A Bessel delta-method and exponential sums for $GL(2)$

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Abstract. In this paper, we introduce a simple Bessel $\delta$-method to the theory of exponential sums for $GL_2$. Some results of Jutila on exponential sums are generalized in a less technical manner to holomorphic newforms of arbitrary level and nebentypus. In particular, this gives a short proof for the Weyl-type subconvex bound in the $t$-aspect for the associated $L$-functions.

1. Introduction

It is a classical problem to estimate exponential sums involving the Fourier coefficients of a modular form. Let $g \in S_k^1(M, \xi)$ be a holomorphic cusp newform of level $M$, weight $k$, nebentypus character $\xi$, with the Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n) n^{(k-1)/2} e(nz), \quad e(z) = e^{2\pi iz},$$

for $\text{Im} z > 0$. For example, it is well-known that for any real $\gamma$ and $N \geq 1$,

$$\sum_{n \leq N} \lambda_g(n) e(\gamma n) \sim_{\gamma, N} N^{1/2} \log(2N),$$

with the implied constant depending only on $g$ (cf. [Iwa, Theorem 5.3]). This is a classical estimate due to Wilton. This type of estimates with uniformity in $\gamma$ was generalized by Stephen D. Miller to cusp forms for $GL_3(\mathbb{Z})$ in [Mil] and further by the fourth-named author to arbitrary number fields for both $GL_2$ and $GL_3$ in [Qi1].

In this paper, we consider the following exponential sum (and its variants),

$$S^f(N) = S^f_j(N) = \sum_{N \leq n \leq 2N} \lambda_g(n) e(f(n)),$$

for a phase function $f$ of the form

$$f(x) = T \phi(x/N) + \gamma x,$$

where $\phi$ is real-valued and smooth (see Theorem 1.1), $\gamma$ is real, and $N, T \geq 1$ are large parameters. We assume here that $\phi$ is not a linear function, as otherwise the sum is already estimated in (1.1). As usual, we shall be mainly investigating the smoothed exponential sum

$$S(N) = S_j(N) = \sum_{n=1}^{\infty} \lambda_g(n) e(f(n)) V \left( \frac{n}{N} \right),$$

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for a certain smooth weight function $V \in C^\infty_c(0, \infty)$ supported in $[1, 2]$ as described in Theorem 1.1.

This type of exponential sums (with $\gamma = 0$) for modular forms $g$ of level $M = 1$ were first studied by Jutila [Jut1], using Farey fractions, the Voronoï summation formula, and stationary phase analysis. See also [Hux] §10 for an account of Jutila’s method.

Thanks to the Rankin–Selberg theory, we know that $|\lambda_g(n)|$’s obey the Ramanujan conjecture on average:

$$\sum_{n \leq N} |\lambda_g(n)|^2 \ll_g N.$$  

Moreover, by the work of Deligne, the Ramanujan conjecture for holomorphic cusp forms is now well-known:

$$\lambda_g(n) \ll n^\varepsilon.$$  

An application of the Cauchy–Schwarz inequality followed by (1.5) yields the trivial bounds $S(N), S^2(N) \ll_g N$. Thus one aims to improve over these trivial bounds or, in other words, to show that there is no correlation between $\lambda_g(n)$ and $e(f(n))$.

The primary purpose of this paper is to find a $\delta$-method which is analytically richer so that the stationary phase analysis at later stages becomes cleaner. It turns out that an added benefit of our pursuit is a generalization of some results in Jutila’s treatise [Jut1] to modular forms of arbitrary level and nebentypus. An application amongst others is the Weyl-type subconvex bound for the associated $L$-functions in the $t$-aspect.

The main novelty of our work is a simple Bessel $\delta$-method to be described as follows. Another feature of our approach is the use of a version of the Voronoï summation formula in which the test function is not necessarily compactly supported.

**A simple Bessel $\delta$-method.** As usual, let $e(x) = e^{2\pi i x}$ and let $J_\nu(x)$ be the $J$-Bessel function of order $\nu$. For a condition $C$, let $\delta(C)$ denote the Kronecker $\delta$ that detects $C$.

We fix a smooth bump function $U$ in $C^\infty_c(0, \infty)$. Our Bessel $\delta$-method is based on the observation that for a prime $p$, some large parameters $N, X$, and integers $r, n \in [N, 2N]$, one has

$$\frac{1}{p} \sum_{a \equiv (n-r) \pmod{p}} e\left(\frac{a(n-r)}{p}\right) \cdot \int_0^\infty e\left(\frac{2\sqrt{rx}}{p}\right) J_{k-1}\left(\frac{4\pi \sqrt{nx}}{p}\right) U\left(\frac{x}{X}\right) dx
$$

$$= \delta(r \equiv n \pmod{p}) \cdot \delta(|r-n| < X^\varepsilon p \sqrt{N/X}) \cdot \text{“some factor” + “error”}
$$

$$= \delta(r = n) \cdot \text{“some factor” + “error”},$$

provided that $N < X^{1-\varepsilon}$ and $p^2 < NX$. This is made explicit in Lemma 3.3. The merit of this Bessel $\delta$-identity is that it arises naturally from the Voronoï summation formula, for the Bessel integral may be interpreted as the Hankel transform of $e(2\sqrt{rx}/p) U(x/X)$.

As explained in §4.3, there is a vague but interesting connection between the Bessel integral above and the formula

$$\int_0^\infty J_{k-1}(4\pi a \sqrt{x}) J_{k-1}(4\pi b \sqrt{x}) dx \frac{\delta(a-b)}{8\pi^2 b},$$

where $\delta(a-b)$ is now the Dirac $\delta$-distribution. Thus the use of $\delta$ is justified from a different perspective.

**Main results.**
Theorem 1.1. Let \( \varepsilon > 0 \) be an arbitrarily small constant. Let \( N, T, \Delta \geq 1 \) be parameters such that

\[
N^\varepsilon \Delta \leq T.
\]

Let \( V(x) \in C_c^\infty((0, \infty)) \) be a smooth function with support in \([1, 2]\). Assume that its total variation \( \text{Var}(V) \leq 1 \) and that \( V^{(j)}(x) \ll j! \Delta^j \) for \( j \geq 0 \). For \( \gamma \) real, and \( \phi(x) \in C_c^\infty(1/2, 5/2) \) satisfying \(|\phi''(x)| \gg 1\) and \( \phi^{(j)}(x) \ll j \) for \( j \geq 1 \), define \( f(x) = T\phi(x/N) + \gamma x \). Let \( g \in \mathcal{S}_x^\ast(M, \xi) \) and \( \lambda_n(n) \) be its Fourier coefficients. Then

\[
\sum_{n=1}^N \lambda_n(n)e(f(n))V\left(\frac{n}{N}\right) \ll T^{1/3}N^{1/2+\varepsilon} + \frac{N^{1+\varepsilon}}{T^{1/6}},
\]

with the implied constant depending only on \( g, \phi \) and \( \varepsilon \).

Corollary 1.2. Let \( \phi, f \) and \( g \) be as above. Let \( N^{1+\varepsilon}/T \leq H \leq N \). We have

\[
\sum_{N \leq n \leq N+H} \lambda_n(n) e(f(n)) \ll_{g, \phi, \varepsilon} T^{1/3}N^{1/2+\varepsilon} + \frac{N^{1+\varepsilon}}{T^{1/6}}.
\]

As a consequence,

\[
S_f^\varepsilon(N) = \sum_{N \leq n \leq 2N} \lambda_n(n) e(f(n)) \ll_{g, \phi, \varepsilon} T^{1/3}N^{1/2+\varepsilon} + \frac{N^{1+\varepsilon}}{T^{1/6}}.
\]

Jutila’s estimate for \( S_f^\varepsilon(N) \), say, for modular forms \( g \) of level \( M = 1 \) and for phase functions \( f(x) = T\phi(x/N) \) (cf. [Hux §10]) is as follows,

\[
S_f^\varepsilon(N) \ll_{g, \phi, \varepsilon} T^{1/3}N^{1/2+\varepsilon},
\]

provided that \( N^{3/4} < T < N^{3/2} \).

Corollary 1.2 may be regarded as a generalization of Theorem 4.6 of Jutila [Jut1] in several aspects. First of all, the modular form \( g \) here is of arbitrary level and nebentypus. Secondly, the estimate in (1.10) is non-trivial as long as \( N^\varepsilon < T < N^{3/2-\varepsilon} \), while it is assumed in [Jut1] that \( N^{3/4} < T < N^{3/2} \). Note that our estimate is weaker than Jutila’s when \( N^{3/4} < T < N \). Nevertheless, we are usually more concerned with the case when \( N^{1-\varepsilon} < T < N^{3/2-\varepsilon} \), for example, in the subconvexity problem; our estimate is the same as Jutila’s in this case. Thirdly, our phase function \( f(x) \) contains an additional linear term \( \gamma x \).

For ease of exposition, only holomorphic modular forms are considered here, but our approach also works for Maass forms with some efforts. It seems also possible to generalize our method to number fields as in [Qil].

Examples. A typical and simple choice of \( \phi(x) \) is the power function \( \pm x^\beta \) so that \( f(x) = ax^\beta + \gamma x \) \((T = |a||N^\beta|)\). Let

\[
S_{a, \beta, \gamma}^\varepsilon(N) = \sum_{n \leq N} \lambda_n(n) e(an^\beta + \gamma n).
\]

For modular forms \( g \) of level \( M = 1 \), there are abundant works on this type of exponential sums in the literature (usually, with \( \gamma = 0 \)).

As alluded to above, the first non-trivial bound for \( S_{a, \beta, 0}^\varepsilon(N) \) was obtained by Jutila (cf. [Jut1 Theorem 4.6]) for the range \( 3/4 < \beta < 3/2, \beta \neq 1 \), as follows,

\[
S_{a, \beta, 0}^\varepsilon(N) \ll_{g, a, \beta, \varepsilon} N^{\gamma + \frac{\varepsilon}{2} + \varepsilon}.
\]
When $\beta = 1/2$, $\alpha = -2 \sqrt{q}$ for integer $q > 0$, and $\gamma = 0$, it was first shown by Iwaniec, Luo and Sarnak [ILS] (C.17) that the smoothed sum

$$\sum_{n=1}^{\infty} \lambda_q(n)e(-2\sqrt{qn})V\left(\frac{n}{N}\right)$$

has a main term of size $N^{3/4}$.

The first non-trivial bound towards $S_{a,\beta,\gamma}(N)$ for all $0 < \beta < 1$ is due to X. Ren and Y. Ye [RY1], who refined the aforementioned result of Iwaniec, Luo and Sarnak for $\beta = 1/2$, and proved for $\beta \neq 1/2$ that

$$S_{a,\beta,0}(N) \ll N^{3/4+\epsilon}.\tag{1.13}$$

This was improved into $N^{1/2+\epsilon}$ in [SW] for $0 < \beta < 1/2$ (the Maass form case is also considered there). Note that Jutila’s estimate (1.12) is stronger than (1.13) for $3/4 < \beta < 1$.

It should be mentioned that Q. Sun [Sun] obtained the bound $N^{1/2+\epsilon}$ for $S_{a,\beta,\gamma}(N)$ in the range $0 < \beta \leq 1/2$. Her bound was improved into $N^{3/4+\epsilon}$ by Godber [God] (for $0 < \beta < 1$). For $\gamma = 0$, these bounds are both weaker than (1.13).

There is also a very distinguishable result—Pitt’s uniform estimate for $S_{a,2,\gamma}(N)$ with quadratic phase in [Pit],

$$S_{a,2,\gamma}(N) \ll N^{3/4+\epsilon}, \tag{1.14}$$

where the implied constant depends only on $g$ and $\epsilon$. The exponent $15/16$ was later improved into $7/8$ by K. Liu and X. Ren [LR].

More generally, one can also consider analogous exponential sums of Fourier coefficients of Maass cusp forms for $GL_m$. Some similar results for $GL_3$ and $GL_m$ were obtained later by X. Ren and Y. Ye in [RY2, RY3]. Recently, Kumar et al. [KMS] had some improvement over the results in [RY2], by using the $\delta$-symbol method of Duke–Friedlander–Iwaniec [DFI] together with a conductor-lowering trick which was first introduced by Munshi [Mun1].

A direct consequence of Corollary 1.2 is the following estimates for $S_{a,\beta,\gamma}(N)$ for modular forms $g \in S_{a,1}^\ast(M,\xi)$.

**Corollary 1.3.** Let $g \in S_{a,1}^\ast(M,\xi)$ and $\lambda_q(n)$ be its Fourier coefficients. For real $\alpha, \gamma$ and $\beta$ with $\alpha \neq 0$, $\beta \neq 1$, we have

$$\sum_{n \leq N} \lambda_q(n) e(\alpha n^\beta + \gamma n) \ll_{\alpha,\beta,\gamma} |\alpha|^{3/4} N^{3/8+\epsilon} + N^{-1/4} N^{1/4+\epsilon}. \tag{1.15}$$

In particular,

$$\sum_{n \leq N} \lambda_q(n) e(\alpha n^\beta + \gamma n) \ll_{\alpha,\beta,\gamma} N^{3/4+\epsilon} + N^{1/4+\epsilon}. \tag{1.16}$$

Note that the estimate (1.16) is non-trivial for $0 < \beta < 3/2$. Though weaker for $3/4 < \beta < 1$, it is the same as Jutila’s estimate (1.12) for $1 < \beta < 3/2$. At any rate, our estimate is an extension of Jutila’s result (for $1 < \beta < 3/2$, literally) to modular forms of general level.

Also note that (1.16) is better than Ren and Ye’s estimate (1.13) as long as $\beta > 6/7$. However, our bound is worse than theirs for $\beta \leq 6/7$. This is due to the nature and the limitation of our Bessel $\delta$-method or any $\delta$- or circle method. For if $\beta$ is relatively small then $e(\alpha n^\beta)$ is not quite oscillatory, and it would not benefit much to separate the oscillations of $e(\alpha n^\beta)$ and $\lambda_q(n)$ by the $\delta$-method. The approach in [RY1] works far better in this
situation, where the Voronoi summation (with modulus 1) is applied directly, followed by stationary phase arguments.

In [Pit], the δ-method of Duke–Friedlander–Iwaniec, along with Diophantine approximation, is used to prove the estimate in (1.14) in the quadratic case β = 2. However, this approach does not work with fractional β.

Application: Weyl-type subconvex bound in the t-aspect. For g ∈ S_k^\ast(M,ξ) with Fourier coefficients λ_g(n), the associated L-function is given by

\[ L(s, g) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s}, \quad \text{Re } s > 1.\]

This L-series has an analytic continuation to the whole complex plane. The Phragmén–Lindelöf principle implies the t-aspect convex bound

\[ L(1/2 + it, g) \ll_{g,ε} (1 + |t|)^{1/2+ε} \]

for any ε > 0. Any improvement on the exponent on the right-hand side of the inequality is referred to as a subconvex bound, and in general it requires significant amount of work to achieve it.

When M = 1, the following Weyl-type subconvex bound was first proven by Good [Goo],

\[ L(1/2 + it, g) \ll_{g,ε} (1 + |t|)^{1/3+ε}, \]

by appealing to the spectral theory of automorphic functions. Later, the same bound was obtained by Jutila using his method developed in [Jut1]. See [Meu, Jut2] for the extension of these methods to the Maass-form case.

There has been much progress lately, due to new methods, especially variants of the δ-symbol or circle method become available. For example, Munshi [Mun1] solved the t-aspect subconvexity problem for L-functions on GL_3 by adopting Kloosterman’s version of the circle method. He also invented a GL_2 δ-method and used it in a series of papers [Mun2]–[Mun4] for various subconvexity problems. These methods were applied in [AS, AKMS] to obtain the Weyl bound in the GL_2 setting. In a recent preprint [Mun5], Munshi was even able to break the long standing Weyl-bound barrier by introducing extra variants into the GL_2 δ-method approach.

Recently, there are Weyl-type subconvexity results for cusp forms of general level by Booker et al. [BMN] and the first-named author [Agg]. Booker et al. [BMN] generalized Huxley’s treatment of Jutila’s method by using a Voronoï formula with arbitrary additive twists to obtain their result. On the other hand, Aggarwal [Agg] used a simple δ-symbol method and followed Munshi’s approach [Mun1]. This treatment allowed him to use the Voronoï formula of Kowalski–Michel–VanderKam to get the Weyl-type bound, along with an explicit dependence on the level of the cusp form.

By applying Theorem 1.4 with \( \phi(x) = -\log x \), we shall derive in §6 the Weyl subconvex bound for g ∈ S_k^\ast(M,ξ).

**Theorem 1.4.** Let g ∈ S_k^\ast(M,ξ). Then

\[ L(1/2 + it, g) \ll (1 + |t|)^{1/3+ε}. \]

with the implied constant depending only on g and ε.

This work is of the same theme as [Agg], but it is technically simpler here, for our Bessel δ-method is more intimate to the Voronoï summation formula than his trivial δ-method. Moreover, our argument by the Bessel δ-method is very short compared to that by the Jutila method generalized in [BMN].
A motivation of our work is from [AHLS], in which, together with Q. Sun, the first three named authors investigated subconvex bounds for $L(1/2, g \otimes \chi)$, where $\chi$ is a primitive Dirichlet character of prime conductor $q$. They were able to use a ‘trivial’ delta method to give a simpler proof for the Burgess bound in the $q$-aspect, 
\[ L(1/2, g \otimes \chi) \ll_{g, \epsilon} q^{3/8 + \epsilon}. \]

The Bessel $\delta$-method is an outcome of our search for a similar simple approach to strong subconvex bounds in the $t$-aspect. It seems natural that the argument of this paper can be combined with the approach in [AHLS] to obtain a uniform subconvexity bound for $L(1/2 + it, g \otimes \chi)$ in both the $q$ and $t$ aspects.

**Notation.** Let $p$ always stand for prime. The notation $n \sim N$ or $p \sim P$ is used for integers or primes in the dyadic segment $[N, 2N]$ or $[P, 2P]$, respectively.

2. The Voronoï summation

We shall use a version of the Voronoï summation formula, slightly more general than [KMV], Theorem A.4, in which the test function is not necessarily vanishing near zero.

Let $S_k^{\ast}(M, \xi)$ denote the set of primitive newforms of level $M$, weight $k$ and nebentypus $\xi$. The term “primitive” means that the form is Hecke-normalized so that its Fourier coefficients and Hecke eigenvalues coincide.

**Definition 2.1.** Let $C^\infty([0, \infty))$ denote the space of smooth functions $F(x)$ on $(0, \infty)$ which admit an asymptotic expansion $F(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$. Let $\mathcal{S}([0, \infty))$ denote the space of functions in $C^\infty_0([0, \infty))$ which are Schwartz at $\infty$, namely, with derivatives decaying faster than any negative power of $x$ as $x \to \infty$. For integer $k \geq 1$, define
\[ \mathcal{S}_k^{\text{sis}}([0, \infty)) = \left\{ x^{\frac{k}{2}} F(x) : F(x) \in \mathcal{S}([0, \infty)) \right\}. \]

**Lemma 2.2 (The Voronoï Summation Formula).** Let $g$ be a primitive holomorphic newform in $S_k^{\ast}(M, \xi)$. Let $a, \alpha, c$ be integers such that $c \geq 1$, $(a, c) = 1$, $\alpha a \equiv 1 \pmod{c}$ and $(c, M) = 1$. Let $F(x)$ be a function in $\mathcal{S}_k^{\text{sis}}([0, \infty))$ defined as in Definition 2.1. Then there exists a complex number $\eta_k(M)$ of modulus 1 and a newform $g^\ast \in S_k^{\ast}(M, \xi)$ such that
\[ \sum_{n=1}^{\infty} \lambda_{g^\ast}(n) e\left(\frac{an}{c}\right) F(n) = \xi(-c) \frac{\eta_k(M)}{c \sqrt{M}} \sum_{n=1}^{\infty} \lambda_g(n) e\left(-\frac{an}{c}\right) \tilde{F}(\frac{n}{c^2 M}). \]

where $\tilde{F}(y)$ is the Hankel transform of $F(x)$ defined by
\[ \tilde{F}(y) = 2\pi y \int_0^\infty F(x) J_{k-1}(4\pi \sqrt{xy}) \, dx \]

**Lemma 2.2** may be easily deduced from [KMV] Theorem A.4] via approximating $F(x) \in \mathcal{S}_k^{\text{sis}}([0, \infty))$ by functions in $C^\infty([0, \infty))$. It is known that the Hankel transform is an invertible map (indeed an isometry in certain sense) on the space $\mathcal{S}_k^{\text{sis}}([0, \infty))$ via the Hankel inversion formula (3.10). This may be proven by using (3.1), (3.2) and (3.3) in the same way of analyzing the Fourier transform.

The $\mathcal{S}_k^{\ast}$ space is introduced by Miller and Schmid in their work on the Voronoï summation formula for $\text{GL}_n(\mathbb{Z})$ [MS1, MS2, MS3] and further investigated in [O2] for both $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$ (the subscript “sis” stands for “simple singularity” at zero). Note that the $\mathcal{S}_k^{\ast}$ space is already used in their $\text{GL}_3$ Voronoï summation as in [MS2, Theorem 1.18].

The Voronoï summation formula in [KMV] Theorem A.4] is more general, where it is only required that $(c, M, M/(c, M)) = 1$. However, in our setting $c = p$ will be a large prime while $M$ is fixed, so our condition $(c, M) = 1$ in Lemma 2.2 is justified. For
comparison, we remark that, Jutila’s method requires the \( a/c \) to be every fraction, so this Voronoï works only if \( M \) is square-free; thus in [BMN], they need a more general Voronoï even without the restriction \(((c, M), M/(c, M)) = 1\).

### 3. A Bessel \( \delta \)-method

#### 3.1. Basics of Bessel functions.

For complex \( \nu \), let \( J_\nu(z) \) be the Bessel function of the first kind ([Wat]), defined by the series

\[
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z^2/4)^n}{n! \Gamma(n + 1)}.
\]

Moreover, we may write (cf. [WVW §16.12, 16.3, 17.5] or [Wat §7.2])

\[
J_\nu(x) = \frac{1}{\sqrt{2\pi x}} \left( e^{ix} W_{\nu, +}(x) + e^{-ix} W_{\nu, -}(x) \right),
\]

with

\[
x^j W_{\nu, j}(x) \ll_{\nu, j} 1, \quad x \gg 1.
\]

#### 3.2. Asymptotic of a Bessel integral.

For a fixed (non-negative valued) bump function \( U \in C^\infty_c(0, \infty) \), say with support in \([1, 2]\), \( a, b > 0 \) and \( X > 1 \), consider the Bessel integral

\[
I_k(a, b; X) = \int_0^\infty U(x/X) e(2a \sqrt{x}) J_{k-1}(4\pi b \sqrt{x}) dx.
\]

By [GR 6.699 1, 2], we have

\[
\int_0^\infty e^{iax} J_\nu(bx)x^{\mu-1} dx = \frac{e^{i(\nu+\mu)/2} b^{\nu 2} \Gamma(\nu + 1)}{2^{\nu + \mu} \Gamma(\nu + 1)} F \left( \frac{\nu + \mu + 1}{2}, \frac{\nu + 1}{2} \right) \left( \frac{b^2}{a^2} \right)
\]

for \( b > a > 0 \) and \(-Re \nu < Re \mu < 3/2\). By appealing to the Gaussian formula (cf. [MOS §2.1])

\[
F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \quad Re(\alpha + \beta - \gamma) < 0, \gamma \neq 0, 1, 2, ..., \]

and the duplication formula for the gamma function, we obtain

\[
\int_0^\infty e^{iax} J_\nu(ax)x^{\mu-1} dx = \frac{e^{i(\nu+\mu)/2} \Gamma(\nu + 1/2 - \mu)}{\sqrt{\pi(2a)^\mu} \Gamma(\nu - \mu + 1)} \left( -Re \nu < Re \mu < \frac{1}{2} \right),
\]

after letting \( b \to a \). Note that the limit \( b \to a \) is legitimate because both the integral on the left and the hypergeometric series on the right are absolutely and uniformly convergent for \(-Re \nu < Re \mu < 1/2\) (cf. [MOS §2.1]).

We first consider \( I_k(a, a; X) \) as defined in (3.4). By Mellin inversion

\[
I_k(a, a; X) = \frac{X}{2\pi i} \int_{(\sigma)} \hat{U}(s) \int_0^\infty 2e(a \sqrt{X} x) J_{k-1}(4\pi a \sqrt{X} x)x^{1 - 2s} dx ds,
\]

where \( \hat{U}(s) \) denotes the Mellin transform of the function \( U \), and \( (\sigma) \) stands for the contour \( Re s = \sigma \) as usual. Applying (3.5) to evaluate the inner integral, we infer that

\[
I_k(a, a; X) = \frac{X}{2\pi i} \int_{(\sigma)} \hat{U}(s) \frac{2^{k-1} \Gamma(k - 2s + 1) \Gamma(2s - 3/2)}{\sqrt{\pi} \Gamma(k + 2s - 2)} ds,
\]

for \( 3/4 < \sigma < (k + 1)/2 \). Assume that \( a^2 X > 1 \). By shifting the contour of integration to \( Re s = 0 \), say, and collecting the residues at \( s = 3/4 \) and \( 1/4 \), we obtain the following asymptotic for \( I_k(a, a; X) \).
Lemma 3.1. We have

\[ I_k(a, a; X) = \frac{(1 + i)^{k-1} \hat{U}(3/4)X}{4\pi(a^2X)^{1/4}} + O\left(\frac{X}{(a^2X)^{3/4}}\right), \]

with the implied constant depending only on \( k \) and \( U \).

We now consider \( I_k(a, b; X) \) as in (3.4) for \( a \neq b \). For this, we assume that \( b^2X > 1 \) so that \( J_{k-1}(4\pi b \sqrt{x}) \) is oscillatory. In view of (3.2) and (3.3), the lemma below is a direct consequence of Lemma A.1.

Lemma 3.2. Suppose that \( b^2X > 1 \). Then \( I_k(a, b; X) = O(X^{-A}) \) for any \( A \geq 0 \) if \( |a - b| \sqrt{X} > X^\varepsilon \).

3.3. Remarks on the Bessel integral. After suitable changes, Weber’s second exponential integral formula in [Wat 13.31 (1)] may be written as

\[ \int_0^\infty \exp(-2\pi x/X) J_{k-1}(4\pi a \sqrt{x}) J_{k-1}(4\pi b \sqrt{x}) \, dx \]

\[ = \left(\frac{X}{2\pi}\right) I_{k-1}(4\pi abX) \exp(-2\pi(a^2 + b^2)X), \]

for \( a, b, X > 0 \). Since \( J_{k-1}(4\pi a \sqrt{x}) \) and \( e(2a \sqrt{x}) \) have the same type of oscillation (see (3.2) or [Wat 7.21 (1)]), the Weber integral in (3.7) may be viewed as a variant of the Bessel integral in (3.4). However, the exponential function \( \exp(-2\pi x/X) \) is not as nice as the compactly supported function \( \hat{U}(x/X) \) from the perspective of Fourier analysis—the Fourier transform of \( \exp(-2\pi x/X) (x \in (0, \infty)) \) decays at \( \infty \) only to the first order.

The connection between the Weber integral and the Dirac \( \delta \)-distribution might be of its own interest. This justifies the use of \( \delta \) in another way.

According to [Wat 7.23 (2)], we have the asymptotic \( I_{k-1}(x) \sim \exp(x)/\sqrt{2\pi x} \) as \( x \to \infty \), so if one let \( X \to \infty \) then the right-hand side of (3.7) is asymptotic to

\[ \frac{\sqrt{2X} \exp(-2\pi(a - b)^2X)}{8\pi^2 \sqrt{ab}} = \frac{N(a - b, 1/\sqrt{4\pi X})}{8\pi^2 \sqrt{ab}} \cdot \frac{\delta(a - b)}{8\pi^2 b}, \]

where \( N(a - b, 1/\sqrt{4\pi X}) \) is the Gaussian distribution of variance \( 1/\sqrt{4\pi X} \) and \( \delta(a - b) \) is the Dirac \( \delta \)-distribution. Thus the limiting form of (3.7) is

\[ \int_0^\infty J_{k-1}(4\pi a \sqrt{x}) J_{k-1}(4\pi b \sqrt{x}) \, dx = \frac{\delta(a - b)}{8\pi^2 b}, \]

or

\[ \int_0^\infty J_{k-1}(ax) J_{k-1}(bx) \, dx = \frac{\delta(a - b)}{b}, \]

while this is equivalent to the Hankel inversion formula (cf. [Wat 14.3 (3), 14.4 (1)])

\[ \int_0^\infty x \, dx \int_0^\infty F(a) J_{k-1}(ax) J_{k-1}(bx) \, da = F(b), \]

for \( F(a) \in C^\infty(0, \infty) \) subject to the condition

\[ \int_0^\infty |F(a)| \sqrt{a} \, da < \infty. \]

3.4. A Bessel \( \delta \)-method. By Lemma 3.1 and 3.2 we have the following asymptotic \( \delta \)-identity.
Lemma 3.3. Let \( p \) be prime and \( N, X > 1 \) be such that \( X > p^2/N \) and \( X^{1-\varepsilon} > N \). Let \( r, n \) be integers in the dyadic interval \([N, 2N]\). For any \( A \geq 0 \), we have

\[
\frac{2\pi C_U r^{1/4}}{p^{1/2} N^{3/4}} \sum_{a \equiv (a \bmod p)} \frac{1}{p} e \left( \frac{a(n-r)}{p} \right) I_k \left( \frac{\sqrt{r}}{p}, \sqrt{n}; X \right) = \delta(r = n) \left( 1 + O_{k, U} \left( \frac{p}{\sqrt{NX}} \right) \right) + O_{k, U, A} (X^{-A}),
\]

(3.12)

where \( C_U = (1+i)/\tilde{U}(3/4) \), the \( \delta(r = n) \) is the Kronecker \( \delta \) that detects \( r = n \), and the implied constants depend only on \( k, U \) and \( A \).

Proof. Lemma 3.1 yields the \( \delta \)-term, while Lemma 3.2 implies that \( I_k(\sqrt{r}/p, \sqrt{n}/p; X) \) is negligibly small unless \( |r - n| \leq X^c \sqrt{N/X} \). On the other hand, the exponential sum in (3.12) gives us \( r \equiv n \pmod{p} \). Consequently, (3.12) follows immediately for \( X^c \sqrt{N/X} < p \) as assumed.

Q.E.D.

Remark 3.4. By the remarks after Lemma 3.2 \( F(x) = I_k(\sqrt{r}/p, \sqrt{n}/p; X) \) is in the space \( \mathcal{S}_k^\infty(0, \infty) \), so the Voronoï summation is applicable. When \( g^* = g \), it would be more convenient to apply the Voronoï summation in the reversed direction, so that one may avoid appealing to the Hankel inversion or working on the space \( \mathcal{S}_k^\infty(0, \infty) \).

Remark 3.5. We should point out that the identity

\[
\frac{1}{p} \sum_{a \equiv (a \bmod p)} e \left( \frac{a(n-r)}{p} \right) = \delta(n \equiv r \pmod{p})
\]

plays a key role in the work [AHLS]. In fact, the approach therein is based on the observation:

\[
\sum_{r \sim N} g(r) \sum_{n \sim X} \lambda_g(n) S(r, n; c) \approx X \sum_{n \sim N} \lambda_g(n) \chi(n),
\]

where the modulus \( c \) is chosen to be \( c = pq \gg N^{1+\varepsilon} \) and \( X = p^2 q^2 / N \); \( \chi \) is a primitive Dirichlet character modulo \( q \).

Here the Bessel-exponential integral \( I_k(\sqrt{r}/p, \sqrt{n}/p; X) \) serves the role of “lowering” the conductor of the underlying problem.

4. Application of the Bessel \( \delta \)-method and the Voronoï summation

We start with separating oscillations by writing

\[
S_0(N) = \sum_{n=1}^\infty \lambda_g(n) e(f(n)) V \left( \frac{n}{N} \right) = \sum_{r=1}^\infty e(f(r)) V \left( \frac{r}{N} \right) \sum_{n=1}^\infty \lambda_g(n) \delta(r = n).
\]

Applying the \( \delta \)-method identity (3.12) in Lemma 3.3 and dividing the \( a \)-sum according as \( (a, p) = 1 \) or not, we have

\[
S_0(N) = S_0^*(N, X) + S_0^0(N, X) + R_p(N, X) + O(X^{-A}),
\]

with

\[
S_0^*(N, X) = \frac{2\pi M^{1/2} N^{1/4}}{i^k \eta_k(M) p^{3/2} X^{3/4}} \sum_{r=1}^\infty e(f(r)) V \left( \frac{r}{N} \right) \sum_{a \equiv (a \bmod p)} e \left( \frac{ar}{p} \right) \sum_{n=1}^\infty \lambda_g(n) e \left( \frac{an}{p} \right) I_k \left( \frac{\sqrt{r}}{p}, \sqrt{n}; X \right).
\]

(4.1)
\[ S_p^0(N, X) = \frac{2\pi M^{1/2}N^{1/4}}{p^2 \eta_p(M) p^{3/2}X^{3/4}} \sum_{r=1}^\infty e\left(\frac{f(r)}{N}\right) \sum_{n=1}^\infty \lambda_p(n) I_k \left(\frac{\sqrt{r}}{p}, \frac{\sqrt{n}}{p}, X\right), \]

where \( V_q(x) = C_U \eta_q(M) M^{-1/2} \cdot x^{1/4} V(x) \) and \( \sum^* \) means that the \( a \)-sum is subject to \( (a, p) = 1 \), and

\[ R_p(N, X) = O\left(\frac{p}{\sqrt{N X}} \sum_{n=1}^N |\lambda_q(n)|\right) = O\left(p \sqrt{\frac{N}{X}}\right). \]

Assuming \( p > M \), we now apply the Voronoï summation in Lemma 3.2 to the \( n \)-variable. By applying Hankel’s inversion (3.10) to the integral (3.4), we infer that

\[ 2\pi k \int_0^\infty \left(\frac{\sqrt{r}}{p}, \frac{\sqrt{x}}{p}, X\right) J_{k-1} \left(\frac{4\pi \sqrt{n x}}{p \sqrt{M}}\right) dx = \frac{p^2}{2\pi} e\left(\frac{2\sqrt{nr}}{\sqrt{M} p}\right) U\left(\frac{n}{MX}\right). \]

Consequently, Lemma 2.2 yields (see Remark 3.4)

\[ S_p^0(N, X) = \frac{\xi(-p)N^{1/4}}{p^1/2X^{3/4}} \sum_{r=1}^\infty e\left(\frac{f(r)}{N}\right) V_q\left(\frac{r}{N}\right) \sum_{n=1}^\infty \lambda_p(n) S(n, r; p) e\left(\frac{2\sqrt{nr}}{\sqrt{M} p}\right) U\left(\frac{n}{MX}\right), \]

where, as usual, \( S(n, r; p) \) is the Kloosterman sum

\[ S(n, r; p) = \sum_{a \pmod{p}} e\left(\frac{an + \overline{ar}}{p}\right). \]

Similarly,

\[ S_p^0(N, X) = \frac{p^{1/2}N^{1/4}}{X^{3/4}} \sum_{r=1}^\infty e\left(\frac{f(r)}{N}\right) V_q\left(\frac{r}{N}\right) \sum_{n=1}^\infty \lambda_p(n) \frac{2\sqrt{nr}}{\sqrt{M} p} U\left(\frac{n}{MX}\right), \]

after Voronoï. Estimating trivially, we find that

\[ S_p^0(N, X) \ll \frac{N^{5/4}X^{1/4}}{p^{3/2}}. \]

Finally, we introduce an average over primes \( p \) in \([P, 2P]\) for a large parameter \( P \). The results that we have established are summarized as follows.

**Proposition 4.1.** Let parameters \( N, X, P > N^\varepsilon \) be such that

\[ P^2/N < X, \quad N < X^{1-\varepsilon}. \]

Let \( P^* \) be the number of primes in \([P, 2P]\), satisfying \( P^* = P/\log P \). We have

\[ S(N) = \sum_{n=1}^\infty \lambda_p(n) e(f(n)) V\left(\frac{n}{N}\right) = S(N, X, P) + O\left(\frac{P \sqrt{N}}{\sqrt{X}} + \frac{N^{5/4}X^{1/4}}{P^{3/2}}\right), \]

with

\[ S(N, X, P) = \frac{N^{1/4}}{P^*X^{3/4}} \sum_{p=P^*}^\infty \sum_{r=1}^\infty e\left(\frac{f(r)}{p}\right) V_q\left(\frac{r}{N}\right) \sum_{n=1}^\infty \lambda_p(n) S(n, r; p) e\left(\frac{2\sqrt{nr}}{\sqrt{M} p}\right) U\left(\frac{n}{MX}\right), \]

where \( V_q(x) = C_U \eta_q(M) M^{-1/2} \cdot x^{1/4} V(x) \) is again supported in \([1, 2]\), satisfying \( \text{Var}(V_q) \ll 1 \) and \( V_q^{(j)}(x) \ll j A^j \).
5. Application of the Poisson summation and the Cauchy inequality

In view of Proposition 1.1 to study $S(N)$ it suffices to consider the sum $S(N, X, P)$ defined in (4.9). For convenience of our analysis, we let
\begin{equation}
X = P^2 K^2 / N, \quad N^\varepsilon < K < T^{1-\varepsilon},
\end{equation}
with the parameter $K$ to be optimized later. Then the first assumption in (4.7) is justified, while the second assumption $N < X^{1-\varepsilon}$ amounts to
\begin{equation}
P > N^{1+\varepsilon}/K.
\end{equation}

5.1. First application of the Poisson summation. Recall that \( f(r) = T\phi(r) + \gamma r \)
(as in (1.3)). By applying Poisson summation to the $r$-sum in (4.9), we have
\begin{equation}
\sum_{r=1}^{\infty} e(f(r))S(n, r; p)e \left( \frac{2\sqrt{n}r}{\sqrt{M}p} \right) V_n \left( \frac{r}{N} \right) = N \sum_{(r, p) = 1} e \left( -\frac{\gamma n}{p} \right) \bar{g}(n, r, p),
\end{equation}
where
\begin{equation}
\bar{g}(y, r, p) = \int_0^\infty V_n(x) e \left( T\phi(x) + \gamma Nx + \frac{2\sqrt{Nxy}}{\sqrt{M}p} - \frac{rNx}{p} \right) dx.
\end{equation}
Recall that \( \gamma (= n) \sim MX \). Lemma A.1 implies that \( \bar{g}(y, r, p) \) is negligibly small if \( N|r/p - y| \gg \max \{ T, \sqrt{N}/p \} = \max \{ T, K \} = T \), provided that \( \phi^{(j)}(x) \ll 1 \) \( (j \geq 1) \) and that \( V^{(j)}(x) \ll J \) for \( J < T/N^{\varepsilon} \). Accordingly, set
\begin{equation}
R = PT/N.
\end{equation}
So if we assume that
\begin{equation}
A < T/N^{\varepsilon},
\end{equation}
then we can effectively truncate the sum at \( |r - \gamma p| = R \), at the cost of a negligible error. Note that (5.5) amounts to the condition (1.7) in Theorem 1.1

Moreover, the second derivative test in Lemma A.2 immediately yields the following estimate for \( \bar{g}(y, r, p) \).

**Lemma 5.1.** Suppose that \( |\phi''(x)| \gg 1 \). Then, for \( 1 \leq y/MX \leq 2 \), we have
\begin{equation}
\bar{g}(y, r, p) \ll 1/\sqrt{T}.
\end{equation}

**Proof.** The second derivative of the phase function in (5.3) is equal to
\begin{equation}
T\phi''(x) - \frac{\sqrt{Ny}}{2p \sqrt{Mxx}} = T\phi''(x) + O(K).
\end{equation}
By our assumptions, \( |\phi''(x)| \gg 1 \) and \( K < T^{1-\varepsilon} \), the estimate above follows easily from Lemma A.2. Q.E.D.

Consequently, (4.9) is transformed into
\begin{equation}
S(N, X, P) = \frac{N^2}{P^2 (PK)^{3/2}} \sum_{n=1}^{\infty} \lambda_{\phi^n}(n) U \left( \frac{n}{MX} \right) \sum_{p \sim \sqrt{N}} \sum_{\{r, p \} = 1} e \left( -\frac{\gamma n}{p} \right) \bar{g}(n, r, p)
\end{equation}
\begin{equation}
+ O(N^{-A}).
\end{equation}

5.2. Application of the Cauchy inequality and the second Poisson summation.

Next we apply Cauchy and the Ramanujan bound on average for the Fourier coefficients
\( \lambda_p(n) \) as in (1.5). Thus,
\[
S(N, X, P) \ll \frac{N^{3/2}}{P^{1/2}} \left( \sum_{n=1}^{\infty} \sum_{p \sim P} \sum_{(r,p)=1}^{\infty} e \left( \frac{-\tau_n}{p} \right) \mathcal{J}(n, r, p) \right)^2 U \left( \frac{n}{MX} \right)^{1/2}.
\]

Opening the square and switching the order of summations, the square of the right-hand side is
\[
\frac{N^3}{P^{1/2} \sqrt{P K}} \sum_{p_1, p_2 \sim P} \sum_{(r, p_1)=1}^{\infty} \sum_{|r-\gamma p_1| \ll R} \sum_{n=1}^{\infty} e \left( \frac{\tau_{p_1 n}}{p_2} - \frac{\tau_{p_1 n}}{p_1} \right) \mathcal{J}(n, r_1, p_1) \mathcal{J}(n, r_2, p_2) U \left( \frac{n}{MX} \right) \right)^{1/2}.
\]

Remark 5.2. To keep in mind some representative cases, we notice that the diagonal contribution \( (p_1, r_1) = (p_2, r_2) \) towards \( S(N, X, P) \) is
\[
\frac{N^{3/2}}{P^{1/2}} \left( \sum_{n=1}^{\infty} \sum_{p \sim P} \sum_{(r, p)=1}^{\infty} |\mathcal{J}(n, r, p)|^2 U \left( \frac{n}{MX} \right) \right)^{1/2} \ll \sqrt{NK \log P},
\]

where \( R = PT/N \) as in (5.4) and we have used the bound in Lemma 5.1 for \( \mathcal{J}(n, r, p) \).

It is therefore important to introduce the extra average over \( p \) as in (4.19) because without it the diagonal contribution would be \( O(\sqrt{pNK}) \) instead.

We then apply Poisson summation with modulus \( p_1, p_2 \) (note that \( p_1 \) and \( p_2 \) need not be distinct) to the \( n \)-sum in (5.8), getting
\[
\sum_{n=1}^{\infty} e \left( \frac{\tau_{p_2 n}}{p_2} - \frac{\tau_{p_1 n}}{p_1} \right) \mathcal{J}(n, r_1, p_1) \mathcal{J}(n, r_2, p_2) U \left( \frac{n}{MX} \right)
\]
\[
= \frac{MX}{p_1 p_2} \sum_{n=-\infty}^{\infty} \sum_{a(\mod p_1 p_2)} e \left( \frac{a \tau_{p_2 n}}{p_2} - \frac{a \tau_{p_1 n}}{p_1} + \frac{a n}{p_1 p_2} \right) \cdot \mathcal{L} \left( \frac{MX r_1}{p_1 p_2}; r_2, p_2 \right),
\]

with
\[
\mathcal{L}(x) = \mathcal{L}(x; r_1, r_2, p_1, p_2) = \int_{0}^{\infty} U(y) \mathcal{J}(MX y, r_1, p_1) \mathcal{J}(MX y, r_2, p_2) e \left( -xy \right) dy.
\]

Recall that \( \sqrt{NX} = PK \) as in (5.1) and that \( \mathcal{J}(MX y, r, p) \) is defined as in (5.3). We have
\[
\mathcal{L}(x) = \int_{0}^{\infty} \int_{0}^{\infty} V_{x}(v_1) V_{x}(v_2) e \left( T \left( \phi(v_1) - \phi(v_2) \right) + \gamma N(v_1 - v_2) - \frac{N r_1 v_1}{p_1} + \frac{N r_2 v_2}{p_2} \right) \cdot \int_{0}^{\infty} U(y) e \left( 2PK \left( \frac{\sqrt{v_1}}{p_1} - \frac{\sqrt{v_2}}{p_2} \right) \sqrt{y} - xy \right) dy dv_1 dv_2.
\]

We note that the \( a \)-sum in (5.9) yields the congruence condition
\[
n \equiv \tau_{p_1 p_2} - \tau_{p_1 p_2} (\mod p_1 p_2),
\]
where \( \tau_1 \) and \( \tau_2 \) denote the multiplicative inverses of \( r_1 \) and \( r_2 \) modulo \( p_1 \) and \( p_2 \) respectively. Thus the right-hand side of (5.9) is simplified to
\[
\sum_{n=\tau_1 p_1 - \tau_2 p_2 (\mod p_1 p_2)}^{\tau_1 p_2 - \tau_2 p_1} \mathcal{L} \left( \frac{MX n}{p_1 p_2}; r_1, r_2, p_1, p_2 \right).
\]

5.3. Analysis of the integral \( \mathcal{L}(x) \). This section is dedicated to the analysis of the integral \( \mathcal{L}(x) \) as defined in (5.10) or (5.11).
By applying Lemma 5.4 to the integral in (5.10), we obtain the following (trivial) estimate.

LEMMA 5.3. We have

\[ \mathcal{L}(x) \ll 1/T. \]

Further, we wish to improve the estimate above by examining the triple integral in (5.11), especially the \( y \)-integration.

LEMMA 5.4. For real \( w, x \), with \( |w| \leq \sqrt{2} - 1/2 < 1 \), define

\[ \mathcal{K}(wK, x) = \int_0^\infty U(y)e(2wK\sqrt{y} - xy)\,dy \]

so that the \( y \)-integral in the second line of (5.11) is equal to \( \mathcal{K}(wK, x) \) with \( w = \sqrt{v_1}P/p_1 - \sqrt{v_2}P/p_2 \).

1. We have \( \mathcal{K}(wK, x) = O(N^{-A}) \) if \( |x| \geq K \).
2. For \( |x| > N^\varepsilon \), we have \( \mathcal{K}(wK, x) = O(N^{-A}) \) unless \( 2/3 < wK/x < 3/2 \), say, and for \( 1/2 < wK/x < 2 \), if we let \( \lambda = K^2w^2/x \) and \( W(\lambda) = W(\lambda, x) = e(-\lambda)\mathcal{K}(\sqrt{x}x, x) \) then

\[ \lambda^{j}\mathcal{K}(\lambda) \ll j/|x|. \]

3. \( \mathcal{K}(wK, 0) = W_0(2wK) \) for some Schwartz function \( W_0 \).

PROOF. The statements in (1) and the first part of (2) follow from Lemma A.1. Since \( W_0 \) is the Fourier transform of \( -2yU(y)^2 \), (3) is also clear. It is left to prove (5.16) for \( 1/4 < \lambda/x < 4 \). For this, we change the variable \( y \) to \( \lambda y/x = w^2K^2y/x^2 \) in (5.15) so that

\[ W(\lambda, x) = \frac{\lambda^j}{x} \int_0^{\infty} U(\lambda y/x)e(-\lambda(1 - 2\sqrt{y} + y))\,dy. \]

Then the estimates in (5.16) follow from Lemma A.4. Q.E.D.

LEMMA 5.5. Let \( K \ll T \). Suppose that \( \phi(1)(v) \ll 1 \) for \( j = 2, 3 \) and that \( |\phi''(v)| \geq 1 \) for all \( v \in (1/2, 5/2) \). Let the integral \( \mathcal{L}(x) \) be as in (5.11).

1. We have \( \mathcal{L}(x) = O(N^{-A}) \) if \( |x| \geq K \).
2. Assume that \( K^2/T > N^\varepsilon \). For \( K^2/T \ll |x| < K \), we have

\[ \mathcal{L}(x) \ll 1/(T\sqrt{|x|}). \]

For \( |x| \ll K^2/T \), we have

\[ \mathcal{L}(x) \ll 1/T. \]

3. Let \( p_1 = p_2 = p \). Then

\[ \mathcal{L}(0) \ll \min \left\{ \frac{1}{T}, \frac{PN\varepsilon}{KNr_1 - r_2} \right\}. \]

PROOF. The statement in (1) is obvious in view of Lemma 5.4(1).

We then turn to the proof of (2) in the first case when \( K^2/T \ll |x| < K \). First of all, by Lemma 5.4(2), we may write the integral in (5.11) as below,

\[ \mathcal{L}(x) = \frac{1}{\sqrt{|x|}} \int_0^1 V_0(v_1)V_0(v_2)W_0(wK/x)e(f(v_1, v_2))\,dv_1\,dv_2 + O(N^{-A}), \]

where \( w = \sqrt{v_1}P/p_1 - \sqrt{v_2}P/p_2 \), \( W_0(y) = \sqrt{|x|}W(xy^2)F(y) \) for \( W \) defined as in Lemma 5.4(2) and \( F \) a smooth function supported in \([1/2, 2]\), with \( F \equiv 1 \) on \([2/3, 3/2]\), and

\[ f(v_1, v_2) = T\left(\phi(v_1) - \phi(v_2)\right) + \gamma N(v_1 - v_2) - N\left(\frac{r_1v_1}{p_1} - \frac{r_2v_2}{p_2}\right). \]
where $V$.

Now assume that $x$.

and $p$.

Thus, we may further write

$\mathcal{L}(x) = \frac{1}{\sqrt{|x|}} \int_{-N}^{N} \hat{W}(v) \int \mathcal{V}(v_1) \mathcal{V}(v_2) e(f(v_1, v_2; v)) dv_1 dv_2 + O(N^{-A})$.

with

$f(v_1, v_2; v) = f(v_1, v_2) + \frac{Kp^2}{x} \left( \frac{\sqrt{v_1}}{p_1} - \frac{\sqrt{v_2}}{p_2} \right)$.

We have

$\partial^2 f(v_1, v_2; v)/\partial v_1^2 = T \phi''(v_1) + \frac{K^2 p^2}{2x p_1 p_2} \sqrt{v_2} - \frac{Kp^2}{4x p_1 \sqrt{v_1 v_1}}$.

$\partial^2 f(v_1, v_2; v)/\partial v_2^2 = -T \phi''(v_2) + \frac{K^2 p^2}{2x p_1 p_2} \sqrt{v_2} + \frac{Kp^2}{4x p_2 \sqrt{v_2 v_2}}$.

$\partial^2 f(v_1, v_2; v)/\partial v_1 \partial v_2 = -\frac{K^2 p^2}{2x p_1 p_2 \sqrt{v_1 v_2}}$.

Since $\phi''(v) \gg 1$, when $K^2 / |x| \ll T$, it is clear that

$|\partial^2 f/\partial v_1^2|, |\partial^2 f/\partial v_2^2| \gg T, |\partial^2 f/\partial v_1 \partial v_2| \ll K^2 / |x|, \det f'' \gg T^2$

for $1 \leq v_1, v_2 \leq 2$ and $|v| \leq N$. We obtain the estimate in (5.17) by applying the two-dimensional second derivative test in Lemma A.3 with $\lambda = \rho = T$.

In the second case in (2) when $|x| \ll K^2 / T$ is small, the estimate in (5.18) is just (5.14) in Lemma 5.3.

Finally, let us consider (3). The bound $\mathcal{L}(0) \ll 1/T$ is already contained in (5.18).

Now assume that $|r_1 - r_2| > PT/N^{\frac{1}{2}}$. In view of Lemma 5.4(3), we may write

$\mathcal{L}(0) = \int_{-N/K}^{N/K} W_0(2wK) \int_1^2 V_0(w, v_2) e(f_0(w, v_2)) dv_2 dw + O(N^{-A})$.

where $V_0(w, v_2) = (2p/P)(pw/P + \sqrt{v_2}) V_1((pw/P + \sqrt{v_2})^2) V_2(v_2)$, satisfying

$\var(V_0(w, v_2)) = \int_1^2 |\partial V_0(w, v_2)/\partial v_2| dv_2 \ll 1$,

and

$f_0(w, v_2) = T \left( \phi((pw/P + \sqrt{v_2})^2) - \phi(v_2) \right) - \frac{N}{p} (r_1 - r_2) v_2 - \frac{2N(r_1 - \gamma p)}{P} \sqrt{v_2} w - \frac{Np(r_1 - \gamma p)}{p^2} w^2$. 
Recall that $\phi^{(j)}(v) \ll 1$ $(j = 2, 3)$. For $|r_1 - \gamma p| \ll R = PT/N$ (see (5.24)), $|w| < N^\varepsilon/K$, we have

$$
\partial f_0(w, v_2)/\partial v_2 = -\frac{N}{p}(r_1 - r_2) - \frac{N(r_1 - \gamma p)}{P}\sqrt{v_2}
$$

$$
+ \frac{T}{\sqrt{v_2}} (\phi'((pw/P + \sqrt{v_2})^2) - \sqrt{v_2}\phi'(v_2))
$$

$$
= -\frac{N}{p}(r_1 - r_2) + O\left(\frac{TN^\varepsilon}{K}\right),
$$

and, similarly,

$$
\partial^2 f_0(w, v_2)/\partial v_2^2 = O(TN^\varepsilon/K).
$$

It follows from $|r_1 - r_2| > PT/KN^{1-\varepsilon}$ that $|\partial f_0(w, v_2)/\partial v_2| \gg N|r_2 - r_1|/P$. By partial integration (the first derivative test), we infer that $\mathcal{L}(0) \ll P/KN^{1-\varepsilon}|r_1 - r_2|$, as desired. Q.E.D.

### 5.4. Estimates for $S(N, X, P)$

Combining (5.8), (5.9) and (5.13), along with Lemma 5.5, we conclude that

$$
S(N, X, P) \ll \sqrt{S_{\text{diag}}^2(N, X, P)} + \sqrt{S_{\text{off}}^2(N, X, P)} + N^{-A},
$$

with

$$
S_{\text{diag}}^2(N, X, P) = \frac{N^3X}{P^2P^2K} \sum_{r = p} \sum_{r_1, r_2 = 1}^{P^2} \min\left\{\frac{1}{T}, \frac{PN^\varepsilon}{KN|r_1 - r_2|}\right\},
$$

and

$$
S_{\text{off}}^2(N, X, P) = \frac{N^3X}{P^2P^2K} \sum_{p_1, p_2 = 1}^{P^2} \sum_{r, \gamma p \equiv r_1 (\mod p)} \left( \sum_{n = 1}^{T} \frac{\sqrt{p_1 p_2}}{T \sqrt{X/n}} \right)
$$

$$
+ \sum_{0 < |n| < NT} \left( \sum_{n = 1}^{T} \frac{1}{T} \right),
$$

in correspondence to the cases where $n = 0$ and $n \neq 0$ in (5.13), respectively.

Note that in the case $n = 0$ the congruence condition in (5.12) would imply $p_1 = p_2$ ($= p$, say) and $r_1 \equiv r_2 (\mod p)$. Moreover, when applying the estimates (5.17) and (5.18) to $\mathcal{L}(MXN/p_1 p_2)$, note that $K^2/T \ll |x| < K$ or $|x| \ll K^2/T$ amounts to $NT \ll |n| \ll N/K$ or $|n| \ll N/T$, respectively, for $X = P^2K^2/N$ (see (5.1)). We record here the condition in Lemma 5.5(2):

$$
K > \sqrt{TN^\varepsilon}.
$$

For $S_{\text{diag}}^2(N, X, P)$, we split the sum over $r_1$ and $r_2$ according as $r_1 = r_2$ or not,

$$
S_{\text{diag}}^2(N, X, P) = \frac{N^3X}{P^2P^2K} \left( \sum_{r = p} \sum_{r_1, r_2 = 1} \frac{1}{T} + \sum_{r = p} \sum_{r_1, r_2 = 1} \sum_{r_1, r_2 (\mod p)} \frac{PN^\varepsilon}{KN|r_1 - r_2|} \right),
$$

where $r_1, r_2 = 1$.
and hence
\begin{equation}
S^2_{\text{diag}}(N, X, P) \leq \frac{N^3 X}{P^2 - 2KP} \left( \frac{P^* R}{T} + P^* R \frac{N^c}{KN} \right) \leq (KN + T)N^c.
\end{equation}

Recall here that $NX = P^2K^2$ and $R = PT/N$ as in (5.1) and (5.4).

To deal with $S^2_{\text{off}}(N, X, P)$, we first note that necessarily $p_1 \neq p_2$. Otherwise, if $p_1 = p_2 = p$, then the congruence $n \equiv T_1p - T_2p \pmod{p^2}$ would imply $p|n$. This is impossible, in view of our assumption $N^{1+\varepsilon}/K < P$ in (5.2) and the length $N/K$ of the $n$-sum. We now interchange the sum over $n$ and the sums over $r_1, r_2$. Note that for fixed $n$, the congruence $n \equiv T_1p_2 - T_2p_1 \pmod{p_1p_2}$ splits into $r_1 = \pi p_2 \pmod{p_1}$ and $r_2 = -\pi p_1 \pmod{p_2}$, so
\begin{equation}
S^2_{\text{off}}(N, X, P) = \frac{N^3 X}{P^2 - 2KP} \sum_{p_1 \neq p_2} \sum_{p_1 \sim P} \sum_{\substack{p_1 \sim P \sim P \mid \mid |N/|N/K| |r_1 - \gamma p_1| |r_1 - \gamma p_2| < R \mid \mid p_1 \cdot \pi p_2 \pmod{p_1} \mid \mid p_2 \cdot -\pi p_1 \pmod{p_2} \mid \mid 0 < |n| < |N/T| |r_1 - \gamma p_1| |r_2 - \gamma p_2| < R \mid \mid r_1 = \pi p_2 \pmod{p_1} \mid \mid r_2 = -\pi p_1 \pmod{p_2} \mid \mid 1 \right).
\end{equation}

When $T \geq N$ so that $R \geq P$, we have
\begin{equation}
S^2_{\text{off}}(N, X, P) \leq \frac{N^3 X}{P^2 - 2KP} P^2 \left( \frac{P}{T \sqrt{X}} \sqrt{\frac{N}{K}} + \frac{N}{T^2} \right) R^2 = \frac{NT}{\sqrt{K}} + KN.
\end{equation}

When $T < N$, the $(R/P)^2$ in (5.25) needs to be replaced by 1. In other words, we lose $(R/P)^2 = (N/T)^2$. However, the loss may be reduced to $N/T$ if we rearrange the sum $S^2_{\text{off}}(N, X, P)$ as follows
\begin{equation}
\frac{N^3 X}{P^2 - 2KP} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\substack{p_2 \sim P \mid \mid |N/|N/K| |r_1 - \gamma p_1| |r_2 - \gamma p_2| < R \mid \mid p_2 = \pi r \pmod{p_1} \mid \mid p_2 = -\pi r \pmod{p_2} \mid \mid 0 < |n| < |N/T| |r_1 - \gamma p_1| |r_2 - \gamma p_2| < R \mid \mid r_1 = \pi r \pmod{p_1} \mid \mid r_2 = -\pi r \pmod{p_2} \mid \mid 1 \right).
\end{equation}

Thus for $T < N$, we have
\begin{equation}
S^2_{\text{off}}(N, X, P) \leq \frac{N^3 X}{P^2 - 2KP} P^* R \left( \frac{P}{T \sqrt{X}} \sqrt{\frac{N}{K}} + \frac{N}{T^2} \right) \leq \left( \frac{NT}{\sqrt{K}} + KN \right) \frac{N}{T} \log P.
\end{equation}

Combining (5.25) and (5.26), we have
\begin{equation}
S^2_{\text{off}}(N, X, P) \leq \left( \frac{NT}{\sqrt{K}} + KN \right) \left( 1 + \frac{N}{T} \right) \log P.
\end{equation}

We conclude from (5.20), (5.24) and (5.27) that
\begin{equation}
S(N, X, P) \leq \left( \sqrt{T} + \left( \sqrt{KN} + \frac{\sqrt{NT}}{K^{1/4}} \right) \left( 1 + \sqrt{\frac{N}{T}} \right) \right)^{N^c}.
\end{equation}

5.5. Conclusion. In view of (4.8) in Proposition 4.1 and (5.28), we have
\begin{equation}
S(N) \leq \sqrt{T} N^c + \left( \sqrt{KN} + \frac{\sqrt{NT}}{K^{1/4}} \right) \left( 1 + \sqrt{\frac{N}{K}} + \frac{N}{P} \right).
For the estimate in (1.8) to be non-trivial, we assume that $N^\epsilon < T < N^{3/2-\epsilon}$. Then
\[ S(N) \ll T^{1/3} N^{1/2+\epsilon} \left( 1 + \frac{N^{1/2}}{T^{1/2}} \right)^4 N T^{1/3} \ll T^{1/3} N^{1/2+\epsilon} + \frac{N^{1+\epsilon}}{T^{1/6}}, \]
on choosing $K = T^{2/3}$ and $P \geq N/T^{1/3}$. The required conditions in (5.1), (5.2) and (5.23) are well justified for our choice of $K$ and $P$. This proves Theorem 1.1.

For Corollary 1.2 define
\[ S_H(N) = \sum_{N \leq n \leq N+H} \lambda_n(n) e(f(n)). \]
Let the smooth function $V$ in Theorem 1.1 be supported on $[1, 1 + H/N]$ with $V(x) \equiv 1$ on $[1 + 1/\delta, 1 + H/N - 1/\delta]$. For this, it is necessary that $\delta \geq 2N/H$. By the Deligne bound (1.6), we would have
\[ S_H(N) = S(N) + O(N^{1+\epsilon}/\delta). \]
Then Corollary 1.2 follows from Theorem 1.1 upon choosing $\delta = T/N^\epsilon$.

6. Proof of the Weyl-type subconvex bound

For $g \in S_1^*(M, \xi)$ with Fourier coefficients $\lambda_{\chi}(n)$, let $\bar{g} \in S_1^*(M, \bar{\xi})$ be its dual form with Fourier coefficients $\lambda_{\bar{\chi}}(n) = \lambda_{\chi}(n)$, and let $\epsilon_{\chi}$ be the root number of $L(s, g)$ satisfying the functional equation
\[ \Lambda(s, g) = \epsilon_{\chi} \Lambda(1 - s, \bar{g}), \]
with
\[ \Lambda(s, g) = M^{1/2}(2\pi)^{-s} \frac{\Gamma(s)}{\Gamma(s+1)} L(s, g). \]
From this one may deduce the following Approximate Functional Equation (cf. [Har], Theorem 2.5) and [BMN] Lemma 2.1).

Lemma 6.1 (Approximate Functional Equation). Let $F : (0, \infty) \to \mathbb{R}$ be a smooth function satisfying $F(x) + F(1/x) = 1$ and with derivatives decaying faster than any negative power of $x$ as $x \to \infty$. Then
\[ L(1/2 + it, g) = \sum_{n=1}^{\infty} \frac{\lambda_{\chi}(n)}{n^{1/2+it}} F \left( \frac{n}{\sqrt{C}} \right) + \epsilon_{\chi}(2\pi)^{2it} \frac{\Gamma(k + it)}{\Gamma(k + 1)} \sum_{n=1}^{\infty} \frac{\lambda_{\bar{\chi}}(n)}{n^{1/2-it}} F \left( \frac{n}{\sqrt{C}} \right) + O_{\epsilon,F} \left( M^{1/2} C^{1/4-\epsilon} \right), \]
where $C = C(g, t)$ is the analytic conductor defined by
\[ C = \frac{M}{\pi^2} \left| \frac{k + 1}{2} + it \right| \left| \frac{k + 3}{2} + it \right|. \]

Let $t > 1$ be large. By applying a dyadic partition of unity to the approximate functional equation (6.1), we infer that
\[ L(1/2 + it, g) \ll t^\epsilon \left( \frac{|S(N)|}{\sqrt{N}} + \frac{1}{\sqrt{t}} \right) \]
for some $N < t^{1+\epsilon}$, where
\[ S(N) = \sum_{n=1}^{\infty} \lambda_{\chi}(n) n^{-it} V \left( \frac{n}{N} \right), \]
and $V(x)$ is some function in $C_\infty^\infty(0, \infty)$ supported on $[1, 2]$, satisfying $V^{(j)}(x) \ll j$.
Recall that the Rankin–Selberg estimate in \[1.5\] yields the trivial bound \( S(N) \ll N \). Therefore it suffices to beat the trivial bound in the range \( t^{1-\delta+\varepsilon} < N < t^{1+\varepsilon} \) for some \( \delta > 0 \). For the Weyl subconvex bound, we would need \( \delta = 1/3 \).

Note that \( e(f(n)) = N^\delta n^{-\varepsilon} \) if we choose \( \phi(x) = -\log x, T = t/2\pi \) and \( \gamma = 0 \) in \((1.3)\). Consequently, Theorem 1.1 implies that for \( t^{2/3+\varepsilon} < N < t^{1+\varepsilon} \) the sum \( S(N) \) has the following bound:

\[
\frac{S(N)}{\sqrt{N}} \ll t^{1/3+\varepsilon},
\]
as desired.

**Appendix A. Stationary phase**

Firstly, we have Lemma 8.1 in [BKY] with some improvements.

**Lemma A.1.** Let \( w(x) \) be a smooth function with support on \((a, b)\) and \( f(x) \) be a real smooth function on \([a, b]\). Suppose that there are parameters \( Q, U, Y, Z, R > 0 \) such that

\[
f^{(i)}(x) \ll iY^i/Q^i, \quad w^{(j)}(x) \ll jZ^j/U^j,
\]

for \( i \geq 2 \) and \( j \geq 0 \), and

\[
|f'(x)| \geq R.
\]

Then for any \( A \geq 0 \) we have

\[
\int_a^b e(f(x))w(x)dx \ll (b-a)Z \left( \frac{Y}{R^2Q^2} + \frac{1}{RU} + \frac{1}{RQ} \right)^A.
\]

**Proof.** In the proof of Lemma 8.1 in [BKY], one can actually impose an additional condition \( \gamma_2 + \gamma_3 + \ldots = \nu - n \) to the inner sum in (8.5) so that the \( Y^{(\nu-n)/2} \) may be replaced by \( Y^\nu \) in (8.6) and the sum over \( \mu \) should be only up to \( 2n - \nu \). In this way, their condition \( Y \geq 1 \) becomes unnecessary and their estimate in (8.3) may be improved as above.

For the reader’s convenience, we record here the one- and two-dimensional second derivative tests (cf. [Hux] Lemma 5.1.3, [Mun] Lemma 4).

**Lemma A.2.** Let \( f(x) \) be a real smooth function on \((a, b)\). Let \( w(x) \) be a real smooth function with support in \((a, b)\) and let \( V \) be its total variation. If \( f''(x) \geq \lambda > 0 \) on \((a, b)\), then

\[
\left| \int_a^b e(f(x))w(x)dx \right| \leq \frac{4V}{\sqrt{\pi\lambda}}.
\]

**Lemma A.3.** Let \( f(x, y) \) be a real smooth function on \((a, b) \times (c, d)\) with

\[
|\partial^2 f/\partial x^2| \gg \lambda > 0, \quad |\partial^2 f/\partial y^2| \gg \rho > 0,
\]

\[
|\det f''| = |\partial^2 f/\partial x \partial y^2 - \partial^2 f/\partial x^2 \partial y| \gg \lambda, \rho,
\]
on the rectangle \((a, b) \times (c, d)\). Let \( w(x, y) \) be a real smooth function with support in \((a, b) \times (c, d)\) and let

\[
V = \int_a^b \int_c^d \left| \frac{\partial^2 w(x, y)}{\partial x \partial y} \right| dx dy.
\]

\(^1\)Since \( w(x) \) is supported in \((a, b)\), we do not need to add its maximum modulus to \( V \) as in [Hux] Lemma 5.1.3.
Then
\[ \int_a^b \int_{-\delta}^{\delta} e(f(x,y))w(x,y)\,dx\,dy \leq \frac{V}{\sqrt{\lambda \rho}}, \]
with an absolute implied constant.

Finally, the following stationary phase estimate is from [Sog. Theorem 1.1.1].

**Lemma A.4.** Let \( Z > 0 \) and \( \lambda \geq 1 \). Let \( w(x;\lambda) \) be a smooth function with support in \((a,b)\) for all \( \lambda \), and \( f(x) \) be a real smooth function on an open neighborhood of \([a,b]\). Suppose that \( \lambda f''(x_0) = f'(x_0) = 0 \) at a point \( x_0 \in (a,b) \), with \( f'''(x_0) \neq 0 \) and \( f'(x) \neq 0 \) for all \( x \in [a,b] \setminus \{x_0\} \). Then
\[ \frac{d^n}{dx^n} \int_a^b e(\lambda f(x))w(x;\lambda)\,dx \ll_j \frac{Z}{\lambda^{1/2+j}}. \]

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**References**

[Agg] K. Aggarwal. Weyl bound for GL(2) in \( t \)-aspect via a trivial delta method. *preprint*, [arXiv:1810.10479] 2018.

[AHL] K. Aggarwal, R. Holowinsky, Y. Lin, and Q. Sun. The Burgess bound via a trivial delta method. *preprint*, [arXiv:1803.00542] 2018.

[AKM] R. Acharya, S. Kumar, G. Maiti, and S. K. Singh. Subconvexity bound for GL(2) \( t \)-functions via a trivial delta method. *preprint*, [arXiv:1805.04972] 2018.

[AS] K. Aggarwal and S. K. Singh. \( t \)-aspect subconvexity bound for GL(2) \( L \)-functions. *preprint*, [arXiv:1706.04977] 2017.

[BKY] V. Blomer, R. Khan, and M. P. Young. Distribution of mass of holomorphic cusp forms. *Duke Math. J.*, 162(14):2609–2644, 2013.

[BMN] A. R. Booker, M. B. Milinovich, and N. Ng. Subconvexity for modular form \( L \)-functions in the \( t \) aspect. *Adv. Math.*, 341:299–335, 2019.

[DFI] W. Duke, J. Friedlander, and H. Iwaniec. Bounds for automorphic \( L \)-functions. *Invent. Math.*, 112(1):1–8, 1993.

[God] D. Godber. Additive twists of Fourier coefficients of modular forms. *J. Number Theory*, 133(1):83–104, 2013.

[Goo] A. Good. The square mean of Dirichlet series associated with cusp forms. *Mathematika*, 29(2):278–295 (1983), 1982.

[GR] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Elsevier/Academic Press, Amsterdam, Seventh edition, 2007.

[Har] G. Harcos. Uniform approximate functional equation for principal \( L \)-functions. *Int. Math. Res. Not.*, (18):923–932, 2002.

[Hux] M. N. Huxley. *Lattice Points, and Exponential Sums*. Volume 13 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[ILS] H. Iwaniec, W. Luo, and P. Sarnak. Low lying zeros of families of \( L \)-functions. *Inst. Hautes Études Sci. Publ. Math.*, (91):55–131 (2001), 2000.

[Iwa] H. Iwaniec. *Topics in Classical Automorphic Forms*, volume 17 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1997.

[Jut1] M. Jutila. *Lectures on a Method in the Theory of Exponential Sums*, volume 80 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Published for the Tata Institute of Fundamental Research, Bombay, by Springer-Verlag, Berlin, 1987.

[Jut2] Matti Jutila. Mean values of Dirichlet series via Laplace transforms. In *Analytic number theory (Kyoto, 1996)*, volume 247 of *London Math. Soc. Lecture Note Ser.*, pages 169–207. Cambridge Univ. Press, Cambridge, 1997.

[KMS] S. Kumar, K. Mallesham, and S. K. Singh. Non-linear additive twist of Fourier coefficients of GL(3) Maass forms. *preprint*, [arXiv:1905.13109] 2019.
E. Kowalski, P. Michel, and J. VanderKam. Rankin-Selberg L-functions in the level aspect. Duke Math. J., 114(1):123–191, 2002.

K. Liu and X. Ren. On exponential sums involving Fourier coefficients of cusp forms. J. Number Theory, 132(1):171–181, 2012.

T. Meurman. On the order of the Maass L-function on the critical line. In Number theory, Vol. I (Budapest, 1987), volume 51 of Colloq. Math. Soc. János Bolyai, pages 325–354. North-Holland, Amsterdam, 1990.

S. D. Miller. Cancellation in additively twisted sums on GL. Amer. J. Math., 128(3):699–729, 2006.

W. Magnus, F. Oberhettinger, and R. P. Soni. Formulas and Theorems for the Special Functions of Mathematical Physics. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

S. D. Miller and W. Schmid. Distributions and analytic continuation of Dirichlet series. Trans. Amer. Math. Soc., 347(9):3481–3494, 1995.

W. Magnus, F. Oberhettinger, and R. P. Soni. Formulas and Theorems for the Special Functions of Mathematical Physics. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

K. Liu and X. Ren. On exponential sums involving Fourier coefficients. Duke Math. J., 132(1):171–181, 2006.

S. D. Miller and W. Schmid. Automorphic distributions, L-functions, and Voronoi summation for GL(3). Ann. of Math. (2), 164(2):423–488, 2006.

S. D. Miller and W. Schmid. A general Voronoi summation formula for GL(n). Duke Math. J., 164(2):423–488, 2006.

G. N. Watson. A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge, 1922.

E. T. Whittaker and G. N. Watson. A Course of Modern Analysis. Fourth edition. Reprinted. Cambridge University Press, Cambridge, 1962.

Z. Qi. Theory of fundamental Bessel functions of high rank. arXiv:1612.03552 [math.DG] to appear in Mem. Amer. Math. Soc., 2016.

Q. Sun. On cusp form coefficients in exponential sums. Q. J. Math., 52(4):485–497, 2001.

Z. Qi. Cancellation in the additive twists of Fourier coefficients for GL(2) and GL(3) over number fields. arXiv:1604.08000 [math.NT], 2016.

N. J. E. Pitt. On cusp form coefficients in exponential sums. Q. J. Math., 52(4):485–497, 2001.

Y. Wu. Exponential sums involving Maass forms. Front. Math. China, 9(6):1349–1366, 2014.

G. N. Watson. A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944.

E. T. Whittaker and G. N. Watson. A Course of Modern Analysis. Fourth edition. Reprinted. Cambridge University Press, New York, 1962.