Multilinear algebra in the context of diffeology

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Abstract

This note is dedicated to the details of multilinear algebra on diffeological vector spaces; most of them are the to-be-expected corollaries of standard constructions and various facts of diffeology collected elsewhere. Most of the attention is paid to the notion of the diffeological dual and the implications of the resulting absence of isomorphism-by-duality.

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Introduction

The aim of this work is rather modest; it is intended as a (self-contained) supplement to [3], detailing the basics of multilinear algebra on diffeological vector spaces. Most of what is to be found here is rather simple and easily follows from the ingredients already collected in the excellent and comprehensive source [2]; the main idea is to spell out the definitions and their motivations that stem from the intention to extend the notion of the Riemannian metric to the diffeological context. It is necessary to say at this point that the very notion of diffeological tangent bundle has only just appeared in the very recent [1], as the internal tangent bundle with the des diffeology; this seems to be the first notion of a diffeological tangent bundle where each fibre is a diffeological vector space (the property that other notions of a tangent bundle of a diffeological space do not generally enjoy; do note that there are other approaches, some of which are summarized in [1], Section 3.4, and also references therein). The appearance of this new notion makes the attempt to define, by analogy with the notion of a Riemannian metric, a kind of diffeological metric a reasonable one; obviously, in order to carry it out, the main issue is to construct a suitable diffeological bundle such that its smooth sections yield, at each point, a scalar product on the internal tangent space at the point (and include all “reasonable” collections of scalar products on internal tangent bundle...)

What this latter request means precisely, is the motivation for the present paper. Indeed, again by analogy with Riemannian metrics, it is natural to look for a bundle whose fibres be covariant 2-tensors on the fibres of the internal tangent space. This requires, as a preliminary, the construction of the proper internal cotangent bundle, that is, with fibres the duals of internal tangent spaces; and at this point, as we will see below, the diffeological setting diverges from the usual smooth one (at least a priori), by virtue of the fact that the usual dual of a vector space may not be the same as the diffeologically smooth dual of a diffeological vector space; other similar phenomena follow. This, in brief, is what calls for putting down the details of multilinear algebra on diffeological vector spaces, spelling out what diffeology the dual space or the tensor product must have... None of this claims to be particularly original (it is all easily assembled from the ingredients already known); perhaps the main point is to make, and explain the choices made, in reference to the case of those diffeological vector spaces that can appear as fibres of a diffeological tangent bundle as opposed to the a priori case of an abstract diffeological vector space, and to provide a suitable selection of examples whenever could appear desirable.

Finally, we shall note that a description of a tensor bundle construction for diffeological spaces is available in [6], in a rather concise form from the multilinear algebra point of view. We make suitable references/comparisons as we go along, possibly reminding the notions defined therein.

¹The description given in [6] applies essentially to the so-called fine diffeological vector space (see Section 2); as is shown in [1], those are not sufficient for the study of tangent spaces.
The content  Our final aim is to obtain constructions where the result be a diffeological vector space with a “reasonable” diffeology. The measure of such reasonableness is a compatibility with the known properties of (fibres of) internal tangent bundles of diffeological spaces, considered with the *dvs diffeology* $^1$ All constructions come from the “elementary” diffeologies described in detail in $^2$, Chapter 1, and recalled in Section 1, that are applied to the usual constructions of multilinear algebra. The material thus collected is simple but probably (in part) new (at least I was not able to find a reference where it be spelled out). As already mentioned, some of the definitions were given in $^3$, Section 2.3.

Other considerations that influence the content of the paper are, for many if not all constructions of multilinear algebra there is more than one way to carry them out; each of these ways finds its own reflection in elementary constructions of diffeology. From the linear algebra point of view all these different ways to define, let’s say, the tensor product are equivalent; but would the same be true also for diffeologies at which we arrive by these different methods? And if not, what does the difference depend on, and most importantly, which is the most reasonable one to prefer? $^3$

Finally, it is mentioned in the preface to Chapter 3 of $^2$ that many of the possible connections between the notion of a diffeological space and various kinds of vector spaces that naturally appear in (linear algebra) applications have not yet been explored; by providing whenever possible “characteristic” examples (even if some of them are aprioristic rather than practical), we hope to give a small contribution to filling that void.

The structure  In the first two sections we collect the necessary notions regarding *diffeological structures* (in the first section) and *diffeological vector spaces* (in the second one). The other three sections are where the main content lies; therein, we describe the typical constructions of multilinear algebra from the diffeological point of view. In particular, we concentrate on the difference between the diffeological dual and the usual dual, and the consequences thereof; and discuss the (im)possibility to carry over into the diffeological setting some of the classic equivalences of multilinear algebra (such as various ways to define the tensor product, etc).

Acknowledgments  I came across the notion of diffeology only recently (but I do have to thank xxx.lanl.gov for that), but the time I’ve spent wondering the ways of mathematical research will soon constitute a half of my total lifetime. This might in part explain why I feel this work owes a lot to a colleague of mine who is not a mathematician and who, at the time of these words being written, has no idea that it exists. The name of my colleague is Prof. Riccardo Zucchi; what this work owes to him is inspiration and motivation, first of all those needed to be brave enough as to invest time and effort into something destined to be imperfect, and to do what, correctly or not, seems right or necessary at one particular moment only, independently of what the next step would be.

1 Background on diffeological spaces

This section is devoted to a short background on diffeological spaces, introducing the concepts that we will need in what follows.

The concept  We start by giving the basic definition of a diffeological space, following it with the definition of the *standard diffeology* on a smooth manifold; it is this latter diffeology that allows for the natural inclusion of smooth manifolds in the framework of diffeological spaces.

Definition 1.1. ($^5$) A *diffeological space* is a pair $(X, D_X)$ where $X$ is a set and $D_X$ is a specified collection of maps $U \to X$ (called plots) for each open set $U$ in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ the following three conditions are satisfied:

2In brief, the total space of the internal tangent bundle is the disjoint union of all the internal tangent spaces; the dvs diffeology is the smallest diffeology such that the corresponding sub-diffeology on each fibre makes it a diffeological vector space.

3We do not really endeavour to give a precise answer to the latter question; rather, we are interested in spelling out explicitly all the different versions, perhaps indicating which one seems preferable in some abstract sense. The true answer (if there exists the true answer) to the question, which one is better, would depend on the context, which the present paper does not provide.
1. (The covering condition) Every constant map \( U \to X \) is a plot;

2. (The smooth compatibility condition) If \( U \to X \) is a plot and \( V \to U \) is a smooth map (in the usual sense) then the composition \( V \to U \to X \) is also a plot;

3. (The sheaf condition) If \( U = \bigcup_i U_i \) is an open cover and \( U \to X \) is a set map such that each restriction \( U_i \to X \) is a plot then the entire map \( U \to X \) is a plot as well.

Typically, we will simply write \( X \) to denote the pair \((X, D_X)\). Such \( X \)'s are the objects of the category of diffeological spaces; naturally, we shall define next the arrows of the category, that is, say which maps are considered to be smooth in the diffeological sense. The following definition says just that.

**Definition 1.2.** ([5]) Let \( X \) and \( Y \) be two diffeological spaces, and let \( f : X \to Y \) be a set map. We say that \( f \) is smooth if for every plot \( p : U \to X \) of \( X \) the composition \( f \circ p \) is a plot of \( Y \).

As is natural, we will call an isomorphism in the category of diffeological spaces a diffeomorphism. The typical notation \( C^\infty(X, Y) \) will be used to denote the set of all smooth maps from \( X \) to \( Y \).

**The standard diffeology on a smooth manifold** Every smooth manifold \( M \) can be canonically considered a diffeological space with the same underlying set, if we take as plots all maps \( U \to M \) that are smooth in the usual sense. With this diffeology, the smooth (in the usual sense) maps between manifolds coincide with the maps smooth in the diffeological sense. This yields the following result (see Section 4.3 of [2]).

**Theorem 1.3.** There is a fully faithful functor from the category of smooth manifolds to the category of diffeological spaces.

**Comparing diffeologies** Given a set \( X \), the set of all possibile diffeologies on \( X \) is partially ordered by inclusion (with respect to which it forms a complete lattice). More precisely, a diffeology \( D \) on \( X \) is said to be finer than another diffeology \( D' \) if \( D \subseteq D' \) (whereas \( D' \) is said to be coarser than \( D \)). Among all diffeologies, there is the finest one, which turns out to be the natural discrete diffeology and which consists of all locally constant maps \( U \to X \); and there is also the coarsest one, which consists of all possible maps \( U \to X \), for all \( U \subseteq \mathbb{R}^n \) and for all \( n \in \mathbb{N} \). It is called the coarse diffeology (or indiscrete diffeology by some authors).

**Generated diffeology and quotient diffeology** One notion that will be crucial for us is the notion of a so-called generated diffeology. Specifically, given a set of maps \( A = \{ U_i \to X \}_{i \in I} \), the diffeology generated by \( A \) is the smallest, with respect to inclusion, diffeology on \( X \) that contains \( A \). It consists of all maps \( f : V \to X \) such that there exists an open cover \( \{ V_j \} \) of \( V \) such that \( f \) restricted to each \( V_j \) factors through some element \( U_i \to X \) in \( A \) via a smooth map \( V_j \to U_i \). Note that the standard diffeology on a smooth manifold is generated by any smooth atlas on the manifold, and that for any diffeological space \( X \), its diffeology \( D_X \) is generated by \( \cup_{n \in \mathbb{N}} C^\infty(\mathbb{R}^n, X) \).

Note that one useful property of diffeology as concept is that the category of diffeological spaces is closed under taking quotients. To be more precise, let \( X \) be a diffeological space, let \( \cong \) be an equivalence relation on \( X \), and let \( \pi : X \to Y := X/\cong \) be the quotient map. The quotient diffeology ([2]) on \( Y \) is the diffeology in which \( p : U \to Y \) is the diffeology in which \( p : U \to Y \) is a plot if and only if each point in \( U \) has a neighbourhood \( V \subseteq U \) and a plot \( \tilde{p} : V \to X \) such that \( p|_V = \pi \circ \tilde{p} \).

**Sub-diffeology and inductions** Let \( X \) be a diffeological space, and let \( Y \subseteq X \) be its subset. The sub-diffeology on \( Y \) is the coarsest diffeology on \( Y \) making the inclusion map \( Y \hookrightarrow X \) smooth. It consists of all maps \( U \to Y \) such that \( U \to Y \hookrightarrow X \) is a plot of \( X \). This definition allows also to introduce the following useful term: for two diffeological spaces \( X, X' \) a smooth map \( f : X' \to X \) is called an induction if it induces a diffeomorphism \( X \to \text{Im}(f) \), where \( \text{Im}(f) \) has the sub-diffeology of \( X \).
Pushforward of diffeology For any diffeological space $X$, any set $X'$, and any map $f : X \to X'$ there exists a finest diffeology on $X'$ that makes the map $f$ smooth. It is this diffeology that is called the pushforward of the diffeology of $X$ by the map $f$; it is denoted by $f_*(\mathcal{D})$ where $\mathcal{D}$ stands for the diffeology of $X$. The pushforward diffeology can be characterized as follows. A map $f' : U \to X'$ defined on a domain $U$ is a plot of $f_*(\mathcal{D})$ if and only if it satisfies the following condition: for every $r \in U$ there exists an open neighbourhood $V$ of $r$ such that either $f'|_V$ is a constant map or there exists a plot $g : V \to X$ such that $f'|_V = f \circ g$.

Pullbacks of diffeologies Let $X$ be a set, let $X'$ be a diffeological space with diffeology $\mathcal{D}'$, and let $f : X \to X'$ be a map. The pullback of the diffeology $\mathcal{D}'$ by the map $f$ is the coarsest diffeology on $X$ such that $f$ is smooth; this pullback diffeology is usually denoted by $f^*(\mathcal{D}')$. Note that $p : U \to X$ is a plot for $f^*(\mathcal{D}')$ if and only if $f \circ p$ is a plot for $\mathcal{D}'$.

Subductions Let $X$ and $X'$ be two diffeological spaces, and let $f : X \to X'$ be some map; this map is said to be a subduction if it satisfies the following conditions: 1) it is surjective, and 2) the diffeology $\mathcal{D}'$ of $X'$ is the pushforward of the diffeology $\mathcal{D}$ of $X$, i.e. $\mathcal{D}' = f_*(\mathcal{D})$. An equivalent description of what it means that $f$ be a subduction is, $f$ must be a smooth surjection such that for every plot $g' : U \to X'$ and for every $x \in U$ there exist an open neighbourhood $V$ of $x$ and a plot $g : V \to X$ such that $g'|_V = f \circ g$.

Sums of diffeological spaces Let $\{X_i\}_{i \in I}$ be a collection of diffeological spaces, with $I$ being some set of indices. The sum, or the disjoint union, of $\{X_i\}_{i \in I}$ is defined as

$$X = \coprod_{i \in I} X_i = \{(i, x) | i \in I \text{ and } x \in X_i\}.$$  

The sum diffeology on $X$ is the finest diffeology such that the natural injections $X_i \to \coprod_{i \in I} X_i$ are smooth for each $i \in I$. The plots of this diffeology are maps $U \to \coprod_{i \in I} X_i$ that are locally plots of one of the components of the sum.

The diffeological product Let, again, $\{X_i\}_{i \in I}$ be a collection of diffeological spaces, and let $\mathcal{D}_i$, $i \in I$, be their respective diffeologies. The product diffeology $\mathcal{D}$ on the product $X = \prod_{i \in I} X_i$ is the coarsest diffeology such that for each index $i \in I$ the natural projection $\pi_i : \prod_{i \in I} X_i \to X_i$ is smooth.

Functional diffeology Let $X$, $Y$ be two diffeological spaces, and let $C^\infty(X, Y)$ be the set of smooth maps from $X$ to $Y$. Let $ev$ be the evaluation map, defined by

$$ev : C^\infty(X, Y) \times X \to Y \text{ and } ev(f, x) = f(x).$$

The words “functional diffeology” stand for any diffeology on $C^\infty(X, Y)$ such that the evaluation map is smooth; note, for example, that the discrete diffeology is a functional diffeology. However, they are typically used, and we also will do that from now on, to denote the coarsest functional diffeology.

2 Diffeological vector spaces

In this section we treat with some detail the notion of a diffeological vector space.

The concept and some basic constructions Let $V$ be a vector space over $\mathbb{R}$. The vector space diffeology on $V$ is any diffeology of $V$ such that the addition and the scalar multiplication are smooth, that is,

$$[(u, v) \mapsto u + v] \in C^\infty(V \times V, V) \text{ and } [(\lambda, v) \mapsto \lambda v] \in C^\infty(\mathbb{R} \times V, V),$$

where $V \times V$ and $\mathbb{R} \times V$ are equipped with the product diffeology. A diffeological vector space over $\mathbb{R}$ is any vector space $V$ over $\mathbb{R}$ equipped with a vector space diffeology.

In the diffeological context, we find all the usual constructions of linear algebra, such as spaces of (smooth) linear maps, products, subspaces, and quotients; we now describe these. First of all, given two
diffeological vector spaces $V$ and $W$, we can speak of the space of smooth linear maps between them; this space is denoted by $L^\infty(V,W)$ and is defined simply as:

$$L^\infty(V,W) = L(V,W) \cap C^\infty(V,W);$$

this is an $\mathbb{R}$-linear subspace of $L(V,W)$. Next, a subspace of a diffeological vector space $V$ is any vector subspace of $V$ endowed with the sub-diffeology. Finally, if $V$ is a diffeological vector space and $W \subseteq V$ is a subspace of it then the quotient $V/W$ is a diffeological vector space with respect to the quotient diffeology.

**Direct product of diffeological vector spaces** Let $\{V_i\}_{i \in I}$ be a family of diffeological vector spaces. Consider the usual direct product $V = \prod_{i \in I} V_i$ of this family; then $V$, equipped with the product diffeology, is a diffeological vector space.

**Euclidean structure on diffeological vector spaces** The notion of a Euclidean diffeological vector space does not differ much from the usual notion of the Euclidean vector space. A diffeological vector space $V$ is Euclidean if it is endowed with a scalar product that is smooth with respect to the diffeology of $V$ and the standard diffeology of $\mathbb{R}$; that is, if there is a fixed map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that has the usual properties of bilinearity, simmetricity, and definite-positiveness and that is smooth with respect to the diffeological product structure on $V \times V$ and the standard diffeology on $\mathbb{R}$.

**Fine diffeology on vector spaces** The fine diffeology on a vector space $\mathbb{R}$ is the finest vector space diffeology on it; endowed with such, $V$ is called a fine vector space. Note that any linear map between two fine vector spaces is smooth.

The fine diffeology admits a more or less explicit description of the following form: its plots are maps $f : U \to V$ such that for all $x_0 \in U$ there exist an open neighbourhood $U_0$ of $x_0$, a family of smooth maps $\lambda_\alpha : U_0 \to \mathbb{R}$, and a family of vectors $v_\alpha \in U_0$, both indexed by the same finite set of indices $A$, such that $f|_{U_0}$ sends each $x \in U_0$ into $\sum_{\alpha \in A} \lambda_\alpha(x)v_\alpha$:

$$f(x) = \sum_{\alpha \in A} \lambda_\alpha(x)v_\alpha \text{ for } x \in U_0.$$

A finite family $(\lambda_\alpha, v_\alpha)_{\alpha \in A}$, with $\lambda \in C^\infty(U_0, \mathbb{R})$ and $v_\alpha \in V$, defined on some domain $U_0$ and satisfying the condition just stated, is called a local family for the plot $f$.

Some generating sets for the fine diffeology can also be described explicitly. Let $L(\mathbb{R}^n, V)$ be the set of all linear maps from $\mathbb{R}^n$ into $V$, and let $L^*(\mathbb{R}^n, V)$ be the set of all injective linear maps from $\mathbb{R}^n$ into $V$. The following two families both generate the fine diffeology of $V$:

$$\mathcal{F} = \cup_{n \in \mathbb{N}} L(\mathbb{R}^n, V) \text{ and } \mathcal{F}^* = \cup_{n \in \mathbb{N}} L^*(\mathbb{R}^n, V).$$

**Example 2.1.** It is easy to see that $\mathbb{R}^n$ with the standard diffeology is a fine vector space. Note, furthermore, that any fine vector space is isomorphic to the standard $\mathbb{R}^n$ ([2], p. 71).

The following statement should be quite important for our discussion (as we will see in the next section):

**Lemma 2.2.** ([2], 3.9) Let $V$ be a fine diffeological vector space. Then for any other diffeological vector space $W$ every linear map from $V$ to $W$ is smooth, i.e.,

$$L^\infty(V,W) = L(V,W).$$

Finally, regarding the examples of diffeological vector spaces, we have already mentioned an obvious one, which is any $\mathbb{R}^n$ with the standard diffeology. We also observe a fact of which we will make use shortly, that any vector space endowed with the coarse diffeology is a diffeological vector space. We will consider various other examples, as we go along.
Existing notions of the tensor product There have already been treatments of the multilinear algebra in the diffeological context, for instance in [6], Section 2.3, which we briefly recall. Doing this requires the notion of the weak vector space diffeology, which is a generalization given in [6] of the above-mentioned fine vector space diffeology. For reasons of terminology, we recall the following definition given in [6].

**Definition 2.3.** ([6], Definition 2.2.1) Let \( V \) be a vector space, and let \( \mathcal{D}_V \) be a diffeology on its underlying set. The **weak vector space diffeology on \( V \)** generated by \( \mathcal{D}_V \) is the weakest diffeology generated by the collection of maps of form \( U \ni u \mapsto \sum_{i=1}^{n} \lambda_i(u)\gamma_i(u) \), i.e. finite sums with smooth functional coefficients of plots of \( \mathcal{D}_V \).

In other words, it is the finest vector space diffeology on \( V \) containing the given one.

**Definition 2.4.** ([6], Definition 2.3.2) Let \( V_1, \ldots, V_n \) be diffeological vector spaces, let \( V_1 \otimes \ldots \otimes V_n \) be their tensor product as vector spaces, and let \( \phi : V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n \) be the universal multilinear map. The **tensor product diffeology** on \( V_1 \otimes \ldots \otimes V_n \) is the weak vector space diffeology generated by the diffeology that is the pushforward by \( \phi \) of the product diffeology on \( V_1 \times \ldots \times V_n \).

Modifying a bit the notation in [6], we denote this diffeology by \( \sum \mathcal{D}_{V_i} \); it is then claimed ([6], Theorem 2.3.5) that, if \( F \) is another diffeological vector space, \( V_1 \otimes \ldots \otimes V_n \) is endowed with the diffeology \( \sum \mathcal{D}_{V_i} \), and \( V_1 \times \ldots \times V_n \) is endowed with the product diffeology, the space of all smooth linear maps \( V_1 \otimes \ldots \otimes V_n \to F \) is diffeomorphic to the space of all smooth multilinear maps \( V_1 \times \ldots \times V_n \to F \), these two spaces being considered each with the corresponding functional diffeology.

## 3 Smooth linear and bilinear maps

In this section we begin our treatment of the multilinear algebra on diffeological vector spaces; the first step is to consider the various possibilities to define the diffeological dual (this is actually done in the section that follows), which immediately imposes that we consider, more generally, the issue of linear maps and smooth linear maps. As we show, replacing the former with the latter makes *a priori* a significant difference.

### 3.1 Linear maps and smooth linear maps

The sometimes significant difference between the two notions mentioned in the title is illustrated by the following example; we present it immediately, so as to motivate the reasoning that follows.

**Example 3.1.** Let us see \( V \) such that \( L^\infty(V, \mathbb{R}) \) is a proper subspace of \( L(V, \mathbb{R}) \). Set \( V = \mathbb{R}^n \) equipped with the coarse diffeology; we claim that the only smooth linear map \( V \to \mathbb{R} \) is the zero map. Indeed, let \( f : V \to \mathbb{R} \) be a linear map; recall that, by definition, for \( f \) to be smooth, the composition \( f \circ p \) must be a plot of \( \mathbb{R} \) for any plot \( p \) of \( V \). What this means is that \( f \circ p \) must be a smooth map \( U \to \mathbb{R} \) for some domain \( U \) of some \( \mathbb{R}^k \); but by definition of the coarse diffeology, \( p \) is allowed to be any set map \( U \to V \), so it might not even be continuous. Already by this observation it is intuitively clear that we will find numerous plots \( p \) such that \( f \circ p \) is not smooth; but let us be precise.

Choose some basis \( \{v_1, \ldots, v_n\} \) of \( V = \mathbb{R}^n \), and a basis \( \{v\} \) of \( \mathbb{R} \). With respect to these, \( f \) is given by \( n \) real numbers, more precisely, by the matrix \( (a_1 \ldots a_n) \). Let us choose \( n \) specific plots, that we call \( p_i \) for \( i = 1, \ldots, n \), by setting \( p_i : \mathbb{R} \to V \) and \( p_i(x) = |x|v_i \); then \( (f \circ p_i)(x) = a_i |x|v \). The only way for this latter map to be smooth is to have \( a_i = 0 \); recalling again that if \( f \) is smooth then the composition \( f \circ p_i \) must be smooth for all \( i \), we conclude that we must have \( a_i = 0 \) for all \( i = 1, \ldots, n \), whence our claim.

As we have already indicated, one of the main issues (possibly, the main one) that presents itself when one tries carry the (multi)linear algebra over to the diffeological context is the issue of smoothness of linear maps: the fact (that we have already mentioned a few times) is that, for two arbitrary diffeological spaces \( V, W \), it might *a priori* occur that \( L^\infty(V, W) \prec L(V, W) \). True, this would require some rather surprising vector spaces/diffeologies for this happen; but diffeology was designed for dealing with surprising, or at

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*Obviously, the respective canonical bases would do the job just fine.*
least unusual from the Differential Geometry point of view, objects\(^5\) and so some “weird examples” should be welcome.

In fact, such surprises are easy to find, as we show by elaborating on the Example 3.1 and this we do in a rather obvious fashion. We stress that what comes in play here is the very essence of what diffeology aims to add to the “usual” setting of Differential Geometry; which is the flexibility of what can be called smooth. In particular, the fact that any given map \(\mathbb{R}^k \supset U \to X\) can be a plot for some diffeology on a given set \(X\) (namely, for the diffeology generated by this map plus, for example, some already existing diffeology on \(X\)) easily gives rise to some surprising instances of, say, diffeological vector spaces, as we have already seen in the Example 3.1 and as we discuss in more detail below.

**Example 3.2.** Once again, consider \(V = \mathbb{R}^n\) and some basis \(\{v_1, \ldots, v_n\}\) of \(V\); endow it with the vector space diffeology generated by all smooth maps plus the map \(p_i\) already mentioned, that is, the map \(p_i : \mathbb{R} \to V\) acting by \(p_i(x) = |x|v_i\). Let \(v\) be a generator of \(\mathbb{R}\) (i.e., any non-zero vector). Using the same reasoning as in Example 3.1 one can show that if \(f : V \to \mathbb{R}\) is linear and, with respect to the bases chosen has matrix \((a_1 \ldots a_n)\), then for it to be smooth we must have \(a_i = 0\); hence the (usual vector space) dimension \(L^\infty(V, \mathbb{R})\) (i.e., that of the diffeological dual of \(V\)) is at most \(n - 1\).

This reasoning can be further extended by choosing some natural number \(1 < k < n\) and a set of \(k\) indices \(1 \leq i_1 < i_2 < \ldots < i_k \leq n\), and endowing \(V\) with the vector space diffeology generated by all smooth maps plus the set \(\{p_{i_1}, \ldots, p_{i_k}\}\). Arguing as above, we can easily conclude that \(\dim(L^\infty(V, \mathbb{R}))\) is at most \(n - k\).

The examples just cited show that the diffeological dual can be much different from the usual one. Given the importance of the isomorphism-by-duality in the usual multilinear algebra, the implications of this difference call for some care; which is why below we try to explore them as thoroughly as possible.

### 3.2 Bilinear maps and smooth bilinear maps

In this section we consider the question analogous to the one considered for linear maps in the previous section: given two diffeological vector spaces, what is the difference between the set of all bilinear maps on one of them with values in the other, and the set of all such bilinear maps that in addition are smooth?

**Smooth bilinear maps** Let \(V, W\) be two diffeological spaces. From the linear algebra point of view, a \(W\)-valued bilinear map can be interpreted in two ways. One is the straightforward definition of it as a map \(V \times V \to W\) linear in each argument. In the diffeological context we restrict ourselves to the maps that are smooth (with respect to the product diffeology on \(V \times V\)), thus facing again the possibility that the set of smooth maps is strictly smaller than that of bilinear maps.

Let us first fix some notation. Given \(V, W\) two diffeological vector spaces, let \(B(V, W)\) be the set of bilinear maps on \(V\) with values in \(W\), and let \(B^\infty(V, W)\) be the set of those bilinear maps that are smooth with respect to the product diffeology on \(V \times V\) and the given diffeology on \(W\).

**Example 3.3.** The examples seen in the previous section provide readily the instances of \(V\) and \(W\) such that \(B^\infty(V, W)\) is a proper subspace of \(B(V, W)\). Indeed, let us take \(V = \mathbb{R}^n\) equipped with the coarse diffeology, and let \(W = \mathbb{R}\) considered with the standard diffeology. It does not take much to extend the reasoning of Example 3.1 to show that for these two spaces \(B^\infty(V, W) = 0\).

Once again, take a basis \(\{v_1, \ldots, v_n\}\) of \(V\) and a basis \(\{w\}\) of \(W\); let \(f \in B^\infty(V, W)\). Then with respect to the bases chosen \(f\) is defined by the matrix \((a_{ij})_{n \times n}\) where \(f(v_i, v_j) = a_{ij}w\). For each \(i = 1, \ldots, n\) consider the already-mentioned map \(p_i : \mathbb{R} \to V\) given by \(p_i(x) = |x|v_i\); this map is a plot of \(V\) by definition of the coarse diffeology (that includes all set maps from domains of various \(\mathbb{R}^k\) to \(V\)). Now call \(p_{ij}\) the product map \(p_{ij} : \mathbb{R} \to V \times V\), i.e. the map given by \(p_{ij}(x) = (p_i(x), p_j(x))\); it is obviously a plot for the product diffeology on \(V \times V\). Putting everything together, we get that \((f \circ p_{ij})(x) = a_{ij}|x|w\); recalling that for \(f\) to be smooth this composition must be a plot of \(\mathbb{R}\), which is equivalent to being smooth, we conclude

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\(^5\)This is a story beautifully told in the Preface and Afterword to the excellent book \(^6\).

\(^6\)One can actually show that \(f\) is smooth if and only if \(a_i = 0\), and so \(\dim(L^\infty(V, \mathbb{R}))\) is precisely \(n - 1\); we do not elaborate on this, since we mostly interested in showing that it can be strictly smaller (by any admissible value, as we see below).

\(^7\)Artificial as they may appear.
that \( a_{ij} = 0 \). The indices \( i, j \) being arbitrary, we conclude that the only way for \( f \) to be smooth is for it be the zero map, whence the conclusion.

The example just given stresses the importance of making a distinction between bilinear maps and smooth bilinear maps, showing that the two families can be \( (a \text{ priori}) \) quite different, and motivates the next paragraph.

**The function spaces** \( B^\infty(V, W) \) and \( L^\infty(V, L^\infty(V, W)) \)

As is well-known, in the usual setting each bilinear map can be viewed as a linear map \( V \to L(V, W) \). In the diffeological context, since \( a \text{ priori} \) we might have \( L^\infty(V, W) < L(V, W) \), we need to consider the question of whether any smooth bilinear map can be seen as a smooth map \( V \to L^\infty(V, W) \), where the latter is endowed with the functional diffeology. Indeed this follows rather easily from the definition of the product diffeology and that of the functional diffeology, as we now show, starting with the following lemma:

**Lemma 3.4.** Let \( V, W \) be two diffeological vector spaces, let \( f : V \times V \to W \) be a bilinear map smooth with respect to the product diffeology on \( V \times V \) and the given diffeology on \( W \), and let \( G : V \to L^\infty(V, W) \) be a linear map that is smooth with respect to the given diffeology on \( V \) and the functional diffeology on \( L^\infty(V, W) \). Then:

- for every \( v \in V \) the linear map \( F(v) : V \to W \) given by \( F(v)(v') = f(v, v') \) is smooth;
- the bilinear map \( g : V \times V \to W \) given by \( g(v, v') = G(v)(v') \) is smooth.

*Proof.* Let us prove the first statement. Fix a \( v \in V \). Recall that \( F(v) \) is smooth if and only if for every plot \( p : U \to V \) the composition \( F(v) \circ p \) is a plot of \( W \); recall also that \( \tilde{p} : U \to V \times V \) is a plot for product diffeology if and only if \( \pi_1 \circ \tilde{p} : U \to V \) and \( \pi_2 \circ \tilde{p} : U \to V \) are both plots of \( V \) (where \( \pi_1 \) and \( \pi_2 \) are the two natural projections).

Fix now a plot \( p : U \to V \) of \( V \); define \( \tilde{p} : U \to V \times V \) by setting \( \tilde{p}(x) = (v, p(x)) \) for all \( x \in V \). Observe that \( \tilde{p} \) is a plot for the product diffeology: indeed, \( \pi_1 \circ \tilde{p} \) is a constant map in \( V \) (and all such maps are plots of any diffeology by its definition), while \( \pi_2 \circ \tilde{p} = p \), which is a plot by assumption. It is obvious by construction that \( F(v) \circ p = f \circ \tilde{p} \); the latter map is a plot of \( W \) since \( f \) is smooth by assumption. Since \( p \) is arbitrary, this proves that \( F(v) \) is smooth.

To prove the second statement, it suffices to observe that \( g \) writes as the composition \( g = \text{ev} \circ (G \times \text{Id}_V) \); the map \( \text{Id}_V \) being obviously smooth, \( G \) being smooth by assumption, their product being smooth by definition of the product diffeology, and, finally, the evaluation map \( \text{ev} \) being smooth by the definition of the functional diffeology, we get the conclusion. \( \square \)

What the above lemma gives us are the following two maps:

- the map \( \tilde{F} : B^\infty(V, W) \to L(V, L^\infty(V, W)) \) that assigns to each \( f \in B^\infty(V, W) \) the map \( F \) of the lemma \( (i.e., \text{the specified map that to each } v \in V \text{ assigns the smooth linear map } F(v) : V \to W) \). Observe that \( F \) now writes as \( F = \tilde{F}(f) \) and that the following relation holds: \( f = \text{ev} \circ (F \times \text{Id}_V) \);

- the map \( \tilde{G} : L^\infty(V, L^\infty(V, W)) \to B^\infty(V, W) \) that assigns to each \( G \in L^\infty(V, L^\infty(V, W)) \) the map \( g = \text{ev} \circ (G \times \text{Id}_V) \). This latter map now writes as \( g = \tilde{G}(G) \).

Before going further, we cite the following statement, which we will use immediately afterwards:

**Proposition 3.5.** \([2], 1.57\) Let \( X, Y \) be two diffeological spaces, and let \( U \) be a domain of some \( \mathbb{R}^n \). A map \( p : U \to C^\infty(X, Y) \) is a plot for the functional diffeology of \( C^\infty(X, Y) \) if and only if the induced map \( U \times X \to Y \) acting by \( (u, x) \mapsto p(u)(x) \) is smooth.

We are now ready to prove the following lemma:

**Lemma 3.6.** The following statements hold:

1. The map \( \tilde{F} \) takes values in \( L^\infty(V, L^\infty(V, W)) \); furthermore, it is smooth with respect to the functional diffeologies of \( B^\infty(V, W) \) and \( L^\infty(V, L^\infty(V, W)) \).

2. The map \( \tilde{G} \) is smooth with respect to the functional diffeologies of \( L^\infty(V, L^\infty(V, W)) \) and \( B^\infty(V, W) \).
3. The maps $\tilde{F}$ and $\tilde{G}$ are inverses of each other.

**Proof.** 1. Let us first prove that $F : V \to L^\infty(V,W)$ is smooth. Consider an arbitrary plot $p : U \to V$; by definition of a smooth map, we need to show that $F \circ p$ is a plot for the functional diffeology on $L^\infty(V,W)$. Applying Proposition 3.5 to the composition $F \circ p$, we consider the induced map $U \times V \to W$ that acts by the assignment $(u,v') \mapsto (F \circ p)(u)(v') = F(p(u))(v') = f(p(u),v') = f \circ (p \times \text{Id}_V)(u,v')$.

Since $p \times \text{Id}_V$ is obviously a plot for the product diffeology on $V \times V$ and $f$ is smooth, $f \circ (p \times \text{Id}_V)$ is a plot of $W$, so it is naturally smooth. The Proposition then allows us to conclude that $F \circ p$ is a plot for $L^\infty(V,W)$, which, $p$ being arbitrary, means that $F$ is a smooth map.

Let us now show that $\tilde{F} : B^\infty(V,W) \to L^\infty(V,L^\infty(V,W))$ is smooth; taking $p : U \to B^\infty(V,W)$ a plot of $B^\infty(V,W)$, we need to show that $\tilde{F} \circ p$ is a plot of $L^\infty(V,L^\infty(V,W))$. To do this, we apply again Proposition 3.5 it suffices to consider the map $U \times V \to L^\infty(V,W)$ acting by $(u,v) \mapsto (\tilde{F} \circ p)(u)(v) = \tilde{F}(p(u))(v) = \text{Ev} \circ ((F \circ p) \times \text{Id}_V)(u,v)$. Having already established that $F$ is smooth, we can now conclude that $\tilde{F}$ is smooth as well.

2. Let us now prove that $\tilde{G} : L^\infty(V,L^\infty(V,W)) \to B^\infty(V,W)$ is smooth, i.e., taking an arbitrary plot $p : U \to L^\infty(V,L^\infty(V,W))$, we need to show that $\tilde{G} \circ p$ is a plot of $B^\infty(V,W)$. Applying again Proposition 3.5 we consider the map $U \times (V \times V) \to W$ defined by $(u,(v,v')) \mapsto (\tilde{G} \circ p)(u)(v,v') = (\text{Ev} \circ (p \times \text{Id}_V \times \text{Id}_V))(u,(v,v'))$, which allows us to conclude that the map is smooth, and therefore, $\tilde{G} \circ p$ is a plot of $B^\infty(V,W)$; whence the conclusion.

3. This follows immediately from the definitions of the two maps. 

We now get the desired conclusion, which does mimic what happens in the usual linear algebra case:

**Theorem 3.7.** Let $V$ and $W$ be two diffeological vector spaces, let $B^\infty(V,W)$ be the space of all smooth bilinear maps $V \times V \to W$ considered with the functional diffeology, and let $L^\infty(V,L^\infty(V,W))$ be the space of all smooth linear maps $V \to L^\infty(V,W)$ endowed, it as well, with the functional diffeology. Then the spaces $B^\infty(V,W)$ and $L^\infty(V,L^\infty(V,W))$ are diffeomorphic as diffeological vector spaces.

**Proof.** The desired diffeomorphism as diffeological spaces is given by the maps $\tilde{F}$ and $\tilde{G}$ of Lemma 3.6. It remains to note that these two maps are also linear (actually, as vector spaces maps they coincide with the usual constructions), and that all the functional diffeologies involved are vector space diffeologies.

4 The diffeological dual

In this section we consider the various ways to define the diffeological dual; this discussion stems from the previous section, which illustrates how the function spaces of linear maps change at the passage to smooth maps. The final notion remains the most natural one, but we discuss also other possibilities and their implications.

4.1 The dual as the set of smooth linear maps

We first discuss the most obvious notion of the dual of a diffeological vector space, obtained by adding “diffeological” (or “smooth”) wherever possible. Recall that for an arbitrary vector space $V$ its dual space $V^*$ is defined as the (vector) space of all linear maps $V \to \mathbb{R}$, that is, $V^* = L(V,\mathbb{R})$. Now suppose that $V$ is a diffeological vector space; then defining the dual in the usual manner just stated could a priori take us out of the category of diffeological spaces. The latter would occur in the case $L^\infty(V,\mathbb{R}) < L(V,\mathbb{R})$ (with $\mathbb{R}$ being considered with the standard diffeology). That this can actually occur is illustrated by Example 3.1.

On the other hand, the equality $L^\infty(V,\mathbb{R}) = L(V,\mathbb{R})$ does hold for fine diffeological vector spaces, so we should say why we do not restrict the discussion to those. This is due to our interest in tangent spaces of diffeological spaces, not all of which are fine (see Example 4.22).

What has just been said thus justifies the following definition:

**Definition 4.1.** Let $V$ be a diffeological vector space. The **diffeological dual** of $V$, denoted by $V^*$, is the set $L^\infty(V,\mathbb{R})$ of all smooth linear maps $V \to \mathbb{R}$. 

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*There are however ways to do it, of which we speak later on.*
Lemma 4.2. Let $V$ be a diffeological vector space. Then its diffeological dual $V^*$ equipped with the functional diffeology is a diffeological vector space.

Proof. As is shown in [2], Section 3.3, $V^*$ is a vector space; it is furthermore a subspace of both $L(V, \mathbb{R})$ and $C^\infty(V, \mathbb{R})$, the latter being a vector space with respect to the pointwise addition and multiplication by a scalar. By [2], Section 1.58, the functional diffeology on $V^*$ coincides with the sub-diffeology of the functional diffeology on $C^\infty(V, \mathbb{R})$, and, as mentioned in [2], pp. 66-67, the latter is a diffeological vector space. By 3.5, [2], we conclude that $V^*$ is a diffeological vector space.

[Proof]

Unless specified otherwise, the diffeological dual will always be considered with its functional diffeology.

Observation 4.3. As follows from the example above, the diffeological dual $V^*$ a priori is not diffeomorphic to $V$, even if the dimension of $V$ is finite. We stress that this occurs because in the case described $V^* = L^\infty(V, \mathbb{R})$ is a proper subspace of $L(V, \mathbb{R}) \cong V$, which obviously implies that it has a smaller dimension (zero in the case illustrated); so they are not isomorphic even as usual vector spaces.

It is natural to wonder at this point: suppose that $V$ is a (say, finite-dimensional) diffeological vector space such that $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$; does this imply that $V$ is also diffeomorphic to $V^* = L^\infty(V, \mathbb{R})$?

The following proposition provides a positive answer to this question.

Proposition 4.4. Let $V$ be a finite-dimensional diffeological vector space such that $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$, i.e., such that every real-valued linear map from $V$ is smooth. Then $V$ is diffeomorphic to $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$, i.e., to its diffeological dual.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$, and let $\pi_i : V \to \mathbb{R}$ the projection of $V$ onto the $i$-th coordinate for all $i = 1, \ldots, n$ (i.e., if $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ then $\pi_i(v) = \alpha_i$). Note that each $\pi_i$, being a linear map from $V$ to $\mathbb{R}$, is smooth by assumption.

Observe, furthermore, that the following map is smooth: $\tilde{\pi}_1 : V \times V \to \mathbb{R}$ given by $\tilde{\pi}_1(v_1, v_2) = \pi_1(v_1)\pi_1(v_2)$, the usual product of $\pi_1$ by itself; this follows from the definition of the product diffeology. Indeed, let $p : U \to V \times V$, written as $p(x) = (p_1(x), p_2(x))$, be a plot of $V \times V$; we need to show that $\tilde{\pi}_1 \circ p$ is a plot of $\mathbb{R}$, that is, that it is a usual smooth map $U \to \mathbb{R}$. Observe that each of $p_1, p_2$ is a smooth map $U \to V$, hence each of $\pi_1 \circ p_1, \pi_1 \circ p_2$ is a smooth map $U \to \mathbb{R}$. The diffeology of $\mathbb{R}$ being the standard one, the usual product map is smooth. This product map being $\tilde{\pi}_1 \circ p$, we get the conclusion.

Let us now prove the main statement. Take the dual basis $\{v^1, \ldots, v^n\}$ of $V^*$ and define $F : V \to V^*$ by setting $F(v_i) = v^i$ for each $i$ and extending by linearity. The map $F$ is obviously an isomorphism of vector spaces; we need to show that it is also smooth. To do this, take an arbitrary plot $p : U \to V$ of $V$; we need to show that $F \circ p$ is a plot for the functional diffeology of $V^* = L^\infty(V, \mathbb{R})$. By Proposition 3.5, this is equivalent to the induced map $U \times V \to \mathbb{R}$, given by $(u, v) \mapsto F(p(u))(v)$, being smooth. By the definition of $F$, we have $F(p(u))(v) = \sum_{i=1}^n \pi_i(p(u))\pi_i(v) = \sum_{i=1}^n \tilde{\pi}_i(p \times \text{Id}_V)(u, v)$. Since $p \times \text{Id}_V$ is smooth by definition of the product diffeology (on $U \times V$) and $\tilde{\pi}_i$ has already been shown to be smooth, we conclude that $F \circ p$ is smooth, which proves the proposition.

Observe that the definition of the diffeological dual given in [6] coincides with ours; but the discussion in [6] does not deal with the issue of the existence of diffeological spaces such that not every linear real-valued map be smooth: $L^\infty(V, \mathbb{R}) < L(V, \mathbb{R})$ (or, more generally, that $L^\infty(V, W) < L(V, W)$; see Example 5.1).

In the following two sections we turn to considering other a priori possibilities for defining the diffeological dual; in particular, we explore those alternatives that provide hope to preserve as much of the usual isomorphism-by-duality as possible.

4.2 The dual as the set of linear maps with functional diffeology

In this section we treat the following option:

- let $V^*$ be the usual dual of $V$, i.e., $V^* = L(V, \mathbb{R})$, endowed with the corresponding functional diffeology.

\[3\text{We will see shortly other examples of similar kind.}\]

\[4\text{Which eventually turns out to be unfounded, from the diffeological viewpoint.}\]
What this option gives us is a dual that is isomorphic as a vector space to the initial $V$ (provided that $V$ has finite dimension). But this is not the only issue in our context; what we should see next is whether it is also diffeomorphic to it. The following example shows that the answer is a priori negative.

**Example 4.5.** What follows stems naturally from an example already seen (Example [3.4]). Indeed, consider again $V = \mathbb{R}^n$ with the coarse diffeology; let $\{e_1, \ldots, e_n\}$ be its canonical basis, and let $\langle \cdot | \cdot \rangle$ be the canonical scalar product. We have already called $V^* = L(\mathbb{R}^n, \mathbb{R})$ the space of all linear maps $\mathbb{R}^n \to \mathbb{R}$ considered with the functional diffeology. The canonical scalar product defines an isomorphism $f : V \to V^*$ acting by $f(v)(v') = \langle v | v' \rangle$; we now show that $f$ is not smooth.

Consider, once again, the map $p_1 : \mathbb{R} \to V$ acting by $p_1(x) = |x| e_i$; for $f$ to be smooth it is necessary that $f \circ p_1$ be a plot of $V^* = L(\mathbb{R}^n, \mathbb{R})$. By Proposition 4.4, this is equivalent to the smoothness of the map $\varphi : \mathbb{R} \times V \to \mathbb{R}$ given by $\varphi(x, v) = (f \circ p_1)(x)(v) = \langle p(x)|v \rangle = |x| v^i$ (where $v^i$ is the $i$th component of $v$).

Let us show that $\varphi$ is not smooth. Indeed, for it to be so, for any plot $p$ of the product diffeology on $\mathbb{R} \times V$ the composition $\varphi \circ p$ must be a plot of $\mathbb{R}$ with the standard diffeology, i.e. it must be smooth in the usual sense, as a map $\mathbb{R} \to \mathbb{R}$. We fix $w \in V$ and define $p$ to be $p = \text{Id}_\mathbb{R} \times e_w$, where $e_w$ is the constant map that sends everything to $w$; this is obviously a plot for the product diffeology on $\mathbb{R} \times V$. Then $(\varphi \circ p)(x) = |x| w^i$, with $w^i$ being a constant; it is obviously not smooth and in particular it is not a plot for $\mathbb{R}$ with the standard diffeology. Hence $\varphi$ is not a smooth map; thus $f \circ p_1$ is not a plot of $V^*$, hence $f$ is not smooth.

Now note that the same reasoning holds for any other isomorphism between $V$ and $V^*$: indeed, any such isomorphism is defined by a (non-degenerate) pairing. Let us sketch the details: let $g : V \to \hat{V}^*$ be an isomorphism; denoting by $\{e_1, \ldots, e^n\}$ the dual basis, we obtain that $g$ is a linear map given by the matrix $(a^i_j)$ in the usual way, $g(e_i) = a^1_i e^1 + \cdots + a^n_i e^n$. As before, it is sufficient to show that $\psi_i : \mathbb{R} \times V \to \mathbb{R}$ defined by $\psi_i(x, v) = (g \circ p_i)(x)(v)$ is not smooth (for some $i$). Let us now take $n$ plots $\mathbb{R} \to U \times V$, denoted by $s_j$ and defined by $s_j = \text{Id}_\mathbb{R} \times e_j$, where $e_j$ is the constant map that sends everything to $e_j$; we obtain that $(\psi_i \circ s_j)(x) = |x| a^i_j$, so it is not smooth as a map $\mathbb{R} \to \mathbb{R}$, as soon as $a^i_j \neq 0$. Hence the conclusion desired.  

The example just made does not in and of itself invalidate the notion proposed later on (when we speak of the tensor product) we provide some comments regarding the implications of this different notion.

### 4.3 The dual as the set of linear maps with pushforward diffeology

Another way to define the dual is take, once again, the usual space of all linear maps $V \to \mathbb{R}$ and endow it with the diffeology obtained using an isomorphism of (finite-dimensional) $V$ with its usual dual, as is stated more precisely below:

- let $V$ be a finite-dimensional diffeological vector space, and let $\hat{V}^*$ be the usual vector space dual of $V$ endowed with the following diffeology: choose an isomorphism $f : V \to \hat{V}^*$ and denote by $D_f$ the pushforward of the diffeology of $V$ by the map $f$.

The definition as posed presents the (somewhat formal) question of being well-posed, i.e., whether the diffeology obtained depends on the choice of the isomorphism.

**Lemma 4.6.** Let $V$ be a finite-dimensional diffeological vector space, let $f : V \to \hat{V}^*$ and $g : V \to \hat{V}^*$ be two vector space isomorphisms of $V$ with its dual, and let $D_f$ and $D_g$ be the corresponding pushforward diffeologies. Then $D_f = D_g$.

**Proof.** It is sufficient to show that the composition map $g \circ f^{-1} : (\hat{V}^*, D_f) \to (\hat{V}^*, D_g)$ is smooth with respect to the pushforward diffeology. Let $p : U \to (\hat{V}^*, D_f)$ be a plot of $(\hat{V}^*, D_f)$; we need to show that

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11The proof that we have just spelled out is actually much more than what we need for the purposes at hand (it would have been sufficient to start by choosing $i$ and $j$ such that $a^i_j \neq 0$ and speak of those only); it allows us to conclude that the only smooth linear map from $V$ to $\hat{V}^*$ is the zero map.

12Although one can object to it from the categorical point of view.

13Even if in the case of an isomorphism it does not make a difference, we mention that instead of a pushforward of the diffeology of $V$ we could speak of its pullback by the inverse isomorphism.
we get that \( f \) from the analogous expression for \( \psi \) Proposition 4.8.

In the second case we give an example.

Proof. \((g \circ f^{-1}) \circ p \) is also a plot, of \((\hat{V}^*, \mathcal{D}_g)\). By definition of the pushforward diffeology, \( p \) being a plot of \((\hat{V}^*, \mathcal{D}_f)\) implies that (up to passing to a smaller neighbourhood) there exists a plot \( p' : U \to V \) of \( V \) such that \( p = p' \circ f \). Since \( f \) is invertible, we can write now \((g \circ f^{-1}) \circ p = g \circ (f^{-1} \circ p) = g \circ p' \); and the latter map is by definition a plot of the pushforward diffeology on \( \hat{V}^* \) by \( g \) (that is, it is a plot of \((\hat{V}^*, \mathcal{D}_g))\), whence the conclusion. \(\square\)

Now, of course, there are obvious limitations to this approach; the biggest one is that not all vector spaces are isomorphic to their duals. Nevertheless later on we will make some comments regarding the possibility of using this notion in the finite-dimensional case.

Observation 4.7. One might wonder if there is a non-a priori reason to not limit ourselves to finite-dimensional spaces only. Indeed, there is one: it was shown in [4] that even in a rather simple example, such as \( \mathbb{R}^n \) with the so-called wire diffeology, the internal tangent space \( (0) \) is infinite-dimensional.

4.4 The dual map

In this section we speak of the dual maps of linear maps; recall that, given two vector spaces \( V, W \), and a linear map \( f : V \to W \) between them, then the dual map is \( f^* : W^* \to V^* \) acting by \( f^*(g)(v) = g(f(v)) \).

Now suppose that \( V \) and \( W \) are two diffeological vector spaces, and that \( f \in L^{\infty}(V, W) \). Then, for each of the three possible notions of the dual space, there is the question whether the corresponding \( f^* \) is a smooth map. It turns out that the answer is positive for \( V^* \) and \( \hat{V}^* \), but \( a \ priori \) it is negative for \( \hat{V}^* \).

Let us prove these two statements. In the first case, we give the full proof for \( V^* \) only, noting that it essentially uses the fact that it is endowed with the functional diffeology (so can be carried over to \( \hat{V}^* \)). In the second case we give an example.

Proposition 4.8. Let \( V, W \) be two diffeological vector spaces, and let \( f : V \to W \) be a smooth linear map. Let \( f^* : W^* \to V^* \) be the dual map between the diffeological duals, \( f^*(g)(v) = g(f(v)) \). Then \( f^* \) is smooth.

Proof. Let \( p \) be a plot of \( W^* \); we need to show that \( f^* \circ p \) is a plot of \( V^* \). The diffeology of \( W^* \) being functional, by Proposition 3.5 \( p \) being a plot is equivalent to the smoothness of the map \( \psi : U \times W \to \mathbb{R} \) acting by \( \psi(u, w) = p(u)(w) \); now, for \( \psi \) to be smooth, we must have for any plot \( (p_1, p_W) : U' \to U \times W \) (where \( p_1 \) and \( p_W \) are plots of \( U' \) and \( W \) respectively) that \( \psi \circ (p_1, p_W) \) is a plot of \( \mathbb{R} \). By definition of the pushforward diffeology, \( \psi \circ (p_1, p_W) \) is a smooth map \( U' \to \mathbb{R} \). For future use, let us write explicitly that \( \psi \circ (p_1, p_W)(u') = p_1(u')(\psi(p_W(u'))) \).

Now, to prove that \( f^* \circ p \) is a plot of \( V^* \), we need to show that the map \( \varphi : U \times V \to \mathbb{R} \) given by \( \varphi(u, v) = p(v)(f(u)) \) is smooth, that is, that for any plot \( (p_1, p_V) : U' \to U \times V \) (where \( p_1 \) and \( p_V \) are plots of \( U' \) and \( V \) respectively) the composition \( \varphi \circ (p_1, p_V) \) is a plot of \( \mathbb{R} \). Writing explicitly \( \varphi \circ (p_1, p_V)(u') = p_1(u')(\varphi(p_V(u'))) \) and observing that \( f \) being smooth and \( p_V \) being a plot of \( V \), we get that \( f \circ p_V \) is a plot of \( W \), so setting \( p_W = f \circ p_V \), we deduce immediately the desired conclusion from the analogous expression for \( \psi \) (smooth by assumption) and \( p_W \). \(\square\)

We now briefly describe an example that shows that the dual map \( f^* : \hat{W}^* \to \hat{V}^* \) may not be smooth.

Example 4.9. Let \( V \) be \( \mathbb{R}^n \) with the fine diffeology, and let \( W \) be \( \mathbb{R}^n \) with the coarse diffeology. Observe that this implies that \( \hat{V}^* \) and \( W^* \) have respectively, the fine and the coarse diffeology. Let \( f : V \to W \) be any linear map (it is automatically smooth); then \( f^* \) is a map from \( \mathbb{R}^n \) with coarse diffeology to the one with fine (that is, standard) diffeology. It suffices to choose, as a plot \( p \) of \( W^* \), any non-smooth map to \( \mathbb{R}^n \), to get that \( f^* \circ p \) is not a plot of \( \hat{V}^* \), thus disproving the smoothness of \( f^* \).

4.5 Conclusion

To summarize the discussion carried out up to now in the present section, we settle for the first definition of the diffeological dual, \( i.e. \), that of the set of all smooth linear maps:

\[
V^* = L^{\infty}(V, \mathbb{R}).
\]
Apart from it being the most natural (and obvious) from the categorical point of view, the other two notions present their own limitations, which preclude a universal choice. To be more precise, we have already said that the main drawback of $V^*$ is the (frequent) failure of the isomorphism-by-duality by Proposition 4.3, its existence (in the finite-dimensional case) is equivalent to the equality $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$, so can be guaranteed only for fine diffeological vector spaces. This drawback however is not escaped by taking $V^*$, which is indeed isomorphic to $V$ whenever the usual dual is so, but it may not be diffeomorphic to it, as Example 4.5 shows. Finally, this issue is resolved by $\hat{V}^*$, which however, in addition to not being always defined, lacks the advantages of being considered with its most natural diffeology, the functional one: as is noted above, this in some instances implies that the dual map of a smooth linear map may not be smooth. Therefore our choice: although we do not pretend that other notions cannot be considered.

5 The tensor product

In this section we discuss the definition (and the relative properties) of the tensor product, starting with some alternatives, discussing in detail the case of two factors. While there is a definition that we prefer, we briefly mention 15 based on the discussion in the previous section.

5.1 The tensor product of two diffeological vector spaces

We now begin to consider the tensor product, first treating in detail the case of the tensor product of just two diffeological vector spaces; it seems convenient to do so before going into the full generality, as, on one hand, all the main issues are already visible at this point, and, on the other hand, these can be seen in a very concrete fashion.

The tensor product: definition Let us first look at the case of the tensor product between two diffeological vector spaces $V$ and $W$, following a somewhat naive approach. Namely, let us describe step-by-step the direct construction of the tensor product and how it reflects itself in the construction of the corresponding diffeology.

- Consider first the direct product $V \times W$ and endow it with the product diffeology (so it, too, becomes a diffeological vector space).

- Next, recall that the tensor product $V \otimes W$ can be defined as the quotient of $V \times W$ by the subspace $Z$ generated by all elements of the form $(\alpha_1 v_1 + \alpha_2 v_2, w) - \alpha_1 (v_1, w) - \alpha_2 (v_2, w)$, $(v, \beta_1 w_1 + \beta_2 w_2) - \beta_1 (v, w_1) - \beta_2 (v, w_2)$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ run over $\mathbb{R}$, $v, v_1, v_2$ run over $V$, and $w, w_1, w_2$ run over $W$. Endow $Z$ with the sub-diffeology of $V \times W$.

- Finally, endow $V \otimes W = (V \times W)/Z$ with the quotient diffeology, which we denote by $\mathcal{D}_{V \otimes W}$.

The procedure just described certainly yields a diffeology on $V \otimes W$, that makes it a diffeological vector space, and possesses the first properties of the tensor product, such as being commutative14 and having $V \otimes \mathbb{R} \cong V$ (as diffeological vector spaces) for any $V$; but it is not the only possible and, what is more important, it might not be the same as the one arising from other ways of defining the tensor product. Whether it is, or it is not, is what we shall see next.

Comparison with the tensor product diffeology of [6] Let $V_1$, $V_2$ be two diffeological vector spaces; let us make a comparison between the diffeology $\sum \mathcal{D}_{V_i}$ of [6] and the diffeology $\mathcal{D}_{V_1 \otimes V_2}$ described in the previous paragraph.

Lemma 5.1. For any two diffeological spaces $V_1, V_2$ the diffeologies $\sum \mathcal{D}_{V_i}$ and $\mathcal{D}_{V_1 \otimes V_2}$ on $V_1 \otimes V_2$ coincide.

14 Which we actually want to be a diffeomorphism
15 Due to alternative definitions of the diffeological dual.
16 Meaning that $V \otimes W$ and $W \otimes V$ are isomorphic as diffeological vector spaces, as follows from the commutativity of the product diffeology.
Proof. This is a direct consequence of the definitions: in fact the diffeology \( \mathcal{D}_{V_1 \otimes V_2} \) is the quotient diffeology with respect to the kernel of the standard projection \( \phi : V_1 \times V_2 \to V_1 \otimes V_2 \); this is the same as the pushforward of the product diffeology by this projection. The latter is actually the diffeology \( \sum \mathcal{D}_{V_i} \) of \( \mathbb{G} \) (note that by \( \mathbb{G} \), Section 3.5, this pushforward is already a vector space diffeology). \( \square \)

We anticipate that this statement holds just the same (and with the same proof) for the tensor product of more than two spaces, the fact that we will state when we arrive to discussing that case. Note that this means that the analogue of the usual universality result (Theorem 2.3.5 of \( \mathbb{G} \)) holds for our (description of the) tensor product, that is:

\[
(V \otimes W)^* = L^\infty(V \otimes W, \mathbb{R}) \cong \text{Mult}^\infty(V \times W, \mathbb{R}),
\]

where \( \text{Mult}^\infty(V \times W, \mathbb{R}) \) is the space of all smooth (for the product diffeology) bilinear maps \( V \times W \to \mathbb{R} \).

The tensor product of maps Let us consider two (smooth) linear maps between diffeological vector spaces, \( f : V \to V' \) and \( g : W \to W' \). As usual, we have the tensor product map \( f \otimes g : V \otimes W \to V' \otimes W' \), defined by \( (f \otimes g)(\sum v_i \otimes w_i) = \sum f(v_i) \otimes g(w_i) \). We observe that \( f \otimes g \) is a smooth map (with respect to the tensor product diffeologies on \( V \otimes W \) and \( V' \otimes W' \)) due to the properties of the product and the quotient diffeologies.

The tensor product and the direct sum Let \( V_1, V_2, V_3 \) be vector spaces; recall that in the usual linear algebra the tensor product is distributive with respect to the direct sum, \( i.e. \):

\[
V_1 \otimes (V_2 \oplus V_3) \cong (V_1 \otimes V_2) \oplus (V_1 \otimes V_3),
\]

via a canonical isomorphism, which we denote by \( T_{\ominus, \oplus} \). Now, if \( V_1, V_2, V_3 \) are diffeological vector spaces, then so are \( V_1 \otimes (V_2 \oplus V_3) \) and \( (V_1 \otimes V_2) \oplus (V_1 \otimes V_3) \). The map \( T_{\ominus, \oplus} \) is still an isomorphism of the underlying vector spaces; we shall deal with the question whether it is also a diffeomorphism.

Lemma 5.2. Let \( V_1, V_2, V_3 \) be diffeological vector spaces, and let \( T_{\ominus, \oplus} : V_1 \otimes (V_2 \oplus V_3) \to (V_1 \otimes V_2) \oplus (V_1 \otimes V_3) \) be the standard isomorphism. Then \( T_{\ominus, \oplus} \) is smooth.

Proof. By the properties of the quotient diffeology, it is sufficient to show that the covering map \( \tilde{T}_{\ominus, \oplus} : V_1 \times (V_2 \oplus V_3) \to (V_1 \times V_2) \oplus (V_1 \times V_3) \) is smooth. Let \( p : U \to V_1 \times (V_2 \oplus V_3) \) be a plot; we must show that \( \tilde{T}_{\ominus, \oplus} \circ p \) is a plot of \( (V_1 \times V_2) \oplus (V_1 \times V_3) \). Let \( \pi_1 : V_1 \times (V_2 \oplus V_3) \to V_1 \) and \( \pi_{2,3} : V_1 \times (V_2 \oplus V_3) \to (V_2 \oplus V_3) \) be the natural projections; observe that by definition of the sum diffeology, \( \pi_{2,3} \) writes (at least locally) as \( \pi_{2,3} = p_2 \oplus p_3 \), where \( p_2 \) is a plot of \( V_2 \) and \( p_3 \) is a plot of \( V_3 \).

Write now \( \tilde{T}_{\ominus, \oplus} \circ p \) as \( \tilde{T}_{\ominus, \oplus} \circ p = p' \oplus p'' \); observe that \( p' = (\pi_1 \circ p, p_2) \), while \( p'' = (\pi_1 \circ p, p_3) \). These are plots for the sum diffeology on \( (V_1 \times V_3) \oplus (V_2 \times V_3) \), hence the conclusion. \( \square \)

The tensor product \( V \otimes W \) as a function space As is known, in the usual linear algebra context the tensor product of two finite-dimensional vector spaces \( V \otimes W \) is isomorphic to the spaces \( L(V^*, W) \), the space of linear maps \( V^* \to W \), and \( L(W^*, V) \), the space of linear maps \( W^* \to V \). Recall that, somewhat naively, these isomorphisms are given as:

- For \( f \in V^*, v \in V \), and \( w \in W \) we set \( (v \otimes w)(f) = f(v)w \), extending by linearity;
- For \( g \in W^*, v \in V \), and \( w \in W \) we set \( (v \otimes w)(g) = g(w)v \), extending by linearity.

The question that we consider now is whether these isomorphisms continue to exist if all spaces we consider are (finite-dimensional) diffeological vector spaces (in particular, all duals are meant in the diffeological sense), all linear maps are smooth, and all function spaces are endowed with their functional diffeologies. The observations made regarding the sometimes substantial difference between a diffeological vector space \( V \) and its diffeological dual \( V^* \) (in particular, that it might be zero even for \( V \) “quite large”, for instance, with total space any \( \mathbb{R}^n \)) suggest that we start by considering again one of our examples.

\[\text{By considering, in addition to the tensor product diffeology, the sum diffeology, whenever appropriate.}\]
Example 5.3. Once again, consider $V = \mathbb{R}^n$ for $n \geq 2$ with the coarse diffeology and $W = \mathbb{R}$ with the standard diffeology. Then, as shown in Example 5.3, the diffeological dual of $V$ is trivial: $V^* = \{0\}$; this obviously implies that $L^\infty(V^*, W) = \{0\}$. Recall also that, the diffeology of $W$ being fine, its dual is isomorphic to $W$, so we have $W \cong W^* \cong \mathbb{R}$; furthermore, as it occurs for all fine diffeological vector spaces (see Section 2), we have $L^\infty(W^*, V) = L(W^*, V) \cong V$.

On the other hand, the total space of the diffeological tensor product $V \otimes W$ is the same as that of the usual tensor product, i.e., it is isomorphic to $V$. This implies right away that there is not an isomorphism between $V \otimes W$ and $L^\infty(V^*, W)$, the two spaces being different as sets.

On the other hand, $L^\infty(W^*, V)$ and $V \otimes W$ are isomorphic as usual vector spaces; it is easy to see that they are also diffeomorphic (this follows from the fact that $V$ has the coarse diffeology).

The example just made shows that in general, at least one of these classical isomorphisms might fail to exist (and at a very basic level). We may wish however to see what could be kept of the standard Proposition 5.4.

Proposition 5.4. Let $V$, $W$ be two finite-dimensional diffeological vector spaces. Then:

1. If $\hat{F} : V \otimes W \to L(V^*, W)$ is the map defined, via linearity, by $v \otimes w \mapsto [\hat{F}(v \otimes w)](f) = f(v)w$ then $\hat{F}$ takes values in $L^\infty(V^*, W)$. Furthermore, as a map $V \otimes W \to L^\infty(V^*, W)$ between diffeological spaces, it is smooth;

2. If $\hat{G} : V \otimes W \to L(W^*, V)$ is the map defined, via linearity, by $v \otimes w \mapsto [\hat{G}(v \otimes w)](g) = g(w)v$ then $\hat{G}$ takes values in $L^\infty(W^*, V)$. Furthermore, as a map $V \otimes W \to L^\infty(W^*, V)$ between diffeological spaces, it is smooth.

Proof. Let us prove 1; we will quite liberally avail ourselves of the commutativity of all the products. We need to show that $\hat{F}$ is a smooth map that takes values in $L^\infty(V^*, W)$. To prove the latter, it is enough to show that $\hat{F}(v \otimes w)$ is smooth, for any $v \in V$ and $w \in W$. Let us fix $v \in V$ and $w \in W$; we need to show that for any plot $p : U \to V^*$ the composition $\hat{F}(v \otimes w) \circ p$ is a plot of $W$. Writing explicitly $(\hat{F}(v \otimes w) \circ p)(u) = \hat{F}(v \otimes w)(p(u)) = p(u)v$, we recall that any constant map on a domain is a plot for any diffeology, so the map $c_w : U \to W$ that sends everything in $w$ is a plot of $W$. Finally, the map $(u, v) \mapsto p(u)v$ is a smooth map to $\mathbb{R}$, by Proposition 5.5 and because $p$ is a plot of $V^* = L^\infty(V, \mathbb{R})$ whose diffeology is functional; recalling that multiplication by scalar is smooth for any diffeological vector space, we get the conclusion.

Let us now prove that $\hat{F}$ is a smooth map $V \otimes W \to L^\infty(V^*, W)$; by Proposition 5.4 we need to prove that the induced map $V^* \times U \to W$ is smooth. This map acts by sending each $(f, u)$ (where $f \in V^*$) to $(\hat{F} \circ p)(u)(f)$ and so it writes as $(f, u) \mapsto ([e_{V^*} \otimes Id_W](Id_{V^*} \times p)(f, u))$; the diffeology of $V^*$ being functional, so that the evaluation map is smooth, we conclude that $\hat{F} \circ p$ is smooth, so the conclusion.

The proof of 2 is completely analogous, so we omit it.

Observation 5.5. As a final remark to this paragraph, we observe that already Example 5.3 tells us that, in general, there is not an analogue of the classical isomorphism $V^* \otimes V \cong L^\infty(V, V)$: it suffices to consider the same $V$, that is, $\mathbb{R}^n$ with the coarse diffeology. Then the product on the left is the trivial space, $V^*$ being the trivial space, whereas the space on the right consists of all linear maps $V \to V$ (since the coarse diffeology includes any map into $V$; all of these maps are automatically smooth).

Tensor product of duals and the dual of a tensor product. Recall, once again, that for usual vector spaces there is a standard isomorphism $V^* \otimes W^* \cong (V \otimes W)^*$; we are now interested in the question whether the existence of this isomorphism extends to the diffeological context, i.e., whether the corresponding map is (always) smooth.

\footnote{Consider the obvious map $F : V \to L(\mathbb{R}, V) = L^\infty(\mathbb{R}, V)$ given by $F(v)(x) = xv$; it is obviously bijective, and it is smooth by Proposition 5.4. Indeed, for any plot $p : U \to V$ we need that $F \circ p$ be a plot, which is equivalent to the map $U \times \mathbb{R} \to V$ given by $(u, x) \mapsto (F \circ p)(u)(x) = xp(u)$ being smooth. But simply due to the fact that it is a map in $V$, that has the coarse diffeology, it is a plot of it, so the conclusion.}

\footnote{Note the change in the order of factors, for formal purposes.}

\footnote{This question becomes quite important when one comes to considering scalar products, as is our intention.}
The standard isomorphism $V^* \otimes W^* \to (V \otimes W)^*$, which in this paragraph we denote by $F$ is defined by setting:

$$F(\sum_i f_i \otimes g_i)(\sum_j v_j \otimes w_j) = \sum_{i,j} f_i(v_j)g_i(w_j).$$

The first thing that we need to check is whether it does take values in $(V \otimes W)^*$, that is, if, fixed some $f \otimes g \in V^* \otimes W^*$ it actually defines a smooth (and not just linear) map $V \otimes W \to \mathbb{R}$.

**Lemma 5.6.** Let $V$, $W$ be diffeological vector spaces, and let $f \in V^*$, $g \in W^*$. Then the map $F(f \otimes g) : V \otimes W \to \mathbb{R}$ is smooth.

**Proof.** By Proposition 3.5 we need to check that for any plot $p : U \to V \otimes W$ the composition $F(f \otimes g) \circ p$ is a smooth map $U \to \mathbb{R}$. Recall that locally (so we assume that $U$ is small enough, so as to avoid complicating the notation) $p$ writes as a composition $p = \pi_\otimes \circ \tilde{p}$, where $\pi_\otimes$ is the natural projection $V \times W \to V \otimes W$ and $\tilde{p} : U \to V \times W$ is a plot for the product diffeology; furthermore, $\tilde{p}$ writes as $\tilde{p} = (p_V, p_W)$, where $p_V$ is a plot of $V$ and $p_W$ is a plot of $W$.

Putting all of this together, we write $(F(f \otimes g) \circ p)(u) = f(p_V(u))g(p_W(u))$, that is, $F(f \otimes g) \circ p$ is the usual product in $\mathbb{R}$ of two maps, $f \circ p_V$ and $g \circ p_W$. Now, $f$ being smooth by its choice and $p_V$ being a plot of $V$, their composition $f \circ p_V$ is a smooth map in $\mathbb{R}$. The same holds also for $g \circ p_W$; the product of two smooth maps being smooth, we get the desired conclusion. \hfill \Box

By the lemma just proven, $F$ is an injective linear map from the tensor product of the diffeological duals $V^*$, $W^*$ into the diffeological dual of the tensor product $V \otimes W$. We should check next whether it is smooth.

**Proposition 5.7.** Let $V$, $W$ be diffeological vector spaces, and let $F : V^* \otimes W^* \to (V \otimes W)^*$ be the already defined map between the diffeological duals. Then $F$ is smooth.

**Proof.** For the map $F$ to be smooth, it is required that, for any plot $p : U \to V^* \otimes W^*$ the composition $F \circ p$ be a plot of $(V \otimes W)^*$. Recall that the latter is equivalent to the following map being smooth:

- $\Phi : U \times (V \otimes W) \to \mathbb{R}$ such that $\Phi(u, \sum v_j \otimes w_j) = F(p(u))(\sum v_j \otimes w_j)$.

The map $\Phi$ being smooth is equivalent to:

- for any map $(p_V, p_{V \otimes W}) : U' \to U \times (V \otimes W)$ such that $p_V : U' \to U$ is smooth and $p_{V \otimes W} : U' \to V \otimes W$ is a plot of $V \otimes W$ the composition $\Phi \circ (p_V, p_{V \otimes W})$ is a smooth map $U' \to \mathbb{R}$.

We write explicitly:

$$(\Phi \circ (p_V, p_{V \otimes W}))(u') = F((p \circ p_u)(u'))(p_{V \otimes W}(u')).$$

This is the map of which we need to establish the smoothness.

To do so, let us write explicitly what it means that $p : U \to V^* \otimes W^*$ is a plot of the second space. First of all, by definition of the quotient diffeology we have:

- for $U$ small enough $p$ lifts to a smooth map $\tilde{p} : U \to V^* \times W^*$, that is, $p = \pi_{V^* \otimes W^*} \circ \tilde{p}$, where $\pi_{V^* \otimes W^*}$ is the natural projection (smooth by definition); moreover, $\tilde{p}$ writes as $\tilde{p} = (p_{V^*}, p_{W^*})$, where $p_{V^*}$ is a plot of $V^*$ and $p_{W^*}$ is a plot of $W^*$.

Recall that $p_{V^*}$ being a plot of $V^*$ means that the map $\varphi_V : U \times V \to \mathbb{R}$ given by $\varphi_V(u, v) = p_{V^*}(u)(v)$ is smooth; accordingly, $p_{W^*}$ being a plot of $W^*$ means that the map $\varphi_W : U \times W \to \mathbb{R}$ given by $\varphi_W(u, w) = p_{W^*}(u)(w)$ is smooth.

In addition, we should say what it means for $p_{V \otimes W}$ to be a plot:

- for $U$ small enough, $p_{V \otimes W}$ lifts to $\tilde{p}_{V \otimes W}$, a plot of $V \times W$, that is, $p_{V \otimes W}$ writes as $p_{V \otimes W} = \pi_\otimes \circ \tilde{p}_{V \otimes W}$ for the appropriate natural projection $\pi_\otimes$; furthermore, $\tilde{p}_{V \otimes W}$ writes as $\tilde{p}_{V \otimes W} = (p_V, p_W)$, where $p_V$ is a plot of $V$ and $p_W$ is a plot of $W$.

\footnote{Extending by linearity is smooth by definition of a diffeological vector space.}
\footnote{Which we can always assume}
Assume now that the domain $U$ is small enough so that all of the above be valid; then we can write, by definition of $F$, that

$$(\Phi \circ (p_U, p_{V \otimes W}))(u') = p_U(u')(p_{V}(u')) \cdot p_{W}(u')(p_{W}(u')) = ev(p_{V}, p_{V})(u') \cdot ev(p_{W}, p_{W})(u'),$$

where $(p_{V}, p_{V}) : U' \to V^* \times V$ and $(p_{W}, p_{W}) : U' \to W^* \times W$ are the obvious maps. By definition of the product diffeology they are plots for, respectively, $V^* \times V$ and $W^* \times W$; furthermore, each diffeological dual carrying the functional diffeology, each evaluation map $ev$ is obviously smooth. It follows that $\Phi \circ (p_U, p_{V \otimes W}) : U' \to \mathbb{R}$ writes as the product of two smooth maps $U' \to \mathbb{R}$; the diffeology of $\mathbb{R}$ being the standard one, these maps are smooth in the usual sense, hence so is their product. This implies that $\Phi$ is a smooth map, therefore $F$ is smooth, and the Proposition is proven.

We are now ready to prove the following statement:

**Theorem 5.8.** Let $V, W$ be two finite-dimensional diffeological vector spaces. Then $F : V^* \otimes W^* \to (V \otimes W)^*$ is a diffeomorphism.

**Proof.** It remains to check that $F$ is surjective with smooth inverse, i.e., that for any smooth linear map $f : V \otimes W \to \mathbb{R}$ its pre-image $F^{-1}(f)$ (which a priori belongs to the tensor product of the usual duals) actually belongs to the tensor product of the differentials. By definition of $F$, it is sufficient to observe that $f$ being smooth means that for any plot $p : U \to V \otimes W$ the composition $f \circ p : U \to \mathbb{R}$ is a (usual) smooth map; furthermore, for $U$ small enough $p$ writes as $p = \pi \circ (p_{V}, p_{W})$, where $\pi : V \times W \to V \otimes W$ is the natural projection, $p_{V} : U \to V$ is a plot of $V$, and $p_{W} : U \to W$ is a plot of $W$, hence $f \circ p$ actually writes as $(f \circ p)(u) = f(p_{V}(u) \otimes p_{W}(u))$. Note that $F^{-1}(f)$ writes as $F^{-1}(f) = \sum f_{i} \otimes g_{i}$, with $f_{i}$ belonging to the usual dual of $V$, and $g_{i}$ belonging to the usual dual of $W$; we obtain that $(F^{-1}(f) \circ (p_{V}, p_{W}))(u) = \sum (f_{i} \circ p_{V})(u)(g_{i} \circ p_{W})(u)$, and we can draw the desired by choosing the appropriate $p_{V}, p_{W}$.

**5.2 Possible alternatives**

Since there are more ways to look at the tensor product in the usual (multi)linear algebra, there are a priori more ways to define a diffeology on $V \otimes W$ (the usual vector space). We now briefly describe these other possibilities.

**The diffeology stemming from $V \otimes W \cong L(V, \mathbb{R}), W$** Let $V$ and $W$ be two finite-dimensional diffeological vector spaces; the existence of classical (multi)linear algebra isomorphisms $V \otimes W \cong L(L(V, \mathbb{R}), W) \cong L(H(W, \mathbb{R}), V)$ could suggest to define the diffeological tensor product by taking the usual tensor product $V \otimes W$ and endowing it with the functional diffeology of either $L(L(V, \mathbb{R}), W)$ or $L(L(W, \mathbb{R}), V)$, where $L(V, \mathbb{R})$ and $L(W, \mathbb{R})$ are considered with their respective functional diffeologies. More precisely, the idea is to follow the two steps below:

- let $\hat{V}^*$ be the space defined in Section 4.2 Consider $\bar{L}(\hat{V}^*, W)$ and endow it with the functional diffeology;
- endow $V \otimes W$ with the diffeology $D_{\hat{V} \otimes \hat{W}}$ that is the pullback of the above diffeology by the standard map $\hat{V} \otimes \hat{W} \to \bar{L}(\hat{V}^*, W)$.

The map just referenced is the map analogous to $\hat{F}$ of the previous section, i.e., acting by the rule

$$\hat{F} (\sum v_{i} \otimes w_{i}) (f) = \sum f(v_{i}) w_{i} \text{ for } f : V \to \mathbb{R} \text{ linear}.$$ 

The construction just described is of course possible; the specific reason why we do not employ it stems from the proof of Lemma 5.4 for which the fact that the elements of the dual are smooth maps is significant. The Lemma being a prerequisite for (for example) Theorem 5.8 we discard the option just described.
The space $\hat{V}^*$  Another a priori option (once again, in the case of finite-dimensional spaces) is to apply the construction of the previous paragraph, taking instead of $\bar{V}^*$ the space $\hat{V}^*$ defined in Section 4.3. We shall avoid following this path, the motivation being the already-mentioned indications (such as the potential non-smoothness of the dual map) that this construction is too artificial.

5.3 Scalar products

The motivation for this work stemming from wishing to have an analogue of Riemannian metric on diffeological bundles with fibres diffeological vector spaces, we wish to pay particular attention to various ways of viewing scalar products on the latter. Recall that in the usual context a scalar product on a vector space $V$, being a bilinear map (with some extra properties), can be see also as an element of $V^* \otimes V^*$; in the diffeological context, it is a smooth bilinear map (symmetric and definite positive) and the tensor product is that of the diffeological duals. Thus, a priori there is the question whether similar identification continues to hold. This follows from Theorem 2.3.5 of [6] and Theorem 5.8. Indeed, the former implies that the diffeological space of all smooth bilinear maps $V \times V \to \mathbb{R}$ is diffeomorphic to the space of all smooth linear maps $V \otimes V \to \mathbb{R}$, that is, to the diffeological dual $(V \otimes V)^*$ of the tensor product of $V$ with itself. Theorem 5.8 then shows that $(V \otimes V)^*$ is diffeomorphic to the tensor product $V^* \otimes V^*$ of the diffeological dual of $V$ with itself.

Remark 5.9. We thus get a work-in-progress conclusion that a prospective notion of a diffeological metric can use a bundle with fibre $(V \otimes V)^*$, or a bundle with fibre $V^* \otimes V^*$: there would not be any difference (at least, on the level of the total space).

5.4 The tensor product of $n$ spaces

We now provide quickly the definition of the tensor product of more than two spaces; this construction is easily generalized, and in the most obvious manner, from the case of $n = 2$. Let $V_1, \ldots, V_n$ be diffeological vector spaces, let $T : V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n$ be the universal map onto their tensor product as vector spaces, and let $Z \subseteq V_1 \times \ldots \times V_n$ be the kernel of $T$. We denote by $D_{\otimes}$ the following diffeology on $V_1 \otimes \ldots \otimes V_n$:

- endow $V_1 \times \ldots \times V_n$ with the product diffeology, and $Z$ with the corresponding subspace diffeology;
- let $D_{\otimes}$ be the quotient diffeology on $(V_1 \times \ldots \times V_n)/Z = V_1 \otimes \ldots \otimes V_n$.

As has already been mentioned, the diffeology $D_{\otimes}$ can also be described as the pushforward by $T$ of the product diffeology on $V_1 \times \ldots \times V_n$.

This construction obviously includes the usual $p$-covariant and $q$-contravariant tensors, defined as elements of the tensor product $V^* \otimes \ldots \otimes V^* \otimes V \otimes \ldots \otimes V$, where $V$ is a diffeological vector space and $V^*$ its diffeological dual. As usual, the notation $T^p_q(V)$ is used for the above diffeological tensor product.

5.5 Symmetrization and antisymmetrization

Considering the space $T^p(V) = T^p_0(V) = V^* \otimes \ldots \otimes V^*$, the usual symmetrization and antisymmetrization operators are defined; and there are their invariant subspaces, the space $S^p(V)$ of symmetric tensors and the space $A^p(V)$ of antisymmetric tensors. These spaces are endowed with the subspace diffeology, which, due to the smoothness of the operations of a diffeological vector space, ensures that the two operators are smooth.$^23$

$^23$We omit the proof.
5.6 The exterior product

We finally mention the exterior product of diffeological vector spaces. Recall that, given \( A \in \Lambda^p(V) \) and \( B \in \Lambda^q(V) \), their exterior product is defined as \( A \wedge B = \frac{(p+q)!}{p!q!} \text{Alt}(A \otimes B) \). The exterior product is smooth as a map \( \Lambda^p(V) \times \Lambda^q(V) \to \Lambda^{p+q}(V) \) (the former space being considered with the product diffeology), as follows from the definition of the diffeological tensor product and the smoothness of the antisymmetrization operator.

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