Quantum Cryptography in Noisy Channels

Hoi-Kwong Lo* and H. F. Chau†

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540

(April 11, 2018)

Abstract

We provide a complete proof of the security of quantum cryptography against any eavesdropping attack including coherent measurements even in the presence of noise. Polarization-based cryptographic schemes are shown to be equivalent to EPR-based schemes. We also show that the performance of a noisy channel approaches that of a noiseless one as the error rate tends to zero. (i.e., the secrecy capacity $C_s(\epsilon) \to C_s(0)$ as $\epsilon \to 0$.) One implication of our results is that one can double the efficiency of a most well-known quantum cryptographic scheme proposed by Bennett and Brassard simply by assigning vastly different probabilities to the two conjugate bases.

PACS Numbers: 89.70.+c, 02.50.-r, 03.65.Bz, 89.80.+h

AMS 1991 Subject Classification: (Primary) 68P25, 81P15, 94A17, 94A24, 94A60

(Secondary) 68P20, 81V99

Keywords: channel capacity, coherent measurement, quantum cryptography, quantum information theory, secrecy capacity

*e-mail: hkl@sns.ias.edu

†e-mail: chau@sns.ias.edu
I. INTRODUCTION

Cryptography is the art of providing secure communication over insecure (i.e., subject to eavesdropping) communication channels. The security of a conventional cryptosystem often lies on a relatively short secret value known as the key that has to be agreed on by the two legitimate users before secure communication can be started. For this reason, secure key distribution is a crucial issue in cryptography. Unfortunately, classical cryptography provides no tools to guarantee the security of the key distribution because classical signals are vulnerable to passive interception: A passive wiretapper can simply make copies (clones) of the carrier of information and read off from those copies the value of the key. Since the original carrier of information can be resent to the legitimate user without alteration, there is no way for the two users to check whether the carrier has been intercepted.

In quantum mechanics, any measurement that does not disturb a complete set of non-orthogonal states also fails to yield any information distinguishing them \[1\]. (See Appendix for a proof.) In particular, it is impossible for the eavesdropper to clone non-orthogonal states. Therefore, coding based on non-orthogonal states can be used to detect any eavesdropping attempt \[2\]. The feasibility of secure quantum key distribution over long distance by optical fiber has been recently demonstrated: A prototype system at BT laboratories is capable of key transfer over 10 km in optical fiber at date rates of 20 kbit s\(^{-1}\) \[3\]. The investigation on the foundations of quantum cryptography is thus timely.

Noise is inevitable in any real communication channel. It is, therefore, crucial to demonstrate the security of quantum cryptography when the communication channel is noisy. Various eavesdropping strategies have been investigated in the literature \[4\]. In order to acquire any appreciable amount of information about the transmitted signals, they are all shown to introduce a substantial change in the error rate. Therefore, quantum cryptography is generally conjectured to be secure. Unfortunately, it has not yet been ruled out
that still more sophisticated use of quantum physics might defeat quantum cryptography. This is hardly a comforting situation: In the long history of classical cryptography, there were numerous instances of unexpected failures of cryptographic schemes (e.g., the knapsack scheme) that were once believed to be unbreakable. Such failures often led to dramatic and even disastrous consequences. To ensure that quantum cryptography does not follow the same trail, it is, therefore, essential for us to establish rigorously its absolute security. The first goal of this paper is to give such a proof.

The most general eavesdropping strategy available to an eavesdropper, traditionally called Eve, is for her to coherently manipulate all the transmitted particles by coupling them as a single entity with a probe (an ancilla). Eve may subsequently perform measurements on the ancilla to acquire information about the transmission. In Section II, we prove the security of quantum cryptography in a noisy channel by showing that it is unbreakable even by coherent manipulations performed by the eavesdropper.

After establishing the security of quantum cryptography, we come to the next question: how much information can be securely transmitted through a noisy quantum communication channel? Both eavesdropping and the intrinsic noise of the system introduce errors (including decoherence) in the channel and it is often difficult to distinguish between the two sources. Therefore, a conservative user may assume that all the errors are due to the wiretapping. Since wiretapping in a quantum channel necessarily leads to errors in the transmission, the legitimate users can put an upper bound to the extent of wiretapping by determining the error rate of the channel. Standard techniques such as error-correcting codes and privacy amplification can then be applied to the partly secret raw signals to distill a shorter but absolutely secure sequence of bits which can then be used as the key for subsequent classical

1Deutsch et al. [5] have suggested a purification scheme in which the two legitimate users perform coherent manipulations on the transmitted particles. Such a scheme is asymptotically unconditionally safe against any attack. Unfortunately, its efficiency is very limited.
communication. An insecure but unjammable (i.e., subject to wiretapping but not alteration of messages) classical channel can be used for the public discussion between the two users during the distillation process.

The maximal number of secure bits that can be distilled from each raw signal transmitted through a quantum channel is clearly a function of the error rate. It is defined as the secrecy capacity \[ E \] of the quantum channel. The second goal of this paper is to find out the properties of this function.

However, we face two problems in our investigation. The first difficulty is that, unlike classical information theory \[ 7 \] which is a mature field, a quantum theory of information is still being developed. For example, despite much past effort, a most basic problem in quantum information theory—the classical information-carrying capacity of non-orthogonal quantum signals—is still generally unsolved \[ 8 \]. Our inability to answer this basic question makes the issue of secrecy capacity even more intractable. The second problem is the fact that, even within the context of classical information theory, no simple expression has been found for the secrecy capacity in general \[ 6 \]. Only lower and upper bounds have been obtained. Going to the quantum regime will almost certainly not make things any easier. Despite these two difficulties, we shall see in this paper that much about the secrecy capacity can still be learned.

The organization of this paper is as follows. In Section II, we introduce a simple spherical symmetric EPR-based cryptographic scheme as a toy model and establish its security against eavesdropping even in the presence of noise. By deriving a lower bound to its security capacity, we demonstrate that as the error rate tends to zero, the performance of such a noisy quantum channel approaches that of a noiseless one. In Section III, we generalize our toy model results to more realistic cryptographic schemes. First, the assumption of spherical symmetry can be relaxed and any choice of two or more non-orthogonal measurement bases suffices to guarantee the security of an EPR-based scheme (provided that the error rate is sufficiently small). Second, we show that polarization based cryptographic schemes are conceptually equivalent to EPR-based schemes. Hence, the proof of the security of quantum
cryptography and the discussion about secrecy capacity for our toy model trivially carry over to polarization based schemes. One interesting implication of our results is that one can essentially double the efficiency of a most well known cryptographic scheme proposed by Bennett and Brassard simply by assigning vastly different probabilities to the two conjugate bases.

II. COHERENT MEASUREMENTS

In this Section, we shall establish the security of the following cryptographic scheme against any attack even in the presence of noise and investigate its secrecy capacity. Consider two legitimate users, traditionally called Alice and Bob who make use of $N$ EPR pairs to transmit secret messages. (Here $N$ is supposed to be large.) We assume that there is also another public, unjammable, classical communication channel between them. That is, anyone including the eavesdropper can listen to the signals without worrying about being detected. However, alterations of the signals are forbidden. The procedure goes as follows: Alice prepares $N$ EPR pairs and sends one member of each pair to Bob, keeping the other member. After receiving all $N$ transmitted particles, Bob publicly acknowledges his reception. For each particle that she has kept, Alice chooses a random axis independently to measure its spin. Afterwards, she publicly announces the axes that she has chosen for her measurements, but not the results. Bob then performs a measurement on the spin of the other member of the pair along the axis chosen by Alice. Ideally, the combined state of each pair should be a singlet. Thus, the measurement results of Alice and Bob should be antiparallel. Of course, errors are inevitable due to the presence of noise in the communication channel. Nonetheless, most of the spin measurements for the two members of the various pairs should remain antiparallel. (Of course, errors may also occur due to the measurement process itself. For example, a misalignment of the measurement bases used by Alice and Bob will lead to an increased error rate. However, in this paper we will not consider this type of errors.)
A more serious problem is the following: it is conceivable that some of the errors are
due to an eavesdropping attempt by an eavesdropper, traditionally called Eve. To estimate
the extent of eavesdropping, Alice and Bob may choose randomly a subset of \( m \) pairs and
declare their measurement results in public. By doing so, they can compute the error rate
for the \( m \) test pairs. If the error rate is found to be unreasonably large, they assume that
eavesdropping has occurred. Thus, they should reject the whole run and go through the
procedure again. Otherwise, they assume that no successful eavesdropping attempt has
been made. Now they share the remaining \( N - m \) bits which may well be corrupted by the
noise and Eve’s wiretapping attempt. So, the hope is that, at sufficiently small error rates,
Alice and Bob can use well-known schemes of error correction and privacy amplification to
distill out a shorter key of absolute security against Eve’s attack.

The question that we would like to answer is the following: For a noisy quantum communica-
tion channel, will Eve be able to obtain a large amount of the information shared between
Alice and Bob without exposing her eavesdropping attempt by coherently interacting the
\( N \) transmitted particles with an ancilla? This is in fact the most general eavesdropping
strategy.

Before going into the question of coherent manipulation, let us consider a single EPR pair.
The Hilbert space of an EPR pair is spanned by the singlet \( |\psi_0\rangle \) and the three other states
\( |\psi_1\rangle, |\psi_2\rangle, \) and \( |\psi_3\rangle \). Only the singlet state is guaranteed to give the desirable antiparallel
result for the measurement along any axis chosen by Alice and Bob. For a noisy channel,
the output will generally be a mixed state which may be described by a density matrix \( M \).
One can define the *fidelity* as \( F = \langle \psi_0 | M | \psi_0 \rangle \). It is the probability of the mixed state
for passing a test for being a singlet state. (Thus, \( 0 \leq F \leq 1 \)). Being so, it is invariant
under simultaneous rotations of the two particles. Given an ensemble of identical pairs each
described by \( M \), one can estimate its fidelity by the following process. For each pair, Alice
picks a random axis to measure the spin of a member of the pair. Bob performs a similar
measurement along the same axis on the other member. Notice that for a given mixed
state of an EPR pair described by \( M \) and a random axis of measurement chosen by Alice,
the probability that Alice and Bob's measurements give antiparallel results is \((1 + 2F)/3\). The physical reason is simple. If the two members of the \(i\)-th pair are measured along the \(z\)-axis and found to be antiparallel, a state in the subspace spanned by the singlet state and \((|10\rangle_z + |01\rangle_z)/\sqrt{2}\) is consistent with this result but those in orthogonal complement of this subspace are not. If they are measured along the \(x\)-axis instead, an antiparallel result will only be consistent with a state in the subspace spanned by the singlet and \((|10\rangle_x + |01\rangle_x)/\sqrt{2}\) but not with a state in its orthogonal complement. As we are considering a random axis, there is spherical symmetry. Since only one of the three non-singlet states will give an antiparallel result, we have 

\[ P(\text{antiparallel}) = F + [(1 - F)/3] = (1 + 2F)/3. \]

Intuitively, this means that when an EPR pair is in a non-singlet state, there is a probability of \(2/3\) of failing to give an antiparallel result. As discussed before, Alice and Bob estimate the error rate of the channel by publicly announcing the results of their measurements for \(m\) pairs. For a communication channel with a small error rate, it would therefore be unwise for Eve to cheat by substituting non-singlet EPR pairs into the communication channel. Any amount of substitution with the number of non-singlets higher than \(3/2\) of the original error rate of the channel is highly likely to lead to an abnormally high error rate in the \(m\) test bits and consequently detection by Alice and Bob. The curious fact is that, in what follows this simple observation will play a crucial role in our argument for the case of coherent manipulation.

**A. Security of our EPR Based Scheme**

To prove the security of the above EPR based scheme, first note that the most favorable scenario for an eavesdropper Eve would be to allow her to prepare the states for the \(N\) EPR pairs. Any (more realistic) situation will involve environmental noises and can be regarded as a special case in which Eve does not utilize the full control she has on the EPR states. The most general state that Eve can prepare is of the form

\[ \sum a_{i_1i_2\ldots i_N} |\psi_{i_1}\rangle |\psi_{i_2}\rangle \cdots |\psi_{i_N}\rangle |R_{i_1i_2\ldots i_N}\rangle, \]

(1)
where $|R_{i_1i_2...i_N}\rangle$ is the state of the ancilla. $|R_{i_1i_2...i_N}\rangle$ are normalized but need not be orthogonal to one another.

Suppose Eve is eavesdropping an ideal channel. Alice and Bob may draw $m$ pairs randomly out of the $N$ transmitted pairs and publicly compare their measurement results. They will regard the transmission of $N$ particles as untampered only if all the $m$ drawn pairs show antiparallel results in their measurements. The only way for Eve to guarantee this is to have all the $N$ pairs in the singlet state. Therefore, Eve must set all $a_{i_1i_2...i_N}$ to be zero except for one state (the tensor product of singlets). Thus, she will not be able to obtain any information about an ideal channel.

What about a noisy channel? Suppose, based on previous communication experience, Alice and Bob know that the actual channel error rate in the absence of eavesdropping is $\epsilon$. (Except for Subsection III D, we will only be interested in the regime $\epsilon \ll 1$ in this paper.) They again draw $m$ pairs randomly from the $N$ transmitted pairs. We assume $m \ll N$ but $m$ is still large enough for an accurate estimation of the error rate. In the limit $N \to \infty$, we let $m \to \infty$ but $m/N \to 0$. Alice and Bob may agree that the channel error rate is acceptable if and only if the number of errors found is in the region say $[(\epsilon - c\epsilon^2)m, (\epsilon + c\epsilon^2)m]$ where $c = O(1)$.

The key observation is that most basis vectors in Eq. (1) are highly unlikely to give an error rate in this region. Even if we are generous enough to extend the acceptable error range to $[0, (\epsilon + c\epsilon^2)m]$, our conclusion does not change. Consider a vector of the form $|\psi_{i_1}\rangle|\psi_{i_2}\rangle \cdots |\psi_{i_N}\rangle$ where $Na$ of the $i_j$’s (for $j = 1, 2, \cdots, N$) are nonzero (i.e., non-singlet). Since the measurement axes are chosen randomly for the $m$ test samples, such a state on average gives a parallel (i.e., incorrect) result for $2ma/3$ pairs which is much larger than the maximal tolerable number $(\epsilon + c\epsilon^2)m$ for say $a \geq 2\epsilon > \epsilon + c\epsilon^2$. Since we assume $\epsilon \ll 1$, most of the basis vectors in Eq. (1) contain far more than $2N\epsilon$ non-singlet states in a tensor product decomposition with respect to each particle and tend to give abnormally high error rates. Therefore, inspired by Shannon [10], we divide up the Hilbert space of the $N$ pairs into a ‘typical’ subspace and its orthogonal complement, an ‘atypical’ subspace. A typical
subspace is one whose states have exponentially small probabilities to give an acceptable error rate \[7\]. A vector in an atypical subspace may fare better. An example of a typical subspace \(\Lambda\) may be spanned by vectors of the form
\[
|\psi_{i_1}\rangle|\psi_{i_2}\rangle\ldots|\psi_{i_N}\rangle
\]
where the number of non-singlet \(i_j\)’s (i.e., \(i_j \neq 0\)) (here, \(j = 1, 2, \cdots N\)) are larger than or equal to \(2N\epsilon\). Its orthogonal complement is the atypical subspace. It is spanned by vectors of the form
\[
|\psi_{i_1}\rangle|\psi_{i_2}\rangle\ldots|\psi_{i_N}\rangle
\]
where the number of \(i_j\)’s (\(j = 1, 2, \cdots N\)) that are non-singlet (i.e., \(i_j \neq 0\)) are less than \(2N\epsilon\). Notice that, given a state, the number of \(i_j\)’s that are non-zero has an invariant meaning. We shall only consider simultaneous rotations of the two particles in each pair. Under arbitrary and independent rotations of all pairs, such a state transforms into a linear superposition of states with the same number of non-zero \(i_j\)’s.

Here comes another important observation: the atypical subspace has a small dimension (as compared to \(2^N\), the dimension which gives the \(N\) classical bit of information shared between Alice and Bob). To be more precise, one can give the following generous bound to the dimension of the atypical subspace

\[
\dim_{\text{atypical}} \leq \sum_{a,b,c=0}^{2N\epsilon-1} \binom{N}{a} \binom{N-a}{b} \binom{N-a-b}{c}
\]

\[
\leq (2N\epsilon)^3 \binom{N}{2N\epsilon} \binom{N}{2N\epsilon} \binom{N}{2N\epsilon}
\]

\[
\leq (2N\epsilon)^{32^{3NH(2\epsilon)}}
\]

\[
< 2^{N[6H(\epsilon)+\mu]}
\]

\[
< 2^{-Nk\epsilon \log_2 \epsilon},
\]

(2)

where \(k\) is a positive constant, \(\mu\) a small number of order \(\log N/N\), and \(H(x) = -[x \log_2 x + (1 - x) \log_2 (1 - x)]\) is the entropy function. Note that the inequality \[7\]

\[
\binom{N}{r} \leq 2^{NH(r/N)}
\]

(3)

and the concavity of \(H(x)\) have been used in the third and the fourth lines of Eq. (2) respectively. For our purposes, Eq. (2) is good enough because \(2^{-Nk\epsilon \log_2 \epsilon}\) is clearly exponentially
smaller than \(2^N\), the dimension that gives \(N\) bits of information. Nonetheless, we remark on passing that a much more refined bound could be found.

Suppose Eve prepare the state

\[
\sum_i a_i|\text{typical}_i\rangle|R_i\rangle + \sum_j b_j|\text{atypical}_j\rangle|R_j\rangle
\]

(4)

for the combined system of \(2N\) particles and ancilla. Alice and Bob will only accept a run of \(N\) pairs if \(m\) randomly chosen samples give a reasonable error rate. If we average over all the random axes, the probability of passing such a test

\[
P(\text{passing}) \leq \sum_i |a_i|^2 \exp \left(-f(m)\right) + \sum_j |b_j|^2,
\]

(5)

where \(f(m)\) the minimal exponential suppression factor for vectors in the typical subspace to pass such a test [7]. Notice that \(f(m) \to \infty\) as \(m \to \infty\). Thus, the contribution from the typical subspace is bounded above by \(\exp \left(-f(m)\right)\) which goes to 0 as \(m \to \infty\).

**B. Eve’s Dilemma**

Now the dilemma that Eve faces is clear. In order to have even just an exponentially small probability \(\exp \left(-f(m)/2\right)\) of passing the sample testing, the contribution from the atypical subspace must exponentially dominate that from the typical subspace. Without much loss of generality, one can assume that the whole typical space simply drops out whenever the testing of the \(m\) samples is passed. Therefore, effectively, the dimension of the Hilbert space is reduced to that of the atypical space. But the atypical subspace has a small number of dimension and is incapable of giving Eve much information.

In case the above discussion is still not transparent, in this paragraph, we show how this selection effect comes about in more detail. Let us specify the measurement axes for the \(m\) test samples. An outcome is the results (up or down) of the \(2m\) measurements made by Alice and Bob. Suppose the initial state of the combined ancilla-particles system is given by \(|u_0\rangle\). According to the conventional interpretation of quantum mechanics, if a measurement
gives a outcome $j$ (a state $|v_j\rangle$ for the $m$ test pairs), the state of the system will be projected onto $|j\rangle = (|v_j\rangle \langle v_j| \otimes I_{\text{other}}) |u_0\rangle$, where $I_{\text{other}}$ is the identity operator for the other degrees of freedom (i.e., the $N - m$ remaining pairs and the ancilla). The probability $p_j$ of this outcome $j$ occurring is given by $\sum_i |\langle v_j| \langle r_i| |u_0\rangle|^2$ where $\langle r_i|$ denotes the state of the other degrees of freedom and the sum is over a complete basis. Since Alice and Bob will reject all measurement results with abnormally high error rates, most outcomes $j$ will be rejected. Under the assumption that the $m$ samples pass the test, the state of the combined system after the test will be described by a density matrix

$$
\rho_c = \sum_j' \left( \frac{p_j}{\sum_i p_i} \right) |j\rangle\langle j|,
$$

where the sums are over those outcomes that pass the test. The crucial insight is, however, that all basis vectors are not created equal. As noted before, if we consider a tensor product state of $n$ non-singlets and $N - n$ singlets, under arbitrary and independent rotations of all pairs, it will transform into a linear superposition of states that are also made of $n$ non-singlets and $N - n$ singlets. (Here, particles in the same pair are only allowed to be rotated by the same amount because we are only interested in measurements that are done along the same axis on the two members of a pair.) The likelihood of a state in passing the test depends on $n$. Vectors in the typical space have a large $n$ and are exponentially unlikely (as a function of $m$) to pass the test while those in atypical space may fare better. Therefore, any realistic chance of passing the test is due to the atypical space (which consists of vectors of small $n$). This selection effect effectively eliminates the whole typical space from our consideration.

Moreover, the atypical space has a small dimension $2^{-N k \epsilon \log \epsilon}$ as given by Eq. (2). An upper bound on the amount of information that Eve can acquire by measuring the ancilla is given by the Holevo’s theorem [11]:

$$
I_{\text{max}}^{\text{eve}} = S(\rho_R) = -\text{Tr} \rho_R \log \rho_R,
$$

where
\[ \rho_R = \text{Tr}_{\text{particles}} \rho_c = \text{Tr}_{\text{particles}} \sum_j' (p_j / \sum_l' p_l) |j\rangle \langle j|, \]  

is the reduced density matrix for the ancilla given that the \( m \) samples pass the test. We obtain an upper bound

\[ I_{\text{max}}^{\text{eve}} \leq N[-k\epsilon \log \epsilon + \theta], \]  

where \( \theta \) is a small correction term coming from the typical space. Asymptotically, \( \theta \) can be made as small as one is pleased by taking \( m \to \infty \).

Eve may well have some a priori information about the measurements. The point is that the probability of getting an “up” in Alice’s (or Bob’s) measurement may well depend on the orientation of the axis chosen. The probability that the spin measurements by Alice and Bob are antiparallel can also have such an orientation dependence. Thus, the mutual information shared by Alice and Bob may actually be smaller than \( N \) bits. Nevertheless, any correction term must be of the order \( -N\epsilon \log \epsilon \). For sufficiently small error rate \( \epsilon \), the secrecy capacity \( C_s \) of the channel (per EPR pair) therefore satisfies

\[ C_s > [1 - k'\epsilon \log \epsilon], \]  

where \( k' \) is some constant. Notice that as the fidelity \( F \to 1 \), \( \epsilon \to 0 \) and \( C_s \to 1 \). Therefore, an arbitrarily small error rate implies a secrecy capacity arbitrarily close to the ideal channel capacity (which is one bit per EPR pair). Notice that Eve can still mess up the results of say \( O(\log N) \) pairs without worrying about being detected because the portion of pairs tested \( m/N \to 0 \). However, this has no effect on the secrecy capacity. What we have shown is that any attempt to obtain \( O(N) \) bits of information will be almost surely detected.

What is the principle underlying the security of an EPR-based cryptographic scheme? Ekert [12] suggested that it comes Bell’s theorem. However, Bennett, Brassard and Mermin [1] later proved the security of EPR-based schemes without invoking the Bell’s theorem. Nevertheless, both works only addressed noiseless channels. Here, we would like to propose an alternative viewpoint which remains useful even for noisy channels. From an information-theoretic point of view, the security of an EPR-based quantum cryptographic scheme can
be traced back to the observation that in quantum physics, knowing completely the state of a composite system does not guarantee complete knowledge of the states of the individual constituents because of the presence of entanglement entropy. For instance, the entropy of each member of a perfect EPR pair is non-zero even though the total entropy of the pair is zero. Consequently, two observers are able to use an EPR pair to transmit a random but secret bit of classical information. Heuristically, in the presence of noise, we expect that transmission of secret information is still possible as long as the “entanglement entropy” remains larger than the entropy of the composite system.

III. GENERALIZATIONS

For simplicity, we have discussed only the case in which Alice chooses the bases for her measurement randomly. Some generalizations of the above result are possible. A moment of thought will convince the reader that a similar proof can be formulated to the case when say only two non-orthogonal bases are used.

Let us consider another modification of the procedure. What if Alice performs the measurement on the particles in her share before sending Bob the other member of the EPR pair? Since the operators Alice uses in the measuring process only act on the particles in her share, they must commute with any operators (which may be used by Eve and Bob) that act on the particles in Bob’s control. It is, thus, immaterial whether Alice performs the measurement first and sends out the rest second or the other way round provided Alice announces her basis only after Bob’s public acknowledgment of his reception of all $N$ particles in his share. Since all particles are already in Bob’s hand, it is too late for Eve to do anything. Notice that if Alice announces her basis too early (say she announces her basis each time Bob acknowledges his reception of one particle), this argument does not preclude an intelligent

\(^2\)Incidentally, a related idea is used in the context of quantum computing in a recent preprint by Griffiths and Niu.
eavesdropper from obtaining a substantial amount of information about Alice and Bob’s measurement results.

A. Polarization Based Schemes

Remarkably, the above simple observation—that it is immaterial whether Alice measures the spins first and sends out the other particles second or vice versa—has deep consequences. So far our discussion has been concentrated on EPR based cryptographic schemes. However, another class of schemes that are based on the polarization of photons has been discussed in the literature. For instance, in 1984 Bennett and Brassard [14] proposed a scheme (BB84) in which the key distribution between Alice and Bob is done by sending photons over optical fiber. To detect eavesdropping, Alice chooses randomly with equal probability between the rectilinear basis (i.e., horizontal and vertical) and the diagonal basis (45° and 135°). A horizontally polarized photon can represent a 0 and vertical a 1. Similarly, a 45° polarized photon can represent a 0 and 135° a 1. 0’s and 1’s are chosen with equal probability. Alice then transmits a photon in the basis of her choice. Similarly, Bob performs a measurement along the rectilinear basis and the diagonal basis with equal probability. Afterwards, both Alice and Bob publicly announce the bases that they have chosen, but not the results of their measurements. Their bases will therefore agree with each other only half of the time. As in the case of EPR based schemes, they can then choose a subset of those measurements that are done in the same bases and compare the results in public. From the error rate of the m test samples, they can estimate the error rate for the whole run and hence the degree of eavesdropping. They can then decide whether to accept the run or to reject the run and do it again.

As argued by Bennett, Brassard and Mermin [1], the two classes of schemes (EPR based and polarization based) are conceptually equivalent. The point is that Alice could have prepared each photon by producing an EPR pair of photons and measuring one member along a random axes (rectilinear or diagonal), letting the other particle, now in a known
random one of the four states, pass to Bob. We remark that this argument remains valid even for a noisy channel.

There is still, however, one minor difference. So far we have assumed that, after the transmission of the $N$ EPR pairs, Alice informs Bob of her basis of measurement for her particle in each pair and Bob is supposed to measure the spin of the corresponding partner along the same basis. It is of course experimentally difficult for Bob to store up a large number of photons to wait for Alice’s announcement of her bases. Allowing Alice and Bob to choose between the two bases (rectilinear and diagonal) independently as in the BB84 scheme is definitely more realistic. Is this going to affect our conclusions?

The answer is no. Recall that all we need in our proof is to use a small subset of the EPR pair to estimate the error rate when the axes Alice and Bob used do agree. Whether the axes for other measurements agree or not is irrelevant. Our proof of security of EPR based cryptography, therefore, automatically implies the security of polarization based schemes even if Alice and Bob choose their measurement axes independently.

There are two alternative points of view regarding the underlying principles governing the security of quantum cryptography. The first and more well publicized point of view, which has been discussed in Section I, is that measurements performed on non-orthogonal states in quantum mechanics generally lead to disturbance. For a noiseless channel, it leads to the generalized “no-cloning” theorem (see the Appendix). In our opinion, the trade-off between information gain and disturbance by an eavesdropper in a noisy channel remains to be studied in more detail. The second point of view, which has been discussed in the last paragraph of Section II, is that the security of quantum cryptography lies on the possibility in quantum mechanics of the “entanglement entropy” between two subsystems being larger than the entropy of the whole system. The equivalence between EPR and polarization based schemes suggests that these two alternative points of view are in fact equivalent.
B. Doubling Efficiency in BB84

In the BB84 scheme, Alice and Bob choose their measurement axes from two conjugate bases (rectilinear and diagonal) independently and with equal probability. A drawback of such a scheme is the reduction of the ideal secrecy capacity to half of the optimal value (i.e., 1/2 bit per pair vs 1 bit per pair because only half of time will the two independently chosen bases by Alice and Bob agree). However, we would like to remark that the restriction of equal probability in choosing the two bases is totally redundant. Conceptually, the probability of choosing the rectilinear basis can be made much larger than the probability for the diagonal basis. At small error rates, this will lead to a secrecy capacity which is almost double of the original BB84 scheme.

More explicitly, given a noisy channel with error rate $\epsilon \ll 1$. Suppose they choose the rectilinear and diagonal bases with probabilities $1 - \omega$ and $\omega$ respectively (where $\omega \ll 1$). For the transmission of $N$ photons, there are on average $N\omega^2$ photons for which both Alice and Bob measure along the diagonal basis. They can for example publicly compare their measurement results for those $N\omega^2$ photons. In addition, they also randomly choose $N\omega^2$ photons from the set for which they both measure along the rectilinear axis. They can decide that the error rate is acceptable if and only if it is less than $2\epsilon$. Now given any $\omega \ll 1$, there exists an $N_0$ such that, for the transmission of $N > N_0$ photons, any eavesdropping attempt to get more than $O(-N\epsilon \log \epsilon)$ bits of information about the state of the transmitted particles will almost surely be detected. Thus, this scheme with different probabilities for the two bases is clearly secure. Furthermore, it has the benefit that, in the limit $\omega \rightarrow 0$, we obtain essentially double of the efficiency of the scheme proposed by Bennett and Brassard. It is of practical interest to investigate whether this observation will lead to the design of more efficient protocols for say quantum oblivious transfer and quantum bit commitment.
C. Other Generalizations

Of course, in practical applications, the quantum signals used in BB84 are low-intensity light pulses rather than ideal single photon pulses. In that case, we must consider the possibility of beamsplitting attack. We shall, however, pursue this problem no further in this paper.

There are other protocols of polarization based cryptographic schemes. For instance, two rather than four non-orthogonal quantum states are used in a scheme proposed by Bennett [16]. We believe that the techniques developed in this paper can be used to prove the security of this kind of schemes as well.

D. Properties of Secrecy Capacity

Let us return to the subject of secrecy capacity. So far, we have only derived a lower bound to the secrecy capacity of a quantum communication channel. Can we derive an upper bound? At first sight, the answer is a simple yes: One can just choose an eavesdropping strategy and compute the amount of information acquired through it. On second thought, it is not so simple. Given an eavesdropping strategy, Alice and Bob may counteract by changing their procedure. Instead of measuring the state of each carrier of quantum signal and perform classical error correction and privacy amplification as assumed before, they may manipulate the state of a number of the particles coherently [5]. It is not entirely inconceivable that such quantum processing of signals might give the users more information than any classical methods. This subject deserves further investigations.

Baring coherent manipulation by the users, one can show that the secrecy capacity is a convex function of the error rate. In other words, \( C_s(ax + by) \leq aC_s(x) + bC_s(y) \) for all \( a, b \leq 1 \) such that \( a + b = 1 \). The idea is the following: given strategies \( S_x \) and \( S_y \) that correspond to the preparation of the ancilla-particles states \( |u_x\rangle \) and \( |u_y\rangle \) with error rates \( x \) and \( y \) respectively, Eve can construct the tensor product state \( |u_x\rangle \otimes |u_x\rangle \otimes \cdots \otimes |u_x\rangle \otimes \cdots \)
\(|u_y\rangle \otimes |u_y\rangle \otimes \cdots \otimes |u_y\rangle\) (with arbitrary numbers of \(|u_x\rangle\)’s and \(|u_y\rangle\)’s and permutations of the particles involved if desired) to give an error rate \(ax + by\). If the legitimate users knew about the decomposition of the channel, the secrecy capacity they could achieve would be 
\(aC_s(x) + bC_s(y)\). Their ignorance certainly makes things worse. Hence, 
\(C_s(ax + by) \leq aC_s(x) + bC_s(y)\).

Another interesting question is the following: For the EPR based scheme discussed in section II, what is the minimal value of \(\epsilon\) (call it \(\epsilon_{\text{min}}\)) such that 
\(C_s(\epsilon) = 0\)? That is to say the channel is too noisy to be of any use. Clearly, 
\(C_s(1/2) = 0\) (because \(P(\text{parallel}) = P(\text{antiparallel}) = 1/2\)) and thus \(\epsilon_{\text{min}} \leq 1/2\). The value of \(\epsilon_{\text{min}}\) is an interesting open question.

The EPR scheme introduced in Section II is designed to be spherical symmetric so that the problem can be characterized by just one parameter, namely the fidelity or the error rate. This is why the secrecy capacity is a function of one variable. In a more general setting, more than one parameters may be needed for the characterization of the noise level of a quantum communication channel. Consequently, the secrecy capacity will be a function of multiple variables. As far as the legitimate users are concerned, the output of a communication channel is related to the input by a superscattering matrix. The goal of the users is to choose their inputs so as to maximize the information of the output and minimize the information leakage to the environment at the same time.

E. Conclusions

We have proved that an EPR based quantum cryptographic scheme is secure against coherent measurements by eavesdroppers. Our proof relies on the law of large number. The dimension of the space of states that are consistent with a small rate is exponentially smaller than the dimension of the whole Hilbert space, Thus, by testing the error rate for a small subset of signals, one can effectively eliminate most dimensions. Consequently, an eavesdropper is unable to get much information. Moreover, we prove that a polarization based cryptographic scheme is conceptually equivalent to an EPR based scheme. Our proof
of the security of quantum cryptography therefore carries over to the former. The secrecy capacity of a quantum channel is also investigated.

On a conceptual level, our work suggests that the two alternative points of view (namely (1) the “no-cloning” theorem for non-orthogonal quantum states and (2) that the entanglement entropy between subsystems being larger than the entropy of the whole system) concerning the principles underlying the security of quantum cryptography are in fact equivalent. One practical implication of our results is that one can double the efficiency of the cryptographic scheme proposed by Bennett and Brassard [14] (BB84) simply by assigning vastly different probabilities to the two conjugate bases. Finally, we remark that the beamsplitting attack remains to be addressed in future investigations.

IV. ACKNOWLEDGMENT

H.-K. L. thanks J. Preskill for introducing him to the subject of quantum computation and R. Josza for providing references. We also thank F. Wilczek for critical comments. This work is supported by DOE grant DE-FG02-90ER40542.

APPENDIX: GENERALIZED QUANTUM “NO-CLONING” THEOREM

Suppose we are given a particle that can be in either one of the two non-orthogonal states, $|u_1\rangle$ or $|u_2\rangle$ of a two-dimensional Hilbert space. Here, we prove that it is impossible to obtain information distinguishing between the two possibilities without perturbing its state. A simple proof goes as follows. An eavesdropper may generally couple an ancilla in the state $|\Psi\rangle$ to the particle and evolve the combined system. To avoid detection, the final state of the signal has to remain unchanged. Now suppose $U(|u_i\rangle|\Psi\rangle) = |u_i\rangle|\Phi_i\rangle$. Since $U$ is unitary,

$$\langle u_1|u_2\rangle = \langle \Psi|\langle u_1|u_2\rangle|\Psi\rangle = \langle u_1|u_2\rangle\langle \Phi_1|\Phi_2\rangle. \quad (A1)$$
Since $\langle u_1 | u_2 \rangle \neq 0$, it follows that $\langle \Phi_1 | \Phi_2 \rangle = 1$. Thus, $|\Phi_1 \rangle = |\Phi_2 \rangle$ and no information can be obtained.
REFERENCES

[1] See for example, C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. 68, 557 (1992).

[2] This idea was first proposed by Wiesner around 1970 but the paper went unpublished until 83: S. Wiesner, SIGART News 15, 78 (1983). For an introduction, see C. H. Bennett, G. Brassard and A. K. Ekert, Sci. Am. (Oct. 1992), 50.

[3] See for example, J. D. Franson and H. Ilves, Appl. Opt. 33, 2949 (1994).

[4] C. H. Bennett et al., J. of Cryptography 5, 3 (1992); M. J. Werner and G. J. Milburn, Phys. Rev. A47, 639 (1993); S. M. Barnett and S. J. D. Phoenix, Phys. Rev. A48, 5 (1993); A. K. Ekert et al., Phys. Rev. A50, 1047 (1994).

[5] C. H. Bennett et al., “Purification of Noisy Entanglement, and Faithful Teleportation via Noisy Channels.” (unpublished manuscript); D. Deutsch et al., “Entanglement-based quantum cryptography is unconditionally secure.” (unpublished manuscript).

[6] For classical channels, see for example, U. M. Maurer, IEEE Trans. Inform. Theory 39, 733 (1993); R. Ahlswede and Z. Zhang, IEEE Trans. Inform. Theory 41, 1040 (1995). For quantum channels, see A. K. Ekert et al., Phys. Rev. D50, 1047 (1994).

[7] See for example, T. M. Cover and J. A. Thomas, Elements of Information Theory (Wiley, New York, 1991) p. 151.

[8] C. A. Fuchs and C. M. Caves, Phys. Rev. Lett. 73, 3047 (1994) and references cited therein.

[9] B. Schumacher, Phys. Rev. A51, 2738 (1995); R. Jozsa and B. Schumacher, J. Mod. Opt. 41, 2343 (1994).

[10] See for example, C. E. Shannon and W. W. Weaver, The Mathematical Theory of Communication (University of Illinois Press, Urbana, IL, 1949).
[11] A. S. Holevo, Probl. Inform. Transm. 9, 177 (1973).

[12] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).

[13] R. B. Griffiths and C.-S. Niu, Los Alamos preprint quant-ph:9511007 (1995).

[14] C. H. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India (IEEE, 1984), p. 175.

[15] C. H. Bennett et al., in Advances in Crytology: Proceedings of Crypto ’91, Lecture Notes in Computer Science 576, 351 (Springer-Verlag, 1992); G. Brassard and C. Crépeau, in Advances in Crytology: Proceedings of Crypto ’90, Lecture Notes in Computer Science 537, 49 (Springer-Verlag, 1991).

[16] C. H. Bennett, Phys. Rev. Lett. 68, 3121 (1992).