Rational vertex operator algebras are finitely generated

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Abstract

It is proved that any vertex operator algebra for which the image of the Virasoro element in Zhu’s algebra is algebraic over complex numbers is finitely generated. In particular, any vertex operator algebra with a finite dimensional Zhu’s algebra is finitely generated. As a result, any rational vertex operator algebra is finitely generated.

Although many well known vertex operator algebras are finitely generated, but whether or not an arbitrary rational vertex operator algebra is finitely generated has been a basic problem in the theory of vertex operator algebra. In this paper we give a positive answer to this problem and our result justifies the assumption in the physics literature that any rational conformal field theory is finitely generated.

A systematic study of generators for an arbitrary vertex operator algebra was initiated in [L], [KL]. A vertex operator algebra $V$ is called $C_1$-cofinite if $V = V_0 + V_1 \cdots$ with $V_0$ being 1-dimensional and $V/C_1(V)$ is finite dimensional where $C_1(V)$ is a subspace of $V$ spanned by vectors $u_{-1}v, u_{-2}1$ for $u, v \in V^+ = \sum_{n>0} V_n$ and $u_n$ is the component operator of $Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$. It is proved in [L] that if a vertex operator algebra $V$ is $C_1$-cofinite then it is finitely generated. In fact, if $\{x_i, i \in I\}$ is a set of vectors of $V$ such that $x_i + C_1(V)$ for $i \in I$ form a basis of $V/C_1(V)$, then $V$ is generated by this set of vectors and $V$ has a PBW-like spanning set [KL].

Another important finiteness for a vertex operator algebra is the $C_2$-cofiniteness introduced by Zhu [Z] in the proof of modular invariance of the $q$-characters of irreducible modules for a rational vertex operator algebra. A vertex operator algebra $V$ is called $C_2$-cofinite if $V/C_2(V)$ is finite dimensional where $C_2(V)$ is a subspace of $V$ spanned by $u_{-2}v$ for $u, v \in V$. It is proved in [Z] that the span of the $q$-characters of irreducible modules for a rational, $C_2$-cofinite vertex operator algebra affords a representation of the modular group $SL(2, \mathbb{Z})$. So many results in the theory of vertex operator algebra using the modularity of the $q$-characters of the irreducible modules need both $C_2$-cofiniteness and rationality (see [DLM3], [DM1], [DM2], [DM3]). It is shown in [GN] that a $C_2$-cofinite vertex operator algebra $V$ is finitely generated with a better PBW-like spanning set. Again one can choose a set $X$ of homogeneous vectors of $V$ such that $x + C_2(V)$ for $x \in X$ form a basis of $V/C_2(V)$, then $V$ is spanned by

$$x_{-n_1}^1 \cdots x_{-n_k}^k 1$$

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where \( x^i \in X \) and \( n_1 > n_2 > \cdots > n_1 > 0 \). This result has been extended in [Bu] to give a spanning set for a weak module generated by one vector.

The connection between the generators for a vertex operator algebra and the \( C_1 \) or \( C_2 \)-cofiniteness is not surprising as both \( C_1 \)-cofinite and \( C_2 \)-cofinite properties are internal conditions on a vertex operator algebra. The mentioned results say that \( C_1(V) \) or \( C_2(V) \) can be generated by the other "basic vectors."

A vertex operator algebra is called rational if the admissible module category is semisimple [Z], [DLM2], also see below. It is clear that the rationality is an external condition. So the question is how to connect the rationality and generators for a vertex operator algebra. The bridge is the associative algebra \( A(V) \) introduced in [Z]. We use the fact that the algebra \( A(V) \) for rational vertex operator algebra \( V \) is a finite dimensional semisimple associative algebra in this paper to prove that any rational vertex operator algebra is finitely generated.

It turns out that a much weaker condition on \( V \) is good enough to guarantee that \( V \) is finitely generated. Let \( \omega \) be the Virasoro vector of \( V \). Then the image \([\omega]\) of \( \omega \) in \( A(V) \) is a central element. It is proved in this paper that if \([\omega]\) is algebraic over \( \mathbb{C} \) in \( A(V) \) then \( V \) is finitely generated. As corollaries, rational vertex operator algebras are finitely generated and their automorphism groups are algebraic groups [DG].

It is worthy to point out that although we can prove such vertex operator algebra is finitely generated, we do not know how to find a minimal set of generators for a rational vertex operator algebra.

We first review various notion of modules for a vertex operator algebra, following [FLM], [Z] and [DLM1].

Let \( V = (V, Y, 1, \omega) \) be a vertex operator algebra [B], [FLM]. A weak \( V \) module is a vector space \( M \) equipped with a linear map \( Y_M : V \rightarrow \text{End}(M)[[z, z^{-1}]] \)

\[
v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}
\]

and \( v_n \in \text{End}(M) \). In addition \( Y_M \) satisfies the following:
1) \( v_n w = 0 \) for \( n >> 0 \) where \( v \in V \) and \( w \in M \)
2) \( Y_M(1, z) = \text{Id}_M \)
3) The Jacobi Identity holds:

\[
z_0^{-1} \delta(\frac{z_1 - z_2}{z_0}) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta(\frac{z_2 - z_1}{-z_0}) Y_M(v, z_2) Y_M(u, z_1)
= z_2^{-1} \delta(\frac{z_1 - z_0}{z_2}) Y_M(Y(u, z_0)v, z_2).
\]

An admissible \( V \) module is a weak \( V \) module which carries a \( \mathbb{Z}_+ \) grading, \( M = \bigoplus_{n \in \mathbb{Z}_+} M(n) \), such that if \( v \) is homogeneousthen \( v_M M(n) \subseteq M(n + wt - m - 1) \). Since a uniform degree shift gives an isomorphic module we will assume \( M(0) \neq 0 \) if \( M \) is nonzero.
An ordinary $V$ module is a weak $V$ module which carries a $\mathbb{C}$ grading, $M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$, such that: 1) $\text{dim}(M_{\lambda}) < \infty$, 2) $M_{\lambda+n=0}$ for fixed $\lambda$ and $n << 0$, 3) $L(0)w = \lambda w = wt(w)w$, for $w \in M$ where $L(0)$ is the component operator of $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$.

It is easy to see that an ordinary module is admissible. One of the main results in [DLM2] and [Z] says that if $V$ is rational then there are only finitely many irreducible admissible modules up to isomorphism and each irreducible admissible module is ordinary.

A vertex operator algebra is called regular if every weak module is a direct sum of irreducible ordinary modules (see [DLM1]). It is clear that the regularity implies the rationality. One of the most important problems in the theory of vertex operator algebra is to understand the relations among rationality, regularity and $C_2$-cofiniteness. It is shown that any regular vertex operator algebra is $C_2$-cofinite and is finitely generated [L]. Conversely, a rational and $C_2$-cofinite vertex operator algebra is regular [ABD]. It is also known that a $C_2$-cofinite vertex operator algebra is not necessarily rational [A]. It is suspected that rationality implies the $C_2$-cofiniteness but there is no any progress in this direction so far.

For the purpose of the main result we also need to review the theory of associative algebra $A(V)$ from [Z].

For any homogeneous vectors $a \in V$, and $b \in V$, we define

$$a * b = \text{Res}_z \left( \frac{1+z}{z} \right)^{\text{wt}a} Y(a, z)b,$$

$$a \circ b = \text{Res}_z \left( \frac{1+z}{z} \right)^{\text{wt}a} Y(a, z)b,$$

and extend to $V \times V$ bilinearly. Denote by $O(V)$ the linear span of $a \circ b$ ($a, b \in V$) and set $A(V) = V/O(V)$. Set $[a] = a + O(V)$ for $a \in V$. Let $M$ be a weak $V$-module. For a homogeneous $a \in V$ we write $o(a)$ for the operator $a_{\text{wt}a-1}$ on $M$. Extend the notation $o(a)$ to all $a \in V$ linearly. From the definition of admissible module we see that $o(a)M(n) \subset M(n)$ for all $n \in \mathbb{Z}$ if $M$ is an admissible module.

The following theorem is due to [Z] (also see [DLM2]).

**Theorem 1** Let $V$ be a vertex operator algebra. Then

1. The bilinear operation $*$ induces $A(V)$ an associative algebra structure. The vector $[1]$ is the identity and $[\omega]$ is in the center of $A(V)$.
2. Let $M = \bigoplus_{n=0}^{\infty} M(n)$ be an admissible $V$-module with $M(0) \neq 0$. Then the linear map

$$o : V \to \text{End}M(0), \ a \mapsto o(a)|_{M(0)}$$

induces an algebra homomorphism from $A(V)$ to $\text{End}M(0)$. Thus $M(0)$ is a left $A(V)$-module.
3. The map $M \mapsto M(0)$ induces a bijection from the set of equivalence classes of irreducible admissible $V$-modules to the set of equivalence classes of irreducible $A(V)$-modules.
4. $[\omega]$ acts on any irreducible $A(V)$-module as a constant.
5. If $V$ is rational then $A(V)$ is a finite dimensional semisimple associative algebra.

A vector $u \in A(V)$ is called algebraic if there exits a nonzero polynomial $f(x) \in \mathbb{C}[x]$ such that $f(u) = 0$. Note that if $A(V)$ is finite dimensional then any $u \in A(V)$ is algebraic.
We now in a position to prove that main result of this paper.

**Theorem 2** If $[\omega] \in A(V)$ is algebraic then $V$ is finitely generated. In particular, if $A(V)$ is finite dimensional, $V$ is finitely generated. Furthermore, any rational vertex operator algebra is finitely generated.

**Proof:** Since $[\omega] \in A(V)$ is algebraic then $[\omega]$ satisfies

$$([\omega] - \lambda_1) \cdots ([\omega] - \lambda_k) = 0$$

for some complex numbers $\lambda_i$. That is, there exist homogeneous vectors $u^i, v^i \in V$ for $i = 1, \ldots, p$ such that

$$(\omega - \lambda_1) \ast \cdots \ast (\omega - \lambda_k) = \sum_i u^i \circ v^i.$$ 

Let $n$ be a positive integer such that $n > \lambda_j$ if $\lambda_j \in \mathbb{Z}$ and that $n$ is greater than or equal to the weights of $u^i, v^i$ for all $i$. We can also assume that $n \geq 2$. Let $U$ be a vertex operator subalgebra generated by $\sum_{m \leq n} V_m$. We claim that $V = U$.

If $U \neq V$, let $t$ be the minimal positive integer such that $V_t \neq U_t$. Then $V/U = \oplus_{m \geq t} V_m/U_m$ is a $U$-module where $U_m = V_m \cap U$. Note that $V_t/U_t$ is an $A(U)$-module and $[\omega]_{U} = \omega + O(U)$ acts on $V_t/U_t$ as $t$. On the other hand, $[\omega]_{U}$ is also algebraic in $A(U)$ and satisfies the same relation

$$([\omega]_{U} - \lambda_1) \cdots ([\omega]_{U} - \lambda_k) = 0.$$ 

That is, if $[\omega]_{U}$ acts on an $A(U)$-module as a constant, this constant must be one of $\lambda_j$. Since $t$ is different from $\lambda_j$ for all $j$ we have a contradiction. □

It is worthy to note that if $V$ is $C_2$-cofinite, then $A(V)$ is finite dimensional. In fact, it is easy to verify that if $x_i \in V$ ($i \in I$) be a set of homogeneous vectors such that $x_i + C_2(V)$ for $i \in I$ form a basis of $V/C_2(V)$ then $x_i + O(V)$ for $i \in I$ give a spanning set of $A(V)$. In particular, $\dim A(V) \leq \dim V/C_2(V)$. So an immediate corollary is that a $C_2$-cofinite vertex operator algebra is finitely generated.

We can strengthen Theorem 2 in the following way.

**Corollary 3** Let $V$ be a vertex operator algebra and $U$ a vertex operator subalgebra with the same Virasoro algebra. If $\omega + O(U)$ is algebraic in $A(U)$ then $V$ is finitely generated. In particular, any vertex operator algebra which has a rational vertex operator subalgebra is finitely generated.

**Proof:** It is clear that $O(U)$ is a subspace of $O(V)$. So the embedding from $U$ to $V$ induces an algebra homomorphism. So $[\omega]$ is also algebraic in $A(V)$. By Theorem 2 we have the result. □

Another application of Theorem 2 is on the automorphism group of a rational vertex operator algebra. Recall that an automorphism $g$ of a vertex operator algebra $V$ is a linear isomorphism from $V$ to $V$ such that $g1 = 1$, $gw = \omega$ and $gY(v, z)g^{-1} = Y(gv, z)$ for all $v \in V$. Let $\text{Aut}(V)$ denote the full automorphism group.
**Corollary 4** If $A(V)$ is finite dimensional then $\text{Aut}(V)$ is an algebraic group. In particular, the automorphism group of any rational vertex operator algebra is an algebraic group.

**Proof:** The corollary follows from Theorem 2 and a result in [DG] which says the automorphism group of any finitely generated vertex operator algebra is an algebraic group. □

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