Generalized Riesz Products on the Bohr Compactification of $\mathbb{R}$

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Abstract We study a class of generalized Riesz products connected to the spectral type of some class of rank one flows on $\mathbb{R}$. Applying Kac’s central limit theorem, we exhibit a large class of singular generalized Riesz products on the Bohr compactification of $\mathbb{R}$.

Keywords Generalized Riesz products · Almost periodic functions · Bohr compactification · Kakutani’s criterion · Bourgain’s singularity criterion · Flat polynomials

Mathematics Subject Classification 42A05 · 42A55 · 11L03 · 42A61

1 Introduction

The purpose of this paper is to extend and to study the notion of generalized Riesz product in the setting of the Bohr compactification of $\mathbb{R}$. This notion was formulated in the same manner as Peyrière in [11]. Therein, the authors proved that the spectral type of some class of rank one flows is given by some kind of generalized Riesz product on $\mathbb{R}$ normalized by some class of kernels. Here, following Peyrière’s suggestion [23], we give an alternative extension using the Bohr compactification of $\mathbb{R}$.
It is usual that the extension of some notions from the periodic setting to the almost periodic ones requires the consideration of the Bohr compactification of $\mathbb{R}$, which plays in the almost periodic case the same role played by the torus $\mathbb{T} \overset{\text{def}}{=} \{ t \in \mathbb{C}, |t| = 1 \}$ in the periodic case. As opposed to the torus, the Bohr compactification is often a non-separable compact topological space and this lack of separability is a source of difficulties in trying to adapt the arguments from the periodic context to the almost periodic one. Peyrière also mentioned those difficulties in [23].

Our analysis is also motivated by the recent growing interest in the problem of the flat polynomials in the real setting suggested in [24]. It turns out that the main idea developed in [24] can not be adapted in ours. For a recent account on the problem of the flat polynomials we refer the reader to [26].

The paper is organized as follows. In Sect. 2, we recall some standard facts on almost periodic functions and the Bohr compactification of $\mathbb{R}$. In Sect. 3, we introduce generalized Riesz products on the Bohr compactification of $\mathbb{R}$ and state our main result concerning the singularity of a large class of these generalized Riesz products. In Sect. 4, we summarize and extend the relevant material on Kakutani’s criterion and Bourgain’s criterion on the singularity of the generalized Riesz products introduced in Sect. 3. In Sect. 5, we state and prove the central limit theorem due to M. Kac. Finally, in Sect. 6, we apply Kac’s central limit theorem to prove our main result.

2 The Bohr Compactification of $\mathbb{R}$

The Bohr compactification of $\mathbb{R}$ is based on the theory of almost periodic functions initiated by H. Bohr [3] in connection with the celebrated Riemann’s $\zeta$-function. In this section we are going to recall the basic ingredients of this theory. For the classical presentation we refer the reader to [1,3,18, Chap V. Section 2].

The space of all almost periodic functions is denoted by $\mathcal{AP}(\mathbb{R})$. It is well known that $\mathcal{AP}(\mathbb{R})$ is a subspace of the space of bounded continuous functions on $\mathbb{R}$. An important characterization of almost periodic functions is due to Bohr and it can be stated as follows

**Theorem 2.1** (Bohr) A bounded continuous function $f$ is almost periodic function if, and only if, $f$ is uniformly approximated by finite linear combinations of functions in the set $\{ \cos(tx), \sin(tx) \}_{t \in \mathbb{R}}$.

We will denote by $\mathfrak{b}\mathbb{R}$ the Bohr compactification of $\mathbb{R}$, and by $C(\mathfrak{b}\mathbb{R})$ the space of continuous functions on $\mathfrak{b}\mathbb{R}$. We remind that one can define $\mathfrak{b}\mathbb{R}$ as follows:

**Theorem 2.2** [13] The group $\mathbb{R}$, equipped with the usual addition operation, may be embedded as a dense subgroup of a compact abelian group $\mathfrak{b}\mathbb{R}$ such that $\mathcal{AP}(\mathbb{R})$ is the family of all restrictions functions $f|_{\mathbb{R}}$ to $\mathbb{R}$ of functions $f \in C(\mathfrak{b}\mathbb{R})$. The operator $f \mapsto f|_{\mathbb{R}}$ is an isometric $\ast$-isomorphism of $C(\mathfrak{b}\mathbb{R})$ onto $\mathcal{AP}(\mathbb{R})$. Moreover, the addition operation $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ extends uniquely to the continuous group operation of $\mathfrak{b}\mathbb{R}$, $+: \mathfrak{b}\mathbb{R} \times \mathfrak{b}\mathbb{R} \rightarrow \mathfrak{b}\mathbb{R}$. The group $\mathfrak{b}\mathbb{R}$ is called the Bohr compactification of $\mathbb{R}$.

For simplicity of notation, for any $f$ in $\mathcal{AP}(\mathbb{R})$, we use the same letter $f$ for its canonical extension to $\mathfrak{b}\mathbb{R}$. We denote by $h$ the Haar measure on $\mathfrak{b}\mathbb{R}$, normalized to be a probability measure, and by $dt$ the usual Lebesgue measure on $\mathbb{R}$.
It is obvious that the continuous characters of \( b\mathbb{R} \) are the functions \( e^{i\lambda t} : b\mathbb{R} \rightarrow \mathbb{T} \). For \( f \in \mathcal{AP}(\mathbb{R}) \) we denote by \( \int_{b\mathbb{R}} f(t)dh(t) \) the asymptotic mean value of \( f \), given by

\[
\int_{b\mathbb{R}} f(t)dh(t) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt.
\]

Following [18], for any \( f \in \mathcal{AP}(\mathbb{R}) \), we put

\[
\hat{f}(\{\lambda\}) = \int_{b\mathbb{R}} f(t)e^{-i\lambda t}dh(t).
\]

\( (\hat{f}(\{\lambda\}))_{\lambda \in \mathbb{R}} \) are the Fourier coefficients of \( f \) relative to orthonormal family \( \{e^{i\lambda t}\}_{\lambda \in \mathbb{R}} \); the inner product is given by

\[
(f, g) = \int_{b\mathbb{R}} f(t)\overline{g}(t)dh(t).
\]

Furthermore, for any \( p \geq 1 \), we denote by \( \| \cdot \|_p \) the norm in \( L^p(b\mathbb{R}, h) \). We remind that the sequence of probability measures \( (\mu_n) \) on \( b\mathbb{R} \) converge in the weak-* topology to some probability measure \( \mu \) if, for any \( \lambda \in \mathbb{R} \),

\[
\hat{\mu_n}(\lambda) = \int_{b\mathbb{R}} e^{-i\lambda t}d\mu_n(t) \quad \xrightarrow{n\to+\infty} \quad \hat{\mu}(\lambda).
\]

### 3 Generalized Riesz Products on \( b\mathbb{R} \)

Riesz products was introduced in [28]. Therein, F. Riesz construct a continuous measure on the torus whose Fourier coefficients do not vanish at infinity. Roughly speaking, Riesz products are measures on \( \mathbb{T} \) which are constructed inductively. This powerful construction can be used to produce examples of measures with desired properties. Later, Riesz’s construction was extended in [29, p.208].

As shown in [21] Riesz products can be realized as a spectral type of some dynamical systems. Specific Riesz products are the right tool to describe the spectrum of a class of dynamical systems arising from the substitution [25,27]. A large class of Riesz products was realized in [15] as the maximal spectral type of the unitary operator associated to a non-singular dynamical system and a cocycle over it. In [4], the author established a connection between some class of generalized Riesz products on the circle and the maximal spectral type of rank one maps. Alternative proofs were given in [6,20,22].

Generalized Riesz products analogous to Peyrière–Riesz products were realized in [11] as a spectral type of some class of rank one flows.

Here, our aim is to extend the notion of generalized Riesz products to \( b\mathbb{R} \). Let \( (p_k)_{k \in \mathbb{N}} \) be a sequence of positive integers greater than 2 and \( (s_{k,j})_{j=0,\ldots,p_k} \) be
a sequence of finite sequence of positive real numbers with $s_{k,0} = 0$ for any $k \in \mathbb{N}$. Put

$$P_k(t) = \frac{1}{\sqrt{p_k}} \sum_{j=0}^{p_k-1} e^{it(j h_k + s_{k,j})},$$

(1)

where $s_{k,0} = 0$, and $s_{k,j} = \sum_{l=0}^{j} s_{k,l}$, $1 \leq j \leq p_k - 1$. The sequence $(h_k)$ is defined inductively by

$$h_0 = 1 \text{ and } h_{k+1} = p_k h_k + \sum_{l=0}^{p_k-1} s_{k,l} = p_k h_k + s_{k,p_k-1}. \quad (2)$$

For $p, q \in \{0, \cdots, p_k - 1\}$, we introduce the following sequence of positive real numbers

$$\overline{s_{k,p,q}} = \max(p,q) \sum_{j=\min(p,q)+1}^{\max(p,q)} s_{k,j} = (\overline{s_{k,p}} - \overline{s_{k,q}}) \text{ sgn}(p-q).$$

Finally, for any real number $t$, we set $e(t) = e^{it}$.

**Theorem 3.1** (Generalized Riesz Products on $\mathfrak{B}\mathbb{R}$) Let $(P_n)_{n \in \mathbb{N}}$ be a family of trigonometric polynomials given by (1) and $\sigma_n = \prod_{k=0}^{n} P_k(t) \, dh(t)$, $n = 0, 1, 2, \cdots$. Then $(\sigma_n)_{n \geq 0}$ converge in the weak-$\star$ topology to some probability measure $\sigma$ on $\mathfrak{B}\mathbb{R}$. $\sigma$ will be denoted by

$$\prod_{k=0}^{\infty} \left| P_k(t) \right|^2.$$

**Proof** Let $R_n(t) = \left| P_0(t) \cdots P_n(t) \right|^2$. Then $\sigma_n = R_n(t) \, dh(t)$, and by the definition of $P_n$, we have

$$\left| P_n(t) \right|^2 = 1 + \Delta_n(t) \text{ with } \Delta_n(t) = \frac{1}{p_n} \sum_{p \neq q} e^{\left( ((p-q)h_n + \overline{s_{n,p,q}} \text{ sgn}(p-q))t \right)}.$$

Hence, obviously

$$\int_{\mathfrak{B}\mathbb{R}} \left| P_n(t) \right|^2 \, dh(t) = 1.$$

Put

$$W_n = \{(b-a)h_n + \overline{s_{n,b}} - \overline{s_{n,a}} \mid b \neq a \in \{0, \cdots, p_n - 1\}\}.$$
By expanding the product of the $|P_k(t)|^2$, we can write
\[
\int_{\mathbb{R}} R_n(t) dh(t) - 1,
\]
as a sum of terms of the type
\[
\frac{1}{p_0 \cdots p_n} \int_{\mathbb{R}} e \left( \sum_{j=1}^{n} \epsilon_j w_j t \right) dh(t),
\]
where each $\epsilon_j$ is 0 or 1 (not all = 0) and each $w_j$ belongs to $W_j$. To show that
\[
\int_{\mathbb{R}} R_n(t) dh(t) - 1 = 0,
\]
it is sufficient to prove that each of these terms is null. For that, it is sufficient to prove that each of the numbers $\sum_{j=1}^{k} \epsilon_j w_j$ has absolute value $\geq 1$.

We consider one of these expressions $\sum_{j=1}^{n} \epsilon_j w_j$ and denote by $j_0$ the greater index $j$ such that $\epsilon_j \neq 0$. Then, we have
\[
\left| \sum_{j} \epsilon_j w_j \right| \geq |w_{j_0}| - \sum_{j < j_0} |w_j|.
\]
Moreover, for all $j$,
\[
h_j \leq |w_j| \leq (p_j - 1)h_j + s_{j,p_{j-1}}.
\]
Hence, to conclude, it is sufficient to prove that
\[
h_{j_0} - \sum_{j=0}^{j_0-1} \left( (p_j - 1)h_j + s_{j,p_{j-1}} \right) \geq 1.
\]
The equality
\[
h_n = p_{n-1}h_{n-1} + s_{n-1,p_{n-1}-1}
\]
implies by induction
\[
h_n - \sum_{k=0}^{n-1} (p_k - 1)h_k + s_{k,p_k-1} = 1,
\]
whence
\[
\int_{\mathbb{R}} R_n(t) dh(t) = 1. \tag{3}
\]
Thus $\sigma_n$ is a probability measure on $b\mathbb{R}$. In addition, for any $t \in \mathbb{R}$, we have

$$R_{n+1}(t) = R_n(t) |P_n(t)|^2 = R_n(t) + R_n(t) \Delta_n(t),$$

since $\hat{R}_n$ and $\hat{\Delta}_n$ are $\geq 0$ on $\mathbb{R}$

$$\hat{\sigma}_{n+1}(\lambda) = \hat{R}_{n+1}(\{\lambda\}) = \hat{R}_n(\{\lambda\}) \ast \hat{\Delta}_n(\lambda) \geq \hat{R}_n(\{\lambda\}) = \hat{\sigma}_n(\lambda).$$

Consequently the limit $r_\lambda$ of the sequence $(\hat{\sigma}_n(\lambda))$ exists. Now, since $b\mathbb{R}$ is a compact space, and $(\sigma_n)$ is a sequence of probability measures on $b\mathbb{R}$, we can extract a subnet $(\sigma_{n_k})$ which converges weakly to some probability measure on $b\mathbb{R}$. This gives that the limit of $(\sigma_n)$ exists in the weak-$\ast$ topology and the proof is complete. \hfill \square

The proof above is largely inspired by Lemma 2.1 in [20]; it gives more, namely, the polynomials $P_n$ given in (1) can be chosen with positive coefficients and satisfying

$$\int_{b\mathbb{R}} |P_n(t)|^2 dh(t) = 1 \text{ and } \int_{b\mathbb{R}} \prod_{j=1}^n |P_j(t)|^2 dh(t) = 1.$$

We further mention that we have

$$\int_{b\mathbb{R}} \prod_{j=1}^k |P_{n_j}(t)|^2 dh(t) = 1,$$

for any given sequence of positive integers $n_1 < n_2 < \cdots < n_k$, $k \in \mathbb{N}^\ast$. The proof is the same as the proof of (3).

We are now able to formulate our main result.

**Theorem 3.2** (Main result) *Let $(p_m)_{m \in \mathbb{N}}$ be a sequence of positive integers greater than 1 and $((s_m,j)_{j=0}^{p_m-1})_{m \in \mathbb{N}}$ be a sequence of positive real numbers. Defining $(h_m)$ as in (2) and assuming that there exists a sequence of positive integers $m_1 < m_2 < \cdots$, such that, the numbers $h_{m_j}, s_{m_j,0}, \cdots, s_{m_j,p_{m_j}-1}, j = 1, 2, \cdots$, are rationally independent. Then the generalized Riesz product

$$\mu = \prod_{k=0}^{+\infty} |P_k(t)|^2$$

where $P_k$ is given by (1), is singular with respect to the Haar measure on $b\mathbb{R}$.***

We recall that the real numbers $\lambda_1, \lambda_2, \cdots, \lambda_r, r \geq 2$, are rationally independent if they are linearly independent over $\mathbb{Z}$, i.e. for all $n_1, \cdots, n_r \in \mathbb{Z},$

$$n_1\lambda_1 + \cdots + n_r\lambda_r = 0 \implies n_1 = \cdots = n_r = 0.$$
We say that the set of real numbers \( \{ \lambda_n, \ n \geq 1 \} \) is rationally independent if for any finite subsequence \( m_1, m_2, \ldots, m_r \), the real numbers \( \lambda_{m_1}, \lambda_{m_2}, \ldots, \lambda_{m_r} \) are rationally independent.

### 4 On Kakutani’s Criterion and Bourgain’s Singularity Criterion in the Setting of Generalized Riesz Products on \( b\mathbb{R} \)

The famous dichotomy theorem of Kakutani has a rather long history. In his 1948 paper [17], Kakutani established a purity law for infinite product measures. More precisely, he proved that if \( P = \bigotimes_{i=1}^{+\infty} P_i \) and \( Q = \bigotimes_{i=1}^{+\infty} Q_i \) are infinite product measures, where \( P_i, Q_i \) are probability measures such that \( P_i \) is absolutely continuous with respect to \( Q_i \), for each positive integer \( i \). Then

\[
P \ll Q \text{ or } P \perp Q \text{ according as } \prod_i \int \left( \frac{dP_i}{dQ_i} \right) dQ_i \text{ converges or diverges.}
\]

There are several proofs of Kakutani’s criterion in literature (see [5] and the references therein). For a proof based on the Hellinger’s integral we refer the reader to [8, p. 60].

Here, applying Bourgain’s methods combined with the central limit tools introduced in [12], we obtain a new extension of Kakutani’s theorem in the setting of generalized Riesz products on \( b\mathbb{R} \). Indeed, we will see that the independence along subsequences allows us to prove the singularity.

We start by stating and proving the Bourgain’s singularity criterion in the setting of \( b\mathbb{R} \). We remind that this criterion in the periodic setting follows from the initial remarks in the proof of Proposition 1 in [4, Equations (2.15) and (2.22)].

**Theorem 4.1** *(\( b\mathbb{R} \) version of Bourgain’s criterion)* The following are equivalent

1. \( \int_{b\mathbb{R}} \prod_{k=0}^{N} |P_k| \, dh \xrightarrow[N \to +\infty]{\rightarrow} 0. \)
2. \( \mu \) is singular with respect to the Haar measure.
3. \( \inf \left\{ \int_{b\mathbb{R}} \prod_{\ell=1}^{L} |P_{n_\ell}| \, dh : L \in \mathbb{N}, \ n_1 < n_2 < \ldots < n_L \right\} = 0. \)

**Proof** \( (i) \implies (ii) \): it suffices to show that for any \( \epsilon > 0 \), there is a Borel set \( E \subset b\mathbb{R} \) with \( h(E) < \epsilon \) and \( \mu(E^c) < \epsilon \). Let \( 0 < \epsilon < 1 \), by \( (i) \) there exists \( N_0 \) such that

\[
\int_{b\mathbb{R}} \prod_{k=0}^{N} |P_k| \, dh < \epsilon^2.
\]
The set $E = \{ \lambda \in b^R : \prod_{k=0}^{N_0} |P_k(\lambda)| \geq \epsilon \}$ satisfies:

$$h(E) \leq \frac{1}{\epsilon} \left\| \prod_{k=0}^{N_0} P_k \right\|_1 \leq \epsilon^2 / \epsilon = \epsilon,$$

and, since $E^c$ is an open set, it follows by the portmanteau theorem that

$$\mu(E^c) \leq \liminf_{M \to +\infty} \int_{E^c} \prod_{k=0}^{N_0} |P_k|^2 \, dh \leq \liminf_{M \to +\infty} \int_{E^c} \prod_{k=0}^{N_0} |P_k|^2 \, dh \leq \epsilon^2 \lim_{M \to +\infty} \int_{b^R} \prod_{k=N_0+1}^{M} |P_k|^2 \, dh = \epsilon^2 < \epsilon.$$

For $(ii) \implies (i)$: given $0 < \epsilon < 1$, there exists a continuous function $\varphi$ on $b^R$ such that:

$$0 \leq \varphi \leq 1, \quad \mu(\{ \varphi \neq 0 \}) \leq \epsilon \quad (*) \quad \text{and} \quad h(\{ \varphi = 1 \}) \leq \epsilon \quad (**)$$

Indeed, by hypothesis, there exists a Borel set $E \subset b^R$ such that $\mu(E) = h(E^c) = 0$. Let $K \subset E$ be a compact set such that $h(K^c) < \epsilon$. Since $\mu(K) = 0$, we can choose a compact set $L \subset K^c$, such that $\mu(L^c) < \epsilon$. Then, by Urysohn’s Lemma [10, pp.47], there exists a continuous function $\varphi : b^R \to [0, 1]$, taking the value 0 at all points of $L$, and the value 1 at all points of $K$. It is easy to check that $\varphi$ satisfy $(*)$ and $(**)$.

Let $f_N = \prod_{k=1}^{N} |P_k|$. By Cauchy–Schwarz inequality, we have

$$\int_{b^R} f_N \, dh = \int_{\{ \varphi \neq 1 \}} f_N \, dh + \int_{\{ \varphi = 1 \}} f_N \, dh \leq h(\{ \varphi \neq 1 \})^{1/2} \left( \int_{b^R} f_N^2 \, dh \right)^{1/2} + \left( \int_{\{ \varphi = 1 \}} f_N^2 \, dh \right)^{1/2} h(\{ \varphi = 1 \})^{1/2} \leq \sqrt{\epsilon} + \left( \int_{b^R} f_N^2 \varphi \, dh \right)^{1/2}.$$

Since $\mu$ is the weak limit of $f_N^2 \, dh$, we obtain

$$\lim_{N \to \infty} \int_{b^R} f_N^2 \, dh = \int_{b^R} \varphi \, d\mu \leq \mu(\{ \varphi \neq 0 \}) \leq \epsilon.$$
Thus, \( \lim \sup \int_{\mathbb{R}} f_N \, dh \leq 2\sqrt{\epsilon} \). Since \( \epsilon \) is arbitrary, we get \( \lim_{N \to \infty} \int_{\mathbb{R}} f_N \, dh = 0 \).

Now (i) obviously implies (iii) and by a simple application of Cauchy–Schwarz inequality, (iii) implies (i). Indeed, consider \( n_1 < n_2 < \ldots < n_L, \, N \geq n_L \) and denote \( \mathcal{N} = \{n_1 < n_2 < \ldots < n_L\} \) with \( \mathcal{N}^c \) its complement in \( \{1, \ldots, N\} \). Then

\[
\int_{\mathbb{R}} \prod_{k=0}^{N} |P_k| \, dh = \int_{\mathbb{R}} \prod_{k \in \mathcal{N}} |P_k|^{\frac{1}{2}} \times \prod_{k \in \mathcal{N}^c} |P_k|^{\frac{1}{2}} \prod_{k=0}^{N} |P_k|^{\frac{1}{2}} \, dh
\]

\[
\leq \left( \int_{\mathbb{R}} \prod_{k \in \mathcal{N}} |P_k| \, dh \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \prod_{k \in \mathcal{N}^c} |P_k| \, dh \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \prod_{k=0}^{N} |P_k|^2 \, dh \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}} \prod_{k=0}^{N} |P_k|^2 \, dh \right)^{\frac{1}{4}}
\]

Therefore

\[
\int_{\mathbb{R}} \prod_{k=0}^{N} |P_k| \, dh \leq \left( \int_{\mathbb{R}} \prod_{k \in \mathcal{N}} |P_k| \, dh \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \prod_{k \in \mathcal{N}^c} |P_k|^2 \, dh \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}} \prod_{k=0}^{N} |P_k|^2 \, dh \right)^{\frac{1}{4}}
\]

\[
= \left( \int_{\mathbb{R}} \prod_{k \in \mathcal{N}} |P_k| \, dh(t) \right)^{\frac{1}{2}}.
\]

The last equality follows from (4).

From now, let us fix a sequence \( \mathcal{M} \) of positive integers for which the set \( \{h_m, s_m, 0, \ldots, s_m, p_m-1, \, m \in \mathcal{M}\} \) is linearly independent over the rational numbers. Let \( k \in \mathbb{N} \) and let \( \mathcal{N} = \{n_1 < n_2 < \ldots < n_k\} \) be a subsequence of \( \mathcal{M} \). Put

\[
Q_k(t) = \prod_{i=1}^{k} P_{n_i}(t).
\]

Our strategy in the proof of our main theorem is to construct a subsequence \( \{n_1 < n_2 < \ldots\} \) of \( \mathcal{M} \) for which the Bourgain’s criterion (Theorem 4.1) holds.

Having in mind applications beyond the context of this paper, we state and prove a sufficient condition for the existence of an absolutely continuous component with respect to the Haar measure for a given Riesz product on \( \mathbb{R} \). In the case of the torus, the result is due to Mélanie Guenais [14].

**Proposition 4.2** If \( \sum_{k=1}^{+\infty} \sqrt{1 - \left( \int_{\mathbb{R}} |P_k| \, dh \right)^2} < \infty \), then \( \mu \) admits an absolutely continuous component.
Proof Write $v_k^2 = 1 - \|P_k\|^2$. Then $\sum_{k=1}^{\infty} v_k < \infty$, equivalently $\prod_{k=1}^{\infty} \|P_k\| > 0$. For all functions $f, g \in L^2(b\mathbb{R}, dh)$, Cauchy-Schwarz inequality gives

$$|\|f \cdot g\|_1 - \|f\|_1 \|g\|_1| \leq (\|f\|_2^2 - \|f\|_1^2)^{1/2}(\|g\|_2^2 - \|g\|_1^2)^{1/2}.$$ 

Fix an integer $n_0 > 1$ and let $k > n_0$. Then

$$\left| \left\| \prod_{j=n_0}^{k} P_j \right\|_1 - \sum_{j=n_0}^{k-1} \|P_j\|_1 \right| \leq \left( \left\| \prod_{j=n_0}^{k-1} P_j \right\|_2^2 - \left\| \prod_{j=n_0}^{k-1} P_j \right\|_1^2 \right)^{1/2} \left( \|P_k\|_2^2 - \|P_k\|_1^2 \right)^{1/2} \leq v_k.$$ 

hence

$$\left| \left\| \prod_{j=n_0}^{k} P_j \right\|_1 - \sum_{j=n_0}^{k-1} \|P_j\|_1 \right| \leq v_k,$$

and this gives

$$\left| \left\| \prod_{j=n_0}^{k-1} P_j \right\|_1 - \sum_{j=n_0}^{k-2} \|P_j\|_1 \right| \leq v_{k-1} \|P_k\|_1 \leq v_{k-1},$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\left| \left\| P_{n_0+1} \right\|_1 - \sum_{j=n_0}^{n_0+2} \|P_j\|_1 \right| \leq v_{n_0+1},$$

since, for any $j \in \mathbb{N}$, $\|P_j\|_1 \leq 1$ by (4). On adding the above inequalities:

$$\left| \left\| \prod_{j=n_0}^{k} P_j \right\|_1 - \sum_{j=n_0}^{k-1} \|P_j\|_1 \right| \leq \sum_{j=n_0}^{k} v_j.$$
Since $\prod_{j=1}^{\infty} \|P_j\|_1 > 0$ and $\sum_{j=1}^{\infty} v_k < \infty$, we see that $\limsup_{k \to \infty} \| \prod_{j=1}^{k} P_j \|_1 > 0$.

Whence, by Bourgain’s criterion for singularity (Theorem 4.1), we see that $\mu$ is not singular to Haar measure on $b\mathbb{R}$. \hfill \square

5 On Kac’s Central Limit Theorem

Kac’s central limit theorem in the setting of $b\mathbb{R}$ is stated and proved in [16]. For the sake of completeness we prove it here using standard probability arguments.

We will need the following classical central limit theorem [9, p.81]. For our purpose, we state it in the following form

**Theorem 5.1** (Complex CLT) Let $(Z_k)_{k \in \mathbb{N}}$ be a sequence of independent and identically distributed complex random variables. Suppose that the real and imaginary parts of $Z_1$ have variance $\frac{1}{2}$ and covariance $0$. Then, the sequence of random variables

$$\left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Z_k \right)_{n \geq 1}$$

converges in distribution to the complex gaussian measure $\mathcal{N}_\mathbb{C}(0, 1)$ on $\mathbb{C}$.

Now let us state and prove Kac’s central limit theorem.

**Theorem 5.2** (Kac’s CLT [16]) Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of rationally independent real numbers. Then the functions $\cos(\lambda_n t) + i \sin(\lambda_n t)$, $n = 1, \cdots$, are stochastically independent and identically distributed under the Haar measure of $b\mathbb{R}$ and converges in distribution to the complex gaussian measure $\mathcal{N}_\mathbb{C}(0, 1)$ on $\mathbb{C}$.

**Proof** By Theorem 5.1, it suffices to prove that the random variables $e(\lambda_n t)$, $n = 1, \cdots$ are stochastically independent and identically distributed under the Haar measure of $b\mathbb{R}$. For that, notice that for any positive integer $k$ and for a given positive integers $l_1, l_2, \cdots, l_k$, we have

$$\int_{b\mathbb{R}} e(\lambda_1 t)^{l_1} e(\lambda_2 t)^{l_2} \cdots e(\lambda_k t)^{l_k} \, dh(t)$$

$$= \int_{b\mathbb{R}} e(\lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k) \, dh(t)$$

$$= \int_{b\mathbb{R}} e(\lambda_1 t)^{l_1} \, dt \int_{b\mathbb{R}} e(\lambda_2 t)^{l_2} \, dt \cdots \int_{b\mathbb{R}} e(\lambda_k t)^{l_k} \, dh(t).$$

Indeed, $\int_{b\mathbb{R}} e(\lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k) \, dh(t) = 1$ is equivalent to $\lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k = 0$, and by assumption, this is equivalent to $l_1 = l_2 = \cdots = l_k = 0$.  

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Therefore, if \( \lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k = 0 \), then

\[
\int_{b\mathbb{R}} e((\lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k)t)dh(t) = 1 = \prod_{j=1}^{k} \int_{b\mathbb{R}} e(\lambda_j t)^0 dh(t),
\]

and if \( \lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k \neq 0 \), then

\[
\int_{b\mathbb{R}} e((\lambda_1 l_1 + \lambda_2 l_2 + \cdots + \lambda_k l_k)t)dh(t) = 0 = \prod_{j=1}^{k} \int_{b\mathbb{R}} e(\lambda_j l_j t)dh(t).
\]

Hence, the random variables \( e(\lambda_n t), n = 1, \cdots \), are stochastically independent under the Haar measure on \( b\mathbb{R} \). It remains to prove that \( e(\lambda_n t), n = 1, \cdots \), are identically distributed.

Let \( s_1, s_2 \in \mathbb{R} \) and let us compute the characteristic function of the random variable \( e(\lambda t) \), with \( \lambda \in \{\lambda_n, n = 1, 2, \cdots \} \). We have

\[
\int_{b\mathbb{R}} e\left(\text{Re}\left((s_1 + is_2)e(-\lambda t)\right)\right)dh(t) = \sum_{n=0}^{+\infty} \frac{i^n}{n!} \int_{b\mathbb{R}} \text{Re}\left((s_1 + is_2)e(-\lambda t)\right)^n dh(t) = \sum_{n=0}^{+\infty} \frac{i^n}{n!} \int_{b\mathbb{R}} \left(s_1 \cos(\lambda t) + s_2 \sin(\lambda t)\right)^n dh(t).
\]

Write

\[
\cos(\lambda t) = \frac{1}{2}\left(e(\lambda t) + e(-\lambda t)\right), \quad \sin(\lambda t) = \frac{1}{2i}\left(e(\lambda t) - e(-\lambda t)\right),
\]

and notice that we have

\[
\int_{b\mathbb{R}} e(\alpha t)dh(t) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e(\alpha t)dt = \begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{if not.} \end{cases}
\]

Therefore

\[
\int_{b\mathbb{R}} \left(s_1 \cos(\lambda t) + s_2 \sin(\lambda t)\right)^n dh(t) = \sum_{k=0}^{n} \binom{n}{k} (s_2 + is_1)^k (-s_2 + is_1)^{n-k} \left(\int_{b\mathbb{R}} e((2k-n)\lambda t) dh(t)\right).
\]

Whence

\[
\int_{b\mathbb{R}} \left(s_1 \cos(\lambda t) + s_2 \sin(\lambda t)\right)^n dh(t) = \begin{cases} \binom{n}{k} (-1)^{\frac{n}{2}} \frac{(s_2^2 + s_1^2)^{\frac{n}{2}}}{(2i)^n} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}
\]
We thus get
\[
\int_{b}^{\infty} e^{\left(\text{Re}(s_1 + is_2) e(-\lambda t)\right)} dh(t) = \sum_{l=0}^{+\infty} \frac{(-1)^l}{4^l(l)!^2} (s_1^2 + s_2^2)^l = J_0(|s_1 + is_2|)
\]
\[
= \frac{1}{\pi} \int_{0}^{\pi} \cos(|s_1 + is_2| \sin(t)) dt,
\]
where \( J_0 \) is the familiar Bessel function. We conclude that the hypotheses of central limit theorem 5.1 are satisfied, and the sequence
\[
\left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e(\lambda_k t) \right)_{n \geq 1}
\]
converges in distribution to the complex gaussian measure \( \mathcal{N}_\mathbb{C}(0, 1) \) on \( \mathbb{C} \). In particular, by Theorem 5.3 below,
\[
\int_{b}^{\infty} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e(\lambda_k t) \right| dh(t) \overset{n \to +\infty}{\longrightarrow} \frac{1}{2} \sqrt{\pi}.
\]
\( \square \)

It is well-known that the convergence in distribution or probability does not in general imply that the moments converge (even if they exist). The useful condition to ensure the convergence of the moments is the uniform integrability. Indeed, we have

**Theorem 5.3** If the sequence of random variables \( \{X_n\} \) converges in distribution to some random variable \( X \) and if for some \( p > 0 \), \( \sup_{n \in \mathbb{N}} (\mathbb{E}(|X_n|^p)) = M < +\infty \), then for each \( r < p \),
\[
\lim_{n \to +\infty} \mathbb{E}\left(|X_n|^r\right) = \mathbb{E}\left(|X|^r\right).
\]

For the proof of Theorem 5.3 we refer to [2, pp. 32–33] or [7, p. 100]. We remind that the condition
\[
\sup_{n \in \mathbb{N}} \left( \mathbb{E}(|X_n|^{1+\varepsilon}) \right) < +\infty,
\]
for some \( \varepsilon \) positive, implies that \( \{X_n\} \) are uniformly integrable.

Notice that our proof yields that for any sequence \( (\lambda_j)_{j \in \mathbb{N}} \) of rationally independent numbers, we have
\[
\int_{b}^{\infty} \left| \frac{1}{\sqrt{q_n}} \sum_{j=0}^{q_n-1} e(\lambda_j t) \right| dt \overset{n \to +\infty}{\longrightarrow} \frac{1}{2} \sqrt{\pi}.
\]
(5)
6 Proof of the Main Result (Theorem 3.2)

Using the analogue of Féjer’s lemma combined with the CLT methods introduced in [12], we will give a direct proof of the singularity of a large class of generalized Riesz products on $\mathbb{R}$. Therefore, our strategy is slightly different from the strategy used by many authors in the case of the torus [4,12,19,20]. More precisely, they showed that the weak limit point of the sequence $\left(\left|P_m \right|^2 - 1\right)$ is bounded below by a positive constant and it is well-known that this implies the singularity of the generalized Riesz products (see for instance [12] or [19]).

Let us start our proof with the following analogue of Féjer’s lemma [29, Theorem 4.15, p. 52]

**Lemma 6.1** Under the notations of Theorem 3.2, let $n_1 < n_2 < \cdots < n_k \in \mathcal{M}$ and $Q = \prod_{j=1}^{k} P_{n_j}$. Then, for any $m > n_k$, $m \in \mathcal{M}$, we have

$$\int_{\mathbb{R}} \left|Q\right| P_m dh = \left(\int_{\mathbb{R}} |Q| dh\right) \left(\int_{\mathbb{R}} |P_m| dh\right).$$

**Proof** By our assumption $(h_k, (s_k,p_k-1)) \in \mathcal{M}$ are rationally independent. Hence, by Kac’s theorem (Theorem 5.2), for $m > n_k$, $m \in \mathcal{M}$, $Q$ and $P_m$ are stochastically independent since $Q$ is the sum of the random variables $e(\lambda t)$ with $\lambda \in \{j h_{n_i} + \overline{s_{n_i,j}}, j = 0, \cdots, p_{n_i} - 1, i = 1, \cdots, k\}$ and $P_m$ is the sum of the random variables $e(\lambda t)$ with $\lambda \in \{j h_{m} + \overline{s_{m,j}}, j = 0, \cdots, p_m - 1\}$. \hfill \Box

**Proof of Theorem 3.2** By application of Lemma 6.1 combined with (5), we proceed inductively to construct a subsequence $\{n_k\}$ of $\mathcal{M}$ such that, for any $k \geq 1$, we have

$$\int_{\mathbb{R}} \prod_{j=1}^{k+1} \left|P_{n_j}\right| dh \leq \frac{51}{100} \sqrt{\pi} \int_{\mathbb{R}} \prod_{j=1}^{k} \left|P_{n_j}\right| dh. \quad (6)$$

Denote $u_k = \int_{\mathbb{R}} \prod_{j=1}^{k} \left|P_{n_j}\right| dh$. Since $\frac{51}{100} \sqrt{\pi} < 1$, the sequence $(u_k)$ will be decreasing with limit 0. Hence, $\mu$ is singular with respect to the Haar measure by Bourgain’s criterion (Theorem 4.1).

By our assumption combined with Theorem 5.2, $(P_m)_{m\in\mathcal{M}}$ converges in distribution to the complex gaussian measure $\mathcal{N}_{\mathbb{C}}(0, 1)$. Hence, from Theorem 5.3 applied with $p = 2$ and $r = 1$ we get

$$\lim_{m \to +\infty} \int_{\mathbb{R}} \left|P_m\right| dh = \int_{\mathbb{C}} |z| d\mathcal{N}_{\mathbb{C}}(0, 1)(z) = \frac{\sqrt{\pi}}{2}. \quad (7)$$

Recall that the density of the standard complex normal distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$ is given by

$$f(z) = \frac{1}{\pi} e^{-|z|^2}.$$
Let us assume that we have already constructed $n_1 < n_2 < n_3 < \cdots < n_k$ satisfying $u_k \leq \frac{51}{100} \sqrt{\pi} u_{k-1}$. By (7) there exists $m > n_k$, $m \in M$, such that

$$\left| \int_{b\mathbb{R}} |P_m| dh - \frac{\sqrt{\pi}}{2} \right| < \frac{\sqrt{\pi}}{100}.$$ 

Therefore, by Lemma 6.1, we can write

$$\int_{b\mathbb{R}} |Q||P_m| dh \leq \frac{51}{100} \sqrt{\pi} \int_{b\mathbb{R}} |Q| dh,$$

and set $n_{k+1} = m$. This implies that the inequality (6) holds which finish the proof of the theorem.

**Remark 6.2** The argument in the above proof strongly depends on the assumption that along subsequence the set of positive real numbers $\{h_m, s_{m,0}, \cdots, s_{m,p_m-1}, m \geq 1\}$ is linearly independent over the rationales. In the general case, one may use the methods of [4], [19], [20] and [12] to establish the singularity of a large class of generalized Riesz products on $b\mathbb{R}$, in particular when $(p_m)$ is bounded. In a forthcoming paper, we will see how to extend classical results from the torus and real line settings to the Riesz products on $b\mathbb{R}$.

In view of (5) one may ask

**Question 1** Let $n \in \mathbb{N}^*$ and put

$$J_{\lambda,n} = \left\{ P \in \mathcal{P} : P(t) = \sum_{k=0}^{n} a_k e^{i\lambda k t}, a_0 = 1, a_k \in \{0, 1\} \right\}.$$

where $\mathcal{P}$ is the subspace of polynomials on $b\mathbb{R}$. The polynomials in $J_{\lambda,n}$ are called Newman polynomials. Is it true that have for any increasing sequence $(\omega_j)_{j \in \mathbb{N}}$ of real numbers, we have

$$\sup_{n \geq 1} \left\{ \sup_{P \in J_{\omega,n}} \left( \frac{\|P(t)\|_1}{\|P(t)\|_2} \right) \right\} < 1?$$

In the contrast of the previous question, one may ask

**Question 2** Does there exist a sequence of $L^1$-flat Newman polynomials on $b\mathbb{R}$? That is, does there exist a sequence of polynomials $P_n \in J_{\lambda,n}$ such that

$$\lim_{n \to +\infty} \frac{\|P_n(t)\|_1}{\|P_n(t)\|_2} = 1?$$

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References

1. Besicovitch, A.S.: Almost Periodic Functions. Cambridge University Press, Cambridge (1932)
2. Billingsley, P.: Convergence of Probability Measures. Wiley Series in Probability and Statistics: Probability and Statistics, 2nd edn. Wiley, New York (1999)
3. Bohr, H.: Almost Periodic Functions. Chelsea Pub. Co., New York (1947)
4. Bourgain, J.: On the spectral type of Ornstein class one transformations. Isr. J. Math. 84, 53–63 (1993)
5. Brown, G., Moran, W.: Products of random variables and Kakutani’s criterion for orthogonality of product measures. J. Lond. Math. Soc. (2) 10, 401–405 (1975)
6. Choksi, J.R., Nadkarni, M.G.: The maximal spectral type of rank one transformation. Can. Math. Bull. 37(1), 29–36 (1994)
7. Chung, K.L.: A Course in Probability Theory, 3rd edn. Academic Press Inc, San Diego (2001)
8. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge (1992)
9. D. Dacunha-Castelle and M. Duflo, Probabilités et statistiques. Tome 2, (French) [Probability and statistics. Vol. 2], Problèmes à temps mobile [Movable-time problems], Collection Mathématiques Appliquées pour la Maîtrise [Collection of Applied Mathematics for the Master’s Degree], Masson, Paris, 1983
10. Dudley, R.M.: Real Analysis and Probability. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove (1989)
11. El Abdalaoui, E.H., Lemańczyk, M., Lesigne, E., Ulcigrai, C.: Spectral disjointness in some class of rank one flows, preprint
12. El Abdalaoui, E.H.: A new class of rank-one transformations with singular spectrum. Ergod. Theory Dynam. Syst. 27(5), 1541–1555 (2007)
13. Gelfand, I.M., Raikov, D.A., Chilov, G.E.: Commutative Normed Rings. Chelsea Pub. Co., New York (1964)
14. Guenais, M.: Étude spectrale de certains produits gauches en théorie ergodique. Thèse de doctorat, Paris 13, (1997)
15. Host, B., Mélia, J.-F., Parreau, F.: Nonsingular transformations and spectral analysis of measures. Bull. Soc. Math. France 119(1), 33–90 (1991)
16. Kac, M.: Statistical Independence in Probability, Analysis and Number Theory. The Carus Mathematical Monographs. No. 12 Published by the Mathematical Association of America. Wiley, New York (1959)
17. Kakutani, S.: On equivalence of infinite product measures. Ann. of Math. (2) 49, 214–224 (1948)
18. Katznelson, Y.: An Introduction to Harmonic Analysis, 3rd edn. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2004)
19. Klemes, I.: The spectral type of staircase transformations. Thohoku Math. J. 48, 247–258 (1994)
20. Klemes, I., Reinhold, K.: Rank one transformations with singular spectre type. Isr. J. Math. 98, 1–14 (1997)
21. Ledrappier, F.: Des produits de Riesz comme mesures spectrales (French, English summary). Ann. Inst. Henri Poincaré Sect. B (N.S.) 6, 335–344 (1970)
22. Nadkarni, M.G.: Spectral Theory of Dynamical Systems. Birkhäuser, Cambridge (1998)
23. Peyrière, J.: Étude de quelques propriétés des produits de Riesz. Ann. Inst. Fourier (Grenoble) 25(2), 127–169 (1975)
24. Prikhod’ko, A.A.: Littlewood polynomials and their applications to the spectral theory of dynamical systems. (Russian) Mat. Sb. 204 (6), 135–160 (2013); translation in Sb. Math. 204 (5–6), 910–935 (2013)
25. Queffélec, M.: Mesures spectrales associées à certaines suites arithmétiques (French). Bull. Soc. Math. France 107(4), 385–421 (1979)
26. Queffélec, H., Saffari, B.: On Bernstein’s inequality and Kahane’s ultraflat polynomials. J. Fourier Anal. Appl. 2(6), 519–582 (1996)
27. Queffélec, M.: Substitution Dynamical Systems-Spectral Analysis. Lecture Notes in Mathematics, vol. 1294, 2nd edn. Springer, Berlin (2010)
28. Riesz, F.: Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung (German). Math. Z. 2(3–4), 312–315 (1918)
29. Zygmund, A.: Trigonometric Series, vol. I, 2nd edn. Cambridge University Press, Cambridge (1959)