Self-bound Bose–Fermi liquids in lower dimensions

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Abstract
We study weakly interacting mixtures of ultracold atoms composed of bosonic and fermionic species in 2D and 1D. When interactions between particles are appropriately tuned, self-bound quantum liquids can be formed. We show that while formation of these droplets in 2D is due to the higher order correction terms contributing to the total energy and originating in quantum fluctuations, in 1D geometry the quantum fluctuations have a negligible role on formation of the self-bound systems. The leading mean-field interactions are then sufficient for droplet formation in 1D. We analyze energetic stability for 2D and 1D systems and predict values of equilibrium densities of droplets.

1. Introduction

Although the physical space is three-dimensional, lower dimensional spaces are quite often considered on a theoretical ground. This is because they are simpler for analysis, nonetheless able to capture the basic physics of the three-dimensional space. On the other hand the Hall effect, fractional statistics, Berezinskii–Kosterlitz–Thoules transition, Tonks–Girardeau gas are only some examples of physical phenomena attributed exclusively to 2D or 1D spaces.

It is possible to impose constraints on real 3D systems such that from the kinematic point of view they behave as low dimensional. In a case of ultracold quantum gases such tight confinement in one or two spatial directions can be obtained by a proper shaping of external fields forming a trap, making them highly anisotropic. If the excitation energy in the tightly confined direction(s) is the largest energy scale of the problem, the lower dimensional physics comes into play.

Recently quantum liquids attracted a lot of attention of both experimentalists [1–7] and theorists [8–12]. Quantum self-bound droplets can be formed in dipolar condensates or two-component Bose–Bose mixtures [8]. Recently conditions for formation of droplets in a Bose–Fermi mixture were found [13]. The life-time of droplets is limited by three-body collisions and at present experiments it is of the order of tens milliseconds. In lower dimensions it is expected that this life-time can be extended because of reduced phase-space available to colliding atoms. Low dimensional systems are therefore of the key interest [14–17]. In this work we study a possibility of formation of the self-bound low dimensional liquid droplets in a two component mixture of ultracold bosonic and fermionic atoms.

2. Uniform 2D mixture

The mean-field energy density of an uniform Bose–Fermi mixture is given by

$$\varepsilon_{MF} = \frac{1}{2} \beta n_{B}^{2} + \frac{1}{2} g_{BB} n_{B}^{2} + g_{BF} n_{B} n_{F},$$

where \(n_{B}\) and \(n_{F}\) are bosonic and fermionic densities, respectively, and \(\beta = 2\pi\hbar^{2}/m_{p}\). The first term in equation (1) corresponds to kinetic energy of fermions. The next two terms account for energy of intra-species...
(boson–boson) and inter-species (boson–fermion) interactions. The coupling constant $g_{\sigma\tau}$ is related to the 2D scattering lengths $a_{\sigma\tau}$ by $g_{\sigma\tau} = \frac{2\hbar^2}{\mu_{\sigma\tau} \ln\left(\frac{a_{\sigma\tau}}{a_{\sigma\tau}}\right)}$, where $\kappa$ is a cut-off momentum, $\epsilon = 4\exp(-2\gamma)$, $\gamma$ is Euler’s constant, and $\mu_{\sigma\tau} = m_\sigma m_\tau / (m_\sigma + m_\tau)$ is the reduced mass [14, 18]. Here $\sigma \in \{B, F\}$.

2D scattering has few key differences in comparison to the 3D case—the 2D scattering length, $a_{\sigma\tau}$, is always positive valued; scattering amplitude is energy dependent for all possible values of $a_{BB}$ and $a_{BF}$. The dependence of energy is weak, so it is always possible to use a fixed value of coupling strength provided that high energy contributions are eliminated by a proper choice of a value of a cut-off momentum $\kappa$. The weakly-interacting regime requires $a_{\sigma\tau} \ll \kappa$ (weak repulsion) or $a_{\sigma\tau} \gg \kappa$ (weak attraction). Here $\kappa$ is inter-particle separation. We show that it is possible to find such energy range of colliding particles, and thus to choose a suitable value for the cut-off momentum, for which the intra-species interaction is weakly repulsive and interspecies interaction is weakly attractive.

In the case of a two-dimensional Bose gas we deal with a quasicondensate. In such a case the lowest order correction to the mean field energy is given by [19]

$$\Delta_{\text{LHY}} = E_{\text{LHY}} / \Omega = \frac{1}{2} \sum_{k < \kappa} \left[ \omega_k - g_{BB} n_B - \frac{\hbar^2 k^2}{2m_B} \right],$$

(2)

where $\Delta_{\text{LHY}}$ is the Lee–Huang–Yang (LHY) [20] energy density, $\Omega$ is a volume entity, $\omega_k = (g_{BB} n_B \hbar^2 k^2 / m_B + (\hbar^2 k^2 / 2m_B)^2)^{1/2}$. The contribution to the bosonic repulsive energy, $E_{\text{LHY}} = \Omega \Delta_{\text{LHY}}$, due to the zero-point energy can be obtained via standard Bogoliubov theory.

The above expression, representing interacting Bose gas, was obtained by treating ultraviolet divergences using the discreet lattice model as a regularization scheme. Therefore integration in equation (2) over momentum is carried up to the cut-off momentum $\kappa$. The 2D integration gives the following contribution:

$$\Delta_{\text{LHY}} = \frac{g_{BB}^2 n_B^2 m_B}{8\pi \hbar^2} \ln \left( \frac{g_{BB} m_B \epsilon}{\hbar^2 \kappa^2 / m_B} \right).$$

(3)

The higher-order correction in the Bose–Fermi coupling in a three-dimensional mixture were estimated via several theoretical approaches [21–23], the results from which agree within appropriate validity regimes. In order to evaluate the correction to the Bose–Fermi interaction due to coupling between density fluctuations in the two species, we begin with the relevant term given in [22]:

$$\delta E_{\text{BF}} = -\frac{n_B n_B}{\Omega^2} \sum_{|k| < \kappa, |q| < \kappa} \left( u_k + v_q \right)^2 \ln \left( \frac{n_B^2 (1 - n_B^2) + n_F^2 (1 - n_F^2)}{\omega_k + \epsilon_q + \epsilon_{\epsilon} - \epsilon_q} \right).$$

(4)

Similarly as for Bose system we use here a discreet lattice model as the regularization scheme, and thus integration over momenta are carried up to the cut-off momentum determined by the lattice. Therefore the term used in [22] that counters ultraviolet divergence has been dropped. Here $\epsilon_{\epsilon} = \frac{\hbar^2 k^2}{2m_B}$, $\omega_k = ((\kappa / k)^2 + 2g_{BB} n_B \epsilon_k)^{1/2}$ and $n_\epsilon = \Theta(k_F - q)$. The Fermi momentum is given by $k_F = \sqrt{4\pi n_F}$. The coefficients $u_k$ and $v_q$ follow from the relations: $u_k^2 = 1 + v_q^2 = (\epsilon_k^2 + 2g_{BB} n_B (\omega_k + \epsilon_k) / 2\omega_k$ and $2u_k v_q = -g_{BB} n_B / 2\omega_k$.

Finally, we obtain

$$\delta E_{\text{BF}} = -\frac{2\pi m_B g_{BB}^2 2m_B k_F^2}{(2\pi)^4 \hbar^2} T_c,$$

(5)

where

$$T_c = \int_0^k d k \int_0^q d q \int_0^{2\pi} d \theta \frac{k q (1 - \Theta(1 - \sqrt{k^2 + q^2 + 2\omega q \cos \theta}))}{\sqrt{k^2 + \alpha (\sqrt{k^2 + \alpha + w k + 2w q \cos \theta})}},$$

(6)

Here $k_c = \kappa / k_0$ and $\alpha = 2w (g_{BB} n_B / \epsilon_q)$, where $\epsilon_q$ is the Fermi energy, and $w = m_B / m_F$.

Let $\varepsilon_B = g_{BB} n_B^2 / 2 + \varepsilon_{\text{LHY}}$ and $\varepsilon_{BB} = g_{BB} n_B n_F + \delta E_{\text{BF}}$. The total energy density of the system:

$$\varepsilon_{2D} = \beta n_F^2 / 2 + \varepsilon_B + \varepsilon_{BB},$$

formally depends on a cut-off momentum $\kappa$ via coupling constants and to specify the system the value of $\kappa = \kappa_c$ has to be set. If chosen correctly the total energy density of the system is almost independent of the choice.

Guided by analogous analysis of Petrov and Astrakharchik [14] for Bose–Bose droplet in two-dimensions, we observe that the energy density reaches a maximal value when varying the cut-off at fixed values of bosonic and fermionic densities, $\partial \varepsilon_{2D} / \partial \kappa_{\kappa_c = \kappa_c} = 0$. $\kappa_c$ is our choice for the cut-off momentum. Evidently, it depends on the densities of the species. Two comments are rather obvious: (i) close to the maximum there is no dependence on $\kappa$, (ii) for correctly chosen cut-off, the beyond-mean field terms should be small corrections to the leading mean-field energy.
To better justify this choice for the Bose–Fermi system we consider a limit of infinitely weak attraction between the two components of the mixture, i.e. $n_B a_{BF}^2 \to \infty$. In this limit bosonic and fermionic atoms become independent. Energy of bosonic subsystem in the limit of vanishing density, $n_B \to 0$, should approach the energy of a two-dimensional single component Bose gas, given by the famous Schick formula [24]. In appendix A we show that this is indeed the case.

The cut-off depends on densities, but on the other hand, the densities of species forming a droplet are fixed by interactions, thus depend (weakly) on the cut-off. The problem of choosing the cut-off must be solved in a self-consistent way. Our strategy is the following. If the cut-off is set then the interactions’ couplings are fixed. Stability conditions of the self-bound system give equilibrium densities of a droplet. Having those, the cut-off corresponding the maximum of energy, for fixed atomic densities, can be determined. The self-consistency can be easily reached. The procedure converges very fast and gives both, the optimal cut-off and the densities of Bose and Fermi components.

We focus here at the weakly interacting regime assuming Bose–Fermi attraction, i.e. $a_{BF}^2 n_B n_F \gg 1$, and Bose–Bose repulsion, i.e. $a_{BF}^2 n_B n_F \ll 1$. The initial guess for the cut-off momentum is chosen from the bare minimum stability condition of the mixture, similarly as in [14]. Determinant of the second derivatives of mean-field energies must vanish at the stability edge: $(\partial^2 \varepsilon_{MF}/\partial n_B^2)(\partial^2 \varepsilon_{MF}/\partial n_F^2)_{\kappa = \kappa_c} - (\partial^2 \varepsilon_{MF}/\partial n_B \partial n_F)_{\kappa = \kappa_c}^2 = 0$. This leads us to the following equation:

$$2 \ln^2 \left(\frac{\epsilon}{a_{BF}^2 \kappa_c^2}\right) = \frac{(1 + w)^2}{w} \ln \left(\frac{\epsilon}{a_{BF}^2 \kappa_c^2}\right),$$

which can be solved for $\kappa_c$ and the above-mentioned iterative procedure can be initiated. The initial value of $\kappa_c$ happens to be a quite good guess.

The necessary condition for the appearance of a liquid droplet is vanishing pressure i.e.

$$p = \varepsilon_{2D} - \mu_F n_F - \mu_B n_B = 0,$$

where the chemical potentials $\mu_F = \partial \varepsilon_{2D}/\partial n_F$. Due to the quadratic form of the dominant mean-field energy of the Bose–Fermi mixture in 2D, at the edge of the mean-field instability, i.e. for a small and negative $\partial^2 \varepsilon_{2D}/\partial n_F^2$, we can use arguments of [18] and show that the equilibrium densities are close to the line $n_B^\text{eq}/n_F^\text{eq} \approx \sqrt{\beta/\delta_B}$, which turns out to be

$$n_B^\text{eq}/n_F^\text{eq} \approx \left(\frac{w \ln(\epsilon/(a_{BF}^2 \kappa_c^2))}{2}\right)^{1/2}.$$  

The above equation can be further approximated if instead of finding iteratively the maximum of energy density, we use the initial guess for the cut-off as given (8). This choice of cut-off combined with approximate expression (10) gives:

$$\frac{n_B^\text{eq}/n_F^\text{eq}}{w^2} \approx \frac{w}{2}(r(1 - \sqrt{2p/r + 1}) + p),$$

where $r = (1 + w^2)/(4w)$ and $p = \ln(a_{BF}^2/a_{BB}^2)$.

In the following we consider a weakly interacting mixture of $^{133}\text{Cs}$ and $^6\text{Li}$, corresponding to the mass ratio $w = 22.09$, with varied $a_{BF}/a_{BB}$. Figure 1(a) demonstrates a graphical representation of the numerical solution from equation (9) for representative cases of $^{133}\text{Cs}$–$^6\text{Li}$ with $a_{BF}/a_{BB} = 10^4$, $2 \times 10^4$ and $4 \times 10^4$. The zero-pressure lines form closed contours on $n_B$–$n_F$ plane. The droplets form at equilibrium densities, for which the energy density of the system constrained by the zero–pressure requirement (equation (9)) attains the minimal value. This implies

$$\mu_B \frac{\partial p}{\partial n_F} - \mu_F \frac{\partial p}{\partial n_B} = 0.$$  

The energy minimas are marked by dots in figure 1(a). The variation of the energy density, $\varepsilon_{2D}$, along the zero-pressure line is shown for a particular case with $a_{BF}/a_{BB} = 10^4$ in the inset of figure 1(a). The particular density ratio $n_B/n_F$, for which the system’s total energy density is minimal, corresponds to the dot of the corresponding zero–pressure contour.

In figure 1(b), we examine the flatness of various components of the total energy density functional around the chosen cut-off momentum, $a_{BB} \kappa_c = 0.034$ obtained via the self-consistent approach. One can see that close to $\kappa_c$, once the higher–order corrections are accounted for along with the mean–field contributions, the energy density functionals show only a slight variance with the cut-off momentum. This suggests existence of a plateau around $\kappa = \kappa_c$, where droplet equilibrium densities are almost independent of the choice of a particular cut-off.  

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6 The universality of such treatment adopted for 2D system can be surmised from the work in [14], which treats the Bose–Bose system on an equivalent ground, but further back it up by evidences supported by Monte-Carlo analysis.
Figure 2. 2D case of $^{133}$Cs–$^6$Li mixture. (a) Red dashed, blue dotted and green solid lines show the zero-pressure contours for $a_{BB}/a_{BF} = 10^4$, $2 \times 10^4$ and $4 \times 10^4$, respectively. The equilibrium densities are marked by the dots. Inset: energy density, $\varepsilon_{2D}$, as a function of $n_B/n_F$, subjected to the zero-pressure constraint for $a_{BF}/a_{BB} = 10^4$. The vertical axis is rescaled by the equilibrium energy density, $|\varepsilon_0|$. The minimum corresponds to the red dot on the zero-pressure contour. (b) Variation of various contribution to the total energy density as a function of the cut-off momentum, $\kappa$, around the chosen cut-off momentum $\kappa_c$ obtained via iterative procedure for $a_{BF}/a_{BB} = 10^4$. The blue solid line shows fermionic kinetic energy density. The red dotted line and the green dashed–dotted line correspond to $\varepsilon_B$ and $\varepsilon_{BF}$, respectively. The total energy density of the system (equation (7)) is marked by black dashed line.

In figure 2(a), we show the equilibrium density ratios of liquid droplets in $^{133}$Cs–$^6$Li mixture with $a_{BF}/a_{BB}$. The solid line shows the approximate results governed by equation (11), which corresponds to initial choice for our cut-off momentum obtained via equation (8). This is subjected to further refinement via iterative process. The circles represent the full solutions corresponding to the iteratively obtained final cut-off momenta and direct inspections of energy minima on the zero-pressure contours as explained in figure 1. The agreement between the complete and approximate solutions remains very good, implying equation (11), although simplified, is very effective as it gives (with a very good accuracy) the ratio of densities of both species in terms of physical parameters—the mass and the scattering length ratios of the fermionic and bosonic components. Figure 2(b) presents the trends of the iteratively obtained cut-off momentum, which exhibit that $\kappa_c/\kappa_0$ increases with increasing $a_{BF}/a_{BB}$. Figures 2(c) and (d) respectively display the bosonic and fermionic equilibrium density as a function of $a_{BF}/a_{BB}$.

3. Finite 2D mixture

Now we investigate the density profile of the Bose–Fermi droplets with finite number of particles. We follow the similar treatment adopted in context of the 3D Bose–Fermi droplets [13]. In order to incorporate the surface effects, it is required to consider additional density gradient terms in system energy. This is done within local density approximation via inclusion of the kinetic energy $E^B_k = \int d^3r \epsilon^B_k$ with $\epsilon^B_k = (\hbar^2/2m_B)(\nabla \sqrt{n_B})^2$ for bosonic component and the Weizsäcker correction [25] to the kinetic energy, $E^F_k = \int d^3r \epsilon^F_k$, where $\epsilon^F_k = \xi (\hbar^2/8m_F)(\nabla n_F)^2/n_F$, for the fermionic component. For a 3D Fermi gas $\xi = 1/9$ [26, 27], for a 2D case (equation (7)), however, parameter $\xi$ starts to depend on the number of fermions [28]. The dependence is rather weak and for the number of $^6$Li atoms considered by us here it is fairly to put $\xi \approx 0.04$. The total energy of a finite
Bose–Fermi droplet is then given by
\[ E_{\text{BF}}(\mathbf{r}) = \left[ \frac{h^2 \pi^2 n^2}{6m} + g_{\text{BF}}^2 n^2_B n_F + g_{\text{BF}}^2 n_B n_F \right] \]. Both of the system's constituents, bosonic and fermionic clouds, can be regarded as fluids that can be treated within standard quantum hydrodynamics \cite{29, 30} by introducing the density and the velocity fields. Hydrodynamic equations can be reworked into a form of coupled set of Schrödinger-like equations via inverse Madelung transformation \cite{31–33}. In order to find the density profiles of the bosonic and fermionic species, the coupled set of Schrödinger equations of motion is solved by employing imaginary time propagation technique \cite{34}. See appendices B and C for details.

Figure 3 shows the ground state densities for \(^{133}\text{Cs} – ^{6}\text{Li} \) mixture for three different numbers of bosons and fermions. With increasing number of particles, as the droplets grow in size, the surface effects diminish as expected, and consequently, the peak densities approach the ones predicted by the analysis based on a uniform mixture in thermodynamic limit.

4. 1D mixture

We now look at the possibilities for droplet formation in the case of attractive inter- and repulsive intraspecies interactions of a 1D Bose–Fermi mixture. The mean-field energy density of 1D uniform mixture is given by
\[ \varepsilon_{\text{MF}}^{\text{1D}} = \frac{\hbar^2 \pi^2 n^2}{6m} + \frac{1}{2} g_{\text{BF}}^2 n^2_B n_F + g_{\text{BF}}^2 n_B n_F \].

The weakly interacting regime requires \[ |a_{\text{sc}}^{\text{1D}}|^2 n_F \gg 1 \].

The 1D LHY correction for the Bose–Fermi mixture can be obtained from equation (2). Unlike the 2D case we no longer require to introduce a cut-off. The integration over the entire momentum range can be performed analytically. We find \[ \varepsilon_{\text{LHY}}^{\text{1D}} = -2g_{\text{BF}} n_B (m_B g_{\text{BF}} n_B n_F)^{1/2} / (3 \pi \hbar) \]. The correction to the Bose–Fermi interaction in 1D, as it can be followed from equation (4), turns out to be \[ \delta \varepsilon_{\text{BF}}^{\text{1D}} = -(m_B g_{\text{BF}}^2 n_B n_F) / (2 \pi^2 \hbar^2), \] where
Finally, combining all the contributions, the total energy of the 1D system

$$\epsilon_{1D} = \epsilon_{MF}^{1D} + \epsilon_{LHY}^{1D} + \delta\epsilon_{BF}^{1D}. \quad (14)$$

The investigation for the possibility of a stable droplet formation in $^{133}$Cs–$^6$Li follows the same route discussed already in context of the 2D systems. The contours forming closed loops, where pressure vanishes in the $n_B$–$n_F$ plane, for certain choices of $a_{BF}^{1D}/|a_{BB}^{1D}|$ shown in the left panel of figure 4. The red circles in right panel of figure 4 show the equilibrium densities of $^{133}$Cs as function of $a_{BF}^{1D}/|a_{BB}^{1D}|$. The equilibrium densities of corresponding $^6$Li are shown by blue circles in the inset. One can see that the smaller the value $a_{BF}^{1D}/|a_{BB}^{1D}|$, the greater the quantity $|a_{BF}^{1D}|/n_\epsilon$, which must be much larger than unity in order to ensure the applicability of our theory.

Interestingly, we find that an effectively attractive mixture satisfying all conditions for droplet formation exist within the mean-field description itself. Analytical expressions for the equilibrium densities of individual species in 1D within mean-field treatment can be obtained by solving equations (9) and (12), where the 2D quantities have to be suitably modified by their 1D counterparts. The equilibrium density ratio in the thermodynamic limit is given by $n_{B,0}^{1D}/n_{F,0}^{1D} = (1 + w) |a_{BB}^{1D}|^2$, and their quantitative values are predicted to be

$$n_{B,0}^{1D} = \frac{3(1 + w)^3 |a_{BB}^{1D}|}{8\pi^2 w (a_{BF}^{1D})^3},$$

$$n_{F,0}^{1D} = \frac{3(1 + w)^3 |a_{BB}^{1D}|}{4\pi^2 w (a_{BF}^{1D})^2}. \quad (15)$$

Comparisons between the mean-field solutions with that from full solutions obtained after accounting for higher-order corrections are shown in figure 4. In contrast to the 2D and 3D Bose–Fermi droplets and also to the Bose–Bose droplets, the tiny correction terms, although present, are negligible and do not play any crucial role in formation of a stable droplet in 1D.

In figure 5 we show the bosonic and fermionic densities of a finite one-dimensional ‘mean-field’ droplet for two separate cases with $\delta_{BF}/\delta_{BB} = -80 (a_{BF}^{1D}/|a_{BB}^{1D}| = 0.144)$ and $\delta_{BF}/\delta_{BB} = -280 (a_{BF}^{1D}/|a_{BB}^{1D}| = 0.041)$ and different numbers of atoms. Since the number of fermions considered is small, we use here an approach which treats fermionic atoms individually. We use the Hartree–Fock formalism. Assigning a single-particle orbital to each fermionic atom, $\phi_i^F(r)$, and assuming all the bosons to occupy the same state $\phi_B(r)$, we solve the following set of time-dependent Hartree–Fock equations.

**Figure 3.** Radial densities (solid and dashed lines for bosons and fermions, respectively) of the 2D Bose–Fermi droplets for $^{133}$Cs–$^6$Li mixture for $a_{BB}/a_{BB} = 10^3$ and the initial number of bosons (fermions) equal to 1000 (100), 4000 (400), and 10000 (1000). The horizontal lines are the bosonic and fermionic densities coming from the analysis ignoring the surface effects. Clearly, the surface effects for larger droplets can be neglected.
\[ i\hbar \frac{\partial \phi_j}{\partial t} = \left[ -\frac{\hbar^2}{2m_B} \nabla^2 + g_B n_B + g_{BF} n_F \right] \phi_j \]

\[ i\hbar \frac{\partial \phi_F}{\partial t} = \left[ -\frac{\hbar^2}{2m_F} \nabla_j^2 + g_{BF} n_B \right] \phi_F \]

Figure 4. (Upper panel) zero pressure isobars for 1D $^{133}\text{Cs} - ^{6}\text{Li}$ mixture. (a) Black dashed, green solid and red dashed–dotted–dotted lines show the zero-pressure contours for $a_{BF}^F/|a_{BB}^B| = 0.041, 0.06$ and $0.08$, respectively. The equilibrium densities are marked by the dots. The red dotted line, which almost coincide with Black dashed line, represent zero pressure contour from the mean-field equation, equation (14), for $a_{BF}^F/|a_{BB}^B| = 0.041$. (Middle panel) the red circles represent the equilibrium densities of the bosonic species in a 1D $^{133}\text{Cs} - ^{6}\text{Li}$ mixture as a function of $a_{BF}^F/|a_{BB}^B|$ for the energy density including the higher-order correction terms (equation (14)). The mean-field predictions are represented by the black solid lines. (Lower panel) the equilibrium densities for the corresponding fermionic species are shown by red circles. The black solid line shows the fermionic density from mean-field theory.

where $j = 1, \ldots, N_F$ and the total fermionic and bosonic densities are $n_F = \sum_{j=1}^{N_F} |\phi_j|^2$ and $n_B = N_B |\phi_B|^2$, as well as its time-independent version (see [13] for details). In this way we find self-consistently the single-fermion orbitals, i.e. one-particle eigenstates of the effective potential created by both bosonic and fermionic components and hence densities. Clearly, as figure 5 suggests, going to larger samples the solutions change their character from the soliton–like (black solid and dashed lines, left corner) reported by us already some time ago [35] to the one with fully developed flat part typical for droplets (green solid and dashed lines). Noticeably, appearance of a train of Bose–Fermi solitons for appropriately tuned Bose–Fermi interaction, as claimed in [35], has been confirmed in recent experiment [36]. In 1D the fermions can be considered as impurities—much smaller in fraction in comparison to 2D and 3D [13] cases, yet essential for the droplet formation.
5. Discussion

In conclusion, in this work we show that higher-order quantum corrections may lead an ultracold weakly interacting gas of Bose–Fermi mixture towards a self-bound liquid state of matter. We find that in contrast to the 3D case [13], where droplets emerge above certain critical value of $a_{BF}/a_{BB}$, 2D ultradilute liquids are formed for arbitrarily small intra-species and inter-species interactions. For small $a_{BF}/a_{BB}$ in 2D, eventually the diluteness criteria will be invalid, and our theory can not be applied in this limit. Interestingly, in 1D geometry no quantum fluctuations are needed to form self-bound droplets. Our theory in 1D is not suitable in the limit of large $a_{BF}/a_{BB}$, where the diluteness criteria is not well satisfied.

Bose–Fermi systems are exotic due to the non-trivial interplay between dimension dependent scaling of kinetic energy of fermions, mean-field interactions and their higher-order contribution. The role of quantum pressure of fermions decreases while going from 3D to 1D geometry. Unlike in 3D systems, where fermionic kinetic energy scales like $n_F^{5/3}$, in 2D space the kinetic energy density scales as $n_F^2$, and in 1D the scaling is $n_F^3$. This is why self-bound droplets in 2D can be formed for almost vanishing mean-field energy what is not the case for the 3D system. While recent experiments with Bose–Bose droplets have vindicated the role of beyond-mean-field LHY correction, the quantum Bose–Fermi droplets promises to be an ideal platform for probing another higher order quantum correction in Bose–Fermi interaction originating from the density fluctuations of bosonic and fermionic species.

In future, it will be interesting to understand the nature of higher-order quantum correction in Bose–Fermi interaction, and the role they play in liquid formation, throughout the entire dimensional crossovers [16].

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Appendix A. Justification for the choice of the cut-off

For a two-dimensional Bose gas alone, in order to reproduce low-energy scattering properties of the exact potential, one has [14, 19]

$$\frac{1}{g_{BB}} = \frac{m_B}{2\pi\hbar^2}(\ln 2/(\kappa a_{BB}) - \gamma), \quad (A.1)$$

where $\kappa$ is a cut-off momentum. The formula is equivalent to the one used in the main text. Energy of a two-dimensional Bose gas is then given by

$$E_B = \frac{1}{2}g_{BB} n_B N_B + \frac{1}{2} \sum_{k<\kappa} \left( \sqrt{\hbar^2 k^2/2m_B (\hbar^2 k^2/2m_B + 2g_{BB} n_B)} ight) - \frac{\hbar^2 k^2}{2m_B} - g_{BB} n_B). \quad (A.2)$$

After integrating over momenta, the energy density becomes

$$\varepsilon_B = \frac{1}{2}g_{BB} n_B^2 + \frac{(g_{BB} n_B)^2 m_B}{8\pi\hbar^2} \ln \left( \frac{g_{BB} n_B \sqrt{e}}{\hbar^2 \kappa^2 / m_B} \right) \quad (A.3)$$

and the first (mean-field) and the second (LHY correction) terms can already be identified in equations (1) and (3), respectively, in the main text. By introducing $g_{BB}$ from (A.1) into equation (A.3) one gets

$$\frac{\varepsilon_B}{n_B} \frac{m_B a_{BB}^2}{2\pi \hbar^2} = \frac{x}{\ln y} + \frac{x}{\ln y} \ln \left( \frac{1}{y \ln y} \right)^{1/2}, \quad (A.4)$$

where $x = n_B a_{BB}^2$ and $y = (\kappa a_{BB}/2e^{-\gamma})^2$ are dimensionless parameters. In the limit of small values of parameter $x$, the energy per boson is given by Schick formula [24]

$$\varepsilon_{\text{Schick}} / n_B = \frac{\hbar^2}{m_B} \frac{2 \pi n_B}{\ln(n_B a_{BB}^2)} = \frac{2 \pi \hbar^2}{m_B} \frac{x}{n_B a_{BB}^2} \ln x^1. \quad (A.5)$$

Therefore

$$\frac{\varepsilon_B / \varepsilon_{\text{Schick}} = \frac{\ln x}{\ln x^2} + \frac{\ln x}{\ln x^2} \ln \left( \frac{1}{z \ln x^2} \right)^{1/2}. \quad (A.6)$$

where a new, density dependent, cut-off $z = y / x \propto \kappa^2 / n_B$ has been introduced. Obviously, the second term in (A.6) vanishes in the limit of small $x$, whereas the first one approaches one. The low density limit for the Bose gas is then recovered. The energy becomes flat in a wider range of the cut-off parameter $z$ when $x$ gets smaller.

However, in this paper we consider a two-dimensional Bose–Fermi mixture and description of boson–fermion interaction in two-dimensions also requires introducing an appropriate cut-off momentum. We equalize both cut-offs and propose an iterative procedure to equalize both cut-offs and propose an iterative procedure to

\[\text{utilize the condition } \frac{\partial E_{2D}}{\partial \kappa^2} = 0 \text{ to get a new value of } \kappa. \text{This is because the total energy of the system should only weakly depend on the cut-off. Therefore, the value of the cut-off should be close to the position of the wide maximum of the total energy, } E_{2D}(\kappa) \text{ for fixed densities. Then the next iterations are performed until the bosonic and fermionic densities, as well as position of maximum do not change. The value of the cut-off found in this way is } a_{BB} \kappa = 0.0340 \text{ and the densities are: } n_B = 3.97 \times 10^{-5} a_{BB}^2 \text{ and } n_F = 4.10 \times 10^{-6} a_{BB}^2, \text{ see figure 1.} \]

We expect that even after adding fermions to bosons, when mutual attraction becomes infinitely small, $n_B a_{BB} \rightarrow \infty$, the energy of bosonic component will still satisfy the limit of low densities, the Schick formula. It is indeed the case, as shown in figure A1. The bosonic energy becomes more and more flat when the bosonic density gets smaller and the value of cut-off approaches the position corresponding to the maximum of $\varepsilon_B / \varepsilon_{\text{Schick}}$ which in turn approaches the value of one as expected (red dot in figure A1). This way we show that the cut-off we choose allows to recover the standard expression of energy of weakly interacting Bose gas in 2D.

thus a good guess. Then the bosonic and fermionic densities are obtained based on equations (8) and (12). Next, we utilize the condition $\frac{\partial E_{2D}}{\partial \kappa^2} = 0$ to get a new value of $\kappa$. This is because the total energy of the system should only weakly depend on the cut-off. Therefore, the value of the cut-off should be close to the position of the wide maximum of the total energy, $E_{2D}(\kappa)$ for fixed densities. Then the next iterations are performed until the bosonic and fermionic densities, as well as position of maximum do not change. The value of the cut-off found in this way is $a_{BB} \kappa = 0.0340$ and the densities are: $n_B = 3.97 \times 10^{-5} a_{BB}^2$ and $n_F = 4.10 \times 10^{-6} a_{BB}^2$, see figure 1.

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Appendix B. Hydrodynamic equations

Hydrodynamic equations for a gas of neutral fermionic or bosonic atoms can be derived based on quantum kinetic equations for reduced density matrices [37–39]. From the whole hierarchy of equations, the one for the one-particle density matrix is of most practical interest

\[
\frac{i\hbar}{\partial t} \rho_1(\vec{r}_1, \vec{r}_2, t) = -\frac{\hbar^2}{2m}(\vec{\nabla}_1^2 - \vec{\nabla}_2^2) \rho_1(\vec{r}_1, \vec{r}_2, t) \\
+ \int d\vec{r}' [V(\vec{r}_1 - \vec{r}') - V(\vec{r}_2 - \vec{r}')] \rho_2(\vec{r}_1', \vec{r}_2', t) \\
+ [V_{\text{ext}}(\vec{r}_1, t) - V_{\text{ext}}(\vec{r}_2, t)] \rho_1(\vec{r}_1, \vec{r}_2, t),
\]

(B.1)

where \( V(\vec{r}_1 - \vec{r}_2) \) is the two-particle interaction term and \( V_{\text{ext}}(\vec{r}, t) \) is the external potential. Equation (B.1) involves the two-particle density matrix \( \rho_2 \). In the limit \( \vec{r}_1 \to \vec{r}_2 (=\vec{r}) \), equation (B.1) results in the continuity equation

\[
\frac{\partial n(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot [n(\vec{r}, t) \vec{v}(\vec{r}, t)] = 0,
\]

(B.2)

where the density and velocity fields are defined as follows:

\[
n(\vec{r}, t) = \lim_{\vec{r}_1 \to \vec{r}_2} \rho_1(\vec{r}_1, \vec{r}_2, t) \\
\vec{v}(\vec{r}, t) = \frac{\hbar}{2m} \lim_{\vec{r}_1 \to \vec{r}_2} (\vec{\nabla}_1 - \vec{\nabla}_2) \chi(\vec{r}_1, \vec{r}_2, t)
\]

and \( \chi(\vec{r}_1, \vec{r}_2, t) \) is the phase of the one-particle density matrix \( \rho_1(\vec{r}_1, \vec{r}_2, t) = \sigma(\vec{r}_1, \vec{r}_2, t) \exp[i \chi(\vec{r}_1, \vec{r}_2, t)]. \)

Equation (B.1) can be rewritten by introducing the center-of-mass, \( \vec{r} = (\vec{r}_1 + \vec{r}_2)/2 \), and the relative position, \( \vec{s} = \vec{r}_1 - \vec{r}_2 \), coordinates. Then, by taking the derivative of equation (B.1) with respect to the coordinate \( \vec{s} \), the hydrodynamic Euler-type equation of motion is obtained in the limit \( \vec{s} \to 0 \)

\[
\frac{\partial \vec{v}(\vec{r}, t)}{\partial t} = -\frac{\vec{\nabla}}{m n(\vec{r}, t)} \cdot \left[ \frac{\vec{v}(\vec{r}, t) \cdot \vec{\nabla}}{m n(\vec{r}, t)} \right] \vec{v}(\vec{r}, t) + \frac{\vec{F}(\vec{r}, t)}{m n(\vec{r}, t)} + \frac{\vec{\nabla} V_{\text{ext}}(\vec{r}, t)}{m},
\]

(B.4)

where the kinetic-energy stress tensor, \( T_{kl} \), is given by

\[
T_{kl} = -\frac{\hbar^2}{m} \lim_{\vec{s} \to 0} \frac{\partial}{\partial s_k} \sigma(\vec{r}, \vec{s}, t)
\]

(B.5)

and the force, \( \vec{F}(\vec{r}, t) \), resulting from interactions between atoms is

\[
\vec{F}(\vec{r}, t) = -\int d\vec{r}' \vec{\nabla} V(\vec{r}' - \vec{r}') \rho_2(\vec{r}', \vec{r}', t).
\]

The kinetic-energy stress tensor for fermions can be easily calculated within the Thomas–Fermi approximation. The one-particle Wigner function within the Thomas–Fermi approximation in a three-dimensional space is given by

Figure A1. Energy of bosonic component of a two-dimensional Bose–Fermi mixture as a function of density dependent cut-off momentum \( z \). Left frame: \( a_{BB} = 10^{10} a_{BB} \) and \( n_{BB} a_{BB}^2 = 3.97 \times 10^{-5} \). Right frame: \( a_{BB} = 10^{10} a_{BB} \) and \( n_{BB} a_{BB}^2 = 4.07 \times 10^{-13} \). The blue solid line is the formula (A.6) whereas red dots show the value of energy density at the cut-off momentum found via iterative procedure.
\[ w(\vec{r}, \vec{p}) = \eta(\hbar^2 [6\pi^2 m_0(\vec{r})]^{2/3} - \frac{\vec{p}^2}{2}), \]

where \( \eta() \) is the unit step function. The one-particle density matrix is calculated according to

\[ \rho_{fi}(\vec{r}, \vec{s}) = \int \frac{d^3p}{(2\pi\hbar)^3} w(\vec{r}, \vec{p}) e^{\vec{p}\vec{s}/\hbar}, \]

which implies that

\[ \rho_{fi}(\vec{r}, \vec{s}) = \frac{2}{(2\pi\hbar)^3} \left[ -\frac{\hbar p_x}{s} \cos \left( \frac{p_x s}{\hbar} \right) + \frac{\hbar^2}{s^2} \sin \left( \frac{p_x s}{\hbar} \right) \right], \]

where \( p_x(\vec{r}) = (\hbar^2 / 30\pi^2 m_0) \) is the local Fermi momentum. Then, from equation (B.5) one gets the kinetic-energy stress tensor \( T_{\text{fi}} = \left( \hbar^2 / 30\pi^2 m_0 \right) \delta_{\text{fi}} \). Assuming the case of non-interacting fermions (spin-polarized atoms at low temperature), the equations (B.2) and (B.4) become a closed set of hydrodynamic equations

\[ \frac{\partial n_{\text{fi}}}{\partial t} + \vec{v}_F \cdot (n_{\text{fi}} \vec{v}_F) = 0, \]
\[ \frac{\partial \vec{v}_F}{\partial t} + (\vec{v}_F \cdot \nabla) \vec{v}_F + \nabla \left( \frac{\hbar^2}{2m_{\text{fi}}} (6\pi^2)^{2/3} n_{\text{fi}}^{2/3} + \frac{V_{\text{ext}}}{m_{\text{fi}}} \right) = 0. \]

(B.6)

There is, of course, a space for improvements here. Commonly, gradient corrections are locally added to the Thomas–Fermi kinetic energy density (diagonal part of the kinetic–energy stress tensor), the simplest one being the Weizsäcker correction in the form of \( T_W = \xi (\hbar^2 / 8m_{\text{fi}}) (\nabla n_{\text{fi}})^2 / n_{\text{fi}}, \) with \( \xi = 1/9 \) [25–27]. Then the last term in the second equation of equations (B.6) is modified by adding the derivative \( \delta T_W / \delta n_{\text{fi}} \).

We further assume that the fermionic flow is irrotational, i.e. the vorticity vanishes, \( \vec{\nabla} \times \vec{v}_F = 0 \). By using one of the vector identities, \( (\vec{v}_F \cdot \nabla) \vec{v}_F = (\nabla \times \vec{v}_F) \times \vec{v}_F + \nabla (\vec{v}_F^2) / 2 \), equations (B.6) are turned into the following ones

\[ \frac{\partial n_{\text{fi}}}{\partial t} = -\vec{\nabla} \cdot (n_{\text{fi}} \vec{v}_F), \]
\[ m_{\text{fi}} \frac{\partial \vec{v}_F}{\partial t} = -\nabla \left( \frac{\hbar^2}{2m_{\text{fi}}} (6\pi^2)^{2/3} n_{\text{fi}}^{2/3} + \frac{\delta T_W}{\delta n_{\text{fi}}} + \frac{m_{\text{fi}} \vec{v}_F^2}{2} + V_{\text{ext}} \right). \]

(B.7)

The set of equations (B.6) was first used by Ball et al [30] to study the oscillations of electrons in a many-electron atom induced by ultraviolet and soft x-ray photons.

For a gas of neutral bosonic atoms, in the mean-field approximation when the system’s many-body wave function is \( \varphi(\vec{r}) \varphi(\vec{r}) \ldots \varphi(\vec{r}_N) \), the one-particle density matrix is given by

\[ \rho_{\text{bi}}(\vec{r}, \vec{s}) = \varphi^* (\vec{r} + \frac{1}{2}\vec{s}) \varphi (\vec{r} - \frac{1}{2}\vec{s}). \]

The kinetic-energy stress tensor possesses now off-diagonal elements and leads to the so-called quantum pressure term in the equation of motion. Assuming only contact interactions between bosonic atoms (with \( a \) being the scattering length), the equations (B.2) and (B.4) become the set of the following equations

\[ \frac{\partial n_{\text{bi}}}{\partial t} = -\vec{\nabla} \cdot (n_{\text{bi}} \vec{v}_{\text{bi}}), \]
\[ m_{\text{bi}} \frac{\partial \vec{v}_{\text{bi}}}{\partial t} = -\nabla \left( \frac{4\pi \hbar^2 a_{\text{bi}}}{m_{\text{bi}}} n_{\text{bi}} + \frac{\vec{v}_{\text{bi}}^2}{2} + V_{\text{ext}} - \frac{\hbar^2}{2m_{\text{bi}}} \sqrt{n_{\text{bi}}} \right). \]

(B.8)

Equations (B.8) are, in fact, the hydrodynamic representation of the Gross–Pitaevskii equation. They can be improved with respect to the boson–boson interactions by considering the beyond mean-field correction—the LHY correction.

The mixture of bosonic and fermionic atoms is then described by equations (B.7) and (B.8) modified by a term representing the interactions between bosons and fermions. This term, of the form of \( \delta E_{\text{BF}} / \delta n_{\text{bi}} \) for fermions and \( \delta E_{\text{BF}} / \delta n_{\text{fi}} \) for bosons, where \( E_{\text{BF}} \) is the total Bose–Fermi interaction energy, acts as an additional potential in the Euler-like equations of motion. What is crucial for the existence of Bose–Fermi droplets, \( E_{\text{BF}} \) energy includes the beyond mean-field correction. Although the above derivation of hydrodynamics of the Bose–Fermi mixture was performed explicitly in a three-dimensional space, it can be repeated in two- and one-dimensional cases as well. The changes which should be done are related to the local fermionic kinetic energy (for instance, in a two-dimensional space \( T_W = (\hbar^2 / 4m_{\text{fi}}) n_{\text{fi}}^2 / 4a_{\text{b}} \) and \( T_{\text{bf}} = \xi (\hbar^2 / 8m_{\text{bi}}) (\nabla n_{\text{bi}})^2 / n_{\text{bi}} \) with \( \xi = 0.04 \)) and beyond mean-field corrections for Bose–Bose and Bose–Fermi interaction terms.
Appendix C. Inverse Madelung transformation

As shown in the previous section, the Bose–Fermi mixture can be treated by using equations like (B.7) and (B.8). The very convenient way to further tackle equations (B.7) and (B.8) is to put them in a form of the Schrödinger-like equations by using the inverse Madelung transformation [29]. This is just a mathematical transformation which introduces the single complex function instead of density and velocity fields used in a hydrodynamic description. Both treatments are equivalent provided the velocity field is irrotational (vanishing vorticity). This assumption is obviously fulfilled for bosonic component since we consider that bosons populate a single quantum state. It is also true for fermions in a stationary case (zero velocity field) and in the case of dynamics studied by us while the trap confining mixture is adiabatically removed (in this case we a posteriori check that vorticity is zero).

After the inverse Madelung transformation is applied:
\[ \partial \psi(\vec{r}, t) = \sqrt{n_B(\vec{r}, t)} e^{-i\lambda(\vec{r}, t)}, \]

where \( \vec{v}(\vec{r}, t) = \nabla \lambda(\vec{r}, t) \), equations (B.7) and (B.8) are turned into

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m_f} \nabla^2 + \frac{\xi \hbar^2}{2m_f} \nabla^2 |\psi_f| \right] + \beta |\psi_f|^2 + g_{BF} |\psi_f|^2 \]
\[ + C I_C(\omega, \alpha) + C |\psi_f|^2 \frac{\partial I_C}{\partial \alpha} \frac{\partial |\psi_f|^2}{\partial m_f} \psi_f, \]

\[ i\hbar \frac{\partial \psi_B}{\partial t} = \left[ -\frac{\hbar^2}{2m_b} \nabla^2 + \alpha |\psi_f|^2 + g_{BF} |\psi_f|^2 + A |\psi_f|^2 \right] \]
\[ + 2A |\psi_f|^2 \ln(B |\psi_f|^2) + C |\psi_f|^2 \frac{\partial A}{\partial \alpha} \frac{\partial |\psi_f|^2}{\partial m_B} \psi_B, \]

where now the main objects to be solved are the bosonic wave function, \( \psi_B \), and the fermionic pseudo-function, \( \psi_f \). The coefficients are: \( \xi = 1 - \xi = 0.96, A = \frac{\hbar^2 g_{BF}}{m_B |\psi_f|^2}, B = \frac{\hbar^2 g_{BF}}{m_B |\psi_f|^2}, \) and \( \beta = -\frac{\alpha}{(\hbar |\psi_f|^2)^2} \). The bosonic wave function and the fermionic pseudo-wave function are normalized as \( N_{BF} = \int |\psi_B|^2 \). The integral \( I_C \) and all the parameters \( \omega, \alpha, \beta \) and those appearing in coefficients \( A, B, \) and \( C \) are defined in the main text.

Note that equations (C.2) describe two-dimensional Bose–Fermi mixture with beyond mean-field LHY correction for boson–boson interactions and analogous correction for the interaction between bosons and fermions.

References

[1] Kadju H, Schmitt M, Wenzel M, Wink C, Maier T, Ferrier-Barbut I and Pfau T 2016 Observing the Rosensweig instability of a quantum ferrofluid Nature 530 194
[2] Ferrier-Barbut I, Kadju H, Schmitt M, Wenzel M and Pfau T 2016 Observation of quantum droplets in a strongly dipolar Bose gas Phys. Rev. Lett. 116 215301
[3] Schmitt M, Wenzel M, Böttcher B, Ferrier-Barbut I and Pfau T 2016 Self-bound droplets of a dilute magnetic quantum liquid Nature 539 259
[4] Chomaz L, Baier S, Petter D, Mark M J, Wächtler F, Santos L and Ferlaino F 2016 Quantum–fluid crossover in quasi-one-dimensional dipolar Bose–Fermi condensates Science 359 101
[5] Chineny P, Cabrera C R, Tanzi L, Naylor B, Thomas P, Cheiney P and Tarruell L 2018 Quantum liquid droplets in a mixture of Bose–Einstein condensates Science 359 101
[6] Chomaz L, Baier S, Mark M J, Wächtler F, Santos L and Ferlaino F 2016 Quantum–fluid crossover in quasi-one-dimensional dipolar Bose–Einstein condensates Science 359 101
[7] Cui X 2018 Spin–orbit coupling induced quantum droplet in ultracold Bose–Fermi mixtures Phys. Rev. A 98 023630
[8] Petrov D S 2013 Quantum mechanical stabilization of a collapsing Bose–Bose mixture Phys. Rev. Lett. 115 153302
[9] Wächtler F and Santos L 2016 Quantum filaments in dipolar Bose–Einstein condensates Phys. Rev. A 93 061603(R)
[10] Baillie D, Wilson R M, Bisset R N and Blakie P B 2015 Self-bound dipolar droplet: a localized matter wave in free space Phys. Rev. A 94 021602(R)
[11] Oldziejewski R and Jacobymski K 1996 Properties of strongly dipolar Bose gases beyond the Born approximation Phys. Rev. A 94 063638
[12] Cui X 2018 Spin–orbit coupling induced quantum droplet in ultracold Bose–Fermi mixtures Phys. Rev. A 98 023630
[13] Rakshit D, Karpuk T, Brevecykay M and Gajda M 2019 Quantum Bose–Fermi droplets SciPost Phys. 6 079
[14] Petrov D S and Astrakharchik G 2016 Ultradiute low-dimensional liquids Phys. Rev. Lett. 117 100401
[15] Chomaz L, Baier S, Wächtler F, Nath R, Sinha S and Santos L 2017 Quantum fluctuations in quasi-one-dimensional dipolar Bose–Einstein condensates Phys. Rev. Lett. 119 050403
[16] Zin P, Pylab M, Wasak T, Gajda M and Idriszek Z 2018 Quantum Bose–Bose droplets at a dimensional crossover, quantum Bose–Bose droplets at a dimensional crossover Phys. Rev. A 98 051603
[17] Ilg T, Kaulin J, Santos L, Petrov D S and Büchler H P 2018 Dimensional crossover for the beyond-mean-field correction in Bose gases Phys. Rev. A 98 051604
[18] Popov V N 1971 To the theory of superfluidity of the twodimensional and one-dimensional Bose systems *Teor. Mat. Fiz.* **11** 354
Popov V N 1972 *Theor. Math. Phys.* **11** 565

[19] Mora C and Castin Y 2003 Extension of Bogoliubov theory to quasicondensates *Phys. Rev. A* **67** 053615

[20] Lee T D, Huang K and Yang C N 1957 Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties *Phys. Rev.* **106** 1135

[21] Albus A P, Gardiner S A, Illuminati F and Wilkens M 2002 Quantum field theory of dilute homogeneous Bose–Fermi mixtures at zero temperature: general formalism and beyond mean-field corrections *Phys. Rev. A* **65** 053607

[22] Viverit L and Giorgini S 2002 Ground-state properties of a dilute Bose–Fermi mixture *Phys. Rev. A* **66** 063604

[23] Yu Z-Q, Zhang S and Zhai H 2011 Stability condition of a strongly interacting boson-fermion mixture across an interspecies Feshbach resonance *Phys. Rev. A* **83** 041603(R)

[24] Schick M 1971 Two dimensional system of hard-core bosons *Phys. Rev. A* **3** 1067

[25] Weizsäcker C F 1935 *Z. Phys.* **96** 431

[26] Kirznits D A 1957 *Sov. Phys.—JETP* **5** 64

[27] Oliver G L and Perdew J P 1979 Spin-density gradient expansion for the kinetic energy *Phys. Rev. A* **20** 397

[28] van Zyl B P, Zaremba E and Pisarski P 2013 Thomas–Fermi–von Weizsäcker theory for a harmonically trapped, two-dimensional, spin-polarized dipolar Fermi gas *Phys. Rev. A* **87** 043614

[29] Madelung E 1927 Z. Phys. **40** 322

[30] Ball J A, Wheeler J A and Fireman E L 1973 Photoabsorption and charge oscillation of the Thomas–Fermi atom *Rev. Mod. Phys.* **45** 333

[31] Dey B K and Deb B M 1998 Femtosecond quantum fluid dynamics of helium atom under an intense laser field *Int. J. Quantum Chem.* **70** 441

[32] Domps A, Reinhard P-G and Suraud E 1998 Time-dependent Thomas–Fermi approach for electron dynamics in metal clusters *Phys. Rev. Lett.* **80** 5520

[33] Grochowski P T, Karpiuk T, Brewczyk M and Rzążewski K 2017 Unified description of dynamics of a repulsive two-component Fermi gas *Phys. Rev. Lett.* **119** 215303

[34] Gawryluk K, Karpiuk T, Gajda M, Rzążewski K and Brewczyk M 2017 Unified way for computing dynamics of Bose–Einstein condensates and degenerate Fermi gases *Int. J. Comput. Math.* **95** 2143–61

[35] Karpiuk T, Brewczyk M, Ospelkaus-Schwarzer S, Bongs K, Gajda M and Rzążewski K 2004 Soliton trains in Bose–Fermi mixtures *Phys. Rev. Lett.* **93** 100401

[36] DeSalvo B J, Patel K, Cai G and Chin C 2019 Fermion-mediated interactions between bosonic atoms *Nature* **568** 61

[37] Frolich H 1967 *Physica* **37** 215

[38] Wong C Y and McDonald J A 1977 Dynamics of nuclear fluid: III. General considerations on the kinetic theory of quantum fluids *Phys. Rev. C* **16** 1396

[39] March N H and Deb B M 1987 *The Single-Particle Density in Physics and Chemistry* (London: Academic)