The hidden subgroup problem and permutation group theory

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Abstract

We employ concepts and tools from the theory of finite permutation groups in order to analyse the Hidden Subgroup Problem via Quantum Fourier Sampling (QFS) for the symmetric group. We show that under very general conditions both the weak and the random-strong form (strong form with random choices of basis) of QFS fail to provide any advantage over classical exhaustive search. In particular we give a complete characterisation of polynomial size subgroups, and of primitive subgroups, that can be distinguished from the identity subgroup with the above methods. Furthermore, assuming a plausible group theoretic conjecture for which we give supporting evidence, we show that weak and random-strong QFS for the symmetric group have no advantage whatsoever over classical search.

1 Introduction

In the last decade quantum computation has provided us with powerful tools to solve problems not known to be classically efficiently solvable, like factoring [Sho94] and discrete log [Kit95]. Nearly all the problems in which a quantum computer excels more than quadratically with respect to its classical counterpart can be cast into the framework of the Hidden Subgroup Problem (HSP). Let $G$ be a finite group and $H \leq G$ a subgroup. Given a function $f : G \to S$ that is constant on (left)-cosets $gH$ of $H$ and takes different values for different cosets, determine a set of generators for $H$. The decision version of this problem is to determine whether there is a non-identity hidden subgroup or not.

The reason that quantum computers seem to provide a speed-up for this type of problem is that it is possible to implement the Fourier transform over certain groups efficiently on a quantum computer. This in turn allows to sample the Fourier components efficiently (this technique is referred to as the “standard method”). In the case of Abelian groups $G$ (appearing in factoring and discrete log) the hidden subgroup can be reconstructed with only a polynomial (in $\log |G|$) number of queries to the function and a polynomial number of measurements (samplings in the Fourier basis) and postprocessing steps.

Addressing the HSP in the non-Abelian case is considered to be one of the most important challenges at present in quantum computing. A positive answer to the question whether quantum computers can efficiently solve the Hidden Subgroup Problem over non-Abelian groups would have several important implications for the solution of problems in NP, which are neither known to be NP-complete nor in P; and which are good candidates
for a quantum speed-up. Among the most prominent such problems is Graph Isomorphism, where the group in question is the symmetric group.

For many non-Abelian groups it is possible to implement the Fourier transform on a quantum computer efficiently [EH99, RB98, PRB99, GSVV01, MRR04], and in particular explicit constructions exist for the symmetric group $S_n$ [Bea97]. This fact and the prominence of the problems involved make it very desirable to get a handle on the power of Quantum Fourier Sampling (QFS) to solve the HSP or its decision version for general groups.

In this paper we focus on the question which hidden subgroups can be distinguished from the identity via QFS with special attention to the symmetric group. Several partial positive results have been obtained previously for groups that are in some ways “close” to Abelian, like some semidirect products of Abelian groups [EH99, RB98, Kup03, MRRS04], in particular the Dihedral group; Hamiltonian groups [HRT00], groups with small commutator groups [IMS01], and solvable groups of constant exponent and constant length derived series [EIM+03]. Often in these cases the irreducible representations are known and can be analysed in a relatively straightforward way. For instance the Dihedral group $D_n$, the first non-Abelian group to be analysed in this context by Ettinger and Hoyer [EH99], is “nearly” Abelian in the sense that all of its irreducible representations (irreps) have degree at most two. Indeed hidden reflections of $D_n$ can be distinguished from identity with only polynomial Quantum Fourier Samplings, similar to the Abelian case (where all irreducible representations are one-dimensional).\(^1\)

The holy grail of the field is the symmetric group $S_n$, which seems much harder to analyse, partly because to this day there is still only partial explicit knowledge about its irreducible representations and character values [Sag01], because most of its subgroups are far from normal (have many conjugate subgroups), because most of its irreducible representations have very large dimension ($2^{\Theta(n \log n)}$) and the number of different irreducible representations is an exponentially small fraction of the size of the group, to name just some of the difficulties. The structure of distinguishable versus indistinguishable subgroups of $S_n$ has remained highly elusive.

In this work we provide a substantial step towards a complete classification of subgroups of the symmetric group for which the decision version of the HSP can be solved efficiently via Quantum Fourier Sampling. We bring into play classical notions and results from the theory of finite permutation groups, which have not been employed before in quantum computing, and seem to be relevant in these investigations. These include notions such as minimal degree, rank, subdegrees, and primitivity, which played a key role in permutation group theory since the days of Jordan in the 19th century. Moreover, recent advances in finite permutation groups, due to sophisticated work by Babai [Bab81, Bab82] and others on the one hand, and due to the Classification of Finite Simple Groups (CFSG) on the other hand (see Cameron [Cam81]), provide us with very powerful machinery.

Using these notions and machinery we present several new results, which incorporate existing results in the area, and we provide a toolbox for further investigations. We are able to give both upper bounds and, for the first time in this context, lower bounds on the total variation distance of the relevant distributions, and to derive many applications.

\(^1\)Note, however, that the computational version of HSP seems much harder: even though a polynomial number of samples suffice to distinguish hidden reflections information theoretically, no efficient reconstruction procedure beyond sophisticated exhaustive search is known.
Outline of results: In a nutshell, our main results show that under various conditions on the hidden subgroup $H \leq S_n$, the following statement is true for the hidden subgroup $H \leq S_n$:

$\blacklozenge$ $H$ can be distinguished from the identity subgroup with either the weak standard method or the strong standard method with random basis only if it contains an element of constant support (i.e. a permutation in which all but a constant number of points are fixed).

Since there is only a polynomial number of such elements in $S_n$, the statement implies that the standard method both in its weak and strong form with random basis provides no advantage over classical exhaustive search.

- Statement $\blacklozenge$ is true if $H$ is of polynomial size.
- It is also true for an important class of subgroups known as primitive groups. These subgroups, which can be superpolynomial in size, are considered the building blocks of permutation groups.
- We exhibit a family of even larger subgroups, of exponential order, for which Statement $\blacklozenge$ is true.

The cases we study seem to suggest that Statement $\blacklozenge$ might hold for all subgroups $H$. Surprisingly we show that this is indeed true under a plausible group theoretic conjecture, for which we provide further evidence. Assuming the conjecture, it follows that Quantum Fourier Sampling (with random basis) provides no advantage whatsoever over classical exhaustive search.

Main results: We focus on the weak form of the standard method (see Section 2), since the strong form with random choices of basis does not provide any non-negligible additional information for the symmetric group and the subgroups we consider [GSVV01]. In particular we are not aware of any cases where there is a good basis for the strong method, but a random choice of basis does not also solve the HSP.

To state our main results, let $G$ be a finite group, $H \leq G$ a subgroup, and let $D_H$ denote the total variation distance between the distributions on the irreducible representations of $G$ induced by $H$ and by the identity respectively, when sampled with the weak standard method. We say that $H$ is distinguishable (using the weak standard method) if $D_H \geq \log |G|^{-c}$ for some constant $c$, and is indistinguishable otherwise.

Our starting point is a general result providing both upper and lower bounds on the total variation distance $D_H$ in terms of the same group theoretic data.

**Theorem 1.1** Let $C_1, \ldots, C_k$ denote the non-identity conjugacy classes of $G$. Then

$$
\sum_{i=1}^{k} |C_i \cap H|^2 |H|^{-1} |C_i|^{-1} < D_H \leq \sum_{i=1}^{k} |C_i \cap H||C_i|^{-\frac{1}{2}}.
$$

[1] The strong standard method sometimes provides substantially more information than its weak counterpart, and is indeed necessary to efficiently solve HSP in the case of groups like the Dihedral group [EH99, Kup03] and other semidirect product groups [MRRS04]. An irrep, and hence the non-Abelian QFT, is given only up to the choice of basis. Grigni et al. show that for a random basis the additional information provided by the strong method is exponentially small, provided the group is sufficiently non-Abelian and the hidden subgroup sufficiently small, as is the case for all groups we analyse here. It remains to be seen whether judicious choices of basis for each irrep can give more information in the case where random choices don’t help; but this is believed to be unlikely.
Applying the upper bound with $|H| = 2$ gives the result obtained previously by Hallgren et al. and Grigni et al. [HR'00, GSVV'01]. No lower bounds seem to exist in the literature.

This theorem has a wide range of applications. For example, it enables us to characterise distinguishable subgroups $H \leq G$ of polylogarithmic order (see Theorem 3.2 below).

Specialising to $G = S_n$ we show that the minimal degree of $H$ is a crucial notion in the study of the distinguishability of $H$. The minimal degree $m(H)$ of a permutation group $H$ is defined to be the minimal number of points moved by a non-identity element of $H$. In other words, for $g \in S_n$ let $\text{fix}(g)$ be the number of fixed points of $g$, and let $\text{supp}(g) = n - \text{fix}(g)$ be the support of $g$. Then

$$m(H) = \min\{\text{supp}(h) : 1 \neq h \in H\}.$$

This notion goes back to the 19th century, and plays an important role in the theory of finite permutation groups since the days of Jordan [Jor73, Jor75]. It is intriguing that it plays some role in the HSP as well, giving a complete characterisation of distinguishable polynomial size subgroups:

**Theorem 1.2** Let $H \leq S_n$ with $|H| \leq n^c$ for some constant $c$. Then $H$ is distinguishable if and only if its minimal degree $m(H)$ is constant.

For instance we cannot distinguish a group generated by a cycle of non-constant length or an involution with non-constant number of transpositions (implying the result in [HR'00, GSVV'01]). Note that the strength of this theorem comes from the “if-and-only-if”: The distinguishable subgroups must contain an element of constant support. Since there are only polynomially many such elements in $S_n$ we can just exhaustively query them.

This also has implications for the Graph Isomorphism (GI) problem. Recall that to solve GI for two graphs $G_1, G_2$, it suffices to distinguish a hidden subgroup of the automorphism group $\text{Aut}(G_1 \cup G_2)$ of the form $H_1 \times H_2$ (not $G_1 \cong G_2$), where $H_i = \text{Aut}(G_i)$, from a subgroup of the form $H \cup \sigma H$ ($G_1 \cong G_2$), where $H = H_1 \times H_2$ and $\sigma$ maps $G_1$ to $G_2$ (see e.g. [Joz00]). If the automorphism group of each graph is of polynomial size our results imply that we cannot distinguish each of the two possible cases from identity, and hence (using the triangle inequality) we cannot distinguish them from each other unless $\text{Aut}(G_i)$ contains an element of constant support. Thus QFS provides no advantage here.

Our next result concerns primitive subgroups. A permutation group is called primitive if it is transitive (has only one orbit) and does not preserve a non-trivial partition (block system) of the permutation domain. Primitive permutation groups are considered the building blocks of finite permutation groups in general, and were extensively studied over the past 130 years. We note that if $H \leq S_n$ is primitive and $H \neq A_n, S_n$ then Babai showed that $m(H) \geq (\sqrt{n} - 1)/2$ and $|H| \leq n^{4\sqrt{n}\log n}$. Using the Classification of Finite Simple Groups the latter bound can be somewhat improved to $|H| \leq 2n^{\sqrt{n}}$, which is essentially best possible [CamSI]; in particular the order of $H$ can be much more than polynomial, and so Theorem 1.2 above does not apply.

However, we obtain the following somewhat surprising general result:

**Theorem 1.3** Let $H \neq A_n, S_n$ be a primitive subgroup. Then $H$ is indistinguishable.

As the hidden subgroups get large we would suppose that it becomes easier to distinguish them from the identity. However, we show below that $H$ can get extremely large and yet cannot be distinguished with the weak standard method:
Theorem 1.4 Let $\varepsilon(n)$ be a sequence of real numbers which tend to zero as $n \to \infty$. Then for all sufficiently large $n$ there is an indistinguishable subgroup $H < S_n$ of size $|H| \geq |S_n|^{\varepsilon(n)}$.

In particular, there are indistinguishable subgroups $H$ of exponential order.

To prove Theorems 1.3, 1.4 we give a somewhat technical group theoretic criterion for indistinguishability of subgroups of non-constant minimal degree (Proposition 4.1). We conjecture that this criterion applies universally. This implies that every distinguishable subgroup has a non-identity element of constant support.

It is interesting that permutation-theoretic data is relevant to the distinguishability problem even when the group $G$ in question is not $S_n$, but an arbitrary finite group. Indeed, given $H \leq G$ there is a standard way to view $G$ (or $G/N$ where $N$ is the normal core of $H$) as a transitive permutation group on the set $X = G/H$ of (right) cosets of $H$ in $G$ (where $g \in G$ acts by right multiplication). Recall that a suborbit of $G$ in this action is an orbit of $H$ on $X$, and the rank $r_X(G)$ of $G$ is defined to be the number of suborbits of $G$. The subdegrees of $G$ are the sizes of its suborbits. Thus the average subdegree of $G$ is $|G : H|/r_X(G)$. Setting $H^g = gHg^{-1}$ it is easy to see that the subdegrees of $G$ have the form $|H : H \cap H^g|$ for $g \in G$. Using this data, Theorem 1.4 and classical permutation theoretic tools, we obtain the following positive result.

Theorem 1.5 Suppose $|H|$ is not polylogarithmic, but the average subdegree of $G$ on $G/H$ is polylogarithmic. Then $H$ is distinguishable. In particular this holds when $|H : H \cap H^g| \leq (\log |G|)^c$ for all $g \in G$.

This theorem extends the result of [HR 00] showing that if $H$ is normal in $G$ then $H$ is distinguishable (since in this case $|H : H \cap H^g| = 1$ for all $g$). It also implies the easy observation that subgroups of size at least $|G|/(\log |G|)^c$ are always distinguishable.

Our methods also allow us to examine the more general case of distinguishing between two subgroups $H$ and $K$ of $G$, see Section 3 for some details.

Further Related Work: The HSP plays a central role in most known quantum algorithms and the efficient algorithm for the Abelian case using Fourier Sampling is folklore. The non-Abelian HSP has received a lot of attention in recent years, due to its connection to several candidate problems in NP like Graph Isomorphism (for the symmetric group) and lattice problems [Reg 02] (for the Dihedral group); we mention only the work relevant to ours.

Despite a lot of progress for various non-Abelian groups [EH 99, RB 98, MRRS 04, IMS 01, FIM 03] the results on the symmetric group are very sparse. Grigni et al. [GSVV 01] show that sampling the row index in the strong standard method provides no additional information. They also show that the additional information provided by the strong method in a random basis scales with $\sqrt{|H|^2k(G)/|G|}$ where $k(G)$ is the number of conjugacy classes of the group $G$ and $|H|$ the size of the hidden subgroup. Both Hallgren et al. and Grigni et al. [HRT 00, GSVV 01] show that hidden subgroups of $S_n$ of size $|H| = 2$, generated by involutions with large support, cannot be distinguished from identity; exactly the task that needs to be solved for Graph Automorphism.

Hallgren et al. [HRT 00] also point out that the weak standard method cannot distinguish between conjugate subgroups. In [HRT 00, GSVV 01] it is shown that the weak standard method allows us to efficiently determine the normal core of a hidden subgroup $H$ and hence in particular normal subgroups.
2 Preliminaries and notation

Fix a finite group $G$ and a subgroup $H \leq G$. We denote states of the vector space $\mathbb{C}[G]$, spanned by the group elements, with a $| \cdot \rangle$, as is standard in quantum computation\textsuperscript{3}.

The Quantum Fourier Transform (QFT) over a group $G$ is the following unitary transformation on $\mathbb{C}[G]$:

$$|g\rangle \rightarrow \frac{1}{\sqrt{|G|}} \sum_{\rho, i, j} \sqrt{d_\rho} \rho(g)_{ij} |\rho, i, j\rangle$$

where $\rho$ labels an irreducible representation of $G$, $d_\rho$ is its dimension and $1 \leq i, j \leq d_\rho$. The $|\rho, i, j\rangle$ span another basis of $\mathbb{C}[G]$, the so called Fourier basis.

The standard method of Quantum Fourier Sampling is the following: The state is initialised in a uniform superposition over all group elements; a second register is initialised to $|0\rangle$. Then the function $f$ is applied reversibly over both registers (i.e. $f : |g\rangle|0\rangle \rightarrow |g\rangle|f(g)\rangle$). Finally the second register is measured, which puts the first register into the superposition of a (left)-coset of $H$, i.e. in the state $|gH\rangle := \frac{1}{\sqrt{|H|}} \sum_{h \in H} |gh\rangle$ for some random $g \in G$. Finally the QFT over $G$ is performed, yielding the state

$$\frac{1}{\sqrt{|G||H|}} \sum_{\rho, i, j} \sqrt{d_\rho} \sum_{h \in H} \rho_{ij}(gh) |\rho, i, j\rangle.$$ 

A basis measurement now gives $(\rho, i, j)$ with probability $P_{gH}(\rho, i, j) = \frac{d_\rho}{|G||H|} |\sum_{h \in H} \rho_{ij}(gh)|^2$.

Since we do not know $g$ and $g$ is distributed uniformly, we sample $(\rho, i, j)$ with probability $P_H = \frac{1}{|G|} \sum_{g} P_{gH}$. The strong standard method samples both $\rho$ and its entries $i, j$. In the weak standard method only the character $\chi_\rho$ is measured (but not the entries $i, j$, which are averaged over)\textsuperscript{4}. The probability to measure $\rho$ in the weak case is

$$P_H(\rho) = \frac{d_\rho}{|G|} \sum_{h \in H} \chi_\rho(h).$$

Let $Irr(G)$ be the set of irreducible characters of $G$. Then $P_H$ is a distribution on $Irr(G)$.

To solve HSP we need to infer $H$ from the resulting distribution. Distinguishing the trivial subgroup $\{e\}$ from a larger subgroup $H$ efficiently using the standard method is possible if and only if the $L_1$ distance $D_H$ between $P_{\{e\}}$ and $P_H$ is larger than some inverse polynomial in $\log |G|$. The $L_1$ distance (also known as the total variation distance) is given as

$$D_H = \frac{1}{|G|} \sum_{\rho} d_\rho \sum_{h \in H, h \neq e} |\chi_\rho(h)|.$$ \hspace{1cm} (2)

We say that $H$ is distinguishable (using the weak standard method) if $D_H \geq (\log |G|)^{-c}$ for some constant $c$, and indistinguishable otherwise. If $K \leq G$ is another subgroup we let $D(H, K) = |P_H - P_K|_1$ be the $L_1$ distance between the distributions $P_H$ and $P_K$.

We also need some group theoretic notation. For $x \in G$ we let $x^G$ denote the conjugacy class of $x$ in $G$. Let $C_1, \ldots, C_k$ denote the non-identity conjugacy classes of $G$. For an irreducible character $\chi_\rho \in Irr(G)$ we let $\chi_\rho(C_i)$ denote the common value of $\chi_\rho(x)$ for elements $x \in C_i$.

\textsuperscript{3}For the necessary background in quantum computation see e.g. [NC00].

\textsuperscript{4}It is easy to see [HRT00, GSVV01] that for the weak standard method the probability to sample $\rho$ is independent of the coset of $H$ we happen to land in.
3 Arbitrary groups

In this section we discuss results for arbitrary groups $G$, providing some proofs when space allows.

Proof of Theorem 1.1. For each irreducible representation $\rho$ of $G$ we have

\[ |\sum_{h \in H, h \neq e} \chi_\rho(h)| \leq \sum_{h \in H, h \neq e} |\chi_\rho(h)| \leq \sum_{h \in H, h \neq e} d_\rho < |H| d_\rho. \]

Hence $d_\rho > |H|^{-1} \sum_{h \in H, h \neq e} \chi_\rho(h)$. Substituting this in (2) we obtain

\[ D_H > \frac{1}{|G||H|} \sum_\rho \sum_{h \in H, h \neq e} |\rho(h)|^2. \]

Note that $\chi_\rho(h) = \chi_\rho(C_i)$ if $h \in H \cap C_i$. This yields $\sum_{h \in H, h \neq e} \chi_\rho(h) = \sum_{i=1}^k |H \cap C_i| \chi_\rho(C_i)$, and so

\[ D_H > \frac{1}{|G||H|} \sum_\rho \sum_{i=1}^k |H \cap C_i| \chi_\rho(C_i)^2. \]

Now,

\[ k \sum_{i=1}^k |H \cap C_i| \chi_\rho(C_i)^2 = \sum_{i=1}^k |H \cap C_i|^2 |\chi_\rho(C_i)|^2 + \sum_{i \neq j} |H \cap C_i||H \cap C_j| \chi_\rho(C_i) \chi_\rho(C_j). \]

Using the generalised orthogonality relations we observe that

\[ \sum_\rho \sum_{i=1}^k |H \cap C_i|^2 |\chi_\rho(C_i)|^2 = \sum_{i=1}^k |H \cap C_i|^2 |G|/|C_i|, \]

and

\[ \sum_\rho \sum_{i \neq j} |H \cap C_i| |H \cap C_j| \chi_\rho(C_i) \chi_\rho(C_j) = 0. \]

It follows that

\[ D_H > \frac{1}{|G||H|} \sum_{i=1}^k |H \cap C_i|^2 |G|/|C_i| = \sum_{i=1}^k |H \cap C_i|^2 |H|^{-1} |C_i|^{-1}. \]

This completes the proof of the lower bound. To prove the upper bound, write

\[ D_H |G| = \sum_\rho d_\rho \sum_{h \in H, h \neq e} \chi_\rho(h) \leq \sum_\rho d_\rho \sum_{h \in H, h \neq e} |\chi_\rho(h)| = \sum_\rho \sum_{h \in H, h \neq e} d_\rho |\chi_\rho(h)|. \tag{3} \]

Fix $h \in H$ and choose $i$ such that $h \in C_i$. Using the Cauchy-Schwarz inequality we obtain

\[ \sum_\rho d_\rho |\chi_\rho(h)| \leq \left( \sum_\rho d_\rho^2 \right)^{1/2} \left( \sum_\rho |\chi_\rho(h)|^2 \right)^{1/2}, \]

giving (using the orthogonality relations) $\sum_\rho d_\rho |\chi_\rho(h)| \leq |G|^{1/2} (|G|/|C_i|)^{1/2} = |G||C_i|^{-1/2}$. Summing over non-identity elements $h \in H$, and observing that the upper bound above occurs $|H \cap C_i|$ times, we obtain $\sum_{h \in H, h \neq e} d_\rho |\chi_\rho(h)| \leq \sum_{i=1}^k |H \cap C_i||G||C_i|^{-1/2}$. Combining this with (3) we obtain $D_H \leq \sum_{i=1}^k |H \cap C_i||C_i|^{-1/2}$, as required. \(\square\)

The following is an immediate consequence of Theorem 1.1.
Corollary 3.1 Let $C_{\text{min}}$ denote a non-identity conjugacy class of minimal size intersecting $H$ non-trivially. Then we have

$$|H|^{-1}|C_{\text{min}}|^{-1} < D_H \leq (|H| - 1)|C_{\text{min}}|^{-1/2}.$$ 

We can now characterise distinguishable subgroups of polylogarithmic order in an arbitrary group $G$. Assuming $|H|$ is polylogarithmic Corollary 3.1 shows that $D_H^{-1}$ is polylogarithmic if and only if $|C_{\text{min}}|$ is. In other words we have proved the following.

**Theorem 3.2** Suppose $|H| \leq (\log |G|)^c$ for some constant $c$. Then $H$ is distinguishable if and only if $|C_{\text{min}}|$ is. In other words we have proved the following.

**Theorem 3.2** Suppose $|H| \leq (\log |G|)^c$ for some constant $c$. Then $H$ is distinguishable if and only if $H$ has a non-identity element $h$ such that $|h^G| \leq (\log |G|)^{c'}$ for some constant $c'$.

There is an interesting reformulation of the lower bound in Theorem 1.1 in terms of fixed points. Regard $G$ as a permutation group on $X = G/H$. Denote by $\text{fix}_X(g)$ the number of fixed points of $g \in G$ in this action. Let $r = r_X(G)$ denote the rank of $G$ in this action, namely, the number of orbits of the point stabilizer $H$ on $X$.

**Corollary 3.3** With the above notation we have

(i) $D_H > \frac{1}{|G|} \sum_{h \in H, h \neq e} \text{fix}_X(h)$;
(ii) $D_H > r_X(G)/|X| - 1/|H|$.

**Proof:** It is well known that, if $g \in C_i$, then $\text{fix}_X(g)/|X| = |H \cap C_i|/|C_i|$. Therefore

$$\sum_{i=1}^{k} |H \cap C_i|^2 |H|^{-1}|C_i|^{-1} = |H|^{-1} \sum_{h \in H, h \neq e} \frac{\text{fix}_X(h)}{|X|} = |G|^{-1} \sum_{h \in H, h \neq e} \text{fix}_X(h).$$

Combining this with Theorem 1.1 we deduce part (i).

To prove part (ii) we use the well known lemma of Frobenius that the number of orbits of $H$ on $X$ equals the average number of fixed points of $h \in H$ on $X$ (see for instance Theorem 1.7A of [DM96]). This shows that

$$|H|^{-1} \sum_{h \in H, h \neq e} \text{fix}_X(h) = r_X(G) - |X|/|H|.$$

Dividing both sides by $|X|$ we deduce part (ii) from part (i). □

**Proof of Theorem 1.2** This follows easily from part (ii) of the above Corollary. □

We close this section by considering the more general problem of distinguishing between two arbitrary subgroups $H$ and $K$ of $G$. Obviously, if the total variation distance $D(H, K)$ between the respective distributions is zero then the weak standard method cannot distinguish between $H$ and $K$, even if superpolynomial complexity is allowed. This gives rise to the fundamental problem of characterising subgroups $H, K$ of distance zero. It has already been observed that conjugate subgroups have distance zero [HR100], does the converse hold?

To solve these problems, recall that the permutation representation of $G$ on $G/H$ gives rise to a linear representation of $G$ in dimension $G/H$ (which can be realised using the corresponding permutation matrices). We can now state
Theorem 3.4 The following are equivalent for subgroups $H, K \leq G$.

(i) $D(H, K) = 0$.

(ii) For each conjugacy class $C$ of $G$ we have $|H \cap C| = |K \cap C|$.

(iii) The permutation representations of $G$ on $G/H$ and $G/K$ give rise to equivalent linear representations.

Moreover, there exist finite groups $G$ and non-conjugate subgroups $H, K \leq G$ such that $D(H, K) = 0$.

Our proof of the equivalence of (i)-(iii) is elementary, based on the fact that the characters in $\text{Irr}(G)$ form a base for the class functions on $G$. The proof of the last assertion is deeper and will be omitted in this version.

4 Symmetric groups

Let us now focus on the case $G = S_n$.

Proof of Theorem 1.2. Let $g \in S_n$ with $\text{supp}(g) = k$. Then it is straightforward to verify that \binom{n}{k} \leq |g^S_n| \leq n^k. As a consequence we see that a conjugacy class $C$ in $S_n$ has polynomial order if and only if it consists of elements of constant support. This observation, when combined with Theorem 3.2, completes the proof of Theorem 1.2.

The proofs of Theorems 1.3 and 1.4 are longer and less elementary, and so we will only sketch them here. In the heart of both proofs lies the following somewhat technical result.

Proposition 4.1 Let $H \leq S_n$ be a subgroup with non-constant minimal degree. Suppose that, for each $k \leq n$, $H$ has at most $n^{k/7}$ elements of support $k$. Then $H$ is indistinguishable.

Proof: Apply the upper bound of Theorem 1.1, written in the form

\[ D_H \leq \sum_{1 \neq h \in H} |h^G|^{-1/2}. \]

To evaluate this sum we use a result from [LS01], showing that, for $G = S_n$ and for $h \in G$ of support $k$ we have $|h^G| > n^{ak}$ for any real number $a < 1/3$ and $n$ large enough (given $a$). Setting

\[ H_k = \{ h \in H : \text{supp}(h) = k \}, \]

we obtain

\[ D_H < \sum_{k \geq m(H)} |H_k|n^{-bk}, \]

for any real number $b < 1/6$ and sufficiently large $n$. Fix $b$ with $1/7 < b < 1/6$, and set $c = b - 1/7$, $m = m(H)$. Then

\[ D_H < \sum_{k \geq m} n^{k/7} n^{-bk} = \sum_{k \geq m} n^{-ck} \leq 2n^{-cm}. \]

Since $m = m(H)$ is non-constant, we see that $D_H$ is smaller than than any fixed negative power of $n$, and so $H$ is indistinguishable. □
Now, for Theorem 1.3 we use Babai’s lower bound on the minimal degree of primitive subgroups $H \neq A_n, S_n$ [Bab81], showing that

$$m(H) \geq (\sqrt{n} - 1)/2.$$  \hfill (4)

Furthermore, we apply a theorem of Cameron [Cam81] (which in turns relies on the Classification of Finite Simple Groups) describing all primitive groups of ‘large’ order. In particular it follows from that description that, for all large $n$, and for a primitive subgroup $H \neq A_n, S_n$, either

(i) $|H| \leq n^{cn^{1/3}}$, or 
(ii) $n = \binom{l}{2}$ for some $l$, and $H \leq S_l$ acting on 2-subsets of $\{1, \ldots, l\}$, or 
(iii) $n = l^2$ for some $l$, and $H \leq S_l \wr S_2$ acting on $\{1, \ldots, l\}^2$ in the so called product action.

We claim that for all large $n$ and for all $k$ we have $|H_k| \leq n^{k/7}$.

To show this it suffices to consider $k \geq (\sqrt{n} - 1)/2$, otherwise $|H_k| = 0$ by (ii). Now, if $H$ satisfies condition (i) above then the claim follows trivially using $|H_k| \leq |H|$. So it remains to consider groups $H$ in cases (ii) and (iii). Here a detailed computation based on the known actions of $H$, which we omit from this version, completes the proof of the claim.

At this point Proposition 4.1 can be applied, and we conclude that $H$ is indistinguishable. In fact our argument shows that, for some $\varepsilon > 0$, all primitive subgroups $H \neq A_n, S_n$ satisfy $D_H < n^{-\varepsilon \sqrt{n}}$.

Finally, to prove Theorem 1.4 we construct $H$ as the full symmetric group on $\lfloor n/r \rfloor$ blocks of size $r$, where $r = r(n)$ tends very slowly to infinity. Then $m(H) = 2r$, which is non-constant. A detailed computation shows that the second assumption of Proposition 4.1 also holds, which yields the desired conclusion. The details are left to the reader.

We end this paper with

**Conjecture ♠:** Suppose $H \leq S_n$ is distinguishable. Then its minimal degree $m(H)$ is constant.

**Conjecture ♣:** Every subgroup $H \leq S_n$ with non-constant minimal degree has at most $n^{k/7}$ elements of support $k$.

Proposition 4.1 shows that Conjecture ♣ implies Conjecture ♠. We regard Conjecture ♣ as a plausible group theoretic conjecture, for which we have mounting evidence.

First note that by the above discussion Conjecture ♣ holds for primitive groups. Secondly we can show that the set of subgroups satisfying the conjecture is closed under direct products. Thirdly we can prove the conjecture for wreath products $K \wr L$, if $K$ satisfies the conjecture. Recall that all transitive imprimitive groups are subgroups of wreath products $W = S_k \wr S_l$ and the maximal ones are the full wreath product $W$.

Hopefully the methods introduced in this paper and the group theoretic reductions will lead to a full classification of distinguishable subgroups of $S_n$ and of other groups.

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