A curious general relativistic sphere

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We present a solution to the Einstein field equations that has interesting properties. It can mimic the gravitational lensing of a singular isothermal sphere. The standard energy conditions are satisfied. The geometry is hyperconical everywhere, and light and matter follow degenerate trajectories on exactly solvable geodesics. We find two possible physical origins for the source of the associated energy-momentum tensor: a very large number of strings through a point and a magnetic monopole in the strong field limit of Born-Infeld theory.

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We present an exact solution to the Einstein field equations with several interesting properties. It can mimic the gravitational lensing of a singular isothermal sphere. The energy-momentum tensor corresponding to this geometry could arise from a very large collection of vacuum cosmic strings through a point or as a magnetic monopole in the strong field limit of the Born-Infeld electrodynamics. The null, weak, strong, and dominant energy conditions are all satisfied. It has an unusual hypercone geometry with exactly solvable geodesics where matter and radiation follow the same trajectories. Our solution belongs to the Kerr-Schild class featuring the Schwarzschild and de Sitter geometries and may be easily combined with other members of the class. In particular, our solution extends the family of asymptotically de Sitter solutions beyond the usual combinations of Schwarzschild, Reissner-Nordstrom, and de Sitter.

The energy-momentum tensor of our solution has special symmetries under boosts and rotations. Several symmetries of energy-momentum tensors show up frequently in general relativity. For instance, perfect fluids have $T_{\mu\nu}$ with three degenerate spatial eigenvalues (pressures) and are invariant at each point with respect to spatial rotations. In Segre notation (see [1]), these energy-momentum tensors are $[(111), 1]$. Cosmological constant or vacuum energy $T_{\mu\nu}$ have the three spatial eigenvalues and the temporal eigenvalue (minus energy density) all degenerate, such that they are symmetric for rotations about, and boosts along, all spatial axes. The Segre notation for vacuum energy-momentum tensors is $[(111, 1)]$. One slightly less famous, but still important, class of $T_{\mu\nu}$ has two spatial eigenvalues degenerate with each other, and the third spatial eigenvalue degenerate with the temporal eigenvalue, their Segre type is $[(111, 1)]$. At each spacetime point, these systems have the symmetry of rotations about and boosts along a single axis, and the axis direction can vary from point to point and change with time. We refer to these systems as “uniaxial.” Explicitly, these energy-momentum tensors can be written

$$T_{\mu\nu} = (\rho + p_\perp)(u_\mu u_\nu - n_\mu n_\nu) + p_\perp g_{\mu\nu},$$

where $u^\mu$ is a timelike unit vector, $n^\mu$ is a spacelike unit vector along the symmetry axis and orthogonal to $u^\mu$, $\rho$ is an energy density, and $p_\perp$ is the pressure in the spacelike plane orthogonal to $u^\mu$ and $n^\mu$. The eigenvalues of the mixed index diagonal tensor $T^\mu_\nu$ are $-\rho, -\rho, p_\perp, p_\perp$. Examples of physical systems with uniaxial behavior are a non-null electromagnetic field for which $p_\perp = \rho$ [1], a Nambu string [2] including vacuum cosmic strings [3] for which $p_\perp = 0$, a regular black hole of Kerr-Schild type as in [6], and, as a degenerate case, a cosmological constant for which $p_\perp = -\rho$.

In this paper, we consider static spherically symmetric systems with a uniaxial $T_{\mu\nu}$. Spherical symmetry implies that the pressure $p_\perp$ along the two angular directions equals $p_\perp$. Therefore the pressure $p_\perp$ along the radial direction must be $-\rho$. The metric of a static spherically symmetric uniaxial system can always be put in the form

$$ds^2 = -\left(1 - \frac{2Gm(r)}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2)$$
In fact, using spherical coordinates \((t, r, \theta, \phi)\), where \(r\) is the area radius, a general static spherically symmetric metric has a line element

\[
 ds^2 = -e^{2\Phi(r)} c^2 dt^2 + \frac{dr^2}{1 - \frac{2Gm(r)}{c^2 r}} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \tag{3}
\]

Using \(p_r = -\rho\), Einstein’s equations give

\[
 \frac{c^2 dm}{dr} = 4\pi r^2 \rho, \tag{4}
\]

\[
 \frac{d\Phi}{dr} = \frac{1}{2} \frac{d}{dr} \log \left(1 - \frac{2Gm}{c^2 r}\right), \tag{5}
\]

\[
 \frac{r}{2} \frac{d\rho}{dr} + \rho = -p_r. \tag{6}
\]

Equation (6) can be integrated to give \(\Phi\). Rescaling the \(t\) coordinate to absorb the constant of integration, we can write the metric with \(-g_{tt}\) and \(g_{rr}\) as inverses and obtain Eq. (2).

The metric in Eq. (2) is of the Kerr-Schild class, which means it can be written

\[
 ds^2 = -c^2 dt^2 + \frac{dr^2}{1 - \frac{2Gm(r)}{c^2 r}} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \tag{7}
\]

The first four terms are the Minkowski metric, and \(k_\mu dx^\mu = cdt - dr\).

Equation (8) is a special case of the anisotropic Tolman-Oppenheimer-Volkoff (TOV) equation, in which the gravitational force term does not appear because it is proportional to \(\rho + p_r\) = 0. Thus the density profile for spherical uniaxial systems is the same as if there were no gravity.

In addition, the TOV allows for superposition of spherical systems with uniaxial symmetry about a shared center without alteration of either system. This is related to the result that Kerr-Schild systems sharing the null vector \(k^\mu\) obey a form of superposition principle in the sense that their \(m\) functions add. Two physically relevant examples of \(m\) functions are a “point mass” \(m = M\) and a cosmological constant \(m = \Lambda r^3/6G\). Well-known solutions where this Kerr-Schild superposition behavior manifests itself are the Reissner-Nordstrom solution with \(m = M - Q^2/(8\pi\epsilon c^2 r)\), representing the superposition of a point mass and a Coulomb field, and the deSitter-Schwarzschild solution with \(m = M + \Lambda r^3/6G\), representing the superposition of a point mass and a cosmological constant.

**Solution for \(p_\perp = 0\)**. One way to differentiate between spherically symmetric uniaxial systems is to specify an equation of state for \(p_\perp\). For this paper we use

\[
 p_\perp = 0, \tag{9}
\]

which is, for example, the equation of state for Nambu strings. With this equation of state, plus the condition \(\rho = -p_r \geq 0\), the null, weak, strong, and dominant energy conditions are all satisfied since \(|p_\perp| \leq \rho, |p_\perp| \leq \rho, \rho + p_r + 2p_\perp \geq 0\). Zero is the lowest value \(p_\perp\) can have and still satisfy the strong energy condition. The density profile dictated by Eqs. (6) and (9) is

\[
 \rho = \frac{\lambda \epsilon^2}{4\pi r^2}, \tag{10}
\]

where \(\lambda\) is a constant with units mass per length. If we assume \(M = 0\) (there is no “point mass” at the origin), Eq. (4) gives

\[
 m = \lambda r, \tag{11}
\]

and the metric becomes

\[
 ds^2 = -\kappa^2 c^2 dt^2 + \frac{1}{\kappa^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{12}
\]
where \( \kappa = \sqrt{1 - 2G\lambda/c^2} \). Notice that the \( g_{tt} \) and \( g_{rr} \) metric functions are constant everywhere. This metric describes a hypercone in four dimensions in that the circumference of a circle of proper radius \( r^* \) is \( 2\pi\kappa r^* \). Demanding that the \( t \) coordinate remains timelike requires \( 2G\lambda/c^2 < 1 \).

**Geodesics.** The geodesics in this hyperconical metric Eq. (12) obey the equation

\[
g_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} = -\epsilon c^2, \tag{13}\]

where \( \epsilon = 1 \) and \( \tau \) is proper time for timelike geodesics, and \( \epsilon = 0 \) and \( \tau \) is an affine parameter for null geodesics. Because of the spherical symmetry and static metric, any plane through the origin is equivalent so we describe trajectories in the \( \theta = \pi/2 \) equatorial plane. A further consequence of the symmetry is the existence of conserved quantities \( cL = r^2 \frac{d\phi}{d\tau} \) and \( E = \kappa^2 \frac{dt}{d\tau} \). Equation (13) becomes

\[
E^2 = \frac{1}{c^2} \left( \frac{dr}{d\tau} \right)^2 - V_{eff}(r), \tag{14}\]

with the effective potential \( V_{eff}(r) = \kappa^2 (\epsilon + L^2/r^2) \). There is no minimum of \( V_{eff} \) so there are no bound orbits. We can eliminate \( \tau \) and obtain a differential equation for the trajectories

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{E^2 - \kappa^2 (L^2 u^2 + \epsilon)}{L^2}, \tag{15}\]

where \( u = 1/r \). It is useful to take a \( \phi \) derivative of Eq. (15), which gives

\[
\frac{d^2u}{d\phi^2} = -\kappa^2 u. \tag{16}\]

Equation (16) is a simple harmonic oscillator equation and can be easily solved to give trajectories

\[
r \cos(\kappa \phi) = \kappa b, \tag{17}\]

where \( b \) is the impact parameter.

There are two special properties of trajectories for our solution. The first is that \( \epsilon \) drops out, so lightlike and massive particles follow the same trajectories. The second is that the deflection angle

\[
\alpha = \pi \frac{1 - \kappa}{\kappa} \tag{18}\]

is independent of the impact parameter. This independence occurs in vacuum cosmic string solutions \[3, 5, 8\] and singular isothermal spheres. When \( \kappa \) goes to 1 we recover Minkowski space, Eq. (17) describes a straight line, and Eq. (18) dictates the deflection angle is 0. For \( \alpha > \pi \), trajectories wrap around the origin and intersect themselves. This happens when \( \kappa < 1/2 \) or equivalently \( G\lambda/c^2 > 3/8 \). Figure 1 shows examples of trajectories.
Figure 1. Trajectories of geodesics in the equatorial plane for various values of \( \lambda \). The shape of trajectories is independent of \( \epsilon \), and the impact parameter \( b \) only acts as a scaling factor. Note that as \( \lambda \) increases the trajectories become more bent.

An asymptotically de Sitter combination. Because of the superposition behavior of systems with uniaxial symmetry, we can examine a combined system of our solution with a point mass \( M > 0 \) and cosmological constant \( \Lambda > 0 \). Then

\[
-g_{tt} = \frac{1}{g_{rr}} = \left( 1 - \frac{2GM}{c^2 r} - \frac{2G\lambda}{c^2} - \frac{\Lambda r^2}{3c^2} \right).
\]

This system has a Schwarzschild horizon at small \( r \) and de Sitter horizon at large \( r \). The \( r \to 0 \) and \( r \to \infty \) behavior is dominated by the Schwarzschild and de Sitter terms, respectively (unless \( M \) or \( \Lambda \) is 0). The presence of our solution will cause the Schwarzschild horizon to move outward (to \( R = 2GM/c^2 \kappa \) if \( \Lambda \) is neglected) and de Sitter horizon to move inward (to \( R = \sqrt{3}/\Lambda c \kappa \) if \( M \) is neglected).

Concerning orbits Eq. 14 applies with \( E = -g_{tt} \frac{dt}{dr} \) and

\[
V_{eff} = \left( \kappa^2 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3c^2} \right) \left( \frac{L^2}{r^2} + \epsilon \right).
\]

This potential has bound orbits as follows from studying the sign of \( \frac{dV_{eff}}{dr} \). One particular interesting example is the \( \Lambda \) negligible regime. Circular orbits occur at radii \( r \) corresponding to an equivalent Schwarzschild-de Sitter solution with mass \( M/\kappa^2 \) and cosmological constant \( \Lambda/\kappa^2 \).

Discussion. There are other systems where the mass increases linearly with radius like Eq. (11), and it is useful to draw comparisons.

It is intriguing that if one were to truncate our solution at radius \( R \) and embed it in a vacuum, the total mass of the system would be \( M = \lambda R \). In the limit \( \kappa \to 0^+ \), equivalently \( G\lambda/c^2 \to 0.5^- \), we get \( 2GM/c^2 = R \), which is the horizon radius of a Schwarzschild black hole.

One system well known for having a density proportional to \( 1/r^2 \) is the singular isothermal sphere. In particular, the Newtonian singular isothermal sphere has a mass profile defined in terms of a velocity dispersion \( \sigma_v \) as

\[
m(r) = \frac{2\sigma_v^2 r}{G}.
\]

This corresponds to Eq. (11) when \( 2\sigma_v^2 = G\lambda \). The deflection angle for singular isothermal spheres in gravitational lensing is

\[
\alpha = \frac{4\pi\sigma_v^2}{c^2},
\]
which is independent of impact parameter, as in our solution. In the low $\lambda$ limit, the deflection angle for our solution Eq. (18) becomes

$$\alpha \approx \pi \frac{G \lambda}{c^2}.$$  \hfill (23)

Thus the deflection angle of our solution is equivalent to that of a Newtonian isothermal sphere that is half as dense. Therefore, if an object akin to our solution was observed in a lensing survey, it would appear the same as a galaxy, without containing baryonic or dark matter. Unlike a standard dark matter halo, our solution Eq. (12) does not, by itself, support bound orbits. Adding $M$ and $\Lambda$ as in Eq. (19) does not give flat rotation curves either.

Another system with a density profile of the form $1/r^2$ is a TOV star with a linear equation of state $p = k \rho$ (see [9] for a general treatment and [10] for a specific physically interesting case). Because of its equation of state and density profile, this system is sometimes called a general relativistic isothermal sphere. The mass function and $g_{rr}$ metric function are the same as for our solution, with $G \lambda/c^2 = 2k/(1 + 6k + k^2)$. The largest possible $\lambda$ for general relativistic singular isothermal spheres is $G \lambda/c^2 = 1/4$ when $k = 1$, which is half the value our solution takes in the $\kappa \to 0^+$ limit. Also, the $g_{tt}$ metric function is different than our solution because Einstein’s equations only give Eq. (5) when $p_r = -\rho$.

A physical system that can correspond to our solution is a collection of a very large number of strings originating from a point, like a koosh ball. The energy-momentum tensor of a vacuum cosmic string has $p = -\rho$ and $p_{\perp} = 0$.

Further, if we had a collection of strings with linear energy density $\mu$ emerging from a point, and the number of strings per unit solid angle was $n$, then the average density of the system would be

$$\rho = \frac{\mu n}{r^2}.$$  \hfill (24)

This becomes Eq. (10) when $4\pi \mu n = \lambda$.

There is another interesting system that could act as a source for the energy-momentum tensor we consider. Many uniaxial spherical objects arise in nonlinear electrodynamics theories [11, 12], and are commonly used to construct nonsingular black holes [13–18]. In particular, our $p_{\perp} = 0$ system has the same energy-momentum tensor as a magnetic monopole in a theory with the Lagrangian

$$\mathcal{L} = -b \sqrt{\frac{F^{\mu \nu} F_{\mu \nu}}{2}},$$  \hfill (25)

Where $F^{\mu \nu}$ is the electromagnetic field strength tensor. Eq. (25) is the strong magnetic field limit to the famous Born-Infeld [19] Lagrangian

$$\mathcal{L} = b^2 \left(1 - \sqrt{1 + \frac{F^{\mu \nu} F_{\mu \nu}}{2b^2} - \left(\tilde{F}^{\mu \nu} F_{\mu \nu}\right)^2}\right)$$  \hfill (26)

with $\tilde{F}^{\mu \nu} = 1/2 \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}$ (for magnetic monopoles the term with $\tilde{F}^{\mu \nu}$ is zero). Although the Born-Infeld theory was originally proposed to regularize the self-energy of electric point charges, it also arises in string theory as the effective action for gauge fields on a D-brane [20]. It was previously noted [12] that under certain circumstances Born-Infeld solutions lead to a conical singularity and $\rho \propto 1/r^2$ as $r \to 0$. If we were to consider the source of $T^{\mu \nu}$ as a magnetic monopole, it would be natural to add a point mass term to the $m$ function, making the object a black hole.

One final point worth considering is if there were multiple nonconcentric systems resembling our solution. This is a question connected to the formation of our object. One possibility is that the uniaxial character of the total energy-momentum tensor would be maintained, and there would be a well-defined axis at every point along which $p = -\rho$. One could think of this as strings which line up smoothly or nonlinear magnetic field lines. Another possibility is that the superposition would lead to an isotropization of the pressure due to axes nearby pointing in many directions, as in tangled strings, leading to a net value of $p = -\rho/3$.

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