Some functorial properties of Schatten classes

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Abstract
In this paper, we begin with the study of elements in $C^*$-algebras which are mapped to Schatten class ideals through the faithful left regular representation. We further give some functorial properties of Schatten classes on the category of representations of a $C^*$-algebra and category of unitary representations of a group.

Keywords Schatten classes · Representations · Categories · Functors

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1 Introduction

We aim to explore how Schatten classes on Hilbert spaces influence the representations of $C^*$-algebras or locally compact Hausdorff groups. We begin our study in the second section with the left regular representation of $L^\infty(X)$ on $L^2(X)$, where $(X, \mathcal{B}(X), \mu)$ is a $\sigma$-finite measure space. We give a few conditions on measure properties of $X$ such that the pullback of $p^{th}$-Schatten ideal is not trivial (zero). This establishes the fact that these pullbacks are not trivially zero. Then in the third section, we give a nice functor on category of representations of a $C^*$-algebra and category of representations of a group which is associated to Schatten classes. Let us begin with a few preliminaries.
1.1 Measure space

Let $(X, \mu)$ be a measure space. A set $A \subset X$ is said to be atom of $\mu$ if $\mu(A) > 0$ and for every $B \subset A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Subsets of $X$ which are not atoms are said to be diffuse. A measure is said to be atomic if there exists at least one atom, else the measure is said to be without atom.

1.2 Schatten classes on Hilbert spaces

Let $H$ be Hilbert space and $T \in B(H)$. For $p \in [1, \infty)$ define Schatten p-norm of $T$ as

$$||T||_p = \text{Tr}(|T|^p)$$

The $p^{th}$ Schatten class operators are those $T \in B(H)$ for which $||T||_p < \infty$. Let $S_p(H)$ denotes the collection of all p-th Schatten class operators in $B(H)$. Then $S_p(H)$ is an ideal of $B(H)$ and $|| \cdot ||_p$ is a norm on $S_p(H)$ which makes it a Banach $*$-algebra (involutions are adjoint maps). Further, we have the containment $S_p(H) \subset S_q(H)$ for each $q > p$. The Schatten p-norm satisfies $|| \cdot ||_q \leq || \cdot ||_p$ for all $q \geq p$. Operators in $S_p(H)$ are compact, and $S_p(H)$ contains all finite rank operators. $S_p(H)$ is reflexive for $1 < p < \infty$ since they are uniformly convex. We also know that $S_p(H)^* = S_q(H)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The ideal $S_2(H)$ is a Hilbert space with inner product $\langle A, B \rangle = \text{Tr}(B^*A)$. A detailed discussion about these facts can be found in [7]. We will use the notation $S_\infty(H) = B_0(H)$ for the algebra of compact operators.

1.3 Representations of $C^*$ algebras and locally compact groups.

A representation of a $C^*$-algebra $A$ is a continuous $*$-homomorphism $A \rightarrow B(H)$ for some Hilbert space $H$. Injective representations are called faithful representations. Gelfand Naimark theorem states that every $C^*$-algebra possess a faithful representation on some Hilbert space [6, Ch.3].

A unitary representation of a locally compact Hausdorff group $G$ is a homomorphism $G \rightarrow U(H)$ for some Hilbert space $H$, continuous with respect to strong topology on $U(H)$, where $U(H)$ denotes the group of unitary operators on $H$ [4, Ch.3].

1.4 Categories

By $\text{Ban}_1$, we denote the category whose objects are Banach spaces and morphisms are contractive linear maps. $\text{Rep}(\mathcal{A})$ denotes the category whose objects are representations of $C^*$-algebra $\mathcal{A}$ and natural transformations generated by intertwining operators define morphisms, i.e. $\phi : \pi_1 \rightarrow \pi_2$ is a morphism between two representations $\pi_1 : \mathcal{A} \rightarrow B(H_1)$ and $\pi_2 : \mathcal{A} \rightarrow B(H_2)$, then there exists an
operator \( U : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) such that for each \( x \in \mathcal{A} \), the diagram in the following figure commutes.

\[
\begin{array}{ccc}
\mathcal{H}_1 & \xrightarrow{\pi_1(x)} & \mathcal{H}_1 \\
\downarrow U & & \downarrow U \\
\mathcal{H}_2 & \xrightarrow{\pi_2(x)} & \mathcal{H}_2
\end{array}
\]

Similarly, we denote the category of unitary representations of a group \( G \) by \( \text{Rep}(G) \) where natural transformations generated by intertwining operators define morphisms.

An assignment \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \), where \( \mathcal{C}, \mathcal{D} \) are any two categories, is said to be a functor if it assigns to each object \( C \in \mathcal{C} \) a unique object \( \mathcal{F}(C) \in \mathcal{D} \) and to each morphism \( f : C_1 \rightarrow C_2 \) in \( \mathcal{C} \) a unique morphism \( \mathcal{F}(f) : \mathcal{F}(C_1) \rightarrow \mathcal{F}(C_2) \) in \( \mathcal{D} \) such that \( \mathcal{F}(\text{Id}_C) = \text{Id}_{\mathcal{F}(C)} \) and \( \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f) \) for any pair of morphisms \( f, g \) in \( \mathcal{C} \) (given that they can be composed).

## 2 Schatten type functions in \( L^\infty(X) \)

Let \( X \) be a measure space with \( \sigma \)-finite Borel measure \( \mu \). Consider the faithful representation of \( C^* \)-algebra \( L^\infty(X) \) by

\[
\pi : L^\infty(X) \rightarrow B(L^2(X))
\]
such that \( \pi(f)g = f \cdot g \) for each \( f \in L^\infty(X) \) and \( g \in L^2(X) \). For each \( 1 \leq p \leq \infty \), we define the set

\[
S_p(X) = \pi^{-1}(S_p(L^2(X))),
\]

which is a collection of all the functions in \( L^\infty(X) \) which are mapped to Schatten \( p \)-class operators on Hilbert space \( L^2(X) \). Clearly, \( S_p(X) \) is an ideal in \( L^\infty(X) \).

**Lemma 1** For group \( \mathbb{Z} \) with Haar measure (counting measure), \( S_p(\mathbb{Z}) = \ell^p(\mathbb{Z}) \) for all \( 1 \leq p \leq \infty \).

**Proof** Consider the standard orthonormal basis \( \{e_i\}_{i \in \mathbb{Z}} \) of \( \ell^2(\mathbb{Z}) \). Let \( f \in \ell^\infty \). Then \( f \in S_p(\mathbb{Z}) \) if and only if

\[
||\pi(f)||_p = \sum_{n \in \mathbb{Z}} |f(n)|^p e_n, e_n > = \sum_{n \in \mathbb{Z}} |f(n)|^p < \infty
\]

Hence \( S_p(\mathbb{Z}) = \ell^p(\mathbb{Z}) \) for all \( 1 \leq p < \infty \). Case \( p = \infty \) is trivial. \( \square \)

**Remark 1** The collection \( S_p(X) \) is not independent of the faithful representation. For example, consider the faithful representation \( \pi \otimes 1 : \ell^2(\mathbb{Z}) \rightarrow B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})) \). In this case, none of the sequences is mapped to trace class operator.
Lemma 2  For group $\mathbb{R}$ with Haar measure, $S_p(\mathbb{R}) = \{0\}$ for all $1 \leq p < \infty$ and $S_\infty(\mathbb{R}) = L^\infty(\mathbb{R})$. Even for compact group $\mathbb{S}^1$ with Haar measure, $S_p(\mathbb{S}^1) = \{0\}$ for $1 \leq p < \infty$ and $S_\infty(\mathbb{S}^1) = L^\infty(\mathbb{S}^1)$.

Proof Consider the Gabor orthonormal basis $\{e_{n,m}\}_{n,m}^\infty$ of $L^2(\mathbb{R})$, where $e_{n,m}(x) = e^{2\pi i mx} \chi_{[n,n+1]}$. Let $f \in L^\infty(\mathbb{R})$ and $1 \leq p < \infty$. Then $f \in S_p(\mathbb{R})$ if and only if

$$||\pi(f)||_p = \sum_{n,m \in \mathbb{Z}} \int_{\mathbb{R}} |f|^p e_{n,m} \cdot \overline{e_{n,m}} d\mu = \sum_{n,m} \int_{\mathbb{R}} |f|^p d\mu < \infty.$$  

This is unconditional sum of positive numbers which converges if and only if $\int_{\mathbb{R}} |f|^p d\mu = 0$ and hence if and only if $f = 0$.

Similarly, using the orthonormal basis $\{e^{2\pi imx}\}_{n \in \mathbb{Z}}$ for $L^2(\mathbb{S}^1)$, one can prove that $f \in S_p(\mathbb{S}^1)$ if and only if $f = 0$. The $p = \infty$ situation is easily seen to be true in both cases. \hfill $\square$

Remark 2 The above examples may lead us to the hypothesis that $S_p(X)$ is non-trivial if and only if $X$ is discrete, which is not true as proved below.

Theorem 1 For a measure space $X$ with a $\sigma$-finite Borel measure $\mu$, the ideal $S_p(X)$ for $1 \leq p < \infty$ is trivial(zero) if and only $\mu$ is without atom.\footnote{This proof is motivated by the idea provided by Prof. Martin Argerami.}

Proof Suppose $(X, \mu)$ is atomic. Then we have $X = F \cup (\bigcup_{x \in D} \{x\})$, where $F$ is the diffuse set(without atoms) and $D$ is the set of all atoms (atoms in regular Borel measure are always almost singleton and countable [3, 2.2]). Let $f = \sum_{x \in D} \beta_x \chi_x$ such that $\sum_{x \in D} |\beta_x|^p < \infty$. Notice that the set $\left\{ \frac{1}{\mu(x)^{1/2}} \chi_x \right\}$ is orthonormal . Hence,

$$||\pi(f)||_p = \sum_{x \in D} |\beta_x|^p < \infty.$$  

It follows that $f \in S_p(X)$ and

$$S_p(X) = \{ f \in L^\infty(X) : \text{supp}(f) \subset D \text{ and } \sum_{x \in D} |f(x)|^p < \infty \}$$  

(1)

for all $1 \leq p < \infty$. Conversely, suppose $S_p(X)$ is non-empty for all $1 \leq p < \infty$. Let $f \in S_p(X)$ for a fixed $p$. Then $\pi(f)$ is compact(all Schatten class operators are compact) and $\sigma(\pi(f)) = \mathcal{F}(\mathbb{X})$ must be finite or a set with limit point 0. So $f$ must be a simple function, $f = \sum_j \gamma_j \chi_{E_j}$. Here projection $\pi(\chi_{E_j})$ must be finite rank except when $\chi_j = 0$. Now, if the set $D$ of atoms is empty then $X = F$. Hence there exists some $E_j$ which has positive measure and can be decomposed into disjoint union of infinite subsets. This forces that $\pi(\chi_{E_j})$ is not finite rank. Thus, $D$ must be non-empty. \hfill $\square$
Remark 3 Notice that $S_p(\mathbb{R})$ and $S_p(\mathbb{S}^1)$ were empty for $1 \leq p < \infty$ because the Haar measure on $\mathbb{R}$ and $\mathbb{S}^1$ are non-atomic. $S_p(\mathbb{Z})$ was non-empty because $\mathbb{Z}$ with counting measure is atomic (singletons are atoms). There are in fact compact non-discrete groups which are atomic. For example, consider the infinite product $\Pi_f G$ with counting measure, where $G$ is a finite group with discrete topology. Clearly, $\Pi_f G$ is a non-discrete compact group which is atomic (singletons are the atoms, and diffuse is empty). For groups with left/right invariant measures, even stronger result holds.

Theorem 2 Let $G$ be a group equipped with left/right invariant $\sigma$-finite Borel measure $\mu$. Then $S_p(G)$ for $1 \leq p < \infty$ is non-empty if and only if $\mu$ is counting measure (up to product with scalars) and $G$ is a countable group.

Proof From Theorem 1, we know that $S_p(G)$ is non-empty if and only if $\mu$ is atomic. Let $\{x\}$ be any atom. Then $\mu(\{x\}) > 0$. By left/right translation invariance we know that $\mu(\{g\}) = \mu(\{x\})$ for all $g \in G$. Hence each singleton in $G$ is an atom. However, atoms in a $\sigma$-finite measure space are countable [3, 2.2, ex.11]. Thus $G$ must be a countable group in which all singletons are atoms of the same measure. Hence $\mu$ must be some scalar multiple of counting measure.

3 Functors associated to Schatten classes

3.1 Category of representations of a $C^*$-algebra

The above constructions characterize the family of spaces(with measure) for which $S_p(X)$ is non-empty. Studying $S_p(X)$ for these spaces results in some abstract nonsense. We take a short exact sequence of Banach spaces, as defined in [2]. Consider the category of Banach spaces, $\text{Ban}_1$. For each $1 \leq p < \infty$, we get an exact sequence $E_p$

$$0 \longrightarrow S_p(X) \longrightarrow \pi \longrightarrow S_p(L^2(X)) \longrightarrow S_p(L^2(X)) \longrightarrow \pi(S_p(X)) \longrightarrow 0 \quad (2)$$

Also for each pair $(p, q)$ such that $1 \leq p < q < \infty$, we have the natural contractive injections -

$$i_{p,q} : S_p(L^2(X)) \rightarrow S_q(L^2(X)) \text{and } \pi^{-1} \circ i_{p,q} \circ \pi : S_p(X) \rightarrow S_q(X) \quad (3)$$

It is easy to verify that $\{E_p\}_{p \geq 1}$ forms a directed system in the category of exact sequences of Banach spaces (morphism between two exact sequences $E_p$ and $E_q$ for $q > p$, denoted by $\phi_{p,q}$, are required to be contractions between the corresponding nodes).

Lemma 3 The pair $\{E_p, \phi_{p,q}\}$ forms a directed system on the set $[1, \infty)$ in the category of exact sequences of Banach spaces.
Proof This is just an easy verification exercise. Figure 1 gives us a complete picture. Vertical columns are short exact sequences and each row, $D_0, D_1$ and $D_2$ forms a directed system in the category of Banach spaces. Notice that each rectangular block commutes.

As we mentioned in Remark 1, the definition of $S_p(X)$ depends on the left regular representation of $L^\infty(X)$. Therefore, it leads us to study the more general case. Let $A$ be a $C^*$-algebra, and $\pi : A \to B(H_\pi)$ be a representation(not necessarily faithful) of $A$. We define

$$S_p(X) = \pi^{-1}(S_p(H_\pi)).$$

Now, we have an exact sequence $E_p^\pi$, and for $p > q$ we have the following morphism between $E_p^\pi \rightarrow E_q^\pi$ arising through natural maps.

$$0 \rightarrow S_p^\pi(A) \rightarrow S_p(H_\pi) \rightarrow S_p(H_\pi) / S_p^\pi(A) \rightarrow 0 \quad E_p^\pi$$

$$0 \rightarrow S_q^\pi(A) \rightarrow S_q(H_\pi) \rightarrow S_q(H_\pi) / S_q^\pi(A) \rightarrow 0 \quad E_q^\pi$$

\[ \hfill (4) \hfill \]

Theorem 3 Schatten classes on Hilbert spaces gives a functor from the category $\mathcal{R}ep(A)$ of faithful representations of $C^*$-algebra $A$ to the category $DS_{[1,\infty]}$ of directed systems on the directed set $\{ p : 1 \leq p < \infty \}$.

Proof The assignment $S : \mathcal{R}ep(A) \rightarrow DS_{[1,\infty]}$ given by

$$S(\pi) = E^\pi$$

is a well-defined functor, where $E^\pi$ is the directed system $\{ E_p^\pi \}_{p \geq 1}$

\[ \begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
D_2 & S_1(L^2(X)) & \pi(S_1(X)) & \rightarrow & S_p(L^2(X)) & \pi(S_p(X)) & \rightarrow \ B_0(L^2(X)) & \pi(C_0(X)) & \rightarrow \ B(L^2(X)) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
D_1 & S_1(L^2(X)) & \rightarrow & S_p(L^2(X)) & \rightarrow & B_0(L^2(X)) & \rightarrow & B(L^2(X)) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
D_0 & S_1(X) & \rightarrow & S_p(X) & \rightarrow & C_0(X) & \rightarrow & L^\infty(X) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0 & 0 \\
\end{array} \]

$$E_1 \rightarrow E_p \rightarrow E_0 \rightarrow E_\infty$$

Fig. 1 Directed system of exact sequences
and the morphism between two directed systems is a family of morphism between corresponding objects of both the directed systems. It is easy, albeit cumbersome, to check that such assignment is functorial.

\[ E_1^p \rightarrow \ldots \rightarrow E_p^p \rightarrow \ldots \]

\[ E_1^p \rightarrow \ldots \rightarrow E_p^p \rightarrow \ldots \]

3.2 Category of group representations.

Let \( G \) be a locally compact Hausdorff group. Consider the unitary representation \( \pi : G \rightarrow U(\mathcal{H}_\pi) \) of \( G \) on a Hilbert space \( \mathcal{H}_\pi \). Consider the induced non-degenerate representation \( \pi : L^1(G) \rightarrow B(\mathcal{H}_\pi) \) of Banach-* algebra \( L^1(G) \) [4, 3.2], which is given by

\[ \langle \pi(f)u, v \rangle = \int_G f(x) \langle \pi(x)u, v \rangle \, dx. \]

Now, we define

\[ T^\pi_p(G) = \pi^{-1}(S_p(\mathcal{H}_\pi)) \tag{5} \]

the pullback (through \( \pi \)) of Schatten p-class operators on \( \mathcal{H}_\pi \). Clearly, \( T^\pi_p(G) \) is an ideal in \( L^1(G) \). With this notion, we get a functor associated to Schatten classes as described in next theorem.

**Theorem 4** Let \( G \) be a locally compact Hausdorff group. Associated to Schatten classes, there is a functor \( S : \text{Rep}(G) \rightarrow DS_{[1,\infty)} \).

**Proof** This is again an easy, albeit cumbersome, verification that the assignment \( \pi \rightarrow \{E^\pi_p\}_{p \geq 1} \) is functorial. Where \( E^\pi_p \) is the exact sequence given by

\[ 0 \rightarrow T^\pi_p(G)/\ker(\pi) \rightarrow S_p(\mathcal{H}_\pi) \rightarrow \pi(S_p(\mathcal{H}_\pi)) \rightarrow 0 \tag{6} \]

and the morphisms are again the natural ones similar to the ones considered in Theorem 3.

3.3 Some problems

We end this note with a few problems.

1. Can we give a condition on topology of \( G \), similar to the condition on measure on \( X \) in Theorem 1, for the left regular representation \( \pi \) of a locally compact Hausdorff group \( G \) such that the \( \pi^{-1}(S_p(L^2(G))) \) is non-empty?

2. It is shown in [1, Th. 4.1] that direct limits of a directed system of Banach spaces exist in the category \( \text{Ban}_1 \). Its also been shown in the remark after [5, Def. 2.4] that the filtered limits may not be exact. Does filtered limit from Fig. 1...
preserve the exactness? If yes then what about the same problem with arbitrary representation \( \pi \)?

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