Cayley graph on symmetric groups with generating block transposition sets

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Abstract
This paper deals with the Cayley graph Cay(Sym$_n$, S$_n$), where the generating set consists of all block transpositions. A motivation for the study of these particular Cayley graphs comes from current research in Bioinformatics. We prove that Aut(Cay(Sym$_n$, S$_n$)) is the product of the right translation group by N $\times$ D$_{n+1}$, where N is the subgroup fixing S$_n$ element-wise and D$_{n+1}$ is a dihedral group of order 2($n+1$). We conjecture that N is trivial. We also prove that the subgraph $\Gamma$ with vertex-set S$_n$ is a 2($n-2$)-regular graph whose automorphism group is D$_{n+1}$. Furthermore, $\Gamma$ has as many as $n+1$ maximum cliques of size 2. Also, its subgraph $\Gamma(V)$ whose vertices are those in these cliques is a 3-regular, Hamiltonian, and vertex-transitive graph.

1 Introduction

Block transpositions are well known sorting operations with applications in Bioinformatics which act on a string by removing a block of consecutive entries and inserting it somewhere else. In terms of the symmetric group Sym$_n$ of degree $n$, the strings are the permutations on $[n] = \{1, 2, \ldots, n\}$. For any three integers $i, j, k$ with $0 \leq i < j < k \leq n$, the block transposition $\sigma(i, j, k)$ with cut points $(i, j, k)$ turns the permutation $\pi = [\pi_1 \ldots \pi_n]$ to the permutation $\pi' = [\pi_1 \ldots \pi_i \pi_{j+1} \ldots \pi_k \pi_{i+1} \ldots \pi_j \pi_{k+1} \ldots \pi_n]$. This action of
σ(i, j, k) on π can also be expressed as the composition \( π' = π \circ σ(i, j, k) \) whenever \( σ(i, j, k) \) is identified with the permutation defined in (1). The set \( S_n \) of all block transpositions has size \((n + 1)n(n - 1)/6\) and is an inverse closed generating set of \( \text{Sym}_n \). The arising Cayley graph \( \text{Cay}(\text{Sym}_n, S_n) \) is a very useful tool in Bioinformatics since “sorting a permutation by block transpositions” is equivalent to finding the shortest paths between vertices in \( \text{Cay}(\text{Sym}_n, S_n) \); see [5, 6, 8, 10].

Although the definition of a block transposition arose from a practical need, \( \text{Cay}(\text{Sym}_n, S_n) \) presents some remarkable features, the most interesting being the existence of automorphisms other than the right translations.

The subgraph \( \Gamma \) with vertex-set \( S_n \), named \emph{block transposition graph}, has especially nice properties. As we prove in this paper, \( \Gamma \) is a \((2n - 2)\)-regular graph whose automorphism group is a dihedral group \( D_{n+1} \) of order \( 2(n+1) \). This group \( D_{n+1} \) arises from the toric equivalence in \( \text{Sym}_n \) and the reverse permutation, and hence is an automorphism group of \( \text{Cay}(\text{Sym}_n, S_n) \) as well. Therefore, the automorphism group \( \text{Aut}(\text{Cay}(\text{Sym}_n, S_n)) \) is the product of the right permutation group \( R(\text{Cay}(\text{Sym}_n, S_n)) \) by \( N \rtimes D_{n+1} \), where \( N \) is the subgroup fixing every block transposition. Computer aided computation carried out for \( n \leq 8 \) supports our conjecture that \( N \) is trivial, equivalently that \( \text{Aut}(\text{Cay}(\text{Sym}_n, S_n)) = R(\text{Cay}(\text{Sym}_n, S_n)) \rtimes D_{n+1} \).

Furthermore, we show that \( \Gamma \) has as many as \( n+1 \) maximum cliques of size 2, and look inside the subgraph \( \Gamma(V) \) whose vertices are the \( 2(n+1) \) vertices of these cliques. We prove that \( \Gamma(V) \) is 3-regular. We also prove that \( \Gamma(V) \) is Hamiltonian and \( D_{n+1} \) is an automorphism group of \( \Gamma(V) \) acting transitively (and hence regularly) on \( V \). This confirms the Lovász conjecture for \( \Gamma(V) \).

For basic facts on Cayley graphs and combinatorial properties of permutations the reader is referred to [1, 3].

### 2 Background on block transpositions

Throughout the paper, \( n \) denotes a positive integer. In our investigation the cases \( n \leq 3 \) are trivial. Our results for \( n = 4 \) are different, in some cases, from the general ones, and are presented in Section 7. Since some proofs are performed by induction on \( n \), the smallest cases, \( n = 5, 6 \), are thoroughly worked out in Section 7 by a computer aided exhaustive search. All computation in the paper are performed by using the package “grape” of GAP [11].
For a set $X$ of size $n$, we denote by $\text{Sym}_n$ the set of all permutations on $X$. For the sake of simplicity, $[n] = \{1, 2, \ldots, n\}$ is usually taken for $X$. We mostly adopt the functional notation for permutations so that if $\pi \in \text{Sym}_n$, then $\pi = [\pi_1\pi_2 \cdots \pi_n]$ with $\pi(t) = \pi_t$ for every $t \in [n]$. In particular, the reverse permutation is $\omega = [n\ n-1 \cdots 1]$ and $\iota = [1 2 \cdots n]$ is the identity permutation. For any $\pi, \rho \in \text{Sym}_n$, $\pi \circ \rho$ is carried out by $\pi(\rho(t))$ for every $t \in [n]$.

For any three integers, named cut points, $i, j, k$ with $0 \leq i < j < k \leq n$, we define the block transposition (transposition; see [7]) $\sigma(i, j, k)$ as a function $[n]$:

$$\sigma(i, j, k)_t = \begin{cases} 
  t, & 1 \leq t \leq i \\
  t + j - i, & i + 1 \leq t \leq k - j + i \\
  t + j - k, & k - j + i + 1 \leq t \leq k.
\end{cases}$$

(1)

Actually, $\sigma(i, j, k)$ can also be represented as the permutation

$$\sigma(i, j, k) = \left\{ \begin{array}{ll}
  [1 \cdots i \ j + 1 \cdots k \ i + 1 \cdots j \ k + 1 \cdots n], & 1 \leq i, k < n, \\
  [j + 1 \cdots k \ 1 \cdots j \ k + 1 \cdots n], & i = 0, k < n, \\
  [1 \cdots i \ j + 1 \cdots n \ i + 1 \cdots j], & 1 \leq i, k = n, \\
  [j + 1 \cdots n \ 1 \cdots j], & i = 0, k = n
\end{array} \right.$$

(2)

such that the action of $\sigma(i, j, k)$ on $\pi$ is defined as the product

$$\pi \circ \sigma(i, j, k) = [\pi_1 \cdots \pi_i \ \pi_{j+1} \cdots \pi_k \ \pi_{i+1} \cdots \pi_j \ \pi_{k+1} \cdots \pi_n].$$

Therefore, applying a block transposition on the right of $\pi$ consists in switching two adjacent subsequences of $\pi$, namely blocks, without changing the order of the integers within each block. This may also be expressed by

$$[\pi_1 \cdots \pi_i | \pi_{i+1} \cdots \pi_j | \pi_{j+1} \cdots \pi_k | \pi_{k+1} \cdots \pi_n].$$

From now on, $S_n$ denotes the set of all block transpositions on $[n]$. The size of $S_n$ is equal to $n(n + 1)(n - 1)/6$. Obviously, $S_n$ is not a subgroup of $\text{Sym}_n$. Nevertheless, $S_n$ is power and inverse closed. For any cut points $i, j, k$,

$$\sigma(i, j, k)^{-1} = \sigma(i, k - j + i, k), \quad \sigma(i, i + 1, k)^{j-i} = \sigma(i, j, k).$$

(3)

Also, for any two integers $i, k$ with $0 \leq i < k \leq n$ the subgroup generated by $\sigma(i, i + 1, k)$ consists of all $\sigma(i, j, k)$ together with the identity. In particular, $\sigma(0, 1, n)$ generates a subgroup of order $n$ that often appears in our
arguments. Throughout the paper, $\beta = \sigma(0,1,n)$ and $B$ denotes the set of non-trivial elements of the subgroup generated by $\beta$.

We introduce some subsets in $S_n$ that plays a relevant role in our study. Every permutation $\bar{\pi}$ on $\{n-1\}$ extends to a permutation $\pi$ on $[n]$ such that $\pi_t = \bar{\pi}_t$ for $1 \leq t \leq n-1$ and $\pi_n = n$. Hence, $S_{n-1}$ is naturally embedded in $S_n$ since every $\sigma(i,j,k) \in S_n$ with $k \neq n$ is identified with the block transposition $\bar{\sigma}(i,j,k)$. On the other side, every permutation $\pi'$ on $\{2,3,\ldots,n\}$ extends to a permutation on $[n]$ such that $\pi_t = \pi'_t$, for $2 \leq t \leq n$ and $\pi_1 = 1$. Thus, $\sigma(i,j,k) \in S_n$ with $i \neq 0$ is identified with the block transposition $\sigma'(i,j,k)$. The latter block transpositions form the set $S_{\triangle}^{\nabla} = \{\sigma(i,j,k) | i \neq 0\}$.

3 The toric equivalence in the symmetric group

The definition of toric (equivalence) classes in Sym$_n$ requires to consider permutation on $[n]^0 = \{0,1,\ldots,n\}$ and recover the permutations $\pi = [\pi_1 \cdots \pi_n]$ on $[n]$ in the form $[0 \pi]$, where $[0 \pi]$ stands for the permutation $[\pi_0 \pi_1 \cdots \pi_n]$ on $[n]^0$ with $\pi_0 = 0$. Let $\alpha = [1 2 \ldots n 0]$.

For any two integers $r, s$ with $0 \leq r, s \leq n$, the permutation $\alpha^s \circ [0 \pi] \circ \alpha^r$ on $[n]^0$ fixes 0 if and only if $s + \pi_r = n + 1$; see [3, Lemma 5.1]. This gives rise to toric maps $f_r$ on Sym$_n$ with $0 \leq r \leq n$, defined by

$$f_r(\pi) = \rho \iff [0 \rho] = \alpha^{n+1-\pi_r} \circ [0 \pi] \circ \alpha^r.$$ (4)
The **toric class** of $\pi$ is

$$F(\pi) = \{ f_r(\pi) | r = 0, 1, \ldots, n \}. \quad (5)$$

Since

$$(f_r(\pi))_t = \pi_{r+t} - \pi_r \quad \text{for every } t \in [n], \quad (6)$$

where the indices are taken mod($n+1$), (5) formalizes the intuitive definition of toric classes introduced in [3] by Eriksson and his coworkers. In general, the toric class of $\pi$ comprises $n+1$ permutations, but it may consist of a smaller number of permutations and can even collapse to a unique permutation. The latter case occurs when $\pi$ is the identity permutation or the reverse permutation. The number of elements in a toric class is always a divisor of $n+1$, and there are exactly $\varphi(n+1)$ classes that have only one element, where $\varphi$ is the Euler function; see [4].

From (4), $f_s \circ f_r = f_{s+r}$, where the indices are taken mod($n+1$). Hence, $f_r = f^r$ with $f = f_1$, and the set

$$F = \{ f_r | r = 0, 1, \ldots, n \}$$

is a cyclic group of order $n+1$ generated by $f$.

If $\pi \in \text{Sym}_n$ and $0 \leq r \leq n$, then

$$f_r^{-1}(\pi) = f_{\pi_r}(\pi^{-1}). \quad (7)$$

In particular, $f_r^{-1}(\pi) = f_r(\pi^{-1})$ provided that $\pi_r = r$.

The **reverse map** $g$ on $\text{Sym}_n$ is defined by

$$g(\pi) = \rho \iff [0 \rho] = [0 w] \circ [0 \pi] \circ [0 w]. \quad (8)$$

$g$ is an involution, and

$$(g(\pi))_t = n+1 - \pi_{n+1-t} \quad \text{for every } t \in [n]. \quad (9)$$

Also, for all $0 \leq r \leq n$,

$$g \circ f_r \circ g = f_{n+1-r} \quad (10)$$

since $[0 w] \circ \alpha^r \circ [0 w] = \alpha^{n+1-r}$.

The following result states the invariance of $S_n$ under the action of toric maps and the reverse map; see [8] Lemma 5.1.
Proposition 3.1. Toric maps and the reverse map take any block transposition to a block transposition.

Proof. For toric maps, the assertion is known; see [8, Lemma 5.1]. For the reverse map, (2) yields
\[ g(\sigma(i, j, k)) = \sigma(n - k, n - j, n - i), \]
whence the assertion follows. \(\square\)

In the proof of [8, Lemma 5.1], several equations linking \(\alpha\) and block transpositions are given. Some of these are also useful for the present investigation and listed below in terms of toric maps.

Lemma 3.2. For any positive integer \(r \leq n\),

(i) \(f_r(\sigma(i, j, k)) = \sigma(i - r, j - r, k - r)\) if \(0 \leq i - r < k - j + i - r < k - r \leq n\);

(ii) \(f_r(\sigma(i, j, k)) = \sigma(k - j + i - r, n + 1 + 2i - j - r, n + 1 + i - r)\) if \(0 \leq k - j + i - r < k - r < n + 1 + i - r \leq n\);

(iii) \(f_r(\sigma(i, j, k)) = \sigma(k - r, 2k - j - r, n + 1 + k - j + i - r)\) if \(0 \leq k - r < n + 1 + i - r < n + 1 + k - j + i - r \leq n\);

(iv) \(f_r(\sigma(i, j, k)) = \sigma(n + 1 + i - r, n + 1 + j - r, n + 1 + k - r)\) if \(0 \leq n + 1 + i - r < n + 1 + j - r < n + 1 + k - r \leq n\).

4 Cayley graph of the symmetric group with generating block transpositions

Since \(S_n\) is an inverse closed generator set of \(\text{Sym}_n\) which does not contain \(\iota\), the Cayley graph \(\text{Cay}(\text{Sym}_n, S_n)\) is an undirected simple graph, where \(\{\pi, \rho\}\) is an edge if and only if \(\rho = \pi \circ \sigma(i, j, k)\) for some \(\sigma(i, j, k) \in S_n\). Obviously, the vertices of \(\text{Cay}(\text{Sym}_n, S_n)\) adjacent to \(\iota\) are exactly the block transpositions.

By a result of Cayley, every \(h \in \text{Sym}_n\) defines a right translation \(h\) which is the automorphism of \(\text{Cay}(\text{Sym}_n, S_n)\) that takes the vertex \(\pi\) to the vertex \(\pi \circ h\), and hence the edge \(\{\pi, \rho\}\) to the edge \(\{\pi \circ h, \rho \circ h\}\). These automorphisms form the right translation group \(R(\text{Cay}(\text{Sym}_n, S_n))\) of \(\text{Cay}(\text{Sym}_n, S_n)\).
Clearly, \( \text{Sym}_n \cong R(\text{Cay}(\text{Sym}_n, S_n)) \). Furthermore, since \( R(\text{Cay}(\text{Sym}_n, S_n)) \) acts regularly on \( \text{Sym}_n \), every automorphism of \( \text{Cay}(\text{Sym}_n, S_n) \) is the product of a right translation by an automorphism fixing \( \iota \).

One may ask if there is a non-trivial automorphism of \( \text{Cay}(\text{Sym}_n, S_n) \) fixing \( \iota \). The answer is affirmative by the following results.

**Lemma 4.1.** For every \( \pi, \rho \in \text{Sym}_n \),

\begin{enumerate}[(i)]
    \item \( f(\pi \circ \rho) = f_{\rho_1}(\pi) \circ f(\rho) \);
    \item \( g(\pi \circ \rho) = g(\pi) \circ g(\rho) \).
\end{enumerate}

**Proof.** (i) From (4), \( f(\pi \circ \rho) = \mu \) with

\[ [0 \mu] = \alpha^{n+1-(\pi \circ \rho)} \circ [0 \pi] \circ [0 \rho] \circ \alpha \]

\[ = \alpha^{n+1-(\pi \circ \rho)} \circ [0 \pi] \circ \alpha^{\rho_1} \circ \alpha^{n+1-\rho_1} [0 \rho] \circ \alpha. \]

Now, the first assertion follows from (4).

(ii) A similar argument depending on (8) shows that the second assertion holds true for \( g \). \( \square \)

**Proposition 4.2.** Toric maps and the reverse map are automorphisms of \( \text{Cay}(\text{Sym}_n, S_n) \).

**Proof.** Let \( \pi, \rho \in \text{Sym}_n \) be any two adjacent vertices of \( \text{Cay}(\text{Sym}_n, S_n) \). Then, \( \rho = \pi \circ \sigma \) for some \( \sigma = \sigma(i, j, k) \in S_n \). Here, Lemma 4.1 yields \( f(\rho) = f_{\sigma_1}(\pi) \circ f(\sigma) \). Therefore, the first assertion for \( f \) follows from Proposition 3.1.

By induction on \( r \), this holds true for all toric maps.

A similar argument can be used to show the first assertion for the reverse map. This completes the proof. \( \square \)

By (10), the set consisting of \( F \) and its coset \( F \circ g \) is a dihedral group \( D_{n+1} \) of order \( 2(n+1) \). Clearly, \( D_{n+1} \) fixes \( \iota \). Now, Proposition 4.2 has the following corollary.

**Theorem 4.3.** The automorphism group of \( \text{Cay}(\text{Sym}_n, S_n) \) contains a dihedral subgroup \( D_{n+1} \) of order \( 2(n+1) \) fixing the identity permutation.

From now on, the term of toric-reverse group stands for \( D_{n+1} \), and \( G \) denotes the stabilizer of \( \iota \) in the automorphism group of \( \text{Cay}(\text{Sym}_n, S_n) \). By Theorem 4.3, the problem arises whether \( D_{n+1} \) is already \( G \). We state our result on this problem.
Clearly, $G$ preserves the subgraph of $\text{Cay}(\text{Sym}_n, S_n)$ whose vertices are the block transpositions. We call this subgraph $\Gamma$ the block transposition graph and denote $R$ its automorphism group. The kernel of the permutation representation of $G$ on $S_n$ is a normal subgroup $N$, and the factor group $G/N$ is a subgroup of $R$. Since $D_{n+1}$ and $N$ have trivial intersection, the toric-reverse group can be regarded as a subgroup of $G/N$. Our main result in this paper is a proof of the theorem below.

**Theorem 4.4.** The automorphism group of $\Gamma$ is the toric-reverse group.

As a corollary, $G = N \rtimes D_{n+1}$. From this the following result is obtained.

**Theorem 4.5.** The automorphism group of $\text{Cay}(\text{Sym}_n, S_n)$ is the product of the right multiplicative group by $N \rtimes D_{n+1}$.

**Remark 4.6.** Computation shows that $N$ is trivial for $n \leq 8$. This motivates to make the following conjecture.

**Conjecture 4.7.** The automorphism group of $\text{Cay}(\text{Sym}_n, S_n)$ is the product of the right multiplicative group by the toric-reverse group.

The proof of Theorem 4.4 depends on several results on combinatorial properties of $\Gamma$, especially on the set of its maximal cliques of size 2. These results of independent interest are stated and proven in the next sections.

## 5 Combinatorial properties of the block transposition graph

We begin with some results on the components of the partition in Lemma 2.1. Since $B$ consists of all nontrivial elements of a subgroup of $S_n$ of order $n$, the block transpositions in $B$ are the vertices of a complete graph of size $n - 1$. Lemma 2.1 and 1.1 give the following property.

**Lemma 5.1.** The reverse map preserves both $B$ and $S_{n-2}^\Delta$ while it switches $L$ and $F$.

**Lemma 5.2.** No edge of $\text{Cay}(\text{Sym}_n, S_n)$ has one endpoint in $B$ and the other in $S_{n-2}^\Delta$. 

8
Proof. Suppose on the contrary that \( \{\sigma(i', j', k'), \sigma(0, j, n)\} \) with \( i' \neq 0 \) and \( k' \neq n \) is an edge of Cay(Sym\(_n\), S\(_n\)). By (3), \( \rho = \sigma(0, n - j, n) \circ \sigma(i', j', k') \in S_n \). Also, \( \rho \in B \) as \( \rho_1 \neq 1 \) and \( \rho_n \neq n \). Since \( B \) together with the identity is a group, \( \sigma(0, j, n) \circ \rho \) is also in \( B \). This yields \( \sigma(i', j', k') \in B \), a contradiction with Lemma 2.1. \( \square \)

The proofs of the subsequent properties use a few more equations involving block transpositions which are stated in the following two lemmas.

**Lemma 5.3.** In each of the following cases \( \{\sigma(i, j, k), \sigma(i', j', k')\} \) is an edge of Cay(Sym\(_n\), S\(_n\)).

(i) \( (i', j') = (i, j) \);
(ii) \( (i', j') = (j, k) \) if \( k < k' \);
(iii) \( (j', k') = (j, k) \);
(iv) \( (j', k') = (i, j) \) if \( i' < i \);
(v) \( (i, k) = (i', k') \) if \( j < j' \).

Proof. (i) W.l.g. \( k' < k \). By (2), \( \sigma(i, j, k) = \sigma(i', j', k') \circ \sigma(k' - j + i, k', k) \).

(ii) W.l.g. \( i' < i \). From (2), \( \sigma(i, j, k) = \sigma(i', j', k') \circ \sigma(i', k - j + i', k) \).

In the remaining cases, from (2),

\[
\begin{align*}
\sigma(i, j, k) &= \sigma(j, k, k') \circ \sigma(i, k' - k + j, k'), \\
\sigma(i, j, k) &= \sigma(i', i, j) \circ \sigma(i', j - i + i', k), \\
\sigma(i, j, k) &= \sigma(i, j', k) \circ \sigma(i, k - j + j', k).
\end{align*}
\]

Hence the statements hold. \( \square \)

The proof of the lemma below is straightforward and requires only (2).

**Lemma 5.4.** The following equations hold.

(i) \( \sigma(i, j, n) = \sigma(0, j, n) \circ \sigma(0, n - j, n - j + i) \) for \( i \neq 0 \);
(ii) \( \sigma(i, j, n) = \sigma(0, i, j) \circ \sigma(0, j - i, n) \) for \( i \neq 0 \);
(iii) \( \sigma(0, j, n) = \sigma(i, j, n) \circ \sigma(0, i, n - j + i) \);
(iv) \( \sigma(0, j, n) = \sigma(0, j, j + i) \circ \sigma(i, j + i, n) \) for \( i \neq 0 \).
Lemma 5.5. Let $i \neq 0$.

(i) If $\sigma(i, j, n) = \sigma(0, \bar{j}, n) \circ \sigma(i', j', k')$, then $\bar{j} = j$.

(ii) If $\sigma(i, j, n) = \sigma(i', j', k') \circ \sigma(0, \bar{j}, n)$, then $\bar{j} = i - j$.

Proof. (i) Assume $\bar{j} \neq j$. From Lemma 5.4 (i) and (3),

$$\sigma(i', j', k') = \sigma(0, j^*, n) \circ \sigma(0, n - j, n - j + i),$$

(12)

where $j^*$ denotes the smallest positive integer such that $j^* \equiv j - \bar{j} \pmod{n}$. First we prove $i' = 0$. Suppose on the contrary, $(\sigma(0, j^*, n) \circ \sigma(0, n - j, n - j + i))_1 = 1$. On the other hand, $\sigma(0, n - j, n - j + i)_1 = n - j + 1$ and $\sigma(0, j^*, n)_{n-j+1} = n - \bar{j} + 1$ since $\sigma(0, j^*, n)_t = t + j^* \pmod{n}$ by (1). Thus, $n - \bar{j} + 1 = 1$, a contradiction since $\bar{j} < n$.

Now, from (12), $\sigma(0, j', k') \neq n$. Hence $k' = n$. Therefore, $\sigma(0, n - j, n - j + i) = \sigma(0, j + \bar{j}, n) \circ \sigma(0, j', n) \in B$. A contradiction since $i \neq j$. This proves the assertion.

(ii) Taking the inverse of both sides of the equation in (ii) gives by (3)

$$\sigma(i, n - j + i, n) = \sigma(0, n - j, n) \circ \sigma(i', j', k')^{-1}.$$  

Now, from (i), $n - \bar{j} = n - j + i$, and the assertion follows. \hfill \square

Proposition 5.6. The bipartite graphs arising from the components of the partition in Lemma 2.1 have the following properties.

(i) In the bipartite graph $(L \cup F, B)$, every vertex in $L \cup F$ has degree 1 while every vertex of $B$ has degree $n - 2$.

(ii) The bipartite graph $(L, F)$ is a $(1, 1)$-biregular graph.

Proof. (i) Lemma 5.5 (i) together with Lemma 5.4 (i) show that every vertex in $F$ has degree 1. Lemma 5.1 ensures that this holds true for $L$.

For every $1 \leq j \leq n - 1$, Lemma 5.4 (iii) shows that there exist at least $j - 1$ edges incident with $\sigma(0, j, n)$ and a vertex in $L$. Furthermore, from Lemma 5.4 (iv), there exist at least $n - j - 1$ edges incident with $\sigma(0, j, n)$ and a vertex in $F$. Therefore, at least $n - 2$ edges incident with $\sigma(0, j, n)$ have a vertex in $L \cup F$. On the other hand, this number cannot exceed $n - 2$ since $|L \cup F| = (n - 1)(n - 2)$ from Lemma 2.1. This proves the first assertion.

(ii) From Lemma 5.4 (ii), there exists at least one edge incident a vertex in $F$ and a vertex in $L$. Also, Lemma 5.3 (ii) ensures the uniqueness of such an edge. \hfill \square
Corollary 5.7. \( \Gamma(B) \) is the unique maximal clique of \( \Gamma \) of size \( n - 1 \) containing an edge of \( \Gamma(B) \).

Proof. Proposition 5.6 (i) together with Lemma 2.1 show that the endpoints of an edge of \( \Gamma(B) \) do not have a common neighbor outside \( B \).

Computation shows that \( \Gamma \) is a 6-regular subgraph for \( n = 5 \) and 8-regular subgraph for \( n = 6 \); see Section 7. This generalizes to the following result.

Proposition 5.8. \( \Gamma \) is a \( 2(n - 2) \)-regular graph.

Proof. Since \( \Gamma(B) \) is a maximal clique of size \( n - 1 \), every vertex of \( B \) is incident with \( n - 2 \) edges of \( \Gamma(B) \). From Proposition 5.6 (i), as many as \( n - 2 \) edges incident with a vertex in \( B \) have an endpoint in \( L \cup F \). Thus, the assertion holds for the vertices in \( B \).

In \( \Gamma(F) \) every vertex has degree \( 2(n - 1) - 4 = 2n - 6 \), by induction on \( n \). This together with Proposition 5.6 (ii) show that every vertex of \( \Gamma(F) \) has degree \( 2n - 5 \) in \( \Gamma(L \cup F) \). By Lemma 5.1, this holds true for every vertex of \( \Gamma(L) \). The degree increases to \( 2n - 4 \) when we also count the unique edge in \( \Gamma(B) \), according to the first assertion of Proposition 5.6 (i).

In \( \Gamma(S_{n-2}^\Delta) \) every vertex has degree \( 2n - 8 \), by induction on \( n \). Furthermore, in \( \Gamma(L \cup S_{n-2}^\Delta) \) every vertex has degree \( 2n - 6 \) by induction on \( n \), and the same holds for \( \Gamma(F \cup S_{n-2}^\Delta) \). This together with Lemma 5.2 show that every vertex in \( S_{n-2}^\Delta \) is the endpoint of exactly \( 2(2n - 6) - (2n - 8) \) edges in \( \Gamma \).

Our next step is to determine the set of all maximal cliques of size 2 in \( \Gamma \). According to Lemma 5.3 (v), let \( \Lambda \) be the set of all edges
\[
e_l = \{ \sigma(l, l + 1, l + 3), \sigma(l, l + 2, l + 3) \},
\]
where \( l \) ranges over \{0, 1, \ldots, n - 3\}. From (3), the endpoints of each such edge are the inverse of one of the other. Let denote \( v_1 \) and \( v_2 \) the endpoints of \( e_{n-4} \), that is,

\[
v_1 = \sigma(n - 4, n - 3, n - 1), \quad v_2 = \sigma(n - 4, n - 2, n - 1).
\]

Proposition 5.9. The maximal cliques of \( \Gamma \) of size 2 are the edges in \( \Lambda \) together with three more edges, namely
\[
e_{n-2} = \{ \sigma(0, n - 2, n - 1), \sigma(0, n - 2, n) \};
\]
\[
e_{n-1} = \{ \sigma(0, 2, n), \sigma(1, 2, n) \};
\]
\[
e_n = \{ \sigma(1, n - 1, n), \sigma(0, 1, n - 1) \}.
\]

(13)
Also, all these edges are pairwise disjoint.

Proof. The above edges are pairwise disjoint since \( n \geq 5 \) is assumed.

The edge \( e_{n-4} \) is a maximal clique of size 2 by the proof of Corollary 5.7.

From Lemma 3.2 (i),

\[
\begin{align*}
\sigma(l, l + 1, l + 3) &= f_{n-4-l}(v_1); \\
\sigma(l, l + 2, l + 3) &= f_{n-4-l}(v_2);
\end{align*}
\]

for every \( 0 \leq l \leq n - 4 \). From Lemma 3.2 (ii),

\[
\begin{align*}
\sigma(1, n - 1, n) &= f_{n-3}(v_1); \\
\sigma(0, n - 2, n - 1) &= f_{n-2}(v_1); \\
\sigma(0, n - 2, n) &= f_{n-3}(v_2).
\end{align*}
\]

From Lemma 3.2 (iii),

\[
\begin{align*}
\sigma(0, 2, n) &= f_{n-1}(v_1); \\
\sigma(0, 1, n - 1) &= f_{n-1}(v_2); \\
\sigma(1, 2, n) &= f_{n-2}(v_2).
\end{align*}
\]

From Lemma 3.2 (iv),

\[
\begin{align*}
\sigma(n - 3, n - 2, n) &= f_{n}(v_1); \\
\sigma(n - 3, n - 1, n) &= f_{n}(v_2).
\end{align*}
\]

We have already pointed out that \( e_{n-4} \) is a maximal clique of size 2. Since \( F \) is an automorphism group of \( \Gamma \) and \( v_1 \) and \( v_2 \) are the endpoints of \( e_{n-4} \), the above equations suffice to complete the proof.

For now on \( V \) denotes the set of the vertices of the edges \( e_m \) with \( m \) ranging over \( \{0, 1, \ldots, n\} \).

Lemma 5.10. Toric maps and the reverse map preserve \( V \). Furthermore, the toric-reverse group is regular on \( V \), and \( \Gamma(V) \) is a vertex-transitive graph.

Proof. We have seen in the proof of Proposition 5.9 that \( V \) is the union of the toric classes of \( v_1 \) and \( v_2 \). This proves the vertex-transitivity of \( \Gamma(V) \) and the first assertion for toric maps. This holds true for the the reverse map since it interchanges \( v_1 \) and \( v_2 \) by (11). Therefore, the toric-reverse group acts transitively on \( V \). Since \( V \) and the toric-reverse group have the same size, the action is regular. \( \square \)
Our next step is to show that the \( n + 1 \) edges \( e_m \) are the unique maximal cliques of size 2 in \( \Gamma \). Computation shows that the assertion is true for \( n = 5, 6 \); see Section 7.

**Lemma 5.11.** Every maximal clique of size 2 in \( \Gamma \) is one of the edges \( e_m \) with \( 0 \leq m \leq n \).

**Proof.** On the contrary take a maximal clique \( e \) of size 2 in \( \Gamma \) other than the edges \( e_m \). By induction on \( n \), \( e \) is not an edge in \( \Gamma(L \cup S_n^\Delta) \) or in \( \Gamma(F \cup S_n^\Delta) \). Therefore, one of the endpoints of \( e \) is in \( B \), say \( \sigma(0, j, n) \). By Lemma 5.2, the other endpoint of \( e \) is in \( F \cup L \). By Lemma 5.1, that endpoint may be assumed to be \( \sigma(i', j, n) \) \( \in F \). Then, \( j' = j \) by Lemma 5.5 (i). Also, by the proof of the first assertion of Proposition 5.6 (i), the vertex \( \sigma(0, j, n) \) is adjacent exactly to \( \sigma(\bar{i}, j, n) \) for any \( 0 \leq \bar{i} < j \). So take any \( 0 < \bar{i} < j \) with \( \bar{i} \neq i' \). Then \( \sigma(0, j, n), \sigma(i', j, n), \) and \( \sigma(\bar{i}, j, n) \) are three pairwise adjacent vertices, a contradiction. \( \Box \)

Lemma 5.11 shows that \( V \) consists of the endpoints of the edges of \( \Gamma \) which are maximal cliques of size 2. Thus \( \Gamma(V) \) is relevant for the study of \( \text{Cay}(\text{Sym}_n, S_n) \). We prove some properties of \( \Gamma(V) \).

**Proposition 5.12.** \( \Gamma(V) \) is a 3-regular graph.

**Proof.** First we prove the assertion for the endpoint \( v_3 = \sigma(0, 2, n) \) of \( e_{n-1} \). By Lemma 5.3 (i) (iii) (v), \( \sigma(0, 2, 3), \sigma(1, 2, n), \) and \( \sigma(0, n-2, n) \) are neighbors of \( v_3 \). Since \( \sigma(1, 2, n) \in F \) and \( \sigma(0, 2, 3) \in L \), from the first assertion of Proposition 5.6 (i), \( v_3 \) is not adjacent to any more vertices in either \( V \cap F \) or \( V \cap L \). Hence, each other vertex of \( V \) is in \( S_n^\Delta \). On the other hand, since \( v_3 \in B \), Lemma 5.2 yields that no vertex in \( V \cap S_n^\Delta \) is adjacent to \( \sigma(0, 2, n) \). Thus, \( v_3 \) has degree 3 in \( \Gamma(V) \).

Now, from the first equation in (16), \( f_{n-1}(v_1) = v_3 \) which shows that \( v_1 \) has also degree 3 in \( \Gamma(V) \). The reverse map takes \( v_3 \) to the endpoint \( v'_3 = \sigma(0, n-2, n) \) of \( e_{n-2} \). By the second assertion of Lemma 5.10, \( v'_3 \) also has degree 3 in \( \Gamma(V) \). From the third equation in (15), \( f_{n-1}(v_2) = v'_3 \) yielding that \( v_2 \) has also degree 3 in \( \Gamma(V) \).

From the proof of Proposition 5.9, each vertex in \( V \) is the image of either \( v_1 \) or \( v_2 \) by a toric map. By the first assertion of Lemma 5.10, every vertex in \( V \) has the same degree as \( v_1 \) or \( v_2 \). This completes the proof. \( \Box \)

**Proposition 5.13.** \( \Gamma(V) \) is a Hamiltonian graph.
Proof. We start by exhibiting a path $P$ in $V$ beginning with $\sigma(0,2,3)$ and ending with $v_1$ that visits all vertices $\sigma(l,l+1,l+3), \sigma(l,l+2,l+3) \in \Lambda$ with $0 \leq l \leq n-4$.

For $n = 5$, $v_1 = \sigma(1,2,4)$, and $P = \sigma(0,2,3), \sigma(0,1,3), \sigma(1,3,4), v_1$.

Assume $n > 5$. For every $l$ with $0 \leq l \leq n-4$, Lemma 5.3 (ii) (v) show that both edges below are incident to $\sigma(l,l+1,l+3)$:

$$\{\sigma(l,l+1,l+3), \sigma(l+1,l+3,l+4)\}, \{\sigma(l,l+2,l+3), \sigma(l,l+1,l+3)\}.$$ 

Therefore, $\sigma(0,2,3), \sigma(0,1,3), \sigma(1,3,4), \ldots, \sigma(l+1,l+3,l+4), \ldots, v_1$

is a path $P$ with the requested property.

By Lemma 5.3 there also exists a path $P'$ beginning with $v_1$ and ending with $\sigma(0,2,3)$ which visits the other vertices of $V$, namely

$$v_1, \sigma(n-3,n-1,n), \sigma(n-3,n-2,n), \sigma(0,n-2,n), \sigma(0,n-2,n-1), \sigma(0,1,n-1), \sigma(1,n-1,n), \sigma(1,2,n), \sigma(0,2,n), \sigma(0,2,3).$$

By Theorem 5.9 the union of $P$ and $P'$ is a cycle in $V$ that visits all vertices. This completes the proof. 

Remark 5.14. By a famous conjecture of Lovász, every finite connected vertex-transitive graph contains a Hamiltonian cycle except the five known counterexamples; see [9, 2]. The second assertion of Lemma 5.10 and Proposition 5.13 show that the Lovász conjecture holds for the graph $\Gamma(V)$.

6 The automorphism group of the block transposition graph

We are in a position to give a proof for Theorem 4.4. From Proposition 3.1, the toric-reverse group $D_{n+1}$ is a subgroup $R$, the automorphism group of $\Gamma$. Also, $D_{n+1}$ is regular on $V$, by the second assertion of Lemma 5.10. Therefore, Theorem 4.4 is a corollary of the following lemma.

Lemma 6.1. The identity is the only automorphism of $\Gamma$ which fixes a vertex of $V$. 

14
Proof. We prove the assertion by induction on $n$. Computation shows that the assertion is true for $n = 5, 6$. Therefore, we assume $n \geq 7$.

First we prove that any automorphism of $\Gamma$ fixing a vertex $v \in V$ is an automorphism of $\Gamma(V)$ as well. Since $D_{n+1}$ is regular on $V$, we may limit ourselves to take $\sigma(0, 2, n)$ for $v$. Let $H$ be the subgroup of $R$ which fixes $\sigma(0, 2, n)$.

We look inside the action of $H$ on $\Gamma(V)$ and show that $H$ fixes the edge $\{\sigma(0, 2, n), \sigma(0, n - 2, n)\}$. By Proposition 5.12, $\Gamma(V)$ is 3-regular. More precisely, the endpoints of the edges of $\Gamma(V)$ which are incident with $\sigma(0, 2, n)$ are $\sigma(0, 2, 3)$, $\sigma(1, 2, n)$, and $\sigma(0, n - 2, n)$; see Lemma 5.4 (i) (iii) (v). By Proposition 5.9, the edge $e_{n-1} = \{\sigma(0, 2, n), \sigma(1, 2, n)\}$ is a maximal clique of $\Gamma$ of size 2, and no two distinct maximal cliques of $\Gamma$ of size 2 have a common vertex. Thus, $H$ fixes $\sigma(1, 2, n)$. Now, from Corollary 5.7 the edge $\{\sigma(0, 2, n), \sigma(0, n - 2, n)\}$ lies in a unique maximal clique of size $n - 1$. By Lemma 5.3 (i), the edge $\{\sigma(0, 2, n), \sigma(0, 2, 3)\}$ lies on a clique of size $n - 2$ whose set of vertices is $\{\sigma(0, 2, k) \mid 3 \leq k \leq n\}$. Here, we prove that any clique $C$ of size $n - 2$ containing the edge $\{\sigma(0, 2, n), \sigma(0, 2, 3)\}$ is maximal. By the first assertion of Proposition 5.6 (i), $\sigma(0, 2, 3)$ is adjacent to a unique vertex in $B$, namely $\sigma(0, 2, n)$. On the other hand, among the $2(n - 2)$ neighbors of $\sigma(0, 2, n)$, only as many as $n - 3$ vertices are off $B$, by the proof of Proposition 5.8. Then $C$ does not extend to a clique of size $n - 1$. Therefore, $H$ cannot interchange the edges $\{\sigma(0, 2, n), \sigma(0, n - 2, n)\}$ and $\{\sigma(0, 2, n), \sigma(0, 2, 3)\}$ but fixes both.

Here, we prove that $H$ fixes $e_{n-2} = \{\sigma(0, n - 2, n), \sigma(0, n - 2, n - 1)\}$ and $e_{n-3} = \{\sigma(n - 3, n - 2, n), \sigma(n - 3, n - 1, n)\}$. By Proposition 5.12 and Lemma 5.4 (i) (iii), $\sigma(0, n - 2, n)$ is adjacent to two more vertices, namely $\sigma(0, n - 2, n - 1)$ and $\sigma(n - 3, n - 2, n)$. Since $e_{n-2}$ is a maximal clique of $\Gamma$ of size 2, $H$ fixes $e_{n-2}$. This together with what we have proven so far shows that $H$ fixes $\sigma(n - 3, n - 2, n)$, and then the maximal clique $e_{n-3}$.

Since the edge $\{\sigma(0, 2, n), \sigma(0, n - 2, n)\}$ is in $\Gamma(B)$, Corollary 5.7 implies that $H$ preserves $B$. By Lemma 2.1, this implies that $H$ preserves $S_{n-2}^\triangle$, and hence it induces an automorphism group of $\Gamma(S_{n-2}^\triangle)$. Actually, we prove that $H$ fixes every vertex of $V \cap S_{n-2}^\triangle$. Since $H$ fixes $\{\sigma(0, 2, n), \sigma(0, 2, n)\}$, then the maximal clique $e_0 = \{\sigma(0, 1, 3), \sigma(0, 2, 3)\}$ is preserved by $H$. Therefore, two of the vertices adjacent to $\sigma(0, 2, 3)$, namely $\sigma(0, 2, n)$ and $\sigma(0, 1, 3)$, are fixed by $H$. From Proposition 5.12 and Lemma 5.4 (ii), the third edge incident to $\sigma(0, 2, 3)$ in $\Gamma(V)$ has its endpoint $\sigma(2, 3, 5)$. Hence, $H$ fixes $\sigma(2, 3, 5)$. Now, by the inductive hypothesis, $H$ fixes every block transpositions in $S_{n-2}^\triangle$. 

15
Therefore, $H$ fixes all the vertices in $V \cap S_{n-2}^\triangle$, namely all vertices in $\Lambda$ with $0 < l < n - 3$.

This together with what proven so far show that $H$ fixes all vertices of maximum cliques of $\Gamma$ of size 2 with only two possible exceptions, namely the endpoints of the edge $e_n = \{\sigma(0, 1, n-1), \sigma(1, n-1, n)\}$. In this exceptional case, $H$ would swap $\sigma(0, 1, n-1)$ and $\sigma(1, n-1, n)$. Actually, this exception cannot occur since $\sigma(0, 1, n-1)$ and $\sigma(1, n-1, n)$ do not have a common neighbor. Therefore, $H$ fixes every vertex in $V$. Hence, $H$ is the kernel of the permutation representation of $R$ on $V$. Thus $H$ is a normal subgroup of $R$.

Our next step is to prove that the block transpositions in $L \cup B$ are also fixed by $H$. Take any block transposition $\sigma(0, j, k)$. Here, we show that the toric class of $\sigma(0, j, k)$ contains a block transposition $\sigma(i', j', k')$ from $S_{n-2}^\triangle$. In fact, from Lemma 3.2

\begin{align*}
  f_{k+2}(\sigma(0, j, k)) &= \sigma(n - k - 1, n + j - k - 1, n - 1), \\
  f_{n-2}(\sigma(0, j, n-1)) &= \sigma(1, n - j, n - j + 2), \quad j \neq 2, \\
  f_{n-4}(\sigma(0, 2, n-1)) &= \sigma(1, 3, 5), \\
  f_{n-1}(\sigma(0, j, n)) &= \sigma(1, n - j + 1, n - j + 2), \quad j \neq 1, 2, \\
  f_{n-2}(\sigma(0, 1, n)) &= \sigma(1, 2, 3), \\
  f_{n-3}(\sigma(0, 2, n)) &= \sigma(1, 2, 4),
\end{align*}

for any positive integer $k$ with $k \neq n - 1, n - 2$. From what already proven, $\sigma(i', j', k')$ is fixed by $H$. Since $H$ is a normal subgroup of $R$, for every $h \in H$ there exists $h' \in H$ such that $f_{r}^{-1} \circ h' = h \circ f_{r}^{-1}$. From (18),

$$
\sigma(0, j, k) = (f_{r}^{-1} \circ h')(\sigma(i', j', k')) = h(\sigma(0, j, k)).
$$

This shows that $H$ fixes every block transposition in $L \cup B$. This holds true for $F$ by the second assertion of Proposition 5.6. Therefore, $H$ fixes every vertex of $\Gamma$.  

\section{The cases $n = 4, 5, 6$}

\subsection{n=4}

The 10 block transpositions of $\text{Sym}_4$ are listed below.

\begin{align*}
  1 &= \sigma(0, 1, 2), & 2 &= \sigma(0, 1, 3), & 3 &= \sigma(0, 1, 4), & 4 &= \sigma(0, 2, 3), \\
  5 &= \sigma(0, 2, 4), & 6 &= \sigma(0, 3, 4), & 7 &= \sigma(1, 2, 3), & 8 &= \sigma(1, 2, 4), \\
  9 &= \sigma(1, 3, 4), & 10 &= \sigma(2, 3, 4).
\end{align*}
The edges of the block transposition graph \( \Gamma \) of \( \text{Cay}(\text{Sym}_4, S_4) \) are

\[
\{1, 2\}, \{1, 4\}, \{1, 8\}, \{1, 10\}, \{2, 3\}, \{2, 5\}, \{2, 8\}, \{3, 4\}, \{3, 5\}, \{3, 9\}, \{4, 6\}, \{4, 10\}, \{5, 6\}, \{5, 7\}, \{6, 7\}, \{6, 8\}, \{7, 9\}, \{7, 10\}, \{8, 9\}, \{9, 10\}.
\]

\( \Gamma \) is a 4-regular. The full automorphism group of \( \Gamma \) is the dihedral group \( D_5 \) of order 10. The toric classes are

\[
\{1, 3, 6, 10, 7\}, \\
\{2, 5, 9, 4, 8\}.
\]

The maximal cliques of \( \Gamma \) of size 2 are

\[
\{4, 5\}, \{2, 4\}, \{5, 8\}, \{8, 9\}, \{2, 9\}.
\]

\( \Gamma(V) \) is a Hamiltonian and 2-regular graph. The full automorphism group of \( \Gamma(V) \) has order 10. The full automorphism group \( \text{Aut}(\text{Cay}(\text{Sym}_4, S_4)) \) has order 240.

### 7.2 \( n=5 \)

The 20 block transpositions of \( \text{Sym}_5 \) are listed below.

\[
1 = \sigma(0,1,2), \quad 2 = \sigma(0,1,3), \quad 3 = \sigma(0,1,4), \quad 4 = \sigma(0,1,5), \\
5 = \sigma(0,2,3), \quad 6 = \sigma(0,2,4), \quad 7 = \sigma(0,2,5), \quad 8 = \sigma(0,3,4), \\
9 = \sigma(0,3,5), \quad 10 = \sigma(0,4,5), \quad 11 = \sigma(1,2,3), \quad 12 = \sigma(1,2,4), \\
13 = \sigma(1,2,5), \quad 14 = \sigma(1,3,4), \quad 15 = \sigma(1,3,5), \quad 16 = \sigma(1,4,5), \\
17 = \sigma(2,3,4), \quad 18 = \sigma(2,3,5), \quad 19 = \sigma(2,4,5), \quad 20 = \sigma(3,4,5).
\]

The edges of the block transposition graph \( \Gamma \) of \( \text{Cay}(\text{Sym}_5, S_5) \) are

\[
\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 11\}, \{1, 12\}, \{1, 13\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \\
\{2, 14\}, \{2, 15\}, \{3, 4\}, \{3, 6\}, \{3, 8\}, \{3, 16\}, \{4, 7\}, \{4, 9\}, \{4, 10\}, \\
\{5, 6\}, \{5, 7\}, \{5, 11\}, \{5, 17\}, \{5, 18\}, \{6, 7\}, \{6, 8\}, \{6, 12\}, \{6, 19\}, \\
\{7, 9\}, \{7, 10\}, \{7, 13\}, \{8, 9\}, \{8, 14\}, \{8, 17\}, \{8, 20\}, \{9, 10\}, \{9, 15\}, \\
\{9, 18\}, \{10, 16\}, \{10, 19\}, \{10, 20\}, \{11, 12\}, \{11, 13\}, \{11, 17\}, \\
\{11, 18\}, \{12, 13\}, \{12, 14\}, \{12, 19\}, \{13, 15\}, \{13, 16\}, \{14, 15\}, \\
\{14, 17\}, \{14, 20\}, \{15, 16\}, \{15, 18\}, \{16, 19\}, \{16, 20\}, \{17, 18\}, \\
\{17, 20\}, \{18, 19\}, \{19, 20\}.
\]
\( \Gamma \) is a 6-regular graph. The full automorphism group of \( \Gamma \) is the dihedral group \( D_6 \) of order 12. The toric classes are

\[
\{1, 4, 10, 20, 17, 11\}, \\
\{2, 7, 16, 8, 18, 12\}, \\
\{3, 9, 19, 1, 4, 5, 13\}, \\
\{6, 15\}.
\]

The maximal cliques of \( \Gamma \) of size 2 are

\[
\{2, 5\}, \{13, 7\}, \{8, 9\}, \{12, 14\}, \{3, 16\}, \{18, 19\}.
\]

\( \Gamma(V) \) is a Hamiltonian and 3-regular graph. The full automorphism group of \( \Gamma(V) \) has order 48. The full automorphism group \( \text{Aut}(\text{Cay}(\text{Sym}_5, S_5)) \) has order 1440.

### 7.3 \( n=6 \)

The 35 block transpositions of \( \text{Sym}_6 \) are listed below.

\[
1 = \sigma(0, 1, 2), \quad 2 = \sigma(0, 1, 3), \quad 3 = \sigma(0, 1, 4), \quad 4 = \sigma(0, 1, 5), \\
5 = \sigma(0, 1, 6), \quad 6 = \sigma(0, 2, 3), \quad 7 = \sigma(0, 2, 4), \quad 8 = \sigma(0, 2, 5), \\
9 = \sigma(0, 2, 6), \quad 10 = \sigma(0, 3, 4), \quad 11 = \sigma(0, 3, 5), \quad 12 = \sigma(0, 3, 6), \\
13 = \sigma(0, 4, 5), \quad 14 = \sigma(0, 4, 6), \quad 15 = \sigma(0, 5, 6), \quad 16 = \sigma(1, 2, 3), \\
17 = \sigma(1, 2, 4), \quad 18 = \sigma(1, 2, 5), \quad 19 = \sigma(1, 2, 6), \quad 20 = \sigma(1, 3, 4), \\
21 = \sigma(1, 3, 5), \quad 22 = \sigma(1, 3, 6), \quad 23 = \sigma(1, 4, 5), \quad 24 = \sigma(1, 4, 6), \\
25 = \sigma(1, 5, 6), \quad 26 = \sigma(2, 3, 4), \quad 27 = \sigma(2, 3, 5), \quad 28 = \sigma(2, 3, 6), \\
29 = \sigma(2, 4, 5), \quad 30 = \sigma(2, 4, 6), \quad 31 = \sigma(2, 5, 6), \quad 32 = \sigma(3, 4, 5), \\
33 = \sigma(3, 4, 6), \quad 34 = \sigma(3, 5, 6), \quad 35 = \sigma(4, 5, 6).
\]
The edges of the block transposition graph $\Gamma$ of $\text{Cay}(\text{Sym}_6, S_6)$ are

\[
\{1, 2\}, \{1, 4\}, \{1, 8\}, \{1, 10\}, \{1, 18\}, \{1, 20\}, \{1, 33\}, \{1, 35\}, \{2, 3\},
\{2, 5\}, \{2, 8\}, \{2, 15\}, \{2, 18\}, \{2, 30\}, \{2, 33\}, \{3, 4\}, \{3, 5\}, \{3, 9\},
\{3, 15\}, \{3, 19\}, \{3, 30\}, \{3, 34\}, \{4, 6\}, \{4, 10\}, \{4, 16\}, \{4, 20\}, \{4, 31\},
\{4, 35\}, \{5, 6\}, \{5, 7\}, \{5, 11\}, \{5, 15\}, \{5, 26\}, \{5, 30\}, \{6, 7\}, \{6, 8\},
\{6, 11\}, \{6, 16\}, \{6, 26\}, \{6, 31\}, \{7, 9\}, \{7, 10\}, \{7, 11\}, \{7, 17\}, \{7, 26\},
\{7, 32\}, \{8, 9\}, \{8, 12\}, \{8, 18\}, \{8, 27\}, \{8, 33\}, \{9, 10\}, \{9, 12\}, \{9, 19\},
\{9, 19\}, \{9, 27\}, \{9, 34\}, \{10, 13\}, \{10, 20\}, \{10, 28\}, \{10, 35\}, \{11, 12\},
\{11, 13\}, \{11, 14\}, \{11, 21\}, \{11, 26\}, \{12, 13\}, \{12, 14\}, \{12, 15\},
\{12, 21\}, \{12, 27\}, \{13, 14\}, \{13, 16\}, \{13, 18\}, \{13, 21\}, \{13, 28\},
\{14, 17\}, \{14, 19\}, \{14, 20\}, \{14, 21\}, \{14, 29\}, \{15, 16\}, \{15, 17\},
\{15, 22\}, \{15, 30\}, \{16, 17\}, \{16, 18\}, \{16, 22\}, \{16, 31\}, \{17, 19\},
\{17, 20\}, \{17, 22\}, \{17, 32\}, \{18, 19\}, \{18, 23\}, \{18, 33\}, \{19, 20\},
\{19, 23\}, \{19, 34\}, \{20, 24\}, \{20, 35\}, \{21, 22\}, \{21, 23\}, \{21, 24\},
\{21, 25\}, \{22, 23\}, \{22, 24\}, \{22, 25\}, \{22, 26\}, \{23, 24\}, \{23, 27\},
\{23, 30\}, \{24, 25\}, \{24, 28\}, \{24, 31\}, \{24, 33\}, \{25, 29\}, \{25, 32\}, \{25, 34\}, \{25, 35\}, \{26, 27\}, \{26, 28\}, \{26, 29\}, \{27, 28\},
\{27, 29\}, \{27, 30\}, \{28, 29\}, \{28, 31\}, \{28, 33\}, \{29, 32\}, \{29, 34\},
\{29, 35\}, \{30, 31\}, \{30, 32\}, \{31, 32\}, \{31, 33\}, \{32, 34\}, \{32, 35\},
\{33, 34\}, \{34, 35\}.
\]

$\Gamma$ is a 8-regular graph. The full automorphism group of $\Gamma$ is the dihedral group $D_7$ of order 14. The toric classes are

\[
\{1, 2, 5, 11, 21, 25, 35\},
\{3, 6, 12, 22, 29, 20, 33\},
\{4, 8, 15, 26, 14, 24, 34\},
\{7, 13, 23, 32, 10, 18, 30\},
\{9, 16, 27, 17, 28, 19, 31\}.
\]

The maximal cliques of $\Gamma$ of size 2 are

\[
\{3, 4\}, \{6, 8\}, \{12, 15\}, \{14, 29\}, \{22, 26\}, \{24, 20\}, \{33, 34\}.
\]

$\Gamma(V)$ is a Hamiltonian and 3-regular graph. The full automorphism group of $\Gamma(V)$ has order 336. The full automorphism group $\text{Aut}(\text{Cay}(\text{Sym}_6, S_6))$ has order 10080.
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