Four-Derivative Quantum Gravity Beyond Perturbation Theory

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In this work we investigate the ultraviolet behavior of Euclidean four-derivative quantum gravity beyond perturbation theory. In addition to a perturbative fixed point, we find an ultraviolet fixed point that is non-trivial in all couplings and is described by only two free parameters. This result is in line with the asymptotic safety scenario in quantum gravity. In particular, it supports the conjecture that the full theory is described by a finite number of free parameters.

I. INTRODUCTION

The quantum description of spacetime is one of the fundamental open questions in theoretical physics. An approach that satisfies the principle of minimal assumptions about unknown physics is the asymptotic safety scenario, which was introduced by Steven Weinberg in 1976, [1]. The underlying idea is that the renormalization group flow of quantum gravity approaches a non-trivial, i.e., non-Gaussian, ultraviolet fixed point with a finite dimensional UV-critical surface. The first property ensures finiteness, while the latter makes the theory predictive in the sense that it contains only a finite number of free parameters. It is well-known that the standard perturbative renormalization programme for general relativity fails: the theory turns out to be perturbatively non-renormalizable [2, 3]. Weinberg’s idea, however, is not limited to perturbative quantization of general relativity, but provides the possibility for a non-perturbative UV-completion of the theory. Nevertheless, the theory is described by a quantum field theory where it is only assumed that there is an underlying diffeomorphism symmetry and that the degrees of freedom are carried by a spin-two field. Therefore, there are no further assumptions about unknown new physics at some high energy scale.

The explicit framework to search for asymptotic safety in quantum gravity was set up by Reuter [4] based on the functional renormalization group and the Wetterich equation, [5]. During the last two decades, numerous publications investigated the ultraviolet behavior of the renormalization group flow of quantum gravity and compelling hints for the existence of a non-Gaussian fixed point were found, [6–43]. This also extends to theories where gravity is coupled to matter and gauge fields, [44–58]. For reviews on asymptotic safety in quantum gravity see [59–68]. The general idea of asymptotic safety is of course not restricted to quantum gravity, but non-Gaussian fixed points are of general interest in quantum field theory. Asymptotically safe theories without gravity are studied e.g. in [69–79].

A particularly interesting type of gravity theories are four-derivative theories, i.e. theories with an action that includes not only Einstein-Hilbert terms but also operators with mass dimension four. The couplings of these operators are dimensionless in four spacetime dimensions and the propagator has a $1/p^4$ falloff, which makes the theory perturbatively renormalizable, [80]. Using a complete basis of four-derivative operators, it was shown explicitly in one-loop calculations that the theory is asymptotically free in the coupling of the Weyl-squared tensor, while it exhibits a non-trivial UV-fixed point in the other couplings, [81, 82]. Such theories were also studied in the context of the asymptotic safety scenario, [10, 13, 16, 83]. In [13], also a purely non-Gaussian fixed point was discovered in the four-derivative theory. Despite being perturbatively renormalizable, higher derivative theories did not receive a lot of attention as candidates for a fundamental quantum theory of gravitation during the last decades. The reason is that squares of the Ricci tensor induce an additional pole with negative residue in the graviton propagator around flat background. It is believed that this feature spoils unitarity of the resulting quantum field theory. However, one can offer several objections against this claim. First of all, the faith of this pole in a resummed graviton propagator is not clear, and second, the non-perturbative relation between poles in the Euclidean propagator and unitarity of the theory in real-time, i.e. the spectral reconstruction, is highly non-trivial. Hence, the question of unitarity should be considered as an open issue, but cannot be used to abandon four-derivative theories right away.

In this work we address several aspects of the asymptotic safety scenario in quantum gravity. We use the formalism for vertex expansions developed in [22]. In this approach the effective action is expanded around a flat background, i.e. the spectral reconstruction, is highly non-trivial. Hence, the question of unitarity should be considered as an open issue, but cannot be used to abandon four-derivative theories right away.
we present several technical advances in the context of
flow equations. The non-Gaussian fixed point in the Einstein-Hilbert
truncation, where the set of $\beta$ functions is determined by
the gravitational constant $G$ and the cosmological con-
stant $\Lambda$, is characterized by two relevant directions. This
implies that the theory is described by two free parame-
ters. As predictivity is encoded in a finite number of rele-
vant directions, the asymptotic safety conjecture heavily
relies on the identification of a pattern that guarantees irrelevance of higher order operators. In this work we find
a non-Gaussian fixed point in four dimensional coupling
space that exhibits only two relevant directions. Thus,
the theory has not only a well-defined ultraviolet limit,
but the classically marginal four-derivative operators do
not induce further relevant directions. This is a very en-
couraging structure, as it suggests that quantum fluctua-
tions do not turn irrelevant into relevant operators and
that the UV-critical surface is indeed finite dimensional.
On the technical side, we generalize the setup in [22, 23]
in the presence of higher order operators, including the
projection on the coupling constants. Moreover, we use
a gauge fixing condition that is different from the usual
choice used in four-derivative gravity, but is found to
be the natural choice in the present setup. Indeed, it
has been shown in [84] that this particular choice for
the gauge-fixing functional induces a decoupling of gauge
fluctuations.

II. FUNCTIONAL RENORMALIZATION AND
VERTEX CONSTRUCTION

A. The Wetterich Equation and Vertex Flow
Equations

The basic ingredients for an investigation of the asym-
ptotic safety conjecture are the $\beta$-functions beyond stan-
dard perturbative expansions. The functional renormal-
ization group is a non-perturbative approach to contin-
uum quantum field theory and in particular its formulation
for the 1PI effective action with the Wetterich equa-
tion [5] has proven to be a very powerful method. It is
based on the Wilsonian idea of coarse graining by suc-
cessively integrating out infinitesimal momentum shells.
This idea is implemented with a regulator term in the
path integral, which introduces a cutoff-scale $k$. This
finally leads to a functional differential equation that de-
termines the scale-dependence of the quantum effective
action $\Gamma_k$, which now depends on the RG-scale $k$. An
additional complication in gravity is that this regulator
necessitates the introduction of a background field $\bar{g}$. Besides technical reasons, a background metric $\bar{g}$ in
the regulator is needed in order to construct a differen-
tial operator that defines via its spectrum the meaning of
large and small momenta. In addition to that, the Wet-
terich equation is formulated in terms of propagators.
Therefore one needs to work with a gauge fixed theory,
which in turn requires a background field. As a result,
the quantum effective action $\Gamma[\bar{g}, \phi]$ depends on the back-
ground $\bar{g}$ and a fluctuation super-field $\phi = (h, \bar{c}, c)$. In
gravity this super-field contains the dynamical gravita-
ton field $h$, as well as the Faddeev-Popov ghost fields $\bar{c}$
and $c$. With these ingredients the Wetterich equation for
quantum gravity reads

$$
\partial_t \Gamma_k[\bar{g}, \phi] = \frac{1}{2} \text{Tr} \left( \left( \Gamma_k^{(2h)} + R_{k,h} \right)^{-1} \partial_t R_{k,h} \right) [\bar{g}, \phi] 
- \text{Tr} \left( \left( \Gamma_k^{(c)} + R_{k,c} \right)^{-1} \partial_t R_{k,c} \right) [\bar{g}, \phi],
$$

(1)

and we use the abbreviation

$$
\Gamma_k^{(\phi_1 \ldots \phi_n)}[\bar{g}, \phi] := \frac{\delta^n \Gamma_k[\bar{g}, \phi]}{\delta \phi_1 \cdots \delta \phi_n}
$$

(2)

for functional derivatives. In the above functional dif-
erential equation $R_k$ is the regulator function that or-
ganizes local momentum integration of fluctuations with
$q \approx k$. Moreover, $t$ is the logarithmic RG-scale $t := \log(k/k_0)$ with an arbitrary reference scale $k_0$. The Tr
in the above flow equation denotes a summation over all
discrete indices and an integration over continuous ones.
We will also use the notation $\partial_t f(k) =: \dot{f}(k)$ for any func-
tion $f$.

There are several important issues concerning the role
the background field in the flow equation. According to the
general principles of gravity, physical observables should
be independent of an auxiliary background $\bar{g}$ that needs
to be introduced for technical reasons. In the present for-
malism with the two fields $\bar{g}$ and $\phi$, the effective action
$\Gamma[\bar{g}, \phi]$ is truly a functional of two fields, and the depen-
dence cannot be combined into a single, physical metric
g = $\bar{g}$ + $\phi$. In particular, it follows that the vertex func-
tions $\Gamma^{(n)}$ are explicitly background dependent. How-
ever, these correlation functions are not directly related
to observables and their explicit background dependence
is indeed necessary in order to guarantee background in-
dependence of physical observables. The separate de-
dependence on the two fields $\bar{g}$ and $\phi$ is encoded in non-
trivial Nielsen identities, also called split-Ward-identities
[24, 28, 42, 84–90]. In the standard background field ap-
proximation one evaluates the Wetterich equation (1) at
vanishing fluctuation field $\phi = 0$. However, this does not
lead to a closed equation as

$$
\left. \frac{\delta^2 \Gamma_k[\bar{g}, \phi]}{\delta \bar{h}^2} \right|_{\phi=0} \neq \left. \frac{\delta^2 \Gamma_k[\bar{g}, 0]}{\delta \bar{g}^2} \right|_{\phi=0}.
$$

(3)

Consequently, by using this approximation one does not
calculate correlation functions of the fluctuation field $\phi$,
but correlations of the background field $\bar{g}$. In order to
circumnavigate this problem, we can make use of the in-
finite hierarchy of flow equations that is generated by
the master equation (1). This hierarchy is obtained by
The above notation is symbolic and indices and its contractions as well as spacetime integrals are suppressed. In this work we choose flat spacetime as the expansion point. The dressed graviton propagator is represented by the double line, the ghost propagator by the dashed line, while a dressed vertex is denoted by a dot. The regulator insertion is indicated by the crossed circle.

Taking functional derivatives of the Wetterich equation,

$$\delta^n \Gamma_k [\bar{g}, h] = \text{Flow}^{(n)}[\Gamma^{(2)}, ..., \Gamma^{(n+2)}],$$  \hspace{1cm} (4)

where \text{Flow}^{(n)} denotes the \( n \)-th functional derivative of the RHS of (1). It is important to note that the flow of the vertex function of order \( n \) depends on the vertex functions of order two up to order \( n+2 \). With these relations we are equipped with equations for the \( n \)-th moments of the effective action, which define the quantum field theory. This approach has the additional advantage that one gains access to the momentum dependence of the vertex functions \( \Gamma^{(n)} \) and that their dependence on the RG-scale \( k \) can be studied separately. In this work we will study the flow equation for the inverse propagator \( \Gamma^{(2)} \), which has the diagrammatic representation shown in Figure 1.

### B. Vertex Functions

In this section we turn to the construction of the vertex functions, which are the essential building blocks in the flow equation. The general setup is based on the formalism presented in [22], but is generalized to tensor structures beyond Einstein-Hilbert. Our goal is to construct an approximation based on a vertex expansion, i.e. a functional Taylor expansion of the effective action in powers of the fluctuation field \( h \) according to

$$\Gamma[\bar{g}, h] = \sum_{n=0}^N \frac{1}{n!} \delta^n \Gamma[\bar{g}, h] \bigg|_{h=0} h^n.$$  \hspace{1cm} (5)

The above notation is symbolic and indices and its contractions as well as spacetime integrals are suppressed. In this work we choose flat spacetime as the expansion point of the effective action, i.e., \( g_{\mu\nu} = 1 \). The most general form of the vertex functions is not unique, but one can choose different parameterizations. A canonical form is given by

$$\Gamma^{(n)}_i (p_1, ..., p_n) = \sum_i g^{(n)}_i (k, p_1, ..., p_n) T^{(n)}_i (p_1, ..., p_n),$$  \hspace{1cm} (6)

where \( T^{(n)}_i \) are tensor structures that form a basis in the relevant tensor space. The \( g^{(n)}_i \) are parameters, which in general depend not only on the RG-scale \( k \), but also on all external momenta \( p_1, ..., p_n \). However, this is obviously far too general for practical computations. In order to construct approximations to this most general form, there are several guiding principles that underlie the following construction of vertex functions. First of all, it is important to note that the expansion (5) is not diffeomorphism invariant nor background independent. In particular, the vertex functions \( \Gamma^{(n)} \) inherit this property. Nevertheless, we want to restrict their tensor structures to the ones that originate from functional derivatives of diffeomorphism invariant operators. This is motivated by the conjecture that diffeomorphism invariance is broken only weakly, which is observed in [23] and [91]. This also what is expected in semi-perturbative regimes. In the present work, we are interested in four-derivative gravity, and thus our tensor structures are generated by the action

$$S_{G,\Lambda,a,b} [g] = \frac{1}{16\pi} \int_x \left( \frac{2\Lambda}{G} - \frac{R}{G} + a R^2 + b R_{\mu\nu}^2 \right),$$  \hspace{1cm} (7)

where we have defined \( \int_x := \int d^4 x \sqrt{\text{det} g} \). Moreover, we work in Euclidean spacetime throughout this work. Action (7) is the most general diffeomorphism invariant action that contains up to four derivatives of the metric. More precisely, the operators \( R^2 \) and \( \text{Ricc}^2 \) are a basis of local diffeomorphism invariants in four spacetime dimensions if we drop boundary terms, i.e. terms that are total derivatives and do not contribute to local physics. Hence, we ignore a \( \Delta R \) term in the action. Moreover, the Riemann-tensor squared \( \text{Riem}^2 \) can be written as a linear combination of \( R^2 \), \( \text{Ricc}^2 \) and a topological invariant due to the generalized Gauss-Bonnet theorem in four spacetime dimensions. Additionally, the \( \text{Ricc}^2 \) term can always be traded for the square of the Weyl-tensor \( \text{C}^2 \). In the basis with the \( \text{C}^2 \) operator, the most common parameterization reads

$$S_{G,\Lambda,\omega,s} [g] = \frac{1}{16\pi} \int_x \left( \frac{2\Lambda}{G} - \frac{R}{G} + \frac{\omega}{s} R^2 + \frac{1}{s} \text{C}^2 \right),$$  \hspace{1cm} (8)

where the four-derivative operators have the common coupling \( 1/s \) and the relative interaction strength of \( R^2 \) and \( \text{C}^2 \) is encoded in the coupling \( \omega \). The different couplings in (7) and (8) are related by simple algebraic equations. Most of the time we will use the parameterization (7).

The vertex construction used in this paper is based on a quantum deformation of the classical vertices, which
reduces to the latter in the perturbative limit. The verti-
ces are obtained as follows. Using the above assump-
tion about weak breaking of diffeomorphisms invari-
ance we first expand the classical action in analogous fashion
to (5), which leads to
\[ S[\bar{g}, h] = \sum_{n=0}^{N} \frac{1}{n!} \frac{\delta^n S[\bar{g}, h]}{\delta h^n} \bigg|_{h=0} h^n, \]  
(9)
with \( S \) given by (7). Again, the notation is symbolic and
indices and its contractions as well as spacetime integ-
als are suppressed. Introducing an explicit notation, the
quadratic part takes the form
\[ \frac{\delta^2 S[\bar{g}, h]}{\delta h^2} \bigg|_{h=0} = \int_{\Delta x_1, x_2} (S^{(2)})^{A_1 A_2} h_{A_1} h_{A_2}, \]  
(10)
with the super-index \( A_i = \{ \mu, \nu, i, \} \) and the classical
two-point function
\[ (S^{(2)})^{A_1 A_2} = \frac{\delta^2 S[\bar{g}, h]}{\delta h^{\mu_1 \nu_1}(x_1) \delta h^{\mu_2 \nu_2}(x_2)} \bigg|_{h=0}, \]  
(11)
where we have explicitly written out the single compo-
ents of the super-index \( A_i \).
We proceed by choosing a linear split of the metric
\[ g_{\mu \nu} = \bar{g}_{\mu \nu} + \sqrt{G} h_{\mu \nu}, \]  
(12)
with \( \bar{g}_{\mu \nu} = 1 \). Moreover, the graviton field \( h \) acquires
the usual mass dimension one for a bosonic field and a
canonical kinetic term due to the factor of \( \sqrt{G} \) in
the definition. The classical vertex functions in the above
expansion acquire a scale dependence due to quantum
fluctuations, which turns the gravitational coupling
\( G \) into a scale-dependent running coupling \( G_k \). This also
leads to a dressing of the vertex functions \( \Gamma^{(n)} \) with an
overall coupling \( G_k^{\frac{n}{2} - 1} \). Moreover, we account for quan-
tum contributions with a scale-dependent wave-function
renormalization \( Z_k \) for the graviton field. From general
reparameterization invariance of the effective action, it
follows that if the field scales with \( Z_k^{1/2} \), then the vertex
functions \( \Gamma^{(n)} \) scale as \( Z_k^{n/2} \). As a consequence, the vertex
function \( \Gamma^{(2)} \) has an overall renormalization factor \( Z_k \),
which determines the anomalous scaling. In summary,
this procedure can be reformulated as a rescaling of the
classical graviton field according to
\[ h \rightarrow (Z_k G_k)^{\frac{1}{2}} h. \]  
(13)
So far, this procedure fixes the overall scale-dependence
of vertex functions. In addition to that, we have to take
into account the relative scale-dependence of the cou-
plings associated to different tensor structures, i.e. in our
case the couplings of the different diffeomorphism invari-
ants. Therefore, we also allow for a scale-dependence of
the couplings associated with the four-derivative interac-
tions, which corresponds to \( a \rightarrow a_k \) and \( b \rightarrow b_k \), as
well as a running of the cosmological constant \( \Lambda_k \). In
addition to this, it is important to note that due to the
non-diffeomorphism invariance and background depend-
ence of the vertex functions \( \Gamma^{(n)} \), all of the above dress-
ings with a scale dependence are in principle different for
each order \( n \), i.e. there is an overall gravitational coupling
\( G_k^{(n)} \) related to the \( n \)-graviton vertex and similarly running
couplings \( a_k^{(n)} \), \( b_k^{(n)} \) and \( \Lambda_k^{(n)} \). This reflects the fact that
due to the lack of gauge invariance, the couplings that
belong to different orders of the correlation functions are
not protected in the sense that gauge invariance enforces
simple relations amongst them. This is exactly the same
problem as in Yang-Mills theory, where the perturbative
beta-function extracted from the one-loop effective
action is the same as the one extracted from one-loop ap-
proximations of the two-, three, or four-point function.
However, in the non-perturbative regime this not true and
different couplings that agree in perturbation theory
must be distinguished. In the present approximation
we set \( G_k^{(n)} = G_k \) for \( n \geq 3 \) and for the constant terms
\( \Lambda_k^{(n)} = 0 \) for \( n \geq 3 \). The constant term of the two-point
function is conveniently written as \( M_k^2 = -2\Lambda_k^{(2)} \) and is
called the effective graviton mass parameter. We demon-
strate this construction explicitly for the transverse trace-
less component of the graviton two-point function, which
is now an effective, dressed correlator and is transformed
in momentum space according to
\[ S^{(2)}_{TT}(p_1, p_2) = \frac{1}{32\pi} (-2 \frac{\Lambda}{G} + p^2 + b p^4) \delta(p_1 + p_2), \]  
\[ \downarrow \]
\[ \Gamma^{(2)}_{TT}(p_1, p_2) = \frac{Z_k}{32\pi} (M_k^2 + p^2 + G_k b_k p^4) \delta(p_1 + p_2), \]  
(14)
where first, all couplings in \( S^{(2)} \) are dressed with a
\( k \)-dependence, a wave-function renormalization fac-
tor of \( Z_k \) is attached according to the above general
renormalization group arguments, and the entire ex-
pression is multiplied by \( G_k \). Similarly, the three-
point function is then obtained from the classical vertex
\[ S^{(3)}_{A_1 A_2 A_3}(A, a, b; p_1, p_2, p_3) \] as
\[ \Gamma^{(3)}_{A_1 A_2 A_3}(Z_k, G_k, a_k b_k; p_1, p_2, p_3) = \]
\[ Z_k^{\frac{3}{2}} G_k^{\frac{3}{2}} S^{(3)}_{A_1 A_2 A_3}(1, 0, G_k a_k, G_k b_k; p_1, p_2, p_3). \]  
(15)
In general a vertex function of order \( n \geq 3 \) reads
\[ \Gamma^{(n)}_{A_1 A_2 \ldots A_n}(Z_k, G_k, a_k b_k; (p)) = \]
\[ Z_k^{\frac{n}{2}} G_k^{\frac{n}{2} - 1} S^{(n)}_{A_1 A_2 \ldots A_n}(1, 0, G_k a_k, G_k b_k; (p)), \]  
(16)
where \( (p) = (p_1, ..., p_n) \).
Crucially, all the coupling constants above are fluctuation
field couplings, as they are related to functional derivatives with respect to the fluctuation field. Although we identify $G_k^{(n)} \equiv G_k$ for $n \geq 3$, we resolve the important difference between the graviton wave-function renormalization $Z_k$ and Newton's coupling $G_k$ as well as the difference between the mass parameter $M_k^2$ of the fluctuation field propagator and the background cosmological constant, which is given by $\Lambda_k^{(0)}$. Finally we mention, that while the fluctuation field couplings constitute the dynamical set of parameters of the theory, they are not directly related to observables. It is also for that reason, that $M_k^2 \neq 0$ is not to be confused with a model of massive gravity.

In such an expansion around a flat background, the flow equation for the propagator is given by a momentum integral over propagators and vertices according to

$$\text{Flow}_{\alpha,\beta;\mu,\nu}^{(2h)} =$$

$$-\frac{1}{2} \int_{\mathbb{R}^4} \frac{d^4q}{(2\pi)^4} \Gamma^{(4h)}_{\alpha_\beta\gamma\tau;\kappa\mu\nu}(p,q,-p,q) \left( G \dot{RG} \right)_{hh}^{kk\gamma\tau}(q)$$

$$+ \int_{\mathbb{R}^4} \frac{d^4q}{(2\pi)^4} \Gamma^{(3h)}_{\alpha_\beta\gamma;\tau\kappa\mu\nu}(p,q,-p,q) \left( G \dot{RG} \right)_{hh}^{kk\gamma\tau}(q)$$

$$\times \Gamma^{(3h)}_{\mu\nu;\kappa\rho\sigma}(p,-p,q,p+q) \left( G \dot{RG} \right)_{hh}^{kk\gamma\tau}(q)$$

$$- 2 \int_{\mathbb{R}^4} \frac{d^4q}{(2\pi)^4} \Gamma^{(hcc)}_{\alpha\beta\gamma\tau}(p,q,-p,q) \left( G \dot{RG} \right)_{hh}^{kk\gamma\tau}(q)$$

$$\times \Gamma^{(hcc)}_{\mu\nu;\kappa\rho\sigma}(p,-p,q,p+q) \left( G \dot{RG} \right)_{hh}^{kk\gamma\tau}(q), \quad (17)$$

where $G := (\Gamma_k^{(2)} + R_k)^{-1}$ is the regularized, full propagator. Equation (17) is just the explicit form of the diagrammatic representation depicted in Figure 1.

We end this section with some remarks on the running couplings involved in the flow equations within the present approximation. On the left hand side of (17) there appears the scale derivative of the two-point function, which in turn contains scale derivatives of the wave function renormalization $Z_k$, the mass parameter $M_k^2$ and the four derivative couplings $a_k$ and $b_k$. This means that the set

$$(\dot{Z}_k, M_k^2, a_k, b_k), \quad (18)$$

will define the beta-function of the theory. The gravitational coupling $G_k$ will also enter the LHS of equation (17) for the two-point function, but only in combination with the four-derivative couplings $a_k$ and $b_k$. Moreover, it is important to note that the wave-function renormalization $Z_k$ is not an essential coupling and its dependence will appear only through the anomalous dimension $\eta_k := -\dot{Z}_k/Z_k$, which will obey an algebraic rather than a differential equation. However, the RHS in Figure 1 contains the three- and the four-point vertex, which are proportional to $G_k$, whose scale derivative is not determined by the equation for the two-point function but by the higher order vertex functions. This reflects the fact that in the infinite hierarchy (4) the flow equation of order $n$ depends on vertex functions up to order $n+2$. Therefore, we can treat the coupling $G_k$ as a free parameter in the beta-functions and study the dependence parametrically, or we can use equations obtained from the three-point function with Einstein-Hilbert tensor structures, which amounts to neglecting the feedback of the higher derivative couplings. Both possibilities will be taken into account.

**C. Gauge fixing**

The standard way of gauge fixing in four-derivative quantum gravity is by choosing a gauge fixing condition that is also fourth order in derivatives, see e.g. [10, 13, 92]. In this work, we present a different gauge fixing condition, which is second order in derivatives. This is sufficient in order to define an invertible two-point function and therefore a gauge-fixed propagator. We use the two-parameter family of gauge fixing conditions given by

$$F_\mu = \nabla^\nu h_{\mu\nu} - \frac{1 + \beta}{4} \nabla_\mu h^\nu, \quad (19)$$

which results via Faddeev-Popov quantization in the gauge-fixing action

$$S_{GF} = \frac{1}{2\alpha} \int d^4x \sqrt{\det \tilde{g}} \tilde{g}^{\mu\nu} F_\mu F_\nu. \quad (20)$$

The Landau limit $\alpha \rightarrow 0$ corresponds to a sharp implementation of the gauge fixing condition and is a fixed point of a scale-dependent gauge-fixing parameter $\alpha$. Moreover, as we will see below, the choice $\beta = -1$ diagonalizes the propagator-matrix in the Landau-gauge. It has also been argued recently that the choice $\alpha \rightarrow 0$ and $\beta \rightarrow -1$ corresponds to the “physical gauge-fixing” as it acts on true gauge fluctuations only. [84, 93, 94]. The diagonal structure of the propagator in this gauge will be made explicit in the next section. Moreover, exponentiation of the Faddeev-Popov determinant introduces the Grassmann-valued ghost fields $\bar{c}$ and $c$. Their action is given by

$$\int d^4x \sqrt{\det \tilde{g}} \bar{c}^\mu M_{\mu\nu} c^\nu, \quad (21)$$

with the Faddeev-Popov operator

$$M_\mu^\nu = \tilde{g}^{\mu\alpha} \nabla_\alpha (g_{\alpha\beta} \nabla_\beta + g_{\beta\nu} \nabla_\alpha) - \frac{1}{2} (1 + \beta) \tilde{g}^{\alpha\beta} \nabla_\alpha g_{\beta\nu} \nabla_\nu. \quad (22)$$

**D. Two-Point Function, Regulator and Propagator**

With the vertex construction introduced in the previous section and the full basis of four-derivative operators
we can now derive the components of the graviton-two-point function. In what follows, we drop the subscript \( k \) for the scale-dependent couplings for better readability. Unless stated otherwise, all couplings and vertex functions are from now on scale-dependent. We expand the two-point function in a complete set of projectors according to

\[
\Gamma^{(2)} = \sum_{i=1}^{6} \Gamma_i^{(2)} P_i ,
\]

(23)

The pseudo-projectors \( P_i \) are introduced in the appendix, section A. For general gauge fixing parameters the different components of the graviton two-point functions are then given by

\[
\Gamma_1^{(2)} = \Gamma^{(2)}_{TT} = \frac{Z}{32\pi} \left( M^2 + p^2 + G b p^4 \right) ,
\]

(24)

\[
\Gamma_2^{(2)} = \Gamma^{(2)}_V = \frac{Z}{32\pi} \left( M^2 + \frac{p^2}{\alpha} \right) ,
\]

(25)

\[
\Gamma_3^{(2)} = \frac{Z}{32\pi} \frac{1}{8\alpha} \left( -4\alpha M^2 + p^2 (-16\alpha + 3(\beta + 1)^2) + 32p^4\alpha(3a + b) \right) ,
\]

(26)

\[
\Gamma_4^{(2)} = \frac{Z}{32\pi} \frac{1}{8\alpha} \left( 4\alpha M^2 + p^2 (\beta - 3)^2 \right) ,
\]

(27)

\[
\Gamma_5^{(2)} = \Gamma_6^{(2)} = \frac{Z}{32\pi} \frac{\sqrt{3}}{8\alpha} \left( -4\alpha M^2 + 2p^2 (\beta - 3)(\beta + 1) \right) .
\]

(28)

As it is well-known, the contributions of the \( R^2 \) operator to the \( TT \) component vanish, and therefore the \( p^4 \) coefficient originates only from the \( R_{\mu\nu} R^{\mu\nu} - 1/3 R^2 \), term.

In the physical gauge the inverse two-point function takes the form

\[
\left( \Gamma^{(2)} \right)^{-1}_1 = \left( \Gamma^{(2)}_1 \right)^{-1}_1 P_1 + \left( \Gamma^{(2)} \right)^{-1}_3 P_3 ,
\]

(29)

i.e. all but the \( TT \)- and one scalar component of the graviton propagator vanish. Explicitly, these components read

\[
\left( \Gamma^{(2)} \right)^{-1}_1 = 32\pi Z \frac{1}{M^2 + p^2 + G b p^4} ,
\]

(30)

\[
\left( \Gamma^{(2)} \right)^{-1}_3 = 32\pi Z \frac{2}{-M^2 - 4p^2 + 8G p^4(3a + b)} ,
\]

(31)

where we immediately identify the well-known factor \( (3a + b) \) attributed to the conformal combination \( R_{\mu\nu} R^{\mu\nu} - 1/3 R^2 \).

The choice of the regulator is a crucial ingredient in the construction of the renormalization group flow. We choose a regulator that enables us to do completely analytical calculations. Such a regulator is given by the Litim regulator

\[
\left( R_k(q^2) \right)_i = \left( \Gamma^{(2)}_{1i} \right)_i \left( k - \Gamma^{(2)}_{1i}(q) \right) \theta(k^2 - q^2) P_i ,
\]

(32)

adjusted for each component of the two-point function \( \Gamma^{(2)}_{1i}(q^2) \). This can easily be rewritten as

\[
\left( R_k(q^2) \right)_i = \Gamma^{(2)}_{1i} r_k(q^2) P_i
\]

(33)

with the dimensionless shape function

\[
r_k(q^2) = \left( \frac{\Gamma^{(2)}_{1i}(k^2)}{\Gamma^{(2)}_{1i}(q^2)} - 1 \right) \theta(k^2 - q^2) .
\]

(34)

It is worth noting, that the Landau limit \( \alpha \to 0 \) cannot be taken in the regulator \( R \), but the propagator \( \mathcal{G} \) and the product of propagators with the scale-derivative regulator \( \mathcal{G} R \mathcal{G} \) that enter the flow equations are finite in this limit.

### III. BETA-FUNCTIONS

With the construction of the vertices, the propagator and the regularization of the last sections, we are now in a position to derive the \( \beta \)-functions \( \beta_{M^2}, \beta_a, \beta_b \) and the anomalous dimension \( \eta \) via a suitable projection of the flow equation (17).

#### A. Projection

From the equations (24, 25, 26, 27, 28) for the two-point functions one can see that the \( TT \)-mode is independent of the gauge fixing, as is the \( p^4 \) coefficient of the trace-mode. Moreover, as we have already pointed out, the \( p^4 \) coefficient of the former receives contributions from the \( \text{Ric}^2 \) operator only, while the latter also contains contributions from the \( R^2 \) term. We exploit this fact for the definition of the projections on the running couplings. Introducing the operator \( \circ \) that denotes full contraction of tensor indices, we obtain

\[
\partial_t \mu = -2 \mu + \mu \eta + \frac{32\pi}{5} \lim_{p \to 0} \frac{\partial^2}{\partial p^2} \left( \mathcal{P}_{TT} \circ \text{Flow}^{(2)} \right) ,
\]

(35)

for the running of the mass parameter,

\[
\eta = -\frac{16\pi}{5} \lim_{p \to 0} \frac{\partial^2}{\partial p^2} \left( \mathcal{P}_{TT} \circ \text{Flow}^{(2)} \right) ,
\]

(36)

for the anomalous dimension \( \eta \) and

\[
\partial_t a = -\frac{\partial b}{3} + \frac{(3a + b)(g(2 + \eta) - \partial_t g)}{3g} + \frac{\pi k^2}{9g} \lim_{p \to 0} \frac{\partial^4}{\partial p^4} \left( \mathcal{P}_{Tr} \circ \text{Flow}^{(2)} \right) ,
\]

(37)
as well as
\[ \partial b = \frac{b}{g} (g(2 + \eta) - \partial_t g) + \frac{4\pi k^2}{15 g} \lim_{p \to 0} \frac{\partial^4}{\partial p^4} \left( P_{TT} \circ \text{Flow}^{(2)} \right), \]

for the four-derivative couplings.

There are some subtleties concerning the above momentum projection. As described in the previous section we employ a Litim cutoff. It is well-known that this cutoff does not allow an expansion in powers of $p^2$ of the right-hand side of the flow equation, but with such a cutoff the flow is an expansion in the absolute value $p$, where odd powers of $p$ appear beyond quadratic order [95]. However, we expect that this is not a problem here, as we can always introduce a smooth version of the Litim-cutoff by the replacement $\theta$, with $\theta \to \theta$ in the limit $\epsilon \to 0$, i.e. just a smeared version of the $\theta$-function. For any finite $\epsilon$ the flow does allow an expansion in $p^2$ and the limit $\epsilon \to 0$ exists on both sides of the flow equation. Therefore, we expect that there should be no qualitative difference between the Litim regulator and its smoothened counterpart. Nevertheless, it is certainly true that the cutoff employed in this work is not optimized in the present fourth-order approximation. Optimization and the convergence of the derivative expansion is discussed in [87, 96, 97].

There are further technical difficulties arising due to derivatives of the $\theta$-function. The momentum derivatives of the $\theta$-function immediately produce $\delta$-functions and derivatives thereof. In the limit $p \to 0$ these distributional products are not well-defined. However, with a proper treatment these ill-defined terms do not appear. These mathematical problems are solved in the appendix, section B.

B. Fixed Points of Non-Perturbative Beta-functions

The construction described in the last sections leads to the set of $\beta$ functions for the couplings $a$, $b$ and $M^2$, as well as an algebraic equation for the anomalous dimension. These equations also depend on the gravitational constant $G$, which is in principle obtained from the three–point function $\Gamma^{(3)}$. First, we close this system of equations by using an equation for the beta-function $\beta_G$ of the gravitational coupling obtained from the flow equation for $\Gamma^{(3)}$ in a vertex expansion with Einstein-Hilbert tensor structures, [23]. With this $\beta$-function for the gravitational coupling we ignore the direct feedback of the higher derivative couplings $a$ and $b$ to the running of the gravitational coupling. More precisely, in a fully consistent calculation where the gravitational coupling is obtained from a three–point function including higher derivative structures, there will be terms proportional to $a$ and $b$ in the $\beta$-function for $g$. Here, the feedback is only indirect via the dependence of $\mu$ and $\eta$ on $a$ and $b$. The fixed point analysis is then formulated for the dimensionless couplings, and we define $g = Gk^2$ and $\mu = M^2/k^2$. In total, we obtain the set of equations
\[ \left( \dot{g}, \dot{\tilde{Z}}, \dot{\mu}, \dot{a}, \dot{b} \right). \]

These $\beta$-functions are all derived in closed analytic form, however the expressions are way too bulky in order to be given explicitly in this paper.

The fixed point condition for this set of couplings reads $\beta_a = \beta_b = \beta_\eta = \beta_\mu = 0$, whereas the anomalous dimension $\eta$ takes a value dynamically determined by the fixed point values of the couplings. The number of relevant and irrelevant directions is determined by the properties of the linearized flow around the fixed point, which, in turn is characterized by the eigenvalues $\theta_i$ of the stability matrix $B$. The stability matrix is given by
\[ B_{ij} = \left. \frac{\partial^2 \beta_{g_i}}{\partial \theta_j} \right|_{g_i = g_i^*}, \]

where $\{g_i\}$ represents the set of all coupling constants. Negative eigenvalues $\theta_i$ of the stability matrix indicate a relevant direction, while positive eigenvalues belong to an irrelevant direction. An irrelevant direction implies that one parameter is fixed by the asymptotic safety condition. More precisely, one initial condition of the flow is fixed and therefore the evolution of the coupling with energy is determined by the theory. Consequently, the value of this coupling constant at an arbitrary energy scale is a prediction of the theory.

An analytical solution of the full equations is not possible, but numerically the system exhibits several fixed points. However, only one fixed point has eigenvalues $\theta_i < 10$ of the stability matrix and obeys the constraints $g_i > 0$ and that all couplings are real valued. This fixed point has the coordinates
\[ (g^*, \mu^*, a^*, b^*, \eta^*) = (0.43, -0.34, -0.41, 0.91, 0.77) \]

with eigenvalues of the stability matrix given by
\[ (\theta_1, \theta_2, \theta_3, \theta_4) = (-1.5 - 2.7i, -1.5 + 2.7i, 2.4, 8.3) \]

A very important property of this fixed point is that it is characterized by two irrelevant and two relevant directions. In all Einstein-Hilbert like approximations, i.e. where one retains only two couplings, one finds two negative eigenvalues, i.e. two relevant directions. In our case, we included the four-derivative couplings, but the UV-critical surface remains two-dimensional. Interestingly, our result differs in this respect from the structure of the non-Gaussian fixed point in four-derivative gravity found within background field flows, where the UV-critical surface is three-dimensional, [13]. However, we note that including the full feedback of the higher derivative couplings into the $\beta$-function for the gravitational coupling

can in principle turn an irrelevant direction into a relevant one. In the light of the asymptotic safety conjecture, our result is very encouraging, as predictivity is encoded in a finite dimensional UV-critical surface. The four-derivative couplings $a$ and $b$ are classically marginal, but quantum corrections turn them into irrelevant couplings. Couplings related to even higher derivatives are classically irrelevant, and increasingly large quantum fluctuations would be necessary in order to form further relevant directions. Based on our results, it is reasonable that this does not happen in the present case, as even the marginal couplings turn irrelevant. Moreover, higher order polynomial expansions of $f(R)$ truncations with background field flows show near-Gaussian scaling for higher order operators $[29, 32]$, and it is reasonable that this pattern translates also to the fluctuation field flow equations.

In the non-perturbative beta-functions above, resummations to infinite order are included by the anomalous dimensions on the right-hand side of the flow equations, as well as by the non-trivial $g$-dependence in the propagators. Both aspects turn the right-hand side into a infinite power series in $g$, whose convergence depends on the value of the coupling. There are two aspects one needs to take into account. First of all, the complicated structure of the resummations obviously induces many fixed points, which might either have unphysical properties or are truncation artefacts. This is particularly important as the resummations are all regulator dependent. The potentially dangerous ones are the latter ones, as they appear at first sight as fixed points with viable physical properties, such as positive Newton coupling and real-fixed point values of all couplings. Nevertheless, some of them might be only present in the truncated theory and will disappear once further improvements of the truncation are taken into account. Sometimes such fixed points reveal themselves by very large critical exponents. However, it has also been observed, that by including higher order operators and new couplings, the critical exponents can grow quite large, but converge to smaller values once even further improvements of the truncation are taken into account $[29, 32]$. This makes it difficult to ensure that a fixed point does not fall into this class. We rush to mention that the fixed point found in the previous section (41) with the moderate eigenvalues (42) is not in this class, nevertheless, it is important to check the reliability of this fixed point. One way of doing this is to expand the right hand-sides of the $\beta$-functions in powers of the resummation parameter, i.e. in powers of the coupling $g$ in our case. This removes artificial zeroes of the beta functions, however, in principle it can also remove fixed points that are induced by non-perturbative effects. In addition to that, this procedure makes sense only by expanding to lowest order, as at higher orders artificial zeros are created again, and for obvious reasons polynomial expansions are particularly dangerous in that respect.

Therefore, we expand the full beta functions (35, 36, 37, 38) to first order in $g$ and solve the fixed point equations. As a result we find

\[(g^*, \mu^*, a^*, b^*, \eta^*) = (0.59, -0.29, -0.35, 0.51, 0.56)\]

with critical exponents

\[(\theta_1, \theta_2, \theta_3, \theta_4) = (-2.2 + 2.3 i, -2.2 - 2.3 i, 2, 2.5)\]

which agrees rather well with the fixed point (41). Interestingly, all other fixed points have at least one coupling constant with a complex fixed point value and are therefore unphysical. This result provides evidence that the fixed point of the full equations including all resummations is not just a truncation artefact.

### C. Stability Check: Expansion of Threshold Functions and Corresponding Fixed Points

In the fixed point analysis presented above we have used an equation for $\beta_g$ from the flow of the three-point function, but with Einstein-Hilbert tensor structures. We have already mentioned that this is an approximation where there is no direct feedback of the four-derivative couplings to $\beta_g$, but only an indirect feedback via $\eta$ and $\mu$. Therefore, it is interesting to study the behavior of the fixed point by treating $g$ as a free parameter. This means that we solve the system of equations given by $\beta_a = \beta_b = \beta_\mu = 0$ and the algebraic equation for the anomalous dimension $\eta$. We find a continuous deformation of the fixed point (43) in the range $g \in [0, \approx 0.7]$, where in the limit $g \rightarrow 0$ the fixed point turns into the Gaussian fixed point, whereas for $g \gtrsim 0.7$ it turns into a pair of complex conjugate fixed points.

### D. Parametric Study of the Fixed Point

As we have already discussed, a classical theory of gravity based on the higher derivative action (7) is perturbatively renormalizable due to the $p^{-4}$-propagator. Moreover, the coupling constants $a$ and $b$ are dimensionless, and therefore universal, in the sense that they are independent of the regularization. The perturbative 1-loop running of the $Ric^2$ coupling $b$ is given by the beta-function

\[\beta_b^{\text{pert}} = \frac{1}{16\pi^2} \frac{133}{16} \approx 0.084.\]  

As this beta function is constant there is no fixed point on the perturbative one-loop level in the coupling $b$. However, by changing the basis in the space of four derivative
classically marginal operators to the parameterization (8), one finds in four dimensions \( b = s^{-1} \), and therefore

\[
\beta_{s}^{\text{pert}} = -\frac{1}{(4\pi)^2} \frac{133}{10} s^2, \tag{46}
\]

i.e. the coefficient is negative and therefore the coupling is asymptotically free. This result is found within perturbative calculations for the one-loop effective action [16, 81, 82] and with the Functional Renormalization Group and background field flows [10, 13]. In the present setup, the calculation differs from these, as we calculate the fluctuation field couplings. In our setup the one-loop coefficient reads

\[
\beta_{b}^{\text{pert}} = \frac{29}{32\pi^2} \approx 0.092, \tag{47}
\]

which differs from (45) by a factor of 1.1. Therefore, the one-loop coefficient is not universal in the sense that background and fluctuation field calculations agree. The reason for this disagreement is most likely rooted in the non-triviality of Nielsen identities at this fixed point. Also gauge- and parameterization dependencies might play a role here. This very important issue is under current investigation and the results will be reported elsewhere.

**IV. CONCLUSIONS**

In this work we have studied the graviton propagator of four-derivative quantum gravity non-perturbatively. We used an expansion of the effective action in terms of the dynamical graviton field and flow equations for the vertex functions derived from the Wetterich equation. This enabled us to determine the scale dependence of the Einstein-Hilbert and four-derivative couplings as well as the graviton anomalous dimension. We analyzed the resulting \( \beta \)-functions and found a non-Gaussian fixed point, which can be understood as an extension of the well-known fixed point in Einstein-Hilbert gravity. An important property of this fixed point is that there are only two relevant directions, i.e. the same number as in the Einstein-Hilbert truncation. This means that the two classically marginal operators \( R^2 \) and \( R_{\mu\nu}^2 \) do not generate further relevant directions. This is of particular interest as a relevant direction always corresponds to a free parameter, which needs to be fixed by external input, e.g. some measurement. Therefore, a fixed point describes a predictive theory only if the number of relevant directions is finite.

Based on this work, there are several important directions that one can pursue. First of all, it would be interesting to include even further invariants in order to test the conjecture that classically irrelevant operators stay irrelevant. The present work provides the first step in systematically including more tensor structures in vertex expansions. Moreover, a genuine flow of the gravitational coupling \( g \) from the graviton three-point and four-point function along the lines presented in [23] but with higher derivative operators would provide very valuable insights. These two studies are of major importance in order to test an apparent convergence in vertex expansions, see also [91]. Furthermore, the stability of derivative expansion around \( p = 0 \) should be tested in a systematic way. In addition to that, the flow away from the fixed point towards the infrared encodes the information whether the non-trivial ultraviolet fixed point is indeed connected to infrared physics that describes general relativity. This is essential for the question if the non-Gaussian fixed point is a viable candidate for the construction of the continuum limit. In order to connect with reality, it is also inevitable to study the coupling to gauge and matter fields.

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**Appendix A: Tensor decomposition of the graviton propagator**

As the two-point function needs to be inverted in order to obtain the propagator, we represent \( \Gamma^{(2)} \) in a quasi-projector basis given by the six projection operators \( P_{TT} = P_{1} \), \( P_{V} = P_{2}, P_{3}, P_{4}, P_{5}, P_{6} \). The index \( TT \) indicates that this operator projects on the transverse-traceless tensor structure of a symmetric rank two tensor, and analogously \( V \) refers to the vector mode, while \( P_{3} \) and \( P_{4} \) project on two scalar components. The operators \( P_{5} \) and \( P_{6} \) generate mixings in the scalar sector, as we will see below. In terms of the well-known transverse and longitudinal projectors

\[
\Pi_{TT,\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}, \tag{A1}
\]

and

\[
\Pi_{L,\mu
u}(p) = \frac{p_{\mu}p_{\nu}}{p^2} \tag{A2}
\]

the above operators read

\[
P_{1,\mu\nu\alpha\beta}(p) = \frac{1}{2} (\Pi_{TT,\mu\alpha}(p)\Pi_{TT,\nu\beta}(p) + \Pi_{TT,\mu\beta}(p)\Pi_{TT,\nu\alpha}(p))
- \frac{1}{3} \Pi_{TT,\mu\nu}(p)\Pi_{TT,\alpha\beta}(p), \tag{A3}
\]

\[
P_{2,\mu\nu\alpha\beta}(p) = \frac{1}{2} (\Pi_{TT,\mu\alpha}(p)\Pi_{L,\nu\beta}(p) + \Pi_{TT,\mu\beta}(p)\Pi_{L,\nu\alpha}(p)
+ \Pi_{TT,\nu\alpha}(p)\Pi_{L,\mu\beta}(p) + \Pi_{TT,\nu\beta}(p)\Pi_{L,\mu\alpha}(p)), \tag{A4}
\]

\[
P_{3,\mu\nu\alpha\beta}(p) = \frac{1}{3} \Pi_{TT,\mu\nu}(p)\Pi_{TT,\alpha\beta}(p), \tag{A5}
\]
\[ P_{4,\mu\nu\alpha\beta}(p) = \Pi_{L,\mu\nu}(p)\Pi_{L,\alpha\beta}(p), \quad (A6) \]
\[ P_{5,\mu\nu\alpha\beta}(p) = \frac{1}{\sqrt{3}} \Pi_{T,\mu\nu}(p)\Pi_{L,\alpha\beta}(p), \quad (A7) \]
\[ P_{6,\mu\nu\alpha\beta}(p) = \frac{1}{\sqrt{3}} \Pi_{L,\mu\nu}(p)\Pi_{T,\alpha\beta}(p). \quad (A8) \]

There is an orthogonal subset of these projectors given by
\[ P_iP_j = \delta_{ij}P_j \quad \text{with} \quad i = 1, 2, 3, 4. \quad (A9) \]

The relations for the transfer operators \( P_5 \) and \( P_6 \) are a bit more complicated. It is advantageous to introduce two more indices and map the old index \( i \in (3, 4, 5, 6) \) to the index set \((a, b) \in ((3, 3), (3, 4), (4, 3), (4, 4))\) such that the operators in the scalar sector can be grouped into a two-by-two matrix. The mapping is done such that \( P_3 = P_2, P_4 = P_3, P_5 = P_3, P_6 = P_3 \). The relations of the scalar projectors in this language read
\[ P^{ab}P^{cd} = \delta^{bc}P^{aa} \quad \forall d \quad \text{and} \quad a \neq b, c \neq d \quad (A10) \]
\[ P^{aa}P^{bc} = \delta^{ab}P^{ac} \quad \text{with} \quad b \neq c \quad (A11) \]
\[ P^{ab}P^{cc} = \delta^{bc}P^{ac} \quad \text{with} \quad a \neq b, \quad (A12) \]
and all scalar operators are orthogonal to \( P_1 \) and \( P_2 \). As one can see from these operator relations, the off-diagonal operators \( P_5 = P_3^4 \) and \( P_6 = P_3^4 \) induce a mixing in the scalar sector and therefore they are also called the spin-zero transfer operators. As the operators \( P_i \) form a complete set, the two point function can be expanded as
\[ \Gamma^{(2)} = \sum_{i=1}^{6} \Gamma_i^{(2)} P_i. \quad (A13) \]

Obviously, this operator set is not orthogonal in the scalar sector due to the non-trivial algebra. The coefficients in the above expansion are obtained as follows. We define a four-by-four matrix \( a \) with diagonal elements according to
\[ \Gamma_i^{(2)} = a_{ii} = \frac{\text{Tr} (P_i \Gamma^{(2)} P_j)}{\text{Tr} (P_i P_j)} \quad \text{with} \quad i \in (1, 2, 3, 4). \quad (A14) \]

and off-diagonal elements \( a_{34} \) and \( a_{43} \). Using the operator relations above one can easily obtain that \( a_{34} = \Gamma_5^{(2)} = \text{Tr} P_5 P_3 \Gamma^{(2)} P_4 \) and \( a_{43} = \Gamma_6^{(2)} = \text{Tr} P_6 P_4 \Gamma^{(2)} P_3 \). This four-by-four matrix can then be inverted and it is easy to show that
\[ \left( \Gamma^{(2)} \right)^{-1} = \sum_{i=1}^{6} \left( \Gamma_i^{(2)} \right)^{-1} P_i, \quad (A15) \]
with the coefficients \( (\Gamma_i^{(2)})^{-1} \) obtained from the inverse coefficient matrix \( a^{-1} \), is indeed the inverse of \( \Gamma^{(2)} \).

**Appendix B: Momentum Derivatives and Distributions**

In this appendix we will derive a general formula for projecting on the \( p^2 \) and the \( p^4 \) coefficients of the flow at zero momentum and with a Litim regulator. The derivation is readily generalized for higher order derivatives. Moreover, the following derivation can be applied for general theories, not only for gravity. In order to emphasize the structure of the following calculation, we write the propagator in the form
\[ G(p) = \frac{1}{Z(M^2 + \alpha p^2 + \beta G p^4)(1 + r(p^2 / k^2))}, \quad (B1) \]
with a mass term \( M^2 \) and coefficients \( \alpha \) and \( \beta \). The regulator shape function for the Litim regulator and the above propagator then takes the form
\[ r \left( \frac{p^2}{k^2} \right) = \left( \frac{M^2 + \alpha k^2 + \beta G k^4}{M^2 + \alpha p^2 + \beta G p^4} - 1 \right) \theta(k^2 - p^2). \quad (B2) \]

All the components of the graviton propagator are of the form \( (B1) \) times a tensor structure. Therefore, it is sufficient for the following general analysis to assume such a form of the propagator. The main goal is to find a closed expression for
\[ \lim_{p \to 0} \frac{\partial^n}{\partial p^n} \left( P \circ \text{Flow}^{(2)} \right), \quad (B3) \]
where \( n = 2, 4 \) and \( P \) is an operator that projects on a component of the propagator, which is of the form \( (B1) \). We will also see that the generalization to arbitrary \( n \) is straightforward.
The RHS of equation (17) is proportional to \( \dot{R} \), which is given by

\[
\dot{R}(q^2) = \left( \Gamma^{(2)}(k) - \hat{\Gamma}^{(2)}(q) \right) \theta(1 - \frac{q^2}{k^2}) + 2 \frac{q^2}{k^2} \left( \Gamma^{(2)}(k) - \Gamma^{(2)}(q) \right) \delta(1 - \frac{q^2}{k^2}) = \left( \Gamma^{(2)}(k) - \hat{\Gamma}^{(2)}(q) \right) \theta(1 - \frac{q^2}{k^2}), \tag{B4}
\]

where the last equals sign is of course understood in the distributional sense. Therefore, the RHS of the flow equation will always be proportional to \( \theta(1 - \frac{q^2}{k^2}) \). Consequently, all theta functions \( \theta(1 - \frac{q^2}{k^2}) \) that appear in the propagators can be set to one, as the loop integral vanishes for \( q^2 > k^2 \). This structure makes the tadpol diagram in the flow equation (17) very easy, as there is no momentum-dependent propagator and the momentum derivatives hit only the vertex. The derivatives of the contracted tadpol diagram can then be written as

\[
\int_{\mathbb{R}^4} \frac{d^4q}{(2\pi)^4} \Theta(k^2 - q^2) \frac{1}{(M^2 + \alpha k^2 + \beta G k^4)^2} \frac{\partial^n}{\partial p^n} f_1(p, q), \tag{B5}
\]

where \( f_1 \) depends on the vertex and is a regular function of its arguments. Moreover, \( f_1 \) also depends on the couplings and the anomalous dimensions, but this dependence is irrelevant here. In the self-energy diagram, there are momentum dependent propagators \( G(p + q) \) and corresponding \( \theta \) functions. These are treated with a case-by-case analysis, i.e., we distinguish the cases where the \( \theta \)-function parameterized as (B1) takes the form

\[
f_2(p, q) \times \begin{cases} \left( M^2 + \alpha k^2 + \beta G k^4 \right)^{-1} & \text{if } \theta(k^2 - (p_\mu + q_\mu)^2) = 1 \\ \left( M^2 + \alpha k^2 + \beta G k^4 \right)^{-1} & \text{if } \theta(k^2 - (p_\mu + q_\mu)^2) = 0 \end{cases}, \tag{B6}
\]

where \( f_2 \) is again a function that depends on the vertices and is a regular. Due to this regularity, the explicit form of \( f_2 \) is not relevant for the following. We then introduce the definitions

\[
c_1 := \frac{1}{(M^2 + \alpha k^2 + \beta G k^4)^2}, \tag{B7}
\]

and

\[
c_2 = \hat{c}_2 \hat{c}_2 \quad \text{and} \quad c_3 = \hat{c}_3 \hat{c}_3, \tag{B8}
\]

with

\[
\hat{c}_2(q) := \frac{1}{(M^2 + \alpha k^2 + \beta G k^4)^2} \tag{B9}
\]

and

\[
\hat{c}_2(p, q, x) := \Theta \left( k^2 - (p_\mu + q_\mu)^2 \right) \tag{B10}
\]

Analogously

\[
\hat{c}_3(p, q, x) := \frac{1}{(M^2 + \alpha k^2 + \beta G k^4)^2 \left( M^2 + \alpha (p_\mu + q_\mu)^2 + \beta G (p_\mu + q_\mu)^4 \right)} \tag{B11}
\]

and

\[
\hat{c}_3(p, q, x) := \Theta \left( (p_\mu + q_\mu)^2 - k^2 \right) \tag{B12}
\]

where \( x = \cos(\theta) \), and \( \theta \) is the angle between the external momentum \( p_\mu \) and the loop momentum \( q_\mu \), whose absolute values are denoted as \( p \) and \( q \) respectively. The \( p \)-dependent terms in the \( q \)-integrals in the flow, i.e., those which depend on the external momentum, are then given by \( f_1, f_2, \hat{c}_2, \hat{c}_3, \hat{c}_3 \).

The \( n \)-th momentum derivative of the flow is then expressed as

\[
\lim_{p \to 0} \frac{\partial^n}{\partial p^n} \left( \mathcal{P} \circ \text{Flow}^{(2)} \right) \sim \lim_{p \to 0} \frac{\partial^n}{\partial p^n} \int_0^\infty \int_{-1}^1 dq \, dq \, q^3 \sqrt{1 - x^2} \theta(k^2 - q^2) \left( -\frac{1}{2} f_1 c_1 + f_2 (\hat{c}_2 \hat{c}_2 + \hat{c}_3 \hat{c}_3) \right), \tag{B13}
\]
where we have transformed to 4-d spherical coordinates. First we will evaluate this for \( n = 2 \). In order to do so, we interchange the \( p \)-derivatives with the \( q \)-integral at fixed, finite \( p \), which is perfectly well-defined. Taking two derivatives of the integrand yields

\[
\partial_p^2 \left\{ -\frac{1}{2} f_1 c_1 + f_2 (\dot{c}_2 \ddot{c}_2 + \dddot{c}_3) \right\}
\]

\[
= \partial_p \left\{ -\frac{1}{2} f''_1 c_1 + f'_2 (\dot{c}_2 \dddot{c}_2 + \dot{c}_3 \dddot{c}_3 + \dddot{c}_3) \right\}
\]

\[
= \partial_p \left\{ -\frac{1}{2} f''_1 c_1 + f'_2 (\dot{c}_2 \dddot{c}_2 + \dot{c}_3 \dddot{c}_3) + f_2 (\dddot{c}_3) \right\}
\]

\[
= -\frac{1}{2} f''_1 c_1 + f'_2 (c_2 + c_3) + f_2 (\dddot{c}_3)
\]

\[
= -\frac{1}{2} f''_1 c_1 + f'_2 (c_2 + c_3) + f_2 (\dddot{c}_3)
\]

\[
(B14)
\]

where we used the weak identity \( \dddot{c}_3 = -\dddot{c}_2 \ddot{c}_2 \) and the convention \( \partial_p f \equiv f' \) for any function \( f \). In the above equation there is one term proportional to \( \dddot{c}_3 \sim \delta((p_\mu + q_\mu)^2) - k^2 \). Already in second order derivative expansion this term is problematic as one cannot simply interchange the \( p \rightarrow 0 \) limit with the \( q \)-function as this would produce terms proportional to \( \sim \Theta(k^2 - q^2) \times \delta(q^2 - k^2) \), which is not well defined since the contribution of the delta function is exactly at the discontinuity of the Heaviside function. Moreover, subsequent \( p \)-derivatives will generate \( \delta' \)-terms. The trick in order to deal with these terms is that we evaluate the angular integrals for finite \( p \), such that there are no more \( \delta \)-distributions involved. Then, we rewrite the terms such that the limit \( p \rightarrow 0 \) can be taken safely. For the \( p^4 \) coefficient the limit is taken after further \( p \)-derivatives. More precisely, this trick works as follows. First we note that all the terms containing \( \delta \)-functions will be of the form

\[
\int_0^\infty \int_{-1}^1 dq dx q^3 \sqrt{1 - x^2} \Theta(k^2 - q^2) f^{(n)}(p,q,x) \dddot{c}_3(p,q,x),
\]

with the standard notation for the \( n \)-th derivative with respect to \( p \). We rewrite the \( \delta \)-function as a function of the angular integration variable \( x \) according to

\[
\delta(f(x)) = \delta(q^2 + p^2 + 2pqx - k^2) = \frac{1}{2pq} \delta(x - \frac{k^2 - p^2 - q^2}{2pq}).
\]

Then, we perform the \( p \)-derivatives acting on the various factors in the terms, before killing the angular integration with the \( \delta \) function. After this we are left with terms of the form

\[
\lim_{p \rightarrow 0} \int_0^\infty \int_{-1}^1 dq dx q^3 \sqrt{1 - x^2} \Theta(k^2 - q^2) f^{(n)}(p,q,x) \dddot{c}_3(p,q,x) \frac{2p + 2qy}{2pq} \delta(x - \frac{k^2 - p^2 - q^2}{2pq})
\]

\[
= \lim_{p \rightarrow 0} \int_{k-p}^k dq q^3 \left( \frac{k^2 - p^2 - q^2}{2pq} \right)^2 f^{(n)}(p,q,x) \frac{2p + 2qy}{2pq} \delta(x - \frac{k^2 - p^2 - q^2}{2pq})
\]

\[
\times \dddot{c}_3(p,q,x = \frac{k^2 - p^2 - q^2}{2pq}) \frac{1}{2pq} \left( \frac{k^2 - p^2 - q^2}{2pq} \right),
\]

where the new domain of integration of the \( q \)-integral arises due to the condition that the contribution of the \( \delta \)-function is in the domain of integration, which is equivalent to

\[
\frac{k^2 - p^2 - q^2}{2pq} \in (-1, 1).
\]

The best way to solve these integrals is to keep in mind that we are interested in the limit \( p \rightarrow 0 \). Therefore, we want to exploit the fact that the domain of integration vanishes proportional to \( p \) and that all terms of order larger than \( p^{-1} \) will vanish after integration. Terms of order \( p^{-n} \) with \( n > 1 \) cannot occur in these expressions. We proceed by writing \( q = k - \epsilon \) with \( \epsilon = yp \) and transform the integral according to

\[
\lim_{p \rightarrow 0} \int_{k-p}^k dq F(p,q) = -\lim_{p \rightarrow 0} \int_0^1 dyF(p,k - yp).
\]

\[
(B19)
\]
Interchanging the limit $p \rightarrow 0$ with the $y$ integration is now trivial and perfectly well defined. Now we can proceed with the derivation of a master formula for the fourth momentum-derivative of the flow. Taking two further derivatives of (B14) with respect to $p$ and using $\hat{\epsilon}_3^\prime \hat{\epsilon}_3^\prime = -\hat{\epsilon}_2^\prime \hat{\epsilon}_2^\prime$ after each derivative, we arrive at

$$
\partial_p^2 \left\{ -\frac{1}{2} f_1 c_1 + f_2 (\hat{\epsilon}_2 \hat{\epsilon}_2 + \hat{\epsilon}_3 \hat{\epsilon}_3) \right\} = -\frac{1}{2} f_1^{(4)} c_1 + f_2^{(4)} (c_2 + c_3) + 4 f_2^{(3)} \epsilon_3 \hat{\epsilon}_3 + 6 f_2^{(2)} \epsilon_3^\prime \hat{\epsilon}_3 + 3 f_2^{(2)} \epsilon_3^\prime \hat{\epsilon}_3^\prime + 4 f_2^{(1)} \epsilon_3 \hat{\epsilon}_3 \\
+ 3 f_2^{(1)} \epsilon_3^\prime \hat{\epsilon}_3^\prime + f_2 \epsilon_3 \hat{\epsilon}_3 + f_2 \epsilon_3 \hat{\epsilon}_3 + \partial_p^2 (f_2 \epsilon_3 \hat{\epsilon}_3) + 2 \partial_p (f_2 \epsilon_3 \hat{\epsilon}_3) + \partial_p (f_2 \epsilon_3 \hat{\epsilon}_3) .
$$

(B20)

In the limit $p \rightarrow 0$ obviously $\hat{\epsilon}_3 \rightarrow 0$. Due to the overall $\theta(k^2 - q^2)$, these terms are exactly zero except for $q = k$. However, this is just one point, i.e. it is a domain with zero measure, and the whole integrand is finite at $q = k$. Consequently, the limit of vanishing momentum can taken before integration and all the terms proportional to $\hat{\epsilon}_3$ vanish. In the terms proportional to $c_1$ and $c_2$ the limit is also unproblematic, but the contributions are finite. Hence, we are left with

$$
\int_0^\infty \int_{\mathbb{R}^4} dq \, dx \, q^3 \sqrt{1-x^2 (k^2 - q^2)} \left\{ -\frac{1}{2} f_1^{(4)} (0, q) c_1 + f_2^{(4)} (0, q) c_2 \right\} + \lim_{p \rightarrow 0} \partial_p^2 \int_{\mathbb{R}^4} dq \, dx \, q^3 \sqrt{1-x^2 f_2 \epsilon_3 \hat{\epsilon}_3 \theta(k^2 - q^2)} \\
+ 2 \lim_{p \rightarrow 0} \partial_p \int_{\mathbb{R}^4} dq \, dx \, q^3 \sqrt{1-x^2 f_2 \epsilon_3 \hat{\epsilon}_3} + \lim_{p \rightarrow 0} \partial_p \int_{\mathbb{R}^4} dq \, dx \, q^3 \sqrt{1-x^2 (k^2 - q^2)} f_2 \epsilon_3 \hat{\epsilon}_3 \\
\lim_{p \rightarrow 0} \int_{\mathbb{R}^4} dq \, dx \, q^3 \sqrt{1-x^2 (k^2 - q^2)} (3 f_2^{(2)} \epsilon_3 \hat{\epsilon}_3 + 3 f_2^{(2)} \epsilon_3 \hat{\epsilon}_3 + f_2 \epsilon_3 \hat{\epsilon}_3) .
$$

(B21)

The terms proportional to $\hat{\epsilon}_3$ can be integrated according to the above prescription in order to eliminate the $\delta$-function, and the residual differentiation with respect to $p$ can then be taken afterward, see equations (B17) and (B19). As a check, this formula was applied to scalar field-theory where we found the correct $p^4$ coefficient.
