FREE TRANSFORMATIONS OF $S^1 \times S^n$
OF SQUARE-FREE ODD PERIOD

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1. Introduction

Let $\ell > 1$ be an integer. Consider the $\ell$-periodic homeomorphism without fixed points:

$$T_\ell : S^1 \times S^n \to S^1 \times S^n ; \ (z, x) \mapsto (\zeta_\ell z, x) \quad \text{where} \quad \zeta_\ell := e^{2\pi i / \ell} \in \mathbb{C}.$$ 

Write $\mathcal{A}_\ell^\ell$ for the set of conjugacy classes $(C)$ in Homeo($S^1 \times S^n$) of those subgroups $C$ of order $\ell$ without fixed points. The classification of $\mathcal{A}_\ell^\ell$ was done by Jahren–Kwasik [JK11].

Recall the Euler totient function $\varphi$ is the number of units modulo a given natural number. Let $d > 1$. A partition $Q_d^\omega$ of $\mathbb{Z}_d$ is given by $[q] = [q']$ if $d q \equiv \pm q' \pmod{d}$ for some $a$. The map $(g \mapsto g^{-1})$ on the cyclic group $C_d$ induces an involution $i$ on the projective class group $Wh_0(C_d) := K_0(\mathbb{Z}C_d)/K_0(\mathbb{Z})$ with coinvariants $H_0(C_2; Wh_0(C_d)) := Wh_0(C_d)/(1-i)$.

**Theorem 1.1** (Classification Theorem). Let $\ell > 1$ be square-free odd. Then $\mathcal{A}_\ell^k = \{(T_\ell)\}$ for all $k > 0$ and $\mathcal{A}_\ell^1 = \{(T_\ell)\}$. Otherwise for any $k > 1$, there is a finite-to-one surjection

$$\bigcup_{1 \leq d | \ell} Q_d^k \times \mathbb{Z}^{(d-1)/2} \times H_0(C_2; Wh_0(C_d)) \longrightarrow \mathcal{A}_\ell^{2k-1} - \{(T_\ell)\}.$$ 

The $d$-indexed terms in the disjoint union have disjoint images. In the $d$-th image, each point-preimage has cardinality dividing $8 \gcd(k, \varphi(d)/2)$, which has bounded growth in $\ell$.

Different preimages have different cardinalities $(6.8)$. For $n = 3$, this theorem answers the existence part of [Sch85, Problem 6.14]; indeterminacy in the uniqueness is at most 16.

**Corollary 1.2.** Let $p \neq 2$ be prime. Then $\mathcal{A}_p^k = \{(T_p)\}$ for all $k > 0$ and $\mathcal{A}_p^1 = \{(T_p)\}$. Otherwise for any given $k > 1$, there is a finite-to-one surjection

$$Q_p^k \times \mathbb{Z}^{(p-1)/2} \times H_0(C_2; Cl_p) \longrightarrow \mathcal{A}_p^{2k-1} - \{(T_p)\}.$$ 

Each preimage has cardinality dividing $8 \gcd(k, (p-1)/2)$, which is bounded in $p$. \qed
Here, \( \text{Cl}_p \) is the ideal class group of \( \mathbb{Z}[\zeta_p] \), the involution \( \iota \) is induced by \( (\zeta_p \mapsto \zeta_p^{-1}) \). The three parts are understood using the quotient manifold \( M \) of the free \( C_p \)-action, specifically invariants of the infinite cyclic cover \( \overline{M} \), as follows. The \( \mathbb{Q}_p^\times \)-part is the first Postnikov invariant of \( \overline{M} \). The \( \mathbb{Z}((p^{-1})/2 \)-part is a projective \( p \)-invariant of \( \overline{M} \). The \( \text{Cl}_p \)-part is the Wall finiteness obstruction of \( \overline{M} \). The indeterminacy \( 8 \text{gcd}(k, (p-1)/2) \) is due to ineffective action of the group (which grows quadratically in \( p \)) of self-homotopy equivalences of \( M \).

**Remark 1.3.** Consider the ideal class group \( \text{Cl}_p^+ \) of the real subring \( \mathbb{Z}[\zeta_p + \zeta_p^{-1}] \) of \( \mathbb{Z}[\zeta_p] \). Write \( G \) for the Galois group of \( \mathbb{Q}(\zeta_p) \) over \( \mathbb{Q} \). The induced \( \mathbb{Z}[G] \)-module map \( \text{Cl}_p^+ \rightarrow \text{Cl}_p \) is injective ([Was97, Theorem 4.14]). The norm map \( N := 1 + \iota : \text{Cl}_p \rightarrow \text{Cl}_p^+ \) is surjective ([Was97, Proof 10.2]). Since the fixed field of the automorphism \( \iota \in G \) is \( \mathbb{Q}(\zeta_p + \zeta_p^{-1}) \), \( \iota \) induces the identity on \( \text{Cl}_p^+ \). Then \( \iota \) induces negative the identity on \( \text{Cl}_p := \text{Cl}_p^+ / \text{Cl}_p^+ \), since

\[
\iota(I) = N(I) - I \equiv -I \pmod{\text{Cl}_p^+}.
\]

Therefore we obtain an exact sequence of \( \mathbb{Z}[G/\iota] \)-modules:

\[
\begin{array}{cccc}
2(\text{Cl}_p^+) & \xrightarrow{1 - \iota} & \text{Cl}_p^+ & \xrightarrow{\iota} & H_0(C_2; \text{Cl}_p) & \xrightarrow{\iota} & \text{Cl}_p^+ / 2 & \rightarrow 0.
\end{array}
\]

Here \( 2A : = \{a \in A \mid 2a = 0 \} \) denotes the exponent-two subgroup of any abelian group \( A \), and \( 1 - \iota := (1 - \iota) \circ s \) is a well-defined homomorphism via a setwise section \( s : \text{Cl}_p \rightarrow \text{Cl}_p \).

**Remark 1.4.** The \( \text{Cl}_p^+ / 2 \) are only known for \( p < 500 \) [Sch98]. Even worse, the \( \text{Cl}_p^+ \) are only known for \( p \leq 151 \). The \( \text{Cl}_p^+ \) are conditionally known for \( 157 \leq p \leq 241 \) [Mil15], which we denote by *\^\#", under the Generalized Riemann Hypothesis for the zeta function of the Hilbert class field of \( \mathbb{Q}(\zeta_p + \zeta_p^{-1}) \). We list these new results of R Schoof and J C Miller:

| Table 1. \( \text{Cl}_p^+ \): [Mil15, Theorem 1.1] | \( \text{Cl}_p^+ / 2 \): [Sch98, Table 4.4] |
|--------------------------------------------------|-------------------------------------|
| \( H_0(C_2; \text{Cl}_p) \): This group vanishes* for the 46 primes \( p \leq 241 \) not listed. |
| \( p \) | \( \text{Cl}_p^+ \) | \( \text{Cl}_p^+ / 2 \) | \( H_0(C_2; \text{Cl}_p) \) |
| 29 | 0 | (2, 2, 2) | (2, 2, 2) |
| 113 | 0 | (2, 2, 2) | (2, 2, 2) |
| 163 | (2, 2)* | (2, 2) | \( 4 \leq \text{order} \leq 16^* \) |
| 191 | (11)* | 0 | (11)* |
| 197 | 0* | (2, 2, 2) | (2, 2, 2)* |
| 229 | (3)* | 0 | (3)* |
| 239 | 0* | (2, 2, 2) | (2, 2, 2)* |

**Remark 1.5.** The products of \( S^1 \) with the classical lens spaces \( \Lambda \) of type \( (p; q_1, \ldots, q_k) \) and \( \Lambda' \) of type \( (p; q_1', \ldots, q_k') \) are distinguished in Corollary 1.2, first by homotopy type in the first factor, and then by homeomorphism type in the other factors, as follows. First, \( \Lambda \) has the homotopy type of \( L_{p,q} \), where \( q := q_1 \cdots q_k \), and similarly for \( \Lambda' \) with \( q' := q_1' \cdots q_k' \). Furthermore, these types are equal if and only if \( [q] = [q'] \) in the set \( \mathbb{Q}_p^\times \) [Coh73, (29.4)].

Now assume \( [q] = [q'] \), so there exists a homotopy equivalence \( f : \Lambda' \rightarrow \Lambda \).

Second, assume \( 0 = \rho(\Lambda', f) = \rho(\Lambda') - \rho(\Lambda) \), which is independent of the choice of \( f \). Indeed, \( \rho \) is an invariant of the \( h \)-bordism class of \( (\Lambda', f) \) [AS68, 7.5]. Then \( [\Lambda', f] = [S^1 \times \Lambda', \text{id}_{S^1 \times \Lambda}] \in S^0_{\text{TOP}}(S^1 \times \Lambda) \) maps to zero in \( S^0_{\text{TOP}}(S^1 \times \Lambda) \cong \mathbb{Z}((p^{-1})/2 \); see Corollary 3.6.
This kernel is identified with the kernel of \( \overline{\mathcal{L}}_{2k}^1(C_p) \rightarrow \overline{\mathcal{L}}_{2k}^1(C_p) \), which is further identified with the following cokernel \( \mathcal{K}(C_p) \): arising in the Ranicki–Rothenberg sequence \([\text{Bak} 78]\):

\[
\mathcal{K}(C_p) := \text{Cok} \left( \overline{H}_0(C_2; K_0(\mathbb{Z}C_p, \mathbb{Q}C_p)) \rightarrow \overline{H}_0(C_2; C_l)) \right).
\]

So the structure \([\Lambda', f]\) lies in the subquotient \( \mathcal{H}(C_p) \) of the third factor, \( H_0(C_2; C_l) \).

Third, assume the given 2-torsion element \([\Lambda', f]\) of \( S^1_{\text{top}}(\Lambda) \) vanishes in \( H_0(C_2; C_l) \). Then \( f : \Lambda' \rightarrow \Lambda \) is \( h \)-bordant to the identity map. In particular, \( \Lambda' \) is \( h \)-cobordant to \( \Lambda \). Therefore, they are isometric \([\text{Mil} 66, 12.12]; \) equivalently \( \Lambda \) and \( \Lambda' \) are homeomorphic.

Theorem 1.1 follows from Theorems 1.6 and 1.7 below. Consider complex coordinates

\[
S^{2k-1} = \{ u \in \mathbb{C}^k : u \cdot \overline{u} = 1 \}.
\]

For any \( q \) coprime to any \( d > 1 \), there is a linear isometry of \( S^{2k-1} \) giving a free \( C_d \)-action:

\[
\Phi_{d,q} : S^{2k-1} \rightarrow S^{2k-1} ; (u_1, u_2, \ldots, u_k) \mapsto (\zeta_d q u_1, \zeta_d q u_2, \ldots, \zeta_d q u_k).
\]

The quotient manifold \( L_{d,q}^{2k-1} := S^{2k-1}/\Phi_{d,q} \) is called the lens space of type \((d; q, 1, \ldots, 1)\).

For any closed manifold \( X \), consider the set \( \mathcal{M}_{\text{top}}^{h/s}(X) \) of closed topological manifolds \( M \) homotopy equivalent to \( X \) up to homeomorphism. The calculation of \( \mathcal{A}_\ell \) reduces to \( M \).

**Theorem 1.6.** Let \( \ell \) be square-free odd. Then \( \mathcal{A}_\ell^{d,q} = \{(T_\ell)\} \) for all \( k > 0 \) and \( \mathcal{A}_\ell^1 = \{(T_\ell)\} \).

Otherwise, for all \( k > 1 \), passage to orbit spaces induces a bijection

\[
\mathcal{A}_\ell^{d,q} \rightarrow \mathcal{M}_{\text{top}}^{h/s}(S^1 \times L_{d,q}^{2k-1}).
\]

We calculate these \( M \) by methods of surgery theory and express them with \( K \)-theory.

**Theorem 1.7.** Let \( d \) be square-free odd, \( q \) coprime to \( d \), and \( k > 1 \). There is a surjection

\[
\mathbb{Z}^{(d-1)/2} \times H_0(C_2; \text{Wh}_0(C_d)) \rightarrow \mathcal{M}_{\text{top}}^{h/s}(S^1 \times L_{d,q}^{2k-1}).
\]

Any preimage has cardinality dividing \( 8 \gcd(k, \varphi(d))/2 \), which has bounded growth in \( d \).

Theorem 1.6 and Theorem 1.7 are proven in Section 2 and Section 6, respectively. The difficulty in generalizing Theorem 1.1 to all odd \( \ell \) comes from the proof of Theorem 1.6. When \( d > 1 \) is not square-free, say \( d = pp \), the groups \( NK_1(\mathbb{Z}[C_{pp}]) \) are huge: they are closely related to infinitely generated modules over the Verschiebung algebra of \( \mathbb{F}_p[I] \). Nonetheless, there would be two difficulties in handling elements of \( NK_1 \) in this paper: topologically there would be a ‘relaxation’ obstruction to making Proposition 2.2 work, and algebraically there would be a ‘homotopy’ obstruction to making Lemma 4.1(1) work.

### 2. Classification of homotopy types

The first stage is the homotopy classification of orbit spaces, then analysis of conjugacy.

**Proposition 2.1.** Let \( S^1 \times S^n \) be an \( \ell \)-fold regular cyclic cover of a topological space \( M \), with \( n \geq 1 \) and odd \( \ell > 1 \). Then \( M \) is homotopy equivalent to \( S^1 \times S^n \) or \( S^1 \times L_{d,q}^{2k-1} \) with \( d|\ell \).

The degree \( \ell \) must be odd, or else the Klein bottle \( M = \mathbb{R}P^2 \# \mathbb{R}P^2 \) is a counterexample.

**Proof.** The regular covering map \( S^1 \times S^n \rightarrow M \) has degree \( \ell > 1 \). Since \( \ell \) is odd, the quotient manifold \( M \) is oriented. If \( n = 1 \) then \( M \) must be homeomorphic to the torus \( S^1 \times S^1 \). If \( n = 2 \) then \( M \) must be homotopy equivalent to \( S^1 \times S^2 \). So now assume \( n \geq 3 \).
The covering map $S^1 \times S^n \rightarrow M$ has covering group $C_{\ell}$. Write $\Gamma := \pi_1(M)$ for the fundamental group of the quotient space. The exact sequence of homotopy groups contains

$$1 \longrightarrow C_\infty \overset{i}{\longrightarrow} \Gamma \overset{\varphi}{\longrightarrow} C_{\ell} \longrightarrow 1.$$ 

Write $T \in \Gamma$ for the image under $i$ of a generator of $C_\infty$. Select an element $S \in \Gamma$ such that $S$ maps under $\varphi$ to a generator $s$ of $C_{\ell}$. Define a setwise section

$$\sigma : C_{\ell} \longrightarrow \Gamma ; \; s^b \mapsto S^b \text{ for all } 0 \leq b < \ell.$$ 

In general for a group extension equipped with a setwise section, one has $\Gamma = (\text{Im} \, i)(\text{Im} \, \sigma)$. Then, for each $x \in \Gamma$, we obtain the normal form $x = T^a S^b$ for some $a \in \mathbb{Z}$ and $0 \leq b < \ell$. Note $S^{-1} T S \in (T, T^{-1})$. If $S^{-1} T S = T^{-1}$ then $S^{-1} T S^f = T^{(-1)f}$, but $S^f \in \text{Ker} \varphi = \text{Im} \, i$ and $f$ is odd, so $T = T^{-1}$, a contradiction. Hence $TS = ST$. Therefore $\Gamma$ is abelian. Hence $\pi_1(M) = \Gamma \cong C_\infty \times C_d$ for some divisor $d$ of $\ell$ (this includes the case of $d = 1$).

There is a corresponding infinite cyclic cover $\overline{M}$ with covering translation $t : \overline{M} \longrightarrow \overline{M}$. There is a bundle sequence $\mathbb{R} \longrightarrow \overline{M} \rightarrow S^1 \longrightarrow M$, with total space the mapping torus of $t$.

Observe that $\overline{M}$ is a $PD_n$-complex, since the $PD_n$-complex $\mathbb{R} \times S^n$ is its universal cover with finite covering group $\pi_1(\overline{M}) = C_d$. For any $PD_n$-complex $X$ with $n \geq 3$ and $\overline{X} \cong S^n$, Wall showed that the first Postnikov invariant $k_1(X) : K(\pi_1 X, 1) \rightarrow K(\mathbb{Z}, n+1)$ is a generator of the abelian group $H^{n+1}(\pi_1 X; \mathbb{Z})$ and that the oriented homotopy type of $X$ is uniquely determined by the orbit $[k_1(X)]$ under action of the group $\text{Out}(\pi_1 X)$ [Wal67, Theorem 4.3].

If $d = 1$ then $\overline{M}$ is homotopy equivalent to $S^n$. Otherwise assume $d > 1$. Since $d$ is odd, $\text{Ker}(C_d, 1)$ has 2-periodic cohomology. However $C_d$ acts freely on $\mathbb{R} \times S^n \cong S^n$, so a standard argument with the Leray–Serre spectral sequence shows that $\text{K}(C_d, 1)$ has $(n+1)$-periodic cohomology. Hence $n = 2k - 1$ for some $k > 1$. Write $q \in H^{2k}(C_d; \mathbb{Z}) = \mathbb{Z}/d$ for the first Postnikov invariant of $\overline{M}$; we have $\gcd(d, q) = 1$. The lens space $L(d; q, 1, \ldots, 1)$ also has first Postnikov invariant $q$, so $\overline{M}$ must be homotopy equivalent to $L^{2k-1}_{d,q} = L(d; q, 1, \ldots, 1)$.

In any of these cases of $d$ and $q$, there exist a closed $n$-manifold $L$ and a homotopy equivalence $h : L \longrightarrow \overline{M}$. Write $h^{-1} : \overline{M} \longrightarrow L$ for the homotopy inverse; consider the oriented homotopy equivalence $\alpha := h^{-1} \circ i \circ h : L \longrightarrow L$. By cyclic permutation of composition, $L \times_S S^1 \cong \overline{M} \times_S S^1 \cong M$. Then $C_d \times_{\alpha_q} C_\infty \cong C_d \times C_\infty$, where $\alpha_q \in \text{Out}(C_d)$ is the induced automorphism on $\pi_1(L)$. So $\alpha_q = \text{id}$. Hence $\alpha = \text{id}$ by [Coh73, (29.5A)].

The linking form on the $(k - 1)$-st integral homology group of the infinite cyclic cover $\overline{M}$ determines the integer $q$, which is unique up to its equivalence class $[q]$ in the partition $Q_d^{2k}$. In the sequel, we shall fix $k > 1$ and consider the latter, closed $2k$-dimensional manifold

$$X_{d,q} := S^1 \times L^{2k-1}_{d,q}.$$ 

The following definition generalizes the homeomorphism of Jahren–Kwasik [JK11, §4]. Write $r$ and $s$ for the usual generators of $C_\infty$ and $C_d$, respectively. Note $(r^s, s^r) \mapsto (r^s, s^{r+s})$ in $\text{Aut}(C_\infty \times C_d)$ is induced by the well-defined self-homeomorphism (like a Dehn twist):

$$e : X_{d,q} \longrightarrow X_{d,q} ; \; (z, [u_1, u_2, \ldots, u_k]) \mapsto (z, [z^{|u_1|} u_1, z^{|u_2|} u_2, \ldots, z^{|u_k|} u_k]).$$

This is multiplication by the path $[0, 2\pi] \longrightarrow GL_d(\mathbb{C}) ; \; \theta \mapsto \text{diag}(e^{i|u_1| \theta}, e^{i|u_2| \theta}, \ldots, e^{i|u_k| \theta})$. 


Proposition 2.2. Let \( f : M \to X_{d,q} \) be a homotopy equivalence with \( M \) a closed manifold. There exists \( \delta \in \text{Homeo}(M) \) satisfying a homotopy commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & X_{d,q} \\
\delta \downarrow & & \\
M & \xrightarrow{f} & X_{d,q}'.
\end{array}
\]

Later, in Section 4, we prove Proposition 2.2 based on surgery-theoretic calculations.

Notice that the fundamental group \( \pi_1(X_{d,q}) = C_\infty \times C_d \) does not have a unique subgroup \( Z \) of index \( d \), rather there are exactly \( d \) such subgroups (generated by \( ts^d \) with \( 0 \leq r < d \)). Although each \( Z \) is normal, none is characteristic: Aut\((C_\infty \times C_d)\) acts transitively on them.

Corollary 2.3. Let \( M \) be a closed manifold in the homotopy type of \( X_{d,q} \). For any pair of subgroups \( Z \) and \( Z' \) of index \( d \) in \( \pi_1(M) \), there exists \( \delta' \in \text{Homeo}(M) \) such that \( \delta'(Z) = Z' \).

Proof. Select a homotopy equivalence \( f : M \to X_{d,q} \). There are integers \( a \) and \( b \) such that \( f_0(Z) \) and \( f_0(Z') \) are generated respectively by \( ts^a \) and \( ts^b \) in \( \pi_1(X_{d,q}) \). By Proposition 2.2, there is \( \delta \in \text{Homeo}(M) \) with \( f \circ \delta = \delta^2 \circ f \). Define \( \delta' := \delta^{(b-a)(1-d)/2} \in \text{Homeo}(M) \). Note

\[
\delta'(f_0^{-1}(ts^a)) = f_0^{-1}(s_0^{(b-a)(1-d)}(ts^b)) = f_0^{-1}(ts^b) = f_0^{-1}(ts^b).
\]

Proof of Theorem 1.6. Conjugate subgroups of \( \text{Homeo}(S^1 \times S^n) \) give homeomorphic orbit spaces. Then, by Proposition 2.1, we can define a function \( \Phi \) given by homeomorphism classes of homotopy types of orbit spaces:

\[
\Phi : \mathcal{A}_T \to \mathcal{M}_{\text{TOP}}^{b/s}(S^1 \times S^n) \cup \{ \emptyset \} \quad \text{if } n = 1 \text{ or } n = 2k
\]
\[
\bigcup_{1 < d \leq \ell} \bigcup_{q \in \mathbb{Q}_d} \mathcal{M}_{\text{TOP}}^{b/s}(X_{d,q}) \quad \text{if } n = 2k - 1 \geq 3.
\]

Note \( \Phi([T_\ell]) = \{[S^1 \times S^n]\} = \mathcal{M}_{\text{TOP}}^{b/s}(S^1 \times S^n) \), where the latter equality follows from: classification of surfaces if \( n = 1 \), Thurston’s Geometrization Conjecture if \( n = 2 \) (see [And04]), and the topological surgery sequence [KS77] if \( n \geq 3 \) (use [FQ90] if \( n = 3 \)).

First suppose \( n = 1 \). Then, as noted above, \( \Phi \) is constant hence surjective. (Since \( \ell \) is odd, only the torus \( S^1 \times S^1 \) has \( \ell \)-fold cover \( S^1 \times S^1 \). That is, \( \Phi(\mathcal{A}_T) = \{[S^1 \times S^1]\} \).)

Let \( C \in \mathcal{A}_T \). There exists a choice of homeomorphism \( h : (S^1 \times S^1)/C \to S^1 \times S^1 \). Under the quotient map \( S^1 \times S^1 \to (S^1 \times S^1)/C \) composed with \( h \), the image of the fundamental group of \( S^1 \times S^1 \) is a subgroup \( Z \) of index \( \ell \) in \( \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} \). There exists a nontrivial homomorphism \( \phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/\ell \) such that \( Z = \text{Ker}(\phi) \). Write \( a := \phi(1, 0) \) and \( b := \phi(0, 1) \). Post-composition with an automorphism of \( \mathbb{Z}/\ell \) preserves the kernel \( Z \), so we may assume that either \( a = 1 \) or \( (a, b) = (0, 1) \). If \( a = 1 \) then define \( A := \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \). If \( (a, b) = (0, 1) \) then define \( A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). In any case, the unimodular matrix \( A \in GL_2(\mathbb{Z}/\ell) \) carries \((a, b)\) to \((1, 0)\). Observe \((1, 0)\) corresponds to the index \( \ell \) subgroup \( \ell \mathbb{Z} \times \mathbb{Z} \). There is \( \delta' \in \text{Homeo}(S^1 \times S^1) \) inducing \( A \) on fundamental group. Write \( h' := \delta' \circ h \). Then, by the lifting property of covering spaces, there exists a commutative diagram

\[
\begin{array}{rcl}
S^1 \times S^1 & \xrightarrow{\delta'} & S^1 \times S^1 \\
\downarrow /C & & \downarrow /T_\ell \\
(S^1 \times S^1)/C & \xrightarrow{h'} & S^1 \times S^1.
\end{array}
\]

The element \( h' \in \text{Homeo}(S^1 \times S^1) \) conjugates \( T_\ell \) into \( C \). Therefore \( \Phi \) is injective.
Now suppose \( n > 1 \) and that the orbit space of \((C) \in A^p_{\ell}\) is homeomorphic to \( S^1 \times S^n \), say by a homeomorphism \( h \). Since \( \pi_1(S^1 \times S^n) = C_{\infty} \) has a unique subgroup of index \( \ell \), by the lifting property of covering spaces, there exists a commutative diagram

\[
\begin{array}{ccc}
S^1 \times S^n & \xrightarrow{\tilde{h}} & S^1 \times S^n \\
\downarrow{C} & & \downarrow{T_\ell} \\
(S^1 \times S^n)/C & \xrightarrow{h} & S^1 \times S^n.
\end{array}
\]

In other words, there is \( \tilde{h} \in \text{Homeo}(S^1 \times S^n) \) that conjugates \( T_\ell \) into \( C \). Thus \( \Phi \) restricts to

\[
\Phi : A^p_{\ell} \setminus \{(T_\ell)\} \longrightarrow \begin{cases}
\emptyset & \text{if } n = 1 \text{ or } n = 2k \\
\prod_{1 \leq d \leq \ell} \prod_{q \in G_q} M^{h/s}_{\text{TOP}}(X_{d,q}) & \text{if } n = 2k - 1 \geq 3.
\end{cases}
\]

Next, we show that \( \Phi \) is surjective if \( n = 2k - 1 \geq 3 \). Let \( M \) be a closed manifold in the homotopy type of some example \( X_{d,q} \), say by a homotopy equivalence \( f \). There is a pullback diagram of covering spaces

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & S^1 \times S^n \\
\downarrow & & \downarrow{T_\ell} \\
M & \xrightarrow{f} & X_{d,q}.
\end{array}
\]

Let \( T \neq \text{id} \) be a covering transformation of \( \tilde{M} \). Since \( M^{h/s}_{\text{TOP}}(S^1 \times S^n) = \{ [S^1 \times S^n] \} \), there is a homeomorphism \( h : \tilde{M} \rightarrow S^1 \times S^n \). Then \( T_M := h \circ T \circ h^{-1} \) is an element of \( \text{Homeo}(S^1 \times S^n) \) of order \( d \) without fixed points. Hence \( M = \Phi(T_M) \) and \( \Phi \) is surjective.

Finally, we show that \( \Phi \) is injective if \( n = 2k - 1 \geq 3 \). Let \((C), (C') \in A^p_{\ell}\) have orbit spaces \( M, M' \) in the homotopy type of some example \( X_{d,q} \). Suppose there is a homeomorphism \( h : M' \rightarrow M \). Write \( \Pi := \pi_1(S^1 \times S^n) \). Consider the lifting problem

\[
\begin{array}{ccc}
S^1 \times S^n & \xrightarrow{\text{----}} & S^1 \times S^n \\
\downarrow{p'} & & \downarrow{p} \\
M' & \xrightarrow{h} & M.
\end{array}
\]

By Corollary 2.3, there exists \( \delta' \in \text{Homeo}(M) \) such that \( \delta'_p((h \circ p')_w(\Pi)) = p_w(\Pi) \). Note \( h' := \delta' \circ h : M' \rightarrow M \) satisfies \( (h' \circ p')_w(\Pi) = p_w(\Pi) \). Then, by the lifting property, there is \( \tilde{h}' \in \text{Homeo}(S^1 \times S^n) \) covering \( h' \) that conjugates \( C' \) to \( C \). Therefore \( \Phi \) is injective. \( \Box \)

See [Tha10] for the homotopy types of free \( C_p \)-actions on products of 1-connected spheres.

3. Classification of \( h \)-cobordism types

For the second stage, consider the subgroup \( S \ell(X) \) of \( \text{Wh}_1(\pi_1X) \) consisting of the Whitehead torsions of strongly inertial \( h \)-cobordisms, that is, the torsion \( \tau(W \rightarrow X) \) of any \( h \)-cobordism \( (W; X, X') \) such that the map \( X' \hookrightarrow W \rightarrow X \) is homotopic to a homeomorphism.

**Theorem 3.1.** Let \( M \) and \( X \) be closed connected topological manifolds of dimension \( n \geq 4 \). If \( n = 4 \) then assume \( \pi_1X \) is good in the sense of Freedman–Quinn [FQ90]. If \( M \) is homotopy equivalent to \( X \), then \( S \ell(M) \cong S \ell(X) \) as subgroups of \( \text{Wh}_1(\pi_1M) \cong \text{Wh}_1(\pi_1X) \).
This theorem is an affirmative answer to a question raised by Jahren–Kwasik [JK, §7]. Later, in Section 5, we shall develop the techniques needed to prove this theorem.

Next, for any compact manifold $X$, write $S^{b/s}_{\text{TOP}}(X)$ for the set of pairs $(M, f)$, where $M$ is a compact topological manifold and $f : M \to X$ is a homotopy equivalence that restricts to a homeomorphism $\partial f : \partial M \to \partial X$, taken up to $s$-bordism relative to $\partial X$. Assuming that the $s$-cobordism theorem applies, then $[M, f] = [M', f']$ if and only if $f'$ is homotopic to $f \circ h$ relative to $\partial X$ for some homeomorphism $h : M' \to M$. Then observe

$$A^{b/s}_{\text{TOP}}(X) = \hMod(X) \setminus S^{b/s}_{\text{TOP}}(X).$$

Here $S^{b/s}_{\text{TOP}}(X)$ has a canonical left action by the group $\hMod(X)$, which consists of homotopy equivalences $X \to X$ restricting to the identity on $\partial X$, taken up to homotopy rel $\partial X$.

The first step in proving Theorem 1.7 is an observation of Jahren–Kwasik [JK, §3]. In the definition of $S^{b/s}_{\text{TOP}}(X)$, weaken the equivalence relation “$s$-bordism” to “$h$-bordism.” Then the resulting set $S^{h}_{\text{TOP}}(X)$ has the structure of an abelian group, according to Ranicki [Ran92]. Hence $S^{h}_{\text{TOP}}(X)$ is more calculable; it also has a left setwise action of $\hMod(X)$.

**Proposition 3.2** (Jahren–Kwasik). Let $X$ be a closed connected topological manifold of dimension $n > 4$. If $n = 4$ then assume $\pi_1 X$ good in the sense of Freedman–Quinn [FQ90]. The set $S^{h}_{\text{TOP}}(X)$ has a canonical right action of the Whitehead group $\Wh_1(\pi_1 X)$, so that

$$S^{h}_{\text{TOP}}(X) = S^{h}_{\text{TOP}}(X) / \Wh_1(\pi_1 X).$$

The isotropy group of any element $[M, f] \in S^{h}_{\text{TOP}}(X)$ is the subgroup $S I(M)$. Also, the forgetful map $S^{h}_{\text{TOP}}(X) \to S^{h}_{\text{TOP}}(X)$ is equivariant with respect to the left action of $\hMod(X)$.

Only the isotropy group for $[M, f] = [X, \text{id}]$ is asserted in [JK, §3]; we prove the others.

**Proof.** Recall the canonical left action. Let $\gamma \in \hMod(X)$ and $[M, f] \in S^{b/s}_{\text{TOP}}(X)$. Define

$$\gamma \cdot [M, f] := [M, \gamma \circ f].$$

The left action on $S^{b/s}_{\text{TOP}}(X)$ has the same formula, so the forgetful map is equivariant.

Next, recall the canonical right action. Let $[M, f] \in S^{b/s}_{\text{TOP}}(X)$ and $\alpha \in \Wh_1(\pi_1 X)$. By realization, there is an $h$-cobordism $(W; M, M')$ with torsion $\tau(W \to M) = f_*^{-1}(\alpha)$. Define

$$[M, f] \cdot \alpha := [M', f \circ (M \leftarrow W \leftarrow M')]$$

This is well-defined in $S^{b/s}_{\text{TOP}}(X)$ since $(W; M, M')$ is unique up to homeomorphism rel $M$. Thus the forgetful map induces a function $S^{b/s}_{\text{TOP}}(X)/\Wh_1(\pi_1 X) \to S^{h}_{\text{TOP}}(X)$, a bijection.

Finally, we determine isotropy groups of the right action. Clearly $SI(M)$ fixes $[M, f]$. Suppose $[M, f] \cdot \alpha = [M, f]$. Abbreviate the homotopy equivalence $g_a := (M \leftarrow W \leftarrow M')$. Then $f \circ g_a$ is $s$-bordant to $f$. By the $s$-cobordism theorem, there exists a homeomorphism $h : M' \to M$ such that $f \circ g_a$ is homotopic to $f \circ h$. By post-composition with a homotopy inverse $f^{-1} : X \to M$ of $f$, we have $g_a$ is homotopic to $h$. Therefore $f_*^{-1}(\alpha) \in S I(M)$. □

In general, the above $h$-structure group decomposes when $X = S^1 \times Y$ [PR80, §4].

**Proposition 3.3** (Pedersen–Ranicki). Let $Y$ be a topological space, and let $m$ be an integer. There is a functorial isomorphism of Ranicki structure groups:

$$S^b_m(S^1 \times Y) \cong S^b_m(Y) \oplus S^b_{m-1}(Y).$$
Further suppose that $Y$ is a closed connected topological manifold of dimension $n - 1$. The total surgery obstruction of Ranicki [Ran92, Theorem 18.5] gives the identifications

$$S^h_{\text{top}}(S^1 \times Y) \xrightarrow{\sigma} S^h_{n+1}(S^1 \times Y) \quad \text{and} \quad S^h_{\text{top}}(I \times Y) \xrightarrow{\sigma} S^h_{n+1}(Y).$$

Since $\sigma$ exists for all dimensions $n$, by the Five Lemma applied to the 4-dimensional surgery sequence [FQ90, §11.3], we also have these bijections when $n = 4$ and $\pi_1 Y$ is finite.

The next two lemmas determine certain $S_*(Y)$ when $Y$ is a lens space of odd order.

**Lemma 3.4.** Let $d > 1$ be odd, select $q$ coprime to $d$, and let $k > 1$. Then $S^{kh}_{2k+1}(L_{d,q}^{2k-1}) = 0$.

**Proof.** Write $L^n := L_{d,q}^{2k-1}$. Consider the $s$- or $h$-algebraic surgery exact sequence [Ran92]:

$$L^{kh}_{2k+1}(C_d) \xrightarrow{\sigma^h_{2k}} S^{kh}_{2k+1}(L^n) \xrightarrow{H_{2k}(L^n; \mathbb{L}(1))} \xrightarrow{\sigma^h_{2k}} L^{kh}_{2k}(C_d).$$

First, since $d$ is odd, $L^{kh}_{2k+1}(C_d) = 0$ by Bak’s vanishing result [Bak75]. Next, we apply the Atiyah–Hirzebruch spectral sequence to the homological version of the normal invariants:

$$E^2_{i,j} = H_i(L^n; \mathbb{L}(1)) \Rightarrow H_{i+j}(L^n; \mathbb{L}(1)).$$

The coefficient group $L(1)$ vanishes for $j \leq 0$ or $j$ odd. Otherwise, it either is $\mathbb{Z}$ if $j \equiv 0 \pmod{4}$ or is $\mathbb{Z}/2$ if $j \equiv 2 \pmod{4}$. Note that $H_{even}(L^n; \mathbb{Z}) = 0$ and, since $d$ is odd, that $H_{even}(L^n; \mathbb{Z}/2) = 0$. Thus the diagonal entries $i + j = \text{even}$ are zero except along $i = 0$. Also note that $H_{odd}(L^n; \mathbb{Z}) \in \{0, \mathbb{Z}/d, \mathbb{Z}\}$ and, since $d$ is odd, that $H_{odd}(L^n; \mathbb{Z}/2) = 0$. Therefore, since the image of an odd-order group in either $\mathbb{Z}$ or $\mathbb{Z}/2$ is zero, in summary we obtain:

$$H_{2k}(L^n; \mathbb{L}(1)) = E^\infty_{0,2k} = E^0_{0,2k} = L(1)_{2k} = L_{2k}(1).$$

Thus the assembly map is injective, $\sigma^h_{2k} : L_{2k}(1) \rightarrow L^{kh}_{2k}(C_d)$. Hence $S^{kh}_{2k+1}(L^n) = 0$. □

**Lemma 3.5.** Let $d > 1$ be odd, select $q$ coprime to $d$, and let $k > 1$. Then $S^p_{2k}(L_{d,q}^{2k-1})$ is free abelian of rank $(d - 1)/2$. Moreover, $L^p_{2k}(C_d) \rightarrow S^p_{2k}(L_{d,q}^{2k-1})$ is injective with finite index.

**Proof.** Write $L^n := L_{d,q}^{2k-1}$. Consider the $p$-algebraic surgery exact sequence [Ran92]:

$$H_{2k}(L^n; \mathbb{L}(1)) \xrightarrow{\sigma^p} L^p_{2k}(C_d) \xrightarrow{S^p_{2k}(L^n)} H_{2k-1}(L^n; \mathbb{L}(1)) \xrightarrow{\sigma^p_{2k-1}} L^p_{2k-1}(C_d).$$

From the proof of Lemma 3.4, the edge map $L_{2k}(1) \rightarrow H_{2k}(L^n; \mathbb{L}(1))$ is an isomorphism, so $\sigma^p_{2k}$ is split injective. Also $\sigma^p_{2k-1}$ is zero, since it factors through $L^p_{2k-1}(C_d) = 0$ above. So we obtain an exact sequence of abelian groups:

$$0 \rightarrow \overline{L}^p_{2k}(C_d) \rightarrow S^p_{2k}(L^n) \rightarrow H_{2k-1}(L^n; \mathbb{L}(1)) \rightarrow 0.$$

Since $\mathbb{R}C_d = \mathbb{R} \times \prod^{(d-1)/2} \mathbb{C}$ as rings, the reduced $L$-group $\overline{L}^p_{2k}(C_d)$ is free abelian of rank $(d - 1)/2$, and it is detected by the projective multi-signature [Bak78]. From the same Atiyah–Hirzebruch spectral sequence as in the proof of Lemma 3.4, since $d$ is odd, note:

$$E^2_{i,j} = H_i(L^n; \mathbb{L}(1)) = \begin{cases} \mathbb{Z} & \text{if } i = 2k - 1, \text{ and } 4 \text{ divides } j > 0 \\ \mathbb{Z}/d & \text{if } 0 < i < 2k - 1 \text{ odd, } 4 \text{ divides } j > 0 \Rightarrow H_{i+j}(L^n; \mathbb{L}(1)) \\ 0 & \text{otherwise} \end{cases}$$

Then each $E^\infty_{i,j}$ is either zero or $\mathbb{Z}/\delta$ with $\delta | d$. Thus it follows that $H_{2k-1}(L^n; \mathbb{L}(1))$ is a finite abelian group of odd order. Therefore it remains to show that $S^p_{2k}(L^n)$ has no odd torsion.

---

1A more detailed analysis can show furthermore that $H_{2k-1}(L^n; \mathbb{L}(1)) \rightarrow H_{2k-1}(L^n; ko[\zeta^{1/4}])$ is an isomorphism.
The function $S_{\text{TOP}}^1(L^n) \to \mathbb{Q} \otimes_{\mathbb{Z}} R_{\hat{G}}^{1-1/p}$, defined by the difference of $p$-invariants, was shown by Wall to be injective [Wal99, Theorem 14E.7]. Later, Macko–Wegner promoted this function to a homomorphism of abelian groups and reproved its injectivity [MW11, Theorem 5.2]. So $S_{\text{TOP}}^1(L^n)$ is free abelian. By the Ranicki–Rothenberg exact sequences [Ran92, p327], $S_{25}^2(L^n) \to S_{25}^2(L^n)$ and $S_{25}^0(L^n) \to S_{25}^0(L^n)$ have kernels and cokernels of exponent two. Hence $S_{2k}^0(L^n)$ has no odd torsion; it is free abelian of rank $(d-1)/2$. □

Corollary 3.6. Let $d > 1$ be odd, select $q$ coprime to $d$, and let $k > 1$. Then the group $S_{\text{TOP}}^0(S^1 \times L_{d,q}^{2k-1})$ is free abelian of rank $(d-1)/2$. Moreover, the component homomorphism $\hat{L}_{2k}^p(C_d) \to L_{2k+1}^n(\pi_1 X_{d,q}) \to S_{\text{TOP}}^0(X_{d,q})$ of Wall realization is injective with finite index.

Proof. This is immediate from Proposition 3.3, Lemma 3.4, and Lemma 3.5. □

4. APPLICATION TO THE ‘DEHN TWIST’ HOMEOMORPHISM

Fix $n = 2k-1 \geq 3$. Recall the self-homeomorphism $\varepsilon$ of $X_{d,q} = S^1 \times L_{d,q}^n$ in Equation (1).

Lemma 4.1. Let $d > 1$ be an odd integer, and select $q$ coprime to $d$.

1. The self-map $\varepsilon$ induces the identity map on $\text{Wh}(\pi_1 X_{d,q})$ if $d$ is square-free.

2. The self-map $\varepsilon$ induces the identity map on $S_{\text{TOP}}^1(X_{d,q})$.

3. The self-map $\varepsilon^2$ induces the identity map on $S_{\text{TOP}}^2(X_{d,q})$.

The $d = 2$ case for Part (2) was a key technical assertion of Jahren–Kwasik [JK11, §4].

Proof of Lemma 4.1(1). On the fundamental group $\pi_1(X_{d,q}) = C_{\infty} \times C_d$, recall that $\varepsilon$ induces $(t^q, s^q) \mapsto (t^q, s^{q+q})$; it is the identity on the subgroup $C_d$, which is generated by $s$. Then, by Proposition 5.2(1), we obtain a commutative diagram whose rows are split exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Wh}_1(C_d) & \longrightarrow & \text{Wh}_1(\pi_1 X_{d,q}) & \longrightarrow & \text{Wh}_0(C_d) & \longrightarrow & 0 \\
& & \downarrow{\text{id}} & & \varepsilon & & \varepsilon & & \\
0 & \longrightarrow & \text{Wh}_1(C_d) & \longrightarrow & \text{Wh}_1(\pi_1 X_{d,q}) & \longrightarrow & \text{Wh}_0(C_d) & \longrightarrow & 0.
\end{array}
$$

Here $R := \mathbb{Z}[C_d]$, and $\varepsilon : R[t, t^{-1}] \to R[t, t^{-1}]$ restricts to ring maps $\varepsilon : R[t^{\pm 1}] \to R[t^{\pm 1}]$.

Now, the splitting of the epimorphism $\partial$ of Bass–Heller–Swan [Bas68, XII:7.4] is

$h : \text{Wh}_0(C_d) \to \text{Wh}_1(\pi_1 X_{d,q}) : [P] \mapsto [t : P[t, t^{-1}] \to P[t, t^{-1}]]$.

Here $P$ is a finitely generated projective $R$-module. Then note

$\varepsilon_*[P] = (\varepsilon \circ \partial \circ h)[P] = (\partial \circ \varepsilon_*)[t : P[t, t^{-1}] \to P[t, t^{-1}]]$.

Since $\varepsilon(t) = st$, and since $\varepsilon(s) = s$ implies $(R \leftarrow R[t, t^{-1}] \xrightarrow{\varepsilon} R[t, t^{-1}]) = (R \leftarrow R[t, t^{-1}])$,

$\varepsilon_*[t : P[t, t^{-1}] \to P[t, t^{-1}]] = [st : P[t, t^{-1}] \to P[t, t^{-1}]]$.

Recall the map $\partial$ in the localization sequence for $R[t] \to R[t, t^{-1}]$ [Bas68, IX:6.3]:

$\varepsilon_*[P] = \partial[st : P[t, t^{-1}] \to P[t, t^{-1}]] = [\text{Cok}(st : P[t] \to P[t])] = [P]$.

So $\varepsilon_* = \text{id}$ on $\text{Wh}_0(C_d)$. Moreover, in $\text{Wh}_1(\pi_1 X_{d,q})$ note

$\varepsilon_*[h[P]] = [s : P \to P] \in \text{Wh}_1(C_d)$

$d \cdot [s : P \to P] = [sd = 1 : P \to P] = 0$. □
Thus, since $\text{Wh}_1(C_d)$ is torsion-free [Mil66, Theorem 6.4], we obtain

$$\varepsilon_* = \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}$$
onumber

on $\text{Wh}_1(\pi_1X_{d,q}) = \text{Wh}_1(C_d) \oplus \text{Wh}_0(C_d)$.

Therefore $\varepsilon$ induces the identity automorphism on $\text{Wh}_1(\pi_1X_{d,q})$. \hfill \Box

**Proof of Lemma 4.1(2).** By Corollary 3.6, it suffices to show that $\varepsilon_* = \text{id}$ on $L^P_{2k}(C_d)$. Its definition is $\varepsilon_* := B \circ \varepsilon_0 \circ T$, in terms of the induced automorphism $\varepsilon_* : L^h_{2k+1}(C_\infty \times C_d) \to L^h_{2k+1}(C_\infty \times C_d)$, and $L^h_{2k+1}(C_\infty \times C_d)$, the epimorphism $B : L^h_{2k+1}(C_\infty \times C_d) \to L^h_{2k}(C_d)$, and its algebraic splitting $\mathcal{B} : L^h_{2k}(C_d) \to L^h_{2k+1}(C_\infty \times C_d)$ of Ranicki [Ran73a, Theorem 1.1]. Then, heavily using Ranicki’s notation and slightly modifying his proof of splitting [Ran73b, p. 134], note:

$$\varepsilon_*[\theta_1, \theta_2] = \varepsilon_*[\theta_1] \varepsilon_*[\theta_2]$$

Here, the equivalence classes are of various quadratic forms and formations. We only used that the $\mathbb{Z}[C_d]$-algebra map $\varepsilon_* : \mathbb{Z}[C_d][t, t^{-1}] \to \mathbb{Z}[C_d][t, t^{-1}]$ is graded of degree 0. \hfill \Box

**Proof of Lemma 4.1(3).** Observe $\varepsilon_*$ respects the Ranicki–Rothenberg exact sequence

$$\tilde{H}^n(C_2; \text{Wh}_1X_{d,q}) \to S^n_{\text{TOP}}(X_{d,q}) \to S^n_{\text{TOP}}(X_{d,q}) \to \tilde{H}^{n+1}(C_2; \text{Wh}_1X_{d,q}).$$

In particular, by Corollary 3.6, this restricts to an exact sequence

$$(3) \quad 0 \to H \to S^n_{\text{TOP}}(X_{d,q}) \to K \to 0$$

with $H$ finite abelian and $K$ free abelian. By Lemma 4.1(1,2), $\varepsilon_* = \text{id}$ on $H$ and $K$. Hence

$$\varepsilon_* = \begin{pmatrix} \text{id}_H & \nu \\ 0 & \text{id}_K \end{pmatrix}$$

on $S^n_{\text{TOP}}(X_{d,q}) = H \oplus \nu K$,

where $\nu : K \to H$ is a component of $\varepsilon_*$ and $\nu : K \to S^n_{\text{TOP}}(X_{d,q})$ is a choice of right-inverse of $S^n_{\text{TOP}}(X_{d,q}) \to K$. Since $2H = 0$, note $2\nu = 0$. Hence $\varepsilon_*^2 = \text{id}$ on $S^n_{\text{TOP}}(X_{d,q})$. \hfill \Box

We show that the homotopy-theoretic order of $\varepsilon$ divides $2dd$; see more in Proof of 6.4.

**Lemma 4.2.** The homeomorphism $\varepsilon^{2dd}$ is homotopic to the identity on $X_{d,q} = \mathbb{S}^1 \times L^n$.

**Proof.** Observe that the $d$-th power of $\varepsilon$ induces the identity on fundamental group:

$$\varepsilon^d : \mathbb{S}^1 \times L^n \to \mathbb{S}^1 \times L^n ; (z, [u_1, u_2, \ldots, u_k]) \mapsto (z, [z^du_1, zu_2, \ldots, zu_k]).$$

Each $1 \leq j \leq k$ has an isotopy of diffeomorphisms that lifts the generator of $\pi_1(SO_3) = C_2$:

$$(4) \quad \rho_j : \mathbb{S}^1 \times L^n \to \mathbb{S}^1 \times L^n ; (z, [u_1, \ldots, u_j, \ldots, u_k]) \mapsto (z, [u_1, \ldots, zu_j, \ldots, u_k]).$$

In the proof of [HJ83, Proposition 3.1], Hsiang–Jahren showed that each homotopy class $[\rho_j]$ has order $2d$ in the group $\pi_1(\text{Map} L^n, \text{id})$. Since $\mathbb{S}^1$ is a co-$H$-space and $\text{Diff} L^n$ is an $H$-space, the two multiplications on $\pi_1(\text{Diff} L^n, \text{id})$ are equal (and abelian), so

$$[\varepsilon^d] = [\rho_1]^d \circ [\rho_2] \circ \cdots \circ [\rho_k] = [\rho_1]^d [\rho_2] \cdots [\rho_k] \in \pi_1(\text{Diff} L^n, \text{id}).$$

Therefore $[\varepsilon^{2dd}] = [\varepsilon^d]^{2dd} = [\rho_1]^{2dd} [\rho_2]^{2dd} \cdots [\rho_k]^{2dd} = 1$ in $\pi_1(\text{Map} L^n, \text{id})$. \hfill \Box

Structure sets quantify homeomorphism types within a homotopy type, so we can start:
Proof of Proposition 2.2. Consider the homotopy equivalence \( \alpha := f^{-1} \circ e^2 \circ f : M \rightarrow M \), where \( f^{-1} \) denotes a homotopy inverse for \( f \). By the composition formula for Whitehead torsion [Mil66, Lemma 7.8], by topological invariance [Cha74], and by Lemma 4.1(1),

\[
\tau(\alpha) = \tau(f^{-1}) + f_*^{-1}(\tau(e^2) + e^2_\ast \tau(f)) = -f_*^{-1}(\tau(f) + f^{-1}_\ast(0 + \tau(f))) = 0 \in \text{Wh}_1(\pi_1 M).
\]

That is, \( \alpha \) is a simple homotopy equivalence, hence it defines an element \( [M, \alpha] \in S^l_{\text{TOP}}(M) \).

On the other hand, by Lemma 4.1(3) and Lemma 4.2, note

\[
\alpha_s = f_*^{-1} \circ e^2_\ast \circ \alpha_s = f_*^{-1} \circ \text{id} \circ \alpha_s = \text{id} : S^e_{\text{TOP}}(M) \rightarrow S^e_{\text{TOP}}(M)
\]

\[
\alpha^{dd} = f^{-1} \circ e^{2dd} \circ f = f^{-1} \circ \text{id} \circ f \cong \text{id} : M \rightarrow M.
\]

Then, by Ranicki’s composition formula for simple structure groups [Ran09], note

\[
\dd[M, \alpha] = \sum_{j=0}^{dd-1} [M, \alpha] = \sum_{j=0}^{dd-1} (\alpha_s)^j[M, \alpha] = [M, \alpha^{dd}] = [M, \text{id}] = 0 \in S^l_{\text{TOP}}(M).
\]

By Equation (3) and Corollary 3.6, \( S^l_{\text{TOP}}(M) \cong S^l_{\text{TOP}}(X_{d,q}) \) is a sum of copies of \( \mathbb{Z}/2 \) and \( \mathbb{Z} \). So, since \( d \) is odd, we must have \( [M, \alpha] = 0 \). That is, \( \alpha \) is \( s \)-bordant to the identity. Therefore, by the \( s \)-cobordism theorem, \( \alpha \) is homotopic to a self-homeomorphism \( \delta \). \( \square \)

5. Classification of homeomorphism types

We resume with the calculation of the isotropy subgroups \( SI(M) \) from Proposition 3.2.

Proposition 5.1. Let \( M \) be a closed connected topological manifold of dimension \( n \geq 4 \). If \( n = 4 \) then assume \( \pi_1 M \) is good in the sense of Freedman–Quinn [FQ90].

1. With respect to the standard involution on \( \text{Wh}_1(\pi_1 M) \) given by \( (g \mapsto g^{-1}) \),

\[
(-1)^\ast \text{-evens} \subset SI(M) \subset (-1)^\ast \text{-symmetrics}.
\]

Hence \( SI(M)/( -1)^\ast \text{-evens} \cong \hat{H}^N(C_2; \text{Wh}_1(\pi_1 M)) \), which is a sum of copies of \( \mathbb{Z}/2 \).

2. This quotient is expressible in structure groups (stacking in the \( I \)-coordinate):

\[
\frac{S^b_{\text{TOP}}(M \times I)}{S^l_{\text{TOP}}(M \times I)} \overset{\text{tors}}{\rightarrow} \frac{SI(M)}{(-1)^\ast \text{-evens}}.
\]

This quantification generalizes a specific argument given by Jahren–Kwasik [JK, §7]. Our structure sets are ‘rel \( \partial \)’ (homeomorphism on the unspecified boundary [Wal99, §0]).

Proof of Proposition 5.1(1). Let \( \alpha \in SI(M) \). There is a strongly inertial \( h \)-cobordism \((W; M, M')\) such that \( \alpha = \tau(W \rightarrow M) \). By the addition formula [Mil66, Lemma 7.8],

\[
0 = \tau(M \leftrightarrow W \rightarrow M) = \tau(W \rightarrow M) + (W \rightarrow M), \tau(M \leftrightarrow W).
\]

By Milnor duality [Mil66, §10], note

\[
\tau(M' \leftrightarrow W) = (-1)^\ast \tau(M \leftrightarrow W)^\ast.
\]

Then, since the \( h \)-cobordism is strongly inertial, by Chapman’s topological invariance of Whitehead torsion [Cha74], and by the addition formula again, note:

\[
0 = \tau(M' \leftrightarrow W \rightarrow M) = \tau(W \rightarrow M) + (W \rightarrow M), \tau(M' \leftrightarrow W) = \alpha + (-1)^\ast(W \rightarrow M), \tau(M \leftrightarrow W)^\ast = \alpha - (-1)^\ast \alpha^\ast.
\]

Thus \( SI(M) \leq (-1)^\ast \text{-symmetrics} \) in \( \text{Wh}_1(\pi_1 M) \).
Let $\beta \in \text{Wh}_1(\tau_1 M)$. There exists an $h$-cobordism $(W'; M, M'')$ with $\beta = \tau(W' \to M)$. Consider the untwisted double $W := W' \cup_{M'} -W'$. Note $(W; M, M)$ is a strongly inertial $h$-cobordism, since

$$(M \leftrightarrow W \to M) = (M \leftrightarrow -W' \to M'' \leftrightarrow W' \to M)$$

$$\simeq (M \leftrightarrow -\text{flip}_W W' \to M)$$

$$\simeq (M \overset{id} \to M).$$

This doubled $h$-cobordism has Whitehead torsion

$$\tau(W \to M) = \tau(W \to W' \to M)$$

$$= \tau(W' \to M) + (W' \to M), \tau(W \to W'')$$

$$= \beta + (M'' \leftrightarrow W' \to M), \tau(-W' \to M'')$$

$$= \beta + (-1)^n \tau(W' \to M)$$

$$= \beta + (-1)^n \beta''.$$ 

Thus $SI(M) \geq (-1)^n$-evens in $\text{Wh}_1(\tau_1 M)$.

\textit{Proof of Proposition 5.1(2).} Let $f : (W; M, M) \to M \times (I; 0, 1)$ be a homotopy equivalence of manifold triads such that the restriction $\partial f : \partial W \to M \times \partial I$ is the identity map. Since $f : W \to M \times I$ represents the retraction $W \to M$, the $h$-cobordism $(W; M, M)$ is strongly inertial. Then $\tau(f) = \tau(W \to M) \in SI(M)$.

Now, let $F : (V; W, W') \to M \times I \times (I; 0, 1)$ be an $h$-bordism, relative to $M \times \partial I \times I$, from $f$ to another homotopy equivalence $f' : (V'; M, M) \to M \times (I; 0, 1)$. By the addition formula [Mil66, Lemma 7.8], note:

$$\tau(M \leftrightarrow W \leftrightarrow V) = \tau(W \leftrightarrow V) + (W \leftrightarrow V), \tau(M \leftrightarrow W)$$

$$\tau(M \leftrightarrow W' \leftrightarrow V) = \tau(W' \leftrightarrow V) + (W' \leftrightarrow V), \tau(M \leftrightarrow W').$$

Also from the addition formula, as in the proof of Proposition 5.1(1), note:

$$\tau(M \leftrightarrow W) = -(M \leftrightarrow W), \tau(f)$$

$$\tau(M \leftrightarrow W') = -(M \leftrightarrow W'), \tau(f').$$

By Milnor duality [Mil66, §10], note

$$\tau(W' \leftrightarrow V) = (-1)^{n+1} \tau(W \leftrightarrow V).$$

Then

$$\tau(W \leftrightarrow V) - (M \leftrightarrow V), \tau(f) = (-1)^{n+1} \tau(W \leftrightarrow V) - (M \leftrightarrow V), \tau(f')$$

$$\tau(f) - \tau(f') = (M \leftrightarrow V), (-1)^{n+1} \tau(W \leftrightarrow V).$$

Thus we obtain a homomorphism of abelian groups, where addition in this relative structure set is given by stacking homotopy equivalences in the $I$-coordinate:

$$S^h_{\text{TOP}}(M \times I) \overset{\text{tors}}\to SI(M), (-1)^n \text{-evens} : [f] \mapsto [\tau(f)].$$

Let $\alpha \in SI(M)$. There is an $h$-cobordism $(W; M, M)$ such that $M \leftrightarrow W \to M$ is homotopic to the identity map. The Homotopy Extension Property produces a homotopy equivalence $f : (W; M, M) \to M \times (I; 0, 1)$ such that $\partial f : \partial W \to M \times \partial I$ is the identity map and $f : W \to M \times I$ represents the retraction $W \to M$. Then $[f] \in S^h_{\text{TOP}}(M \times I)$ and $\alpha = \tau(W \to M) = \tau(f)$. Therefore tors is surjective.
Finally, $\text{tors}[f] = 0$ if and only if $f : W \to M \times I$ is $h$-bordant to a simple homotopy equivalence. Thus the kernel of tors is the image of $S^1_{\text{TOP}}(M \times I)$.

The homotopy invariance of the subgroup $SI(X) \leq \text{Wh}_1(\pi_1 X)$ is now a corollary.

**Proof of Theorem 3.1.** The function tors is a homomorphism with respect to Ranicki’s abelian group structure on the structure sets. This follows from the commutative diagram with exact rows (using Proposition 5.1 and [Ran92, Theorem 18.5]):

\[
\begin{array}{cccccc}
S^e_{\text{TOP}}(X \times I) & \longrightarrow & S^0_{\text{TOP}}(X \times I) & \longrightarrow & SI(X)/(-1)^n\text{-evens} \\
\downarrow \cong & & \downarrow \cong & & \\
S^e_{n+2}(X \times I) & \longrightarrow & S^h_{n+2}(X \times I) & \longrightarrow & \widehat{H}^n(C_2; \text{Wh}_1(\pi_1 X)). \\
\uparrow \delta & & \uparrow \delta & & \uparrow \cong \\
L_e^h(X \times I) & \longrightarrow & L^h_{n+2}(X \times I) & \longrightarrow & \widehat{H}^{n+2}(C_2; \text{Wh}_1(\pi_1 X))
\end{array}
\]

The bottom two squares consist of homotopy-invariant functors from the category of spaces to the category of abelian groups; that is, if continuous functions of spaces are homotopic, then these functors induce equal homomorphisms of abelian groups.

Therefore, for any homotopy class of continuous function $f : M \to X$, the induced homomorphism $f_* : \widehat{H}^n(C_2; \text{Wh}_1(\pi_1 M)) \to \widehat{H}^n(C_2; \text{Wh}_1(\pi_1 X))$ restricts to a homomorphism $f_* : SI(M)/(-1)^n\text{-evens} \to SI(X)/(-1)^n\text{-evens}$. Hence the induced map $f_* : \text{Wh}_1(\pi_1 M) \to \text{Wh}_1(\pi_1 X)$ restricts to a map $f_* : SI(M) \to SI(X)$. If $f$ is a homotopy equivalence, then all of these induced maps are isomorphisms.

The following proposition is not original; it is merely a record. Recall $X_{d,q} = S^1 \times L^2_{d,q}$.

**Proposition 5.2.** Let $d > 1$ be a square-free odd integer. Select an integer $q$ coprime to $d$.

1. There is a canonical identification

\[
\text{Wh}_1(\pi_1 X_{d,q}) = \text{Wh}_1(C_d) \oplus \text{Wh}_0(C_d).
\]

2. The standard involution $(g \mapsto g^{-1})$ on $\text{Wh}_1(\pi_1 X_{d,q})$ restricts to the standard involution on $\text{Wh}_1(C_d)$ and to negative the standard involution on $\text{Wh}_0(C_d)$.

3. Furthermore, with respect to these restricted involutions:

\[
\frac{\text{Wh}_1(C_d)}{\text{symmetrics}} = 0 \quad \text{and} \quad \frac{\text{Wh}_0(C_d)}{\text{skew-evens}} = H_0(C_2; \text{Wh}_0(C_d)).
\]

**Proof.** Part (1) is the fundamental theorem of algebraic $K$-theory [Bas68, XII:7.3, 7.4b] combined with the vanishing of $NK_1(Z[C_d])$ for $d$ square-free [Har87].

Part (2) is the analysis of the restriction of the overall involution done in [Ran73b, p21].

For Part (3), by [Mil66, Lemma 6.7], the standard involution $(g \mapsto g^{-1})$ on $\mathbb{Z}[C_d]$ induces the identity on $\text{Wh}_1(C_d)$. Therefore $\text{Wh}_1(C_d)/\text{symmetrics} = 0$. The assertion about $\text{Wh}_0(C_d)$ is simply the definition of $H_0(C_2; \text{Wh}_0(C_d))$.

**Corollary 5.3.** Let $d > 1$ be square-free odd, select an integer $q$ coprime to $d$, let $k > 1$. Let $M$ be any closed topological manifold in the homotopy type of $X_{d,q}$. We can identify

\[
\frac{\text{Wh}_1(\pi_1 X_{d,q})}{SI(M)} = H_0(C_2; \text{Wh}_0(C_d)).
\]
Proof. By Theorem 3.1, we have $SI(M) = SI(X_{d,g})$ as subgroups of $\text{Wh}_1(\pi_1 X_{d,g})$.

The surgery exact sequence for $X_{d,g} \times I$ rel $\partial$ admits forgetful maps of decorations. Consider the commutative diagram with exact rows, which we write schematically:

$$
\begin{array}{cccccc}
H_{2k+2} & \longrightarrow & L^s_{2k+2} & \longrightarrow & S^s & \longrightarrow & H_{2k+1} \longrightarrow & L^s_{2k+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H_{2k+2} & \longrightarrow & L^h_{2k+2} & \longrightarrow & S^h & \longrightarrow & H_{2k+1} \longrightarrow & L^h_{2k+1}.
\end{array}
$$

(6)

By Ranicki’s version of Shaneson’s thesis [Ran73a], Bak’s vanishing result [Bak75], and Bak–Kolster’s vanishing result [BK82, Corollary 4.7], note the computations:

- $L^s_{2k+2}(C_\infty \times C_d) = L^s_{2k+2}(C_d) \oplus L^h_{2k+1}(C_d) = L^s_{2k+2}(C_d)$
- $L^h_{2k+2}(C_\infty \times C_d) = L^h_{2k+2}(C_d) \oplus L^h_{2k+1}(C_d) = L^h_{2k+2}(C_d)$
- $L^s_{2k+1}(C_\infty \times C_d) = L^s_{2k+1}(C_d) \oplus L^h_{2k}(C_d) = L^h_{2k}(C_d)$
- $L^h_{2k+1}(C_\infty \times C_d) = L^h_{2k+1}(C_d) \oplus L^h_{2k}(C_d) = L^h_{2k}(C_d)$.

Substituting, we may now consider the following commutative diagram of groups:

$$
\begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & L^s_{2k+2}(C_d)/H_{2k+2} & \longrightarrow & S^s & \longrightarrow & H_{2k+1}/L_{2k}(1) \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & L^h_{2k+2}(C_d)/H_{2k+2} & \longrightarrow & S^h & \longrightarrow & H_{2k+1}/L_{2k}(1) \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \widehat{H}^{2k+2}(C; \text{Wh}_1(C_d)) & \longrightarrow & S^h/\Sigma^s & \longrightarrow & 0 \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

(7)

Clearly, the right column of (7) is exact. Next, Bass showed that $\text{Wh}_1(C_d)$ is a free abelian group ([Mil66, Theorem 6.4]) and that the standard involution ($g \mapsto g^{-1}$) on $\mathbb{Z}[C_d]$ induces the identity on $\text{Wh}_1(C_d)$ ([Mil66, Lemma 6.7]). Then the subgroup of skew-symmetries in $\text{Wh}_1(C_d)$ is zero. So $\widehat{H}^{2k+3}(C; \text{Wh}_1(C_d)) = 0$. Recall the vanishing result above: $L^s_{2k+1}(C_d) = 0$. Therefore, by the Rothenberg sequence, the left column of (7) is exact. Then, finally, a diagram chase in (6) shows that the middle column of (7) is exact.

The generalized homology of a space cross a circle admits a canonical decomposition:

$$
H_{2k+1} = H_{2k+1}(X_{d,g}; \mathbb{L}(1)) = H_{2k+1}(L^s_{d,q} \mathbb{L}(1)) \oplus H_{2k}(L^s_{d,q} \mathbb{L}(1)).
$$

By naturality, the assembly map $H_{2k+1} \longrightarrow L^h_{2k+1}$ for $X_{d,g}$ is the direct sum of the assembly maps $H_{2k+1} \longrightarrow L^s_{2k+1}$ for $X_{d,g}$ and $H_{2k} = L_{2k}(1) \longrightarrow L^h_{d,q}$ for $L^s_{d,q}$ by (2). So the kernel of the assembly map $H_{2k+1} \longrightarrow L^h_{2k+1}$ for $X_{d,g}$ is the summand $H_{2k+1}(L^s_{d,q} \mathbb{L}(1)) \cong H_{2k+1}/L_{2k}(1)$. Therefore, by exactness of rows in (6), the top and middle rows of (7) are exact.
So, by the Nine Lemma, the bottom row of (7) is exact. Then, by Proposition 5.1,
\[ \frac{SI(X_{d,q})}{\text{evens}} = \mathbb{H}^{2k+2}(C_2; \text{Wh}_1(C_d)) = \text{symmetrics in Wh}_1(C_d) \]
Therefore, we obtain the formula
\[ SI(X_{d,q}) = \text{symmetrics in Wh}_1(C_d) \oplus \text{skew-evens in Wh}_0(C_d). \]
The calculation of Wh$_1(X_{d,q})/SI(X_{d,q})$ now follows from Proposition 5.2.

**Remark 5.4.** Proposition 3.2, Corollary 3.6, and Corollary 5.3 produce a based bijection
\[ \mathbb{Z}^{(d-1)/2} \times H_0(C_2; \text{Wh}_0(C_d)) \xrightarrow{\sim} \text{Sym}^{\text{b/s}}_0(X_{d,q}). \]

**6. Computation of the action of the group of self-equivalences**

For any topological space $Z$, write Map($Z$) for the topological monoid of continuous self-maps $Z \to Z$. Recall that $h\text{Mod}(Z) \subset \pi_0\text{Map}(Z)$ is the group of homotopy classes of self-homotopy equivalences. A pair $(X_1, X_2)$ of based topological spaces satisfies the Induced Equivalence property if $[f] \in h\text{Mod}(X_1 \times X_2)$ implies $[p_j \circ f \circ i_j] \in \text{hMod}(X_j)$ for both $j = 1, 2$, with based inclusion $i_j : X_j \to X_1 \times X_2$ and projection $p_j : X_1 \times X_2 \to X_j$. We slightly simplify the following result of P I Booth and P R Heath [BH90, Corollary 2.8].

**Theorem 6.1** (Booth–Heath). Let $X$ be a connected CW complex equipped with a co-H-space structure. Let $Y$ be a based connected CW complex with $[Y, X]_0 = 0 = [X \wedge Y, X]_0$. If $(X, Y)$ satisfies the Induced Equivalence Property, there is a split exact sequence of groups:

\[ 1 \longrightarrow [X, \text{Map}(Y)]_0 \longrightarrow \text{hMod}(X \times Y) \longrightarrow \text{hMod}(X) \times \text{hMod}(Y) \longrightarrow 1. \]

**Corollary 6.2.** Let $Y$ be a nonempty connected CW complex. Suppose that $\pi_1(Y)$ is finite. Then there is a natural decomposition of groups:

\[ \pi_1(\text{hMod}(S^1 \times Y)) = \pi_1(\text{Map}(Y)) \to (\text{hMod}(\text{S}^1 \times Y)). \]

Hence, each element of $\text{hMod}(S^1 \times Y)$ is splittable: it restricts to a self-equivalence of $1 \times Y$.

This is false without the hypothesis, since $\text{hMod}(S^1 \times S^1) = \text{GL}_2(\mathbb{Z}) \not\cong \mathbb{Z} \times (\mathbb{Z}/2 \times \mathbb{Z})$.

**Proof of Corollary 6.2.** The circle $X = S^1$ is a co-H-space, and it is a model of $K(\mathbb{Z}, 1)$. Note $[Y, X]_0 = H^1(Y; \mathbb{Z}) = 0$ and $[X \wedge Y, X]_0 = H^1(S^1 \wedge Y; \mathbb{Z}) = \tilde{H}_0(Y; \mathbb{Z}) = 0$. By Theorem 6.1, it remains to show that $(S^1, Y)$ satisfies the Induced Equivalence Property. Let $f : S^1 \times Y \to S^1 \times Y$ be a based homotopy equivalence.

On the one hand, to prove that $p_1 \circ f \circ i_1 : S^1 \to S^1$ is a homotopy equivalence, we must show that induced map on the Hopfian group $\pi_1(S^1) = \mathbb{Z}$ is surjective. Since $f_b$ is surjective, there exists $(a, b) \in \pi_1(S^1) \times \pi_1(Y)$ such that $f_b(a, b) = (t, 1)$, where $t$ generates $\pi_1(S^1)$. Then, since $\text{Hom}(\pi_1(Y, \pi_1S^1) = 1$, note $(p_1)_b(f_b(1, b)) = 1$. So $(p_1)_b(f_b(1, b)) = 1.$

On the other hand, $f$ induces an isomorphism on $\pi_n(S^1 \times Y) = \pi_n(Y)$ for all $n > 1$. Since $Y$ is a CW complex, by the Whitehead theorem, it remains to show that $p_2 \circ f \circ i_2$ is injective on the co-Hopfian group $\pi_1(Y)$. For all $b \in \pi_1(Y)$, recall $(p_1)_b(f_b(1, b)) = 1$. Then $(p_2 \circ f \circ i_2)_b(b) = 1$ if and only if $f_b(1, b) = 1$, and $b = 1$ if and only if $b = 1$ since $f_b$ is injective. \(\square\)

**Remark 6.3.** The corollary below is parallel to $p = 2$; Jahren–Kwasik [JK11, 3.5] showed

\[ \text{hMod}(S^1 \times \mathbb{R}_{\text{loc}}^{2k-1}) = \begin{cases} C_2 \times (C_2)^2 & \text{if } k \equiv 0 \pmod{2} \\ C_2 \times C_4 & \text{if } k \equiv 1 \pmod{2} \end{cases} \times (C_2 \times C_2). \]
Unlike below, the first factor (the $C_2$ on the left) is not represented by a diffeomorphism. The very last $C_2$ factor is represented by the diffeomorphism of $\mathbb{RP}^n$ that reflects in $\mathbb{RP}^{n-1}$.

**Corollary 6.4.** Let $d > 1$ be odd, $q$ coprime to $d$, and $k > 1$. We have a metabelian group

$$h\text{Mod}(S^1 \times L_{d,k}^{2d-1}) = A \rtimes (C_2 \times C_2),$$

where $A$ is abelian of order $2dd$ and $e := \gcd(2k, \varphi(d))$. Furthermore, the subgroup $A \rtimes C_2$ is generated by the three diffeomorphisms

$$
\begin{align*}
\rho : (z, [u]) &\mapsto (z, [u_1, u_2, \ldots, u_k]) \\
\sigma : (z, [u]) &\mapsto (z, [z^{e_1/d}u_1, z^{e_2/d}u_2, \ldots, z^{1/d}u_k]) \\
\tau &\mapsto \text{id}_{L_{d,k}}.
\end{align*}
$$

**Proof.** Since the fundamental group $\pi_1(L^n) = C_d$ is finite, by Corollary 6.2, we have

$$h\text{Mod}(S^1 \times L^n) = \pi_1\text{Map}(L^n) \rtimes (h\text{Mod}S^1 \times h\text{Mod}L^n).$$

The subgroup $h\text{Mod}(S^1)$ is generated by the homotopy class of the diffeomorphism $\tau \times \text{id}_{L^n}$. Since $d$ is odd, by [Coh73, (29.5)], any homotopy equivalence $h : L^n \to L^n$ is classified uniquely by the induced automorphism $h_0 : s \mapsto s^d$ on $\pi_1(L^n)$ where $d^k \equiv \deg(h) \pmod d$ and $\deg(h) = \pm 1$; any $a$ with $a^k \equiv \pm 1 \pmod d$ is induced by an equivalence $h_0 : L^n \to L^n$. That is, the homomorphism $\# : h\text{Mod}(L^n) \to \text{Out}(\pi_1(L^n)) = \text{Out}(C_d) = C_2$ is injective with image the subgroup $C_2$, since $a^2 \equiv \pm 1 \pmod d$ if and only if $a^2k \equiv 1 \pmod d$.

Consider the fibring sequence $\text{Map}_0(L^n) \to \text{Map}(L^n) \to L^n$, where $\text{Map}_0 \subseteq \text{Map}$ is the topological submonoid of basepoint-preserving self-maps. Since $\pi_2(L^n) = 0$, and since any unbased homotopy between two based self-maps of a connected CW complex is relatively homotopic to a based homotopy, there is an exact sequence of abelian groups:

$$1 \longrightarrow \pi_1\text{Map}_0(L^n) \longrightarrow \pi_1\text{Map}(L^n) \longrightarrow \pi_1(L^n) \longrightarrow 1.$$

On the one hand, Hsiang–Jahren [HJ83, Proposition 3.1] showed that the forgetful map $\pi_1\text{Diff}_0(L^n) \to \pi_1\text{Map}_0(L^n)$ is surjective with image of order $2d$ generated by the based homotopy class $[\rho]_0$ of the diffeomorphism $\rho$. On the other hand, since $e_0(t) = ts$, the unbased homotopy class $[e]$ of the diffeomorphism $e$ maps to the generator $s$ of $\pi_1(L^n)$. Therefore $\pi_1\text{Map}(L^n)$ is an abelian group of order $2dd$ generated by $[\rho]_0$ and $[e]$. \qed

The following ‘relative’ lemma is extracted from the proof of [HJ83, Proposition 3.1].

**Lemma 6.5** (Hsiang–Jahren). Consider any lens space $L^n := L(\ell; q_1, \ldots, q_k)$ with $\ell$ odd. Suppose $f, g : S^1 \times L^n \to L^n$ are maps with null-homotopic restrictions $S^1 \times S^1 \to S^1$. Then $f$ and $g$ are homotopic if and only if: the restrictions $1 \times L^n \to L^n$ are homotopic and the induced maps $f, g : S^1 \times S^n \to S^n$ of based covers are homotopic. \footnote{Observe that $\tilde{\rho} : S^1 \times S^n \to S^n$ is exactly the composition of the Gauss map $S^1 \times S^n \to S^{1+n}$ and the $(n-2)$-suspension of the Hopf map $\eta : S^1 \to S^2$. Therefore $\rho : S^1 \times L^n \to L^n$ is not homotopic to projection.}

**Corollary 6.6.** Let $n \geq 3$. The maps $\rho_j : S^1 \to \text{Map}(L^n)$ of Equation (4) are homotopic. \footnote{However, the maps $\rho_j$ are not smoothly isotopic, as follows. Consider the $n = 3$ case. As topological groups, $U(1) \times U(1) \to \text{Isom}Ld, q \to \text{Diff}Ld, q$. According to [HKMR12], the left inclusion has finite index, and the right inclusion is a homotopy equivalence. The $U(1) \times U(1)$ subgroup has coordinates $\rho_1, \rho_2 : S^1 \to \text{Isom}Ld, q$ given by $\rho_1(z), \rho_2(z) : (u, v) \mapsto (zu, v), (u, zv)$. Therefore $\rho_1, \rho_2 : S^1 \to \text{Diff}Ld, q$ are not homotopic.}
Proof. Let \( 1 \leq j \leq k \). The restriction \( \rho_j : S^1 \times 1 \rightarrow L^n \) is null-homotopic, since it lifts to a loop in the universal cover: \( z \mapsto (z, 0, \ldots, 0) \) if \( j = 1 \), or \((1, 0, \ldots, 0)\) otherwise if \( j \neq 1 \). Also, the restriction \( \rho_j : 1 \times L^n \rightarrow L^n \) is the identity. The induced map on based covers is 
\[ \hat{\rho}_j : S^1 \times S^n \rightarrow S^n ; (z, u) \mapsto (u_1, \ldots, u_j, \ldots, u_k). \]
Recall any two coordinates in \( C^k \) can be permuted isometrically and slowly \((0 \leq \theta \leq 1)\): 
\[ \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \left( e^{i \theta n} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 1 + e^{i \theta n} & 1 - e^{i \theta n} \\ 1 - e^{i \theta n} & 1 + e^{i \theta n} \end{pmatrix}. \]
Therefore, by Lemma 6.5, the maps \( \rho_j : S^1 \times L^n \rightarrow L^n \) are homotopic to one another. □

**Remark 6.7.** We can get a formula for the order of \([\varepsilon]\) in \( A \). Recall \([\varepsilon]^d = [\rho_1]^d[\rho_2] \cdots [\rho_k] \) by Equation (5). Also \([\rho_j]\) = \([\rho_1]\) in \( \pi_1 \text{Map}(L^n) \) by Corollary 6.6. Since \( \rho_1 = \rho \), we obtain 
\[ [\varepsilon]^d = [\rho]^{\gcd(q+k-1, 2d)}. \]

To find \( S^{h/2}_{\text{TOP}}(X_{d,q}) \), we now compute the action of the group \( \text{hMod}(X_{d,q}) \) on \( S^{h/2}_{\text{TOP}}(X_{d,q}) \).

**Proof of Theorem 1.7.** First, we show the order \( d \) subgroup of \( \text{hMod}(X_{d,q}) \) acts trivially. By Proof 6.4, this subgroup is generated by the classes \([\rho^2]\) and \([\varepsilon^2]\) of diffeomorphisms. Let \((M, f) \in S^{h/2}_{\text{TOP}}(X_{d,q});\) write \( f^{-1} : X_{d,q} \rightarrow M \) for a homotopy inverse of \( f : M \rightarrow X_{d,q} \).

For any element \([\phi] \in \text{hMod}(X_{d,q}),\) consider the pullback \( f^*[\phi] := [f^{-1} \circ \phi \circ f] \in \text{hMod}(M) \). Recall, by Proposition 2.2, that each pullback \( f^*[\varepsilon^2] \) is represented by a homeomorphism. Thus \([\varepsilon^2]\) acts trivially on the hybrid structure set \( S^{h/2}_{\text{TOP}}(X_{d,q}) \).

The overall argument for \([\rho^2]\) is similar but slightly simpler to that of \([\varepsilon^2]\) in Section 4. By the composition formula for Whitehead torsion [Mil66, Lemma 7.8], and since \( \rho_{\psi} = \text{id} \), \( \tau(f^* \rho) = \tau(f^{-1}) + f^{-1} \circ \tau(\rho) \circ f = 0 \in \text{Wh}_1(\pi_1 M) \).

So \((M, f^* \rho) \in S^h_{\text{TOP}}(M) \). Similar to Proposition 3.3, there is a direct sum decomposition 
\[ S^h_{\text{TOP}}(X_{d,q}) \cong S^h_{\text{TOP}}(1 \times L^n) \oplus S^h_{\text{TOP}}(L^n). \]

Since \( \rho \) restricts to \( \text{id} \) on \( 1 \times L^n \subset S^1 \times L^n \), there is an induced commutative diagram 
\[ \begin{array}{ccc}
0 & \longrightarrow & S^h_{\text{TOP}}(1 \times L^n) \\
\downarrow{[\rho_1]} & \searrow{[\rho_1]} & \downarrow{[\rho_1]} \\
0 & \longrightarrow & S^h_{\text{TOP}}(L^n)
\end{array} \]

The decomposition is compatible with those of \( L^*_1(\pi_1 X_{d,q}) \) and \( H^*_1(X_{d,q}; \mathbb{L}(1)) \), inducing 
\[ \begin{array}{ccc}
0 & \longrightarrow & \tilde{L}^h_{2k}(C_d) \\
\downarrow{[\rho_1]} & \searrow{[\rho_1]} & \downarrow{[\rho_1]} \\
0 & \longrightarrow & \tilde{L}^h_{2k}(L^n)
\end{array} \]

Recall from Proof 3.5 that \( H_{2k-1}(L^n; \mathbb{L}(1)) \) is an abelian group annihilated by a power of \( d \). A similar argument to that proof shows that \( S^h_{\text{TOP}}(L^n) \) has no \( \langle d \rangle \)-torsion.\(^5\) So \([\rho_1] = \text{id} \) on \( S^h_{\text{TOP}}(L^n) \). But \( S^h_{\text{TOP}}(1 \times L^n) = 0 \) by Lemma 3.4. Therefore \( \rho_1 = \text{id} \) on \( S^h_{\text{TOP}}(X_{d,q}) \). Then 
\[ (f^* \rho)^d = f^{-1} \circ \rho^2 \circ f = f^{-1} \circ \text{id} \circ f = \text{id} : S^h_{\text{TOP}}(M) \longrightarrow S^h_{\text{TOP}}(M) \]

\(^5\)This lack of \( \langle d \rangle \)-torsion is true for the \( h\)-structure group, despite that \( \tilde{L}^h_{2k}(C_d) \) may now have some 2-torsion.
Then, by Ranicki’s composition formula for simple structure groups [Ran09], note
\[
d[M, f^* \rho p] = \sum_{j=0}^{d-1} [M, f^* \rho p] = \sum_{j=0}^{d-1} (f^* \rho p)^j[M, f^* \rho p] = [M, (f^* \rho p)^d] = 0 = S^d_{\text{TOP}}(M).
\]
By Equation (3) and Corollary 3.6, \(S^d_{\text{TOP}}(M) \cong S^d_{\text{TOP}}(X, \omega)\) is a sum of copies of \(\mathbb{Z}/2 \oplus \mathbb{Z}\). So \([M, f^* \rho p] = 0\) since \(d\) is odd. That is, \(f^* \rho p\) is \(s\)-bordant to \(id\). By the \(s\)-cobordism theorem, \(f^* \rho p\) is homotopic to a homeomorphism. Thus \([p^2]\) acts trivially on \(S^d_{\text{TOP}}(X, \omega)\). Therefore, from Corollary 6.4, the order \(dd\) subgroup of \(h\text{Mod}(X, \omega)\) acts trivially.

Now, this induces a left action of the quotient group \(C_2 \times C_2 \times C_2\) on \(S^d_{\text{TOP}}(X, \omega)\). Thus, by Remark 5.4, we are done, since this group has order \(4e = 8 \gcd(k, \varphi(d)/2)\). □

**Remark 6.8.** Let \(p \neq 2\) be prime. This quotient group does not act with uniform isotropy, unlike the order \(pp\) subgroup. To conclude, we discuss the three generators of \(C_2 \times C_2 \times C_2\).

1. The above methods show that post-composition with \(\rho^p\) is the identity on the \(h\)-cobordism structure group. There may be a ‘cross-effect’ on the \(s\)-cobordism structure group, that is, a nonzero component of \(\rho^p\) from the free part of \(h\text{Mod}(X, \omega)\) to the \(2\)-torsion part. The author is unaware of the effect within \(H_0(C_2; \text{Cl}_p)\)-orbits.

2. Since complex conjugation \(-\) reverses orientation on the symmetric Poincaré complex \(\sigma^*(\mathbb{S}^1) \in L(C_\infty)\), post-composition with the diffeomorphism \(- \times \text{id}_{L_p}\) is negation\(^6\) on the \(h\)-cobordism structure group \(S^d_{\text{TOP}}(X, \omega) \cong S^p_{\text{TOP}}(L, \omega) = \mathbb{Z}^{(p-1)/2}\). Then \(- \times \text{id}_{L_p}\) must act freely away from the \(H_0(C_2; \text{Cl}_p)\)-orbit of the basepoint \([X, \omega, \text{id}]\) of \(S^d_{\text{TOP}}(X, \omega)\). But \(- \times \text{id}_{L_p}\) must fix \([X, \omega, \text{id}]\), since any two homeomorphisms \(M \longrightarrow X_{p, q}\) are \(s\)-bordant.\(^7\) So \(- \times \text{id}_{L_p}\) acts non-uniformly on \(S^d_{\text{TOP}}(X, \omega)\).

3. Let \(a\) be a primitive \(e\)-th root of unity in the field \(\mathbb{F}_p\). Recall, from Proposition 6.4, that the homotopy equivalence \(h_a : L_{p, q} \longrightarrow L_{p, q}\) uniquely induces \(s \longmapsto s^a\) on fundamental group. Note \(\text{id}_{L_p} \times h_a : X_{p, q} \longrightarrow X_{p, q}\) has zero Whitehead torsion, by the product formula, but the author suspects that \(\text{id}_{L_p} \times h_a\) is often non-representable by a homeomorphism of \(X_{p, q}\).\(^8\) On the other hand, the automorphism of \(S^d_{\text{TOP}}(X, \omega)\) induced by \(\text{id}_{L_p} \times h_a\) is identified with the automorphism of \(S^p_{\text{TOP}}(L, \omega) = \mathbb{Z}^{(p-1)/2}\) induced by \(h_a\), given by a permutation matrix \(\Pi_a\) of order \(e/2\) determined by \(a\). Both these issues complicate the systematic use of Ranicki’s composition formula:

\[
[(\text{id}_{L_p} \times h_a) \circ (f : M \longrightarrow X_{p, q})] = [\text{id}_{L_p} \times h_a] + \Pi_a[f] \in S^d_{\text{TOP}}(X, \omega) = \mathbb{Z}^{(p-1)/2}.
\]

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\(^6\)The error in the proof comes from the false assertion: Ranicki’s \(L\)-orientation of a manifold is preserved by tangential homotopy equivalences. Call a manifold \(w_1\text{-oriented}\) if an orientation is chosen on the \(\text{Ker}(w_1)\)-cover. The correction is: the \(L\)-orientation of a \(w_1\text{-oriented}\) manifold is preserved by \(w_1\text{-oriented}\) tangential homotopy equivalences. For example, the diffeomorphism \(- \times \text{id}_{\mathbb{R}^n}\) is tangential with \(\mu = +1\) but reverses \(w_1\)-orientation.

\(^7\)Suppose there exists \(\alpha \neq 0 \in H_0(C_2; \text{Cl}_p)\), for example if \(p = 29\) by Remark 1.4. It is unlikely that \(- \times \text{id}_{L_p}\) fixes \([X, \omega, \text{id}]\) since the \(h\)-cobordism \(W_{\alpha}\) on \(X_{p, q}\) with torsion \(\alpha \in W_1(C_\infty \times C_\infty)\) has projection \(\alpha \neq 0 \in W_0(\text{Cl}_p) = \text{Cl}_p\). So the \(h\)-cobordism is unlikely splitable along \(1 \times L_{p, q}\); compare with [FH73, 6.1, 6.3].

\(^8\)Using a splitting argument along \(1 \times L_{p, q}\); if \(\text{id}_{L_p} \times h_a\) is homotopic to a homeomorphism, then \(h_a\) is \(h\)-bordant to a homeomorphism, if and only if the Whitehead torsion \(\tau(h_a)\) is divisible by two in \(W_1(\text{Cl}_p) = \mathbb{Z}/2^{(p-3)/2}\). Note \(h_a\) is homotopic to a homeomorphism if and only if \(\tau(h_a) = 0\) [Coh73, §31], if and only if \(e = 2\) [Coh73, (30.1)].
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