Matter Accretion by Brane-World Black Holes

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Abstract

The brane-world description of our universe entails a large extra dimension and a fundamental scale of gravity that might be lower by several orders of magnitude compared to the Planck scale. An interesting consequence of the brane-world scenario is in the nature of spherically symmetric vacuum solutions to the brane gravitational field equations, with properties quite distinct as compared to the standard black-hole solutions of general relativity. We consider the spherically symmetric accretion of matter onto brane-world black holes in terms of relativistic hydrodynamics by assuming that the inflowing gas obeys a polytropic equation of state. As a first step in this study, we consider the accretion process in an arbitrary static, spherically symmetric space-time, and show that the relativistic equations require a transition to a supersonic flow in the solution. The velocity, temperature, and density profiles are obtained for the case of the polytropic equation of state. We apply the general formalism to the study of the accretion properties of several classes of brane-world black holes, and we obtain the distribution of the main physical parameters of the gas. The astrophysical determination of these physical quantities could discriminate, at least in principle, between the different brane-world models, and place some constraints on the existence of the extra dimensions.

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I. INTRODUCTION

The idea, proposed in Refs. [1] and [2], that our four-dimensional Universe might be a three-brane, embedded in a five-dimensional space-time (the bulk) has attracted a considerable interest in the past few years. According to the brane-world scenario, the physical fields (electromagnetic, Yang-Mills, etc.) in our four-dimensional Universe are confined to the three brane. These fields are assumed to arise as fluctuations of branes in string theories. Only gravity can freely propagate in both the brane and bulk space-times, with the gravitational self-couplings not being significantly modified. This model originated from the study of a single 3-brane embedded in five dimensions, with the 5D metric given by

$$ds^2 = e^{-f(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

which, due to the appearance of the warp factor, could produce a large hierarchy between the scale of particle physics and gravity. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. Hence, this model allows the presence of large, or even infinite, non-compact extra dimensions. Our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space-time.

Due to the correction terms coming from the extra dimensions, significant deviations from the Einstein theory occur in brane-world models at very high energies [3, 4]. Gravity is largely modified at the electro-weak scale of 1 TeV. The cosmological and astrophysical implications of the brane-world theories have been extensively investigated in the literature [5, 6].

Several classes of spherically symmetric solutions of the static gravitational field equations in the vacuum on the brane have been obtained [7, 8, 9, 10]. As a possible physical application of these solutions, the behavior of the angular velocity $v_{tg}$ of the test particles in stable circular orbits has been considered [8, 9, 10]. The general form of the solution, together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances, and for a particular set of values of the integration constants, the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies and is usually explained by postulating the existence of dark matter. The exact galactic metric, the dark radiation, the dark pressure, and the lensing in the flat rotation curves region in the brane-world scenario has been obtained [10].

For standard general relativistic spherical compact objects, the exterior space-time is
described by the Schwarzschild metric. In the five dimensional brane-world models, the high-energy corrections to the energy density, together with the Weyl stresses from bulk gravitons, imply that on the brane, the exterior metric of a static star is no longer the Schwarzschild metric \[11\]. The presence of the Weyl stresses also means that the matching conditions do not have a unique solution on the brane; the knowledge of the five-dimensional Weyl tensor is needed as a minimum condition for uniqueness.

Static, spherically symmetric exterior vacuum solutions of the brane-world models have been proposed first in Ref. \[11\] and in Ref. \[12\]. The first of these solutions \[11\], has the mathematical form of the Reissner-Nordstrom solution of the standard general relativity, in which a tidal Weyl parameter plays the role of the electric charge of the general relativistic solution. The solution was obtained by imposing the null energy condition on the 3-brane for a bulk having non-zero Weyl curvature, and it can be matched to the interior solution corresponding to a constant-density brane-world star. A second exterior solution, which also matches a constant density interior, has been derived \[12\].

Two families of analytic solutions of the spherically symmetric vacuum brane world model equations (with $g_{tt} \neq -1/g_{rr}$), parameterized by the Arnowitt-Deser-Misner (ADM) mass and a Parameterized Post-Newtonian (PPN) parameter $\beta$ have been obtained \[13\]. Non-singular black-hole solutions in the brane-world model have been considered by relaxing the condition of the zero scalar curvature, but retaining the null energy condition \[14\]. The four-dimensional Gauss and Codazzi equations for an arbitrary static spherically symmetric star in a Randall–Sundrum type-II brane-world have been completely solved on the brane in \[15\]. The on-brane boundary can be used to determine the full 5-dimensional space-time geometry. The procedure can be generalized to solid objects, such as planets.

A method to extend into the bulk asymptotically-flat static spherically symmetric brane-world metrics has been proposed \[16\]. The exact integration of the field equations along the fifth coordinate was done by using a multipole $(1/r)$ expansion. The results show that the shape of the horizon of the brane-world black-hole solutions is very likely a flat “pancake” for astrophysical sources.

The general solution to the trace of the 4-dimensional Einstein equations for static, spherically-symmetric configurations has been used as a basis for finding a general class of black-hole metrics, containing one arbitrary function $g_{tt} = A(r)$, which vanishes at some $r = r_h > 0$ (the horizon radius) \[17\]. Under certain reasonable restrictions, black-hole
metrics are found, with or without matter. Depending on the boundary conditions, the metrics can be asymptotically flat or can have any other prescribed asymptotic structure. For a review of the black hole properties and of the lensing in the brane world models, see Ref. [18].

It is generally expected that most of the astrophysical objects grow substantially in mass via accretion. Recent observations suggest that around most of the active galactic nuclei (AGN’s) or black hole candidates, there exist gas clouds surrounding the central compact object, and an associated accretion disc, on a variety of scales from a tenth of a parsec to a few hundred parsecs [19]. These clouds are assumed to form a geometrically and optically thick torus (or warped disc), which absorbs most of the ultraviolet radiation and the soft X-rays. The gas exists in either a molecular or an atomic phase. The most powerful evidence for the existence of super-massive black holes comes from the VLBI imaging of molecular H$_2$O masers in the active galaxy NGC 4258 [20]. This imaging, produced by Doppler shift measurements assuming Keplerian motion of the masering source, has allowed a quite accurate estimate of the mass of the central object, which has been found to be a $3.6 \times 10^7 M_\odot$, a super massive dark object within 0.13 parsecs. Hence, important astrophysical information can be obtained from the observation of the motion of the gas streams in the gravitational field of compact objects.

The history of the theoretical study of the accretion of an ideal fluid onto a compact object begins with Bondi’s classic paper [21]. A relativistic generalization of the Newtonian accretion model was proposed by Michel [22] and further considered in Refs. [23, 24, 25, 26, 27, 28]. In particular, Shapiro and Teukolsky [26] gave a general relativistic version of the Bondi [21] model, which is known as the $(p - n)$ model. Another relativistic accretion model, called the $(p - \rho)$ model, was proposed [29] and further developed [30].

It is not yet clear which equation of state is appropriate in the description of the relativistic collapsing gas. There are two commonly used polytropic equations of state: $p = K \rho^n$ (the $(p - \rho)$ model [26]) and $p = C n^\Gamma$ (the $(p - n)$ model [29]). Here $p$ is the pressure, $\rho$ is the density and $n$ is the baryonic mass density, while $K$ and $C$ are constants. Numerical calculations show that the predictions of the models are similar in most aspects. However, in the ultra-relativistic regime the allowed band of the asymptotic speed of sound and the mass accretion rate can be markedly different [29].

The determination of the accretion rate for an astrophysical object can give strong evi-
dence for the existence of a surface for the object. A model in which Sgr A*, the $3.7 \times 10^6 M_\odot$ super massive black hole candidate at the Galactic center, may be a compact object with a thermally emitting surface was considered \[31\]. For very compact surfaces within the photon orbit, the thermal assumption is likely to be a good approximation because of the large number of rays that are strongly gravitationally lensed back onto the surface. Given the very low quiescent luminosity of Sgr A* in the near-infrared, the existence of a hard surface, even in the limit in which the radius approaches the horizon, places a severe constraint on the steady mass accretion rate onto the source, $\dot{M} \leq 10^{-12} M_\odot \text{yr}^{-1}$. This limit is well below the minimum accretion rate needed to power the observed submillimeter luminosity of Sgr A*, $\dot{M} \geq 10^{-10} M_\odot \text{yr}$. Thus, from the determination of the accretion rate, it follows that Sgr A* does not have a surface, that is, it must have an event horizon. Therefore, the study of the accretion processes by compact objects is a powerful indicator of their physical nature.

The stationary, spherically-symmetric accretion of dark energy onto a Schwarzschild black hole was considered in terms of relativistic hydrodynamics \[32\]. To model the dark energy, the approximation of an ideal fluid was used. Constraints on the gravastar models from accreting black holes were obtained \[33\]. In the study, two black hole candidates known to have extraordinarily low luminosities, the super massive black hole in the galactic center, Sagittarius A*, and the stellar-mass black hole XTE J1118+480 have been used. The observational results show that the length scale for modifications of standard general relativity for the gravastar models must be sub-Planckian.

It is the purpose of the present paper to study the matter accretion by brane-world black holes. As a first step, we derive the general equations for the spherically symmetric steady accretion of a fluid in an arbitrary space-time. These equations have a critical (sonic point), and only solutions passing through it correspond to material falling into (or flowing out of) the central accreting object, with monotonically increasing velocity along the particle trajectory. The velocities at the sonic point are obtained in a general form. In order to close the system of equations of motion, we need to impose an equation of state describing the thermodynamics properties of the inflowing matter. By assuming that the equation of state is polytropic in the limit of low temperature, we can solve the equations of motions, and we can obtain the velocity, density, and temperature profiles of the matter being accreted by the black hole as functions of the components of the metric tensor in a general form. This allows the velocity, temperature, and density of the matter at the black hole’s event horizon...
By using the general formalism of accretion, we analyze the accretion process for several brane-world black-hole solutions, which have been previously obtained. Thus, we consider the velocity, density, and temperature profiles of the inflowing gas, the velocities at the sonic point, and the values of the velocity and of the thermodynamical parameters at the event horizon for three classes of vacuum solutions of the brane world models, which have been obtained in Refs. [11], [13] and [17], respectively. Due to the differences in the space-time metrics, the velocity, density, and temperature profiles are different in all these cases and different from the predictions of the standard general relativistic Schwarzschild accretion model. Hence, the study of the accretion processes may provide an effective method to constrain the existence of the extra-dimensions and to test the predictions of the brane-world models.

The present paper is organized as follows: We derive the general equations for spherically symmetric steady accretion in Section II. The case of a polytropic equation of state is analyzed in Section III. The general formalism is applied to several classes of brane-world black holes in Section IV. We discuss and conclude our results in Section V.

II. STEADY SPHERICAL ACCRETION IN ARBITRARY SPHERICALLY-SYMMETRIC SPACETIMES

Let us consider the stationary, spherically symmetric accretion of an ideal fluid in an arbitrary static, spherically symmetric space-time, with a metric given by

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

(1)

Here $r$ is the radial coordinate, and $\theta$ and $\varphi$ are the angular spherical coordinates, respectively. The metric tensor components are assumed to be functions of the radial distance only so that $\nu = \nu (r)$ and $\lambda = \lambda (r)$. We model the fluid as an ideal fluid with the energy-momentum tensor

$$T_{\mu\nu} = (\mu c^2 + p) u_\mu u_\nu - pg_{\mu\nu},$$

(2)

where $\mu$ is the total energy density, $p$ is the pressure, and $u^\mu = dx^\mu/ds$ is the four-velocity. Generally, the total energy density can be represented as $\mu c^2 = \rho c^2 + \epsilon$, where $\rho$ is the matter density and $\epsilon$ is the thermal energy. The pressure is assumed to be an arbitrary function
of the density, \( p = p(\rho) \). In the case of a radial flow, the components of the four-velocity are \( u^0 = c dt/ds \), \( u_0 = e^\nu u^0 \), and \( u^1 = u = dr/ds \). The components of the four-velocity are normalized so that \( u_\mu u^\mu = 1 \) or \( u_0 u^0 + u_1 u^1 = e^{-\nu} u_0^2 - e^{\lambda} u^2 \).

The basic equations of motion of the fluid are the conservation of the mass flux \( J_\mu = \rho u^\mu \), \( J_\mu = 0 \), and the conservation of the energy flux \( T_{0\mu} = 0 \), where the semicolons denote the covariant derivative \([22]\). For a steady spherically symmetric flow, the conservation equations have the form

\[
\frac{d}{dr} \left( \frac{\rho c^2 u^\mu u_\mu r^2}{r^2} \right) = 0, \tag{3}
\]

and

\[
\frac{d}{dr} \left[ \left( \mu c^2 + p \right) u_0 u^\mu u_\mu r^2 \right] = 0, \tag{4}
\]

respectively, giving

\[
\rho c^2 u^\mu u_\mu r^2 = C_1, \tag{5}
\]

and

\[
(\mu c^2 + p) u_0 u^\mu u_\mu r^2 = C_2, \tag{6}
\]

where \( C_1 \) and \( C_2 \) are constants of integration. By dividing Eq. \((5)\) by Eq. \((5)\) and squaring gives

\[
\left( \frac{\mu c^2 + p}{\rho c^2} \right)^2 \left( e^\nu + e^{\nu+\lambda} u^2 \right) = C_3, \tag{7}
\]

where \( C_3 = (C_2/C_1)^2 \). By differentiating Eqs. \((5)\) and \((6)\) and eliminating \( d\ln \rho c^2 \), we find

\[
\left( r\frac{\nu' + \lambda'}{2} + 2 \right) \left\{ 2V^2 - \frac{r \left[ \nu' e^\nu + (\nu' + \lambda') e^{\nu+\lambda} u^2 \right]}{(r\nu' + r\lambda'/2 + 2) (e^\nu + e^{\nu+\lambda} u^2)} \right\} \frac{dr}{r} + 2 \left[ V^2 - \frac{e^{\nu+\lambda} u^2}{e^\nu + e^{\nu+\lambda} u^2} \right] \frac{du}{u} = 0, \tag{8}
\]

where we have denoted

\[
V^2 = \frac{d\ln (\mu c^2 + p)}{d\ln \rho c^2} - 1. \tag{9}
\]

In the case of the Schwarzschild metric, we have \( e^\nu = e^{-\lambda} = 1 - 2m/r \), with \( m = GM/c^2 \) being the total mass of the accreting object. In this case, Eq. \((8)\) reduces to the basic equation for steady spherically symmetric accretion onto compact objects, first derived in Ref. \([22]\).
If one or the other of the bracketed factors in Eq. (8) vanishes, one has a turn-around point, and the solutions are double-valued in either \( r \) or \( u \). Only solutions passing through a critical point correspond to material falling into (or flowing out of) the object with monotonically increasing velocity along the particle trajectory. The critical point (also called the sonic point) is located where all bracketed factors in Eq. (8) vanish \[22\]. Thus, in an arbitrary space-time, the conditions for the existence of a critical point can be formulated as

\[
2V^2 \left( r \frac{\nu' + \lambda'}{2} + 2 \right) - \frac{r \left[ \nu' e^\nu + (\nu' + \lambda') e^{\nu + \lambda} u^2 \right]}{(e^\nu + e^{\nu + \lambda} u^2)} = 0, \tag{10}
\]

and

\[
V^2 - \frac{e^{\nu + \lambda} u^2}{e^\nu + e^{\nu + \lambda} u^2} = 0, \tag{11}
\]

respectively. In the case of the Schwarzschild metric, we obtain \( u_c^2 = m/2r_c \) and \( V_c^2 = u_c^2/(1 - 3u_c^2) \), respectively \[22\]. Substituting Eq. (11) into Eq. (10) gives the expression of the velocity at the sonic point as

\[
u_c^2 = \left. \frac{r \nu' e^{-\lambda}}{4} \right|_{r=r_c}. \tag{12}
\]

Then, Eq. (11) gives

\[
V_c^2 = \frac{e^{\lambda(r_c)} u_c^2}{1 + e^{\lambda(r_c)} u_c^2}. \tag{13}
\]

It is interesting to note that the tangential velocity of a particle in a stable circular orbit in the space-time with the metric given by Eq. (1) is given by \( v_t^2/c^2 = \nu' / 2 \). Therefore, we obtain the following relations for the velocities at the sonic point and the tangential velocity of the particles:

\[
u_c^2 = \left. \frac{v_t^2 e^{-\lambda}}{2c^2} \right|_{r=r_c}, \tag{14}
\]

and

\[
V_c^2 = \frac{v_t^2(r_c) / 2c^2}{1 + v_t^2(r_c) / 2c^2}, \tag{15}
\]

respectively.
III. ACCRETION MODEL WITH A POLYTROPIC EQUATION OF STATE

In order to study the accretion processes in the brane-world models, we need to specify the equation of state \( p = p(\rho) \) of the inflowing matter, which we assume to be in the form of a gas, and the metric of the space-time. As for the equation of state, we adopt the polytropic equation of state so that

\[
p = K\rho^\Gamma,
\]

with \( K \) and \( \Gamma \) being constants [22]. The temperature \( T \) of the gas can be obtained from the ideal gas equation of state \( p = \rho k_B T/\mu m_p \), where \( k_B \) is Boltzmann’s constant, \( \mu \) is the mean molecular weight, and \( m_p \) is the mass of the proton. As a function of temperature, the pressure and the density are given by

\[
\rho = \left(\frac{c^2}{K}\right)^n T_p^n, \quad p = K \left(\frac{c^2}{K}\right)^{n+1} T_p^{n+1},
\]

where

\[
n = \frac{1}{\Gamma - 1}, \quad T_p = \frac{k_B T}{\mu m_p c^2} = \frac{T}{\mu \times 1.09 \times 10^{13} \text{K}}.
\]

For the polytropic equation of state, we have [22]

\[
p + \epsilon = (n + 1)p.
\]

With the use of the equation of state of the gas, Eq. (9) can be written as

\[
V^2 = \left(\frac{(n + 1)T_p}{n[1 + (n + 1)T_p]}\right).
\]

By estimating this equation at the sonic point, where the gas temperature is \( T_{pc} \), and comparing with Eq. (13) gives

\[
T_{pc} = \frac{n}{n + 1 + (1 - n)e^{\lambda(r_\infty)}u_e^2}.
\]

The equations of motion of the inflowing gas with a polytropic equation of state are given by

\[
c^2 \left(\frac{c^2}{K}\right)^n T_p^n u^2 e^{(\nu + \lambda)/2} = B_1.
\]
and
\[ [1 + (n + 1) T_p]^2 (e^\nu + e^{\nu + \lambda} u^2) = B_3, \]
respectively, with \( B_1 \) and \( B_3 \) being constants of integration. Evaluating Eqs. (60) and (61) at \( r = r_\infty \), where \( e^\nu(r_\infty) = e^\nu_\infty \), \( e^\lambda(r_\infty) = e^\lambda_\infty \) and the velocity and the temperature are \( u(r_\infty) = u_\infty \) and \( T_p(r_\infty) = T_{p\infty} \), respectively, gives
\[ B_1 = c^2 \left( \frac{c^2}{K} \right)^n T_{p\infty} u_\infty r_\infty^2 e^{(\nu_\infty + \lambda_\infty)/2}, \]
and
\[ B_3 = [1 + (n + 1) T_{p\infty}]^2 (e^{\nu_\infty} + e^{\nu_\infty + \lambda_\infty} u_\infty^2), \]
respectively.

Therefore, the two equations in the unknowns \( T_p \) and \( u \) describing the motion of the polytropic gas in an arbitrary static, spherically symmetric metric are
\[ T_p u r^2 e^{(\nu + \lambda)/2} = T_{p\infty} u_\infty r_\infty^2 e^{(\nu_\infty + \lambda_\infty)/2}, \]
and
\[ [1 + (n + 1) T_p]^2 (e^\nu + e^{\nu + \lambda} u^2) = [1 + (n + 1) T_{p\infty}]^2 (1 + e^{\nu_\infty + \lambda_\infty} u_\infty^2). \]
Evaluating Eq. (26) at the sonic point \( r = r_c \) gives an algebraic equation for the determination of \( r_c \),
\[ T_{pc}^{5/2} e^{\nu(r_c)} \sqrt{u'} \bigg|_{r=r_c} = 2 T_{p\infty} u_\infty r_\infty^2 e^{(\nu_\infty + \lambda_\infty)/2}. \]
By assuming that \( T_p << 1 \) and \( e^{\nu_\infty + \lambda_\infty} u_\infty^2 << 1 \), Eq. (27) immediately gives
\[ u(r) \approx (1 - e^\nu)^{1/2} e^{-(\nu + \lambda)/2}. \]
By substituting this expression of the velocity in Eq. (26), we obtain the temperature and the density profiles of the cold accreting gas as
\[ T_p(r) \approx T_{p\infty} \left[ \frac{u_\infty r_\infty^2 e^{(\nu_\infty + \lambda_\infty)/2}}{r^{2/n} (1 - e^\nu)^{1/2 n}} \right]^{1/n}, \]
and
\[ \rho(r) \approx \left( \frac{c^2}{K} \right) \frac{u_\infty r_\infty^2 e^{(\nu_\infty + \lambda_\infty)/2}}{r^2 (1 - e^\nu)^{1/2 n}}. \]
respectively. At the event horizon of the black hole, \( r = r_h \). Therefore, we obtain the velocity \( u_h = u(r_h) \), the temperature \( T_{ph} = T_p(r_h) \) and the density \( \rho_h = \rho(r_h) \) of the gas at the event horizon of the black hole as

\[
    u_h \approx \left[ 1 - e^{\nu(r_h)} \right]^{1/2} e^{-[\nu(r_h) + \lambda(r_h)]/2},
\]

\[
    T_{ph} \approx T_p \ \frac{u^2 \Gamma_p}{r^{2/n}_h \left[ 1 - e^{\nu(r_h)} \right]^{1/2}},
\]

\[
    \rho_h \approx \left( \frac{\Gamma T}{K} \right) \frac{u^2 \Gamma_p}{r^{2/n}_h \left[ 1 - e^{\nu(r_h)} \right]^{1/2}},
\]

respectively.

If \( e^{[\nu(r_h) + \lambda(r_h)]} = 1 \), as is the case for the standard Schwarzschild solution of general relativity, and because at the event horizon \( e^{[\nu(r_h)]} = 0 \), the gas particles reach the ”surface” (event horizon) of the black hole with a four-velocity equal to the speed of light, \( u_h \approx 1 \). However, for black holes for which the event horizon is located so that \( e^{[\nu(r_h) + \lambda(r_h)]} \neq 1 \), \( e^{[-\lambda(r_h)]} = 0 \), and \( e^{[\nu(r_h)]} \neq 0 \), we have \( u_h \approx 0 \). Such black holes occur in the brane world models.

An important physical quantity in the description of the accretion is the speed of sound \( a^2 = \partial p/\partial \rho = \Gamma p/\rho = c^2 \Gamma T_p \). Hence, the speed of sound at infinity is related to the temperature at infinity by the simple relation \( T_{p \infty} = a^2_\infty/\Gamma c^2 \). At the sonic point, the speed of sound is

\[
    a_c = c \sqrt{\frac{e^{\lambda(r_c)} u_c^2}{1 + (1 - n) e^{\lambda(r_c)} u_c^2}} \leq c.
\]

Sometimes an alternative form of the equation of state of the gas is used, by assuming that the pressure \( p \) is related to the baryon number density \( n_B \) by the polytropic relation \( p = Kn_B^{\Gamma} \). \( n_B \) and \( \rho \) can be related by the general relation \( n_B = \exp \left[ \int \frac{d\rho}{(\rho^2 + K \rho^{\Gamma})} \right] \), which can be integrated to give \( n_B = \rho \left( \frac{c^2}{K} \right)^{-n} (1 + a^2/\Gamma c^2)^{-n} \). Therefore, in the present model, the baryon number density is given as a function of the radial distance \( r \) by

\[
    n_B \approx \left( \frac{c^2}{K} \right)^{1-n} \frac{u^2 \Gamma_p}{r^{2/n}_h \left[ 1 - e^{\nu(r_h)} \right]^{1/2}} \left( 1 + a^2/\Gamma c^2 \right)^n.
\]
IV. GRAVITATIONAL FIELD EQUATIONS IN THE BRANE-WORLD MODELS

In the present section, we briefly describe the basic mathematical formalism of the brane-world models, and we present the spherically symmetric static vacuum field equations. The solutions of the vacuum field equations on the brane physically describe the brane-world black holes.

A. Gravitational Field Equations on the Brane

We start by considering a five-dimensional (5D) spacetime (the bulk) with a single four-dimensional (4D) brane, on which matter is confined. The 4D brane world \((^{(4)}M, g_{\mu\nu})\) is located at a hypersurface \((B (X^A) = 0)\) in the 5D bulk spacetime \((^{(5)}M, g_{AB})\), whose coordinates are described by \(X^A, A = 0, 1, ..., 4\). The induced 4D coordinates on the brane are \(x^\mu, \mu = 0, 1, 2, 3\). The action of the system is given by \(S = S_{\text{bulk}} + S_{\text{brane}}\), where

\[
S_{\text{bulk}} = \int^{(5)}_M \sqrt{-^{(5)}g} \left[ \frac{1}{2k_5^2}^{(5)}R +^{(5)}L_m + \Lambda_5 \right] d^5X, \tag{37}
\]

and

\[
S_{\text{brane}} = \int^{(4)}_M \sqrt{-^{(5)}g} \left[ \frac{1}{k_5^2}K^\pm + L_{\text{brane}} (g_{\alpha\beta}, \psi) + \lambda_b \right] d^4x, \tag{38}
\]

where \(k_5^2 = 8\pi G_5\) is the 5D gravitational constant, \(^{(5)}R\) and \(^{(5)}L_m\) are the 5D scalar curvature and the matter Lagrangian in the bulk, \(L_{\text{brane}} (g_{\alpha\beta}, \psi)\) is the 4D Lagrangian, which is given by a generic functional of the brane metric \(g_{\alpha\beta}\) and of the matter fields \(\psi\), \(K^\pm\) is the trace of the extrinsic curvature on either side of the brane, and \(\Lambda_5\) and \(\lambda_b\) (the constant brane tension) are the negative vacuum energy densities in the bulk and on the brane, respectively.\[3\]

The Einstein field equations in the bulk are given by \[3\]

\[
^{(5)}G_{IJ} = k_5^2^{(5)}T_{IJ}, \quad ^{(5)}T_{IJ} = -\Lambda_5^{(5)}g_{IJ} + \delta(B) \left[-\lambda_b^{(5)}g_{IJ} + T_{IJ} \right], \tag{39}
\]

where \(^{(5)}T_{IJ} \equiv -2\delta^{(5)}L_m/\delta^{(5)}g^{IJ} + \delta^{(5)}g_{IJ}^{(5)L_m}\) is the energy-momentum tensor of bulk matter fields while \(T_{\mu\nu}\) is the energy-momentum tensor localized on the brane, which is defined by \(T_{\mu\nu} \equiv -2\delta L_{\text{brane}}/\delta g^{\mu\nu} + g_{\mu\nu} L_{\text{brane}}\). The delta function \(\delta(B)\) denotes the localization of the brane contribution. In the 5D spacetime, a brane is a fixed point of the \(Z_2\) symmetry. The basic equations on the brane are obtained by projections onto the brane world. The induced
4D metric is \( g_{IJ} = (5) g_{IJ} - n_I n_J \), where \( n_I \) is the space-like unit vector field normal to the brane hypersurface \( ^{(4)}M \). In the following we assume \( ^{(5)}L_m = 0 \). In the brane-world models only gravity can probe the extra dimensions.

Assuming a metric of the form \( ds^2 = (n_I n_J + g_{IJ}) dx^I dx^J \), with \( n_I dx^I = d\chi \) being the unit normal to the \( \chi = \text{constant} \) hypersurfaces and \( g_{IJ} \) being the induced metric on \( \chi = \text{constant} \) hypersurfaces, the effective 4D gravitational equation on the brane takes the form [3]:

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu},
\]

(40)

where \( S_{\mu\nu} \) is the local quadratic energy-momentum correction

\[
S_{\mu\nu} = \frac{1}{12} TT_{\mu\nu} - \frac{1}{4} T^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3 T^{\alpha\beta} T_{\alpha\beta} - T^2),
\]

(41)

and \( E_{\mu\nu} \) is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor \( C_{IJAB} \), \( E_{IJ} = C_{IJAB} n^A n^B \), with the property \( E_{\mu\nu} \rightarrow E_{\mu\nu} \delta_\mu^\sigma \delta_\nu^\rho \) as \( \chi \rightarrow 0 \). We have also denoted \( k_4^2 = 8\pi G \), with \( G \) being the usual 4D gravitational constant. The 4D cosmological constant, \( \Lambda \), and the 4D coupling constant, \( k_4 \), are related by \( \Lambda = k_5^2 (\Lambda_5 + k_5^2 \lambda_b^2 / 6) / 2 \) and \( k_4^2 = k_5^4 \lambda_b^2 / 6 \), respectively. In the limit \( \lambda_b^{-1} \rightarrow 0 \) we recover standard general relativity [3].

The Einstein equation in the bulk and the Codazzi equation also imply conservation of the energy-momentum tensor of the matter on the brane, \( D_\nu T^\nu_\mu = 0 \), where \( D_\nu \) denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane, it follows that the projected Weyl tensor obeys the constraint \( D_\nu E^\nu_\mu = k_4^2 D_\nu S_{\mu\nu} \).

The symmetry properties of \( E_{\mu\nu} \) imply that, in general, we can decompose it irreducibly with respect to a chosen 4-velocity field \( u^\mu \) as \( E_{\mu\nu} = -k^4 \left[ U (u_\mu u_\nu + \frac{1}{3} h_{\mu\nu}) + P_{\mu\nu} + 2 Q_{(\mu} u_{\nu)} \right] \), where \( k = k_5 / k_4 \), \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) projects orthogonal to \( u^\mu \), the “dark radiation” term \( U = -k^{-4} E_{\mu\nu} u^\mu u^\nu \) is a scalar, \( Q_{\mu} = k^{-4} h^\alpha_{\mu} E_{\alpha\beta} u^\beta \) is a spatial vector and \( P_{\mu\nu} = -k^{-4} \left[ h_{(\mu} \alpha_{\nu)\beta} - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta} \right] E_{\alpha\beta} \) is a spatially, symmetric and trace-free tensor [11].

In the case of the vacuum state, we have \( \rho = p = 0 \) and \( T_{\mu\nu} \equiv 0 \), consequently, \( S_{\mu\nu} \equiv 0 \). Therefore, the field equation describing a static brane takes the form

\[
R_{\mu\nu} = -E_{\mu\nu} + \Lambda g_{\mu\nu},
\]

(42)

with the trace \( R \) of the Ricci tensor \( R_{\mu\nu} \) satisfying the condition \( R = R^\mu_\mu = 4\Lambda \).
In the vacuum case $E_{\mu\nu}$ satisfies the constraint $D_{\nu}E_{\mu}^{\nu} = 0$. In an inertial frame at any point on the brane, we have $u^\mu = \delta_0^\mu$ and $h_{\mu\nu} = \text{diag}(0, 1, 1, 1)$. In a static vacuum, $Q_\mu = 0$, and the constraint for $E_{\mu\nu}$ takes the form \[12\]

$$
\frac{1}{3} D_\mu U + \frac{4}{3} U A_\mu + D^\nu P_{\mu\nu} + A^\nu P_{\mu\nu} = 0,
$$

where $A_\mu = u^\nu D_\nu u_\mu$ is the 4-acceleration. In the static spherically symmetric case, we may choose $A_\mu = A(r) r_\mu$ and $P_{\mu\nu} = P(r) \left(r_\mu r_\nu - \frac{1}{3} h_{\mu\nu}\right)$, where $A(r)$ and $P(r)$ (the “dark pressure”) are some scalar functions of the radial distance $r$, and $r_\mu$ is a unit radial vector \[11\].

B. Brane-world Black Holes

In the following, we will restrict our study to the static and spherically symmetric metric given by

$$
\frac{1}{3} D_\mu U + \frac{4}{3} U A_\mu + D^\nu P_{\mu\nu} + A^\nu P_{\mu\nu} = 0,
$$

(43)

where $A_\mu = u^\nu D_\nu u_\mu$ is the 4-acceleration. In the static spherically symmetric case, we may chose $A_\mu = A(r) r_\mu$ and $P_{\mu\nu} = P(r) \left(r_\mu r_\nu - \frac{1}{3} h_{\mu\nu}\right)$, where $A(r)$ and $P(r)$ (the “dark pressure”) are some scalar functions of the radial distance $r$, and $r_\mu$ is a unit radial vector \[11\].
Equation (45) can immediately be integrated to give

\[ e^{-\lambda} = 1 - \frac{C_1}{r} - \frac{GM_U(r)}{r} - \frac{\Lambda}{3} r^2, \]

(49)

where \( C_1 \) is an arbitrary constant of integration, and we have used \( GM_U(r) = 3\alpha \int_0^r U(r) r^2 dr \). The function \( M_U \) is the gravitational mass corresponding to the dark radiation term (the dark mass). For \( U = 0 \), the metric coefficient given by Eq. (49) must tend to the standard general relativistic Schwarzschild metric coefficient, which gives \( C_1 = 2GM \), where \( M = \text{constant} \) is the baryonic (usual) mass of a gravitating system.

By substituting \( r' \) given by Eq. (48) into Eq. (46) and using Eq. (49), we obtain the following system of differential equations satisfied by the dark radiation term \( U \), the dark pressure \( P \), and the dark mass \( M_U \), describing the vacuum gravitational field exterior to a massive body, in the brane-world model [7]:

\[ \frac{dM_U}{dr} = \frac{3\alpha}{G} r^2 U, \]  

(50)

\[ \frac{dU}{dr} = -\left( \frac{2U + P}{r^2} \right) \left[ 2GM + GM_U - \frac{2}{3} \Lambda r^3 + \alpha \left( U + 2P \right) r^3 \right] - 2 \frac{dp}{d\theta} - 6P, \]  

(51)

To close the system, a supplementary functional relation between one of the unknowns, \( U \), \( P \) or \( M_U \), is needed. Generally, this equation of state is given in the form \( P = P(U) \). Once this relation is known, Eqs. (50)–(51) give a full description of the geometrical properties of the vacuum on the brane.

In the following we will restrict our analysis to the case \( \Lambda = 0 \). Then, the system of equations Eqs. (50) and (51), can be transformed to an autonomous system of differential equations by means of the transformations \( q = 2GM/r + GM_U/r, \mu = 3\alpha r^2 U, p = 3\alpha r^2 P \), and \( \theta = \ln r \) where \( \mu \) and \( p \) are the “reduced” dark radiation and pressure, respectively. With the use of the new variables, Eqs. (50) and (51) become

\[ \frac{dq}{d\theta} = \mu - q, \]  

(52)

\[ \frac{d\mu}{d\theta} = -\frac{(2\mu + p) \left[ g + \frac{1}{3} (\mu + 2p) \right]}{1 - q} - 2 \frac{dp}{d\theta} + 2\mu - 2p. \]  

(53)

Equations. (50) and (51) or, equivalently, Eqs. (52) and (53), are called the structure equations of the vacuum on the brane [7]. In order to close the system of Eqs. (52) and (53),
V. ACCRETION BY BRANE-WORLD BLACK HOLES

The braneworld description of our universe entails a large extra dimension and a fundamental scale of gravity that might be lower by several orders of magnitude compared to the Planck scale \([1, 2]\). It is known that the Einstein field equations in five dimensions admit more general spherically-symmetric black holes on the brane than four-dimensional general relativity. Hence, an interesting consequence of the brane-world scenario is in the nature of spherically-symmetric vacuum solutions to the brane gravitational field equations, which could represent black holes with properties quite distinct from those of ordinary black holes in four dimensions. Such black holes are likely to have very diverse cosmological and astrophysical signatures. In the present section, we consider the accretion properties of several brane-world black holes, which have been obtained by solving the vacuum gravitational field equations. There are many black-hole-type solutions on the brane, and in the following, we analyze three particular examples. In all cases, we assume that the inflowing gas obeys the polytropic equation of state.

A. The DMPR Brane-world Black Hole

The first brane-world black hole we consider is the solution of the vacuum field equations obtained by Dadhich, Maartens, Papadopoulos and Rezania \([11]\), which represents the simplest generalization of the Schwarzschild solution of general relativity. We call this type of brane-world black hole as the DMPR black hole. For this solution, the metric tensor components are given by

\[
e^\nu = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{Q}{r^2},
\]

where \(Q\) is the so-called tidal charge parameter. In the limit \(Q \to 0\), we recover the usual general relativistic case. The metric is asymptotically flat, with \(\lim_{r \to \infty} \exp(\nu) =\)
\[
\lim_{r \to \infty} \exp(\lambda) = 1. \text{ There are two horizons, which are given by }
\]
\[
r_h^{+, -} = m \pm \sqrt{m^2 - Q}.
\]
Both horizons lie inside the Schwarzschild horizon \( r_s = 2m \), \( 0 \leq r_h^- \leq r_h^+ \leq r_s \). In the brane-world models, there is also the possibility of a negative \( Q < 0 \), which leads to only one horizon \( r_{h+} \) lying outside the Schwarzschild horizon,
\[
r_{h+} = m + \sqrt{m^2 + Q} > r_s.
\]
In this case, the horizon has a greater area than its general relativistic counterpart, so that bulk effects act to increase the entropy and decrease the temperature and to strengthen the gravitational field outside the black hole.

For the matter inflowing onto the black hole, we adopt again the equations of state given by Eqs. (16), (17) and (18). For the velocities at the sonic point, we immediately obtain
\[
u^2_c = \frac{1}{2} \left( \frac{m}{r_c^2} - \frac{Q}{r_c^2} \right),
\]
and
\[
V_c^2 = \frac{u_c^2}{1 - 3u_c^2 - Q/r_c^2},
\]
respectively. With the use of Eq. (20), we obtain the temperature of the gas at the sonic point as
\[
T_{pc} = \frac{nu_c^2}{(n + 1) \left[ 1 - (3 + n) u_c^2 - Q/r_c^2 \right]} = \frac{n (m/r_c - Q/r_c^2)}{(n + 1) \left[ 2 - (3 + n)m/r_c + (1 + n)Q/r_c^2 \right]}. \tag{59}
\]
For the adopted metric, the equations describing the steady flow of the matter onto the black hole are given by
\[
\left( \frac{c^2}{K} \right)^n T_p^m u r^2 = D_1, \tag{60}
\]
and
\[
[1 + (n + 1) T_p]^2 \left( 1 - \frac{2m}{r} + \frac{Q}{r^2} + u^2 \right) = D_3, \tag{61}
\]
respectively, with \( D_1 \) and \( D_3 \) being constants of integration. Evaluating Eqs. (60) and (61) at \( r = r_\infty \), where \( e^\nu = e^{-\lambda} \approx 1 \) and the velocity is \( u(r_\infty) = u_\infty \), gives
\[
D_1 = \left( \frac{c^2}{K} \right)^n T_{p\infty}^n u_\infty r_\infty^2, \tag{62}
\]
and
\[ D_3 = [1 + (n + 1) T_{p\infty}]^2 (1 + u_\infty^2), \] (63)
respectively. Therefore, the two algebraic equations of motion of the gas in the unknowns \( T_p \) and \( u \) are
\[ T_p^n u^2 = T_{p\infty}^n u_\infty^2 r_\infty^2, \] (64)
and
\[ [1 + (n + 1) T_p]^2 \left( 1 - \frac{2m}{r} + \frac{Q}{r^2} + u^2 \right) = [1 + (n + 1) T_{p\infty}]^2 (1 + u_\infty^2). \] (65)

By assuming that \( T_p \ll 1 \) and \( u_\infty \ll 1 \), Eq. (65) immediately gives
\[ u^2(r) \approx \frac{2m}{r} - \frac{Q}{r^2}. \] (66)

By substituting this expression of the velocity into Eq. (64), we obtain the temperature profile of the gas as
\[ T_p(r) \approx T_{p\infty} \frac{(u_\infty r_\infty^2)^{1/n}}{r^{1/n} (2m)^{1/2n} (r - Q/2m)^{1/2n}}. \] (67)

The density of the gas varies as a function of the radial distance as
\[ \rho(r) \approx \frac{\rho_0}{r \sqrt{r - Q/2m}}, \] (68)
where we have denoted
\[ \rho_0 = \left( \frac{c^2}{K} \right)^n T_{p\infty}^n \frac{u_\infty^2 r_\infty^2}{\sqrt{2m}}. \] (69)

At the sonic point, the speed of sound \( a_c \) is given by
\[ a_c = \frac{c u_c}{\sqrt{1 - (3 + n) u_c^2 - Q/r_c^2}} = \frac{c \sqrt{m/r_c - Q/r_c^2}}{\sqrt{2 - (3 + n)m/r_c + (1 + n)Q/r_c^2}}. \] (70)

From an observational point of view, it is important to estimate the physical properties of the gas at the event horizon. At the event horizon, the gas is moving at the speed of light, \( u_h = u(r_h) \approx 1 \). By taking \( r = r_h^+ \), we obtain for the surface temperature of the black hole
\[ T_p \left( r_h^+ \right) \approx T_{p\infty} \frac{(u_\infty r_\infty^2)^{1/n}}{(m + \sqrt{m^2 \pm Q})^{2/n}}. \] (71)
The density of the gas traveling through the event horizon is given by

$$\rho (r_h) \approx \left( \frac{c^2}{K} \right)^n \frac{T_{p,\infty} u_{\infty} r_{\infty}^2}{(m + \sqrt{m^2 \pm Q})^2}. \tag{72}$$

On a qualitative level, the DMPR brane-world black hole displays all the typical accretion properties of the standard general relativistic black holes. In particular, the speed of the gas at the event horizon equals the speed of light, and this value is independent of the mass of the black hole. The temperature and the density of the gas at the event horizon are modified due to the presence of the tidal charge $Q$.

B. The CFM Brane-world Black Hole

Two families of analytic solutions in the brane-world model, which are parameterized by the ADM mass and the PPN parameters $\beta$ and $\gamma$ and which reduce to the Schwarzschild black hole for $\beta = 1$, have been found by Casadio, Fabbri, and Mazzacurati \cite{13}. We call the corresponding brane-world black holes as the CFM black holes.

The first class of solutions is given by

$$e^\nu = 1 - \frac{2m}{r}, \tag{73}$$

and

$$e^\lambda = \frac{1 - \frac{3m}{r}}{\left(1 - \frac{2m}{r}\right) \left[1 - \frac{3m}{2r} \left(1 + \frac{4}{9} \eta \right)\right]}, \tag{74}$$

respectively, where $\eta = \gamma - 1 = 2(\beta - 1)$. As in the Schwarzschild case, the event horizon is located at $r = r_h = 2m$. The solution is asymptotically flat; that is, $\lim_{r \to \infty} e^\nu = e^{\nu,\infty} = \lim_{r \to \infty} e^\lambda = e^{\lambda,\infty} = 1$. Then, Eqs. \((12), (13), \text{ and } (35)\) give the velocities at the sonic point as

$$u^2_c = \frac{m}{2r_c} \frac{1 - \frac{3m}{2r_c} \left(1 + \frac{4}{9} \eta \right)}{1 - \frac{3m}{2r_c}}, \tag{75}$$

$$V^2_c = \frac{m}{2r_c - 3m}, \tag{76}$$

and

$$\frac{a^2_c}{c^2} = \frac{m}{2r_c - (n + 3)m}. \tag{77}$$
The temperature of the gas at the sonic point is
\[
T_{pc} = \frac{nm}{(n+1)[2r_c - (n+3)m]} = \frac{n}{n+1} \frac{a_c^2}{c^2}.
\] (78)

The velocity, temperature, and density profiles of the inflowing gas are represented by
\[
u(r) \approx \sqrt{\frac{2m}{r}} \sqrt{\frac{1 - \frac{3m}{2r}(1 + \frac{4}{3}\eta)}{1 - \frac{3m}{r}}},
\] (79)
\[
T_{p}(r) \approx T_{p\infty} \left(\frac{u_{\infty}r_{\infty}^2}{(2m)^{1/2n}r^{3/2n}}\right)^{1/n},
\] (80)
and
\[
\rho(r) \approx \left(\frac{c^2}{K}\right)^n \frac{T_{p\infty}^n u_{\infty}r_{\infty}^2}{\sqrt{2m}r^{3/2}},
\] (81)
respectively. At the event horizon,
\[
u_h \approx \sqrt{1 - \frac{4}{3}\eta}, T_{ph} \approx T_{p\infty} \left(\frac{u_{\infty}r_{\infty}^2}{(2m)^{2/n}}\right)^{1/n},
\] (82)
\[
\rho_h \approx \left(\frac{c^2}{K}\right)^n \frac{T_{p\infty}^n u_{\infty}r_{\infty}^2}{(2m)^2}.
\] (83)

The accretion properties of this brane-world black hole are very similar to the standard general relativistic ones. The temperature and the density distribution of the gas at the event horizon are the same as in the Schwarzschild case. The speed of the gas at the event horizon is very slightly modified by a term proportional to the small parameter \(\eta\).

The second class of solutions corresponding to brane-world black holes \[13\] has the metric tensor components given by
\[
e^\nu = \left[\eta + \sqrt{1 - \frac{2m}{r}(1 + \eta)}\right]^2, \] (84)
and
\[
e^\lambda = \left[1 - \frac{2m}{r}(1 + \eta)\right]^{-1}, \] (85)
respectively. The metric is asymptotically flat. In the case \(\eta > 0\), the only singularity in the metric is at \(r = r_0 = 2m(1 + \eta)\), where all the curvature invariants are regular.
$r = r_0$ is a turning point for all physical curves. For $\eta < 0$, the metric is singular at $r = r_h = 2m/(1 - \eta)$ and at $r_0$, with $r_h > r_0$. $r_h$ defines the event horizon.

For this brane-world black hole, the velocities at the sonic point are given by

$$u_c^2 = \frac{m(1 + \eta)}{2r_c} \sqrt{\frac{1 - \frac{2m}{r_c}(1 + \eta)}{1 + \frac{2m}{r_c}(1 + \eta)}}, \quad (86)$$

$$V^2 = \frac{m(1 + \eta)}{2r_c \left[ 1 + \eta \sqrt{1 - \frac{2m}{r_c}(1 + \eta)} \right] - 3m(1 + \eta)}, \quad (87)$$

$$a_c = c \frac{m(1 + \eta)}{\sqrt{2r_c \left[ 1 + \eta \sqrt{1 - \frac{2m}{r_c}(1 + \eta)} \right] - (n + 3)m(1 + \eta)}}. \quad (88)$$

The sonic temperature is $T_{pc} = na_c^2/(n+1)c^2$. The velocity, temperature and density profiles are

$$u(r) \approx (1 + \eta) \sqrt{\frac{1 - \left[ \eta + \sqrt{1 - \frac{2m}{r}(1 + \eta)} \right]^2}{1 + \eta}} \sqrt{\frac{1 - \frac{2m}{r}(1 + \eta)}{\eta + \sqrt{1 - \frac{2m}{r}(1 + \eta)}}}, \quad (89)$$

$$T_p(r) \approx T_{p\infty} \frac{(u_\infty r_\infty^2)^{1/n}(1 + \eta)^{1/n}}{r^{2/n} \left\{ (1 + \eta)^2 - \left[ \eta + \sqrt{1 - \frac{2m}{r}(1 + \eta)} \right]^2 \right\}^{1/2n}}, \quad (90)$$

$$\rho(r) \approx \left( \frac{c^2}{K} \right)^n T_{p\infty}^{n} \frac{(u_\infty r_\infty^2)(1 + \eta)}{r^2 \sqrt{(1 + \eta)^2 - \left[ \eta + \sqrt{1 - \frac{2m}{r}(1 + \eta)} \right]^2}}. \quad (91)$$

In order to estimate the physical quantities at the surface of the black hole, we have to consider separately the cases $\eta > 0$ and $\eta < 0$. For $\eta > 0$ at the singular point $r_0$, we obtain

$$u_0 = u(r_0) \approx 0, \eta > 0 \quad (92)$$

$$T_{p0} = T_p(r_0) = T_{p\infty} \frac{(u_\infty r_\infty^2)^{1/n}}{(2m)^{2/n}(1 + \eta)^{1/n}(1 + 2\eta)^{1/2n}}, \eta > 0, \quad (93)$$

$$\rho_0 = \rho(r_0) \approx \left( \frac{c^2}{K} \right)^n T_{p\infty}^{n} \frac{u_\infty r_\infty^2}{(2m)^{2}/(1 + \eta)(1 + 2\eta)^{1/2}}, \eta > 0. \quad (94)$$

For this brane-world black hole, the four-velocity of the inflowing gas at the singular point $r = r_0$ is zero. At this point, all the curvature invariants are regular \[13\]. However, the
temperature and the density distribution of the gas at the singularity are very similar to the Schwarzschild black hole case.

For \( \eta < 0 \), the physical parameters at the event horizon have the values

\[
u_h \to \infty, \eta < 0, \quad (95)\]

\[
T_{ph} \approx T_{p\infty} \frac{(u_{\infty} r_{\infty}^{2/3})^{1/n} (1 + \eta)^{2/n}}{(2m)^{2/n}}, \eta < 0, \quad (96)\]

\[
\rho_h \approx \left( \frac{e^2}{K} \right)^{n} T_{p\infty} \frac{u_{\infty} r_{\infty}^{2} (1 + \eta)^{2}}{(2m)^{2}}, \eta < 0. \quad (97)\]

For this brane-world black hole model, the four-velocity of the gas diverges at the event horizon. A typical trajectory approaching and possibly entering the horizon is such that the physical radius always decreases (as in the Schwarzschild case) and hits the singularity at \( r_h > 0 \) \[13\]. However, the divergence of the gas velocity at \( r_h \) may indicate that the case \( \eta < 0 \) is unphysical.

C. The BMD Brane-world Black Hole

Several classes of brane world black hole solutions have been obtained by Bronnikov, Melnikov, and Dehnen \[17\] (for short the BMD black holes). In the following, we analyze the accretion properties of a particular class of these models, with a metric given by

\[
e^{-\nu} \left( 1 - \frac{2m}{r} \right)^{2/s} e^{\lambda} = \left( 1 - \frac{2m}{r} \right)^{-2}, \quad (98)\]

where \( s \in \mathbb{N} \). The metric is asymptotically flat, and at \( r = r_h = 2m \), these solutions have a double horizon.

For the sonic velocities and temperature, we obtain

\[
u_c^2 = \frac{1}{s r_c} \left( 1 - \frac{2m}{r_c} \right), \quad (99)\]

\[
V_c^2 = \frac{m \left( 1 - \frac{2m}{r_c} \right)}{s r_c + m \left( 1 - \frac{2m}{r_c} \right)}, \quad (100)\]
The velocity, temperature, and density profiles are given by

\[
T_p(r) \approx T_{p\infty} \frac{(u_{\infty} r_{\infty}^2)^{1/n}}{r^{2/n} \left[1 - (1 - 2m/r)^{2/s}\right]^{1/2}},
\]

\[
\rho(r) \approx \left(\frac{c^2}{K}\right)^n T_{p\infty}^n \frac{u_{\infty} r_{\infty}^2}{r^2 \left[1 - (1 - 2m/r)^{2/s}\right]^{1/2}}.
\]

For the physical quantities at the event horizon \( r = r_h = 2m \), we obtain

\[
u_h \approx 1, s = 1,
\]

\[
u_h \approx 0, s > 1,
\]

and

\[
T_{ph} \approx T_{p\infty} \frac{(u_{\infty} r_{\infty}^2)^{1/n}}{(2m)^{2/n}}, \rho_h \approx \left(\frac{c^2}{K}\right)^n T_{p\infty}^n \frac{u_{\infty} r_{\infty}^2}{(2m)^2}.
\]

For these classes of brane-world black holes, the temperature and the density distribution of the gas at the event horizon is identical to the Schwarzschild case. The behavior of the four-velocity at \( r_h \) depends on the value of \( s \). For \( s = 1 \), the gas four-velocity tends to the speed of light while for \( s > 1 \), it tends to zero.

**VI. DISCUSSIONS AND FINAL REMARKS**

In the present paper, we have considered the basic physical properties of matter forming a thin accretion disc in the space-time metric of brane-world black holes. The basic physical parameters of the inflowing gas—the temperature, density and velocity profiles—have been explicitly obtained for several classes of brane-world black holes and for several values of the parameters characterizing the vacuum solution of the generalized field equations in the
brane-world models. All the astrophysical quantities related to the observable properties of the accretion processes can be obtained from the black hole metric.

Due to the differences in the space-time structure, the brane-world black holes present some important differences with respect to the accretion properties, as compared to the standard general relativistic Schwarzschild case. Therefore, the study of accretion processes by compact objects is a powerful indicator of their physical nature. Since the temperature and the density distributions, as well as the location of the sonic points and the velocity distributions, in the case of the brane-world black holes are different from those for the standard general relativistic case, the astrophysical determination of these physical quantities could discriminate, at least in principle, between the different brane-world models and give some constrains on the existence of extra dimensions.

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