Directional dimensions of ergodic currents on \( \mathbb{CP}(2) \)

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Abstract

Let \( f \) be a holomorphic endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). We estimate the local directional dimensions of closed positive currents \( S \) with respect to ergodic dilating measures \( \nu \). We infer several applications. The first one shows that the currents \( S \) containing a measure of entropy \( h_\nu > \log d \) have a directional dimension \( > 2 \), which answers a question by de Thélin-Vigny. The second application asserts that the Dujardin’s semi-extremal endomorphisms are close to suspensions of one-dimensional Lattès maps. Finally, we obtain an upper bound for the dimension of the equilibrium measure, towards the formula conjectured by Binder-DeMarco.

Key words: dimension theory, positive closed currents, invariant measures, Lyapunov exponents, normal forms.

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1 Introduction

This article concerns the ergodic properties of holomorphic endomorphisms of \( \mathbb{P}^2 \), see [16]. Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). The Green current \( T \) is defined as \( T := \lim_{n \rightarrow \infty} f^{\star n} \omega \), where \( \omega \) is the Fubini-Study form of \( \mathbb{P}^2 \). The equilibrium measure is defined as \( \mu := T \wedge T \), this is an ergodic measure of entropy \( \log d^2 \), its Lyapunov exponents are \( \geq \frac{1}{2} \log d \), see [6, 16].

We say that an ergodic measure \( \nu \) is dilating if its Lyapunov exponents are positive. The ergodic measures of entropy \( h_\nu > \log d \) are dilating: their exponents are larger than or equal to \( \frac{1}{2} (h_\nu - \log d) \), see [11, 20]. It is known that the support of every ergodic measure \( \nu \) of entropy \( h_\nu > \log d \) is contained in the support of \( \mu \), see [10, 14]. The article [20] constructs such measures by using coding techniques.

1.1 Directional dimensions

Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). The algebraic subsets of \( \mathbb{P}^2 \) do not contain any ergodic measure \( \nu \) of entropy \( h_\nu > \log d \): this comes from the Gromov’s iterated graph argument and from the relative variational principle, see [17] and [16 Section 1.7]. In this Section, we quantify that property thanks to the Lyapunov exponents.
of the measures $\nu$. We shall work in the more general setting of $(1,1)$ closed positive currents $S$. Those currents are as big as the algebraic subsets, since we have:

$$\forall x \in \mathbb{P}^2, \quad d_S(x) := \liminf_{r \to 0} \frac{\log S \wedge \omega(B_x(r))}{\log r} \geq 2,$$

which comes from $S \wedge \omega(B_x(r)) \leq c(x)r^2$, see [13, Chapitre 3]. We shall use the notation $d_S(x)$ for the lim sup. A drawback of the trace measure $S \wedge \omega(B_x(r))$ is that it does not distinguish any specific direction. If $Z$ is a holomorphic coordinate in the neighbourhood of $x \in \mathbb{P}^2$, one defines the lower local directional dimension of $S$ with respect to $Z$ by

$$d_{S,Z}(x) := \liminf_{r \to 0} \frac{\log \left[ S \wedge (\frac{1}{2} dZ \wedge d\overline{Z})(B_x(r)) \right]}{\log r},$$

we shall denote the lim sup by $d_{S,Z}(x)$. Geometrically, the positive measure $S \wedge (\frac{1}{2} dZ \wedge d\overline{Z})$ is the average with respect to Lebesgue measure of the slices of the current $S$ transversaly to the direction $Z$, see Proposition 9.4. If $(Z,W)$ are holomorphic coordinates near $x$, the directional dimensions of $S$ are related to the dimension of $S$ by

$$d_S(x) = \min \left\{ d_{S,Z}(x), d_{S,W}(x) \right\}, \quad \overline{d_S}(x) = \min \left\{ \overline{d_{S,Z}}(x), \overline{d_{S,W}}(x) \right\},$$

see Proposition 9.2. We start with showing lower and upper bounds for the directional dimensions of the Green current $T$ (Theorems 1.1 and 1.2). Next we display our result concerning the general closed positive currents $S$.

**Directional dimensions of the Green current**

The invariance property of the current $T$ allows to obtain estimates on the directional dimensions $\nu$-almost everywhere. In what follows, the functions $O(\epsilon)$ only depend on $\epsilon$, the degree $d$ of the endomorphism, the exponents and the entropy of $\nu$. They tend to 0 when $\epsilon$ tends to 0 and are positive for $\epsilon$ small enough. We shall say that the exponents of $\nu$ do not resonate if $\lambda_1 \neq k\lambda_2$ for every $k \geq 2$.

**Theorem 1.1.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $\nu$ be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. Then there exist functions $O_1(\epsilon), O_2(\epsilon)$ satisfying the following properties. For every $\epsilon > 0$ and for $\nu$-almost every $x$, there exist holomorphic coordinates $(Z,W)$ in a neighbourhood of $x$ such that

$$\overline{d_{T,Z}}(x) \geq 2 + \frac{h_\nu - \log d}{\lambda_2} - O_1(\epsilon),$$

$$\overline{d_{T,W}}(x) \geq 2 \frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2} - O_2(\epsilon).$$

This result applies to the measure $\mu$ and allows to improve a classical lower bound concerning the dimension of the Green current $T$. This current has local $\gamma$-Hölder $psh$ potentials for every $\gamma < \gamma_0 := \min\{1, \log d/\log d_\infty\}$, where $d_\infty := \lim_n ||Df^n||_\infty^{1/n}$, see
[16] Proposition 1.18. This implies that \( T \land \omega(B_x(r)) \leq c_\gamma(x)r^{2+\gamma} \) for every \( x \in \mathbb{P}^2 \) and every \( \gamma < \gamma_0 \), see [21 Théorème 1.7.3]. We deduce that for every \( x \in \mathbb{P}^2 \) and for every local holomorphic coordinates \((Z,W)\) in a neighbourhood of \( x \):

\[
\min\{d_{T,Z}(x), d_{T,W}(x)\} = d_T(x) \geq 2 + \gamma_0. \tag{1.2}
\]

Now, since \( h_\mu = \log d^2 \) and \( \lambda_1 \leq \log d_\infty \), we get:

\[
\frac{h_\mu - \log d}{\lambda_2} = \frac{\log d}{\lambda_2} > \frac{\log d}{\lambda_1} \geq \frac{\log d_\infty}{\log d_\infty} \geq \gamma_0.
\]

The lower bound in the direction \( Z \) provided by Theorem [1.1] is therefore better than (1.2) when \( \epsilon \) is small enough.

Now we show directional upper bounds for the current \( T \) with respect to every dilating measure \( \nu \) whose support is contained in the support of \( \mu \).

**Theorem 1.2.** Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). Let \( \nu \) be an ergodic dilating measure whose exponents \( \lambda_1 > \lambda_2 \) do not resonate. We assume that the support of \( \nu \) is contained in the support of \( \mu \). Then there exist functions \( O_3(\epsilon), O_4(\epsilon) \) satisfying the following properties. For every \( \epsilon > 0 \) and for \( \nu \)-almost every \( x \), there exist holomorphic coordinates \((Z,W)\) in a neighbourhood of \( x \) such that

\[
d_{T,Z}(x) \leq \frac{\log d}{\lambda_2} + 2 + O_3(\epsilon), \\
d_{T,W}(x) \leq \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon).
\]

By combining these two theorems, we manage to separate coordinates \((Z,W)\) in terms of local dimensions.

**Corollary 1.3.** Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). Let \( \nu \) be an ergodic dilating measure whose exponents \( \lambda_1 > \lambda_2 \) do not resonate. For every \( \epsilon > 0 \) and for \( \nu \)-almost every \( x \), there exist holomorphic coordinates \((Z,W)\) in a neighbourhood of \( x \) such that

\[
d_{T,W}(x) \leq \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon), \\
\frac{\log d}{\lambda_2} + 2 - O_1(\epsilon) \leq d_{T,Z}(x).
\]

In the three preceding results, the coordinates \((Z,W)\) come from a normal form Theorem for the inverse branches of \( f^n \), see Section 2. They depend on \( \hat{x} \) in the natural extension \((\hat{f}, \hat{\nu})\), we shall denote them by \((Z_{\hat{x}}^0, W_{\hat{x}}^0)\). The coordinates \((Z,W)\) at a point \( x \) are coordinates of type \((Z_{\hat{x}}^\epsilon, W_{\hat{x}}^\epsilon)\) where \( \pi_0(\hat{x}) = x \). The coordinate \( W_{\hat{x}}^\epsilon \) is always invariant by the shift \( \hat{f} \), \( Z_{\hat{x}}^\epsilon \) is invariant when the exponents do not resonate. Since the current \( T \) is \( f \)-invariant, the functions \( d_{T,Z}(\hat{x}) \) and \( d_{T,W}(\hat{x}) \) are \( \hat{f} \)-invariant, hence \( \hat{\nu} \)-almost everywhere constant, see Proposition 2.4. We shall denote them by \( d_{T,Z}(\nu) \) and \( d_{T,W}(\nu) \). We have the same properties for the upper dimensions.
Dimension of ergodic currents $S$

**Theorem 1.4.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $S$ be a $(1,1)$-closed positive current on $\mathbb{P}^2$. We assume that the support of $S$ contains an ergodic measure $\nu$ of entropy $h_\nu > \log d$ whose exponents satisfy $\lambda_1 > \lambda_2$ and do not resonate. Then there exists a function $O_5(\epsilon)$ satisfying the following properties. For every $\epsilon > 0$, there exist $x \in \text{Supp}\nu$ and a holomorphic coordinate $Z$ in the neighbourhood of $x$ such that:

$$d_{S,Z}(x) \geq 2 + \frac{h_\nu - \log d}{\lambda_2} - O_5(\epsilon).$$

In particular, $S$ has a local directional dimension $> 2$ at some $x \in \text{Supp}\nu$.

This result is localized at a point $x$ because $S$ is not assumed to be $f$-invariant. For every closed positive current $S$ containing an ergodic dilating measure $\nu$ with exponents $\lambda_1 \geq \lambda_2$, de Thélin-Vigny proved in [12] that there exists $x \in \text{supp}(\nu)$ such that

$$d_S(x) \geq 2\frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2}.$$

(1.3)

Theorem 1.4 improves this estimate by replacing $\lambda_2/\lambda_1$ by 1 for a coordinate $Z$. When $\lambda_1 = \lambda_2$, (1.1) shows that this substitution is valid for every coordinate $Z$. Our lower bound answers a question of [12] in a directional way. In the framework of invertible and meromorphic mappings, the preprint [9] gives a lower bound $> 2$ for the dimensions of the Green currents $T^\pm$ by using coding techniques and laminar properties of $T^\pm$.

1.2 Dimension of dilating measures

Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let $\nu$ be an ergodic measure. We refer to [23] and [26] for the beginning of this Section. The dimension of $\nu$ is defined by

$$\dim_H(\nu) := \inf \left\{ \dim_H(A) \mid A \text{ Borel set of } \mathbb{P}^2, \nu(A) = 1 \right\}.$$ 

The lower and upper local dimensions of $\nu$ are

$$d_- \nu := \liminf_{r \to 0} \frac{\log(\nu(B_x(r)))}{\log r}, \quad d_+ \nu := \limsup_{r \to 0} \frac{\log(\nu(B_x(r)))}{\log r},$$

these limits are $\nu$-almost everywhere constant, by ergodicity of $\nu$. If $a \leq d_- \nu \leq d_+ \nu \leq b$, then $a \leq \dim_H(\nu) \leq b$. If $\nu$ is dilating, we have the classical inequalities $\frac{\lambda_1}{\lambda_2} \leq d_- \nu \leq d_+ \nu \leq \frac{\lambda_2}{\lambda_1}$ where $\lambda_1 \geq \lambda_2$ are the exponents of $\nu$. For the equilibrium measure $\mu$, Binder-DeMarco [4] conjectured the formula:

$$\dim_H(\mu) = \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2}$$

(1.4)

which generalizes the one-dimensional Mañé’s formula [22]. The article [4] proves the following upper bound for polynomial mappings

$$\dim_H(\mu) \leq 4 - \frac{2(\lambda_1 + \lambda_2) - \log d^2}{\lambda_1},$$

(1.5)
the article [15] extends this upper bound in a meromorphic context. For every dilating measure \( \nu \), we have at our disposal the following lower bound proved in [19]:

\[
\dim H(\nu) \geq d_\nu \geq \frac{\log d}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_2}.
\]

(1.6)

We obtain a new upper bound which comes closer to Binder-DeMarco’s conjecture (1.4).

**Theorem 1.5.** Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). Let \( \nu \) be an ergodic dilating measure, of exponents \( \lambda_1 > \lambda_2 \) and whose support is contained in the support of \( \mu \). Then

\[
d_\nu \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2 \left(1 - \frac{\lambda_2}{\lambda_1}\right).
\]

Moreover, if the exponents do not resonate, then:

\[
d_\nu \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2 \min \left(1 - \frac{\lambda_2}{\lambda_1}, \frac{\lambda_1}{\lambda_2} - 1\right).
\]

The proof extends the arguments of Theorem 1.2.

### 1.3 Dimension of semi-extremal endomorphisms

We say that \( f \) is *extremal* if the exponents \( \lambda_1 \geq \lambda_2 \) of its equilibrium measure \( \mu \) satisfy \( \lambda_1 = \lambda_2 = \frac{1}{2} \log d \). The articles [1, 3, 15] characterize these endomorphisms by the equivalent properties:

1. \( \dim_H(\mu) = 4 \).
2. \( \mu << \text{Leb}_{\mathbb{P}^2} := \omega \wedge \omega \).
3. \( T \) is a \((1, 1)\) positive smooth form on an open set of \( \mathbb{P}^2 \).
4. \( f \) is a Lattès map: there exist a complex torus \( \mathbb{C}^2/\Lambda \), an affine dilation \( D \) on this torus and a finite galoisian covering \( \sigma : \mathbb{C}^2/\Lambda \to \mathbb{P}^2 \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{C}^2/\Lambda & \xrightarrow{D} & \mathbb{C}^2/\Lambda \\
\sigma \downarrow & & \sigma \downarrow \\
\mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2
\end{array}
\]

There exist such applications for every degree \( d \geq 2 \), see for instance [18].

We say that \( f \) is *semi-extremal* if the exponents \( \lambda_1 \geq \lambda_2 \) of its equilibrium measure \( \mu \) satisfy \( \lambda_1 > \lambda_2 = \frac{1}{2} \log d \). Using (1.5) and (1.6) one sees that the Binder-DeMarco’s conjecture (1.4) holds for these mappings:

\[
\dim_H(\mu) = 2 + \frac{\log d}{\lambda_1}.
\]

(1.7)

Classical examples of semi-extremal endomorphisms are suspensions of one-dimensional Lattès maps, they satisfy \( \mu << T \wedge \omega \). More generally,
Theorem 1.6 (Dujardin [17]). If $\mu \ll T \land \omega$, then $f$ is semi-extremal.

In [17] Dujardin asked if $\mu \ll T \land \omega$ implies the existence of a one-dimensional Lattès factor for $f$. Theorem 1.7 below provides one step in that direction. It shows that, from a theoretical dimensional point of view, these endomorphisms look like suspensions of one-dimensional Lattès maps: the dimension of $T$ is maximal equal to 4 in a coordinate $Z$, and equal to $2 + \log d/\lambda_1$ (the dimension of $\mu$, see (1.7)) in a coordinate $W$. The functions $O_1(\epsilon), O_2(\epsilon)$ come from Theorem 1.1.

**Theorem 1.7.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. We assume that $\mu \ll T \land \omega$ and that $d\mu = d\mu$. We also assume that the exponents $\lambda_1 > \lambda_2 = \frac{1}{2} \log d$ of $\mu$ do not resonate. For every $\epsilon > 0$ and for $\mu$-almost every $x \in \mathbb{P}^2$, there exist holomorphic coordinates $(Z,W)$ in a neighbourhood of $x$ such that

$$
4 - O_1(\epsilon) \leq d_{T,Z}(x) \quad \text{and} \quad 2 + \log d/\lambda_1 - O_2(\epsilon) \leq d_{T,W}(x) \leq 2 + \log d/\lambda_1.
$$

**1.4 Organization of the article**

Sections 2, 3, and 4 are devoted to normal forms, the geometry of inverse branches and separated sets. Sections 5 and 6 establish Theorems 1.1, 1.4 and 1.7, the proofs rest on Theorem 5.2 which relies on the lower local dimension $d_\nu$ (the lower bound (1.6) is therefore crucial to deduce these results). Theorems 1.2 and 1.5 are proved in Sections 7 and 8. Section 9 brings together technical results.

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**2 Normal forms and Oseledec-Poincaré coordinates**

**2.1 Natural extension and normal forms**

Let $\mathcal{C}_f$ be the critical set of $f$, this is an algebraic subset of $\mathbb{P}^2$. If $\nu$ is an ergodic dilating measure, then $x \mapsto \log |\text{Jac}_f(x)| \in L^1(\nu)$, which implies $\nu(\mathcal{C}_f) = 0$. Let $X$ be the $f$-invariant Borel set $\mathbb{P}^2 \setminus \cup_{n \in \mathbb{Z}} f^n(\mathcal{C}_f)$ and let

$$
\hat{X} := \left\{ \hat{x} = (x_n)_{n \in \mathbb{Z}} \in X^\mathbb{Z}, \ x_{n+1} = f(x_n) \right\}.
$$

Let $\hat{f}$ be the left shift on $\hat{X}$ and $\pi_0(\hat{x}) := x_0$. There exists a unique $\hat{f}$-invariant measure $\hat{\nu}$ on $\hat{X}$ such that $(\pi_0)_* \hat{\nu} = \nu$. We set $\hat{x}_n := \hat{f}^n(\hat{x})$ for $n \in \mathbb{Z}$. A function $\alpha : \hat{X} \to [0, +\infty]$ is $\epsilon$-tempered if $\alpha(\hat{f}^{\pm 1}(\hat{x})) \geq e^{-\epsilon} \alpha(\hat{x})$. For every $\hat{x} \in X$ we denote by $f^{\hat{x}}_{-n}$ the inverse branch of $f^n$ defined in a neighbourhood of $x_0$ with values in a neighbourhood of $x_{-n}$. The articles [2] and [21] provide normal forms for these mappings.
Theorem 2.1. ([3], Proposition 4.3]) Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $v$ be an ergodic dilating measure with exponents $\lambda_1 > \lambda_2$. Let $\epsilon > 0$.

There exists an $\hat{f}$-invariant Borel set $\hat{F} \subset \hat{X}$ such that $\hat{v}(\hat{F}) = 1$ and satisfying the following properties. There exist $\epsilon$-tempered functions $\eta, \rho : \hat{F} \to ]0,1]$ and $\beta, L, M : \hat{F} \to ]1, +\infty[$ and for every $\hat{x} \in \hat{F}$, there exists an injective holomorphic mapping

$$\xi^\epsilon : B_{x_0}(\eta(\hat{x})) \to \mathbb{D}^2(\rho(\hat{x}))$$

such that the following diagram commutes for every $n \geq n_\epsilon(\hat{x})$:

$$
\begin{array}{ccc}
B_{x_{-n}}(\eta(\hat{x}_{-n})) & \xrightarrow{f_{\hat{x}}^{-n}} & B_{x_0}(\eta(\hat{x})) \\
\downarrow \xi^\epsilon_{-n} & & \downarrow \xi^\epsilon \\
\mathbb{D}^2(\rho(\hat{x}_{-n})) & \xrightarrow{R_{n,\hat{x}}} & \mathbb{D}^2(\rho(\hat{x}))
\end{array}
$$

and such that

1. $\forall (p, q) \in B_{x_0}(\eta(\hat{x}))$, $\frac{1}{\eta}(p, q) \leq |\xi^\epsilon_p(p) - \xi^\epsilon_q(q)| \leq \beta(\hat{x})d(p, q)$.
2. $\text{Lip}(f_{\hat{x}}^{-n}) \leq L(\hat{x})e^{-n\lambda_2 + \epsilon n}$ on $B_{x_0}(\eta(\hat{x}))$.
3. if $\lambda_1 \notin \{k\lambda_2, k \geq 2\}$, $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}, \beta_{n,\hat{x}})$,
   
   if $\lambda_1 = k\lambda_2$ where $k \geq 2$, $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}, \beta_{n,\hat{x}}) + (\gamma_{n,\hat{x}}w^k, 0)$, with
   
   (a) $e^{-n\lambda_1 + \epsilon n} \leq |\alpha_{n,\hat{x}}| \leq e^{-n\lambda_1 + \epsilon n}$ and $|\beta_{n,\hat{x}}| \leq M(\hat{x})e^{-n\lambda_1 + \epsilon n}$,
   
   (b) $e^{-n\lambda_2 + \epsilon n} \leq |\beta_{n,\hat{x}}| \leq e^{-n\lambda_2 + \epsilon n}$.

Remark 2.2. The diagram commutes for every $n \in \{1, \ldots, n_\epsilon(\hat{x})\}$ for the germs of the mappings, see [3]. The integer $n_\epsilon(\hat{x})$ is the smallest integer such that $L(\hat{x})e^{-n\lambda_2 + \epsilon n} \leq e^{-\epsilon n}$, so that $L(\hat{x})e^{-n\lambda_2 + \epsilon n} \eta_k(\hat{x}) \leq e^{-\epsilon n} \eta_k(\hat{x}) \leq \eta_k(\hat{x}_{-n})$. Item 2 thus ensures that $f_{\hat{x}}^{-n}(B_{x_0}(\eta(\hat{x}_{-n}))) \subset B_{x_{-n}}(\eta(\hat{x}_{-n}))$.

We shall need the following Lemma. Let $n_1(L)$ be the smallest integer $n$ satisfying $L/4 \leq e^{\epsilon n}$. The first item uses the upper bound for $\text{Lip}(f_{\hat{x}}^{-n})$ provided by Theorem 2.1. The second item comes from [15] Proposition 3.1].

Lemma 2.3. Let $\hat{x} \in \hat{F}$ such that $\eta_k(\hat{x}) \geq \eta$ and $L(\hat{x}) \leq L$. If $n \geq n_1(L)$ and $r \leq \eta$,

1. $f_{\hat{x}}^{-n}(B_{x_0}(r/4)) \subset B_{x_0}(r\epsilon^{-n\lambda_2 + 3\epsilon n})$ and $f_{\hat{x}}^{-n}(B_{x_0}(r/4)) \subset B_{x_{-n}}(r\epsilon^{-n\lambda_2 + 3\epsilon n})$.
2. $B_{x_0}(r\epsilon^{-n\lambda_1 - 4\epsilon n}) \subset f_{\hat{x}}^{-n}(B_{x_0}(\frac{\xi^\epsilon}{4}e^{-2\epsilon n}))$. 

2.2 Oseledec-Poincaré coordinates

Let \( \nu \) be an ergodic dilating measure of exponents \( \lambda_1 > \lambda_2 \). We assume that the exponents do not resonate, which means that \( \lambda_1 \notin \{k\lambda_2, k \geq 2\} \). Let \( \epsilon > 0 \), and let us apply Theorem 2.1. For every \( \hat{x} \in \hat{F} \) we denote by \((Z'_{\hat{x}}, W'_{\hat{x}})\) the coordinates of \( \xi'_{\hat{x}} \).

The commutative diagram given by Theorem 2.1 implies:

\[
Z'_{\hat{x}} \circ f^{-n}_{\hat{x}} = \alpha_{n, \hat{x}} \times Z'_{\hat{x}}, \quad W'_{\hat{x}} \circ f^{-n}_{\hat{x}} = \beta_{n, \hat{x}} \times W'_{\hat{x}}.
\]

Hence, \( f^{-n}_{\hat{x}} \) multiplies the first coordinate by \( e^{-n\lambda_1 \pm \epsilon} \) and multiplies the second coordinate by \( e^{-n\lambda_2 \pm \epsilon} \). Let us note that the second property holds in the resonant case \( \lambda_1 \in \{k\lambda_2, k \geq 2\} \). We shall name the collection of local holomorphic coordinates

\[
(Z, W) := (Z'_{\hat{x}}, W'_{\hat{x}})_{\hat{x} \in \hat{F}}
\]

Oseledec-Poincaré coordinates for \((f, \nu)\). Using (2.1) and the fact that the Green current is \( \hat{f} \)-invariant, we obtain the following Proposition.

**Proposition 2.4.** Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \) and let \( T \) be its Green current. Let \( \nu \) be an ergodic dilating measure of exponents \( \lambda_1 > \lambda_2 \). Let \((Z, W)\) be Oseledec-Poincaré coordinates for \((f, \nu)\). Then there exists a \( \hat{f} \)-invariant Borel set \( \hat{\Lambda}_T \subset \hat{F} \) of \( \hat{\nu} \)-measure 1 such that

1. \( \hat{x} \mapsto \overline{d_{T,W}(\hat{x})} \) and \( \hat{x} \mapsto \overline{d_{T}(\hat{x})} \) are \( \hat{f} \)-invariant on \( \hat{\Lambda}_T \).
2. \( \hat{x} \mapsto \overline{d_{T,Z}(\hat{x})} \) and \( \hat{x} \mapsto \overline{d_{T,Z}(\hat{x})} \) are \( \hat{f} \)-invariant on \( \hat{\Lambda}_T \) if \( \lambda_1 \notin \{k\lambda_2, k \geq 2\} \).

In particular, if the exponents do not resonate, these functions are constant \( \hat{\nu} \)-almost everywhere. We shall denote them by

\[
\overline{d_{T,Z}(\nu)}, \overline{d_{T,Z}(\nu)}, \overline{d_{T,W}(\nu)}, \overline{d_{T,W}(\nu)}.
\]

**Proof.** We prove the invariance of \( \overline{d_{T,W}(\hat{x})} \), the same arguments hold for the other functions. For every \( z \in \mathbb{P}^2 \setminus \mathcal{C}_f \) we denote

\[
a(z) := \frac{1}{2}||(D_z f)^{-1}||^{-1}, \quad \gamma(z) := \min\{a(z) ||f||^{-1}_{L^2(\mathbb{P}^2)}, 1\}.
\]

Then [6] Lemme 2 asserts that \( f \) is injective on \( B_z(\gamma(z)) \) and

\[
\forall r \in [0, \gamma(z)], \quad B_f(z)(a(z)r) \subset f(B_z(r)).
\]

Let \( \hat{x} \in \hat{F} \). Since \( x_n \notin \mathcal{C}_f \) for every \( n \in \mathbb{Z} \), we obtain for every \( r \leq \gamma(x_0) \):

\[
T \wedge \left( \frac{i}{2} dW_{f(\hat{x})} \wedge d\overline{W_{f(\hat{x})}} \right) \left[B_f(x_0)(a(x_0)r)\right] \leq T \wedge \left( \frac{i}{2} dW_{f(\hat{x})} \wedge d\overline{W_{f(\hat{x})}} \right) \left[f(B_{x_0}(r))\right].
\]

(2.2)
Since \( f \) is injective on \( B_{x_0}(r) \), we can change the variables to get:

\[
T \wedge \left( \frac{i}{2} dW_{f(x)}^z \wedge dW_{f(x)}^z \right) [f(B_{x_0}(r))] = \int_{B_{x_0}(r)} f^* T \wedge f^* \left( \frac{i}{2} dW_{f(x)}^z \wedge dW_{f(x)}^z \right). \quad (2.3)
\]

Now let us recall that

\[
f^* T = dT \quad \text{and} \quad f^* \left( \frac{i}{2} dW_{f(x)}^z \wedge dW_{f(x)}^z \right) = |c(x)| \frac{d}{dW_{x}^z} \wedge dW_{x}^z,
\]

where the second equality comes from (2.1) by setting \( c(x) = c(\hat{x})^{-1} := \beta_{1,f(x)} \), it is valid near \( x_0 \) according to Remark 2.2. By combining (2.2), (2.3) and (2.4) we deduce:

\[
T \wedge \left( \frac{i}{2} dW_{f(x)}^z \wedge dW_{f(x)}^z \right) \left[ B_{f(x_0)} (a(x_0)) r \right] \leq |c(x)|^2 T \wedge \left( \frac{i}{2} dW_{x}^z \wedge dW_{x}^z \right) [B_{x_0}(r)]
\]

for every \( r \) small enough. Taking the logarithm and dividing by \( \log(a(x_0)r) < 0 \), we get \( \bar{d}_{T,W}(\hat{x}) \geq \bar{d}_{T,W}(\hat{x}) \) by taking limits. Since \( \nu \) is ergodic, the function \( \bar{d}_{T,W}(\hat{x}) \) is constant on a Borel set \( \Lambda_T \) of \( \nu \)-measure 1 (see [25, Chapter 1.5]). One can replace it by \( \bigcap_{n \in \mathbb{Z}} \hat{f}^n(\Lambda_T) \) to obtain an invariant set. \(\square\)

**Proposition 2.5.** Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) and let \( T \) be its Green current. Let \( \nu \) be an ergodic measure. Then the functions \( x \mapsto d_T(x) \) and \( x \mapsto \bar{d}_T(x) \) are invariant, hence \( \nu \)-almost everywhere constant. We denote them by \( d_T(\nu) \) and \( \bar{d}_T(\nu) \).

**Proof.** The arguments follow the proof of Proposition 2.4. In this case we study the measure \( T \wedge \omega \), and we replace the second equality in (2.1) by \( f^* \omega \leq \rho(x_0) \omega \) on \( B_{x_0}(\gamma(x_0)) \), where \( \rho(x_0) > 0 \) is a large enough positive constant. \(\square\)

### 3 Geometry of the inverse branches and uniformizations

Let \( \nu \) be an ergodic dilating measure of exponents \( \lambda_1 > \lambda_2 \). Let \( \epsilon > 0 \) and let \( (Z,W)_\epsilon \) be Oseledec-Poincaré coordinates for \( (f,\nu) \). Our aim is to construct, for every \( \delta > 0 \), a Borel set \( \Lambda_\delta \subset X \) satisfying \( \nu(\Lambda_\delta) \geq 1 - \delta/2 \) which provides convenient uniformizations.

#### 3.1 Dynamical balls

The dynamical distance is defined by \( d_n(x,y) := \max_{0 \leq k \leq n} d(f^k(x), f^k(y)) \). We denote by \( B_n(x,r) \) the ball centered at \( x \) and of radius \( r \) for the distance \( d_n \).

**Lemma 3.1.** There exist \( r_0 > 0 \), \( n_2 \geq 1 \) and \( C \subset \mathbb{P}^2 \) such that \( \nu(C) \geq 1 - \delta/8 \) and satisfying the following properties: for every \( x \in C \) and every \( n \geq n_2 \):

\[
\nu(B_n(x,r_0/8)) \geq e^{-n\nu-\epsilon n}.
\]

\[
\forall r \leq r_0, \quad \nu(B_n(x,5r)) \leq \nu(B_n(x,5r_0)) \leq e^{-n\nu+\epsilon n}.
\]
Proof. Brin-Katok Theorem [8] ensures that there exists $C_1 \subset \mathbb{R}^2$ of full $\nu$-measure such that for every $x \in C_1$:

$$\lim_{r \to 0} \left( \liminf_{n \to +\infty} -\frac{1}{n} \log \nu(B_n(x, r)) \right) = \lim_{r \to 0} \left( \limsup_{n \to +\infty} -\frac{1}{n} \log \nu(B_n(x, r)) \right) = h_\nu.$$ 

Hence, for every $x \in C_1$ there exists $r_0(x)$ such that $r \leq r_0(x)$ implies

$$\liminf_{n \to +\infty} -\frac{1}{n} \log(\nu(B_n(x, 5r))) \geq h_\nu - \epsilon/2$$

and

$$\limsup_{n \to +\infty} -\frac{1}{n} \log(\nu(B_n(x, r/8))) \leq h_\nu + \epsilon/2.$$ 

Let $r_0$ such that $C_2 := \{ x \in C_1 , r_0(x) \geq r_0 \}$ satisfies $\nu(C_2) \geq 1 - \delta/16$. For every $x \in C_2$, there exists $n_2(x)$ such that $n \geq n_2(x)$ implies

$$\nu(B_n(x, r_0/8)) \geq e^{-n h_\nu - \epsilon \nu}.$$ 

Let $n_2 \geq 1$ such that $C := C_2 \cap \{ x \in C_1 , n_2(x) \leq n_2 \}$ satisfies $\nu(C) \geq 1 - \delta/8.$

For every $L > 0$, let $m_L$ be the smallest integer $m$ such that $Le^{-m(\lambda + \epsilon)} \leq 1$ and let $n_3(L)$ be the smallest integer larger than $m_L$ such that $e^{-n \nu} \leq M^{-m_L}$, where $M := \max\{\|Df\|_{\infty, \mathbb{R}^2}, 1\}$.

Lemma 3.2. Let $\hat{x} \in \hat{F}$ such that $\eta_e(\hat{x}) \geq \eta$ and $L_e(\hat{x}) \leq L$. For every $n \geq n_3(L)$ and every $r \leq \eta$,

$$f_{\hat{x}}^{-n}(B_{x_n}(re^{-2n\nu})) \subset B_{x_0}(r).$$

Proof. Let us observe that for every $0 \leq k \leq n$, $f^k$ is injective on $f_{\hat{x}}^{-n}(B_{x_n}(re^{-2n\nu}))$ and that $f^k f_{\hat{x}}^{-n} = f_{\hat{x}}^{-n+k}$. By setting $p = n - k$, it suffices to show that

$$\forall p \in [0, n] \quad f_{\hat{x}}^{-p}(B_{x_n}(re^{-2n\nu})) \subset B_{x_{n-p}}(r). \quad (3.1)$$

To simplify let us set $m := m_L$ et $n_3 := n_3(L)$. We immediately have

$$\forall n \geq n_3, \forall p \in [0, n], \quad f_{\hat{x}}^{-p}(B_{x_n}(re^{-2n\nu})) \subset f_{\hat{x}}^{-p}(B_{x_n}(\frac{r}{M^m}e^{-n\nu})). \quad (3.2)$$

To verify (3.1), we shall consider separately the cases $p \leq m$ and $p > m$. We know that for every $p$, $\text{Lip} f_{\hat{x}}^{-p} \leq L(\hat{x})e^{-\lambda_2 + \epsilon} \leq Le^{\nu e^{-\lambda_2 + \epsilon}}$ on $B_{x_n}(\eta_e(\hat{x}_n))$, which contains $B_{x_n}(\eta e^{-n\nu})$. Hence for every $n \geq n_3 \geq m$, $p \in [m, n]$ and $r \leq \eta$:

$$f_{\hat{x}}^{-p}(B_{x_n}(re^{-n\nu})) \subset B_{x_{n-p}}(re^{-n\nu}Le^{-\lambda_2 + \epsilon}) = B_{x_{n-p}}(rLe^{-\lambda_2 + \epsilon}) \subset B_{x_{n-p}}(r).$$

Since $M^m \geq 1$ this implies for every $n \geq n_3 \geq m$, $p \in [m, n]$ and $r \leq \eta$:

$$f_{\hat{x}}^{-p}(B_{x_n}(\frac{r}{M^m}e^{-n\nu})) \subset B_{x_{n-p}}(\frac{r}{M^m}). \quad (3.3)$$
Thus, by using (3.2) and $M^m \geq 1$:

$$\forall p \in [m, n], \quad f_{\xi_n}^{-p}(B_{x_n}(re^{-2ne})) \subset B_{x_{n-p}}(r).$$  \hfill (3.4)

We have proved (3.1) for $p \in [m, n]$. Let us show this inclusion for $p \in [0, m]$. For every $p \in [0, m]$, let us set $p = m - p'$ where $p' \in [0, m]$. Then

$$f_{\xi_n}^{-p}(B_{x_n}(\frac{r}{M^m}e^{-ne})) = f^{p'}(f_{\xi_n}^{-m}(B_{x_n}(\frac{r}{M^m}e^{-ne}))) \subset f^{p'}(B_{x_{n-m}}(\frac{r}{M^m})),
$$

where the inclusion comes from (3.3) with $p = m$. We deduce:

$$\forall p \in [0, m], \quad f_{\xi_n}^{-p}(B_{x_n}(re^{-2ne})) \subset B_{x_{n-m+p'}}(\frac{r}{M^m}M^p) \subset B_{x_{n-p}}(r).$$  \hfill (3.5)

We finally obtain (3.1) by combining (3.4) and (3.5).

\section*{3.2 Pullback of the Fubini-Study form $\omega$}

Let $\nu$ be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$. Let $(Z, W)_c$ be Oseledec-Poincaré coordinates for $(f, \nu)$. Let $n_4(\beta)$ be the smallest integer $n$ such that $e^{-ne} \leq \beta^{-1}$.

\begin{proposition}
Let $\hat{x} \in \hat{F}$ such that $\eta_n(\hat{x}) \geq \eta$ and $\beta_n(\hat{x}) \leq \beta$. If $n \geq \max\{n_4(\beta), n_\epsilon(\hat{x}_n)\}$ and if $r \leq \eta$, then we have on $f_{\xi_n}^{-n}(B_{x_n}(re^{-ne}))$:

1. $(f^n)^*\omega \geq e^{-4n\epsilon + 2n\lambda_1} \left(\frac{i}{2} dZ^\epsilon_z \wedge d\overline{Z}^\epsilon_{\overline{z}}\right)$ if the exponents do not resonate.

2. $(f^n)^*\omega \geq e^{-4n\epsilon + 2n\lambda_2} \left(\frac{i}{2} dW^\epsilon_z \wedge d\overline{W}^\epsilon_{\overline{z}}\right)$.

The remainder of this Section is devoted to the proof. Theorem 2.4 gives

$$(f^n)^*\omega = (\xi^\epsilon)^*((R_{n,x_n})^{-1})^*((\xi_{\xi_n})^{-1})^*\omega$$

on $f_{\xi_n}^{-n}(B_{x_n}(re^{-ne}))$. Let $\omega_0 := \frac{i}{2} dz \wedge d\overline{z} + \frac{i}{2} dw \wedge d\overline{w}$ be the standard form on $\mathbb{D}^2$.

\begin{lemma}
Let $\hat{x} \in \hat{F}$ such that $\eta_n(\hat{x}) \geq \eta$ and $\beta_n(\hat{x}) \leq \beta$. For every $n \geq n_4(\beta)$ and $r \leq \eta$, we have on $\xi^\epsilon_x(B_{x_n}(re^{-ne}))$:

$$e^{-2ne}\omega_0 \leq ((\xi^\epsilon_{\xi_n})^{-1})^*\omega \leq 2\omega_0.$$

Proof. For every $p = (z, w)$ and $p' = (z', w')$ in $\xi^\epsilon_x(B_{x_n}(re^{-ne}))$, we have

$$e^{-ne}\beta^{-1}d(p, p') \leq \left|((\xi^\epsilon_{\xi_n})^{-1}(p) - (\xi^\epsilon_{\xi_n})^{-1}(p'))\right| \leq 2d(p, p').$$

This implies for every $n \geq n_4(\beta)$ and $(z, w) \in \xi^\epsilon_x(B_{x_n}(re^{-ne}))$:

$$\forall u \in \mathbb{C}^2, \quad e^{-2ne} |u| \leq \left|D_{(z, w)}((\xi^\epsilon_{\xi_n})^{-1}(u))\right| \leq 2 |u|.$$

This provides the desired estimates. \hfill \Box
Lemma 3.5. Let \( \hat{x} \in \hat{F} \). If \( n \geq n_\epsilon(\hat{x}_n) \), then

1. \( ((R_{n,\hat{x}_n})^{-1})^x \omega_0 \geq e^{2(n\lambda_1-n\eta_0)} \frac{1}{2} dz \wedge d\pi \) if the exponents do not resonate.

2. \( ((R_{n,\hat{x}_n})^{-1})^x \omega_0 \geq e^{2(n\lambda_2-n\eta_0)} \frac{1}{2} dw \wedge d\nu_0 \).

**Proof.** We use the fact that the linear part of \( R_{n,\hat{x}_n} \) is diagonal with coefficients \( e^{-n\epsilon-n\lambda_1} \leq |\alpha_{n,\hat{x}_n}| \leq e^{n\epsilon-n\lambda_1} \) and \( e^{-n\epsilon-n\lambda_2} \leq |\beta_{n,\hat{x}_n}| \leq e^{n\epsilon-n\lambda_2} \) (see Theorem 2.1) and the fact that the \((1,1)\)-forms \( \frac{1}{2} dz \wedge d\pi \) and \( \frac{1}{2} dw \wedge d\nu \) are positive.

To end the proof of Proposition 3.3 we observe that for every \( \hat{x} \in \hat{F} \):

\[
(\xi^t_\hat{x})^*(\frac{i}{2} dz \wedge d\pi) = (\frac{i}{2} dZ^r_\hat{x} \wedge d\pi_{\hat{x}} \hat{z}) \quad (\xi^t_\hat{x})^*(\frac{i}{2} dw \wedge d\nu) = (\frac{i}{2} dW^r_\hat{x} \wedge dW_{\hat{x}} \hat{z})
\]

which follows from the definitions of \( Z^r_\hat{x} \) and \( W^r_\hat{x} \).

### 3.3 Uniformizations

Let \( \nu \) be an ergodic dilating measure of exponents \( \lambda_1 > \lambda_2 \) and let \( \delta > 0 \). Let \( \epsilon > 0 \) and let \((Z,W)_\epsilon\) be Oseledec-Poincaré coordinates for \((f,\nu)\).

**Measure of dynamical balls**

We apply Lemma 3.3. There exist \( r_0 > 0 \), \( n_2 \geq 1 \) and \( C \subseteq \mathbb{P}^2 \) such that \( \nu(C) \geq 1 - \delta/8 \) and for every \( x \in C \) and \( n \geq n_2 \):

\[
\nu(B_n(x,r_0/8)) \geq e^{-nh_\nu-\epsilon n}, \quad \forall r \leq r_0, \nu(B_n(x,5r)) \leq \nu(B_n(x,5r_0)) \leq e^{-nh_\nu+\epsilon n}.
\]

We denote \( \Lambda^{(1)} := \pi_{\hat{x}}^{-1}(C) \cap \hat{F} \).

**Control of the functions \( n_\epsilon, \rho_\epsilon, L_\epsilon, \eta_\epsilon, \beta_\epsilon \) of Theorem 2.1**

Let \( n_0, \rho_0 > 0 \), \( L_0 \geq 0 \), \( \eta_0 > 0 \) and \( \beta_0 > 0 \) such that

\[
\Lambda^{(2)} := \{ \hat{x} \in \hat{F}, n_\epsilon(\hat{x}) \leq n_0, \rho_\epsilon(\hat{x}) \geq \rho_0, L_\epsilon(\hat{x}) \leq L_0, \eta_\epsilon(\hat{x}) \geq \eta_0, \beta_\epsilon(\hat{x}) \leq \beta_0 \} \quad (3.6)
\]

satisfies \( \hat{\nu}(\Lambda^{(2)}) \geq 1 - \delta/8 \).

**Uniformization of the dimension of the current.**

Let \( S \) be a positive closed current on \( \mathbb{P}^2 \) whose support contains the support of \( \nu \). Let \( r_1 > 0 \) such that

\[
\Lambda^{(3)} := \{ \hat{x} \in \hat{F}, \forall r \leq r_1, \frac{r_{ds,Z}(\hat{x})+\epsilon}{(S \wedge (\frac{i}{2} dZ^r_\hat{x} \wedge d\pi_{\hat{x}} \hat{z}))(B_{x_0}(r))} \leq \frac{r_{ds,Z}(\hat{x})-\epsilon}{(S \wedge (\frac{i}{2} dW^r_\hat{x} \wedge dW_{\hat{x}}))(B_{x_0}(r))} \}
\]

satisfies \( \hat{\nu}(\Lambda^{(3)}) \geq 1 - \delta/8 \). In the case of the Green current \( T \), the functions \( \hat{d}_{T,Z}, \hat{d}_{T,Z} \) and \( \hat{d}_{T,W} \) are \( \hat{\nu} \)-almost everywhere constant and denoted \( \hat{d}_{T,Z}(\nu), \hat{d}_{T,Z}(\nu) \) and \( \hat{d}_{T,W}(\nu) \).
Uniformization of the dimension of the measure
The lower dimension $d_0$ is defined in Section 1.4. Let $r_2 > 0$ such that
$$D := \{x \in \mathbb{P}^2, \forall r \leq r_2, \nu(B_x(r)) \leq r^{d_0 - \epsilon}\}$$
satisfies $\nu(D) \geq 1 - \delta/8$. We set $\Lambda^{(4)} := \pi_0^{-1}(D) \cap \hat{F}$.

Definition of $\hat{\Lambda}_c, \eta_1$ and $N_c$.
The integers $n_1(L), n_3(L)$ and $n_4(\beta)$ were defined before Lemma 2.3, 3.2 and Proposition 3.3. Let $n_5$ be the smallest integer $n$ such that $e^{-n\epsilon} \leq 1/2$ and $2e^{-n(\lambda + \epsilon)} < 1$.
$$\hat{\Lambda}_c := \Lambda^{(1)} \cap \Lambda^{(2)} \cap \Lambda^{(3)} \cap \Lambda^{(4)},$$
$$\eta_1 := \min\{\eta_0, r_0, r_1, r_2\},$$
$$N_c := \max\{n_0, n_1(L_0), n_2, n_3(L_0), n_4(\beta_0), n_5\}.$$
We have $\nu(\hat{\Lambda}_c) \geq 1 - \delta/2$.

Definition of $\hat{\Lambda}_c^n$.
We set
$$\forall n \geq N_c, \; \hat{\Lambda}_c^n := \hat{F} \cap \hat{f}^{-n}\{n_c(\hat{x}) \leq n\} = \{\hat{x} \in \hat{F}, \; n(\hat{x}_n) \leq n\}.$$Since $\nu$ is $\hat{f}$-invariant and $\Lambda^{(3)} \subset \{n_c(\hat{x}) \leq n\}$, we have $\nu(\hat{\Lambda}_c^n) \geq \nu(\Lambda^{(3)}) \geq 1 - \delta/8$. Hence
$$\forall n \geq N_c, \; \nu(\hat{\Lambda}_c \cap \hat{\Lambda}_c^n) \geq 1 - \delta.$$4 Separated sets
A subset $\{x_1, \ldots, x_N\} \subset \mathbb{P}^2$ is $r$-separated if $d(x_i, x_j) \geq r$ for every $i \neq j$. For $A \subset \mathbb{P}^2$, a subset $\{x_1, \ldots, x_N\} \subset A$ is maximal $r$-separated with respect to $A$ if it is $r$-separated and if for every $y \in A$, there exists $i \in \{1, \ldots, N\}$ such that $d(y, x_i) < r$. We use similar definitions for the distance $d_n$, in which case we say that the subsets are $(n, r)$ separated.

4.1 Elementary separation
Lemma 4.1. Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let $\nu$ be an ergodic measure. Let $A \subset \pi_0(\hat{\Lambda}_c)$ such that $\nu(A) > 0$ and let $c \in [0, 1]$. Let $n \geq N_c$ and let $\{x_1, \ldots, x_N\} \subset A$ be maximal $(n, c\eta_1)$-separated with respect to $A$. Then
1. for every $i \neq j$, $B_n(x_i, c\eta_1/2) \cap B_n(x_j, c\eta_1/2) = \emptyset$.
2. $A \subset \bigcup_{i=1}^N B_n(x_i, c\eta_1)$.
3. $\nu(B_n(x_i, c\eta_1)) \leq e^{-nh_0 + nc}$.
4. \( e^{-nh_{e} - nc} \leq \nu(B_n(x_i, c \eta_1)) \) so \( c \geq 1/8 \).

5. \( N_n \geq \nu(A)e^{nh_{e} - nc} \).

**Proof.** Item 1 comes from separation, Item 2 from the maximal property, Items 3 and 4 from Section 3. Let \( n \geq N_e \), \( c \eta_1 \leq \eta_1 \) and \( x_i \in C \). Items 2 and 3 then imply \( \nu(A) \leq \sum_{i=1}^{N_n} \nu(B_n(x_i, c \eta_1)) \leq N_n e^{-nh_{e} + nc} \), which gives Item 5.

### 4.2 Concentrated separation

Lemma [4.1] applied with \( c = 1/4 \) gives \( \nu(B_n(x_i, \eta_1/4)) \geq e^{-nh_{e} - nc} \) for every \( x_i \) in a maximal \((n, \eta_1/4)\)-separated subset of \( A \). We shall see that it is possible to select a large number of \( x_i \) so that

\[
\nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_{e} - 2nc}.
\]

We take the arguments of de Thélin-Vigny in [12, Section 6]. Let \( n_{\delta} \) be the smallest integer \( n \) such that \( e^{-nc} \leq \delta/2 \).

**Lemma 4.2.** Let \( A \subset \pi_0(\hat{\Lambda}_e) \) such that \( \nu(A) \geq \delta \). For every \( n \geq \max\{N_e, n_{\delta}\} \), there exists a \((n, \eta_1/4)\)-separated subset \( \{x_1, \ldots, x_{N_{n,2}}\} \) of \( A \) such that

1. for every \( i \neq j \), \( B_n(x_i, \eta_1/8) \cap B_n(x_j, \eta_1/8) = \emptyset \).
2. for every \( 1 \leq i \leq N_{n,2} \), \( \nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_{e} - 2c}\).
3. \( N_{n,2} \geq \nu(A)e^{nh_{e} - 2nc} \).
4. \( e^{-nh_{e} - nc} \leq \nu(B_n(x_i, c \eta_1)) \) so \( c \geq 1/8 \).

**Proof.** Let us apply Lemma [4.1] with \( c = 1/4 \) and \( n \geq \max\{N_e, n_{\delta}\} \). There exists a maximal \((n, \eta_1/4)\)-separated subset \( \{x_1, \ldots, x_{N_{n,1}}\} \) of \( A \) satisfying:

- for every \( i \neq j \), \( B_n(x_i, \eta_1/8) \cap B_n(x_j, \eta_1/8) = \emptyset \),
- \( A \subset \bigcup_{i=1}^{N_{n,1}} B_n(x_i, \eta_1/4) \),
- \( e^{-nh_{e} - nc} \leq \nu(B_n(x_i, \eta_1/8)) \),
- \( N_{n,1} \geq \nu(A)e^{nh_{e} - nc} \).

Let us set \( I := \{1 \leq i \leq N_{n,1} \mid \nu(B_n(x_i, \eta_1/4) \cap A) \geq e^{-nh_{e} - 2c}\} \). Let \( N_{n,2} \) be the cardinality of \( I \), and assume that \( I = [1, N_{n,2}] \) (we may adapt the sums below if \( N_{n,2} = 0 \)). We want to bound \( N_{n,2} \) from below. We know that \( A \subset \bigcup_{i=1}^{N_{n,1}} B_n(x_i, \eta_1/4) \), hence

\[
\nu(A) \leq \sum_{i=1}^{N_{n,2}} \nu(B_n(x_i, \eta_1/4) \cap A ) + \sum_{i=N_{n,2}+1}^{N_{n,1}} \nu(B_n(x_i, \eta_1/4) \cap A).
\]

If \( i \notin [1, N_{n,2}] \), we have \( \nu(B_n(x_i, \eta_1/4) \cap A) < e^{-nh_{e} - 2c} \) by definition of \( I \). Otherwise, \( \nu(B_n(x_i, \eta_1/4) \cap A) \leq e^{-nh_{e} + c} \) since \( x_i \in C \). This implies

\[
\nu(A) \leq N_{n,2}e^{-nh_{e} + c} + (N_{n,1} - N_{n,2})e^{-nh_{e} - 2c}.
\] (4.1)
Lemma 4.3. \[ e^{nh_{\nu}+\epsilon n} \geq N_{n,1} \geq N_{n,1} - N_{n,2}. \]

Combining this and (4.1), we obtain
\[ \nu(A) \leq N_{n,2} e^{-nh_{\nu}+\epsilon n} + e^{-\epsilon n}. \]

Since \( n \geq n_{5} \), we have \( e^{-\epsilon n} \leq \frac{\delta}{2} \leq \nu(A)/2 \), and hence \( N_{n,2} \geq \nu(A)e^{nh_{\nu}+\epsilon n}/2 \). Finally \( N_{n,2} \geq \nu(A)e^{nh_{\nu}-2\epsilon n} \) since \( n \geq N_{\epsilon} \geq n_{5} \).

Now we put in \( B_{n}(x, \eta_{1}/2) \) a lot of balls whose centers are in \( B_{n}(x, \eta_{1}/4) \cap A \).

Lemma 4.3. Let \( A \subset \pi_{0}(\hat{A}_{\epsilon}) \) such that \( \nu(A) > 0 \). Let \( x \in A \) and let \( n \geq N_{\epsilon} \) such that
\[ \nu(B_{n}(x, \eta_{1}/4) \cap A) \geq e^{-nh_{\nu}-2\epsilon n}. \]

Let \( \{y_{1}, \ldots, y_{M_{n}}\} \) be a maximal \( 2\eta_{1}e^{-n\lambda_{1}-4\epsilon n} \)-separated subset in \( B_{n}(x, \eta_{1}/4) \cap A \).

1. for every \( i \neq j \), \( B(y_{i}, \eta_{1}e^{-n\lambda_{1}-4\epsilon n}) \cap B(y_{j}, \eta_{1}e^{-n\lambda_{1}-4\epsilon n}) = \emptyset \).
2. \( B_{n}(x, \eta_{1}/4) \cap A \subset \bigcup_{i=1}^{M_{n}} B(y_{i}, 2\eta_{1}e^{-n\lambda_{1}-4\epsilon n}) \).
3. \( B(y_{i}, \eta_{1}e^{-n\lambda_{1}-4\epsilon n}) \subset B_{n}(x, \eta_{1}/2) \).
4. \( M_{n} \geq e^{-nh_{\nu}-2\epsilon n} \left( \frac{1}{2\eta_{1}} e^{n\lambda_{1}+4\epsilon n} \right)^{2_{\epsilon}} \).

Proof. Item 1 comes from separation, Item 2 from the maximal property. Lemmas 2.3 then (3.2) give
\[ B(y_{i}, \eta_{1}e^{-n\lambda_{1}-4\epsilon n}) \subset f^{-1}_{\eta_{1}, n}(B_{y_{i}, n}(\eta_{1}e^{-2\epsilon n}/4) \subset B_{n}(y_{i}, \eta_{1}/4). \]

Since \( y_{i} \in B_{n}(x, \eta_{1}/4) \), we have \( B_{n}(y_{i}, \eta_{1}/4) \subset B_{n}(x, \eta_{1}/2) \), which yields Item 3. Item 2 implies
\[ \nu(B_{n}(x, \eta_{1}/4) \cap A) \leq \sum_{i=1}^{M_{n}} \nu(B(y_{i}, 2\eta_{1}e^{-n\lambda_{1}-4\epsilon n})). \]

By assumption, the left hand side is larger than \( e^{-nh_{\nu}-2\epsilon n} \). For the right hand side, since \( n \geq N_{\epsilon} \geq n_{5} \), we have \( 2\eta_{1}e^{-n\lambda_{1}-\epsilon n} < \eta_{1} \leq r_{2} \) and thus
\[ \nu(B(y_{i}, 2\eta_{1}e^{-n\lambda_{1}-4\epsilon n})) \leq (2\eta_{1}e^{-n\lambda_{1}-4\epsilon n})^{2_{\epsilon}} \]
by using \( y_{i} \in A \subset \pi_{0}(\hat{A}_{\epsilon}) \subset D \). This shows \( e^{-nh_{\nu}-2\epsilon n} \leq M_{n}(2\eta_{1}e^{-n\lambda_{1}-4\epsilon n})^{2_{\epsilon}}. \) \( \square \)
5 Lower bounds for the directional dimensions of $T$

Let $\nu$ be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. Let $\epsilon > 0$ and let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for $(f,\nu)$. We have $d_{T,Z}(\nu) := d_{T,Z}(\hat{x})$ and $d_{T,W}(\nu) = d_{T,W}(\hat{x})$ for $\hat{\nu}$-almost every $\hat{x}$ (see Proposition 2.4). In this Section we prove Theorems 5.1 and 5.2. These results specify in a directional way Theorems A and B of de Thélin-Vigny [12] concerning the dimension of $T$. We use below arguments of [12] by replacing the lower bound (obtained in [12] by slicing arguments) $(f^n)^* \omega \geq e^{2n\lambda_2 e^{-\epsilon} \omega}$ by the two lower bounds (obtained by normal forms Theorem 2.1) $(f^n)^* \omega \geq e^{2n\lambda_1 e^{-\epsilon} dZ} \wedge dZ$ and $(f^n)^* \omega \geq e^{2n\lambda_2 e^{-\epsilon} dW} \wedge dW$.

Theorem 5.1 uses elementary separation (Lemma 4.1), Theorem 5.2 uses concentrated separation (Lemmas 4.2 and 4.3). Theorem 5.2 implies Theorem 1.1 (via the lower bound (1.6) for $d\nu$) and Theorem 1.7.

5.1 First lower bounds for the upper dimensions $d_{T,Z}$ et $d_{T,W}$

**Theorem 5.1.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $\nu$ be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. There exist functions $O_5(\epsilon), O_6(\epsilon)$ satisfying the following properties. Let $\epsilon > 0$ and let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for $(f,\nu)$. Then

$$d_{T,Z}(\nu) \geq 2 + \frac{h_\nu - \log d}{\lambda_1} - O_5(\epsilon)$$

$$d_{T,W}(\nu) \geq 2 \frac{\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_1} - O_6(\epsilon).$$

**Proof.** Let us denote $d_{T,Z} := d_{T,Z}(\nu)$. For first estimate, we are going to show

$$(\lambda_1 + 4\epsilon)(d_{T,Z} - 2 + \epsilon) + 13\epsilon \geq h_\nu - \log d$$

which provides

$$d_{T,Z} \geq 2 + \frac{h_\nu - \log d - 13\epsilon}{\lambda_1 + 4\epsilon} - \epsilon =: 2 + \frac{h_\nu - \log d}{\lambda_1} - O_5(\epsilon).$$

Let $\delta > 0$. Let $\hat{\Lambda}_c$ and $N_c$ be given by Section 4.5. For every $n \geq N_c$ we set $A_n := \pi_0(\hat{\Lambda}_c \cap \hat{\Delta}^n)$, it satisfies $\nu(A_n) \geq \hat{\nu}(\hat{\Lambda}_c \cap \hat{\Delta}^n) \geq 1 - \delta > 0$. Lemma 4.1 applied with $c = 1$ yields a maximal $(n,\eta_1)$-separated subset $\{x_1, \ldots, x_{N_n}\}$ of $A_n$ with

$$N_n \geq \nu(A_n)e^{nh_\nu - ne} \geq (1 - \delta)e^{nh_\nu - ne}.$$
For every $i$, let us choose $\hat{x}_i \in \hat{\Lambda}_e \cap \hat{\Delta}_n^\epsilon$ such that $\pi_0(\hat{x}_i) = x_i$. From Proposition 9.3 we get $d^n = \int_{\mathbb{P}^2} [(f^n)_*T] \wedge \omega$. Therefore

$$d^n \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (f^n)_*(1_B_{n(x_i,\eta_1/2)}T) \wedge \omega \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (1_B_{n(x_i,\eta_1/2)}T) \wedge (f^n)^*\omega.$$  

By using Lemmas 2.3 and 3.2 with $\hat{x}_i \in \hat{\Lambda}_e$, we get

$$B_{\hat{x}_i}(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon}) \subset f_{\hat{x}_i,n}^{-n}(B_{\hat{x}_i,n}(\frac{\eta_1}{8} e^{-2n\epsilon})) \subset B_n(x_i, \frac{\eta_1}{8}).$$  

(5.3)

Since $T \wedge (f^n)^*\omega$ is a positive measure, we deduce

$$d^n \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (1_B_{n(x_i,\eta_1/2)}T) \wedge (f^n)^*\omega.$$  

Thanks to the first inclusion of (5.3) and $\hat{x}_i \in \hat{\Delta}_n^\epsilon$ (which implies $n \geq n_\epsilon(\hat{x}_i,n)$), we can use Proposition 3.3 to bound $(f^n)^*\omega$ from below. We obtain

$$d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left( T \wedge \frac{i}{2} dZ_{\hat{x}_i} \wedge dZ_{\hat{x}_i} \right) (B_{\hat{x}_i}(\frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon})).$$  

Since $\hat{x}_i \in \hat{\Lambda}_e \subset \Lambda^{(3)}$ and $\eta_1 \leq r_1$, we deduce

$$d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left( \frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon} \right) \overline{d_{T,W} + \epsilon}.$$  

Finally, we use the estimate (5.2):

$$d^n \geq \nu(A_n) \left( \frac{\eta_1}{2} \right) \overline{d_{T,W} + \epsilon} e^{(h_\nu - (\lambda_1 + 4\epsilon)(d_{T,W} - 2\epsilon) - 13\epsilon)},$$  

where $\nu(A_n) \geq (1 - \delta)$. By taking logarithm and dividing by $n$, we get (5.1) when $n \to \infty$. The second estimate concerning the coordinate $W$ is proved in a similar way, by using Proposition 3.3 to bound $(f^n)^*\omega$ from below. We precisely get

$$(\lambda_1 + 4\epsilon)(d_{T,W} + \epsilon) + 5\epsilon \geq h_\nu - \log d + 2\lambda_2$$  

(5.4)

which yields

$$\overline{d_{T,W}} \geq \frac{2\lambda_2}{\lambda_1 + 4\epsilon} + \frac{h_\nu - \log d - 5\epsilon}{\lambda_1 + 4\epsilon} - \epsilon =: \frac{2\lambda_2}{\lambda_1} + \frac{h_\nu - \log d}{\lambda_1} - O_6(\epsilon).$$  

This completes the proof of Theorem 5.1. \qed
5.2 Proof of Theorem [1.1]

Theorem [1.1] is a consequence of Theorem 5.2 below and of the lower bound [1.6].

**Theorem 5.2.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $\nu$ be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. There exist functions $O_1(\epsilon), O_2(\epsilon)$ satisfying the following properties. Let $\epsilon > 0$ and let $(Z,W)_e$ be Oseledec-Poincaré coordinates for $(f,\nu)$. Then

$$\bar{d}_{T,Z}(\nu) \geq 2 + d_e - \frac{\log d}{\lambda_1} - O_1(\epsilon),$$

$$\bar{d}_{T,W}(\nu) \geq 2 \frac{\lambda_2}{\lambda_1} + d_e - \frac{\log d}{\lambda_1} - O_2(\epsilon).$$

**Proof.** Let us set $\bar{d}_{T,Z} := \bar{d}_{T,Z}(\nu)$. We are going to show

$$(\lambda_1 + 4\epsilon)(\bar{d}_{T,Z} - d_e + 2\epsilon) + 8\epsilon \geq 2\lambda_1 - \log d. \quad (5.5)$$

This yields as desired

$$\bar{d}_{T,Z} - d_e \geq \frac{2\lambda_1}{\lambda_1 + 4\epsilon} - \frac{\log d - 8\epsilon}{\lambda_1 + 4\epsilon} - 2\epsilon =: 2 - \frac{\log d}{\lambda_1} - O_1(\epsilon). \quad (5.6)$$

Let $\delta > 0$. Let $\hat{\Lambda}$ and $N_e$ be given by Section 4.3. For every $n \geq N_e$, we set $A_n := \pi_0(\hat{\Lambda} \cap \hat{\Lambda}_n)$, it satisfies $\nu(A_n) \geq 1 - \delta$. Let $\{x_1, \ldots, x_{N_{n,2}}\}$ be a $(n, \eta_1/4)$-separated subset of $A_n$ provided by Lemma 4.2. Then for every $x_i$, we set a $2\eta_1 e^{-n\lambda_1 - 4n\epsilon}$-separated subset $\{y_{i1}^1, \ldots, y_{iM_n}^1\}$ given by Lemma 4.3. For every $i$ we choose $\hat{x}_i \in \hat{\Lambda} \cap \hat{\Lambda}_n$ such that $\pi_0(\hat{x}_i) = x_i$, and for every $j$ we choose $\hat{y}_j^i \in \hat{\Lambda} \cap \hat{\Lambda}_n$ such that $\pi_0(\hat{y}_j^i) = y_j^i$. According to Proposition 3.3, we have $d^n = \int_{\mathbb{P}^2}(f^n)_eT \land \omega$, thus

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} \int_{\mathbb{P}^2}(f^n)_e(1_{B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}}) T) \land \omega = \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} \int_{B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon})} (f^n)_e T \land (f^n)_e \omega.$$

Lemma 2.3 with $\hat{y}_j^i \in \hat{\Lambda}$ implies

$$B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \subset \hat{y}_j^{-n} (B(y_{j,n}, \eta_1 e^{-2n\epsilon})). \quad (5.7)$$

Since $\hat{y}_j^i \in \hat{\Lambda}_n$, we can apply Proposition 3.3 to bound $(f^n)_e \omega$ from below:

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 4n\epsilon} \left( T \land \frac{i}{2} dZ_{y_j^i} \land dZ_{\hat{y}_j^i}^\epsilon \right) (B(y_j^i, \eta_1 e^{-n\lambda_1 - 4n\epsilon}) \omega). \quad (5.8)$$

Now $\hat{y}_j^i \in \hat{\Lambda} \subset \Lambda^{(3)}$ and $n \geq N_e$, hence

$$d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 4n\epsilon} \eta_1 e^{-n\lambda_1 - 4n\epsilon} \bar{d}_{T,Z} + \epsilon.$$

Finally, we use the lower bounds for $M_n$ (Lemma 4.3) and for $N_{n,2}$ (Lemma 4.2). We obtain for every $n \geq \max\{N_\epsilon, n_{1-\delta}\}$:

$$d^n \geq (1 - \delta)e^{nh_{\nu} - 2ne} \cdot e^{-nh_{\nu} - 2ne} \left(\frac{1}{2\eta_1}e^{n\lambda_1 + 4ne}\right)^{d_{\nu} - \epsilon} \cdot e^{2n\lambda_1 - 4ne}(\eta_1 e^{-n\lambda_1 - 4ne})^{d_{T,Z}}.$$

Let us note that the entropy $h_{\nu}$ disappear for the benefit of $d_{\nu}$, and we get

$$d^n \geq c e^{-8ne} \left(e^{n\lambda_1 + 4ne}\right)^{d_{\nu} - \epsilon} e^{2n\lambda_1},$$

where $c := (1 - \delta)\eta_1/(2\eta_1)$. Taking logarithm and then dividing by $n$, we obtain (5.9) when $n \to +\infty$. Similarly, we can prove

$$(\lambda_1 + 4\epsilon)(d_{T,W} - d_{\nu} + 2\epsilon) + 8\epsilon \geq 2\lambda_2 - \log d$$

by using again Proposition 3.3 to bound $(f^n)^*\omega$ from below. □

### 6 Currents $S$ and semi-extremal endomorphisms

#### 6.1 Proof of Theorem 1.4

Let $S$ be a $(1,1)$ closed positive current of $\mathbb{P}^2$. If $S$ does not satisfy $f^*S = dS$, the directional dimensions may be not $\hat{\nu}$-almost everywhere constant (see Proposition 2.4). In this case, in the manner of de Thélin-Vigny [12], we take on an adapted definition and obtain the following result. The functions $O_5(\epsilon), O_6(\epsilon)$ were defined in Theorem 5.1.

**Theorem 6.1.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $S$ be a $(1,1)$ closed positive current of $\mathbb{P}^2$, of mass 1. Let $\nu$ be an ergodic dilating measure whose exponents $\lambda_1 > \lambda_2$ do not resonate. We assume that $\text{Supp}(\nu) \subset \text{Supp}S$. Let $\epsilon > 0$ and let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for $(f,\nu)$. For every $\Lambda \subset \bar{F}$ such that $\hat{\nu}(\Lambda) > 0$, we set

$$d_{S,Z}(\Lambda) := \sup_{\hat{x} \in \Lambda} d_{S,Z}(\hat{x}), \quad d_{S,W}(\Lambda) := \sup_{\hat{x} \in \Lambda} d_{S,W}(\hat{x}).$$

Then

$$d_{S,Z}(\hat{\Lambda}) \geq 2 + \frac{h_{\nu} - \log d}{\lambda_1} - O_5(\epsilon),$$

$$d_{S,W}(\hat{\Lambda}) \geq 2 + \frac{\lambda_2}{\lambda_1} + \frac{h_{\nu} - \log d}{\lambda_1} - O_6(\epsilon).$$

**Proof.** Let $2\delta := \hat{\nu}(\hat{\Lambda})$. We construct $\hat{\Lambda}_\epsilon$ and $N_\epsilon$ for the current $S$ as in Section 5.3. We have $\hat{\nu}(\hat{\Lambda}_\epsilon \cap \hat{\Delta}_n^\omega) \geq 1 - \delta$ for every $n \geq N_\epsilon$, thus $\hat{\nu}(\hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}_n^\omega) \geq \delta > 0$. We follow the arguments of Theorem 5.1. Lemma 4.11 applied to $A_n := \pi_0(\Lambda \cap \hat{\Lambda}_\epsilon \cap \hat{\Delta}_n^\omega)$ and $c = 1$ provides a maximal $(n,\eta_1)$-separated subset $\{x_1, \ldots, x_{N_n}\}$ of $A_n$. For every $i$, let us
choose \( \hat{x}_i \in \hat{\Lambda} \cap \hat{\Lambda}_\epsilon \cap \hat{\Lambda}^n \) such that \( \pi_0(\hat{x}_i) = x_i \). According to Proposition 9.3 we have \( d^n = \int_{\mathbb{P}^2} (f^n)_* S \wedge \omega \), hence

\[
d^n = \int_{\mathbb{P}^2} ((f^n)_* S) \wedge \omega \geq \sum_{i=1}^{N_n} \int_{\mathbb{P}^2} (1_{B_n(x_i, \eta_1/2)} S) \wedge (f^n)^* \omega.
\]

Then we use the inclusions and the lower bound for \( (f^n)^* \omega \) to obtain:

\[
d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left( S \wedge \frac{i}{2} dZ_{\hat{x}_i} \wedge d\bar{Z}_{\hat{x}_i} \right) \left( B_{x_i} \left( \frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon} \right) \right).
\]

Since \( \hat{x}_i \in \hat{\Lambda}_\epsilon \) and \( n \geq N_{\epsilon} \), we get

\[
d^n \geq \sum_{i=1}^{N_n} e^{2n\lambda_1 - 4n\epsilon} \left( \frac{\eta_1}{2} e^{-n\lambda_1 - 4n\epsilon} \right) \overline{d_{S,Z}(\hat{x}_i)} + \epsilon.
\]

Now we use the adapted definition of \( \overline{d_{S,Z}}(\hat{\Lambda}) \) and the lower estimate (5.2) to obtain

\[
d^n \geq \nu(A_n) \left( \frac{\eta_1}{2} \overline{d_{S,Z}(\hat{\Lambda})} + \epsilon \right) e^{nh - 13n\epsilon - n(\lambda_1 + 4\epsilon)(\overline{d_{S,Z}(\hat{\Lambda})} - 2 + \epsilon)}, \tag{6.1}
\]

where \( \nu(A_n) \geq \delta \). The lower bound concerning \( W \) is proved in a similar way. \( \square \)

**Remark 6.2.** Theorem 6.1 is the counterpart of Theorem 5.1 for currents \( S \). Similarly, the counterpart of Theorem 5.2 can be proved with \( n \geq \max\{N_{\epsilon}, n_{\delta} \} \) in the proof.

### 6.2 Proof of Theorem 1.7

Since \( T \) is \( f \)-invariant and \( \nu \) is ergodic, we have \( \overline{d_T}(x) = \overline{d_T}(\mu) \) for \( \mu \)-almost every \( x \), see Proposition 2.5. According to Proposition 9.1 \( \mu << \sigma_T \) implies

\[
\overline{d_T}(\mu) \leq \overline{d}(\mu). \tag{6.2}
\]

Let us analyse these quantities. On the one hand, Proposition 9.2 yields \( \overline{d_T}(\mu) = \min \{ \overline{d_{T,Z}(x)}, \overline{d_{T,W}(x)} \} \) for \( \mu \)-almost every \( x \in \mathbb{P}^2 \) and for every holomorphic coordinates \( (Z,W) \) near \( x \). On the other hand, since \( d(\mu) = \overline{d}(\mu) \), then \( d(\mu) = \overline{d}(\mu) = \dim_H(\mu) \), which is equal to \( 2 + \frac{\log d(\lambda_1)}{\lambda_1} \) by (1.7). One deduces from (6.2) that if \( \mu << \sigma_T \), then

\[
\min \{ \overline{d_{T,Z}(x)}, \overline{d_{T,W}(x)} \} \leq 2 + \frac{\log d}{\lambda_1}. \tag{6.3}
\]

Now we use Theorem 5.2. Let \( \epsilon > 0 \) such that \( 4 - O_1(\epsilon) > 2 + \frac{\log d}{\lambda_1} \), where the function \( O_1(\epsilon) \) is defined by (5.6). Let \( (Z,W)_\epsilon \) Oseledec-Poincaré coordinates for \( (f, \mu) \).
First we bound $d_{T,Z}(\mu)$ from below, then we establish the formula for $d_{T,W}(\mu)$ modulo the function $O_2(\epsilon)$. If $\mu << \sigma_T$, then $\lambda_2 = \frac{1}{2} \log d$ by Theorem 1.3. We deduce from (1.6) that $2 + \frac{\log d}{\lambda_1} \leq d_{\mu}$. Theorem 5.2 then provides

$$d_{T,Z}(\mu) \geq 4 - O_1(\epsilon), \quad d_{T,W}(\mu) \geq 2 + \frac{\log d}{\lambda_1} - O_2(\epsilon).$$

Finally, using $4 - O_1(\epsilon) > 2 + \frac{\log d}{\lambda_1}$ and (6.3) with Oseledec-Poincaré coordinates $(Z,W)_\epsilon$, we get $d_{T,W}(\mu) \leq 2 + \frac{\log d}{\lambda_1}$ as desired.

## 7 Upper bounds for the directional dimensions of $T$

In this Section we show Theorem 1.2. Let $\nu$ be an ergodic dilating measure such that $\text{Supp}(\nu) \subset \text{Supp}(\mu)$ and whose exponents $\lambda_1 > \lambda_2$ do not resonate. Let $\epsilon > 0$ and let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for $(f,\nu)$. We want to prove

$$d_{T,Z}(\nu) \leq \frac{\log d}{\lambda_2} + 2 \frac{\lambda_1}{\lambda_2} + O_3(\epsilon) \quad \text{and} \quad d_{T,W}(\nu) \leq \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon).$$

We shall directly obtain these upper bounds for $\hat{\nu}$-almost every $\hat{x}$, by using the jacobians of $T \wedge dZ^\epsilon_x \wedge d\overline{Z}^\epsilon_x$ and $T \wedge dW^\epsilon_x \wedge d\overline{W}^\epsilon_x$ with respect to $f$. In particular, we shall not use separated subsets. The Monge-Ampère equation $\mu = T \wedge T$ will be crucial.

### 7.1 Dimensions of the Green current on the equilibrium measure

The following Proposition is proved in Section 7.3.

**Proposition 7.1.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let $T$ be its Green current. Let $x \in \text{Supp} \mu$ and let $Z$ be a local holomorphic coordinate (submersion) in a neighbourhood $V$ of $x$. Then $T \wedge (\frac{i}{2}dZ \wedge d\overline{Z})$ is not the zero measure on $V$.

This implies:

**Proposition 7.2.** Let $f$ be an endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $\nu$ be an ergodic dilating measure of exponents $\lambda_1 > \lambda_2$ and whose support is contained in the support of $\mu$. Let $\epsilon > 0$ and let $(Z,W)_\epsilon$ be Oseledec-Poincaré coordinates for $(f,\nu)$. We recall that for every $\hat{x} \in \hat{F}$, $(Z^\epsilon_x,W^\epsilon_x)$ is defined on $B_{z_0}(\eta_\epsilon(\hat{x}))$. Then, for every $0 < r < \eta_\epsilon(\hat{x})$,

$$ \left[ T \wedge \frac{i}{2}dZ^\epsilon_x \wedge d\overline{Z}^\epsilon_x \right] (B_x(r)) > 0 \quad \text{and} \quad \left[ T \wedge \frac{i}{2}dW^\epsilon_x \wedge d\overline{W}^\epsilon_x \right] (B_x(r)) > 0.$$
Proof. The first part immediately follows from Proposition 7.1. To prove the second part, let \( m_0 \geq 1 \) and \( L_0 \geq 1 \) be such that \( \nu \left\{ \eta \geq \frac{1}{m_0} \right\} \cap \{ L \leq L_0 \} \geq 1 - \delta/2 \). Then we choose \( q_0 \) large enough so that \( \nu(\hat{\Omega}_n) \geq 1 - \delta \).

We define for every \( n \geq 1 \):

\[
\hat{\Omega}_n := \hat{\Omega} \cap \hat{f}^{-n}(\hat{\Omega}_n).
\]

Since \( \nu \) is invariant, we have:

\[
\nu(\hat{\Omega}_n) \geq 1 - 2\delta. \tag{7.1}
\]

The following Proposition will be useful to prove Theorems 1.2 and 1.5. \( L_0 \) is defined in Proposition 7.2 and \( n_1(L_0) \geq 1 \) is defined before Lemma 2.3.

**Proposition 7.3.** Let \( f \) be an endomorphism of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). Let \( \nu \) be a ergodic dilating measure of exponents \( \lambda_1 > \lambda_2 \) and such that \( \text{Supp}(\nu) \subset \text{Supp}(\mu) \). For every \( n \geq n_1(L_0) \) and \( \hat{x} \in \hat{\Omega}_n \), we have:

\[
\left[ T \wedge \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}} \right] (B_x \left( \frac{1}{m_0}e^{-n\lambda_2+3n\epsilon} \right)) \geq \frac{1}{d^n}e^{-2n\lambda_1-2ne} \frac{1}{q_0} \quad \text{if } \lambda_1 \notin \{k\lambda_2, k \geq 2 \},
\]

\[
\left[ T \wedge \frac{i}{2}dW^x_{\hat{x}} \wedge dW^x_{\hat{x}} \right] (B_x \left( \frac{1}{m_0}e^{-n\lambda_2+3n\epsilon} \right)) \geq \frac{1}{d^n}e^{-2n\lambda_2-2ne} \frac{1}{q_0} \quad \text{for every } \lambda_1 > \lambda_2.
\]

**Proof.** Let \( \hat{x} \in \hat{\Omega}_n \) and let

\[
E_n := f_{\hat{x}}^{-n}(B_{\hat{x}}(\frac{1}{4m_0})).
\]

The inverse branch \( f_{\hat{x}}^{-n} \) is well defined on \( B_{\hat{x}}(\frac{1}{4m_0}) \) since \( \hat{x} \in \hat{\Omega}_n \). Let \( g_n \) be the restriction of \( f^n \) on \( E_n \). By using \( f_{\hat{x}}^{-n} \circ g_n = Id_{E_n} \) and \( T = \frac{1}{d}g_n^*T \) on \( E_n \), we obtain

\[
T \wedge \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}} = \frac{1}{d^n}g_n^*T \wedge g_n^*(f_{\hat{x}}^{-n})^* \left( \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}} \right)
\]

\[
= \frac{1}{d^n}g_n^* \left[ T \wedge \frac{i}{2}(dZ^x_{\hat{x}} \circ (f_{\hat{x}}^{-n}) \wedge d(Z^x_{\hat{x}} \circ (f_{\hat{x}}^{-n})) \right]
\]

on the open subset \( E_n \). Now we use (2.1) to write \( Z^x_{\hat{x}} \circ (f_{\hat{x}}^{-n}) = \alpha_n, \hat{x}, Z_{\hat{x}}. \) Since \( |\alpha_n, \hat{x},| \geq e^{-2n\lambda_1-2ne} \), we get on \( E_n \):

\[
T \wedge \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}} \geq \frac{1}{d^n}e^{-2n\lambda_1-2ne} g_n^* \left[ T \wedge \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}} \right]. \tag{7.2}
\]

We are going to bound from above the left hand side and to bound from below the right hand side (applied to \( E_n \)). Using Lemma 2.3 with \( r = 1/m_0 \) and \( n \geq L_0 \) we obtain \( E_n \subset B_x \left( \frac{1}{m_0}e^{-n\lambda_2+3n\epsilon} \right) \), hence

\[
T \wedge \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}} \left( B_x \left( \frac{1}{m_0}e^{-n\lambda_2+3n\epsilon} \right) \right) \geq T \wedge \frac{i}{2}dZ^x_{\hat{x}} \wedge dZ^x_{\hat{x}}(E_n) \tag{7.3}
\]
For the right hand side, since \( g_n \) is injective on \( E_n \) and \( g_n(E_n) = B_{x_n} \left( \frac{1}{4m_0} \right) \), we get
\[
g_n \left[ T \wedge \frac{i}{2}(dZ^x_{\hat{x}_n} \wedge d\overline{Z}^x_{\hat{x}_n}) \right](E_n) = \left[ \frac{T}{4m_0} \right] \left( B_{x_n} \left( \frac{1}{4m_0} \right) \right) \geq \frac{1}{q_0}, \tag{7.4}
\]
where the inequality comes from \( \hat{x}_n \in \hat{\Omega}_e \). By combining (7.2), (7.3) and (7.4) we obtain
\[
\left[ T \wedge \frac{i}{2}dZ^x_{\hat{x}_n} \wedge d\overline{Z}^x_{\hat{x}_n} \right] \left( B_{x_n} \left( \frac{1}{4m_0} e^{-l_{p}\lambda_2 + 3\epsilon} \right) \right) \geq \frac{1}{d^n} e^{-2n\lambda_1 - 2n\epsilon} \frac{1}{q_0}.
\]
We use \( W_{x_n}^e (f_{\hat{x}_n}^{-1}) = \beta_{n,\hat{x}} W_{x_n} \) and \( |\beta_{n,\hat{x}}|^2 \geq e^{-2n\lambda_2 - 2n\epsilon} \) to prove the other lower bound.

### 7.2 Proof of Theorem 1.2

We take the notations of Section 7.1. Let \( \hat{\Omega}_e := \limsup_{n \in \mathbb{N}} \hat{\Omega}_e \cap f^{-n} (\hat{\Omega}_e) = \limsup_{n \in \mathbb{N}} \hat{\Omega}_e^n \).

We have \( \hat{\nu}(\hat{\Omega}_e) \geq 1 - 2\delta \) according to (7.1). Let \( \hat{x} \in \hat{\Omega}_e \). Then there exists an increasing sequence of integers \( (l_p) \) such that
\[
\hat{x} \in \hat{\Omega}_e \cap f^{-l_p}(\hat{\Omega}_e) = \hat{\Omega}_e^{l_p}
\]
for every \( p \geq 0 \). Proposition 7.3 then asserts for \( p \) large enough:
\[
\left[ T \wedge \frac{i}{2}dZ^x_{\hat{x}_n} \wedge d\overline{Z}^x_{\hat{x}_n} \right] \left( B_{x_n} \left( \frac{1}{m_0}, e^{-l_{p}\lambda_2 + 3\epsilon} \right) \right) \geq \frac{1}{d^n} e^{-2l_{p}\lambda_1 - 2l_{p}\epsilon} \frac{1}{q_0}.
\]
If \( p \) is also large enough so that \( e^{l_{p}\epsilon} \geq \frac{1}{m_0} \) and \( \frac{1}{q_0} \geq e^{-l_{p}\epsilon} \), we obtain with \( r_p := e^{-l_{p}(\lambda_2 - 4\epsilon)} \):
\[
\left[ T \wedge \frac{i}{2}dZ^x_{\hat{x}_n} \wedge d\overline{Z}^x_{\hat{x}_n} \right] \left( B_{x_n} (r_p) \right) \geq e^{-l_{p}(\log d + 2\lambda_1 + 3\epsilon)} = \frac{1}{r_p \lambda_2 - 4\epsilon}.
\]
Since \( (r_p)_p \) tends to 0 and \( \hat{\nu}(\hat{\Omega}_e) > 0 \), we get
\[
\frac{d_{T,\mathcal{Z}}}{}(\nu) \leq \frac{\log d + 2\lambda_1 + 3\epsilon}{\lambda_2 - 4\epsilon} =: \frac{\log d}{\lambda_2} + \frac{2\lambda_1}{\lambda_2} + O_3(\epsilon).
\]
One can prove
\[
\frac{d_{T,\mathcal{W}}}{}(\nu) \leq \frac{\log d + 2\lambda_2 + 3\epsilon}{\lambda_2 - 4\epsilon} =: \frac{\log d}{\lambda_2} + 2 + O_4(\epsilon)
\]
in a similar way.
7.3 Monge-Ampère mass

We prove Proposition 7.1. Let $x \in \text{Supp}\mu$, let $V$ be a neighbourhood of $x$ and let $Z : V \to \mathbb{C}$ be a holomorphic coordinate (submersion) on $V$. We want to prove that the positive measure $T \wedge \frac{i}{2}dZ \wedge d\overline{Z}$ is not the zero measure on $V$. With no loss of generality, we can assume that $x = (0,0)$, $V = \mathbb{D}(2) \times \mathbb{D}(2)$ and $Z(z, w) = z$. Let also $T = 2i\partial\overline{\partial}G$ on $V$, where $G$ is a continuous psh function. We denote $\sigma_z(u) := (z, u)$.

**Lemma 7.4.** If $(T \wedge \frac{i}{2}dZ \wedge d\overline{Z})(\mathbb{D}(2) \times \mathbb{D}(2)) = 0$, then $G \circ \sigma_z$ is harmonic on $\mathbb{D}$ for every $z \in \mathbb{D}$.

**Proof.** Let $z_0 \in \mathbb{D}$ and let $\varphi \in C_0^\infty(\mathbb{D})$ be a test function. Let $\psi \in C_0^\infty(\mathbb{D}^2)$ such that $\psi \circ \sigma_{z_0} = \varphi$ on $\mathbb{D}$. According to Proposition 9.4, we have

$$\left( T \wedge \frac{i}{2}dZ \wedge d\overline{Z} \right)(\psi) = \int_{z \in \mathbb{D}} \left( \int_{w \in \mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \ d\text{Leb}(w) \right) \ d\text{Leb}(z),$$

which is equal to zero by our assumption. Since the measurable function

$$z \mapsto \int_{w \in \mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \ d\text{Leb}(w)$$

is non negative, there exists $A \subset \mathbb{D}$ such that $\text{Leb}(A) = \text{Leb}(\mathbb{D})$ and

$$\forall z \in A, \quad \int_{w \in \mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \ d\text{Leb}(w) = 0. \quad (7.5)$$

Let us extend this property to every $z \in \mathbb{D}$. Since $A$ is a dense subset of $\mathbb{D}$, there exists a sequence $(z_n)_{n}$ of points in $A$ which converges to $z$. Using $(7.5)$, we get

$$\forall n \geq 1, \quad \int_{w \in \mathbb{D}} (G \circ \sigma_{z_n})(w) \times \Delta(\psi \circ \sigma_{z_n})(w) \ d\text{Leb}(w) = 0. \quad (7.6)$$

Since $G$ is continuous on $\overline{\mathbb{D}^2}$ and $\psi$ is smooth on $\overline{\mathbb{D}^2}$, $G$ and $2i\partial\overline{\partial}\psi$ are uniformly continuous on $\overline{\mathbb{D}^2}$. This implies that $G \circ \sigma_{z_n}$ uniformly converges to $G \circ \sigma_z$ on $\mathbb{D}$ and that $\Delta(\psi \circ \sigma_{z_n})$ uniformly converges to $\Delta(\psi \circ \sigma_z)$ on $\mathbb{D}$. Taking the limits in $(7.6)$, we get

$$\forall z \in \mathbb{D}, \quad \int_{w \in \mathbb{D}} (G \circ \sigma_z)(w) \times \Delta(\psi \circ \sigma_z)(w) \ d\text{Leb}(w) = 0.$$

In particular, we obtain using $\psi \circ \sigma_{z_0} = \varphi$:

$$\int_{w \in \mathbb{D}} (G \circ \sigma_{z_0})(w) \times \Delta\varphi(w) \ d\text{Leb}(w) = 0.$$

This holds for every $\varphi \in C_0^\infty(\mathbb{D})$, hence the function $G \circ \sigma_{z_0}$ is harmonic on $\mathbb{D}$. \hfill $\square$

We prove Theorem 7.5 (Briend). Let $G$ be a continuous psh function on $\mathbb{D}(2) \times \mathbb{D}(2)$. Let $E$ be the set of points $p \in \mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4})$ such that there exists a holomorphic disc $\sigma_p : \mathbb{D} \to \mathbb{D}(2) \times \mathbb{D}(2)$ satisfying

**Theorem 7.5 (Briend).** Let $G$ be a continuous psh function on $\mathbb{D}(2) \times \mathbb{D}(2)$. Let $E$ be the set of points $p \in \mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4})$ such that there exists a holomorphic disc $\sigma_p : \mathbb{D} \to \mathbb{D}(2) \times \mathbb{D}(2)$ satisfying
1. the boundary of \( \sigma_p \) is outside \( \mathbb{D}(\frac{1}{2}) \times \mathbb{D}(\frac{1}{2}) \),

2. \( G \circ \sigma_p \) is harmonic on \( \mathbb{D} \).

Then \( (2i\partial \bar{\partial} G \wedge 2i\partial \bar{\partial} G)(E) = 0 \).

In our situation, \( \mathbb{D}(\frac{1}{2}) \times \mathbb{D}(\frac{1}{2}) = E \) since one can take for \( \sigma_p \) the discs \( \sigma_z : \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{D}, u \mapsto (z, u) \). Indeed, the boundary of \( \sigma_z \) is contained in \( \{ z \} \times \partial \mathbb{D} \) and \( G \circ \sigma_z \) is harmonic on \( \mathbb{D} \) according to Lemma 7.4. Theorem 7.5 then gives:

\[
(2i\partial \bar{\partial} G \wedge 2i\partial \bar{\partial} G)(\mathbb{D}(\frac{1}{4}) \times \mathbb{D}(\frac{1}{4})) = 0,
\]
which contradicts \( x = 0 \in \text{Supp} \mu = \text{Supp}(2i\partial \bar{\partial} G \wedge 2i\partial \bar{\partial} G) \).

## 8 Upper bound for the dimension of dilating measures

We prove Theorem 1.3. We shall take the proof of Theorem 5.2 and use Proposition 7.3. Let \( \epsilon > 0 \) and let \( \hat{\Lambda}_n \) and \( \hat{\Delta}_n \) be the sets defined in Section 3.3. The set \( \hat{\Omega}_n \) has been defined in Section 7.1; it satisfies \( \nu(\hat{\Omega}_n) \geq 1 - 2\delta \). Hence we have \( \nu(\hat{\Lambda}_n \cap \hat{\Delta}_n \cap \hat{\Omega}_n) \geq 1 - 3\delta \) for every \( n \geq N \). Now let \( K_n \) be the unique integer satisfying

\[
\eta_1 e^{-n\lambda_1 - 4\epsilon} e^{-\lambda_2 + 3\epsilon} \leq \frac{1}{m_0} e^{-K_n \lambda_2 + 3K_n \epsilon} \leq \eta_1 e^{-n\lambda_1 - 4\epsilon}.
\]

so that \( K_n \approx n\lambda_1 / \lambda_2 \). Let \( \{ x_1, \ldots, x_{N_{n,2}} \} \) be a \((n, \eta_1/4)\)-separated subset of \( A_n := \pi_0(\hat{\Lambda}_n \cap \hat{\Delta}_n \cap \hat{\Omega}_n) \) provided by Lemma 4.2. We have for every \( n \geq \max\{ N_e, n_1 - 3\delta \} \):

\[
N_{n,2} \geq \nu(A_n) e^{nh_\nu - 2\epsilon} \geq (1 - 3\delta) e^{nh_\nu - 2\epsilon}.
\]

Then, for every \( x_i \), let \( \{ y_{i,1}^j, \ldots, y_{M_n}^j \} \) be a \( 2 \epsilon^{-n\lambda_1 - 4\epsilon} \)-separated subset of \( B_n(x_i, \eta_1/4) \cap A_n \) provided by Lemma 4.3. The cardinality of this set satisfies:

\[
M_n \geq e^{-nh_\nu - 2\epsilon} \left( \frac{1}{2m_0} e^{n\lambda_1 + 4\epsilon} \right)^{\frac{d - \epsilon}{e}}.
\]

For every \( j \in \{ 1, \ldots, M_n \} \), we set \( \hat{y}_j \in \hat{\Lambda}_n \cap \hat{\Delta}_n \cap \hat{\Omega}_n \) such that \( y_j = \pi_0(\hat{y}_j) \). Then we follow the proof of Theorem 5.2 until the inequality (5.8):

\[
d^n \geq \sum_{i=1}^{N_{n,2}} \sum_{j=1}^{M_n} e^{2n\lambda_1 - 4\epsilon} \left[ T \wedge \left( \frac{i}{2} d\hat{Z}_{\hat{y}_j} \wedge d\bar{\hat{Z}}_{\hat{y}_j} \right) \right] (B_{\hat{y}_j} (\eta_1 e^{-n\lambda_1 - 4\epsilon})).
\]

We want to apply Proposition 7.3. According to (8.1),

\[
B_{\hat{y}_j} (\eta_1 e^{-n\lambda_1 - 4\epsilon}) \supset B_{\hat{y}_j} \left( \frac{1}{m_0} e^{-K_n \lambda_2 + 3K_n \epsilon} \right).
\]
We apply the positive measure $T \wedge \left( \frac{1}{2} dZ_{y_j}^t \wedge dZ_{y_j}^s \right)$ to this inclusion. Since $y_j \in \tilde{\Omega}_{d}^{K_n}$, we deduce from Proposition 7.3 that for every $n$ satisfying $n \geq N_{\epsilon}$ and $K_n \geq N_{\epsilon}$:

$$T \wedge \left( \frac{1}{2} dZ_{y_j}^t \wedge dZ_{y_j}^s \right) \left( B_{y_j}^t \eta_1 e^{-n\lambda_1 - 4n\epsilon} \right) \geq \frac{1}{dK_n} e^{-2K_n \lambda_1 - 2K_n \epsilon} \frac{1}{q_0}.$$  

We infer from (8.4) that for every $n$ satisfying $n \geq N_{\epsilon}$ and $K_n \geq N_{\epsilon}$,

$$d^n \geq N_{n,2} \cdot M_n \cdot e^{2n\lambda_1 - 4n\epsilon} \cdot \frac{1}{dK_n} e^{-2K_n \lambda_1 - 2K_n \epsilon} \frac{1}{q_0}. \quad (8.5)$$

Now we use the upper bounds for $N_{n,2}$ and $M_n$ given by (8.2) and (8.3):

$$d^{n+K_n} \geq (1 - 3\delta)e^{nh - 2n\epsilon} \cdot e^{-nh - 2n\epsilon} \left( \frac{1}{2\eta_1} e^{n\lambda_1 + 4n\epsilon} \right) d_n - \epsilon \cdot e^{2n\lambda_1 - 4n\epsilon} e^{-2K_n \lambda_1 - 2K_n \epsilon} \frac{1}{q_0}.$$  

If $C_1(\epsilon) := \frac{K_n}{n} / q_0 (2\eta_1)^{\frac{d - \epsilon}{\lambda_1}}$, we get:

$$\log d + \frac{\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon} \log d \geq \frac{1}{n} \log C_1(\epsilon) - 8\epsilon + (\lambda_1 + 4\epsilon)(\frac{d_n}{\lambda_1} - \epsilon) + 2\lambda_1 - \frac{2K_n}{n}(\lambda_1 + \epsilon).$$

By using (8.1), we have

$$\log d + \frac{\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon} \log d \geq \frac{1}{n} \log C_2(\epsilon) - 8\epsilon + (\lambda_1 + 4\epsilon)(\frac{d_n}{\lambda_1} - \epsilon) + 2\lambda_1 - \frac{2\lambda_1 + 4\epsilon}{\lambda_2 - 3\epsilon} (\lambda_1 + \epsilon),$$

where $C_2(\epsilon)$ is another constant. Letting $n$ tend to $+\infty$ and then $\epsilon$ to $0$, we get

$$d_n \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2 \left( \frac{\lambda_1}{\lambda_2} - 1 \right).$$

To obtain the other upper bound, we use the analogue of (8.4) for $W$. Applying Proposition 7.3 with respect to $W$, we obtain instead of (8.5):

$$d^n \geq N_{n,2} \cdot M_n \cdot e^{2n\lambda_2 - 4n\epsilon} \cdot \frac{1}{dK_n} e^{-2K_n \lambda_2 - 2K_n \epsilon} \frac{1}{q_0}.$$  

Then we get

$$d_n \leq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2} + 2 \left( 1 - \frac{\lambda_2}{\lambda_1} \right),$$

which completes the proof of Theorem 1.5.

9 Appendix

9.1 Dimension of measures

Proposition 9.1. Let $\nu_1$ and $\nu_2$ be two probability measures on $\mathbb{P}^2$ such that $\nu_1 \ll \nu_2$. Then for $\nu_1$-almost every $x \in \mathbb{P}^2$, we have:

$$d_{\nu_1}(x) \geq d_{\nu_2}(x) \quad \text{and} \quad d_{\nu_1}(x) \geq d_{\nu_2}(x).$$
Proof. Let \( \varphi \in L^1(\nu_2) \) such that \( \nu_1(A) = \int_A \varphi \nu_2 \) for every Borel set \( A \) of \( \mathbb{P}^2 \). Using the dominated convergence Theorem,
\[
\lim_{M \to -\infty} \int_{\mathbb{P}^2} 1_{\{\varphi \leq M\}} \varphi \, d\nu_2 = \int_{\mathbb{P}^2} \varphi \, d\nu_2 = 1.
\]
For every \( n \geq 1 \), we let \( M_n \) satisfy \( \int_{\mathbb{P}^2} 1_{\{\varphi \leq M_n\}} \varphi \, d\nu_2 \geq 1 - \frac{1}{n} \). By the Lebesgue density Theorem, for \( \nu_1 \)-almost every \( x \) in \( \{ \varphi \leq M_n \} \), we have
\[
\lim_{r \to 0} \frac{\nu_1(B_x(r) \cap \{ \varphi \leq M_n \})}{\nu_1(B_x(r))} = 1.
\]
Then for every \( r \) small enough, we have
\[
\frac{1}{2} \nu_1(B_x(r)) \leq \nu_1(B_x(r) \cap \{ \varphi \leq M_n \}) = \int_{B_x(r) \cap \{ \varphi \leq M_n \}} \varphi \, d\nu_2 \leq M_n \int_{B_x(r)} d\nu_2.
\]
And thus \( \nu_1(B_x(r)) \leq 2M_n \nu_2(B_x(r)) \). We deduce that
\[
d\nu_1(x) \geq d\nu_2(x)
\]
for \( \nu_1 \)-almost every \( x \in \{ \varphi \leq M_n \} \). We end with \( \nu_1(\cup_{n \in \mathbb{N}} \{ \varphi \leq M_n \}) = 1. \square
\]

Now we take the notations of Section 1.1.

**Proposition 9.2.** Let \( S \) be a \((1,1)\)-closed positive current on \( \mathbb{P}^2 \). Let \( x \in \mathbb{P}^2 \) and let \((Z,W)\) be holomorphic coordinates near \( x \). Then
\[
d_S(x) = \min \left\{ d_{S,Z}(x), d_{S,W}(x) \right\} , \quad d_S(x) = \min \left\{ d_{S,Z}(x), d_{T,W}(x) \right\} .
\]

**Proof.** Let us set \( \sigma_{S,Z} = S \wedge \left( \frac{i}{2}dz \wedge d\overline{z} \right) \) and \( \sigma_{S,W} = S \wedge \left( \frac{i}{2}dW \wedge d\overline{W} \right) \). There exists \( c > 0 \) such that \( \frac{1}{c}(\sigma_{S,Z} + \sigma_{S,W}) \leq \sigma_S \leq c(\sigma_{S,Z} + \sigma_{S,W}) \) on a neighbourhood of \( x \), see [13] Chapter III, §3. We deduce for every \( r \) small enough
\[
\frac{1}{c} \max \{ \sigma_{S,Z}(B_x(r)), \sigma_{S,W}(B_x(r)) \} = \sigma_S(B_x(r)) \leq 2c \max \{ \sigma_{S,Z}(B_x(r)), \sigma_{S,W}(B_x(r)) \} .
\]

We finish by observing that the local dimension of the maximum of two measures is equal to the minimum of these two dimensions, since one divides by \( \log r \) which is negative. \( \square \)

### 9.2 Cohomology and slices

We refer to Sections 1.2 and A.3 of Dinh-Sibony’s book [16].

**Proposition 9.3.** Let \( S \) be a \((1,1)\)-closed positive current of \( \mathbb{P}^2 \) of mass 1. Let \( \omega \) be the Fubini-Study form on \( \mathbb{P}^2 \) and let \( f : \mathbb{P}^2 \to \mathbb{P}^2 \) be an endomorphism of degree \( d \). Then,
\[
\int_{\mathbb{P}^2} (f^n)^* S \wedge \omega = \int_{\mathbb{P}^2} S \wedge (f^n)^* \omega = d^n.
\]
Proof. The first equality comes from the definition of duality. We show the second one. By using \( f^* \omega = d \cdot \omega + 2i \partial \bar{\partial} u \), where \( u \) is a smooth function on \( \mathbb{P}^2 \), we obtain by induction

\[
(f^n)^* \omega = d^n \omega + 2i \partial \bar{\partial} v_n,
\]

where \( v_n := (d^{n-1} \cdot u + \cdots + d \cdot u \circ f^{n-2} + u \circ f^{n-1}) \). Hence

\[
\int_{\mathbb{P}^2} S \wedge (f^n)^* \omega = \int_{\mathbb{P}^2} S \wedge \left( d^n \omega + 2i \partial \bar{\partial} v_n \right).
\]

Since \( S \) is a closed current of mass 1, we have \( \int_{\mathbb{P}^2} S \wedge 2i \partial \bar{\partial} v_n = 0 \) and \( \int_{\mathbb{P}^2} S \wedge d^n \omega = d^n \).

**Proposition 9.4.** Let \( G \) be a continuous psh function on \( \mathbb{D}^2 \) and let \( \mathcal{S} = 2i \partial \bar{\partial} G \). Let \((Z,W)\) be the coordinates on \( \mathbb{D}^2 \) and let \( \phi \in C^\infty_0(\mathbb{D}^2) \). Then

\[
\mathcal{S} \wedge \frac{i}{2} dZ \wedge d\overline{Z}(\phi) = 2i \partial \bar{\partial} G(\phi) \frac{i}{2} dZ \wedge d\overline{Z} = \int_{\mathbb{D}^2} G.2i \partial \bar{\partial} (\phi \times \frac{i}{2} dZ \wedge d\overline{Z}).
\]

The computation

\[
2i \partial \bar{\partial} (\phi \times \frac{i}{2} dZ \wedge d\overline{Z}) = 4(\frac{\partial^2 \phi}{\partial w \partial \overline{w}}) \frac{i}{2} dW \wedge d\overline{W} \wedge \frac{i}{2} dZ \wedge d\overline{Z} = 4(\frac{\partial^2 \phi}{\partial w \partial \overline{w}}) \ d\text{Leb}(z,w)
\]

allows to write

\[
S \wedge \frac{i}{2} dZ \wedge d\overline{Z}(\phi) = \int_{(z,w) \in \mathbb{D}^2} G(z,w) \times 4 \frac{\partial^2 \phi}{\partial w \partial \overline{w}}(z,w) \ d\text{Leb}(z,w)
\]

\[
= \int_{z \in \mathbb{D}} \left( \int_{w \in \mathbb{D}} G_z(w) \times \Delta \phi_z(w) \ d\text{Leb}(w) \right) \ d\text{Leb}(z).
\]

Finally, the quantity in brackets is equal to \((\Delta G_z)(\phi_z) = (\sigma_z^* \mathcal{S})(\phi_z)\). \( \square \)

**References**

[1] F. Berteloot and C. Dupont. Une caractérisation des endomorphismes de Lattès par leur mesure de Green. *Comment. Math. Helv.*, 80(2):433–454, 2005.

[2] F. Berteloot, C. Dupont, and L. Molino. Normalization of bundle holomorphic contractions and applications to dynamics. *Ann. Inst. Fourier (Grenoble)*, 58(6):2137–2168, 2008.

[3] F. Berteloot and J.-J. Loeb. Une caractérisation géométrique des exemples de Lattès de \( \mathbb{P}^k \). *Bull. Soc. Math. France*, 129(2):175–188, 2001.
[4] I. Binder and L. DeMarco. Dimension of pluriharmonic measure and polynomial endomorphisms of $\mathbb{C}^n$. *Int. Math. Res. Not.*, 2003(11):613–625, 2003.

[5] J.-Y. Briend. *Exposants de Liapounoff et points périodiques d’endomorphismes holomorphes de $\mathbb{C}P(k)$.* PhD thesis, Université Paul Sabatier de Toulouse, 1997.

[6] J.-Y. Briend and J. Duval. Exposants de Liapounoff et distribution des points périodiques d’un endomorphisme de $\mathbb{C}P^k$. *Acta Math.*, 182(2):143–157, 1999.

[7] J.-Y. Briend and J. Duval. Deux caractérisations de la mesure d’équilibre d’un endomorphisme de $\mathbb{P}^k(\mathbb{C})$. *Publ. Math. Inst. Hautes Études Sci.*, 2001(93):145–159, 2001.

[8] M. Brin and A. Katok. On local entropy. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 30–38. Springer, Berlin, 1983.

[9] H. de Thélin. Minoration de la dimension de haudorff du courant de Green. https://arxiv.org/abs/1709.01356.

[10] H. de Thélin. Un phénomène de concentration de genre. *Math. Ann.*, 332(3):483–498, 2005.

[11] H. de Thélin. Sur les exposants de Lyapounov des applications méromorphes. *Invent. Math.*, 172(1):89–116, 2008.

[12] H. de Thélin and G. Vigny. On the measures of large entropy on a positive closed current. *Math. Z.*, 280(3-4):919–944, 2015.

[13] J.-P. Demailly. *Complex Analytic and Differential Geometry.* En ligne, 2012. disponible à https://www-fourier.ujf-grenoble.fr/de-mailly/manuscripts/agbook.pdf.

[14] T.-C. Dinh. Attracting current and equilibrium measure for attractors on $\mathbb{P}^k$. *J. Geom. Anal.*, 17(2):227–244, 2007.

[15] T.-C. Dinh and C. Dupont. Dimension de la mesure d’équilibre d’applications méromorphes. *J. Geom. Anal.*, 14(4):613–627, 2004.

[16] T.-C. Dinh and N. Sibony. *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, volume 1998 of *Lecture Notes in Math.*, pages 165–294. Springer, Berlin, 2010.

[17] R. Dujardin. Fatou directions along the Julia set for endomorphisms of $\mathbb{C}P^k$. *J. Math. Pures Appl. (9)*, 98(6):591–615, 2012.

[18] C. Dupont. Exemples de Lattès et domaines faiblement sphériques de $\mathbb{C}^n$. *Manuscripta Math.*, 111(3):357–378, 2003.
[19] C. Dupont. On the dimension of invariant measures of endomorphisms of $\mathbb{C}P(k)$. *Math. Ann.*, 349(3):509–528, 2011.

[20] C. Dupont. Large entropy measures for endomorphisms of $\mathbb{C}P^k$. *Israel J. Math.*, 192(2):505–533, 2012.

[21] M. Jonsson and D. Varolin. Stable manifolds of holomorphic diffeomorphisms. *Invent. Math.*, 149:409–430, August 2002.

[22] R. Mañé. The Hausdorff dimension of invariant probabilities of rational maps. In *Dynamical systems, Valparaiso 1986*, volume 1331 of *Lecture Notes in Math.*, pages 86–117. Springer, Berlin, 1988.

[23] Y. B. Pesin. *Dimension theory in dynamical systems*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997. Contemporary views and applications.

[24] N. Sibony. Dynamique des applications rationnelles de $\mathbb{P}^k$. In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.

[25] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[26] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynamical Systems*, 2(1):109–124, 1982.

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