Controlling a nonlinear Fokker-Planck equation via inputs with nonlocal action

Ştefana-Lucia Aniţa

“Octav Mayer” Institute of Mathematics of the Romanian Academy, Iaşi 700506, E-mail: stefi_anita@yahoo.com

Abstract

This paper concerns an optimal control problem $(P)$ related to a nonlinear Fokker-Planck equation. The problem is deeply related to a stochastic optimal control problem $(P_h)$ for a McKean-Vlasov equation. The existence of an optimal control is obtained for the deterministic problem $(P)$. The existence of an optimal control is established and necessary optimality conditions are derived for a penalized optimal control problem $(P_h)$ related to a backward Euler approximation of the nonlinear Fokker-Planck equation (with a constant discretization step $h$). Passing to the limit ($h \to 0$) one derives the necessary optimality conditions for problem $(P)$.

Key words: stochastic/deterministic optimal control problem; nonlinear Fokker-Planck equation; $m$-accretive operator; weak/mild solution; existence of an optimal control; necessary optimality conditions; McKean-Vlasov SDE

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1 Formulation of the problem

Consider the following optimal control problem:

\[(P) \quad \text{Minimize } \int_0^T \int_{\mathbb{R}^d} G(t,x) \rho^*(t,x) \, dx \, dt + \int_{\mathbb{R}^d} G_T(x) \rho^*(T,x) \, dx + \int_{\mathbb{R}^d} Q(x, \zeta(x)) \, dx,\]

where \( T \in (0, +\infty), \ d \in \mathbb{N}, \ d \geq 3, \) and \( \rho^* \) is the weak solution to the following nonlinear Fokker-Planck equation:

\[
\frac{\partial \rho}{\partial t} (t,x) = -\nabla \cdot (u(x) b(\rho(t,x)) \rho(t,x)) + \Delta \beta(\rho(t,x)), \quad t \in [0,T], \ x \in \mathbb{R}^d
\]

\[
\rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}^d.
\]

Here \( u(x) = -\nabla \left( \int_{\mathbb{R}^d} G_R(x-x') \zeta(x') \, dx' \right) = K(\zeta)(x), \)

\( \zeta \in \mathcal{M} = \{ \theta \in L^\infty(\mathbb{R}^d); 0 \leq \theta(x) \leq \tilde{M}_0 \ \text{a.e. } x \in \mathbb{R}^d, \ \theta(x) = 0 \ \text{a.e. } |x|_d > R_0 \}; \)

\( \tilde{M}_0, R, R_0 \) are positive constants, and we assume that \( G_R \in C^1_0(\mathbb{R}^d), \ G_R(x) > 0 \) if \( |x|_d < R \) and \( G_R(x) = 0 \) if \( |x|_d \geq R \). We have that \( K : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)^d \) is defined by

\[
K(\zeta)(x) = -\nabla (G_R * \zeta)(x)
\]

(where \( G_R * \zeta \) is the convolution product of \( G_R \) and \( \zeta \)).

The weak solution to (P) may be viewed as the probabilistic density of a population while \( \zeta(x) \) is the density at \( x \in \mathbb{R}^d \) of a second population (or of another entity, which may be a substance or a signal) which produces a stimulus to the first population. This second population is time-independent, imobile, located in the first population which are at a distance less than the so-called “generalized gradient” (nonlocal gradient) with kernel \( G_R \).

Actually, \( u(x) = -\nabla (G_R * \zeta)(x) = K(\zeta)(x) \) describes the nonlocal action (effect) of \( \zeta \) (the second population) towards the individuals of the first population located at \( x \in \mathbb{R}^d \). The term \( -\nabla \cdot (u(x) b(\rho(t,x)) \rho(t,x)) \) describes a cross-diffusion (see e.g. [19, 20]), and leads to a particular form of Keller-Segel’s models (see e.g. [7, 18]).

Problem (P) asks to optimally displace a population via the repeller action produced by a second population of density \( \zeta \). It is a natural requirement (due to logistic constraints) to place this second population in a bounded domain (here we have considered it to be a ball) and to have bounded density.

Notice that the case of a local repeller action at \( x \in \mathbb{R}^d \) is a limit case of the nonlocal one, where

\[ G_R = \delta \]

(\( \delta \) is the Dirac distribution) and

\[ u(x) = -\nabla \zeta(x). \]
If the second population attracts the first one, then we have to take \( u(x) = \nabla (G_R * \zeta)(x) \) instead (and \( u(x) = \nabla \zeta(x) \) in the case of local action).

Assume that the following hypotheses hold:

**H1** \( \beta \in C^2(\mathbb{R}), \beta(0) = 0 \) and \( \exists 0 < \gamma_0 < \gamma_1 \) such that
\[
\gamma_0 \leq \beta'(r) \leq \gamma_1, \forall r \in \mathbb{R}, \quad \text{and } \Psi \in C^2(\mathbb{R}), \text{ where } \Psi(r) = \frac{\beta(r)}{r};
\]

**H2** \( b \in C^1(\mathbb{R}), b \) is bounded, \( b(r) \geq 0, \forall r \geq 0 \) and
\[
|b^*(r) - b^*(\tilde{r})| \leq \alpha |\beta(r) - \beta(\tilde{r})|, \quad \forall r, \tilde{r} \in \mathbb{R},
\]
where \( \alpha > 0 \) and \( b^*(r) = b(r)r \);

**H3** \( \rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \rho_0(x) \geq 0 \text{ a.e. } x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \rho_0(x)dx = 1; \)

**H4** \( G \in C_b([0, T] \times \mathbb{R}^d) \cap L^2((0, T) \times \mathbb{R}^d), G_T \in C_b(\mathbb{R}^d) \cap H^1(\mathbb{R}^d), Q : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}, Q|_{B(0, R_0) \times [0, M_0]} \in C_b^{0, 1}(B(0, R_0) \times [0, M_0]), \)
\[
0 \leq G(t, x), \forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad 0 \leq G_T(x), \forall x \in \mathbb{R}^d,
\]
\[
0 \leq Q(x, z), \forall (x, z) \in \mathbb{R}^d \times [0, M_0], \quad Q(x, z) = 0, \forall x \in \mathbb{R}^d, |x|_d > R_0, \forall z \in [0, M_0],
\]
and the mappings \( z \mapsto Q(x, z) \) are convex with respect to \( z \in [0, M_0] \) for any \( x \in \mathbb{R}^d \).

The following notations \( \nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d} \right), \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \) will be used. Sometimes we will use the simplified notation \( \| \cdot \|_{L^k} \) (instead of \( \| \cdot \|_{L^k(\mathbb{R}^d)} \) or \( \| \cdot \|_{L^k(\mathbb{R}^d; \mathbb{R}^d)} \)) for the usual norm of \( L^k(\mathbb{R}^d) \) or of \( L^k(\mathbb{R}^d; \mathbb{R}^d) \).

For some recent results concerning the optimal control of Fokker-Planck equations we refer to [23, 4, 5, 9, 22]. The difficulty of our present study is mainly due to the nonlinearities in the Fokker-Planck equation [11].

In section 2 we will recall that there exists a unique weak solution to the nonlinear Fokker-Planck equation [11] and that our optimal control problem is deeply related to a stochastic optimal control problem \( (P_S) \) related to a McKean-Vlasov equation. In section 3 one proves the existence of an optimal control for problem \( (P) \). The existence of an optimal control is established and necessary optimality conditions are derived for a penalized optimal control problem \( (P_k) \) related to a backward Euler scheme for the nonlinear Fokker-Planck equation (with a constant discretization step \( h \)) in section 4. Using the optimality conditions for \( (P_k) \) one derives the necessary optimality conditions for problem \( (P) \) in section 5. Section 6 contains final remarks and comments.
2 The nonlinear Fokker-Planck equation and its relationship to a
McKean-Vlasov equation

For any \( \zeta \in \mathcal{M} \) we consider the equation \((1)\) as the following Cauchy problem in \( L^1(\mathbb{R}^d) \):

\[
\begin{cases}
\frac{d\rho}{dt}(t) + A^\zeta \rho(t) = 0, & t \in [0, T] \\
\rho(0) = \rho_0,
\end{cases}
\]

(2)

where \( A^\zeta : D(A^\zeta) \subset L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d) \) is an \( m \)-accretive operator in \( L^1(\mathbb{R}^d) \) given by

\[
D(A^\zeta) = \{ \theta \in L^1(\mathbb{R}^d); -\Delta \beta(\theta) + \nabla \cdot (\mathcal{K}(\zeta)b(\theta)\theta) \in L^1(\mathbb{R}^d) \},
\]

\[
A^\zeta(\theta) = -\Delta \beta(\theta) + \nabla \cdot (\mathcal{K}(\zeta)b(\theta)\theta) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d), \quad \theta \in D(A^\zeta)
\]

(see Proposition 2.3 in [14]). Moreover, \( \overline{D(A^\zeta)} = L^1(\mathbb{R}^d) \) and \( (I + \lambda A^\zeta)^{-1}(\mathcal{P}) \subset \mathcal{P}, \quad \forall \lambda > 0 \), where \( \mathcal{P} \) is the set of all probability densities on \( \mathbb{R}^d \).

**Definition 2.1.** Let \( \zeta \in \mathcal{M} \). A function \( \rho^\zeta \in C([0, T]; L^1(\mathbb{R}^d)) \) is called a weak/mild solution to \((1)/(2)\) if

\[
\rho^\zeta(t) = \lim_{h \to 0^+} \tilde{\rho}^\zeta_h(t) \text{ in } L^1(\mathbb{R}^d), \quad \text{uniformly for } t \in [0, T],
\]

where for any \( h > 0 \), \( \tilde{\rho}_h^\zeta : [0, T] \rightarrow L^1(\mathbb{R}^d) \) is given by

\[
\tilde{\rho}_h^\zeta(t) = \rho_h^\zeta,i \quad \text{if } t \in [ih, (i + 1)h), t \leq T, \quad i = 0, 1, \ldots, N = \left[ \frac{T}{h} \right]
\]

and \( \rho_h^\zeta,i \) is the solution to

\[
\begin{cases}
\rho_h^\zeta,0 = \rho_0, & \rho_h^\zeta,i \in L^1(\mathbb{R}^d), \\
\rho_h^\zeta,i+1 - h \Delta \beta(\rho_h^\zeta,i+1) + h \nabla \cdot (\mathcal{K}(\zeta)b(\rho_h^\zeta,i)\rho_h^\zeta,i) = \rho_h^\zeta,i & \text{in } \mathcal{D}'(\mathbb{R}^d), \quad i = 0, 1, \ldots, N - 1.
\end{cases}
\]

(3)

The existence and uniqueness of such a weak/mild solution \( \rho^\zeta \) to \((1)/(2)\) follow via Theorem 2.1 in [14] (which is based on Crandall-Liggett’s existence theorem (see [7])). Moreover, by Theorem 2.1 in [14] we get that \( \rho^\zeta \) is also a distributional solution to \((1)\). Furthermore, we have that \( \rho^\zeta(t) \in \mathcal{P}, \quad \forall t \in [0, T] \) and \( \rho^\zeta \in L^\infty((0, T) \times \mathbb{R}^d) \).

**Remark 2.1.** If we consider \( N \in \mathbb{N}^* \) and \( h = \frac{T}{N} \), then we get that

\[
\rho^\zeta(t) = \lim_{h \to 0^+} \rho_h^\zeta(t) = \lim_{N \to +\infty} \rho_N^\zeta(t) \text{ in } L^1(\mathbb{R}^d), \quad \text{uniformly for } t \in [0, T],
\]

where \( \rho_h^\zeta : [0, T] \rightarrow L^1(\mathbb{R}^d) \) is given by

\[
\rho_h^\zeta(t) = \begin{cases}
\rho_0, & \text{if } t = 0, \\
\rho_h^{\zeta,i+1}, & \text{if } t \in (ih, (i + 1)h), \quad i = 0, 1, \ldots, N - 1.
\end{cases}
\]
Indeed, if \( t = ih \ (i \in \{0, 1, ..., N\}) \), then \( \rho_h^\zeta(t) = \rho_h^{\zeta,i} = \bar{\rho}_h^\zeta(t) \) and so
\[
\|\rho_h^\zeta(t) - \rho^\zeta(t)\|_{L^1(\mathbb{R}^d)} = \|\bar{\rho}_h^\zeta(t) - \rho^\zeta(t)\|_{L^1(\mathbb{R}^d)}.
\]
On the other hand, if \( t \in (ih, (i+1)h), \ i \in \{0, 1, ..., N - 1\} \), then \( \rho_h^\zeta(t) = \rho_h^\zeta((i+1)h) \) and so
\[
\|\rho_h^\zeta(t) - \rho^\zeta(t)\|_{L^1(\mathbb{R}^d)} = \|\rho_h^\zeta((i+1)h) - \rho^\zeta(t)\|_{L^1(\mathbb{R}^d)} = \|\bar{\rho}_h^\zeta((i+1)h) - \rho^\zeta(t)\|_{L^1(\mathbb{R}^d)}
\]
\[
\leq \|\bar{\rho}_h^\zeta((i+1)h) - \rho^\zeta((i+1)h)\|_{L^1(\mathbb{R}^d)} + \|\rho^\zeta((i+1)h) - \rho^\zeta(t)\|_{L^1(\mathbb{R}^d)}.
\]
Taking into account the convergence of \( \bar{\rho}_h^\zeta(t) \) to \( \rho^\zeta(t) \) in \( L^1(\mathbb{R}^d) \), uniformly for \( t \in [0, T] \) and the fact that \( \rho^\zeta \in C([0, T]; L^1(\mathbb{R}^d)) \), we get the conclusion.

Consider now the following controlled McKean-Vlasov SDE on \( \mathbb{R}^d \):
\[
\begin{cases}
  dX(t) = f \left( X(t), \mathcal{K}(\zeta)(X(t)), \frac{d\mathcal{L}_Y}{dx}(X(t)) \right) dt + \sigma \left( X(t), \frac{d\mathcal{L}_Y}{dx}(X(t)) \right) dW(t), & t \in [0, T] \\
  X(0) = X_0,
\end{cases}
\]
(4)
where \( \mathcal{L}_Y \) is the law/distribution of the random variable \( Y \) and \( \frac{d\mathcal{L}_Y}{dx} \) is its density (when it exists) with respect to the Lebesgue measure \( dx \), and
\[
f(x, u, r) = ub(r), \quad \sigma(x, r) = \left( \frac{2\beta(r)}{r} \right)^{\frac{1}{2}}.
\]
Since for any \( \zeta \in \mathcal{M} \), \( \rho^\zeta \) is a distributional solution to the nonlinear Fokker-Planck equation, which is also \( t \)-narrowly continuous, then by superposition’s principle (see [21, 26, 24]) we conclude that there exists a unique (in law) probabilistically weak solution \( (X^\zeta(t))_{t \in [0, T]} \) to (4) (which has \( \rho^\zeta(t) \) as a probability density). Actually, the weak solution \( X^\zeta \) corresponds to a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with normal filtration \( (\mathcal{F}_t)_{t \in [0, T]} \) and \( (W(t))_{t \in [0, T]} \) is an \( \mathbb{R}^d - (\mathcal{F}_t)_{t \in [0, T]} \) Brownian motion.

Hence,
\[
\mathbb{E} \left[ \int_0^T G(t, X^\zeta(t))dt \right] + \mathbb{E}[G_T(X^\zeta(T))] = \int_0^T \int_{\mathbb{R}^d} G(t, x)\rho^\zeta(t, x)dx \ dt + \int_{\mathbb{R}^d} G_T(x)\rho^\zeta(T, x)dx.
\]
It follows that the next stochastic optimal control problem
\[
(\mathcal{P}_S) \quad \text{Minimize} \quad \mathbb{E} \left[ \int_0^T G(t, X^\zeta(t))dt \right] + \mathbb{E}[G_T(X^\zeta(T))] + \int_{\mathbb{R}^d} Q(x, \zeta(x))dx,
\]
is equivalent to the deterministic optimal control problem \( (P) \).

We mention that lately there is a high interest in McKean-Vlasov SDEs and in stochastic optimal control problems. Some recent and important results concerning the McKean-Vlasov
can be found in [9, 12, 13, 16]. As concerns problem \((P_S)\) (which is a stochastic optimal control problem related to McKean-Vlasov SDE) we see that it is deeply related to an optimal control problem \((P)\) for a nonlinear Fokker-Planck equation. A similar approach, reducing the investigation of a stochastic optimal control problem to a deterministic optimal control problem, has been recently used for the case of feedback controllers independent of time in [1, 3, 8] and in [2, 22] for controllers dependent of time as well, for optimal control problems with other cost functionals and related to stochastic differential equations with drift and diffusion independent of the probabilistic density. For other approaches for stochastic optimal control problems see [10, 11, 15] while for standard results concerning the control of stochastic differential equations we refer to the monographs [23, 25].

3 Existence of an optimal control for problem \((P)\)

Let \(m^* = \inf_{\zeta \in \mathcal{M}} I(\zeta) \in [0, +\infty)\), where

\[
I(\zeta) = \int_0^T \int_{\mathbb{R}^d} G(t, x) \rho^\zeta(t, x) dx \, dt + \int_{\mathbb{R}^d} G_T(x) \rho^\zeta(T, x) dx + \int_{\mathbb{R}^d} Q(x, \zeta(x)) dx.
\]

For any \(\zeta \in \mathcal{M}\) we have that \(\mathcal{K}(\zeta) \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)\),

\[
supp \mathcal{K}(\zeta) \subset B(0_d; R + R_0),
\]

\[
|\mathcal{K}(\zeta)(x)|_d \leq \tilde{M}_0 \int_{\mathbb{R}^d} |\nabla G_R(-x')|_d dx' = M, \quad \forall x \in \mathbb{R}^d,
\]

\[
|\nabla \mathcal{K}(\zeta)(x)|_{d \times d} \leq \tilde{M}_1, \quad \forall x \in \mathbb{R}^d,
\]

where \(M_1\) is a constant independent of \(\zeta \in \mathcal{M}\).

For any sequence \(\{\zeta_k\}_{k \in \mathbb{N}^*} \subset \mathcal{M}\), there exists a subsequence \(\{\zeta_{k_l}\}_{l \in \mathbb{N}^*}\) and \(\zeta^* \in \mathcal{M}\) such that

\[
\zeta_{k_l} \rightharpoonup \zeta^* \quad \text{weakly * in } L^\infty(\mathbb{R}^d).
\]

This implies that

\[
\mathcal{K}(\zeta_{k_l})(x) = - \int_{\mathbb{R}^d} \nabla G_R(x-x') \zeta_{k_l}(x') dx' \rightarrow - \int_{\mathbb{R}^d} \nabla G_R(x-x') \zeta^*(x') dx' = \mathcal{K}(\zeta^*)(x), \quad \forall x \in \mathbb{R}^d,
\]

and that

\[
\nabla \mathcal{K}(\zeta_{k_l})(x) \rightarrow \nabla \mathcal{K}(\zeta^*)(x), \quad \forall x \in \mathbb{R}^d.
\]

Let us notice that by Arzelà’s compactness theorem we may infer that there exists a subsequence \(\{\mathcal{K}(\zeta_{k_r})\}_{r \in \mathbb{N}^*}\) such that

\[
\mathcal{K}(\zeta_{k_r}) \rightarrow \tilde{u}, \quad \text{uniformly for } x \in B(0_d; R + R_0).
\]

Since \(\mathcal{K}(\zeta_{k_r})(x) = 0, \forall |x|_d > R + R_0\), we get that

\[
\mathcal{K}(\zeta_{k_r}) \rightarrow \tilde{u}, \quad \text{uniformly for } x \in \mathbb{R}^d.
\]
(where $\tilde{u}$ has been extended by the value $0_d$ outside $\overline{B(0_d; R + R_0)}$). Since $\mathcal{K}(\zeta_k) \rightarrow \mathcal{K}(\zeta^*)$, $\forall x \in \mathbb{R}^d$, we conclude that

$$\mathcal{K}(\zeta_k) \rightarrow \mathcal{K}(\zeta^*)$$

uniformly for $x \in \mathbb{R}^d$.

We postpone for the time being proof of the next auxiliary result:

**Lemma 3.1.** If $\{\zeta_n\}_{n \in \mathbb{N}^*} \subset \mathcal{M}$, $\zeta_n \rightarrow \zeta \in \mathcal{M}$ weakly * in $L^\infty(\mathbb{R}^d)$,

$$\mathcal{K}(\zeta_n) = -\nabla (G_R * \zeta_n) \rightarrow -\nabla (G_R * \zeta) = \mathcal{K}(\zeta),$$

uniformly for $x \in \mathbb{R}^d$,

then for any $h > 0$ sufficiently small and for any $f \in L^1(\mathbb{R}^d)$ we have that

$$y_n \rightarrow y \text{ in } L^1(\mathbb{R}^d),$$

where $y_n, y \in L^1(\mathbb{R}^d)$,

$$y_n - h\Delta \beta(y_n) + h \nabla \cdot (\mathcal{K}(\zeta_n)b^*(y_n)) = f \text{ in } \mathcal{D}'(\mathbb{R}^d) \tag{5}$$

and

$$y - h\Delta \beta(y) + h \nabla \cdot (\mathcal{K}(\zeta)b^*(y)) = f \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{6}$$

**Remark 3.1.** By Lemma 2.5 in [14] we have that equations (5) and (6) have unique solutions, respectively. Moreover, if one uses more accurate notations: $y_n(f)$ for the solution to (5), and $y(f)$ for the solution to (6), then

$$\|y_n(f) - y_n(g)\|_{L^1(\mathbb{R}^d)} \leq \|f - g\|_{L^1(\mathbb{R}^d)},$$

for any $f, g \in L^1(\mathbb{R}^d)$. By Lemma 3.1 one obtains that

$$\|y(f) - y(g)\|_{L^1(\mathbb{R}^d)} \leq \|f - g\|_{L^1(\mathbb{R}^d)}, \quad f, g \in L^1(\mathbb{R}^d).$$

Using now Trotter-Kato’s theorem (Theorem 2.1-p.241, [6]) we may conclude that

**Lemma 3.2.** If $\{\zeta_n\}_{n \in \mathbb{N}^*} \subset \mathcal{M}$, $\zeta_n \rightarrow \zeta \in \mathcal{M}$ weakly * in $L^\infty(\mathbb{R}^d)$, $\mathcal{K}(\zeta_n) \rightarrow \mathcal{K}(\zeta)$, uniformly for $x \in \mathbb{R}^d$, then

$$\rho^{\zeta_n} \rightarrow \rho^{\zeta} \text{ in } C([0, T]; L^1(\mathbb{R}^d)).$$

**Theorem 3.3.** Problem $(P)$ has at least one optimal control.

**Proof.** Let $\{\zeta_k\}_{k \in \mathbb{N}^*} \subset \mathcal{M}$ such that

$$m^* \leq I(\zeta_k) < m^* + \frac{1}{k}, \quad k \in \mathbb{N}^*.$$
Since \( \{\zeta_k\}_{k \in \mathbb{N}^*} \subset \mathcal{M} \) it follows that there exists a subsequence (also denoted by \( \{\zeta_k\} \)) such that
\[ \zeta_k \rightharpoonup \zeta^* \in \mathcal{M} \text{ weakly * in } L^\infty(\mathbb{R}^d), \quad \mathcal{K}(\zeta_k) \rightharpoonup \mathcal{K}(\zeta^*), \text{ uniformly for } x \in \mathbb{R}^d. \]

Using the convexity of \( Q \) with respect to \( \zeta \), we obtain that
\[ I(\zeta_k) \geq \int_0^T \int_{\mathbb{R}^d} G(t, x)k^\alpha(t, x)dx + \int_{\mathbb{R}^d} G_T(x)k^\alpha(T, x)dx \]
\[ + \int_{\mathbb{R}^d} Q(x, \zeta^*(x))dx + \int_{\mathbb{R}^d} \frac{\partial Q}{\partial z}(x, \zeta^*(x))(\zeta_k(x) - \zeta^*(x))dx. \]

The last inequality implies via Lemma 3.2 that
\[ m^* \geq \liminf_{k \to +\infty} I(\zeta_k) \geq I(\zeta^*), \]
and so \( \zeta^* \) is an optimal control for problem (P).

**Proof of Lemma 3.1.** Consider for the beginning the function \( \Phi \) defined in the proof of Lemma 3.3 in [16]. Actually, we recall that this particular function satisfies \( \Phi \in C^2(\mathbb{R}^d) \), \( 1 \leq \Phi(x) \), \( \forall x \in \mathbb{R}^d \) and
\[ \lim_{|x|_d \to +\infty} \Phi(x) = +\infty, \quad \nabla \Phi \in L^\infty(\mathbb{R}^d; \mathbb{R}^d), \quad \Delta \Phi \in L^\infty(\mathbb{R}^d). \]

**Case 1.** Let \( f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} |f(x)|\Phi(x)dx < +\infty \).

From the proof of Lemma 2.5 in [14] we have that \( y_n \), the unique solution to (5), satisfies \( \beta(y_n) \in H^1(\mathbb{R}^d) \).

By equation (3) we get that
\[ \int_{\mathbb{R}^d} y_n \beta(y_n)dx + h \int_{\mathbb{R}^d} |\nabla \beta(y_n)|_d^2dx = h \int_{\mathbb{R}^d} \mathcal{K}(\zeta_n)b(y_n)y_n \cdot \nabla \beta(y_n)dx = \int_{\mathbb{R}^d} f \beta(y_n)dx. \]

Notice that \( \beta(y_n) = \beta(y_n) - \beta(0) = \beta'(\zeta_n)y_n \) and \( \nabla \beta(y_n) = \beta'(y_n)\nabla y_n \) (where \( \zeta_n(x) \) is an intermediate point). It follows that for \( h > 0 \) sufficiently small, \( \{\beta(y_n)\}_{n \in \mathbb{N}^*} \) is bounded in \( H^1(\mathbb{R}^d) \) and \( \{y_n\}_{n \in \mathbb{N}^*} \) is bounded in \( H^1(\mathbb{R}^d) \) as well.

It follows that there exists a subsequence \( y_{n_k} \rightharpoonup \tilde{y} \) in \( L^2(B(0d; 1)) \), there exists a subsequence \( y_{n_{k_j}} \rightharpoonup \tilde{y} \) in \( L^2(B(0d; 2)) \), ...

Taking the diagonal subsequence (also denoted by \( \{y_n\}_{n \in \mathbb{N}^*} \)) we get that \( y_n \rightharpoonup \tilde{y} \) in \( L^2(\text{loc.}(\mathbb{R}^d)). \) Repeating the argument in Lemma 3.3 in [16] we obtain that there exists a nonnegative constant \( C \) such that
\[ \int_{\mathbb{R}^d} |y_n(x)|\Phi(x)dx \leq C, \quad \forall n \in \mathbb{N}^*. \]

Using Fatou’s lemma we obtain
\[ \int_{\mathbb{R}^d} |\tilde{y}(x)|\Phi(x)dx \leq C. \]
Since \( \int_{\mathbb{R}^d} [y_n \varphi - h \beta(y_n) \Delta \varphi - h K(\zeta_n) b^*(y_n) \cdot \nabla \varphi] dx = \int_{\mathbb{R}^d} f \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d) \), we get that
\[
\int_{\mathbb{R}^d} [\tilde{y} \varphi - h \beta(\tilde{y}) \Delta \varphi - h K(\zeta) b^*(\tilde{y}) \cdot \nabla \varphi] dx = \int_{\mathbb{R}^d} f \varphi dx.
\]
So, \( \tilde{y} \) satisfies (\( \text{II} \)) in distributional sense.

On the other hand, by Lemma 2.5 in [14] we have that \( \int_{\mathbb{R}^d} |y_n| dx \leq \int_{\mathbb{R}^d} |f| dx \) and consequently \( \int_{\mathbb{R}^d} |\tilde{y}| dx \leq \int_{\mathbb{R}^d} |f| dx \).

We may infer that \( \tilde{y} \in L^1(\mathbb{R}^d), \beta(\tilde{y}) \in L^1(\mathbb{R}^d) \) (and so \( -\Delta \beta(\tilde{y}) + \nabla (K(\zeta) b^*(\tilde{y})) \in L^1(\mathbb{R}^d) \)).

So, \( \tilde{y} \) is solution to (\( \text{II} \)) and by Lemma 2.5 in [14] we conclude that \( \tilde{y} = y \).

Let \( \Phi_R = \min\{\Phi(x); |x|_d \geq R\}, \ R > 0 \). This yields
\[
\int_{\mathbb{R}^d} |y_n - y| dx = \int_{B(0_d; R)} |y_n - y| dx + \int_{|x|_d \geq R} |y_n - y| dx.
\]

Since
\[
\int_{|x|_d \geq R} |y_n| \Phi_R dx \leq \int_{|x|_d \geq R} |y_n| \Phi(x) dx \leq \int_{\mathbb{R}^d} |y_n| \Phi(x) dx \leq C,
\]
it follows that
\[
\int_{|x|_d \geq R} |y_n| dx \leq \frac{C}{\Phi_R} \quad \text{and so} \quad \int_{|x|_d \geq R} |y| dx \leq \frac{C}{\Phi_R}.
\]

On the other hand, since \( \Phi_R \to +\infty \) as \( R \to +\infty \), it follows that for any \( \varepsilon > 0 \), there exists for \( R > 0 \) sufficiently large such that
\[
\int_{|x|_d \geq R} |y_n - y| dx \leq \int_{|x|_d \geq R} (|y_n| + |y|) dx \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon,
\]
and so
\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^d} |y_n - y| dx \leq \varepsilon, \quad \forall \varepsilon > 0, \quad \text{and} \quad y_n \to y \quad \text{in} \ L^1(\mathbb{R}^d).
\]

**Remark 3.2.** The conclusion holds not only for a subsequence of \( \{y_n\}_{n \in \mathbb{N}^*} \), but for the sequence itself. We argue by contradiction.

Assume that there exists a subsequence \( \{y_{l\in \mathbb{N}^*}\} \) such that \( \|y_{l\in \mathbb{N}^*} - y\|_{L^1(\mathbb{R}^d)} \to 0 > 0 \). The first part of the proof allows us to conclude that on a subsequence \( \{y_{n_{l\in \mathbb{N}^*}}\} \) we have that \( y_{n_{l\in \mathbb{N}^*}} \to y \) in \( L^1(\mathbb{R}^d) \) and so \( \|y_{n_{l\in \mathbb{N}^*}} - y\|_{L^1(\mathbb{R}^d)} \to 0 \); absurd.

Case 2. Let \( f \in L^1(\mathbb{R}^d) \) and \( \{f_k\}_{k \in \mathbb{N}^*} \) satisfying the hypotheses of \( f \) in case 1 and satisfying \( f_k \to f \) in \( L^1(\mathbb{R}^d) \). Let
\begin{itemize}
  \item \( y_n^k \) solution to (\( \text{II} \)) with \( \zeta := \zeta_n, \ f := f_k \);
\end{itemize}
• $y_n$ solution to (6) with $\zeta := \zeta_n$;
• $y^k$ solution to (6) with $f := f_k$.

We have that
\[
\|y_n - y\|_{L^1(\mathbb{R}^d)} \leq \|y_n - y^k\|_{L^1(\mathbb{R}^d)} + \|y^k - y\|_{L^1(\mathbb{R}^d)}.
\]

Since
\[
\|y_n - y^k\|_{L^1(\mathbb{R}^d)} \leq \|f - f_k\|_{L^1(\mathbb{R}^d)}, \quad \|y^k - y\|_{L^1(\mathbb{R}^d)} \leq \|f - f_k\|_{L^1(\mathbb{R}^d)},
\]
we conclude that
\[
\|y_n - y\|_{L^1(\mathbb{R}^d)} \leq 2\|f - f_k\|_{L^1(\mathbb{R}^d)} + \|y^k_n - y^k\|_{L^1(\mathbb{R}^d)}.
\]

For a fixed $k$ we have that $y^k_n - y^k \to 0$ in $L^1(\mathbb{R}^d)$. Hence
\[
\limsup_{n \to +\infty} \|y_n - y\|_{L^1(\mathbb{R}^d)} \leq 2\|f - f_k\|_{L^1(\mathbb{R}^d)}, \forall k \in \mathbb{N}^*,
\]
and so
\[
\limsup_{n \to +\infty} \|y_n - y\|_{L^1(\mathbb{R}^d)} \leq 0 \text{ and } y_n \to y \text{ in } L^1(\mathbb{R}^d).
\]

### 4 A penalized optimal control problem $(P_h)$

For any $N \in \mathbb{N}^*$ we consider $h = \frac{T}{N}$ and the following optimal control problem:

\[
(P_h) \quad \text{Minimize} \quad \int_0^T \int_{\mathbb{R}^d} G(t, x) \rho_h^\zeta(t, x) dx \, dt + \int_{\mathbb{R}^d} G_T(x) \rho_h^\zeta(T, x) dx
\]
\[
+ \int_{\mathbb{R}^d} Q(x, \zeta(x)) dx + \frac{1}{2} \int_{\mathbb{R}^d} |\zeta(x) - \zeta^*(x)|^2 dx,
\]
where $\rho_h^\zeta$ and $\rho_h^{\zeta,i}$ were defined in section 2, and $\zeta^*$ is an optimal control for problem $(P)$.

**Remark 4.1.** The following equality holds
\[
\int_0^T \int_{\mathbb{R}^d} G(t, x) \rho_h^{\zeta_t}(t, x) dx \, dt + \int_{\mathbb{R}^d} G_T(x) \rho_h^{\zeta}(T, x) dx
\]
\[
= \sum_{i=1}^N \int_{\mathbb{R}^d} \tilde{G}^i(x) \rho_h^{\zeta,i}(x) dx + \int_{\mathbb{R}^d} G_T(x) \rho_h^{\zeta N}(x) dx,
\]
where
\[
\tilde{G}^i(x) = \int_{(i-1)h}^{ih} G(t, x) dt, \quad i = 1, 2, ..., N.
\]

For the remainder of the section we consider $h = \frac{T}{N}$ to be sufficiently small.
4.1 Existence of an optimal control for \((P_h)\)

**Lemma 4.1.** If \(\{\zeta_n\}_{n \in \mathbb{N}^*} \subset \mathcal{M}, \zeta_n \rightarrow \zeta \in \mathcal{M}\) weakly * in \(L^\infty(\mathbb{R}^d)\), \(K(\zeta_n) \rightarrow K(\zeta)\), uniformly for \(x \in \mathbb{R}^d\), and if \(\{f_n\}_{n \in \mathbb{N}^*} \subset L^1(\mathbb{R}^d)\), \(f_n \rightarrow f\) in \(L^1(\mathbb{R}^d)\), then

\[
y^n_n \rightarrow y \quad \text{in} \quad L^1(\mathbb{R}^d),
\]

where \(y^n_k, y_n\) where introduced in section 3 (in the proof of Lemma 3.1).

**Proof.** By Lemma 2.5 in [14] we get that

\[
\|y^n_n - y\|_{L^1(\mathbb{R}^d)} \leq \|y^n_n - y_n\|_{L^1(\mathbb{R}^d)} + \|y_n - y\|_{L^1(\mathbb{R}^d)} \leq \|f_n - f\|_{L^1(\mathbb{R}^d)} + \|y_n - y\|_{L^1(\mathbb{R}^d)}.
\]

Using the convergence of \(\{f_n\}_{n \in \mathbb{N}^*}\) and that \(y_n \rightarrow y\) in \(L^1(\mathbb{R}^d)\) (by Lemma 3.1) we get the conclusion.

**Lemma 4.2.** If \(\{\zeta_n\}_{n \in \mathbb{N}^*} \subset \mathcal{M}, \zeta_n \rightarrow \zeta \in \mathcal{M}\) weakly * in \(L^\infty(\mathbb{R}^d)\), \(K(\zeta_n) \rightarrow K(\zeta)\), uniformly for \(x \in \mathbb{R}^d\), and if \((\rho^n_{h,i})_{i=0,1,...,N}\) and \((\rho^i_{h,j})_{i=0,1,...,N}\) are defined by [3], then

\[
\rho^n_{h,i} \rightarrow \rho^i_{h,j} \quad \text{in} \quad L^1(\mathbb{R}^d), \quad \forall i \in \{0, 1, ..., N\}.
\]

**Proof.** This follows by finite induction and Lemma 4.1.

**Theorem 4.3.** Problem \((P_h)\) has at least one optimal control \(\zeta_h^*\).

**Proof.** Denote by \(I_h\) the cost functional associated to problem \((P_h)\). Let \(m_h^* = \inf_{\zeta \in \mathcal{M}} I_h(\zeta) \in [0, +\infty)\), and let \(\{\zeta_n\}_{n \in \mathbb{N}^*} \subset \mathcal{M}, \zeta_n \rightarrow \zeta_h^* \in \mathcal{M}\) weakly * in \(L^\infty(\mathbb{R}^d)\),

\[
K(\zeta_n) \rightarrow K(\zeta_h^*), \quad \text{uniformly for} \quad x \in \mathbb{R}^d,
\]

such that:

\[
m_h^* \leq I_h(\zeta_n) < m_h^* + \frac{1}{n}, \quad n \in \mathbb{N}^*.
\]

By Lemma 4.2 we have that

\[
\int_{\mathbb{R}^d} G_T\rho^n_{h,i} dx \rightarrow \int_{\mathbb{R}^d} G_T\rho^i_{h,j} dx,
\]

and

\[
\sum_{i=1}^N \int_{\mathbb{R}^d} \tilde{G}^i \rho^n_{h,i} dx \rightarrow \sum_{i=1}^N \int_{\mathbb{R}^d} \tilde{G}^i \rho^i_{h,j} dx.
\]

On the other hand (from the convexity of \(Q\) with respect to \(\zeta\))

\[
\int_{\mathbb{R}^d} Q(x, \zeta_n(x)) dx + \frac{1}{2} \int_{\mathbb{R}^d} |\zeta_n(x) - \zeta^*(x)|^2 dx \geq \int_{\mathbb{R}^d} Q(x, \zeta_h^*(x)) dx
\]

\[
+ \int_{\mathbb{R}^d} \frac{\partial Q}{\partial z}(x, \zeta_h^*(x)) (\zeta_n(x) - \zeta_h^*(x)) dx.
\]

This implies that

\[
m_h^* \geq \lim_{n \rightarrow +\infty} I_h(\zeta_n) = \liminf_{n \rightarrow +\infty} I_h(\zeta_n) \geq I(\zeta_h^*) \geq m_h^*.
\]

We conclude that \(\zeta_h^*\) is an optimal control for problem \((P_h)\).
4.2 Necessary optimality conditions for \((P_h)\)

Let \(\zeta^*_h\) be an optimal control for problem \((P_h)\). Let \(\xi \in L^\infty(\mathbb{R}^d)\) be such that \(\zeta^*_h + \varepsilon \xi \in \mathcal{M}\), for any \(\varepsilon > 0\) sufficiently small.

For any \(\zeta \in \mathcal{M}\), let \(z^{\zeta,i+1}\) be the weak solution to

\[
\begin{aligned}
  &z^{\zeta,i+1} - h\Delta (\beta'(\rho_h^{\zeta,i+1})z^{\zeta,i+1}) + h\nabla \cdot (\mathcal{K}(\zeta)(b^*)'(\rho_h^{\zeta,i+1})z^{\zeta,i+1}) + h\nabla \cdot (\mathcal{K}(\xi)b^*(\rho_h^{\zeta,i+1})) = z^{\zeta,i}, \\
  &z^{\zeta,0} = 0.
\end{aligned}
\]

Actually, \(z^{\zeta,i+1}\) is a weak solution to the equation in (7) if \(\bar{z}^{i+1} \in H^1(\mathbb{R}^d)\) is the unique weak solution to

\[
\frac{1}{\beta'(\rho_h^{\bar{z}^{i+1}})} \bar{z}^{i+1} - h\Delta \bar{z}^{i+1} + h\nabla \cdot \left(\frac{\mathcal{K}(\bar{z}^{i+1})}{\beta'(\rho_h^{\bar{z}^{i+1}})} \bar{z}^{i+1}\right) = -h\nabla \cdot (\mathcal{K}(\xi)b^*(\rho_h^{\bar{z}^{i+1}})) + \frac{1}{\beta'(\rho_h^{\bar{z}^{i}})} \bar{z}^{i}
\]

and \(\bar{z}^{0} = 0\). Here \(\bar{z}^{i} = \beta'(\rho_h^{\bar{z}^{i}})z^{\zeta,i}\). The existence and uniqueness of a weak solution to this equation follows via Lax-Milgram’s lemma (for sufficiently small \(h\)).

We postpone the proof of the following lemma.

**Lemma 4.4.** The following convergences hold for any \(i \in \{1, 2, \ldots, N\}\):

(i) \(\frac{\zeta^*_h + \varepsilon \xi}{\varepsilon} \rightarrow \rho_h^{\zeta,i} \) in \(L^2(\mathbb{R}^d)\), as \(\varepsilon \rightarrow 0+\);

(ii) \(\frac{\zeta^*_h + \varepsilon \xi - \rho_h^{\zeta,i}}{\varepsilon} \rightarrow z^{\zeta,i} \) in \(L^2(\mathbb{R}^d)\), as \(\varepsilon \rightarrow 0+\) (on a subsequence).

**Theorem 4.5.** If \(p_h^{\zeta,i+1}\) is the unique weak solution to

\[
\begin{aligned}
  &p^{\zeta,i} - h\beta'(\rho_h^{\zeta,i}) \Delta p^{\zeta,i} - h\mathcal{K}(\zeta)(b^*)'(\rho_h^{\zeta,i+1}) \cdot \nabla p^{\zeta,i} + \bar{G}^{i+1} = p^{\zeta,i+1}, \\
  &p^{\zeta,N} = -G_T,
\end{aligned}
\]

for any \(\zeta := \zeta^*_h\), then

\[
\int_{\mathbb{R}^d} \mathcal{K}(\xi) \cdot \left[\sum_{i=1}^{N} [hb^*(\rho_h^{\zeta,i}) \nabla p_h^{\zeta,i-1}] \right] dx - \int_{\mathbb{R}^d} \xi \left[\frac{\partial Q}{\partial z}(x, \zeta^*_h(x)) + (\zeta^*_h(x) - \zeta^*(x))\right] dx \leq 0,
\]

for any \(\xi \in L^\infty(\mathbb{R}^d)\) such that \(\zeta^*_h + \varepsilon \xi \in \mathcal{M}\), for any \(\varepsilon > 0\) sufficiently small.

We say that \(p^{\zeta,i} \in H^1(\mathbb{R}^d)\) is a weak solution to the equation in (8) if it is a weak solution to

\[
\frac{1}{\beta'(\rho_h^{\zeta,i+1})} p^{\zeta,i} - h\Delta p^{\zeta,i} - h\mathcal{K}(\zeta)(b^*)'(\rho_h^{\zeta,i+1}) \cdot \nabla p^{\zeta,i} + \frac{1}{\beta'(\rho_h^{\zeta,i+1})} \bar{G}^{i+1} = \frac{1}{\beta'(\rho_h^{\zeta,i+1})} p^{\zeta,i+1}.
\]
The existence and uniqueness of a weak solution follows via Lax-Milgram’s lemma (for sufficiently small $h$).

**Remark 4.2.** We have that $\zeta_h^*(x) = 0$ a.e. $x \in Ext(B(0_d; R_0))$. On the other hand for almost any $x \in B(0_d; R_0)$ we have that

$$\zeta_h^*(x) = \begin{cases} 0, & \text{if } \int_{\mathbb{R}^d} \nabla G_R(x' - x) \cdot \sum_{i=1}^N [hb^*(\rho_{h,i}) \nabla \rho_{h,i-1}^*(x')]dx' + \frac{\partial Q}{\partial z}(x, \zeta_h^*(x)) + (\zeta_h^*(x) - \zeta^*(x)) > 0 \\
M_0, & \text{if } \int_{\mathbb{R}^d} \nabla G_R(x' - x) \cdot \sum_{i=1}^N [hb^*(\rho_{h,i}) \nabla \rho_{h,i-1}^*(x')]dx' + \frac{\partial Q}{\partial z}(x, \zeta_h^*(x)) + (\zeta_h^*(x) - \zeta^*(x)) < 0. \end{cases}$$

**Proof of Theorem 4.5.** Since $\zeta_h^*$ is an optimal control for problem $(P_h)$, then $I_h(\zeta_h^*) \leq I_h(\zeta_h^* + \varepsilon \xi)$, for any $\varepsilon > 0$ sufficiently small. After an easy evaluation and using the Lemma 4.4 we get that

$$0 \leq \sum_{i=1}^N \int_{\mathbb{R}^d} \tilde{G}_{i,i}^* \xi_{i,i}^* dx + \int_{\mathbb{R}^d} G_T \xi_{i,N}^* dx + \int_{\mathbb{R}^d} \mathcal{K}(\xi) \cdot \nabla \xi_{i,i}^* dx. \tag{10}$$

We multiply (8) by $z_{i,i+1}^*$ and integrate over $\mathbb{R}^d$. After some evaluation and taking into account (7) we obtain that

$$\int_{\mathbb{R}^d} \tilde{G}_{i+1,i}^* z_{i,i+1}^* dx = \int_{\mathbb{R}^d} p_{h,i+1}^* z_{i,i+1}^* dx + \int_{\mathbb{R}^d} p_{h,i}^* [\zeta_{i,i}^* + h \nabla \mathcal{K}(\xi) b^*(\rho_{h,i})] dx.$$ 

By summation and using that $z_{i,i,0}^* = 0$ we obtain that

$$\sum_{i=1}^N \int_{\mathbb{R}^d} \tilde{G}_{i,i}^* \xi_{i,i}^* dx = -\int_{\mathbb{R}^d} G_T \xi_{i,N}^* dx - \sum_{i=1}^N \int_{\mathbb{R}^d} hb^*(\rho_{h,i}) \mathcal{K}(\xi) \cdot \nabla \rho_{h,i-1}^* dx$$

and using (10) we get (9).

**Proof of Lemma 4.4.** By Lemma 3.1 in [16] it follows that there exists $C > 0$ such that

$$\|\rho_{h,\xi,i}^*\|_{L^\infty(\mathbb{R}^d)} \leq C, \quad \forall 0 \leq \varepsilon \leq \varepsilon_0, \quad i = 0, 1, \ldots, N.$$ 

Since $\rho_{h,\xi,i}^* \rightarrow \rho_{h,i}^*$ in $L^1(\mathbb{R}^d)$, $i = 0, 1, \ldots, N$, as $\varepsilon \rightarrow 0+$ (by Lemma 4.2) we conclude that (i) is satisfied.

Let $y^\varepsilon,i = \beta(\rho_{h,\xi,i}^*), \quad y^i = \beta(\rho_{h,i}^*), \quad \eta^i = \beta'(\rho_{h,i}^*) z_{i,i}^* + \eta^\varepsilon,i - y^i, \quad \overline{\eta}^i = \frac{\eta^i}{\varepsilon}.$

We have that $\zeta_{i,i}^*$ is the weak solution to

$$\beta^{-1}(y^\varepsilon,i+1) - \beta^{-1}(y^{i+1}) - h \Delta \eta^i_{\varepsilon,i+1} + h \nabla \cdot (\mathcal{K}(\xi)^*(b^*(\beta^{-1}(y^{\varepsilon,i+1})) - b^*(\beta^{-1}(y^{i+1}))))$$
$+ \varepsilon h \nabla \cdot (K(\xi) b^*(\beta^{-1}(y_{\varepsilon,i+1}))) = \beta^{-1}(y_{\varepsilon,i}) - \beta^{-1}(y^i)$.

If we denote by $w_\varepsilon^i = \tilde{\eta}_\varepsilon^i - \eta^i$, then we have that $w_\varepsilon^{i+1}$ is the unique weak solution to

$$\frac{\beta^{-1}(y_{\varepsilon,i+1}) - \beta^{-1}(y^i)}{\varepsilon} - \frac{1}{\beta'(\beta^{-1}(y^i))} \eta^{i+1} - h\Delta w_\varepsilon^{i+1}$$

$$+ h \nabla \cdot \left( K(\xi^*_h) \frac{b^*(\beta^{-1}(y_{\varepsilon,i+1})) - b^*(\beta^{-1}(y^i))}{\varepsilon} - \frac{(b^*)'(\beta^{-1}(y^i))}{\beta'(\beta^{-1}(y^i))} \eta^{i+1} \right)$$

$$= \frac{\beta^{-1}(y_{\varepsilon,i}) - \beta^{-1}(y^i)}{\varepsilon} - \frac{1}{\beta'(\beta^{-1}(y^i))} \eta^i.$$ 

If we apply the Lagrange’s mean value theorem we get after some evaluation and by finite induction that for a subsequence (also indexed by $\varepsilon$):

$$w_\varepsilon^i \longrightarrow 0 \quad \text{in } H^1(\mathbb{R}^d), \quad \forall i = 0, 1, \ldots, N,$$

and so

$$\frac{\beta(\rho_{t+h}^\varepsilon, \xi, i) - \beta(\rho_{t+\varepsilon}^\varepsilon, \xi, i)}{\varepsilon} \longrightarrow \beta'(\rho_{t+h}^\varepsilon, \xi, i) z_{t,h}^\varepsilon, i \quad \text{in } L^2(\mathbb{R}^d), \quad \text{as } \varepsilon \longrightarrow 0+, \quad \forall i = 0, 1, \ldots, N.$$

If we apply once more the Lagrange’s mean value theorem we get that on a subsequence (ii) holds.

**Remark 4.3.** Note that for any $\zeta \in \mathcal{M}$, (i) is a backward Euler scheme for (1). Moreover, for any $\zeta \in \mathcal{M}$ we get that

$$\rho_h^\zeta(t) \longrightarrow \rho^\zeta(t) \quad \text{in } L^1(\mathbb{R}^d), \quad \text{as } h \longrightarrow 0,$$

uniformly for $t \in [0, T]$.

### 5 Relationship between problems (P) and (P_h). Optimality conditions for (P)

For any $h = \frac{T}{N}, N \in \mathbb{N}^*$, we consider $\zeta_h^* \in \mathcal{M}$ an optimal control for problem (P_h).

**Lemma 5.1.** The sequence $\{\rho_h^\zeta\}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}^d))$, uniformly for $\zeta \in \mathcal{M}$ and $\{\nabla \rho_h^\zeta\}$ is bounded in $L^2(0, T; L^2(\mathbb{R}^d))$, uniformly for $\zeta \in \mathcal{M}$, for $h > 0$ sufficiently small.

**Proof.** By (3) we obtain that for any $\zeta \in \mathcal{M}$:

$$\int_{\mathbb{R}^d} |\rho_{h+1}^{\xi,i+1}|^2 dx + h \int_{\mathbb{R}^d} \nabla \beta(\rho_{h+1}^{\xi,i+1}) \cdot \nabla \rho_{h+1}^{\xi,i+1} dx = \int_{\mathbb{R}^d} \rho_{h+1}^{\xi,i+1} \rho_{h+1}^{\xi,i+1} dx + h \int_{\mathbb{R}^d} K(\xi) b^*(\rho_{h+1}^{\xi,i+1}) \cdot \nabla \rho_{h+1}^{\xi,i+1} dx,$$

and so

$$\frac{1}{2} \int_{\mathbb{R}^d} |\rho_{h+1}^{\xi,i+1}|^2 dx + h \gamma_0 \int_{\mathbb{R}^d} |\nabla \rho_{h+1}^{\xi,i+1}|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\rho_{h+1}^{\xi,i+1}|^2 dx + h t_1 \int_{\mathbb{R}^d} |\rho_{h+1}^{\xi,i+1}| |\nabla \rho_{h+1}^{\xi,i+1}| dx.$$
where $\tilde{M}_1$ is a nonnegative constant. We immediately obtain that there exists a positive constant $M_2$ such that
\[
\frac{1}{2} \int_{\mathbb{R}^d} |\rho_h^{\xi,i+1}|^2 dx + \frac{h \gamma_0}{2} \int_{\mathbb{R}^d} |\nabla \rho_h^{\xi,i+1}|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\rho_h^{\xi,i}|^2 dx + \frac{h M_2}{2} \int_{\mathbb{R}^d} |\rho_h^{\xi,i+1}|^2 dx,
\]
and consequently for $h > 0$ sufficiently small
\[
(1 - h M_2) \int_{\mathbb{R}^d} |\rho_h^{\xi,i+1}|^2 dx + h \gamma_0 \int_{\mathbb{R}^d} |\nabla \rho_h^{\xi,i+1}|^2 dx \leq \int_{\mathbb{R}^d} |\rho_h^{\xi,i}|^2 dx.
\]
After an easy evaluation we get that
\[
(1 - h M_2)^N \|\rho_h^{\xi}(t)\|^2_{L^2} + \gamma_0 (1 - h M_2)^N \int_0^t \int_{\mathbb{R}^d} |\nabla \rho_h^{\xi}(s, x)|^2 dx ds \leq \|\rho_0\|^2_{L^2},
\]
for any $t \in [0, T]$, $\zeta \in M$. Since $h = \frac{T}{N}$ we get the conclusion of the lemma.

We postpone the proof of the following result.

**Lemma 5.2.** If for a sequence \{\(\zeta_h\)\}_{h>0} \subset M we have that \(\zeta_h \rightharpoonup \zeta \in M\) weakly * in \(L^\infty(\mathbb{R}^d)\),
\[
\mathcal{K}(\zeta_h) \rightharpoonup \mathcal{K}(\zeta) \text{ uniformly in } \mathbb{R}^d, \quad \nabla \mathcal{K}(\zeta_h)(x) \rightharpoonup \nabla \mathcal{K}(\zeta)(x), \quad \forall x \in \mathbb{R}^d,
\]
as $h \to 0$, then
\[
\rho_h^{\xi}(t) \rightharpoonup \rho^\xi(t) \text{ in } L^1(\mathbb{R}^d), \text{ uniformly for } t \in [0, T].
\]

**Theorem 5.3 (The relationship between \((P_h)\) and \((P)\)).** The following convergences hold
\[
(j) \quad \zeta_h \rightharpoonup \zeta^\ast \text{ in } L^2(\mathbb{R}^d);
(ii) \quad I_h(\zeta_h) \rightharpoonup I(\zeta^\ast);
(iii) \quad I(\zeta_h) \rightharpoonup I(\zeta^\ast), \text{ as } h \to 0.
\]

Proof. Since \{\(\zeta^\ast_h\)\} \subset M it follows that there exists a subsequence (also denoted by \{\(\zeta^\ast_h\)\}) such that \(\zeta_h \rightharpoonup \tilde{\zeta} \in M\) weakly * in \(L^\infty(\mathbb{R}^d)\) and
\[
\rho_h^{\xi_h}(t) \rightharpoonup \rho^{\tilde{\xi}}(t) \text{ in } L^1(\mathbb{R}^d), \text{ uniformly for } t \in [0, T].
\]
This yields
\[
\begin{aligned}
I_h(\zeta^\ast) &\geq I_h(\zeta^\ast_h) \geq \int_0^T \int_{\mathbb{R}^d} G(t, x) \rho_h^{\xi_h}(t, x) dx dt + \int_{\mathbb{R}^d} G_T(x) \rho_h^{\xi_h}(T, x) dx \\
&+ \int_{\mathbb{R}^d} Q(x, \tilde{\zeta}(x)) dx + \int_{\mathbb{R}^d} \frac{\partial Q}{\partial z}(x, \tilde{\zeta}(x)) (\zeta^\ast_h(x) - \tilde{\zeta}(x)) dx + \frac{1}{2} \int_{\mathbb{R}^d} |\zeta^\ast_h(x) - \zeta^\ast(x)|^2 dx.
\end{aligned}
\]
By Crandall-Liggett’s theorem we have that \(\rho_h^{\xi_h}(t) \rightharpoonup \rho^{\xi}(t)\) in \(L^1(\mathbb{R}^d)\), uniformly for \(t \in [0, T]\), and so we get that
\[
I_h(\zeta^\ast) \rightharpoonup I(\zeta^\ast),
\]
and
\[ I(\zeta^*) \geq \limsup_{h \to 0} I_h(\zeta_h^*) \geq \liminf_{h \to 0} I_h(\zeta_h^*) \geq I(\tilde{\zeta}) + \frac{1}{2} \int_{\mathbb{R}^d} |\tilde{\zeta}(x) - \zeta^*(x)|^2 dx \geq I(\zeta^*). \]

Since \( \zeta^* \) is an optimal control for problem (P) we may infer that \( \tilde{\zeta}(x) = \zeta^*(x) \) a.e. \( x \in \mathbb{R}^d \) and that on a subsequence (j) and (jj) hold.

Arguing by contradiction we get that (j) and (jj) hold in general (not only on a subsequence).

On the other hand, using that \( \rho^{\zeta^*}(t) \to \rho^{\zeta}(t) \) in \( L^1(\mathbb{R}^d) \), uniformly for \( t \in [0,T] \) (by Lemma 3.2), we get that (jjj) holds.

**Theorem 5.4 (Necessary optimality conditions for (P)).** If \( p \in W^{1,2}([0,T]; L^2(\mathbb{R}^d)) \cap L^2(0,T; H^2(\mathbb{R}^d)) \) satisfies
\[
\begin{cases}
\frac{dp}{dt}(t) = -\beta'(\rho^{\zeta^*}(t))\Delta p(t) - K(\zeta^*)(b^*)(\rho^{\zeta^*}(t)) \cdot \nabla p(t) + G(t), & t \in [0,T] \\
p(T) = -G_T,
\end{cases}
\]
then
\[
\int_{\mathbb{R}^d} \int_0^T K(\xi) \cdot b^*(\rho^{\zeta^*}) \nabla p \ dt \ dx - \int_{\mathbb{R}^d} \xi(x) \frac{\partial Q}{\partial z}(x, \zeta^*(x)) dx \leq 0,
\]
for any \( \xi \in L^\infty(\mathbb{R}^d) \) such that \( \zeta^* + \varepsilon \xi \in \mathcal{M} \), for any \( \varepsilon > 0 \) sufficiently small.

**Proof.** We shall use the results obtained for problem \( (P_h) \) in the previous section.

For the sake of clarity we shall denote by \( \zeta_h \) (instead of \( \zeta_h^* \)) an optimal control for \( (P_h) \), and assume that \( G \) is time-independent. Let
\[
p_{h}^{\zeta_h}(t, x) = \begin{cases} 
-G_T(x), & \text{if } t = T \\
p_{h}^{\zeta_h,i-1}(x), & \text{if } t \in [(i-1)h, ih), \ i = 1, 2, ..., N.
\end{cases}
\]

Multiplying (8) by \( -\Delta p_{h}^{\zeta_h,i} \) we get that
\[
\int_{\mathbb{R}^d} |\nabla p_{h}^{\zeta_h,i}|^2 dx + h \int_{\mathbb{R}^d} \beta'(p_{h}^{\zeta_h,i})|\Delta p_{h}^{\zeta_h,i}|^2 dx = \int_{\mathbb{R}^d} \nabla p_{h}^{\zeta_h,i} \cdot \nabla p_{h}^{\zeta_h,i+1} dx + h \int_{\mathbb{R}^d} G(x, \zeta_h(x)) |\Delta p_{h}^{\zeta_h,i}|^2 dx - h \int_{\mathbb{R}^d} \nabla(\zeta_h(x))(b^*)(\rho_{h}^{\zeta_h,i+1}) \cdot \nabla p_{h}^{\zeta_h,i} \Delta p_{h}^{\zeta_h,i} dx,
\]
and so there exist two positive constants \( M_1^0, M_2^0 \) such that
\[
\frac{1}{2} \int_{\mathbb{R}^d} |\nabla p_{h}^{\zeta_h,i}|^2 dx + \frac{h\gamma_0}{2} \int_{\mathbb{R}^d} |\Delta p_{h}^{\zeta_h,i}|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla p_{h}^{\zeta_h,i+1}|^2 dx + \frac{hM_1^0}{2} \int_{\mathbb{R}^d} G(x)^2 dx.
\]
It follows that
\[
(1 - hM_1^0) \int_{\mathbb{R}^d} |\nabla p_{h}^{\zeta_h,i}|^2 dx + h\gamma_0 \int_{\mathbb{R}^d} |\Delta p_{h}^{\zeta_h,i}|^2 dx \leq \int_{\mathbb{R}^d} |\nabla p_{h}^{\zeta_h,i+1}|^2 dx.
\]
After an easy evaluation we have that for $h > 0$ sufficiently small

$$(1 - hM_0^0)^N \int_{\mathbb{R}^d} |\nabla p_{h}^{\gamma,i}|^2 dx + h\gamma_0(1 - hM_1^0)^N \sum_{i=0}^{N-1} \int_{\mathbb{R}^d} |\Delta p_{h}^{\gamma,i}|^2 dx \leq \int_{\mathbb{R}^d} |\nabla G_T|^2 dx$$

$$+ hM_2^0[1 + (1 - hM_0^0) + (1 - hM_1^0)^2 + \cdots] \int_{\mathbb{R}^d} G(x)^2 dx.$$ 

After an easy evaluation we have that for $h > 0$ sufficiently small

$$+ hM_2^0[1 + (1 - hM_0^0) + (1 - hM_1^0)^2 + \cdots] \int_{\mathbb{R}^d} G(x)^2 dx.$$ 

This implies that there exists a nonnegative constant $\tilde{M}$ such that for any $t \in [0, T]$

$$\int_{\mathbb{R}^d} |\nabla p_{h}^{\gamma}(t, x)|^2 dx + \int_0^T \int_{\mathbb{R}^d} |\Delta p_{h}^{\gamma}(t, x)|^2 dx dt \leq \tilde{M}.$$ 

So, $\{\nabla p_{h}^{\gamma}\}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}^d)^d)$ and $\{\Delta p_{h}^{\gamma}\}$ is bounded in $L^2(0, T; L^2(\mathbb{R}^d))$ (for $h > 0$ sufficiently small).

If we multiply now (8) by $p_{h}^{\gamma,i}$ and integrate over $\mathbb{R}^d$ we obtain in the same manner that $\{p_{h}^{\gamma}\}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}^d))$.

We may infer that there exists a subsequence (also denoted by $\{p_{h}^{\gamma}\}$) and a function $p$ such that

$$\begin{cases} 
  p_{h}^{\gamma,i} \rightarrow p & \text{weakly in } L^2(0, T; L^2(\mathbb{R}^d)), \\
  \nabla p_{h}^{\gamma,i} \rightarrow \nabla p & \text{weakly in } L^2(0, T; L^2(\mathbb{R}^d)^d), \\
  \Delta p_{h}^{\gamma,i} \rightarrow \Delta p & \text{weakly in } L^2(0, T; L^2(\mathbb{R}^d)).
\end{cases}$$

By (8) we have that

$$p_{h}^{\gamma,i}(x) = -G_T(x) + \int_{ith}^{T} \beta'(\rho_{h}^{\gamma}(s, x)) \Delta p_{h}^{\gamma}(s, x) ds + \int_{ith}^{T} \mathcal{K}(\zeta_h)(x)(b^*)(\rho_{h}^{\gamma}(s, x)) \cdot \nabla p_{h}^{\gamma}(s, x) ds$$

$$- (T - ith)G(x) \in L^2(\mathbb{R}^d).$$

Since $\zeta_h \rightarrow \zeta^*$ a.e. $x \in \mathbb{R}^d$ (on a subsequence), we may conclude that for any $t \in [0, T]$

$$p(t) = -G_T + \int_{t}^{T} \beta'(\rho^*)(s) \Delta p(s) ds + \int_{t}^{T} \mathcal{K}(\zeta^*)(b^*)(\rho^*(s)) \cdot \nabla p(s) ds$$

$$- \int_{t}^{T} G(x) ds \in L^2(\mathbb{R}^d).$$

We may conclude that $p \in W^{1,2}([0, T]; L^2(\mathbb{R}^d) \cap L^2(0, T; H^2(\mathbb{R}^d)))$ and that $p$ satisfies (11). The uniqueness of the solution to (11) follows in a standard manner.

By (9) we have that

$$\int_{\mathbb{R}^d} \int_{0}^{T} \mathcal{K}(\xi) \cdot b^*(\rho_{h}^{\gamma}) \nabla p_{h}^{\gamma} dt dx - \int_{\mathbb{R}^d} \xi \frac{\partial Q}{\partial z}(x, \zeta_h(x)) + (\zeta_h^*(x) - \zeta^*(x)) dx \leq 0,$$
for any $\xi \in L^\infty(\mathbb{R}^d)$ such that $\zeta_h^* + \varepsilon \xi \in \mathcal{M}$, for any $\varepsilon > 0$ sufficiently small, which immediately implies (12).

**Proof of Lemma 5.2.** We have that
\[
\rho_h^{\zeta_h,i+1} - h\Delta \beta(\rho_h^{\zeta_h,i+1}) + h\nabla \cdot (\mathcal{K}(\zeta_h)b^*(\rho_h^{\zeta_h,i+1})) = \rho_h^{\zeta_h,i}
\]
and
\[
\rho_h^{\zeta,i+1} - h\Delta \beta(\rho_h^{\zeta,i+1}) + h\nabla \cdot (\mathcal{K}(\zeta_h)b^*(\rho_h^{\zeta,i+1})) = \rho_h^{\zeta,i} + h\nabla \cdot ((\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta))b^*(\rho_h^{\zeta,i+1})).
\]
Using the properties of $A^{\zeta_h}$, $A^\zeta$ we get that
\[
||\rho_h^{\zeta_h,i+1} - \rho_h^{\zeta,i+1}||_{L^1} \leq ||\rho_h^{\zeta_h,i} - \rho_h^{\zeta,i} - h\nabla \cdot ((\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta))b^*(\rho_h^{\zeta,i+1}))||_{L^1} \leq ||\rho_h^{\zeta_h,i} - \rho_h^{\zeta,i}||_{L^1} + h||\nabla \cdot (\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta))b^*(\rho_h^{\zeta,i+1})||_{L^1} + h||(\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta))(b^*)'(\rho_h^{\zeta,i+1}) \cdot \nabla \rho_h^{\zeta,i+1}||_{L^1}.
\]
By summation we obtain that for any $t \in [0,T]$: \[
\|\rho_h^\zeta(t) - \rho_h^\zeta(t)\|_{L^1} \leq ||\nabla \cdot (\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta))|| \int_0^T |(b^*)'(\rho_h^\zeta(s))|ds||_{L^1} + \tilde{M}_1 ||\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta)||_{L^2} + \tilde{M}_2 ||\nabla \cdot (\mathcal{K}(\zeta_h) - \mathcal{K}(\zeta))||_{L^2},
\]
where $\tilde{M}_1$, $\tilde{M}_2$ are positive constants. Using now Lemma 5.1 we may infer that
\[
\rho_h^\zeta(t) - \rho_h^\zeta(t) \rightarrow 0 \quad \text{in } L^1(\mathbb{R}^d), \text{ uniformly for } t \in [0,T].
\]
On the other hand
\[
\rho_h^\zeta(t) - \rho^\zeta(t) \rightarrow 0 \quad \text{in } L^1(\mathbb{R}^d), \text{ uniformly for } t \in [0,T]
\]
via Crandall-Liggett theorem. We may infer that
\[
\rho_h^\zeta(t) \rightarrow \rho^\zeta(t) \quad \text{in } L^1(\mathbb{R}^d), \text{ uniformly for } t \in [0,T].
\]

**Remark 5.1.** By (12) we get that for almost any $x \in B(0; R_0)$ we have that
\[
\zeta^*(x) = \begin{cases}
0, & \text{if } \int_{\mathbb{R}^d} \nabla G_R(x' - x) \cdot \int_0^T [b^*(\rho^\zeta^*)\nabla p](t, x') dt dx' + \frac{\partial Q}{\partial z}(x, \zeta^*(x)) > 0 \\
\tilde{M}_0, & \text{if } \int_{\mathbb{R}^d} \nabla G_R(x' - x) \cdot \int_0^T [b^*(\rho^\zeta^*)\nabla p](t, x') dt dx' + \frac{\partial Q}{\partial z}(x, \zeta^*(x)) < 0.
\end{cases}
\]
6 Final comments

Problem (P) has an obvious meaning if we consider $G$ and $G_T$ to be approximations of the characteristic functions of two compact subsets $K \subset [0, T] \times \mathbb{R}^d$ and $K_T \subset \mathbb{R}^d$.

Remark 6.1. The results in the present paper can be easily extended to the case of

$$\zeta \in \mathcal{M}_0 = \{ \theta \in L^\infty(\mathbb{R}^d); \ 0 \leq \theta(x) \leq \tilde{M}_0 \text{ a.e. } x \in \mathbb{R}^d, \ \theta(x) = 0 \text{ a.e. } x \in Ext(D) \},$$

where $D$ is a bounded and open subset of $\mathbb{R}^d$.

Remark 6.2. Due to Lemma 5.1 it follows that the results in the previous sections 3-5 hold if we consider for $G$ the weaker hypotheses

$$G \in L^2((0, T) \times \mathbb{R}^d), \ 0 \leq G(t, x) \text{ a.e. } (t, x) \in (0, T) \times \mathbb{R}^d.$$ 

Let us briefly discuss the situation when $Q \equiv 0$ (i.e. the cost of the control $\zeta$ is negligible). Problem (P) becomes

$$(\hat{P}) \quad \text{Minimize} \quad \int_0^T \int_{\mathbb{R}^d} G(t, x)\rho^\zeta(t, x)dx \ dt + \int_{\mathbb{R}^d} G_T(x)\rho^\zeta(T, x)dx,$$

where $\rho^\zeta$ is the weak solution to the following nonlinear Fokker-Planck equation (1). For any $h = \frac{T}{N}$ (where $N \in \mathbb{N}^*$) sufficiently small we consider the following approximating optimal control problem

$$(\hat{P}_h) \quad \text{Minimize} \quad \int_0^T \int_{\mathbb{R}^d} G(t, x)\rho^\zeta_h(t, x)dx \ dt + \int_{\mathbb{R}^d} G_T(x)\rho^\zeta_h(T, x)dx,$$

where $\rho^\zeta_h$ and $\rho^\zeta_{h^t}$ were defined in section 2.

Since problem (P) is a particular case of problem (P), it follows that there exists at least one optimal control for (P). The existence of an optimal control $\tilde{\zeta}_h$ for problem ($\hat{P}_h$) is proven in the same way as for problem ($P_h$).

The following results follow in an analogous manner to Theorem 5.3 and Theorem 4.5.

Theorem 6.1 (The relationship between ($\hat{P}_h$) and (P)). If $\tilde{\zeta}^*$ is a weak * accumulation point in $L^\infty(\mathbb{R}^d)$ for $\{\tilde{\zeta}_h^*\}$ (and denote the convergent subsequence by $\{\tilde{\zeta}_h^*\}$ as well), then

(l) $\tilde{\zeta}^*$ is an optimal control for problem (P);

(ll) $\tilde{I}_h(\tilde{\zeta}_h^*) \rightarrow \tilde{I}(\tilde{\zeta}^*)$;

(lll) $\tilde{I}(\tilde{\zeta}_h^*) \rightarrow \tilde{I}(\tilde{\zeta}^*)$, as $h \rightarrow 0$.

Here $\tilde{I}$ and $\tilde{I}_h$ are the cost functionals for problems (P) and ($\hat{P}_h$), respectively.

If we replace the assumption $G_T \in C_b(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ by the weaker one

$$G_T \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$
then the next result holds

**Theorem 6.2.** If \( p_{\tilde{h}_{\xi}^{s,1+1}} \) is the unique weak solution to (5) corresponding to \( \zeta := \tilde{h}_{\xi}^{s} \), then

\[
\int_{\mathbb{R}^d} K(\xi) \cdot \left[ \sum_{i=1}^{N} [h^*(\rho_{\xi}^{s,1,1} \nabla p_{\xi}^{s,1,1}] \right] dx \leq 0,
\]

for any \( \xi \in L^\infty(\mathbb{R}^d) \) such that \( \tilde{h}_{\xi}^{s} + \varepsilon \xi \in \mathcal{M} \), for any \( \varepsilon > 0 \) sufficiently small.

**Remark 6.3.** We can adapt the proofs in the previous sections to investigate the more general optimal control problem

\[
(P_0) \quad \text{Minimize} \quad \int_{0}^{T} \int_{\mathbb{R}^d} G(t, x, \rho^c(t, x)) dx \ dt + \int_{\mathbb{R}^d} G_T(x, \rho^c(T, x)) dx + \int_{\mathbb{R}^d} Q(x, \zeta(x)) dx,
\]

where \( \rho^c \) is the weak solution to (1) if appropriate assumptions on \( G, G_T \) are considered.

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