The supersymmetric Ruijsenaars-Schneider model

O. Blondeau-Fournier *1, P. Desrosiers †1,2, and P. Mathieu ‡1

1Département de physique, de génie physique et d’optique, Université Laval, Québec, Canada, G1V 0A6.
2 CRIUSMQ, 2601 de la Canadière, Québec, Canada, G1J 2G3

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Abstract

An integrable supersymmetric generalization of the trigonometric Ruijsenaars-Schneider model is presented whose symmetry algebra includes the super Poincaré algebra. Moreover, its Hamiltonian is showed to be diagonalized by the recently introduced Macdonald superpolynomials.

1 Introduction

In this Letter we resolve a long-standing problem about the existence of a supersymmetric and integrable generalization of the Ruijsenaars-Schneider (RS) models.

The RS models are famous integrable $N$-body dynamical systems [1] [2]. They are the relativistic generalization of the Calogero-Moser-Sutherland (CMS) models [3], in which $N$ particles interact pairwise in one spatial dimension through a long-range potential in $1/r^2$ and a coupling constant $\beta > 0$. “Relativistic” here means that the models contain in addition to $\beta$, another parameter, denoted $c$, and playing the role of the speed of light. This interpretation is supported by the presence of the Poincaré algebra in the algebraic structure of the models. The integrability of the CMS and RS models is valid in both the classical and the quantum contexts. In what follows however, we focus on quantum systems.

Quite unexpectedly, the CMS models have a wide variety of applications. For instance, they attracted considerable attention in statistical mechanics since they describe, via Dyson’s Brownian motions, the “dynamics” of the eigenvalues of random matrices [4]. One case of particular importance is the trigonometric
CMS model – also called the Calogero-Sutherland (CS) model – in which the particles evolve on a circle \[5\]. The eigenfunctions for the CS Hamiltonian were found to be of the form \((\text{ground state}) \times (\text{Jack polynomials})\). Nowadays, the Jack polynomials (Jacks) are considered as fundamental objects of algebraic combinatorics \[7\] and representation theory \[8\]. In condensed matter physics, their clustering properties have been used to modelize quantum fractional Hall states \[9\]. The Jacks were also used in high energy physics in order to give explicit formulas for the singular vectors in 2D Conformal Field Theory (CFT) (see \[10\] and references therein) and to prove AGT-type conjectures \[11\], which relate 4D supersymmetric gauge theories to 2D CFT.

The trigonometric RS (tRS) model is also notorious mainly for the importance of its eigenfunctions, which take the form \((\text{ground state}) \times (\text{Macdonald polynomials})\).\[2\] The Macdonald polynomials (Macs) \[7\] are natural generalizations of many basic symmetric functions, including the Jacks, and like the latter, they are relevant in representation theory \[14\]. In more physical contexts, they furnish a representation of the singular vectors in a \(q\)-deformation of the Virasoro algebra \[15\] and appear in the framework of the five dimensional equivalent of the AGT conjecture \[16\]. Very recently, their remarkable combinatorial properties inspired the introduction of the Macdonald processes \[17\] as generalizations of solvable probabilistic problems related to the KPZ universality class, which describes interfacial growth phenomena \[18\].

The supersymmetric generalization of the CMS models were discovered at the beginning of the 1990s \[19\]. The complete understanding of their symmetry algebra and eigenfunctions appeared a decade later \[20\]. In the trigonometric case (to be referred to as the sCS model), this led to the introduction of superspace analogues of the Jacks, the Jack superpolynomials (sJacks), which were studied in depth in \[21\]. Recently, the sJacks were used in superconformal field theory \[22\] and were shown to possess clustering properties similar to those observed for the quantum fractional Hall states \[23\].

Despite the practical and theoretical importance of the integrable generalizations of the CMS models, finding supersymmetric RS models has remained an open problem for two decades. The rational behind the difficulties encountered in supersymmetrizing the RS model is that the standard techniques of supersymmetric quantum mechanics \[24\] no longer apply in the relativistic setting.\[3\]
Here we prove the existence of a new generalization of the tRS model that is both supersymmetric and integrable. The supersymmetry is explicit since the Hamiltonian and the total momentum are generated by fermionic charges, all together forming the $\mathcal{N} = 1$ super Poincaré algebra. We moreover show that the eigenfunctions of the supersymmetric tRS (stRS) model are built in terms of the Macdonald superpolynomials (sMacs). The latter were recently introduced, by using purely mathematical tools from combinatorics, as a superspace generalization of the Macs [25, 26]. Our results thus link the sMacs to a supersymmetric many-body problem.

2 The tRS model

Before constructing the supersymmetric model, let us recall important features of the tRS model [2, 27]. The model involves $N$ bosonic particles interacting on a ring of length $L$ via a coupling constant $\beta$. Their dynamics is described with the help of the rapidity variables $\zeta_j (j = 1, \ldots, N)$ and the canonical conjugate momenta $\eta_j (j = 1, \ldots, N)$. The former are real variables while the latter are chosen in their differential representation, $\eta_j = -i\partial/\partial \zeta_j$, which guarantees that $[\zeta_j, \eta_k] = i\delta_{jk}$. Note that the masses and $\hbar$ are set equal to 1, but the “speed of light” $c$ remains arbitrary. The interaction between the particles is induced via the “potential” functions $V_j = \prod_{k \neq j} \hbar^{-1/2} - (\zeta_{jk})^{1/2}$ and $W_j = \prod_{k \neq j} \hbar^{1/2} + (\zeta_{jk})^{1/2}$, where $\zeta_{jk} = \zeta_j - \zeta_k$ and

$$h^\pm(x) = \frac{\sin[(\pi/L)(x \pm i\beta/c)]}{\sin[(\pi/L)x]}.$$  \hfill (1)

The Hamiltonian and total momentum are respectively

$$H_{\text{tRS}} = \sum_j \left( V_j e^{\frac{i}{c} \eta_j} W_j + W_j e^{-\frac{i}{c} \eta_j} V_j \right)$$

and

$$P_{\text{tRS}} = \sum_j \left( V_j e^{\frac{i}{c} \eta_j} W_j - W_j e^{-\frac{i}{c} \eta_j} V_j \right).$$  \hfill (2)

Together with the Lorentz boost, $B = -\sum_{i=1}^N \zeta_i$, the Hamiltonian and the momentum form the Poincaré algebra in 1+1 dimensions (with $H_{\text{tRS}} = H, P_{\text{tRS}} = P$):

$$[H, P] = 0, \quad [H, B] = iP, \quad [P, B] = iH/c^2.$$  \hfill (3)

3 Supersymmetric generalization

We now supersymmetrize the tRS model, proceeding in five steps.
First step. To each rapidity $\zeta_j$, which is a bosonic degree of freedom, we associate a fermionic partner $\theta_j$. The fermionic variables are anticommuting: $\{\theta_j, \theta_k\} = 0$, where $\{\cdot, \cdot\}$ stands for the anticommutator. Note especially that $\theta_j^2 = 0$. Any function $\Psi$ depending upon the bosonic variables $\zeta_j$ and the fermionic variables $\theta_j$ (to be referred to as a superfunction), and upon the parameters $\beta$ and $c$, decomposes as follows:

$$\Psi(\zeta, \theta; \beta, c) = \sum_{I \subseteq \{1, \ldots, N\}} \Psi_I(\zeta; \beta, c) \theta_I,$$

(4)

where the sum extends over all sequences of $m$ indices $I = (i_1, \ldots, i_m)$ such that $1 \leq i_1 < \ldots < i_m \leq N$, with $0 \leq m \leq N$. Moreover, each $\Psi_I$ is a complex valued function and $\theta_I = \theta_{i_1} \cdots \theta_{i_m}$. The value of $m$ is called the fermionic degree.

Second step. We define states as functions $\Psi = \Psi(\zeta, \theta; \beta, c)$ that are periodic in $\zeta_j$, with period $L$, and such that $\sum_I \int_T w_I \Psi_I \overline{\Psi_I} d\zeta < \infty$, where $T = [0, L]^N$, $d\zeta = d\zeta_1 \cdots d\zeta_N$, and the “bar” operation stands for the transformation such that $\bar{i} = -i$, $\bar{\zeta} = \zeta$, $\bar{\beta} = \beta$, and $\bar{c} = -c$. Moreover, the weight functions $w_I = w_I(\zeta; \beta, c)$ is equal to $\prod_{j \in I} V_j^2$. The set of all states forms a vector space $\mathcal{H}$ that is naturally equipped with the following scalar product, denoted $\langle \cdot | \cdot \rangle$. Let $\Psi, \Phi$ be two states in $\mathcal{H}$ written as (4). Then, we define

$$\langle \Psi | \Phi \rangle := \sum_I \int_T w_I \Psi_I(\zeta; \beta, c) \overline{\Phi_I(\zeta; \beta, c)} d\zeta.$$

(5)

Note that for two states of different fermionic degree, the scalar product automatically vanishes and the two states are thus orthogonal. This scalar product reduces to that of the sCS (resp. tRS) model in the non-relativistic (resp. non-supersymmetric) limit. Figure 1 summarizes the various limits. Within the state space $\mathcal{H}$, we focus on the subspace $\mathcal{H}^{SN}$ formed by all symmetric states, namely, states that are invariant under any simultaneous exchange of pairs of partners $(\zeta_j, \theta_j) \leftrightarrow (\zeta_k, \theta_k)$.

At this point, it is convenient to change the variables and redefine the parameters as follows [27]:

$$x_j = e^{2\pi i \zeta_j / L}, \quad q = e^{-2\pi / Lc}, \quad t = q^\beta.$$

(6)

This allows us to recall Macdonald’s notation [7]. Let

$$A_i(t) = \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \quad \text{and} \quad \tau_j = qt^j \partial_{x_j}.$$

(7)

We have $\prod_{j \neq i} h_{\pm} (\zeta_{ij}) = t^{\pm(N-1)/2} A_i(t^{\pm 1})$ and so the function $w_I$ is equal to $t^{m(N-1)/2} \prod_{j \in I} A_j(1/t)$, where $m$ represents the cardinality of $I$ (cf. below Eq.
The $q$-shift operator $\tau_i$ is then equal to $e^{-\frac{1}{2} \eta_i}$ in (2) and is such that $\tau_i f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, qx_i, \ldots, x_N)$ for any smooth function $f$.

The presence of the weight function $w_I$ makes the calculation of the adjoint operators somewhat more subtle than in the non-supersymmetric or non-relativistic cases. For instance, the adjoint of $\theta_i$ (multiplication operator from the left) is

$$\theta_i^\dagger = t^{\frac{N-1}{2}} A_i (1/t) \partial \theta_i,$$

where $\partial \theta_i$ is the fermionic left derivative, that is, $\{\partial \theta_i, \theta_j\} = \delta_{ij}$ and $\{\partial \theta_i, \partial \theta_j\} = 0$. Thus, the fermionic variables and their adjoints generate a novel and interesting deformation of the Clifford algebra,

$$\{\theta_i, \theta_j^\dagger\} = t^{\frac{N-1}{2}} A_i (1/t) \delta_{ij},$$

which reduces to the usual one as $c \to \infty$ (i.e., $t \to 1$). As a second example, consider the adjoint of the operator $\tau_i^{-1}$, which in the non-supersymmetric case, is simply $\tau_i$. When fermions are involved, the calculation of $\tau_i^{-1}$’s adjoint requires the introduction of a projection operator,

$$\pi_I = \prod_{i \in I} \theta_i \partial \theta_i \prod_{j \notin I} (1 - \theta_j \partial \theta_j),$$

which is such that $\pi_I (\theta_j) = \theta_I \delta_{IJ}$. One then finds that

$$\left(\tau_i^{-1}\right) = \sum_{I \subseteq \{1, \ldots, N\}} w_I^{-1} \tau_i w_I \pi_I,$$

where

$$w_I^{-1} \tau_i w_I = Z_{(I,i)} \left[ \prod_{j \neq i} \frac{(x_i - x_j)(qx_i - tx_j)}{(qx_i - x_j)(tx_i - x_j)} \right] \chi(I \in I) \tau_i$$

with $\chi(\cdot) = 1$ if its argument is true and 0 otherwise, and

$$Z_{(I,i)} = \prod_{j \in I, j \neq i} \frac{(qx_i - x_j)(tx_i - x_j)}{(qx_i - x_j)(tx_i - x_j)}.$$
Third step. We introduce the supersymmetry charge of the stRS model:

\[ Q_\pm = c \sum_i \theta_i (a_i - 1), \quad a_i = V_i^{-1} \tau_i^{-1} W_i. \]  

Its adjoint with respect to the scalar product is

\[ Q_\dagger = -c \sum_i (\tau_i^{-1} - 1) \theta_i, \quad a_i = W_i (\tau_i^{-1})^1 V_i^{-1}. \]

These charges are fermionic. Moreover, one easily checks that \([a_i, a_j] = 0\). This readily implies that \(Q_\pm^2 = 0\), which in turn implies that \((Q_\dagger)^2 = 0\).

The states annihilated by \(Q_\pm\) are called supersymmetric. For instance, any state of the form \(\theta_1 \cdots \theta_N \Phi_1(\ldots, \Phi_N) (\zeta; \beta, c)\) is supersymmetric. Most importantly, the ground state \(\psi_0\) of the non-supersymmetric tRS model is also supersymmetric. This can be understood as follows. One can show that the ground state wave function is given by

\[ \psi_0 = C_0^{-1/2} \Delta_N^{1/2}, \]

where

\[ \Delta_N = \prod_{i \neq j} (x_i / x_j; q)_\infty, \quad C_0 = \prod_{i \neq j} (q; q)_\infty (q; q)_\infty \]

with \((a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)\). Note that \(\overline{\psi_0} = \psi_0\). Since \(a_i \psi_0 = \psi_0\) for all \(i\), one gets that \(Q_\pm(\psi_0) = 0\), while \(Q_\dagger \psi_0 = 0\) immediately follows from the definition.

The two charges \(Q_\pm\) and \(Q_\dagger\) allow us to define the Hamiltonian of the stRS model:

\[ H = \frac{1}{2} \{Q_\pm, Q_\dagger\} + \varepsilon_+ \]

where \(\varepsilon_+ = c^2 \sum_{i=1}^{N} \cosh[\pi \beta (2i - N - 1) / Lc]\) is a constant introduced for later convenience. By construction, the Hamiltonian is self-adjoint, its spectrum is bounded from below by \(\varepsilon_+\), and it is supersymmetric:

\[ [H, Q_\pm] = [H, Q_\dagger] = 0. \]

Moreover, \(H\) generalizes both the Hamiltonian of the tRS model and that of the sCS model. Indeed, one can check that \(H \Psi = H_{tRS} \Psi\) whenever \(\Psi\) does not depend on fermionic variables \(\theta_i\). One can also show that, up to an additive constant, \(\lim_{c \to \infty} (H - Nc^2)\) is given by

\[ -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial \zeta_i} \right)^2 + \sum_{1 \leq i < j \leq N} \frac{\beta \beta - 1 + (\theta_i - \theta_j) (\partial \theta_i - \partial \theta_j)}{(L/\pi)^2 \sin^2(\pi \zeta_{ij} / L)}. \]

To our knowledge, that \(\Delta_N^{1/2}\) corresponds to the ground state of the tRS model has not been explicitly demonstrated. Let us write the tRS model's hamiltonian in a positive definite form as \(H_{tRS} = \frac{1}{2} \sum_i k_i^1 k_i + \varepsilon_+\) where \(k_i = c(\tau_i^{-1/2} W_i - \tau_i^{-1/2} V_i)\) and \(k_i^1 = c(V_i \tau_i^{-1/2} W_i V_i^{-1})\). The constant \(\varepsilon_+\) is given by \(\varepsilon_+ = c^2 \sum \cosh[\pi \beta (2i - N - 1) / Lc]\). It is then easily checked that \(k_i \Delta_N^{1/2} = 0, \forall i\). As a result, any eigenvalue \(\varepsilon\) of the \(H_{tRS}\) eigenstates must satisfy \(\varepsilon \geq \varepsilon_+\). In addition, the "non-relativistic" limit of \(\Delta_N^{1/2}\) is (up to a normalization) found to be the CS models ground state: with \(t = q^3\), then \(\lim_{t \to 1} \Delta_N^{1/2} = 2^3 \sqrt{N(N-1)/2} \prod_{k < j} \left| \sin(\pi \zeta_{kj} / L) \right|^2\).
which is precisely the sCS model’s Hamiltonian \[19, 20\].

**Fourth step.** We introduce another charge:

\[
Q_+ = \sum_i \theta_i (a_i + 1).
\]

This allows us to define the momentum operator:

\[
P = \frac{1}{2} \{ Q_-, Q_+^\dagger \} + \varepsilon_-,
\]

where \(\varepsilon_- = c \sum_{i=1}^N \sinh[\pi \beta(2i - N - 1)/Lc]\). In addition, we have the following commutation relations:

\[
\frac{\sigma}{2} \{ Q_+, Q_+^\dagger \} = H + \varepsilon_+, \quad \frac{1}{2} \{ Q_+, Q_-^\dagger \} = P + \varepsilon_-,
\]

\[
\{ Q_\pm, Q_\mp^\dagger \} = \{ Q_\pm, Q_\mp \} = 0,
\]

\[
[H, P] = 0.
\]

**Fifth step.** Here we build the full supersymmetry algebra. For this, we choose the same Lorentz boost as in the non-supersymmetric case: \(B = -\sum_j \zeta_j\). We then commute \(B\) with all previously introduced operators:

\[
\{ Q_\pm, B \} = \frac{1}{2c} (c^\mp 1 Q_\mp + Q_\pm), \quad [H, B] = iP,
\]

\[
\{ Q_\pm^\dagger, B \} = \frac{1}{2c} (c^\mp 1 Q_\mp^\dagger - Q_\pm^\dagger), \quad [P, B] = iH/c^2.
\]

From Eqs \((17), (20)–(22)\), we conclude that \(Q_\pm, Q_\pm^\dagger, H, P, B\) do indeed form a super Poincaré algebra.

At first sight, the model seems to have \(N = 2\) supersymmetries since the above superalgebra involve two pairs of fermionic charges. However, if we let

\[
Q_1 = \frac{i}{2} (Q_- + Q_-^\dagger), \quad Q_2 = \frac{i}{2} (Q_+ + Q_+^\dagger),
\]

\[
B = B + \frac{c}{2} \sum_j \theta_j \partial \theta_j,
\]

(note that these operators are self-adjoint), then we find

\[
\{ Q_a, Q_b \} = \sigma_{ab} H + (1 - \delta_{ab}) ic P - \delta_{ab} \varepsilon_+,
\]

\[
[Q_a, B] = \frac{i}{2c} \epsilon_{ab} Q_b, \quad [Q_a, H] = [Q_a, P] = 0,
\]

\[
[H, P] = 0, \quad [H, B] = iP, \quad [P, B] = iH/c^2,
\]

where \(a, b \in \{1, 2\}\), \(\epsilon_{ab}\) is the Levi-Civita symbol, and \(\sigma = \text{diag}(1, -1)\). Eq. \((24)\) is the \(N = 1\) super Poincaré algebra in 1+1 dimensions (see for instance \[28\] and references therein).
4 Macdonald superpolynomials

We now solve explicitly the supersymmetric model just constructed. We concentrate on the space $\mathcal{H}^{SN}$ of symmetric states, which are states that are invariant under all simultaneous exchanges $(x_i, \theta_i) \leftrightarrow (x_j, \theta_j)$. A possible basis for $\mathcal{H}^{SN}$ is provided by the set of all symmetric eigenstates shared by $H$ and $P$. The ground state wave function $\psi_0$ defined before Eq. (16) is a good example of such eigenstate. Actually, in order to simplify our search, we decompose each symmetric eigenstate as $\psi_0 \times f$, for some symmetric function $f = f(x, \theta; q, t)$.

Let us characterize $f$. For this, we first write the Hamiltonian in (17) and the momentum in (20) as

$$
H = \frac{\sigma^2}{2}(H_1 - H_2), \quad P = \frac{\sigma}{2}(H_1 - H_2),
$$

where

$$
H_{\pm 1} = t^{N(N-1)/2} \Delta_1^{1/2} D_{\pm 1} \Delta_N^{-1/2}.
$$

Thus, $\psi_0 \times f$ is an eigenfunction common to $H$ and $P$ if and only if $f$ is an eigenfunction common to $D_{-1}$ and $D_{+1}$. The latter are obviously supersymmetric since $D_{\pm 1} = \{Q_{2\pm 1}, \tilde{Q}_{3\pm 1}\}$, where

$$
\begin{align*}
\tilde{Q}_1 &= \sum \theta_i \tau_i^{-1}, & \tilde{Q}_2 &= \sum A_i (1/t) \partial \theta_i, \\
\tilde{Q}_3 &= \sum A_i (t) Z_i \tau_i \pi_1 \partial \theta_i, & \tilde{Q}_4 &= \sum \theta_i.
\end{align*}
$$

One can show that the symmetric eigenfunctions of $D_{\pm 1}$ are of the form $\sum \theta_i f_i$, where each $f_i$ is a Laurent polynomial in the variables $x_i$ with coefficients that are rational in $q, t$. However, whenever $g$ is an eigenfunction of $D_{\pm 1}$, then so is $(x_1 \cdots x_N)^k g$ for any integer $k$. Thus, the only relevant eigenfunctions are those that are polynomial (not Laurent-type) in $x_i$ and $\theta_i$. Such eigenfunctions for $D_{\pm 1}$ were recently introduced [25] [26]; they are the Macdonald superpolynomials, sMacs for short.

Like any symmetric superpolynomial, the sMacs are labelled by superpartitions. A superpartition, denoted $\Lambda$, is a pair of partitions $\Lambda = (\lambda; \mu)$ such that $\lambda$ is a strictly decreasing partition and $\mu$ is a (regular) non-increasing partition. A superpartition is said to be of degree $(n|m)$ if $n = |\lambda| + |\mu|$ and $\lambda$ has exactly $m$ parts (counting one possible part equal to 0). We define $\Lambda^*$ as the partition obtained from the superpartition $\Lambda$ by removing the semi-colon and reordering the parts, and $\Lambda^\circ$ obtained similarly but with $\lambda_i \rightarrow \lambda_i + 1$.

The sMacs, denoted $P_{\Lambda} = P_\Lambda(x, \theta; q, t)$, for superpartitions of degree $(n|m)$ form a basis for the space of all symmetric superpolynomials of homogeneous degree $n$ in the variables $x$ and fermionic degree $m$. The sMacs are the unique superpolynomials that satisfy the following two conditions [26]. (1) They decompose triangularity on monomials: $P_{\Lambda} = \sum_{\Omega < \Lambda} c_{\Lambda\Omega} (q, t) m_{\Omega}$, where $m_{\Lambda} = \sum_{\omega \in \mathcal{S}_n} \theta_{\omega(1)} \cdots \theta_{\omega(m)} x_{\omega(1)}^{\lambda_1} \cdots x_{\omega(N)}^{\lambda_N}$, with the prime indicating that the sum is over distinct terms. The order is such that $\Omega < \Lambda$ iff both $\Lambda, \Omega$ have same degree and satisfies $\Omega^* < \Lambda^*$ and $\Omega^\circ < \Lambda^\circ$, where the latter comparison is the
usual dominance order for partitions\(^2\) (2) The sMacs satisfy \(\langle P_{\lambda}|P_{\Omega}\rangle' \propto \delta_{\lambda\Omega}\), i.e., they are orthogonal w.r.t. \(q\).

This scalar product can be rewritten as \(\langle f|g\rangle' = \int_T \psi_0^2 f(x)g(x) dx\), where it is understood that \(\tilde{\theta}_1 = \theta_1\) and that \(\tilde{\theta}_1\) is acting on \(\tilde{g}\). As mentioned above, the \(P_k\)'s are the symmetric eigenfunctions of \(D_\pm\).

More generally, \(D_\pm(x_1 \cdots x_N)^k P_{\lambda}\) is equal to \(\epsilon_{A\pm k}(q^{\pm 1}, t^{\pm 1})(x_1 \cdots x_N)^k P_{\lambda}\), where \(\epsilon_{A_k}(q,t) = \sum_t q^{tN-i}, k \in \mathbb{Z}\), and \(\kappa\) is the \(N\)-vector with all components equal to \(k\). Thus, the set of all the states \(\Psi_{A,k}\), such that \(\Psi_{A,k} = (x_1 \cdots x_N)^k \psi_0 P_{\lambda}\), diagonalizes the Hamiltonian \(H\) and momentum \(P\). An orthogonal basis for \(\mathcal{H}^S_\infty\) is then easily formed by making use of \(\langle \Psi_{A,k}|\Psi_{A,k}\rangle = \langle P_{\lambda}|P_{\Omega}\rangle' \propto \delta_{\lambda\Omega}\) for \(k \leq \ell\), where \(\Omega_\lambda\) is the superpartion obtained by replacing each element \(\Omega_i\) of \(\Omega\) by \(\Omega_i - k + \ell\).

5 Integrability of the stRS model.

The proof of integrability relies on the construction of the sMacs in terms of the non-symmetric Macs [26], themselves eigenfunctions of the Cherednik operators. These operators are constructed out of the Hecke algebra generators [24]

\[
T_i = t + \frac{tx_i - x_{i+1}(s_i - 1)}{x_i - x_{i+1}}, \quad i = 1, \ldots, N - 1, \tag{28}
\]

where the \(s_i\) are the elementary transpositions such that \(x_i \leftrightarrow x_{i+1}\). The inverse of \(T_i\) reads \(T_i^{-1} = t^{-1} - 1 + t^{-1}T_i\). The Cherednik operators are [23]

\[
Y_i = t^{-N+i}T_i \cdots T_{N-1} \omega T_1^{-1} \cdots T_{i-1}^{-1}, \tag{29}
\]

\(^2\)Recall that two partitions \(\lambda, \mu\) satisfies \(\lambda > \mu\) in the dominance order iff \(\sum_{i=1}^k (\lambda_i - \mu_i) > 0\) for all \(k\).

\(^6\)The orthogonality relation for the sMacs given in this Letter is slightly more general than what was previously published. To understand how the latter implies the former, suppose not only that \(q = e^{-2\pi i/|L|}\) and \(t = t^d\), but also that \(\beta \in \mathbb{Z}_+\). We then have another scalar product: \((f|g)'' = L^{N-1} t^{-d} C.T. \{f(x, \theta^1; q, t)g(x, \theta^1; q, t)\}, \) where C.T. \{\} is the constant term of the Laurent series in \(x\) and \(d = N(\beta - 1)(N - 1)/4\). From [26 Prop. 22] and basic properties involving the non-symmetric Macs and C.T., we also have that \(\langle P_{\lambda}|P_{\Omega}\rangle'' = F_{A,\Omega}(\beta, c)\) for \(\beta \in \mathbb{Z}_+\), where \(F_{A,\Omega}(\beta, c)\) is known explicitly and is \(\propto \delta_{\lambda\Omega}\). The last orthogonality relation and the residue theorem then lead to \(\langle P_{\lambda}|P_{\Omega}\rangle = \langle P_{\lambda}|P_{\Omega}\rangle'' = F_{A,\Omega}(\beta, c)\) for \(\beta \in \mathbb{Z}_+\). Now, as functions of \(\beta \in \mathbb{C}\), both the integral \(\langle P_{\lambda}|P_{\Omega}\rangle'\) and \(F_{A,\Omega}(\beta, c)\) are entire functions of exponential type. This allows us to invoke Carlson’s theorem and conclude that \(\langle P_{\lambda}|P_{\Omega}\rangle' = F_{A,\Omega}(\beta, c)\) for all values of \(\beta\).

\(^7\)The Hecke algebra relations are \((T_i - t)(T_i + 1) = 0, T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}\) and \(T_i T_j = T_j T_i \) if \(|i - j| > 1\).
for $i = 1, \ldots, N$ and $\omega = s_{N-1} \cdots s_1 \tau_1$. Importantly, these operators satisfy $[Y_i, Y_j] = 0$ for all $i, j$. Now, let

$$
\Gamma(u; v) = \prod_{i=1}^{m}(1 + uvY_i) \prod_{i=m+1}^{N}(1 + uY_i),
$$

$$
G(u; v) = \sum_{\sigma \in S_N} \sigma \left( \prod_{\ell=1}^{m} \frac{\alpha_\ell \Gamma(u; v)}{\alpha_1^{\pi(i, \ldots, m)}} \right),
$$

(30)

where $\alpha_\ell = \prod_{i<j}(x_i - x_j)$ and the permutation $\sigma \in S_N$ is such that $(x_i, \theta_i) \mapsto (x_{\sigma(i)}, \theta_{\sigma(i)})$. In [26], it was showed that the generating functions $G(u; 1) = \sum_{n=1}^{N} u^n D_n$ and $G(u; q) = \sum_{n=1}^{N} u^n I_n$ contain $2N$ independent commuting quantities whose common eigenfunctions are the sMacs of fermionic degree $m$. The commutativity $[D_i, D_j] = [I_i, I_j] = [D_i, I_j] = 0$ follows from that of the Cherednik operators. Let us define the operators $D_n$ and $I_n$ with $n \leq -1$ by replacing $Y_i$ by their inverse in $G(u; v)$ (the resulting $2N$ new operators of course are not independent conserved quantities). Since the stRS Hamiltonian is a combination of the $D_i$’s, it follows that $[H, D_i] = [H, I_i] = 0$ and the integrability of the stRS models is proved, at least in the subspace $\mathcal{H}^{S_N}$. The extension of this conclusion to the full space $\mathcal{H}$ relies on the generalization of the argument of [12, App. C].

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