Hydrodynamic description of long-distance spin transport through noncollinear magnetization states: the role of dispersion, nonlinearity, and damping

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Nonlocal compensation of magnetic damping by spin injection has been theoretically shown to establish dynamic, noncollinear magnetization states that carry spin currents over micrometer distances. Such states can be generically referred to as dissipative exchange flows (DEFs) because spatially diffusing spin currents are established by the mutual exchange torque exerted by neighboring spins. Analytical studies to date have been limited to the weak spin injection assumption whereby the equation of motion for the magnetization is mapped to hydrodynamic equations describing spin flow and then linearized. Here, we analytically and numerically study easy-plane ferromagnetic channels subject to spin injection of arbitrary strength at one extremum under a unified hydrodynamic framework. We find that DEFs generally exhibit a nonlinear profile along the channel accompanied by a nonlinear frequency tunability. At large injection strengths, we fully characterize a novel magnetization state we call a contact-soliton DEF (CS-DEF) composed of a stationary soliton at the injection site, which smoothly transitions into a linear DEF and exhibits a negative frequency tunability. The transition between a DEF and a CS-DEF occurs at the maximum precessional frequency and coincides with the Landau criterion: a subsonic to supersonic flow transition. Leveraging the hydraulic-electrical analogy, the current-voltage characteristics of a nonlinear DEF circuit are presented. Micromagnetic simulations of nanowires that include magnetocrystalline anisotropy and non-local dipole fields are in qualitative agreement with the analytical results. The magnetization states found here along with their characteristic profile and spectral features provide quantitative guidelines to pursue an experimental demonstration of DEFs in ferromagnetic materials and establishes a unified description for long-distance spin transport.

I. INTRODUCTION

Noncollinear magnetization states represent a new paradigm for the transport of spin currents over micrometer distances\textsuperscript{1–10}. A key concept that has enabled the study of these states is the hydrodynamic interpretation of magnetization dynamics, proposed in the seminal paper by Halperin and Hohenberg\textsuperscript{11} in the context of spin wave dispersion relation in ferromagnets and antiferromagnets. Almost four decades later, a similar fluid-like interpretation was used to identify the relationship between an infinite-length, static noncollinear magnetization state in easy-plane ferromagnets and dissipationless spin transport\textsuperscript{12}. These states were characterized by a homogeneous normal-to-plane magnetization and a winding in-plane magnetization. More importantly, energy dissipation via damping was inoperative because the texture was assumed to be static. As a consequence, the mutual exchange torque exerted by neighboring spins could be interpreted as an equilibrium spin current or exchange flow\textsuperscript{12} that did not exhibit any dissipation.

While the prospect of a dissipationless spin current is tantalizing for novel energy-efficient applications\textsuperscript{11–18}, any magnetization dynamics are subject to dissipation via magnetic damping\textsuperscript{19}. An example is the interface between a magnetic material and a spin sink that results in spin pumping\textsuperscript{20}. To circumvent this problem, it is necessary to introduce energy into the system. From an analysis of the linearized hydrodynamic equations for a ferromagnet, it was predicted that spin injection at one extremum of a one-dimensional channel could sustain a dynamic, noncollinear magnetization state that was termed a spin superfluid\textsuperscript{1,2}. Despite the fact that this is a solution to the linearized, long-wavelength hydrodynamic equations, the magnetization vector itself exhibits fully nonlinear spatio-temporal excursions in the form of complete planar rotations. As we will later show, this solution results from a linearized WKB analysis of the equations of motion. The usage of the term superfluid was borrowed from a similarity between the order parameters that describe spin transport in a magnet and mass transport in, e.g., superfluid He\textsuperscript{4} as well as the fact that the normal-to-plane magnetization is approximately constant along the channel, although very small. However, this so-called spin superfluid experiences energy loss via a spatially diffusing spin current, yet its uniform precessional frequency and linearly decaying spin current profile present potential advantages to the exponential decay property of magnons. Similar states have been predicted for antiferromagnets\textsuperscript{7,8,21,22} and their experimental evidence in such materials has been recently presented\textsuperscript{23,24}.

In order to avoid potential misinterpretation of the term spin superfluid and to emphasize the nonlocal compensation of damping along the channel by the exchange torque that originates from spin injection at the device boundary, we will refer to spin superfluids and their generalizations as dissipative exchange flows or DEFs for short.

A more realistic setting for easy-plane ferromagnetic materials must consider the effect of in-plane anisotropy that breaks axial symmetry. For this configuration, it was shown that the hydrodynamic equations of motion map to a damped sine-Gordon equation, with a nonlinear term proportional to the in-plane anisotropy strength.
Because of the broken symmetry imposed by in-plane anisotropy, the structure of a DEF is that of a translating train of Néel domain walls or a soliton lattice with the same chirality and whose inter-wall spacing increases as each domain wall propagates from the spin injection edge to the opposite free spin edge. In the limit of vanishing anisotropy, the train of domain walls smooths into a sinusoidal profile, equivalent to the previously studied, axially symmetric case.

The most striking feature of a DEF is that its spatial structure and coherent precessional frequency depend on the length of the channel. It is a solution to a boundary value problem whereby the channel’s extrema are subject to spin injection and spin pumping or free spin boundary conditions, respectively. As a result, these solutions exhibit peculiar characteristics of technological relevance, namely: the spin injection threshold is proportional to the square root of the in-plane anisotropy field for long channels and the homogeneous frequency is inversely proportional to damping and the channel’s length. For comparison, spin waves excited on a homogeneous magnetization background exhibit a spin injection threshold that is proportional to damping, a frequency proportional to both spin injection and the magnet’s internal field, and an exponential decay rate that is proportional to damping. The exponential decay of spin waves imposes the ultimate limitation on their propagation length and the coherent spin transport, although detection at micrometer lengths scales has been achieved in low-damping materials such as YIG, amorphous YIG, and haematite.

The analytical predictions and characteristics of DEFs are promising for long-distance spin transport. However, the required spin injection has emerged as a practical barrier for their experimental realization. In recent experimental studies, spin injection was realized from quantum Hall edge states in antiferromagnetic graphene and the spin-Hall effect in Pt. A recent numerical study proposes an alternative spin-injection mechanism based on the spin-transfer torque effect, which excites magnetization precession. This method allows for large spin injection magnitudes, breaking the weak injection assumption that has been analytically assumed to date.

Signature of distinct nonlinear, dispersive dynamics exhibiting solitonic features were observed in micromagnetic simulations that include non-local dipole fields. More recently, micromagnetic simulations that incorporate spin-transfer torque along a confined, central strip of a ferromagnet have similarly shown evidence of strongly nonlinear features including a soliton nucleated at the injection site in the large injection regime termed a soliton screened spin superfluid.

While the numerical studies to date by a variety of groups unambiguously demonstrate that long-range spin transport can in principle be achieved with noncollinear magnetization states in magnetic materials, an analysis that incorporates short-wavelength dispersion and large-amplitude nonlinearities—such as those necessarily present for the existence of a soliton—as well as a description of the effect of damping on spin flows is lacking. Here, we provide a unified analytical framework in the context of a dispersive hydrodynamic (DH) formulation of magnetization dynamics. This formulation is an exact transformation of the Landau-Lifshitz equation and, therefore, captures the essential physics that are relevant to describe fully nonlinear, noncollinear magnetization states: exchange, anisotropy, and damping.

The DH formulation gives rise to two equations of motion for a longitudinal spin density and its associated fluid velocity that are analogous to the Navier-Stokes’ mass and momentum equations for a compressible fluid. From a fluid perspective, exchange, anisotropy, and damping give rise to dispersion, nonlinearity, and viscosity, respectively. In contrast to typical fluids, the equivalent magnetic fluid exhibits a non-conserved density, i.e., the mass can be lost. Therefore, noncollinear magnetization states—DEFs—can be interpreted as forced fluid flows that compensate the density and viscous losses manifesting in a profile that balances dispersion and nonlinearity.

In this paper, we find that DEFs are generally characterized by a nonlinear profile in both density and fluid velocity. In the weak spin injection regime, the DH equations reduce to the forced diffusion equation and lead to a linear DEF solution that is equivalent to a spin superfluid. Using boundary layer theory in the strong spin injection regime, we find a novel dynamical state characterized by the nucleation of a stationary soliton at the injection site that smoothly transitions into a linear DEF. We term this dynamical solution as a contact soliton DEF or CS-DEF, which is an analytical representation of a soliton screened spin superfluid. From a hydrodynamic perspective, the soliton nucleated at the injection site occurs precisely when the injection crosses the subsonic to supersonic flow boundary, equivalent to the Landau criterion. Moreover, transition between a DEF and a CS-DEF corresponds to the maximum precessional frequency achieved by spin injection, setting an upper bound to the efficiency of DEF-mediated spin transport. Thus, further spin injection enhances the coherent, superfluid-like soliton at the expense of larger spin transport, which is in sharp contrast to classical fluids where strong channel flows are subject to drag at the boundaries that, above a critical Reynolds number, develop into an incoherent, turbulent state.

Our analytical study also indicates that, for the physically relevant case of magnetic materials with low damping, DEFs can be interpreted as an adiabatic evolution of conservative dynamic solutions, previously termed uniform hydrodynamic states (UHSs) in order to highlight their non-dissipative but flowing character. DEF magnetization states sustained in channels subject to subsonic spin injection conditions can be conveniently represented as curves of constant frequency in the UHS phase space of spin density and fluid velocity. From an applications perspective, the fluid interpretation also lends itself to a circuit analogy, from which we can define the current-
voltage ($I$-$V$) characteristics of the coherent states studied here. Micromagnetic simulations support the analytical results even in the presence of in-plane anisotropy and non-local dipole fields in a thin film.

The remainder of the paper is organized as follows. In Sec. II, we summarize the dispersive hydrodynamic formulation and main features of uniform hydrodynamic states. In Sec. III, we introduce the boundary value problem that describes a channel subject to spin injection at one extremum and derive analytical expressions for linear DEFs, DEFs, and CS-DEFs. In the same section, we study the DEF to CS-DEF transition in the context of a subsonic to supersonic flow transition. In Sec. IV, we establish that the hydrodynamic states sustained in channels realize a nonlinear resistor in the hydraulic analogy of electrical circuits. Micromagnetic simulations of nanowires incorporating STT as a spin injection mechanism, in-plane magnetocrystalline anisotropy, and non-local dipole fields are discussed in Sec. V. Finally, we provide our concluding remarks in Sec. VI.

II. DISPERSIVE HYDRODYNAMIC FORMULATION AND UNIFORM HYDRODYNAMIC STATES

Magnetization dynamics in a continuum approximation can be described by the Landau-Lifshitz (LL) equation

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} - \alpha \mathbf{m} \times \mathbf{m} \times \mathbf{h}_{\text{eff}}, \quad (1a)$$

$$\mathbf{h}_{\text{eff}} = \frac{\Delta \mathbf{m}}{\text{exchange}} - \frac{m_z \hat{z}}{\text{dipole}}, \quad (1b)$$

where $\mathbf{m} = (m_x, m_y, m_z)$ is the magnetization vector normalized to the saturation magnetization $M_s$, $\alpha$ is the phenomenological Gilbert damping parameter, and $\mathbf{h}_{\text{eff}}$ is an effective field, normalized by $M_s$, that incorporates exchange and local (zero-thickness) dipole field as a minimal model for dispersion and nonlinearity, respectively. The dimensionless form of Eq. (1a) is achieved by scaling time by $|\gamma|\mu_0 M_s$ and space by $\lambda_{\text{ex}}$, where $\gamma$ is the gyromagnetic ratio, $\mu_0$ is the vacuum permeability, and $\lambda_{\text{ex}}$ is the exchange length. A dispersive hydrodynamic representation of Eqs. (1a) and (1b) can be achieved by mapping the magnetization vector into hydrodynamic variables $n$ and $u$, namely, a longitudinal spin density $n = m_z$ and a fluid velocity $u = -\nabla |\varphi| = -\nabla \arctan (m_y/m_x)$. In this work, we are interested in effectively one-dimensional dynamics along a channel whose length is oriented in the $\hat{x}$ direction. Therefore, the fluid velocity can be written as a scalar quantity $u = u \cdot \hat{x}$ and the spatial derivatives taken only along $\hat{x}$. The resulting dispersive hydrodynamic equations are

$$\partial_t n = (1 + \alpha^2) \partial_x [(1 - n^2)u] + \alpha(1 - n^2) \partial_x \varphi, \quad (2a)$$

$$\partial_t \varphi = -\frac{(1 - u^2) n}{1 - n^2} \frac{\partial_{xx} n}{1 - n^2} + \frac{n \partial_{xx} n}{1 - n^2} \frac{(1 - u^2)^2}{(1 - n^2)^2} - \frac{\alpha}{1 - n^2} \partial_x [(1 - n^2)u]. \quad (2b)$$

The simplest solutions to Eq. (2a) and (2b) are spin-density waves (SDWs). These wave states parametrized by a constant density and fluid velocity, $(n_0, u_0)$. SDWs are magnetization states that support dissipationless spin transport. A dynamic SDW can be only obtained as a transient state or in the conservative limit, where $\alpha = 0$ and $\partial t \varphi \neq 0$. We refer to this state as a uniform hydrodynamic state (UHS). For both SDWs and UHSs, the density is limited by its deviation from the magnetization’s unit sphere poles ($n_0 \pm 1$ corresponds to vacuum) while the fluid velocity is an unbounded quantity. However, it was shown in Ref. 3 that modulational instability (the exponential growth of perturbations) ensues when $|u_0| > 1$, i.e., for SDWs and UHSs with sub-exchange length in-plane magnetization rotation wavelengths. Therefore, modulationally stable SDWs and UHSs are defined in the phase space spanned by $|n_0| < 1$ and $|u_0| < 1$. UHSs exhibit a precessional frequency given by

$$\Omega_0 = \partial_t \varphi = - \left(1 - u_0^2 / n_0\right) n_0, \quad (3)$$

obtained directly from Eq. (2b). The negative sign of the frequency indicates that the precession is clockwise about the $\hat{z}$ direction.

It is important to emphasize that UHSs are dynamic, textured magnetization states. This is markedly dis-
fferent from small-amplitude perturbations about a homogenous state that are typically associated with spin waves. Interestingly, UHSs support small-amplitude perturbations that exhibit a dispersion relation that is non-reciprocal for \( n \neq 0 \). This nonreciprocity leads to conditions where perturbations can propagate in either two directions or one direction with respect to the UHS fluid velocity \( u_0 \) and can be hydrodynamically interpreted as subsonic or supersonic flow, respectively. The transition between subsonic and supersonic flow is known as the sonic curve. For UHSs, the sonic curve is given by

\[
|u_0| = \sqrt{\frac{1 - n^2}{1 + 3n^2}},
\]

and it is shown in Fig. 1 by a solid black curve in the UHS phase space. Isofrequency contours determined from Eq. (3) are shown by dashed black curves. As we will demonstrate below, the UHS phase space provides information regarding the form of dynamic magnetization states in ferromagnetic channels sustained by spin injection.

III. BOUNDARY VALUE PROBLEM FOR EASY-PLANE FERROMAGNETIC CHANNELS

The steady magnetization states sustained by spin injection can be analytically obtained by solving Eqs. (2a) and (2b) subject to appropriate boundary conditions (BCs). For this, we consider a channel of length \( L \) and introduce spin injection at \( x = 0 \) and free spin boundary conditions at \( x = L \). For simplicity, we disregard spin pumping.

We seek steady, precessional solutions to

\[
0 = (1 + \alpha^2)\frac{d}{dx} [(1 - n^2) u] + \alpha(1 - n^2) \Omega, \tag{5a}
\]

\[
\Omega = -(1 - u^2)n + \frac{1}{1 - n^2} \frac{d^2n}{dx^2}
+ \frac{n}{(1 - n^2)^2} \left( \frac{dn}{dx} \right) - \frac{\alpha}{1 - n^2} \frac{d}{dx} [(1 - n^2)u],
\tag{5b}
\]

with BCs

\[
\frac{dn}{dx}(0) = 0, \quad \frac{dn}{dx}(L) = 0, \tag{6a}
\]

\[
u(0) = \bar{u}, \quad u(L) = 0, \tag{6b}
\]

where \( \bar{u} \) is proportional to the injected spin current \( I \). These boundary conditions are enforced upon \( n = n(x) \), \( u = u(x) \) by introducing the homogeneous precessional frequency \( \Omega = \partial_t \Phi \). Below, we find solutions of this boundary value problem (BVP) with nonlinearity, dispersion, and damping.

A. Linear DEFs

We begin our analysis by revisiting the weak spin injection regime \( 0 < |\bar{u}| \ll \min(1, \alpha L) \), first presented in [2].

For this, we assume that \( u \) is small, \( n \) is constant, and the channel is long (\( L \gg \min(1, \alpha/\pi) \)) in Eqs. (5a) and (5b), so that the linearized equations are

\[
\alpha \bar{\Omega} = -\frac{du}{dx}, \tag{7a}
\]

\[
\bar{\Omega} = -n, \tag{7b}
\]

where \( \bar{\Omega} = \Omega/(1 + \alpha^2) \).

Noting that \( u = -\partial_x \Phi \) and \( \Omega = \partial_t \Phi \), we can rewrite Eqs. (7) as the diffusion equation

\[
\frac{\alpha}{1 + \alpha^2} \partial_t \Phi = \partial_{xx} \Phi, \tag{8}
\]

subject to the boundary conditions

\[
\partial_x \Phi(0) = -\bar{u}, \quad \partial_x \Phi(L) = 0. \tag{9}
\]

For weak damping, \( 1 + \alpha^2 \sim 1 \), Eq. (8) is the linearized hydrodynamic equation for easy-plane ferromagnets from previous studies [2]. By direct integration, Eq. (8) subject to Eq. (9) exhibit the linear DEF solution

\[
u_{\text{DEF}} = \bar{u}(1 - \frac{x}{L}), \quad \bar{\Omega}_{\text{DEF}} = -n_{\text{DEF}} = \frac{\bar{u}}{\alpha L} \tag{10}
\]

that exhibits a linear decay profile in the fluid velocity, which corresponds to the algebraic diffusion of spin current across the channel. Importantly, this approximate solution exhibits a spatially homogeneous frequency and density.

It is important to emphasize that damping plays a fundamental role in the stabilization of the linear DEF solution. It is for this reason that we refer to the solution as a dissipative exchange flow. In fact, in the conservative case where \( \alpha = 0 \), the solution to Eq. (7a) \( (u = \text{const}) \) cannot satisfy both boundary conditions (9).

B. Nonlinear DEFs

We now consider nonlinear but spatially smooth solutions, i.e., slowly varying relative to the exchange length. Consequently, the dispersive terms in Eq. (2b) can be neglected (both \( d^2n/dx^2 \) and \( (dn/dx)^2 \)). Upon simple algebraic manipulation, Eqs. (5a) and (5b) reduce to

\[
\alpha(1 - n^2)\bar{\Omega} = -\frac{d}{dx} [(1 - n^2)u], \tag{11a}
\]

\[
\bar{\Omega} = -(1 - u^2)n. \tag{11b}
\]

Inserting \( n \) from Eq. (11b) into (11a) leads, after some algebra, to the differential equation

\[
\alpha \bar{\Omega} = \frac{du}{dx} \left[ \frac{(\alpha \bar{\Omega} u)^2}{(1 - u^2)(u^2 - 2u^2 + 1 - \bar{\Omega}^2)} - 1 \right], \tag{12}
\]

that relates the fluid velocity to the precessional frequency. By integration, we obtain an implicit equation
FIG. 2. (color online) Magnetization states in a channel of length \( L = 100 \) and \( \alpha = 0.01 \) subject to an injection \( \bar{u} \) at the left edge, \( x = 0 \). In (a) and (b), the panels represent the density \( n \), fluid velocity \( u \), and \( m_x \) magnetization component at an instant of time. (a) For an injection \( \bar{u} = 0.4 \), the numerical solution shown by solid black curves is in good agreement to a DEF shown by dashed red curves. For comparison, the corresponding linear DEF solution is shown by dashed blue curves. (b) For an injection \( \bar{u} = 0.7 \), the numerical solution shown by solid black curves is in good agreement to a CS-DEF shown by dashed red curves. (c) Precessional frequency as a function of injection for a linear DEF (dashed black line), numerical solution of the BVP (solid black curve), DEF (dashed blue curve), and CS-DEF (dashed red curve). The DEF and CS-DEF frequencies define asymptotes for the numerical frequency. The numerical maximum \( \tilde{\Omega}_{\text{max}} = 0.44 \) is found at \( \bar{u} = 0.57 \).

for the fluid velocity (see Appendix A)

\[
\alpha L \tilde{\Omega}_{\text{DEF}} \left( 1 - \frac{x}{L} \right) = u_{\text{DEF}} + 4 \tanh^{-1} (u_{\text{DEF}}) - 2 \left[ N^- (u_{\text{DEF}}) + N^+ (u_{\text{DEF}}) \right],
\]

where

\[
N^\pm (\kappa) = \sqrt{1 \pm \tilde{\Omega}_{\text{DEF}} \tanh^{-1} \left( \frac{\kappa}{\sqrt{1 \pm \tilde{\Omega}_{\text{DEF}}}} \right)}. \tag{13}
\]

The precessional frequency is obtained by evaluating Eq. (13) at \( x = 0 \), \( u_{\text{DEF}}(0) = \bar{u} \), implying the equation for the DEF’s frequency

\[
\alpha L \tilde{\Omega}_{\text{DEF}} = \bar{u} + 4 \tanh^{-1} (\bar{u}) - 2 \left[ N^- (\bar{u}) + N^+ (\bar{u}) \right], \tag{15}
\]

while the density is obtained directly from Eq. (11) as

\[
n_{\text{DEF}} = - \frac{\tilde{\Omega}_{\text{DEF}}}{1 - u_{\text{DEF}}^2}. \tag{16}
\]

Equations (13), (15), and (16) indicate that the DEFs’ spatial profile is, in general, nonlinear and the frequency is a nonlinear function of the spin injection \( \bar{u} \). A numerical solution for a nonlinear DEF is shown by dashed red curves in Fig. 2(a) for the injection \( \bar{u} = 0.4 \), a channel of length \( L = 100 \), and \( \alpha = 0.01 \). The top and center panels show the hydrodynamic variables \( n(x) \) and \( u(x) \), respectively, while the bottom panel shows the \( x \) magnetization component, \( m_x(x,t) = \sqrt{1 - n(x)^2} \cos \Phi(x) \) at a given instant of time (recall that \( \partial_t \Phi \neq 0 \)). Good agreement is found between the analytical solution and numerical solution of the full BVP in Eqs. (2a), (2b), (6a), and (6b), shown by solid black curves. The BVP is numerically solved by a collocation method (MATLAB’s bvp5c).

An important consequence of the DEF nonlinear profile is the concomitant precessional frequency that is a nonlinear function of the injection, \( \bar{u} \), shown by a dashed blue curve in Fig. 2(c). The frequency obtained by solving the full BVP is shown by a solid black curve. Good agreement to Eq. (15) is found up to the maximum frequency \( \tilde{\Omega}_{\text{max}} = 0.44 \) at \( \bar{u}_{\text{max}} = 0.57 \), indicated by a black circle. For \( \bar{u} > \bar{u}_{\text{max}} \), the nonlinear solution does not describe the frequency dependence. As we show below, this qualitative change indicates the initiation of supersonic flow and of a stationary soliton.

The linear DEF solution can be obtained from the nonlinear DEF solution in the weak injection regime. For this, we note that \( \tanh^{-1} (\kappa) \approx \kappa \) when the argument is small and \( N^\pm (\kappa) \approx \kappa \). Introducing these approximations in Eqs. (13), (15), and (16) leads to Eq. (10).

The linear DEF approximation is shown by dashed blue curves in Fig. 2(a) for the same parameters as the DEF and numerical solutions. It is interesting that while the difference between the linear and nonlinear spatial profiles for the fluid velocity (middle panel) is imperceptible, the density in a linear approximation cannot describe the spatial profile. A consequence is that the linear DEF frequency tunability is a linear function of the injection and quantitatively agrees with the nonlinear solution up to \( \bar{u} \approx 0.3 \) for \( L = 100 \) and \( \alpha = 0.01 \), shown in Fig. 2(c) shown by a dashed black line.
C. Contact soliton DEFs

The qualitative change in the frequency dependence observed in Fig. 2(c) is an indication that the inclusion of nonlinearity is not sufficient to describe DEF solutions sustained at an arbitrary injection strength. In such a regime, dispersive terms must be taken into account in Eqs. (17a) and (17b). An analytical methodology for this task is boundary layer theory. This method allows one to separate the system into regimes dominated by different physics that can be asymptotically matched. Below we outline the most important features and results obtained from the calculation while details can be found in Appendix B.

For Eqs. (5a) and (5b) subject to the BCs (6a) and (6b), it is possible to find two regimes: a dispersion-dominated, rapidly varying region close to the injection site, we term inner region; and a long, slowly varying region subject to damping we term the outer region. Continuity is invoked to obtain a smooth solution across both regions. Mathematically, this is achieved by introducing BCs for the inner region

\[
\frac{d}{dx}n_{in}(0) = 0, \quad \lim_{x \to \infty} n_{in}(x) = n_{\infty}, \quad u_{in}(0) = \bar{u}, \quad \lim_{x \to \infty} u_{in}(x) = u_{\infty},
\]

and the outer region,

\[
\lim_{x \to 0} n_{out}(x) = n_{\infty}, \quad \frac{d}{dx}n_{out}(L) = 0, \quad \lim_{x \to 0} u_{out}(x) = u_{\infty}, \quad u_{out}(L) = 0,
\]

where \(n_{\infty}\) and \(u_{\infty}\) are matching conditions to be determined.

The equations of motion for the inner region are dominated by dispersion so that the dissipative terms are neglected

\[
0 = \frac{d}{dx} \left[ (1 - n^2)u \right],
\]

\[
\tilde{\Omega} = -(1 - u^2)n + \frac{1}{1 - n^2 \frac{d^2 n}{dx^2}} + \frac{n}{(1 - n^2)^2} \left( \frac{dn}{dx} \right)^2.
\]

The solution of this system of differential equations involves a series of steps detailed in Appendix B. Ultimately, Eqs. (19a) and (19b) can be integrated to obtain the soliton solution, e.g., see Ref. 33

\[
n_{in} = \frac{\nu_1 \tanh^2(\theta x) + \nu_2 (n_{\infty} - a)}{\operatorname{atanh}^2(\theta x) + \nu_2},
\]

\[
u_1 = u_{\infty} \frac{1 - n_{\infty}^2}{1 - n_{in}^2},
\]

\[
\tilde{\Omega}_{in} = -n_{\infty}(1 - u_{\infty}^2),
\]

with two free parameters: \(n_{\infty}, u_{\infty}\). The coefficients \(\nu_1, \nu_2, \theta,\) and \(a\) are given in Appendix B and all BCs in Eqs. (17a) and (17b) were used. In other words, Eqs. (20a) and (20b) describe, respectively, solitons of density amplitude \(a\) on a nonzero density background \(n_{\infty}\) and fluid velocity background \(u_{\infty}\).

In contrast, the slowly varying outer region is dominated by damping, leading to Eqs. (11a) and (11b) with DEF solutions given by Eqs. (13a) and (13b) we term \(u_{out}\) and \(n_{out}\), respectively. We note that this solution is obtained by evaluating the BCs of Eqs. (15a) and (15b) at \(x = L\), yielding a two-parameter family of solutions

\[
n_{out} = -\frac{\tilde{\Omega}_{out}}{1 - u_{out}^2},
\]

\[
\alpha L \tilde{\Omega}_{out} \left( 1 - \frac{x}{L} \right) = u_{out} + 4 \tanh^{-1} (u_{out}) - 2 [N^- (u_{out}) + N^+ (u_{out})].
\]

To apply boundary layer theory, the inner and outer solutions must asymptotically match and exhibit a single precessional frequency \(\tilde{\Omega}_{cs} = \tilde{\Omega}_{in} = \tilde{\Omega}_{out}\). For the left edge of the channel subject to spin injection, we evaluate the inner region solution, Eqs. (18a) and (18b) at \(x = 0\), to obtain

\[
\bar{u} = u_{\infty} \frac{1 - n_{\infty}^2}{1 - (n_{\infty} - a)^2}.
\]

Then, we evaluate the matching conditions, Eqs. (18a) and (18b). Assuming that \(u_{\infty}\) is a small quantity, the trigonometric terms of Eq. (13) cancel out and we obtain the linear DEF relations

\[
u_{cs} = \frac{u_{in} + u_{DEF} - u_{\infty}}{n_{cs} = n_{in} + n_{DEF} - n_{\infty}},
\]

and describes a soliton located at the injection site smoothly connected to a linear DEF. We call this solution a contact soliton DEF (CS-DEF). The matching conditions \(n_{\infty}\) and \(u_{\infty}\), as well as the precessional frequency \(\tilde{\Omega}_{cs}\) are obtained by solving the system of equations given by Eqs. (21a), (21b), and (21c).

A CS-DEF is shown by dashed red curves in Fig. 2(b) for an injection \(\bar{u} = 0.7\), a channel of length \(L = 100\), and \(\alpha = 0.01\). The numerical solution of the full BVP is shown by solid black curves and it is in excellent quantitative agreement to the boundary layer approach. The frequency dependence to the injection \(\bar{u}\) is shown by a dashed red curve in Fig. 2(c). In contrast to the DEF frequency tunability, the CS-DEF precessional frequency is a decreasing function of \(\bar{u}\). Additionally, we observe that the numerically obtained frequency tunability, solid black line, asymptotically approaches the CS-DEF frequency above \(\bar{u}_{max}\). This indicates that the full profile as a function of injection transitions from a DEF into...
a CS-DEF. In the following section, we investigate this transition and its hydrodynamic interpretation.

Qualitatively, CS-DEFs are similar to the soliton screened spin superfluid observed recently in micromagnetic simulations. In particular, our matching conditions indicate that only the linear terms in a DEF solution match to the soliton. An important difference is that our free-spin boundary conditions model a perfect spin sink so that magnon reflections are inhibited.

D. DEF to CS-DEF transition

In the previous section, a transition from a DEF into a CS-DEF was evidenced as a qualitative change of the frequency tunability to injection. In particular, it is observed in Fig. 2(c) that the full numerical solution (solid black curve) asymptotically approaches the DEF and CS-DEF frequency tunabilities in the small and large injection magnitudes limit, respectively. Whereas a first-order transition is not observed, it is insightful to find an analytical expression for a practical observable, such as the maximum precessional frequency, \( \bar{\Omega}_{\text{max}} \). For this, we can utilize the implicit equation for a DEF fluid velocity profile, Eq. (15), to take the derivative with respect to \( u \) and equate \( \partial_u \bar{\Omega}_{\text{DEF}} = 0 \). Because Eq. (15) is implicit, the maximum frequency will be an implicit equation as well. Utilizing Eq. (16) we can remove \( \bar{\Omega}_{\text{DEF}} \) from the equation and, after some algebra, we find the maximum injection as a function of the maximum input density, \( \bar{n}_{\text{max}} \),

\[
|\bar{u}_{\text{max}}| = \sqrt{1 - \frac{\bar{n}_{\text{max}}^2}{1 + 3\bar{n}_{\text{max}}^2}} \tag{25}
\]

Interestingly, this is exactly the sonic curve, Eq. (3). This expression is a central result of this work.

There are three physical implications of Eq. (25). First, the relation bounds the phase space for DEFs to the UHS subsonic regime. Second, it suggests that DEFs can be interpreted as an adiabatic evolution through a family of UHSs parametrized by spatially-dependent densities and fluid velocities. An adiabatic interpretation is valid as long as \( \alpha^2 \ll 1 \), which is physically true for magnetic materials of interest. Third, exceeding \( \bar{u}_{\text{max}} \) implies supersonic flow and coincides with the development of a soliton at the injection site.

A consequence of the adiabatic interpretation of DEF solutions is that the solution’s profiles can be visualized in a UHS phase space. In Fig. 3(a), we show numerical solutions of the BVP for \( L = 100 \) and \( \alpha = 0.01 \) by solid blue curves. The input conditions for each case are marked by blue circles. The solid and dashed gray curves represent the UHS sonic curve and isofrequency contours, respectively. We observe that the density and fluid velocity of several DEFs follow the UHS isofrequency contours. When the injection and its corresponding density enter the supersonic regime, CS-DEFs ensue and the adiabatic interpretation breaks down. Numerical solutions for CS-DEFs visualized in the UHS phase space are shown by a solid and dashed gray curves, respectively. The DEFs follow the isofrequency contours, in agreement with an adiabatic interpretation through a family of UHSs. CS-DEFs behave markedly different when the parameters are in the supersonic regime. (b) injection (left axis, solid black curve) and frequency (right axis, dashed black curve) at which a DEF transitions into a CS-DEF as a function of the channel length \( L \) and setting \( \alpha = 0.01 \).

From a hydrodynamic perspective, the UHS phase space visualization emphasizes a remarkable quality of CS-DEFs. In classical fluids, a supersonic regime near boundaries is subject to instabilities that result in turbulent flow, i.e., characteristic spatial scales become smaller downstream. Instead, the soliton established at the injection site is a coherent structure that expands the spatial
scales to a slowly varying DEF, precluding turbulence and ultimately establishing a subsonic flow. This feature is possible at the expense of reducing the homogeneous precessional frequency and, consequently, the magnitude of spin currents pumped into a reservoir located, e.g., at the right edge of the channel.

As discussed above, the distinction between DEFs and CS-DEFS from a hydrodynamic perspective can be linked to the flow conditions at the injection site. However, Eq. (25) is expressed as a function of \( u_{\text{max}} \), which is an unknown quantity that is determined by solving for a DEF. In other words, Eq. (25) cannot predict which isofrequency contour in Fig. 3(a) will be followed by a DEF given only the injection \( \vec{u} \). A practical consequence is that the actual maximum injection and precessional frequency will depend on \( L \) and \( \alpha \). By numerically solving the BVP as a function of \( L \) and setting \( \alpha = 0.01 \), we find the maximum injection \( u_{\text{max}} \) and frequency \( \Omega_{\text{max}} \) shown, respectively, by solid and dashed black curves in Fig. 3(b). These results have a clear physical interpretation. For short channels, the problem limits to a local balance between injection and damping. Therefore, the energy introduced into the system is primarily invested in establishing a DEF to compensate damping nonlocally and \( u_{\text{max}} \) is large. We emphasize that neither anisotropy nor non-local dipole fields have been considered so far. For short channels, these fields will most likely change the easy axis direction which can destroy the onset of magnetization textures. However, for long channels, it has been shown that such symmetry-breaking fields primarily introduce a threshold for the onset of DEFs. This implies that the large injections required to trigger a transition into a CS-DEF will be negligibly affected, as recently observed by simulations. In section V, we explore this transition by micromagnetic simulations in nanowires where the injection is parametrized by STT.

IV. ELECTRICAL CIRCUIT ANALOGY

An alternative interpretation that captures the behavior of the channel subject to injection as a two-terminal device is the hydraulic analogy of electrical circuits. This analogy allows one to classify the DEFs and CS-DEFS in the context of electrical elements that provide building blocks to construct devices with a given functionality. For this, we define hydrodynamic quantities that are analogous to a voltage and a current, and from which the I-V characteristics of the device can be obtained.

In the hydraulic analogy, a voltage maps to pressure difference. Using the hydrodynamic formulation of magnetization dynamics, the spatially-dependent pressure \( P(x) \) was derived in Ref. 3 as

\[
2P(x) = \left[ 1 + n(x)^2 \right] \left[ 1 + |u(x)|^2 \right] - 1, \tag{26}
\]

from which the pressure difference or voltage \( V \) in a channel of length \( L \) subject to BCs (6b) is

\[
V = (n_L^2 - n_0^2) + (1 + n_0^2)\bar{u}^2, \tag{27}
\]

where \( n_L = n(x = L) \) and \( n_0 = n(x = 0) \) are the densities at the channel’s extrema.

A current \( I \) is equivalent to a density flow rate. In the steady state modes studied here, the density flow rate corresponds to the precessional frequency, Eq. (3). Note that the precessional frequency is the only spatially-homogeneous quantity of both DEFs and CS-DEFS, just as a current is an equilibrium, constant quantity in electric circuits. Additionally, in the case of a neighboring spin reservoir, the precessional frequency is linearly dependent to the pumped spin current that can give rise to a transverse charge current by inverse spin-Hall effect.

Using Eq. (27) and Eq. (3), we numerically calculate the I-V characteristics shown in Fig. 4 for a channel of length \( L = 100 \) and \( \alpha = 0.01 \). The gray and white areas indicate the regions where, respectively, DEFs and CS-DEFS are sustained.

![FIG. 4. I-V characteristics for a channel of length \( L = 100 \) and \( \alpha = 0.01 \) subject to a spin injection \( u_0 \) at \( x = 0 \). The gray and white area indicate the regions where, respectively, DEFs and CS-DEFS are sustained.](image)
characteristic is positive everywhere, the negative differential conductivity of CS-DEFs implies that these states can potentially amplify oscillatory inputs. The study of oscillatory inputs is outside the scope of this paper and will be presented elsewhere.

V. MICROMAGNETIC SIMULATIONS

In this section, we explore the DEF solutions established in a nanowire by micromagnetic simulations including both non-local dipole fields and magnetocrystalline anisotropy. We utilize the GPU-based code mumax3. We consider material parameter of Py, namely, $M_s = 790$ kA/m, exchange stiffness $A = 10$ pJ/m, in-plane anisotropy field $H_A = 400$ A/m, and $\alpha = 0.01$. The corresponding exchange length to these parameters is $\lambda_{ex} = 5.05$ nm.

We simulate a nanowire of dimensions 512 nm $\times$ 100 nm $\times$ 1 nm. Spin injection is achieved by STT acting on a 10 nm $\times$ 100 nm contact located at the left extremum of the nanowire. Therefore, the nanowire length subject to spin injection is 502 nm that corresponds to a dimensionless length of $L = 99.4$. We use a symmetric STT with polarization $P = 0.65$ and assume that the charge current is spin-polarized along the $\hat{z}$ direction, e.g., by a magnetic material with perpendicular magnetic anisotropy. From a previous study, it was found that DEFs can be excited by STT in the presence of symmetry-breaking terms by charge current densities on the order of $10^{11}$ A/m$^2$. We numerically find a threshold of $\bar{J} = 4 \times 10^{11}$ A/m$^2$.

To explore the dynamical regimes discussed in Sec. III, we vary the charge current density at the left contact, between $1 \times 10^{11}$ A/m$^2$ and $10 \times 10^{11}$ A/m$^2$ in steps of $1 \times 10^{11}$ A/m$^2$. The simulation was set to run for 20 ns for each current, which was found to be sufficient to stabilize a steady state regime.

The results can be visualized in the UHS phase space shown in Fig. 5(a). Because of the oscillations and transverse non-uniformity introduced by anisotropy and non-local dipole fields, respectively, we plot averaged densities and fluid velocities. The average is performed both in space across the width of the nanowire and in time for the range 15 ns to 20 ns. The input hydrodynamic parameters are calculated at the edge of the left contact, where the magnetization is not subject to STT. A current density threshold for the stabilization of hydrodynamic states is observed. At sub-threshold current densities, a partial domain wall is formed at the injection site, evidenced by a solid black vertical line at $n = 0$.

We observe a remarkable qualitative agreement between the micromagnetic simulations and the analytical results shown in Fig. 4. In particular, we observe DEFs that follow the UHS isofrequency contours obtained in Sec. III without non-local dipole and in-plane anisotropy (solid blue curves) and CS-DEFs when the injection conditions are supersonic (solid red curves). Only three CS-DEFs are shown for clarity. The corresponding frequencies are shown in Fig. 5(b) in physical units as a function of $\bar{J}$ and color-coded as panel (a). We emphasize that a linear relation between $\bar{J}$ and $\bar{u}$ is not possible to obtain because of the particularities in the energy landscape imposed by the magnetization texture, anisotropy, and non-local dipole fields. Nonetheless, a maximum frequency is observed at the transition between DEFs and CS-DEFs.

VI. CONCLUSIONS

In this paper, we analytically determined the form and qualitative features of magnetization states sustained by spin injection of arbitrary strength in ferromagnetic channels with easy-plane anisotropy. For this, we utilize a dispersive hydrodynamic formulation that captures the
necessary physical terms without approximations while being analytically tractable. Our analytical study fully characterizes the possible solutions that support long-distance spin transport under a unified framework.

We find that DEFs are generally nonlinear in profile and frequency tunability. Additionally, we characterize a novel solution, a CS-DEF, composed of a stationary soliton nucleated at the injection site that smoothly transitions into a linear DEF. A notable consequence of the onset of CS-DEFs is that the frequency redshifts to injection. This feature is important for spintronic applications because it leads to a saturation of frequency and, therefore, of spin current magnitudes pumped in adjacent spin reservoirs. It is numerically found that the maximum frequency monotonically decays to the channel’s length, indicating the increased energy that must be invested in the nonlocal compensation of damping to sustain DEFs.

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The spatial derivative of $\alpha$ is obtained from Eq. (6b)

\[ \frac{d\alpha}{dx} = 2\alpha \frac{d\Omega}{dx} = 2\alpha \frac{\Omega}{1 - u^2} \frac{du}{dx}. \]  

\[ \frac{dn}{dx} = \frac{d}{dx} \left[ \frac{\Omega}{1 - u^2} \right] = -\frac{2\Omega u}{(1 - u^2)^2} \frac{du}{dx}. \]

Substituting Eq. (A2) into Eq. (A1), we obtain

\[ \alpha \tilde{\Omega} = \frac{du}{dx} \left[ \frac{(\alpha \Omega u)^2}{(1 - u^2)(u^4 - 2u^2 + 1 - \Omega^2)} - 1 \right]. \]  

which is Eq. (12) in the main text. To integrate this expression, we perform partial fraction decomposition. The resulting differential equation is

\[ \alpha \tilde{\Omega} = \frac{du}{dx} \left[ -1 + \frac{4}{u^2 - 1} - \frac{2(1 - \tilde{\Omega})}{u^2 - \sqrt{1 - \tilde{\Omega}}} - \frac{2(1 + \tilde{\Omega})}{u^2 - \sqrt{1 + \tilde{\Omega}}} \right]. \]

Integrate each fractional term that is known and has the form

\[ \int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{2} \tanh^{-1} \frac{x}{a}, & |x| < |a| \\ -\frac{1}{a} \coth^{-1} \frac{x}{a}, & |x| > |a| \end{cases} \]  

\[ \sqrt{1 - \Omega}, \sqrt{1 + \Omega}. \]

The case of interest is that of long channels, $\alpha L \geq 1$. From the linear DEF solution, Eq. (10), this implies that the precessional frequency is smaller than the injection, $\Omega < \bar{u}$. Thus, the case $|x| < |a|$ in Eq. (A3) can be used. We obtain

\[ \alpha \tilde{\Omega} x + C = -u - 4 \tanh^{-1} u \]

\[ + 2\sqrt{1 - \tilde{\Omega}} \tanh^{-1} \left( \frac{u}{\sqrt{1 - \tilde{\Omega}}} \right) \]

\[ + 2\sqrt{1 + \tilde{\Omega}} \tanh^{-1} \left( \frac{u}{\sqrt{1 + \tilde{\Omega}}} \right). \]

Evaluating the BC $\alpha(L, t) = 0$, we directly obtain

\[ \alpha \tilde{\Omega} L + C = 0 \]  

Replacing $C$ into Eq. (A6), we obtain the implicit solution for the fluid velocity Eq. (13).

Appendix B: CS-DEF solution

To solve Eqs. (13), (15), and (16) we use boundary layer theory. For this, we consider two regions: an inner region close to the injection site and an outer region that extends to the unforced edge of the channel. To mathematically distinguish between the relevant physics to consider in each region, we introduce a small parameter $\epsilon \ll L$ and rescale space. In all cases, we seek steady-state solutions so that $\partial_t n = 0$ and $\partial_t \Phi = \Omega$.

1. Inner region

In the inner region, we assume that the spatial profile changes rapidly. Consequently, space is rescaled as $y =
The hydrodynamic equations upon rescaling are
\[
\frac{d}{dy} \left[ (1 - n^2)u \right] = -\alpha(1 - n^2)\tilde{\Omega},
\]
with
\[
\tilde{\Omega} = -(1 - u^2)n + \frac{1}{\epsilon^2} \frac{d^2n}{dy^2} + \frac{1}{\epsilon^2 (1 - n^2)^2} \left( \frac{dn}{dy} \right)^2.
\]

To continue, we assume that \( \epsilon^{-1} \ll \alpha \) so that the right hand side of Eq. (B1a) can be neglected. This implies that in the inner region, the dynamics are effectively conservative. We will further assume that Eq. (B1b) is balanced to order 1. Noting that
\[
\frac{d}{dy} \left[ \frac{1}{1 - n^2} \left( \frac{dn}{dy} \right)^2 \right] = 2 \frac{dn}{dy} \left[ \frac{1}{1 - n^2} \frac{d^2n}{dy^2} + \frac{n}{(1 - n^2)^2} \left( \frac{dn}{dy} \right)^2 \right],
\]
the equations of motion that describe the spatial profile of the inner region are
\[
\frac{d}{dy} \left[ (1 - n^2)u \right] = 0, \quad \text{B3a}
\]
subject to the BCs
\[
\frac{d}{dy} n(0) = 0, \quad \lim_{\epsilon \to 0} \frac{d}{dy} n(L/\epsilon) = n_\infty, \quad \text{B4a}
\]
\[
u(0) = \bar{u}, \quad \lim_{\epsilon \to 0} u(L/\epsilon) = u_\infty, \quad \text{B4b}
\]
with \( n_\infty \) and \( u_\infty \) to be determined.

To continue, we integrate Eq. (B3a) and substitute \( u \) into Eq. (B3b)
\[
2\tilde{\Omega} \frac{dn}{dy} + 2n \frac{dn}{dy} - 2 \frac{C^2}{(1 - n^2)^2} \frac{dn}{dy} = \frac{d}{dy} \left[ \frac{1}{1 - n^2} \left( \frac{dn}{dy} \right)^2 \right],
\]
where \( C \) is a constant of integration. Every term in Eq. (B5) is a simple integral. Therefore, upon integration, we obtain an elliptic integral with four roots
\[
\left( \frac{dn}{dy} \right)^2 = -n^4 - 2\tilde{\Omega} n^3 + (1 - K_2) n^2 + 2\tilde{\Omega} n + K_1 + K_2,
\]
where \( K_1 \) and \( K_2 \) are constants of integration. Elliptic integrals have soliton solutions, e.g. see Ref. 33. Evaluating the right-edge (free spin) BCs, we obtain the two parameter family of soliton solutions
\[
\begin{align*}
n_{in} &= \alpha_1 \tanh^2(\theta y) + \alpha_2 (n_\infty - a), \\
u_{in} &= a - n_\infty^2, \\
u_{in} &= \frac{1}{\epsilon^2} \frac{d^2n}{dy^2} = \frac{1}{\epsilon^2} \frac{d^2n}{dy^2} = \frac{1}{\epsilon^2 (1 - n^2)^2} \left( \frac{dn}{dy} \right)^2.
\end{align*}
\]
where \( \nu_1 = a - n_\infty - 2n_\infty^2 u_\infty^2 \), \( \nu_2 = a - 2n_\infty - 2n_\infty u_\infty^2 \), \( \theta = \sqrt{1 - u_\infty^2 + n_\infty^2} \), and \( a = n_\infty (1 + u_\infty^2) + \sqrt{(1 - u_\infty^2)(1 - n_\infty^2 u_\infty^2}) \).

To find a unique solution, we evaluate the left edge (forced) BCs to obtain
\[
\bar{u} = u_\infty \frac{1 - n_\infty^2}{1 - (n_\infty - a)^2}.
\]

The soliton established in the inner region is therefore given by Eqs. (B7), (B8), and (B9), reported in the main text.

2. Outer region

For the outer region, we assume that the spatial profile varies smoothly and we rescale space by \( z = \epsilon x \). We obtain
\[
\frac{d}{dz} \left[ (1 - n^2)u \right] = -\alpha(1 - n^2)\tilde{\Omega},
\]
\[
\tilde{\Omega} = -(1 - u^2)n + \epsilon^2 \frac{1}{1 - n^2} \frac{d^2n}{dz^2} + \epsilon^2 \frac{n}{(1 - n^2)^2} \frac{dn}{dz} = \frac{d^2n}{dz^2}.
\]

To leading order, the dispersive terms in Eq. (B9b) are negligible and Eqs. (B9a) and (B9b) reduce to Eqs. (11a) and (11b) subject to the BCs
\[
\lim_{\epsilon \to 0} \frac{d}{dz} n(\epsilon) = n_\infty, \quad \frac{d}{dz} n(L) = 0, \quad \text{B10a}
\]
\[
\lim_{\epsilon \to 0} u(\epsilon) = u_\infty, \quad u(L) = 0, \quad \text{B10b}
\]

The solution to these equations is outlined in Appendix A and we label them \( n_{out} \) and \( u_{out} \) in the context of boundary layer theory.

3. Matching

The full solution for a CS-DEF is obtained by enforcing continuity between the inner region spatial profile and the outer region spatial profile. In other words, we asymptotically match the BCs. We require three equations to find a unique solution for \( n_\infty, u_\infty \), and \( \tilde{\Omega} \).

For the inner region, we found Eq. (B11) and (B8). For the outer region, we evaluate Eq. (A6) at \( \lim_{\epsilon \to 0} \epsilon x \) and apply the BC (B10b) to obtain
\[
u_\infty = \alpha L \tilde{\Omega},
\]
where we assumed that \( u_\infty \) is small so that the trigonometric terms in Eq. (A8) cancel out. In other words, the outer solution is approximated by the linear DEF solution. This directly leads to the third equation
\[
\tilde{\Omega} = -n_\infty.
\]
By solving Eqs. [18], [B11], and [B12], we match the inner and outer spatial profiles so that the CS-DEF is given by

$$u_{cs} = u_{in} + u_{out} - u_{\infty},$$  \hspace{1cm} (B13a) \\
$$n_{cs} = n_{in} + n_{out} - n_{\infty},$$  \hspace{1cm} (B13b) \\

and obtain the CS-DEF precessional frequency.
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