Quadrangularity and Strong Quadrangularity in Tournaments

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Abstract

The pattern of a matrix $M$ is a $(0,1)$-matrix which replaces all non-zero entries of $M$ with a $1$. A directed graph is said to support $M$ if its adjacency matrix is the pattern of $M$. If $M$ is an orthogonal matrix, then a digraph which supports $M$ must satisfy a condition known as quadrangularity. We look at quadrangularity in tournaments and determine for which orders quadrangular tournaments exist. We also look at a more restrictive necessary condition for a digraph to support an orthogonal matrix, and give a construction for tournaments which meet this condition.

1 Introduction

A directed graph or digraph, $D$, is a set of vertices $V(D)$ together with a set of ordered pairs of the vertices, $A(D)$, called arcs. If $(u,v)$ is an arc in a digraph, we say that $u$ beats $v$ or $u$ dominates $v$, and typically write this as $u \rightarrow v$. If $v \in V(D)$ then we define the outset of $v$ by,

$$O_D(v) = \{ u \in V(D) : (v,u) \in A(D) \}.$$ 

That is, $O_D(v)$ is all vertices in $D$ which $v$ beats. Similarly, we define the set of all vertices in $D$ which beat $v$ to be the inset of $v$, written,

$$I_D(v) = \{ u \in V(D) : (u,v) \in A(D) \}.$$ 

The closed outset and closed inset of a vertex $v$ are $O_D[v] = O_D(v) \cup \{v\}$ and $I_D[v] = I_D(v) \cup \{v\}$ respectively. The in-degree and out-degree of a vertex $v$ are $d_D^-(v) = |I_D(v)|$ and $d_D^+(v) = |O_D(v)|$ respectively. When it is clear to which digraph $v$ belongs, we will drop the subscript. The minimum out-degree (in-degree) of $D$ is the smallest out-degree (in-degree) of any vertex in $D$ and is represented by $\delta^+(D)$ ($\delta^-(D)$). Similarly, the maximum out-degree (in-degree) of $D$ is the largest out-degree (in-degree) of any vertex in $D$ and is represented by $\Delta^+(D)$ ($\Delta^-(D)$).

A tournament $T$ is a directed graph with the property that for each pair of distinct vertices $u, v \in V(T)$ exactly one of $(u,v), (v,u)$ is in $A(T)$. An $n$-tournament is a tournament on $n$ vertices. If $T$ is a tournament and $W \subseteq V(T)$ we denote by $T[W]$ the subtournament of $T$ induced on $W$. The dual of a tournament $T$, which we denote by $T^\text{r}$, is the tournament on the same vertices as $T$ with $x \rightarrow y$ in $T^\text{r}$ if and only if $y \rightarrow x$ in $T$. If $X, Y \subseteq V(T)$ such that $x \rightarrow y$ for all $x \in X$ and $y \in Y$, then we write $X \Rightarrow Y$. If $X = \{x\}$ or $Y = \{y\}$ we write $x \Rightarrow Y$ or $X \Rightarrow y$ respectively for $X \Rightarrow Y$. A vertex $s \in V(T)$ such that $s \Rightarrow V(T) - s$ is called a transmitter. Similarly a receiver is a vertex $t$ of $T$ such that $V(T) - t \Rightarrow t$.

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We say that a tournament is regular if every vertex has the same out-degree. A tournament is called near regular if the largest difference between the out-degrees of any two vertices is 1. Let $S$ be a subset of $\{1,2,\ldots,2k\}$ of order $k$ such that if $i,j\in S$, $i+j \not\equiv 0 \pmod{2k+1}$. The tournament on $2k+1$ vertices labeled $0,1,\ldots,2k$, with $i\to j$ if and only if $j-i \pmod{2k+1} \in S$ is called a rotational tournament with symbol $S$. If $p \equiv 3 \pmod{4}$ is a prime and $S$ is the set of quadratic residues modulo $p$, then the rotational tournament whose symbol is $S$ is called the quadratic residue tournament of order $p$, denoted $QR_p$. We note that $|O(x)\cap O(y)|=|I(x)\cap I(y)|=k$ for all distinct $x,y\in V(QR_p)$ where $p=4k+3$. For more on tournaments the reader is referred to [2], [11], and [12].

Let $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ be $n$-vectors over some field (While the following definitions hold over any field, we are interested only in those of characteristic 0). We use $\langle x,y \rangle$ to denote the usual euclidean inner product of $x$ and $y$. We say that $x$ and $y$ are combinatorially orthogonal if $|\{i:x_i y_i \neq 0\}| \neq 1$. Observe, this is a necessary condition for $x$ and $y$ to be orthogonal, for if there were a unique $i$ so that $x_i y_i \neq 0$, then $\langle x,y \rangle = x_i y_i \neq 0$. We say a matrix $M$ is combinatorially orthogonal if every two rows of $M$ are combinatorially orthogonal and every two columns of $M$ are combinatorially orthogonal. In [4] Beasley, Bruudli and Shader study matrices with the combinatorial orthogonality property to obtain a lower bound on the number of non-zero entries in a fully indecomposable orthogonal matrix.

Let $M$ be an $n \times n$ matrix. The pattern of $M$ is the $(0,1)$-matrix whose $i,j$ entry is 1 if and only if the $i,j$ entry of $M$ is non-zero. If $D$ is the directed graph whose adjacency matrix is the pattern of $M$, we say that $D$ supports $M$ or that $D$ is the digraph of $M$. We say a digraph $D$ is out-quadrangular if for all distinct $u,v \in V(D)$, $|O(u)\cap O(v)| \neq 1$. Similarly, if for all distinct $u,v \in V(D)$, $|I(u)\cap I(v)| \neq 1$, we say $D$ is in-quadrangular. If $D$ is both out-quadrangular and in-quadrangular, then we say $D$ is quadrangular. It is easy to see that if $D$ is the digraph of $M$, then $D$ is quadrangular if and only if $M$ is combinatorially orthogonal. So, if $D$ is the digraph of an orthogonal matrix, $D$ must be quadrangular. In [4], Gibson and Zhang study an equivalent version of quadrangularity in undirected graphs. In [10], Lundgren, Severini and Stewart study quadrangularity in tournaments. In the following section we expand on the results in [10], and in section 3 we consider another necessary condition for a digraph to support an orthogonal matrix.

### 2 Known orders of quadrangular tournaments

In this section we determine for exactly which $n$ there exists a quadrangular tournament on $n$ vertices. We first need some results from [10].

**Theorem 2.1** [10] Let $T$ be an out-quadrangular tournament and choose $v \in V(T)$. Let $W$ be the subtournament of $T$ induced on the vertices of $O(v)$. Then $W$ contains no vertices of out-degree 1.

**Theorem 2.2** [10] Let $T$ be an in-quadrangular tournament and choose $v \in V(T)$. Let $W$ be the subtournament of $T$ induced on $I(v)$. Then $W$ contains no vertices of in-degree 1.

**Corollary 2.1** [10] If $T$ is an out-quadrangular tournament with $\delta^+(T) \geq 2$, then $\delta^+(T) \geq 4$.

**Corollary 2.2** [10] If $T$ is a quadrangular tournament with $\delta^+(T) \geq 2$ and $\delta^-(T) \geq 2$, then $\delta^+(T) \geq 4$ and $\delta^-(T) \geq 4$.

Note that the only tournament on 4 vertices with no vertex of out-degree 1 is a 3-cycle together with a receiver. Similarly, the only tournament on 4 vertices with no vertex of in-degree 1 is a 3-cycle with a receiver. Thus, if a quadrangular tournament $T$ has a vertex $v$ of out-degree 4, $T[I(u)]$ must be a 3-cycle with a receiver, and if $u$ has in-degree 4, $T[I(u)]$ must be a 3-cycle with a transmitter.

**Theorem 2.3** There does not exist a quadrangular near regular tournament of order 10.
Proof. Suppose $T$ is such a tournament and pick a vertex $x$ with $d^+(x) = 5$. So $d^-(x) = 4$. Therefore $I(x)$ must induce a subtournament comprised of a 3-cycle, and a transmitter. Call this transmitter $u$. If a vertex $y$ in $O(x)$ has $O(y) = I(x)$, then $|O(y) \cap O(w)| = 1$ for all $w \neq u$ in $I(x)$. This contradicts $T$ being quadrangular, so $O(y) \neq I(x)$ for any $y \in O(x)$. Since every vertex in $O(x)$ beats at most 3 vertices outside of $O(x)$, and since $T$ is near regular we have that $\delta^+(T |O(x)|) \geq 1$. Thus, by Theorem 2.1 we have $\delta^+(T |O(x)|) \geq 2$. This means that $T |O(x)|$ must be the regular tournament on 5 vertices.

Consider the vertex $u$ which forms the transmitter in $T |I(x)|$. Since $u$ beats $I[x] - u$, and $T$ is near regular, $u$ can beat at most one vertex in $O(x)$. If $u \to z$ for any $z \in O(x)$, then $|O(u) \cap O(x)| = |\{z\}| = 1$ which contradicts $T$ being quadrangular. Thus, $z \to u$ for all $z \in O(x)$. Since $T$ is near regular, it has exactly 5 vertices of out-degree 5, one of which is $x$. So, there can be at most four vertices in $O(x)$ with out-degree 5. Thus, there exists some vertex in $O(x)$ with out-degree 4, call it $v$. Since $x \to v$, $v$ beats 2 vertices in $O(x)$ and $v \to u$ there is exactly one vertex $r \in I(x) - u$ such that $v \to r$. Since $O(u) = I[x] - u$, we have $|O(v) \cap O(u)| = |\{r\}| = 1$. Therefore, $T$ is not quadrangular, and so such a tournament does not exist.

Given a digraph $D$, and set $S \subseteq V(D)$, we say that $S$ is a dominating set in $D$ if each vertex of $D$ is in $S$ or dominated by some vertex of $S$. The size of a smallest dominating set in $D$ is called the domination number of $D$, and is denoted by $\gamma(D)$. In [10] a relationship is shown to hold in certain tournaments between quadrangularity and the domination number of a subtournament.

Lemma 2.1 If $T$ is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$, then $T$ is near regular. Further, if $d^-(x) = 3$, then $I(x)$ induces a 3-cycle, and if $d^+(y) = 3$, then $O(y)$ induces a 3-cycle.

Proof. Let $T$ be such a tournament. If $T$ has a vertex $a$ with $d^-(a) = 0$ or 1, then $I[a]$ would form a dominating set of size 1 or 2 respectively. If $T$ had a vertex $b$ with $I(b) = \{u,v\}$, where $u \to v$, then $\{u,v\}$ forms a dominating set of size 2. So $d_T(x) \geq 3$ for all $x \in V(T)$. Similarly, $d_T(x) \geq 3$ for all $x \in V(T)$. Thus,

$$3 \leq d_T^-(x) = d_T^+(x) = 8 - 1 - d_T(x) \leq 7 - 3 = 4$$

for all $x \in V(T)$. That is $3 \leq d_T^+(x) \leq 4$ for all $x \in V(T)$, and $T$ is near regular. Now, pick $x \in V(T)$ with $d^-(x) = 3$. If $I(x)$ induces a transitive triple with transmitter $u$, then $\{u,x\}$ would form a dominating set in $T$. Thus, $I(x)$ must induce a 3-cycle. By duality we have that $O(y)$ induces a 3-cycle for all $y$ with $d^+(y) = 3$.

Up to isomorphism there are 4 tournaments on 4 vertices, and exactly one of these is strongly connected. We refer to this tournament as the strong 4-tournament, and note that it is also the only tournament on 4 vertices without a vertex of out-degree 3 or 0.

Lemma 2.2 Suppose $T$ is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$. Then if $x \in V(T)$ with $d^+(x) = 4$, $O(x)$ induces the strong 4-tournament.

Proof. By Lemma 2.1 $T$ is near regular so pick $x \in V(T)$ with $d^+(x) = 4$, and let $W$ be the subtournament induced on $O(x)$. If there exists $u \in V(W)$ with $d^+_W(u) = 0$, then since $d^+_T(u) \geq 3$, $u \to I(x)$ and $\{u,x\}$ forms a dominating set in $T$. This contradicts $\gamma(T) \geq 3$, so no such $u$ exists. Now assume there exists a vertex $v \in V(W)$ with $d^+_W(v) = 3$. If $d^+_T(v) = 4$, then $v \to y$ for some $y \in I(x)$. So, $I(v) = I[x] - y$. However, $I(v) = I[x] - y$ forms a transitive triple, a contradiction to Lemma 2.1. So $d^+_T(v) = 3$. Now, since $\delta^+(W) > 0$, the vertices of $W - v$ all have out-degree 1 in $W$. If some $z \in V(W) - v$ had $d^+_v(z) = 4$, then $z \to I(x)$ and $\{x,z\}$ would form a dominating set of size 2. Therefore, all $z \in V(W)$ have $d^+_v(z) = 3$. Since $T$ is near regular, this implies that every vertex of $I[x]$ must have out-degree 4. Further, since $d^+_v(3) = 3$, $O(x) \subseteq O(x)$ and so $I(x) \Rightarrow v$. So, each vertex of $I(x)$ dominates $x$, $v$ and another vertex of $I(x)$. Thus, each vertex of $I(x)$ dominates a unique vertex of $O(x) - v$. Further each vertex of $O(x) - v$ has out-degree 3 in $T$ and so must
be dominated by a unique vertex of $I(x)$. So label the vertices of $I(x)$ as $y_1, y_2, y_3$ and the vertices of $O(x) - v$ as $w_1, w_2, w_3$ so that $y_i \rightarrow w_i$, and $w_i \rightarrow y_j$ for $i \neq j$. Since $I(x)$ and $O(x) - v$ form 3-cycles we may also assume that $y_1 \rightarrow y_2 \rightarrow y_3, y_3 \rightarrow y_1$ and $w_1 \rightarrow w_2 \rightarrow w_3$ and $w_3 \rightarrow w_1$. So, $O(w_1) = \{w_2, y_2, y_3\}$ which forms a transitive triple a contradiction to Lemma 2.1. Hence, no such $v$ exists and $1 \leq \delta^+(W) \leq \Delta^+(W) \leq 2$ and $W$ is the strong 4-tournament.

\textbf{Theorem 2.4} Let $T$ be a tournament on 8 vertices. Then $\gamma(T) \leq 2$ or $\gamma(T') \leq 2$.

\textbf{Proof.} Suppose to the contrary that $T$ is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T') \geq 3$. By Lemma 2.1 we know that $T$ is near regular. Let $W$ be the subtournament of $T$ induced on the vertices of out-degree 4. We can always choose $x$ in $W$ with $d_W^+(x) \geq 2$. So pick $x \in V(T)$ with $d_T^+(x) = 4$ so that it dominates at most one vertex of out-degree 4. By Lemma 2.2 $O(x)$ induces the strong 4-tournament. By our choice of $x$, at least one of the vertices with out-degree 2 in $T[O(x)]$ has out-degree 3 in $T$. Call this vertex $x_1$. Label the vertices of $O(x_1) \cap O(x)$ as $x_2$ and $x_3$ so that $x_2 \rightarrow x_3$, and label the remaining vertex of $O(x)$ as $x_0$. Note since $T[O(x)]$ is the strong 4-tournament, we must have $x_3 \rightarrow x_0$ and $x_0 \rightarrow x_1$. Since $d_T^+(x_1) = 3$, $x_1$ must dominate exactly one vertex in $I(x)$, call it $y_1$. Recall $I(x)$ must induce a 3-cycle by Lemma 2.1 so we can label the remaining vertices of $I(x)$ as $y_2$ and $y_3$ so that $y_1 \rightarrow y_2 \rightarrow y_3$ and $y_3 \rightarrow y_1$. Note since $O(x_1) \cap I(x) = y_1, y_2 \rightarrow x_1$ and $y_3 \rightarrow x_1$. Also, by Lemma 2.1 $O(x_1)$ forms a 3-cycle, so $x_3 \rightarrow y_1$ and $y_1 \rightarrow x_2$.

Now, assume to the contrary that $y_1 \rightarrow x_0$. Then $O(y_1) = \{x_0, x_2, x, y_2\}$. Now, since $O(x_1) \cap O(x) = \{x_0\}$, $d_T^+(x_3) = 3$ or else $x_3 \Rightarrow I(x)$ and $\{x, x_3\}$ forms a dominating set of size 2. So, $x_3$ dominates exactly one of $y_2$ or $y_3$. If $x_3 \rightarrow y_2$ then $y_3 \rightarrow x_3$ and since $y_3 \rightarrow x_1$, $\{y_1, y_3\}$ forms a dominating set of size 2. So, assume $x_3 \rightarrow y_2$ and $y_2 \rightarrow x_3$. Then, $x, y_3, x_1, x_3 \in O(y_2)$ and $\{y_2, y_1\}$ forms a dominating set of size 2. Thus $x_0 \rightarrow y_1$.

If $x_3 \rightarrow y_2$, then $\{x_3, x\}$ forms a dominating set of size 2, a contradiction. So, $y_2 \rightarrow x_3$. Now, if $x_3 \rightarrow y_3$ then $O(x_3) = \{y_1, y_3, x_0\}$. However, $y_3 \rightarrow y_1$ and $x_0 \rightarrow y_1$, so $O(x_3)$ forms a transitive triple, a contradiction to Lemma 2.1. Thus $y_3 \rightarrow x_3$. Since $d_T^+(y_3) \leq 4$ and $y_1, x, x_1, x_3 \in O(y_3)$, these are all the vertices in $O(y_3)$. So, $x_0 \rightarrow y_3$.

If $x_0 \rightarrow y_2$ then $x_0 \Rightarrow I(x)$ and $\{x, x_0\}$ form a dominating set of size 2, so $y_2 \rightarrow x_0$. So, $x_0, y_3, x \in O(y_2)$ and $y_1, x, x_3 \in O(x_1)$, and so $\{y_2, x_1\}$ forms a dominating set of size 2. Therefore, such a tournament cannot exist. \hfill $\square$

\textbf{Theorem 2.5} No tournament $T$ on 9 vertices with $\delta^+(T) \geq 2$ is out-quadrangular.

\textbf{Proof.} Suppose to the contrary $T$ is such a tournament. Since $T$ is out-quadrangular, and $\delta^+(T) \geq 2$, by Corollary 2.1 $\delta^+(T) \geq 4$. Since the order of $T$ is 9, this means $T$ must be regular. Pick a vertex $x \in V(T)$. Then $O(x)$ must induce a subtournament which is a 3-cycle together with a receiver. Call the receiver of this subtournament $y$. Since $T$ is regular, $d^+(y) = 4$. Since $I(y) = O[x] - y$, this means $O(y) = I(x)$. So, $O(y) = I(x)$ must induce a subtournament which is a 3-cycle together with a receiver vertex. Call this receiver $z$. Since $d^+(z) = 4$, $y \rightarrow z$ and $I(x) - z$ dominate $z$, $O(z) = O[x] - y$. Now, $x \Rightarrow O(x) - y$ and $O(x) - y$ is a 3-cycle so $T[O(z)]$ must contain a vertex of out-degree 1. Hence, by Theorem 2.1 $T$ is not out-quadrangular. Thus, no such tournament exists. \hfill $\square$

\textbf{Corollary 2.3} No tournament $T$ on 9 vertices with $\delta^-(T) \geq 2$ is in-quadrangular.

\textbf{Proof.} Let $T$ be a tournament on 9 vertices with $\delta^-(T) \geq 2$. Then $T'$ is not out-quadrangular by Theorem 2.5. Thus $T$ is not in-quadrangular. \hfill $\square$

We now state a few more results from [10].

\textbf{Theorem 2.6} [10] Let $T$ be a tournament on 4 or more vertices with a vertex $x$ of out-degree 1, say $x \rightarrow y$. Then, $T$ is quadrangular if and only if
1. \( O(y) = V(T) - \{x, y\} \),
2. \( \gamma(T - \{x, y\}) > 2 \),
3. \( \gamma((T - \{x, y\})^r) > 2 \).

**Theorem 2.7** \[10\] Let \( T \) be a tournament on 3 or more vertices with a transmitter \( s \) and receiver \( t \). Then \( T \) is quadrangular if and only if both \( \gamma(T - \{s, t\}) > 2 \) and \( \gamma((T - \{s, t\})^r) > 2 \).

**Theorem 2.8** \[10\] Let \( T \) be a tournament with a transmitter \( s \) and no receiver. Then \( T \) is quadrangular if and only if, \( \gamma(t - s) > 2 \), \( T - s \) is out-quadrangular, and \( \delta^+(T - s) \geq 2 \).

**Corollary 2.4** \[10\] Let \( T \) be a tournament with a receiver \( t \) and no transmitter. Then \( T \) is quadrangular if and only if \( \gamma((T - t)^r) > 2 \), \( T - t \) is in-quadrangular, and \( \delta^-(T - t) \geq 2 \).

**Corollary 2.5** No quadrangular tournament of order 10 exists.

Proof. By Corollaries 2.3 and 2.4 and by Theorems 2.6, 2.7 and 2.8 a quadrangular tournament \( T \) must satisfy one of the following.

1. \( \delta^+(T) \geq 4 \), and hence \( T \) is near regular.
2. \( T \) has a transmitter \( s \) and receiver \( t \) such that \( \gamma(T - \{s, t\}) > 2 \) and \( \gamma((T - \{s, t\})^r) > 2 \).
3. \( T \) contains an arc \((x, y)\) such that \( O(y) = I(x) = V(T) - \{x, y\} \) and \( \gamma(T - \{x, y\}) > 2 \) and \( \gamma((T - \{x, y\})^r) > 2 \).
4. \( T \) has a transmitter \( s \) and \( T - s \) is out-quadrangular with \( \delta^+(T - s) \geq 2 \).
5. \( T \) has a receiver \( t \) and \( T - t \) is in-quadrangular with \( \delta^-(T - t) \geq 2 \).

Note, Theorem 2.3 implies that case 1 is impossible. If 2 or 3 were satisfied, then there would be a tournament on 8 vertices such that it and its dual have domination number at least 3, which contradicts Theorem 2.4. If 4 were satisfied, then \( T - s \) would be of order 9 and out-quadrangular, a contradiction to Theorem 2.5. Similarly, 5 contradicts Corollary 2.3. Thus, no quadrangular tournament on 10 vertices exists.

For the construction in Theorem 2.10 we need the following theorem from [10].

**Theorem 2.9** \[10\] Let \( T \) be a rotational tournament on \( n \geq 5 \) vertices, with symbol \( S \). Then, \( T \) is quadrangular if and only if for all integers \( m \) with \( 1 \leq m \leq \frac{n - 1}{2} \) there exist distinct subsets \( \{i, j\}, \{k, l\} \subseteq S \) such that \( (i - j) \equiv (k - l) \equiv m \pmod{n} \).

**Theorem 2.10** There exist quadrangular tournaments of order 11, 12 and 13.

Proof. Consider the quadratic residue tournament of order 11, \( QR_{11} \). For all \( u, v \in V(QR_{11}) \), recall that \( |O(u) \cap O(v)| = |I(u) \cap I(v)| = \frac{n - 1}{2} = 5 \). Thus, \( QR_{11} \) is quadrangular. Further, this implies that for any two vertices \( u, v \in V(QR_{11}) \) there exists a vertex which dominates both \( u \) and \( v \), so \( \gamma(QR_{11}) > 2 \). Also, since \( QR_{11} \) is regular, \( \delta^+(QR_{11}) = 5 \geq 2 \). Let \( W \) be the tournament formed by adding a transmitter to \( QR_{11} \). Then by Theorem 2.8 \( W \) is quadrangular.

Now, let \( T \) be the rotational tournament on 13 vertices with symbol \( S = \{1, 2, 3, 5, 6, 9\} \). The following table gives the subsets of \( S \) which satisfy Theorem 2.9. Thus, \( T \) is quadrangular.

| \( m \) | subsets       |
|-------|--------------|
| 1     | \{2, 1\}, \{3, 2\} |
| 2     | \{3, 1\}, \{5, 3\} |
| 3     | \{5, 2\}, \{6, 3\} |
| 4     | \{6, 2\}, \{9, 5\} |
| 5     | \{6, 1\}, \{1, 9\} |
| 6     | \{9, 3\}, \{2, 9\} |
Theorem 2.11 There exists a quadrangular tournament of order 14.

Proof. Construct $T$ of order 14 in the following way. Start with a set $V$ of 14 distinct vertices. Partition $V$ into 7 sets of order 2 labeled $V_0, V_1, V_2, \ldots, V_6$. Each $V_i$ is to induce the 2-tournament, and $V_i \rightarrow V_j$ if and only if $j - i \pmod{7}$ is one of 1, 2, 4. We show that the resulting 14-tournament, $T$, is quadrangular.

Note that the condensation of $T$ on $V_0, \ldots, V_6$ is just the quadratic residue tournament on 7 vertices, $QR_7$. Now, $QR_7$ has the property that $|O(x) \cap O(y)| = 1$ for all $x, y \in V(QR_7)$. Thus, if $u, v \in V(T)$ such that $u \in V_i$, $v \in V_j$ for $i \neq j$, $|O(u) \cap O(v)| = 2$. Further, since $QR_7$ is regular of degree 3, if $u, v \in V(T)$ with $u, v \in V_i$ then $|O(u) \cap O(v)| = 6$. Thus, $|O(u) \cap O(v)| \neq 1$ for all $u, v \in V(T)$, and so $T$ is out-quadrangular. Further, since $QR_7$ is isomorphic to its dual, a similar argument shows that $T$ is in-quadrangular and hence quadrangular.

Theorem 2.12 If $n \geq 15$, then there exists a quadrangular tournament on $n$ vertices.

Proof. Pick $n \geq 15$. Let $a_1, a_2, a_3, \ldots, a_l$ be a sequence of at least 3 integers such that $a_i \geq 5$ for each $i$, and $\sum_{i=1}^{l} a_i = n$. Pick $l$ regular or near regular tournaments $T_1, T_2, \ldots, T_l$ such that $|V(T_i)| = a_i$ for each $i$. Let $T'$ be a tournament with $V(T') = \{1, 2, 3, \ldots, l\}$ such that $T'$ has no transmitter or receiver. Construct the tournament $T$ on $n$ vertices as follows. Start with a a set $V$ of $n$ vertices, and partition $V$ into sets $S_1, S_2, \ldots, S_l$ of size $a_1, a_2, \ldots, a_l$ respectively. Place arcs in each $S_i$ to form $T_i$. Now, add arcs such that $S_i \rightarrow S_j$ if and only if $i \rightarrow j$ in $T'$. We claim that the resulting tournament, $T$, is quadrangular.

Pick $u, v \in V(T)$. We consider two possibilities. First, suppose that $u, v \in S_i$ for some $i$. By choice of $T'$, $i \rightarrow j$ for some $j$. Thus

$$|O(u) \cap O(v)| \geq |S_j| = a_j \geq 5 > 1.$$  

Now, suppose that $u \in S_i$ and $v \in S_j$ for $i \neq j$. Since $T'$ is a tournament either $i \rightarrow j$ or $j \rightarrow i$. Without loss of generality, assume that $i \rightarrow j$. Then

$$|O(u) \cap O(v)| \geq |O(v) \cap S_j| \geq \frac{k - 1}{2} \geq 2 > 1.$$  

This shows that $T$ is out-quadrangular. The proof that $T$ is in-quadrangular is similar. Thus, $T$ is a quadrangular tournament of order $n \geq 15$.

Observe that if $T'$ in the construction is strong, then $T$ is strong. Further, if $a_i = k$ for all $i$ and $T'$ is regular, then $T$ is regular or near regular depending on if $k$ is odd or even. We now characterize those $n$ for which there exist a quadrangular tournament of order $n$.

Theorem 2.13 There exists a quadrangular tournament of order $n$ if and only if $n = 1, 2, 3, 9$ or $n \geq 11$.

Proof. Note that the single vertex, the single arc, and the 3-cycle are all quadrangular. Now, recall that the smallest tournament with domination number 3 is $QR_7$ (For a proof of this see [7]). Further, $QR_7$ is isomorphic to its dual, so $\gamma(QR_7) = 3$. This fact together with Theorems 2.8 and 2.7 tells us that the smallest quadrangular tournament, $T$, on $n \geq 4$ vertices with $\delta^+(T) = \delta^-(T) = 0$ or $\delta^+(T) = 1$ or $\delta^-(T) = 1$ has order 9.

Theorem 2.8 and Corollary 2.4 together with the fact that $QR_7$ is the smallest tournament with domination number 3 imply that a quadrangular tournament with just a transmitter or receiver must have at least 8 vertices. However, $QR_7$ is the only tournament on 7 vertices with domination number 3 and a quick check shows that $QR_7$ is neither out-quadrangular nor in-quadrangular. So, $QR_7$ together with a transmitter or receiver is not quadrangular, and hence any quadrangular tournament with just a transmitter or receiver must have order 9 or higher.
Corollary 2.2 states that if \( \delta^+(T) \geq 2 \) and \( \delta^-(T) \geq 2 \), then \( \delta^+(T) \geq 4 \) and \( \delta^-(T) \geq 4 \). The smallest tournament which meets these requirements is a regular tournament on 9 vertices. Thus, there are no quadrangular tournaments of order 4, 5, 6, 7 or 8. The result now follows from Corollary 2.8 and Theorems 2.10, 2.11 and 2.12.

It turns out that quadrangularity is a common (asymptotic) property in tournaments as the following probabilistic result shows.

**Theorem 2.14** Almost all tournaments are quadrangular.

**Proof.** Let \( P(n) \) denote the probability that a random tournament on \( n \) vertices contains a pair of distinct vertices \( x \) and \( y \) so that \( |O(x) \cap O(y)| = 1 \). We now give an over-count for the number of labeled tournaments on \( n \) vertices which contain such a pair, and show \( P(n) \to 0 \) as \( n \to \infty \).

There are \( \binom{n}{2} \) ways to pick distinct vertices \( x \) and \( y \), and the arc between them can be oriented so that \( x \to y \) or \( y \to x \). There are \( n - 2 \) vertices which can play the role of \( z \) where \( \{z\} = O(x) \cap O(y) \). For each \( w \notin \{x, y, z\} \) there are 3 ways to orient the arcs from \( x \) and \( y \) to \( w \), namely \( w \Rightarrow x, y \), \( w \to x \) and \( y \to w \), or \( w \to y \) and \( x \to w \). Also, there are \( n - 3 \) such \( w \). The arcs between all other vertices are arbitrary. So there are \( 2^{(n-2)} \) ways to finish the tournament. When orienting the remaining arcs we may double count some of these tournaments, so all together there are at most

\[
2 \binom{n}{2} (n - 2)3^{n-3}2^{(n-2)^2}
\]
tournaments containing such a pair of vertices. Now, there are \( 2^{(n-2)} \) total labeled tournaments so,

\[
0 \leq P(n) \leq \frac{2^{(n-2)}(n - 2)3^{(n-3)}2^{(n-2)^2}}{2^{(n-2)}}
\]
\[
= n(n - 1)(n - 2)3^{(n-3)}2^{(n-2)^2}
\]
\[
= \frac{n(n - 1)(n - 2)3^{n-3}}{2^{(n-2)^2 + n - 2 + n - 1}}
\]
\[
= \frac{n(n - 1)(n - 2)3^{n-3}}{2^{2(n-3)2^4}}
\]
\[
= \frac{n(n - 1)(n - 2)}{8} \left( \frac{3}{4} \right)^{n-3}
\]
\[
= \frac{1}{4} n(n - 1)(n - 2) \left( \frac{3}{4} \right)^{n-3}.
\]

Since this value tends to 0 as \( n \) tends to \( \infty \), it must be that \( P(n) \to 0 \) as \( n \to \infty \).

From duality we have that the probability that vertices \( x \) and \( y \) exists such that \( |I(x) \cap I(y)| = 1 \) also tends to 0 as \( n \) tends to \( \infty \). Thus, the probability that a tournament is not quadrangular tends to 0 as \( n \) tends to \( \infty \). That is, almost all tournaments are quadrangular. \( \square \)

### 3 Strong Quadrangularity

In this section we define a stronger necessary condition for a digraph to support an orthogonal matrix, and give a construction for a class of tournaments which satisfy this condition. Let \( D \) be a digraph. Let \( S \subseteq V(D) \) such that for all \( u \in S \), there exists \( v \in S \) such that \( O(u) \cap O(v) \neq \emptyset \), and let \( S' \subseteq V(D) \) such that for all \( u \in S' \), there exists \( v \in S' \) such that \( I(u) \cap I(v) \neq \emptyset \). We say that \( D \) is strongly quadrangular if for all such sets \( S \) and \( S' \),
Let and define the set of neighbors of \( A \) must contain a Hamiltonian path. So, label the elements of \( B \).

Theorem 3.1

\[ kl \]

Pick an orthogonal matrix. To see that this is in fact a more restrictive condition consider the following.

In [15], Severini showed that strong quadrangularity is a necessary condition for a digraph to support an orthogonal matrix. To see that this is in fact a more restrictive condition consider the following.

In 

Lemma 3.1

So \( T \) is not strongly quadrangular. We now construct a class of strongly quadrangular tournaments, but first observe the following lemma.

**Lemma 3.1** Let \( T \) be a tournament on \( n \geq 4 \) vertices. Then there must exist distinct \( a, b \in V(T) \) such that \( O(a) \cap O(b) \neq \emptyset \).

**Proof.** Pick a vertex \( a \) of maximum out-degree in \( T \). As, \( n \geq 4 \), \( d^+(a) \geq 2 \). Pick a vertex \( b \) of maximum out-degree in the subtournament \( W \) induced on \( O(a) \). As \( d^+(a) \geq 2 \), \( d^+_w(b) \geq 1 \). Thus, 

\[
\left| \bigcup_{u,v \in S} (O(u) \cap O(v)) \right| = \left| (O(0) \cap O(1)) \cup (O(0) \cap O(5)) \cup (O(1) \cap O(5)) \right|
\]

\[
= \left| \{2, x\} \cup \{2, x\} \cup \{2, x\} \right|
\]

\[
= 2
\]

\[
< |S|.
\]

So \( T \) is not strongly quadrangular. We now construct a class of strongly quadrangular tournaments.

**Theorem 3.1** Pick \( l \geq 1 \). Let \( T' \) be a strong tournament on the vertices \( \{1, 2, \ldots, l\} \), and let \( T_1, T_2, \ldots, T_l \) be regular or near-regular tournaments of order \( k \geq 5 \). Construct a tournament \( T \) on \( kl \) vertices as follows. Let \( V \) be a set of \( kl \) vertices. Partition the vertices of \( V \) into \( l \) subsets \( V_1, \ldots, V_l \) of size \( k \) and place arcs to form copies of \( T_1, T_2, \ldots, T_l \) on \( V_1, \ldots, V_l \) respectively. Finally, add arcs so that \( V_i \Rightarrow V_j \) if and only if \( i \rightarrow j \) in \( T' \). Then the resulting tournament, \( T \), is a strongly quadrangular tournament.

**Proof.** Pick \( S \subseteq V(T) \). Define the set 

\[ A = \{V_i : \exists u \neq v \in S \ni u, v \in V_i\}, \]

and define the set 

\[ B = \{V_i : \exists u \in S \ni u \in V_i\}. \]

Let \( \alpha = |A| \), and \( \beta = |B| \). Then, since each \( V_i \) has \( k \) vertices, \( k\alpha + \beta \geq |S| \). Consider the subtournaments of \( T' \) induced on the vertices corresponding to \( A \) and \( B \). These are tournaments and so must contain a Hamiltonian path. So, label the elements of \( A \) and \( B \) so that \( A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_n \) and \( B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_\beta \). By definition of \( A \), each \( A_i \) contains at least two vertices of \( S \), and so if \( x, y \in S \) and \( x, y \in A_i \), \( i \leq \alpha - 1 \), then \( A_{i+1} \subseteq O(x) \cap O(y) \). Thus, 

\[
\left| \bigcup_{u,v \in S} (O(u) \cap O(v)) \right| \geq k(\alpha - 1).
\]

We now consider three cases depending on \( \beta \).
First assume that \( \beta \geq 2 \). Consider the vertices of \( S \) in \( B \) we see that if \( x, y \in S \) and \( x \in B_i \) and \( y \in B_{i+1} \) then \( O(y) \cap B_{i+1} \subseteq O(x) \cap O(y) \). Thus, \( |O(x) \cap O(y)| \geq \frac{k-1}{2} \), and so

\[
\left| \bigcup_{u, v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \frac{k-1}{2}(\beta - 1) \geq k(\alpha - 1) + 2\beta - 2 \geq k(\alpha - 1) + \beta.
\]

Now, since \( T' \) is a tournament, either \( A_1 \Rightarrow B_1 \) or \( B_1 \Rightarrow A_1 \). If \( A_1 \Rightarrow B_1 \), then for vertices \( x, y \in A_1 \) we know \( B_1 \subseteq O(x) \cap O(y) \). Since no vertex of \( B_1 \) had been previously counted, we have that

\[
\left| \bigcup_{u, v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \beta + k = k\alpha + \beta \geq |S|.
\]

So, assume that \( B_1 \Rightarrow A_1 \). Then for the single vertex of \( S \) in \( B_1 \), \( u \), and a vertex \( v \) of \( S \) in \( A_1 \) \( O(v) \subseteq O(u) \cap O(v) \). This adds \( \frac{k-1}{2} \) vertices which were not previously counted. Also, since \( T' \) is strong, some \( A_i \Rightarrow V_j \) for some \( V_j \not\in A \). We counted at most \( \frac{k-1}{2} \) vertices in \( V_j \) before, and since \( A_i \) contains at least two vertices \( x, y \) from \( S \) these vertices add at least \( \frac{k+1}{2} \) vertices which were not previously counted, so

\[
\left| \bigcup_{u, v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \beta + \frac{k-1}{2} + \frac{k+1}{2} = k\alpha + \beta \geq |S|.
\]

Now assume that \( \beta = 1 \). Since \( T' \) is strong we know that \( A_i \Rightarrow V_j \) for some \( V_j \not\in A \). So,

\[
\left| \bigcup_{u, v \in S} O(u) \cap O(v) \right| \geq k\alpha.
\]

Now, if \( |S| \leq k\alpha \), then we are done, so assume that \( |S| = k\alpha + 1 \). So, for every \( A_i \in A, A_i \subseteq S \). So by Lemma 3.1 we can find two vertices of \( S \) in \( A_1 \) which compete over a vertex of \( A_i \), adding one more vertex to our count, and

\[
\left| \bigcup_{u, v \in S} O(u) \cap O(v) \right| \geq k\alpha + 1 \geq |S|.
\]

For the last case, assume that \( \beta = 0 \). Then since \( T' \) is strong we once again have that some \( A_i \Rightarrow V_j \) for some \( V_j \not\in A \). Thus,

\[
\left| \bigcup_{u, v \in S} O(u) \cap O(v) \right| \geq k\alpha \geq |S|.
\]

Note that the dual of \( T' \) will again be strong, and the dual of each \( T_i \) will again be regular. Thus, by appealing to duality in \( T \) we have that for all \( S \subseteq V(T) \),

\[
\left| \bigcup_{u, v \in S} I(u) \cap I(v) \right| \geq |S|,
\]

and so \( T \) is a strongly quadrangular tournament.

\( \square \)
Recall that strong quadrangularity is a necessary condition for a digraph to support an orthogonal matrix. To emphasize this, consider the strongly quadrangular tournament, $T$, which the construction in the previous theorem gives on 15 vertices. For this tournament, $T_1, T_2$ and $T_3$ are all regular of order 5, and $T'$ is the 3-cycle. Note that up to isomorphism, there is only one regular tournament on 5 vertices, so without loss of generality, assume that $T_1, T_2$ and $T_3$ are the rotational tournament with symbol $\{1, 2\}$. We now show that $T$ cannot be the digraph of an orthogonal matrix.

Let $J_5$ denote the $5 \times 5$ matrix of all 1s, $O_5$ the $5 \times 5$ matrix of all 0s and set

$$RT_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$ 

Then the adjacency matrix $M$ of $T$ is

$$M = \begin{pmatrix} RT_5 & J_5 & O_5 \\ O_5 & RT_5 & J_5 \\ J_5 & O_5 & RT_5 \end{pmatrix}.$$ 

Now, suppose to the contrary that there exists an orthogonal matrix $U$ whose pattern is $M$. Let $R_i$ and $C_j$ denote the $i^{th}$ rows and columns of $U$ respectively for each $i = 1, \ldots, 15$, and let $U_{i,j}$ denote the $i,j$ entry of $U$. Observe from the pattern of $U$ that the only entries of $U$ which contribute to $\langle C_i, C_j \rangle$ for $i = 1, \ldots, 5$, $j = 6, \ldots, 10$ are in the first five rows. So, $(C_1, C_j) = U_{4,1}U_{4,j} + U_{5,1}U_{5,j}$ for $j = 6, \ldots, 10$. Thus, since $0 = \langle C_1, C_j \rangle$ for each $j \neq 1$,

$$U_{4,1} = -\frac{U_{5,1}U_{5,6}}{U_{4,6}} = -\frac{U_{5,1}U_{5,7}}{U_{4,7}} = -\frac{U_{5,1}U_{5,8}}{U_{4,8}} = -\frac{U_{5,1}U_{5,9}}{U_{4,9}} = -\frac{U_{5,1}U_{5,10}}{U_{4,10}}.$$ 

Since $U_{5,1} \neq 0$ this gives,

$$-\frac{U_{4,1}}{U_{5,1}} = \frac{U_{5,6}}{U_{4,6}} = \frac{U_{5,7}}{U_{4,7}} = \frac{U_{5,8}}{U_{4,8}} = \frac{U_{5,9}}{U_{4,9}} = \frac{U_{5,10}}{U_{4,10}}.$$ 

So, the vectors $(U_{4,1}, \ldots, U_{4,10})$ and $(U_{5,6}, \ldots, U_{5,10})$ are scalar multiples of each other. Now, note that for $j = 6, \ldots, 10$, we have $0 = \langle C_2, C_j \rangle = U_{1,2}U_{1,j} + U_{5,2}U_{5,j}$. So, by applying the same argument, we see that $(U_{5,6}, U_{5,7}, U_{5,8}, U_{5,9}, U_{5,10})$ is a scalar multiple of $(U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10})$.

So, $(U_{4,6}, U_{4,7}, U_{4,8}, U_{4,9}, U_{4,10})$ is a scalar multiple of $(U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10})$. Now, from the pattern of $U$ we see that only the $6^{th}$ through $10^{th}$ columns of $U$ contribute to $\langle R_1, R_4 \rangle$. So, since linearly dependent vectors cannot be orthogonal,

$$\langle R_1, R_4 \rangle = \langle (U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10}), (U_{4,6}, U_{4,7}, U_{4,8}, U_{4,9}, U_{4,10}) \rangle \neq 0.$$ 

This contradicts our assumption that $U$ is orthogonal. So, $T$ is not the digraph of an orthogonal matrix.

### 4 Conclusions

The problem of determining whether or not there exist tournaments (other than the 3-cycle) which support orthogonal matrices has proved to be quite difficult. As we have seen in sections 2 and 3, for large values of $n$ we can almost always construct examples of tournaments which meet our necessary conditions. Knowing that almost all tournaments are quadrangular and having a construction for an infinite class of strongly quadrangular tournaments, one may believe that there will exist a tournament which supports an orthogonal matrix. However, attempting to find an orthogonal matrix whose digraph is a given tournament has proved to be a difficult task. In general, aside from the 3-cycle, the existence of a tournament which supports an orthogonal matrix is still an open problem. We conclude this section with a result that may lead one to believe non-existence is the answer to this problem.
Theorem 4.1 Other than the 3-cycle, there does not exist a tournament on 10 or fewer vertices which is the digraph of an orthogonal matrix.

Proof. By Theorem 2.13 there exists a quadrangular $n$-tournament for $n \leq 10$ if and only if $n$ is 1, 2, 3 or 9. Note, in the case $n = 1$ and $n = 2$, the only tournaments are the single vertex and single arc, both of whose adjacency matrices have a column of zeros. Since orthogonal matrices have full rank, these cannot support an orthogonal matrix. When $n = 3$, the 3-cycle is the only quadrangular tournament. The adjacency matrix for this tournament is a permutation matrix and hence orthogonal. Now consider $n = 9$. By Theorem 2.3 if $T$ is quadrangular, $\delta^+(T) \leq 1$. If $\delta^+(T) = 0$, then $T$’s adjacency matrix will have a row of zeros, and $T$ cannot be the digraph of an orthogonal matrix. So we must have $\delta^+(T) = 1$. So by Theorem 2.6 $T$ has an arc $(x,y)$ with $O(y) = I(x) = V(T) - \{x,y\}$ and $\gamma(T - \{x,y\}) > 2$. The only 7-tournament with domination number greater than 2 is $QR_7$, thus $T - \{x,y\} = QR_7$. However, in section 3 we observed that this tournament is not strongly quadrangular. Thus, other than the 3-cycle, no tournament on 10 or fewer vertices can be the digraph of an orthogonal matrix. 

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