Starting with the well-defined product of quantum fields at two spacetime points, we explore an associated Poisson structure for classical field theories within the deformation quantization formalism. We realize that the induced star-product is naturally related to the standard Moyal product through the causal Green functions connecting points in the space of classical solutions to the equations of motion. Our results resemble the Peierls-DeWitt bracket analyzed in the multisymplectic context. Once our star-product is defined we are able to apply the Wigner-Weyl map in order to introduce a generalized version of Wick’s theorem. Finally, we include a couple of examples to explicitly test our method: the real scalar field and the bosonic string. For both models we have encountered causal generalizations of the creation/annihilation relations, and also a causal generalization of the Virasoro algebra in the bosonic string case.

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I. INTRODUCTION

Standard quantization procedures for field theories rely to some extent on a Poisson structure at the classical level. Even though a classical field theory may be completely understood at either the Lagrangian or the Hamiltonian formalisms and, in spite of the mathematical elegance of both approaches, a covariant Poisson formulation for classical fields has not been completely embraced by a vast community of physicists. As it may be suspected, this is at the very heart of most of the relevant physical systems, including all of the fundamental interactions within the standard model, gravitation and string theory, to mention some [1, 2]. In this way, one naturally starts by considering a covariant classical field theory for which one may apply certain standard rules in order to get a quantized version. However, this rules impose at some point a preferred foliation of spacetime in order to fulfil the quantization programme, thus apparently hiding the covariant character of a given field theory.

In this direction, the deformation quantization approach was introduced in [3] as an alternative procedure for standard quantization. The deformation quantization programme has shown to be a mathematical consistent tool for the understanding of quantum systems ranging form standard quantum mechanics to quantum aspects of general Lie algebraic structures. In this formalism, quantizing a classical system simply consists on a deformation of the corresponding algebraic structures such as the algebra of smooth functions defined on the classical phase space. For details, we refer the reader to the seminal papers [3, 4], and the reviews [5–9, and references therein] for a wide range of applications and recent developments.

Our major objective in this paper is to develop, within the deformation quantization formalism, a legitimate algebraic causal Poisson bracket for classical field theories, and also to establish the relation of this causal Poisson bracket to the covariant Poisson structure determined by the Peierls-DeWitt bracket. In order to obtain this we adopt the familiar Kirchhoff representation [10–14] which states that given a field and its normal derivative at a given hypersurface we may find the value of the field at any causally connected point in the chronological future of the original point. This result is based on the construction of an appropriate Green’s function and, in principle, holds even for curved spacetimes. Thus our claim is that to the unambiguous well-defined product of two quantum field operators evaluated at different spacetime points it is possible to assign a correspondent classical causal Poisson structure. Certainly, the Wigner function allows us to implement the Kirchhoff representation in order to map both quantum field operators to the same spacetime points by means of the Stratonovich-Weyl quantizer which admits not only continuous differentiable functions but also distributions. This is the main
motivation to originally introduce the deformation quantization programme as, for example, distributions are not allowed in standard Poisson bracket, therefore precluding the implementation of the Kirchhoff representation in a straightforward manner. Further, the so-called correspondence principle, indicates that the resulting star-product is interrelated to a well-defined classical causal Poisson bracket at two different spacetime points given in terms of the causal Green’s function associated to the field equations. The introduced bracket also reduces to the standard Poisson bracket whenever we consider the two spacetime points in the same spatial hypersurface, that is, in the equal-time limit of field theory.

This causal Poisson bracket results equivalent to the covariant Peierls-DeWitt bracket [15–17] as far as linear field theories are considered (see also [18] for an excellent review on this topic). Examples of these linear field theories are given by non-interacting theories, harmonic Lagrangians and self-adjoint functionals, examples which encompass a huge amount of physically interesting field theories [19–22]. Peierls-DeWitt bracket covariant structure emerges naturally within the multisymplectic framework through either the Poincaré-Cartan form or the DeDonder-Weyl approach in the Lagrangian or the Hamiltonian formalisms, respectively [20–22]. Nonetheless, whenever we consider nonlinear field theories our bracket diverges from the Peierls-DeWitt bracket, as the causal Poisson bracket introduced is only related to the first variation of the action, and thus do not depend on a linearized version of the field equations. In this way, the difference among the brackets may be clearer if we bear in mind that for the Peierls bracket the involved causal Green’s functions turn out to be Jacobi fields, while for the causal bracket the causal Green’s functions are not necessarily Jacobi fields. In this sense, for nonlinear field theories, the causal bracket is not compulsorily covariant but it preserves the causal structure. Similar causal Poisson bracket structures has been implemented in a variety of contexts, including the conformal field theoretical WZNW model, the causal algebras, the localization of particles in QFT, to mention some [27–29].

In the case of field theories with interactions, a perturbative approximation must be considered, as in standard quantum field theory. However, our developed star-product lead us to obtain a generalization of Wick’s theorem for the product of field operators at different space-time points. This generalization involves convenient contractions of the field operators with the causal Green’s functions involved. Besides, we are able, by means of an isomorphism between star-products, to introduce a relation between our causal Green’s function and Feynman’s propagator, thus interpolating both approaches. These results resemble analogous developments found in deformation quantization from an algebraic quantum field theory perspective [32–33].

Finally, we test the causal Poisson bracket formalism for two widely known examples. Firstly, we analyze in detail the real scalar field. In particular, we find that the classical Poisson brackets may be extended to allow relations among the annihilation and creation coefficients at different spacetime hypersurfaces, generalizing the conventional relations at the equal-time limit analyzed in canonical quantization. Secondly, we also investigate the bosonic string. In this case, we also find a causal version of the Poisson brackets for the mode expansion coefficients, which in turn lead us to a generalized version of the Virasoro algebra at two different spatial hypersurfaces.

The paper is organized as follows. In Section II we give a brief review of deformation quantization in order to set the notational conventions used in the subsequent sections. In Section III we introduce the causal Poisson structure for field theory, and study its relevant properties. We test the causal Poisson bracket by developing in detail the analysis for the real scalar field and the bosonic string in Section IV. We include some concluding remarks in Section V. Finally, we leave technical demonstrations of some mathematical properties of the causal bracket to Appendix A.

II. DEFORMATION QUANTIZATION FOR FIELD THEORY

In classical mechanics [30], the phase space is given by a Poisson manifold $\mathcal{M}$, together with an antisymmetric Poisson tensor $\alpha^{ij}$, which endows the commutative algebra of complex-valued smooth functions $C^\infty(\mathcal{M})$ with a Lie algebraic structure by means of the bracket $\{\cdot,\cdot\}: C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ explicitly given by

$$\{f,g\} = \alpha^{ij}\partial_i f \partial_j g,$$

which, besides skew-symmetry and bilinearity, satisfies the Jacobi identity

$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\},$$

and the compatibility Leibnizian condition

$$\{f,gh\} = \{f,g\}h + g\{f,h\},$$

showing that the Poisson bracket is a derivation under both, the Poisson bracket itself and the standard commutative product of functions, respectively. Whenever the Poisson tensor $\alpha^{ij}$ is non-degenerate, the manifold $\mathcal{M}$ is said to be a symplectic manifold. Non-degenerate Poisson tensors mainly comprises systems without local symmetries, although for gauge invariant systems a symplectic manifold may be constructed in the so-called reduced phase space. For symplectic manifolds, the Jacobi identity turns out to have an immediate geometrical meaning, since it is equivalent to the closedness of the 2-form

$$\omega = \frac{1}{2} \omega_{ij}dx^i \wedge dx^j,$$

where $\omega_{ij}$ denotes the inverse matrix to $\alpha^{ij}$. 
A deformation quantization is an associative algebraic structure \( \mathcal{A} := (A(\mathcal{M}), \star) \) on the space \( A(\mathcal{M}) := C^\infty(\mathcal{M})[[\hbar]] \) of formal power series in a formal parameter \( \hbar \) with respect to an associative product, the so-called star-product \( \star \), satisfying for each \( f, \ g \in C^\infty(\mathcal{M}) \) the following properties

1. Locality property:
   \[
   f \star g = \sum_{k=0}^{\infty} \left( \frac{i\hbar}{2} \right)^k C_k(f, g),
   \]
   where \( C_k(f, g) \) are a sequence of bidifferential operators.

2. Deformation property: The star-product is a formal associative deformation of the classical commutative product, that is
   \[
   C_0(f, g) = fg.
   \]

3. Correspondence principle: The star-commutator allows us to define a formal deformation of the Poisson bracket
   \[
   C_1(f, g) - C_1(g, f) = i\hbar \{f, g\}.
   \]

Besides, two star-products, \( \star \) and \( \star' \), are said to be equivalent if there is an isomorphism between the algebras \( \mathcal{A} = (A, \star) \) and \( \mathcal{A}' = (A, \star') \) given by a formal differential operator \( T = 1 + \sum_r \hbar^r T_r \), where \( 1 \) stands for the identity operator and each \( T_r \) is a differential operator which is null on constants, and such that the differential operator \( T \) follows

\[
T(f \star' g) = (Tf) \star (Tg).
\]

This equivalence is related to the operator ordering ambiguity in ordinary quantum mechanics. In this way, defining a new star-product may be interpreted as a change in the ordering prescription in a quantum theory. Besides, this equivalence will be also relevant to understand the connection between our causal propagator and the Feynman propagator in standard quantum field theory, as we will see below.

In deformation quantization, the algebra of quantum observables turns out to be particularly simple, since it is made up by the set of real-valued functions on the phase space. In this manner there is no need of a Hilbert space as in the traditional operator approach, hence avoiding the more difficult problem about domains of unbounded operators. Furthermore, a very important point in deformation quantization comes from the existence of the Kontsevich theorem which provides a universal procedure to construct a well defined star-product starting with an arbitrary classical system, as it states that an arbitrary Poisson manifold admits a deformation quantization \([37–40]\).

Hereinafter, let us specialize our considerations so far to the case of an arbitrary field theory on four-dimensional Minkowski spacetime \( \mathcal{M} \). We will follow as close as possible the notation in references \([41, 42]\). As customarily, we will denote the canonical variables as \( \Phi(x) \) and \( \Pi_I(x) \), where the index \( I \) stands for the set of internal indices, and depends on the nature of each field (and may be omitted when possible), and spacetime points \( x = (x^0, x^i) \in \mathcal{M}, i = 1, 2, 3 \) standing for spatial indices. In deformation quantization, a common starting point will be to define either the Weyl map, or its inverse, the quasi-probabilistic Wigner function, both setting a relation between classical observables and quantum operators \([43, 44]\). We will thus start by constructing the Weyl map. Let \( F[\Phi, \Pi] \) be an arbitrary functional defined on the phase space \( \Gamma(\mathcal{M}) \) associated to \( \mathcal{M} \). We define its Fourier transformation by

\[
\hat{F}[\lambda, \mu] = \int \mathcal{D}\Phi\mathcal{D}\Pi \exp \left\{ -i \int dx (\lambda(x) \cdot \Phi(x) + \mu(x) \cdot \Pi(x)) \right\} F[\Phi, \Pi],
\]

where the formal functional measures are given by \( \mathcal{D}\Phi = \prod_x d\Phi(x) \) and \( \mathcal{D}\Pi = \prod_x d\Pi(x) \), respectively, and the central dot stands for contraction on the appropriate indices. Thus, the Weyl map in phase space is given by the quantum operator \( \hat{F} \) associated to \( F[\Phi, \Pi] \)

\[
\hat{F} := W(F[\Phi, \Pi])
\]

where \( \hat{U}[\lambda, \mu] \) stands for the unitary operator

\[
\hat{U}[\lambda, \mu] = \exp \left\{ i \int dx (\lambda(x) \cdot \hat{\Phi}(x) + \mu(x) \cdot \hat{\Pi}(x)) \right\}
\]

being \( \hat{\Phi} \) and \( \hat{\Pi} \) field operators satisfying \( \hat{\Phi}(x) \langle \Phi(x) \rangle = \langle \Phi(x) \rangle \hat{\Phi}(x) \) and \( \hat{\Pi}(x) \langle \Pi(x) \rangle = \langle \Pi(x) \rangle \hat{\Pi}(x) \), respectively. As shown in \([42]\), by employing the completeness relations \( \int \mathcal{D}\Phi \langle \Phi \rangle = \hat{1} \) and \( \int \mathcal{D}(\frac{d\Phi}{2\pi}) \langle \Pi \rangle = \hat{1} \), it is easy to check that the operator \( \hat{U}[\lambda, \mu] \hat{U}^\dagger[\lambda', \mu'] \) obeys the two very important properties

\[
\text{Tr} \left( \hat{U}[\lambda, \mu] \right) = \int \mathcal{D}\Phi \langle \Phi \rangle \hat{U}[\lambda, \mu| \Phi] = \delta \left( \frac{h\lambda}{2\pi} \right) \delta(\mu) \langle 9 \rangle
\]

\[
\text{Tr} \left( \hat{U}^\dagger[\lambda, \mu] \hat{U}[\lambda', \mu'] \right) = \delta \left( \frac{h}{2\pi}(\lambda - \lambda') \right) \delta(\mu - \mu') \langle 10 \rangle
\]

where the \( \delta \)'s stand for Dirac deltas. Relations \([9, 10]\)
and \( \Omega \) will be relevant in order to construct the quasi-probabilistic Wigner function, which assigns a classical observable to a given quantum operator. Before constructing the Wigner function we note that the Weyl quantization rule \( \hat{\Omega} \) may be written as

\[
\hat{F} = W(F[\Phi, \Pi]) = \int D\Phi D\Pi \frac{\Pi}{2\pi\hbar} F[\Phi, \Pi] \hat{\Omega}[\Phi, \Pi],
\]

(11)

where the operator \( \hat{\Omega}[\Phi, \Pi] \) denotes the standard Stratonovich-Weyl quantizer for quantum field theory.

Bearing in mind relations (9) and (10), it is straightforward to check that the Stratonovich-Weyl quantizer \( \hat{\Omega}[\Phi, \Pi] \) satisfies the identities

\[
\hat{\Omega}^\dagger[\Phi, \Pi] = \hat{\Omega}[\Phi', \Pi'], \quad \text{Tr} \left( \hat{\Omega}[\Phi, \Pi] \right) = 1,
\]

(13)

and

\[
\text{Tr} \left( \hat{\Omega}[\Phi, \Pi] \hat{\Omega}[\Phi', \Pi'] \right) = \delta(\Phi - \Phi') \delta \left( \frac{\Pi - \Pi'}{2\pi\hbar} \right).
\]

(15)

In this notation, the Wigner function simply reads

\[
F[\Phi, \Pi] = W^{-1}(\hat{F}) = \text{Tr} \left( \hat{\Omega}[\Phi, \Pi] \hat{F} \right).
\]

(16)

The next step is to construct a star-product which encloses an specific ordering prescription, as discussed before. We will follow here the standard Weyl-Moyal ordering \[35, 41\]. In order to define the field theoretical Moyal star-product, let \( F_1 = F_1[\Phi, \Pi] \) and \( F_2 = F_2[\Phi, \Pi] \) be some functionals on the phase space \( \Gamma(M) \), whose corresponding field operators are obtained through the Weyl map \( \Omega \) are \( F_1 = W(F_1) \) and \( F_2 = W(F_2) \), respectively. Thus, the Moyal product is defined by means of the convolution relation

\[
W(F_1 \ast F_2) = W(F_1)W(F_2),
\]

(17)

setting the functional corresponding to the product of two field operators via the Wigner function \( \Omega \) as

\[
(F_1 \ast F_2)[\Phi, \Pi] = W^{-1}(W(F_1)W(F_2)) = W^{-1}(\hat{F}_1 \hat{F}_2) = \text{Tr} \left( \hat{\Omega}[\Phi, \Pi] \hat{F}_1 \hat{F}_2 \right),
\]

(18)

which may be explicitly written in its integral representation as

\[
(F_1 \ast F_2)[\Phi, \Pi] = \int D\Phi' D\Phi'' D\Pi'/\pi\hbar D\Pi''/\pi\hbar F_1[\Phi', \Pi'] F_2[\Phi'', \Pi''] \\
\times \exp \left\{ \frac{2i}{\hbar} \int dx \left( (\Phi - \Phi') \cdot (\Pi - \Pi'') - (\Phi - \Phi') \cdot (\Pi - \Pi') \right) \right\}.
\]

(19)

Finally, using the Taylor series expansion for the functionals \( F_1 \) and \( F_2 \) we obtain the well-known expression

\[
(F_1 \ast F_2) = F_1[\Phi, \Pi] \exp \left\{ \frac{i\hbar}{2} \mathcal{P} \right\} F_2[\Phi, \Pi],
\]

(20)

where \( \mathcal{P} \) stands for the bidirectional functional derivative operator

\[
\mathcal{P} = \int dx \left( \frac{\delta}{\delta \Phi(x)} \cdot \frac{\delta}{\delta \Pi(x)} - \frac{\delta}{\delta \Phi(x)} \cdot \frac{\delta}{\delta \Pi(x)} \right).
\]

(21)

It is straightforward to prove that the Moyal star-product \( \mathcal{P} \) follows properties (i) to (iii) stated before in this section.

III. CAUSAL POISSON STRUCTURE FOR FIELD THEORY

Our main aim in this section will be to find a classical Poisson structure which corresponds to the product of two field operators evaluated at different spacetime points \( \Phi(x_1) \Phi(x_2) \), where \( x_1, x_2 \in M \). In order to achieve this, we will adopt the well-known Kirchhoff representation \[10\] appearing in the theory of partial differential equations \[11, 45\] and in the general analysis of relativistic wave equations \[12, 14\]. To start, let us consider a spacelike hypersurface \( \Sigma \), and suppose that the values for a field \( \Phi(x') \) satisfying the Lagrange equations of motion at a point \( x' \in M \), and its normal derivative
\[
\Phi^I(x) = - \int_\Sigma \left( \tilde{G}^{IJ}(x, x') \nabla^\alpha \Phi^J(x') - \Phi^J(x') \nabla^\alpha \tilde{G}^{IJ}(x, x') \right) d\Sigma_{\alpha'}, \tag{22}
\]

where \( \Phi^I(x) \) and \( \Phi^I(x') \) stands for the fields evaluated at two causally connected points \( x, x' \in \mathcal{M} \), respectively, and \( d\Sigma_{\alpha'} \) is the surface element defined on \( \Sigma \). It follows that if \( n_{\alpha'} \) is the future time-like unit normal, then 
\[ d\Sigma_{\alpha'} = n_{\alpha'} dS, \]
with \( dS \) representing the invariant volume element defined on \( \Sigma \).

The causal Green’s function \( \tilde{G}^{IJ}(x, x') \) is given by
\[
\tilde{G}^{IJ}(x, x') := G^{IJ}(x, x') - G^{-IJ}(x, x'), \tag{23}
\]
where \( G^{IJ}(x, x') \) and \( G^{-IJ}(x, x') \) denote the advanced and retarded Green’s function associated to the Euler-Lagrange operator, respectively. The causal Green’s function \( \tilde{G}^{IJ}(x, x') \) and its complex conjugate \((\tilde{G}^{IJ}(x, x'))^\ast\) follow the symmetry relations
\[
\tilde{G}^{IJ}(x, x') = -\tilde{G}^{JI}(x', x), \tag{24}
\]
\[(\tilde{G}^{IJ})^\ast(x, x') = \tilde{G}^{IJ}(x, x'). \tag{25}\]

These relations may be checked straightforwardly as a consequence of the reciprocity conditions of the advanced and retarded Green’s function
\[
G^{\pm JI}(x, x') = G^{\mp JI}(x', x), \tag{26}
\]
as discussed in [17]. At this point, it is important to mention that for a field theory on a flat spacetime the causal properties of the advanced (retarded) Green’s functions
\[
\hat{\Phi}^I(x_1) \hat{\Phi}^J(x_2) = W \left[ \Phi^I(x_1) \right] W \left[ \Phi^J(x_2) \right]
= W \left[ - \int_{\Sigma_2} \left( \tilde{G}^{IK}(x_1, x_2) \frac{\partial \Phi^K(x_2)}{\partial x_2^j} - \Phi^K(x_2) \frac{\partial \tilde{G}^{IK}(x_1, x_2)}{\partial x_2^j} \right) d\Sigma_2 \right] W \left[ \Phi^J(x_2) \right]. \tag{27}\]

By using the Wigner function [16] and the explicit representation of the Weyl quantization rule in terms of the
Stratonovich-Weyl quantizer [12] we straightforwardly obtain the following star-product
\[
\Phi^I(x_1) \ast \Phi^J(x_2) = \text{Tr} \left\{ \int \mathcal{D}\Phi'' \mathcal{D}(\Pi'') \left[ \frac{\Pi'}{2\pi \hbar} \right] \tilde{\Omega}(\Phi, \Pi) \tilde{\Omega}(\Phi', \Pi') \right\} \times \left[ - \int_{\Sigma_2} \left( \tilde{G}^{IK}(x_1, x_2) \frac{\partial \Phi^K(x_2)}{\partial x_2^0} - \Phi^K(x_2) \frac{\partial \tilde{G}^{IK}(x_1, x_2)}{\partial x_2^0} \right) d\Sigma_2 \right] \Phi^J(x_2). \tag{28}\]

1 For simplicity, and without loss of generality, we will consider the point \( x \) lying in the interior of the future light cone of the point \( x' \). This consideration, however is not fundamental, as the causal Green’s function considered in our Kirchoff representation completely determines the causal structure of the theory.
where we have used the Kirchhoff representation \(^{22}\) to map both field operators to the same spacetime point, \(x_2 \in \mathcal{M}\). Applying the trace identities of the Stratonovich-Weyl quantizer \(^{18-21}\), we see that expression \(^{28}\) is reduced to

\[
\Phi^I(x_1) \star \Phi^J(x_2) = \Phi^I(x_1) \Phi^J(x_2) + \frac{i\hbar}{2} \tilde{G}^{IJ}(x_1, x_2).
\]  

(29)

This result encompass the general behaviour of the Moyal product, and may be used to define the star-commutator

\[
\left[ \Phi^I(x_1), \Phi^J(x_2) \right] := \Phi^I(x_1) \star \Phi^J(x_2) - \Phi^J(x_2) \star \Phi^I(x_1),
\]

(30)

which in our case simply reduces to

\[
\left[ \Phi^I(x_1), \Phi^J(x_2) \right] = i\hbar \tilde{G}^{IJ}(x_1, x_2).
\]  

(31)

Of course, star-commutator \(^{31}\) must follow the deformation quantization axioms (i) to (iii) of Section II. Properties (i) and (ii) are directly satisfied. However, property (iii), the so-called correspondence principle, indicates that this star-commutator is interrelated to a classical Poisson structure at two different spacetime points given by

\[
\left( \Phi^I(x_1), \Phi^J(x_2) \right) = \tilde{G}^{IJ}(x_1, x_2).
\]  

(32)

Here we used round brackets instead of curly brackets in order to make a distinction from the standard Poisson bracket. It is important to mention that our classical functional bracket \(^{32}\) results, by construction, consistent with the product of two field operators at different spacetime points. Further, in analogy with the calculation above, it is straightforward to generalize our classical functional bracket to arbitrary functionals \(F_1(\Phi(x_1)), F_2(\Phi(x_2))\) of the fields variables at two causally connected points \(x_1, x_2 \in \mathcal{M}\) by the relation

\[
F_1(\Phi(x_1)) \star F_2(\Phi(x_2)) = F_1(\Phi(x_1)) \exp \left\{ \frac{i\hbar}{2} \tilde{K} \right\} F_2(\Phi(x_2)),
\]

(33)

where the bidifferential operator \(\tilde{K}\) is explicitly given by

\[
\tilde{K} := \exp \left\{ \frac{i\hbar}{2} \oint dxdx' \left( \frac{\delta}{\delta \Phi^M(x)} \tilde{G}^{MN}(x, x') \frac{\delta}{\delta \Phi^N(x')} \right) \right\}.
\]

(34)

Once again, by the correspondence principle (iii), this star-product leads to a well-defined Poisson structure given by

\[
(F_1[\Phi(x_1)], F_1[\Phi(x_2)]) := \int_{\mathcal{M}} dxdx' \frac{\delta F_1[\Phi(x_1)]}{\delta \Phi^M(x)} \tilde{G}^{MN}(x, x') \frac{\delta F_2[\Phi(x_2)]}{\delta \Phi^N(x')},
\]

(35)

which, as it is expected, turns out to be skew-symmetric, bilinear and obey both, the Jacobi identity \(^{2}\) and the Leibniz condition \(^{33}\). That the functional bracket \(^{35}\) is indeed a causal Poisson structure following these properties is shown in A2. It should also be noted that the causal Poisson bracket \(^{35}\) may be extended naturally to include gradients of the fields

\[
\begin{align*}
\left( \partial_\mu \Phi^I(x), \Phi^J(x') \right) &= \partial_\mu \tilde{G}^{IJ}(x, x'), \\
\left( \Phi^I(x), \partial_\mu \Phi^J(x') \right) &= \partial_\mu \tilde{G}^{IJ}(x, x'), \\
\left( \partial_\mu \Phi^I(x), \partial_\nu \Phi^J(x') \right) &= \partial^2_{\mu \nu} \tilde{G}^{IJ}(x, x').
\end{align*}
\]

(36)

The first limit holds from to the definition of the causal Green’s function in terms of the advanced and retarded Green’s functions. The second limit simply states the discontinuity of the causal Green’s function. Finally, the third limit holds since the second derivative of both \(G^{+IJ}\) and \(G^{-IJ}\) are proportional to a Dirac delta distribution.

Furthermore, it is possible to show that the causal Poisson bracket \(^{35}\) is a covariant bracket for the case of linear field theories. Indeed, let us consider two arbitrary observable functionals \(F_1 := F_1[\Phi(x_1)]\) and \(F_2 := F_2[\Phi(x_2)]\) at two causally connected points \(x_1, x_2 \in \mathcal{M}\). If we consider a variation of the action such that it leaves the action invariant, the bracket of these functionals transforms under this variation according to the law

\[
\delta (F_1, F_2) = 0,
\]

(38)
as demonstrated in [2, 24]. In this sense, this Poisson structure results in a covariant structure since it remains explicitly invariant under the symmetries of the action.

Another interesting feature emerging naturally for the functional causal Poisson bracket constructed from the deformation quantization approach is, as far as we consider linear field theories, that it is actually equivalent to the covariant Poisson structure derived from multisymplectic geometry [23, 24]. Indeed, for linear field theories, the so-called covariant phase space \( S \) consisting on the set of critical points of the action functional

\[
S := \{ \Phi \in \mathcal{C} : \delta S[\Phi] = 0 \},
\]

which are precisely the kernel of the Jacobi operator and, in this sense, the causal Green’s function defined above in [24], turns out to be a set of Jacobi fields. Of course, the covariant phase space is a subset of the configuration space, \( \mathcal{C} \). Formally, \( S \) can be thought of as a submanifold of \( \mathcal{C} \), whose tangent space at any \( \Phi \), agree with the tangent space \( T_{\Phi} S \) [12, 24, 47]. It follows that, this covariant phase space \( S \) carries a well-defined symplectic form \( \Omega \), which can be directly derived from the multisymplectic Poincaré-Cartan form \( \omega_I \) in the Lagrangian approach, and from the De Donder-Weyl multisymplectic form \( \omega_H \) in the corresponding Hamiltonian approach. Specifically, within the Lagrangian framework the symplectic form \( \Omega \) may be written as

\[
\Omega(\delta \Phi_1, \delta \Phi_2) = \int_\Sigma j^1(\Phi, \partial \Phi)^* \omega_L (\delta \Phi_1, \partial \delta \Phi_1, \delta \Phi_2, \partial \delta \Phi_2),
\]

where \( j^1(\Phi, \partial \Phi) \) is the first order jet prolongation of the configuration space, that is, the first order extension of the configuration space including the derivatives of the fields, and \( \delta \Phi, \partial \delta \Phi \) corresponds, respectively, to their induced variations [24] for further details. Then, it is possible to construct specific (distributional) solutions \( X^f_\Phi \) of the linearized field equations for any smooth functional \( F \) on the covariant phase space through the causal Green’s function

\[
X^f_\Phi = \int_M dx dy \frac{\delta F[\Phi]}{\delta \Phi^k},
\]

such that the Poisson bracket of two functional \( F_1 \) and \( F_2 \), can be written in complete analogy with the standard symplectic approach, given by

\[
\{ F_1, F_2 \} = \int_M dx \frac{\delta F_1[\Phi]}{\delta \Phi^k(x)} X^K_{F_2} - \int_M dx \frac{\delta F_2[\Phi]}{\delta \Phi^k(x)} X^K_{F_1},
\]

\[
+ \int_M dx dy \frac{\delta F_1[\Phi]}{\delta \Phi^M(x)} \tilde{\Omega}^{MN}(x, x') \frac{\delta F_2[\Phi]}{\delta \Phi^N(x')},
\]

which results equivalent to the functional bracket we have found through deformation quantization. Therefore, it is possible to construct a Poisson bracket not only related to the symplectic form \( \Omega \) and defined on a covariant phase space, but also suitably adapted to the multisymplectic approach and the infinite dimensional framework of field theories. This Poisson structure results precisely the Peierls-DeWitt bracket [12, 17], whose geometric form is given by the expression [24].

For the case of theories involving interacting fields we are confined to a perturbative framework, then we are interested in the star product of \( n \) fields, where due to the properties of the causal Poisson bracket, and the combinatorics of all contractions, this product becomes into a generalized version of the Wick’s theorem [1, 2]

\[
\Phi^{I_1}(x_1) \ast \Phi^{I_2}(x_2) \ast \cdots \ast \Phi^{I_n}(x_n) = \Phi^{I_1}(x_1) \Phi^{I_2}(x_2) \cdots \Phi^{I_n}(x_n)
\]

\[
+ \left( \frac{i\hbar}{2} \right) \sum_{\text{single pairs}} \left[ \tilde{G}^{I_1 I_2}(x_1, x_2) \Phi^{I_1}(x_1) \cdots \Phi^{I_2}(x_2) \cdots \Phi^{I_n}(x_n) \right]
\]

\[
+ \left( \frac{i\hbar}{2} \right)^2 \sum_{\text{double pairs}} \left[ \tilde{G}^{I_1 I_2}(x_1, x_2) G^{I_3 I_4}(x_3, x_4) \Phi^{I_1}(x_1) \cdots \Phi^{I_2}(x_2) \cdots \Phi^{I_3}(x_3) \cdots \Phi^{I_4}(x_4) \cdots \Phi^{I_n}(x_n) \right]
\]

\[
+ \cdots,
\]

where \( \Phi^{I_i}(x_i) \) denotes that the field \( \Phi^{I_i}(x_i) \) has been removed from the summation. The first sum runs over single contractions of pairs, while the second sum runs over double contractions, and so on. If \( n \) is even, the product ends with terms only consisting of products of casual Green’s functions. By making use of the equivalence of star products stated by means of isomorphisms between star algebras in [43], it is possible to write the time ordered product of quantum field operators through the normal ordering map \( \Theta_N \) [41]. This normal ordering map sends any functional defined on the phase space, \( F \) to the associated normal ordering operator \( \Theta_N[F] \). Then
Here, \( G_{F}^{MN}(x, x') \) stands for the Feynman propagator. We can observe that the time ordered product \( T \) do not correspond to the Weyl transform of a causal star-product since, by definition, the time ordered product is fully symmetric in its arguments while the causal star-product have skew-symmetry properties inherited from the construction of the causal Green’s function. Further, we see that the causal and Feynman Green’s functions may be constructed in terms of different combination of primitive Green’s functions as:

\[
\tilde{G} = G^{+} - G^{-} = G^{(+)} + G^{(-)}, \quad G_{F} = G^{-} + G^{(-)} = G^{+} - G^{(+)},
\]

(45)

where \( J(x) \) denotes an external source \[34]. Expanding equation (46) in powers of \( J \), we note that this term corresponds to the perturbation expansion of the scattering operator in quantum field theory, which has been derived entirely under the deformation quantization framework.

IV. EXAMPLES

In this section we put our previously obtained results at work by exploring the widely known examples of a real scalar field and of the bosonic string.

A. Real scalar field

We will work on Minkowski spacetime \( \mathcal{M} \). The action for a real scalar field \( \phi : \mathcal{M} \to \mathbb{R} \) reads \[1, 2\]

\[
S_{KG}[\phi] = -\int_{\mathcal{M}} d^{4}x \frac{1}{2} \left( (\partial_{\mu} \phi)(\partial^{\mu} \phi) - m^{2}\phi^{2} \right),
\]

(47)

where \( \mu = 0, 1, 2, 3 \) denote spacetime indices, and \( m \) is a constant mass term. Motion of the field is given by the well-known Klein-Gordon equation

\[
(\partial_{\mu} \partial^{\mu} + m^{2}) \phi = 0
\]

(48)

for which we may associate the usual advanced and retarded Green’s functions

\[
G^{+}(x, y) = \frac{-i}{(2\pi)^{3}} \int \frac{d^{3}k}{2\omega(k)} e^{i(\omega(k)(x^{0} - y^{0}) - k(x - y))},
\]

(49)

\[
G^{-}(x, y) = \frac{-i}{(2\pi)^{3}} \int \frac{d^{3}k}{2\omega(k)} e^{i(-\omega(k)(x^{0} - y^{0}) - k(x - y))},
\]

(50)

respectively. Here we have written \( \omega(k) = \pm \sqrt{k^{2} + m^{2}} \). Given two spacetime points \( x, y \in \mathcal{M} \), by relation (23) we construct the causal Green function \( \tilde{G}(x, y) \) as

\[
\tilde{G}(x, y) = \frac{1}{(2\pi)^{3}} \int \frac{d^{3}k}{\omega(k)} \sin \left[ \omega(k)(x^{0} - y^{0}) \right] e^{-i(\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}) + i\sigma(k^{2} + m^{2})/\hbar)}. \]

(51)

It may be easily shown that this causal Green’s function follows the equal time limits

\[
\lim_{x^{0} \to y^{0}} \tilde{G}(x, y) = 0,
\]

\[
\lim_{x^{0} \to y^{0}} \frac{\partial \tilde{G}(x, y)}{\partial x^{0}} = \delta(\mathbf{x} - \mathbf{y}) = -\lim_{x^{0} \to y^{0}} \frac{\partial \tilde{G}(x, y)}{\partial y^{0}},
\]

\[
\lim_{x^{0} \to y^{0}} \frac{\partial^{2} \tilde{G}(x, y)}{\partial x^{0}\partial y^{0}} = 0,
\]

(52)

which are relevant in our formulation in order to recover in this limit the standard Poisson bracket for the field \( \phi(x) \) and its conjugate momentum \( \pi(x) := (\partial \mathcal{L}/\partial \dot{\phi})(x) = \dot{\phi}(x) \), where the dot means derivative with respect to the \( x_{0} \) parameter. Thus, the causal Green’s function \( \tilde{G}(x, y) \) is used to establish the Kirchhoff representation

\[
\phi(x) = -\int_{\Sigma} \left( \tilde{G}(x, y)\pi(y) - \phi(y)\frac{\partial \tilde{G}(x, y)}{\partial y^{0}} \right) d^{3}y
\]

(53)

\[
\pi(x) = -\int_{\Sigma} \left( \frac{\partial \tilde{G}(x, y)}{\partial x^{0}} \pi(y) - \phi(y)\frac{\partial^{2} \tilde{G}(x, y)}{\partial x^{0}\partial y^{0}} \right) d^{3}y,
\]

where integrals are taken over a given hypersurface \( \Sigma \). From this representation, and by considering the causal
Poisson brackets introduced in (35), we find the elementary causal brackets

\[(\phi(x), \phi(y)) = \tilde{G}(x, y),\]

\[(\phi(x), \pi(y)) = \frac{\partial \tilde{G}(x, y)}{\partial x^0},\]

\[(\pi(x), \phi(y)) = \frac{\partial \tilde{G}(x, y)}{\partial y^0},\]

\[(\pi(x), \pi(y)) = \frac{\partial^2 \tilde{G}(x, y)}{\partial x^0 \partial y^0}.\] (54)

As stated before, by considering the limits (52) we see that these causal brackets simplify to the standard equal-time classical Poisson brackets at two different spatial points on a given hypersurface Σ.

Next, the real scalar field, \(\phi(x)\), and its conjugate momentum, \(\pi(x)\), may be written in terms of the annihilation and creation coefficients, \(a(k, x^0)\) and \(a^*(k, x^0)\), respectively, as

\[
\phi(x) = \frac{1}{(2\pi)^3} \int d^3 k \left( \frac{\hbar}{2\omega(k)} \right)^{1/2} \left( a(k, x^0) e^{ikx} + a^*(k, x^0) e^{-ikx} \right),
\]

\[
\pi(x) = \frac{i}{(2\pi)^3} \int d^3 k \left( \frac{\hbar \omega(k)}{2} \right)^{1/2} \left( -a(k, x^0) e^{ikx} + a^*(k, x^0) e^{-ikx} \right),
\] (55)

(56)

where \(a(k, x^0) := a(k) e^{-i\omega(k)x^0}\). As usual, relations (55) and (56) may be inverted in order to find the coefficients

\[
a(k, x^0) = \frac{1}{(2\hbar \omega(k))^{1/2}} \int d^3 x e^{-ikx} \left( \omega(k) \phi(x) + i\pi(x) \right),
\]

\[
a^*(k, x^0) = \frac{1}{(2\hbar \omega(k))^{1/2}} \int d^3 x e^{ikx} \left( \omega(k) \phi(x) - i\pi(x) \right).\] (57)

By repeatedly applying the causal bracket relations (54) we may obtain the classical commutation rules

\[
(a(k, x^0), a^*(k', y^0)) = \frac{i}{(4\omega(k) \omega(k'))^{1/2}} e^{i\omega(k)(x^0-y^0)} \delta(k-k'),
\]

\[
(a(k, x^0), a(k', y^0)) = -\frac{i}{(4\omega(k) \omega(k'))^{1/2}} e^{i\omega(k)(x^0-y^0)} \delta(k-k'),
\]

\[
(a^*(k, x^0), a^*(k', y^0)) = \frac{i}{(4\omega(k) \omega(k'))^{1/2}} e^{-i\omega(k)(x^0-y^0)} \delta(k-k').\] (58)

(59)

(60)

where we have substituted the causal Green’s function (51) and its derivatives, and we have explicitly performed the involved integrals. From these classical commutators we note that, due to the \(\omega(k)-\omega(k')\) factor and to the Dirac delta \(\delta(k-k')\), that the last two bracket are vanishing in a distributional sense, indicating the non-
interacting nature of the annihilation and creation coefficients at different times. This may be interpreted as a manifestation of energy conservation at two different spatial hypersurfaces. Also, the first classical commutation rule \[45\] generalizes the standard Poisson bracket allowing annihilation and creation coefficients at different spatial hypersurfaces. Clearly, these classical commutation rules reduce to the standard Poisson bracket at the equal-time limit.

This example may be also relevant for the analysis of the free electromagnetic field given by the action

\[ S_{\text{EM}}[A] := -\frac{1}{4} \int d^2x \, F_{\mu\nu} F^{\mu\nu} \quad (61) \]

where the electromagnetic field \( F_{\mu\nu} \) may be written in a common way in terms of the potential vector field \( A(x) \) by the relation \( F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \) (\( \mu, \nu = 0, 1, 2, 3 \)). Indeed, by fixing the radiation gauge \( A^i = 0 \), for example, the field equations for the spatial components reduce to \( \partial_\mu \partial^\mu A^i = 0 \) which may be thought of as a non-massive Klein-Gordon equation for each of the spatial components \( A^i \). In this sense, the electromagnetic case may be interpreted as three independent real scalar fields, as formulated in \[42\], and thus we may, in principle, extrapolate the results obtained here to the electromagnetic field.

### B. Bosonic string theory

As it is well known, the relativistic boson string may be described by the Nambu-Goto action. Variation of this action leads to non-linear equations of motion for the string due to the complexity of the momenta involved. In order to avoid this issues, we will then start with the classically equivalent Polyakov action \[48\] \[50\]

\[ S_P[X] = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{\mu\nu} \partial_\mu X^\nu \partial_\sigma X^\nu \eta_{\mu\nu} \quad (62) \]

Here the world-sheet \( \Sigma \) swept out by the string is parametrized by \( (\sigma, \tau) \), and \( X^\mu \) is the embedding of the world-sheet \( \Sigma \) into the spacetime manifold \( \mathcal{M} \) coupled to two-dimensional gravity with metric \( \gamma^{\alpha\beta}(\sigma, \tau) \) (\( \gamma = \text{det}(\gamma_{\alpha\beta}) \)). Finally, \( \alpha' \) is a parameter associated to the string scale squared, and may be commonly thought of as proportional to the inverse of the string tension. Equations of motion for the Polyakov action may be substantially reduced if one considers the choice \( \gamma_{\alpha\beta}(\sigma, \tau) = \eta_{\alpha\beta} e^{2\phi} \), where \( \eta \) is a two-dimensional Minkowski metric and \( e^\phi \) is the conformal factor for the spacetime function \( \phi \). In this way, the equations of motion are simply given by

\[ \left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^\mu = 0, \quad (63) \]

that is, the two-dimensional wave equation.

In order to construct the causal Green function, \( \tilde{G}(\sigma, \sigma'; \tau, \tau') \), we need first to impose appropriate boundary conditions: For an open string the total derivative term \( \partial_\tau X^\mu \) fix the boundary contributions, while periodicity conditions on the parameter \( \sigma \) must also be considered for a closed string. In this sense, we will consider boundary conditions \( X^\mu(\tau, 0) = X^\mu(\tau, \pi) \) for the open string, \( X^\mu(\tau, -\infty) = X^\mu(\tau, \infty) \) for the infinitely open string, and \( X^\mu(\tau, 0) = X^\mu(\tau, \pi), \quad X^\mu(\tau, 0) = X^\mu(\tau, \pi) \) together with \( \gamma^{\alpha\beta}(\tau, 0) = \gamma^{\alpha\beta}(\tau, \pi) \), for the closed string, respectively. Here \( X^\mu(\sigma, \tau) \) denotes derivative with respect to the parameter \( \sigma \). Thus, depending on these boundary conditions, we may find in a complete standard manner the causal Green’s functions:

\[ \tilde{G}_{\text{open}}(\sigma, \sigma'; \tau, \tau') = \sum_n \frac{1}{2n} \sin 2n(\tau - \tau') \cos 2n(\sigma - \sigma'), \quad (64) \]

\[ \tilde{G}_{\infty}(\sigma, \sigma'; \tau, \tau') = \theta[(\tau - \tau') - (\sigma - \sigma')], \quad (65) \]

\[ \tilde{G}_{\text{closed}}(\sigma, \sigma'; \tau, \tau') = \sum_n \frac{1}{2n} \sin 2n(\tau - \tau') \cos 2n\sigma \cos 2n\sigma', \quad (66) \]

for the open, the infinitely open, and closed strings, respectively \[61\]. In \[65\], the \( \theta \) stands for the Heaviside step-function. It is easy to see that for the three cases \[64\], \[65\], \[66\], the corresponding causal Green’s func-

\[ \lim_{\tau \to \tau'} \tilde{G}(\sigma, \sigma'; \tau, \tau') = 0, \]

\[ \lim_{\tau \to \tau'} \frac{\partial \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau} = \delta(\sigma - \sigma'), \]

\[ \lim_{\tau \to \tau'} \frac{\partial \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau'} = -\delta(\sigma - \sigma'), \]

\[ \lim_{\tau \to \tau'} \frac{\partial^2 \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau \partial \tau'} = 0, \quad (67) \]
which are very important in order to guarantee the standard Poisson bracket limit, as stated in Section III.

By considering the associated momenta $\Pi^\mu := \partial L / \partial (\dot{X}^\mu)$, for the fields $X^\mu$, where the dot means derivative with respect to the parameter $\tau$, and by using the Kirchhoff representation, we obtain

$$X^\mu(\sigma, \tau) = -\int d\sigma' \left( 2\pi\alpha' \tilde{G}(\sigma, \sigma'; \tau, \tau') \Pi^\mu(\sigma', \tau') \right) - \frac{\partial \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau'} X^\mu(\sigma', \tau')$$

$$\Pi^\mu(\sigma, \tau) = -\frac{1}{2\pi\alpha'} \int d\sigma' \left( 2\pi\alpha' \frac{\partial \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau} \Pi^\mu(\sigma', \tau') \right) - \frac{\partial^2 \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau \partial \tau'} X^\mu(\sigma', \tau') \right) , \quad (68)$$

for any of the causal Green’s functions $\tilde{G}$ (64)-(66). Thus, for these variables we construct the fundamental covariant brackets

$$(X^\mu(\sigma, \tau), X^\nu(\sigma', \tau')) = 2\pi\alpha' \eta^{\mu\nu} \tilde{G}(\sigma, \sigma'; \tau, \tau') ,$$

$$(X^\mu(\sigma, \tau), \Pi^\nu(\sigma', \tau')) = \eta^{\mu\nu} \frac{\partial \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau} ,$$

$$(\Pi^\mu(\sigma, \tau), X^\nu(\sigma', \tau')) = \eta^{\mu\nu} \frac{\partial^2 \tilde{G}(\sigma, \sigma'; \tau, \tau')}{\partial \tau \partial \tau'} . \quad (69)$$

As discussed before, these brackets reduce in the equal-time limit reduce to the standard Poisson brackets. Furthermore, the general solution to the wave equation (68) is given by $X^\mu(\sigma, \tau) = X^\mu_0(\tau + \sigma) + X^\mu_1(\tau - \sigma)$, for which we may write explicitly

$$X^\mu_0(\tau + \sigma) = \frac{1}{2} x^\mu + \alpha^\mu \pi^\mu(\tau + \sigma) + i \frac{(\alpha')}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-2i(n+\sigma)} ,$$

$$X^\mu_1(\tau - \sigma) = \frac{1}{2} x^\mu + \alpha^\mu \pi^\mu(\tau - \sigma) + i \frac{(\alpha')}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-2i(n-\sigma)} , \quad (70)$$

relations

$$x^\mu := \frac{1}{2\pi} \int X^\mu(\sigma, 0) d\sigma ,$$

$$\pi^\mu := \int \Pi^\mu(\sigma, 0) d\sigma . \quad (71)$$

Also, in a standard manner we find the coefficients $\alpha^\mu_n(\tau) := e^{-2i\sigma} \alpha^\mu_n$ and $\tilde{\alpha}^\mu_n(\tau) := e^{2i\sigma} \tilde{\alpha}^\mu_n$ in terms of $X^\mu$ and $\Pi^\mu$ by

$$\alpha^\mu_n(\tau) = \left( \frac{2}{\alpha'} \right)^{1/2} \int \left( \frac{i}{2\pi} X^\mu(\sigma, \tau) + \frac{\alpha'}{2} \Pi^\mu(\sigma, \tau) \right) e^{-2i\sigma} d\sigma ,$$

$$\tilde{\alpha}^\mu_n(\tau) = \left( \frac{2}{\alpha'} \right)^{1/2} \int \left( \frac{i}{2\pi} X^\mu(\sigma, \tau) + \frac{\alpha'}{2} \Pi^\mu(\sigma, \tau) \right) e^{2i\sigma} d\sigma . \quad (72)$$

Green’s function for an open string (63), and by repeated application of brackets (69), we are able to evaluate the causal brackets for these coefficients

$$\left( \alpha^\mu_n(\tau), \alpha^\mu_m(\tau') \right) = -im\delta_{n+m} e^{-2i(n-\tau')} \eta^{\mu\nu} ,$$

$$\left( \tilde{\alpha}^\mu_n(\tau), \tilde{\alpha}^\mu_m(\tau') \right) = -im\delta_{n+m} e^{-2i(n-\tau')} \eta^{\mu\nu} ,$$

$$\left( \alpha^\mu_n(\tau), \tilde{\alpha}^\mu_m(\tau') \right) = m \sin 2n(\tau - \tau') \delta_{nm} \eta^{\mu\nu} . \quad (73)$$

This solution stands for the open string. For the closed string we may also consider the relations $\alpha^\mu_n = \tilde{\alpha}^\mu_n$ in order to preserve the appropriate boundary conditions.
Note that these relations reduce to the standard relations in the equal-time limit. In addition, we may define the familiar classical observables

\[
L_n(\tau) := \frac{1}{2} \sum_{l=-\infty}^{\infty} \alpha^a_{n-l}(\tau) \alpha^a_l(\tau)
\]

\[
\tilde{L}_n(\tau) := \frac{1}{2} \sum_{l=-\infty}^{\infty} \tilde{\alpha}^a_{n-l}(\tau) \tilde{\alpha}^a_l(\tau)
\]

for which we find, after repeatedly applying (73) and using the Leibnizian rule for the causal Poisson bracket, a two-time generalization of the Virasoro algebra which resembles the standard Feynman propagator [54],

\[\mathcal{G}_{\text{open}}(\sigma, \sigma'; \tau, \tau') := \tau - \tau' - \frac{1}{4i} \ln \left(1 - e^{2i((\tau-\tau')+(\sigma-\sigma'))}\right) - \frac{1}{4i} \ln \left(1 - e^{2i((\tau-\tau')-(\sigma-\sigma'))}\right) + \frac{1}{4i} \ln \left(1 - e^{-2i((\tau-\tau')+(\sigma-\sigma'))}\right) + \frac{1}{4i} \ln \left(1 - e^{-2i((\tau-\tau')-(\sigma-\sigma'))}\right),\]  

setting the logarithmic behaviour of \(\mathcal{G}_{\text{open}}(\sigma, \sigma'; \tau, \tau')\) which resembles the standard Feynman propagator [54], \(G_F(\sigma, \sigma'; \tau, \tau')\), as stated at the end of Section [III]

V. CONCLUDING REMARKS

In quantum field theory, the product of field operators at different spacetime points is well-defined. This product, from the perspective of deformation quantization, may be extended to a star-product from which one defines the commutator of two quantum field operators at different spacetime points. This may be done in a complete covariant way. Thus, taking the deformation quantization as our guiding programme, we have focused on the construction of a classical Poisson structure inherited from this quantum commutator. To this end, we have considered the widely known Kirchhoff representation in order to map fields to a single spacetime point. In this sense, our choice of working within the deformation quantization scheme is justified, as neither the Wigner function nor the Stratonovich-Weyl quantizer confront problems whenever distributions are involved. Therefore, we have found a well-defined star-product for the fields at two different spacetime points. This star-product induces a classical causal bracket which follows the axioms of a Poisson structure, and may be extended trivially to obtain a bracket in the appropriate phase space. Also, the classical bracket introduced reduces to the standard Poisson bracket on the assumption that our two spacetime points lie on the same spatial hypersurface, that is, in the equal-time limit. Further, as far as we consider linear field theories, the introduced bracket results tantamount to the Peierls-DeWitt bracket, hence recovering a covariant classical Poisson structure.

For the case of theories involving interacting fields, we have encountered a generalization of Wick’s theorem for the star-product of fields at different spacetime points. This generalization introduces contractions with the causal Green’s function introduced in the Kirchhoff representation. Besides, the connection of our formalism with standard Feynman propagator was encountered by an appropriate isomorphism between star-algebras.

We have tested our formalism for two typical models. On the one side, we analyze the real scalar field for which we have deduced a generalization of the Poisson bracket relations of the classical coefficients associated to the quantum creation and annihilation operators at two different spatial hypersurfaces. This generalization may be straightforwardly extended to the quantum counterpart. On the other side, we also have studied the bosonic string. In this case we have encountered a generalization of the known Virasoro algebra at two different spacetime points. For both models, the introduced causal generalizations reduce to the standard results found in the literature at the equal-time limit.

Despite our results, further work has to be done in the direction of nonlinear field theories, for which a comparison of the developed causal bracket and the covariant Peierls-DeWitt bracket has to be analyzed in detail. Another interesting direction will be to implement the causal bracket for the case of singular Lagrangians.
strained systems in the context of quantization deformation were analyzed in 53, 56. Also, a recent proposal that incorporates the Peierls-DeWitt bracket for constrained systems may be found at reference [18]. These results may be promising, form our point of view, as they pave the way to understand the way the Kirchhoff representation might be introduced for singular systems. This will be done elsewhere.

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Appendix A: Mathematical properties of the classical causal bracket

In this Appendix we include some technical proofs of the mathematical properties followed by the functional causal Poisson bracket 35, 17, 57. Let \( F_a := F_a[\Phi(x_k)] \) denote the \( a \)-th functional on covariant phase space \( S \) attached to the spacetime point \( x_k \in \mathcal{M} \). In this Appendix, we will adopt the short notation for the classical causal Poisson bracket defined through deformation quantization in 35:

\[
0 = (F_1, F_2 + \alpha F_3) = F_1, M \tilde{G}^{MN} (F_2 + \alpha F_3), N
= F_1, M \tilde{G}^{MN} F_2, N + \alpha F_1, M \tilde{G}^{MN} F_3, N
= (F_1, F_2) + \alpha (F_1, F_3),
\]

for \( \alpha \) constant. The Leibniz rule is also directly obtained from the Leibniz rule property of the functional derivative

\[
(F_1, F_2 F_3) = F_1, M \tilde{G}^{MN} (F_2 F_3), N
= F_1, M \tilde{G}^{MN} (F_2, N F_3 + F_2 F_3, N)
= (F_1, M \tilde{G}^{MN} F_2, N) F_3 + F_2 (F_1, M \tilde{G}^{MN} F_3, N)
= (F_1, F_2) F_3 + F_2 (F_1, F_3).
\]

Finally, the Jacobi identity which in the adopted short notation reads

\[
P(F_1, F_2, F_3) := (F_1, (F_2, F_3)) + (F_2, (F_3, F_1)) + (F_3, (F_1, F_2))
= F_1, L P F_2, M F_3, N \left( \tilde{G}^{LM} \tilde{G}^{NM} + \tilde{G}^{MP} \tilde{G}^{NL} \right)
+ F_1, L F_2, M P F_3, N \left( \tilde{G}^{MN} \tilde{G}^{LP} + \tilde{G}^{NP} \tilde{G}^{LM} \right)
+ F_1, L F_2, M F_3, N P \left( \tilde{G}^{NL} \tilde{G}^{MP} + \tilde{G}^{LP} \tilde{G}^{MN} \right).
\]

may be demonstrated by using the skew-symmetry of the causal Green’s function \( \tilde{G}^{MN} \), together with the commutativity of the functional derivatives, \( F_{MN} = F_{NM} \), for all functional \( F \). Thus, it is straightforward to show that all expressions in the last equality are vanishing. For example, the first line in the last equality is

\[
\int_{\mathcal{M}} dxx' \frac{\delta F_1[\Phi(x_1)]}{\delta \Phi^M(x)} \tilde{G}^{MN}(x, x') \frac{\delta F_2[\Phi(x_2)]}{\delta \Phi^N(x')}.
\]
Thus, Jacobi identity also holds. As the four properties have been shown, we conclude that our causal bracket is a genuine Poisson bracket.

2. General covariance

When we are dealing with a linear field theory, as specified in the main text, our claim is that the classical causal bracket turns out to be covariant under symmetry transformations leaving the action invariant. To prove this, we start by noting that the transformation law of the causal bracket is given by

\[ \delta (F_1, F_2) = (\delta F_{1,M}) \tilde{G}^{MN} F_{2,N} + F_{1,M} \left( \delta \tilde{G}^{MN} \right) F_{2,N} + F_{1,M} \tilde{G}^{MN} (\delta F_{2,N}), \]

where \( \delta F_{1,M} \) (analogously for \( \delta F_{2,N} \)) explicitly reads

\[ \delta F_{1,M} = F_{1,MP} \delta \Phi^P = F_{1,MP} Q^P_{\alpha,M} \delta \xi^\alpha = -F_{1,P} Q^P_{\alpha,M} \delta \xi^\alpha, \]

thus demonstrating our claim.

being \( Q_\alpha \) a set of vector fields which leave the action invariant, that is, \( Q_\alpha S = 0 \). Also, we note that using the transformation laws for the causal Green’s function we get

\[ \delta \tilde{G}^{MN} = \delta G^{+MN} - \delta G^{-MN} = \left( Q^M_{\alpha,P} \tilde{G}^{PN} + Q^N_{\alpha,P} \tilde{G}^{MP} \right) \delta \xi^\alpha. \]

Then by virtue of these properties we conclude

\[ \delta (F_1, F_2) = \left[ -F_{1,P} Q^P_{\alpha,M} \tilde{G}^{MN} F_{2,N} + F_{1,P} Q^P_{\alpha,M} \tilde{G}^{MN} F_{2,N} + F_{1,M} Q^N_{\alpha,P} \tilde{G}^{MP} F_{2,N} - F_{1,M} Q^N_{\alpha,N} \tilde{G}^{MN} F_{2,P} \right] \delta \xi^\alpha = 0, \]

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