$S_4$ modular symmetry, seesaw mechanism and lepton masses and mixing

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Abstract. We report on two works which exploit the approach to charged lepton and neutrino masses, neutrino mixing and leptonic CP violation based on the modular $S_4$ symmetry. The light neutrino masses are assumed to be generated via the type I seesaw mechanism. Realistic models without flavons and compatible with the data are constructed and the predicted correlations between neutrino mixing observables are studied.

1. Introduction

The peculiar patterns of fermion masses and mixing raise the question of whether there is some organising principle behind them. The large mixing in the lepton sector (see e.g. [1]), in particular, suggests that some non-Abelian discrete symmetry is at work [2–5]. A broken modular symmetry may play a role analogous to that of traditional flavour symmetries, via the action of a modular group quotient [6]. Very constrained functions of the spurion $\tau$ parameterising the breaking – the modular forms – are needed, instead of flavons.

This approach has so far been explored for the discrete quotient groups $\Gamma_2 \simeq S_3$ [7, 8], $\Gamma_3 \simeq A_4$ [6–12], $\Gamma_4 \simeq S_4$ [13, 14], and $\Gamma_5 \simeq A_5$ [15], in a bottom-up approach. Recent top-down studies include Refs. [16–19]. Here, we report on the two studies focusing on the modular $S_4$ symmetry in the lepton sector. While in [13] neutrino masses arise from an agnostic Weinberg operator, we will restrict here our attention to the type I seesaw completions systematically explored in [14].

2. Framework

The modular group $\Gamma$ is the projective special linear group $PSL(2, \mathbb{Z})$ of $2 \times 2$ matrices with integer coefficients and unit determinant, the matrices $M$ and $-M$ being identified. The group $\Gamma$ acts faithfully on the complex upper-half plane with fractional linear transformations:

$$\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \to \frac{a \tau + b}{c \tau + d}, \quad \text{where} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \quad \text{and} \quad \text{Im} \tau > 0. \quad (1)$$

The group $\Gamma$ is generated by two elements

$$S = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \tau \to -\frac{1}{\tau}, \quad T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \to \tau + 1, \quad (2)$$
which satisfy $S^2 = (ST)^3 = 1$.

The modular group $\Gamma$ contains a series of so-called principle congruence subgroups

$$\Gamma(N) \equiv \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \mid \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where $N$ is natural. Taking the quotient $\Gamma_N \equiv \Gamma / \Gamma(N)$, one obtains a finite modular group, with the presentation $S^2 = (ST)^3 = T^N = 1$. For $N \leq 5$ the finite modular groups are then isomorphic to permutation groups: $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$.

The modular group $\tilde{\Gamma}$ arises in certain string theory compactifications [20, 21]. The corresponding low-energy effective field theory has to be invariant under the modular transformations, which act on the chiral superfields $\chi_I$ as

$$\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma} : \chi_I \rightarrow (c \tau + d)^{-k_I} \rho_I(\gamma) \chi_I,$$

where $\rho_I$ is a unitary representation of the quotient modular group $\Gamma_N$, and $-k_I$ is the so-called modular weight. The superpotential $W$ of such theory can be written as a sum of the chiral field coupling terms $(Y(\tau) \chi_I_1 \ldots \chi_I_n)_{1}$, where $Y(\tau)$ is a multiplet of holomorphic functions of $\tau$, and $1$ stands for a singlet combination of the field components and $Y(\tau)$. One can show that modular invariance of $W$ requires $Y(\tau)$ to transform in the same way as the fields, i.e.

$$Y(\tau) \rightarrow (c \tau + d)^{k_Y} \rho_Y(\gamma) Y(\tau),$$

with $k_Y = k_1 + \ldots + k_n$. Such functions are called modular forms of weight $k_Y$ and level $N$. Non-trivial modular forms exist only for positive even weights $k_Y$. For each level $N$ and weight $k_Y$ they form a finite-dimensional linear space, which decomposes into irreps of $\Gamma_N$. This specific dependence of the superpotential couplings on $\tau$ allows to constrain significantly the form of the Yukawa couplings in a SUSY setting, leading to predictive models of lepton masses and mixing.

In this contribution, we consider the case of $N = 4, \Gamma_N = \Gamma_4 \simeq S_4$. Modular forms of level 4 and lowest non-trivial weight 2 have been explicitly constructed in Ref. [13] in terms of the Dedekind eta function. They are organised into a doublet $Y^{(2)}_2(\tau)$ and a triplet $Y^{(2)}_3(\tau)$ of $S_4$.

In the group representation basis of Ref. [13] (see Appendix A therein) they can be cast as:

$$Y^{(2)}_2(\tau) = \frac{3\pi i}{8} \begin{pmatrix} b_1 - b_3 + 3b_5 \\ b_1 + b_3 + 3b_5 \end{pmatrix}, \quad Y^{(2)}_3(\tau) = -\frac{\pi i}{4} \begin{pmatrix} b_1 - 2b_2 + 2b_4 - b_5 \\ b_1 + (1 + \sqrt{3})b_2 - (1 - \sqrt{3})b_4 - b_5 \\ b_1 + (1 - \sqrt{3})b_2 - (1 + \sqrt{3})b_4 - b_5 \end{pmatrix},$$

where the $b_{1...5} \equiv b_{1...5}(\tau)$ are basis vectors of the aforementioned finite-dimensional linear space of functions. One may obtain $q$-expansions for the $b_i(\tau)$ from the SageMath algebra system [22] in a way analogous to that described in Appendix A.1 of Ref. [15]:

$$b_1(\tau) = 1 + 24 e^{4\pi i} + 24 e^{8\pi i} + 96 e^{12\pi i} + \ldots,$n\)3.

Higher-weight modular forms can be obtained from the tensor products of the lowest-weight modular forms.

As in Ref. [3], the $3'$ representation of $S_4$ is taken to be the 3D irrep for which the character of the conjugacy class of period 4 (which includes $T$) equals +1.
3. Models

Using the results of Ref. [13] on modular forms of level $N = 4$, condensed in eqs. (6) and (7), in Ref. [14] seesaw type I models based on broken modular $S_4$ symmetry in the lepton sector have been systematically explored. Models which are free of flavons have been considered and one has adhered to the principle of minimality by having Higgs multiplets transform trivially and by considering the lowest possible modular weights (and thus the lowest number of free parameters) which lead to a phenomenologically viable scenario. The three generations of lepton $SU(2)$ doublets $L_i$ and singlets $E_i^c$ ($i = 1, 2, 3$) are arranged into $S_4$ triplets ($\rho_L \sim 3$ or $3'$) and singlets ($\rho_i \sim 1$ or $1'$), respectively. An $S_4$ triplet of right-handed neutrino chiral superfields $N_i^c$ ($\rho_N \sim 3$ or $3'$) is also added to the model field content. The relevant superpotential reads

$$W = \alpha_i \left( Y_{\alpha_i} \tau \right) E_i L H_d + g \left( Y_g \tau \right) N^c L H_u + \Lambda \left( Y_\Lambda \tau \right) N^c N^c,$$

where a sum over possible independent singlet contributions is implied in the first line, and the modular multiplets have (non-negative and even) weights $k_{\alpha_i} = k_i + k_L$, $k_g = k_N + k_L$ and $k_\Lambda = 2k_N$, in a straightforward notation. After integrating out the $N^c$ and after electroweak symmetry breaking, the charged lepton mass matrix and the light neutrino Majorana mass matrix are given by $M_c = v_d \lambda^j$ and $M_\nu = -v_u^2 \gamma^jM^{-1}\gamma^j$, with $v_u \equiv \langle H_u^0 \rangle$ and $v_d \equiv \langle H_d^0 \rangle$.

While corrections from SUSY breaking and renormalisation group running are expected, it has been shown that regimes exist where they are negligible [9]. In the present approach, one is then able to determine 12 observables, namely 6 lepton masses, 3 mixing angles and 3 CP violation (CPV) phases. In order to have a relatively small number ($\leq 8$) of free parameters and retain predictive power, both $k_\Lambda$ and $k_g$ are taken to be $\leq 2$.

The form of $\lambda$ depends on the $k_{\alpha_i}$. The most economic choice allowing for a fit to the charged lepton sector is, without loss of generality, $k_{\alpha_1} = 2$ and $k_{\alpha_2} = k_{\alpha_3} = 4$, leading to

$$\lambda = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} Y_3 & Y_5 & Y_4 \\ Y_1 Y_4 - Y_2 Y_5 & Y_1 Y_3 - Y_2 Y_4 & Y_1 Y_5 - Y_2 Y_3 \\ Y_1 Y_4 + Y_2 Y_5 & Y_1 Y_3 + Y_2 Y_4 & Y_1 Y_5 + Y_2 Y_3 \end{pmatrix},$$

The value of $k_g$ determines the shape of $\mathcal{Y}$. One has

$$\mathcal{Y} = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ for } k_g = 0;$$

$$\mathcal{Y} = g \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & 0 & 0 \\ Y_2 & Y_1 & 0 \end{pmatrix} + g' \begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ -Y_4 & Y_3 & 0 \end{pmatrix}, \text{ for } k_g = 2 \text{ and } \rho_N = \rho_L;$$

$$\mathcal{Y} = g \begin{pmatrix} 0 & -Y_1 & Y_2 \\ Y_1 & 0 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + g' \begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix}, \text{ for } k_g = 2 \text{ and } \rho_N \neq \rho_L.$$

Finally, the form of $M$ depends on $k_\Lambda$. One has

$$M = 2\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ for } k_\Lambda = 0; \quad M = 2\Lambda \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & 0 & Y_2 \\ Y_2 & Y_1 & 0 \end{pmatrix}, \text{ for } k_\Lambda = 2.$$
Figure 1. Fundamental domain $\mathcal{D}$ of the modular group $\Gamma$: $\Im \tau > 0$, $|\Re \tau| \leq 1/2$ and $|\tau| \geq 1$. Red dots signal the values of $\tau$ corresponding to the five pairs of local minima of the chi-squared function, which arise for the weight choices $(k_{a1}, k_{a2}, k_{a3}) = (2, 4, 4)$ and $(k_g, k_\Lambda) = (2, 0)$.

In the above formulae we have used the shorthands $Y_{2}^{(2)} \equiv (Y_1, Y_2)^T$ and $Y_{3'}^{(2)} \equiv (Y_3, Y_4, Y_5)^T$. It should be noted that these seesaw models are subcases of the Weinberg operator models of [13] only when $k_\Lambda = 0$, as analyticity in $\tau$ is then kept after integrating out the heavy neutrinos, while the Weinberg models are always analytic by construction (see also Appendix E of [14]).

4. Results

By fitting the models to the accurately known observable quantities (charged lepton masses, neutrino mass-squared differences and neutrino mixing angles), it is possible to obtain predictions for the absolute neutrino mass scale, Dirac and Majorana CPV phases, and thus for the neutrinoless double beta ($\beta\beta^0\nu$-) decay effective Majorana mass.

It is found through numerical search that models with $k_g = 0$ and models with $k_g = k_\Lambda = 2$ cannot reproduce lepton data. For the remaining, viable choice of weights – namely $k_g = 2$ and $k_\Lambda = 0$ – the set of 8 real and independent free parameters may be taken to be:

$$\{ \Re \tau, \Im \tau, v_u^2 g^2 / \Lambda, \Re g'/g, \Im g'/g, v_d \alpha, \beta / \alpha, \gamma / \alpha \} \ . \ (12)$$

The modulus $\tau$ is restricted to the fundamental domain $\mathcal{D}$ of $\Gamma$ depicted in Fig. 1, since values of $\tau$ related by modular transformations are physically equivalent. Apart from that, one can show that conjugation of complex parameters defined as $\tau \rightarrow -\tau^*$, $g'/g \rightarrow (g'/g)^*$ leaves all observables unchanged, except for the CPV phases, which flip their signs. In particular, phenomenologically viable models come in pairs with the opposite CPV phases.

Scanning over the parameter space (12), one finds five pairs of distinct local minima of the chi-squared function consistent with the data at $3\sigma$ confidence level. These viable pairs of models are denoted as A and A*, B and B*, etc. The corresponding values of $\tau$ are shown in Fig. 1. For cases A(*) and B(*) one has $\rho_N \neq \rho_L$, while for the remaining cases $\rho_N = \rho_L$. 

Figure 2. Correlations between $\sin^2 \theta_{23}$ and the Dirac CPV phase, the sum of neutrino masses, and the effective Majorana mass in ($\beta\beta$)$_{0\nu}$-decay, for cases A and A* at 2$\sigma$ (green), 3$\sigma$ (yellow) and 5$\sigma$ (red) CL. The plus (minus) sign of $\delta$ refers to the case without (with) an asterisk.

The absolute neutrino mass scale $m_{\text{min}}$, the sum of neutrino masses $\Sigma_i m_i$ and the ($\beta\beta$)$_{0\nu}$-decay effective Majorana mass $\langle |m| \rangle$ are predicted to lie in the following ranges at 3$\sigma$ confidence level, combined for all the five pairs of viable models:

$$m_{\text{min}} \sim 0.003 - 0.024 \text{ eV}, \quad \Sigma_i m_i \sim 0.077 - 0.121 \text{ eV}, \quad \langle |m| \rangle \sim 0.006 - 0.046 \text{ eV}. \quad (13)$$

The predicted range for the sum of neutrino masses is consistent with the cosmological bound reported by the Planck collaboration [23], which reads $\Sigma_i m_i < 0.12 - 0.16 \text{ eV}$ at 95% confidence level, depending on the data set used as input. The predicted range for the effective Majorana mass will be probed in the next-generation ($\beta\beta$)$_{0\nu}$-decay experiments aiming at the $\langle |m| \rangle \sim 5 \cdot 10^{-3} - 10^{-2} \text{ eV}$ frontiers.

One also finds that, in all the viable models, the value of $\sin^2 \theta_{23}$ is correlated with the values i) of the Dirac phase $\delta$, ii) of $\Sigma_i m_i$ and iii) of $\langle |m| \rangle$. As an example, these correlations for cases A and A* are reported in Fig. 2.

In the above-described setup, $\tau$ is treated as a free parameter determined by fits to the data. Deriving its value from a potential is an open question, which has been previously studied in the context of string compactifications and supergravity (see, e.g., [24–26]). In Ref. [26], considering the most general non-perturbative effective $\mathcal{N} = 1$ supergravity action in four dimensions, invariant under modular symmetry, it has been conjectured that all extrema of such potential lie on the boundary of the fundamental domain of $\Gamma$ and on the imaginary axis ($\text{Re} \tau = 0$). Interestingly, from Fig. 1 one observes that 6 out of 10 local minima indeed lie almost on the fundamental domain boundary. Furthermore, all of them correspond to a spectrum with normal ordering (NO). The four points which are relatively far from the boundary (C, C*, D, and D*) correspond instead to inverted ordering (IO) of neutrino masses, and are strongly disfavoured by present data. If it indeed turns out that NO is realised in Nature, this could be considered as an indication in favour of the modular symmetry approach to flavour.

5. Summary

We have here reported on two studies [13, 14] which develop the modular symmetry approach to flavour proposed in [6]. This approach connects modular-invariant SUSY actions to the observed low-energy flavour structures. We have focused on minimal models, without flavons, where couplings are obtained from modular forms of level 4. These forms, together with the matter superfields, transform in irreps of the finite modular quotient group $\Gamma_4 \simeq S_4$. One has
derived predictions for the absolute neutrino mass scale and CPV phases, as well as correlations between observables, to be tested in future experiments at the intensity and cosmic frontiers.

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