On Symbolic Powers of Prime Ideals

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Abstract. Let \((R, m)\) be a regular local ring with prime ideals \(p\) and \(q\) such that \(\sqrt{p+q} = m\) and \(\dim(R/p) + \dim(R/q) = \dim(R)\). It has been conjectured by Kurano and Roberts that \(p^m \cap q \subseteq m^{m+1}\) for all positive integers \(m\). We discuss this conjecture and related conjectures. In particular, we prove that this conjecture holds for all regular local rings if and only if it holds for all localizations of polynomial algebras over complete discrete valuation rings. In addition, we give an example showing that a certain generalization to nonregular rings does not hold.

1. Introduction

All rings in this paper are assumed commutative and Noetherian, and all modules unital.

Recent years have seen a renewed interest in the properties of symbolic powers of prime ideals, especially in regular local rings. For instance, see [5, 11, 12, 14, 15, 23, 24, 25]. Often these properties, or conjectured properties, are related to the behavior of the symbolic powers \(p^m\) of a prime ideal \(p\) with respect to the maximal ideal \(m\). These properties frequently emerge from seemingly unrelated questions in algebra and geometry. For instance, we have the following. ("SP" stands for "Symbolic Powers").

Conjecture 1.1. Let \((R, m)\) be a regular local ring with prime ideals \(p\) and \(q\) such that \(\sqrt{p+q} = m\) and \(\dim(R/p) + \dim(R/q) = \dim(R)\).

(SP-1): \(p^m \cap q \subseteq m^{m+1}\) for all \(m \geq 1\).

(SP-2): \(p^m \cap q^n \subseteq m^{n+m}\) for all \(m, n \geq 1\).

(SP-1) is a conjecture of Kurano-Roberts [15, 3.1], motivated by their work on Serre’s Positivity Conjecture; we briefly discuss the connection between these two conjectures in Section 2 below. Kurano and Roberts verify (SP-1) in [15, 3.2] for rings containing a field, and the current author gives a different proof in [24]. (SP-2) was conjectured and verified for rings containing a field by the current author in [25].
Each conjecture is open in mixed characteristic. In our main theorem we show that the ramified case of each conjecture follows from the unramified case.

The current author (op. cit.) has also made the following conjecture, which is fundamentally related to Conjecture 1.1. First, we explain a notational convention. In general, regular rings are denoted by $R$, and their prime ideals by Fraktur characters, e.g., $p$ and $q$. Rings that need not be regular are denoted by $A$ and their prime ideals by Roman characters, e.g., $P$ and $Q$; the main exception being the maximal ideal $\mathfrak{n}$. We give the definitions of technical terms in the conjecture after the statement. ("ID" stands for "intersection dimension").

**Conjecture 1.2.** Let $(A, \mathfrak{n})$ be a quasi-unmixed local ring with prime ideals $P$ and $Q$ such that $\sqrt{P} + Q = \mathfrak{n}$.

(ID-1): If $e(A) = e(A_P)$ and $A/P$ is analytically unramified, then

$$\dim(A/P) + \dim(A/Q) \leq \dim(A).$$

(ID-2): If $e(A) < e(A_P) + e(A_Q)$ and both $A/P$ and $A/Q$ are analytically unramified, then

$$\dim(A/P) + \dim(A/Q) \leq \dim(A).$$

In the statement, $e(A)$ is the Hilbert-Samuel multiplicity of the local ring $A$ with respect to its maximal ideal. A ring $A$ is *quasi-unmixed* if the completion $A^*$ is equidimensional, i.e., for every minimal prime ideal $P^*$ of $A^*$ we have $\dim(A^*/P^*) = \dim(A)$. A theorem of Ratliff [20] says that $A$ is quasi-unmixed if and only if it is equidimensional and universally catenary. In particular, when $A$ is excellent, then $A$ is quasi-unmixed if and only if it is equidimensional. The quotient ring $A/P$ is analytically unramified if $(A/P)^* = A^*/PA^*$ is reduced, i.e., the ideal $PA^*$ is an intersection of prime ideals of $A^*$. This is automatic if $A$ is excellent.

For rings that contain a field, the current author has verified both (ID-1) [23, 3.2] and (ID-2) [25, 2.2]. As with Conjecture 1.1, each conjecture (ID-i) is open in mixed characteristic. In our main theorem below, we show that the general case of each conjecture follows from the case of a hypersurface over an unramified regular local ring.

Some relations between our conjectures are straightforward to verify. We summarize them in the following diagram

\[
\begin{array}{ccc}
(ID-2) & \Rightarrow & (SP-2) \\
\downarrow & & \downarrow \\
(ID-1) & \Rightarrow & (SP-1)
\end{array}
\]

and defer explanation to Section 2. In our main theorem, which we now state, we show that the converses of the horizontal implications in this diagram hold.

**Main Theorem.** Fix $i = 1$ or 2. With the notations of Conjectures 1.1 and 1.2 the following conditions are equivalent.

(a) (SP-i) holds for all regular local rings.

(b) (SP-i) holds for all unramified regular local rings that are essentially of finite type over a discrete valuation ring of mixed characteristic.

(c) (ID-i) holds for all quasi-unmixed local rings.
(d)\ (ID-i) holds for all hypersurfaces over unramified regular local rings that are essentially of finite type over a discrete valuation ring of mixed characteristic.

Here we summarize the contents of this paper. In Section 2 we give some background material. In Subsection 2.1 we discuss the history of our conjectures. In Subsection 2.2 we verify some implications between the conjectures. In Subsection 2.3 we include definitions and tools relevant to our results. In Subsection 2.4 we discuss the technical assumptions in the conjectures. In Section 3, we prove our main theorem and verify a weak version of (ID-2) in mixed characteristic. Finally, in Section 4 we give an example to show that a potential generalization of (SP-1) does not hold.

2. Background

We begin this section by discussing the connection between Serre’s Positivity Conjecture and (SP-1), and the other evidence supporting our conjectures.

2.1. Evidence for the conjectures. Let \((\mathcal{R}, \mathfrak{m})\) be a regular local ring of dimension \(d\), and \(\mathfrak{p}, \mathfrak{q}\) prime ideals of \(\mathcal{R}\) such that \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\). Serre [26, Théorème 3 of Chapitre V] proved that

\[
\dim(\mathcal{R}/\mathfrak{p}) + \dim(\mathcal{R}/\mathfrak{q}) \leq d.
\]

We shall refer to this result as Serre’s Intersection Theorem. Serre defined the intersection multiplicity of \(\mathcal{R}/\mathfrak{p}\) and \(\mathcal{R}/\mathfrak{q}\) by the formula

\[
\chi(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}) = \sum_{i=0}^{d} (-1)^i \text{len}(\text{Tor}_i(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}))
\]

where \(\text{len}(T)\) is the length of the \(\mathcal{R}\)-module \(T\). For \(\mathcal{R}\) unramified, Serre proved the following: (i) \(\chi(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}) \geq 0\), and (ii) \(\chi(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}) = 0\) if and only if \(\dim(\mathcal{R}/\mathfrak{p}) + \dim(\mathcal{R}/\mathfrak{q}) < d\). He conjectured that these results hold even when \(\mathcal{R}\) is ramified.

Gillet-Soulé [6, Théorème 1] and Roberts [21, Theorem 1] independently proved the Vanishing Conjecture: if \(\dim(\mathcal{R}/\mathfrak{p}) + \dim(\mathcal{R}/\mathfrak{q}) < d\), then \(\chi(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}) = 0\). Gabber [2], [8], [22] proved the Nonnegativity Conjecture: \(\chi(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}) \geq 0\). The Positivity Conjecture is the converse of the Vanishing Conjecture, and is still open.

The following theorem of Kurano-Roberts [15, 3.2] shows the connection between the Positivity Conjecture and (SP-1). Essentially, it says that, for rings that are equicharacteristic or ramified, the Positivity Conjecture implies (SP-1). It is proved by applying Gabber’s methods to the Positivity Conjecture.

**Theorem 2.1.1.** Let \((\mathcal{R}, \mathfrak{m})\) be a regular local ring with prime ideals \(\mathfrak{p}\) and \(\mathfrak{q}\) such that \(\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}\) and \(\dim(\mathcal{R}/\mathfrak{p}) + \dim(\mathcal{R}/\mathfrak{q}) = \dim(\mathcal{R})\). If \(\mathcal{R}\) is ramified or contains a field and \(\chi(\mathcal{R}/\mathfrak{p}, \mathcal{R}/\mathfrak{q}) > 0\), then \(\mathfrak{p}^m \cap \mathfrak{q} \subseteq \mathfrak{m}^{m+1}\) for all \(m \geq 1\).

This theorem shows that (SP-1) holds in equal characteristic, because the same is true of the Positivity Conjecture. Because Kurano and Roberts expect the Positivity Conjecture to hold, this theorem motivated them to formulate their conjecture, (SP-1). It is worth noting that Kurano and Roberts [15, 3.4] verify the conjecture directly for rings containing a field of characteristic 0, with no reference to positivity.

The current author [25, 2.3] has verified (SP-1) and (SP-2) when \(\mathcal{R}\) contains a field of arbitrary characteristic, also with no reference to positivity. This is
accomplished by first proving that (ID-2) holds for rings containing a field, c.f., \[25, 2.2\]. Then one verifies that (ID-2) implies (SP-2), as follows. If \(R\) is a regular local ring, \(p\) a prime ideal of \(R\) and \(f\) a nonzero element of \(p\), then \(e(R_p/(f)) = m\) if and only if \(f \in p^{(m)} \setminus p^{(m+1)}\). If \(p\) and \(q\) are as in the statement of (SP-2) and \(f \in p^{(m)} \cap q^{(n)} \setminus m^{m+n}\), then we may apply (ID-2) to the hypersurface \(R/(f)\) to arrive at a contradiction.

The evidence in support of (ID-i) is the fact that it holds in equal characteristic. It is worth noting that (ID-i) generalizes Serre’s Intersection Theorem because, if \(A\) is regular, then \(e(A) = e(A_P) = e(A_Q) = 1\). One can take this as support for the conjecture as well.

2.2. Relations between the conjectures. In this subsection we discuss the implications in diagram (1). We have already seen how (ID-2) implies (SP-2), and a similar argument shows that (ID-1) implies (SP-1). Clearly, (SP-2) implies (SP-1). It is not immediately clear that (ID-2) implies (ID-1), because \(A/Q\) need not be analytically unramified in (ID-1). However, (ID-2) implies (ID-1) for complete domains, and the proof of our main theorem shows that we can reduce (ID-1) to the case of a complete domain. It follows that (ID-2) implies (ID-1).

At this time, we do not know if the converses of the vertical implications in diagram (1) hold. With Theorem 2.1.1 in mind, we remark that we do not know whether Serre’s Positivity Conjecture implies any of our conjectures, except (SP-1) in the ramified case. Also, we do not know if any of our conjectures implies Positivity.

2.3. Definitions and tools. In this subsection, we give the definition of the Hilbert-Samuel multiplicity and catalog the properties we will use in the proof of our results.

Let \((A, n)\) be a local ring, \(M\) a nonzero, finitely generated \(A\)-module of dimension \(d\), and \(I\) is an ideal of \(A\) such that \(M/IM\) has finite length. The Hilbert function \(P(n) = \text{len}_A(M/I^nM)\) agrees with a polynomial \(Q(n) \in \mathbb{Q}[n]\) of degree \(d\), for \(n \gg 0\). The Hilbert-Samuel multiplicity is the positive integer \(e_A(I, M)\) such that \(Q(n) = \frac{1}{d!}e_A(I, M)n^d + \text{(lower degree terms)}\). When there is no ambiguity, we shall write \(e(I, M)\). If \(I = n\), we write \(e_A(M)\) or \(e(M)\).

The following is a list of properties satisfied by the Hilbert-Samuel multiplicity that are needed for the proof of our main theorem. Let \(A\) be a ring and \(M\) a finite \(A\)-module.

2.3.1. (Additivity Formula) Let \(A\) be a local ring. Then
\[
e_A(M) = \sum P \text{len}(M_P)e(A/P)
\]
where the sum is taken over all primes \(P\) of \(A\) such that \(\dim(A/P) = \dim(M)\). This sum is finite because we need only take the sum over such primes which are also in the support of \(M\). (C.f., Bruns-Herzog 34.4.7.)

2.3.2. Let \((R, m)\) be a local subring of \(A\) such that the extension \(R \rightarrow A\) is module-finite. Then \(A\) is a semilocal ring such that \(\dim(A) = \dim(R)\). Let \(\{n_1, \ldots, n_n\}\) be the set of maximal ideals of \(A\) such that \(n_i \cap R = m\) and \(\text{ht}(n_i) = \dim(A)\). Then
\[
e_R(m, A) = \sum_i [A/n_i : R/m]e_{A_{n_i}}(mA_{n_i}, A_{n_i}).
\]
2.3.3. Let $R$ be a subring of $A$ such that the extension $R \to A$ is module-finite, and let $\mathfrak{s}$ be a prime ideal of $R$. Then there are finitely many prime ideals of $A$ that contract to $\mathfrak{s}$ in $R$. Let $\{S_1, \ldots, S_j\}$ be the set of prime ideals of $A$ such that $S_i \cap R = \mathfrak{s}$ and $\text{ht}(S_i) = \text{ht}(\mathfrak{s})$. Then

$$e_{R_\mathfrak{s}}(\mathfrak{s}R_\mathfrak{s}, A_\mathfrak{s}) = \sum_{i=1}^j [\kappa(S_i) : \kappa(\mathfrak{s})] e_{A_{S_i}}(\mathfrak{s}A_{S_i}, A_{S_i})$$

where $\kappa(\mathfrak{s})$ and $\kappa(S_i)$ are the residue fields of $R_\mathfrak{s}$ and $A_{S_i}$, respectively. (Apply [23.2] to the extension $R_\mathfrak{s} \to A_\mathfrak{s}$.)

2.3.4. Let $A \to A'$ be a flat, local homomorphism of local rings $(A, \mathfrak{n})$ and $(A', \mathfrak{n}')$ such that $\mathfrak{n}A' = \mathfrak{n}'$. Then $e(A') = e(A)$. (C.f., Herzog [7, 2.3].)

2.3.5. Let $A$ be a local ring and $I$ an ideal of $A$ such that $A/I$ has finite length. If $M$ is an $A$-module of positive rank $r$, then $e_A(I, M) = e_A(I, A) \cdot r$. (C.f., [8, 4.7.9])

2.3.6. If $A$ is local with infinite residue field, then there is an ideal $I$ generated by a system of parameters for $A$ such that $e(I, A) = e(A)$. (C.f., Northcott-Rees [18, Theorem 1 of §6].)

2.3.7. Assume that $(A, \mathfrak{n})$ is local and equidimensional, and contains an excellent local domain $(R, \mathfrak{m})$ such that the extension $R \to A$ is module-finite. Let $P$ and $Q$ be prime ideals of $A$ such that $\sqrt{P + Q} = \mathfrak{n}$, and let $\mathfrak{p} = P \cap R$ and $\mathfrak{q} = Q \cap R$. If $e_R(\mathfrak{m}, A) < e(A_P) + e(A_Q)$, then $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$. (C.f., [25, 2.1].)

2.4. Technical assumptions in the conjectures. From the proof of our main theorem below, one can see why we make certain assumptions in (SP-i) and (ID-i). Some of these assumptions are necessary; we are not sure of the necessity of others. In this subsection, we give some explanation and historical precedence for some of these assumptions.

Straightforward examples show that the following assumptions in (SP-i) are necessary: (i) $R$ is regular, (ii) $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, and (iii) $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. When we assume (i) and (ii), Serre’s Intersection Theorem implies that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$. That is, in (iii) we are requiring the sum $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q})$ to achieve its maximal value. Similar examples show that we must assume that $\sqrt{P + Q} = \mathfrak{n}$ in (ID-i).

In (ID-i), we assume that $A$ is quasi-unmixed. There are straightforward examples of complete local rings that are not equidimensional for which the (ID-i) both fail, so the requirement that $A$ be equidimensional is certainly necessary. Since our arguments involve passing to the completion $A^\ast$, we must assume that $A^\ast$ is equidimensional. We do not know whether the full strength of the quasi-unmixedness assumption is needed. However, if one is inclined to assume that one’s rings are excellent, then “quasi-unmixed” is equivalent to “equidimensional”, as we noted above, and is therefore necessary.

In (ID-i) we assume that $A/P$ is analytically unramified. The purpose of this assumption is to guarantee that the multiplicity $e(A_P)$ is well-behaved under passing to the completion $A^\ast$. To place this in an historical context, note that our proof is very similar to Nagata’s proof of the following result.
Theorem 2.4.1. ([17 (40.1)]) Let $P$ be a prime ideal of a local ring $A$. If $A/P$ is analytically unramified and if $\text{ht}(P) + \dim(A/P) = \dim(A)$, then $e(A_P) \leq e(A)$.

Regarding the analytically unramified assumption in this theorem, Nagata ([17 A2]) writes the following: “It is not yet known to the writer’s knowledge whether or not (40.1) is true without assuming that $P$ is analytically unramified.” If it is shown that this condition can be omitted from the statement of Theorem 2.4.1, then one will probably be able to show that the corresponding conditions should be omitted from the statements (ID-$i$).

3. Main Results

The following theorem includes the Main Theorem announced in Section 1.

Theorem 3.1. With the notation of Conjectures 1.1 and 1.2, fix $i = 1$ or $2$. Then the following conditions are equivalent.

1. $(\text{SP-}i)$ holds for all regular local rings.
2. $(\text{SP-}i)$ holds for all complete, unramified regular local rings of mixed characteristic.
3. For every prime number $p$, every complete $p$-ring $(V, pV)$ and every $d \geq 1$, $(\text{SP-}i)$ holds for $V[X_1, \ldots, X_d](p, X_1, \ldots, X_d)$.
4. $(\text{ID-}i)$ holds for all quasi-unmixed local rings.
5. $(\text{ID-}i)$ holds for all hypersurfaces over an arbitrary complete, unramified regular local ring of mixed characteristic.
6. For every prime number $p$, every complete $p$-ring $(V, pV)$ and every $d \geq 1$, $(\text{ID-}i)$ holds for all hypersurfaces over $V[X_1, \ldots, X_d](p, X_1, \ldots, X_d)$.

In the statement, a $p$-ring is a discrete valuation ring $V$ whose maximal ideal is generated by the prime number $p$.

Proof. We summarize the implications we shall prove in the following diagram.

```
(1) ←(2) →(3) ←(4)
(5) ←(6) →(1)
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The implications “$(1) \implies (2)$” and “$(1) \implies (6)$” are obvious. The implication “$(2) \implies (3)$” follows by passing to the completion $R^*$ of $R$; use the faithful flatness of $R \to R^*$ as in the first step of “$(3) \implies (2)$”. The implication “$(5) \implies (6)$” also follows by passing to the completion; argue as in the proof of [25 2.3]. As noted in the introduction, if $R$ is a regular local ring with prime ideal $p$ and $f$ is a nonzero element of $p$, then $f \in p^{(m)} \setminus p^{(m+1)}$ if and only if $e(R_p/(f)) = m$. From this the implications “$(1) \implies (1)$”, “$(2) \iff (3)$” and “$(5) \iff (6)$” follow easily.

“$(6) \implies (1)$”. We verify this implication in the case $i = 2$; the case $i = 1$ is similar. Let $(A, \mathfrak{n})$ be a quasi-unmixed, local ring with prime ideals $P$ and $Q$ such that $A/P$ and $A/Q$ are analytically unramified, $\sqrt{P+Q} = \mathfrak{n}$, and $e(A) < e(A_P) + e(A_Q)$.

We now show that we may assume without loss of generality that $A$ is a complete local domain of mixed characteristic with infinite residue field.
Step 1. Pass to the ring $A(X) = A[X]_{nA[X]}$ to assume that the residue field of $A$ is infinite. Let $n(X) = nA(X)$, $P(X) = PA(X)$ and $Q(X) = QA(X)$. Then $A(X)$ is a local ring with maximal ideal $P$. If $e$ is a minimal prime over $P$ and $I$ is a complete domain with infinite residue field, it suffices to verify that there is a minimal prime $P_A$ such that $P_A \subseteq P$. Since $A/P$ is analytically unramified, $PA^* = P^*$. The fact that the extension $A_P \to A^*$ is flat therefore implies that $e(A_P) = e(A_P)$ by \[2.3.4\] Similarly, $e(A^*_Q) = e(A_Q)$. If the result holds for $A^*$, then it holds for $A$, as in the previous step.

Step 2. Pass to the completion $(A^*, n^*)$ to assume that $A$ is complete and equidimensional with infinite residue field. Let $P^*$ be a prime ideal of $A^*$ that is minimal over $PA^*$ such that $\text{ht} (P^*) = \text{ht} (P)$, and similarly for $Q^*$. Since $A/P$ is analytically unramified, $PA^*_P = P^*PA^*_P$. The fact that the extension $A_P \to A^*_P$ is flat therefore implies that $e(A^*_P) = e(A_P)$ by \[2.3.4\] Similarly, $e(A^*_Q) = e(A_Q)$. If the result holds for $A^*$, then it holds for $A$, as in the previous step.

Step 3. Pass to the quotient $A/I$ for a suitably chosen minimal prime $I$, to assume that $A$ is a complete domain with infinite residue field. To make this reduction, it suffices to verify that there is a minimal prime $I$ of $A$ contained in $P \cap Q$ such that $e(A/I) < e(A_P/I_A) + e(A/Q/I_A)$.

First, we show how this gives the desired reduction. Let $A' = A/I$ with prime ideals $n' = nA$, $P' = PA$ and $Q' = QA$. Since $A$ is equidimensional, $\dim (A') = \dim (A)$, and since $I \subseteq P \cap Q$, it follows that $\dim (A'/P') = \dim (A/P)$ and $\dim (A'/Q') = \dim (A/Q)$. Therefore, as in Step 1, we may pass to $A'$.

Now we prove that such a minimal prime $I$ exists. Let $\{I_1, \ldots, I_g\} = \min (A)$. Suppose that $e(A/I_j) \geq e(A_P/I_A) + e(A/Q/I_A)$ for every $j$ such that $I_j \subseteq P \cap Q$. (This supposition includes the hypothetical possibility that no $I_j$ is contained in $P \cap Q$.) By \[2.3.4\] (40.1), $e(A_P/I_A) + e(A/Q/I_A)$ for every $I_j$ contained in $P$, and similarly for $I_j$ contained in $Q$. The Additivity Formula then implies that

$$e(A_P) + e(A_Q) = \sum_{I_j \subseteq P} e(A_P/I_A) \text{len}(A_{I_j}) + \sum_{I_j \subseteq Q} e(A_Q/I_A) \text{len}(A_{I_j})$$

$$= \sum_{I_j \subseteq P, I_j \subseteq Q} e(A_P/I_A) \text{len}(A_{I_j}) + \sum_{I_j \subseteq Q} e(A_Q/I_A) \text{len}(A_{I_j})$$

$$+ \sum_{I_j \subseteq P \cap Q} [e(A_P/I_A) + e(A_Q/I_A)] \text{len}(A_{I_j})$$

$$\leq \sum_{I_j \subseteq P} e(A/I_j) \text{len}(A_{I_j}) + \sum_{I_j \subseteq Q} e(A/I_j) \text{len}(A_{I_j})$$

$$+ \sum_{I_j \subseteq P \cap Q} e(A/I_j) \text{len}(A_{I_j})$$

$$\leq \sum_{I_j} e(A/I_j) \text{len}(A_{I_j}) = e(A).$$
This clearly contradicts the assumption that \( e(A) < e(A_P) + e(A_Q) \). Thus, there is a minimal prime \( I \) of \( A \) such that \( I \subseteq P \cap Q \) and \( e(A/I) < e(A_P/IA_P) + e(A_Q/IA_Q) \), as claimed.

Since (ID-\( i \)) has been verified for rings containing a field, we assume without loss of generality that \( A \) has characteristic 0 and that \( k = A/n \) has characteristic \( p > 0 \).

Let \( V \) be a coefficient ring for \( A \), which is a complete \( p \)-ring. Since the residue field of \( A \) is infinite, \( 2.3.6 \) implies that there exists system of parameters \( x = x_1, \ldots, x_d \) of \( A \) such that the ideal \( J = xA \) satisfies \( e(J, A) = e(A) \). Let \( R = V[x_1, \ldots, x_d] \) which is a complete domain contained in \( A \) such that the extension \( R \to A \) is module finite, and the induced map on residue fields is an isomorphism. Let \( m = (p, x_1, \ldots, x_d)R = n \cap R, p = P \cap R \) and \( q = Q \cap R \). Since \( J = xA \subseteq mA \subseteq n \), we have \( e(A) = e(J, A) \geq e(mA, A) \geq e(n, A) = e(A) \). Therefore, 

\[
e(A) = e_A(mA, A)[A/n, R/m] = e_R(m, A) = \text{rank}_R(A)e(R)
\]

by \( 2.3.2 \). Since \( e(A) < e(A_P) + e(A_Q) \), \( 2.3.7 \) implies that \( \sqrt{p + q} = m \).

If \( R \) is a regular local ring, then Serre’s Intersection Theorem implies that

\[
\dim(A/P) + \dim(A/Q) = \dim(R/p) + \dim(R/q) \leq \dim(R) = \dim(A)
\]
as desired. Therefore, we assume that \( R \) is not regular. Let \( X_1, \ldots, X_d \) be indeterminates and let \( B = V[X_1, \ldots, X_d] \). Then \( B \) surjects onto \( R \) via the natural homomorphism \( \pi \) which sends \( X_i \) to \( x_i \). Since \( \dim(R) = d = \dim(B) - 1 \) and \( R \) is a domain, the kernel of \( \pi \) is a height 1 prime of \( B \) and is therefore principal. Thus, \( R \) is a hypersurface over \( B \). If we can show that \( e(R) < e(R_p) + e(R_q) \), then our assumptions imply that \( \dim(R/p) + \dim(R/q) \leq \dim(R) \), completing the proof.

Let \( r = \text{rank}_R(A) = \text{rank}_{R_p}(A_p) > 0 \) so that \( e(A) = re(R) \). We claim that \( re(R_p) \geq e(A_P) \). As before,

\[
re(R_p) = \text{rank}_{R_p}(A_P)e(pR_p, R_p) = e(pR_p, A_p) = \sum_{P_i \cap R = p} e_{R_p}(pA_{P_i}, A_{P_i})[\kappa(P_i) : \kappa(p)]
\]

(where the sum is taken over all primes \( P_i \) of \( A \) contracting to \( p \) in \( R \))

\[
\geq e_{A_P}(pA_P, A_P) \geq e_{A_P}(PA_P, A_P) = e(A_P).
\]

Similarly, \( re(R_q) \geq e(A_Q) \). Thus,

\[
re(R) = e(A) < e(A_P) + e(A_Q) \leq r(e(R_p) + e(R_q))
\]

and since \( r > 0 \), we have \( e(R) < e(R_p) + e(R_q) \), as desired.

“\( 3 \) \( \implies \) \( 2 \)”: The main ideas for this proof come from Hochster \( 9 \) and Dutta \( 4 \), each of which relies heavily on Artin approximation \( 14 \) and on the work of Peskine-Szpiro \( 19 \).

Let \( V \) be a complete \( p \)-ring and \( d \geq 1 \), and suppose that there were a counterexample to (SP-2) in \( R = V[X_1, \ldots, X_d] \). That is, there exist prime ideals \( p, q \) in \( R \) and an element \( f \in R \) such that \( \sqrt{p + q} = m \), \( \dim(R/p) + \dim(R/q) = d + 1 \).
and $f \in p^{(m)} \cap q^{(n)} \setminus m^{m+n}$. Since our conjectures are easily verified for rings of dimension less than 2, we have $\dim(R) \geq 2$.

We may assume without loss of generality that $k = V/pV$ is algebraically closed, as follows. Let $k' \to V$ since going-down holds between $p$ is flat and local, so we may fix prime ideals $m \in R$ dimension less than 2, we have $\dim(R/m) \geq e$ and solutions modulo $\sqrt{2}$ will then yield a solution to these equations in $R$. Since there are finitely many variables, the solution will lie平凡化 for coefficients, and the variables $Y_1, \ldots, Z_d$ already designated, we can express this information in $d + 1$ equations.

Let $R_0 = V^{[X_1, \ldots, X_d]}_{p, X_1, \ldots, X_d}$ with maximal ideal $m_0$. We shall show that the counterexample in $R$ gives rise to a counterexample in $R_0$. More specifically, we show that there exist prime ideals $p_0, q_0$ in $R_0$ and $g \in R_0$ such that $\sqrt{p_0 + q_0} = m_0$, $\dim(R_0/p_0) = \dim(R/p)$, $\dim(R_0/q_0) = \dim(R/q)$, and $g \in p_0^{(m)} \cap q_0^{(n)} \setminus m_0^{m+n}$.

First, we show how the given counterexample gives rise to a counterexample in $R_0$ and therefore in a pointed étale neighborhood $R_1$ of $R_0$. This is accomplished by showing that the essential data describing the counterexample is determined by a finite number of polynomial equations (necessarily in a finite number of variables) with coefficients in $R_0$. Artin’s approximation theorem will then yield a solution to these equations in $R_0$. Furthermore, given an integer $e \geq 1$ we can guarantee that the solutions in $R_0^e$ will be congruent to the original solutions modulo $m^e$. Since there are finitely many variables, the solution will lie in a pointed étale neighborhood $R_1$ of $R_0$. Note that $R_0^e$ and each pointed étale neighborhood $R_1$ of $R_0$ is an unramified regular local ring of dimension $d + 1$ with regular system of parameters $p, X_1, \ldots, X_d$. Let $y_1, \ldots, y_d \in p$ be a minimal set of generators of $p$ and $z_1, \ldots, z_b \in q$ a minimal set of generators of $q$.

1. A finite number of equations determine the minimal free resolutions of $R/p$ and $R/q$; c.f., [19, 6.2]. Let $Y_1, \ldots, Y_a, Z_1, \ldots, Z_b$ be the variables that “keep track” of the elements $y_1, \ldots, y_a, z_1, \ldots, z_b$. Applying Artin approximation will yield ideals $I, J$ in a pointed étale neighborhood $(R_1, m_1)$ of $R_0$ that are generated by the solutions in $R_0$ that are substituted for the variables $Y_1, \ldots, Y_a, Z_1, \ldots, Z_b$.

2. A finite number of equations determine the fact that $\sqrt{p + q} = m$. There exists $q \geq 1$ such that $p^q, X_j^q \in p + q$ for $j = 1, \ldots, d$, so we may write

$$p^q = \sum_{i=1}^a c_i y_i + \sum_{i=1}^b c_i^j z_i \quad \text{and} \quad X_j^q = \sum_{i=1}^a c_{j,i} y_i + \sum_{i=1}^b c_{j,i}^j z_i$$

for $j = 1, \ldots, d$ and $c_i, c_{j,i}, c_i^j, c_{j,i}^j \in R$. Using variables $C_1, C_{j,i}, C_i, C_{j,i}^j$ for the coefficients, and the variables $Y_i, Z_i$ already designated, we can express this information in $d + 1$ equations.
3. A finite number of equations determine the dimension of $R/p$ and $R/q$. This is a result of Hochster [9]. A proof can be found in [3] 8.4.4. See [4] 3.5] for a different proof.

4. A finite number of equations determine the fact that a given sequence $\alpha_1, \ldots, \alpha_r$ is a system of parameters for $R/p$. We can keep track of $\dim(R/p)$. We can keep track of the fact that the $\alpha_i$ are elements of $m$ and that the sequence has $\dim(R/p)$ elements. Also we can keep track of the fact that $(R/p)/(\alpha_1, \ldots, \alpha_r)$ has dimension 0.

It follows that we can keep track of whether a given element $\alpha_1 \in m$ is part of a system of parameters for $R/p$ by extending it to a full system of parameters.

5. A finite number of equations determine the fact that $f \in \mathfrak{p}^m$. There are two cases.

Case 1. If $f \in \mathfrak{p}^m \setminus \mathfrak{p}^n$, then there exists $s \in \mathfrak{m} \setminus \mathfrak{p}$ such that $sf \in \mathfrak{p}^n$. That is, we can write

$$sf = \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} y_1^{i_1} \cdots y_n^{i_n}$$

where the sum is finite and the coefficients are in $R$. Since $s \in \mathfrak{m} \setminus \mathfrak{p}$, $s$ is part of a system of parameters for $R/p$; we have already noted that we can keep track of this fact with a finite number of equations. By using variables $C_{i_1, \ldots, i_n}$ we can also keep track of the fact that $sf \in \mathfrak{p}^n$. Let $S, F$ be the variables we use to keep track of the elements $s, f$.

Case 2. If $f \in \mathfrak{p}^n$, then we can keep track of the fact that $f \in \mathfrak{p}^n$ with a single equation, as in case 1. Using $s = 1$ and the equation $S = 1$ keeps this case consistent with case 1.

In our application of Artin approximation, we may then conclude that there are elements $s_1, f_1 \in R_1$ such that $s_1 f_1 \in \mathfrak{I}^m$ and either $s_1 = 1$ or $s_1$ is part of a system of parameters for $R_1/I$. Similarly, there is an element $t_1 \in R_1$ such that $t_1 f_1 \in \mathfrak{J}^n$ and either $t_1 = 1$ or $t_1$ is part of a system of parameters for $R_1/J$.

6. Choosing $e \geq m + n$, we require that the solutions in $R_1$ are congruent to the original solutions in $R$ modulo $m_1^{m+n}$. In particular, since $f \notin m_1^{m+n}$, it follows that $f_1 \notin m_1^{m+n}$ and, therefore, that $f_1 \notin \mathfrak{m}_1^{m+n}$.

To summarize, we have shown the existence of the following:

(a) a pointed étale neighborhood $(R_1, m_1)$ of $(R_0, m_0)$ and an element $f_1 \in R_1 \setminus m_1^{m+n}$;

(b) ideals $I, J \subset R_1$ such that $\sqrt{I + J} = m_1$, $\dim(R_1/I) = \dim(R/p)$ and $\dim(R_1/J) = \dim(R/q)$;

(c) an element $s_1 \in R_1$ such that $s_1 f_1 \in \mathfrak{I}^m$ and either $s_1 = 1$ or $s_1$ is part of a system of parameters for $R_1/I$;

(d) an element $t_1 \in R_1$ such that $t_1 f_1 \in \mathfrak{J}^n$ and either $t_1 = 1$ or $t_1$ is part of a system of parameters for $R_1/J$.

In particular, $R_1$ is a regular local ring of dimension $d + 1$, essentially of finite type and smooth over $V$, $m_1 = m_0 R_1$, and $R_1/m_1 = V/pV$. Let $p_1$ be a minimal prime of $R_1/I$ such that $\dim(R_1/p_1) = \dim(R_1/I)$, and let $q_1$ be a minimal prime of $R_1/J$ such that $\dim(R_1/q_1) = \dim(R_1/J)$. Then $\sqrt{p_1 + q_1} = m_1$ and $\dim(R_1/p_1) + \dim(R_1/q_1) = \dim(R_1)$. Since $s_1$ is either part of a system of parameters on $R_1/I$ or $s = 1$, it follows that $s_1 \notin p_1$. Since $s_1 f_1 \in \mathfrak{I}^m \subseteq \mathfrak{p}^m$, it follows that $f_1 \in \mathfrak{p}^m$. Similarly, $f_1 \in q_1^{(n)}$. Thus, we have a counterexample in $R_1$, as desired.
Next, we show that we may assume without loss of generality that $f_1$ is irreducible in our conjecture. Since $R$ is a unique factorization domain, write $f_1 = F_1 \cdots F_v$ with each $F_i \in m_1$ irreducible. For $i = 1, \ldots, v$ let $m_i, n_i \geq 0$ be the unique integers such that $F_i \in p_1^{(m_i)} \smallsetminus p_1^{(m_i+1)}$ and $F_i \in q_1^{(n_i)} \smallsetminus q_1^{(n_i+1)}$. In other words, let $m_i = e((R_1)_{p_1}/(F_i))$ and $n_i = e((R_1)_{q_1}/(F_i))$. Assuming that $f_1 \in p_1^{(m)} \cap q_1^{(n)} \subset m_1^{m+n}$, we claim that there exists an integer $i$ such that $1 \leq i \leq v$ and $F_i \not\in m_1^{m+n}$. This will show that our counterexample gives rise to an irreducible counterexample. Suppose that each $F_i \in m_1^{m+n}$. (If $m_i = 0$ or $n_i = 0$, this is automatic by [17 (38.3)]). From this it follows that

$$f_1 = F_1 \cdots F_v \in m_1^{\sum(m_i+n_i)}.$$  

The Additivity Formula implies that

$$e((R_1)_{p_1}/(f_1)) = \sum_i e((R_1)_{p_1}/(F_i)) = \sum_i m_i,$$

or, in other words,  

$$f_1 \in p_1^{(\sum_i m_i)} \smallsetminus p_1^{(1+\sum_i m_i)}.$$  

Since $f_1 \in p_1^{(m)}$, it follows that $m \leq \sum_i m_i$. Similarly, $n \leq \sum_i n_i$. Therefore, 

$$m + n \leq \sum_i (m_i + n_i),$$

and it follows that

$$f_1 \in m_1^{\sum_i (m_i+n_i)} \subseteq m_1^{m+n}$$

which contradicts our hypothesis.

We also note that $f_1 \in m_2^2$, as follows. Suppose not. Since $f_1 \in p_1 \cap q_1 \subseteq m_1$, the fact that $f_1 \not\in m_2^2$ implies that $R' = R_1/(f_1)R_1$ is a regular local ring of dimension $d$ with prime ideals $p' = p_1/(f_1)R_1$ and $q' = q_1/(f_1)R_1$ such that $\sqrt{p'} + q' = m'$, and

$$\dim(R'/p') + \dim(R'/q') = \dim(R_1/p_1) + \dim(R_1/q_1) = d + 1 > \dim(R').$$

This contradicts Serre’s Intersection Theorem. Observe that $f \not\in pR_1$ because $f_1$ is irreducible and $f \in m_2^2$.

Finally, we show how the following theorem of Dutta [4, 1.3] gives rise to a counterexample in $R_0$, thus completing our proof.

**Theorem 3.2.** Let $(A, M, K)$ be a regular local ring of dimension $d + 1$, essentially of finite type and smooth over an excellent discrete valuation ring $(U, \pi)$ such that $K$ is separably generated over $U/\pi U$. Let $a(\neq 0) \in M^2$ be such that $a \not\in \pi A$ (i.e., $\{\pi, a\}$ form an $A$-sequence). Then there exists a regular local ring $(B, N, K) \subset (A, M, K)$ such that

1. The ring $B$ is a localization of a polynomial ring $W[Y_1, \ldots, Y_d]$ at a maximal ideal of the type $(\pi, h(Y_1), Y_2, \ldots, Y_d)$ where $h$ is a monic irreducible polynomial in $W[Y_1]$ and $(W, \pi)$ is an excellent discrete valuation ring contained in $A$; moreover, $A$ is an étale neighborhood of $B$.

2. There exists an element $g$ in $B \cap aA$ such that $B/gB \to A/aA$ is an isomorphism. Furthermore $gA = aA$.

We apply this theorem to the ring $A = R_1$ and the element $a = f_1$. The discrete valuation ring $W$ whose existence is guaranteed by the theorem is constructed by choosing elements $y_1, \ldots, y_1 \in A$ such that their residues modulo $M$ form a transcendence basis of $K$ over $U/\pi U$ and then setting $W = V[y_1, \ldots, y_1]_{\pi V[y_1, \ldots, y_1]}$. The fact that $V/pV \cong R_1/m_1$ implies that we may take $W = V$ in the conclusion.
Since $V/pV$ is algebraically closed, the maximal ideal $(\pi, h(Y_1), Y_2, \ldots, Y_d)$ will be of the form $(\pi, Y_1 - z_1, Y_2, \ldots, Y_d)$ for some element $z_1 \in V$; therefore, $B \cong R_0$.

Without loss of generality, we may assume that $B = R_0$. Let $p_0 = R_0 \cap p_1$ and $q_0 = R_0 \cap q_1$. Then $g \in p_0 \cap q_0$. Since the induced map $R_0/gR_0 \to R_1/f_1R_1$ is an isomorphism, we see that $R_0/p_0 \cong R_1/p_1$ and $R_0/q_0 \cong R_1/q_1$. Furthermore, under this isomorphism, $(p_0/gR_0) + (q_0/gR_0) \cong (p_1/f_1R_1) + (q_1/f_1R_1)$, which implies that $\sqrt{p_0 + q_0} = \sqrt{p_0 + q_0 + gR_0} = m_0$. Also, $(R_0/gR_0)_{p_0} \cong (R_1/f_1R_1)_{p_1}$, so that $e((R_0/gR_0)_{p_0}) = e((R_1/f_1R_1)_{p_1})$; in particular, $g \in p_0^{(m)}$. Similarly, $g \in q_0^{(n)}$ and $g \not\in m_0^{m+n}$. Thus, our counterexample in $R_1$ gives rise to a counterexample in $R_0$, completing the proof. □

Using methods similar to those employed in the proof of Theorem 3.1 we verify the following weaker version of (ID-2).

**Theorem 3.3.** Let $(A, n)$ be a quasi-unmixed local ring with prime ideals $P$ and $Q$ such that $\sqrt{P + Q} = n$. If $e(A) < e(A_P) + e(A_Q)$, e.g., if $e(A) = e(A_P)$, then $\dim(A/P) + \dim(A/Q) \leq \dim(A) + 1$.

**Proof.** Let $A, n, P, Q$ be as in the statement of the theorem. As in the proof of the implication "$\Rightarrow$" in Theorem 3.1 we may assume that $A$ is a hypersurface over the ring $B = V[X_1, \ldots, X_d]$, with surjection $\pi : B \to A$. Let $\mathfrak{M} = \pi^{-1}(n)$ which is the maximal ideal of $B$, $\mathfrak{P} = \pi^{-1}(P)$ and $\mathfrak{Q} = \pi^{-1}(Q)$. Then $\mathfrak{P} + \mathfrak{Q}$ is $\mathfrak{M}$-primary, so by Serre’s Intersection Theorem,

$$\dim(A/P) + \dim(A/Q) = \dim(B/\mathfrak{P}) + \dim(B/\mathfrak{Q}) \leq \dim(B) = \dim(A) + 1$$

as desired. □

### 4. An example

With Conjecture 1.1 in mind, it is natural to ask, “What is the correct conjecture to make for symbolic powers in nonregular rings?” This is a tricky business, as symbolic powers can behave badly in general. For example, in the ring $k[X, Y]/(XY) = k[x, y]$, the prime ideal $p = (x)$ satisfies $p^{(m)} = p$ for all $m \geq 1$.

However, a few interesting suggestions have been offered in this direction. For instance, we have Question 4.1 below, which is motivated by the work of Hochster-Huneke [13]. There, the authors use tight closure methods to investigate symbolic powers.

For an ideal $I$ in a ring $R$ of prime characteristic $p > 0$, let $I^*$ denote the tight closure of $I$. A thorough introduction to the theory of tight closure can be found in [3, Chapter 10] the monograph of Huneke [13].

**Question 4.1.** Let $k$ be an algebraically closed field of prime characteristic $p > 0$ and

$$R = k[X_1, \ldots, X_d]/(f) = k[x_1, \ldots, x_d]$$

for some prime element $f \neq 0$ and $m = (x_1, \ldots, x_d)R$. Let $p, q$ prime ideals of $R$ such that $p + q = m$ and $\dim(R/p) + \dim(R/q) = \dim(R) + 1$. Does the containment $p^{(m)} \cap q \subseteq (m^{m+1})^*$ hold for all $n \geq 1$?

For $n \leq 3$, the answer to this question is “yes”. However, for $n = 4$ the answer is “no”, as the following example shows.
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Example 4.2. Fix an integer \( s \geq 3 \), and let
\[
R = k[X, Y, Z, U]/(XY(Z + U) - U^sZ) = k[x, y, z, u].
\]
Applying Eisenstein’s criterion to \( f = XY(Z + U) - U^sZ \in k[Y, Z, U][X] \) shows that \( f \) is prime. Let \( m = (x, y, z, u)r \), \( p = (x, u)r \) and \( q = (y, z)r \). Then \( p + q = m \) and \( \dim(R/p) + \dim(R/q) = 4 = \dim(R) + 1 \). By Hochster-Huneke [10] (6.2), every element in the Jacobian ideal has a power that is a test element. In this example, the Jacobian ideal is
\[
J = (y(u + z), x(u + z), xy - su^{s-1}z, xy - u^s)R.
\]
Fix an integer \( q \geq 1 \) such that \((xy - u^s)^q\) is a test element and \( 2q \) is divisible by \( s - 1 \). Set
\[
m = \frac{2q}{s-1} + 2 > \frac{2q+1}{s-1} > 0.
\]
We claim that \( x^m y \in (p^{(ms)} \cap q) \setminus (m^{ms+1})^s \).

First, we show that \( x^m y \in p^{(ms)} \cap q \). It suffices to show that \( x^m \in p^{(ms)} \). This follows from the fact that \( y^m(z + u)^m \notin p \) and \( x^m y^m(z + u)^m = u^{ms}z^m \in p^{ms} \).

Next, we show that \( x^m y \notin (m^{ms+1})^s \). As \((xy - u^s)^q\) is a test element, it suffices to show that \((xy - u^s)^q x^m y \notin m^{ms+1}\). Expanding \((xy - u^s)^q x^m y \) yields
\[
(xy - u^s)^q x^m y = x^{q+m} y^{q+1} + [-q x^{q+m-1} y^q u^s + \cdots + (-1)^q x^m y u^m q].
\]
The quantity in brackets is in \( m^{ms+1} \) by our choice of \( m \). It remains to show that \( x^{q+m} y^{q+1} \notin m^{ms+1} \). This is straightforward and can be verified by the interested reader.

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