Abstract

Weitzman (1979) introduced the Pandora’s Box problem as a model for sequential search with inspection costs, and gave an elegant index-based policy that attains provably optimal expected payoff. In various scenarios, the searching agent may select an option without making a costly inspection. Doval (2018) studied a version of Pandora’s problem that allows this, and showed that the index-based policy and various other simple policies are no longer optimal. Beyhaghi and Kleinberg (2019) gave the first non-trivial approximation algorithm for the problem, showing a simple policy with expected payoff at least a \((1 - \frac{1}{e})\)-fraction that of the optimal policy. No hardness result for the problem was known.

In this work we show that it is \textsc{NP}-hard to compute an optimal policy for Pandora’s problem with nonobligatory inspection. We also give a polynomial-time scheme that computes policies with expected payoff at least \((0.8 - \epsilon)\)-fraction of the optimal, for arbitrarily small \(\epsilon > 0\).

1 Introduction

The Pandora’s Box problem was introduced by Weitzman [18] as a model for sequential search with inspection costs. An agent is to make a selection from \(n\) options, viewed as locked boxes. Each box \(i\) contains a value \(v_i\) drawn independently from a known distribution \(F_i\), but \(v_i\) is revealed only if the agent opens box \(i\), incurring a search cost \(c_i\). At any step, the agent may choose to either select a box that has been opened and quit, or to open another box. Her goal is to maximize the expected value of the box selected, minus the search costs accrued along the way.

A policy for such a stochastic sequential problem may conceivably be adaptive in intricate ways. It may therefore come as a surprise that [Weitzman] showed the problem to admit a simple and elegant optimal policy: there are indices, one for each box \(i\), computable from \(F_i\) and \(c_i\), such that a ranking policy based on these indices maximizes the expected payoff.

Weitzman’s formulation and the index-based policy have been highly influential, and serve as the basis for many models that involve search frictions (see e.g. [2] for a survey and see [6] for a recent example). It was later recognized that the index-based policy was in fact a special case of Gittins [12]’s optimal algorithm for Bayesian bandits, an algorithm that is important in its own right in the computer science literature. (Gittins’s work predated Weitzman’s, but the two works were independent.)

In Weitzman [18]’s motivating scenario, one searches for a technology among various alternatives; to be able to adopt any technology, research expenditure (the search cost) must be incurred before one sees the technology’s benefit. In many other scenarios, however, it is more natural to allow
the agent the possibility to select an option without a costly inspection. For example, a student making a school choice may not always pay a campus visit when she is confident enough that a choice is superior. Doval [7], who pointed this out, pioneered the study of Pandora's problem with non-obligatory inspection (PNOI). Besides other results, she showed that various simple ranking policies are not optimal — in fact, optimal policies may be truly adaptive, in the sense that the order in which two options are inspected should depend on the outcome of the inspection of a third option.

In the absence of evidence that an efficient algorithm exists for computing optimal policies for PNOI, Beyhaghi and Kleinberg [3] initiated the study of approximation algorithms for the problem. One may easily attain at least \( \frac{1}{2} \) of the optimal payoff by using the better of two simple policies: (i) Weitzman’s index-based policy, and (ii) selecting the box with the highest expected value without any inspection. It is not hard to show that this strategy does not guarantee more than \( \frac{1}{2} \) of the optimal payoff. Beyhaghi and Kleinberg [3] gave the first non-trivial approximation algorithm for the problem, showing, for any PNOI instance, a simple, polynomial-time computable policy with payoff at least \( (1 - \frac{1}{e}) \) fraction of the optimal. Beyhaghi and Kleinberg showed rich algorithmic possibilities for the problem, although it was unresolved whether computing optimal policies is computationally intractable — the problem could be anywhere between \( \mathcal{P} \) and \( \mathcal{PSPACE} \)-complete.

In this work, we prove the first computational hardness result for PNOI. We also give computationally tractable policies with improved approximation ratios.

**Computational Hardness for PNOI**

We show that computing optimal policies for PNOI is \( \mathcal{NP} \)-hard (Theorem 3.2). We remark that it is more desirable to obtain hardness result for computing optimal policies than for computing the optimal payoff, because in many applications, what matters more is to have a step-by-step policy to follow, rather than to know the expected optimal payoff. There are stochastic optimization problems where computing the optimal expected outcome is demonstrably harder than implementing an optimal policy (see e.g. [13]).

To this end, we first identify a class of PNOI instances (Definition 3.1) that are simple enough to guarantee that the optimal policies are succinctly representable, by orderings of the \( n \) boxes. In these instances, each value distribution \( F_i \) is supported on \( \{0, \frac{1}{2}, 1\} \), with expectation less than \( \frac{1}{2} \), and index (in Weitzman’s sense) at least \( \frac{1}{2} \). It is easy to see that, if any box yields value 1 upon inspection, the search should stop. One can further show that, if no box opened so far has yielded a positive value, the search should continue, unless only one box remains, in which case it should be taken without inspection. The interesting case is when value \( \frac{1}{2} \) is discovered in a box: it turns out that one should henceforth relinquish the option to take a box without inspection; as a consequence, one should switch to running Weitzman’s index-based policy on the remaining boxes, taking the revealed value \( \frac{1}{2} \) as a free outside option. Such a policy is not pre-committed to an ordering, because its search order may change once a box yields value \( \frac{1}{2} \); however, the policy is still fully represented by the order in which it opens the boxes when all of them yield 0.

The possibility of changing search order after each inspection adds difficulty to calculating the policy’s expected payoff, but such a calculation is necessary for a reduction. Key to our analysis is to observe that a closely related policy has the non-exposed property, a property that was first crystallized by Kleinberg, Waggoner and Weyl [2016] and has been instrumental in several works in optimal search (e.g. [17] [10] [3]). This property allows us to derive a relatively clean expression for the difference between a policy’s expected payoff and that of Weitzman’s index-based policy (Lemma 3.6). Computing an optimal policy boils down to finding an ordering of the boxes whose corresponding policy minimizes this difference.

Finally, we give a fairly technical reduction from the classical Partition problem: given a set \( S \)
of \( n \) positive integers, decide whether they can be partitioned into two subsets with equal sums. We embed the \( n \) integers in the parameters of \( n \) boxes, and add two auxiliary boxes, \( B_{n+1} \) and \( B_{n+2} \). Box \( B_{n+2} \) has both high index and high cost, so designed that \( B_{n+2} \) is the one box possibly selected without inspection, but is the first box to be inspected if a value \( \frac{1}{2} \) is found. This creates an exquisite balance between, on one hand, the saving in cost when a high-cost box is selected without inspection, and, on the other, the motive to inspect a high-index box early on. The time point at which to switch to the index-based policy is affected by the position of \( B_{n+1} \). We are able to set the parameters so that the best partition of the integers in \( S \) is realized in the ordering of the optimal policy for the corresponding PNOI instance: the boxes before \( B_{n+1} \) and those after form the partition. The reduction is fairly involved technically due to the need of various approximations — the expected payoff even for such simple instances of PNOI is still complex and takes a fair amount of massaging to bear some resemblance to the sums in the Partition problem.

**Improved Approximation Algorithms.** Our second main result is an approximation scheme that, for any \( \epsilon > 0 \), computes in polynomial time a policy for PNOI with expected payoff at least \((0.8 - \epsilon)\)-fraction of the optimal (Theorem 4.6). This improves upon the approximation ratio of \( 1 - \frac{1}{e} \) by Beyhaghi and Kleinberg \cite{3}. As an intermediary step, for PNOI instances where the value distributions have \( O(1) \) support sizes, we give a polynomial-time approximation scheme (PTAS) — for any \( \epsilon > 0 \), we can compute in polynomial time a policy with expected payoff at least \((1 - \epsilon)\)-fraction of the optimal.

The workhorse behind these improved approximation algorithms is a framework, due to Fu et al. \cite{8}, for constructing PTAS for a broad class of stochastic sequential optimization problems. The main idea of the framework, to put it very roughly, is to start by considering systems with only an \( O(1) \) number of possible states. For such systems, one can show that, in the decision tree of a policy, nodes may be grouped into a small number of blocks — within each block, the system’s state remains the same and the ordering of the actions matters little for the eventual objective. One may therefore use a dynamic programming to exhaustively optimize over decision trees consisting of such blocks, with a loss of only a small fraction of the payoff. A natural way to cast PNOI in this framework is to let the state of the system at a time step be the highest value revealed up to that time. Further manipulations enable us to inherit the main theorem of Fu et al. and to obtain a PTAS for PNOI when there are only \( O(1) \) possible values (Theorem B.2).

A natural way to generalize from these restricted instances is to consider truncating and discretizing the values before applying the PTAS framework. While discretization works as one may expect, it is not clear if truncating values does not drastically hurt the payoff. It is conceivable that a significant portion of the optimal payoff comes from very high values that occur very rarely. Instead, we show that the policy resulting from the PTAS after value truncation and discretization, when combined with two simple policies, yields at least \((\frac{2}{3} - \epsilon)\)-fraction of the optimal payoff. That is, at least one of these three policies attains this approximation ratio, and they are all computable in polynomial time. The two simple policies are precisely the two ingredients for the simple \( \frac{1}{2} \)-approximation; as we mentioned above, using only these two policies yields no more than \( \frac{1}{2} \) of the optimal payoff in worst cases.

We further improve the approximation ratio to \( 0.8 - \epsilon \), by improving two of the policies we used: the truncated policy and the policy that takes the maximum expected value without inspection. For the truncated policy, we allow it to open one more box, the one with the maximum expected value, after it sees a value greater than the threshold of truncation. We show that the framework of \cite{8} can be massaged to compute an approximation to the best such “smartly truncated” policy. For the policy that takes the maximum expected value without inspection, we now allow it to first inspect
boxes with higher Weitzman’s indices; this makes the policy one of those proposed by Beyhaghi and Kleinberg [3]. We derive constraints that describe quantitatively the way in which these policies complement each other. These constraints form a factor revealing optimization problem. Solving this problem yields the worst case approximation ratio of 0.8.

1.1 Additional Related Works

Algorithms for natural variants of the Pandora box problem have received much attention lately. To name a few examples, Boodaghians et al. [4] studied the optimal search problem when there are constraints on the order in which the boxes may be inspected; Chawla et al. [5] studied the case when values in the boxes are correlated; Fu et al. [8] and Segev and Singla [16] used Pandora box problem with commitment as an application for their frameworks of designing PTAS and EPTAS, respectively, for stochastic combinatorial optimization problems. Singla [17] generalized the optimal search problem to settings known as Price of Information, where the set of options that may be selected is governed by combinatorial feasibility systems; Gamlath et al. [10] and Fu et al. [9] studied such settings when the feasibility systems are given by matchings in graphs.

Hardness for computing online optimal policies in Bayesian selection problems is relatively sparse in the literature, but has been gaining attention recently. Agrawal et al. [1] showed \( \text{NP} \)-hardness for choosing the optimal ordering in an online stopping problem. Papadimitriou et al. [15] showed \( \text{PSPACE} \)-hardness for the online stochastic bipartite matching problem.

2 Preliminaries

In an instance of the classical Pandora box problem, we are given \( n \) sealed boxes, each box \( i \) labeled with a search cost \( c_i \geq 0 \) and a distribution \( F_i \). Box \( i \) contains a value \( v_i \geq 0 \), initially hidden, drawn independently from \( F_i \), and one may open the box at cost \( c_i \) to reveal \( v_i \). At any point, a policy may (adaptively) choose to open a sealed box, or to take the highest value revealed so far and quit. Upon quitting, the payoff is the value taken minus the costs incurred along the way. Our goal is to maximize the expected payoff, where expectation is taken over the values and the possible randomness in the policy.

In a problem of Pandora box with non-obligatory inspection (PNOI), it is allowed to take a box that has not been opened before quitting the game.

Weitzman [18] showed that an index-based, deterministic policy yields maximum expected payoff. Kleinberg et al. [14] gave an elegant new proof for the optimality of Weitzman’s policy. Both the index-based policy and Kleinberg et al.’s rederivation play crucial roles in our work, so we reproduce them in some detail below.

Weitzman’s Index-based Policy. For box \( i \), define its index \( \tau_i \) to the unique solution to the equation \( E_{v_i \sim F_i}[(v_i - \tau_i)_+] = c_i \), where \( (v_i - \tau_i)_+ \) denotes \( \max\{0, v_i - \tau_i\} \). Let \( \kappa_i \) be \( \min\{v_i, \tau_i\} \).

We now present the index-based policy. Initialize by writing on each box its index. If all indices are negative, quit without taking anything. Otherwise, enter iterations: in each iteration, open box \( i \) whose index \( \tau_i \) is the highest among the sealed boxes, and replace the index written on it by the newly revealed \( v_i \); if now the highest number written on all boxes is a value (and not an index), take that box and quit; otherwise enter the next iteration.

**Theorem 2.1** [18]. The index-based policy maximizes the expected payoff in the classical Pandora box problem.

The following proof is due to Kleinberg et al. [14].
Proof. Consider any policy for an instance of the classical Pandora box problem. We first derive an upper bound on the policy’s expected payoff, and then show that the index-based policy achieves the upper bound.

For each \( i \), let random variables \( I_i \) and \( A_i \) be the indicator variables for the events that the policy opens box \( i \) and takes box \( i \), respectively. As the policy is not allowed to take a sealed box, we have \( A_i \leq I_i \) with probability 1. The policy’s expected payoff is 

\[
E\left[ \sum_i (A_i v_i - I_i c_i) \right].
\]

Note that the policy’s decision to open box \( i \) is independent from \( v_i \), therefore \( I_i \) and \( v_i \) are independent. This allows us to rewrite the expected payoff using the definition of the indices:

\[
E\left[ \sum_i (A_i v_i - I_i c_i) \right] = E\left[ \sum_i [A_i v_i - (v_i - \tau_i)_+] \right] = E\left[ \sum_i A_i \kappa_i \right] \leq E \left[ \max_i \kappa_i \right].
\]

where in the first inequality we used \( A_i \leq I_i \), in the ensuing equality we used the definition \( \kappa_i \min \{v_i, \tau_i\} \), and the last inequality follows from the constraint that \( \sum_i A_i \leq 1 \) with probability 1.

Let us see that the index-based policy’s payoff is precisely this upper bound. Consider properties of a policy that would turn the two inequalities in the chain into equalities:

1. \( I_i (v_i - \tau_i)_+ = A_i (v_i - \tau_i)_+ \) with probability 1 if, whenever the policy opens a box \( i \) and finds \( v_i > \tau_i \), the policy takes box \( i \);
2. \( \sum_i A_i \kappa_i = \max_i \kappa_i \) with probability 1 if the policy always takes the box with the maximum \( \kappa_i \).

It is straightforward to verify that the index-based policy have both properties, and hence attains a payoff equal to the upper bound \( E[\max_i \kappa_i] \). The index-based policy therefore achieves maximum payoff among all policies.

This rederivation by Kleinberg et al. [14] has proved powerful and has inspired multiple algorithmic works [e.g. 17, 10, 3]. Part of its power is to isolate property (1) in the proof above. It has been known as the non-exposed property.

Definition 2.2. A policy is non-exposed if, when it opens a box \( i \) and finds \( v_i > \tau_i \), it is guaranteed to take box \( i \). More formally, a policy is non-exposed if \( \Pr[(I_i - A_i)(v_i - \tau_i)_+] = 0 \) with probability 1, for each \( i \).

Additional notations. We use \( \mathbb{P}(A, \pi) \) to denote the expected profit of policy \( A \) on a PNOI instance \( \pi \). When the instance is clear from the context, we omit the second argument and write \( \mathbb{P}(A) \). For a given instance of PNOI, we let \( \text{OPT} \) denote the maximum expected profit achievable by any policy.

3 Hardness of PNOI

In this section, we show that computing optimal policies for PNOI is \( \text{NP} \)-hard. Note the difference between computing optimal policies and computing the expected payoffs of optimal policies — there are stochastic problems where the optimal step-by-step policy/mechanism is computationally tractable but calculating the optimal expected objective is hard (see e.g. Gopalan et al. [13]).

A policy for a PNOI instance can be represented by a decision tree, whose size, however, is generally exponential in the size of input. One way to get around this is to formulate the problem
as a single-step problem: given a PNOI instance and box $i$, decide whether there is an optimal policy that opens box $i$ next. Any PNOI instance can be solved by $O(n^2)$ calls to an oracle that solves this single-step problem. We take an alternative approach, and choose to focus on a simple class of PNOI instances that provably admit succinctly representable optimal policies.

**Definition 3.1.** An instance of PNOI is a low-cost-low-return-support-3 (LCLRS3) instance if the following conditions hold:

1. each value distribution $F_i$ is supported on $\{0, \frac{1}{2}, 1\}$, with probability masses $p_i := \Pr[v_i = 1] > 0$, $q_i := \Pr[v_i = \frac{1}{2}]$, $r_i := 1 - p_i - q_i = \Pr[v_i = 0]$;
2. for each box $i$, the cost $c_i > 0$, and the expected value $E[v_i] = p_i + \frac{q_i}{2} < \frac{1}{2}$;
3. for each box $i$, the index $\tau_i \geq \frac{1}{2}$, which implies $\tau_i = 1 - \frac{c_i}{p_i}$.

In Section 3.1 we show that optimal policies for LCLRS3 instances are succinctly represented by permutations on $[n]$, and in Section 3.2 we reduce the partition problem to the problem of giving a permutation that corresponds to an optimal policy for LCLRS3.

**Theorem 3.2.** Optimal policies for LCLRS3 instances admit polynomial-length descriptions, and it is NP-hard to compute optimal policies for LCLRS3 instances of PNOI.

It is immediate from the theorem that the “single-step decision problem” mentioned above is also NP-hard. Before proving the theorem, we quickly remark that the value $\frac{1}{2}$ in the support is necessary for a hardness result.

**Proposition 3.3.** There is a polynomial-time computable optimal policy for PNOI instances where all value distributions are supported on $\{0, 1\}$.

**Proof.** A few observations are in order.

1. A box $i$ to be taken without inspection can be seen as a box with deterministic value $E[v_i]$ with no search cost. Therefore, by the optimality of the index-based policy in the classic Pandora box problem, one should never take a box $i$ without inspection if some other unopened box has index larger than $E[v_i]$.

2. When a box yields value 1 upon inspection, it is payoff-optimal to select the box immediately and quit.

3. For any policy satisfying (2), the way in which it opens boxes is completely described by a subset $S \subseteq [n]$ and a permutation $\pi$ on $S$. The policy opens boxes in $S$ in the order specified by $\pi$: if a box yields value 1, the box is taken immediately and the search terminates; otherwise this goes till all boxes in $S$ are opened and yield value 0, at which point the algorithm may terminate or take a box not in $S$ without inspection.

From these observations, it is without loss of generality to consider policies that: (i) commit to a certain box $i$ that is possibly selected without inspection; (ii) inspect boxes with indices at least $E[v_i]$ in decreasing order of their indices, and if a value 1 is found, select that box and quit; (iii) when all boxes in step (ii) yield value 0, take box $i$ without inspection and quit.

There are altogether $n$ such policies (up to tie-breaking in step (ii), which does not matter). We can enumerate them and choose the best one in polynomial time.

The policies considered in the proof of Proposition 3.3 are called committing policies by Beyhaghi and Kleinberg [3]. This proof amounts to showing that committing policies are optimal for PNOI instances with Bernoulli value distributions.
3.1 Normal Policies and Their Payoffs

In this section we show that optimal policies for LCLRS3 instances are of a format that we call normal (Definition 3.4) and admit succinct representations. We then make use of ideas from Kleinberg et al. [14]'s proof for the index-based policy, and derive an expression for normal policies’ payoff (Lemma 3.6), which plays a crucial role in the reduction we present in Section 3.2.

Definition 3.4. A policy $A$ for a LCLRS3 instance is said to be normal if

- If $A$ sees 0 in the first $n-1$ boxes it opens, then $A$ takes the last box without inspection; this is the only situation in which $A$ exercises the option to bypass inspection.
- Whenever $A$ opens a box and sees value 1 in it, it immediately takes the box and stops.
- Whenever $A$ opens a box and sees value $\frac{1}{2}$ in it, it forsakes the option to take a box without inspection; on the remaining boxes, $A$ runs the index-based policy, with an outside option of value $\frac{1}{2}$.

Lemma 3.5. For any LCLRS3 instance, an optimal policy $A$ is normal.

Proof. We prove the three properties of a normal policy in order.

- It is straightforward to see that $A$ should take the last box without inspection if all previous boxes yield value 0. To see that this is the only situation $A$ should bypass inspection, recall by observation (1) in the proof of Proposition 3.3 that an optimal policy should not take a box without inspection if there are other unopened boxes with higher indexes. Note that, for any two boxes $i$ and $j$, $E[v_i] < \frac{1}{2} \leq \tau_j$ by definition of LCLRS3 instances.
- It is straightforward that $A$ stops when it sees value 1 — no other box can yield higher values in an LCLRS3 instance, and opening more boxes strictly diminishes the payoff.
- Since any unopened box has expected value strictly smaller than $\frac{1}{2}$, $A$ should never take an unopened box without inspection if a value $\frac{1}{2}$ is already seen. In other words, with value $\frac{1}{2}$ seen, $A$ ignores the option to bypass inspection, and the problem degenerates to the classical Pandora box problem for the remaining boxes. Therefore, after a value $\frac{1}{2}$ is seen, $A$ runs the index-based policy on the remaining boxes.

A normal policy $A$ corresponds to a permutation $\sigma_A$ on $[n]$: let $\sigma_A(1)$ be the first box opened by $A$; then, for $k = 1, \ldots, n-2$, let $\sigma_A(k+1)$ be the box opened by $A$ if the first $k$ boxes opened by $A$ all yield value 0; let $\sigma_A(n)$ be the box that $A$ takes without opening when all the other boxes yield 0. Conversely, it is straightforward to see that any permutation of $[n]$ also fully represents a normal policy. The problem of finding an optimal policy for an LCLRS3 instance thus amounts to finding a permutation $\sigma$ that represents one.

For a given LCLRS3 instance and policy $A$ on it, let us denote the expected payoff of $A$ as $P(A)$. We now characterize $P(A)$ when $A$ is normal. Our critical observation is that the difference between $P(A)$ and the payoff of the classical index-based policy admits a relatively clean expression that involves the permutation $\sigma_A$. This is by considering an intermediate policy $A'$, whose payoff admits simplifications using ideas from Kleinberg et al. [14]'s proof (Theorem 2.1).

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1That is, up to tie-breaking for boxes with equal indices after some box yields value $\frac{1}{2}$. It is clear that this tie breaking does not affect the expected payoff of a normal policy.
As in the proof of Theorem 2.1, define \( \kappa_i := \min\{v_i, \tau_i\} \). Recall that \( \mathbb{P}(\mathcal{A}) \) denotes the expected payoff of a policy \( \mathcal{A} \).

**Lemma 3.6.** Given an LCLRS3 instance and a normal policy \( \mathcal{A} \) for it, let \( \sigma \in S_{[n]} \) be the corresponding permutation. For box \( i \), let \( T_\sigma(i) \) be the set of boxes ordered after box \( i \) by \( \sigma \), with Gittins indices strictly larger than \( \tau_i \); that is, \( T_\sigma(i) := \{ j \in [n] : \sigma^{-1}(j) > \sigma^{-1}(i), \tau_j > \tau_i \} \). For \( i \in [n] \) and \( T \subseteq [n] \), define \( g(i, T) := \mathbb{E}[\{(\max_{j \in T} \kappa_j - \tau_i)\}^2] \). Let \( \mathcal{A}_P \) be the index-based policy on the instance. Then

\[
\mathbb{P}(\mathcal{A}_P) = \mathbb{P}(\mathcal{A}) + \sum_i p_i g(i, T_\sigma(i)) \prod_{j=1}^{\sigma^{-1}(i)-1} r_{\sigma(j)} - c_{\sigma(n)} \prod_{i=1}^{n-1} r_{\sigma(i)}. \tag{1}
\]

**Proof.** We first modify \( \mathcal{A} \) to obtain another policy \( \mathcal{A}' \). \( \mathcal{A}' \) has the same behavior as \( \mathcal{A} \) in all situations, except that when \( \mathcal{A} \) takes the last box \( \sigma(n) \) without opening it, \( \mathcal{A}' \) opens box \( \sigma(n) \), pays the cost \( c_{\sigma(n)} \), and takes it. Since \( \mathcal{A} \) is normal, it takes box \( \sigma(n) \) only when all the other boxes have yielded value 0, which happens with probability \( \prod_{i=1}^{n-1} r_{\sigma(i)} \). So we have

\[
\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{A}') + c_{\sigma(n)} \prod_{i=1}^{n-1} r_{\sigma(i)}. \tag{2}
\]

Next we derive a simple expression for \( \mathbb{P}(\mathcal{A}') \) and then show that the difference between \( \mathbb{P}(\mathcal{A}') \) and \( \mathbb{P}(\mathcal{A}_P) \) gives rise to the second term in (1). We claim \( \mathbb{P}(\mathcal{A}') = \sum_i \mathbb{E}[\mathcal{A}'(\kappa_i)] \), where \( \mathcal{A}'_i \) is the indicator variable for the event that \( \mathcal{A}' \) takes box \( i \). To see this, we make two observations, which amounts to showing that \( \mathcal{A}' \) is non-exposed (Definition 2.2). Let \( I'_i \) be the indicator variable for the event that \( \mathcal{A}' \) opens box \( i \).

- \( \mathcal{A}' \) never exercises the option to take a box without opening it, so \( \mathcal{A}'_i \leq I'_i \) with probability 1, for all \( i \).

- Whenever \( \mathcal{A}' \) opens box \( i \) and sees \( v_i > \tau_i \), \( \mathcal{A}' \) immediately takes box \( i \). To see this, if \( i = \sigma(n) \), by definition \( \mathcal{A}' \) takes the box after opening it. For \( i \neq \sigma(n) \), by definition of LCLRS3, \( \tau_i \geq \frac{1}{2} \), so \( v_i > \tau_i \) implies \( v_i = 1 \). Since \( \mathcal{A} \) is normal, it immediately takes box \( i \) when seeing \( v_i = 1 \); \( \mathcal{A}' \) copies the behavior of \( \mathcal{A} \) for \( i \neq \sigma(n) \), and hence also immediately takes it.

From the second observation, we have \( (v_i - \tau_i)_+I'_i = (v_i - \tau)_+A'_i \) with probability 1. Therefore

\[
\mathbb{P}(\mathcal{A}') = \sum_i \mathbb{E} [v_iA'_i - c_i I'_i] = \sum_i \mathbb{E} [v_iA'_i - (v_i - \tau_i)_+I'_i] = \sum_i \mathbb{E} [v_iA'_i - (v_i - \tau_i)_+A'_i] = \sum_i \mathbb{E} [A'_i(\kappa_i)] .
\]

We now compare \( \mathbb{P}(\mathcal{A}_P) \) and \( \mathbb{P}(\mathcal{A}') \). By Theorem 2.1, \( \mathbb{P}(\mathcal{A}_P) = \mathbb{E}[\max_i \kappa_i] \). For every realization of \( \kappa_i \)’s, \( \max_i \kappa_i \geq \sum_i \mathcal{A}'(\kappa_i) \). The inequality is strict only when \( \mathcal{A}' \) takes some box \( i \) with \( \kappa_i < \max_j \kappa_j \). This can happen only when all boxes opened before \( i \) yield value 0, and \( v_i = 1 \), in which case \( \kappa_i = \tau_i \). This happens with probability \( p_i \prod_{j=1}^{\sigma^{-1}(i)-1} r_{\sigma(j)} \); the expected contribution to the difference

\[g(i, 0) := 0.\]
between $\max_i \kappa_i$ and $\tau_i$ conditioning on this happening is $E[\max_{j, \sigma^{-1}(j) > \sigma^{-1}(i)} \tau_j - \tau_i]$, which is just $g(i, T_\sigma(i))$ as defined in the statement of the lemma. Therefore, overall, we have

\[
\mathbb{P}(A_P) - \mathbb{P}(A') = \mathbb{E} \left[ \max_i \kappa_i \right] - \sum_i \mathbb{E} \left[ A'_i \kappa_i \right]
\]

\[
= \sum_i \Pr[A'_i = 1] \cdot \mathbb{E} \left[ \max_j \kappa_j - \kappa_i \mid A'_i = 1 \right]
\]

\[
= \sum_i p_i g(i, T_\sigma(i)) \prod_{j=1}^{\sigma^{-1}(i)-1} r_{\sigma(j)}. \tag{3}
\]

Combining (2) and (3), we have

\[
\mathbb{P}(A_P) = \mathbb{P}(A) + \sum_i p_i g(i, T_\sigma(i)) \prod_{j=1}^{\sigma^{-1}(i)-1} r_{\sigma(j)} - c_{\sigma(n)} \prod_{i=1}^{n-1} r_{\sigma(i)}. \]

\[\square\]

3.2 Reduction

In this section we give a polynomial-time reduction from the classical partition problem to PNOI.

**Definition 3.7** (Partition Problem). *Given a multiset $S$ of positive integers $s_1, \cdots, s_n$, decide whether $S$ can be partitioned into two subsets $S_1$ and $S_2$ such that the sum of the numbers in $S_1$ equals the sum of the numbers in $S_2$.***

It is well-known that Partition problem is NP-complete [see, e.g., [11]]. It is also not difficult to show that the problem is still NP-hard when $1 \leq s_1 \leq \cdots \leq s_n \leq 2^n$. We assume so in the following reduction. We first formally give the reduction, and explain the intuition below.

**Reduction from LCLRS3 to Partition.** Given the multiset $S = \{s_1, \ldots, s_n\}$ of integers between 1 and $2^n$, fix two constants $\Gamma = 2^{3n}$ and $\Delta = 2^{-7n}$. We construct an LCLRS3 instance with $n + 2$ boxes, denoted as $B_1, \cdots, B_{n+1}, B_{n+2}$.

For box $B_{n+1}$, set $p_{n+1} = 1/\Gamma, q_{n+1} = 1 - 41/\Gamma$ and $c_{n+1} = p_{n+1}/2$. This makes $\tau_L := \tau_{n+1} = \frac{1}{2}$ and $\tau_{n+1} = 40/\Gamma$.

For box $B_{n+2}$, set $p_{n+2} = q_{n+2} = 1/8, c_{n+2} = 1/32$ and thus $\tau_{n+2} = 3/4$.

For each $i \in [n]$, set $p_i = q_i = s_i/\Gamma$. Set a constant $\tau_H = 3/4 - O(\Delta)$, whose precise value is to be determined later (see Claim 3.13). Set $c_i$ to make

\[
\tau_i = \tau_H + \frac{p_i p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L)}{2p_{n+2}} = \tau_H + O(\Delta^2).
\]

Note that $p_i \leq \Delta$ for any $i \in [n + 1]$, since we assumed $s_i \leq 2^n$. The construction ensures $\tau_{n+2} > \tau_i > \tau_H > \tau_L = 1/2$ for any $i \in [n]$.

Recall that the optimal solution to any LCLRS3 instance can be represented by a permutation $\sigma$. Given the permutation $\sigma$, the position of $B_{n+1}$ in the permutation plays a crucial role in the following analysis. Let the position of $B_{n+1}$ in $\sigma$ be $\xi$, i.e. $\sigma(\xi) = n + 1$. Thus $B_1, \cdots, B_n$ are
partitioned into two sets in $\sigma$: those before $B_{n+1}$ and those after, which we denote by $T_{Bef}$ and $T_{Aft}$, respectively. Formally, $T_{Bef} := \{ i : 1 \leq i \leq n, \sigma^{-1}(i) < \xi \}$ and $T_{Aft} := \{ i : 1 \leq i \leq n, \sigma^{-1}(i) > \xi \}$.

The next key lemma builds the bridge between the partition problem and LCLRS3 instances:

**Lemma 3.8.** The answer to the Partition problem with input $S$ is Yes if and only if $\sum_{i \in T_{Aft}} p_i = \sum_{i \in T_{Bef}} p_i$ in the permutation $\sigma^*$ that corresponds to an optimal policy for the LCLRS3 instance.

Theorem 3.2 follows immediately from Lemma 3.8 and the fact that Partition problem is NP-hard. It remains to prove Lemma 3.8.

**Intuition of the Reduction and Proof Overview.** By Lemma 3.5 and Lemma 3.6, giving an optimal policy for the LCLRS3 instance boils down to finding a permutation $\sigma$ that maximizes the objective value

$$
\text{Utility}(\sigma) := c_{\sigma(n+2)} \prod_{i=1}^{n+1} r_{\sigma(i)} - \sum_{i} p_i g(i, T_{\sigma}(i)) \prod_{j=1}^{\sigma^{-1}(i)-1} r_{\sigma(j)}. \tag{4}
$$

We would like to focus on the more complex second term (which we call the loss term $[5]$). The role of box $n + 2$ is to fix the first term: $c_{n+2}$ is a constant whereas $c_1, \ldots, c_{n+1}$ are exponentially small. $c_{n+2}$ is so large compared with all other terms in (4) that any reasonable policy must leave box $n + 2$ till the end:

**Claim 3.9.** Let $\sigma^*$ be a permutation which maximizes (4). Then $\sigma^*(n + 2) = n + 2$.

$B_{n+2}$ has the highest index $\tau_{n+2}$. Therefore, once a box yields value $1/2$, a normal policy next opens box $B_{n+2}$. Intuitively, box $B_{n+2}$ is better to be opened earlier if it will be opened. It can be shown that, if all the $n + 1$ boxes before it have the same index $\tau$, the ones with higher ratios of $q_i/p_i$ should be opened early to maximize the utility. On the other hand, opening boxes with higher indices earlier also helps with the utility. The special box $B_{n+1}$ has a lower index than boxes $B_1, \ldots, B_n$, but a much higher ratio of $q_{n+1}/p_{n+1}$. There is therefore a non-trivial trade-off for deciding the position of $B_{n+1}$ in $\sigma$.

Much technical work is needed to substantiate this intuition. Let $S_B$ and $S_T$ be the sum of $p_i$’s for the boxes in $T_{Bef}$ and $T_{Aft}$, respectively. A major technical step in our proof is to show that the
loss term in (4) is approximated by a single-variable quadratic function \(9\), which (roughly) depends on the ratio \(S_B/S_T\). By tuning the parameters, we make the quadratic function achieve its minimum when \(S_B/S_T = 1\), so that, if \(S\) admits a partition with equal sums, the optimal ordering \(\sigma^*\) must yield equal \(S_B\) and \(S_T\), and vice versa. We give the details in Section 3.3.

### 3.3 Correctness of Reduction: A Sketch

From the discussion above, an optimal policy must leave box \(n + 2\) to the last in its permutation \(\sigma\), and the ordering of the other boxes must minimize the loss term:

\[
\text{Loss}(\sigma) := \sum_i p_i g(i, T_\sigma(i)) \prod_{j=1}^{\sigma^{-1}(i)-1} r_\sigma(j),
\]

(5)

We show the key lemma by using a function with \(\sum_{i \in T_{\text{Aft}}} p_i + p_i^2\) as the single variable to approximate Equation (5). For ease of notation, define

\[
y := \sum_{i \in S} \frac{s_i}{\Gamma} \quad \text{and} \quad y := \sum_{i \in T_{\text{Aft}} \cup T_{\text{Bef}}} p_i + p_i^2,
\]

\[
x := \sum_{i \in T_{\text{Bef}}} p_i + p_i^2.
\]

Note that \(y\) is fixed once \(S\) is given, whereas \(x\) is a function of \(T_{\text{Bef}}\) and hence of \(\sigma\).

**Lemma 3.10.** The parameters of the instance can be set up so that

\[
h(x) - O(n^2 \Delta^4) \leq \frac{\text{Loss}(\sigma) - C \pm O(n^2 \Delta^4)}{k_1} \leq h(x) + O(n \Delta^3),
\]

(6)

where

\[
h(x) := e^{-2x} \left(1 - \frac{k_2}{k_1} e^{-y+x}\right),
\]

with \(C, k_1, k_2\) as constants independent of \(\sigma\):

\[
k_1 := \frac{1}{2} p_{n+2}(\tau_{n+2} - \tau_H)(p_{n+1} + q_{n+1}) + p_{n+1}[(1 - p_{n+2})(\tau_H - \tau_L) + p_{n+2}(\tau_{n+2} - \tau_L)],
\]

(7)

\[
k_2 := p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L),
\]

(8)

\[
C := \frac{1}{2} p_{n+2}(\tau_{n+2} - \tau_H) \left(1 - \prod_{i \in [n+1]} r_i\right) + \frac{1}{2} k_2 \sum_{i=1}^{n} p_i^2.
\]

(9)

We give a road map for the proof once we have Lemma 3.10. The minimum value of \(h(x)\) is taken at \(x^* = y - \ln(2k_1/k_2)\). When \(k_1/k_2\) is near \(2e^{y/2}\), \(x^*\) is close to \(y/2\). Our goal is to have the most even partition of \(S\) be the \(T_{\text{Bef}}\) and \(T_{\text{AR}}\) of an optimal policy, which in turn should have \(x\) as close to \(y/2\) as possible. Even with the approximation given in Lemma 3.10, a few obstacles still stand in the way: \(x\) is not \(\sum_{i \in T_{\text{Bef}}} p_i\), nor is \(y\) equal to \(\sum_{i \in [n]} p_i\); both of them have second-order terms, which cause further distortion in the objective through the fact that \(x\) and \(y\) appear in the exponents in \(h\).
We overcome these difficulties by carefully controlling the order of errors throughout our calculation: $p_i$’s are so small that the second-order terms in $x$ and $y$ are negligible; analytical properties of $h$ (Claim 3.16) guarantee that, around its optimum, $h$ is sensitive enough to perturbations, so that suboptimal solutions can be told from the optimal.

Much of the proof of Lemma 3.10, which is fairly technical, is relegated to Appendix A. We mention a tool instrumental in simplifying the calculations, which also explains our setting $p_i = q_i$ for all $i \in [n]$:

**Lemma 3.11.** Given two sequences of positive real numbers $p_1, p_2, \cdots, p_n$ and $r_0, r_1, \cdots, r_n$. Let $r_0 = 1$. If there exists a constant $c > 0$ such that $p_i/(1 - r_i) = c$ for each $1 \leq i \leq n$, then we have

$$\sum_{i=1}^{n} p_i \prod_{j=0}^{i-1} r_j = c \left(1 - \prod_{i=1}^{n} r_i \right).$$

After much simplification, the main terms of $\text{Loss}(\sigma)$ are given in Claim 3.12, before we apply analytical tools and turn products to sums in the exponent (Fact 3.14, Claim 3.15), which leads to Lemma 3.10. Note that we have to appeal to second-order approximations of the exponential function for the required precision in the proof. The setup of the parameter $\tau_H$ is given in Claim 3.13

**Claim 3.12.** For a non-empty set $T \subseteq [n]$, let $f(T) := \prod_{i \in T} r_i = \prod_{i \in T}(1 - 2p_i)$ and $g(T) := \prod_{i \in T}(1 - p_i)$. Also let $f(\emptyset) = g(\emptyset) = 1$. Then

$$\text{Loss}(\sigma) = k_1 f(T_{\text{Bel}}) - k_2 f(T_{\text{Bel}})g(T_{\text{Aft}}) - k_2 \sum_{i \in T_{\text{Aft}}} p_i^2 / 2 + C + O(n^2 \Delta^4). \quad (10)$$

**Claim 3.13.** If we choose $t$ so that $|t - 2e^{y/2}| \leq O(\Delta^2)$, and set $\tau_H$ as follows, then $k_2/k_1 = t$:

$$\tau_H = \frac{-3t\Gamma + 28 + 94t}{-4\Gamma + 56 + 104t}. \quad (11)$$

**Fact 3.14.** For $0 \leq x \leq 1/2$, we have $1-x \leq e^{-x} \leq 1 - x + x^2/2$, and $1-x \leq e^{-x-x^2/2} \leq 1-x + O(x^3)$.

**Claim 3.15.** For any subset $T$ of the first $n$ boxes, one has

$$e^{-\sum_{i \in T} (2p_i + p_i^2)} \geq f(T) \geq e^{-\sum_{i \in T} (2p_i + p_i^2)} - O(n\Delta^3),$$

$$e^{-\sum_{i \in T} (p_i + p_i^2/2)} \geq g(T) \geq e^{-\sum_{i \in T} (p_i + p_i^2/2)} - O(n\Delta^3).$$

With the approximation in Lemma 3.10 in hand, we are almost ready to prove Lemma 3.8. The next lemma shows that the function $h$ is sensitive enough to perturbations around its minimum.

**Claim 3.16.** If $|k_2/k_1 - 2e^{y/2}| \leq O(\Delta^2)$, $\epsilon \in \mathbb{R}$ is such that $2^{-6n} \geq |\epsilon| \geq 1/\Gamma = 2^{-8n}$, let $x^* \in [0, 1/2]$ be where $h(x)$ takes its minimum value, then $|x^* - \frac{y}{2}| \leq O(\Delta^2)$, $h(x^* + \epsilon) \geq h(x^*) + \epsilon^2/2$.

The “if” part is obvious: if the permutation $\sigma^*$ of a policy yields a partition $T_{\text{Bel}}^*$ and $T_{\text{Aft}}^*$, with $\sum_{i \in T_{\text{Bel}}^*} p_i = \sum_{i \in T_{\text{Aft}}^*} p_i$, then since $p_i = s_i/\Gamma$ for each $i \in [n], (T_{\text{Bel}}^*, T_{\text{Aft}}^*)$ certifies that $S$ is a YES instance of Partition.

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For the “only if” part, suppose $S$ can be partitioned into disjoint subsets $S_1$ and $S_2$ with $\sum_{s \in S_1} s = \sum_{s \in S_2} s$, we show that any policy whose corresponding $T_{\text{bef}}$ and $T_{\text{aft}}$ is not an even partition of $S$ must be suboptimal. By our setting of parameters and Claim 3.13, we know $|x^* - y/2| \leq O(\Delta^2)$. For any permutation $\sigma$ whose corresponding sets $T_{\text{bef}}, T_{\text{aft}}$ are such that $\sum_{i \in T_{\text{aft}}} p_i \neq \sum_{i \in T_{\text{bef}}} p_i$, we have $|x - y/2| \geq 1/\Gamma - \sum_{i \in T_{\text{aft}} \cup T_{\text{bef}}} p_i^2 \geq 1/\Gamma - n\Delta^2$, and hence $|x - x^*| \geq 1/\Gamma - n\Delta^2/2$. Hence, one has

$$\text{Loss}(\sigma) - C + O(n^2\Delta^4) \geq h(x) - O(n^2\Delta^4) > h(x^*) + \Omega(1/\Gamma^2)$$

where the second inequality follows from Claim 3.16, the first and last inequality follow from Lemma 3.10. Hence $\text{Loss}(\sigma) > \text{Loss}(\sigma^*)$.

4 Improved Approximation Algorithms

In this section we give improved approximation algorithms for PNOI. In Section 4.1 we first give a simpler $(2/3 - \epsilon)$-approximation algorithm; in Section 4.2 we introduce more sophistication and present an improved $(4/5 - \epsilon)$-approximation algorithm. In Section 4.3 we state PTAS for restricted cases of PNOI. All missing proofs in this section can be found in Appendix B.

4.1 A $(2/3 - \epsilon)$-Approximation Algorithm for PNOI

Fu et al. [8] introduced a powerful framework for obtaining PTAS’s for many stochastic optimization problems. Key to its application is that the “state space” should be small; in natural adaptations of PNOI instances to this framework, this requires the supports of the value distributions to be of constant size. Unfortunately, in general this is not true, nor does crude discretization seem to resolve this issue. We show, nonetheless, that an adaptation of the framework to truncated policies (to be defined below), combined with two simple algorithms, yields improved approximation algorithms. This section is an appetizer and introduces some main ideas; in Section 4.2 we further refine the technique and obtain a $(0.8 - \epsilon)$ approximation algorithm.

**Theorem 4.1.** For any constant $\epsilon > 0$, there is a polynomial-time computable policy with expected payoff at least $(2/3 - \epsilon) \cdot \text{OPT}$.

**Definition 4.2.** For $h > 0$, a policy is $h$-truncated if, whenever it opens a box and sees a value at least $h$, it takes that box and terminates.

For a given $\epsilon \in (0, \frac{1}{2})$, we consider $\theta$-truncated policies where $\theta$ is an arbitrary number between $\frac{1}{4\epsilon} \text{OPT}$ and $\frac{1}{\epsilon} \text{OPT}$. One may first use the simple 2-approximation algorithm to compute $\text{OPT} \in [\frac{1}{\epsilon} \text{OPT}, \text{OPT}]$ and set $\theta = \frac{1}{\epsilon} \text{OPT}$. Hereafter we refer to $\theta$-truncated policies simply as truncated policies.

We apply Fu et al.’s framework and show that truncated policies for PNOI can be approximated to arbitrary precision in polynomial time. Let $\mathcal{A}^*_T$ be an optimal truncated policy.

**Lemma 4.3.** For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $\mathcal{A}_T$ with expected payoff at least $\mathbb{P}(\mathcal{A}^*_T) - \epsilon \cdot \text{OPT}$.
We do not know whether $A_T$ already approximates $\text{OPT}$, and deem it an interesting open question to either prove or refute this. Instead, we combine $A_T$ with two other simple policies to obtain a policy with improved performance. Given a PNOI instance, let $A_I$ be Weitzman’s index-based policy, and $A_E$ the policy that takes the box with the largest expectation without inspection. The key observation is the following relationship among these policies.

**Lemma 4.4.** Given a PNOI instance $\pi$, let $A^*$ be an optimal policy. Let $p_T$ be the probability with which $A^*$ sees in an opened box a value at least $\theta$ during its execution on $\pi$. Then,

1. $\mathbb{P}(A_T^*) + p_T \cdot \mathbb{P}(A_I) \geq \text{OPT}$;
2. $\mathbb{P}(A_I) + (1 - p_T) \cdot \mathbb{P}(A_E) \geq \text{OPT}$;

**Proof.** Once $A^*$ sees a value at least $\theta$, $A^*$ will never take a box without inspection, since $\max_i E[v_i] \leq \text{OPT} \leq \theta$. Instead, $A^*$ must execute the index-based policy on the remaining boxes, taking the seen value as an outside option. Therefore, the expected additional gain on the payoff of $A^*$, after seeing a value at least $\theta$, is at most $\mathbb{P}(A_I)$. Consider a truncated policy $B$ that copies $A^*$’s behavior except when $A^*$ sees a value at least $\theta$, at which point $B$ terminates. It follows from the argument above that $\mathbb{P}(B) + p_T \cdot \mathbb{P}(A_I) \geq \text{OPT}$. Then observing $\mathbb{P}(A_T^*) \geq \mathbb{P}(B)$ yields the first part of the lemma.

It also follows from the reasoning above that, the probability with which $A^*$ ever takes a box without inspection is upper bounded by $(1 - p_T)$. Consider replacing every “take without inspection” action in $A^*$ by doing nothing. Denote this new policy by $A'$. By the definition of $A'$, we have

$$\mathbb{P}(A') \geq \text{OPT} - (1 - p_T) \cdot \max_{i \in [n]} E[v_i] = \text{OPT} - (1 - p_T) \cdot \mathbb{P}(A_E).$$

But note that $A'$ is a valid policy for the original Pandora problem (that enforces inspection). Therefore, we have

$$\mathbb{P}(A_I) \geq \mathbb{P}(A') \geq \text{OPT} - (1 - p_T) \cdot \mathbb{P}(A_E).$$

This gives the second part of the lemma. \qed

**Corollary 4.5.** For any PNOI instance $\pi$,

$$\max\{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A_T^*)\} \geq \frac{2}{3} \cdot \text{OPT}.$$

**Proof Sketch.** Consider the following optimization problem. Normalize OPT to be 1, and let $\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A_T^*) \geq 0$ and $p_T \in [0, 1]$ be variables. The goal is to minimize $\max\{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A_T^*)\}$ under the constraints given in Lemma 4.4. This optimization problem can be solved by elementary analysis, which gives $\frac{2}{3}$ as the optimal objective, achieved when $\mathbb{P}(A_I) = \mathbb{P}(A_T^*) = \mathbb{P}(A_E) = \frac{2}{3}$ and $p_T = 1/2$. We omit the details of calculation here. \qed

**Proof of Theorem 4.1.** Combining Lemma 4.3 and Corollary 4.5 the best one among $\{A_T, A_I, A_E\}$ has approximation ratio at least $\frac{2}{3} - \epsilon$. Furthermore, the expected payoff of each of these three policies can be calculated efficiently by a dynamic programming on their decision trees (See the formal definition of decision tree in Appendix B.1). Therefore, we obtain an algorithm, which first calculates the expected payoff for $\{A_T, A_I, A_E\}$, and then executes the best one among them. This policy has expected payoff at least $(\frac{2}{3} - \epsilon) \cdot \text{OPT}$. \qed
4.2 Improving the Approximation Ratio to $0.8 - \epsilon$

In this section, we give a polynomial-time algorithm that computes a policy with improved approximation ratio of $0.8 - \epsilon$ for any $\epsilon > 0$.

**Theorem 4.6.** For any constant $\epsilon > 0$, there is a polynomial time algorithm with expected payoff at least $(0.8 - \epsilon) \cdot \text{OPT}$.

The improvement is by replacing two component policies in the proof of Theorem 4.1 with two other more powerful policies.

- **Policy $A_{CE}$**: Let $i_E$ be the box with the largest expected value (breaking ties arbitrarily), and $\text{MAXE}$ its expected value. Recall that $A_E$ is the policy that takes $i_E$ without any inspection. This is wasteful, because at least $i_E$ may be treated as a box with deterministic value $\text{MAXE}$ with zero search cost, and an index-based policy that treats box $i_E$ this way outperforms $A_E$ by a margin of $E[(\max \kappa_i - \text{MAXE})_+].$ Such a policy we denote as $A_{CE}$. We note that $A_{CE}$ is an example of committing policies introduced by Beyhaghi and Kleinberg [3].

- **Policy $A^*_TE$**: Recall that a truncated policy terminates once it sees a value $v$ no less than $\theta$. An optimal policy should execute the index-based policy on the remaining boxes, with $v$ as an outside option. The following family of policies take a small step toward this: we say a policy is *smartly truncated* if it does the following after seeing a value $v \geq \theta$:
  1. if box $i_E$ has not been opened yet, and its index $\tau_{i_E} > v$, then the policy opens box $i_E$ and then takes the higher between $v$ and $v_{i_E}$, and terminates;
  2. otherwise, the policy takes the value $v$ and terminates.

Given any truncated policy, allowing it one more chance to check box $i_E$ weakly improves its payoff. We denote by $A^*_TE$ an optimal one among all smartly truncated policies.

It turns out that Fu et al.’s framework can be further massaged to compute, in polynomial time, policies with payoff that approximates $\mathbb{P}(A^*_TE)$ to arbitrary precision. We relegate the details to Appendix B.3.

**Lemma 4.7.** For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $A_{TE}$ with expected payoff at least $\mathbb{P}(A^*_TE) - \epsilon \cdot \text{OPT}$.

Recall that $A_I$ is the classical index-based policy. We show below that the best among $A_I, A^*_TE$ and $A_{CE}$ gives a 0.8-approximation, from which Theorem 4.6 immediately follows. We provide some intuition why this is the case.

- We have argued above that $A_{CE}$ improves upon $A_E$ by a margin of $E[(\max \kappa_i - \text{MAXE})_+]$. In the proof of Corollary 4.5 we saw that the approximation ratio of $\max \{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A^*_T)\}$ is $\frac{2}{3}$ only when the three components are all $\frac{2}{3} \text{OPT}$. Therefore if $\max \{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A^*_T)\}$ is not already better than $\frac{2}{3} \text{OPT}$, and if $\mathbb{P}(A_{CE})$ is still not significantly better, it must be that $E[(\max \kappa_i - \text{MAXE})_+]$ is negligible.

- Next, consider the smartly truncated version of the optimal policy $A^*$, which we denote as $O_{TE}$: $O_{TE}$ mimics $A^*$, except that whenever it sees a value $v \geq \theta$, it opens box $i_E$ if it is not opened yet and if $\tau_{i_E} > v$, and takes the better of $v$ and $v_{i_E}$; otherwise it just takes $v$ and terminates. Consider the difference between $A^*$ and $O_{TE}$: the two differ only when $A^*$ sees a value $v \geq \theta$,.
in which case $A^*$ performs an index-based search on all remaining boxes with indices larger than $v$, whereas $O_{TE}$ may open only box $i_E$, so the conditional expected difference between the two is at most $\mathbb{E}[(\max_i \kappa_i - \text{MaxE})_+].$ Therefore, if $\mathbb{E}[(\max_i \kappa_i - \text{MaxE})_+]$ is negligible, $\mathbb{P}(O_{TE})$ must be close to $\text{OPT}$. Since $A^*_{TE}$ is optimal among all smartly truncated policies, $\mathbb{P}(A^*_{TE}) \geq \mathbb{P}(O_{TE}).$

The next lemma formalizes this intuition.

**Lemma 4.8.** Fixing a PNOI instance $\pi$, define $p_T$ as the probability that the optimal policy $A^*$ sees in an opened box a value greater than $\theta$ during its execution on $\pi$. Then we have

$$\mathbb{P}(A^*_{TE}) + p_T \cdot [\mathbb{P}(A_{CE}) - \mathbb{P}(A_E)] \geq \text{OPT}.$$  

**Proof.** Let $P_I(S,v) := \mathbb{E}[(\max_i \kappa_i - v)_+]$ be the expected additional payoff gained from performing an index-based policy on a set $S$ of unopened boxes, with an outside option of $v$. The following properties of the function $P_I$ are straightforward:

1. $P_I([n]\{i_E\}, P(A_E)) = P(A_{CE}) - P(A_E)$;
2. $P_I(S, v_1) \geq P_I(S, v_2)$, if $v_1 \leq v_2$;
3. $P_I(S_1, v) \leq P_I(S_2, v)$, if $S_1 \subset S_2$;
4. $P_I(S \cup \{i\}, v) \leq P_I(S, v) + P_I(\{i\}, v)$.

Consider the utility loss of policy $O_{TE}$ compared with the optimal, i.e., $\text{OPT} - \mathbb{P}(O_{TE})$. The loss could only occur when a box with value $v^* \geq \theta$ is opened. Once this happens (with probability $p_T$), let $T$ be the set of boxes that have not been opened yet.

- If $i_E \notin T$, the utility loss is $P_I(T, v^*)$, which is upper bounded by $P_I([n]\{i_E\}, P(A_E))$, because $P(A_E) \leq \text{OPT} < \theta \leq v^*$.
- If $i_E \in T$, the utility loss is

$$P_I(T, v^*) - P_I(\{i_E\}, v^*) \leq P_I(T\{i_E\}, v^*) \leq P_I([n]\{i_E\}, P(A_E)) = P(A_{CE}) - P(A_E).$$

Therefore, $\text{OPT} - \mathbb{P}(O_{TE}) \leq p_T \cdot (P(A_{CE}) - P(A_E))$. The lemma follows by observing that $\mathbb{P}(A^*_{TE}) \geq \mathbb{P}(O_{TE}).$ 

By combining the constraints given in Lemma 4.4 and Lemma 4.8, we show a better approximation ratio of 0.8, which is achieved by taking $\mathbb{P}(A_I) = \mathbb{P}(A^*_{TE}) = \mathbb{P}(A_{CE}) = 0.8$, $\mathbb{P}(A_E) = 0.4$ and $p_T = 1/2$. The detailed calculation is in Appendix B.3

**Corollary 4.9.** For any fixed PNOI instance $\pi$,

$$\max\{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A^*_{TE}), \mathbb{P}(A_{CE})\} \geq 0.8 \cdot \text{OPT}$$

Now, let us wrap everything up.

**Proof of Theorem 4.6.** From Lemma 4.7 we could replace policy $A^*_{TE}$ by $A_{TE}$ with a loss of $\epsilon \text{OPT}$. We obtain an algorithm, which first calculates the expected payoff for $\{A_{TE}, A_I, A_E, A_{CE}\}$, and then executes the best one among them. This policy has expected payoff at least $(0.8 - \epsilon) \cdot \text{OPT}$. 

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4.3 Conditional PTAS’s

It is worth mentioning that, under various conditions, the value space of an PNOI instance can be effectively discretized without provably negligible loss in the payoff. When this happens, Fu et al.’s framework does yield a PTAS.

Theorem 4.10. For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $A$ with expected payoff at least $(1 - \epsilon) \cdot \text{OPT}$ for PNOI instances that satisfy at least one of the following three conditions:

1. the union of the supports of $F_1, \ldots, F_n$ has $O(1)$ size;
2. there is a constant $c_\epsilon$, which only depends on $\epsilon$, such that $v_i \leq c_\epsilon \cdot \text{OPT}$ with probability 1, for all $i$;
3. there is a constant $c_\epsilon$, which only depends on $\epsilon$, such that each value distribution $F_i$ is discrete and, for any value $v$ in the support, has $F_i(v) \geq 1/c_\epsilon$.

The proof is presented in Appendix B.1 and Appendix B.2.

5 Conclusion and Open Problems

In this work we proved the first computational hardness result for PNOI and improved the state-of-the-art approximation algorithm. From a computational point of view, one may wonder what the best possible approximation ratio is. Our hardness result does not exclude the existence of a PTAS for the problem, although direct adaptation of existing approximation schemes such as Fu et al. [8]’s seems challenging.

There are other online decision problems where hardness results are missing and approximation algorithms have been developed in the absence of a tractable optimal algorithm. The price of information setting for bipartite matching is one such example [17, 10].

While our algorithms yield policies for PNOI with better approximation guarantees than that of Beyhaghi and Kleinberg [3], the latter has the advantage of being simple. The so-called committing policies are almost non-adaptive: they are pre-committed to possibly inspecting a subset of boxes in a pre-committed ordering. Beyhaghi and Kleinberg showed that the best committing policy is a $0.8$-approximation when there are two boxes, and conjectured that it is a $0.8$-approximation in general. We find it interesting that our approximation algorithm, which incorporates one committing policy (the policy $A_{TE}$), achieves essentially the approximation ratio in the conjecture. It remains an attractive conjecture whether committing policies alone give a $0.8$-approximation.

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A Omitted Proofs from Section 3

Claim 3.9. Let $\sigma^*$ be a permutation which maximizes (4). Then $\sigma^*(n + 2) = n + 2$.

Proof. By (4), it is easy to verify that when $n \geq 1$ and $\sigma^*(n + 2) = n + 2$ one has

$$\text{Utility}(\sigma^*) \geq c_{n+2} r_{n+1} (1 - 2^{-6n})^n - np_{n+2} \max_i p_i(\tau_H - \tau_i) - p_{n+1} \left( (\tau_H - \tau_L)p_{n+2} + n \max_{i \in [n]} p_i(\tau_i - \tau_L) \right)$$

$$\geq \frac{40}{32\Gamma} (1 - 2^{-6n})^n - \frac{1}{32\Gamma} - O(n\Delta^2)$$

$$\geq \frac{38}{32\Gamma}.$$

For any permutation $\sigma$ with $\sigma(n + 2) \in [n]$, we have

$$\text{Utility}(\sigma) \leq r_{n+1} \max_{i \in [n]} c_i \leq O(\Delta^2)$$

as $c_i = \frac{p_i}{\tau_i + 1} \leq p_i$.

For any permutation $\sigma$ with $\sigma(n + 2) = n + 1$, we have

$$\text{Utility}(\sigma) \leq r_{n+2} c_{n+1} \leq \frac{3}{8\Gamma} < \frac{38}{32\Gamma}.$$

Hence if some permutation $\sigma^*$ maximize Equation (4), then $\sigma^*(n + 2) = n + 2$. \qed

For ease of presentation, we introduce the following notations.
Definition A.1.

\[ g_H := p_{n+2}(\tau_{n+2} - \tau_H); \]
\[ g_L := \left[ 1 - \prod_{i \in T_{\text{An}}} (1 - p_i) \right] (1 - p_{n+2})(\tau_H - \tau_L) + p_{n+2}(\tau_{n+2} - \tau_L); \]
\[ g_i := g(i, T_\sigma(i)), \quad \text{for } i = 1, \ldots, n + 1. \]

Claim A.2.

\[ g_i = g_H - \frac{p_i p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L)}{2} \pm O(n \Delta^3), \quad \text{for } i = 1, \ldots, n; \]
\[ g_{n+1} = g_L \pm O(n \Delta^3). \]

Proof. Recall that \(g(i, T) := E(\max_{j \in T} \kappa_j - \tau_i)\) and \(\kappa_i = \min\{v_i, \tau_i\}\). Also, \(\tau_{n+2} > \tau_i \geq \tau_H > \tau_L = 1/2\) for all \(i \in [n]\) in our LCLR3 instance. Therefore, for each \(j \neq i\),

\[ (\kappa_j - \tau_i)_+ = \begin{cases} 0, & \text{if } v_j \leq 1/2; \\ (\tau_j - \tau_i)_+, & \text{if } v_j = 1. \end{cases} \]

Since \(\tau_{n+2}\) is by far the largest among all indices, and \(\tau_{n+1} = \tau_L\) is the lowest index, we have for each \(i \in [n]\),

\[ g_i = p_{n+2}(\tau_{n+2} - \tau_i) + (1 - p_{n+2}) \mathbb{E} \left[ \max_{j \in T_\sigma(i) \setminus \{n+2\}} (\kappa_j - \tau_i)_+ \right] \]
\[ = p_{n+2}(\tau_{n+2} - \tau_H) - \frac{1}{2} p_i p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L) + (1 - p_{n+2}) \mathbb{E} \left[ \max_{j \in T_\sigma(i) \setminus \{n+2\}} (\kappa_j - \tau_i)_+ \right] \]
\[ \leq g_H - \frac{1}{2} p_i p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L) + (1 - p_{n+2}) \sum_{j \in T_\sigma(i) \setminus \{n+2\}} p_j (\tau_j - \tau_i)_+ \]
\[ \leq g_H - \frac{1}{2} p_i p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L) \pm O(n \Delta^3). \]

The last inequality is from the fact that \(\tau_i = \tau_H + O(\Delta^2)\) for any \(i \in [n]\).

Similarly,

\[ g_{n+1} = p_{n+2}(\tau_{n+2} - \tau_L) + (1 - p_{n+2}) \mathbb{E} \left[ \max_{j \in T_{\text{An}}} (\kappa_j - \tau_L)_+ \right] \]
\[ \leq p_{n+2}(\tau_{n+2} - \tau_L) + (1 - p_{n+2})(1 - \prod_{i \in T_{\text{An}}} (1 - p_i)) \max_{j \in T_{\text{An}}} (\tau_j - \tau_L) \]
\[ \leq p_{n+2}(\tau_{n+2} - \tau_L) + (1 - p_{n+2})(1 - \prod_{i \in T_{\text{An}}} (1 - p_i)) (\tau_H \pm O(\Delta^2) - \tau_L) \]
\[ = g_L \pm O(n \Delta^3). \]

\[ \square \]

Lemma 3.11. Given two sequences of positive real numbers \(p_1, p_2, \ldots, p_n\) and \(r_0, r_1, \ldots, r_n\). Let \(r_0 = 1\). If there exists a constant \(c > 0\) such that \(p_i / (1 - r_i) = c\) for each \(1 \leq i \leq n\), then we have

\[ \sum_{i=1}^{n} p_i \prod_{j=0}^{i-1} r_j = c \left( 1 - \prod_{i=1}^{n} r_i \right). \]
Proof. We prove this lemma by induction on $n$. When $n = 1$, we get $p_1 r_0 = p_1 = c(1 - r_1)$ by the assumption.

Suppose one can have $\sum_{i=1}^n p_i \prod_{j=0}^{i-1} r_j = c(1 - \prod_{i=1}^n r_i)$, then

$$\sum_{i=1}^{n+1} p_i \prod_{j=0}^{i-1} r_j = \sum_{i=1}^n p_i \prod_{j=0}^{i-1} r_j + p_{n+1} \prod_{j=0}^n r_j$$

$$= c(1 - \prod_{i=1}^n r_i) + p_{n+1} \prod_{j=0}^n r_j$$

$$= c(1 - \prod_{i=1}^n r_i) + c(1 - r_{n+1}) \prod_{j=0}^n r_j$$

$$= c(1 - \prod_{i=1}^{n+1} r_i).$$

\[ \square \]

**Claim 3.12.** For a non-empty set $T \subseteq [n]$, let $f(T) := \prod_{i \in T} r_i = \prod_{i \in T} (1 - 2p_i)$ and $g(T) := \prod_{i \in T} (1 - p_i)$. Also let $f(\emptyset) = g(\emptyset) = 1$. Then

$$\text{Loss}(\sigma) = k_1 f(T_{\text{Bef}}) - k_2 f(T_{\text{Bef}}) g(T_{\text{Aft}}) - k_2 \sum_{i \in T_{\text{Aft}}} p_i^2/2 + C + O(n^2 \Delta^4). \quad (10)$$

**Proof.** First, we show

$$\text{Loss}(\sigma) = \frac{g_H}{2} \left( 1 - \prod_{i \in T_{\text{Bef}}} r_i \right) + g_L p_{n+1} \prod_{i \in T_{\text{Bef}}} r_i + \frac{g_H r_{n+1}}{2} \prod_{i \in T_{\text{Bef}}} r_i \left( 1 - \prod_{i \in T_{\text{Aft}}} r_i \right)$$

$$+ \frac{1}{2} \sum_{i \in T_{\text{Bef}}} p_i^2 p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \pm O(n^2 \Delta^4).$$

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The claim follows by plugging in the parameters. One has

\[
\text{Loss}(\sigma) = \sum_i p_i g(i, T_\sigma(i)) \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)}
\]

\[
= \sum_{i \in \mathcal{T}_\text{bef}} p_i g_i \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)} + p_{n+1} g_{n+1} \prod_{i \in \mathcal{T}_\text{bef}} r_i + \sum_{i \in \mathcal{T}_\text{Alt}} p_i g_i \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)}
\]

\[
= \sum_{i \in \mathcal{T}_\text{bef}} p_i \left( g_H - \frac{1}{2} p_i p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \pm O(n \Delta^3) \right) \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)}
\]

\[
+ p_{n+1} (g_L \pm O(n \Delta^3)) \prod_{i \in \mathcal{T}_\text{bef}} r_i + \sum_{s \in \mathcal{T}_\text{bef}} r_s \sum_{i \in \mathcal{T}_\text{Alt} \ j = |\mathcal{T}_\text{bef} + 2|} r_{\sigma(j)} p_i g_i \pm O(n^2 \Delta^4)
\]

\[
= \sum_{i \in \mathcal{T}_\text{bef}} p_i g_H \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)} - \sum_{i \in \mathcal{T}_\text{bef}} \left( \frac{1}{2} p_i^2 p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \right) \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)}
\]

\[
+ p_{n+1} g_L \prod_{i \in \mathcal{T}_\text{bef}} r_i \pm O(n^2 \Delta^4)
\]

where in the last equality we used \( p_{n+1} = 1/\Gamma = O(\Delta) \). Further analyzing the last term, we have

\[
\prod_{s \in \mathcal{T}_\text{bef}} r_s \sum_{i \in \mathcal{T}_\text{Alt} \ j = |\mathcal{T}_\text{bef} + 2|} r_{\sigma(j)} p_i \left( g_H - \frac{1}{2} p_i p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \right)
\]

\[
= \prod_{s \in \mathcal{T}_\text{bef}} r_s \sum_{i \in \mathcal{T}_\text{Alt} \ j = |\mathcal{T}_\text{bef} + 2|} r_{\sigma(j)} p_i g_H - O(n \Delta^4).
\]

Note that \( \prod_{j=1}^{\sigma_1(i)-1} r_{\sigma(j)} = 1 - O(n \Delta) \) for each \( i \in \mathcal{T}_\text{bef} \) and \( r_{n+1} = 3/\Gamma = O(\Delta) \). Besides, notice
that, for each \( i \in [n] \), we have \( \frac{p_i}{1 - r_i} = \frac{s_i/I}{2s_i/I} = \frac{1}{2} \). Therefore, Lemma 3.11 applies, and we have

\[
\text{Loss}(\sigma) = \frac{gH}{2} \left( 1 - \prod_{i \in T_{\text{Bet}}} r_i \right) + (1 - O(n\Delta)) \sum_{i \in T_{\text{Bet}}} \frac{1}{2} p_i^2 p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) + p_{n+1} gL \prod_{i \in T_{\text{Bet}}} r_i \\
+ r_{n+1} \prod_{s \in T_{\text{Bet}}} r_s \sum_{i \in T_{\text{Att}}} \prod_{j \in T_{\text{Bet}} + 2} r_{\sigma(j)} p_i g_H \pm O(n^2 \Delta^4)
\]

\[
= \frac{gH}{2} \left( 1 - \prod_{i \in T_{\text{Bet}}} r_i \right) + \sum_{i \in T_{\text{Bet}}} \frac{1}{2} p_i^2 p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) + p_{n+1} gL \prod_{i \in T_{\text{Bet}}} r_i \\
+ \frac{1}{2} g_H r_{n+1} \prod_{s \in T_{\text{Bet}}} r_s \left( 1 - \prod_{i \in T_{\text{Att}}} r_i \right) \pm O(n^2 \Delta^4).
\]

Rewrite the equation in terms of \( f, g \) and \( k_1, k_2 \), and we have

\[
\text{Loss}(\sigma) = \left( -\frac{gH}{2} + g_L p_{n+1} + \frac{g_H r_{n+1}}{2} \right) \prod_{i \in T_{\text{Bet}}} r_i + \frac{1}{2} p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \sum_{i \in T_{\text{Bet}}} p_i^2 \\
+ \frac{gH}{2} - \frac{1}{2} gH \prod_{i \in [n+1]} r_i \pm O(n^2 \Delta^4)
\]

\[
= \left( -\frac{gH(p_{n+1} + q_{n+1})}{2} \right) + p_{n+1}((1 - p_{n+2}) (\tau_H - \tau_L) + p_{n+2} (\tau_{n+2} - \tau_L)) \prod_{i \in T_{\text{Bet}}} r_i \\
- p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \prod_{i \in T_{\text{Bet}}} r_i \prod_{j \in T_{\text{Att}}} (1 - p_j) + \frac{1}{2} p_{n+1} (1 - p_{n+2}) (\tau_H - \tau_L) \sum_{i \in T_{\text{Bet}}} p_i^2 \\
+ \frac{p_{n+2} (\tau_{n+2} - \tau_H)}{2} \left( 1 - \prod_{i \in [n+1]} r_i \right) \pm O(n^2 \Delta^4)
\]

\[
= k_1 f(T_{\text{Bet}}) - k_2 f(T_{\text{Bet}}) g(T_{\text{Att}}) - \frac{1}{2} k_2 \sum_{i \in T_{\text{Att}}} p_i^2 + C \pm O(n^2 \Delta^4).
\]

\[\square\]

**Claim 3.15.** For any subset \( T \) of the first \( n \) boxes, one has

\[
e^{-\sum_{i \in T} 2(p_i + p_i^2)} \geq f(T) \geq e^{-\sum_{i \in T} 2(p_i + p_i^2)} - O(n\Delta^3),
\]

\[
e^{-\sum_{i \in T} (p_i + p_i^2/2)} \geq g(T) \geq e^{-\sum_{i \in T} (p_i + p_i^2/2)} - O(n\Delta^3).
\]

**Proof.** Consider \( f(T) \) first. By Fact 3.14, as \( p_i \leq 2^n/|\Gamma| = \Delta \), we know \( e^{-2(p_i + p_i^2/2)} = 1 - 2p_i + O(p_i^3) \).

Then \( f(T) = \prod_{i \in T} (1 - 2p_i) = \prod_{i \in T} \left( e^{-2(p_i + p_i^2)} - O(p_i^3) \right) \geq \prod_{i \in T} e^{-2(p_i + p_i^2)} - |T| \max_i O(p_i^3) \).

As for the other side, again by Fact 3.14 we have \( 1 - 2p_i \leq e^{-2(p_i + p_i^2)} \), which directly implies \( e^{-\sum_{i \in T} 2(p_i + p_i^2)} \geq f(T) \).

The other inequality is based on \( e^{-(p_i + p_i^2/2)} - O(p_i^3) \leq 1 - p_i \leq e^{-(p_i + p_i^2/2)} \), therefore it holds by the same argument. \[\square\]
Claim 3.13. If we choose $t$ so that $|t - 2e^{y/2}| \leq O(\Delta^2)$, and set $\tau_H$ as follows, then $\frac{k_2}{k_1} = t$:

$$\tau_H = \frac{-3\Gamma + 28 + 94t}{-4t\Gamma + 56 + 104t}.$$  \hspace{1cm} (11)

Proof. We would like

$$k_2 = tk_1 = tk_2 + t[-p_{n+2}(\tau_{n+2} - \tau_H)(p_{n+1} + q_{n+1})/2 + p_{n+1}p_{n+2}(\tau_{n+2} - \tau_L)],$$

which is equivalent to

$$(1-t)p_{n+1}(1-p_{n+2})(\tau_H - \tau_L) = t[-p_{n+2}(\tau_{n+2} - \tau_H)(p_{n+1} + q_{n+1})/2 + p_{n+1}p_{n+2}(\tau_{n+2} - \tau_L)].$$

Rearranging the terms, we get

$$[(1-t)p_{n+1}(1-p_{n+2}) - t p_{n+2}(p_{n+1} + q_{n+1})/2] \tau_H = t[-p_{n+2}(\tau_{n+2} - \tau_H)(p_{n+1} + q_{n+1})/2 + p_{n+1}p_{n+2}(\tau_{n+2} - \tau_L)] + (1-t)p_{n+1}(1-p_{n+2})\tau_L.$$

Recall that $p_{n+2} = 1/8$, $\tau_{n+2} = 3/4$, $p_{n+1} = 1/\Gamma$, $q_{n+1} = 1 - 41/\Gamma$, $\tau_L = 1/2$ and thus

$$\tau_H = \frac{\left[7(1-t)\Gamma - t\Gamma\right]}{16(1 - \frac{40}{\Gamma})} = \frac{7(1-t)}{16} - \frac{3t}{64}\left(1 - \frac{40}{\Gamma}\right).$$

We need $1/2 < \tau_H < 3/4$, which means

$$\frac{1}{2} < \frac{-3\Gamma + 28 + 94t}{-4t\Gamma + 56 + 104t} < \frac{3}{4}.$$ 

For the first inequality, noting that $|t - 2| \leq O(y^\Delta) = O(\Delta^2)$ and $\Delta = 2^{-7n}$, we have

$$-4t\Gamma + 56 + 104t > -6t\Gamma + 2(28 + 94t),$$

which means the first inequality holds.

As for the second inequality, we need

$$-12t\Gamma + 4(28 + 94t) > -12t\Gamma + 3(56 + 104t),$$

$$\iff 64t > 56,$$

which holds by our choice of $t$. \hfill \Box

Lemma 3.10. The parameters of the instance can be set up so that

$$h(x) - O(n^2\Delta^4) \leq \frac{\text{Loss}(\sigma) - C \pm O(n^2\Delta^4)}{k_1} \leq h(x) + O(n\Delta^3),$$

$$\frac{k_2}{k_1} = 2e^{y/2} \pm O(\Delta^2),$$

where

$$h(x) := e^{-2x}\left(1 - \frac{k_2}{k_1}e^{-y+x}\right),$$

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with \( C, k_1, k_2 \) as constants independent of \( \sigma \):

\[
k_1 := -\frac{1}{2}p_{n+2}(\tau_{n+2} - \tau_H)(p_{n+1} + q_{n+1}) + p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L) + p_{n+2}(\tau_{n+2} - \tau_L),
\]

\[
k_2 := p_{n+1}(1 - p_{n+2})(\tau_H - \tau_L),
\]

\[
C := \frac{1}{2}p_{n+2}(\tau_{n+2} - \tau_H) \left( 1 - \prod_{i \in [n+1]} r_i \right) + \frac{1}{2}k_2 \sum_{i=1}^{n} p_i^2.
\]

**Proof.** By the setup of the parameters, we have \( k_1, k_2 > 0 \) and \(|k_2/k_1 - 2e^{y/2}| \leq \Delta^2\). Combining Claim 3.12 and Claim 3.15 gives

\[
k_1 \left( e^{-\sum_{i \in \mathcal{T}} 2(p_n + p_i^2)} \right) \left( 1 - \frac{k_2}{k_1} e^{-\sum_{i \in \mathcal{T}} (p_n + p_i^2/2)} \right)
\]

\[
\leq \text{Loss}(\sigma) - C + \frac{1}{2}k_2 \sum_{i \in \mathcal{T}} p_i^2 \pm O(n^2 \Delta^4),
\]

\[
\leq k_1 \left( e^{-\sum_{i \in \mathcal{T}} 2(p_n + p_i^2)} - O(n^3) \right) \left( 1 - \frac{k_2}{k_1} \left( e^{-\sum_{i \in \mathcal{T}} (p_n + p_i^2/2)} - O(n^3) \right) \right)
\]

\[
\leq k_1 \left( e^{-\sum_{i \in \mathcal{T}} 2(p_n + p_i^2)} \right) \left( 1 - \frac{k_2}{k_1} e^{-\sum_{i \in \mathcal{T}} (p_n + p_i^2/2)} \right) + O(|k_1|n^3).
\]

Recall that \( y := \sum_{i \in S} s_i/\Gamma + (s_i/\Gamma)^2 = \sum_{i \in \mathcal{T} \cup \mathcal{T}_B} p_i + p_i^2 \), \( x := \sum_{i \in \mathcal{T}_B} p_i + p_i^2 \) and we define \( z := \sum_{i \in \mathcal{T}} p_i^2/2 \) for analysis. We know that \( e^{-\sum_{i \in \mathcal{T}_B} 2(p_n + p_i^2)} = e^{-2x} \) and \( e^{-y+x+z} = e^{-\sum_{i \in \mathcal{T}} (p_n + p_i^2/2)} \). Note that \( 0 < k_1 = O(1) \) and we can rewrite the equations above as

\[
e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x+z} \right)
\]

\[
\leq \frac{\text{Loss}(\sigma) - C + k_2 \sum_{i \in \mathcal{T}} p_i^2/2}{k_1} \pm O(n^2 \Delta^4)
\]

\[
\leq e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x+z} \right) + O(n^3).
\]

Note that \( 1 + z \leq e^z \leq 1 + z + z^2 \) as \( 0 \leq z \leq 1/4 \) and \( k_2/k_1 \approx 2 \). On one hand, we know that

\[
e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x+z} \right) \leq e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x}(1 + z) \right).
\]

\[
\leq e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x} \right) - \frac{k_2}{k_1} z + O(xz).
\]

On the other hand,

\[
e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x+z} \right) \geq e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x}(1 + z + z^2) \right)
\]

\[
\geq e^{-2x} \left( 1 - \frac{k_2}{k_1} e^{-y+x} \right) - \frac{k_2}{k_1} z - O(z^2).
\]

Noting that \( O(xz) = O(n^3) \) and \( O(z^2) \leq O(n^2 \Delta^4) \) completes the proof. \( \square \)
Claim 3.16. If $|k_2/k_1 - 2e^{y/2}| \leq O(\Delta^2)$, $\epsilon \in \mathbb{R}$ is such that $2^{-6n} \geq |\epsilon| \geq 1/\Gamma = 2^{-8n}$, let $x^* \in [0, 1/2]$ be where $h(x)$ takes its minimum value, then $|x^* - y/2| \leq O(\Delta^2)$, 
\[
h(x^* + \epsilon) \geq h(x^*) + \epsilon^2/2.
\]

Proof. Recall $h(x) = e^{-2x}(1 - k_2/k_1 e^{-y+x})$. We have $\frac{dh(x)}{dx} = -2e^{-2x} + \frac{k_2}{k_1} e^{-y-x}$, and $\frac{d^2h(x)}{dx^2} = 4e^{-2x} - \frac{k_2}{k_1} e^{-y-x} \in [1, 4]$ for $-2^{-6n} \leq x \leq 1/2$. Therefore, $\frac{dh(x)}{dx} |_{x^*} = 0$. Hence by strong convexity, we know $h(x^* + \epsilon) \geq h(x^*) + \epsilon^2/2$.

Now we prove $|x^* - y/2| \leq O(\Delta^2)$. We know the derivative $|h'(y/2)| = | -2e^{-y} + \frac{k_2}{k_1} e^{-3y/2} | \leq O(\Delta^2)$, which means $|x^* - y/2| \leq O(\Delta^2)$ by the strong convexity.

B Omitted Proofs from Section 4

B.1 Fu et al.’s Framework and Its Application to PNOI

We first introduce Fu et al. [8]’s framework, which give polynomial-time approximation schemes (PTAS) for a class of stochastic sequential decision problems. Some simplifications are made in this presentation. We then adapt the framework to the PNOI problem and obtain a PTAS in the constant-support case.

A stochastic sequential decision process (SSDP) is given by a 6-tuple $(V, A, f, g, I_0, R)$. $V$ is the set of states of the process. $I_0 \in V$ is the initial state; we denote by $I_t$ the state at round $t$. $A$ is the set of actions. The process could last for at most $R$ rounds. In each round, an action $a \in A$ is taken and each action could only be taken once during the process. A policy could also choose to end the process at any time before the $R$-th rounds.

Function $f$ is the stochastic state transition function, which maps a (current_state, action) pair to the new state in the next round. Function $g$ calculates the marginal payoff in each round. Assuming an action $a \in A$ is taken at the current state $I$, the process yields a random payoff of $g(I, a)$.

The total payoff of the process is the sum of all marginal payoff incurred in each round, and the goal is to maximize the expected payoff for this decision process. We define MAX as the maximum expected payoff of the SSDP if hypothetically it could choose to start with any state $I_0 \in V$, not just $I_0$.

Note that for any randomized policy $A^R$, there is a deterministic policy that achieves expected payoff no less than $A^R$. Therefore we assume in this section that policies are deterministic.

Theorem B.1 (Essentially from [8]). Given an SSDP $(V, A, f, g, I_0, R)$, if the following conditions are satisfied,

1. The number of possible states is a constant, i.e., $|V| = O(1)$.
2. The state space $V$ could be ordered such that $f(I, a) \geq I$ for any $I \in V, a \in A$, i.e., the state is non-decreasing.
3. There exists an optimal policy which never takes an action with negative expected marginal payoff in any round.
4. MAX = $O(OPT)$.

Then, for any fixed $\epsilon > 0$, there is an policy $A$ computable in time $n^{2O(\epsilon^{-3})}$, with expected payoff at least $(1 - \epsilon) \cdot OPT$. 

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We here present a high-level idea for Theorem B.1; readers interested in the details are referred to Fu et al. [8]. Consider the decision tree of an optimal policy, in which an action is taken on each node. We first simplify this decision tree by a method called block adaptive policy, which squeezes a sequence of actions into a single node, i.e., a block. Exploiting the fact that the number of different states is constant, the new decision tree for the block adaptive policy could be compressed to constant depth, while having little loss on the payoff. We can then enumerate all possible topologies of the decision tree for block adaptive policies, as the depth and the maximum degree of such a tree are bounded by constants.

After we find an optimal topology, there are still exponentially many ways to fill actions into each block. To find the optimal way to do so, we define a signature vector to represent each block of actions, so that every two blocks having the same signature are roughly equivalent; also, there are only a polynomial number of possible signatures, i.e., they could be efficiently enumerated. Finally, after fixing the topology and then the signatures, we use a dynamic programming approach to check whether the given set of signatures could be achieved with all the actions available.

**SSDP Model for PNOI**

We present PNOI in the general framework of SSDP and apply Theorem B.1 to get a PTAS for PNOI in the constant-support case.

**Theorem B.2.** If the union of the supports of $F_1, \ldots, F_n$ has $O(1)$ size, then for any fixed $\epsilon > 0$, there is a polynomial-time algorithm that computes a policy with expected payoff at least $(1 - \epsilon) \cdot \text{OPT}$.

Define the set of all possible states $V$ as the union of the supports of $F_1, \cdots, F_n$. Let $\text{MaxV}$ to be the maximum value that is possibly taken by $v_1, \ldots, v_n$, which is also the largest state in $V$ by definition. We set the round limit $T := n$.

Let $a_i^0$ be the action that we open the box $v_i$ and $a_i^1$ be the action that we take box $v_i$ without open it. We thus have $2n$ possible actions, i.e., $A := \{a_1^0, a_1^1, \ldots, a_n^0, a_n^1\}$. Note that at most one of the actions $a_i^0, a_i^1$ could be chosen during the process by the definition of PNOI problem. This conflicting constraint is not covered by the standard SSDP framework. Fortunately, [8] provides a simple way to handle it in the dynamic programming step and Theorem B.1 still works with this additional constraint.

We set $I_0 = 0$ and define $I_t$ as the largest value ever opened in the first $t$ rounds. Specially, we set $I_t$ equal to $\text{MaxV}$ once a box is taken away without opening it. This could prevent the policy from taking further actions after selecting a box.

By the definition above, the state transition function $f$ works as follow:

$$f(I, a_i^0) = \max\{I, v_i\};$$
$$f(I, a_i^1) = \text{MaxV}.$$

We amortize the payoff into each round and define the following *marginal payoff* function $g$,

$$g(I, a_i^0) = (\max\{I, v_i\} - I) - c_i;$$
$$g(I, a_i^1) = \mathbb{E}[v_i] - I.$$

It’s easy to verify that the total payoff in each round is exactly the payoff defined in the PNOI problem.

---

4 Actually, similar conflicting constraints also arose in Committed ProbeTop-k Problem and Committed Pandora Problem in [8].
**Decision Tree**  We take a detour to introduce the language of decision tree, which is our major tool to analyze PNOI policy in SSDP model.

Let \( A^0 = \{ a^0_1, \ldots, a^0_n \} \) to be the set of actions that open the boxes; let \( A^1 = \{ a^1_1, \ldots, a^1_n \} \) to be the set of actions that take a box without opening it. We denote the set of all actions by \( A = A^0 \cup A^1 \). For any action \( a \in A^0 \) and state \( I_1, I_2 \in V \), we define \( \Phi_a(I_1, I_2) := \Pr[f(I_1, a) = I_2] \) and \( G(I, a) := \mathbb{E}[g(I, a)] \) for brevity.

Given an instance \( \pi \) of PNOI, the process of applying a deterministic policy \( \mathcal{A} \) could be fully described by a decision tree \( T \). For each node \( v \) on \( T \), we have the following attributes:

1. an action \( a_v \in A \cup \{ \text{end} \} \), which is going to be taken at node \( v \); \( a_v = \text{end} \) iff node \( v \) is a leaf node;
2. the current state \( I_v \) before taking action \( a_v \);
3. a set of children nodes \( \text{Ch}(v) \) (for non-leaf node); each child node \( u \in \text{Ch}(v) \) corresponds to a different possible state of \( f(I_v, a_v) \).

We define \( G(v) := G(I_v, a_v) \) as the expected marginal payoff at node \( v \). Specially, \( G(v) = 0 \) if \( v \) is a leaf node with \( a_v = \text{end} \).

For each leaf node \( u \), define \( H(u) \) as the accumulated payoff at node \( u \), i.e.,

- if there is an ancestor node \( v \) of \( u \) such that \( a_v = a^1_i \in A^1 \), \( H(u) \) is set to \( \mathbb{E}[v_i] \) minus the total cost on the path to node \( v \);
- otherwise, \( H(u) \) equals to the maximum value ever opened along the path to \( v \) minus the total cost on the path to node \( v \);

Let \( \Phi(v) \) denote the probability that node \( v \) will be reached during execution. Now, the expected payoff of policy \( \mathcal{A} \) on the instance \( \pi \) could be represented in two ways,

\[
\mathbb{P}(\mathcal{A}) = \sum_{v \in T} G(v) \cdot \Phi(v) = \sum_{u \in T, u \text{ is leaf}} H(u) \cdot \Phi(u). \tag{12}
\]

**Analysis**  Now we are ready to prove the following non-trivial property of our model.

**Lemma B.3.** *Any optimal PNOI policy would never take action with negative marginal payoff in any round.*

**Proof.** In the language of decision tree, it is equivalent to prove that in a decision tree \( T^* \) of any optimal policy \( \mathcal{A}^* \), there isn’t any node \( v \in T^* \) with negative marginal payoff \( G(v) \).

We prove by contradiction. Assume there is a node \( u \in T^* \) with negative marginal payoff \( G(u) \). If \( a_u \in A^1 \), replacing \( a_u \) with \( a_s \) will strictly improve the expected payoff, which leads to a contradiction.

When \( a_u = a^0_i \in A^0 \) for some \( i \in [n] \), we construct a slightly modified policy \( \mathcal{A}' \). \( \mathcal{A}' \) uses the same decision tree \( T^* \). The only difference is that when policy \( \mathcal{A}' \) reaches the node \( u \), \( \mathcal{A}' \) will not actually probe the variable \( v_i \), but sample a value \( v'_i \) from the same distribution of \( v_i \) using internal randomness, and pretend the value \( v'_i \) to be the realized value of \( v_i \). By definition, \( \mathcal{A}^* \) and \( \mathcal{A}' \) will reach every node \( v \in T^* \) with same probability.

Define \( I'_v \) as the actual state at node \( v \) when executing policy \( \mathcal{A}' \), which is a random variable smaller or equal to \( I_v \). Similarly, define \( G'(v) \) to be the actual expected marginal payoff at node \( v \), i.e., \( G'(u) = 0 \) and \( G'(v) = G(I'_v, a_v) \) for any \( v \neq u \). From the definition of function \( g \) in Section [B.1], we know that \( G'(v) \geq G(v) \) for any node \( v \in T^* \), and in particular, \( 0 = G'(u) > G(u) \). From Equation [12] we conclude that \( \mathbb{P}(\mathcal{A'}) > \mathbb{P}(\mathcal{A}^*) \), which is a contradiction. \( \square \)
Finally, we prove Theorem B.2 by checking each condition required by Theorem B.1

Proof of Theorem B.2. The first three conditions are trivially satisfied by the definition of our SSDP model; the third condition is promised by Lemma B.3.

For the last condition, if the process starts with an initial state $I_0' > 0$, it’s equivalent to have a PNOI instance with boxes $v_1', \ldots, v_n'$ where $v_i' = v_i - I_0'$. Thus, the payoff of the new instances is smaller or equal to the original instance with $I_0 = 0$, i.e., $\text{MAX} = \text{OPT}$.

\section*{B.2 Discretization}

When the supports of the value distributions are not of constant sizes, we apply similar discretization technique as in Fu et al. \cite{8} to satisfy the condition $|V| = O(1)$. We show in this section that there is a PTAS for PNOI with milder conditions.

\begin{theorem}
For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $A$ with expected payoff at least $(1 - \epsilon) \cdot \text{OPT}$ for PNOI instances that satisfy at least one of the following two conditions:

\begin{enumerate}
  \item there is a constant $c_\epsilon$, which only depends on $\epsilon$, such that $v_i \leq c_\epsilon \cdot \text{OPT}$ with probability 1, for all $i$;
  \item there is a constant $c_\epsilon$, which only depends on $\epsilon$, such that each value distribution $F_i$ is discrete and, for any value $v$ in the support, has $F_i(v) \geq 1/c_\epsilon$.
\end{enumerate}
\end{theorem}

Let $\widetilde{\text{OPT}}$ be an estimation for $\text{OPT}$ satisfying $(1 - 1/e) \cdot \text{OPT} \leq \widetilde{\text{OPT}} \leq \text{OPT}$. Such an estimation is computable in polynomial time, e.g. using the approximation algorithm by Beyhaghi and Kleinberg \cite{9}. Define discretization function $D_\epsilon(x) := \floor[\frac{x}{\text{OPT} \cdot \epsilon}] \cdot \widetilde{\text{OPT}} \cdot \epsilon$. Let $\tilde{v}_i := D_\epsilon(v_i)$ be the discretized version of $v_i$.

We say a policy $A$ is discretized, if the decisions made by $A$ only depends on the discretized values $\tilde{v}_1, \ldots, \tilde{v}_n$.

\begin{lemma}
Let $\pi$ be an instance of PNOI and $\tilde{\pi}$ the discretized instance of $\pi$, in which each value $v_i$ is replaced by $\tilde{v}_i$. Then,

\begin{enumerate}
  \item for any policy $A$, there is a discretized policy $\tilde{A}$ such that $\mathbb{P}(\tilde{A}, \tilde{\pi}) \geq \mathbb{P}(A, \pi) - O(\epsilon) \cdot \text{OPT}$;
  \item for any discretized policy $\tilde{A}$, $\mathbb{P}(\tilde{A}, \pi) \geq \mathbb{P}(\tilde{A}, \tilde{\pi})$.
\end{enumerate}
\end{lemma}

Proof. Fix an arbitrary (deterministic) policy $A$ and denote its decision tree for instance $\pi$ by $T$. We construct a randomized discretized policy $\tilde{A}$ which simulates $A$ when running on the discretized instance $\tilde{\pi}$. More specifically, $\tilde{A}$ will take same actions to the corresponding discretized variables. Also, once probing a discretized variable $\tilde{v}_i$, $\tilde{A}$ will first randomly sample a value $x_i'$ from the distribution of $v_i$ conditioned on the realized value of $\tilde{v}_i$. $\tilde{A}$ will then use $x_i'$ as the realized value for $v_i$ to simulate $A$.

It’s easy to verify that $\tilde{A}$ could be represent by the same decision tree $T$. Moreover, $\tilde{A}$ will reach each node in $T$ with the same probability as $A$ did. Let $\tilde{H}(v)$ to be the actual accumulated payoff at node $v \in T$ when $\tilde{A}$ is executing on $\tilde{\pi}$. By definition, we have $\tilde{H}(v) \geq H(v) - \epsilon \cdot \text{OPT}$. Finally, from Equation \cite{12} we derive that

$$\mathbb{P}(\tilde{A}, \tilde{\pi}) = \sum_{u \in T \atop u \text{ is leaf}} \tilde{H}(u) \cdot \Phi(u) \geq \sum_{u \in T \atop u \text{ is leaf}} (H(u) - \epsilon \cdot \text{OPT}) \cdot \Phi(u) \geq \mathbb{P}(A, \pi) - O(\epsilon) \cdot \text{OPT}.$$
This first statement is now proved.

For the second statement, notice that for any discretized policy $\tilde{A}$, the decision tree is exactly same when running on either $\pi$ or $\tilde{\pi}$; also, each node the on decision tree will be reached with same probability. As we always have $v_i \geq \tilde{v}_i$, there is

$$P(\tilde{A}, \tilde{\pi}) \leq P(\tilde{A}, \pi).$$

Theorem B.4 then follows from Lemma B.5 and Theorem B.2.

**Proof of Theorem B.4.** We first show that the second condition implies the first. Given the second condition, for any box $i$ and value $v$ in the support of $v_i$, we have

$$\text{OPT} \geq \Pr[v_i = v] \cdot v \geq \frac{1}{c} \cdot v.$$ 

Therefore $v \leq c \cdot \text{OPT}.$

Now we show the lemma under the first condition. Let $A^*$ denote an optimal policy for the original instance $\pi$, i.e., $P(A^*, \pi) = \text{OPT}$. The first part of Lemma B.5 guarantees the existence of a discretized policy $\tilde{A}^*$ with

$$P(\tilde{A}^*, \tilde{\pi}) \geq P(A^*, \pi) - O(\epsilon) \cdot \text{OPT} = (1 - O(\epsilon)) \cdot \text{OPT}.$$

We denote the value space of the discretized instance $\tilde{\pi}$ by $\tilde{V} := \{0, \epsilon \cdot \text{OPT}, 2\epsilon \cdot \text{OPT}, \ldots, D_\epsilon(\text{MaxV})\}$. Since $\text{MaxV} \leq c \cdot \text{OPT}$, we have $|\tilde{V}| \leq \text{MaxV}/\text{OPT} = O(1)$. Thus, Theorem B.2 applies to the discretized instance $\tilde{\pi}$ and yields a discretized policy $\tilde{A}$ such that

$$P(A, \tilde{\pi}) \geq (1 - O(\epsilon))P(\tilde{A}^*, \tilde{\pi}).$$

Finally, using the second part of Lemma B.5, we conclude that

$$P(A, \pi) \geq (1 - O(\epsilon)) \cdot \text{OPT}.$$

**Corollary B.6.** For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $A$ with expected payoff at least $\text{OPT} - \epsilon \cdot \text{MaxV}$. 

**Proof.** When $\text{MaxV} \geq (1/\epsilon \cdot \text{OPT})$, the statement is trivially true. Otherwise, the first condition of Theorem B.4 is satisfied, and there is a polynomial-time computable policy $A$ with expected payoff at least

$$(1 - \epsilon) \cdot \text{OPT} \geq \text{OPT} - \epsilon \cdot \text{MaxV}.$$
B.3 Proof of Lemma 4.3 and Lemma 4.7

**Lemma 4.3.** For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $A_T$ with expected payoff at least $\mathbb{P}(A^*_T) - \epsilon \cdot \text{OPT}$.

**Proof.** Define a truncated version of PNOI called T-PNOI, such that T-PNOI will immediately end once a value greater than $\theta$ is revealed. In other words, any valid policy for the T-PNOI problem is a valid truncated policy for the PNOI problem and vice versa.

T-PNOI could be modeled as an SSDP, with following modifications to the previous SSDP model of PNOI in Section B.1,

1. All the states with value greater than $\theta$ are replaced by a new state $I_T := \theta$.

2. The state transition function $f_T$ in T-PNOI will be truncated accordingly, namely,

$$
\begin{align*}
    f_T(I, a^0_i) &= \min\{\max\{I, v_i\}, I_T\}; \\
    f_T(I, a^1_i) &= I_T.
\end{align*}
$$

3. The marginal payoff function $g_T$ is set to 0 once reaching the state $I_T$, namely,

$$
g_T(I_T, \cdot) = 0.
$$

Therefore, we could apply Theorem B.2 and the discretization technique to the T-PNOI model, and get a polynomial-time computable policy $A_T$ with expected payoff at least $\text{OPT}_T - \epsilon \cdot \text{OPT}$, where $\text{OPT}_T$ is the optimal expected payoff of the T-PNOI problem.

Noticing that $\mathbb{P}(A^*_T) \leq \text{OPT}_T$ and $\text{OPT}_T \leq \text{OPT}$, we conclude that

$$
\mathbb{P}(A_T) \geq \text{OPT}_T - \epsilon \cdot \text{OPT} \geq \mathbb{P}(A^*_T) - \epsilon \cdot \text{OPT}.
$$

\[\square\]

**Lemma 4.7.** For any fixed $\epsilon > 0$, there is a polynomial-time computable policy $A_{TE}$ with expected payoff at least $\mathbb{P}(A^*_T) - \epsilon \cdot \text{OPT}$.

**Proof.** Similar to the case of $A_T$, we define a truncated version of PNOI called TE-PNOI. Recall that box $i_E$ is the one with maximal expected value. Once a value greater than $\theta$ is revealed, an algorithm for TE-PNOI could either stop immediately, or open box $i_E$ and then stop, if box $i_E$ still available.

We could modify the SSDP model for T-PNOI to get an SSDP for TE-PNOI.

1. Each state $I$ is now extended to a pair $(y, e)$, where $y$ is the maximal value ever revealed and $e \in \{0, 1\}$ represents whether box $i_E$ has been taken. The value of $y$ is always truncated to the threshold $\theta$.

2. The state transition function is mostly the same, while tracking whether box $i_E$ has been taken. Formally, denote $f_{TE} := (f_{TE}^y, f_{TE}^e)$ as the state transition function in TE-PNOI. Then,

$$
\begin{align*}
    f_{TE}^y((y, e), \cdot) &= f_T(y, \cdot); \\
    f_{TE}^e((y, e), a) &= e \lor 1[a = a_{0}^{i_E}].
\end{align*}
$$

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3. The marginal payoff function \( g_{TE} \) has to plan one step further, i.e., calculating the payoff of opening box \( i_E \) after receiving a value greater than \( \theta \). Formally, recall that \( \mathbb{P}_I(S,v) := \mathbb{E}[(\max_{i \in S} \kappa_i - v)_+] \) is the expected additional payoff gained from performing an index-based policy on a set \( S \) of unopened boxes, with an outside option of \( v \). Then, for any \( i \neq i_E \),

\[
g_{TE}(v,0,a_i^0) = g_T(v,a_i^0) + \Pr[v_i > \theta] \cdot \mathbb{E}[\mathbb{P}_I(\{i_E\},v_i) \mid v_i > \theta].
\]

\( g_{TE} \) is equal to \( g_T \) in any other cases by simply ignoring \( e \) in the state.

Using the same argument for \( A_{TE} \), we have

\[
\mathbb{P}(A_{TE}) \geq \mathbb{P}(A_{TE}^*) - \epsilon \cdot \text{OPT}.
\]

### B.4 Proof of Corollary 4.9

**Corollary 4.9.** For any fixed PNOI instance \( \pi \),

\[
\max\{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A_{TE}^*), \mathbb{P}(A_{CE})\} \geq 0.8 \cdot \text{OPT}
\]

**Proof.** We use \( I, E, CE, TE \) to indicate \( \mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A_{CE}), \mathbb{P}(A_{TE}^*) \) and also replace \( \text{OPT} \) by 1 w.l.o.g. for brevity. Now we rewrite the constraints in Lemma 4.4 and Lemma 4.8 in below.

\[
I + (1 - p_T) \cdot E \geq 1 \quad (13)
\]

\[
TE + p_T \cdot (CE - E) \geq 1 \quad (14)
\]

Note that we only include the second constraint in Lemma 4.4 here, which could only make the approximation ratio smaller. Recall that when taking

\[
I = TE = CE = 0.8, E = 0.4, p_T = 0.5,
\]

we have \( \max\{I, TE, CE, E\} = 0.8 \). I.e., in an optimal solution, there is \( \max\{I, TE, CE, E\} = X \leq 0.8 \).

Using inequality 13 and inequality 14, we could infer that

1. \( p_T \in [0.2, 0.8] \);
2. \( CE - E \geq 0.2 \);
3. \( E \geq 0.2 \).

**Claim:** Both inequality 13 and inequality 14 must be tight in the optimal solution.

We prove this claim by contradiction.

- If \( I + (1 - p_T) \cdot E = 1 + \epsilon \) for any \( \epsilon > 0 \), we could take

\[
I' = I - 0.1\epsilon, E' = E - 0.5\epsilon, CE' = CE - 0.1\epsilon, TE' = TE - 0.01\epsilon.
\]

It is easy to verify that \( (I', E', CE', TE', p_T) \) is valid and strictly better, a contradiction!
• If $TE + p_T \cdot (CE - E) = 1 + \epsilon$ for any $\epsilon > 0$, we could take

\[
I'' = I - 0.01\epsilon, E'' = E + 0.5\epsilon, CE'' = CE - 0.1\epsilon, TE'' = TE - 0.1\epsilon.
\]

It is easy to verify that $(I'', E'', CE'', TE'', p_T)$ is valid and strictly better, again, a contradiction.

Note that by taking $I = CE = TE = x^*$, the solution is still valid and optimal. Therefore, w.l.o.g., we could replace $I, CE, TE$ with $x^*$. Now we get,

\[
x^* + (1 - p_T) \cdot E = 1 \quad (15)
\]
\[
x^* + p_T \cdot (x^* - E) = 1 \quad (16)
\]

By subtracting equality (15) from equality (16), we get $E = p_T \cdot x^*$. Plugging it into equality (16), there is $x^* \cdot (1 + p_T - p_T^2) = 1$. Recall that $p_T \in [0.2, 0.8]$, we know $(1 + p_T + p_T^2) \leq 1.25$, which means $x^* \geq 0.8$. We conclude that $0.8 \cdot \text{OPT}$ is the minimum possible value of $\max\{\mathbb{P}(A_I), \mathbb{P}(A_E), \mathbb{P}(A_{CE}), \mathbb{P}(A_{TE})\}$.

\[\square\]