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On locally conformally Kähler metrics on Oeljeklaus–Toma manifolds

Received: 4 March 2022 / Accepted: 18 May 2022 / Published online: 10 June 2022

Abstract. We show that Oeljeklaus–Toma manifolds $X(K, U)$ where $K$ is a number field of signature $(s, t)$ such that $s \geq 1$, $t \geq 2$ and $s \geq 2t$ admit no locally conformally Kähler metric. Combined with the earlier results by Dubickas (N Y J Math 20:257–274, 2014) and Oeljeklaus and Toma (Ann Inst Fourier Grenoble 55(1):161–171, 2005) this completely solves the problem of existence of locally conformally Kähler metrics on Oeljeklaus–Toma manifolds.

1. Introduction

Oeljeklaus–Toma manifolds were introduced by Oeljeklaus and Toma [6], as a generalization to higher dimensions of the Inoue surfaces $S_M$ [4]. Very briefly, their construction goes as follows. Fix a number field $K$ having $s \geq 1$ real embeddings and $2t \geq 2$ complex ones, and label its embeddings such that $\sigma_1, \ldots, \sigma_s$ are the real ones, while $\sigma_{s+t+i} = \sigma_{s+i}$ for any $i = 1, \ldots, t$. Let $O_K$ be the ring of integers of $K$, $O_K^*$ the group of units of $O_K$ and $O_K^{*,+}$ the subgroup of $O_K^*$ of totally positive units, that is elements $u \in O_K^*$ such that $\sigma_i(u) > 0$ for all $i = 1, \ldots, s$. Letting $\mathbb{H} := \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$, we see there are natural actions of $O_K$ and respectively of $O_K^{*,+}$ on $\mathbb{H}^s \times \mathbb{C}^t \subset \mathbb{C}^{s+t}$ by

$$a \cdot (x_i)_{i=1,...,s+t} := (x_i + \sigma_i(a))_{i=1,...,s+t}, \quad \forall a \in O_K$$

and respectively

$$u \cdot (x_i)_{i=1,...,s+t} := (\sigma_i(u)x_i)_{i=1,...,s+t}, \quad \forall u \in O_K^{*,+}.$$ 

The combined resulting action of $O_K^{*,+} \rtimes O_K$ is however not discrete in general. Still, in [6] it is shown that one can always find subgroups $U \subset O_K^{*,+}$ such that the action of $U \rtimes O_K$ is discrete and cocompact: the resulting compact complex
manifold is usually denoted $X(K, U)$ and is called an *Oeljeklaus–Toma manifold* (OT, for short). The Inoue surfaces $S_M$ are corresponding to the particular case when $K$ has degree 3 and $U \subseteq \mathcal{O}_K^{*+}$ is any subgroup of finite index.

The OT manifolds have a number of very interesting properties. For instance, they are non-Kählerian [6], do not respect Hodge symmetry nor satisfy the $\partial \bar{\partial}$-lemma (see [5] or [6]), and for appropriate (but rather generic) choices of the group of units $U$, they have no closed proper complex analytic subspaces [7].

As these manifolds do not admit Kähler metrics, it is natural to ask whether other natural metrics (do) exist on them: for a detailed account on this problem, see e.g. [1]. One of the most interesting ones are the *locally conformally Kähler* metrics (LCK, for short): these are those whose associated $(1, 1)$-forms $\omega$ has the property

$$d \omega = \theta \wedge \omega$$

for some closed 1-form $\theta$ (for more details see [2]). The existence of such metrics on OT manifolds $X(K, U)$ can be read off the Galois properties of the group of units $U$. More precisely, it was shown (see [3], appendix by L. Battisti) that:

**Proposition 1.** An Oeljeklaus–Toma manifold $X(K, U)$ admits an LCK metric if and only if for any unit $u \in U$ one has

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)|$$

Already since these manifolds were introduced in [6], it was shown that such metrics exist on $X(K, U)$ when $t = 1$ and do not exist when $s = 1$ and $t \geq 2$. For the remaining possibilities for $(s, t)$, the second-named author showed in [8] that for a certain number of cases, LCK metrics do not exist. The result was widely extended by Dubickas [3]. More exactly he proves

**Proposition 2.** An OT manifold $X(K, U)$ with $t \geq 2$ does not admit LCK metrics except possibly when $s = (2t + 2m)q - 2t$ with $q \geq 2, m \geq 0$.

The goal of this note is to show the non-existence of LCK metrics in these remaining cases.

**2. The results**

**Lemma 1.** Let $\Lambda \subset \mathbb{R}^n$ be a discrete lattice. Then $\Lambda$ cannot be written as a finite union

$$\Lambda = \bigcup_{i=1}^{m} \Lambda_i$$

of sublattices of smaller rank, $\text{rank}_\mathbb{Z}(\Lambda_i) < \text{rank}_\mathbb{Z}(\Lambda), \forall i = 1, \ldots, m$. 


**Proof.** For a lattice \( \Lambda \subset \mathbb{R}^n \) freely generated by some vectors \( e_1, \ldots, e_N \) we let

\[
\Lambda \mathbb{Q} := \left\{ \sum_{j=1}^{N} q_j e_i \mid q_i \in \mathbb{Q}, \forall i \right\};
\]

then \( \Lambda \mathbb{Q} \) is a \( \mathbb{Q} \)-vector space and if \( \Lambda \) is discrete then \( \dim_{\mathbb{Q}}(\Lambda \mathbb{Q}) = \text{rank}_{\mathbb{Z}}(\Lambda) \).

Next, we infer that (3) implies

\[
\Lambda \mathbb{Q} = \bigcup_{i=1}^{m} \Lambda_i \mathbb{Q}
\]

The only inclusion to see here is “\( \subset \)”. Take a vector \( v \in \Lambda \mathbb{Q} \); then \( v = \sum_{j=1}^{N} q_j e_i \) hence \( v = \frac{1}{M} \sum_{j=1}^{N} a_i e_i \) for some \( M \in \mathbb{N}^* \) and \( a_i \in \mathbb{Z} \). But then the vector \( w := \sum_{j=1}^{N} a_i e_i \) is in \( \Lambda \), hence by our assumption \( w \) belongs to some \( \Lambda_i \); it follows that \( v \in \Lambda_i \mathbb{Q} \).

But decomposition (4) leads to a contradiction, since a vector space over an infinite field cannot be written as a finite union of subspaces of smaller dimension. Q.E.D.

**Notations 1.** Let \( K \) be a number field with \( s \) real embeddings and \( 2t \) complex ones; we suppose we labeled the embeddings \( \sigma_i \) \( (i = 1 \ldots s + 2t) \) of \( K \) such that \( \sigma_1, \ldots, \sigma_s \) are the real embeddings and such that \( \sigma_{s+k} = \sigma_{s+k+t} \) for all \( k = 1, \ldots, t \). We denote by \( \Lambda_K \) the image of the units of \( K \) under the logarithmic embedding

\[
l(u) := (\log |\sigma_1(u)|, \ldots, \log |\sigma_{s+t}(u)|) \subset \mathbb{R}^{s+t}.
\]

Dirichlet’s unit theorem tells us that \( \Lambda_K \) is a discrete (and complete) lattice in the hyperplane \( \mathcal{H}_{\text{Dir}} \) given by

\[
(\mathcal{H}_{\text{Dir}}) : x_1 + \cdots + x_s + 2x_{s+1} + \cdots + 2x_{s+t} = 0
\]

If \( L \subset K \) is a number subfield, we will similarly denote by \( \Lambda_L \) the image of the units of \( L \) under the previous embedding.

**Proposition 3.** Let \( K \) be a number field of signature \( (s, t) \). If \( s \geq 1 \), \( t \geq 2 \) and \( s \geq 2t \) then \( \mathcal{O}_K^* \) has no subgroup \( U \) of rank \( s \) such that

\[
\sigma_{s+1}(u)\sigma_{s+t+1}(u) = \cdots = \sigma_{s+t}(u)\sigma_{s+2t}(u)
\]

holds good for any \( u \in U \).

**Proof.** Assume such an \( U \) would exist. First notice that the logarithmic image \( l(U) \) of \( U \) lives on the intersection of the hyperplane \( \mathcal{H}_{\text{Dir}} \) above with the \( t-1 \) hyperplanes \( \mathcal{H}_i, i = 1, \ldots, t-1 \) given by

\[
x_{s+i} = x_{s+i+1}.
\]

Notice that the linear variety \( \mathcal{H}_{\text{Dir}} \cap \left( \bigcap_{i=1}^{t-1} \mathcal{H}_i \right) \) is of dimension \( s \). Q.E.D.
We will prove that there are finitely many sublattices $\Lambda' \subset \Lambda_K$ with $\text{rank}(\Lambda') < s$ such that any element $l(u) \in l(U)$ lives in (at least) one such $\Lambda'$, getting henceforth a contradiction with the Lemma 1. So take an arbitrary element $u \in U$.

If $\deg(u) < [K : \mathbb{Q}]$ then there exists some proper subfield $L \subsetneq K$ such that $u \in L$; in particular $l(u) \in \Lambda_L$. Call $(s', t')$ the signature of $L$; then $\text{rank}(\Lambda_L) = s' + t' - 1$. Letting $d := [K : L]$ we have $s + 2t = d(s' + 2t')$. Hence $s' + 2t' = \frac{s + 2t}{d}$ so

$$s' + t' - 1 = \frac{s + 2t}{d} - t' - 1.$$

Now

$$\frac{s + 2t}{d} - t' - 1 < s \Leftrightarrow s + 2t < ds + d(t' + 1) \Leftrightarrow 2t - d(t' + 1) < (d - 1)s.$$

But $2t - d(t' + 1) < 2t$ and $(d - 1)s \geq s$ as $d \geq 2$ (since $L \subsetneq K$). Hence $\text{rank}(\Lambda_L) < s$.

We are left with the case when $u$ has maximal degree. As $t \geq 2$ the relation

$$\sigma_{s+1}(u)\sigma_{s+t+1}(u) = \sigma_{s+2}(u)\sigma_{s+t+2}(u) \quad (5)$$

holds good. As the absolute Galois group $G_{\mathbb{Q}/\mathbb{Q}}$ acts transitively on the Galois conjugates of $\sigma_{s+1}(u)$, we see there exists some $\varphi$ in the Galois group of the normal closure of $K$ such that $\varphi(\sigma_{s+1}(u)) = \sigma_1(u)$. Applying $\varphi$ to relation (5) we get

$$\sigma_1(u)\sigma_j(u) = \sigma_k(u)\sigma_1(u) \quad (6)$$

for some $j, k, l \in \{2, \ldots, s+2t\}$. Taking absolute values we see that the logarithmic image of $u$ lives in the hyperplane $\mathcal{H}_{j/k'}$ of $\mathbb{R}^n$ given by

$$(\mathcal{H}_{j/k'}): x_1 + x_{j'} = x_{k'} + x_{l'}$$

where $j', k', l'$ equals respectively $j, k, l$ if they are $\leq s + t$ or $j - t, k - t, l - t$ otherwise. Since $s \geq 1$ we see that

$$\dim \left( \mathcal{H}_{j'/k'} \cap \mathcal{H}_{Dir} \cap \left( \bigcap_{i=1}^{t-1} \mathcal{H}_i \right) \right) = s - 1 < s$$

so

$$\Lambda_{j'/k'} := \Lambda_K \cap \left( \mathcal{H}_{j'/k'} \cap \mathcal{H}_{Dir} \cap \left( \bigcap_{i=1}^{t-1} \mathcal{H}_i \right) \right)$$

is a lattice of rank $< s$ since it is discrete.

We conclude that any $l(u), u \in U$ lives either in a lattice of the form $\Lambda_L$ with $L \subsetneq K$ a proper subfield or in a lattice of the form $\Lambda_{j'/k'}$ as above; as all these lattices are of rank $< s$ and they are finitely many, we got our contradiction.

**Corollary 1.** If an Oeljeklaus–Toma manifold $X(K, U)$ admits an LCK metric, then $K$ has exactly $2t = 2$ complex embeddings.
The proof follows at once from the above Propositions 1, 2 and 3.

Acknowledgements The authors thank Liviu Ornea for useful discussions and valuable suggestions and Alexandra Otiman for a careful reading of a previous version of the present note. We also thank the anonymous referee for pointing us some unclarities.

Data availability The manuscript has no associated data.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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