Determination of Collapse Load of Engineering Structures using Iterative Node-based Smoothed Finite Element Analysis Method

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Abstract. This paper presents an iterative limit analysis scheme modeled within a three-node node-based smoothed finite element framework incorporating a so-called modified elastic compensation method to determine the ultimate bearing capacity of ductile structures. More explicitly, the approach iteratively reduces those elemental Young’s modulus associated with a high-stress intensity such that the inelastic behavior of the elastic-perfectly plastic material model is taken into account. The approach provides good accuracy and convergence of solution through a series of standard linear elastic analyses that make possible the application in large-scale engineering structures, even the one with complex geometric configurations, and hence eminently accessible to structural engineers. As a result, complex inelastic analyses are eliminated and computational cost is significantly reduced. Several benchmark examples, one of which reported, illustrates the efficiency and robustness of the iterative limit analysis approach in tracing the load multipliers of in-plane ductile structures from the beginning until convergence of collapse load solution or failure state occurs. The yield areas at the ultimate state gradually appear in the graph of elastic modulus distribution.

1. Introduction
Recently, the ductile fracture of civil structures is famous among failure modes in infrastructure systems, such as building, bridge, etc. In engineering practice, the consideration of ductile structures in the plastic deformation range after the working stress reaches the yield stress is essential and needs to put more attention on. The reason is that if the external loads exceed a critical load value, ductile structures can still carry more load on which the plastic deformation range will rapidly enlarge and finally lead to the failure of the whole structure. Consequently, the determination of exact ultimate bearing capacity, particularly complex structures, is of importance for engineers in practical design to remaining safe.

The limit analysis is the most frequently used method to compute the collapse load limit of engineering structures underpinning the two well-known theorems, namely upper (kinematic) and lower (static) bound theorems associated with the statically admissible stress and the displacement fields, respectively. The elastic compensation method (or ECM) proposed by [1] that a series of conventional linear elastic analyses successively modify the Young’s modulus (or elastic modulus) of critical elements identified by the intensity of stresses. The standard ECM, however, was not a
guaranteed method for effectively obtaining the limit load of complex structures (namely ones containing flaws or stress singularities). The modified ECM version is later developed by [2] to enhance the computational precision by adding an adjustable factor. The constant strain finite element (or linear triangular element) is amongst widely used finite elements in engineering mechanics applications owing to its simplicity, economy and flexible application to complex geometric configurations. However, the conventional finite element method (FEM) adopting the ordinary isoparametric element formulation has exhibited an “overly-stiff” response in the plastic deformation range detailed in [3]. The smoothed finite element method (S-FEM) was introduced by [4] in which the standard FEM is combined with some meshfree techniques to form various S-FEM models, namely edge-, nodal- and cell-based S-FEM models, for 2D solid mechanics problems. Each of the smoothed finite element (S-FE) models exhibits different outstanding properties based on inherent “softening” effects over the “overly-stiff” property existing in the original FEM model. The node-based smoothed finite element method (or NS-FEM) stands out as a reliable numerical method for engineering mechanics applications owing to its superb accuracy and superconvergence in stress resultants, free from the volumetric locking induced by the nearly incompressible material property, and possibly applied for n-sided elements. These properties make NS-FEM an eminently suitable candidate for lower bound limit analysis [5, 6]. Various advanced techniques for the standard finite element method [7-12]; the boundary element method [13]; mesh-free method [14] have been devised to address difficulties in the limit analysis procedure. Recently, the S-FEM has been also applying to limit analysis [15-17]. Most of which incorporating mathematical programming (e.g. either for linear or non-linear programming) for solving minimization problems underpinning the upper bound limit analysis theorem. The application of S-FEM into a lower bound limit analysis theorem has not been studied yet. The present study introduces an algorithm that performing a series of standard linear elastic finite element analyses within a discrete model adopting node-based finite elements (or NS-FEs). The approach adopting the modified ECM under the lower bound theorem easily but effectively determine the maximum bearing capacity of in-plane inelastic engineering structures. The present method adopts the simple three-node triangle element that is eminently suitable for complex geometries, simple and hence computationally efficient. Several benchmark examples, including ones associated with the volumetric locking situation in plasticity range or dominated by in-plane bending, highlight the robustness and accuracy of the proposed analysis scheme in evaluating collapse load limit for the whole redistribution process of the inelastic stress field until collapse state of engineering structures is obtained.

2. Formulations of generic three-node node-based finite element (NS-FE) model
This section summarizes the novel formulations that describe the elastic stiffness analysis of in-plane structures modeled within the three-node NS-FE framework. The shape functions employed in the three-node NS-FE model are identical to those used the standard three-node FE model on a common discrete model using $N_t$ three-node triangle elements. The NS-FEM, however, constructs so-called “smoothed” strains enveloped by local smoothing domains in place of the origin compatible strain fields $\bar{e} = \nabla \bar{u}$ A set of $N_t = N_n$ (namely $N_t$ the number of node-based smoothing domains and $N_n$ is the number of nodes (triangle vertices) in the three-node triangle NS-FE model) for both “non-overlap” and “no-gap” smoothing domains $\Omega_k$ represents the whole structural domain (and $\Omega = \bigcup_{k=1}^{N_t} \Omega_k \cap \Omega_m = \emptyset$ | n ≠ m). The node-based smoothing domains (see shaded areas in Figure 1) encompass the areas bounded by linking the mid-edge points to the central points of surrounding triangle elements (e.g. smoothing domain for the node $k$ as depicted in Figure 1). Each three-node triangle element therefore include three quadrilaterals known as sub-domains. The strains developed within every NS-FEs are smoothed over the smoothing domains and are denoted by $\varepsilon_k$ of a generic smoothing domain as $\Omega_k$.
\[ \varepsilon_k = \int_{\Omega_k} \varepsilon(x) W(x) d\Omega = \int_{\Omega_k} \nabla \mu(x) W(x) d\Omega \]  

where \( W(x) \geq 0 \) is a Heaviside-type weight smoothing function, especially in this case a constant smoothing function considered as

\[ W(x) = \begin{cases} 
1/A_k', & x \in \Omega_k', \\
0 , & x \notin \Omega_k', 
\end{cases} \quad \int_{\Omega_k'} W(x) d\Omega = 1 \]  

\[ A_k' = \int_{\Omega_k'} d\Omega \text{ an area of a smoothing domain } \Omega_k' \text{ (viz. } A_k' = \sum_{j=1}^{i} (A_j / 3) \text{ for the domain over the node } k \text{ as in Figure 1}), \text{ and } A_j / 3 \text{ the area of individual sub-domain within the } j\text{-th element around the node } k. \]

In Figure 1, the smoothing domain encompassing the node \( K \), \( \Omega_k' \), includes the quadrilateral-shaped sub-domains as \( KSGH, KHIJ, KJLM, KMON, KOPQ \), and \( KQRS \).

Substituting Equation (2) into Equation (1) and rewriting the formulation in the matrix form, the smoothed strains for the smoothing domain \( \Omega_k' \) then read

\[ \varepsilon_k = B(x_k) d = \sum_{i \in N_{n,k}} B_i(x_k) d_i \]  

Where \( N_{n,k} \) is the number of triangle vertices contributing to a smoothing domain \( \Omega_k' \) of the common node \( k \) (see e.g. \( N_{n,k} = 6, \{ K, A, B, C, E, F \} \) corresponding to the smoothing domain associated with the node \( K \) Figure 1), \( d_i \) and \( B_i(x_k) \) is the nodal displacements and the smoothed strain-displacement matrix of the \( i\)-th node in the set of nodes \( N_{n,k} \) encompassing the smoothing domain \( \Omega_k' \), respectively.

**Figure 1.** The discretization of an NS-FE model with smoothing (shaded) domains

For the case of using linear triangular elements, the smoothed strain-displacement matrix \( B_i(x_k) \) of the \( i\)-th node for the smoothing domain \( \Omega_k' \) reduces to an area-weighted average formulation as given by [5]

\[ B_i(x_k) = \frac{1}{A_k'} \sum_{j=1}^{N_{e,k}} \frac{1}{3} \bar{B}_j A_j \]  

where \( N_{e,k} \) is the number of triangle elements sharing the common node \( k \), for example, \( N_{e,K} = 6 \) for \( \Omega_k' \) (viz. triangle elements of \( KAB, KBC, KCD, KDE, KEF \) and \( KFA \) as in Figure 1), and a matrix
$\bar{B}_j = \sum_{i=1}^{(N-3)} \bar{B}_i$ is the compatible strain-displacement matrix for the $j$-th triangle element surrounding the node $k$ constituted from the compatible strain-displacement matrices $\bar{B}_i(x)$ of the nodes in the set three vertices of that triangle element. Due to the constant solution of $\bar{B}_i(x)$ inside each triangle element, $B_i(x_k)|_{k=1,...,N_1}$ is therefore also piecewise constant over each smoothing domain.

Assume a two-dimensional solid body $\Omega$ loaded by body forces $b$, external traction forces $t$ on the boundary $\Gamma$, and displacement boundary conditions $u = u_d$ on the boundary $\Gamma_d$. The smoothed Galerkin weak form [4] then reads

$$\sum_{i=1}^{N_1} A'_k \delta u_k^T D e_i - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma} \delta u^T t d\Gamma = 0$$

(5)

where $D$ is the elemental elastic stiffness matrix given as in Equations (8.2) and (9.2). After substituting Equation (3) into Equation (5), the governing elastic stiffness equations for the three-node NS-FE model [5] read

$$K d = f$$

(6)

$$K = \sum_{i=1}^{N_1} K_i = \sum_{i=1}^{N_1} \left[ \int_{\Omega_i} B_i^T D B_i d\Omega \right] = \sum_{i=1}^{N_1} \left( A'_k B^T d B_k \right)$$

(7)

where the global elastic stiffness matrix $K$ is assembled using the domain stiffness matrices $K_i$ with the information of degrees of freedom associated with each smoothing domain $\Omega_i$, $k = 1, ..., N_1$, rather than triangle element in Equation (7). The unknown displacements $d$ and the externally applied forces $f$ are identically defined by the standard three-node FE analysis procedures.

3. An iterative NS-FE limit analysis procedure.

3.1. Modified elastic compensation method within NS-FE model

The present approach successively performs the conventional linear elastic analysis using NS-FEs for two-dimensional ductile structures to capture the maximum bearing capacity at failure state. This analysis procedure performs the modified elastic compensation algorithm under a lower bound limit analysis theory. At the first iteration, one step of linear elastic analysis is conducted using the initial properties of the material. Then, Young’s modulus of highly stressed triangle elements in localized regions, among Young’s modulus field $E'_j$ ($j = 1$ to $N_1$) of all triangle elements, will be modified based on the previous stress resultants, e.g., the current step $r$ for linear elastic analysis is performed by using the stress results obtained from the previous analysis step $r-1$.

The equivalent von Mises stress associated with node-based smoothing domains in three-node NS-FE model at the step $r-1$ is defined by

For plane stress von Mises, the yield condition reads

$$\sigma_k^{r-1} = \left( (\sigma_x - \sigma_y)^2 + \sigma_x \sigma_y + 3 \tau_{xy}^2 \right)^{1/2} \leq \sigma_{y,k}^{r-1}, \text{ for all } k = 1 \text{ to } N_1$$

(8.1)

and

$$D = EM = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}$$

(8.2)
For plane strain von Mises, the yield condition reads

\[
\sigma_{\text{e}} = \left( \frac{3}{2} \left( \left( \sigma_{x} - \sigma_{y} \right)^2 + 4\tau_{xy}^2 \right) \right)^{1/2} \leq \sigma_{y,k}; \quad \text{for all } k = 1 \text{ to } N
\]  

(9.1)

and

\[
D = EM = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & 1 - 2\nu/2
\end{bmatrix}
\]

(9.2)

\( E \) and \( \nu \) are the Young’s (elastic) modulus and Poison’s ratio of the material adopted, respectively.

The elemental equivalent stress in each triangle element is the maximum of sub-elemental equivalent stresses (namely three smoothing sub-domains for the three-node NS-FE model) contributing to that element at step \( r-1 \) (e.g. \( \sigma_{\text{e}}^{r-1} = \max(\sigma_{\text{KAB}}^{r-1}; \sigma_{\text{KHIJ}}^{r-1}; \sigma_{\text{ATIH}}^{r-1}) \) see Figure 1).

Then, the elastic modulus of triangle elements, of which the equivalent stress \( \sigma_{\text{e}}^{r-1} \) higher than the nominal stress \( \sigma_{\text{e}}^{r-1} \) in Equation (11) at step \( r-1 \), will be adjusted to redistribute the inelastic stress at the current analysis step \( r \) as follows:

\[
E_{\text{e}}' = \begin{cases} 
E_{\text{e}}^{r-1} \frac{\sigma_{\text{e}}^{r-1}}{\sigma_{\text{e}}^{r-1}} & \text{for } \sigma_{\text{e}}^{r-1} > \sigma_{\text{e}}^{r-1} \\
E_{\text{e}}^{r-1} & \text{for } \sigma_{\text{e}}^{r-1} \leq \sigma_{\text{e}}^{r-1}
\end{cases}
\]

(10)

\[
\sigma_{\text{e}}^{r-1} = \sigma_{\text{e}}^{r-1} - \lambda \left( \sigma_{\text{e}}^{r-1} - \sigma_{\text{e}}^{r-1} \right)
\]

(11)

Where the maximum and minimum elemental equivalent stresses \( \sigma_{\text{e}}^{r-1} \) and \( \sigma_{\text{e}}^{r-1} \), respectively reads

\[
\begin{cases} 
\sigma_{\text{e}}^{\max} = \max(\sigma_{j}^{r-1}); \quad (j = 1 \text{ to } N) \\
\sigma_{\text{e}}^{\min} = \min(\sigma_{j}^{r-1}); \quad (j = 1 \text{ to } N)
\end{cases}
\]

(12)

\( \lambda \) is a modification factor indicating that a small value will take more iterations, while the convergence of the solution will be obtained. In contrast, a large value \( \lambda \) may not lead to convergence.

Each analysis iteration \( r \) results into an admissible stress field and the load multiplier \( \alpha' \) that ensures yield conformity in the problem domain as follows

\[
\alpha' = \min \left( \frac{\sigma_{y,1}}{\sigma_{1}^{r}}, ..., \frac{\sigma_{y,N}}{\sigma_{N}^{r}} \right)
\]

(13)

It is worth noting that the elastic modulus \( E \) in matrix \( D \) is no longer constant over each smoothing domain \( \Omega_{j}^{r} \). It is piecewise constant over sub-domain contributing to the smoothing domain \( \Omega_{j}^{r} \).

Equation (7) that calculates the global elastic stiffness matrix \( K \) at the iteration \( r \) becomes

\[
K' = \sum_{j=1}^{N} K_{j} = \sum_{j=1}^{N} \left( \int_{\Omega_{j}^{r}} B_{j}^{T} (E_{j}^{r-1} M) B_{j} d\Omega \right) = \sum_{j=1}^{N} \left( B_{j}^{T} MB_{j} \int_{\Omega_{j}^{r}} E_{j}^{r-1} d\Omega \right)
\]

(14)
E.g. for the node \( k \) in Figure 1, \( K'_k = B^T MB \left( \int E^{r-1} d\Omega \right) = B^T MB \begin{pmatrix} E^{r-1}_{KHIJ}A_{KHIJ} + \ldots + E^{r-1}_{KJHG}A_{KJHG} \end{pmatrix} \)

After a series of standard linear elastic analysis, assumed “\( r_{max} \)” iterations, the maximum value of load multipliers is considered as the collapse load solution of the structure as follows

\[ \alpha_{col} = \max \left( \alpha^{1},\ldots,\alpha^{\max} \right) \] (15)

3.2. Algorithm implementation

In this section, the step-by-step numerical implementation is summarized through key steps that perform the proposed iterative limit analysis procedure within the NS-FE framework. The algorithm is presented as follows:

I. Initialization
- Discretize the problem domain with a coarse mesh of three-node triangular elements and construct the information matrices for node-based smoothing domains.
- At the first iteration \( r = 0 \), initialize: the maximum number of analysis iterations \( r_{max} \), Young’s modulus \( E_{j}^{y,0} \), yield limit \( \sigma_{y,k} \), Poisson’s ratio \( \nu \), and the modification factor of \( \lambda \in (0,0.5] \)

II. Iterative NS-FE limit analyses
+ For \( r = 1 \) to \( r_{max} \)
  - Assemble the global stiffness matrix \( K' \) (step \( r \)) from the element stiffness matrices obtained from the previous elastic modulus field \( E_{j}^{r-1} \) (step \( r-1 \))
  - Solve the discretized linear system of equations and compute the equivalent von Mises stress field for node-based smoothing domains and then for triangle elements using Equations (8.1) and (9.1)
  - Update the new Young’s modulus field for all \( N_t \) triangle elements from Equations (9) to (11)
  - Calculate the load multiplier using Equation (13)

III. Termination at \( r = r_{max} \)
- Determine the collapse load through Equation (15)
- Plot the \((\alpha -\text{DOF})\) and \((\alpha - r)\) responses, elastic modulus distributions captured at some analysis steps

Note that: to make it simple and concise, we do not present some basic steps in the algorithm

4. Illustrative example: Cook’s problem

Several illustrative examples, one of which presented in this section, have been successfully solved to validate the simplicity and high accuracy of the proposed iterative limit analysis method in capturing the collapse load of in-plane ductile structures. The modified ECM running within a numerical model of the NS-FEs, denoted by mECM-NSFEM, is adopted. The iterative analyses, adopting the modification factor of \( \lambda = 0.05 \) based on our experience, are terminated at \( r_{max} = 400 \), the maximum number of computing iterations. The whole structures have been modeled and encoded within a MATLAB programming environment.

This example consists of the skew beam, or Cook’s problem in plane-stress condition, loaded at the right end by a distributed vertical force \( \alpha \) in Figure 2. An elastic-perfectly plastic material model, obeying the von Mises criterion, as presented in [18, 19] includes \( E = 2\times10^4 \) and \( \sigma_y = 1 \), the thickness of \( t = 1 \). The Poisson’s ratio of \( \nu = 0.4999 \) (viz. giving a nearly incompressible plastic deformation in plastic range of material model), well-known as a volumetric locking test, is used.

Four different meshes, including 128 (162 DOFs) in Figure 3 where solid lines denote to displacement constraints, 512 (578 DOFs), 2048 (2178 DOFs) and 8192 (8450 DOFs) three-node triangle elements, are adopted. The analytical solution of the limit load is not available The results obtained by the three-field mixed FEM formulation of [18] are used for comparison in which the limit load converged and reached the lowest value at \( \alpha_{col} = 0.3956 \) for a discrete model of 1024 (2178 DOFs) quadrilateral elements in Figure 4a. On the same graph, the collapse load solutions \( \alpha_{col} \), obtained from mECM-
NSFE model, converge to the lower bound limit and hit $\alpha_{col} = 0.3943$ with the mesh of 8192 (8450 DOFs) triangle elements, which is around 0.32% difference of reference value [18]. It is believed that the exact collapse load value is between the above limit values

\[ \alpha = 0.3943 \]

![Figure 2. Cook’s problem: geometry and loading](image)

![Figure 3. Cook’s problem: a discrete model with 128 elements](image)

The response of load multipliers $\alpha'$ to the number of computing iterations $r$ ($r = 1$ to 400 in this example) as plotted in Figure 4b for the NS-FE model of 2048 elements illustrates the convergence, accuracy, and stability of the present numerical approximating solutions. The modification factor of $\lambda = 0.05$ is small enough to produce stable numerical solutions of load multipliers. However, it slowly converges and costs much computing resources compared to the choice of higher values.

It is worth noting that the proposed analysis approach effectively tackles the difficulties associated with bending domination as illustrated by providing a collapse load solution close to the literature. Lower converged collapse load limit also demonstrates the mECM-NSFE model as a free volumetric locking computing model and hence satisfying the lower bound limit analysis theory.

The elastic modulus distributions captured at some analysis iterations illustrate the areas undergoing plasticity using the mesh of 8192 (8450 DOFs) elements as depicted in Figure 5. The yield areas progressively appear and become similar to yield patterns presented by [18]. A smaller and hence more accurate plasticity zone is obtained with an increasing number of elements.

![Figure 4. Cook’s problem: (a) collapse load solutions $\alpha_{col}$ with an increasing number of DOFs (b) load multiplier responses $\alpha_{col}$ associated with analysis iterations $r$ for the mesh of 2048 elements](image)
5. Conclusions

For the results stated, some concluding remarks are drawn as follows:

The NS-FEM with a “soften effect” into the conventional FEM has produced a super accuracy and superconvergence solution in every elastic analysis using a relatively coarsen mesh of simple three-node triangle elements.

An iterative elastic analysis scheme for structures made of perfectly plastic material has been presented. The NS-FE model incorporates the modified ECM to efficiently and robustly determine the collapse load of ductile engineering structures. Other material models are also applicable.

The present approach is easy-to-implement by using only a series of standard linear elastic analyses to determine the maximum load capacity of inelastic engineering structures at the plastic collapse state. More complicated inelastic analyses will be avoided and computational cost then is significantly reduced by using the proposed approach. It is eminently feasible to develop the present limit analysis procedure to solve large-scale and three-dimensional engineering structures.

Several examples, one of which has been reported, illustrates the simplicity, solution accuracy and numerical stability of the proposed limit analysis approach. The well-known volumetric locking associated with a nearly incompressible material model in the plastic range has also been eliminated by adopting the NS-FEM as shown to be its superior feature in the literature. The distributions of elastic modulus, which is a by-product of the iterative elastic analysis procedure, describe those localized plastic regions that agree well with the literature.

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