Triply periodic monopoles and difference modules on elliptic curves

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Abstract

We explain the correspondences between monopoles with Dirac type singularity and polystable mini-holomorphic bundles with Dirac type singularity of degree 0 on a 3-dimensional torus. We also explain that they are equivalent to polystable parabolic difference modules of degree 0 on elliptic curves.

MSC: 53C07, 58E15, 14D21, 81T13

1 Introduction

We studied the Kobayashi-Hitchin correspondences for singular monopoles with periodicity in one direction [4] or two directions [5]. In this paper, we study singular monopoles with periodicity in three directions. In the analytic aspect, this case is much simpler than the other cases because a 3-dimensional torus is compact. But, there still exist interesting correspondences with algebro-geometric objects.

1.1 Triply periodic monopoles with Dirac type singularity

Let $Y$ be an oriented 3-dimensional $\mathbb{R}$-vector space with an Euclidean metric $g_Y$. Let $\Gamma$ be a lattice of $Y$. We set $M := Y/\Gamma$, which is equipped with the induced metric $g_M$. Let $Z$ be a finite subset of $M$. Let $E$ be a $C^\infty$-vector bundle on $M \setminus Z$ with a Hermitian metric $h$, a unitary connection $\nabla$ and an anti-self-adjoint endomorphism $\phi$. The tuple $(E, h, \nabla, \phi)$ is called a monopole on $M \setminus Z$ if the Bogomolny equation 

$$F(\nabla) = *\nabla \phi$$

is satisfied, where $F(\nabla)$ denotes the curvature of $\nabla$, and $*$ denotes the Hodge star operator with respect to $g_M$. A point of $P \in Z$ is called a Dirac type singularity of the monopole $(E, h, \nabla, \phi)$ if $|\phi_Q|_h = O(d(Q, Z)^{-1})$ for any $Q \in M \setminus Z$, where $d(Q, Z)$ denotes the distance between $Q$ and $Z$. Note that the notion of Dirac type singularity was originally introduced by Kronheimer [3]. The above condition is equivalent to the original one, according to [6].

1.2 Mini-holomorphic bundles with Dirac type singularity

Let us explain a correspondence between monopoles with Dirac type singularity and polystable mini-holomorphic bundles with Dirac type singularity on a 3-dimensional torus. It was formulated by Kontsevich and Soibelman [2].

1.2.1 Mini-complex structure

We take a mini-complex structure on $M \setminus Z$. Namely, we take a linear coordinate system $(x_1, x_2, x_3)$ on $Y$ compatible with the orientation such that $g_Y = \sum dx_i \cdot dx_i$, and we set $t := x_1$ and $w = x_2 + \sqrt{-1}x_3$. They induce the complex vector fields $\partial_t$ and $\partial_w$ on $M$. A $C^\infty$-function $f$ on an open subset of $M$ is called mini-holomorphic if $\partial_t f = \partial_w f = 0$. Let $\mathcal{O}_{M \setminus Z}$ denote the sheaf of mini-holomorphic functions on $M \setminus Z$. 


1.2.2 Mini-holomorphic bundles with Dirac type singularity

Let $\mathcal{V}$ be a locally free $\mathcal{O}_{\mathcal{M}\setminus Z}$-module. Let $P$ be a point of $Z$. We take a lift $(t_0, w_0) \in Y$. Set $B^*_w(\delta) := \{ w \in \mathbb{C} \mid 0 < |w - w_0| < \delta \}$ for any $\delta > 0$. For any $t \in [t_0 - \delta, t_0 + \delta]$, the restriction $\mathcal{V}|_{(t) \times B^*_w(\delta_2)}$ is naturally a locally free $\mathcal{O}_{B^*_w(\delta_2)}$-module. Because mini-holomorphic functions are constant in the $t$-direction, we obtain an isomorphism of $\mathcal{O}_{B^*_w(\delta_2)}$-modules $\mathcal{V}|_{(t_0 - \delta_1) \times B^*_w(\delta_2)} \cong \mathcal{V}|_{(t_0 + \delta_1) \times B^*_w(\delta_2)}$. If it is meromorphic at $w_0$, then $P$ is called a Dirac type singularity of $\mathcal{V}$. If every point of $Z$ is Dirac type singularity, then $\mathcal{V}$ is called a mini-holomorphic bundle with Dirac type singularity on $(\mathcal{M}; Z)$.

1.2.3 Stability condition

Kontsevich and Soibelman [2] introduced a sophisticated way to define a stability condition for mini-holomorphic bundles with Dirac type singularity on $(\mathcal{M}; Z)$.

Let $H^j(\mathcal{M}\setminus Z)$ denote the cohomology group of $\mathcal{M}\setminus Z$ with $\mathbb{R}$-coefficient. Let $H_j(\mathcal{M}, Z)$ denote the relative $j$-th homology group of $(\mathcal{M}, Z)$ with $\mathbb{R}$-coefficient. Note that there exists the natural isomorphism

$$\Phi_Z : H^2(\mathcal{M}\setminus Z) \cong H_1(\mathcal{M}, Z).$$

Let $\mathcal{T}$ denote the space of left invariant vector fields on $\mathcal{M}$, and let $\mathcal{T}^\vee$ denote the left invariant 1-forms on $\mathcal{M}$. Let $\sigma$ denote the image of $1$ via the canonical morphism $\mathbb{R} \rightarrow \mathcal{T} \otimes \mathcal{T}^\vee$. It is described as $\sigma = \sum_{i=1,2,3} \partial_{x_i} \otimes dx_i$.

For any mini-holomorphic bundle with Dirac type singularity $\mathcal{V}$ on $(\mathcal{M}; Z)$, we obtain $c_1(\mathcal{V}) \in H^2(\mathcal{M}\setminus Z)$, and hence $\Phi_Z(c_1(\mathcal{V})) \in H_1(\mathcal{M}, Z)$. Then, we obtain the following:

$$\int_{\Phi_Z(c_1(\mathcal{V}))} \sigma = \sum_{i=1,2,3} \left( \int_{\Phi_Z(c_1(\mathcal{V}))} dx_i \right) \partial_{x_i} \in \mathcal{T}.$$

Kontsevich and Soibelman found that $\int_{\Phi_Z(c_1(\mathcal{V}))} \sigma$ is a scalar multiplication of $\partial_1 = \partial_{x_1}$, and they define the degree $\deg^{KS}(\mathcal{V})$ for $\mathcal{V}$ as follows:

$$\int_{\Phi_Z(c_1(\mathcal{V}))} \sigma = \deg^{KS}(\mathcal{V}) \partial_1.$$

They introduced the following stability condition.

**Definition 1.1** A mini-holomorphic bundle with Dirac type singularity $\mathcal{V}$ on $(\mathcal{M}; Z)$ is called stable (resp. semistable) if

$$\deg^{KS}(\mathcal{V}')/\text{rank}(\mathcal{V}') < \deg^{KS}(\mathcal{V})/\text{rank}(\mathcal{V}) \quad \text{resp.} \quad \deg^{KS}(\mathcal{V}')/\text{rank}(\mathcal{V}') \leq \deg^{KS}(\mathcal{V})/\text{rank}(\mathcal{V})$$

for any locally free $\mathcal{O}_{\mathcal{M}\setminus Z}$-submodule $\mathcal{V}'$ of $\mathcal{V}$ such that $0 < \text{rank}(\mathcal{V}') < \text{rank}(\mathcal{V})$. It is called polystable if it is semistable and a direct sum of stable submodules.

1.2.4 Kobayashi-Hitchin correspondence

It is a standard fact that a monopole with Dirac type singularity on $\mathcal{M}\setminus Z$ induces a mini-holomorphic bundle with Dirac type singularity on $(\mathcal{M}; Z)$. The following theorem was formulated by Kontsevich and Soibelman [2].

**Theorem 1.2 (Theorem 2.7, Proposition 3.2)** The procedure induces an equivalence between monopoles with Dirac type singularity on $\mathcal{M}\setminus Z$ and polystable mini-holomorphic bundles with Dirac type singularity of degree 0 on $(\mathcal{M}; Z)$.

We shall relate the degree of Kontsevich-Soibelman with the analytic degree defined in terms of Hermitian metrics (Proposition 3.2). Then, Theorem 1.2 follows from the fundamental theorem due to Simpson [7] as we shall explain in the proof of Theorem 2.7, which is an analogue of a result due to Charbonneau and Hurtubise [1] for singular monopoles on 3-dimensional manifolds obtained as the product of $S^1$ and a compact Riemann surface.

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1.3 Parabolic difference modules on elliptic curves

Let us give a complement on correspondences between mini-holomorphic bundles with Dirac type singularity on a 3-dimensional torus and parabolic difference modules on elliptic curves.

1.3.1 Parabolic difference modules on elliptic curves and a stability condition

Let \( \Gamma_0 \) be a lattice of \( \mathbb{C} \). We set \( T := \mathbb{C}/\Gamma_0 \). Let \( a \in \mathbb{C} \). Let \( \Phi : T \to T \) be the morphism induced by \( \Phi(z) = z + a \). Let \( D \subset T \) be a finite subset. A parabolic \( a \)-difference module on \( T \) consists of the following data \( V_\ast = (V, (\tau_P, \mathcal{L}_P)_{P \in D}) \):

- A locally free \( \mathcal{O}_T \)-module \( V \).
- An isomorphism of \( \mathcal{O}_T(\ast D) \)-modules \( V(\ast D) \simeq (\Phi^\ast)^{-1}(V)(\ast D) \).
- A sequence \( 0 \leq \tau_{P,1} < \tau_{P,2} < \cdots < \tau_{P,m(P)} < 1 \) for each \( P \in D \).
- Lattices \( \mathcal{L}_{P,i} \) \((i = 1, \ldots, m(P) - 1)\) of \( V(\ast D) \). We formally set \( \mathcal{L}_{P,0} := V_P \) and \( \mathcal{L}_{P,m(P)} := (\Phi^\ast)^{-1}(V)_P \).

When we fix \((\tau_P)_{P \in D}\), it is called a parabolic \( a \)-difference module on \((T, (\tau_P)_{P \in D})\).

The degree of a parabolic \( a \)-difference module \((V, (\tau_P, \mathcal{L}_P)_{P \in D})\) is defined as follows:

\[
\deg(V, (\tau_P, \mathcal{L}_P)_{P \in D}) := \deg(V) + \sum_{P \in D} \sum_{i=1}^{m(P)} (1 - \tau_{P,i}) \deg(\mathcal{L}_{P,i}, \mathcal{L}_{P,i-1}).
\]

Here, \( \deg(\mathcal{L}_{P,i}, \mathcal{L}_{P,i-1}) := \text{length}(\mathcal{L}_{P,i}/\mathcal{L}_{P,i-1} \cap \mathcal{L}_{P,i}) - \text{length}(\mathcal{L}_{P,i-1}/\mathcal{L}_{P,i-1} \cap \mathcal{L}_{P,i}) \). It is easy to see that \( \sum_{P \in D} \sum_{i=1}^{m(P)} \deg(\mathcal{L}_{P,i}, \mathcal{L}_{P,i-1}) = 0 \). We also set

\[
\mu(V, (\tau_P, \mathcal{L}_P)_{P \in D}) := \deg(V, (\tau_P, \mathcal{L}_P)_{P \in D})/\text{rank}V.
\]

For any \( \mathcal{O}_T(\ast D) \)-module \( 0 \neq V' \subset V \) such that \( V'(\ast D) \simeq (\Phi^\ast)^{-1}(V')(\ast D) \), we obtain lattices \( \mathcal{L}'_{P,i} \) of \( V'(\ast D)_P \) by setting \( \mathcal{L}'_{P,i} := \mathcal{L}_{P,i} \cap V'(\ast D)_P \) in \( V(\ast D)_P \), and we obtain a parabolic \( a \)-difference module \((V', (\tau_P, \mathcal{L}'_P)_{P \in D})\). Such \((V', (\tau_P, \mathcal{L}'_P)_{P \in D})\) is called a parabolic \( a \)-difference submodule of \((V, (\tau_P, \mathcal{L}_P)_{P \in D})\).

**Definition 1.3** \((V, (\tau_P, \mathcal{L}_P)_{P \in D})\) is called stable (resp. semistable) if

\[
\mu(V', (\tau_P, \mathcal{L}'_P)_{P \in D}) < \mu(V, (\tau_P, \mathcal{L}_P)_{P \in D}) \quad \text{(resp. } \mu(V', (\tau_P, \mathcal{L}'_P)_{P \in D}) \leq \mu(V, (\tau_P, \mathcal{L}_P)_{P \in D})\text{)}
\]

for any parabolic \( a \)-difference submodules such that \( 0 < \text{rank}V' < \text{rank}V \). It is called polystable if it is semistable and a direct sum of stable objects.

1.3.2 Equivalence

We return to the situation in [12]. We take a generator \( e_i = (a_i, \alpha_i) \) \((i = 1, 2, 3)\) of \( \Gamma \subset \mathbb{R}_t \times \mathbb{C}_u = Y \), which is compatible with the orientation of \( Y \). We also assume that \( \alpha_1 \) and \( \alpha_2 \) generate a lattice in \( \mathbb{C} \). Let \( \Gamma_0 \) denote the lattice, and we set \( T := \mathbb{C}/\Gamma_0 \). We set

\[
\gamma := \frac{a_1 \overline{a}_2 - a_2 \overline{a}_1}{\alpha_1 \alpha_2 - \alpha_1 \overline{a}_2}, \quad t := a_3 + \text{Re}(\gamma a_3), \quad a := a_3.
\]

It is easy to see that \( t > 0 \). We define the isomorphism \( F : \mathbb{R}_t \times \mathbb{C}_u \simeq \mathbb{R}_s \times \mathbb{C}_u \) by

\[
s = t + 2 \text{Re}(\gamma w), \quad u = w.
\]

Note that the induced action of \( \Gamma \) on \( \mathbb{R}_s \times \mathbb{C}_u \) is expressed as follows:

\[
e_i(s, u) = (s, u + \alpha_i) \quad (i = 1, 2), \quad e_3(s, u) = (s + t, u + \alpha_3).
\]

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Let $Z_Y$ be the pull back of $Z$ by $Y \to \mathcal{M}$. Let $D$ denote the image of the composite of the following maps:

$$F(Z_Y) \cap ([0, t] \times \{u_0\}) \subset \mathbb{R}_s \times \mathbb{C}_u \to \mathbb{C}_u \to T.$$ 

For any $P \in D$, we take $u_0 \in \mathbb{C}$ which is mapped to $P$. We obtain a sequence $0 \leq s_{P,1} < s_{P,2} < \cdots < s_{P,m(P)} < t$ by the condition:

$$\{ (s_{P,i}, u_0) | i = 1, \ldots, m(P) \} = F(Z_Y) \cap ([0, t] \times \{u_0\}).$$

It is independent of the choice of $u_0$. We set $\tau_{P,i} := s_{P,i}/t$.

**Proposition 1.4 (Proposition 4.1, Proposition 4.2)** There exists an equivalence between parabolic difference modules on $(T, (\tau_P)_{P \in D})$ and mini-holomorphic bundles with Dirac type singularity on $(\mathcal{M}; Z)$. The equivalence preserves the degree up to the multiplication of a positive constant. As a result, the equivalence preserves the (poly)stability condition. 

See [4,2] for the explicit correspondence. As a consequence of Theorem 1.2 and Proposition 1.4, we obtain the following theorem.

**Theorem 1.5** We have the equivalence of the following objects:

- Monopoles with Dirac type singularity on $\mathcal{M} \setminus Z$.
- Polystable mini-holomorphic bundles with Dirac type singularity of degree 0 on $(\mathcal{M}, Z)$.
- Polystable parabolic difference modules of degree 0 on $(T, (\tau_P)_{P \in D})$.

Here, $Z$ and $(\tau_P)_{P \in D}$ are related as above.

**Acknowledgement** The author thanks Maxim Kontsevich and Yan Soibelman for the communication and for sending the preprint [2]. Indeed, this study grew out of my answer to one of their questions. I hope that this would be useful for their project. I owe much to Carlos Simpson whose ideas on the Kobayashi-Hitchin correspondence are fundamental in this study. I thank Masaki Yosino for discussions.

I am partially supported by the Grant-in-Aid for Scientific Research (S) (No. 17H06127), the Grant-in-Aid for Scientific Research (S) (No. 16H06335), and the Grant-in-Aid for Scientific Research (C) (No. 15K04843), Japan Society for the Promotion of Science.

## 2 Monopoles and analytically stable mini-holomorphic bundles

### 2.1 Mini-holomorphic bundles with Dirac type singularity

**2.1.1 Mini-complex spaces**

We take a base $(a_i, \alpha_i) \ (i = 1, 2, 3)$ of the $\mathbb{R}$-vector space $\mathbb{R} \times \mathbb{C}$. Let $Y := \mathbb{R} \times \mathbb{C}$ with the Riemannian metric $dt \, dt + dw \, d\overline{w}$. We also consider the mini-complex structure on $Y$ induced by the coordinate system $(t, w)$. We consider the action of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ on $Y$ given by

$$e_i(t, w) = (t, w) + (a_i, \alpha_i) \ \ (i = 1, 2, 3).$$

Let $\mathcal{M}$ denote the quotient space of $Y$ by the action of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$. It is equipped with a naturally induced mini-complex structure.
2.1.2 Analytic stability condition for mini-holomorphic bundles with a Hermitian metric

Let \( Z \) be a finite subset of \( \mathcal{M} \). Let \((E, \overline{\mathcal{J}}_E)\) be a mini-holomorphic bundle on \( \mathcal{M} \setminus Z \). (See [4] §2.2 for the notion of mini-holomorphic bundle.) Let \( h \) be a Hermitian metric of \( E \). We obtain the Chern connection \( \nabla_h \) and the Higgs field \( \phi_h \) as explained in [4] §2.2.3. Recall that we set

\[
G(h) := \left[ \nabla_{h,w}, \nabla_{h,w} \right] - \frac{\sqrt{-1}}{2} \nabla_{h,t} \phi_h.
\]

**Definition 2.1** If \( \text{Tr} G(h) \) is expressed as a sum of an \( L^1 \)-function and a non-positive function, then we set
\[
\deg(E, \overline{\mathcal{J}}_E, h) := \int_{\mathcal{M} \setminus Z} \text{Tr} G(h) \text{dvol}_M \in \mathbb{R} \cup \{-\infty\}.
\]

We also set \( \mu(E, \overline{\mathcal{J}}_E, h) := \deg(E, \overline{\mathcal{J}}_E, h)/\text{rank}(E) \).

Suppose that \( |G(h)|_h \) is \( L^1 \). Let \( E' \) be any mini-holomorphic subbundle of \( E \). Let \( h_{E'} \) be the induced metric of \( E' \). Let \( p_{E'} \) be the orthogonal projection of \( E \) onto \( E' \) with respect to \( h \). Recall the Chern-Weil formula (see [4] §2.8.2):

\[
\int \text{Tr} G(h_{E'}) = \int \text{Tr}(G(h_E) \cdot p_{E'}) - \int |\partial_E \overline{\mathcal{J}}_E p_{E'}|^2 - \frac{1}{4} \int |\partial_E t p_{E'}|^2.
\]

Hence, \( \deg(E', h_{E'}) \) is defined in \( \mathbb{R} \cup \{-\infty\} \).

**Definition 2.2** Suppose that \( G(h) \) is \( L^1 \). Then, \((E, \overline{\mathcal{J}}_E, h)\) is called analytically stable if \( \mu(E', \overline{\mathcal{J}}_E, h_{E'}) < \mu(E, \overline{\mathcal{J}}_E, h) \) for any mini-holomorphic subbundle \( E' \subset E \) with \( 0 < \text{rank}(E') < \text{rank}(E) \).

2.1.3 Adapted metrics of mini-holomorphic bundles with Dirac type singularity

Let \((E, \overline{\mathcal{J}}_E)\) be a mini-holomorphic bundle with Dirac type singularity on \((\mathcal{M}; Z)\). Let \( P \in Z \). Let \((t_0, w_0) \in Y \) be a lift of \( P \). We set \( t_P := t - t_0 \) and \( w_P := w - w_0 \). Then, \((t_P, w_P)\) induces a mini-complex coordinate system on a neighbourhood \( U_P \) of \( P \). By using the coordinate system, we may regard \( U_P \) as an open neighbourhood of \((0,0)\) in \( \mathbb{R} \times \mathbb{C} \). Let \( \varphi : \mathbb{C}^2 \to \mathbb{R} \times \mathbb{C} \) be given by \( \varphi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 z_2) \). Let \( U_P \) be the pull back of \( U_P \) by \( \varphi \). The mini-holomorphic bundle \((E, \overline{\mathcal{J}}_E)|_{U_P \setminus \{P\}}\) induces an \( S^1 \)-equivariant holomorphic vector bundle \((\tilde{E}_P, \overline{\mathcal{J}}_{\tilde{E}_P})\) on \( \tilde{U}_P \setminus \{(0,0)\} \), which is uniquely extended to an \( S^1 \)-equivariant holomorphic vector bundle \((\tilde{E}_P, \overline{\mathcal{J}}_{\tilde{E}_P})\) on \( \tilde{U}_P \). (See [3] §2.2 for a more detailed explanation.)

**Definition 2.3** A Hermitian metric \( h \) of \( E \) is called adapted at \( P \) if \( h_{|U_P \setminus \{P\}} \) induces a \( C^\infty \)-metric of \( \tilde{E}_P \). We say that \( h \) is adapted if \( h \) is adapted at any \( P \in Z \).

Note that if \( h \) is an adapted metric at \( P \), then \( G(h)_P = O(d(P, Q)^{-1}) \) [4] Lemma 2.35] on \( U_P \setminus \{P\} \), where \( d(P, Q) \) denotes the distance of \( P \) and \( Q \in U_P \setminus \{P\} \).

**Lemma 2.4** Let \((E, \overline{\mathcal{J}}_E)\) be a mini-holomorphic bundle with Dirac type singularity on \((\mathcal{M}; Z)\). Let \( E' \neq 0 \) be a mini-holomorphic subbundle of \( E \). Let \( h \) and \( h' \) be adapted Hermitian metrics of \( E \) and \( E' \), respectively. Let \( h_{E'} \) be the metric of \( E' \) induced by \( h \). Then, \( \deg(E', h_{E'}) = \deg(E', h') \).

**Proof** It can be proved by the argument in the proof of [4] Proposition 9.4.

2.1.4 Analytic stability condition for mini-holomorphic bundles with Dirac type singularity

Let \((E, \overline{\mathcal{J}}_E)\) be a mini-holomorphic bundle with Dirac type singularity on \((\mathcal{M}; Z)\). We set

\[
\deg^a(E, \overline{\mathcal{J}}_E) := \deg(E, \overline{\mathcal{J}}_E, h), \quad \mu^a(E, \overline{\mathcal{J}}_E) := \deg^a(E, \overline{\mathcal{J}}_E)/\text{rank}(E)
\]

for an adapted Hermitian metric \( h \) of \( E \), which is independent of the choice of \( h \).

**Definition 2.5** Let \((E, \overline{\mathcal{J}}_E)\) be a mini-holomorphic bundle with Dirac type singularity on \((\mathcal{M}; Z)\). We say that \((E, \overline{\mathcal{J}}_E)\) is analytically stable if \( \mu^a(E', \overline{\mathcal{J}}_E) < \mu^a(E, \overline{\mathcal{J}}_E) \) holds for any mini-holomorphic subbundle \( E' \subset E \) with \( 0 < \text{rank}(E') < \text{rank}(E) \). It is called polystable if \((E, \overline{\mathcal{J}}_E) = \bigoplus (E_i, \overline{\mathcal{J}}_{E_i})\), where each \((E_i, \overline{\mathcal{J}}_{E_i})\) is stable such that \( \mu^a(E_i, \overline{\mathcal{J}}_{E_i}) = \mu^a(E, \overline{\mathcal{J}}_E) \).
We obtain the following lemma from Lemma 2.4

Lemma 2.6 A mini-holomorphic bundle with Dirac type singularity \((E, \overline{\mathcal{D}}_E)\) on \((\mathcal{M}; Z)\) is analytically stable if and only if \((E, \overline{\mathcal{D}}_E, h)\) is analytically stable for an adapted Hermitian metric \(h\) of \(E\).

2.2 Monopoles with Dirac type singularity

2.2.1 Statements

For a monopole \((E, \overline{\mathcal{D}}_E, \nabla, \phi)\) on \(\mathcal{M} \setminus Z\), the differential operators \(\nabla_t\) and \(\nabla_{\pi}\) induce a mini-holomorphic structure \(\overline{\mathcal{D}}_E\) on the bundle \(E\). Conversely, let \((E, \overline{\mathcal{D}}_E)\) be a mini-holomorphic bundle on \(\mathcal{M} \setminus Z\) with a Hermitian metric \(h\) of \(E\). The induced tuple \((E, h, \nabla_h, \phi_h)\) is a monopole if and only if \(G(h) = 0\). (See [4] §2.3.2.) In that case, \((E, \overline{\mathcal{D}}_E, h)\) is also called a monopole. Recall that a point \(P\) of \(Z\) is called a Dirac type singularity of a monopole \((E, h, \nabla, \phi)\) if the following conditions are satisfied [3]:

- Each point of \(Z\) is a Dirac type singularity of the underlying mini-holomorphic bundle \((E, \overline{\mathcal{D}}_E)\).
- \(h\) is an adapted metric of \((E, \overline{\mathcal{D}}_E)\) at \(P\).

By definition, a monopole with Dirac type singularity \((E, h, \nabla, \phi)\) on \(\mathcal{M} \setminus Z\) induces a mini-holomorphic bundle with Dirac type singularity \((E, \overline{\mathcal{D}}_E)\) on \((\mathcal{M}; Z)\). The following is a variant of the correspondence in [1] on the basis of [7].

Theorem 2.7 The above construction induces an equivalence between monopoles with Dirac type singularity on \(\mathcal{M} \setminus Z\) and analytically polystable mini-holomorphic bundles with Dirac type singularity of degree 0 on \((\mathcal{M}; Z)\).

More precisely, Theorem 2.7 consists of Proposition 2.8, Proposition 2.9 and Proposition 2.12 below.

2.2.2 Polystability

Let \((E, \overline{\mathcal{D}}_E, h)\) is a monopole with Dirac type singularity on \(\mathcal{M} \setminus Z\).

Proposition 2.8 \((E, \overline{\mathcal{D}}_E)\) is analytically polystable satisfying \(\text{deg}^{\text{an}}(E, \overline{\mathcal{D}}_E) = 0\).

Proof Clearly, we have \(\text{deg}^{\text{an}}(E, \overline{\mathcal{D}}_E) = \text{deg}(E, \overline{\mathcal{D}}_E, h) = 0\). Let \(E'\) be a mini-holomorphic subbundle of \(E\). Let \(h_{E'}\) be the metric of \(E'\) induced by \(h\). By the Chern-Weil formula [1] and Lemma 2.4, we have

\[
\text{deg}^{\text{an}}(E', \overline{\mathcal{D}}_{E'}) = \int \text{Tr} G(h_{E'}) = -\int |\partial_{E', \pi} p_{E'}|^2 - \frac{1}{4} \int |\partial_{E', t} p_{E'}|^2 \leq 0.
\]

If \(\text{deg}^{\text{an}}(E', \overline{\mathcal{D}}_{E'}) = 0\), we obtain \(\partial_{E', \pi} p_{E'} = \partial_{E', t} p_{E'} = 0\). We obtain that the orthogonal complement \(E'^\perp\) is also a mini-holomorphic subbundle of \(E\). Let \(h_{E'^\perp}\) be the metric of \(E'^\perp\) induced by \(h\). Thus, we obtain a decomposition of monopoles \((E, \overline{\mathcal{D}}_E, h) = \left(E', \overline{\mathcal{D}}_{E'}, h_{E'}\right) \oplus \left(E'^\perp, \overline{\mathcal{D}}_{E'^\perp}, h_{E'^\perp}\right)\). Hence, we obtain the polystability of \((E, \overline{\mathcal{D}}_E)\) by an easy induction.

2.2.3 Construction of monopoles

Let \((E, \overline{\mathcal{D}}_E)\) be a stable mini-holomorphic bundle with Dirac type singularity on \((\mathcal{M}; Z)\) of \(\text{deg}^{\text{an}}(E, \overline{\mathcal{D}}_E) = 0\).

Proposition 2.9 There exists a Hermitian metric \(h\) of \((E, \overline{\mathcal{D}}_E)\) such that \((E, \overline{\mathcal{D}}_E, h)\) is a monopole with Dirac type singularity on \(\mathcal{M} \setminus Z\).

Proof As a preliminary, let us consider the rank one case. Note that the stability condition is trivial in the rank one case.

Lemma 2.10 Assume that \(\text{rank} E = 1\). Then, there exists a Hermitian metric \(h\) of \((E, \overline{\mathcal{D}}_E)\) such that \((E, \overline{\mathcal{D}}_E, h)\) is a monopole with Dirac type singularity on \(\mathcal{M} \setminus Z\).
Proof We take a Hermitian metric $h_0$ of $E$ such that the following holds:

- Each $P \in Z$ has a neighbourhood $U_P$ in $\mathcal{M}$ such that (i) $G(h_0) = 0$ on $U_P \setminus \{ P \}$; (ii) $P$ is Dirac type singularity of the monopole $(E, \overline{\partial}_E, h|_{U_P \setminus \{ P \}})$.

Let $f$ be any $C^\infty$-function on $\mathcal{M}$. Note that $G(h_0 e^f) - G(h_0) = -4^{-1} \Delta f$. (See [4, §2.8.4].) Because $\int_{\mathcal{M}} G(h_0) = 0$, there exists an $\mathbb{R}$-valued $C^\infty$-function $f$ such that $\Delta f = -4G(h_0)$. Thus, we obtain the claim of Lemma 2.11.

Let us study the general case. On $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, we use the real coordinate system $(s, t, x, y)$ and the complex coordinate system $(z, w)$ given by $z = s + \sqrt{-1} t$ and $w = x + \sqrt{-1} y$.

Let $\tilde{\Gamma}$ denote the lattice of $\mathbb{R}^4 = \mathbb{R} \times (\mathbb{R} \times \mathbb{C})$ generated by $(1, 0, 0)$ and $(0, a_i, \alpha_i)$ $(i = 1, 2, 3)$. We consider the action of $\tilde{\Gamma}$ on $\mathbb{R}^4$ induced by the natural $\mathbb{Z}$-action on $\mathbb{R}$ and the $\Gamma$-action on $\mathbb{R} \times \mathbb{C}$. Let $(X, g_X)$ denote the Kähler manifold obtained as the quotient of $(\mathbb{C}^2, dz d\bar{z} + dw d\bar{w})$ by the $\tilde{\Gamma}$-action. We have the natural projection $p : X \to \mathcal{M}$.

We set $\tilde{E} := p^{-1}(E)$ on $X \setminus p^{-1}(Z)$. It is equipped with the complex structure $\overline{\partial}_E$ determined by

$$\overline{\partial}_{\tilde{E}, \overline{\partial}_E} p^{-1}(f) = p^{-1}(\overline{\partial}_E \partial_E f), \quad \overline{\partial}_{\tilde{E}, \overline{\partial}_E} p^{-1}(f) = p^{-1}(2^{-1} \phi \cdot f + 2^{-1} \sqrt{-1} \overline{\partial}_{\tilde{E}, \overline{\partial}_E} f)$$

for sections $f$ of $E$. For any adapted Hermitian metric $h_0$ of $E$, set $\tilde{h}_0 := p^{-1}(h_0)$.

Let $F(\tilde{h}_0)$ denote the curvature of the Chern connection of $(\tilde{E}, \overline{\partial}_{\tilde{E}, \overline{\partial}_E}, \tilde{h}_0)$. Let $\Lambda$ denote the contraction from $(1, 1)$-forms to $(0, 0)$-forms with respect to the Kähler form of $(X, g_X)$. Then, $\sqrt{-1} \Lambda F(\tilde{h}_0) = p^{-1} G(h_0)$ holds.

For any saturated coherent $O$-submodule $\tilde{E}' \subset \tilde{E}$, we have a closed complex analytic subset $W \subset X \setminus p^{-1}(Z)$ with codimension 2 such that $\tilde{E}'$ is a subbundle of $\tilde{E}$ outside of $W$. We have the induced metric $\tilde{h}_{0, \tilde{E}'}$ of $\tilde{E}'|_{X \setminus p^{-1}(Z) \setminus W}$. We define

$$\deg(\tilde{E}', \tilde{h}_0) := \sqrt{-1} \int \Lambda F(\tilde{h}_{0, \tilde{E}'}) \ d\text{vol}_X.$$ 

Because of the Chern-Weil formula, it is well defined in $\mathbb{R} \cup \{-\infty\}$ as explained in [7]. Then, $(\tilde{E}, \overline{\partial}_{\tilde{E}, \overline{\partial}_E}, \tilde{h}_0)$ is defined to be analytically stable with respect to the $S^1$-action if

$$\frac{\deg(\tilde{E}', \tilde{h}_0)}{\text{rank} \tilde{E}'} < \frac{\deg(\tilde{E}, \tilde{h}_0)}{\text{rank} \tilde{E}}$$

holds for any $S^1$-invariant saturated subsheaf $\tilde{E}' \subset \tilde{E}$ with $0 < \text{rank} \tilde{E}' < \text{rank} \tilde{E}$. The following is clear.

**Lemma 2.11** $(\tilde{E}, \overline{\partial}_{\tilde{E}, \overline{\partial}_E}, \tilde{h}_0)$ is analytically stable with respect to the $S^1$-action if and only if $(E, \overline{\partial}_E, h_0)$ is analytically stable.

Suppose that $(E, \overline{\partial}_E)$ is analytically stable of degree 0. We take an adapted Hermitian metric $h_0$ such that each $P \in Z$ has a neighbourhood $U_P$ such that $G(h_0)|_{U_P \setminus \{ P \}} = 0$. We may also assume that $\text{Tr} G(h_0) = 0$ by the construction in the rank one case in [2.2.3]. By Lemma 2.11 $(\tilde{E}, \overline{\partial}_{\tilde{E}, \overline{\partial}_E}, \tilde{h}_0)$ is analytically stable with respect to the $S^1$-action. We also have $\text{Tr} F(\tilde{h}_0) = 0$. According to a theorem of Simpson [7, Theorem 1], there exists an $S^1$-invariant metric $\tilde{h}$ of $\tilde{E}$ such that (i) $\det(\tilde{h}) = \det(\tilde{h}_0)$, (ii) $\Lambda F(\tilde{h}) = 0$, (iii) $\tilde{h}$ and $\tilde{h}_0$ are mutually bounded. We obtain the corresponding metric $h$ of $E$, for which $G(h) = 0$ holds. Because $h$ and $\tilde{h}_0$ are mutually bounded, each $P \in Z$ is a Dirac type singularity of $(E, \overline{\partial}_E, h)$ which is implied by [6, Theorem 3].

**2.2.4 Uniqueness**

The uniqueness is also standard.

**Proposition 2.12** Let $(E, \overline{\partial}_E)$ be a mini-holomorphic bundle with Dirac type singularity on $(\mathcal{M}; Z)$. Let $h_1$ and $h_2$ be adapted Hermitian-metrics of $E$ such that $G(h_i) = 0$. Then, there exists a decomposition $(E, \overline{\partial}_E) = \bigoplus (E_i, \overline{\partial}_{E_i})$ such that (i) it is orthogonal with respect to both $h_1$ and $h_2$, (ii) we have $a_i > 0$ such that $h_{2i} E_i = a_i h_{1i} E_i$. 

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Proof Let $s$ be the automorphism of $E$ determined by $h_2 = h_1 s$. By [14 Corollary 2.30], we have the following inequality on $\mathcal{M} \setminus Z$:

$$\left(-\left(\partial_w \partial_w + \frac{1}{4} \partial_\tau \right) \right) \text{Tr}(s) = -|\partial_{E,h_1} \pi(s) s^{-1/2}|_{h_1}^2 - \frac{1}{4} |\partial_{E,h_1,t}(s) s^{-1/2}|_{h_1}^2 \leq 0.$$ 

By the assumption, $\text{Tr}(s) \geq 0$ is bounded. Then, the inequality holds on $\mathcal{M}$ in the sense of distributions. (See the proof of [2 Proposition 2.2].) Hence, we obtain that $\text{Tr}(s)$ is constant, and $\partial_{E,h_1} \pi(s) = \partial_{E,h_1,t}(s) = 0$. Because $s$ is self-adjoint with respect to $h_1$, we also obtain that $\partial_{E,h_1,w}(s) = \partial_{E,h_1,t}(s) = 0$. (See [32 §2.3.2] for $\partial'_{E,h_1,t}$.) We obtain that the eigenvalues of $s$ are constant, and the eigen decomposition $E = \bigoplus E_i$ is compatible with the mini-holomorphic structure. Then, the claim of the proposition follows.

3 A more sophisticated formulation of the stability condition

We explain a different way to define the stability condition of mini-holomorphic bundles introduced by Kontsevich and Soibelman [14, which we already mentioned in §1.2]. This section is devoted to explain their idea.

3.1 Preliminary

3.1.1 Closed 1-forms and 1-homology

Let $A$ be a 3-dimensional manifold. Let $Z^1_{\text{DR}}(A)$ denote the space of closed 1-form $\tau$ on $A$. Let $B$ be finite subset of $A$. Let $H_j(A,B)$ denote the relative $j$-th homology group with $\mathbb{R}$-coefficient.

Let $\gamma$ be any element of $H_1(A,B)$. We take a representative of $\gamma$ by a smooth 1-chain $\tilde{\gamma}$. For any $\omega \in Z^1_{\text{DR}}(A)$, the number $\int_\gamma \omega$ is independent of the choice of a representative $\tilde{\gamma}$. They are denoted by $\int_\gamma \omega$.

Let $C^\infty(A,B)$ denote the space of $C^\infty$-functions $f$ on $A$ such that $f(P) = 0$ for any $P \in B$. Let $Z^1_{\text{DR}}(A)$ denote the space of closed 1-forms on $A$. Let $B^1_{\text{DR}}(A,B)$ denote the image of $d : C^\infty(A,B) \rightarrow Z^1_{\text{DR}}(A)$. Because $\int_\gamma df = 0$ for any $f \in C^\infty(A,B)$, we obtain the well defined map

$$\int_\gamma : Z^1_{\text{DR}}(A)/B^1_{\text{DR}}(A,B) \rightarrow \mathbb{R}.$$ 

3.1.2 Duality

Suppose that $A$ is compact and oriented. Let $H^j(A \setminus B)$ denote the $j$-th de Rham cohomology group of $A \setminus B$. Let $H^j_c(A \setminus B)$ denote the $j$-th de Rham cohomology group with compact support. We have the non-degenerate pairing between $H^2(A \setminus B)$ and $H^1_c(A \setminus B)$ induced by the cup product and the integration. We also have the non-degenerate pairing between $H^1_c(A \setminus B)$ and $H_1(A,B)$ induced by the integration. Hence, we obtain the isomorphism

$$\Phi_{A,B} : H^2(A \setminus B) \simeq H_1(A,B),$$

By definition, for any $a \in H^2(A \setminus B)$ and $b \in H^1_c(A \setminus B)$, the following holds:

$$\int_{\Phi_{A,B}}(a) = \int_A a \wedge b.$$ 

Take any Riemannian metric $g_A$ of $A$. For any $j$-form $\tau$ on $A \setminus B$, let $|\tau|_{g_A}$ denote the function on $A \setminus B$ obtained as the norm of $\tau$ with respect to $g_A$.

Lemma 3.1 Let $\tau \in Z^2_{\text{DR}}(A \setminus B)$ such that $|\tau|_{g_A}$ is an $L^1$-function on $A$. Then, the following holds for any $\rho \in Z^1_{\text{DR}}(A)$:

$$\int_{\Phi_{A,B}}(|\tau|) = \int_A \rho \wedge \tau.$$ 

Here, $[\tau] \in H^2_{\text{DR}}(A \setminus B)$ denotes the cohomology class of $\tau$. 


Proof For any point \( P \in Z \), we take a small coordinate neighbourhood \((A_P, x_P, 1, x_P, 2, x_P, 3)\) of \( P \) such that (i) \( P \) corresponds to \((0, 0, 0)\), (ii) \( A_P \simeq \{(x_i, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 < 1\} \) by the coordinate system. Set \( \|x_P\| := (x_P^2 + x_P^2 + x_P^2)^{1/2} \). Then, there exists a \( C^\infty \)-function \( f_P \) on \( A_P \) such that (i) \( df_P = \rho \) on \( \|x_P\| < 1/2 \), (ii) \( f_P(P) = 0 \), (iii) \( f_P(Q) = 0 \) for \( Q \in \{\|x_P\| > 2/3\} \). We naturally regard \( f_P \) as a \( C^\infty \)-function on \( A \). Then, the following holds:

\[
\int_{\Phi_{A,B}((\tau))} \rho = \int_{\Phi_{A,B}((\tau))} (\rho - \sum_{P \in B} df_P) = \int_A (\rho - \sum_{P \in B} df_P) \wedge \tau = \int_A \rho \wedge \tau - \sum_P \int_A (df_P \wedge \tau).
\]

We set \( S_P^2(r) := \{\|x_P\| = r\} \) with the orientation as the boundary of \( \{\|x_P\| < r\} \). Then, we obtain the following

\[
\int_A df_P \wedge \tau = -\lim_{\epsilon \to 0} \int_{S_P^2(\epsilon)} f_P \tau.
\]

Note that the limit exists because \( df_P \wedge \tau \) is integrable. Because \( |\tau|_{g_A} \) is \( L^2 \), we have \( \int dr \int_{S_P^2(\epsilon)} |\tau|_{g_A} < \infty \), and hence there exists a sequence \( r_i \to 0 \) such that \( r_i \int_{S_P^2(r_i)} |\tau|_{g_A} \to 0 \). Because \( |f_P| = O(\|x_P\|) \), we obtain that \( (2) \) is 0.

### 3.2 Relation between degrees of mini-holomorphic bundles

We may naturally regard \( M \) as a 3-dimensional abelian Lie group. Let \( \mathfrak{g} \) denote the space of the invariant vector fields on \( M \). Let \( \mathfrak{g}^\gamma \) denote the space of the invariant 1-forms on \( M \). We have the natural non-degenerate paring \( \mathfrak{g} \otimes \mathfrak{g}^\gamma \to \mathbb{R} \). We have the dual morphism \( \mathbb{R} \to \mathfrak{g}^\gamma \otimes \mathfrak{g} \). Let \( \sigma \) denote the image of 1. If we take a base \( e_i \) (\( i = 1, 2, 3 \)) of \( \mathfrak{g} \) and the dual frame \( e_i^\gamma \) (\( i = 1, 2, 3 \)), then \( \sigma = \sum e_i^\gamma \otimes e_i \).

Let \( E \) be a vector bundle on \( M \setminus Z \). Kontsevich and Soibelman [2] introduced the following element:

\[
\int_{\Phi_{M,Z}(c_1(E))} \sigma \in \mathfrak{g}.
\]

**Proposition 3.2** Let \((E, \overline{\nabla}_E)\) be a mini-holomorphic bundle with Dirac type singularity on \((M; Z)\). Then,

\[
\int_{\Phi_{M,Z}(c_1(E))} \sigma = \frac{1}{\pi} \deg_{an}(E) \cdot \partial_t.
\]

**Proof** Let \( h \) be an adapted metric of \((E, \overline{\nabla}_E)\). For each \( P \in Z \), we take any small neighbourhood \( U_P \) of \( P \) in \( M \). Let \( d(P, Q) \) denote the distance of \( P \) and \( Q \in U_P \). Then, the following holds for \( Q \in U_P \setminus \{P\} \) (for example, see [3] §5):

\[
|\phi_{h, Q}|_h = O(d(P, Q)^{-1}), \quad |(\nabla \phi_{h})|_h, g_M = O(d(P, Q)^{-2}), \quad |F(h)|_h, g_M = O(d(P, Q)^{-2}).
\]

By Lemma 3(A), it is enough to prove the following equality:

\[
\frac{\sqrt{-1}}{2} \int_M \text{Tr} F(h) \cdot \sigma = \int_M \text{Tr} G(h) \text{dvol}_M \cdot \partial_t.
\]

For \( \kappa = t, x, y \), we obtain the following by the Stokes formula and \( |\phi_{h, Q}|_h = O(d(P, Q)^{-1}) \):

\[
\int \text{Tr} (\nabla_{h, \kappa} \phi_{h}) dt \, dx \, dy = 0.
\]

Note that \( F(h)_{tx} = \nabla_{h, y} \phi_{h} \) and \( F(h)_{yt} = \nabla_{h, x} \phi_{h} \) holds. Hence, we obtain

\[
\int \text{Tr} F(h)_{tx} dt \, dx \, dy = 0.
\]

We also obtain the following from (4):

\[
\int \text{Tr} F(h)_{xy} dt \, dx \, dy = 0.
\]

Then, we obtain (3), and the proof of Proposition 3.2 is completed.
Remark 3.3 Kontsevich and Soibelman [2] formulated the stability condition for mini-holomorphic bundles in terms of the coefficient of $\partial_t$ in $\int_{\Phi^\ast(c_1(E))} \sigma$, as explained in §1.2. 

4 Parabolic difference modules on elliptic curves

We use the notation in §2.1.1. We assume that (i) the tuple $(a_i, \alpha_i)$ $(i = 1, 2, 3)$ is an oriented base of $\mathbb{R} \times \mathbb{C}$, (ii) $\alpha_1$ and $\alpha_2$ are linearly independent. Let $\Gamma_0 \subset \mathbb{C}$ be the lattice generated by $\alpha_1$ and $\alpha_2$. The projection $Y \rightarrow \mathbb{C}$ induces a morphism $\mathcal{M} \rightarrow T := \mathbb{C}/\Gamma_0$.

Let $\mathcal{M}^{\text{cov}}$ denote the quotient space of $Y$ by the action of $\mathbb{Z}_1 \oplus \mathbb{Z}_2$. We have the natural isomorphism $\mathcal{M}^{\text{cov}}/\mathbb{Z}_3 \simeq \mathcal{M}$.

4.1 Another mini-complex coordinate system

We introduce another mini-complex coordinate system $(s, u)$ on $Y$. We set

$$\gamma := \frac{a_1\overline{a}_2 - a_2\overline{a}_1}{a_1\alpha_2 - a_1\alpha_2}.$$ 

We introduce another mini-complex coordinate system $(s, u)$ on the mini-complex manifold $Y$ as follows:

$$s := t + 2\Re(\gamma w) = t + \overline{\gamma}w + \gamma w, \quad u := w.$$ 

Then, we have $e_i(s, u) = (s, u + a_i)$ for $i = 1, 2$. We also have $e_3(s, u) = (s + t, u + a)$:

$$t := a_3 + \Re(\gamma a_3), \quad a := a_3.$$ 

Note that $t > 0$ because the tuple $\{(a_i, \alpha_i)\}_{i=1,2,3}$ is an oriented base of $\mathbb{R} \times \mathbb{C}$. We have the following relations of complex vector fields:

$$\partial_{\overline{w}} = \partial_t + \overline{\gamma} \partial_s, \quad \partial_w = \partial_u + \gamma \partial_s, \quad \partial_t = \partial_s.$$ 

The product $\mathbb{R}_s \times T$ is equipped with the natural mini-complex structure. The mini-complex coordinate system $(s, u)$ induces an isomorphism of mini-complex manifolds $\mathcal{M}^{\text{cov}} \simeq \mathbb{R}_s \times T$.

4.2 Mini-holomorphic bundles and difference modules

Let $Z$ be a finite subset in $\mathcal{M}$. Let $Z^{\text{cov}} \subset \mathcal{M}^{\text{cov}} \simeq \mathbb{R}_s \times T$ denote the pull back of $Z$. We take $\epsilon > 0$ such that $([-\epsilon, 0[ \times T) \cap Z^{\text{cov}} = \emptyset$. Let $D$ be the image of $Z^{\text{cov}} \cap ([\epsilon, \epsilon[ \times T)$ via the projection $\mathbb{R}_s \times T \rightarrow T$. For each $P \in D$, we obtain the sequence $0 \leq s_{P,1} < s_{P,2} < \cdots < sp_m(P) \leq 1$ by the condition:

$$\{(s_{P,i}, P) \mid i = 1, \ldots, m(P)\} = ([0, \epsilon[ \times \{P\}) \cap Z^{\text{cov}}.$$

We set $\tau_{P,i} := s_{P,i}/t$.

Let $(E, \overline{\partial}_E)$ be a mini-holomorphic bundle on $\mathcal{M} \setminus Z$ with Dirac type singularity at $Z$. Let us observe that $(E, \overline{\partial}_E)$ induces a parabolic difference module $\Upsilon(E, \overline{\partial}_E)$ over $(T, (\tau_P)_P \in D))$.

Let $V$ be the locally free $O_T$-module obtained as $E_{\langle -\epsilon \rangle \times T}$. It is independent of the choice of $\epsilon$ as above, up to canonical isomorphisms.

Let $\Phi : T \rightarrow T$ be the morphism induced by $\Phi(u) = u + a$. We have the natural isomorphism $\Phi^\ast(E^{\text{cov}}_{\langle -\epsilon \rangle \times T}) \simeq E^{\text{cov}}_{\langle -\epsilon \rangle \times T}$.

The scattering map induces an isomorphism

$$E^{\text{cov}}_{\langle -\epsilon \rangle \times T}(\ast D) \simeq E^{\text{cov}}_{\langle -\epsilon \rangle \times T}(\ast D).$$ 

Hence, $V$ is equipped with an isomorphism $V(\ast D) \simeq (\Phi^\ast)^{-1}(V)(\ast D)$. 

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For each $P \in D$ and for $i = 1, \ldots, m(P) - 1$, we take $s_{P,i} < b_{P,i} < s_{P,i+1}$. Let $(E^\text{cov}_{\{(-\epsilon)\times T\}})_P$ denote the $O_T, P$-module obtained as the stalk of the sheaf of holomorphic sections of $E^\text{cov}_{\{(-\epsilon)\times T\}}$ at $P$. Similarly, $(E^\text{cov}_{\{(b_{P,i})\times T\}})_P$ denote the $O_T, P$-module obtained as the stalk of the sheaf of holomorphic sections of $E^\text{cov}_{\{(b_{P,i})\times T\}}$ at $P$. The scattering map induces isomorphisms of $O_T(*P)_P$-modules:

$$(E^\text{cov}_{\{(-\epsilon)\times T\}})_P(*P) \simeq (E^\text{cov}_{\{(b_{P,i})\times T\}})_P(*P).$$

Hence, $(E^\text{cov}_{\{(b_{P,i})\times T\}})_P (i = 1, \ldots, m(P) - 1)$ induce a sequence of lattices $L_{P,i} (i = 1, \ldots, m(P))$ of $V(*D)_P$. Thus, we obtain the following parabolic $a$-difference module on $(T, (\tau_P)_{P \in D})$:

$$\Upsilon(E, \overline{\mathcal{E}}) := (V, (\tau_P, L_P)_{P \in D}).$$

The following proposition is easy to see.

**Proposition 4.1** $\Upsilon$ induces an equivalence between holomorphic bundles of Dirac type singularity on $(M; Z)$ and parabolic $a$-difference modules on $(T, (\tau_P)_{P \in D})$.

### 4.3 Comparison of stability conditions

Let $(E, \overline{\mathcal{E}})$ be a mini-holomorphic bundle with Dirac type singularity on $(M; Z)$.

**Proposition 4.2** We have $\deg^m(E, \overline{\mathcal{E}}) = \tau \pi \deg \Upsilon(E, \overline{\mathcal{E}})$. As a result, $(E, \overline{\mathcal{E}})$ is analytically (poly)stable if and only if $\Upsilon(E, \overline{\mathcal{E}})$ is (poly)stable.

**Proof** We consider the real vector field $v := 2\tau \partial_w + 2\gamma \partial_{\overline{w}} - \left(2|\gamma|^2 - \frac{1}{2}\right) \partial_t$ on $M$. Let $h$ be any Hermitian metric of $E$.

**Lemma 4.3** $G(h) = [\partial_{E,h,u}, \partial_{E,\overline{w}}] - \sqrt{-1} \nabla_{h,\overline{w}} \phi_h$ holds.

**Proof** The following holds:

$$\partial_{E,\overline{w}} = \nabla_{h,\overline{w}} - \tau (\nabla_{h,t} + \sqrt{-1} \phi_h), \quad \partial_{E,h,u} = \nabla_{h,w} - \gamma (\nabla_{h,t} + \sqrt{-1} \phi_h).$$

Hence, we obtain

$$[\partial_{E,h,u}, \partial_{E,\overline{w}}] = [\nabla_{h,w}, \nabla_{h,\overline{w}}] - \tau [\nabla_{h,w}, \nabla_{h,t}] + \tau \sqrt{-1} \nabla_{h,w} \phi_h + \gamma [\nabla_{h,\overline{w}}, \nabla_{h,t}] + \gamma \sqrt{-1} \nabla_{h,\overline{w}} \phi - 2\sqrt{-1}|\gamma|^2 \nabla_{h,t} \phi_h. \quad (5)$$

Recall that $[\nabla_{h,w}, \nabla_{h,t}] = -\sqrt{-1} \nabla_{h,w} \phi_h$ and $[\nabla_{h,\overline{w}}, \nabla_{h,t}] = \sqrt{-1} \nabla_{h,\overline{w}} \phi_h$ (see [4, §2.3]). Hence, we obtain

$$[\partial_{E,h,u}, \partial_{E,\overline{w}}] = [\nabla_{h,w}, \nabla_{h,\overline{w}}] + 2\sqrt{-1}\gamma \nabla_{h,w} \phi + 2\sqrt{-1}\gamma \nabla_{h,\overline{w}} \phi - 2\sqrt{-1}|\gamma|^2 \nabla_{h,t} \phi.$$

Then, we obtain the claim of the lemma.

Let $h$ be an adapted metric of $(E, \overline{\mathcal{E}})$. Then, $G(h), F(h)$ and $\nabla_{h,\overline{w}} \phi_h$ are $L^1$. Hence, we obtain

$$\deg^m(E) = \int_M \text{Tr} G(h) = \int_M \text{Tr} \left[\partial_{E,\overline{w}}, \partial_{E,h,u}\right] - \int_M \sqrt{-1} \nabla_{h,\overline{w}} \phi_h.$$

By the Stokes theorem, we obtain that $\int_M \text{Tr} \nabla_{h,\overline{w}} \phi_h = 0$. (See the proof of Proposition 3.2) Note that the volume form of $M$ is equal to $\frac{1}{2} ds du d\overline{w}$. By the Fubini theorem, we obtain that

$$\int_M \text{Tr} \left[\partial_{E,\overline{w}}, \partial_{E,h,u}\right] = \int_0^4 ds \int_{\{s\} \times T} \text{Tr} \left[\partial_{E,\overline{w}}, \partial_{E,h,u}\right] \frac{\sqrt{-1}}{2} du d\overline{w} = \int_0^4 ds \int_{\{s\} \times T} \pi c_1(E^\text{cov}_{\{s\} \times T}) = \tau \pi \deg \Upsilon(E, \overline{\mathcal{E}}). \quad (6)$$

Thus, we obtain the claim of Proposition 4.2.
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