Completing NLO QCD Corrections for Tree Level Non-Leptonic $\Delta F = 1$ Decays Beyond the Standard Model

Andrzej J. Buras$^{a,b}$ and Jennifer Girrbach$^{b,c}$

$^a$ Physik Department, Technische Universität München, D-85748 Garching, Germany
$^b$ TUM-IAS, Lichtenbergstr. 2a, D-85748 Garching, Germany
$^c$ Excellence Cluster Universe, TUM, Boltzmannstraße 2, D-85748 Garching

Abstract

In various extensions of the Standard Model (SM) tree level non-leptonic decays of hadrons receive contributions from new heavy gauge bosons and scalars. Prominent examples are the right-handed $W'$ bosons in left-right symmetric models and charged Higgs ($H^\pm$) particles in models with extended scalar sector like two Higgs doublet models and supersymmetric models. Even in the case of decays with four different quark flavours involved, to which penguin operators cannot contribute, twenty linearly independent operators, instead of two in the SM, have to be considered. Anticipating the important role of such decays at the LHCb, KEKB and Super-B in Rome and having in mind future improved lattice computations, we complete the existing NLO QCD formulae for these processes by calculating $O(\alpha_s)$ corrections to matching conditions for the Wilson coefficients of all contributing operators in the NDR-MS scheme. This allows to reduce certain unphysical scale and renormalization scheme dependences in the existing NLO calculations. Our results can also be applied to models with tree-level heavy neutral gauge boson and scalar exchanges in $\Delta F = 1$ transitions and constitute an important part of NLO analyses of those non-leptonic decays to which also penguin operators contribute.
1 Introduction

In the Standard Model (SM) the non-leptonic $\Delta F = 1$ decays of mesons are governed by the $(V - A) \times (V - A)$ structure of the leading four-quark operators originating in the tree-level $W^\pm$ exchanges. If all the four flavours of the participating quarks are different from each other the only possible diagrams contributing to these decays in the SM and in any of its extensions are the current-current ones: penguin diagrams are absent. Decays of this type are theoretically cleaner than the ones in which also penguin diagrams and penguin operators contribute. As such they are well suited for the determination of the CKM parameters, in particular the angles $\gamma$ and $\beta$ in the unitarity triangle \cite{1,2,3}.

While non-leptonic decays are subject to significant non-perturbative uncertainties originating in hadronic matrix elements of four-quark operators, the QCD factorization approach to non-leptonic two-body decays \cite{4} combined with advanced lattice calculations could one day promote non-leptonic two-body meson decays to precise tools in testing the SM and its extensions. In these studies renormalization group short distance QCD effects play an important role. In the SM they are known including the NLO corrections and in a few processes at the NNLO level. An up-to-date review can be found in \cite{5}.

Beyond the SM new local four-quark operators with different Dirac structures can be generated. The simplest example are $(V + A) \times (V + A)$ operators originating in the exchange of $W^\pm$ gauge bosons in the left-right symmetric models. However, also right-handed (RH) couplings of the SM $W^\pm$ gauge bosons can be generated in various extensions of the SM like left-right symmetric models and generally also models with vectorial heavy quarks that mix with the SM chiral quarks. In this case also $(V - A) \times (V + A)$ operators contribute. The latter operators generate through QCD corrections $(S - P) \times (S + P)$ operators present also in models with charged $(H^\pm)$ Higgs particles. In the latter models also $(S \pm P) \times (S \pm P)$ operators are present. Needless to say all these statements also apply to neutral gauge bosons and scalars mediating $\Delta F = 1$ transitions.

The full set of twenty linearly independent dimension six four-quark operators with four different flavours in all extensions of the SM has been listed in \cite{6,7,8}, where also two-loop QCD anomalous dimensions of these operators have been calculated. However, the full NLO QCD renormalization group analysis of non-leptonic decays requires also the calculation of $\mathcal{O}(\alpha_s)$ corrections to matching conditions for the Wilson coefficients of the operators in question. While such corrections are known within the SM \cite{9,10} at the NLO level, to our knowledge a complete analysis of these corrections including all operators in any extension of the SM is absent in the literature.

In a recent paper \cite{11} we have calculated $\mathcal{O}(\alpha_s)$ corrections to matching conditions for the Wilson coefficients relevant for $\Delta F = 2$ processes mediated by heavy colourless neutral gauge bosons and scalars reducing thereby certain unphysical scale and renormalization scheme dependences present in the absence of such corrections. Similar unphysical scale and renormalization scheme dependences are present in the absence of $\mathcal{O}(\alpha_s)$ matching corrections in $\Delta F = 1$ amplitudes generated by tree-level gauge boson and scalar exchanges and it is desirable to reduce them as well.

The main goal of our paper is the calculation of $\mathcal{O}(\alpha_s)$ corrections to matching conditions
for the Wilson coefficients of all dimension six four-quark operators with four different flavours contributing to $\Delta F = 1$ decays mediated by colourless gauge bosons and scalars in the NDR-\overline{MS} scheme. As the two-loop anomalous dimensions for these operators have been already calculated in this scheme in [8] our calculations complete the NLO QCD analysis of the decays in question.

Our paper is organized as follows. In Section 2 we recall the general structure of the effective Hamiltonians for $\Delta F = 1$ processes in question and we give the full list of four-fermion operators that contribute to these transitions. Subsequently we collect their one- and two-loop anomalous dimension matrices. In Section 3 we calculate $\mathcal{O}(\alpha_s)$ corrections to the amplitudes in the full theory and in Section 4 the corresponding results for the matrix elements of operators are presented. This allows us in Section 5 to present the Wilson coefficients of all twenty operators including $\mathcal{O}(\alpha_s)$ corrections. In Section 6 combining our results with the known renormalization group evolution matrices we arrive at a complete NLO formulae for the Wilson coefficients of the involved operators. In Section 7 we demonstrate the scale independence of the physical amplitudes analytically and in Section 8 we investigate the removal of this scale dependence numerically. We conclude with a brief summary in Section 9.

# 2 Theoretical Framework

## 2.1 Local Operators

While in the SM only two current-current operators contribute to each $\Delta F = 1$ transition, the list of current-current operators beyond the SM is much longer. As in [8] we choose the operators in such a manner that all the four flavours they contain are different. In such a case, the only possible diagrams are the current–current ones. In what follows we will fix the four flavours to be $b, u, c, d$ but other choices are clearly possible without changing our results.

Twenty linearly independent operators can be built out of four different quark fields. They can be split into eight separate sectors, between which there is no mixing. The operators belonging to the first two sectors (VLL, VLR), that are relevant for gauge boson contributions, are given as follows

\begin{align}
Q_{1}^{\text{VLL}} &= (\bar{b}^\alpha \gamma_\mu P_L u^\beta)(\bar{c}^\beta \gamma_\mu P_L d^\alpha), \\
Q_{2}^{\text{VLL}} &= (\bar{b}^\alpha \gamma_\mu P_L u^\alpha)(\bar{c}^\beta \gamma_\mu P_L d^\beta), \\
Q_{1}^{\text{VLR}} &= (\bar{b}^\alpha \gamma_\mu P_L u^\beta)(\bar{c}^\beta \gamma_\mu P_R d^\alpha), \\
Q_{2}^{\text{VLR}} &= (\bar{b}^\alpha \gamma_\mu P_L u^\alpha)(\bar{c}^\beta \gamma_\mu P_R d^\beta),
\end{align}

where $\alpha, \beta$ denote quark colours. In the case of scalar contributions the following operators
have to be considered:

\[ Q_{1}^{\text{SLR}} = (\bar{b}^{\alpha} P_{L} u^{\beta})(\bar{c}^{\beta} P_{R} d^{\alpha}), \]  
\[ Q_{2}^{\text{SLR}} = (\bar{b}^{\alpha} P_{L} u^{\alpha})(\bar{c}^{\beta} P_{R} d^{\beta}), \]  
\[ Q_{1}^{\text{SLL}} = (\bar{b}^{\alpha} P_{L} u^{\beta})(\bar{c}^{\beta} P_{L} d^{\alpha}), \]  
\[ Q_{2}^{\text{SLL}} = (\bar{b}^{\alpha} P_{L} u^{\alpha})(\bar{c}^{\beta} P_{L} d^{\beta}), \]  
\[ Q_{3}^{\text{SLL}} = (\bar{b}^{\alpha} \sigma_{\mu \nu} P_{L} u^{\beta})(\bar{c}^{\beta} \sigma^{\mu \nu} P_{L} d^{\alpha}), \]  
\[ Q_{4}^{\text{SLL}} = (\bar{b}^{\alpha} \sigma_{\mu \nu} P_{L} u^{\alpha})(\bar{c}^{\beta} \sigma^{\mu \nu} P_{L} d^{\beta}). \]

The operators belonging to the four remaining sectors (VRR, VRL, SRL and SRR) are obtained from the above by interchanging \( P_{L} \) and \( P_{R} \). Obviously, it is sufficient to calculate the Wilson coefficients only for the VLL, VLR, SLR and SLL sectors. The “mirror” operators in the VRR, VRL, SRL and SRR sectors will have exactly the same properties under QCD renormalization.

The two-loop anomalous dimensions for these operators have been calculated in the NDR-\( \overline{\text{MS}} \) scheme in [8] with a particular choice of the evanescent operators. As discussed there, while these operators are essential for the correct evaluation of two-loop matrix elements, the virtue of the formulation of the NDR-\( \overline{\text{MS}} \) scheme introduced in [10] is that the evanescent operators defined in this scheme influence only two-loop anomalous dimensions. By definition they do not contribute to the matching and to the finite gluon corrections to the matrix elements of renormalized operators calculated by us. They are simply subtracted away in the process of renormalization. This issue is summarized in Section 6.9.4 of [12], where further references can be found. A very important paper in this respect is also the one of Herrlich and Nierste [13], where this issue is discussed in full generality. Therefore effectively the one-loop calculations presented here and in [11] are based on the projections of various Dirac structures on physical operators that are consistent with [8, 10] but otherwise the evanescent operators can be dropped from the beginning.

### 2.2 Renormalization Group Functions

#### 2.2.1 Running of QCD Coupling and Running Masses

For the complete NLO renormalization group analysis we need a number of renormalization group functions that we recall here for completeness.

In particular we need the QCD \( \beta \) function at the two-loop level

\[ \beta(g) = -\beta_{0} \frac{g^{3}}{16\pi^{2}} - \beta_{1} \frac{g^{5}}{(16\pi^{2})^{2}}, \]  
\[ (4) \]
where
\[ \beta_0 = \frac{11N - 2f}{3}, \quad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nf - 2C_F f, \quad C_F = \frac{N^2 - 1}{2N}. \tag{5} \]

\( f \) is number of flavours and \( N \) the number of colours.

Similarly, the two-loop expression for the quark mass anomalous dimension can be written as
\[ \gamma_m(\alpha_s) = \gamma_m^{(0)} + \frac{\alpha_s}{4\pi} \gamma_m^{(1)} \left( \frac{\alpha_s}{4\pi} \right)^2, \tag{6} \]
where
\[ \gamma_m^{(0)} = 6C_F \quad \gamma_m^{(1)} = C_F \left( 3C_F + \frac{97}{3}N - \frac{10}{3}f \right). \tag{7} \]

### 2.2.2 One-Loop and Two-Loop Anomalous Dimension Matrices of Operators

The most important ingredients of any renormalization group analysis in weak decays are the anomalous dimension matrices that we define in general form as follows
\[ \hat{\gamma}(\alpha_s) = \hat{\gamma}^{(0)} + \frac{\alpha_s}{4\pi} \hat{\gamma}^{(1)} \left( \frac{\alpha_s}{4\pi} \right)^2. \tag{8} \]

In particular the one-loop anomalous dimension matrices \( \hat{\gamma}^{(0)} \) will play important role in our discussion of the removal of the unphysical \( \mu \) dependences at NLO in Section [7]

The one-loop matrices for all operators considered in this paper read
\[ \hat{\gamma}^{(0)}_{\text{VLL}} = \begin{pmatrix} -\frac{6}{N} & 6 \\ \frac{6}{N} & -\frac{6}{N} \end{pmatrix}, \tag{9} \]
\[ \hat{\gamma}^{(0)}_{\text{VLR}} = \begin{pmatrix} -6N + \frac{6}{N} & 0 \\ -6 & \frac{6}{N} \end{pmatrix}, \tag{10} \]
\[ \hat{\gamma}^{(0)}_{\text{SLR}} = \begin{pmatrix} \frac{6}{N} & -6 \\ 0 & -6N + \frac{6}{N} \end{pmatrix}, \tag{11} \]
\[ \hat{\gamma}^{(0)}_{\text{SLL}} = \begin{pmatrix} \frac{6}{N} & -6 & \frac{N}{2} - \frac{1}{N} & \frac{1}{2} \\ 0 & -6N + \frac{6}{N} & \frac{N}{2} - \frac{1}{N} & \frac{1}{2} \\ \frac{48}{N} & 24N & \frac{24}{N} & \frac{1}{2} - 4N \\ 48 & \frac{48}{N} & 0 & 2N - \frac{2}{N} \end{pmatrix}. \tag{12} \]

The two-loop matrices for all operators in the NDR-\( \overline{\text{MS}} \) scheme used in this paper read
\[
\hat{\gamma}^{(1)}_{\text{VLL}} = \left( -\frac{23}{3} - \frac{57}{2N^2} - \frac{2}{3N} f \begin{array}{l}
-\frac{19}{6} N + \frac{39}{N} + \frac{2}{3} f \\
-\frac{22}{3} - \frac{57}{2N^2} - \frac{2}{3N} f
\end{array} \right),
\]

(13)

\[
\hat{\gamma}^{(1)}_{\text{VLR}} = \left( -\frac{203}{6} N^2 + \frac{479}{2} + \frac{15}{N^2} + \frac{10}{3} N f - \frac{22}{3} f \begin{array}{l}
-\frac{71}{2} N - \frac{18}{N} + 4 f \\
\frac{137}{6} + \frac{15}{2N^2} - \frac{22}{3N} f
\end{array} \right),
\]

(14)

\[
\hat{\gamma}^{(1)}_{\text{SLR}} = \left( \frac{137}{6} + \frac{15}{2N^2} - \frac{22}{3N} f \begin{array}{l}
-\frac{71}{2} N - \frac{18}{N} + 4 f \\
-\frac{203}{6} N^2 + \frac{479}{2} + \frac{15}{2N^2} + \frac{10}{3} N f - \frac{22}{3N} f
\end{array} \right),
\]

(15)

\[
\hat{\gamma}^{(1)}_{\text{SLL}} = \begin{array}{l}
-\frac{N^2}{2} + \frac{148}{3} - \frac{107}{2N^2} - 2N f - \frac{10}{3N} f \\
-\frac{178}{3} N + \frac{64}{N} + \frac{16}{N} f \\
\frac{107}{3} N^2 - \frac{71}{18} - \frac{4}{N^2} - \frac{4}{18} N f + \frac{f}{9N} \\
-\frac{109}{30} N + \frac{8}{N} - \frac{f}{18} \\
-26N + \frac{104}{N} \\
-\frac{203}{6} N^2 + \frac{28}{3} - \frac{107}{2N^2} + \frac{10}{3} N f - \frac{10}{3N} f \\
\frac{89}{18} N + \frac{2}{N} - \frac{1}{9} f \\
\frac{53}{18} - \frac{4}{N^2} + \frac{1}{9N} f \\
\frac{676}{3} N^2 - \frac{1880}{3} - \frac{320}{N^2} - \frac{88}{3} N f + \frac{176}{3N} f \\
\frac{820}{3} N + \frac{448}{N} - \frac{88}{3} f \\
-\frac{257}{18} N^2 - \frac{116}{9} + \frac{21}{2N^2} + \frac{22}{9} N f + \frac{2}{9N} f \\
\frac{50}{3} N - \frac{8}{3} f \\
\frac{488}{3} N + \frac{416}{N} - \frac{176}{3} f \\
\frac{776}{3} - \frac{320}{N^2} + \frac{176}{3N} f \\
\frac{22}{3} N - \frac{40}{N} + \frac{8}{3} f \\
\frac{343}{18} N^2 + \frac{28}{9} + \frac{21}{2N^2} - \frac{26}{9} N f + \frac{2}{9N} f.
\end{array}
\]

(16)
2.3 Effective Hamiltonian

The effective Hamiltonian for $\Delta F = 1$ transitions can be written in a general form as follows:

$$\mathcal{H}_{\text{eff}}^{\Delta F=1} = \kappa \sum_{i,a} C_i^a(\mu) Q_i^a,$$

(17)

where $Q_i^a$ are the operators given in Eqs. (1), (2) and (3) and $C_i^a(\mu)$ their Wilson coefficients evaluated at a scale $\mu$ at which the hadronic matrix elements are evaluated. The scale $\mu$ can be the low energy scale $\mu_L$ at which actual lattice calculations are performed or any other scale, in particular the matching scale $\mu_{\text{in}}$. In this case the matrix elements are obtained by evolving lattice results by means of renormalization group (RG) equations from $\mu_L$ to $\mu_{\text{in}}$. The resulting matrix elements of the effective Hamiltonian, that are directly related to decay amplitudes, can then be written generally as follows:

$$\langle \mathcal{H}_{\text{eff}}^{\Delta F=1} \rangle = \kappa \sum_{i,a} C_i^a(\mu_{\text{in}}) \langle Q_i^a(\mu_{\text{in}}) \rangle.$$

(18)

The overall factor $\kappa$ in our analysis depends on the exchanged boson (vector or scalar) and will be chosen such that for non-vanishing Wilson coefficients $C_i^a(\mu_{\text{in}}) = 1$ in LO.

Evidently the matrix elements $\langle Q_i^a(\mu_{\text{in}}) \rangle$ depend on the matching scale $\mu_{\text{in}}$ and also on the renormalization scheme for operators. In order to remove these unphysical dependences from physical amplitudes one has to calculate $O(\alpha_s)$ corrections to $C_i^a(\mu_{\text{in}})$. This is the goal of Sections 3-5.

2.4 Procedure for Matching

Before entering the details let us recall that the calculations of $O(\alpha_s)$ QCD corrections to Wilson coefficients are by now standard and have been described in several papers. In particular in [14, 12] in the case of the SM. In [11] this procedure has been used to calculate $O(\alpha_s)$ corrections to the Wilson coefficients of operators contributing to $\Delta F = 2$ FCNC processes mediated by tree level neutral gauge boson and scalar exchanges. Here we recall briefly this procedure that we will use in the case of $\Delta F = 1$ decays.

**Step 1**

We first calculated the amplitudes in the full theory. This amounts to the calculation of the diagrams in Figs. 1 and 2 in the case of a gauge boson exchange and a scalar exchange, respectively. In the presence of massless gluons one encounters infrared divergences. We have regulated these divergences by a common external momentum $p$ with $-p^2 > 0$ for all external massless fields as we did in [11]. The ultraviolet divergences present in diagrams in Fig. 1 and 2 with gluon corrections to vertices have been regulated using dimensional regularization with anti-commuting $\gamma_5$ in $4 - 2\epsilon$ dimensions.

\[^1\text{In what follows we drop for simplicity } h.c.\]
Figure 1: Tree level diagram and one-loop QCD corrections to $\Delta B = 1$ transition mediated by a gauge boson in the full theory. Mirror diagrams are not shown.

Figure 2: Tree level diagram and one-loop QCD corrections to $\Delta B = 1$ transition mediated by a scalar particle in the full theory. Mirror diagrams are not shown.

Step 2

We have calculated the matrix elements of contributing operators by evaluating the diagrams in Fig. 3 making the same assumptions about the external fields as in the first step. In contrast to step 1 one has to renormalize the operators. This we do in the $\overline{\text{MS}}$ renormalization scheme with anti-commuting $\gamma_5$, which corresponds to the NDR scheme used also in [8] and [15].
Step 3

We finally inserted the results of the two steps above into a formula like the one in Eq. (18) and comparing the coefficients of operators appearing on the l.h.s (full theory) and r.h.s (effective theory) we found the coefficients $C_{\mu}(\mu_{\text{in}})$. As these coefficients cannot depend on the infrared behaviour of the theory, the dependences on $p^2$ found in the first two steps have to cancel each other in the evaluation of $C_{\mu}(\mu_{\text{in}})$. Indeed we verified this explicitly.

The interested reader can do this as well by inspecting our intermediate results that we present in Sections 3 and 4, respectively.

Very often in analyses of new physics contributions the overall factor in front of the sum in Eq. (18) is chosen as in the SM. However, in our analysis it will be more convenient to use in each case the normalization in which the Wilson coefficient of the leading operator evaluated at the matching scale is equal to unity in the absence of QCD corrections. In this manner the applications of our formulae in various new physics (NP) models will be facilitated.

After this preparation we are ready to present our calculations in three steps in question.

3 Amplitudes in the Full Theory

Using the Feynman rules in Fig. [4] we find for the colourless gauge boson exchange (after quark wave function renormalization)
Figure 4: Feynman rules for colourless charged gauge boson $A^+$ with mass $M_A$, and charged colourless scalar particle $H^+$ with mass $M_H$, where $i (j)$ denotes an up-type (down-type) quark flavour with charge $+\frac{2}{3}$ ($-\frac{1}{3}$) and $\alpha, \beta$ are colour indices.

\begin{align*}
A_{VLL} &= \left(\frac{\Delta_{ub}^{ab}(A)}{M_A^2}\right)^* \Delta_{cd}^{cd}(A) \left[ \left( 1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} \right) Q_{2}^{VLL} \\
&\quad + \frac{\alpha_s}{4\pi} \left( \frac{1}{2} + \log \frac{M_A^2}{-p^2} \right) \left[ \frac{3}{2} N Q_{2}^{VLL} - 3Q_{1}^{VLL} \right] \right] \tag{19} \\
A_{VLR} &= \left(\frac{\Delta_{ub}^{ab}(A)}{M_A^2}\right)^* \Delta_{cd}^{cd}(A) \left[ \left( 1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2} \right) Q_{2}^{VLR} \\
&\quad + \frac{\alpha_s}{4\pi} \left( \frac{1}{2} + \log \frac{M_A^2}{-p^2} \right) \left[ -\frac{3}{2} N Q_{2}^{VLR} + 3Q_{1}^{VLR} \right] \right] \tag{20} \\
\end{align*}

For the colourless scalar exchange (after quark wave function and quark mass renormalizations) we find

\begin{align*}
A_{SLR} &= - \frac{\Delta_{L}^{ub}(H)}{M_H^2} \frac{\Delta_{cd}^{cd}(H)}{M_H^2} \left[ \left( 1 + 8C_F \frac{\alpha_s}{4\pi} \left( 1 + \log \frac{\mu^2}{-p^2} \right) \right) Q_{2}^{SLR} \right. \\
&\quad + \frac{\alpha_s}{4\pi} \left( \frac{1}{2} + \log \frac{M_H^2}{-p^2} \right) \left[ \frac{1}{2} N Q_{4}^{SLR} - \frac{1}{2} Q_{3}^{SLR} \right] \right] \tag{21} \\
A_{SLL} &= - \frac{\Delta_{L}^{ub}(H)}{2M_H^2} \frac{\Delta_{cd}^{cd}(H)}{2M_H^2} \left[ \left( 1 + 8C_F \frac{\alpha_s}{4\pi} \left( 1 + \log \frac{\mu^2}{-p^2} \right) \right) Q_{2}^{SLL} \right. \\
&\quad + \frac{\alpha_s}{4\pi} \left( \frac{1}{2} + \log \frac{M_H^2}{-p^2} \right) \left[ \frac{1}{2} N Q_{4}^{SLL} - \frac{1}{2} Q_{3}^{SLL} \right] \right] \tag{22} \\
\end{align*}

4 Matrix Elements of Operators

Calculating the diagrams in Fig. 3 we find after quark wave function renormalization and operator renormalization
\[
\langle Q_{VLL}^1 \rangle = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right] Q_{VLL}^1 + \frac{\alpha_s}{4\pi} \left(\log \frac{\mu^2}{-p^2} + \frac{7}{3}\right) \left[\frac{3}{N} Q_{VLL}^1 - 3 Q_{VLL}^2\right]
\]

\[
\langle Q_{VLL}^2 \rangle = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right] Q_{VLL}^2 + \frac{\alpha_s}{4\pi} \left(\log \frac{\mu^2}{-p^2} + \frac{7}{3}\right) \left[\frac{3}{N} Q_{VLL}^2 - 3 Q_{VLL}^1\right]
\]

\[
\langle Q_{VLR}^1 \rangle = \left[1 + 8C_F \frac{\alpha_s}{4\pi} \left(1 + \log \frac{\mu^2}{-p^2}\right)\right] Q_{VLR}^1 + \frac{\alpha_s}{4\pi} \left[\frac{3}{N} Q_{VLR}^1 - 3 Q_{VLR}^2\right]
\]

\[
\langle Q_{VLR}^2 \rangle = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \left(1 + \log \frac{\mu^2}{-p^2}\right)\right] Q_{VLR}^2 + \frac{\alpha_s}{4\pi} \left(\frac{1}{3} + \log \frac{\mu^2}{-p^2}\right) \left[-\frac{3}{N} Q_{VLR}^2 + 3 Q_{VLR}^1\right]
\]

\[
\langle Q_{SLR}^1 \rangle = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \log \frac{\mu^2}{-p^2}\right] Q_{SLR}^1 + \frac{\alpha_s}{4\pi} \left(\frac{1}{3} + \log \frac{\mu^2}{-p^2}\right) \left[-\frac{3}{N} Q_{SLR}^1 + 3 Q_{SLR}^2\right]
\]

\[
\langle Q_{SLR}^2 \rangle = \left[1 + 8C_F \frac{\alpha_s}{4\pi} \left(1 + \log \frac{\mu^2}{-p^2}\right)\right] Q_{SLR}^2 + \frac{\alpha_s}{4\pi} \left[\frac{3}{N} Q_{SLR}^2 - 3 Q_{SLR}^1\right]
\]

\[
\langle Q_{SLL}^1 \rangle = \left[1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{3}{2} + \log \frac{\mu^2}{-p^2}\right)\right] Q_{SLL}^1 + \frac{\alpha_s}{4\pi} \left(\frac{5}{6} + \log \frac{\mu^2}{-p^2}\right) \left[-\frac{3}{N} Q_{SLL}^1 + 3 Q_{SLL}^2\right]
\]

\[
\langle Q_{SLL}^2 \rangle = \left[1 + 8C_F \frac{\alpha_s}{4\pi} \left(1 + \log \frac{\mu^2}{-p^2}\right)\right] Q_{SLL}^2 + \frac{\alpha_s}{4\pi} \left(2 + \log \frac{\mu^2}{-p^2}\right) \left[-\frac{2 - N^2}{4N} Q_{SLL}^2 - \frac{1}{4} Q_{SLL}^4\right]
\]

\[
\langle Q_{SLL}^3 \rangle = \left[1 + \frac{\alpha_s}{4\pi} 3N \left(\log \frac{\mu^2}{-p^2} + \frac{5N^2 + 4}{6N^2}\right)\right] Q_{SLL}^3 - \frac{\alpha_s}{4\pi} 3 \left(\frac{3}{2} + \log \frac{\mu^2}{-p^2}\right) Q_{SLL}^4
\]

\[
\langle Q_{SLL}^4 \rangle = Q_{SLL}^4 + 24 \frac{\alpha_s}{4\pi} \left(\frac{1}{3} + \log \frac{\mu^2}{-p^2}\right) \left[\frac{1}{N} Q_{SLL}^2 - Q_{SLL}^1\right]
\]

\[
+ \frac{\alpha_s}{4\pi} \left[\frac{2}{N} Q_{SLL}^4 - 2 Q_{SLL}^3\right]
\]
We remark that for the matching performed in this paper only QCD corrections to the matrix elements \( \langle Q_{VLL}^2 \rangle, \langle Q_{VLR}^2 \rangle, \langle Q_{SLR}^2 \rangle \) and \( \langle Q_{SLL}^2 \rangle \) are required. The QCD corrections to the matrix elements of the remaining operators would enter the NLO analysis if the exchanged gauge bosons and scalars were coloured. They would also enter our analysis if we included \( \mathcal{O}(\alpha_s^2) \) corrections. We include these additional matrix elements for completeness.

5 Results for the Wilson Coefficients

In what follows we will list the general structure of the effective Hamiltonian in each case and subsequently we will list our results for the Wilson coefficients (we drop \( h.c. \) in what follows).

5.1 Colourless gauge boson

\[
\mathcal{H}_{\text{eff}}^\text{gauge} = \frac{(\Delta_{L}^b(A))^{*} \Delta_{L}^d(A)}{M_A^2} [C_{1}^{VLL}(\mu)Q_{L}^{VLL} + C_{2}^{VLL}(\mu)Q_{2}^{VLL}] \\
+ \frac{(\Delta_{L}^b(A))^{*} \Delta_{R}^d(A)^2}{M_A^2} [C_{1}^{VLR}(\mu)Q_{1}^{VLR} + C_{2}^{VLR}(\mu)Q_{2}^{VLR}] + L \leftrightarrow R. 
\]  

(33)

We find for an arbitrary number of colours \( N \)

\[
C_{1}^{VLL}(\mu) = \frac{\alpha_s}{4\pi} \left( -3 \log \frac{M_A^2}{\mu^2} + \frac{11}{2} \right),
\]

(34)

\[
C_{2}^{VLL}(\mu) = 1 + \frac{\alpha_s}{4\pi} \left( \frac{3}{N} \log \frac{M_A^2}{\mu^2} - \frac{11}{2N} \right) = 1 + \frac{\alpha_s}{4\pi} \left( \log \frac{M_A^2}{\mu^2} - \frac{11}{6} \right),
\]

(35)

\[
C_{1}^{VLR}(\mu) = \frac{\alpha_s}{4\pi} \left( 3 \log \frac{M_A^2}{\mu^2} + \frac{1}{2} \right),
\]

(36)

\[
C_{2}^{VLR}(\mu) = 1 + \frac{\alpha_s}{4\pi} \left( -\frac{3}{N} \log \frac{M_A^2}{\mu^2} - \frac{1}{2N} \right) = 1 + \frac{\alpha_s}{4\pi} \left( -\log \frac{M_A^2}{\mu^2} - \frac{1}{6} \right).
\]

(37)
5.2 Colourless scalar

\[
\mathcal{H}_{\text{scalar}}^{\text{eff}} = - \frac{\left(\Delta^a_{L}(A)\right)^* \Delta^a_{L}(A)}{M^2_H} \left[C^1_{\text{SLR}}(\mu) Q^1_{\text{SLR}} + C^2_{\text{SLR}}(\mu) Q^2_{\text{SLR}}\right]
\]
\[
- \frac{\left(\Delta^a_{L}(A)\right)^* \Delta^a_{L}(A)}{M^2_H} \left[C^3_{\text{SLR}}(\mu) Q^3_{\text{SLR}} + C^4_{\text{SLR}}(\mu) Q^4_{\text{SLR}}\right]
\]
\[
- \frac{\left(\Delta^a_{L}(H)\right)^* \Delta^a_{R}(H)}{M^2_H} \left[C^1_{\text{SLR}}(\mu) Q^1_{\text{SLR}} + C^2_{\text{SLR}}(\mu) Q^2_{\text{SLR}}\right] + L \leftrightarrow R
\]

We find for an arbitrary number of colours \( N \)

\[
C^1_{\text{SLR}}(\mu) = 3 \frac{\alpha_s}{4 \pi},
\]
\[
C^2_{\text{SLR}}(\mu) = 1 - \frac{\alpha_s}{4 \pi} \frac{3}{N} = 1 - \frac{\alpha_s}{4 \pi},
\]
\[
C^1_{\text{SLL}}(\mu) = 0,
\]
\[
C^2_{\text{SLL}}(\mu) = 1,
\]
\[
C^3_{\text{SLL}}(\mu) = \frac{\alpha_s}{4 \pi} \left(- \frac{1}{2} \log \frac{M^2_H}{\mu^2} + \frac{3}{4}\right),
\]
\[
C^4_{\text{SLL}}(\mu) = \frac{\alpha_s}{4 \pi} \left(\frac{1}{2N} \log \frac{M^2_H}{\mu^2} - \frac{3}{4N}\right) = \frac{\alpha_s}{4 \pi} \left(\frac{1}{6} \log \frac{M^2_H}{\mu^2} - \frac{1}{4}\right).
\]

We emphasize that the Wilson coefficients \( C^a_i \) of the “mirror” operators \( (P_L \leftrightarrow P_R) \) as defined by us are equal to the ones presented above. The formulae presented in this section are the main results of our paper.

6 Master Formulae for NLO Wilson Coefficient functions

With these results at hand and the known one-loop and two-loop anomalous dimension matrices that we have listed in Section 2, we can complete the NLO renormalization group analysis which can give us the Wilson coefficients at the low energy scales at which hadronic matrix elements are evaluated. In the case of the \( B \) decays the final result for the Wilson coefficients in each of the four sectors of operators with no mixing between different sectors can be written as follows:

\[
\tilde{C}(\mu_b) = \tilde{U}(\mu_b, \mu_{\text{in}}) \tilde{C}(\mu_{\text{in}}),
\]

(45)
where \( \hat{U} \) are evolution matrices and \( \vec{C} \) column vectors. For each VLL (VRR), VLR (VRL), SLR (SRL) sector \( \hat{U} \) is a \( 2 \times 2 \) matrix, while it is a \( 4 \times 4 \) matrix in the SLL (SRR) sector. Similarly for each VLL (VRR), VLR (VRL), SLR (SRL) sector \( \vec{C} \) is a two-dimensional column vector, while it is a four-dimensional one in the SLL (SRR) sector.

General formulae for the evolution matrix at the NLO level have been derived in [16]. In order to make our paper self-contained we recall these formulae in what follows:

\[
\hat{U}(\mu_b, \mu_{\text{in}}) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi}\hat{j}\right) \hat{U}^{(0)}(\mu_b, \mu_{\text{in}}) \left(1 - \frac{\alpha_s(\mu_{\text{in}})}{4\pi}\hat{j}\right).
\]  

(46)

Here \( \hat{U}^{(0)} \) is the evolution matrix in leading logarithmic approximation and the matrix \( \hat{J} \) expresses the NLO corrections to this evolution. We have

\[
\hat{U}^{(0)}(\mu_b, \mu_{\text{in}}) = \hat{V} \left(\begin{bmatrix} \alpha_s(\mu_{\text{in}}) \\ \alpha_s(\mu_b) \end{bmatrix} \hat{\gamma}^{(0)}_D \right) \hat{V}^{-1},
\]  

(47)

where \( \hat{V} \) diagonalizes \( \hat{\gamma}^{(0)\top} \)

\[
\hat{\gamma}^{(0)}_D = \hat{V}^{-1} \hat{\gamma}^{(0)\top} \hat{V}
\]  

(48)

and \( \hat{\gamma}^{(0)} \) is the vector containing the diagonal elements of the diagonal matrix \( \hat{\gamma}^{(0)}_D \).

If we define

\[
\hat{G} = \hat{V}^{-1} \hat{\gamma}^{(1)\top} \hat{V}
\]  

(49)

and a matrix \( \hat{H} \) whose elements are

\[
H_{ij} = \delta_{ij} \gamma_i^{(0)} \frac{\beta_1}{2\beta_0^2} - \frac{G_{ij}}{2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)}},
\]  

(50)

the matrix \( \hat{J} \) is given by

\[
\hat{j} = \hat{V} \hat{H} \hat{V}^{-1}.
\]  

(51)

We next write our results for the Wilson coefficients at the matching scale in a general form as follows

\[
\vec{C}(\mu_{\text{in}}) = \vec{C}_0 - \frac{\alpha_s(\mu_{\text{in}})}{4\pi} \vec{C}_1.
\]  

(52)

Finally combining these initial values with the evolution matrix (46) we obtain

\[
\vec{C}(\mu_b) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi}\hat{j}\right) \hat{U}^{(0)}(\mu_b, \mu_{\text{in}}) \left(1 - \frac{\alpha_s(\mu_{\text{in}})}{4\pi} \left(\vec{C}_1 + \hat{j} \vec{C}_0\right)\right).
\]  

(53)

We recall that when using this evolution down to low energy scales one has to insert the correct number of effective flavours. As this procedure is by now standard, we refer to Section IIIE in [17] for details.

We end this section by recalling certain features of the fundamental formula (53):

\[
\vec{C}(\mu_b) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi}\hat{j}\right) \hat{U}^{(0)}(\mu_b, \mu_{\text{in}}) \left(1 - \frac{\alpha_s(\mu_{\text{in}})}{4\pi} \left(\vec{C}_1 + \hat{j} \vec{C}_0\right)\right).
\]  

(53)
• The renormalization scheme dependence of the matrix \( \hat{J} \) on the left-hand side of the LO evolution matrix is cancelled by the one of hadronic matrix elements.

• This scheme dependence on the right-hand side of the LO evolution matrix is cancelled by the one of \( \vec{C}_1 \) calculated by us: \( \vec{C}_1 + \hat{J} \vec{C}_0 \) is renormalization scheme independent.

• The dependence on the precise choice of the scale \( \mu_b \) in the evolution matrix is cancelled by the one present in the hadronic matrix elements. In the case of \( \Delta F = 2 \) transitions in the SM, in which only a single operator is present this allows to introduce renormalization scheme and renormalization scale invariant parameters like \( B_K \).

• The dependence on the precise choice of the scale \( \mu_{in} \) in the evolution matrix is cancelled by the logarithmic terms in \( \vec{C}_1 \) that we calculated in this paper.

We will now look in more details at the last issue.

7 Renormalization Scale Dependence

One of the main virtues of our calculation of \( \mathcal{O}(\alpha_s) \) corrections to Wilson coefficients at the high energy matching scale \( \mu_{in} \) is the cancellation of the \( \mu_{in} \) dependence of the renormalization group evolution matrix by the \( \mu_{in} \) dependence of the Wilson coefficients in question. This cancellation requires particular values of the coefficients of the \( \log(M^2/\mu_{in}^2) \) in \( C_i(\mu_{in}) \) where \( M \) stands for the mass of a heavy gauge boson or heavy scalar involved. As this cancellation constitutes an important test of our results it is useful to derive a general condition on the coefficients of \( \log(M^2/\mu_{in}^2) \) in \( C_i(\mu_{in}) \).

To this end let us look at the evolution matrix in Eq. (45). Expanding this matrix around the two fixed scales \( m_b \) and \( M \) keeping only the logarithmic terms one obtains

\[
\hat{U}(\mu_b, \mu_{in}) = \left( 1 + \frac{\alpha_s(\mu_b)}{4\pi} \frac{\hat{\gamma}^{(0)\top}}{2} \log \frac{\mu_b^2}{m_b^2} \right) \hat{U}(m_b, M) \left( 1 + \frac{\alpha_s(\mu_{in})}{4\pi} \frac{\hat{\gamma}^{(0)\top}}{2} \log \frac{M^2}{\mu_{in}^2} \right),
\]

where \( \hat{\gamma}^{(0)} \) is the coefficient of \( \alpha_s \) in the one loop anomalous dimension matrix that describes the mixing of operators. The \( \hat{\gamma}^{(0)} \) matrices for VLL (VRR), VLR (VRL), SLR (SRL) and SLL (SRR) sectors have been collected in Section [2] Note that it is \( \hat{\gamma}^{(0)\top} \) and not \( \hat{\gamma}^{(0)} \) that enters Eq. (54). Moreover, in the study of the \( \mu_{in} \) dependence in the case of the scalar exchange one has to take into account that in this case the \( m^2(\mu_{in}) \) dependence is hidden in the coefficients \( (\Delta L_{ij}^{L/R}(H))^* \Delta L_{iL}^{R/L}(H) \).

Considering then the cases of colourless gauge bosons and scalars we find that the following quantities should be \( \mu_{in} \) - independent:

\[
R^{\text{gauge}} = \hat{U}(\mu_b, \mu_{in})\vec{C}(\mu_{in}),
\]

\[
R^{\text{scalar}} = \hat{U}(\mu_b, \mu_{in})\vec{C}(\mu_{in})m^2(\mu_{in}).
\]
For each VLL (VRR), VLR (VRL), SLR (SRL) sector $\vec{C}(\mu)$ is a two-dimensional column vector, while it is four-dimensional for the SLL (SRR) sector.

We write next in each case

$$\vec{C}(\mu_{in}) = \vec{C}_0 - \frac{\alpha_s(\mu_{in})}{4\pi} K \log \frac{M^2}{\mu_{in}^2},$$

where we suppressed $\mu_{in}$ independent $O(\alpha_s)$ terms and

$$m^2(\mu_{in}) = m^2(M) \left( 1 + \frac{\alpha_s(\mu_{in})}{4\pi} \gamma_m^{(0)} \log \frac{M^2}{\mu_{in}^2} \right),$$

with $\gamma_m^{(0)}$ governing the scale dependence of quark masses in QCD.

Imposing Eq. (55), the conditions for $K$ to ensure $\mu_{in}$ independence of resulting amplitudes in these two cases read

$$K^{\text{gauge}} = \frac{\gamma_m^{(0)}T}{2} \vec{C}_0,$$

$$K^{\text{scalar}} = \frac{\gamma_m^{(0)}T}{2} \vec{C}_0 + \gamma_m^{(0)} \vec{C}.$$ (58a, 58b)

Thus the coefficients of logarithms in $\vec{C}(\mu_{in})$ can be found without the calculation of the loop diagrams in Fig. [1] but formulae in Eq. (58) serve as a useful check of our results for logarithmic terms. These terms are renormalization scheme independent and while cancelling the $\mu_{in}$ dependence of $\hat{U}(\mu_b, \mu_{in})$ in perturbation theory cannot remove its renormalization scheme dependence at the NLO level. To this end as discussed in Section [6] the $O(\alpha_s)$ non-logarithmic terms have to be calculated which constitutes the main new result of our paper.

Inserting the formulae for the one-loop anomalous dimension matrices and $\gamma_m^{(0)}$, that we listed in Section [2] into Eq. (58) one can verify that the resulting coefficients $K^{\text{gauge}}$ and $K^{\text{scalar}}$ equal the coefficients of the logarithmic terms calculated by us. This implies that the inclusion of $O(\alpha_s)$ corrections in question remove the unphysical dependence on the precise value of the matching scale.

The manner in which the $\mu_{in}$-dependence is removed resembles the one which we encountered in the case of $\Delta F = 2$ transitions [11]:

- In the gauge boson case the $\mu_{in}$-dependence of $\hat{U}(\mu_b, \mu_{in})$ can only be cancelled by the corrections calculated by us.

- The case of scalar exchange with SLR couplings is quite different. Here the $\mu_{in}$-dependence of $\hat{U}(\mu_b, \mu_{in})$ is totally cancelled by the one of the $m^2(\mu_{in})$ so that even without our corrections the amplitudes are $\mu_{in}$ independent. The role of our calculation in this case is then the removal of the renormalization scheme dependence.

- In the case of SLL operators both the running quark masses and the corrections calculated by us are required for the removal of the unphysical matching scale dependence present in LO.
8 Numerical Analysis

We will now compute the size of unphysical $\mu_{\text{in}}$-dependence present in the LO expressions and we will demonstrate their reduction after the inclusion of $O(\alpha_s)$ corrections.

Compared to our analysis of $\Delta F = 2$ processes in [11] a numerical analysis of left-over unphysical scale dependences in the decay amplitudes is complicated by three facts:

- The analogues of $P_i^a$ factors [15] that summarize the renormalization group effects between high energy and low energy scales and include also the values of hadronic matrix elements do not exist in the literature for decays discussed here although obviously complete NLO analyses for the VLL case are known.

- The hadronic matrix elements of new operators contain larger theoretical uncertainties than in the case of $\Delta F = 2$ decays.

- The increased number of operators relative to the ones in the case of $\Delta F = 2$ transitions makes the analysis in questions very involved.

Leaving a detailed analysis in a concrete model for the future, we nevertheless illustrate the size of unphysical $\mu_{\text{in}}$-dependence present in the LO expressions in the sectors VLL, VLR and SLL. In the SLR sector our corrections have no impact on the cancellation of the $\mu_{\text{in}}$ dependence as we discussed above.

We will solve the problem of poorly known hadronic elements in the following manner. We will go to the operator basis in which the one-loop anomalous dimension matrices are diagonal. At LO in each of the VLL, VLR and SLR sectors there are then two-operators that do not mix with each other and with other operators of other sectors. There are four such operators in the SLL sector. This property remains true at NLO only in the VLL sector as only in this sector the one-loop and two-loop anomalous dimension matrices are diagonalized by the same matrix $\hat{V}$. For VLR and also scalar sectors one has to use the general formulae presented in Section 6.

Yet, as we have seen in the previous section in order to study the cancellation of the $\mu_{\text{in}}$-dependence present in the LO expressions it is sufficient to study the evolution matrix at the leading order and keep only the leading logarithms in $\tilde{C}^{(1)}$. Diagonalizing then the matrices $\hat{\gamma}^{(0)}$ in all four sectors we can factor out the hadronic matrix element in each case so that the issue of the study of the $\mu_{\text{in}}$-dependence as expected can be investigated transparently without any hadronic uncertainties.

Denoting by $C_{\pm}^a$ the Wilson coefficients of the operators $Q_{\pm}^a$ corresponding to the eigenvalues of the two anomalous dimension matrices $a = \text{VLL, VLR}$ and normalizing $C_{\pm}^a$ so that they are equal unity at the matching scale at LO we find:

VLL Case

$$Q_{\text{VLL}}^+ = \frac{Q_{2,\text{VLL}}^+ + Q_{1,\text{VLL}}^+}{2}, \quad C_{\text{VLL}}^+ = C_{2,\text{VLL}}^+ + C_{1,\text{VLL}}^+, \quad \gamma^{(0)+}_{\text{VLL}} = -\frac{6}{N} + 6,$$  (59)
\[ Q_{VLL}^+ = \frac{Q_2^{VLL} - Q_1^{VLL}}{2}, \quad C_{VLL}^+ = C_2^{VLL} - C_1^{VLL}, \quad \gamma_{VLL}^{(0)+} = \frac{6}{N} - 6. \] (60)

**VLR Case**

\[ Q_{VLR}^+ = Q_2^{VLR} - \frac{Q_1^{VLR}}{N}, \quad C_{VLR}^+ = C_2^{VLR}, \quad \gamma_{VLR}^{(0)+} = \frac{6}{N}, \] (61)

\[ Q_{VLR}^- = \frac{Q_1^{VLR}}{N}, \quad C_{VLR}^- = C_2^{VLR} + NC_1^{VLR}, \quad \gamma_{VLR}^{(0)-} = \frac{6}{N} - 6N. \] (62)

Here \( \gamma_a^{(0)\pm} \) denote the anomalous dimensions of the operators \( Q_a^\pm \). Then the quantities \((a = VLL, VLR)\)

\[ R_a^\pm = \left[ \frac{\alpha_s(\mu_{in})}{\alpha_s(M)} \right]^{\gamma_a^{(0)\pm}}_{\gamma_a(0)} \left( 1 + \frac{\alpha_s(\mu_{in})}{4\pi} \left[ -K_a^\pm \log \frac{M^2}{\mu_{in}^2} + r_a^\pm \right] \right), \] (63)

where \( r_a^\pm \) denote the non-logarithmic \( \mathcal{O}(\alpha_s) \) corrections in the new basis should be \( \mu_{in} \) independent at \( \mathcal{O}(\alpha_s) \). Here \( M \) is the mass of the exchanged gauge boson that we will set to 1 TeV in our numerical calculations.

We should emphasize that in addition to \( \mathcal{O}(\alpha_s) \) corrections proportional to \( r_a^\pm \) there are also corrections at this order that come from the evolution matrix. We have denoted them by \( \tilde{J} \) in Section 6. We do not include them in our analysis in order to exhibit the size of the corrections calculated here but they have to be included in any phenomenological analysis in order to obtain renormalization scheme independent results.

From our discussion of the previous section and the diagonalization performed here we find that

\[ K_a^\pm = \frac{\gamma_a^{(0)\pm}}{2}, \quad a = VLL, VLR \] (64)

\[ r_{VLL}^+ = \frac{11(N - 1)}{2N} = \frac{11}{3}, \quad r_{VLL}^- = -\frac{11(N + 1)}{2N} = -\frac{22}{3}; \] (65a)

\[ r_{VLR}^+ = -\frac{1}{2N} = -\frac{1}{6}, \quad r_{VLR}^- = \frac{N^2 - 1}{2N} = \frac{4}{3}; \] (65b)

and indeed then the \( R_a^\pm \) should be equal unity after the inclusion of only the \( \mathcal{O}(\alpha_s) \) logarithmic corrections \( K_a^\pm \) calculated here. The departure from unity at NLO signals the presence of \( r_a^\pm \) terms.

In Fig. 5 we plot \( R_a^\pm \) for \( a = VLL, VLR \) as functions of the matching scales setting as an example the masses of gauge bosons to 1 TeV for three cases: Without the \( \mathcal{O}(\alpha_s) \) corrections in the Wilson coefficients (dashed blue line), with the contribution proportional to \( K_a^\pm \) (dotted green line) and including both logarithmic \( K_a^\pm \) and non-logarithmic \( r_a^\pm \).
Figure 5: The quantities \( R^\pm_a \) (\( a = \text{VLL, VLR} \)) defined in Eq. (63) as a function of \( \mu_{\text{in}} \) for \( M = 1 \text{ TeV} \). The LO result (removing contribution proportional to \( K^\pm_a \) and \( r^\pm_a \)) is shown by the dashed blue line, the dotted green line is the NLO result including only logarithmic \( \mathcal{O}(\alpha_s) \) corrections \( K^\pm_a \) and the solid red line shows the NLO result including both \( K^\pm_a \) and \( r^\pm_a \).

corrections (solid red line). While the dashed blue lines exhibit a significant \( \mu_{\text{in}} \) dependence, the dotted green lines and solid red lines stay nearly constant over the considered \( \mu_{\text{in}} \) range. The dotted green lines that include only logarithmic \( \mathcal{O}(\alpha_s) \) corrections are equal to 1 at \( \mu_{\text{in}} = 1 \text{ TeV} \) as expected. For the solid red lines a small shift relative to the dotted green ones occurs which is due to the non-logarithmic corrections \( r^\pm_a \). Only in the VLL sector which has larger \( r^\pm_a \) than the VLR sector a slight \( \mu_{\text{in}} \) dependence occurs. This dependence can only be cancelled by NNLO corrections.

**SLL Case**

The diagonalization in the SLL case is a bit more involved since four operators are present. Furthermore, in the scalar case we also have to take into account the \( \mu_{\text{in}} \) dependence of the quark masses (see Eq. (55)). Thus instead of Eq. (63) the following quantities

\[
R^j_{\text{SLL}} = \left[ \frac{\alpha_s(\mu_{\text{in}})}{\alpha_s(M)} \right]^{\gamma(0)_{\text{SLL}} \gamma(0)_{\text{SLL}}} \left( 1 + \frac{\alpha_s(\mu_{\text{in}})}{4\pi} \left[ -K^j_{\text{SLL}} \log \frac{M^2}{\mu_{\text{in}}^2} + r^j_{\text{SLL}} \right] \right) \frac{m^2(\mu_{\text{in}})}{m^2(M)}, \quad j = \pm, \pm
\]

should be \( \mu_{\text{in}} \) independent at \( \mathcal{O}(\alpha_s) \). For simplicity we set in the following \( N = 3 \) in order
Figure 6: The quantities $R_{SLL}^j$ ($j = \pm\pm, \pm$) as a function of $\mu_{\text{in}}$ for $M = 1$ TeV. The LO result (removing contribution proportional to $K_{SLL}^j$ and $r_{SLL}^j$) is shown by the dashed blue line, the dotted green line is the NLO result including only logarithmic $\mathcal{O}(\alpha_s)$ corrections $K_{SLL}^j$ and the solid red line shows the NLO result including both $K_{SLL}^j$ and $r_{SLL}^j$.

to shorten the formulae. We find

$$K_{SLL}^j = \frac{\gamma_{\text{SLL}}^{(0)j}}{2} + \gamma_{\text{in}}^{(0)}, \quad j = \pm\pm, \pm,$$

with

$$\gamma_{\text{SLL}}^{(0)\pm\pm} = \frac{2}{3}(1 \pm \sqrt{241}), \quad \gamma_{\text{SLL}}^{(0)\pm} = \frac{2}{3}(-17 \pm \sqrt{241}).$$

The non-logarithmic corrections in the basis in which $\gamma_{\text{SLL}}^{(0)}$ is diagonal and LO Wilson coefficients $C_{SLL}^j$ ($j = \pm\pm, \pm$) are normalized to 1 read

$$r_{SLL}^{\pm\pm} = \frac{1}{2}(25 \pm \sqrt{241}), \quad r_{SLL}^{\pm} = \frac{1}{2}(7 \pm \sqrt{241}).$$

In the Appendix A we list the Wilson coefficients and operators in this new basis.

In Fig. 6 we show the results for the SLL sector, again for the three different cases: LO (dashed blue), NLO with only logarithmic corrections (dotted green) and NLO with both logarithmic and non-logarithmic corrections (solid red). As expected from the values of $K_{SLL}^j$ the dashed blue lines show a very strong $\mu_{\text{in}}$ dependence in the case of $\pm\pm$ and $\pm$. 
The inclusion of the logarithmic $\mathcal{O}(\alpha_s)$ terms calculated by us practically removes this dependence (dotted green lines). However, the large size of the non-logarithmic terms $\gamma^{++/+}_{\text{SLR}}$ implies significant left-over $\mu_{\text{in}}$ dependence in these two cases at the NLO level (solid red lines) that can only be removed by NNLO corrections.

SLR case

At last we discuss the special case SLR where no logarithms appear in the matching conditions. Diagonalizing $(\gamma^{(0)}_{\text{SLR}})^\top$ we get

$$Q^\text{SLR}_+ = Q^\text{SLR}_2 - NQ^\text{SLR}_1, \quad C^\text{SLR}_+ = -\frac{1}{N} C^\text{SLR}_1, \quad \gamma^{(0)+}_{\text{SLR}} = \frac{6}{N}, \quad (70)$$

$$Q^\text{SLR}_- = Q^\text{SLR}_2, \quad C^\text{SLR}_- = \frac{1}{N} C^\text{SLR}_1 + C^\text{SLR}_2, \quad \gamma^{(0)-}_{\text{SLR}} = -6N + \frac{6}{N}. \quad (71)$$

We can read off that the coefficient $C^\text{SLR}_+$ cannot be unity in leading order. Only $C^\text{SLR}_-$ can be normalized to 1 and its $\mu_{\text{in}}$ dependence related to the anomalous dimension of $Q^\text{SLR}_-$ is exactly cancelled by the $\mu_{\text{in}}$ dependence of quark masses. This means simply that $\gamma^{(0)-}_{\text{SLR}} = -2\gamma^{(0)}_m$ as seen explicitly in Eq. (71).

9 Summary

In various extensions of the Standard Model (SM) tree level non-leptonic decays of hadrons receive contributions from new heavy gauge bosons and scalars. Prominent examples are the right-handed $W'$ bosons in left-right symmetric models and charged Higgs ($H^\pm$) particles in models with extended scalar sector like two Higgs doublet models and supersymmetric models. Of particular interest are the decays in which the contributing local operators involve four different flavours so that penguin operators cannot contribute and the decays are simpler to analyse theoretically. Anticipating the important role of such decays at the LHCb, KEKB and Super-B in Rome and having in mind future improved lattice calculations combined with the QCD factorization approach, we have completed the existing NLO QCD calculations of these processes by calculating $\mathcal{O}(\alpha_s)$ corrections to matching conditions for the Wilson coefficients of all contributing operators in the NDR-\overline{MS} scheme in any extension of the SM. The main results of our paper can be found in Section 5.

Our calculation allowed to reduce certain unphysical scale and renormalization scheme dependences in the existing NLO calculations. Our results can also be applied to models with tree-level heavy neutral gauge boson and scalar exchanges in $\Delta F = 1$ decays and constitute an important part of NLO analyses of those non-leptonic decays to which also penguin operators contribute.

For completeness we have collected all the relevant formulae necessary to perform the full NLO renormalization group analysis that would require the evaluation of the hadronic matrix elements in the the NDR-\overline{MS} scheme. They can be found in Section 6 with all ingredients given in Sections 2 and 5.
We are aware of the fact that present hadronic uncertainties in the decays considered are significantly larger than corrections calculated by us. However, one should recall that twenty years ago similar comments were made in connection with NLO QCD corrections within the SM \cite{5}. During the last decade after significant progress in lattice calculation has been made, the results of the 1990’s constitute an important part of any phenomenological analysis of weak decays in the SM. We expect that this will be the case of our results in due time.

**Acknowledgements**

This research was done in the context of the ERC Advanced Grant project “FLAVOUR” (267104) and was partially supported by the DFG cluster of excellence “Origin and Structure of the Universe”.

### A  Change of basis: SLL case

For our numerics in Section \[8\] we had to diagonalize the anomalous dimension matrix in the SLL sector. In this case the situation is a bit more intricate as four operators are involved. The eigenvalues $\gamma_{\text{SLL}}^{(0)}(j = \pm\pm, \pm)$ are given in Eq. (68) for $N = 3$. For completeness we list here the Wilson coefficients and operators in the basis in which $\gamma_{\text{SLL}}^{(0)}$ is diagonal and where LO Wilson coefficients $C_{\text{SLL}}^{j}(j = \pm\pm, \pm)$ are normalized to 1. We recall, as found in Section \[5\] that in the original basis only the coefficient $C_{\text{SLL}}^{2}$ is non-vanishing in the LO.

\begin{align}
Q_{\pm\pm}^{\text{SLL}} &= \left(-\frac{1}{8} \pm \frac{19}{8\sqrt{241}}\right)Q_{1}^{\text{SLL}} + \left(\frac{1}{4} \mp \frac{4}{\sqrt{241}}\right)Q_{2}^{\text{SLL}} + \left(\frac{1}{32} \mp \frac{15}{32\sqrt{241}}\right)Q_{3}^{\text{SLL}} \\
&\pm \frac{1}{16\sqrt{241}}Q_{4}^{\text{SLL}},
\end{align}

\begin{align}
C_{\pm\pm} &= \frac{1}{2}(15 \pm \sqrt{241})C_{1}^{\text{SLL}} + C_{2}^{\text{SLL}} + 2(19 \pm \sqrt{241})C_{3}^{\text{SLL}} + 4(16 \pm \sqrt{241})C_{4}^{\text{SLL}},
\end{align}

\begin{align}
Q_{\pm}^{\text{SLL}} &= \left(\frac{1}{8} \mp \frac{1}{8\sqrt{241}}\right)Q_{1}^{\text{SLL}} + \left(\frac{1}{4} \pm \frac{4}{\sqrt{241}}\right)Q_{2}^{\text{SLL}} + \left(\frac{1}{32} \pm \frac{21}{32\sqrt{241}}\right)Q_{3}^{\text{SLL}} \\
&\mp \frac{5}{16\sqrt{241}}Q_{4}^{\text{SLL}},
\end{align}

\begin{align}
C_{\pm} &= \frac{1}{10}(21 \pm \sqrt{241})C_{1}^{\text{SLL}} + C_{2}^{\text{SLL}} + \frac{2}{5}(1 \pm \sqrt{241})C_{3}^{\text{SLL}} + \frac{4}{5}(-16 \mp \sqrt{241})C_{4}^{\text{SLL}}.
\end{align}
References

[1] R. Fleischer, *In Pursuit of New Physics with B Decays*, [arXiv:1105.5998](http://arxiv.org/abs/1105.5998).

[2] G. Buchalla, *B Physics Theory for Hadron Colliders*, [arXiv:0809.0532](http://arxiv.org/abs/0809.0532).

[3] M. Antonelli, D. M. Asner, D. A. Bauer, T. G. Becher, M. Beneke, *et. al.*, *Flavor Physics in the Quark Sector*, *Phys.Rept.* **494** (2010) 197–414, [arXiv:0907.5386](http://arxiv.org/abs/0907.5386).

[4] M. Beneke, G. Buchalla, M. Neubert, and C. T. Sachrajda, *QCD factorization for B → Kπ, ππ decays: Strong phases and CP violation in the heavy quark limit*, *Phys.Rev.Lett.* **83** (1999) 1914–1917, [hep-ph/9905312](http://arxiv.org/abs/hep-ph/9905312).

[5] A. J. Buras, *Climbing NLO and NNLO Summits of Weak Decays*, [arXiv:1102.5650](http://arxiv.org/abs/1102.5650).

[6] M. Ciuchini, E. Franco, V. Lubicz, G. Martinelli, I. Scimemi, *et. al.*, *Next-to-leading order QCD corrections to ΔF = 2 effective Hamiltonians*, *Nucl.Phys.* **B523** (1998) 501–525, [hep-ph/9711402](http://arxiv.org/abs/hep-ph/9711402).

[7] M. Ciuchini, V. Lubicz, L. Conti, A. Vladikas, A. Donini, *et. al.*, *ΔM_K and ǫ_K in SUSY at the next-to-leading order*, *JHEP* **9810** (1998) 008, [hep-ph/9808328](http://arxiv.org/abs/hep-ph/9808328). Erratum added online, Mar/29/2000.

[8] A. J. Buras, M. Misiak, and J. Urban, *Two loop QCD anomalous dimensions of flavor changing four quark operators within and beyond the standard model*, *Nucl.Phys.* **B586** (2000) 397–426, [hep-ph/0005183](http://arxiv.org/abs/hep-ph/0005183).

[9] G. Altarelli, G. Curci, G. Martinelli, and S. Petrarca, *QCD Nonleading Corrections to Weak Decays as an Application of Regularization by Dimensional Reduction*, *Nucl.Phys.* **B187** (1981) 461.

[10] A. J. Buras and P. H. Weisz, *QCD Nonleading Corrections to Weak Decays in Dimensional Regularization and ’t Hooft-Veltman Schemes*, *Nucl.Phys.* **B333** (1990) 66.

[11] A. J. Buras and J. Girrbach, *Complete NLO QCD Corrections for Tree Level ΔF = 2 FCNC Processes*, [arXiv:1201.1302](http://arxiv.org/abs/1201.1302).

[12] A. J. Buras, *Weak Hamiltonian, CP violation and rare decays*, [hep-ph/9806471](http://arxiv.org/abs/hep-ph/9806471). To appear in ’Probing the Standard Model of Particle Interactions’, F.David and R. Gupta, eds., 1998, Elsevier Science B.V.

[13] S. Herrlich and U. Nierste, *Evanescent operators, scheme dependences and double insertions*, *Nucl.Phys.* **B455** (1995) 39–58, [hep-ph/9412375](http://arxiv.org/abs/hep-ph/9412375).

[14] A. J. Buras, M. Jamin, and P. H. Weisz, *Leading and next-to-leading QCD corrections to ǫ parameter and B^0 − ̅B^0 mixing in the presence of a heavy top quark*, *Nucl. Phys.* **B347** (1990) 491–536.
[15] A. J. Buras, S. Jager, and J. Urban, *Master formulae for $\Delta F = 2$ NLO QCD factors in the standard model and beyond*, Nucl.Phys. **B605** (2001) 600–624, [hep-ph/0102316](http://arxiv.org/abs/hep-ph/0102316).

[16] A. J. Buras, M. Jamin, M. Lautenbacher, and P. H. Weisz, *Effective Hamiltonians for $\Delta S = 1$ and $\Delta B = 1$ nonleptonic decays beyond the leading logarithmic approximation*, Nucl.Phys. **B370** (1992) 69–104.

[17] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, *Weak decays beyond leading logarithms*, Rev.Mod.Phys. **68** (1996) 1125–1144, [hep-ph/9512380](http://arxiv.org/abs/hep-ph/9512380).