Explicit Symplectic Methods in Black Hole Spacetimes

Xin Wu1,2,3, Ying Wang1,2, Wei Sun1,2, Fu-Yao Liu1, and Wen-Biao Han1,5,6,7

1 School of Mathematics, Physics and Statistics, Shanghai University of Engineering Science, Shanghai 201620, People’s Republic of China; wuxin_1134@sina.com, wangying424524@163.com, sunweiy@163.com, liufuyao2017@163.com
2 Center of Application and Research of Computational Physics, Shanghai University of Engineering Science, Shanghai 201620, People’s Republic of China
3 Guangxi Key Laboratory for Relativistic Astrophysics, Guangxi University, Nanning 530004, People’s Republic of China
4 Shanghai Astronomical Observatory, Chinese Academy of Sciences, Shanghai 200030, People’s Republic of China; wbhan@shao.ac.cn
5 Hangzhou Institute for Advanced Study, University of Chinese Academy of Sciences, Hangzhou 310124, People’s Republic of China
6 School of Astronomy and Space Science, University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
7 Shanghai Frontiers Science Center for Gravitational Wave Detection, 800 Dongchuan Road, Shanghai 200240, People’s Republic of China

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Abstract

Many Hamiltonian problems in the solar system are separable into two analytically solvable parts, and thus serve as a great chance to develop and apply explicit symplectic integrators based on operator splitting and composing. However, such constructions are not in general available for curved spacetimes in general relativity and modified theories of gravity because these curved spacetimes correspond to nonseparable Hamiltonians without the two-part splits. Recently, several black hole spacetimes such as the Schwarzschild black hole were found to allow for the construction of explicit symplectic integrators, since their corresponding Hamiltonians are separable into more than two explicitly integrable pieces. Although some other curved spacetimes including the Kerr black hole do not have such multipart splits, their corresponding appropriate time-transformation Hamiltonians do. In fact, the key problem in obtaining symplectic analytically integrable decomposition algorithms is how to split these Hamiltonians or time-transformation Hamiltonians. Considering this idea, we develop explicit symplectic schemes in curved spacetimes. We introduce a class of spacetimes whose Hamiltonians are directly split into several explicitly integrable terms. For example, the Hamiltonian of a rotating black ring has a 13-part split. We also present two sets of spacetimes whose appropriate time-transformation Hamiltonians have the desirable splits. For instance, an eight-part split exists in a time-transformed Hamiltonian of a Kerr–Newman solution with a disfomal parameter. In this way, the proposed symplectic splitting methods can be used widely for long-term integrations of orbits in most curved spacetimes we know of.

Unified Astronomy Thesaurus concepts: Black hole physics (15); Computational methods (1965); Computational astronomy (293); Celestial mechanics (211)

1. Introduction

Symplectic integration methods (Hairer et al. 1999; Feng & Qin 2009) preserve the phase-space structure of Hamiltonian dynamics and do not cause certain constants of motion (e.g., energy) to have unphysical drifts over large time spans. They yield numerical solutions, which inherit the qualitative properties of the exact solutions. Because of these good properties, symplectic integrators are widely used for long-term numerical integrations of various dynamical evolution problems in molecular dynamics and quantum and celestial mechanics.

Near integrable Hamiltonian systems involving planetary N-body problems in the solar system are separable from the variables or can be split into two analytically solvable parts in general. Hence, symplectic analytically integrable decomposition algorithms, as explicit symplectic integration algorithms based on splits and compositions, are easily available. The second-order leapfrog splitting method of Wisdom & Holman (1991) is one of the most efficient symplectic methods for the long-term numerical simulation of planetary dynamics in Jacobi coordinates. High-order methods (Yoshida 1990; Chambers & Merrison 2000; Blanes & Moan 2002) can be developed by composing the exact flows of the two parts. Besides the two-part split, multipart splits have been applied to the construction of symplectic analytically integrable decomposition algorithms for Hamiltonian systems that split into more than two parts (Malhotra 1991; Duncan et al. 1998; Levison & Duncan 2000; Wu et al. 2003). High-order multipart-split symplectic integrators by composing many different operators have also appeared in the literature (Blanes et al. 2008, 2010; Skokos et al. 2014).

Although geodesic orbits in several standard general relativity curved spacetime backgrounds such as a Schwarzschild metric and a Kerr metric are integrable, the known integrability only shows the solutions in terms of quadratures rather than in terms of elementary functions. Numerical integration schemes are essential to study these geodesics. If magnetic fields are included as extra sources in the curved spacetimes, the orbits become more complicated and are even unintegrable and chaotic in many circumstances. Numerical integrations are more important to solve such unintegrable orbits. In general, the motions of photons or test particles in general relativity or modified theories of gravity can be described in terms of Hamiltonian systems; therefore, symplectic methods are suitable for use without a doubt.

The Hamiltonians obtained from curved spacetimes are nonseparable, or cannot be decomposed into two explicitly integrable pieces. This created an obstacle to the implementation of symplectic analytically integrable decomposition algorithms for a long time. Instead, implicit symplectic
methods were occasionally used in some relativistic astrophysics studies. The implicit midpoint rule was regarded as a variational-symplectic integrator for application to general relativity and other constrained Hamiltonian systems (Brown 2006). This variational integrator was developed to solve general nonconservative systems (Tsang et al. 2015). The implicit Gauss–Legendre Runge–Kutta symplectic method was employed to detect a transition from regular to chaotic circulation in magnetized coronae near rotating black holes (Kopáček et al. 2010). Symmetric, symplectic Gauss–Runge–Kutta collocation methods with step-size controllers were used to integrate geodesic orbits in spacetime backgrounds corresponding to unintegrable Hamiltonian systems (Seyrich & Lukes-Gerakopoulos 2012). These integrators preserve the symplectic form, conserve Noether charges, and exhibit excellent long-term energy behavior. They are implicit and therefore are numerically more expensive to solve than explicit integration schemes. They are directly applied to Hamiltonian systems that do not need any splits, and all phase-space variables are completely implicitly solved. In this sense, they belong to completely implicit algorithms. Nevertheless, splitting and composition methods are used in some implicit integration schemes. In the context of splitting a Hamiltonian into two or more parts, some individual parts have explicit solutions obtained from analytical methods or an explicit leapfrog integrator, while others have implicit solutions given by the implicit midpoint rule. By the composition of the explicit solutions of the subsystems and the implicit solutions of the other subsystems, explicit and implicit mixed symplectic splitting integrators are obtained. An explicit and implicit mixed symplectic integrator with adaptive time steps was used to calculate post-Newtonian effects of the Kerr metric in the Galactic center region (Preto & Saha 2009). Lubich et al. (2010) composed a noncanonical explicit and implicit mixed symplectic integration scheme for a post-Newtonian Hamiltonian of a spinning black hole binary. The method is based on a splitting of the Hamiltonian into an orbital contribution with numerical solutions, and two spin (spin–orbit and spin–spin) contributions with analytical solutions. Such an integrator can become canonical when the conjugate spin coordinates of Wu & Xie (2010) are adopted. More intensive studies on this topic were given by Zhong et al. (2010), Mei et al. (2013a), and Mei et al. (2013b). These explicit and implicit mixed symplectic splitting methods should be numerically less expensive to solve than the completely implicit symplectic nonsplitting algorithms.

Considering the superiority of explicit integrators in terms of computational efficiency, several authors have attempted to develop explicit methods for nonseparable Hamiltonian systems such as those in curved spacetimes. Chin (2009) designed explicit symplectic integrators for a selected class of nonseparable Hamiltonians, which are product forms of functions with respect to momenta and functions versus position coordinates. Although these explicit integrators do not need any splits of the Hamiltonians, their applications are limited to only the selected Hamiltonians. To present explicit leapfrog splitting methods for an inseparable Hamiltonian system, Pihajoki (2015) obtained an extended phase-space new Hamiltonian, which is the sum of the original Hamiltonian depending on the original momenta and new coordinates and identical copy depending on the original coordinates and new momenta. Clearly, the newly extended Hamiltonian has a two-part split, although the original Hamiltonian is inseparable. The two-part split leapfrog method shows good long-term stability and error behavior. However, it is not symplectic in the original phase space and the extended phase space because it is combined with coordinate mixing transformations. The phase-space mixing maps were improved by Liu et al. (2016) and Luo et al. (2017). In particular, the midpoint permutations between the coordinates and those between the momenta were regarded as the best choice of the phase-space mixing maps (Luo et al. 2017; Li & Wu 2017; Li et al. 2017; Wu & Wu 2018). Tao (2016) did not use any mixing maps and established three-part-split explicit methods for nonseparable Hamiltonians in extended phase spaces. These algorithms are symplectic in the extended phase space but not in the original phase space. Jayawardana & Ohsawa (2023) and Ohsawa (2022) proposed semiexplicit symplectic integrators for nonseparable Hamiltonian systems. This method, as a combination of explicit methods and implicit ones, is symplectic in the original phase space and the extended phase space.

Recently, multipart-split methods were applied to several individual curved spacetimes so as to successfully construct explicit symplectic analytically integrable decomposition algorithms based on splitting and composition. Splitting the Hamiltonian for the description of charged particles moving near a Schwarzschild black hole with an external magnetic field into four terms, Wang et al. (2021a) designed four-part-split explicit symplectic integrators. A Reissner–Nordström black hole corresponds to a Hamiltonian separable into five terms and allows for the use of explicit symplectic methods (Wang et al. 2021b). The Hamiltonian describing charged particles moving near a magnetized Reissner–Nordström anti–de Sitter black hole separable into six terms is required (Wang et al. 2021c). McLachlan (2022) showed that such multipart-split methods in these curved spacetimes are very appropriate for application to the high-order symplectic partitioned Runge–Kutta–Nystrom optimized methods of Blanes & Moan (2002). Zhou et al. (2022) claimed that the splitting methods of the Hamiltonians associated with curved spacetimes are not unique but have various options. In addition, the number of splitting pieces should be as small as possible so that round-off errors are reduced. However, such multipart splits are not applicable to the Hamiltonian of the Kerr metric. Wu et al. (2021) found that an appropriate time-transformed Hamiltonian has five splitting parts and allows for the construction of explicit symplectic integrators. The five-part split is also suited for the Hamiltonian for the description of charged particles moving near the Kerr black hole (Sun et al. 2021a).

Are there any other curved spacetimes allowing for the application of explicit symplectic integrators besides the abovementioned individual curved spacetimes? Which Hamiltonians of curved spacetimes have multipart splits? Of those that do not, which have appropriate time-transformation Hamiltonians that do? To solve these problems, we shall introduce a class of curved spacetimes that correspond to Hamiltonians with multipart splits and two sets of curved spacetimes that correspond to time-transformation Hamiltonians with multipart splits. Such a large extension to the application of explicit symplectic integrators in curved spacetimes is the main aim of this paper.

The remainder of this paper is organized as follows. In Section 2, we briefly introduce symplectic splitting and composition methods for a Hamiltonian with multisplit parts.
in the literature. In Section 3, we demonstrate how to directly split Hamiltonians in a class of curved spacetimes. In Section 4, we provide two sets of curved spacetimes whose corresponding Hamiltonians are not directly split in several explicitly integrable pieces but whose time-transformed Hamiltonians are. Finally, the main results are presented in Section 5.

2. Symplectic Splitting and Composition Methods

Splitting and composition methods are a main path to obtain explicit symplectic integrators. Suppose a 2n-dimensional Hamiltonian system with n-dimensional momentum $p$ and n-dimensional coordinate $q$ is decomposed into many pieces:

$$H(p, q) = \sum_{i=1}^{k} H_i(p, q),$$

(1)

where all sub-Hamiltonians $H_i$ can be integrated exactly and have analytical solutions as explicit functions of time. A series of operators $\varphi_i$ are analytical solvers of the sub-Hamiltonians $H_i$. The terms $\varphi_i$ are symplectic operators, and Equation (1) is a symplectic splitting method of the total Hamiltonian $H$. The exact solution of each of the sub-Hamiltonians from the starting solution $z_0 = (p(0), q(0))$ through a time step $h$ is expressed as $z = (p(h), q(h)) = \varphi^{h}_{0}(z_0)$.

Combining these solutions produces a first-order approximation to the exact solution of the Hamiltonian system $H$:

$$\chi_h = \varphi^{h}_{1} \times \cdots \times \varphi^{h}_{1}. \quad (2)$$

Its adjoint reads

$$\chi^*_h = \varphi^{*h}_{1} \times \cdots \times \varphi^{*h}_{1}. \quad (3)$$

The ordering of terms in the two flow operators may affect the accuracy of the two flows (McLachlan 2022). The two operators can symmetrically compose a second-order explicit symplectic scheme for $H$:

$$S_2(h) = \chi_{h/2} \times \chi^*_{h/2}. \quad (4)$$

That is, the Hamiltonian system $H$ has a second-order approximation solution $z = S_2(h, z_0)$. Increasing the order of such an integrator by composition yields a fourth-order explicit symplectic method of Yoshida (1990) as follows:

$$S_4 = S_2(h\gamma_1) \times S_2(h\gamma_2) \times S_2(h\gamma_1), \quad (5)$$

where $\gamma_1 = 1/(1 - \frac{1}{2})$ and $\gamma_2 = 1 - 2\gamma_1$. An optimal fourth-order explicit symplectic Runge–Kutta–Nystrom (RKN) method can also be obtained by a symmetric composition of the two operators $\chi$ and $\chi^*$. It is expressed as

$$\text{RKN}_4 = \chi_{h\alpha_{12}} \times \chi^*_{h\alpha_{11}} \times \chi_{h\alpha_{21}} \times \chi^*_{h\alpha_{22}} \times \chi_{h\alpha_{31}} \times \chi^*_{h\alpha_{32}} \times \chi_{h\alpha_{41}} \times \chi^*_{h\alpha_{42}},$$

(6)

The related time coefficients for $k = 2$ in the Hamiltonian of Equation (1) were given by Blanes & Moan (2002). For $k > 2$ in the Hamiltonian of Equation (1), Zhou et al. (2022) gave the time coefficients of Equation (6):

$$\alpha_1 = \alpha_{12} = 0.082984402775764, \quad \alpha_2 = \alpha_{11} = 0.162314549088478, \quad \alpha_3 = \alpha_{10} = 0.233995243906975, \quad \alpha_4 = \alpha_9 = 0.370877400040627, \quad \alpha_5 = \alpha_8 = -0.409933704882860, \quad \alpha_6 = \alpha_7 = 0.059762109071016. \quad (7)$$

Here, the optimization requires that the number of time coefficients should be more than that of the order conditions, and the coefficients should be determined by minimizing the sum of the square of coefficients of the fifth-order truncation error terms. In addition, several high-order three-part-split symplectic integrators were presented by Skokos et al. (2014).

Clearly, such splitting and composition methods for the construction of explicit symplectic integrators acting on a Hamiltonian consist of three steps. They need to split the Hamiltonian into two or more pieces in an appropriate way, solve an exactly analytical solution for each piece,\textsuperscript{8} and combine these solutions to construct various approximations for the Hamiltonian. These splitting methods belong to an important class of geometric numerical integrators, which preserve structural properties of the exact solution. In what follows, we consider the application of splitting and composition methods to curved spacetimes.

3. Direct Splitting Methods in a Set of Curved Spacetimes

At first we provide a family of curved spacetimes whose Hamiltonians can be directly split in the form of Equation (1) so as to allow for the application of explicit symplectic integrators such as Equations (4)–(6). Then, we list several examples of this kind of spacetime metrics.

3.1. A Class of Curved Spacetimes

The number $k = 2$ corresponding to the splitting of Hamiltonian pieces in Equation (1) is suitable for numerous Newtonian gravitational problems in the solar system. However, it is not in general appropriate for relativistic gravitational problems in curved spacetimes. Recently, our group found in a series of works that the splitting forms with $k > 2$ are probably admissible in curved spacetimes. The Hamiltonian associated with the Schwarzschild spacetime can be decomposed into four (i.e., $k = 4$) integrable parts having analytical solutions as explicit functions of proper time (Wang et al. 2021a). It also accepts the number of splitting terms $k = 3$ (Zhou et al. 2022). The splitting number is $k = 5$ for the Reissner–Nordström black hole (Wang et al. 2021b) and a magnetized modified gravity Schwarzschild spacetime (Yang et al. 2022). The splitting number is $k = 6$ for the Reissner–Nordström–(anti–)de Sitter black hole (Wang et al. 2021c). These splitting methods in the three examples are dependent on concrete black hole metrics. Which black hole metrics have the direct splitting forms of Equation (1)? Let us seek a set of universal spacetime metrics meeting this requirement.

\textsuperscript{8} The splitting Hamiltonian method is said to be appropriate only when the analytical solution is an explicit function of time.

\textsuperscript{9} These structural features involve symplectivity, volume, time-symmetry, and first integrals.
Setting \( x^\mu = (t, u, v, w) \) as spacetime coordinates, we consider a generic spacetime metric

\[
ds^2 = -f_0(u, v)dt^2 + 2f_0(u, v)dudv + f_3(u, v)dwd^2
\]

\[
+ \frac{f_{11}(v)}{f_{12}(u)}du^2 + \frac{f_{31}(u)}{f_{22}(v)}dv^2.
\]

(8)

Here, \( f_0, f_0, \) and \( f_3 \) are functions of \( u \) and \( v \); \( f_{11} \) is a function of \( v \), and \( f_{31} \) is a function of \( u \). Functions \( f_{12} \) and \( f_{22} \) are supposed to have the expressions

\[
f_{12} = \sum_{i=0}^{1} b_i u^{\kappa_i} + \sum_{i=0}^{1} d_i (u + \kappa_i) v^{\kappa_i},
\]

(9)

\[
f_{22} = \sum_{i=0}^{1} b_i v^{\kappa_i} + \sum_{i=0}^{1} d_i (v + \kappa_i) v^{\kappa_i},
\]

(10)

where \( a_i, b_i, c_i, d_i, \) \( \kappa_i \), \( \bar{b}_i, \bar{c}_i, \bar{d}_i, \) \( \kappa_2 \), and \( \kappa_2 \) are constant parameters expressed in terms of real numbers. This metric corresponds to the Lagrangian formulism

\[
\mathcal{L} = \frac{1}{2} \frac{ds^2}{dt^2} = -\frac{1}{2} f_0(u, v)\dot{t}^2 + f_0(u, v)\dot{u}\dot{v} + \frac{1}{2} f_1(u, v)\dot{w}^2
\]

\[
+ \frac{1}{2} \frac{f_{11}(v)}{f_{12}(u)}\dot{u}^2 + \frac{1}{2} \frac{f_{31}(u)}{f_{22}(v)}\dot{v}^2,
\]

(11)

where the four velocities \((i, u, v, w)\) are derivates of spacetime coordinates \( x^\mu = (t, u, v, w) \) with respect to proper time \( \tau \). Based on the Lagrangian \( \mathcal{L} \), generalized momenta are defined as \( p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \), that is,

\[
p_i = -\frac{1}{2} f_0(u, v)\dot{t} + f_0(u, v)\dot{u},
\]

(12)

\[
p_u = \frac{f_{11}(v)}{f_{12}(u)}\dot{u},
\]

(13)

\[
p_v = \frac{f_{31}(u)}{f_{22}(v)}\dot{v},
\]

(14)

\[
p_w = f_3(u, v)\dot{w} + f_0(u, v)\dot{t} = L.
\]

(15)

Here, \( E \) is a conserved energy of a test particle moving the gravitational field, and \( L \) is also a constant of motion of a test particle. This Lagrangian is exactly equivalent to the Hamiltonian formulism

\[
H = \frac{1}{2} \left( \frac{L^2}{f_3} - \frac{E^2}{f_0} \right) + f_{03}(Ef_3 + Lf_{03}) \frac{(Lf_0 - Ef_{03})}{(f_0 f_3 + f_{03}^2)}
\]

\[
+ \frac{1}{2} \frac{f_{12}(u)}{f_{11}(v)}p_u^2 + \frac{1}{2} \frac{f_{32}(v)}{f_{31}(u)}p_v^2.
\]

(16)

The Hamiltonian has two degrees of freedom and a four-dimensional phase space. If the particle is time-like, the Hamiltonian is always identical to a given constant

\[
H = -\frac{1}{2},
\]

(17)

because the four velocities satisfy the relation \( \dot{x}^\mu \dot{x}_\mu = -1 \). Here, the speed of light is taken to be one geometric unit, \( c = 1 \). The constant of gravity also uses one geometric unit, \( G = 1 \).

Now, the Hamiltonian of Equation (16) can be directly separated in the form

\[
H = H_1 + \sum_{i=0}^{1} H_{bi} + \sum_{i=0}^{1} H_{di},
\]

(18)

\[
H_1 = f_{03}(Ef_3 + Lf_{03}) \frac{(Lf_0 - Ef_{03})}{(f_0 f_3 + f_{03}^2)}
\]

\[
+ \frac{1}{2} \left( \frac{L^2}{f_3} - \frac{E^2}{f_0} \right),
\]

(19)

\[
H_{bi} = \frac{1}{2} \frac{f_{12}(u)}{f_{11}(v)}p_u^2,
\]

(20)

\[
H_{di} = \frac{1}{2} \frac{f_{32}(v)}{f_{31}(u)}p_v^2,
\]

(21)

\[
H_{di} = \frac{1}{2} \frac{f_{32}(v)}{f_{31}(u)}p_v^2.
\]

(22)

Obviously, each of the sub-Hamiltonians in Equations (19)–(23) is analytically solvable and its solutions are explicit functions of proper time \( \tau \). In other words, the Hamiltonian of Equation (18) resembles Equation (1), where \( q = (u, v) \) and \( p = (p_u, p_v) \). Thus, the explicit symplectic integrators such as Equations (4)–(6) are applicable to the spacetime metric of Equation (8).

Consider that an asymptotically uniform electromagnetic field exists in the vicinity of the central body. This electromagnetic field is assumed to have a four-vector potential with two nonzero components \( A_i \) and \( A_w \) as functions of \( u \) and \( v \). The motion of a test particle with charge \( e \) around the central body is represented by the Hamiltonian

\[
H_e = \left( \frac{L - eA_w}{2f_3} \right)^2 - \left( \frac{E + eA_i}{2f_0} \right)^2 + f_{03}(L - eA_w)f_{03}
\]

\[
- (E + eA_i)f_0[(L - eA_w)f_0 + (E + eA_i)f_{03}]
\]

\[
\times (f_0 f_3 + f_{03}^2) - \frac{1}{2} \left( \frac{f_{12}(u)}{f_{11}(v)}p_u^2 + \frac{f_{32}(v)}{f_{31}(u)}p_v^2 \right)
\]

\[
+ \frac{1}{2} \frac{f_{12}(u)}{f_{11}(v)}p_u^2.
\]

(24)

This Hamiltonian still allows for the splitting form in Equation (18), where only minor modifications are given to Equation (19).

Two notable points are given here. Splitting methods in these curved spacetimes involve four steps: (i) obtaining the Hamiltonian in terms of the metric; (ii) splitting the Hamiltonian; (iii) exactly solving each of the split pieces; and (iv) combining these solutions. The black hole spacetimes that allow such splitting methods in the previous studies (Wang et al. 2021a, 2021b, 2021c) are several examples of the metric family of Equation (8). A modified gravity Schwarzschild black hole solution based on the scalar-tensor-vector modified gravitational theory (Yang et al. 2022) resembles one of the metric family of Equation (8). Brane-world black holes (Deng 2020a; Hu & Huang 2022) are also an example of the
same metric family. It was shown in the previous works that the explicit symplectic integrators have an advantage over the implicit symplectic methods, and the implicit and explicit mixed symplectic methods at the same order in computational efficiency. The explicit integrators have such a good computational efficiency regardless of the type of Hamiltonian systems considered. In addition to these mentioned spacetimes, other black hole metrics belonging to the metric family of Equation (8) are present. Two of them are listed in what follows.

3.2. Reissner–Nordström Spacetime with Extra Sources

Boyer–Lindquist coordinates \((t, r, \theta, \phi)\) corresponding to the spacetime coordinates \((t, u, v, w)\) in Equation (8) are chosen. In this coordinate system, a spherically symmetric static Reissner–Nordström–(de Sitter)–Anti–de Sitter black hole surrounded by extra sources such as quintessence and a cloud of strings has a covariant metric (Kiselev 2003)

\[
d\tau^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2,
\]

where four nonzero metric components are

\[
g_{tt} = \frac{1}{f_0(r)},
\]

\[
g_{rr} = \frac{1}{f_0(r)},
\]

\[
g_{\theta\theta} = r^2,
\]

\[
g_{\phi\phi} = r^2 \sin^2 \theta.
\]

The function \(f_0(r)\) is expressed as

\[
f_0(r) = \left(1 - b_c - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 - \frac{\alpha_q}{r^{3\omega_q+1}}\right).
\]

The related notations in Equation (30) are given below.

\(M\) and \(Q\) are the mass and charge of the black hole. \(\Lambda\) denotes a cosmological constant. The cosmological constant associated with the vacuum energy can provide a negative pressure responsible for the accelerating expansion of the universe (Perlmutter et al. 1999). The term \(\omega_q\) represents a quintessential state parameter, and \(\alpha_q\) stands for a quintessence parameter. The quintessence parameter equating to the ratio of the pressure and density is the so-called quintessential state equation. The quintessential state parameter can characterize a dark energy and therefore is termed the quintessential dark energy. The quintessence is regarded as another origin of the negative pressure causing the accelerating expansion of the universe. The quintessence field demands \(\alpha_q > 0\) and \(\omega_q < 0\).

The ranges of quintessential state parameter are \(\omega_q < -1\) for the phantom energy and \(1 < \omega_q < -1/3\) for the quintessence. Obviously, the existence of the vacuum energy or the quintessence changes the asymptotic structure of black hole but still allows for the presence of cosmological horizons. In fact, the quintessence field is obtained via the Einstein gravity coupled to a scalar field, and is an alternative or extension of the standard Einstein gravity. See the paper of Toledo & Bezerra (2020) for more information on the quintessence matter surrounding a black hole. In addition, \(b_c\) is a parameter for measuring the intensity of a cloud of strings around the black hole. The cloud formed by strings can be viewed as a source of the gravitational field, where the universe is described by a collection of extended objects corresponding to one-dimensional strings, but is not represented by a collection of point particles (Letelier 1979). In short, Equation (30) includes all these different gravitational sources, including the cosmological constant, the quintessence matter, and the cloud of strings. Equation (30) is obtained based on the assumption that the energy–momentum tensor is a linear superposition of the energy–momentum tensors associated with each one of the sources.

The black hole metric of Equation (25) as one of the metric family of Equation (8) corresponds to the Hamiltonian

\[
H = \sum_{i=1}^{7} H_i,
\]

where the sub-Hamiltonian parts are

\[
H_1 = -\frac{E^2}{2f_0(r)} + \frac{L^2}{2r^2 \sin^2 \theta},
\]

\[
H_2 = \frac{1}{2} \frac{b_c}{p_r^2},
\]

\[
H_3 = \frac{M}{r} p_r^2,
\]

\[
H_4 = \frac{Q^2}{2r^2} p_r^2,
\]

\[
H_5 = \frac{\Lambda}{6} r^2 p_r^2,
\]

\[
H_6 = \frac{\alpha_q p_r^2}{2r^{3\omega_q+1}},
\]

\[
H_7 = \frac{p_r^2}{2r^2}.
\]

\(L\) in Equation (32) stands for the particle’s angular momentum. Equations (33)–(37) stem from the splits of the term \(f_0(r) p_r^2 / 2\). The seven pieces have analytical solutions as explicit functions of proper time \(\tau\). According to the result of Cao et al. (2022), the sum of the seven split pieces, i.e., the Hamiltonian \(H\) of Equation (31), is integrable. Notice that each of the seven split pieces is integrable or not, irrespective of whether the sum of the seven split pieces is integrable. Each piece is integrable, but the sum may be unintegrable. Now, explicit symplectic algorithms like Equations (4)–(6) can work in the metric of Equation (25).

The splitting method with several explicitly integrable pieces is not unique, as was claimed in the work of Zhou et al. (2022). For instance, the number of split pieces in Equation (31) is six when the sum of \(H_2\) and \(H_7\) is considered.

3.3. Rotating Black Ring

Emparan & Reall (2002) gave a solution of the vacuum Einstein equations in five dimensions for a rotating black ring. In ring coordinates \((t, x, y, \phi, \psi)\) with \(|x| \leq 1\) and \(y \leq -1\), the
solution was written in the paper of Igata et al. (2011) as

\[
\begin{align*}
ds^2 &= -\frac{F(y)}{F(x)}\left(dt - CR\frac{1 + y}{F(y)}d\psi\right)^2 \\
&\quad + \frac{R^2 F(x)}{(x - y)^2} \left(-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)}\right) \\
&\quad + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2.
\end{align*}
\]

\[R > 0\] is a parameter representing the black ring’s radius. \(C\) is also a parameter characterizing the rotation velocity \(\lambda\) and the thickness \(\nu\) of the ring and is expressed as \(C = [\lambda(\lambda - \nu)(1 + \lambda)/1 - \lambda)]^{1/2}\) with \(0 < \nu \leq \lambda < 1\). Two functions are

\[
F(z) = 1 + \lambda z, \quad G(z) = (1 - z^2)(1 + \nu z).
\]

(40)

The black ring metric is stationary, asymptotically flat, and has an event horizon of nonspherical topology. Although \(y = -1/\nu\) is the position of the event horizon, it is not when the polar coordinates \((y, \psi)\) are transformed into Cartesian coordinates. Regularity of the full metric at the ring axis and the equatorial plane exists for the condition \(2\nu/1 + \nu^2\). This means that the spacetime can be completely regular on and outside an event horizon of nonspherical topology in this case.

The black ring metric of Equation (39) exactly corresponds to the Hamiltonian

\[
H = H_1 + \frac{1}{2}g^{xx}p_x^2 + \frac{1}{2}g^{yy}p_y^2,
\]

\[
H_1 = \frac{1}{2}(g^{xx}E^2 + g^{yy}l^2_y + g^{yy}l^2_y) + 2g^{yy}El_y,
\]

(42)

where \(-E, l_x,\) and \(l_y\) are constant conjugate momenta, and these contravariant metric components are

\[
g^{xx} = \frac{(x - y)^2 G(x)}{R^2 F(x)},
\]

\[
g^{yy} = \frac{G(y)}{R^2 G(x)},
\]

\[
g^{yy} = \frac{-G(y)}{R^2 G(x)},
\]

\[
g^{yy} = \frac{-G(y)}{R^2 G(x)},
\]

\[
g^{yy} = \frac{-G(y)}{R^2 G(x)},
\]

(48)

The Hamiltonian of Equation (41) contains two degrees of freedom and its phase space has four dimensions. Igata et al. (2011) found that the black ring geometry does not allow for the separation of variables in the Hamilton–Jacobi equation for Equation (41) but does allow for the presence of chaotic bound orbits. This indicates the absence of an additional constant of motion except for the conserved Hamiltonian of Equation (17) and the constants \(E, l\phi,\) and \(l\psi\) associated with the Killing vectors. In spite of this, the Hamiltonian of Equation (41) exists as a separable form similar to Equation (18).

Splitting the Hamiltonian in Equation (41) requires splitting \(g^{xx}\) and \(g^{yy}\) in Equations (44) and (45). Because \(g^{yy}\) takes \(y\) as a variable and \(x\) as a constant, it is simply split into the form

\[
g^{yy} = \frac{-1}{R^2 F(x)}(x^2 + x(\nu x - 2)y) + (1 - 2\nu x - x^2)y^2 + (\nu + 2x - \nu x^2)y^3 + (2\nu x - 1)y^4 - \nu^2 y^5.
\]

(49)

The third term of Equation (41) consists of six explicitly solvable parts

\[
H_2 = \frac{x^2 p_x^2}{2R^2 F(x)},
\]

\[
H_3 = \frac{xy(\nu x - 2)}{2R^2 F(x)} p_y^2,
\]

\[
H_4 = \frac{y^2 p_y^2}{2R^2 F(x)}(1 - 2\nu x - x^2),
\]

\[
H_5 = \frac{y^2 p_y^2}{2R^2 F(x)}(\nu + 2x - \nu x^2),
\]

\[
H_6 = \frac{y^2 p_y^2}{2R^2 F(x)}(2\nu x - 1),
\]

\[
H_7 = \frac{1}{2R^2 F(x)}.
\]

(55)

As far as \(g^{xx}\) is concerned, \(x\) is a variable and \(y\) is a constant. Compared with \(g^{yy}\), \(g^{xx}\) has a more complicated splitting form. Setting \(\xi = 1 + \lambda x\), i.e., \(x = (\xi - 1)/\lambda\), we have

\[
g^{xx} = \frac{1}{R^2} \sum_{i=1}^{6} G_i
\]

where

\[
G_1 = -\frac{x^2}{\lambda^2} \xi^4,
\]

\[
G_2 = \left(\frac{5\nu}{\lambda^2} + \frac{2\nu y - 1}{\lambda^2}\right) \xi^3,
\]

\[
G_3 = \left[-\frac{10\nu}{\lambda^2} + \frac{4}{\lambda^2}(1 - 2\nu y) + \frac{1}{\lambda^2}(\nu + 2y - \nu y^2)\right] \xi^2,
\]

\[
G_4 = \left[\frac{10\nu}{\lambda^2} - \frac{6}{\lambda^2}(1 - 2\nu y) - \frac{3}{\lambda^2}(\nu + 2y - \nu y^2) + \frac{1}{\lambda^2}(1 - 2\nu y - y^2)\right] \xi,
\]

\[
G_5 = \left[-\frac{5\nu}{\lambda^2} + \frac{4}{\lambda^2}(1 - 2\nu y) + \frac{3}{\lambda^2}(\nu + 2y - \nu y^2) - \frac{2}{\lambda^2}(1 - 2\nu y - y^2) + \frac{y}{\lambda}(\nu y - 2)\right].
\]

(61)
Thus, the Hamiltonian of Equation (41) can be split into 13 explicitly solvable parts

\[
H = \sum_{i=1}^{13} H_i. \tag{69}
\]

Explicit symplectic algorithms such as Equations (4)–(6) are available for the spacetime metric of Equation (39).

In short, the Hamiltonians corresponding to the metric in Equation (8) have the splitting forms of Equation (1). However, the spacetimes whose Hamiltonians have such splitting forms are not restricted to the metric family of Equation (8).

4. Indirect Splitting Methods in Two Types of Curved Spacetimes

Hamiltonians for some other curved spacetimes such as the Kerr metric are not directly split into Equation (1). However, their time-transformed Hamiltonians have the splitting form of Equation (1), as was claimed by several authors (Wu et al. 2021; Sun et al. 2021a, 2021b; Zhang et al. 2021, 2022). In what follows, two such types of curved spacetimes are given.

4.1. Type 1: Inseparable Parts as Functions of One Variable

The metric of Equation (8) is slightly modified as

\[
ds^2 = -f_0(u, v)dt^2 + 2f_{03}(u, v)dtdw + f_1(u, v)dudw^2 + \frac{f_{11}(v)}{f_{12}(u)}du^2 + \frac{f_{21}(u)}{f_{22}(v)}dv^2, \tag{70}
\]

where \(f_{12}\) and \(f_{22}\) are separable parts given by Equations (9) and (10), but \(e(u)\) is a function of the variable \(u\). The Hamiltonian of Equation (16) is also slightly altered as

\[
H = \frac{1}{2} \left( \frac{L}{f_0} - \frac{E}{f_0} \right) + f_{03}(E f_1 + L f_{03}) \left( \frac{L f_0 - E f_{03}}{f_0 f_3 + f_{03}^2} \right)^2 + \frac{e(u) f_{12}(u)}{2 f_{11}(v)}|p_u|^2 + \frac{1}{2} f_{22}(v)|p_v|^2. \tag{71}
\]

Here, the function \(e(u)\) is chosen so that the third term of Equation (71) is inseparable or is not split in the form of Equation (1). In this case, the explicit symplectic integrators of Equations (4)–(6) are not appropriate for the numerical integration of the Hamiltonian in Equation (71). Wu et al. (2021) successfully constructed the explicit symplectic methods for the Kerr metric by following the idea of Mikkola (1997), who introduced time transformation to improve the efficiency of a Wisdom–Holman-like symplectic algorithm for various hierarchical few-body problems. The time-transformed explicit symplectic algorithms for the Kerr metric are similarly extended to the Hamiltonian of Equation (71). The implementation of time-transformed explicit symplectic algorithms for the Hamiltonian of Equation (71) is briefly described as follows.

Taking \(\tau = \sigma\) as a new coordinate together the corresponding conjugate momentum \(p_0 = -H = 1/2\), we obtain an extended phase space \((\sigma, u, v, p_0, p_u, p_v)\). A new Hamiltonian in the extended phase space is

\[
\mathcal{H} = g(u)(H + p_0), \tag{72}
\]

where \(g(u)\) is a time-transformation function (or a time-step function) from the proper time \(\tau\) to a fictitious time \(\sigma\) in the form

\[
d\tau = g(u) d\sigma. \tag{73}
\]

\(\mathcal{H} = 0\) for any new time \(\sigma\). When the time-step function is chosen as

\[
g(u) = \frac{1}{e(u)}, \tag{74}
\]

the Hamiltonian of Equation (72) is separable. Its splitting is similar to Equation (18), but the differences in Equations (19), (22), and (23) are as follows: \(H_1 \rightarrow g(u)(H_1 + p_0)\), \(H_6 \rightarrow g(u)H_6\), and \(H_7 \rightarrow g(u)H_7\). Hence, operator splitting techniques can be used to derive explicit symplectic integration algorithms such as Equations (4)–(6) for the Hamiltonian of Equation (72). These constructions are not directly applicable to the nonseparable Hamiltonian of Equation (71) but act on the separable time-transformed Hamiltonian of Equation (72). They are called indirect splitting methods. An important role of the time-transformation function \(g\) is to eliminate the inseparable terms in the numerators or denominators of the metric functions.

Some notable points are given here. Indirect splitting methods in these curved spacetimes involve several steps: (i) obtain the Hamiltonian in terms of the metric; (ii) extend the phase space of the Hamiltonian; (iii) find a time-transformation function that eliminates the inseparable terms in the numerators or denominators of the metric functions, and write a time-transformed Hamiltonian; and (iv) apply the splitting and composition methods introduced in Section 2 to the time-
transformed Hamiltonian. A constant step size is used for the new time $\sigma$ in the proposed algorithms acting on the time-transformed Hamiltonian of Equation (72), but the proper time step will vary according to Equation (73). The constant step size can ensure the good long-term behavior of such symplectic methods for the time-transformed Hamiltonian. The varying time steps are useful to improve the efficiency of the leapfrog method for various few-body problems with large eccentricities. If time transformations are used to obtain the desirable splitting of the time-transformed Hamiltonian but are not used for a consideration of adaptive time-step control to the proposed symplectic integrators, then specific choices of the time-step function are $g(u)\approx 1$. In the next discussions, we list several examples of the metric family of Equation (70).

4.1.1. Rotating Black Ring

We have shown in Section 3.3 that the Hamiltonian of Equation (41) without time transformation is directly split into 13 explicitly integrable parts and allows for the construction of the explicit symplectic integrators. We also use time transformations to simply establish our algorithms.

One path is the time-step function given by

$$g(x) = F(x).$$

(75)

This leads to the elimination of the function $F(x)$ in Equations (44) and (45). All the functions $F(x)$ in Equations (50)–(55) are also eliminated. The second term of Equation (41) is still separated into six explicitly integrable pieces because the numerator of $g^{xx}$ in Equation (44) is a quintic polynomial of $x$. That is to say, the Hamiltonian of Equation (41) is still required to have the 13 desirable split parts so that it is suitable for the application of explicit symplectic integrators. A variable proper time step $\Delta \tau = g(x)\Delta \sigma = hg(x)$ is the range of $(1 - \lambda)h \leq \Delta \tau \leq (1 + \lambda)h$.

Another path is the time-step function chosen as

$$g(x, y) = -\frac{\nu y^5 F(x)}{G(x)G(y)(y - y')^2}.$$  

(76)

The Hamiltonian of Equation (41) corresponds to the time-transformed Hamiltonian with three analytically solvable parts

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3,$$

(77)

where

$$\mathcal{H}_1 = g(x, y)(H_1 + p_0),$$  

(78)

$$\mathcal{H}_2 = \frac{1}{2} \frac{\nu y^5}{R^2 G(y)} p_1^2,$$  

(79)

$$\mathcal{H}_3 = \frac{1}{2} \frac{\nu y^5}{R^2 G(x)} p_2^2.$$  

(80)

This means that $k = 3$ in Equations (2) and (3). Thus, the explicit symplectic methods of Equations (4)–(6) are easily available. The choice of the time-step function of Equation (76) causes the proper time step $\Delta \tau$ to slightly vary in the vicinity of the fixed new time step $h$.

The above demonstrations show that the two choices of the time-step function yield the explicit symplectic methods. In fact, the time-step function has various choices. A suitable choice of the time-step function can bring a simple construction of the algorithms.

4.1.2. Regular Black Holes

Ayón-Beato & García (1998) gave a spherically symmetric black hole with mass $M$ and charge $Q$ in Schwarzschild coordinates $(t, r, \theta, \phi)$:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(81)

where the metric function is

$$f(r) = 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}} + \frac{Q^2 r^2}{(r^2 + Q^2)^2}.$$  

(82)

This metric has the event horizon singularity but lacks curvature singularities and is regular everywhere. Such a nonsingular metric solution satisfies the Einstein field equation coupled with suitable nonlinear electromagnetic fields. That is, it is obtained from modified or alternative theories of gravity. It is also viewed as a Reissner–Nordström black hole with variable mass and charge. The metric of Equation (81) corresponds to the Hamiltonian

$$H = H_1 + \frac{f(r)}{2} p_1^2 + \frac{1}{2} r^2 p_0^2,$$

(83)

$$H_1 = \frac{1}{2} \left( \frac{L^2}{r^2 \sin^2 \theta} - \frac{E^2}{f(r)} \right).$$  

(84)

The second term of Equation (83) does not have the desirable splitting due to the presence of two fractions appearing in $f(r)$. Taking the time-transformation function

$$g(r) = \frac{1}{f(r)},$$  

(85)

we have the time-transformed Hamiltonian

$$\mathcal{H} = \frac{H_1 + p_0}{f(r)} + \frac{p_1^2}{2} + \frac{p_0^2}{2r^2 f(r)}.$$  

(86)

The three-part split is what we want. Zhang et al. (2022) gave a similar splitting to another regular black hole metric.

There are other regular black holes. Balart & Vagenas (2014) found a regular black hole solution that has the metric of Equation (81) with the metric function of Equation (82), being

$$f(r) = 1 - \frac{2M}{r} \left[ \frac{2}{\exp(Q^2 / 4Mr) + 1} \right]^4,$$

(87)

where $Q = 1.153M$. The metric function in Equation (87) is an exponential function and is unlike the metric function of Equation (82), being a fractional function. In spite of this, the time-step function obtaining Equation (85) with Equation (87) still meets the requirement. The black-bounce–Reissner–Nordström spacetime (Zhang & Xie 2022a, 2022b) is globally regular, too.

4.1.3. Gauss–Bonnet Black Hole

The Gauss–Bonnet black hole (Zeng et al. 2020) is a spherically symmetric black hole whose metric is Equation (81)
but whose metric function is
\[
f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{8\alpha M}{r^3}} \right),
\]
where \(\alpha\) represents the Gauss–Bonnet coupling constant. Two horizons exist for \(\alpha > 0\), while only one horizon exists for \(\alpha < 0\). Although the metric function of Equation (88) is a radical rather than a fractional function in Equation (82), the same method induces the time-transformed Hamiltonian that resembles Equation (86).

A similar example is hairy black holes in Einstein–scalar–Gauss–Bonnet theories (Gao & Xie 2021). The Kehagias–Sfetsos asymptotically flat black hole solution of the modified Hořava–Lifshitz geometry in external magnetic fields (Abdujabbarov et al. 2011; Stuchlík et al. 2014; Toshmatov et al. 2015) also allows the time-transformed Hamiltonians similar to Equation (86) to be obtained. Some other examples include 4D Einstein–Lovelock black holes (Lin & Deng 2021), quantum-corrected Schwarzschild black holes (Deng 2020b; Gao & Deng 2021; Lu & Xie 2021), and an Einstein–Lovelock ultracompact object (Gao & Xie 2022).

4.2. Type 2: Inseparable Parts as Functions of Two Variables

The metric of Equation (88) is slightly modified as
\[
ds^2 = -f_0(u, v) dt^2 + 2f_3(u, v) dt dw + f_1(u, v) dw^2
+ j(u, v) \left( f_{12}(u) du^2 + f_{22}(u) dv^2 \right),
\]
where \(f_{12} \) and \(f_{22}\) are separable parts given by Equations (9) and (10), but \(j(u, v)\) is a function of the two variables \(u\) and \(v\) and \(1/j(u, v)\) is inseparable.

Taking the time-step function
\[
g(u, v) = j(u, v)
\]
derives the time-transformed Hamiltonian of Equation (72), which is consistent with Equation (18), but \(H_1\) in Equation (19) should be \(j(u, v)(H_1 + p_0)\). Hence, such a time-transformed Hamiltonian meets the requirement of splitting and composition methods. The time-step function of Equation (76) for the rotating black ring is an example of the time-step function of Equation (90). Other examples are used to show the implementation of the algorithms in the following discussions.

4.2.1. Majumdar–Papapetrou Dihole Spacetime

The Majumdar–Papapetrou dihole black holes (Hartle & Hawking 1972) are two fixed-charged black holes in equilibrium under their gravitational and electrical forces. The Majumdar–Papapetrou geometry is described in polar coordinates \((r, \phi, z)\) by the metric (Nakashi & Igata 2019)
\[
ds^2 = -\frac{dt^2}{U^2} + U^2(d\rho^2 + \rho^2 d\phi^2 + dz^2),
\]
where \(U\) is a function of \(\rho\) and \(z\) in the form
\[
U = 1 + \frac{M_1}{\sqrt{\rho^2 + (z - a)^2}} + \frac{M_2}{\sqrt{\rho^2 + (z + a)^2}}.
\]
\(M_1\) and \(M_2\) are masses of the two black holes at \(z = \pm a\) \((a > 0)\).

Choosing the time-step function
\[
g(\rho, z) = U^2,
\]
we obtain the time-transformed Hamiltonian with two splitting pieces
\[
\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,
\]
\[
\mathcal{H}_1 = \frac{1}{2} \left( \frac{L^2}{\rho^2} - U^4 E^2 \right) + p_0 U^2,
\]
\[
\mathcal{H}_2 = \frac{1}{2}(p_\rho^2 + p_z^2).
\]
Thus, \(\chi\) and \(\chi^*\) in Equations (2) and (3) contain two operators associated with \(\mathcal{H}_1\) and \(\mathcal{H}_2\) in Equations (95) and (96). In this way, the explicit symplectic methods of Equations (4)–(6) are applicable to the time-transformed Hamiltonian \(\mathcal{H}\) of Equation (94).

4.2.2. Reissner–Nordström–Melvin Black Holes

The Reissner–Nordström–Melvin black holes are a family of electrovacuum-type solutions of the Einstein–Maxwell equations with scalar field perturbations. They describe the Reissner–Nordström black holes permeated by uniform magnetic fields in the metric (Gibbons et al. 2013)
\[
ds^2 = \Theta \left( -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\theta^2 \right)
+ \frac{r^2}{\Theta} (d\varphi - \Omega dt)^2 \sin^2 \theta,
\]
\[
\Delta = r^2 - 2Mr + Q^2,
\]
\[
\Theta = 1 + \frac{1}{2} B^2 (r^2 \sin^2 \theta + 3Q^2 \cos^2 \theta)
+ \frac{1}{16} B^4 (r^2 \sin^2 \theta + Q^2 \cos^2 \theta)^2,
\]
\[
\Omega = -\frac{2}{r} QB + \frac{r}{2} QB^2 \left( 1 + \frac{\Delta}{r^2} \cos^2 \theta \right).
\]
\(M\) is the mass of the black hole, and \(Q\) is the charge of the black hole. \(B\) stands for the strength of the magnetic field. \(\Omega\) is a dragging potential proportional to the coupling \(QB\) because the interaction between the charge \(Q\) and the magnetic field \(B\) serves as a rotating source for rotation; namely, it directly arises from the charge. See also the paper of Santos & Herdeiro (2021) for more details on the metric.

This metric has two Killing vectors associated with stationarity and axisymmetry, which correspond to constant energy \(E\) and angular momentum \(L\) of a test particle. The two constants satisfy the relations
\[
i = \frac{r^2}{\Theta \Delta} (E - \Omega L),
\]
\[
\varphi = \frac{r^2 \Omega}{\Theta \Delta} E + \left( \frac{\Theta}{r^2 \sin^2 \theta} - \frac{r^2 \Omega^2}{\Theta \Delta} \right) L.
\]
The spacetime determines the Hamiltonian
\[
H = H_1 + \frac{\Delta p_r^2}{2r^2 \Theta} + \frac{p_0^2}{2r^2 \Theta}.
\]
\[ H_1 = -\frac{r^2}{2\Theta \Delta} (E - \Omega L)^2 + \frac{\Theta L^2}{2r^2 \sin^2 \vartheta}. \]  

(101)

If \( Q = 0 \), the Hamiltonian of Equation (100) is unintegrable. In this case, chaos was shown by Li & Wu (2019). When \( Q \neq 0 \), the system should also be unintegrable.

Given the time transformation
\[ g(r, \vartheta) = \Theta, \]

(102)

the time-transformed Hamiltonian is
\[ \mathcal{H} = \mathcal{H}_1 + \frac{\Delta P^2}{2r^2} + \frac{P^2}{2r^2}, \]
\[ \mathcal{H}_1 = p_0 \Theta - \frac{r^2}{2\Delta} (E - \Omega L)^2 + \frac{\Theta L^2}{2r^2 \sin^2 \vartheta}. \]

(103)

The second term of Equation (103) contains three solvable parts, as was shown by Wang et al. (2021b). Hence, the Hamiltonian of Equation (103) has five solvable parts and the explicit symplectic schemes of Equations (4)–(6) can work.

4.2.3. Relativistic Core–Shell Models

Core–shell models describe black holes or neutron stars surrounded by axially symmetric shell of dipoles, quadrupoles, and octopoles. Vieira & Letelier (1999) gave these models in the Schwarzschild coordinates \((t, r, \vartheta, \phi)\):
\[ ds^2 = -\left(1 - \frac{2}{r}\right)e^\varphi dt^2 + e^{\vartheta - p}\left(1 - \frac{2}{r}\right)^{-1} dr^2 + r^2 d\vartheta^2 + e^{-p} r^2 \sin^2 \vartheta d\phi^2, \]

(104)

where \( Q \) and \( P \) are two complicated functions of \( r \) and \( \vartheta \) consisting of multipoles.

We easily establish our explicit symplectic algorithms for the obtained time-transformation Hamiltonian by taking the time-step function
\[ g(r, \vartheta) = e^{\vartheta - p}. \]

(105)

4.2.4. Kerr–Newman Solution With Disformal Parameter

Let a disformal parameter \( \beta \) describe the deviation of modified vector tensor theory from the usual Einstein–Maxwell gravity. The action of such a modified gravity can derive a Kerr–Newman solution (Filippini & Tasinato 2018):
\[ ds^2 = -\frac{1}{\Sigma^2} (dt - a \sin^2 \theta d\phi)^2 (\Delta \Sigma + \beta^2 \Sigma^2 r^2) \]
\[ + \frac{\sin^2 \theta}{\Sigma} [adt - (a^2 + r^2) d\phi]^2 \]
\[ + \Sigma \left( \frac{\Sigma dr}{\Delta \Sigma - \beta^2 \Sigma^2 r^2} + d\theta^2 \right) \]
\[ \Delta = a^2 + r^2 - 2Mr + Q^2, \]
\[ \Sigma = r^2 + a^2 \cos^2 \theta. \]

(106)

Note that \( M, Q, \) and \( a \) are the black hole mass, charge, and spin, respectively. In addition, \( t \in [0, \infty), r \in (0, \infty), \theta \in (0, \pi), \) and \( \phi \in [0, 2\pi). \)

In the motion of a particle around the black hole, there are constant energy \( E \) and angular momentum \( L \):
\[ i = \frac{E g_{\phi\phi} + L g_{\phi t}}{g_{tt} - g_{t\phi}}, \]
\[ \dot{\phi} = \frac{E g_{\phi\phi} + L g_{\phi t}}{g_{tt} - g_{t\phi}}, \]

(107)

(108)

where \( g_{tt}, g_{t\phi}, \) and \( g_{\phi\phi} \) are metric components. The long expressions of \( i \) and \( \dot{\phi} \) can be found in the paper of Nazar et al. (2019). We obtain the Hamiltonian
\[ H = H_1 + \frac{(\Delta \Sigma - \beta^2 \Sigma^2 r^2) P^2}{2\Sigma^2}, \]
\[ \mathcal{H}_1 = -\frac{1}{2\Sigma^2} [i - a \sin^2 \theta \dot{\phi}]^2 (\Delta \Sigma + \beta^2 \Sigma^2 r^2) \]
\[ + \frac{\sin^2 \theta}{2\Sigma} [at - (a^2 + r^2) \dot{\phi}]^2. \]

(109)

(110)

Taking the time-transformation function
\[ g(r, \vartheta) = \frac{\Sigma^2}{r^4}, \]

(111)

we have the following time-transformation Hamiltonian
\[ \mathcal{H} = \mathcal{H}_1 + \frac{p^2}{2r_4^4} (\Delta \Sigma - \beta^2 \Sigma^2 r^2) + \frac{\Sigma p_0^2}{2r^4}, \]
\[ \mathcal{H}_1 = -\frac{1}{2r^4} [i - a \sin^2 \theta \dot{\phi}]^2 (\Delta \Sigma + \beta^2 \Sigma^2 r^2) \]
\[ + \frac{\Sigma \sin^2 \theta}{2r^4} [at - (a^2 + r^2) \dot{\phi}]^2 \]
\[ + \frac{p_0^2}{r^4}. \]

(112)

(113)

The second term of Equation (112) can be split into five integrable parts:
\[ \mathcal{H}_2 = \frac{P_r^2}{2}, \]
\[ \mathcal{H}_3 = -\frac{M}{r} P_r^2, \]
\[ \mathcal{H}_4 = \frac{P_r^2}{2r^4} [a^2(1 + \cos^2 \theta) \]
\[ + Q^2(1 - \beta^2)], \]
\[ \mathcal{H}_5 = -\frac{M p_r^2}{r^3} a^2 \cos^2 \theta, \]
\[ \mathcal{H}_6 = \frac{p_r^2}{2r^4} a^2(a^2 + Q^2) \cos^2 \theta. \]

(114)

(115)

(116)

(117)

(118)

The third term of Equation (112) is \( \Gamma = \Sigma p_{\phi}^2 / (2r^4) \). It seems to be simple, but is solved in a somewhat complicated way. The
Hamiltonian $\Gamma$ is rewritten as

$$
\Gamma = \frac{p_\theta^2}{2r^4}(r^2 + a^2 \cos^2 \theta)
$$

$$
= \frac{p_\theta^2}{2r^4}[r^2 + a^2(1 - \sin^2 \theta)]
$$

$$
= \frac{p_\theta^2}{2r^4}(r^2 + a^2) + \left( -\frac{p_\theta^2}{2r^4}a^2 \sin^2 \theta \right)
$$

$$
= \mathcal{H}_T + \mathcal{H}_S. \tag{119}
$$

$\mathcal{H}_T$ is easily solved. Now, let us focus on solving $\mathcal{H}_S$. This Hamiltonian has the canonical equations

$$
\frac{dr}{d\sigma} = 0,
$$

$$
\frac{d\theta}{d\sigma} = -\frac{p_\theta}{r^4}a^2 \sin^2 \theta,
$$

$$
\frac{dp_\theta}{d\sigma} = \frac{p_\theta^2}{r^4}a^2 \cos \theta \sin \theta,
$$

$$
\frac{dp_r}{d\sigma} = -\frac{2p_\theta^2}{r^3}a^2 \sin^2 \theta. \tag{123}
$$

Their analytical solutions are explicit functions of the new time $\sigma = \sigma_0 + h$:

$$
c_1 = p_{0,\theta} \sin \theta_0, \tag{124}
$$

$$
c_2 = \tan \left( \frac{\theta_0}{2} \right), \tag{125}
$$

$$
r = r_0, \tag{126}
$$

$$
\theta = 2 \arctan \left( c_2 e^{-\frac{c_1 \epsilon \sqrt{2}}{\epsilon}} \right), \tag{127}
$$

$$
p_\theta = \frac{c_1}{\sin \theta}, \tag{128}
$$

$$
p_r = p_{0,\theta} - \frac{2h}{c_0}a^2 \epsilon c_1. \tag{129}
$$

Here, $r_0$, $\theta_0$, $p_{0,\theta}$, and $p_{0,\theta}$ are the values of $r$, $\theta$, $p_\theta$, and $p_r$ at the new time $\sigma_0$. Two problems are worth noting. Why is $\cos^2 \theta$ replaced with its equivalent form $1 - \sin^2 \theta$ in Equation (119)? If $\cos^2 \theta$ is still used, $\sin \theta$ becomes $\cos \theta$ in Equation (128). When $\theta = \pi/2$, the computation of $c_1/\cos \theta$ does not continue. Why is $\cos^2 \theta$ not replaced with its other equivalent form $[1 + \cos(2\theta)]/2$ in Equation (119)? If it is, no explicitly analytical solutions are given to the Hamiltonian $\Gamma$.

It is clear that the Hamiltonian of Equation (112) has eight explicitly solvable pieces. Thus, the Hamiltonian is typically suitable for the application of the explicit symplectic methods of Equations (4)–(6). In such a similar way, these constructions can be generalized to a rotating non-Kerr black hole immersed in a uniform magnetic field (Abdujabbarov et al. 2013). They are also applicable to a nonaxisymmetrical system of a rotating black hole in an external magnetic field (Kopáček & Karas 2014), and to a modification to the Kerr–Newman black holes of general relativity in Eddington-inspired Born–Infeld gravity (Guerrero et al. 2020).

Wu et al. (2021) confirmed that the fourth-order explicit algorithm $S_4$ for the Kerr black hole is superior to the fourth-order implicit symplectic method and the fourth-order explicit and implicit mixed symplectic method in computational efficiency. The efficiency superiority of the application of the explicit algorithms to the other black hole spacetimes should not be altered.

5. Summary

Following our previous works (Wang et al. 2021a, 2021b, 2021c; Wu et al. 2021; Sun et al. 2021a), we have developed explicit symplectic algorithms for the long-term numerical integration of orbits in general relativity and modified theories of gravity. We mainly address one problem, of which Hamiltonians of curved spacetimes are directly split into multiple explicitly integrable terms. We also solve another problem, of which Hamiltonians of curved spacetimes are not splitable, but the time-transformation Hamiltonians of curved spacetimes are. The key problem of how to split these Hamiltonians or time-transformation Hamiltonians is particularly considered.

For the spacetimes given in Equation (8), their corresponding Hamiltonians are directly split into the desirable forms and naturally allow for the application of explicit symplectic integrators. Without loss of generality, these spacetimes include the Schwarzschild black hole, the Reissner–Nordström anti–de Sitter black hole, and the Reissner–Nordström–(de Sitter)–Anti–de Sitter black hole surrounded by quintessence and a cloud of strings and rotating black ring, etc. In particular, the Hamiltonian of a rotating black ring is shown to have 13 explicitly analytically solvable split parts.

However, the Hamiltonians of most metrics such as Equations (70) and (89) are not directly separable into several explicitly integrable pieces. Instead, appropriate time-transformation Hamiltonians to the Hamiltonians are. In this way, explicit symplectic schemes are still available for these types of spacetimes. The established symplectic algorithms use fixed time steps in the new time, but might adopt adaptive time steps in the original proper time. Some of the spacetimes meeting this requirement are the rotating black ring, regular black holes, the Gauss–Bonnet black hole, the Kerr black hole, Majumdar–Papapetrou dihole spacetime, Reissner–Nordström–Melvin black holes, core–shell models, and the Kerr–Newman solution with disformal parameter, etc. For example, an eight-part split is given to the time-transformed Hamiltonian of the Kerr–Newman solution with disformal parameter.

The splitting methods of the Hamiltonians or time-transformed Hamiltonians associated with curved spacetimes are not altered in general when external magnetic fields surround the central bodies. Although various splitting methods can be given to a Hamiltonian, the number of split Hamiltonian pieces should be as small as possible to reduce round-off errors. Many time-transformation functions can also be given to a Hamiltonian.

The multipart-split explicit symplectic integrators for Equations (8), (70), and (89) are appropriate for most of the spacetimes we know of. This brings a great extension to the application of explicit symplectic methods for integrations of orbits in curved spacetimes. Such algorithms provide an effective means to numerically study various dynamical problems in general relativity and modified theories of gravity. They are suited for studying the transition from regular to chaotic dynamics of charged test particles moving near black holes immersed in external magnetic fields, such as a rotating black hole surrounded by an external nonaxisymmetrical
magnetic field (Kopáček & Karas 2014). Extreme mass ratio inspirals are important sources for the space-borne gravitational-wave detectors. Their orbits need to be integrated very accurately (Zhang & Han 2021; Zhang et al. 2021), and these explicit symplectic methods may be useful to this kind of dynamical system. It should be good to use the explicit symplectic integrators rather than nonsymplectic Runge–Kutta methods in ray-tracing codes on black hole shadows (Pu et al. 2016).

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ORCID iDs
Xin Wu © https://orcid.org/0000-0002-1223-8978
Wen-Biao Han © https://orcid.org/0000-0002-2039-0726

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