On Heilbronn’s exponential sum *

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Annotation.

In the paper we prove a new upper bound for Heilbronn’s exponential sum and obtain some applications of our result to distribution of Fermat quotients.

1 Introduction

Let $p$ be a prime number. Heilbronn’s exponential sum is defined by

$$S(a) = \sum_{n=1}^{p} e^{\frac{2\pi i an^p}{p^2}}.$$  \hspace{1cm} (1)

In papers [3], [4] (see also [14]) a nontrivial upper bound for the sum $S(a)$ was obtained.

**Theorem 1** Let $p$ be a prime, and $a \neq 0 \pmod{p}$. Then

$$|S(a)| \ll p^{\frac{7}{8}}.$$

Igor Shparlinski asked to the author about the possibility of an improvement of Theorem 1. The main result of the paper is

**Theorem 2** Let $p$ be a prime, and $a \neq 0 \pmod{p}$. Then

$$|S(a)| \ll p^{\frac{59}{68}} \log^{\frac{5}{34}} p.$$

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Heilbronn’s exponential sum is connected (see e.g. [1], [2], [5], [6], [11], [12]) with so–called Fermat quotients defined as
\[ q(n) = \frac{n^{p-1} - 1}{p}, \quad n \neq 0 \pmod{p}. \]

Our main result has some applications to the distribution of such quotients.

By \( l_p \) denote the smallest \( n \) such that \( q(n) \neq 0 \pmod{p} \). In [1] an upper bound for \( l_p \) was obtained.

**Theorem 3** One has
\[ l_p \leq (\log p)^{\frac{463}{252} + o(1)} \]
as \( p \to \infty \).

We improve the constant \( \frac{463}{252} \) above, see Theorem 12 of section 4.

Another applications are:
- discrepancy of Fermat quotients from [6], Theorems 18–19,
- new bound for the size of the image of \( q(n) \), see [11], Theorem 1,
- estimates for Ihara sum, [12],
- better bounds for the sums \( \sum_{n=1}^{k} \chi(q(n)) \), \( \sum_{n=1}^{k} \chi(nq(n)) \), see [2], Theorems 3.1, 3.2, 5.4.

Let us say few words about the proof. Clearly, sum (1) can be considered as the sum over the following multiplicative subgroup
\[ \Gamma = \{ m^p : 1 \leq m \leq p - 1 \} \subseteq \mathbb{Z}/(p^2\mathbb{Z}) \] (2)
(see the discussion at the beginning of section 3). Recently, some progress in estimating of exponential sums over ”large” subgroups (but in \( \mathbb{Z}/p\mathbb{Z} \) not in \( \mathbb{Z}/p^2\mathbb{Z} \)) such as (2) was attained (see [10]). So it is natural to try to use the approach from the paper to obtain some new upper bound for (1). Unfortunately, the methods from [10] cannot be applied directly in the case. The reason is that we know much less about distribution of Heilbronn’s subgroup (2) then about subgroups in \( \mathbb{Z}/p\mathbb{Z} \) as well as about looking similar convex–type sets (see sections 6, 7 from [10]).

The only we know is Lemma 6 below, which gives, roughly speaking, a nontrivial upper bound for the number of the solutions of the equation \( x - y \equiv c \pmod{p^2} \) for fixed \( c \in \mathbb{Z}/(p^2\mathbb{Z}), c \neq 0 \) and \( x, y \in \Gamma \) as well as upper bounds for the moments of such quantities. Nevertheless the size of \( \Gamma \) is large and the ordinary Fourier transformation methods (see Lemma 5), combining with the approach from [10], namely, so–called the eigenvalues method allows us to prove Theorem 2.

The author is grateful to Igor Shparlinski for useful discussions as well as pointing some applications of the main result.

## 2 Definitions

Let \( G \) be an abelian group. If \( G \) is finite then denote by \( N \) the cardinality of \( G \). It is well–known [7] that the dual group \( \hat{G} \) is isomorphic to \( G \) in the case. Let \( f \) be a function from \( G \) to
We denote the Fourier transform of \( f \) by \( \hat{f} \),
\[
\hat{f}(\xi) = \sum_{x \in G} f(x) e(-\xi \cdot x),
\]
where \( e(x) = e^{2\pi i x} \). We rely on the following basic identities
\[
\sum_{x \in G} |f(x)|^2 = \frac{1}{N} \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2.
\]
\[
\sum_{y \in G} \left| \sum_{x \in G} f(x) g(y-x) \right|^2 = \frac{1}{N} \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2 |\hat{g}(\xi)|^2.
\]
and
\[
f(x) = \frac{1}{N} \sum_{\xi \in \hat{G}} \hat{f}(\xi) e(\xi \cdot x).
\]
If
\[
(f * g)(x) := \sum_{y \in G} f(y) g(x-y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in G} f(y) g(y+x)
\]
then
\[
\hat{f} * \hat{g} = \hat{f} \hat{g} \quad \text{and} \quad \hat{f} \circ \hat{g} = \hat{f} \hat{g} = \hat{f} \hat{g},
\]
where for a function \( f : G \to \mathbb{C} \) we put \( f^c(x) := f(-x) \). Clearly, \((f * g)(x) = (g * f)(x)\) and \((f \circ g)(x) = (g \circ f)(-x), x \in G\).

We use in the paper the same letter to denote a set \( S \subseteq G \) and its characteristic function \( S : G \to \{0, 1\} \). Write \( E(A, B) \) for the additive energy of two sets \( A, B \subseteq G \) (see e.g. [13]), that is
\[
E(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.
\]
If \( A = B \) we simply write \( E(A) \) instead of \( E(A, A) \). Clearly,
\[
E(A, B) = \sum_{x} (A * B)(x)^2 = \sum_{x} (A \circ B)(x)^2 = \sum_{x} (A \circ A)(x)(B \circ B)(x).
\]
and by (5),
\[
E(A, B) = \frac{1}{N} \sum_{\xi} |\hat{A}(\xi)|^2 |\hat{B}(\xi)|^2.
\]
Let
\[
E_k(A) = \sum_{x \in G} (A \circ A)(x)^k,
\]
and
\[
E_k(A, B) = \sum_{x \in G} (A \circ A)(x)(B \circ B)(x)^{k-1}
\]
be the higher energies of \( A \) and \( B \). Similarly, we write \( E_k(f, g) \) for any complex functions \( f \) and \( g \). Quantities \( E_k(A, B) \) can be written in terms of generalized convolutions (see [9]).
Definition 4 Let $k \geq 2$ be a positive number, and $f_0, \ldots, f_{k-1} : G \rightarrow \mathbb{C}$ be functions. Write $F$ for the vector $(f_0, \ldots, f_{k-1})$ and $x$ for vector $(x_1, \ldots, x_{k-1})$. Denote by $C_k(f_0, \ldots, f_{k-1})(x_1, \ldots, x_{k-1})$ the function

$$C_k(F)(x) = C_k(f_0, \ldots, f_{k-1})(x_1, \ldots, x_{k-1}) = \sum_z f_0(z)f_1(z+x_1)\ldots f_{k-1}(z+x_{k-1}).$$

Thus, $C_2(f_1, f_2)(x) = (f_1 \circ f_2)(x)$. If $f_1 = \cdots = f_k = f$ then write $C_k(f)(x_1, \ldots, x_{k-1})$ for $C_k(f_1, \ldots, f_k)(x_1, \ldots, x_{k-1})$.

For a positive integer $n$, we set $[n] = \{1, \ldots, n\}$. All logarithms used in the paper are to base 2. By $\ll$ and $\gg$ we denote the usual Vinogradov’s symbols. If $N$ is a positive integer then write $\mathbb{Z}_N$ for $\mathbb{Z}/N\mathbb{Z}$ and if $p$ is a prime then put $\mathbb{Z}_p^*$ for $(\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$.

3 Preliminaries

Put

$$\Gamma = \{m^p : 1 \leq m \leq p-1\} \subseteq \mathbb{Z}_p^2.$$

It is easy to see that $\Gamma$ is a subgroup and that

$$\Gamma = \{x^p : x \in \mathbb{Z}_p^2, x \neq 0\} = \{x \in \mathbb{Z}_p^2 : x^p \equiv 1 \pmod{p^2}\}$$

because of $x \equiv y \pmod{p}$ implies $x^p \equiv y^p \pmod{p^2}$.

Put

$$f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{p-1}}{p-1} \in \mathbb{Z}_p[x].$$

Put also

$$\mathcal{F}(u) = |\{x \in \mathbb{Z}_p : f(x) = u\}|.$$

We prove a simple lemma which is connecting the numbers $\mathcal{F}(u)$ and the convolutions of the subgroup $\Gamma$.

Lemma 5 Let $0 \leq a, b \leq p - 1$. Then

$$(\Gamma \circ \Gamma)(a + bp) = \begin{cases} 
\mathcal{F}(aq(a) - b), & \text{if } a \neq 0, \\
|\Gamma|, & \text{if } a = b = 0, \\
0, & \text{otherwise}.
\end{cases}$$

Proof. To calculate $(\Gamma \circ \Gamma)(a + bp)$ consider the equation

$$m_1^p - m_2^p \equiv a + bp \pmod{p^2}, \quad (12)$$

where $m_1, m_2 \in \Gamma$. From (12) it follows that $m_1 - m_2 \equiv a \pmod{p}$. If $a \equiv 0 \pmod{p}$ then $b \equiv 0 \pmod{p}$. Suppose that $a \neq 0 \pmod{p}$. Then

$$m_1 \equiv av \pmod{p} \quad \text{and} \quad m_2 \equiv a(v-1) \pmod{p}$$

where $v 

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for some \(v \neq 0 \pmod{p}\). It follows that

\[
m_1^p - m_2^p \equiv a^p(v^p - (v - 1)^p) \equiv a^p \cdot \sum_{l=1}^{p} (-1)^{l-1}v^{p-l}\binom{p}{l} \equiv a^p \cdot (1 - pf(v)) \pmod{p^2}
\]

and hence

\[
a + bp \equiv a^p \cdot (1 - pf(v)) \pmod{p^2}
\]

This completes the proof. \(\square\)

The following lemma was proved in [4].

**Lemma 6** Let \(U \subseteq \mathbb{Z}_p\) be a set. Then

\[
\sum_{u \in U} F(u) \ll p^{2/3}|U|^{2/3}.
\]

Using Lemma 5 and Lemma 6, one can easily deduce upper bounds for moments of convolution of \(\Gamma\). These estimates are the same as in the case of multiplicative subgroups in \(\mathbb{Z}_p\) (see, e.g. [8]).

**Corollary 7** We have

\[
E(\Gamma) \ll |\Gamma|^{5/2}, \quad E_3(\Gamma) \ll |\Gamma|^3 \log |\Gamma|, \tag{13}
\]

and for all \(l \geq 4\) the following holds

\[
E_l(\Gamma) = |\Gamma|^l + O(|\Gamma|^{2l+3}) \tag{14}
\]

We need in a lemma about Fourier coefficients of an arbitrary set which is invariant under the action of a subgroup (see e.g. [8]).

**Lemma 8** Let \(\Gamma \subseteq \mathbb{Z}_N\) be a multiplicative subgroup, and \(Q\) be an \(\Gamma\)-invariant subset of \(\mathbb{Z}_N\), that is \(Q\Gamma = Q\). Then for any \(\xi \neq 0\) the following holds

\[
|\hat{Q}(\xi)| \leq \min\left\{ \left(\frac{|Q||N|^{1/2}}{|\Gamma|} \right)^{1/2}, \frac{|Q|^{3/4}N^{1/4}E^{1/4}(\Gamma)}{|\Gamma|}, N^{1/8}E^{1/8}(\Gamma)E^{1/8}(Q)\left(\frac{|Q|}{|\Gamma|}\right)^{1/2}\right\}. \tag{15}
\]

Finally, we formulate the lemma from [10], see Theorem 57, section 7. This is the key new ingredient of our proof.
Lemma 9 Let $A, D \subseteq G$ be two sets, and $D = -D$. Then
\[
\mu^2 \cdot \mathcal{E}(A, f) \leq \sum_{x, y, z \in A} D(x - y)D(x - z)(A \circ A)(y - z),
\]
where
\[
\mu \geq \frac{1}{|A|} \sum_{x \in D} (A \circ A)(x),
\]
and $f$ is a nonnegative function such that $\|f\|_2 = 1$, supp $f \subseteq A$, and
\[
\mu f(x) = A(x)(D \ast f)(x).
\]

4 The proof of the main result

Let $\Gamma$ be the subgroup from (2).

Theorem 10 We have
\[
\mathcal{E}(\Gamma) \ll |\Gamma|^\frac{44}{17} \log^{\frac{19}{17}} |\Gamma|.
\]

Proof. Let $P = p^2$, $t = |\Gamma|$, $E = E(\Gamma) = t^3/K$, $K \geq 1$, and $E_3 = E_3(\Gamma)$. By Lemma 6 and simple average arguments, we have
\[
E \ll \sum_{x \neq 0 : 2^{-1}tK^{-1} < (\Gamma \circ \Gamma)(x) \leq cK} (\Gamma \circ \Gamma)^2(x),
\]
where $c > 0$ is some absolute constant. Let
\[
D_j = \{x \neq 0 : c2^{-j}K < (\Gamma \circ \Gamma)(x) \leq c2^{-j+1}K\}, \quad j \leq \log(2cK^2t^{-1}).
\]
Put $l = [\log(2cK^2t^{-1})]$. By (17) there is $j \in [l]$ such that
\[
\frac{2^j E}{lK} \ll \sum_{x \in D_j} (\Gamma \circ \Gamma)(x).
\]
Put $D = D_j$. Clearly, $D = -D$ and $D$ is $\Gamma$–invariant set. Note also
\[
|D| \ll \frac{2^j E}{t}.
\]
Put
\[
\sigma := \sum_{x \in D} (\Gamma \circ \Gamma)(x)
\]
Using Lemma [9] with $A = \Gamma$, $D = D$ and also inequality (18), we obtain
\[ \frac{2^j E^2}{t^2 K^2 t^2} \cdot E(\Gamma, f) \ll \frac{\sigma^2}{t^2} \cdot E(\Gamma, f) \ll \mu^2 \cdot E(\Gamma, f) \leq \sum_{x, y, z \in \Gamma} D(x - y)D(x - z)(\Gamma \circ \Gamma)(y - z), \]
where
\[ \mu f(x) = \Gamma(x)(D \ast f)(x). \quad (20) \]
Because of $D$ is $\Gamma$–invariant it is easy to see that any solution $f$ of equation (20) coincide with a character on $\Gamma$. We know that $f(x) \geq 0$, so $f$ is the main character, i.e. $f(x) = \Gamma(x)/t^{1/2}$ (for more details see [9] or [10]). Thus
\[ \frac{2^j E^5}{t^2 t^9} \ll \frac{2^j E^3}{t^6} \cdot E(\Gamma, f) \ll \sum_{x, y, z \in \Gamma} D(x - y)D(x - z)(\Gamma \circ \Gamma)(y - z). \quad (21) \]
If $y = z$ then by (18), (21) and the definition of $\sigma$ the following holds
\[ \frac{2^j E^5}{t^2 t^9} \ll \sum_{\alpha \neq \beta} D(\alpha)D(\beta)(\Gamma \circ \Gamma)(\alpha - \beta)C_3(\Gamma)(\alpha, \beta). \]
Using Cauchy–Schwarz, we get
\[ \frac{2^j E^{10}}{t^{4/3} E^3} \ll \sum_{\alpha \neq \beta} D(\alpha)D(\beta)(\Gamma \circ \Gamma)^2(\alpha - \beta). \quad (22) \]
Put
\[ S_i = \{ x \neq 0 : c' t^{2/3} 2^{-i} < (\Gamma \circ \Gamma)(x) \leq c' t^{2/3} 2^{-i+1} \}, \]
where $c' > 0$ is an absolute constant such that $(\Gamma \circ \Gamma)(x) \leq c' t^{2/3}$ for all $x \neq 0$. Such constant exists by Lemma [9] Trivially
\[ |S_i|t^{2/3} 2^{-i} c' \leq t^2, \quad (23) \]
and
\[ |S_i|t^{4/3} 2^{-2i} c'^2 \leq E. \quad (24) \]
By the definition of the sets $S_i$, we have
\[ \frac{2^j E^{10}}{t^{4/3} E^3} \ll t^{4/3} \sum_i 2^{-2i} \sum_x S_i(x)(D \circ D)(x). \]
Certainly, each set $S_i$ is $\Gamma$–invariant. Thus, using Lemma [9] Fourier transform and Parseval, we obtain
\[ \frac{2^j E^{10}}{t^{4/3} E^3} \ll t^{1/3} \sum_i 2^{-2i}|S_i|^{3/4}|D|^{1/4} E^{1/4} + t^{4/3} \sum_i 2^{-2i} |S_i||D|^2/\sigma. \quad (25) \]
Let us estimate the second term from (25). Using (24) and (19), we get

\[ t^{4/3} \sum_i 2^{-2i}|S_i||D|^2 \leq \frac{2^{2j}E^3}{t^2P}. \]

If

\[ \frac{2^{4j}E^{10}}{t^4t^{18}E^3} \leq \frac{2^{2j}E^3}{t^2P} \]

then \( E \leq t^{17/7} \log^{5/7} t \) and the result follows. Thus

\[ \frac{2^{4j}E^{10}}{t^4t^{18}E^3} \leq t^{1/3} \sum_i 2^{-2i} |S_i|^{3/4} |D| P^{1/4} E^{1/4}. \]  \hspace{1cm} (26)

By Lemma 6 and inequalities (23), (24), we have

\[ |S_i| \leq \min \{ t^{2^{3i}}, E^{t^{-4/3}2^{2i}}, t^{4^{3/2}2i} \}. \]

Applying the first bound for \( 2^{i} \leq t^{-7/3} \), the second one for \( E t^{-7/3} \leq 2^{i} \leq t^{8/3} E^{-1} \) and the third bound for other \( i \), we get by (26)

\[ \frac{2^{4j}E^{10}}{t^4t^{18}E^3} \leq t^{1/3} |D| P^{1/4} E^{1/4} \left( t^{3/4} \left( \frac{E}{t^{7/3}} \right)^{1/4} + \frac{E^{3/4}}{t^{1/2}} \left( \frac{t^{7/3}}{E} \right)^{1/2} + t \left( \frac{E}{t^{8/3}} \right)^{5/4} \right) = \]

\[ = t^{1/3} |D| P^{1/4} E^{1/4} \left( E^{1/4} t^{1/6} + E^{5/4} t^{-7/3} \right) = t^{1/2} |D| P^{1/4} E^{1/2} \left( 1 + \frac{E}{t^{5/2}} \right) \leq t^{1/2} |D| P^{1/4} E^{1/2}, \]

where Corollary 7 has been used. Applying the last bound and inequality (19) after some calculations, we obtain (16). This completes the proof.

\[ \square \]

**Corollary 11** Let \( p \) be a prime, \( a \neq 0 \) (mod \( p \)), and \( M, N \) be positive integers, \( N \leq p \). Then

\[ \left| \sum_{n=M}^{N+M} e \left( \frac{an^p}{p^2} \right) \right| \leq p^{50} N^{5} \log^{5} p. \]  \hspace{1cm} (27)

In particular

\[ |S(a)| \leq p^{50} \log^{5} p. \]  \hspace{1cm} (28)

**Proof.** One can get (28) just using Theorem 10 and Lemma 8. To obtain (27) write \( P = [M, \ldots, N+M] \) and note that by Fourier transform or the completing method (see 3 or 4 for details) combining with Hölder, we get

\[ \left| \sum_{n=M}^{N+M} e \left( \frac{an^p}{p^2} \right) \right| \leq \frac{1}{p} \sum_{x} |\hat{f}(x)||\hat{P}(x)| \leq E^{1/4}(\Gamma) \cdot \left( \frac{1}{p} \sum_{x} |\hat{P}(x)|^{4/3} \right)^{3/4}. \]  \hspace{1cm} (29)
By assumption $N \leq p$. Using Theorem [10] to estimate $E(\Gamma)$ and a well–known estimate (see e.g. [4])

$$\sum_x |\hat{P}(x)|^{4/3} \ll pN^{1/3},$$

we have (27). This completes the proof. \qed

Using the arguments from [1] and Theorem [10] we obtain the following result about Fermat quotients.

**Theorem 12** One has

$$l_p \leq (\log p)^{7829_{4284} + o(1)}$$

as $p \to \infty$.

Note that

$$\frac{463}{252} = 1.83730\ldots \quad \text{and} \quad \frac{7829}{4284} = 1.82749\ldots$$

Theorem [12] has a consequence (see [5]).

**Corollary 13** For every $\varepsilon > 0$ and a sufficiently large integer $n$, if $a^{n-1} \equiv 1 \pmod{n}$ for every positive integer $a \leq (\log p)^{7829_{4284} + \varepsilon}$ then $n$ is squarefree.

Similar improvement of constant $\frac{463}{252}$ of Theorem 1 from [11] as well further applications from [12] can be obtained exactly the same way. New values of constants in paper [2] follows from the proposition below (compare with Proposition 2.1 of article [2]) and our Theorem [10].

**Proposition 14** For $\xi \in \mathbb{Z}_p$ define

$$u(\xi) = |\{x \in [p] : x^p - x \equiv \xi \pmod{p^2}\}|.$$

Then for any $\varepsilon > 0$ and all sufficiently large $p$ the following holds

$$\sum_{\xi} u^2(\xi) \leq p^{\frac{1}{3}+\varepsilon}E^{1/2}(\Gamma).$$

**References**

[1] J. Bourgain, K. Ford, S. V. Konyagin, I. E. Shparlinski, *On the Divisibility of Fermat Quotients*, Michigan Math. J. **59** (2010), 313–328.

[2] M.-C. Chang, *Short character sums with Fermat quotients*, Acta Arith. **152** (2012), 23–38.

[3] D. R. Heath–Brown, *An estimate for Heilbronn’s exponential sum*, Analytic number theory vol. 2, (Allerton Park, IL 1995), Progr. Math., 1 39, Birkhäuser, Boston (1996), 451–463.
[4] D. R. Heath–Brown, S. V. Konyagin, *New bounds for Gauss sums derived from kth powers, and for Heilbronn’s exponential sum*, Quart. J. Math. **51** (2000), 221–235.

[5] H. W. Lenstra, *Miller’s primality test*, Inform. Process. Lett. **8** (1979), 86–88.

[6] A. Ostafe and I. E. Shparlinski, *Pseudorandomness and dynamics of Fermat quotients*, SIAM J. Discr. Math. **25** (2011), 50–71.

[7] W. Rudin, *Fourier analysis on groups*, Wiley 1990 (reprint of the 1962 original).

[8] T. Schoen, I. D. Shkredov, *Additive properties of multiplicative subgroups of $\mathbb{F}_p$*, to appear in Quart. J. Math.

[9] T. Schoen, I. D. Shkredov, *Higher moments of convolutions*, arXiv:1110.2986v1 [math.CO] 13 Oct 2011.

[10] I. D. Shkredov, *Some new inequalities in additive combinatorics*, arXiv:1208.2341v2 [math.CO].

[11] I. E. Shparlinski, *On the value set of Fermat quotients*, Proc. Amer. Math. Soc. **140** (2012), 1199–1206.

[12] I. E. Shparlinski, *On vanishing Fermat quotients and a bound of the Ihara sum*, Kodai Math. J. (to appear).

[13] T. Tao, V. Vu, *Additive combinatorics*, Cambridge University Press 2006.

[14] H. B. Yu, *Note on Heath–Brown estimate for Heilbronn’s exponential sum*, Proc. AMS **127**:7 (1999), 1995–1998.

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