Golden Binomials and Carlitz Characteristic Polynomials

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Abstract

The golden binomials, introduced in the golden quantum calculus, have expansion determined by Fibonomial coefficients and the set of simple zeros given by powers of Golden ratio. We show that these golden binomials are equivalent to Carlitz characteristic polynomials of certain matrices of binomial coefficients. It is shown that trace invariants for powers of these matrices are determined by Fibonacci divisors, quantum calculus of which was developed very recently.

1 Introduction

The golden quantum calculus, based on the Binet formula for Fibonacci numbers $F_n$ as $q$-numbers, was introduced in [1]. In this calculus, the finite-difference $q$-derivative operator is determined by two Golden ratio bases $\phi$ and $\phi'$, while the golden binomial expansion, by Fibonomial coefficients. The coefficients are expressed in terms of Fibonacci numbers, while zeros of these binomials are given by powers of Golden ratio $\phi$ and $\phi'$. It was observed that similar polynomials were introduced by Carlitz in 1965 from different reason, as characteristic polynomials of certain matrices of binomial coefficients [3]. The goal of the present paper is to show equivalence of Carlitz characteristic polynomials with golden binomials. In addition, the proof and interpretation
of main formulas for trace of powers of the matrix $A_{n+1}$ in terms of Fibonacci divisors and corresponding quantum calculus, developed recently in [4] would be given.

2 Golden Binomials

2.1 Fibonomials and Golden Pascal Triangle

The binomial coefficients defined by

$$\binom{n}{k}_F = \frac{[n]_F!}{[n-k]_F! [k]_F!} = \frac{F_n!}{F_{n-k}! F_k!},$$

with $n$ and $k$ being non-negative integers, $n \geq k$, are called the Fibonomials. Using the addition formula for Fibonacci numbers [1],

$$F_{n+m} = \varphi^n F_m + \varphi^m F_n$$

we have following expression

$$F_n = F_{n-k+k} = \left(-\frac{1}{\varphi}\right)^k F_{n-k} + \varphi^{n-k} F_k.$$  \hfill (3)

By using

$$\varphi^n = \varphi F_n + F_{n-1}, \quad \varphi^m = \varphi' F_n + F_{n-1},$$

it can be rewritten as follows

$$F_n = F_{n-k-1} F_k + F_{n-k} F_{k+1}$$

$$= F_{n-k} F_{k-1} + F_{n-k+1} F_k.$$  \hfill (5)

With the above definition (1) it gives recursion formula for Fibonomials in two forms,

$$\binom{n}{k}_F = \frac{(-1/\varphi)^k [n-1]_F!}{[k]_F! [n-k-1]_F!} + \frac{\varphi^{n-k} [n-1]_F!}{[n-k]_F! [k-1]_F!}$$

$$= \left(-\frac{1}{\varphi}\right)^k \binom{n-1}{k}_F + \varphi^{n-k} \binom{n-1}{k-1}_F$$

$$= \varphi^k \binom{n-1}{k}_F + \left(-\frac{1}{\varphi}\right)^{n-k} \binom{n-1}{k-1}_F.$$  \hfill (6)

These formulas, for $1 \leq k \leq n-1$, determine the Golden Pascal triangle for Fibonomials [1].
2.2 Golden Binomial

The Golden Binomial is defined as [1]:

\[(x + y)_F^n = (x + \varphi^{n-1}y)(x + \varphi^{n-2}\varphi'y)\ldots(x + \varphi\varphi^{n-2}y)(x + \varphi^{n-1}y) \quad (8)\]

or due to \(\varphi\varphi' = -1\) it is

\[(x + y)_F^n = (x + \varphi^{n-1}y)(x - \varphi^{n-3}y)\ldots(x + (-1)^{n-1}\varphi^{-n+1}y). \quad (9)\]

It has n-zeros at powers of the Golden ratio

\[\frac{x}{y} = -\varphi^{n-1}, \quad \frac{x}{y} = -\varphi^{n-3}, \quad \ldots, \quad \frac{x}{y} = -\varphi^{-n+1}.\]

For Golden binomial the following expansion in terms of Fibonomials is valid [1]

\[(x + y)_F^n = \sum_{k=0}^{n} \left[\frac{n}{k}\right]_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k \quad (10)\]

The proof is easy by induction and using recursion formulas (6), (7). In terms of Golden binomials we introduce the Golden polynomials

\[P_n(x) = \frac{(x - a)^n}{n!F_n}, \quad (11)\]

where \(n = 1, 2, \ldots\), and \(P_0(x) = 1\). These polynomials satisfy relations

\[D^n_F P_n(x) = P_{n-1}(x), \quad (12)\]

where the Golden derivative is defined as

\[D^n_F P_n(x) = \frac{P_n(x) - P_n(x')}{(\varphi' - \varphi)x}, \quad (13)\]

For even and odd polynomials we have different products

\[P_{2n}(x) = \frac{1}{F_{2n}!} \prod_{k=1}^{n} (x - (-1)^{n+k}\varphi^{2k-1}a)(x + (-1)^{n+k}\varphi^{-2k+1}a), \quad (14)\]
\[ P_{2n+1}(x) = \frac{(x - (-1)^n a)}{F_{2n+1}!} \prod_{k=1}^{n} (x - (-1)^{n+k} \varphi^{2k} a)(x - (-1)^{n+k} \varphi^{-2k} a). \] (15)

By using (4) it is easy to find
\[ \varphi^{2k} + \frac{1}{\varphi^{2k}} = F_{2k} + 2F_{2k-1}, \] (16)
\[ \varphi^{2k+1} - \frac{1}{\varphi^{2k+1}} = F_{2k+1} + 2F_{2k}. \] (17)

Then we can rewrite our polynomials in terms of Fibonacci numbers
\[ P_{2n}(x) = \frac{1}{F_{2n}!} \prod_{k=1}^{n} (x^2 - (-1)^{n+k}(F_{2k-1} + 2F_{2k-2})xa - a^2), \] (18)
\[ P_{2n+1}(x) = \frac{(x - (-1)^n a)}{F_{2n+1}!} \prod_{k=1}^{n} (x^2 - (-1)^{n+k}(F_{2k} + 2F_{2k-1})xa + a^2). \] (19)

The first few odd polynomials are
\[ P_1(x) = (x - a), \] (20)
\[ P_3(x) = \frac{1}{2}(x + a)(x^2 - 3xa + a^2), \] (21)
\[ P_5(x) = \frac{1}{2 \cdot 3 \cdot 5}(x - a)(x^2 + 3xa + a^2)(x^2 - 7xa + a^2), \] (22)
\[ P_7(x) = \frac{1}{2 \cdot 3 \cdot 5 \cdot 8 \cdot 13}(x + a)(x^2 - 3xa + a^2)(x^2 + 7xa + a^2)(x^2 - 18xa + a^2), \] (23)
and the even ones
\[ P_2(x) = (x^2 - xa - a^2), \] (24)
\[ P_4(x) = \frac{1}{2 \cdot 3}(x^2 + xa - a^2)(x^2 - 4xa - a^2), \] (25)
\[ P_6(x) = \frac{1}{2 \cdot 3 \cdot 5 \cdot 8}(x^2 - xa - a^2)(x^2 + 4xa - a^2)(x^2 - 11xa - a^2). \] (26)
2.3 Golden analytic function

By golden binomials in complex domain, the golden analytic function can be derived, which is complex valued function of complex argument, not analytic in usual sense \[2\]. The complex golden binomial is defined as

\[
(x + iy)^n_F = (x + i\varphi^{n-1}y)(x - i\varphi^{n-3}y)\ldots(x + i(-1)^{n-1}\varphi^{1-n}y)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{k(k-1)/2} x^{n-k} i^k y^k.
\]  

(27)

(28)

It can be generated by the golden translation

\[
E^{iyD^x_F} x^n = (x + iy)^n_F,
\]

where

\[
E^x_F = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} x^n F_n!.
\]

The binomials determine the golden analytic function

\[
f(z, F) = E^{iyD^x_F} f(x) = \sum_{n=0}^{\infty} a_n \frac{(x + iy)^n}{F_n!},
\]

satisfying the golden $\bar{\partial}_F$ equation

\[
\frac{1}{2}(D^x_F + iD^y_F) f(z; F) = 0,
\]

(29)

where $D^x_F = (-1)^{\frac{n(n+1)}{2}} D^x_F$. For $u(x, y) = \cos_F(yD^x_F) f(x)$ and $v(x, y) = \sin_F(yD^x_F) f(x)$, the golden Cauchy-Riemann equations are

\[
D^x_F u(x, y) = D^y_F v(x, y), \quad D^y_F u(x, y) = -D^x_F v(x, y),
\]

(30)

and the golden-Laplace equation is

\[
(D^x_F)^2 u(x, y) + (D^y_F)^2 u(x, y) = 0.
\]

(31)

2.4 Particular Case

The golden binomial $(x - a)^n_F$ can be also generated by the golden translation

\[
E^{aD^x_F} x^n = (x - a)^n_F.
\]

(32)
In particular case \( a = 1 \) we have
\[
(x - 1)_m^F = (x - \varphi^{m-1})(x + \varphi^{m-3})...(x - (-1)^{m-1}\varphi^{-m+1}).
\] (33)

First few binomials are
\[
(x - 1)_1^F = x - 1, \quad (34)
\]
\[
(x - 1)_2^F = (x - \varphi)(x - \varphi'), \quad (35)
\]
\[
(x - 1)_3^F = (x - \varphi^2)(x + 1)(x - \varphi^2), \quad (36)
\]
\[
(x - 1)_4^F = (x - \varphi^3)(x + \varphi)(x + \varphi')(x - \varphi^3), \quad (37)
\]

and corresponding zeros
\[
m = 1 \Rightarrow x = 1 \quad (38)
\]
\[
m = 2 \Rightarrow x = \varphi, x = \varphi' \quad (39)
\]
\[
m = 3 \Rightarrow x = \varphi^2, x = -1, x = \varphi^2 \quad (40)
\]
\[
m = 4 \Rightarrow x = \varphi^3, x = -\varphi, x = -\varphi', x = \varphi^3. \quad (41)
\]

For arbitrary even and odd \( n \) we have following zeros of Golden binomials
\[
n = 2k \Rightarrow (x - 1)_{2k}^F : \varphi^{n-1}, \varphi^{m-1}, -\varphi^{n-3}, -\varphi^{m-3}, ..., \pm \varphi, \pm \varphi'; \quad (42)
\]
\[
n = 2k + 1 \Rightarrow (x - 1)_{2k+1}^F : \varphi^{n-1}, \varphi^{m-1}, -\varphi^{n-3}, -\varphi^{m-3}, ..., \pm 1. \quad (43)
\]

3 Carlitz Polynomials

In Section 2 we have introduced the Golden binomials. Now we are going to relate these binomials with characteristic equations for some matrices, constructed from binomial coefficients by Carlitz \[3\].

**Definition 3.0.1** We define an \((n+1) \times (n+1)\) matrix \(A_{n+1}\) with binomial coefficients,
\[
A_{n+1} = \begin{bmatrix} \binom{r}{n-s} \end{bmatrix},
\]
where \(r, s = 0, 1, 2, ..., n\). Here,
\[
\binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!k!}, & \text{if } k \leq n; \\ 0, & k > n. \end{cases}
\] (45)
First few matrices are,

\[ n = 0 \Rightarrow r = s = 0 \Rightarrow A_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = (1) \]

\[ n = 1 \Rightarrow r, s = 0, 1 \Rightarrow A_2 = \begin{pmatrix} r \\ 1 - s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \]

\[ n = 2 \Rightarrow r, s = 0, 1, 2 \Rightarrow A_3 = \begin{pmatrix} r \\ 2 - s \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \]

Continuing, the general matrix \( A_{n+1} \) of order \((n + 1)\) can be written as,

\[
A_{n+1} = \begin{pmatrix}
\ldots & 0 & 0 & 0 & 1 \\
\ldots & 0 & 0 & 1 & 1 \\
\ldots & 0 & 1 & 2 & 1 \\
\ldots & 0 & 1 & 3 & 3 & 1 \\
\ldots & 1 & 4 & 6 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}_{(n+1)\times(n+1)}
\]

where the lower triangular matrix is build from Pascal’s triangle. We notice that trace of first few matrices \( A_{n+1} \) gives Fibonacci numbers. As would be shown, it is valid for any \( n \) (Theorem [4.0.5] equation [60]).

**Definition 3.0.2** Characteristic polynomial of matrix \( A_{n+1} \) is determined by,

\[
Q_{n+1}(x) = \det(xI - A_{n+1}). \tag{46}
\]

First few polynomials explicitly are

\[ n = 0 : \quad Q_1(x) = x - 1, \]

\[ n = 1 : \quad Q_2(x) = \det(xI - A_2) = \begin{vmatrix} x & -1 \\ -1 & x - 1 \end{vmatrix} = x^2 - x - 1, \]

\[ n = 2 : \quad Q_3(x) = \det(xI - A_3) = \begin{vmatrix} x & 0 & -1 \\ 0 & x - 1 & -1 \\ -1 & -2 & x - 1 \end{vmatrix} = x^3 - 2x^2 - 2x + 1, \]

\[ n = 3 : \]
\[ Q_4(x) = \det(xI - A_4) = \begin{vmatrix} x & 0 & 0 & -1 \\ 0 & x & -1 & -1 \\ 0 & -1 & x - 2 & -1 \\ -1 & -3 & -3 & x - 1 \end{vmatrix} = -x^4 + 3x^3 + 6x^2 - 3x - 1. \]

Corresponding eigenvalues are represented by powers of \( \varphi \) and \( \varphi' \);

\begin{align*}
  n = 0 & \Rightarrow x_1 = 1, \\
  n = 1 & \Rightarrow x_1 = \varphi, \ x_2 = \varphi', \\
  n = 2 & \Rightarrow x_1 = \varphi^2, \ x_2 = -1, \ x_3 = \varphi^2, \\
  n = 3 & \Rightarrow x_1 = \varphi^3, x_2 = -\varphi, x_3 = -\varphi', x_4 = \varphi^3.
\end{align*}

Comparing zeros of first few characteristic polynomials, with zeros of Golden Binomial (33), we notice that they coincide. According to this, we have following conjecture.

**Conjecture:** The characteristic equation (46) of matrix \( A_{n+1} \) coincides with Golden Binomial;

\[ Q_{n+1}(x) = \det(xI - A_{n+1}) = (x - 1)^{n+1}. \quad (47) \]

To prove this conjecture, firstly we represent Golden binomials in the product form.

**Proposition 3.0.3** The Golden binomial can be written as a product,

\[ (x - 1)^{n+1}_F = \prod_{j=0}^{n} \left( x - \varphi^j \varphi'^{n-j} \right). \quad (48) \]

**Proof 3.0.4** Starting from Golden binomial in product representation

\[ (x + y)^n_F \equiv \prod_{j=0}^{n-1} \left( x - (-1)^{j-1} \varphi^{n-1-j} \varphi^{-2j} y \right) \quad (49) \]

by using

\[ \varphi^{-2j} = \left( \frac{1}{\varphi} \right)^{2j} = \left( -\frac{1}{\varphi} \right)^{2j} = \varphi^{2j}, \quad (50) \]

after substitution \( y = -1 \) we have

\[ (x - 1)^n_F \equiv \prod_{j=0}^{n-1} \left( x - (-1)^j \varphi^{n-1-j} \varphi^{2j} \right). \]
By shifting \( n \rightarrow n + 1 \),

\[
(x - 1)^{n+1}_F = \prod_{j=0}^{n} \left( x - (-1)^j \varphi^n \varphi'^{2j} \right)
\]

\[
= \prod_{j=0}^{n} \left( x - (-1)^j \varphi^n \frac{(-1)^{2j}}{\varphi^j \varphi'^j} \right)
\]

\[
= \prod_{j=0}^{n} \left( x - \varphi^n \left( -\frac{1}{\varphi} \right)^j \frac{1}{\varphi^j} \right)
\]

\[
= \prod_{j=0}^{n} \left( x - \varphi^{n-j} \varphi'^j \right)
\]

and substituting \( j = n - m \) we get,

\[
(x - 1)^{n+1}_F = \prod_{m=0}^{n} \left( x - \varphi^m \varphi'^{n-m} \right).
\]

The formula shows explicitly that zeros of Golden binomial in (42) and (43) are given by powers of \( \varphi \) and \( \varphi' \).

**Corollary 3.0.5** The eigenvalues of matrix \( A_{n+1} \) are the numbers,

\[
\varphi^n, \varphi^{n-1} \varphi', \varphi^{n-2} \varphi'^2, \ldots, \varphi \varphi^{m-1}, \varphi^m.
\]  

(51)

As it was shown by Carlitz [3], this product formula is just characteristic equation (46) for matrix \( A_{n+1} \). Since zeros of two polynomials \( \text{det}(xI - A_{n+1}) \) and \( (x - 1)^{n+1}_F \) coincide, then the conjecture is correct and we have following theorem.

**Theorem 3.0.6** Characteristic equation for combinatorial matrix \( A_{n+1} \) is given by Golden binomial:

\[
Q_{n+1}(x) = \text{det}(xI - A_{n+1}) = (x - 1)^{n+1}_F.
\]  

(52)

## 4 Powers of \( A_{n+1} \) and Fibonacci Divisors

**Proposition 4.0.1** Arbitrary \( n^{th} \) power of \( A_2 \) matrix is written in terms of Fibonacci numbers,

\[
A_n^2 = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.
\]  

(53)
Proof 4.0.2 Proof will be done by induction. For $n = 1$,

\[ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}, \]

and for $n = 2$,

\[ A_2^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}. \]

Suppose for $n = k$,

\[ A_k^2 = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}, \]

then

\[ A_{k+1}^2 = A_k^2 A_2 = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_{k+2} \end{pmatrix}. \]

This result can be understood from observation that eigenvalues of matrix $A_2$ are $\varphi$ and $\varphi'$, and eigenvalues of $A_2^n$ are powers $\varphi^n$, $\varphi'^n$ related with Fibonacci numbers.

As we have seen, eigenvalues of matrix $A_3$ are $\varphi^2$, $\varphi'^2$, $-1$. It implies that for $A_3^n$, eigenvalues are $\varphi^{2n}$, $\varphi'^{2n}$, $(-1)^n$, and the matrix can be expressed by Fibonacci divisor $F_n^{(2)}$ conjugate to $F_2$, due to [4],

\[ (\varphi^k)^n = \varphi^k F_n^{(k)} + (-1)^{k+1}F_{n-1}^{(k)}, \]
\[ (\varphi'^k)^n = \varphi'^k F_n^{(k)} + (-1)^{k+1}F_{n-1}^{(k)}, \]

where $F_n^{(k)} = F_{nk}/F_k$.

Proposition 4.0.3 Arbitrary $n^{th}$ power of $A_3$ matrix can be expressed in terms of Fibonacci divisors $F_n^{(2)}$,

\[ A_3^n = \frac{1}{5} \begin{pmatrix} (2F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) & (2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n) & (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) \\ (F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n) & (6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n) & (4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n) \\ (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) \end{pmatrix} \]
Proof 4.0.4 Let’s diagonalize the matrix $A_3$, 

$$\phi_3 = \sigma_3^{-1} A_3 \sigma_3,$$

where $\phi_3$ is the diagonal matrix and 

$$A_3 = \sigma_3 \phi_3 \sigma_3^{-1}.$$

Taking the $n^{th}$ power of both sides gives, 

$$A_3^n = (\sigma_3 \phi_3 \sigma_3^{-1}) (\sigma_3 \phi_3 \sigma_3^{-1}) \ldots (\sigma_3 \phi_3 \sigma_3^{-1}) (\sigma_3 \phi_3 \sigma_3^{-1})$$

Therefore, 

$$A_3^n = \sigma_3 \phi_3^n \sigma_3^{-1}. \quad (57)$$

By using the diagonalization principle, $\sigma_3$ and $\sigma_3^{-1}$ matrices can be obtained as, 

$$\sigma_3 = \frac{1}{2} \begin{pmatrix} -\varphi' & \frac{4}{3} & -\varphi \\ 1 & \frac{3}{3} & 1 \\ \varphi & -\frac{4}{3} & \varphi' \end{pmatrix}$$

and, 

$$\sigma_3^{-1} = \begin{pmatrix} \frac{2(\varphi' + 2)}{5(\varphi - \varphi')} & -\frac{4(\varphi' + 2)}{5\varphi' (\varphi - \varphi')} & \frac{2(2\varphi' - 1)}{5\varphi' (\varphi - \varphi')} \\ \frac{3}{5} & \frac{\varphi}{\varphi' (\varphi - \varphi')} & -\frac{3}{5} \\ -\frac{2(\varphi + 2)}{5(\varphi - \varphi')} & \frac{4(\varphi' + 2)}{5\varphi' (\varphi - \varphi')} & \frac{2(1 - 2\varphi)}{5\varphi' (\varphi - \varphi')} \end{pmatrix} = \frac{2}{5\sqrt{5}} \begin{pmatrix} \varphi' + 2 & -2(1 - 2\varphi) & (2 + \varphi) \\ \frac{3\sqrt{5}}{2} & \frac{3\sqrt{5}}{2} & -\frac{3\sqrt{5}}{2} \\ -2(1 - 2\varphi) & 2(1 - 2\varphi') & -(2 + \varphi') \end{pmatrix}$$

Since eigenvalues of matrix $A_3$ are $\varphi^2, -1, \varphi^2$, the diagonal matrix $\phi_3$ is, 

$$\phi_3 = \begin{pmatrix} \varphi^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \varphi^2 \end{pmatrix}, \quad (58)$$

and an arbitrary $n^{th}$ power of this matrix is, 

$$\phi_3^n = \begin{pmatrix} (\varphi^2)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (\varphi^2)^n \end{pmatrix}. \quad (59)$$
Finally by using (57), \( A_3^n = \)

\[
\frac{1}{5} \begin{pmatrix}
(2F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n) & (2F_n^{(2)} + 2F_{n-1}^{(2)} + 2(-1)^n) & (3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) \\
(F_n^{(2)} + F_{n-1}^{(2)} + (-1)^n) & (6F_n^{(2)} - 4F_{n-1}^{(2)} + (-1)^n) & (4F_n^{(2)} - F_{n-1}^{(2)} - (-1)^n) \\
(3F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (8F_n^{(2)} - 2F_{n-1}^{(2)} - 2(-1)^n) & (7F_n^{(2)} - 3F_{n-1}^{(2)} + 2(-1)^n)
\end{pmatrix}
\]

is obtained.

As we can expect, these results can be generalized to arbitrary matrix \( A_{n+1} \). Since eigenvalues of \( A_{n+1} \) are powers \( \varphi^n, \varphi^m, \ldots \), for \( A_{n+1}^N \) eigenvalues are \( \varphi^{nN}, \varphi^{mN}, \ldots \). But these powers can be written in terms of Fibonacci divisors as in (55), (56), and the matrix \( A_{n+1}^N \) itself can be represented by Fibonacci divisors \( F_N^{(n)} \).

For powers of matrix \( A_{n+1} \) we have the following identities.

**Theorem 4.0.5** Invariants of \( A_{n+1}^k \) matrix are found as,

\[
\begin{align*}
\text{Tr} \left( A_{n+1}^k \right) &= \frac{F_{kn+k}}{F_k} = F_{n+1}^{(k)}, \\
\text{det} \left( A_{n+1}^k \right) &= (-1)^k \frac{n(n+1)}{2}.
\end{align*}
\]

For \( k = 1 \), it gives

\[
\begin{align*}
\text{Tr} \left( A_{n+1} \right) &= F_{n+1}, \\
\text{det} \left( A_{n+1} \right) &= (-1)^{\frac{n(n+1)}{2}}.
\end{align*}
\]

**Proof 4.0.6** Let’s diagonalize the general matrix \( A_{n+1} \) as,

\[
\phi_{n+1} = \sigma_{n+1}^{-1} A_{n+1} \sigma_{n+1}
\]

where \( \phi_{n+1} \) is diagonal and

\[
A_{n+1} = \sigma_{n+1} \phi_{n+1} \sigma^{-1}_{n+1}.
\]

Taking the \( k \)th power of both sides gives,

\[
A_{n+1}^k = \left( \sigma_{n+1} \phi_{n+1} \sigma^{-1}_{n+1} \right) \left( \sigma_{n+1} \phi_{n+1} \sigma^{-1}_{n+1} \right) \ldots \left( \sigma_{n+1} \phi_{n+1} \sigma^{-1}_{n+1} \right)
\]

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and

\[ A^k_{n+1} = \sigma_{n+1}^k \phi_{n+1}^k \sigma_{n+1}^{-1}. \]  \hspace{1cm} (62)

By taking trace from both sides and using the cyclic permutation property of trace,

\[ Tr(A^k_{n+1}) = Tr(\sigma_{n+1}^k \phi_{n+1}^k \sigma_{n+1}^{-1}) = Tr(\sigma_{n+1}^{-1} \sigma_{n+1} \phi_{n+1}^k) = Tr(I \phi_{n+1}^k) = Tr(\phi_{n+1}^k) \]

we get

\[ Tr(A^k_{n+1}) = Tr(\phi_{n+1}^k). \]

The eigenvalues of matrix \( A_{n+1} \) in (51), allows one to construct the diagonal matrix \( \phi_{n+1} \) and calculate

\[
\begin{pmatrix}
\varphi^n & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \varphi^{n-1}\varphi' & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \varphi^{n-2}\varphi'^2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \varphi^2\varphi'^{n-2} & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & \varphi\varphi'^{n-1} & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & \varphi^n
\end{pmatrix}^k
\]

It gives

\[
\begin{pmatrix}
(\varphi^n)^k & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & (\varphi^{n-1}\varphi')^k & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & (\varphi^{n-2}\varphi'^2)^k & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (\varphi^2\varphi'^{n-2})^k & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & (\varphi\varphi'^{n-1})^k & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & (\varphi^n)^k
\end{pmatrix}
\]

and

\[ Tr(A^k_{n+1}) = (\varphi^n)^k + (\varphi^{n-1}\varphi')^k + \ldots + (\varphi\varphi'^{n-1})^k + (\varphi^n)^k, \]  \hspace{1cm} (63)

or

\[ Tr(A^k_{n+1}) = (\varphi^k)^n + (\varphi^k)^{n-1}\varphi'^k + \ldots + \varphi^k(\varphi'^k)^{n-1} + (\varphi'^k)^n. \]  \hspace{1cm} (64)
The powers \((\varphi^k)^n\) and \((\varphi'^k)^n\) substituted from equations \((55)\) and \((56)\) give \(\text{Tr}(A_{n+1}^k) = \)

\[
= (\varphi^k F_n^k) + (-1)^{k+1} F_{n-1}^k + (\varphi^k F_{n-1}^k + (-1)^{k+1} F_{n-2}^k) \varphi'^k + \ldots \\
+ (\varphi^k F_1^k + (-1)^{k+1} F_0^k) (\varphi'^k)^n + (\varphi'^k)^n \\
= \varphi^k (F_n^k + F_{n-1}^k (\varphi'k) + F_{n-2}^k (\varphi'^k)^2 + \ldots + F_1^k (\varphi'^k)^n) \\
+ (-1)^{k+1} \left( F_{n-1}^k + F_{n-2}^k (\varphi'^k) + F_{n-3}^k (\varphi'^k)^2 + \ldots + F_0^k (\varphi'^k)^n \right) \\
+ (\varphi'^k)^n \\
= F_{kn}^k F_k^k + F_{(n-1)k}^k F_k^k (-1)^k + F_{(n-2)k}^k F_k^k (-1)^k (\varphi'^k) + \ldots + F_k^k (\varphi'^k)^n \\
+ F_{(n-1)k}^k F_k^k (-1)^{k+1} + F_{(n-2)k}^k F_k^k (-1)^{k+1} (\varphi'^k) + F_{(n-3)k}^k (-1)^{k+1} (\varphi'^k)^2 \\
+ \ldots + F_0^k (-1)^{k+1} (\varphi'^k)^n + (\varphi'^k)^n \\
= F_{kn}^k F_k^k \varphi^k + F_{(n-1)k}^k F_k^k (-1)^k + F_{(n-2)k}^k F_k^k (-1)^k (\varphi'^k) + \ldots \\
+ F_{(n-(n-1))k}^k F_k^k (-1)^{k+1} (\varphi'^k)^n - 2 + F_{(n-1)k}^k F_k^k (-1)^{k+1} + F_{(n-2)k}^k F_k^k (-1)^{k+1} (\varphi'^k)^k \\
+ F_{(n-3)k}^k F_k^k (-1)^{k+1} (\varphi'^k)^2 + \ldots + F_k^k (-1)^{k+1} (\varphi'^k)^n - 2 + (\varphi'^k)^n \\
= F_{kn}^k F_k^k \varphi^k + F_{(n-1)k}^k F_k^k \left( (-1)^k + (-1)^{k+1} \right) \\
+ F_{(n-2)k}^k F_k^k \left( (-1)^k (\varphi'^k) + (-1)^{k+1}\varphi'^k \right) + F_{(n-3)k}^k F_k^k \left( (-1)^k (\varphi'^k)^2 + (-1)^{k+1} (\varphi'^k)^k \right) \\
+ \ldots + F_k^k \left( (-1)^k (\varphi'^k)^n - 2 + (-1)^{k+1} (\varphi'^k)^n - 2 \right) + (\varphi'^k)^n \\
= F_{kn}^k F_k^k \varphi^k + F_{(n-1)k}^k F_k^k (-1)^k (1 - 1) + F_{(n-2)k}^k F_k^k (-1)^k \varphi'^k (1 + (-1))
\[
\begin{split}
&\frac{F_{(n-3)k}}{F_k}(-1)^k(\varphi'^k)^2(1 + (-1)) + \ldots + \frac{F_k}{F_k}(-1)^k(\varphi'^k)^n(1 + (-1)) \\
&+ (\varphi'^k)^n \\
&= \frac{F_{kn}}{F_k} \varphi^k + (\varphi'^k)^n \\
\text{(56)} &= \frac{F_{kn}}{F_k} \varphi^k + \varphi'^k \frac{F_{n}^{(k)}}{F_k} + (-1)^{k+1} \frac{F_{n-1}^{(k)}}{F_k} \\
&= \frac{1}{F_k} \left( \frac{F_{kn} \varphi^k + \varphi'^k F_{kn} + (-1)^{k+1} F_{k(n-1)}}{F_k} \right) \\
&= \frac{1}{F_k} \left[ \left( \frac{\varphi^k}{\varphi} - \frac{\varphi'^k}{\varphi'} \right) \left( \frac{\varphi^k}{\varphi} + \varphi'^k \left( \frac{\varphi^k}{\varphi} - \frac{\varphi'^k}{\varphi'} \right) \right) + (-1)^{k+1} \left( \frac{\varphi^{(n-1)k} - \varphi'^{(n-1)k}}{\varphi'} \right) \right] \\
&= \frac{1}{F_k} \frac{1}{\varphi - \varphi'} \left[ \varphi^{k(n+1)} - \varphi'^{k(n+1)} - (-1)^{kn} \varphi^{k(1-n)} + (-1)^{k} \varphi^{k(n-1)} - (-1)^{k} \varphi^{k(n-1)} \\
&+ (-1)^{k(-1)} \varphi^{k(1-n)} \right] \\
&= \frac{1}{F_k} \frac{1}{\varphi - \varphi'} \left[ \varphi^{k(n+1)} - \varphi'^{k(n+1)} - (-1)^{kn} \varphi^{k(1-n)} + (-1)^{k} \varphi^{k(n-1)} - (-1)^{k} \varphi^{k(n-1)} \\
&+ (-1)^{k(-1)} \varphi^{k(1-n)} \right] \\
&= \frac{1}{F_k} \frac{1}{\varphi - \varphi'} \left[ \varphi^{k(n+1)} - \varphi'^{k(n+1)} \right] \\
&= \frac{1}{F_k} \frac{\varphi^{k(n+1)} - \varphi'^{k(n+1)}}{\varphi - \varphi'} \\
&= \frac{1}{F_k} F_{k(n+1)} \\
&= \frac{F_{k(n+1)}}{F_k}.
\end{split}
\]
To prove the relation for $\text{det} \left( A_{n+1}^k \right)$, we take the determinant from both sides in (62),

$$\text{det} \left( A_{n+1}^k \right) = \text{det} \left( \sigma_{n+1} \phi_{n+1}^k \sigma_{n+1}^{-1} \right). \quad (65)$$

By using property of determinants,

$$\text{det}(AB) = \text{det}(A) \text{det}(B) \quad (66)$$

we obtain,

$$\begin{align*}
\text{det} \left( A_{n+1}^k \right) &= \text{det} \left( \sigma_{n+1} \right) \text{det} \left( \phi_{n+1}^k \right) \text{det} \left( \sigma_{n+1}^{-1} \right) \\
\text{det} \left( A_{n+1}^k \right) &= \text{det} \left( \sigma_{n+1} \right) \text{det} \left( \sigma_{n+1}^{-1} \right) \text{det} \left( \phi_{n+1}^k \right) \\
\text{det} \left( A_{n+1}^k \right) &= \text{det} \left( \sigma_{n+1} \sigma_{n+1}^{-1} \right) \text{det} \left( \phi_{n+1}^k \right) \\
\text{det} \left( A_{n+1}^k \right) &= \text{det} \left( \sigma_{n+1} \right) \text{det} \left( \sigma_{n+1}^{-1} \right) \text{det} \left( \phi_{n+1}^k \right) \\
\text{det} \left( A_{n+1}^k \right) &= \text{det} \left( \phi_{n+1}^k \right)
\end{align*}$$

Since the matrix $\phi_{n+1}^k$ is known, the above equation becomes,

$$\begin{align*}
\text{det} \left( A_{n+1}^k \right) &= (\varphi^n)^k (\varphi^{n-1})^k (\varphi^{n-2})^k \ldots (\varphi^2)^k (\varphi^2)^k (\varphi^{n-1})^k (\varphi^n)^k \\
&= (\varphi^k)^{\varphi(n-1)k} (\varphi^{n-2k})^k \ldots (\varphi^{n-2k})^k (\varphi^{n-2k})^k (\varphi^{n-2k})^k \\
&= (\varphi^{2k})^{\varphi(n-1)k} (\varphi^{2k})^{\varphi(n-1)k} (\varphi^{2k})^{\varphi(n-1)k} \\
&= \varphi^{k\left\lfloor n(n-1)+n-2+2k+1 \right\rfloor} \varphi^{k\left\lfloor 1+2+\ldots+(n-2)+(n-1)+n \right\rfloor} \\
&= \varphi^{k\left\lfloor \frac{n(n+1)}{2} + \frac{n(n+1)}{2} \right\rfloor} \\
&= \left( \varphi^\varphi \right)^k \left( \frac{n(n+1)}{2} \right)^k \\
&= (-1)^k \left( \frac{n(n+1)}{2} \right)^k
\end{align*}$$

The above Theorem represents Fibonacci divisors $F_{n+1}^{(k)}$ in terms of combinatorial matrix $A_{n+1}$. Quantum calculus for such divisors was constructed recently in [4]. As was shown, it is related with several problems from hydrodynamics, quantum integrable systems and quantum information theory. This is why results of the present paper can be useful in the studies of this calculus and its applications.
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