CONVERGENCE ESTIMATES FOR THE MAGNUS EXPANSION I.
BANACH ALGEBRAS

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Abstract. We review and provide simplified proofs related to the Magnus expansion, and improve convergence estimates. Observations and improvements concerning the Baker–Campbell–Hausdorff expansion are also made.

In this Part I, we consider the general Banach algebraic setting. We show that the (cumulative) convergence radius of the Magnus expansion is 2; and of the Baker–Campbell–Hausdorff series is $C_2 = 2.89847930\ldots$.

Introduction

General introduction. The Baker–Campbell–Hausdorff expansion and its continuous generalization, the Magnus expansion, attract attention from time to time. This is probably partly due to their aesthetically pleasing nature. There are many rediscoveries in this area: For example, broadly related to the Magnus expansion, see Magnus [20] and Chen [7]; Mielnik, Plebański [22] and Strichartz [34] and Vinokurov [36]; Goldberg [15] and Helmstetter [17].

In this series of articles, we review and provide simplified proofs related to the Magnus expansion, and improve convergence estimates. Observations and improvements concerning the Baker–Campbell–Hausdorff expansion are also made. See Blanes, Casas, Oteo, Ros [1] for a general review of the Magnus expansion for aspects we do not discuss.

We refer to Bourbaki [3], Reutenauer [29], and Bonfiglioli, Fulci [2] for general background in Lie theory, including the Baker–Campbell–Hausdorff expansion, the Poincaré–Birkhoff–Witt theorem, the Dynkin–Specht–Wever lemma, and the Dynkin formula; for those we do not make particular references. Especially [2] can be useful: It provides a detailed discussion of the above mentioned topics and a guide to further literature, but the discussion itself ends around the same place where our discussion begins. Dunford, Schwartz [13] still provides a reasonably good reference for functional analysis. We cite Flajolet, Sedgewick [14] for generating function techniques, including a discussion of Pringsheim’s theorem. We refer to Coddington, Levinson [9], Hsieh, Sibuya [18], Teschl [34] for the theory of ordinary differential equations, including existence and uniqueness theorems, and some standard solution techniques. Otherwise, the discussion aims to be quite self-contained.

Introduction to the Banach algebraic setting. In this part, we aim to deal with general Banach algebras, although some liberties are taken to extend the range of algebras. We sometimes use the locally convex algebras $F_K^{1, loc}$ for computational purposes. Moreover, we may also consider the “algebraic” topology of formal power series, which we use for algebraic purposes without much comment.

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Regarding earlier results for convergence in general Banach algebras, the most naive power series based estimate for the BCH series shows convergence for cumulative norm (i.e. the norm variation of the noncommutative mass to be expanded) less than \( \log 2 \), cf. Dynkin [11], which bound extends to the Magnus case without difficulty, cf. remarks in Pechukas, Light [26], or Karasev, Mosolova [19]. Beyond such naive power series estimates, the relevant papers are Strichartz [33], Thompson [35], Day, So, Thompson [10], Vinokurov [36], Moan, Oteo [25]. In particular, although their results are formulated in slightly weaker settings, Thompson [35] (1989) and Moan, Oteo [25] (2001) essentially state that the BCH expansion and the Magnus expansion, respectively, converge if the cumulative norm is less than 2.

Divergence was studied by Wei [37] and Michel [21], but the most relevant information is given by Vinokurov [36] (1991), that there are counterexamples for the convergence of the BCH expansion with cumulative norm greater than \( \pi \); and by Schäffer [31] (1964) (indirectly), and by Moan [23] (2002). Moan, Niesen [24] (2008) that there is a counterexample for the convergence of the Magnus expansion with cumulative norm \( \pi \). These counterexamples are, however, in the \( C^\ast \)-algebraic setting where stronger convergence results hold; and they turn out not to be representative with respect to the general Banach algebraic setting.

The above mentioned papers put the guaranteed convergence radius of the Magnus expansion (and the BCH expansion) in the Banach setting between 2 and \( \pi \).

Outline of content. In this article, we show that the Magnus expansion converges if the cumulative norm is \(< 2\); moreover, it converges even if the cumulative norm is 2 and the integrand is of Lebesgue–Bochner type; and there are counterexamples with cumulative norm 2, and there are counterexamples of multivariable BCH type with cumulative norm \( > 2 \). In the convergent cases we show that the exponential formula holds even in the stronger logarithmic sense. In establishing these results, we use the resolvent technique of Mielnik, Plebański [22] (although their original results are purely formal). We also show that the Baker–Campbell–Hausdorff series converges if the cumulative norm is less than \( C_2 = 2.89847930 \ldots \), and there are counterexamples with cumulative norm \( C_2 \).

More specifically: Section 1 provides an introduction to the Magnus expansion in the setting of Banach algebras. Apart from the discussion of divergence, the emphasis is on recovering some known results, organized efficiently. Section 2 uses the resolvent technique to refine these results; we prove the logarithmic version of the Magnus expansion theorem, and we discuss the critical case when the cumulative norm is 2. Section 3 deals with the Baker–Campbell–Hausdorff expansion, and with counterexamples. In Appendix A we discuss measure-theoretic terminology. In Appendix B we describe the spaces \( F^{1,\text{loc}}(Ω, \mathcal{F}, \omega) \). These are essentially formal power series with continuous variables. In Appendix C we supply some information regarding some classical formulae of Schur for future reference.

Terminology and notation. The principal point of the Magnus expansion is that (in appropriate circumstances) the (sum of the) time-ordered exponential expansion of an “ordered non-commutative mass” \( \phi \) is the ordinary exponential of the (sum of the) Magnus expansion this “ordered non-commutative mass” \( \phi \). Traditionally such a \( \phi \) is presented as

\[
\phi(t) = f(t) \, dt |_{[α, β]};
\]

where \( f \) is a continuous or at least Lebesgue–Bochner integrable Banach-algebra (or Banach–Lie algebra) valued function. In what follows, somewhat more generally, we...
take any continuous Banach-algebra (or Banach–Lie algebra) valued ordered measure of finite variation for $\phi$.

In what follows, all Banach algebras will be (real or complex) unital algebras (although much of the discussion extends to the non-unital case in appropriate form). For the purposes of spectral calculus, it is customary to complexify real Banach algebras. If $\mathfrak{A}$ is a real Banach algebra with norm $| \cdot |_{\mathfrak{A}}$, then for the complexification $\mathbb{C} \mathfrak{A}$, our choice of norm is

$$|x|_{\mathbb{C} \mathfrak{A}} = \inf \left\{ \sum_{i \in I} |t_i| \cdot |x_i|_{\mathfrak{A}} : x = \sum_{i \in I} t_i x_i, x_i \in \mathfrak{A}, t_i \in \mathbb{C} \right\}.$$  

This is the maximal (i.e. most pessimistic) isometric complexification of $(\mathfrak{A}, | \cdot |_{\mathfrak{A}})$. Other complexifications are possible, and, in fact, useful in certain places. (For example, the function on that disk.

$r(z)$ is identified with $R$}

On the complex plane, $D(z_0, r)$ denotes the (possibly degenerate) closed disk with center $z_0$ and radius $r$. $D(z_0, r)$ denotes the corresponding open disk. Sometimes $\mathbb{C}$ is identified with $\mathbb{R}^2$, which yields the notation $D((a, b), r)$. $\mathbb{1}$ denotes the Lebesgue measure. $\Sigma_S$ denotes the permutations of the set $S$, $\Sigma_n$ denotes the permutations of $\{1, \ldots, n\}$. Later, some definitions will be given in statements, but this practice is perhaps acceptable in the sense that the later usage of these definitions requires the understanding of those places in question.

When we consider generating functions, we may understand them either in formal, real function theoretic, or analytic sense. For example, the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n$ can be understood as a formal power series; or as a real function which is equal to $\frac{2}{1+\sqrt{1-x}}$ for $0 \leq x \leq 1$ and equal to $+\infty$ for $x > 2$; or as the analytic function $\frac{2}{1+\sqrt{1-x}}$ defined at least on $D(0,1)$, but rather on $\mathbb{C} \setminus \{1, \infty\}$. We try to be clear in what sense we understand the generating function at hand, but, of course, much of the power of generating functions comes from change between various viewpoints.

Using power series, one often encounters the “problem of specifcation”. The most typical problem is as follows. When a formal series is given not in series form, but, say, as a result of division, e.g. $\frac{e^x-1}{x}$, then it can be well-defined as a formal power series, but substituting a zero divisor to the place of $x$ naively will lead to an expression which does no make sense as such. For that reason we prefer to use concrete function names, e.g. $\alpha(x)$ in the previous case. We do not do this consistently, but we do this when the a divisional form might cause confusion. However, we do not eschew the divisional arithmetic of formal power series (or the use of analytical methods) either, as they are very useful to establish identies.

If $A(X) = \sum_{n=0}^{\infty} a_n X^n, B(X) = \sum_{n=0}^{\infty} b_n X^n$ are formal power series, then we use the notation

$$A(X) \preceq B(X)$$

to indicate that $a_n \leq b_n$ for all $n$. This notation also extends to the case when $A$ has several, possibly noncommuting variables. If $A(X_\omega : \omega \in \Omega)$ is a formal power series in several noncommuting variables, then we might consider its version $A(x_\omega : \omega \in \Omega)$
with commuting variables. If \( A(X_\omega : \omega \in \Omega) \) has only nonnegative coefficients, then we can consider \( A_{\text{real}}(x_\omega : \omega \in \Omega) \) with \( x_\omega \in [0, \infty] \), taking possibly infinite values. For us, by default, power series are formal; but we can substitute elements from a more general algebra \( \mathfrak{A} \), in which case existence is understood as at least order-independent convergence in the topology of \( \mathfrak{A} \), but preferably absolute convergence (in locally convex spaces); except in the (nonnegative) ‘real’ case, when we allow \( +\infty \) as the result or even as the argument.

Regarding the Magnus expansion, most original results in the literature were stated for matrices and/or for (piecewise) continuous functions as integrands in the Magnus expansion, and then extended (and often simplified) by other authors. Sometimes the very same phenomenon is viewed in different formalisms. Thus, proper attribution of results may be somewhat intricate, but I try to list the significant contributions. (But for generalizations requiring no particular effort, I stick to the original names.) In certain cases other authors arrive to results of same mathematical strength or clearly possess the knowledge to answer the relevant questions easily but they do not do this in the context of the Magnus expansion. ‘Notes’ and ‘Examples’ are integral parts of the text. ‘Remarks’, however, can be omitted from the formal development of the paper.

1. THE COMBINATORIAL APPROACH

Let \( \mathfrak{A} \) be a Banach algebra (real or complex). For \( X_1, \ldots, X_k \in \mathfrak{A} \), we define the Magnus commutator (or Dynkin commutator) by

\[
\mu_k(X_1, \ldots, X_k) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \log(\exp(t_1 X_1) \cdots \exp(t_k X_k)) \bigg|_{t_1 = \ldots = t_k = 0};
\]

where \( \exp \) is defined as usual, and \( \log \) can be defined either by power series around 1, or with spectral calculus, the complex plane cut along the negative real axis. Algebraically, in terms of formal power series,

\[
\mu_k(X_1, \ldots, X_k) = \log(\exp(X_1) \cdots \exp(X_k)) \text{ the variables } X_1, \ldots, X_k \text{ has multiplicity 1},
\]

i. e. the part of \( \log(\exp(X_1) \cdots \exp(X_k)) \) where every variable \( X_i \) has multiplicity 1. This implies that, in terms of formal power series,

\[
\log(\exp(X_1) \cdots \exp(X_k)) = \sum_{\{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}} \mu_l(X_{i_1}, \ldots, X_{i_l}) + H(X_1, \ldots, X_k),
\]

where \( H(X_1, \ldots, X_k) \) collects the terms where some variable have multiplicity more than one. Taking the exponential, and detecting the terms where every variable has multiplicity 1, this yields

\[
X_1 \cdots X_k = \sum_{I_1 \cup \cdots \cup I_s = \{1, \ldots, k\} \atop I_j = \{i_{j,1}, \ldots, i_{j,l_j}\} \neq \emptyset \atop i_{j,1} < \ldots < i_{j,l_j}} \frac{1}{s!} \cdot \mu_{t_1}(X_{i_{1,1}, \ldots, X_{i_{1,t_1}}}) \cdots \cdot \mu_{t_s}(X_{i_{s,1}, \ldots, X_{i_{s,t_s}}}),
\]

where we sum over ordered partitions.

A continuous recast of the identities (1) is given by
Theorem 1.1 (Magnus [20], Chen [6], 1954). Let $\phi$ be a continuous $\mathfrak{A}$-valued measure of finite variation on the interval $I$. If $T$ is a formal variable, then

$$1 + \sum_{k=1}^{\infty} T^k \int_{t_1 \leq \cdots \leq t_k \in I} \phi(t_1) \cdots \phi(t_k) = \exp \sum_{k=1}^{\infty} T^k \int_{t_1 \leq \cdots \leq t_k \in I} \mu_k(\phi(t_1), \ldots, \phi(t_k)).$$

Proof. Take the exponential on the RHS, and contract the terms of order $k$ according to (4). □

Theorem 1.2 (Magnus [20], Chen [6], 1954). Let $\phi$ be a continuous $\mathfrak{A}$-valued measure of finite variation on the interval $I$. If

$$\sum_{k=1}^{\infty} \left| \int_{t_1 \leq \cdots \leq t_k \in I} \mu_k(\phi(t_1), \ldots, \phi(t_k)) \right| < +\infty,$$

then

$$1 + \sum_{k=1}^{\infty} \int_{t_1 \leq \cdots \leq t_k \in I} \phi(t_1) \cdots \phi(t_k) = \exp \sum_{k=1}^{\infty} \int_{t_1 \leq \cdots \leq t_k \in I} \mu_k(\phi(t_1), \ldots, \phi(t_k)).$$

Proof. Substitute $T = 1$ in (5). □

Note. More precisely, Magnus [20] has the information of this amount, however he is more interested in the commutator recursion for $\mu_k$ than starting with (1) directly. Thus (7), as it is here, is more in the spirit of Chen [6] (but which is a bit over-algebraized). On the other hand, the idea, without integrals, is already present in Dynkin [12] (1949). Magnus [20] is superior in the sense that it has explicit commutator recursions using Schur’s argument, while Chen [7] (1957) exhibits that later. In what follows, we will simply use the term ‘Magnus expansion’ for the series in the RHS of (7) after ‘$\exp$’. △

Using a more compact notation, the right Magnus expansion of $\phi$ is

$$\mu_R(\phi) = \sum_{k=1}^{\infty} \mu_{k,R}(\phi),$$

where

$$\mu_{k,R}(\phi) = \int_{t_1 \leq \cdots \leq t_k \in I} \mu_k(\phi(t_1), \ldots, \phi(t_k));$$

and the theorem says that it exponentiates to the right time-ordered exponential

$$\exp_R(\phi) = 1 + \sum_{k=1}^{\infty} \int_{t_1 \leq \cdots \leq t_k \in I} \phi(t_1) \cdots \phi(t_k)$$

of $\phi$, as long as the Magnus expansion (8) is absolute convergent as a series.

One can, of course, formulate similar statements in terms of the left Magnus expansion $\mu_L(\phi)$ and left time-ordered exponential $\exp_L(\phi)$ by reversing the ordering on $I$. The two formalisms have the same capabilities (we just have to reverse the measures $\phi \leftrightarrow \phi^\dagger$), but the ‘R’ formalism is better suited for purely algebraic manipulations, and the ‘L’ formalism is better suited to the analytic viewpoint of differential equations.

In what follows, instead of ‘continuous measure on an interval’ we may simply write ‘ordered measure’. (Cf. Appendix A.)
We can give an estimate for the convergence of the Magnus expansion as follows: Let \( \Theta_k \) be \( \frac{1}{k!} \) times the sum of the absolute value of the coefficients in the monomial expansion of \( \mu_k(X_1, \ldots, X_k) \). (Cf. (10) later.) We define the absolute Magnus characteristic as

\[
\Theta(x) = \sum_{k=1}^{\infty} \Theta_k x^k,
\]

i. e. as the exponential generating function associated to the sum of the absolute value of the coefficients in the monomial expansion of the Magnus commutators. Then

\[
|\mu_{k,R}(\phi)| \leq \int_{t_1 \leq \ldots \leq t_k \in I} |\mu_k(\phi(t_1), \ldots, \phi(t_k))| \leq \Theta_k \cdot \left( \int |\phi| \right)^k,
\]

and, consequently,

\[
\sum_{k=1}^{\infty} |\mu_{k,R}(\phi)| \leq \sum_{k=1}^{\infty} \int_{t_1 \leq \ldots \leq t_k \in I} |\mu_k(\phi(t_1), \ldots, \phi(t_k))| \leq \Theta_{\text{real}} \left( \int |\phi| \right).
\]

Note that we have equalities in the case \( \phi = c \cdot Z_{[0,1]}, c \in [0,\infty) \), the ordered totally noncommutative continuous mass of norm \( c \). See the Appendix B for a detailed explanation of this measure.

**Remark 1.3.** One can formulate Theorem 1.2 and the subsequent discussion for not necessarily continuous measures, too. In this case, we have to insert the multiplicity terms

\[
\int_{t_1 \leq \ldots \leq t_k \in I} \cdots \sim \int_{t_1 \leq \ldots \leq t_k \in I} \frac{1}{\text{mul}(t_1, \ldots, t_n)!} \cdots
\]

everywhere. This is in accordance to taking the continuous blowup \( \phi^* \) of \( \phi \). Thus, it is left to the reader to decide whether he wants to incorporate non-continuous measures under the name of ‘ordered measure’, or even to relax the intervals to ordered sets. △

\( \Theta(x) \) can be worked out as follows. First, one has a direct expression for \( \mu_k \). In order to have it, let us introduce some terminology. If \( \sigma = (\sigma(1), \ldots, \sigma(k)) \) is a finite sequences of real numbers, let asc(\( \sigma \)) denote the number of its ascents, i. e. the number of pairs such that \( \sigma(i) < \sigma(i+1) \); and let des(\( \sigma \)) denote the number of its descents, i. e. the number of pairs such that \( \sigma(i) > \sigma(i+1) \). This naturally applies in the special case when \( \sigma \) is a permutation from the symmetric group \( \Sigma_k \). Then asc(\( \sigma \)) + des(\( \sigma \)) = \( k - 1 \).

**Theorem 1.4** (Mielnik, Plebański [22] (1970), Dynkin [12] (1949)).

\[
\mu_k(X_1, \ldots, X_k) = \sum_{\sigma \in \Sigma_k} (-1)^{\text{asc}(\sigma)} \frac{\text{asc}(\sigma)! \text{des}(\sigma)!}{k!} X_{\sigma(1)} \cdots X_{\sigma(k)}.
\]

**Proof.** (After Strichartz [33] (1987), Dynkin [12] (1949).) Considering (2), and the power series of log, one can see that \( \mu_k(X_1, \ldots, X_k) \) is a sum of terms

\[
(\frac{-1}{j}) \cdot \cdots \cdot \frac{1}{\text{mul}(t_1, \ldots, t_n)!} X_{\sigma(1)} \cdots X_{\sigma(t_1)} | X_{\sigma(t_1+1)} \cdots X_{\sigma(t_{j-1})} | X_{\sigma(t_{j-1}+1)} \cdots X_{\sigma(k)},
\]

where separators show the ascendingly indexed components which enter into power series of log. Considering a given permutation \( \sigma \), the placement of the separators is
either necessary (in case $\sigma(i) > \sigma(i+1)$), or optional (in case $\sigma(i) < \sigma(i+1)$). Summing over the $2^{\text{asc}(\sigma)}$ many optional possibilities, the coefficient of $X_{\sigma(1)} \cdots X_{\sigma(k)}$ is

$$
\sum_{p=0}^{\text{asc}(\sigma)} \frac{(-1)^{\text{des}(\sigma)+p}}{\text{des}(\sigma)+1+p} \binom{\text{asc}(\sigma)}{p}.
$$

This simplifies according to the combinatorial identity $\sum_{p=0}^{r} \frac{(-1)^p}{d+1+p} \binom{r}{p} = \frac{r!d!}{(d+1+r)!}$. (But see the approach of Mielnik and Plebański later.) □

\textbf{Note.} Results of Mielnik, Plebański [22] were announced in Białynicki-Birula, Mielnik, Plebański [4] (1969). Actually, the RHS of (10) appears already in Solomon [32] (1968), but not directly in the context of the Magnus expansion. Applied to free noncommutative algebras, however, it has the same combinatorial strength. Dynkin [12] (1949) has certainly priority for the formula, and it is entirely in the spirit of the way we introduced Magnus commutators, but at that time he applies them only in the setting of the Baker–Campbell–Hausdorff expansion. Thus, the ‘Magnus commutator’ could also be called as ‘Dynkin commutator’ or ‘Dynkin–Magnus commutator’. The fact is that [12] has not been particularly well known until lately; and that work was not expressly in context of the Magnus expansion (although very closely related); thus in this series of papers we simply use the term ‘Magnus commutator’. △

The following is very classical, cf. Comtet [8], Graham, Knuth, Patashnik [16], or Petersen [27]. However, we indicate its proof, because it is a prototype argument.

\textbf{Theorem 1.5} (Euler, 1755). Let $A(n,m)$ denote the number of permutations $\sigma \in \Sigma_n$ such that $\text{des}(\sigma) = m$. (These are the Eulerian numbers.) Consider the exponential generating function

$$
G(u,v) \equiv \sum_{0 \leq m < n} \frac{A(n,m)}{n!} u^{n-1} v^m.
$$

Then the following hold:

(a) As formal power series in $u,v$,

$$
G(u,v) = \frac{\tanh \frac{u-v}{2}}{1 - u + v \tanh \frac{u-v}{2}} = \frac{1 - \frac{1}{3} (\frac{u-v}{2})^2 + \frac{2}{15} (\frac{u-v}{2})^4 - \frac{17}{315} (\frac{u-v}{2})^6 + \ldots}{1 - (\frac{u+v}{2}) + \frac{1}{3} (\frac{u+v}{2})^3 - \frac{2}{15} (\frac{u+v}{2})^5 (\frac{u-v}{2})^4 + \ldots}
$$

$$
= \frac{\sinh \frac{u-v}{2}}{\cosh \frac{u-v}{2} - u + v \sinh \frac{u-v}{2}} = \frac{1 + \frac{1}{3} (\frac{u-v}{2})^2 + \frac{1}{45} (\frac{u-v}{2})^4 + \ldots}{1 - (\frac{u+v}{2}) + \frac{1}{3} (\frac{u+v}{2})^3 - \frac{2}{15} (\frac{u+v}{2})^5 (\frac{u-v}{2})^2 + \ldots}
$$

$$
= \frac{e^u - e^v}{ue^v - ve^u} = \frac{\left(\frac{u-v}{2}\right) \left(1 + \frac{u+v}{2} + \frac{u^2 + uv + v^2}{3!} + \frac{u^3 + u^2v + uv^2 + v^3}{4!} + \ldots\right)}{\left(\frac{u-v}{2}\right) \left(1 + \frac{u+v}{2} + \frac{uv(u+v)}{3!} + \frac{uv^2 + u^2v^2 + v^3}{4!} + \ldots\right)}.
$$

(b) The generating function is analytically equal to, i. e. it has the same Taylor expansion at $(0,0)$ as, the meromorphic function

$$
G_{\text{an}}(u,v) = \begin{cases} 
\frac{e^u - e^v}{u^v - v^u} & u \neq v, \\
\frac{1}{1-u} & u = v,
\end{cases}
$$
This function is analytic around \((0,0)\), with poles in the real quadrant \(u, v > 0\) at
\[
 u + v = \begin{cases} 
 \frac{u+v}{e-1} \log \frac{u}{v} & \frac{u}{v} \neq 1, \\
 2 & \frac{u}{v} = 1,
\end{cases}
\]
using the ‘rational polar’ coordinates \(u + v, \frac{u}{v}\).

(c) Regarding absolute convergence, for \(u, v \geq 0\), the generating function gives the real function \(G_{\text{real}}(u, v)\), which is finite and equal to \(G_{\text{an}}(u, v)\) if
\[
 u = 0 \quad \text{or} \quad v = 0 \quad \text{or} \quad u + v < \begin{cases} 
 \frac{u+v}{e-1} \log \frac{u}{v} & \frac{u}{v} \neq 1, \\
 2 & \frac{u}{v} = 1,
\end{cases}
\]
and \(G_{\text{real}}(u, v) = +\infty\) otherwise.

Sketch of proof. Consider the extended generating function
\[
 G(u, v; x) = \sum_{0 \leq m < n} \frac{A(n, m)}{n!} u^{n-1-m} v^m x^n.
\]
Based on the image of "1" in the permutations, one can develop a recursion for the Eulerian numbers, which, in terms of the generating function, formally reads as the differential equation (in \(x\))
\[
 G'(u, v; x) = (1 + uG(u, v; x)(1 + vG(u, v; x))),
\]
\(G(u, v; 0) = 0\).

Now, \(G(u, v; x) = xG(ux, vx)\) solves this differential equation not only formally but also analytically for small \((ux, vx)\). Substituting \(ux \mapsto u, vx \mapsto v, x \mapsto 1\) formally gives the indicated \(G(u, v)\) as formal power series, but also analytically for small \((u, v)\). The nature of the poles follows from elementary considerations. The statement about absolute convergence follows from restricting in radial directions and considering the general properties of power series with nonnegative coefficients. (Originally, Euler computes \(1 + tG(t, tz) = \frac{1}{1 - z e^t (z - 1)} - z\), but uses the same idea.) \(\square\)

Remark 1.6. Using (the meromorphic extension of) the notation \((60)\), we have the more compact and less singular presentation
\[
 G(u, v) = \frac{T \tan \left(-\left(\frac{u-v}{2}\right)^2\right)}{1 - \frac{u+v}{2} T \tan \left(-\left(\frac{u-v}{2}\right)^2\right)}.
\]
Yet, such compact presentations might also obscure the arithmetical combinatorial nature of the given objects. Thus they are used somewhat less than possible. \(\triangle\)

Mielnik and Plebański, who have this generating function \(G(u, v)\), are not interested in metric estimates, thus they miss to give the following

Theorem 1.7. The absolute Magnus characteristic is given by
\[
 \Theta(x) = \int_{\lambda=0}^{1} xG(\lambda x, (1 - \lambda)x) \, d\lambda;
\]
and
\[
 \Theta_{\text{real}}(x) = \int_{\lambda=0}^{1} xG_{\text{real}}(\lambda x, (1 - \lambda)x) \, d\lambda = \int_{y=0}^{x} G_{\text{real}}(y, x-y) \, dy.
\]
In particular, \(\Theta_{\text{real}}(x) < +\infty\) if \(0 \leq x < 2\); and \(\Theta_{\text{real}}(x) = +\infty\) if \(2 \leq x\).
Proof. According to (10),
\[ \Theta(x) = \sum_{0 \leq m < n} ^{\infty} \frac{A(n, m)}{n!} \cdot \frac{(n - 1 - m)!m!}{n!} x^n. \]
Then the first equality in the statement follows from the beta function identity
\[ \frac{(n - 1 - m)!m!}{n!} = \int_0^1 \lambda^{n-1-m} (1 - \lambda)^m d\lambda. \]
In the real case, summation is valid, as the terms are nonnegative. Now, \( \Theta_{\text{real}}(x) < +\infty \) if \( 0 \leq x < 2 \), because the integrand is finite and continuous (it is easy to see that for the poles \( u + v \geq 2 \)). On the other hand,
\[ \Theta_{\text{real}}(2) = \int_0^1 2G_{\text{real}}(2\lambda, 2(1 - \lambda)) d\lambda = \int_{\nu = -1}^1 G_{\text{real}}(1 + \nu, 1 - \nu) d\nu. \]
The integrand is continuous away from \( \nu = 0 \), but it has a double pole there,
\[ G_{\text{real}}(1 + \nu, 1 - \nu) = \frac{\tanh \nu}{\nu - \tanh \nu} \sim 3\nu^{-2} + \text{holomorphic in } \nu. \]
This implies \( \Theta_{\text{real}}(2) = +\infty. \) □

Corollary 1.8 (Moan, Oteo [25], 2001). In the Banach algebra setting, the Magnus expansion converges absolutely (even as a time-ordered integral, cf. (9)) if \( \int |\phi| < 2 \). □

Note. In effect, Moan and Oteo generalize the estimate of Thompson [35], who is using the results of Goldberg [15] in the BCH case, to the Magnus case but using the results of Mielnik, Plebański [22]. They formulate convergence on the unit interval, for \( L^\infty \) norm < 2. △

Corollary 1.9. In the Banach algebra setting, the Magnus expansion diverges if \( \phi = 2 \cdot Z_{[0,1]} \), in which case \( \int |\phi| = 2 \).

Proof. In this case the Magnus expansion converges in \( F^1_{\text{loc}}([0,1]) \) but to an element whose conventional norm is \(+\infty\). This implies divergence in \( F^1_{\text{loc}}([0,1]) \). □

Remark. This counterexample is a measure and a not nice function (times the Lebesgue measure). This is not accidental; it will be addressed subsequently. △

The \( n \)-summand in (11),
\[ G_n(u, v) \equiv \sum_{m=0}^{n-1} \frac{A(n, m)}{n!} u^{n-1-m} v^m, \]
is, by definition, an Eulerian polynomial (in bivariate form) divided by \( n! \). It gives the individual coefficient
\[ \Theta_k = \int_0^1 G_k(\lambda, 1 - \lambda) d\lambda. \]

For later reference, we give here some terms:
\[ \Theta(x) = x + \frac{1}{2} x^2 + \frac{2}{9} x^3 + \frac{7}{72} x^4 + \frac{13}{300} x^5 + \frac{71}{3600} x^6 + \frac{67}{7350} x^7 + O(x^8). \]
We also give a crude but explicit estimate for \( \Theta_k \).
For \( \lambda \in [0,1] \), let
\[ \Theta^{(\lambda)}(x) = xG(\lambda x, (1 - \lambda)x). \]
Note that $\Theta(\lambda)(x)$ solves the differential equation (IVP)
\begin{equation}
\Theta^{(\lambda)'}(x) = (1 + \lambda \Theta^{(\lambda)}(x)) (1 + (1 - \lambda) \Theta^{(\lambda)}(x)),
\end{equation}
$$\Theta^{(\lambda)}(0) = 0.$$
(Of course, $\Theta^{(\lambda)}(x)$ could be expressed completely explicitly, cf. later.)

In this notation, as for a formal power series in $x$,
$$\Theta(x) = \int_{\lambda=0}^{1} \Theta^{(\lambda)}(x) d\lambda.$$

Lemma 1.10. For $\lambda \in [0, 1]$,
\begin{equation}
\Theta^{(\lambda)}(x) \leq \Theta^{(1/2)}(x).
\end{equation}
Consequently,
\begin{equation}
\Theta(x) \leq \Theta^{(1/2)}(x) \equiv \frac{x}{1 - \frac{1}{2} x},
\end{equation}
which can be written in other terms as
\begin{equation}
\Theta_k \leq 2^{1-k}.
\end{equation}

Proof. Then $\Theta^{(\lambda)}(x)$ solves the differential equation (IVP)
\begin{equation}
\Theta^{(\lambda)'}(x) = 1 + \Theta^{(\lambda)}(x) + \lambda (1 - \lambda) \Theta^{(\lambda)}(x)^2,
\end{equation}
$$\Theta^{(\lambda)}(0) = 0.$$

Thinking about this as a recursion for the nonnegative coefficients of the formal variable $x$, it is easy to prove that the greatest growth is achieved when $\lambda(1 - \lambda) \in [0, \frac{1}{4}]$ is maximal, which is at $\lambda = \frac{1}{2}$. This proves (16). Integrated, it yields (17). In terms of the individual coefficients of the power series, we obtain (18).

We will not use the individual estimate (18) particularly, but it is easy to remember. (It appears in Moan, Oteo [25] first; they use it to prove Corollary 1.8.)

This closes the discussion of known results regarding the convergence of the Magnus expansion in the Banach algebraic case. In Section 2, using the resolvent method, we reobtain these results in sharper form.

After discussing the convergence of the Magnus expansion, let us consider the relationship to the Baker–Campbell–Hausdorff expansion. A special case of the Magnus expansion is when $\phi = X1_{[0, 1)}, Y1_{[0, 1)}$, i.e., when we take the constant function $X$ on one unit interval, take the constant function $Y$ on another one, and concatenate them. (The first term, $X1_{[0, 1)}$ comes in lower, the second term $Y1_{[0, 1)}$ comes in higher in the ordering of the new interval.) Then the Magnus integral of order $n$ immediately specifies to
\begin{equation}
BCH_n(X, Y) = \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \mu_n(X, \ldots, X, Y, \ldots, Y).
\end{equation}

(Another equivalent viewpoint is that we consider measures supported only at two points, one with mass $X$ and one with mass $Y$.) Thus, from the Magnus formula we obtain
$$\exp(X) \exp(Y) = \exp \left( \sum_{n=1}^{\infty} BCH_n(X, Y) \right),$$
which is valid, as long as the sum in the RHS converges absolutely. This situation also adapts to formal variables $X, Y$; just take certain nilpotent elements in truncated associative free algebras to see that

$$\log(\exp(X) \exp(Y)) = \text{BCH}(X, Y) \equiv \sum_{n=1}^{\infty} \text{BCH}_n(X, Y)$$

in formal sense. $\text{BCH}_n(X, Y)$ collects exactly the terms of degree $n$. (We must note that formula [19] was already obtained by Dynkin [12].) This yields some explicit formulas for the terms of the BCH expansion, see Theorem 1.12.

One can discuss the BCH expansion $\text{BCH}(X_1, \ldots, X_k) = \log((\exp X_1) \ldots (\exp X_k))$ with several variables in similar manner; the formulas are analogous.

Even without the explicit knowledge of the terms $\text{BCH}(X_1, \ldots, X_k)$, we can derive some qualitative statements about the convergence of the Magnus and BCH expansions.

(O) Let $\Gamma(X, Y)$ be the same noncommutative formal series as $\text{BCH}(X, Y)$ but all coefficients turned into nonnegative. We can apply this series with commutative variables, or with nonnegative real variables. Note that

$$\Gamma_{\text{real}}(x_1, \ldots, x_k) < +\infty$$

$(x_i \in [0, +\infty))$ implies the $\text{BCH}(X_1, \ldots, X_k)$ will surely converge as long $|X_i| \leq x_i$. On the other hand, if

$$\Gamma_{\text{real}}(x_1, \ldots, x_k) = +\infty,$$

then there is counterexample, where $|X_i| = x_i$, and $\text{BCH}(X_1, \ldots, X_k)$ does not converge; in fact there is no element whose exponential is the product $(\exp X_1) \ldots (\exp X_k)$. Indeed, $X_i = x_i Y_i$ can be taken in the setting of $F_R^1[Y_i : 1 \leq i \leq k]$.

(I) The first observation, related to convergence of the Magnus expansion, is that the (multivariable) BCH expansions are special cases of the Magnus expansion, thus the estimates of the Magnus expansion also apply to them. In particular, the formal expansion of $\log(\exp(X) \exp(Y))$ converges absolutely in the Banach algebra setting if $|X| + |Y| < 2$.

(II) The second observation, related to divergence of the Magnus expansion, is that while the counterexample $\phi = 2 \cdot Z_{[0,1]}$ is only a general measure (and not a function times a Lebesgue measure), it is possible to divide it to by small intervals into parts $2 \cdot Z_{[c_i, c_{i+1}]}$, and then replace those parts by $\mu_R(2 \cdot Z_{[c_i, c_{i+1}]} \cdot 1_{[c_i, c_{i+1}]}$. In that way, the Magnus expansion of resulted measure $\phi_C$ is still convergent in $F_R^{1,\text{loc}}([0,1])$ to the very same element as of $\phi$, thus it is divergent $F_R^{1,\text{loc}}([0,1])$. Then $\phi_C$ is of (multivariable) BCH type, which can be actually be smoothed out completely; yet the total variation of $\phi_C$ is $\sum_i \Theta_{\text{real}}(2(c_{i+1} - c_i))$, which can be arbitrarily close to 2, as $\Theta_{\text{real}}(x) \sim x$ for small $x$. Thus we have a nice set of counterexamples for Magnus expansion although not with cumulative norm 2 but with $2 + \varepsilon$.

In particular, a numerical consequence is

**Lemma 1.11.** For $x_i \in [0, +\infty),$ $\Gamma_{\text{real}}(x_1, \ldots, x_k) \leq \Theta_{\text{real}}(x_1 + \ldots + x_k) \leq \Gamma_{\text{real}}(\Theta_{\text{real}}(x_1), \ldots, \Theta_{\text{real}}(x_k)).$

**Proof.** The first inequality follows from applying the Magnus expansion estimate to $Y_1[0,x_1] \ldots, Y_k 1_{[x_1 + \ldots + x_{k-1}, x_1 + \ldots + x_k]}$ in $F_R^{1,\text{loc}}[Y_i : 1 \leq i \leq k]$. The second one follows from the trivial $\Gamma$-estimate for the BCH expansion with respect to $X_1 = \mu_R(Z_{[0,x_1]}^1), \ldots, X_k = \mu_R(Z_{[x_1 + \ldots + x_{k-1}, x_1 + \ldots + x_k]}^1)$ in $F_R^{1,\text{loc}}([0, x_1 + \ldots + x_k]).$ \qed
Taking a closer look, from the explicit form of Magnus expansion one can obtain

**Theorem 1.12** (Goldberg [15], 1956). *The coefficient of the monomial
\[ M = (X \lor Y)^{k_1} \cdots X^{k_i} Y^{k_{i+1}} \cdots (X \lor Y)^{k_p} \]
\((X \text{ and } Y \text{ alternating}) \) in \( \log(\exp(X) \exp(Y)) \) is
\[ c_M = \int_{t=0}^{1} t^{\text{asc}(M)}(t-1)^{\text{des}(M)} G_{k_1}(t, t-1) \cdots G_{k_p}(t, t-1) \, dt, \]
where \( \text{asc}(M) \) is number of consecutive \( XY \) pairs in \( M \), \( \text{des}(M) \) is number of consecutive \( YX \) pairs in \( M \), and \( G_n(u, v) \) is as in (12).

*Proof.* (After Helmstetter, [17].) Let \( \deg_X(M) \) be sum of the exponents \( k_i \) belonging to \( X \), and let \( \deg_Y(M) \) be sum of the exponents \( k_i \) belonging to \( Y \). Consider (10) in the case when the first \( \deg_X(M) \) many variables are substituted by \( X \), and the remaining \( \deg_Y(M) \) many variables are substituted by \( Y \). Examine those permutations which lead to \( M \), and compute the generating polynomial of their ascents \( (u) \) and descents \( (v) \). \( M \) itself introduces ordered partitions of the variables. There are \( \frac{\deg_X(M)! \deg_Y(M)!}{k_1! \cdots k_p!} \) many possible partitions. Inside each partition set coming from \( (X \lor Y)^{k_i} \), the generating polynomial is \( k_i! G_{k_i}(u, v) \); and there are \( u^{\text{asc}(M)} \) and \( v^{\text{des}(M)} \) coming from the boundaries between the partitions. Thus the generating polynomial is
\[ \deg_X(M)! \deg_Y(M)! u^{\text{asc}(M)} v^{\text{des}(M)} G_{k_1}(u, v) \cdots G_{k_p}(u, v). \]

We obtain the coefficient of \( M \) in \( \mu_k(X, \ldots, X, Y, \ldots, Y) \) by replacing the terms \( u^a v^b \) with \( (-1)^b \frac{a! b!}{(a+b)!} \), respectively. According to the beta function identity, this corresponds exactly to integration of the \( u \mapsto t \) and \( v \mapsto t-1 \) substituted expression as in the statement. Then, according to (11), we have to divide by \( \deg_X(M)! \deg_Y(M)! \) in order to get the coefficients in the BCH expansion. \( \square \)

Considering the explicit nature of (20), and the fact that Goldberg [15] also computes the generating functions of the coefficients (with fixed \( p \)), in theory, one could obtain much sharper estimates for \( \Gamma_{\text{real}}(x, y) \). However, in practice, this is not entirely straightforward. For example, Thompson [35] (in some sense, the predecessor of Moan, Oteo [25]) obtains convergence for \( |X|, |Y| < 1 \) only. Arguments in Section 2 will already imply that the bound 2 is not sharp in the BCH case, but, in Section 3 we obtain more precise information about the convergence domain.

2. **The resolvent approach**

As we have seen, in the Banach algebraic setting, the Magnus expansion can be treated quite directly. In particular, the arguments of Mięńk, Plebański [22], and their symbolic functional calculus for time-ordered products are not needed in their full power. However, their ideas come handy when we inquire about finer analytical details.

In Banach algebras, we can define the logarithm of \( A \in \mathfrak{A} \) by
\[ \log A = \int_{\lambda=0}^{1} \frac{A - 1}{\lambda + (1 - \lambda)A} \, d\lambda = \int_{s=-\infty}^{0} \frac{A - 1}{(1 - s)(A - s)} \, ds, \]
which we consider well-defined if and only if \( \text{sp}(A) \) is disjoint from the closed negative real axis. For the sake of simplicity, we call \( A \) log-able, if it has this spectral property. It did not really matter before, but here we clarify that we consider (21) as our official definition of log. This definition also works perfectly well in the formal setting.
Notice the (modified) resolvent expression

\[(22) \quad R^{(\lambda)}(A) \equiv \frac{A - 1}{\lambda + (1 - \lambda)A}.\]

(Here we have used \(\lambda\) to parametrize the resolvent. Technically, \(\nu = 2\lambda - 1\) or \(\xi = 1 - \lambda\) are equally good choices, or even better; but the use of \(\lambda\) fits to the earlier discussions.)

**Remark 2.1.** The resolvent can be defined as any element \(R^{(\lambda)}(A)\) such that

\[(23) \quad (\lambda + (1 - \lambda)A)R^{(\lambda)}(A) = A - 1 \quad \text{and} \quad R^{(\lambda)}(A)(\lambda + (1 - \lambda)A) = A - 1\]

holds (“equational” definition). However, in those cases, it is easy to show that

\((\lambda + (1 - \lambda)A)(1 - (1 - \lambda)R^{(\lambda)}(A)) = 1 \quad \text{and} \quad (1 - (1 - \lambda)R^{(\lambda)}(A))(\lambda + (1 - \lambda)A) = 1\)

holds. Hence, \((\lambda + (1 - \lambda)A)^{-1}\) exists. It commutes with \(A - 1\); and the quantity \(R^{(\lambda)}(A)\) is also given as the product of \(A - 1\) and \((\lambda + (1 - \lambda)A)^{-1}\). Thus, the “equational” definition \((23)\) is equivalent to \((22)\).

\(\triangle\)

**Remark 2.2.** The identities

\[(1 - (1 - \lambda)R^{(\lambda)}(A)) = \frac{1}{\lambda + (1 - \lambda)A}\]

and

\[(1 - \lambda R^{(\lambda)}(A)) = \frac{A}{\lambda + (1 - \lambda)A}\]

are useful. The first equation, in particular shows (again) that the existence of \(R^{(\lambda)}(A)\) is equivalent to invertibility of \(\lambda + (1 - \lambda)A\) (i.e. the denominator of our resolvent expression).

It is well-known that if \(A\) is a formal perturbation of 1, and \(\varepsilon\) is a formal perturbation of 0, then

\[(A + \varepsilon)^{-1} - A^{-1} = -A^{-1}\varepsilon A^{-1} + O(\varepsilon)^2,\]

where \(O(\varepsilon)^2\) simply means terms which can be expressed with higher multiplicative frequency in \(\varepsilon\). In similar manner, one can prove

\[R^{(\lambda)}(A + \varepsilon) - R^{(\lambda)}(A) = (1 + (\lambda - 1)R^{(\lambda)}(A))\varepsilon(1 + (\lambda - 1)R^{(\lambda)}(A)) + O(\varepsilon)^2.\]

Further, substituting, \(\varepsilon \mapsto A\varepsilon\), one finds

\[(24) \quad R^{(\lambda)}(A(1 + \varepsilon)) - R^{(\lambda)}(A) = (1 + \lambda R^{(\lambda)}(A))\varepsilon(1 + (\lambda - 1)R^{(\lambda)}(A)) + O(\varepsilon)^2.\]

Applying this to \(A = \exp(X_1) \cdot \ldots \cdot \exp(X_{k-1})\) and \(\varepsilon = \exp(X_k) - 1\), we find

\[R^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_k)) =
\]

\[= R^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_{k-1}))
+ X_k
+ \lambda R^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_{k-1}))X_k
+ (\lambda - 1)X_kR^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_{k-1}))
+ \lambda(\lambda - 1)R^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_{k-1}))X_kR^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_{k-1}))
+ H(X_1, \ldots, X_k),\]

where \(H(X_1, \ldots, X_k)\) contains some terms where some variables \(X_i\) have multiplicity more than 1.
Using this as an induction step, one can prove that, in terms of formal variables,

\begin{equation}
R^{(\lambda)}(\exp(X_1) \cdot \ldots \cdot \exp(X_k)) = \sum_{i=(i_1,\ldots,i_l) \in \{1,\ldots,k\}^l, i_a \neq i_b, l \geq 1} \lambda^{\asc(i)}(\lambda - 1)^{\des(i)}X_{i_1} \cdot \ldots \cdot X_{i_l} + H(X_1,\ldots,X_k),
\end{equation}

where \( H(X_1,\ldots,X_k) \) collects the terms with multiplicities in the variables.

This is the starting point of the arguments of Mielnik, Plebański [22]. If we integrate (25) in \( \lambda \in [0,1] \), then the beta function identity yields

\begin{equation}
\log(\exp(X_1) \cdot \ldots \cdot \exp(X_k)) = \sum_{i=(i_1,\ldots,i_l) \in \{1,\ldots,k\}^l, i_a \neq i_b, l \geq 1} (-1)^{\des(i)} \frac{\asc(i)! \des(i)!}{l!} X_{i_1} \cdot \ldots \cdot X_{i_l} + H(X_1,\ldots,X_k).
\end{equation}

Comparing this to (3), gives a proof of (10).

Let \( \mu^{(\lambda)}(X_1,\ldots,X_n) = \sum_{\sigma \in \Sigma_n} \lambda^{\asc(\sigma)}(\lambda - 1)^{\des(\sigma)}X_{\sigma(1)} \cdot \ldots \cdot X_{\sigma(n)} \).

Assume that \( \phi \) is a continuous \( \mathcal{A} \)-valued measure of finite variation on the interval \( I \).

Let us define the Mielnik–Plebański integrals

\begin{equation}
\mu_{k,R}(\phi) = \int_{t=(t_1,\ldots,t_k) \in I^k} \lambda^{\asc(t)}(\lambda - 1)^{\des(t)}\phi(t_1) \cdot \ldots \cdot \phi(t_k)
= \int_{t_1 \leq \ldots \leq t_k} \mu_{k}(\phi(t_1),\ldots,\phi(t_k)),
\end{equation}

analogously to the terms of the Magnus expansion. Then

\begin{equation}
\mu_{k,R}(\phi) = \int_{\lambda=0}^{1} \mu_{k,R}^{(\lambda)}(\phi) \, d\lambda.
\end{equation}

**Theorem 2.3.** (The resolvent formula of Mielnik, Plebański [22], analytic version.) Let \( \phi \) be a continuous \( \mathcal{A} \)-valued measure of finite variation on the interval \( I \). If

\begin{equation}
\sum_{k=1}^{\infty} \left| \mu_{k,R}^{(\lambda)}(\phi) \right| < +\infty,
\end{equation}

then

\begin{equation}
R^{(\lambda)}(\exp_R(\phi)) = \sum_{k=1}^{\infty} \mu_{k,R}^{(\lambda)}(\phi).
\end{equation}

**Proof.** Indeed, (25) supplies sufficiently many identities to prove that \( (1 - \lambda) + \lambda \exp_R(\phi) \) times the RHS of (30) contracts to \( \exp_R(\phi) - 1 \). The “equational” definition is satisfied. (Note that we need only the multiplicity-free part of the identities.) And this implies the statement.

**Remark 2.4.** Let us apply the previous statement with \( \phi = X_{1,0,1} \) in the formal setting. \( X \) is a formal variable, the algebra is either with the topology of formal power series.
Remark 2.5. which shows that the naive majorizing estimate obtained from (24) is actually exact.

(32) \( \Theta(\lambda) = \sum_{n=1}^{\infty} G_n(\lambda, \lambda - 1)X^n \).

Also, we know this from before. Considering \( XG(uX, vX) \) with \( u = \lambda, v = \lambda - 1 \), Theorem 1.5.a yields

\[
\sum_{n=1}^{\infty} G_n(\lambda, \lambda - 1)X^n = XG(\lambda X, (\lambda - 1)X) = \frac{(\exp X) - 1}{\lambda + (1 - \lambda)\exp X} \equiv \mathcal{R}(\lambda)(\exp X).
\]

In fact, we can view this phenomenon as an alternative proof for Theorem 1.5.a. \( \triangle \)

Let us take another look to the time-ordered resolvent expression. Let \( \Theta_n(\lambda) \) be \( 1/n! \) times the sum of the absolute value of the coefficients in \( \mu^{(\lambda)}(X_1, \ldots, X_n) \); and let \( \Theta^{(\lambda)}(x) = \sum_{n=1}^{\infty} \Theta_n^{(\lambda)}(x) \) be the associated generating function. Note that

\[
(31) \quad \Theta^{(\lambda)}(x) = xG(|\lambda|x, |1 - \lambda|x).
\]

This a conservative extension of the notation \( \Theta \) from \( \lambda \in [0, 1] \). (But, beyond a trickery, it has now a direct meaning concerning the resolvent expressions.)

Also note that \( \Theta^{(\lambda)}(x) \) solves the differential equation (IVP)

\[
(32) \quad \Theta^{(\lambda)'}(x) = (1 + |\lambda|\Theta^{(\lambda)}(x))(1 + |1 - \lambda|\Theta^{(\lambda)}(x)), \quad \Theta^{(\lambda)}(0) = 0;
\]

which shows that the naive majorizing estimate obtained from (24) is actually exact.

**Remark 2.5.** If \( \lambda \in [0, 1] \), then \( \Theta^{(\lambda)}(x) = xG(\lambda x, (1 - \lambda)x) \), and

\[
\Theta^{(\lambda)}(x) = \begin{cases} 
\frac{\tanh \frac{2\lambda - 1}{2} x}{1 - \frac{1}{2} \tanh \frac{2\lambda - 1}{2} x} & \text{if } \lambda \in [0, 1] \setminus \{\frac{1}{2}\}, \\
\frac{x}{1 - \frac{1}{2} x} & \text{if } \lambda = \frac{1}{2}.
\end{cases}
\]

Furthermore, in terms of Taylor series,

\[
\Theta^{(\lambda)}(x) = x + \frac{1}{2} x^2 + \frac{1 + 2\lambda(1 - \lambda)}{6} x^3 + \ldots \quad \text{if } \lambda \in [0, 1]. \quad \triangle
\]

Recall that \( \Theta^{(\lambda)}(x) = xG(|1 - \lambda|x, |\lambda|x) \). Thus

\[
\Theta^{(\lambda)}(x) = \sum_{0 \leq m < n} \frac{A(n, m)}{n!} |\lambda|^{n-1-m} |1 - \lambda|^m x^n,
\]

cf. Theorem 1.5. Then, analogously to the Magnus case,

\[
\left| \mu^{(\lambda)}_{k; R}(\phi) \right| \leq \int_{t_1 \leq \ldots \leq t_k \in I} \left| \mu^{(\lambda)}_{k}(\phi(t_1), \ldots, \phi(t_k)) \right| \leq \Theta^{(\lambda)}_k \left( \int |\phi| \right)^k,
\]

and,

\[
(33) \quad \sum_{k=1}^{\infty} \left| \mu^{(\lambda)}_{k; R}(\phi) \right| \leq \sum_{k=1}^{\infty} \int_{t_1 \leq \ldots \leq t_k \in I} \left| \mu^{(\lambda)}_{k}(\phi(t_1), \ldots, \phi(t_k)) \right| \leq \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi| \right).
\]
Theorem 2.6. (The logarithmic version of the Magnus formula.)

Let $\phi$ be a continuous $A$-valued measure of finite variation on the interval $I$. If

\[(34) \quad \sum_{k=1}^{\infty} |\mu_{k,R}^{(\lambda)}(\phi)\rangle \quad \text{is bounded on } \lambda \in [0,1], \text{ then } \sum_{n=1}^{\infty} |\mu_{n,R}(\phi)| < +\infty \text{ holds, } \exp_R(\phi) \text{ is log-able, and } \]

\[(35) \quad \mu_R(\phi) = \log \exp_R(\phi). \]

Proof.

\[(36) \quad \sum_{k=1}^{\infty} |\mu_{k,R}(\phi)| \equiv \sum_{k=1}^{\infty} \left| \int_{t_1 \leq \ldots \leq t_k \in I} \mu_k(\phi(t_1), \ldots, \phi(t_k)) \right| \]

\begin{align*}
&= \sum_{k=1}^{\infty} \left| \int_{\lambda=0}^{1} \int_{t_1 \leq \ldots \leq t_k \in I} \mu_k^{(\lambda)}(\phi(t_1), \ldots, \phi(t_k)) \, d\lambda \right| \\
&\leq \sum_{k=1}^{\infty} \int_{\lambda=0}^{1} \left| \int_{t_1 \leq \ldots \leq t_k \in I} \mu_k^{(\lambda)}(\phi(t_1), \ldots, \phi(t_k)) \right| \, d\lambda \\
&= \int_{\lambda=0}^{1} \sum_{k=1}^{\infty} \left| \int_{t_1 \leq \ldots \leq t_k \in I} \mu_k^{(\lambda)}(\phi(t_1), \ldots, \phi(t_k)) \right| \, d\lambda \\
&= \int_{\lambda=0}^{1} \sum_{k=1}^{\infty} |\mu_{k,R}^{(\lambda)}(\phi)| \, d\lambda;
\end{align*}

and this is bounded by the assumption. Consequently, and similarly,

\[(37) \quad \sum_{k=1}^{\infty} \mu_{k,R}(\phi) = \int_{\lambda=0}^{1} \sum_{k=1}^{\infty} \mu_{k,R}^{(\lambda)}(\phi) \, d\lambda \]

also holds. As the requirements of the previous theorem hold, we see that we integrate the resolvents, as in the definition of log.

(Remark: in (36) and (37), we used two types of Fubini theorems, which are unrelated. The first type is on $[0,1] \times I^k$ applied to linear combinations of linear product measures with coefficients polynomial in $\lambda$, which are barely more than interval function integrals, which can easily be controlled in terms of variation. The second type is interchangeability of sums and integrals, that is Beppo Levi’s theorem, also valid in the Lebesgue-Bochner setting, although we use it only for functions continuous in $\lambda$.) \(\square\)

At first sight, the statement (35) looks relatively innocent, but, through the existence of the resolvents, it tells something significant about the spectral properties of the objects in question. It implies $\text{sp}(\exp_R(\phi)) \cap (-\infty,0] = \emptyset$ and $\text{sp}(\mu_R(\phi)) \subset \{z \in \mathbb{C} : |\text{Im } z| < \pi\}$.

Remark 2.7. As we see,

\[\int_{\lambda=0}^{1} \sum_{k=1}^{\infty} |\mu_{k,R}^{(\lambda)}(\phi)| \, d\lambda < +\infty\]

is actually sufficient for the absolute convergence of the Magnus expansion. If we can guarantee the existence of resolvents (the finiteness of the integrand for any $\lambda \in [0,1]$ suffices), then (35) also holds. \(\triangle\)
Corollary 2.8. If $\phi$ is a continuous $\mathfrak{A}$-valued measure, and $\int |\phi| < 2$, then $\exp_R(\phi)$ is log-able, and $\mu_R(\phi) = \log \exp_R(\phi)$.

Proof. In this case, for $\lambda \in [0, 1]$,

$$\sum_{k=1}^{\infty} |t_{k,R}(\phi)| \leq \Theta_{\text{real}}^{(\lambda)}(x) = \left( \int |\phi| \right) \cdot G_{\text{real}} \left( \lambda \int |\phi|, (1 - \lambda) \int |\phi| \right),$$

which is bounded. (Cf. Theorem 1.5.) Thus the conditions of the previous theorem hold.

Lemma 2.9. For $\lambda \in [0, 1]$, the convergence radius of $\Theta^{(\lambda)}(x)$ around $x = 0$ is

$$C_{\infty}^{(\lambda)} = \begin{cases} 2 & \text{if } \lambda = \frac{1}{2} \\ \frac{2 \arctanh(1 - 2\lambda)}{1 - 2\lambda} & \text{if } \lambda \in (0, 1) \setminus \left\{ \frac{1}{2} \right\} \\ +\infty & \text{if } \lambda \in \{0, 1\}. \end{cases}$$

This is a strictly convex, nonnegative function in $\lambda \in (0, 1)$, symmetric for $\lambda \mapsto 1 - \lambda$; its minimum is $C_{\infty}^{(1/2)} = 2$. In particular, in $\lambda \in [0, 1]$, it yields a $[2, +\infty]$-valued strictly convex continuous function.

For $\lambda \in [0, 1]$, let

$$w^{(\lambda)} = 1/C_{\infty}^{(\lambda)}.$$

In $\lambda \in [0, 1]$, it is a $[0, 1/2]$-valued strictly convex continuous function, symmetric for $\lambda \mapsto 1 - \lambda$; its maximum is $w^{(1/2)} = 1/2$.

Proof. $C_{\infty}^{(\lambda)}$ is just the convergence radius of $G(\lambda x, (1 - \lambda)x)$ in $x$, cf. Theorem 1.5. The rest is elementary calculus. \hfill \Box

Remark 2.10. We see that in the critical case $\int |\phi| = 2$, the possible divergence of the Magnus expansion comes from around $G(1, 1)$; corresponding to $R^{(1/2)}(\exp_R(\phi))$, which is $-2$ times the real involutive Cayley transform of $\exp_R(\phi)$. If we manage to provide absolute convergence on that area, the (logarithmic) Magnus formula will converge better. \hfill \triangle

Remark 2.11. Similarly to the Magnus expansion, one can formulate Theorem 2.8 and the subsequent discussion for not necessarily continuous measures, too. Then, in (27), we have insert the multiplicity terms

$$\int_{t_1 \leq \ldots \leq t_n \in I} \cdots \sim \int_{t_1 \leq \ldots \leq t_n \in I} \frac{1}{\text{mul}(t_1, \ldots, t_n)!} \cdots$$

or, depending on viewpoint,

$$\int_{t=(t_1, \ldots, t_n) \in I^n} \lambda^{\text{asc}(t)}(\lambda - 1)^{\text{des}(t)} \cdots \sim \int_{t=(t_1, \ldots, t_n) \in I^n} \lambda^{\text{asc}(t)}(\lambda - 1)^{\text{des}(t)} G_{\text{mul}(t_1, \ldots, t_n)}(\lambda, \lambda - 1) \cdots$$

throughout, where

$$G_{(m_1, \ldots, m_a)}(u, v) = G_{m_1}(u, v) \cdots G_{m_a}(u, v)$$

by definition. Again, this is in accordance to taking the continuous blowup $\phi^*$ of $\phi$. 


According to this viewpoint, the Goldberg presentation reads as the expression for the Magnus expansion via [27]/[28] in the extreme case when the measure is supported in two points.

For $A \in \mathfrak{A}$, we define its Magnus exponent as

$$\mathcal{M}_\mathfrak{A}(A) := \inf \left\{ \int |\phi| : \exp_R(\phi) = A \right\}.$$  

Here it was left unexplained what kind ordered measures $\phi$ we consider at all. Now, the point is that as long as we consider ordered measures of finite variation (and this definition aims those) those measures can be modified to multivariable BCH type, as it was explained in point (II), at arbitrarily small expense. Those can be smoothed out, etc. Ultimately, the Magnus exponent is the same for reasonable measure classes.

**Theorem 2.12.** If $\mathcal{M}_\mathfrak{A}(A) < 2$, then $A$ is log-able, and

$$|\log A| \leq \Theta_{\text{real}}(\mathcal{M}_\mathfrak{A}(A)).$$

**Proof.** Whenever $A = \exp_R(\phi)$, $\int |\phi| < 2$, by Theorems 2.6 and 1.7, we know that $|\log A| = |\mu_R(\phi)| \leq \Theta_{\text{real}}(\int |\phi|)$ holds. \hfill \Box

**Remark 2.13.** As Mielnik, Plebański [22] notes, having control on the resolvent expressions allows to proceed with a more general holomorphic function calculus on $\exp_R(\phi)$. E. g., the square root function

$$\sqrt{A} = \frac{1}{\pi} \int_{\lambda=0}^{1} \frac{A}{\lambda + (1-\lambda)A} \frac{d\lambda}{\sqrt{\lambda(1-\lambda)}} = 1 + \frac{1}{\pi} \int_{\lambda=0}^{1} \frac{A - 1}{\lambda + (1-\lambda)A} \sqrt{\frac{\lambda}{1-\lambda}} \frac{d\lambda}{\lambda}$$

allows very similar estimates. This is due to the fact that more or less the same spectral cut can be used to define as in the case of log. \hfill \Box

**2.14.** As a formal noncommutative power series, let us define

$$M^\lambda(X, Y) = X(1 - \lambda(\lambda - 1)YX)^{-1} + Y(1 - \lambda(\lambda - 1)XY)^{-1} + \lambda XY(1 - \lambda(\lambda - 1)XY)^{-1} + (\lambda - 1)YX(1 - \lambda(\lambda - 1)XY)^{-1}$$

$$= \sum_{k=0}^{\infty} \left( \lambda^k(\lambda - 1)^kY(\lambda - 1)^kX + \lambda^k(\lambda - 1)^kX(\lambda - 1)^kY \right)$$

and then, inductively,

$$M^\lambda(X_1, \ldots, X_n) = M^\lambda(M^\lambda(X_1, \ldots, X_{n-1}), X_n).$$

Then it is easy to show that

$$M^\lambda(X_1, \ldots, X_n) = \sum_{i_1, \ldots, i_s \in \{1, \ldots, n\}}^{\text{asc}(i_1, \ldots, i_s)} (\lambda - 1)^{\text{des}(i_1, \ldots, i_s)} X_{i_1} \cdots X_{i_s};$$

and $M^\lambda$ is an associative operation formally.

Let $P^\lambda(X_1, \ldots, X_n)$ denote $M^\lambda(X_1, \ldots, X_n)$ but here every coefficient of the monomials $X_{j_1} \cdots X_{j_s}$ is replaced by its absolute value. It is notable that in the expansion of [35] there are no monomials where a variable occurs twice consecutively, and there
are no multiplicities among the monomials coming from the four terms of (38). This allow us to conclude that
\[
P(\lambda)(X, Y) = Y(1 - |\lambda| \cdot |\lambda - 1|XY)^{-1} + (1 - |\lambda| \cdot |\lambda - 1|XY)^{-1}X
\]
\[
+ |\lambda|XY(1 - |\lambda| \cdot |\lambda - 1|XY)^{-1} + |\lambda - 1|Y(1 - |\lambda| \cdot |\lambda - 1|XY)^{-1}X;
\]
and
\[
P(\lambda)(X_1, \ldots, X_n) = P(\lambda)(P(\lambda)(X_1, \ldots, X_{n-1}), X_n);
\]
and \(P(\lambda)\) is still an associative operation formally. Putting the formal commutative variables \(x, y\) into the place of \(X, Y\) we find
\[
P(\lambda)(x, y) = \frac{x + y + (|\lambda| + |1 - \lambda|)xy}{1 - |\lambda| \cdot |1 - \lambda|xy},
\]
If \(\lambda \in [0, 1]\), then
\[
P(\lambda)(x, y) = \frac{x + y + xy}{1 - \lambda(1 - \lambda)xy}.
\]
\[\blacktriangle\]

**Lemma 2.15.** (a) \(P(\lambda)(x_1, \ldots, x_n)\) applied with commutative variables is symmetric.

(b)
\[
P^{(1/2)}(x, \ldots, x) = \frac{2nx}{2 - (n - 1)x}.
\]

**Proof.** (a) follows from the identities \(P(\lambda)(P(\lambda)(x_1, x_2), x_3) = P(\lambda)(x_1, P(\lambda)(x_2, x_3))\) and \(P(\lambda)(x_1, x_2) = P(\lambda)(x_2, x_1)\). (b) is easy to prove by induction. \(\square\)

**Remark 2.16.** These are also implied by the formula \(P(\lambda)(\Theta(\lambda)(x_1), \ldots, \Theta(\lambda)(x_n)) = \Theta(\lambda)(x_1 + \ldots + x_n)\), cf. Theorem 2.18 later.

Consider \(\mu^{(\lambda)}_{k+n}(X_1, \ldots, X_k, Y_1, \ldots, Y_n)\). Tentatively, it has shape
\[
(39) \sum \lambda^{n/k+\text{steps} \cdot (\lambda - 1)^{n/k+\text{steps} \cdot \ldots \cdot \mu_{k+1}(X_{i+1}, \ldots)\mu_{k+2}(X_{i+2}, \ldots, \ldots)
\]
where \(X\)-blocks and \(Y\)-blocks follow each other, the variables are with increasing indexing in each \(X\)-block and \(Y\)-block, respectively; and, multiplicatively, every variable has multiplicity 1. So, this decomposition, in its patterns, corresponds to \(M^{(\lambda)}(X, Y)\). Similar picture applies in the case when the variables are partitioned to \(p\) many segments. Then it is related to \(M^{(\lambda)}\) with \(p\) many variables.

**Theorem 2.17.** Suppose that \(\phi_1, \phi_2\) are \(\mathfrak{A}\)-valued ordered measures of finite variation. Then
\[
\sum_{n=1}^{\infty} \left| \mu^{(\lambda)}_{n,R}(\phi_1, \phi_2) \right| \leq P^{(\lambda)}_{\text{real}} \left( \sum_{n=1}^{\infty} \left| \mu^{(\lambda)}_{n,R}(\phi_1) \right|, \sum_{n=1}^{\infty} \left| \mu^{(\lambda)}_{n,R}(\phi_2) \right| \right).
\]
If the RHS is finite, then
\[
\mu^{(\lambda)}_{R}(\phi_1, \phi_2) = M^{(\lambda)} \left( \mu^{(\lambda)}_{R}(\phi_1), \mu^{(\lambda)}_{R}(\phi_2) \right),
\]
meaning so that the sum obtained from the formal power series of \(M^{(\lambda)}\) is absolute convergent. Analogous statements hold decomposition to several parts.

**Proof.** This corresponds to the decomposition (39) in the integrand. The multidecomposition case can be proven analogously. \(\square\)
Theorem 2.18. As formal commutative power series,
\[ \Theta^{(\lambda)}(x + y) = P^{(\lambda)} \left( \Theta^{(\lambda)}(x), \Theta^{(\lambda)}(y) \right). \]

Consequently,
\[ \Theta_{\text{real}}^{(\lambda)}(x + y) = P_{\text{real}}^{(\lambda)} \left( \Theta_{\text{real}}^{(\lambda)}(x), \Theta_{\text{real}}^{(\lambda)}(y) \right). \]

Analogous statements hold with several variables.

Proof. Let \( \Theta^{(\lambda)}(x, y) = \sum_{k,n=0}^{\infty} \Theta_{k,n}^{(\lambda)} x^k y^n \), where \( \Theta_{k,n}^{(\lambda)} = \frac{1}{k! n!} \) times the sum of the absolute value of the coefficients in \( \mu^{(\lambda)}(X_1, \ldots, X_k, Y_1, \ldots, Y_n) \). Based on the placement of \( X_1 \) in the corresponding monomials, one has the formal differential equation / IVP

\[ \frac{\partial}{\partial x} \Theta^{(\lambda)}(x, y) = (1 + |\lambda| \Theta^{(\lambda)}(x, y))(1 + |1 - \lambda| \Theta^{(\lambda)}(x, y)), \]

\[ \Theta^{(\lambda)}(0, y) = \Theta^{(\lambda)}(y). \]

Comparison to \( [32] \) yields \( \Theta^{(\lambda)}(x, y) = \Theta^{(\lambda)}(x + y) \). On the other hand, the nature of decomposition \( [39] \) implies \( \Theta^{(\lambda)}(x, y) = P^{(\lambda)} \left( \Theta^{(\lambda)}(x), \Theta^{(\lambda)}(y) \right) \) (the multinomial identity is applies with respect to the distribution of the \( X_i \) and \( Y_j \), respectively). The multivariable case follows by induction. \( \square \)

Alternative proof. Let \( x, y \geq 0 \) but \( x + y < 2 \). Let us apply Theorem 2.18 to \( \phi_1 = Z^{1}_{x,y} \) and \( \phi_2 = Z^{1}_{x+y} \). Taking norm at both sides yields the equality \( \Theta_{\text{real}}^{(\lambda)}(x + y) = P_{\text{real}}^{(\lambda)} \left( \Theta_{\text{real}}^{(\lambda)}(x), \Theta_{\text{real}}^{(\lambda)}(y) \right) \) for \( x + y < 2 \). This, however, already implies the equality of the formal series. The multivariable case can be proven analogously. \( \square \)

Third proof. We know the \( \Theta^{(\lambda)} \) and \( P^{(\lambda)} \) in question explicitly, see \( [31] \) and Theorem 1.3 \( \{ \} \) thus we can check the statement arithmetically. The multivariable case follows by induction. \( \square \)

Corollary 2.19.

\[ P^{(\lambda)} \left( \Theta^{(\lambda)}(x), y \right) = \Theta^{(\lambda)} \left( x + \left( \Theta^{(\lambda)} \right)^{-1} (y) \right); \]

and

\[ P_{\text{real}}^{(\lambda)} \left( \Theta_{\text{real}}^{(\lambda)}(x), y \right) = \Theta_{\text{real}}^{(\lambda)} \left( x + \left( \Theta_{\text{real}}^{(\lambda)} \right)^{-1} (y) \right). \]

Proof. In Theorem 2.18 substitute \( \left( \Theta^{(\lambda)} \right)^{-1} (y) \) or \( \left( \Theta_{\text{real}}^{(\lambda)} \right)^{-1} (y) \) to the place of \( y \), respectively. \( \square \)

Remark 2.20. For \( \lambda \in [0, 1] \),

\[ \left( \Theta^{(\lambda)} \right)^{-1} (y) = \begin{cases} \log \frac{1 + \lambda y}{1 + (1 - \lambda) y} & \text{if } \lambda \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}, \\ \frac{2 \text{artanh} (2\lambda - 1)y}{2\lambda - 1} - \frac{2 + y}{2\lambda - 1} & \text{if } \lambda = \frac{1}{2}. \end{cases} \]

\( \triangle \)
Theorem 2.21. Suppose that \(\phi_1, \phi_2, \phi_3\) are \(\mathfrak{A}\)-valued ordered measures of finite variation; \(N \in [0, +\infty)\). If
\[
\sum_{n=1}^{\infty} |\mu^{(\lambda)}_{n,R}(\phi_2)| \leq N,
\]
then
\[
\sum_{n=1}^{\infty} |\mu^{(\lambda)}_{n,R}(\phi_1, \phi_2, \phi_3)| \leq \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| + \int |\phi_3| + \left( \Theta^{(1/2)} \right)^{-1}(N) \right).
\]
If \(\lambda \in [0,1]\), then this is
\[
\leq \Theta^{(1/2)} \left( \int |\phi_1| + \int |\phi_3| + \left( \Theta^{(1/2)} \right)^{-1}(N) \right);
\]
which is finite, if
\[
\int |\phi_1| + \int |\phi_3| < \frac{4}{2 + N}.
\]

Proof. According Theorem 2.17 (multidecomposition version),
\[
\sum_{n=1}^{\infty} |\mu^{(\lambda)}_{n,R}(\phi_1, \phi_2, \phi_3)| \leq P^{(\lambda)}_{\text{real}} \left( \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| \right), N, \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_3| \right) \right)
= P^{(\lambda)}_{\text{real}} \left( P^{(\lambda)}_{\text{real}} \left( \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| \right), \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_3| \right) \right), N \right)
= P^{(\lambda)}_{\text{real}} \left( \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| + \int |\phi_3| \right), N \right).
\]
For \(\lambda \in [0, 1]\), \(\Theta^{(\lambda)}(x) \leq \Theta^{(1/2)}(x)\) (see 17) and \(P^{(\lambda)}(x, y) \leq P^{(1/2)}(x, y)\) (commutative variables, it is easy to see) implies
\[
\leq P^{(1/2)}_{\text{real}} \left( \Theta^{(1/2)}_{\text{real}} \left( \int |\phi_1| + \int |\phi_3| \right), N \right).
\]
Taking Corollary 2.19 into account, we obtain the statement. \(\square\)

Corollary 2.22. Suppose that \(\phi_1, \phi_2, \phi_3\) are \(\mathfrak{A}\)-valued ordered measures of finite variation, \(\phi = \phi_1, \phi_2, \phi_3; N \in [0, +\infty)\). If
\[
\sup_{\lambda \in [0,1]} \sum_{n=1}^{\infty} |\mu^{(\lambda)}_{n,R}(\phi_2)| \leq N,
\]
and
\[
\int |\phi_1| + \int |\phi_3| < \frac{4}{2 + N},
\]
then the boundedness of (41) holds, thus the (logarithmic) Magnus formula holds for \(\phi\). In particular, the Magnus series is absolutely convergent. \(\square\)

Lemma 2.23. Suppose that \(\phi\) is a \(\mathfrak{A}\)-valued measure of finite variation, which arises as a Lebesgue-Bochner integrable function times the Lebesgue measure on an interval. Assume \([\lambda_0, 1 - \lambda_0] \subset (0, 1)\) and \(\int |\phi| > 0\). Then there is an \(\eta > 0\) such that
\[
|\mu^{(\lambda)}_{2,R}(\phi)| + \eta \leq \Theta^{(\lambda)}_2 \cdot \left( \int |\phi| \right)^2
\]
holds for any $\lambda \in [\lambda_0, 1 - \lambda_0]$. Consequently,
\[
\sum_{n=1}^{\infty} |\mu_{n,R}^{(\lambda)}(\phi)| + \eta \leq \Theta^{(\lambda)} \left( \int |\phi| \right)
\]
also holds for $\lambda \in [\lambda_0, 1 - \lambda_0]$.

**Proof.** Suppose that $\phi(t) = h(t)1_I$. Let $\varepsilon > 0$. Then $h(t)$ can be approximated by stepfunctions (in $L^1$ norm) arbitrarily well. This implies that there is a nontrivial subinterval $I' \subset I$ of nonzero length and a nonzero element $a \in \mathcal{A}$ such that $\int_{I'} |h(t) - a| \leq \varepsilon |a| \cdot |I'|$.

Let $k(t) = h(t) - a$. Note that
\[
\mu_2^{(\lambda)}(X_1, X_2) = \lambda X_1 X_2 - (1 - \lambda) X_1 X_2.
\]

Thus
\[
\left| \int_{t_1 \leq t_2 \in I'} \mu_2^{(\lambda)}(h(t_1), h(t_2)) \, dt_1 \, dt_2 \right| \leq 
\]
\[
\min(\lambda, 1 - \lambda) \int_{t_1, t_2 \in I'} |a| \, |k(t_2)| + |k(t_1)| \, |a| + |k(t_1)| \, |k(t_2)| \, dt_1 \, dt_2 
\]
\[
+ \frac{1}{2} \left( \frac{\lambda}{1 - \lambda} \right) \int_{t_1, t_2 \in I'} |a|^2 + |a| \, |k(t_2)| + |k(t_1)| \, |a| + |k(t_1)| \, |k(t_2)| \, dt_1 \, dt_2 
\]
\[
\leq \frac{1}{2} |a|^2 |I'|^2 \left( \frac{1}{2} \left( 1 - \frac{\lambda}{1 - \lambda} \right) + 2\varepsilon + \varepsilon^2 \right).
\]

On the other hand, in the formal estimate we count at least
\[
\int_{t_1, t_2 \in I'} |a|^2 - |a| \, |k(t_2)| - |k(t_1)| \, |a| - |k(t_1)| \, |k(t_2)| \, dt_1 \, dt_2 \geq \frac{1}{2} |a|^2 |I'|^2 (1 - 2\varepsilon - \varepsilon^2).
\]

Between the two estimates there is a gap
\[
\frac{1}{2} |a|^2 |I'|^2 (1 - 2\varepsilon - \varepsilon^2) - \frac{1}{2} |a|^2 |I'|^2 \left( \frac{1}{2} \left( 1 - \frac{\lambda}{1 - \lambda} \right) + 2\varepsilon + \varepsilon^2 \right).
\]

Assume that $\varepsilon > 0$ was chosen originally so that
\[
1 - 2 \left( \frac{1}{2} \left( 1 - \frac{\lambda}{1 - \lambda} \right) \right) - 4\varepsilon - 2\varepsilon^2 = \delta > 0.
\]

Then, for $\lambda \in [\lambda_0, 1 - \lambda - \delta]$
\[
\left| \mu_{2,R}^{(\lambda)}(\phi_{I'}) \right| + \frac{1}{2} |a|^2 |I'|^2 \delta \leq \Theta_2^{(\lambda)} \left( \int |\phi_{I'}| \right)^2.
\]

Extending to estimate to $I$ trivially, we find
\[
\left| \mu_{2,R}^{(\lambda)}(\phi) \right| + \frac{1}{2} |a|^2 |I'|^2 \delta \leq \Theta_2^{(\lambda)} \left( \int |\phi| \right)^2.
\]

This proves the statement with $\eta = \frac{1}{2} |a|^2 |I'|^2 \delta$. \hfill $\square$

**Lemma 2.24.** Suppose that $\phi$ is a $\mathcal{A}$-valued measure of finite variation which arises as a Lebesgue-Bochner integrable function times the Lebesgue measure on an interval. Assume $[\lambda_0, 1 - \lambda_0] \subset (0, 1)$ and $0 < \int |\phi| \leq 2$. Then there is an $0 < \delta < \int |\phi|
\[
\sum_{n=1}^{\infty} |\mu_{n,R}^{(\lambda)}(\phi)| \leq \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi| - \delta \right)
\]
also holds for \( \lambda \in [\lambda_0, 1 - \lambda_0] \).

**Proof.** First assume that \( 0 < \int |\phi| < 2 \). Apply Lemma 2.23. Then, due to the continuity of \( \Theta^{(\lambda)}_{\text{real}} \), there is a \( 0 < \delta < \int |\phi| \), such that

\[
\Theta^{(\lambda)}_{\text{real}} \left( \int |\phi| \right) - \eta \leq \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi| - \delta \right).
\]

This implies the statement.

If \( \int |\phi| = 2 \), then take any decomposition \( \phi = \phi_1, \phi_2 \) such that \( 0 < \int |\phi_1| < 2 \). As we know the statement for \( \phi_1 \), we can assume that \( 0 < \delta < |\phi_1| \) is such that

\[
\sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi_1) \right| \leq \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| - \delta \right).
\]

Apply Theorem 2.17. It yields

\[
\sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi) \right| \equiv \sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi_1, \phi_2) \right| \leq P^{(\lambda)}_{\text{real}} \left( \sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi_1) \right|, \sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi_2) \right| \right)
\]

\[
\leq P^{(\lambda)}_{\text{real}} \left( \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| - \delta \right), \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_2| \right) \right)
\]

\[
= \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi_1| - \delta + \int |\phi_2| \right) \equiv \Theta^{(\lambda)}_{\text{real}} \left( \int |\phi| - \delta \right);
\]

which proves the statement. \( \square \)

**Theorem 2.25.** Suppose that \( \phi \) is a \( \mathbb{A} \)-valued measure, which arises as a Lebesgue-Bochner integrable function times the Lebesgue measure on an interval. Suppose that \( \int |\phi| = 2 \). Then the boundedness of (31) holds, thus the (logarithmic) Magnus formula holds. In particular, the Magnus series is absolutely convergent.

**Proof.** Pick \( 0 < \lambda_0 < 1/2 \) arbitrarily. Then, for \( \lambda \in [0, \lambda_0] \cup [1 - \lambda_0, 1] \),

\[
\sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi) \right|
\]

is bounded by \( \Theta^{(\lambda_0)}(2) \). Applying Lemma 2.23 we find it is bounded for \( \lambda \in [\lambda_0, 1 - \lambda_0] \) by \( \Theta^{(1/2)}(2 - \delta) \).

\( \square \)

**Remark 2.26.** In Lemma 2.23 Lemma 2.24 Theorem 2.25 it is sufficient that \( \phi \) is Lebesgue-Bochner integrable times the Lebesgue measure only on a subinterval but with nonzero variation there. This implies that those statements extend automatically to the case of noncontinuous measures: In that case the blowup has a subinterval with the required properties. \( \triangle \)

**Remark 2.27.** In similar manner, in Corollary 2.22 boundedness, and thus the convergence of the Magnus expansion, extends to the case \( \int |\phi_1| + \int |\phi_3| = \frac{1}{2N} \). \( \triangle \)

**Remark 2.28.** However, if we ask only that \( \phi \) is a Lebesgue-Bochner integral not in uniform but in strong sense, then one cannot make a similar statement.

Let us consider the measures

\[
\Xi_n = X_1 1_{[0,1/2^n]} \ldots X_{2^n} 1_{[0,1/2^n]},
\]

\( 2^n \) terms
where the $X_j$ are formal noncommutative variables with $\ell_1$ norm. If $c \geq 0$, then $|c \Xi_n| = c$, and it is easy to show that $n \nearrow \infty$ yields $|\mu_R(c \Xi_n)| \nearrow \Theta(c)$.

Taking an infinite direct sum of the $2 \cdot \Xi_n$, we obtain a strong Lebesgue-Bochner construction of variation 2, whose Magnus expansion, however, cannot converge due to the Banach-Steinhaus theorem. \hfill \triangle

At this point we can argue that 2 can be improved as the cumulative convergence radius of the Baker–Campbell–Hausdorff expansion. Consider $x_1, x_2 \in [0, +\infty)$. Let $\phi_{x_1, x_2} = x_1X_1I_{(0, 1)} + x_2X_2I_{[0, 1)}$, where $X_1, X_2$ are formal variables with $\ell_1$ norm 1. According to arguments from the proof of Lemma 2.21 and Lemma 2.24 there is $\delta > 0$ such that if $0 \leq x_1 + x_2 \leq 2$, $\lambda \in [\lambda_0, 1 - \lambda_0] \subset [0, 1]$, then $\sum_{n=1}^{\infty} \left| \mu_{n,R}(\phi_{x_1, x_2}) \right| \leq \Theta_{\text{real}}(2 - \delta)$.

(The statements themselves say this only for individual pairs $x_1, x_2$, but the argument can be modified for a “compact variety”.) Then, as in Theorem 2.21 it easy to see that uniform boundedness (and convergence) still holds if $0 \leq x_1 + x_2 \leq 2 + \delta'$ with some $\delta' > 0$. In order to quantify this $\delta'$, however, it is better not to pursue the present line of arguments but to apply direct estimates.

3. THE DISCRETE RESOLVENT APPROACH

Here we take the “resolvent Goldbergian” approach. We try to make this discussion independent from the previous section, as much it is possible, in order to avoid the measure terminology; although we do not redevelop the functions $M^{(\lambda)}$ and $P^{(\lambda)}$ here.

**Lemma 3.1.** In a real or complex algebra, if $\lambda$ is a scalar, then

$$R = \frac{A - 1}{\lambda + (1 - \lambda)A} \quad \Leftrightarrow \quad A = \frac{1 + \lambda R}{1 - (1 - \lambda)R};$$

and in those cases

$$\frac{1}{\lambda + (1 - \lambda)A} = 1 - (1 - \lambda)R \quad \text{and} \quad \frac{1}{1 - (1 - \lambda)R} = \lambda + (1 - \lambda)A.

**Proof.** Elementary computation. \hfill \Box

**Lemma 3.2.** Assume that $R^{(\lambda)}(A)$ and $R^{(\lambda)}(B)$ exist. Then

(i) $R^{(\lambda)}(AB)$ exists

(ii) $R^{(\lambda)}(BA)$ exists

(iii) $(1 + \lambda(1 - \lambda)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1}$ exists

(iv) $(1 + \lambda(1 - \lambda)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1}$ exists

are equivalent to each other.

In those cases,

\begin{align*}
R^{(\lambda)}(AB) = & R^{(\lambda)}(A)(1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1} \\& \\
& + R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1} \\
& + \lambda R^{(\lambda)}(A)R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1} \\
& + (\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A)(1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1}.
\end{align*}

**Note.** Using the identities

$$R^{(\lambda)}(A)(1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1} = (1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1}R^{(\lambda)}(A)$$

and

$$R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1} = (1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1}R^{(\lambda)}(B),$$

\begin{align*}
R^{(\lambda)}(AB) = & R^{(\lambda)}(A)(1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1} \\& \\
& + R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1} \\
& + \lambda R^{(\lambda)}(A)R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1} \\
& + (\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A)(1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1}.
\end{align*}
one can write (42) somewhat flexibly.

Proof. (i) ⇔ (iii) follows from the identity

\[
1 + \lambda (1 - \lambda) R^{(\lambda)}(A) R^{(\lambda)}(B) = \left(1 - (1 - \lambda) R^{(\lambda)}(A)ight) \left(1 - (1 - \lambda) R^{(\lambda)}(B)\right)
\]

\[
\left(\lambda + (1 - \lambda) \frac{1 + \lambda R^{(\lambda)}(A)}{1 - (1 - \lambda) R^{(\lambda)}(A)} \frac{1 + \lambda R^{(\lambda)}(B)}{1 - (1 - \lambda) R^{(\lambda)}(B)}\right)
\]

\[
\left(\lambda + (1 - \lambda) R^{(\lambda)}(A) R^{(\lambda)}(B)\right)
\]

(ii) ⇔ (iv) is similar. (iii) ⇔ (iv) follows from the general identity

\[
(1 - H G)^{-1} = 1 - H (1 - G H)^{-1} G.
\]

Similarly, the identities in the remark are consequences of the general identity

\[
(1 - G H)^{-1} G = G (1 - H G)^{-1}.
\]

It remains to prove (42). Now,

\[
R^{(\lambda)}(A) + R^{(\lambda)}(B) + (2\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B) = \left(1 - (1 - \lambda) R^{(\lambda)}(A)\right)
\]

\[
\left(1 - (1 - \lambda) R^{(\lambda)}(B)\right)
\]

Comparing (44) and (43), we find

\[
(1 - (1 - \lambda) R^{(\lambda)}(A)) R^{(\lambda)}(AB) = (R^{(\lambda)}(A) + R^{(\lambda)}(B) + (2\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B))
\]

\[
(1 + \lambda (1 - \lambda) R^{(\lambda)}(A) R^{(\lambda)}(B))^{-1}
\]

\[
(1 - (1 - \lambda) R^{(\lambda)}(A)).
\]

Expanding the to 3 × 1 × 2 = 6 terms, it yields

\[
(g) = + R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B))^{-1}
\]

\[
(b) + R^{(\lambda)}(B)(1 - \lambda(\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B))^{-1}
\]

\[
(cf) + (2\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B)(1 - \lambda(\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B))^{-1}
\]

\[
(e) - (1 - \lambda) R^{(\lambda)}(A) R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A))^{-1}
\]

\[
(d) - (1 - \lambda) R^{(\lambda)}(B) R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A))^{-1}
\]

\[
(ahj) - (1 - \lambda)(2\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B) R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A))^{-1}
\]

This can be decomposed further as

\[
(a) = + R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A))^{-1}
\]

\[
(b) + R^{(\lambda)}(B)(1 - \lambda(\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B))^{-1}
\]

\[
(c) + \lambda R^{(\lambda)}(A) R^{(\lambda)}(B)(1 - \lambda(\lambda - 1) R^{(\lambda)}(A) R^{(\lambda)}(B))^{-1}
\]

\[
(d) + (\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A))^{-1}
\]

\[
(e) - (1 - \lambda) R^{(\lambda)}(A) R^{(\lambda)}(A)(1 - \lambda(\lambda - 1) R^{(\lambda)}(B) R^{(\lambda)}(A))^{-1}
\]
(f) $\ - (1 - \lambda)R^{(\lambda)}(A)R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1}$

(g) $\ - (1 - \lambda)\lambda R^{(\lambda)}(A)R^{(\lambda)}(A)R^{(\lambda)}(B)(1 - \lambda(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B))^{-1}$

(h) $\ - (1 - \lambda)(\lambda - 1)R^{(\lambda)}(A)R^{(\lambda)}(B)R^{(\lambda)}(A)(1 - \lambda(\lambda - 1)R^{(\lambda)}(B)R^{(\lambda)}(A))^{-1}$

\[ \begin{array}{c}
\text{(i)} \\
\text{(j)}
\end{array} \]

\[ A, \quad - A. \]

(For example, line (ahj) decomposes to lines (a)(h)(j).) Let temporarily $R^{(\lambda)}(A, B)$ denote the RHS of (42). Then lines (i) and (j) cancel each other, while lines (a)–(h) yield the $2 \times 4 = 8$ terms of $(1 - (1 - \lambda)R^{(\lambda)}(A))R^{(\lambda)}(A, B)$. Thus,

\[ (1 - (1 - \lambda)R^{(\lambda)}(A))R^{(\lambda)}(AB) = (1 - (1 - \lambda)R^{(\lambda)}(A))R^{(\lambda)}(A, B), \]

which implies $R^{(\lambda)}(AB) = R^{(\lambda)}(A, B)$. \qed

**Lemma 3.3.** If $X, Y$ are formal variables, then

\[ R^{(\lambda)}((\exp X)(\exp Y)) = \sum_{k=0}^{\infty} \lambda^k(\lambda - 1)^kR^{(\lambda)}((\exp Y)R^{(\lambda)}((\exp X))R^{(\lambda)}((\exp X)))^k \]

\[ + \lambda^k(\lambda - 1)^k(\exp X)R^{(\lambda)}((\exp Y)R^{(\lambda)}((\exp X)))^k \]

\[ + \lambda^{k+1}(\lambda - 1)^kR^{(\lambda)}((\exp X)R^{(\lambda)}((\exp Y)))^k+1 \]

\[ + \lambda^k(\lambda - 1)^k+1R^{(\lambda)}((\exp Y)R^{(\lambda)}((\exp X)))^k+1 \].

One can replace $\exp X$ and $\exp Y$ by other formal perturbations of 1.

**Proof.** $R^{(\lambda)}((\exp X) = X + \ldots$ is purely formal, so the previous Lemma can be applied. $(1 - \lambda(\lambda - 1))R^{(\lambda)}((\exp X))R^{(\lambda)}((\exp Y))^{-1}$ expand as

\[ \sum_{k=0}^{\infty} \lambda^k(\lambda - 1)^k(\exp X)R^{(\lambda)}((\exp Y)))^k, \]

etc. The general case is similar, or follows by a change of variables. \qed

**Theorem 3.4.** Suppose that $X_1, \ldots, X_n$ are formal variables. Then

\[ R^{(\lambda)}((\exp X_1) \cdot \ldots \cdot (\exp X_n)) = \sum_{i_1, \ldots, i_s \in \{1, \ldots, n\}}^{s \geq 1, i_j \neq i_{j+1}} \lambda^{\text{asc}(i_1, \ldots, i_s)(\lambda - 1)^{\text{des}(i_1, \ldots, i_s)}}R^{(\lambda)}((\exp X_{i_1}) \cdot \ldots \cdot (\exp X_{i_s})). \]

One can replace $\exp X$ and $\exp Y$ by other formal perturbations of 1.

**Proof.** Splitting up $(\exp X_1) \cdot \ldots \cdot (\exp X_n) = ((\exp X_1) \cdot \ldots \cdot (\exp X_{n-1})) \cdot (\exp X_n)$, let us apply Lemma 3.3 inductively. Again, the general case is similar, or follows by a change of variables. \qed

**Remark 3.5.** We can give an alternative proof for the previous theorem based on Section 2. Apply Theorem 2.3 and Remark 2.11 with respect to a non-continuous measure which is supported at $n$ points with masses $X_1, \ldots, X_n$. (We should do this in the $F_{1/n}$ topology, or, alternatively, we should give $X_i$ arbitrarily small norm.) This yields
On the other hand, by the same argument (or cf. Remark 2.4),

\[
\sum_{\substack{i_1, \ldots, i_s \in \{1, \ldots, n\} \\ j_i \neq i_{j_i+1}}} \lambda^{\text{asc}(i_1, \ldots, i_s)} (\lambda - 1)^{\text{des}(i_1, \ldots, i_s)} G_{m_1} (\lambda, \lambda - 1) X_{i_1}^{m_1} \cdots G_{m_s} (\lambda, \lambda - 1) X_{i_s}^{m_s}.
\]

which allows to collect the corresponding terms. The replaceability of the exponentials follows from a simple change of variables.

Another possibility to present the same argument is to apply Theorem 2.17 to the measure \(X_1 1_{\{0,1\}} \cdots X_n 1_{\{0,1\}}\); cf. the forthcoming discussion.

We will not need this right now, but, as an illustration for the resolvent expansion, we give a somewhat peculiar proof for

**Theorem 3.6** (F. Schur [30] (1893), Poincaré [28] (1899)). Let us consider the (formal) power series

\[
\beta(x) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \beta_n x^n.
\]

Then the coefficient of \(X^{n_1} Y X^{n-n_1}\) in \(\text{BCH}(X, Y)\) is

\[
(-1)^n \beta_n \cdot (-1)^{n-n_1} \binom{n}{n_1}.
\]

Similarly, the coefficient of \(Y^{n_1} X Y^{n-n_1}\) in \(\text{BCH}(X, Y)\) is

\[
(-1)^n \beta_n \cdot (-1)^{n_1} \binom{n}{n_1}.
\]

**Proof.** We will prove only the first statement. Let us consider \(\text{BCH}(X, Y)\); replace the terms \(X^{n_1} Y X^{n-n_1}\) by \(x_1^{n_1} x_2^{n-n_1}\), and ignore the other terms. Then, according Lemma 3.3 and the logarithm formula, the generating function to obtain is

\[
\int_{\lambda=0}^{1} \left( 1 + \lambda (\lambda - 1) \mathcal{R}^{(\lambda)}(\exp x_1) \mathcal{R}^{(\lambda)}(\exp x_2) + \lambda \mathcal{R}^{(\lambda)}(\exp x_1) + (\lambda - 1) \mathcal{R}^{(\lambda)}(\exp x_2) \right) \mathrm{d}\lambda
\]

\[
= \int_{\lambda=0}^{1} \frac{1}{\lambda e^{-x_1} + (1 - \lambda) \lambda + (1 - \lambda) e^{x_2}} \mathrm{d}\lambda = \left[ \frac{1}{e^{x_2-x_1} - 1} \log \left( \frac{\lambda + (1 - \lambda) e^{x_1}}{\lambda + (1 - \lambda) e^{x_2}} \right) \right]_{\lambda=0}^{1}
\]

\[
= \frac{x_2 - x_1}{e^{x_2-x_1} - 1} = \beta(x_2 - x_1) = \sum_{n=0}^{\infty} (-1)^n \beta_n \cdot (x_1 - x_2)^n.
\]

The last expression for the generating function expresses the statement to prove. \(\square\)

**Note.** Alternatively, using commutator notation, we can write the result above as

\[
\text{BCH}(X, Y)_{\text{deg}X,Y=(n,1)} = (-1)^n \beta_n [X, [X, \ldots [X, [X, Y], \ldots], Y], \ldots],
\]

One obtains

\[
\text{BCH}(X, Y)_{\text{deg}X,Y=(1,n)} = (-1)^n \beta_n [[\ldots [X, Y], \ldots], Y]
\]

\[\text{n times}\]
similarly. (Here \((-1)^n \beta_n\) makes difference only as \((-1)^1 \beta_1 = \frac{1}{2}\).)

**Remark.** Goldberg [15] contains more on generating functions. Furthermore, Mielnik, Plebański [22] contains, in general, an array of advanced arguments of analytical combinatorial nature, which are able to upset any hierarchy of arguments.

**Remark 3.7.** The Bernoulli numbers \(B_j\) are defined by the expansion

\[
\beta(x) = \frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \beta_j x^j = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j \quad (|x| < 2\pi).
\]

Then

\[
(49) \quad B_j = \sum_{k=0}^{j} \binom{j}{k} B_k \quad (j \geq 2);
\]
\[
(50) \quad B_j = -\sum_{k=0}^{j-1} \frac{B_k}{j+1-k} \quad (j \geq 2);
\]
\[
(51) \quad B_{2j+1} = 0 \quad (j \geq 1);
\]
\[
(52) \quad \tan x = \sum_{j=1}^{\infty} 2^{2j}(2^{2j} - 1)(-1)^{j+1} \frac{B_{2j}}{(2j)!} x^{2j-1} \quad (|x| < \pi);
\]
\[
(53) \quad \text{sgn } B_{2j} = (-1)^{j+1} \quad (j \geq 1);
\]
\[
(54) \quad \tanh x = \sum_{j=1}^{\infty} 2^{2j}(2^{2j} - 1) \frac{B_{2j}}{(2j)!} x^{2j-1} \quad (|x| < \pi);
\]
\[
(55) \quad B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66};
\]
\[
(56) \quad B_{2j} = (-1)^{j+1} \frac{(2j)!}{(2\pi)^{2j}} 2\zeta(2j) = (-1)^{j+1} \frac{(2j)!}{(2\pi)^{2j}} \sum_{N=1}^{\infty} \frac{1}{N^{2j}} \quad (j \geq 1).
\]

(The power series above are to be interpreted analytically.) Properties (49–55) are relatively straightforward to prove using elementary analysis, while (56) follows from Euler’s formula

\[
\pi \cot \pi x = \frac{1}{x} + \sum_{N=-\infty}^{\infty} \left( \frac{1}{x-N} + \frac{1}{N} \right).
\]

Thus, the Bernoulli numbers turn up in some relevant power series.

Let us recall the formal power series \(M^{(\lambda)}(X_1, \ldots, X_n)\) and \(P^{(\lambda)}(X_1, \ldots, X_n)\) from point 2.14. With this terminology, (43) reads as

\[
\mathcal{R}^{(\lambda)}((\exp X_1) \cdot \ldots \cdot (\exp X_n)) = M^{(\lambda)}(\mathcal{R}^{(\lambda)}(\exp X_1), \ldots, \mathcal{R}^{(\lambda)}(\exp X_n)).
\]

Let \(\Gamma^{(\lambda)}(X_1, \ldots, X_n)\) denote \(\mathcal{R}^{(\lambda)}((\exp X_1) \cdot \ldots \cdot (\exp X_n))\) but where every coefficient of the monomials \(X_{j_1} \cdot \ldots \cdot X_{j_s}\) is replaced by its absolute value. Then, we find

\[
(57) \quad \Gamma^{(\lambda)}(X_1, \ldots, X_n) = P^{(\lambda)}(\Gamma^{(\lambda)}(X_1), \ldots, \Gamma^{(\lambda)}(X_n)).
\]
The connection to the Baker–Campbell–Hausdorff expansion is as follows. Formally (just taking the logarithm),

\[ \text{BCH}(X_1, \ldots, X_n) = \int_{\lambda=0}^{1} R^{(\lambda)}((\exp X_1) \cdot \ldots \cdot (\exp X_n)) \, d\lambda \]

Thus, taking absolute values in the power series,

\[ \Gamma(X_1, \ldots, x_n) \leq \int_{\lambda=0}^{1} \Gamma^{(\lambda)}(x_1, \ldots, x_n) \, d\lambda. \]

In particular, applying this to nonnegative variables,

\[ \Gamma_{\text{real}}(x_1, \ldots, x_n) \leq \int_{\lambda=0}^{1} \Gamma^{(\lambda)}_{\text{real}}(x_1, \ldots, x_n) \, d\lambda. \]

In fact, if \( \Gamma^{(\lambda)}_{\text{real}}(x_1, \ldots, x_n) \) is bounded on \( \lambda \in [0, 1] \), then the BCH expansion not only converges, but the resolvents exists, and for \( |X_i| \leq x_i \) the formula (58) actually yields

\[ \text{BCH}(X_1, \ldots, X_n) = \log((\exp X_1) \ldots (\exp X_n)) \]

is strict analytical sense.

**Remark 3.8.** Even at this point, we can improve on the convergence range of the BCH formula easily:

For \( \lambda \in [0, 1] \), let \( \rho^{(\lambda)}(x) \) be the power series expansion of \( R^{(\lambda)}(\exp x) \) around \( x = 0 \). \( R^{(\lambda)}(\exp x) \) satisfies the recursion / formal IVP

\[ \rho^{(\lambda)'}(x) = 1 - (1 - 2\lambda)\rho^{(\lambda)}(x) - \lambda(1 - \lambda)\rho^{(\lambda)}(x)^2, \]

\[ \rho^{(\lambda)}(0) = 0. \]

Thus, thinking combinatorially, we can estimate the absolute value of the coefficients in \( \rho^{(\lambda)}(x) \) by the coefficients of the solution of the recursion / formal IVP

\[ g^{(\lambda)'}(x) = 1 + |1 - 2\lambda|g^{(\lambda)}(x) + \lambda(1 - \lambda)g^{(\lambda)}(x)^2, \]

\[ g^{(\lambda)}(0) = 0. \]

The actual solution (until it blows up) is given by

\[ g^{(\lambda)}(x) = \frac{x \tan \left( \frac{x^2}{4} \left( 8\lambda(1 - \lambda) - 1 \right) \right)}{1 - |1 - 2\lambda|^\frac{1}{2} x \tan \left( \frac{x^2}{4} (8\lambda(1 - \lambda) - 1) \right)}, \]

where

\[ \tan(z) = \begin{cases} \tan \sqrt{z} & \text{if } z > 0, \\ \sqrt{z} & \text{if } z = 0, \\ \tanh \sqrt{-z} & \text{if } z > 0. \end{cases} \]

Thus, according to the previous discussion,

\[ \Gamma^{(\lambda)}(x) \leq g^{(\lambda)}(x). \]

(This is exact for \( \lambda = 0, \frac{1}{2}, 1 \), cf. Lemma 3.10.)
Consequently, 
\[
\Gamma^{(\lambda)}(x, y) \leq P^{(\lambda)}(g^{(\lambda)}(x), g^{(\lambda)}(y)) = \frac{g^{(\lambda)}(x) + g^{(\lambda)}(y) + g^{(\lambda)}(x)g^{(\lambda)}(y)}{1 - \lambda(1 - \lambda)g^{(\lambda)}(x)g^{(\lambda)}(y)}.
\]

Let 
\[
h(\lambda) = \begin{cases} 
\frac{2}{|1 - 2\lambda|} \text{AT} \left( \frac{8\lambda(1 - \lambda) - 1}{|1 - 2\lambda|^2} \right) & \text{if } \lambda \neq 0, \\
0 & \text{if } \lambda = 1/2,
\end{cases}
\]

where
\[
\text{AT}(x) = \begin{cases} 
\arctan \frac{\sqrt{x}}{x} & \text{if } x > 0 \\
1 & \text{if } x = 0 \\
\text{artanh} \frac{\sqrt{-x}}{-x} & \text{if } -1 < x < 0.
\end{cases}
\]

One can check that if \(x, y \geq 0\) and \(x + y = h(\lambda)\), then
\[
\lambda(1 - \lambda)g^{(\lambda)}(x)g^{(\lambda)}(y) = 1.
\]

Let
\[
\tilde{C}_2 = \min_{x \in [0, 1]} h(\lambda) = \min_{x \in [0, 1/2]} h(\lambda).
\]

Now,
\[
\tilde{C}_2 = 2.701428513 \ldots,
\]
and
\[
\tilde{C}_2 = h(\lambda_0),
\]
where \(\lambda_0 = 0.258744139 \ldots\). Here \(\lambda_0\) is the unique solution
\[
h(\lambda) \cdot 4\lambda(1 - \lambda)|1 - 2\lambda| = 1
\]
for \(\lambda \in [0, 1/2]\). (As such, it can be computed with Newton iterations very efficiently.)

Consequently, if \(x, y \geq 0\) and \(x + y < \tilde{C}_2\), then
\[
\sup_{\lambda \in [0,1]} \lambda(1 - \lambda)g^{(\lambda)}(x)g^{(\lambda)}(y) < 1.
\]

Thus,
\[
\Gamma^{(\lambda)}_{\text{real}}(x, y) \leq P^{(\lambda)}(g^{(\lambda)}(x), g^{(\lambda)}(y)) = \frac{g^{(\lambda)}(x) + g^{(\lambda)}(y) + g^{(\lambda)}(x)g^{(\lambda)}(y)}{1 - \lambda(1 - \lambda)g^{(\lambda)}(x)g^{(\lambda)}(y)}.
\]

Hence, if \(X, Y \in \mathfrak{A}\), and
\[
|X| + |Y| < \tilde{C}_2,
\]
then the (logarithmic) BCH formula holds.

This \(\tilde{C}_2 = 2.701428513 \ldots\) is already a significant improvement compared to 2. \(\triangle\)

**Lemma 3.9.** For \(\lambda \in [0, 1]\), the convergence radius of \(R^{(\lambda)}(\exp X)\) around \(X = 0\) is given by
\[
C^{(\lambda)}_{\text{e}} = \left| \log \frac{\lambda}{\lambda - 1} \right| = \begin{cases} 
\sqrt{\pi^2 + \left( \log \frac{\lambda}{1 - \lambda} \right)^2} & \text{if } \lambda \in (0, 1), \\
+\infty & \text{if } \lambda \in \{0, 1\}.
\end{cases}
\]
For \( \lambda \in (0,1) \), taken in complex sense, \( X \mapsto R^{(\lambda)}(\exp X) \) has just two simple poles on the boundary of the convergence disk.

\( C^{(\lambda),\varepsilon}_\infty \) is a strictly convex, nonnegative function in \( \lambda \in (0,1) \), symmetric for \( \lambda \mapsto 1-\lambda \); its minimum is \( C^{(1/2)}_\infty = \pi \). In particular, in \( \lambda \in [0,1] \), it yields a \( [\pi, +\infty] \)-valued strictly convex continuous function. For \( \lambda \in [0,1] \), let

\[
\omega^{(\lambda),\varepsilon} = 1/C^{(\lambda),\varepsilon}_\infty.
\]

In \( \lambda \in [0,1] \), \( \omega^{(\lambda),\varepsilon} \) is a \( [0,1/\pi] \)-valued strictly convex continuous function, symmetric for \( \lambda \mapsto 1-\lambda \); its maximum is \( \omega^{(1/2),\varepsilon} = 1/\pi \).

**Proof.** The case \( \lambda \in \{0,1\} \) is trivial. For \( 0 < \lambda < 1 \), formally,

\[
R^{(\lambda)}(\exp X) = \frac{(\exp X) - 1}{\lambda + (1-\lambda)(\exp X)}
\]

\[
= \frac{1}{\sqrt{\lambda(1-\lambda)} \cosh \frac{X}{2} \left( X - \log \frac{\lambda}{1-\lambda} \right)}
\]

\[
= \frac{2\lambda - 1}{2\lambda(1-\lambda)} + \frac{1}{2\lambda(1-\lambda)} \tanh \frac{1}{2} \left( X - \log \frac{\lambda}{1-\lambda} \right).
\]

Analytically, this function has simple poles at \( X = \log \frac{\lambda}{1-\lambda} + (2k+1)\pi \), \( k \in \mathbb{Z} \) (with residue \( \frac{1}{\lambda(1-\lambda)} \)). The rest is elementary calculus. \( \square \)

**Lemma 3.10.** For \( \lambda = 0, \frac{1}{2}, 1 \), we have

\[
R^{(0)}(\exp X) = 1 - (\exp -X);
\]

\[
R^{(1/2)}(\exp X) = 2 \tanh \frac{X}{2};
\]

\[
R^{(1)}(\exp X) = (\exp X) - 1;
\]

\[
\Gamma^{(0)}(X) = \exp X - 1;
\]

\[
\Gamma^{(1/2)}(X) = 2 \tan \frac{X}{2};
\]

\[
\Gamma^{(1)}(X) = \exp X - 1.
\]

**Proof.** It follows from the power series expansions. \( \square \)

Firstly, we draw conclusions regarding the Cayley transform case, i.e. \( \lambda = \frac{1}{2} \), \(-\frac{1}{2}R^{(1/2)}(A)\) being the real involutive Cayley transform \( \frac{1-A}{1+A} \).

**Theorem 3.11.** If \( x_1, \ldots, x_n \in [0, \infty) \) \( (n \geq 2) \) and

\[
x_1 + \ldots + x_n \leq 2n \arctan \frac{1}{n-1},
\]

then

\[
\Gamma^{(1/2)}_{\text{real}}(x_1, \ldots, x_n) \leq \frac{2n \tan \frac{x_1 + \ldots + x_n}{2n}}{1 - (n-1) \tan \frac{x_1 + \ldots + x_n}{2n}} < +\infty.
\]

However

\[
\Gamma^{(1/2)}_{\text{real}} \left( \underbrace{\frac{2 \arctan \frac{1}{n-1}, \ldots, 2 \arctan \frac{1}{n-1}}{n \text{ terms}}} \right) = +\infty.
\]

(For \( n = 1 \), \( 2 \arctan \frac{1}{n-1} = \pi \) could be taken.)

**Proof.** Lemma 2.15 and Lemma 3.10 allow us to compute \( \Gamma^{(1/2)}(X_1, \ldots, X_n) \) formally. In particular, we can compute \( \Gamma^{(1/2)}_{\text{real}}(X_1, \ldots, X_n) \) explicitly if \( n = 2 \) or \( x_1 = \ldots = x_n \).
Simple analysis shows that if $c \in [0, \pi]$, then the function $x \mapsto \Gamma^{(1/2)}(x, c - x)$ is concave on the interval $x \in [0, c]$. Using the associativity of $P^{(\lambda)}$, this allows us to make the estimate

$$\Gamma^{(1/2)}_{\text{real}}(x_1, \ldots, x_n) \leq \Gamma^{(1/2)}_{\text{real}}\left(\frac{x_1 + \ldots + x_n}{n}, \ldots, \frac{x_1 + \ldots + x_n}{n}\right);$$

and the RHS is known according to the previous discussion.

\[ \square \]

**Theorem 3.12.** If $|X_1| + \ldots + |X_n| < 2n \arctan \frac{1}{n - 1}$, then (45) specified to $\lambda = \frac{1}{2}$, i.e. (divided by 2)

$$\frac{1}{2} R^{(1/2)}((\exp X_1) \cdot \ldots \cdot (\exp X_n)) = \sum_{\substack{i_1, \ldots, i_s \in \{1, \ldots, n\} \\ s \geq 1, i_j \neq i_{j+1}}} (-1)^{\text{des}(i_1, \ldots, i_s)} \frac{1}{2^s} R^{(1/2)}(\exp X_{i_1}) \cdot \ldots \cdot R^{(1/2)}(\exp X_{i_s}),$$

is well-defined and absolute convergent.

This is, however, not necessarily true if $|X_1| = \ldots = |X_n| = 2 \arctan \frac{1}{n - 1}$, then the LHS might not even exist.

**Proof.** The convergence statement follows from the previous theorem immediately. For the non-convergence statement, let us consider the Banach algebra of noncommutative power series of $Y_1, \ldots, Y_n$ with $\ell^1$ norm of the coefficients, and apply $X_n = 2 \arctan \frac{1}{n - 1}Y_n$. The power series expansion of the LHS is algebraically determined, yet we know that it would have norm $+\infty$. \[ \square \]

**Corollary 3.13.** If $|X| + |Y| < \pi$, then $(\exp X) + (\exp Y)$ is invertible.

**Proof.** The previous statement yields $\frac{1}{(\exp X)(\exp Y) + 1} = \frac{1}{2} - \frac{1}{2} (\exp X)(\exp Y) - 1$, that is $((\exp X)(\exp Y) + 1)^{-1}$. Substituting $Y = -Y$, and multiplying by $(\exp Y)^{-1}$ on the left yields $((\exp X) + (\exp Y))^{-1}$. \[ \square \]

The convergence of the BCH formula requires more work:

**Lemma 3.14.** (a) The function

$$\tilde{\Gamma}^{[1]}_{\text{real}} : [0, 1] \times [0, +\infty] \to [0, +\infty]$$

$$(\lambda, x) \mapsto \Gamma^{(\lambda)}_{\text{real}}(x)$$

is continuous. $\tilde{\Gamma}^{[1]}_{\text{real}}(\lambda, x)$ is monotone increasing in the variable $x$, and it is strictly increasing in the variable $x$ on $\left(\Gamma^{[1]}_{\text{real}}\right)^{-1}(0, +\infty))$.

(b) $\tilde{\Gamma}^{[1]}_{\text{real}}(\lambda, x) < +\infty$ if and only if $x < \sqrt{\pi^2 + \left(\log \frac{\lambda}{x}\right)^2} = \left|\log \frac{\lambda}{x^-}\right|" (the latter one is defined as $+\infty$ for $\lambda = 0, 1$). Some special values are given by

$$\tilde{\Gamma}^{[1]}_{\text{real}}(\lambda, 0) = 0;$$

and

$$\tilde{\Gamma}^{[1]}_{\text{real}}(0, x) = (\exp x) - 1 \quad \text{if} \quad x < +\infty;$$
\[
\tilde\Gamma^{[1]}_{\text{real}} \left( \frac{1}{2}, x \right) = 2 \tan \frac{x}{2} \quad \text{if } x < \pi;
\]

\[
\tilde\Gamma^{[1]}_{\text{real}}(1, x) = (\exp x) - 1 \quad \text{if } x < +\infty.
\]

**Proof.** (b) Then the finiteness statement follows from Lemma 3.9 and simple principles of complex analysis. The rest is straightforward, cf. Lemma 3.10.

(a) The Cauchy formula can be applied to give locally uniform (in \(\lambda\)) estimates for the power series coefficients. This leads to continuity on the open set \((\tilde\Gamma^{[1]}_{\text{real}})^{-1}((0, +\infty))\). The monotonicity statements are immediate from the nonnegative power series expansion and that \(\Gamma^{(\lambda)}(x) = x + \ldots\). Separate continuity (i.e. continuity for fixed \(\lambda\)) is straightforward from the pole structure above. Separate continuity, continuity on the finite part and monotonicity implies global continuity. \(\square\)

**Lemma 3.15.** (a) The function

\[
P^{[2]}_{\text{real}} : [0, 1] \times [0, +\infty] \times [0, +\infty] \to [0, +\infty]
\]

\[(\lambda, x, y) \mapsto P^{(\lambda)}_{\text{real}}(x, y) = \begin{cases} 
x + y + xy & \text{if } x, y < +\infty, \lambda(1 - \lambda)xy < 1, \\
+\infty & \text{otherwise}
\end{cases}
\]

is continuous. \(P^{[2]}_{\text{real}}(\lambda, x, y)\) is monotone increasing in the variables \(x, y\); and \(P^{[2]}_{\text{real}}(\lambda, x, y)\) is strictly increasing in the variables \(x, y\) on \((P^{[2]}_{\text{real}})^{-1}((0, +\infty))\).

(b) Special values are given by

\[
P^{[2]}_{\text{real}}(\lambda, x, 0) = x, \quad P^{[2]}_{\text{real}}(\lambda, 0, y) = y,
\]

\[
P^{[2]}_{\text{real}}(\lambda, +\infty, y) = P^{[2]}_{\text{real}}(\lambda, x, +\infty) = +\infty.
\]

**Proof.** This is just elementary analysis. However, we remark the following. Consider the function \(\iota : [0, 1] \to [0, +\infty]\) such that

\[
\iota(x) = \begin{cases} 
x & \text{if } 0 \leq x < 1, \\
+\infty & \text{if } x = 1.
\end{cases}
\]

Then reparametrizing the second and third variables and the range in \(P^{[2]}_{\text{real}}\), it yields

\[
\iota^* P^{[2]}_{\text{real}}(\lambda, x, y) = \min \left(1, \frac{x + y - xy}{1 - \lambda(1 - \lambda)xy} \right),
\]

which is particularly transparent regarding continuity. \(\square\)

**Lemma 3.16.** The function

\[
\tilde\Gamma^{[n]}_{\text{real}} : [0, 1] \times [0, +\infty] \times \ldots \times [0, +\infty] \to [0, +\infty]
\]

\[(\lambda, x_1, \ldots, x_n) \mapsto \Gamma^{(\lambda)}_{\text{real}}(x_1, \ldots, x_n)
\]

is continuous. \(\tilde\Gamma^{[n]}_{\text{real}}(\lambda, x_1, \ldots, x_n)\) is monotone increasing in the variables \(x_1, \ldots, x_n\); and \(\tilde\Gamma^{[n]}_{\text{real}}(\lambda, x_1, \ldots, x_n)\) is strictly increasing in the variables \(x_1, \ldots, x_n\) on \((\tilde\Gamma^{[n]}_{\text{real}})^{-1}((0, +\infty))\).

**Proof.** This follows from combining Lemma 3.14a and Lemma 3.15a. \(\square\)
For \( x_1, \ldots, x_n \in [0, +\infty] \), let
\[
\Gamma_{\text{real}}(x_1, \ldots, x_n) = \sup_{\lambda \in [0,1]} \Gamma_{\text{real}}^{(\lambda)}(x_1, \ldots, x_n).
\]

(In contrast to previous definitions, this is not derived from a power series but of real function theoretic nature.)

**Corollary 3.17.** The function
\[
\Gamma_{\text{real}}^{[n]} : \underbrace{[0, +\infty] \times \ldots \times [0, +\infty]}_{n \text{ terms}} \to [0, +\infty]
\]
\[
(x_1, \ldots, x_n) \mapsto \Gamma_{\text{real}}(x_1, \ldots, x_n) = \sup_{\lambda \in [0,1]} \Gamma_{\text{real}}^{(\lambda)}(x_1, \ldots, x_n).
\]
is continuous. \( \Gamma_{\text{real}}^{[n]}(x_1, \ldots, x_n) \) is monotone increasing in the variables \( x_1, \ldots, x_n \); and \( \Gamma_{\text{real}}^{[n]}(x_1, \ldots, x_n) \) is strictly increasing in the variables \( x_1, \ldots, x_n \) on \( \Gamma_{\text{real}}^{[n]} \).\(^{-1}(0, +\infty)\).

**Proof.** This follows from general properties of continuous functions on compact sets. \( \square \)

**Lemma 3.18.** Suppose that \( x_1, \ldots, x_n \in [0, +\infty) \), \( x_1 + \ldots + x_{k-1} > 0 \), \( x_k + \ldots + x_n > 0 \). Then the following conditions are equivalent:

(i) \( \Gamma_{\text{real}}(x_1, \ldots, x_n) = +\infty \), but for \( 0 \leq y_j \leq x_j \),
\[
y_1 + \ldots + y_n < x_1 + \ldots + x_n \quad \Rightarrow \quad \Gamma_{\text{real}}(y_1, \ldots, y_n) < +\infty;
\]

(ii) \( \Gamma_{\text{real}}(x_1, \ldots, x_n) = +\infty \), but
\[
0 \leq t < 1 \quad \Rightarrow \quad \Gamma_{\text{real}}(tx_1, \ldots, tx_n) < +\infty;
\]

(iii) \[
\sup_{\lambda \in [0,1]} \lambda(1 - \lambda)\Gamma_{\text{real}}^{(\lambda)}(x_1, \ldots, x_{k-1})\Gamma_{\text{real}}^{(\lambda)}(x_k, \ldots, x_n) = 1.
\]

**Proof.** (i) \( \Rightarrow \) (ii) is obvious, and (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i) follows from the formal identity
\[
\Gamma^{(\lambda)}(x_1, \ldots, x_n) = \frac{\Gamma^{(\lambda)}(x_1, \ldots, x_{k-1}) + \Gamma^{(\lambda)}(x_k, \ldots, x_n) + \Gamma^{(\lambda)}(x_1, \ldots, x_{k-1})\Gamma^{(\lambda)}(x_k, \ldots, x_n)}{1 - \lambda(1 - \lambda)\Gamma^{(\lambda)}(x_1, \ldots, x_{k-1})\Gamma^{(\lambda)}(x_k, \ldots, x_n)}
\]
and the monotonicity properties. \( \square \)

An immediate consequence of (59) is
\[
\Gamma_{\text{real}}(x_1, \ldots, x_n) \leq \Gamma_{\text{real}}(x_1, \ldots, x_n).
\]

Consider a sequence \( \Xi_1, \Xi_2, \ldots \) with values in \( D(0,1) \subset \mathbb{C} \). We define an operation \( \Xi \) on formal power series as follows. Consider the monomial \( X_{i_1}^{m_1} \cdot \ldots \cdot X_{i_s}^{m_s} \), where \( i_j \neq i_{j+1} \), and \( m_j \geq 1 \). On this monomial the effect of \( \Xi \) is given by
\[
\Xi_{\leq} \left( X_{i_1}^{m_1} \cdot \ldots \cdot X_{i_s}^{m_s} \right) = (-1)^{\text{des}(i_1, \ldots, i_s)}\Xi_{m_1} \cdot \ldots \cdot \Xi_{m_s} X_{i_1}^{m_1} \cdot \ldots \cdot X_{i_s}^{m_s}.
\]

Then extend \( \Xi_{\leq} \) linearly. Notice that every monomial gets multiplied by an element of \( D(0,1) \). In particular, this operation is “contractive”.

Let us apply \( \Xi_{\leq} \) to \( R^{(\lambda)}((\exp X_1) \cdot \ldots \cdot (\exp X_n)) \) in the light of (45). We find that the negative signs from the descents vanish, and \( R^{(\lambda)}(\exp X_j) \) gets replaced by \( \Xi_{\leq} R^{(\lambda)}(\exp X_j) \). Thus, formally,
\[
\Xi_{\leq} R^{(\lambda)}((\exp X_1) \cdot \ldots \cdot (\exp X_n)) = P^{(\lambda)}(\Xi_{\leq} R^{(\lambda)}(\exp X_1), \ldots, \Xi_{\leq} R^{(\lambda)}(\exp X_n)).
\]
Let $\Xi^{[\lambda]}(X_1, \ldots, X_n)$ denote the power series $\Xi \in \mathcal{R}^{[\lambda]}((\exp X_1) \cdots (\exp X_n))$. Then, due to the contractivity property, we find that $x_1, \ldots, x_n \in \mathbb{R}$ the value $\Xi^{[\lambda]}(x_1, \ldots, x_n)$ is well-defined (absolute convergent), as long as $\Gamma^{[\lambda]}_{\text{real}}(|x_1|, \ldots, |x_n|) < +\infty$.

**Theorem 3.19.**

$$
\Xi \Gamma^{[n]} : (\Gamma^{[n]})^{-1}(0, +\infty) \to \mathbb{C}
$$

is a smooth function. It satisfies

$$
\lambda \to \Xi \Gamma^{[\lambda]}(x_1, \ldots, x_n)
$$

**Proof.** First, we see that the statement holds for $n = 1$, as we have the locally uniform (in $\lambda$) estimates for the coefficients of the power series expansion of not only for $\mathcal{R}^{[\lambda]}(\exp X)$ but for $\frac{d\mathcal{R}^{[\lambda]}(\exp X)}{d\lambda} = (\mathcal{R}^{[\lambda]}(\exp X))^2$, and other higher derivatives. After that one can use induction to extend the result. \hfill \square

**Theorem 3.20.** Assume that $x_1, \ldots, x_n \in [0, \infty)$, and at least two them is positive. Then

$$
\Gamma_{\text{real}}(x_1, \ldots, x_n) < +\infty
$$

if and only if

$$
\Gamma_{\text{real}}(x_1, \ldots, x_n) < +\infty.
$$

**Proof.** ($\Rightarrow$) is obvious; we have prove ($\Leftarrow$). We assume that $x_i \neq 0 (n \geq 2)$. Assume, indirectly, that $\Gamma(x_1, \ldots, x_n) < +\infty$, but $\Gamma(x_1, \ldots, x_n) = +\infty$. We can also assume that for any $t \in [0, 1)$

$$
\sup_{\lambda \in [0, 1]} \lambda (1 - \lambda) \Gamma^{[\lambda]}_{\text{real}}(tx_1) \Gamma^{[\lambda]}_{\text{real}}(tx_2, \ldots, tx_n) < 1,
$$

however, the set

$$
\Lambda_0 = \{ \lambda : \lambda (1 - \lambda) \Gamma^{[\lambda]}_{\text{real}}(x_1) \Gamma^{[\lambda]}_{\text{real}}(x_2, \ldots, x_n) = 1 \}
$$

is non-empty. Consider, say, $\lambda_0 = \inf \Lambda_0 \in \Lambda_0$. Let

$$
\Xi_k = \text{sgn}\left( \text{the coefficient of } X^k \text{ in } \mathcal{R}^{[\lambda]}(\exp X, \lambda_0) \right).
$$

Then

$$
f(\lambda, t) = \lambda (1 - \lambda) \Xi^{[\lambda]}(tx_1) \Xi^{[\lambda]}(tx_2, \ldots, tx_n)
$$

is smooth on $[0, 1] \times [0, 1]$, and $|f(\lambda, t)| \leq 1$. Due to its definition, $f(\lambda_0, 1) = 1$.

Notice that $f(\lambda, t) = 1$ implies $t = 1$, $\lambda \in \Lambda$, and $\Xi^{[\lambda]}(x_1), \Xi^{[\lambda]}(x_2, \ldots, x_n) > 0$. Indeed, $t = 1$ is implied by the supremum property; then

$$
(\lambda - 1)^2 \Xi^{[\lambda]}(tx_1) \Xi^{[\lambda]}(tx_2, \ldots, tx_n) = 1.
$$

We have to rule out $\Gamma^{[\lambda]}(x_1), \Gamma^{[\lambda]}(x_2, \ldots, x_n) < 0$. As the starting terms in the power series of $\Gamma^{[\lambda]}(x_1)$ and $\Gamma^{[\lambda]}(x_2, \ldots, x_n)$ are $x_1$ and $x_2 + \ldots + x_n$, respectively; negative signs imply $|\Xi^{[\lambda]}(x_1)| < \Gamma^{[\lambda]}_{\text{real}}(x_1)$ and $|\Xi^{[\lambda]}(x_2, \ldots, x_n)| < \Gamma^{[\lambda]}_{\text{real}}(x_2, \ldots, x_n)$ in contradiction to (61).

Consider

$$
F(\lambda, t) = \frac{\Gamma^{[\lambda]}(tx_1) + \Xi^{[\lambda]}(tx_2, \ldots, tx_n) + \Gamma^{[\lambda]}(tx_1) \Xi^{[\lambda]}(tx_2, \ldots, tx_n)}{1 - \lambda (1 - \lambda) \Xi^{[\lambda]}(tx_1) \Xi^{[\lambda]}(tx_2, \ldots, tx_n)}.
$$
It is well-defined and equal to \( \Xi \Gamma(\lambda) (tx_1, tx_2, \ldots, tx_n) \) for \( 0 \leq t < 1 \). However, according to the previous discussion it extends continuously to \( t = 1 \) by setting \( f(\lambda, 1) = +\infty \) for \( f(\lambda, 1) = 1 \), and the standard arithmetical value otherwise; the negative part is bounded. Then
\[
\lim_{t \to 1} \Xi \Gamma(t x_1, \ldots, t x_n) \equiv \lim_{t \to 1} \int_{\lambda=0}^{1} F(\lambda, t) \, d\lambda = \int_{\lambda=0}^{1} F(\lambda, 1) \, d\lambda.
\]
On the other hand, the RHS is \(+\infty\) because the smoothness of \( 1 - f(\lambda, 1) \) at \( \lambda = \lambda_0 \) implies a nonintegrable singularity in \( F(\lambda, 1) \). Thus
\[
\lim_{t \to 1} \Xi \Gamma(t x_1, \ldots, t x_n) = +\infty.
\]
But \( \Xi \Gamma(t x_1, \ldots, t x_n) \leq \Gamma_{\text{real}}(t x_1, \ldots, t x_n) \) implies \( \Gamma_{\text{real}}(x_1, \ldots, x_n) = +\infty \).

**Theorem 3.21.** There exits \( \varepsilon > 0 \), such that
\[
\Gamma_{\text{real}}\left(\frac{1}{2}, \frac{1}{n-1}, \ldots, \frac{1}{2} \arctan \frac{1}{n-1}\right) = +\infty.
\]

**Proof.** It is easy to show that
\[
\Gamma_{\text{real}}\left(\frac{2\arctan}{n-1}, \ldots, \frac{2\arctan}{n-1}\right) = +\infty.
\]

Indeed, \( \Gamma_{\text{real}}^{(1/2)}(\ldots) = +\infty \Rightarrow \Gamma_{\text{real}}(\ldots) = +\infty \Rightarrow \Gamma_{\text{real}}(\ldots) = +\infty \).

Next, it is sufficient to show that the supremum of
\[
(62) \quad \lambda(1 - \lambda) \Gamma_{\text{real}}^{(1)} \left(\frac{2\arctan}{n-1}, \ldots, \frac{2\arctan}{n-1}\right) \Gamma_{\text{real}}^{(1)} \left(\frac{1}{n-1}\right)
\]
(\( \lambda \in [0, 1] \)) is not taken at \( \lambda = \frac{1}{2} \). (The actual value is 1 for \( \lambda = 1/2 \).) Now,
\[
(63) \quad \frac{d}{d\lambda} R^{(1)}(\exp X) \bigg|_{\lambda=\frac{1}{2}} = \left( 2 \tanh \frac{X}{2} \right)^2 = 4 \left( 1 - \frac{d}{dX} \left( 2 \tanh \frac{X}{2} \right) \right).
\]

Note that \( (63) \) is supported in even degrees, in contrast to \( R^{(1)}(\exp X) \bigg|_{\lambda=\frac{1}{2}} = 2 \tanh \frac{X}{2} \), which is supported in odd degrees. Knowing the power series expansion of \( \tanh X \), it is a matter of elementary analysis to show that for \( 0 \leq x < \pi \),
\[
\lim_{\lambda \to \frac{1}{2} \pm} \frac{\Gamma_{\text{real}}^{(1)}(x) - \Gamma_{\text{real}}^{(1/2)}(x)}{\lambda - 1/2} = \pm \left( 2 \tan \frac{x}{2} \right)^2.
\]

As \( (62) \) is built up from \( \Gamma_{\text{real}}^{(1)}(x) \) (with \( x = 2 \arctan \frac{1}{n-1} \)) and \( \lambda(1 - \lambda) \) as a positive infinite series, and \( \frac{d}{d\lambda} \lambda(1 - \lambda) \bigg|_{\lambda=\frac{1}{2}} = 0 \); we see that \( (62) \) has a strict local minimum at \( \lambda = 1/2 \).

Once we know that the supremum of \( (62) \) is beyond 1, then, using continuity, we can slightly contract the variables to the same effect. \( \square \)
Thus, the (cumulative) convergence radius of the $n$-term BCH-formula is at most $2n \arctan \frac{1}{n^2 - 1}$. As $n \rightarrow \infty$, we have $2n \arctan \frac{1}{n^2 - 1} \leq 2$. In particular, this demonstrates that the convergence radius of the Magnus expansion cannot be improved beyond 2 in the general setting of Banach algebras.

**Definition 3.22.** Let $C_1 = +\infty$. For $n \geq 2$, let
\[
C_n = \min\{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in [0, +\infty), \Gamma_{\text{real}}(x_1, \ldots, x_n) = +\infty\},
\]
(equivalent with $\Gamma_{\text{real}}(x_1, \ldots, x_n) = +\infty$ and $\Gamma^{[n]}_{\text{real}}$ is continuous), that is the cumulative convergence radius of the Baker–Campbell–Hausdorff formula with $n$ terms.

**Theorem 3.23.** Let $\phi = X_1 1_{[0,1]} \ldots X_n 1_{[0,1]}$, $n \geq 2$, and $\lambda \in [0,1]$. Then
\[
\sum_{n=1}^{\infty} \left| \mu_{n,R}^{(\lambda)}(\phi) \right| \leq \Gamma_{\text{real}}^{(\lambda)}(\{X_1\}, \ldots, \{X_n\}) \leq \Gamma_{\text{real}}(\{X_1\}, \ldots, \{X_n\}),
\]
where every term is well-defined, and they are finite for $|X_1| + \ldots + |X_n| < C_n$.

Thus, if $|X_1| + \ldots + |X_n| < C_n$ holds, then the conditions of Theorem 2.6 are satisfied, and the (stronger) logarithmic version of the Baker–Campbell–Hausdorff formula (with $n$ variables) holds.

On the other hand, there is a Banach algebra $\mathfrak{A}$ with $X_1, \ldots, X_n \in \mathfrak{A}$ such that $|X_1| + \ldots + |X_n| = C_n$, but there is no element $X$ such that $\exp X = (\exp X_1) \ldots (\exp X_n)$ holds.

**Proof.** This a consequence of the previous discussion. □

**Theorem 3.24.** If $n \geq 2$, then $2 < C_n < 2n \arctan \frac{1}{n^2 - 1}$. As $n$ increases, $C_n$ strictly decreases.

**Proof.** Regarding the lower bound, we always have convergence for $x_1 + \ldots + x_n \leq 2$ according to Theorem 2.23. Thus, thus, on that domain, $\Gamma(x_1, \ldots, x_n)$ is finite. Then, by continuity, also in a neighborhood. Regarding the upper bound, this follows from Theorem 3.21.

Regarding the monotonicity of $C_n$, note that if $0 < \Gamma_{\text{real}}^{(\lambda)}(x_k) < +\infty$ and
\[
\lambda(1 - \lambda)\Gamma_{\text{real}}^{(\lambda)}(x_1, \ldots, x_{k-1})\Gamma_{\text{real}}^{(\lambda)}(x_k) = 1, \quad x_k > 0
\]
implies
\[
\lambda(1 - \lambda)\Gamma_{\text{real}}^{(\lambda)}(x_1, \ldots, x_{k-1})\Gamma_{\text{real}}^{(\lambda)}\left(\frac{x_k}{2}, \frac{x_k}{2}\right) > 1;
\]
because in this case $\lambda \neq 0, 1$, thus $\Gamma_{\text{real}}^{(\lambda)}\left(\frac{x_k}{2}, \frac{x_k}{2}\right) > \Gamma_{\text{real}}^{(\lambda)}(x_k)$. (See $R^{(\lambda)}((\exp X)(\exp Y)) = \lambda XY + (\lambda - 1)YX + \ldots$ for cancellations.) □

For $n = 2$, this gives $2 < C_2 < \pi$ for the (cumulative) convergence radius of the BCH expression. One may wonder what is the actual value. Due to Theorem 3.20 this can be considered as a numerical problem.

**Theorem 3.25.**
\[
C_2 = 2.89847930\ldots
\]
Furthermore, if $\Gamma_{\text{real}}^{(\lambda)}(x, y) = +\infty$, $x + y = C_2$, then $x = y = \frac{1}{2}C_2$, and
\[
0.35865 < \min(\lambda, 1 - \lambda) < 0.35866.
\]

**Proof.** First, we would like estimate the value of $\Gamma^{(\lambda)}(x)$. 
Lemma 3.26. Let $0 < \lambda < 1$. We will use the notation $|\log \frac{\lambda}{1-\lambda}| = \sqrt{\pi^2 + \left(\log \frac{\lambda}{1-\lambda}\right)^2}$.

Let

$$\gamma^{(\lambda)}(X) = \left(2 + |2\lambda - 1|X + 4 \frac{X^2}{\log \frac{\lambda}{1-\lambda}} \left(\log \frac{\lambda}{1-\lambda} - X\right) \sqrt{\lambda(1-\lambda)} + \frac{X^2}{6\pi^2(3\pi - X)}\right) \sinh \frac{X}{2}.$$ 

Then

$$\Gamma^{(\lambda)}(X) \leq \gamma^{(\lambda)}(X).$$

Proof. Due to

$$R^{(\lambda)}(\exp X) = \frac{1}{\sqrt{\lambda(1-\lambda)} \cosh \frac{X}{2} \left(X - \log \frac{\lambda}{1-\lambda}\right)},$$

it is sufficient to estimate the sech term. Using the complex analytic formula

$$\frac{1}{\cosh \frac{X}{2}} = \sum_{N=1}^{\infty} \left(\frac{(-1)^N 2i}{x - (2N - 1)i} - \frac{(-1)^N 2i}{x + (2N - 1)i}\right),$$

we find that

$$\frac{1}{\cosh \frac{X}{2} \left(X - \log \frac{\lambda}{1-\lambda}\right)} \leq \frac{2}{\sqrt{\lambda(1-\lambda)}} + |2\lambda - 1| \sqrt{\lambda(1-\lambda)} X + 4 \sum_{r=2}^{\infty} \frac{X^r}{\sqrt{\pi^2 + \left(\log \frac{\lambda}{1-\lambda}\right)^2 + 1}} + 4 \sum_{r=2}^{\infty} \sum_{N=2}^{\infty} \frac{X^r}{(2N - 1)\pi^{r+1}}.$$ 

Using the estimate

$$\sum_{N=2}^{\infty} \frac{1}{((2N - 1)\pi)^{r+1}} = \frac{(1 - 2^{r+1})\zeta(r + 1) - 1}{\pi^{r+1}} \leq \frac{3/2}{(3\pi)^{r+1}},$$

and combining the expressions, we arrive to the statement.

For $x \in [0, \pi)$, this allows to make the upper and lower estimates

$$\Gamma_{k\text{-th \ X-Taylor}}(x) \leq \Gamma_{\text{real}}^{(\lambda)}(x) \leq \Gamma_{k\text{-th \ X-Taylor}}^{(\lambda)}(x) - \gamma^{(\lambda)}(x) - \gamma^{(\lambda)}_{\text{real}}(x).$$

One can, of course, use other series instead of $\gamma^{(\lambda)}(X)$. Also note that in this manner we can estimate $\frac{d}{dx} \Gamma_{\text{real}}^{(\lambda)}(x)$, etc.

Having good estimates allows to zero in on $C_2$:

Lemma 3.27. \[2.8984 < C_2 < 2.8985\]

Moreover, if $\Gamma_{\text{real}}^{(\lambda)}(x, y) = +\infty$, i.e.

$$\sup_{\lambda \in [0,1]} \lambda(1-\lambda)\Gamma_{\text{real}}^{(\lambda)}(x)\Gamma_{\text{real}}^{(\lambda)}(y) \geq 1;$$
and $x + y = C_2$, (that is $x + y$ is minimal form the property above, and in particular, the supremum is actually 1), then
\[0.358 < \min(\lambda, 1 - \lambda) < 0.359,\]
and
\[1.445 < x, y < 1.455.\]

Proof. This is a series of finer and finer estimates locating critical behaviour. \qed

Lemma 3.28. If
\[2.8984 < c < 2.8985\]
and
\[0.358 < \lambda < 0.359\]
and
\[1.445 < x < 1.455,\]
then the function
\[x \to \Gamma^{(\lambda)}_{\text{real}}(x)\Gamma^{(\lambda)}_{\text{real}}(c - x)\]
is concave in $x$. In particular, due to symmetry, its maximum is taken at $x = c/2$.

Proof. For a fixed $\lambda$, the function $\Gamma^{(\lambda)}_{\text{real}}(x)$ is smooth in $x$; and the first and second derivatives can be estimated as in (64). The second derivative of $\Gamma^{(\lambda)}_{\text{real}}(x)\Gamma^{(\lambda)}_{\text{real}}(c - x)$ is
\[\Gamma^{(\lambda)\prime\prime}_{\text{real}}(x)\Gamma^{(\lambda)}_{\text{real}}(c - x) - 2\Gamma^{(\lambda)\prime}_{\text{real}}(x)\Gamma^{(\lambda)}_{\text{real}}(c - x) + \Gamma^{(\lambda)}_{\text{real}}(x)\Gamma^{(\lambda)\prime\prime}_{\text{real}}(c - x).\]
Thus, we should use upper estimators in the first and third terms, and we should use lower estimators in the middle term. When we do this, the second derivative turns out to be smaller than $-1$ on the given range. \qed

Theorem 3.25 but with weaker constants follows from a combination of the previous two lemmas. However, once we know that the maximum is taken for $x = y = 1/2C_2$, we can search for $C_2$ more effectively. \[\text{End of proof for Theorem 3.25.}\]

Remark 3.29. Note that $C_2$ is not taken at $\lambda = 1/2$. In fact, combining arguments from Theorem 3.11 and Theorem 3.24, one can show that this is neither the case for higher $C_n$. On the other hand, for a fixed $x \geq 0,$
\[
\lim_{x_1, \ldots, x_n = x, x_i \geq 0, \max\{x_1, \ldots, x_n\} \to 0} \Gamma^{(\lambda)}_{\text{real}}(x_1, \ldots, x_n) = \Theta^{(\lambda)}_{\text{real}}(x).
\]
From monotonicity with respect to refinements, and the actual shape of $\Theta^{(\lambda)}_{\text{real}}(x)$, one can conclude that the critical $\lambda$ must converge to 1/2. \qed

Appendix A. Ordered measures

General measures. Suppose that $\mathfrak{a}$ is a Banach space. An $\mathfrak{a}$-valued measure on a bounded interval $I$ is a $\sigma$-additive $\mathfrak{a}$-valued function on the set of (possible degenerate, even empty) subintervals of $I$. If $\phi$ is $\mathfrak{a}$-valued measure on an interval $I$, and $J \subset I$ is a subinterval, then we can define the variation by
\[|\phi|(J) := \sup \left\{ \sum_{s \in S} |\phi(J_s)| : \text{the } J_s \text{ are finitely many disjoint intervals in } J \right\}.\]
This yields a $[0, \infty]$-valued measure on $I$. We will be interested only in measures $\phi$ of finite variation, i.e. for which $\int |\phi| = |\phi|(I) < +\infty$.

An $\mathfrak{A}$-valued measure $\phi$ on the interval $I$ is absolutely continuous if there exists a nonnegative Lebesgue-integrable function $H$ on $I$ such that $|\phi|(J) \leq \int_{t \in J} H(t) \, dt$ for any subinterval $J \subset I$. For us, absolute continuity has little importance in itself. However, the most notable examples are absolutely continuous:

**Example A.1.** Suppose that $I \subset \mathbb{R}$ is an interval, $h$ is an $\mathfrak{A}$-valued Lebesgue-Bochner integrable function on $I$. Let $\phi$ be the measure which is $h$ times the Lebesgue measure restricted to $I$; i.e. for any subinterval $J \subset I$

$$\phi(J) = \int_{t \in J} h(t) \, dt.$$ 

Then $\phi$ is an $\mathfrak{A}$-valued measure of finite variation; moreover $|h|$ is Lebesgue-integrable on $I$, and for any subinterval $J \subset I$

$$|\phi|(J) = \int_{t \in J} |h(t)| \, dt.$$ 

(Or, in other terms, $|\phi| = |h|1_{.}) \quad \diamondsuit

**Example A.2.** Let $\mathfrak{A} = \mathfrak{B}(\mathfrak{X})$ be the set of bounded operators on a Banach space $\mathfrak{X}$. Suppose $h$ is an $\mathfrak{A}$-valued strongly Lebesgue-Bochner integrable function on $I$; i.e. for any $y \in \mathfrak{X}$ the function $hy : I \to \mathfrak{X} : t \to h(t)y \in \mathfrak{X}$ is Lebesgue-Bochner integrable. We define $\phi(J)$ as the operator given by

$$\phi(J) y = \int_{t \in J} h(t)y \, dt.$$ 

(Notice that that $\phi(J)$ is not necessarily in $\mathfrak{A} = \mathfrak{B}(\mathfrak{X})$.) Assume that

$$\sup \left\{ \sum_{s \in S} |\phi(J_s)| : \text{the } J_s \text{ are finitely many disjoint intervals in } J \right\} < +\infty.$$ 

Then $\phi$ is an $\mathfrak{A}$-valued measure of finite variation. Moreover, there is a nonnegative Lebesgue-integrable function $[h]$ on $I$ such that for any subinterval $J \subset I$,

$$|\phi|(J) = \int_{t \in J} [h](t) \, dt.$$ 

In fact, $[h](t)$ is the supremum of the set of functions $\{ |h(t)y| : y \in \mathfrak{X}, |y| = 1 \}$ in $\leq_{\text{almost everywhere}}$ sense (which is an object whose existence is not a priori guaranteed).

Conversely, if $h$ is an $\mathfrak{A}$-valued strongly Lebesgue-Bochner integrable function on $I$, and $H$ is a nonnegative Lebesgue integrable function such that for any $y \in \mathfrak{X}$, $|y| = 1$ the inequality $|h(t)y| \leq_{\text{almost everywhere in } t} H(t)$ holds, then (65) holds, and for any subinterval $J \subset I$, the inequality $|\phi|(J) \leq \int_{t \in J} H(t) \, dt$ holds. \quad \diamondsuit

An $\mathfrak{a}$-valued measure $\phi$ on the interval $I$ is continuous if for any $\varepsilon > 0$, and any subinterval $J \subset I$, $J$ is a countable union of subintervals $J_\xi$, such that $|\phi|(J_\xi) < \varepsilon$. (Thus, continuity is an absolute notion and absolute continuity is a relative notion.)

Whenever $\phi$ is an $\mathfrak{a}$-valued measure on the interval $I$, and $f$ is, say, a real-valued function on $I$, and $A \subset I$, then by $\int_A f \phi \equiv \int_{t \in A} f(t) \phi(t)$ we mean $\int_I (\chi_A f)(t) \phi(t) = \int_{t \in I}(\chi_A(t)f(t)) \phi(t)$, where $\chi_A$ is the characteristic function of the set $A$. I.e., we integrate over the original measure and not over a restriction, or an extension, or both. (Although this will not matter in our cases.)
In the previous discussion, \( I \) can replaced by an abstract set \( \Omega \), and the system of subintervals can be replaced by a semiring \( \mathfrak{F} \). For the sake of simplicity, we require that \( \Omega \) should be a countable union of elements from \( \mathfrak{F} \). (I. e. \( \mathfrak{F} \) is \( \sigma \)-finite.) Then \( \phi : \mathfrak{F} \to a \) is should be \( \sigma \)-additive. The Lebesgue-measure can be replaced by any measure (i.e. \( \sigma \)-additive function) \( \omega : \mathfrak{F} \to [0, \infty) \). Lebesgue-integrable and Lebesgue–Bochner-integrable should be replaced by Lebesgue-integrable and Lebesgue–Bochner-integrable with respect to \( \omega \). Otherwise, the same applies.

The only case where we need this greater generality is when it rises from the product construction from the interval case. A generalized interval \( \mathcal{I} \) (generalized) interval \( \phi \) form a semiring, etc. Suppose that \( \mathcal{I} \) is should be replaced by Lebesgue-integrable and Lebesgue–Bochner-integrable. Otherwise, the same applies.

Ordered measures. An \( a \)-valued measure \( \phi \) of finite variation on the interval \( I \) is continuous if for any \( \phi \) leading to a measure \( \mathfrak{m} \) is a continuous \( n \)-linear continuous function with norm \( |F| \). Then we can define \( \phi = F(\phi_1, \ldots, \phi_n) \) by

\[
\phi(J_1 \times \ldots \times J_n) = F(\phi_1(J_1), \ldots, \phi_n(J_n)).
\]

This yields an \( a \)-valued measure \( \phi \) of finite variation \( \int |\phi| \leq |F| \cdot (\int |\phi_1|) \ldots \cdot (\int |\phi_n|) \).

(\( \phi \)-valued measure of finite variation on the (generalized) interval \( I_j \) for \( j = 1, \ldots, n \). Also assume that \( F : a_1 \times \ldots \times a_n \to a \) is an \( n \)-linear continuous function with norm \( |F| \). Then we can define \( \phi = F(\phi_1, \ldots, \phi_n) \) by

\[
\phi(J_1 \times \ldots \times J_n) = F(\phi_1(J_1), \ldots, \phi_n(J_n)).
\]

This yields an \( a \)-valued measure \( \phi \) of finite variation \( \int |\phi| \leq |F| \cdot (\int |\phi_1|) \ldots \cdot (\int |\phi_n|) \).

(Ordered measures. An \( a \)-valued measure \( \phi \) of finite variation on the interval \( I \) is continuous if for any \( x \in I \), the equality \( \phi(\{x\}) = 0 \) holds. In general, if \( \phi \) is of finite variation, then \( \phi(\{x\}) = 0 \) holds for all except countably many \( x \in I \). These points, quite canonically, can be blown up to closed intervals with continuous uniform distribution, leading to a measure \( \phi^* \) on an interval \( I^* \). In many arguments, it is advantageous to pass from \( \phi \) to \( \phi^* \).

For an ordered \( n \)-tuple \( (t_1, \ldots, t_n) \), we define its multiplicity sequence \( \text{mul}(t_1, \ldots, t_n) \) as the sequence \( (m_1, \ldots, m_n) \) of positive integers, where

\[
m_1 + \ldots + m_n = n
\]

and

\[
t_i \neq t_{i+1} \text{ if and only if } i \in \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \ldots + m_{s-1}\}.
\]

We also use the multindex factorial \( (m_1, \ldots, m_n)! = m_1! \cdot m_2! \cdot \ldots \cdot m_n! \).

If \( \phi \) is an \( a \)-valued measure of finite variation on the interval \( I \subset \mathbb{R} \), and \( F : a^n \to a \) is a continuous \( n \)-linear function, then we can consider the right-ordered evaluation

\[
F_R(\phi) := \int_{(t_1, \ldots, t_n) \in I^n} \frac{1}{\text{mul}(t_1, \ldots, t_n)!} F(\phi(t_1), \ldots, \phi(t_n)) dt_1 \ldots dt_n;
\]

and the left-ordered evaluation

\[
F_L(\phi) := \int_{(t_1, \ldots, t_n) \in I^n} \frac{1}{\text{mul}(t_1, \ldots, t_n)!} F(\phi(t_n), \ldots, \phi(t_1)) dt_1 \ldots dt_n.
\]

The reason behind this definition is as follows. The definition is uncontroversial if \( \phi \) is continuous. Indeed, in this case \( \text{mul}(t_1, \ldots, t_n)! = 1 \) except on negligible subset. On the other hand, if \( \phi \) is not necessarily continuous, but \( \phi^* \) is a continuous blowup of \( \phi \), then \( F_R(\phi) = F_R(\phi^*) \), and \( F_L(\phi) = F_L(\phi^*) \) holds. Moreover, the definition behaves naturally with respect to blow-ups. This allows to pass between \( \phi \) and \( \phi^* \) in many arguments.

In general, we consider the case of continuous measures as the basic one, where expressions are slightly simpler; however, in some situations, it is instructive to see how the general case formulates.

Even more generally, \( I \) can be replaced by a fully ordered set \( \Omega \) which contains a countable dense set in the order topology; and \( \mathfrak{F} \) can be replaced by a semiring of some
possibly degenerate subintervals of \( \Omega \). (\( \mathcal{F} \) is still required to be \( \sigma \)-finite.) This generality is required only if we want to concatenate measures abstractly. Otherwise, one should restrict to half-open intervals, or to measures which are continuous at the endpoints. In those cases the concatenation of measures on intervals is unproblematic.

**Appendix B. \( L^1 \) spaces of noncommutative power series**

**\( L^1 \) spaces of formal power series.** Let \( K \) be \( \mathbb{R} \) or \( \mathbb{C} \). In the case of the BCH expansion, the most important Banach algebras with respect to our investigations are the Banach algebras of formal noncommutative power series with the \( \ell^1 \) norm:

If \( A \) is a formal power noncommutative series with variables from \( \{Y_\lambda : \lambda \in \Lambda\} \) with coefficients from \( K \), then let \( |A|_{\ell^1} \) denote the sum of the absolute value of the coefficients. This may be infinite, but the elements \( A \) with \( |A|_{\ell^1} < +\infty \) form the Banach algebra \( \mathcal{F}_K^1[Y_\lambda : \lambda \in \Lambda] \). A slightly bigger space but still an algebra is \( \mathcal{F}^1_{K,\text{loc}}[Y_\lambda : \lambda \in \Lambda] \); this contains those formal power series \( A \) such that \( |\pi_k A|_{\ell^1} < +\infty \) for any \( k \), where \( \pi_k \) denotes the projection to the component of formal degree \( k \). If \( A \) is a locally convex algebra with seminorms \( |\cdot|_{\ell^1} \). This is a useful space, because some expressions might converge to an \( A \) in \( \mathcal{F}^1_{K,\text{loc}}[Y_\lambda : \lambda \in \Lambda] \) but with \( |A|_{\ell^1} = +\infty \), which precludes convergence in \( \mathcal{F}_K^1[Y_\lambda : \lambda \in \Lambda] \). If \( \tilde{\sim} : \mathcal{F}_K^1[Y_\lambda : \lambda \in \Lambda] \rightarrow \mathcal{F}_K^1[Y_\lambda : \lambda \in \Lambda] \) is the map which replaces power series coefficients with absolute values, then it is norm-preserving and weakly contractive (1-Lipschitz), i.e. \( |\tilde{A} - \tilde{B}|_{\ell^1} \leq |A - B|_{\ell^1} \). If \( \sim \) is an equivalence relation on \( \Lambda \), and \( \tilde{\sim} : \mathcal{F}^1_{K,\text{loc}}[Y_\lambda : \lambda \in \Lambda] \rightarrow \mathcal{F}^1_{K,\text{loc}}[Y_\lambda : \lambda \in \Lambda/\sim] \) is the map which identifies variables according to \( \sim \), then it is also weakly contractive (1-Lipschitz). One can also factorize by simply sending some variables to 0. Similar observations apply to \( \mathcal{F}^1_{K,\text{loc}}[Y_\lambda : \lambda \in \Lambda] \).

In the case of the Magnus expansion, a continuous version of the previous example plays a similar role: Consider the noncommutative polynomial algebra

\[
\mathcal{F}_{K}^1(\mathbb{R}) = \mathcal{F}_K[Z_{(a,b)} : a < b \in \mathbb{R}],
\]

i.e. the free noncommutative algebra generated by the symbols \( Z_{(a,b)} \) where \( a, b \in \mathbb{R}, \ a < b \). For a noncommutative polynomial \( P(Z_{[a_i,b_i]} : 1 \leq i \leq s) \) we define its norm as follows. Take any finite set \( C = \{c_1, \ldots, c_k\} \subset \mathbb{R} \) such that \( c_1 < \ldots < c_k \) and

\[
\{a_1,\ldots,a_s,b_1,\ldots,b_s\} \subset C.
\]

We can assume that \( a_i = c_{A_i} \) and \( b_i = c_{B_i} \). Then we define

\[
|P(Z_{(a_i,b_i)} : 1 \leq i \leq s)|_{\ell^1} := \left| \left( P \left( \sum_{r=A_i}^{B_i-1} (c_{r+1} - c_r) Y_{(c_r,c_{r+1})} : 1 \leq i \leq s \right) \right) \right|_{\ell^1},
\]

where the latter \( \ell^1 \) is understood with respect to \( \mathcal{F}^1_{K,\text{loc}}[Y_{(c_r,c_{r+1})} : 1 \leq i < k] \). It is easy to see that further refinements of \( C \) do not change \( |\cdot|_{\ell^1} \). Thus it is well-defined and \( |Z_{(a,b)}|_{\ell^1} = b - a \). It is also easy to see that \( |\cdot|_{\ell^1} \) is a seminorm on \( \mathcal{F}^1_{K}(\mathbb{R}) \) with the submultiplicative property. However, it is not a norm, because \( |\cdot|_{\ell^1} \) cannot distinguish between \( Z_{(a,b)} \) and \( Z_{(a,c)} + Z_{(c,b)} \) for \( a < c < b \); not even if they are part of larger polynomials. The seminorm \( |\cdot|_{\ell^1} \) actually descends to the factor algebra

\[
\mathcal{F}_K(\mathbb{R}) = \mathcal{F}_K[Z_{(a,b)} : a < b \in \mathbb{R}] / \{Z_{(a,b)} = Z_{(a,c)} + Z_{(c,b)} : a < c < b \in \mathbb{R}\}.
\]

There it induces a norm \( |\cdot|_{\ell^1} \), because if (66) yields 0, then it means that, using an appropriate \( C \), the refined expansion of \( P(Y_{(a_i,b_i)} : 1 \leq i \leq s) \) yields 0, thus it is 0.
This normed space can be completed (still \(| \cdot |_{1}\)), yielding the Banach space \(F_{K}^{1}(\mathbb{R})\). More generally, it can be considered as \(F_{K}^{*}(\mathbb{R})\) factorized by elements of seminorm \(0\), and completed. This construction is effectively based on the semiring of the intervals \([a, b]\), and common refinements of them, and the Lebesgue measure (interval length).

This construction can be formulated in greater generality. Suppose that \(\Omega\) is a set, \(\mathfrak{F}\) is semiring on it, and \(\omega : \mathfrak{F} \rightarrow [0, +\infty)\) is a measure (i.e. \(\sigma\)-additive). Then consider the noncommutative polynomial algebra

\[
F_{K}^{*}(\Omega, \mathfrak{F}, \omega) = F_{K}[Z_{A} : A \in \mathfrak{F}].
\]

For a noncommutative polynomial \(P(Z_{A_{i}} : 1 \leq i \leq s)\), take a finite set \(C \subset \mathfrak{F}\), such that any \(C\) contains pairwise disjoint, nonempty sets, and any element \(A_{i}, 1 \leq i \leq s\), is the disjoints union of some elements from \(C\). Then we define

\[
|P(Z_{A_{i}} : 1 \leq i \leq s)|_{\ell^{1}} := \left| P\left( \sum_{A \in C, A \cap A_{i} \neq \emptyset} \omega(A)Y_{A} : 1 \leq i \leq s \right) \right|_{\ell^{1}},
\]

where the latter \(\ell^{1}\) is understood with respect to \(F_{K}^{1}[Y_{A} : A \in C]\). Further refinements of \(C\) do not change \(| \cdot |_{\ell^{1}}\), thus it is well-defined. In particular, \(|Z_{A_{i}}|_{\ell^{1}} = \omega(A)\).

Again, \(| \cdot |_{\ell^{1}}\) is a seminorm on \(F_{K}^{*}(\Omega, \mathfrak{F}, \omega)\) with the submultiplicative property. Then we factorize \(F_{K}^{*}(\Omega, \mathfrak{F}, \omega)\) with those elements whose seminorm is 0, and then we complete it to a Banach algebra \(F_{K}^{1}(\Omega, \mathfrak{F}, \omega)\). Actually, we can simply describe the algebra after the factorization. It is given by

\[
F_{K}(\Omega, \mathfrak{F}, \omega) = F_{K}[Z_{A} : A \in \mathfrak{F}] / \{ Z_{A} = 0 \text{ if } \omega(A) = 0, Z_{A} = Z_{A_{1}} + \ldots + Z_{A_{s}} \text{ if } A = A_{1} \cup \ldots \cup A_{s}\}.
\]

Indeed, these factorizations are valid, yet if (67) yields 0, then it means that the corresponding polynomial can be expanded to 0 using them. Thus \(F_{K}^{1}(\Omega, \mathfrak{F}, \omega)\) is the completion of \(F_{K}(\Omega, \mathfrak{F}, \omega)\). (We still use the notation \(| \cdot |_{\ell^{1}}\) for the norm.) We will use the notation \(Z_{A}^{\prime}\) for the corresponding generators of the ring (68). The resulted spaces are naturally graded, and for any \(A \in F_{K}^{1}(\Omega, \mathfrak{F}, \omega)\),

\[
|A|_{\ell^{1}} = \sum_{k=0}^{\infty} \pi_{k}A|_{\ell^{1}},
\]

where \(\pi_{k}A\) is the component of grade \(k\) of \(A\).

If we use the construction where \(\Omega = \mathbb{R}\), \(\mathfrak{F} = \mathfrak{I}\) is the set of bounded intervals \([a, b]\), and \(\omega\) is the restriction of the Lebesgue measure (i.e. interval length), then this yields the construction of the previous example. If we use the semiring \(\mathfrak{F}' = \mathfrak{I}\) of the possibly degenerate (not necessarily half-open) intervals, then it yields the same construction. In fact, they are the same even on the pre-completed algebraic level \(F_{K}(\Omega, \mathfrak{F}, \omega)\). Thus we will use the notation \(F_{K}^{1}(\mathbb{R})\) without particularly mentioning the semiring. The same applies for \(F_{K}^{1}([0, 1]), F_{K}^{1}([0, 1]),\) etc. In fact, we can even use the semiring \(\mathfrak{F}''\) of the subsets of \(\mathbb{R}\) of finite Lebesgue measure. This will be different on the algebraic level, but essentially the same on the completed level; see the following measure theoretic interpretation:

Let \((\Omega^{\mathbb{N}}, \mathfrak{F}^{\mathbb{N}}, \omega^{\mathbb{N}}) = \bigcup_{n \in \mathbb{N}}(\Omega^{n}, \mathfrak{F}^{n}, \omega^{n})\). Then \(F_{K}^{1}(\Omega, \mathfrak{F}, \omega)\) is naturally isomorphic to \(L^{1}(\Omega^{\mathbb{N}}, \mathfrak{F}^{\mathbb{N}}, \omega^{\mathbb{N}})\) as a base space with the appropriate norm; the grading corresponds to the power decomposition, the product is induced from taking exterior direct of function in homogeneous components, and \(Z_{A} (A \in \mathfrak{F})\) corresponds to the
characteristic function of $A$ on $\Omega$. Indeed, if we map $Z_{A_1} \ldots Z_{A_s}$ into the class of $(t_1, \ldots, t_n) \mapsto \chi_{A_1}(t_1) \ldots \chi_{A_s}(t_s)$, then $F^*_K(\Omega, \mathfrak{F}, \omega)$ (and $F_K(\Omega, \mathfrak{F}, \omega)$) gets mapped into a dense subset of $L^1(\Omega^{\mathbb{N}^r}, \mathfrak{F}^{\mathbb{N}^r}, \omega^{\mathbb{N}^r})$ isometrically. We now that the latter one is a Banach algebra, thus completion yields an isomorphism between $F^*_K(\Omega, \mathfrak{F}, \omega)$ and $L^1(\Omega^{\mathbb{N}^r}, \mathfrak{F}^{\mathbb{N}^r}, \omega^{\mathbb{N}^r})$. Thus the “completion picture” and the “exterior $L^1$ algebra picture” are equivalent to each other. At this point the second one looks simpler because it is constructed using pre-built elements. However, if we are to take more sophisticated norms on formal power series, the first one is simpler to handle. In general, we use the terminology of the completion picture. (Such spaces often occur in functional analysis and probability theory, but with ‘\$\otimes\$’ as the product. This is to indicate the we deal with a “free noncommutative” construction. But, this latter notation for the product is quite unnecessary.)

We define the tautological measure $Z^1_{(\Omega, \mathfrak{F}, \omega)} : \mathfrak{F} \to F^*_K(\Omega, \mathfrak{F}, \omega)$ as the $F^*_K(\Omega, \mathfrak{F}, \omega)$-valued measure given by

$$Z^1_{(\Omega, \mathfrak{F}, \omega)}(A) = Z^1_A \in F^*_K(\Omega, \mathfrak{F}, \omega)$$

for $A \in \mathfrak{F}$. The variation measure of $Z_{(\Omega, \mathfrak{F}, \omega)}$ is the measure $\omega$. If $f : \Omega^n \to \mathbb{K}$ is Lebesgue integrable with respect to $\omega^n$, then

$$\int_{(t_1, \ldots, t_n) \in \Omega^n} f(t_1, \ldots, t_n) Z^1_{(\Omega, \mathfrak{F}, \omega)}(t_1) \ldots Z^1_{(\Omega, \mathfrak{F}, \omega)}(t_n) \in F^*_K(\Omega, \mathfrak{F}, \omega)$$

exists and corresponds to “$f$” in the exterior $L^1$ algebra picture. In particular, it has norm $\|f\|_{L^1(\Omega^n, \mathfrak{F}^n, \omega^n)}$. We use the notation $Z^1_{[0,1]}$, etc in the case of $F^*_K(\mathbb{R})$, $F^*_K([0,1])$, etc., respectively. These latter measure has approximately the same level of individuality as the Lebesgue measure. Thus it is fair to say that $c \cdot Z^1_{[0,1]}$, with $c \in (0, +\infty)$, is the totally noncommutative continuous mass of norm $c$.

The space $F^*_{K,loc}(\Omega, \mathfrak{F}, \omega)$ can be defined similarly, as in the discrete case. There is a map $\hat{\cdot} : F^*_K(\Omega, \mathfrak{F}, \omega) \to F^*_K(\Omega, \mathfrak{F}, \omega)$ induced by replacing power series coefficients with absolute values. (In the exterior $L^1$ algebra picture it is taking the absolute value of the representing function.) This is also norm-preserving and weakly contractive. Regarding factorizations, we are interested only in the simplest cases. Assume that $A_1 \cup \ldots \cup A_s \subset \Omega$, $A_i \in \mathfrak{F}$. Then there is a map $\hat{\cdot} : F^*_K(\Omega, \mathfrak{F}, \omega) \to F^*_K(\omega \hat{A}_i : 1 \leq i \leq s)$ induced by $Z^*_K \mapsto \sum_{i=1}^s \omega(A \cap A_i) Y_{A_i}$. This corresponds to restricting to $A_1 \cup \ldots \cup A_s$, and coarsening into finitely many points. This also yields a weak contraction. Similar observations apply to $F^*_{K,loc}(\Omega, \mathfrak{F}, \omega)$.

Part of the reason for the relative detailed discussion here was that later the same constructions will be applied to formal Lie power series where the process is analogous. The main point is that it is sufficient have a good norm for free algebras as long as a certain refinement invariance property holds.

**Appendix C. The Banach algebraic form of Schur’s formulae**

In many approaches to the Magnus / Baker–Campbell–Hausdorff expansions, Schur’s formulae play the role of a stepping stone. In our discussion they have not played a critical role but their formal versions have already been encountered in Theorem 3.6. For the sake of completeness, and also for further reference, we give the form valid in general Banach algebras. The development given here is the standard one using functional calculus; it is independent from the (combinatorial) discussion before.
In the setting of Banach algebras, for \( X \in \mathfrak{A} \), let \( \text{ad} X \) denote the operator \( \text{ad} X : \mathfrak{A} \to \mathfrak{A} \), given by \( Y \mapsto [X,Y] \). Consider the meromorphic function

\[
\beta(x) = \frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \beta_j x^j.
\]

Note that this function has poles at \( 2\pi i \mathbb{Z} \setminus \{0\} \).

**Theorem C.1** (F. Schur [30] (1893), Poincaré [28] (1899)). If \( |x| < \pi \), or \( \text{sp}(X) \subset \{ z \in \mathbb{C} : |z| < \pi \} \), or \( \text{sp}(X) \subset \{ z \in \mathbb{C} : |\Re z| < \pi \} \), then \( \text{ad} X \) is analytic, or \( \text{sp}(X) \subset \{ z \in \mathbb{C} : |z| < 2\pi \} \), or \( \text{sp}(X) \subset \{ z \in \mathbb{C} : |\Re z| < 2\pi \} \), respectively. In particular, \( \beta(\text{ad} X) : \mathfrak{A} \to \mathfrak{A} \) makes sense as an absolutely convergent power series (first two cases) or as a homomorphic function of \( \text{ad} X \).

In these cases, for \( Y \in \mathfrak{A} \),

\[
\begin{align*}
\frac{d}{dt} \log(\exp(tY) \exp(X)) \bigg|_{t=0} &= \beta(\text{ad} X)Y \\
\frac{d}{dt} \log(\exp(tY) \exp(X)) \bigg|_{t=0} &= \beta(- \text{ad} X)Y
\end{align*}
\]

hold; with the usual log branch cut along the negative real axis.

Before giving a proof let us comment on the nature of the objects involved above. First of all, note that the left multiplication \( X_L \) by \( X \) and right multiplication \( X_R \) by \( X \) on \( \mathfrak{A} \) are isospectral to \( X \). In this terminology, \( \text{ad} X = X_L - X_R \). Also, \( X_R \) and \( X_L \) commute.

**Lemma C.2.** (a) For the spectrum of \( \text{ad} X \),

\[
\text{sp}(\text{ad} X) \subset \text{sp}(X_L) + \text{sp}(-X_R) = \text{sp}(X) - \text{sp}(X).
\]

More in term of formulas, the following are true:

(b) Suppose that \( \gamma = \bigcup_i \gamma_i \) is a finite disjoint union of simple cycles encircling, altogether, every point of \( \text{sp}(X_L) \) exactly \( +1 \) times; and \( \delta = \bigcup_j \delta_j \) is a finite disjoint union of simple cycles encircling, altogether, every point of \( \text{sp}(-X_R) \) exactly \( +1 \) times. Assume that \( \gamma \) and \( \delta \) bound \( R_1 \) and \( R_2 \), respectively; i.e. \( \gamma \in \partial R_1 \) and \( \delta = \partial R_2 \). Then for any \( \lambda \notin R_1 + R_2 \),

\[
\frac{1}{X_L + (-X)_R - \lambda \text{Id}} = \frac{1}{(2\pi i)^2} \int_{\gamma} \int_{\delta} \frac{1}{z + w - \lambda} \frac{dz}{z \text{Id} - X_L} \frac{dw}{w \text{Id} - (-X)_R}
\]

holds.

In particular, if \( \gamma = \bigcup_i \gamma_i \) in the \( \varepsilon_1 \)-neighborhood of \( \text{sp}(X_L) \), \( \delta = \bigcup_j \delta_j \) in the \( \varepsilon_2 \)-neighborhood of \( \text{sp}(-X_R) \) (and such \( \gamma \) and \( \delta \) can indeed be chosen for any \( \varepsilon_1, \varepsilon_2 > 0 \)), then \( \text{[72]} \) holds for any \( \lambda \), not in the \( \varepsilon_1 + \varepsilon_2 \)-neighborhood of \( \text{sp}(X_L) + \text{sp}(-X_R) \).

(c) Assume that \( \text{sp}(X) \subset \{ z \in \mathbb{C} : - \frac{|\Im \lambda|}{2} < \Im z < \frac{|\Im \lambda|}{2} \} \).

\[
\frac{1}{X_L + (-X)_R - \lambda \text{Id}} = \frac{\text{sgn} \Im \lambda}{2\pi i} \int_{u \in \mathbb{R}} \frac{1}{(\frac{1}{2} + u) \text{Id} - X_L} \cdot \frac{1}{(\frac{1}{2} - u) \text{Id} + X_R} du.
\]

(This formula, via linear transformations \( A \mapsto cA + d \), can easily be transcribed to bands not around the real axis.)
Proof. These are standard applications of the spectral calculus of commuting operators (applied to $X_L$ and $-X_R$). Indeed, behind (a), the standard machinery is (b). (c) follows considering certain (infinite) rectangles $R$ and $(1-\varepsilon)R$, resolving the integral along the inner contour, and pushing the loose boundary piece to the infinity. 

Remark. Case (c) can also be established by a more arithmetical (although technically not much easier) argument:

$$(X_L + (\lambda)X_R - \lambda Id)\cdot \frac{1}{2\pi i} \int_{u \in \mathbb{R}} \frac{1}{(\frac{\lambda}{2} + u) Id - X_L} \cdot \frac{1}{(\frac{\lambda}{2} - u) Id + X_R} \, du$$

$$= \frac{1}{2\pi i} \int_{u \in \mathbb{R}} \left(- \frac{1}{(\frac{\lambda}{2} + u) Id - X_L} + \frac{1}{(\frac{\lambda}{2} - u) Id + X_R} \right) \, du$$

$$= \lim_{N \to +\infty} \frac{1}{2\pi i} \int_{u = -N}^{N} \left(- \frac{1}{(\frac{\lambda}{2} + u) Id - X_L} \right) \, du + \lim_{N \to +\infty} \frac{1}{2\pi i} \int_{u = -N}^{N} \left(\frac{1}{(\frac{\lambda}{2} - u) Id + X_R} \right) \, du$$

$$= (\text{sgn Im} \lambda) \frac{1}{2} \text{Id} + (\text{sgn Im} \lambda) \frac{1}{2} \text{Id}_R = (\text{sgn Im} \lambda) \text{Id}.$$ 

This is equivalent to (73). Considering the linear transforms, this is already sufficient to establish $\text{sp(ad} X) \subset \text{conv(sp(X)}L + \text{sp(-X)}R)) = \text{conv(sp}(X) - \text{sp}(X))$.

Now, $\beta(\text{ad} X)$ can be constructed as long as $\text{sp(ad} X)$ is disjoint $2\pi i(Z \setminus \{0\})$. On the other hand, for the entire function

$$\alpha(x) = \frac{e^x - 1}{x} = \sum_{j=0}^{\infty} \frac{1}{(j+1)!} x^j,$$

the map $\alpha(\text{ad} X)$ is well-constructed anyway. The significance of $\beta(\text{ad} X)$ is actually that, it provides an inverse to $\alpha(\text{ad} X)$ (for good $X$).

The significance of $\alpha(\text{ad} X)$, in turn, comes from the derivative the exponential function. Using power series it easy to see that

$$\frac{d}{dt} \exp(X + tZ) \bigg|_{t=0} = \exp(-X) = \int_{s=0}^{1} \exp(sX)Z \exp(-sX) \, ds = \alpha(\text{ad} X)Z$$

and

$$\exp(-X) \frac{d}{dt} \exp(X + tZ) \bigg|_{t=0} = \int_{s=0}^{1} \exp(-sX)Z \exp(sX) \, ds = \alpha(-\text{ad} X)Z.$$ 

Proof of Theorem C.1. The arguments regarding the norms are trivial, and the more general spectral behaviour follows from Lemma C.2. Regarding (70) and (71): In those spectral ranges, $\exp A$ (defined for $\text{sp}(A) \subset \{ z \in \mathbb{C} : |\text{Re} z| < \pi \}$) and $\log A$ (defined for $\text{sp}(A) \subset \mathbb{C} \setminus (-\infty, 0)$) are smooth and inverses of each other.

More specifically, for fixed $X$ with $\text{sp}(X) \subset \{ z \in \mathbb{C} : |\text{Re} z| < \pi \}$, the maps

$$\hat{Z} \mapsto \hat{Y} = \log(\exp(\hat{Z}) \exp(-X))$$

and

$$\hat{Y} \mapsto \hat{Z} = \log(\exp(\hat{Y}) \exp(X))$$

are inverses of each other (and smooth) for $\hat{Z} \sim X$ and $\hat{Y} \sim 0$. In the first case, as the differential of log at 1 is the identity map, the linear derivative at $\hat{Z} = X$ is given by $Z \mapsto \alpha(\text{ad} X)Z$. In the second case, this forces the linear derivative at $\hat{Y} = 0$ to be the inverse map $Y \mapsto \beta(\text{ad} X)Y$. This establishes (70).
Analogously, the maps
\[ \hat{Z} \mapsto \hat{Y} = \log(\exp(-X) \exp(\hat{Z})) \]
and
\[ \hat{Y} \mapsto \hat{Z} = \log(\exp(X) \exp(\hat{Y})) \]
are inverses of each other (and smooth) for \( \hat{Z} \sim X \) and \( \hat{Y} \sim 0 \). In the first case, the linear derivative at \( \hat{Z} = X \) is given by \( Z \mapsto \alpha(-\text{ad } X) \). In the second case, this forces the linear derivative at \( \hat{Y} = 0 \) to be the inverse map \( Y \mapsto \beta(-\text{ad } X) \). This establishes (71).

\[\text{Corollary C.3 (F. Schur \cite{Schur}, Poincaré \cite{Poincare} ).} \]
If \( X \) and \( Y \) are formal noncommutative variables, then
\[ \log(\exp(Y) \exp(X)) \text{ the multiplicity of } Y = 1 = \beta(\text{ad } X)Y \]
and
\[ \log(\exp(X) \exp(Y)) \text{ the multiplicity of } Y = 1 = \beta(-\text{ad } X)Y; \]
where \( \beta(\text{ad } X) \) is understood in the sense of formal power series.

**Proof.** This follows even from the general matrix case of Theorem C.1. \(\square\)

By this, we have rederived Theorem 3.6 (In turn, we could have obtained Theorem C.1 from Theorem 3.6 / Corollary C.3 by analytic extension.)

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