ON PARAMETER DEPENDENCE OF EXPONENTIAL STABILITY OF EQUILIBRIUM SOLUTIONS IN DIFFERENTIAL EQUATIONS WITH A SINGLE CONSTANT DELAY

JUNYA NISHIGUCHI

Department of Mathematics, Kyoto University
Kitashirakawa Oiwake-cho, Sakyo-ku
Kyoto 606-8502, Japan

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Abstract. A transcendental equation \( \lambda + \alpha - \beta e^{-\lambda \tau} = 0 \) with complex coefficients is investigated. This equation can be obtained from the characteristic equation of a linear differential equation with a single constant delay. It is known that the set of roots of this equation can be expressed by the Lambert W function. We analyze the condition on parameters for which all the roots have negative real parts by using the “graph-like” expression of the W function. We apply the obtained results to the stabilization of an unstable equilibrium solution by the delayed feedback control and the stability condition of the synchronous state in oscillator networks.

1. Introduction. In this paper, we investigate how the exponential stability of equilibrium solutions of a delay differential equation (DDE) in \( \mathbb{R}^n \) \((n \geq 2)\)

\[
x'(t) = f(x(t), x(t-\tau)) \quad (t \geq 0)
\]

is determined by this equation. Here \( f(x(t), x(t-\tau)) \) is a smooth (nonlinear) function of \( (x(t), x(t-\tau)) \in \mathbb{R}^n \times \mathbb{R}^n \), and \( \tau > 0 \) is a “delay” parameter. By the exponential stability of an equilibrium solution \( x^*(t) \equiv p \in \mathbb{R}^n \), we mean that there exist \( \delta, C \) and \( \gamma > 0 \) such that

\[
|x(t) - p| \leq Ce^{-\gamma t} \quad \text{for all } t \geq 0
\]

holds for any solutions \( x(t) \) satisfying an initial condition

\[
|x(t) - p| < \delta \quad \text{for all } t \in [-\tau, 0],
\]

where \( | \cdot | \) denotes the Euclidean norm.

We have the following stability result called the principle of linearized stability.

Fact 1 (e.g. Diekmann et al. [4]). We consider the linearized equation

\[
x'(t) = -Ax(t) + Bx(t-\tau) \quad (A = -D_1f(p,p), B = D_2f(p,p))
\]

of \( (1) \) along \( x^*(t) \equiv p \) and its characteristic equation

\[
\det(I + A - e^{-\lambda \tau} B) = 0 \quad (I \text{ is the } n \times n \text{ identity matrix}),
\]

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where $D_1 f(p, p)$ and $D_2 f(p, p)$ denote the partial Jacobian matrices of $f$ at $(p, p)$. Then the following statements hold.

(i) $x^*(t)$ is exponentially stable if $\Re(\lambda) < 0$ for all characteristic roots $\lambda$,
(ii) $x^*(t)$ is unstable if $\Re(\lambda) > 0$ for some characteristic root $\lambda$.

1.1. Transcendental equations and Lambert W function. Our concern in this paper is how to analyze the characteristic equation (2) to obtain the parameter dependence. See Stépán [14] for general results about the characteristic equations in DDEs. If matrices $A$ and $B$ have some (generalized) eigenvectors in common, then there is a chance that (2) is reduced to a transcendental equation

$$\lambda + \alpha - \beta e^{-\lambda \tau} = 0,$$

where $\alpha$ and $\beta$ are some eigenvalues of $A$ and $B$, respectively. For the case $\alpha, \beta \in \mathbb{R}$, Hayes [7] obtained a necessary and sufficient condition on parameters for which all the roots have negative real parts. The condition on $\tau > 0$ for each fixed $\alpha$ and $\beta$ is easily obtained (cf. Cooke & Grossman [1]) from this. However, to the best of the author’s knowledge, no condition on $\alpha$, $\beta$ and $\tau$ where $\alpha, \beta \in \mathbb{C}$ has been obtained.

We note that (3) can be viewed as a transcendental equation with real coefficients

$$\lambda^2 + q - r(\lambda)e^{-\lambda \tau} + s(\lambda) e^{-\lambda \tau} \cdot e^{(\lambda+\alpha)\tau} = 0,$$

where $p = \alpha + \bar{\alpha}$, $q = |\alpha|^2$, $r(\lambda) = (\beta + \bar{\beta})\lambda + (\alpha \bar{\beta} + \bar{\alpha} \beta)$ and $s = |\beta|^2$ by multiplying (3) and the equation with complex conjugate coefficients.

To achieve our objective, we use the Lambert W function that is the multi-valued inverse of a complex function $z \mapsto z e^z$ (see Corless et al. [3]), that is

$$W(\zeta) = \{ z \in \mathbb{C} : z e^z = \zeta \} \quad (\zeta \in \mathbb{C}).$$

By using the W function, the set of roots of (3) can be represented as

$$\frac{1}{\tau} W(\tau \beta e^{\lambda \tau}) - \alpha$$

because (3) is equivalent to

$$(\lambda + \alpha) \tau \cdot e^{(\lambda+\alpha)\tau} = \tau \beta e^{-\lambda \tau} \cdot e^{(\lambda+\alpha)\tau}.$$ 

The idea of this paper is to study this set by investigating the complex branches of the W function. The “graph-like” expression of these branches in some coordinate system of the complex plane $\mathbb{C}$ (Corollary 1 in Section 2) is the ingredient of this approach.

Definition 1.1. We call a complex function $f$ to be graph-like on a subset $D$ if there exists a real-valued function $g$ such that we have a formula

$$f(z) = x + ig(x, y) \quad \text{for all } z = x + iy \in D.$$ 

Here $i$ is the imaginary unit.

The main result is the following:

Theorem 1.2. Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$, and $\tau > 0$. Then all the roots of (3) have negative real parts if and only if $\alpha$, $\beta$ and $\tau$ satisfy the following (a) or (b):

(a) $\Re(\alpha) > |\beta|$, 
(b) $|\beta| < \Re(\alpha) \leq |\beta|$ and

$$Z(\Im(\alpha) \tau + \Arg(\beta)) > \Arccos(\Re(\alpha)/|\beta|) + \tau \sqrt{|\beta|^2 - \Re(\alpha)^2}. \quad (4)$$

The notation is given in Subsection 3.1.
1.2. Applications to delayed feedback control and oscillator networks.

1.2.1. Stabilization of equilibrium solutions by delayed feedback control. The delayed feedback control (DFC) proposed by Pyragas [12] is a method to stabilize an orbitally unstable periodic solution of a given autonomous smooth ordinary differential equation (ODE).

Assume that we have a smooth ODE \( x'(t) = f(x(t)) \) in \( \mathbb{R}^n \) and there is an orbitally unstable periodic solution \( \gamma(t) \). Then we consider a DDE

\[
x'(t) = f(x(t)) + K \cdot (x(t - \tau) - x(t)),
\]

where \( K \) is a constant \( n \times n \) real matrix and \( \tau > 0 \) is an integer multiple of the minimal period of \( \gamma(t) \). We note that \( \gamma(t) \) is also a periodic solution of (5). The objective of the DFC is to find \( K \) and \( \tau \) so that \( \gamma(t) \) is an orbitally stable periodic solution of (5). Pyragas showed the possibility of the stabilization by the DFC applying this to the Rössler system numerically, but this problem remains open mathematically. There is a positive result by Fiedler et al. [6] in which they showed that it is possible to stabilize unstable periodic orbits of the normal form of a subcritical Hopf bifurcation that emanate from the stable equilibrium by the DFC.

Instead of a periodic solution, one can consider the stabilization of an unstable equilibrium solution by the DFC, in which we can choose an arbitrarily \( \tau > 0 \) (e.g., see Hövel & Schöll [9]). By applying the results obtained from Theorem 1.2, we show that it is possible to stabilize an unstable equilibrium solution by the DFC on some assumption of its equilibrium (Theorem 4.1 in Subsection 4.1).

1.2.2. Stability criterion of synchronous state in oscillator networks. Earl & Strogatz [5] studied networks of identical phase oscillators with delayed coupling in which the in-degrees of each vertices of the associated graph are uniform. For these oscillator networks, they obtained an analytic criterion of the linear stability of the synchronous state which does not depend on their connection topologies.

We consider a network of identical phase oscillators with delayed coupling. For this network, we assume that the associated graph \( \Gamma \) does not have self loops and the in-degrees of each vertices of \( \Gamma \) are equal. This model is obtained from a system of DDEs

\[
\dot{\theta}_i(t) = \omega + \frac{K}{k} \sum_{j=1}^{N} a_{ij} f(\theta_j(t - \tau) - \theta_i(t)) \quad (1 \leq i \leq N),
\]

where \( \theta_i \) is the phase variable of the \( i \)th oscillator, \( \omega \) is its natural frequency, \( K \) is the coupling strength, and \( f \) is a nonlinear \( 2\pi \)-periodic function. On the connection topology of \( \Gamma \), \( k \) is the in-degree of vertices, and \( A = (a_{ij})_{i,j} \) is the adjacency matrix. The above assumptions mean that \( a_{ij} = 0 \) or 1, \( a_{ii} = 0 \), and

\[
\sum_{j=1}^{N} a_{ij} = k
\]

holds for each \( i \).

We treat the exponential stability of the synchronous state \( \theta_i(t) = \Omega t \) \( (1 \leq i \leq N) \), where \( \Omega \) satisfies

\[
\Omega = \omega + K f(-\Omega \tau).
\]

Earl & Strogatz [5] claimed the following statement.
Claim (5). Assume that $K f'(-\Omega \tau) \neq 0$. Then the synchronous state $\theta_i(t) = \Omega t$ $(1 \leq i \leq N)$ is linearly stable if and only if $K f'(-\Omega \tau) > 0$.

Here $f'(-\Omega \tau)$ denotes the derivative of $f$ at $-\Omega \tau$.

This statement means that the linear stability of the synchronous state does not depend on the connection topology of $\Gamma$, that is, the in-degree $k$ and the adjacency matrix $A$ of $\Gamma$.

We show that this condition in fact depends on the connection topology by applying Theorem 1.2 (Theorem 4.7 in Subsection 4.2).

This paper is organized as follows. In Section 2, we show the “graph-like” expressions of the complex branches of the Lambert W function (Corollary 1). Section 3 is divided into two subsections. In Subsection 3.1, we prove Theorem 1.2 which gives a necessary and sufficient condition on $\alpha$, $\beta$ and $\tau$ for which all the roots of (3) have negative real parts. In Subsection 3.2, we investigate a condition on $\tau$ for each fixed $\alpha$ and $\beta$ by using Theorem 1.2 and obtain Theorems 3.2 and 3.3. In Section 4, we apply these results to the stabilization of unstable equilibrium solutions by the delayed feedback control and the stability criterion of the synchronous states of oscillator networks. In Appendix A, some lemmas are proved.

2. Lambert W function. To study an equation $ze^z = \zeta$ of $z$, Wright [16] considered the (multi-valued) inverse of a complex function $z \mapsto z + \log(z)$, where $\log$ is the principal branch of the complex logarithm. Wright claimed that this inverse function is single-valued except for the branch cut. Corless & Jeffrey [2] named this inverse the Wright omega function.

In this section, we redefine the Wright omega function and investigate the relationship between the W function and the omega function to investigate the graph-like expression of the W function. The proofs of lemmas are given in Appendix A.

We first define two multi-valued functions.

Definition 2.1. For $z \in \mathbb{C} \setminus \{0\}$, we define $\text{Arg}_W(z)$ and $\text{Log}_W(z)$ as

$$\text{Arg}_W(z) = \begin{cases} \{\pi, -\pi\} & (z = -1), \\ \{-\pi\} & (z < -1), \\ \{\text{Arg}(z)\} & \text{(otherwise)} \end{cases}$$

and $\text{Log}_W(z) = \ln|z| + i \text{Arg}_W(z)$, respectively. Here $\text{Arg}(z)$ represents the principal value of the argument of $z \in \mathbb{C} \setminus \{0\}$, i.e., $\text{Arg}(z) \in (-\pi, \pi]$ and $z = |z| \cdot e^{i \text{Arg}(z)}$. We call the multi-valued inverse $\omega(\cdot)$ of

$$z \mapsto z + \text{Log}_W(z) := \{z + w : w \in \text{Log}_W(z)\}$$

the Wright omega function.

The changes in definition of $\text{Arg}_W(z)$ are brought by the branch structure of the W function.

We define $h(z)$ as

$$h(z) = \text{Arg}_W(z) + \Im(z),$$

and by using this we introduce a family $(R_m)_{m \in \mathbb{Z}}$ of subsets of $\mathbb{C} \setminus \{0\}$ as

$$R_m = \{z \in \mathbb{C} \setminus \{0\} : h(z) \cap ((-\pi, \pi] + 2m\pi) \neq \emptyset\},$$

where $(-\pi, \pi] + 2m\pi = ((2m - 1)\pi, (2m + 1)\pi]$. By definition of $\text{Arg}_W(z)$, $R_i \cap R_j = \{-1\}$ for $(i, j) = (-1, 0)$ or $(0, -1)$, otherwise $R_i \cap R_j = \emptyset$. 


Lemma 2.2 (cf. Corless & Jeffrey [2]). For each $\zeta \in \mathbb{C}\setminus\{0\}$ and $m \in \mathbb{Z}$, we have
\[ W(\zeta) \cap R_m = \omega(\text{Log}(\zeta) + 2m\pi i). \]

Lemma 2.2 shows the omega function contains all information about the $W$ function. Next, we study the omega function. Let
\[ \gamma(x, \rho) = \sqrt{(\rho e^{-x})^2 - x^2}, \]
\[ H(\rho, x) = h(x + i\gamma(x, \rho)) \cup h(x - i\gamma(x, \rho)) \]
for $x \in \mathbb{R}$ and $\rho > 0$ satisfying $-\rho \leq xe^x \leq \rho$.

Lemma 2.3. We denote the multi-valued inverse of $H(\rho, \cdot)$: $x \mapsto H(\rho, x)$ by $G(\rho, \cdot)$. Then for each $z = x + iy$ and $\zeta = \xi + i\eta \in \mathbb{C}$, the following conditions are equivalent:
(a) $z \in \omega(\zeta)$,
(b) $x \in G(e^\xi, \eta)$ and $y = \text{sgn}(\eta) \cdot \gamma(x, e^\xi)$.

Here $\text{sgn}$ represents the signum function, that is, $\text{sgn}(\eta) = \eta/|\eta|$ for $\eta \neq 0$ and $\text{sgn}(0) = 0$.

Function $G(\rho, \cdot)$ is indeed single-valued and has monotonicity property.

Lemma 2.4. $G(\rho, \cdot)$ is a single-valued function defined on $\mathbb{R}$, which is decreasing on $[0, +\infty)$ and increasing on $(-\infty, 0]$.

Theorem 2.5. The omega function is single-valued, and $\omega(\zeta)$ is expressed as follows:
\[ \omega(\zeta) = G(e^\xi, \eta) + \text{sgn}(\eta) \cdot i\gamma(G(e^\xi, \eta), e^\xi) \quad (\zeta = \xi + i\eta \in \mathbb{C}). \]

Proof. The proof is obvious from Lemmas 2.3 and 2.4. \qed

Note that Theorem 2.5 gives a “graph-like” expression of the omega. In view of Lemma 2.2 and Theorem 2.5, the branches of the $W$ function can be defined.

Definition 2.6 (cf. Corless et al. [3]). For each $m \in \mathbb{Z}$, we call the complex function $W_m(\cdot)$ defined as
\[ \{W_m(\zeta)\} = W(\zeta) \cap R_m \quad \text{for} \quad \zeta \in \mathbb{C}\setminus\{0\}, \]
that is, $W_m(\zeta) = \omega(\text{Log}(\zeta) + 2m\pi i)$ the $m$-th branch of the $W$ function.

The following is a corollary of Theorem 2.5.

Corollary 1. For each $\zeta \in \mathbb{C}\setminus\{0\}$ and $m \in \mathbb{Z}$, $W_m(\zeta)$ is expressed as follows:
\[ W_m(\zeta) = G(|\zeta|, \text{Arg}(\zeta) + 2m\pi) + \text{sgn}(\text{Arg}(\zeta) + 2m\pi) \cdot i\gamma(G(|\zeta|, \text{Arg}(\zeta) + 2m\pi), |\zeta|). \]

By monotonicity of $G(\rho, \cdot)$, the above expression gives another proof of relations
\[ \Re(W_m(\zeta)) \geq \Re(W_{m+1}(\zeta)) \quad \text{and} \quad \Re(W_m(\zeta)) \geq \Re(W_{m-1}(\zeta)), \]
where $m \geq 0$ and $m \leq 0$ respectively, which are stated in Shinozaki & Mori [13]. Note that
\[ \Re(W_0(\zeta)) = \max_{m \in \mathbb{Z}} \Re(W_m(\zeta)) \quad (8) \]
holds in particular.
3. A necessary and sufficient condition. In this section, we find a necessary and sufficient condition on parameters for which all the roots of transcendental equation (3) have negative real parts.

The following lemma is key for this purpose.

**Lemma 3.1.** For each \( \zeta \in \mathbb{C} \setminus \{0\} \) and \( x \in \mathbb{R} \), an inequality \( \Re(W_0(\zeta)) < x \) holds if and only if

\[ (i) \quad xe^x > |\zeta|, \text{ or} \]
\[ (ii) \quad -|\zeta| < xe^x \leq |\zeta| \quad \text{and} \quad |\text{Arg}(\zeta)| > \text{Arg}(x + i\gamma(x, |\zeta|)) + \gamma(x, |\zeta|) \]

hold.

**Proof.** From Corollary 1,

\[ \Re(W_0(\zeta)) = G(|\zeta|, \text{Arg}(\zeta)) \]

holds. We have the following three cases:

(I) \( x > G(|\zeta|, 0) \),

(II) \( G(|\zeta|, \pi) < x \leq G(|\zeta|, 0) \),

(III) \( x \leq G(|\zeta|, \pi) \).

We note that

\[ \max_{\eta \in (-\pi, \pi]} G(|\zeta|, \eta) = G(|\zeta|, 0) \quad \text{and} \quad \min_{\eta \in (-\pi, \pi]} G(|\zeta|, \eta) = G(|\zeta|, \pi) \]

hold by monotonicity. Therefore,

- \( G(|\zeta|, \text{Arg}(\zeta)) < x \) trivially holds for (I),
- \( G(|\zeta|, \text{Arg}(\zeta)) < x \) does not hold for (III).

Let \( f(x) = xe^x \). We investigate the condition in (I).

**Claim 1.** An equivalence \( x > G(|\zeta|, 0) \iff xe^x > |\zeta| \) holds.

Since \( x \mapsto f(x) \) is monotonically increasing on \([-1, +\infty)\), it is enough to show that,

\[ G(|\zeta|, 0) > 0 \quad \text{and} \quad f(G(|\zeta|, 0)) = |\zeta| \]

By definition, \( H(|\zeta|, G(|\zeta|, 0)) \neq 0 \). Let \( h(\rho, x) = \text{Arg}(x + i\gamma(x, \rho)) + \gamma(x, \rho) \). From the expressions of \( H(\rho, x) \) in the proof of Lemma 2.4 in Appendix A.1, we have

\[ h(|\zeta|, G(|\zeta|, 0)) = 0. \]

Therefore, \( G(|\zeta|, 0) > 0 \) and \( \gamma(G(|\zeta|, 0), |\zeta|) = 0. \)

Next, we consider case (II).

**Claim 2.** \( x \) and \( \zeta \) satisfy (II) and \( \Re(W_0(\zeta)) < x \) if and only if the condition (ii) in this lemma holds.

(Only-if-part). (II) holds if and only if there is \( \eta \in [0, \pi) \) such that \( x = G(|\zeta|, \pm \eta) \), which implies \( h(|\zeta|, x) = \eta \) because \( \eta \geq 0 \). Thus, for \( \text{Arg}(\zeta) \neq 0 \) we have

\[ G(|\zeta|, \text{Arg}(\zeta)) < x \iff \begin{cases} \text{Arg}(\zeta) > \eta & (\text{Arg}(\zeta) > 0) \\ \text{Arg}(\zeta) < -\eta & (\text{Arg}(\zeta) < 0). \end{cases} \]

(If-part). We should reverse the argument.

This completes the proof. \( \square \)
3.1. **Condition on \( \alpha, \beta \) and \( \tau \).** To study (3), we only have to consider the case \( \beta \neq 0 \). Denote by \( \text{Arccos} \) the inverse of cosine which is restricted on \([0, \pi]\). Let

\[ Z(x) = \text{Arccos}(\cos(x)). \]

Then \( x \mapsto Z(x) \) is a \( 2\pi \)-periodic real function which satisfies

\[ Z(x) = |x| \quad \text{for all} \quad x \in [-\pi, \pi]. \]

**Proof of Theorem 1.2.** From the representation of the set of roots of (3) by the \( W \) function, all the roots of (3) have negative real parts if and only if \( \alpha, \beta, \) and \( \tau \) satisfy

\[ \Re(W_0(\beta \tau e^{\alpha \tau})) < \Re(\alpha) \tau \]

because \( W_0 \) is the right-most branch from (8). The conclusion is obtained from Lemma 3.1.

3.2. **Condition on \( \tau \).** For each \( \alpha \) and \( \beta \), let

\[ I(\alpha, \beta) = \{ \tau > 0 : \text{all the roots of (3) have negative real parts} \}. \]

The following questions arise.

**Question 1.** (Q1) *What is the condition for which \( I(\alpha, \beta) \) is nonempty?* (Q2) *How is \( I(\alpha, \beta) \) expressed as a function of \( \alpha \) and \( \beta \)?*

The next corollary is a direct consequence of Theorem 1.2.

**Corollary 2** (Theorem 2.9 (i) (1), (iii), and (ii) (1) in Wei & Zhang [15]). *If \( \alpha \) and \( \beta \) satisfy \( |\Re(\alpha)| \geq |\beta| \), then

\[ I(\alpha, \beta) = \begin{cases} (0, +\infty) & (\Re(\alpha) > |\beta|), \\ \{ \tau > 0 : \Im(\alpha)\tau + \text{Arg}(\beta) \notin 2\pi\Z \} & (\Re(\alpha) = |\beta|), \\ \emptyset & (\Re(\alpha) \leq -|\beta|) \end{cases} \]

hold.*

From Corollary 2, we only have to consider the case \( |\Re(\alpha)| < |\beta| \). The case for \( \Im(\alpha) = 0 \) is not difficult.

**Corollary 3** (Theorem 2 in Matsunaga [11], cf. Theorem 2.9 (i) (2) and (ii) (1) in [15]). *Assume \( |\Re(\alpha)| < |\beta| \). If \( \Im(\alpha) = 0 \), then \( I(\alpha, \beta) \neq \emptyset \) if and only if \( \alpha \) and \( \beta \) satisfy

\[ \text{Arg}(\beta) \neq 0 \quad \text{and} \quad \alpha > \Re(\beta). \]

*The expression of \( I(\alpha, \beta) \) is

\[ I(\alpha, \beta) = \left( 0, \frac{|\text{Arg}(\beta)| - \text{Arccos}(\alpha/|\beta|)}{\sqrt{|\beta|^2 - \alpha^2}} \right). \]

**Proof.** By assumption, inequality (4) becomes

\[ Z(\text{Arg}(\beta)) > \text{Arccos}(\alpha/|\beta|) + \tau \sqrt{|\beta|^2 - \alpha^2}. \]

Therefore, \( I(\alpha, \beta) \neq \emptyset \) if and only if \( Z(\text{Arg}(\beta)) > \text{Arccos}(\alpha/|\beta|) \). Since

\[ Z(\text{Arg}(\beta)) = |\text{Arg}(\beta)| = \text{Arccos}(\Re(\beta)/|\beta|), \]

the condition is obtained.
3.2.1. Statements of theorems. We hereafter assume that $\alpha$ and $\beta$ satisfy
$$|\Re(\alpha)| < |\beta| \text{ and } \Im(\alpha) \neq 0.$$ 
In the following, we consider two cases (i) $|\alpha| \leq |\beta|$ and (ii) $|\alpha| > |\beta|$. Note that
$$I(\alpha, \beta) = I(\bar{\alpha}, \bar{\beta})$$
holds since $\lambda = \lambda_0$ is a root of (3) if and only if $\lambda = \bar{\lambda}_0$ is a root of $\lambda + \bar{\alpha} - \bar{\beta}e^{-\lambda\tau} = 0$. Therefore, it is enough to consider the case
$$\Im(\alpha) > 0.$$

**Remark 1.** $\Arg(\bar{\alpha}) = -\Arg(\alpha)$ always holds because $\alpha$ is not real, but $\Arg(\bar{\beta}) = \Arg(\beta)$ if $\beta$ is a negative real number.

We introduce some notation:
$$\sigma^\pm(\alpha, \beta) = \Arg(\beta) \mp \Arccos(\Re(\alpha)/|\beta|),$$
$$\omega^\pm(\alpha, \beta) = \pm \sqrt{|\beta|^2 - \Re(\alpha)^2 - \Im(\alpha)},$$
$$E(\rho, x) = \frac{\pi - \Arccos(x/\rho)}{\sqrt{\rho^2 - x^2}} \quad \text{for } \rho > 0,$$
$$\tau_j(\alpha, \beta) = \frac{j\pi - \Arg(\beta)}{\Im(\alpha)} \quad \text{for } j \in \mathbb{Z}.$$

**Theorem 3.2.** Assume $|\Re(\alpha)| < |\beta|$ and $\Im(\alpha) > 0$. If $|\alpha| \leq |\beta|$, then $I(\alpha, \beta) \neq \emptyset$ if and only if
$$\Arg(\beta) \neq 0 \text{ and } \Re(\alpha) > \Re(\beta).$$
In this case,
$$I(\alpha, \beta) = \begin{cases} 
0, \frac{\sigma^-(\alpha, \beta) - 2\pi}{\omega^-(\alpha, \beta)} & (\Arg(\beta) > 0 \text{ and } \tau_1(\alpha, \beta) < E(|\beta|, \Re(\alpha))), \\
0, \frac{\sigma^+(\alpha, \beta)}{\omega^+(\alpha, \beta)} & (\Arg(\beta) > 0 \text{ and } \tau_1(\alpha, \beta) \geq E(|\beta|, \Re(\alpha))), \\
0, \frac{\sigma^-(-\alpha, \beta)}{\omega^-(\alpha, \beta)} & (\Arg(\beta) < 0).
\end{cases}$$

**Theorem 3.3.** Assume $|\Re(\alpha)| < |\beta|$ and $\Im(\alpha) > 0$. If $|\alpha| > |\beta|$, then $I(\alpha, \beta) \neq \emptyset$ if and only if $\alpha$ and $\beta$ satisfy
(i) $-\pi < \Arg(\beta) < 0$ and $\Re(\alpha) > \Re(\beta)$, or
(ii) There exists a positive odd number $N = 2m + 1$ such that
$$\tau_N(\alpha, \beta) < E(|\beta|, \Re(\alpha)).$$

$I(\alpha, \beta)$ is expressed as follows:
(I) Case: (i) holds but (ii) does not hold.
$$I(\alpha, \beta) = \left(0, \frac{\sigma^-(-\alpha, \beta)}{\omega^-(\alpha, \beta)}\right).$$

(II) Case: (i) and (ii) hold.
$$I(\alpha, \beta) = \left(0, \frac{\sigma^-(-\alpha, \beta)}{\omega^-(\alpha, \beta)}\right) \cup \bigcup_{\ell=0}^m \left(\frac{\sigma^+(\alpha, \beta) - 2\ell\pi}{\omega^+(\alpha, \beta)}, \frac{\sigma^-(-\alpha, \beta) - 2(\ell + 1)\pi}{\omega^-(\alpha, \beta)}\right).$$
(III) Case: (i) does not hold but (ii) holds. If \( \sigma^+(\alpha, \beta) \geq 0 \) or \( \text{Arg}(\beta) = \pi \), then

\[
I(\alpha, \beta) = \left( 0, \frac{\sigma^-(\alpha, \beta) - 2\pi}{\omega^-(\alpha, \beta)} \right) \cup \bigcup_{\ell = 1}^{m} \left( \frac{\sigma^+(\alpha, \beta) - 2\ell\pi}{\omega^+(\alpha, \beta)}, \frac{\sigma^-(\alpha, \beta) - 2(\ell + 1)\pi}{\omega^-(\alpha, \beta)} \right).
\]

If \( \text{Arg}(\beta) < \pi \) and \( \sigma^+(\alpha, \beta) < 0 \), then

\[
I(\alpha, \beta) = \bigcup_{\ell = 0}^{m} \left( \frac{\sigma^+(\alpha, \beta) - 2\ell\pi}{\omega^+(\alpha, \beta)}, \frac{\sigma^-(\alpha, \beta) - 2(\ell + 1)\pi}{\omega^-(\alpha, \beta)} \right).
\]

**Remark 2.** In the statement of Theorem 3.3, we suppose that \( N \) satisfies

\[
\tau_N(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \leq \tau_{N+2}(\alpha, \beta).
\]

If \( \tau_N(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \) holds for all \( N \), then we can consider the above \( m \) to be infinity.

**Remark 3.** Wei & Zhang [15] investigated equation (3) with \( \alpha, \beta \in \mathbb{C} \) by using the so-called \( \tau \)-decomposition method, which is widely used for the investigation of transcendental equations containing a delay parameter \( \tau > 0 \). However, one can not say that [15] answered Question 1 by the following reasons:

(i) the explicit expression of \( I(\alpha, \beta) \) is not found,
(ii) the condition on \( \alpha \) and \( \beta \) for which \( I(\alpha, \beta) \neq \emptyset \) is missing when \( \alpha \) and \( \beta \) satisfy \(|\alpha| > |\beta|\).

3.2.2. **Proofs of theorems.** From Theorem 1.2, on the assumption of \(|\Re(\alpha)| < |\beta|\)

\[
I(\alpha, \beta) = \{ \tau : \text{inequality (4) holds} \}.
\]

For each \( \ell \in \mathbb{Z} \), there are two cases for inequality (4)

- \( 3(\alpha)\tau + \text{Arg}(\beta) \in \left[-\pi, 0\right] + 2\ell\pi \), i.e., \( \tau_{2\ell - 1}(\alpha, \beta) \leq \tau \leq \tau_{2\ell}(\alpha, \beta) \),
- \( 3(\alpha)\tau + \text{Arg}(\beta) \in \left[0, \pi\right] + 2\ell\pi \), i.e., \( \tau_{2\ell}(\alpha, \beta) \leq \tau \leq \tau_{2\ell+1}(\alpha, \beta) \).

We define \( I^\pm(\alpha, \beta) \) as

\[
I^-(\alpha, \beta) = I(\alpha, \beta) \cap [\tau_{2\ell-1}(\alpha, \beta), \tau_{2\ell}(\alpha, \beta)],
\]

\[
I^+(\alpha, \beta) = I(\alpha, \beta) \cap [\tau_{2\ell}(\alpha, \beta), \tau_{2\ell+1}(\alpha, \beta)].
\]

Note that \( I^\pm(\alpha, \beta) \) are empty for \( \ell \leq -1 \) because \( \tau_j(\alpha, \beta) < 0 \) for \( j \leq -1 \). Therefore, \( I(\alpha, \beta) \) is decomposed into

\[
I(\alpha, \beta) = \bigcup_{\ell \geq 0} (I^-_\ell(\alpha, \beta) \cup I^+_\ell(\alpha, \beta)),
\]

where

\[
I^-\ell(\alpha, \beta) = \{ \tau > 0 : \omega^-(\alpha, \beta)\tau > \sigma^-(\alpha, \beta) - 2\ell\pi \} \cap [\tau_{2\ell-1}(\alpha, \beta), \tau_{2\ell}(\alpha, \beta)],
\]

\[
I^+_\ell(\alpha, \beta) = \{ \tau > 0 : \omega^+(\alpha, \beta)\tau < \sigma^+(\alpha, \beta) - 2\ell\pi \} \cap [\tau_{2\ell}(\alpha, \beta), \tau_{2\ell+1}(\alpha, \beta)],
\]

respectively since \( Z(x) = |x - 2\ell\pi| \) if \( x \in [-\pi, \pi] + 2\ell\pi \).

The proofs of Theorems 3.2 and 3.3 are obtained by studying \( I^\pm(\alpha, \beta) \), whose properties are investigated in the following lemmas.

**Lemma 3.4.** Assume \(|\Re(\alpha)| < |\beta|\) and \( 3(\alpha) > 0 \). Then

1. \( I_0^-(\alpha, \beta) \neq \emptyset \) if and only if \( -\pi < \text{Arg}(\beta) < 0 \) and \( \Re(\alpha) > \Re(\beta) \). In this case, \( I_0^-(\alpha, \beta) = (0, \sigma^-(\alpha, \beta)/\omega^-(\alpha, \beta)) \).
2. for all \( \ell \geq 1 \), \( I^{-}_I(\alpha, \beta) \neq \emptyset \) if and only if \( \tau_{2\ell-1}(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \). In this case,

- for \( \ell = 1 \),
  \[
  I^{-}_I(\alpha, \beta) = \begin{cases} (0, (\sigma^-(-\alpha, \beta) - 2\pi)/\omega^-(\alpha, \beta)) & (\Arg(\beta) = \pi), \\ [\tau_1(\alpha, \beta), (\sigma^-(-\alpha, \beta) - 2\pi)/\omega^-(\alpha, \beta)) & (\Arg(\beta) < \pi), \\ \end{cases}
  \]
- for \( \ell \geq 2 \),
  \[
  I^{-}_I(\alpha, \beta) = [\tau_{2\ell-1}(\alpha, \beta), (\sigma^-(-\alpha, \beta) - 2\ell\pi)/\omega^-(\alpha, \beta))].
  \]

Remark 4. In this lemma, \( \omega^-(\alpha, \beta) < 0 \) because \( \Im(\alpha) > 0 \).

Lemma 3.5. Assume \( |\Re(\alpha)| < |\beta| \) and \( \Im(\alpha) > 0 \). If \( \omega^+(\alpha, \beta) \geq 0 \), then

1. \( I^+_I(\alpha, \beta) = \emptyset \) for all \( \ell \geq 1 \),
2. \( I^+_0(\alpha, \beta) \neq \emptyset \) if and only if \( 0 < \Arg(\beta) < \pi \) and \( \Re(\alpha) > \Re(\beta) \). In this case,

\[
I^+_0(\alpha, \beta) = \{(0, \sigma^+(\alpha, \beta)/\omega^+(\alpha, \beta)) : (\tau_1(\alpha, \beta) = E(|\beta|, \Re(\alpha)) \}
\]

hold.

Lemma 3.6. Assume \( |\Re(\alpha)| < |\beta| \) and \( \Im(\alpha) > 0 \). If \( \omega^+(\alpha, \beta) < 0 \), then

1. \( I^+_0(\alpha, \beta) \neq \emptyset \) if and only if \( \Arg(\beta) < \pi \) and \( \alpha, \beta \) satisfy
   - \( \sigma^+(\alpha, \beta) \geq 0 \), or
   - \( \sigma^+(\alpha, \beta) < 0 \) and \( \tau_1(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \),

where

\[
I^+_0(\alpha, \beta) = \begin{cases} (0, \tau_1(\alpha, \beta)) & (\sigma^+(\alpha, \beta) \geq 0), \\ (\sigma^+(\alpha, \beta)/\omega^+(\alpha, \beta), \tau_1(\alpha, \beta)) & (\sigma^+(\alpha, \beta) < 0). \\ \end{cases}
\]

2. for all \( \ell \geq 1 \), \( I^+_I(\alpha, \beta) \neq \emptyset \) if and only if \( \tau_{2\ell+1}(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \). In this case,

\[
I^+_I(\alpha, \beta) = [\sigma^+(\alpha, \beta) - 2\ell\pi)/\omega^+(\alpha, \beta), \tau_{2\ell+1}(\alpha, \beta)]
\]

hold.

Now we prove Theorems 3.2 and 3.3 by using the above lemmas. The theorems are connected with the lemmas by the following propositions.

Proposition 1. Suppose \( |\Re(\alpha)| < |\beta| \) and \( \Im(\alpha) > 0 \). Then we have equivalences

- (i) \( \omega^+(\alpha, \beta) > 0 \) \iff \( |\alpha| < |\beta| \),
- (ii) \( \omega^+(\alpha, \beta) = 0 \) \iff \( |\alpha| = |\beta| \),
- (iii) \( \omega^+(\alpha, \beta) < 0 \) \iff \( |\alpha| > |\beta| \).

Proof. It is enough to prove (i). \( \omega^+(\alpha, \beta) > 0 \) is equivalent to \( |\beta|^2 - \Re(\alpha)^2 > \Im(\alpha)^2 \) because \( \Im(\alpha) > 0 \).

Proposition 2. The following equivalences hold:

- (i) \( \sigma^+(\alpha, \beta) > 0 \) \iff \( \Arg(\beta) > 0 \) and \( \Re(\alpha) > \Re(\beta) \),
- (ii) \( \sigma^+(\alpha, \beta) = 0 \) \iff \( \Arg(\beta) > 0 \) and \( \Re(\alpha) = \Re(\beta) \),
- (iii) \( \sigma^+(\alpha, \beta) < 0 \) \iff \( \Arg(\beta) \leq 0 \), or \( \Arg(\beta) > 0 \) and \( \Re(\alpha) < \Re(\beta) \).

Proof. The equivalences are obtained from

\[
\Arg(\beta) = \Arccos(\Re(\beta)/|\beta|)
\]

for \( \Arg(\beta) > 0 \).
Proof of Theorem 3.2. Theorem 3.2 corresponds to Lemmas 3.4 and 3.5 from the above proposition, and \( I(\alpha, \beta) \neq \emptyset \) if and only if
\[
\begin{aligned}
1. & \quad I_0^-(\alpha, \beta) \neq \emptyset, \\
2. & \quad I_0^+(\alpha, \beta) \neq \emptyset, \\
3. & \quad I_\ell^-(\alpha, \beta) \neq \emptyset \text{ for some } \ell \geq 1.
\end{aligned}
\]
For these conditions, we have
\[
\begin{aligned}
1. & \quad \Leftrightarrow -\pi < \text{Arg}(\beta) < 0 \text{ and } \Re(\alpha) > \Re(\beta), \\
2. & \quad \Leftrightarrow 0 < \text{Arg}(\beta) < \pi \text{ and } \Re(\alpha) > \Re(\beta), \\
3. & \quad \Leftrightarrow \tau_{2\ell-1}(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \quad (\ell \geq 1).
\end{aligned}
\]

Now we find the equivalent condition.
(If-part). Suppose \( \text{Arg}(\beta) \neq 0 \) and \( \Re(\alpha) > \Re(\beta) \). When \( \text{Arg}(\beta) = \pi \), \( \tau_1(\alpha, \beta) = 0 \). Therefore, \( I_1^- (\alpha, \beta) \neq \emptyset \). When \( \text{Arg}(\beta) < \pi \), \( I_0^- (\alpha, \beta) \neq \emptyset \) or \( I_0^+ (\alpha, \beta) \neq \emptyset \).
(Only-if-part). The following claims are used.

\text{Claim 3.} If \(|\alpha| \leq |\beta|\), then \( \tau_{2\ell-1}(\alpha, \beta) \geq E(|\beta|, \Re(\alpha)) \) holds for each \( \ell \geq 2 \).

Note that one can consider \( E(|\beta|, \Re(\alpha)) \) to be a function of \( \Re(\alpha) \) for fixed \( \beta \) and \( \Im(\alpha) \) satisfying \(|\alpha| \leq |\beta|\), where \( \Re(\alpha) \) satisfies \(|\Re(\alpha)| \leq \sqrt{|\beta|^2 - \Im(\alpha)^2}\). Since \( E(\rho, \cdot) : x \mapsto E(\rho, x) \) is monotonically increasing,

\[
E(|\beta|, \Re(\alpha)) \leq E(|\beta|, \sqrt{|\beta|^2 - \Im(\alpha)^2})
\]

holds. We investigate an inequality \( \tau_{2\ell-1}(\alpha, \beta) \geq E(|\beta|, \sqrt{|\beta|^2 - \Im(\alpha)^2}) \). A simple calculation shows that this is equivalent to

\[
\text{Arccos}\left(\sqrt{|\beta|^2 - \Im(\alpha)^2}/|\beta|\right) \geq \text{Arg}(\beta) - 2(\ell - 1)\pi,
\]

where the right hand side is negative. Therefore, the inequality holds.

\text{Claim 4.} If \(|\alpha| \leq |\beta|\), then \( \text{Arg}(\beta) > 0 \) and \( \tau_1(\alpha, \beta) < E(|\beta|, \Re(\alpha)) \) imply \( \Re(\alpha) > \Re(\beta) \).

By using

\[
\tau_1(\alpha, \beta) = \frac{\pi - \text{Arg}(\beta)}{\Im(\alpha)} \geq \frac{\pi - \text{Arg}(\beta)}{\sqrt{|\beta|^2 - \Re(\alpha)^2}},
\]

we get \( \sigma^+(\alpha, \beta) > 0 \), which is equivalent to the condition from Proposition 2.

This completes the proof of the first part. The second part about the expression of \( I(\alpha, \beta) \) is easily obtained from Lemmas 3.4 and 3.5.

\text{Proof of Theorem 3.3.} Theorem 3.3 corresponds to Lemmas 3.4 and 3.6, and \( I(\alpha, \beta) \neq \emptyset \) if and only if
\[
\begin{aligned}
1. & \quad I_0^- (\alpha, \beta) \neq \emptyset, \\
2. & \quad I_0^+(\alpha, \beta) \neq \emptyset, \\
3. & \quad I_\ell^- (\alpha, \beta) \neq \emptyset \text{ for some } \ell \geq 1, \text{ or} \\
4. & \quad I_\ell^+ (\alpha, \beta) \neq \emptyset \text{ for some } \ell \geq 1.
\end{aligned}
\]
For these conditions, we have

1. \( \iff -\pi < \text{Arg}(\beta) < 0 \) and \( \Re(\alpha) > \Re(\beta) \),

2. \( \iff \text{Arg}(\beta) < \pi \), and \( \left\{ \begin{array}{c} \sigma^+(\alpha, \beta) \geq 0, \\
\sigma^+(\alpha, \beta) < 0 \end{array} \right\} \), or \( \tau_1(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \),

3. \( \iff \tau_{2\ell-1}(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \) (\( \ell \geq 1 \)),

4. \( \iff \tau_{2\ell+1}(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \) (\( \ell \geq 1 \)).

(If-part). Suppose (i) or (ii) hold in Theorem 3.3. When (i) holds, \( I_0^- (\alpha, \beta) \neq \emptyset \). Assume (ii) holds. If \( \tau_1(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \), then \( I_1^+ (\alpha, \beta) \neq \emptyset \). If \( \tau_N(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \) for \( N = 2m + 1 \geq 1 \), then \( I_n^+(\alpha, \beta) \) and \( I_{n+1}^-(\alpha, \beta) \) are nonempty.

(Only-if-part). We use the following claim.

Claim 5. If \( |\alpha| > |\beta| \), then \( \sigma^+(\alpha, \beta) \geq 0 \) implies \( \tau_1(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \).

By assumption, we have \( \Im(\alpha) > |\beta|^2 - \Re(\alpha)^2 \), and an inequality

\[
\text{E}(|\beta|, \Re(\alpha)) > \frac{\pi - \text{Arccos}(\Re(\beta)/|\beta|)}{\Im(\alpha)} = \tau_1(\alpha, \beta)
\]

holds from \( \text{Arg}(\beta) > 0 \) and \( \Re(\alpha) \geq \Re(\beta) \).

This completes the proof of the necessary and sufficient condition.

Next, we prove the expressions of \( I(\alpha, \beta) \). (i) is equivalent to \( I_0^- (\alpha, \beta) \neq \emptyset \), and

\[
I_0^- (\alpha, \beta) = \left( 0, \frac{\sigma^-(\alpha, \beta)}{\omega^-(\alpha, \beta)} \right).
\]

(ii) is equivalent to the following conditions:

- \( I_{\ell}^- (\alpha, \beta) \neq \emptyset \) for \( 1 \leq \ell \leq m + 1 \),
- \( I_{\ell}^+ (\alpha, \beta) \neq \emptyset \) for \( 1 \leq \ell \leq m \).

(I) \( \sigma^+(\alpha, \beta) < 0 \) from Proposition 2, but \( \tau_1(\alpha, \beta) < \text{E}(|\beta|, \Re(\alpha)) \) does not hold. Therefore, \( I_0^+ (\alpha, \beta) = \emptyset \), and

\[
I(\alpha, \beta) = I_0^- (\alpha, \beta).
\]

(II) \( I_0^+ (\alpha, \beta) \neq \emptyset \) since \( \sigma^+(\alpha, \beta) < 0 \), and

\[
I(\alpha, \beta) = I_0^- (\alpha, \beta) \cup \bigcup_{\ell=0}^{m} (I_{\ell}^- (\alpha, \beta) \cup I_{\ell+1}^+ (\alpha, \beta)),
\]

where

\[
I_{\ell}^- (\alpha, \beta) \cup I_{\ell+1}^+ (\alpha, \beta) = \left( \frac{\sigma^+(\alpha, \beta) - 2\ell\pi}{\omega^+(\alpha, \beta)} - \frac{\sigma^-(\alpha, \beta) - 2(\ell + 1)\pi}{\omega^-(\alpha, \beta)} \right)
\]

because

\[
I_{\ell}^- (\alpha, \beta) = \left[ \tau_{2\ell-1}(\alpha, \beta), \frac{\sigma^-(\alpha, \beta) - 2\ell\pi}{\omega^-(\alpha, \beta)} \right] \quad (1 \leq \ell \leq m + 1),
\]

\[
I_{\ell}^+ (\alpha, \beta) = \left( \frac{\sigma^+(\alpha, \beta) - 2\ell\pi}{\omega^+(\alpha, \beta)}, \tau_{2\ell+1}(\alpha, \beta) \right) \quad (0 \leq \ell \leq m).
\]

(III) If \( \text{Arg}(\beta) = \pi \), then \( I_0^+ (\alpha, \beta) = \emptyset \). Otherwise, \( I_0^+ (\alpha, \beta) \neq \emptyset \). By combining the expressions of \( I_{\ell}^- (\alpha, \beta) \) (\( 1 \leq \ell \leq m \)) and \( I_{\ell}^+ (\alpha, \beta) \) (\( 0 \leq \ell \leq m \)), the expression of \( I(\alpha, \beta) \) is obtained.

This completes the proof. \( \square \)
4. Applications. In this section, we apply the results obtained in Section 3 to the delayed feedback control and the stability of synchronous states in oscillator networks.

4.1. Delayed feedback control. Assume that the given ODE $x'(t) = f(x(t))$ has an equilibrium solution $x^*(t) \equiv 0 \in \mathbb{R}^n$, which is also an equilibrium solution of (5). The linearization of (5) along $x^*(t)$ gives the characteristic equation

$$\det(\lambda I - \{M + (e^{-\lambda \tau} - 1)K\}) = 0 \quad (M = Df(0)).$$

(10)

In the following, we see how (10) can be reduced to a transcendental equation (3).

4.1.1. Reduction to transcendental equation. We take an eigenvalue $\mu$ of $M$ and suppose that $\mu$ satisfies one of the following conditions:

(E1) $\mu$ is not real,
(E2) $\mu$ is a real eigenvalue whose algebraic multiplicity is even.

Let $E_\mu$ be the sum $\tilde{E}(\mu) + \tilde{E}(\bar{\mu})$ of generalized eigenspaces $\tilde{E}(\mu)$ and $\tilde{E}(\bar{\mu})$ associated to $\mu$ and $\bar{\mu}$, respectively. Then the generalized eigenspace decomposition gives

$$\mathbb{C}^n = E_\mu \oplus V_0,$$

where $V_0$ is the sum of the generalized eigenspaces associated to the other eigenvalues. Let $2m = \dim(E_\mu)$. Note that we can take a real basis of $E_\mu$ in both cases (E1) and (E2).

We choose an $n \times n$ real matrix $K$ so that

$$\sigma(K|_{E_\mu}) = \{\kappa, \bar{\kappa}\} \quad \text{for some } \kappa \in \mathbb{C} \text{ and } \sigma(K|_{V_0}) = 0,$$

and the characteristic equation (10) becomes

$$\det(\lambda I_{|V_0} - M_{|V_0}) \cdot [\lambda - \{\mu + (e^{-\lambda \tau} - 1)\kappa\}]^m \cdot [\lambda - \{\bar{\mu} + (e^{-\lambda \tau} - 1)\bar{\kappa}\}]^m = 0,$$

where $I_{|V_0}$ is the identity transformation on $V_0$. We denote by $\mathcal{K}_\mu$ the set of such $K$ having the above properties. Here $\sigma(T)$ represents the set of eigenvalues of a linear transformation $T$.

Lastly, (10) is reduced to

$$\lambda - \{\mu + (e^{-\lambda \tau} - 1)\kappa\} = 0,$$

which is an equation (3) with

$$\alpha = \alpha(\kappa) := -\mu + \kappa \text{ and } \beta = \beta(\kappa) := \kappa$$

(12)

if $\mu$ and $\bar{\mu}$ are only eigenvalues satisfying $\Re(\mu) \geq 0$.

**Theorem 4.1.** Suppose that the set of the eigenvalues of the equilibrium 0 whose real part is nonnegative is

$$\{\mu, \bar{\mu}\} \quad \text{for some } \mu \in \mathbb{C}.$$

Then the following statements hold:

(i) If $\mu$ satisfies (E1), then there exist $K \in \mathcal{K}_\mu$ and $\tau > 0$ such that $x^*(t)$ is an exponentially stable solution of (5),

(ii) If $\mu$ satisfies (E2), then $x^*(t)$ is an unstable solution of (5) for any $K \in \mathcal{K}_\mu$ and $\tau > 0$. 

Remark 5. As mentioned in Kokame et al. [10], if $M$ has an odd number of positive eigenvalues counted with multiplicity, then (10) has a positive root for any $K \in \mathbb{R}^{n \times n}$ and $\tau > 0$ because
\[
\det \left( \lambda I - \{ M + (e^{-\lambda \tau} - 1)K \} \right)_{\lambda=0} = \det(-M) \geq 0
\]
holds.

4.1.2. Proof of Theorem 4.1. Since (11) has a root $\lambda = 0$ if $\mu = 0$, we suppose $\mu \neq 0$. Without loss of generality, we may assume $\Im(\mu) \geq 0$. Theorem 4.1 are proven by Theorems 4.3, 4.5, and 4.6 which are stated and proved in the following argument.

Let $I(\kappa) = I(\alpha(\kappa), \beta(\kappa))$. From Corollary 2, the following statements hold:

- Since $\Re(\alpha(\kappa)) \leq \Re(\beta(\kappa)) \leq |\beta(\kappa)|$, there are no $\kappa \in \mathbb{C}$ such that $I(\kappa) = (0, +\infty)$.
- If $\Re(\mu) = 0$ and $\kappa > 0$, then
\[
I(\kappa) = \{ \tau > 0 : \Im(\mu)\tau \notin 2\pi\mathbb{Z} \}
\]
because this condition is equivalent to $\Re(\alpha(\kappa)) = |\beta(\kappa)|$.

We next investigate conditions in Theorem 3.3 since the condition
\[
\arg(\beta) \neq 0 \text{ and } \Re(\alpha) > \Re(\beta)
\]
in Corollary 3 and Theorem 3.2 does not hold for $\alpha = \alpha(\kappa)$ and $\beta = \beta(\kappa)$. Let $\sigma^\pm(\kappa) = \sigma^\pm(\alpha(\kappa), \beta(\kappa))$ and $\omega^\pm(\kappa) = \omega^\pm(\alpha(\kappa), \beta(\kappa))$.

\[
\sigma^\pm(\kappa) = \arg(\kappa) \mp \arccos(\Re(-\mu + \kappa)/|\kappa|),
\]
\[
\omega^\pm(\kappa) = \pm \sqrt{|\kappa|^2 - \Re(-\mu + \kappa)^2 - \Im(-\mu + \kappa)}.
\]

We define a subset $D$ of $\mathbb{C} \setminus \{0\}$ as
\[
D = \{ \kappa \in \mathbb{C} \setminus \{0\} : |\Re(\alpha(\kappa))| < |\beta(\kappa)|, \Im(\alpha(\kappa)) \neq 0 \text{ and } |\alpha(\kappa)| > |\beta(\kappa)| \}.
\]
In the following propositions, we study the above inequalities. Let
\[
r_0(\theta) = \frac{\Re(\mu)}{1 + \cos \theta} \quad \text{for } \theta \in (-\pi, \pi).
\]

Proposition 3. $|\Re(\alpha(\kappa))| < |\beta(\kappa)|$ is equivalent to

(i) $\kappa \notin \mathbb{R}$ if $\Re(\mu) = 0$, and
(ii) $\kappa \notin (-\infty, 0)$ and $|\kappa| > r_0(\arg(\kappa))$ if $\Re(\mu) > 0$.

Proof. If $\Re(\mu) = 0$, then the inequality becomes $-|\kappa| < \Re(\kappa) < |\kappa|$. When $\Re(\mu) > 0$, an inequality $-\Re(\mu) + \Re(\kappa) < |\kappa|$ always holds. Therefore, the inequality is equivalent to $\Re(\mu) < |\kappa|(1 + \cos \arg(\kappa))$. \qed

Let
\[
r_1(\theta) = \begin{cases} 
|\mu| & \text{if } \cos(\theta - \arg(\mu)) > 0, \\
\frac{|\mu|}{2 \cos(\theta - \arg(\mu))} & \text{if } \cos(\theta - \arg(\mu)) \leq 0.
\end{cases}
\]
$\bar{\mu} \cdot \bar{\kappa}$ denotes the standard inner product between two dimensional vectors
\[
\bar{\mu} = (\Re(\mu), \Im(\mu)) \quad \text{and} \quad \bar{\kappa} = (\Re(\kappa), \Im(\kappa)).
\]

Proposition 4. $|\alpha(\kappa)| > |\beta(\kappa)|$ is equivalent to $\bar{\mu} \cdot \bar{\kappa} < |\mu|^2/2$, which is also equivalent to $|\kappa| < r_1(\theta)$.

Proof. We use an identity $\mu \bar{\kappa} + \bar{\mu} \kappa = 2(\bar{\mu} \cdot \bar{\kappa})$. From this, we obtain the first inequality. The second inequality is obtained from the trigonometric identity. \qed
We first consider the case \( \Re(\mu) = 0 \).

**Lemma 4.2.** If \( \Re(\mu) = 0 \), then
\[
D = \{ \kappa \in \mathbb{C} \setminus \{0\} : \Im(\kappa) \neq 0 \text{ and } \Im(\kappa) < \Im(\mu)/2 \}.
\]
Furthermore, \( \Im(\alpha(\kappa)) < 0 \) for all \( \kappa \in D \).

**Proof.** Since \( \arg(\mu) = \pi/2 \),
\[
r_1(\arg(\kappa)) = \frac{\Im(\mu)}{2 \sin \arg(\kappa)} \quad \text{and} \quad \cos(\arg(\kappa) - \arg(\mu)) = \sin \arg(\kappa).
\]
From Propositions 3 and 4, the expression of \( D \) is obtained. \( \Im(\alpha(\kappa)) < 0 \) is clear. \( \square \)

**Theorem 4.3.** If \( \Re(\mu) = 0 \) and \( \Im(\mu) > 0 \), then all the roots of (11) have negative real parts if and only if one of the following conditions hold:

(i) \( \kappa > 0 \) and \( \Im(\mu) \tau \notin 2\pi \mathbb{Z} \),

(ii) for some \( N = 2m + 1 \geq 1 \),
\[
0 < \Im(\kappa) < \frac{\Im(\mu)}{2(m + 1)} \cdot \frac{\pi - \arg(\kappa)}{\pi} \quad \text{and} \quad \tau \in \bigcup_{\ell=0}^{m} \left( \frac{2 \arg(\kappa) + 2\ell \pi}{\Im(\mu) - 2\Im(\kappa)}, \frac{2(\ell + 1)\pi}{\Im(\mu) - 2\Im(\kappa)} \right),
\]

(iii) for some \( N = 2m + 1 \geq 1 \),
\[
0 > \Im(\kappa) > -\frac{\Im(\mu)}{2m} \cdot \frac{\pi + \arg(\kappa)}{\pi} \quad \text{and} \quad \tau \in \bigcup_{\ell=0}^{m} \left( \frac{2\ell \pi}{\Im(\mu)}, \frac{2 \arg(\kappa) + 2(\ell + 1)\pi}{\Im(\mu) - 2\Im(\kappa)} \right).
\]

**Proof.** (i) follows from (13).

We suppose \( \kappa \in D \). Since \( \Im(\alpha(\kappa)) < 0 \) from Lemma 4.2, we apply Theorem 3.3 as \( \alpha = \tilde{\alpha}(\kappa) \) and \( \beta = \tilde{\beta}(\kappa) \), where \( \alpha(\kappa) \) and \( \beta(\kappa) \) are abbreviated as \( \tilde{\alpha}(\kappa) \) and \( \tilde{\beta}(\kappa) \), respectively. We investigate an inequality
\[
\tau_N(\tilde{\alpha}(\kappa), \tilde{\beta}(\kappa)) < E(|\tilde{\beta}(\kappa)|, \Re(\tilde{\alpha}(\kappa)), \quad (14)
\]
which is inequality (9) with \( \alpha = \tilde{\alpha}(\kappa), \beta = \tilde{\beta}(\kappa) \). Note that \( \arg(\kappa) = \arg(\mu) \).

We only consider the case \( \Im(\kappa) > 0 \) since the case \( \Im(\kappa) < 0 \) is also proved in the same way. Then (14) becomes
\[
\frac{N \pi + \arg(\mu)}{\Im(\mu) - \Im(\kappa)} < \frac{\pi - \arg(\kappa)}{\Im(\kappa)},
\]
from which one can obtain the inequalities of \( \Im(\kappa) \) in Theorem 4.3 by a simple calculation. We note here that
\[
\frac{\Im(\mu)}{N + 1} \cdot \frac{\pi - \arg(\kappa)}{\pi} < \frac{1}{2} \Im(\mu)
\]
holds.

We next find the expression of \( I(\kappa) \). By using
\[
\sigma^{+}(\kappa) = 0 \quad \text{and} \quad \sigma^{-}(\kappa) = 2 \arg(\kappa),
\]
\[
\omega^{+}(\kappa) = \Im(\mu) \quad \text{and} \quad \omega^{-}(\kappa) = \Im(\mu) - 2\Im(\kappa),
\]
we have
\[
\sigma^{+}(\tilde{\alpha}(\kappa), \tilde{\beta}(\kappa)) = -\sigma^{-}(\kappa) < 0.
\]
Lemma 4.4. Therefore, \( \mathbb{I} \) holds. In particular, \( r_0(\theta) = r_1(\theta) \) if and only if \( \theta = 2 \text{Arg}(\mu) \).

Proof. A simple calculation and the trigonometric identities show
\[
\left| r_0(\theta) \right| = \left| 1 - \cos(2 \text{Arg}(\mu) - \theta) \right| \geq 0,
\]
which leads us to the conclusion.

Lemma 4.4. If \( \Re(\mu) > 0 \), then
\[
D = \{ \kappa \in \mathbb{C} \setminus \{0\} : \kappa \notin (-\infty, 0), \mathbb{I}(\kappa) \neq \mathbb{I}(\mu) \text{ and } r_0(\text{Arg}(\kappa)) < |\kappa| < r_1(\text{Arg}(\kappa)) \}.
\]
Furthermore, we have
\[
\mathbb{I}(\alpha(\kappa)) \begin{cases} > 0 & (\text{Arg}(\kappa) > 2 \text{Arg}(\mu)), \\ = 0 & (\text{Arg}(\kappa) = 2 \text{Arg}(\mu)), \\ < 0 & (\text{Arg}(\kappa) < 2 \text{Arg}(\mu)). \end{cases}
\tag{15}
\]
for \( \kappa \) satisfying \( r_0(\text{Arg}(\kappa)) \leq |\kappa| \leq r_1(\text{Arg}(\kappa)) \).

Proof. The expression of \( D \) is obtained from Propositions 3, 4 and (15).

We now prove (15). The trivial cases \( \text{Arg}(\mu) = 0 \) or \( \text{Arg}(\kappa) \leq 0 \) are skipped. We estimate \( \mathbb{I}(\alpha(\kappa)) \) from below and above:
\[
\mathbb{I}(\kappa) \geq r_0(\text{Arg}(\kappa)) \cdot \sin \text{Arg}(\kappa) = \frac{\sin \text{Arg}(\kappa)}{1 + \cos \text{Arg}(\kappa)} \cdot \Re(\mu),
\]
\[
\mathbb{I}(\kappa) \leq r_1(\text{Arg}(\kappa)) \cdot \sin \text{Arg}(\kappa) = \frac{\sin \text{Arg}(\kappa)}{2 \cos(\text{Arg}(\kappa) - \text{Arg}(\mu))} \cdot |\mu|.
\]
Therefore,
\[
\mathbb{I}(\kappa) > \frac{\sin(2 \text{Arg}(\mu))}{1 + \cos(2 \text{Arg}(\mu))} \cdot \Re(\mu) = \mathbb{I}(\mu) \text{ for } \text{Arg}(\kappa) > 2 \text{Arg}(\mu),
\]
\[
\mathbb{I}(\kappa) < \frac{\sin(2 \text{Arg}(\mu))}{2 \cos \text{Arg}(\mu)} \cdot |\mu| = \mathbb{I}(\mu) \text{ for } \text{Arg}(\kappa) < 2 \text{Arg}(\mu),
\]
and for \( \text{Arg}(\kappa) = 2 \text{Arg}(\mu) \), we have \( \mathbb{I}(\kappa) = \mathbb{I}(\mu) \).

Let \( C_0 \) and \( C_1 \) be curves in the complex plane \( \mathbb{C} \) defined as
\[
C_0 = \{ r_0(\theta) e^{i\theta} : -\pi < \theta < \pi \},
\]
\[
C_1 = \{ r_1(\theta) e^{i\theta} : \theta \in (-\pi, \pi) \text{ satisfying } \cos(\theta - \text{Arg}(\mu)) > 0 \}.
\]
Lemma 4.4 shows that \( D \) is the region enclosed by \( C_0 \) and \( C_1 \) except the line \( \mathbb{I}(\kappa) = \mathbb{I}(\mu) \).
On the assumptions of $\kappa \in D$ and $\tau > 0$, there are two cases for inequality (9) from Lemma 4.4:

(i) $\alpha = \alpha(\kappa)$, $\beta = \beta(\kappa)$ if $\arg(\kappa) > 2 \arg(\mu)$, and

(ii) $\alpha = \bar{\alpha}(\kappa)$, $\beta = \bar{\beta}(\kappa)$ if $\arg(\kappa) < 2 \arg(\mu)$.

Let $r = |\kappa|$ and $\theta = \arg(\kappa)$. Then (9) becomes

$$\frac{\pm N\pi - \theta}{-\Im(\mu)/r + \sin \theta} < E(1, -\Re(\mu)/r + \cos \theta),$$

where $+$ and $-$ correspond to $\theta > 2 \arg(\mu)$ and $\theta < 2 \arg(\mu)$, respectively. The right-hand side of (16) is a monotonically increasing function of $r > 0$. The left-hand side is

- monotonically decreasing if $\theta > 2 \arg(\mu)$,
- monotonically increasing if $\theta < 2 \arg(\mu)$

since $N \geq 1$.

**Proposition 6.** If $\theta \in (-\pi, \pi)$ satisfies $\cos(\theta - \arg(\mu)) > 0$, then we have

$$\lim_{r \to r_1(\theta)} E(1, -\Re(\mu)/r + \cos \theta) = \frac{\theta - 2 \arg(\mu)}{\sin(\theta - 2 \arg(\mu))}.$$ (17)

**Proof.** By definition, the above limits are equal to

$$-2 \cos(\arg(\mu)) \cos(\theta - \arg(\mu)) + \cos \theta,$$

$$-2 \sin(\arg(\mu)) \cos(\theta - \arg(\mu)) + \sin \theta,$$

respectively. The conclusions are obtained from the trigonometric identities. \qed

From (17), we have

$$\lim_{r \to +\infty} E(1, -\Re(\mu)/r + \cos \theta) = \frac{\theta - 2 \arg(\mu)}{\sin(\theta - 2 \arg(\mu))}.$$ (19)

**Theorem 4.5.** If $\Re(\mu) > 0$ and $\Im(\mu) = 0$, then (11) has a root with nonnegative real part for any $\kappa \in \mathbb{C}$ and $\tau > 0$.

**Proof.** We have

$$D = \{ \kappa \in \mathbb{C} \setminus \{0\} : \Im(\kappa) \neq 0 \text{ and } r_0(\arg(\kappa)) < |\kappa| < r_1(\arg(\kappa)) \}$$

from Lemma 4.4, where

$$r_0(\theta) = \frac{\mu}{1 + \cos \theta} \quad \text{and} \quad r_1(\theta) = \frac{\mu}{2 \cos \theta}.$$

In this case, the left-hand side of (16) does not depend on $r > 0$.

- When $\cos \theta \leq 0$, $r > 0$ satisfies $r_0(\theta) < r < +\infty$, and we have

  $$\lim_{r \to +\infty} E(1, -\Re(\mu)/r + \cos \theta) = \frac{\pm \pi - \theta}{\sin \theta}.$$  

- When $0 < \cos \theta < 1$, $r > 0$ satisfies $r_0(\theta) < r < r_1(\theta)$, and from (19)

  $$\lim_{r \to r_1(\theta)} E(1, -\Re(\mu)/r + \cos \theta) = \frac{\theta}{\sin \theta}.$$  

Therefore, (16) does not hold. \qed
We define a real function $g_j$ for each $j \in \mathbb{Z}$ as
\[ g_j(\theta) = \frac{\theta + \sin \theta + j\pi}{1 + \cos \theta} \quad \text{for } \theta \in (-\pi, \pi). \]

**Theorem 4.6.** Suppose that $\Re(\mu), \Im(\mu) > 0$.
1. If $\kappa \notin D$, or $\kappa \in D$ and $\Arg(\kappa) > 2\Arg(\mu)$, then (11) has a root with nonnegative real part for any $\tau > 0$.
2. Suppose $\kappa \in D$ and $\Arg(\kappa) < 2\Arg(\mu)$. If $|\kappa|$ is sufficiently large, then (11) has a root with nonnegative real part for any $\tau > 0$. If $\Arg(\kappa)$ satisfies \[ g_N(\Arg(\kappa)) < \Im(\mu)/\Re(\mu) \quad \text{for some } N = 2m + 1, \] then all the roots of (11) have negative real parts for $\kappa$ and $\tau$ satisfying
\[ |\kappa| - r_0(\Arg(\kappa)) > 0 \quad \text{is sufficiently small,} \]
\[ \tau \in \bigcup_{l=0}^m \left( \frac{\sigma^-(\kappa) + 2\ell\pi}{\omega^-(\kappa)}, \frac{\sigma^+(\kappa) + 2(\ell + 1)\pi}{\omega^+(\kappa)} \right). \]

**Proof.** We use
\[ \lim_{r \to \infty} E(1, -\Re(\mu)/r + \cos \theta) = \begin{cases} \frac{(\pi - \theta)/\sin \theta}{\sin \theta} & (\theta > 0) \\ \frac{(-\pi - \theta)/\sin \theta}{\sin \theta} & (\theta < 0), \end{cases} \]
which is also used in the proof of Theorem 4.5. In the following, let $N$ be a positive odd number.

(i) Suppose $\theta \geq \Arg(\mu) + \pi/2$. Then $\cos(\theta - \Arg(\mu)) \leq 0$, and
\[ \lim_{r \to \infty} \frac{N\pi - \theta}{-\Im(\mu)/r + \sin \theta} = \frac{N\pi - \theta}{\sin \theta} \geq \frac{\pi - \theta}{\sin \theta} = \lim_{r \to \infty} E(1, -\Re(\mu)/r + \cos \theta) \]
hold.

(ii) Suppose $2\Arg(\mu) < \theta < \Arg(\mu) + \pi/2$. Then it is easily checked that
\[ \lim_{r \to \infty} \frac{N\pi - \theta}{-\Im(\mu)/r + \sin \theta} = \frac{N\pi - \theta}{\sin(\theta - 2\Arg(\mu))} \geq \frac{\theta - 2\Arg(\mu)}{\sin(\theta - 2\Arg(\mu))} = \lim_{r \to \infty} E(1, -\Re(\mu)/r + \cos \theta) \]
hold by the assumption.

From (i) and (ii), if $\theta > 2\Arg(\mu)$, then (16) does not hold for all $r$ from monotonicity. Therefore, 1 follows.

(iii) Suppose $\theta < 2\Arg(\mu)$. Then the following inequalities are verified:
\[ \lim_{r \to \infty} \frac{-N\pi - \theta}{-\Im(\mu)/r + \sin \theta} \geq \lim_{r \to \infty} E(1, -\Re(\mu)/r + \cos \theta) \quad \text{for } \theta \geq \Arg(\mu) - \pi/2, \]
\[ \lim_{r \to \infty} \frac{-N\pi - \theta}{-\Im(\mu)/r + \sin \theta} \geq \lim_{r \to \infty} E(1, -\Re(\mu)/r + \cos \theta) \quad \text{for } \theta < \Arg(\mu) - \pi/2. \]

Therefore, if $r > 0$ is sufficiently large, then (16) does not hold. For the limit as $r \to r_0(\theta)$,
\[ \lim_{r \to r_0(\theta)} \frac{-N\pi - \theta}{-\Im(\mu)/r + \sin \theta} < \lim_{r \to r_0(\theta)} E(1, -\Re(\mu)/r + \cos \theta) \]
is equivalent to \( g_N(\theta) < \Im(\mu)/\Re(\mu) \).

The case (iii) corresponds to \( \Im(\alpha(\kappa)) < 0 \), and
\[
\sigma^+(\bar{\alpha}(\kappa), \bar{\beta}(\kappa)) = -\sigma^-(\kappa) = -\text{Arg}(\kappa) - \text{Arccos}(\Re(-\mu + \kappa)/|\kappa|) < 0.
\]

Therefore, \( I(\kappa) \) consists of intervals
\[
\left( \frac{\sigma^+(\bar{\alpha}(\kappa), \bar{\beta}(\kappa)) - 2\ell \pi}{\omega^+(\bar{\alpha}(\kappa), \bar{\beta}(\kappa))}, \frac{\sigma^-(\kappa) - 2(\ell + 1)\pi}{\omega^-(\kappa)} \right)
\]
from Theorem 3.3.

Simple calculations show that \( g_1 \) satisfies
- \( g_1 \) is monotonically increasing,
- \( \lim_{\theta \to -\pi} g_1(\theta) = 0 \) and \( \lim_{\theta \to \pi} g_1(\theta) = +\infty \).

Furthermore, we have
\[
g_1(2 \text{Arg}(\mu)) > \frac{\sin \theta}{1 + \cos \theta} \bigg|_{\theta = 2 \text{Arg}(\mu)} = \frac{\Im(\mu)}{\Re(\mu)}.
\]
This completes the proof of Theorem 4.1.

4.2. Oscillator networks.

4.2.1. Reduction to transcendental equations. The first variational equation of (6) along \( \theta_i(t) = \Omega t \ (1 \leq i \leq N) \) becomes
\[
\dot{v}_i(t) = \frac{K}{\kappa} f'(-\Omega \tau) \sum_{j=1}^{N} a_{ij} [v_j(t - \tau) - v_i(t)] \quad (1 \leq i \leq N),
\]
where the in-degree condition (7) are used. Note that (20) is a system of a constant coefficient linear DDEs. The characteristic equation of (20) is
\[
\det \left( \lambda I + K f'(-\Omega \tau)I - e^{-\lambda \tau} \cdot \frac{K}{\kappa} f'(-\Omega \tau)A \right) = 0,
\]
where \( I \) is the \( N \times N \) identity matrix. Let \( (\sigma_1, \ldots, \sigma_N) \) be the spectrum of the adjacency matrix \( A \). Then (21) becomes
\[
\prod_{\ell=1}^{N} \left( \lambda + K f'(-\Omega \tau) - \frac{K}{\kappa} f'(-\Omega \tau) \sigma_\ell \cdot e^{-\lambda \tau} \right) = 0,
\]
from which we obtain (3) with
\[
\alpha = K f'(-\Omega \tau) \quad \text{and} \quad \beta = \frac{K}{\kappa} f'(-\Omega \tau) \sigma_\ell.
\]
4.2.2. Stability criterion.

**Theorem 4.7.** Assume that \( Kf'(-\Omega \tau) \neq 0 \). Then all the roots of \((22)\) have negative real parts if and only if \( Kf'(-\Omega \tau) > 0 \) and \( \sigma_\ell \) is not a positive real number for \( \sigma_\ell \) satisfying \( |\sigma_\ell| = k \).

**Example 1.** Let \( N = 2 \). Then the associated graph \( \Gamma \) is undirected, \( k = 1 \) and the adjacency matrix \( A \) is
\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Since the eigenvalues of \( A \) are 1 and \(-1\), the condition in Theorem 4.7 is not satisfied.

**Proof of Theorem 4.7.** We consider transcendental equation \((3)\) with \( \alpha, \beta \) in \((23)\). Then
\[
\Re(\alpha) = Kf'(-\Omega \tau) \text{ and } |\beta| = \frac{|\sigma_\ell|}{k} \cdot |Kf'(-\Omega \tau)|.
\]
Note that \( |\sigma_\ell|/k \leq 1 \) holds by the Gershgorin circle theorem (see Horn & Johnson [8]). Therefore, if \( Kf'(-\Omega \tau) < 0 \), then
\[
|\beta| = \frac{|\sigma_\ell|}{k} \cdot (-Kf'(-\Omega \tau)) \leq -\Re(\alpha),
\]
that is, \( \Re(\alpha) \leq -|\beta| \). From Theorem 1.2, \((3)\) has a root with positive real part.

We hereafter assume \( Kf'(-\Omega \tau) > 0 \). Then we have \( \Re(\alpha) \geq |\beta| \), and \( \Re(\alpha) = |\beta| \) holds if and only if \( |\sigma_\ell| = k \). In this case, the condition in Theorem 1.2 becomes
\[
Z(\text{Arg}(\beta)) > 0,
\]
that is, \( \beta \) is not positive real. This completes the proof.

**Appendix A. Proof of lemmas.**

A.1. Lemmas in Section 2.

**Proof of Lemma 2.2.** Let us consider the following conditions on \( z \):
\begin{itemize}
  \item[(a)] \( z \in W(\zeta) \cap R_m \),
  \item[(b)] \( z + \log W(z) \ni \log(\zeta) + 2m\pi i \),
  \item[(c)] \( h(z) \cap (-\pi, \pi] + 2m\pi \neq \emptyset \), |\(ze^z| = |\zeta|\), and \( \text{Arg}(ze^z) = \text{Arg}(\zeta) \),
  \item[(d)] \( |ze^z| = |\zeta| \) and \( h(z) \ni \text{Arg}(\zeta) + 2m\pi \),
\end{itemize}
where (b) is equivalent to \( z \in \omega(\log(\zeta) + 2m\pi i) \) by definition of the omega function.

Then equivalences (a) \( \Leftrightarrow \) (c) and (b) \( \Leftrightarrow \) (d) hold. We see that (c) \( \Leftrightarrow \) (d) holds by considering the following cases:
\begin{itemize}
  \item[(i)] If \( z = -1 \), then (c) and (d) are both equivalent to \( \zeta = -1/e \), and \( m = 0 \) or \( m = -1 \) since \( h(-1) = \{\pi, -\pi\} \).
  \item[(ii)] If \( z < -1 \), then (c) and (d) are both equivalent to \( |ze^z| = |\zeta| \), \( \text{Arg}(\zeta) = -\pi \) and \( m = -1 \) since \( h(z) = \{-\pi\} \).
  \item[(iii)] Otherwise, the equivalence is clear since \( \text{Arg}_W(z) = \{\text{Arg}(z)\} \).
\end{itemize}
This completes the proof.

**Proof of Lemma 2.3.** We consider a condition
\begin{itemize}
  \item[(c)] \( \sqrt{x^2 + y^2 e^z} = e^z \) and \( h(x + iy) \ni \eta \).
\end{itemize}
Then (a) $z \in \omega(\zeta)$ and (c) are equivalent by definition of the omega function. We prove an equivalence (b) $\iff$ (c) by using

$$h(x + iy) = \{\text{Arg}(x + iy) + y\} \quad \text{and} \quad \text{sgn}(\text{Arg}(x + iy) + y) = \text{sgn}(y)$$

for $y \neq 0$.

(c) $\Rightarrow$ (b). From (c), we have $y \in \{\gamma(x, e^\xi), -\gamma(x, e^\xi)\}$ and $\eta \in H(e^\xi, x)$. Therefore, $x \in G(e^\xi, \eta)$ holds. Since $h(x + iy) \geq \eta$,

(i) $y = 0$ and $x > 0$ if $\eta = 0$,
(ii) $y \geq 0$ if $\eta > 0$, and
(iii) $y \leq 0$ if $\eta < 0$.

This implies $y = \text{sgn}(\eta) \cdot \gamma(x, e^\xi)$.

(b) $\Rightarrow$ (c). From $y = \text{sgn}(\eta) \cdot \gamma(x, e^\xi)$, $\sqrt{x^2 + y^2}e^x = e^\xi$ holds. By definition, we have $\eta \in H(e^\xi, x)$, and we can obtain $\eta \in h(x + iy)$ by considering the each cases of $\eta$ stated above. This completes the proof. $\square$

**Proof of Lemma 2.4.** By definition of $H(\rho, x)$,

$$H(\rho, x) = [\text{Arg}_W(x + i\gamma(x, \rho)) + \gamma(x, \rho)] \cup [\text{Arg}_W(x - i\gamma(x, \rho)) - \gamma(x, \rho)].$$

Since

- $\gamma(x, \rho) = 0 \iff xe^x = \pm \rho$, and
- for $\rho \leq 1/e\: xe^x = -\rho$ has solutions in $(-\infty, -1]$ and $[-1, 0)$,

we can obtain the following expressions of $H(\rho, x)$:

(i) For each $\rho > 1/e$, $H(\rho, x) = \{h(\rho, x), -h(\rho, x)\}$,

(ii) For each $\rho = 1/e$,

$H(\rho, x) = \begin{cases} \{h(\rho, x), -h(\rho, x)\} & (x \neq -1), \\
\{\pi, -\pi\} & (x = -1), \end{cases}$

(iii) For each $\rho < 1/e$,

$H(\rho, x) = \begin{cases} \{h(\rho, x), -h(\rho, x)\} & (xe^x \neq -\rho), \\
\{\pi\} & (xe^x = -\rho \text{ and } x > -1), \\
\{-\pi\} & (xe^x = -\rho \text{ and } x < -1). \end{cases}$

Next, we investigate the behavior of function $h(\rho, \cdot)$. The derivative is

$$\frac{d}{dx} h(\rho, x) = -\frac{(1 + x)^2 + \gamma(x, \rho)^2}{\gamma(x, \rho)} < 0 \quad \text{for} \: \gamma(x, \rho) \neq 0,$$

and we have $\lim_{x \to -\infty} h(\rho, x) = +\infty$. Therefore, $h(\rho, \cdot)$ is a monotonically decreasing function whose range is $[0, +\infty)$. This implies the conclusion. $\square$

**A.2. Lemmas in Subsection 3.2.**

**Proof of Lemma 3.4.** The conclusions are obtained by the following claims.

**Claim 6.** We have $\sigma^{-}(\alpha, \beta) < 0 \iff \text{Arg}(\beta) < 0$ and $\Re(\alpha) > \Re(\beta)$.

This claim is proved in the same way of the proof of Proposition 2.

**Claim 7.** Let $\ell \geq 0$. Then an inequality

$$\frac{\sigma^{-}(\alpha, \beta) - 2\ell\pi}{\omega^{-}(\alpha, \beta)} < \tau_{2\ell}(\alpha, \beta)$$

holds for all $\alpha$ and $\beta$.
The inequality is equivalent to
\[-[2\ell \pi - \text{Arg}(\beta)] + \arccos\left(\frac{\Re(\alpha)}{|\beta|}\right) > \left(-\sqrt{|\beta|^2 - \Re(\alpha)^2} / \Im(\alpha) \right) - 1\] \[2\ell \pi - \text{Arg}(\beta)],\]
which always holds.

**Claim 8.** For all \(\alpha, \beta\) and \(\ell\), we have an equivalence
\[\tau_{2\ell-1}(\alpha, \beta) < \frac{\sigma^-(\alpha, \beta) - 2\ell \pi}{\omega^-(\alpha, \beta)} \iff \tau_{2\ell-1}(\alpha, \beta) < E(|\beta|, \Re(\alpha)).\]

The left hand side is equivalent to
\[\left(-\sqrt{|\beta|^2 - \Re(\alpha)^2} / \Im(\alpha) \right) - 1\] \[(2\ell - 1)\pi - \text{Arg}(\beta)] > -[(2\ell - 1)\pi - \text{Arg}(\beta)] - \left[\pi - \arccos\left(\frac{\Re(\alpha)}{|\beta|}\right)\right],\]
from which the right hand side is obtained.

**Proof of Lemma 3.5.**
1. \(\sigma^+(\alpha, \beta) - 2\ell \pi\) is negative for all \(\ell \geq 1\) because \(\sigma^+(\alpha, \beta) \leq \pi\).
2. \(I^+_0(\alpha, \beta) \neq \emptyset\) if and only if \(\tau_1(\alpha, \beta) > 0\) and \(\sigma^+(\alpha, \beta) > 0\). \(\tau_1(\alpha, \beta) > 0\) is equivalent to \(\text{Arg}(\beta) < \pi\).

Therefore, we obtain a necessary and sufficient condition for \(I^+_0(\alpha, \beta) \neq \emptyset\). The expression of \(I^+_0(\alpha, \beta)\) is obtained by the following claim.

**Claim 9.** If \(\omega^+(\alpha, \beta) > 0\), then an equivalence
\[\tau_1(\alpha, \beta) < \frac{\sigma^+(\alpha, \beta)}{\omega^+(\alpha, \beta)} \iff \tau_1(\alpha, \beta) < E(|\beta|, \Re(\alpha)).\]
holds.

The left hand side is equivalent to
\[\left(\sqrt{|\beta|^2 - \Re(\alpha)^2} / \Im(\alpha) \right) - 1\] \[(\pi - \text{Arg}(\beta)) < -(\pi - \text{Arg}(\beta)) + \left[\pi - \arccos\left(\frac{\Re(\alpha)}{|\beta|}\right)\right],\]
from which the inequality is obtained.

The case \(\omega^+(\alpha, \beta) = 0\) is included since this is equivalent to \(|\alpha| = |\beta|\), and in this case \(\tau_1(\alpha, \beta) < E(|\beta|, \Re(\alpha))\) always holds.

**Proof of Lemma 3.6.** The conclusions are obtained by the following claims.

**Claim 10.** If \(\omega^+(\alpha, \beta) < 0\), then an equivalence
\[\frac{\sigma^+(\alpha, \beta) - 2\ell \pi}{\omega^+(\alpha, \beta)} < \tau_{2\ell+1}(\alpha, \beta) \iff \tau_{2\ell+1}(\alpha, \beta) < E(|\beta|, \Re(\alpha))\]
holds for all \(\ell\).

The left hand side is equivalent to
\[-[(2\ell + 1)\pi - \text{Arg}(\beta)] + \left[\pi - \arccos\left(\frac{\Re(\alpha)}{|\beta|}\right)\right] > \left(\sqrt{|\beta|^2 - \Re(\alpha)^2} / \Im(\alpha) \right) - 1\] \[(2\ell + 1)\pi - \text{Arg}(\beta)],\]
from which the right hand side is obtained.

**Claim 11.** Let $\ell \geq 1$. If $\omega^+(\alpha, \beta) < 0$, then an inequality

$$\tau_{2\ell}(\alpha, \beta) < \frac{\sigma^+(\alpha, \beta) - 2\ell \pi}{\omega^+(\alpha, \beta)}$$

always holds.

This inequality is equivalent to

$$\left( \sqrt{|\beta|^2 - \Re(\alpha)^2} \right) - 1 \left[ 2\ell \pi - \text{Arg}(\beta) \right] > -\left[ 2\ell \pi - \text{Arg}(\beta) \right] - \text{Arccos} \left( \frac{\Re(\alpha)}{|\beta|} \right),$$

which always holds.

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_E-mail address: j-nishi@math.kyoto-u.ac.jp_