Solving a puzzle in the rank 2 $\mathcal{N} = 2$ classification by Argyres and Martone

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Abstract

Argyres and Martone have produced a beautiful and deep classification of the scale invariant Special Geometries in rank 2. They get a puzzle: the scale-invariant geometries with Coulomb dimensions $\{2,2\}$ appear to depend on four free complex parameters, while on physical grounds we expect only two marginal deformations. We show that the isoclasses of $\{2,2\}$ Special Geometries are indeed parametrized by a complex space of dimension 2, in facts by a non-singular del Pezzo surface of degree 5, a result which exactly matches the physical expectation by Gaiotto. This solves the puzzle.
1 Introduction

In [1] Argyres and Martone have given a nice, deep, and explicit classification of scale invariant Special Geometries in rank 2. However they get a puzzling result. The geometries with Coulomb dimensions \{2, 2\} depend on four complex free parameters (see eq.(3.1) below) while these geometries are expected to describe the $SU(2) \times SU(2)$ superconformal gauge theory with a bi-fundamental hypermultiplet and two fundamental hypermultiplets for each gauge factor, which has only two exactly marginal deformations. Indeed this SCFT is the theory of class $S[A_1]$ with Gaiotto curve [2] the sphere with five punctures, so its conformal manifold should be the moduli space of 5-punctured spheres, $M_{0,5}$ (whose canonical compactification is the del Pezzo surface of degree 5), which has complex dimension 2 not 4.

Indeed physics dictates the number of marginal deformations in a rank-$r$ 4d $\mathcal{N} = 2$ SCFT to be equal to the multiplicity of 2 in the list of Coulomb dimensions, so finding a dimension 4 deformation space for \{2, 2\} special geometries would be a kind of a counterexample. This would be quite surprising since this statement about the dimension of the conformal manifold can be proven directly in Special Geometry without any appeal to physical arguments. It seems that we have got a purely geometric paradox.

In this short note we show that the space which parametrizes the inequivalent \{2, 2\} special geometries constructed by Argyres and Martone in [1] is indeed $M_{0,5}$ exactly as predicted by Gaiotto.

The geometric techniques we use are more interesting than the result itself. This is perhaps a justification for writing this otherwise “obvious” note.

2 Review of scale invariant special geometry

We review the relevant geometric facts mainly to fix the notation.

A (rigid) special geometry is a holomorphic fibration $\pi: \mathcal{X} \to \mathcal{C}$, with section $s$, where $\mathcal{X}$ is a holomorphic symplectic manifold, with holomorphic $(2, 0)$ form $\Omega$, while the smooth fibers of $\pi$ are Lagrangian submanifolds as well as polarized Abelian varieties. The rank of the geometry is the complex dimension of the base $\mathcal{C}$, called the Coulomb branch (it is an affine variety). The geometry is scale invariant if, in addition, it admits an holomorphic Euler vector $\mathcal{E}$ whose exponential is an automorphism for all the implied geometric structures, i.e. for all $t \in \mathbb{C}$ the exponential $\exp(t\mathcal{E}): \mathcal{X} \to \mathcal{X}$ is an automorphism of the complex manifold $\mathcal{X}$ which maps fibers into fibers and fixes (set-wise) the section $s$, while

$$\mathcal{L}_\mathcal{E} \Omega = \Omega.$$  \hspace{1cm} (2.1)

A non-singular scale invariant geometry has a base $\mathcal{C}$ which is biholomorphic to $\mathbb{C}^r$ with
coordinates \( \{u_1, \cdots, u_r\} \) of definite scaling dimension \( \Delta_i \) \(^1\)

\[
\mathcal{L}_u u_i = \Delta_i u_i \tag{2.2}
\]

The \( r \)-tube of numbers \( \{ \Delta_1, \cdots, \Delta_r \} \) is called the \textit{Coulomb dimensions}. If the geometry describes an unitary \( \mathcal{N} = 2 \) SCFT without free subsectors, the \( \Delta_i \)'s are rational numbers \( > 1 \). In any given rank \( r \) only a finite list of \( r \)-tuples \( \{ \Delta_i \} \) are consistent with the geometric structures underlying a special geometry [3]. The holomorphic differential

\[
\lambda \overset{\text{def}}{=} \iota_{\mathcal{E}} \Omega \tag{2.3}
\]

is called the Seiberg-Witten differential.

Let \( \mathcal{D} \subset \mathbb{C}^r \) be the subset of points with a singular fiber. It is a closed analytic subset of pure codimension 1 called the \textit{(reduced) discriminant}. We write \( \mathcal{D}_i \) for its irreducible components and write \( \mathcal{D} = \sum_i \mathcal{D}_i \) (as divisors in \( \mathbb{C}^r \)).

The fiber \( \mathcal{X}_u \) over a “good” point \( u \in \mathbb{C}^r \setminus \mathcal{D} \) is a polarized Abelian variety. For simplicity we assume the polarization to be \textit{principal}, although this is not really necessary. Then we have a period map

\[
p: \mathbb{C}^r \setminus \mathcal{D} \to \mathcal{M}_r, \tag{2.4}
\]

where

\[
\mathcal{M}_r \overset{\text{def}}{=} Sp(2r, \mathbb{Z})/Sp(2r, \mathbb{R})/U(r) \tag{2.5}
\]

is the moduli space of principally polarized Abelian varieties of dimension \( r \) (the Siegel variety). The map \( p \) sends a “good” point \( u \) of the Coulomb branch to the \textit{isoclass} of its Abelian fiber \( \mathcal{X}_u \).

Consider the automorphism \( \exp(t\mathcal{E}): \mathcal{X} \to \mathcal{X} \) with \( t \in \mathbb{C} \). Identifying the Coulomb branch with the image of \( s \) (preserved by the automorphism), we get its action on the Coulomb branch \( \mathbb{C}^r \)

\[
\exp(t\mathcal{E}): (u_1, \cdots, u_r) \mapsto (e^{t\Delta_1} u_1, \cdots, e^{t\Delta_r} u_r). \tag{2.6}
\]

It is convenient to write [4]

\[
\{ \Delta_1, \cdots, \Delta_r \} = \lambda \{ d_1, \cdots, d_r \}, \tag{2.7}
\]

where \( \{ d_1, \cdots, d_r \} \) are the unique positive integers with \( \gcd(d_i) = 1 \) which represent the point \( \{ \Delta_1, \cdots, \Delta_r \} \in \mathbb{P}^{r-1}(\mathbb{Q}) \). This shows that the group acting effectively on \( \mathbb{C}^r \) is \( \mathbb{C}^\times \)

\[
(u_1, \cdots, u_r) \mapsto (\zeta^{d_1} u_1, \cdots, \zeta^{d_r} u_r), \quad \zeta \equiv e^{t\lambda} \in \mathbb{C}^\times. \tag{2.8}
\]

Each connected component \( \mathcal{D}_i \) of the discriminant is preserved by this \( \mathbb{C}^\times \) action, i.e. \( \mathcal{D}_i \) is

\(^1\) In eq.(2.2) \( u_i \) should be understood as the function \( \pi^* u_i \) on \( \mathcal{X} \).
the closure of the union of all the $\mathbb{C}^\times$-orbits it contains. The origin $0 \in \mathbb{C}^r$ – the unique point in the Coulomb branch where the superconformal symmetry is not spontaneously broken and also the unique closed $\mathbb{C}^\times$-orbit – then belongs to $D_i$ for all $i$.

**The projective Abelian family $\mathcal{F}$.** Let $u \in \mathbb{C}^r \setminus D$ be a “good” point. Since $\exp(tE)$ is an automorphism, the fibers $X_u$ and $X_{\exp(tE)u}$ are isomorphic as polarized Abelian varieties, i.e. the period map $p$ takes the same value on all points of each $\mathbb{C}^\times$-orbit in $\mathbb{C}^r$. Said differently: the period map $p$ factors through the projective period map $\varpi$

$$\varpi: (\mathbb{C}^r \setminus D)/\mathbb{C}^\times \to M_r.$$

Now

$$(\mathbb{C}^r \setminus D)/\mathbb{C}^\times = \mathbb{P}(d_1, \cdots, d_r) \setminus S_D$$

where $\mathbb{P}(d_1, \cdots, d_r)$ is the weighted projective space of weights $(d_1, \cdots, d_r)$ [5] and $S_D$ is the weighted projective hypersurface whose quasi-cone is the discriminant $D$ [5].

In other words, any scale invariant special geometry with given Coulomb dimensions $\{\Delta_i\}$ defines a family $\mathcal{F}$ of (principally) polarized Abelian varieties of dimension $r$ parametrized by the complement of a hypersurface $S$ in the $(r-1)$-dimensional weighted projective space $\mathbb{P}(d_1, \cdots, d_r)$. We claim that we have a dichotomy:

- either the family $\mathcal{F}$ is isotrivial, i.e. $\varpi$ is the constant map;
- or the family $\mathcal{F}$ is rigid in the sense of Faltings-Peters [6, 7].

The validity of the claim for ranks $r \leq 7$ is obvious [8], and since we are mainly interested in the $r = 2$ case, we shall not pursue this point any further.

We may ask the inverse question: given a rigid family $\mathcal{F}$ of polarized Abelian $r$-varieties parametrized by the complement of a hypersurface $S$ in $\mathbb{P}(d_1, \cdots, d_r)$, can we find a special geometry over $\mathbb{C}^r$ which corresponds to this family?

Usually the answer is NO. We can, of course, pull back the family $\mathcal{F}$ to a family parametrized by the “good locus” $\mathbb{C}^r \setminus D$, but then we must extend its total space over the discriminant locus, while requiring the resulting space to have a regular holomorphic symplectic structure $\Omega$. The last requirement is very strong and “almost all” families $\mathcal{F}$ do not admit such a symplectic extension. The existence and regularity of $\Omega$ are quite formidable constraints.

On the other hand, in the rare situation where our family $\mathcal{F}$ does have a symplectic extension to a special geometry, we expect this extension to be unique. This is the typical situation, and very plausibly it is true in general.
Assuming uniqueness of the symplectic extension, we get the following useful result:

Assume that we have two scale-invariant special geometries with the same $r$-tuple \{\(d_1, \ldots, d_r\)\} and suppose that their respective projective families \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are isomorphic as holomorphic families of polarized Abelian varieties. Then their parent scale-invariant special geometries are also isomorphic.

3 The \{2,2\} geometry

The authors of [1] exploit the fact that an Abelian variety of dimension 2 is either the Jacobian of a smooth hyperelliptic curve of genus 2 or the product of two elliptic curves. They replace the Abelian fibration \(\pi: \mathcal{X} \to \mathbb{C}^2\) by a corresponding fibration \(\tilde{\pi}: \mathcal{Y} \to \mathbb{C}^2\) whose smooth fibers over \(\mathbb{C}^2 \setminus \mathcal{D}\) are genus 2 hyperelliptic curves. The original fibration \(\pi: \mathcal{X} \to \mathbb{C}^2\) is then recovered as the Jacobian fibration of \(\tilde{\pi}: \mathcal{Y} \to \mathbb{C}^2\). Concretely, they write explicit hyperelliptic equations of degree 6 or 5 with coefficients which are polynomials in the Coulomb branch coordinates \(u, v\). For instance, for dimensions \{2,2\} their equation is [1]

\[
y^2 = (ux - v)(x^5 + \tau_1x^3 + \tau_2x^2 + \tau_3x + \tau_4)
\]

(3.1)

where \(\tau_1, \tau_2, \tau_3, \tau_4\) are free complex parameters. By construction the hyperelliptic curve over the point \((u, v) \in \mathbb{C}^2 \setminus \mathcal{D}\) is isomorphic to the one over \((\lambda u, \lambda v)\), so that we get a family \(\mathcal{G}\) of genus 2 curves over \(\mathbb{P}^1 \setminus \mathcal{S}\) whose Jacobian fibration is our family \(\mathcal{F}\). The symplectic form may be shown to be

\[
\Omega = du \wedge \frac{x dx}{y} + dv \wedge \frac{dx}{y}.
\]

(3.2)

Equation (3.1) leads to a puzzle. Physically we expect this geometry to correspond to the \(\mathcal{N} = 2\) theory of class \(\mathcal{S}[A_1]\) with Gaiotto curve the sphere with five punctures [2], whose conformal manifold should be \(\mathcal{M}_{0,5}\), the moduli space of the 5-punctured sphere, whose complex dimension is 2 not 4. Why 2 is easy to understand: physically the marginal deformations should be the elements of \(\mathbb{C}[u, v]\) of scaling dimension 2, and there are only two of them for dimensions \{2,2\}.

How we reconcile these physical facts with the findings in [1] that the geometry (3.1) depends on four parameters?

Solving the puzzle. The crucial ingredient is eq.(A.31) of [1] which yields the discriminant for this geometry. The reduced discriminant \(\mathcal{D}\) is

\[
\prod_{i=1}^{5} (u + \alpha_i(\tau)v)
\]

(3.3)
where $\alpha_i(\tau)$ are certain functions of the four parameters $\tau_1, \tau_2, \tau_3, \tau_4$ (called collectively $\tau$).

For a given value $\tau$ of the parameters, the base of the family $\mathcal{F}(\tau)$ is then

$$B(\tau) = \mathbb{P}^1 \setminus \{-1/\alpha_1(\tau), \ldots, -1/\alpha_5(\tau)\} \quad (3.4)$$

which is a 5-punctured sphere. The monodromy around each puncture may be read from table 1 of [1]; each one of them is conjugated over $Sp(4, \mathbb{Z})$ to the unipotent element

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.5)$$

Thus the family $\mathcal{F}$ may be completed to a semistable fibration $\tilde{\mathcal{F}} \to \mathbb{P}^1$; the Seiberg-Witten differential yields a holomorphic contact structure on its total space.

In particular the geometry is non-isotrivial, hence rigid.\(^2\) By rigidity\(^3\) the family $\mathcal{F}(\tau)$ is uniquely defined by the five special points $-1/\alpha_i(\tau) \in \mathbb{P}^1$ and the conjugacy classes of the local monodromies around them. Since the monodromy classes are fixed, two families $\mathcal{F}(\tau')$ and $\mathcal{F}(\tau)$ are isomorphic if and only if $B(\tau') \simeq B(\tau)$ as 5-punctured spheres, i.e. iff they correspond to the same point in $\mathcal{M}_{0,5}$, that is, iff the two sets of five points $\{\alpha_i(\tau')\}$ and $\{\alpha_i(\tau)\}$ have the same cross-ratios.

We conclude that the inequivalent families are parametrized by $\mathcal{M}_{0,5}$. Using the remark (\(\ast\)) at the end of the previous section, we conclude that:

- the non-isomorphic scale-invariant special geometries with dimensions $\{2, 2\}$ are parametrized by the two-dimensional complex space $\mathcal{M}_{0,5}$.

Up to isomorphism, the special geometry does not depend on the four parameters $\tau_a$ individually, but only on two cross-ratios of their functions $\alpha_i(\tau)$.

We stress that the above result is precisely Gaiotto’s prediction [2] for the conformal manifold of the $SU(2) \times SU(2)$ superconformal gauge theory with a bi-fundamental hypermultiplet as well as two fundamental hypermultiplets for each gauge factor.

References

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\(^2\) This also follows from the fact that the family $\mathcal{G}$ is rigid by Arakelov theorem [9].

\(^3\) Here we are cheating a little bit. We need a slightly stronger notion of rigidity: what we are really using here is that the periods of the Seiberg-Witten differential, i.e. the multivalued special coordinates of Special Geometry, have the form $\sqrt{u} f(v/u)$ where the functions $f(z)$ are Pochhammer transcendentals.
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