Some bijections for lattice paths

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Abstract

We present three bijections, the first between little Schröder paths and a class of growth-constrained integer sequences, the second between lattice paths consisting of steps with nonnegative slope and another class of growth-constrained sequences, the third between a class of lattice paths with steps $(1,1)$ and $(1,-j)$, $j \geq 1$, and a class with steps $(k,1)$, $k \geq 1$, and $(1,-1)$.

1 Introduction

The purpose of this paper is to present three bijections involving combinatorial objects counted by sequences $A001003$, $A049600$, and $A090345$ respectively in The On-Line Encyclopedia of Integer Sequences (OEIS) [1]. Lattice paths are a common feature and the bijections are presented in the next 3 sections.

2 A manifestation of the little Schröder numbers

The little Schröder numbers $(s_n)_{n \geq 1} = (1,3,11,45,197,\ldots)$ form sequence $A001003$ in OEIS. They have many combinatorial manifestations (see e.g., [2, Sec. 6.2.8 and Ex. 6.39]) including little Schröder paths, defined below. In this section we give a bijective proof for another one: $s_n = |\mathcal{S}_n|$ where $\mathcal{S}_n$ is the set of sequences of nonnegative integers $(u_1,u_2,\ldots,u_n)$ such that (i) $u_1 = 1$, (ii) $u_i \leq i$ for all $i$, and (iii) the nonzero $u_i$ are weakly increasing. For instance, $\mathcal{S}_2 = \{10, 11, 12\}$.

A Schröder path of semilength $n$ is a lattice path of steps east $E = (1,0)$, north $N = (0,1)$ and northeast (diagonal) $D = (1,1)$ from the origin to $(n,n)$ that never rises above the diagonal line $y = x$. Such a path has equal numbers of $E$ and $N$ steps and its semilength is the number of $E$ steps plus the number of $D$ steps. A little Schröder path is one that has no $D$ steps on the diagonal $y = x$. Let $\mathcal{LS}$ denote them. For instance, $\mathcal{LS}_2 = \{EENN, EDN, ENEN\}$. 

1
Here is a bijection from $\mathcal{L}S_n$ to $S_n$ for $n \geq 1$. Given a little Schröder path, say $EDENEDNDDNEN \in \mathcal{L}S_8$ as in Figure 1 in blue, record the heights above the horizontal line $y = -1$ of the terminal points of each $E$ and $D$ step, marking the height of each diagonal step with a superscript $^d$ as in $h^d$.

The example produces the height sequence $1 \ 2^d \ 2^d \ 3 \ 4^d \ 6^d \ 7^d \ 8$. Clearly, this height sequence determines the path and a valid height sequence begins with an unmarked 1, is weakly increasing and strictly so if the second height is marked, and the $i$-th height is $\leq i$ for all $i$.

Now, say a marked height $h^d$ is lonely if its predecessor height, $g$ or $g^d$, satisfies $h \geq g + 2$ (this occurs when a $D$ in the path is immediately preceded by an $N$). The main step in the bijection is iterative: as long as a lonely $h^d$ is present, interchange the first such $h^d$ (although any such $h^d$ would do) with its predecessor and decrement $h$ to
$h - 1$. The example yields

$$
\begin{array}{cccccccc}
1 & 2^d & 2 & 3 & 4^d & 6^d & 7^d & 8 \\
& 5^d & 4^d & \\
& 4^d & 3 & \\
3^d & 2 & \\
1 & 2^d & 3^d & 2 & 3 & 6^d & 4^d & 8 \\
& 5^d & 3 & \\
1 & 2^d & 3^d & 4^d & 2 & 3 & 4^d & 8,
\end{array}
$$

where blank spaces indicate no change from the previous line. When done, each marked height $h^d$ will be preceded by $(h - 1)^d$ or $h - 1$. The last step is to replace each marked height with 0, yielding 1 0 0 2 3 0 8 in our example.

The iterative steps in the bijection trade “lonely” marked heights for “offending” marked heights, where $h^d$ is offending if it has a right neighbor $j$ or $j^d$ with $h > j$. It should now be clear how to reverse the map by eliminating offending marked heights.

In case there are no diagonal steps in a little Schröder path, we have a (rotated) Dyck path, and the above bijection produces a well known manifestation of the Catalan numbers: sequences of positive integers $(u_1, u_2, \ldots, u_n)$ such that $u_i \leq i$ for all $i$ and the $u_i$ are weakly increasing.

### 3 Kimberling paths

A *Kimberling path*, after A049600, is a lattice path from the origin $O = (0, 0)$ to a point $(i, j)$ that consists of steps with finite nonnegative slope. Thus vertical steps (with infinite slope) are not allowed. Let $K(i, j)$ denote the Kimberling paths that terminate at $(i, j)$ (such paths exist only for $i \geq 1$ and $j \geq 0$). The 3 paths in $K(2, 1)$ are shown in Figure 2.

![Figure 2](image)

The paths in $K(2, 1)$: the middle path has vertex set $\{(0, 0), (1, 1), (2, 1)\}$

Choosing the $x$-coordinates of the interior vertices of a Kimberling path and then the
y-coordinates, the number of paths in $K(i,j)$ with $k$ interior vertices is easily seen to be $(i-1)\binom{j+k}{k}$.

For $i, j \geq 0$, let $L(i,j)$ denote the set of sequences of positive integers $(u_1, u_2, \ldots, u_i)$ such that (i) $u_k \leq j+2$ for all $k$, and (ii) $u_k \geq \max\{u_1, \ldots, u_{k-1}\} - 1$ for $k \geq 2$. By convention, $L(0,j)$ consists of just one sequence, the empty sequence $\epsilon$ of length 0, and for example, $L(1,j) = \{1,2,\ldots,j+2\}$ and $L(2,1) = \{11,12,13,21,22,23,32,33\}$, where each sequence is presented as a contiguous string of digits.

The set $L(i,j)$ is equinumerous with $K(i+1,j)$ and we now define a bijection $\phi_j : L(i,j) \to K(i+1,j)$. The subscript $j$ on $\phi$ is needed because, while the value of $i$ can be inferred from an integer sequence in $L$ (being its length), the value of $j$ cannot. The bijection is defined recursively by induction on both $i$ and $j$. For brevity, we write $\phi_j(u_1,\ldots,u_i)$ for $\phi_j(u)$ with $u = (u_1,\ldots,u_i) \in L(i,j)$.

For the base cases, with $i = 0$, $\phi_j(\epsilon)$ is the path of one step from $(0,0)$ to $(1,j)$, and with $i = 1$,

$$\phi_j(1) = \{(0,0), (2,j)\} \text{ while } \phi_j(k) = \{(0,0), (1,k-2), (2,j)\} \text{ for } 2 \leq k \leq j+2.$$ 

Now suppose $i \geq 2$ and $u_1 = k$. Then $\phi_j(k,u_2,\ldots,u_i)$ is defined as follows. If $k \leq 2$, add $(1,0)$ to each vertex of $\phi_j(u_2,\ldots,u_i)$, that is, shift each vertex one unit east and then

- if $k = 1$, replace vertex $(1,0)$ by $(0,0)$ so the image path has no vertex on the vertical line $x = 1$,

- if $k = 2$, prepend $(0,0)$ so that the image path begins $\{(0,0), (1,0), \ldots\}$.

On the other hand, if $k \geq 3$, add $(1,k-2)$ to each vertex of $\phi_{j-(k-2)}(u_2-(k-2),\ldots,u_i-(k-2))$ and then prepend $(0,0)$ so that $(1,k-2)$ is a vertex on the path for $3 \leq k \leq j+2$. For example, $\phi_3(1,4,3,5) = \{(0,0),(2,2),(4,3),(5,3)\}$ because $\phi_3(4,3,5) = \{(0,0),(1,2),(3,3),(4,3)\}$ which follows from $\phi_1(1,3) = \{(0,0),(2,1),(3,1)\}$.

It is not hard to recursively define an inverse for $\phi$, looking first at where a Kimberling path intersects the line $x = 1$.

4 Deutsch paths and Ramírez paths

A Deutsch path [3] is a lattice path of upsteps $(1,1)$ and downsteps $(1,-j)$, $j \geq 1$, that starts at the origin and never goes below the $x$-axis, and it is closed if it ends on the $x$-axis.
Let \( P_n \) denote the set of \( n \)-step closed Deutsch paths with short valley downsteps, that is, each downstep followed by an upstep thereby creating a valley is of the from \((1, -1)\), the shortest possible. For example, \( P_4 \) consists of \( UUU3, UU11, U1U1 \), where an upstep is denoted by \( U \) and a downstep \((1, -j)\) is denoted by \( j \) so that a number immediately preceding a \( U \) must be \( 1 \), and \( P_5 \) consists of \( UUUU4, UUU12, UUU21, UU1U2, U1UU2 \).

A Ramírez path, after A090345, is a lattice path of upsteps \((k, 1)\) with \( k \geq 1 \) and downsteps \((1, -1)\) that starts at the origin, never goes below the \( x \)-axis, and ends on the \( x \)-axis. Its size is the \( x \)-coordinate of its terminal point. Let \( \mathcal{R}_n \) denote the set of Ramírez paths of size \( n \). For example, \( \mathcal{R}_4 \) consists of \( 1D1D, 11DD, 3D \) where a downstep is denoted by \( D \) and an upstep \((k, 1)\) is denoted by \( k \), and \( \mathcal{R}_5 \) consists of \( 1D2D, 12DD, 2D1D, 21DD, 4D \).

We will show that \( P_n \) and \( \mathcal{R}_n \) are equinumerous. Note that \( P_0 \) and \( \mathcal{R}_0 \) both consist of the empty path, \( P_1 \) and \( \mathcal{R}_1 \) are both empty, and \( P_2 \) and \( \mathcal{R}_2 \) both consist of the single path \( UD \). We now give a simple graphical description of a bijection from \( P_n \) to \( \mathcal{R}_n \) for \( n \geq 3 \). So suppose given a path \( P \in P_n \) as in Figure 3a) below. The image path \( Q \in \mathcal{R}_n \) is obtained as follows. First replace each “long” downstep \((1, -j), j \geq 2 \) with \( j \) short downsteps \((1/j, -1)\) and color the last \( j - 1 \) of them blue along with their matching upsteps as in Figure 3b).

\[ P = UUUU1UUU313 \in P_{14} \]

A closed Deutsch path with short valley downsteps

Figure 3a)  \hspace{1cm} Figure 3b)
Each (maximal) run of blue upsteps in Figure 3b) is necessarily followed by an (uncolored) upstep; color this upstep blue also as in Figure 4a). Lastly, replace each run of $k$ blue upsteps by a step $(k, 1)$, delete all the blue downsteps, and change $(1/j, -1)$ downsteps to regular $(1, -1)$ downsteps, as in Figure 4b).

We leave the reader to devise the inverse map.

References

[1] The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2021.

[2] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999.

[3] Helmut Prodinger, Deutsch paths and their enumeration, arXiv:2003.01918v2 [math.CO] 14 Apr 2020.

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