Positive divisors in symplectic geometry

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Abstract In this paper, we first prove a vanishing theorem of relative Gromov-Witten invariant of \(P^1\)-bundle. Based on this vanishing theorem and degeneration formula, we obtain a comparison theorem between absolute and relative Gromov-Witten invariant under some positive condition of the symplectic divisor.

Keywords Gromov-Witten invariant, degeneration formula, blow-up

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1 Introduction

Divisors or codimension 2 symplectic submanifolds play an important role in Gromov-Witten theory and symplectic geometry [7,8]. For example, there is now a well-known degeneration operation to decompose a symplectic manifold \(X\) into \(X_1 \cup_Z X_2\), a union of \(X_1, X_2\), along a common divisor \(Z\). To utilize the above degeneration to compute Gromov-Witten invariants inductively, one needs to develop the relative Gromov-Witten theory of the pair \((X, Z)\). Such a theory and its degeneration formula were first constructed by Li and Ruan [6] (see [2] for a different version and [4,5] for an algebraic treatment).

Maulik and Pandharipande [9] systematically studied the degeneration formula of Gromov-Witten invariant for the degeneration \(X \rightarrow X \cup_Z P_Z\), where \(P_Z\) is the projective closure of the normal bundle \(N_Z\). By introducing a certain partial order on relative invariants, they interpreted the degeneration formula as a “correspondence”, a complicated upper triangular linear map from relative invariants to absolute invariants. One consequence of their correspondence is that the absolute and relative invariants determine each other. Such a correspondence is a very powerful tool in determining the totality of Gromov-Witten theory. But it is not effective in determining any single invariant. A natural question is if a divisor with a stronger condition will give us a much stronger correspondence. We answer this affirmatively for so-called positive symplectic divisors. We call a symplectic divisor \(Z\) positive if for some tamed almost complex structure \(J\), \(C_1(N_Z)(A) > 0\) for any \(A \in H_2(Z, Z)\) represented by a non-trivial \(J\)-sphere in \(Z\). This is a generalization of ample divisor from algebraic geometry. We call \(0 \neq A \in H_2(Z, Z)\) stably effective if there is a nonzero genus zero Gromov-Witten invariant of \(Z\) with class \(A\). Let \(C_{\text{min}}(Z) = \min \{C_1(N_Z)(A) \mid A \in H_2(Z, Z)\text{ is stably effective}\}\).

Suppose that \(\alpha_i \in H^*(X, \mathbb{R}), 1 \leq i \leq l\), and \(A \in H_2(X, Z)\). Denote by \langle \alpha_1, \ldots, \alpha_l \rangle_A^X\text{ the genus zero Gromov-Witten invariant of degree }A\text{ with insertions }\alpha_1, \ldots, \alpha_l.\text{ For a nonnegative integer }k, \text{ suppose...
that $Z$ is a smooth symplectic divisor and the $k$-tuple $(t_1, \ldots, t_k)$ of positive integers is the partition of $Z \cdot A$. Let $T = \{(t_1, \beta_1), \ldots, (t_k, \beta_k)\}$ be a weighted partition of $Z \cdot A$ weighted by cohomology classes $\beta_j \in H^\ast(Z, \mathbb{R})$. Denote by $(\alpha_1, \ldots, \alpha_l) \rightarrow (T)_A^{X,Z}$ the relative genus zero Gromov-Witten invariant of degree $A$ with tangent contact multiplicities $(t_1, \ldots, t_k)$ at the relative marked points. An important issue here is to find the relations between absolute and relative Gromov-Witten invariants. McDuff [10] obtained some absolute/relative comparison theorems. Denote by $(k + 1)_0(\beta)$ the genus zero completed cycle weighted by the cohomology class $\beta$, see Section 4 or [12]. Our main theorem is the following comparison theorem.

**Theorem 1.1.** Suppose that $Z$ is a positive divisor and $C_{\min}(Z) \geq \sum d_i$. Then for $A \in H_2(X, \mathbb{Z})$, $\alpha_i \in H^\ast(X, \mathbb{R})$, $1 \leq i \leq \mu$, and $\beta_j \in H^\ast(Z, \mathbb{R})$, $1 \leq j \leq l$, such that the product of any two classes $\beta_j$ vanishes, we have

$$\langle \alpha_1, \ldots, \alpha_\mu, \tau_{d_1-1} \bigcdot (\beta_1), \ldots, \tau_{d_l-1} \bigcdot (\beta_l) \rangle_X^A$$

$$= \langle \alpha_1, \ldots, \alpha_\mu | \{(d_1)_0(\beta_1), \ldots, (d_l)_0(\beta_l), (1, [Z]), \ldots, (1, [Z])\} \rangle_X^{A,Z},$$

where $(d_i)_0(\beta_i)$ is the corresponding weighted genus zero completed cycle and the number of insertions $(1, [Z])$ depends on the completed cycles.

McDuff [10] also obtained similar comparison result in the case without descendant classes. Zinger [17] considered the similar comparison theorem for Gromov-Witten invariants.

The paper is organized as follows. In Section 2, we first review Gromov-Witten theory and its degeneration formula to set up the notation. In Section 3, we prove some vanishing results for relative Gromov-Witten invariants of $\mathbb{P}^1$-bundles. In Section 4, we prove a comparison theorem between absolute and relative Gromov-Witten invariants.

## 2 Preliminaries

In this section, we want to review briefly the constructions of virtual integration in the definitions of the absolute and relative GW invariants, which are the main tool of our paper. We refer to [6,13,15] for the details.

### 2.1 GW-invariants

Suppose that $(X, \omega)$ is a compact symplectic manifold and $J$ is a tamed almost complex structure.

**Definition 2.1.** A stable $J$-holomorphic map is an equivalence class of pairs $(\Sigma, f)$. Here $\Sigma$ is a connected nodal marked Riemann surface of genus $g$ with $k$ smooth marked points $x_1, \ldots, x_k$, and $f : \Sigma \rightarrow X$ is a continuous map whose restriction to each component of $\Sigma$ (called a component of $f$ in short) is $J$-holomorphic. Furthermore, it satisfies the stability condition: if $f|_{\Sigma_i}$ is constant (called a ghost bubble) for some component $\Sigma_i$, then the $\Sigma_i$ satisfies $2g_i + k_i \geq 3$, where $g_i$ is the arithmetic genus of $\Sigma_i$ and $k_i$ is the number of special points (marked points or nodal points) on $\Sigma_i$. $(\Sigma, f)$, $(\Sigma', f')$ are equivalent, or $(\Sigma, f) \sim (\Sigma', f')$, if there is a biholomorphic map $h : \Sigma' \rightarrow \Sigma$ such that $f' = f \circ h$.

An essential feature of Definition 2.1 is that, for a stable $J$-holomorphic map $(\Sigma, f)$, the automorphism group

$$\text{Aut}(\Sigma, f) = \{ h \mid h \circ (\Sigma, f) = (\Sigma, f) \}$$

is finite. We define the moduli space $\mathcal{M}_{g,k}(X, A, J)$ to be the set of equivalence classes of stable $J$-holomorphic maps of genus $g$ with $k$ marked points such that $f_*[\Sigma] = \Delta \in H_2(X, \mathbb{Z})$. The virtual dimension of $\mathcal{M}_{g,k}(X, A, J)$ is computed by index theory,

$$\text{virdim}_{g,k}(X, A, J) = 2c_1(A) + 2(n - 3)(1 - g) + 2k,$$

where $n$ is the complex dimension of $X$. 

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Unfortunately, $\overline{M}_{g,k}(X, A, J)$ is highly singular and may have larger dimension than the virtual dimension. There are several ways to extract invariants, the one we use is the virtual neighborhood method in [13].

First of all, we drop the $J$-holomorphic condition from the previous definition and require only each component of $f$ be smooth. We call the resulting object a stable map or a $C^\infty$-stable map. Denote the corresponding space of equivalence classes by $\mathcal{B}_{g,k}(X, A, J)$. $\mathcal{B}_{g,k}(X, A, J)$ is clearly an infinite-dimensional space. It has a natural stratification given by the topological type of $\Sigma$ together with the fundamental classes of the components of $f$. The stability condition ensures that $\mathcal{B}_{g,k}(X, A, J)$ has only finitely many strata such that each stratum is a Frechet orbifold. Secondly, one can use the pregluing construction to define a topology on $\mathcal{B}_{g,k}(X, A, J)$ which is Hausdorff and makes $\overline{M}_{g,k}(X, A, J)$ a compact subspace (see [13]).

We can define another infinite-dimensional space $\Omega^{0,1}$ together with a map

$$\pi : \Omega^{0,1} \rightarrow \mathcal{B}_{g,k}(X, A, J)$$

such that the fiber is $\pi^{-1}(\Sigma, f) = \Omega^{0,1}(f^*TX)$. The Cauchy-Riemann operator

$$\overline{\partial}_J : \mathcal{B}_{g,k}(X, A, J) \rightarrow \Omega^{0,1}$$

is now interpreted as a section of $\pi : \Omega^{0,1} \rightarrow \mathcal{B}_{g,k}(X, A, J)$ with $\overline{\partial}_J^{-1}(0)$ nothing but $\overline{M}_{g,k}(X, A, J)$.

At each $(\Sigma, f) \in \overline{M}_{g,k}(X, A, J)$, there is a canonical decomposition of the tangent space of $\Omega^{0,1}$ into the horizontal piece and the vertical piece. Furthermore, by choosing a compatible Riemannian metric on $X$ we can linearize $\overline{\partial}_J$ with respect to deformations of stable maps and project to the vertical piece to obtain an elliptic complex over $\mathcal{B}_{g,k}(X, A, J)$,

$$L_{\Sigma, J} : \Omega^0(f^*TX) \rightarrow \Omega^{0,1}(f^*TX).$$ (2.1)

**Remark 2.2.** Let $\nabla$ be the Levi-Civita connection compatible with the Riemannian metric on $X$. $L_{\Sigma, f}$ is the linearization of $\overline{\partial}$ at $[\Sigma, f] \in \overline{M}_{g,k}(X, A, J)$. In general, when $\Sigma$ is smooth, $L_{\Sigma, f} = \nabla + N$, where $\nabla$ is the projection of $\nabla$ onto the $(0,1)$-factor and $N$ is the Nijenhuis tensor of $J$, see Section 3 of [11] for their explicit expressions. When $\Sigma$ is a nodal curve, we need to modify $L_{\Sigma, f}$ by inducing some conditions on nodal points and gluing the operator $L_{\Sigma, f_i}$ on each smooth component $(\Sigma_i, f_i)$, see [13] for the detail. In particular, if $J$ is integrable, $L_{\Sigma, f} = \nabla$.

Since $\overline{M}_{g,k}(X, A, J)$ is compact, we can construct a bundle $E$ over a neighborhood $U$ of $\overline{M}_{g,k}(X, A, J)$ together with a bundle map $\eta : E \rightarrow \Omega^{0,1}$ supported in $U$.

**Remark 2.3.** In the constructions of $E$ and $\eta$, the essential feature is that when restricting to a neighborhood of $[\Sigma, f] \in \overline{M}_{g,k}(X, A, J)$, $\eta$ must map onto the cokernel, $\text{Coker}L_{\Sigma, f}$, of $L_{\Sigma, f}$.

Consider the finite-dimensional vector bundle $p : E|_U \rightarrow U$ over $U$. The stabilizing equation $\overline{\partial}_J + \eta$ can be interpreted as a section of the bundle $p^*\Omega^{0,1} \rightarrow E|_U$. By construction this section

$$\overline{\partial}_J + \eta : E|_U \rightarrow p^*\Omega^{0,1}$$

is transverse to the zero section of $p^*\Omega^{0,1} \rightarrow E|_U$.

The set $U^X_\Sigma = (\overline{\partial}_J + \eta)^{-1}(0)$ is called a virtual neighborhood in [13]. The heart of [13] is to show that $U^X_\Sigma$ has the structure of a $C^1$-manifold.

Notice that $U^X_\Sigma \subset E|_U$. Over $U^X_\Sigma$ there is the tautological bundle

$$E_X = p^*(E|_U)|_{U^X_\Sigma}.$$  

It comes with the tautological inclusion map

$$S_X : U^X_\Sigma \rightarrow E_X, \quad ((\Sigma', f'), e) \rightarrow e,$$
which can be viewed as a section of $E_X$. It is easy to check that

$$S_X^{-1}(0) = \overline{\mathcal{M}}_{g,k}(X, A, J).$$

Furthermore, one can show that $S_X$ is a proper section.

There are evaluation maps

$$ev_i : \overline{F}_{g,k}(X, A, J) \rightarrow X, \quad (\Sigma, f) \mapsto f(x_i),$$

for $1 \leq i \leq k$. $ev_i$ induces a natural map from $U_{S_n} \rightarrow X^k$, which can be shown to be smooth.

Let $\Theta$ be the Thom form of the finite-dimensional bundle $E_X \rightarrow U_{S_n}$.

**Definition 2.4.** The (primitive and primary) GW invariant is defined as

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g,A}^X = \int_{U_{S_n}} S_X^* \Theta \wedge \Pi_i ev_i^* \alpha_i,$$

where $\alpha_i$ are classes in $H^*(X; \mathbb{R})$ and are called primary insertions.

For each marked point $x_i$, we denote by $L_i$ the universal cotangent orbifold complex line bundle over $E_{g,k}(X, A, J)$ whose fiber at $[\Sigma, f]$ is $T^*_x \Sigma$. Such a line bundle can be pulled back to $U_{S_n}^X$ (still denoted by $L_i$). Denote $c_1(L_i)$, the first Chern class of $L_i$, by $\psi_i$.

**Definition 2.5.** The descendant genus zero GW invariant is defined as

$$\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_k} \alpha_k \rangle_{g,A}^X = \int_{U_{S_n}^X} S_X^* \Theta \wedge \Pi_i \psi_i^{d_i} \wedge ev_i^* \alpha_i,$$

where $\alpha_i \in H^*(X; \mathbb{R})$.

**Remark 2.6.** The invariants are independent of the choice of the obstruction bundle $E$ and the stabilization map $\eta$, see [13] for a proof.

**Remark 2.7.** For each $\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_k} \alpha_k \rangle_{g,A}^X$, we can conveniently associated a simple graph $\Gamma$ of one vertex decorated by $(g, A)$ and a tail for each marked point. We then further decorate each tail by $(d_i, \alpha_i)$ and call the resulting graph $\Gamma((\{d_i, \alpha_i\}))$ a weighted graph. Using the weighted graph notation, we denote the above invariant by $\langle \Gamma((\{d_i, \alpha_i\})) \rangle^X$. We can also consider the disjoint union $\Gamma^\bullet$ of several such graphs and use $A_{\Gamma^\bullet, g\bullet}$ to denote the total homology class and total arithmetic genus, respectively. Here the total arithmetic genus is $1 + \Sigma(g_i - 1)$. Then, we define $\langle \Gamma^\bullet((\{d_i, \alpha_i\})) \rangle^X$ as the product of GW invariants of the connected components.

### 2.2 Relative GW-invariants

In this section, we will review the relative GW-invariants. The readers can find more details in the reference [6].

Let $Z \subset X$ be a real codimension 2 symplectic submanifold. Suppose that $J$ is an $\omega$-tamed almost complex structure on $X$ preserving $TZ$, i.e., making $Z$ an almost complex submanifold. The relative GW invariants are defined by counting the number of stable $J$-holomorphic maps intersecting $Z$ at finitely many points with prescribed tangency. More precisely, fix a $k$-tuple $T_k = (t_1, \ldots, t_k)$ of positive integers, consider a marked pre-stable curve

$$(\Sigma, x_1, \ldots, x_m, y_1, \ldots, y_k)$$

and stable $J$-holomorphic maps $f : \Sigma \rightarrow X$ such that the divisor $f^* Z$ is

$$f^* Z = \sum_i t_i y_i.$$

To form a compact moduli space of such maps we must allow the target $X$ to degenerate as well (compare with [4]). Let $Q$ be the projective completion of the normal bundle $N_{Z|X}$, i.e., $Q = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. 


Then $Q$ has a zero section $Z_0 = \mathbb{P}(N_{Z/X} \otimes 0)$ and an infinity section $Z_\infty = \mathbb{P}(0 \oplus \mathbb{C})$. For any non-negative integer $m$, construct $Q_m$ by gluing together $m$ copies of $Q$, where the infinity section of the $i$-th component is glued to the zero section of the $(i + 1)$-th component for $1 \leq i \leq m$. Denote the zero section of the $i$-th component by $Z_{i,0}$ and the infinity section by $Z_{i,\infty}$, so $\text{Sing} Q_m = \bigcup_{i=1}^{m-1} Z_{i,\infty}$. We will also denote $Z_{m,\infty}$ by $Z_\infty$ if there is no possible confusion. Define $X_m$ by gluing $X$ to $Q_m$ along $Z \subset X$ and $Z_{1,0} \subset Q_m$. In particular, $X_0 = X$ will be referred to as the root component and the other irreducible components will be called the bubble components.

$Z \subset X$ can be thought of as the infinity section $Z_{0,\infty}$ of the 0-th component (which is $X$), thus $\text{Sing} X_m = \bigcup_{i=0}^{m-1} Z_{i,\infty}$. Let $\text{Aut}_Z Q_m$ be the group of automorphisms of $Q_m$ preserving $Z_{i,0}$, $Z_{i,\infty}$, and the morphism to $Z$. And let $\text{Aut}_Z X_m$ be the group of automorphisms of $X_m$ preserving $X$ (and $Z$) and with restriction to $Q_m$ being contained in $\text{Aut}_Z Q_m$ (so $\text{Aut}_Z X_m = \text{Aut}_Z Q_m \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the $i$-th $\mathbb{P}^1$-bundle). Denote by $\pi_m : X_m \longrightarrow X$ the map which is the identity on the root component $X_0$ and contracts all the bubble components to $Z = Z_{1,0}$ via the fiber bundle projections.

Now consider a nodal curve $C$ mapped into $X_m$ by $f : \Sigma \longrightarrow X_m$ with specified tangency $Z_{m,\infty}$. There are two types of marked points:

(i) absolute marked points whose images under $f$ lie outside $Z_{m,\infty}$ labeled by $x_i$;
(ii) relative marked points which are mapped into $Z_{m,\infty}$ by $t$ labeled by $y_j$.

A relative $J$-holomorphic map $f : \Sigma \longrightarrow X_m$ is said to be pre-deformable if $f^{-1}(Z_{i-1,\infty} = Z_{i,0})$ consists of a union of nodes so that for each node $p \in f^{-1}(Z_{i-1,\infty} = Z_{i,0})$, $i = 1, 2, \ldots, m$, the two branches at the node are mapped to different irreducible components of $X_m$ and the orders of contact to $Z_{i-1,\infty} = Z_{i,0}$ are equal.

An isomorphism of two such $J$-holomorphic maps $f$ and $f'$ to $X_m$ consists of a diagram

$$
\begin{array}{ccc}
(\Sigma, x_1, \ldots, x_l, y_1, \ldots, y_k) & \xrightarrow{f} & X_m \\
\downarrow h & & \downarrow t \\
(\Sigma', x'_1, \ldots, x'_l, y'_1, \ldots, y'_k) & \xrightarrow{f'} & X_m,
\end{array}
$$

where $h$ is an isomorphism of marked curves and $t \in \text{Aut}_Z(X_m)$. With the preceding understood, a relative $J$-holomorphic map to $X_m$ is said to be stable if it has only finitely many automorphisms.

We introduced the notion of a weighted graph in Remark 2.7. We need to refine it for relative stable maps to $(X, Z)$. A (connected) relative graph $\Gamma$ consists of the following data:

(i) a vertex decorated by $A \in H_2(X; \mathbb{Z})$ and genus $g$;
(ii) a tail for each absolute marked point;
(iii) a relative tail for each relative marked point.

**Definition 2.8.** Let $\Gamma$ be a relative graph with $k$ (ordered) relative tails and $T_k = (t_1, \ldots, t_k)$, a $k$-tuple of positive integers forming a partition of $A \cdot [Z]$. A relative $J$-holomorphic map to $(X, Z)$ with type $(\Gamma, T_k)$ consists of a marked curve $(\Sigma, x_1, \ldots, x_l, y_1, \ldots, y_k)$ and a map $f : \Sigma \longrightarrow X_m$ for some non-negative integer $m$ such that

(i) $\Sigma$ is a connected curve (possibly reducible) of arithmetic genus $g$;
(ii) the map

$$
\pi_m \circ f : \Sigma \longrightarrow X_m \longrightarrow X
$$

satisfies $(\pi_m \circ f)_*[\Sigma] = A$;
(iii) the $x_i, 1 \leq i \leq l$, are the absolute marked points;
(iv) the $y_j, 1 \leq i \leq k$, are the relative marked points;
(v) $f^{-1}(Z_{m,\infty})$ consists of precisely the points $\{y_1, \ldots, y_k\}$ and $f$ has the tangency order $t_i$ at each $y_i$.

Let $\overline{M}_{\Gamma, T_k}(X, Z, J)$ be the moduli space of equivalence classes of pre-deformable relative stable $J$-holomorphic maps with type $(\Gamma, T_k)$. Notice that for an element $f : \Sigma \to X_m$ in $\overline{M}_{\Gamma, T_k}(X, Z, J)$ the intersection pattern with $\text{Sing} X_m$ is only constrained by the genus condition and the pre-deformability condition.
Consider the configuration space $\overline{\cal M}_{\Gamma,T_k}(X,Z,J)$ of equivalence classes of smooth pre-deformable relative stable maps to $X_m$ for all $m \geq 0$. Here, for each $m$, we still take the equivalence class under $\text{Aut}_Z X_m$. In particular, the subgroup of $\text{Aut}_Z X_m$ fixing such a map is required to be finite. The maps are required to intersect the $Z_{i,\infty}$ only at finitely many points in the domain curve. Furthermore, at these points, the map is required to have a holomorphic leading term in the normal Taylor expansion for any local chart of $X$ taking $D$ to a coordinate hyperplane and being holomorphic in the normal direction along $D$. Thus the notion of contact order still makes sense, and we can still impose the pre-deformability condition and contact order condition at the $y_i$ being governed by $T_k$.

With the preceding understood, by choosing a unitary connection on the normal complex line bundles of the $Z_{i,\infty}$, we can define the analog of (2.1),

$$L_{\Sigma,f}^{X,Z} \oplus \bigoplus_i T_i : \Omega^0 \to \Omega^0_{r,1} \oplus \bigoplus_i \mathcal{J}^{t_i}.$$ 

Here an element $u \in \Omega^0$ is an element of $\Omega^0(f^*TX)$ or $\Omega^0(f^*TP(N_{Z/X} \oplus \mathbb{C}))$ with the following property: Choose a unitary connection on $N$ so that we can decompose the tangent bundle of $P(N_{Z/X} \oplus \mathbb{C})$ into tangent and normal directions to $Z$. Near $f(y_i)$, $u(y_i)$ can be decomposed into $(u_Z, u_N)$, where $u_Z$, $u_N$ are tangent and normal components, respectively. Now we require that $u_N$ vanish at $y_i$ with order $t_i$. When $\Sigma$ consists of two components joined at one point, we require their $u_Z$ components be the same at the intersection point. Each summand

$$\mathcal{J}^{t_i} \cong \bigoplus_{j=1}^{t_i-1} \text{Hom}((T_{y_i}, \Sigma)^j, N_{f(y_i)})$$

is the $(t_i - 1)$-jet space, and the map $T_i(f)$ is the $(t_i - 1)$-jet of $f$ at $y_i$, i.e., the first $(t_i - 1)$ terms of the Taylor polynomial.

We may apply the virtual neighborhood technique to construct $U_{S_\Sigma}^{X,Z}, E_{X,Z}, S_{X,Z}$ as in Subsection 2.1, see Section 4 in [6] for the details.

**Remark 2.9.** Similar to the case of absolute Gromov-Witten invariants, the essential feature of $E_{X,Z}$ is that it locally dominates the cokernel bundle of $L_{\Sigma,f}^{X,Z} \oplus \bigoplus_i T_i$.

In addition to the evaluation maps on $\overline{\cal M}_{\Gamma,T_k}(X,Z,J)$,

$$ev_i^X : \overline{\cal M}_{\Gamma,T_k}(X,Z,J) \to X, \quad 1 \leq i \leq l,$$

$$(\Sigma, x_1, \ldots, x_l, y_1, \ldots, y_k, f) \mapsto \pi_m \circ f(x_i),$$

there are also the evaluations maps

$$ev_j^Z : \overline{\cal M}_{\Gamma,T_k}(X,Z,J) \to Z, \quad 1 \leq j \leq k,$$

$$(\Sigma, x_1, \ldots, x_l, y_1, \ldots, y_k, f) \mapsto f(y_j),$$

where $Z = Z_{m,\infty}$ if the target of $f$ is $X_m$.

**Definition 2.10.** $\alpha_i \in H^*(X; \mathbb{R})$, $1 \leq i \leq l$, $\beta_j \in H^*(Z; \mathbb{R})$, $1 \leq j \leq k$. Define the relative GW invariant

$$\langle \Pi_T \tau_{\alpha} | \Pi_J \beta \rangle_{\Gamma,T_k}^{X,Z} = \frac{1}{|\text{Aut}(T_k)|} \int_{U_{S_\Sigma}^{X,Z}} S_{X,Z}^* \Theta \wedge \Pi_J \psi_i^{\alpha_i} \wedge (ev_i^X)^* \alpha_i \wedge \Pi_J (ev_j^Z)^* \beta_j,$$

where $\Theta$ is the Thom class of the bundle $E_{X,Z}$ and $\text{Aut}(T_k)$ is the symmetry group of the partition $T_k$. Denote by $T_k = \{(t_j, \beta_j) \mid j = 1, \ldots, k\}$ the weighted partition of $A \cdot \{Z\}$. If the vertex of $\Gamma$ is decorated by $(g, A)$, we will sometimes write

$$\langle \Pi_T \tau_{\alpha} | \Pi_J \beta \rangle_{T_k}^{X,Z}$$

for $\langle \Pi_T \tau_{\alpha} | \Pi_J \beta \rangle_{\Gamma,T_k}^{X,Z}$. 


Remark 2.11. In [6], Li and Ruan proved that the invariant does not depend on the choice of the bundle $E_{X,Z}$. Only invariants without descendant classes were considered. But it is straightforward to extend the definition of [6] to include absolute descendant classes. Similar to the case of absolute Gromov-Witten invariants, the relative invariant is also independent of the choice of the virtual neighborhood, see [6].

We can decorate the tail of a relative graph $\Gamma$ by $(d_i, \alpha_i)$ as in the absolute case. We can further decorate the relative tails by the weighted partition $T_k$. Denote the resulting weighted relative graph by $\Gamma\{\{(d_i, \alpha_i)\}|T_k\}$. In [6] the source curve is required to be connected. We will also need to use a disconnected version. For a disjoint union $\Gamma^\bullet$ of weighted relative graphs and a corresponding disjoint union of partitions, still denoted by $T_k$, we use $\Gamma^\bullet\{\{(d_i, \alpha_i)\}|T_k\}^{X,Z}$ to denote the corresponding relative invariants with a disconnected domain, which is simply the product of the connected relative invariants. Notice that although we use $\bullet$ in our notation following [9], our disconnected invariants are different. The disconnected invariants there depend only on the genus, while ours depend on the finer graph data.

2.3 Degeneration formula

Now we describe the degeneration formula of GW-invariants under symplectic cutting [3,14].

As an operation on topological spaces, the symplectic cut is essentially collapsing the circle orbits in the hypersurface $H^{-1}(0)$ to points in $Z$. Thus we have a continuous map

$$h : X \to X^+ \cup_Z X^-.$$ 

As for the symplectic forms, we have $\omega^+ |_Z = \omega^- |_Z$. Hence, the pair $(\omega^+, \omega^-)$ defines a cohomology class of $X^+ \cup_Z X^-$, denoted by $[\omega^+ \cup_Z \omega^-]$. It is easy to observe that

$$h^*([\omega^+ \cup_Z \omega^-]) = [\omega].$$

(2.2)

Let $B \in H_2(X;\mathbb{Z})$ be in the kernel of

$$h_* : H_2(X;\mathbb{Z}) \longrightarrow H_2(X^+ \cup_Z X^-;\mathbb{Z}).$$

By (2.2) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. The geometric description of the vanishing cycle is rim tori. For each simple closed curve $\gamma$ in $Z$, $h^{-1}(\gamma)$ is a torus in $H^{-1}(0)$, i.e., rim tori. It is easy to see that each vanishing cycle can be represented by a rim tori. In particular, in the case of blow-up along a complex codimension two submanifold $S$, $H^{-1}(0)$ is the sphere bundle of the normal bundle to $S$ in $X$. Since the fiber is simply connected, there are no rim tori. Therefore, there are no vanishing cycle. For $A \in H_2(X;\mathbb{Z})$ define $[A] = A + \text{Ker}(h_*)$ and

$$\langle \Pi_1 \tau_{d_i} \alpha_i \rangle_{g,[A]}^{X^+} = \sum_{B \in [A]} \langle \Pi_1 \tau_{d_i} \alpha_i \rangle_{g,B}^{X^-}.$$ 

(2.3)

Notice that $\omega$ has constant pairing with any element in $[A]$. It follows from the Gromov compactness theorem that there are only finitely many such elements in $[A]$ represented by $J$-holomorphic stable maps. Therefore, the summation in (2.3) is finite.

The degeneration formula expresses $\langle \Pi_1 \tau_{d_i} \alpha_i \rangle_{g,[A]}^{X^+}$ in terms of relative invariants of $(X^+, Z)$ and $(X^-, Z)$ possibly with disconnected domains.

To begin with, we need to assume that the cohomology class $\alpha_i$ is of the form

$$\alpha_i = h^*{(\alpha_i^+ \cup_Z \alpha_i^-)}.$$ 

(2.4)

Here $\alpha_i^\pm \in H^*(X^\pm;\mathbb{R})$ are classes with $\alpha_i^+ |_Z = \alpha_i^- |_Z$ so that they give rise to a class $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(X^+ \cup_Z X^-;\mathbb{R})$.

Next, we proceed to write down the degeneration formula. We first specify the relevant topological type of a marked Riemann surface mapped into $X^+ \cup_Z X^-$ with the following properties:
(i) Each connected component is mapped either into $X^+$ or $X^-$ and carries a respective degree 2 homology class.

(ii) The marked points are not mapped to $Z$.

(iii) Each point in the domain mapped to $Z$ carries a positive integer (representing the order of tangency).

By abusing language we call the above data a $(X^+, X^-)$-graph. Such a graph gives rise to two relative graphs of $(X^+, Z)$ and $(X^-, Z)$, each possibly being disconnected. We denote them by $\Gamma^+$ and $\Gamma^-$ respectively. From (iii) we also get two partitions $T^+$ and $T^-$. We call a $(X^+, X^-)$-graph a degenerate $(g, A, l)$-graph if the resulting pairs $(\Gamma^+, T^+)$ and $(\Gamma^-, T^-)$ satisfy the following constraints: the total number of marked points is $l$, the relative tails are the same, i.e., $T^+ = T^-$, and the identification of relative tails produces a connected graph of $X$ with total homology class $\pi_*[A]$ and arithmetic genus $g$.

Let $\{\beta_a\}$ be a self-dual basis of $H^*(Z; \mathbb{R})$ and $\eta^{ab} = \int_Z \beta_a \cup \beta_b = \delta_{ab}$. Given $g, A$ and $l$, consider a degenerate $(g, A, l)$-graph. Let $T_k = T^+ = T^-$ and $\tilde{T}_k$ be a weighted partition $\{t_j, \beta_{a_j}\}$. Let $\tilde{T}_k = \{t_j, \beta_{a'_j}\}$ be the dual weighted partition.

The degeneration formula for $\langle \pi_t, \alpha_i \rangle^X_{g, \{\beta_a\}}$ then reads as follows,

$$
\langle \pi_t, \alpha_i \rangle^X_{g, \{\beta_a\}} = \sum_j \langle \Gamma^+ \{ (d_i, \alpha_i^+) \} | T_k \rangle^{X^+, Z} \Im \langle \Gamma^- \{ (d_i, \alpha_i^-) \} | T_k \rangle^{X^-, Z},
$$

where the summation is taken over all degenerate $(g, A, l)$-graphs, and

$$
\Im \langle T_k \rangle = \prod_j t_j |\text{Aut}(T_k)|.
$$

See Theorem 5.7 in [6] for the precise degeneration formula.

### 3 Relative GW-invariants of $\mathbb{P}^1$-bundles

Suppose that $Z$ is a symplectic submanifold of $X$ of codimension 2. When applying the degeneration formula, we often need to express the absolute Gromov-Witten invariants of $X$ in terms of relative Gromov-Witten invariants. Thus if we want to obtain a comparison theorem of Gromov-Witten invariant by the degeneration formula, the point will be how to compute the relative Gromov-Witten invariants of a $\mathbb{P}^1$-bundle. In this section, we will prove a vanishing theorem and a nonvanishing theorem for genus zero relative Gromov-Witten invariants of the $\mathbb{P}^1$-bundle $Y$ relative to the infinity section and compute some genus zero two-point relative fiber class $GW$ invariants of the $\mathbb{P}^1$-bundle $Y$.

Suppose that $L$ is a line bundle over $Z$ and $Y = \mathbb{P}_Z(L \oplus \mathbb{C})$. Let $D = \mathbb{P}(0 \oplus \mathbb{C})$, $Z \cong \mathbb{P}(L \oplus 0)$ be the infinity section and zero section of $Y$, respectively. Let $\beta_1, \ldots, \beta_{m_Z}$ be a self-dual basis of $H^*(Z, \mathbb{Q})$ containing the identity element. We will often denote the identity by $\beta_{id}$. The degree of $\beta_i$ is the real grading in $H^*(Z, \mathbb{Q})$. We view $\beta_i$ as an element of $H^*(Y, \mathbb{Q})$ via the pull-back by $\pi : Y = \mathbb{P}(L \oplus \mathbb{C}) \longrightarrow Z$. Let $[Z], [D] \in H^2(Y, \mathbb{Q})$ denote the cohomology classes associated to the divisors. Define classes in $H^*(Y, \mathbb{Q})$ by

$$
\gamma_i = \beta_i,
$$

$$
\gamma_{m_Z+i} = \beta_i \cdot [Z],
$$

$$
\gamma_{2m_Z+i} = \beta_i \cdot [D].
$$

We will use the following notation:

$$
\gamma_i^\beta = \beta_i \mod m_Z,
$$

$$
\gamma_i^D = 1, [Z], or [D].
$$

The second assignment depends upon the integer part of $(i - 1)/m_Z$. The set $\{\gamma_1, \ldots, \gamma_{2m_Z}\}$ is a basis of $H^*(Y, \mathbb{Q})$.

Since $D \cong Z$ as topological manifolds, $H^*(D, \mathbb{Q})$ is isomorphic to $H^*(Z, \mathbb{Q})$. If there is no confusion, we will also use $\beta_1, \ldots, \beta_{m_Z}$ as a self-dual basis of $H^*(D, \mathbb{Q})$. 

3.1 A vanishing theorem

In this subsection, we will prove a vanishing result for some relative Gromov-Witten invariants of $\mathbb{P}^1$-bundle, in particular, for some non-fiber homology class invariants.

Let $\Gamma_0$ be a relative graph with the following data:
(i) a vertex decorated by a homology class $A \in H_2(Y, \mathbb{Q})$ and genus zero;
(ii) $l + q$ tails associated to $l + q$ absolute marked points;
(iii) $k$ relative tails associated to $k$ relative marked points.

Denote by $A$ the homology class of the relative stable map $(\Sigma, f)$ to $(Y, D)$ and by $F$ the homology class of a fiber of $Y$. Let $T_k = \{t_1, \ldots, t_k\}$ be a partition of $D \cdot A$ and $d_i$, $1 \leq i \leq l$, be positive integers. Denote $d = \sum_{i=1}^l d_i$. Denote by $\iota : Z \to Y$ the inclusion of $Z$ into $Y$ via the zero section of $Y$. Then for any $\beta \in H^*(Z, \mathbb{R})$, the inclusion map $\iota$ pushes forward the class $\beta$ to a cohomology class $i^! (\beta) \in H^*(Y, \mathbb{Q})$, determined by the pull-back map $i^!$ and Poincaré duality.

**Proposition 3.1.** Suppose $A \neq sF$ or $k + l + q \geq 3$. Assume that $Z^*(A) \geq \sum d_i$ and $c_1(L)(C) \geq 0$ for any $J$-holomorphic curve $C$ into $Z$. Then for any $\beta_i \in H^*(Z, \mathbb{Q})$, $1 \leq i \leq l$, and any weighted partition $T_k = \{ (t_i, \delta_i) \}$ of $D \cdot A$, we have

$$\langle \varpi, \tau_{d_i-1} i^! (\beta_1), \ldots, \tau_{d_i-1} i^! (\beta_l) | T_k \rangle_{\Gamma_0, T_k} = 0,$$

where $\varpi$ consists of insertions of the form $\pi^* \alpha_1, \ldots, \pi^* \alpha_q$ for some $\alpha_i \in H^*(Z, \mathbb{R})$.

**Proof.** The projection $\pi : Y = \mathbb{P}(L \oplus \mathbb{C}) \to Z$ induces a map between the moduli spaces, denoted also by $\pi$,

$$\pi : \mathcal{M}_{\Gamma_0, T_k}(Y, D, J) \to \mathcal{M}_{0, k+l+q}(Z, \pi_* (A), J),$$

where $\pi$ contracts the unstable rational component whose image is a fiber. $\pi$ is well defined if $A \neq sF$ or $k + l + q \geq 3$. By the definition, $\pi$ commutes with the evaluation map. Furthermore, there is also a natural map (denoted by $\pi$ as well) on $\Omega^{0,1}$ commuting with the map on configuration spaces. Moreover, $\overline{\partial}$ commutes with $\pi$. Hence, it induces a map from $\text{Coker} \ L^Y_{\pi} \to \text{Coker} \ L^Z_{\pi}$.

We claim that $\pi$ induces a map on a virtual neighborhood.

Let $\omega$ be an integral symplectic form on $Z$. Let $(\Sigma, f)$ be a relative stable map of $(Y, D)$. Using Siebert’s construction [16], we can construct a bundle $\mathcal{E}$ dominating the local obstruction bundle generated by $\text{Coker} \ L^Z_{\Sigma, f}$. $\pi^* \mathcal{E}$ is a bundle over $\mathcal{M}_{\Gamma_0, T_k}(Y, D, J)$. We want to show that $\pi^* \mathcal{E}$ dominates its local obstruction bundle. First of all, we have

**Lemma 3.2.** $\text{Coker} \ L^Y_{\Sigma, f}$ is isomorphic to $\text{Coker} \ L^Z_{\Sigma, f}$.

We will prove this lemma after the proof of Proposition 3.1. Since we have identified the obstruction spaces, we first choose a stabilization term $\eta_i$ on $\mathcal{M}_{0, k+l+q}(Z, \pi_* (A), J)$ to dominate the local obstruction bundle generated by $\text{Coker} \ L^Z_{\pi \Sigma, f}$. Then, we pull back $\eta_i$ over $\mathcal{M}_{\Gamma_0, T_k}(Y, D, J)$. By Lemma 3.2, it dominates $\text{Coker} \ L^Y_{\Sigma, f}$. This implies that $\pi$ induces a smooth map on virtual neighborhood and a commutative diagram on obstruction bundles

$$\begin{array}{ccc}
\mathcal{E}_{Y, D} & \to & \mathcal{E}_Z \\
\downarrow & & \downarrow \\
\pi_{\mathcal{S}_a} : U_{\mathcal{S}_a}^{Y, D} & \to & U_{\mathcal{S}_a}^Z.
\end{array}$$

Furthermore, the proper sections $S_{Y, D}$, $S_Z$ commute with the above diagram. $\pi_{\mathcal{S}_a}$ commutes with the evaluation map for those $\beta_i$ classes. Choose a Thom form $\Theta$ of $\mathcal{E}_Z$. Its pullback is the Thom form of $\mathcal{E}_{Y, D}$ (still denoted by $\Theta$).

Note that

$$\dim U_{\mathcal{S}_a}^{Y, D} = \text{rank} \ E_Z + 2 \left( c^Y_1 (A) + n - 3 + l + q + k - \sum t_i \right),$$

$$\dim U_{\mathcal{S}_a}^Z = \text{rank} \ E_Z + 2 (c^Z_1 (\pi_* (A)) + n - 1 - 3 + l + q + k).$$
However,
\[ c_1^Y(A) = c_1^Z(\pi_s(A)) + c_1(L)(\pi_s(A)) + 2 \sum t_i = c_1^Z(\pi_s(A)) + Z^*(A) + \sum t_i. \]
Hence,
\[ \dim U^Y_{S_c} - \dim U^Z_{S_c} = 2(Z^*(A) + 1). \]
By definition and \( Z^*(A) \geq d \), we have
\[ \deg(\Theta + \sum_i \deg(\beta_i) + \deg \varpi + \sum_j \deg(\delta_j) = \dim U^Y_{S_c} - 2d > \dim U^Z_{S_c}, \quad (3.3) \]
where \( \deg \varpi = \sum \deg \alpha_j \). Then, from (3.3), we have
\[
\int_{U^Y_{S_c}} (S_{Y,D})^*\Theta \prod \psi_i^{d_i-1} ev^*_i \beta_i \wedge ev^* \varpi \wedge \prod \psi_i^* \delta_j \\
= \int_{U^Y_{S_c}} (S_{Y,D})^*\Theta \prod \psi_i^{d_i-1}(\pi_{S_c})^* \left( ev^*_i \beta_i \wedge \prod ev^* \varpi \wedge \prod ev^*_j \delta_j \right) = 0.
\]
In the last equality, we use \( ev^*_i \beta_i \wedge \prod ev^* \varpi \wedge \prod ev^*_j \delta_j = 0 \) on \( U^Z_{S_c} \). Hence, the relative invariant is zero. This completes the proof of Proposition 3.1. \( \square \)

**Remark 3.3.** McDuff also proved the same result in the case without insertion classes \( \beta_i \) by a totally different method; see Lemmas 2.7 and 2.8(i) in [10].

**Proof of Lemma 3.2.** It is well known that a stable map can be naturally decomposed into connected components lying outside of \( D \) (rigid factors) or completely inside \( D \) (rubber factors). Let \( (\Sigma, f) \) be a rigid factor or a rubber factor with relative marked points \( x_1, \ldots, x_r \) such that \( f(x_i) \in Z \) or \( D \) with order \( k_i \). In both cases, it is a stable map into \( Y \). Without loss of generality, we assume that \( (\Sigma, f) \) is a rigid factor, i.e., \( f(x_i) \in D \). The rubber case may be dealt with similarly. We take the complex as
\[
\tilde{L}^Y_{\Sigma, f} \times \Sigma T_x^k : \{ u \in \Omega^0(f^*TY) \mid u(x_i) \in TD \} \rightarrow \Omega^{0,1}(f^*TY \otimes \mathcal{O}_\Sigma(x_i)) \oplus \mathcal{I}_x^k.
\]
Next, we study the cokernel of \( \tilde{L}^Y_{\Sigma, f} \). Choose a Hermitian metric on \( TY \) and a unitary connection on the line bundle \( L \) which induces a unitary connection on \( TY \) such that the following sequence
\[ 0 \rightarrow V \rightarrow TY \rightarrow \pi^* TZ \rightarrow 0, \quad (3.4) \]
is exact where \( V \) is the vertical tangent bundle on \( Y \). Then \( f^* V \) is a holomorphic line bundle over \( \Sigma \). This will induce a short exact sequence
\[ 0 \rightarrow f^* V(-\Sigma k_i x_i) \rightarrow f^* TY(-\log D) \rightarrow f^* \pi^* TZ \rightarrow 0. \quad (3.5) \]
(3.5) induces a long exact sequence in cohomology
\[
0 \rightarrow H^0(f^* V(-\Sigma k_i x_i)) \rightarrow H^0(f^* TY(-\log D)) \\
\rightarrow H^0(f^* \pi^* TZ) \rightarrow H^1(f^* V(-\Sigma k_i x_i)) \\
\rightarrow H^1(f^* TY(-\log D)) \rightarrow H^1(f^* \pi^* TZ) \rightarrow 0. \quad (3.6)
\]
It is easy to see that
\[ H^0(f^* V(-\Sigma k_i x_i)) = \{ v \in H^0(f^* V) \mid v(x_i) = 0 \}. \]
We claim that \( H^1(f^* V(-\Sigma k_i x_i)) = 0 \). Note that since \( \Sigma \) is a tree of \( \mathbb{P}^1 \)'s, we see that \( H^1(L') = 0 \) for any line bundle \( L' \) on \( \Sigma \) satisfying \( \deg(L'\overline{\Sigma}) \geq 0 \) for any irreducible component \( \overline{\Sigma} \) of \( \Sigma \).
A simple calculation shows \( c_1(V) = c_1(L) + 2D \). Therefore we have
\[
\deg(f^* V(-\Sigma k_i x_i)\overline{\Sigma}) = f_s(\overline{\Sigma}) \cdot c_1(V) - f_s(\overline{\Sigma}) \cdot D \\
= f_s(\overline{\Sigma}) \cdot (\pi^* c_1(L) + 2D) - f_s(\overline{\Sigma}) \cdot D
\]
Applying $L' = f^*V(-\Sigma k_i x_i)$, we conclude that

$$H^1(f^*V(-\Sigma k_i x_i)) = 0. \quad (3.7)$$

Next, we show that

$$\bigoplus_i T_{x_i}^k : H^0(f^*TY(-\log D)) \to \bigoplus_i T_{x_i}^k \quad (3.8)$$

is surjective. It is enough to show that the restriction to $H^0_L(f^*V(-\Sigma k_i x_i))$ is surjective. Consider the exact sequence

$$0 \to f^*V \otimes_1 \mathcal{O}(-k_i x_i) \to f^*V \to \bigoplus_i f^*V_{k_i x_i} \to 0.$$

It induces a long exact sequence

$$H^0(f^*V) \to \bigoplus_i H^0(f^*V_{k_i x_i}) \to H^1(f^*V \otimes_1 \mathcal{O}(-k_i x_i)).$$

It follows from (3.7) that

$$H^0(f^*V) \to \bigoplus_i H^0(f^*V_{k_i x_i})$$

is surjective. Now dropping the constant term in $H^0(f^*V_{k_i x_i})$, (3.8) becomes

$$H^0(f^*V(-\Sigma k_i x_i)) \to \bigoplus_i T_{x_i}^k, \quad (3.9)$$

which is obviously surjective. By (3.6), we have proved that $\text{Coker}(L_{\Sigma f}^Y \times \sum T_{x_i}^k)$ is isomorphic to $H^1(f^*\pi^*TZ)$. Then, we argue that $H^1(f^*\pi^*TZ)$ is isomorphic to $H^1(\pi(f)^*TZ)$. This is obvious if $\pi(f)$ contracts an unstable component $\mathbb{P}^1$, $\pi \circ f(\mathbb{P}^1)$ = constant and $\mathbb{P}^1$ has one or two special points. Moreover, $\pi(\Sigma)$ is obtained by contracting $\mathbb{P}^1$. Note that $f^*\pi^*TZ|_{\mathbb{P}^1}$ is trivial.

The space of meromorphic 1-forms on $\mathbb{P}^1$ with a simple pole at one or two points is zero or 1-dimensional. If $\mathbb{P}^1$ has only one special point, the residue at the special point has to be zero. We can simply contract this component and remove the pole at the other component which $\mathbb{P}^1$ is connected to. If $\mathbb{P}^1$ has two special points, the residues at the two points have to be the same. Then we can remove this component and the joint residue at the two special points. Then we identify $H^1(f^*\pi^*TZ)$ and $H^1(\pi(f)^*TZ)$.

Suppose that $(\Sigma, f)$ has more than one subfactor. Both $\text{Coker} L_{\Sigma f}^Y$ and $\text{Coker} L_{\pi(\Sigma), f}^Z$ are obtained by requiring the residues at the new marked points to be opposite to each other. Then our proof also extends to this case. Then we finish the proof of Lemma 3.2.

\[\square\]

### 3.2 Fiber class invariants

In this subsection, we mainly compute some genus zero relative GW invariants of $\mathbb{P}^1$-bundles with a fiber class. According to [9], we may transfer the computation of this invariant on the $\mathbb{P}^1$-bundle into that of some associated invariants on $\mathbb{P}^1$.

Let $\Gamma$ be the relative graph with the following data:

(i) a vertex decorated by $A = sF \in H_2(Y, \mathbb{Z})$ and genus zero;
(ii) $k$ relative tails;
(iii) $l$ absolute tails.

Similarly, denote still by $\Gamma$ the relative graph for $(\mathbb{P}^1, p_1)$ obtained by replacing the decoration on the vertex by degree of the map. We denote by $\mathcal{M}_\Gamma(\mathbb{P}^1, p_1, T_1)$ the moduli space of relative stable maps to $(\mathbb{P}^1, p_1)$, see [4] for its definition.

Next, we first review Maulik-Pandharipande’s algorithm [9] which reduces the relative Gromov-Witten invariant of $(Y, D)$ of fiber class to that of $(\mathbb{P}^1, p_1)$. Note that the moduli space of stable relative maps

$$\mathcal{M}_Y = \mathcal{M}_{\Gamma, T_k}(Y, D)$$
where the interior push-forward with fiber isomorphic to the moduli space of maps of degree \( s \) to \( \mathbb{P}^1 \) relative to the infinity point \( p_1 \) with tangency order \( s \):

\[
\overline{\mathcal{M}}_{\mathbb{P}^1} = \overline{\mathcal{M}}_{\mathbb{P}^1, T_1}(\mathbb{P}^1, p_1).
\]

In fact, \( \overline{\mathcal{M}}_Y \) is the fiber bundle constructed from the principal \( S^1 \)-bundle associated to \( L \) and a standard \( S^1 \)-action on \( \overline{\mathcal{M}}_{\mathbb{P}^1} \).

By integrating along the fiber, we can compute the relative Gromov-Witten invariants of \( Y \) by computing the equivariant integrations in the relative Gromov-Witten theory of \( \mathbb{P}^1 \); see [9] for the details.

Let \( T_k = \{(t_i, \beta_i)\} \) be the cohomology weighted partition of \( s \). From the proof of Proposition 3.1, we may have, also see [9] for the details,

\[
\langle \tau_{d_1-1} \gamma_1, \ldots, \tau_{d_l-1} \gamma_l \mid T_k \rangle^Y_D = \int_{\overline{\mathcal{M}}_Y} \prod_{i=1}^l \psi_i^{d_i-1} \nu_i^* \gamma_i \wedge \prod_j \psi_j^* \beta_j = \frac{1}{|\text{Aut}(T_k)|} \int_{\mathcal{M}} \left( \prod_i \psi_i^{d_i-1} \nu_i^* (\gamma_i^{Z}) \cap [M_Y]_{\text{vir}} \right),
\]

where the interior push-forward

\[
\pi_* \left( \prod_i \psi_i^{d_i-1} \nu_i^* (\gamma_i^{Z}) \cap [M_Y]_{\text{vir}} \right)
\]

is obtained from the corresponding Hodge integral in the equivariant Gromov-Witten theory of \((\mathbb{P}^1, p_1)\) after replacing the hyperplane class on \( \mathbb{C} \mathbb{P}^\infty \) by \( C_1(L) \).

Therefore, via (3.11), we may reduce the computation of relative Gromov-Witten invariants \( \langle \tau_{d_1-1} \gamma_1, \ldots, \tau_{d_l-1} \gamma_l \mid T_k \rangle^Y_D \) to that of

\[
\langle \tau_{d_1-1} \delta_1, \ldots, \tau_{d_l-1} \delta_l \mid [pt], \ldots, [pt] \rangle^Y_D = \frac{1}{|\text{Aut}(T_k)|} \int_{\mathcal{M}} \left( \prod_i \psi_i^{d_i-1} \nu_i^* (\gamma_i^{Z}) \cap [M_Y]_{\text{vir}} \right),
\]

where \( \delta_i \in H^*(\mathbb{P}^1, \mathbb{Q}) \), \( 1 \leq i \leq l \).

About the two point genus zero relative Gromov-Witten invariant of \((\mathbb{P}^1, p_1)\), we have

**Lemma 3.4.** Let \( \varpi \in H^2(\mathbb{P}^1, \mathbb{Q}) \). Then

(i) If \( d \neq s \), then \( \langle \tau_{d-1} \varpi \mid (s, [pt]) \rangle^Y_{\mathbb{P}^1, p_1} = 0 \).

(ii) For \( s > 0 \), we have

\[
\langle \tau_{s-1} \varpi \mid (s, [pt]) \rangle^Y_{\mathbb{P}^1, p_1} = \frac{1}{s!}
\]

The proof of (i) follows from a simple dimension calculation and (ii) of the lemma is Lemma 1.4 of [12]. In [1], the authors generalized the result to general projective space \( \mathbb{P}^n \) via localization techniques.

**Proposition 3.5.** Let \( s > 0 \).

(i) Let \( T_k = \{(t_i, \beta_i)\} \) be a cohomology weighted partition of \( s \). Then

\[
\langle \nu^* \alpha_1, \ldots, \nu^* \alpha_q, \beta_1 \cdot [Z], \ldots, \beta_l \cdot [Z] \mid T_k \rangle^Y_{Y, F} = 0
\]

except for \( s = k = 1 \) and \( q = 0 \).

(ii) For \( s > 0 \), we have the two-point relative invariant

\[
\langle \tau_{d-1} (\beta_0 \cdot [Z]) \mid (s, \beta_\infty) \rangle^Y_{Y, F} = \begin{cases} \frac{1}{s!} \int_Z \beta_0 \wedge \beta_\infty, & d = s, \\ 0, & d \neq s, \end{cases}
\]

where \( \beta_0 \in H^*(Z, \mathbb{Q}) \) and \( \beta_\infty \in H^*(D, \mathbb{Q}) \).

(iii) For \( s = k = 1 \), we have

\[
\langle \iota_i^* (\beta_1), \ldots, \iota_i^* (\beta_l) \mid (1, \gamma) \rangle^Y_{F} = \int_Z \beta_1 \wedge \cdots \wedge \beta_l \wedge \gamma.
\]
Proof. (i) From (3.11), we are reduced to a relative Gromov-Witten invariant of $\mathbb{P}^1$ of the form
\[
\langle \mathbb{P}^1, \ldots, \mathbb{P}^1, [pt], \ldots, [pt] \mid (t_1, [pt]), \ldots, (t_k, [pt]) \rangle_{\mathbb{P}^1, [pt]}.
\] A dimension count shows that this invariant of $\mathbb{P}^1$ is nonzero only if $s + k = 2 - q$. Since $s > 0$ and $k > 0$, the only possibility is $s = k = 1$ and $q = 0$.

The proof of (ii) directly follows from (3.11) and Lemma 3.4. (iii) From (3.11), we have
\[
\langle \ell, \ldots, \ell, (1, \gamma) \rangle^{Y, D}_{\mathbb{P}^1, [pt]} = \int_{\mathbb{P}^1} \beta_1 \wedge \cdots \wedge \beta_\ell \wedge \gamma \langle [pt], \ldots, [pt] \mid (1, [pt]) \rangle_{\mathbb{P}^1, [pt]}.
\] It remains to prove $\langle [pt], \ldots, [pt] \mid (1, [pt]) \rangle_{\mathbb{P}^1, [pt]} = 1$. In fact, we consider the absolute invariant of $\mathbb{P}^1$ with $l + 1$ point insertions: $\langle [pt], \ldots, [pt] \rangle_{\mathbb{P}^1}$. First of all, by divisor axiom, we know that this absolute invariant equals 1. We apply the degeneration formula to this invariant of $\mathbb{P}^1$ and distribute one point insertion to one side and another $l$ point insertions to other side. Then we have
\[
1 = \langle [pt], \ldots, [pt] \rangle_{\mathbb{P}^1}
= \langle [pt], \ldots, [pt] \mid (1, [pt]) \rangle_{\mathbb{P}^1} \langle [pt] \mid (1, [pt]) \rangle_{\mathbb{P}^1}
= \langle [pt], \ldots, [pt] \mid (1, [pt]) \rangle_{\mathbb{P}^1}.
\]
In the last equality, we used Lemma 3.4. This proved (iii). \qed

\section{4 A comparison theorem}

Let $X$ be a compact symplectic manifold and $Z \subset X$ be a smooth symplectic submanifold of codimension 2. $\iota : Z \longrightarrow X$ is the inclusion map. The cohomological push-forward\[\iota^* : H^*(Z, \mathbb{R}) \longrightarrow H^*(X, \mathbb{R})\]
is determined by the pullback $\iota^*$ and Poincaré duality.

**Definition 4.1.** A symplectic divisor $Z$ is said to be *positive* if for some tamed almost complex structure $J$, $C_1(N_{Z;X})(A) > 0$ for any $A$ represented by a non-trivial $J$-sphere in $Z$.

This is a generalization of ample divisor from algebraic geometry. Define
\[
C_{\text{min}}(Z) := \min\{C_1(N_{Z;X})(A) > 0 \mid A \in H_2(Z, \mathbb{Z}) \text{ is a stably effective class}\}.
\]

To formulate our comparison theorem, we need to introduce *weighted genus zero completed cycles*. The completed cycles was first introduced by Okounkov and Pandharipande [12]. For an integer $k$, one defines a partition $\mu$ of $k$ to be a sequence of integers
\[\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0),\]
where $|\mu| = \sum \mu_i = k$. Let $\ell(\mu)$ denote the length of the partition $\mu$ and denote
\[\delta(\mu) := \prod \mu_i |\text{Aut}(\mu)|.
\]
Since we only consider the genus zero Gromov-Witten invariant in this paper, we introduce the genus zero completed cycle as follows.

**Definition 4.2.** For an integer $k$, define the genus zero completed cycle $(k+1)_0$ as
\[\langle (k+1)_0 \rangle := \sum_{\mu} k! \delta(\mu) \langle \tau_k \varpi \mid \mu \rangle_{0, k} \mu,
\]
where the summation runs over all the partitions of $k$ and the $\varpi \in H^2(\mathbb{P}^1, \mathbb{R})$ is the cohomology class of a point.
Proof.

On the completed cycles.

The homology class of a fiber of \( \pi \)

Witten invariants

\( W \)

submanifold

Suppose that degeneration formula of Gromov-Witten invariants. The central theorem of this section is Gromov-Witten invariants, which we call as a comparison theorem. The main tool of this section is the normal bundle of the divisor, we will give an explicit relation between the absolute and relative new invariants: the relative Gromov-Witten theory of \((X, Z)\).

In other words, we can choose \( \delta \)

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\( 1 \)

\( \ell \)

\( Z \)

Now we apply the degeneration formula for invariants \( Y \) \( D \):

Now we denote the weighted genus zero completed cycle by

\[ \Psi(\alpha_i, \mu, \tau_d - i^! (\beta_i), \cdots, \tau_d - i^! (\beta_j))^{X}_{A} \]

where \( (d_i)_{\delta}(\beta_i) \) is the corresponding weighted completed cycle and the number of insertions \( (1, [Z]) \) depends on the completed cycles.

\[ \Psi_C = \Psi(\mu) \langle \alpha_i, \mu, \tau_d - i^! (\beta_i), \cdots, \tau_d - i^! (\beta_j) \rangle^{X}_{A} \]

and express it as a summation of products of relative invariants of \((X, Z)\) and \((Y, D)\). Moreover, from the degeneration formula, each summand may consist of a product of relative Gromov-Witten invariants with disconnected domain curves of both \((X, Z)\) and \((Y, D)\).

On the side of \( Y \), there may be several disjoint components. Let \( A' \) be the total homology class. Then, from our assumption, we have \( Z \cdot A' = Z \cdot A \geq \sum d_i \). Suppose that we have a nonzero summand \( \Psi_C \neq 0 \) in the right-hand side of the above degeneration expression. Then \( \Psi_C \) is of the form

\[ \Psi_C = \Psi(\mu) \langle \alpha_i, \mu, \tau_d - i^! (\beta_i), \cdots, \tau_d - i^! (\beta_j) \rangle^{X}_{A} \]

\[ \Psi_C = \Psi(\mu) \langle \alpha_i, \mu, \tau_d - i^! (\beta_i), \cdots, \tau_d - i^! (\beta_j) \rangle_{\mu}^{Y, D} \]

where the superscript \( \bullet \) means that it may be a disjoint relative invariants.

From Proposition 3.1 and the assumption \( C_{\min}(Z) \geq \sum d_i \), the nonzero factor of the relative Gromov-Witten invariants \( \langle \alpha_i^+, i^! (\beta_i) \rangle_{\mu}^{Y, D} \) of \((Y, D)\) must be the fiber class relative invariants. Denote by \( F \) the homology class of a fiber of \( Y \), then \( A_2 = |\mu|F \).
From Proposition 3.5, we know that the nonzero factor in \( \langle \alpha^+_{i_1}, \ell^i(\beta_i) | \mu \rangle_{\mu, |\mu|}^{Y,D} \) must be of the form

\[
\langle \tau_{d_i-1}^i(\beta_i) | \{(\mu_1, \delta_1), \ldots, (\mu_{\ell(\mu)}, \delta_{\ell(\mu)})\} \rangle_{|\mu|}^{Y,D} \quad \text{with} \quad |\mu| < d_i,
\]

or

\[
\langle \{[1, |pt]\} \rangle_{F}^{Y,D},
\]

where \( \delta_1, \ldots, \delta_{\ell(\mu)} \) is a dual decomposition of \( \beta_i \), i.e., \( \int_Z \beta_i \wedge \delta_1 \wedge \cdots \wedge \delta_{\ell(\mu)} = 1 \).

From (3.11), for each \( i, 1 \leq i \leq l \), we have

\[
\langle \tau_{d_i-1}^i(\beta_i) | \{(\mu_1, \delta_1), \ldots, (\mu_{\ell(\mu)}, \delta_{\ell(\mu)})\} \rangle_{0, |\mu|}^{Y,D} = \langle \tau_{d_i-1}^i(\beta_i) \rangle_{0, |\mu|}^{Y,D} = \langle \tau_{d_i-1}^i(\beta_i) \rangle_{F}^{Y,D},
\]

where \( \beta_i \in H^2(\mathbb{P}^1, \mathbb{R}) \) is the cohomology class of a point. Moreover, if some \( \beta_i = [pt] \), then the marked point must be in a two-point component and the nonzero relative invariant must be \( \langle \ell^i([pt]) \rangle_{F}^{Y,D} = 1 \).

Since there are no vanishing two-cycles in this case, from the degeneration formula and Definition 4.2, we may write down the summation as follows.

\[
\langle \alpha_1, \ldots, \alpha_{\mu}, \tau_{d_i-1}^i(\beta_1), \ldots, \tau_{d_i-1}^i(\beta_i) \rangle_{A}^{X,Z} = \langle \alpha_1, \ldots, \alpha_{\mu}, \alpha_{l} | (\overline{d_i})_0(\beta_1), \ldots, (\overline{d_i})_0(\beta_i) \rangle_{A}^{X,Z},
\]

where \( \overline{d_i}(\beta_i) \) is the weighted completed cycle. We complete the proof of our comparison theorem.

**Corollary 4.4.** Under the assumption of Theorem 4.3, if \( d_i = 1 \) for all \( i \), then we have

\[
\langle \alpha_1, \ldots, \alpha_{\mu}, \ell^i(\beta_1), \ldots, \ell^i(\beta_i) \rangle_{A}^{X,Z} = \langle \alpha_1, \ldots, \alpha_{l} | T \rangle_{A}^{X,Z},
\]

where \( T = \{(1, \beta_1), \ldots, (1, \beta_i), (1, [Z]), \ldots, (1, [Z])\} \) is a weighted partition of \( Z \cdot A \).

**Remark 4.5.** McDuff also obtained the similar comparison result in the case without descendant classes, see Proposition 3.2 of [10].

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