Approximate kinematic synthesis of the four-bar mechanism by two given positions of the links

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Abstract. Planar four-bar linkages are widely used in various automatic devices and equipment; therefore, kinematical synthesis methods should be developed in order to reduce the amount of initial data and derived the exact solution using modern mathematical methods. The method of the best quadratic approximation of a function was applied to derive the optimality criterion through the well-known equation for closing a vector contour for four-bar linkage. The obtained mathematical model of kinematical synthesis can be used also to determine the singular positions of the mechanism that can cause a spontaneous change in the law of motion of the output link. Modeling of the proposed kinematic synthesis method using Mathcad confirmed the correctness of the mathematical model and allowed plotting the graphical diagram of connecting rod curves.

1. Introduction
The planar four-bar linkages are used as an actuator in various machines and devices. For example, they can be found in the rotating mechanism of the bicycle and the solar panel, as well as the mechanism for opening automatic garage doors, in the steering mechanism of an car, and the medical equipment etc., where it is necessary to transform the rotation of the input link into the given law of motion of the output link.

Despite the fact that there are numerous research methods, the synthesis methods namely the determination the geometric parameters of links that provide a given continuous motion of the output link and satisfy the restrictions on the working space and kinematic and dynamic conditions for the transmission motion, are still approximate and mostly of them are graphic. Using the graphical software simplifies the solution of the synthesis problem but it takes too much time to clarify parameters of the mechanism as a result of the kinematical analysis. In different versions of kinematic synthesis problem as the initial data for calculating unknown parameters of the mechanism, three given positions of the input and output links as well as the length of the crank or the fixed link are required.

Levitskiy N. I. [1] established that there is one exact solution to the kinematical synthesis problem of the four-bar linkage if three positions of the input and output links are given. In other cases, the number of possible solutions is infinite. Modern methods of kinematical synthesis of the four-bar linkages are based on the quadratic approximation of the function and reduce the mathematical model to either a system of quadratic equations or an iterative procedure for solving several systems of the linear equations. Even in these cases, the problem of choosing the initial values of the unknown parameters is still very complicated. In practice, the task of kinematical synthesis becomes more complicated if, in accordance with the condition, the special point, as a rule, the vertex of the triangular link with based on the connecting rod has to move along the given trajectory.
The article describes a method for optimizing kinematical synthesis of a planar crank four-bar mechanism. This method allows to automate the calculation of the parameters of the mechanism and to reduce the number of the necessary initial conditions for an exact solution. It belongs to the group of analytical methods using the best Chebishev’s approximation. The values of the free parameters, namely, two given positions of input crank and output rocker as well as the length of the fixed link are set arbitrary. Increasing their number leads an increase in the degree of approximation function and, as a sequence, complicates the mathematical model and its solution.

2. **Formulation of the problem**

In general case, the kinematical parameters of the planar four-bar linkages are calculated as a result of solving the kinematical synthesis problem in accordance with the given positions of links and geometric restrictions of the working space as well as the kinematic and dynamic conditions of motion transmission. It should be noted the exact movement of the output link along a certain trajectory can be perform only in some special cases, therefore, its several positions are more widely which simplifies the task.

The driving link of the designed mechanism is the crank AB, rotating uniformly with the constant angular velocity $\omega$:

$$\varphi_1 = \omega t, \quad \omega = \text{const},$$

the rocker of the mechanism oscillates between two critical positions.

The kinematical scheme of the four-bar linkage is shown in the Fig. 1. The Cartesian coordinate system $xAy$ with the origin coinciding with the joint A and the abscissa axis directed along the fixed link AD is used. The coordinates of the joint D are chosen arbitrary based on the requirement for the working space. The positions of input crank AB and corresponding positions of the output link CD are described by angular coordinates $\left( \varphi_1^1, \varphi_1^3 \right)$ and $\left( \varphi_2^1, \varphi_2^3 \right)$, which are measured counterclockwise from axis $Ax$, as shown in the Fig. 1. The lengths $l_1, l_2, l_4$ of moving links AB, BC and CD, respectively, are calculated with accuracy $\varepsilon = 0,01$, and the complete rotation of the input link AB must also be performed.

![Figure 1. The scheme of the crank four-bar linkage](image)

3. **The main part**

To obtain a design model, we will use the scheme of the planar four-bar linkage in the Fig. 1 taking into account the designation of lengths and angles of rotation of the links, as well as the Cartesian coordinate system. The distance between the fixed joints A and D is denotes as $l_0$. Law of motion shows the relationship between the rotation angles of output CD and input AB links is denoted as $\varphi_3 \left( \varphi_1 \right)$. It is set at two points $\left( \varphi_1^1, \varphi_3^1 \right)$ and $\left( \varphi_2^1, \varphi_3^3 \right)$ respectively.

Using Zinoviev’s method, the equation of closed vector loop ABCD is derived as:

$$\vec{l}_1 + \vec{l}_2 + \vec{l}_3 - \vec{l}_0 = 0,$$

(1)

The vectors $\vec{l}_0$, $\vec{l}_1$, $\vec{l}_2$ and $\vec{l}_3$ are directed along the links $AD$, $AB$, $BC$, and $CD$ as shown in the Fig.
1 and its modules are equal to the lengths of the moving links.

The vector $\vec{l}_i$ is the product of the module $|l_i|$ and the unit vector $\vec{e}_i, i = 0, ..., 3$ coinciding with the links $AD, AB, BC$ respectively so $\vec{l}_i = |l_i| \vec{e}_i$. The coordinates of the unit vectors in the given interpolation nodes of the position function of the four-bar linkage ABCD, taking into account the previously introduced notation, are described as follows:

- at the first position $\varphi_1^1, \varphi_2^1$: $\vec{e}_1 = (\cos \varphi_1^1, \sin \varphi_1^1)$ and $\vec{e}_1^1 = (\cos \varphi_1^1, \sin \varphi_1^1)$;
- at the second position $\varphi_3^3, \varphi_4^3$: $\vec{e}_1^2 = (\cos \varphi_3^3, \sin \varphi_3^3)$ and $\vec{e}_1^3 = (\cos \varphi_4^3, \sin \varphi_4^3)$.

According with the technique of the best quadratic approximation of a function, the closed-loop vector equation (1) will be used to estimate the deviation from the given positions:

$$I = (\vec{l}_1 + \vec{l}_2 + \vec{l}_3 - \vec{l}_0)^2 \rightarrow \min.$$ (2)

Substitute the expression for vectors $\vec{l}_0, \vec{l}_1, \vec{l}_2$ and $\vec{l}_3$ in the formula (2) and replace the variable $z_i = |l_i|/|l_0|$ to reduce the number of unknowns; at the end, the final function was derived:

$$I = 1 + z_1^2 + z_2^2 + z_3^2 + 2z_1 z_2 \vec{e}_2 \vec{e}_1 + 2z_1 z_3 \vec{e}_3 \vec{e}_1 + 2z_2 z_3 \vec{e}_3 \vec{e}_2 - 2z_1 \vec{e}_0 \vec{e}_1 - 2z_2 \vec{e}_0 \vec{e}_2 - 2z_3 \vec{e}_0 \vec{e}_3 \rightarrow \min.$$ (3)

The scalar product of unit vectors $\vec{e}_i, i, j = 0 \div 3$ coincides with the links of the design mechanism.

Since the function $I$ is non-negative definite quadratic function of variables $z_1, z_2, z_3$, it tends to a minimal value. The necessary condition for the existence of the extreme value is equality of zero partial derivatives:

$$\frac{\partial I}{\partial z_i} = 0, \quad i = 1, 3.$$ (4)

As a result of differentiation, a system of the uniform linear equations with respect to the unknowns $z_1, z_2, z_3$ was written:

$$\begin{align*}
z_1 + e_1 e_2 z_2 + e_1 e_3 z_3 - e_0 e_1 &= 0; \\
e_1 e_2 z_1 + z_2 + e_2 e_3 z_3 - e_0 e_2 &= 0; \\
e_1 e_3 z_1 + e_3 e_2 z_2 + z_3 - e_0 e_3 &= 0.
\end{align*}$$ (5)

To estimate the linear independence of the variables of the system of equations (5) and, as consequence, the presence of unique non-trivial solution of the optimization synthesis problem the determinant composed of the coefficient for $z_1, z_2, z_3$ was checked:

$$\Delta = \begin{vmatrix} 1 & e_1 e_2 & e_1 e_3 \\ e_3 e_2 & 1 & e_3 e_1 \\ e_1 e_3 & e_2 e_1 & 1 \end{vmatrix} = 1 + 2(e_1 e_2)(e_3 e_1) - (e_1 e_3)^2 - (e_2 e_1)^2 - (e_3 e_2)^2.$$ (6)

Without breaking the generality of reasoning we assume that with a certain combination of the geometric parameters of a mechanism, all links lie on the common line coinciding with the horizontal axis $Ax$ of the Cartesian coordinate system. Thus, the angular coordinates will be equal to
\( \varphi_3 = 0^\circ, \varphi_2 = 180^\circ \) and \( \varphi_1 = 180^\circ \). According to the theory of singular positions, in this way the spontaneous changes of the kinematic scheme due to gaps in the joints will be possible.

For the combination of the angular coordinates mentioned above, the determinant of matrix (6) is identically equal to zero; therefore, the parameters \( z_1, z_2, z_3 \) are linearly dependent and the kinematical synthesis problem has an infinity number of solutions.

As additional restriction on unknown variables \( z_1, z_2, z_3 \), the Grassgof’s law is used according to which the existence of a crank four-bar linkage depends on the following inequality:

\[
p + q > l + s, \quad (7)
\]

The shortest and the longest links are denoted as \( s \) and \( l \) correspondently. The other two different links are \( p \) and \( q \).

Group the equations of system (5) as follows:

\[
\begin{align*}
z_2 \bar{e}_1 e_2 + e_0 e_2 z_3 &= e_0 e_1 - z_1; \\
z_2 + z_4 e_2 e_3 &= e_0 e_2 - e_1 e_2 z_4.
\end{align*}
\]

\[
\begin{align*}
e_1 e_2 z_2 + e_1 e_3 z_3 &= e_0 e_1 - z_1; \\
z_2 e_2 e_3 + z_3 &= e_0 e_3 - e_1 e_3 z_4.
\end{align*}
\]

The unknowns in system (8) and (9) are not only the scalar values \( z_1, z_2, z_3 \) but also the unit vector \( \bar{e}_2 \). Using Kramer method, the expressions for \( z_2 \) and \( z_3 \) were obtained:

\[
z_2 = \frac{(\bar{e}_1 e_0 - z_1) \bar{e}_2 e_3 - e_1 e_1 (\bar{e}_2 e_0 - \bar{e}_1 e_0 z_1)}{(\bar{e}_2 e_1) \cdot (\bar{e}_3 e_2) - e_3 e_1};
\]

\[
z_3 = \frac{(\bar{e}_2 e_0 - e_1 e_0 z_1) \bar{e}_3 e_3 - (\bar{e}_1 e_0 - z_1)}{(\bar{e}_2 e_1) \cdot (\bar{e}_3 e_2) - e_3 e_1}.
\]

Similarly, the formulas for \( z_2 \) and \( z_3 \) was defined from the system (9):

\[
z_2 = \frac{(\bar{e}_1 e_0 - z_1) \bar{e}_2 e_3 - (\bar{e}_3 e_0 - \bar{e}_1 e_0 z_1) e_3 e_1}{\bar{e}_1 e_2 - (\bar{e}_1 e_3) \cdot (\bar{e}_3 e_3)};
\]

\[
z_3 = \frac{(\bar{e}_3 e_0 - \bar{e}_3 e_1 z_4) \bar{e}_2 e_1 - e_3 e_2 (\bar{e}_1 e_0 - z_1)}{\bar{e}_1 e_2 - (\bar{e}_1 e_3) \cdot (\bar{e}_3 e_3)}.
\]

Pairwise equating the equations (10) and (12) as well as (11) and (13) respectively, we find out the functions for calculating the unknown scalar parameter \( z_1 \).
\[
\begin{align*}
\mathbf{z}_1 &= \frac{\mathbf{e}_1 \cdot \left( \mathbf{B} \cdot \mathbf{e}_0 - \mathbf{A} \cdot \mathbf{e}_0 \right) - \mathbf{B} \cdot \mathbf{e}_0 \cdot \mathbf{e}_1}{(\mathbf{e}_1 \cdot \mathbf{e}_0 - \mathbf{e}_1 \cdot \mathbf{e}_1)} + \mathbf{A} \\
\mathbf{z}_1 &= \frac{(A - B) \mathbf{e}_1 \cdot \mathbf{e}_0 - \mathbf{e}_2 \cdot \mathbf{e}_1 \left( \mathbf{A} \cdot \mathbf{e}_0 - \mathbf{B} \cdot \mathbf{e}_0 \right)}{\mathbf{e}_2 \cdot \mathbf{e}_0 - \mathbf{A} \cdot \mathbf{e}_1} + (A - B).
\end{align*}
\]

The variables \( A = (\mathbf{e}_1 \cdot \mathbf{e}_2) \cdot (\mathbf{e}_2 \cdot \mathbf{e}_3) - \mathbf{e}_1 \cdot \mathbf{e}_3 \) and \( B = \mathbf{e}_2 \cdot \mathbf{e}_1 - (\mathbf{e}_3 \cdot \mathbf{e}_1) \) are used to simplify the result formulae.

Based on the expressions (14) and (15) we derived an equation for the unknown unit vector \( \mathbf{e}_2 = (\cos \varphi_2, \sin \varphi_2) \)

\[
\begin{align*}
\left[ \mathbf{e}_1 \cdot \left( \mathbf{B} \cdot \mathbf{e}_0 - \mathbf{A} \cdot \mathbf{e}_0 \right) - \mathbf{B} \cdot \mathbf{e}_0 \cdot \mathbf{e}_1 \right] \cdot [\mathbf{e}_1 \cdot \left( \mathbf{B} \cdot \mathbf{e}_0 - \mathbf{A} \cdot \mathbf{e}_0 \right) + (A - B)] = \\
= \left[ (A - B) \mathbf{e}_1 \cdot \mathbf{e}_0 - \mathbf{e}_2 \cdot \mathbf{e}_1 \left( \mathbf{A} \cdot \mathbf{e}_0 - \mathbf{B} \cdot \mathbf{e}_0 \right) \right] \cdot \left[ \mathbf{e}_1 \cdot \mathbf{e}_0 - \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{A} \right]
\end{align*}
\]

Substituting the given positions namely \( \mathbf{e}_1 = (\cos \varphi_1, \sin \varphi_1) \); \( \mathbf{e}_2 = (\cos \varphi_2, \sin \varphi_2) \) and \( \mathbf{e}_3 = (\cos \varphi_3, \sin \varphi_3) \) in formula (16) we solved the trigonometric equations and obtained two values of the unit vector \( \mathbf{e}_0 = (\cos \varphi_0, \sin \varphi_0) \) or angle of rotation the rod BC. It should be mentioned that the coordinates of unit vector \( \mathbf{e}_0 = (\cos \varphi_0, \sin \varphi_0) \) are also known and in accordance with the Cartesian coordinate system used in the Fig. 1 they are equal to \( \mathbf{e}_0 = (1, 0) \).

Expanding the scalar product of unit vectors and grouping the similar elements contained the common multiplier \( \sin \varphi_2 \) and \( \cos \varphi_2 \) of the equation (16), we calculated the required parameters \( (\varphi_1', \varphi_2') \) at the given positions of links of the designed four-bar linkage ABCD. Then substituting the results to the expressions (14) and (15) we counted the dimensionless parameter \( z_1 \). Calculations are repeated until the required accuracy is achieved, it is estimates as follows:

\[
|z_1^1 - z_1^2| \leq \varepsilon.
\]

After that, the other unknowns \( z_2 \) and \( z_3 \) are found out. Using the given length of the fixed link \( l_0 \) we converted the relative units to absolute values length of the links of the designed four-bar linkage.

4. Numerical experience

The effectiveness of the proposed method of the kinematical synthesis will be demonstrated by the next example. Suppose that it is necessary to design the planar four-bar linkage whose input link makes a complete revolution in one cycle of movement and also the length of the fixed link \( l_0 \) is equal to 50 in dimensionless units, in accordance with the accepted notations in the Fig. 1, and the movable links AB and CD pass through two predetermined positions described by the angular coordinates \( (\varphi_1, \varphi_3) : (60^\circ, 270^\circ) \) and \( (120^\circ, 280^\circ) \).
Analytical dependencies for calculating the angle of rotation $\theta_2$ of the rod BC in accordance with the first predetermined position

The initial value of the parameter $\theta_2 = \frac{10}{180}$

The unit vector $\mathbf{e}_2$ coinciding with the link BC of the designed mechanism

$$\mathbf{e}_2(\theta_2) = (\cos(\theta_2), \sin(\theta_2)) \quad \mathbf{e}_2(\theta_2) = (0.985, 0.174)$$

$$A_1(\theta_2) = \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_2(\theta_2)\right)^T \left(\mathbf{e}_2(\theta_2) \cdot \mathbf{e}_1(\theta_2)^T\right) - \mathbf{e}_1(\theta_2) \cdot \mathbf{e}_1(\theta_2)^T \quad A_1(\theta_2) = 0.754$$

$$B_1(\theta_2) = \mathbf{e}_2(\theta_2) \cdot \mathbf{e}_1(\theta_2)^T \left(\mathbf{e}_3(\theta_2) \cdot \mathbf{e}_1(\theta_2)^T\right) \left(\mathbf{e}_3(\theta_2) \cdot \mathbf{e}_2(\theta_2)^T\right)^T \quad B_1(\theta_2) = 0.492$$

$$C_2 = \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(B_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T - A_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)$$

$$D_2 = (A_1(\theta_2) - B_1(\theta_2)) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(B_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T - A_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) + (A_1(\theta_2) - B_1(\theta_2))$$

Equation for calculating the angle of rotation of the rod BC

$$F_1(\theta_2) = C_1(\theta_2) - D_1(\theta_2) \quad F_1(\theta_2) = -0.163$$

$$\theta_2 = \arctan(F_1(\theta_2), F_2(\theta_2)) \quad \theta_2 = \frac{180}{\pi} \quad \theta_2 = 22.934$$

The procedure for calculating parameters in the second predetermined position is similar. Then the values are clarified to provide the required accuracy.

$$z_1 = \left(\frac{A_1(\theta_2) - B_1(\theta_2)}{A_1(\theta_2) - B_1(\theta_2)} \cdot \mathbf{e}_3(\theta_2)^T\right) \left(B_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T - A_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) + (A_1(\theta_2) - B_1(\theta_2))$$

The length of the fixed link 10 is used as a scale factor

$$z_2 = \frac{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_3(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_2(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_2(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_3(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}$$

$$z_3 = \frac{\left\|\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right\| \cdot \left\|\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right\| - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}$$

$$z_4 = \frac{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_3(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}$$

$$z_5 = \frac{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}$$

$$z_6 = \frac{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \cdot \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right)}{\left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) \left(\mathbf{e}_1(\theta_2) \cdot \mathbf{e}_3(\theta_2)^T\right) - \left(\math_2 \cdot \mathbf{e}_2(\theta_2)^T\right) \cdot \left(\math_2 \cdot \mathbf{e}_2(\theta_2)^T\right)}$$
In the application for solving the kinematical synthesis problem, the mathematical model for the kinematical analysis has been added. As a result, the connecting rods curves were plotted, shown in the Fig. 2.

The kinematic scheme of the mechanism in the given positions and the trajectory of joints moving

Coordinates of point B

\[
x_B(f_1) = l_1 \cdot \cos(f_1)
y_B(f_1) = l_1 \cdot \sin(f_1)
\]

The length of the diagonal BD

\[
BD(f_1) = \sqrt{\left(l_1^2 + l_2^2 - 2 \cdot l_1 \cdot l_2 \cdot \cos(f_1)\right)}
\]

\[
\delta(f_1) = \tan\left(\frac{l_1 \cdot \sin(f_1)}{10 - l_1 \cdot \cos(f_1)}\right)
\]

\[
\lambda(f_1) = \cos\left(\frac{l_1^2 + l_2^2 - l_1^2}{2l_1(l_1 - l_2)}\right)
\]

\[
\gamma(f_1) = z - \delta(f_1) - \lambda(f_1)
\]

Coordinates of point C

\[
x_C(f_1) = 10 + l_3 \cdot \cos(\gamma(f_1))
y_C(f_1) = 13 \cdot \sin(\gamma(f_1))
\]

The coordinates of the middle point S of the link BC

\[
x_S(f_1) = 0.5 \cdot x_C(f_1) + 0.5 \cdot x_B(f_1)
y_S(f_1) = 0.5 \cdot y_C(f_1) + 0.5 \cdot y_B(f_1)
\]

The positions of the input crank AB for the certain predetermined value of the angle of rotation are used to plot the kinematical scheme of the mechanism.

- \( \phi_1 = 120\) deg
- \( x_0 = 0 \), \( x_1 = x_B(f_1) \), \( x_2 = x_S(f_1) \), \( x_3 = x_C(f_1) \), \( x_4 = 10 \)
- \( y_0 = 0 \), \( y_1 = y_B(f_1) \), \( y_2 = y_S(f_1) \), \( y_3 = y_C(f_1) \), \( y_4 = 0 \)

Thus, the plotted diagrams in Fig. 2 confirmed the correctness the mathematical models and the fact that the designed mechanism is a planar crank linkage and its output rocker CD passes through two given positions.

![Figure 2](image.png)

**Figure 2.** The kinematic diagrams of the movable links of the studied mechanism

5. **Conclusion**

In the article the planar crank four-bar linkage is discussed. Such mechanisms are widely used as an
actuator in a different device and machinery. The kinematical synthesis problem for these linkages is about the estimation of geometric parameters namely the length of the links under the given restrictions on the workspace and on the kinematic and dynamic characteristics of the quality of motion transmission. As a result of the analysis of scientific publications, it figured out that the most synthesis methods are partially graphic and also in approximate formulation this problem has an infinity number of solutions.

The proposed optimization method for the kinematic synthesis of the planar four-bar linkage made it possible to formulate the mathematical model in accordance with the squared approximation of a function based on the well-known equation of close vector loop. In terms of minimizing of the optimal criterion, the system of linear equation is obtained for unknown geometric parameters in dimensionless units. It is established that the discriminant of a matrix composed from the coefficient for unknowns is equal to zero for the certain combination of the angular coordinates describing the position of the links. So there is the linear dependence between parameters, and this fact is confirmed by Grasgof’s law. Thus, the solution of kinematical synthesis problem is multivariate and an iterative procedure must be used to achieve the given accuracy.

The numerical example of the synthesis implemented in Mathcad demonstrated the effectiveness of the method and the satisfactory convergence of the computational procedure. Solved kinematic analysis problem of the planar crank four-bar linkage for calculated values of the link length confirms the operability of mechanism i.e. its crank makes complete revolution and the movable links passes through two given positions and also the requirements for the quality of transmission of motion are fulfilled.

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