POLYGONS IN MINKOWSKI SPACE AND GELFAND-TSETLIN FOR PSEUDOUNITARY GROUPS

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ABSTRACT. We study the symplectic geometry of the moduli spaces of polygons in the Minkowski 3-space. These spaces naturally carry completely integrable systems with periodic flows. We extend the Gelfand-Tsetlin method to pseudo-unitary groups and show that the action variables are given by the Minkowski lengths of non-intersecting diagonals.

1. Introduction

The geometry of moduli spaces of polygons in the Euclidean 3-space has been studied by many authors, notably by Klyachko [9], Kapovich and Millson [8], Haussman and Knutson [5], among others, and many interesting results in symplectic geometry have been obtained. In this paper we study polygons in the Minkowski 3-space and obtain a variety of results, similar in spirit, but which are, on the other hand, considerably different, and the differences are illuminated by the use of pseudounitary groups $U(p, q)$ and their coadjoint orbits.

Polygons in the Minkowski 3-space were briefly considered by Millson in [10]. Original application of the Gelfand-Tsetlin method to integrable systems on coadjoint orbits is due to Guillemin and Sternberg [4].

2. Polygons in Minkowski space

A surface $\mathcal{H}_R$ in $\mathbb{R}^3$ defined by the equation $t^2 - x^2 - y^2 = R^2$ is called a *pseudosphere*, since the Minkowski metric of signature (2,1) restricts to the constant curvature Riemannian metric on it. Alternatively, we can think of a pseudosphere as a set of points equidistant from the origin in $\mathbb{R}^3$ with respect to the Minkowski metric. The connected component $\mathcal{H}_R^+$ corresponding to $t > 0$ will be called a future pseudosphere, and $\mathcal{H}_R^-$ corresponding to $t < 0$ a past pseudosphere respectively. Note that the group $SU(1, 1)$ acts transitively on each connected component, since we can think of $\mathbb{R}^3$ as $\mathfrak{su}(1, 1)^*$, where these connected components can be thought of as elliptic coadjoint orbits. Therefore each has a natural symplectic structure, invariant under the action of the group. The metric

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is invariant as well, since SU(1, 1) acts by isometries. Each connected component is also a Kähler manifold, since it is isomorphic to the hyperbolic plane SU(1, 1)/U(1).

The purpose of this section is to study the geometry of the symplectic quotients of the product of several future and past pseudospheres with respect to the diagonal SU(1, 1)-action. These spaces have a natural interpretation as polygon spaces in Minkowski 3-space. Let us start by fixing notation. Let $\mathbf{r} = (r_1, ..., r_n)$ be an $n$-tuple of positive real numbers, and let us fix two positive integers $p \geq q$ such that $p + q = n$. In the language of polygons, this will mean that we have the first $p$ sides in the future timelike cone and the last $q$ in the past. The Minkowski length of the $i$-th side is equal to $r_i$, and the space of closed polygons, i.e. those where the sum of the first $p$ sides in the future timelike cone equals the negative of the sum of the last $q$ sides in the past timelike cone, is identified with the zero level set of the moment map:

$$\mu : \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \to \mathfrak{su}(1, 1)^*.$$ 

Here $\mathcal{O}_i \simeq \mathcal{H}_{r_i}^+$ is a future pseudosphere of radius $r_i$ if $1 \leq i \leq p$ and past $\mathcal{O}_i \simeq \mathcal{H}_{r_i}^-$ if $p + 1 \leq i \leq n$ with its coadjoint orbit symplectic structure. Note that the triangle inequalities in the future (or past) timelike cone are reversed from the usual ones. If $v_1$ and $v_2$ are two equally directed timelike vectors, then $||v_1 + v_2|| \geq ||v_1|| + ||v_2||$. For convenience, let us fix the perimeter of the polygon to be equal to 2. This means that $\sum_{i=1}^n r_i = 2$. Note that each $\mathcal{O}_i$ can itself be naturally interpreted as the symplectic quotient of $\mathbb{C}^2$ with complex coordinates $(z, w)$, and symplectic form $\sqrt{-1 \over 2} (dz \wedge d\bar{z} - dw \wedge d\bar{w})$ for $1 \leq i \leq p$ and negative of that for $p + 1 \leq i \leq n$, at the level $r_i$, with respect to the diagonal circle action.

Let us denote by $M_\mathbf{r}$ the quotient $\mu^{-1}(0)/SU(1, 1)$. This is a quotient of a non-compact space, in general, by the action of a non-compact Lie group. Therefore, questions of its topology and geometry require careful consideration. However, notice that for a generic choice of $\mathbf{r}$, i.e. such that $M_\mathbf{r}$ is non-empty and $r_1 + \cdots + r_p \neq r_{p+1} + \cdots + r_n$, every point in the moduli space represents a polygon with a trivial stabilizer. This means that the group PSU(1, 1) acts freely and properly on $\mu^{-1}(0)$. An easy computation shows that with our assumptions, 0 is a regular value for the map $\mu$, and therefore [1 Theorem 1.11.4], the quotient space $M_\mathbf{r}$ has the structure of a smooth manifold of dimension $2n - 6$. In a later section we will see a different approach to $M_\mathbf{r}$ through a fixed point set of an involution in a hyper-Kähler manifold.

One of the powerful tools in dealing with polygons in a compact setting proved to be reduction in stages, or symplectic Gelfand-MacPherson correspondence. In fact, an appropriate modification of this method proves to be useful for our purposes as well.
Let us consider the space $\mathbb{C}^{2n}$ with complex coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_n)$ and symplectic form $\Omega$ given by

$$2\sqrt{-1}\Omega = \sum_{i=1}^{p}(dz_i \wedge d\bar{z}_i - dw_i \wedge d\bar{w}_i) - \sum_{j=p+1}^{n}(dz_j \wedge d\bar{z}_j - dw_j \wedge d\bar{w}_j).$$

We introduce two elements $z = (z_1, \ldots, z_n)^T$ and $w = (w_1, \ldots, w_n)^T$ of $\mathbb{C}^n$ and comprise an $n \times 2$ matrix $M = (z \ w)$, representing an element of $\mathbb{C}^{2n}$. There is a natural left action of $U(p, q)$ on $\mathbb{C}^{2n}$ given by left multiplication $M \mapsto AM$, where $A \in U(p, q)$ and $M$ as before, and similarly a natural right action of $U(1, 1)$. Both are hamiltonian actions with respective moment maps

$$\eta : \mathbb{C}^{2n} \rightarrow u(p, q)^*, \quad \eta(M) = MJ_{1,1}M^*J_{p,q},$$

where $M^* = M^T$, as usual, and

$$\nu : \mathbb{C}^{2n} \rightarrow u(1, 1)^*, \quad \nu(M) = J_{1,1}M^*J_{p,q}M,$$

where $J_{1,1} = \text{diag}(1, -1)$ and $J_{p,q} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$.

To make it more explicit, we note that

$$\eta(M) = \left( \begin{array}{cc}
||z||^2 & \langle w, z \rangle \\
-\langle z, w \rangle & -||w||^2
\end{array} \right),$$

where the norm and the pairing come from the standard pseudohermitian structure on $\mathbb{C}^n$ of signature $(p, q)$.

We notice that the left action of the diagonal $\mathbb{T}^n \subset U(p, q)$ commutes with the right action of $U(1, 1)$. Now, shifting for convenience by the identity matrix (i.e. the central matrix with the trace equal to the perimeter), we can look at the level set of $\nu$ corresponding to the identity $2 \times 2$ matrix $I_2$ in $u(1, 1)^*$, i.e. the orthonormal pairs of vectors $(z, w)$ in $\mathbb{C}^n$ such that $z$ is timelike and $w$ is spacelike. The quotient of this level set by the aforementioned action of $U(1, 1)$ is naturally isomorphic to the semisimple symmetric space

$$X_{p,q,1,1} \simeq U(p, q)/U(1, 1) \times U(p-1, q-1)$$

(recall that we are working under the assumption that $p \geq q \geq 1$). This space is a pseudo-hermitian symmetric space; it has invariant complex and compatible symplectic structures.

The residual hamiltonian action of $\mathbb{T}^n$ on $X_{p,q,1,1}$ has as a moment map $\eta_T$ the projection of $\eta(M)$ onto the diagonal, i.e.

$$\eta_T(M) = (|z_1|^2 - |w_1|^2, \ldots, |z_p|^2 - |w_p|^2, |w_{p+1}|^2 - |z_{p+1}|^2, \ldots, |w_n|^2 - |z_n|^2).$$
Naturally, the quotient of the level set of \( \eta_T \) corresponding to \((r_1, ..., r_n)\) is the moduli space of polygons in question \( M_r \).

This correspondence between the two symplectic quotients helps to understand the following important feature of the space \( M_r \):

**Proposition 2.1.** If \( q = 1 \), the space \( M_r \) is compact, and if \( q > 1 \), the space \( M_r \) is not compact.

**Proof.** The coadjoint orbit of \( \eta(M) \) is elliptic, and passes through

\[
\Lambda = \text{diag}(1, 0, ..., 0, 1, 0, ..., 0)
\]

We will show that for \( q > 1 \) there is no positive adapted system of roots, in terminology of [12, Definition VII.2.6], for which \( \Lambda \) is admissible. Then [loc.cit., Theorem VIII.1.8] would immediately imply that the map \( \eta_T : X_{p,q,1,1} \to t^* \) is not proper. Given that \( r \) is chosen generic in the image, we will be able to conclude the statement. However, one characterization of the positive adapted root system, [loc.cit., Proposition VII.2.12] valid for quasihermitian Lie algebras, implies that the condition of \( \Delta^+ \) being adapted is equivalent to the system \( \Delta^+ \) of positive non-compact roots being invariant under the baby Weyl group. Now it is easy to see that the latter is possible if and only if \( q = 1 \), in which case \( \Delta^+ \) should be taken the negative of the standard subset of positive roots. Q.E.D.

Another, more visual way of seeing that \( M_r \) is only compact when \( q = 1 \), can be found using polygons. First, let us explain compactness for \( q = 1 \). The last side of the polygon, \( e_n \), of Minkowski length \( r_n \) can be represented, after applying the action by an element of \( SU(1,1) \) by a vector in \( \mathbb{R}^3 \) with coordinates \((0, 0, -r_n)\). Therefore, the \((n-1)\) future timelike sides of the polygon should add up to \((0, 0, r_n)\), since the only degree of symmetry left is the circle rotation around the \( t \)-axis. Clearly, this space is bounded and closed, therefore compact. On the contrary, when \( q > 1 \), the space \( M_r \) is not compact. Let us explain this in the simplest example \( p = q = 2 \) and \( r_1 = r_2 = r_3 = r_4 = 1/2 \). Note that again, using the action of \( SU(1,1) \), we can assume that \( e_4 = (0, 0, -1/2) \), and the only degree of symmetry left is again the rotation about the \( t \)-axis. We will produce an explicit sequence of points in \( M_r \) with no limit. Let \( x_n \) be the closed polygon corresponding to \( e_1 = -e_4 = (0, 0, 1/2), e_2 = (n, 0, \sqrt{n^2 + 1/4}), \) and \( e_3 = -e_2 \). Clearly, the sequence \( (x_n) \) has no limit points in \( M_r \) and thus \( M_r \) is not compact.

Let us denote by \( d_i \) the length of \( i \)-th diagonal, i.e. the diagonal connecting the first and the \((i + 1)\)-st vertex. By our convention, \( d_1 = r_1, d_{n-1} = r_n, \) and \( d_n = 0 \). In the next section we will show that similarly to the compact situation, the lengths of the \((n-3)\) varying diagonals \( d_2, ..., d_{n-2} \) define a completely integrable system on \( M_r \), and are action variables for \((n-3)\) periodic flows.
3. Symplectic structure on the moduli space

In this section we will spell out the elementary definition of the symplectic structure on
the space $M_r$, in quite a similar way to [S Section 3]. First of all, let us define the follow-
ing two operations on $\mathbb{R}^3$, with coordinate functions $(x, y, t)$. For two vectors $v_1 = (x_1, y_1, t_1)^T$ and $v_2 = (x_2, y_2, t_2)^T$ we define the Minkowski cross
product $\times$ and the Minkowski dot product $\circ$ as follows:

$$v_1 \times v_2 = \det \begin{pmatrix} -i & -j & k \\ x_1 & y_1 & t_1 \\ x_2 & y_2 & t_2 \end{pmatrix}, \quad \text{and} \quad v_1 \circ v_2 = -x_1 x_2 - y_1 y_2 + t_1 t_2,$$

where $i, j, \text{and} \ k$ are the usual unit vectors in the positive directions of the $x$-, $y$-, and $t$-axes respectively. Note that $\circ$ is non-degenerate and positive definite in the timelike cone.

These operations satisfy the usual properties of the dot and cross products in $\mathbb{R}^3$:

$$a \times b = -b \times a$$
$$\left( a \times b \right) \times c + \left( b \times c \right) \times a + \left( c \times a \right) \times b = 0$$
$$a \times (b \times c) = b (a \circ c) - c (a \circ b)$$
$$a \circ (b \times c) = \det(a \ b \ c)$$
$$\left( a \times b \right) \times (c \times d) = \det(a \ b \ d)c - \det(a \ b \ c)d$$

The first two properties show that $(\mathbb{R}^3, \times)$ is a Lie algebra, in fact isomorphic to $\mathfrak{su}(1, 1)$ under the following map:

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} \sqrt{-1} \cdot t & x + \sqrt{-1} \cdot y \\ x - \sqrt{-1} \cdot y & \sqrt{-1} \cdot t \end{pmatrix}. $$

Under this identification, $\circ$ corresponds to $-2\text{Tr}(AB)$.

Now the description of the symplectic two-form $\omega$ on the hyperboloid $\mathcal{H}_R$ given by the equation $t^2 - x^2 - y^2 = R^2$ is given by

$$\omega_u(v_1, v_2) = \frac{1}{R^2} u \circ (v_1 \times v_2),$$

where $u$ is a point on the hyperboloid, and $v_1$ and $v_2$ are elements of $T_u \mathcal{H}_R$. Here we think of $T_u \mathcal{H}_R$ as the linear subspace of $\mathbb{R}^3$ orthogonal to $u$ with respect to $\circ$. Similarly to [S Lemma 3.1] we see that the map

$$\mathcal{H}_{r_1} \times \cdots \times \mathcal{H}_{r_n} \to \mathbb{R}^3$$
$$\left( u_1, \ldots, u_n \right) \mapsto u_1 + \cdots + u_n$$

is the moment map with respect to the diagonal $\text{SU}(1, 1)$-action and the product symplectic structure.
Now we will describe the hamiltonian flow $\phi_i(t)$ on the space $M_r$ corresponding to the hamiltonian function $d_i$ - the Minkowski length of the $i$-th diagonal of the polygon, connecting the first and the $(i+1)$-st vertices. Note that

$$d_i^2 = d_i \dot{d}_i = (u_1 + \cdots + u_{i+1}) \dot{d}(u_1 + \cdots + u_{i+1})$$

is a positive real number, since the vector in parenthesis is in the (future) timelike cone, by our assumptions, so we take $d_i$ real positive as well. Note that if we place the first vertex at the origin of $\mathbb{R}^3$ and use the action of $SU(1,1)$ to move the $(i+1)$-st vertex to a position on the $t$-axis, then the corresponding bending flow is easy to describe as rotation of vertices numbered $2, \ldots, i$ about the $t$-axis with a constant angular speed, since the Hamiltonian vector field in this case is given by

$$(3.1) \quad (d_i \dot{x}u_1, \ldots, d_i \dot{x}u_i, 0, \ldots, 0) = d_i(y_1i - x_1j, \ldots, y_i i - x_i j, 0, \ldots, 0).$$

The general statement follows from the equivariance with respect to the $SU(1,1)$-action. This shows that the flows are indeed periodic with periods equal to $2\pi/d_i$.

Next, we wish to describe the angle variables, which, however clear are from the preceding description, can be further illuminated by the formula analogous to [2, Equation 7.1.1] for the compact case:

$$\cos \phi_i = \frac{(d_i \dot{x}u_i) \dot{d}(d_i \dot{x}u_{i+1})}{||d_i \dot{x}u_i|| \cdot ||d_i \dot{x}u_{i+1}||}$$

Let us explain why this formula is true. We can assume, as before, that the diagonal $d_i$ is aligned with the positive direction of the $t$-axis, i.e. $d_i = d_i k$. According to formula [3.1], both Minkowski cross products in the numerator are in the $xy$-plane, and thus the Minkowski dot product is just the negative of the usual Euclidean dot product. Now, the denominator has Minkowski norms of two vectors in the $xy$-plane, each of which equals $\sqrt{-1}$ times the Euclidean norm. Therefore, the expression yields the cosine of the oriented dihedral angle between the two planes, which is the $i$-th angle variable. Obviously, this formula holds in general as well, since it is invariant under the action of $SU(1,1)$.

4. GELFAND-TSETLIN SYSTEM FOR $U(p,q)$

Let $p$ and $q$ be positive integers, $p \geq q$, $n = p + q$. For the Lie algebra $\mathfrak{g} = \mathfrak{u}(p,q)$ we use the form $\text{Tr}(AB)$ to identify its dual space $\mathfrak{u}(p,q)^*$ with $\sqrt{-1} \cdot \mathfrak{u}(p,q)$, i.e. the space of $n \times n$ matrices $A$ such that $JA$ is Hermitian symmetric, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
Let $A$ be the block matrix \[ A = \begin{pmatrix} H_p & B \\ -B^T & H_q \end{pmatrix}, \]
where $H_p$ and $H_q$ are $p \times p$ and $q \times q$ Hermitian symmetric matrices respectively and $B$ is a complex $p \times q$ matrix. The complexification of $u(p, q)$ is, as usual, $g_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$, and let us denote by $\mathfrak{h}$ the diagonal Cartan subalgebra, the complexification of the compact Cartan $\mathfrak{t}$ in $u(p, q)$. Let $\Delta$ be the root system with respect to $(g, \mathfrak{h})$, let $\Delta^+$ be the standard subset of positive roots of the form $e_{ij}$, $1 \leq i < j \leq n$. We note that the positive roots $e_{ij}$ are compact if $1 \leq i, j \leq p$ or $p < i, j \leq n$ and non-compact if $1 \leq p < j$. Denote by $\Delta^+_n$ the subset of positive non-compact roots, by $\mathcal{W}$ the Weyl group and by $\mathcal{W}_c$ the baby Weyl group corresponding to the pair $(\mathfrak{t}, \mathfrak{h})$, where $\mathfrak{t} = u(p) \times u(q)$ is the standard maximal compact subalgebra in $g$.

Let us now choose an $n$-tuple of real numbers
\[ \Lambda = (\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \]
satisfying the following conditions: $\lambda_i \leq \lambda_j$ for $i < j$ and $\mu_i \leq \mu_j$ also for $i < j$. Besides, we will require that $\lambda_1 > \mu_q$. We will refer to this property as $\Lambda$ being admissible. The condition of admissibility is sufficient to guarantee that the Gelfand-Tsetlin variables for the elliptic coadjoint orbit $O_\Lambda := G.\Lambda$ are all real (here we consider $\Lambda$ as a diagonal matrix representing an element of $\mathfrak{t}^* \subset g^*$). Later on, we will consider a particular case, relevant to polygons, where this condition does not hold, but where the Gelfand-Tsetlin variables are still real. By [6, Theorem 5.17], the projection of $O_\Lambda$ onto $\mathfrak{t}^*$ is the sum of the convex polyhedron $\text{conv}(\mathcal{W}_c.\Lambda)$ and the convex polyhedral cone defined by the non-compact positive roots.

**Remark.** It is an interesting separate question to find a necessary condition, which would force the eigenvalues of a truncated matrix to be real as well. A mere requirement that $\Lambda$ is elliptic does not guarantee this, as we found using a computer algebra software, already in the case of SU(2, 2).

Let $\mathfrak{g}_{n-1}$ be the subalgebra of $\mathfrak{g}$ corresponding to the left upper principal submatrix of size $(n-1) \times (n-1)$. The algebra $\mathfrak{g}_{n-1}$ is isomorphic to $u(p, q - 1)$. Denote by $p_n$ the projection $\mathfrak{g}^* \to \mathfrak{g}_{n-1}^*$. The image $p_n(O_\Lambda)$ is the union of certain coadjoint orbits of $U(p, q - 1)$ in $\mathfrak{g}_{n-1}^*$, which we will describe next. For convenience, we denote
\[ \mathbf{x}^\dagger = (J\mathbf{x})^T. \]
Proposition 4.1. If a coadjoint $U(p,q-1)$-orbit $\mathcal{O}$ is in the image $p_n(\mathcal{O}_\Lambda)$, then it is admissible. If we arrange the real eigenvalues of $\mathcal{O}$ in the non-decreasing order

$$\mu_1^{n-1} \leq \mu_2^{n-1} \leq \ldots \leq \mu_{q-1}^{n-1} \leq \lambda_1^{n-1} \leq \lambda_2^{n-1} \leq \ldots \leq \lambda_p^{n-1},$$

then the following interlacing conditions hold: for $1 \leq i \leq q-1$, $\mu_i \leq \mu_{i+1}^{n-1}$, also for $1 \leq j \leq p-1$, $\lambda_j \leq \lambda_{j+1}^{n-1}$ and $\lambda_p^{n-1} \geq \lambda_p$.

Proof. The admissibility directly follows from [6, Theorem 5.17]. To show interlacing, one should adapt the Courant-Fischer Theorem [7] separately to the timelike cone and to the spacelike cone. For example, if we let $v_i$ be the eigenvector in the timelike cone of $\mathbb{C}^n$ for the eigenvalue $\lambda_i$ and $w_j$ in the spacelike cone for $\mu_j$, then for

$$x = \alpha_1 v_1 + \cdots + \alpha_p v_p + \beta_1 w_1 + \cdots + \beta_q w_q$$

the Rayleigh quotient modifies to

$$R_A(x) = \frac{x^t A x}{x^t x} = \frac{\sum_{i=1}^p |\alpha_i|^2 \lambda_i - \sum_{j=1}^q |\beta_j|^2 \mu_j}{\sum_{i=1}^p |\alpha_i|^2 - \sum_{j=1}^q |\beta_j|^2},$$

from which one concludes that in the timelike and spacelike cones $\mathbb{C}^n_+$ and $\mathbb{C}^n_-$ we respectively have

$$\lambda_1 = \min_{x \in \mathbb{C}^n_+} R_A(x) \quad \text{and} \quad \mu_q = \max_{x \in \mathbb{C}^n_-} R_A(x).$$

Continuing with the standard minmax arguments for timelike and spacelike cones, one gets the full set of interlacing conditions. In order to show that each interlacing pattern can be obtained this way, one can just suitably adapt the arguments of [7, Theorem 4.3.10] to the pseudo-Hermitian case at hand. Q.E.D.

Remark. The Gelfand-Tsetlin patterns for the unitary representations of $U(p,q)$ with highest weights were studied many years ago by e.g. Todorov [16], Olshanskii [13], Molev [11] and others. The pattern described in the above Proposition corresponds to the partition $p = p + 0$ in the terminology of [16].

By repeating verbatim the arguments for the compact case, one shows that the Gelfand-Tsetlin variables for the chain of subalgebras

$$u(1) \subset u(2) \subset \cdots \subset u(p) \subset u(p,1) \subset \cdots \subset u(p,q-1) \subset u(p,q)$$

yield a complete family of hamiltonians in involution on any admissible coadjoint orbit of $U(p,q)$, which all have periodic flows.

Now, we will show the direct derivation of the Gelfand-Tsetlin pattern, analogous to [2, Proposition 6.1.3]. First, let $e$ be a $2 \times 2$ matrix representing an element of $u(1,1)^*$. Let $\delta$ and $\gamma$ be the eigenvalues of $e$ with corresponding orthonormal eigenvectors $u$ and $v$ respectively. We assume at this moment that $u$ is timelike and $v$ is spacelike. They are

\[1\] I thank Hermann Flaschka for explaining this to me.
mutually orthogonal with respect to the pseudohermitian form of signature \((1, 1)\). Note that if \(u = (a \ b)^T\), then \(v = (\bar{b} \ \bar{a})^T\). Now, for a unit timelike vector \(w\) and a real number \(r \in \mathbb{R}\), we set

\[
L = e + rw \otimes w^\dagger.
\]

If we decompose \(w = \alpha u + \beta v\), then we compute:

\[
\frac{\det(\lambda I - L)}{\det(\lambda I - e)} = 1 - r \frac{|\alpha|^2}{\lambda - \delta} + r \frac{|\beta|^2}{\lambda - \gamma}.
\]

By analyzing the function in the right hand side, we see that if \(\delta > \gamma\) and \(r > 0\), then one of its zeroes is going to be in the interval \((-\infty, \gamma)\) and the other in \((\delta, +\infty)\). On the contrary, if \(r < 0\), then the two zeroes are only possible in the interval \((\gamma, \delta)\). Note that when \(\gamma r |\alpha|^2 - \delta r |\beta|^2 \leq -\gamma \delta\), there indeed will be zeroes in this interval. This observation can directly produce the Gelfand-Tsetlin pattern, which we discuss in the next Section.

5. Action variables

Let us first describe the Gelfand-Tsetlin pattern for the coadjoint orbit of

\[
\Lambda = \text{diag}(1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0).
\]

Let \(A \in O_\Lambda\) be a matrix in the coadjoint orbit of \(\Lambda\) and let \(A_\ell\) be the principal left upper \(\ell \times \ell\) submatrix of \(A\) with real eigenvalues \(\gamma_\ell \leq \delta_\ell\) complementing the \((\ell - 2)\) zero eigenvalues (since the rank of \(A_\ell\) is at most 2). Then the Gelfand-Tsetlin pattern is as follows:

\[
\begin{align*}
\gamma_\ell &\leq \gamma_{\ell+1}, \quad \delta_\ell \geq \delta_{\ell+1} \quad \text{for} \quad p \leq \ell \leq n - 1 \\
\gamma_\ell &\geq \gamma_{\ell+1}, \quad \delta_\ell \leq \delta_{\ell+1} \quad \text{for} \quad 1 \leq \ell \leq p - 1 \\
\gamma_\ell &\leq 0 \quad \text{for} \quad \ell \leq p
\end{align*}
\]

Here we think \(\gamma_n = \delta_n = 1\) and \(\gamma_1 = 0\) (since we only need one non-trivial eigenvalue, \(\delta_1\), of the \(1 \times 1\) matrix \(A_1\)).

Remark. Note that our pattern is in agreement with the pattern (3.6) in [16], for the decomposition \(p = (p - 1) + 1\).

With this pattern in mind, we define \(M_\ell\), to be the \(2 \times \ell\) complex matrix obtained from the \(n \times 2\) matrix \(M = (z \ w)\) by removing the last \((n - \ell)\) rows. Let also \(\gamma_i \leq \delta_i\) be the eigenvalues of the \(\ell \times \ell\) matrix \(A_\ell = \eta(M_\ell) = MJ_{1,1}M^*J_{p,\ell-p}\), complementing the \((\ell - 2)\) zero eigenvalues, which are the same as the eigenvalues of the \(2 \times 2\) matrix \(\nu(M) = J_{1,1}M^*J_{p,\ell-p}M\).

Just by analyzing the traces, one can see that \(\gamma_\ell + \delta_\ell = \sum_{i=1}^\ell r_i\), and by repeating the arguments in (5.1) of [5], one finds that \(\delta_\ell - \gamma_\ell\) yields the length of the \(\ell\)-th diagonal, \(d_\ell\).

The \((n - 3)\) functions on \(M_\ell\), namely \(d_2, \ldots, d_{n-2}\), yield a completely integrable system with periodic flows. The flow, corresponding to the hamiltonian \(d_\ell\) can be visualized
similarly to the Euclidean case, as follows. We use the action of the group U(1, 1) to move the \((\ell + 1)\)-st vertex to the \(t\)-axis. Then we consider the \(S^1\)-action on the polygon, which revolves the vertices numbered \(i + 2, \ldots, n\) around the \(t\)-axis, while not moving all the other vertices (we assume, as usual, that the first vertex is placed at the origin).

The triangle inequalities for the Minkowski space imply the following inequalities being imposed on the lengths \(r_i\)’s and \(d_i\)’s:

\[
\begin{align*}
    d_\ell &\geq d_{\ell-1} + r_\ell & \text{for } & 1 \leq \ell \leq p \\
    d_\ell &\geq d_{\ell+1} + r_{\ell+1} & \text{for } & p \leq \ell \leq n
\end{align*}
\]

Note that this is in complete agreement with the Gelfand-Tsetlin pattern described in the beginning of this Section.

6. MODULI SPACES OF POLYGONS AS LAGRANGIAN LOCI IN COMPLEX QUOTIENTS

In this section we will further justify considering \(M_r\) as a reasonable geometric object. We will show that it is a Hausdorff topological space, and, moreover, for a generic choice of \(r\), it has a structure of smooth manifold. In the spirit of [15], one should always view quotients by real reductive groups as being homeomorphic to quotients of certain minimal loci by the maximal compact subgroup. In our situation, we can go a little further as in [3], since we have compatible involutions on groups and spaces in question at our disposal.

Let \(\tau\) stand for the complex conjugate involution of \(\mathfrak{gl}(2, \mathbb{C})\) defining the real form \(\mathfrak{u}(1, 1)\). Without any fear of confusion, we will denote by \(\tau\) also the induced involution on the space of traceless matrices \(\mathfrak{sl}(2, \mathbb{C})\) as well as on the corresponding dual vector spaces. By using the Killing form, we identify \(\mathfrak{su}(1, 1)^*\) as the subspace of \(\mathfrak{sl}(2, \mathbb{C})^*\), which is the fixed point set of \(\tau\).

Let us consider the product of complex coadjoint orbits \(\mathcal{O}_1^C \times \cdots \times \mathcal{O}_n^C\) corresponding for \(1 \leq i \leq n\) respectively to integral points \(\text{diag}(m_i, -m_i)\) in \(t^*\). Note that with our convention, these are fixed by \(\tau\). The choice of \(m_i\)'s leads to a choice of polarization on the orbits, and therefore, we can consider the GIT quotient \(Y = (\mathcal{O}_1^C \times \cdots \times \mathcal{O}_n^C) / / \text{SL}(2, \mathbb{C})\). Note that the quotient map \(\mathcal{O}_j^C \to \mathbb{C}P^1\) by the action of the maximal unipotent subgroup \(N\) is equivariant with respect to the \(\text{SL}(2, \mathbb{C})\)-action and therefore \(Y\) fibers over the moduli space of euclidean polygons \(P_{m_1, \ldots, m_n}\), which is a smooth projective variety, for a generic choice of \(m_i\)'s, with contractible fibers.

The involution \(\tau\) descends onto the space \(Y\) and its fixed point set by [15] is homeomorphic to \(M_r\).

Note that from the polygonal consideration, the isotropy subgroup of a polygon will be trivial if we have \(\sum_{i=1}^p r_i \neq \sum_{j=p+1}^n r_j\) (in which case we can visualize the degenerate \(n\)-gon as being aligned along the \(t\)-axis with \(p\) forward-tracks and \(q\) backtracks). Barring this situation, a polygon will represent a smooth point in the moduli space.
7. Final remarks

It would be interesting to study further the topology of these moduli spaces, in particular compute their cohomology rings. Also, one can be interested in extending these results to minimal elliptic orbits of more general Lie groups, the same way the Flaschka-Millson spaces [2] extend the moduli spaces of spatial Euclidean polygons.

Another interesting question is to understand the relationship between the lattice points in the convex polyhedral set $P$ defined by [5.1] and decompositions of the tensor product of representations of $SU(1,1)$. In the case $q = 1$ this question was answered by Millson in [10], where he showed that the inequalities, restricting the lengths of the diagonals, exactly match the analogue [14] of the Clebsch-Gordan formula for the discrete series representations of $SU(1,1) \simeq SL(2,\mathbb{R})$.

Finally, unlike in the Euclidean case, the bending torus action on $M_r$ appears to be globally defined, which raises another interesting question, whether for an integral choice of $r$, the space $M_r$ has the structure of a toric variety, corresponding to the polyhedral set $P$.

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