THE CLIFFORD TWIST

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Abstract. This is an elementary exposition of the twisted group algebra rep-
resentation of simple Clifford algebras.

1. Clifford Algebra

Clifford Algebra is an algebra defined on a potentially infinite set $e_1, e_2, e_3, \ldots$ of linearly independent unit vectors, their finite products (called multi-vectors) and the unit scalar 1 (denoted $e_0$). Every element of the algebra is a linear combination of these basis elements over some ring, usually the real numbers.

The vectors $e_1, e_2, e_3, \ldots$ are referred to as ‘1-blades.’ A product of two vectors is called a ‘2-blade.’ three vectors a ‘3-blade’ and so forth. The scalar $e_0$ is a ‘0-blade.’ An $n$-blade multi-vector is said to be of grade $n$.

There are four fundamental multiplication properties of 1-blades.

1. The square of 1-blades is $\mu$ (where $\mu^2 = 1$).
2. The product of 1-blades is anti-commutative.
3. The product of 1-blades is associative.
4. Every $n$-blade can be factored into the product of $n$ distinct 1-blades.

The product of $e_i$ and $e_j$ is denoted $e_{ij}$ if $i < j$ and by $-e_{ij}$ if $i > j$. Likewise for higher order blades. For example, if $i < j < k$ then $e_ie_je_k = e_{ijk}$.

Any two $n$-blades may be multiplied by first factoring them into 1-blades. For example, the product of $e_{134}$ and $e_{23}$, is computed as follows:

$$e_{134}e_{23} = e_1e_3e_4e_2e_3$$
$$= -e_1e_4e_3e_2e_3$$
$$= e_1e_4e_2e_3e_3$$
$$= \mu e_1e_4e_2$$
$$= -\mu e_1e_2e_4$$
$$= -\mu e_{124}$$

2. Representing Clifford Algebra as a twisted group algebra

Each of the basis elements of Clifford algebra $1, e_1, e_2, e_{12}, e_3, \ldots$ can be associated with an element of the set $G$ of non-negative integers.

Each vector $e_k$ is associated with the integer $2^{k-1}$ and the scalar $e_0$ is associated with 0. A multi-vector is associated with the sum of the integers associated with its vector factors. Thus, for example, the multi-vector $e_{134}$ is associated with the sum $2^0 + 2^2 + 2^3 = 13$. Notice that the binary representation of 13 is 1101 with

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bits 1, 3 and 4 set. We will represent the sequence $1, e_1, e_2, e_1 e_2, e_1 e_3, \cdots$ by the sequence $i_0, i_1, i_2, i_3, i_4, \cdots$ where the subscript of $i$ is the number associated with the corresponding vector or multi-vector.

Notice that, since the square of a vector $e_k$ is $\mu$ which is either 1 or $-1$, the product of two basis elements $i_p$ and $i_q$ will always be either $i_r$ or $-i_r$ where $r$ is the XOR (exclusive or) of the binary representations of integers $p$ and $q$. The set $G$ of non-negative integers is a group under XOR. For brevity, we will denote the operation $p$ XOR $q$ by simple concatenation $pq$. Thus there is a function $\phi$ mapping $G \times G$ into $\{-1, 1\}$ such that if $p, q \in G$ then

$$i_p i_q = \phi(p, q)i_{pq}$$

thereby representing Clifford algebra as a twisted group algebra.

Let $2p$ denote the double of $p$. Notice that the vector factors of $i_{2p}$ are the successors of the vector factors of $i_p$ in the sense that $e_k$ is a vector factor of $i_{2p}$ if and only if $e_k-1$ is a vector factor of $i_p$. For example, $i_{13} = e_{134}$ and $i_{26} = e_{245}$. This is more intuitive if the subscripts are represented in binary. $13 = 1101_B$ with bits 1, 3 and 4 set, and $2(13) = 26 = 11010_B$ with bits 2, 4 and 5 set. Multiplying by 2 in binary shifts bits to the left and appends a 0 on the right.

The next two lemmas are then immediately obvious.

**Lemma 2.1.**

$e_1 i_{2p} = i_{2p+1}$

**Lemma 2.2.**

$e_1 i_{2p+1} = \mu i_{2p}$

Let $\beta(p)$ denote the sum of the bits of $p$. Then $\beta(p)$ is the grade of $i_p$. The remaining lemmas follow from the fact that $i_{2p}$ contains exactly $\beta(p)$ vector factors and $e_1$ must be ‘commuted’ with each of them to ‘find its place’ so to speak.

**Lemma 2.3.**

$i_{2p} e_1 = (-1)^{\beta(p)} i_{2p+1}$

**Lemma 2.4.**

$i_{2p+1} e_1 = (-1)^{\beta(p)} \mu i_{2p}$

**Theorem 2.5.** There is a twist $\phi(p, q)$ mapping $G \times G$ into $\{-1, 1\}$ such that if $p, q \in G$, then $i_p i_q = \phi(p, q)i_{pq}$.

**Proof.** Let $G_n = \{p \mid 0 \leq p < 2^n\}$ with group operation “bit-wise exclusive or.”

To begin with, $i_0 i_0 = \phi(0, 0)i_0 = 1$ provided $\phi(0, 0) = 1$.

This defines the twist for $G_0$.

If $p$ and $q$ are in $G_{n+1}$, then there are elements $u$ and $v$ in $G_n$ such that one of the following is true:

1. $p = 2u$ and $q = 2v$
2. $p = 2u$ and $q = 2v + 1$
3. $p = 2u + 1$ and $q = 2v$
4. $p = 2u + 1$ and $q = 2v + 1$

Assume $\phi$ is defined for $u, v \in G_n$, then consider these four cases in order.
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(1) \( p = 2u \) and \( q = 2v \)

\[ i_p i_q = i_{2u} i_{2v} = \phi(u, v) i_{2uv} = \phi(2u, 2v) i_{(2u)(2v)} = \phi(p, q) i_{pq} \]

provided \( \phi(2u, 2v) = \phi(u, v) \).

(2) \( p = 2u \) and \( q = 2v + 1 \)

\[ i_p i_q = i_{2u} i_{2v+1} = i_{2u} e_1 i_{2v} = (-1)^{\beta(u)} e_1 i_{2u} i_{2v} = (-1)^{\beta(u)} e_1 \phi(2u, 2v) i_{2uv} = (-1)^{\beta(u)} \phi(u, v) i_{2uv+1} = \phi(2u, 2v + 1) i_{2uv+1} = \phi(p, q) i_{pq} \]

provided \( \phi(2u, 2v + 1) = (-1)^{\beta(u)} \phi(u, v) \).

(3) \( p = 2u + 1 \) and \( q = 2v \)

\[ i_p i_q = i_{2u+1} i_{2v} = e_1 i_{2u} i_{2v} = e_1 \phi(u, v) i_{2uv} = \phi(u, v) i_{2uv+1} = \phi(2u + 1, v) i_{2uv+1} = \phi(p, q) i_{pq} \]

provided \( \phi(2u + 1, 2v) = \phi(u, v) \).

(4) \( p = 2u + 1 \) and \( q = 2v + 1 \)

\[ i_p i_q = i_{2u+1} i_{2v+1} = e_1 i_{2u} e_1 i_{2v} = (-1)^{\beta(u)} e_1 e_1 i_{2u} i_{2v} = (-1)^{\beta(u)} e_1 \phi(u, v) i_{2uv} = \phi(2u + 1, 2v + 1) i_{2uv} = \phi(p, q) i_{pq} \]

provided \( \phi(2u + 1, 2v + 1) = (-1)^{\beta(u)} e \phi(u, v) \).

Corollary 2.6. Assume \( p, q \in G_n \). The Clifford algebra twist can be defined recursively as follows:

(1) \( \phi(0, 0) = 1 \)

(2) \( \phi(2p, 2q) = \phi(2p + 1, 2q) = \phi(p, q) \)

(3) \( \phi(2p, 2q + 1) = (-1)^{\beta(p)} \phi(p, q) \)
\[
\phi(2p + 1, 2q + 1) = (-1)^{\beta(p)} \mu \phi(p, q)
\]

Stated another way

\[
\begin{bmatrix}
\phi(2p, 2q) & \phi(2p, 2q + 1) \\
\phi(2p + 1, 2q) & \phi(2p + 1, 2q + 1)
\end{bmatrix}
= \phi(p, q)
\begin{bmatrix}
1 & (-1)^{\beta(p)} \\
1 & (-1)^{\beta(p)} \mu
\end{bmatrix}
\]

3. Recursive generation of twist matrices for higher dimensions

The twist matrix for dimension one is found when \( p = q = 0 \)

\[
\begin{bmatrix}
1 & 1 \\
1 & \mu
\end{bmatrix}
\]

For two dimensions, \( 0 \leq p \leq 1, 0 \leq q \leq 1 \), the twist matrix is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \mu & 1 & \mu \\
1 & -1 & \mu & -\mu \\
1 & -\mu & \mu & -1
\end{bmatrix}
\]

For \( \mu = -1 \) these coincide with the twist tables for complex numbers and quaternions. For dimension 3, however, we do not get the twist table for the octonions, rather

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \mu & 1 & \mu & 1 & \mu & \mu & 1 \\
1 & -1 & \mu & -\mu & 1 & -1 & \mu & -\mu \\
1 & -\mu & \mu & -1 & 1 & -\mu & \mu & -1 \\
1 & -1 & -1 & 1 & \mu & -\mu & -\mu & \mu \\
1 & -\mu & -1 & \mu & \mu & -1 & -\mu & 1 \\
1 & 1 & -\mu & -\mu & \mu & \mu & -1 & -1 \\
1 & \mu & -\mu & -1 & \mu & 1 & -1 & -\mu
\end{bmatrix}
\]

For dimension four the twist matrix is too large to represent in this form, so we make the following substitutions:

\[
A = \begin{bmatrix}
1 & 1 \\
1 & \mu
\end{bmatrix}
\]

(3.1)

\[
B = \begin{bmatrix}
1 & -1 \\
1 & -\mu
\end{bmatrix}
\]

(3.2)

The matrices \( A \) and \( B \) are simply the values of \( M(p) = \begin{bmatrix} 1 & (-1)^{\beta(p)} \\ 1 & (-1)^{\beta(p)} \mu \end{bmatrix} \) when \((-1)^{\beta(p)}\) is positive and negative, respectively.

Then the dimension 4 twist table can be represented as follows.
The twist tables for the various dimensions can be generated recursively beginning with \(A\) for dimension 1, then making the following replacements to generate the twist table for each successively higher dimension:

\[
\begin{bmatrix}
A & A & A & A & A & A & A & A \\
B & \mu B & B & \mu B & B & \mu B & B & \mu B \\
B & -B & \mu B & -\mu B & B & -B & \mu B & -\mu B \\
A & -\mu A & \mu A & -A & A & -\mu A & \mu A & -A \\
B & -B & -B & B & \mu B & -\mu B & -\mu B & B \\
A & -\mu A & -A & \mu A & \mu A & -A & -A & A \\
B & \mu B & -\mu B & -B & \mu B & B & -B & -\mu B 
\end{bmatrix}
\]

The twist tables for the various dimensions can be generated recursively beginning with \(A\) for dimension 1, then making the following replacements to generate the twist table for each successively higher dimension:

\[
A \Rightarrow \begin{bmatrix} A & A \\ B & \mu B \end{bmatrix}
\]

\[
B \Rightarrow \begin{bmatrix} B & -B \\ A & -\mu A \end{bmatrix}
\]

4. A tree for computing the Clifford twist

In [3] a tree for computing the Cayley-Dickson twist is described. The same procedure applies to the Clifford twist.

The tree consists of only four components which repeat indefinitely, beginning at node \(A\). There are two versions, one for each value of \(\mu\).

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{clifford_twist_tree.png}
\caption{Clifford twist tree for \(\mu = 1\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{clifford_twist_tree_neg.png}
\caption{Clifford twist tree for \(\mu = -1\).}
\end{figure}

Let us illustrate the use of the tree to compute the product \(i_{2636}i_{1143}\) given \(\mu = -1\).
(1) Convert the subscripts to binary notation. $2636 = 101001101100_B$ and $1143 = 10001110111_B$.
(2) Pair the bits of the first subscript with the bits of the second by placing one over the other. Pad the smaller with zero bits if necessary.
(3) Each binary pair is an instruction for traversing one of the four tree components. A zero is an instruction to move down a left branch and a one is an instruction to move down a right branch. The result is the following path.

\[
\begin{array}{c|c}
A & \stackrel{1}{\rightarrow} B \\
0 & \stackrel{0}{\rightarrow} -B \\
1 & \stackrel{0}{\rightarrow} -A \\
0 & \stackrel{0}{\rightarrow} -A \\
1 & \stackrel{0}{\rightarrow} -A \\
0 & \stackrel{1}{\rightarrow} B \\
0 & \stackrel{0}{\rightarrow} -B \\
1 & \stackrel{0}{\rightarrow} B \\
1 & \stackrel{1}{\rightarrow} A \\
1 & \stackrel{0}{\rightarrow} -B \\
0 & \stackrel{1}{\rightarrow} B \\
0 & \stackrel{0}{\rightarrow} -B \\
0 & \stackrel{1}{\rightarrow} -B
\end{array}
\]

Since the result is $-B$, $\phi(2636, 1143) = -1$. Whenever the result is $-A$ or $-B$, $\phi = -1$ and whenever the result is $A$ or $B$, $\phi = +1$. Since $101001001100 \text{ XOR } 010001110111 = 111000111011 = 3643$ the result is

\[i_{2636} \cdot i_{1143} = -i_{3643}\]

or

\[e_{347ac} \cdot e_{123576b} = -e_{12456abc}\]

References

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