Theory of Bessel Functions of High Rank - I: 
Fundamental Bessel Functions

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Abstract. In this article we introduce a new category of special functions called fundamental Bessel functions. The fundamental Bessel functions of rank one and two are the oscillatory exponential functions $e^{\pm i x}$ and the classical Bessel functions respectively. The function $x \cdot \frac{x}{2} e^{\rho x}$, with $\rho$ a $2n$-th root of unity, is a very illustrative example of the fundamental Bessel functions of rank $n$. The main implements and subjects of our study of fundamental Bessel functions are their formal integral representations and Bessel equations.

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1. Introduction

1.1. Background. The Hankel transform (of high rank) is introduced as an important constituent of the Voronoï summation formula by Miller and Schmid in [MS04a, MS06, MST1]. This summation formula is a fundamental analytic tool in number theory and has its root from representation theory.

In this article, we shall develop the analytic theory of fundamental Bessel functions. These Bessel functions constitute the integral kernel of the Hankel transform. Thus, to motivate our study of fundamental Bessel functions, we shall start with introducing the Hankel transform and its number theoretic and representation theoretic background.

1.1.1. Two expressions of the Hankel transform. Let $n$ be a positive integer, and let \((\lambda, \delta) = (\lambda_1, \ldots, \lambda_n, \delta_1, \ldots, \delta_n) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n\). The first expression of the Hankel transform of rank $n$ associated with $(\lambda, \delta)$ is based on the signed Mellin transform as follows.

Let $\mathcal{S}(\mathbb{R})$ denote the space of Schwartz functions on $\mathbb{R}$. For $\lambda \in \mathbb{C}$, $j \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $\eta \in \mathbb{Z}/2\mathbb{Z}$, let $\nu$ be a smooth function on $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ such that $\text{sgn}(x)^\eta (\log |x|)^{-j} |x|^{-\lambda} \nu(x) \in \mathcal{S}(\mathbb{R})$. For $\delta \in \mathbb{Z}/2\mathbb{Z}$, the Mellin transform with sign $\delta$ of $\nu$ is defined by

\[
\mathcal{M}_\delta \nu(s) = \int_{\mathbb{R}^\times} \nu(x) \text{sgn}(x)^\delta |x|^s d^\times x.
\]

The Bessel functions studied here are called fundamental in order to be distinguished from the Bessel functions for $\text{GL}_n(\mathbb{R})$. Throughout this article, we shall drop the adjective fundamental for brevity. Moreover, the usual Bessel functions will be referred to as classical Bessel functions.

Some evidences show that fundamental Bessel functions are actually the building blocks of the Bessel functions for $\text{GL}_n(\mathbb{R})$. 

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Here \(d^*x = |x|^{-1}dx\) is the standard multiplicative Haar measure on \(\mathbb{R}^x\). The Mellin inversion formula is

\[
u(x) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \frac{\text{sgn}(x)^\delta}{4\pi i} \int_{(\sigma)} \mathcal{M}_\delta \nu(s) |x|^{-s} ds, \quad \sigma > -\Re \lambda,
\]

where the integral contour is the vertical line \(\{s : \Re s = \sigma\}\) from \(\sigma - i\infty\) to \(\sigma + i\infty\).

Let \(\mathcal{S}(\mathbb{R}^x)\) denote the space of smooth functions on \(\mathbb{R}^x\) whose derivatives are rapidly decreasing at both zero and infinity. We associate to \(\nu \in \mathcal{S}(\mathbb{R}^x)\) a function \(\Upsilon\) on \(\mathbb{R}^x\) satisfying the following two identities

\[
(1.2) \quad \mathcal{M}_\delta \Upsilon(s) = \left( \prod_{\ell=1}^n G_{\delta+\delta}(s-\lambda_\ell) \right) \mathcal{M}_\delta \nu(1-s), \quad \delta \in \mathbb{Z}/2\mathbb{Z},
\]

where \(G_\delta(s)\) denotes the gamma factor

\[
(1.3) \quad G_\delta(s) = i^\delta \pi^{\frac{s}{2}} \frac{\Gamma \left( \frac{1}{2}(s+\delta) \right)}{\Gamma \left( \frac{1}{2}(1-s+\delta) \right)} = \begin{cases} 2(2\pi)^{-\frac{s}{2}} \Gamma(s) \cos \left( \frac{\pi s}{2} \right), & \text{if } \delta = 0, \\ 2i(2\pi)^{-\frac{s}{2}} \Gamma(s) \sin \left( \frac{\pi s}{2} \right), & \text{if } \delta = 1, \end{cases}
\]

where for the second equality we apply the duplication formula and Euler's reflection formula of the Gamma function,

\[
\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma(s)\Gamma \left( s + \frac{1}{2} \right) = 2^{1-2s} \sqrt{\pi}\Gamma(2s).
\]

\(\Upsilon\) is called the Hankel transform of \(\nu\) of index \((\lambda, \delta)\). According to [MS06] §6, \(\Upsilon\) is smooth on \(\mathbb{R}^x\) and decays rapidly at infinity, along with all its derivatives. At the origin, \(\Upsilon\) has singularities of some very particular type. Indeed, \(\Upsilon(x) \in \sum_{\ell=1}^n |x|^{-\lambda_\ell} \text{sgn}(x)^\delta \mathcal{S}(\mathbb{R})\) when no two components of \(\lambda\) differ by an integer, and in the nongeneric case powers of \(\log |x|\) are included.

By the Mellin inversion,

\[
(1.4) \quad \Upsilon(x) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \frac{\text{sgn}(x)^\delta}{4\pi i} \int_{(\sigma)} \left( \prod_{\ell=1}^n G_{\delta+\delta}(s-\lambda_\ell) \right) \mathcal{M}_\delta \nu(1-s) |x|^{-s} ds,
\]

for \(\sigma > \max \{\Re \lambda_\ell\}\).

In [MS11] there is an alternative description of \(\Upsilon\) defined by the Fourier-type transform, in symbolic notion, as follows

\[
(1.5) \quad \Upsilon(x) = \frac{1}{|x|} \int_{\mathbb{R}^x} \left( \frac{\text{sgn}(x)\delta \mathcal{M}_\delta \nu(s) |x|^{-s} ds}{(2\pi)^2} \right) \left( \prod_{\ell=1}^n \left( \text{sgn}(x_\ell)^\delta \right) |x_\ell|^{-\lambda_\ell} e(x_\ell) \right) \ dx_\ell x_{n-1}...dx_1,
\]

with \(e(x) = e^{2\pi i x}\). The integral in (1.5) converges when performed as iterated integral in the order \(dx_0 dx_{n-1}...dx_1\), starting from \(x_0\), then \(x_{n-1},...,\) and finally \(x_1\), provided \(\Re \lambda_1 > ... > \Re \lambda_{n-1} > \Re \lambda_n\), and it has meaning for arbitrary values of \(\lambda \in \mathbb{C}^n\) by analytic continuation.

\(\dagger\)Note if \(\nu\) is the \(f\) in [MS11] then \(|x|^{\Upsilon((-)^\nu) x}\) is their \(F(x)\).
According to [MS11], though less suggestive than (1.5), the expression (1.4) of the Hankel transform is more useful in applications. Indeed, all the applications of the Voronoï summation formula so far are based on (1.4) with exclusive use of Stirling’s asymptotic formula of the Gamma function. On the other hand, there is no occurrence of the Fourier-type integral transform (1.5) in the literatures other than Miller and Schmid’s foundational work.

**Assumption.** Subsequently, we shall always assume that the index $\lambda$ satisfies $\sum_{\ell=1}^{m} \lambda_{\ell} = 0$. Accordingly, we define the complex hyperplane $L_{n-1} = \{ \lambda \in \mathbb{C}^n : \sum_{\ell=1}^{m} \lambda_{\ell} = 0 \}$.

1.1.2. **Background of Hankel transforms in number theory and representation theory.**

For $n = 1$, the number theoretic background lies on the local theory in Tate’s thesis at the real place. Actually, in view of (1.5), the Hankel transform of rank one associated with $(\lambda, \delta) = (0, \delta)$ is essentially the (inverse) Fourier transform, (1.6) $\Upsilon(x) = \int_{\mathbb{R}} v(y) \text{sgn}(xy)^{\delta} e(xy)dy$.

The Voronoï summation formula of rank one is the summation formula of Poisson. Recall that Riemann’s proof of the functional equation of his $\zeta$-function relies on the Poisson summation formula, whereas Tate’s thesis reinterprets this using the Poisson summation formula for the adele ring.

For $n = 2$, the Hankel transforms associated to $GL_2$-automorphic forms have been present in the literatures as part of the Voronoï summation formula for $GL_2$ for decades. See, for instance, [HM06, Proposition 1] and the references there. According to [HM06, Proposition 1] (see also Remark 2.8), we have

(1.7) $\Upsilon(x) = \int_{\mathbb{R}^+} v(y) J_F(xy)dy, \quad x \in \mathbb{R}^+$,

where, if $F$ is a Maaß form with eigenvalue $\frac{1}{4} + t^2$ and weight $k$,

\[
J_F(x) = -\frac{\pi}{\cosh(\pi t)} \left( Y_{2\nu}(4\pi \sqrt{x}) + Y_{-2\nu}(4\pi \sqrt{x}) \right)
= \frac{\pi t}{\sinh(\pi t)} \left( J_{2\nu}(4\pi \sqrt{x}) - J_{-2\nu}(4\pi \sqrt{x}) \right)
= \pi i \left( e^{-\pi H_{2\nu}^{(1)}(4\pi \sqrt{x})} - e^{\pi H_{2\nu}^{(2)}(4\pi \sqrt{x})} \right),
\]

(1.8) $J_F(-x) = 4 \cosh(\pi t) K_{2\nu}(4\pi \sqrt{x})$

\[
= \frac{\pi i}{\sinh(\pi t)} \left( J_{2\nu}(4\pi \sqrt{x}) - J_{-2\nu}(4\pi \sqrt{x}) \right), \quad x > 0,
\]

for $k$ even,

\[
J_F(x) = -\frac{\pi}{\sinh(\pi t)} \left( Y_{2\nu}(4\pi \sqrt{x}) - Y_{-2\nu}(4\pi \sqrt{x}) \right)
\]

\[\text{III} \text{This condition is just a matter of normalization. Equivalently, the corresponding representations of } GL_n(\mathbb{R}) \text{ are trivial on the positive component of the center. With this condition on } \lambda, \text{ the associated Bessel functions can be expressed in a simpler way.}\]
The identity in the Voronoï summation formula for $GL_n$ is also interpreted from their perspective in

We shall call $(3, 4, 5, 6), 3.7 (6)$ are applied

Thus the integral kernel $J_F$ has an expression in Bessel functions for various $F$, where, in standard notation, $J_v, Y_v, H_v^{(1)}, H_v^{(2)}, I_v$ and $K_v$ are the various Bessel functions (see for instance [Wat44]). Here, in (1.8) (1.9) the following connection formulae ([Wat44 3.61 (3, 4, 5, 6), 3.7 (6)]) are applied

$$J_F(x) = \frac{1}{\pi} J_{k-1}(4\pi \sqrt{x}), \quad J_F(-x) = 0, \quad x > 0.$$  

(1.10)

for $k$ odd and if $F$ is a holomorphic cusp form with weight $k$,

$$K_v(x) = \frac{\pi (I_v(x) - I_{-v}(x))}{2 \sin(\pi v)}.$$

(1.13)

The theory of Bessel functions has been extensively studied since the early 19th century, and we refer the reader to Watson’s beautiful book [Wat44] for an encyclopedic treatment.

For $n \geq 3$, the Hankel transform is formulated in Miller and Schmid [MS06, MS11], given that $(\lambda, \delta)$ is a certain parameter of a cuspidal $GL_n(\mathbb{Z})$-automorphic representation of $GL_n(\mathbb{R})$. It is the archimedean ingredient which relates the weight functions on two sides of the identity in the Voronoï summation formula for $GL_n(\mathbb{Z})$. For $n = 1, 2$ the Poisson and the Voronoï summation formula are also interpreted from their perspective in [MS04b].

Using the global theory of $GL_n \times GL_1$-Rankin-Selberg $L$-functions, Inchino and Templier [IT13] extend Miller and Schmid’s work and prove the Voronoï summation formula for any irreducible cuspidal automorphic representation of $GL_n$ over an arbitrary number field for $n \geq 2$. According to [IT13], the defining identity (1.2) of the Hankel transform follows from renormalizing the local functional equation of the $GL_n \times GL_1$-Rankin-Selberg zeta integrals over $\mathbb{R}$.

1.1.3. Bessel kernels. In the case $n \geq 3$, when applying the Voronoï summation formula, it might have been realized by many authors that, similar to (1.6) (1.7), the Hankel transform should also admit an integral kernel, that is,

$$\Upsilon(x) = \int_{\mathbb{R}^{+}} u(y) J_{(\lambda, \delta)}(xy) dy.$$  

(1.14)

We shall call $J_{(\lambda, \delta)}$ the (fundamental) Bessel kernel of index $(\lambda, \delta)$.

\footnote{For this case there are two insignificant typos in [HM06 Proposition 1].}
Actually, the formula (1.14) and the expression of $J_{(A,B)}$ in terms of the Gamma function may be easily seen from the expression (1.4) of the Hankel transform (see §2.1). Moreover, in the literatures, we have seen applications of the asymptotic expansion of $J_{(A,B)}$ derived from Stirling’s asymptotic formula of the Gamma function (see Appendix A). When the dependence on the index $\lambda$ is taken into account, the error bound in the asymptotic obtained in this way does not meet from our expectations. In this direction from (1.4), it seems that we can not proceed any further.

In this article, we shall take the approach to the Bessel kernel $J_{(A,B)}$ starting from the expression (1.5) of the Hankel transform. This approach is more accessible, at least in symbolic notions, in view of the simpler form of (1.5) compared to (1.4). Once we can make sense of the symbolic notions in (1.5), some well-developed methods from analysis and differential equations can be exploited in our study.

1.2. Outline of article.

1.2.1. Bessel functions and their formal integral representations. First of all, in §2.1 we formulate the Bessel kernel $J_{(A,B)}$ into a signed sum of certain functions $J(x; \varsigma, \lambda)$ called the Bessel function of rank $n$ associated with $\varsigma \in \{+, -\}^n$ and $\lambda \in \mathbb{L}^{n-1}$.

With some manipulations on the Fourier-type expression (1.5) of the Hankel transform in a symbolic manner, we obtain in §2.2 a formal integral representation of the Bessel function $J(x; \varsigma, \lambda)$. Denote $\mathbb{R}_+ = (0, \infty)$ and define $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ by $\nu_\ell = \lambda_\ell - \lambda_n$, with $\ell = 1, ..., n - 1$. The formal integral is given by

$$J_\nu(x; \varsigma) = \int_{\mathbb{R}_+^{n-2}} \left( \prod_{\ell=1}^{n-1} \nu_{\ell-1} \right) e^{i x (\varsigma_1 + ... + \sum_{\ell=1}^{n-1} \varsigma_{\ell-1})} \, dt_{n-1}...dt_1. \tag{1.15}$$

Justification of this formal integral representation is the main subject of §§3 and 4. For this, we partition the formal integral $J_\nu(x; \varsigma)$ according to some partition of unity on $\mathbb{R}_+^{n-1}$, and then repeatedly apply two kinds of partial integration operators on each resulting integral. In this way, $J_\nu(x; \varsigma)$ is transformed into a finite sum of absolutely convergent multiple integrals. This sum of integrals is regarded as the rigorous definition of $J_\nu(x; \varsigma)$. However, the simplicity of the expression (1.15) is sacrificed after these technical procedures. Furthermore, it is shown that

$$J(x; \varsigma, \lambda) = J_\nu(x; \varsigma). \tag{1.16}$$

1.2.2. Asymptotics via stationary phase. In §3 we shall either adapt techniques or directly apply results from the method of stationary phase to study the asymptotic behaviour of $J_\nu(x; \varsigma)$ at large argument.

When all the components of $\varsigma$ are identically $\pm$, we denote $J(x; \varsigma, \lambda)$, respectively $J_\nu(x; \varsigma)$, by $H^\pm(x; \lambda)$, respectively $H^\pm_\nu(x)$, and call it an $H$-Bessel function. This pair of $H$-Bessel functions will be of paramount significance in our treatment.

\footnote{If a statement or a formula includes $\pm$ or $\mp$, then it should be read with $+$ and $-$ or $-$ and $+$.}
It is shown that \( H^\pm(x; \lambda) = H_\mp^\pm(x) \) admits an analytic continuation from \( \mathbb{R}_+ \) onto the half-plane \( \mathbb{H}^\pm = \{ z \in \mathbb{C} \setminus \{0\} : 0 < \pm \arg z < \pi \} \). We have the asymptotic

\[
H^\pm(z; \lambda) = n^{-\frac{1}{2}}(\pm 2\pi i)^{-\frac{\lambda}{\pi}}e^{\pm \frac{\lambda}{2}|z|}e^{\pm \frac{\lambda}{2}|z|} \left( \mathcal{C} |z|^{-\frac{\lambda}{2}} \right),
\]

(1.17)

for all \( z \in \mathbb{H}^\pm \) such that \( |z| \geq C \), where \( \mathcal{C} = \max \{|A\ell|\} + 1, \Re = \max \{|\Re \lambda|\}, M \geq 0, B_m(\lambda) \in \mathbb{Q}[\lambda] \) is some symmetric polynomial in \( \lambda \) of degree \( 2m \), and \( B_0(\lambda) = 1 \). In particular, these two \( H \)-Bessel functions oscillate and decay proportionally to \( x^{-\frac{1}{2}} \) on \( \mathbb{R}_+ \).

The other Bessel functions are called \( K \)-Bessel functions and are shown to be Schwartz functions at infinity.

1.2.3. Bessel equations. The differential equation satisfied by the Bessel function \( J(x; \varsigma, \lambda) \) is discovered in [6]. Such equations are called the Bessel equations.

Given \( \lambda \in \mathbb{L}^{n-1} \), there are exactly two Bessel equations

\[
\sum_{j=1}^{n} V_{n,j}(\lambda) x^j w^{(j)} + (V_{n,0}(\lambda) - \varsigma(n) x^0) w = 0, \quad \varsigma \in \{+,-\},
\]

(1.18)

where \( V_{n,j}(\lambda) \) is some symmetric polynomial in \( \lambda \) of degree \( n - j \) that can be explicitly given. We call \( \varsigma \) the parity of the Bessel equation (1.18). \( J(x; \varsigma, \lambda) \) satisfies the Bessel equation of parity \( S_n(\varsigma) = \prod_{i=1}^{n} \varsigma_i \).

The entire \([7]\) is devoted to the study of Bessel equations. Let \( \mathbb{U} \) denote the Riemann surface \( \mathbb{U} \) associated with \( \log z \), that is, the universal cover of \( \mathbb{C} \setminus \{0\} \). Replacing \( x \) by \( z \) to stand for complex variable in (1.18), we extend the domain from \( \mathbb{R}_+ \) to \( \mathbb{U} \). According to the theory of linear ordinary differential equations with analytic coefficients, \( J(x; \varsigma, \lambda) \) admits an analytic continuation onto \( \mathbb{U} \).

Firstly, since zero is a regular singularity, the Frobenius method may be exploited to find a solution \( J_\ell(z; \varsigma, \lambda) \) of (1.18), for each \( \ell = 1, ..., n \), defined by the following series,

\[
J_\ell(z; \varsigma, \lambda) = \sum_{m=0}^{\infty} \frac{(\varsigma(n_m) )^{m} e^{-\ell|\alpha|}}{\Gamma(\ell + m + 1)}.
\]

(1.19)

\( J_\ell(z; \varsigma, \lambda) \) are called Bessel functions of the first kind, since they generalize the Bessel functions \( J_\ell \), and the modified Bessel functions \( I_\ell \), of the first kind.

We show that each \( J(z; \varsigma, \lambda) \) may be expressed in terms of \( J_\ell(z; \varsigma, \lambda) \) for \( \varsigma = S_n(\varsigma) \).

This leads to the following connection formula

\[
J(z; \varsigma, \lambda) = e^\left( \frac{\pm \sum_{\ell \in L_\pm(\varsigma)} \alpha_\ell}{2} \right) H^\pm \left( \frac{\varsigma(n_\pm(\varsigma))}{n_\pm(\varsigma)} z; \lambda \right),
\]

(1.19)

where \( L_\pm(\varsigma) = \{ \ell : \varsigma_\ell = \pm \} \) and \( n_\pm(\varsigma) \) is \( |L_\pm(\varsigma)| \). Thus the Bessel function \( J(z; \varsigma, \lambda) \) is determined up to a constant by the pair of integers \( (n_+(\varsigma), n_-(\varsigma)) \) called the signature of either \( \varsigma \) or \( J(z; \varsigma, \lambda) \).
Secondly, $\infty$ is an irregular singularity of rank one. The formal solutions at infinity serve as the asymptotic expansions of some actual solutions of Bessel equations. This gives us the second approach to the asymptotics of Bessel functions.

Let $\xi$ be an $n$-th root of $\zeta$. There exists a unique formal solution $\hat{J}(z; \lambda; \xi)$ of the Bessel equation of parity $\varsigma$ of the following form

$$\hat{J}(z; \lambda; \xi) = e^{in\xi}z^{-\frac{\varsigma-1}{2}} \sum_{m=0}^{\infty} B_m(\lambda; \xi) z^{-m},$$

where $B_m(\lambda; \xi)$ is a symmetric polynomial in $\lambda$ of degree $2m$, with $B_0(\lambda; \xi) = 1$. The coefficients of $B_m(\lambda; \xi)$ depend only on $m, \xi$ and $n$. There exists a unique solution $J(z; \lambda; \xi)$ of the Bessel equation of parity $\varsigma$ possessing $\hat{J}(z; \lambda; \xi)$ as its asymptotic expansion on the sector

$$\mathbb{S}_\xi = \left\{ z \in \mathbb{U} : \left| \arg z - \arg(\xi) \right| < \frac{\pi}{n} \right\},$$

or any of its open subsector.

The study of the theory of asymptotic expansions for ordinary differential equations traces back to Poincaré, and there are abundant references on this topic, for instance, [CL55 Chapter 5], [Was65 Chapter III-V] and [Olv74 Chapter 7]. However, the author is not aware of any error analysis with dependence on the index $\lambda$ desired by analytic number theorist in the literatures except for differential equations of second order in [Olv74]. Nevertheless, with some effort, we are able to find a very satisfactory error bound in §7.4.

For $0 < \theta < \frac{\pi}{2}$ define

$$\mathbb{S}_\xi'(\theta) = \left\{ z \in \mathbb{U} : \left| \arg z - \arg(\xi) \right| < \pi + \frac{\pi}{n} - \theta \right\}.$$

The following asymptotic is established in §7.4

$$(1.20) \quad J(z; \lambda; \xi) = e^{in\xi}z^{-\frac{\varsigma-1}{2}} \left( \sum_{m=0}^{M-1} B_m(\lambda; \xi) z^{-m} + O(M, \mathbb{U}) \right)$$

for all $z \in \mathbb{S}_\xi'(\theta)$ with $|z| \gg M, \theta, n \in \mathbb{C}^2$.

For a $2n$-th root of unity $\xi$, $J(z; \lambda; \xi)$ is called a Bessel function of the second kind. We have the following formula that relates all the the Bessel functions of the second kind to either $J(z; \lambda; 1)$ or $J(z; \lambda; -1)$ upon rotating the argument by a $2n$-th root of unity,

$$(1.21) \quad J(z; \lambda; \xi) = (\pm \xi)^{\frac{\varsigma-1}{2}} J(\pm \xi; \lambda; \pm 1).$$

So, roughly speaking, there is only one Bessel function of the second kind!

1.2.4. Connections between $J(z; \varsigma, \lambda)$ and $J(z; \lambda; \xi)$. Comparing the asymptotic expansions of $H^\pm(\lambda; \xi)$ and $J(z; \lambda; \pm 1)$ in (1.17) and (1.20), we obtain the identity

$$(1.22) \quad H^\pm(\lambda; \xi) = n^{-\frac{1}{2}}(\pm 2\pi i)^{\frac{\varsigma-1}{2}} J(z; \lambda; \pm 1).$$

It follows from (1.19) and (1.21) that

$$J(z; \varsigma, \lambda) = \frac{(\pm 2\pi i)^{\frac{\varsigma-1}{2}}}{\sqrt{n}} e^{\pm \frac{\varsigma-1}{2} \pi i} e^{\pm \frac{(n-1)\varsigma(\varsigma+1)}{4n}} \sum_{\ell \in \ell_\pm(\varsigma)} \frac{\lambda_\ell^2}{2} J\left( z; \lambda; \mp e^{\pi i} \varsigma(\varsigma+1) \right).$$
Thus (1.20) may be applied to improve the error bound in the asymptotic expansion of the $H$-Bessel function $H^\pm(x; \lambda)$ when $x \gg \xi^2$ and also to show the exponential decay of $K$-Bessel functions.

1.2.5. Connections between $J_\ell(pz; \varsigma, \lambda q)$ and $J_p(z; \varsigma, \lambda q)$. The identity (1.22) also yields connection formulae between the two kinds of Bessel functions, $J_\ell(pz; \varsigma, \lambda q)$ and $J_p(z; \varsigma, \lambda q)$. A Vandermonde matrix occurs here in a very natural way.

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2. Bessel functions and their formal integral representations

Using the formula (1.4) of the Hankel transform we shall derive the definition of the Bessel function $J_p(x; \varsigma, \lambda q)$, with $\varsigma P t^\ell$, $\eta P L_\ell$ and $\lambda P 1$. On the other hand, The Fourier-type expression (1.5) of the Hankel transform yields, in a symbolic manner, a formal integral representation of $J_p(x; \varsigma, \lambda q)$. In §2.3 we shall see where $J_p(x; \varsigma, \lambda q)$ and its formal integral representation stand in the theory of classical Bessel functions when $n = 2$. Without loss of generality, we assume $\nu(-y) = (-)^n \nu(y)$, with $\eta \in \mathbb{Z}/2\mathbb{Z}$.

2.1. The definition of the Bessel function $J(x; \varsigma, \lambda)$. We start with reformulating (1.3) as

$$ G_x(s) = (2\pi)^{-n} \Gamma(s) \left( e^{\frac{\varsigma s}{4}} + (-)^n e^{-\frac{s}{4}} \right) . $$

Inserting this formula of $G_x$, (1.4) may be rewritten as

\[
\Upsilon(x) = \text{sgn}(x)^\eta \sum_{\varsigma \in \{+\}^n} \left( \prod_{\ell=1}^n \varsigma_\ell^{\mathfrak{m}_\ell + \eta} \right) \Upsilon(|x|; \varsigma),
\]

where $\varsigma = (\varsigma_1, ..., \varsigma_n) \in \{+\}^n$,

\[
\Upsilon(x; \varsigma) = \frac{1}{2\pi i} \int_{(\mathfrak{m})} \int_0^\infty y^{-1} dy \cdot G(s; \varsigma, \lambda)((2\pi)^n x)^{-s} ds, \quad x > 0,
\]

and

\[
G(s; \varsigma, \lambda) = \prod_{\ell=1}^n \Gamma(s - \lambda_\ell) e^{\left( \varsigma_\ell (s - \lambda_\ell) / 4 \right)} .
\]

Since all the derivatives of $\nu$ rapidly decay at both zero and infinity, repeating partial integrations yields the bound

\[
\int_0^\infty \nu(y)y^{-s}dy \ll_{\mathfrak{m}, \lambda, \nu} ((|\text{Im } x| + 1)^{-A} \quad \text{for any } A \geq 0.
\]

Hence the iterated double integral in (2.2) is convergent due to Stirling’s formula.

Choose $\sigma' < -\frac{1}{2} - \frac{1}{2\pi}$ so that $\sum_{\ell=1}^n (\sigma' - \Re \lambda_\ell - \frac{1}{2}) < -1$. Shift the contour of integration $\{s : \Re s = \sigma\}$ without passing through the poles of $G(s; \lambda)$ to a contour $C$ on
which \( \Re s = \sigma' \) if \( |\Im s| - \max \{ |\Im \lambda_f| \} \gg 1 \). After this contour shift, the double integral in (2.2) becomes absolutely convergent by Stirling’s formula, and therefore we are able to change the order of integrals. Then,

\[
\Upsilon(x; \varsigma) = \int_0^\infty v(y)J(2\pi xy; \varsigma, \lambda) dy, \tag{2.4}
\]

where

\[
J(x; \varsigma, \lambda) = \frac{1}{2\pi i} \int_C G(s; \varsigma, \lambda) x^{-s} ds. \tag{2.5}
\]

We view \( J(x; \varsigma, \lambda) \) as the inverse Mellin transform of \( G(s; \varsigma, \lambda) \). For \( \lambda \in \mathbb{L}^{n-1} \) and \( \varsigma \in \{+, -\}^n \), we call the function \( J(x; \varsigma, \lambda) \) defined by (2.5) a Bessel function.

Suitably choosing the integral contour \( C \), it may be verified that \( J(x; \varsigma, \lambda) \) is a smooth function of \( x \) and is analytic with respect to \( \lambda \).

**Remark 2.1.** The contour of integration \( \{ s : \Re s = \sigma \} \) does not need modification if the components of \( \varsigma \) are not identical. For further discussions of the integral in the definition (2.5) of \( J(x; \varsigma, \lambda) \) see Remark 7.10.

**Remark 2.2.** If we drop the parity condition on \( \nu \), then

\[
\Upsilon(x) = \int_{\mathbb{R}^n} v(y)J_{(\alpha, \delta)}(xy) dy, \quad x \in \mathbb{R}^n,
\]

for any \( v \in \mathcal{S}(\mathbb{R}^n) \), where the Bessel kernel \( J_{(\alpha, \delta)} \) is given by

\[
J_{(\alpha, \delta)}(\pm x) = \frac{1}{2} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} (-1)^\delta \sum_{\varsigma \in \{+,-\}^n} \left( \prod_{f=1}^n \varsigma_f^{\delta_f} \right) J(2\pi x^{\pm}; \varsigma, \lambda), \quad x \in \mathbb{R}_+.
\]

We have

\[
J_{(\alpha, \delta)}(x) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \frac{\text{sgn}(x)^\delta}{4\pi i} \int_C \left( \prod_{f=1}^n G_{\delta_f + \delta}(s - \lambda_f) \right) |x|^{-s} ds. \tag{2.8}
\]

### 2.2. The Formal Integral Representation of \( J(x; \varsigma, \lambda) \)

In this section, we assume \( n \geq 2 \). We shall manipulate the Fourier-type integral transform (1.5) only in a symbolic manner, and therefore the restrictions on the index \( \lambda \) that guarantee the convergence of the iterated integral in (1.5) will not be imposed here.

With the parity condition on \( \nu \), the integral in (1.5) may be written as

\[
\frac{\text{sgn}(x)^\eta}{|x|} \sum_{\varsigma \in \{+,-\}^n} \left( \prod_{f=1}^n \varsigma_f^{\delta_f + \eta} \right) \int_{\mathbb{R}^n_+} \nu \left( \frac{X_1...X_n}{|x|} \right) \left( \prod_{f=1}^n x_f^{\lambda_f} e^{(\varsigma_f X_f)} \right) dX_n dX_{n-1}...dX_1.
\]
Compare this signed sum with \( e^{ix} \) and change variables \( x_n = |x|y(x_1...x_{n-1})^{-1}, x_\ell = y_\ell^{-1}, \ell = 1,..., n-1 \), then we see that

\[
\Gamma(x; \varsigma) = \int_{\mathbb{R}^+} v(y) (xy)^{-\lambda_n} \left( \prod_{\ell=1}^{n-1} y_\ell^{-1} \right) e^{\left( \varsigma_n xy_1...y_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell y_\ell^{-1} \right)} dy_{n-1}...dy_1.
\]

(2.9)

In view of (2.4), if we formally change the order of integrals, which rigorously is not permissible since the integral is not absolutely convergent, then \( J(x; \varsigma, \lambda) \) can be expressed as a symbolic integral as bellow,

\[
J(2\pi x; \varsigma, \lambda) = x^{-n} \int_{\mathbb{R}^+} \left( \prod_{\ell=1}^{n-1} y_\ell^{-1} \right) e^{\left( \varsigma_n x y_1...y_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell y_\ell^{-1} \right)} dy_{n-1}...dy_1.
\]

(2.10)

Making another substitution \( y_\ell = t_\ell x^{-1} \) and applying \( \sum_{\ell=1}^{n} \lambda_\ell = 0 \) yield

\[
J(x; \varsigma, \lambda) = \int_{\mathbb{R}^+} \left( \prod_{\ell=1}^{n-1} t_\ell^{-1} \right) e^{i x (\varsigma_n t_1...t_{n-1} + \sum_{\ell=1}^{n-1} \varsigma_\ell t_\ell^{-1})} dt_{n-1}...dt_1.
\]

The above integral is only in symbolic notation, and we call it the formal integral representation of \( J(x; \varsigma, \lambda) \).

**Remark 2.3.** Before realizing its connection with the Fourier-type transform (1.5), the formal integral representation (2.10) of \( J(x; \varsigma, \lambda) \) was derived by the author from (1.4) based on a symbolic application of the product-convolution principle of the Mellin transform together with the following formula (GR07 3.764)

\[
\Gamma(s) e\left( \pm \frac{s}{4} \right) = \int_0^{\infty} e^{\pm is x} x^s \, dx, \quad 0 < \Re s < 1.
\]

(2.11)

Though not specified, this principle is implicitly suggested in Miller and Schmid’s work, especially [MS04a, Theorem 4.12, Lemma 6.19] and [MS06 (5.22, 5.26)].

**2.3. The classical cases.**

**2.3.1. The case \( n = 1 \).**

**Proposition 2.4.** Suppose \( n = 1 \). Let \( \mathcal{C} \) be the contour chosen in (2.1) such that \( \Re s = \sigma' \) if \( s \in \mathcal{C} \) and \( |\Im s| \gg 1 \), with \( \sigma' < -\frac{1}{2} \), and all the nonpositive integers lie on the left side of \( \mathcal{C} \). We have

\[
e^{\pm is} = \int_{\mathcal{C}} \Gamma(s) e\left( \pm \frac{s}{4} \right) x^{-s} \, ds.
\]

(2.12)

Therefore

\[
J(x; \pm, 0) = e^{\pm is}.
\]

\[\text{VI} \]

Observe that there are \( 2^n \) linear equations if we let \( \delta \) vary. These equations constitute a linear system whose associated \( 2^n \times 2^n \) matrix of entries \( \pm 1 \) is invertible. This justifies our comparison.
PROOF. Let $z \in \mathbb{C}$ with $\Re z > 0$. We have for $\Re s > 0$
\[ \Gamma(s)z^{-s} = \int_0^\infty e^{-zx}x^s \, dx, \]
where the integral is absolutely convergent. Using the Mellin inversion formula and mak-
ing a contour shift from a certain vertical line in the positive right half plane to $\mathcal{C}$, it follows
that
\[ e^{-zx} = \int_{\mathcal{C}} \Gamma(s)z^{-s}x^{-s} \, ds. \]
Choose $z = e^{\pi(\frac{1}{4} - \epsilon)}$, $\pi > \epsilon > 0$. Since the convergence of the integral above is uniform
in $\epsilon$ due to Stirling’s formula, we obtain (2.12) by letting $\epsilon \to 0$. Q.E.D.

REMARK 2.5. Observe that the integral in (2.11) is only conditionally convergent, the
Mellin inversion formula does not apply. Nevertheless, (2.12) should be view as the Mellin
inversion of (2.11).

REMARK 2.6. From the proof of Proposition 2.4 we actually have for $a \in \left[ -\frac{1}{4}, \frac{1}{4} \right]$
(2.13)
\[ e^{-e(a)x} = \int_{\mathcal{C}} \Gamma(s)e(-as)x^{-s} \, ds. \]

2.3.2. The case $n = 2$.

PROPOSITION 2.7. Let $\lambda \in \mathbb{C}$. Then
\[ J(x; \pm, \mp, \lambda, -\lambda) = \pi i e^{\pi i \lambda} H_{2\lambda}^{(1)}(2x), \]
\[ J(x; -, -, \lambda, -\lambda) = -\pi i e^{-\pi i \lambda} H_{2\lambda}^{(2)}(2x), \]
\[ J(x; \pm, \mp, \lambda, -\lambda) = 2e^{\mp \pi i \lambda} K_{2\lambda}(2x). \]
Here $H_{\nu}^{(1)}$ and $H_{\nu}^{(2)}$ are Bessel functions of the third kind, also known as Hankel functions,
whereas $K_{\nu}$ is the modified Bessel function of the second kind, occasionally called the
$K$-Bessel function.

Proof. We have the following identities following from [GR07, 6.561 (14-16)] and
Euler’s reflection formula of the Gamma function.
\[ \pi \int_0^\infty J_\nu(2\sqrt{x})x^{s-1} \, dx = \Gamma\left(s + \frac{\nu}{2}\right) \Gamma\left(s - \frac{\nu}{2}\right) \sin\left(\pi\left(s - \frac{\nu}{2}\right)\right) \]
for $-\frac{1}{2} < \Re s < \frac{1}{2}$.
\[ -\pi \int_0^\infty Y_\nu(2\sqrt{x})x^{s-1} \, dx = \Gamma\left(s + \frac{\nu}{2}\right) \Gamma\left(s - \frac{\nu}{2}\right) \cos\left(\pi\left(s - \frac{\nu}{2}\right)\right) \]
for $\frac{1}{2} < |\Re s| < \frac{1}{2}$, and
\[ 2 \int_0^\infty K_\nu(2\sqrt{x})x^{s-1} \, dx = \Gamma\left(s + \frac{\nu}{2}\right) \Gamma\left(s - \frac{\nu}{2}\right) \]
for $\Re s > \frac{1}{2}|\Re \nu|$. 


Observe the absolute convergence of the above integrals for \( \Re s \in \) in the given ranges. Using the Mellin inversion formula, the above identities imply

\[
J(x; \pm, \pm, \lambda, -\lambda) = \pm \pi \epsilon^{\pm \pi i \lambda} (J_{2 \lambda}(2x) \pm iY_{2 \lambda}(2x)), \quad |\Re \lambda| < \frac{1}{4};
\]

\[
J(x; \pm, \mp, \lambda, -\lambda) = 2 \pi \epsilon^{\pm \pi i \lambda} K_{2 \lambda}(2x).
\]

The first formula is still valid even for \( |\Re \lambda| \geq \frac{1}{4} \) due to the analyticity in \( \lambda \).

We conclude the proof by recollecting the definitions \( H_v^{(1)}(x) = J_v(x) + iY_v(x) \) and \( H_v^{(2)}(x) = J_v(x) - iY_v(x) \).

**Remark 2.8.** Let \( \lambda = \text{it} \) if \( F \) is a Maass form with eigenvalue \( \frac{1}{2} + t^2 \) and weight \( k \), and \( \lambda = \frac{k-1}{2} \) if \( F \) is a holomorphic cusp form with weight \( k \). Then \( F \) is parametrized by \( (\lambda, \vartheta) = (\lambda, -\lambda, k(\text{mod} \, 2), 0) \) and \( F = J_{(\lambda, \vartheta)} \). From the expression (2.1) of the Bessel kernel, we have

\[
J_{(\lambda, \vartheta)}(x) = J(2\pi \sqrt{x}; +, +, \lambda, -\lambda) + (-)^\vartheta J(2\pi \sqrt{x}; -, -, \lambda, -\lambda),
\]

\[
J_{(\lambda, \vartheta)}(-x) = J(2\pi \sqrt{x}; +, -, \lambda, -\lambda) + (-)^\vartheta J(2\pi \sqrt{x}; -, +, \lambda, -\lambda).
\]

Thus, Proposition 2.7 implies (1.8, 1.9, 1.10).

When \( |\Re v| < 1 \), we have the following integral representations of Bessel functions ([Wat44] 6.21 (10, 11), 6.22 (13))

\[
H_v^{(1)}(x) = \frac{2 \epsilon^{-\frac{i}{2} \pi v}}{\pi i} \int_0^\infty e^{i x \cosh r} \cosh(vr) dr,
\]

\[
H_v^{(2)}(x) = \frac{2 \epsilon^{\frac{i}{2} \pi v}}{\pi i} \int_0^\infty e^{i x \cosh r} \cosh(vr) dr,
\]

\[
K_v(x) = \frac{1}{\cos \left( \frac{1}{2} \pi v \right)} \int_0^\infty \cos(x \sinh r) \cosh(vr) dr,
\]

for \( x > 0 \). On changing variables \( t = x' \), some calculations show

\[
\int_0^\infty t^{r-1} e^{i x (t+r-1)} dt = \pi \epsilon^{\frac{i}{2} \pi v} H_v^{(1)}(2x),
\]

\[
\int_0^\infty t^{r-1} e^{-i x (t+r-1)} dt = -\pi \epsilon^{-\frac{i}{2} \pi v} H_v^{(2)}(2x),
\]

\[
\int_0^\infty t^{r-1} e^{i x (t-r-1)} dt = 2 \epsilon^{\frac{i}{2} \pi v} K_v(2x).
\]

In view of Proposition 2.7 and (2.10), these formulae with \( v = 2 \lambda \) should be interpreted as the formal integral representations in (2.10) for \( n = 2 \). Observe that these integrals are not formal, since they conditionally converge if \( |\Re \lambda| < \frac{1}{2} \).

**2.4. A prototypical example.** According to [Wat44] 3.4 (3, 6), 3.71 (13)),

\[
J_{\frac{1}{2}}(x) = \left( \frac{2}{\pi x} \right)^{\frac{1}{2}} \sin x, \quad J_{-\frac{1}{2}}(x) = \left( \frac{2}{\pi x} \right)^{\frac{1}{2}} \cos x.
\]
Then the connection formulae \([1.12]\) (Wat44 3.61 (5, 6)) imply that
\[
H_\nu^{(1)}(x) = -i \left( \frac{2}{\pi x} \right)^{1/2} e^{ix}, \quad H_\nu^{(2)}(x) = i \left( \frac{2}{\pi x} \right)^{1/2} e^{-ix}.
\]
Moreover, \([\text{Wat44} 3.71 (13)]\) reads
\[
K_\nu(x) = \left( \frac{\pi}{2x} \right)^{1/2} e^{-x}.
\]
Therefore, from the formulae in Proposition 2.7 we have
\[
J(x; \pm, \pm, \frac{1}{2}, -\frac{1}{4}) = \left( \frac{\pi}{x} \right)^{1/2} e^{2ix \pm \frac{1}{2} \pi i}, \quad J(x; \pm, \mp, \frac{1}{2}, -\frac{1}{4}) = \left( \frac{\pi}{x} \right)^{1/2} e^{-2ix \mp \frac{1}{2} \pi i}.
\]
These formulae admit generalizations to arbitrary rank.

**Proposition 2.9.** For \(\mathbf{s} \in \{+, -\}^n\) we define \(L_\pm(\mathbf{s}) = \{\ell : \mathbf{s}_\ell = \pm\}\) and \(n_\pm(\mathbf{s}) = |L_\pm(\mathbf{s})|\). Put \(\xi(\mathbf{s}) = i e^{n\pi \mathbf{s} \cdot (\frac{n-1}{2})} = e^{\mp \pi i n_\pm(\mathbf{s})}.\) Suppose \(\lambda = \lambda(\mathbf{s}) = \frac{1}{n} \left( \frac{n-1}{2}, \ldots, \frac{n-1}{2} \right),\) that is, \(\lambda_\ell = \frac{1}{n} \left( \frac{n+1}{2} - \ell \right).\) Then we have
\[
J(x; \mathbf{s}, \lambda) = c(\mathbf{s}) \left( \frac{2\pi}{x} \right)^{n-1} e^{i\xi(\mathbf{s}) x},
\]
where the constant \(c(\mathbf{s}) = e \left( \mp \frac{n+1}{2} \right) \pm \frac{n_\pm(\mathbf{s})}{2} \sum_{\ell \in L_\pm(\mathbf{s})} \ell \right)\) has norm 1.

**Proof.** Using the multiplication formula of the gamma function
\[
\prod_{k=0}^{n-1} \Gamma \left( s + \frac{k}{n} \right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - n} \Gamma(ns),
\]
some calculations yield
\[
G(s; \mathbf{s}, \lambda) = c_1(\mathbf{s}) \left( 2\pi \right)^{\frac{n-1}{2}} n^{\frac{1}{2} - n} \Gamma \left( n \left( s - \frac{n+1}{2n} \right) \right) \Gamma \left( n \left( s - \frac{n-1}{2n} \right) \right) e \left( \frac{n_+(\mathbf{s}) - n_-(\mathbf{s})}{4n} \cdot s \right),
\]
with \(c_1(\mathbf{s}) = e \left( \mp \frac{n+1}{2n} \right) \pm \frac{n_\pm(\mathbf{s})}{2} \sum_{\ell \in L_\pm(\mathbf{s})} \ell \right).\) Inserting this into the definition (2.5) of \(J(x; \mathbf{s}, \lambda)\) and making the change of variables from \(s\) to \(\frac{s}{2} \left( s + \frac{n+1}{2} \right),\) we arrive at
\[
J(x; \mathbf{s}, \lambda) = c_1(\mathbf{s}) c_2(\mathbf{s}) \left( 2\pi \right)^{\frac{n-1}{2}} \left( 2\pi \right)^{\frac{n+1}{2}} \left( \frac{2\pi}{x} \right)^{n-1} \Gamma(s) e \left( \frac{n_+(\mathbf{s}) - n_-(\mathbf{s})}{4n} \cdot s \right) (nx)^{-s} ds,
\]
with \(c_2(\mathbf{s}) = e \left( \mp \frac{n-1}{2} \right) \pm \frac{n_\pm(\mathbf{s})}{2} \sum_{\ell \in L_\pm(\mathbf{s})} \ell \right).\) Now (2.14) follows from (2.13) upon suitably choosing the contour \(\mathcal{C}.\)

**Q.E.D.**

3. The rigorous interpretation of formal integral representations

We first introduce some new notations. Let \(d = n - 1, \ t = (t_1, \ldots, t_d) \in \mathbb{R}_+^d,\)
\(v = (v_1, \ldots, v_d) \in \mathbb{C}^d\) and \(\mathbf{s} = (s_1, \ldots, s_d, s_{d+1}) \in \{+, -\}^{d+1}.\) For \(a > 0\) define \(\mathbb{S}_d^a = \{v \in \mathbb{C}^d : |\text{Re } v_\ell| < a \text{ for all } \ell = 1, \ldots, d\}.\) For \(v \in \mathbb{C}\) write
\[
[v]_\alpha = \prod_{k=0}^{\alpha-1} (v - k), \quad (v)_\alpha = \prod_{k=0}^{\alpha-1} (v + k) \text{ if } \alpha \geq 1, \quad [v]_0 = (v)_0 = 1,
\]
Denote by $p_\nu$ the power function

$$p_\nu(t) = \prod_{\ell=1}^d t_\ell^{\nu-1},$$

let

$$\theta(t; \varsigma) = \zeta_{d+1} t_1 \ldots t_d + \sum_{\ell=1}^d \zeta_\ell t_\ell^{\nu-1},$$

and define $J_\nu(x; \varsigma)$ to be the formal integral

$$J_\nu(x; \varsigma) = \int_{\mathbb{R}_+^d} p_\nu(t) e^{ix\theta(t; \varsigma)} dt.$$ 

We see that the formal integral representation (2.10) of $J(x; \varsigma, \lambda)$ is equal to $J_\nu(x; \varsigma)$ with $\nu = \lambda_d - \lambda_{d+1}$.

For $d = 1$, we saw in §2.3 that $J_\nu(x; \varsigma)$ is conditionally convergent for $|\Re \nu| < 1$. However, it fails to be absolutely convergent. If $d \geq 2$, we are in a worse scenario. The notion of convergence for multiple integrals is always in the absolute sense. Thus, the integral in (3.1), viewed as a multiple integral, is nonsense by itself since it is clearly not convergent.

In the following, we shall address this fundamental convergence issue of the formal integral (3.1), relying on its structural simplicity, so that it will be provided with mathematically rigorous meanings. Moreover, it will be shown that our rigorous interpretation of $J_\nu(x; \varsigma)$ is a smooth function of $x$ on $\mathbb{R}_+$ as well as an analytic function of $\nu$ on $\mathbb{C}^d$.

### 3.1. Formal partial integration operators.

The most crucial observation is that there are two kinds of formal partial integrations for the integral $J_\nu(x; \varsigma)$. The first kind corresponds to

$$\partial \left( e^{ix\theta_1 t_1^{\nu-1}} \right) = -\zeta_1 t_1^{\nu-2} e^{ix\theta_1 t_1^{\nu-1}} \partial t_1,$$

and the second kind to

$$\partial \left( e^{ix\theta_{d+1} - \lambda_d} \right) = \zeta_{d+1} t_1 \ldots t_d e^{ix\theta_d t_1^{\nu-1}} \partial t_d,$$

where $\partial t_\ell$ means omitting $t_\ell$ from the product. More generally and precisely, we have the following definitions.

**Definition 3.1.** Let

$$\mathcal{F}(\mathbb{R}_+) = \left\{ h \in C^\infty(\mathbb{R}_+) : t^\alpha h(t) \in L^1 1 \text{ for all } \alpha \in \mathbb{N} \right\}.$$
For $h(t) \in \mathbb{S}_d \mathcal{T}(\mathbb{R}_+)$, which means that $h(t)$ is a linear combination of functions of the form $\prod_{\ell=1}^d h_\ell(t_\ell)$, define the formal integral

$$J_\nu(x; \zeta; h) = \int_{\mathbb{R}_+^d} h(t)p_\nu(t)e^{i\omega(t)} dt,$$

and call $J_\nu(x; \zeta; h)$ a J-integral of index $\nu$. Define an auxiliary space

$$\mathcal{F}_\nu(\zeta) = \text{Span}_{\mathbb{C}[x^{-1}]} \left\{ J_\nu(x; \zeta; h) : \nu' \in \nu + \mathbb{Z}^d, h \in \mathbb{S}_d \mathcal{T}(\mathbb{R}_+) \right\}.$$

Here $\mathbb{C}[x^{-1}]$ is the ring of polynomial with variable $x^{-1}$ and complex coefficients. $\mathcal{P}_{+,\ell}$ and $\mathcal{P}_{-,\ell}$ are defined to be $\mathbb{C}[x^{-1}]$-linear operators on the space $\mathcal{F}_\nu(\zeta)$, in symbolic notion, as follows,

$$\mathcal{P}_{+,\ell}(J_\nu(x; \zeta; h)) = \zeta_i s_{d+1} J_{\nu+e^\ell+e_t} (x; \zeta; h)$$

$$- \zeta_i it_{\ell+1}^{-1} x J_{\nu+e^\ell} (x; \zeta; h) - \zeta_i i x^{-1} J_{\nu+e^\ell} (x; \zeta; t_\ell \partial_\ell h),$$

$$\mathcal{P}_{-,\ell}(J_\nu(x; \zeta; h)) = \zeta_i s_{d+1} J_{\nu-e^\ell-e_t} (x; \zeta; h)$$

$$+ \zeta_i (\nu_t - 1) x^{-1} J_{\nu-e^\ell} (x; \zeta; h) + \zeta_i i x^{-1} J_{\nu-e^\ell} (x; \zeta; t_\ell \partial_\ell h),$$

where $e_\ell = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $e^\ell = (1, \ldots, 1)$, and $\partial_\ell h$ is the abbreviated $\partial h/\partial t_\ell$.

The formulation of $\mathcal{P}_{+,\ell}$ is quite involved at a first glance. However, the most essential feature of these operators is simply index-shift! For this, we have the following observation.

**Observation.** After the operation of $\mathcal{P}_{+,\ell}$ on a J-integral of index $\nu$, all indices of the three resulting J-integrals are nondecreasing and the increment of $v_t$ is one greater than the others. The operator $\mathcal{P}_{-,\ell}$ has the effect of decreasing all indices by one except possibly two for $v_t$.

**Lemma 3.2.** Let notations be as above.

1. Let $h(t) = \prod_{\ell=1}^d h_\ell(t_\ell)$. Suppose that $\{1, 2, \ldots, d\}$ splits into two subsets $L_+$ and $L_-$, and that $h_\ell \in \mathcal{T}(\mathbb{R}_+)$ vanishes at infinity if $\ell \in L_-$ and vanishes in a neighbourhood of zero if $\ell \in L_+$. If $\Re v_\ell > 0$ for all $\ell \in L_-$ and $\Re v_\ell < 0$ for all $\ell \in L_+$, then the J-integral $J_\nu(x; \zeta; h)$ absolutely converges.

2. Assume the same conditions in (1). Moreover, suppose that $\Re v_\ell > 1$ for all $\ell \in L_-$ and $\Re v_\ell < -1$ for all $\ell \in L_+$. For $\ell \in L_-$ the resulting integrals in $\mathcal{P}_{+,\ell}(J_\nu(x; \zeta; h))$ remain absolutely convergent and the operation of $\mathcal{P}_{+,\ell}$ on $J_\nu(x; \zeta; h)$ is the actual partial integration of the first kind on the integral over $t_\ell$. Similarly, for $\ell \in L_+$ the operation
of \( \mathcal{P}_{-\ell} \) preserves absolute convergence and is the actual partial integration of the second kind on the integral over \( \tau_t \).

(3). \( \mathcal{P}_{+\ell} \) and \( \mathcal{P}_{-\ell} \) commute with \( \mathcal{P}_{+k} \) and \( \mathcal{P}_{-k} \) if \( \ell \neq k \).

(4). For a nonnegative integer \( \alpha \) we have

\[
\mathcal{P}_{+\ell}^\alpha (J_v(x; \xi; h)) = \mathcal{P}_{+\ell}^\alpha \left[ \mathcal{P}_{+\ell}^{\alpha} \sum_{\alpha_1, \alpha_2, \alpha_3 \geq 0, \alpha_1 + \alpha_2 + \alpha_3 = \alpha} \frac{\alpha!}{\alpha_1! \alpha_2! \alpha_3!} [v_I + 2\alpha - 1]_{\alpha_1} x^{-\alpha + \alpha_1} J_{\nu + \alpha_1, \xi, \xi} t_{\nu, \xi} h \right],
\]

and

\[
\mathcal{P}_{-\ell}^\alpha (J_v(x; \xi; h)) = \mathcal{P}_{-\ell}^\alpha \left[ \mathcal{P}_{-\ell}^{\alpha} \sum_{\alpha_1, \alpha_2, \alpha_3 \geq 0, \alpha_2 + \alpha_3 = \alpha} (-)^{\alpha_2} \frac{\alpha_2!(\alpha_2 - 1)!}{\alpha_2!} [v_I - 1]_{\alpha_3} x^{-\alpha + \alpha_2} J_{\nu - \alpha_2, \xi, \xi} t_{\nu, \xi} h \right].
\]

**Proof.** (1-3) are obvious. The formulae in (4) follow from calculating

\[
(-)^\alpha \mathcal{P}_{+\ell}^\alpha \left[ (-\xi i x)^{-\alpha} h(t) e^{i(x\xi + i\xi t_{\nu, \xi} + 1 + \sum_{\alpha \xi, \alpha_1 \xi} \xi_1)} e^{-i\xi t_{\nu, \xi}} \right] e^{i\xi t_{\nu, \xi}}
\]

and

\[
(-)^\alpha \mathcal{P}_{-\ell}^\alpha \left[ (\xi_{\ell + 1} i x)^{-\alpha} h(t) e^{i(x\xi + i\xi t_{\nu, \xi} + 1 + \sum_{\alpha \xi, \alpha_1 \xi} \xi_1)} e^{-i\xi t_{\nu, \xi}} \right] e^{i\xi t_{\nu, \xi}},
\]

where we recall that for \( \alpha \geq 1 \) and \( \alpha \in \mathbb{C} \)

\[
\frac{d^\alpha (e^{\alpha t_{\nu, \xi}})}{dt^\alpha} = (-)^\alpha \sum_{\beta=1}^{\alpha} \frac{\alpha! (\alpha - 1)!}{\beta! (\beta - 1)!} \beta^\alpha e^{\alpha t_{\nu, \xi}}.
\]

Q.E.D.

3.2. Partitioning the integral \( J_v(x; \xi) \). Let \( I \) be a finite set including \( \{+, -\} \) and

\[
\sum_{\ell \in I} h_\ell(t) = 1, \quad t \in \mathbb{R}_+,
\]

be a partition of unity on \( \mathbb{R}_+ \) such that \( h_\ell \) are smooth functions, \( h_-(t) = 1 \) on a neighbourhood of zero and \( h_+(t) = 1 \) for large \( t \). Put \( h_\ell(t) = \prod_{\ell=1}^d h_\ell(t) \) for \( \ell = (\ell_1, ..., \ell_d) \in I^d \).

We partition the integral \( J_v(x; \xi) \) into a finite sum of integrals

\[
J_v(x; \xi) = \sum_{\ell \in I} J_v(x; \xi; h_\ell),
\]

where

\[
J_v(x; \xi; h) = J_v(x; \xi; h_\ell) = \int_{\mathbb{R}_+} h_\ell(t) p_v(t) e^{i\xi(t, \xi)} dt.
\]
3.3. The definition of $I_\nu(x; \xi)$.

Let $a > 0$ and assume $v \in \mathbb{S}_\alpha$. Let $A \geq a + 2$ be a positive integer. For $\varrho \in \mathbb{D}$ denote $L_\pm(\varrho) = \{ \ell : \varrho_\ell = \pm \}$.

Firstly, we treat $J_\nu(x; \xi; \varrho)$ with $\varrho$ such that both $L_+(\varrho)$ and $L_-(\varrho)$ are nonempty. Let $\mathcal{P}_{+\varrho} = \prod_{\ell \in L_+(\varrho)} \mathcal{P}_{+\ell}$. This is well-defined by commutativity (Lemma 3.2 (3)). By Lemma 3.2 (4) we see that $\mathcal{P}_{+\varrho}(J_\nu(x; \xi; \varrho))$ is a linear combination of

\[
\left( \prod_{\ell \in L_-(\varrho)} [v_\ell + 4A - 1]_{\alpha_1, \ell} \right) x^{-2A|L_-(\varrho)| + \sum_{\ell \in L_-(\varrho)} \alpha_1, \ell}.
\]

(3.2)

\[
J_{\nu+}(\prod_{\ell \in L_-(\varrho)} \mathcal{P}_{+\ell}) e^{2A \sum_{\ell \in L_-(\varrho)} \alpha_1, \ell} (x; \xi; (\prod_{\ell \in L_-(\varrho)} \mathcal{P}_{+\ell}) h_\xi),
\]

with $\alpha_1, \ell, \alpha_2, \ell, \alpha_3, \ell \geq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 2A$ for each $\ell \in L_-(\varrho)$. Then, choose $\ell_+ \in L_+(\varrho)$ and apply $(\sum_{\ell \in L_-(\varrho)} \alpha_1, \ell + A)$-times of $\mathcal{P}_{-\ell_+}$ on the $J$-integral in (3.2). By Lemma 3.2 (4) we obtain a linear combination of

\[
[v_\ell_+ - 1]_{\alpha_1} \left( \prod_{\ell \in L_-(\varrho)} [v_\ell + 4A - 1]_{\alpha_1, \ell} \right) x^{-A|L_-(\varrho)| + 1} + \alpha_1,
\]

(3.3)

\[
J_{\nu-}(\prod_{\ell \in L_-(\varrho)} \mathcal{P}_{-\ell}) e^{2A \sum_{\ell \in L_-(\varrho)} \alpha_1, \ell} (x; \xi; (\prod_{\ell \in L_-(\varrho)} \mathcal{P}_{-\ell}) h_\xi),
\]

with $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \leq \sum_{\ell \in L_-(\varrho)} \alpha_1, \ell + A$. For $\ell \in L_-(\varrho)$ the real part of the $\ell$-th index of the $J$-integral is $\Re v_\ell + A > 0$, whereas for $\ell \in L_+(\varrho)$ this is $\Re v_\ell - A - \alpha_1 \leq \Re v_\ell - A < 0$. Therefore, the $J$-integral in (3.3) is absolutely convergent according to Lemma 3.2 (1). We define the total linear combination of $J$-integrals obtained after all these operations to be $I_\nu(x; \xi; \varrho)$.

For $\varrho$ with $L_-(\varrho) \neq \emptyset$ and $L_+(\varrho) = \emptyset$, we define $I_\nu(x; \xi; \varrho) = \mathcal{P}_{+\varrho}(J_\nu(x; \xi; \varrho))$. This is a linear combination of absolutely convergent integrals,

\[
\left( \prod_{\ell \in L_-(\varrho)} [v_\ell + 2A - 1]_{\alpha_1, \ell} \right) x^{-A|L_-(\varrho)| + \sum_{\ell \in L_-(\varrho)} \alpha_1, \ell}.
\]

(3.4)

\[
J_{\nu+}(\prod_{\ell \in L_-(\varrho)} \mathcal{P}_{+\ell}) e^{A \sum_{\ell \in L_-(\varrho)} \alpha_1, \ell} (x; \xi; (\prod_{\ell \in L_-(\varrho)} \mathcal{P}_{+\ell}) h_\xi),
\]

with $\alpha_1, \ell + \alpha_2, \ell + \alpha_3, \ell = A$.

For $\varrho$ with $L_+(\varrho) \neq \emptyset$ and $L_-(\varrho) = \emptyset$, choose $\ell_+ \in L_+(\varrho)$ and define $I_\nu(x; \xi; \varrho) = \mathcal{P}_{-\ell_+}(J_\nu(x; \xi; \varrho))$. This is a linear combination of absolute convergent integrals,

\[
[v_\ell_+ - 1]_{\alpha_1} x^{\alpha_1 + A} J_{\nu-}(\mathcal{P}_{-\ell_+}^A) (x; \xi; (\mathcal{P}_{-\ell_+}^A) h_\xi),
\]

(3.5)

\[
\text{with } \alpha_1 + \alpha_2 + \alpha_3 \leq A.
\]

Finally, put $I_\nu(x; \xi; \varrho) = J_\nu(x; \xi; \varrho)$ for $\varrho$ with empty $L_-(\varrho)$ and $L_+(\varrho)$.

**Lemma 3.3.** The definition of $I_\nu(x; \xi; \varrho)$ is independent of $A$ and the choice of $\ell_+ \in L_+(\varrho)$.

**Proof.** We treat the case that both $L_+(\varrho)$ and $L_-(\varrho)$ are nonempty. The other cases are similar and simpler.
Since we assumed \( A \geq a + 2 \) (note that \( A \geq a \) is sufficient to guarantee the absolute convergence), we are able to maintain absolute convergence, starting with \( I_\nu(x; \zeta; \varrho) \) defined with \( A \), when performing the following operations in succession for all \( \ell \in L_- (\varrho) \): \( \mathcal{P}_{+, \ell} \) twice, and then \( \mathcal{P}_{-; \ell, -} \) once, twice or three times on the resulting \( J \)-integrals so that the increment of the \( \ell \)-th index is one. Then we obtain \( I_\nu(x; \zeta; \varrho) \) defined with \( A + 1 \) in view of Lemma 3.2 (3). Under our circumstances, \( \mathcal{P}_{+; \ell} \) and \( \mathcal{P}_{-; \ell, +} \) are actual partial integrations (Lemma 3.2 (2)), so the value of integral is preserved in the process. Therefore, \( I_\nu(x; \zeta; \varrho) \) is independent of \( A \).

Suppose \( \ell_+, k_+ \in L_+ (\varrho) \). Repeating the process described in the last paragraph \( A \)-times with \( \ell_+ \) replaced by \( k_+ \), \( I_\nu(x; \zeta; \varrho) \) defined with \( \ell_+ \) turns into an integral with expression symmetric about \( \ell_+ \) and \( k_+ \), where both \( \mathcal{P}_{-; \ell, -} \) and \( \mathcal{P}_{-; \ell, +} \) have been applied \( A \)-times. Interchanging \( \ell_+ \) and \( k_+ \) throughout the arguments above, \( I_\nu(x; \zeta; \varrho) \) defined with \( k_+ \) is transformed into the same integral. Thus we conclude that \( I_\nu(x; \zeta; \varrho) \) is independent of the choice of \( \ell_+ \). 

Putting these together, we define

\[
I_\nu(x; \zeta) = \sum_{\zeta \in \mathcal{I}^d} I_\nu(x; \zeta; \varrho),
\]

and call \( I_\nu(x; \zeta) \) the \textit{rigorous interpretation} of \( J_\nu(x; \zeta) \). The definition of \( I_\nu(x; \zeta) \) is clearly independent of the partition of unity \( \{ \varrho \}_\ell \in \mathcal{I}^d \) on \( \mathbb{R}_+ \).

Uniform convergence implies that \( I_\nu(x; \zeta) \) is an analytic function of \( \nu \) on \( \mathbb{R}_a^d \), and then on the whole \( \mathbb{C}^d \) since \( a \) was arbitrary. Moreover, for \( j \geq 0 \) choose \( A \geq a + j + 2 \), then differentiating \( I_\nu(x; \zeta) \) \( j \)-times under the integral with respect to \( x \) is legitimate. Thus, \( I_\nu(x; \zeta) \) is a smooth function of \( x \).

Henceforth, with ambiguity, we shall write \( I_\nu(x; \zeta) \) as \( J_\nu(x; \zeta) \), and \( I_\nu(x; \zeta; \varrho) \) as \( J_\nu(x; \zeta; \varrho) \), respectively.

4. Equality between \( J_\nu(x; \zeta) \) and \( J(x; \zeta; \lambda) \)

The goal of this section is to show that the Bessel function \( J(x; \zeta; \lambda) \) is indeed equal to the rigorous interpretation of its formal integral representation.

Proposition 4.1. Suppose that \( \lambda \in \mathbb{R}_a^d \) and \( \nu \in \mathbb{C}^d \) satisfy \( \nu_\ell = \lambda_\ell - \lambda_{d+1} \), \( \ell = 1, \ldots, d \). Then

\[
J(x; \zeta; \lambda) = J_\nu(x; \zeta).
\]

We first show how the iterated integral \( \Theta(x; \zeta) \) in (2.9) is interpreted (compare [MS04a §6] and [MS06 §5]).

Suppose that \( \Re \nu_1 > \ldots > \Re \nu_d > 0 \). Let

\[
(4.1) \quad \Theta_{d+1}(x; \zeta; \lambda_{d+1}) = \int_{\mathbb{R}_+} v(y)y^{-\lambda_{d+1}}e(\zeta_{d+1}xy)d\nu, \quad x \in \mathbb{R}_+,
\]
and for each $\ell = 1, \ldots, d$ recursively define
\[
\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu')
\]
\[
= \int_{\mathbb{R}^+} \mathcal{Y}_{\ell+1} (y; \varsigma, \lambda_{d+1}, \nu'^{\ell+1}) y^{\nu_1 - \nu_{\ell+1} - 1} e \left( \varsigma_1 y^{\nu_1 - 1} \right) dy, \quad x \in \mathbb{R}^+, \tag{4.2}
\]
where $\nu' = (\nu_1, \ldots, \nu_d)$, and we denote $(\lambda_{d+1}, \nu'^{\ell+1}) = \lambda_{d+1}$ and $\nu_{d+1} = 0$ for convenience.

The following lemma justifies the definitions (4.1) and (4.2) of $\mathcal{Y}_{d+1} (x; \varsigma, \lambda_{d+1})$ and $\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu')$.

**Lemma 4.2.** Suppose that $\Re v_1 > \ldots > \Re v_d > 0$. Recall the definition of $\mathcal{F}(\mathbb{R}^+)$ given in Definition 3.1 and define the space $\mathcal{F}_x(\mathbb{R}^+)$ of all functions in $\mathcal{F}(\mathbb{R}^+)$ which decay rapidly at infinity, along with all their derivatives. Then $\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu') \in \mathcal{F}_x(\mathbb{R}^+)$ for each $\ell = 1, \ldots, d + 1$.

**Proof.** We apply an induction on $\ell$.

In the case $\ell = d + 1$, $\mathcal{Y}_{d+1} (x; \varsigma, \lambda_{d+1})$ is the Fourier transform of a Schwartz function on $\mathbb{R}$ (supported on $[0, \infty)$), and hence is actually a Schwartz function on $\mathbb{R}$. In particular, $\mathcal{Y}_{d+1} (x; \varsigma, \lambda_{d+1}) \in \mathcal{F}_x(\mathbb{R}^+)$.

Indeed, partial integration on the integral in (4.1) has the effect of dividing $-\varsigma_{d+1} 2\pi i x$ and results in an integral of the same type, so repeating partial integration shows that $\mathcal{Y}_{d+1} (x; \varsigma, \lambda_{d+1})$, and similarly all its derivatives, rapidly decays.

Suppose that $\mathcal{Y}_{\ell+1} (x; \varsigma, \lambda_{d+1}, \nu'^{\ell+1}) \in \mathcal{F}_x(\mathbb{R}^+)$. Partial integration on the integral in (4.2) has the effect of dividing $\varsigma_{d+1} 2\pi i x$ and raising the power of $y$ by one, so repeating this yields the rapid decay of $\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu')$. Differentiation under the integral decreases the power of $y$ by one, so $\alpha$-time differentiating $\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu')$ is legitimate after repeating partial integrations $\alpha$ times, $\alpha \in \mathbb{N}$. From this we show that $\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu') \in \mathcal{F}(\mathbb{R}^+)$. Finally, keeping repeating partial integrations yields the rapid decay of all the derivatives of $\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu')$. Q.E.D.

Changing variables from $y$ to $xy$ in (4.2) yields

\[
\mathcal{Y}_{\ell} (x; \varsigma, \lambda_{d+1}, \nu') = \int_{\mathbb{R}^+} \mathcal{Y}_{\ell+1} (xy; \varsigma, \lambda_{d+1}, \nu'^{\ell+1}) x^{\nu_1 - \nu_{\ell+1}} y^{\nu_{\ell+1} - 1} e \left( \varsigma_1 y^{\nu_1 - 1} \right) dy.
\]

Therefore, $\mathcal{Y}_{1} (x; \varsigma, \lambda_{d+1}, \nu')$ is equal to the following iterated integral

\[
x^{\nu_1} \int_{\mathbb{R}^+} v(y) y^{-\lambda_{d+1}} \left( \prod_{\ell=1}^{d} y_{\nu_{\ell}}^{-1} \right) e \left( \varsigma_{d+1} x y_{1} y_{2} \ldots y_{d} + \sum_{\ell=1}^{d} \varsigma_{\ell} y_{\nu_{\ell}}^{-1} \right) dy_{d} \ldots dy_{1}.
\]

Comparing this with (2.9), we have $\mathcal{Y}(x; \varsigma) = x^{-\nu_{d+1}} \mathcal{Y}_{1} (x; \varsigma, \lambda_{d+1}, \nu')$.

The (actual) partial integration $\mathcal{P}$ on the integral over $dy_{\ell}$ corresponds to $\mathcal{P}_{+, \ell}$, whereas the partial integration $\mathcal{P}_{d+1}$ on the integral over $dy$ has the similar effect as $\mathcal{P}_{-, \ell_{+}}$ of decreasing the powers of all the $y_{\nu}$ by one (see the proof of Lemma 4.2). These observations are crucial to our proof of Proposition 4.1 as follows.

**Proof of Proposition 4.1** Suppose that $\Re v_1 > \ldots > \Re v_d > 0$. We first partition the integral over $dy_{\ell}$ in (4.3), for each $\ell = 1, \ldots, d$, into a sum of integrals according to a
partition of unity \( \{ h_\omega \}_\omega \) of \( \mathbb{R}_+ \). These partitions result in a partition of the integral \((4.3)\) into the sum
\[
T_1(x; \mathcal{C}, \lambda_{d+1}, \nu) = \sum_{\varrho \in \mathcal{P}} T_1(x; \mathcal{C}, \lambda_{d+1}, \nu, \varrho),
\]
with
\[
T_1(x; \mathcal{C}, \lambda_{d+1}, \nu, \varrho) = x^{\nu_1} \int_{\mathcal{P}_{\lambda+1}} u(y) y^{-\lambda_{d+1}} \left( \prod_{\ell=1}^d h_{\nu_\ell}(y_\ell) y_\ell^{\nu_\ell - 1} \right) e^{\left( \zeta_d x y_1 \cdots y_d + \sum_{\ell=1}^d \zeta_\ell y_\ell^{\nu_\ell - 1} \right) dy_d \cdots dy_1}.
\]

We now perform the operations in \((4.3)\) with \( \mathcal{P}_{\lambda+} \) replaced by \( \mathcal{P}_\ell \) and \( \mathcal{P}_{\lambda-} \) by \( \mathcal{P}_{d+1} \) to each integral \( T_1(x; \mathcal{C}, \lambda_{d+1}, \nu, \varrho) \) defined in \((4.3)\). While preserving the value, these (actual) partial integrations turn the iterated integral \( T_1(x; \mathcal{C}, \lambda_{d+1}, \nu, \varrho) \) into an absolutely convergent multiple integral. Then we are able to change the order of integrals and move the innermost integral over \( dy_d \) to the outermost place. The integral over \( dy_d \cdots dy_1 \) becomes now the inner integral. Make a change of variables \( y_\ell = t_\ell(x y_1 \cdots y_\ell)^{-\nu_\ell} \) to the inner integral, then the partial integration \( \mathcal{P}_\ell \) that we did turns into \( \mathcal{P}_{\lambda+} \). By the same arguments in the proof of Lemma \((4.3)\) showing that \( J_\nu(x; \mathcal{C}) \) is independent of the choice of \( \lambda_+ \in L_+(\mathcal{C}) \), the operations of \( \mathcal{P}_{d+1} \) we performed at the beginning may be reversed and substituted by the operations of \( \mathcal{P}_{\lambda-} \). It follows that the inner integral over \( dy_d \cdots dy_1 \) is equal to \( x^{\nu_1 - \lambda_{d+1}} u(y) J_\nu(2\pi(x y_1 \cdots y_d)^{-\nu_1}; \mathcal{C}, \varrho) \), with \( h_\nu(t) = h_\nu(t(x y_1 \cdots y_d)^{-\nu_1}) \). Summing over \( \varrho \in \mathcal{P}_d \), we conclude that
\[
T(x; \mathcal{C}) = x^{\nu_1 - \lambda_{d+1}} T_1(x; \mathcal{C}, \lambda_{d+1}, \nu) = \int_0^{\infty} u(y) \frac{J_\nu(2\pi(x y_1 \cdots y_d)^{-\nu_1}; \mathcal{C})}{y_1 \cdots y_d} dy.
\]
Therefore, in view of \((2.4)\) we have \( J(x; \mathcal{C}, \lambda) = J_\nu(x; \mathcal{C}) \). This equality still holds true for arbitrary \( \nu \) by analyticity.

In view of Proposition \((1.1)\) we shall subsequently let \( \lambda \in \mathbb{R}^d \) and \( \nu \in \mathbb{C}^d \) be related by \( \nu_\ell = \lambda_\ell - \lambda_{d+1}, \ell = 1, \ldots, d \).

5. \( H \)-Bessel functions and \( K \)-Bessel functions

According to Proposition \((2.7)\) \( J_{2\pm}(x; \mathcal{C}, \pm) = J(x; \pm, \pm, \lambda, -\lambda) \) is a Hankel function, and \( J_{2\pm}(x; \pm, \mp) = J(x; \pm, \mp, \lambda, -\lambda) \) is a \( K \)-Bessel function. There is a remarkable difference between behaviours of the Hankel functions and the \( K \)-Bessel function at large argument. The Hankel functions oscillate and decay proportionally to \( \frac{1}{\sqrt{x}} \), whereas the \( K \)-Bessel function exponentially decays. On the other hand, such phenomenons persist in higher rank for the prototypical example shown in Proposition \((2.9)\).

In the following, we shall show that such a categorization stands in general for the Bessel functions \( J_\nu(x; \mathcal{C}) \) of an arbitrary index \( \nu \). For this, we shall analyze each integral \( J_\nu(x; \mathcal{C}, \varrho) \) in the rigorous interpretation of \( J_\nu(x; \mathcal{C}) \) using the method of stationary phase.
First of all, the asymptotic behaviour of \( J_\nu(x; \varsigma) \) at large argument should rely on the existence of a stationary point of the phase function \( \theta(t; \varsigma) \) on \( \mathbb{R}^d_+ \). Since

\[
\theta'(t; \varsigma) = \left( \varsigma_{d+1} t_1 \ldots t_d - \varsigma_t^{-1} \right)^d_{\ell=1},
\]
a stationary point of \( \theta(t; \varsigma) \) exists in \( \mathbb{R}^d_+ \) if and only if \( \varsigma_1 = \ldots = \varsigma_d = \varsigma_{d+1} \), and is equal to \( t_0 = (1, \ldots, 1) \) if it exists.

Terminology 5.1. We write \( H^\pm_\nu(x) = J_\nu(x; \pm, \ldots, \pm) \), \( H^\mp_\nu(x; \lambda) = J(x; \pm, \ldots, \pm, \lambda) \), and call them H-Bessel functions. If two of the \( \varsigma_1, \ldots, \varsigma_d, \varsigma_{d+1} \) are different, then \( J_\nu(x; \varsigma) \), or \( J(x; \varsigma, \lambda) \), is called a K-Bessel function.

Notations and preparations. We shall retain the notations in §3. Moreover, for our purpose we choose a partition of unity \( \{ h_\ell \}_{\ell \in \{-0,+\}} \) on \( \mathbb{R}^+ \) such that \( h_- \), \( h_0 \) and \( h_+ \) are smooth functions supported on \( K_- = (0, \frac{1}{2}] \), \( K_0 = \left[ \frac{1}{2}, 4 \right] \) and \( K_+ = [2, \infty) \) respectively, and that \( r^\ell h_\ell(t) \leq_\alpha 1 \) for any \( \alpha = -, 0, + \) and \( \alpha \in \mathbb{N} \). Put \( K_\sigma = \prod_{\ell=1}^d K_\sigma_\ell \) and \( h_\sigma(t) = \prod_{\ell=1}^d h_\sigma_\ell(t) \) for \( \sigma \in \{-, 0, +\}^d \). Note that \( t_0 \) is enclosed in the central hypercube \( K_0 \). According to this partition of unity, \( J_\nu(x; \varsigma) \) is partitioned into the sum of 3\(^d\) integrals \( J_\nu(x; \varsigma; \sigma) \). In view of (3.3, 3.4, 3.5), \( J_\nu(x; \varsigma; \sigma) \) is a \( \mathbb{C}[x^{-1}] \)-linear combination of absolutely convergent \( J \)-integrals of the form

\[
J_\nu(x; \varsigma, h) = \int_{\mathbb{R}^+} h(t) p_\nu(t) e^{i\theta(t; \varsigma)} dt,
\]
where \( \nu' \in \nu + \mathbb{Z}^d \) satisfies

\[
\Re \nu' \geq \Re \nu + A \text{ if } \ell \in L_-(\sigma), \text{ and } \Re \nu' \leq \Re \nu - A \text{ if } \ell \in L_+(\sigma),
\]
with \( A > \max \{ \| \Re \nu' \| \} + 2 \) and \( h \in \mathcal{D}_d(T(\mathbb{R}^+)) \) supported in \( K_\sigma \).

5.1. Bounds for \( J_\nu(x; \varsigma; \sigma) \) with \( \sigma \neq 0 \). Let

\[
\Theta(t; \varsigma) = \sum_{\ell=1}^d \left( t_\ell \theta(t; \varsigma) \right)^2 = \sum_{\ell=1}^d \left( \varsigma_{d+1} t_1 \ldots t_d - \varsigma_t^{-1} \right)^2.
\]

Lemma 5.2. Let \( \sigma \neq 0 \). We have for all \( t \in K_\sigma \)

\[
\Theta(t; \varsigma) \geq \frac{1}{16}.
\]

Proof. Instead, we shall prove

\[
\max \left\{ \left| \varsigma_{d+1} t_1 \ldots t_d - \varsigma_t^{-1} \right| : t \in \mathbb{R}^d_+ \setminus K_\sigma \text{ and } \ell = 1, \ldots, d \right\} \geq \frac{1}{4}.
\]

If \( t_1 \ldots t_d < \frac{1}{2} \), then there exists \( t_\ell < 1 \) and hence \( \left| \varsigma_{d+1} t_1 \ldots t_d - \varsigma_t^{-1} \right| > 1 - \frac{1}{4} = \frac{1}{4} \). Similarly, if \( t_1 \ldots t_d > \frac{2}{3} \), then there exists \( t_\ell > 1 \) and hence \( \left| \varsigma_{d+1} t_1 \ldots t_d - \varsigma_t^{-1} \right| > \frac{2}{3} - 1 > \frac{1}{4} \). Finally, suppose \( \frac{1}{2} \leq t_1 \ldots t_d \leq \frac{2}{3} \), then for our choice of \( t \) there exists \( \ell \) such that \( t_\ell \notin \left( \frac{1}{4}, 2 \right) \), and therefore we still have \( \left| \varsigma_{d+1} t_1 \ldots t_d - \varsigma_t^{-1} \right| \geq \frac{1}{4} \). Q.E.D.
Using (5.3), we rewrite the $J$-integral $J_\nu(x; \cdot; h)$ in (5.1) as below,

\begin{equation}
\sum_{\ell=1}^{d} \int_{\mathbb{R}^d_+} h(t) \left( \zeta_{\ell+1} p_{\nu' + e_\ell} - \zeta_{\ell} p_{\nu' - e_\ell} \right) \Theta(t; \cdot; \cdot)^{-1} \cdot \hat{\vartheta}(t; \cdot; \cdot) e^{ix(t; \cdot; \cdot)} dt.
\end{equation}

For each $\ell$ we have

$$\hat{\vartheta}(e^{ix(t; \cdot; \cdot)}) = ix \cdot \hat{\vartheta}(t; \cdot; \cdot) e^{ix(t; \cdot; \cdot)} \hat{c}_{\ell},$$

This gives the third kind of partial integration for the integral over $t_\ell$. After applied a partial integration of the third kind for each $\ell$, (5.4) turns into

$$- (ix)^{-1} \sum_{\ell=1}^{d} \int_{\mathbb{R}^d_+} t_\ell \hat{\vartheta} h \left( \zeta_{\ell+1} p_{\nu' + e_\ell} - \zeta_{\ell} p_{\nu' - e_\ell} \right) \Theta^{-1} e^{ix(t; \cdot; \cdot)} dt$$

$$- (ix)^{-1} \sum_{\ell=1}^{d} \int_{\mathbb{R}^d_+} h \left( \zeta_{\ell+1} (\nu'_\ell + 1) p_{\nu' + e_\ell} - \zeta_{\ell} (\nu'_\ell - 1) p_{\nu' - e_\ell} \right) \Theta^{-1} e^{ix(t; \cdot; \cdot)} dt$$

$$+ \zeta_{\ell+1} 2d^2 (ix)^{-1} \int_{\mathbb{R}^d_+} h p_{\nu' + 3e_\ell} \Theta^{-2} e^{ix(t; \cdot; \cdot)} dt$$

$$+ 2(ix)^{-1} \sum_{\ell=1}^{d} \int_{\mathbb{R}^d_+} h (\zeta_{\ell} (1 - 2d) p_{\nu' + 2e_\ell - e_\ell})$$

$$- \zeta_{\ell+1} p_{\nu' + e_\ell - e_\ell} \Theta^{-2} e^{ix(t; \cdot; \cdot)} dt$$

$$+ 4(ix)^{-1} \sum_{1 \leq \ell < k \leq d} \zeta_{\ell+1} \zeta_{k+1} \int_{\mathbb{R}^d_+} h p_{\nu' + e_\ell - e_\ell} \Theta^{-2} e^{ix(t; \cdot; \cdot)} dt,$$

where $\Theta$ and $\theta$ are shorthand notations for $\Theta(t; \cdot; \cdot)$ and $\theta(t; \cdot; \cdot)$. Since the shifts of indices do not exceed 3, it follows from the condition (5.2) on $\nu'$, combined with Lemma 5.2 that all the integrals above absolutely converge provided $A > r + 3$.

Repeating the above manipulations, we obtain the following lemma via a straightforward inductive argument.

**Lemma 5.3.** Let $B$ be a nonnegative integer, and choose $A = |x| + 3B + 3$. Then $J_\nu(x; \cdot; h)$ is a linear combination of \( \left( \frac{1}{2} (d^2 - d) + 7d + 1 \right)^B \) many absolutely convergent integrals of the following form

\( (ix)^{-B} P(\nu') \int_{\mathbb{R}^d_+} t^\alpha \Theta^\varphi h(t) p_{\nu'}(t) \Theta(t; \cdot; \cdot)^{-B-B_2} e^{ix(t; \cdot; \cdot)} dt, \)

with $\alpha \in \mathbb{N}^d$, $|\alpha| + B_1 + B_2 = B$, $P(\nu') \in \mathbb{Z}[\nu']$ a polynomial in $\nu'$ of degree $B_1$ and with coefficients of size $O_{B_1,d}(1)$, and $|\nu'_{\ell'} - \nu'_{\ell}| \leq B + 2B_2$ for all $\ell = 1, \ldots, d$. Recall that in the multi-index notation $|\alpha| = \sum_{\ell=1}^{d} \alpha_{\ell}$, $t^\alpha = \prod_{\ell=1}^{d} t_{\ell}^{\alpha_{\ell}}$ and $\Theta^\varphi = \prod_{\ell=1}^{d} \Theta^\varphi_{\ell}$.

Define $\epsilon = \max \{|\nu'_{\ell}| + 1\}$ and $\tau = \max \{|\Re \epsilon_{\nu'}|\}$. Lemma 5.3 along with (3.2) (3.4) (3.5), implies that for any given $C > 0$

\[ J_\nu(x; \cdot; \cdot) \ll_{x,C,d} \left( \frac{C}{x} \right)^C \text{ for all } x \gg C. \]
Slight modification of the above arguments shows that for any \( j \geq 0 \)

\[
J^{(j)}_x (x; \xi; \varrho) \ll_{r, C, l, d} \left( \frac{\xi}{x} \right)^C \quad \text{for all } x \geq c.
\]  

**Remark 5.4.** Our proof of (5.5) is similar to that of [Hör83, Theorem 7.7.1]. Indeed, \( \Theta(t; \xi) \) plays the same role as \(|f'|^2 + 3mf \) in the proof of [Hör83, Theorem 7.7.1], where \( f \) is the phase function there. Observe that the noncompactness of \( K_\varrho \) prohibits the application of [Hör83, Theorem 7.7.1] to the \( J \)-integral in (5.1).

### 5.2. \( K \)-Bessel functions

Suppose that \( \zeta_{t_0} \neq \zeta_{d+1} \) for some \( t_0 \). Then for any \( t \in K_0 \) we have

\[
|t_0 \hat{\zeta}_{t_0}(\theta(t; \xi))| = \left| \zeta_{d+1}t_1\cdots t_d - \zeta_{t_0}t_0^{-1} \right| \geq \frac{1}{4},
\]

Similar to the arguments in §5.1 repeating the third kind partial integration for the integral over \( t_0 \) yields the same bound (5.5) for \( \varrho = 0 \).

**Remark 5.5.** For this, we may also directly apply [Hör83, Theorem 7.7.1].

**Theorem 5.6.** Let \( \epsilon = \max \{|v_1| + 1\} \) and \( \delta = \max \{ |\Re v_r| \} \). Let \( j \) and \( C \) be nonnegative integers. Suppose that two of the \( \zeta_{d+1}, \ldots, \zeta_d, \zeta_{d+1} \) are different. Then for any \( x \geq c \) we have

\[
J^{(j)}_x (x; \xi) \ll_{r, C, l, d} \left( \frac{\xi}{x} \right)^C.
\]

In particular, \( J_x (x; \xi) \) is a Schwartz function at infinity, namely, all the derivatives \( J^{(j)}_x (x; \xi) \) rapidly decay at infinity.

### 5.3. Asymptotic expansions of \( H \)-Bessel functions

In the following, we shall adopt the convention \((\pm i)^a = e^{\pm \frac{\pi}{2}ia}, \ a \in \mathbb{C}\).

We first introduce the function \( W_r (x; \pm) \), which is related to the classical Whittaker’s function of imaginary argument when \( d = 1 \) (see [WW62, §17.5, 17.6]), defined by

\[
W_r (x; \pm) = (d + 1)\hat{e}_{-i(d+1)x} H^{\pm}_r (x).
\]

Write \( H^\pm_r (x; \varrho) = J_r (x; \pm, \ldots, \pm; \varrho) \) and define

\[
W_r (x; \pm; \varrho) = (d + 1)\hat{e}_{-i(d+1)x} H^{\pm}_r (x; \varrho).
\]

For \( \varrho \neq 0 \), the bound (5.5) for \( H^\pm_r (x; \varrho) \) is also valid for \( W_r (x; \pm; \varrho) \). Therefore, we are left with analyzing \( W_r (x; \pm; 0) \). We have

\[
W^{(j)}_r (x; \pm; 0) = (d + 1)\hat{e}_{(d+1)x} (\pm i)^j \int_{K_0} (\varrho(t) - d - 1) h_0 (t) p_\varrho (t) e^{\pm i(\varrho(t) - d - 1)} dt,
\]

with

\[
\varrho(t) = \varrho(\tau; +, \ldots, +) = \tau_1 \cdots \tau_d + \sum_{\ell=1}^d \tau_{\ell}^{-1}.
\]

To study this integral, we apply the method of stationary phase in Hörmander [Hör83].
PROPOSITION 5.7. [Hör83] Theorem 7.7.5. Let \( K \subset \mathbb{R}^d \) be a compact set, \( X \) an open neighbourhood of \( K \) and \( M \) a nonnegative integer. If \( u(t) \in C^M_0(K) \), \( f(t) \in C^{3M+1}(X) \) and \( \Im f \geq 0 \) in \( X \), \( \Im f(t_0) = 0 \), \( f'(t_0) = 0 \), \( \det f''(t_0) \neq 0 \) and \( f' \neq 0 \) in \( K \setminus \{ t_0 \} \), then for \( x > 0 \)

\[
\left| \int_K u(t)e^{ixf(t)}dt - e^{ixf(t_0)}(2\pi i)^{-d} \det f''(t_0) \right|^1 \sum_{m=0}^{M-1} x^{-m-\frac{d}{2}} \mathcal{L}_m u \leq C x^{-M} \sup_{|\alpha| \leq 2M} |D^\alpha u|.
\]

Here \( C \) depends only on \( M, f, K \) and \( d \). With

\[
g_{t_0}(t) = f(t) - f(t_0) - \frac{1}{2} \langle f''(t_0)(t-t_0), t-t_0 \rangle
\]

which vanishes of third order at \( t_0 \), we have

\[
\mathcal{L}_m u = i^{-m} \sum_{r=0}^{2m} \frac{1}{2m+r(m+r)!} \langle f''(t_0)^{-1} D, D \rangle^{m+r} (g_{t_0}^r u)(t_0).
\]

This is a differential operator of order \( 2m \) acting on \( u \) at \( t_0 \). The coefficients are rational homogeneous functions of degree \(-m\) in \( f''(t_0), \ldots, f(2m+2)(t_0) \) with denominator \( (\det f''(t_0))^{3m} \). In every term the total number of derivatives of \( u \) and of \( f'' \) is at most \( 2m \).

Recall that according to Hörmander \( D = -(i\hat{\epsilon}_1, \ldots, i\hat{\epsilon}_d) \). This is a common notation in harmonic analysis.

According to \((5.6)\) we have

\[
K = K_0 = \left[ \frac{1}{4}, \frac{3}{4} \right]^d, \quad X = \left( \frac{1}{2}, \frac{3}{2} \right)^d,
\]

\[
f(t) = \pm (\theta(t) - d - 1), \quad f'(t) = \pm (t_1 \ldots t_d - t_1^{-2})_{t=1}^d, \quad t_0 = (1, \ldots, 1),
\]

\[
f''(t_0) = \pm \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}, \quad \det f''(t_0) = (\pm)^d(d+1), \quad g_{t_0}(t) = \pm G(t),
\]

\[
f''(t_0)^{-1} = \pm \frac{1}{d+1} \begin{pmatrix} d & -1 & \cdots & -1 \\ -1 & d & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & d \end{pmatrix},
\]

\[
u(t) = (d+1)^{\frac{1}{2}}(\pm 2\pi i)^{-\frac{d}{2}}(\pm i)^{\frac{d}{2}}(\theta(t) - d - 1)^{\frac{d}{2}} p_\nu(t) h_0(t),
\]

where

\[
G(t) = \left( t_1 \ldots t_d + \sum_{i=1}^d (-t_i^2 + (d+1)t_i + t_i^{-1}) \right) \quad \text{for } i \leq j \leq d.
\]

(5.8)
Observe that \( u \equiv (d + 1)^{\frac{\ell}{2}}(\pm 2\pi i)^{-\frac{\ell}{2}}(\pm i)^i (\theta - d - 1)^i p_{\theta} \) on the neighbourhood \([\frac{1}{2}, 2]^d\) of \( t_0 \). Therefore, Proposition 5.7 yields the following asymptotic expansion of \( W^{(j)}_v(x; \pm 0) \),

\[
W^{(j)}_v(x; \pm 0) = \sum_{m=0}^{M-1} (\pm i)^{j-m}B_{m,j}(v)x^{-m-\frac{\ell}{2}} + O_{j,M,d}(c^{2M}x^{-M}), \quad x > 0,
\]

with

\[
B_{m,j}(v) = \sum_{r=0}^{2m} \frac{(-)^{m+r}L^{m+r}G'(\theta - d - 1)^i p_{\theta}}{(2(d + 1))^{m+r}(m + r)!},
\]

where \( L \) is the second-order differential operator given by

\[
L = d \sum_{\ell=1}^{d} \sum_{\ell \leq k \leq d} \partial^2_{x} - 2 \sum_{1 \leq \ell < k \leq d} \partial_{x} \partial_{k}.
\]

The expression in (5.9) is symmetric with respect to \( v \). Since \( \theta - d - 1 \) vanishes of second order at \( t_0 \), \( 2j \) many derivatives are required to remove the zero of \( (\theta - d - 1)^i \) at \( t_0 \). Thus, in view of the descriptions of the differential operator \( L_u \) in Proposition 5.7, we have

\[
B_{m,j}(v) = 0 \quad \text{if} \quad m < j.
\]

Otherwise, \( B_{m,j}(v) \in \mathbb{Q}[v] \) is a symmetric polynomial in \( v \) of degree \( 2m - 2j \). In particular,

\[
B_{m,j}(v) \ll_{j,m,d} c^{2m-2j} \quad \text{if} \quad m \geq j.
\]

We now add those \( W^{(j)}_v(x; \pm 0) \) for all \( 0 \neq \mathfrak{q} \). Such contributions satisfy the bound (5.5) with \( C = 2M \). Then we obtain the following proposition.

**Proposition 5.8.** Let \( j, M \) be nonnegative integers such that \( M \geq j \). Then for \( x \geq \epsilon \) we have

\[
W^{(j)}_v(x; \pm) = \sum_{m=0}^{M-1} (\pm i)^{j-m}B_{m,j}(v)x^{-m-\frac{\ell}{2}} + O_{\epsilon,j,M,d}(c^{2M}x^{-M}).
\]

**Corollary 5.9.** Let \( j, N \) be nonnegative integers such that \( N \geq j \), and let \( \epsilon > 0 \).

1. For \( x \geq \epsilon \) we have \( W^{(j)}_v(x; \pm) \ll_{\epsilon,j,d} c^{j}x^{-j} \).
2. For \( x \geq c^{2+\epsilon} \) we have

\[
W^{(j)}_v(x; \pm) = \sum_{m=0}^{N-1} (\pm i)^{j-m}B_{m,j}(v)x^{-m-\frac{\ell}{2}} + O_{\epsilon,j,N,d}(c^{2N}x^{-N-\frac{\ell}{2}}).
\]

In particular, \( W^{(j)}_v(x; \pm) \ll_{\epsilon,j,x,d} x^{-j-\frac{\ell}{2}} \).

**Proof.** On letting \( M = j \), Proposition 5.8 yields (1). On choosing \( M \) sufficiently large so that \( (2 + \epsilon)(M - N + \frac{\ell}{2}) \geq 2(M - N) \), Proposition 5.8 together with (5.11) yields

\[
W^{(j)}_v(x; \pm) = \sum_{m=0}^{N-1} (\pm i)^{j-m}B_{m,j}(v)x^{-m-\frac{\ell}{2}} + O_{\epsilon,j,N,d}(c^{2N}x^{-N-\frac{\ell}{2}}).
\]
Finally, we consider the asymptotic of $H_{\pm}(x; \lambda) = H_{\pm}^\lambda(x)$. For this, we define $W(x; \pm, \lambda) = \sqrt{n}(2\pi i)^{-\frac{1}{2}} e^{\mp i n x} H_{\pm}(x; \lambda)$ and denote $B_{m,j}(\lambda) = B_{m,j}(\nu)$. By the definitions (2.5, 2.3) we have the symmetry of $H_{\pm}(x; \lambda) = J(\pm x, \ldots, \pm \lambda)$ with respect to $\lambda$, so the coefficients $B_{m,j}(\lambda)$ must be a symmetric polynomial in $\lambda$. As for the error term, define $C = \max \{|\lambda_j|\} + 1$ and $R = \max \{|\Re \lambda_j|\}$. Note that $\frac{1}{2}C \leq C \leq (2 - \frac{1}{n})C$ and $\frac{1}{2}x \leq \Re \leq (2 - \frac{1}{n})\Re$.

**Theorem 5.10.** Define $C = \max \{|\lambda_j|\} + 1$ and $R = \max \{|\Re \lambda_j|\}$. Let $M$ be a nonnegative integer.

1. Let $M \geq j > 0$. Then for $x \geq C$ we have
   
   $W^{(j)}(x; \pm, \lambda) = \sum_{m=j}^{M-1} (\pm i)^{-m} B_{m,j}(\lambda) x^{-m-\frac{1}{2}} + O_{\Re,j,M,n} \left( C^{2M} x^{-M} \right)$.

   Here $B_{m,j}(\lambda) \in \mathbb{Q}[\lambda]$ is a symmetric polynomial in $\lambda$ of degree $2m$, with $B_{0,0}(\lambda) = 1$. The coefficients of $B_{m,j}(\lambda)$ depends only on $m, j$ and $d$.

2. Let $B_m(\lambda) = B_{m,0}(\lambda)$. Then for $x \geq C$ we have
   
   $H_{\pm}(x; \lambda) = n^{-\frac{1}{2}} (2\pi i)^{-\frac{1}{2}} e^{\pm i n x} x^{-\frac{1}{2}} \left( \sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\lambda) x^{-m} + O_{\Re,M,d} \left( C^{2M} x^{-M+\frac{1}{2}} \right) \right)$.

**Corollary 5.11.** Let $M$ be a nonnegative integer, and let $\epsilon > 0$. Then for $x \geq C^{2+\epsilon}$ we have

$H_{\pm}(x; \lambda) = n^{-\frac{1}{2}} (2\pi i)^{-\frac{1}{2}} e^{\pm i n x} x^{-\frac{1}{2}} \left( \sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\lambda) x^{-m} + O_{\Re,M,e,d} \left( C^{2M} x^{-M} \right) \right)$.

**5.4. Concluding remarks.**

5.4.1. On analytic continuations of $H$-Bessel functions. Our observation is that the phase function $\theta$ defined by (5.7) is always positive on $\mathbb{R}_+^d$. It follows that if $x$ is replaced by $z = xe^{i\omega}$ with $x > 0$ and $0 \leq \omega \leq \pi$, then the various integrals in the rigorous interpretation of $H_{\mp}^\lambda(z)$ are still absolutely convergent, uniformly with respect to $z$, since $|e^{\pm i\omega(\theta)}| = e^{\mp x \sin \omega \theta(t)} \leq 1$. Therefore, the resulting integral $H_{\mp}^\lambda(z)$ defines an analytic continuation of $H_{\mp}^\lambda(z)$ onto the half-plane $\mathbb{H}^\pm = \{z \in \mathbb{C} \setminus \{0\} : 0 \leq \pm \arg z \leq \pi\}$. Because of Proposition 4.1 we define $H_{\pm}^\lambda(z; \lambda) = H_{\pm}^\lambda(z)$ and regard it as the analytic continuation of $H_{\pm}(x; \lambda)$ from $\mathbb{R}_+$ onto $\mathbb{H}^\pm$. Furthermore, with slight modifications of the arguments above, where the phase function $f$ is chosen to be $\pm e^{i\omega(\theta - d - 1)}$ in the application of Proposition 5.7, we see that the asymptotic expansion in Theorem 5.10 still holds true for $H_{\pm}(z; \lambda)$ on $\mathbb{H}^\pm$. To be precise, we have the asymptotic

$H_{\pm}(z; \lambda) = n^{-\frac{1}{2}} (2\pi i)^{-\frac{1}{2}} e^{\pm i n z} z^{-\frac{1}{2}} \left( \sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\lambda) z^{-m} + O_{\Re,M,e,d} \left( C^{2M} |z|^{-M+\frac{1}{2}} \right) \right)$.

(5.12)
for all \( z \in \mathbb{H}^\pm \) such that \( |z| \geq C \).

Obviously, the above method of obtaining the analytic continuation of \( H^\pm_v \) does not apply to \( K \)-Bessel functions.

5.4.2. On the asymptotic of the Bessel kernel \( J_{(\lambda, \delta)} \). Since \( J_{(\lambda, \delta)}(\pm x) \) is a combination of \( J(2\pi x^2; \xi, \lambda) \) as in (2.7), its asymptotic follows immediately from Theorems 5.6 and 5.10.

**Theorem 5.12.** Let \( (\lambda, \delta) \in \mathbb{L}^{n-1} \times (\mathbb{Z}/2\mathbb{Z})^n \). Let \( M \) be a nonnegative integer.

1. If \( n \) is even, then for \( x \geq C \) we have
   \[
   J_{(\lambda, \delta)}(x^\alpha) = \sum_{\pm} (\pm) \sum_{i=0}^r \delta_i \frac{e^{(\pm n-1 \pm i)} \pm n x}{\sqrt{4\pi x^2}} \sum_{m=0}^{M-1} (\pm 2\pi i)^{-m} B_m(\lambda) x^{-m} + O_{R,M,n}(\mathfrak{C}^{2M} x^{-M}),
   \]

   and
   \[
   J_{(\lambda, \delta)}(-x^\alpha) = O_{R,C,n}(\mathfrak{C} x^{-C})
   \]

   for any nonnegative integer \( C \).

2. If \( n \) is odd, then for \( x \geq C \) we have
   \[
   J_{(\lambda, \delta)}(±x^\alpha) = \sum_{\pm} (\pm) \sum_{i=0}^r \delta_i \frac{e^{(\pm n-1 \pm i)} \pm n x}{\sqrt{4\pi x^2}} \sum_{m=0}^{M-1} (\pm 2\pi i)^{-m} B_m(\lambda) x^{-m} + O_{M,n}(\mathfrak{C}^{2M} x^{-M}).
   \]

5.4.3. On the implied constants. All the implied constants that occur in this section are of exponential dependence on either \( r \) or \( R \). When \( F \) is a Maass form, our error bounds should be satisfactory, since it is known from [LRS95] that \( R \leq \frac{1}{2} - \frac{1}{n+1} \). However, our error bounds are awful if, for instance, \( F \) is the \( d \)-th symmetric lift of a holomorphic Hecke cusp form of weight \( k \) so that all the parameters \( \lambda \ell, R \) are real and \( R \) is roughly of size \( k \).

In §9.1 we shall further explore the theory of Bessel functions from the differential equation aspect. As a consequence amongst the others, it will be seen in §9.1 that the dependence on \( R \) may be completely removed if \( x \gg \mathfrak{C}^2 \).

5.4.4. On the coefficients in the asymptotics. (5.9) provides an explicit formula of the coefficient \( B_m(\lambda) = B_{m,0}(\nu) \). However, simplifications of this formula can be considerably complicated. For this, we observe that the function \( G \) defined in (5.8) does not merely vanish of third order at \( t_0 \). Actually, \( \partial^\alpha G(t_0) \) vanishes except when \( \nu = (0,...,0,\alpha,...,0) \) with \( \alpha \geq 3 \), where in the exceptional case one has \( \partial^\alpha G(t_0) = (-)^\alpha \alpha! \). To illustrate, we consider the case \( d = 1 \). It turns out that, even in this easiest case, the formula of \( B_m(\pm, \mp) = B_{m,0}(\nu) \) obtained in this way is not of its simplest form.

We have \( \mathcal{L} = (d/dt)^2 \) if \( d = 1 \). For \( 2m \geq r \geq 1 \) we have
\[
(d/dt)^{2m+2r} (G' p_v) (1)
\]
\[
= (2m + 2r)! \sum_{k=0}^{2m-r} \sum_{q=0}^r \binom{r}{q} \alpha_q = 2m + 2r - k, \alpha_q \geq 3 \rangle \frac{(-)^k [v - 1]_k}{k!}
\]
\[
= (2m + 2r)! \sum_{k=0}^{2m-r} \binom{2m-k-1}{r-1} \frac{(1-v)_k}{k!},
\]
It is easy to verify (6.1) and (6.2) using the rigorous interpretation of expression for \( J(x; \psi, \lambda) \) established in [Wat44, 7.2 (1, 2)]

On the other hand, one has the asymptotic expansions of \( H_v^{(1)} \) and \( H_v^{(2)} \) (6.13)

Thus, in view of Proposition 2.7 and Theorem 5.10 we obtain the following combinatoric identity by comparing two expressions of the coefficients in the asymptotic expansion of either \( H_v^{(1)} \) or \( H_v^{(2)} \)

\[
(-)^m \frac{\left(\frac{1}{2} - v\right)_m \left(\psi + v\right)_m}{m!}
\]

\[
= \frac{(1 - v)_{2m}}{m!} + \sum_{r=1}^{2m} (-1)^r(2m + 2r) \frac{(2m - k - 1) (1 - v)_k}{r!}
\]

It seems hard to find an elementary proof of this identity.

6. Recurrence relations and differential equations of Bessel functions

The purpose of this section is to derive the differential equation satisfied by the Bessel function \( J(x; \psi, \lambda) \). This should follow from certain recurrence formulae of \( J(x; \psi, \lambda) \) analogous to those of the classical Bessel functions.

6.1. Recurrence relations. Applying the formal partial integration of either the first or the second kind and the differentiation on the formal integral expression (3.1) of \( J_v(x; \psi) \) yields

\[
\frac{\psi}{ix} J_v(x; \psi) = \zeta J_{v-i}(x; \psi) - \zeta_{d+1} J_{v+i}(x; \psi)
\]

for \( \ell = 1, \ldots, d \), and

\[
J'_v(x; \psi) = \zeta_{d+1} J_{v+i}(x; \psi) + i \sum_{\ell=1}^{d} \zeta_{\ell} J_{v-i}(x; \psi).
\]

It is easy to verify (6.1) and (6.2) using the rigorous interpretation of \( J_v(x; \psi) \) established in §3.3.

Using (6.1) we reformulate (6.2) as below,

\[
J'_v(x; \psi) = \zeta_{d+1} i(d + 1) J_{v+i}(x; \psi) + \frac{\psi}{ix} J_v(x; \psi).
\]
6.2. Differential equations of Bessel functions.

**Lemma 6.1.** Define $e^\ell = (1, \ldots, 1, 0, \ldots, 0)$, $\ell = 1, \ldots, d$, and denote $e^0 = e^{d+1} = (0, \ldots, 0)$ for convenience. Recall that we defined $v_{d+1} = 0$.

(1). For $\ell = 1, \ldots, d + 1$ we have

\[ J_{\nu + e^\ell}^{(1)}(x; \cdot) = \zeta i(d + 1)J_{\nu + e^\ell}^{(1)}(x; \cdot) - \frac{\Lambda_{d-\ell+1}(\nu) + d - \ell + 1}{x}J_{\nu + e^\ell}(x; \cdot), \]

with

\[ \Lambda_{m}(\nu) = -\sum_{k=1}^{d} v_{k} + (d + 1)v_{d-m+1}, \quad 0 \leq m \leq d. \]

(2). For $0 \leq j \leq k \leq d + 1$ define

\[ U_{k,j}(\nu) = \begin{cases} 1, & \text{if } j = k, \\ - (A_{j}(\nu) + k - 1)U_{k-1,j}(\nu) + U_{k-1,j-1}(\nu), & \text{if } 0 \leq j \leq k - 1, \end{cases} \]

with the convention that $U_{k,-1}(\nu) = 0$, and

\[ S_{0}(\cdot) = +, \quad S_{j}(\cdot) = \prod_{m=0}^{j-1} s_{d-m+1} \text{ for } j = 1, \ldots, d + 1. \]

Then

\[ J_{\nu}^{(k)}(x; \cdot) = \sum_{j=0}^{k} S_{j}(\cdot)(i(d + 1))^{j}U_{k,j}(\nu)x^{j-k}J_{\nu + e^{j-k}}^{(1)}(x; \cdot). \]

**Proof.** By (6.3) and (6.1) with $\nu$ replaced by $\nu + e^\ell$ one has

\[ J_{\nu + e^\ell}^{(k)}(x; \cdot) = \zeta \hat{s}_{d+1} i(d + 1)J_{\nu + e^\ell + e^\ell}^{(1)}(x; \cdot) + \frac{\sum_{k=1}^{d} v_{k} + \ell}{x}J_{\nu + e^\ell}(x; \cdot), \]

\[ = i(d + 1)\left( - \frac{\nu + 1}{lx}J_{\nu + e^\ell}(x; \cdot) + \zeta \hat{s}_{d+1} J_{\nu + e^{d+1}}^{(1)}(x; \cdot) \right) + \frac{\sum_{k=1}^{d} v_{k} + \ell}{x}J_{\nu + e^\ell}(x; \cdot), \]

\[ = \zeta i(d + 1)J_{\nu + e^{d+1}}^{(1)}(x; \cdot) + \frac{\sum_{k=1}^{d} v_{k} - (d + 1)v_{d} + \ell - d - 1}{x}J_{\nu + e^\ell}(x; \cdot). \]

This proves (6.4).

(6.5) is trivial when $k = 0$. Suppose that $k \geq 1$ and (6.5) is already proved for $k - 1$.

Inductive hypothesis and (6.4) imply

\[ J_{\nu}^{(k)}(x; \cdot) = \sum_{j=0}^{k-1} S_{j}(\cdot)(i(d + 1))^{j}U_{k-1,j}(\nu)x^{j-k+1} \]

\[ (j - k + 1)x^{-1}J_{\nu + e^{d+j-1}}^{(1)}(x; \cdot), \]

\[ + \zeta \hat{s}_{d+1} i(d + 1)J_{\nu + e^{d+j}}^{(1)}(x; \cdot) - (A_{j}(\nu) + j)x^{-1}J_{\nu + e^{d+j+1}}^{(1)}(x; \cdot) \]

\[ = - \sum_{j=0}^{k-1} S_{j}(\cdot)(i(d + 1))^{j}U_{k-1,j}(\nu)(A_{j}(\nu) + k - 1)x^{j-k}J_{\nu + e^{d+j+1}}^{(1)}(x; \cdot). \]
we shall denote by
\[ (6.8) \]
and the recurrence relation
\[ (6.9) \]
respectively defined via the recurrence relation
\[ (6.7) \]
denote
\[ X_j \]
\[ J_k \]
6.3. Calculations of the coefficients
Lemma 6.1 (2) may be recapitulated as
\[ (6.6) \]
where \( X_v(x; \xi) = (J_v^{(k)}(x; \xi))_{k=0}^{d+1} \) and \( Y_v(x; \xi) = (J_v^{(d+j+1)}(x; \xi))_{j=0}^{d+1} \) are column vectors of functions, \( S(\xi) = \text{diag}(S_j(\xi))(i(d+1))_{j=0}^{d+1} \) and \( D(x) = \text{diag}(x^j)_{j=0}^{d+1} \) are diagonal matrices, and \( U(\nu) \) is the lower triangular unipotent \((d+2) \times (d+2)\) matrix whose \((k+1, j+1)\)-th entry is equal to \( U_{k,j}(\nu) \), with \( 0 \leq j \leq k \leq d+1 \). The inverse matrix \( U(\nu)^{-1} \) is again a lower triangular unipotent matrix. The \((k+1, j+1)\)-th entry of \( U(\nu)^{-1} \), which we shall denote by \( V_{k,j}(\nu) \), is a polynomial in \( \nu \) of degree \( k-j \) and integral coefficients.

Observe that \( J_v^{(d+j+1)}(x; \xi) = J_v^{(d+j)}(x; \xi) = J_v(x; \xi) \). Therefore, \( (6.6) \) implies that \( J_v(x; \xi) \) satisfies the following linear differential equation of order \( d+1 \)
\[ (6.7) \]
6.3. Calculations of the coefficients in the differential equations.

Definition 6.2. Let \( A = \{ A_m \}_{m=0}^{\infty} \) be a sequence of complex numbers.

(1) For \( k, j \geq -1 \) inductively define a double sequence of polynomials \( U_{k,j}(A) \) in \( A \) via the initial conditions
\[ U_{-1,-1}(A) = 1, \quad U_{k,-1}(A) = U_{-1,j}(A) = 0 \text{ if } k, j \geq 0, \]
and the recurrence relation
\[ (6.8) \]
\[ U_{k,j}(A) = -(A_j + k - 1) U_{k-1,j}(A) + U_{k-1,j-1}(A), \quad k, j \geq 0. \]

(2) For \( j, m \geq -1 \) with \((j, m) \neq (-1, -1)\) define a double sequence of integers \( A_{j,m} \) via the initial conditions
\[ A_{-1,0} = 1, \quad A_{-1,m} = A_{1,-1} = 0 \text{ if } m \geq 1, j \geq 0, \]
and the recurrence relation
\[ (6.9) \]
\[ A_{j,m} = jA_{j,m-1} + A_{j-1,m}, \quad j, m \geq 0. \]

(3) For \( k, m \geq 0 \) we define \( \sigma_{k,m}(A) \) to be the elementary symmetric polynomial in \( A_0, \ldots, A_k \) of degree \( m \), with the convention that \( \sigma_{k,m}(A) = 0 \text{ if } m \geq k+2 \). Moreover, we denote
\[ \sigma_{-1,0}(A) = 1, \quad \sigma_{1,-1}(A) = \sigma_{-1,m}(A) = 0 \text{ if } k \geq -1, m \geq 1. \]

Observe that, with the above notations as initial conditions, \( \sigma_{k,m}(A) \) may also be inductively defined via the recurrence relation
\[ (6.10) \]
\[ \sigma_{k,m}(A) = A_k \sigma_{k-1,m-1}(A) + \sigma_{k-1,m}(A), \quad k, m \geq 0. \]
(4). For \( k \geq 0, j \geq -1 \) define

\[
V_{k,j}(A) = \begin{cases} 
0, & \text{if } j > k, \\
\sum_{m=0}^{k-j} A_{j,k-j-m} \sigma_{k-1,m}(A), & \text{if } k \geq j.
\end{cases}
\]  

\[(6.11)\]

**Lemma 6.3.** Let notations be as above.

1. \( U_{k,j}(A) \) is a polynomial in \( A_0, \ldots, A_j \). \( U_{k,j}(A) = 0 \) if \( j > k \), and \( U_{k,k}(A) = 1 \).
2. \( U_{k,0}(A) = [-A_0]_k \) for \( k \geq 0 \).
3. \( A_{j,0} = 1 \), and \( A_{j,1} = \frac{1}{j}j(j+1) \).
4. \( V_{k,j}(A) \) is a symmetric polynomial in \( A_0, \ldots, A_{k-1} \). \( V_{k,k}(A) = 1 \). \( V_{k,-1}(A) = 0 \) and \( V_{k,k-1}(A) = \sigma_{k-1,1}(A) + \frac{k}{k(j-1)} \) for \( k \geq 0 \).

**Proof.** (1-3) are evident from the definitions.

4. \( (6.12) \) is obvious if \( j \geq k \). If \( k > j \), then the recurrence relations \( (6.9, 6.10) \) of \( \sigma_{k,m}(A) \) and \( A_{j,m} \) in conjunction with the definition \( (6.11) \) of \( V_{k,j}(A) \) yield

\[
V_{k,j}(A) = \sum_{m=0}^{k-j} A_{j,k-j-m} \sigma_{k-1,m}(A)
\]

\[
= A_{k-1} \sum_{m=1}^{k-j} A_{j,k-j-m} \sigma_{k-2,m-1}(A) + \sum_{m=0}^{k-j} A_{j,k-j-m} \sigma_{k-2,m}(A)
\]

\[
= A_{k-1} \sum_{m=0}^{k-j-1} A_{j,k-j-m-1} \sigma_{k-2,m}(A)
\]

\[
+ j \sum_{m=0}^{k-j-1} A_{j,k-j-m-1} \sigma_{k-2,m}(A) + \sum_{m=0}^{k-j} A_{j-1,k-j-m} \sigma_{k-2,m}(A)
\]

\[
= (A_{k-1} + j)V_{k-1,j}(A) + V_{k-1,j-1}(A).
\]

Q.E.D.

**Lemma 6.4.** For \( k \geq 0 \) and \( j \geq -1 \) such that \( k \geq j \) we have

\[
\sum_{\ell=j}^{k} U_{k,\ell}(A)V_{\ell,j}(A) = \delta_{k,j},
\]  

\[(6.13)\]

where \( \delta_{k,j} \) denotes Kronecker’s delta symbol.

**Proof.** \( (6.13) \) is obvious if either \( k = j \) or \( j = -1 \). In the proof we may therefore assume that \( k - 1 \geq j \geq 0 \) and that \( (6.13) \) is already proved for smaller values of \( k - j \) as well as for smaller values of \( j \) and the same \( k - j \).
By the recurrence relations (6.8, 6.12) of $U_{k,j}(A)$ and $V_{k,j}(A)$ and the induction hypothesis we have

$$
\sum_{\ell=j}^{k} U_{k,\ell}(A)V_{\ell,j}(A)
$$

$$
= - \sum_{\ell=j}^{k-1} (k-1 + A_{\ell})U_{k-1,\ell}(A)V_{\ell,j}(A) + \sum_{\ell=j}^{k} U_{k-1,\ell-1}(A)V_{\ell,j}(A)
$$

$$
= - (k-1)\delta_{k-1,j} - \sum_{\ell=j}^{k-1} A_{\ell}U_{k-1,\ell}(A)V_{\ell,j}(A) + \sum_{\ell=j+1}^{k} A_{\ell-1}U_{k-1,\ell-1}(A)V_{\ell-1,j}(A)
$$

$$
+ j \sum_{\ell=j+1}^{k} U_{k-1,\ell-1}(A)V_{\ell-1,j}(A) + \sum_{\ell=j}^{k} U_{k-1,\ell-1}(A)V_{\ell-1,j-1}(A)
$$

$$
= - (k-1)\delta_{k-1,j} + 0 + j\delta_{k-1,j} + \delta_{k-1,j-1} = 0.
$$

Thus, (6.13) is proved. Q.E.D.

Finally, we have the following explicit formulae for $A_{jm}$.

**Lemma 6.5.** We have $A_{0,0} = 1$, $A_{0,m} = 0$ if $m \geq 1$, and

$$
(6.14) \quad A_{jm} = \sum_{r=1}^{j} \frac{(-)^{-r+1} \nu^{m+j}}{r!(j-r)!} \quad \text{if } j \geq 1, m \geq 0.
$$

**Proof.** It is easily seen that $A_{0,0} = 1$ and $A_{0,m} = 0$ if $m \geq 1$.

It is straightforward to verify that the sequence defined by (6.14) satisfies the recurrence relation (6.9), so it is left to show that (6.14) holds true for $m = 0$. Initially, $A_{j,0} = 1$, and hence we must verify

$$
\sum_{r=1}^{j} \frac{(-)^{-r+1} \nu^{m+j}}{r!(j-r)!} = 1.
$$

This follows from considering all the identities obtained by differentiating the following binomial identity up to $j$ times and then evaluating at $x = 1$,

$$
(x-1)^j - (-1)^j = j! \sum_{r=1}^{j} \frac{(-)^{-r+1} \nu^{m+j}}{r!(j-r)!} x^r.
$$

Q.E.D.

**6.4. Conclusion.** We first observe that, when $0 \leq j \leq k \leq d + 1$, both $U_{k,j}(A)$ and $V_{k,j}(A)$ are polynomials in $A_0, \ldots, A_d$ according to Lemma 6.3 (1, 3). Put $\Lambda_m = \Lambda_m(\nu)$, $m = 0, \ldots, d$. Then we have $U_{k,j}(\nu) = U_{k,j}(A)$, and it follows from Lemma 6.4 that $V_{k,j}(\nu) = V_{k,j}(A)$. Moreover, the relations $\nu_\ell = \lambda_\ell - A_{d+1}$. $\ell = 1, \ldots, d$, and the assumption $\sum_{d+1}^{d+1} \lambda_\ell = 0$ yield

$$
\Lambda_m(\nu) = (d+1)\Lambda_{d-m+1}.
$$

Thus, we can reformulate (6.7) in the following theorem.
Theorem 6.6. The Bessel function $J(x; \zeta, \lambda)$ satisfies the following linear differential equation of order $d + 1$

$$
\sum_{j=1}^{d+1} V_{d+1,j}(\lambda) x^j w^{(j)} + (V_{d+1,0}(\lambda) - S_{d+1}(\zeta))(i(d + 1))^{d+1} x^{d+1}) w = 0,
$$

where

$$
S_{d+1}(\zeta) = \sum_{\ell=1}^{d+1} \zeta \ell, \quad V_{d+1,j}(\lambda) = \sum_{m=0}^{d-j+1} A_{j,d-j-m+1}(d + 1)^m \sigma_m(\lambda),
$$

$\sigma_m(\lambda)$ denotes the elementary symmetric polynomial in $\lambda$ of degree $m$, with $\sigma_1(\lambda) = 0$, and $A_{j,m}$ is recurrently defined in Definition 6.2(3) and explicitly given in Lemma 6.5. We shall call the equation (6.15) a Bessel equation of index $\lambda$.

We collect some simple facts on $V_{d+1,j}(\lambda)$ in the following lemma. They shall play important roles later in the study of the Bessel equation (6.15).

Lemma 6.7. We have

1. $\sum_{j=1}^{d+1} V_{d+1,j}(\lambda)[-(d + 1)\lambda d + 1] j = 0.$
2. $V_{d+1,d}(\lambda) = \frac{1}{d!}(d + 1).$

Since $S_{d+1}(\zeta) = \prod_{\ell=1}^{d+1} \zeta \ell \in \{+, -\}$, (6.15) only provides two Bessel equations with given index $\lambda$. In other words, $S_{d+1}(\zeta)$ determines the Bessel equation associated with $J(x; \zeta, \lambda)$.

Definition 6.8. We call $S_{d+1}(\zeta) = \prod_{\ell=1}^{d+1} \zeta \ell$ the parity of $J(x; \zeta, \lambda)$ as well as the parity of the Bessel equation (6.15).

Remark 6.9. If we define

$$
J(x; \zeta, \lambda) = J((d + 1)^{-1}x; \zeta, (d + 1)^{-1}\lambda),
$$

then this normalized Bessel function satisfies a differential equation with coefficients free of powers of $(d + 1)$, that is,

$$
\sum_{j=1}^{d+1} V_{d+1,j}(\lambda) x^j w^{(j)} + (V_{d+1,0}(\lambda) - S_{d+1}(\zeta))(d+1)^{d+1} x^{d+1}) w = 0,
$$

with

$$
V_{d+1,j}(\lambda) = \sum_{m=0}^{d-j+1} A_{j,d-j-m+1}(d + 1)^m \sigma_m(\lambda).
$$

In particular, if $d = 1$, $\lambda = (\lambda, -\lambda)$, then the two normalized Bessel equations are

$$
x^2 \frac{d^2 w}{dx^2} + x \frac{dw}{dx} + (-\lambda^2 \pm x^2) w = 0.
$$

These are the Bessel equation and the modified Bessel equation of index $\lambda$. 
7. Bessel equations

The theory of linear ordinary differential equations with analytic coefficients will be employed in this section to study Bessel equations. The reader will observe the structural simplicity of these Bessel equations as well as the abundance of symmetries.

Subsequently, we shall use $z$ instead of $x$ to indicate complex variable. Recall that $n = d + 1$. For $\varsigma \in \{+,-\}$ and $\lambda \in \mathbb{N}^{-1}$ define the Bessel differential operator

\begin{equation}
\nabla_{\varsigma,\lambda} = \sum_{j=1}^{n} V_{n,j}(\lambda) z^j \frac{d^j}{dz^j} + V_{n,0}(\lambda) - \varsigma (in)^n z^n.
\end{equation}

Then the Bessel equation of index $\lambda$ and parity $\varsigma$, or simply the Bessel equation of parity $\varsigma$ if the index $\lambda$ is given, may be written as

\begin{equation}
\nabla_{\varsigma,\lambda}(w) = 0.
\end{equation}

We shall study Bessel equations on the Riemann surface $U$ associated with $\log z$, that is, the universal cover of $\mathbb{C} \setminus \{0\}$. Each element in $U$ is represented by a pair $(x, \omega)$ with modulus $x \in \mathbb{R}_+$ and argument $\omega \in \mathbb{R}$, and will be denoted by $z = xe^{i\omega} = e^{\log z + i\omega}$ with some ambiguity. Conventionally, define $z^t = e^{t \log z}$ for $z \in U$, $\lambda \in \mathbb{C}$, $\overline{z} = e^{-\log z}$, and moreover let $1 = e^0$, $-1 = e^\pi i$ and $\pm i = e^{\pm \frac{1}{2} \pi i}$.

First of all, since Bessel equations are nonsingular on $U$, all the solutions of Bessel equations are analytic on $U$. Each Bessel equation has only two singularities at $z = 0$ and $z = \infty$. According to the classification of singularities, $0$ is a regular singularity, so the Frobenius method may be exploited to obtain solutions of Bessel equations developed in series of ascending powers of $z$, or possibly logarithmic sums of this kind of series, whereas $\infty$ is an irregular singularity of rank one, and therefore we may find certain formal solutions which are the asymptotic expansions of some actual solutions of Bessel equations.

When studying the asymptotic expansions for the Bessel equation (7.2), it is more convenient to consider its corresponding system of differential equations,

\begin{equation}
w' = B(z; \varsigma, \lambda)w,
\end{equation}

with

\[B(z; \varsigma, \lambda) =
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
-V_{n,0}(\lambda) z^{-n} + \varsigma (in)^n \\
-V_{n,1}(\lambda) z^{-n+1} \\
\vdots \\
-V_{n,n-1}(\lambda) z^{-1}
\end{pmatrix}.
\]

Moreover, a simple but important observation is as follows.

\[\text{IX[CT55] Chapter 4, 5] and [Was65] Chapter II-V] are the main references that we follow, and the reader is referred to these books for terminologies and definitions.\]
Lemma 7.1. Let $\zeta \in \{+, -\}$ and $a$ be an integer. If $\varphi(z)$ is a solution of the Bessel equation of parity $\zeta$, then $\varphi(e^{n\pi i}z)$ satisfies the Bessel equation of parity $(-)^a\zeta$.

Variants of Lemma 7.1, Lemma 7.8 and Lemma 7.21 assert that rotations of the argument by a $2n$-th root of unity from a given Bessel function, which will be chosen to be one of the $H$-Bessel functions, yield all the Bessel functions.

### 7.1. Bessel functions of the first kind.

The indicial equation associated with $\nabla \zeta$, $\lambda$ is as below,

$$
\sum_{j=0}^{n} [p_j] V_{n,j}(\lambda) = 0.
$$

Let $P_\lambda(\rho)$ denote the polynomial on the left of this equation. Lemma 6.7(1) and the symmetry of $V_{n,j}(\lambda)$ yield the following identity for each $\ell = 1, \ldots, n$,

$$
\sum_{j=0}^{n} [n\ell_j] V_{n,j}(\lambda) = 0.
$$

Therefore,

$$
P_\lambda(\rho) = \prod_{\ell=1}^{n} (\rho + n\ell).
$$

Consider the formal series

$$
\sum_{m=0}^{\infty} c_m \rho^{m+n},
$$

where the index $\rho$ and the coefficients $c_m$, with $c_0 \neq 0$, are to be determined. It is easy to see that

$$
\nabla \zeta \sum_{m=0}^{\infty} c_m \rho^{m+n} = \sum_{m=0}^{\infty} c_m P_\lambda(\rho + m) \rho^{m+n} - \zeta(n) \sum_{m=0}^{\infty} c_m \rho^{m+n}.
$$

If the following equations are satisfied

$$
c_m P_\lambda(\rho + m) = 0, \quad n > m \geq 1,
$$

$$
c_m P_\lambda(\rho + m) - \zeta(n) c_{m-n} = 0, \quad m \geq n,
$$

then we have

$$
\nabla \zeta \sum_{m=0}^{\infty} c_m \rho^{m+n} = c_0 P_\lambda(\rho) \rho^n.
$$

For each $\ell = 1, \ldots, n$ choose $\rho = -n\ell$ and $c_0 = \prod_{k=1}^{n} \Gamma (\lambda_k - \lambda_{\ell} + 1)^{-1}$. The system of equations (7.4) is uniquely solvable, provided that no two components of $n\lambda$ differ by an integer so that $P_\lambda(-n\lambda_{\ell} + m)$ does not vanish for any $m \geq 1$. Moreover, for such $\lambda$ we have $c_0 \neq 0$. It follows that

$$
\sum_{m=0}^{\infty} \frac{(\xi^n)^m \zeta^{m-n\ell+m}}{\prod_{k=1}^{n} \Gamma (\lambda_k - \lambda_{\ell} + m + 1)}
$$

is a formal solution of the differential equation (7.2).

Suppose now that $\lambda \in \mathbb{L}^{n-1}$ is unrestricted. The series in (7.5) is absolutely convergent, locally uniformly convergent with respect to $\lambda$, and hence defines an analytic function of $z$.
on either the Riemann surface associated with \( z^{-n} \) or its universal cover \( U \), as well as an analytic function of \( \lambda \). We denote by \( J_\ell(z; \varsigma, \lambda) \) the analytic function given by the series (7.5) and call it a Bessel function of the first kind. It is evident that \( J_\ell(z; \varsigma, \lambda) \) is an actual solution of (7.2).

**Definition 7.2.** Let \( \mathbb{D}^{n-1} \) denote the set of \( \lambda \in \mathbb{R}^{n-1} \) such that no two components of \( \lambda \) differ by an integer. We call an index \( \lambda \) generic if \( \lambda \in \mathbb{D}^{n-1} \).

When \( \lambda \in \mathbb{D}^{n-1} \), all the \( J_\ell(z; \varsigma, \lambda) \) constitute a fundamental set of solutions, since the leading term in the expression (7.5) of \( J_\ell(z; \varsigma, \lambda) \) does not vanish. However, this is no longer true if \( \lambda \notin \mathbb{D}^{n-1} \). Indeed, if \( \lambda_\ell - \lambda_k \) is an integer, \( k \neq \ell \), then \( J_\ell(z; \varsigma, \lambda) = (\varsigma^m)_{k=1}^{\ell-1} K_\ell(z; \varsigma, \lambda_k) \). Other solutions are certain logarithmic sums of series of ascending powers of \( z \). Roughly speaking, powers of \( \log z \) may occur in some solutions, with powers not exceeding \( n-1 \). For more details the reader may consult [CL55, §4.8].

Finally, the following lemma follows from the definition (7.5) of \( J_\ell(z; \varsigma, \lambda) \).

**Lemma 7.3.** Let \( a \) be an integer. We have

\[
J_\ell(e^{\pi i \#} z; \varsigma, \lambda) = e^{-\pi i a \lambda} J_\ell(z; (-)^a \varsigma, \lambda).
\]

**Remark 7.4.** If \( n = 2 \), then we have the following formulae according to [Wat44, 3.1 (8), 3.7 (2)],

\[
J_1(z; +, \lambda, -\lambda) = J_{-2\lambda}(2z), \quad J_2(z; +, \lambda, -\lambda) = J_{2\lambda}(2z),
\]

\[
J_1(z; -, \lambda, -\lambda) = J_{-2\lambda}(2z), \quad J_2(z; -, \lambda, -\lambda) = J_{2\lambda}(2z).
\]

### 7.2. The analytic continuation of \( J(x; \varsigma, \lambda) \)

For any given \( \lambda \in \mathbb{R}^{n-1} \), since \( J(x; \varsigma, \lambda) \) satisfies the Bessel equation of parity \( S_\ell(\varsigma) \), it admits a unique analytic continuation \( J(z; \varsigma, \lambda) \) onto \( U \). Recall the definition

\[
J(x; \varsigma, \lambda) = \frac{1}{2\pi i} \int_{\gamma} G(s; \varsigma, \lambda) x^{-s} ds, \quad x \in \mathbb{R}_+,
\]

where \( G(s; \varsigma, \lambda) = \prod_{k=1}^{n} \Gamma(s - \lambda_k) e^{\frac{1}{4} \varsigma_k (s - \lambda_k)} \) and \( \gamma \) is a suitable contour defined in \( \S 2.1 \). Suppose \( \varsigma = S_\ell(\varsigma) \), and for the moment assume that \( \lambda \) is generic. For \( \ell = 1, \ldots, n \) and \( m \geq 0 \), \( G(s; \varsigma, \lambda) \) has a simple pole at \( \lambda_\ell - m \) with residue

\[
(-)^m \frac{1}{m!} e^{\left(-\frac{\varsigma m}{4}\right)} \prod_{k \neq \ell} \Gamma(\lambda_\ell - \lambda_k - m) e^{\left(\frac{\varsigma_k (\lambda_\ell - \lambda_k - m)}{4}\right)}
\]

\[
= (-)^m \frac{1}{m!} e^{\left(-\frac{\sum_{k=1}^{n} \varsigma_k m}{4}\right)} e^{\left(\frac{\sum_{k=1}^{n} \varsigma_k (\lambda_\ell - \lambda_k)}{4}\right)}
\]

\[
\prod_{k \neq \ell} \frac{\pi}{\sin(\pi(\lambda_\ell - \lambda_k - m))} \Gamma(\lambda_k - \lambda_\ell + m + 1)
\]

\[
= \pi^{n-1} e^{\left(-\frac{\sum_{k=1}^{n} \varsigma_k \lambda_k}{4}\right)} e^{\left(\frac{\sum_{k=1}^{n} \varsigma_k \lambda_k}{4}\right)} \left(\prod_{k \neq \ell} \frac{1}{\sin(\pi(\lambda_\ell - \lambda_k))}\right)^{-1}
\]

\[
(\varsigma^m) \prod_{k=1}^{n} \Gamma(\lambda_k - \lambda_\ell + m + 1)^{-1},
\]

where \( \varsigma = S_\ell(\varsigma) \).
where we have used Euler’s reflection formula for the Gamma function and the observation
\((-)^n e\left(-\frac{1}{4}\sum_{k=1}^{n} \xi_k^2\right) = \zeta^n\). Therefore, on applying Cauchy’s residue theorem, \(J(x; \varsigma, \lambda)\) is developed into an absolutely convergent series by shifting the contour \(C\) far left,

\[
\pi^{n-1} e \left(-\frac{\sum_{k=1}^{n} \xi_k \lambda_k}{4}\right) \sum_{\ell=1}^{n} e \left(-\frac{\sum_{k=1}^{n} \xi_k \lambda_k \ell}{4}\right) \left(\prod_{k \neq \ell} \sin (\pi (\lambda_{\ell} - \lambda_k))^{-1}\right)
\]

(7.7)

Indeed, transforming \(G(s; \varsigma, \lambda)\) using Euler’s reflection formula, Stirling’s formula implies that the contour integral of \(G(s; \varsigma, \lambda) x^{-ns}\) is arbitrarily small if the contour is left shifted far enough.

Replace \(x\) by a complex variable \(z\), then (7.7) defines an analytic function of \(z\) on \(U\) that must be identical with \(J(z; \varsigma, \lambda)\). Thus, in view of (7.5), we obtain

\[
J(z; \varsigma, \lambda) = \pi^{n-1} e \left(-\frac{\sum_{k=1}^{n} \xi_k \lambda_k}{4}\right) \sum_{\ell=1}^{n} e \left(-\frac{\sum_{k=1}^{n} \xi_k \lambda_k \ell}{4}\right) \left(\prod_{k \neq \ell} \sin (\pi (\lambda_{\ell} - \lambda_k))^{-1}\right) J_\ell(z; \varsigma, \lambda), \quad z \in U.
\]

(7.8)

Because of the possible vanishing of \(\sin (\pi (\lambda_{\ell} - \lambda_k))\), the right hand side of (7.8) fails to make sense if \(\lambda\) is not generic. However, on the left hand side, \(J(z; \varsigma, \lambda)\) is well-defined on \(U\) for arbitrary \(\lambda \in \mathbb{L}^{n-1}\). Thus, taking the limit of (7.8) yields

\[
J(z; \varsigma, \lambda) = \lim_{\lambda' \to \lambda} \pi^{n-1} e \left(-\frac{\sum_{k=1}^{n} \xi_k \lambda'_k}{4}\right) \sum_{\ell=1}^{n} e \left(-\frac{\sum_{k=1}^{n} \xi_k \lambda'_k \ell}{4}\right) \left(\prod_{k \neq \ell} \sin (\pi (\lambda'_{\ell} - \lambda'_k))^{-1}\right) J_\ell(z; \varsigma, \lambda'), \quad z \in U.
\]

(7.9)

We recollect the definitions of \(L_\pm(\varsigma)\) and \(n_\pm(\varsigma)\) in Proposition 2.9

**Definition 7.5.** Let \(\varsigma \in \{+,-\}^n\). We define \(L_\pm(\varsigma) = \{\ell : \varsigma_\ell = \pm\}\) and \(n_\pm(\varsigma) = |L_\pm(\varsigma)|\). The pair of integers \((n_+(\varsigma), n_-(\varsigma))\) is called the signature of \(\varsigma\), as well as the signature of the Bessel function \(J(z; \varsigma, \lambda)\).

With Definition 7.5, we reformulate (7.8) (7.9) in the following lemma.

**Lemma 7.6.** We have

\[
J(z; \varsigma, \lambda) = \pi^{n-1} e \left(-\frac{\sum_{k \in L_+(\varsigma)} \xi_k + \sum_{k \in L_-(\varsigma)} \lambda_k^2}{4}\right) \sum_{\ell=1}^{n} e \left(-\frac{(n+(\varsigma) - n_-(\varsigma)) \lambda_\ell}{4}\right) \left(\prod_{k \neq \ell} \sin (\pi (\lambda_{\ell} - \lambda_k))^{-1}\right) J_\ell(z; (-)^{n_-(\varsigma)}, \lambda).
\]

When \(\lambda\) is not generic, the right hand side is to be replaced by its limit.
Remark 7.7. In view of Proposition 2.1 and Remark 7.4, Lemma 7.6 amounts to the connection formulae in \([1.12, 1.13]\) (see \([\text{Wat}44, 3.61(5, 6), 3.7(6)]\)).

Combining Lemma 7.3 with Lemma 7.6, we prove the following lemma, which implies that the Bessel function \(J(z; \varsigma, \lambda)\) is determined by its signature up to a constant multiple if \(\lambda\) is given.

**Lemma 7.8.** Define \(H^\pm(z; \lambda) = J(z; \pm, ..., \pm, \lambda)\). Then

\[
J(z; \varsigma, \lambda) = e \left( \pm \sum_{\ell \in \mathbb{Z}} \frac{\lambda_{\ell}}{2} \right) H^\pm \left( e^{\pm \frac{n \pi (\varsigma)}{n}} z, \lambda \right).
\]

Remark 7.9. In the case where \(\lambda_{\ell} = \frac{1}{n} \left( \frac{\pi}{2} - \ell \right)\), the formula in Lemma 7.6 amounts to splitting the Taylor series expansion of \(e^{i \varsigma x} x^{s}\) in \((2.13)\) according to the residue class of indices modulo \(n\). To see this, we require the multiplicative formula \((7.15)\) of the Gamma function as well as the trigonometric identity

\[
\prod_{k=1}^{n-1} \sin \left( \frac{k \pi}{n} \right) = \frac{n}{2^{n-1}}.
\]

Remark 7.10. We have the following Barnes-type integral representation,

\[
J(z; \varsigma, \lambda) = \frac{1}{2 \pi i} \int_{\mathcal{C}'} G(s; \varsigma, \lambda) z^{-n s} ds, \quad z \in \mathbb{C},
\]

where \(\mathcal{C}'\) is a contour which starts from and returns to \(-\infty\) after encircling the poles of the integrand counter-clockwise. Compare \([\text{Wat}44, 6.5]\). Lemma 7.8 may also be seen from this integral representation.

When \(-\frac{n(\varsigma)}{n} \pi < \arg z < \frac{n(\varsigma)}{n} \pi\), the contour \(\mathcal{C}'\) may be opened out to the vertical line \(\{ s : \Re s = \sigma \} \) with \(\sigma > \max \{ \Re \lambda_{\ell} \}\). Thus

\[
J(z; \varsigma, \lambda) = \frac{1}{2 \pi i} \int_{\mathcal{C}'} G(s; \varsigma, \lambda) z^{-n s} ds, \quad -\frac{n(\varsigma)}{n} \pi < \arg z < \frac{n(\varsigma)}{n} \pi.
\]

On the boundary rays \(\arg z = \pm \frac{n(\varsigma)}{n} \pi\), the contour \(\{ s : \Re s = \sigma \} \) may be shifted to \(\mathbb{C}\) defined as in \((7.1)\) in order to secure convergence.

The contour integrals in \((7.10), (7.11)\) absolutely converge, locally uniformly in both \(z\) and \(\lambda\). Again, we have used Stirling’s formula to examine the behaviour of the integrand \(G(s; \varsigma, \lambda) z^{-n s}\) on integral contours, where for \((7.10)\) a transformation of \(G(s; \varsigma, \lambda)\) by Euler’s reflection formula is required.

### 7.3. Asymptotics for Bessel equations and Bessel functions of the second kind

We now proceed to investigate the asymptotics at infinity for Bessel equations. Our main references are \([\text{CL55, Chapter 5}]\) and \([\text{Was65, Chapter IV}]\).

**Definition 7.11.** For a positive integer \(N\) and \(\varsigma \in \{+, -\}\) we let \(\Xi_N(\varsigma)\) denote the set of \(N\)-th roots of \(\varsigma\).

\(\Xi\) Under certain circumstances, it is suitable to view an element \(\xi\) of \(\Xi_N(\varsigma)\) as a point in \(\mathbb{U}\) instead of \(\mathbb{C} \setminus \{0\}\), since some expressions may depend on the choice of \(\arg \xi\). This however should be clear from the context.
Before we delve into our general study, let us consider the prototypical case where \( \lambda = \frac{1}{n} \left( \frac{1}{2}, ... \frac{1}{2} \right) \).

**Proposition 7.12.** For any \( \xi \in \mathbb{C}^{2n}(+) \), the function \( z^{-\frac{1}{2} \xi} e^{i n \xi} \) is a solution of the Bessel equation of index \( \frac{1}{n} \left( \frac{1}{2}, ... \frac{1}{2} \right) \) and parity \( \xi^n \).

**Proof.** Provided that \( \Im \xi \geq 0 \), this can be seen from Proposition 2.9 and Theorem 6.6. The restriction on \( \xi \) can be removed in view of Lemma 7.1. Q.E.D.

7.3.1. **Formal solutions of Bessel equations at infinity.** Following [CL55], Chapter 5, we shall consider the system of differential equations (7.3). We have

\[
B(\infty; \varsigma, \lambda) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\varsigma(in)^n & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

The characteristic equation of \( B(\infty; \varsigma, \lambda) \) is \( \lambda^n - \varsigma(in)^n = 0 \), and hence \( B(\infty; \varsigma, \lambda) \) has distinct eigenvalues \( in \xi_1, ..., in \xi_n \), with \( \xi_k \in \mathbb{C}_n(\varsigma) \). The conjugation by the following matrix diagonalizes \( B(\infty; \varsigma, \lambda) \),

\[
T = \frac{1}{n} \begin{pmatrix}
1 & \left( in \xi_1 \right)^{-1} & \cdots & \left( in \xi_1 \right)^{-n+1} \\
1 & \left( in \xi_2 \right)^{-1} & \cdots & \left( in \xi_2 \right)^{-n+1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \left( in \xi_n \right)^{-1} & \cdots & \left( in \xi_n \right)^{-n+1}
\end{pmatrix},
\]

\[
T^{-1} = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & in \xi_1 & in \xi_2 & \cdots & in \xi_n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\left( in \xi_1 \right)^{n-1} & \left( in \xi_2 \right)^{n-1} & \cdots & \left( in \xi_n \right)^{n-1}
\end{pmatrix}.
\]

Thus, the substitution \( u = Tw \) turns the equation (7.3) into

(7.12)

\[
u' = A(z)u,
\]

where \( A(z) = TB(z; \varsigma, \lambda)T^{-1} \) is a matrix of polynomials in \( z^{-1} \) of degree \( n \),

\[
A(z) = \sum_{j=0}^{n} z^{-j} A_j,
\]

where

(7.13)

\[
A_0 = A = \text{diag} \left( in \xi_k \right) = 1, \\
A_j = -i^{-j+1} n^{-j} \mathbb{V}_{n,n-j}(\lambda) \left( \xi_k \xi_k^{-j} \right)^n, \quad j = 1, ..., n.
\]

For convenience we let \( A_j = 0 \) if \( j > n \). The dependence on \( \varsigma, \lambda \) and the ordering of the eigenvalues have been suppressed in our notations in the interest of brevity.
Suppose $\Phi$ is a formal solution matrix for (7.12) of the form
\[
\hat{\Phi}(z) = P(z)e^{Qz},
\]
where $P$ is a formal power series in $z^{-1}$,
\[
P(z) = \sum_{m=0}^{\infty} z^{-m}P_m,
\]
and $R$, $Q$ are constant diagonal matrix. Since
\[
\hat{\Phi} = p\Phi(z) = P(z)e^{Qz} + \sum_{m=0}^{\infty} P_m e^{Qz} + P(z) e^{Qz} = (P' + z^{-1}PR + P Q) e^{Qz},
\]
the differential equation (7.12) yields
\[
\sum_{m=0}^{\infty} z^{-m-1}P_m(R - m I) + \sum_{m=0}^{\infty} z^{-m}P_mQ = \left( \sum_{j=0}^{\infty} z^{-j}A_j \right) \left( \sum_{m=0}^{\infty} z^{-m}P_m \right),
\]
where $I$ denotes the identity matrix. Comparing the coefficients of various powers of $z^{-1}$, it follows that $\Phi$ is a formal solution matrix for (7.12) if and only if $R$, $Q$ and $P_m$ satisfy the following relations
\[
P_0Q - \Delta P_0 = 0
\]
(7.14)
\[
P_{m+1}Q - \Delta P_{m+1} = \sum_{j=1}^{m+1} A_j P_{m-j+1} + P_m(ml - R), \quad m \geq 0.
\]
A solution of the first equation in (7.14) is given by
\[
Q = A, \quad P_0 = I.
\]
Using (7.15), the second equation in (7.14) for $m = 0$ is
\[
P_1A - \Delta P_1 = A_1 - R.
\]
(7.16)
Since $A$ is diagonal, the diagonal entries of the left side of (7.16) are zero, and hence the diagonal entries of $R$ must be identical with those of $A_1$. In view of (7.13) and Lemma 6.7 (2), we have
\[
A_1 = -\frac{1}{n} V_{n,n-1}(A) \cdot (\xi_k \xi_{\ell}^{-1})^n_{k,\ell=1} = -\frac{n-1}{2} (\xi_k \xi_{\ell}^{-1})^n_{k,\ell=1},
\]
and therefore
\[
R = -\frac{n-1}{2} I.
\]
(7.17)
Let $p_{1,k\ell}$ denote the $(k, \ell)$-th entry of $P_1$. It follows from (7.13) (7.16) that
\[
in(\xi_k - \xi_{\ell})p_{1,k\ell} = -\frac{n-1}{2} \xi_k \xi_{\ell}^{-1}, \quad k \neq \ell.
\]
(7.18)
Since $\xi_k \neq \xi_{\ell}$ if $k \neq \ell$, (7.18) uniquely determines the off-diagonal entries of $P_1$. Therefore, a solution of (7.16) is
\[
P_1 = D_1 + P_1^0,
\]
(7.19)
where $D_1$ is any diagonal matrix and $P_1^1$ is the matrix with diagonal entries zero and $(k, \ell)$-th entry $p_{1,k\ell}, \ k \neq \ell$. To determine $D_1$ we resort to the second equation in (7.14) for $m = 1$ which, in view of (7.15, 7.17, 7.19), may be written as
\[
P_2A - \Delta P_2 - \left( A_1 + \frac{n - 1}{2} \right) D_1 - \frac{n + 1}{2} P_1^m = A_1 P_1^0 + A_2 + D_1,
\]
where all the matrices with zero diagonal entries have been moved to the left side. It follows that $D_1$ must be equal to the diagonal part of \(A_1 P_1^1 - A_2\).

Using (7.15, 7.17), the second equation in (7.14) may be written as
\[
(7.20) \quad P_{m+1}A - \Delta P_{m+1} = \sum_{j=1}^{m+1} A_j P_{m-j+1} + \left( m + \frac{n - 1}{2} \right) P_m, \quad m \geq 0.
\]

Applying (7.20), an induction on $m$ implies that
\[
P_m = D_m + P_m^0, \quad m \geq 1,
\]
where $D_m$ and $P_m^0$ are inductively defined as follows. Put $D_0 = I$. Let $mD_m$ be the diagonal part of
\[
- \sum_{j=2}^{m+1} A_j D_{m-j+1} - \sum_{j=1}^{m} A_j P_{m-j+1}^m,
\]
and let $P_{m+1}^0$ be the matrix with diagonal entries zero such that $P_{m+1}^0 A - \Delta P_{m+1}^0$ is the off-diagonal part of
\[
\sum_{j=1}^{m+1} A_j D_{m-j+1} + \sum_{j=1}^{m} A_j P_{m-j+1}^m + \left( m + \frac{n - 1}{2} \right) P_m^0.
\]

Thus, an inductive construction of the formal solution matrix of (7.12) is completed for the given initial choices $Q = \Delta, P_0 = I$.

Observe that $A_j$ is of degree $j$ in $\lambda$ for $j \geq 2$, whereas $A_1$ is constant. Using an induction, we may show the following lemma.

**Lemma 7.13.** The entries of $P_m$ are symmetric polynomial in $\lambda$. If $m \geq 1$, then the off-diagonal entries of $P_m$ have degree in $\lambda$ at most $2m - 2$, whereas the degree of each diagonal entry is exactly $2m$.

The first row of $T^{-1} \tilde{\Phi}$ constitute a fundamental system of formal solutions of the Bessel equation (7.2) of parity $\varsigma$. Some calculations yield the following proposition, where for the derivatives of order higher than $n-1$ the differential equation (7.2) is applied.

**Proposition 7.14.** Let $\varsigma \in \{+, -\}$ and $\xi \in \mathbb{X}_n(\varsigma)$. There exists a unique sequence of symmetric polynomials $B_m(\lambda; \xi)$ in $\lambda$ of degree $2m$, with coefficients depending only on $m$, $\xi$ and $n$, and with $B_0(\lambda; \xi) = 1$, such that
\[
e^{i\xi \lambda z - \frac{i\pi}{4}} \sum_{m=0}^{\infty} B_m(\lambda; \xi) z^{-m}.
\]
is a formal solution of the Bessel equation (7.2) of parity $\varsigma$. We shall denote the formal series in (7.21) by $\hat{J}(z; \lambda, \xi)$. Moreover, the $j$-th formal derivative $\hat{J}^{(j)}(z; \lambda, \xi)$ is also of the form as (7.21), say

$$e^{i\xi z}z^{\varsigma-\frac{m}{2}} \sum_{m=0}^{\infty} B_{m,j}(\lambda; \xi) z^{-m},$$

for some symmetric polynomial $B_{m,j}(\lambda; \xi)$ in $\lambda$ of degree 2$m$. The coefficients of $B_{m,j}(\lambda; \xi)$ depend only on $m$, $j$, $\xi$ and $n$.

**Remark 7.15.** The above arguments are essentially adapted from the proof of [CL55, Chapter 5, Theorem 2.1]. This construction of the formal solution and Lemma 7.13 will be required later in §7.4 for the error analysis.

However, this method is not the best for the actual computation of the coefficients $B_m(\lambda; \xi)$. We may derive the recurrent relations of $B_m(\lambda; \xi)$ by a more direct but less suggestive approach as follows.

The substitution $w = e^{i\xi z}z^{-\frac{m}{2}} u$ transforms the Bessel equation (7.2) into

$$\sum_{j=0}^{n} W_j(z; \lambda) u^{(j)} = 0,$$

where $W_j(z; \lambda)$ is a polynomial in $z^{-1}$ of degree $n - j$,

$$W_j(z; \lambda) = \sum_{k=0}^{n-j} W_{j,k}(\lambda) z^{-k},$$

with

$$W_{0,0}(\lambda) = (in\xi)^n - \varsigma(in)^n = 0,$$

$$W_{j,k}(\lambda) = \frac{(in\xi)^{n-j-k}}{j!(n-j-k)!} \sum_{r=0}^{k} \frac{(n-r)!}{(k-r)!} \left[- \frac{n-1}{2}\right]_{k-r} V_{n,n-r}(\lambda), \quad (j,k) \neq (0,0).$$

We have

$$W_{0,1}(\lambda) = (in\xi)^{n-1} \left(n - \frac{n-1}{2}\right) V_{n,n}(\lambda) + V_{n,n-1}(\lambda) = 0,$$

and $W_{1,0}(\lambda) = n(in\xi)^{n-1}$ is nonzero. Some calculations show that $B_m(\lambda; \xi)$ satisfy the following recurrence relations

$$(m-1)W_{1,0}(\lambda)B_{m-1}(\lambda; \xi) = \sum_{k=2}^{\min\{n,m\}} W_{0,k}(\lambda)B_{m-k}(\lambda; \xi)$$

$$+ \sum_{j \geq 1, k \geq 0, \in k \leq \min\{n,m-1\}} W_{j,k}(\lambda)[j+k-m]B_{m-j-k}(\lambda; \xi) = 0, \quad m \geq 2.$$

If $n = 2$, for a fourth root of unity $\xi = \pm 1, \pm i$ one may calculate in this way to obtain

$$B_m(\lambda, -\lambda; \xi) = \frac{(\frac{1}{2} - 2\lambda)_m}{(4i\xi)^m} \frac{\left(\frac{1}{2} + 2\lambda\right)_m}{m!}.$$
7.3.2. Bessel functions of the second kind. Bessel functions of the second kind are solutions of Bessel equations defined according to their asymptotic expansions at infinity. We shall apply several results in the asymptotic theory of ordinary differential equations from [Was65], Chapter IV.

Firstly, [Was65] Theorem 12.3 implies the following lemma.

**Lemma 7.16 (Existence of solutions).** Let \( \zeta \in \{+,-\}, \xi \in \mathcal{H}_n(\zeta), \) and \( S \subset U \) be an open sector with vertex at the origin and a positive central angle not exceeding \( \pi \). Then there exists a solution of the Bessel equation (7.2) of parity \( \zeta \) which has the asymptotic expansion \( \hat{J}(z; \lambda; \xi) \) defined in (7.21) on \( S \). Moreover, derivatives of this solution have the formal derivative of \( \hat{J}(z; \lambda; \xi) \) of the same order as their asymptotic expansion.

For two distinct \( \xi, \xi' \in \mathcal{H}_n(\zeta) \), the ray emitted from the origin on which

\[
\Re((i\xi - i\xi')z) = -3m \left((\xi - \xi')z\right) = 0
\]

is called a separation ray.

We first consider the case \( n = 2 \). For \( \zeta = + \) the separation rays are the two real half axes, and thus separate \( \mathbb{C} \setminus \{0\} \) into the upper and the lower half planes. Similarly, for \( \zeta = - \) the two separation rays constitute the imaginary axis, and \( \mathbb{C} \setminus \{0\} \) is separated into the right and the left half planes. Accordingly, we define \( S_{\pm 1} = \{z : \pm 3m z > 0\} \) and \( S_{\pm i} = \{z : \pm \Re z > 0\} \).

In the case \( n \geq 3 \), there are \( 2n \) distinct separation rays in \( \mathbb{C} \setminus \{0\} \) given by the equations

\[
\arg z = \arg(i\xi'), \quad \xi' \in \mathcal{H}_{2n}(\zeta).
\]

These separation rays divide \( \mathbb{C} \setminus \{0\} \) into \( 2n \) open sectors

\[
S^\pm_\xi = \left\{z : 0 < \pm \arg z - \arg(i\xi) < \frac{\pi}{n}\right\}, \quad \xi \in \mathcal{H}_n(\zeta).
\]

In both sectors \( S^+_\xi \) and \( S^-_\xi \) we have

\[
\Re(i\xi z) < \Re(i\xi' z) \quad \text{for all } \xi' \in \mathcal{H}_n(\zeta), \xi' \neq \xi.
\]

Let \( S_\xi \) be the sector on which (7.23) is satisfied. It is evident that

\[
S_\xi = \left\{z : \left|\arg z - \arg(i\xi)\right| < \frac{\pi}{n}\right\}.
\]

**Lemma 7.17.** Let \( \zeta \in \{+,-\} \) and \( \xi \in \mathcal{H}_n(\zeta) \).

1. Existence of asymptotics. If \( n \geq 3 \), all the solutions of the Bessel equation (7.2) of parity \( \zeta \) on \( S^\pm_\xi \) have asymptotic representation a multiple of \( \hat{J}(z; \lambda; \xi') \) for some \( \xi' \in \mathcal{H}_n(\zeta) \). If \( n = 2 \), the same assertion is true with \( S^+_\xi \) replaced by \( S_\xi \).

2. Uniqueness of the solution. There is a unique solution of the Bessel equation of parity \( \zeta \) possessing \( \hat{J}(z; \lambda; \xi) \) as its asymptotic expansion on \( S_\xi \) or any of its open subsector, and we shall denote this solution by \( J(z; \lambda; \xi) \). Moreover, \( J^{(j)}(z; \lambda; \xi) \sim \hat{J}^{(j)}(z; \lambda; \xi) \) on \( S_\xi \) for any \( j \geq 0 \).
Proof: (1) follows directly from [Was65, Theorem 15.1].

For \( n = 2 \), since (7.23) holds for the sector \( S_\xi \), (2) is true according to [Was65, Corollary to Theorem 15.3]. Similarly, if \( n \geq 3 \), (2) is true with \( S_\xi \) replaced by \( S_\xi^\pm \).

Thus, assuming \( n \geq 3 \), there exists a unique solution of the Bessel equation of parity \( \varsigma \) possessing \( \tilde{J}(z; \lambda; \xi) \) as its asymptotic expansion on \( S_\xi^\pm \) or any of its open subsector, and for the moment we denote it by \( J^\pm(z; \lambda; \xi) \). On the other hand, because \( S_\xi \) has central angle \( \frac{\pi}{n} < \pi \), there exists a solution \( J(z; \lambda; \xi) \) with asymptotic \( \tilde{J}(z; \lambda; \xi) \) on a given open subsector \( \mathbb{S} \subset S_\xi \) due to Lemma 7.16. Observe that at least one of \( S \cap S_\xi^+ \) and \( S \cap S_\xi^- \) is a nonempty open sector, say \( S \cap S_\xi^+ \neq \emptyset \), then the uniqueness of \( J(z; \lambda; \xi) \) follows from that of \( J^+(z; \lambda; \xi) \) and analyticity.

Q.E.D.

Proposition 7.18. Let \( \varsigma \in \{+,-\} \), \( \xi \in \mathcal{X}_n(\varsigma) \), \( \theta \) be a small positive constant, say \( 0 < \theta < \frac{\pi}{n} \), and define

\[
S_\xi^\theta(\theta) = \left\{ z : \left| \arg z - \arg(i\xi) \right| < \pi + \frac{\pi}{n} - \theta \right\}.
\]

Then \( J(z; \lambda; \xi) \) is the unique solution of the Bessel equation of parity \( \varsigma \) which has asymptotic expansion \( \tilde{J}(z; \lambda; \xi) \) on \( S_\xi^\theta(\theta) \). Moreover, \( J(j) (z; \lambda; \xi) \sim \tilde{J}(j)(z; \lambda; \xi) \) on \( S_\xi^\theta(\theta) \) for any \( j \geq 0 \).

Proof. Following from Lemma 7.16, there exists a solution of the Bessel equation of parity \( \varsigma \) which has the asymptotic expansion \( \tilde{J}(z; \lambda; \xi) \) on the open sector

\[
S_\xi^\theta(\theta) = \left\{ z : \frac{\pi}{n} - \theta < \left| \arg z - \arg(i\xi) \right| < \pi + \frac{\pi}{n} - \theta \right\}.
\]

On the nonempty open sector \( S_\xi \cap S_\xi^\theta(\theta) \) this solution must be identical with \( J(z; \lambda; \xi) \) by Lemma 7.17 (2) and hence is equal to \( J(z; \lambda; \xi) \) on \( S_\xi \cup S_\xi^\theta(\theta) \) due to analyticity. Therefore, the region of validity of the asymptotic \( J(z; \lambda; \xi) \sim \tilde{J}(z; \lambda; \xi) \) may be widened from \( S_\xi \) into \( S_\xi^\theta(\theta) = S_\xi \cup S_\xi^\theta(\theta) \cup S_\xi^-^\theta(\theta) \). In the same way, Lemma 7.16 and 7.17 (2) also imply that \( J(j) (z; \lambda; \xi) \sim \tilde{J}(j)(z; \lambda; \xi) \) on \( S_\xi^\theta(\theta) \) for any \( j \geq 0 \).

Q.E.D.

Corollary 7.19. Let \( \varsigma \in \{+,-\} \). All the \( J(z; \lambda; \xi) \), with \( \xi \in \mathcal{X}_n(\varsigma) \), form a fundamental set of solutions of the Bessel equation (7.2) of parity \( \varsigma \).

Remark 7.20. If \( n = 2 \), by [Wat44, 3.7 (8), 3.71 (18), 7.2 (1, 2), 7.23 (1, 2)] we have the following formula of \( J(z; \lambda; -\lambda; \xi) \), with \( \xi = \pm 1, \pm i \), and the corresponding sector on which its asymptotic expansion is valid

\[
J(z; \lambda; -\lambda; 1) = \sqrt{\frac{\pi}{2}} e^{n \pi i / 2} H_2^{(1)}(2z), \quad S_\xi^\theta(\theta) = \left\{ z : -\pi + \theta < \arg z < 2\pi - \theta \right\};
\]

\[
J(z; \lambda; -\lambda; -1) = \sqrt{\frac{\pi}{2}} e^{-n \pi i / 2} H_2^{(2)}(2z), \quad S_\xi^\theta(\theta) = \left\{ z : -2\pi + \theta < \arg z < \pi - \theta \right\};
\]

\[
J(z; \lambda; -\lambda; i) = \frac{2}{\sqrt{\pi}} K_{2i}(2z), \quad S_\xi^\theta(\theta) = \left\{ z : \arg z < \frac{\pi}{2} + \theta \right\};
\]

\[
J(z; \lambda; -\lambda; -i) = 2 \sqrt{\pi} I_{2i}(2z) - \frac{2i}{\sqrt{\pi}} e^{2n \pi i} K_{2i}(2z), \quad S_\xi^\theta(\theta) = \left\{ z : \frac{\pi}{2} + \theta < \arg z < \frac{5\pi}{2} - \theta \right\}.
\]
Lemma 7.21. Let $\xi \in \mathbb{R}_{2n(+)}$. We have

$$J(z; \lambda; \xi) = (\pm \xi)^{\lambda/2} J(\pm \xi; \lambda; \pm 1),$$

and $B_m(\lambda; \xi) = (\pm \xi)^{-m} B_m(\lambda; \pm 1)$. Note that $(\pm \xi)^{\lambda/2}$ may depend on the choice of $\text{arg}(\pm \xi)$.

Proof. By Lemma 7.1, $(\pm \xi)^{\lambda/2} J(\pm \xi; \lambda; \pm 1)$ is a solution of one of the two Bessel equations of index $\lambda$. In view of Proposition 7.14 and Lemma 7.17 (2), it must possess $\tilde{J}(z; \lambda; \xi)$ as its asymptotic expansion on $S_{\xi}$, and hence is identical with $J(z; \lambda; \xi)$. This completes the proof. Q.E.D.

Terminology 7.22. For $\xi \in \mathbb{R}_{2n(+)}$, $J(z; \lambda; \xi)$ is called a Bessel function of the second kind.

Remark 7.23. The results in this section do not provide any information on the asymptotics near zero of Bessel function of the second kind, and therefore their connections with Bessel function of the first kind can not be clarified here. We shall nevertheless find the connection formulae between the two kinds of Bessel functions later in §8 appealing to the asymptotic expansion of the H-Bessel function $H^\pm(z; \lambda)$ on the half-plane $\mathbb{H}^\pm$ that we showed earlier in §5.

7.4. Error Bounds for asymptotic expansions. The error bound for the asymptotic expansion of $J(z; \lambda; \xi)$ with dependence on $C = \max \{|\lambda t|\} + 1$ is always desirable for potential applications in analytic number theory. However, the author does not find any general results on the error analysis for differential equations of order higher than two. We shall nevertheless combine and generalize the ideas from [CL55 §5.4] and [Olv74 §7.2] to obtain an almost optimal error estimate for the asymptotic of the Bessel function $J(z; \lambda; \xi)$. Observe that both of their methods have drawbacks for generalizations. [Olv74] hardly uses the viewpoint from differential systems as only the second-order case is treated, whereas [CL55 §5.4] is restricted to the positive real axis for more clarified expositions.

7.4.1. Notations and preparations. We retain the notations from §7.3.1. For $M \geq 1$ denote by $P_{(M)}$ the polynomial in $z^{-1}$,

$$P_{(M)}(z) = \sum_{m=0}^{M} z^{-m} P_m,$$

and by $\tilde{\Phi}_{(M)}$ the truncation of $\tilde{\Phi}$,

$$\tilde{\Phi}_{(M)}(z) = P_{(M)}(z)z^{-\lambda/2} e^{\lambda z}.$$ 

By Lemma 7.13 we have $|z^{-m} P_m| \leq_m C^m |z|^{-m}$, so $P_{(M)}^{-1}$ exists as an analytic function for $|z| > c_1 C^2$, where $c_1$ is some constant depending only on $M$ and $n$. Moreover,

$$|P_{(M)}(z)|, \quad |P_{(M)}^{-1}(z)| = O_{M,n}(1), \quad |z| > c_1 C^2.$$ 

(7.26)
Let \( A(M) \) and \( E(M) \) be defined by
\[
A(M) = \hat{\Phi}'(M) \hat{\Phi}^{-1}(M), \quad E(M) = A - A(M).
\]

\( A(M) \) and \( E(M) \) are clearly analytic for \(|z| > c_1 \mathbb{C}^2\). Since
\[
E(M)P(M) = AP(M) - \left( P'(M) - \frac{n-1}{2}z^{-1}P(M) + P(M)A \right),
\]
it follows from the construction of \( \hat{\Phi} \) in (7.31) that \( E(M)P(M) \) is a polynomial in \( z^{-1} \) of the form \( \sum_{m=M+1}^{\infty} z^{-m}E_m \), and that
\[
E_{M+1} = P_{M+1}^* - AP_{M+1}^*, \quad E_m = \sum_{j=m-M}^{\text{min}(m,n)} A_j P_{m-j}, \quad M + 1 < m \leq M + n.
\]

Therefore, in view of Lemma 7.13, |\( E_{M+1} | \ll M, n \mathbb{C}^{2M} \) and \( |E_m| \ll M, n \mathbb{C}^{M+n} \) for \( M + 1 < m \leq M + n \). It follows that \( |E(M)(z)P(M)(z)| \ll M, n \mathbb{C}^{M+n}z^{-M-1} \) for \( |z| > c_1 \mathbb{C}^2 \), and this, combined with (7.26), yields
\[
|E(M)(z)| = O_{M,n} \left( \mathbb{C}^{2M}|z|^{-M-1} \right). \tag{7.27}
\]

By the definition of \( A(M) \), for \( |z| > c_1 \mathbb{C}^2 \), \( \hat{\Phi}(M) \) is a fundamental matrix of the system (7.12), that is,
\[
u' = A(M)u.
\]
We shall regard the differential system (7.12), that is,
\[
u' = A u = A(M)u + E(M)u,
\]
as a nonhomogeneous system with (7.28) as the corresponding homogeneous system.

7.4.2. Construction of a solution. Now let \( \ell = 1, \ldots, n \), and let
\[
\hat{\Phi}_{M,\ell}(z) = P(M,\ell)(z)z^{\frac{n-1}{2}}e^{i\ell z}
\]
be the \( \ell \)-th column vector of the matrix \( \hat{\Phi}(M) \), where \( P(M,\ell) \) is the \( \ell \)-th column vector of \( P(M) \). Using a version of the variation-of-constants formula and the method of successive approximations, we shall construct a solution \( \varphi(M,\ell) \) of (7.12), for \( z \) in some suitable domain, satisfying
\[
|\varphi(M,\ell)(z)| = O_{M,n} \left( |z|^{-\frac{n-1}{2}}e^{Re(\ell z)} \right), \tag{7.30}
\]
and
\[
|\varphi(M,\ell)(z) - \hat{\varphi}(M,\ell)(z)| = O_{M,n} \left( \mathbb{C}^{2M}|z|^{-M-\frac{n-1}{2}}e^{Re(\ell z)} \right), \tag{7.31}
\]
where the implied constant in (7.31) also depend on the domain that we choose.

**Step 1. Constructing the domain and the contours for the integral equation.** For \( C \geq c_1 \mathbb{C}^2 \) and \( 0 < \theta < \frac{\pi}{2} \) define a domain \( D(C; \theta) \subset \mathbb{U} \) by
\[
D(C; \theta) = \left\{ z : |\arg z| \leq \pi, |z| > C \right\} \cup \left\{ z : \pi < |\arg z| < \frac{3}{2} \pi - \theta, \ Re z < -C \right\}.
\]
For $k \neq \ell$ let $\omega(\ell, k) = \arg(\bar{\xi}_\ell - i\bar{\xi}_k) = \arg(\bar{\xi}_\ell) + \arg(1 - \xi_k \xi^*_\ell)$, and define

$$D_{\xi_\ell}(C; \vartheta) = \bigcap_{k \neq \ell} e^{i\omega(\ell, k)} \cdot D(C; \vartheta).$$

With the observation that

$$\left\{ \arg(1 - \xi_k \bar{\xi}^*_\ell) : k \neq \ell \right\} = \left\{ \left( \frac{1}{2} - \frac{a}{n} \right) \pi : a = 1, \ldots, n - 1 \right\},$$

it is straightforward to show that $D_{\xi_\ell}(C; \vartheta) = i\bar{\xi}_\ell D'(C; \vartheta)$, where $D'(C; \vartheta)$ is defined to be the union of the sector

$$\left\{ z : \arg z \leq \frac{\pi}{2} + \frac{\pi}{n}, |z| > C \right\}$$

and the following two domains

$$\left\{ z : \frac{\pi}{2} + \frac{\pi}{n} < \arg z < \pi + \frac{\pi}{n} - \vartheta, \text{Im} \left( e^{-\frac{i}{2}\pi z} \right) > C \right\},$$

$$\left\{ z : -\pi - \frac{\pi}{n} + \vartheta < \arg z < -\pi - \frac{\pi}{n} + \text{Im} \left( e^{\frac{1}{2}\pi z} \right) < -C \right\}. $$
For $z \in \mathbb{D}(C; \theta)$ we define a contour $\mathcal{C}(z) \subset \mathbb{D}(C; \theta)$ which starts from $\infty$ and ends at $z$; see Figure 2. For $z \in \mathbb{D}(C; \theta)$ with $|\arg z| < \pi$ the contour $\mathcal{C}(z)$ consisting of the part of the positive axis where magnitude exceeds $|z|$ and an arc of the circle centered at the origin of radius $|z|$, angle not exceeding $\pi$ and endpoint $z$. For $z \in \mathbb{D}(C; \theta)$ with $\pi < |\arg z| < \frac{3}{2}\pi - \theta$, the definition of the contour $\mathcal{C}(z)$ is modified so that the circular arc has radius $-\Re z$ instead of $|z|$ and ends at $\Re z$ on the negative real axis, and that $\mathcal{C}(z)$ also consists of a vertical line segment joining $\Re z$ and $z$. The crucial property that $\mathcal{C}(z)$ satisfies is the nonincreasing of $\Re z$ along $\mathcal{C}(z)$.

We also define a contour $\mathcal{C}'(z)$ for $z \in \mathbb{D}'(C; \theta)$ of a similar shape as $\mathcal{C}(z)$ illustrated in Figure 3.

**Step 2. Solving the integral equation via successive approximations.** We first split $\hat{\Phi}^{-1}_{(M)}$ into $n$ parts

$$
\hat{\Phi}^{-1}_{(M)} = \sum_{k=1}^{n} \Psi^{(k)}_{(M)},
$$

where the $j$-th row of $\Psi^{(k)}_{(M)}$ is identical with the $k$-th row of $\hat{\Phi}^{-1}_{(M)}$, or identically zero, according as $j = k$ or not.

The integral equation to be considered is the following

$$
u(z) = \hat{\varphi}_{(M), r}(z) + \sum_{k \neq l} \int_{C_{(M), \theta} e^{i\omega(k, l)}} K_{k}(z, \xi) u(\xi) d\xi + \int_{C_{(M), \theta} e^{i\omega(k, l)}} K_{l}(z, \xi) u(\xi) d\xi,
$$

where

$$
K_{k}(z, \xi) = \hat{\Phi}_{(M)}(z) \Psi^{(k)}_{(M)}(\xi) E_{(M)}(\xi), \quad z, \xi \in \mathbb{D}_{\xi}(C; \theta), k = 1, ..., n,
$$

the integral in the sum is integrated on the contour $C_{(M), \theta} e^{i\omega(k, l)}$, whereas the last integral is on the contour $C_{(M), \theta} e^{-i\omega(k, l)}$. Clearly, all these contours lie in $\mathbb{D}_{\xi}(C; \theta)$. Most importantly, we note that $\Re((i\xi_{1} - i\xi_{2})\xi)$ is a negative multiple of $\Re(e^{-i\omega(k, l)} \xi)$ and hence is nondecreasing along the contour $e^{i\omega(k, l)} C_{(M), \theta} e^{-i\omega(k, l)} z$.

By direct verification, it follows that if $u(z) = \varphi(z)$ satisfies (7.32), where the integrals converge, then $\varphi$ satisfies (7.29).

In order to solve (7.32), define the successive approximations by

$$
\varphi^{0}(z) = 0,
$$

$$
\varphi^{n+1}(z) = \hat{\varphi}_{(M), r}(z) + \sum_{k \neq l} \int_{C_{(M), \theta} e^{i\omega(k, l)}} K_{k}(z, \xi) \varphi^{n}(\xi) d\xi + \int_{C_{(M), \theta} e^{i\omega(k, l)}} K_{l}(z, \xi) \varphi^{n}(\xi) d\xi.
$$

The $(j, r)$-th entry of the matrix $\hat{\Phi}^{-1}_{(M)}(z) \Psi_{(M)}^{(k)}(\xi)$ is given by

$$
\left(\hat{\Phi}_{(M)}(z) \Psi_{(M)}^{(k)}(\xi)\right)_{jr} = (P_{(M)}(z))_{jk} (P^{-1}_{(M)}(\xi))_{kr} \left(\frac{z}{\xi}\right)^{-\frac{n-1}{2}} e^{i\omega_{d}(z-\xi)}.
$$

It follows from (7.26) (7.27) that

$$
|K_{k}(z, \xi)| \leq c_{2} e^{2M|z|^{-\frac{n+1}{2}}} |\xi|^{-M-1+\frac{n+1}{2}} e^{\Re(e^{i\omega_{d}(z-\xi)})},
$$

where $c_{2}$ is a positive constant.
for some constant $c_2$ depending only on $M$ and $n$. Furthermore, we may appropriately choose $c_2$, with dependence only on $M$ and $n$, such that

$$\int_{\xi, \ell} |\xi|^{-M-1} |d\xi| \leq c_2 M^{-M}, \quad k \neq \ell. \tag{7.35}$$

According to (7.33), $\varphi^1(z) = \hat{\varphi}_{(M,\ell)}(z) = p_{(M,\ell)}(z) z^{-\frac{M}{2}} e^{i\pi z}$, so

$$|\varphi^1(z) - \varphi^0(z)| = |\varphi^1(z)| \leq c_2 |z|^{-\frac{M}{2}} e^{i\Re (i\pi z)}, \quad z \in \mathbb{D}_\ell(C; \theta). \tag{7.36}$$

We shall show by induction that for all $z \in \mathbb{D}_\ell(C; \theta)$

$$|\varphi^\alpha(z) - \varphi^{\alpha-1}(z)| \leq c_2 (nc_2^2 \mathbb{C}^2M^{-M})^{\alpha-1} |z|^{-\frac{M}{2}} e^{i\Re (i\pi z)}. \tag{7.36}$$

Let $z \in \mathbb{D}_\ell(C; \theta)$, and assume (7.36) holds. From (7.33) we have

$$|\varphi^{\alpha+1}(z) - \varphi^\alpha(z)| \leq \sum_{k \neq \ell} R_k + R_\ell,$$

with

$$R_k = \int_{\xi, \ell} |K_k(z, \xi)| |\varphi^\alpha(\xi) - \varphi^{\alpha-1}(\xi)| |d\xi|, \quad R_\ell = \int_{\xi, \ell} |K_\ell(z, \xi)| |\varphi^\alpha(\xi) - \varphi^{\alpha-1}(\xi)| |d\xi|. \tag{7.37}$$

It follows from (7.33), (7.36) that $R_k$ has bound

$$c_2^2 \mathbb{C}^2 \left( nc_2^2 \mathbb{C}^2M^{-M} \right)^{\alpha-1} |z|^{-\frac{M}{2}} e^{i\Re (i\pi z)} \int_{\xi, \ell} |\xi|^{-M-1} e^{i\Re (i\pi (\xi - z))} |d\xi|. \tag{7.38}$$

Since $\Re \left( (i\xi - i\xi_k)\xi \right)$ is nondecreasing on the integral contour,

$$R_k \leq c_2^2 \mathbb{C}^2 \left( nc_2^2 \mathbb{C}^2M^{-M} \right)^{\alpha-1} |z|^{-\frac{M}{2}} e^{i\Re (i\pi z)} \int_{\xi, \ell} |\xi|^{-M-1} |d\xi|, \tag{7.39}$$

and (7.35) yields

$$R_k \leq c_2^2 |z|^{-\frac{M}{2}} e^{i\Re (i\pi z)}. \tag{7.40}$$

Similar arguments show that $R_\ell$ has the same bound as $R_k$. Thus (7.36) is true with $\alpha$ replaced by $\alpha + 1$.

Set the constant $C = c \mathbb{C}^2$ such that $c^M \geq 2nc_2^2$. Then $nc_2^2 \mathbb{C}^2M^{-M} \leq \frac{1}{2}$, and therefore the series $\sum_{\alpha=1}^{\infty} (\varphi^\alpha(z) - \varphi^{\alpha-1}(z))$ absolutely and locally uniformly converges. The limit function $\varphi_{(M,\ell)}(z)$ satisfies (7.30) for all $z \in \mathbb{D}_\ell(C; \theta)$. More precisely,

$$|\varphi_{(M,\ell)}(z)| \leq 2c_2 |z|^{-\frac{M}{2}} e^{i\Re (i\pi z)}, \quad z \in \mathbb{D}_\ell(C; \theta). \tag{7.31}$$

Using a standard argument for successive approximations, it follows that $\varphi_{(M,\ell)}(z)$ satisfies the integral equation (7.32) and hence the differential system (7.29).

The proof of the error bound (7.31) is similar. Since $\varphi_{(M,\ell)}(z)$ is a solution of the integral equation (7.32), we have

$$|\varphi_{(M,\ell)}(z) - \hat{\varphi}_{(M,\ell)}(z)| \leq \sum_{k \neq \ell} S_k + S_\ell,$$
where

\[ S_k = \int_{\mathcal{C}^\ell,\iota} |K_k(z, \zeta)| \left| \varphi_{(M),\ell}(\zeta) \right| |d\zeta|, \quad S_\ell = \int_{\mathcal{C}^\ell} |K_\ell(z, \zeta)| \left| \varphi_{(M),\ell}(\zeta) \right| |d\zeta|. \]

With the observation that \( |\zeta| \geq \sin \theta \cdot |z| \) for \( z \in \mathcal{D}_{\ell}(C; \theta) \) and \( \zeta \) on the integral contours given above, we may replace (7.35) by the following

\[ (7.38) \quad \int_{\mathcal{C}^\ell} |\zeta|^{M-1} |d\zeta|, \quad \int_{\mathcal{C}^\ell,\iota} |\zeta|^{M-1} |d\zeta| \leq c_2|z|^{-M}, \quad k \neq \ell, \]

whereas now \( c_2 \) also depends on \( \theta \).

The bounds (7.34, 7.37) of \( K_k(z, \zeta) \) and \( \varphi_{(M),\ell}(z) \) along with (7.38) yield

\[ S_k \leq 2c_2^2 |z|^{2M} |\zeta|^{M-1} e^{\Re (i\ell \zeta)} \int_{\mathcal{C}^\ell,\iota} |\zeta|^{M-1} e^{\Re (i(\ell \zeta - \xi \zeta)(\zeta - z))} |d\zeta| \]

\[ \leq 2c_2^2 |z|^{2M} |\zeta|^{M-1} e^{\Re (i\ell \zeta)} \int_{\mathcal{C}^\ell,\iota} |\zeta|^{M-1} |d\zeta| \]

\[ \leq 2c_2^2 |z|^{2M} |\zeta|^{-M-1} e^{\Re (i\ell \zeta)}. \]

Again, the second inequality follows from the fact that \( \Re (i\ell \zeta - i \xi \zeta) \) is nondecreasing on the integral contour. Similarly, \( S_\ell \) has the same bound as \( S_k \). Thus, (7.31) is proved and can be made precise as below

\[ (7.39) \quad \left| \varphi_{(M),\ell}(z) - \hat{\varphi}_{(M),\ell}(z) \right| \leq 2nc_2^2 |z|^{2M} |\zeta|^{-M-1} e^{\Re (i\ell \zeta)}, \quad z \in \mathcal{D}_{\ell}(C; \theta). \]

7.4. Conclusion. Restricting to the sector \( S^+_{\ell i} \cap \{ z : |z| > C \} \subset \mathcal{D}_{\ell}(C; \theta) \) with \( S^+_{\ell i} \) defined in (7.32) where \( S^+_{\ell i} \) is replaced by \( S_{\ell i} \) if \( n = 2 \), each \( \varphi_{(M),\ell} \) has an asymptotic representation a multiple of \( \hat{\varphi}_k \) for some \( k \) according to Lemma 7.17 (1). Since \( \Re (i \xi \zeta) < \Re (i \ell \zeta) \) for any \( j \neq \ell \), the bounds (7.30, 7.31) forces \( k = \ell \). Therefore, for any \( M \geq 1 \), \( \varphi_{(M),\ell} \) is identical with the unique solution \( \hat{\varphi}_\ell \) of the differential system (7.12) with asymptotic expansion \( \hat{\varphi}_\ell \) on \( S^+_{\ell i} \). Replacing \( \varphi_{(M),\ell} \) by \( \varphi_{\ell} \) and absorbing the \( M \)-th term of \( \hat{\varphi}_{(M),\ell} \) into the error bound, we may reformulate (7.39) as the following error bound for \( \varphi_{\ell} \)

\[ (40) \quad \left| \varphi_{\ell}(z) - \hat{\varphi}_{(M-1),\ell}(z) \right| = O_{M, \theta, n} \left( |z|^{2M} |\zeta|^{-M-1} e^{\Re (i\ell \zeta)} \right), \quad z \in \mathcal{D}_{\ell}(C; \theta). \]

Moreover, recalling the definition of the sector \( S^+_{\ell i}(\theta) \) given in (7.25), we have

\[ (7.41) \quad S^+_{\ell i}(\theta) \cap \left\{ z : |z| > \frac{C}{\sin \theta} \right\} \subset \mathcal{D}_{\ell}(C; \theta). \]

Thus, the following theorem is finally established by (7.40) and (7.41).

**Theorem 7.24.** Let \( \xi \in \{ +, - \}, \xi \in \mathcal{V}_n(\xi), \ M \geq 1 \), \( 0 < \theta < \frac{1}{2} \pi \), and \( S^+_{\ell i}(\theta) \) be the sector defined as in (7.25). Then there exists a constant \( c \), depending only on \( M \), \( \theta \), and \( n \), such that

\[ (7.42) \quad J(z; \lambda; \xi) = e^{i\ell \xi z^{\frac{1}{\ell} \lambda}} \left( \sum_{m=0}^{M-1} B_m(\lambda; \xi) z^{-m} + O_{M, \theta, n} \left( |z|^{-M} \right) \right) \]
for all \( z \in \mathbb{C} \) such that \(|z| > c \mathbb{C}^2\). Similar asymptotic is valid for all the derivatives of \( J(z; \lambda; \xi) \), where the constant \( c \) and the implied constant of the error term are allowed to depend on the order of the derivative.

Finally, we remark that, since \( B_m(\lambda; \xi)z^{-m} \) is of size \( O_m, (\xi 2m|z|^{-m}) \), the error bound in (7.42) is optimal, given that \( \theta \) is fixed.

8. Connections between various types of Bessel functions

Recall from §5.4.1 that the asymptotic expansion in Theorem 5.10 remains valid for the \( H \)-Bessel function \( H^\pm(z; \lambda) \) on the half-plane \( \mathbb{H}^\pm = \{ z : 0 \leq \pm \arg z \leq \pi \} \) (see (5.12)). With the observations that \( H^\pm(z; \lambda) \) satisfies the Bessel equation of parity \((\pm)^n\), that the asymptotic expansions of \( \sqrt{n}(\pm 2\pi i)^{-m} H^\pm(z; \lambda) \) and \( J(z; \lambda; \pm 1) \) have exactly the same form and the same leading term due to Theorem 5.10 and Proposition 7.14, and that \( \xi \) is actually solvable!

As consequences of Theorem 8.1, we can establish the connections between various Bessel functions, that is, \( J(z; \xi; \lambda) \), \( J_L(z; \xi; \lambda) \) and \( J(z; \lambda; \xi) \). Recall that \( J(z; \xi; \lambda) \) was already expressed in terms of Bessel functions of the first kind, namely \( J_L(z; \xi; \lambda) \), in Lemma 7.6

8.1. Relations between \( J(z; \xi; \lambda) \) and \( J(z; \lambda; \xi) \). In view of Lemma 7.8, \( J(z; \xi; \lambda) \) is equal to a multiple of \( H^\pm \left( e^{\pm \pi i \frac{\xi + \lambda}{2}} \right) \). On the other hand, \( J(z; \lambda; \xi) \) is a multiple of \( J(\pm \xi z; \lambda; \pm 1) \) due to Lemma 7.21. The equality up to constant between \( H^\pm(z; \lambda) \) and \( J(z; \lambda; \pm 1) \) has just been built in Theorem 8.1. Some calculations yield the following corollary to Theorem 8.1

**Corollary 8.3.** Recall \( L^\pm(\xi) = \{ \ell : \xi \ell = \pm \} \) and \( n^\pm(\xi) = |L^\pm(\xi)| \) in Definition 7.6. Let \( c(\xi, \lambda) = e^{\left( \pm \frac{n-1}{n} + \frac{(n-1)n(\xi + \lambda)}{4n} + \frac{1}{2} \sum_{\ell \in L^\pm(\xi)} \ell \right)} \) and \( \xi(\xi) = \pm e^{\pm \pi i \frac{\xi + \lambda}{2}} \). Then we have

\[
J(z; \xi; \lambda) = \frac{(2\pi)^{\pm 1} c(\xi, \lambda)}{\sqrt{n}} J(\pm \xi z; \lambda; \pm 1).
\]

Here, it is understood that \( \arg \xi(\xi) = \frac{n}{n} \pi = \pi - \frac{n}{n} \pi \).
Corollary 8.3 shows that \( J(z; \varsigma, \lambda) \) should be categorized in the class of Bessel functions of the second kind. Moreover, the asymptotic behaviours of the Bessel functions \( J(z; \varsigma, \lambda) \) are classified by their signatures \( (n_+(\varsigma), n_-(\varsigma)) \). Therefore, \( J(z; \varsigma, \lambda) \) is uniquely determined by its signature up to a constant multiple.

8.2. Relations connecting the two kinds of Bessel functions. From Lemma 7.21 and Theorem 8.1, we see that \( J(z; \lambda, \xi) \) is a constant multiple of \( H^+(\xi z; \lambda) \). On the other hand, \( H^+(\xi z; \lambda) \) can be expressed in terms of Bessel functions of the first kind in view of Lemma 7.6. Finally, using Lemma 7.3, the following corollary is readily established.

**Corollary 8.4.** Let \( \varsigma \in \{+, -\} \). If \( \xi \in \mathbb{R}(\varsigma) \), then

\[
J(z; \lambda; \xi) = \sqrt{n} \left( -\frac{\pi i \xi}{2} \right) e^{\frac{\pi i n}{2}(n-1)} \sum_{\ell=1}^{\frac{n-1}{2}} (\xi)^{n\ell} \left( \prod_{k \neq \ell} \sin (\pi(\lambda_{\ell} - \lambda_k))^{-1} \right) J_{\ell}(z; \varsigma, \lambda).
\]

Recall that \((-i\xi)^{\frac{n}{2}} = e^\frac{\pi in}{4}(-\frac{i}{2} \pi + i \arg \xi)\) and \( (\xi)^{\frac{n}{2}} = e^\frac{i\pi mn - in\lambda_j}{2} \arg \xi \) according to our convention. When \( \lambda \) is not generic, the right hand side is to be replaced by its limit.

We now fix an integer \( a \) and let \( \varsigma = (-)^a \) and \( \xi_j = e^{\frac{i\pi j}{n}(n-1)} \), with \( j = 1, \ldots, n \). It follows from Corollary 8.4 that

\[
X(z; \lambda) = \sqrt{n} \left( \frac{\pi}{2} \right)^{\frac{n-1}{2}} e^{-\frac{\pi i n(n-1)}{4}} \cdot DV(\lambda) S(\lambda) E(\lambda) Y(z; \lambda),
\]

where

\[
X(z; \lambda) = \left( J(z; \lambda; \xi_j) \right)_{j=1}^n, \quad Y(z; \lambda) = \left( J_{\ell}(z; \varsigma, \lambda) \right)_{\ell=1}^n,
\]

\[
D = \text{diag} \left( (\xi_j)^{\frac{n}{2}} \right)_{j=1}^n, \quad E(\lambda) = \text{diag} \left( e^{\pi i (\frac{n}{2} - \ell)} \right)_{\ell=1}^n,
\]

\[
S(\lambda) = \text{diag} \left( \prod_{k \neq \ell} \sin (\pi(\lambda_{\ell} - \lambda_k))^{-1} \right)_{\ell=1}^n, \quad V(\lambda) = \left( e^{-2\pi i (\frac{1}{2} - \ell)} \right)_{\ell=1}^n.
\]

Observe that \( V(\lambda) \) is a Vandermonde matrix.

**Lemma 8.5.** For an \( n \)-tuple \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n \) we define the Vandermonde matrix

\[
V = \left( x_{j-1}^{\ell-1} \right)_{\ell,j=1}^n.
\]

For \( d = 0, 1, \ldots, n-1 \) and \( m = 1, \ldots, n \), let \( \sigma_{m,d} \) denote the elementary symmetric polynomial in \( x_1, \ldots, x_m \), \( \ldots, x_n \) of degree \( d \), and let \( \tau_m = \prod_{k \neq m} (x_m - x_k) \). If \( \mathbf{x} \) is generic in the sense that all the components of \( \mathbf{x} \) are distinct, then \( V \) is invertible, and furthermore, the inverse of \( V \) is

\[
\left( (-1)^d \sigma_{m,n-d-1} \right)_{m,j=1}^n.
\]

**Proof of Lemma 8.5.** It is a well-known fact that \( V \) is invertible whenever \( \mathbf{x} \) is generic. If we denote by \( w_{m,j} \) the \( (m, j) \)-th entry of \( V^{-1} \), then

\[
\sum_{j=1}^n w_{m,j} x_j^{j-1} = \delta_{m,j}.
\]

The Lagrange interpolation formula implies the following identity of polynomials

\[
\sum_{j=1}^n w_{m,j} x_j^{j-1} = \prod_{k \neq m} \frac{x_m - x_k}{x_m - x_k}.
\]
Identifying the coefficient of $x^{l-1}$ on both sides yields the formula of $w_{m,l}$.

Q.E.D.

Choosing $x_l = e^{-2\pi i l}$ in Lemma\footnote{8.5} we see that if $\lambda$ is generic then the matrix $V(\lambda)$ is invertible and its inverse can be explicitly given. Some straightforward computations yield the following corollary.

**Corollary 8.6.** Let $a$ be a given integer. We let $\zeta = (-)^a$ and for $j = 1, \ldots, n$ define $\xi_j = e^{\pi \frac{2j+1}{n}}$. For $d = 0, 1, \ldots, n - 1$ and $\ell = 1, \ldots, n$, let $\sigma_{\ell,d}(\lambda)$ denote the elementary symmetric polynomial in $e^{-2\pi i l_1}, \ldots, e^{-2\pi i l_n}, \ldots, e^{-2\pi i l_n}$ of degree $d$. Then we have

$$J_\ell(z; \zeta, \lambda) = \frac{e^{\frac{2\pi i(n-1)}{n}}}{\sqrt{\pi}} e^{\pi i \left[\left(\frac{2\ell}{n} + 1\right) + \sum_{j=1}^{n} (-)^{n-j} \xi_j\right]} \sigma_{\ell,n-j}(\lambda) J(z; \lambda, \xi_j).$$

**Remark 8.7.** In view of Proposition\footnote{2.7} Remark\footnote{7.3} and \footnote{7.20} when $n = 2$, Corollary\footnote{8.4} corresponds to the connection formulae \footnote{(Wat44 3.61(5, 6), 3.7 (6))}

\begin{align*}
H^1_\nu(z) &= J_{-\nu}(z) - e^{i\pi \nu} J_{\nu}(z) = \frac{2}{i \sin(\pi \nu)} J_{\nu}(z) - J_{-\nu}(z), \\
H^2_\nu(z) &= e^{i\pi \nu} J_{\nu}(z) - J_{-\nu}(z), \\
K_{\nu}(z) &= \frac{\pi (1 - J_{-\nu}(z))}{2 \sin(\pi \nu)}, \\
\pi I_{\nu}(z) &- ie^{i\pi \nu} K_{\nu}(z) = \frac{\pi i}{2 \sin(\pi \nu)} \left( e^{-i\pi \nu} I_{\nu}(z) - e^{i\pi \nu} I_{-\nu}(z) \right),
\end{align*}

whereas Corollary\footnote{8.6} with $a = 0$ or $1$, amounts to the formulae \footnote{(Wat44 3.61(1, 2), 3.7 (6))}

\begin{align*}
J_{\nu}(z) &= \frac{H^1_\nu(z) + H^2_\nu(z)}{2}, \\
J_{-\nu}(z) &= \frac{e^{i\pi \nu} H^1_\nu(z) + e^{-i\pi \nu} H^2_\nu(z)}{2}, \\
I_{\nu}(z) &= \frac{ie^{i\pi \nu} K_{\nu} + (\pi I_{\nu}(z) - ie^{i\pi \nu} K_{\nu}(z))}{\pi}, \\
I_{-\nu}(z) &= \frac{ie^{-i\pi \nu} K_{\nu} + (\pi I_{\nu}(z) - ie^{i\pi \nu} K_{\nu}(z))}{\pi}.
\end{align*}

9. H-Bessel functions and K-Bessel functions, II

In this concluding section, we apply Theorem\footnote{7.24} to improve the results in \footnote{7.24} on the asymptotics of Bessel functions $J(x; \zeta, \lambda)$ as well as the Bessel kernel $J_{(\lambda, \xi)}(\pm x)$ when $x \gg \xi^2$.

9.1. Asymptotic expansions of H-Bessel functions. As a consequence of Theorem\footnote{7.24} and \footnote{8.1} we have the following proposition.

**Proposition 9.1.** Let $0 < \theta < \frac{1}{2} \pi$.

1. Let $M$ be a positive integer. We have

\begin{equation}
H^\pm(z; \lambda) = n^{-\frac{1}{4}} (\pm 2\pi i)^{-\frac{1}{2}} e^{\pm i\pi \nu} e^{\pm i\pi \nu} \left( \sum_{m=0}^{M-1} (\pm i)^{-m} B_m(\lambda) z^{-m} + O_{M,\theta, n} \left( \xi^{2M |z|^{-M}} \right) \right),
\end{equation}

for $z \in \mathbb{S}_\pm(\theta)$ such that $|z| \gg M, \theta, \xi^2$.

2. Define $W(z; \pm, \lambda) = \sqrt{n} (\pm 2\pi i)^{-\frac{1}{2}} e^{\pm i\pi \nu} H^\pm(z; \lambda)$. Let $M - 1 \geq j \geq 0$. We have

\begin{equation}
W^{(j)}(z; \pm, \lambda) = z^{-\frac{j}{2}} \left( \sum_{m=0}^{M-1} (\pm i)^{-m} B_{m,j}(\lambda) z^{-m} + O_{M,\theta, j, n} \left( \xi^{2M-2j |z|^{-M}} \right) \right),
\end{equation}

for $z \in \mathbb{S}_\pm(\theta)$ such that $|z| \gg M, \theta, \xi^2$. 

9.2. Asymptotic expansions of K-Bessel functions.
for \( z \in S'_{\pm 1}(\theta) \) such that \(|z| \gg M, \theta, n, \mathbb{C}^2\).

Observe that
\[
\mathbb{H}^\pm = \{ z \in \mathbb{C} : 0 \leq \pm \arg z \leq \pi \}
\]
\[
\subseteq S'_{\pm 1}(\theta) = \left\{ z \in \mathbb{U} : -\left(1 - \frac{2}{n}\right) \pi - \theta < \pm \arg z < \left(\frac{3}{2} + \frac{1}{n}\right) \pi + \theta \right\}.
\]
Fixing \( \theta \) and restricting to the domain \( \{ z \in \mathbb{H}^\pm : |z| \gg M, n, \mathbb{C}^2 \} \), Proposition 9.1 improves Theorem 5.10.

9.2. Exponential decay of \( K \)-Bessel functions. Now suppose that \( J(z; \xi, \lambda) \) is a \( K \)-Bessel function so that \( 0 < n_\pm(\xi) < n \). Since \( R_+ \subseteq S'_\xi(\theta) \), Corollary 8.3 and Theorem 7.24 imply that \( J(x; \xi, \lambda) \) is not only a Schwartz function at infinity, which was shown in Theorem 5.6 but also a function with exponential decay on \( R_+ \). Of course this assertion is also valid for all the derivatives of \( J(x; \xi, \lambda) \).

**Proposition 9.2.** If \( J(x; \xi, \lambda) \) is a \( K \)-Bessel function, then
\[
J^{(\ell)}(x; \xi, \lambda) \ll_{j,\ell,n} x^{-\frac{n}{2}} \exp \left( -\pi \Im \Lambda(\xi, \lambda) - nI(\xi)x \right),
\]
for \( x \gg_{j,\ell,n} \mathbb{C}^2 \), where \( \Lambda(\xi, \lambda) = \pm \sum_i \ell \xi_i \lambda_i \) and \( I(\xi) = \Im \xi = \sin \left( \frac{\pi}{\pi} \right) > 0 \).

**Remark 9.3.** Given that \( x \gg \mathbb{C}^2 \), \( \exp \left( -\pi \Im \Lambda(\xi, \lambda) \right) \) is negligible compared to \( \exp(nI(\xi)x) \). Thus, if we choose a small positive constant \( \epsilon \), then
\[
J^{(\ell)}(x; \xi, \lambda) \ll_{j,\ell,n} \exp \left( -\left( nI(\xi) - \epsilon \right) x \right),
\]
for all \( x \gg_{j,\ell,n} \mathbb{C}^2 \).

9.3. The asymptotic of the Bessel kernel \( J_{(\lambda, \delta)} \). In comparison with Theorem 5.12 we have the following proposition.

**Proposition 9.4.** Let \( (\lambda, \delta) \in \mathbb{L}^{n-1} \times \{\mathbb{Z}/2\mathbb{Z}\}^n \). Let \( M \) be a positive integer.

1. If \( n \) is even, then for \( x \gg_{M,n} \mathbb{C}^2 \) we have
\[
J_{(\lambda, \delta)}(x^n) = \sum_{\pm} \left( \frac{\pi}{2} \right)^{\frac{3n}{2}} e^{\left( \frac{\pi}{2} \right) \pm \pi} \frac{1}{\sqrt{\pi x^{\frac{3n}{2}}}} \sum_{m=0}^{M-1} (-1)^{-m} B_m(\lambda) x^{-m} + O_{M,n} \left( \mathbb{C}^{2M} x^{-\frac{3n}{2}} \right),
\]
and for \( x \gg_{n} \mathbb{C}^2 \) we have
\[
J_{(\lambda, \delta)}(-x^n) = O_n \left( x^{-\frac{3n}{2}} \exp \left( \frac{\pi}{\pi} \right) x \right),
\]
with \( \Im = \max \{ |\Im \lambda_i| \} \).

2. If \( n \) is odd, then for \( x \gg_{M,n} \mathbb{C}^2 \) we have
\[
J_{(\lambda, \delta)}(\pm x^n) = \left( \frac{\pi}{2} \right) e^{\left( \frac{\pi}{2} \right) \pm \pi} \frac{1}{\sqrt{\pi x^{\frac{3n}{2}}}} \sum_{m=0}^{M-1} (-1)^{-m} B_m(\lambda) x^{-m} + O_{M,n} \left( \mathbb{C}^{2M} x^{-\frac{3n}{2}} \right).
\]


Appendix A. An alternative approach to asymptotic expansions

When \( n = 3 \), the application of Stirling’s asymptotic formula in deriving the asymptotic expansion of the Hankel transform was first found in [Mil06 §4]. The asymptotic was later formulated more explicitly in [Li09 Lemma 6.1], where the author attributed the arguments in her proof to [Ivi97]. Using similar ideas as in [Mil06, Blo12 simplified the proof of [Li09 Lemma 6.1] (see the proof of [Blo12 Lemma 6.1]). This method using Stirling’s asymptotic formula is however the only known approach so far in the literature.

Closely following [Blo12], we shall prove the asymptotic expansions of \( H^\pm (x; \lambda) \) of any rank \( n \) by means of Stirling’s asymptotic formula.

From the definitions (2.5, 2.3) we have

\[
H^\pm (x; \lambda) = \frac{1}{2\pi i} \int_C e \left( \frac{\pm ns}{4} \right) x^{-ns} ds.
\]

In view of the condition \( \sum_{\ell=1}^n \lambda_\ell = 0 \), Stirling’s asymptotic formula yields

\[
\prod_{\ell=1}^n (s - \lambda_\ell) = \Gamma \left( ns - \frac{n-1}{2} \right) n^{-ns} \exp \left( \sum_{m=0}^{M-1} C_m(A) s^{-m} \right) (1 + R_M(s))
\]

for some constants \( C_m(A) \) and remainder term \( R_M(s) = O_{L_M,n} (|s|^{-M}) \). Using the Taylor expansion for the exponential function and some straightforward algebraic manipulations, the right hand side can be written as

\[
\sum_{m=0}^{M-1} \tilde{C}_m(A) \Gamma \left( ns - \frac{n-1}{2} - m \right) n^{-ns} (1 + \tilde{R}_M(s))
\]

for certain constants \( \tilde{C}_m(A) \) and a similar function \( \tilde{R}_M(s) = O_{L_M,n} (|s|^{-M}) \). Suitably choosing the contour \( C \), it follows from (2.12) that

\[
\frac{1}{2\pi i} \int_C e \left( \frac{\pm ns}{4} \right) (ns)^{-ns} ds
\]

\[
= e \left( \pm \left( \frac{n-1}{8} + \frac{1}{4} m \right) \right) \frac{1}{2\pi i} \int_{n^{\frac{1}{4}+\epsilon} c^{-m}} \Gamma(s)e \left( \frac{s}{4} \right) (nx)^{-s} ds
\]

\[
= e \left( \pm \left( \frac{n-1}{8} + \frac{1}{4} m \right) \right) e^{\pm inx} = n^{-\frac{n+1}{4} - \frac{1}{4} - m} e^{\pm inx} x^{-\frac{n+1}{4} - m}.
\]

As for the error estimate, assume \( x \geq 1 \). Insert the part containing \( \tilde{R}_M(s) \) into (A.1), shift the contour to \( \{ s : \Re s = \frac{M-1-\epsilon}{n} + \frac{1}{2} \} \) for some \( \epsilon > 0 \) so that the integral remains absolutely convergent, then by Stirling’s formula the error term is \( O_{L_M,n} \left( x^{-M+\frac{1}{4} - \frac{1}{2} - \epsilon} \right) \). Absorbing several main terms into the error, we arrive at the following asymptotic expansion

\[
H^\pm (x; \lambda) = e^{\pm inx} x^{-\frac{n+1}{4} - \frac{1}{4}} \left( \sum_{m=0}^{M-1} C_m^\pm (A) x^{-m} + O_{L_M,n} (x^{-M}) \right), \quad x \geq 1,
\]

where \( C_m^\pm (A) \) is some constant depending on \( \lambda \).
Finally, we make some comparisons between the three asymptotic expansions (A.2), (5.12) and (9.1) obtained from
- Stirling’s asymptotic formula,
- the method of stationary phase,
- the asymptotic method of ordinary differential equations.

Recall that \( C = \max \{ |\lambda\ell| \} + 1 \), \( \Re = \max \{ |\Re \lambda\ell| \} \) and \( 0 < \vartheta < \frac{1}{2} \pi \). Firstly, the effective domains of these asymptotic expansions are
\[
\{ x \in \mathbb{R}_+ : x \geq 1 \},
\{ z \in \mathbb{C} : |z| \geq C, 0 \leq \pm \arg z \leq \pi \},
\{ z \in \mathbb{U} : |z| \succ M, \theta, \rho \, \mathbb{C}^2, \omega \left( \frac{1}{2} - \frac{1}{n} \right) \pi - \vartheta < \pm \arg z < \left( \frac{3}{2} + \frac{1}{n} \right) \pi + \vartheta \},
\]
respectively. The range of argument is extending while that of modulus is reducing. Secondly, the sizes of error terms are
\[
O_{\lambda, M, \vartheta, \rho} \left( x^{-M - \frac{1}{n}} \right), \quad O_{\Re, M, \theta} \left( (\mathbb{C}^2 |z|^{-M} \right), \quad O_{M, \theta, \rho} \left( \mathbb{C}^2 M |z|^{-M - \frac{1}{n}} \right),
\]
respectively. The dependence of the implied constant in the error bound on \( \lambda \) is improving in all aspects.

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