Anomalous non-classicality via correlative and entropic Bell inequalities

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Entropic Bell inequalities witness contextual probability distributions on sets of jointly measurable observables. We find that their violation does not entail a violation of the correlative Bell inequality for certain parameters. This anomalous contextuality without non-locality shows that specifying a value range, rather than admitting all theoretically allowed parameters, helps to precisely determine the type of non-classical resource. We then show that the anomaly persists under ‘permissible’ but disappears under ‘forbidden’ permutations of observables. Under more ‘exotic’ modifications of local parties, the anomaly can also persist, giving rise to resources that are non-classical under all non-equivalent operational party assignments in the device-independent approach.

I. INTRODUCTION

It is known that maximally entangled states do not achieve the maximum violation of certain Bell inequalities [1]. This “anomaly of non-locality” [2] indicates that entanglement and non-locality are different non-classical resources. Entropic Bell inequalities [3] witness yet another resource, quantum contextuality [4–6]. We show that they also exhibit an anomaly.

In the bipartite Bell scenario, Fritz and Chaves found a surprising feature of the entropic inequality: measurements leading to its maximum violation are not the ones that produce the maximum violation of the correlative Clauser-Horne-Schimony-Holt (CHSH) inequality [7]. They conclude that, since non-contextual hidden variable models are the same as local hidden variable models, contextuality should necessarily imply non-locality. Even more surprisingly, we find probability distributions that violate the entropic inequality but respect the correlative one: for certain ranges of measurement parameters, contextuality can be detected without non-locality.

This anomaly can be explained by the failure of entropic measures to distinguish between perfectly correlated and anti-correlated variables. We also see its further conceptual significance. Hailed as a “very important recent development” [8], the device-independent approach consists in describing an experiment by specifying its input and output as strings in alphabets of finite cardinality [9, 10]. A party, conventionally conceived of by delimiting its spatial location or by identifying the physical system on which it acts, is now defined by associating algebraic input and output variables over many runs of the experiment. Different assignments lead to non-equivalent settings, some of which may contain causally non-independent or non-local parties; for each assignment, however, it is possible to formulate Bell inequalities that measure non-classicality of the corresponding probability distribution. We demonstrate possible variations in violating the entropic and correlative inequalities as one reassigns inputs and outputs to the parties, leading to the disappearance of the anomaly in some cases. The anomaly can also persist: we give examples of probability distributions that remain contextual under any party assignment. These situations suggest that in the device-independent approach there exist resources that are non-classical in all non-equivalent operational settings.

After introducing the framework in section II, we analyze the difference between correlative and entropic measures of non-classicality in section III. In section IV we explore ‘permissible’, ‘forbidden’, and ‘exotic’ permutations in the device-independent approach.

II. MATHEMATICAL FRAMEWORK

Suppose that two parties, $A$ and $B$, share a quantum state $|\psi\rangle$. At each run of the experiment $A$ and $B$ choose a binary input, resp. $x, y \in \{0,1\}$, and measure a corresponding observable, resp. $A_x, B_y$, obtaining a binary outcome, resp. $a, b \in \{0,1\}$. The presence of non-contextual correlations between $A$ and $B$ is witnessed by a violation of the Bell inequality [11] in the CHSH form [12].

$$CHSH = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2.$$  

(1)

Another measure of non-classicality is based on entropic inequalities. In the bipartite Bell scenario, two observables from different parties, $A_x$ and $B_y$, are jointly measurable for all values of $x$ and $y$ while two observables of the same party, e.g. $A_0$ and $A_1$, are not. Define a marginal scenario consisting of all sets of jointly measurable observables [13]. This scenario is called non-contextual if the joint probability $P(a,b|x,y)$ can be written as a function of hidden variable $\lambda$ with probability distribution $\rho(\lambda)$:

$$P(a,b \mid x,y) = \sum_\lambda \rho(\lambda) P(a|x,\lambda) P(b|y,\lambda).$$  

(2)
TABLE I. Examples of measurement settings leading to the maximum violation of the $CHSH$ and $CHSH_E$ inequalities.

| $\theta_0$ | $\theta_1$ | $\theta_0'$ | $\theta_1'$ | $CHSH$ | $CHSH_E$ |
|-----------------|-----------------|-----------------|-----------------|----------|----------|
| 2.070           | 1.466           | 1.572           | 0.769           | 2.248    | 0.2369   |
| 2.709           | 2.106           | 0.739           | 0.125           | 2.250    | 0.2368   |
| 1.316           | 2.894           | 1.033           | 2.606           | 2.828    | -1.205   |
| 2.050           | 0.486           | 1.877           | 0.294           | 2.828    | -1.210   |

Based on marginal scenarios, Fritz and Chaves proved that the contextuality of $P(a,b|x,y)$ is equivalent to a violation of the entropic inequality [2]

$$CHSH_E = H(A_1, B_1) + H(A_0) + H(B_0) - H(A_0, B_0) - H(A_1, B_1) - H(A_1, B_0) \leq 0,$$  

(3)

where $H$ denotes respective Shannon entropies.

Choose an entangled pure state $|\psi\rangle = \cos \alpha |00\rangle + \sin \alpha |11\rangle$. The maximum violation of inequalities (1) and (3) can be obtained at $\alpha = \pi/4$ by parties $A$ and $B$ performing measurements in the $Y-Z$ plane at angles $\theta_x$ and $\theta_y'$ respectively:

$$A_x = \sin \theta_x \sigma_Y + \cos \theta_x \sigma_Z$$
$$B_y = \sin \theta_y' \sigma_Y + \cos \theta_y' \sigma_Z,$$

where $\sigma_Y$ and $\sigma_Z$ are Pauli matrices. Strong symmetries at $\alpha = \pi/4$ imply $P(1,1|x,y) = P(0,0|x,y)$, $P(0,1|x,y) = P(1,0|x,y)$, and $H(A_x) = H(B_y) = 1$. Quantum theory allows for any values $\theta_x, \theta_y' \in [0, 2\pi]$. The maximum violation of inequality (1) over this entire range is equal to the Tsirelson bound $2\sqrt{2}$ [14], while the violation of (3) reaches at most $CHSH_E \simeq 0.237$ [7]. Note that the measurement angles $\theta_x, \theta_y'$ that maximise $CHSH$ yield non-maximal values of $CHSH_E$, and vice versa (Table I).

III. DISCREPANCY IN EVALUATING NON-CLASSICALITY

Correlative and entropic $CHSH$ inequalities (1) and (3) capture non-classicality in distinct ways. Figure 1 demonstrates this discrepancy on 10 million measurement settings given by a set of four angles $\theta_x, \theta_y'$. Zone 1 contains all settings that do not violate either inequality, i.e. probability distributions that are both local and non-contextual. Zone 3 contains non-classical settings that violate both inequalities, while in zones 2 and 4 only one of the inequalities, resp. $CHSH_E$ and $CHSH$, detects non-classicality. For $\alpha = \pi/4$, approximately 82.5% of measurements belong to zone 1 and the remaining 17.5% demonstrate at least some non-classical behaviour (Table II).

Zone 3 is shown in detail on Figure 2. It vanishes at $CHSH \simeq 2.561$, corresponding to the maximum violation of the correlative inequality (1) that can be reached in a contextual setting. The $CHSH$ values beyond this bound, in particular the ones close to the Tsirelson bound, can only be witnessed by non-contextual probability distributions. Reproducing this analysis for $\theta < \alpha < \pi/4$, we find that zone 3 decreases in size more rapidly than zones 2 and 4, implying that the violations of non-classicality in the correlative and entropic forms are more often witnessed separately.

![Diagram showing zones and values]

FIG. 1. Values of $CHSH$ and $CHSH_E$ for 10 million measurement settings. Non-classicality is detected by entropic contextuality if $CHSH_E > 0$ and/or by non-classical correlations if $CHSH > 2$.

![Diagram showing zone proportions]

TABLE II. Proportions of 10 million measurement settings in zones 1-4.

| Zone   | % of Settings |
|--------|---------------|
| Zone 1 | 82.5%         |
| Zone 2 | 14%           |
| Zone 3 | 2.9%          |
| Zone 4 | 15.2%         |

![Diagram showing zone 3 analysis]

FIG. 2. Zone 3 of the graph in Figure 1. Inequalities (1) and (3) are both violated.

Zone 2 illustrates an effect similar to the “anomaly of non-locality” [2]. If one considers the maximum values of $CHSH$ and $CHSH_E$ over all theoretically allowed
measurement angles, \( \theta_x, \theta_y \in [0, 2\pi] \), then violating inequality (1) is a necessary (but not a sufficient) condition for violating inequality (3). In zone 2, however, only inequality (3) is violated. Measurement settings that belong to this zone yield anomalous entropic contextuality without correlative non-locality in the bipartite setting.

This anomalous behavior is due to the failure of entropic measures to distinguish between perfectly correlated and anti-correlated variables. Its origin can be illustrated by plotting the correlators \( E(x, y) = P(a = b|x, y) - P(a \neq b|x, y) \) for CHSH \( E > 0 \) (Figure 3). One would expect an entropic inequality to be violated when different variables are maximally uncorrelated, i.e. relative entropies are high. In reality, maximum violations of the entropic inequality (3) occur close to all but one ‘corners’ of highly correlated and/or highly anti-correlated outputs. Note that the correlative inequality (1) remains sensitive to this distinction and cannot be violated in some of these ‘corners’.

IV. PERMISSIBLE, FORBIDDEN, AND EXOTIC PERMUTATIONS

Fritz and Chaves discuss permutations that transform an entropic contextuality inequality into an equivalent one [7]. Similarly, one can consider permutations that lead to equivalent correlative inequalities. These ‘permissible’ permutations include permutations of the parties, permutations of the observables of some party, or their arbitrary combinations. An equivalent inequality is obtained by permuting signs in the original formula, for example exchanging \( B_0 \) with \( B_1 \) transforms (3) into

\[
CHSH_E = H(A_0) + H(B_1) - H(A_0B_0) - H(A_0B_1) + H(A_1B_0) - H(A_1B_1) \leq 0. \tag{4}
\]

In any given marginal scenario only one among four such sign-permuted inequalities can witness non-classicality.

In a device-independent approach [15], suppose that each run of the experiment is characterized by a binary string of length four. A homomorphism on these strings, mapping bit value of run \( i \) on bit value of run \( j \) over all runs, defines input and output variables, e.g. \( x, y \) and \( a, b \) respectively in the notation of Section II. The identification of an input variable with an output variable, e.g. \( x \) with \( a \) and \( b \) with \( y \), provides a definition of parties \( A \) and \( B \). Different assignments lead to different interpretations of the experiment in terms of parties (‘local observers that perform local operations’), while physically meaningful interpretations also require additional causal assumptions [16], e.g. the no-signalling condition \( P(a|x, y) = P(a|x), P(b|x, y) = P(b|y) \) typically used in the device-independent models [17].

Since permissible permutations respect joint measurability, they result in eight equivalent measurement settings. More generally, for each given set of four observables \( \{X_i\}_{i=1}^{4} \), e.g. \( A_x \) and \( B_y \) in the notation of Section II, there exist three classes of different marginal scenarios:

Class 1: A has \( X_1, X_2 \); B has \( X_3, X_4 \);
Class 2: A has \( X_1, X_4 \); B has \( X_2, X_3 \);
Class 3: A has \( X_1, X_3 \); B has \( X_2, X_4 \).

These classes yield distinct party assignments, e.g. class 2 corresponds to \( A \) having \( A_0 \) and \( B_1 \), \( B \) having \( A_1 \) and \( B_0 \) in the notation of Section II. The attribution of labels \( A \) and \( B \) to parties is arbitrary, for it is subject to a permissible permutation. The classes are conserved under permissible as well as ‘forbidden’ permutations that do
not preserve the entropic contextuality inequality but respect joint measurability, i.e. they remain within the same marginal scenario. An example of a ‘forbidden’ permutation is given by exchanging $B_0$ and $B_1$ without switching from inequality (3) to (4).

Different classes are connected by ‘exotic’ permutations that change the marginal scenario, for instance exchanging $A_0$ and $B_0$. A physical interpretation under exotic permutations may be complicated by the fact that some parties admit signalling or lack operational meaning as local observers, for example by exhibiting non-local properties in time [13]. It is a legitimate concern to require that an assignment of parties be physically well-defined in order to be empirically realizable; here, we ask whether theoretically allowed probability distributions, connected by forbidden or exotic permutations, exhibit non-classical behaviour in the correlative or entropic sense.

Under some, but not all, forbidden permutations the probability distributions change zones in Figure 1. Three possible situations include staying in zone 1, switching between zones 1 and any other zone, or switching between zones 2 and 4. Thus a forbidden permutation, i.e. an inequivalent assignment of parties that nevertheless respects joint measurability of the observables, may erase non-classicality by bringing the probability distribution from any non-classical zone into zone 1. We find that there always exists precisely one such permutation that will change an anomalous contextual distribution without non-locality (zone 2) into a non-local but also a non-contextual distribution (zone 4). This follows from the fact that no hidden-variable model (2) exists for the setting that violates inequality (3), hence at least one forbidden permutation on the probability distribution that implements the same party assignment for inequality (1) will lead to its violation. Since only one among the non-equivalent sign-permutated inequalities is violated in a given marginal scenario, the forbidden permutation between zones 2 and 4 is unique.

Exotic permutations, contrary to the forbidden ones, can accommodate any change of zone in Figure 1. Anomalous non-classicality can therefore be interpreted as an artefact of unfortunate choice of the setting, in particular of a problematic party assignment: although such an assignment might be physical in the sense of observer locality, it is problematic in the sense of containing a non-classical resource that can be dissolved by permutation.

Interestingly, there exist cases when all three non-equivalent scenarios, i.e. all non-equivalent party assignments on a set of inputs and outputs, are both contextual and non-local. In these cases even ‘exotic’ permutations between classes 1–3 do not alter non-classicality of the probability distribution. Two examples of such triple violation of inequality (3) in zones 2 and 3 are shown in Table III.

| Zone | Angles | Class 1 | Class 2 | Class 3 |
|------|--------|---------|---------|---------|
| 2    | $\theta_0 = 0.40, \theta_1 = 3.02, \theta'_0 = 2.72, \theta'_1 = 2.38$ | 1.71 | 1.71 | 1.42 |
| 3    | $\theta_0 = 1.97, \theta_1 = 1.34, \theta'_0 = 1.22, \theta'_1 = 0.83$ | 2.22 | 2.07 | 2.29 |

TABLE III. CHSH and $CHSH_E$ values for two sets of measurement parameters under all non-equivalent party assignments. Observables in zone 2 exhibit entropic contextuality without correlated non-locality; those in zone 3 exhibit both non-classical properties.

V. CONCLUSION

The question of allowed parameter values for state preparation and measurement has been previously raised in two contexts. On one hand, an axiom of quantum theory stipulates that these parameters vary continuously [19][22]. On the other hand, continuous number fields, e.g. real numbers, may not be an adequate representation of physical reality [23][25]. We add a third aspect.

It is not surprising that the two fundamental criteria of non-classicality in the bipartite Bell scenario, viz. correlated and entropic CHSH inequalities, detect non-classical probability distributions differently. However, under a restriction on the continuous range of parameter values, there exists an anomalous discrepancy corresponding to contextuality without non-locality. This result illustrates that specifying a value range, rather than admitting all theoretically allowed parameters, helps to precisely determine the type of non-classical resource. Mathematically, the anomaly stems from the fact that entropic measures do not distinguish between perfectly correlated and anti-correlated observables. Conceptually, it connects non-classicality with the assignment of inputs and outputs to different parties in a device-independent approach.

Explored here through numerical simulation, the link between non-equivalent party assignments and imposing a restriction on parameter values should be amenable to experimental demonstration. For further work, it seems interesting to probe the zone of anomalous entropic contextuality without correlated non-locality empirically. To obtain a resource that remains non-classical under all non-equivalent party assignments, it would be equally interesting to implement the measurement settings that preserve non-classicality under ‘forbidden’ permutations.

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