ON THE MODIFIED $J$-EQUATION

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Abstract. In this paper, we study the modified $J$-equation introduced by Li–Shi [LS16]. We show that the solvability of the modified $J$-equation is equivalent to the coercivity of the modified $J$-functional on general compact Kähler manifolds. On toric manifolds we give a numerical necessary and sufficient condition for the existence of solutions, extending the results of Collins–Székelyhidi [CS17].

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1. Introduction

Let $(X, \hat{\chi})$ be an $n$-dimensional compact Kähler manifold and $T \subset \text{Aut}_0(X)$ a real torus. Assume that $T$-action on $(X, \hat{\chi})$ is Hamiltonian with moment map $\hat{\mu}: X \to \mathfrak{t} := \text{Lie}(T)$. Let $\omega$ be another $T$-invariant Kähler form, and denote the Kähler classes of $\omega$, Date: July 12, 2022.

2020 Mathematics Subject Classification. Primary 53C55; Secondary 35A01.

Key words and phrases. $J$-equation, Nakai–Moishezon criterion.

arXiv:2207.04953v1 [math.DG] 11 Jul 2022
\( \hat{\chi} \) by \( \alpha, \beta \) respectively. Let \( \mathcal{H}^T \) be the space of \( T \)-invariant \( \hat{\chi} \)-Kähler potentials

\[
\mathcal{H}^T := \{ \phi \in C^\infty(X; \mathbb{R})^T | \chi_\phi := \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi > 0 \}.
\]

Then we observe that for any \( \phi \in \mathcal{H}^T \) the \( T \)-action on \((X, \chi_\phi)\) is also Hamiltonian with moment map

\[
\mu_\phi = \hat{\mu} + d\phi \circ J,
\]

where \( J \) denotes the complex structure of \( X \). Then the complexified torus \( T^c \) also acts on \( X \) in a standard manner. If \( \hat{\chi} \in c_1(L) \) for some ample line bundle \( L \to X \) this is equivalent to say that the \( T \)-action (and \( T^c \)-action as well) has a lift to \( L \). Let \( v \) be a holomorphic vector field with \( \text{Im}(v) \in t \). We define the Hamiltonian \( \theta_v^X(\chi_\phi) \) as a real-valued function uniquely determined by the properties

\[
i_v \chi_\phi = \frac{\sqrt{-1}}{2\pi} \theta_v^X(\chi_\phi), \quad \int_X \theta_v^X(\chi_\phi) \chi_\phi^n = 0.
\]

Or equivalently, the function \( \theta_v^X(\chi_\phi) \) coincides with the natural paring \( \langle \mu_\phi, \text{Im}(v) \rangle \) up to additive constants, and hence is \( T \)-invariant.

Finding canonical metrics has been the main focus of Kähler geometry. Especially several equations including Hamiltonians have been studied extensively in the last few decades (e.g. [BN14, Li19, Mab03, TZ00, Zhu00]). In this paper, we study the modified \( J \)-equation introduced by Li–Shi [LS16]

\[
\omega \wedge \chi^{n-1}_\phi = \frac{1}{n} (c_X + \theta_v^X(\chi_\phi)) \chi^n_\phi,
\]

(1.1)

where the constant \( c_X \) only depends on \( \alpha, \beta \), and determined by

\[
c_X := \frac{n\alpha \cdot \beta^{n-1}}{\beta^n}.
\]

In particular, if \( T \) is trivial then the equation (1.1) matches up with the \( J \)-equation introduced by Donaldson [Don03] and Chen [Che00]. It was shown by Li–Shi [LS16] that such a \( \phi \in \mathcal{H}^T \) exists if and only if there exists \( \phi \in \mathcal{H}^T \) such that

\[
(c_X + \theta_v^X(\chi_\phi)) \chi^{n-1}_\phi - (n-1)\omega \wedge \chi^{n-2}_\phi > 0,
\]

(1.2)

where the positivity of \((n-1, n-1)\)-forms are understood in the sense of [Dem12, Chapter III, Section 1.A]. The solutions to the equation (1.1) are characterized as the critical points of a strictly convex function on \( \mathcal{H}^T \), called the modified \( J \)-functional \( J_{T, \omega} \). The motivation for (1.1) comes from the decomposition formula of the modified \( K \)-energy which plays a crucial role in the study of extremal Kähler metrics (see [LS16] for more details). For later arguments, it is convenient to extend (1.1) to a more general equation of the form:

\[
\text{Tr}_\omega \chi_\phi + b \frac{\omega^n}{\chi_\phi^n} = c + \theta_v^X(\chi_\phi),
\]

(1.3)

where \( c > 0, b \) are constants related by each other by

\[
b = \frac{c\beta^n - n\alpha \cdot \beta^{n-1}}{\alpha^n}.
\]

(1.4)
In most cases, we assume that $b \geq 0$, but sometimes allow $b$ to be slightly negative. First, we extend the result [LS16, Theorem 3.3] to the generalized equation (1.3) as follows:

**Theorem 1.1.** Let $X$ be a compact complex manifold, and $T, v, \omega, \widehat{\chi}$ as above. Then the generalized equation (1.3) with $b \geq 0$ admits a solution if and only if there exists $\phi \in \mathcal{H}^T$ such that

$$
(c + \theta^X_v(\chi_\phi))\chi^{n-1}_\phi - (n - 1)\omega \wedge \chi^{n-2}_\phi > 0.
$$

In Kähler geometry, a fundamental question is whether the solvability of an equation can be related to the properness or coercivity of a certain energy functional. For instance, Tian’s properness conjecture [Tia94] predicted that the existence of Kähler–Einstein metrics is equivalent to the properness of the Mabuchi K-energy. When the manifold has trivial automorphism group, Ding–Tian [DT92] proved one direction (properness $\Rightarrow$ existence), and Tian [Tia97] confirms a converse direction (existence $\Rightarrow$ properness). In general case, Darvas–Rubinstein [DR16] proposed a refined version of Tian’s properness conjecture and proved it by exploiting a general framework in Finsler geometry. For constant scalar curvature Kähler metrics, a version of Tian’s properness conjecture was proved by Berman–Darvas–Lu [BDL20] and Chen–Cheng [CC17, CC18, CC21]. For complex Hessian equations, Collins–Székelyhidi [CS17] obtained a similar result for the $J$-equation, and Chu–Lee [CL21] for the hypercritical deformed Hermitian–Yang–Mills equation.

Motivated by above works, we show the equivalence between the existence and coercivity for the modified $J$-equation:

**Theorem 1.2.** Let $X$ be a compact complex manifold, and $T, v, \omega, \widehat{\chi}$ as above. Then the modified $J$-equation (1.1) admits a solution $\phi \in \mathcal{H}^T$ if and only if the modified $J$-functional $J_{T,\omega}$ is coercive.

For the precise definition of $J_{T,\omega}$ and coercivity, see Section 2. To prove the above theorem, we use the local smoothing and gluing method established in [Che21, CS17]. A new feature arising in the modified case is that the local smoothing argument requires a uniform Dirichlet energy bounds for Kähler potentials (Lemma 4.2). Also we have to take care of the problem that holomorphic coordinates used in the local smoothing are not compatible with the $T$-action, so we can never expect that the resulting potential function is $T$-invariant.

In practice, it seems to be difficult to produce $\phi \in \mathcal{H}^T$ satisfying (1.2). In particular, it is not clear whether the existence of a solution to the equation depends only on the classes $\alpha, \beta$. The next theorem settles this question.

**Theorem 1.3.** Let $X$ be a compact complex manifold, and $T, v, \omega, \widehat{\chi}$ as above. Suppose that there exists a $T$-invariant Kähler form $\chi \in \beta$ satisfying (1.1) with respect to $\omega$ and $\omega' \in \alpha$ is another Kähler form. Then there exists $\chi' \in \beta$ that solves (1.1) with respect to $\omega'$.

In the same split as Demailly–Păun [DP04] (a generalization of the well-known Nakai–Moishezon criterion) we propose the following conjecture for when the equation can be solved.
Conjecture 1.4. Let $X$ be a compact complex manifold, and $T$, $v$, $\omega$, $\hat{\chi}$ as above. Then there exists $\phi \in \mathcal{H}^T$ satisfying (1.1) if and only if
\[ c_X + \min_X \theta^X_v(\chi) > 0, \tag{1.5} \]
and for all $T$-invariant irreducible subvarieties $Y \subset X$ with $p = 1, \ldots, n - 1$ we have
\[ \int_Y \left( (c_X + \theta^X_v(\chi)) \chi^p - p\omega \wedge \chi^{p-1} \right) > 0, \tag{1.6} \]
where $\chi \in \beta$ is any $T$-invariant Kähler form.

Remark 1.5. The conditions (1.5) and (1.6) are independent of the choice of $\chi \in \beta$. Indeed, the term including the Hamiltonian $\theta^X_v(\chi)$ can be rewritten as
\[ \int_Y \theta^X_v(\chi) \chi^p = \frac{1}{p+1} \int_Y (\chi + \theta^X_v(\chi))^{p+1}. \]
Set $d_v := d - 4\pi i \text{Im}(v)$ so that $\chi + \theta^X_v(\chi)$ is $d_v$-closed. If we take another $T$-invariant Kähler form $\chi' + \frac{1}{2\pi} \partial \bar{\partial} \phi$ then a direct computation shows that
\[ \chi' + \theta^X_v(\chi') = \chi + \theta^X_v(\chi) + \frac{1}{4\pi} d_v d_c \phi \]
where $d_c := \sqrt{-1}(\bar{\partial} - \partial)$. So the required statement follows from the equivariant Chern–Weil theory.

When $T$ is trivial, Conjecture 1.4 has been confirmed by Song [Son20], building on a uniform stability result obtained by Chen [Che21]. In Chen’s proof, a crucial step is to construct a solution as a Kähler current whose local smoothing in Euclidean balls with sufficiently small scale satisfies the subsolution condition. However, the Kähler current could have non-trivial Lelong numbers, and hence have infinite Monge–Ampère energy. On the other hand, in the modified $J$-equation case the proof requires a uniform Dirichlet energy bound as mentioned above, so it seems to be hard to apply Chen’s method directly in our case. However, when $X$ is toric we can write down the equation (1.1) in terms of convex functions and its Legendre transform on the complement of the toric divisors $D$. From this point of view, we can establish the local $C^{\ell,\gamma}$-estimates on $X \setminus D$, and proceed the inductive step for dimension of toric subvarieties. Eventually we can obtain the following:

Theorem 1.6. Let $X$ be a compact toric manifold with the standard torus $T$, and $v$, $\omega$, $\hat{\chi}$ as above. Then there exists a solution $\phi \in \mathcal{H}^T$ to the modified $J$-equation (1.1) if and only if
\[ c_X + \min_X \theta^X_v(\chi) > 0, \]
and for all irreducible toric subvarieties $Y \subset X$ with $p = 1, \ldots, n - 1$ we have
\[ \int_Y \left( (c_X + \theta^X_v(\chi)) \chi^p - p\omega \wedge \chi^{p-1} \right) > 0, \]
where $\chi \in \beta$ is any $T$-invariant Kähler form.
We remark that when $v$ is zero Theorem 1.2, Theorem 1.3 and Theorem 1.6 have been obtained in [CS17].

The organization of this paper is as follows. In Section 2 we collect some fundamental properties that will be used throughout this paper. In Section 3 we establish the long-time existence of the modified $J$-flow. In particular we prove the convergence of the flow when $b \geq 0$, which gives a proof of Theorem 1.1. In Section 4 we will show that the existence of solutions is equivalent to the coercivity of the modified $J$-functional $J_{\rho, \omega}$ (Theorem 1.2). As a corollary a proof of Theorem 1.3 is also given. In Section 5 we establish a Nakai–Moishezon type criterion for compact toric manifolds, which gives our main result, Theorem 1.6.

**Acknowledgment.** The author was supported by Grant-in-Aid for Early-Career Scientists (20K14308) from JSPS.

2. Preliminaries

2.1. Hamiltonians. Let $(X, \widehat{\chi})$ be a compact Kähler manifold with a Hamiltonian $T$-action. Take a $T$-invariant reference Kähler form $\widehat{\chi} \in \beta$. We start with the following fact.

**Lemma 2.1** ([Zhu00], Corollary 5.3). For any $\phi \in \mathcal{H}^T$ we have $\theta^X_v(\chi_\phi) = \theta^X_v(\widehat{\chi}) + v(\phi)$. In particular, $\max_X \theta_v(\chi_\phi)$ as well as $\min_X \theta_v(\chi_\phi)$ is independent of the choice of $\phi \in \mathcal{H}^T$. 

**Proof.** Applying $i_v$ to $\chi_\phi = \widehat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$ we observe that $\theta^X_v(\chi_\phi) = \theta^X_v(\widehat{\chi}) + v(\phi)$ modulo additive constants. Thus

$$\theta^X_v(t \chi_\phi) = \theta^X_v(\widehat{\chi}) + tv(\phi) + C_t, \quad t \in [0, 1]$$

for some $t$-dependent constant $C_t$. Differentiating $\int_X \theta^X_v(t \chi_\phi) \chi^n_{t \phi} = 0$ in $t$ and integrating by parts we get

$$0 = \frac{d}{dt} \int_X \theta^X_v(\chi_{t \phi}) \chi^n_{t \phi} = \int_X (v(\phi) + \frac{d}{dt} C_t + \theta^X_v(\chi_{t \phi}) \Delta_{t \phi} \phi) \chi^n_{t \phi} = \beta^n \frac{d}{dt} C_t.$$

Thus we have $C_t = C_0 = 0$ which shows the first assertion. For the second assertion we take a point $x_0 \in X$ where $\theta_v(\chi_\phi)$ achieves the maximum. Then $\partial \theta_v(\chi_\phi)(x_0) = 0$, so the vector $v$ vanishes at $x_0$. Combining with $\theta^X_v(\chi_\phi) = \theta^X_v(\widehat{\chi}) + v(\phi)$ we observe that

$$\max_X \theta^X_v(\chi_\phi) = \theta^X_v(\chi_\phi)(x_0) = \theta^X_v(\widehat{\chi})(x_0) \leq \max_X \theta^X_v(\widehat{\chi}).$$

By changing the role of $\widehat{\chi}$ and $\chi_\phi$ we obtain $\max_X \theta^X_v(\chi_\phi) = \max_X \theta^X_v(\widehat{\chi})$. The proof for $\min_X \theta^X_v(\chi_\phi)$ is similar. □

From the above lemma we know that

$$m_X := c_X + \min_X \theta^X_v$$

is independent of a choice of $\phi \in \mathcal{H}^T$. From the equation (1.1) we can easily see that

**Lemma 2.2.** If there exists $\phi \in \mathcal{H}^T$ satisfying the modified $J$-equation (1.1) then $m_X > 0$. 

We fix a uniform constant $C_\theta > 0$ (independent of $\phi \in \mathcal{H}^T$) such that $|\theta^X_\phi(\chi_\phi)| \leq C_\theta$.

2.2. Convexity properties. Let $\Gamma \subset \mathbb{R}^n$ be the positive orthant. For any constant $b \in \mathbb{R}$ we define
\[
f_b(\lambda) := \sum_{i=1}^{n} \frac{1}{\lambda_i} + \frac{b}{\lambda_1 \cdots \lambda_n}, \quad \lambda \in \Gamma.
\]
Let $\mathcal{M}$ be the space of all $n \times n$ positive definite Hermitian matrices. Set
\[
F_b(A) := f_b(\lambda),
\]
where $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of $A$ (up to ordering). Then at a diagonal matrix $A \in \mathcal{M}$ with eigenvalues $\lambda_i$, the derivative of $F_b$ is given by
\[
\partial_{ij} F_b = \partial_i f_b \cdot \delta_{ij}, \quad \partial_{ij} \partial_{rs} F_b = \partial_i \partial_r f_b \cdot \delta_{ij} \delta_{rs} + \frac{\partial_i f_b - \partial_j f_b}{\lambda_i - \lambda_j} (1 - \delta_{ij}) \delta_{ir} \delta_{js}, \quad (2.1)
\]
where we will regard the quotient appeared in the last term as the limit when $\lambda_i = \lambda_j$ [And94, Ger96]. For Kähler forms $\omega$, $\chi$ defined in an open subset $\Omega$ of a compact complex manifold $X$, we often write $g$, $h$ as the Riemannian metric corresponding to $\omega$, $\chi$ respectively, i.e.
\[
\omega = \sqrt{-1} \sum_{i,j} g_{ij} dz^i \wedge \bar{dz}^j, \quad \chi = \sqrt{-1} \sum_{i,j} h_{ij} dz^i \wedge \bar{dz}^j
\]
in local coordinates. We define the endomorphism $A$ on $T^{1,0}X$ by $A^i_j := g^{ik} h_{jk}$, and set
\[
F_{\omega,b}(\chi) := F_b(A), \quad P_\omega(\chi) := \max_{k=1, \ldots, n} \sum_{i \neq k} \frac{1}{\lambda_k}, \quad Q_\omega(\chi) := F_{\omega,0}(\chi) = \sum_{i=1}^{n} \frac{1}{\lambda_i},
\]
where $\lambda_i$ denotes the eigenvalues of $A$. For a submanifold $Y \subset X$, we consider the restriction of the operator $Q_\omega$ to $Y$:
\[
Q_{Y,\omega}(\chi) := Q_{\omega|Y}(\chi|_Y).
\]
Then we will have:

**Proposition 2.3.** The inequality
\[
Q_{Y,\omega}(\chi) \leq Q_{X,\omega}(\chi)
\]
holds on $\Omega \cap Y$.

**Proof.** Indeed, the required statement follows from the Courant–Fischer–Weyl type min-max principle [Che21, Section 3] when $\dim Y = n - 1$. However one can show the similar formula for arbitrary dimension $\dim Y = k$, by using the convexity property for $\max I \sum_{i \in I} \frac{1}{\lambda_i}$ and the Schur–Horn theorem, where $I$ runs over all subsets $I \subset \{1, \ldots, n\}$ of cardinality $k$. For instance, see [CLT21, Section 2] in deformed Hermitian–Yang–Mills equation case (the same argument also works for the $J$-equation).
2.2.1. The $b = -\varepsilon \leq 0$ case. First we consider the case $b = -\varepsilon \leq 0$. For $K \in (0, \infty]$ we set $\Gamma_K := \{ \lambda \in \Gamma | f_0(\lambda) < K \}$ and $\mathcal{M}_K := \{ A \in \mathcal{M} | F_0(A) < K \}$.

**Proposition 2.4.** For any $K \in (0, \infty]$ there exists $\varepsilon_1 = \varepsilon_1(K) > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, the function $f_{-\varepsilon} : \Gamma_K \to \mathbb{R}$ satisfies the following properties:

1. $f_{-\varepsilon} > 0$.
2. $\partial_i f_{-\varepsilon} < 0$ for all $i$.
3. If $\lambda_i \geq \lambda_j$, then $\partial_i f_{-\varepsilon} \geq \partial_j f_{-\varepsilon}$.
4. $f_{-\varepsilon}$ is convex.

**Proof.** For notational convenience let

$$S_k(\lambda) := \sum_{1 \leq j_1 \leq \ldots \leq j_n \leq \lambda} \lambda_{j_1} \ldots \lambda_{j_n}$$

be the elementary symmetric function of degree $k$, and

$$S_{n-1;i}(\lambda) := \frac{S_n(\lambda)}{\lambda_i}.$$

Since $\lambda_i > K^{-1}$ from the condition $f_0(\lambda) < K$ we have

$$f_{-\varepsilon}(\lambda) = \frac{S_{n-1}(\lambda) - \varepsilon}{S_n(\lambda)} \geq \frac{(n - 1)K^{-n+1} - \varepsilon}{S_n(\lambda)},$$

$$\partial_i f_{-\varepsilon}(\lambda) = \frac{-S_{n-1;i}(\lambda) + \varepsilon}{\lambda_i S_n(\lambda)} \leq \frac{-K^{-n+1} + \varepsilon}{\lambda_i S_n(\lambda)}.$$  

Also if $\lambda_i \geq \lambda_j$ then

$$\partial_i f_{-\varepsilon}(\lambda) - \partial_j f_{-\varepsilon}(\lambda) = \frac{(\lambda_i - \lambda_j)(S_{n-1;i}(\lambda) + S_{n-1;j}(\lambda) - \varepsilon)}{\lambda_i \lambda_j S_n(\lambda)} \geq \frac{(\lambda_i - \lambda_j)(2K^{-n+1} - \varepsilon)}{\lambda_i \lambda_j S_n(\lambda)}.$$

Finally, we compute

$$\partial_i \partial_j f_{-\varepsilon}(\lambda) = \frac{2S_{n-1;i}(\lambda)\delta_{ij} - \varepsilon(1 + \delta_{ij})}{\lambda_i \lambda_j S_n(\lambda)}.$$

Thus for all $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ we have

$$\sum_{i,j} w_i w_j \partial_i \partial_j f_{-\varepsilon}(\lambda) = \frac{2}{S_n(\lambda)} \sum_i \frac{w_i^2}{\lambda_i^2} S_{n-1;i}(\lambda) - \frac{\varepsilon}{S_n(\lambda)} \left( \sum_i \frac{w_i}{\lambda_i} \right)^2 - \frac{\varepsilon}{S_n(\lambda)} \sum_i \frac{w_i^2}{\lambda_i^2} \geq \frac{2K^{-n+1} - \varepsilon(n + 1)}{S_n(\lambda)} \sum_i \frac{w_i^2}{\lambda_i^2}.$$

The above computations show that all of the required statements hold if $\varepsilon$ is sufficiently small. \hfill \Box

In particular, the properties (3), (4) together with (2.1) imply that $F_{-\varepsilon} : \mathcal{M}_K \to \mathbb{R}$ is convex for $\varepsilon \in (0, \varepsilon_1)$. However, we need a strong convexity property as follows:
Proposition 2.5 (CS17, Lemma 9). For $K \in (0, \infty]$ there exists $\varepsilon_2 > 0$ such that for all $\varepsilon \in [0, \varepsilon_2)$, diagonal matrix $A \in \mathcal{M}_K$ with entries $\lambda_i$ and Hermitian matrix $B_{ij}$ we have

$$\sum_{p,q,r,s} B_{rs} B_{pq} \partial_{pq} \partial_{rs} F_{-\varepsilon}(A) + \sum_{i,j} |B_{ij}|^2 \frac{\partial n F_{-\varepsilon}(A)}{\lambda_j} \geq 0.$$ 

We also need the following estimate to deal with the modified $J$-flow when $b = -\varepsilon < 0$. We follow closely to the argument [CS17, Lemma 19].

Lemma 2.6. For any $K \in (0, \infty]$ there exists a constant $\varepsilon_3 = \varepsilon_3(K) > 0$ such that for any $\varepsilon \in [0, \varepsilon_3)$, we have the following: if the continuous path $A_t \in \mathcal{M}$ ($t \in [0, 1]$) satisfies

$$F_0(A_0) < K, \quad F_{-\varepsilon}(A_t) < K + 2 C_\theta \quad (t \in [0, 1]),$$

then $F_0(A_t) < K + 3 C_\theta$ for $t \in [0, 1]$.

Proof. Let $\lambda_i$ be the eigenvalue of $A_t$. Assume that $F_0(A_t) = K + 3 C_\theta$ for some $t \in (0, 1]$. Then AM-GM inequality shows that

$$\frac{1}{\lambda_1 \cdots \lambda_n} \leq n^{-n} \left( \sum_{i=1}^n \frac{1}{\lambda_i} \right)^n = n^{-n} (K + 3 C_\theta)^n.$$

If we set $\varepsilon := \frac{n^n C_\theta}{2(K + 3 C_\theta)^n}$ then for any $\varepsilon \in (0, \varepsilon_3)$ we have

$$F_{-\varepsilon}(A_t) = \sum_{i=1}^n \frac{1}{\lambda_i} - \frac{\varepsilon}{\lambda_1 \cdots \lambda_n} \geq K + 3 C_\theta - \varepsilon n^{-n} (K + 2 C_\theta)^n > K + 2 C_\theta$$

which contradicts the assumption. So we get $F_0(A_t) < K + 3 C_\theta$ for all $t \in [0, 1]$. \hfill $\square$

Lemma 2.7. Let $\omega_1, \omega_2, \chi$ be Kähler forms defined on an open subset $\Omega \subset X$. Assume that $Q_{\omega_2}(\chi) \leq K$. Then there exists $C = C(K, n)$ such that if

$$(1 - \sigma) \omega_1 \leq \omega_2 \leq (1 + \sigma) \omega_1$$

for some $\sigma \in (0, 1)$ then

$$|F_{\omega_1, -\varepsilon}(\chi) - F_{\omega_2, -\varepsilon}(\chi)| \leq C \sigma$$

for any $\varepsilon \in [0, 1)$.

Proof. Recall that

$$F_{\omega_k, -\varepsilon}(\chi) = \text{Tr} \omega_k \chi - \varepsilon \frac{\omega_k^n}{\chi^n}, \quad k = 1, 2.$$ 

Let $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \leq \ldots \leq \mu_n$ be eigenvalues of $\chi$ with respect to $\omega_1$ and $\omega_2$. Then the assumption yields that

$$(1 - \sigma) \lambda_i \leq \mu_i \leq (1 + \sigma) \lambda_i, \quad i = 1, \ldots, n.$$ 

Also from $Q_{\omega_3}(\chi) \leq K$ we know that $\mu_i \geq K^{-1}$. Since the first derivative of the function $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$ is uniformly bounded, we get

$$|\text{Tr} \omega_1 \chi - \text{Tr} \omega_2 \chi| \leq C \sqrt{\sum_{i=1}^n \left( \frac{1}{\lambda_i} - \frac{1}{\mu_i} \right)^2} = C \sqrt{\sum_{i=1}^n \left( \frac{\lambda_i - \mu_i}{\lambda_i \mu_i} \right)^2} \leq C \sqrt{n} K \sigma.$$
Also we have
\[
\frac{\omega_1^n}{\chi^n} - \frac{\omega_2^n}{\chi^n} \leq \frac{(1 + \sigma)^n - 1}{\mu_1 \cdots \mu_n} \leq K^n \sum_{k=1}^{n} \binom{n}{k} \sigma^k,
\]
\[
\frac{\omega_2^n}{\chi^n} - \frac{\omega_1^n}{\chi^n} \leq 1 - \frac{(1 - \sigma)^n}{\mu_1 \cdots \mu_n} \leq K^n \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sigma^k.
\]
Combining all things together we obtain
\[
|F_{\omega_1, -\varepsilon}(\chi) - F_{\omega_2, -\varepsilon}(\chi)| \leq C \sqrt{nK} + \varepsilon K^n \sum_{k=1}^{n} \binom{n}{k} \sigma^k.
\]
Hence we may set \( C(K, n) := C \sqrt{nK} + K^n \sum_{k=1}^{n} \binom{n}{k} \).

**Lemma 2.8.** Let \( \omega, \chi \) be Kähler forms defined on an open subset \( \Omega \subset X \). Assume that \( Q_\omega(\chi) \leq K \). Then there exists \( \varepsilon_4 = \varepsilon_4(K) \) such that for all \( \varepsilon \in [0, \varepsilon_4) \) we have
\[
P_\omega(\chi) \leq F_{\omega, -\varepsilon}(\chi).
\]

**Proof.** Let \( \lambda_1 \leq \ldots \leq \lambda_n \) be the eigenvalues of \( \chi \) with respect to \( \omega \). By using \( \lambda_i \geq K^{-1} \) we compute
\[
F_{\omega, -\varepsilon}(\chi) = P_\omega(\chi) + \frac{1}{\lambda_n} \left( 1 - \varepsilon \frac{1}{\lambda_1 \cdots \lambda_{n-1}} \right) \geq P_\omega(\chi) + \frac{1}{\lambda_n} (1 - \varepsilon K^{n-1}).
\]
So if we set \( \varepsilon_4 = K^{-n+1} \) then for any \( \varepsilon \in (0, \varepsilon_4) \), we have
\[
P_\omega(\chi) \leq F_{\omega, -\varepsilon}(\chi).
\]

**2.2.2. The \( b \geq 0 \) case.** Now we will make a few remarks to the case \( b \geq 0 \), which is relatively easy to handle thanks to the convexity of the function \( \lambda \mapsto \frac{1}{\lambda_1 \cdots \lambda_n} \). Indeed, we can show that:

**Proposition 2.9.** Assume \( b \geq 0 \). Then the function \( f_b : \Gamma \to \mathbb{R} \) satisfies all of the properties in Proposition 2.4. Moreover, the function \( F_b : \mathcal{M} \to \mathbb{R} \) satisfies the strong convexity property as in Proposition 2.5.

**Proof.** One can easily check the desired statements just by replacing \(-\varepsilon\) with \(b\) in the proof of Proposition 2.4 and \([CS17, \text{Lemma 9}]\).

**2.3. Subsolutions.** In this subsection we study general properties for subsolutions (1.2) related to the regularized maximum and averaging on the torus \( T \). Note that the subsolution condition (1.2) is equivalent to
\[
P_\omega(\chi_\phi) - \theta^X_v(\chi_\phi) < c.
\]
Following \([Dem12, \text{Section 5.E}]\) for arbitrary \( \eta = (\eta_1, \ldots, \eta_N) \in (0, \infty)^N \), we define the function \( M_\eta : \mathbb{R}^N \to \mathbb{R} \) by
\[
M_\eta(t_1, \ldots, t_N) := \int_{\mathbb{R}^N} \max\{t_1 + h_1, \ldots, t_N + h_N\} \prod_{j=1, \ldots, N} \theta\left(\frac{h_j}{\eta_j}\right) dh_1 \cdots dh_N,
\]
where \( \theta \) be a non-negative smooth function on \( \mathbb{R} \) with support in \([-1, 1]\) such that 
\[
\int_{\mathbb{R}} \theta(h) dh = 1 \quad \text{and} \quad \int_{\mathbb{R}} h \theta(h) dh = 0.
\]
The fundamental properties for the regularized maximum are summed up as follows:

**Lemma 2.10** ([Dem12], Lemma 5.18). The function \( M_\eta \) satisfies the following:

1. \( M_\eta(t_1, \ldots, t_N) \) is non decreasing in all variables and convex on \( \mathbb{R}^N \).
2. \( \max\{t_1, \ldots, t_N\} \leq M_\eta(t_1, \ldots, t_N) \leq \max\{t_1 + \eta_1, \ldots, t_N + \eta_N\} \).
3. \( M_\eta(t_1, \ldots, t_N) = M(\eta_1, \eta_2, \ldots, \eta_N)(t_1, \ldots, t_N, \eta_1, \eta_2, \ldots, \eta_N) \) if \( t_j + \eta_j \leq \max_{k \neq j} \{t_k - \eta_k\} \),
   where \( f \) means eliminating \( f \).
4. \( M_\eta(t_1 + a, \ldots, t_N + a) = M_\eta(t_1, \ldots, t_N) + a \) for all \( a \in \mathbb{R} \).

By the property (1) and differentiating (4) at \( a = 0 \) we have
\[
\sum_{j=1}^{N} \frac{\partial M_\eta}{\partial t_j} = 1, \quad \frac{\partial M_\eta}{\partial t_j} \geq 0. \quad (2.2)
\]
Moreover, let \( \{B_j\}_{j \in \{1, \ldots, N\}} \) be a finite covering of \( X \) and \( u_j \) smooth functions on \( B_j \) satisfying \( u_j(x) < \max_{k=1, \ldots, N} \{u_k(x)\} \) at every point \( x \in \partial B_j \) (where we extend \( u_j \) on \( X \) so that \( u_j = -\infty \) outside \( B_j \)). In particular, we can choose a sufficiently small vector \( \eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N \) so that \( u_j(x) + \eta_j \leq \max_{k=1, \ldots, N} \{u_k(x) - \eta_k\} \) holds for all \( j \) and \( x \in \partial B_j \). Then as a corollary of the above lemma (see also [Dem12 Corollary 5.19]) we find that the function \( u := M_\eta(u_1, \ldots, u_N) \) is smooth. Moreover, a direct computation shows that the second derivatives in local coordinates \((z^1, \ldots, z^n)\) are given by
\[
\frac{\partial^2 u}{\partial z^a \partial z^b} = \sum_{a, b} \frac{\partial^2 M_\eta}{\partial t_a \partial t_b}(u_1, \ldots, u_N) \cdot \frac{\partial u_a}{\partial z^a} \frac{\partial u_b}{\partial z^b} + \sum_{a} \frac{\partial M_\eta}{\partial t_a}(u_1, \ldots, u_N) \cdot \frac{\partial^2 u_a}{\partial z^a \partial \bar{z}^a}. \quad (2.3)
\]

The following extends the argument in [Che21 Section 4]:

**Proposition 2.11.** Let \( \{B_j\}_{j \in \{1, \ldots, N\}} \) be a finite covering of \( X \) and \( \varphi_j \) smooth functions on \( B_j \) satisfying \( \varphi_j(x) < \max_{k=1, \ldots, N} \{\varphi_k(x)\} \) at every point \( x \in \partial B_j \), and \( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j > 0 \). For a sufficiently small vector \( \eta \in \mathbb{R}^N \) chosen as above, define the regularized maximum \( \varphi := M_\eta(\varphi_1, \ldots, \varphi_N) \) and set \( \chi := \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi \). Then

1. \( \chi \) is Kähler.
2. If each \( \varphi_j \) satisfies
   \[
P_\omega \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j \right) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(\varphi_j) < c
   \]
   on \( B_j \), then
   \[
P_\omega(\chi) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(\varphi) < c
   \]
   on \( X \).
3. If each \( \varphi_j \) satisfies
   \[
Q_\omega \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j \right) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(\varphi_j) < c
   \]
   on \( B_j \), then
   \[
Q_\omega(\chi) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(\varphi) < c
   \]
on $X$.

**Proof.** (1) By the convexity of $M_\eta$, (2.2) and (2.3) we observe that
\[
\chi \geq \sum_j \frac{\partial M_\eta}{\partial t_j} \cdot (\hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j) > 0.
\]

(2) The term $\sum_j \frac{\partial M_\eta}{\partial t_j} \cdot (\hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j)$ is a weighted average of $\hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j$ with $\sum_j \frac{\partial M_\eta}{\partial t_j} = 1$. Thus the monotonicity and convexity property of $P_\omega$ shows that
\[
P_\omega(\chi) \leq \sum_j \frac{\partial M_\eta}{\partial t_j} \cdot P_\omega \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j \right),
\]
and hence
\[
P_\omega(\chi) - \theta^X_v(\hat{\chi}) - \text{Re}(v)(\varphi) \leq \sum_j \frac{\partial M_\eta}{\partial t_j} \cdot \left[ P_\omega \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_j \right) - \theta^X_v(\hat{\chi}) - \text{Re}(v)(\varphi) \right] < c.
\]

(3) The proof of (2) only requires the monotonicity and convexity of the operator $P_\omega$. So the same proof also works for $Q_\omega$. \)

However the resulting Kähler form $\chi$ is not $T$-invariant since we can not take each $(B_j, \varphi_j)$ to be $T$-invariant in general. For this reason we need the averaging argument as follows:

**Proposition 2.12.** For a $\hat{\chi}$-Kähler potential $\varphi$ with $\chi := \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$, set
\[
\tilde{\varphi} := \int_{\tau \in T} \tau^* \varphi dT, \quad \tilde{\chi} := \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{\varphi}
\]
so that $\tilde{\chi} = \int_{\tau \in T} \tau^* \chi dT > 0$, where $dT$ denotes the Haar measure on $T$ normalized by $\int_T dT = 1$. Then we have the following:

1. If $\varphi$ satisfies
\[
P_\omega(\chi) - \theta^X_v(\hat{\chi}) - \text{Re}(v)(\varphi) < c,
\]
then $\tilde{\varphi} \in \mathcal{H}^T$ and
\[
P_\omega(\tilde{\chi}) - \theta^X_v(\tilde{\chi}) < c.
\]

2. If $\varphi$ satisfies
\[
Q_\omega(\chi) - \theta^X_v(\hat{\chi}) - \text{Re}(v)(\varphi) < c,
\]
then $\tilde{\varphi} \in \mathcal{H}^T$ and
\[
Q_\omega(\tilde{\chi}) - \theta^X_v(\tilde{\chi}) < c.
\]

**Proof.** We prove only the property (1) since (2) is similar. Since $\omega$ is $T$ invariant, by the convexity of $P_\omega$ we have
\[
P_\omega(\tilde{\chi}) \leq \int_{\tau \in T} \tau^* P_\omega(\chi) dT.
\]
Taking into account the fact that \( \tau \in T \) commutes with the flow generated by \( \text{Re}(v) \) and \( \theta_v^X(\tilde{\chi}) \) is \( T \)-invariant we obtain
\[
P_\omega(\tilde{\chi}) - \theta_v^X(\tilde{\chi}) = P_\omega(\tilde{\chi}) - \theta_v^X(\tilde{\chi}) - \text{Re}(v)(\tilde{\varphi})
\leq \int_{\tau \in T} \tau^*(P_\omega(\chi) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(\varphi))dT
< c
\]
as desired.

2.4. **Functionals on the space of Kähler potentials.** Let \((X, \tilde{\chi})\) be a compact Kähler manifold with a Hamiltonian \( T \)-action, and \( \omega \in \alpha \) a \( T \)-invariant Kähler form. Define the Aubin’s \( I \)-functional \([\text{Aub84}]\) by the formula
\[
I(\phi) := \int_X \phi(\tilde{\chi}^n - \chi_\phi^n).
\]
Integrating by parts one can see that
\[
I(\phi) = -\int_X \phi \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \wedge (\chi_\phi^{n-1} + \cdots + \tilde{\chi}^{n-1})
= \frac{\sqrt{-1}}{2\pi} \int_X \partial \phi \wedge \bar{\partial} \phi \wedge (\chi_\phi^{n-1} + \cdots + \tilde{\chi}^{n-1}) \tag{2.4}
\geq \int_X |\partial \phi|^2_{\tilde{h}} \tilde{\chi}^n,
\]
where \( \tilde{h} \) denotes the Riemannian metric corresponding to \( \tilde{\chi} \). Also we define the \( J_{T,\omega,b} \)-functional by the variational formula (up to additive constants)
\[
\delta J_{T,\omega,b}|_{\phi}(\psi) := \int_X \psi(F_{\omega,b}(\chi_\phi) - c - \theta_v^X(\chi_\phi))\chi_\phi^n.
\]
If \( b = 0 \) the functional \( J_{T,\omega,b} \) was introduced by Li–Shi \([\text{LS16}]\), which we will denote by \( J_{T,\omega} \) for simplicity. By the definition the critical points of \( J_{T,\omega,b} \) are just given by solutions to the generalized equation \([\text{L.3}]\).

**Proposition 2.13.** If \( b \geq 0 \) the functional \( J_{T,\omega,b} \) is strictly convex along any \( C^{1,1} \) geodesics in \( H^T \).

**Proof.** Indeed, by adjusting constants we can decompose the functional \( J_{T,\omega,b} \) as
\[
J_{T,\omega,b} = J_{T,\omega} + b \int_X \phi \omega^n.
\]
The first term \( J_{T,\omega} \) is strictly convex by \([\text{LS16}] \) Proposition 3.1]. For \( C^2 \) geodesics
\[
\frac{d^2}{dt^2} \int_X \phi_t \omega^n = \int_X |\nabla_t \frac{d}{dt} \phi_t|_{\tilde{h}}^2 \omega^n = 0,
\]
we can easily see that
\[
\frac{d^2}{dt^2} \int_X \phi_t \omega^n = \int_X |\nabla_t \frac{d}{dt} \phi_t|_{\tilde{h}}^2 \omega^n \geq 0.
\]
Then we may approximate \( C^{1,1} \) geodesics by \( \varepsilon \)-geodesics just as in \([\text{Che00}] \) Proposition 2.1].
**Definition 2.14.** We say that the functional $J_{T,\omega,b}$ is coercive if there exist constants $\delta_c > 0$, $B_c > 0$ such that

$$J_{T,\omega,b}(\phi) \geq \delta_c I(\phi) - B_c$$

for all $\phi \in \mathcal{H}^T$.

**Remark 2.15.** We define the Aubin’s $J$-functional [Aub84] by the variational formula (up to scale)

$$\delta J|_\phi(\psi) = -\int_X \psi \chi^n + \int_X \psi \tilde{\chi}^n.$$  

Then it is well-known that there exists a constant $C > 0$ depending on $n$ such that

$$C^{-1} I(\phi) \leq J(\phi) \leq CI(\phi)$$

(e.g. [BBGZ13, Section 2.1]). Thus one can use $J$ instead of $I$ for defining the coercivity. Or one can use $d_1$-distance on $\mathcal{H}^T$ as an equivariant characterization of coercivity (for instance, see [DR16]).

3. The modified $J$-flow

Let $(X, \tilde{\chi})$ be a compact Kähler manifold with a Hamiltonian $T$-action, and $\omega \in \alpha$ a $T$-invariant Kähler form. We define the modified $J$-flow as the gradient flow of $J_{T,\omega,b}$-functional:

$$\frac{d}{dt} \phi_t = -F_b(A_t) + c + \theta^X_t(\chi_t),$$  

(3.1)

where we set $A_t := g^{ik} h_{jk}$, $\chi_t := \tilde{\chi} + \sqrt{-1} \partial \bar{\partial} \phi_t$ for simplify the notations. Note that $\phi_t \in \mathcal{H}^T$ as long as it exists. When $b \geq 0$ we have already seen that $J_{\omega,b}$ is strictly convex along any $C^{1,1}$ geodesics in $\mathcal{H}^T$ (Lemma 2.13). On the other hand it is not the case with $b < 0$ so that we should take $\varepsilon$ sufficiently small according to the initial data in order to obtain the regularity of the flow. In what follows we study the long-time behavior of (3.1) in two cases $b \geq 0$ and $b = -\varepsilon \leq 0$ separately.

3.1. Long-time existence when $b = -\varepsilon \leq 0$. Assume $b = -\varepsilon \leq 0$. Then by (1.4) the constant $c = c_\varepsilon$ is given by

$$c_\varepsilon = \frac{n\alpha \cdot \beta^{n-1} - \varepsilon \alpha^n}{\beta^n} = c_X - \varepsilon \frac{\alpha^n}{\beta^n}.$$  

Fix $\phi_0 \in \mathcal{H}^T$ and take a constant $K_0 > 0$ so that $Q_\omega(\chi_0) = F_0(A_0) < K_0$. We set $\varepsilon_5 := \min\{\varepsilon_1(K_0 + 10C_\theta), \varepsilon_2(K_0 + 10C_\theta), \varepsilon_3(K_0), \varepsilon_4(K_0 + 10C_\theta)\}$ and assume that $\varepsilon \in (0, \varepsilon_5)$. Then the flow

$$\frac{d}{dt} \phi_t = -F_{-\varepsilon}(A_t) + c_\varepsilon + \theta^X_t(\chi_t)$$  

(3.2)

is parabolic at $t = 0$ from the choice of $\varepsilon$, so the short-time existence follows from standard theory. Let $T_{\max}$ be the maximal existence time of $\phi_t \in \mathcal{H}^T$. We introduce the operator $\Box_{L,-\varepsilon}$ by

$$\Box_{L,-\varepsilon} := \frac{d}{dt} + \partial_{ij} F_{-\varepsilon}(A_t) g^{ik} \nabla_j \nabla_k - \text{Re}(v),$$

where $\nabla$ denotes the Chern connection with respect to $\omega$.  

Lemma 3.1. Along the flow \( \phi_t \) for \( t \in [0, T_{\text{max}}) \) we have the following:

1. \( \left| \frac{d}{dt} \phi_t \right| < C. \)
2. \( \left| \phi_t \right| \leq C(t + 1). \)
3. \( F_0(A_t) < K_0 + 3C_\theta. \)

Proof. Differentiating the equation (3.2) we have

\[ \Box_{L, \epsilon} \frac{d}{dt} \phi_t = 0 \]

for all \( t \in [0, T_{\text{max}}) \). So the maximum principle shows that

\[ \inf X \frac{d}{dt} \phi_0 \leq \frac{d}{dt} \phi_t \leq \sup X \frac{d}{dt} \phi_0 \]

for all \( t \in [0, T_{\text{max}}) \) which shows (1), and (2) by integrating in \( t \). Moreover, the lower bound of \( \frac{d}{dt} \phi_t \) implies that

\[ F_{-\epsilon}(A_t) \leq \theta^X_v(\chi_t) + \sup X (F_{-\epsilon}(A_0) - \theta^X_v(\chi_0)) < K_0 + 2C_\theta. \]

Then it follows from Lemma 2.6 that \( F_0(A_t) < K_0 + 3C_\theta \) for all \( t \in [0, T_{\text{max}}) \), hence we have (3). \( \square \)

Lemma 3.2. The modified \( J \)-flow admits a solution \( \phi_t \) for all \( t \in [0, \infty) \).

Proof. It follows from Lemma 3.1 that the function \( F_{-\epsilon} \) is convex as long as the flow exists. Thus by the standard parabolic theory, if \( |\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_t| \) is uniformly bounded on \( [0, T_{\text{max}}) \) then \( \phi_t \) is bounded in \( C^\infty \) in the same interval. We take normal coordinates of \( \omega \) at a point so that \( \chi_t \) is diagonal with eigenvalues \( \lambda_1, \ldots, \lambda_n \). By using (2.1) we compute

\[
\nabla_i \nabla_i F_{-\epsilon}(A_t) = \nabla_i (\partial_{pq} F_{-\epsilon}(A_t) \cdot \nabla_i A_q^p)
\]

\[
= \partial_{pq} F_{-\epsilon}(A_t) \cdot \nabla_i \nabla_i A_q^p + \partial_{pq} \partial_{rs} F_{-\epsilon}(A_t) \cdot \nabla_i A_q^r \cdot \nabla_i A_q^p
\]

\[
= \partial_{pq} F_{-\epsilon}(A_t) \cdot \nabla_i \nabla_i h_{qp} + \partial_{pq} \partial_{rs} F_{-\epsilon}(A_t) \cdot \nabla_i h_{sr} \cdot \nabla_i h_{qp}
\]

\[
= \partial_{pq} F_{-\epsilon}(A_t) \cdot (\nabla_p \nabla \phi h_{q} + \text{Rm} * h) + \partial_{pq} \partial_{rs} F_{-\epsilon}(A_t) \cdot \nabla_i h_{sr} \cdot \nabla_i h_{qp},
\]

where \( \text{Rm} \) denotes the Riemannian curvature of \( \omega \). By the definition of Hamiltonians \( \theta^X_v(\chi_t) \) we have

\[
\nabla_i \nabla_i \theta^X_v(\chi_t) = \nabla_i (v^k h_{ki})
\]

\[
= \nabla_i v^k \cdot h_{ki} + v^k \nabla_i h_{ki}
\]

\[
= \nabla_i v^k \cdot h_{ki} + v^k \nabla_i \hat{h}_{ki} + v^k \nabla_i \phi_{ki}
\]

\[
= \nabla_i v^k \cdot h_{ki} + v^k \nabla_i \hat{h}_{ki} + v^k \nabla_k \phi_{i}\]

\[
\leq C_1 |h| + v(\Delta g \phi) + C_1,
\]
where \( \hat{h} \) denotes the Riemannian metric corresponding to \( \hat{\chi} \), and we used \( h_{\text{max}} = \hat{h}_{\text{max}} + \phi_{\text{max}} \). By \textbf{3.2} we obtain

\[
\frac{d}{dt} \log Tr_\omega \chi_t = \frac{1}{Tr_\omega \chi_t} \nabla_i \nabla_j ( - F_{-\varepsilon}(A_t) + \theta^X_\omega (\chi_t) )
\[
= \frac{1}{Tr_\omega \chi_t} \left( - \partial_{pp} F_{-\varepsilon}(A_t) \cdot \nabla_p \nabla_i h_{ii} - \partial_{pp} F_{-\varepsilon}(A_t) \cdot \nabla_i h_{ss} \cdot \nabla_i h_{qq} + \nabla_i \nabla_j \theta^X_\omega (\chi_t) \right)
\[
\leq \frac{1}{Tr_\omega \chi_t} \left( - \partial_{pp} F_{-\varepsilon}(A_t) \cdot \nabla_p \nabla_i h_{ii} + |\partial_{pp} F_{-\varepsilon}(A_t)||Rm||h| - \partial_{pp} \partial_{rs} F_{-\varepsilon}(A_t) \cdot \nabla_i h_{ss} \cdot \nabla_i h_{qq} + C_1|h| + v(\Delta g\phi) + C_1 \right).
\]

Also a direct computation shows that

\[
\nabla_p \nabla_i \log Tr_\omega \chi_t = \frac{\nabla_p \nabla_i Tr_\omega \chi_t}{Tr_\omega \chi_t} - \frac{|\nabla_p \nabla_i h_{ii}|^2}{(Tr_\omega \chi_t)^2} = \frac{\nabla_p \nabla_i h_{ii}}{Tr_\omega \chi_t} - \frac{|\nabla_i h_{pi}|^2}{(Tr_\omega \chi_t)^2},
\]

\[
v(\log Tr_\omega \chi_t) = \frac{v(\log Tr_\omega \chi_t)}{Tr_\omega \chi_t} = \frac{1}{Tr_\omega \chi_t} (v(\log Tr_\omega \chi_t) + v(\Delta g\phi)) \geq \frac{1}{Tr_\omega \chi_t} (v(\Delta g\phi) - C_2).
\]

By using the strong convexity (Proposition \textbf{2.5}) we observe that

\[
\partial_{pp} \partial_{rs} F_{-\varepsilon}(A_t) \cdot \nabla_i h_{ss} \cdot \nabla_i h_{qq} + \partial_{pp} F_{-\varepsilon}(A_t) \frac{|\nabla_i h_{pi}|^2}{Tr_\omega \chi_t} \geq \partial_{pp} \partial_{rs} F_{-\varepsilon}(A_t) \cdot \nabla_i h_{ss} \cdot \nabla_i h_{qq} + \partial_{pp} F_{-\varepsilon}(A_t) \frac{|\nabla_i h_{pi}|^2}{\lambda_j} \geq 0.
\]

Combining all things together we will have

\[
\square_{L_{-\varepsilon}} \log Tr_\omega \chi_t \leq \frac{1}{Tr_\omega \chi_t} (|\partial_{pp} F_{-\varepsilon}(A_t)||Rm||h| + C_1|h| + C_3),
\]

where \( 0 < -\partial_{pp} F_{-\varepsilon}(A_t) \leq \frac{1}{\lambda_j} \), \( Tr_\omega \chi_t = \sum_i \lambda_i \) and \( |h| = (\sum_i \lambda_i^2)^{1/2} \). Since \( \lambda_i \) is uniformly bounded from blow this shows that the RHS is bounded from above by some uniform constant \( C_4 > 0 \). Therefore if we set \( G := \log Tr_\omega \chi_t - 2C_4 t \) then for any \( T \in (0, T_{\text{max}}) \) and maximum point \((x_0, t_0)\) of \( G \) on \( X \times [0, T] \) with \( t_0 > 0 \) we must have

\[
0 \leq \square_{L_{-\varepsilon}} G \leq -C_4,
\]

but this is impossible. It follows that

\[
Tr_\omega \chi_t \leq e^{2C_4 t + C_5}
\]

on \([0, T_{\text{max}}]\) which gives the uniform upper bound for \( \lambda_i \). \( \square \)
3.2. **Convergence when** \( b \geq 0 \). In this subsection we will give a proof of Theorem 1.1 by showing that the modified \( J \)-flow (3.1)

\[
\frac{d}{dt} \phi_t = -F_b(A_t) + c + \theta^X_v(\chi_t)
\]  

(3.3)

converges smoothly to the potential (1.3) under the existence of a subsolution \( \widehat{\chi} \in \beta \)

\[ P_\omega(\widehat{\chi}) < c + \theta^X_v(\widehat{\chi}) \]

chosen as a reference Kähler form. Define the operator

\[ \square_{L,b} := \frac{d}{dt} + \partial_{ij} F_b(A_t) g^{ik} \nabla_j \nabla_k - \text{Re}(v), \]

and let \( T_{\text{max}} \) be the maximal existence time of the flow.

**Lemma 3.3.** We have \( |\frac{d}{dt} \phi_t| \leq C \) along the flow. In particular, the eigenvalues of \( A_t \) have a uniform positive lower bound.

**Proof.** Differentiating the equation (3.3) in \( t \) yields that \( \square_{L,b} \frac{d}{dt} \phi_t = 0 \). The remaining is similar to the proof of Lemma 3.1.  \( \square \)

**Lemma 3.4.** There exists a uniform constant \( N > 0, C > 0 \) such that

\[
\text{Tr}_\omega \chi_t \leq C e^{N(\phi_t - \inf_{M \times [0,T]} \phi_t)}
\]

(3.4)

for all \( t \in [0,T] \) and \( T \in [0,T_{\text{max}}) \).

**Proof.** From the assumption we can choose a constant \( \epsilon > 0 \) sufficiently small so that

\[ P_\omega(\widehat{\chi}) < c + \theta^X_v(\widehat{\chi}) - 2\epsilon. \]

We choose normal coordinates for \( \omega \) so that \( \chi_t \) is diagonal with entries \( \lambda_1 \leq \ldots \leq \lambda_n \). The metric \( \widehat{\chi} \) may not be diagonal, but we denote its eigenvalues by \( \mu_1, \ldots, \mu_n \). By using Proposition 2.9 and a uniform positive lower bound for \( \lambda_i \) we can compute just as in the proof of Lemma 3.2 to find that

\[ \square_{L,b} \log \text{Tr}_\omega \chi_t \leq C_1. \]

On the other hand,

\[
\square_{L,b} \phi_t = -F_b(A_t) + c + \theta^X_v(\widehat{\chi}) + \partial_{ii} F_b(A_t)(h_{ii} - \widehat{h}_{ii}).
\]

Hence if we set \( G := \log \text{Tr}_\omega \chi_t - N \phi_t \) then

\[ \square_{L,b} G \leq N \epsilon + NF_b(A_t) - Nc - N\theta^X_v(\widehat{\chi}) + N\partial_{ii} F_b(A_t)(\widehat{h}_{ii} - h_{ii}), \]

where the large constant \( N > 0 \) is taken so that \( C_1 \leq N \epsilon \). For any fixed \( T \in (0,T_{\text{max}}) \) the maximum principle shows that

\[ -\epsilon \leq F_b(A_t) - c - \theta^X_v(\widehat{\chi}) + \partial_{ii} F_b(A_t)(\widehat{h}_{ii} - h_{ii}) \]

at the maximum point \( (x_0, t_0) \) of \( G \) on \( X \times [0,T] \) if \( t_0 > 0 \). In particular,

\[ P_\omega(\widehat{\chi}) + \epsilon \leq F_b(A_t) + \partial_{ii} F_b(A_t)(\widehat{h}_{ii} - h_{ii}) \]
from our choice of $\epsilon$, which can be written as
\[
\max_k \sum_{i \neq k} \frac{1}{\mu_i} + \epsilon \leq f_b(\lambda) + \partial_i f_b(\lambda) \cdot (\widehat{h}_{i\overline{i}} - \lambda_i).
\]

Following [CS17, Section 2] we define the function
\[
\tilde{f}_b(\lambda_1, \ldots, \lambda_{n-1}) := \lim_{\lambda_n \to \infty} f_b(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i}.
\]

By using the uniform positive lower bound for $\lambda_i$, one can easily observe that for any constant $\tau > 0$ there exists $K = K(\tau) > 0$ such that if $\lambda_n \geq K$ then
\[
f_b(\lambda_1, \ldots, \lambda_n) \leq \tilde{f}_b(\lambda_1, \ldots, \lambda_{n-1}) + \tau,
\]
\[
\sum_{i=1}^{n-1} (\partial_i f_b(\lambda_1, \ldots, \lambda_n) - \partial_i \tilde{f}_b(\lambda_1, \ldots, \lambda_{n-1})) \cdot (\widehat{h}_{i\overline{i}} - \lambda_i) = \sum_{i=1}^{n-1} \frac{b}{\lambda_1 \cdots \lambda_n} \left(1 - \frac{\widehat{h}_{i\overline{i}}}{\lambda_i}\right) \leq \tau,
\]
\[
\partial_n f_b(\lambda) \cdot (\widehat{h}_{n\overline{n}} - \lambda_n) = \frac{1}{\lambda_n} \left(1 + \frac{b}{\lambda_1 \cdots \lambda_{n-1}}\right) \left(1 - \frac{\widehat{h}_{n\overline{n}}}{\lambda_n}\right) \leq \tau.
\]

Combining with the convexity of $\tilde{f}_b$ yields that
\[
f_b(\lambda) + \partial_i f_b(\lambda) \cdot (\widehat{h}_{i\overline{i}} - \lambda_i)
\leq \tilde{f}_b(\lambda_1, \ldots, \lambda_{n-1}) + \sum_{i=1}^{n-1} \partial_i \tilde{f}_b(\lambda_1, \ldots, \lambda_{n-1}) \cdot (\widehat{h}_{i\overline{i}} - \lambda_i) + 3\tau
\leq \tilde{f}_b(\hat{h}_{1\overline{1}}, \ldots, \hat{h}_{n-1\overline{n-1}}) + 3\tau
\leq \max_k \sum_{i \neq k} \frac{1}{\mu_i} + 3\tau,
\]

where in the last inequality we used the fact that diagonal entries $(\hat{h}_{1\overline{1}}, \ldots, \hat{h}_{n\overline{n}})$ lies in the convex full of the permutations of the $\mu_i$ by the Schur–Horn theorem. Eventually if we choose $\tau = \epsilon/6$ then we have $\lambda_n \leq K(\tau)$, which yields the desired estimate. □

Once we obtain (3.4) then we can show the uniform $C^0$-estimate by using exactly the same argument as [SW08, Wei04, Wei06] since it does not depend on the equation (3.3). The higher order estimate follows from the Evans–Krylov estimate [Kry76, Kry82], Schauder estimate and standard bootstrapping argument.

**Theorem 3.5.** Assume $\hat{\chi} \in \beta$ satisfies the subsolution condition. Then along the modified $J$-flow (3.3) the Kähler form $\chi_{\phi_t}$ converges smoothly to a solution of (1.3) as $t \to \infty$.

**Proof.** For $\phi \in \mathcal{H}^T$ we define
\[
E_{T,\omega,b}(\phi) := \int_X (F_{\omega,b}(\chi_{\phi}) - c - \theta_v^X(\chi_{\phi}))^2 \chi_{\phi}^n.
\]
We will compute the derivative of $E_{T,\omega,b}$ along the flow. To simplify the notations we set
\[ \sigma := \frac{d}{dt} \phi_t = -F_b(A_t) + c + \theta_v^X(\chi_t), \quad \nu := \frac{\omega^n}{\chi_t}. \]

In normal coordinates of $\chi_t$ we compute
\[ \frac{d}{dt}\sigma = h^{i\bar{j}} h^{k\bar{l}} g_{i\bar{j}} \sigma_{k\bar{l}} + b \nu \Delta_t \sigma + \nu(\sigma). \]

Hence we have
\[ \frac{d}{dt} E_{T,\omega,b} = 2 \int_X \sigma \frac{d}{dt}\sigma^n + \int_X \sigma^2 \Delta_t \sigma^n \]
\[ = 2 \int_X \left( \sigma \frac{d}{dt}\sigma - |\nabla \sigma|^2_h \right) \chi_t^n \]
\[ = 2 \int_X \left( h^{i\bar{j}} h^{k\bar{l}} g_{i\bar{j}} \sigma_{k\bar{l}} + b \nu h^{k\bar{l}} \sigma_{k\bar{l}} - \sigma(\nu) \right) \chi_t^n \]
\[ = -2 \int_X \left( h^{i\bar{j}} h^{k\bar{l}} g_{i\bar{j}} \sigma_{k\bar{l}} + h^{i\bar{j}} h^{\tilde{i}\tilde{j}} g_{i\bar{j}} \sigma_{\tilde{i}\tilde{j}} + b \nu h^{k\bar{l}} \sigma_{k\bar{l}} - \sigma(\nu) \right) \chi_t^n \]
\[ = -2 \int_X \left( h^{i\bar{j}} h^{k\bar{l}} g_{i\bar{j}} \sigma_{k\bar{l}} + h^{i\bar{j}} h^{\tilde{i}\tilde{j}} g_{i\bar{j}} \sigma_{\tilde{i}\tilde{j}} + b \nu h^{k\bar{l}} \sigma_{k\bar{l}} - \sigma(\nu) \right) \chi_t^n \]
\[ = -2 \int_X \left( h^{i\bar{j}} h^{k\bar{l}} g_{i\bar{j}} \sigma_{k\bar{l}} + h^{i\bar{j}} h^{\tilde{i}\tilde{j}} g_{i\bar{j}} \sigma_{\tilde{i}\tilde{j}} + b \nu h^{k\bar{l}} \sigma_{k\bar{l}} - \sigma(\nu) \right) \chi_t^n \]
\[ - h^{k\bar{l}} \sigma_{k\bar{l}} (\text{Tr}_{\chi_t} \omega + b - \theta_v^X(\chi_t)) \chi_t^n \]
\[ = -2 \left( \int_X |\nabla \sigma|^2_h \chi_t^n + b \int_X |\nabla \sigma|^2_h \omega^n \right) \leq 0, \]

where the Hermitian metric $\eta$ is defined by $\eta_{i\bar{j}} := g^{k\bar{l}} h_{i\bar{j}} h^{k\bar{l}}$. Since we have already obtained uniform $C^{t,\gamma}$-estimate we know that
\[ \int_0^\infty \left( \int_X |\nabla \sigma|^2_h \chi_t^n + b \int_X |\nabla \sigma|^2_h \omega^n \right) dt \leq +\infty. \]

By passing to a subsequence $t_k$ we get
\[ \int_X |\nabla \sigma(t_k)|^2_h \chi_t^n + 2b \int_X |\nabla \sigma(t_k)|^2_h \omega^n \to 0, \]

and hence $\sigma(t_k) \to 0$ by $\int_X \sigma(t_k) \chi_t^n = 0$. This shows that the limit $\chi_{t_\infty}$ satisfies the generalized equation
\[ \text{Tr}_{\chi_{t_\infty}} \omega + b \frac{\omega^n}{\chi_{t_\infty}^n} = c + \theta_v^X(\chi_{t_\infty}), \]

where the Kähler form $\chi_{t_\infty}$ does not depend on the choice of subsequences due to the strict convexity property of $J_{T,\omega,b}$ (Proposition 2.13). Thus we conclude that $\chi_{\phi_t}$ converges smoothly to the unique solution of (1.3). \[ \square \]

In particular, Theorem 3.5 provides a proof of Theorem 1.1.
4. Coercivity of the Modified J-Functional

4.1. Coercivity implies existence. In this section we will prove Theorem 1.2. As in [CS17 Proposition 21] a crucial observation is that if \( J_{T,\omega} := J_{T,\omega,0} \) is coercive then so is \( J_{T,\omega,-\varepsilon} \) for sufficiently small \( \varepsilon > 0 \). However we need a quantitative estimate including the explicit choice of the coercivity constants in order to obtain a uniform bound for the Dirichlet energy. Now we normalize \( J_{T,\omega,-\varepsilon} \) so that \( J_{T,\omega,-\varepsilon}(0) = 0 \). Also for a fixed \( \omega \in \alpha \) we take a reference Kähler form \( \hat{\chi} \in \beta \) as a solution to the Monge–Ampère equation

\[
\hat{\chi}^n = \frac{\beta^n}{\alpha^n} \omega^n,
\]

which is known to exist by Yau’s theorem [Yau78].

**Proposition 4.1.** Suppose that there exist uniform constants \( \delta_c > 0, B_c > 0 \) such that

\[
J_{T,\omega}(\phi) \geq \delta_c I(\phi) - B_c
\]

for all \( \phi \in \mathcal{H}^T \). Then there exists \( \varepsilon_6 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_6) \) and \( \phi \in \mathcal{H}^T \) we have

\[
J_{T,\omega,-\varepsilon}(\phi) \geq \frac{\delta_c}{2} I(\phi) - B_c.
\]

**Proof.** If we subtract \( J_{T,\omega} \) from \( J_{T,\omega,-\varepsilon} \) the integrands including Hamiltonians cancel each other out. So we can compute in the same way as in [CS17 Proposition 21]. From the normalization we have

\[
J_{T,\omega,-\varepsilon}(\phi) = J_{T,\omega}(\phi) + \varepsilon \int_0^1 \int_X \phi \left( \frac{\alpha^n}{\beta^n} \chi^n_{t\phi} - \omega^n \right) dt
\]

\[
= J_{T,\omega}(\phi) + \varepsilon \frac{\alpha^n}{\beta^n} \int_0^1 \int_X \phi (\chi^n_{t\phi} - \hat{\chi}^n) dt
\]

\[
= J_{T,\omega}(\phi) + \varepsilon \frac{\alpha^n}{\beta^n} \int_0^1 \int_X t\phi \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \wedge (\chi^{n-1}_{t\phi} + \cdots + \hat{\chi}^{n-1}) dt
\]

\[
\geq J_{T,\omega}(\phi) - \varepsilon \frac{\sqrt{-1}}{2\pi} \frac{\alpha^n}{\beta^n} \int_0^1 \int_X \partial \phi \wedge \bar{\partial} \phi \wedge (\chi^{n-1}_{t\phi} + \cdots + \hat{\chi}^{n-1}) dt dt.
\]

By using the coercivity estimate \( J_{T,\omega}(\phi) \geq \delta_c I(\phi) - B_c \) we have

\[
J_{T,\omega,-\varepsilon}(\phi) \geq \delta_c \frac{\sqrt{-1}}{2\pi} \int_X \partial \phi \wedge \bar{\partial} \phi \wedge (\chi^{n-1}_{t\phi} + \cdots + \hat{\chi}^{n-1}) - B_c
\]

\[
- \varepsilon \frac{\sqrt{-1}}{2\pi} \frac{\alpha^n}{\beta^n} \int_0^1 \int_X \partial \phi \wedge \bar{\partial} \phi \wedge (\chi^{n-1}_{t\phi} + \cdots + \hat{\chi}^{n-1}) dt dt - B_c.
\]

(4.1)

Since \( \chi_{t\phi} = (1 - t)\chi + t\chi_{t\phi} \) the integrands including \( \chi_{t\phi} \) can be expressed as

\[
\chi^{n-1-k}_{t\phi} \wedge \hat{\chi}^k = \sum_{i=0}^{n-1-k} p_i(t) \chi^{n-1-k-i}_{t\phi} \wedge \hat{\chi}^{k+i}
\]

for all \( k = 0, 1, \ldots, n - 1 \), where \( p_i(t) := \binom{n-1-k}{i}(1-t)^i t^{n-1-k-i} \). This shows that the second term in the RHS of (4.1) can be absorbed in the first term if \( \varepsilon \) is sufficiently
small. So there exists $\varepsilon_6 > 0$ (depending only on $n, \alpha, \beta, \delta_c$) such that if $\varepsilon \in (0, \varepsilon_6)$ then
\[
J_{T,\varepsilon}(\phi) \geq \frac{\delta_c}{2} I(\phi) - B_c
\]
for all $\phi \in \mathcal{H}^T$. \qed

From now on we assume that $J_{T,\omega}(\phi) \geq \delta_c I(\phi) - B_c$ holds for all $\phi \in \mathcal{H}^T$. Take a large number $K_0 > 0$ so that $Q_\omega(\hat{\chi}) < K_0$ and set $\varepsilon_7 := \min\{\varepsilon_5(K_0), \varepsilon_6(n, \alpha, \beta, \delta_c)\}$. Then by Lemma 3.1 and Lemma 3.2 the modified $J$-flow starting from $0 \in \mathcal{H}^T$ exists for all $t \in [0, \infty)$ with uniform bound
\[
Q_\omega(\chi_{\phi_t}) < K_0 + 3C_\theta. \quad (4.2)
\]
By the coercivity estimate there is a sequence $t_k \to \infty$ such that
\[
\lim_{k \to \infty} \frac{d}{dt} J_{T,\omega,-\varepsilon}(\phi_{t_k}) = 0,
\]
and hence
\[
\lim_{k \to \infty} \int_X \left( F_{\omega,-\varepsilon}(\chi_{\phi_{t_k}}) - c_\varepsilon - \theta^X_\omega(\chi_{\phi_{t_k}}) \right)^2 \omega^n = 0
\]
where we used the uniform lower bound for eigenvalues along the flow $\phi_t$. We normalize $\phi_{t_k}$ as
\[
u_k := \phi_{t_k} - \sup_X \phi_{t_k}.
\]
Then
\[
\lim_{k \to \infty} \int_X \left( F_{\omega,-\varepsilon}(\chi_{\nu_k}) - c_\varepsilon - \theta^X_\omega(\chi_{\nu_k}) \right)^2 \omega^n = 0.
\]
This shows that $F_{\omega,-\varepsilon}(\chi_{\nu_k}) - \Re(v)(u_k) \to c_\varepsilon + \theta^X_\omega(\hat{\chi})$ in $L^2$ when $k \to \infty$ with $\varepsilon$ fixed. On the other hand, by using the coercivity and monotonicity along the flow we observe that
\[
I(u_k) \leq \frac{J_{T,\omega,-\varepsilon}(u_k) + B_c}{\delta_c/2} \leq \frac{J_{T,\omega,-\varepsilon}(0) + B_c}{\delta_c/2} \leq C \quad (4.3)
\]
for some uniform constant $C > 0$ (independent of $\varepsilon$ and $k$). Hence by choosing a subsequence we may assume that $u_k \to u_\infty \in \mathcal{E}^1(X, \hat{\chi})$ in $L^1$ and $\chi_{u_k} \to \chi_{u_\infty} := \hat{\chi} + \frac{\chi}{2\pi i} \partial \bar{\partial} u_\infty$ weakly as $k \to \infty$, where $\mathcal{E}^1(X, \hat{\chi})$ denotes the space of $\hat{\chi}$-PSH function with finite energy (see Remark 2.15 and [BBGZ13, Lemma 3.3]). So by [GZ07, Corollary 1.8] the function $u_\infty$ has zero Lelong numbers. Moreover, by (2.4) and (4.3) we have a uniform bound for the Dirichlet energy (independent of $\varepsilon$ and $k$)
\[
\int_X |\partial u_k|^2 \omega^n \leq C. \quad (4.4)
\]
4.1.1. **Local smoothing.** The Kähler current $\chi_{u_\infty}$ can be regarded as a solution to (1.1) in a weak sense. Now we establish estimates for local smoothing of $\chi_{u_\infty}$ in Euclidean balls. As in [Che21, Section 4] we choose a finite covering $\{B_{j,AR}\}_{j \in J}$ of $X$ satisfying the following properties:

- Each $B_{j,AR}$ is a coordinate chart of $X$ which is biholomorphic to a Euclidean ball $B_{AR}(0)$ of radius $4R$ centered at the origin in $\mathbb{C}^n$.
- $\{B_{j,R}\}_{j \in J}$ is also a covering of $X$, where $B_{j,R} \simeq B_R(0) \subset \mathbb{C}^n$.
- Set $v = \sum_{p=1}^n v^p \frac{\partial}{\partial \bar{z}^p}$. Since $v$ is fixed we may assume that $|v^p| \leq S$ for some uniform constant $S > 0$ (independent of a choice of the covering $\{B_{j,AR}\}_{j \in J}$).
- For any fixed constant $\sigma > 0$ we can choose $R > 0$ and a finite covering $\{B_{j,AR}\}_{j \in J}$ so that

$$
(1 - \sigma)\omega_j \leq \omega \leq (1 + \sigma)\omega_j, \quad |v^p - c^p| \leq \sigma \quad (p = 1, \ldots, 2n)
$$

on each $B_{j,AR}$, where $\omega_j$ denotes the Kähler form with constant coefficients corresponding to the Euclidean metric $g_j$ and $c^p \in \mathbb{R}$. We define a holomorphic vector field $\xi_j$ with constant coefficients by $\xi_j := \sum_{p=1}^n c^p \frac{\partial}{\partial \bar{z}^p}$.

- On each ball $B_{j,AR}$ we take local potential functions $\phi_{\omega,j}, \phi_{\bar{\omega},j}$ as

$$
\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_{\omega,j}, \quad \bar{\omega} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_{\bar{\omega},j}
$$

with properties

$$
|\phi_{\omega,j} - |z|^2| \leq \frac{1}{10000} R^2, \quad |\nabla \theta_v^X(\bar{\omega})| + |\nabla \phi_{\omega,j}| + |\nabla \phi_{\bar{\omega},j}| + |\nabla^2 \phi_{\omega,j}| \leq S.
$$

For each $j \in J$ and any locally $L^1$-integrable function $f$ on $B_{j,AR}$, the smoothing of $f$ with scale $r \in (0, R)$ is defined by

$$
f^{(r)}(z) := \int_{\mathbb{C}^n} r^{-2n} \rho \left( \frac{|y|}{r} \right) f(z - y) d\text{Vol}_{g_j}(y),
$$

where $\rho(t)$ is a smooth non-negative function with support in $[0, 1]$, and is constant on $[0, 1/2]$. Moreover, the function $\rho$ satisfies the normalization condition

$$
\int_{B_1(0)} \rho(|y|) d\text{Vol}_{g_j}(y) = 1.
$$

The derivative bounds for $\phi_{\bar{\omega},j}$ and $\theta_v^X(\bar{\omega})$ imply that there exists a constant $r_0 \in (0, R)$ such that for all $r \in (0, r_0)$ and $j \in J$ we have

$$
|\text{Re}(v)(\phi_{\bar{\omega},j}^{(r)} - \phi_{\bar{\omega},j})| \leq \sigma, \quad |\theta_v^X(\bar{\omega})^{(r)} - \theta_v^X(\bar{\omega})| \leq \sigma.
$$

For each $k$ and $j \in J$ we set

$$
u_{k,j} := \phi_{\bar{\omega},j} + u_k, \quad u_{\infty,j} := \phi_{\omega,j} + u_{\infty}.
$$

These are locally $L^1$-integrable functions satisfying

$$
\chi_{\nu_k} = \bar{\omega} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_k = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}, \quad \chi_{u_{\infty}} = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}.
$$

**Lemma 4.2.** There are constants $\varepsilon_8 \in (0, \varepsilon_7)$ and $\delta > 0$ such that the following holds: for any $\varepsilon \in (0, \varepsilon_8)$ there exists constant $\sigma_0 > 0$ such that if $\sigma \in (0, \sigma_0)$ then
\[(1) \quad P_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) - \theta_v^X (\bar{\chi}) - \text{Re}(v)(u_{\infty,j}^{(r)} - \phi_{\chi,j}) \leq c_X - \delta \varepsilon, \]

\[(2) \quad Q_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) \leq K_0 + 5C_\theta \]

on \( B_{j,3R} \) for all \( j \in J \) and \( r \in (0,r_0) \).

**Proof.** By Lemma 2.7 and (4.2) we know that

\[ \left| Q_{\omega_j} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j} \right) - Q_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j} \right) \right| \leq C_1 \sigma, \quad (4.5) \]

\[ \left| F_{\omega_j,\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j} \right) - F_{\omega,\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j} \right) \right| \leq C_1 \sigma \quad (4.6) \]

for \( C_1 = C_1(K_0, n) \). By using the convexity of \( Q_{\omega_j} \), (4.5) and the fact that the coefficients of \( \omega_j \) are constants we have

\[
\frac{1}{1 + \sigma} Q_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}^{(r)} \right)(z) \leq Q_{\omega_j} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}^{(r)} \right)(z) \\
\leq \int_{\C^n} r^{-2n} \rho \left( \frac{|y|}{r} \right) Q_{\omega_j} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}^{(r)} \right)(z - y)d\text{Vol}_{g_j}(y) \\
\leq \int_{\C^n} r^{-2n} \rho \left( \frac{|y|}{r} \right) Q_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}^{(r)} \right)(z - y)d\text{Vol}_{g_j}(y) + C_1 \sigma \\
\leq K_0 + 3C_\theta + C_1 \sigma.
\]

If \( \sigma \) satisfies \( (1 + \sigma)(K_0 + 3C_\theta + C_1 \sigma) \leq K_0 + 5C_\theta \) then

\[ Q_{\omega_j} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}^{(r)} \right) \leq K_0 + 5C_\theta, \quad Q_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) \leq K_0 + 5C_\theta. \]

By letting \( k \to \infty \) we have

\[ Q_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) \leq K_0 + 5C_\theta, \]
which shows (2). Since coefficients of \( \omega_j, \xi_j \) are constants, by using (4.6) and convexity of \( F_{\omega_j, -\varepsilon} \) we compute

\[
F_{\omega_j, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j}^{(r)} \right)(z) - \text{Re}(\xi_j)(u_k^{(r)}(z)) = \int C_n \rho \left( \frac{|y|}{r} \right) \left[ F_{\omega, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j} \right) (z) - \text{Re}(\xi_j)(u_k)(z) \right] d \text{Vol}_{g_j}(y)
\]

and hence

\[
\int C_n \rho \left( \frac{|y|}{r} \right) \left[ F_{\omega, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{k,j} \right) (z) - \text{Re}(v)(u_k)(z) \right] d \text{Vol}_{g_j}(y) \leq C_2 \sigma.
\]

Thus by taking the limit \( k \to \infty \) we get

\[
F_{\omega_j, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) - \xi_j(u_{\infty}^{(r)}) \leq c_\varepsilon + (\theta^X_v(\hat{\chi}))^{(r)} + C_3 \sigma \leq c_\varepsilon + \theta^X_v(\hat{\chi}) + C_4 \sigma
\]

on \( B_{j,3R} \) for all \( j \in J \) and \( r \in (0, r_0) \). By using (4.4) again we have

\[
|\text{Re}(v - \xi_j)(u_k^{(r)})| \leq C_5 \sigma,
\]

and hence

\[
|\text{Re}(v - \xi_j)(u_{\infty}^{(r)})| \leq C_6 \sigma.
\]

Meanwhile, by using (2) and Lemma 2.7 we find that

\[
\left| F_{\omega_j, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) - F_{\omega_j, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) \right| \leq C_9 \sigma.
\]

Combining (4.7), (4.8) and (4.9) and Lemma 2.8 we obtain

\[
P_{\omega, -\varepsilon} \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} \right) - \theta^X_v(\hat{\chi}) - \text{Re}(v)(u_{\infty}^{(r)}) \leq c_\varepsilon + C_7 \sigma.
\]
from our choice of \( r \in (0, r_0) \). Consequently we obtain
\[
P_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u^{(r)}_{\infty,j} \right) - \theta^X(\bar{x}) - \text{Re}(v)(u^{(r)}_{\infty,j} - \phi_{\infty,j}) \leq c_\varepsilon + C_8 \sigma.
\]
Recalling the fact that \( c_\varepsilon = c_X - \varepsilon \frac{a_n}{\beta^n} \) we will get (1) for some \( \delta > 0 \) if \( \sigma < \varepsilon^2 \) and \( \varepsilon \) is sufficiently small. \( \square \)

4.1.2. **Gluing argument.** For \( z \in B_{j,3R} \) and \( r \in (0, R) \) we set
\[
u_{j,u^{(r)}}(z) := \sup_{B_{j,r}(z)} u^{(r)}_{\infty,j},
\]
where \( B_{j,r}(z) \) denotes a Euclidean ball of radius \( r \) centered at \( z \) in \( B_{j,4R} \). For \( r \in (0, R/2) \) we define
\[
u_{j,u^{(r)}}(z,r) := \frac{u_{j,\frac{3}{4}R}(z) - u_{j,r}(z)}{\log \left( \frac{3}{4}R \right) - \log r}, \quad z \in B_{j,3R}.
\]
As \( r \to 0 \) the quantity \( \nu_{j,u^{(r)}}(z,r) \) converges decreasingly to the Lelong number of \( u^{(r)}_{\infty,j} \) with respect to a reference metric \( \tilde{\chi} \):
\[
\lim_{r \to 0} \nu_{j,u^{(r)}}(z,r) = \nu_{u^{(r)}}(z).
\]
By adapting the idea from Blocki–Kołodziej [BK07], Chen [Che21, Lemma 4.2] proved the following lemma, which says that one can control the difference between \( u_{j,r}(z) \) and \( u^{(r)}_{\infty,j}(z) \) by means of \( \nu_{j,u^{(r)}}(z,r) \).

**Lemma 4.3.** For any \( r \in (0, R/2) \) and \( z \in B_{j,3R} \) we have
\[
\begin{align*}
(1) \quad 0 & \leq u_{j,r}(z) - u_{j,\frac{3}{4}R}(z) \leq (\log 2) \nu_{j,u^{(r)}}(z,r), \\
(2) \quad 0 & \leq u_{j,r}(z) - u^{(r)}_{\infty,j}(z) \leq \eta \nu_{j,u^{(r)}}(z,r), \quad \text{where} \quad \eta > 0 \quad \text{is a universal constant depending only on} \ n.
\end{align*}
\]

Let \( \lambda_1 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u^{(r)}_{\infty,j} \) with respect to \( \omega \). Then Lemma 4.2 yields that
\[
\sum_{i=1}^n \frac{1}{\lambda_i} \leq K_0 + 5C_\theta,
\]
which gives a uniform bound \( \lambda_i \geq (K_0 + 5C_\theta)^{-1} \). By using this we observe that the form \( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u^{(r)}_{\infty,j} - \varepsilon^2 \omega \) is Kähler and
\[
P_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u^{(r)}_{\infty,j} - \varepsilon^2 \omega \right) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} - \varepsilon^2
\]
\[
= \sum_{i=1}^{n-1} \frac{1}{\lambda_i} + \sum_{i} \frac{\varepsilon^2}{(\lambda_i - \varepsilon^2) \lambda_i}
\]
\[
\leq \sum_{i=1}^{n-1} \frac{1}{\lambda_i} + C \varepsilon^2
\]
\[
= P_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u^{(r)}_{\infty,j} \right) + C \varepsilon^2
\]
by decreasing \( \varepsilon_8 \) if necessary, where \( C > 0 \) is a uniform constant depending only on the lower bound for \( \lambda_i \). Also we have

\[
\text{Re}(v)(u_{\infty,j}^{(r)} - \phi_{\tilde{\chi},j} - \varepsilon^2 \phi_{\omega,j}) = \text{Re}(v)(u_{\infty,j}^{(r)} - \phi_{\tilde{\chi},j} - \varepsilon^2 \text{Re}(v)(\phi_{\omega,j}) \geq \text{Re}(v)(u_{\infty,j}^{(r)} - \phi_{\tilde{\chi},j} - C\varepsilon^2)
\]

by using \( |v^\rho| \leq S \) and \( |\nabla \phi_{\omega,j}| < S \). Thus by decreasing \( \varepsilon_8 \) again we get

\[
P_\omega \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_{\infty,j}^{(r)} - \varepsilon^2 \omega \right) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(u_{\infty,j}^{(r)} - \phi_{\tilde{\chi},j} - \varepsilon^2 \phi_{\omega,j}) \leq c_X - \delta\varepsilon + C\varepsilon^2 < c_X.
\]

Now we fix \( \varepsilon \in (0, \varepsilon_8) \), \( \sigma \in (0, \sigma_0) \) and set

\[
\varphi_{j,r} := u_{\infty,j}^{(r)} - \phi_{\tilde{\chi},j} - \varepsilon^2 \phi_{\omega,j}.
\]

**Lemma 4.4.** There exists \( r_1 \in (0, r_0) \) such that for any \( r \in (0, r_1) \) and \( z \in X \) we have

\[
\max_{\{j \in J \mid z \in B_{j,1} \setminus B_{j,2R}^1 \}} \varphi_{j,r}(z) \leq \max_{\{j \in J \mid z \in B_{j,1} \}} \varphi_{j,r}(z) - \frac{\varepsilon^2 R^2}{2}.
\]

**Proof.** Suppose that \( z \in (B_{j,3R} \setminus B_{j,2R}^2) \cap B_{i,R}^1 \). The function \( \nu_{j,u_{\infty,j}}(z, r) \) is increasing with respect to \( r \) (when fixing \( z \)) and upper semi-continuous with respect to \( z \) (when fixing \( r \)). Moreover, the Lelong number \( \nu_{u_{\infty}} \) is zero everywhere. Thus we can apply the Dini–Cartan lemma to know that for a constant \( A > 0 \) determined later, after decreasing \( r_0 \) for any \( r \in (0, r_0) \) and \( z \in B_{j,3R} \) we have

\[
0 \leq \nu_{j,u_{\infty,j}}(z, r) \leq A^{-1}\varepsilon^2.
\]

We compute

\[
u_{j,r}^{(r)}(z) \leq u_{j,r}(z)
\]

\[
\leq u_{j,1}(z) + (\log 2)A^{-1}\varepsilon^2
\]

\[
\leq \sup_{B_{j,2}(z)} (\phi_{\tilde{\chi},j} + u_\infty) + (\log 2)A^{-1}\varepsilon^2
\]

\[
\leq \sup_{B_{j,2}(z)} \phi_{\tilde{\chi},j} + \sup_{B_{j,2}(z)} u_\infty + A^{-1}\varepsilon^2.
\]

It follows that

\[
\varphi_{j,r}(z) = u_{\infty,j}^{(r)}(z) - \phi_{\tilde{\chi},j}(z) - \varepsilon^2 \phi_{\omega,j}(z)
\]

\[
\leq \sup_{B_{j,2}(z)} u_\infty + \sup_{B_{j,2}(z)} \phi_{\tilde{\chi},j} - \phi_{\tilde{\chi},j}(z) + A^{-1}\varepsilon^2 - \varepsilon^2 \phi_{\omega,j}(z)
\]

\[
\leq \sup_{B_{j,2}(z)} u_\infty + Sr + A^{-1}\varepsilon^2 - 3\varepsilon^2 R^2.
\]

On the other hand,

\[
u_{\infty,j}^{(r)}(z) \geq u_{i,r}(z) - \eta A^{-1}\varepsilon^2
\]

\[
= \sup_{B_{i,r}(z)} (\phi_{\tilde{\chi},i} + u_\infty) - \eta A^{-1}\varepsilon^2
\]

\[
\geq \sup_{B_{i,r}(z)} u_\infty + \inf_{B_{j,r}(z)} \phi_{\tilde{\chi},i} - \eta A^{-1}\varepsilon^2.
\]
This shows that
\[
\varphi_{i,r}(z) = u_{\infty,i}^{(r)}(z) - \phi_{\hat{\chi},i}(z) - \varepsilon^2 \phi_{\omega,i}(z) \\
\geq \sup_{B_{i,r}(z)} u_{\infty} + \inf_{B_{i,r}(z)} \phi_{\hat{\chi},i} - \phi_{\hat{\chi},i}(z) - \eta A^{-1} \varepsilon^2 - \varepsilon^2 \phi_{\omega,i}(z) \\
\geq \sup_{B_{i,r}(z)} u_{\infty} - S r - \eta A^{-1} \varepsilon^2 - 2 \varepsilon^2 R^2.
\]

The above computation yields that
\[
\varphi_{i,r}(z) - \varphi_{j,r}(z) \geq \sup_{B_{i,r}(z)} u_{\infty} - \sup_{B_{j,r}(z)} u_{\infty} - 2 S r - (\eta + 1) A^{-1} \varepsilon^2 + \varepsilon^2 R^2.
\]

From the choice of the covering \(\{B_{j,AR}\}_{j \in J}\) one can easily see that \(B_{j,\hat{\tau}}(z) \subset B_{i,r}(z)\). Thus
\[
\varphi_{i,r}(z) - \varphi_{j,r}(z) \geq -2 S r - (\eta + 1) A^{-1} \varepsilon^2 + \varepsilon^2 R^2.
\]

By choosing \(A = \frac{3(\eta+1)}{R^2}\) we have
\[
\varphi_{i,r}(z) - \varphi_{j,r}(z) \geq -2 S r + \frac{2 \varepsilon^2 R^2}{3}.
\]

So if we set \(r_1 = \frac{\varepsilon R^2}{12 S}\) then for any \(r \in (0, r_1)\) we conclude that
\[
\varphi_{i,r}(z) - \varphi_{j,r}(z) \geq \frac{\varepsilon^2 R^2}{2}.
\]

\[\square\]

**Proof of “coercivity \(\Rightarrow\) existence” in Theorem 1.2.** From Lemma 4.4 we know that for any \(r \in (0, r_1)\) the regularized maximum \(\varphi\) of \(\{(B_{j,3R}, \varphi_{j,r})\}_{j \in J}\) (for a sufficiently small vector \(\eta\)) is a smooth function on \(X\). Define \(\chi := \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi\). Then \(\chi\) satisfies
\[
P_\omega(\chi) - \theta^X(\hat{\chi}) - \Re(v)(\varphi) < c_X
\]
by Proposition 2.11. After averaging it over \(T\) (Proposition 2.12) we obtain a \(T\)-invariant Kähler form \(\tilde{\chi} \in \beta\) satisfying
\[
P_\omega(\tilde{\chi}) - \theta^X(\tilde{\chi}) < c_X.
\]

Thus the result follows from Theorem 1.1. \(\square\)

### 4.2. Existence implies coercivity.

We will prove the converse direction of Theorem 1.2 and Theorem 1.3. The proof is essentially due to [CS17 Proposition 22], but we will give it here for the sake of completeness.

**Proof of “existence \(\Rightarrow\) coercivity” in Theorem 1.2.** Assume that there exists a solution \(\hat{\chi} \in \beta\) to the modified \(J\)-equation
\[
\Tr_{\hat{\chi}} \omega = c_X + \theta^X(\hat{\chi}).
\]

Take a sufficiently small \(\delta > 0\) so that \(\omega' := \omega - \delta \hat{\chi} > 0\). Then
\[
\Tr_{\hat{\chi}} \omega' = c_X - n \delta + \theta^X(\hat{\chi}),
\]
i.e. the Kähler form \( \hat{\chi} \) solves the modified \( J \)-equation with respect to \( \omega' \). In particular, we have \( J_{T,\omega'}(\phi) \geq -C \) for all \( \phi \in \mathcal{H}^T \) [LS16, Corollary 3.2]. If we normalize \( J_{T,\omega}, J_{T,\omega'} \) so that \( J_{T,\omega}(0) = 0 \) and \( J_{T,\omega'}(0) = 0 \), then we have

\[
J_{T,\omega}(\phi) = J_{T,\omega'}(\phi) - n\delta \int_0^1 \int_X \phi (\chi_{t\phi}^n - \hat{\chi} \wedge \chi_{t\phi}^{n-1}) dt
\]

for all \( \phi \in \mathcal{H}^T \). Integrating by parts we get

\[
- \int_X \phi (\chi_{t\phi}^n - \hat{\chi} \wedge \chi_{t\phi}^{n-1}) = - \int_X \phi \chi_{t\phi}^{n-1} \left( t \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) = \frac{\sqrt{-1}}{2\pi} \int_X \partial \phi \wedge \bar{\partial} \phi \wedge \chi_{t\phi}^{n-1} = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n-1} \int_X \partial \phi \wedge \bar{\partial} \phi \wedge \chi_i \wedge \hat{\chi}^{n-1-i},
\]

where \( p_i(t) := \binom{n-1}{i} t^{i+1} (1-t)^{n-1-i} \). We take a constant \( \kappa > 0 \) (depending only on \( n \)) sufficiently small so that

\[
\int_0^1 p_i(t) dt \geq \kappa, \quad i = 1, \ldots n-1.
\]

Then we obtain

\[
-n\delta \int_0^1 \int_X \phi (\chi_{t\phi}^n - \hat{\chi} \wedge \chi_{t\phi}^{n-1}) dt = \frac{\sqrt{-1}}{2\pi} n\delta \sum_{i=1}^{n-1} \int_0^1 p_i(t) dt \int_X \partial \phi \wedge \bar{\partial} \phi \wedge \chi_i \wedge \hat{\chi}^{n-1-i} \geq \frac{\sqrt{-1}}{2\pi} n\delta \kappa \int_X \partial \phi \wedge \bar{\partial} \phi \wedge \sum_{i=1}^{n-1} \chi_i \wedge \hat{\chi}^{n-1-i} = n\delta \kappa I(\phi).
\]

Thus we obtain

\[
J_{T,\omega}(\phi) \geq n\delta \kappa I(\phi) - C
\]

for all \( \phi \in \mathcal{H}^T \).

As a corollary of this we can show that the solvability of the \( J \)-equation does not depend on a choice of \( \omega \in \alpha \).

**Proof of Theorem 1.3.** Thanks to Theorem 1.2 it suffices to show that the coercivity of \( J_{T,\omega} \) does not depend on the choice of \( \omega \). If \( \omega, \omega' \in \alpha \) are \( T \)-invariant Kähler forms there exists a \( T \)-invariant smooth function \( \varphi_0 \) such that \( \omega = \omega' + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_0 \). Then by
adjusting constants we obtain
\[ J_{T,\omega}(\phi) - J_{T,\omega}(\phi) = n \int_0^1 \int_X \phi \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_0 \wedge \chi^{n-1}_{t_0} dt \]
\[ = n \int_0^1 \int_X \varphi_0 \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \wedge \chi^{n-1}_{t_0} dt \]
\[ = \int_0^1 \int_X \varphi_0 \frac{d\chi^n_{t_0}}{dt} dt \]
\[ = \int_X \varphi_0 (\chi^n_{t_0} - \hat{\chi}^n), \]
and hence
\[ |J_{T,\omega}(\phi) - J_{T,\omega}(\phi)| \leq \int_X |\varphi_0| \chi^n_{t_0} + C \leq C \]
as desired.

5. A Nakai–Moishezon type criterion for compact toric manifolds

5.1. Local $C^2$-estimate for the generalized equation. Let $X$ be a compact complex manifold and $c > 0$, $b \geq 0$ are constants with relation
\[ b = \frac{c \beta^n \cdot n \alpha \cdot \beta^{n-1}}{\alpha^n}. \]
In this subsection we establish the local $C^2$-estimate of the generalized equation
\[ \text{Tr}_{\chi \phi} \omega + b \frac{\omega^n}{\chi^n_{\phi}} = c + \theta^X_v (\chi_{\phi}). \] (5.1)

Lemma 5.1. Let $\phi \in \mathcal{H}^T$ be a solution to (5.1). Assume that there exists a Kähler form $\tilde{\chi} = \tilde{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi$ defined on the closure $\overline{\Omega}$ of an open subset $\Omega \subset X$ satisfying
\[ P_\omega(\tilde{\chi}) - \theta^X_v (\tilde{\chi}) - \text{Re}(v)(\psi) < c - 2\delta \]
for some $\delta > 0$ (where $\Omega$, $\psi$ and $\tilde{\chi}$ do not have to be $T$-invariant). Then
\[ \text{Tr}_{\omega} \chi_{\phi} \leq C e^{N(\phi - \inf_\Omega \phi)} \]
on $\overline{\Omega}$, where the constants $C > 0$, $N > 0$ depend on $\tilde{\chi}$, $\psi$, $\delta$, an upper bound of $c$ and the maximum of $\text{Tr}_{\omega} \chi_{\phi}$ on $\partial \Omega$.

Proof. The proof is essentially the same as Lemma 3.4. Set $\varphi := \phi - \psi$ so that $\chi_{\phi} = \tilde{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$. Define the operator $\mathcal{L}_b$ by
\[ \mathcal{L}_b := \partial_{i\bar{j}} F_b(A) g^{ik} \nabla_j \nabla_k - \text{Re}(v), \]
where $A = g^{ik} h_{ijk}$ and $\nabla$ denotes the Chern connection with respect to $\omega$. We take normal coordinates of $\omega$ on which $\chi_{\phi}$ is diagonal. From the equation (5.1) we know that $\chi_{\phi} \geq C_1^{-1} \omega$ for some constant $C_1 > 0$ depending on an upper bound of $c$. Then we have
\[ \mathcal{L}_b \log \text{Tr}_{\omega} \chi_{\phi} \leq C_2 \]
(indeed, we may substitute $\frac{d}{dt} \phi_t = 0$ to the computation in Lemma 3.4 since $\phi$ is a stationary point of the modified $J$-flow). Similarly by using (5.1) and $\varphi = \phi - \psi$ a direct computation shows that

$$L_b \varphi = -F_b(A) + c + \theta_v^X(\tilde{\chi}) + \text{Re}(v)(\psi) + \partial_{\bar{i}} F_b(A)(h_{\bar{i}i} - \tilde{h}_{\bar{i}i}),$$

where $\tilde{h}$ denotes the Riemannian metric corresponding to $\tilde{\chi}$. Define the function $G: \Omega \to \mathbb{R}$ by $G := \log \text{Tr}_\omega \chi_\phi - N \varphi$. Then the above computation shows that

$$L_b G \leq C_2 + N F_b(A) - N c - N \theta_v^X(\tilde{\chi}) - N \text{Re}(v)(\psi) + N \partial_{\bar{i}} F_b(A)(\tilde{h}_{\bar{i}i} - h_{\bar{i}i}).$$

We take a large constant $N > 0$ so that $C_2 \leq N \delta$. If the function $G$ takes maximum at some point $x_0 \in \Omega$ then we have

$$-\delta \leq F_b(A) - c - \theta_v^X(\tilde{\chi}) - \text{Re}(v)(\psi) + \partial_{\bar{i}} F_b(A)(\tilde{h}_{\bar{i}i} - h_{\bar{i}i}),$$

and hence

$$P_\omega(\tilde{\chi}) + \delta \leq F_b(A) + \partial_{\bar{i}} F_b(A)(\tilde{h}_{\bar{i}i} - h_{\bar{i}i})$$

at $x_0$. Thus we can apply the same argument in the second half of the proof for Lemma 5.1 just by replacing $\tilde{\chi}$ with $\tilde{\chi}$, and get

$$\text{Tr}_\omega \chi_\phi \leq C e^{N(\varphi - \inf \Phi \varphi)}.$$

Finally by using a bound of $\psi$ we obtain the desired estimate.

5.2. Continuity method and induction for toric subvarieties. In what follows let $X$ be a compact toric manifolds with the standard torus $T$. Let $v$ be a holomorphic vector field with $\text{Im}(v) \in \mathfrak{t}$, and assume that Kähler forms $\omega \in \alpha$, $\tilde{\chi} \in \beta$ are $T$-invariant. For any $p$-dimensional toric subvarieties $Y \subset X$ we define the set $\Gamma_{\omega, \tilde{\chi}}(Y)$ consisting of germs of $\tilde{\chi}$-Kähler potential $\phi$ satisfying

$$Q_{X, \omega} \left( \tilde{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) - \theta_v^X(\tilde{\chi}) - \text{Re}(v)(\phi) < c_X$$

at $Y \subset X$. When $Y$ is irreducible, the standard torus $T_Y$ attached to $Y$ is a quotient of $T$ via the natural surjection morphism $T \to T_Y$. In particular, the $T$-action on $Y$ is just obtained by composing this morphism with the $T_Y$-action on $Y$. Define

$$\mathcal{H}^Y_{T_Y} := \{ \phi \in C^\infty(Y; \mathbb{R})^{T_Y} | \chi_\phi := \tilde{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi > 0 \}.$$ 

Note that the restriction of $\theta_v^X(\tilde{\chi})$ to $Y$ differs from $\theta_v^Y(\tilde{\chi})$ by

$$\theta_v^Y(\tilde{\chi}) = \theta_v^X(\tilde{\chi}) - I_Y,$$

where

$$I_Y := \frac{1}{\beta^p} \int_Y \theta_v^X(\tilde{\chi}) \tilde{\chi}^p.$$

Since $\theta_v^Y(\chi_\phi) = \theta_v^Y(\tilde{\chi}) + v(\phi)$ we have

$$\theta_v^Y(\chi_\phi) = \theta_v^Y(\tilde{\chi}) + v(\phi) - I_Y \quad (5.2)$$

\footnote{In this paper the word "$p$-dimensional subvariety" means that it is reduced, but not necessary to be irreducible nor pure $p$-dimensional.}
for all $\phi \in H^T_Y$.

**Theorem 5.2.** Let $X$, $T$, $v$, $\omega$, $\hat{\chi}$, $\alpha$, $\beta$ as above. Assume $m_X > 0$ and there exists a $T$-invariant Kähler form $\chi \in \beta$ such that

$$\int_Y (((c_X + \theta^X_v(\chi)) \omega^p - p \omega \wedge \chi^{p-1}) > 0$$

for all $p$-dimensional irreducible toric subvarieties $Y \subset X$ ($p = 1, \ldots, n - 1$). Then for any $p$-dimensional toric subvariety $Y \subset X$ ($p = 0, 1, \ldots, n - 1$) we have

$$\Gamma_{\omega, \hat{\chi}}(Y) \neq \emptyset.$$

**Proof.** We will prove by induction of $p = \dim Y$.

**Step 0.** First, we note that if $\dim Y = 0$, i.e. $Y$ is a union of finitely many points, then the result easily follows from $m_X = c_X + \min_X \theta^X_v(\hat{\chi}) > 0$, and the facts that $\text{Re}(v)$ vanishes at $Y$ and every Kähler class in a sufficiently small neighborhood of a point is trivial.

**Step 1.** Assume that $p \geq 1$ and the statement is true for all toric subvarieties $Z \subset X$ of $\dim Z \leq p - 1$. For any $p$-dimensional irreducible toric subvariety $Y \subset X$ let us consider the following continuity path $\chi_{\phi_t} = \hat{\chi} + \frac{1}{2\pi} \partial \bar{\partial} \phi_t \in \beta|_Y$ ($t \in [0, \infty)$) with $\phi_t \in H^T_Y$:

$$\text{Tr}_{\chi_{\phi_t}} \omega + b_t \frac{\omega^p}{\chi_{\phi_t}} = c_t + \theta^Y_v(\chi_{\phi_t}),$$

where

$$c_t := c_X(t + 1) + I_Y \geq c_X + I_Y > \frac{p \alpha \cdot \beta^{p-1}}{\beta^p} > 0,$$

$$b_0 := \frac{(c_X + I_Y) \beta^p - p \alpha \cdot \beta^{p-1}}{\alpha^p} > 0,$$

$$b_t := \frac{c_t \beta^p - p \alpha \cdot \beta^{p-1}}{\alpha^p} = b_0 + c_X \frac{\beta^p}{\alpha^p} t > 0$$

from the assumption. We note that the RHS satisfies

$$c_t + \min_Y \theta^Y_v(\chi_{\phi_t}) = c_t + \min_Y \theta^Y_v(\hat{\chi})$$

$$\geq c_X + I_Y + \min_Y (\theta^X_v(\hat{\chi}) - I_Y)$$

$$\geq c_X + \min_X \theta^X_v(\hat{\chi})$$

$$= m_X.$$

Set

$$\mathcal{T} := \{ t \in [0, \infty) \mid (5.3) \text{ admits a solution } \phi_t \in H^T_Y \}.$$

By Theorem 1.1 the solvability of (5.3) is equivalent to the existence of $\phi \in H^T_Y$ such that

$$P_\omega(\chi_{\phi}) - \theta^Y_v(\chi_{\phi}) < c_t.$$
From the uniform bound of $\theta_v^Y(\chi_\phi)$ we know that any $\phi \in \mathcal{H}^T_Y$ satisfies (5.4) for sufficiently large $t$. It follows that $\mathcal{T}$ is open, and there exists a large constant $T > 0$ such that $[T, \infty) \subset \mathcal{T}$.

Now we will show that $\mathcal{T}$ is closed. Let $t_0 := \inf \mathcal{T}$ and $D = \cup_i D_i$ be union of all toric divisors of $Y$. From the inductive hypothesis there exists an open neighborhood $U_D$ of $D$ in $X$ and a $\hat{\chi}$-Kähler potential $\phi_D$ on $\overline{U_D}$ such that

$$Q_{X,\omega} \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_D \right) - \theta_v^X(\hat{\chi}) - \text{Re}(v)(\phi_D) < c_X - \delta$$

for some $\delta > 0$. By Proposition 2.3 and (5.2) we have

$$Q_{Y,\omega} \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_D \right) - \theta_v^Y(\hat{\chi}) - \text{Re}(v)(\phi_D) < c_X + I_Y - \delta \leq c_{t_0} - \delta$$

on the closure $\overline{U_Y}$ of $U_Y := U_D \cap Y$ in $Y$ since $\text{Re}(v)$ is tangent to $Y$. For $t > t_0$ near $t_0$ we invoke Lemma 5.1 to reduce $C^2$-estimates of $\phi_t$ on $\overline{U_Y}$ to the boundary $\partial U_Y$. However, by Proposition 5.4 we obtain bounds of the form

$$\text{Tr}_{\omega} \chi_{\phi_t} \leq C e^{N(\phi_t - \inf_{Y\setminus U_Y} \phi_t)}$$

on $Y \setminus U_Y$. Eventually we obtain the same inequality globally on $Y$. Thus we can use the argument [SW08, Wei04, Wei06], the Evans–Klyrov theorem and Schauder estimates to obtain higher order estimates. This shows that $\mathcal{T}$ is closed, and hence $\mathcal{T} = [0, \infty)$.

**Step 2.** By Step 1 we have obtained $\phi \in \mathcal{H}^T_Y$ satisfying

$$Q_{Y,\omega} \left( \hat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) - \theta_v^Y(\hat{\chi}) - \text{Re}(v)(\phi) \leq c_X - 2\delta$$

as a solution to (5.3) for $t = 0$ on $Y$, where the existence of a positive constant $\delta > 0$ follows from $b_0 > 0$. Note that the line bundle $\mathcal{O}_Y(D_i)$ has a natural $T_Y$-action (for instance, see [Gon11, Example II.3]). We take a defining section $s_i$ of $D_i$ and $T_Y$-invariant fiber metric $h_i$ on $\mathcal{O}_Y(D_i)$. Then the curvature $\xi_i$ of $h_i$ has an expression

$$\xi_i = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |s_i|^2_{h_i}$$

on $Y \setminus D_i$. In particular we have a uniform bound

$$-C \omega \leq \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |s_i|^2_{h_i} \leq C \omega$$

on $Y \setminus D_i$. Moreover, applying $i_v$ we get

$$i_v \xi_i = -\frac{\sqrt{-1}}{2\pi} \bar{\partial} (\text{Re}(v)(\log |s_i|^2_{h_i})),$$

and hence

$$|\text{Re}(v)(\log |s_i|^2_{h_i})| \leq C$$

on $Y \setminus D_i$. Thus for sufficiently small $\kappa > 0$ the function

$$\psi := \phi + \kappa \sum_i \log |s_i|^2_{h_i}$$
satisfies
\[ Q_{Y,\omega}(\widehat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi) - \theta_v^X(\widehat{\chi}) - \text{Re}(v)(\psi) \leq c_X - \delta \]
on $Y \setminus D$. 

Now let $Y \subset X$ be a $p$-dimensional toric subvariety, $Y_k$ $(k = 1, \ldots, \ell)$ $p$-dimensional irreducible components of $Y'$ and $Y'_k \subset Y_k$ the complement of toric divisors in $Y_k$. By the induction hypothesis for $Z := Y \setminus \bigcup_k Y'_k$ we know that there exist a neighborhood $U$ of $Z \subset X$ and a $\widehat{\chi}$-Kähler potential $\phi_Z$ such that
\[ Q_{X,\omega}(\widehat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_Z) - \theta_v^X(\widehat{\chi}) - \text{Re}(v)(\phi_Z) < c_X \]
on $U$. On the other hand we apply the above construction to each $Y_k$, and know that there exists $\psi_k$ such that
\[ Q_{Y_k,\omega}(\widehat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi_k) - \theta_v^X(\widehat{\chi}) - \text{Re}(v)(\psi_k) \leq c_X - \delta \]
on $Y'_k$ and $\psi_k \to -\infty$ along the toric divisors of $Y_k$. Now we assume that $\ell = 1$ for simplicity (in general, we may just apply the argument to each $Y_k$). We can take open neighborhoods $W \subset V \subset U$ of $Z$ in $X$ such that
\begin{enumerate}
\item $\psi_1 > \phi_Z + 2$ in $Y_1 \cap (U \setminus V)$
\item $\psi_1 < \phi_Z - 2$ in $Y_1 \cap W$
\end{enumerate}
by subtracting a large constant from $\phi_Z$ since $\psi_1 \to -\infty$ along the toric divisors of $Y_1$. We take a smooth extension of $\psi_1$ near $Y_1 \setminus W$ and still denote it by $\psi_1$. We consider
\[ \tilde{\psi}_1 := \psi_1 + Bd_{Y_1}, \]
where $B > 0$ is a constant and $d_{Y_1}$ is the distance function from $Y_1$. By taking $B$ sufficiently large we observe that
\[ Q_{X,\omega}(\widehat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{\psi}_1) - \theta_v^X(\widehat{\chi}) - \text{Re}(v)(\tilde{\psi}_1) < c_X \]
in an open neighborhood $\tilde{U}$ of $Y_1 \setminus W$ in $X$ since $\text{Re}(v)(|d_{Y_1}|^2) = 0$ along $Y_1$. By using the continuity of $\psi_1$, after shrinking $\tilde{U}$ we have
\[ \tilde{\psi}_1 > \phi_Z + 1 \]
in $\tilde{U} \cap (U \setminus V)$ and
\[ \tilde{\psi}_1 < \phi_Z - 1 \]
in $\tilde{U} \cap W$. Let $\varphi$ be the regularized maximum of $(\tilde{U}, \tilde{\psi}_1)$ and $(V, \phi_Z)$. Then from the above observations we conclude that
\[ Q_{X,\omega}(\widehat{\chi} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi) - \theta_v^X(\widehat{\chi}) - \text{Re}(v)(\varphi) < c_X \]
on a small neighborhood of $Y$ in $X$. This completes the proof. □
Proof of Theorem 1.6. We apply Theorem 5.2 to the union of toric divisors $D$ in $X$, and obtain $\Gamma_{\omega,\chi}(D) \neq \emptyset$. Using this we proceed in the same way as Step 2 of the proof of Theorem 5.2 to find that (5.3) has a solution for all $t \in [0, \infty)$. Since $b_0 = 0$ from the definition of $c_X$ the solution $\phi_0 \in \mathcal{H}^T$ for $t = 0$ satisfies

$$\text{Tr}_{\omega} \chi_{\phi_0} = c_X + \theta^X_v(\chi_{\phi_0}).$$

\[\Box\]

5.3. Interior $C^2$-estimates. Let $X$ be an $n$-dimensional compact toric manifold. We identity the open dense $T$-orbit in $X$ with $\mathbb{R}^n \times (\mathbb{S}^1)^n$ and express $\omega, \chi$ as

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f, \quad \chi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} g$$

for some smooth strictly convex function $f, g: \mathbb{R}^n \to \mathbb{R}$. Assume that $f, g$ satisfy the equation

$$g^{ij} f_{ij} + b \frac{\det(D^2f)}{\det(D^2g)} = c + A(\nabla g) - \frac{1}{V} \int_P Ady,$$

where $b \geq 0$, $c > 0$, $V := \text{Vol}(P)$, $A: \mathbb{R}^n \to \mathbb{R}$ is an affine linear function, $P = \nabla g(\mathbb{R}^n)$ is the moment polytope determined uniquely from the Kähler class $\beta$ up to parallel translations. Also we assume that

$$c + A(\nabla g) - \frac{1}{V} \int_P Ady \geq \delta$$

(5.6)

for some uniform constant $\delta > 0$ (independent of $g$). The equation (5.5) as well as the condition (5.6) does not change when adding affine linear functions to $g$. Thus we may assume that $g(0) = 0$ and $\nabla g(0) = 0$. In particular since $0 \in P$ we obtain:

Lemma 5.3. For any compact subset $K \subset \mathbb{R}^n$ there exists $C > 0$ such that $\sup_K |g| < C$.

Let $h: P \to \mathbb{R}$ be the Legendre transform of $g$. Then we observe that

$$\nabla h(y) = (\nabla g)^{-1}(y), \quad D^2 h(y) = (D^2 g(\nabla h(y)))^{-1}.$$

(5.7)

By using (5.7) we can translate the equation (5.5) to

$$f_{ij}(\nabla h(y)) h_{ij} + b \frac{\det(D^2f(\nabla h(y))) \det(D^2h(y))}{\det(D^2g(\nabla h(y)))} = c + A(y) - \frac{1}{V} \int_P Ady$$

(5.8)

and the condition (5.6) to

$$c + A(y) - \frac{1}{V} \int_P Ady \geq \delta, \quad y \in P.$$

(5.9)

Our strategy is essentially the same as [CS17, Section 5.1], i.e. we establish higher order estimates for the Legendre transform $h$. If there is a sequence of equations (5.5) which violates the $C^2$ bound then we take the limit $k \to \infty$ and apply the Bian–Guan’s constant rank theorem [BG09, Theorem 1.1] to get a contradiction.
Proposition 5.4. Let $B \subset \mathbb{R}^n$ be the unit ball, and $f$, $g$ satisfy (5.5) with normalization $\inf_B g = g(0) = 0$. Then there exists $C > 0$ depending only on $\sup_B |g|$, bounds on $c$, $b$, the coefficients of $A$, lower bound $\delta$ in (5.9), $C^{3,\gamma}$ bounds on $f$ and a positive lower bound on the Hessian of $f$ such that

$$ \sup_{\frac{1}{2} B} |g_{ij}| < C. $$

Proof. Suppose that there are sequences $f_k$, $g_k$, $c_k$, $b_k$, $A_k$ satisfying the hypothesis, including $|g_k| < N$, but $|D^2 g_k(x_k)| > k$ for some $x_k \in \frac{1}{2} B$. The moment polytopes $P_k$ of $g_k$ satisfy $0 \in P_k$ and are parallel to each other, so by passing to a subsequence we may assume that $P_k \to P_\infty \subset \mathbb{R}^n$ via parallel translations, and hence $\int_{P_k} A_k dy \to \int_{P_\infty} A_\infty dy$. By shrinking the ball a bit we can assume that $g_k \to g$ uniformly for some strictly convex $g: B \to \mathbb{R}$. Also we have

$$ c_k + A_k(y) - \frac{1}{V} \int_{P_k} A_k dy \geq \delta, \quad y \in P_k $$

from the assumption, and

$$ g_{ij} > \tau \delta_{ij} \quad (5.10) $$

for some $\tau > 0$ independent of $k$ by (5.5), bounds of $c_k$, $A_k$ and a lower bound of $D^2 f_k$. Let $h_k$ be the Legendre transform of $g_k$ and set $U_k := (\nabla g_k)(B) \subset P_k$. Then we have $h_k(0) = 0$, $\nabla h_k(0) = 0$ from the normalization. Moreover, the formula (5.7) together with (5.10) yields an upper bound of $D^2 h_k$, and hence $h_k$ is uniformly bounded in $C^2$. Let $\nabla g$ denote the subdifferential of $g$, i.e.

$$ (\nabla g)(x) := \{ p \in \mathbb{R} | g(x') \geq g(x) + p(x' - x) \text{ for all } x' \in B \}, \quad x \in B $$

and

$$ (\nabla g)(E) := \bigcup_{x \in E} (\nabla g)(x) $$

for any subset $E \subset B$. Then the estimate for subdifferentials [CS17, Lemma 28] implies that for sufficiently large $k$ we have $(\nabla g)(0.9 B) \subset U_k$, and so $(\nabla g)(0.8 B)$ has a definite distance from $\partial U_k$ for large $k$. In addition $h_k$ solves the equation

$$ f_{k,ij}(\nabla h_k(y)) h_{k,ij} + b_k \det(D^2 f_k(\nabla h_k(y))) \det(D^2 h_k(y)) = c_k + A_k(y) - \frac{1}{V} \int_{P_k} A_k dy. $$

We use Proposition 5.5 together with the Schauder estimate, to obtain uniform $C^{3,\gamma}$ bounds for $h_k$ on $\nabla g(0.8 B)$. So combining with bounds for $f_k$, $c_k$, $b_k$, $A_k$, we can extract a convergent subsequence $h_k \to h_\infty$ on $\nabla g(0.8 B)$ in $C^{3,\gamma/2}$ satisfying an equation of the form

$$ f_{\infty,ij}(\nabla h_\infty(y)) h_{\infty,ij} + b_\infty \det(D^2 f_\infty(\nabla h_\infty(y))) \det(D^2 h_\infty(y)) = c_\infty + A_\infty(y) - \frac{1}{V} \int_{P_\infty} A_\infty dy. $$

As in [CS17, Proposition 27] we consider the following two cases separately:

- If we have a positive lower bound on $D^2 h_\infty$ this yields a uniform lower bound on $D^2 h_k$ on $\nabla g(0.8 B)$, and so a uniform upper bound for $D^2 g_k$ at all points for which $\nabla g_k(x) \in \nabla g(0.8 B)$. However, we have $\nabla g_k(0.7 B) \subset \nabla g(0.8 B)$ for
sufficiently large $k$ by [CS17, Lemma 28], and hence an upper bound of $D^2g_k$ on $0.7B$. This is a contradiction.

- If the Hessian $D^2h_\infty$ degenerates as some point then we apply the Bian–Guan’s constant rank theorem [BG09, Theorem 1.1] to know that $D^2h_\infty$ degenerates everywhere. Thus we have

$$\int_{\nabla g(0.8B)} \det(D^2h_\infty) = 0.$$ 

On the other hand,

$$\int_{\nabla g_k(0.7B)} \det(D^2h_k) = \text{Vol}(0.7B).$$

This contradicts the fact that $\nabla g_k(0.7B) \subset \nabla g(0.8B)$ for sufficiently large $k$ and $h_k \to h_\infty$ in $C^{3,\gamma/2}$.

□

Now we will establish the local $C^{2,\gamma}$-estimate for $h$. A main difference from the $J$-equation case [CS17, Proposition 29] is that we have to deal with the affine linear function $A$ in the RHS of (5.8). However we will observe that the function $A$ converges to a uniform positive constant through the blowup argument. So the Liouville rigidity result [CS17, Lemma 31] can also be applied to our case.

**Proposition 5.5.** Suppose that $h$ is a smooth convex function on the unit ball $B \subset \mathbb{R}^n$ satisfying the equation

$$a_{ij}(\nabla h)h_{ij} + b(\nabla h) \det(D^2h) = A,$$  

where $a_{ij}, b \in C^{1,\gamma}$, $b \geq 0$, $\lambda < a_{ij} < \Lambda$, $A$ is an affine linear function with lower bound $A \geq \delta$ for some constant $\delta > 0$. Then we have $\|h\|_{C^{2,\gamma}(\frac{1}{2}B)} < C$ for some constant $C > 0$ depending on $\lambda, \Lambda, \delta$, bound of the coefficients of $A$, $C^{1,\gamma}$ bounds for $a_{ij}, b$ and $C^2$ bounds for $h$.

**Proof.** Let

$$N_h := \sup_{y \in B} d_y|D^3h(y)|,$$

where $d_y := d(y, \partial B)$ denotes the distance from the boundary $\partial B$. Assume that $N_h > 1$ and the supremum is achieved at a point $y_0 \in B$. Define the function

$$\tilde{h}(y) = d_{y_0}^{-2}N_h^2h(y_0 + d_{y_0}N_h^{-1}y) - \ell(y),$$

where the affine function $\ell$ is chosen so that

$$\tilde{h}(0) = 0, \quad \nabla \tilde{h}(0) = 0.$$  

(5.12)

Note that the rescaled function $\tilde{h}$ is defined on the ball $B_{N_h}(0)$. Also a direct computation shows that

$$D^2\tilde{h}(y) = D^2h(y_0 + d_{y_0}N_h^{-1}y), \quad D^3\tilde{h}(y) = d_{y_0}N_h^{-1}D^3h(y_0 + d_{y_0}N_h^{-1}y).$$
In particular, we have $|D^3\tilde{h}(y)| \leq 2$ on $B_{2^{-1}N_h}(0)$. Moreover, since $|D^2\tilde{h}| = |D^2h| < C$ the normalization (5.12) implies that
\[
\|\tilde{h}\|_{C^3(B_{2^{-1}N_h}(0))} < C
\]
for a uniform constant $C > 0$. One can observe that the rescaled function $\tilde{h}$ satisfies an equation of the form
\[
\bar{a}_{ij}(\nabla \tilde{h}(y))\bar{h}_{ij}(y) + \bar{b}(\nabla \tilde{h}(y)) \det(D^2\tilde{h}(y)) = A(y_0 + d_{y_0}N_h^{-1}y),
\]
where the coefficients $\bar{a}_{ij}$, $\bar{b}$ has the same $C^0$ bounds as $a_{ij}$, $b$, whereas
\[
\sup |\nabla \bar{a}_{ij}| \leq d_{y_0}N_h^{-1} \sup |\nabla a_{ij}|, \quad \sup |\nabla \bar{b}| \leq d_{y_0}N_h^{-1} \sup |\nabla b|.
\]
The differentiating the equation (5.13) and using the standard Schauder theory we know that $\nabla \tilde{h}$ is bounded in $C^{2,\gamma}$. Now we argue by contradiction. So we suppose that there exists a sequence of convex functions $h_k$ on $B$ satisfying (5.11) with $a_{k,ij}$, $b_k$, $A_k$ and the associated constant $N_{h_k} > 4k$. For each $k$ we take a point $y_{0,k} \in B$ that attains the supremum of $N_{h_k}$. Then the rescaled function $\tilde{h}_k$ defined on $B_{4k}(0)$ satisfies an equation of the form
\[
\bar{a}_{k,ij}(\nabla \tilde{h}_k(y))\bar{h}_{k,ij}(y) + \bar{b}_k(\nabla \tilde{h}_k(y)) \det(D^2\tilde{h}_k(y)) = A_k(y_{0,k} + d_{y_{0,k}}N_h^{-1}y),
\]
has uniform $C^{3,\gamma}$ bounds on $B_{4k}(0)$ and satisfies $|D^3\tilde{h}_k(0)| = 1$. By passing to a subsequence we may assume that $y_{0,k} \to y_{0,\infty} \in B$. Combining with bounds for $a_{k,ij}$, $b_k$, $A_k$ as well as a standard diagonal argument shows that by taking a subsequence we can extract a convex limit $\tilde{h}_\infty : \mathbb{R}^n \to \mathbb{R}$ in $C^{3,\gamma/2}_{\text{loc}}$, satisfying $|D^3\tilde{h}_\infty(0)| = 1$ and
\[
\bar{a}_{\infty,ij}\tilde{h}_{\infty,ij}(y) + \bar{b}_\infty \det(D^2\tilde{h}_\infty(y)) = A_\infty(y_{0,\infty})
\]
with constant coefficients $\bar{a}_{\infty,ij}$, $\bar{b}_\infty$ due to (5.14). Since $A_\infty(y_{0,\infty})$ is a constant with a positive lower bound $\delta$, after a linear change of coordinates we can apply the Liouville rigidity result [CS17, Lemma 31] to conclude that $\tilde{h}_\infty$ is a quadratic polynomial, which contradicts the fact that $|D^3\tilde{h}_\infty(0)| = 1$. This completes the proof. □
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