Determining the law of ice deformation

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Abstract. A mathematical model of nonlinear deformation of ice samples from distilled water is implemented. An algorithm for numerical solution of corresponding boundary-value problem is proposed and realized. Results of the numerical modeling have been compared to experimental data on the three-point bending of ice beams.

1. Introduction

Many researchers deal with the mechanics of ice in many parts of the world (see, for example, [1–11] and the references therein), since the ice structures are widely used in cold regions. However, ice is “intricate” and insufficiently studied in terms of building material because of many factors. Its physical and mechanical properties heavily depend on prehistory of the natural ice and process of preparation of the test samples, test conditions such as speed of loading and temperature, crystallographic characteristics of the ice, orientation and size of the grains, geometrical dimensions of the test samples, presence of natural or artificially introduced impurities and etc. It is also known that ice may be any one of the 18 known solid crystalline phases of water, or in an amorphous solid state at various densities. As a consequence, for example, published data on the ultimate strength of ice vary greatly [1, 2, 8, 10].

The present paper is devoted to simulating the bending of ice beams made of distilled water. Studies related to the definition of the deformation law of such samples will form the basis for the creation of mathematical models for calculating and analyzing the stress-strain state of composite materials on the basis of an ice matrix. Russia is the wealthiest country in terms of the deposits of natural ice, which is associated with the cold climate of a considerable part of its territory and adjacent northern seas. The development of country is to a large extent associated with the industrial and social development of the Arctic zone. Therefore, it is reasonable to use the ice composite materials in cold regions for constructing various buildings, roads [6], ice crossings, air strips and etc.

2. Statement of the problem and solution strategy

The problem involves determining the law of deformation based on experimental data on the three-point bending of ice samples (parallelepiped with the dimensions of $20 \times 50 \times 145$ mm$^3$). The tests were conducted on a Zwick/Roell Z050 all-purpose electromechanical machine (Germany) equipped with an environmental chamber (All-Russian Scientific Research Institute of Aviation
Figure 1. Experimental P-w diagram for the ice samples from distilled water. The readers are referred to [8,9] for more information about the experiments. Figure 1 shows the experimental data on the three-point bending of different ice samples (P-w diagram). Note that the experimental data range is sufficiently great, which proves the “intricacy” of ice. Additionally, at low values of loading and deformation, the experimental dependences (Fig. 1) strongly depend on the quality of the sample surface, i.e. a region of running in is present. Therefore, we are more interested in the section with sufficiently large loads, where nonlinear behavior is observed. The exact dimensions of these samples are given in Table 1.

The relative error

$$\|E_r\|^2_2 = \sqrt{\frac{\int_0^{P_{\text{max}}} (w_h - w_{ex})^2 dP}{\int_0^{P_{\text{max}}} w_{ex}^2 dP}}$$

was calculated after solving the direct problem for fixed values, where $P_{\text{max}}$ denotes the maximum load before cracking (fracture of the sample), $w_h$ is the numerical solution arising from the least squares collocation (LSC) method (section 3), and $w_{ex}$ is deflection experimental data. Note that the coefficient $a_1$ should not differ much from the initial elastic modulus. Figure 2 shows the solution strategy of this problem.

Figure 2. Solution strategy
3. Mathematical model and solving the direct problem

Let us consider the model of three-point bending of a beam with rectangular cross section \( b \times 2h \) and span \( l \) between the supports used by Amelina, Golushko et al. [12, 13]. Here, \( l = 125 \text{ mm} \) whilst the beam size \( L = 145 \text{ mm} \). Figure 3 shows the scheme of physical model. The left edge of the beam is hinged, while the right one is supported freely. The center of the beam bears a concentrated load. The model neglects the shape of the supports and assumes the occurring load \( P \) and support responses \( R_A \) and \( R_B \) to be concentrated. In addition, the model neglects the possible heterogeneity of the deformations in the direction normal both to the longitudinal direction and to the load direction. In this case, the beam’s upper part undergoes compression strain in the longitudinal direction, the bottom part — tension strain. It is assumed that the material reacts identically to tension and compression. However, this model is capable of taking into account the difference between tensile and compressive moduli of elasticity.

![Figure 3. The scheme of physical model of three-point beam bending with rectangular cross section](image)

Figure 3. The scheme of physical model of three-point beam bending with rectangular cross section

![Figure 4. System of coordinates](image)

Figure 4. System of coordinates

The Euler-Bernoulli beam theory can be used for description of the equilibrium state due to the very low deformation rates (speed of tie beam equals 1 mm/min). Let the reference surface of the beam coincide with the median one \( z = 0 \) (Fig. 4). The corresponding equilibrium equations are written as

\[
\frac{dN}{dx} = 0, \quad \frac{dQ}{dx} = 0, \quad \frac{dM}{dx} = Q, \tag{3.1}
\]

where \( Q(x) \) denotes the shear force, \( M(x) \) is the bending moment, \( N(x) \) is the longitudinal force.

The responses \( R_A \) and \( R_B \) are determined by ratio \( R_A = R_B = P/2 \). The bending moments at the support points equal zero: \( M_A = M_B = 0 \). The solution of (3.1) can be expressed as

\[
N = 0, \quad Q(x) = \begin{cases} 
  P/2, & 0 \leq x \leq l/2, \\
  -P/2, & l/2 \leq x \leq l,
\end{cases} \tag{3.2}
\]

\[
M(x) = \begin{cases} 
  Px/2, & 0 \leq x \leq l/2, \\
  -P(x - l)/2, & l/2 \leq x \leq l.
\end{cases}
\]

Strain distribution for the beam’s thickness can be obtained from the Kirchhoff-Love kinematic hypotheses:

\[
\varepsilon(x, z) = e(x) + z\kappa(x), \tag{3.3}
\]

\[
\kappa(x) = -\frac{d^2w}{dx^2}, \tag{3.4}
\]

where \( \varepsilon(x, z) \) denotes the strain in the beam; \( e(x) \) is the median surface strain, \( \kappa(x) \) is changes in the median surface curvature, and \( w(x) \) is the beam bending. Due to the fact that the deformation law for tension and compression is the same in this study, \( e(x) = 0 \). As mentioned above, the beam undergoes tension and compression strain. In this case for the section area
−h ≤ z ≤ 0, the strain will be negative, and for 0 ≤ z ≤ h positive. The constitutive equations can be expressed as
\[ \sigma^\pm (x, z) = f^\pm (\varepsilon), \] (3.5)
where the superscript “+” refers to the areas with positive strains and “−” — to the area with negative ones; \( f(\varepsilon) \) denotes the relationship between the stress and strain (a linear, quadratic, or cubic function).

Bending moment \( M \) in the beam cross section is determined by equation
\[ M = b \left( \int_{-h}^{0} \sigma^- z \, dz + \int_{0}^{h} \sigma^+ z \, dz \right). \] (3.6)

Having substituted (3.6) with relations (3.3), (3.5) and integrated it over the beam thickness, one obtains the equation to determine \( \kappa \):
\[ M(\kappa, x) = \begin{cases} \frac{P}{2}, & 0 \leq x \leq l/2, \\ -P(x - l)/2, & l/2 \leq x \leq l. \end{cases} \] (3.7)

Equation (3.7) in general case is nonlinear and the Newton method is applied to solve it. Thus, the linearized equation has the form:
\[ M(\kappa_0, x) + \frac{\partial M(\kappa_0, x)}{\partial \kappa}(\kappa - \kappa_0) = M(x), \]
it can be solved for unknown value
\[ \kappa = F(\kappa_0, M(x)), \]
where \( \kappa_0 \) is the initial approximation and \( M(x) \) is determined from (3.2).

As the initial approximation at small values of the load \( P \) the solutions obtained for the linear constitutive equations \( \sigma = a_1 \varepsilon \) were used. Since the computation is performed with a relatively small increment of \( P \), in case of big values of \( P \) one can use the computation results acquired at a previous step as the initial approximation for current step.

Having determined the change of median surface curvature, one can find the beam bend. It is necessary to solve the Dirichlet problem \( w(0) = w(l) = 0 \) for Poisson equation (3.4) in the one-dimensional case. For that purpose, the LSC method has been applied [14, 15]. Given the limits on the length of this paper, here we only note that an approximate solution of the differential problem is reduced to solving an overdetermined system of linear algebraic equations consisting of collocation equations, matching conditions, and boundary conditions in the LSC method.

4. Comparison of the simulation and experimental data
Figures 5–7 compare data of diagram \( P = P(w) \) arising from the experiment and simulation for different approximations \( \sigma = \sigma(\varepsilon) \). Note that \( \sigma^+ = a_1 \varepsilon + a_2^+ \varepsilon^2 + a_3 \varepsilon^3 \) and \( \sigma^- = a_1 \varepsilon + a_2^- \varepsilon^2 + a_3 \varepsilon^3 \), where \( a_2^+ = -a_2^- \) in case when material reacts identically to tension and compression. Applying the linear relationship between the stress and strain has not resulted in adequate approximation (Fig. 5). The error value is smaller in the case of a quadratic relationship compared with linear relationship. However, the application of the quadratic relationship has not resulted in a satisfactory match with experimental data for \( \sigma > 150 \text{ MPa} \) (Fig. 6). It was found out that the part of diagram with \( \sigma > 150 \text{ MPa} \) is poorly caught in case \( \sigma = a_1 \varepsilon + a_2^\pm \varepsilon^2 \) in a large number
of the numerical experiments. There is not much difference between the cubic relationship with and without \(a_2 = 0\) the quadratic term (Fig. 7, Fig. 8).

Table 1 shows the beam dimensions in mm, coefficient values when \(\sigma = a_1 \varepsilon + a_3 \varepsilon^3\), and relative errors \(\|E_r\|_2\). On top of that, this table also shows the average value for these parameters. It is important to note that the average value for \(a_1\) coincides in order with the bending modulus of

| Sample number | \(2h \times b \times L\) (mm\(^3\)) | \(a_1\) (MPa) | \(a_3\) (MPa) | \(\|E_r\|_2\)  |
|---------------|-----------------------------------|---------------|---------------|-------------|
| 1             | 19.8\(\times\)49.7\(\times\)145  | 3.45e+2       | 0             | 5.62e-2     |
| 2             | 20.4\(\times\)49.9\(\times\)145  | 1.3e+2        | 4.5e+6        | 3.32e-2     |
| 3             | 21.6\(\times\)50.5\(\times\)145  | 3.0e+1        | 2.7e+6        | 1.63e-2     |
| 4             | 21.0\(\times\)50.3\(\times\)145  | 3.0e+1        | 2.3e+6        | 1.31e-2     |
| 5             | 20.2\(\times\)50.4\(\times\)145  | 2.1e+2        | 9.3e+6        | 3.34e-2     |
| 6             | 17.6\(\times\)50.2\(\times\)145  | 1.4e+2        | 6.8e+6        | 3.48e-2     |
| 7             | 20.6\(\times\)49.05\(\times\)145 | 2.8e+1        | 4.5e+6        | 4.2e-2      |
| 8             | 19.05\(\times\)49.1\(\times\)145 | 2.4e+2        | 1.65e+7       | 3.34e-2     |
| 9             | 23.1\(\times\)48.9\(\times\)145  | 2.0e+2        | 4.0e+6        | 4.83e-2     |
| 10            | 24.45\(\times\)49.1\(\times\)145 | 4.0e+1        | 1.4e+6        | 2.88e-2     |

**Average value** | 20.78\(\times\)49.71\(\times\)145 | 1.39e+2 | 5.2e+6 | 3.40e-2 |
elasticity, which was determined by the displacement of the walking beam at the section of the diagram of loading from 1.5 to 2 MPa by the least squares method [8]. Interestingly, the linear relationship between the stress and strain is sufficient for sample 1.

5. Conclusions
The results of presented study are the following:
— the algorithm for determining the law of ice deformation based on the solution of the bending problem is proposed;
— the mathematical model for nonlinear flexural deformation of ice beams has been implemented;
— the LSC method has been proposed and realized for the numerical solution of the Dirichlet problem for the Poisson equation.

The mathematical model and numerical method developed have proved to be applicable to the problem of three-point bending of ice beams. A satisfactory match with the results of mechanical tests has been obtained. The range and average values of the coefficients in the calculated laws of deformation for ice beams have been established. It is shown that the best approximation to the bundle of experimental curves present the polynomial third-degree deformation law.

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