The $r$-Dowling–Lah Polynomials

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Abstract. The notions of $r$-Bell polynomials and their generalization, the $r$-Dowling polynomials are due to Mező and Cheon, Jung. Recently, Nyul and Rácz defined the $r$-Lah polynomials, which are close relatives of $r$-Bell polynomials. In the present paper, we introduce the Dowling type generalization of $r$-Lah polynomials, the $r$-Dowling–Lah polynomials. We give a comprehensive study of them using the results of the author and Nyul on $r$-Whitney–Lah numbers, which are the coefficients of these polynomials.

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1. Introduction

Lah numbers, like Stirling numbers, are essential to enumerative combinatorics. Their definition originates from Lah [7,8]. The Lah number $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle$ is the number of partitions of the set \{1,\ldots,n\} into $k$ ordered blocks. If we modify the problem to partition the set \{1,\ldots,n+r\} into $k+r$ ordered blocks in a way that the first $r$ distinguished elements belong to distinct ordered blocks, then we arrive at the notion of $r$-Lah numbers $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_r$, defined and studied by Nyul and Rácz [12], Cheon and Jung [1] further generalized the $r$-Lah numbers, and with the help of $r$-Whitney numbers of the first kind $w_{m,r}(n,k)$ and $r$-Whitney numbers of the second kind $W_{m,r}(n,k)$, they defined the $r$-Whitney–Lah numbers $WL_{m,r}(n,k)$, and proved some of their properties. Recently, Gyimesi and Nyul [6] gave their combinatorial interpretation, and based on this, they derived additional identities. Independently, Ramírez and Shattuck [14] found a similar interpretation when they studied a $(p,q)$-analogue of these numbers.

Bell polynomials, as well as $r$-Bell polynomials by Mező [10], are also well known in enumerative combinatorics, whose coefficients are Stirling and $r$-Stirling numbers of the second kind, respectively. Analogously, Nyul and

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[13] introduced Lah polynomials \( L_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \), \( r \)-Lah polynomials \( L_{n,r}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r x^k \), and studied them in detail.

The \( r \)-Dowling polynomials \( D_{n,m,r}(x) \) are a generalization of the \( r \)-Bell polynomials, whose coefficients are \( r \)-Whitney numbers of the second kind. They were defined and studied by Cheon, Jung [1] and Corcino, Corcino, Aldema [2,3]. Later, Gyimesi and Nyul [5] gave their combinatorial interpretation, based on which further properties were proved. Independently, similar investigations were done by Corcino, Corcino, Mező and Ramírez [4].

About the background of the above enumeration problems, we recommend the reader to consult the introductory book [11].

In this paper, we introduce the Dowling-like generalization of \( r \)-Lah polynomials, with \( r \)-Whitney–Lah numbers as coefficients. These polynomials will be called \( r \)-Dowling–Lah polynomials and denoted by \( DL_{n,m,r}(x) \). Our intent is to fully examine them. Among others, we give their combinatorial interpretation, as well as their Spivey type formula, recurrence relations, Dobinski type formula and exponential generating function. Furthermore, we prove that the roots of \( r \)-Dowling–Lah polynomials are real and non-positive, and show that the sequence of \( r \)-Dowling–Lah numbers \( DL_{n,m,r}(1) \) is log-convex. We note that although combinatorial proofs dominated in [13] and [5], in the majority of cases those ideas do not work here, so we need to use new techniques.

2. Definition and Combinatorial Interpretation

First, we need to introduce the following notions. Let \( n \geq 0 \) and \( m \geq 1 \), then the \( n \)th rising and falling factorials of \( x \) with difference \( m \) are

\[
(x|m)^\pi = \prod_{i=0}^{n-1} (x + im), \quad (x|m)^\mu = \prod_{i=0}^{n-1} (x - im).
\]

If \( m = 1 \), then we use simply the notations \( x^\pi = (x|1)^\pi \) and \( x^\mu = (x|1)^\mu \).

Before we define the \( r \)-Dowling–Lah polynomials and give their combinatorial interpretation, we recall the definitions of the \( r \)-Whitney–Lah coloured partitions and the \( r \)-Whitney–Lah numbers, introduced in [6]: For \( n, r \geq 0, n + r \geq 1 \) and \( m \geq 1 \), we call a partition of \( \{1, \ldots, n + r\} \) an \( r \)-Whitney–Lah coloured partition with \( m \) colours if

- the distinguished elements \( 1, \ldots, r \) belong to distinct ordered blocks,
- the smallest elements of the ordered blocks are not coloured,
- an element in an ordered block containing distinguished element is not coloured if there are no smaller numbers between the distinguished element and this element,
- the remaining elements are coloured with \( m \) colours.

Then, the \( r \)-Whitney–Lah number \( WL_{m,r}(n,k) \) \((0 \leq k \leq n)\) is the number of \( r \)-Whitney–Lah coloured partitions of \( \{1, \ldots, n + r\} \) with \( m \) colours into \( k + r \) ordered blocks, while \( WL_{m,0}(0,0) = 1 \).
These numbers can be computed, for example, using the initial values
\[ WL_{m,r}(n,0) = (2r|m)^n, \quad WL_{m,r}(n,n) = 1 \quad (n \geq 0) \]
and the recurrence relation (see [6, Theorem 3.2])
\[ WL_{m,r}(n+1,k) = WL_{m,r}(n,k-1) + (m(n+k)+2r)WL_{m,r}(n,k) \quad (1 \leq k \leq n), \]
or, assuming \( r \geq 1 \), the explicit formula (see [6, Theorem 3.5])
\[ WL_{m,r}(n,k) = \binom{n}{k}(2r|m)^n \quad (0 \leq k \leq n). \]

Now, we are in position to define the \( r \)-Dowling–Lah polynomials.

**Definition 2.1.** Let \( n,r \geq 0 \) and \( m \geq 1 \). Then we call the polynomial
\[ DL_{n,m,r}(x) = \sum_{k=0}^{n} WL_{m,r}(n,k)x^k \]
an \( r \)-Dowling–Lah polynomial.

A combinatorial interpretation of \( r \)-Dowling–Lah polynomials can be given through the notion of the \( r \)-Whitney–Lah coloured partitions in the following way: For \( c \geq 1 \), \( DL_{n,m,r}(c) \) is the number of \( r \)-Whitney–Lah coloured partitions of \( \{1, \ldots, n+r\} \) with \( m \) colours, where the smallest elements of those ordered blocks which contain no distinguished element are additionally coloured with \( c \) colours. Unfortunately, this interpretation can be used to obtain a purely combinatorial proof only for a minority of our results, we will present them when it possible.

If we choose \( c = 1 \), then the additional colouring with one colour is equivalent to omitting this colouring. In this case, we call the number \( DL_{n,m,r} = DL_{n,m,r}(1) \) an \( r \)-Dowling–Lah number. Though most of the statements and proofs in this paper are about \( r \)-Dowling–Lah polynomials, those can be easily adapted to \( r \)-Dowling–Lah numbers using the previous substitution.

If \( r = 1 \), then the previously defined polynomials can be simply called Dowling–Lah polynomials. Moreover, clearly, if \( m = 1 \), or \( m = 1 \) and \( r = 0 \), then \( DL_{n,1,r}(x) = L_{n,r}(x) \), \( DL_{n,1,0}(x) = L_{n}(x) \), respectively.

### 3. Properties and Identities

First, we derive a Spivey type formula for \( r \)-Dowling–Lah polynomials. To prove the theorem, we will need the following lemma, the Spivey type generalization of [6, Theorem 3.6] for \( r \)-Whitney–Lah numbers.

**Lemma 3.1.** If \( t,n,k,r,s \geq 0 \), \( k \leq t+n \) and \( m,l \geq 1 \), then
\[
\begin{align*}
&\binom{n}{k}^{l-m}WL_{m,r}(t+n,k) = \sum_{i=0}^{\min\{t,k\}} \sum_{j=\max\{0,k-i\}}^{n} WL_{m,r}(t,i) \binom{n}{j}^{l-i}m^{i+j} \\
&\quad \cdot WL_{l,s}(j,k-i)(ml(t+i)+2lr-2ms/ml)^{n-j}.
\end{align*}
\]
Proof. By [6, Theorem 3.1], we obtain
\[(mlx + 2lr|ml)^{t+n} = t^{t+n} (mx + 2r|m)^{t+n} = t^{t+n} \sum_{k=0}^{t+n} WL_{m,r} (t + n, k) m^k x^k.\]

Furthermore, applying again [6, Theorem 3.1] multiple times and using the binomial theorem for rising factorials, we get
\[(mlx + 2lr|ml)^{t+n} = \left(t^t (mx + 2r|m) + 2l + m|t|ml\right)^n\]
\[= t^t \sum_{i=0}^{t} \sum_{j=0}^{n} \sum_{k=0}^{j} WL_{m,r} (t, i) m^i x^i \binom{n}{j} (mlx - mli + 2ms|ml)^j\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j}\]
\[= t^t \sum_{i=0}^{t} \sum_{j=0}^{n} \sum_{k=0}^{j} WL_{m,r} (t, i) m^i x^i \binom{n}{j} m^j (l (x - i) + 2s|l)^j\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j}\]
\[= t^t \sum_{i=0}^{t} \sum_{j=0}^{n} \sum_{k=0}^{j} WL_{m,r} (t, i) \binom{n}{j} m^j \sum_{k=0}^{j} WL_{l,s} (j, k) l^k (x - i)^k\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j}\]
\[= t^t \sum_{i=0}^{t} \sum_{j=0}^{n} \sum_{k=0}^{j} WL_{m,r} (t, i) \binom{n}{j} m^j l^{i+j} WL_{l,s} (j, k)\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j} x^{i+k}\]
\[= t^t \sum_{i=0}^{t} \sum_{j=0}^{n} \sum_{k=0}^{j} WL_{m,r} (t, i) \binom{n}{j} m^j l^{i+j} WL_{l,s} (j, k - i)\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j} x^k\]
\[= t^t \sum_{k=0}^{t+n} \sum_{i=0}^{k} \sum_{j=0}^{n} WL_{m,r} (t, i) \binom{n}{j} l^{k-i} m^{i+j} WL_{l,s} (j, k - i)\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j} x^k.\]

Now the Spivey type formula for $r$-Dowling–Lah polynomials can be proved with the help of the lemma.

Theorem 3.2. If $t, n, r, s \geq 0$ and $m, l \geq 1$, then
\[l^n DL_{t+n,m,r} (mx) = \sum_{i=0}^{t} \sum_{j=0}^{n} WL_{m,r} (t, i) \binom{n}{j} m^j DL_{j,l,s} (lx)\]
\[\cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j} (mx)^i,\]
\[ DL_{t+n,m,r} (x) = \sum_{i=0}^{t} \sum_{j=0}^{n} WL_{m,r} (t, i) \binom{n}{j} DL_{j,m,s} (x) \cdot (m (t + i) + 2r - 2s|m)^{n-j} x^i. \]

**Proof.** Using Lemma 3.1, we have

\[
l^n DL_{t+n,m,r} (mx) = l^n \sum_{k=0}^{t+n} WL_{m,r} (t + n, k) m^k x^k
\]

\[
= \sum_{k=0}^{t+n \min\{t,k\}} \sum_{i=0}^{n} WL_{m,r} (t, i) \binom{n}{j} l^k (m^{i+j} WL_{l,s} (j, k - i)) \cdot (ml (t + i) + 2lr - 2ms|ml)^{n-j} x^k
\]

\[
= \sum_{i=0}^{t} \sum_{j=0}^{n} WL_{m,r} (t, i) \binom{n}{j} i^j (ml (t + i) + 2lr - 2ms|ml)^{n-j}
\]

\[
= \sum_{k=0}^{j} WL_{l,s} (j, k) l^k x^{k+i}
\]

\[
= \sum_{i=0}^{t} \sum_{j=0}^{n} WL_{m,r} (t, i) \binom{n}{j} m^j (ml (t + i) + 2lr - 2ms|ml)^{n-j}
\]

\[
= DL_{j,l,s} (lx) (mx)^i.
\]

The second formula follows by substituting \( l \) with \( m \). \( \square \)

**Remark.** The above theorem reduces to the definition of \( r \)-Dowling–Lah polynomials in case of \( n = 0 \).

For \( t = 0 \) in our formula, we obtain a connection between \( r \)-Dowling–Lah polynomials with \( m \) colours and \( s \)-Dowling–Lah polynomials with \( l \) colours.

**Corollary 3.3.** If \( n, r, s \geq 0 \) and \( m, l \geq 1 \), then

\[
l^n DL_{n,m,r} (mx) = \sum_{j=0}^{n} \binom{n}{j} m^j DL_{j,l,s} (lx) (2lr - 2ms|ml)^{n-j},
\]

\[
DL_{n,m,r} (x) = \sum_{j=0}^{n} \binom{n}{j} DL_{j,m,s} (x) (2r - 2s|m)^{n-j}.
\]

**Proof.** In the case of \( r \geq s \), we can also give a purely combinatorial proof for the second formula of this corollary.
The number of $r$-Whitney–Lah coloured partitions of $\{1, \ldots, n + r\}$ with $m$ colours, where the smallest elements of those ordered blocks which contain no distinguished element are additionally coloured with $c$ colours, is equal to $DL_{n,m,r}(c)$. They can be counted in an alternative way.

Denote by $j$ the number of those non-distinguished elements which are in the ordered blocks of the first $s$ distinguished elements or in an ordered block without a distinguished element ($j = 0, \ldots, n$). They can be chosen in \(\binom{n}{j}\) ways, then we can partition them and the first $s$ distinguished elements according to the $s$-Whitney–Lah rules with $m$ colours and $c$ additional colours in $DL_{j,m,s}(c)$ ways. As the final step, considering the remaining $n - j$ non-distinguished elements in increasing order, they can be put into the ordered blocks of the last $r - s$ distinguished elements and coloured in $(2r - 2s|m)^{n-j}$ ways. Therefore, the number of partitions under enumeration is \(\binom{n}{j} DL_{j,m,s}(c) (2r - 2s|m)^{n-j}\) for a fixed $j$. \hfill \Box

\textbf{Remark.} As special cases of Theorem 3.2 and Corollary 3.3, for $s = r; s = 1; l = 1; l = 1$ and $s = r; l = 1$ and $s = 0$, the $r$-Dowling–Lah polynomials with $m$ colours can be expressed by $r$-Dowling–Lah polynomials with $l$ colours, Dowling–Lah polynomials with $l$ colours, $s$-Lah polynomials, $r$-Lah polynomials, Lah polynomials, respectively. Due to the high number of these formulas, we omit them in this paper.

In case of $t = 1, l = m$ and $s = r$, we obtain the below recurrence relation for $r$-Dowling–Lah polynomials.

\textbf{Corollary 3.4.} If $n, r \geq 0$ and $m \geq 1$, then
\[
DL_{n+1,m,r}(x) = 2r \sum_{j=0}^{n} \binom{n}{j} DL_{j,m,r}(x) m^{n-j} (n-j)! + x \sum_{j=0}^{n} \binom{n}{j} DL_{j,m,r}(x) m^{n-j} (n-j+1)!. 
\]

Finally, from Theorem 3.2 and Corollary 3.3 follows a Carlitz type formula.

\textbf{Corollary 3.5.} If $t, n, r \geq 0, m \geq 1$ and $m$ is even, then
\[
DL_{t+n,m,r}(x) = \sum_{i=0}^{t} WL_{m,r}(t, i) DL_{n,m,r+\frac{m+2r}{2}}(x) x^i. 
\]

Now we give a second-order recurrence relation for $r$-Dowling–Lah polynomials.

\textbf{Theorem 3.6.} If $r \geq 0$ and $n, m \geq 1$, then
\[
DL_{n+1,m,r}(x) = (x + 2mn + 2r) DL_{n,m,r}(x) - mn (mn + 2r - m) DL_{n-1,m,r}(x). 
\]
Proof. By applying Corollary 3.4 twice, and some fundamental properties of binomial coefficients and factorials, we obtain

\[ DL_{n+1,m,r}(x) = 2r \sum_{j=0}^{n} \binom{n}{j} DL_{n-j,m,r}(x) m^j j! + x \sum_{j=0}^{n} \binom{n}{j} DL_{n-j,m,r}(x) m^j (j+1)! \]

\[ = (x + 2r) DL_{n,m,r}(x) + 2nr \sum_{j=1}^{n} \binom{n-1}{j-1} DL_{n-j,m,r}(x) m^j (j-1)! \]

\[ + nx \sum_{j=1}^{n} \binom{n-1}{j-1} DL_{n-j,m,r}(x) m^j j! + x \sum_{j=1}^{n} \binom{n}{j} DL_{n-j,m,r}(x) m^j j! \]

\[ = (x + 2r) DL_{n,m,r}(x) + 2nr \sum_{j=1}^{n-1} \binom{n-1}{j} DL_{n-1-j,m,r}(x) m^j j! \]

\[ + mnx \sum_{j=0}^{n-1} \binom{n-1}{j} DL_{n-1-j,m,r}(x) m^j (j+1)! \]

\[ + x \sum_{j=1}^{n} n^2 DL_{n-j,m,r}(x) m^j \]

\[ = (x + mn + 2r) DL_{n,m,r}(x) + x \sum_{j=1}^{n} n^2 DL_{n-j,m,r}(x) m^j. \]

Finally, using this equation, we have

\[ DL_{n+1,m,r}(x) - mn DL_{n,m,r}(x) \]

\[ = (x + mn + 2r) DL_{n,m,r}(x) + x \sum_{j=1}^{n} n^2 DL_{n-j,m,r}(x) m^j \]

\[ - mn (x + m (n-1) + 2r) DL_{n-1,m,r}(x) - x \sum_{j=2}^{n} n^2 DL_{n-j,m,r}(x) m^j \]

\[ = (x + mn + 2r) DL_{n,m,r}(x) + xmn DL_{n-1,m,r}(x) \]

\[ - mn (x + m (n-1) + 2r) DL_{n-1,m,r}(x). \]

\[ \square \]

The next theorem will be useful in the remaining part of the paper.

**Theorem 3.7.** If \( n, r \geq 0 \) and \( m, l \geq 1 \), then

\[ DL_{n,ml,lr}(lx) = l^n DL_{n,m,r}(x). \]

**Proof.** The statement follows immediately from [6, Theorem 3.8]:

\[ DL_{n,ml,lr}(lx) \]

\[ = \sum_{k=0}^{n} WL_{ml,lr}(n,k) l^k x^k = \sum_{k=0}^{n} l^n WL_{m,r}(n,k) x^k = l^n DL_{n,m,r}(x). \]

\[ \square \]
In the following, we give an addition formula, or in other words, a binomial convolutional identity for \( r \)-Dowling–Lah polynomials with \( m \) colours and \( s \)-Dowling–Lah polynomials with \( l \) colours. Primarily, we give a proof of this equation using algebraic manipulations. However, as we will see, it can be shown with the help of the combinatorial interpretation of \( r \)-Dowling–Lah polynomials, analogously to the proof of [5, Theorem 3.4].

**Theorem 3.8.** If \( n, r, s \geq 0 \) and \( m, l \geq 1 \), then

\[
DL_{n,ml,lr+ms} (ml (x + y)) = \sum_{j=0}^{n} \binom{n}{j} j^l m^{n-j} DL_{j,m,r} (mx) DL_{n-j,l,s} (ly),
\]

\[
DL_{n,m,r+s} (x + y) = \sum_{j=0}^{n} \binom{n}{j} DL_{j,m,r} (x) DL_{n-j,m,s} (y).
\]

**Proof 1.** Applying the binomial theorem and [6, Theorem 3.9], we get

\[
DL_{n,ml,lr+ms} (ml (x + y)) = \sum_{i=0}^{n} WL_{ml,lr+ms} (n, i) m^{i} l^{i} (x + y)^{i}
\]

\[
= \sum_{i=0}^{n} WL_{ml,lr+ms} (n, i) m^{i} l^{i} \sum_{k=0}^{i} \binom{i}{k} x^{k} y^{i-k}
\]

\[
= \sum_{k=0}^{n} \sum_{i=k}^{n} \binom{i}{k} WL_{ml,lr+ms} (n, i) m^{i} l^{i} x^{k} y^{i-k}
\]

\[
= \sum_{k=0}^{n} \sum_{h=0}^{n-k} \binom{k+h}{k} WL_{ml,lr+ms} (n, k+h) m^{k+h} l^{k+h} x^{k} y^{h}
\]

\[
= \sum_{k=0}^{n} \sum_{h=0}^{n-k} \sum_{j=k}^{n} \binom{n}{j} j^{l+m} j^{m-j+k} WL_{m,r} (j, k) WL_{l,s} (n-j, h) x^{k} y^{h}
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} j^{l+m-j} \sum_{k=0}^{j} WL_{m,r} (j, k) m^{k} x^{k} \sum_{h=0}^{n-j} WL_{l,s} (n-j, h) l^{h} y^{h}
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} j^{l+m-j} DL_{j,m,r} (mx) DL_{n-j,l,s} (ly).
\]

The second formula follows from Theorem 3.7 and the substitution of \( l = m \). \( \square \)

**Proof 2.** Let \( c, d \geq 1 \) and consider the \((lr+ms)\)-Whitney–Lah coloured partitions of \( \{1, \ldots, n + lr + ms\} \) with \( ml \) colours, where the smallest elements of those ordered blocks which contain no distinguished element are additionally coloured with \( ml \) \((c + d)\) colours. The number of such coloured partitions is equal to \( DL_{n,ml,lr+ms} (ml (c + d)) \) according to the interpretation. They can be counted in a different way.
Let $j$ be the number of those non-distinguished elements which belong to the ordered blocks of the first $lr$ distinguished elements, or an ordered block with no distinguished element such that the smallest element of this ordered block is coloured by one of the first $mlc$ additional colours ($j = 0, \ldots, n$). They can be chosen in $\binom{n}{j}$ ways. These $j$ elements together with the first $lr$ distinguished elements can be partitioned in $DL_{j,ml,lr}(mlc)$ ways, while the other $n-j$ non-distinguished elements together with the last $ms$ distinguished elements in $DL_{n-j,ml,ms}(mld)$ ways, respectively, in $lr$-, $ms$-Whitney–Lah sense with $ml$ colours and $mlc$, $mld$ additional colours. Consequently, by Theorem 3.7 the number of possibilities is

$$\binom{n}{j} DL_{j,ml,lr}(mlc) DL_{n-j,ml,ms}(mld)$$

for a fixed $j$.

Now we give a Dobinski type formula for $r$-Dowling–Lah polynomials, and show two different proofs. First, we provide a direct proof for the polynomials, then we prove the statement for $r$-Dowling–Lah numbers using probabilistic tools.

**Theorem 3.9.** If $n, r \geq 0$ and $m \geq 1$, then

$$DL_{n,m,r}(x) = \exp \left( -\frac{x^m}{m} \right) \sum_{j=0}^{\infty} \frac{(mj + 2r|m)^{\pi}}{mj!} x^j.$$ 

**Proof 1.** Let $WL_{m,r}(n,k) = 0$ for $k > n$. Applying [6, Theorem 3.1], we have

$$\sum_{k=0}^{n} WL_{m,r}(n,k) m^k j^k = \sum_{k=0}^{\infty} WL_{m,r}(n,k) m^k j^k = \sum_{k=0}^{j} WL_{m,r}(n,k) m^k j^k! = \sum_{k=0}^{j} WL_{m,r}(n,k) m^k j^k! \frac{m^{k-j}}{(j-k)!}.$$ 

It is clear from above that $\left( \frac{(mj + 2r|m)^{\pi}}{mj!} \right)_{j=0}^{\infty}$ is the convolution of $(WL_{m,r}(n,j))_{j=0}^{\infty}$ and $\left( \frac{m^{-j}}{j!} \right)_{j=0}^{\infty}$, whence

$$\sum_{j=0}^{\infty} \frac{(mj + 2r|m)^{\pi}}{mj!} x^j = DL_{n,m,r}(x) \exp \left( \frac{x}{m} \right).$$

**Proof 2.** Let $\lambda > 0$ and $\xi$ be a Poisson distributed random variable with parameter $\lambda$. Apply now [6, Theorem 3.1] again, to obtain

$$E(m\xi + 2r|m)^{\pi}$$
\[= \sum_{j=0}^{\infty} (mj + 2r|m)\pi P(\xi = j)\]

\[= \sum_{j=0}^{\infty} (mj + 2r|m)\pi \frac{\lambda^j}{j!} e^{-\lambda} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \sum_{k=0}^{n} WL_{m,r}(n,k) m^k j^k\]

\[= e^{-\lambda} \sum_{k=0}^{n} WL_{m,r}(n,k) m^k \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} j^k\]

\[= e^{-\lambda} \sum_{k=0}^{n} WL_{m,r}(n,k) m^k \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} (j-k)!\]

\[= e^{-\lambda} \sum_{k=0}^{n} WL_{m,r}(n,k) m^k \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} j^k\]

\[= e^{-\lambda} \sum_{k=0}^{n} WL_{m,r}(n,k) m^k \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} j^k\]

\[= \sum_{k=0}^{n} WL_{m,r}(n,k) m^k \lambda^k = DL_{n,m,r}(m\lambda).\]

By choosing \(\lambda = \frac{1}{m}\), we get

\[DL_{n,m,r} = E(m\xi + 2r|m)\pi = \sum_{j=0}^{\infty} (mj + 2r|m)\pi \frac{1}{m^j j!} e^{-\frac{1}{m}}.\]

\[\square\]

Now we present the exponential generating function of the sequence of \(r\)-Dowling–Lah polynomials with two different proofs.

**Theorem 3.10.** If \(r \geq 0\) and \(m \geq 1\), then

\[\sum_{n=0}^{\infty} \frac{DL_{n,m,r}(x)}{n!} y^n = \exp \left( \frac{xy}{1-my} \right) (1-my)^{-\frac{2r}{m}}.\]

**Proof 1.** Using the exponential generating function of \((WL_{m,r}(n,k))_{n=k}^{\infty}\) (see [1]), we obtain

\[\sum_{n=0}^{\infty} \frac{DL_{n,m,r}(x)}{n!} y^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{WL_{m,r}(n,k) x^k y^n}{n!} = \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} \frac{WL_{m,r}(n,k) y^n}{n!}\]

\[= \sum_{k=0}^{\infty} x^k \frac{(1-my)^{-\frac{2r}{m}}}{k!} \left( \frac{y}{1-my} \right)^k = (1-my)^{-\frac{2r}{m}} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{xy}{1-my} \right)^k\]

\[= (1-my)^{-\frac{2r}{m}} \exp \left( \frac{xy}{1-my} \right).\]

\[\square\]

**Proof 2.** For \(l = 1\) and \(s = 0\), it follows from Corollary 3.3 that \((DL_{n,m,r}(x))_{n=0}^{\infty}\) is the binomial convolution of \((m^n L_n(\frac{x}{m}))_{n=0}^{\infty}\) and \(((2r|m)\pi)_{n=0}^{\infty}$.
The exponential generating function of the sequence of Lah polynomials is known (see [13]), from which the exponential generating function of \((m^n L_n(x/m))_{n=0}^{\infty}\) is \(\exp(xy)\). Furthermore, we can determine the exponential generating function of \((2^r m)\sum_{n=0}^{\infty} \cdots\) with the help of the binomial series, as follows:

\[
\sum_{n=0}^{\infty} \frac{(2^r m)_n}{n!} y^n = \sum_{n=0}^{\infty} m^n \frac{(2^r m + n - 1)_n}{n!} y^n = \sum_{n=0}^{\infty} (-1)^n \binom{2^r m + n - 1}{n} (-my)^n \]

\[
= \sum_{n=0}^{\infty} \left( -\frac{2^r m}{n} \right) (-my)^n = (1 - my)^{-\frac{2^r m}{m}}.
\]

\[\square\]

Now, we show that the roots of r-Dowling–Lah polynomials are real.

**Theorem 3.11.** Let \(r \geq 0\) and \(n, m \geq 1\). Then \(DL_{\nu, n, m, r}(x)\) has simple real roots. If \(r \geq 1\), then all the roots are negative, and if \(r = 0\), then one of the roots is 0 and the others are negative.

**Proof.** We prove the statement for \(r \geq 1\) using induction on \(n\). The case of \(r = 0\) can be proved with similar thoughts.

If \(n = 1\), then \(DL_{1, n, m, r}(x) = x + 2^r m\). Assume that the assertion holds for some \(n\). In case of \(n + 1\), apply special values and the recurrence of r-Whitney–Lah numbers (see [1] and [6, Theorem 3.2]) to derive

\[
DL_{n+1, m, r}(x) = \sum_{k=0}^{n+1} WL_{m, r}(n+1, k) x^k
\]

\[
= \left(2^r m\right)^{n+1} + \sum_{k=1}^{n} (WL_{m, r}(n, k-1) + (m(n+k) + 2^r) WL_{m, r}(n, k)) x^k + x^{n+1}
\]

\[
= x \sum_{k=0}^{n} WL_{m, r}(n, k) x^k + (mn + 2^r) \sum_{k=0}^{n} WL_{m, r}(n, k) x^k + m x \sum_{k=1}^{n} k WL_{m, r}(n, k) x^{k-1}
\]

\[
= x DL_{n, m, r}(x) + (mn + 2^r) DL_{n, m, r}(x) + m x DL'_{n, m, r}(x).
\]

If we multiply this equation by \(e^x x^{mn+2^r-1} DL_{n, m, r}^{m-1}(x)\), then we obtain

\[
e^x x^{mn+2^r-1} DL_{n, m, r}^{m-1}(x) DL_{n+1, m, r}(x) = \left(e^x x^{mn+2^r} DL_{n, m, r}^{m}(x)\right)'.
\]

According to the induction hypothesis, \(DL_{n, m, r}(x)\) has \(n\) simple negative real roots. Then, \(x^{mn+2^r} DL_{n, m, r}^{m}(x)\) has \(n+1\) distinct real roots, where
0 has a multiplicity of \( mn + 2r \), and the remaining roots are negative with multiplicity \( m \).

Since \( e^x > 0 \ (x \in \mathbb{R}) \), \( e^x x^{mn+2r} DL_{n,m,r}^m (x) \) has \( n+1 \) zeros, one of which is 0, the rest of them are negative, and \( \lim_{x \to -\infty} e^x x^{mn+2r} DL_{n,m,r}^m (x) = 0 \).

As per Rolle’s Theorem, its derivative, \( e^x x^{mn+2r-1} DL_{n,m,r}^{m-1} (x) DL_{n+1,m,r} (x) \) has \( n + 1 \) distinct negative zeros, which are different from the roots of \( DL_{n,m,r} (x) \). Therefore, \( DL_{n+1,m,r} (x) \) has \( n + 1 \) distinct negative roots. □

As an immediate consequence of the above theorem, we get the following corollary regarding the log-concavity of \( r \)-Whitney–Lah numbers, previously proved in [6, Theorem 3.12].

**Corollary 3.12.** Let \( r \geq 0 \) and \( n, m \geq 1 \). Then the sequence \( (WL_{m,r} (n, k))^n_{k=0} \) is strictly log-concave, therefore, it is unimodal.

The next theorem describes connections between \( r \)-Dowling–Lah and \( r \)-Dowling polynomials.

**Theorem 3.13.** Let \( n, r, s \geq 0 \) and \( m, l \geq 1 \). Then

\[
DL_{n,ml,lr + ms}^2 (mlx) = \sum_{j=0}^{n} l^{n-j} m^{j} w_{m,r} (n,j) D_{j,l,s}^2 (lx)
\]

if \( lr \) and \( ms \) have the same parity,

\[
DL_{n,m,l + ms}^2 (x) = \sum_{j=0}^{n} w_{m,r} (n,j) D_{j,m,s}^2 (x)
\]

if \( r \) and \( s \) have the same parity;

\[
D_{n,ml,2ms-lr} (mlx) = \sum_{j=0}^{n} (-1)^{n-j} l^{n-j} m^{j} W_{m,r} (n,j) DL_{j,l,s} (lx) \quad \text{if} \ 2ms \geq lr,
\]

\[
D_{n,m,2s-r} (x) = \sum_{j=0}^{n} (-1)^{n-j} W_{m,r} (n,j) DL_{j,m,s} (x) \quad \text{if} \ 2s \geq r.
\]

**Proof.** Using the sixth formula of [6, Theorem 3.10], we get

\[
DL_{n,ml,lr + ms}^2 (mlx)
\]

\[
= \sum_{k=0}^{n} WL_{ml,lr + ms} (n,k) m^{k} l^{k} x^{k}
\]

\[
= \sum_{k=0}^{n} \sum_{j=k}^{n} l^{n-j} m^{j-k} w_{m,r} (n,j) W_{l,s} (j,k) m^{k} l^{k} x^{k}
\]

\[
= \sum_{j=0}^{n} l^{n-j} m^{j} w_{m,r} (n,j) \sum_{k=0}^{j} W_{l,s} (j,k) l^{k} x^{k}
\]

\[
= \sum_{j=0}^{n} l^{n-j} m^{j} w_{m,r} (n,j) D_{j,l,s} (lx).
\]
While, applying the fifth formula of \[6, \text{Theorem 3.10}\], we have

\[
D_{n,m l,2m s - lr} (mlx)
\]

\[
= \sum_{k=0}^{n} W_{ml,2ms - lr} (n, k) m^k l^k x^k
\]

\[
= \sum_{k=0}^{n} \sum_{j=k}^{n} (-1)^{n-j} l^{n-j} m^j x^j \sum_{k=0}^{j} W_{l,s} (j, k) l^k x^k
\]

\[
= \sum_{j=0}^{n} (-1)^{n-j} l^{n-j} m^j W_{m,r} (n, j) \sum_{k=0}^{j} W_{l,s} (j, k) l^k x^k
\]

\[
= \sum_{j=0}^{n} (-1)^{n-j} l^{n-j} m^j W_{m,r} (n, j) DL_{j,l,s} (lx).
\]

The second and fourth identity follows simply for \(l = m\) using Theorem 3.7. \(\square\)

Remark. We note that the substitutions \(l = m\) and \(s = r\) show that the sequence of \(r\)-Dowling–Lah polynomials is the \(r\)-Whitney transform of the first kind of the sequence of \(r\)-Dowling polynomials. This special case could be proved in a purely combinatorial manner.

Finally, we verify that the sequence of \(r\)-Dowling–Lah numbers is log-convex.

**Theorem 3.14.** If \(n, r \geq 0\) and \(m \geq 1\), then

\[
DL^2_{n+1,m} \leq DL_{n+2,m} - m DL_{n,m} (DL_{n+1,m} + DL_{n,m})
\]

consequently \((DL_{n,m})\) is log-convex.

**Proof.** By special values of \(r\)-Whitney–Lah numbers and their recurrence (see [1] and [6, Theorem 3.2]), it follows that

\[
DL_{n+1,m}
\]

\[
= \sum_{k=0}^{n+1} WL_{m,r} (n+1, k)
\]

\[
= (2r|m)^{n+1} + \sum_{k=1}^{n} (WL_{m,r} (n, k-1) + (m (n + k) + 2r) WL_{m,r} (n, k)) + 1
\]

\[
= \sum_{k=0}^{n} WL_{m,r} (n, k) + \sum_{k=0}^{n} (m (n + k) + 2r) WL_{m,r} (n, k)
\]

\[
= DL_{n,m} + \sum_{k=0}^{n} (m (n + k) + 2r) WL_{m,r} (n, k),
\]
and

\[
DL_{n+2,m,r} - DL_{n+1,m,r} \\
= \sum_{k=0}^{n+1} \left( (m(n+1+k) + 2r) WL_{m,r}(n+1,k) \right) \\
= (2m^2)^{n+2} + \sum_{k=1}^{n} \left( (m(n+1+k) + 2r) WL_{m,r}(n,k-1) \right) \\
+ \sum_{k=1}^{n} \left( (m(n+k) + 2r|m)^2 WL_{m,r}(n,k) \right) \\
+ \sum_{k=0}^{n} \left( (m(n+k+2) + 2r) WL_{m,r}(n,k) \right) \\
= \sum_{k=0}^{n} \left( (m(n+k) + 2r|m)^2 WL_{m,r}(n,k) \right) \\
+ \sum_{k=0}^{n} \left( (m(n+k) + 2r) WL_{m,r}(n,k) + 2m \sum_{k=0}^{n} WL_{m,r}(n,k) \right) \\
= \sum_{k=0}^{n} \left( (m(n+k) + 2r|m)^2 WL_{m,r}(n,k) + DL_{n+1,m,r} \right) \\
+ (2m-1) DL_{n,m,r}.
\]

As a consequence of the inequality of weighted arithmetic and quadratic means, it can be shown that

\[
\left( \frac{\sum_{k=0}^{n} (m(n+k) + 2r) WL_{m,r}(n,k)}{DL_{n,m,r}} \right)^\frac{2}{m} \\
\leq \frac{\sum_{k=0}^{n} (m(n+k) + 2r|m)^2 WL_{m,r}(n,k)}{DL_{n,m,r}}.
\]

Now we can use the above relations to express the numerators of these fractions, which implies

\[
(DL_{n+1,m,r} - DL_{n,m,r}) (DL_{n+1,m,r} + (m-1) DL_{n,m,r}) \\
\leq (DL_{n+2,m,r} - 2DL_{n+1,m,r} - (2m-1) DL_{n,m,r}) DL_{n,m,r},
\]

\[
DL_{n+1,m,r}^2 \leq DL_{n+2,m,r} DL_{n,m,r} - m DL_{n,m,r} (DL_{n+1,m,r} + DL_{n,m,r}).
\]

\[\Box\]

Remark. The log-convexity of the sequence of \(r\)-Dowling–Lah numbers can be shown in numerous different ways. On the one hand, due to the log-convexity of \((1)_{n=0}^{\infty}\), it is an immediate consequence of [6, Theorem 3.14], which states that the \(r\)-Whitney–Lah transformation preserves log-convexity. On the other
hand, since the same theorem holds for \( r \)-Whitney transformation of the
first kind, it also comes from the log-convexity of the sequence of \( r \)-Dowling
numbers [5, Theorem 3.7] and the remark after Theorem 3.13. Further, it can
be proved analogously to the first proof of [5, Theorem 3.7], using a theorem
of Davenport and Pólya.

Finally, we point out that a theorem of Liu and Wang [9] directly implies
that if \( r \geq 0 \) and \( m \geq 1 \), then \( (DL_{n,m,r}(q))_{n=0}^{\infty} \) is \( q \)-log-convex.

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