A classification of fibre bundles over 2-dimensional spaces

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22 November 1999

Abstract

The classification problem for principal fibre bundles over two-dimensional CW-complexes is considered. Using the Postnikov factorization for the base space of a universal bundle a Puppe sequence that gives an implicit solution for the classification problem is constructed. In cases, when the structure group $G$ is path-connected or $\pi_1(G) = 0$, the classification can be given in terms of cohomology groups.

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*Talk given at the Meeting New Developments in Algebraic Topology (July 13-14, 1998, Faro, Portugal)
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1 Introduction

In the present contribution we consider the classification problem for principal fibre bundles. We give a solution for the case when $M$ is a two-dimensional path-connected CW-complex.

A motivation for this study came from calculations in two-dimensional quantum Yang-Mills theories in Refs. [AK1], [AK2]. Consider a pure Yang-Mills (or pure gauge) theory on a space-time manifold $M$ with gauge group $G$ which is usually assumed to be a compact semisimple Lie group. The vacuum expectation value of, say, traced holonomy $T_\gamma(A)$ for a closed path $\gamma$ in $M$ is given by the following formal functional integral:

$$< T_\gamma > = \frac{1}{Z(0)} Z(\gamma), \quad (1)$$

$$Z(\gamma) = \int_A \mathcal{D}A \ e^{-S(A)} T_\gamma(A), \quad (2)$$

where $A$ is a local 1-form on $M$, describing the gauge potential, and $S(A)$ is the Yang-Mills action. The form $A$ is obtained from the connection 1-form $w$ in a principal fibre bundle. Its base space is the space-time $M$ and its structure group is the gauge group $G$. We denote such bundles by $\xi = (E, M, G, p)$, where $E$ is called the total space and $p : E \to M$ is known as the projection map (see the definitions in Sect. 2). For a local cross-section $s$ of $\xi$ the 1-form $A$ is obtained from $w$ as the pull-back $A = s^* w$ (see, for example, [NS], [DFN], [D]). Integration in Eq. (2) is performed over the space $A$ of connections in $\xi = (E, M, G, p)$. In quantum field theory often some heuristic measure of integration is assumed, but in the two-dimensional case the functional integral can be defined rigourously (see [AL]). In general, the space $A$ is not connected but may consist of a number of components $A^{(\alpha)}$ labelled by elements $\alpha$ of some index set $B$. Then the functional integral in (2) is given by a sum over the elements of $B$,

$$Z(\gamma) = \sum_{\alpha \in B} \int_{A^{(\alpha)}} \mathcal{D}A^{(\alpha)} \ e^{-S(A^{(\alpha)})} T_\gamma(A^{(\alpha)}), \quad (3)$$

each term being the functional integral over connections in $A^{(\alpha)}$. The set $B$ of the components of $A$ is in 1-1 correspondence with the set $\mathcal{B}_G(M)$ of non-equivalent principal $G$-bundles over the manifold $M$.

This feature has an analog in quantum mechanics. There functional integrals are calculated over the space of paths which connect two given points, say $x_0$ and $x_1$, in a space-time manifold $M$. Consider the case when $x_0 = x_1$. Then the space of integration
is the space of loops based at $x_0 \in M$. We denote it by $\Omega M$. If the space-time $M$ is multiply connected, then the space of loops splits into components labelled by elements

$$\alpha \in \mathcal{B} = \pi_0(\Omega M, \ast) \cong \pi_1(M, x_0),$$

and the functional integral is given by a sum of integrals over these components.

Similar to the case of quantum mechanics, in gauge theory it is useful to have a characterization of the space $\mathcal{B}_G(M)$ in terms of some objects that can be calculated relatively easy, e.g., in terms of homotopy or cohomology groups. This is essentially the classification problem. Usually two aspects of this problem are considered:

1) Characterize the set $\mathcal{B}_G(M)$ of equivalence classes of principal fibre bundles with a given structure group $G$ over a given space $M$ in terms of some algebraic objects that are easy to calculate and that allow to "enumerate" the bundles;

2) Given two non-equivalent bundles $\xi_1$ and $\xi_2$, find a set of characteristics that allows to distinguish them.

In the present article we consider the first aspect of the problem. We would like to mention that a powerful concept, giving a partial solution of the second aspect of the classification problem, is the concept of characteristic classes [MS].

With the motivation, coming from gauge theory, in the present article we address, in fact, a more general case, namely the classification of principal fibre bundles with structure group that is an arbitrary topological group and with base space that is a two-dimensional path-connected CW-complex. All necessary tools have been already developed in the literature. A method, which in many cases gives a solution of the classification problem and which we closely follow here, is discussed in the lectures by Avis and Isham [AI]. Various particular cases were considered in previous works. For example, a classification of principal fibre bundles with $G = U(1)$ over manifolds was studied in Refs. [K], [AB]. In Ref. [Wi] a classification of principal fibre bundles over two-dimensional manifolds in the case when $G$ is a connected Lie group was obtained. There the classification was given in terms of elements of the group $\Gamma$, specifying the global structure of $G$ through the relation $G = \tilde{G}/\Gamma$, where $\tilde{G}$ is the universal covering group of $G$. However, we did not find in the literature a solution of the classification problem for $G$-bundles with more general base spaces $M$ and more general structure groups. In the present contribution we give a solution of this problem in terms of cohomology groups of $M$ in the case when $M$ is a two-dimensional CW-complex. Taking into account the character of the meeting, we tried to make our discussion rather pedagogical. To this end we gather together necessary definitions and explain in some detail main results and theorems from standard books on algebraic topology that are relevant for the classification problem and that we use in our derivation.
The plan of the article is the following. In Sect. 2 we describe the setting of the problem and recall main definitions. In Sect. 3 the Eilenberg-Maclane spaces and the Postnikov decomposition are discussed. In Sect. 4 we construct a short exact sequence which includes the set of equivalence classes of principal fibre bundles as an element. This is the main result of the article. Particular cases, when this set is characterized in terms of a cohomology group, are also discussed. In the Appendix we list some basic definitions and examples of categories and functors and give a formulation of Brown’s representation theorem.

2 Setting of the problem and main definitions

We begin with some definitions [Sp], [Sw]. Here we work in the category $\mathcal{PT}$ of pointed topological spaces and base-point-preserving continuous maps (see the Appendix). Let $(E, e_0)$ and $(X, x_0)$ be pointed topological spaces. In order to simplify notations we omit the base points in some formulas if this does not cause confusion.

**Definition 2.1** A map $p : E \to X$ is said to have a homotopy lifting property with respect to a topological space $Y$ if for every map $\tilde{f} : Y \to E$ and a homotopy $h : Y \times I \to X$ with $h(y, 0) = p(\tilde{f}(y))$ there exists a homotopy $\tilde{h} : Y \times I \to E$ that covers $h$ (i.e., $p \circ \tilde{h} = h$) and such that $\tilde{h}(y, 0) = \tilde{f}(y)$.

**Definition 2.2** (see [Sp], [Sw], [B], [Wh]) A map $p : E \to X$ is called a fibration if it has the homotopy lifting property with respect to any topological space $Y$. A map $p : E \to X$ is called a weak fibration if it has the homotopy lifting property with respect to all disks $D^n$, $n \geq 0$.

Space $F = p^{-1}(x_0)$, where $x_0$ is the base point of $X$, is called the fibre of the fibration $p$. In general $p^{-1}(x)$ for $x \neq x_0$ need not be homeomorphic to $F$, but they all are of the same homotopy type.

Let $I = [0, 1]$ be the closed unit interval.

**Definition 2.3** The space of all continuous maps $w : I \to X$ with $w(0) = x_0$, topologized by the compact-open topology, is called the space of paths in $(X, x_0)$ starting at $x_0$ and is denoted by $(PX, w_0)$.

The base point of $(PX, w_0)$ is the constant map given by $w_0(t) = x_0$ for all $t \in I$. 
**Definition 2.4** The space of all continuous maps \( w : I \to X \) with \( w(0) = w(1) = x_0 \), topologized by the compact-open topology, is called the space of loops in \((X, x_0)\) based at \( x_0 \) and is denoted by \((\Omega X, w_0)\).

The base point \( \omega_0 \) is the trivial loop \( w_0(t) = x_0 \) for all \( t \in I \). If \( f \) is a map from \( X \) to \( Y \), then it induces the map \( \Omega f : \Omega X \to \Omega Y \), defined in a natural way as \( (\Omega f)(w) = f \circ w \) for \( w \in \Omega X \). Higher loopings are defined by induction: \( \Omega^n X = \Omega (\Omega^{n-1} X) \) and \( \Omega^n f = \Omega (\Omega^{n-1} f) \).

An important example of a fibration is the path-loop fibration of a pointed topological space \((X, x_0)\). The fibration map \( \tilde{p} : (P X, w_0) \to (X, x_0) \) is defined by \( \tilde{p}(w) = w(1) \) for a path \( w \) in \( X \).

**Proposition 2.5** \([Sp]\), \([Sw]\) The map \( \tilde{p} : P X \to X \) is a fibration with fibre \( \Omega X \).

**Definition 2.6** A fibre bundle, denoted by \( \xi = (E, B, F, p) \), is a collection consisting of a total space \((E, e_0)\), a base space \((B, b_0)\), a fibre \((F, e_0)\) and a bundle map \( p : (E, e_0) \to (B, b_0) \) such that there exists an open covering \( \{U\} \) of \( B \) and for each open set \( U \in \{U\} \) there is a homeomorphism \( \phi_U : U \times F \to p^{-1}(U) \subset E \) such that the following property of local triviality holds:

\[
(p \circ \phi_U)(b, x) = p(\phi_U(b, x)) = b
\]

for all \((b, x) \in U \times F\).

A particular fibre over \( b \in B \) is \( p^{-1}(b) \), and \( p^{-1}(b) \) is homeomorphic to \( F \) for all \( b \in B \). We also have the inclusion map \( i : (F, e_0) \hookrightarrow (E, e_0) \).

Now we define the notion of a principal fibre bundle. Let \( G \) be a topological group.

**Definition 2.7** \([Sw]\) A principal fibre bundle with structure group \( G \) is a fibre bundle \( \xi = (E, B, G, p) \) with fibre \( G \) such that:

(i) there exists an open covering \( \{U\} \) of \( B \), and for each open set \( U \in \{U\} \) there is a homeomorphism \( \phi_U : U \times G \to p^{-1}(U) \), satisfying (4);

(ii) \( G \) acts on \( E \) with a right action: \( E \times G \to E \) in such a way that, for any \( b \in U \) and \( g \in G \), \( \phi_U(b, g) = \phi_U(b, g_0) \cdot g \), where \( g_0 \) is the unit in \( G \) and the dot denotes the right action of \( G \) on \( E \).

For a principal fibre bundle with structure group \( G \) the term “principal \( G \)-bundle” is often used. Note that under the action of \( G \) points of \( E \) move along the same fibre. Relations between fibre bundles and fibrations are given by the following propositions.
Proposition 2.8 If $\xi = (E, B, F, p)$ is a fibre bundle, then $p: E \to B$ is a weak fibration.

Proposition 2.9 Suppose that the base space of a fibre bundle is paracompact and Hausdorff; then such fibre bundle is a fibration.

It is clear that when we speak about the classification of principal fibre bundles we mean classification of classes of bundles defined by a natural equivalence relation.

Definition 2.10 Two fibre bundles $\xi_1 = (E_1, B, F, p_1)$ and $\xi_2 = (E_2, B, F, p_2)$ with the same base and same fibre are said to be equivalent (this is indicated by writing $\xi_1 \cong \xi_2$) if there exists a homeomorphism $h: E_1 \to E_2$, satisfying the relation $p_2 \circ h = p_1$. If both bundles are principal fibre bundles with structure group $G$, then $h$ has to satisfy the following additional property:

$$h(e \cdot g) = h(e) \cdot g, \quad g \in G, \quad e \in E.$$ 

The pointed set of all equivalence classes of principal $G$-bundles over $M$ is denoted by $B_G(M)$. The base point of this set is the equivalence class of the trivial $G$-bundle $\xi_0 = (M \times G, M, G, \text{pr}_1)$ with the obvious right action of $G$ on $M \times G$. Here $\text{pr}_1: M \times G \to M$ is the projection onto the first factor, i.e., $\text{pr}_1(x, g) = x$ for $x \in M$ and $g \in G$.

Now we recall the definition of an induced fibre bundle. Consider a fibre bundle $\xi = (E, B, F, p)$ and a map $f: M \to B$ from some space $M$.

Definition 2.11 The fibre bundle induced from $\xi$ by the map $f$, which is denoted by $f^*\xi$, is the fibre bundle with base $M$, fibre $F$, total space $E'$, given by

$$E' = \{(x, e) \in M \times E | f(x) = p(e)\} \subset M \times E,$$

and bundle projection $p': E' \to M$ defined as $p'((x, e)) = x$.

In other words, the induced bundle $f^*\xi$ is constructed by glueing copies of the fibre $p^{-1}(b)$, $b \in B$, over all points $x$ of $M$ such that $f(x) = b$. There is also a map $f': E' \to E$ given by $f'((x, e)) = e$. From the definition of the induced bundle it follows that

$$p \circ f' = f \circ p',$$

i.e., the following diagram is commutative:

5
When $\xi$ is a principal $G$-bundle, the induced fibre bundle $\xi' = f^*\xi$ is also a principal $G$-bundle with a $G$-action on $E' \subset M \times E$ defined by $(x, e) \cdot g = (x, e \cdot g)$ for $x \in M$, $e \in E$ and $g \in G$.

The following theorem is important for the classification of principal fibre bundles (see [Sw]).

**Theorem 2.12** Let $\xi = (E, B, G, p)$ be a principal $G$-bundle with base space being a CW-complex. Let $M$ be a CW-complex, and let $f_1, f_2$ be two maps from $M$ into $B$ that are homotopic, $f_1 \simeq f_2$. Then the induced principal $G$-bundles $f_1^*\xi$ and $f_2^*\xi$ are equivalent, $f_1^*\xi \simeq f_2^*\xi$.

It turns out that for any topological group $G$ there exists a space $BG$, called its classifying space, and a principal $G$-bundle $\xi_G = (EG, BG, G, p_G)$, called the universal $G$-bundle, such that every principal $G$-bundle $\xi = (E, M, G, p)$ is induced from $\xi_G$ by some map $f : M \to BG$ [St]. Two homotopic maps $M \to BG$ induce equivalent bundles. The formal definition is the following.

**Definition 2.13** A bundle $\xi = (E, B, G, p)$ is $n$-universal if for any pointed CW-complex $(M, x_0)$ with $\dim M = (n - 1)$ there is a bijection $[M, x_0; B, b_0] \cong B_G(M)$. A $G$-bundle is universal (or $\infty$-universal) provided it is $n$-universal for any $n \geq 1$; in this case the base space $B$ is called a classifying space of $G$.

The universal bundle and the classifying space are unique up to homotopy equivalence.

The result quoted above can be derived in an elegant way if one considers $B_G(M)$ as a result of the action of a cofunctor $B_G$ and uses Brown’s representation theorem (see Theorem A.9 in the Appendix). We would like to discuss this issue in more detail.

From now on we will be considering fibre bundles whose base spaces are path-connected pointed CW-complexes, but we do not restrict their dimension for the time being. Let $G$ be a topological group. Following Ref. [Sw] let us introduce a cofunctor $B_G : \mathcal{PW}' \to \mathcal{PS}$ as follows (we recall that main definitions, some examples and the notation of categories and functors are listed in the Appendix; see in particular examples E.4 and E.8). For a pointed CW-complex $(X, x_0)$ its image $B_G(X)$ is, as before, the pointed set of all equivalence classes of principal $G$-bundles over $X$. For a map $f : (X, x_0) \to (Y, y_0)$, where
(X, x₀) and (Y, y₀) are objects in \( \mathcal{PW}' \), the morphism \( B_G([f]) \) is the map from \( B_G(Y) \) to \( B_G(X) \) such that for a class \([\xi]\) of \( G \)-bundles over \( Y \) the class \( B_G([f])[\xi] = [f^*\xi] \), where \( f^*\xi \) is the induced \( G \)-bundle over \( X \).

The important property of the cofunctor \( B_G \) is given by the following theorem.

**Theorem 2.14** [Sw] The cofunctor \( B_G : \mathcal{PW}' \to \mathcal{PS} \) satisfies the wedge axiom and the Mayer-Vietoris axiom.

These axioms are given in the Appendix (axioms W) and MV)). The property of \( B_G \), described by Theorem 2.14, allows us to apply Brown’s representation theorem (Theorem A.9 in the Appendix) and, as a consequence, obtain the result, we are interested in. Namely, Brown’s theorem implies that for any \( G \) there exists a CW-complex \((BG, \tilde{b}_0)\) and a principal \( G \)-bundle \( \xi_G = (EG, BG, G, p_G) \) such that for any pointed CW-complex \((M, x_0)\) there is an equivalence

\[
[M, x_0; BG, \tilde{b}_0] \cong B_G(M) .
\]

The correspondence is realized by the natural transformation \( T_{\xi_G} \), constructed according to Example E.12 in the Appendix. In the case under consideration for any map \( f : M \to BG \) we have \( T_{\xi_G}(M)([f]) = [f^*\xi_G] \). Brown’s theorem states that for any \( G \) there exists a CW-complex \((BG, \tilde{b}_0)\) and a principal \( G \)-bundle \( \xi_G = (EG, BG, G, p_G) \) such that for any pointed CW-complex \((M, x_0)\) there is an equivalence

\[
[M, x_0; BG, \tilde{b}_0] \cong B_G(M) .
\]

Another property of \( \xi_G \) can be obtained by considering the long homotopy sequence of a bundle. We recall that for a fibre bundle \( \xi = (E, B, G, p) \) the following exact sequence exists:

\[
\ldots \overset{p_*}{\longrightarrow} \pi_{q+1}(B, b_0) \overset{\Delta}{\longrightarrow} \pi_q(F, e_0) \overset{\iota_*}{\longrightarrow} \pi_q(E, e_0) \overset{p_*}{\longrightarrow} \pi_q(B, b_0) \overset{\Delta}{\longrightarrow} \ldots ,
\]

\[ (8) \]
where the maps $p_*$ and $i_*$ are induced by the bundle projection $p$ and the inclusion map $i : (F, e_0) \rightarrow (E, e_0)$ respectively and $\Delta$ is constructed out of the boundary homomorphism in a standard way \[Sp\], \[Sw\], \[St\]. Let us consider long homotopy sequence (8) of the universal bundle $\xi_G$. Since all homotopy groups of its total space $EG$ are trivial, the long homotopy sequence breaks into short exact sequences. The latter imply that

$$\pi_q(BG, \tilde{b}_0) \cong \pi_{q-1}(G, g_0)$$

for $q \geq 1$ \[St\]. We would like to note that the long homotopy sequence also exists for fibrations.

As an immediate application of relation (9) we obtain the classification of principal $G$-bundles over the sphere $S^n$ in terms of the $(n - 1)$th homotopy group of $G$:

$$B_G(S^n) \cong \pi_n(BG, \tilde{b}_0) \cong \pi_{n-1}(G, g_0).$$

\section{Eilenberg-MacLane spaces and Postnikov decomposition}

A general explicit construction of the universal $G$-bundle $\xi_G$ for any topological group $G$ is due to Milnor (see, for example, \[H\]). For the classical groups $G = SO(k), O(k), SU(k), U(k)$ and $Sp(k)$ there exist more convenient constructions of the universal bundles with total spaces being Stiefel manifolds and classifying spaces being Grassmann manifolds \[DFN\], \[H\]. However, in general it is not easy to describe the set of maps $[M; BG]$. It turns out that in the case, when $M$ is of finite dimension, $BG$ can be substituted by some other space, which may be simpler to construct and to characterize. In order to discuss this it is necessary to first introduce the notion of $n$-equivalence.

\begin{definition}
Let $(X, x_0), (Y, y_0)$ be pointed topological spaces. A map $f : (X, x_0) \rightarrow (Y, y_0)$ is called an $n$-equivalence ($n \geq 1$) if it induces a one-to-one correspondence between the path components of $X$ and $Y$ and for all $x_0 \in X$ the induced map

$$f_* : \pi_q(X, x_0) \rightarrow \pi_q(Y, y_0)$$

is an isomorphism for $0 < q < n$ and an epimorphism for $q = n$. A map $f : (X, x_0) \rightarrow (Y, y_0)$ is called an $\infty$-equivalence or weak homotopy equivalence if it is an $n$-equivalence for all $n \geq 1$.
\end{definition}
For CW-complexes a map is a homotopy equivalence if and only if it is a weak homotopy equivalence (see [Sw]). The importance of the notion of $n$-equivalence is seen from the following theorem.

**Theorem 3.2** [Sw] Let $(Y, y_0)$ and $(Z, z_0)$ be pointed topological spaces and let $f : (Y, y_0) \rightarrow (Z, z_0)$ be an $n$-equivalence. Then for any pointed CW-complex $(M, x_0)$ the induced map $f_* : [M, x_0; Y, y_0] \rightarrow [M, x_0; Z, z_0]$ is a bijection if $\dim M < n$ and is a surjection if $\dim M \leq n$.

Applying this result to the classification of principal $G$-bundles over a pointed CW-complex $(M, x_0)$ with $\dim M \leq n$ we conclude that if there exists a CW-complex $BG_n$ and a map $f : BG \rightarrow BG_n$ which is an $(n + 1)$-equivalence, then

$$B_G(M) \cong [M; BG] \cong [M; BG_n],$$

(11)

i.e., all such bundles are in 1-1 correspondence with homotopy classes of maps $M \rightarrow BG_n$.

For the construction of $BG_n$ Eilenberg-MacLane spaces play an important role.

**Definition 3.3** A space $X$, satisfying the properties:

i) $X$ is path connected;

ii) $\pi_q(X, *) = \begin{cases} \pi, & \text{if } q = n, \\ 0, & \text{if } q \neq n, \end{cases}$

where $\pi$ is a group and $n$ is a positive integer, is called an Eilenberg-MacLane space and is denoted by $K(\pi, n)$.

For $n \geq 2$ the group $\pi$ in this definition must be abelian. It can be shown that for any integer $n \geq 1$ and for any group $\pi$ (abelian if $n \geq 2$) the space $K(\pi, n)$ can always be constructed as a CW-complex and is unique up to weak homotopy equivalence [Sp], [Sw], [Wh]. Here follow some examples of Eilenberg-MacLane spaces.

1. $K(Z, 1) = S^1$.
2. $K(Z, 2) = CP^\infty$, where $CP^\infty$ is defined as the direct limit $CP^\infty = \lim \rightarrow CP^n$, i.e. the CW-complex which is the union of the sequence $CP^1 \subset CP^2 \subset \ldots$ of complex projective spaces, topologized by the topology coherent with the collection $\{CP^j\}_{j \geq 1}$.
3. $K(Z_m, 1) = L^\infty(m)$, the $\infty$-dimensional lens space. In particular,

$$K(Z_2, 1) = RP^\infty = \lim \rightarrow RP^n.$$

(12)
In general, Eilenberg-MacLane spaces are infinite dimensional, case 1 above is an exception.

Let us discuss some properties of Eilenberg-MacLane spaces which are needed for the sequel. We first recall the notion of suspension.

**Definition 3.4** A (reduced) suspension \((SX, z_0)\) of a pointed topological space \((X, x_0)\) is defined as the quotient space of \(X \times I\), where points of
\[(X \times \{0\}) \cup (\{x_0\} \times I) \cup (X \times \{1\})\]
are indentified to a single point.

The base point \(z_0\) is the image of \((x_0, 0) \in X \times I\) under the quotient map \(X \times I \to SX\). It can be shown that if \(X = S^n\), the \(n\)-dimensional sphere, then \(S(S^n)\) is homeomorphic to \(S^{n+1}\) for all \(n \geq 0\).

It turns out that for any two pointed topological spaces \((X, x_0)\), \((Y, y_0)\) the sets \([Y, y_0; \Omega X, w_0]\) and \([SX, z_0; Y, y_0]\) possess group structures. This property follows from the fact that the loop space \((\Omega X, \omega_0)\) is an \(H\)-group and the the suspension \((SX, z_0)\) is an \(H\)-cogroup. In brief an \(H\)-group is a pointed topological space with a binary operation, called a multiplication, which satisfies the group axioms up to homotopy. If a multiplication is homotopy commutative, then the \(H\)-group is said to be an abelian \(H\)-group. An \(H\)-cogroup is a pointed topological space with a co-multiplication that is homotopy associative, with a homotopy inverse, and with a homotopy identity map. An \(H\)-cogroup is called abelian if the co-multiplication is homotopy commutative. For rigorous definitions of \(H\)-groups and \(H\)-cogroups we refer the reader to the literature (see, for example, \([Sp]\), \([Sw]\)). The group structure on \([Y, y_0; \Omega X, w_0]\) (resp. \([SX, z_0; Y, y_0]\)) is induced by the \(H\)-group structure of \((\Omega X, w_0)\) (resp. \(H\)-cogroup structure of \((SX, z_0)\)).

The following two propositions are relevant for us (see \([Sp]\), \([Sw]\)). Let \((X, x_0)\) and \((Y, y_0)\) be pointed topological spaces.

**Proposition 3.5** There is an isomorphism of groups
\[A : [SX, z_0; Y, y_0] \to [X, x_0; \Omega Y, \omega_0].\]

This isomorphism is called an adjoint correspondence. Let us remark that \(\Omega\) can be viewed as a functor from the category \(\mathcal{PT}\) of pointed topological spaces to the category of \(H\)-groups and continuous homomorphisms, whereas \(S\) can be considered as a functor from \(\mathcal{PT}\) to the category of \(H\)-cogroups and continuous homomorphisms. The existence of the isomorphism \(A\) means that the functors \(S\) and \(\Omega\) are adjoint.
Proposition 3.6  The set \([SX, z_0; ΩY, w_0]\) is an abelian group.

Using this it can be shown that for \(n \geq 2\) the space \(Ω^n Y\) is an abelian \(H\)-group and \(S^n X\) is an abelian \(H\)-cogroup. An immediate implication of Proposition 3.6 is that

\[
π_n(X, x_0) = [S^n; X] ≅ [S^{n-1}; ΩX] = π_{n-1}(ΩX, w_0).
\]

Applying this result to the Eilenberg-MacLane space \(K(π, n)\) \((n \geq 1)\) we see that

\[
π_q(ΩK(π, n), *) = π_{q+1}(K(π, n), *) = \begin{cases} π, & \text{if } q = n - 1, \\ 0, & \text{if } q \neq n - 1. \end{cases}
\]

It can be proved that these relations imply the existence of a weak homotopy equivalence between \(K(π, n - 1)\) and \(ΩK(π, n)\) (see [Sw]). Therefore

\[
ΩK(π, n) \simeq K(π, n - 1).
\]

(13)

Hence, the set \([M; K(π, n)]\) carries a natural group structure. Moreover, since \(K(π, n) \simeq Ω^2 K(π, n + 2)\), according to Proposition 3.6 this group is abelian. It can be specified further. Consider the \(n\)th cohomology cofunctor \(H^n\) with coefficients in \(π\) (see Example E.11 in the Appendix). It turns out that when restricted to the category \(PW'\) its classifying space (see Definition A.8) is \(Y = K(π, n)\) [Sp]. This means that there is a natural equivalence between the cofunctors \(π^Y\) and \(H^n\), i.e., for any \((M, x_0) \in PW'\)

\[
[M, x_0; K(π, n), *} \simeq H^n(M; π).
\]

(14)

The next important element, which we need for the classification of principal fibre bundles, is the Postnikov decomposition (called also the Postnikov factorization). First let us recall the notion of a simple space.

Definition 3.7  A pointed topological space \((X, x_0)\) is called simple if \(π_1(X, x_0)\) acts trivially on homotopy groups \(π_n(X, x_0)\) for all \(x_0 \in X\) and all \(n \geq 1\).

Since \(π_1(X, x_0)\) acts on itself by conjugation, then for a path-connected \(X\) simplicity implies that \(π_1(X, x_0)\) is abelian.

In order to formulate the theorem on the Postnikov decomposition we first define some class of fibrations and the notion of a Postnikov system.
Definition 3.8 Let \((B, b_0), (B', b'_0)\) be pointed topological spaces, \(\theta : (B, b_0) \to (B', b'_0)\) a base-point-preserving map, and \(p_\theta : E_\theta \to B\) the fibration, induced by \(\theta\) from the path-loop fibration \(p' : PB' \to B'\). The fibration \(p_\theta\) is called the principal fibration induced by \(\theta\). If \(B'\) is an Eilenberg-MacLane space \(K(\pi, n)\), where \(n \geq 1\) and \(\pi\) is abelian, then \(p_\theta\) is called a principal fibration of type \((\pi, n)\).

We recall that the path-loop fibration was defined in Sect. 2. The fibre of \(p_\theta\) is \(\Omega B'\).

Definition 3.9 Let \((X, x_0)\) and \((Y, y_0)\) be pointed topological spaces, and let \(f : (X, x_0) \to (Y, y_0)\) be a map. A Moore-Postnikov decomposition of \(f\) is a sequence
\[
\ldots \to Y_{n+1} \xrightarrow{p_n} Y_n \to \ldots \to Y_2 \xrightarrow{p_1} Y_1 \xrightarrow{p_0} Y_0
\] (15)
of spaces \(Y_n\) and maps \(f_n : X \to Y_n\) with the following properties:

1. for \(n = 0\) the space \(Y_0 = Y\) and \(f_0 = f\);
2. for \(n \geq 1\) the map \(p_n : Y_{n+1} \to Y_n\) is a principal fibration of type \((\pi_n(X, x_0), n + 1)\), and \(p_0 : Y_1 \to Y_0 = Y\) is a fibration;
3. for \(n \geq 0\) the map \(f_{n+1} : X \to Y_{n+1}\) is a covering map of \(f_n : X \to Y_n\), i.e. \(f_n = p_n \circ f_{n+1}\);
4. for \(n \geq 1\) the map \(f_n\) is an \(n\)-equivalence;
5. if \(n \geq 1\) then \(\pi_q(Y_n) = 0\) for \(q \geq n\).

Definition 3.10 Let \((X, x_0)\) be a pointed topological space. A Postnikov system (or Postnikov decomposition) of \(X\) is the Moore-Postnikov decomposition of the map \(f : X \to Y\), where \(Y\) is the set of path-components of \(X\) with the factor topology and \(f\) is the natural projection.

Thus, if \(X\) is path-connected, then \(Y = \{y_0\}\) is the one-point space and \(f\) is the constant map \(f : X \to y_0\).

Theorem 3.11 (see [S1], [B], [W1], [M]) If \(X\) is a simple pointed path-connected topological space, then there exists a Postnikov system of \(X\).
Let \( \pi_n \) denote the \( n \)th homotopy group \( \pi_n(X, x_0) \) of \( X \). For every \( n \geq 1 \) the fibration \( p_n : Y_{n+1} \to Y_n \) is induced by a map \( \theta_n : Y_n \to K(\pi_n, n+1) \) from the path-loop fibration \( PK(\pi_n, n+1) \to K(\pi_n, n+1) \). Its fibre is \( \Omega K(\pi_n, n+1) \cong K(\pi_n, n) \), see Eq. (13). Due to relation (14) the map \( \theta_n \) corresponds to a characteristic class \( k_{n+1}(X) \in H^{n+1}(Y_n, \pi_n) \) called the Postnikov invariant. It is obtained by the transgression of the so called fundamental class of some auxiliary fibration, which appears in this construction. We refer the reader to Refs. [Sp], [MT] for details.

It can be shown that the spaces \( Y_n \) of the Postnikov system are unique up to homotopy equivalence [VT], [WH]. A method to construct the Postnikov system of a given simple pointed path-connected CW-complex \( X \) is described in Ref. [WH]. It appears also that a space can be reconstructed from its Postnikov system [WH]. Namely, suppose we are given a sequence \( \mathcal{P} = \{\pi_n, Y_{n-1}, p_{n-1}|n \geq 1\} \), also called a Postnikov system (or fibred Postnikov system in [WH]), with the properties: 1) \( \pi_n \) is an abelian group; 2) \( Y_0 \) is contractible; 3) \( p_n : Y_{n+1} \to Y_n \) is a fibration with fibre \( F_n \) being an Eilenberg-MacLane space \( K(\pi_n, n) \); 4) the injection \( \pi_n(F_n) \to \pi_n(Y_{n+1}) \) is an isomorphism. Then the inverse limit \( Y = \lim_{\leftarrow} Y_n \) is well-behaved. In general \( Y \) does not have the homotopy type of a CW-complex. Nevertheless, there exists a CW-approximation \( X_{\mathcal{P}} \) of \( Y \) whose Postnikov system is \( \mathcal{P} \). In other words, if \( \mathcal{P} \) is the Postnikov system of a space \( X \), then \( X \) and \( X_{\mathcal{P}} \) are of the same homotopy type.

The Postnikov decomposition will be used for the classification of principal \( G \)-bundles in the next section. There \( X \) will play the role of \( BG \) and the spaces \( Y_n \) will be \( BG_{n-1} \).

4 Classification of \( G \)-bundles over 2-dimensional CW - complexes

In this section we assume that \((M, x_0)\) is a pointed path-connected CW-complex of dimension \( \dim M = 2 \).

We apply Theorem 3.11 to the space \( BG \), the base space of the universal \( G \)-bundle. Since \( \dim M = 2 \), for the classification of principal \( G \)-bundles over \( M \) it suffices to find a 3-equivalence \( BG \to BG_2 \).

As it was explained above, the total space \( EG \) of the universal \( G \)-bundle is \( \infty \)-connected, hence the base \( BG \) is path-connected. For the conditions of the Postnikov factorization theorem to hold we assume that \((BG, \tilde{b}_0)\) is simple. This implies, in particular, that \( \pi_1(BG) \) is abelian. Let us discuss briefly this point. Recall that for a topological group \( G \) we have \( \pi_0(G, g_0) = G/G_0 \), where \( G_0 \) is the path-connected component of the
group unity, and $\pi_0(G, g_0)$ acts on higher homotopy groups of $G_0$ [51]. According to relation (9) we have

$$\pi_1(BG, \tilde{b}_0) \cong \pi_0(G, g_0).$$

(16)

Then the condition of simplicity of $BG$ implies that $\pi_0(G, g_0)$ is abelian. One example, when this condition is trivially fulfilled, is $G$ being a path-connected group. In this case $\pi_1(BG, \tilde{b}_0) \cong \pi_0(G, g_0) = 0$ and $BG$ is simple. Another example is when $G$ is discrete and abelian. In this case $\pi_0(G, g_0) \cong (G, g_0)$ and $\pi_q(G, g_0) = 0$ for all $q \geq 1$.

In order to guarantee that $BG$ is simple we assume that $G$ is such that $\pi_0(G, g_0)$ is abelian and $\pi_1(BG, \tilde{b}_0) \cong \pi_0(G, g_0)$ acts trivially on $\pi_n(G, g_0)$ for $n \geq 1$. In addition we assume that $\pi_0(G, g_0)$ is discrete. The latter condition will be used in the derivation of a short exact sequence which includes $B_G(M)$.

In what follows, $\pi_n$ denotes the $n$th homotopy group $\pi_n(BG, \tilde{b}_0)$ of $BG$. The first non-trivial level of the Postnikov diagram for $BG$ starts with $Y_2$:

\[
\begin{array}{cccc}
Y_3 & \xrightarrow{\theta_3^*} & PK(\pi_2, 3) & \leftarrow \Omega K(\pi_2, 3) \cong K(\pi_2, 2) \\
\downarrow p_2 & & \downarrow \tilde{p}_2 & \\
Y_2 & \xrightarrow{\theta_2^*} & K(\pi_2, 3) & \\
\downarrow p_1 & & & \\
BG & \longrightarrow & Y_0 & \xrightarrow{\theta_1} & K(\pi_1, 2) \\
\end{array}
\]

Recall that here $Y_0 = \{y_0\}$.

Auxiliary path-loop fibrations are

$\tilde{p}_1 : PK(\pi_1, 2) \rightarrow K(\pi_1, 2)$ with fiber $\Omega K(\pi_1, 2) \cong K(\pi_1, 1)$

and

$\tilde{p}_2 : PK(\pi_2, 3) \rightarrow K(\pi_2, 3)$ with fiber $\Omega K(\pi_2, 3) \cong K(\pi_2, 2)$.

The principal fibrations $p_1$ and $p_2$ are induced from them by $\theta_1^*$ and $\theta_2^*$ respectively. Of course, $Y_2 = \{y_0\} \times \Omega K(\pi_1, 2) \cong K(\pi_1, 1)$. The 3-equivalence, we are looking for, is $f_3 : BG \rightarrow Y_3$, so that $Y_3$ plays the role of $BG_2$. Its existence is guaranteed by the Postnikov decomposition Theorem [3.1]. $f_3$ is related to the Postnikov invariant $k^3(BG) \in H^3(Y_2; \pi_2)$, which corresponds to the map $\theta_2 : Y_2 \rightarrow K(\pi_2, 3)$ through the relation $[Y_2; K(\pi_2, 3)] \cong H^3(Y_2; \pi_2)$ (see Sect. [3]). Therefore, the non-trivial fibration, which is important for us, is $p_2 : Y_3 = BG_2 \rightarrow Y_2 \cong K(\pi_1, 1)$ with fiber $K(\pi_2, 2)$.
As the next step we build an exact sequence associated with this bundle. This is a well known construction \cite{Sp}, \cite{Sw}, \cite{Wh} (see also \cite{AI}) which we remind here. Let $p : (E, e_0) \rightarrow (B, b_0)$ be a fibration with fiber $F$, and let $i : (F, e_0) \hookrightarrow (E, e_0)$ be the inclusion of the fibre. Firstly, we define

$$\bar{E} = \{(e, w) \in E \times PB | w(0) = p(e)\} \subset E \times PB,$$

where $(PB, w_0)$ is the space of paths in $(B, b_0)$. Secondly, we define a path lifting function $\lambda : \bar{E} \rightarrow PE$ for $p$ (in Ref. \cite{Wh} this map is called a connection) by the following conditions:

$$\lambda(e, w)(0) = e \quad \text{and} \quad p(\lambda(e, w)) = w.$$

Thus, $\lambda(e, w)$ is a path in $E$ which starts at $e \in E$ and covers the path $w$ in $B$. If $w$ is a closed path in $B$, then $\lambda(e, w)(1)$ is in the same fiber as $e$. In a certain sense a lifting function is a generalization of the notion of a connection in differential geometry.

**Proposition 4.1** \cite{Sp}, \cite{Sw}, \cite{Wh} A map $p : E \rightarrow B$ is a fibration if and only if there is a path lifting function for $p$.

Let us define a map $\rho : \Omega B \rightarrow F$ by the following formula:

$$\rho(w) = \left(\lambda(e_0, w^{-1})\right)(1), \quad w \in \Omega B. \quad (17)$$

The map $\rho$ allows us to define a left exact sequence, characterized by Theorem 4.3, for the fibration $p$.

**Definition 4.2** A sequence

$$\ldots \rightarrow (Y_n, y_n) \xrightarrow{f_n} (Y_{n-1}, y_{n-1}) \rightarrow \ldots \rightarrow (Y_1, y_1) \xrightarrow{f_1} (Y_0, y_0)$$

of pointed topological spaces is called left exact if for any pointed topological space $(X, x_0)$ the sequence

$$\ldots \rightarrow [X, x_0; Y_n, y_n] \xrightarrow{(f_n)_*} [X, x_0; Y_{n-1}, y_{n-1}] \rightarrow \ldots$$

$$\ldots \rightarrow [X, x_0; Y_1, y_1] \xrightarrow{(f_1)_*} [X, x_0; Y_0, y_0]$$

is exact in the category of pointed sets.
Theorem 4.3 \[SW\], \[WP\] If \( p : (E, e_0) \to (B, b_0) \) is a fibration with fibre \((F, e_0)\), and \( \rho : \Omega B \to F \) is the map defined by Eq. (17), then the sequence

\[
\ldots \to \Omega^{n+1}B \xrightarrow{\Omega^n p} \Omega^n F \xrightarrow{\Omega^n i} \Omega^n E \xrightarrow{\Omega^n \theta} \Omega^n B \to \ldots \to \Omega B \xrightarrow{\rho} F \xrightarrow{i} E \xrightarrow{p} B. \tag{18}
\]

is left exact.

Sequence (18) is called the Puppe sequence of the fibration \( p : E \to B \) \[MT\], \[P\].

Now let us consider the Puppe sequence of the principal fibration \( p_2 : Y_3 \to Y_2 \) with fiber \( F \simeq K(\pi_2, 2) \) and apply \([M, x_0; -]\) to it. As it follows from Theorem 4.3, the sequence of pointed sets

\[
\ldots \to \left[M, x_0; \Omega^{n+1}Y_2, *\right] \xrightarrow{(\Omega^n p_2)_*} \left[M, x_0; \Omega^n K(\pi_2, 2), *\right] \xrightarrow{(\Omega^n i)_*} \left[M, x_0; \Omega^n Y_3, *\right] \xrightarrow{(\Omega^n p_2)_*} \left[M, x_0; \Omega^n Y_2, *\right] \to \ldots \to \left[M, x_0; Y_3, *\right] \xrightarrow{(p_2)_*} \left[M, x_0; Y_2, *\right] \tag{19}
\]

is exact. Sequences of this type sometimes are also called Puppe sequences.

We extend this sequence to the right by adding \((\theta_2)_* : [M; Y_2] \to [M; K(\pi_2, 3)]\), induced by \( \theta_2 \). The extended sequence is exact too. The proof of this is rather simple and follows from properties of principal fibrations (see, for example, Ref. \[B\], \[P\]). Indeed, let \( \tilde{p} : PK \to K \) be a path-loop fibration over a space \((K, k_0)\), and consider a map \( \theta : B \to K \) and the fibration \( p : E_\theta \to B \) induced from \( \tilde{p} \) by \( \theta \). Let \( M \) be a pointed topological space. Any map \( f : M \to E_\theta \) gives rise to a pair of maps: 1) \( f_1 = p \circ f : M \to B \) and 2) \( f_2 = \theta \circ f : M \to PK \), where \( \theta : E_\theta \to PK \) is the canonical map appearing in the definition of the induced fibration (see \(B\) for an analogous map in the case of the induced bundle). Of course, \( \theta \circ f_1 = \tilde{p} \circ f_2 \). Let \( p_* \) and \( \theta_* \) be the maps \( p_* : [M; E_\theta] \to [M; B] \) and \( \theta_* : [M; B] \to [M; K] \) induced by \( p \) and \( \theta \) respectively. For any \( x \in M \) \( f_2(x) \) is a path in \( K \) connecting the base point \( k_0 \) with the point

\[
(f_2(x))(1) = \tilde{p}(f_2(x)) = (\theta \circ f_1)(x) \in K.
\]

Hence, the map \( \theta \circ f_1 = \theta_* f_1 \) is homotopically trivial. Since \( f_1 = p_* f \), we conclude that \( \text{im } p_* \subseteq \ker \theta_* \). The inverse inclusion is also easy to prove. Consider a map \( f_1 : M \to B \) such that \( \theta_* f_1 = \theta \circ f_1 \) is homotopically trivial. This means that there is a homotopy \( h : M \times I \to K \) with \( h(x, 0) = k_0 \) and \( h(x, 1) = (\theta_* f_1)(x) \) for all \( x \in M \). It defines a path \( w_x = h(x, -) \) in \( K \), connecting \( w_x(0) = k_0 \) with

\[
w_x(1) = (\theta_* f_1)(x), \tag{20}
\]
and allows to introduce a function \( f_2 : M \to PK \) by the formula \( f_2(x) = w_x \). Next we define a map \( f : M \to E_\theta \subset B \times PK \) by \( f(x) = (f_1(x), f_2(x)) \). Relation (20) gives that \( \tilde{p}(f_2(x)) = w_x(1) = \theta(f_1(x)) \), and this guarantees that the image \( f(x) \in E_\theta \). Since \( p_* f = f_1 \), we conclude that \( \ker \theta_* \subset \mathrm{im} p_* \). This proves that \( \ker \theta_* = \mathrm{im} p_* \).

Thus, for the principal fibration \( p_2 : Y_3 \to Y_2 \) we have the following extended Puppe sequence which is exact:

\[
\ldots \longrightarrow [M, x_0; \Omega BG_2, \ast] \xrightarrow{(\Omega p_2)_*} [M, x_0; \Omega K(\pi_1, 1), \ast] \xrightarrow{(\rho)_*} [M, x_0; K(\pi_2, 2), \ast] \\
\xrightarrow{(i)_*} [M, x_0; BG_2, \ast] \xrightarrow{(p_2)_*} [M, x_0; K(\pi_1, 1), \ast] \xrightarrow{(\theta_2)_*} [M, x_0; K(\pi_2, 3), \ast].
\] (21)

Here we took into account that \( Y_3 = BG_2 \) and \( Y_2 \simeq K(\pi_1, 1) \). Let us specify elements of this sequence.

Firstly, isomorphism (14) gives that

\[
[M, x_0; K(\pi_1, 1), \ast] \cong H^1(M; \pi_1) \quad \text{and} \quad [M, x_0; K(\pi_2, 2), \ast] \cong H^2(M; \pi_2).
\]

Since the CW-complex \( M \) is two-dimensional, the last term in the sequence

\[
[M, x_0; K(\pi_2, 3), \ast] \cong H^3(M; \pi_2) = 0.
\]

This follows from the Universal Coefficient Theorem, see [Sp], [Sw].

Secondly, we use the relation

\[
\Omega K(\pi, 1) \simeq \pi
\] (22)

valid for a discrete group \( \pi \). To derive it take the CW-complex \( X = K(\pi, 1) \) with \( \pi \) discrete. Consider the map \( f : (\Omega X, \ast) \to \pi_1(X, \ast) \) which assigns to a loop \( w \in \Omega X \) its homotopy class \( [w] \in \pi_1(X, \ast) \), and a map \( g : \pi_1(X, \ast) \to (\Omega X, \ast) \) which determines a representative in each class \( [w] \in \pi_1(X, \ast) \). By definition \( f \) is the homotopy inverse with respect to \( g \) and vice versa, i.e., the map \( f \) is a homotopy equivalence. Therefore, \( \Omega K(\pi, 1) \simeq \pi_1(K(\pi, 1), \ast) \cong \pi \).

Since, according to our assumption, \( \pi_1 = \pi_1(BG, \tilde{b}_0) \cong \pi_0(G, g_0) \) is discrete, using result (22) we conclude that

\[
[M, x_0; \Omega K(\pi_1, 1), \ast] \cong [M, x_0; \pi_1, \ast] = 0.
\]

Finally, we recognize the term \([M; BG_2]\) in sequence (21) as the set that serves for the purpose of classification of principal \( G \)-bundles over \( M \) and that we are looking for.
Indeed, according to the Postnikov decomposition theorem there exists a 3-equivalence $f_3 : BG \to Y_3 = BG_2$, so that relation (11) with $n = 2$ is valid. We have

$$B_G(M) \cong [M; BG] \cong [M; BG_2].$$

Taking into account the properties of the terms in sequence (21) and recalling that $\pi_n = \pi_n(BG, b_0) \cong \pi_{n-1}(G, g_0)$, we arrive at the following result.

**Theorem 4.4** Let $(M, x_0)$ be a path-connected pointed CW-complex of dim $M = 2$. Let $G$ be a group such that $\pi_0(G, g_0)$ is abelian and discrete and acts trivially on the higher homotopy groups $\pi_n(G, g_0)$ for $n \geq 1$. Then for the set $B_G(M)$ of equivalence classes of principal $G$-bundles over $M$ there exists the following short exact sequence of pointed sets:

$$0 \to H^2(M, x_0; \pi_1(G, g_0), \ast) \to B_G(M) \to H^1(M, x_0; \pi_0(G, g_0), \ast) \to 0. \quad (23)$$

The maps in this sequence can be read from the extended Puppe sequence (21).

This is our result on the classification of principal $G$-bundles. Further characterization of $B_G(M)$ requires the knowledge of additional details about $M$ or $G$.

Now we consider two particular cases and present a discussion of the result.

1. $\pi_1(G, g_0) = 0$. It follows from (23) that

$$B_G(M) \cong H^1(M; \pi_0(G, g_0)). \quad (24)$$

This case includes the class of discrete groups. If $G$ is discrete, $\pi_0(G) \cong G$ and the formula above simplifies further:

$$B_G(M) \cong H^1(M; G). \quad (25)$$

2. $G$ is path-connected. Then $\pi_0(G, g_0) = 0$ and

$$B_G(M) \cong H^2(M; \pi_1(G, g_0)). \quad (26)$$

The results in particular cases 1 and 2 above can be obtained, in fact, from the following general theorem.

**Theorem 4.5** Let $(X, x_0)$ be a pointed CW-complex, $(Y, y_0)$ an $(n - 1)$-connected pointed space, and $H^{q+1}(X; \pi_q(Y)) = 0 = H^q(X; \pi_q(Y))$ for all $q \geq n$. Then there exists a one-to-one correspondence

$$[X, x_0; Y, y_0] \cong H^n(X; \pi_n(Y)). \quad (27)$$
Let us apply this theorem to \( X = M \) with \( \dim M = 2 \). In the case, when \( Y = BG \) is 0-connected with \( \pi_2(BG) = \pi_1(G) = 0 \), we obtain result (24). If \( Y = BG \) is 1-connected, i.e., \( \pi_1(BG) = \pi_0(G) = 0 \), then (27) gives (26). The latter case is also a result of the Hopf-Whitney classification theorem [Wh].

Relations (24) and (26) become more concrete if further properties of the group \( G \) are known. For completeness of the discussion, we present a list of the first homotopy groups \( \pi_1(G) \) for some connected Lie groups:

1. Simply connected, \( G = SU(n), Sp(n) \): \( \pi_1(G) = 0 \).
2. \( G = SO(n), n = 3 \) and \( n \geq 5 \): \( \pi_1(G) = Z_2 \).
3. \( G = U(n) \): \( \pi_1(G) = Z \).

We would like to mention that for \( G = U(1) \) actually a stronger result is known [AI]. Indeed, consider the Hopf bundle \( \xi_n = (S^{2n-1}, CP^n, U(1), p) \), where the sphere is realized as

\[
S^{2n-1} = \left\{(z_1, \ldots, z_n) \in C^n \mid \sum_{i=1}^{n}|z_i|^2 = 1\right\}
\]

and the bundle projection \( p : S^{2n-1} \rightarrow CP^n \) is given by

\[
p(z_1, \ldots, z_n) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}, \ldots, \frac{z_n}{z_1}\right)
\]

for a neighbourhood corresponding to \( z_1 \neq 0 \), etc. Since \( \pi_q(S^{2n-1}) = 0 \) for \( q < 2n - 1 \), the bundle is \((2n - 2)\)-universal. Then one takes the direct limits \( S^\infty = \lim_{\rightarrow} S^n, CP^\infty = \lim_{\rightarrow} CP^n \). The bundle \( \xi_\infty = (S^\infty, CP^\infty, U(1), p) \) is \( \infty \)-universal. Thus, \( EU(1) = S^\infty \) and \( BU(1) = CP^\infty \). This also can be obtained by using the Milnor construction [H]. In Sect. 3 we have already mentioned that \( CP^\infty = \tilde{K}(Z, 2) \). Then for a CW-complex of any dimension

\[
E_{U(1)}(M) \cong [M; BU(1)] \cong [M; CP^\infty] = [M; K(Z, 2)] \cong H^2(M; Z).
\] (28)

This is, of course, in agreement with our result (26).

Eq. (24), giving the classification of principal \( G \)-bundles with discrete structure group, is in fact a particular case of a more general result valid for CW-complexes \( M \) of any dimension. It follows from the fact that for a discrete structure group \( G \)

\[
BG \simeq K(G, 1).
\] (29)
To show this, consider the universal covering \( UK(G, 1) \to K(G, 1) \) of \( K(G, 1) \). One can prove that it is in fact a universal \( G \)-bundle \([Sw], [St], [P]\). Indeed, \( \pi_1(UK(G, 1), *) = 0 \) by definition, \( \pi_n(UK(G, 1), *) = \pi_n(K(G, 1), *) = 0 \) for \( n \geq 2 \) and the group of automorphisms \( Aut(UK(G, 1)) \cong \pi_1(K(G, 1), *) \cong G \). Hence, \( K(G, 1) \) is the classifying space (see \([St]\) and Theorem 2.15), i.e., we have relation (29). With this we get

\[
B_G(M) \cong [M; BG] \cong [M; K(G, 1)] \cong H^1(M; G).
\]

(cf. Eq. (25)). In particular, the classifying space for \( G = Z_2 \) is \( BZ_2 \cong K(Z_2, 1) = RP^\infty \), as one can see from (12). Using the Milnor construction it can be shown that the total space of the universal \( Z_2 \)-bundle is \( EZ_2 = S^{\infty} \).

If the space \( M \) is of certain type, formulas (23), (24) and (26) become more concrete. Thus, if \( M \) is a two-dimensional differentiable manifold one can use known formulas for cohomology groups. For example, if \( \pi \) is abelian then (see \([B]\))

1) for \( M \) compact and orientable

\[
H^2(M; \pi) \cong \pi,
\]

(30)

2) for \( M \) compact and not orientable

\[
H^2(M; \pi) \cong \pi/2\pi.
\]

(31)

More expressions for cohomology groups of various two-dimensional surfaces can be found in \([GH]\). In particular, for \( M = S^2 \) it is known that its first cohomology group \( H^1(S^2, \pi) = 0 \). Then from Theorem 1.4, Eq. (14) and Definition 3.3 it follows that

\[
B_G(M) \cong H^2(S^2; \pi_1(G, g_0)) \cong [S^2; K(\pi_1(G), 2)] \cong \pi_2(K(\pi_1(G), 2), *) = \pi_1(G).
\]

This is in accordance with Eq. (10).

If the structure group \( G \) is path connected and simply connected, i.e., \( \pi_0(G, g_0) = 0 = \pi_1(G, g_0) \), then (23) implies that the only (up to equivalence) existing principal \( G \)-bundle \( \xi = (E, M, G, p) \) is the trivial one.

To finish our discussion of the result let us mention the classification of principal fibre bundles over two-dimensional compact orientable manifolds obtained by Witten in Ref. \([Wi]\). It is known that any connected Lie group \( G \) can be obtained as a quotient group \( G = \tilde{G}/\Gamma \) (see, for example, \([BR], [FH]\)). Here \( \tilde{G} \) is the unique (up to isomorphism) connected and simply connected Lie group, called the universal covering group. \( \Gamma \) is a
discrete subgroup of the center $Z(\tilde{G})$ of $\tilde{G}$. Witten showed that principal $G$-bundles over $M$ are classified by elements of $\Gamma$, i.e. $\mathcal{B}_G(M) \cong \Gamma$. This agrees with Eq. (26). Indeed, taking into account that $\pi_1(G, g_0) \cong \pi_0(\Gamma) \cong \Gamma$ and using Eq. (30), from (26) we get
\[ \mathcal{B}_G(M) \cong H^2(M; \pi_1(G, g_0)) \cong H^2(M; \Gamma) \cong \Gamma. \]

Acknowledgements

We would like to thank Manolo Asorey, Marco Mackaay, José Mourão and Evgeny Troitsky for many fruitful discussions and useful remarks. Financial support by Fundação para a Ciência e a Tecnologia (Portugal) under grants PRAXIS/2/2.1/FIS/286/94 and CERN/P/FIS/1203/98, from fellowship PRAXIS XXI/BCC/4802/95 and from the Russian Foundation for Basic Research (grant 98-02-16769-a) are acknowledged.

Appendix

In this Appendix we give basic definitions and theorems, as well as some examples from the theory of categories, which we use in the article.

Definition A.1 A category $\mathcal{C}$ consists of

1. a class of objects;

2. sets of morphisms from $X$ to $Y$ for every ordered pair of objects $X$ and $Y$; such sets are denoted by $\text{hom}(X; Y)$; if $f \in \text{hom}(X; Y)$ we write $f : X \to Y$;

3. functions $\text{hom}(Y; Z) \times \text{hom}(X; Y) \to \text{hom}(X; Z)$ for every ordered triple of objects $(X, Y, Z)$; such a function is called a composition; if $f \in \text{hom}(X; Y)$ and $g \in \text{hom}(Y; Z)$, then the image of the pair $(g, f)$ in $\text{hom}(X; Z)$ is denoted by $g \circ f$.

Objects and morphisms satisfy the following two axioms:

C1 Associativity: if $f \in \text{hom}(X; Y)$, $g \in \text{hom}(Y; Z)$ and $f \in \text{hom}(Z; W)$, then $h \circ (g \circ f) = (h \circ g) \circ f \in \text{hom}(X; W)$;

C2 Existence of an identity: for every object $X$ there exists a morphism $1_X \in \text{hom}(X; X)$ such that for every $g \in \text{hom}(Y; X)$ and every $h \in \text{hom}(X; Z)$, for all $Y, Z$, we have $1_X \circ g = g$ and $h \circ 1_X = h$. 

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It can be shown that $1_X$ is unique. Here follow a few examples of categories.

**E.1** Category $\mathcal{S}$ of all sets and all functions.

**E.2** Category $\mathcal{T}$ of all topological spaces and all continuous maps.

**E.3** Category $\mathcal{G}$ of groups and homomorphisms.

**E.4** Category $\mathcal{PS}$ of pointed sets (sets with a distinguished element called a base point) and base-point-preserving functions. In this case the objects are denoted by $(X, x_0)$, where $X$ is a set and $x_0 \in X$ is the base point, and any $f \in \text{hom}(X, x_0; Y, y_0)$ is a function $f : X \to Y$ such that $f(x_0) = y_0$.

**E.5** Category $\mathcal{PT}$ of pointed topological spaces and base-point-preserving continuous maps.

**E.6** Category $\mathcal{T}'$ whose objects are topological spaces and $\text{hom}(X; Y) = [X; Y]$, the set of homotopy classes of continuous maps $f : X \to Y$. The homotopy class of a continuous map $f : X \to Y$ is denoted by $[f]$. For given $[f] \in \text{hom}(X; Y)$ and $[g] \in \text{hom}(Y; Z)$ the composition $[g] \circ [f]$ is defined as $[g \circ f]$.

**E.7** Category $\mathcal{PT}'$ whose objects are pointed topological spaces and $\text{hom}(X, x_0; Y, y_0) = [X, x_0; Y, y_0]$ is the set of homotopy classes of continuous base-point-preserving maps $f : (X, x_0) \to (Y, y_0)$.

**E.8** Homotopy category $\mathcal{PW}'$ whose objects are path-connected pointed CW-complexes and morphisms are homotopy classes of continuous base-point-preserving maps. Let us remind that a CW-complex is a topological space built up of cells, i.e., balls of various dimensions, glued together in a certain way. For a formal definition and detailed description of such spaces we refer the reader to Refs. [SP], [SW], [WH].

**Definition A.2** Two objects $X$ and $Y$ in a category $\mathcal{C}$ are called equivalent if there exist morphisms $f \in \text{hom}(X; Y)$ and $g \in \text{hom}(Y; X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case the morphisms $f$ and $g$ are called equivalences.

We would like to mention that two sets are equivalent in the category $\mathcal{S}$ if and only if there exists a bijection between them. Two topological spaces $X$ and $Y$ are equivalent in $\mathcal{T}$ if and only if they are homeomorphic, and they are equivalent in $\mathcal{T}'$ if and only if they are homotopy equivalent.
Definition A.3 A functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ (we write $F : \mathcal{C} \to \mathcal{D}$) is a function that assigns

1. to each object $X$ in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$;
2. to each morphism $f \in \text{hom}_\mathcal{C}(X;Y)$ in $\mathcal{C}$ a morphism $F(f) \in \text{hom}_\mathcal{D}(F(X);F(Y))$ such that the following axioms are fulfilled:

   $\text{F1}$ for each object $X$ in $\mathcal{C}$ $F(1_X) = 1_{F(X)}$;
   $\text{F2}$ for every $f \in \text{hom}_\mathcal{C}(X;Y)$ and every $g \in \text{hom}_\mathcal{C}(Y;Z)$ in $\mathcal{C}$ $F(g \circ f) = F(g) \circ F(f) \in \text{hom}_\mathcal{D}(F(X);F(Z))$.

The notion of a cofunctor is, in a certain sense, dual to that of a functor.

Definition A.4 A cofunctor $F^*$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ (we write $F^* : \mathcal{C} \to \mathcal{D}$) is a function which assigns

1. to each object $X$ in $\mathcal{C}$ an object $F^*(X)$ in $\mathcal{D}$;
2. to each morphism $f \in \text{hom}_\mathcal{C}(X;Y)$ in $\mathcal{C}$ a morphism $F^*(f) \in \text{hom}_\mathcal{D}(F^*(Y);F^*(X))$, such that the following axioms are fulfilled:

   $\text{CF1}$ for each object $X$ in $\mathcal{C}$ $F^*(1_X) = 1_{F^*(X)}$;
   $\text{CF2}$ for every $f \in \text{hom}_\mathcal{C}(X;Y)$ and every $g \in \text{hom}_\mathcal{C}(Y;Z)$ in $\mathcal{C}$ $F^*(g \circ f) = F^*(f) \circ F^*(g) \in \text{hom}_\mathcal{D}(F^*(Z);F^*(X))$.

Let us consider some examples.

E.9 Let us fix a pointed topological space $(B, b_0)$ in $\mathcal{PT}$ and define the functor $\pi_B : \mathcal{PT} \to \mathcal{PS}$ as follows:

(i) for each object $(X, x_0)$ in $\mathcal{PT}$ the pointed set $\pi_B(X, x_0)$ is

$$\pi_B(X, x_0) = [B, b_0; X, x_0];$$

(ii) for each morphism $f : (X, x_0) \to (Y, y_0)$ in $\mathcal{PT}$ the morphism $\pi_B(f)$ is given by

$$\pi_B(f)[g] = [f \circ g] \in [B, b_0; Y, y_0],$$

where $[g] \in [B, b_0; X, x_0]$. Often $\pi_B(f)$ is denoted $f_*$ or $f_\#$. 

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The base point in the set \( \pi_B(X, x_0) \) is the homotopy class \([f_{x_0}]\) of the constant map \( f_{x_0} : B \to x_0 \in X \).

Let \( f \) and \( f' \) be two continuous base-point-preserving maps \((X, x_0) \to (Y, y_0)\) that are homotopic (written as \( f \simeq f' \)). It is easy to check that then \( \pi_B(f) = \pi_B(f') \). Hence \( \pi_B \) is, in fact, a functor from the category \( \mathcal{PT}' \) to \( \mathcal{PS} \).

**E.10** For a fixed pointed topological space \((B, b_0)\) in \( \mathcal{PT} \) we define the cofunctor \( \pi^B : \mathcal{PT} \to \mathcal{PS} \) as follows:

(i) for each object \((X, x_0)\) in \( \mathcal{PT} \) the pointed set \( \pi^B(X, x_0) \) is

\[ \pi^B(X, x_0) = [X, x_0; B, b_0]; \]

(ii) for each morphism \( f : (X, x_0) \to (Y, y_0) \) in \( \mathcal{PT} \) the morphism \( \pi^B(f) \) is given by

\[ \pi^B(f)[g] = [g \circ f] \in [X, x_0; B, b_0], \]

where \([g] \in [Y, y_0; B, b_0]\). Often \( \pi^B(f) \) is denoted \( f^* \) or \( f^# \).

The base point in \( \pi^B(X, x_0) \) is the homotopy class of the constant map \( X \to b_0 \in B \).

For the same reason as in Example E.8 \( \pi^B \) is in fact a cofunctor from the category \( \mathcal{PT}' \) to \( \mathcal{PS} \).

**E.11** Let \( G \) be a group. The \( n \)th cohomology cofunctor \( H^n \) with coefficients in \( G \) is the cofunctor from \( \mathcal{PT}' \) to \( \mathcal{G} \) defined as follows: for any space \((X, x_0)\) in \( \mathcal{PT}' \) \( H^n(X) = H^n(X, x_0; G) \), the \( n \)th singular cohomology group with coefficients in \( G \).

**Definition A.5** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two categories and \( F_1 \) and \( F_2 \) be two functors from \( \mathcal{C} \) to \( \mathcal{D} \). A natural transformation \( T \) from \( F_1 \) to \( F_2 \) is a function which assigns to each element \( X \) in \( \mathcal{C} \) a morphism \( T(X) \in \text{hom}_\mathcal{D}(F_1(X); F_2(X)) \) in \( \mathcal{D} \) such that for each \( f : X \to Y \) in \( \mathcal{C} \) the following relation holds:

\[ T(Y) \circ F_1(f) = F_2(f) \circ T(X). \]  

(A.1)

According to this relation the diagram
\[
\begin{array}{ccc}
F_1(X) & F_1(f) & F_1(Y) \\
\downarrow T(X) & \downarrow T(Y) & \\
F_2(X) & F_2(f) & F_2(Y)
\end{array}
\]

is commutative. The definition of the natural transformation between two cofunctors is obtained by inversion of the horizontal arrows in this diagram. Relation \((A.1)\) then takes the form

\[\text{T}(X) \circ F_1^*(f) = F_2^*(f) \circ \text{T}(Y).\]

**Definition A.6** Let \(\mathcal{C}\) and \(\mathcal{D}\) be two categories and let \(F_1, F_2 : \mathcal{C} \rightarrow \mathcal{D}\) be two functors (cofunctors). A natural transformation \(T\) from \(F_1\) to \(F_2\) is called a natural equivalence if \(T(X)\) is an equivalence in the category \(\mathcal{D}\) for each object \(X\) in \(\mathcal{C}\).

The following example is important for us.

**E.12** Consider a cofunctor \(H : \mathcal{P}\mathcal{T}' \rightarrow \mathcal{P}\mathcal{S}\) and fix a pointed topological space \((B, b_0)\) and an element \(u \in H(B)\). There is a standard way to introduce a natural transformation from \(\pi^B\) to \(H\). For every \((X, x_0)\) in \(\mathcal{P}\mathcal{T}'\) define a function \(T_u(X) : [X, x_0; B, b_0] \rightarrow H(X)\) as follows:

\[T_u(X) ([g]) = H ([g]) (u) \in H(X)\]

for any \([g] \in [X, x_0; B, b_0]\). It is easy to check that \(T_u\) is a natural transformation.

The problem which arises in many applications is the following: for a given cofunctor \(H\) find a space \((B, b_0)\) and an element \(u \in H(B)\) such that \(T_u\) is a natural equivalence for the cofunctors \(\pi^B\) and \(H\). A solution is given by Brown’s representation theorem discussed below. This theorem plays an important role for the classification of principal fibre bundles.

An important class of cofunctors defined on \(\mathcal{P}\mathcal{W}'\) is formed by those cofunctors \(H\) for which the following two axioms are fulfilled (see \(\text{[Sw]}\)).

**W**) Wedge axiom. Let \(\{X_\alpha\}\) be an arbitrary family of \(CW\)-complexes in \(\mathcal{P}\mathcal{W}'\), and \(\iota_\alpha : X_\alpha \rightarrow \bigvee \alpha X_\alpha\) are the inclusions. Then for every \(\alpha\) the morphism

\[H(\iota_\alpha) : H (\bigvee \alpha X_\alpha) \rightarrow \prod_\alpha H (X_\alpha)\]

is a bijection.
Mayer-Vietoris axiom. Consider a triple \((X; A_1, A_2)\), where \(X\) is a CW-complex and \(A_1, A_2\) are its subcomplexes such that \(A_1 \cup A_2 = X\). Let \(x_1 \in H(A_1)\) and \(x_2 \in H(A_2)\) be arbitrary elements such that \(H(i_1)(x_1) = H(i_2)(x_2)\), where \(i_1 : A_1 \cap A_2 \to A_1\) and \(i_2 : A_1 \cap A_2 \to A_2\) are the inclusions. Then there exists an element \(y \in H(X)\) such that \(H(j_1)(y) = x_1\) and \(H(j_2)(y) = x_2\), where \(j_1 : A_1 \to X\) and \(j_2 : A_2 \to X\) are the inclusions.

**Proposition A.7** \([\text{Sw}]\) For any \((B, b_0)\) in \(\mathcal{PW}'\) the cofunctor \(\pi^B = [-; B, b_0]\) satisfies W) and MV).

To formulate Brown’s representation theorem we need the notions of a universal element and of a classifying space.

**Definition A.8** Let \(H\) be a cofunctor on \(\mathcal{PT}'\). An element \(u \in H(B)\) for some \((B, b_0)\) is called \(n\)-universal \((n \geq 1)\) if

\[
T_u(S^q) : [S^q, s_0; B, b_0] \to H(S^q)
\]

is an isomorphism for \(1 \leq q < n\) and an epimorphism for \(q = n\). An element \(u \in H(B)\) is called universal if it is \(n\)-universal for all \(n \geq 1\). In this case the space \(B\) is called a classifying space for \(H\).

For any cofunctor \(H\) satisfying W) and MV) there exists a classifying space \((Y, y_0)\) which is a CW-complex and a universal element \(u \in H(Y)\). The classifying space for such a cofunctor is unique up to homotopy equivalence. Namely, if \((Y, y_0), (Y', y_0') \in \mathcal{PW}'\) are two classifying spaces for \(H\) and \(u \in H(Y), u' \in H(Y')\) are corresponding universal elements, then there exists a homotopy equivalence \(h : Y \to Y'\), unique up to homotopy, such that \(H(h)(u') = u\). Now we can formulate the theorem.

**Theorem A.9** (Brown’s representation theorem, see \([\text{Sy}]\)) Let \(H : \mathcal{PW}' \to \mathcal{PS}\) be a cofunctor satisfying axioms W) and MV). Then there exists a classifying space \((B, b_0)\) in \(\mathcal{PW}'\) and a universal element \(u \in H(B)\) such that

\[
T_u : \pi^B \to H
\]

is a natural equivalence.
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