A note on the Blum-Hanson property of some contractions on $L^p$-spaces

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Abstract

If $\Omega$ is a measure space, we show that absolute contractions which are selfadjoint on $L^2(\Omega)$ induce contractions on $L^p(\Omega)$ which satisfy the Blum-Hanson property. Our very short argument relies on the use of noncommutative $L^p$-spaces.

1 Introduction

A (linear) contraction $T: X \to X$ on a Banach space $X$ has the Blum-Hanson property if for any $x, y \in X$ such that the sequence $(T^n(x))$ converges to $y$ for the weak topology, the sequence of means

$$\frac{1}{N} \sum_{k=1}^{N} T^{n_k}(x)$$

converges to $y$ in the norm topology for any strictly increasing sequence $(n_k)_{k \geq 1}$ of integers. The space $X$ is said to have the Blum-Hanson property if every contraction on $X$ has the Blum-Hanson property. This property has its origin in the paper [BlH60]. We refer to the survey [Gri19] and to the nice memoir [Oos09] for more information.

It was independently proved in [JoK71] and [AkS72] that a Hilbert space has this property. By [MuT07], the Banach space $\ell^p$ where $1 < p < \infty$ has equally this property. It is an open question if it is true for an arbitrary $L^p$-space $L^p(\Omega)$ with $1 < p < \infty$. A positive contraction $T: L^p(\Omega) \to L^p(\Omega)$ on a $L^p$-space has this property by [AkS75].

Our result is the following theorem.

Theorem 1.1 Let $\Omega$ be a (localizable) measure space. Suppose $1 < p < \infty$. A (not necessarily positive) selfadjoint contraction $T: L^\infty(\Omega) \to L^\infty(\Omega)$ induces a contraction $T_p: L^p(\Omega) \to L^p(\Omega)$ which has the Blum-Hanson property.

The proof relies on a 2x2-matrix argument combined with the result of [YeK79] which says that positive contractions acting on noncommutative $L^p$-spaces have the Blum-Hanson property. Our theorem improves some results of [ChS17]. We refer to [AHR74], [Bel75], [FoS73], [LeM16], [LMP16] and [Net21] for related papers.

Structure of the paper The paper is organized as follows. The next section 2 gives background. In Section 3, we prove our result.

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2 Preliminaries

von Neumann algebras and noncommutative $L^p$-spaces A von Neumann algebra $\mathcal{M}$ is a $\ast$-algebra of bounded operators on a Hilbert space $H$ which is closed in the weak operator topology and which contains the identity operator. If $\Omega$ is a (localizable) measure space then $L^\infty(\Omega)$ can be seen as a von Neumann algebra (of multiplication operators) on the Hilbert space $L^2(\Omega)$. Suppose that $\mathcal{M}$ is equipped with a normal semifinite faithful trace $\tau$ (=noncommutative integral). Let $S_+$ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\text{supp } x) < \infty$, where $\text{supp } x$ denotes the support projection of $x$. Let $\mathcal{S}$ be the linear span of $S_+$. Given $1 \leq p < \infty$, for any $x \in \mathcal{S}$, we define

$$\|x\|_p = \left[\tau(|x|^p)\right]^{1/p}$$

where $|x| = (x^* x)^{1/2}$ is the modulus of $x$. Then $(\mathcal{S}, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative $L^p$-space $L^p(\mathcal{M})$ associated with $(\mathcal{M}, \tau)$. We have the interpolation formula $L^p(\mathcal{M}) = (\mathcal{M}, L^1(\mathcal{M}))_{\frac{1}{p}}$.

In the case $\mathcal{M} = L^\infty(\Omega)$, we have an isometric identification $L^p(\mathcal{M}) = L^p(\Omega)$. If $\mathcal{M} = B(\ell^2)$, we obtain the Schatten class $L^p(\mathcal{M}) = S^p$. We refer to [ArK], [JMX06], [PiX03] and [Pis98] for more information on (vector-valued) noncommutative $L^p$-spaces.

Selfadjoint maps Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace $\tau$. Let $T: \mathcal{M} \to \mathcal{M}$ be a contractive linear map. We say that $T$ is selfadjoint [JMX06, page 49] if for any $x, y \in \mathcal{M} \cap L^1(\mathcal{M})$ we have

$$\tau(T(x)y^*) = \tau(x(T(y))^*).$$

Then, the restriction $T|_{\mathcal{M} \cap L^1(\mathcal{M})}$ extends to a contraction $T_1: L^1(\mathcal{M}) \to L^1(\mathcal{M})$. By complex interpolation, we obtain a contraction $T_p: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ if $1 < p < \infty$. Clearly, the induced map $T_2: L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is selfadjoint in the classical sense.

Property (P) Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace $\tau$. Let $T: \mathcal{M} \to \mathcal{M}$ be a linear map. Following [Kri11, Definition 3], we say that $T$ satisfies (P) if there exist linear maps $v_1, v_2: \mathcal{M} \to \mathcal{M}$ such that the linear map

$$\Phi \overset{\text{def}}{=} \begin{bmatrix} v_1 & T \\ T^\circ & v_2 \end{bmatrix} : M_2(\mathcal{M}) \to M_2(\mathcal{M}), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} v_1(a) & T(b) \\ T^\circ(c) & v_2(d) \end{bmatrix}$$

is completely positive, completely contractive and selfadjoint where $T^\circ(x) \overset{\text{def}}{=} (T(x^*))^*$ and where $M_2(\mathcal{M})$ is equipped with the normal semifinite faithful trace $\text{Tr} \otimes \tau$. An operator $T$ satisfying (P) is necessarily completely contractive and selfadjoint. Moreover, $v_1$ and $v_2$ are contractive and selfadjoint.

For example, by [ArK, Proposition 8.2] a Fourier multiplier $T: VN(G) \to VN(G)$ on the group von Neumann algebra $VN(G)$ of a discrete group $G$ has (P) if and only if $T$ is selfadjoint and contractively decomposable.

The following characterization is [Kri11, Proposition 6.1] without the assumption of weak$^*$ continuity. Indeed, it can be removed with the method of [ArK, Remark 3.29].

Proposition 2.1 Let $\Omega$ be a (localizable) measure space. A linear map $T: L^\infty(\Omega) \to L^\infty(\Omega)$ satisfies (P) if and only if $T$ is contractive and selfadjoint.

1. We do not assume that $T$ is weakly$^*$ continuous here. This assumption seems useless in [JMX06].

2
3 Selfadjoint absolute contractions and the Blum-Hanson property

We show that a map with property (P) induces contractions on the associated $L^p$-spaces with the Blum-Hanson property.

**Proposition 3.1** Let $\mathcal{M}$ be a von Neumann algebra equipped with normal semifinite faithful trace. Let $T: \mathcal{M} \to \mathcal{M}$ be an operator with (P). Suppose $1 < p < \infty$. Let $x, y \in L^p(\mathcal{M})$. Suppose that the sequence $(T^n_p(x))_{n \geq 0}$ is weakly convergent to $y$ in $L^p(\mathcal{M})$. Then, for any strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N T^{n_k}_p(x) = y$$

in the norm topology of $L^p(\mathcal{M})$.

**Proof**: There exists linear maps $v_1, v_2: \mathcal{M} \to \mathcal{M}$ satisfying (2.1). We consider the induced map

$$(3.2) \Phi_p = \begin{bmatrix} v_{1,p} & T_p \\ T_p & v_{2,p} \end{bmatrix} : S^p_2(L^p(\mathcal{M})) \to S^p_2(L^p(\mathcal{M}))$$

on the vector-valued Schatten class $S^p_2(L^p(\mathcal{M}))$ which is completely positive and contractive. For any integer $n$, we have

$$(3.3) \Phi_n^p \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} v_{1,p} & T_p \\ T_p & v_{2,p} \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & T_p(x) \\ 0 & 0 \end{bmatrix}.$$ 

We deduce that the sequence $\left( \Phi_n^p \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right)_{n \geq 0}$ is weakly convergent in $S^p_2(L^p(\mathcal{M}))$ to $\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$. Since positive contractions on noncommutative $L^p$-spaces have Blum-Hanson property by [YeK79, Corollary 3.3], we deduce that the limit in the norm topology of

$$\frac{1}{N} \sum_{k=1}^N \Phi_{n_k}^p \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \frac{1}{N} \sum_{k=1}^N \begin{bmatrix} 0 & T_p(x) \\ 0 & 0 \end{bmatrix}$$

is $\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$ when $N$ goes to infinity. We conclude that (3.1) is true. \[\square\]

By applying Proposition 2.1, we obtain the following result. Note that $T$ is not necessarily positive.

**Corollary 3.2** Let $\Omega$ be a (localizable) measure space. Let $T: L^\infty(\Omega) \to L^\infty(\Omega)$ be a selfadjoint contraction. Suppose $1 < p < \infty$. Let $f, g \in L^p(\Omega)$. Suppose that the sequence $(T^n_p(f))_{n \geq 0}$ is weakly convergent to $g$ in $L^p(\Omega)$. Then, for any strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N T^{n_k}_p(f) = g$$

in the norm topology of $L^p(\Omega)$. 3
Remark 3.3 It would be interesting to find a proof of this result which does not use noncommutative $L^p$-spaces.

Remark 3.4 It would be nice to investigate the case of contractively regular operators acting on (classical or noncommutative) $L^p$-spaces, see [ArK].

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Bibliography

[AcK75] M.A. Akcoglu and L. Sucheston. Weak convergence of positive contractions implies strong convergence of averages. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 (1975), 139–145.

[AcK72] M.A. Akcoglu and L. Sucheston. On operator convergence in Hilbert space and in Lebesgue space. Period. Math. Hungar. 2 (1972), 235–244.

[AHR74] M. A. Akcoglu, J. P. Huneke and H. Rost. A counter example to the Blum Hanson theorem in general spaces. Pacific J. Math. 50 (1974), 305–308.

[ArK] C. Arhancet and C. Kriegler. Projections, multipliers and decomposable maps on noncommutative $L^p$-spaces. Preprint, arXiv:1707.05591.

[Bel75] A. Bellow. An $L^p$-inequality with application to ergodic theory. Houston J. Math. 1 (1975), no. 1, 153–159.

[BiH60] J. R. Blum and D. L. Hanson. On the mean ergodic theorem for subsequences. Bull. Amer. Math. Soc. 66 (1960), 308–311.

[ChS17] V. Chilin and F. Sukochev. Blum-Hanson type ergodic theorems in noncommutative symmetric spaces. J. Funct. Anal. 273 (2017), no. 1, 329–351.

[FoS73] H. Fong and L. Sucheston. On a mixing property of operators in $L_p$-spaces. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 26 (1973/74), 165–171.

[Gri19] S. Grivaux. The Blum-Hanson property. Concr. Oper. 6 (2019), no. 1, 92–105.

[JoK71] L. Jones and V. Kufinec. A note on the Blum-Hanson theorem. Proc. Amer. Math. Soc. 30 (1971), 202–203.

[JMX06] M. Junge, C. Le Merdy and Q. Xu. $H^\infty$ functional calculus and square functions on noncommutative $L^p$-spaces. Astérisque No. 305 (2006).

[Kri11] C. Kriegler. Analyticity angle for non-commutative diffusion semigroups. J. Lond. Math. Soc. (2) 83 (2011), no. 1, 168–186.

[LeM16] P. Lefèvre and É. Matheron. The Blum-Hanson property for $C(K)$ spaces. Pacific J. Math. 282 (2016), no. 1, 203–212.

[LMP16] P. Lefèvre, É. Matheron, and A. Primot. Smoothness, Asymptotic Smoothness and the Blum-Hanson Property. Israel J. Math. 211 (2016), no. 1, 271–309.

[MuT07] V. Müller and Y. Tomilov. Quasisimilarity of power bounded operators and Blum-Hanson property. J. Funct. Anal. 246 (2007), no. 2, 385–399.

[Net21] F. Netillard. Banach spaces with the Blum-Hanson Property. Preprint, arXiv:2110.03089.

[Oos09] M. Van Oosterhout. The Blum-Hanson Property. Master Thesis. Delft University of Technology, 2009.

[Pis98] G. Pisier. Non-commutative vector valued $L_p$-spaces and completely $p$-summing maps. Astérisque, 247 (1998).

[PiX03] G. Pisier and Q. Xu. Non-commutative $L^p$-spaces. In Handbook of the Geometry of Banach Spaces, Vol. II, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, 1459–1517, 2003.

[YeK79] F. J. Yeadon and P. E. Kopp. Inequalities for noncommutative $L^p$-spaces and an application. J. London Math. Soc. (2) 19 (1979), no. 1, 123–128.
