On moments of Pitman estimators: the case of fractional Brownian motion

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Abstract. In some non-regular statistical estimation problems, the limiting likelihood processes are functionals of fractional Brownian motion (fBm) with Hurst’s parameter $H$, $0 < H \leq 1$. In this paper we present several analytical and numerical results on the moments of Pitman estimators represented in the form of integral functionals of fBm. We also provide Monte Carlo simulation results for variances of Pitman and asymptotic maximum likelihood estimators.

Keywords: Pitman estimators, fractional Brownian motion, integral functionals, Riemann-Zeta function.

1. Introduction

Pitman estimators ([19]), also known as Bayesian estimators with a constant prior on the real line ([3]), for parameters of stochastic processes are optimal under various continuous- and discrete-time settings, [10], [12]. For example, one may consider the estimation problem of parameter $\theta$ by observing the diffusion process $X = \{X_t, 0 \leq t \leq T\}$ that is a solution of stochastic differential equation

$$dX_t = s(X_t, t, \theta)dt + \sigma(X_t)dW_t, \ 0 \leq t \leq T,$$

where the drift $s(x, t, \theta)$ is a non-regular function, e.g. $s(x, t, \theta) = |x-\theta|^p$, $p < \frac{1}{2}$, or $s(x, t, \theta) = I\{\theta > t\}$. For such non-regular statistical estimation problems it is a typical situation when the respective limit likelihood process $Z_t$ is generated by a fractional Brownian motion (fBm) $W_t^H$ with Hurst’s parameter $H \in (0, 1]$, namely

$$Z_t = e^{W_t^H - \frac{1}{2}|t|^{2H}}, \ t \in R, \ R = (-\infty, \infty),$$

see [15], [7]. Note that the case $H = \frac{1}{2}$ appears in a study of a change point problem for a Brownian motion (Bm) in [10], [12] and processes with a time delay in [9]. The case $H \neq \frac{1}{2}$ appears in various continuous-time settings, see (Chapter 3, [15]) and [7], and discrete-time frameworks, [12], [12].

Distributional properties of Pitman estimators for large sample sizes have not been studied in much detail. In this paper, in continuation of our results

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from [18], we study the limit distribution of Pitman estimators, which can be defined as the distribution of a random variable

$$\zeta_H = \int_{-\infty}^{\infty} t q_t dt,$$

(1)

where

$$q_t = Z_t \left( \int_{-\infty}^{\infty} Z_u du \right)^{-1},$$

(2)

$\zeta_H$ represents a conditional expectation with respect to postaposterioy density $q_t$. Recall that $W^H = \{W^H_s, s \in \mathbb{R}\}$ is a Gaussian process with continuous trajectories

$$W^0_H = 0, E(W^H_s) = 0, E|W^H_s - W^H_t|^2 = |s - t|^{2H}, s \in \mathbb{R}, t \in \mathbb{R}.$$  

This implies that the covariance function of $W^H_s$ is

$$R(t, s) := E(W^H_t W^H_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}).$$

Note that, even when $H = \frac{1}{2}$, i.e. in the case of a standard Bm neither the distribution nor the moments $\zeta_1$ (greater than 2), are known in an explicit form. For other cases, except $H = 1$, the essential difficulty in studying the functionals of fBm is due to the fact that $W^H$ is not a semimartingale and therefore, standard tools of stochastic calculus (based on the Ito formula) are not applicable.

The case $H = 1$ corresponds to the regular statistical estimation problems where the limit distribution is normal, $\zeta_1 \sim N(0, 1)$. In this paper we obtain several results on the variance and higher moments of $\zeta_H$, $0 < H \leq 1$, using the measure transformation technique and Gaussian property of fBm. In [18] we showed that, for $H > 0.309...$, the random variable (r.v.) $|\zeta|^H$ is exponentially bounded i.e. there exists a constant $\alpha_H > 0$ such that

$$E e^{\delta |\zeta|^H} < \infty \text{ for } \delta < \alpha_H.$$  

(3)

This result implies, of course, finiteness of all moments of $\zeta_H$. In Section 2 we improved this result (see Theorem 1) by showing that (3) does hold for all $H \in (0, 1]$. Improvement is achieved thanks to application of the measure transformation technique, see Lemma 1 in Section 2. Note that Lemma 1 also will be used in the proof of Theorem 2 in Section 3 which presents a general identity for expectations of functions of $\zeta_H$. Then, using the obtained identity, we derive a useful representation for the variance of $\zeta_H$ when $H \in (0, 1]$, (see Corollary 1). Corollary 2 provides a lower bound for the moments $E \zeta_H^k$, $k = 2, 4, ..$. 

In Section 4, Theorem 3, for the case $H \in [\frac{1}{2}, 1)$ (see Theorem 3) we also derive a new representation for $\text{Var}(\zeta_H)$ in terms of the function

$$g(m) := E \log \left( \int_{-\infty}^{\infty} Z_u e^{mu} du \right).$$

(4)
This result was formulated in [18] without proof. Earlier in [18] we derived another expression for \( \text{Var}(\zeta_H) \) in terms of the function

\[
g(m_1, m_2) = E \log \int_0^\infty (Z_u e^{-m_1 u} + Z_{-u} e^{-m_2 u}) du.
\]

In [18] was shown that for the case \( H = \frac{1}{2} \) the function \( g(m_1, m_2) \) (as well as \( g(m) = g(-m, m) \)) can be expressed in terms of the PolyGamma function leading to much shorter derivation of the following result from [20]

\[
\text{Var}(\zeta_{\frac{1}{2}}) = 16 \text{Zeta}[3] \approx 19.23,
\]

(5)

where \( \text{Zeta}[k] \) is the Riemann-zeta function, see details in [18].

In Section 5 we present Monte Carlo simulation results for \( \text{Var}(\zeta_H) \) and the variance of asymptotic maximum likelihood estimator which is the argmax of \( W^H = \{ W^H_s, s \in R \} \).

2. Exponentially boundedness of \( |\zeta_H|^{2H} \)

The following lemma on a measure transformation for Gaussian processes (and hence for fBm \( W^H_t, t \in R \), as well) plays key role in the proof of Theorem 1 below.

We formulate Lemma 1 in terms of a Gaussian system \( (\xi, \{ X_s \}, s \in D) \) (see [25]) defined on probability space \( (\Omega, F, P) \). Recall that it means that \( (\xi, \{ X_t \}, t_i \in D, i = 1, ..., n) \) is a Gaussian vector for any \( n \). We use the upper index, e.g., \( Q \), to indicate that expectations are taken with respect to a measure \( Q \), so \( E^Q(.) \) is used for the expectation with respect to measure \( Q \). Lemma 1 gives a new result which belongs to a group of results broadly known as Cameron-Martin-Girsanov-Maruyama-... type measure transformations.

**Lemma 1.** Let \( (\xi, \{ X_s \}, s \in D) \) be a Gaussian system on a probability space \( (\Omega, F, P) \). Set

\[
E^P(\xi) = 0, \quad \text{Var}^P(\xi) = \sigma^2
\]

and consider the measure transformation

\[
Q(A) = E^P I(A)e^{\xi - \frac{\sigma^2}{2}}.
\]

Then on the probability space \( (\Omega, F, Q) \):

1) the system \( (\xi, \{ X_s \}, s \in D) \) is Gaussian;

2) \( E^Q X_s = E^P X_s + \text{Cov}^P(\xi, X_s), \quad \text{Cov}^Q(X_t, X_s) = \text{Cov}^P(X_t, X_s) \).

**Proof.** Property 1) is a consequence of the definition of a Gaussian system and the fact that any linear transformation of a Gaussian vector is a Gaussian vector.
To check the second property one should write out the joint moment generating function of $X_s$ and $X_t$ with respect to measure $Q$ for $t, s \in \mathbb{R}$

$$E^Q e^{z_1 X_s + z_2 X_t} =$$

$$= \exp \{ z_1 E^P X_s + z_2 E^P X_t - \frac{\sigma^2}{2} + \frac{1}{2} Var^P(\xi + z_1 X_s + z_2 X_t) \}. \quad (6)$$

Since

$$\frac{1}{2} Var^P(\xi + z_1 X_s + z_2 X_t) = \frac{1}{2} (\sigma^2 + z_1^2 Var^P(X_s) + z_2^2 Var^P(X_t)) +$$

$$+ z_1 Cov^P(\xi, X_s) + z_2 Cov^P(\xi, X_t) + z_1 z_2 Cov^P(X_t, X_s),$$

differentiating in (6) with respect to $z_1$ and $z_2$ we obtain

$$E^Q X_s = \frac{\partial}{\partial z_1} E^Q e^{z_1 X_s + z_2 X_t} \bigg|_{z_1 = z_2 = 0} = E^P X_s + Cov^P(\xi, X_s),$$

$$E^Q X_t X_s = \frac{\partial^2}{\partial z_1 \partial z_2} E^Q e^{z_1 X_s + z_2 X_t} \bigg|_{z_1 = z_2 = 0}$$

$$= (E^P X_s + Cov^P(\xi, X_s))(E^P X_t + Cov^P(\xi, X_t)) + Cov^P(X_t, X_s) = E^Q X_t E^Q X_s + Cov^P(X_t, X_s).$$

This completes the proof.

While dealing with integrals of a Gaussian process $X_s$ we assumed that there is a progressively measurable modification of $X_s$ such that integrals are well defined.

For proving Theorem 1 we also need the following lemma.

**Lemma 2.** Let $X_s$ be a Gaussian process with $EX_s = 0$. Then for any $t > 0$ and $r > 0$

$$E \left( \int_0^t e^{X_s} ds \right)^{-r} \leq \frac{1}{t^r} \exp \left\{ \frac{r^2}{2t^2} Var(\int_0^t X_s ds) \right\}. \quad (7)$$

**Proof.** Applying Jensen’s inequality we obtain for any $t > 0$

$$\int_0^t e^{X_s} ds \geq t \exp \left\{ \frac{1}{t} \int_0^t X_s ds \right\},$$

and hence for any $t > 0$ and $r > 0$

$$E(\int_0^t e^{X_s} ds)^{-r} \leq \frac{1}{t^r} E \exp \left\{ -\frac{r}{t} \int_0^t X_s ds \right\} = \frac{1}{t^r} \exp \left\{ \frac{r^2}{2t^2} Var(\int_0^t X_s ds) \right\}.$$

**Remark 1.** Integrals of type $E(\int_0^t e^{X_s} ds)^{-1}$ were considered in [14], [17] and [13].

**Theorem 1.** For any $H \in (0, 1]$, there exists a positive number $\alpha_H$ such that

$$E e^{\delta |\xi_H|^2} < \infty \text{ for } \delta < \alpha_H.$$  \quad (8)
Proof. If $H = 1$ then it is $\alpha_1 = \frac{1}{2}$ in (8) because it is well known that in this case $\zeta_1 \sim N(0, 1)$ and therefore

$$
Ee^{\delta \zeta_1^2} < \infty \text{ for } \delta < \frac{1}{2}; \quad Ee^{\frac{1}{2} \zeta_1^2} = \infty.
$$

In the case $H \in (0, 1)$ we need to find a proper estimate for the expectation of $E_q t$ which leads to (8).

Note that the function $e^{\delta |x|^{2H}}$ is a convex function for $H \geq \frac{1}{2}$, and for $H \in (0, \frac{1}{2})$ it is dominated by convex function $\max(C_{\delta, H}, e^{\delta |x|^{2H}})$, where $C_{\delta, H}$ is a sufficiently large number. The random process $q_t$ represents a density function as it is a normalised nonnegative function of $t$ such that $\int_{-\infty}^{\infty} q_t dt = 1$. Thus by Jensen’s inequality, for any $\delta \geq 0$, from (1) we have

$$
e^{\delta \zeta_{H, u}^{2H}} \leq C_{\delta, H} + \int_{-\infty}^{\infty} e^{\delta |t|^{2H}} q_t dt \leq C_{\delta, H} + \int_{-1}^{1} e^{\delta |t|^{2H}} q_t dt + \int_{-\infty}^{\infty} I\{|t| > 1\} e^{\delta |t|^{2H}} q_t dt \leq C_{\delta, H} + e^{\delta} + \int_{-\infty}^{\infty} I\{|t| > 1\} e^{\delta |t|^{2H}} q_t dt,
$$

where the constant $C_{\delta, H} \geq 0$, and $C_{\delta, H} = 0$ in the case $H \geq \frac{1}{2}$; above we also used the fact that $\int_{-\infty}^{\infty} q_t dt = 1$. In view of the symmetry property of fBm in distributional sense

$$
\{W_u^H, u \geq 0\} \stackrel{d}{=} \{W_{-u}^H, u \geq 0\},
$$

we have

$$
Ee^{\delta \zeta_{H, u}^{2H}} \leq c_{\delta} + e^{\delta} + 2 \int_{-1}^{1} e^{\delta |t|^{2H}} E_q t dt.
$$

For finding a proper upper bound for $E_q t$ for $t \geq 1$ we use Lemma 1 with

$$
\xi = \lambda W_1^H, \quad X_s = W_s^H,
$$

where $\lambda$ is a real number. Then

$$
E^P(X_s) = 0, \quad \sigma^2 = \frac{\lambda^2 2H}{2}; \quad E^Q(W_1^H) = \lambda R(t, s), \quad R^Q(t, s) = R(t, s).
$$

This means that with respect to the measure $Q$ the process $\{W_t^H - \lambda R(t, s), t \in R\}$ is a (standard) fBm. Using this fact we obtain

$$
E_q t = E e^{\lambda W_t^H - \frac{\lambda^2}{2} t^{2H}} e^{(1-\lambda)W_t^H - \frac{(1-\lambda)^2}{2} t^{2H}} \left( \int_{-\infty}^{\infty} e^{W_s^H - \frac{\alpha s^{2H}}{2} ds} \right)^{-1} =
$$

$$
E^Q e^{(1-\lambda)W_t^H - \frac{(1-\lambda)^2}{2} t^{2H}} \left( \int_{-\infty}^{\infty} e^{W_s^H - \frac{\alpha s^{2H}}{2} ds} \right)^{-1}.
$$
Applying Lemma 1 we have

\[ Eqt = e^{(1-\lambda)W_t^H+\lambda(1-\lambda)t^{2H}} \int_{-\infty}^{\infty} e^{W_s^H+\lambda R(t,s)-\frac{|u|^{2H}}{2}} ds -1 = \]

\[ e^{-(\lambda -1)u^{2H}/2} e^{(1-\lambda)W_t^H} \int_{-\infty}^{\infty} e^{W_s^H+t^{2H} f(s/t) ds} -1, \]

where

\[ t^{2H} f(s/t) := \lambda R(t, s) - |s|^{2H}/2. \]

Note that

\[ f(u) = \frac{\lambda + (\lambda -1)u^{2H} - \lambda|1-u|^{2H}}{2}, \quad f(0) = 0, \quad f(1) = \lambda - 1/2. \]

To simplify the exposition of the proof we choose \( \lambda = \frac{1}{2} \) (although it seems that somewhat better estimator for \( \alpha_H \) can be obtained with a proper choice of \( \lambda \) depending on \( H \)). Then assuming \( \lambda = \frac{1}{2} \) we can rewrite (11) as follows

\[ Eq_t = e^{-\frac{1}{2}t^{2H}} e^{\frac{1}{2}W_t^H} \int_{-\infty}^{\infty} e^{W_s^H+t^{2H} f(s/t) ds} -1, \]

where

\[ f(u) = f(1-u) = \frac{1 - |u|^{2H} - |1-u|^{2H}}{4}, \quad f(0) = 0, \quad f(1) = 0. \]

It is easy to see that for \( H \in (0,1/2) \) the function \( f(u), 0 < u < 1 \) is negative; for \( H \in (1/2,1) \) the function \( f(u), 0 < u < 1 \) is positive.

Applying the inequality \( e^{x/2} \leq (1 + e^x)/2 \) for the term \( e^{W_t^H/2} \) in (13) and reducing the range of integration to \( s \in [0,t] \) instead of \( s \in R \) for the integral we obtain

\[ Eq_t \leq \frac{1}{2} e^{-\frac{1}{2}t^{2H}} \left[ E(\int_0^t e^{W_s^H+t^{2H} f(s/t) ds} -1 + E(\int_0^t e^{W_s^H-W_t^H+t^{2H} f(s/t) ds} -1). \right] \]

In view of following translation-invariance property of fBm

\[ \{W_s^H - W_t^H, s \in R\} \overset{d}{=} \{W_{t-s}^H, s \in R\}, \]

the expectation \( E(\int_0^t e^{W_s^H-W_t^H+t^{2H} f(s/t) ds} -1 \) is equal to \( E(\int_0^{t-s} e^{W_s^H+t^{2H} f(1-s/t) ds} -1 \) after change of variable \( t-s \) to \( s \). Thus we get the estimate

\[ Eq_t \leq \frac{1}{2} e^{-\frac{1}{2}t^{2H}} \left[ E(\int_0^t e^{W_s^H+t^{2H} f(s/t) ds} -1 + E(\int_0^{t-s} e^{W_s^H+t^{2H} f(1-s/t) ds} -1). \right] \]

Since \( f(u) = f(1-u) \) the above inequality can be now rewritten as follows

\[ Eq_t \leq e^{-\frac{1}{2}t^{2H}} e^{\int_0^t e^{W_s^H+t^{2H} f(s/t) ds} -1. \]
Consider now the case $H \in [1/2, 1)$. Then it is easy to see $f(s/t) \geq 0$ and therefore by Lemma 2 for $t \geq 1$

$$E q_t \leq e^{-\frac{t}{2}H^2} E(\int_0^1 e^{W_H^s ds})^{-1} < \infty.$$ 

Combining all estimates obtained above for $H \in [1/2, 1)$ and $\delta < \frac{1}{8}$ we have

$$E e^{\delta |\zeta|^2 H} \leq c_\delta + e^\delta + 2E(\int_0^1 e^{W_H^s ds})^{-1} \int_1^\infty e^{-(\frac{1}{8} - \delta) t^2} dt < \infty.$$ 

Since $E(\int_0^1 e^{W_H^s ds})^{-1}$ is finite due to the result of Lemma 2 we have proved Theorem 1 with $\alpha_H \geq \frac{1}{8}$.

The case $H \in (0, 1/2]$.

Obviously, the function $f(u)$ (defined in \[14\]) is decreasing on the interval $u \in (0, \frac{1}{2})$. This fact implies that $f(s/t) \geq f(\varepsilon)$ for all $s \in (0, ct)$ and any $\varepsilon \in (0, \frac{1}{2})$, where $f(\varepsilon) = -(e^{2H}/4(1 + o(1))$ as $\varepsilon \to 0$. Thus, from \[16\] we have

$$E q_t \leq e^{-(\frac{1}{8} + f(\varepsilon) - \delta)} E(\int_0^{ct} e^{W_H^s ds})^{-1}.$$ 

Again, the fact that $E(\int_0^{ct} e^{W_H^s ds})^{-1}$ is a bounded and decreasing function of $t \in [1, \infty)$ easily follows from Lemma 2.

Combining all estimates obtained above for $H \in (0, \frac{1}{2}]$ we have the following estimate

$$E e^{\delta |\zeta|^2 H} \leq c_\delta + e^\delta + 2E(\int_0^{ct} e^{W_H^s ds})^{-1} \int_1^\infty e^{-(\frac{1}{8} + f(\varepsilon) - \delta) t^2} dt$$ 

where the right-hand side is obviously finite when $\varepsilon$ is small enough and $\delta < \frac{1}{8}$.

This completes the proof of Theorem 1 with $\alpha_H \geq \frac{1}{8}$ for all $H \in (0, 1]$.

Remark 2. Using a different approach in \[18\] we found that

$$\alpha_H \geq \frac{4H^2 + 2H - 1}{2(2H + 2)(2H + 1)} \text{ for } H > (\sqrt{5} - 1)/4 = 0.3090....$$ 

Comparing this estimate with that one obtained in the proof of Theorem 1 we get the following lower bounds:

$$\alpha_H \geq \frac{1}{8} \text{ for } H \in (0, H_0), \quad \alpha_H \geq \frac{4H^2 + 2H - 1}{2(2H + 2)(2H + 1)} \text{ for } H \in [H_0, 1),$$

where $H_0 = (\sqrt{73} - 1)/12 = 0.6287...$, $H_0$ is the largest root of the equation

$$\frac{4H^2 + 2H - 1}{2(2H + 2)(2H + 1)} = \frac{1}{8}.$$ 

Recall that for $H = 1$ the index $\alpha_1 = \frac{1}{2}$, see \[9\].
Conjecture. There exists an index $\alpha_H$ such that
\[ E e^{\delta |\zeta_H|^{2H}} < \infty \quad \text{for } \delta < \alpha_H, \quad E e^{\delta |\zeta_H|^{2H}} = \infty \quad \text{for } \delta > \alpha_H. \]

Remark 3. This conjecture is motivated by the result of Theorem 1 and the following result on the limit distribution of Maximum Likelihood Estimator (MLE) $\xi_H$ for the case $H = 1/2$.

It is well known that the distribution of $\xi_{\frac{1}{2}}$ coincides with the distribution of a location of maximum of two-sided Brownian motion and is
\[ P(|\xi_{\frac{1}{2}}| > t) = (t + 5)\Phi\left(-\frac{\sqrt{t}}{2}\right) - \sqrt{\frac{2t}{\pi}} e^{-\frac{t}{2}} - 3e^{\frac{3\sqrt{t}}{2}}, \quad t \to \infty, \tag{17} \]
where $\Phi(t)$ is a standard normal distribution. This result can be easily derived from the papers [24] and [23]. Using the well known formula $\Phi(-x) = \sqrt{\frac{1}{2\pi}} e^{-\frac{x^2}{2}} (1 + o(1)), \ x \to \infty$, we have
\[ P(|\xi_{\frac{1}{2}}| > t) = \sqrt{\frac{32}{\pi t}} e^{-\frac{t}{2}} (1 + o(1)), \ t \to \infty \]
and, hence,
\[ E e^{\delta |\xi_{\frac{1}{2}}|} < \infty \quad \text{for } \delta < \frac{1}{8}, \quad E e^{|\xi_{\frac{1}{2}}|} = \infty. \]

Note that, from (17) one can directly obtain
\[ Var(\xi_{\frac{1}{2}}) = 26. \tag{18} \]
This result appeared in [21] for the first time.

Remark 4. The reviewer of this paper indicated that the existence of exponential moments for $|\zeta_H|^{2H}$ can be retraced from the general results of Ibragimov-Hasminski theory (see [12]) in combination with some results from [6].

3. Identities for expectations of functions of $\zeta_H$.

Theorem 2. Let $G(\zeta_H)$ be a measurable bounded function of $\zeta_H$, $H \in (0, 1]$. Then
\[ EG(\zeta_H) = \int_{-\infty}^{\infty} EG(\zeta_H - t) q_t dt. \tag{19} \]

Proof. Using (1) and Lemma 1 with
\[ \xi = W^H_t, \ X_s = W^H_s, \]
we obtain
\[ \int_{-\infty}^{\infty} EG(\zeta_H - t) q_t dt = \int_{-\infty}^{\infty} E^QG(\zeta_H - t)(\int_{-\infty}^{\infty} Z_s ds)^{-1} dt \]

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(using Lemma 1 after simplifying we get)

\[
\int_{-\infty}^{\infty} \text{EG} \left( \int_{-\infty}^{\infty} e^{W_t + \frac{1}{2}|t-s|^{2H}} ds \left( \int_{-\infty}^{\infty} e^{W_s + \frac{1}{2}|s-t|^{2H}} ds \right)^{-1} - t \right) \times \left( \int_{-\infty}^{\infty} e^{W_t + \frac{1}{2}|t-s|^{2H}} ds \right)^{-1} dt.
\]

Next, using (15) we have

\[
\int_{-\infty}^{\infty} \text{EG}(\zeta_H - t)q_t dt =
\int_{-\infty}^{\infty} \text{EG} \left( \int_{-\infty}^{\infty} e^{W_{s-t} + \frac{1}{2}|s-t|^{2H}} ds \left( \int_{-\infty}^{\infty} e^{W_{s-t} + \frac{1}{2}|s-t|^{2H}} ds \right)^{-1} - t \right) \times e^{W_{s-t} + \frac{1}{2}|s-t|^{2H}} \left( \int_{-\infty}^{\infty} e^{W_{s-t} + \frac{1}{2}|s-t|^{2H}} ds \right)^{-1} dt
\]

(setting \( s - t = u \))

\[
= \int_{-\infty}^{\infty} \text{EG}(\zeta_H)q_t dt = \text{EG}(\zeta_H) \int_{-\infty}^{\infty} q_t dt = \text{EG}(\zeta_H).
\]

This completes the proof.

**Remark 5.** A discrete-time analog of (19) for independent identically distributed (iid) observations can be found in [4], Lemma 2.18.1, [8].

Further we use notations:

\[
B_i = \int_{-\infty}^{\infty} t^i Z_t dt, \ i = 0, 1, 2... \\
A_p = \int_{-\infty}^{\infty} |t|^p q_t dt, \ p > 0.
\]

Due to the fact that \( q_t = Z_t(\int_{-\infty}^{\infty} Z_u du)^{-1} = \frac{Z_t}{B_0} \) is a density function and due to the Holder inequality we obtain from (11) that for any \( p \geq 1 \)

\[
|\zeta_H|^p \leq A_p.
\]
Corollary 1. For any $H \in (0, 1]$

$$\text{Var}(\zeta_H) = E\zeta_H^2 = \frac{1}{2}EA_2. \quad (21)$$

**Proof.** Let $G(\zeta_H) = \min(|\zeta_H|^2, K)$ with a finite parameter $K > 0$. Then in view of Theorem 1 and passing to the limit as $K \to \infty$ (using the Lebesgue theorem and Fatou’s lemma) we obtain

$$E\zeta_H^2 = \int_{-\infty}^{\infty} E(\zeta_H - t)^2 q_\eta dt.$$

Note that by (20) for any $H \in (0, 1]$

$$E\zeta_H^2 = EA_1^2 < \infty, \quad \int_{-\infty}^{\infty} t^2 Eq_\eta dt < \infty.$$

Expanding $(\zeta_H - t)^2 = \zeta_H^2 - 2\zeta_H t + t^2$ this implies

$$E\zeta_H^2 = \int_{-\infty}^{\infty} E(\zeta_H^2 - 2\zeta_H t + t^2) \frac{Z_t}{B_0} dt = E\zeta_H^2 - 2E\zeta_H A_1 + EA_2$$

$$= E\zeta_H^2 - 2E\zeta_H^2 + EA_2.$$

After simplifying we get (21).

**Remark 6.** Originally the identity (21) was proved in [8] for $H > \frac{1}{2}$. The method used in [8] was based on the fact that a similar identity is valid for Pitman estimators of a location parameter for independent identically distributed observations.

Theorem 1 can be used for derivation of various useful properties of the distribution of $\zeta_H$. As another example we present the following result.

**Corollary 2.** For any $H \in (0, 1]$ and $k = 2, 4, 6, \ldots$ there exist constants $c_k > 0$ such that

$$E\zeta_H^k \geq c_k^k EA_k, \quad (22)$$

where $c_k$ is the unique positive root of the equation

$$(x + 1)^k - (x - 1)^k + 2k(x^k - x^{k-1}) = 2. \quad (23)$$

**Proof.** The validity of the result for $k = 2$ can be seen from Corollary 1.

For the case $k \geq 4$ we apply Theorem 1 with the polynomial $G(x) = x^k$. Then we obtain

$$E\zeta_H^k = E\zeta_H^k - kE\zeta_H^{k-1} A_1 + \sum_{i=2}^{k-1} (-1)^i C_k^i E\zeta_H^{k-i} A_i + EA_k,$$

where $C_k^i$ are binomial coefficients. This implies

$$kE\zeta_H^k = \sum_{i=2}^{k-1} (-1)^i C_k^i E\zeta_H^{k-i} A_i + EA_k$$

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and hence
\[ kE(\zeta_H^k) \geq \sum_{i=odd \geq 3}^{k-1} C_k^i E|\zeta_H^{k-i}|A_i + EA_k. \]

By the Holder inequality
\[ E|\zeta_H^{k-i}| A_i \leq (E\zeta_H^k)^{1-i/k}(EA_k^{i/k})^{i/k} \leq (E\zeta_H^k)^{1-i/k}(EA_k)^{i/k}, \]
we have
\[ kE\zeta_H^k \geq EA_k - \sum_{i=odd \geq 3}^{k-1} C_k^i(E\zeta_H^k)^{1-i/k}(EA_k)^{i/k}. \]

Set \( x^k = (E\zeta_H^k)/EA_k \). Then
\[ (E\zeta_H^k)^{1-i/k}(EA_k)^{1-i/k} = (E\zeta_H^k/EA_k)^{1-i/k} = x^{k-i} \]
and therefore the last inequality is equivalent to
\[ kx^k + \sum_{i=odd \geq 3}^{k-1} C_k^i x^{k-i} \geq 1. \]

We can find a short expression for \( \sum_{i=odd \geq 3}^{k-1} C_k^i x^{k-i} \) using the following elementary identity
\[ 2 \sum_{i=odd \geq 3}^{k-1} C_k^i x^{k-i} = (x + 1)^k - (x - 1)^k - 2kx^{k-1}. \]

Hence we obtain
\[ (x + 1)^k - (x - 1)^k + 2k(x^k - x^{k-1}) \geq 2. \]

This implies the result.

**Remark 7.** One can easily verify that
\[ c_k = \frac{D}{k}(1 + o(1)), \ k \to \infty, \] (24)
where \( D = \ln(1 + \sqrt{2}) \) is the unique positive root of the equation
\[ \sinh(D) = 1. \]

The derivation of (24) is elementary and is omitted. We restrict ourselves to illustration of accuracy of the approximation (24) for \( k = 100, D/100 \approx 8.813 \times 10^{-3} \), and in this case the exact solution of (23) is \( c_{100} = 8.8412 \times 10^{-3} \).
4. Representation for \( \text{Var}(\zeta_H) \).

In this section for the case \( H \in [\frac{1}{2}, 1] \) we derive another representation for \( \text{Var}(\zeta_H) \) in terms of the function \( g(m) \) defined above in (4).

Furthermore we use the following parametrised random functions:

\[
\alpha(m) = \int_{-\infty}^{\infty} u e^{mu} Z_u du, \quad \beta(m) = \int_{-\infty}^{\infty} e^{mu} Z_u du
\]

where \( m \) is an auxiliary parameter. In these notations we have

\[
\int_{-\infty}^{\infty} Z_u du = \beta(0), \quad \int_{-\infty}^{\infty} u Z_u du = \alpha(0), \quad \alpha(m) = \frac{\partial}{\partial m} \beta(m)
\]

and

\[
\zeta_H = \frac{\alpha(0)}{\beta(0)}.
\]

Note that due to the symmetry property of fBm we have \( g(m) = g(-m) \) and from inequality \( \log(a + b) \leq \log(a + 1) + \log(b + 1), \ (a > 0, \ b > 0) \), we have

\[
g(m) \leq E \log(\int_{-\infty}^{0} e^{mu} Z_u du + 1) + E \log(\int_{0}^{\infty} e^{mu} Z_u du + 1). \tag{25}
\]

Let \( m > 0 \). The finiteness of the first expectation in the RHS of (25) is obvious due to the inequality \( \log(x + 1) \leq x \) and the equality \( EZ_u = 1 \).

The finiteness of the second expectation in the right-hand side (RHS) of (25) for \( m > 0 \) can be shown as follows.

Note

\[
\log(\int_{0}^{\infty} e^{mu} Z_u du + 1) = \int_{0}^{\infty} (\int_{0}^{s} e^{mu} Z_u du) \frac{ds}{e^{ms}} \leq \int_{1}^{\infty} e^{ms} (\int_{0}^{s} Z_u du)^{-1} ds + e^{m} \int_{0}^{1} Z_s (\int_{0}^{s} Z_u du)^{-1} ds
\]

Since \( EZ_u (\int_{0}^{s} Z_u du)^{-1} = Eq_u \leq Ce^{-\delta s^{2H}} \) for \( s > 1 \) and \( \delta < \frac{1}{7} \) (see the proof of Theorem 1 and \( [3] \), from now on \( C \) is a generic constant) we can claim that \( g(m), 0 < m < \frac{1}{8} \) is finite (recall that we assumed \( H \in [\frac{1}{2}, 1] \)). Obviously, \( g(m) \) is a continuous function.

**Theorem 3.** Let \( H \in [\frac{1}{2}, 1] \). Then the function \( g(m) \) is twice continuously differentiable on the interval \( m \in (-1/8, 1/8) \) and

\[
\text{Var}(\zeta_H) = \frac{\partial^2 g(m)}{\partial m^2} |_{m=0}.
\]

**Proof.** Using the notation introduced above we have

\[
\text{Var}(\zeta_H) = E\frac{\alpha^2(0)}{\beta^2(0)} = \lim_{m \to 0} E\frac{\alpha^2(m)}{\beta^2(m)}.
\]
The last equality can be justified by (3) and the estimate

$$E\frac{\alpha^2(m)}{\beta^2(m)} = E\left(\int_{-\infty}^{\infty} u e^{\mu u} Z_u du\right)^2 \leq E \int_{-\infty}^{\infty} u^2 e^{\mu u} Z_u du < \infty.$$ 

By direct calculations we obtain for $m > 0$ that

$$\frac{\partial^2 \log \beta(m)}{\partial m^2} = -\frac{\alpha^2(m)}{\beta^2(m)} + \frac{\int_{-\infty}^{\infty} u^2 e^{\mu u} Z_u du}{\beta(m)}.$$ 

Applying the expectation to both sides of the last equality and using well-known theorems about differentiability of expectations with respect to a parameter we obtain

$$\frac{\partial^2 g(m)}{\partial m^2} = E\frac{\partial^2 \log \beta(m)}{\partial m^2} = -E\frac{\alpha^2(m)}{\beta^2(m)} + E\frac{\int_{-\infty}^{\infty} u^2 e^{\mu u} Z_u du}{\beta(m)}, \quad (26)$$

where the RHS is a continuous function of $m$. This implies $\frac{\partial^2 g(m)}{\partial m^2}$ is a continuous function for $m \in (0, 1/8)$ and (due to symmetry) also for $m \in (-1/8, 0)$. Passing to the limit in (26) as $m \to 0$ we obtain

$$\left.\frac{\partial^2 g(m)}{\partial m^2}\right|_{m=0} = -E\frac{\alpha^2(0)}{\beta^2(0)} + E\frac{\int_{-\infty}^{\infty} u^2 Z_u du}{\beta(0)} = -\text{Var}(\zeta_H) + 2\text{Var}(\zeta_H) = \text{Var}(\zeta_H).$$

This completes the proof.

5. Modelling results.

To the best of our knowledge the problem of evaluation of integral functionals numerically remains unresolved. The only known explicit result is given by formula (5). These integral functionals can be modelled using Monte-Carlo simulation method. The results of Monte-Carlo modelling for variances of Pitman estimator $\zeta_H$ and asymptotic MLE $\xi_H$ for $H \in [0, 4/1)$ are given in the Table 1.

For simulation of increments of fBm we implemented the "Circulant embedding method" (see [22]) which is recognised as one of the fastest methods for simulation of stationary Gaussian processes.

The graphs of $\text{Var}(\zeta_H)$ and $\text{Var}(\xi_H)$ versus $H \in [0, 4, 1)$ are plotted in Figure 1. Both $\text{Var}(\zeta_H)$ and $\text{Var}(\xi_H)$ are monotone functions taking larger values for small values of $H$, $\text{Var}(\zeta_H) < \text{Var}(\xi_H)$. The results of calculations agree well with formulae (5) and (18). Detailed discussion of accuracy of $\text{Var}(\zeta_H)$ and $\text{Var}(\xi_H)$ are provided in [16].

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Table 1: Monte Carlo estimates for $\text{Var}(\zeta_H)$ and $\text{Var}(\xi_H)$ using $10^6$ trajectories. Each trajectory is generated with $2^{18}$ equally-spaced discretisation points on the interval $(-10^5, 10^5)$.

| $H$ | $\text{Var}(\zeta_H)$ | $SE\text{Var}(\zeta_H)$ | $\text{Var}(\xi_H)$ | $SE\text{Var}(\xi_H)$ |
|-----|----------------|----------------|----------------|----------------|
| 0.4 | 109.682 | 0.4698 | 151.707 | 0.2145 |
| 0.5 | 19.2544 | 0.0350 | 25.964 | 0.0367 |
| 0.6 | 6.52596 | 0.0163 | 8.63501 | 0.0386 |
| 0.7 | 3.16871 | 0.0066 | 4.08858 | 0.0182 |
| 0.81 | 1.82699 | 0.0032 | 2.24197 | 0.0101 |
| 0.91 | 1.28289 | 0.002 | 1.47782 | 0.0066 |

Figure 1: Values of $\text{Var}(\zeta_H)$ (solid line) and $\text{Var}(\xi_H)$ (dashed line) for $H \in [0.4, 1]$ are given on a logarithmic axis.
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