Non-monotonic Keplerian velocity profiles around near-extreme braneworld Kerr black holes

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Abstract
We study the non-monotonic Keplerian velocity profiles related to locally non-rotating frames (LNRF) in the field of near-extreme braneworld Kerr black holes and naked singularities in which the non-local gravitational effects of the bulk are represented by a braneworld tidal charge $b$ and the 4D geometry of the spacetime structure is governed by the Kerr–Newman geometry. We show that positive tidal charge has a tendency to restrict the values of the black hole dimensionless spin $a$ admitting the existence of the non-monotonic Keplerian LNRF-velocity profiles; the non-monotonic profiles exist in the black hole spacetimes with tidal charge smaller than $b = 0.410\,05$ (and spin larger than $a = 0.768\,08$). With decreasing value of the tidal charge (which need not be only positive), both the region of spin allowing the non-monotonicity in the LNRF-velocity profile around braneworld Kerr black hole and the velocity difference in the minimum–maximum parts of the velocity profile increase implying growing astrophysical relevance of this phenomenon.

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1. Introduction

Fast rotating black holes play a crucial role in understanding processes observed in quasars and active galactic nuclei (AGN) or in microquasars. It has been shown that supermassive black holes in AGN evolve into states with dimensionless spin $a \sim 1$ due to accretion from thin discs [1, 2]. This statement is supported by the analysis of profiled x-ray (Fe56) lines observed in some AGN (e.g. in MCG-6-30-15) [3–5] and in some microquasars (e.g. GRS 1915+105) [6]. Evidence for the existence of near-extreme Kerr black holes comes from high-frequency quasi-periodic oscillations (QPOs) of the observed x-ray flux in some microquasars [7, 8]. A fast rotating black hole could also be located in the galaxy centre source Sgr A* [9–11].
It is widely accepted that the phenomena observed in AGN and microquasars are related to accretion discs orbiting Kerr black holes. However, we can also consider the possibility of explaining these phenomena by Kerr superspinars with the external field described by the geometry of Kerr naked singularity spacetime [12]. Then both accretion and related optical effects and the QPO effects enable us to find clear signature of the Kerr superspinar’s presence [13–18].

Properties of accretion discs can be appropriately represented by circular orbits of test particles or fluid elements orbiting black holes (superspinars). The local properties can be efficiently expressed when related to the locally non-rotating frames (LNRFs), since these frames corotate with the spacetime in a way that enables us to cancel the frame-dragging effects as much as possible [19]. A new phenomenon related to the LNRF-velocity profiles of matter orbiting near-extreme Kerr black holes has been found by Aschenbach [9, 20, 21], namely a non-monotonicity in the velocity profile of the Keplerian motion in the field of Kerr black holes with dimensionless spin $a > 0.9953$. Such a hump in the LNRF-velocity profile of the corotating orbits is a typical and relatively strong feature in the case of Keplerian motion in the field of Kerr naked singularities, but in the case of Kerr black holes it is a very small effect appearing for near-extreme black holes only—see figure 1. In the naked singularity case, we call the orbits to be of the first family rather than corotating, since these can be retrograde relative to the LNRF in vicinity of the ring singularity for small values of spin $(a < 5/3)$, while they are corotating for larger values of the spin [15]; the humpy character of the LNRF-velocity profile ceases for naked singularities with $a > 4.0014$—as demonstrated in figure 1. A study of non-Keplerian distribution of specific angular momentum $(l = \text{const})$, related to geometrically thick discs of perfect fluid, has shown that the ‘humpy’ LNRF-velocity profile appears for near-extreme Kerr black holes with $a > 0.9998$ [21]. The humpy LNRF-velocity profile emerges in the ergosphere of near-extreme Kerr black holes, at the vicinity of the marginally stable circular orbit. The maximal velocity difference between the local minimum and maximum of the humpy Keplerian velocity profiles is $\Delta v \approx 0.07 c$ and takes place for $a = 1$ [22].

Using the idea of ‘hump-induced’ oscillations related to humpy LNRF-velocity profiles, the extended orbital resonance model (EXORM) was developed and applied to explain complex
QPO patterns observed in some black hole sources [22–24]. In the EXORM, the resonant phenomena between oscillations with orbital frequencies (Keplerian and epicyclic—radial or vertical) and the so-called humpy frequency, given by the maximal slope of the LNRF-velocity profile in its humpy part, are assumed to appear at the radius where the ‘humpy’ oscillations are expected to be generated by the non-monotonicity of the LNRF-velocity profile [23]. The EXORM is able to explain all five high-frequency QPOs observed in the microquasar GRS 1915+105 by the humpy frequency, the radial (and vertical) epicyclic frequencies and their simple combinations taken at the common ‘humpy’ radius, implying the black hole parameters of the source $M = 14.8 M_\odot$, $a = 0.9998$ [22], in good agreement with estimates given by different methods [6, 20]. This model can give interesting results also in the case of the x-ray binary system XTE J1650-500 [24], and an ULX candidate system NGC 5408 X-1. On the other hand, QPOs observed in Sqr A∗ [9] cannot be explained by the EXORM [25].

The humpy LNRF-velocity profiles for both Keplerian and $l = \text{const}$ specific angular momentum distributions of orbiting matter were studied in the Kerr–de Sitter and Kerr–anti-de Sitter black hole spacetimes and values of the black hole spin $a$ allowing for the existence of the humpy profiles were found in dependence on the value of the cosmological constant [26, 27].

The last decade gave rise to a plenty of models modifying the 4D Einstein general relativity due to hidden dimensions; therefore, it is interesting to investigate the Aschenbach effect in rotating black hole (naked singularity) spacetimes allowed in alternative gravitational theories. The string theories, describing gravity as the higher-dimensional interaction, appearing to be effectively 4D at low energies, inspired braneworld models assuming the observable universe to be a 3-brane, i.e. the ‘domain wall’, to which the non-gravitational matter fields are confined, while gravity enters the extra spatial dimensions that could be much larger than $l_p \sim 10^{-33}$ cm. The model of Randall and Sundrum (RS model) [28] allows gravity localized near the brane with an infinite size extra dimension while the warped spacetime satisfies the 5D Einstein equations with a negative cosmological constant. An arbitrary energy–momentum tensor could then be allowed on the brane and effective 4D Einstein equations have to be satisfied on the brane. The RS model implies standard 4D Einstein equations in the low energy limit, but significant deviations occur at high energies, near black holes or compact stars. The combination of high-energy (local) and bulk stress (non-local) effects alters the matching problem on the brane in comparison with the standard 4D gravity [29]. The bulk gravity stresses imply that the matching conditions do not have a unique solution on the brane and the 5D Weyl tensor is needed as a minimum condition for uniqueness.

No exact solution of the 5D braneworld Einstein equations is known at present, but a numerical solution has been found quite recently [30]. On the other hand, the 4D stationary and axisymmetric vacuum solution describing a braneworld rotating black hole has been found by solving the braneworld constrained equations under an assumption of the specialized form of the metric (namely of the Kerr–Schild form) [31]. Of course, it is not an exact solution satisfying the full system of 5D equations, but in the framework of the constrained equations it represents a consistent rotating black hole solution reflecting the influence of the extra dimension through a single braneworld parameter. The braneworld rotating black holes are described by the metric tensor of the Kerr–Newman form with the braneworld tidal charge $b$ determining the 5D non-local gravitational coupling between the brane and the bulk [31]. For non-rotating braneworld black holes, the metric is reduced to the Reissner–Nordström form containing the tidal charge [32]. This spacetime can also represent the external field of braneworld neutron stars described by the uniform density internal spacetime [29]. The influence of the braneworld tidal charge on physical processes has been extensively investigated for both the black holes [33–38] and neutron stars [39–43], or in the weak field
limit [44, 45]. In the case of microscopic black holes, experimental evidence is assumed in the LHC [46].

The braneworld tidal charge can be, in principle, both positive and negative, but the negative values are probably more relevant [32]. Note that for \( b > 0 \), the braneworld spacetime can be identified with the Kerr–Newman spacetime by \( b \to Q^2 \), where \( Q^2 \) is the squared electric charge; however, it is not the Kerr–Newman background since the electromagnetic part of this background is missing. Some astrophysically relevant restrictions on the value of the tidal charge \( b \) were obtained both in the weak-field limit [44] and in the strong-field limit. Here, we will study the existence of the humpy LNRF-velocity profiles in the field of braneworld rotating black holes considering both negative and positive values of the braneworld tidal charge. Our results related to \( b > 0 \) are relevant also in the case of the standard Kerr–Newman spacetimes (with \( b \to Q^2 \)) for uncharged particles. We restrict our attention to the Keplerian LNRF-velocity profiles postponing the study of perfect fluid configurations to future work.

2. Effective gravitational equations in braneworld models

In the 5D warped space models of Randall and Sundrum, the gravitational field equations in the bulk can be expressed in the form [32, 47]

\[
\tilde{G}_{AB} = \tilde{k}^2 \left[ -\tilde{\Lambda} \delta_{AB} + \delta(\chi)(-\lambda g_{AB} + T_{AB}) \right],
\]

where the fundamental 5D Planck mass \( \tilde{M}_P \) enters via \( \tilde{k}^2 = \frac{8\pi}{\tilde{M}_P^3} \), \( \lambda \) is the brane tension, and \( \tilde{\Lambda} \) is the negative bulk cosmological constant; \( g_{AB} = \tilde{g}_{AB} - n_A n_B \) is the induced metric on the brane, with \( n_A \) being the unit vector normal to the brane.

The effective gravitational field equations induced on the brane are determined by the bulk field equations (1), the Gauss–Codazzi equations and the generalized matching Israel conditions. They can be expressed in the form of modified Einstein’s equations containing additional terms reflecting bulk effects onto the brane [32, 47]

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + \tilde{k}^2 T_{\mu\nu} + \tilde{k}^2 S_{\mu\nu} - E_{\mu\nu},
\]

where \( \tilde{k}^2 = \frac{8\pi}{\tilde{M}_P^2} \), with \( \tilde{M}_P \) being the braneworld Planck mass. The relations of the energy scales and cosmological constants are given in the form

\[
M_P = \sqrt{\frac{3}{4\pi}} \left( \frac{\tilde{M}_P^2}{\tilde{\Lambda}} \right)^{\frac{1}{3}} \tilde{M}_P; \quad \Lambda = \frac{4\pi}{\tilde{M}_P^2} \left[ \tilde{\Lambda} + \frac{4\pi}{3\tilde{M}_P^2} \lambda^2 \right].
\]

Local bulk effects on the matter are determined by the ‘squared energy–momentum’ tensor \( S_{\mu\nu} \), while the non-local bulk effects are given by the tensor \( E_{\mu\nu} \).

Assuming zero cosmological constant on the brane (\( \Lambda = 0 \)), we arrive to the condition

\[
\tilde{\Lambda} = -\frac{4\pi \lambda^2}{3\tilde{M}_P^2}.
\]

In the vacuum case, \( T_{\mu\nu} = 0 = S_{\mu\nu} \), the effective gravitational field equations on the brane reduce to the form [47]

\[
R_{\mu\nu} = -E_{\mu\nu}, \quad R_{\mu}^{\mu} = 0 = E_{\mu}^{\mu}
\]

implying divergence constraint [47]

\[
\nabla_{\mu} E_{\mu\nu} = 0,
\]

where \( \nabla_{\mu} \) denotes the covariant derivative on the brane.
Equation (6) represents Bianchi identities on the brane, i.e. an integrability condition for the field equations $R_{\mu \nu} = -\mathcal{E}_{\mu \nu}$ [31]. For stationary and axisymmetric (or static, spherically symmetric) solutions, (5) and (6) form a closed system of equations on the brane.

The 4D general relativity energy–momentum tensor $T_{\mu \nu}$ (with $T_{\mu \mu} = 0$) can be formally identified to the bulk Weyl term on the brane due to the correspondence

$$k^2 T_{\mu \nu} \leftrightarrow -\mathcal{E}_{\mu \nu}.$$  

The general relativity conservation law $\nabla^\mu T_{\mu \nu} = 0$ then corresponds to the constraint equation on the brane (6). This behaviour indicates that the Einstein–Maxwell solutions in standard general relativity should correspond to constrained braneworld vacuum solutions. This was indeed shown in the case of braneworld (Reissner–Nordström and Kerr–Newman) black hole solutions [31]. In both of these solutions, the influence of the non-local gravitational effects of the bulk on the brane is represented by a single ‘braneworld’ parameter $b$. The $1/r^2$ behaviour of the second term in the Newtonian potential

$$\Phi = -\frac{M}{M^2 r^2} + \frac{b}{2r^2}$$  

inspired the name ‘tidal charge’ for the parameter $b$ [32].

### 3. Orbital motion in the braneworld Kerr spacetimes

The motion of test particles in the field of braneworld rotating black holes is given by the geodesic structure of the Kerr–Newman spacetimes with the tidal charge $b$. The braneworld parameter reflects the tidal effects of the bulk space and has no influence on the motion of charged particles. The geodesic structure given by the Carter equations [48] is relevant for both uncharged and charged test particles. The circular test particle orbits of the braneworld Kerr black holes are identical to the circular geodesics of the Kerr–Newman spacetime with properly chosen charge parameter.

We will study the Aschenbach effect, i.e. we look for the non-monotonicity (humps) in the LNRF-velocity profiles of Keplerian discs orbiting near-extreme braneworld Kerr black holes or naked singularities.

#### 3.1. Geometry

Using standard Boyer–Lindquist coordinates $(t, r, \theta, \phi)$ and geometric units ($c = G = 1$), we can write the line element of rotating (Kerr) black hole on the 3D brane in the form

$$ds^2 = -\left(1 - \frac{2Mr - b}{\Sigma}\right)dt^2 - \frac{2a(2Mr - b)}{\Sigma}\sin^2\theta \, dt \, d\phi$$

$$+ \frac{\Sigma}{\Delta} \, dr^2 + \Sigma \, d\theta^2 + \left(\frac{2Mr - b}{\Sigma} - a^2 \sin^2\theta\right) \sin^2\theta \, d\phi^2,$$

where

$$\Delta = r^2 - 2Mr + a^2 + b,$$

$$\Sigma = r^2 + a^2 \cos^2\theta,$$

$M$ and $a = J/M$ are the mass parameter and the specific angular momentum of the background, while the braneworld parameter $b$, called the ‘tidal charge’, represents the imprint of non-local (tidal) gravitational effects of the bulk space [31]. The physical ‘ring’ singularity of the
braneworld rotating black holes (and naked singularities) is located at $r = 0$ and $\theta = \pi/2$, as in the Kerr spacetimes.

The form of metric (9) is the same as that of the standard Kerr–Newman solution of the 4D Einstein–Maxwell equations with tidal charge $b$ being replaced by the squared electric charge $Q^2$ [49]. The stress tensor on the brane $\mathcal{E}_{\mu\nu}$ takes the form

$$
\mathcal{E}_t^t = -\mathcal{E}_\varphi^\varphi = -\frac{b}{\Sigma} \left[ \Sigma - 2(r^2 + a^2) \right],
$$

$$
\mathcal{E}_r^r = -\mathcal{E}_\theta^\theta = -\frac{b}{\Sigma},
$$

$$
\mathcal{E}_\varphi^t = -\frac{2ab}{\Sigma} (r^2 + a^2) \sin^2 \theta,
$$

that is, fully analogical ($b \rightarrow Q^2$) to components of the electromagnetic energy–momentum tensor of the Kerr–Newmann solution in Einstein’s general relativity [31]. For negative values of the tidal charge ($b < 0$), the values of the black hole spin $a > M$ are allowed. Such a situation is forbidden for the standard 4D Kerr black holes. In the following, we put $M = 1$ in order to work with completely dimensionless formulae.

### 3.2. LNRFs and orbital motion

The orbital velocity of matter orbiting a braneworld Kerr black hole along circular orbits is given by appropriate projections of its 4-velocity $U = (U^t, 0, 0, U^\varphi)$ onto the tetrad of a LNRF [19]:

$$
e^{(t)} = (\omega^2 g_{\varphi\varphi} - g_{tt}) \frac{1}{2} dr,
$$

$$
e^{(\varphi)} = (g_{\varphi\varphi}) \frac{1}{2} (d\varphi - \omega dt),
$$

$$
e^{(r)} = \left( \frac{\Sigma}{\Lambda} \right) \frac{1}{2} dr,
$$

$$
e^{(\theta)} = \frac{1}{\Sigma} \frac{1}{2} d\theta,
$$

where $\omega$ is the angular velocity of the LNRF relative to distant observers and reads

$$
\omega = -\frac{g_{tt}}{g_{\varphi\varphi}} = \frac{a(2r - b)}{\Sigma(r^2 + a^2) + (2r - b)a^2 \sin^2 \theta}.
$$

For the circular motion, the only non-zero component of the 3-velocity measured locally in the LNRF is the azimuthal component that is given by

$$
\mathcal{V}^{(\varphi)} = \frac{[\Omega - \omega]}{\sqrt{(\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}})}} = \frac{[(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] \sin \theta (\Omega - \omega)}{\Sigma \sqrt{\Lambda}},
$$

where

$$
\Omega = \frac{U^{\varphi}}{U^t} = -\frac{l g_{tt} + g_{\varphi\varphi}}{l g_{tt} + g_{\varphi\varphi}} \quad \text{and} \quad l = -\frac{U^t}{U},
$$

is the angular velocity of the orbiting matter relative to distant observers and
is its specific angular momentum; \( U_r, U_\phi \) are the covariant components of the 4-velocity field of the orbiting matter.

Using (9) we arrive to the formula

\[
\Omega = \frac{(1 - \frac{2r-b}{\Sigma}) l + \frac{a(2r-b)}{\Sigma} \sin^2 \theta}{(r^2 + a^2 + \frac{2r-b}{\Sigma} a^2 \sin^2 \theta) \sin \theta}.
\]

(22)

4. Velocity profiles for the Keplerian distribution of the specific angular momentum

4.1. Circular geodesics

The motion of test particles following circular geodetical orbits in the equatorial plane \( (\theta = \pi/2) \) is described by the Keplerian distribution of the specific angular momentum, which in the braneworld Kerr backgrounds take the form

\[
l_{K\pm}(r; a, b) = \pm \frac{(r^2 + a^2) \sqrt{r - b} \mp a(2r - b)}{r^2 - 2r + b \pm a \sqrt{r - b}};
\]

(23)

the signs \( \pm \) refer to two distinct families of orbits in the Kerr braneworld spacetimes. From the LNRF point of view, the second family, or minus-family (given by the lower sign), represents retrograde orbits, while the first family, or plus-family (given by the upper sign), represents direct orbits in the black hole spacetimes, but can represent both direct and retrograde orbits in the naked singularity spacetimes. We can find a similar situation for Kerr spacetime in [15, 16]. For both families, we find a formal limit of the Keplerian motion to be located at \( r = b \). In the black hole spacetimes, it is located under the inner horizon and is irrelevant from the astrophysical point of view. In the naked singularity spacetimes, it is relevant for \( b > 0 \), while for \( b < 0 \), it is located under the ring singularity at \( r = 0 \) which represents the limit on the location of circular orbits.

The corresponding Keplerian angular velocity related to distant observers is given by the relation

\[
\Omega_{K\pm}(r; a, b) = \pm \frac{1}{r^2 / \sqrt{r - b} \pm a},
\]

(24)

and the Keplerian orbital velocity related to the LNRF in braneworld Kerr backgrounds is thus given by the relation

\[
V_{K\pm}^{(\phi)}(r; a, b) = \pm \frac{\sqrt{r - b}(r^2 + a^2) \mp a(2r - b)}{(r^2 \pm a \sqrt{r - b}) \sqrt{\Delta}}.
\]

(25)

Clearly, there is a limit on the existence of circular equatorial geodesics at \( r = b \) that is relevant (and located above the ring singularity) for spacetimes with a positive tidal charge.

Like in the Kerr backgrounds \( (b = 0) \), also in the braneworld Kerr backgrounds only the plus-family orbits exhibit the ‘minimum–maximum’ humpy structure of the LNRF-related orbital velocity profiles \( V_{K\pm}^{(\phi)}(r) \). On the other hand, in braneworld Kerr naked-singularity spacetimes with a positive tidal charge \( (b > 0) \), there is always at least one local extreme in the orbit velocity profiles (maximum in \( V_{K+}^{(\phi)} \) and minimum in \( V_{K-}^{(\phi)} \) profiles) where the orbital velocity gradient changes its sign. This non-monotonicity, however, does not correspond to the Aschenbach effect.

We illustrate the typical character of \( V_{K\pm}^{(\phi)}(r; a, b) \) velocity profiles in the braneworld rotating black hole and naked singularity spacetimes with a negative tidal charge in figure 2—in this case, the profiles are extended down to \( r = 0 \), but here we do not consider the region of \( r < 0 \). For special values of parameters of naked singularity spacetimes, the Aschenbach
effect can be related to the retrograde region of first family orbits (the humpy structure contains orbits with $V_{K+}^{(\phi)} < 0$).

For the positive tidal charge, the characteristic profiles of $V_{K+}^{(\phi)}$ are presented in figure 3—in this case, the profiles finish their validity at $r = b$. The velocity profile under the inner horizon of the black hole is also depicted. Numerical calculations indicate that for those kinds of velocity profiles there is no Aschenbach effect, so for the black hole spacetimes we will focus our attention on the behaviour of the Keplerian profiles $V_{K+}^{(\phi)}(r; a, b)$ above the outer horizon.

For both positive and negative tidal charges, the profiles of $V_{K+}^{(\phi)}$ in black hole and naked singularity spacetimes are presented in figure 4. These profiles do not exhibit any ‘minimum–maximum’ structures. Therefore, in the following, we will focus our attention on the plus-family orbits only and use the notation $V_{K+}^{(\phi)}(r; a, b)$ instead of $V_{K+}^{(\phi)}(r; a, b)$.
We have to consider the function \( V_K^{(\phi)}(r; a, b) \) (and also the functions \( l_K(r; a, b) \), \( \Omega_K(r; a, b) \)) within the range of definition of the Keplerian motion [33]. The range is governed by the radii of the photon circular geodesics \( r_{ph} \) given implicitly by the relation

\[
a = a_{ph}(r; b) \equiv \frac{r(3 - r) - 2b}{2\sqrt{r - b}},
\]

and by the radii of the marginally stable circular geodesics \( r_{ms} \), implicitly given by

\[
a = a_{ms}(r; b) \equiv \frac{4(r - b)^{3/2} + r\sqrt{3r^2 - 2r(1 + 2b) + 3b}}{3r - 4b}.
\]

The functions \( a_{ph}(r; b) \) and \( a_{ms}(r; b) \) are illustrated in figure 5—for a detailed discussion of the properties of photon and marginally stable orbits, see [33, 39]. Above the black hole outer horizon, the stable orbits are located in the interval of \( r_{ms} < r < \infty \), while unstable orbits are located in the interval \( r_{ph} < r < r_{ms} \). Note that for Kerr naked singularity, the situation is generally more complicated; see, for example, [33]. For \( r \to r_{ph} \), there is \( V_K^{(\phi)}(r; a, b) \to 1 \). The stable circular orbits are relevant for Keplerian accretion discs; therefore, it is reasonable to put the limits on the physical relevance of the Aschenbach effect to the region of stable circular geodesics.

It is well known that for standard 4D Kerr black holes, the Aschenbach effect, i.e. the non-monotonic LNRF-related orbital velocity profile, appears for the strongly limited class of near-extreme black holes [9, 21]. It appears in the regions where the LNRF-related velocity of Keplerian motion reaches relatively large magnitude \( V_K^{(\phi)} \sim 0.5–0.6 \), but the velocity difference of the minimum–maximum hump is much smaller (\( \Delta V_K^{(\phi)} \sim 0.01 \)). These are the reasons why the effect was overlooked for a relatively long time. On the other hand, the Aschenbach effect is much stronger for Kerr naked singularities and is manifested for a large range of spin \( 1 < a < 4.0005 \). Moreover, for Kerr naked singularities with spin close to the extreme black hole state (\( a = 1 \)), the Aschenbach effect is connected to another interesting effect related to circular geodesics—namely the retrograde character of the first family circular geodesics related to the LNRF. The counter-rotating orbits of the first family can constitute a part of the non-monotonic Keplerian profile (see figures 1 and 2). The ‘retrograde’ Kerr naked
singularity manifesting (strongly) the Aschenbach effect can be determined by the relation (implied by the condition $V_{\text{K}}^{(\phi)} = 0$)

$$a = a_{R\pm}(r) = \sqrt{r(1 - \sqrt{1 - r})}.$$  \hfill (28)

It is illustrated in figure 5 (the case $b = 0$)—we see that the retrograde first-family orbits exist at radii $0 < r < 1$ for spin parameters $1 < a_{R+} < 3\sqrt{3}/4$; the Keplerian LNRF-related velocity profile touches $V_{\text{K}}^{(\phi)} = 0$ at $r = 1$ for $a = 1$.

For braneworld Kerr naked singularities, the retrograde motion of plus-family circular geodesics appears for spin determined by the condition $a_{R-} < a < a_{R+}$, where

$$a_{R\pm}(r; b) = \frac{(2r - b) \pm \sqrt{D}}{2\sqrt{r} - b},$$  \hfill (29)

where

$$D = b^2 - 4r(1 - r)b + 4r^2(1 - r).$$  \hfill (30)

Functions $a_{R\pm}(r; b)$ are illustrated in figure 5. The conditions $D \geq 0$ and $r > b$ put a limit on the radii where for the given tidal charge $b$ the retrograde plus-family orbits can exist, as illustrated in figure 6.
4.2. The Aschenbach effect in braneworld spacetimes

The character of the LNRF-related velocity profile of the Keplerian (equatorial) circular motion is determined by the behaviour of the velocity gradient that can be expressed in the form

\[ \frac{\partial V^{(\psi)}_K}{\partial r} = \frac{A_1 A_4}{A_2 A_3^2}, \]  
(31)

where

\[ A_1 = \sqrt{r - b} \left( \frac{r^2 + a^2}{2(r - b)} + 2r \right) - 2a, \]
(32)

\[ A_2 = (a\sqrt{r - b + r^2})\sqrt{\Delta}, \]
(33)

\[ A_3 = \sqrt{r - b}(r^2 + a^2) - a(2r - b), \]
(34)

\[ A_4 = \sqrt{\Delta} \left[ \left( \frac{a}{2\sqrt{r - b}} + 2r \right) + \frac{r - 1}{\Delta} (a\sqrt{r - b + r^2}) \right]. \]
(35)

Considering the braneworld Kerr black holes, we restrict our attention to the region above the event horizon at \( r > r_+ = 1 + \sqrt{1 - a^2 - b} \). The rotation (spin) parameter of black hole spacetimes is limited by

\[ a_{\text{ex}} = \sqrt{1 - b} \]
(36)

for a given tidal charge \( b \). When braneworld Kerr naked singularities are considered, the region \( r > b \) has to be studied for the existence of the humpy LNRF-velocity profiles when \( b > 0 \), while for \( b < 0 \) we have to analyse the whole region of \( r > 0 \) above the ring singularity.

The local extrema of \( V^{(\psi)}_K(r) \) profiles (giving their ‘minimum–maximum’ humpy parts) are determined by the condition \( \partial V^{(\psi)}_K / \partial r = 0 \), i.e. by zero points of the function (31) that are identical to the roots of the polynomial

\[ g(Z) : \sqrt{r}(h_1 Z^4 + h_2 Z^3 + h_3 Z^2 + h_4 Z + h_5) = 0, \]
(37)

where

\[ h_1 = r[3r^2 + 2(1 - 2b)r - 3b], \]
(38)
We demonstrate this dependence in figure 8 presenting the function $\Delta \alpha(b)$, which denotes the spin interval of the black holes (with tidal charge $b$) allowing for the Aschenbach effect. Since the maximal black hole spin $a_{\text{ex}}$ also depends on $b$, we give the dependence of the ratio $\Delta \alpha / a_{\text{ex}}$ on $b$ for completeness. Note that for tidal charge $b = -1$, the spin interval $\Delta \alpha$ (and even its relative magnitude given by $\Delta \alpha / a_{\text{ex}}$) increases by almost one order as compared to the case of $b = 0$. In the naked singularity spacetimes, the Aschenbach effect appears in the wide region of $a, b$ parameters—both for positive and negative tidal charges. The allowed naked singularity region is limited by $b = 2.14$ and $a = 5.99$—the point $Q$ in figure 8.

The behaviour of humpy LNRF-velocity profiles of Keplerian orbits in the field of braneworld Kerr black holes is represented by a series of figures for both positive and negative tidal charges. In order to clearly illustrate all the aspects of the Aschenbach effect in dependence on the black hole parameters, we give figures with both fixed value of the tidal...
Figure 8. Spin range of braneworld Kerr black holes allowing for the Aschenbach effect given as a function of the tidal charge $b$. It is illustrated by the function $\Delta a(b)$ and the relative spin range determined by the function $\Delta a(b)/a_{\text{ex}}$. The darkest area represents the most interesting case (from astrophysical point of view) when the radius of marginally stable orbit $r_{\text{ms}}$ is less than the local minimum of the Keplerian velocity profile.

Figure 9. Non-monotonic LNRF-related velocity profiles for braneworld Kerr black hole backgrounds given for some values of the tidal charge $b$ and appropriately chosen values of the $a$. The black points denote the loci of $r_{\text{ms}}$. (The boundary of stable orbits given by the marginally stable orbit is represented by a point on all the profiles.) Moreover, we include the dependences of the LNRF velocities in the minimum and maximum of the humpy charge $b$ and fixed value of the rotation parameter $a$, see figure 9. (The boundary of stable orbits given by the marginally stable orbit is represented by a point on all the profiles.) Moreover, we include the dependences of the LNRF velocities in the minimum and maximum of the humpy
Figure 10. (a)–(d) Function $V_{K,ex}^{(v)}$, which determines values of the local minimum and maximum of the function $V_{K}^{(v)}$. The upper line represents the local maximum, while the lower line represents the local minimum. These curves demonstrate that maximum of the velocity difference is reached for the extreme black hole states. (d) An indication for the definition of extremal velocity difference (occurring for the extreme black hole states) as a function of the tidal charge $b$, function $\Delta V_{K,ex}^{(v)}$ giving maximal difference between value of the local maximum and local minimum of the function $V_{K,ex}^{(v)}$, considered with fixed $b$, as in figure 11. (e)–(h) Function $V_{K}^{(v)}(r=r_{\text{ms}},a,b) - V_{K}^{(v)}(r=r_{\text{min}},a,b)$ which defines the difference between $V_{K}^{(v)}$ with radius for the marginally stable orbit and with radius for the local minimum. The dark grey region represents more astrophysically interesting cases, when the radius of the marginally stable orbit $r_{\text{ms}}$ is lower than the radius for the local minimum $r_{\text{min}}$, see figure 8.
Figure 11. Function $\Delta V_{K,ex}^{(\phi)}$ giving maximal difference between values of the local maximum and local minimum of the function $V_{K,ex}^{(\phi)}$ for a fixed $b$.

Figure 12. (a) Sequence of non-monotonic LNRF-related velocity profiles for the near-extreme black hole background. In the sequence, there is $a = 0.996 \ a_{ex}$ and we change the parameter $b$. (b) Sequence of non-monotonic LNRF-related velocity profiles for the naked singularity background close to the extreme black hole state given by $a_{ex} = \sqrt{1 - b}$. In the sequence, there is $a = 1.05 \ a_{ex}$, and we can see the transform between the retrograde and purely co-rotating velocity profiles. For comparison, the velocity profile constructed for the extreme Kerr black hole is included.

profile on the black hole parameters (see figure 10). These extremal LNRF-related velocities decrease with the tidal charge descending, but their difference increases significantly—for fixed $b$, the depth of the humpy profile grows with black holes spin approaching $a_{ex} = \sqrt{T - b}$. Finally, in figure 11, there is the dependence of the velocity difference at the minimum–maximum part of the humpy profile on the tidal charge $b$ for extreme braneworld Kerr black holes. We can see that for $b = -1$ the extremal difference increases by a factor $\sim 2$ as compared to the case of $b = 0$.

We also demonstrate there, how the Aschenbach effect disappears about the point $Q$ for positive tidal charges, and how the LNRF-related Keplerian profiles in the naked singularity spacetime can be reduced into those for the corresponding extreme black hole spacetime,
when the retrograde naked singularity profile is transformed into a discontinuity occurring at the radius $r = 1$, and the negatively valued part under the horizon becomes physically irrelevant. The behaviour of Keplerian velocity profiles in both black hole and naked singularity backgrounds corresponding to near-extreme black hole cases is shown in figure 12.

5. Conclusions

We have shown that the Aschenbach effect is a typical feature of the circular geodetical motion in the field of both standard and braneworld Kerr naked singularities with a relatively large interval of spins above the extreme black hole limit. For naked-singularity spin sufficiently close to the extreme black hole state, the Aschenbach effect is manifested by the retrograde plus-family circular orbits. For black hole spacetimes, such retrograde orbits can appear under the inner horizon, thus being irrelevant from the astrophysical point of view. In the field of near-extreme rotating black holes, the Aschenbach effect located above the outer black hole horizon can thus be considered as a small remnant of the typical naked singularity phenomenon.

We have demonstrated that in the braneworld near-extreme Kerr black hole spacetimes, the non-monotonic LNRF-related orbital velocity profiles of the Keplerian motion are suppressed for positive tidal charges as compared with the standard Kerr black holes, and disappear for the tidal charge $b > 0.410\,05$, while they are strongly enlarged for decreasing negative tidal charge. The spin range of black holes allowing for the Aschenbach effect increases significantly with decreasing tidal charge, strengthening possible relevance of the Aschenbach effect in astrophysical phenomena (see, e.g., [50]). This possibility is further supported by increasing the magnitude of the velocity difference in the minimum–maximum part of the LNRF-related Keplerian profile with decreasing negative tidal charge. Recent observations of high-frequency quasiperiodic oscillations observed in the x-ray spectra of some neutron-star binaries [51–54] that could be used to test braneworld models in the strong-field limits imply negatively valued dimensionless tidal charges as high as $b = −2$ when the Aschenbach effect could be quite significant and well observed [50].

We conclude that the Aschenbach effect can play a significant role in explaining a variety of physical phenomena (optical effects and related line profiles, explanation of a variety of high-frequency quasi-periodic oscillations observed in some microquasars and active galactic nuclei, etc) expected to appear in the strong gravitational field of braneworld Kerr black holes, especially in the case of negative tidal charge.

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