NODAL AREA DISTRIBUTION FOR ARITHMETIC RANDOM WAVES

VALENTINA CAMMAROTA

Abstract. We obtain the limiting distribution of the nodal area of random Gaussian Laplace eigenfunctions on \( T^3 = \mathbb{R}^3 / \mathbb{Z}^3 \) (three-dimensional “arithmetic random waves”). We prove that, as the multiplicity of the eigenspace goes to infinity, the nodal area converges to a universal, non-Gaussian distribution. Universality follows from the equidistribution of lattice points on the sphere. Our arguments rely on the Wiener chaos expansion of the nodal area: we show that, analogous to the two-dimensional case addressed by Marinucci et al., the fluctuations are dominated by the fourth-order chaotic component. The proof builds upon recent results from Benatar and Maffiucci that establish an upper bound for the number of nondegenerate correlations of lattice points on the sphere. We finally discuss higher-dimensional extensions of our result.

1. Introduction and framework

1.1. Toral eigenfunctions and nodal volume. Let \( f : T^d = \mathbb{R}^d / \mathbb{Z}^d \rightarrow \mathbb{R}, d \geq 2 \), be the real-valued functions satisfying the eigenvalue equation

\[
\Delta f + Ef = 0,
\]

where \( E > 0 \) and where \( \Delta \) is the Laplace-Beltrami operator on \( T^d \); the spectrum of \( \Delta \) is totally discrete.

The nodal set of a function is the zero set. Nodal sets for eigenfunctions of the Laplacian on smooth compact Riemannian manifolds have been studied intensively; it is known \([7]\) that except for a subset of lower dimension, the nodal sets of eigenfunctions are smooth manifolds of codimension 1 in the ambient manifold, and hence the nodal volume

\[
\text{Vol}(f^{-1}(0))
\]

of \( f \) is well defined. A fundamental conjecture of Yau \([30,31]\) asserts that for any smooth compact Riemannian manifold \( \mathcal{M} \) there exist constants \( 0 < c_1(\mathcal{M}) \leq c_2(\mathcal{M}) \) such that

\[
c_1(\mathcal{M}) \sqrt{E} \leq \text{Vol}(f^{-1}(0)) \leq c_2(\mathcal{M}) \sqrt{E}.
\]

Yau’s conjecture was proven for real-analytic metrics by Donnelly and Fefferman \([9]\), and the lower bound in Yau’s conjecture was recently established for general smooth manifolds by Logunov \([21]\).

Received by the editors February 24, 2018, and, in revised form, December 10, 2018.

2010 Mathematics Subject Classification. Primary NUMBER(S).

The research leading to these results received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007–2013)/ERC grant agreement no. 335141.
For \( \mathcal{M} = \mathbb{T}^d \) the eigenspaces of the Laplacian are related to the theory of lattice points on \((d - 1)\)-dimensional spheres. Let 
\[
S = \{ n \in \mathbb{Z} : n = n_1^2 + \cdots + n_d^2 \text{ for } n_1, \ldots, n_d \in \mathbb{Z} \}
\]
be the collection of all numbers expressible as a sum of \(d\) squares. The sequence of eigenvalues, or energy levels, of (1.1) are all numbers of the form 
\[
E_n = 4\pi^2 n, \quad n \in S.
\]
In order to describe the Laplace eigenspace corresponding to \(E_n\), we introduce the set of frequencies \(\Lambda_n\): for \(n \in S\) let 
\[
\Lambda_n = \{ \lambda \in \mathbb{Z}^d : ||\lambda||^2 = n \}.
\]
\(\Lambda_n\) is the frequency set corresponding to \(E_n\). Using the notation \(e(\langle \lambda, \cdot \rangle) = e^{2\pi i \langle \lambda, \cdot \rangle})\), the \(L^2\)-spanned by the \(\Lambda_n\) is the centered Gaussian random field with covariance function 
\[
E[f_n(x)f_n(y)] = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi \langle \lambda, x - y \rangle).
\]
Note that the normalizing factor in (1.3) is chosen so that \(f_n\) has unit-variance.
1.3. Prior work on this model. Our object of study is the nodal volume, i.e., the sequence \( \{V_n\}_{n \in S} \) of all random variables of the form
\[
V_n = \text{Vol}(f_n^{-1}(0)).
\]
The expected value of \( V_n \) was computed in [32] to be, for every \( d \geq 2 \),
\[
E[V_n] = \mathcal{I}_d \sqrt{\frac{E_n}{4\pi^2}}, \quad \mathcal{I}_d = \sqrt{\frac{4\pi}{d} \frac{\Gamma(d+1)}{\Gamma(d/2)}},
\]
in agreement with Yau’s conjecture (1.2). The more challenging question of the asymptotic behavior of the variance was also addressed in [32], where the following bound for the variance was computed for every dimension \( d \geq 2 \):
\[
\text{Var}(V_n) = O\left( \frac{E_n \sqrt{\mathcal{N}_n}}{\mathcal{N}_n} \right),
\]
and it was conjectured that the stronger bound
\[
\text{Var}(V_n) = O\left( \frac{E_n}{\mathcal{N}_n} \right)
\]
should hold.

1.3.1. Nodal length. In [18] was derived the precise asymptotic behavior of the variance of the nodal length \( L_n \) of the random eigenfunctions on \( T^2 \). For \( d = 2 \) the set \( \Lambda_n \) induces a discrete probability measure \( \mu_n \) on the unit circle \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) by defining
\[
\mu_n = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \mathcal{N}_n} \delta_{\lambda},
\]
where \( \delta_x \) is the Dirac delta centered at \( x \in S^1 \). It was shown [18, Theorem 1.1] that if \( \{n_i\}_{i \geq 1} \) is any sequence of elements in \( S \) such that \( \mathcal{N}_{n_i} \to \infty \), then
\[
\text{Var}(L_n) = c_{n_i} \frac{E_{n_i}}{\mathcal{N}_{n_i}^2} (1 + o(1)), \quad c_{n_i} = \frac{1 + \hat{\mu}_{n_i}^2(4)}{512},
\]
where \( \hat{\mu}_n(k) \in [-1, 1] \) is the Fourier transform of \( \mu_n \). The positive real numbers \( c_{n_i} \) in the leading constant depend on the angular distribution of \( \Lambda_{n_i} \); i.e., in dimension \( d = 2 \) the asymptotic behavior of the variance is nonuniversal.

Also remarkably, the order of magnitude of (1.5) is much smaller than expected (1.4) since the terms of order \( E_n/\mathcal{N}_n \) in the asymptotic expression for the nodal length variance cancel perfectly. This effect was called arithmetic Berry cancellation after the cancellation phenomenon observed by Berry in [3]. The limiting distribution of the nodal length was derived in [23, Theorem 1.1], where it is proved that the normalized nodal length converges to a nonuniversal, non-Gaussian, limiting distribution, depending on the angular distribution of lattice points. For \( \{n_i\}_{i \geq 1} \in S \) such that \( \mathcal{N}_{n_i} \to \infty \) and \( |\hat{\mu}_{n_i}(4)| \to \eta \in [0, 1] \), one has
\[
\frac{L_n - E[L_n]}{\sqrt{\text{Var}(L_n)}} \overset{\text{law}}{\to} \frac{1}{2\sqrt{1 + \eta^2}}[2 - (1 + \eta)X_1^2 - (1 - \eta)X_2^2],
\]
where \( X_1, X_2 \) are i.i.d. standard Gaussian. A quantitative version of (1.6) was derived in [28].
1.3.2. Nodal area. The asymptotic behavior of the nodal area variance on $\mathbb{T}^3$ has recently been analyzed in [1]: the variance of the nodal area has the following precise asymptotic behavior as $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$:

$$\text{Var}(A_n) = \frac{n}{N_n^2} \left[ \frac{32}{375} + O \left( \frac{1}{n^{1/28-\omega(1)}} \right) \right].$$

The condition $n \not\equiv 0, 4, 7 \pmod{8}$ implies that $N_n \to \infty$ (see Section 3.1).

In particular, the three-dimensional torus exhibits arithmetic Berry cancellation like the two-dimensional torus. However, unlike the two-dimensional case, the leading-order term does not fluctuate; this is due to the equidistribution of lattice points on the sphere (see Section 3.2).

1.3.3. Chaotic cancellation phenomenon. As observed in [8,23], the noncentral and nonuniversal behavior of second-order fluctuations originates from the chaotic cancellation phenomenon: in the Wiener chaos expansion of $\mathcal{L}_n$ the projection on the second chaos vanishes, and the limiting fluctuations of $\mathcal{L}_n$ are completely determined by its projection on the fourth Wiener chaos. Should the second projection of $\mathcal{L}_n$ not disappear in the limit, then the order of the variance would be $E_n/N_n$.

In dimension 2 the asymptotic dominance of the fourth-order chaos, and the consequent lower order of the variance, was observed for the nodal lines on both the torus and the sphere [23,24]. Similarly in [25] it was proved that, for the defect of random spherical harmonics, the second term in the chaotic expansion vanishes and all other summands are relevant. The results in [6], on the asymptotic variance of the number of critical points of random spherical harmonics, indicate that the chaotic cancellation phenomenon could be proved also for other geometric functionals like the number of critical points and the Euler characteristic of the excursions above 0.

It is natural to ask whether the chaotic cancellation phenomenon still holds for geometric functionals of Laplace eigenfunctions in arbitrary dimensions $d > 2$. In this paper we consider for the first time the case $d = 3$, and we prove that the fluctuations of the nodal area of random Gaussian Laplace eigenfunctions on $\mathbb{T}^3$ are dominated by the fourth-order chaotic component.

We stress that for generic manifolds there are no spectral degeneracies, so it is necessary to consider linear Gaussian combinations of eigenfunctions. The two most studied models are the so-called long energy window and the short energy window (monochromatic random waves). The expected nodal volume on generic manifolds has been extensively studied in [2,34].

2. Main results and outline of the proof

Our principal result is the asymptotic distribution, as $N_n \to \infty$, of the sequence of normalized nodal areas.

**Theorem 1.** Let $\chi$ be a chi-square with 5 degrees of freedom. As $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$,

$$\frac{A_n - E[A_n]}{\sqrt{\text{Var}(A_n)}} \overset{\text{law}}{\to} \frac{1}{\sqrt{5 \cdot 2}} (5 - \chi).$$

The first step in the proof of Theorem 1 is the derivation of the Wiener chaos expansion of the nodal area $A_n$. 


In particular, in Lemma 4.1 we derive the Wiener chaos expansion of the nodal volume for every dimension $d \geq 2$:

$$V_n = \mathbb{E}[V_n] + \sum_{q=1}^{\infty} V_n[q];$$

here $V_n[q]$, $q = 1, 2, \ldots$, denotes the orthogonal projection of $V_n$ onto the so-called Wiener chaos of order $q$ (see Section 4.1). The proof of Lemma 4.1 (see Appendix A) is a $d$-dimensional generalization of the Wiener chaos expansion of the nodal length performed in [23] (see also [17] for analogous computations involving the length of level curves for Gaussian fields on the Euclidean plane). As for the nodal length, we obtain that both the second-order projection and all odd-order projections vanish.

A precise analysis of the fourth-order chaos $V_n[4]$ allows as to show the following:

$$V_n[4] = \sqrt{\frac{n \pi}{d}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\sqrt{N_n}} \left[4 \frac{d-1}{d+2} + \frac{2}{d+2} W^2(n) - \frac{2d}{d+2} \sum_{j,k} W_{j,k}^2(n) + X(n) + 2 \sum_k X_{k,k}(n) - \frac{d}{d+2} \sum_{j,k} X_{k,j,j}(n) + o_P(1)\right],$$

where

$$W(n) = \frac{1}{\sqrt{N_n}} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1),$$

$$W_{j,k}(n) = \frac{1}{n\sqrt{N_n}} \sum_{\lambda \in \Lambda_n} \lambda_{(k)} \lambda_{(j)} (|a_\lambda|^2 - 1),$$

$$X(n) = \frac{1}{N_n} \sum_{\lambda_1, \ldots, \lambda_4 \in X_n(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$$X_{k,k}(n) = \frac{1}{n^2 N_n} \sum_{\lambda_1, \ldots, \lambda_4 \in X_n(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$$X_{k,j,j}(n) = \frac{1}{n^2 N_n} \sum_{\lambda_1, \ldots, \lambda_4 \in X_n(4)} \lambda_{1,(k)} \lambda_{2,(j)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},$$

$\lambda_{(k)}$ denotes the $k$th component of $\lambda$, and $X_n(4)$ is the set of $d$-dimensional lattice point, nondegenerate, 4-correlations defined in Section 3.3 (see also [1] Section 1.4). We remark that in dimension $d = 2$

$$|X_n(4)| = 0,$$

for all $n \in S$, which may be seen by noting that two circles intersect in at most two points (Zygmund’s trick), so the asymptotic behavior of the nodal length studied in [23] comes from a precise analysis of the asymptotic behavior of the first three terms in (2.1):

$$L_n[4] = \pi \sqrt{\frac{n}{512 N_n}} \left[2 + W^2(n) - 2 \sum_{j,k} W_{j,k}(n) + o_P(1)\right].$$

Our proof of Theorem 1 relies on recent results by Benatar and Maffucci [1] (see Lemma 3.4), showing that, in dimension $d = 3$, the tuples that cancel pairwise
dominate the nondegenerate tuples in the high frequency limit. In particular, in Lemma 5.2 we prove that as $n \to \infty$, $n \not\equiv 0, 4, 7 \mod 8$,

$$X(n), X_{k,k}(n), X_{k,k,j,j}(n) \overset{L^2}{\to} 0.$$ 

This implies that

$$A_n[4] = \frac{\sqrt{n}}{5\sqrt{3}N_n} \left[ 4 + W^2(n) - 3 \sum_{j,k} W_{j,k}^2(n) + o_P(1) \right].$$

To prove Theorem 1, we first show that the normalized fourth-order projection of the nodal area converges to a universal, non-Gaussian distribution: as $N_n \to \infty$,

$$\frac{A_n[4]}{\sqrt{\text{Var}(A_n[4])}} \overset{\text{law}}{\to} \frac{1}{\sqrt{5 \cdot 2}} (5 - \chi),$$

where $\chi$ is a chi-square with 5 degrees of freedom. The derivation of such a limiting distribution requires a precise analysis of the asymptotic behavior of the covariance matrix of the $W_{j,k}(n)$, i.e., the asymptotic behavior of the following quantities:

$$\frac{1}{n^2N_n} \sum_{\lambda \in \Lambda_n} \lambda^4(k) = \frac{1}{5} + O \left( \frac{1}{n^{1/28-o(1)}} \right), \quad \frac{1}{n^2N_n} \sum_{\lambda \in \Lambda_n} \lambda^2(k) \lambda^2(j) = \frac{1}{3 \cdot 5} + O \left( \frac{1}{n^{1/28-o(1)}} \right).$$

This is obtained in Lemma 3.3 as an application of the equidistribution of lattice points on spheres proved by Duke and Schulze-Pillot [10,11] and Golubeva and Fomenko [13] (see [27, Lemma 8] or Lemma 3.2).

The last step in the proof of Theorem 1 is the observation that the nodal area is dominated by its fourth-order chaos component: using the asymptotic behavior of the variance in (1.7), it is easy to check the equivalence

$$\text{Var}(A_n) \sim \text{Var}(A_n[4])$$

as $N_n \to \infty$. Theorem 1 follows since different chaotic projections are orthogonal in $L^2$.

In dimension $d \geq 5$ the set of nondegenerate tuples $X_n(4)$ is much larger than $D_n(4)$ [Section 3.3.2], as opposed to what happens in dimensions 2 and 3. This implies that the derivation of the asymptotic behavior of variance and limiting distribution of the nodal volume requires a precise analysis of the structure of the nondegenerate tuples $X_n(4)$, which seems to be very technically demanding. The problem is still open in dimension $d = 4$.

2.1. Notation. For functions $f$ and $g$ we will use Landau’s asymptotic notation,

$$f = O(g),$$

to denote that $f \leq Cg$ for some constant $C$. With $\overset{\text{law}}{\to}$ we denote weak convergence of probability measures, and we use the symbol $o_P(1)$ to denote a sequence of random variables converging to 0 in probability.

We will use $\lambda, \lambda_1, \lambda_2, \ldots$ and in general $\lambda_i, i = 1, 2, \ldots$ to denote elements of $\Lambda_n$, while $\lambda_{(k)}$ and $\lambda_{i,(k)}$ with $k = 1, \ldots, d$, will denote the kth component of the vectors $\lambda$ and $\lambda_i \in \Lambda_n$, respectively. The indices $j, k$ always run from 1 to $d$.

For $k = 1, \ldots, d$ we denote with $\partial_k f_n(x)$ the derivative of $f_n(x)$ with respect to $x_k$,

$$\partial_k f_n(x) = \frac{2\pi i}{\sqrt{N_n}} \sum_{\lambda \in \Lambda_n} a_{\lambda} \lambda_{(k)} e(\langle \lambda, x \rangle);$$
in view of [32 Lemma 2.3] (see formula (3.1)) the random field \( \partial_k f_n \) has variance
\[
\text{Var}(\partial_k f_n(x)) = \frac{2^2 \pi^2}{N_n} \sum_{\lambda_1, \lambda_2 \in \Lambda_n} \mathbb{E}[a_{\lambda_1} a_{\lambda_2}] \lambda_1, \lambda_2 (k) \, \epsilon ((\lambda_1, x)) \, \epsilon ((\lambda_2, x)) = \frac{2^2 \pi^2}{N_n} \sum_{\lambda \in \Lambda_n} \lambda^2 (k) = \frac{2^2 \pi^2 n}{d}.
\]

We introduce then the normalized derivative \( f_{n,k}(x) \) defined by
\[
f_{n,k}(x) = \frac{\partial_k f_n(x)}{2\pi \sqrt{d}} = i \sqrt{\frac{d}{nN_n}} \sum_{\lambda \in \Lambda_n} \lambda_k a_{\lambda} e(\langle \lambda, x \rangle).
\]

Note that \( f_{n,k}(x) \) is real valued since \( f_{n,k}^2(x) = f_{n,k}(x) f_{n,k}(\bar{x}) \). We note that conditions (i) and (ii) in (1.3) immediately imply that
\[
\text{Var}(a_{\lambda}^2) = \mathbb{E}[a_{\lambda}^2] = \mathbb{E}[\text{Re}(a_{\lambda})] - \mathbb{E}[\text{Im}(a_{\lambda})] = 0,
\]
and that \( 2 |a_{\lambda}|^2 \) has a chi-squared distribution with 2 degrees of freedom:
\[
\mathbb{E}[|a_{\lambda}|^2] = 1, \quad \mathbb{E}[|a_{\lambda}|^2 - 1]^2 = \text{Var}(|a_{\lambda}|^2) = 1, \quad \mathbb{E}[|a_{\lambda}|^4] = 2.
\]

We also define
\[
R(n) = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} |a_{\lambda}|^4, \quad R_{k,j}(n) = \frac{1}{n^2 N_n} \sum_{\lambda \in \Lambda_n} \lambda^2_{(k)} \lambda^2_{(j)} |a_{\lambda}|^4.
\]

3. Lattice points on spheres: Spectral correlations and equidistribution

3.1. Properties of the frequency set. The dimension \( N_n = \dim \mathcal{E}_n \) is the number of ways of expressing \( n \) as a sum of \( d \) integer squares. In dimension \( d \geq 5 \) \( N_n \) grows roughly as \( n^{d/2 - 1} \) as \( n \to \infty \) [16 Theorem 20.9]. For \( d \leq 4 \) the dimension of the eigenspace need not grow with \( n \), and the behavior of \( N_n \) is more erratic.

If \( d = 2 \) \( N_n \) is given in terms of the prime decomposition of \( n \) as follows [14 Section 16.9]: for
\[
n = 2^\alpha \prod_j p_j^{\beta_j} \prod_k q_k^{2\gamma_k},
\]
where \( p_j, q_k \) are odd primes \( p_j \equiv 1 \mod 4 \) and \( q_k \equiv 3 \mod 4 \) and \( \alpha, \beta_j, \gamma_k \) are positive integers, then \( N_n = 4 \prod_j (\beta_j + 1) \), or otherwise \( n \) is not a sum of two squares and \( N_n = 0 \). \( N_n \) is subject to large and erratic fluctuations; it grows on average [19], over integers which are sums of two squares, as \( \text{const} \cdot \sqrt{\log n} \), but it could be as small as 8 for an infinite sequence of prime numbers \( p \equiv 1 \mod 4 \), or as large as a power of \( \log n \).

In dimension \( d = 3 \), by a classical result of Legendre and Gauss, \( n \) is a sum of three squares if and only if \( n \neq 4^a (8b + 7) \). The behavior of \( N_n \) is very subtle [4 Section 1], and it was shown in the 1930s that \( N_n \) goes to infinity with \( n \), assuming that \( n \) is square free (if \( n = 4^a \), then there are only six solutions). It is known that \( N_n \ll n^{1/2 + o(1)} \). If there are primitive lattice points, which happens if and only if \( n \neq 0, 4, 7 \) (mod 8), then there is a lower bound \( N_n \gg n^{1/2 - o(1)} \).

The frequency set \( \Lambda_n \) is invariant under the group \( W_d \) of signed permutations, consisting of coordinate permutation and sign-change of any coordinate. In particular, \( \Lambda_n \) is symmetric under \( \lambda \to -\lambda \), and since \( 0 \notin \Lambda_n \), \( N_n \) is even. Using invariance under \( W_d \) in [32 Lemma 2.3], the following lemma is proved.
Lemma 3.1. For any subset $O \subset \Lambda_n$ which is invariant under the group $W_d$, we have

\[ \sum_{\lambda \in O} \lambda_j \lambda_k = |O| \frac{n}{d} \delta_{j,k}. \tag{3.1} \]

We note that, using the invariance of $\Lambda_n$ under the group $W_d$, we also immediately obtain that

\[ \sum_{\lambda \in \Lambda_n} \prod_{i=1}^{d} \lambda_{\alpha_i} = 0 \tag{3.2} \]

if at least one of the exponents $\alpha_i$ is odd.

3.2. Equidistribution of lattice points on spheres. The classical Linnik problem \[12\] about the distribution of lattice points on a sphere was first introduced and discussed by Linnik in \[20\].

In his book Linnik asked whether the points $\Lambda_n/\sqrt{n}$, obtained by projecting the set $\Lambda_n = \{ \lambda \in \mathbb{Z}^3 : ||\lambda||^2 = n \}$ to the unit sphere $S^2$, become equidistributed with respect to the (normalized) Lebesgue measure $d\sigma$ on $S^2$ as $n \to \infty$, subject to the condition that $n \not\equiv 0, 4, 7 \pmod{8}$.

Linnik was able to solve the problem using his ergodic method and assuming the generalized Riemann hypothesis. The Linnik problem was solved unconditionally by Duke and Schulze-Pillot \[10,11\] and Golubeva and Fomenko \[13\], following a breakthrough by Iwaniec \[15\] on modular forms.

As a consequence, we may approximate a summation over the lattice point set by an integral over the unit sphere as follows.

Lemma 3.2 \([27, \text{Lemma } 8]\). Letting $g \in C^\infty(S^2)$, for every $n \not\equiv 0, 4, 7 \pmod{8}$ we have

\[ \frac{1}{N} \sum_{\lambda \in \Lambda_n} g\left( \frac{\lambda}{||\lambda||} \right) = \int_{S^2} g(u) \, d\sigma(u) + O\left( \frac{1}{n^{1/28-o(1)}} \right). \tag{3.3} \]

From Lemma 3.2 the next lemma immediately follows.

Lemma 3.3. For $k, j = 1, 2, 3$, $k \neq j$, we have

\[ \frac{1}{n^2 N} \sum_{\lambda \in \Lambda_n} \lambda^4_{(k)} = \frac{1}{5} + O\left( \frac{1}{n^{1/28-o(1)}} \right), \tag{3.4} \]

\[ \frac{1}{n^2 N} \sum_{\lambda \in \Lambda_n} \lambda^2_{(k)} \lambda^2_{(j)} = \frac{1}{3 \cdot 5} + O\left( \frac{1}{n^{1/28-o(1)}} \right). \tag{3.5} \]

Proof. By invariance under the group $W_3$ the integrand, on the right-hand side of (3.3), is a function of only one angle. We apply Lemma 3.2 and obtain

\[ \frac{1}{n^2 N} \sum_{\lambda \in \Lambda_n} \lambda^4_{(k)} = \frac{1}{N} \sum_{\lambda \in \Lambda_n} \left( \frac{\lambda_{(k)}}{||\lambda||} \right)^4 = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \cos^4 \phi_1 \sin \phi_1 + O\left( \frac{1}{n^{1/28-o(1)}} \right) = \frac{2\pi^2}{4\pi} \frac{2}{5} + O\left( \frac{1}{n^{1/28-o(1)}} \right). \]
The proof of (3.5) is similar. □

Malyshev [22] and Pommerenke [29] have established the analogue of Lemma 3.2 for integral positive quadratic forms in more than three variables.

3.3. Spectral correlations. For \( \ell \geq 2 \) we denote by \( \mathcal{C} = \mathcal{C}_n(\ell) \) the set of \( d \)-dimensional \( \ell \)-correlations:

\[
\mathcal{C}_n(\ell) = \{(\lambda_1, \ldots, \lambda_\ell) \in \Lambda_n : \sum_{i=1}^\ell \lambda_i = 0\}.
\]

The set of nondegenerate \( \ell \)-correlations \( \mathcal{X} = \mathcal{X}_n(\ell) \) is the subset of \( \mathcal{C} \) defined by

\[
\mathcal{X}_n(\ell) = \{(\lambda_1, \ldots, \lambda_\ell) \in \mathcal{C}_n(\ell) : \forall \mathcal{H} \subset \{1, \ldots, \ell\}, \sum_{i \in \mathcal{H}} \lambda_i \neq 0\},
\]

and we denote by \( \mathcal{D} = \mathcal{C} \setminus \mathcal{X} \) the set of degenerate correlations. For \( d > 2 \) a summation over \( \mathcal{C}(4) \) may be treated by separating it as follows:

\[
\sum_{\mathcal{C}(4)} = \sum_{\lambda_1 = -\lambda_2} + \sum_{\lambda_2 = -\lambda_3} + \sum_{\lambda_3 = -\lambda_4} - \sum_{\lambda_1 = -\lambda_2 = -\lambda_3} - \sum_{\lambda_1 = -\lambda_2 = -\lambda_4} - \sum_{\lambda_1 = -\lambda_3 = -\lambda_4} + \sum_{\mathcal{X}(4)}.
\]

Note that if \( d = 2 \), the set \( \mathcal{X}(4) \) is empty. The next lemma deals with the three-dimensional setting and provides an estimate for the number of nondegenerate correlations.

Lemma 3.4 (\cite{1} Theorem 1.5). Letting \( n \to \infty \), one has the estimate

\[
|\mathcal{X}_n(4)| \ll \mathcal{N}_n^{7/4+\Theta(1)}.
\]

4. Chaotic expansion of \( \mathcal{V}_n \)

4.1. Wiener chaos expansion. The celebrated Wiener chaos expansion [33] concerns the representation of square integrable random variables in terms of an infinite orthogonal sum. In this section we recall briefly some basic facts on Wiener chaotic expansion for nonlinear functionals of Gaussian fields. We refer the reader to [26] for an exhaustive discussion.

Denote by \( \{H_k\}_{k \geq 0} \) the Hermite polynomials on \( \mathbb{R} \), defined as follows:

\[
H_0 = 1, \quad H_k(t) = (-1)^k \gamma^{-1}(t) \frac{d^k}{dt^k} \gamma(t), \quad k \geq 1,
\]

where \( \gamma(t) = e^{-t^2/2} \sqrt{2\pi} \) is the standard Gaussian density on the real line; \( \mathbb{H} = \{H_k/\sqrt{k!} : k \geq 0\} \) is a complete orthogonal system in

\[
L^2(\gamma) = L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma(t)dt).
\]

The random eigenfunctions \( f_n \) defined in (1.3) are a byproduct of the family of complex-valued, Gaussian random variables \( \{a_n\} \), defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Following the discussion in [23] we define the space \( \mathcal{A} \) to be the closure in \( L^2(\mathbb{P}) \) generated by all real, finite, linear combinations of random variables of the
form $za_{\lambda} + \pi a_{-\lambda}$, $z \in \mathbb{C}$; the space $A$ is a real, centered, Gaussian Hilbert subspace of $L^2(\mathbb{P})$.

For each integer $q \geq 0$ the $q$th Wiener chaos $\mathcal{H}_q$ associated with $A$ is the closed linear subspace of $L^2(\mathbb{P})$ generated by all real, finite, linear combinations of random variables of the form

$$H_q(a_1) \cdot H_q(a_2) \cdot \cdots H_q(a_k)$$

for $k \geq 1$, where the integers $q_1, q_2, \ldots, q_k \geq 0$ satisfy $q_1 + q_2 + \cdots + q_k = q$ and $(a_1, a_2, \ldots, a_k)$ is a real, standard, Gaussian vector extracted from $A$. In particular, $\mathcal{H}_0 = \mathbb{R}$.

As well-known Wiener chaos $\{\mathcal{H}_q, q = 0, 1, 2, \ldots\}$ are orthogonal [26, Theorem 2.2.4], i.e., $\mathcal{H}_q \perp \mathcal{H}_p$ for $p \neq q$ (the orthogonality holds in the sense of $L^2(\mathbb{P})$), and the following decomposition holds: every real-valued function $F \in A$ admits a unique expansion of the type

$$F = \sum_{q=0}^{\infty} F[q],$$

where the projections $F[q] \in \mathcal{H}_q$ for every $q = 0, 1, 2, \ldots$, and the series converges in $L^2(\mathbb{P})$. Note that $F[0] = \mathbb{E}[F]$.

4.2. Chaotic expansion of $\mathcal{V}_n$. In this section we derive the explicit form for the projections in the chaos decomposition of the nodal volume $\mathcal{V}_n$.

4.2.1. Approximating the nodal volume. Let $1_{[-\varepsilon, \varepsilon]}$ be the indicator function of the interval $[-\varepsilon, \varepsilon]$, and let $|| \cdot ||$ be the standard Euclidean norm in $\mathbb{R}^d$. We define for $\varepsilon > 0$

$$\mathcal{V}_n^\varepsilon = \frac{1}{2\varepsilon} \int_{\mathbb{T}^d} 1_{[-\varepsilon, \varepsilon]}(f_n(x)) ||\nabla f_n(x)|| dx.$$  

It was shown in [32, Lemma 3.1] that a.s.

$$\mathcal{V}_n = \lim_{\varepsilon \to \infty} \mathcal{V}_n^\varepsilon,$$

and that [32, Lemma 3.2] $\mathcal{V}_n^\varepsilon$ is uniformly bounded, that is,

$$\mathcal{V}_n^\varepsilon \leq 6d\sqrt{E_n}.$$  

The dominated convergence theorem and the uniform bound (4.3) imply that the convergence in (4.2) is in $L^2(\mathbb{P})$; i.e., for every $n \in S$

$$\lim_{\varepsilon \to 0} \mathbb{E}[|\mathcal{V}_n^\varepsilon - \mathcal{V}_n|^2] = 0.$$  

4.2.2. Chaos expansion. In view of (4.4) we first compute the chaotic expansion of $\mathcal{V}_n^\varepsilon$, then the expansion of $\mathcal{V}_n$ in Lemma 4.1 follows by letting $\varepsilon \to 0$. In Lemma 4.1 we prove that all odd-order chaotic components in the Wiener chaos expansion of $\mathcal{V}_n$ vanish, and we derive an explicit expression for the even-order chaotic components.

**Lemma 4.1.** The Wiener chaos expansion of $\mathcal{V}_n$ is $\mathcal{V}_n = \mathbb{E}[\mathcal{V}_n] + \sum_{q=1}^{\infty} \mathcal{V}_n[q]$ in $L^2(\mathbb{P})$, where $\mathcal{V}_n[2q+1] = 0$ for $q \geq 1$, and for $q > 1$

$$\mathcal{V}_n[2q] = 2\pi \sqrt{\frac{\beta_2}{d}} \sum_{p=0}^{q} \frac{\beta_{2q-2p}}{(2q-2p)!} \sum_{x \in \mathbb{Z}^d} a(2s) \int_{\mathbb{T}^d} H_{2q-2p}(f_n(x)) \prod_{j=1}^{d} H_{2s_j}(f_{n,j}(x)) dx,$$ where $\beta_2$.
with
\[ \beta_{2m-2p} = \frac{1}{\sqrt{2\pi}} H_{2m-2p}(0), \]
\[ a(2q) = \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\sqrt{2}}{\pi} \right)^{i+\frac{1}{2}} \Gamma \left( \frac{1}{2} + i \right) \right]^{q_1 \cdots q_d} (-1)^{q_1 \cdots q_d-j_1 \cdots j_d}
\times \left( j_1, \ldots, j_d \right) \left( q_1 - j_1 \right)! \cdots \left( q_d - j_d \right)! 2^{q_1 \cdots q_d-j_1 \cdots j_d}. \]
\[ (4.5) \]

The proof of Lemma 4.1 is postponed to Appendix A.

4.2.3. Second- and fourth-order chaos. For \( j, k = 1, \ldots, d \) we denote by \( s(j) \) the vector in \( \mathbb{R}^d \) with \( j \)th component equal to 1 and all of the other components equal to 0, and we denote by \( s(j,k) \) the vector in \( \mathbb{R}^d \) with \( j \)th and \( k \)th components equal to 1, and all of the other components equal to 0. To evaluate the second- and fourth-order chaos, we need the following lemma. Q4

Lemma 4.2.
\[ \beta_0 = \frac{1}{\sqrt{2\pi}}, \quad \beta_2 = -\frac{1}{\sqrt{2\pi}}, \quad \beta_4 = \frac{3}{\sqrt{2\pi}}. \]

For \( k = 1, \ldots, d \)
\[ a(0) = \sqrt{\frac{2}{\pi}} \Gamma \left( \frac{d+1}{2} \right), \quad a(2s(k)) = \frac{\Gamma \left( \frac{d+1}{2} \right)}{2 \sqrt{2} \Gamma \left( \frac{d+2}{2} \right)}, \quad a(4s(k)) = -\frac{1}{2^4 \sqrt{2} \Gamma \left( \frac{d+4}{2} \right)}, \]
and for \( j \neq k, j, k = 1, \ldots, d \)
\[ a(2s(j,k)) = -\frac{1}{2^3 \sqrt{2} \Gamma \left( \frac{d+4}{2} \right)}. \]

The proof of Lemma 4.2 is postponed to Appendix B. Using Lemma 4.2, we easily see that we can rewrite the second-order chaos and the fourth-order chaos as follows:
\[ V_n[2] = \sqrt{\frac{n\pi}{d}} \Gamma \left( \frac{d+1}{2} \right) \left[ -\frac{1}{\Gamma \left( \frac{d}{2} \right)} \int_{\mathbb{T}^d} H_2(f_n(x)) dx + \frac{1}{2} \frac{1}{\Gamma \left( \frac{d+2}{2} \right)} \sum_{k=1}^{d} \int_{\mathbb{T}^d} H_2(f_{n,k}(x)) dx \right], \]
\[ (4.7) \]
\[ V_n[4] = \sqrt{\frac{n\pi}{d}} \Gamma \left( \frac{d+1}{2} \right) \left[ \frac{1}{2^2 \Gamma \left( \frac{d+2}{2} \right)} \int_{\mathbb{T}^d} H_4(f_n(x)) dx - \frac{1}{2^4 \Gamma \left( \frac{d+4}{2} \right)} \sum_{k=1}^{d} \int_{\mathbb{T}^d} H_2(f_n(x)) H_2(f_{n,k}(x)) dx \right. \\
- \frac{1}{2^4 \Gamma \left( \frac{d+4}{2} \right)} \sum_{k=1}^{d} \int_{\mathbb{T}^d} H_4(f_{n,k}(x)) dx \right. \\
\left. - \frac{1}{2^3 \Gamma \left( \frac{d+4}{2} \right)} \sum_{j<k} \int_{\mathbb{T}^d} H_2(f_{n,j}(x)) H_2(f_{n,k}(x)) dx \right]. \]

We first prove that the second-order chaotic projection vanishes, i.e., \( V_n[2] = 0 \). This is an immediate consequence of the following lemma. Q5
Lemma 4.3.

\[ \int_{T^d} H_2(f_n(x))dx = \frac{1}{\sqrt{N_n}}W(n), \]
\[ \sum_{k=1}^{d} \int_{T^d} H_2(f_{n,k}(x))dx = \frac{d}{\sqrt{N_n}}W(n). \]

The proof of Lemma 4.3 is postponed to Appendix B. The precise analysis of the fourth-order chaotic component also relies on the following representation lemma.

Lemma 4.4.

\[ \int_{T^d} H_4(f_n(x))dx = \frac{3}{N_n}W^2(n) - \frac{3}{N_n}R(n) + \frac{1}{N_n}X(n), \]
\[ \sum_{k=1}^{d} \int_{T^d} H_2(f_n(x))H_2(f_{n,k}(x))dx = \frac{d}{N_n}W^2(n) - \frac{d}{N_n}R(n) - \frac{d}{N_n} \sum_{k=1}^{d} X_{k,k}(n), \]
\[ \int_{T^d} H_4(f_{n,k}(x))dx = \frac{3d^2}{N_n}W_{k,k}(n) - \frac{3d^2}{N_n}R_{k,k}(n) + \frac{d^2}{N_n}X_{k,k,k,k}(n), \]
\[ \sum_{j \neq k} \int_{T^d} H_2(f_{n,j}(x))H_2(f_{n,k}(x))dx = \frac{d^2}{N_n}W^2(n) \]
\[ - \frac{d^2}{N_n} \sum_{k=1}^{d} W_{j,k}(n) + \frac{2d^2}{N_n} \sum_{j \neq k} W_{j,k}(n) \]
\[ - \frac{3d^2}{N_n} \sum_{j \neq k} R_{k,j}(n) + \frac{d^2}{N_n} \sum_{j \neq k} X_{k,k,j,j}(n). \]

Lemma 4.4 is proved in Appendix B. The key tool in the proof of Lemma 4.4 is Lemma 4.5, where we evaluate summations over \( C(4) \) using the structure in (3.6).

Lemma 4.5.

\[ \sum_{\lambda(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = 3 - 3 \sum_{\lambda(4)} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \sum_{\lambda(4)} |a_{\lambda}|^4, \]
\[ \sum_{\lambda(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = - \sum_{\lambda(4)} \lambda_{1,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 + \sum_{\lambda(4)} \lambda_{2,(k)}^2 |a_{\lambda}|^4 \]
\[ + \sum_{\lambda(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}, \]
\[ \sum_{\lambda(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = 3 \sum_{\lambda(4)} \lambda_{1,(k)}^2 \lambda_{2,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 \]
\[ - 3 \sum_{\lambda(4)} \lambda_{1,(k)}^4 |a_{\lambda_1}|^4 + \sum_{\lambda(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}, \]
\[ \sum_{\lambda(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = \sum_{\lambda(4)} \lambda_{1,(k)}^2 \lambda_{2,(j)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 \]
\[ + 2 \left[ \sum_{\lambda(4)} \lambda_{(j)} |a_{\lambda}|^2 - 1 \right]^2 - 3 \sum_{\lambda(4)} \lambda_{(j)}^2 |a_{\lambda}|^4 \]
The proof of Lemma 4.5 is in Appendix B. Combining (4.7) and Lemma 4.4 leads to the following representation of the fourth-order chaotic component of the nodal volume $V_n[4]$,

$$V_n[4] = \sqrt{\frac{n\pi}{d}} \frac{1}{4N_n \Gamma\left(\frac{d+1}{2}\right)} \left[ \frac{2}{d+2} \sum_{j,k} W_{j,k}^2(n) - R(n) + \frac{3d}{d+2} \sum_{j,k} R_{k,j}(n) + X(n) + 2 \sum_k X_{k,k}(n) - \frac{d}{d+2} \sum_{j,k} X_{k,k,j,j}(n) \right].$$

5. Proof of Theorem 1

5.1. Asymptotic behavior of the fourth-order chaos in dimension $d = 3$. The aim of this section is the analysis of the asymptotic behavior, as $n \to \infty$, $n \not\equiv 0, 4, 7 \,(\text{mod} \,8)$, of the sequence $\frac{A_n[4]}{\sqrt{\text{Var}(A_n[4])}}$.

From (4.14) we know that

$$A_n[4] = \sqrt{\frac{n}{5\sqrt{3} N_n}} \left[ W_{j,k}^2(n) - 3 \sum_{j,k} W_{j,k}^2(n) - \frac{5}{2} R(n) + \frac{3^2}{2} \sum_{j,k} R_{k,j}(n) + \frac{5}{2} X(n) + 5 \sum_k X_{k,k}(n) - \frac{3}{2} \sum_{j,k} X_{k,k,j,j}(n) \right].$$

In Lemma 5.1 we prove that

$$-\frac{5}{2} R(n) + \frac{3^2}{2} \sum_{j,k} R_{k,j}(n) = 4 + o_p(1),$$

and in Lemma 5.2 we obtain that

$$\frac{5}{2} X(n) + 5 \sum_k X_{k,k}(n) - \frac{3}{2} \sum_{j,k} X_{k,k,j,j}(n) = o_P(1).$$

**Lemma 5.1.** As $n \to \infty$, $n \not\equiv 0, 4, 7 \,(\text{mod} \,8)$,

$$R(n) \overset{p}{\to} 2,$$

$$R_{k,j}(n) \overset{p}{\to} \begin{cases} \frac{2}{7} & \text{if } k = j, \\ \frac{2}{3} & \text{if } k \neq j. \end{cases}$$

**Proof.** Since $\Lambda_n$ is symmetric under $\lambda \to -\lambda$ and $N_n$ is even, we rewrite $R(n)$ as follows:

$$R(n) = \frac{2}{N_n} \sum_{\lambda \in \Lambda_n/\pm} |a_\lambda|^4,$$
where $\Lambda_n/\pm$ denotes the representatives of the equivalence class of $\Lambda_n$ under $\lambda \to -\lambda$; $R(n)$ is then written in terms of a sum of independent and identically distributed random variables with $\mathbb{E}[|a_\lambda|^4] = 2$. The limit in (5.1) follows from the law of large numbers. To prove (5.2), we can apply again the law of large numbers since $\lambda$ distributed random variables with

\[ R_{k,j}(n) = \frac{1}{n^2 N_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_\lambda|^4 \]

\[ = \frac{2}{n^2 N_n} \sum_{\lambda \in \Lambda_n/\pm} \lambda_{(k)}^2 \lambda_{(j)}^2 |a_\lambda|^4 \xrightarrow{p} \lim_{n \to \infty} \frac{2}{n^2 N_n} \sum_{\lambda \in \Lambda_n} \lambda_{(k)}^2 \lambda_{(j)}^2. \]

Formula (5.2) follows from Lemma 5.3. \qed

Lemma 5.2. As $n \to \infty$, $n \not\equiv 0, 4, 7 \pmod{8}$, we can rewrite (5.3) as follows:

\[ X(n), X_{k,k}(n), X_{k,k,j,j}(n) \xrightarrow{L^2} 0. \]

Proof.

\[ \mathbb{E}(|X(n)|^2) = \frac{1}{N_n^2} \mathbb{E} \left[ \sum_{\lambda \in \lambda_{(4)}} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \sum_{\lambda \in \lambda_{(4)}} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} \right] = \frac{1}{N_n^2} O(|X_{\lambda}(4)|). \]

In view of Lemma 5.3 we have

\[ \mathbb{E}(|X(n)|^2) = \frac{1}{N_n^2} O(N_n^{7/4+o(1)}) = O(N_n^{-1/4+o(1)}). \]

Exactly the same holds for $X_{k,k}(n), X_{k,k,j,j}(n)$ once we observe that $\lambda_{(k)} \lambda_{(j)} / n^2 \leq 1$ and $\lambda_{(k)} \lambda_{(j)} / n^2 \leq 1$. \qed

Then we can write the fourth-order chaos of the nodal area $\mathcal{A}_n[4]$ as follows:

\[ \mathcal{A}_n[4] = \frac{\sqrt{n}}{5 \sqrt{3} N_n} \left[ 4 + W^2(n) - 3 \sum_{j,k} W_{j,k}(n) + o_\varepsilon(1) \right]. \]

Now we note that $W(n) = \sum_{k=1}^d W_{k,k}(n)$, so we can rewrite (5.3) as follows:

\[ \mathcal{A}_n[4] = \frac{\sqrt{n}}{5 \sqrt{3} N_n} \left[ 4 - (W_{1,1}(n) - W_{2,2}(n))^2 \right. \]

\[ - (W_{1,1}(n) - W_{3,3}(n))^2 - (W_{2,2}(n) - W_{3,3}(n))^2 \]

\[ - 6(W_{1,2}(n) + W_{1,3}(n) + W_{2,3}(n)) + o_\varepsilon(1) \].

Let $\mathcal{W}(n)$ be the seven-dimensional vector with components

\[ \mathcal{W}(n) = (W_{1,1}(n), W_{1,2}(n), W_{1,3}(n), W_{2,2}(n), W_{2,3}(n), W_{3,3}(n)). \]

Lemma 5.3. As $n \to \infty$, $\mathcal{W}(n) \xrightarrow{d} V$, where $V$ is a centered Gaussian vector with covariance matrix

\[ \Sigma = \begin{pmatrix}
\frac{2}{5} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} \\
0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} \\
0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3}
\end{pmatrix}. \]
Proof. In Appendix C, we prove that the covariance matrix $\Sigma(n)$ of $W(n)$ converges to $\Sigma$. Since for every fixed integer $n$ each component of $W(n)$ belongs to the second Wiener chaos, in view of [26, Theorem 6.2.3] the following two conditions are equivalent:

1. $W(n)$ converges in law to $V$.
2. Each component of $W(n)$ converges in distribution to a one-dimensional centered Gaussian random variable.

We prove (2): we observe that for $\lambda \in \Lambda_n/\pm$ the random variables $|a_\lambda|^2$ are independent and identically distributed with mean and variance equal to 1, and we write

$$W_{k,j}(n) = 2 \sum_{\lambda \in \Lambda_n/\pm} (Q_\lambda - \mu_\lambda),$$

where

$$Q_\lambda = \frac{\lambda(k)\lambda(j)}{n\sqrt{N_n}} |a_\lambda|^2, \quad \mu_\lambda = \frac{\lambda(k)\lambda(j)}{n\sqrt{N_n}},$$

and we note that the $Q_\lambda$ are independent random variables, each with expected value $\mu_\lambda$ and variance $\mu_\lambda^2$. We also note that they have a finite expected value and a finite variance since $\lambda(k), \lambda(j) \leq \sqrt{n}$.

Define

$$s_\lambda^2 = \sum_{\lambda \in \Lambda_n/\pm} \text{Var}(Q_\lambda) = \begin{cases} \frac{1}{25} + O\left(\frac{1}{n^{1/28-o(1)}}\right), & k = j, \\ \frac{1}{28} + O\left(\frac{1}{n^{1/28-o(1)}}\right), & k \neq j. \end{cases}$$

We apply now the Lyapunov condition, so we need to prove that

$$\lim_{n \to \infty} \frac{1}{s_\lambda^2} \sum_{\lambda \in \Lambda_n/\pm} \mathbb{E}[|Q_\lambda - \mu_\lambda|^4] = 0.$$ 

To do that, we first evaluate the fourth central moment of $Q_\lambda$; since $Q_\lambda \sim \frac{1}{2} \mu_\lambda \chi_2$, where $\chi_2$ is a chi-square with 2 degrees of freedom, we need the moments

$$\mathbb{E}[\chi_2^m] = 2^m \Gamma(m + 1)$$

so that

$$\mathbb{E}[(Q_\lambda - \mu_\lambda)^4] = \mathbb{E}[Q_\lambda^4] - 4\mu_\lambda \mathbb{E}[Q_\lambda^3] + 6\mu_\lambda^2 \mathbb{E}[Q_\lambda^2] - 4\mu_\lambda^3 \mathbb{E}[Q_\lambda] + \mu_\lambda^4 = \mu_\lambda^4.$$ 

We finally note that

$$0 \leq \lim_{n \to \infty} \frac{1}{O(1)} \sum_{\lambda \in \Lambda_n/\pm} \mu_\lambda^4 \leq \lim_{n \to \infty} \frac{1}{O(1)} \sum_{\lambda \in \Lambda_n/\pm} \frac{1}{N_n^2} = \lim_{n \to \infty} \frac{1}{O(1)} \frac{1}{2N_n} = 0.$$ 

So

$$\frac{1}{s_\lambda} W_{k,j}(n) \xrightarrow{d} 2N(0, 1);$$

that is, $W_{k,j}(n)$ converges in distribution to a centered Gaussian with variance $2/5$ if $k = j$, and $2/(3 \cdot 5)$ if $k \neq j$. $\square$

Note that the covariance matrix $\Sigma$ is nonsingular. The multidimensional central limit theorem stated in Lemma 5.3 implies that

$$4 - (W_{1,1}(n) - W_{2,2}(n))^2 - (W_{1,1}(n) - W_{3,3}(n))^2 - (W_{2,2}(n) - W_{3,3}(n))^2 - 6(W_{1,2}(n) + W_{1,3}(n) + W_{2,3}(n)) + o_p(1)$$

implies

$$\frac{1}{s_\lambda} W_{k,j}(n) \xrightarrow{d} 2N(0, 1);$$

that is, $W_{k,j}(n)$ converges in distribution to a centered Gaussian with variance $2/5$ if $k = j$, and $2/(3 \cdot 5)$ if $k \neq j$. $\square$
\[
\text{law} \quad 4 - (V_{1,1} - V_{2,2})^2 - (V_{1,1} - V_{3,3})^2 - (V_{2,2} - V_{3,3})^2 - 6(V_{1,2}^2 + V_{1,3}^2 + V_{2,3}^2)
\]
\[
= 4 - \sum_{i=1}^{3} X_i^2 - 6 \sum_{i=1}^{3} Y_i^2,
\]
where \( X = (X_1, X_2, X_3) \) and \( Y = (Y_1, Y_2, Y_3) \) are two independent centered Gaussian vector with covariance matrices
\[
\Sigma_X = \frac{2^3}{3 \cdot 5} \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}, \quad \Sigma_Y = \frac{2}{3 \cdot 5} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
respectively. The covariance matrix \( \Sigma_X \) is singular; hence we consider the transformation \( X_1 = Z_1, X_2 = \alpha Z_1 + \beta Z_2, X_3 = \gamma Z_1 + \delta Z_2 \), where \( Z = (Z_1, Z_2) \) is a centered Gaussian vector with covariance matrix \( \Sigma_Z = \frac{2^3}{3 \cdot 5} I_2 \). We note that
\[
\text{Cov}(X_1, X_2) = \frac{1}{2} \frac{2^3}{3 \cdot 5} = \text{Cov}(Z_1, \alpha Z_1 + \beta Z_2) = \frac{2^3}{3 \cdot 5} \alpha,
\]
\[
\text{Cov}(X_1, X_3) = -\frac{1}{2} \frac{2^3}{3 \cdot 5} = \text{Cov}(Z_1, \gamma Z_1 + \delta Z_2) = \frac{2^3}{3 \cdot 5} \gamma,
\]
\[
\text{Cov}(X_2, X_3) = \frac{1}{2} \frac{2^3}{3 \cdot 5} = \text{Cov}(\alpha Z_1 + \beta Z_2, \gamma Z_1 + \delta Z_2) = \frac{2^3}{3 \cdot 5} (\alpha \gamma + \beta \delta),
\]
\[
\text{Var}(X_2) = \frac{2^3}{3 \cdot 5} = \text{Var}(\alpha Z_1 + \beta Z_2) = \frac{2^3}{3 \cdot 5} (\alpha^2 + \beta^2);
\]
this implies that \( \alpha = -\gamma = 1/2 \) and \( \delta = \beta = \sqrt{3}/2 \). We write
\[
(5.4)
\]
\[
4 - \sum_{i=1}^{3} X_i^2 - 6 \sum_{i=1}^{3} Y_i^2 = 4 - \frac{6}{4} \sum_{i=1}^{2} Z_i^2 - 6 \sum_{i=1}^{3} Y_i^2 = 4 - \frac{6}{3 \cdot 5} \sum_{i=1}^{5} U_i^2 = 4 - \frac{4}{5} \chi,
\]
where \( U \) is a five-dimensional centered standard Gaussian vector and \( \chi \) is a central chi-square with 5 degrees of freedom.

**5.2. Proof of Theorem 1** In view of (5.4)
\[
\text{Var}(A_n[4]) = \frac{n}{5^2 \cdot 3 \cdot N_n^2} \frac{4^2}{5^2} \cdot 5 \cdot 2 + o \left( \frac{n}{N_n^2} \right) = \frac{n}{N_n^2} \frac{2^5}{5^3} \cdot 3 + o \left( \frac{n}{N_n^2} \right)
\]
and
\[
\frac{A_n[4]}{\sqrt{\text{Var}(A_n[4])}} \xrightarrow{\text{law}} \frac{1}{\sqrt{5 \cdot 2}} (5 - \chi).
\]

Theorem 1 follows immediately by observing that \( \text{Var}(A_n) \sim \text{Var}(A_n[4]) \), and that different chaotic components are orthogonal in \( L^2 \).

**APPENDIX A. CHAOTIC EXPANSION OF \( \mathcal{V}_n \): PROOF OF LEMMA 1**

**A.1. Hermite expansion of \( \frac{1}{2 \epsilon} 1_{[-\epsilon, \epsilon]}(f_n) \).** We first expand the function \( \frac{1}{2 \epsilon} 1_{[-\epsilon, \epsilon]}(\cdot) \) into Hermite polynomials. Using completeness and orthonormality of the set \( \mathbb{H} \) in \( L^2(\gamma) \), the following decomposition holds:
\[
\frac{1}{2 \epsilon} 1_{[-\epsilon, \epsilon]}(\cdot) = \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k H_k(\cdot),
\]
Therefore the probability density function of $\mathbf{f}(x)$ can be expanded into a series of Hermite polynomials; we have

$$
\beta_k^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^\varepsilon \gamma(t) dt, \quad \beta_k^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^\varepsilon \gamma(t) H_k(t) dt, \quad k \geq 1.
$$

From (A.1) and observing that $H_k$ is an even function if $k$ is even, we easily obtain

$$
\beta_{2k-1}^\varepsilon = 0, \quad \beta_{2k}^\varepsilon = -\frac{1}{\varepsilon} \gamma(\varepsilon) H_{2k-1}(\varepsilon).
$$

(A.1)

**A.2. Hermite expansion of $||\nabla f_n(x)||$.** We first consider a standard Gaussian vector $Z$ in $\mathbb{R}^d$, and the random variable $||Z||$ is square integrable, so it can be expanded into a series of Hermite polynomials; we have

$$
||Z|| = \sum_{p=0}^{\infty} \sum_{s \in \mathbb{N}^d; s_1 + \cdots + s_d = p} a(s) \prod_{j=1}^{d} H_{s_j}(Z_j),
$$

where

$$
a(s) = \frac{1}{s_1! s_2! \cdots s_d!} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} ||z||^d \prod_{j=1}^{d} H_{s_j}(z_j) e^{-\frac{||z||^2}{2}} dz.
$$

We note that $e^{-\frac{||x||^2}{2}}$ and $||z||$ are even functions of $z \in \mathbb{R}^d$, so $a(s)$ vanishes if one of the $s_j$ is odd.

We need to evaluate the integrals $a(2s)$. As in [23], we introduce the Gaussian vector $U$ in $\mathbb{R}^d$ of independent random variables with unit variance and mean $\mu \in \mathbb{R}^d$. It is known that $||U||^2$ is distributed according to a noncentral chi-squared distribution with $d+2i$ degrees of freedom. Therefore the probability density function of $||U||$ is $f_{||U||}(t) = 2t f_{||U||^2}(t^2)$, and its expectation is

$$
E[||U||] = \int_{0}^{\infty} t f_{||U||}(t) dt = e^{-\frac{||\mu||^2}{2}} \sum_{i=0}^{\infty} \frac{1}{i!} ||\mu||^{2i} \frac{1}{2} t^{2i} \frac{\Gamma(d/2 + i + 1/2)}{\Gamma(d/2 + i)}.
$$

We Taylor expand the exponential, and we apply Newton’s formula

$$
e^{-\frac{||\mu||^2}{2}} = \sum_{k_1, \ldots, k_d = 0}^{\infty} \frac{(-1)^{k_1 + \cdots + k_d}}{k_1! \cdots k_d!} \frac{1}{2k_1 + \cdots + k_d} \mu_1^{2k_1} \cdots \mu_d^{2k_d},
$$

$$
||\mu||^{2i} = \sum_{j_1 + \cdots + j_d = i} \frac{i^i}{j_1! \cdots j_d!} \mu_1^{2j_1} \cdots \mu_d^{2j_d}
$$

to rewrite (A.3) as follows:

$$
E[||U||] = \sum_{l_1, \ldots, l_d = 0}^{\infty} \mu_1^{2l_1} \cdots \mu_d^{2l_d} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\sqrt{2\Gamma(d/2 + i + 1/2)}}{\Gamma(d/2 + i)}.
$$

(A.4)
we finally obtain

\[ \sum_{j_1 + \ldots + j_d = i, j_1 \leq l_1, \ldots, j_d \leq l_d} \left( \frac{i}{j_1! \ldots j_d!} \right) \frac{(-1)^{l_1 - j_1 + \ldots + l_d - j_d}}{(l_1 - j_1)! \cdots (l_d - j_d)!} 2^{j_1 - j_1 + \ldots + l_d - j_d}, \]

where we set \( l = k + j \). On the other hand, we can rewrite the Gaussian expectation \( E[||U||] \) as

\[ E[||U||] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} ||u|| e^{-\frac{||u-x||^2}{2}} du, \]

and using the Hermite expansion of the exponential [26 Proposition 1.4.2]

\[ e^{cx - \frac{x^2}{2}} = \sum_{l=0}^{\infty} \frac{c^l}{l!} H_l(x), \quad c \in \mathbb{R}, \]

we get

\[ E[||U||] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} ||u|| \prod_{j=1}^d e^{-\frac{u_j^2}{2} + \mu_j u_j} du \]

\begin{equation}
\tag{A.5}
= \frac{1}{(2\pi)^{d/2}} \sum_{l_1, \ldots, l_d = 0} \frac{\mu_1^{l_1} \cdots \mu_d^{l_d}}{l_1! \cdots l_d!} \int_{\mathbb{R}^d} ||u|| \prod_{j=1}^d H_j(u_j) du \int_{\mathbb{R}^d} ||u|| \prod_{j=1}^d H_{2l_j}(u_j) du
\end{equation}

since if one of the \( l_j \) is odd, then the integral vanishes. Combining [A.4] and [A.5], we finally obtain

\[ \frac{1}{(2\pi)^{d/2}} \frac{1}{(2l_1)! \cdots (2l_d)!} \int_{\mathbb{R}^d} ||u|| \prod_{j=1}^d H_{2l_j}(u_j) du \]

\[ \sum_{i=0}^{\infty} \frac{1}{i! 2^i} \frac{\sqrt{2\Gamma(d/2 + i + 1/2)} \Gamma(d/2 + i)}{\Gamma(d/2 + i)} \times \sum_{j_1 + \ldots + j_d = i} \left( \frac{i}{j_1! \ldots j_d!} \right) \frac{(-1)^{l_1 - j_1 + \ldots + l_d - j_d}}{(l_1 - j_1)! \cdots (l_d - j_d)!} 2^{j_1 - j_1 + \ldots + j_d - j_d}, \]

and this implies that

\[ a(2s) = \frac{1}{(2\pi)^{d/2}} \frac{\sqrt{2\Gamma(d/2 + i + 1/2)} \Gamma(d/2 + i)}{\Gamma(d/2 + i)} \times \sum_{j_1 + \ldots + j_d = i} \left( \frac{i}{j_1! \ldots j_d!} \right) \frac{(-1)^{s_1 - j_1 + \ldots + s_d - j_d}}{(s_1 - j_1)! \cdots (s_d - j_d)!} 2^{s_1 - j_1 + \ldots + s_d - j_d}. \]

To obtain the Hermite expansion of \( ||\nabla f_n(x)|| \), we note that

\[ ||\nabla f_n(x)|| = 2\pi \sqrt{\frac{n}{d}} \left( \sum_{j=1}^d f_{n,j}^2(x) \right)^{1/2}, \]
where \( f_{n,j}(x) \) is the normalized derivative defined in (2.2), and where, for every fixed \( x \in \mathbb{T}^d \), \( f_n(x) \) and \( \partial_j f_n(x) \) are stochastically independent. Then we have
\[
\frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(f_n(x)) \| \nabla f_n(x) \|
= \sum_{k=0}^{\infty} \left( \frac{1}{2k} \right)! \beta_{2k} H_{2k}(f_n(x)) \frac{1}{\sqrt{2\pi}} \sum_{p=0}^{\infty} \sum_{s_1+\cdots+s_d=p} a(2s) \prod_{j=1}^{d} H_{2s_j}(f_{n,j}(x)),
\]
i.e., the projection onto each odd-order chaos vanishes, whereas the projection onto the chaos of order \( 2q \) for \( q \geq 1 \) is
\[
\mathcal{V}_{n}[2q] = 2\pi \sqrt{d} \sum_{p=0}^{q} \left( \frac{1}{2(2q-2p)} \right) \beta_{2q-2p} H_{2q-2p}(f_n(x)) \sum_{s_1+\cdots+s_d=p} sa(2s) \prod_{j=1}^{d} H_{2s_j}(f_{n,j}(x)),
\]
where we set \( 2q = 2k + 2p \) i.e. \( 2k = 2q - 2p \).

### A.3. Evaluation of the coefficients \( \beta_{2q-2p} \).

In view of the \( L^2(\mathbb{P}) \) convergence in (4.3), the chaotic expansion of \( \mathcal{V}_n \) follows by letting \( \varepsilon \to 0 \):
\[
\beta_0 = \lim_{\varepsilon \to 0} \beta_0^\varepsilon = \frac{1}{\sqrt{2\pi}}, \quad \beta_{2k} = \lim_{\varepsilon \to 0} \beta_{2k}^\varepsilon = \frac{1}{\sqrt{2\pi}} \gamma(\varepsilon) H_{2k-1}(\varepsilon) = \frac{1}{\sqrt{2\pi}} H_{2k}(0).
\]

### Appendix B. Second- and fourth-order chaos

For \( j, k = 1, \ldots, d \) we denote by \( s(j) \) the vector in \( \mathbb{R}^d \) with \( j \)th component equal to 1 and all of the other components equal to 0, and we denote by \( s(j, k) \) the vector in \( \mathbb{R}^d \) with \( j \)th and \( k \)th components equal to 1, and all of the other components equal to 0. To evaluate the second- and fourth-order chaos, we need the following lemma.

**Proof of Lemma 4.2.** From (4.3) we immediately derive \( \beta_0, \beta_2, \) and \( \beta_4 \). To evaluate \( a(0) \), we note that, in view of (A.2), \( a(0) = \mathbb{E}(\| Z \|) \), where \( Z \) is a standard Gaussian vector in \( \mathbb{R}^d \). Then from (A.3) with \( \mu = 0 \) we have
\[
a(0) = \mathbb{E}(\| Z \|) = \sqrt{2} \frac{\Gamma\left(\frac{d}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.
\]
Now letting \( \gamma(i) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\Gamma\left(\frac{d}{2} + \frac{1}{2}\right)}}{\Gamma\left(\frac{d}{2}\right)} \), we have \( \gamma(0) = \frac{\sqrt{2\pi}(\frac{d}{2} + \frac{1}{2})}{\Gamma\left(\frac{d}{2}\right)} \), \( \gamma(1) = \frac{\sqrt{2\pi}(\frac{d}{2} + \frac{3}{2})}{\Gamma\left(\frac{d}{2} + 1\right)} \), and \( \gamma(2) = \frac{\sqrt{2\pi}(\frac{d}{2} + \frac{5}{2})}{\Gamma\left(\frac{d}{2} + 2\right)} \). To evaluate \( a(2s(k)) \), we apply (4.6) and obtain
\[
a(2s(k)) = \sum_{i=0}^{1} \gamma(i) \sum_{j_k=1}^{(i)} \left( \frac{(-1)^{q_k-j_k}}{(q_k-j_k)!} \frac{1}{2q_k-j_k} \right) = -\frac{\gamma(0)}{2} + \frac{\gamma(1)}{2} = \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{d}{2} + \frac{3}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)}.
\]
And similarly we derive \( a(4s(k)) \) by observing that
\[
a(4s(k)) = \sum_{i=0}^{2} \gamma(i) \sum_{j_k=1}^{(i)} \left( \frac{(-1)^{q_k-j_k}}{(2-j_k)!} \frac{1}{2^{2-j_k}} \right) = \frac{\gamma(0)}{2^4} + \frac{\gamma(1)}{2^4} + \frac{\gamma(2)}{2^4} = \frac{\Gamma\left(\frac{d}{2} + \frac{5}{2}\right)}{2^4 \sqrt{2\pi} \Gamma\left(\frac{d}{2} + 2\right)}.
\]
Finally, for \( j \neq k \), we have
\[
a(2s(j,k)) = 2 \sum_{i=0}^{2} \gamma(i) \sum_{j_1+j_2=i} \binom{i}{j_1,j_2} \frac{(-1)^{1-j_1+1-j_2}}{(1-j_1)!(1-j_2)!} \frac{1}{2^{1-j_1+1-j_2}}
\]
\[
= \gamma(0) - \gamma(1) + 2\gamma(2) = -\frac{1}{2^3 \sqrt{2} \Gamma(\frac{3}{2} + 2)}. \quad \square
\]

**Proof of Lemma 4.3** We apply the orthogonality relations of exponentials
\[
\int_{T_d} e((\mu,x)) dx = \begin{cases} 
1, & \mu = 0, \\
0, & \mu \neq 0.
\end{cases}
\]

Note that \( H_2(x) = x^2 - 1 \), so we have
\[
\int_{T_d} H_2(f_n(x)) dx = \int_{T_d} \left( \int_{T_d} f_n^2(x) - 1 \right) dx = \frac{1}{N_n} \sum_{\lambda_1,\lambda_2} a_{\lambda_1} a_{\lambda_2} \int_{T_d} e((\lambda_1, x)) e((\lambda_2, x)) dx - 1
\]
\[
= \frac{1}{N_n} \sum_{\lambda_1+\lambda_2=0} a_{\lambda_1} a_{\lambda_2} - 1 = \frac{1}{N_n} \sum_{\lambda} a_{\lambda} \bar{a}_{\lambda} - 1 = \frac{1}{N_n} \sum_{\lambda} |a_{\lambda}|^2 - 1
\]
\[
= \frac{1}{N_n} \sum_{\lambda} (|a_{\lambda}|^2 - 1).
\]

For \( k = 1, \ldots, d \)
\[
\int_{T_d} H_2(f_{n,k}(x)) dx = \int_{T_d} \left( \int_{T_d} f_{n,k}^2(x) - 1 \right) dx
\]
\[
= -\frac{d}{nN_n} \sum_{\lambda_1,\lambda_2} a_{\lambda_1} a_{\lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} \int_{T_d} e((\lambda_1, x)) e((\lambda_2, x)) dx - 1
\]
\[
= -\frac{d}{nN_n} \sum_{\lambda_1+\lambda_2=0} a_{\lambda_1} a_{\lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} - 1 = \frac{d}{nN_n} \sum_{\lambda} |a_{\lambda}|^2 \lambda_{(k)}^2 - 1
\]
\[
= \frac{d}{nN_n} \sum_{\lambda} \lambda_{(k)}^2 (|a_{\lambda}|^2 - 1).
\]

where in the last step we applied (3.11). \( \square \)

**Proof of Lemma 4.3** Formulas (4.10) and (4.12) follow immediately from (3.6). To prove (4.11), we note that
\[
\sum_{C(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} = \sum_{\lambda_1,\lambda_2} \lambda_{1,(k)}^2 a_{\lambda_1}^2 \lambda_{2,(k)}^2 a_{\lambda_2}^2 + 2 \sum_{\lambda_1,\lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 + \sum_{\lambda} \lambda_{(k)}^2 |a_{\lambda}|^4 + \sum_{\lambda \in C(4)} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},
\]

where the second term cancels since \( a_{\lambda} = \bar{a}_{\lambda} \), and then
\[
\sum_{\lambda} \lambda_{(k)} |a_{\lambda}|^2 = 0.
To prove the identity \((4.13)\), we apply again \((3.6)\) to get
\[
\sum_{C(4)} \lambda_1(k) \lambda_2(k) \lambda_3(j) \lambda_4(j) a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}
\]
\[
= \sum_{\lambda_1, \lambda_3} \lambda_1^2 \lambda_3^2 |a_{\lambda_1}|^2 |a_{\lambda_3}|^2
\]
\[
+ 2 \sum_{\lambda_1, \lambda_2} \lambda_1(k) \lambda_2(k) \lambda_1(j) \lambda_2(j) |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \sum_{\lambda} \lambda^2 \lambda^2 |a_{\lambda}|^4
\]
\[
+ \sum_{\lambda} \lambda_1(k) \lambda_2(k) \lambda_3(j) \lambda_4(j) a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4},
\]
where we note that, in view of \((3.1)\), we can write
\[
\sum_{\lambda_1, \lambda_2} \lambda_1(k) \lambda_2(k) \lambda_1(j) \lambda_2(j) |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 = \left[ \sum_{\lambda} \lambda(k) \lambda(j) (|a_{\lambda}|^2 - 1) \right]^2.
\]

\(\square\)

**Proof of Lemma** \((4.4)\). We use \(H_4(x) = x^4 - 6x^2 + 3\) and formula \((4.8)\) to write
\[
\int_{T^d} H_4(f_n(x)) \, dx = \int_{T^d} f_n^4(x) \, dx - 6 \int_{T^d} f_n^2(x) \, dx + 3
\]
\[
= \frac{1}{N_n^2} \sum_{C(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{6}{N_n^2} \sum_{\lambda} |a_{\lambda}|^2 + 3.
\]
In view of formula \((4.10)\) we have
\[
\int_{T^d} H_4(f_n(x)) \, dx = \frac{3}{N_n^2} \sum_{\lambda_1, \lambda_2} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - \frac{3}{N_n^2} \sum_{\lambda} |a_{\lambda}|^4
\]
\[
+ \frac{1}{N_n^2} \sum_{\lambda} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{6}{N_n^2} \sum_{\lambda} |a_{\lambda}|^2 + 3
\]
\[
= \frac{3}{N_n^2} \left[ \sum_{\lambda} (|a_{\lambda}|^2 - 1) \right]^2 - \frac{3}{N_n^2} \sum_{\lambda} |a_{\lambda}|^4 + \frac{1}{N_n^2} \sum_{C(4)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
\]
We evaluate now
\[
\sum_{k=1}^{d} \int_{T^d} H_2(f_n(x)) H_2(f_{n,k}(x)) \, dx
\]
\[
= \sum_{k=1}^{d} \int_{T^d} \left( f_n^2(x) - 1 \right) \left( f_{n,k}^2(x) - 1 \right) \, dx
\]
\[
= \sum_{k=1}^{d} \left\{ \int_{T^d} f_n^2(x) f_{n,k}(x) \, dx - \int_{T^d} f_n^2(x) \, dx - \int_{T^d} f_{n,k}(x) \, dx + 1 \right\}
\]
\[
= \sum_{k=1}^{d} \left\{ \frac{d}{n N_n^2} \sum_{C(4)} \lambda_1(k) \lambda_2(k) \lambda_3(k) \lambda_4(k) a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - \frac{1}{N_n^2} \sum_{\lambda} |a_{\lambda}|^2 - \frac{1}{n N_n^2} \sum_{\lambda} \lambda^2 |a_{\lambda}|^2 + 1 \right\}
\]
where we applied (4.8) and (4.9) and in the last step we use the fact that $\sum_k \lambda^2_{(k)} = n$; from formulas (4.11) and (3.1) we have

$$
\sum_{k=1}^{d} \int_{T^d} H_2(f_n(x)) H_2(f_{n,k}(x)) dx
$$

$$
= \frac{d}{N_n^2} \sum_{\lambda_1, \lambda_2} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - \frac{d}{N_n^2} \sum_{\lambda} |a_\lambda|^4
$$

$$
- \frac{d}{nN_n^2} \sum_{k=1, \chi(4)}^{d} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - 2 \frac{d}{N_n^2} \sum_{\lambda} |a_\lambda|^2 + d
$$

$$
= \frac{d}{N_n^2} \left[ \sum_{\lambda} (|a_\lambda|^2 - 1) \right]^2 - \frac{d}{N_n^2} \sum_{\lambda} |a_\lambda|^4
$$

$$
- \frac{d}{nN_n^2} \sum_{k=1, \chi(4)}^{d} \lambda_{1,(k)} \lambda_{2,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
$$

We also have that

$$
\int_{T^d} H_4(f_{n,k}(x)) dx
$$

$$
= \frac{d^2}{n^2N_n^2} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - 6 \int_{T^d} f^2_{n,k}(x) dx + 3
$$

$$
= 3 \frac{d^2}{n^2N_n^2} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(k)}^2 |a_{\lambda_1}|^2 |a_{\lambda_2}|^2 - 3 \frac{d^2}{n^2N_n^2} \sum_{\lambda} \lambda_{(k)}^4 |a_\lambda|^4
$$

$$
+ \frac{d^2}{n^2N_n^2} \sum_{\chi(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} - 6 \frac{d^2}{N_n^2} \sum_{\lambda} |a_\lambda|^2 + 3
$$

$$
= 3 \frac{d^2}{n^2N_n^2} \left[ \sum_{\lambda} \lambda_{(k)}^2 (|a_\lambda|^2 - 1) \right]^2 - 3 \frac{d^2}{n^2N_n^2} \sum_{\lambda} \lambda_{(k)}^4 |a_\lambda|^4
$$

$$
+ \frac{d^2}{n^2N_n^2} \sum_{\chi(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(k)} \lambda_{4,(k)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
$$

And finally,

$$
\int_{T^d} H_2(f_{n,j}(x)) H_2(f_{n,k}(x)) dx
$$

$$
= \int_{T^d} \left( f^2_{n,j}(x) - 1 \right) \left( f^2_{n,k}(x) - 1 \right) dx
$$

$$
= \int_{T^d} \left( f^2_{n,j}(x) f^2_{n,k}(x) - f_{n,j}(x) f_{n,k}(x) - f^2_{n,j}(x) - f^2_{n,k}(x) + 1 \right) dx
$$

$$
= \frac{d^2}{n^2N_n^2} \sum_{\chi(4)} \lambda_{1,(k)} \lambda_{2,(k)} \lambda_{3,(j)} \lambda_{4,(j)} a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4}.
$$
In view of Lemma 4.5, formula (4.13), we apply Lemmas 3.2 and 3.1 and equation (3.2), and we use so now we note that

\[
- \frac{d}{nN_n} \sum_{\lambda} \lambda^2_{(j)} |a_{\lambda_1}|^2 - \frac{d}{nN_n} \sum_{\lambda} \lambda^2_{(k)} |a_{\lambda}|^2 + 1.
\]

In view of Lemma 4.5, formula (4.15),

\[
\int_{T^d} H_2(f_{n,j}(x))H_2(f_{n,k}(x))dx = \frac{d^2}{n^2N_n^2} \sum_{\lambda_1, \lambda_2} \lambda_{1,(k)}^2 \lambda_{2,(j)}^2 \sum_{\lambda} \left( \sum_{j \neq \lambda_2} \lambda^2_{(j)} \right) \left( |a_{\lambda_1}|^2 - 1 \right) \left( |a_{\lambda_2}|^2 - 1 \right) + \frac{2d^2}{n^2N_n^2} \left( \sum_{\lambda} \lambda_{(j)} \lambda_{(k)} \left( |a_{\lambda}|^2 - 1 \right) \right)^2
\]

now we note that

\[
W_{k,k}(n) \sum_{j \neq k} W_{j,j}(n) = \frac{1}{n^2N_n} \sum_{\lambda_1 \in \Lambda_n} \lambda_{1,(k)}^2 \left( |a_{\lambda_1}|^2 - 1 \right) \sum_{j \neq \lambda_2} \sum_{\lambda_2 \in \Lambda_n} \lambda_{2,(j)}^2 \left( |a_{\lambda_2}|^2 - 1 \right)
\]

\[
= \frac{1}{n^2N_n} \sum_{\lambda_1 \in \Lambda_n} \lambda_{1,(k)}^2 \left( |a_{\lambda_1}|^2 - 1 \right) \sum_{\lambda_2 \in \Lambda_n} (n - \lambda_{2,(k)}^2) \left( |a_{\lambda_2}|^2 - 1 \right)
\]

\[
= W(n)W_{k,k}(n) - W_{k,k}^2(n),
\]

so

\[
\sum_{k=1}^{d} W_{k,k}(n) \sum_{j \neq k} W_{j,j}(n) = W(n) \sum_{k=1}^{d} W_{k,k}(n) - \sum_{k=1}^{d} W_{k,k}^2(n) = W^2(n) - \sum_{k=1}^{d} W_{k,k}^2(n).
\]

\[
\square
\]

**Appendix C. Covariance matrices \( \Sigma(n) \) and \( \Sigma \)**

In this section we compute the covariance matrix \( \Sigma(n) \) and its limiting matrix \( \Sigma \); we apply Lemmas 3.2 and 3.1 and equation (3.2), and we use

\[
\mathbb{E}[(|a_{\lambda_1}|^2 - 1)(|a_{\lambda_2}|^2 - 1)] = \begin{cases} 1, & \text{if } \lambda_1 = \pm \lambda_2, \\ 0, & \text{otherwise.} \end{cases}
\]

We note that

\[
\mathbb{E}[W_{k,l}(n)W_{j,m}(n)]
\]
\[ \frac{1}{n^2 N_n} \sum_{\lambda_1, \lambda_2} \lambda_1(k) \lambda_1(i) \lambda_2(j) \lambda_2(m) E[|\alpha_{\lambda_1}|^2 - 1]|\alpha_{\lambda_2}|^2 - 1] \]

\[ \frac{1}{n^2 N_n} \sum_{\lambda} (\lambda(k) \lambda(i) \lambda(j) \lambda(m) E[|\alpha_{\lambda}|^2 - 1]^2] + \lambda(k) \lambda(i) (-\lambda(j)) (-\lambda(m)) E[|\alpha_{\lambda}|^2 - 1]^2] \]

\[ \frac{2}{n^2 N_n} \sum_{\lambda} \lambda(k) \lambda(i) \lambda(j) \lambda(m) E[|\alpha_{\lambda}|^2 - 1]^2] \]

\[ \begin{cases} \frac{2}{3} + O\left(\frac{1}{n^{1/2} - o(1)}\right), & k = l, j = m, k \neq j \text{ or } k = j, l = m, k \neq l \text{ or } k = m, l = j, k \neq l, \\ \frac{2}{5} + O\left(\frac{1}{n^{1/2} - o(1)}\right), & k = l = j = m, \\ 0 & \text{otherwise}. \end{cases} \]

Acknowledgments

The author wishes to thank Igor Wigman for drawing her attention to this Q11 problem and for the stimulating discussions. The author is grateful to Jacques Benatar, Domenico Marinucci, and Ze’ev Rudnick for the insightful remarks.

References

[1] J. Benatar and R. W. Maffiucci, Random waves on $\mathbb{T}^3$: Nodal area variance and lattice points correlations, Int. Math. Res. Not. IMRN (to appear), DOI 10.1093/imrn/rnx220.

[2] P. Bérard, Volume des ensembles nodaux des fonctions propres du laplacien, Bony-Sjöstrand-Meyer seminar, 1984-1985, École Polytech., Palaiseau, 1985, pp. Exp. No. 14, 10 (French). MR19780

[3] M. V. Berry, Statistics of nodal lines and points in chaotic quantum billiards: Perimeter corrections, fluctuations, curvature, J. Phys. A 35 (2002), no. 13, 3025–3038, DOI 10.1088/0305-4470/35/13/301. MR1913853

[4] J. Bourgain, P. Sarnak, and Z. Rudnick, Local statistics of lattice points on the sphere, Modern trends in constructive function theory, Contemp. Math., vol. 661, Amer. Math. Soc., Providence, RI, 2016, pp. 269–282, DOI 10.1090/conm/661/13287. MR3489563

[5] V. Cammarota and D. Marinucci, A quantitative central limit theorem for the Euler-Poincaré characteristic of random spherical eigenfunctions, Ann. Probab. 46 (2018), no. 6, 3188–3228, DOI 10.1214/17-AOP1245. MR3857854

[6] V. Cammarota and I. Wigman, Fluctuations of the total number of critical points of random spherical harmonics, Stochastic Process. Appl. 127 (2017), no. 12, 3825–3869, DOI 10.1016/j.spa.2017.02.013. MR3718098

[7] S. Y. Cheng, Eigenfunctions and nodal sets, Comment. Math. Helv. 51 (1976), no. 1, 43–55, DOI 10.1007/BF02568142. MR0397805

[8] F. Dalmao, I. Nourdin, G. Peccati, and M. Rossi, Phase singularities in complex arithmetic random waves, arXiv:1608.05631 (2016).

[9] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93 (1988), no. 1, 161–183, DOI 10.1007/BF01393691. MR943927

[10] W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, Invent. Math. 92 (1988), no. 1, 73–90, DOI 10.1007/BF01393993. MR931205

[11] W. Duke and R. Schulze-Pillot, Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids, Invent. Math. 99 (1990), no. 1, 49–57, DOI 10.1007/BF01234411. MR1029390

[12] W. Duke, An introduction to the Linnik problems, Equidistribution in number theory, an introduction, NATO Sci. Ser. II Math. Phys. Chem., vol. 237, Springer, Dordrecht, 2007, pp. 197–216, DOI 10.1007/978-1-4020-5404-4_10. MR2290500
[13] E. P. Golubeva and O. M. Fomenko, *Asymptotic distribution of lattice points on the threedimensional sphere*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987), no. Anal. Teor. Chisel i Teor. Funktsi˘ı, 8, 54–71, 297, DOI 10.1007/BF02342921 (Russian); English transl., J. Soviet Math. **52** (1990), no. 3, 3036–3048. MR906844

[14] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 5th ed., Clarendon Press, Oxford University Press, New York, 1979. MR568909

[15] H. Iwaniec, *Fourier coefficients of modular forms of half-integral weight*, Invent. Math. **87** (1987), no. 2, 385–401, DOI 10.1007/BF01389423. MR870736

[16] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214

[17] M. F. Kratz and J. R. Le´on, *Central limit theorems for level functionals of stationary Gaussian processes and fields*, J. Theoret. Probab. **14** (2001), no. 3, 639–672, DOI 10.1023/A:1017588905727. MR1860517

[18] M. Krishnapur, P. Kurlberg, and I. Wigman, *Nodal length fluctuations for arithmetic random waves*, Ann. of Math. (2) **177** (2013), no. 2, 699–737, DOI 10.4007/annals.2013.177.2.8. MR3010810

[19] E. Landau, *¨Uber die einteilung der positiven zahlen nach vier klassen nach der mindestzahl der zu ihrer addition zusammensetzung erforderlichen quadrate*, Archiv der Mathematik und Physics III (1908).

[20] Yu. V. Linnik, *Ergodic properties of algebraic fields*, Translated from the Russian by M. S. Keane. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 45, Springer-Verlag, New York, 1968. MR0238801

[21] A. Logunov, *Nodal sets of Laplace eigenfunctions: Proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture*, Ann. of Math. (2) **187** (2018), no. 1, 241–262, DOI 10.4007/annals.2018.187.1.5. MR3792232

[22] D. Marinucci, G. Peccati, M. Rossi, and I. Wigman, *Non-universality of nodal length distribution for arithmetic random waves*, Geom. Funct. Anal. **26** (2016), no. 3, 926–960, DOI 10.1007/s00039-016-0376-5. MR3540457

[23] A. Palczewski, J. Schneider, and A. V. Bobylev, *A consistency result for a discrete-velocity model of the Boltzmann equation*, SIAM J. Numer. Anal. **34** (1997), no. 5, 1865–1883, DOI 10.1137/S0036142995299807. MR1472201

[24] G. Peccati and M. Rossi, *Quantitative limit theorems for local functionals of arithmetic random waves*, arXiv:1702.03765 (2017).

[25] Z. Rudnick and I. Wigman, *On the volume of nodal sets for eigenfunctions of the Laplacian on the torus*, Ann. Henri Poincar´e **9** (2008), no. 1, 109–130, DOI 10.1007/s00023-007-0352-6. MR2398892

[26] N. Wiener, *The Homogeneous Chaos*, Amer. J. Math. **60** (1938), no. 4, 897–936, DOI 10.2307/2371268. MR1507356
[34] S. Zelditch, *Real and complex zeros of Riemannian random waves*, Spectral analysis in geometry and number theory, Contemp. Math., vol. 484, Amer. Math. Soc., Providence, RI, 2009, pp. 321–342, DOI 10.1090/conm/484/09482. MR1500155

Department of Mathematics, King’s College London, London, England; and Dipartimento di Scienze Statistiche, Università degli Studi di Roma “La Sapienza”, Rome, Italy

Email address: valentina.cammarota@uniroma1.it
QUERIES

Q1: Please note that, as the abstracts in AMS papers must be self-contained, the two citations in the Abstract were removed and the text reworded. Please review to ensure that your intended meaning has been preserved.

Q2: Please supply 2010 Mathematics Subject Classification numbers for the paper, making sure to indicate whether each number provided is a “Primary” or a “Secondary” one.

Q3: Please note that AMS style allows only one use of quotation marks or italics for the emphasis of a given term per paper. When quotation marks or italics were used multiple times for the same term in this paper, only the first mention was set in quotation marks or italics.

Q4: Please consider adding some introductory text before the display equations at the start of Lemma 4.2.

Q5: Please consider adding some introductory text before the two display equations at the start of Lemma 4.3.

Q6: Please consider adding introductory text before the display equations that begin Lemma 4.4.

Q7: Please consider adding some introductory text before the display equations that begin Lemma 4.5.

Q8: Please consider adding introductory text before the display equation at the beginning of the proof of Lemma 5.2.

Q9: The acronym “CLT” was replaced by its definition, in accordance with TRAN style guidelines, as the acronym appeared only once in the text. Please ensure that the acronym was properly defined as “central limit theorem”.

Q10: Please ensure that the two additions of “distribution” to the sentence beginning “It is known” conveys your intended meaning. Nouns were required at the points of insertion.

Q11: Please note that the information pertaining to grant support was removed from the Acknowledgments and reset as a first-page footnote, in accordance with TRAN style guidelines.

Q12: Please supply volume and page numbers for [19].

Q13: Please review the city and country details that were added to the author affiliations.