Lovelock Gravity with Spontaneous Dimensional Breaking

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Abstract

We consider $D$-dimensional Lovelock gravity with only one term of higher-order Lovelock Lagrangian densities, and show that a product of Minkowski space-time and $n$-spheres is its vacuum solution. The most interesting feature of our model is that the spontaneous compactification of the extra dimensions results in reproduction of the Einstein gravity with no cosmological constant.

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§1. Introduction

Lovelock gravity is a natural extension of Einstein gravity to higher dimensions, and it is of great interest in theoretical physics as it describes a wide class of models. The equations of motion of Lovelock gravity do not contain more than second order derivatives with respect to metric, as in the case of general relativity.

Recently, Maeda and Dadhich presented vacuum solutions of the form $\mathcal{M}^m \times \mathcal{H}^n$ in Gauss Bonnet gravity theory. Cai, Cao, and Ohta generalized the solutions in $D$-dimensional Lovelock gravity. Here, $\mathcal{M}^m$ is the $m$-dimensional manifold and $\mathcal{H}^n$ is a negative constant curvature space.

In this paper, we consider $D$-dimensional Lovelock gravity with only one term of higher-order Lovelock Lagrangian densities, and show that a product of Minkowski space-time and $n$-spheres (positive constant curvature space) is its vacuum solution. This vacuum solution indicates the possibility that compactification of the extra dimensions takes place spontaneously. We show that the usual Einstein gravity with no cosmological constant is reproduced as a result of the spontaneous compactification.

To illustrate our idea, let us consider the simplest example. We take the following Lagrangian

$$S = \frac{1}{2\kappa^2} \int d^6x \sqrt{-g} L^{(6)}_{GB},$$  \hspace{1cm} (1.1)

where $L^{(6)}_{GB}$ is the six-dimensional Gauss-Bonnet term (second order Lovelock Lagrangian density) $L^{(6)}_{GB} \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$. Let us compactify the six-dimensional space-time to a product of a four-dimensional manifold and a two-sphere, that is, $\mathcal{M}^D \rightarrow \mathcal{M}^4 \times S^2$.

By substituting $R_{abcd} = r^2 (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc})$ in the two-sphere part, eqn. (1.1) can be rewritten as

$$S = \frac{V^{(2)}}{2\kappa^2} \int d^4x \sqrt{-g} R^{(4)},$$  \hspace{1cm} (1.2)

where $V^{(2)}$ represents the volume of $S^2$ and we have used $\int d^4x \sqrt{-g} L^{(4)}_{GB} = \int d^4x \sqrt{-g} L^{(2)}_{GB} = 0$. Note that we obtained the Einstein-Hilbert action and (1.2) does not include the cosmological constant.

This example could not reproduce the Standard Model, because it could only yield the gauge symmetry of $SU(2)$, which comes from $S^2$, when compactified down to four dimensions. In the next section, we generalize the above treatment to higher dimensions.
§2. The $D$-dimensional model

We consider the following Lagrangian

$$ S = \frac{1}{2\kappa^2(D)} \int d^D x \sqrt{-g} \mathcal{L}^{(D)} + S_{\text{matter}}, $$

where $p \leq [(D - 1)/2]$ ($[N]$ denotes the integral part of the number $N$), $\kappa(D) \equiv \sqrt{8\pi G(D)}$ where $G(D)$ is the $D$-dimensional gravitational constant and $\mathcal{L}^{(D)}$ is the $D$-dimensional $p$-th order Lovelock Lagrangian density

$$ \mathcal{L}^{(D)} = \frac{1}{2^p} \delta^{\mu_1 \nu_1 \cdots \mu_p \nu_p} R_{\mu_1 \nu_1} \cdots R_{\mu_p \nu_p} - \frac{1}{2} \delta_{\mu}^{\nu} \mathcal{L}^{(D)} \quad (2.2) $$

The $\delta$ symbol denotes a totally anti-symmetric product of Kronecker deltas, normalized to take values 0 and $\pm 1$.

Varying the action (2.1), we obtain the equations of motion as

$$ G^{(D)}(p)_{\mu \nu} = \kappa^2(D) T_{\mu \nu}, $$

where $T_{\mu \nu}$ is the energy-momentum tensor of the matter field derived from $S_{\text{matter}}$ and

$$ G^{(D)}(p)_{\mu \nu} \equiv \delta^{\mu_1 \nu_1 \cdots \mu_p \nu_p} R_{\mu_1 \nu_1} \cdots R_{\mu_p \nu_p} - \frac{1}{2} \delta_{\mu}^{\nu} \mathcal{L}^{(D)} $$

$$ \quad = - \frac{1}{2^{p+1}} \delta^{\mu_1 \nu_1 \cdots \mu_p \nu_p} R_{\mu_1 \nu_1} \cdots R_{\mu_p \nu_p}. \quad (2.4) $$

Let us compactify the $D$-dimensional space-time to a product of an $m$-dimensional manifold and an $n$-sphere, that is, $\mathcal{M}^D \to \mathcal{M}^m \times S^n$. For $S^n$, the Riemann curvature can be calculated as

$$ R_{abce} = r^2(\eta_{ae} \eta_{bd} - \eta_{ad} \eta_{be}), \quad R^{ab}_{cd} = \frac{1}{r^{(n)}} \delta^{ab}_{cd}. \quad (2.5) $$

where $r^{(n)}$ is the constant radius of the $n$-sphere. We indicate the vectors in the $m$-dimensional manifold by subscripted Roman letters $i$, $j$, $k$, etc. and the vectors in the $n$-sphere by subscripted Roman letters $a$, $b$, $c$, etc. Greek letters such as $\mu$ and $\nu$ are used to label $D$-dimensional space-time vectors. Substituting (2.5) into (2.4), we can decompose (2.4) into $m$-dimensional and $n$-dimensional parts

$$ G^{(D)}(D)_{i j} = - \frac{1}{2^{p+1}} \delta^{\mu_1 \nu_1 \cdots \mu_p \nu_p} R_{\mu_1 \nu_1} \cdots R_{\mu_p \nu_p} $$

$$ \quad = - \sum_{t=0}^{p} \binom{p}{t} \frac{1}{2^{p+1}} \delta_{\mu_1 \nu_1 \cdots \mu_p \nu_p} R^m_{m_1 m_2 \cdots m_t} R^m_{m_1 m_2 \cdots m_t} $$

$$ \times R^{e_1 \cdots e_{k-t}} \delta_{k-t} \cdot \cdot \cdot R^{e_{k-t}} \delta_{k-t} $$

$$ = \sum_{t=0}^{[m-1]/2} \binom{p}{t} \frac{n!}{(n - 2(p - t))!} \left( \frac{1}{r^{2(n)}} \right)^{p-t} \times G^{(m)}(t)_{i j}. \quad (2.6) $$
\begin{align}
G^{(m)i}_{(l)j} & = -\frac{1}{2^{t+1}} \delta^{ki_1 \cdots ki_t} R^{m_{i_1} \cdots m_{i_t}}_{k_1 l_1} \cdots R^{m_{i_t}}_{k_{t+1} l_t}, \quad (2.7) \\
G^{(D)\alpha}_{(p) \beta} & = -\frac{1}{2^{p+1}} \delta^{\mu_1 \nu_1 \cdots \mu_p \nu_p} \lambda_1 \sigma_1 \cdots \lambda_p \sigma_p \lambda_1 \sigma_1 \cdots \lambda_p \sigma_p \\
& = -\sum_{t=0}^{p} \left( \begin{array}{c} p \\ t \end{array} \right) \frac{1}{2^{t+1}} \delta_{\alpha 1}^{\alpha_1 \cdots \alpha_t \cdots \alpha_{t+1}} R^{\mu_1 \nu_1 \cdots \mu_t \nu_t \cdots \mu_{t+1} \nu_{t+1}}_{\beta_1 \sigma_1 \cdots \beta_t \sigma_t \cdots \beta_{t+1} \sigma_{t+1}} \\
& \times R^{e_1 f_1} \cdots R^{e_{t-1} f_{t-1}} \\
& = -\frac{1}{2} \delta \left( \begin{array}{c} m/2 \\ t \end{array} \right) \frac{(n-1)!}{(n-1-2(p-t))!} \left( \frac{1}{r^{(n)}} \right)^{p-t} \times \mathcal{L}^{(m)}_{(l)}, \quad (2.8) \\
\mathcal{L}^{(m)}_{(l)} & = \frac{1}{2} \delta^{ki_1 \cdots ki_t} R^{m_{i_1} \cdots m_{i_t}}_{k_1 l_1} \cdots R^{m_{i_t}}_{k_{t+1} l_t}, \quad (2.9)
\end{align}

where \( p - t \geq 1 \), and we have used the identities for the Lovelock tensors \( \delta \).

\begin{align}
G^{(m)i}_{(l)j} & \equiv 0 \text{ for } m \leq 2t, \quad \mathcal{L}^{(m)}_{(l)} \equiv 0 \text{ for } m \leq 2t - 1, \quad (2.10)
\end{align}

and the identity

\begin{align}
\delta^{\mu_1 \cdots \mu_p - 1 \mu_p} \delta^{\nu_1 \cdots \nu_p - 1 \nu_p} = 2[r - (p - 1)][r - (p - 2)] \delta^{\mu_1 \cdots \mu_{p-2}}_{\nu_1 \cdots \nu_{p-2}} \quad (p \geq 2), \quad (2.11)
\end{align}

where \( r \) denotes the range of the index \( r = m \) for \( \mathcal{M}^m \) and \( r = n \) for \( S^n \) and \( \delta^{\mu_1 \cdots \mu_{p-2}}_{\nu_1 \cdots \nu_{p-2}} \equiv 1 \) for \( p = 2 \). The other components (such as \( G^{(m)a}_{(l)i} \)) automatically vanish.

The Lagrangian (2.22) can be decomposed as

\begin{align}
\mathcal{L}^{(D)}_{(p)} = \sum_{t=0}^{[m/2]} \left( \begin{array}{c} p \\ t \end{array} \right) \frac{n!}{(n-2(p-t))!} \left( \frac{1}{r^{(n)}} \right)^{p-t} \times \mathcal{L}^{(m)}_{(l)}, \quad (2.12)
\end{align}

where \( p - t \geq 1 \).

Let us consider the most important case \( \mathcal{M}^4 \times S^n \). By substituting \( m = 4 \) in (2.10), (2.8), and (2.12), we obtain

\begin{align}
G^{(D)i}_{(p)j} = n(n-1) \cdots (n-2p+1) \left( \frac{1}{r^{(n)}} \right)^{p} G^{(4)i}_{(0)j} \\
+ n(n-1) \cdots (n-2p+3) \left( \frac{1}{r^{(n)}} \right)^{p-1} G^{(4)i}_{(1)j}, \quad (2.13)
\end{align}

\begin{align}
G^{(D)a}_{(p) b} = -\frac{1}{2} \delta^{a} \left( \begin{array}{c} n-1 \cdots (n-2p) \\ n \end{array} \right) \left( \frac{1}{r^{(n)}} \right)^{p} \mathcal{L}^{(4)}_{(0)}, \quad (2.14)
\end{align}
\[ +p(n-1)\cdots(n-2p+2)\left(\frac{1}{r^2(n)}\right)^{p-1}\mathcal{L}^{(4)}_{(1)} \]
\[ +\frac{p(p-1)}{2}(n-1)\cdots(n-2p+4)\left(\frac{1}{r^2(n)}\right)^{p-2}\mathcal{L}^{(4)}_{(2)}, \quad (2.14) \]

\[ \mathcal{L}^{(p)}_{(p)} = n(n-1)\cdots(n-2p+1)\left(\frac{1}{r^2(n)}\right)^{p}\mathcal{L}^{(4)}_{(0)} \]
\[ +pm(n-1)\cdots(n-2p+3)\left(\frac{1}{r^2(n)}\right)^{p-1}\mathcal{L}^{(4)}_{(1)} \]
\[ +\frac{p(p-1)}{2}n(n-1)\cdots(n-2p+5)\left(\frac{1}{r^2(n)}\right)^{p-2}\mathcal{L}^{(4)}_{(2)}. \quad (2.15) \]

Note that the usual Einstein gravity with no cosmological constant can be reproduced when
\[ n + 1 \leq 2p < n + 3. \quad (2.16) \]

For example, when \( \mathcal{M}^{13} \to \mathcal{M}^4 \times S^9 \) \((n = 9)\) and \( p = 5 \), by using \( (2.13) \) and \( (2.14) \), the equations of motion \( (2.3) \) can be decomposed as
\[ R^{(4)}_{ij} - \frac{1}{2}g_{ij}R^{(4)} = \kappa_{(4)}^2 T_{ij} \quad (2.17) \]
for the four-dimensional parts, and
\[ -\delta^a_b\left(R^{(4)} + \kappa_{(9)}^2\mathcal{L}^{(4)}_{(2)}\right) = 18\kappa_{(4)}^2 T^a_{\ b} \quad (2.18) \]
for the nine-dimensional parts. By using \( (2.15) \), \( (2.1) \) can be rewritten as
\[ S = \frac{V^{(n)}}{2\kappa_{(4)}^2} \int d^4x\sqrt{-g}R^{(4)} + S_{\text{matter}}, \quad (2.19) \]
where \( V^{(n)} \) is the volume of \( S^n \) and
\[ \kappa_{(4)}^2 = \frac{\kappa_{(9)}^2}{5 \times 9!} \kappa_{(13)}^2. \quad (2.20) \]

Incidentally, the space-time \( \mathcal{M}^4 \times S^9 \) has a gauge symmetry of \( SO(10) \) which comes from \( S^9 \).

In the case of vacuum \( (T_{\mu\nu} = 0) \), Minkowski space-time is a trivial solution of \( (2.17) \) and \( (2.18) \), namely, the product of a four-dimensional Minkowski and an \( n \)-sphere is a solution.
of (2.3). This vacuum solution indicates the possibility that compactification of the extra dimensions takes place spontaneously.

Next, let us consider the compactification $\mathcal{M}^{13} \rightarrow \mathcal{M}^4 \times S^2 \times S^2 \times S^5$. We set $p = 5$. Because $SO(3) = SU(2)$ and $SO(6) = SU(4)$, the space-time $\mathcal{M}^4 \times S^2 \times S^2 \times S^5$ has the gauge symmetry of $SU(4) \otimes SU(2) \otimes SU(2)$ as the internal symmetry. First, we compactify $\mathcal{M}^{13}$ to $\mathcal{M}^8 \times S^5$. By using (2.6), the eight-dimensional parts of $G^{(13)}_{(5)\mu\nu}$ is decomposed as

$$G^{(13)}_{(5)ij} = \frac{10 \times 5!}{r_5^4} G^{(8)}_{(3)ij}. \quad (2.21)$$

Next, we compactify $\mathcal{M}^8$ to $\mathcal{M}^6 \times S^2$. Then the six-dimensional parts of $G^{(8)}_{(3)ij}$ are decomposed as

$$G^{(8)}_{(3)ij} = \frac{6}{r_2^4} G^{(6)}_{(2)ij}. \quad (2.22)$$

Lastly, we compactify $\mathcal{M}^6$ to $\mathcal{M}^4 \times S^2$. The four-dimensional parts of $G^{(6)}_{(2)ij}$ are decomposed as

$$G^{(6)}_{(2)ij} = \frac{4}{r_2^4} G^{(4)}_{(1)ij}. \quad (2.23)$$

Combining (2.21), (2.22), and (2.23), we obtain the four-dimensional parts of $G^{(13)}_{(5)\mu\nu}$ as

$$G^{(13)}_{(5)ij} = \frac{28800}{r_5^4 r_2^4} G^{(4)}_{(1)ij}. \quad (2.24)$$

Similarly, by using (2.8) and (2.12), the five-dimensional parts of $G^{(13)}_{(5)\mu\nu}$ can be decomposed as

$$G^{(13)}_{(5)ab} = -\frac{2880}{r_5^4 r_2^4} \delta^a_b \left( \mathcal{L}^{(4)}_{(1)} + \frac{r_5^2}{2} \mathcal{L}^{(4)}_{(2)} \right) \quad (2.25)$$

and the two two-dimensional parts of $G^{(13)}_{(5)\mu\nu}$ can be decomposed as

$$G^{(13)}_{(2)xy} = 0, \quad G^{(13)}_{(2)x'y'} = 0, \quad (2.26)$$

where the vectors in $S^5$ are indicated by subscripted Roman letters $a$, $b$, etc.; the vectors in $S^2$ by subscripted Roman letters $x$, $y$, etc.; and the vectors in $S^2$ by subscripted Roman letters $x'$, $y'$, etc. The equations of motion (2.3) can therefore be rewritten as

$$R^{(4)}_{ij} - \frac{1}{2} g_{ij} R^{(4)} = \kappa_4^2 T_{ij}, \quad (2.27)$$

$$- \delta^a_b \left( R^{(4)} + \frac{r_5^2}{2} \mathcal{L}^{(4)}_{(2)} \right) R^{(4)} = 10 \kappa_4^2 T^a_b. \quad (2.28)$$
\[ 0 = T_{xy}, \quad 0 = T_{x'y'}, \quad (2.29) \]

where
\[ \kappa^2 = \frac{r^4 r'(2)^2 r''(2)}{28800} \kappa^2_{(13)}, \quad (2.30) \]

Thus, we have retrieved the Einstein gravity with no cosmological constant.

§3. Conclusion

In this paper, we considered \( D \)-dimensional Lovelock gravity with only one term of the higher-order Lovelock Lagrangian densities. We show that a product space of Minkowski space-time and \( n \)-spheres is its vacuum solution. Furthermore, we showed that the Einstein gravity with no cosmological constant can be obtained as a result of spontaneous compactification of the extra dimensions. It is remarkable that without introducing negative constant curvature spaces and adjusted coefficients, we were able to obtain the compactified vacuum solutions and the zero cosmological constant.

References

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