \in \out(v)} q(e) - \sum_{e \in \in(v)} q(e) \right)$$

$$= -\lambda + \sum_{e \in E} \left( -\hat{g}(q|e) + (\lambda + \mu_v - \mu_{v'})q(e) \right),$$

where \(v\) and \(v'\) are the origin and destination of \(e\), i.e., \(e = (v, v')\).

According to the first order KKT conditions, we have

$$\hat{g}'(q^*|e) = \lambda^* + \mu^*_v - \mu^*_{v'}.$$ 

The above variables can be interpreted as follows:

- \(\hat{g}'(q^*|e)\): the marginal contribution (of flow) along edge \(e\).

- \(\lambda^*\): the system-wise marginal contribution of driver population. Note that by the complementary slackness, if \(\lambda^* > 0\), the sum of the total flow must be 1, meaning all drivers are busy. Otherwise, there would be some idle drivers.

- \(\mu_v\): per period potential contribution of unit driver at node \(v\). One can think that if we allow the out flow from node \(v\) to be slightly more than the in flow to node \(v\), then \(\mu_v\) is the contribution of injecting some drivers into the system at node \(v\).
7 Generalizations

Despite of the simple structure of our model, it is flexible enough to extend to many practically important generalizations.

Non-uniform distances As we mentioned while introducing our model, the uniform distance assumption is enforced to simplify the presentation and is not necessary for our techniques to work. More concretely, we show how to construct an instance of our simplified model from any given instance with more general conditions, such as non-uniform distances.

Formally, suppose the traveling time along \( e = (s, t) \) is \( d > 1 \). Then we can add \( d - 1 \) auxiliary nodes into the graph, i.e., \( v_1^e, \ldots, v_{d-1}^e \), and the directed edges connecting them, i.e.,

\[
E' = \{(s, v_1^e), (v_1^e, v_2^e), \ldots, (v_{d-2}^e, v_{d-1}^e), (v_{d-1}^e, t)\}
\]

To ensure the routing behavior being consistent with the non-uniform distance instance, we need to split the objective of traveling along \( e \) to the new edges, i.e., each new edge has objective function

\[
\hat{g}(q|e') = \frac{1}{d} \hat{g}(q|e), \forall e' \in E'
\]

It is not hard to verify that by the above reduction, we can get an instance of our simplified model with size at most \( d' \) times of the original graph, where \( d' \) is the maximum traveling time along one edge in the non-uniform distance instance. In particular, there is a straightforward bijection between the dispatching behaviors of the original and constructed instances. Hence we can always recover the optimal solution on the constructed instance to an optimal solution on the original instance.

Dynamic driver population Similar to the uniform distance assumption, in our framework, we can safely remove the constraint on the fixed driver population. Instead, we can apply different constraints on the driver population in different periods, or one single constraint that specifies the total driver population accumulated overtime. In particular, we find the following special cases quite interesting:

- **Fixed total driver population accumulated overtime.** To do this, one can simply replace the driver population constraints on each period to one single constraint on the total accumulated driver population. In fact, such a constraint can be interpreted as that there is a fixed driver work hour supply. According to [8], it is a very realistic assumption given that most drivers have their fixed target working time per week.

- **Unlimited driver supply, but with increasing marginal cost.** In stead of placing a hard constraint on the driver supply, it is more realistic to allow the platform to have unlimited power of attracting more drivers but at increasing marginal cost. To incorporate this with our convex program, one can simply remove the hard driver population constraints and subtract the additional marginal cost of attracting more drivers from the objective. As long as the marginal cost is weakly increasing, the new program is still convex.
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