A single quantum particle in a sufficiently strong disorder potential does not explore the full phase space at given energy, but remains Anderson localized to a finite spatial region due to quantum interference. Over the last decade it has been shown that such broken ergodicity and absence of transport persist in many-body systems of finite density, if disorder is sufficiently strong and interactions are weak enough. At non-zero temperature this phenomenon, known as "many-body localization", comes along with a non-extensive bipartite entanglement entropy in highly excited eigenstates, and, in well localized regimes, with a complete set of quasi-local conserved quantities that inhibit transport.

In almost all many-body systems studied so far, quenched disorder is central to stabilize the localized phase: it ensures that local rearrangements are typically associated with significant energy mismatches, which appear as large denominators in perturbation theory, and suppress real decay processes. In contrast, it has been suggested early on in the context of defect diffusion in solid Helium crystals that localization effects could also be induced solely by sufficiently strong interactions, without any quenched disorder. Several recent works have revived this idea, focusing on the question of genuine many-body localization in low-dimensional systems, such as Bose-Hubbard models, mixtures of heavy and light interacting particles and quantum spin chains. In such systems, a tendency to localize arises from the configurational disorder present in generic inhomogeneous initial conditions. In Ref. it was conjectured that another notion of localization, as evinced by an incomplete volume law entanglement, could exist in systems without disorder.

Such an interaction-induced localization contrasts in an important way with the more standard scenario, in which many-body localization simply embodies the survival of the Anderson-localized phase despite dephasing interactions, cf. Fig. In the absence of disorder, the interactions take a completely different role: they create a rough energy landscape in which weak quantum fluctuations are unable to restore ergodic dynamics, similarly to what happens in classical glasses. Despite this analogy, in our quantum models classical frustration plays no role. This distinguishes them from other quantum systems which inherit their non-ergodicity from a classically glassy counterpart. At the same time, the role of temperature is opposite to that in disorder-dominated localization, where it enhances the phase space for scattering, dephasing and dissipation. When interactions dominate instead, the higher the en-
ergy density, the stronger the configurational disorder and hence the localization tendency [15].

Localization due to interactions has the experimentally appealing aspect of being an unambiguous many-body effect, since it cannot be ascribed to disorder. Standard many-body localization manifests itself in the absence of transport and thermalization. In contrast, in the disorder-free context its most natural manifestation lies in the dynamical persistence of initial inhomogeneities in the density of particles or energy. Such an effect could in principle be observed in experiments with binary mixtures of cold atoms [27, 28]. Indeed, while in a finite system with periodic boundary conditions (pbc) every inhomogeneous initial condition eventually restores translational invariance upon taking time averages, in the localized phase one expects the time scale for this relaxation to grow exponentially with system size, being divergent in the thermodynamic limit.

In this Letter, we discuss quantitatively the phenomenology of interaction-induced disorder-free localization in a class of models of experimental relevance. We analyze the dynamics in the limit of small quantum fluctuations, and show that the relaxation of an initial inhomogeneity remains incomplete up to times which are exponentially large in the system size. By extrapolating our result to larger quantum fluctuations we will obtain an analytical estimate (upper bound) for the boundary of the localized phase. Caveats related to non-perturbative effects [18, 19], which might reinstall weak diffusion in very large systems, will be thoroughly discussed.

We consider a class of models describing one-dimensional hard-core particles, with repulsive interactions between subsequent particles on the line:

$$H = -t \sum_{j=1}^{L} \left( e^{i\Phi} e^{|j+1} c_{j} + e^{-i\Phi} c_{j+1} \right) + U \sum_{j,l=1}^{L} v(l) n_{j} n_{j+l} \prod_{k=1}^{l-1} (1 - n_{j+k}) ,$$

where $n_{j} = c_{j}^{|j}$, and the constraint $\sum_{j} n_{j} = N = \rho L$ is imposed. We put a magnetic flux $\phi$ through the ring, in order to break spatial reflection symmetry. This removes the degeneracy in the spectrum, which simplifies our subsequent analysis. We set $v(1) = 1$ and assume a power-law decrease, $v(l) = t^{-\beta}$, with an exponent $\beta > 0$. Such models arise, for example, upon considering a quasi-one dimensional binary mixture of atoms with very different masses [20] and integrating out the fast particles [20]. Note also that the above model maps onto a Bose-Hubbard model if particles are numbered and considered as sites, while distances between the particles are seen as site occupation numbers. The phenomena discussed below are indeed much more generic than the specific model [1] discussed here.

Using pbc, the model is translation invariant. Hence, for any finite system size $L$ the many-body eigenstates can be chosen to be eigenvectors of total momentum. For infinitesimal hopping $t$, the eigenstates organize in momentum minibands. These are essentially formed by hybridizations of a classical particle configuration $|C\rangle$ with all its translations around the ring, $T^{j} |C\rangle$, for $j = 0, 1, 2, ..., L - 1$, where $T$ translates the state by one lattice site. Typical states correspond to configurations $|C\rangle$ where sites are filled randomly with probability $\rho$. The eigenstates of such minibands take the form

$$|C, P_{n}\rangle \approx \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{ijP_{n}} T^{j} |C\rangle ,$$

where $P_{n}$ is the total momentum. The hopping Hamiltonian connects typical configurations $|C\rangle$ and its translations only at very high order of perturbation theory, since to translate the system by one site one needs to move all $N = \rho L$ particles. This leads to an exponentially small dispersion of the band

$$\varepsilon_{n} = -2t_{\text{eff}} \cos P_{n} , \quad P_{n} = (2\pi n + \phi)/L ,$$

where $t_{\text{eff}}$ is the effective hopping of the center of mass of this state. For small $t$ it is exponentially small in the system size, and will be calculated in Eq. [12] below. This has important consequences for the dynamics: after preparing the system in an inhomogeneous initial configuration, the time scale to relax to a homogeneous state (if averaged over time) is proportional to $t_{\text{eff}}^{-1}$. In the thermodynamic limit relaxation is suppressed, and hence translation symmetry is dynamically broken.

The description of Eq. (2) is oversimplified, as it neglects the presence of resonances, i.e., hybridizations with configurations $|C'\rangle$, that are not translations of $|C\rangle$, but have the same unperturbed energy $O(U)$. The simplest examples are shown in Fig. 2: the configurations $C_{1}, C'_{1}$ formed by three particles at mutual distances $l$ and $l + 1$ are classically degenerate. Their degeneracy is lifted at first order in perturbation theory. In configuration $C_{2}$ the two intervals of length $l, l + 1$ are separated by a third one, and hence the degeneracy is lifted at second order. In both cases, two hybridized states are formed:

$$|C, P_{\pm}\rangle \approx \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{ijP_{\pm}} T^{j} \left( \frac{|C_{1}\rangle \pm |C'_{1}\rangle}{\sqrt{2}} \right) .$$

![FIG. 2: Examples of resonances. The configurations $C_{1}, C'_{1}$ (top) hybridize at first order of degenerate perturbation theory in t, while $C_{2}$ and $C'_{2}$ (bottom) hybridize at second order. Moving the middle particle(s) to the right costs no energy.](image-url)
Such states can be seen as the admixture of two of the miniband described by Eq. (2). It is straightforward to generalize this expression to the case in which \( n \) resonances are present in the initial configuration, each hybridizing \( r_i \) states. The eigenstates then take the form

\[
|C, P, \{\alpha_i\} \rangle \approx \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i P k} T_k \prod_{i=1}^{n} \left( \sum_{m_i=1}^{r_i} \psi_{m_i}^{\alpha_i} R_k^{(m_i)} \right) |C \rangle,
\]

where the \( \{\alpha_i\} \) label the possible states of the \( i \)th resonance, which are described by amplitudes \( \psi_{m_i}^{\alpha_i} \) multiplying local operators \( R_k^{(m_i)} \) that rearrange the classical configuration at the resonant spot. Eigenstates take such a form only for very small \( t \). At larger hopping, states with finite energy differences of \( O(t) \) hybridize as well. However, the crucial point is that at the perturbative level in \( t/U \) no system spanning hybridizations are expected.

Let us now consider the time evolution from a classical initial configuration \( C \). We first restrict to the case where \( |C \rangle \) has no resonant spots, which allows for exact calculations. To characterize the relaxation process, we define the average spatial density inhomogeneity,

\[
\Delta \rho^2_\psi (\tau) \equiv \frac{1}{L} \sum_{j=1}^{L} \left[ \langle \psi (\tau) | (n_{j+1} - n_j) \psi (\tau) \rangle \right]^2,
\]

where \( \psi (\tau) \equiv e^{-i H \tau} |C \rangle \). This observable vanishes for any translation invariant state. Below, we will also consider its time average, \( \langle \Delta \rho^2_\psi \rangle (T) \equiv T^{-1} \int_0^T d\tau \Delta \rho^2_\psi (\tau) \), in order to avoid dealing with quantum revivals when looking at finite systems. In the absence of resonances, the relevant eigenstates and energies are given by Eqs. (23), and one finds (23).

\[
\Delta \rho^2_\psi (\tau) = \frac{1}{L^2} \sum_{m \neq m'} \sum_{n \neq n'} e^{-i \tau [(\varepsilon_m + \varepsilon_n') - (\varepsilon_m + \varepsilon_n)]} \sum_{k \neq k'} e^{i q \cdot (m-n) k} e^{i 2 \pi \rho (m'-n') k'} \cdot \left( 2 G (k - k') - G (k - k' - 1) - G (k - k' + 1) \right),
\]

with the auto-correlation function of the initial density,

\[
G (k - k') \equiv \frac{1}{L} \sum_{j=1}^{L} \langle C | n_{j+k} | C \rangle \langle C | n_{j+k'} | C \rangle.
\]

In the thermodynamic limit, we can take a continuum limit and measure time naturally in units of \( t_{\text{eff}} \). Assuming an essentially random initial configuration of particles of density \( \rho \), we further have \( G (k - k') = \rho (1 - \rho) \delta_{k-k',a} + \rho^2 \). After some manipulations one finds the inhomogeneity to relax according to

\[
\frac{\Delta \rho^2_\psi (\tau)}{\Delta \rho^2_\psi (0)} = \int \frac{dq}{2\pi} J_0^2 (4 \pi t_{\text{eff}} |q|) \sin^2 q,
\]

where \( J_0 \) denotes the Bessel function of the first kind. For times \( \tau \ll t_{\text{eff}}^{-1} \) one finds essentially no relaxation,

\[
\frac{\Delta \rho^2_\psi (\tau)}{\Delta \rho^2_\psi (0)} = 1 - 6 (\tau t_{\text{eff}})^2 + O ((\tau t_{\text{eff}})^4), \quad \tau \ll t_{\text{eff}}.
\]

For large times, if no time average is taken the inhomogeneity oscillates, with an envelope decaying as \( \Delta \rho^2_\psi (\tau) \propto \tau^{-1} \).

In Fig. 3 we compare the above calculations with numerical data from exact diagonalizations of finite systems, initialized in a resonance-free, but otherwise random configuration \( C \) of \( N = \rho L \) particles, with \( \rho = 1/3 \). We used very small hopping \( t = 10^{-3} U \), and interactions decaying with an exponent \( \beta = 2 \). For each data set, time is rescaled with the appropriate effective center of mass hopping, \( t_{\text{eff}} (C) \). In finite systems, the long-time average of \( \langle \Delta \rho^2_\psi \rangle (T) \) is finite. For a non-degenerate spectrum a simple calculation yields \( \langle \Delta \rho^2_\psi \rangle (\infty) = \Delta \rho^2_\psi (0) / L \), which is subtracted in Fig. 3 so that all curves asymptotically tend to zero. Despite the small sizes, the agreement with Eq. (3) for the thermodynamic limit is very good.

The inset of Fig. 3 illustrates the long-time plateau of inhomogeneity, whose length diverges in the thermodynamic limit. The latter is due to the exponential smallness of \( t_{\text{eff}} \). For small \( t \) it can be calculated in \( N \)th order perturbation theory, as a the sum over all ways in which the \( N = \rho L \) particles can be moved rightwards by one site. In general, this sum is difficult to evaluate. However, in the dilute limit, \( \rho \ll 1 \), the relevant energy denominators are well approximated as sums of uncorrelated energy shifts \( \Delta V_i \) due to moving the \( i \)th particle,

\[
\Delta V_i = U \cdot |p' (r_i - r_{i-1}) - p' (r_{i+1} - r_{i})|.
\]

In the absence of resonances, the sum over all \( N! \) paths can be resummed nearly exactly. To logarithmic accuracy one obtains the estimate

\[
\log \left( \frac{t_{\text{eff}}}{t} \right)_{\text{typ}} \approx (N - 1) \left( \log \left( \frac{t}{\Delta V_i} \right) \right).
\]
where the bracket indicates an average over the particles $i$. In a random state, we may instead average $\log(\Delta V_i)$ over the Poissonian distribution of interparticle distances $P(l) = \rho (1 - \rho)^{l - 1}$,
\[
\langle \log \Delta V \rangle \approx \rho^2 \sum_{l_1 \neq l_2 + 1} (1 - \rho)^{l_1 + l_2 - 2} \log U |v'(l_1) - v'(l_2)|.
\]
As we explain below, the singular point $l_1 = l_2 + 1$ is excluded since it corresponds to a resonant configuration.

Despite the fact that the above estimate is valid only for very small $t$, we can use it to estimate the hopping $t_c$ at which typical random states delocalize. We expect
\[
t_c \lesssim \exp[\langle \log \Delta V \rangle],
\]
to be an upper bound, as resonances proliferate with increasing $t$. For a power law interaction $v(l) = l^{-\beta}$, one finds $t_c \sim U \rho^{\beta+1}$ to be proportional to the typical force with which particles repel each other. For $\beta = 2$ (the value we used in our simulations) numerical evaluation of the sum, approximated as an integral, yields $t_c \lesssim 70 U \rho^3$ for $\rho \ll 1$. This bound holds at very small density only. To extract the value of $t_c$ at $\rho = 1/3$, we fitted the size dependence of the numerically evaluated $t_{\text{eff}}$ as $t_{\text{eff}} \propto (t/t_c)^{\rho L}$, obtaining $t_c (\rho = 1/3) \approx 0.2 U$.

Let us now discuss the role of resonances, that is, compact subsets of the configuration $C$, which, when moved by one lattice spacing to the right lead to a new configuration $C'$ with the same classical energy as $C$. At low density the dominant type of resonances is shown in Fig. 2. We assume that resonances are not crossing other resonant configurations of shorter distance (otherwise, only the shorter of the two should be considered as producing a small denominator). They result in fast, partial relaxation processes through resonant admixture. This diminishes the inhomogeneity plateau by an amount proportional to the density $\rho$ of resonating configurations. Moreover, resonances increase $t_{\text{eff}}$ by a factor $O((U/t)^{\sqrt{\pi \rho N}})$, as compared to the non-resonant estimate $t_{\text{eff}} \sim t^N$ from Eq. (11).

It has recently been argued \cite{18, 19} that, in addition to these resonances, rare but nearly ergodic bubbles with very low particle and energy density exist in the thermodynamic limit. Rather than dissolving in their denser surrounding, those might instead be mobile due to resonant delocalization, and thus restore transport at any finite hopping $t$. In this scenario, the localized regions in the phase diagram would have to be understood as merely ”asymptotically localized”, Griffiths’-like phases with strongly suppressed transport due to rare regions that act as mobile baths. In order to clarify the relevance of our predictions for experimental systems, we have estimated \cite{20} the density $n_B$ of such rare bubbles for $t \ll t_c$ as
\[
n_B \lesssim \exp \left\{ -2 \left( \frac{t_c}{t} \right)^{\frac{1}{\beta}} \frac{\log \frac{U}{\rho}}{\log \left( \frac{1}{\rho} \left( \frac{t_c}{t} \right)^{\frac{1}{\beta+1}} \right) + 1} \right\},
\]
which tends to zero very rapidly as $t \to 0$. For the parameters used in our numerics, we estimate $n_B \sim O(10^{-10})$, showing that such effects can safely be neglected for realistic system sizes, deep enough in the localized phase.

The simplest experimental realizations, in which to observe the phenomenology described here, are strongly interacting cold atomic gases in one-dimensional optical lattices \cite{27, 28}. While our calculation assumed periodic boundary conditions, the essence of interaction-induced localization will also be present in dense but randomly distributed cold atoms in a confining trap, which prevents the escape of particles at the boundaries. The response of their center of mass to a tilt of the trap should be strongly suppressed, being governed by an exponentially small $t_{\text{eff}}$. Related systems can be realized in highly anisotropic spin chains or ladders, which could be analyzed via hole burning techniques \cite{30}. The same physics can be realized in higher dimensions, even though localization is harder to achieve.

In conclusion, we have shown that relaxation in an interacting quantum system without disorder can be exponentially slow in the system size, implying that in the thermodynamic limit the dynamics become truly non-ergodic. The ensuing quantum glass phase persists up to the hopping strengths of the order of the typical energy change incurred by displacing a single particle. The latter tends to be increased by thermal disorder. Therefore, temperature has a localizing tendency, in contrast to its dephasing role in disorder-dominated localization.

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Note added: After completion of this work we became aware of a related study [31], which finds an exponentially growing time scale for relaxation, in agreement with our results. The authors further report a scale-dependent relaxation time. We conjecture that the latter is a specific property of linear response, which is absent in our relaxation dynamics from random initial conditions and the dynamics studied in Ref. [16]. Both are concerned with strongly non-linear perturbations with respect to a homogeneous state.

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SUPPLEMENTARY MATERIAL: DYNAMICS IN MANY-BODY LOCALIZED QUANTUM SYSTEMS WITHOUT DISORDER

Reducing models of light and heavy particles to the Hamiltonian of Eq. 1

In previous work [20], a one-dimensional model of two fermionic species was studied: a “fast” (light) species a, and a “slow” (heavy) species c. The light particles are not allowed to hop over the heavy particles, which therefore are referred to as “barriers”. The model is described by the following Hamiltonian:

\[ H = -J \sum_{j=1}^{L} \left( a_{j+1}^{\dagger} a_{j} + a_{j}^{\dagger} a_{j+1} \right) + \left( 1 - c_{j+1}^{\dagger} c_{j} \right) - \lambda \sum_{j=1}^{L} \left( c_{j+1}^{\dagger} c_{j} + c_{j}^{\dagger} c_{j+1} \right). \]  

(S1)

The barriers move with very small kinetic energy \( \lambda \ll J \) as compared with the hopping strength \( J \) of the fast particles. We further impose pbc, identifying sites 1 and \( L+1 \), in order to ensure the full translation invariance of the Hamiltonian. Note that the barriers could equally well be taken to be hard-core bosons. This choice does not affect the spectrum or localization properties, but only the non-local (in space and time) correlation functions.

To simplify the model and to reduce the Hilbert space, the presence of fast particles can be emulated by an effective distance-dependent repulsion \( v(l) \) between the heavy barriers, which leads to the effective model of Eq. 1. Assuming a single fast particle between every pair of barriers, forced to remain in its ground state, the effective repulsion decays as a power law, \( v(l) = \frac{U}{l^2} \). Note that in this picture \( U \) scales as the hopping, or inverse mass, of the fast particles. The exponent 2 can be substituted by a generic exponent \( \beta > 0 \) without qualitatively changing the physics of the problem.

Temporal decay of spatial inhomogeneity

We characterize the spatial inhomogeneity of the system by

\[ \Delta \rho_{\psi}^2 (\tau) \equiv \frac{1}{L} \sum_{j=1}^{L} \left[ \langle \psi(\tau) | (n_{j+1} - n_j) | \psi(\tau) \rangle \right]^2, \]  

(S2)

where \( |\psi(\tau)\rangle = \exp[-iH\tau] |C\rangle \) is the state time evolved from the classical initial configuration \( C \). For small hopping \( t \), if we restrict \( C \) to configurations without resonances, the only eigenstates with significant overlap with \( C \) are the states in the miniband described by Eq. 2. Expanding into such eigenstates, labelled by \( n, m \), we obtain

\[ \Delta \rho^2 (\tau) = \frac{1}{L} \sum_{j=1}^{L} \left[ \sum_{n,m} e^{i(\varepsilon_n - \varepsilon_m)\tau} \langle C | n \rangle \langle m | C \rangle \langle n | \Delta \rho_j | m \rangle \right]^2 = \frac{1}{L} \sum_{j=1}^{L} \sum_{n,m,n',m'} e^{i(\varepsilon_n + \varepsilon_{n'}) - (\varepsilon_m + \varepsilon_{m'})\tau} \langle C | n \rangle \langle m | C \rangle \langle n | \Delta \rho_j | m \rangle \langle C | n' \rangle \langle m' | C \rangle \langle n' | \Delta \rho_j | m' \rangle, \]  

(S3)

where the energies \( \varepsilon_n \) are given by Eq. 3, and

\[ \langle C | m \rangle = \frac{1}{\sqrt{L}}. \]  

(S4)

Further, since the operators \( n_i \) are diagonal in the basis of classical configurations, the matrix elements of the site occupations are

\[ \langle n | n_j | m \rangle = \frac{1}{L} \sum_{k,k'=0}^{L-1} e^{i\frac{2\pi}{L}(m+n)k} e^{-i\frac{2\pi}{L}(n+m)k'} \langle C | T^{-k} n_j T^{k} | C \rangle = \frac{1}{L} \sum_{k=0}^{L-1} e^{i\frac{2\pi}{L}(m-n)k} \langle C | n_{j+k} | C \rangle, \]  

(S5)
where \( T \) is the translation operator. Then the expression for the inhomogeneity becomes

\[
\Delta \rho_\psi^2(T) = \frac{1}{L^2} \sum_{j=1}^{L} \sum_{m,n=0}^{L-1} \sum_{n',m'=0}^{L-1} e^{i[(\varepsilon_n + \varepsilon_{n'}) - (\varepsilon_m + \varepsilon_{m'})]T} e^{i\frac{2\pi}{L}(m-n)k} e^{i\frac{2\pi}{L}(n'-m')k'} \langle C| (n_{j+k+1} - n_{j+k}) |C\rangle \langle C| (n_{j+k'+1} - n_{j+k'}) |C\rangle .
\]

(S6)

This leads to Eq. (6) of the main text, upon using the density autocorrelation,

\[
G(k - k') = \frac{1}{L} \sum_{j} \langle C| n_{j+k} |C\rangle \langle C| n_{j+k'} |C\rangle ,
\]

(S7)

in the initial state.

**Enhancement of the effective tunneling due to resonances**

At very small \( t \), the effective hopping of the center of mass of a generic configuration \( C \) scales as \( t_{eff} \propto t^n \rho L \). The exponent \( \alpha \) differs from the naive expectation \( \alpha = 1 \) due to resonances. Here we determine \( \alpha \) to leading order in the density \( \rho \ll 1 \). The effective hopping can be computed in perturbation theory in two equivalent ways: namely as a sum over all possible ways in which all particles can be moved by one lattice site, either rightwards or leftwards. In either way of constructing the perturbative matrix element for the effective hopping, the maximal number of resonances, i.e., the number of small denominators which are met, is the same. (This can be understood as follows: if a particle can be moved by one site to the right without energy cost, it can be moved to the left at no cost too, after all other particles have been moved.)

The simplest type of resonance is a pair of two consecutive intervals with lengths \( (l, l+1) \) or \( (l+1, l) \).

The probability of finding an interval of length \( l \) in a random configuration of density \( \rho \) is

\[
P(l) = \rho (1 - \rho)^{l-1} .
\]

(S9)

There are about

\[
N_{1res} = 2N \rho^2 \sum_{l=1}^{\infty} (1 - \rho)^{2l-1} + O(\rho^2) = \rho N + O(\rho^2)
\]

(S10)

resonances in a typical configuration \( C \), where we neglect corrections due to overlapping pairs. The factor of 2 accounts for both possibilities \( (l, l+1) \) and \( (l+1, l) \). Local configurations like this hybridize at first order in perturbation theory. Accordingly they reduce the power of \( t \) in the effective tunneling by one each, which yields

\[
(\Delta \alpha)_{1res} = -\rho + O(\rho^2) .
\]

(S11)

However, the dominant reduction in \( \alpha \) is due to sequences of interval lengths of the form

\[
(l, p_1, ..., p_m, l+1) ,
\]

(S12)

where the \( p_i = 1, ..., m \notin \{l-1, l, l+1\} \) are non-resonant with \( l \) or \( l+1 \). If \( m > 1 \), such configurations do not lead to strong hybridizations though, and thus they do not contribute significantly to the partial fast relaxation occurring before the long-time plateau. However, they increase the effective hopping by introducing a small denominator in perturbation theory. Such denominator is generically of order \( t^2 \) due to self-energies that arise in second order of perturbation theory. Those typically lift the degeneracy which is present at the classical level. (For further discussion of higher order degeneracies, see Ref. [20]). If two separated pairs of \( l, l+1 \) and \( l', l'+1 \) are interlaced, only one of them can be used to create a small denominator, however. The maximal number of resonances encountered in perturbation theory is likely obtained by retaining the shorter of the two pairs.
Let us thus compute the total number of resonant pairs of the form \[ (S12) \], which are not interlaced by shorter resonances. The probability of finding such a sequence formed by \( m + 2 \) intervals can be estimated to leading order as \( \rho \), multiplied by the probability that there are no resonant sequences of shorter length which interlace it. To compute such probability, we first impose that the interval of length \( l + 1 \) is not in resonance with the \( m \) intervals that follow it, yielding a factor \( (1 - \rho/2)^m \). Next we impose that the interval \( p_m \) is not in resonance with either \( l + 1 \), nor any of the subsequent \( m - 1 \) intervals, yielding another factor \( (1 - \rho/2)^m \). The preceding interval \( p_{m-1} \) can be in resonance with interval \( p_m \) (since such a resonance would be nested inside the considered one) but not with \( l + 1 \) or the following \( m - 2 \) intervals. This yields a factor \( (1 - \rho/2)^{m-1} \). We iterate this procedure up to interval \( p_1 \), and then square the resulting probability since the same conditions must be imposed also on the left of the sequence. This leads to

\[
N_{2\text{res}} \approx N \rho \left(1 - \frac{\rho}{2}\right)^{2m} \prod_{j=1}^{m} \left(1 - \frac{\rho}{2}\right)^{2j} = N \rho \left(1 - \frac{\rho}{2}\right)^{m^2+3m} \approx N \rho e^{-\frac{2}{\sqrt{\pi \rho}}} (m^2+3m).
\]  

(S13)

The corresponding reduction in the exponent \( \alpha \) can be estimated by summing the above over \( m \) and approximating the sum as an integral:

\[
(\Delta \alpha)_{2\text{res}} \simeq -2\rho \int_{1}^{\infty} dm e^{-\frac{2}{\sqrt{\pi \rho}}} (m^2+3m) = -\sqrt{2\pi \rho} + O(\rho),
\]

(S14)

where the factor of 2 is due to the fact that each resonance typically increases the effective hopping by a factor \( O(t^{-2}) \). This yields the dominant reduction of the tunneling exponent, \( \alpha = 1 - \sqrt{2\pi \rho} \), as quoted in the main text.

**Density of rare, nearly ergodic bubbles**

In a recent work \[18, 19\] it has been conjectured that in the thermodynamic limit delocalization might occur at any value of the hopping \( t \), due to non-perturbative rare events. We briefly reproduce the argument here. It starts from the observation that a random initial state will contain large, but very rare regions where the particle and energy density are so low that a bulk system with the same parameters would be delocalized and ergodic. One then diagonalizes the Hamiltonian within such a bubble (considering it decoupled from the outside) to obtain effectively ergodic internal states. Further, one estimates the matrix element to displace the bubble by one site, at second order in the coupling Hamiltonian within such a bubble (considering it decoupled from the outside) to obtain effectively ergodic internal states. This is suggested by considering the motion of rare hot and nearly ergodic bubbles which always exist in typical low temperature states. \[13\]. This is supported by considering the motion of rare hot and nearly ergodic bubbles which always exist in typical low temperature states. \[13\].

Here we do not aim to confirm or disprove this conjecture, but simply estimate the minimal bubble size that the argument would require. This will show that for small enough hopping \( t \), such bubbles are exceedingly rare, and are thus irrelevant for any realistic finite size system, even if in the thermodynamic limit there might be exponentially weak transport.

We consider an initial random state which includes an ”ergodic bubble” where the local energy density is below the critical threshold for localization (cf. Fig. 2). We assume the global density of particles \( \rho \) to be small, and \( t \) sufficiently smaller than the delocalization threshold \( t_c(\rho) \), as estimated for states with roughly homogeneous density distributions. Recalling that \( t_c \propto \rho^{\frac{3}{2}+1} \), the density \( \rho_B \) in the ”ergodic bubble” should be smaller than

\[
\frac{\rho_B}{\rho} < \left( \frac{t}{t_c} \right) \frac{1}{\rho^{\frac{3}{2}+1}}.
\]

(S15)
Calling $L_B$ the number of sites in the bubble, the dimension of the Hilbert space $\mathcal{H}_B$ of internal states with $\rho_B L_B$ particles is

$$\dim(\mathcal{H}_B) = \left( \frac{L_B}{\rho_B L_B} \right) \approx \exp[\rho_B (1 - \log(\rho_B)) L_B] \equiv \kappa^{L_B}, \quad \rho_B \ll 1.$$ \hfill (S16)

Since we assume $\rho_B < \rho$ to be very small, $\kappa$ is very close to one, such that the phase space of such bubbles grows slowly with their size. Consequently, very large regions are necessary to obtain small enough level spacings that might potentially induce delocalization of the bubble.

The minimal size $L_B$ is estimated from the hybridization between an initial bubble state $\psi_i$ and a final state $\psi_f$ in which the bubble has moved by one site. Delocalization may potentially occur if the admixture of $\psi_f$ to $\psi_i$ is large in second order in perturbation theory, i.e., if

$$\sum_{\psi_B} t^2 \langle \psi_f | O | \psi_B \rangle \langle \psi_B | O | \psi_i \rangle (E_i - E_B) (E_B - E_f) \gtrsim 1,$$ \hfill (S17)

where $| \psi_B \rangle$ runs over intermediate states, and $t \cdot O$ is the part of the hopping Hamiltonian that couples the bubble to the surrounding degrees of freedom. Let us first estimate the matrix elements of the hopping: making the generous assumption that the bubble is internally fully ergodic and that its eigenstates satisfy the Eigenstate Thermalization Hypothesis [34], matrix elements with a generic local operator can be argued to scale as

$$\langle \phi | O | \chi \rangle \sim \frac{1}{\sqrt{\dim(\mathcal{H}_B)}} \sim \kappa^{\frac{L_B}{2}},$$ \hfill (S18)

where $\phi, \chi$ label generic internal eigenstates. In order to minimize the energy denominators in (S17), one should optimize the intermediate and final states, which yields

$$\min \chi |E_\chi - E_\phi| \sim \frac{U}{\dim(\mathcal{H}_B)} \sim U \kappa^{-L_B}.$$ \hfill (S19)

Inserting these estimates into Eq. (S17), we obtain a condition on $L_B$:

$$L_B \gtrsim 2 \frac{\log(U/t)}{\log \kappa} \approx \frac{2}{\rho_B} \frac{\log(U/t)}{\log(1/\rho_B) + 1}.$$ \hfill (S20)

Note that the required length diverges logarithmically in the limit $t \to 0$, implying that these bubbles are non-perturbative in nature. In this aspect they bear some resemblance to rare regions in Griffiths’ phases.

The density $n_B$ of such large bubbles is given by the probability of finding only $\rho_B L_B$ particles in a region of length $L_B$, while the global density is $\rho$. For small $\rho$ and $t$ this is given by:

$$n_B \approx \left( \frac{L_B}{\rho_B L_B} \right) \rho^{\rho_B L_B} (1 - \rho)^{L_B (1 - \rho_B)} \approx \exp \left[ -L_B \left( \rho - \rho_B - \rho_B \log \frac{\rho}{\rho_B} \right) \right].$$ \hfill (S21)

In the regime $t \ll t_c$ (and thus $\rho_B \ll \rho$) this can be approximated as $n_B \approx \exp(-\rho L_B)$. Using the bound on $\rho_B$ from Eq. (S15) we find an upper bound on the density of ergodic bubbles,

$$n_B \lesssim \exp[-\rho L_B] \lesssim \exp \left\{ -2 \left( \frac{t_c}{t} \right)^{\frac{1}{2 \beta - 1}} \frac{\log \left( \frac{t_c}{t} \right)}{\log \left( \frac{t_c}{t} \right)^{2 \beta - 1} + 1} \right\},$$ \hfill (S22)

which is the expression of Eq. (13). This is exponentially small and non-perturbative in the limit $t \to 0$. For system sizes $L \ll 1/n_B$, such effects are irrelevant, since a typical realization will not contain any such bubbles.