LIMITS OF MODIFIED HIGHER $q,t$-CATALAN NUMBERS

KYUNGYONG LEE, LI LI, AND NICHOLAS A. LOEHR

Abstract. The $q,t$-Catalan numbers can be defined using rational functions, geometry related to Hilbert schemes, symmetric functions, representation theory, Dyck paths, partition statistics, or Dyck words. After decades of intensive study, it was eventually proved that all these definitions are equivalent. In this paper, we study the similar situation for higher $q,t$-Catalan numbers. We compute the limits of several versions of the modified higher $q,t$-Catalan numbers and show that these limits equal the generating function for integer partitions. This result supports the conjecture that the combinatorial definitions of higher $q,t$-Catalan numbers are indeed equivalent to the algebraic and geometric definitions.

1. Introduction

The purpose of this paper is to study the different definitions of higher $q,t$-Catalan numbers and provide support to the conjecture that they are indeed equivalent. We begin by reviewing seven ways of defining the higher $q,t$-Catalan numbers in (a)–(g) below. We then indicate which of these definitions are already known to be equivalent.

(a) Suppose $\lambda$ is an integer partition with Ferrers diagram $\text{dg}(\lambda)$. Define $\text{area}(\lambda) = |\lambda| = |\text{dg}(\lambda)|$, the number of cells in the diagram of $\lambda$. For a cell $x \in \text{dg}(\lambda)$, define the leg $l(x)$, the arm $a(x)$, the coleg $l'(x)$, and the coarm $a'(x)$ to be the distances shown in Figure 1. Let $\text{Par}^{(m)}_n$ be the set of partitions $\lambda$ such that $\text{dg}(\lambda)$ fits in the triangle with vertices $(0,0)$, $(mn,n)$, and $(0,n)$.
(0, n), and (mn, n) when drawn as shown in the figure. For such partitions, define \( \text{area}^c(\lambda) = m \binom{n}{2} - \text{area}(\lambda) \) and

\[
c_m(\lambda) = |\{ x \in \text{dg}(\lambda) : ml(x) \leq a(x) \leq ml(x) + m \}|.
\]

For example, when \( m = 2 \) and \( \lambda = (7, 5, 4) \), we have \( |\lambda| = 16 \) and \( c_2(\lambda) = 13 \).

Define the partition version of the higher \( q, t \)-Catalan numbers by

\[
PC_n^{(m)}(q, t) = \sum_{\lambda \in \text{Par}^{(m)}} q^{\text{area}^c(\lambda)} t^{c_m(\lambda)}.
\]

For example, \( PC_2^{(3)}(q, t) = q^3 + qt^2 + t^3 \) and

\[
PC_2^{(3)}(q, t) = q^6 + q^5t + q^4t^2 + q^4t + q^3t^3 + q^3t^2 + q^2t^2 + q^2t^3 + qt^4 + q^2t^4 + qt^5 + t^6.
\]

(b) An \( m \)-Dyck word is a sequence \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \) such that \( \gamma_i \in \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \gamma_0 = 0 \), and \( \gamma_{i+1} \leq \gamma_i + m \) for \( 0 \leq i < n - 1 \). Let \( \Gamma_n^{(m)} \) be the set of \( m \)-Dyck words of length \( n \). For such a word \( \gamma \), define \( \text{area}(\gamma) = \sum_{i=0}^{n-1} \gamma_i \). As in [12], define \( \text{dinv}_m(\gamma) = \sum_{0 \leq i < j \leq n} s_{cm}(\gamma_i - \gamma_j) \), where

\[
s_{cm}(p) = \begin{cases} 
m + 1 - p, & \text{if } 1 \leq p \leq m; \\
m + p, & \text{if } -m \leq p \leq 0; \\
0, & \text{for all other } p.
\end{cases}
\]

For example, \( \gamma = (0, 2, 0, 1, 1) \in \Gamma_5^{(2)} \) has \( \text{area}(\gamma) = 4 \) and \( \text{dinv}_2(\gamma) = 13 \).

Define the word version of the higher \( q, t \)-Catalan numbers by

\[
WC_n^{(m)}(q, t) = \sum_{\gamma \in \Gamma_n^{(m)}} q^{\text{area}(\gamma)} t^{\text{dinv}_m(\gamma)}.
\]

(c) An \( m \)-Dyck path of order \( n \) is a lattice path \( \pi \) from \((0, 0)\) to \((mn, n)\) using north and east steps such that the path never goes below the diagonal line segment with endpoints \((0, 0)\) and \((mn, n)\). Let \( D_n^{(m)} \) be the set of such \( m \)-Dyck paths. For such a path \( \pi \), let \( \text{area}(\pi) \) be the number of complete unit squares between \( \pi \) and the diagonal. Define the \( m \)-bounce statistic \( b_m(\pi) \) as follows. Set \( v_i = 0 \) for all negative integers \( i \). Starting from \((0, 0)\), construct a bounce path by induction on \( i \geq 0 \). In the \((i+1)\)th step, move north from the current position \((u, v)\) until hitting an east step of the \( m \)-Dyck path that starts on the line \( x = u \), and define the distance traveled to be \( v_i \). Then move east from this position \( v_i + v_{i-1} + \cdots + v_{i-m+1} \) units. Continue bouncing until reaching \((mn, n)\). (In fact, it suffices to stop once we reach the horizontal line \( y = n \).) Then \( b_m(\pi) = \sum_{k \geq 0} k v_k \). For example, the path \( \pi \in D_5^{(2)} \) in Figure 2 has \( \text{area}(\pi) = 4 \), \((v_0, v_1, \ldots, v_5) = (2, 0, 1, 1, 1, 0)\), and \( b_2(\pi) = 9 \).

Define the Dyck path version of the higher \( q, t \)-Catalan numbers by

\[
DC_n^{(m)}(q, t) = \sum_{\pi \in D_n^{(m)}} q^{b_m(\pi)} t^{\text{area}(\pi)}.
\]
(d) The $q,t$-Catalan numbers may be defined using symmetric functions, as follows. This discussion assumes the reader is familiar with the elementary symmetric functions $e_n$, the modified Macdonald polynomials $\tilde{H}_\mu$, and the Hall scalar product $\langle \cdot, \cdot \rangle$ on symmetric functions; see [1] or [3, §3.5.5] for details. For any integer partition $\mu$, define $n(\mu) = \sum_{x \in \text{dg}(\mu)} l(x)$ and $n(\mu') = \sum_{x \in \text{dg}(\mu')} l(x) = \sum_{x \in \text{dg}(\mu)} a(x)$, where $\mu'$ denotes the transpose of $\mu$. Define $T_\mu = q^{n(\mu')} t^{n(\mu)}$. Let $\Lambda$ denote the ring of symmetric functions with coefficients in the field $F = \mathbb{Q}(q,t)$. The Bergeron-Garsia nabla operator [1] is the unique $F$-linear map $\nabla$ on $\Lambda$ that acts on the modified Macdonald basis via $\nabla(\tilde{H}_\mu) = T_\mu \tilde{H}_\mu$ for all partitions $\mu$. For $m \in \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $\nabla^m$ denotes the composition of $m$ copies of the operator $\nabla$. We now define the symmetric function version of the higher $q,t$-Catalan numbers by

$$SC_n^{(m)}(q,t) = \langle \nabla^m(e_n), e_n \rangle.$$  

(e) The higher $q,t$-Catalan numbers were originally defined by Garsia and Haiman in [3] as sums of rational functions in $\mathbb{Q}(q,t)$ constructed from integer partitions. Recall that $\mu \vdash n$ means that $\mu$ is an integer partition of $n$. With $T_\mu$ defined as in (d), we further define

$$B_\mu = \sum_{x \in \text{dg}(\mu)} q^{a(x)} t^{l(x)}, \quad \Pi_\mu = \prod_{x \in \text{dg}(\mu), \{(0,0)\}} (1 - q^{a(x)} t^{l(x)}),$$  

$$w_\mu = \prod_{x \in \text{dg}(\mu)} [(q^{a(x)} - t^{l(x)+1}) (t^{l(x)} - q^{a(x)+1})].$$

Then the rational function version of the higher $q,t$-Catalan numbers is defined by

$$RC_n^{(m)}(q,t) = \sum_{\mu \vdash n} (1 - q)(1 - t) T_{\mu}^{m+1} B_\mu \Pi_\mu / w_\mu.$$  

(f) For fixed $n \in \mathbb{N}^+$, consider the polynomial ring $\mathbb{C}[x,y] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$. $S_n$ acts diagonally on this ring by the rule $w \cdot x_i = x_{w(i)}$, $w \cdot y_i = y_{w(i)}$ for $w \in S_n$ and $1 \leq i \leq n$. A polynomial $f \in \mathbb{C}[x,y]$ is called alternating iff $w \cdot f = \text{sgn}(w)f$ for all $w \in S_n$. Let $I$ be the ideal in $\mathbb{C}[x,y]$ generated by all alternating polynomials, and let $m$ be the maximal ideal generated by $x_1, y_1, \ldots, x_n, y_n$. We write $I = I_n$ and $m = m_n$ if it is necessary to indicate the number of variables. Let $M^{(m)} = I^m / mf^m$ for $m \in \mathbb{N}$, and for simplicity, let $M = M^{(1)}$. Given a monomial $f = x_1^{d_1} y_1^{b_1} \cdots x_n^{d_n} y_n^{b_n} \in \mathbb{C}[x,y]$, we define the bidegree of $f$ to be the ordered pair $(\sum_{i=1}^n d_i, \sum_{i=1}^n b_i)$. We say that a polynomial in $\mathbb{C}[x,y]$ is bihomogeneous of bidegree $(d_1, d_2)$ if all its monomials have the same bidegree $(d_1, d_2)$. Then $I^m$ and $M^{(m)}$ become doubly-graded $S_n$-modules by taking bidegrees in the $x$-variables and
Upon setting \( PC_{n}^{(m)}(1,1) \) and \( SC_{n}^{(m)}(1,1) \), we see from \([3, \text{Corollary 4.1}]\) and \([12, \S 3.3]\) that
\[
RC_{n}^{(m)}(q,1) = RC_{n}^{(m)}(1,q) = \sum_{\pi \in D_{n}^{(m)}} q^{\text{area}(\pi)} = PC_{n}^{(m)}(q,1) = DC_{n}^{(m)}(1,q).
\]

Upon setting \( t = 1/q \), we see from \([3, \text{Corollary 4.1}]\) and \([12, \S 3.3]\) that
\[
RC_{n}^{(m)}(q,1/q)q^{m(2)}(q) = WC_{n}^{(m)}(q,1/q)q^{m(2)}(q) = DC_{n}^{(m)}(q,1/q)q^{m(2)}(q) = \frac{1}{[mn+1]_{q}} \left[ mn + n \atop n \right]_{q}.
\]

This paper provides more evidence of the conjectured equality of the seven definitions of \( q, t \)-Catalan numbers. Specifically, we will study the limiting behavior, as \( n \) tends to infinity, of the “modified” \( q, t \)-Catalan numbers given by
\[
q^{m(2)} PC_{n}^{(m)}(q^{-1}, t), \quad q^{m(2)} DC_{n}^{(m)}(t, q^{-1}), \text{ and } q^{m(2)} AC_{n}^{(m)}(q^{-1}, t).
\]

We will show that all of these polynomials have as their limit the famous generating function \( \prod_{i=1}^{\infty} (1 - tq^{i})^{-1} \), which enumerates integer partitions by area and number of parts. (Here we are taking limits in a formal power series ring, which means that for each fixed monomial \( q^{a}t^{b} \), the coefficient of this monomial becomes stable for sufficiently large \( n \).) We state our main theorem more precisely as follows.
Theorem 1.1. For any positive integer \( m \), we have
\[
\lim_{n \to \infty} q^{m(n)} PC_n^{(m)}(q^{-1}, t) = \lim_{n \to \infty} q^{m(n)} DC_n^{(m)}(t, q^{-1}) = \lim_{n \to \infty} q^{m(n)} AC_n^{(m)}(q^{-1}, t)
\]
\[
= \prod_{i=1}^{\infty} (1 - tq^i)^{-1} = \sum_{\mu \in \text{Par}} q^{\text{area}(\mu)} \ell(\mu),
\]
where \( \text{Par} \) is the set of all integer partitions, and \( \ell(\mu) \) is the number of parts of \( \mu \).

The paper is organized as follows. In §2 we introduce further notation, background, and preliminary results. In §3 we prove the combinatorial part of our main theorem. In §4 we prove the algebraic part of the main theorem. In §5, we extend the method used in the proof of the main theorem to identify certain coefficients of \( AC_n^{(m)}(q, t) \) as partition numbers. In §6 we give some related conjectures. In §7 we indicate the proofs of the equalities stated in (1.1) and (1.2).

2. Background and Notation

2.1. Notation.

- For \( k, b \in \mathbb{N}^+ \), denote the set of integer partitions of \( k \) by \( \text{Par}(k) \), and denote the set of integer partitions of \( k \) into at most \( b \) parts by \( \text{Par}(b, k) \). More explicitly, \( \text{Par}(k) = \{ \nu = (\nu_1, \nu_2, \ldots, \nu_\ell) | \nu_i \in \mathbb{N}^+, \nu_1 \leq \nu_2 \leq \cdots \leq \nu_\ell, \nu_1 + \nu_2 + \cdots + \nu_\ell = k \} \) and \( \text{Par}(b, k) = \{ \nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \text{Par}(k) | \ell \leq b \} \). By convention, \( \text{Par}(0) = \{0\} \), \( \text{Par}(0, k) = \emptyset \) for \( k > 0 \), and \( \text{Par}(h, 0) = \{0\} \) for all \( h \geq 0 \) (where \( \{0\} \) is a set with one element). Let \( p(k) \) and \( p(b, k) \) be the cardinalities of \( \text{Par}(k) \) and \( \text{Par}(b, k) \), respectively. In other words, \( p(k) \) is the number of partitions of \( k \) and \( p(h, k) \) is the number of partitions of \( k \) into at most \( h \) parts. By the above conventions, \( p(0) = 1 \), \( p(0, k) = 0 \) for \( k > 0 \), and \( p(h, 0) = 1 \) for all \( h \geq 0 \).

- Let \( \mathbb{C}[\nu] = \mathbb{C}[\nu_1, \nu_2, \ldots] \) be the polynomial ring with countably many variables \( \nu_i \), for \( i \in \mathbb{N}^+ \). As a convention, we set \( \nu_0 = 1 \). For a partition \( \nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \text{Par}(k) \), define \( \nu_w = \nu_{i_1} \nu_{i_2} \cdots \nu_{i_\ell} \in \mathbb{C}[\nu] \). Define the weight of a monomial \( c \nu_{i_1} \cdots \nu_{i_\ell} \) (where \( c \in \mathbb{C} \setminus \{0\} \)) to be \( i_1 + \cdots + i_\ell \). For \( w \in \mathbb{N} \), define \( \mathbb{C}[^w] \) to be the subspace of \( \mathbb{C}[\nu] \) spanned by monomials of weight \( w \). For \( f \in \mathbb{C}[\nu] \), there is a unique expression \( f = \sum_{w=0}^{\infty} \{f\}_w \) with \( \{f\}_w \in \mathbb{C}[^w] \), and we call \( \{f\}_w \) the weight-\( w \)-part of \( f \).

- For \( P = (a, b) \in \mathbb{N} \times \mathbb{N} \), we write \( |P| = a + b \), \( |P|_x = a \), and \( |P|_y = b \).

- For \( n \in \mathbb{N}^+ \), define \( \mathcal{D}_n = \{D \subset \mathbb{N} \times \mathbb{N} : |D| = n\} \).

For \( D \in \mathcal{D}_n \), we write \( D = \{P_1, P_2, \ldots, P_n\} \) where each \( P_i = (a_i, b_i) \in \mathbb{N} \times \mathbb{N} \). Unless otherwise specified, we always choose notation so that \( P_1, \ldots, P_n \) are in increasing graded lexicographic order. This means that \( P_1 < P_2 < \cdots < P_n \), where
\[
(a, b) < (a', b') \text{ if } a + b < a' + b', \text{ or if } a + b = a' + b' \text{ and } a < a'.
\]
To visualize a set \( D \in \mathcal{D}_n \), we can draw a square grid on which we plot the \( n \) ordered pairs in \( D \). For example, in the following picture, the horizontal and vertical bold lines represent the \( x \)-axis and \( y \)-axis, and \( D = \{(0, 0), (1, 0), (1, 1), (2, 0), (3, 0)\} \).
Given $D = \{P_1, \ldots, P_n\} \in \mathfrak{D}_n$, define the total degree, $x$-degree, $y$-degree, and bidegree of $D$ to be $\sum_{i=1}^n (|P_1|_x + |P_1|_y)$, $\sum_{i=1}^n |P_i|_x$, $\sum_{i=1}^n |P_i|_y$, and $(\sum_{i=1}^n |P_i|_x, \sum_{i=1}^n |P_i|_y)$, respectively. Then the $x$-degree (resp. $y$-degree) of $D$ will be denoted by $d_1(D)$ (resp. $d_2(D)$). Let $k(D) = (\binom{n}{2} - d_1(D) - d_2(D)$.

- The diagonal ideal $I$ of $\mathbb{C}[x,y]$ and the bigraded $\mathbb{C}$-vector space $M = \bigoplus_{d_1,d_2 \in \mathbb{N}} M_{d_1,d_2}$ were defined in §1(f). The ideal generated by all homogeneous elements in $I$ of total degree less than $d$ is denoted by $I_{<d}$.

- For $D = \{(a_1, b_1), \ldots, (a_n, b_n)\} \in \mathfrak{D}_n$, the alternating polynomial $\Delta(D) \in \mathbb{C}[x,y]$ is defined by

$$
\Delta(D) = \det[x_i^{a_j} y_i^{b_j}]_{1 \leq i, j \leq n} = \det \begin{bmatrix}
  x_1^{a_1} y_1^{b_1} & x_1^{a_2} y_1^{b_2} & \cdots & x_1^{a_n} y_1^{b_n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_n^{a_1} y_n^{b_1} & x_n^{a_2} y_n^{b_2} & \cdots & x_n^{a_n} y_n^{b_n}
\end{bmatrix}.
$$

Note that $\Delta(D)$ is bihomogeneous of bidegree equal to the bidegree of $D$.

- Given two polynomials $f, g \in I^m$ of the same bidegree $(d_1, d_2)$, let $\bar{f}, \bar{g}$ be the corresponding elements in $M^{(m)}_{d_1,d_2}$. For $m = 1$, we say that

$$
\bar{f} \equiv \bar{g} \pmod{\text{lower degrees}}
$$

if $\bar{f} = \bar{g}$ in $M_{d_1,d_2}$, or, equivalently, if $f - g$ is in $I_{<d_1+d_2}$.

### 2.2. Properties of the module $M_{d_1,d_2}$

To be self-contained, we review a few definitions given in [11]. The reader is suggested to look at Example 2.4 while reading these definitions.

**Definition-Proposition 2.1** ([11] Definition-Proposition 6). Let $D = \{P_1, \ldots, P_n\} \in \mathfrak{D}_n$, and write $P_i = (a_i, b_i)$. Then there is an $n \times n$ matrix $S$ whose $(i,j)$-entry is

$$
\begin{cases}
  0, & \text{if } i \leq |P_j|; \\
  z_{i_1} z_{i_2} \cdots z_{i_{|P_j|}}, & \text{where } z_{i_\ell} \text{ is either } x_i - x_\ell \text{ or } y_i - y_\ell, \text{ otherwise,}
\end{cases}
$$

for all $1 \leq i, j \leq n$, such that $\det(S) \equiv \Delta(D)$ (modulo lower degrees). We call $S$ a staircase form of $D$.

**Definition 2.2.** Let $D$ and $S$ be defined as in Definition-Proposition 2.1. Consider the set $\{j : |P_j| = j - 1\} = \{r_1 < r_2 < \cdots < r_\ell\}$ and define $r_{\ell+1} = n + 1$. For $1 \leq t \leq \ell$, define the $t$-th block $B_t$ of $S$ to be the square submatrix of $S$ of size $(r_{t+1} - r_t)$ whose upper-left corner is the $(r_t, r_t)$-entry. Define the block diagonal form $B(S)$ of $S$ to be the block diagonal matrix $\text{diag}(B_1, \ldots, B_\ell)$.

**Definition 2.3.** Let $S$ be a staircase form, $B(S)$ be its block diagonal form with blocks $B_1, \ldots, B_\ell$. For $1 \leq t \leq \ell$, let $\mu_t$ be the number of nonzero entries in block $B_t$ that are strictly above the diagonal, i.e., the number of nonzero $i,j$-entries in $B_t$ where $j > i$. Eliminating zeros in $(\mu_1, \ldots, \mu_\ell)$ and then rearranging the sequence in ascending order, we obtain a partition of $k$, denoted by $\mu(S)$. We say that $S$ is of partition type $\mu(S)$. We call a block $B_t$ minimal if every $(i,j)$-entry $(j > i + 1)$ that lies in $B_t$ is zero. We call $S$ a minimal staircase
form if all the blocks in $B(S)$ are minimal. We say $D \in \mathfrak{D}_n$ is of partition type $\mu(S)$ if $S$ is a staircase form of $D$. (Note that the partition type does not depend on the choice of $S$.)

**Example 2.4.** (i) Let $D = \{(0, 0), (0, 1), (0, 2), (1, 1)\} \in \mathfrak{D}_4$. We list here $\Delta(D)$ and a possible staircase form $S$ together with the corresponding block diagonal forms $B(S)$. In this example, $D$ is of partition type (1), $S$ is a minimal staircase form, and $B(S)$ has two blocks of size 1 and one block of size 2.

\[
\Delta(D) = \begin{bmatrix}
1 & y_1 & y_1^2 & x_1 y_1 \\
1 & y_2 & y_2^2 & x_2 y_2 \\
1 & y_3 & y_3^2 & x_3 y_3 \\
1 & y_4 & y_4^2 & x_4 y_4
\end{bmatrix},
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & y_2 & 0 & 0 \\
1 & y_3 & y_3 y_2 & x_3 y_3 \\
1 & y_4 & y_4 y_2 & x_4 y_4
\end{bmatrix},
B(S) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & y_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & y_4 & y_4 y_2
\end{bmatrix},
\]

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$.

(ii) Let $D = \{(0, 0), (0, 1), (1, 0), (1, 2), (2, 1), (3, 0)\} \in \mathfrak{D}_6$. A staircase form $S$ and the corresponding block diagonal form $B(S)$ are given below. Then $D$ is of partition type (1, 3), $B(S)$ has three blocks of sizes 1, 2, 3 respectively, and $S$ is not a minimal staircase form (because the (4, 6)-entry of $B(S)$ is $x_{41} x_{42} x_{43} \neq 0$, therefore the block $B_3$ is not minimal).

\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & y_2 & 0 & 0 & 0 & 0 \\
1 & y_3 & 0 & 0 & 0 & 0 \\
1 & y_4 & y_4 x_{41} & y_4 x_{42} & y_4 x_{43} & 0 \\
1 & y_5 & y_5 x_{51} & y_5 x_{52} & y_5 x_{53} & y_5 x_{54} \\
1 & y_6 & y_6 x_{61} & y_6 x_{62} x_{63} & y_6 x_{62} x_{63} & y_6 x_{62} x_{63}
\end{bmatrix},
B(S) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & y_2 & 0 & 0 & 0 & 0 \\
0 & 0 & y_3 & 0 & 0 & 0 \\
0 & 0 & 0 & y_4 & y_4 x_{43} & y_4 x_{42} x_{43} \\
0 & 0 & 0 & 0 & y_5 & y_5 x_{53} \\
0 & 0 & 0 & 0 & 0 & y_6 x_{62} x_{63}
\end{bmatrix},
\]

**Theorem 2.5** ([11] Theorem 5]). Let $n$ be a positive integer, and let $d_1, d_2, k$ be non-negative integers such that $k = \binom{n}{2} - d_1 - d_2$. Define $\delta = \min(d_1, d_2)$. Then $\dim M_{d_1, d_2} \leq p(\delta, k)$, and equality holds if and only if $k \leq n - 3$, or $k = n - 2$ and $\delta = 1$, or $\delta = 0$.

Recall some definitions in [11]. For $b \in \mathbb{N}$ and $w \in \mathbb{Z}$, define

\[
h(b, w) = \left\{(1 + \rho_1 + \rho_2 + \cdots)^b\right\}_w.
\]

For $D = \{P_1, \ldots, P_n\} \in \mathfrak{D}_n$, define $\varphi(D)$ to be

\[
(-1)^{k(D)} \det \begin{bmatrix}
h(b_1, -|P_1|) & h(b_1, 1 - |P_1|) & h(b_1, 2 - |P_1|) & \cdots & h(b_1, n - 1 - |P_1|) \\
h(b_2, -|P_2|) & h(b_2, 1 - |P_2|) & h(b_2, 2 - |P_2|) & \cdots & h(b_2, n - 1 - |P_2|) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h(b_n, -|P_n|) & h(b_n, 1 - |P_n|) & h(b_n, 2 - |P_n|) & \cdots & h(b_n, n - 1 - |P_n|)
\end{bmatrix}.
\]

It is proved in [11] Lemma 47 that $\varphi$ induces a well-defined linear map $\bar{\varphi} : M_{d_1, d_2} \rightarrow \mathbb{C}[\rho]^\binom{n}{2} - d_1 - d_2$. We have conjectured that $\bar{\varphi}$ is injective and proved the injectivity under the condition $\binom{n}{2} - d_1 - d_2 \leq n - 3$ and $d_2 \leq d_1$ [11] Conjecture 48, Theorem 43, Theorem 44]. With a slight modification, we can prove the injectivity under the sole condition $\binom{n}{2} - d_1 - d_2 \leq n - 3$ without the constraint $d_2 \leq d_1$. (We briefly explain the modification using the terminology in [11]: assume now $d_2 > d_1$. It suffices to prove that, for each $\nu \in \Pi_{d_1, k}$, there exists an alternating polynomial $g_\nu$ such that the leading monomial $\text{LM}(\varphi(g_\nu)) = \rho_\nu$. In fact, such a $g_\nu$ can be obtained, up to a sign, by switching $x$- and $y$-coordinates of the $f_\nu$ constructed for $M_{d_2, d_1}$ in [11] Theorem 44].)
Lemma 2.6. Suppose \( 0 \leq \binom{n-1}{2} - d'_1 - d'_2 \leq n - 4, \) \( d'_1 \leq d_1, d'_2 \leq d_2, \) and \( d'_1 + d'_2 + (n - 1) = d_1 + d_2. \) Let \( M'_{d'_1,d'_2} \) and \( M_{d_1,d_2} \) be the indicated bigraded components of \( I_{n-1}/m_{n-1}I_{n-1} \) and \( I_n/m_nI_n, \) respectively. Let

\[
 f_0 = \prod_{i=1}^{d_1 - d'_1} (x_n - x_i) \cdot \prod_{i=d_1 - d'_1 + 1}^{n-1} (y_n - y_i).
\]

Then the linear map \( h : M'_{d'_1,d'_2} \to M_{d_1,d_2} \) that maps \( \bar{f} \) to \( f_0 \bar{f} \) is injective.

Proof. First observe that \( \binom{n-1}{2} - d'_1 - d'_2 = \binom{n}{2} - d_1 - d_2, \) which we denote by \( k. \) It is also easy to check that \( h \) is well-defined.

We now explain that the following triangle is commutative:

\[
 \begin{array}{ccc}
 M'_{d'_1,d'_2} & \xrightarrow{h} & M_{d_1,d_2} \\
 \downarrow{\varphi'} & & \downarrow{\varphi} \\
 \mathbb{C}[\rho]_k & & \\
 \end{array}
\]

i.e., \( \varphi'(\bar{f}) \) is identical with \( \varphi(f_0 \bar{f}). \) Indeed, for \( D' \in \mathcal{D}_{n-1} \) of bidegree \( (d_1, d_2), \) let \( f = \Delta(D'), \) then \( f_0 f = \Delta(D) \) for \( D = D' \cup \{(d_1 - d'_1, n - 1 - d_1 + d'_1)\} \in \mathcal{D}_n. \) Then \( k(D') = k(D) = k. \) Let \( A' \) (resp. \( A \)) be the \( (n-1) \times (n-1) \) matrix (resp. \( n \times n \) matrix) in the definition of \( \varphi'(f) \) (resp. \( \varphi(f_0 \bar{f}) \)). Since \( A' \) is the first \( (n-1) \times (n-1) \) minor of \( A \) and the last row of \( A \) is \( (0, \ldots, 0, 1), \) \( \det(A') = \det(A). \) Therefore \( \varphi'(\bar{f}) = (-1)^k \det(A') = (-1)^k \det(A) = \varphi(f_0 \bar{f}). \)

Now since \( \varphi' : M'_{d'_1,d'_2} \to \mathbb{C}[\rho]_k \) is injective for \( k \leq (n-1) - 3, \) \( h \) is also injective. \( \square \)

Definition 2.7. Given \( d_1 + d_2 = \binom{n}{2}, \) take arbitrary \( D = \{P_1, \ldots, P_n\} \in \mathcal{D}_n \) of bidegree \( (d_1, d_2) \) such that \( \lvert P_i \rvert = i - 1. \) Define \( f_{d_1,d_2} \) to be the equivalence class of \( \Delta(D) \) in \( M_{d_1,d_2}. \) By [11, Lemma 16], this equivalence class is independent of the choice of \( D. \)

3. Limits of the Modified Combinatorial Higher \( q, t \)-Catalan Numbers

We recall the following theorem [13, Thm. 3].

Theorem 3.1. For \( \lambda \in \text{Par} \) and \( m \in \mathbb{R}^+, \) define \( h^+_m(\lambda) \) to be the number of cells \( x \in \text{dg}(\lambda) \) such that \( \frac{a(x)}{a(x)+1} \leq m < \frac{a(x)+1}{a(x)}. \) Then

\[
 \sum_{\lambda \in \text{Par}} q^{\text{area}(\lambda)} t^{h^+_m(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{1 - tq^i} = \sum_{\mu \in \text{Par}} q^{\text{area}(\mu)} t^{\ell(\mu)}.
\]

Proposition 3.2. (i) For all \( m, n \in \mathbb{N}^+, \)

\[
 q^m \binom{n}{2} PC^{(m)}_n(q^{-1}, t) = \sum_{\lambda \in \text{Par}^+(m)} q^{\text{area}(\lambda)} t^{e_m(\lambda)}.
\]
Proposition 3.3. (i) For sufficiently large \( n \),

\[
\lim_{n \to \infty} q^{m(n)} PC_n^{(m)}(q^{-1}, t) = \sum_{\lambda \in \text{Par}} q^{\text{area}(\lambda)} t^{c_m(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{1 - t q^i}.
\]

Proof. (i) is straightforward. For (ii), if we increase \( n \) by 1, a partition \( \lambda \) in the \( mn \times n \) triangle will also fit into the \( m(n + 1) \times (n + 1) \) triangle, and the two statistics area(\( \lambda \)) and \( c_m(\lambda) \) do not change with \( n \). Since all integer partitions of a fixed area will fit in the triangle for sufficiently large \( n \), the first equality follows from (i). The second equality follows from Theorem \( 3.1 \) and the observation that \( h_m^+(\lambda) = c_m(\lambda) \).

Proposition 3.3. (i) For all \( m, n \in \mathbb{N}^+ \),

\[
q^{m(n)} DC_n^{(m)}(t, q^{-1}) = \sum_{\pi \in D_n^{(m)}} q^{\text{area}^c(\pi)} b_m(\pi),
\]

where \( \text{area}^c(\pi) \) is the number of lattice squares in the \( mn \times n \) triangle above \( \pi \).

(ii) For all \( m \in \mathbb{N}^+ \),

\[
\lim_{n \to \infty} q^{m(n)} DC_n^{(m)}(t, q^{-1}) = \sum_{\lambda \in \text{Par}} q^{\text{area}(\lambda)} t^{\ell(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{1 - t q^i}.
\]

Proof. (i) is straightforward. For (ii), note that each \( m \)-Dyck path \( \pi \in D_n^{(m)} \) determines an integer partition \( \lambda = \lambda(\pi) \) whose diagram consists of the squares above \( \pi \) in the \( mn \times n \) triangle. For each \( n \), there is an injection \( D_n^{(m)} \to D_{n+1}^{(m)} \) that adds one north step to the beginning of \( \pi \) and adds \( m \) east steps to the end of \( \pi \). This injection preserves both \( \lambda(\pi) \) and \( \text{area}^c(\pi) = \text{area}(\lambda(\pi)) \), but the value of the bounce statistic \( b_m(\pi) \) may change. However, as \( n \) continues to increase, the bounce statistic will eventually stabilize. More specifically, once \( n \geq 2\text{area}^c(\pi) \), it is routine to check that the bounce path will satisfy \( v_0 = n - \ell(\lambda(\pi)) \), \( v_1 = \ell(\lambda(\pi)) \), and \( v_i = 0 \) for all \( i > 2 \); the key observation is that the first horizontal move (of length \( v_0 \)) moves to the right of all the squares in \( \text{dg}(\lambda(\pi)) \). It follows that \( b_m(\pi) = \ell(\lambda(\pi)) \) for such \( n \). By fixing the area of the partition outside \( \pi \) and taking \( n \) larger than twice this area, we see as in the previous proposition that the indicated limit holds.

4. Limits of the Modified Algebraic Higher \( q, t \)-Catalan Numbers

Lemma 4.1. Let \( D \in \mathcal{D}_n \), let \( S \) be a staircase form of \( D \), and let \( B(S) \) be the block diagonal form of \( S \). Then the number of \( 1 \times 1 \) blocks in \( B(S) \) is at least \( n - 2k(D) \).

Proof. Suppose the number of size-1 blocks in \( B(S) \) is \( t \), and the other blocks have sizes \( s_1, \ldots, s_r \). On one hand, \( t + \sum_{i=1}^r s_i = n \). On the other hand, a block of size \( s_i \) contributes at least \( s_i - 1 \) to \( k(D) \), hence \( \sum_{i=1}^r (s_i - 1) \leq k(D) \). Since \( s_i \geq 2 \), we have \( s_i \leq 2(s_i - 1) \) and \( t = n - \sum_{i=1}^r s_i \geq n - \sum_{i=1}^r 2(s_i - 1) \geq n - 2k(D) \).
Lemma 4.2 ([11] Lemma 15)]. Let \( D = \{P_1, \ldots, P_n\} \in \mathcal{D}_n \) and \( P_i = (a_i, b_i) \) be as in \( \S 2 \). Let \( i, j \) be two integers satisfying \( 1 \leq i \neq j \leq n, |P_i| = i - 1, |P_{i+1}| = i, |P_j| = j - 1, |P_{j+1}| = j \), \( b_i > 0, a_j > 0 \) (we define \( |P_{n+1}| = n \)). Let \( D' \) be obtained from \( D \) by moving \( P_i \) to southeast and \( P_j \) to northwest, i.e.,
\[
D' = \{P_1, \ldots, P_{i-1}, P_i + (1, -1), P_{i+1}, \ldots, P_{j-1}, P_j + (-1, 1), P_{j+1}, \ldots, P_n\}.
\]
Then \( \Delta(D) \equiv \Delta(D') \) (modulo lower degrees).

Theorem 4.3. Assume \( n, m, k, d_1, d_2 \in \mathbb{N} \) satisfy \( n \geq 3, m > 0, k = m \binom{n}{2} - d_1 - d_2 < n/2 - 1 \), and \( d_2 < n/2 - 1 \). For each \( \mu \in \text{Par}(d_2, k) \), let \( S_{\mu} \) be an arbitrary minimal staircase form of bidegree \( (d_1 - (m - 1) \binom{n}{2}, d_2) \) and partition type \( \mu \). Then \( M_{d_1, d_2}^{(m)} \) is generated as a vector space by
\[
\left\{ (\det S_{\mu}) \prod_{1 \leq i < j \leq n} (x_i - x_j)^{m-1} \right\}_{\mu \in \text{Par}(d_2, k)}.
\]
Consequently,
\[
\dim M_{d_1, d_2}^{(m)} \leq p(d_2, k).
\]

Proof. We use induction on \( m \). The base case \( m = 1 \) is done in [11]. Let us briefly sketch a proof for the base case.

For each \( \mu \in \text{Par}(d_2, k) \), the assumption \( k < n/2 - 1 \) implies that there exists a minimal staircase form, say \( S_{\mu} \), of bidegree \( (d_1 - (m - 1) \binom{n}{2}, d_2) \) and partition type \( \mu \). Let \( D_{\mu} \) be an element in \( \mathcal{D}_n \), whose staircase form is \( S_{\mu} \). Since \( \bar{\varphi}(\Delta(D_{\mu})) = \rho_{\mu} \) and \( \bar{\varphi} \) is injective in this case, \( M_{d_1, d_2} \) is generated by \( \Delta(D_{\mu})(\equiv \det S_{\mu}) \).

Now assume that \( m \geq 2 \). Note that \( M_{d_1, d_2}^{(m)} \) is generated by products \( \prod_{i=1}^{m} \Delta(D_i) \), where \( D_i \in \mathcal{D}_n, \sum_{i=1}^{m} d_1(D_i) = d_1 \) and \( \sum_{i=1}^{m} d_2(D_i) = d_2 \). So we only need to prove that each such product is a linear combination of \( \{\det(S_{\mu}) \prod_{1 \leq i < j \leq n} (x_i - x_j)^{m-1}\}_{\mu \in \text{Par}(d_2, k)} \) modulo lower degrees. Define \( k' = k(D_1), d'_2 = d_2(D_1), k'' = k - k', d''_2 = d_2 - d'_2 \). By inductive assumption, \( \prod_{i=1}^{m} \Delta(D_i) \) is a linear combination of \( \{\det(S_{\lambda}') \prod_{1 \leq i < j \leq n} (x_i - x_j)^{m-2}\}_{\lambda \in \text{Par}(d''_2, k'')} \) modulo lower degrees, and \( \Delta(D_1) \) is a linear combination of \( \{\det(S_{\nu}') \}_{\nu \in \text{Par}(d'_2, k') \} \) modulo lower degrees. Hence \( \prod_{i=1}^{m} \Delta(D_i) \) is a linear combination of \( \{\det(S_{\nu}') \det(S_{\lambda}') \prod_{1 \leq i < j \leq n} (x_i - x_j)^{m-2}\}_{\nu \in \text{Par}(d''_2, k''), \lambda \in \text{Par}(d'_2, k')} \) modulo lower degrees. So to prove the theorem, it suffices to show the following statement:
\[
(\ast) \quad \det(S_{\nu}') \det(S_{\lambda}') \text{ is a linear combination of } \{\det(S_{\mu}) \prod_{1 \leq i < j \leq n} (x_i - x_j)\}_{\mu \in \text{Par}(d_2, k)} \text{ modulo lower degrees}.
\]

Since \( S_{\nu}' \) and \( S_{\lambda}' \) can be arbitrary minimal staircase forms of fixed bidegree and fixed partition type, we may assume that all the \( 1 \times 1 \) blocks but the first one in the block diagonal form \( B(S_{\nu}') \) are below bigger blocks, and that all the \( 1 \times 1 \) blocks in the block diagonal form \( B(S_{\lambda}') \) are above bigger blocks. Let \( T' \) (resp. \( T'' \)) be the product of determinants of the blocks of size greater than 1 in the block diagonal form \( B(S_{\nu}') \) (resp. \( B(S_{\lambda}') \)). We have
\[
\det(S_{\nu}') = T' \prod_{j=a}^{n} \prod_{i=1}^{j-1} z_{ij}^{(1)}, \quad \det(S_{\lambda}') = T'' \prod_{j=2}^{n} \prod_{i=1}^{j-1} z_{ij}^{(2)} \quad T''.
\]
where \( z_{ij}^{(t)} = x_i - x_j \) or \( y_i - y_j \) for \( t = 1, 2 \). The numbers of size-1 blocks in \( B(S'_\nu) \) and \( B(S''_\lambda) \) are \( n - a + 2 \) and \( b \), respectively. We assume without loss of generality that \( S'_\nu \) has no more size-1 blocks than \( S''_\lambda \), in other words, that \( n - a + 2 \leq b \). By Lemma 4.2, \( n - a + 2 \geq n - 2k' \) and \( b \geq n - 2k'' \). Then

\[
2b \geq (n - a + 2) + b \geq 2n - 2k' - 2k'',
\]

therefore \( b - a \geq n - 2 - 2k > n - 2 - (n - 2) = 0 \) and \( b \geq n - k > n/2 + 1 \). Since \( d_2 < n/2 - 1 < b - 1 \), we can use Lemma 4.2 to adjust the first \( b \) columns in \( S''_\lambda \) without changing \( \det(S''_\lambda) \) (modulo lower degrees), so that \( z_{ij}^{(2)} = x_i - x_j \) for \( 1 \leq i < j \leq b - 1 \). Note that \( z_{ib}^{(2)} \) can be either \( x_i - x_j \) or \( y_i - y_j \) for \( 1 \leq i < b \). Similarly, we can adjust the last \( n - b + 2 \) columns in \( S'_\nu \) such that \( z_{ij}^{(1)} = x_i - x_j \) for \( b \leq j \leq n - 1 \) and \( 1 \leq i < j \). Then

\[
\det(S'_\nu) = T' \prod_{j=n}^{b-1} \prod_{i=1}^{j-1} z_{ij}^{(1)} \prod_{j=b}^{n} \prod_{i=1}^{j-1} (x_i - x_j), \quad \det(S''_\lambda) = \left( \prod_{j=2}^{b-1} \prod_{i=1}^{j-1} (x_i - x_j) \right) \left( \prod_{i=1}^{b-1} z_{ib}^{(2)} \right) T'',
\]

and

\[
\det(S'_\nu) \det(S''_\lambda) = A \prod_{1 \leq i < j \leq n} (x_i - x_j), \quad \text{where } A = T' \left( \prod_{j=a}^{b-1} \prod_{i=1}^{j-1} z_{ij}^{(1)} \right) \left( \prod_{i=1}^{b-1} z_{ib}^{(2)} \right) T''.
\]

One verifies that \( A \) is a polynomial of bidegree \((d_1 - (m - 1)\binom{n}{2}, d_2)\) in \( I \). Applying the base case \( m = 1 \), we conclude that \( \det(S'_\nu) \det(S''_\lambda) \) is a linear combination of

\[
\left\{ \det(S_\mu) \prod_{1 \leq i < j \leq n} (x_i - x_j) \right\}_{\mu \in \Par(d_2, k)}
\]

modulo lower degrees. This proves (*) \( \square \).

**Lemma 4.4.** Let \( a \) be a positive integer. Then \( \sum_{i=0}^{a} p(i, a - i) = p(a) \).

**Proof.** Given a partition \( \nu = (\nu_1, \ldots, \nu_\ell) \) of \( a \) satisfying \( \nu_1 \leq \cdots \leq \nu_\ell \), we let \( i = \nu_\ell \), and send \( \nu \) to the transpose of the partition \( (\nu_1, \ldots, \nu_{\ell-1}) \), which is a partition of \( a - i \) into at most \( i \) parts. This gives a one-to-one correspondence from \( \Par(a) \) to \( \bigcup_{i=0}^{a} \Par(i, a - i) \). Counting the cardinalities of the two sets gives the stated equality. \( \square \)

**Corollary 4.5.** Let \( n, m, k, d_2 \in \mathbb{N} \) satisfy \( n \geq 3 \), \( m > 0 \), and \( k + d_2 < n/2 - 1 \). Define \( d_1 = m\binom{n}{2} - k - d_2 \). Then the coefficient of \( q^{d_1} t^{d_2} \) in \( AC_n^{(m)}(q, t) \) is

\[
\dim M_{d_1, d_2}^{(m)} = p(d_2, k).
\]

**Proof.** We recalled in the introduction that \( AC_n^{(m)}(q, 1) = RC_n^{(m)}(q, 1) = \sum_{\pi \in D_n^{(m)}} q^{\text{area}(\pi)} \). Then for each \( d_1 \), \( \sum_{d_2=0}^{\infty} \dim M_{d_1, d_2}^{(m)} \) is the number of \( m \)-Dyck paths \( \pi \in D_n^{(m)} \) with \( \text{area}(\pi) = d_1 \). Each such \( m \)-Dyck path uniquely determines a Ferrers diagram of size \( m\binom{n}{2} - d_1 \) consisting of the set of boxes above the \( m \)-Dyck path in the \( mn \times n \) triangle. On the other hand, since any Ferrers diagram of size less than \( n \) determines an \( m \)-Dyck path and \( m\binom{n}{2} - d_1 = k + d_2 <
n, each Ferrers diagram of size \( m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1 \) determines an \( m \)-Dyck path in \( \mathcal{D}_n^{(m)} \). Therefore the number of such \( m \)-Dyck paths is equal to the partition number \( p \left( m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1 \right) \), and

\[
\sum_{d_2=0}^{m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1} \dim M_{d_1,d_2}^{(m)} = p \left( m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1 \right) = \sum_{d_2=0}^{m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1} p \left( d_2, m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1 - d_2 \right),
\]

where the second equality is because of Lemma 4.4. On the other hand, Theorem 4.3 asserts that

\[
\dim M_{d_1,d_2}^{(m)} \leq p \left( d_2, m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1 - d_2 \right).
\]

Therefore each inequality is actually an equality. This implies \( \dim M_{d_1,d_2}^{(m)} = p(d_2,k) \). □

**Proof of Theorem 1.1.** For any fixed nonnegative integers \( k, h \), whenever \( n > 2(k + h + 1) \), the coefficient of \( q^{m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - k - h} \) in \( AC_n^{(m)}(q,t) \) is equal to \( p(h,k) \) by Corollary 4.5. The theorem follows from this observation and Propositions 3.2 and 3.3. □

### 5. Comparison of Coefficients of \( AC_n^{(m)}(q,t) \) to Partition Numbers.

In this section, we extend the method in the previous section to obtain a theorem that identifies the dimensions of certain submodules \( M_{d_1,d_2}^{(m)} \) (i.e., the coefficients of certain terms \( q^{d_1} t^{d_2} \) in \( AC_n^{(m)}(q,t) \)) as partition numbers.

**Theorem 5.1.** Let \( n \geq 6 \) and \( m \) be positive integers, and let \( k, d_1, d_2 \) be nonnegative integers such that \( k = m\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) - d_1 - d_2 \leq n - 6 \). Let \( \delta = \min(d_1,d_2) \). Then

\[
\dim M_{d_1,d_2}^{(m)} = p(\delta,k).
\]

We need a few lemmas.

**Lemma 5.2** (Grafting Lemma). Let \( D_1 = \{P_1,\ldots,P_n\} \) and \( D_2 = \{Q_1,\ldots,Q_n\} \) be in \( \mathcal{D}_n \), where the \( P_i \) and \( Q_i \) are listed in increasing graded lexicographic order. Suppose \( |P_r| = |Q_r| = r - 1 \). Let \( D'_1 = \{P_1,\ldots,P_{r-1},Q_r,\ldots,Q_n\} \) and \( D'_2 = \{Q_1,\ldots,Q_{r-1},P_r,\ldots,P_n\} \). Then

\[
\Delta(D_1) \cdot \Delta(D_2) \equiv \Delta(D'_1) \cdot \Delta(D'_2)
\]

in \( M^{(2)} \). □

We omit the proof of the lemma. The following example illustrates the idea of the proof.

**Example 5.3.** Consider \( D_1, D_2, D'_1, \) and \( D'_2 \) pictured below.

![Diagram](image-url)
Then
\[ \Delta(D_1) \cdot \Delta(D_2) = \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{21} & 0 & 0 & 0 \\ 0 & 0 & x_{31} y_{32} & x_{31} x_{32} & 0 \\ 0 & 0 & x_{41} y_{42} & x_{41} x_{42} & x_{41} x_{42} x_{43} \\ 0 & 0 & x_{51} y_{52} & x_{51} x_{52} & x_{51} x_{52} x_{53} \end{bmatrix} \cdot \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{21} & 0 & 0 & 0 \\ 0 & 0 & y_{31} y_{32} & x_{31} y_{32} & x_{31} x_{32} \\ 0 & 0 & y_{41} y_{42} & x_{41} y_{42} & x_{41} x_{42} \\ 0 & 0 & y_{51} y_{52} & x_{51} y_{52} & x_{51} x_{52} \end{bmatrix}, \]

and
\[ \Delta(D'_1) \cdot \Delta(D'_2) = \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{21} & 0 & 0 & 0 \\ 0 & 0 & y_{31} y_{32} & x_{31} y_{32} & x_{31} x_{32} \\ 0 & 0 & y_{41} y_{42} & x_{41} y_{42} & x_{41} x_{42} \\ 0 & 0 & y_{51} y_{52} & x_{51} y_{52} & x_{51} x_{52} \end{bmatrix} \cdot \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & y_{21} & 0 & 0 & 0 \\ 0 & 0 & y_{31} y_{32} & x_{31} x_{32} & 0 \\ 0 & 0 & x_{41} y_{42} & x_{41} x_{42} & x_{41} x_{42} x_{43} \\ 0 & 0 & y_{51} y_{52} & x_{51} y_{52} & x_{51} x_{52} \end{bmatrix}. \]

One readily verifies that the two products are equal.

**Definition 5.4.** For \( k \leq n - 4 \) and \( d_1 + d_2 + k = 2(\binom{n}{2}) \), define a subspace \( N_{d_1,d_2} \) of \( M_{d_1,d_2}^{(2)} \) by

\[ N_{d_1,d_2} = \begin{cases} M_{d_1-(\binom{n}{2}),d_2} \cdot f_{\binom{n}{2},0} & \text{if } d_2 \leq k; \\ M_{d_1,d_2-(\binom{n}{2})} \cdot f_{0,\binom{n}{2}} & \text{if } d_1 \leq k; \\ M_{d_1+d_2-(\binom{n}{2}),k} \cdot f_{\binom{n}{2}-d_2+k,d_2-k} & \text{otherwise}. \end{cases} \]

**Lemma 5.5** (Higher Transfactor Lemma). Suppose \( k \leq n - 4 \), \( d'_1 \leq d_1 \), \( d'_2 \leq d_2 \), \( d_1 + d_2 + k = 2(\binom{n}{2}) \), and \( d'_1 + d'_2 + (\binom{n}{2}) = d_1 + d_2 \).

(i) If \( d'_2 < d_2 \) and \( d'_1 \geq k + 1 \), then

\[ M_{d'_1,d'_2} \cdot f_{d_1-d'_1,d_2-d'_2} \subseteq M_{d'_1-1,d'_2+1} \cdot f_{d_1-d'_1+1,d_2-d'_2-1} \]

as subspaces of \( M_{d_1,d_2}^{(2)} \).

(ii) If \( d'_1 < d_1 \) and \( d'_2 \geq k + 1 \), then

\[ M_{d'_1,d'_2} \cdot f_{d_1-d'_1,d_2-d'_2} \subseteq M_{d'_1+1,d'_2-1} \cdot f_{d_1-d'_1-1,d_2-d'_2+1} \]

as subspaces of \( M_{d_1,d_2}^{(2)} \).

(iii) \( M_{d'_1,d'_2} \cdot f_{d_1-d'_1,d_2-d'_2} \) is a subspace of \( N_{d_1,d_2} \). Moreover, if \( d'_1,d'_2 \geq k \), then \( M_{d'_1,d'_2} \cdot f_{d_1-d'_1,d_2-d'_2} \) is equal to \( N_{d_1,d_2} \).

**Proof.** (i) Let

\[ P_n = \begin{cases} (n - 1,0) & \text{if } d'_2 < k; \\ (n - 1 - d'_2 + k,d'_2 - k) & \text{if } k \leq d'_2 \leq n - 2 + k; \\ (1,n - 2) & \text{if } d'_2 > n - 2 + k. \end{cases} \]

Then there exists a basis \( \{\Delta(D_i)\} \) of \( M_{d_1,d_2} \) such that the last point of each \( D_i \) is \( P_n \). Indeed, consider the first case \( d'_2 < k \). Let \( M_{d'_1-(n-1),d'_2} \) be the indicated graded piece of \( I_{n-1}/m_{n-1} I_{n-1} \). Let \( \{\Delta(D'_i)\} \) be a basis of \( M_{d'_1-(n-1),d'_2} \), and let \( D_i \) be obtained from \( D'_i \) by adding the point \( P_n \). Since \( M_{d'_1,d'_2} \) and \( M_{d_1,d_2} \) have the same dimension \( p(d_2,k) \), Lemma 2.6 implies that \( \{\Delta(D_i)\} \) forms a basis of \( M_{d_1,d_2} \). The other two cases can be proved similarly.
Now for each $D_i = \{P_1, \ldots, P_n\}$, define $D'_i = \{P_1, \ldots, P_{n-1}, P_n + (-1, 1)\}$. By the Grafting Lemma 5.2 we have $\Delta(D_i) \cdot f_{d_1-d_1',d_2-d_2'} \equiv \Delta(D'_i) \cdot f_{d_1-d_1',d_2-d_2'}$ in $M^{(2)}_{d_1,d_2}$. Then the inclusion stated in (i) follows immediately.

(ii) This is symmetric to (i).

(iii) This follows from (i) and (ii). \qed

**Lemma 5.6.** Assume $n, d'_1, d'_2, k', d''_1, d''_2, k'' \in \mathbb{N}$ satisfy $n \geq 6$, $k' = \binom{n}{2} - d'_1 - d'_2$, $k'' = \binom{n}{2} - d''_1 - d''_2$, $k' + k'' \leq n - 6$, and $(d'_1, d'_2) + (d''_1, d''_2) = (d_1, d_2)$. Then

$$M_{d'_1,d'_2} \cdot M_{d''_1,d''_2} \subseteq N_{d_1,d_2}$$

as subspaces of $M^{(2)}_{d_1,d_2}$.

**Proof.** Define $n' = k' + 3$. First, we claim that $M_{d'_1,d'_2}$ has a basis consisting of elements of the form

$$\Delta(D') = \Delta(\{P'_1, \ldots, P'_n\})$$

where $|P'_i| = i - 1$ for $n' + 1 \leq i \leq n$, and $M_{d''_1,d''_2}$ has a basis consisting of elements of the form

$$\Delta(D'') = \Delta(\{P''_1, \ldots, P''_n\})$$

where $|P''_i| = i - 1$ for $0 \leq i \leq n'$. Indeed, one may find a pair of integers $(e'_1, e'_2)$ such that

$$\min(d'_1, k') \leq e'_1 \leq d'_1 \leq e'_1 + \binom{n}{2} - \binom{n'}{2},$$

$$\min(d'_2, k') \leq e'_2 \leq d'_2 \leq e'_2 + \binom{n}{2} - \binom{n'}{2},$$

and $e'_1 + e'_2 = \binom{n'}{2} - k'$. Then we choose $P'_{n'+1}, \ldots, P'_n$ such that $|P'_i| = i - 1$ for $n' + 1 \leq i \leq n$, and the sum of their bidegrees is $(d'_1 - e'_1, d'_2 - e'_2)$. Choose a basis $\{\Delta(D')\}$ of $(I_{n'}/m_{n'I_{n'}})_{e'_1,e'_2}$ and replace each $D' = \{Q_1, \ldots, Q_{n'}\}$ by

$$D' = \{Q_1, \ldots, Q_{n'}, P'_{n'+1}, \ldots, P'_n\}.$$ 

In this way, we obtain a basis for $M_{d'_1,d'_2}$. On the other hand, one can verify that there exist a pair of integers $(e''_1, e''_2)$ and a nonnegative integer $c \leq n'$ such that

$$\min(d''_1, k'') \leq e''_1 \leq d''_1 - (n - n')c \leq e''_1 + \binom{n'}{2},$$

$$\min(d''_2, k'') \leq e''_2 \leq d''_2 - (n - n')(n' - c) \leq e''_2 + \binom{n'}{2},$$

and $e''_1 + e''_2 = \binom{n - n'}{2} - k''$. 

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Then we choose $P''_1, \ldots, P''_n$ such that $|P'_i| = i - 1$ for $1 \leq i \leq n'$, and the sum of their bidegrees is $(d''_1 - (n - n')c - e''_1, d''_2 - (n - n')(n' - c) - e''_2)$. Take a basis $\{\Delta(D'')\}$ of $(I_{n-n'}/m_{n-n'}I_{n-n'})e'_1,e'_2$, and replace each $D'' = \{Q_1, \ldots, Q_{n-n'}\}$ by

$$D'' = \{P''_1, \ldots, P''_n, Q_1 + (c, n' - c), Q_2 + (c, n' - c), \ldots, Q_{n-n'} + (c, n' - c)\}.$$ In this way, we obtain a basis for $M_{d'_1,d'_2}$. Next, using the Grafting Lemma 5.2,

$$\Delta(D')\Delta(D'') = \Delta(\{P'_1, \ldots, P'_{n'}, P''_{n'+1}, \ldots, P''_n\})\Delta(\{P''_1, \ldots, P''_{n'}, P''_{n'+1}, \ldots, P''_n\}),$$

hence is in $N_{d_1,d_2}$ by Lemma 5.5(iii).

\textbf{Proof of Theorem 5.7.} Without loss of generality, we assume $d_1 \geq d_2$. After applying Lemma 3.6 successively, we can conclude that

$$M_{d_1,d_2}^{(m)} = M_{d_1-a,d_2-b} \cdot g_{a,b}$$

for some nonnegative integers $a, b$, where $a + b = (m - 1){n \choose 2}$, and $g_{a,b} = \prod_{i=1}^{m-1} f_{a,b_i}$ has bidegree $(a, b)$. Moreover, by inspecting the proof of Lemma 5.6 carefully, we can assume $b = \max(0, d_2 - k)$. Therefore

$$\dim M_{d_1,d_2}^{(m)} = \dim(M_{d_1-a,d_2-b} \cdot g_{a,b}) \leq \dim M_{d_1-a,d_2-b} \leq p(d_2,k).$$

Now we prove $\dim M_{d_1,d_2}^{(m)} \geq p(d_2,k)$. Take a sufficiently large integer $\tilde{n} > n$ such that $k, d_2 < \tilde{n}/2 - 1$. Let $\tilde{M}$ be $I_{\tilde{n}}/m_{\tilde{n}}I_{\tilde{n}}$. Let

$$\tilde{f}_0 = \prod_{j = n+1}^{\tilde{n}} \prod_{i=1}^{j-1}(x_j - x_i).$$

Define $\tilde{d}_1 = d_1 + (n + \tilde{n} - 1)(\tilde{n} - n)/2$. By applying Lemma 2.6 successively, we conclude that the linear map $h : M_{d_1-a,d_2-b} \to \tilde{M}_{d_1-a,d_2-b}$ that sends $f$ to $f \cdot \tilde{f}_0$ is injective. Moreover, since $k \leq n - 6$, the domain and the codomain of $h$ have the same dimension $p_{d_2,k}$. So $h$ is also surjective. Consider the following commutative diagram:

$$\begin{array}{ccc}
M_{d_1-a,d_2-b} & \xrightarrow{h} & \tilde{M}_{d_1-a,d_2-b} \\
\psi_1 \downarrow & & \psi_1 \downarrow \\
M_{d_1-a,d_2-b} \cdot g_{a,b} & \xrightarrow{\tilde{g}_{a,b}} & \tilde{M}_{d_1-a,d_2-b} \\
\psi_2 \downarrow & & \psi_2 \downarrow \\
M_{d_1,d_2}^{(m)} & \xrightarrow{\tilde{M}_{d_1,d_2}^{(m)}} & \tilde{M}_{d_1,d_2}^{(m)}
\end{array}$$

where $\tilde{g}_{a,b} = g_{a,b} \cdot (\tilde{f}_0)^{m-1}$, $\psi_1(f) = f \cdot g_{a,b}$, $\psi_1(f) = f \cdot \tilde{g}_{a,b}$, and both the middle and bottom horizontal maps are given by $f \mapsto f \cdot (\tilde{f}_0)^{m}$. Since $h$ and $\psi_1$ are surjective and $\psi_2$ is an isomorphism, the bottom horizontal map is surjective. By Corollary 4.3

$$\dim M_{d_1,d_2}^{(m)} \geq \dim \tilde{M}_{d_1,d_2}^{(m)} = p(d_2,k).$$
Thus the theorem is proved.

In fact, we expect a stronger statement to hold:

**Conjecture 5.7.** Let \( n \geq 2, m \geq 2, d_1, d_2, k \) be positive integers such that \( k = m(n_2) - d_1 - d_2 \).
Define \( \delta = \min(d_1, d_2) \). Then \( \dim M^{(m)}_{d_1, d_2} \leq p(\delta, k) \). Moreover, equality holds if and only if \( k \leq n - 2 \).

6. Conjectures

**Conjecture 6.1.** For \( \pi \in D^{(m)}_n \) and \( 1 \leq i \leq mn \), let \( a_i(\pi) \) be the number of full squares in the \( i \)'th column below \( \pi \) and above the line \( my = x \), and let \( b_i(\pi) \) be the number of full squares \( w \) in the \( i \)'th column which are above \( \pi \) and satisfy \( m \cdot l(w) \leq a(w) \leq m(l(w) + 1) \).

For \( \pi \in D^{(m)}_n \) and \( 1 \leq j \leq m \), let
\[
D_j(\pi) = \{(a_j(\pi), b_j(\pi)), (a_{j+m}(\pi), b_{j+m}(\pi)), \ldots, (a_{j+m(n-1)}(\pi), b_{j+m(n-1)}(\pi))\} \subset \mathbb{N} \times \mathbb{N}.
\]
Then \( \prod_{j=1}^{m} \det(D_j(\pi)) : \pi \in D^{(m)}_n \) generates the \( m \)-th power \( I^m_n \) of the ideal \( I_n \) generated by alternating polynomials in \( \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] \).

**Conjecture 6.2.** Let \( I_n \) be the ideal generated by alternating polynomials in \( R = \mathbb{C}[x, y] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] \). Consider the minimal free resolution of \( I_n \):
\[
0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow R \rightarrow R/I_n \rightarrow 0.
\]
Then for each \( 1 \leq i \leq n \), the bigraded Hilbert series of
\[
\text{Tor}_i(R/I_n, \mathbb{C}) = \text{Tor}_i(R/I_n, R/m)
\]
is equal to
\[
(-1)^{i-1} \sum_{\lambda \vdash n} \langle (s_1)^n, s_\lambda \rangle \langle \nabla(s_\lambda), s_{(1^n)} \rangle.
\]
(Recall that \( \langle (s_1)^n, s_\lambda \rangle = f_\lambda \), the number of standard Young tableaux of shape \( \lambda \). For definition of \( \text{spin} \), see p.6 in [14].)

The conjecture is verified for \( n \leq 6 \). As a special case, we have:

**Conjecture 6.3.** The bigraded Hilbert series of \( I_n \) is
\[
\frac{1}{(1-q)^n(1-t)^n} \langle \nabla(s_1^n), s_{(1^n)} \rangle.
\]

Conjecture 6.3 follows from Conjecture 6.2 because \( s_1^n = \sum_{\lambda \vdash n} \langle (s_1)^n, s_\lambda \rangle s_\lambda \).
7. Appendix: Comparison of Definitions of Higher $q, t$-Catalan Numbers

In the Introduction, we gave seven definitions (a)–(g) of the higher $q, t$-Catalan numbers. Here we explain the known relations among these definitions.

(a)⇔(b): There is an obvious bijection between partitions $\gamma \in \text{Par}^{(m)}_n$ and $m$-Dyck words $\gamma \in \Gamma^{(m)}_n$, defined as follows. Given the partition $\lambda$, embed the diagram of $\lambda$ in an $mn \times n$ triangle as shown in Figures 1 and 2. For $0 \leq i < n$, let $\gamma_i$ be the number of complete squares to the right of $\lambda$ and to the left of the diagonal in the $(i+1)$'th row from the bottom. For example, when $m = 2, n = 5$, and $\lambda = (7,5,4)$, we see from Figure 2 that the associated 2-Dyck word is $\gamma = (0,2,0,1,1)$. It is routine to verify that this process defines a bijection from $\text{Par}^{(m)}_n$ onto $\Gamma^{(m)}_n$ such that $\text{area}(\lambda) = \text{area}(\gamma)$. It is less routine to prove that $c_m(\lambda) = \text{dinv}_m(\gamma)$; see [5] Lemma 6.3.3] for the proof. (Note that what we call $c_m(\lambda)$ is called $b_m(\lambda)$ in [5].)

(b)⇔(c): See [12, §2.5] for a bijection from $\Gamma^{(m)}_n$ to $D_n^{(m)}$ such that if $\gamma$ maps to $\pi$ under the bijection, then $\text{area}(\gamma) = b_m(\pi)$ and $\text{dinv}_m(\gamma) = \text{area}(\pi)$. This proves $WC_n^{(m)}(q,t) = DC_n^{(m)}(q,t)$. On the other hand, it is an open problem to define a bijection $\gamma \mapsto \pi$ from $\Gamma^{(m)}_n$ to $D_n^{(m)}$ satisfying $\text{area}(\gamma) = \text{area}(\pi)$ and $\text{dinv}_m(\gamma) = b_m(\pi)$. This problem is equivalent to proving bijectively that the combinatorial definitions (a), (b), and (c) are symmetric in $q$ and $t$.

(d)⇔(e): One can use well-known facts about Macdonald polynomials to prove that $SC_n^{(m)}(q,t) = RC_n^{(m)}(q,t)$ (cf. [3] and [2]). Indeed, since $e_n = \sum_{\mu \vdash n} (1-q)(1-t)B_{\mu} / w_{\mu} \tilde{H}_{\mu}$ and $\nabla(\tilde{H}_{\mu}) = T_{\mu} \tilde{H}_{\mu}$, linearity of $\nabla$ gives $\nabla^m(e_n) = \sum_{\mu \vdash n} (1-q)(1-t)T_{\mu}^m B_{\mu} / w_{\mu} \tilde{H}_{\mu}$. Since $\langle \tilde{H}_{\mu}, e_n \rangle = T_{\mu}$, we can conclude that $\langle \nabla^m(e_n), e_n \rangle = \sum_{\mu \vdash n} (1-q)(1-t)T_{\mu}^{m+1} B_{\mu} / w_{\mu}$, as desired.

(d)⇔(f): Let $J$ be the ideal in $\mathbb{C}[x, y]$ generated by polarized power sums $\sum_{i=1}^n x_i^h y_i^k$ ($h + k \geq 1$). One can also describe $J$ as the ideal generated by all $S_n$-invariant polynomials without constant term, where $S_n$ acts diagonally [7]. Let $\varepsilon$ be the sign representation of $S_n$. It is proved in [5, Proposition 6.1.1] that

$$\nabla^m(e_n(z_1, z_2, \ldots)) = J_{\varepsilon^{m-1} \otimes I^{m-1} / JI^{m-1}}(z_1, z_2, \ldots; q, t),$$

where the right side denotes the Frobenius series of $\varepsilon^{m-1} \otimes I^{m-1} / JI^{m-1}$. (Note that the meanings of $I$ and $J$ are switched in [5].) On the other hand, one may check that the $S_n$-alternating part $\left(\varepsilon^{m-1} \otimes I^{m-1} / JI^{m-1}\right)^{\varepsilon}$ is isomorphic to $\varepsilon^{m-1} \otimes I^m / \mathfrak{m}I^m$. We can extract the $S_n$-alternating part from the Frobenius series by taking the scalar product with $e_n = s_{(1^n)}$. Therefore,

$$SC_n^{(m)}(q,t) = \langle \nabla^m(e_n), e_n \rangle = \sum_{u,v \geq 0} q^u t^v \dim(\varepsilon^{m-1} \otimes I^{m-1} / \mathfrak{m}I^m)_{u,v}$$

$$= \sum_{u,v \geq 0} q^u t^v \dim(I^m / \mathfrak{m}I^m)_{u,v} = \sum_{u,v \geq 0} q^u t^v \dim M_{u,v}^{(m)} = AC_n^{(m)}(q,t).$$
(e)⇔(g): Haiman showed the identity
\[ RC_n^{(m)}(q,t) = \sum_{i=0}^{n-1} (-1)^i tr H^i(Z_n, O(m))(q,t) \]
in [9, §3, Theorem 2]. Then he showed that for \( i > 0 \) and \( l \geq 0 \), \( H^i(Z_n, P \otimes B \otimes l) = 0 \), where \( P \) and \( B \) are the vector bundles defined in [10, §2]. In particular, this implies \( H^i(Z_n, O(k)) = 0 \) for \( i > 0 \) [10, Introduction and Theorem 2.2]. Therefore \( RC_n^{(m)}(q,t) = tr H^0(Z_n, O(m))(q,t) \), which is exactly \( GC_n^{(m)}(q,t) \).

REFERENCES

[1] F. Bergeron, A. Garsia, Science fiction and Macdonald polynomials, CRM Proceedings and Lecture Notes AMS 6 (1999), 363–429.
[2] M. Can, N. Loehr, A proof of the q,t-square conjecture, J. Combin. Theory Ser. A 113 (2006), no. 7, 1419–1434.
[3] A. Garsia, M. Haiman, A remarkable q, t-Catalan sequence and q-Lagrange inversion, J. Algebraic Combin. 5 (1996), no. 3, 191–244.
[4] J. Haglund, The q, t-Catalan Numbers and the Space of Diagonal Harmonics, with an Appendix on the Combinatorics of Macdonald Polynomials, AMS University Lecture Series, 2008.
[5] J. Haglund, M. Haiman, N. Loehr, J. Remmel, A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, Duke Math. J. 126 (2005), no. 2, 195–232.
[6] M. Haiman, Combinatorics, symmetric functions, and Hilbert schemes, Current developments in mathematics, 2002, 39–111, Int. Press, Somerville, MA, 2003.
[7] M. Haiman, Conjectures on the quotient ring by diagonal invariants, J. Algebraic Combin. 3 (1994), 17–76.
[8] M. Haiman, Hilbert schemes, polygraphs and the Macdonald positivity conjecture, J. Amer. Math. Soc. 14 (2001), no. 4, 941–1006.
[9] M. Haiman, t,q-Catalan numbers and the Hilbert scheme, Selected papers in honor of Adriano Garsia (Taormina, 1994). Discrete Math. 193 (1998), no. 1-3, 201–224.
[10] M. Haiman, Vanishing theorems and character formulas for the Hilbert scheme of points in the plane, Invent. Math. 149 (2002), no. 2, 371–407.
[11] K. Lee, L. Li, q,t-Catalan numbers and generators for the radical ideal defining the diagonal locus of \((C^2)^n\), Electron. J. Combin. 18(1) (2011), Research Paper 158, 34pp.
[12] N. Loehr, Conjectured statistics for the higher q,t-Catalan sequences, Electron. J. Combin. 12 (2005), Research Paper 9, 54pp.
[13] N. Loehr, G. Warrington, A continuous family of partition statistics equidistributed with length, J. Combin. Theory Ser. A 116 (2009), no. 2, 379–403.
[14] N. Loehr, G. Warrington, Nested quantum Dyck paths and \( \nabla(s_{\lambda}) \), Int. Math. Res. Not. (2008), no. 5, Art. ID rnn 157, 29pp.

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202

E-mail address: klee@math.wayne.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, OAKLAND UNIVERSITY, ROCHESTER, MI 48309

E-mail address: li2345@oakland.edu
DEPARTMENT OF MATHEMATICS, VIRGINIA TECH, BLACKSBURG, VA 24061

E-mail address: nloehr@math.vt.edu