FREE BOUNDARY REGULARITY NEAR THE FIXED BOUNDARY FOR THE FULLY NONLINEAR OBSTACLE PROBLEM

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ABSTRACT. The interior free boundary theory for linear elliptic operators in higher dimensions was developed by Caffarelli [Caf77] in the low regularity context. In these notes, the up-to-the boundary free boundary regularity is discussed for nonlinear elliptic operators based on a different approach.

1. INTRODUCTION

Caffarelli proved that if \( L \) is a linear uniformly elliptic operator and \( u \geq 0 \) solves
\[
L(D^2u) = \chi_{\{u>0\}} \quad \text{in } B_1
\]
then for \( x \in \Gamma = \partial\{u>0\} \cap B_1 \) with positive Lebesgue density for \( \{u = 0\} \), i.e. satisfying
\[
\liminf_{r \to 0^+} \frac{|B_r(x) \cap \{u = 0\}|}{|B_r(x)|} > 0
\]
there is a Lipschitz function \( g \) such that \( \Gamma \cap B_s(x) \) admits a representation with respect to \( g \) in a coordinate system for some \( s > 0 \). The Lipschitz regularity can be improved to \( C^1 \) and higher regularity follows (up to analyticity) via a theorem of Kinderlehrer and Nirenberg [KN77]. Caffarelli’s theorem is optimal in the sense that there exists a solution when \( L = \Delta \) for which there is a free boundary point with zero Lebesgue density for \( \{u = 0\} \) and in a neighborhood

\[\frac{1}{|B_r(x) \cap \{u = 0\}|} \geq ar^n \text{ for } r > 0 \text{ small enough is usually assumed for convenience; } \limsup_{r \to 0^+} \frac{|B_r(x) \cap \{u = 0\}|}{|B_r(x)|} > 0 \text{ is sufficient.} \]
of the point the free boundary develops a cusp singularity and is not a graph in any system of coordinates [Sch77].

In a recent work [Ind19], the author proved that for solutions of (1.1) with zero Dirichlet boundary data, with $L$ replaced by a convex fully nonlinear uniformly elliptic operator $F$, if $x \in \partial B_1 \cap \Gamma$, then $\Gamma$ can be represented as the graph of a $C^1$ function in a neighborhood of $x$. There are two surprising differences between the interior and boundary result: first, there are no density assumptions in the boundary case (in particular, cusp-type singularities do not exist); second, there is an example which generates a free boundary which is $C^1$ with a specific Dini modulus of continuity for the free normal (see e.g. [PSU12, Remark 8.8]).

In his original approach, Caffarelli estimated pure second derivatives from below. The linear approach developed thereafter to handle regularity near the fixed boundary involves specific barrier constructions involving the operator and monotonicity formulas [Ura96, SU03, And07]. The nonlinear method is based on understanding a maximal mixed partial derivative along a preferred direction.

In what follows, $F$ satisfies

- $F(0) = 0$.
- $F$ is uniformly elliptic with ellipticity constants $\lambda_0, \lambda_1 > 0$ such that

$$\mathcal{P}^-(M - N) \leq F(M) - F(N) \leq \mathcal{P}^+(M - N),$$

where $M$ and $N$ are symmetric matrices and $\mathcal{P}^\pm$ are the Pucci operators

$$\mathcal{P}^-(M) := \inf_{\lambda_0 \leq N \leq \lambda_1} \text{tr}(NM), \quad \mathcal{P}^+(M) := \sup_{\lambda_0 \leq N \leq \lambda_1} \text{tr}(NM).$$

- $F$ is convex and $C^1$.

Let $\Omega$ be an open set and $B^+_r = \{x : |x| < r, x_n > 0\}$. A continuous function $u$ belongs to $P^+_r(0, M, \Omega)$ if $u$ satisfies in the viscosity sense

- $F(D^2 u) = \chi_{\Omega}$ in $B^+_r$;
• \(||u||_{L^\infty(B^+_1)} \leq M;\)

• \(u = 0\) on \(\{x_n = 0\} \cap \overline{B^+_1} =: B'_1.\)

In [IM16a] it was shown that \(W^{2,p}\) solutions are \(C^{1,1}\) (see also [FS14, IM16b] for the interior case). Furthermore, given \(u \in P^+_r(0, M, \Omega),\) the free boundary is denoted by \(\Gamma = \partial \Omega \cap B^+_1.\)

A blow-up limit of \(\{u_j\} \subset P^+_1(0, M, \Omega)\) is a limit of the form

\[
\lim_{k \to \infty} \frac{u_{j_k}(s_k x)}{s_k^2},
\]

where \(\{j_k\}\) is a subsequence of \(\{j\}\) and \(s_k \to 0^+.\)

In §2 non-transversal intersection is shown for \(\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}\) and a problem in superconductivity is discussed in which \(\Omega = \{\nabla u \neq 0\} \cap \{x_n > 0\};\) in §3 \(C^1\) regularity is proved when \(u \geq 0;\) last, some of the technical details are shown in the appendix §4.

2. Non-transversal intersection and classification of blow-up limits

One of the main results discussed in this section is the following.

**Theorem 2.1.** There exists \(r_0 > 0\) and a modulus of continuity \(\omega\) such that

\[
\Gamma(u) \cap B^+_r \subset \{x : x_n \leq \omega(|x'|)|x'|\}
\]

for all \(u \in P^+_1(0, M, \Omega)\) provided \(0 \in \overline{\Gamma(u)}\) and \(\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}.\)

If one varies the boundary data, then non-transversal intersection may not hold [And07] see Examples 3 & 4]. The difficulty in the fully nonlinear context is that monotonicity formulas are not available and a classification of blow-up limits requires a new approach: if blow-up limits are not half-space solutions, then a certain regularity property holds. More precisely:
Proposition 2.2. Suppose $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}$ and $\{u_j\} \subset P_1^+(0,M,\Omega)$. If $0 \in \{u_j \neq 0\}$ and $\nabla u_j(0) = 0$, then one of the following is true:

(i) all blow-up limits of $\{u_j\}$ at the origin are of the form $u_0(x) = bx_n^2$ for $b > 0$;

(ii) there exists $\{u_k_j\} \subset \{u_j\}$ such that for all $R \geq 1$, there exists $j_R \in \mathbb{N}$ such that for all $j \geq j_R$,

$$u_k_j \in C^{2,\alpha}(B_{r_j}^+),$$

where the sequence $\{r_j\}$ depends on $\{u_j\}$.

The proof relies on the fact that if not all blow-up solutions are half-space solutions, then one can construct a specific sequence producing a limit of the form $ax_1x_n + bx_n^2$.

Proposition 2.3. Let $\{u_j\} \subset P_1^+(0,M,\Omega)$ and suppose $0 \in \{u_j \neq 0\}$, $\{\nabla u_j \neq 0\} \cap \{x_n > 0\} \subset \Omega$, $\nabla u_j(0) = 0$. Then one of the following is true:

(i) all blow-up limits of $\{u_j\}$ at the origin are of the form $u_0(x) = bx_n^2$ for some $b > 0$;

(ii) there exists a blow-up limit of $\{u_j\}$ of the form $ax_1x_n + bx_n^2$ for $a \neq 0$, $b \in \mathbb{R}$.

Proof. Let

$$N := \limsup_{|x|\to 0, x_n > 0} \sup_{x_n \in \{u_j\}} \sup_{e \in S^{n-2} \cap e_n^+} \partial_e u(x),$$

and consider a sequence $\{x^k\}_{k \in \mathbb{N}}$ with $x^k_n > 0$, $u_{jk} \in \{u_j\}$, and $e^k \in S^{n-2} \cap e_n^+$ such that the previous limit is given by

$$\lim_{k \to \infty} \frac{1}{x^k_n} \partial_{e^k} u_{jk}(x^k).$$

Note that $N < \infty$ by $C^{1,1}$ regularity for the class $P_1^+(0,M,\Omega)$ and the boundary condition (see [IM16a]). By compactness, $e^k \to e_1 \in S^{n-2}$ (along a subsequence) so that up to a rotation,

$$N = \lim_{k \to \infty} \frac{1}{x^k_n} \partial_{e_1} u_{jk}(x^k).$$
Next, if

$$\tilde{u}_j(x) := \frac{u_{k_j}(s_j x)}{s_j^2} \rightarrow u_0(x)$$

for some sequence $s_j \rightarrow 0^+$, where the convergence is in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}_+^n)$ for any $\alpha \in [0, 1)$, $u_0 \in C^{1,1}(\mathbb{R}_+^n)$ satisfies the following PDE in the viscosity sense

$$\begin{cases}
F(D^2 u_0) = 1 & \text{a.e. in } \mathbb{R}_+^n \cap \Omega_0 \\
|\nabla u_0| = 0 & \text{in } \mathbb{R}_+^n \setminus \Omega_0 \\
u = 0 & \text{on } \mathbb{R}_+^{n-1},
\end{cases}$$

where $\Omega_0 = \{\nabla u_0 \neq 0\} \cap \{x_n > 0\}$. Note that

$$\frac{N}{\sqrt{v}} \geq \lim_{k \rightarrow \infty} \frac{\partial_{x_i} u_{k_j}(s_j x)}{s_j^{x_n}} \geq \lim_{k \rightarrow \infty} \frac{\partial_{x_i} \tilde{u}_j(x)}{x_n} = \frac{\partial_{x_i} u_0(x)}{x_n}$$

for all $i \in \{1, \ldots, n-1\}$. If $N = 0$, then $\partial_{x_i} u_0 = 0$ for all $i \in \{1, \ldots, n-1\}$ so that $u_0(x) = u_0(x_n)$ and the conditions readily imply $u_0(x_n) = bx_n^2$. Since $N$ does not depend on the sequence $\{s_j\}$ it follows that in this case all blow-up limits have the previously stated form. Suppose that $N > 0$, let $r_k = |x^k|$, and consider the re-scaling of $u_{j_k}$ with respect to $r_k$. Note that along a subsequence, $y^k := \frac{x^k}{r_k} \rightarrow y \in S^{n-1}$. By the choice of $r_k$,

$$\lim_{k \rightarrow \infty} \frac{v(y^k)}{y_n^k} = \lim_{k \rightarrow \infty} \frac{\partial_{x_1} u_k(y^k)}{y_n^k} = \lim_{k \rightarrow \infty} \frac{\partial_{x_1} u_{j_k}(r_k y^k)}{r_k y_n^k} = N,$$

where $v = \partial_{x_1} u_0$. In particular,

$$v(y) = Ny_n$$

and by an argument in [IM16a] (involving the boundary Harnack inequality), $u_0(x) = ax_1 x_n + bx_n^2$ with $a \neq 0$. \hfill \Box

**Proof of Proposition 2.2.** Either all blow-up limits are of the form $u_0(x) = bx_n^2$ or there exists a subsequence

$$\tilde{u}_j(x) = \frac{u_{k_j}(r_j x)}{r_j^2}$$
producing a limit of the form \( u_0(x) = ax_1x_n + bx_n^2 \) for \( a > 0 \) (up to a rotation).

Let \( c = c(a, b) \) be the constant from Lemma 4.4 and note that since \( \tilde{u}_j \to u_0 \) in \( C^{1,\alpha}_{loc} \), there exists \( j_0 = j_0(a, R) \in \mathbb{N} \) such that for every cylinder \( S_{(\alpha, \beta)}(e_1) \) there exists \( x \in S_{(\alpha, \beta)}(e_1) \cap B_R^+ \) such that \( |\nabla \tilde{u}_j(x)| \geq \frac{c}{2} \) for all \( j \geq j_0 \), where \( R \geq 1 \).

Choose a constant \( C_0 = C_0(a, b, R) > 0 \) such that
\[
C_0 \partial_{x_1} u_0 - u_0 \geq 0
\]
in \( B_R^+ \) and \( j' \geq j_0 \) for which
\[
(2.3) \quad C_0 \partial_{x_1} \tilde{u}_j - \tilde{u}_j \geq 0 \quad \text{in } B_{\frac{R}{2}}^+
\]
whenever \( j \geq j' \) by Lemma 4.1. Now fix \( j \geq j' \) and suppose \( z \in \Gamma_i(\tilde{u}_j) \cap B_{\frac{R}{2}}^+ \).

Then there exists a ball \( B \subset \text{int}\{\tilde{u}_j = 0\} \cap B_{\frac{R}{2}}^+ \) and a cylinder \( S \) in the \( e_1 \)-direction generated by \( B \). Now select \( x \in S \cap B_R^+ \) for which \( |\nabla \tilde{u}_j(x)| > 0 \) and \(-R < x_1 < -R/2\). In particular, there exists a small ball around \( x \), say \( \tilde{B} \) such that \( F(D^2 \tilde{u}_j) = 1 \) in \( \tilde{B} \) and one may assume \( \tilde{B} \subset \{\tilde{u}_j \neq 0\} \). Note that \( \tilde{B} \) is contained in the cylinder \( S \) and let \( E_t = \tilde{B} + te_1 \) for \( t \in \mathbb{R} \). If \( t > 0 \) is such that \( \overline{E_t} \cap \{\tilde{u}_j = 0\} \neq \emptyset \), and for all \( 0 \leq s < t \), \( E_s \cap \{\tilde{u}_j = 0\} = \emptyset \), choose \( y \in \overline{E_t} \cap \{\tilde{u}_j = 0\} \). If \( \tilde{u}_j > 0 \) in \( \tilde{B} \), then by (2.3) it follows that \( \tilde{u}_j \) is strictly positive at a point in \( \{\tilde{u}_j = 0\} \), a contradiction. Thus \( \tilde{u}_j < 0 \) in \( \tilde{B} \). By convexity of \( F \)
\[
a_{kl} \partial_{kl} \tilde{u}_j \geq 0 \quad \text{in } E_t.
\]
Since \( 0 = \tilde{u}_j(y) > \tilde{u}_j(x) \) for \( x \in E_t \) and \( y \) satisfies an interior ball condition, then Hopf’s lemma implies that \( \frac{\partial}{\partial n} \tilde{u}_j(y) > 0 \), where \( n \) is the outer normal to the ball at \( y \). If there exists \( z \in B_\delta(y) \) such that \( \tilde{u}_j(z) > 0 \), then this contradicts the monotonicity, if \( \delta > 0 \) is sufficiently small: \( \overline{E_{\eta}} \subset B \subset \text{int}\{\tilde{u}_j = 0\} \) for \( \eta > 0 \) large enough and since \( \tilde{u}_j(z) > 0 \), the monotonicity (2.3) implies that \( \tilde{u}_j(z + e_1s) > 0 \), for some \( s > 0 \) such that \( z + e_1s \in \{\tilde{u}_j = 0\} \). Hence, \( \tilde{u}_j \leq 0 \) on \( B_\delta(y) \) and thus
$\nabla \tilde{u}_j(y) = 0$, a contradiction. The conclusion is that for $j \geq j_0'$,

$$\Gamma_i(\tilde{u}_j) \cap B_{R_j}^+ = \emptyset.$$ 

In particular, $(B_{R_j}^+ \setminus \Omega_j)^\circ = \emptyset$ and non-degeneracy implies that $|B_{R_j}^+ \setminus \Omega_j| = 0$. Thus the $C^{1,1}$ function $\tilde{u}_j$ satisfies $F(D^2\tilde{u}_j) = 1$ in $B_{R_j}^+$ in the viscosity sense and the up to the boundary Evans-Krylov theorem (see e.g. [Saf94]) implies that $\tilde{u}_j \in C^{2,\alpha}(B_{R_j}^+)$. In particular, $u_{k_j} \in C^{2,\alpha}(B_{R_{k_j}}^+)$.

\textbf{Theorem 2.4.} Suppose $u \in P_1^+(0, M, \Omega)$ and $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}$. If $0 \in \{u \neq 0\}$ and $\nabla u(0) = 0$, then the blow-up limit of $u$ at the origin has the form

$$u_0(x) = ax_1x_n + bx_n^2$$

for $a, b \in \mathbb{R}$.

Proof. By Proposition 2.2, either $u_0(x) = bx_n^2$ or $D^2u(0)$ exists and the rescaling of $u$ is given by

$$u_j(x) = \frac{u(r_jx)}{r_j^2} = \langle x, D^2u(0)x \rangle + o(1).$$

Since $u_0(x', 0) = 0$ for $x' \in \mathbb{R}^{n-1}$, it follows that $u_0$ has the claimed form (up to a rotation).

\textbf{Theorem 2.5.} Suppose $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}$, $0 \in \Gamma$, and $\{u_j\} \subset P_1^+(0, M, \Omega)$. Then the blow-up limit of $\{u_j\}$ at the origin has the form

$$u_0(x) = bx_n^2$$

for $b > 0$.

Proof. By Proposition 2.2, either $u_0(x) = bx_n^2$ or there exists a subsequence

$$u_{k_j}(x) \in C^{2,\alpha}(B_{R_{k_j}}^+).$$
which contradicts that $F$ is continuous (consider a sequence of points approaching the free boundary from the set where the equation is satisfied with the right-hand-side being equal to one and from the complement).

Remark 2.1. There exist global solutions which are not blow-up solutions (at contact points).

**proof of Theorem 2.1.** It suffices to show that for any $\epsilon > 0$ there exists $\rho_\epsilon > 0$ such that $\Gamma(u) \cap B_{1/\rho_\epsilon}^+ \subset B_{1/\rho_\epsilon}^+ \setminus C_\epsilon$, where $C_\epsilon = \{ x_n > \epsilon | x' | \}$. If not, then there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $u_k \in P_1^+(0, M, \Omega)$ with

$$\Gamma(u_k) \cap B_{1/k}^+ \cap C_\epsilon \neq \emptyset,$$

where $0 \in \overline{\Gamma(u_k)}$. If all blow-ups of $\{u_k\}$ are half-space solutions. Let $x_k \in \Gamma(u_k) \cap B_{1/k}^+ \cap C_\epsilon$ and set $y_k = \frac{x_k}{r_k}$ with $r_k = | x_k |$. Consider $\tilde{u}_k(x) = \frac{u_k(r_k x)}{r_k^2}$ so that $y_k \in \Gamma(\tilde{u}_k)$, $\tilde{u}_k \to bx_n^2$, $y_k \to y \in \partial B_1 \cap C_\epsilon$ (up to a subsequence), and $y \in \Gamma(u_0)$, a contradiction. Second, select a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that for all $j \geq j_2$, $u_{k_j} \in C^{2,\alpha}(B_{2j_2}^+)$, where $j_2 \in \mathbb{N}$ and the sequence $\{r_j\}$ depends on $\{u_k\}$. Since $0 \in \overline{\Gamma(u_{k_j})}$, there exists

$$x_j \in \Gamma(u_{k_j}) \cap B_{2j_2}^+$$

which contradicts the continuity of $F$ (consider a sequence of points approaching the free boundary from the set where the equation is satisfied with the right-hand-side being equal to one and from the complement). □

2.1. **An obstacle problem in superconductivity.** Equations of the type

$$F(D^2 u, x) = g(x, u)\chi_{\{ \nabla u \neq 0 \}}$$

have been investigated in [CS02] and are based on physical models, e.g. the stationary equation for the mean-field theory of superconducting vortices when the scalar stream is a function of the scalar magnetic potential [Cha95, CRS96, ESS98]. It is shown that in certain configurations in two dimensions, the set
\{(\nabla u = 0)\} is convex. In a recent paper [Ind18], the author proved non-transversal intersection for \( \Omega = \{(\nabla u \neq 0) \cap \{x_2 > 0\} \). If \( \{u < 0\} \) has sufficiently small density, non-transversal intersection follows from the techniques discussed above without a dimension restriction: suppose
\[
\frac{|\{u < 0\} \cap B_r^+|}{|B_r^+|} \to 0
\]
as \( r \to 0^+ \). A limit of the form
\[
u_0(x) = \lim_{k \to \infty} \frac{u_{jk}(s_kx)}{s_k^2}
\]
satisfies \( u_0 \geq 0 \) and therefore cannot be \( ax_1x_n + bx_n^2 \) for \( a \neq 0 \). In particular, it must be a half-space solution by Proposition 2.3 and the non-transversal intersection follows as before. The assumption on the negativity set appeared in [MM04] where the authors considered the non-transversal intersection subject to additional assumptions on the operator and solution.

3. \( C^1 \) Regularity

In the physical case when \( u \geq 0 \), the free boundary is \( C^1 \) without density assumptions.

**Theorem 3.1.** Let \( u \in P_1^+(0, M, \Omega) \) be non-negative, \( \Omega = \{(u \neq 0) \cup \{\nabla u \neq 0\}\} \cap \{x_n > 0\} \), and \( 0 \in \overline{\Gamma(u)} \). There exists \( r_0 > 0 \) such that \( \Gamma \) is the graph of a \( C^1 \) function in \( B_{r_0}^+ \).

**Proof.** First, for any \( \epsilon > 0 \) there exists \( r(\epsilon, M) > 0 \) such that if \( x^0 \in \Gamma(u) \cap B_{1/2}^+ \) and \( d = x_n^0 < r \), then
\[
\sup_{B_{2d}(x^0)} |u - h| \leq \epsilon d^2, \quad \sup_{B_{2d}(x^0)} |\nabla u - \nabla h| \leq \epsilon d,
\]
where
\[
h(x) = b[(x_n - d)^+]^2,
\]
and \( b > 0 \) depends on the ellipticity constants of \( F \). If not, then there exists \( \epsilon > 0 \), non-negative \( u_j \in P_1^+(0, M, \Omega) \), and \( x^j \in \Gamma(u_j) \cap B_{1/2}^+ \) with \( d_j = x^j_n \to 0 \), for which

\[
\sup_{B_{2d_j}(x^j)^+} |u_j - b[(x_n - d_j)^+]^2| > \epsilon d_j^2,
\]
or

\[
\sup_{B_{2d_j}(x^j)^+} |\nabla u_j - 2b(x_n - d_j)^+| > \epsilon d_j.
\]

Let \( \tilde{u}_j(x) = \frac{u_j((x^j)'+d_jx)}{d_j^2} \) so that in particular

\[
||\tilde{u}_j - h||_{C^1(B_2^+(e_n))} \geq \epsilon,
\]

where \( h(x) = b[(x_n - 1)^+]^2 \). Since \( \tilde{u}_j(e_n) = |\nabla \tilde{u}_j(e_n)| = 0 \), the \( C^{1,1} \) regularity of \( \tilde{u}_j \) implies that \( |\tilde{u}_j(x)| \leq C|x - e_n|^2 \). By passing to a subsequence, if necessary,

\[
\tilde{u}_j \to u_0
\]

where \( u_0 \in C^{1,1}(\mathbb{R}_n^+) \) satisfies the following PDE in the viscosity sense

\[
\begin{aligned}
F(D^2 u_0) &= 1 \quad \text{a.e. in } \mathbb{R}_n^+ \cap \Omega_0, \\
|\nabla u_0| &= 0 = u_0 \quad \text{in } \mathbb{R}_n^+ \setminus \Omega_0, \\
u_0 &= 0 \quad \text{on } \mathbb{R}_n^{n-1}.
\end{aligned}
\]

Now let

\[
N = \lim\sup_{|x| \to 0, x_n > 0} \frac{1}{x_n} \sup_{x_n \in \xi^+_1 \cap \{u \geq 0\}} \sup_{e \in S^{n-2} \cap e_n^\perp} \sup_{y \in B_{1/2}^+ \cap \{x_n = 0\}} \partial_e u(x + y)
\]

and note that \( N < \infty \) by \( C^{1,1} \) regularity and the boundary condition: for any \( e \in S^{n-2} \cap e_n^\perp \) and \( y \in B_{1/2}^+ \cap \{x_n = 0\} \), it follows that \( \partial_e u(x' + y) = 0 \). Furthermore,

\[
N \geq \lim_{j} \left| \frac{\partial_{x_j} u_j(d_j x + (x^j)')}{{d_j}^2} \right| = \lim_{j} \left| \frac{\partial_{x_j} \tilde{u}_j(x)}{x_n} \right| = \left| \frac{\partial_{x_n} u_0(x)}{x_n} \right|
\]
for all $i \in \{1, \ldots, n-1\}$. In particular, let $v = \partial_{x_i} u_0$ so that in $\mathbb{R}^n_+$,

\begin{equation}
|v(x)| \leq N x_n.
\end{equation}

If $N = 0$, then $\partial_{x_i} u_0 = 0$ for all $i \in \{1, \ldots, n-1\}$ and therefore $u_0(x) = u_0(x_n)$. Since $e_n$ is a free boundary point, it follows that $u_0 = h$, a contradiction. Thus $N > 0$ and there is a sequence $\{x^k\}_{k \in \mathbb{N}}$ with $x^k_n > 0$, $u_k \in P^+_1(0, M, \Omega)$, $u_k \geq 0$, $y^k \in \overline{B}_{1/2} \cap \{x_n = 0\}$, and $e^k \in S^{n-2} \cap e^+_n$ such that

$$N = \lim_{k \to \infty} \frac{1}{x^k_n} \partial_{e_k} u_k(x^k + y^k).$$

By compactness, $e^k \to e_1 \in S^{n-2}$ (along a subsequence) so that up to a rotation,

$$N = \lim_{k \to \infty} \frac{1}{x^k_n} \partial_{x_1} u_k(x^k + y^k).$$

Let

$$\tilde{u}_k(x) = \frac{u_k(y^k + r_k x)}{r^2_k},$$

where $r_k = |x^k|$, $z^k = \frac{x^k}{r_k}$, and note that along a subsequence $z^k \to z \in S^{n-1}$ and $\tilde{u}_k \to u_0$. It follows that $\partial_{x_1} u_0(z) = N z_n$ and proceeding as in [IM16a] one deduces that $u_0(x) = ax_1 x_n + c x_n + \tilde{b} x^2_n$ for $a \neq 0$ and $c, \tilde{b} \in \mathbb{R}$, contradicting that $u \geq 0$. This implies that in a neighborhood of the origin, there is a cone of fixed opening that can be placed below and above each free boundary point; therefore, the free boundary is Lipschitz continuous and thus $C^1$ by interior results [FS14, Theorem 1.3]. Since the intersection of $\Gamma$ and the origin occurs non-transversally, and

$$\sup_{B^+_{2d}(x^0)} |u - h| \leq \epsilon d^2, \quad \sup_{B^+_{2d}(x^0)} |\nabla u - \nabla h| \leq \epsilon d,$$

the aperture of the cones can be taken arbitrarily close to $\pi$. \hfill \Box

**Acknowledgement** The author wishes to thank Donatella Danielli and Irina Mitrea for organizing the AMS Special Session “Harmonic Analysis and Partial Differential Equations” at Northeastern University.
4. Appendix

**Lemma 4.1.** Let \( u \in P^+_{r}(0, M, \Omega) \) where \( \{u \neq 0\} \subset \Omega, e \in \mathbb{S}^{n-2} \cap \mathbb{S}^{n-1} \), and suppose there exist non-negative constants \( \epsilon_0, C_0 \) such that \( C_0 \partial_e u - u \geq -\epsilon_0 \) in \( B^+_r \). Then there exists \( c = c(n, \Lambda, r) > 0 \) such that if \( \epsilon_0 \leq c \), then \( C_0 \partial_e u - u \geq 0 \) in \( B^+_r \).

**Proof.** By convexity of \( F \), there exist measurable uniformly elliptic coefficients \( a_{ij} \) such that

\[
F(D^2 u(x + he)) - F(D^2 u(x)) \geq a_{ij} (\partial_{ij} u(x + he) - \partial_{ij} u(x))
\]

if \( x \in \Omega \) provided \( h \) is small enough. Therefore,

\[
0 \geq a_{ij} \partial_{ij} u \text{ in } \Omega.
\]

Convexity also yields

\[
a_{ij} \partial_{ij} u \geq F(D^2 u(x)) - F(0) = 1 \text{ in } \Omega.
\]

Suppose now that there exists \( y \in B^+_r \) for which \( C_0 \partial_e u(y) - u(y) < 0 \). Let \( w(x) = C_0 \partial_e u(x) - u(x) + \frac{|x - y|^2}{2n\Lambda} \). Since \( \Lambda Id \leq (a_{ij}) \leq \Lambda Id \), it follows by the above that \( Lw \leq 0 \) in \( \Omega \) where \( L = a_{ij} \partial_{ij} \). The maximum principle implies \( \min_{\partial(\Omega \cap B^+_r)} w = \min_{\Omega \cap B^+_r} w < 0 \). Note that \( w \geq 0 \) on \( \partial \Omega \) and likewise on \( \{x_n = 0\} \). Therefore, the minimum occurs on \( \partial B_r \) and thus \( 0 > -\epsilon_0 + \frac{1}{8n\Lambda} r^2 \), a contradiction if \( \epsilon_0 \) is small enough. \( \square \)

**Remark 4.2.** One may take \( \epsilon_0 = cr^2 \), where \( c > 0 \) depends only on the dimension and ellipticity constants of \( F \).

**Remark 4.3.** If \( u \geq 0 \), then \( \partial_{e_n} u \geq 0 \) on \( \{x_n = 0\} \cap B_r \) and Lemma 4.1 holds therefore in this case for all \( e \in \mathbb{S}^{n-1} \) such that \( e \cdot e_n \geq 0 \).
Lemma 4.4. Let $u_0(x) = ax_1x_n + bx_n^2$ with $a \neq 0$ and $R \geq 1$. Then there exists $c = c(a, b) > 0$ such that
\[
\inf_D |\nabla u_0(x)| \geq c,
\]
where $D = \{x = (x_1, x'', x_n) : R > |x| > R/2, |x''| \leq \delta(R)\}$ for some $\delta(R) > 0$.

Proof. Note $|\nabla u_0(x)|^2 = a^2x_n^2 + a^2x_1^2 + 2abx_1x_n + 4b^2x_n^2$ so that if $|x_n| > \frac{1}{3}$, then $|\nabla u_0(x)|^2 \geq \frac{a^2}{9}$. If $|x_n| \leq \frac{1}{3}$, then for points that satisfy $|x''| \leq \sqrt{\frac{5}{72}}R$, where $x'' = (x_2, x_3, \ldots, x_{n-1})$, it follows that
\[
x_1^2 > \frac{5}{72}R^2.
\]
If $b \neq 0$, let $\epsilon^2 \in (\frac{1}{a^2 + 4b^2}, \frac{1}{b^2})$. Then
\[
|\nabla u_0(x)|^2 \geq (a^2 + 4b^2 - \frac{1}{\epsilon^2})x_n^2 + (a^2 - \epsilon^2a^2b^2)x_1^2
\]
\[
> (a^2 - \epsilon^2a^2b^2)(\frac{5}{72}R^2).
\]
□

Lemma 4.5. Let $u_0(x) = ax_1x_n + bx_n^2$ with $a > 0$ and $R \geq 1$. Then there exists $C_0 = C_0(a, b, R) > 0$ such that
\[
C_0 \partial x_1u_0(x) - u_0(x) \geq 0
\]
in $B^+_R$.

Proof. The condition is equivalent to $ax_n(C_0 - x_1) \geq bx_n^2$. Since $x_1 \leq R$ and $0 \leq x_n \leq R$, it follows that any $C_0 \geq \frac{b}{a}R + R$ satisfies the condition. □

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