An algebraic proof of Bogomolov-Tian-Todorov theorem

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Abstract. We give a completely algebraic proof of the Bogomolov-Tian-Todorov theorem. More precisely, we shall prove that if $X$ is a smooth projective variety with trivial canonical bundle defined over an algebraically closed field of characteristic 0, then the $L_{\infty}$-algebra governing infinitesimal deformations of $X$ is quasi-isomorphic to an abelian differential graded Lie algebra.

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Introduction

Let $X$ be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic 0, with tangent sheaf $\Theta_X$. Given an affine open cover $\mathcal{U} = \{U_i\}$ of $X$, we can consider the Čech complex $\check{C}(\mathcal{U}, \Theta_X)$. By classical deformation theory [Kod86, Se06], the group $H^1(\mathcal{U}, \Theta_X)$ classifies first order deformations of $X$, while $H^2(\mathcal{U}, \Theta_X)$ is an obstruction space for $X$.

Moreover, as a consequence of the results contained in [Hin97, HiS97a, HiS97b, FMM08], there exists a canonical sequence of higher brackets on $\check{C}(\mathcal{U}, \Theta_X)$, defining an $L_{\infty}$ structure and governing deformations of $X$ over local Artinian $\mathbb{K}$-algebras, via Maurer-Cartan equation. When $\mathbb{K} = \mathbb{C}$, such $L_{\infty}$ structure is canonically quasi-isomorphic to the Kodaira-Spencer differential graded Lie algebra of $X$, whose Maurer-Cartan equation corresponds to the integrability condition of almost complex structures [GM90].

Assume now that $X$ has trivial canonical bundle, the well known Bogomolov-Tian-Todorov (BTT) theorem states that $X$ has unobstructed deformations. This was first proved by Bogolomov in [Bo78] in the particular case of complex hamiltonian manifolds; then, Tian [Ti87] and Todorov [To89] proved independently the theorem for compact
Kähler manifolds with trivial canonical bundle. Their proofs are transcendental and make a deep use of the underlying differentiable structure as well of the $\partial \bar{\partial}$-Lemma.

More algebraic proofs of BTT theorem, based on $T^1$-lifting theorem and degeneration of the Hodge spectral sequence, were given in [Ra92] for $\mathbb{K} = \mathbb{C}$ and in [Kaw92, FM99] for any $\mathbb{K}$ as above.

For $\mathbb{K} = \mathbb{C}$, the BTT theorem is also a consequence of the stronger result [GM90, Ma04b] that the Kodaira-Spencer differential graded Lie algebra of $X$ is quasi-abelian, i.e., quasi-isomorphic to an abelian differential graded Lie algebra. This result, also proved with transcendental methods, is important for applications to Mirror symmetry [BK98].

The main result of this paper is to give a purely algebraic proof of the quasi-abelianity of the $L_\infty$-algebra $\mathcal{C}(U, \Theta_X)$ when $X$ is projective with trivial canonical bundle. This is achieved using the degeneration of the Hodge-de Rham spectral sequence, proved algebraically by Faltings, Deligne and Illusie [Fa88, DeIl87], and the $L_\infty$ description of the period map [FiMa08].

More precisely, we prove the following theorems.

**Theorem (A).** Let $X$ be a smooth projective variety of dimension $n$ defined over an algebraically closed field of characteristic 0. If the contraction map

$$H^*(\Theta_X) \xrightarrow{\Delta} \text{Hom}^*(H^*(\Omega^n_X), H^*(\Omega^{n-1}_X))$$

is injective, then for every affine open cover $U$ of $X$, the $L_\infty$-algebra $\mathcal{C}(U, \Theta_X)$ is quasi-abelian.

**Theorem (B).** Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic 0. If the canonical bundle of $X$ is trivial or torsion, then the $L_\infty$-algebra $\mathcal{C}(U, \Theta_X)$ is quasi-abelian.

The paper goes as follows: the first section is intended for the non expert reader and is devoted to recall the basic notions of differential graded Lie algebras, $L_\infty$-algebras and their role in deformation theory.

In Section 2, we review the construction of the Thom-Whitney complex associated with a semicosimplicial complex.

In Section 3, following [FMM08], we introduce semicosimplicial differential graded Lie algebras, the associated Thom-Whitney DGLAs and the $L_\infty$ structure on the associated total complexes. We also investigate some properties of mapping cones associated with a morphism of DGLAs.

In Sections 4, we collect some technical results about Cartan homotopies and contractions.

In Section 5, we give the definition of the deformation functor $H^1_{\text{sc}}(\exp g^\Delta)$, associated with a semicosimplicial Lie algebra $g^\Delta$, introduced essentially in [Hin97, Pr03] and described in more detailed way in [FMM08]. Moreover, following [FMM08], we prove, in a complete algebraic way, that the infinitesimal deformations of a smooth variety $X$, defined over a field of characteristic 0, are controlled by the $L_\infty$-algebra $\mathcal{C}(U, \Theta_X)$, where $U$ is an open affine cover of $X$.

Section 6 is devoted to the algebraic proof of the previous Theorems A and B together with some applications to deformation theory.

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The first author wish to thank the Mathematical Department “G. Castelnuovo” of Sapienza Università di Roma for the hospitality. Thanks also to D. Fiorenza and E. Martinengo for useful discussions on the subject of this paper. We are grateful to the referee for improvements in the exposition of the paper.

1. Review of DGLAs and $L_\infty$-algebras

Let $\K$ be a fixed algebraically closed field of characteristic zero. A **differential graded vector space** is a pair $(V, d)$, where $V = \bigoplus_i V^i$ is a $\mathbb{Z}$-graded vector space and $d: V^i \to V^{i+1}$ is a differential of degree $+1$. For every integer $n$, we define a new differential graded vector space $V[n]$ by setting

$$V[n]^i = V^{n+i}, \quad d_{V[n]} = (-1)^n d_V.$$

A **differential graded Lie algebra** (DGLA for short) is the data of a differential graded vector space $(L, d)$ together with a bilinear map $[-,-]: L \times L \to L$ (called bracket) of degree $0$ such that:

1. **(graded skewsymmetry)** $[a, b] = -(-1)^{\deg(a) \deg(b)} [b, a].$
2. **(graded Jacobi identity)** $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a) \deg(b)} [b, [a, c]].$
3. **(graded Leibniz rule)** $d [a, b] = [da, b] + (-1)^{\deg(a)} [a, db].$

In particular, the Leibniz rule implies that the bracket of a DGLA $L$ induces a structure of graded Lie algebra on its cohomology $H^*(L) = \bigoplus_i H^i(L)$.

**Example 1.1.** Let $(V, d_V)$ be a differential graded vector space and $\Hom^i(V, V)$ the space of morphisms $V \to V$ of degree $i$. Then, $\Hom^* (V, V) = \bigoplus_i \Hom^i(V, V)$ is a DGLA with bracket

$$[f, g] = fg - (-1)^{\deg(f) \deg(g)} gf,$$

and differential $d$ given by

$$d(f) = [d_V, f] = d_V f - (-1)^{\deg(f)} f d_V.$$

For later use, we point out that there exists a natural isomorphism

$$H^* (\Hom^*(V, V)) \xrightarrow{\sim} \Hom^* (H^*(V), H^*(V)).$$

A morphism of differential graded Lie algebras $\varphi: L \to M$ is a linear map that preserves degrees and commutes with brackets and differentials. A **quasi-isomorphism** of DGLAs is a morphism that induces an isomorphism in cohomology. Two DGLAs $L$ and $M$ are said to be **quasi-isomorphic** if they are equivalent under the equivalence relation generated by: $L \sim M$ if there exists a quasi-isomorphism $\phi : L \to M$.

Next, denote by $\text{Set}$ the category of sets (in a fixed universe) and by $\textbf{Art}_\K$ the category of local Artinian $\K$-algebras with residue field $\K$. Unless otherwise specified, for every objects $A \in \textbf{Art}$, we denote by $m_A$ its maximal ideal. Given a DGLA $L$, we define the Maurer-Cartan functor $MC_L : \textbf{Art} \to \text{Set}$ by setting $[\text{Ma}99]$

$$MC_L(A) = \left\{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2} [x, x] = 0 \right\},$$

where the DGLA structure on $L \otimes m_A$ is the natural extension of the DGLA structure on $L$. The gauge action $*: \exp(L^0 \otimes m_A) \times MC_L(A) \to MC_L(A)$ may be defined by the explicit formula

$$e^a * x := x + \sum_{n \geq 0} \frac{[a, -]^n}{(n+1)!} ([a, x] - da).$$
The deformation functor \( \text{Def}_L : \text{Art} \to \text{Set} \) associated to a DGLA \( L \) is:

\[
\text{Def}_L(A) = \frac{\text{MCL}(A)}{\text{gauge}} = \{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0 \}/\exp(L^0 \otimes m_A).
\]

**Remark 1.2.** Every morphism of DGLAs induces a natural transformation of the associated deformation functors. A basic result asserts that if \( L \) and \( M \) are quasi-isomorphic DGLAs, then the associated functor \( \text{Def}_L \) and \( \text{Def}_M \) are isomorphic [SS79, GM88, GM90, Ma02, [Kon03, Get04, FiMa07] and [Ma04b] Corollary 5.52].

Next, we briefly recall the definition of an \( L_\infty \) structures on a graded vector space \( V \). For a more detailed description of such structures we refer to [SS79, LS93, LM95, Ma02, Fu03, [Kon03, Get04, FiMa07] and [Ma04b] Chapter IX].

Let \( V \) be a graded vector space: we denote by \( \bigodot^n V \) its graded symmetric \( n \)-th power. Given \( v_1, \ldots, v_n \) homogeneous elements of \( V \), for every permutation \( \sigma \) we have

\[
v_1 \odot \cdots \odot v_n = \epsilon(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)},
\]

where \( \epsilon(\sigma; v_1, \ldots, v_n) \) is the Koszul sign. When the sequence \( v_1, \ldots, v_n \) is clear from the context, we simply write \( \epsilon(\sigma) \) instead of \( \epsilon(\sigma; v_1, \ldots, v_n) \).

**Definition 1.3.** Denote by \( \Sigma_n \) the group of permutations of the set \( \{1, 2, \ldots, n\} \). The set of *unshuffles* of type \( (p, n - p) \) is the subset \( S(p, n - p) \subset \Sigma_n \) of permutations \( \sigma \) such that \( \sigma(1) < \sigma(2) < \cdots < \sigma(p) \) and \( \sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(n) \).

**Definition 1.4.** An \( L_\infty \) structure on a graded vector space \( V \) is a sequence \( \{q_k\}_{k \geq 1} \) of linear maps \( q_k \in \text{Hom}^1(\bigodot^k(V[1]), V[1]) \) such that the map

\[
Q : \bigoplus_{n \geq 1} \bigodot^n V[1] \to \bigoplus_{n \geq 1} \bigodot^n V[1],
\]

is a codifferential on the reduced symmetric graded coalgebra \( \bigoplus_{n \geq 1} \bigodot^n V[1] \); in other words, the datum \((V, q_1, q_2, q_3, \ldots)\) is called an \( L_\infty \)-algebra if \( QQ = 0 \).

If \((V, q_1, q_2, q_3, \ldots)\) is an \( L_\infty \)-algebra, then \( q_1 q_1 = 0 \) and therefore \((V[1], q_1)\) is a differential graded vector space. The relation between DGLAs and \( L_\infty \)-algebras is given by the following example.

**Example 1.5 (Qui69).** Let \((L, d, [\cdot, \cdot])\) be a differential graded Lie algebra and define:

\[
q_1 = -d : L[1] \to L[1],
\]

\[
q_2 \in \text{Hom}^1(\bigodot^2(L[1]), L[1]), \quad q_2(v \odot w) = (-1)^{\deg(v)}[v, w],
\]

and \( q_k = 0 \) for every \( k \geq 3 \). Then \((L, q_1, q_2, 0, \ldots)\) is an \( L_\infty \)-algebra.

An \( L_\infty \)-morphism \((V, q_1, q_2, \ldots) \to (W, p_1, p_2, \ldots)\) of \( L_\infty \)-algebras is a sequence \( f_\infty = \{f_n\} \) of degree zero linear maps

\[
f_n : \bigodot^n V[1] \to W[1], \quad n \geq 1,
\]
such that the (unique) morphism of graded coalgebras
\[ F : \bigoplus_{n \geq 1} V[1] \to \bigoplus_{n \geq 1} W[1], \]
lifting \( \sum_n f_n : \bigoplus_{n \geq 1} \bigoplus^n V[1] \to W[1] \), commutes with the codifferentials. This condition implies that the linear part \( f_1 : V[1] \to W[1] \) of an \( L_\infty \)-morphism \( f_\infty : (V, q_1, q_2, \ldots) \to (W, p_1, p_2, \ldots) \) satisfies the condition \( f_1 \circ q_1 = p_1 \circ f_1 \), and therefore \( f_1 \) is a map of differential complexes \( (V[1], q_1) \to (W[1], p_1) \).

An \( L_\infty \)-morphism \( f_\infty = \{f_n\} \) is said to be linear if \( f_n = 0 \), for every \( n \geq 2 \). Notice that an \( L_\infty \)-morphism between two DGLAs is linear if and only if it is a morphism of differential graded Lie algebras.

A quasi-isomorphism of \( L_\infty \)-algebra is an \( L_\infty \)-morphism, whose linear part is a quasi-isomorphism of complexes. Two \( L_\infty \)-algebras are quasi-isomorphic if they are equivalent under the equivalence relation generated by the relation: \( L \sim M \) if there exists a quasi-isomorphism \( \phi : L \to M \).

**Definition 1.6.** An \( L_\infty \)-algebra \( (V, q_1, q_2, \ldots) \) is called abelian if \( q_i = 0 \) for every \( i \geq 2 \). An \( L_\infty \)-algebra is called quasi-abelian if it is quasi-isomorphic to an abelian \( L_\infty \)-algebra.

Given an \( L_\infty \)-algebra \( (V, q_1, q_2, \ldots) \) and a differential graded commutative algebra \( A \), there exists a natural \( L_\infty \) structure on the tensor product \( V \otimes A \). The Maurer-Cartan functor \( MC_V \) associated with the \( L_\infty \)-algebra \( V \) is the functor \( \text{MC}_V : \text{Art} \to \text{Set} \):

\[
\text{MC}_V(A) = \left\{ \gamma \in V[1]^0 \otimes m_A \left| \sum_{j \geq 1} q_j(\gamma^{(j)}) \frac{j!}{j!} = 0 \right. \right\}.
\]

Two elements \( x \) and \( y \in \text{MC}_V(A) \) are homotopy equivalent if there exists \( g(s) \in \text{MC}_{V \otimes [s,d]}(A) \) such that \( g(0) = x \) and \( g(1) = y \). Then, the deformation functor \( \text{Def}_V \) associated with the \( L_\infty \)-algebra \( V \) is

\[
\text{Def}_V : \text{Art} \to \text{Set}, \quad \text{Def}_V(A) = \frac{\text{MC}_V(A)}{\text{homotopy}}.
\]

**Example 1.7.** Given an abelian \( L_\infty \)-algebra \( (V, q_1, 0, 0, \ldots) \), for every \( A \in \text{Art} \) there exists a canonical isomorphism

\[
\text{Def}_V(A) = H^0(V[1], q_1) \otimes m_A.
\]

**Remark 1.8.** If \( L \) is a DGLA, then the deformation functor associated with \( L \), viewed as an \( L_\infty \)-algebra, is isomorphic to the previous one (Maurer-Cartan modulo gauge equivalence) \( \text{SS79, Ma02, Fu03, Kon03} \).

**Remark 1.9.** As for the DGLAs, every morphism of \( L_\infty \)-algebras induces a natural transformation of the associated deformation functors. If two \( L_\infty \)-algebras are quasi-isomorphic, then there exists an isomorphism between the associated deformation functors \( \text{Fu03, Kon03, Ma04} \) Corollary IX.22. In particular, the deformation functor of a quasi-abelian \( L_\infty \)-algebra is unobstructed.

**Lemma 1.10.** Let \( (V, q_1, q_2, \ldots) \), \( (W, r_1, r_2, \ldots) \) be \( L_\infty \)-algebras with \( W \) quasi-abelian and \( f_\infty : V \to W \) an \( L_\infty \)-morphism. If \( f_1 \) is injective in cohomology, then \( V \) is quasi-abelian.
PROOF. According to homotopy classification of \(L_\infty\)-algebras (see e.g. [Kon03]), if \(H\) is the cohomology of the complex \((W, r_1)\), there exists an \(L_\infty\) structure on \(H\) and a surjective quasi-isomorphism of \(L_\infty\)-algebras \(p_\infty: W \to H\). The \(L_\infty\) structure on \(H\) depends, up to isomorphism, by the quasi-isomorphism class of \(W\) and therefore every bracket on \(H\) is trivial.

Replacing \(W\) with \(H\) and \(f_\infty\) with \(p_\infty f_\infty\) it is not restrictive to assume \(r_i = 0\) for every \(i\). There exist a graded vector space \(K\) and a morphism of graded vector spaces \(\beta: W[1] \to K[1]\) such that the composition

\[
(V[1], q_1) \xrightarrow{f_\infty} (W[1], 0) \xrightarrow{\beta} (K[1], 0)
\]

is a quasi-isomorphism of complexes. Then, \(\beta\) is a linear \(L_\infty\)-morphism and the composition

\[
(V, q_1, q_2, \ldots) \xrightarrow{f_\infty} (W, 0, 0, \ldots) \xrightarrow{\beta} (K, 0, 0, \ldots)
\]

is a quasi-isomorphism of \(L_\infty\)-algebras. \(\square\)

2. The Thom-Whitney complex

Let \(\Delta_{\text{mon}}\) be the category whose objects are the finite ordinal sets \(\{0, 1, \ldots, n\}\), \(n = 0, 1, \ldots\), and whose morphisms are order-preserving injective maps among them. Every morphism in \(\Delta_{\text{mon}}\), different from the identity, is a finite composition of coface morphisms:

\[
\partial_k : [i-1] \to [i], \quad \partial_k(p) = \begin{cases} p & \text{if } p < k, \\ p+1 & \text{if } p \leq k, \end{cases} \quad k = 0, \ldots, i.
\]

The relations about compositions of them are generated by

\[
\partial_l \partial_k = \partial_{k+1} \partial_l, \quad \text{for every } l \leq k.
\]

According to [EZ50, We94], a semicosimplicial object in a category \(C\) is a covariant functor \(A^\Delta: \Delta_{\text{mon}} \to C\). Equivalently, a semicosimplicial object \(A^\Delta\) is a diagram in \(C\):

\[
A_0 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_2 \xrightarrow{\partial} \cdots
\]

where each \(A_i\) is in \(C\), and, for each \(i > 0\), there are \(i+1\) morphisms

\[
\partial_k : A_{i-1} \to A_i, \quad k = 0, \ldots, i,
\]

such that \(\partial_l \partial_k = \partial_{k+1} \partial_l\), for any \(l \leq k\).

Given a semicosimplicial differential graded vector space

\[
V^\Delta: \quad V_0 \xrightarrow{\partial} V_1 \xrightarrow{\partial} V_2 \xrightarrow{\partial} \cdots
\]

the graded vector space \(\bigoplus_{n \geq 0} V_n[-n]\) has two differentials

\[
d = \sum_n (-1)^n d_n, \quad \text{where } d_n \text{ is the differential of } V_n,
\]

and

\[
\partial = \sum_i (-1)^i \partial_i, \quad \text{where } \partial_i \text{ are the coface maps}.
\]

More explicitly, if \(v \in V^i_n\), then the degree of \(v\) is \(i+n\) and

\[
d(v) = (-1)^n d_n(v) \in V^{i+1}_n, \quad \partial(v) = \partial_0(v) - \partial_1(v) + \cdots + (-1)^{n+1} \partial_{n+1}(v) \in V^{i+1}_n.
\]

Since \(d\partial + \partial d = 0\), we define \(\text{Tot}(V^\Delta)\) as the graded vector space \(\bigoplus_{n \geq 0} V_n[-n]\), endowed with the differential \(d + \partial\).
Example 2.1. Let $\mathcal{U} = \{U_i\}$ be an affine open cover of a smooth variety $X$, defined over an algebraically closed field of characteristic 0; denote by $\Theta_X$ the tangent sheaf of $X$. Then, we can define the Čech semicosimplicial Lie algebra $\Theta_X(\mathcal{U})$ as the semicosimplicial Lie algebra $\Theta_X(U_i)$.

\begin{equation}
\Theta_X(U) : \prod_i \Theta_X(U_i) \xrightarrow{\partial} \prod_{i<j} \Theta_X(U_{ij}) \xrightarrow{\partial} \prod_{i<j<k} \Theta_X(U_{ijk}) \xrightarrow{\partial} \cdots,
\end{equation}

where the coface maps $\partial_h : \prod_{i_0<\cdots<i_{k-1}} \Theta_X(U_{i_0\cdots i_{k-1}}) \to \prod_{i_0<\cdots<i_k} \Theta_X(U_{i_0\cdots i_k})$ are given by

\begin{equation}
\partial_h(x)_{i_0\cdots i_k} = x_{i_0}\cdots\hat{x}_{i_h}\cdots x_{i_{k-1}}, \quad \text{for } h = 0, \ldots, k.
\end{equation}

Since every Lie algebra is, in particular, a differential graded vector space (concentrated in degree 0), it makes sense to consider the total complex $\text{Tot}(\Theta_X(\mathcal{U}))$, which coincides with the Čech complex $\check{C}(\mathcal{U}, \Theta_X)$.

Example 2.2. Let $\Omega^*_X$ be the algebraic de Rham complex of a smooth variety $X$ of dimension $n$:

\begin{equation}
\Omega^*_X : \quad 0 \to \mathcal{O}_X = \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \to 0.
\end{equation}

Given an affine open cover $\mathcal{U} = \{U_i\}$ of $X$, we can define a semicosimplicial differential graded vector space $\Omega^*_X(\mathcal{U})$:

\begin{equation}
\Omega^*_X(U) : \quad \prod_i \Omega^*_X(U_i) \xrightarrow{\partial} \prod_{i<j} \Omega^*_X(U_{ij}) \xrightarrow{\partial} \prod_{i<j<k} \Omega^*_X(U_{ijk}) \xrightarrow{\partial} \cdots,
\end{equation}

where $\prod_{i_0<\cdots<i_k} \Omega^*_X(U_{i_0\cdots i_k})$ is the complex

\begin{equation}
\prod_{i_0<\cdots<i_k} \Omega^*_X(U_{i_0\cdots i_k}) \xrightarrow{d} \prod_{i_0<\cdots<i_{k-1}} \Omega^*_X(U_{i_0\cdots i_{k-1}}) \xrightarrow{d} \cdots \xrightarrow{d} \prod_{i_0<\cdots<i_n} \Omega^*_X(U_{i_0\cdots i_n}).
\end{equation}

Here the total complex $\text{Tot}(\Omega^*_X(\mathcal{U}))$ is the Čech complex $\check{C}(\mathcal{U}, \Omega^*_X)$ of $\Omega^*_X$, with respect to the affine cover $\mathcal{U}$.

Let $V^\Delta$ be a semicosimplicial differential graded vector space and $(A_{PL})_n$ the differential graded commutative algebra of polynomial differential forms on the standard $n$-simplex $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1\}$ [PHT01]:

\begin{equation}
(A_{PL})_n = \frac{\mathbb{R}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(1 - \sum t_i, \sum dt_i)}.
\end{equation}

For every $n, m$ the tensor product $V_n \otimes (A_{PL})_n$ is a differential graded vector space and then also $\prod_n V_n \otimes (A_{PL})_n$ is a differential graded vector space.

Denoting by

\begin{equation}
\delta^k : (A_{PL})_n \to (A_{PL})_{n-1}, \quad \delta^k(t_i) = \begin{cases} t_i & \text{if } 0 \leq i < k \\ 0 & \text{if } i = k \\ t_{i-1} & \text{if } k < i \end{cases}, \quad k = 0, \ldots, n,
\end{equation}

the face maps, for every $0 \leq k \leq n$, there are well-defined morphisms of differential graded vector spaces

\begin{equation}
Id \otimes \delta^k : V_n \otimes (A_{PL})_n \to V_n \otimes (A_{PL})_{n-1},
\end{equation}

\begin{equation}
\partial_h \otimes Id : V_{n-1} \otimes (A_{PL})_{n-1} \to V_n \otimes (A_{PL})_{n-1}.
\end{equation}
The Thom-Whitney differential graded vector space $\text{Tot}_{TW}(V^\Delta)$ of $V^\Delta$ is the differential graded subvector space of $\prod_n V_n \odot (APL)_n$, whose elements are the sequences $(x_n)_{n \in \mathbb{N}}$ satisfying the equations

$$(Id \otimes \delta^k)x_n = (\partial_k \otimes Id)x_{n-1}, \text{ for every } 0 \leq k \leq n.$$ 

In [Whi57], Whitney noted that the integration maps

$$\int_{\Delta^n} \otimes Id: (APL)_n \otimes V_n \to C[n] \otimes V_n = V_n[n]$$

give a quasi-isomorphism of differential graded vector spaces $I: (\text{Tot}_{TW}(V^\Delta), d_{TW}) \to (\text{Tot}(V^\Delta), d_{\text{Tot}})$.

Moreover, there exist an explicit injective quasi-isomorphism of differential graded vector spaces

$$E: \text{Tot}(V^\Delta) \to \text{Tot}_{TW}(V^\Delta)$$

and an explicit homotopy

$$h: \text{Tot}_{TW}(V^\Delta) \to \text{Tot}_{TW}(V^\Delta)[-1],$$

such that

$$IE = \text{Id}_{\text{Tot}(V^\Delta)}; \quad EI - \text{Id}_{\text{Tot}_{TW}(V^\Delta)} = h d_{TW} + d_{TW} h.$$ 

Moreover, the morphisms $I, E, h$ are functorial and commute with morphisms of semicosimplicial differential graded vector spaces. For more details and explicit description of $E$ and $h$, we refer to [Dup76, Dup78, NaA87, Get04, FMM08, CG08].

Let $V^\Delta$ and $W^\Delta$ be two semicosimplicial graded vector spaces, with degeneracy maps $\partial_{k,n}$ and $\partial'_{k,n}$, respectively. Then, we define the tensor product $(V \otimes W)^\Delta$ as the semicosimplicial graded vector space, such that $(V \otimes W)^\Delta_n = V_n \otimes W_n$ and the degeneracy maps are defined on each factor, i.e., $\partial_{k,n} = \partial'_{k,n} \otimes \partial''_{k,n}: (V \otimes W)^\Delta_{n-1} \to (V \otimes W)^\Delta_n$.

**Remark 2.3.** The Thom-Whitney construction is compatible with tensor product, i.e., there exists a natural transformation

$$\Phi: \text{Tot}_{TW}(V^\Delta) \otimes \text{Tot}_{TW}(W^\Delta) \to \text{Tot}_{TW}((V \otimes W)^\Delta)$$

Indeed, we have to prove that $x \otimes y \in \text{Tot}_{TW}((V \otimes W)^\Delta)$, for every pair of sequences $x = (x_n)_{n \in \mathbb{N}} \in \text{Tot}_{TW}(V^\Delta)$ and $y = (y_n)_{n \in \mathbb{N}} \in \text{Tot}_{TW}(W^\Delta)$.

We have

$$\partial_k \otimes Id(x_n \otimes y_n) = (\partial_k \otimes Id(x_n)) \otimes (\partial_k \otimes Id(y_n)),$$

and,

$$Id \otimes \delta^k(x_{n+1} \otimes y_{n+1}) = (Id \otimes \delta^k(x_{n+1})) \otimes (Id \otimes \delta^k(y_{n+1})).$$

It is sufficient to observe that the right parts of the above two equation are the same for every $k \leq n$.

In particular, every bilinear map of semicosimplicial graded vector spaces $V^\Delta \times W^\Delta \to Z^\Delta$ induces a bilinear map $\text{Tot}_{TW}(V^\Delta) \times \text{Tot}_{TW}(W^\Delta) \to \text{Tot}_{TW}(Z^\Delta)$.
3. Semicosimplicial differential graded Lie algebras and mapping cones

Let

\[ g^\Delta : g_0 \longrightarrow g_1 \longrightarrow g_2 \longrightarrow \cdots , \]

be a semicosimplicial differential graded Lie algebra. Every \( g_i \) is a DGLA and so, in particular, a differential graded vector space, thus we can consider the total complex \( \text{Tot}(g^\Delta) \).

**Example 3.1.** Every morphism \( \chi : L \to M \) of differential graded Lie algebras can be interpreted as the semicosimplicial DGLA

\[ \chi^\Delta : L \longrightarrow M \longrightarrow 0 , \quad \partial_0 = \chi , \ \partial_1 = 0 , \]

and the total complex \( \text{Tot}(\chi^\Delta) \) coincides with the mapping cone of \( \chi \), i.e.,

\[ \text{Tot}(\chi^\Delta)^i = L^i \oplus M^{i-1} , \quad d(l, m) = (dl, \chi(l) - dm) . \]

Even in the case of \( L \) and \( M \) Lie algebras, it is not possible to define a canonical bracket on the mapping cone, making \( \text{Tot}(\chi^\Delta) \) a DGLA and the projection \( \text{Tot}(\chi^\Delta) \to L \) a morphism of DGLAs. To see this it is sufficient to consider \( L = M \) the Lie subalgebra of \( sl(2, \mathbb{K}) \) generated by the matrices

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \]

and \( \chi \) equal to the identity. If \( \text{Tot}(\chi^\Delta) \) is a DGLA then, by functoriality, for every \( x \in L \) the subspace generated by \( x, \chi(x) \) is a subalgebra and, for every \( \lambda \in \mathbb{K} \), the linear map \( B \mapsto B, \ A \mapsto \lambda A \) is an automorphism of DGLA. It is an easy exercise to prove that these properties imply the failure of Jacobi identity.

Even if the complex \( \text{Tot}(g^\Delta) \) has no natural DGLA structure, it can be endowed with a canonical \( L_\infty \) structure by homological perturbation theory [FiMa07, FMM08].

Indeed, as in the previous section, we can consider the Thom-Whitney construction also for semicosimplicial differential graded Lie algebras: it is evident that in such case we have \( \text{Tot}_{TW}(V^\Delta) \) a differential graded Lie algebra. In this case, the morphisms \( I, E, h \) are functorial and commute with morphisms of semicosimplicial DGLAs. Moreover, these morphisms \( I, E, h \) can be used to apply the following basic result about \( L_\infty \)-algebras, dating back to Kadeishvili’s work on the cohomology of \( A_\infty \) algebras [Kad82]; see also [HK91].

**Theorem 3.2** (Homotopy transfer). Let \( (V, q_1, q_2, q_3, \ldots) \) be an \( L_\infty \)-algebra and \( (C, \delta) \) a differential graded vector space. Assume we have two morphisms of complexes

\[ \pi : (V[1], q_1) \to (C[1], \delta[1]) , \quad \iota_1 : (C[1], \delta[1]) \to (V[1], q_1) \]

and a linear map \( h \in \text{Hom}^{-1}(V[1],V[1]) \) such that \( h q_1 + q_1 h = \iota_1 \pi - \text{Id} \).

Then, there exist a canonical \( L_\infty \)-algebra structure \( (C, \langle \rangle_1, \langle \rangle_2, \ldots) \) on \( C \) extending its differential complex structure, and a canonical \( L_\infty \)-morphism \( \iota_\infty : (C, \langle \rangle_1, \langle \rangle_2, \ldots) \to (V, q_1, q_2, q_3, \ldots) \) extending \( \iota_1 \). In particular, if \( \iota_1 \) is an injective quasi-isomorphism of complexes, then also \( \iota_\infty \) is an injective quasi-isomorphism of \( L_\infty \)-algebras.

**Proof.** See [FiMa07] and references therein: explicit formulas for the quasi-isomorphism \( \iota_\infty \) and the brackets \( \langle \rangle_n \) have been described by Merkulov in [Me99]; then, it has been remarked by Kontsevich and Soibelman in [KS00, KS01] (see also [Fu03]) that Merkulov’s formulas can be nicely written as summations over rooted trees. □
Corollary 3.3 (FiMa07, FMM08). There exists a canonical $L_\infty$-algebra structure $\overline{\text{Tot}}(g^\Delta)$ on the differential graded vector space $\text{Tot}(g^\Delta)$, together with an injective quasi-isomorphism $E_\infty: \overline{\text{Tot}}(g^\Delta) \to \text{Tot}_{T W}(g^\Delta)$.

Proof. It is sufficient to apply Theorem 5.2 to the morphisms $I, E, h$, in order to define a canonical $L_\infty$-algebra structure $\overline{\text{Tot}}(g^\Delta)$ on $\text{Tot}(g^\Delta)$, and an injective quasi-isomorphism $E_\infty: \overline{\text{Tot}}(g^\Delta) \to \text{Tot}_{T W}(g^\Delta)$ extending $E$.

Notice that $E_\infty$ induces an isomorphism of functors $\text{Def}_{\overline{\text{Tot}}(g^\Delta)} \xrightarrow{\cong} \text{Def}_{\text{Tot}_{T W}(g^\Delta)}$. □

Let $\chi: L \to M$ be a morphism of differential graded Lie algebras over a field $\mathbb{K}$ of characteristic 0. We have already seen, in Example 3.1, that $\chi$ can be interpreted as the semicosimplicial DGLA

$$\chi^\Delta: L \xrightarrow{0} M \xrightarrow{\mu} 0 \cdots,$$

and therefore we have a canonical $L_\infty$ structure on the total complex $\text{Tot}(\chi^\Delta) = \bigoplus_i \text{Tot}(\chi^\Delta)^i$, $\text{Tot}(\chi^\Delta)^i = L^i \oplus M^{i-1}$.

The brackets $\mu_n: \bigwedge^n \text{Tot}(\chi^\Delta) \to \text{Tot}(\chi^\Delta)[2 - n]$, $n \geq 1$, have been explicitly described in [FiMa07]. Namely, one has

$$\mu_1(l, m) = (dl, \chi(l) - dm), \quad l \in L, m \in M,$$

$$\mu_2((l_1, m_1) \wedge (l_2, m_2)) = \left[ [l_1, l_2], \frac{1}{2} [m_1, \chi(l_2)] + \frac{(-1)^{\deg(l_1)}}{2} [\chi(l_1), m_2] \right]$$

and for $n \geq 3$

$$\mu_n((l_1, m_1) \wedge \cdots \wedge (l_n, m_n)) = \pm \frac{B_{n-1}}{(n - 1)!} \sum_{\sigma \in S_n} \varepsilon(\sigma)[m_{\sigma(1)}, \cdots, [m_{\sigma(n-1)}, \chi(l_{\sigma(n)})] \cdots]].$$

Here the $B_n$’s are the Bernoulli numbers, $\varepsilon$ is the Koszul sign and we refer to [FiMa07] for the exact determination of the overall $\pm$ sign in the formulas (it will not be needed in the present paper).

Proposition 3.4. In the notation above, assume that:

1. $\chi: L \to M$ is injective,
2. $\chi: H^*(L) \to H^*(M)$ is injective.

Then, the $L_\infty$-algebra $\overline{\text{Tot}}(\chi^\Delta)$ is quasi-abelian.

Proof. Since $H^0(\text{Hom}^*(L, L)) = \text{Hom}^0(H^*(L), H^*(L))$ and $H^0(\text{Hom}^*(M, L)) = \text{Hom}^0(H^*(M), H^*(L))$, the surjective morphism

$$\text{Hom}^*(M, L) \to \text{Hom}^*(L, L), \quad \phi \mapsto \phi \chi,$$

induces surjective maps

$$H^0(\text{Hom}^*(M, L)) \to H^0(\text{Hom}^*(L, L)), \quad Z^0(\text{Hom}^*(M, L)) \to Z^0(\text{Hom}^*(L, L))$$

and then the identity on $L$ can be lifted to a morphism $\pi: M \to L$ of differential graded vector spaces. Denoting $V = \ker(\pi)$, we have a direct sum decomposition of differential graded vector spaces $M = \chi(L) \oplus V$. 

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Denote by $d$ the differential on $M$, by assumption $d(V) \subset V$ and the inclusion $V[-1] \hookrightarrow \text{Tot}(\chi^\Delta)$ is an injective quasi-isomorphism. Let $H$ be a graded vector space, endowed with trivial differential, and $q: H \to V[-1]$ an injective morphism of differential graded vector spaces, inducing an isomorphism $H \xrightarrow{\sim} H^*(V[-1])$. The linear map

$$f: H \to \text{Tot}(\chi^\Delta), \quad f(h) = (0, g(h)),$$

is annihilated by every bracket of $\text{Tot}(\chi^\Delta)$ and then it is an injective quasi-isomorphism of $L_\infty$-algebras. \hfill \square

**Example 3.5.** Let $W$ be a differential graded vector space and let $U \subset W$ be a differential graded subspace. Assume that the induced morphism $H^*(U) \to H^*(W)$ is injective, then the injective morphism of DGLAs

$$\chi: \{ f \in \text{Hom}^*(W, W) \mid f(U) \subset U \} \to \text{Hom}^*(W, W)$$

satisfies the hypothesis of Proposition 3.4. In fact, the same argument used above shows that there exists a direct sum decomposition of differential graded vector spaces $W = U \oplus V$. Next, consider the subspace

$$K = \{ f \in \text{Hom}^*(W, W) \mid f(W) \subset V, f(V) = 0 \}.$$

It is straightforward to check that $K$ is a complementary subcomplex of the image of $\chi$ inside $\text{Hom}^*(W, W)$.

**Remark 3.6.** Let $W$ be a differential graded vector space and let $U \subset W$ be a differential graded subspace. It is showed in [FiMa06], that the deformation functor associated with the morphism of DGLAs

$$\chi: \{ f \in \text{Hom}^*(W, W) \mid f(U) \subset U \} \to \text{Hom}^*(W, W),$$

i.e., the deformation functor associated to the $L_\infty$-algebra $\text{Tot}(\chi^\Delta)$, has a natural interpretation as the local structure of the derived Grassmannian of $W$ at the point $U$. Moreover, Proposition 3.4 implies that the derived Grassmannian of $W$ is smooth at the points corresponding to subspaces $U$ such that $H^*(U) \to H^*(W)$ is injective.

4. **Semicosimplicial Cartan homotopies**

The abstract notion of Cartan homotopy has been introduced in [FiMa06, FiMa08] as a powerful tool for the construction of $L_\infty$ morphisms.

**Definition 4.1.** Let $L$ and $M$ be two differential graded Lie algebras. A linear map of degree $-1$

$$i: L \to M$$

is called a *Cartan homotopy* if, for every $a, b \in L$, we have:

$$i_{[a,b]} = [i_a, d_M i_b] + i_{d_L b} \quad \text{and} \quad [i_a, i_b] = 0.$$  

For every Cartan homotopy $i$, it is convenient to consider the map

$$l: L \to M, \quad l_a = d_M i_a + i_{d_L a}.$$  

It is straightforward to check that $l$ is a morphism of DGLAs and the conditions of Definition 4.1 become

$$i_{[a,b]} = [i_a, l_b] \quad \text{and} \quad [i_a, i_b] = 0.$$  

As a morphism of complexes, $l$ is homotopic to 0 (with homotopy $i$).
Example 4.2. Let $X$ be a smooth algebraic variety, $\Theta_X$ the tangent sheaf and $(\Omega^*_X, d)$ the algebraic de Rham complex. Then, for every open subset $U \subset X$, the contraction

$$\Theta_X(U) \otimes \Omega_X^k(U) \xrightarrow{\iota} \Omega_X^{k-1}(U)$$

induces a linear map of degree $-1$

$$\iota: \Theta_X(U) \rightarrow \text{Hom}^*(\Omega_X^*(U), \Omega_X(U)), \quad \iota_\xi(\omega) = \xi \omega$$

that is a Cartan homotopy. In fact, the differential of $\Theta_X(U)$ is trivial and the differential on the DGLA $\text{Hom}^*(\Omega_X^*(U), \Omega_X^*(U))$ is $\phi \mapsto [d, \phi] = d\phi - (-1)^{\deg(\phi)}\phi d$. Therefore, since $\iota_a$ has degree $-1$, we have that $I_a = di_a + i_a d$ is the Lie derivative and the above conditions reduce to the classical Cartan’s homotopy formulas:

1. $[i_a, i_b] = 0$;
2. $i_{[a, b]} = I_a i_b - I_b i_a = [I_a, i_b] = [i_a, I_b]$.

The relation between Cartan’s homotopy and $L_\infty$ algebras is given by the following theorem.

Theorem 4.3 ([FiMa08, Corollary 3.7]). Let $\chi: N \rightarrow M$ be a morphism of DGLAs and $\iota: L \rightarrow M$ be a Cartan homotopy. Assume that $\phi: L \rightarrow N$ is a morphism of DGLAs, such that $\chi \phi = l = d_M i + i d_L$. Then, the linear map

$$\Phi: L \rightarrow \text{Tot}(\chi^\Delta) \quad \Phi(a) = (\phi(a), i_a)$$

is a linear $L_\infty$-morphism. In particular, if $N$ is a subalgebra of $M$ containing $L(L)$ and $\chi$ is the inclusion, then the map

$$L \rightarrow \text{Tot}(\chi^\Delta) \quad a \mapsto (I_a, i_a)$$

is a linear $L_\infty$-morphism.

Proof. Straightforward consequence of the explicit description of the $L_\infty$ structure of $\text{Tot}(\chi^\Delta)$. \qed

Remark 4.4. It is plain from definition that Cartan homotopies are stable under composition with morphisms of DGLAs. More precisely, if $f: L' \rightarrow L$ and $g: M \rightarrow M'$ are morphisms of differential graded Lie algebras and $\iota: L \rightarrow M$ is a Cartan homotopy, then also $gf: L' \rightarrow M'$ is a Cartan homotopy.

Lemma 4.5. Let $\iota: L \rightarrow M$ be a Cartan homotopy and $A$ be a differential graded commutative algebra. Then the map

$$\iota \otimes \text{Id}: L \otimes A \rightarrow M \otimes A$$

$$(x \otimes a) \mapsto \iota_x \otimes a$$

is a Cartan homotopy.

Proof. By definition, for any $x \otimes a$ and $y \otimes b \in L \otimes A$, we have

$$[(\iota \otimes \text{Id})_{x \otimes a}, (\iota \otimes \text{Id})_{y \otimes b}] = [\iota_x \otimes a, \iota_y \otimes b] = (-1)^{\deg(a)(\deg(y)-1)}[\iota_x, \iota_y] \otimes ab = 0.$$ 

Moreover, denoting $l = d_M i + i d_L$ and $\tilde{l} = d_M A (\iota \otimes \text{Id}) + (\iota \otimes \text{Id}) d_L A$ we have

$$l_{x \otimes a} = d_M A (\iota_x \otimes a) + (\iota \otimes \text{Id}) (d_L x \otimes a) + (-1)^{\deg(x)} x \otimes d_A a =$$

$$= d_M l_x \otimes a - (-1)^{\deg(x)} x \otimes d_A a + \iota_d x \otimes a + (-1)^{\deg(x)} x \otimes d_A a = l_x \otimes a.$$ 

Thus, for any $x \otimes a$ and $y \otimes b \in L \otimes A$, we get

$$(\iota \otimes \text{Id})_{x \otimes a, y \otimes b} = (\iota \otimes \text{Id})_{(-1)^{\deg(a)\deg(y)} x, y} \otimes ab = (-1)^{\deg(a)\deg(y)} \iota_{x, y} \otimes ab =$$
\(-1)^{\deg(a)\deg(b)}[i_x, l_y] \otimes ab = [i_x \otimes a, l_y \otimes b] = [(i \otimes \text{Id})x \otimes a, \tilde{l}_y \otimes b] \). 

**Definition 4.6.** Let \( L \) be a differential graded Lie algebra and \( V \) a differential graded vector space. A bilinear map 
\[
L \times V \longrightarrow j \longrightarrow V 
\]
of degree \(-1\) is called a *contraction* if the induced map 
\[
i: L \rightarrow \text{Hom}^\ast(V, V), \quad i_l(v) = l \downarrow v,
\]
is a Cartan Homotopy.

The notion of contraction is stable under scalar extensions, more precisely:

**Lemma 4.7.** Let \( V \) be a differential graded vector space and 
\[
L \times V \longrightarrow j \longrightarrow V
\]
a contraction. Then, for every differential graded commutative algebra \( A \), the natural extension 
\[
(L \otimes A) \times (V \otimes A) \longrightarrow (V \otimes A) \quad (l \otimes a) \downarrow (v \otimes b) = (-1)^{\deg(a)\deg(v)}l \downarrow v \otimes ab,
\]
is a contraction.

**Proof.** According to Remark 4.4, Lemma 4.5 and Definition 4.6, it is sufficient to prove that the natural map 
\[
\alpha: \text{Hom}^\ast(V, V) \otimes A \rightarrow \text{Hom}^\ast(V \otimes A, V \otimes A), \quad \alpha(\phi \otimes a)(v \otimes b) = (-1)^{\deg(a)\deg(v)}\phi(v) \otimes ab,
\]
is a morphism of DGLAs. This is completely straightforward and it is left to the reader. \( \square \)

The notions of Cartan homotopy and contraction extend naturally to the semicosimplicial setting. Here, we consider only the case of contractions.

**Definition 4.8.** Let \( g^\Delta \) be a semicosimplicial DGLA and \( V^\Delta \) a semicosimplicial differential graded vector space. A semicosimplicial contraction 
\[
g^\Delta \times V^\Delta \longrightarrow V^\Delta,
\]
is a sequence of contractions \( g_n \times V_n \longrightarrow V_n, n \geq 0 \), commuting with coface maps, i.e., 
\[
\partial_k(l \downarrow v) = \partial_k(l) \downarrow \partial_k(v), \text{ for every } k.
\]

**Proposition 4.9.** Every semicosimplicial contraction 
\[
g^\Delta \times V^\Delta \longrightarrow V^\Delta
\]
extends naturally to a contraction 
\[
\text{Tot}_{TW}(g^\Delta) \times \text{Tot}_{TW}(V^\Delta) \longrightarrow \text{Tot}_{TW}(V^\Delta).
\]

**Proof.** By definition, for every \( n \), we have a Cartan homotopy 
\[
i: g_n \rightarrow \text{Hom}^\ast(V_n, V_n),
\]
and so, by Lemma 4.7, a Cartan homotopy 
\[
i: g_n \otimes (A_{PL})_n \rightarrow \text{Hom}^\ast(V_n \otimes (A_{PL})_n, V_n \otimes (A_{PL})_n).
\]

Therefore, it is enough to prove that \( i_{x^y}(y) \in \text{Tot}_{TW}(V^\Delta) \), for every pair of sequences \( x = (x_n)_{n \in \mathbb{N}} \in \text{Tot}_{TW}(g^\Delta) \) and \( y = (y_n)_{n \in \mathbb{N}} \in \text{Tot}_{TW}(V^\Delta) \); it follows from Remark 2.3. \( \square \)
5. Semicosimplicial Lie algebras and deformations of smooth varieties

Let $g^\Delta$ be a semicosimplicial Lie algebra, then we can apply the construction of Section 3 in order to construct the Thom-Whitney DGLA $\text{Tot}_T \text{W}(g^\Delta)$, the $L_\infty$-algebra $\text{Tot}(g^\Delta)$ and their associated (and isomorphic) deformation functors $\text{Def}_{\text{Tot}}(g^\Delta) \simeq \text{Def}_{\text{Tot}_T \text{W}}(g^\Delta)$.

Beyond this way, there is another natural, and more geometric, way to define a deformation functor, see [Pr03 Definitions 1.4 and 1.6] and [FMM08 Section 3]. More precisely, if $g^\Delta$ is a semicosimplicial Lie algebra, then we denote

$$Z^1_{sc}(\exp g^\Delta) : \text{Art} \to \text{Set}$$

as

$$Z^1_{sc}(\exp g^\Delta)(A) = \{ x \in g_1 \otimes m_A \mid e^{\partial_i(x)} e^{-\partial_i(x)} e^{\partial_2(x)} = 1 \},$$

and

$$H^1_{sc}(\exp g^\Delta) : \text{Art} \to \text{Set}$$

such that

$$H^1_{sc}(\exp g^\Delta)(A) = Z^1_{sc}(\exp g^\Delta)(A)/\sim,$$

where $x \sim y$ if and only if there exists $a \in g_0 \otimes m_A$, such that $e^{-\partial_i(a)} e^x e^{\partial_i(a)} = e^y$.

**Example 5.1.** Let $\mathcal{L}$ be a sheaf of Lie algebras on a paracompact topological space $X$, and $\mathcal{U}$ an open covering of $X$; it is naturally defined the Čech semicosimplicial Lie algebra $\mathcal{L}(\mathcal{U})$

$$\mathcal{L}(\mathcal{U}) : \prod_i \mathcal{L}(U_i) \xrightarrow{\partial_i} \prod_{i<j} \mathcal{L}(U_{ij}) \xrightarrow{\partial_{ij}} \prod_{i<j<k} \mathcal{L}(U_{ijk}) \xrightarrow{\partial_{ijk}} \cdots,$$

and, for every $A \in \text{Art}$, the set $H^1_{sc}(\exp \mathcal{L}(\mathcal{U}))(A)$ is exactly the cohomology set $H^1(\mathcal{U}, \exp(\mathcal{L} \otimes m_A))$ [Hir78].

The relation between the above functors is given by the following theorem.

**Theorem 5.2.** Let $g^\Delta$ be a semicosimplicial Lie algebra. Then, for every $A \in \text{Art}$,

$$MC_{\text{Tot}}(g^\Delta)(A) = Z^1_{sc}(\exp g^\Delta)(A),$$

as subsets of $g_1 \otimes m_A$. Moreover, we have natural isomorphisms of deformation functors

$$\text{Def}_{\text{Tot}_T \text{W}}(g^\Delta) \simeq \text{Def}_{\text{Tot}}(g^\Delta) \simeq H^1_{sc}(\exp g^\Delta).$$

**Proof.** For the proof, we refer to [FMM08]. □

Next, assume that $X$ is a smooth algebraic variety over a field $\mathbb{k}$ of characteristic 0, with tangent sheaf $\Theta_X$, and let $\mathcal{U} = \{ U_i \}_{i \in I}$ be an affine open covering of $X$.

Since every infinitesimal deformation of a smooth affine scheme is trivial [Se06 Lemma II.1.3], every infinitesimal deformation $X_A$ of $X$ over $\text{Spec}(A)$ is obtained by gluing the trivial deformations $U_i \times \text{Spec}(A)$ along the double intersections $U_{ij}$, and therefore it is determined by the sequence $\{ \theta_{ij} \}_{i<j}$ of automorphisms of sheaves of $A$-algebras

$$\mathcal{O}(U_{ij}) \xrightarrow{\theta_{ij}} \mathcal{O}(U_{ij}) \otimes A \xrightarrow{\theta_{ij}} \mathcal{O}(U_{ij}) \otimes A$$
Theorem 6.1. Let $\Theta_X$ be a smooth projective variety of dimension $n$, defined over an algebraically closed field of characteristic $0$. If the contraction map

$$H^*(\Theta_X) \longrightarrow \text{Hom}^*(H^*(\Omega^*_X, H^*(\Omega^*_X^{-1})))$$

is injective, then, for every affine open cover $\mathcal{U}$ of $X$, the DGLA $\text{Tot}_{TW}(\Theta_X(\mathcal{U}))$ is quasi-abelian.

**Proof.** According to Lemma [1.10], it is sufficient to prove that there exist a quasi-abelian $L_\infty$-algebra $H$ and a morphism $\text{Tot}_{TW}(\Theta_X(\mathcal{U})) \rightarrow H$ that is injective in cohomology.

Let $n$ be the dimension of $X$ and denote by $\Omega^*_X$ the algebraic de Rham complex. For every $i \leq n$, let $\check{C}(\mathcal{U}, \Omega^*_X)$ be the Cech complex of the coherent sheaf $\Omega^*_X$, with respect to the affine cover $\mathcal{U}$, and $\check{C}(\mathcal{U}, \Omega^*_X)$ the total complex of the semicosimplicial differential graded vector space $\Omega^*_X(\mathcal{U})$ (Example [2.2]). Notice that

$$\check{C}(\mathcal{U}, \Omega^*_X)^i = \bigoplus_{a+b=i} \check{C}(\mathcal{U}, \Omega^*_X)^b.$$
and $\hat{C}(U, \Omega^\infty_X)$ is a subcomplex of $\hat{C}(U, \Omega^\infty_X)$.

Then, we have a commutative diagram of complexes with horizontal quasi-isomorphisms:

\[
\begin{array}{ccc}
\hat{C}(U, \Omega^\infty_X) = \text{Tot}(\Omega^\infty_X(U)) & \xrightarrow{E} & \text{Tot}_{TW}(\Omega^\infty_X(U)) \\
\downarrow & & \downarrow \\
\hat{C}(U, \Omega^\infty_X) = \text{Tot}(\Omega^\infty_X(U)) & \xrightarrow{E} & \text{Tot}_{TW}(\Omega^\infty_X(U)).
\end{array}
\]

Since $\mathbb{K}$ has characteristic 0 and $X$ is smooth and proper, the Hodge spectral sequence degenerates at $E_1$ (we refer to [Fa88, DeI87] for a purely algebraic proof of this fact). Therefore, we have injective maps

\[
H^*(X, \Omega^\infty_X) = H^*(\hat{C}(U, \Omega^\infty_X)) \hookrightarrow H^*(\hat{C}(U, \Omega^\infty_X)) = H^*_{DR}(X/\mathbb{K}).
\]

Thus, the natural inclusions of complexes

\[
\text{Tot}_{TW}(\Omega^\infty_X(U)) \to \text{Tot}_{TW}(\Omega^\infty_X(U)),
\]

\[
\text{Tot}_{TW}(\Omega^{n-1}_X(U)) \to \frac{\text{Tot}_{TW}(\Omega^\infty_X(U))}{\text{Tot}_{TW}(\Omega^\infty_X(U))},
\]

are injective in cohomology.

According to Example 4.2, for every open subset $U \subset X$, the contraction of vector fields with differential forms defines a Cartan homotopy:

\[
i: \Theta_X(U) \to \text{Hom}^*(\Omega^\infty_X(U), \Omega^\infty_X(U)), \quad i_\xi(\omega) = \xi \lrcorner \omega.
\]

Since the contraction $\lrcorner$ commutes with restrictions to open subsets, we have a semi-cosimplicial contraction

\[
\Theta_X(U) \times \Omega^\infty_X(U) \xrightarrow{\cdot} \Omega^\infty_X(U),
\]

and, by Proposition 4.9, this induces naturally a Cartan homotopy

\[
i: \text{Tot}_{TW}(\Theta_X(U)) \to \text{Hom}^*(\text{Tot}_{TW}(\Omega^\infty_X(U)), \text{Tot}_{TW}(\Omega^\infty_X(U))).
\]

Notice that, for every $\xi \in \text{Tot}_{TW}(\Theta_X(U))$ and every $i$, we have

\[
i_\xi(\text{Tot}_{TW}(\Omega^i(U))) \subset \text{Tot}_{TW}(\Omega^{i-1}(U)),
\]

\[
i_\xi(\text{Tot}_{TW}(\Omega^i(U))) \subset \text{Tot}_{TW}(\Omega^i(U)), \quad i_\xi = di_\xi + i_d\xi.
\]

Moreover, the assumption of the theorem, together with [NaA87, 3.1], implies that the map

\[
\text{Tot}_{TW}(\Theta_X(U)) \xrightarrow{i} \text{Hom}^*(\text{Tot}_{TW}(\Omega^\infty_X(U)), \text{Tot}_{TW}(\Omega^{n-1}_X(U)))
\]

is injective in cohomology.

Next, consider the differential graded Lie algebras

\[
M = \text{Hom}^*(\text{Tot}_{TW}(\Omega^\infty_X(U)), \text{Tot}_{TW}(\Omega^\infty_X(U))),
\]

\[
L = \{ f \in M \mid f(\text{Tot}_{TW}(\Omega^\infty_X(U))) \subset \text{Tot}_{TW}(\Omega^\infty_X(U)) \},
\]

and let $\chi: L \to M$ be the inclusion. According to Example 8.3, the $L_\infty$-algebras $\text{Tot}(\chi^\Delta)$ is quasi-abelian.

Moreover, we have $\mathfrak{l}(\text{Tot}_{TW}(\Theta_X(U))) \subset L$ and so, by Theorem 4.3, there exists a linear $L_\infty$-morphism

\[
\text{Tot}_{TW}(\Theta_X(U)) \xrightarrow{(l, i)} \text{Tot}(\chi^\Delta), \quad x \mapsto (l_x, i_x).
\]
Since the map $\chi$ in injective, its mapping cone $\text{Tot}(\chi^\Delta)$ is quasi-isomorphic to its cokernel

$$\text{Coker } \chi = \text{Hom}^*(\text{Tot}_W(\Omega^n_X(U)), \text{Tot}_W(\Omega^{n-1}_X(U)))$$

and we have a commutative diagram of complexes

$$\begin{array}{ccc}
\text{Tot}_W(\Theta_X(U)) & \xrightarrow{(l, i)} & \text{Tot}(\chi^\Delta) \\
\downarrow i & & \downarrow q-\text{iso} \\
\text{Hom}^*(\text{Tot}_W(\Omega^n_X(U)), \text{Tot}_W(\Omega^{n-1}_X(U))) & \xrightarrow{\alpha} & \text{Coker } \chi.
\end{array}$$

Since both $i$ and $\alpha$ are injective in cohomology, also the $L_\infty$-morphism $(l, i)$ is injective in cohomology.

\[\square\]

**Theorem 6.2.** Let $U = \{U_i\}$ be an affine open cover of a smooth projective variety $X$ defined over an algebraically closed field of characteristic 0. If the canonical bundle of $X$ is trivial or torsion, then the DGLA $\text{Tot}_W(\Theta_X(U))$ is quasi-abelian.

**Proof.** Assume first that $X$ has trivial canonical bundle. If $n$ is the dimension of $X$, the cup product with a nontrivial section of the canonical bundle gives the isomorphisms $H^i(\Theta_X) \cong H^i(\Omega^{n-1}_X)$ and the conclusion follows immediately from Theorem 6.1. If $X$ has torsion canonical bundle we may consider the canonical cyclic cover $\pi: Y \to X$ and the affine open cover $V = \{\pi^{-1}(U_i)\}$. Now the variety $Y$ has trivial canonical bundle and then the $L_\infty$-algebra $\text{Tot}(\Theta_Y(V))$ is quasi-abelian. Since $\pi$ is an unramified cover the natural injective map $\tilde{\text{Tot}}(\Theta_X(U)) \to \tilde{\text{Tot}}(\Theta_Y(V))$ is also injective in cohomology and we conclude the proof by using the same argument of Theorem 6.1.

\[\square\]

**Remark 6.3.** When $\mathbb{K} = \mathbb{C}$, the previous theorems together with Remark 5.2 implies that the Kodaira-Spencer DGLA is quasi abelian, for a projective manifold with trivial or torsion canonical bundle.

**Theorem 6.4.** Let $X$ be a smooth projective variety of dimension $n$ defined over an algebraically closed field of characteristic 0. Then, the obstructions to deformations of $X$ are contained in the kernel of the contraction map

$$H^2(\Theta_X) \to \prod_p \text{Hom}(H^p(\Omega^n_X), H^{p+2}(\Omega^{n-1}_X)).$$

**Proof.** We have seen in the proof of Theorem 6.1 that, for every affine open cover $U$ of $X$, there exists an $L_\infty$-morphism $\tilde{\text{Tot}}(\Theta_X(U)) \to \tilde{\text{Tot}}(\chi^\Delta)$ and that $\tilde{\text{Tot}}(\chi^\Delta)$ is quasi-abelian. Therefore, we are in the condition to apply the general strategy used in [Ma04a, Ma09, Ia07]: the deformation functor associated to $\tilde{\text{Tot}}(\chi^\Delta)$ is unobstructed and the obstructions of $\text{Def}_X \simeq \text{Def}_{\tilde{\text{Tot}}(\Theta_X(U))}$ are contained in the kernel of the obstruction map $H^2(\tilde{\text{Tot}}(\Theta_X(U))) \to H^2(\tilde{\text{Tot}}(\chi^\Delta)).$

\[\square\]

**Corollary 6.5.** Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic 0. If the canonical bundle of $X$ is trivial, then $X$ has unobstructed deformations.

**Proof.** The previous Corollary 6.2 implies that $\tilde{\text{Tot}}(\Theta_X(U))$ is quasi-abelian and so $\text{Def}_{\tilde{\text{Tot}}(\Theta_X(U))}$ is smooth. By Theorem 5.3 $\text{Def}_X \cong \text{Def}_{\tilde{\text{Tot}}(\Theta_X(U))}$.

\[\square\]
Remark 6.6. Transcendental proofs of the analogue of Theorem 6.4 for compact Kähler manifolds can be found in [Ma04a, Cle05, Ma09], while we refer to [Ia07, Ma09] for the proof that the $T^1$-lifting is definitely insufficient for proving Theorem 6.4.

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