Converse Lyapunov Functions and Converging Inner Approximations to Maximal Regions of Attraction of Nonlinear Systems

Morgan Jones, Matthew M. Peet

Abstract—This paper considers the problem of approximating the “maximal” region of attraction (the set that contains all asymptotically stable sets) of any given set of locally exponentially stable nonlinear Ordinary Differential Equations (ODEs) with a sufficiently smooth vector field. Given a locally exponential stable ODE with a differentiable vector field, we show that there exists a globally Lipschitz continuous converse Lyapunov function whose 1-sublevel set is equal to the maximal region of attraction of the ODE. We then propose a sequence of $d$-degree Sum-of-Squares (SOS) programming problems that yields a sequence of polynomials that converges to our proposed converse Lyapunov function uniformly from above in the $L^1$ norm. We show that each member of the sequence of 1-sublevel sets of the polynomial solutions to our proposed sequence of SOS programming problems are certifiably contained inside the maximal region of attraction of the ODE, and moreover, we show that this sequence of sublevel sets converges to the maximal region of attraction of the ODE with respect to the volume metric. We provide numerical examples of estimations of the maximal region of attraction for the Van der Pol oscillator and a three dimensional servomechanism.

I. INTRODUCTION

For a given equilibrium point, a Region of Attraction (ROA) of a nonlinear Ordinary Differential Equation (ODE) is defined as a set of initial conditions for which the solution map of the ODE tends to that equilibrium point. The maximal ROA of an equilibrium point, meanwhile, is defined as the ROA which contains all other ROAs of that equilibrium point. Specifically, for an ODE $\dot{x}(t) = f(x(t))$, we denote the solution map (known to exist when $f$ is Lipschitz continuous) of the ODE by $\phi_f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ which satisfies

$$\frac{d}{dt} \phi_f (x, t) = f(\phi_f (x, t))$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$,

$$\phi_f (x, 0) = x$$

for all $x \in \mathbb{R}^n$,

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(0) = 0$. The maximal ROA is then defined as

$$\text{ROA}_f := \{ x \in \mathbb{R}^n : \lim_{t \to +\infty} ||\phi_f (x, t)||_2 = 0 \}.$$

The problem of computing sets which accurately approximate the maximal ROA with respect to some set metric plays a central role in the stability analysis of many engineering applications. For instance, knowledge of the ROA provides a metric for the susceptibility of the F/A-18 Hornet aircraft experiencing an unsafe out of control flight departure phenomena, called falling leaf mode [1], [2].

If the matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz (the real part of the eigenvalues of $A$ are all negative) then the associated linear system, with vector field $f(x) = Ax$, has a maximal ROA that can be found exactly as $\text{ROA}_f = \mathbb{R}^n$. In the more general case of nonlinear systems there is no known general analytical formula for $\text{ROA}_f$. However, for particular nonlinear systems, such as those arising from gradient flow dynamics, the maximal ROA can be expressed analytically [3]. In the absence of an analytical formula for $\text{ROA}_f$ in recent years there has been considerable interest in discovering numerical methods for approximating $\text{ROA}_f$ rather than finding $\text{ROA}_f$ exactly.

Lyapunov’s second method is arguably the most widely used technique for finding ROAs associated with an ODE [4]. Rather than solving the ODE directly to find a closed form expression of the solution map, ROAs can be computed indirectly by searching for a “generalized energy function”, called a Lyapunov function. A Lyapunov function of an ODE is any function that is positive everywhere, apart from the origin where it is zero, and is strictly decreasing along the solution map of the ODE. Specifically, if we can find a function $V$ such that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$, then if $\forall (x) = \frac{d}{dt} V(x)$ is negative over the sublevel set $\{ x \in \mathbb{R}^n : V(x) \leq a \}$ we have that $\{ x \in \mathbb{R}^n : V(x) \leq a \} \subseteq \text{ROA}_f$ is a ROA [5]. For linear systems, $f(x) = Ax$ where $A \in \mathbb{R}^{n \times n}$, a necessary and sufficient condition for $\text{ROA}_f = \mathbb{R}^n$ is that there exists a quadratic Lyapunov function of form $V(x) = x^T P x$ where $P > 0$. Thus, in this case, the problem of finding the maximal ROA of a linear system is reduced to solving the Linear Matrix Inequality (LMI) $A^T P + PA < 0$ for $P > 0$.

In the case of nonlinear systems a common approach for finding Lyapunov functions has been to generalize the search from quadratic functions, $V(x) = x^T P x$, to Sum-of-Square (SOS) polynomials functions, $V(x) = Z_d(x)^T P Z_d(x)$ where $Z_d$ is the degree $d \in \mathbb{N}$ monomial vector. Then, to find a Lyapunov function we must solve an SOS optimization problem, rather than solving an LMI (as was the case for linear systems). Over the years, many SOS optimization problems have been proposed for ROA estimation [6], [7], [8]. Recently in [10], SOS was used to estimate the region of attraction of an uncrewed aircraft; in [11] an SOS based algorithm was proposed to construct a rational Lyapunov function that yields an estimate of the ROA; in [12] a recursive procedure for constructing the polynomial Lyapunov functions was proposed.

Despite the recent success of modern techniques to find accurate approximations of the maximal ROA, to the best of our knowledge, a numerical algorithm that can be proven to provide an approximation of the maximal ROA arbitrarily well with respect to any set metric has yet to be proposed. Many of the current numerical methods for finding ROAs use SOS programming to find polynomial Lyapunov functions. However, barring any assumptions on the existence of a sufficiently smooth Lyapunov function, it is currently unknown how well
The rest of this paper is organized as follows. In Section III we define the maximal region of attraction of an ODE in terms of the solution map of the ODE. In Section IV we formulate the problem of approximating the region of attraction as an optimization problem. In Section V we propose a globally Lipschitz continuous Lyapunov function that characterizes the maximal region of attraction. In Section VI we propose a convex optimization problem for the approximation of our proposed converse Lyapunov function in the $L^1$-norm. In Section VII we tighten this optimization problem to an SOS programming problem. Finally, several numerical examples are given in Section VIII and our conclusion is given in Section IX.

A. Set Notation

We denote the power set of $\mathbb{R}^n$, the set of all subsets of $\mathbb{R}^n$, as $P(\mathbb{R}^n) = \{X : X \subset \mathbb{R}^n\}$. For two sets $A, B \in \mathbb{R}^n$ we denote $A/B = \{x \in A : x \notin B\}$. For $x \in \mathbb{R}^n$ we denote $||x||_p = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$. For $\eta > 0$ and $y \in \mathbb{R}^n$ we denote the set $B_\eta(y) = \{x \in \mathbb{R}^n : ||x-y||_2 < \eta\}$. For a set $X \subset \mathbb{R}^n$ we say $x \in X$ is an interior point of $X$ if there exists $\epsilon > 0$ such that $\{y \in \mathbb{R}^n : ||y-x|| < \epsilon\} \subset X$. We denote the set of all interior points of $X$ by $X^\circ$. The point $x \in X$ is a limit point of $X$ if for all $\epsilon > 0$ there exists $y \in \{y \in \mathbb{R}^n/\{x\} : ||y-x|| < \epsilon\}$ such that $y \in X$; we denote the set of all limit points of $X$, called the closure of $X$, as $(X)^\circ$. Moreover, we denote the boundary of $X$ by $\partial X = (X)^\circ$/$/X$. For $A \subset \mathbb{R}^n$ we denote the indicator function by $1_A : \mathbb{R}^n \rightarrow \mathbb{R}$ that is defined as $1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$ For $B \subset \mathbb{R}^n$, $\mu(B) := \int_B 1_B(x)dx$ is the Lebesgue measure of $B$. Let us denote bounded subsets of $\mathbb{R}^n$ by $\mathcal{B} := \{B \subset \mathbb{R}^n : \mu(B) < \infty\}$. If $M$ is a subspace of a vector space $X$ we denote equivalence relation $\sim_M$ for $x, y \in X$ by $x \sim_M y$ if $x-y \in M$. We denote quotient space by $X$ (mod $M$) := $\{\{x \in X : y \sim_M x\} : x \in X\}$. For an open set $\Omega \subset \mathbb{R}^n$ and $\sigma > 0$ we denote $\Omega_{\sigma} := \{x \in \Omega : (B(x, \sigma) \subset \Omega)\}$.

B. Continuity Notation

Let $C(\Omega, \Theta)$ be the set of continuous functions with domain $\Omega \subset \mathbb{R}^n$ and image $\Theta \subset \mathbb{R}^n$. We denote the set of locally and uniformly Lipschitz continuous functions on $\Theta_1$ and $\Theta_2$, Defn. 3, by $LocLip(\Theta_1, \Theta_2)$ and $Lip(\Theta_1, \Theta_2)$ respectively. For $\alpha \in \mathbb{N}^n$ we denote the partial derivative $D^\alpha f(x) := \Pi_{i=1}^n \frac{\partial^\alpha f}{\partial x_i^{\alpha_i}}(x)$ where by convention if $\alpha = [0, \ldots, 0]^T$ we denote $D^\alpha f(x) := f(x)$. We denote the set of $i$th continuously differentiable functions by $C^i(\Omega, \Theta) := \{f \in C(\Omega, \Theta) : D^\alpha f \in C(\Omega, \Theta) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that } \sum_{i=1}^n \alpha_i \leq i\}$. For $V \subset C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ we denote $\nabla V := (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n})^T$ and $\nabla \sigma = \frac{\partial \sigma}{\partial x}$. We denote the essential supremum by $\text{esssup}_{x \in X} f(x) := \inf\{a \in \mathbb{R} : \mu(\{x \in X : f(x) > a\}) = 0\}$.

C. Sobolev Space Notation

For an open set $\Omega \subset \mathbb{R}^n$ and $p \in [1, \infty)$ we denote the set of $p$-integrable functions by $L^p(\Omega, \mathbb{R}) := \{f : \Omega \rightarrow \mathbb{R} \text{ measurable : } f_\Omega ||f||_p < \infty\}$, in the case $p = \infty$ we denote $L^\infty(\Omega, \mathbb{R}) := \{f : \Omega \rightarrow \mathbb{R} \text{ measurable : } \text{esssup}_{x \in \Omega} f(x) < \infty\}$. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we denote the Sobolev space of functions with weak derivatives (Defn. 4) by $W^{k,p}(\Omega, \mathbb{R}) :=$...
throughout the paper we will assume there exists a unique $f$. We say the set $U$ is a solution map of the ODE (1) over $D$. For simplicity throughout this paper we will assume $f(0) = 0$ so the origin is an equilibrium point.

Given $D \subset \mathbb{R}^n$, $I \subset [0, \infty)$, and an ODE (1) we say any function $\phi_f : D \times I \rightarrow \mathbb{R}^n$ satisfying

$$\frac{\partial \phi_f}{\partial t} = f(\phi_f(x,t)) \text{ for } (x,t) \in D \times I,$$

then we say $U \subset \mathbb{R}^n$ is an exponentially stable set of the ODE (1).

**Definition 2.** The (Maximal) Region of Attraction (ROA) of the ODE (1) is defined as the following set:

$$ROA_f := \{ x \in \mathbb{R}^n : \lim_{t \to \infty} ||\phi_f(x,t)||_2 = 0 \}. \tag{4}$$

The ROA of the ODE (1) can be thought of as the "maximal" asymptotically stable set. That is if $U \subset \mathbb{R}^n$ is an asymptotically stable set of the ODE (1) then $U \subseteq ROA_f$. Moreover, as we will show next, the ROA is an open set.

**Lemma 1** (Lemma 8.1 [19]). Consider an ODE of Form (1). The set $ROA_f$ (Defined in Eqn. (4)) is open.

Before proceeding we introduce some useful notation for the $\eta$-ball set entry times of solution maps. For a given function $\phi_f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, $x \in ROA_f$, and $\eta > 0$ we denote

$$F_\eta(x) := \inf \{ t \geq 0 : \phi_f(x,t) \in B_\eta(0) \}. \tag{5}$$

We now state two important properties of solution maps used in many of the proofs presented in this paper.

**Lemma 2** (Exponential divergence of solution maps. Page 392 [20]). Suppose $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and there exists $\theta, R > 0$ such that $||D^2 f(x)||_2 < \theta$ for all $x \in B_R(0)$ and any $||\alpha||_1 \leq 2$, where $\alpha \in \mathbb{R}^n$. Then the solution map satisfies the following inequality

$$||\phi_f(x,t) - \phi_f(y,t)||_2 \leq e^{\theta t}||x - y||_2 \text{ for } t \geq 0 \text{ and } x, y \in ROA_f. \tag{6}$$

**Lemma 3** (Smoothness of the solution map. Page 149 [20]). If $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then the solution map is such that $\phi_f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.
“closest” to ROA\(f\), we find the “closest” \(d\)-degree polynomial to \(W\) with respect to the \(L^1\) norm. Thus we consider the following “algebraic” problem:
\[
P_d \in \arg \min_{P_d \in \mathcal{P}_d([\mathbb{R}^n, \mathbb{R}])} \int_{\lambda} |J(x) - W(x)| \, dx
\]

(8)
such that \(W(x) \leq J(x)\) for all \(x \in \Omega\),
where ROA\(f\) ⊆ \(\lambda \subseteq \Omega \subset \mathbb{R}^n\). Then, Cor. 4 (found in Appendix XI) can be used to show that \(\{x \in \lambda : P_d(x) < 1\}\) converges to \(\{x \in \lambda : W(x) < 1\} = \text{ROA} f\) as \(d \to \infty\) with respect to the volume metric (given in Eqn. (46)).

Solving the optimization problem given in Eqn. (8) has the following challenges:

1) Does there exist a converse Lyapunov function \(W : \mathbb{R}^n \to \mathbb{R}\) such that \(\text{ROA} f = \{x \in \mathbb{R}^n : W(x) < 1\}\)?

2) Can the constraint, \(W(x) \leq J(x)\) for all \(x \in \Omega\), be tightened to a convex constraint without necessarily having an analytical formula for \(W\)?

3) Does the solution, \(P_d\), tend towards \(W\) with respect to the \(L^1\) norm as \(d \to \infty\)?

In the next section we tackle the first of these challenges. We propose a converse Lyapunov function, \(W\), whose 1-sublevel set is equal to ROA\(f\). Then, in Sec. VI we tackle the second challenge; we propose a sufficient condition, in the form of a linear partial differential inequality, that when satisfied by a function \(J\) implies \(W(x) \leq J(x)\) for all \(x \in \Omega\). Finally, in Appendix X, we tackle the third challenge of showing that there exists a sequence of \(d\)-degree polynomials, feasible to Opt. (8) for \(d \in \mathbb{N}\), that converges to \(W\) with respect to the \(L^1\) norm. For implementation purposes Opt. (8) is then tightened to an SOS optimization problem, given in Eqn. (43), that can be efficiently numerically solved. The main result of the paper is then given in Theorem 1, showing that our proposed family of \(d\)-degree SOS Optimization Problems (43) yields a sequence of sets that converge to the region of attraction of a given locally exponentially stable ODE with respect to the volume metric as \(d \to \infty\).

V. A Globally Lipschitz Continuous Converse Lyapunov Function that Characterizes the ROA

In [21] a converse Lyapunov function, called the maximal Lyapunov function, was proposed. It was shown that for any given asymptotically stable ODE there exists a maximal Lyapunov function whose \(\infty\)-sublevel set is equal to the region of attraction of the ODE. However, since by definition any maximal Lyapunov function is unbounded outside of the region of attraction it cannot be approximated arbitrarily well (with respect to any norm) by a polynomial over any compact set that contains points outside of the region of attraction (since polynomials are bounded over compact sets). Thus, it is not possible to design an SOS based algorithm that can approximate maximal Lyapunov functions arbitrarily well. To overcome this challenge we propose a new converse Lyapunov function (found in Eqn. (11)) whose 1-sublevel set is equal to ROA\(f\), is globally bounded, and is globally Lipschitz continuous. Before introducing our new converse Lyapunov function let us recall the definition of Lipschitz continuity.

Definition 3. Consider sets \(\Theta_1 \subset \mathbb{R}^n\) and \(\Theta_2 \subset \mathbb{R}^n\). We say the function \(F : \Theta_1 \to \Theta_2\) is locally Lipschitz continuous on \(\Theta_1\) and \(\Theta_2\), denoted \(F \in \text{LocLip}(\Theta_1, \Theta_2)\), if for every compact set \(X \subseteq \Theta_1\) there exists \(K_X > 0\) such that for all \(x, y \in X\)
\[
||F(x) - F(y)||_2 \leq K_X ||x - y||_2.
\]

(9)
If there exists \(K > 0\) such that Eqn. (9) holds for all \(x, y \in \Theta_1\) we say \(F\) is globally Lipschitz continuous, denoted \(F \in \text{Lip}(\Theta_1, \Theta_2)\).

We consider two different types of converse Lyapunov functions. The first converse Lyapunov function (given in Eqn. (10)) is a special case of those first found in [22] that have the form \(V_1(x) := \int_0^\infty G(||\phi_1(x,t)||_2) \, dt\) for some class K function, \(G : [0, \infty) \to [0, \infty)\) (class K is the class of functions which monotonically approach zero at zero). Later, in [21] it was shown that for a locally stable ODE, the \(\infty\)-sublevel set of \(V_1\) is equal to the region of attraction of the ODE; this Lyapunov function was named the maximal Lyapunov function. In this paper we only consider locally exponentially stable systems and hence may restrict ourselves to the special case when \(G(y) = y^{2\beta}\) for some \(\beta \in \mathbb{N}\).

The second converse Lyapunov function we consider (found in Eqn. (11)) can be thought of as a nonlinear transformation of the first converse Lyapunov function. A function of a similar structure was previously considered in [23] and took the form \(V_2(x) := \exp(-\lambda \int_0^\infty ||\phi_2(x,t)||_2^2) - 1\). Although [23] used \(V_2\) to certify the stability of a system, \(V_2\) is not a Lyapunov function in the classical sense since it is not positive everywhere (unlike our proposed converse Lyapunov function in Eqn. (11)). We note that [23] did establish the globally continuity of \(V_2\) but did not show the stronger result that \(V_2\) is Lipschitz continuous.

Now, consider \(f \in \text{LocLip}(\mathbb{R}^n, \mathbb{R}^n)\), \(\lambda > 0\) and \(\beta \in \mathbb{N}\). Let us denote the functions \(V_\beta : \text{ROA} f \to \mathbb{R}\) and \(W_{\lambda, \beta} : \mathbb{R}^n \to \mathbb{R}\) where
\[
V_\beta(x) := \int_0^\infty ||\phi_\beta(x,t)||_2^2 \, dt,
\]
\[
W_{\lambda, \beta}(x) := \begin{cases} 1 - \exp(-\lambda \int_0^\infty ||\phi_\beta(x,t)||_2^2) \, dt & \text{when } x \in \text{ROA}_f \\ 1 & \text{otherwise}. \end{cases}
\]

(11)
A. Converse Lyapunov Functions that Characterize the ROA

The function, \(V_\beta\), given in Eqn. (10) is a special case of a class of Lyapunov functions called maximal Lyapunov functions [21]. In the following lemma we will show that \(V_\beta\) tends to infinity for sequences of points approaching the boundary of the region of attraction and is finite inside the region of attraction.

Lemma 4. Consider \(f \in \text{LocLip}(\mathbb{R}^n, \mathbb{R})\), \(\beta \in \mathbb{N}\) and \(V_\beta\) given in Eqn. (10). Suppose there exists \(R, \eta > 0\) such that \(\text{ROA}_f \subset B_R(0)\) and \(B_\eta(0)\) is an exponentially stable set (Defn. 1) of the ODE (1). Then the following holds.

1) For any sequence \(\{x_k\}_k \subset \text{ROA}_f\) such that \(\lim_{k \to \infty} x_k \in \partial\text{ROA}_f\) we have that
\[
\lim_{k \to \infty} V_\beta(x_k) = \infty.
\]

(12)
2) We have that
\[ x \in \text{ROA}_f \text{ if and only if } V_\beta(x) < \infty. \]  
\[ \eta \geq \lim_{t \to \infty} \text{finite limit the same limit we have } \lim_{t \to \infty} \text{B}_\eta(x_k) \text{ (where } F_\eta(x) \text{ is given in Eqn. (5)). Since } x_k \in \text{ROA}_f \text{ it follows } T_k < \infty \text{ for all } k \in \mathbb{N}. \]

Now,
\[ V_\beta(x_k) = \int_0^{T_k} ||\phi_f(x_k,t)||_2^2 \, dt + \int_{T_k}^{\infty} ||\phi_f(x_k,t)||_2^2 \, dt \]
\[ \geq \int_0^{T_k} ||\phi_f(x_k,t)||_2^2 \, dt \geq \eta_1^2 T_k. \]

We will now show \( T_k \to \infty \) as \( k \to \infty \) and thus Eqn. (14) shows Eqn. (12). For contradiction suppose \( \lim_{k \to \infty} T_k \neq 0 \), then there exists a bounded subsequence \( T_{k_n}, n \in \mathbb{N} \subset \{ T_k \}_{k \in \mathbb{N}} \). By Theorem 6 there exists a subsequence \( T_{k_n}, n \in \mathbb{N} \), we denote by \( T_n \), that converges to a finite limit \( T^* := \lim_{n \to \infty} T_n < \infty \). Let us denote the corresponding subsequence of \( \{ x_{k_n} \} \), \( \{ x_n \} \), and every subsequence of a convergent sequence must converge to the same limit we have \( \lim_{n \to \infty} x_n = x^* \).

Since \( \phi_f(x^*,T^*)_2 = \lim_{t \to \infty} ||\phi_f(x_t,t)||_2 \leq \eta_1 < \eta, \) and since \( B_\eta(0) \) is an exponentially stable set we have that
\[ \lim_{t \to \infty} ||\phi_f(x^*,T^*+t)||_2 = ||\phi_f(x^*,T^*+t)||_2 \leq \mu e^{-\eta \delta t}, \]
\[ \leq \mu^2 e^{-\delta t} \eta \eta_1^2 \leq \mu^2 \eta_2 \eta \eta_1^2. \]

Therefore, Eqn. (15) implies that
\[ \lim_{t \to \infty} ||\phi_f(x^*,t)||_2 = \lim_{t \to \infty} ||\phi_f(x^*,T^*+t)||_2 = \lim_{t \to \infty} \mu \eta e^{-\delta t} = 0, \]
thus showing \( x^* \in \text{ROA}_f \). Now ROA\(_f\) is an open set (by Lemma 1). Therefore if \( x^* \in \text{ROA}_f \) then \( x^* \notin \partial \text{ROA}_f \), providing a contradiction that \( x^* \in \partial \text{ROA}_f \). Hence, Eqn. (12) holds.

We now Statement 2) in Lem. 4 by showing Eqn. (13) holds. First suppose \( x \in \text{ROA}_f \). We will now show \( V_\beta(x) < \infty \). Since \( x \in \text{ROA}_f \) we have that \( \lim_{t \to \infty} ||\phi_f(x,t)||_2 = 0 \) and thus it follows there exists \( T < \infty \) such that \( ||\phi_f(x,t)||_2 < \eta \) for all \( t \geq T \) implying \( F_\eta(x) \leq T < \infty \). Moreover, by properties of the set entry time we have that \( ||\phi_f(x,F_\eta(x))||_2 \leq \eta \) and since \( B_\eta(0) \) is an exponentially stable set we have that
\[ ||\phi_f(x,t)||_2 = ||\phi_f(F_\eta(x),F_\eta(x)+t-F_\eta(x))|| \leq \mu \eta e^{-\delta t} \]
\[ \leq \mu^2 \eta^2 \eta \eta_1^2 < \infty. \]

Therefore, using the fact that \( \text{ROA}_f \subset B_\beta(0) \) together with Eqn. (16) we get that,
\[ V_\beta(x) = \int_0^{F_\eta(x)} ||\phi_f(x,t)||_2^2 \, dt + \int_{F_\eta(x)}^{\infty} ||\phi_f(x,t)||_2^2 \, dt \]
\[ \leq F_\eta(x) R^2 \eta^2 + \mu^2 \eta^2 \eta \eta \eta_1^2 < \infty. \]

Now, on the other hand let us now suppose \( x \in \mathbb{R}^n \) is such that \( V_\beta(x) < \infty \). We will show \( x \in \text{ROA}_f \). For contradiction suppose \( x \notin \text{ROA}_f \). Then \( \lim_{t \to \infty} ||\phi_f(x,t)||_2 \neq 0 \). Therefore, there exists \( \epsilon > 0 \) such that \( ||\phi_f(x,t)||_2 > \epsilon \) for all \( t \geq 0 \). Thus
\[ V_\beta(x) = \int_0^{\infty} ||\phi_f(x,t)||_2^2 \, dt \geq \int_0^{\infty} \epsilon^2 \, dt = \infty, \]
providing a contradiction that \( V_\beta(x) < \infty \). Hence, Eqn. (13) holds.

As we will show next, the function, \( W_{\lambda, \beta} \), given in Eqn. (11), can also characterize \( \text{ROA}_f \) as its 1-sublevel set.

**Corollary 1.** Consider \( f \in \text{LocLip}(\mathbb{R}^n, \mathbb{R}), \beta \in \mathbb{N}, \lambda > 0 \) and \( W_{\lambda, \beta} \) given in Eqn. (11). Suppose there exists \( R, \eta > 0 \) such that \( \text{ROA}_f \subset B_R(0) \) and \( B_\eta(0) \) is an exponentially stable set (Defn. 1) to the ODE (1). Then the following holds.

1) For any sequence \( \{ x_k \}_{k \in \mathbb{N}} \subset \text{ROA}_f \) such that \( \lim_{k \to \infty} x_k \in \partial \text{ROA}_f \) we have that
\[ \lim_{k \to \infty} W_{\lambda, \beta}(x_k) = 1. \]

2) We have that
\[ \text{ROA}_f = \{ x \in \mathbb{R}^n : W_{\lambda, \beta}(x) < 1 \}. \]

**Proof.** We first show Statement 1) in Cor. 1 by showing Eqn. (17) holds. For \( x \in \text{ROA}_f \) we have that \( W_{\lambda, \beta}(x) = 1 - e^{-\lambda \eta \delta t} \beta. \) Moreover, \( e^t \) is a continuous function of \( x \in \mathbb{R}. \) Therefore, by Lemma 4, for \( \{ x_k \}_{k \in \mathbb{N}} \subset \text{ROA}_f \) we have that
\[ \lim_{k \to \infty} W_{\lambda, \beta}(x_k) = 1 - \exp(-\lambda \lim_{k \to \infty} \text{V}(x_k)) = 1. \]

We next show Statement 2) in Cor. 1 by showing Eqn. (18) holds. If \( x \in \text{ROA}_f \) then by Lemma 4 we have that \( V_\beta(x) < \infty \) and thus \( e^{-\lambda \eta \delta t} \beta < 1 \). Therefore, \( \text{ROA}_f \subset \{ x \in \mathbb{R}^n : W_{\lambda, \beta}(x) < 1 \}. \) On the other hand if \( y \in \{ x \in \mathbb{R}^n : W_{\lambda, \beta}(x) < 1 \} \) then \( a := 1 - W_{\lambda, \beta}(y) > 0. \) Thus,
\[ W_{\lambda, \beta}(y) = 1 - \frac{1}{\ln(a)} = \frac{1}{\ln(a)} \leq \text{lim}_{k \to \infty} W_{\lambda, \beta}(x_k) = 1. \]

**B. A Globally Lipschitz Continuous Lyapunov Function**

The function \( V_\beta \) is only defined over the set \( \text{ROA}_f \) and is unbounded. Such properties make approximating \( V_\beta \) by polynomials challenging. On the other hand \( W_{\lambda, \beta} \) is defined over the whole of \( \mathbb{R}^n \) and is bounded by 1. What is more, we next show in Prop. 1 that \( W_{\lambda, \beta} \) is globally Lipschitz continuous. One may intuit this continuity property by considering the similarity in structure between \( W_{\lambda, \beta} \) and the standard mollifier given in Eqn. (56); a function known to be infinitely differentiable.

**Proposition 1.** Consider \( f \in C^2(\mathbb{R}^n, \mathbb{R}) \) and \( W_{\lambda, \beta} \) as in Eqn. (11) where \( \lambda > 0 \) and \( \beta \in \mathbb{N}. \) Suppose there exists \( \theta, \eta, R > 0 \) such that \( ||Df(x)||_2 < \theta \) for all \( x \in B_R(0) \) and \( ||Df(x)||_2 \leq 2, B_\eta(0) \) is an exponentially stable set (Defn. 1) to the ODE (1), and \( \text{ROA}_f \subset B_R(0). \) Then if \( \lambda > \theta \eta^{-2 \beta} \) and \( \beta > \frac{\theta}{2 \beta} + \frac{1}{2} \) we have that \( W_{\lambda, \beta} \in \text{Lip}(\mathbb{R}^n, \mathbb{R}). \) Moreover, the
Lipschitz constant of $W_{\lambda, \beta}$ is less than or equal to $K > 0$, where

$$K := 2\lambda \max \left\{ \frac{2\beta R^{2\beta -1}}{\theta}, \frac{2\beta (\mu \eta)^{2\beta -1}}{\delta(2\beta -1) - \theta} \right\}. \quad (19)$$

Proof. To prove $W_{\lambda, \beta} \in Lip(\mathbb{R}^n, \mathbb{R})$ we will now show

$$|W_{\lambda, \beta}(x) - W_{\lambda, \beta}(y)| < K||x-y||_2 \text{ for all } x, y \in \mathbb{R}^n, \quad (20)$$

where $K > 0$ is given in Eqn. (19).

**Case 1:** $x, y \in ROA_f$. Since $B_R(0)$ is an exponentially stable set of the ODE (1) and by applying a similar argument in the derivation of Eqn. (16), it follows that there exists $\delta, \mu > 0$ such that

$$||\phi_f(x,t)||_2 \leq \mu \eta e^{-\delta(t-F_{\eta}(x))} \text{ for all } t > F_{\eta}(x),$$

$$||\phi_f(y,t)||_2 \leq \mu \eta e^{-\delta(t-F_{\eta}(x))} \text{ for all } t > F_{\eta}(y).\quad (21)$$

Without loss of generality we will assume $F_{\eta}(x) \geq F_{\eta}(y)$ (otherwise we can relabel $x$ and $y$).

Now,

$$|W_{\lambda, \beta}(x) - W_{\lambda, \beta}(y)| = \exp \left( -\lambda \int_0^\infty ||\phi_f(x,t)||_2^{2\beta} dt \right)$$

$$-\exp \left( -\lambda \int_0^\infty ||\phi_f(y,t)||_2^{2\beta} dt \right)$$

$$= \exp \left( -\lambda \int_0^\infty ||\phi_f(x,t)||_2^{2\beta} dt \right)$$

$$\times \left| 1 - \exp \left( -\lambda \int_0^\infty ||\phi_f(y,t)||_2^{2\beta} - ||\phi_f(x,t)||_2^{2\beta} dt \right) \right|$$

$$\leq \exp \left( -\lambda \int_0^\infty ||\phi_f(x,t)||_2^{2\beta} dt \right)$$

$$\times \lambda \exp \left( -\lambda V_{\beta}(x) \right) |V_{\beta}(x) - V_{\beta}(y)|,$$

where the inequality in Eqn. (22) follows by the exponential inequality given in Eqn. (90) in Lemma 8 and the function $V_{\beta}$ is as in Eqn. (10).

We first derive a bound for $|V_{\beta}(x) - V_{\beta}(y)|$.

$$|V_{\beta}(x) - V_{\beta}(y)| = \int_0^\infty ||\phi_f(x,t)||_2^{2\beta} - ||\phi_f(y,t)||_2^{2\beta} dt \quad (23)$$

$$\leq \int_0^\infty ||\phi_f(x,t)||_2^{2\beta} - ||\phi_f(y,t)||_2^{2\beta} dt$$

$$\times \left( \sum_{i=0}^{2\beta -1} ||\phi_f(x,t)||_2^{2\beta -1-i} ||\phi_f(y,t)||_2^{2\beta -1-i} dt \right)$$

$$\leq \int_{F_{\eta}(x)}^\infty ||\phi_f(x,t)| - \phi_f(y,t)||_2$$

$$\times \left( \sum_{i=0}^{2\beta -1} ||\phi_f(x,t)||_2^{2\beta -1-i} ||\phi_f(y,t)||_2^{2\beta -1-i} dt \right)$$

$$+ \int_{F_{\eta}(x)}^\infty ||\phi_f(x,t)| - \phi_f(y,t)||_2$$

$$\times \left( \sum_{i=0}^{2\beta -1} ||\phi_f(x,t)||_2^{2\beta -1-i} ||\phi_f(y,t)||_2^{2\beta -1-i} dt \right)$$

We now derive a bound for the two terms that appear in the right hand side of Eqn. (23). Using the fact $||\phi_f(x,t)||_2 < R$ and $||\phi_f(y,t)||_2 < R$ since $ROA_f \subset B_R(0)$, and using Lemma 2, we get,

$$\int_{F_{\eta}(x)}^\infty ||\phi_f(x,t)| - \phi_f(y,t)||_2$$

$$\times \left( \sum_{i=0}^{2\beta -1} ||\phi_f(x,t)||_2^{2\beta -1-i} ||\phi_f(y,t)||_2^{2\beta -1-i} dt \right)$$

$$\leq 2\beta R^{2\beta -1} \int_{F_{\eta}(x)}^\infty ||\phi_f(x,t)| - \phi_f(y,t)||_2 dt$$

$$\leq 2\beta R^{2\beta -1} ||x-y||_2 \int_{F_{\eta}(x)}^\infty e^{\eta t} dt$$

$$= 2\beta R^{2\beta -1} \left( e^{-\eta F_{\eta}(x)} - 1 \right) ||x-y||_2.$$
Therefore, since \(|z_g - x| = |\beta| |x - y|\), it follows that \(z_g \in ROF\) for all \(\beta \in [0, \varepsilon/|x - y|]\). Thus \(\sigma := \sup \{\beta : z_g \in ROF\} \geq \varepsilon/|x - y| > 0\). Moreover, \(\sigma \leq 1\) as \(z_1 = y \notin ROA\).

Consider \(a_n := \sigma(1 - 1/n)\) and denote the sequence of points \(w_n := \beta a_n\). It follows that \(\{w_n\} \subset ROA\) and \(w^* := \lim_{n \to \infty} w_n \in \partial ROA\). By Lemma 4, we have that \(\lim_{n \to \infty} V_\beta(w_n) = \infty\). Therefore, there exists \(N \in \mathbb{N}\) such that
\[
\exp(-\lambda V_\beta(w_n)) < {K/2} |x - y|^2 \quad \text{for all } n > N. \tag{28}
\]
Moreover, since \(y \notin ROA\) we have that \(W_{\lambda, \beta}(y) = 1\). Thus by Eqn. (28) we have that
\[
|W_{\lambda, \beta}(w_n) - W_{\lambda, \beta}(y)| = |1 - \exp(-\lambda V_\beta(w_n)) - 1| = \exp(-\lambda V_\beta(w_n)) \leq {K/2} |x - y|^2 \quad \text{for all } n > N. \tag{29}
\]
Furthermore, for any \(n > N\) we have that \(w_n \in ROA\) and \(x \in ROF\) and thus Case 1 shows that
\[
|W_{\lambda, \beta}(x) - W_{\lambda, \beta}(w_n)| < {K/2} |x - w_n|^2. \tag{30}
\]
Thus, by Eqns (29) and (30) and selecting any \(n > N\), it now follows that
\[
|W_{\lambda, \beta}(x) - W_{\lambda, \beta}(y)| = |W_{\lambda, \beta}(x) - W_{\lambda, \beta}(w_n)| + |W_{\lambda, \beta}(w_n) - W_{\lambda, \beta}(y)| \\
\leq {K/2} |x - w_n|^2 + \exp(-\lambda V_\beta(w_n)) \\
\leq {K/2} \left(1 - \frac{1}{n}\right) |x - y|^2 + {K/2} |x - y|^2 \\
\leq K |x - y|^2,
\]
where the third inequality follows since \(\sigma(1 - 1/n) < 1\) for all \(n \in \mathbb{N}\). Therefore, Eqn. (20) holds when \(x \in ROA\) and \(y \notin ROF\).

**Case 3:** \(y \in ROA\) and \(x \notin ROF\). It follows by a similar argument to Case 2 that
\[
|W_{\lambda, \beta}(x) - W_{\lambda, \beta}(y)| \leq K |x - y|^2, \tag{31}
\]
and thus Eqn. (20) holds when \(y \in ROA\) and \(x \notin ROF\).

**Case 4:** \(x, y \notin ROF\). We have that \(W_{\lambda, \beta}(x) = W_{\lambda, \beta}(y) = 1\) for all \(x, y \notin ROA\) and thus,
\[
|W_{\lambda, \beta}(x) - W_{\lambda, \beta}(y)| = 0 \leq K |x - y|^2, \tag{32}
\]
and Eqn. (20) holds when \(x, y \notin ROF\).

**C. The Converse Lyapunov Function Satisfies a PDE**

Proposition 1 shows \(W_{\lambda, \beta}\) is a Lipschitz continuous function when \(\lambda > 0\) and \(\beta \in \mathbb{N}\) are sufficiently large. Rademacher’s Theorem (Theorem 4 found in Appendix XII) shows that Lipschitz continuous functions are differentiable almost everywhere. Therefore, \(W_{\lambda, \beta}\) must satisfy some Partial Differential Equation (PDE) almost everywhere. We next derive this PDE by showing \(W_{\lambda, \beta}\) satisfies Eqn. (31).

**Proposition 2.** Consider \(f \in C^2(\mathbb{R}^n, \mathbb{R})\) and \(W\) as in Eqn. (11). Suppose there exists \(\theta, \eta, R > 0\) such that \(\|D^2f(x)\|_2 < \theta\) for all \(x \in B_R(x)\) and \(\|x\|_2 \leq 2, B_\eta(0)\) is an exponentially stable set (Defn. 1) of the ODE (1), and \(ROA \subset B_R(0)\). If \(\lambda > \eta^{-2\beta}\) and \(\beta > \frac{\theta}{2\beta} + \frac{1}{2}\), then
\[
\nabla W_{\lambda, \beta}(x)^T f(x) = -\lambda |x|^2 \left(1 - W_{\lambda, \beta}(x)\right) \tag{33}
\]
for almost every \(x \in \mathbb{R}^n\).

**Proof.** By Prop. 1 we have that \(W_{\lambda, \beta} \in Lip(\mathbb{R}^n, \mathbb{R})\). Therefore by Rademacher’s Theorem (stated in Theorem 4 and found in Appendix XII) \(W_{\lambda, \beta}\) is differentiable almost everywhere. Moreover, \(\phi_f\) is differentiable by Lemma 3. Since the composition of differentiable functions is itself differentiable it follows by the chain rule that,
\[
\frac{d}{dt} W_{\lambda, \beta}(\phi_f(x(t))) \bigg|_{t=0} = \nabla W_{\lambda, \beta}(x)^T \frac{d}{dt} \phi_f(x(t)) \bigg|_{t=0} \tag{34}
\]
\[
= \nabla W_{\lambda, \beta}(x)^T f(x) \quad \text{for almost every } x \in \mathbb{R}^n.
\]
On the other hand, if \(x \in ROA\) it follows \(\phi_f(x(t)) \in ROA\) for all \(t \geq 0\) and thus,
\[
W_{\lambda, \beta}(\phi_f(x(t))) = 1 - \exp \left( -\lambda \int_0^t \|\phi_f(x,s)\|_{L^2}^2 ds \right) \tag{35}
\]
By the fundamental theorem of calculus and the fact \(\phi_f(x,0) = x\) for all \(x \in \mathbb{R}^n\) we have that,
\[
\frac{d}{dt} W_{\lambda, \beta}(\phi_f(x(t))) \bigg|_{t=0} = 0 = -\lambda \int_0^t \|\phi_f(x,s)\|_{L^2}^2 ds \bigg|_{t=0} = -\lambda |x|_{L^2}^2 \left(1 - W_{\lambda, \beta}(x)\right) \quad \text{for } x \in ROA. \tag{36}
\]
If \(x \notin ROA\) then clearly \(\phi_f(x(t)) \notin ROA\) for all \(t \geq 0\). Thus \(W(\phi_f(x(t))) = 1\) for all \(x \notin ROA\) and \(t \geq 0\). Therefore,
\[
\frac{d}{dt} W_{\lambda, \beta}(\phi_f(x(t))) \bigg|_{t=0} = \frac{d}{dt} \bigg|_{t=0} = 0 = -\lambda |x|_{L^2}^2 \left(1 - 1\right) \tag{37}
\]
\[
= -\lambda |x|_{L^2}^2 \left(1 - W_{\lambda, \beta}(x)\right) \quad \text{for } x \notin ROA.
\]
Hence, Eqns (32), (34) and (35) prove that the PDE given in Eqn. (31) holds.

**VI. A CONVEX OPTIMIZATION PROBLEM FOR APPROXIMATING THE CONVERSE LYAPUNOV FUNCTION**

We have reduced the problem of approximating the region of attraction to solving the optimization problem given in Eqn. (8), where \(W = W_{\lambda, \beta}\) is given in Eqn. (11). Unfortunately, no analytical formula for \(W_{\lambda, \beta}\) is known. Therefore, the optimization problem given in Eqn. (8) cannot be solved in its current form.

Fortunately, the unknown function \(W_{\lambda, \beta}\) can be removed from the objective function of Opt. (8). To see this note that if \(J(x) \geq W_{\lambda, \beta}(x)\) for all \(x \in \Lambda \subseteq \Omega\), then minimizing \(\int_\Lambda J(x) - W_{\lambda, \beta}(x) dx\) is equivalent to minimizing \(\int_\Lambda J(x) dx\). Thus, Opt. (8) is equivalent to the following optimization problem,
\[
P_\beta \in \arg \min_{J \in \mathcal{P}_\beta(\mathbb{R}^n, \mathbb{R})} \int_\Lambda J(x) dx \tag{38}
\]
such that \(J(x) \geq W_{\lambda, \beta}(x)\) for all \(x \in \Omega\).
Unfortunately, the constraint of Opt. (36) still involves the unknown function \( W_{\lambda, \beta} \). In the absence of an analytical formula for \( W_{\lambda, \beta} \) we propose in Prop. 3 conditions, in the form of the linear partial differential inequalities given in Eqs. (37), (38) and (39), that when satisfied by some function \( J \in C^1(\mathbb{R}^n, \mathbb{R}) \) implies that \( W_{\lambda, \beta}(x) \leq J(x) \). Thus, any \( J \) satisfying Eqs. (37), (38) and (39) is feasible to Opt. (36).

A. Bounding The Converse Lyapunov Function From Above

**Proposition 3.** Consider \( f \in C^1(\mathbb{R}^n, \mathbb{R}) \), \( \beta \in \mathbb{R} \) and \( \lambda > 0 \). Suppose there exists \( J \in C^1(\Omega, \mathbb{R}) \), \( \lambda > 0 \) that satisfies

\[
\nabla J(x)^T f(x) \leq -\lambda \|x\|_2^2 (1 - J(x)) \quad \text{for all } x \in \Omega, 
\]

\[
J(x) \geq 1 \quad \text{for all } x \in \partial \Omega, 
\]

\[
J(0) \geq 0, 
\]

where \( \Omega \subset \mathbb{R}^n \) is a compact set. Then \( W_{\lambda, \beta}(x) \leq J(x) \) for all \( x \in \Omega \), where \( W_{\lambda, \beta} \) is as in Eqs. (11).

**Proof.** Consider \( x \in \Omega \). Let us consider the time the solution map exits the set \( \Omega \subset \mathbb{R}^n \), denoted by \( T_x := \sup \{ t \geq 0 : \phi_f(x, t) \in \partial \Omega \} \). Furthermore, let us denote \( u(t) := J(\phi_f(x, t)) - 1 \) and \( \alpha(t) := \lambda \|\phi_f(x, t)\|_2^2 \). It follows from Eqn. (37) that

\[
\frac{d}{dt} u(t) \leq \alpha(t) u(t) \quad \text{for all } t \in [0, T_x].
\]

Therefore by Lemma 9 it follows that

\[
uu = u(0) \exp \left( \int_0^t \alpha(s) ds \right) \quad \text{for all } t \in [0, T_x],
\]

and thus selecting \( t = T_x \) we have that

\[
J(\phi_f(x, T_x)) - 1 \leq (J(0) - 1) \exp \left( \lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right). \tag{40}
\]

By rearranging Eqn. (40) we get that,

\[
J(x) \geq 1 - (1 - J(\phi_f(x, T_x))) \exp \left( -\lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right). \tag{41}
\]

**Case 1:** \( T_x < \infty \). In this case the solution map exits the set \( \Omega \) in some finite time. Since \( \phi_f \in C(\mathbb{R}^n \times [0, \infty), \mathbb{R}^n) \) (by Lemma 3) it is clear that \( \phi_f(x, T_x) \in \partial \Omega \). Therefore by Eqn. (38) we have that \( J(\phi_f(x, T_x)) \geq 1 \). Hence,

\[
(1 - J(\phi_f(x, T_x))) \exp \left( -\lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right) \leq 0.
\]

Thus, by Eqn. (41) we have that

\[
J(x) \geq 1 - (1 - J(\phi_f(x, T_x))) \exp \left( -\lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right) \geq 1 \geq W_{\lambda, \beta}(x),
\]

since \( W_{\lambda, \beta}(x) \leq 1 \).

**Case 2a:** \( T_x = \infty \) and \( x \in \text{ROA}_f \). In this case we have \( \lim_{t \to \infty} \|\phi_f(x, t)\|_2 = \|\phi_f(x, T_x)\|_2 = 0 \) since \( x \in \text{ROA}_f \). Moreover, since \( J(\phi_f(x, T_x)) = J(0) \geq 0 \) (by Eqn. (39)) and \( \exp(x) \geq 0 \) for all \( x \in \mathbb{R}^n \) it follows from Eqn. (41) that

\[
J(x) \geq 1 - (1 - J(0)) \exp \left( -\lambda \int_0^{\infty} \|\phi_f(x, s)\|_2^2 ds \right) \geq 1 \exp \left( -\lambda \int_0^{\infty} \|\phi_f(x, s)\|_2^2 ds \right) = W_{\lambda, \beta}(x).
\]

**Case 2b:** \( T_x = \infty \) and \( x \in \Omega / \text{ROA}_f \). If \( x \in \Omega / \text{ROA}_f \) we have that \( W(x) = 1 \). Moreover, if \( T_x = \infty \) then the solution map never exists the set \( \Omega \), that is \( \phi_f(x, t) \notin \Omega \) for all \( t \geq 0 \). Since \( J \) is differentiable and \( \Omega \) is compact we have that \( J \) is bounded, that is, there exists \( M > 0 \) such that \( |J(\phi_f(x, t))| < M \) for all \( t \geq 0 \). Since \( x \notin \text{ROA}_f \) we have that \( \phi_f(x, t) \notin \text{ROA}_f \) for all \( t \geq 0 \). This there exists \( \epsilon > 0 \) such that \( |\phi_f(x, t)|_2^2 \geq \epsilon^2 \) for all \( t \geq 0 \). Thus, since \( |J(\phi_f(x, t))| < M \) for all \( t \geq 0 \), we have that

\[
(1 - J(\phi_f(x, t))) \exp \left( -\lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right) = \lim_{T \to \infty} \left( 1 - J(\phi_f(x, T)) \right) \exp \left( -\lambda \int_0^T \|\phi_f(x, s)\|_2^2 ds \right) \leq \lim_{T \to \infty} \left( M + 1 \exp (-T \lambda \epsilon^2) \right) = 0,
\]

implying \( (1 - J(\phi_f(x, T_x))) \exp \left( -\lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right) = 0 \).

It is now clear by Eqn. (41) that

\[
J(x) \geq 1 - (1 - J(\phi_f(x, T_x))) \exp \left( -\lambda \int_0^{T_x} \|\phi_f(x, s)\|_2^2 ds \right) \geq 1 = W_{\lambda, \beta}(x).
\]

**Corollary 2.** Consider \( f \in C^1(\mathbb{R}^n, \mathbb{R}) \), \( \beta \in \mathbb{R} \) and \( \lambda > 0 \). Suppose there exists \( J \in C^1(\Omega, \mathbb{R}) \) that satisfies Eqs. (37), (38) and (39) for some compact set \( \Omega \). Then \( J(x) \geq 0 \) for all \( x \in \Omega \).

**Proof.** By Prop. 3 we have that \( J(x) \geq W_{\lambda, \beta}(x) \geq 0 \), where \( W_{\lambda, \beta} \) is as in Eqs. (11).

B. Tightening The Problem of Approximating Our Proposed Converse Lyapunov Function

Using Prop. 3 we now tighten the optimization problem given in Eqn. (36). For given \( f \in C^1(\mathbb{R}^n, \mathbb{R}) \), \( \lambda > 0 \), \( \beta \in \mathbb{R} \), \( R > 0 \) and \( \Lambda \leq \Omega \subset \mathbb{R}^n \) consider the following optimization problem,

\[
P_d = \arg \min_{J \in P_d(\mathbb{R}^n, \mathbb{R})} \int_\Lambda J(x) dx \tag{42}
\]

such that \( J \) satisfies (37), (38), and (39).

Clearly the Opt. (42) is a tightening of the Opt. (8) since if \( J \) is feasible to Opt. (42) then by Prop. 3 we have that \( J \) is also feasible to Opt. (8). Moreover, Opt. (42) is a convex optimization problem since it is linear in its decision variable, \( J \), in both the constraints and objective function. In the next section we further tighten Opt. (42) to an SOS Optimization Problem (43) that can be tractably solved. For implementation purposes we select \( \Omega = B_R(0) \), where \( R > 0 \), and \( \Lambda \subset \mathbb{R}^n \) as some rectangular set (of form \([a_1, b_1] \times \ldots \times [a_2, b_2] \subset \mathbb{R}^n \)).

VII. AN SOS OPTIMIZATION PROBLEM FOR ROA APPROXIMATION

For a given ODE (1) we next propose a sequence of convex Sum-of-Squares (SOS) optimization problems, indexed by \( d \in \)
We show that the sequence of solutions, \( \{P_d\}_d \in \mathbb{N} \), yields a sequence of sublevel sets which are contained inside the region of attraction of the ODE and which converge to the region of attraction of the ODE with respect to the volume metric as \( d \to \infty \).

For given \( f \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) \), \( \lambda > 0 \), \( \beta \in \mathbb{N} \), \( R > 0 \) and integration region \( \Lambda \subset \mathbb{R}^n \) consider the following sequence of SOS optimization problems indexed by \( d \in \mathbb{N} \):

\[
P_d = \arg \min_{J \in \mathcal{P}_d(\mathbb{R}^n, \mathbb{R})} c^T \alpha \tag{43}
\]

\[
J(x) = c^T Z_d(x),
\]

\[
J(k, k_2, s) \in \mathcal{P}_d(\mathbb{R}^n, \mathbb{R})
\]

\[
\lambda(0) \geq 0,
\]

\[
k_1(x) = -(J(x) - 1) - p(x)(R^2 - ||x||^2_2),
\]

\[
k_2 = (J(x) - 1) - p(x)(R^2 - ||x||^2_2),
\]

where \( \alpha = \int_{\Lambda} Z_d(x) dx \), recalling \( Z_d : \mathbb{R}^n \to \mathbb{R}^d \) is the vector of monomials of degree \( d \) in \( \mathbb{N} \) and \( \nu_d = (\nu_d^\alpha) \).

We will show next, in Cor. 3, that the family of SOS optimization problems given in Eqn. (43) yields an inner approximation of ROA\(_f\) for each \( d \in \mathbb{N} \) (an approximation certifiably contained inside of ROA\(_f\)).

**Corollary 3.** Consider \( f \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \), \( \lambda > 0 \), \( \beta \in \mathbb{N} \), \( R > 0 \) and \( \Lambda \subset \mathbb{R}^n \). Suppose ROA\(_f\) \( \subset B(0) \) and there exists \( \eta > 0 \) such that \( B(0) \) is an exponentially stable set. Then we have that

\[
\{ x \in B(0) : P_d(x) < 1 \} \subset \text{ROA}_f \text{ for all } d \in \mathbb{N}, \tag{44}
\]

where \( P_d \) is any solution to the SOS Problem (43) for \( d \in \mathbb{N} \).

**Proof.** Suppose \( P_d \) is any solution to the SOS Problem (43) for \( d \in \mathbb{N} \). Then \( P_d \) satisfies the constraints of the SOS Problem (43) and thus satisfies Eqs (37), (38), and (39) for \( \Omega = B(0) \). Therefore, \( W_{\lambda, \beta}(x) \leq P_d(x) \) for all \( x \in B(0) \) by Prop. 3. Hence, it is clear that

\[
\{ x \in B(0) : P_d(x) < 1 \} \subset \{ x \in B(0) : W_{\lambda, \beta}(x) < 1 \}. \tag{45}
\]

Moreover, Cor. 1 shows \( \{ x \in B(0) : W_{\lambda, \beta}(x) < 1 \} = \text{ROA}_f \) and thus Eqn. (44) holds.

Cor. 3 implies that solution maps initialized inside our ROA\(_f\) approximation asymptotically cover the origin. That is for any \( d \in \mathbb{N} \) and for all \( y \in \{ x \in B(0) : P_d(x) < 1 \} \) we have that lim\(_{d \to \infty}\)\(|\phi_d(y)|\) = 0, where \( P_d \) is any solution to the SOS Problem (43) for \( d \in \mathbb{N} \) (note this does not rule out the possibility that \( \{ x \in B(0) : P_d(x) < 1 \} = \emptyset \)).

Further to Cor. 3, we will next show, in Theorem 1, that for sufficiently large \( \lambda > 0 \) and \( \beta \in \mathbb{N} \) the sequence of SOS optimization problems given in Eqn. (43) yields a sequence of sets that tend to ROA\(_f\) with respect to the volume metric as \( d \to \infty \). We first define the volume metric. For sets \( A, B \subset \mathbb{R}^n \), we denote the volume metric as \( D_V(A, B) \), where

\[
D_V(A, B) := \mu((A \setminus B) \cup (B \setminus A)). \tag{46}
\]

We note that \( D_V \) is a metric (Defn. 5), as shown in Lem. 6 (found in Appendix XI).

**Theorem 1.** Consider \( f \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \) and integration region \( \Lambda \subset \mathbb{R}^n \). Suppose there exists \( \theta, \eta, \lambda > 0 \) such that \( \|D^\beta f(x)\|_2 < \theta \) for all \( x \in B(0) \) and \( \|\alpha\|_1 \leq 2, B(0) \) is an exponentially stable set (Defn. 1) of the ODE (1), and ROA\(_f\) \( \subset B(0) \). Then if ROA\(_f\) \( \subset \Lambda \subset B(0) \), \( \lambda > \eta \frac{2^\beta}{1+\frac{1}{2}} \) and \( \beta > \frac{\eta}{2^\beta} + 1 \) we have that

\[
\lim_{d \to \infty} D_V(\text{ROA}_f, \{ x \in \Lambda : P_d(x) < 1 \}) = 0, \tag{47}
\]

where \( P_d \) is any solution to the Problem (43) for \( d \in \mathbb{N} \).

**Proof.** By Cor. 1 we have that ROA\(_f\) = \( \{ x \in \mathbb{R}^n : W_{\lambda, \beta}(x) < 1 \} \). Moreover, since \( P_d \) satisfies the constraints of the SOS Problem (43) it follows that \( P_d \) satisfies Eqs (37), (38), and (39) for \( \Omega = B(0) \). Therefore, \( W_{\lambda, \beta}(x) \leq P_d(x) \) for all \( x \in B(0) \) by Prop. 3. Thus, by Cor. 4 (found in Appendix XI) it follows that Eqn. (47) holds if lim\(_{d \to \infty}\)\(\|P_d - W_{\lambda, \beta}\|_{L^1(\Lambda, \mathbb{R})} = 0 \). To show lim\(_{d \to \infty}\)\(\|P_d - W_{\lambda, \beta}\|_{L^1(\Lambda, \mathbb{R})} = 0 \) we must show for all \( \epsilon > 0 \) there exists \( D \in \mathbb{N} \) such that

\[
\int_\Lambda |P_d(x) - W_{\lambda, \beta}(x)| dx < \epsilon \text{ for all } d > D. \tag{48}
\]

Now, let \( \epsilon > 0 \). Then Theorem 3 shows there exists \( J \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \) such that

\[
\sup_{x \in B(0)} |J(x) - W_{\lambda, \beta}(x)| < \frac{\epsilon}{\mu(\Lambda) + 1}. \tag{49}
\]

\[
\nabla J(x)^T f(x) - \lambda (1 - J(x)) \|\alpha\|_p^\beta - s_1(x) (R^2 - \|x\|^2_2) = s_2(x),
\]

for all \( x \in \mathbb{R}^n \).

\[
J(x) - 1 - (s_3(x) - s_4(x)) (R^2 - \|x\|^2_2) = s_5(x),
\]

for all \( x \in \mathbb{R}^n \).

Let \( D := \max\{\max_{i=1,\ldots,5} (\text{deg}(s_i)), \text{deg}(J)\} \). Then from Eqs (50) and (51) and since \( J > 0 \) it follows that \( J \) is feasible to the SOS Problem (43) for any \( d > D \). Since, \( P_d \) is the optimal solution to the SOS Problem (43) it follows that the objective function of the SOS Problem (43) evaluated at \( P_d \) is less than or equal to the objective function evaluated at the feasible solution \( J \) for \( d > D \). That is by writing \( P_d \) and \( J \) with respect to the monomial vector, \( P_d(x) = c_d^T Z_d(x) \) and \( J(x) = b^T Z_{\text{deg}(J)}(x) \), we have that

\[
\int_\Lambda |P_d(x) - W_{\lambda, \beta}(x)| dx < \epsilon \text{ for all } d > D, \tag{52}
\]

where \( \alpha = \int_{\Lambda} Z_d(x) dx \), and \( \gamma = \int_{\Lambda} Z_{\text{deg}(J)}(x) dx \). We now show Eqn. (48). Using the fact \( W_{\lambda, \beta}(x) \leq P_d(x) \) for all \( x \in \Lambda \) together with Eqns (49) and (52) we get that

\[
\int_\Lambda |P_d(x) - W_{\lambda, \beta}(x)| dx = \int_\Lambda P_d(x) dx - \int_\Lambda W_{\lambda, \beta}(x) dx \leq \int_\Lambda J(x) dx - \int_\Lambda W_{\lambda, \beta}(x) dx \leq \mu(\Lambda) \sup_{x \in \Lambda} \{ |J(x) - W_{\lambda, \beta}(x)| \} < \epsilon \text{ for all } d > D.
\]
Hence by Cor. 4 (found in Appendix XI) it follows that Eqn. (47) holds.

VIII. NUMERICAL EXAMPLES

We now present several numerical examples that demonstrate that by solving the SOS problem, given in Eqn. (43), we are able to approximate the region of attraction of a nonlinear system. Note that for numerical implementation it is best to choose $\lambda > 0$ as small as possible. This is because the Lipschitz constant (given in Eqn. (19)) of $W_{\lambda, \beta}$ (given in Eqn. (10)) grows as $\lambda > 0$ increases.

**Example 1.** Consider the Van der Pol oscillator defined by the ODE:

\[
\begin{align*}
\dot{x}_1(t) &= -x_2(t) \\
\dot{x}_2(t) &= x_1(t) - x_2(t)(1 - x_1^2(t)).
\end{align*}
\]

In Fig. 1 we have plotted our estimation of the region of attraction of the ODE (53). Our estimation is given by the 1-sublevel set of the solution to the SOS optimization problem given in Eqn. (43) for $d = 12$, $\lambda = 0.05$, $\beta = 2$, $R = \sqrt{2^2 + 2.7^2} \approx 3.36$, $\Lambda = [-2, 2] \times [-2.7, 2.7]$, and $f = [-x_2, x_1 + x_2(x_1^2)]^T$.

**Example 2.** Consider the third order servomechanism with multiplicative feedback control found in [24] given by the following ODE:

\[
T \ddot{y} + \frac{dy}{dt} + K_2(1 - K_3 y^2)\frac{dy}{dt} + K_1 y = 0,
\]

where $T \in \mathbb{R}$ is a time constant and $K_1, K_2, K_3 \in \mathbb{R}$ are gain constants. We consider the case $T = K_2 = 1$ and $K_1 = K_3 = 1$. The ODE (54) can be represented in the form $\dot{x}(t) = f(x(t))$ with

\[
f(x) = [x_2, x_3, (1/T)(-x_3 - K_2(1 - K_3 x_1^2)x_2 - K_1x_1)]^T.
\]

Through numerical experiments using Matlab’s ODE45 function it was found that the ODE with vector field given in Eqn. (55) appeared to have unbounded region of attraction. Therefore, for this system, Theorem 1 does not show that the sequence of sublevel sets to the solution to the SOS problems given in Eqn. (43) converges to the region of attraction as $d \to \infty$. Nevertheless, in Fig. 2 we have plotted the 1-sublevel set of the solution to the SOS optimization problem given in Eqn. (43) for $d = 10$, $\lambda = 0.5$, $\beta = 2$, $R = \sqrt{3}$, $\Lambda = [-1, 1]^3$ and $f$ given in Eqn. (55). Fig. 2 indicates that even for systems with unbounded regions of attraction, our proposed SOS algorithm can provide arbitrarily good inner estimations of ROA $\cap \Lambda$, where $\Lambda \subset \mathbb{R}^n$ is some compact set. Through Monte Carlo simulation the volume of ROA $\cap \Lambda$ was estimated to be 0.3372 whereas the volume of our ROA approximation was found to be 0.2806, an error of 0.0566.

IX. CONCLUSION

For a given locally exponentially stable dynamical system, described by an ODE, we have proposed a family SOS optimization problems that yields a sequence of sublevel sets that converge to the region of attraction of the ODE with respect to the volume metric. In order to facilitate this result we proposed a new converse Lyapunov function that was shown to be globally Lipschitz continuous. We have provided several numerical examples of practical interest showing how our proposed family of SOS problems can provide arbitrarily good approximations of regions of attraction. In future work we aim extend this work to systems with weaker forms of stability and investigate systems with unbounded regions of attraction.

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X. APPENDIX A: APPROXIMATION OF LIPSCHITZ CONVERSE LYAPUNOV FUNCTIONS IN SOBOLEV SPACE

In this section we introduce aspects of mollification and polynomial approximation theory used in our proof of Theorem 3; that there exists a polynomial arbitrarily “close” to the converse Lyapunov function $W_{\beta}$ (given in Eqn. (11)) and is also a feasible solution to some $d \in \mathbb{N}$ instantiation of the family of SOS optimization problems given in Eqn. (43). Theorem 3 is a key ingredient in the proof of Theorem 1 (the main result of the paper).

A. Approximating Lipschitz Functions by Infinitely Differentiable Functions

For an overview of approximation by mollification we refer to [25].

a) Mollifiers: The standard mollifier, $\eta \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is defined as

$$\eta(x) = \begin{cases} C \exp \left(\frac{1}{||x||^2-1}\right) & \text{when } ||x|| < 1, \\ 0 & \text{when } ||x|| \geq 1, \end{cases}$$

where $C > 0$ is chosen such that $\int_{\mathbb{R}^n} \eta(x) dx = 1$.

For $\sigma > 0$ we denote the scaled standard mollifier by $\eta_\sigma \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$\eta_\sigma(x) := \frac{1}{\sigma^n} \eta \left( \frac{x}{\sigma} \right).$$

Note, clearly $\eta_\sigma(x) = 0$ for all $x \notin B_{\sigma}(0)$.

b) Mollification of a Function (Smooth Approximation): Recall from Section II that for open sets $\Omega \subset \mathbb{R}^n$ and $\sigma > 0$ we denote $< \Omega >_\sigma := \{ x \in \Omega : B_\sigma(x) \subset \Omega \}$. Now, for each $\sigma > 0$ and function $V \in L^1(\Omega, \mathbb{R})$ we denote the $\sigma$-mollification of $V$ by $[V]_\sigma := < \Omega >_\sigma \rightarrow \mathbb{R}$, where

$$[V]_\sigma(x) := \int_{\mathbb{R}^n} \eta_\sigma(x-z)V(z) dz = \int_{B_{\sigma}(0)} \eta_\sigma(z)V(x-z) dz. \tag{57}$$

To calculate the derivative of a mollification we next introduce the concept of weak derivatives.

Definition 4. For $\Omega \subset \mathbb{R}^n$ and $F \in L^1(\Omega, \mathbb{R})$ we say any $H \in L^1(\Omega, \mathbb{R})$ is the weak inverse derivative of $F$ if

$$\int_{\Omega} F(x) \frac{\partial}{\partial x_i} \alpha(x) dx = - \int_{\Omega} H(x) \alpha(x) dx, \text{ for } \alpha \in C^\infty(\mathbb{R}^n, \mathbb{R}).$$
Weak derivatives are “essentially unique”. That is if $H_1$ and $H_2$ are both weak derivatives of a function $F$ then the set of points where $H_1(x) \neq H_2(x)$ has measure zero. If a function is differentiable then its weak derivative is equal to its derivative in the “classical” sense. We will use the same notation for the derivative in the “classical” sense and in the weak sense.

In the next proposition we state some useful properties about Sobolev spaces and mollifications taken from [25].

**Proposition 4** (25). For $1 \leq p < \infty$ and $k \in \mathbb{N}$ we consider $V \in W^{k,p}(E, \mathbb{R})$, where $E \subset \mathbb{R}^n$ is an open bounded set, and its $\sigma$-mollification $[V]_{\sigma}$. Recalling from Section II that for an open set $\Omega \subset \mathbb{R}^n$ and $\sigma \geq 0$ we denote $\sigma = \{x \in \Omega : B(x, \sigma) \subset \Omega\}$, the following holds:
1. For all $\sigma > 0$ we have $\sup_{x \in \Omega} |[V]_{\sigma} (x) - V(x)| < \epsilon$.

2. For all $\sigma > 0$ we have $\sup_{x \in \Omega} \left| \frac{\partial}{\partial x} [V]_{\sigma}(x) - \frac{\partial}{\partial x} V(x) \right| < \epsilon$.

3. If $x \in C(E, \mathbb{R})$ then for any compact set $K \subset E$ we have $\lim_{\sigma \to 0} \sup_{x \in K} \left| [V]_{\sigma}(x) - V(x) \right| = 0$.

**B. Weighted Polynomial Approximation in Sobolev Space**

We next state a result that can be thought of as a generalization of the Weierstrass approximation theorem. It proves there exists a polynomial that can approximate a sufficiently smooth function arbitrarily well with respect to the $W^{1,\infty}$ norm weighted by a function of form $w(x) = 1/\|x\|_2^2$. This result was first presented in the case of $\beta = 1$ in [15] and then later extended to the general case of $\beta \in \mathbb{N}$ in [16].

**Theorem 2** (Weighted Polynomial Approximation [16]). Let $E \subset \mathbb{R}^n$ be an open set, $\sigma \in \mathbb{N}$ and $V \in C^{2\beta+2}(E, \mathbb{R})$. For any compact set $K \subset E$ and $\epsilon > 0$ there exists a polynomial function $p$ of degree $2\beta$ such that

$$|V(x) - p(x)| < \epsilon \|x\|_2^{2\beta} \text{ for all } x \in K,$$

$$\|\nabla V(x) - \nabla p(x)\|_2 < \epsilon \|x\|_2^{2\beta} \text{ for all } x \in K.$$

**C. Approximation of Lyapunov Functions**

In this section we show in Theorem 3 that there exists a polynomial arbitrarily “close” to the converse Lyapunov function $W_{\lambda, \beta}$ (given in Eqn. (11)) and also a feasible solution to some $d \in \mathbb{N}$ instantiation of the family of SOS optimization problems given in Eqn. (43). We take the following steps:

(A) In Lemma 5 we take the mollification of $W_{\lambda, \beta}$ to show there exists an infinitely differentiable function that satisfies Eqns (58), (59) and (60).

(B) In Prop. 5 we use Lemma 5 together with partitions of unity (Theorem 7) to show there exists an infinitely differentiable function that satisfies Eqns (63), (64) and (65).

(C) In Theorem 3 we use Prop. 5 together with the polynomial approximation results in Theorem 2 to show there exists a polynomial function that satisfies Eqns (75), (76) and (77).

**Lemma 5.** Consider $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and $W$ as in Eqn. (11). Suppose there exists $\theta, \eta, R > 0$ such that $\|D^2 f(x)\|_2 < \theta$ for all $x \in B_R(x)$ and $\|\alpha\|_1 \leq 2$, $B_\eta(0)$ is an exponentially stable set (Defn. 1) of the ODE (1), and ROA $f \subset B_R(0)$. If $\lambda > \theta \eta^{2\beta}$ and $\beta > \frac{\theta}{2\beta} + \frac{1}{2}$ then for any $\epsilon > 0$ and $R_1 > R$ there exists $J \in C^\infty(B_{R_1}(0), \mathbb{R})$ such that

$$\sup_{x \in B_{R_1}(0)} |J(x) - W_{\lambda, \beta}(x)| < \epsilon,$$  \hspace{1cm} (58)

$$\nabla J(x)^T f(x) < -\lambda (1 - J(x))|x|_2^{2\beta} + \epsilon \text{ for all } x \in B_{R_1}(0),$$  \hspace{1cm} (59)

$$J(x) = 1 \text{ for all } x \in \partial B_{R_1}(0) \text{ and } J(0) \geq 0.$$  \hspace{1cm} (60)

**Proof.** Let $\epsilon > 0$ and $R_2 > R_1 > R$. Since $W_{\lambda, \beta} \in Lip(\mathbb{R}^n, \mathbb{R})$ (by Prop. 1) we know that by Theorem 4 that $W_{\lambda, \beta} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R})$.

For $\sigma > 0$ we let us denote the $\sigma$-mollification of $W_{\lambda, \beta}$ by $J_\sigma(x) := [W_{\lambda, \beta}]_{\sigma}(x)$. We note that the domain of $W_{\lambda, \beta}$ is $\mathbb{R}^n$. However, for mollification purposes we consider $W_{\lambda, \beta}$ over the restricted domain $B_{R_1}(0) \subset \mathbb{R}^n$.

Let $\sigma_0 := \frac{R_2 - R_1}{2}$. It is clear that $B_{R_1}(0) \subset B_{R_2}(0) > \sigma_0$ for all $0 < \sigma < \sigma_0$. Therefore, by Prop. 4 we have that $J_\sigma \in C^\infty(B_{R_2}(0), \mathbb{R}) \subset C^\infty(B_{R_1}(0), \mathbb{R})$ for all $0 < \sigma < \sigma_0$.

We will now show there exists $\sigma > 0$ such that Eqns (58), (59) and (60) hold.

First we show Eqn. (58) holds. By Prop. 4 we know that there exists $\sigma_2 > 0$ such that for all $0 < \sigma < \sigma_2$ we have that

$$\sup_{x \in B_{R_1}(0)} |J_\sigma(x) - W_{\lambda, \beta}(x)| < \epsilon.$$  \hspace{1cm} (61)

We now show Eqn. (59) holds. Let us denote $r(x) := |x|_2^{2\beta}$. It is clear using the triangle inequality and the fact that $|x - z| < 2R_1$ for all $x, z \in B_{R_1}(0)$ that

$$r(x) - r(x - z) = (|x|_2 - |x - z|_2) \sum_{k=0}^{2\beta - 1} |x|_2^{2\beta - 1 - k} |x - z|_2^{k} \leq \left( \sum_{k=0}^{2\beta - 1} 2^{k} \right) |z|_2.$$  \hspace{1cm} (62)

Let $\sigma_3 := \frac{\epsilon}{KL_f + \lambda \left( \sum_{k=0}^{2\beta - 1} 2^{k} \right)}$, where $K$ (given in Eqn. (19)) is the Lipschitz constants of $W_{\lambda, \beta}$ and $L_f$ is the Lipschitz constants of $f$. For $0 < \sigma < \sigma_3$, using Prop. 4 and the fact
Consider the sets \( \mathcal{U}_m = B_{R_1}(0)/(B_{1/m}(0))^c \) for \( m \in \mathbb{N} \). It is clear \( \{ \mathcal{U}_m \}_{m \in \mathbb{N}} \) form an open cover (Defn. 6) of \( B_{R_1}(0)/\{0\} \), that is \( \bigcup_{m \in \mathbb{N}} \mathcal{U}_m = B_{R_1}(0)/\{0\} \). By Theorem 7 (found in Appendix X) there exists a partition of unity, we denote by \( \{ \psi_m \}_{m \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n, \mathbb{R}) \), subordinate to the open cover \( \{ \mathcal{U}_m \}_{m \in \mathbb{N}} \).

Let \( \varepsilon > 0 \). For each \( m \in \mathbb{N} \) it was shown in Lemma 5 that there exists \( J_m \in C^\infty(B_{R_1}(0), \mathbb{R}) \) such that

\[
\sup_{x \in B_{R_1}(0)} |J_m(x) - W_{\lambda, \beta}(x)| < \varepsilon
\]  

(66)

\[
\nabla J_m(x)^T f(x) < -\lambda (1 - J_m(x)) ||x||^2 + \varepsilon ||x||^{2\beta}
\]  

(67)

\[J_m(x) = 1 \text{ for all } x \in \partial B_{R_1}(0) \text{ and } J_m(0) = 0.
\]  

(68)

Note, \( \sup_{x \in B_{R_1}(0)} |\nabla \psi_m(x)^T f(x)| < \infty \) for each \( m \in \mathbb{N} \) since \( \mathcal{U}_m \) is bounded and the function \( \psi_m(x)^T f(x) \) is continuous in \( x \).

We now consider the function \( J(x) := \sum_{m=1}^{\infty} \psi_m(x) J_m(x) \). We first note that \( J \in C^\infty(B_{R_1}(0), \mathbb{R}) \). Since \( J \) is bounded and \( \psi(x)^T f(x) \) is continuous in \( x \), it thus follows for any \( x \in \mathbb{R}^n \) that \( J(x) \) is a finite sum of \( C^\infty(B_{R_1}(0), \mathbb{R}) \) functions over \( S \) and thus differentiable at \( x \). Since \( x \in \mathbb{R}^n \) was arbitrarily chosen it follows \( J \in C^\infty(B_{R_1}(0), \mathbb{R}) \).

We now show \( J \) satisfies Eqs. (63). Using the fact \( \sum_{m=1}^{\infty} \psi_m(x) = 1 \) for all \( x \in B_{R_1}(0)/\{0\} \) and \( \sum_{m=1}^{\infty} \psi_m(x) = 0 \) together with Eqs. (66) we have that

\[
\left|J(x) - W_{\lambda, \beta}(x)\right| = \left|\sum_{m=1}^{\infty} \psi_m(x) J_m(x) - W_{\lambda, \beta}(x)\right|
\leq \sum_{m=1}^{\infty} \psi_m(x) \left|J_m(x) - W_{\lambda, \beta}(x)\right|
\leq \sum_{m=1}^{\infty} \psi_m(x) \varepsilon \frac{2}{m^{2\beta}} < \varepsilon
\]  

for \( x \in B_{R_1}(0) \).

We now show \( J \) satisfies Eqs. (64). Before doing so we note that \( \sum_{m=1}^{\infty} \psi_m(x) = 1 \) for all \( x \in B_{R_1}(0)/\{0\} \). Since only finitely many \( \psi_m \)'s are non-zero for each \( x \in B_{R_1}(0)/\{0\} \) it follows \( \sum_{m=1}^{\infty} \psi_m(x) \) is a finite sum of infinitely differentiable functions. Therefore, we can interchange the derivative and the summation to show \( \frac{\partial}{\partial \tau} 1 = \frac{\partial}{\partial \tau} \sum_{m=1}^{\infty} \psi_m(x) = \sum_{m=1}^{\infty} \frac{\partial}{\partial \tau} \psi_m(x) \) for all \( x \in B_{R_1}(0)/\{0\} \) and \( i \in \{1, \ldots, n\} \). Thus it follows

\[
\sum_{m=1}^{\infty} \nabla \psi_m(x)^T f(x) = 0 \text{ for all } x \in B_{R_1}(0)/\{0\}\]  

(69)

\[\nabla J(x)^T f(x) \leq -\lambda (1 - J(x)) ||x||^2 + \varepsilon ||x||^{2\beta}\]  

(70)

Therefore,

\[
\sup_{m \in I} \left\{ \frac{1}{m^{2\beta}} \right\} \leq ||x||^{2\beta}\]  

for all \( x \in B_{R_1}(0)/\{0\} \).
Moreover, for \( x \in B_R(0)/\{0\} \) and \( m \notin I_\epsilon \) we have that \( x \notin U_m \). Thus, since \( \{ x \in \mathbb{R}^n : \psi_m(x) \neq 0 \} \subset U_m \) (by Theorem 7 found in Appendix XII) we have that
\[
\psi_m(x) = 0 \quad \text{for all} \quad x \in B_R(0)/\{0\} \quad \text{and} \quad m \notin I_\epsilon. \tag{71}
\]

Now, using Eqns (66), (67), (69), (70) and (71), and the fact \( \sum_{m=1}^{\infty} \frac{1}{m^n} = 1 \) we have that,
\[
\nabla J(x)^T f(x) + \lambda(1 - J(x))||x||^2_2
\]
\[
= \sum_{m=1}^{\infty} \psi_m(x) \left( \nabla J_m(x)^T f(x) + \lambda(1 - J_m(x))||x||^2_2 \right) + \sum_{m=1}^{\infty} J_m(x)\nabla \psi_m(x)^T f(x) - W_\lambda(x) \sum_{m=1}^{\infty} \nabla \psi_m(x)^T f(x)
\]
\[
= \sum_{m \in I_\epsilon} \psi_m(x) \left( \nabla J_m(x)^T f(x) + \lambda(1 - J_m(x))||x||^2_2 \right) + \sum_{m \notin I_\epsilon} (J_m(x) - W_\lambda(x))\nabla \psi_m(x)^T f(x)
\]
\[
\leq \sum_{m \in I_\epsilon} \psi_m(x) \frac{\varepsilon}{2m^{2\beta}} + \sum_{m \notin I_\epsilon} \frac{\varepsilon}{2m^{1+2m^{2\beta}}}
\]
\[
\leq \varepsilon \sup_{m \in I_\epsilon} \left\{ \frac{1}{m^{2\beta}} \right\} \left( \frac{1}{2} \sum_{m \notin I_\epsilon} \psi_m(x) + \frac{1}{2} \sum_{m \notin I_\epsilon} \frac{1}{m} \right)
\]
\[
\leq \varepsilon \sup_{m \in I_\epsilon} \left\{ \frac{1}{m^{2\beta}} \right\} \leq \varepsilon ||x||^2_2 \quad \text{for all} \quad x \in B_{R_1}(0)/\{0\}. \tag{72}
\]

Eqn. (72) shows \( J \) satisfies Eqn. (64) for \( x \in B_{R_1}(0)/\{0\} \). We still need to show \( J \) satisfies Eqn. (64) for \( x = 0 \). Let us denote the function \( F(x) := \nabla J(x)^T f(x) + \lambda(1 - J(x))||x||^2_2 \). To show \( J \) satisfies Eqn. (64) at \( x = 0 \) we must show \( F(0) = 0 \). We first note that \( F \in C^2(B_{R_1}(0)/\mathbb{R}) \) since \( F \in C^2(B_{R_1}(0)/\mathbb{R}) \) and \( ||x||^2_2 \in C^2(\mathbb{R}^n, \mathbb{R}) \). Then \( F \in \text{LocLip}(\mathbb{R}^n, \mathbb{R}) \).

Therefore,
\[
|F(0) - F(x)| \leq L_F ||x||_2 \quad \text{for all} \quad x \in B_{R_1}(0), \tag{73}
\]
where \( L_F \) is the Lipschitz constant of \( F \). Then, Eqn. (72) together with Eqn. (73) implies that
\[
F(0) \leq L_F ||x||_2 + F(x) \leq L_F ||x||_2 + \varepsilon ||x||_2^{2\beta} \tag{74}
\]
for all \( x \in B_{R_1}(0)/\{0\} \).

Now, for contradiction suppose the negation of \( F(0) \leq 0 \), that is there exists \( a > 0 \) such that \( F(0) \geq a \). Considering \( x = \min\{ \frac{a}{3L_F + 1}, \sqrt[n]{\frac{\lambda}{\varepsilon a}}, \frac{R_1}{\sqrt[n]{\beta}} \} \in B_{R_1}(0)/\{0\} \) and using Eqn. (74) we have that
\[
a < F(0) \leq \frac{2}{3} a,
\]
providing a contradiction. Therefore, \( F(0) \leq 0 \) and \( J \) satisfies Eqn. (64) for all \( x \in B_{R_1}(0)/\{0\} \).

We now show \( J \) satisfies Eqn. (65). Let \( x \notin \partial B_{R_0}(0) \). By Eqn. (68) we have that \( J_m(x) = 1 \) for all \( m \in \mathbb{N} \). Therefore, using the fact \( \sum_{m=1}^{\infty} \psi_m(x) = 1 \) for all \( x \in B_{R_1}(0)/\{0\} \) and \( \partial B_{R_0}(0) \subset B_{R_1}(0)/\{0\} \) since \( R_1 > R_0 \), we have that
\[
J(x) = \sum_{m=1}^{\infty} \psi_m(x)J_m(x) = \sum_{m=1}^{\infty} \psi_m(x) = 1.
\]

Moreover, \( 0 \notin B_{R_1}(0)/\{0\} \) so \( \psi_m(x) = 0 \) for all \( m \in \mathbb{N} \). Hence \( J(0) = \sum_{m=1}^{\infty} \psi_m(x)J_m(x) = 0 \).

\[\square\]

**Theorem 3.** Consider \( f \in C^2(\mathbb{R}^n, \mathbb{R}) \) and \( W \) as in Eqn. (11). Suppose there exists \( \theta, \eta, R > 0 \) such that \( ||D^2f(x)||_2 \leq \theta \) for all \( x \in B_R(x) \) and \( ||x||_1 \leq 2 \), \( B_R(0) \) is an exponentially stable set (Defn. 1) of the ODE (1), and \( ROA_f \subset B_R(0) \). If \( \lambda > \theta \eta^{-2\beta} \) and \( \beta > \frac{\epsilon}{\lambda R} + \frac{1}{4} \) then for any \( \epsilon > 0 \) there exists \( P \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \) such that
\[
\sup_{x \in B_{R_0}(0)} |P(x) - W_{\lambda, \beta}(x)| < \epsilon, \tag{75}
\]
\[
\sup_{x \in B_{R_1}(0)} |J(x) - W_{\lambda, \beta}(x)| < \frac{\epsilon}{a}, \tag{78}
\]
\[
\sup_{x \in B_{R_0}(0)} |\nabla P(x) + f(x) - W_{\lambda, \beta}^T(x)| < \frac{\epsilon}{a}, \tag{79}
\]
\[
\sup_{x \in B_{R_0}(0)} |\nabla J(x) + f(x) - W_{\lambda, \beta}^T(x)| < \frac{\epsilon}{a}. \tag{80}
\]

**Proof.** Let \( \epsilon > 0 \) and \( R_1 > R \). By Prop. 5 there exists \( J \in C^\infty(B_{R_1}(0), \mathbb{R}) \) that satisfies
\[
J(x) = 1 \quad \text{for all} \quad x \in \partial B_R(0) \quad \text{and} \quad J(x) = 0, \tag{81}
\]
where
\[
a := \max \left\{ 3, \frac{\sup_{x \in B_{R_1}(0)} ||f(x)||_2}{\lambda R} + R^{-2\beta} + \frac{1}{\lambda} + 2 \right\}. \tag{82}
\]

Now, Theorem 2, found in Section X, shows there exists \( \tilde{P} \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \) such that
\[
|J(x) - \tilde{P}(x)| < \frac{\epsilon}{aR^{2\beta}} ||x||_2^{2\beta} \quad \text{for all} \quad x \in (B_R(0))^{cl}, \tag{83}
\]
\[
|\nabla J(x) - \nabla \tilde{P}(x)||_2 < \frac{\epsilon}{aR^{2\beta}} ||x||_2^{2\beta} \quad \text{for all} \quad x \in (B_R(0))^{cl}. \tag{84}
\]

Let \( P(x) := \tilde{P}(x) + \frac{a - 2}{a} \epsilon \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \). We will now show \( P \) satisfies Eqns (75), (76) and (77).

We first show Eqn. (75) holds. Using the triangle inequality along with Eqns (78) and (82) we have that
\[
|P(x) - W_{\lambda, \beta}(x)| \leq |\tilde{P}(x) - W_{\lambda, \beta}(x)| + \frac{a - 2}{a} \epsilon
\]
\[
\leq |\tilde{P}(x) - J(x)| + |J(x) - W_{\lambda, \beta}(x)| + \frac{a - 2}{a} \epsilon
\]
\[
\leq \frac{\epsilon}{a} + \frac{\epsilon}{a} + \frac{a - 2}{a} \epsilon = \epsilon.
\]

We now show Eqn. (76) holds. Using Eqns (79), (81), (82),
and (83) we have that
\[
\nabla P(x)^T f(x) + \lambda (1 - P(x)) ||x||^2 - \frac{\varepsilon (a - 2)}{a} ||x||^2
\]
\[
= \nabla \tilde{P}(x)^T f(x) - \lambda (1 - J(x)) ||x||^2 + \frac{\varepsilon}{a} ||x||^2
\]
\[
= (\nabla \tilde{P}(x) - \nabla J(x))^T f(x) + \lambda (J(x) - \tilde{P}(x)) ||x||^2 + \frac{\varepsilon}{a} (1 - \lambda (a - 2)) ||x||^2
\]
\[
\leq \left( \frac{\sup_{x \in B_0} ||f(x)||}{R} + \lambda R^{-2\beta} + 1 - \lambda (a - 2) \right) \frac{\varepsilon}{a} ||x||^2 \leq 0.
\]

We now show Eqn. (77) holds. From Eqn. (82) we have that \( \tilde{P}(x) > J(x) - \frac{\varepsilon}{a \beta} ||x||^2 \) for all \( x \in (B_R(0))^c \). Moreover, Eqn. (80) we have that \( J(0) = 0 \). Therefore \( \tilde{P}(x) - \tilde{P}(0) + \frac{\varepsilon}{a} x - \frac{\varepsilon}{a \beta} R > 1 + \frac{\varepsilon}{a} R > 1 + \frac{\varepsilon}{a} \beta > 1 \). Also from Eqn. (82) we have that \( \tilde{P}(0) = J(0) \). From Eqn. (80) we have that \( \lambda J(0) = 0 \). Therefore \( \tilde{P}(0) = \tilde{P}(0) + \frac{\varepsilon}{a} \beta > 0 \). □

XI. APPENDIX B: SUBLEVEL SET APPROXIMATION

This appendix is concerned with the volume metric \( D_V \) in Eqn. (46). The sublevel approximation results presented in this appendix are required in the proof of Theorem 1.

**Definition 5.** \( D : X \times X \rightarrow \mathbb{R} \) is a metric if the following is satisfied for all \( x, y \in X \),

\[
\begin{align*}
D(x, y) & \geq 0, & D(x, y) = D(y, x), \\
D(x, y) & = 0 \text{ if } x = y, & D(x, z) \leq D(x, y) + D(y, z).
\end{align*}
\]

**Lemma 6 ([26]).** Consider the quotient space,
\[
X := \mathcal{B} \mod \{X \subset \mathbb{R}^n : X \neq \emptyset, \mu(X) = 0\},
\]
recalling \( \mathcal{B} := \{B \subset \mathbb{R}^n : \mu(B) < \infty\} \) is the set of all bounded sets. Then \( D_V : X \times X \rightarrow \mathbb{R} \), defined in Eqn. (46), is a metric.

**Lemma 7 ([26]).** If \( A, B \in \mathcal{B} \) and \( B \subseteq A \) then
\[
D_V(A, B) = \mu(A - B) = \mu(A) - \mu(B).
\]

**Proposition 6 ([27]).** Consider a set \( A \in \mathcal{B} \), a function \( V \in L^1(\Lambda, \mathbb{R}) \), and a family of functions \( \{J_d \in L^1(\Lambda, \mathbb{R}) : d \in \mathbb{N}\} \) that satisfies the following properties:

1) For any \( d \in \mathbb{N} \) we have \( J_d(x) \leq V(x) \) for all \( x \in \Lambda \).
2) \( \lim_{d \rightarrow \infty} ||V - J_d||_{L^1(\Lambda, \mathbb{R})} = 0 \).

Then for all \( \gamma \in \mathbb{R} \)
\[
\lim_{d \rightarrow \infty} \left( \{x \in \Lambda : V(x) \leq \gamma\}, \{x \in \Lambda : J_d(x) \leq \gamma\} \right) = 0. \quad (84)
\]

**Corollary 4.** Consider a set \( A \in \mathcal{B} \), a function \( V \in L^1(\Lambda, \mathbb{R}) \), and a family of functions \( \{J_d \in L^1(\Lambda, \mathbb{R}) : d \in \mathbb{N}\} \) that satisfies the following properties:

1) For any \( d \in \mathbb{N} \) we have \( J_d(x) \geq V(x) \) for all \( x \in \Lambda \).
2) \( \lim_{d \rightarrow \infty} ||V - J_d||_{L^1(\Lambda, \mathbb{R})} = 0 \).

Then for all \( \gamma \in \mathbb{R} \)
\[
\lim_{d \rightarrow \infty} \left( \{x \in \Lambda : V(x) \geq \gamma\}, \{x \in \Lambda : J_d(x) \geq \gamma\} \right) = 0. \quad (85)
\]

**Proof.** Let us denote \( \tilde{V}(x) = -V(x) \) and \( J_d(x) = -J_d(x) \). It follows that \( J_d(x) \leq \tilde{V}(x) \) for all \( x \in \Lambda \) and \( \lim_{d \rightarrow \infty} \tilde{V} - J_d \mid_{L^1(\Lambda, \mathbb{R})} = 0 \). Therefore, by Prop. 6 we have that
\[
\lim_{d \rightarrow \infty} \left( \{x \in \Lambda : \tilde{V}(x) \leq \gamma\}, \{x \in \Lambda : \tilde{J}_d(x) \leq \gamma\} \right) = 0. \quad (86)
\]

Now, \( \Lambda = \{x \in \Lambda : \tilde{V}(x) < \gamma\} \cup \{x \in \Lambda : \tilde{V}(x) \geq \gamma\} = \{x \in \Lambda : V(x) < \gamma\} \cup \{x \in \Lambda : \tilde{V}(x) \leq \gamma\} \). Therefore
\[
\{x \in \Lambda : V(x) < \gamma\} = \Lambda_1 \cup \{x \in \Lambda : \tilde{V}(x) \leq \gamma\},
\]
and by a similar argument
\[
\{x \in \Lambda : J_d(x) < \gamma\} = \Lambda_1 \cup \{x \in \Lambda : \tilde{J}_d(x) \leq \gamma\}.
\]
Thus, by Lem. 7 and since \( \{x \in \Lambda : \tilde{J}_d(x) \leq \gamma\} \subseteq \Lambda \), we have that
\[
D_V \left( \{x \in \Lambda : V(x) < \gamma\}, \{x \in \Lambda : J_d(x) < \gamma\} \right) \quad (87)
\]
\[
= D_V \left( \Lambda_1 \cup \{x \in \Lambda : \tilde{V}(x) < \gamma\}, \Lambda_1 \cup \{x \in \Lambda : \tilde{J}_d(x) < \gamma\} \right)
\]
\[
= D_V \left( \{x \in \Lambda : V(x) < \gamma\}, \{x \in \Lambda : J_d(x) \leq \gamma\} \right).
\]

Now by Eqns (86) and (87) it follows that Eqn. (85) holds. □

XII. APPENDIX C

In this appendix we present several miscellaneous results required in various places throughout the paper and not previously found in any of the other appendices.

**Lemma 8 (Exponential inequalities).** The following inequalities hold
\[
\exp(-x) \leq 1 \text{ for all } x \geq 0 \quad (88)
\]
\[
-x \exp(-x) \leq 1 \text{ for all } x \in \mathbb{R} \quad (89)
\]
\[
\exp(x) \geq 1 + x \text{ for all } x \in \mathbb{R} \quad (90)
\]

**Lemma 9 (Gronwall’s Inequality [20]).** Consider scalars \( a, b \in \mathbb{R} \) and functions \( u, \beta \in C^1(I, \mathbb{R}) \). Suppose
\[
\frac{d}{dt} u(t) \leq \beta(t) u(t) \text{ for all } t \in (a, b).
\]
Then it follows that
\[
u(t) \leq u(a) \exp \left( \int_a^t \beta(s) ds \right) \text{ for all } t \in [a, b].
\]

**Theorem 4 (Rademacher’s Theorem [28] [25]).** If \( \Omega \subset \mathbb{R}^n \) is an open subset and \( V \in Lip(\Omega, \mathbb{R}) \), then \( V \) is differentiable almost everywhere in \( \Omega \) with point-wise derivative corresponding to the weak derivative almost everywhere; that is the set of points in \( \Omega \) where \( V \) is not differentiable has Lebesgue measure zero. Moreover,
\[
\text{ess sup} \left. \left| \frac{\partial}{\partial x_i} V(x) \right| \right| \leq L_V \text{ for all } 1 \leq i \leq n,
\]

15
where $L_V > 0$ is the Lipschitz constant of $V$ and $\frac{\partial}{\partial x_i} V(x)$ is the weak derivative of $V$.

**Theorem 5** (Putinar’s Positivstellensatz [29]). Consider the semialgebraic set $X = \{ x \in \mathbb{R}^n : g_i(x) \geq 0 \text{ for } i = 1, \ldots, k \}$. Further suppose $\{ x \in \mathbb{R}^n : g_i(x) \geq 0 \}$ is compact for some $i \in \{1, \ldots, k\}$. If the polynomial $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x) > 0$ for all $x \in X$, then there exists SOS polynomials $\{ s_i \}_{i \in \{1, \ldots, m\}} \subset \mathcal{SOS}$ such that,

$$f - \sum_{i=1}^{m} s_ig_i \in \mathcal{SOS}.$$

**Theorem 6** (The Bolzano Weierstrass Theorem). Consider a sequence $\{ x_n \}_{n \in \mathbb{N}} \subset \mathbb{R}^n$. Then the $\{ x_n \}_{n \in \mathbb{N}}$ is a bounded sequence, that is there exists $M > 0$ such that $x_n < M$ for all $n \in \mathbb{N}$, if and only if there exists a convergent subsequence $\{ y_n \}_{n \in \mathbb{N}} \subset \{ x_n \}_{n \in \mathbb{N}}$.

**Definition 6.** Let $\Omega \subset \mathbb{R}^n$. We say $\{ U_i \}_{i=1}^\infty$ is an open cover for $\Omega$ if $U_i \subset \mathbb{R}^n$ is an open set for each $i \in \mathbb{N}$ and $\Omega \subseteq \bigcup_{i=1}^\infty U_i$.

**Theorem 7** (Existence of Partitions of Unity [30]). Let $U \subseteq \mathbb{R}^n$ and let $\{ U_i \}_{i=1}^\infty$ be an open cover of $E$. Then there exists a collection of $C^\infty(\mathbb{R}^n, \mathbb{R})$ functions, denoted by $\{ \psi_i \}_{i=1}^\infty$, with the following properties:

1) For all $x \in U$ and $i \in \mathbb{N}$ we have $0 \leq \psi_i(x) \leq 1$.
2) For all $x \in U$ there exists an open set $S \subseteq U$ containing $x$ such that all but finitely many $\psi_i$ are 0 on $S$.
3) For each $x \in U$ we have $\sum_{i=1}^\infty \psi_i(x) = 1$.
4) For each $i \in \mathbb{N}$ we have $\{ x \in U : \psi_i(x) \neq 0 \} \subseteq U_i$. 
