Dirac submanifolds of Jacobi manifolds

by

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ABSTRACT. The notion of a Dirac submanifold of a Poisson manifold studied by Xu is interpreted in terms of a general notion of tensor fields soldered to a normalized submanifold. This interpretation is used to define the notion of a Dirac submanifold of a Jacobi manifold. Several properties and examples are discussed.

1 Normalized submanifolds

In this section we make some general considerations on submanifolds of a differentiable manifold. These considerations were inspired by the theory of Dirac submanifolds of a Poisson manifold developed in 

Definition 1.1 Let $N^n$ be a submanifold of $M^m$ (indices denote dimensions), and $\iota : N \subseteq M$ the corresponding embedding. A normalization of $N$ by a normal bundle $\nu N$ is a splitting

$$TM|_N = TN \oplus \nu N$$

($T$ denotes tangent bundles). A submanifold endowed with a normalization is called a normalized submanifold.

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\footnote{In this paper everything is of class $C^\infty$ and all submanifolds are embedded.}
Then, if \( X \in \Gamma TM \) (\( \Gamma \) denotes spaces of global cross sections) is a vector field on \( M \) such that \( X|_N \in \Gamma TN \), respectively \( X|_N \in \Gamma \nu N \), \( X \) is said to be tangent, respectively normal, to \( N \).

Let \( (N^n, \nu N) \) be a normalized submanifold of \( M^m \). Let \( \sigma : W \to N \) be a tubular neighborhood of \( N \) such that \( \forall x \in N, T_x(W_x) = \nu_x N \) (\( W_x \) is the fiber of \( W \) and \( \nu_x N \) is the fiber of \( \nu N \) at \( x \)). Then \( W \) is said to be a compatible tubular neighborhood of \( N \); obviously, such neighborhoods exist.

Furthermore, each point \( x \in N \) has a \( \sigma \)-trivializing neighborhood \( U \) endowed with coordinates \( (x^a) (a, b, c, \ldots = 1, \ldots, m - n) \) on the fibers of \( \sigma \) and such that \( x^a|_{N \cap U} = 0 \), and coordinates \( (y^u) (u, v, w, \ldots = m - n + 1, \ldots, m) \) on \( N \cap U \). We will say that \( (x^a, y^u) \) are adapted local coordinates.

With respect to adapted coordinates, \( N \) has the local equations \( x^a = 0 \), and

\[
(2) \quad TN|_{U \cap N} = \text{span} \left\{ \frac{\partial}{\partial y^u} \bigg|_{x^a=0} \right\}, \quad \nu N|_{U \cap N} = \text{span} \left\{ \frac{\partial}{\partial x^a} \bigg|_{x^a=0} \right\}.
\]

Accordingly, the transition functions between two systems of adapted local coordinates must be of the form

\[
(3) \quad \tilde{x}^a = \tilde{x}^a(x^b, y^v), \quad \tilde{y}^u = \tilde{y}^u(y^v),
\]

and satisfy the conditions

\[
(4) \quad \frac{\partial \tilde{x}^a}{\partial y^v} \bigg|_{x^b=0} = 0, \quad \frac{\partial \tilde{y}^u}{\partial x^b} \equiv 0.
\]

The splitting (1) induces a similar relation for the dual bundles:

\[
(5) \quad T^*M|_N = T^*N \oplus \nu^*N,
\]

and, locally, with respect to adapted coordinates one has

\[
(6) \quad T^*N = \text{ann}(\nu N) = \text{span}\{dy^u|_{x^a=0}\}, \quad \nu^*N = \text{ann}(TN) = \text{span}\{dx^a|_{x^a=0}\}
\]

\( (\text{ann}) \) denotes annihilator spaces).

Two normal bundles \( \nu N \) and \( \tilde{\nu} N \) of the same submanifold \( N \) are connected as follows. Let \( p_\nu, p_T \) be the projections defined by the splitting (1), and \( p_{\tilde{\nu}}, p_{\tilde{T}} \) the similar projections of the second normalization. The mapping
$(v \in \nu N) \mapsto p_\nu v$ is an isomorphism $\varphi : \nu N \to \tilde{\nu}N$, and with respect to adapted coordinates we have

$$\tilde{\nu}N = \text{span}\{\varphi \left( \frac{\partial}{\partial x^a} \bigg|_{x^c=0} \right) = X_a|_{x^c=0}\},$$

where

$$X_a = \frac{\partial}{\partial x^a} - \theta^u_a \frac{\partial}{\partial y^u}$$

are vector fields on $U$, and $(\theta^u_a|_{x^c=0})$ is the local matrix of the homomorphism $\psi : \nu N \to TN$ defined by $\psi(v) = \tilde{p}_T(v)$ $(v \in \nu N)$. Notice that $\varphi + \psi$ is the inclusion of $\nu N$ in $TM|_N$, and $\tilde{\nu}N$ is uniquely determined by any of the mappings $\varphi, \psi$.

Some geometric objects of $M$ may have a strong relationship with the normalized submanifold $(N, \nu N)$.

**Definition 1.2** A differential $k$-form $\kappa \in \Omega^k(M)$ ($\Omega$ denotes spaces of differential forms) is *soldered* to $N$ if $\iota^*[L_X\kappa] = 0$ for any vector field $X \in \Gamma TM$ normal to $N$.

Since $\forall f \in C^\infty(M)$ and any vector fields $X, Y_1, \ldots, Y_k$, one has

$$(L_f X \kappa)(Y_1, \ldots, Y_k) = f(L_X \kappa)(Y_1, \ldots, Y_k)$$

$$- \sum_{j=1}^k (-1)^j (Y_j f)[i(X)\kappa](Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_k),$$

it follows that $\kappa$ is soldered iff for any vector field $X \in \Gamma TM$ normal to $N$ one has

$$\iota^*[i(X)\kappa] = 0, \ i^*[L_X\kappa] = 0,$$

or, equivalently,

$$\iota^*[i(X)\kappa] = 0, \ i^*[i(X)\kappa] = 0.$$  

\(^2\)In this paper, we use the Einstein summation convention.
With respect to adapted local coordinates, $\kappa$ has the expression
\[
\kappa = \sum_{s+i=k} \frac{1}{s!i!} \kappa_{a_1 \ldots a_s u_1 \ldots u_i} dx^{a_1} \wedge \cdots \wedge dx^{a_s} \wedge dy^{u_1} \wedge \cdots \wedge dy^{u_i},
\]
and $\kappa$ is soldered to $N$ iff
\[
k_{a_1 \ldots u_{k-1}}|_{x^b=0} = 0, \quad \frac{\partial \kappa_{a_1 \ldots u_k}}{\partial x^a}|_{x^b=0} = 0.
\]
In particular, the space of soldered functions is
\[
C^\infty(M,N,\nu N) = \{f \in C^\infty(M) / (\partial f/\partial x^a)_{x^a=0} = 0\}.
\]

We will denote by $\Omega^k(M,N,\nu N)$ the space of soldered $k$-forms (for $k = 0$ we have the space (14)). Obviously, the soldering conditions (11) are compatible with the exterior product and the exterior differential. Therefore, we get a cohomology algebra $\oplus_k H^*_sdeR(M,N,\nu N)$, which will be called the soldered de Rham cohomology algebra, defined by the cochain spaces $\Omega^k(M,N,\nu N)$ and the operator $d$. The inclusion in the usual de Rham complex induces homomorphisms
\[
\iota^k : H^k_sdeR(M,N,\nu N) \longrightarrow H^k_{deR}(M).
\]
In principle, the spaces $H^k_sdeR(W,N,\nu N)$, where $W$ is a compatible tubular neighborhood, should provide interesting information about the normalized submanifold $(N,\nu N)$.

**Definition 1.3** A $k$-vector field $Q \in V^k(M)$ ($V$ denotes spaces of multivector fields) is soldered to the normalized submanifold $(N,\nu N)$ if: i) for any $(k-1)$-form $\lambda \in \Gamma \wedge^{k-1}[\text{ann}(\nu N)]$ the vector field defined along $N$ by $i(\lambda)Q|_N$ is tangent to $N$; ii) for any vector field $X$ on $M$ normal to $N$, $(L_X Q)|_{\text{ann}(\nu N)} = 0$.

We will denote by $V^k(M,N,\nu N)$ the space of soldered $k$-vector fields. Using adapted local coordinates we see that $V^0(M,N,\nu N)$ is, again, (14), and for $k = 1$ one has
\[
V^1(M,N,\nu N) = \left\{ Y \in V^1(M) / Y|_N = \eta^u \frac{\partial}{\partial y^u}|_N, \quad \frac{\partial \eta^u}{\partial x^a}|_{x^c=0} = 0 \right\}.
\]
Generally, we have an expression of the form

\[ Q = \sum_{s+t=k} \frac{1}{s!t!} Q_{a_1\ldots a_s u_1\ldots u_t} \frac{\partial}{\partial x^{a_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{a_s}} \wedge \frac{\partial}{\partial y^{u_1}} \wedge \ldots \wedge \frac{\partial}{\partial y^{u_t}}, \]  

(17)

and \( Q \in \mathcal{V}^k(M, N, \nu N) \) iff

\[ Q_{a_1\ldots u_{k-1}} \big|_{x^c=0} = 0, \quad \frac{\partial Q_{a_1\ldots u_k}}{\partial x^a} \big|_{x^c=0} = 0. \]  

(18)

The spaces of soldered forms and multivector fields are components of important algebraic structures namely,

**Proposition 1.1**

1. The space \( \mathcal{V}^1(M, N, \nu N) \) is a Herz-Reinhart Lie algebra over \( (\mathbb{R}, C^\infty(M, N, \nu N)) \).
2. The complex of \( N \)-soldered differential forms is a complex over the Lie algebra \( \mathcal{V}^1(M, N, \nu N) \).
3. The triple \( \bigoplus_k \mathcal{V}^k(M, N, \nu N), \wedge, [, ] \), where \([ , ]\) is the Schouten-Nijenhuis bracket is a Gerstenhaber algebra.

**Proof.** For the definition of the algebraic structures above see, for instance, [2] and [3]. The use of adapted local coordinates shows that if \( f \in C^\infty(M, N, \nu N) \) and \( Y, Z \in \mathcal{V}^1(M, N, \nu N) \) then \( fY \) and \([Y, Z]\) belong to \( \mathcal{V}^1(M, N, \nu N) \), which proves 1. Furthermore, for the same \( Y \) and any \( \kappa \in \Omega^k(M, N, \nu N) \), \( i(Y)\kappa \in \Omega^{k-1}(M, N, \nu N) \), and we get 2. Finally, the exterior product of soldered multivector fields obviously is soldered, and in order to get 3, it remains to prove that the Schouten-Nijenhuis bracket\( [ P, Q ] \) of \( P \in \mathcal{V}^p(M, N, \nu N) \) and \( Q \in \mathcal{V}^q(M, N, \nu N) \) belongs to \( \mathcal{V}^{p+q-1}(M, N, \nu N) \). To see that soldering condition i) is satisfied, we look at the known formula (e.g., [7])

\[ i([P, Q])\omega = (-1)^{(p+1)(q+1)}i(P)d[i(Q)\omega] - i(Q)d[i(P)\omega] \]

\[ +(-1)^pi(P \wedge Q)d\omega, \]

(19)

where \( \omega \) is an arbitrary \((p + q - 1)\)-form on \( M \), and use this formula for

\[ \omega = dx^a \wedge dy^{u_1} \wedge \ldots \wedge dy^{u_{p+q-2}}. \]

\( ^3 \)We take this bracket with the sign convention of the axioms of graded Lie algebras, e.g., [3], Proposition 4.21.
Then, soldering condition ii) follows by using (19) to evaluate the terms of the equality

\[ L_X[P, Q] = [L_X P, Q] + [P, L_X Q], \]

where \( X \) is normal to \( N \), on \( dy^{u_1} \wedge \cdots \wedge dy^{u_p+q-1} \). Q.e.d.

The following proposition extends a result proven for Poisson bivector fields in [8]

**Proposition 1.2** If the involutive diffeomorphism \( \varphi : M \to M \) \((\varphi^2 = \text{Id})\) preserves a \( k \)-form \( \kappa \), respectively a \( k \)-vector field \( Q \), then \( \kappa \), respectively \( Q \), is soldered to the fixed point locus \( N \) of \( \varphi \).

**Proof.** It is well known that \( N \) is a submanifold of \( M \), the tangent bundle \( TN \) consists of the \((+1)\)-eigenspaces of \( \varphi_* \) along \( N \), and \( N \) has a normalization with the normal bundle \( \nu N \) defined by the \((-1)\)-eigenspaces of \( \varphi_* \) along \( N \) [8]. This also implies that \( \text{ann}(\nu N) = T^* N \) consists of the \((+1)\)-eigenspaces, and \( \text{ann}(TN) = \nu^* N \) consists of the \((-1)\)-eigenspaces, of \( \varphi^* \) along \( N \). If \( \varphi^* \kappa = \kappa \), if \( X \) is a normal vector field of \( N \) on \( M \) and \( Y_1, \ldots, Y_{k-1} \) are tangent to \( N \), then

\[ \kappa(X, Y_1, ..., Y_{k-1})|_N = (\varphi^* \kappa)(X, Y_1, ..., Y_{k-1})|_N = -\kappa((X, Y_1, ..., Y_{k-1})|_N \]

hence \( i^*[\iota(X)] \kappa|_N = 0 \). The same holds for \( d\kappa \), therefore, \( \kappa \) is soldered to \( N \). The proof for a \( k \)-vector field \( Q \) is similar, and uses the fact that \( \varphi_*(L_X Q) = L_{\varphi_* X} (\varphi_* Q) \circ \varphi \), for any diffeomorphism \( \varphi : M \to N \). Q.e.d.

The \( N \)-soldered differential forms and multivector fields have a nice interpretation by means of the geometry of the tangent bundle \( TM \), and by looking at the normal bundle \( \nu N \) as a submanifold of the former. (See [8] for the case of a Poisson bivector field.)

The tensor fields of the manifold \( M \) may be lifted to \( TM \) by various processes and, in particular, there exists a complete lift, which comes from the lift of the flow of a vector field [9]. In the case of differential forms and multivector fields, the complete lift has the following coordinate expression

\[ \kappa^C = \frac{1}{(k - 1)!} \kappa_{i_1 \ldots i_k} dz^{i_1} \wedge dz^{i_2} \wedge \cdots \wedge dz^{i_k} + \frac{1}{k!} z^k \frac{\partial \kappa_{i_1 \ldots i_k}}{\partial z^k} dz^{i_1} \wedge \cdots \wedge dz^{i_k}, \]

(20)
Proposition 1.3 i) The differential form $\kappa$ is soldered to the normalized submanifold $(N, \nu N)$ of $M$ iff, $\forall Z \in \Gamma T(\nu N)$, the form $i(Z)\kappa^C$ belongs to the ideal generated by $\Gamma[\text{ann}(T(\nu N))]$. ii) The $k$-vector field $Q$ is soldered to $(N, \nu N)$ iff, $\forall \alpha \in \Gamma[\text{ann}(T(\nu N))]$, $i(\alpha)Q^C$ belongs to the ideal generated by $\Gamma T(\nu N)$.

Proof. On $M$, we use $N$-adapted local coordinates and on $TM$ the corresponding natural vector coordinates as described above. Then, the submanifold $\nu N$ has the local equations $x^a = 0, \dot{y}^u = 0$, and the results are immediate consequences of formulas (20) and (21). Q.e.d.

We remark that it is possible to define soldered symmetric tensor fields, similarly. The following proposition provides a nice example:

Proposition 1.4 Let $M$ be a Riemannian manifold with the metric tensor $g$. Then $g$ is soldered to a submanifold $N$ normalized by the normal bundle $\nu N \perp TN$ iff $N$ is a totally geodesic submanifold.

Proof. If $\nabla$ is the Levi-Civita connection of $g$, i.e., $\nabla g = 0$ and $\nabla$ has no torsion, one has

$$(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).$$

Then, if $X$ is normal and $Y, Z$ are tangent to $N$, the restriction of the previous formula to $N$ yields

$$(L_X g)(Y, Z) = 2(g(b(Y, Z), X),$$

where $b$ is the second fundamental form of the submanifold $N$. Thus, the soldering condition of $g$ is equivalent to $b = 0$. Q.e.d.

The notion of soldering has other interesting extensions too. First, for a multivector field $Q \in \mathcal{V}^k(M)$, condition ii) of Definition 1.3 is itself a
geometric condition, since it is easy to check that if it holds for $X$ normal to $N$ it also holds for $fX$, $\forall f \in C^\infty(M)$. If $Q$ satisfies only this condition, we will call it quasi-soldered to $(N,\nu N)$. In fact, this notion extends to any contravariant tensor field. The notion of a soldered differential form also extends to any covariant tensor field but, it implies algebraic conditions like the first condition (10) too.

Then, we may look at objects that only satisfy the algebraic condition of soldering (e.g., the first condition (10), condition i) of Definition 1.3, etc.), and call them algebraically compatible with the normalized submanifold. If the algebraic condition holds, the Lie derivative in the normal directions yields an important object for the submanifold. For instance, a Riemannian metric is algebraically compatible with any submanifold $N$, if the normal bundle is the $g$-orthogonal bundle of $TN$, and formula (22) shows that $(L_Xg)|_{TN}$ is equivalent with the second fundamental form of the submanifold.

Finally, we may add to the algebraic condition a condition that is weaker than soldering. For instance, a tensor field that is algebraically compatible with a normalized submanifold will be called weakly soldered if, along the submanifold, the Lie derivative in the normal directions are proportional to the pullback of the tensor field to the submanifold. For instance, the Riemannian metric $g$ is weakly soldered to a submanifold $N$ with the $g$-normal bundle if, for some 1-form $\alpha$, one has $(L_Xg)|_{TN} = \alpha(X)g|_{TN}$, for any normal vector field $X$, and this happens iff $N$ is a totally umbilical submanifold.

2 Dirac submanifolds of Poisson manifolds

In this section we recall the definition and characteristic properties of the Dirac submanifolds of a Poisson manifold studied by Xu [8], and give a few additional facts. A Dirac submanifold inherits an induced Poisson structure, and the cotangent Lie algebroid of the latter may be seen as a Lie subalgebroid of the cotangent Lie algebroid of the original manifold. We refer to [7] for generalities of Poisson geometry.

**Definition 2.1** A submanifold $N^n$ of a Poisson manifold $M^m$, with the Poisson bivector field $\Pi$, is a Dirac submanifold if $N$ has a normalization (11) with the following properties:

i) $\sharp_\Pi(ann(\nu N)) \subseteq TN$ ($\sharp_\Pi$ is the morphism $T^*M \to TM$ defined by $\Pi$); if this condition holds, we will say that $\nu N$ is algebraically $\Pi$-compatible;
ii) \( \forall x \in N \) there exists an open neighborhood \( U \) of \( x \) in \( M \) such that \( \forall f, g \in C^\infty(U) \) which satisfy the conditions \( df|_{\nu N} = 0, dg|_{\nu N} = 0 \) one has \( d\{f,g\}|_{\nu N} = 0 \). (\( \{f,g\} \) is the Poisson bracket defined by \( \Pi \)).

**Proposition 2.1** With the notation of Definition 2.1, the submanifold \( N \) is a Dirac submanifold iff there exists a normalization (1) such that the Poisson bivector field \( \Pi \) is soldered to \((N,\nu N)\).

**Proof.** \( \Pi \) is soldered to \((N,\nu N)\) iff with the notation and the adapted coordinates as defined in Section 1, one has

\[
(23) \quad \Pi = \frac{1}{2} \Pi^{ab} \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} + \Pi^{au} \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial y^u} + \frac{1}{2} \Pi^{uv} \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v},
\]

where

\[
(24) \quad \Pi^{au}|_{x^c=0} = 0,
\]

\[
(25) \quad \frac{\partial \Pi^{uv}}{\partial x^a}|_{x^c=0} = 0.
\]

On the other hand, if \( \Pi \) is given by (23), condition i) of Definition 2.1 is equivalent to \( \sharp_\Pi(dy^u) \in \operatorname{span}\{\partial/\partial y^v\} \) along \( N \), which means (24), and condition ii) is equivalent with the fact that \( d\{y^u, y^v\}|_{x^c=0} = 0 \), which is (25). Q.e.d.

Furthermore, from the definition of soldered multivector fields we also get

**Proposition 2.2** The submanifold \( N \) of \((M, \Pi)\) is a Dirac submanifold iff \( N \) has a \( \Pi \)-compatible normal bundle \( \nu N \) such that, for any, normal to \( N \), vector field \( X \) of \( M \), one has

\[
(26) \quad (L_X \Pi)|_{\mathfrak{ann}(\nu N)} = 0.
\]

**Corollary 2.1** Let \( \{X_\alpha\} \) be a family of Poisson infinitesimal automorphisms of \((M, \Pi)\). Then, any submanifold \( N \) such that \( \operatorname{span}\{X_\alpha|_N\} \) is a \( \Pi \)-compatible normal bundle of \( N \) is a Dirac submanifold.
**Proof.** The hypotheses of Corollary 2.1 imply the characteristic conditions stated by Proposition 2.2. Q.e.d.

The $\Pi$-compatibility hypothesis of Corollary 2.1 also has the following meaning. A family $\{X_\alpha\}$ of Poisson infinitesimal automorphisms has an associated generalized distribution $D(X_\alpha)$ spanned by the $\Pi$-hamiltonian vector fields of the functions $f \in C^\infty(M)$ that are constant along the orbits of the vector fields $X_\alpha$, and this distribution is involutive. $\text{span}\{X_\alpha|_N\}$ is a $\Pi$-compatible normal bundle of $N$ iff $D(X_\alpha) \subseteq TN$. If the family $\{X_\alpha\}$ reduces to one hamiltonian vector field $X_\Pi^h$, we have no submanifolds $N$ as in Corollary 2.1 since $N$ should be both tangent and normal to $X_\Pi^h$. But, we may have the required type of submanifolds if the family consists of a single infinitesimal automorphism $X$ that is not a hamiltonian vector field. For instance, if $D(X)$ is a regular distribution, it must be a foliation and the leaves of this foliation are Dirac hypersurfaces of $M$.

Before going on with the discussion of Dirac submanifold, let us also consider some of the situations mentioned at the end of Section 1.

**Definition 2.2** A normalized submanifold $(N, \nu N)$ of the Poisson manifold $(M, \Pi)$ will be called an algebraically Poisson-compatible (a.P.c.) submanifold, respectively a quasi-Dirac submanifold, if the Poisson bivector field $\Pi$ is algebraically compatible, respectively quasi-soldered, to the submanifold.

Thus, the a.P.c. property is characterized by i) of Definition 2.1, respectively, by the local condition (24), and the quasi-Dirac property is characterized by the local condition (25).

The a.P.c. and Dirac properties of a submanifold may hold for more than one normal bundle [8]. A second normal bundle $\tilde{\nu} N$ may be defined by (4), and the corresponding local expression of the Poisson bivector field is obtained by switching to the bases $(X_a, \partial/\partial y^u)$ in (23). Accordingly, for $\tilde{\nu} N$, the a.P.c. condition is equivalent to

$$\left(\Pi^{ab}_b \theta^u_a\right)|_N = 0 \iff \psi \circ (\sharp\Pi|_{\text{ann}(TN)}) = 0,$$

and condition ii) of Definition 2.1 is equivalent to

$$X_c[\Pi^{ab}_b \theta^u_a - \Pi^{av}_a \theta^u_v + \Pi^{cv}_c \theta^v_a + \Pi^{cv}_c] |_{x^c=0} = 0.$$

In view of (24), (25), and (27), (28) becomes

$$\left(\frac{\partial \Pi^{ab}}{\partial x^e} \theta^u_a \theta^v_b + \frac{\partial \Pi^{av}}{\partial x^c} \theta^u_a - \frac{\partial \Pi^{au}}{\partial x^c} \theta^v_a - \frac{\partial \Pi^{uv}}{\partial y^w} \theta^w_c\right)_{x^c=0} = 0.$$
Condition (27) shows that, if $\sharp\Pi|_{\text{ann}(TN)}$ is a surjection, therefore, an isomorphism onto $\nu N$ (equivalently, $\det(\Pi^{ab}) \neq 0$), then $\nu N = \sharp\Pi(\text{ann}(\nu N))$ provides the only normalization which makes $N$ an a.P.c. submanifold of $M$. The submanifolds $N$ such that $\sharp\Pi(\text{ann}(\nu N))$ is a complement of $TN$ in $TM|_{N}$ are called cosymplectic submanifolds, and it is known that they are Dirac submanifolds [8]. Indeed, the a.P.c. property follows from the skew symmetry of $\Pi$, and (25) follows from the following component of the Poisson condition $[\Pi, \Pi] = 0$ in the local adapted coordinates of (23) [7]:

$$[\Pi, \Pi]^{auv}|_{x^c = 0} = 2 \left( \Pi^{ab} \frac{\partial \Pi^{uv}}{\partial x^b} \right)_{x^c = 0} = 0. \tag{30}$$

The following result is, obviously, important

**Proposition 2.3** If $N$ is either an a.P.c. or a quasi-Dirac submanifold, the bivector field $\Pi' = p_T(\Pi|_{N})$ is a Poisson bivector field on $N$. Moreover, in the a.P.c. case $\Pi'$ does not depend on the choice of the normal bundle among those which satisfy Definition 2.2.

**Proof.** From (23), it follows

$$\Pi' = \frac{1}{2} \left( \Pi^{uw} \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \right)_{x^c = 0}. \tag{31}$$

Then, from $[\Pi, \Pi] = 0$ and either (24) or (25) we get

$$[\Pi, \Pi]^{u_1u_2u_3}|_{x^c = 0} = 2 \left( \sum_{Cycl(u_1, u_2, u_3)} \Pi^{u_1w} \frac{\partial \Pi^{u_2u_3}}{\partial y^w} \right)_{x^c = 0} = 0. \tag{32}$$

i.e., $\Pi'$ is a Poisson bivector field on $N$.

Finally, (24) shows that $\iota_{\ast}(\sharp\Pi(dy^n)) = \sharp\Pi(dy^n)$ ($\iota : N \subseteq M$), and this proves the last assertion. Q.e.d.

The Poisson structure $\Pi'$ is said to be induced by the Poisson structure $\Pi$, and was defined and studied for Dirac submanifolds in [8]. In the a.P.c. case, the submanifold has a second fundamental form $(L_X\Pi)|_{\text{ann}(\nu N)}$, which vanishes iff $N$ is a Dirac submanifold.
If \( \sigma : W \to N \) is a compatible tubular neighborhood of \( (N, \nu N) \), the induced Poisson structure is characterized by

\[
\{f, g\}_{\Pi'} = \{f \circ \sigma, g \circ \sigma\}_{\Pi} \circ \iota, \tag{33}
\]

\( \forall f, g \in C^\infty(N) \) and \( \iota : N \subseteq M \). With the same tubular neighborhood, the a.P.c. property is equivalent with the fact that \( \forall f \in C^\infty(N) \), the \( \Pi \)-hamiltonian vector field of \( f \circ \sigma \) is tangent to \( N \) or that one has

\[
\iota_* \circ \sharp_{\Pi'} = \sharp_{\Pi} \circ \sigma^*. \tag{34}
\]

Furthermore, \( N \) is a Dirac submanifold if, besides the above, it also has the property that \( \forall x \in N, \forall X \in T_x(W_x), \forall f, g \in C^\infty(N) \) one has

\[
X \{f \circ \sigma, g \circ \sigma\}_{\Pi} = 0. \tag{35}
\]

We summarize the above remarks in

**Proposition 2.4** A submanifold \( N \) of the Poisson manifold \((M, \Pi)\) is an a.P.c. submanifold iff \( N \) is endowed with a Poisson structure \( \Pi' \) and has a tubular neighborhood \( \sigma : W \to N \) such that conditions (33) and (34) hold. Furthermore, \( N \) is a Dirac submanifold iff it is a.P.c. and condition (35) holds too.

**Remark 2.1** In [8] the author uses the independence of the induced Poisson structure on the choice of the normal bundle to define local Dirac submanifolds as submanifolds \( N \) of the Poisson manifold \((M, \Pi)\) such that, \( \forall x \in N \), there exists an open neighborhood \( U \) in \( M \) where \( N \cap U \) is a Dirac submanifold. Proposition 2.3 shows that a local Dirac submanifold also inherits a well defined, global, induced Poisson structure. In fact, Proposition 2.3 shows that local a.P.c. submanifolds may be defined similarly.

**Remark 2.2** Proposition 2.4 suggests considering submanifolds \( N \) of \((M, \Pi)\) which come endowed with a Poisson structure \( \Pi' \) such that, for some tubular neighborhood \( \sigma : W \to N \), \( \sigma \) is a Poisson mapping or, equivalently, the brackets \( \{f \circ \sigma, g \circ \sigma\}_{\Pi} \), which are defined on \( W \), are constant along the fibers of \( \sigma \). Such submanifolds deserve the name of strong Dirac submanifolds. Obviously, they are Dirac submanifolds, and, with respect to adapted coordinates, one must have \( \partial \Pi'_{uv}/\partial x^a \equiv 0 \).
The complete lift of the Poisson bivector field $\Pi$ is a Poisson structure $\Pi^C$ called the tangent Poisson structure of $\Pi$. The tangent structure is exactly the one induced by the Lie algebroid structure of $T^*M$ defined by $\Pi$ on the total space of its dual vector bundle $TM$. From Proposition 1.3, we get the following characteristic property of Dirac submanifolds

**Proposition 2.5** The submanifold $N$ of $(M, \Pi)$ is a Dirac submanifold iff there exists a normal bundle $\nu N$ which is a coisotropic submanifold of $(TM, \Pi^C)$.

The following result is significant for the next section. Recall that a Poisson structure $\Pi$ is homogeneous if there exists a vector field $Z \in \Gamma TM$ such that

$$L_Z \Pi = -\Pi. \tag{36}$$

**Proposition 2.6** Let $N$ be an a.P.c. submanifold of the homogeneous Poisson manifold $(M, \Pi, Z)$ such that $Z|_N \in \Gamma TN$. Then $(N, \Pi', Z|_N)$, where $\Pi'$ is the induced Poisson structure, also is a homogeneous Poisson manifold.

**Proof.** If the homogeneity condition (36) is evaluated on $(d\varphi, d\psi)$ ($\varphi, \psi \in C^\infty(M))$, one gets the equivalent condition

$$\{\varphi, \psi\}_\Pi = Z\{\varphi, \psi\}_\Pi - [Z, X^\Pi_\varphi]_\psi + [Z, X^\Pi_\psi]_\varphi, \tag{37}$$

where $X$ denotes hamiltonian vector fields.

Now, let $\sigma : W \to N$ be a tubular neighborhood of $N$ where the conditions of Proposition 2.4 hold. Then, (37), written for $\varphi = f \circ \sigma, \psi = g \circ \sigma$ ($f, g \in C^\infty(M)$) and composed by $\iota$, provides the similar condition (57) for $\Pi'$. Q.e.d.

Now, we point out the existence of a specific cohomology related with a Dirac submanifold $(N, \nu N)$ of the Poisson manifold $(M, \Pi)$. Namely, since the Poisson bivector field $\Pi$ is soldered to $N$,

$$\mathcal{V}(M, N, \nu N) = (\mathcal{V}^k(M, N, \nu N), \partial = -[\Pi, \cdot]) \tag{38}$$

is a subcomplex of the Lichnerowicz-Poisson cochain complex of $(M, \Pi)$ (e.g., [7]), therefore, it defines cohomology spaces $H_{sP}^k(M, N, \nu N)$ that will be called soldered Poisson cohomology spaces.
Proposition 2.7  The homomorphism \( \sharp_\Pi : T^*M \to TM \) induces homomorphisms

\[
\sharp^k_\Pi : H^k_{sdeR}(M, N, \nu N) \to H^k_{sP}(M, N, \nu N).
\]  

If the Poisson structure \( \Pi \) is defined by a symplectic form \( \omega \), the mappings (39) are isomorphisms.

Proof. Using adapted local coordinates, it is easy to check that, \( \forall \kappa \in \Omega^k(M, N, \nu N) \), \( \sharp_\Pi \kappa \in V^k(M, N, \nu N) \) and, as for the general Poisson cohomology, \( \partial_\Pi \circ \sharp_\Pi = -\sharp_\Pi \circ d \). This justifies the existence of the homomorphisms (39).

In the symplectic case, \( N \) must be a symplectic \((2n)\)-dimensional submanifold of \((M^{2m}, \omega)\), and \( \nu N \) must be the \( \omega \)-orthogonal bundle of \( TN \). Indeed, each point \( x \in N \) has an open neighborhood with coordinates \((x^a, x^{a*}, y^u, y^{u*})\) \( (a = 1, ..., m - n, u = 2m - 2n + 1, ..., 2m - n, a^* = a + m - n, u^* = u + n) \) such that

\[
\omega = \sum_a dx^a \wedge dx^{a*} + \sum_u dy^u \wedge dy^{u*},
\]

and \( N \) has the local equations \( x^a = 0, x^{a*} = 0 \). Obviously, these coordinates also are adapted coordinates with respect to the \( \omega \)-orthogonal bundle \( \nu N \) of \( TN \). The uniqueness of the normal bundle follows from (27). Then, \( \sharp_\Pi \) has the inverse \( -\flat_\omega \), and the latter induces the inverses of the homomorphisms (39). Q.e.d.

3 Dirac submanifolds of Jacobi manifolds

For a detailed study of Jacobi manifolds we refer the reader to [1] and its references. Jacobi manifolds are a natural generalization of Poisson manifolds namely, a Jacobi structure on a manifold \( M^m \) is a Lie algebra bracket \( \{ f, g \} \) on \( C^\infty(M) \), which is given by bidifferential operators. It follows that one must have

\[
\{ f, g \} = \Lambda(df, dg) + fEg - gEf,
\]

where \( E \) is a vector field and \( \Lambda \) is a bivector field on \( M \) such that

\[
[\Lambda, \Lambda] = -2E \wedge \Lambda, \ L_E\Lambda = 0.
\]

\[\text{[1]}\]

\[\text{[2]}\]

\[\text{[3]}\]

\[\text{[4]}\]The minus sign comes from our sign convention for the Schouten-Nijenhuis bracket in Section 1.
Thus, if $E = 0$ we have a Poisson structure.

The Jacobi structure $(M, \Lambda, E)$ is equivalent with the homogeneous Poisson structure

$$
\Pi = e^{-\tau}(\Lambda + \frac{\partial}{\partial \tau} \wedge E), \quad Z = \frac{\partial}{\partial \tau}
$$

on $M \times \mathbb{R}$; $M$ will then be identified with $M \times \{0\}$. For instance [5], let $\Pi$ be a linear Poisson structure on $\mathbb{R}^n \setminus \{0\}$, and consider the diffeomorphism $S^{n-1} \times \mathbb{R} \approx \mathbb{R}^n \setminus \{0\}$ defined by

$$
x^i = e^{-\tau}u^i, \quad \sum_{i=1}^{n} (u^i)^2 = 1, \quad \tau \in \mathbb{R}.
$$

Then, it is easy to check that $\Pi$ must be of the form (43), which provides a Jacobi structure on $S^{n-1}$. We call this structure a Lichnerowicz-Jacobi structure of the sphere.

On a Jacobi manifold, one may define Hamiltonian vector fields

$$
X_f = \sharp_{\Lambda} df + fE \quad (f \in C^\infty(M)),
$$

and they span a generalized foliation $\mathcal{S}$ such that the leaves of $\mathcal{S}$ are either contact or locally conformal symplectic manifolds. For instance, in the case of a Lichnerowicz-Jacobi structure the leaves are the orbits of the quotient coadjoint representation of a connected Lie group $G$ with the Lie algebra $\mathcal{G}$ of structure defined by the corresponding linear Poisson structure $\Pi$ (see above), i.e., the action defined on $S^{n-1}$ by the coadjoint action of $G$ on $\mathcal{G}^* \setminus \{0\} \approx \mathbb{R}^n \setminus \{0\}$ if $S^{n-1}$ is seen as a quotient space of $\mathcal{G}^* \setminus \{0\}$ [4].

Another important fact we want to recall is that, $\forall \varphi \in C^\infty(M)$, the bracket

$$
\{f, g\}^\varphi = e^{-\varphi}\{e^\varphi f, e^\varphi g\}
$$

is a Jacobi bracket said to have been obtained by a conformal change of the original bracket. The tensor fields of the new bracket are

$$
\Lambda^\varphi = e^\varphi \Lambda, \quad E^\varphi = e^\varphi (E + i(d\varphi)\Lambda).
$$

Now, we begin our considerations on submanifolds.
Definition 3.1 Let \((N, \nu N)\) be a normalized submanifold of the Jacobi manifold \((M, \Lambda, E)\). Then: 1) \(N\) is an almost Dirac submanifold if \(\Lambda\) is an \(N\)-soldered bivector field; 2) \(N\) is an algebraically Jacobi-compatible (a.J.c.) submanifold if \(\Lambda\) and \(E\) are algebraically compatible with the normalization of \(N\); 3) \(N\) is a (quasi-) Dirac submanifold if \(\Lambda\) and \(E\) are (quasi-) soldered to \((N, \nu N)\).

Equivalently, \(N\) is an almost Dirac submanifold if each point \(x \in N\) has a neighborhood with adapted coordinates as in Section 1, such that

\[
\Lambda = \frac{1}{2} \Lambda_{ab} \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} + \Lambda_{au} \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial y^u} + \frac{1}{2} \Lambda_{uv} \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v},
\]

where

\[
\Lambda_{au} \big|_{x^c=0} = 0,
\]

\[
\frac{\partial \Lambda_{uv}}{\partial x^a} \bigg|_{x^c=0} = 0.
\]

Then, \(N\) is a Dirac submanifold if, furthermore, the vector field \(E\) is tangent to \(N\) and has the local expression

\[
E = \epsilon^a \frac{\partial}{\partial x^a} + \epsilon^u \frac{\partial}{\partial y^u},
\]

where

\[
\epsilon^a \big|_{x^c=0} = 0, \quad \frac{\partial \epsilon^u}{\partial x^a} \bigg|_{x^c=0} = 0.
\]

For the quasi-Dirac case, we only have the condition (48) and the second equality (49). Finally, \(N\) is an a.J.c. submanifold if (48) and the first condition (51) hold.

Proposition 3.1 Let \((N, \nu N)\) be either an almost Dirac or an a.J.c. or a quasi-Dirac submanifold of the Jacobi manifold \((M, \Lambda, E)\). Then, \([\Lambda' = p_T(\Lambda|_N), E' = p_T(E|_N)]\) is a Jacobi structure on \(N\). Furthermore, in the almost Dirac and the a.J.c. case \(\Lambda'\) does not depend on the choice of the normalization.
Proof. With local adapted coordinates we have

\[
\Lambda' = \frac{1}{2} \left( \Lambda_{uv} \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \right)_{x = 0},
\]

and

\[
[\Lambda, \Lambda]_{u^1u^2u^3} |_{x = 0} = 2 \left( \sum_{\text{Cycl}(u_1, u_2, u_3)} \Lambda^{u_1a} \frac{\partial \Lambda^{u_2u_3}}{\partial x^a} 
+ \sum_{\text{Cycl}(u_1, u_2, u_3)} \frac{\partial \Lambda^{u_1w}}{\partial y^w} \right)_{x = 0} = [\Lambda', \Lambda']_{u^1u^2u^3}.
\]

Hence, in all the cases of the proposition we have

\[
[\Lambda', \Lambda'] = p_T([\Lambda, \Lambda]|_N) = -2E' \wedge \Lambda'.
\]

Then, an examination of the coordinate expression of \( L_E\Lambda \), where \( E \) and \( \Lambda \) are given by (50) and (47), respectively, shows that the conditions (48) and (49), as well as either (48) and the first condition (51) or (49) and the second condition (51), imply

\[
L_E\Lambda' = p_T(L_E\Lambda|_N) = 0.
\]

Finally, where asserted, the independence of \( \Lambda' \) of the normalization follows from \( \iota_*(\sharp \Lambda'(dy^u) = \sharp \Lambda(dy^u), \iota : N \subseteq M \). Q.e.d.

We notice that formula (53) also implies

**Proposition 3.2** Let \((N, \nu N)\) be a normalized submanifold of the Jacobi manifold \((M, \Lambda, E)\) such that \( \Lambda \) is algebraically compatible with the normalization and \( E \) is a normal field of \( N \). Then \( \Lambda' \) is a Poisson structure on \( N \), and it is independent on the choice of \( \nu N \) among all possible choices that contain \( E|_N \).

The Jacobi or Poisson structures defined on \( N \) by \((\Lambda', E')\) are said to be induced by \((\Lambda, E)\). In the cases where only algebraic compatibility holds, invariants of the second fundamental form type \( (L_X\Lambda)|_{\text{ann}(\nu N)}, [X, E]|_{\text{ann}(\nu N)} \), where \( X \) is normal to \( N \), appear.
Proposition 1.2 allows us to give some simple examples. Consider the Jacobi manifold $M = \mathbb{R}^{3n+1}$ with

$$\Lambda = \sum_{i=1}^{n} u_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \left( t \frac{\partial}{\partial t} \right) \wedge \left( \sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j} \right), \quad E = t \frac{\partial}{\partial t}$$

(the variables of (54) are the natural coordinates of $M$). Then the hyperplane $t = 0$ is the fixed point locus of the involution $(u_i, q^i, p_i, t) \rightarrow (u_i, q^i, p_i, -t)$. This involution preserves the tensor fields (54) hence, the hyperplane $t = 0$ is a Dirac submanifold with an induced Poisson structure. For the same $M$, the involution $(u_i, q^i, p_i, t) \rightarrow (-u_i, -q^i, p_i, t)$ also preserves (54) hence, its fixed point locus, which is the $(n+1)$-plane $u_i = 0, q^i = 0$, is a Dirac submanifold with an induced Jacobi structure. Finally, if we restrict $M$ to the domain $p_i > 0, t > 0$ and consider the involution that sends $t$ to $1/t$ and preserves the other coordinates, only $\Lambda$ of (54) is preserved hence, the fixed point locus, which is the hyperplane $t = 1$, is an almost Dirac submanifold. Moreover, the last involution sends $E$ to $-E$, therefore, $E$ is normal to the submanifold, and the induced structure is a Poisson structure.

Another interesting fact is

**Proposition 3.3** Let $(M, \Lambda, E)$ be a Jacobi manifold, and $(M \times \mathbb{R}, \Pi)$, with $\Pi$ defined by (53), the corresponding homogeneous Poisson manifold. Then, $N$ is an a.J.c. or a Dirac submanifold of the former iff $N \times \mathbb{R}$ is an a.J.c., respectively Dirac, submanifold of the latter.

**Proof.** We will use the lift of $\nu N$ to $N \times \mathbb{R}$ as a normal bundle. $\tau$ of (13) is a coordinate along $N \times \mathbb{R}$, and we see that if $\Lambda$ and $E$ are of the local form (47), (50) then $\Pi$ satisfies the conditions (24), (25), and conversely. Q.e.d.

From Proposition 1.3 it follows that the almost Dirac and Dirac submanifolds of a Jacobi manifold may also be characterized by using the tangent bundle $TM$ namely,

**Proposition 3.4** The normalized submanifold $(N, \nu N)$ is almost Dirac iff the submanifold $\nu N$ of $TM$ is such that $\tau_{\Lambda C}(\text{ann}(T\nu N)) \subseteq TVN$, where $\Lambda C$ is the complete lift of $\Lambda$. Furthermore, $N$ is a Dirac submanifold iff besides the previous condition, one also has $E C \in \Gamma(T\nu N)$, where $E C$ is the complete lift of the vector field $E$. 

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Remark 3.1 The tensor fields \((\Lambda^C, E^C)\) do not define a Jacobi structure on \(TM\). A tangent Jacobi structure can be obtained by considering the Poisson structure induced on the manifold \(TM \times \mathbb{R}\) by the Lie algebroid \(J^1M = T^*M \times \mathbb{R}\) of \((\Lambda, E)\) [4]. Namely, with the notation of (20), (21), if we associate with each cross section \((f, \alpha^i dz^i) \in \Gamma J^1M\) the function \(e^-\tau(f + \alpha^i z^i) \in C^\infty(TM \times \mathbb{R})\), the Lie algebroid bracket of \(J^1M\) yields a Poisson bracket of the specified kind of functions, which extends to a Poisson bracket on \(C^\infty(TM \times \mathbb{R})\). Computations show that the Poisson bivector of this structure is

\[
\tilde{\Pi} = e^{-\tau}[\Lambda^C - \Lambda^V - \mathcal{E} \wedge (E^C - E^V) + \frac{\partial}{\partial \tau} \wedge E^C],
\]

where the upper index \(V\) denotes the vertical lift [9], and \(\mathcal{E}\) is the Euler vector field i.e.,

\[
E^V = z^i \frac{\partial}{\partial z^i}, \quad \Lambda^V = \frac{1}{2} \Lambda^{ij} \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^j}, \quad \mathcal{E} = z^i \frac{\partial}{\partial z^i}.
\]

Accordingly \((\Lambda^C - \Lambda^V - \mathcal{E} \wedge (E^C - E^V), E^C)\) is a Jacobi structure on \(TM\), which deserves the name of tangent Jacobi structure.

A particular class of Dirac submanifolds was studied in [1], and we reprove here

**Proposition 3.5** [1] Assume that \(N\) is a submanifold of \((M, \Lambda, E)\) such that \(\sharp_\Lambda(\text{ann}(TN))\) is a normal bundle \(\nu N\) of \(N\). Then \(N\) is a Dirac submanifold iff the vector field \(E\) is tangent to \(N\). Furthermore, there always exists a conformal change (45), with \(\phi\rvert_N = 0\), such that \(N\) is a Dirac submanifold of \((M, \Lambda^\phi, E^\phi)\).

**Proof.** If \(N\) is a Dirac submanifold, \(E \in \Gamma TN\) by definition. For the converse, we use the normal bundle of the hypothesis, and represent \(\Lambda\) and \(E\) by (17) and (54), respectively. Clearly, the choice of \(\nu N\) is such that \(\sharp_\Lambda(dx^a) \in \Gamma \nu N\) along \(N\), which is equivalent to (48). If this condition holds, the \(auv\)-component of the first equality (42) yields

\[
(\Lambda^{ab} \frac{\partial \Lambda^{uv}}{\partial x^b} + E^a \Lambda^{uv})\rvert_N = 0.
\]
Since $\sharp(\Lambda_{\text{ann}(TN)})$ is normal to $N$ iff the matrix $(\Lambda^{ab})$ is non degenerate, we see that $E$ tangent to $N$ implies (19). Furthermore, if $E|_N \in \Gamma TN$, the second equality (12) yields

\[
\Lambda^{ab} \frac{\partial E^u}{\partial x^b}|_N = 0,
\]

therefore (51) holds.

Concerning the last part of the proposition, (16) shows that the required conformal transformation exists if there exists a function $\varphi \in C^\infty(M)$, which vanishes on $N$ and is such that

\[
E^a|_N = \left(\Lambda^{ab} \frac{\partial \varphi}{\partial x^b}\right)|_N.
\]

Since $(\Lambda^{ab})$ is non degenerate, the conditions for $\varphi$ prescribe the 1-jet with respect to the variables $(x^a)$ of $\varphi$ at the points of $N$. Therefore, a required function $\varphi$ exists around every point of $N$. Then, these local solutions may be glued up by a partition of unity along $N$. (See also the argument of [1]). Q.e.d.

We end by a discussion of Dirac submanifolds of the transitive Jacobi manifolds i.e., locally conformal symplectic (l.c.s.) and contact manifolds [1].

**Proposition 3.6** A submanifold $N$ is an almost Dirac submanifold of the l.c.s. manifold $M$ iff it is Dirac, and this happens iff $N$ inherits from $M$ an induced l.c.s. structure. Moreover, there is only one possible normal bundle, the symplectic orthogonal bundle of $TN$.

**Proof.** Recall that the l.c.s. structure of $M$ is a non degenerate 2-form $\Omega$ such that for some open covering $M = \cup \alpha U_\alpha$, $\forall \alpha$, $\Omega|_{U_\alpha} = e^{-\sigma_\alpha} \Omega_\alpha$, where $\sigma_\alpha$ are functions, $\Omega_\alpha$ are 2-forms and $d\Omega_\alpha = 0$. Equivalently, $d\Omega = \omega \wedge \Omega$, where $\omega$ is the closed 1-form defined by gluing up the local forms $d\sigma_\alpha$ ($\omega$ is called the Lee form). It is known that $M$ is a Jacobi manifold with the structure defined by the bivector field $\Lambda$, where $\sharp_\Lambda = \flat_\Omega^{-1}$, and the vector field $E = \sharp_\Lambda \omega$ [1].

Assume that $N$ is an almost Dirac submanifold with the normal bundle $\nu N$. Then, $\sharp_{\Lambda|_{\text{ann}(TN)}}$ is an isomorphism onto $\nu N$, which, just like (27), ensures the uniqueness of $\nu N$, and $\sharp_{\Lambda|_{\text{ann}(\nu N)}}$ is an isomorphism on $TN$, which is equivalent with the fact that $i^*\Omega (i : N \subseteq M)$ is non degenerate and
provides an l.c.s. structure on \( N \). Accordingly, we may use again Marle’s theorem \([6]\), and find local coordinates of \( N \) on some neighborhood \( U_\alpha \) such that, with the notation of (40), one has

\[
\Omega|_{U_\alpha} = e^{-\sigma_\alpha} \left( \sum_a dx^a \wedge dx^{a^*} + \sum_u dy^u \wedge dy^{u^*} \right) .
\]

(60)

Obviously, this expression of \( \Omega \) implies that \( \nu N \) is \( \Omega \)-orthogonal to \( TN \) and that the coordinates used in (60) are adapted coordinates.

Now, condition (49) applied to (60) becomes \((\partial \sigma_\alpha / \partial x^a)_{x^c=0} = 0\) i.e., \( \omega_a|_{x^c=0} = 0 \). Furthermore, one of the conditions that express \( d\omega = 0 \) is \( \partial \omega_a / \partial y^a = \partial \omega_a / \partial x^a \), whence we also get \((\partial \omega_u / \partial x^a)_{x^c=0} = 0\). Therefore, the Lee form \( \omega \) and the vector field \( E \) too, are soldered to \( N \), and \( N \) must be a Dirac submanifold of \( M \). The converse part of the proposition follows from (60). Q.e.d.

**Proposition 3.7** Let \( M^{2m+1} \) be a contact manifold with the contact 1-form \( \theta \). Then a submanifold \( \iota : N \subseteq M \) is a Dirac submanifold iff \( \iota^* \theta \) is a contact form on \( N \). Furthermore, the normal bundle of \( N \) is unique, and it is the \( d\theta \)-orthogonal bundle of \( TN \).

**Proof.** Recall that \( \theta \) is a contact form iff \( \theta \wedge (d\theta)^m \) vanishes nowhere. A contact form produces a Jacobi structure \([1]\), which consists of the vector field \( E \) defined by

\[
i(E)\theta = 1, \quad i(E)d\theta = 0,
\]

(61)

and the bivector field

\[
\Lambda(df, dg) = d\theta(X_f^\theta, X_g^\theta) \quad (f, g \in C^\infty(M)),
\]

(62)

where the *hamiltonian vector field* \( X_f^\theta \) is defined by

\[
i(X_f^\theta)\theta = f, \quad i(X_f^\theta)d\theta = -df + (Ef)\theta.
\]

(63)

From (63) we get

\[
X_f^\theta = \sharp_{d\theta} df + Ef, \quad \Lambda(df, dg) = d\theta(\sharp_{d\theta} df, \sharp_{d\theta} dg).
\]

(64)

If \( M \) is the contact manifold above, \( M \times \mathbb{R} \) has the Poisson bivector \( \Pi \) given by (43), and it also has the symplectic form

\[
\Omega = e^\tau (d\theta + d\tau \wedge \theta).
\]

(65)
An easy computation shows that all the functions of the form $e^\tau f \in C^\infty(M \times \mathbb{R})$ ($\tau \in \mathbb{R}, f \in C^\infty(M)$) have the same hamiltonian vector fields with respect to $\Pi$ and $\Omega$, therefore, $\sharp_\Pi \circ \flat_\Omega = -Id$.

Now, Proposition 3.3 tells that $N$ is a Dirac submanifold of $M$ iff $N \times \mathbb{R}$ is a Dirac submanifold of $M \times \mathbb{R}$. In the present case, this means that $N \times \mathbb{R}$ is a symplectic submanifold of $(M \times \mathbb{R}, \Omega)$, and it follows that $(N, \iota^*\theta)$ must be a contact manifold, and that the normal bundle must be the one indicated by the proposition. Q.e.d.

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