Lelong-Skoda transform for compact Kähler manifolds and self-intersection inequalities

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Abstract

Let $X$ be a compact Kähler manifold of dimension $k$ and $T$ be a positive closed current on $X$ of bidimension $(p, p)$ ($1 \leq p < k - 1$). We construct a continuous linear transform $L_p(T)$ of $T$ which is a positive closed current on $X$ of bidimension $(k - 1, k - 1)$ which has the same Lelong numbers as $T$. We deduce from that construction self-intersection inequalities for positive closed currents of any bidegree.

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1 Introduction

Let $C$ be a singular algebraic curve of degree $d$ in $\mathbb{P}^2$. Bézout’s theorem implies that the number of intersections near all the singularities between $C$ and a small perturbation of $C$ using an automorphism satisfies:

$$\sum d_n(d_n - 1) \leq d^2,$$

where the $d_n$ are the multiplicity of the singularities (we refer the reader to the survey [Dem92a] for more on that topic). Let $(X, \omega)$ be a compact Kähler manifold of dimension $k$. Here $\omega$ is a fixed Kähler form. We want to extend that inequality to the case where $C$ is replaced by a variety of any bidimension. Two main difficulties appear: one cannot perturb the variety, and the self-intersection is not defined for varieties of small dimension. That is why we work on the more general case of positive closed currents on $X$. In the case of currents, the multiplicity of the singularities are replaced by the Lelong numbers.

Let $T$ be a positive closed current of bidimension $(p, p)$ in $X$. Let $E_c$ ($c > 0$) denotes the set of point $z \in X$ where the Lelong number $\nu(T, z)$ of $T$ at $z$ is strictly larger than $c$. Siu’s theorem implies that $E_c$ is an analytic subset of $X$ (possibly empty) of dimension $\leq p$ [Siu74]. Define $b_q := \inf\{c > 0, \dim E_c \leq q\}$ and $b_{q-1} := \max_{x \in X} \nu(T, x)$. Then $0 = b_0 \leq \cdots \leq b_{q-1}$. For $c \in [b_q, b_{q-1}]$, the dimension of $E_c$ is equal to $q$. So at $b_q$, there is a jump in the dimension for the analytic set $E_c$. Let $(Z_{q, n})_{n \geq 1}$ be the at most countable family of irreducible components of dimension $q$ of the $E_c$ for $c \in [b_q, b_{q-1}]$; note that $(Z_{q, n})_{n \geq 1}$ is finite if $b_q > 0$. Let $\nu_{q, n} := \min_{x \in Z_{q, n}} \nu(T, x) \in [b_q, b_{q-1}]$ be the generic Lelong number of $T$ on $Z_{q, n}$. Then we obtain:

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Theorem 1.1 Let $T$ be a positive closed current of bidimension $(p, p)$ on $X$. Then with the above notation, for $q \leq p$ we have:
\[
\sum_{n \geq 1} (\nu_{q,n} - b_p) \ldots (\nu_{q,n} - b_q) \| Z_{q,n} \| \leq C \| T \|^{p+1-q},
\]
where $C$ is a constant which depends only on $(X, \omega)$, $\| T \|$ denotes the mass of the current $T$ and $\| Z_{q,n} \|$ denotes the mass of the current of integration on $Z_{q,n}$.

Of course we can apply that inequality to the case where $T$ is the current of integration on subvarieties of $X$. Note that $\| Z_{q,n} \|$ is equal to $q!$ times the volume of $Z_{q,n}$. We refer to [DS07a] for a recent application of such inequalities in complex dynamics. In that paper, the authors prove that the normalized pull-back by $f^n$ of a generic hypersurface (in the Zariski sense) converges to the Green current of $f$ for a holomorphic endomorphism $f$ of $\mathbb{P}^k$.

For $p = k - 1$, the inequality was proved by Demailly using a regularization of quasi plurisubharmonic functions in [Dem92] (with $C = 1$ when $X = \mathbb{P}^k$). Méo extended the result to the case of any bidimension for $X$ projective [Méo98]. The key point of his proof is that for a current of bidimension $(p, p)$ in $\mathbb{P}^k$, he constructed a positive closed current of bidegree $(1, 1)$ having the same degree than $T$ and the same Lelong numbers. Such currents were constructed in $\mathbb{C}^k$ by Lelong and Skoda in [Lel68] and [Sko72]. They used a kernel in order to express the potential of the bidegree $(1, 1)$ current. Méo used a geometric approach taking advantage of the fact that the family of projective subspaces of dimension $p$ in $\mathbb{P}^k$ is very rich. The case of projective manifolds is then deduced from an embedding of $X$ in some $\mathbb{P}^N$.

Theorem 1.1 generalizes this result to the Kähler case. For that, we use an approach inspired by [DS04a] (see also [BGS94], [GS91] and [DS04b]). Let $T$ be a positive closed current of bidimension $(p, p)$ on $X$ for $1 \leq p < k - 1$. Consider the canonical projections $\pi_1, \pi_2 : X \times X \to X$ and let $\pi : \tilde{X} \times \tilde{X} \to X \times X$ be the blow-up of $X \times X$ along the diagonal $\Delta$ of $X \times X$. We modify the pull-back by $\pi_2$ of the current $T$ by multiplying it by a suitable form $\Theta$ and then we push it forward to $X$ by $\pi_1$. The form $\Theta$ is smooth outside $\Delta$. It is defined as $(\pi_1(\tilde{\Omega}))^{p+1}$ where $\tilde{\Omega}$ is a Kähler form in $\tilde{X} \times \tilde{X}$. This is what we call the Lelong-Skoda transform $\mathcal{L}_p(T)$ of the current $T$. The Lelong-Skoda transform is a continuous linear operator which sends positive closed currents of bidimension $(p, p)$ to positive closed currents of bidegree $(1, 1)$. The Lelong-Skoda transform is linear in the sense that it is linear on the space of currents spanned by positive closed currents. We show in Theorem 2.1 below that $\mathcal{L}_p(T)$ has the same Lelong number than $T$ at every point. We then prove Theorem 1.1 in the same way than in [Méo98]. We will also give some properties of the Lelong-Skoda transform. Finally, we extend the results to the case of harmonic currents.

2 Lelong-Skoda transform

For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, we use the notation $|z| := (\sum_{j} |z_j|^2)^{1/2}$. Consider a compact Kähler manifold $Y$ of dimension $m$ endowed with a Kähler form $\theta$. We denote by $\mathcal{C}_p(Y)$ the cone of positive closed currents of bidimension $(p, p)$ in $Y$ on which we consider the usual weak topology. For $T \in \mathcal{C}_p(Y)$, let $\| T \| := \int_Y T \wedge \theta^p$ be the mass of $T$. Note that this mass depends only on the class of $T$ in the Hodge cohomology.
group $H^{m-p,m-p}(Y,\mathbb{C})$. We refer the reader to the book of Voisin [Vo02] for basics on compact Kähler manifolds.

We recall some basic facts on Lelong numbers. See [Dem93] for proofs and details and also [FS95] for results on intersection theory of currents. For $a \in Y$ and $T \in C_p(Y)$, we consider a chart $V$ around $a$ in which the coordinates are given by $x$. For simplicity, assume $a$ is given by 0 here. Then for $r > 0$, the positive measure $T(x) \wedge (dd^c \log |x|)^p$ is well defined by [Dem93] and we define the quantities:

$$\nu(T,0,r) := \int_{|x|<r} T(x) \wedge (dd^c \log |x|)^p$$

$$\nu(T,0) := \lim_{r \to 0} \int_{|x|<r} T(x) \wedge (dd^c \log |x|)^p.$$  

The quantity $\nu(T,0)$ is called the Lelong number of $T$ at 0 and does not depend on the choice of the coordinates $x$ (so it is well defined on manifolds). It is the mass at 0 of the measure $T(x) \wedge (dd^c \log |x|)^p$. We have the equivalent definition:

$$\nu(T,0) = \lim_{r \to 0} r^{-2p} \int_{|x|<r} T \wedge \frac{1}{2} dd^c |x|^2)^p.$$  

It follows that the function $S \to \nu(S,0)$ is a linear form from the vector space spanned by positive closed currents, so the Lelong number is well defined for a difference of positive closed currents.

We give now a geometrical interpretation of the Lelong number that we will use latter. Let $\iota : \tilde{Y} \to Y$ denote the blow-up of $Y$ at $a$ and $H := \iota^{-1}(a)$ the exceptional divisor. Then $H \simeq \mathbb{P}^{m-1}$ and we put on $H$ the Fubini-Study form $\omega_{FS}$ normalized by $\int_{\mathbb{P}^{m-1}} \omega_{FS}^{m-1} = 1$. Let $V$ and $x$ be as above. Consider a sequence $(T_n)$ of smooth positive closed currents of bidimension $(p, p)$ in $V$ with $T_n \to T$ in the sense of currents in $V$. We can obtain $(T_n)$ using a convolution operator. Let $\tilde{T}$ be a cluster value of the (bounded) sequence $(\iota^*(T_n))$. In bidegree $(1, 1)$, $\tilde{T}$ is unique and does not depend on $(T_n)$, this is not true in higher bidegree. Then we have the characterization:

**Lemma 2.1** With the above notation, the Lelong number $\nu(T,0)$ is the mass of $\tilde{T}$ on $H$. In particular, that mass does not depend on the choice of $(T_n)$.

**Proof.** Observe that $\iota^{-1}(V) = \{(x,[u]) \in V \times \mathbb{P}^{m-1}, \ x \in [u]\}$. Write $x = (x_1, \ldots, x_m)$ and $u = [u_1 : \cdots : u_m]$. Let $j : \iota^{-1}(V) \to V \times \mathbb{P}^{m-1}$ be the canonical holomorphic injection. Pull-backs by $j$ of positive closed smooth forms are positive closed smooth forms. Let $p : V \times \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ be the projection on the second factor and recall that $\omega_{FS}$ is the Fubini-Study form on $\mathbb{P}^{m-1}$. We endow $V \times \mathbb{P}^{m-1}$ with the form $p^*(\omega_{FS})$. We consider the form $j^*(p^*(\omega_{FS}))$ on $\iota^{-1}(V)$, the restriction of $j^*(p^*(\omega_{FS}))$ to $H$ is indeed $\omega_{FS}$ so we write for simplicity $\omega_{FS}([u])$ instead of $j^*(p^*(\omega_{FS}))(x,[u])$, and we drop the $[u]$ when there can be no confusion. In the chart of $\iota^{-1}(V)$ where $u_1 = 1$, write $u = [1 : u_2 : \cdots : u_m]$ and consider the coordinates $(X_1, \ldots, X_m)$ given by $X_1 = x_1$, $X_2 = u_2, \ldots, X_m = u_m$ (so $X_i = x_i/x_1$ for $i > 1$) in which $H$ is given by $(X_1 = 0)$. In these coordinates, the form $\omega_{FS}$ is:

$$\omega_{FS}([u]) = \frac{1}{2} dd^c \log(1 + |X_2|^2 + \cdots + |X_m|^2).$$
Since \( dd^c \log |x| = 0 \) for \( x \neq 0 \) and \( (dd^c \log |x|)^p \) does not charge the point \( x = 0 \) for \( p < k \), we have that:

\[
\begin{align*}
\iota_\star(\omega_{FS}([u])) &= dd^c \log |x| \\
\iota_\star(\omega_{FS}([u])^p) &= (dd^c \log |x|)^p.
\end{align*}
\]

Observe that we consider the push-forward of a smooth form and not \( \iota^\star(dd^c \log |x|) \) which is the pull-back of a current; indeed to have the "good" continuity's properties one should write \( \iota^\star(dd^c \log |x|) = \omega_{FS}([u]) + |H| \). Consider a smooth psh function \( f \) that coincides with \( \log |x| \) for \( |x| > r/2 \). Then Stokes formula gives that \( \nu(T, 0, r') = \|T \wedge (dd^c f)^p\|_{B_{r'}} \) for \( r' > r/2 \) (\( B_{r'} \) is the ball of center \( 0 \) and radius \( r' \)). Then \( T_n \wedge (dd^c f)^p \to T \wedge (dd^c f)^p \) in the sense of measures. In particular, for \( \varepsilon > 0 \) small enough, we have:

\[
\nu(T, 0, (1 - 2 \varepsilon)r) \leq \lim_{n \to \infty} \int_{B_{(1 - \varepsilon)r}} T_n \wedge (dd^c f)^p \leq \nu(T, 0, r).
\]

Again, Stokes formula gives that \( \int_{B_{(1 - \varepsilon)r}} T_n \wedge (dd^c f)^p = \nu(T_n, 0, (1 - \varepsilon)r) \), so:

\[
\nu(T, 0) = \lim_{r \to 0} \lim_{n \to \infty} \nu(T_n, 0, r).
\] (2)

Now, for \( T_n \) smooth the current \( \iota^\star(T_n) \) is a well defined smooth form so:

\[
\nu(T_n, 0, r) = \int_{|x| < r} T_n(x) \wedge (dd^c \log |x|)^p
\]

\[
= \int_{i^{-1}(|x| < r)} \iota^\star(T_n(x)) \wedge \omega_{FS}^p.
\]

By definition of weak convergence we have for a set of \( r \) of full measure that:

\[
\lim_{n \to \infty} \left( \int_{i^{-1}(|x| < r)} \iota^\star(T_n(x)) \wedge \omega_{FS}^p \right) \to \int_{i^{-1}(|x| < r)} \tilde{T} \wedge \omega_{FS}^p.
\]

So:

\[
\nu(T, y) = \lim_{r \to 0} \int_{i^{-1}(|x| < r)} \tilde{T} \wedge \omega_{FS}^p.
\]

The restriction to \( H \) of \( \omega_{FS}^p \) is the Fubini-Study form on \( H \) at the power \( p \). We decompose \( \tilde{T} = \tilde{T}_1 + \tilde{T}_2 \) where \( \tilde{T}_1 \) is the restriction of \( \tilde{T} \) to \( H \) and \( \tilde{T}_2 \) the restriction of \( \tilde{T} \) to \( i^{-1}(V) \setminus H \). \( \tilde{T}_2 \) can also be defined as the trivial extension of \( \iota^\star(T) \) defined outside \( H \). In other words:

\[
\nu(T, 0) = \langle \tilde{T}_1, \omega_{FS}^p \rangle,
\]

so \( \nu(T, 0) \) can be interpreted geometrically as the mass of the current \( \tilde{T} \) on the exceptional divisor. □

Let \( X \) be a compact Kähler manifold of dimension \( k \). Let \( \Delta \) denote the diagonal in \( X \times X \) and \( \pi : X \times X \to X \times X \) the blow-up of \( X \times X \) along \( \Delta \). Then \( X_\times X \) is a compact Kähler manifold by Blanchard [Blu56]. Fix a Kähler form \( \Omega \) on \( X_\times X \)

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and define \( \Omega := \pi_* (\tilde{\Omega}) \). Then \( \Omega \) is a positive closed \((1,1)\) current smooth outside \( \Delta \) and that does not charge \( \Delta \). Observe that for \( p < k - 1 \), \( \Omega^{p+1} = \pi_* (\tilde{\Omega}^{p+1}) \) since those two positive closed currents coincide outside \( \Delta \) and cannot charge \( \Delta \) since they are of bidimension \( > k = \dim (\Delta) \). Define \( \tilde{\Delta} := \pi^{-1} (\Delta) \).

Let \( \pi_1 \) and \( \pi_2 \) denote the canonical projections from \( X \times X \) to its factors and denote \( \tilde{\pi}_i := \pi_i \circ \pi \). Then \( \pi_1, \tilde{\pi}_i \) are submersions (see \cite{DS01b}) so push-forward and pull-back by \( \pi_1 \) and \( \tilde{\pi}_i \) of positive closed currents are well defined operators which are continuous for the topology of currents. The mass of \( \Omega \wedge [\tilde{\pi}_2^{-1} (y)] \) does not depend on \( y \in X \) for cohomological reasons and is positive since \( \tilde{\Omega} \) is a Kähler form. So renormalizing \( \tilde{\Omega} \) if necessary, we can assume that this mass is equal to 1. The following result gives a description of the singularities of \( \Omega \) near \( \Delta \) and explains our choice of \( \Omega \) (see also the definition of Lelong number):

**Proposition 2.2** The form \( \Omega \) admits locally a potential \( \varphi \) that is \( dd^c \varphi = \Omega \) where \( \varphi \) is a psh function such that \( \varphi (\ast) - \log \text{dist}(\ast, \Delta) \) is bounded.

The problem is local so we let \( U \) be a chart of \( X \times X \) in which we consider the coordinates \((z, w)\) such that \( \Delta \) is given by \( z = 0 \) here and \( \pi_2 (z, w) = w \). So \( \text{dist}(\ast, \Delta) \) is equivalent to \( |z| \) here. Then \( \pi^{-1} (U) = \{(z, w, [u]) \in U \times \mathbb{P}^{k-1} \mid z \in [u] \} \) and \( \tilde{\Delta} \) is given by \( \{z = 0\} \) here.

As above, consider on \( \pi^{-1} (U) \) the smooth form \( \omega_{FS} ([u]) \) then \( \tilde{\alpha} := \tilde{\Omega} - \omega_{FS} ([u]) \) is a smooth closed form of bidegree \((1,1)\) such that \( \int_{\tilde{P}_w^{-1}} \tilde{\alpha} \wedge \tilde{\Omega}^{k-2} = 0 \) (where \( \tilde{P}_w^{-1} := \{(0, w, [u]), [u] \in \mathbb{P}^{k-1}\}) \). The idea is to show that \( \tilde{\alpha} \) is exact so it can be written \( dd^c g \) for \( g \) a continuous function. We will need the following lemma:

**Lemma 2.3** Any closed form \( \tilde{\alpha} \) of bidegree \((1,1)\) on \( \pi^{-1} (U) \) such that \( \int_{\tilde{P}_w^{-1}} \tilde{\alpha} \wedge \omega_{FS}^{k-2} = 0 \) for all \( w \) is in fact exact.

**Proof.** We can assume that \( \tilde{\alpha} \) is real. Using Poincaré’s lemma in the non-compact case, the only obstruction would be that there exists a compact oriented surface \( S \) and a smooth function \( f : S \to \pi^{-1} (U) \) such that \( \int_S f^* (\tilde{\alpha}) \neq 0 \). To simplify the notations, assume that \((0,0) \in U \). If \( i : \pi^{-1} (U) \times [0,1] \to \pi^{-1} (U) \) and \( j : \pi^{-1} (U) \to \pi^{-1} (U) \) are the maps defined by \( i(z, w : [u], t) = (tz, tw, [u]) \) and \( j(z, w : [u]) = (0,0, [u]) \), we obtain by homotopy that \( \int_S (j \circ f)^* (\tilde{\alpha}) \neq 0 \) which contradicts our hypothesis. \( \square \)

**End of the proof of Proposition 2.2** Let \( p_w : \mathbb{P}^{k-1} \to \pi^{-1} (U) \) be the canonical injection. We denote by \( \tilde{\alpha}_w \) the form \( p_w^* (\tilde{\alpha}) \) on \( \mathbb{P}^{k-1} \). Observe that \( \int_{\tilde{P}_w^{-1}} \tilde{\alpha} \wedge \omega_{FS}^{k-2} = \int_{\tilde{P}_w^{-1}} \tilde{\alpha}_w \wedge \omega_{FS}^{k-2} \).

Then \( \tilde{\alpha}_w \) is a smooth exact form, hence it is \( dd^c \) exact. So there exists a real smooth function \( v_w \) such that \( dd^c v_w = \tilde{\alpha}_w \) (observe that any solution of \( dd^c v_w = \tilde{\alpha}_w \) is smooth). Since \( v_w \) can be written via a kernel with coefficients in \( L^1 \) smooth outside the diagonal of \( (\mathbb{P}^{k-1})^2 \), we deduce that \( v(z, w, [u]) = v_w ([u]) \) is continuous in all variables and smooth in the \( w \) variable. A more precise analysis of the singularities of the kernel shows that its gradient has also coefficients in \( L^1 \) so \( v(z, w, [u]) \) is in \( C^1 \) (see the proof of Proposition 2.3.1 in \cite{DS07}), we believe that we can choose \( v \) smooth but we do not know how to prove it). Still the current \( dd^c v \) is represented by a continuous form (we use the integral representation of \( v \) and the fact that \( \tilde{\alpha}_w \) is smooth hence continuous in both variables). Replacing \( \tilde{\alpha} \) by \( \tilde{\Omega} - \omega_{FS} ([u]) - dd^c v \) in \( \pi^{-1} (U) \), we can assume that \( \tilde{\alpha}_w = 0 \) for all \( w \in \pi_2 (U) \).

Using the previous lemma, we can write \( \tilde{\alpha} = d\tilde{\beta} \) where \( \tilde{\beta} \) is a form in \( C^1 \). We decompose \( \tilde{\beta} \) as \( \tilde{\beta}_{1,0} + \tilde{\beta}_{0,1} \), its \((1,0)\) and \((0,1)\) components. We deduce from the
equation $\tilde{\alpha} = d\tilde{\beta}$ that $\tilde{\beta}_{0,1}$ is $\partial$-closed and since $\tilde{\alpha}$ is real $\tilde{\beta}_{0,1} = \bar{\beta}_{1,0}$. The form $\pi_w^*(\tilde{\beta}_{0,1})$ is $\partial$-closed and $\partial$-closed since $2\partial\tilde{\beta}_{0,1} = \tilde{\alpha}$ (we use here that $\alpha_w = 0$). So $p_w^*(\tilde{\beta}_{0,1})$ is a closed $(0,1)$-form on $\mathbb{P}^{k-1}$: it is equal to zero.

We work in the chart where $|u_1| > 1/2 \max_i |u_i|$ so we take $u_1 = 1$. We consider the coordinates given by $(Z_1, \ldots, Z_n, w)$ where:

$$Z_1 = z_1, \quad Z_2 = z_2/z_1, \ldots, \quad Z_k = z_k/z_1.$$  

The form $\tilde{\beta}_{0,1}$ can be written as $\sum_i \tilde{\beta}_i d\overline{Z}_i + \sum_i \tilde{\beta}_i^j dw_i$ and since $p_w^*(\tilde{\beta}_{0,1}) = 0$, we have that $\tilde{\beta}_i(0, Z_2, \ldots, Z_k, w) = 0$ for all $i \geq 2$ while the other coefficients are bounded. Since the coefficients are in $C^1$, we have that $|\tilde{\beta}_i| \leq C|Z_i|$ for $i \geq 2$. Let $\beta_{0,1} = \pi_*(\tilde{\beta}_{0,1})$, then in the local coordinates, it can be written:

$$\beta_{0,1} = \tilde{\beta}_1 dz_1 + \sum_{i \geq 2} \tilde{\beta}_i \frac{dz_i}{z_1} + \sum_i \tilde{\beta}_i^j dw_i.$$  

We write $d(z_i/z_1) = dz_i/z_1 - z_i/z_1^2 dz_1$. Using the fact that the coefficients $\tilde{\beta}_i$ are bounded by $C|z_i|$ and $|z_i| < 2|z_1|$ we get that $\beta_{0,1}$ has coefficients in $L^\infty$. In particular, $\beta_{0,1}$ is a well defined (0,1) current $\partial$-closed with $L^\infty$-coefficients. So taking $U$ strictly pseudoconvex, we can solve $\partial u = \beta_{0,1}$ with $u$ a continuous function $[HL84]$ ($L^\infty$ is enough for our purpose). Let $g := -i\pi(u - \bar{u})$ it is a real bounded function and it satisfies the following identities for $\alpha := \pi_*(\alpha)$ and $\beta := \pi_*(\beta)$:

$$dd^c g = \partial\bar{\partial}(u - \bar{u}) = \partial\beta_{0,1} + \partial\overline{\beta}_{1,0} = dd^c\alpha + dd^c\pi_*(v).$$

This implies that $\Omega = dd^c(|\log z| + g - \pi_*(v))$ and gives the proposition ($\pi_*(v)$ is only bounded). □

Let $T \in \mathcal{C}_p(X)$ with $1 \leq p < k - 1$ (there is nothing to be done for $p = k - 1$). The current $(\pi_2)^*(T) \wedge \Omega^{p+1}$ is a well defined element of $\mathcal{C}_{k-1}(\bar{X} \times X)$. It is of finite mass and coincides with $(\pi_2)^*(T) \wedge \Omega^{p+1}$ outside $\bar{\pi}^{-1}(\Delta)$. So $(\pi_2)^*(T) \wedge \Omega^{p+1}$ is well defined on $X \times X$ as the trivial extension over $\Delta$ of the above current. Consequently, the current:

$$T_{LS} := (\pi_1)_*((\pi_2)^*(T) \wedge \Omega^{p+1})$$

belongs to $\mathcal{C}_{k-1}(X)$. That gives us the following notion:

**Definition 2.4** For $p < k - 1$, define the Lelong-Skoda transform $\mathcal{L}_p$ from $\mathcal{C}_p(X)$ to $\mathcal{C}_{k-1}(X)$ by:

$$\mathcal{L}_p(T) := T_{LS}.$$  

Of course, the operator $\mathcal{L}_p$ depends on the choice of $\tilde{\omega}$.

## 3 Properties of the Lelong-Skoda transform

This section is devoted to prove the following result:

**Theorem 3.1** The Lelong-Skoda transform $\mathcal{L}_p$ is a continuous linear operator. In particular, there exists a constant $C > 0$ such that $\|\mathcal{L}_p(T)\| \leq C\|T\|$. Moreover, it preserves the Lelong number, i.e. $\mathcal{L}_p(T)$ and $T$ have the same Lelong number at every point.
Since the set $C_p(X)$ is a convex cone, we should not speak of linear operator. Nevertheless, the transform $\mathcal{L}_p$ can be extended to a linear operator on the space of currents spanned by positive closed currents. For a current $S$ which is the difference of positive closed currents, we can write:

$$||S|| := \inf(||T'|| + ||T''||)$$

where the infimum is taken over all the decompositions $S = T' - T''$ where $T'$ and $T''$ are positive closed currents. We use the following lemma to prove the continuity:

**Lemma 3.2** The transform $\mathcal{L}_p$ satisfies : $\mathcal{L}_p(T) = (\tilde{\pi}_1)_*\left(\left((\tilde{\pi}_2)^*(T) \wedge \tilde{\Omega}^{p+1}\right)\right)$.

**Proof.** Since $\tilde{\Omega}$ is smooth, it is sufficient to show that $(\tilde{\pi}_2)^*(T)$ does not charge $\tilde{\Delta}$. The problem is local so we let $U$ be a chart of $X \times X$ in which we consider the coordinates $(z,w)$ such that $\Delta$ is given by $z = 0$ and $\pi_2(z,w) = w$. Then $\pi^{-1}_1(U) = \{(z,w,[u]) \in U \times \mathbb{P}^{k-1}, \ z \in [u]\}$ and $\Delta$ is given by $z = 0$ here, so $\pi^{-1}_1(U)$ is locally a product. So $\tilde{\omega}_t^2$ of a current is just integration on fibers. We want to know if $(\tilde{\pi}_2)^*(T)$ charges $z = 0$ which is impossible by Fubini’s theorem. More precisely, let $B_r$ denote the ball of center $0$ and radius $r$ in $\mathbb{C}^k$. We can reduce ourselves to the case where $\pi^{-1}_1(U)$ is of the form $\tilde{B}_{r_1} \times B_{r_2}$ where $\tilde{B}_{r_1}$ is the blow-up of $B_{r_1}$ at $z = 0$. Let $\omega_{\tilde{B}_{r_1}}$ and $\omega_{B_{r_2}}$ denote Kähler form on $\tilde{B}_{r_1}$ and $B_{r_2}$. Then:

$$\|\tilde{\pi}_2^*(T)\|_{\tilde{B}_{r_1} \times B_{r_2}} = \left(\frac{k + p}{k}\right) \int_{\tilde{B}_{r_1}} T'(w) \wedge \omega_{\tilde{B}_{r_1}}^k \int_{B_{r_2}} \omega_{B_{r_2}}^k = \left(\frac{k + p}{k}\right) \|T\|_{\tilde{B}_{r_1}} \omega_{\tilde{B}_{r_1}}^k(\tilde{B}_{r_1}).$$

And the lemma follows from letting $r$ goes to $0$. □

The lemma implies that $\mathcal{L}_p$ is a continuous linear operator since $\tilde{\Omega}^{p+1}$ is smooth and pull-back and push-forward by the submersions $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are continuous operators on (positive closed) currents.

**Proof of Theorem 3.1** We want to interpret the mass of $\mu$ at $(0,0)$ as in Lemma 2.1. For that purpose, we will pull-back some integrals on some suitable blow-ups of $X \times X$ in order to desingularize the forms $\Omega$, $dd^c \log |x|$ and $dd^c \log |(x,y)|$.

Let $p_1 : \tilde{X} \times \tilde{X} \to X \times X$ be the blow-up of $X \times X$ at $(0,0)$. Consider the system of local coordinates $(z,w)$ in the neighborhood $\tilde{V} \times \tilde{V}$ of $(0,0)$ in $X \times X$ given by $(z,w) := (x - y, x)$ for $(x,y) \in V \times V$. Then:

$$\tilde{V} \times \tilde{V} := p_1^{-1}(V \times V) = \{(z,w,[u : v]) \in V \times V \times \mathbb{P}^{2k-1}, \ (z,w) \in [u : v]\},$$

where $z = (z_1, \ldots, z_k)$, $w = (w_1, \ldots, w_k)$, $[u : v] = [u_1 : \cdots : u_k : v_1 : \cdots : v_k]$. In $\tilde{V} \times \tilde{V}$, there exists a smooth form that we denote by $\omega_{\tilde{F}\times\tilde{F},2k-1}$ such that $dd^c \log |(z,w)| = (p_1)_*(\omega_{\tilde{F}\times\tilde{F},2k-1})$ (that is what we mean by desingularization). But we cannot do that for $dd^c \log |w|$ nor for $\Omega$ so we need to blow-up once more.

Consider the smooth submanifold $M$ of $\tilde{X} \times \tilde{X}$ given by $\{u = 0\} \cup \{v = 0\}$ in $\tilde{V} \times \tilde{V}$ and by $p_1^{-1}(\Delta \cup ([0] \times \tilde{X}))$ outside $\tilde{V} \times \tilde{V}$. It is the disjoint union of two submanifolds.
which are the strict transform of \((z = 0)\) and \((w = 0)\). In other words, the blow-up \(p_1\) desingularizes the analytic set \(\Delta \cup \{x = 0\}\). Let \(p_2 : \hat{X} \times X \to \hat{X} \times X\) denote the blow-up of \(\hat{X} \times \hat{X}\) along \(M\). So:

\[
\hat{V} \times V := p_2^{-1}(\hat{V} \times V) = \{(z, w, [u : v], [u'], [v']) \in (\hat{V} \times V) \times \mathbb{P}^{2k-1} \times \mathbb{P}^{k-1} \times \mathbb{P}^{k-1}, (z, w) \in [u : v], u \in [u'], v \in [v']\}.
\]

Observe that \(z \in [u']\). So we can define the holomorphic projection \(\tilde{P} : \hat{X} \times \hat{X} \to \hat{X} \times \hat{X}\) which is given in \(\hat{V} \times V\) by:

\[
\tilde{P} : (z, w, [u : v], [u'], [v']) \mapsto (z, w, [u']).
\]

The form \(P^*(\hat{\Omega})\) is a well defined positive smooth form that we will simply write \(\hat{\Omega}\).

Define finally \(\pi' : \hat{X} \times \hat{X} \to X \times X\) the blow-up of \(X \times X\) along \(\{x = 0\}\). Observe that \(w \in [v']\). So we can define the holomorphic projection \(P' : \hat{X} \times \hat{X} \to \hat{X} \times \hat{X}'\) which is given in \(\hat{V} \times V\) by:

\[
P' : (z, w, [u : v], [u'], [v']) \mapsto (z, w, [v']).
\]

Let \(\omega_{FS,k-1}\) denote the smooth form on \((\pi')^{-1}(V \times X)\) which is the pull-pack of the Fubini-Study form on \(\mathbb{P}^{k-1}\) to \((\pi')^{-1}(V \times X)\) (observe that there may not be a global mapping \(\hat{X} \times \hat{X}' \to X \times X \times \mathbb{P}^{k-1}\) so we cannot speak of \(\omega_{FS,k-1}\) on \(\hat{X} \times \hat{X}'\)). Then the form \((P')^*(\omega_{FS,k-1})\) is a well defined positive smooth form on \((p_1 \circ p_2)^{-1}(V \times X)\) that we will simply write \(\omega_{FS,k-1}\).

We define \(E := (p_1 \circ p_2)^{-1}(\{(0,0)\}) = \{(0,0), [u : v], [u'], [v']\} \in \hat{V} \times V\) the fiber of \((0,0)\) in \(\hat{X} \times \hat{X}\). Note that \(E\) can be considered as a blow-up of \(\mathbb{P}^{2k-1} \simeq p_1^{-1}(0,0)\) along two disjoint subspaces of dimension \(k - 1\). Let \(\Pi : E \to \mathbb{P}^{2k-1}\) be the canonical projection defined by:

\[
\Pi : ([u : v], [u'], [v']) \mapsto [u : v].
\]

So \(\Pi\) is just the restriction of \(p_2\) to \(E\) if we identify \(\mathbb{P}^{2k-1}\) to \(p_1^{-1}(0,0)\). Finally let \(\tilde{\pi}_i := \pi_i \circ p_1 \circ p_2\).

We will need to regularize the currents \(T\) and \(T_{L,S}\) in a neighborhood of \(0\). For that we will regularize the current \(T\) and use the continuity of the Lelong-Skoda transform. But in order to do that we have to regularize the current \(T\) in \(X\) and not only in a neighborhood of \(0\) because \(T_{L,S}\) is defined "globally". In the case where \(X = \mathbb{P}^k\), smooth positive closed forms are dense in the space of positive closed currents. It is not true in the case of Kähler manifolds. Nevertheless, in [DS04b], the authors prove the following regularization result:

**Theorem 3.3** For a positive closed current \(T\), there exist smooth positive closed forms of bidegree \((k - p, k - p)\), \(T_+^p\) and \(T_-^p\) such that \(T_+^p - T_-^p\) converges weakly to the current \(T\). Moreover, \(\|T_+^p\| \leq C_X\|T\|\) where \(C_X\) is a constant independent of \(T\).
Let \((T^+_n)\) and \((T^-_n)\) be as in the theorem. Extracting if necessary, we suppose that the sequences converge to \(T^+\) and \(T^-\). By continuity of the transform, we have \(L_p(T^+_n) \to L_p(T^\pm)\). Recall that the function \(S \to \nu(S,0)\) is a linear form on positive closed currents. So proving the theorem for \(T^\pm\) gives the theorem for \(T\). In particular, it is enough to consider the case where \(T\) is the limit in the sense of currents of a sequence \((T_n)\) of smooth positive closed forms. We have \(\|T_n\| \to \|T\|\). Then we have the lemma:

**Lemma 3.4** The sequences \(\|\pi_2^+(T_n)\|\) is bounded. In particular, we can extract a subsequence of \((\pi_2^+(T_n))\) that converges to a cluster value that we denote by \(\hat{T}\).

**Proof.** The mass \(\|\pi_2^+(T_n)\|\) depends only on the cohomology class \([T_n]\). The lemma follows since the cohomology class \([T_n]\) is controlled by the mass of \(T_n\) which is bounded with \(n\). □

The current \(\hat{T}\) is positive closed as a limit of positive closed currents. As in the case of the Lelong number, the current \(\hat{T}\) is not unique. Despite that fact, we will show that the mass of \(\hat{T}\) on the set \(E\) is independent of the choice of \((T_n)\) and \(\hat{T}\). More precisely, let \(\hat{T}_{|E}\) denote the restriction of \(\hat{T}\) at \(E\). We have the following proposition which is the key of our proof:

**Proposition 3.5** The Lelong number \(\nu(T_{LS},0)\) is equal to the mass of the current \((\Pi_{|E}^{\pm}(\hat{T}_{|E}))\) on \(\Pi(E) = p^{-1}_1(0,0)\).

**Lemma 3.6** Let \(S\) be a smooth positive closed form of bidimension \((p,p)\) in \(X\) then \(L_p(S)\) is a continuous positive closed form of bidegree \((1,1)\). Furthermore:

\[L_p(S) = \pi_1^*(\pi_2^+(S) \wedge \hat{\Omega}^{p+1})\]

**Proof.** The first part of the lemma follows from the fact that \(\Omega^{p+1}\) has coefficients in \(L^1\). The second assertion is a consequence of the fact that \(p < k - 1\) hence \(\Omega^{p+1}\) does not charge \(\Delta\). □

We leave as an exercise to the reader the fact that \(T_n\) is in fact smooth: it is a consequence of the fact that \(\pi_1^*\) and \(\pi_1\) are submersion. Nevertheless, continuity is sufficient for our purpose. We have the lemma:

**Lemma 3.7** With the above notations:

\[\nu(T_{LS},0) = \lim_{r \to 0} \lim_{n \to \infty} \nu(L_p(T_n),0,r)\]

**Proof.** We argue as in the proof of Lemma 2.1, we use an approximation of the logarithm, Stokes formula and weak convergence. □

Since \(L_p(T_n)\) is continuous, using Lemma 3.6 we have that:

\[\nu(L_p(T_n),0,r) = \int_{B_r} \pi_1^*(\pi_2^+(T_n) \wedge \hat{\Omega}^{p+1}) \wedge (dd^c \log |x|)^{k-1}\]

We have that \((dd^c \log |x|)^{k-1} = \pi_1^*(\omega_{FS,k-1}^{k-1})\) and \(\pi_1^*((dd^c \log |x|)^{k-1}) = \omega_{FS,k-1}^{k-1}\) outside \((p_1 \circ p_2)^{-1}(x = 0)\) (that is outside \(w = 0\) in the new coordinates). Let
0 < r' < r, we have that:
\[
\int_{B_r \setminus B_{r'}} \pi_1^*(\tilde{\pi}_2^*(T_n) \wedge \tilde{\Omega}^{p+1}) \wedge (ddc \log |x|)^{k-1} = \\
\int_{\tilde{\pi}_1^{-1}(B_r \setminus B_{r'})} \pi_2^*(T_n) \wedge \tilde{\Omega}^{p+1} \wedge \omega_{FS,k}^{k-1}.
\]

So we claim that:
\[
\nu(L_p(T_n), 0, r) = \int_{\tilde{\pi}_1^{-1}(B_r)} \pi_2^*(T_n) \wedge \tilde{\Omega}^{p+1} \wedge \omega_{FS,k}^{k-1}.
\]

Indeed, the current \( \pi_1^*(\tilde{\pi}_2^*(T_n) \wedge \tilde{\Omega}^{p+1}) \wedge (ddc \log |x|)^{k-1} \) does not charge 0 and the current \( \pi_2^*(T_n) \wedge \tilde{\Omega}^{p+1} \wedge \omega_{FS,k}^{k-1} \) does not charge \( w = 0 \).

Consider the smooth form \( L \) defined by:
\[
L := \tilde{\Omega}^{p+1} \wedge \omega_{FS,k}^{k-1}.
\]

As in the proof of Lemma 2.1 letting \( n \to \infty \) in (3) we have:
\[
\int_{\tilde{\pi}_1^{-1}(B_{r'})} \hat{T} \wedge L \leq \lim_{n \to \infty} \nu(L_p(T_n), 0, r) \leq \int_{\tilde{\pi}_1^{-1}(B_{r'})} \hat{T} \wedge L.
\]

In fact we have the equality \( \int_{\tilde{\pi}_1^{-1}(B_{r'})} \hat{T} \wedge L = \lim_{n \to \infty} \nu(L_p(T_n), 0, r) \) for \( r \) generic but we only need the previous inequalities. Combining this with Lemma 3.7 we have the equality:
\[
\nu(T_{LS}, 0) = \lim_{r \to 0} \int_{\tilde{\pi}_1^{-1}(B_{r'})} \hat{T} \wedge L = \|\hat{T} \wedge L\|_{\tilde{\pi}_1^{-1}(0)}.
\]

The next lemma shows that the mass is in fact concentrated on \( E \).

**Lemma 3.8** With the above notations \( \nu(T_{LS}, 0) \) is the mass of \( \hat{T} \wedge L \) on \( E \).

**Proof.** Let \( W \) be a small neighborhood of \( E \) in \( \overline{X \times X} \). It is sufficient to show that the current \( \hat{T} \) does not charge the set \( \tilde{\pi}_1^{-1}(0) \) \( \setminus W \). We argue as in Lemma 3.2 taking advantage of the fact that \( \pi_2^* \) of a current is given here by integration on fibers which are transverse to \( \pi_1^{-1}(0) \). Of course, in \( W \) the geometry is more complicated and there \( \pi_1 \) is not a submersion. \( \square \)

**End of the proof of Proposition 3.2.** On \( \Pi(E) \), the currents \( \Pi_*(L|E) \), \( \Pi_*(\tilde{\Omega}^{p+1}_{E}) \) and \( \Pi_*(\omega_{FS,k-1}^{k-1}|E) \) are well defined and have no mass on \( \Pi(M) \) because \( M \) is of dimension \( k - 1 \) and the bidimension of \( \Pi_*(\tilde{\Omega}^{p+1}_{E}) \) and \( \Pi_*(\omega_{FS,k-1}^{k-1}|E) \) is at least \( k \) and the singularities are in two disjoint subvarieties (\( \Pi_*(\tilde{\Omega}^{p+1}_{E}) \) is smooth where \( \Pi_*(\omega_{FS,k-1}^{k-1}|E) \) is singular and vice versa). Furthermore, \( \Pi_*(L|E) = \Pi_*(\tilde{\Omega}^{p+1}_{E}) \wedge \Pi_*(\omega_{FS,k-1}^{k-1}|E) \). On \( \Pi(E) \), the measure \( \Pi_*(\hat{T}_E \wedge L_E) \) is well defined as its trivial extension over \( E \) of its restriction to \( E \setminus M \). Indeed, \( \hat{T}_E \) does not charge the set \( (p_2)^{-1}(M) \) (it is the same argument as in Lemma 3.2). In particular, the measure is equal to \( \Pi_*(\hat{T}_E) \wedge \Pi_*(L|E) \) and, by Lemma 3.8 its mass is equal to \( \nu(T_{LS}, 0) \).

Then the mass of the measure can be computed in cohomology since for positive closed currents in \( \mathbb{P}^{2k-1} \) the mass of a wedge product is the product of the masses.
Let $F$ be a subspace of $\mathbb{C}^{2k}$ and consider the orthogonal projection from $\mathbb{C}^{2k}$ to $F$. It induces a meromorphic map $\sigma_F : \mathbb{P}^{2k-1} \dashrightarrow \mathbb{P}(F)$. Then for a positive closed current $S$ of mass $m$ on $\mathbb{P}(F)$, the pull-back $\sigma_F^*(S)$ is well defined and of mass $m$ (see Section 1 in [Méo98]). Applying this to $\mathbb{P}(F_1) = \{v = 0\}$ and $\mathbb{P}(F_2) = \{u = 0\}$ we obtain that $\Pi_*(L_{|E}|) = \Pi_*(\tilde{\Omega}_{p+1}^{E}) \wedge \Pi_*((\omega_{FS,k-1}^{P-1})_{|E})$ is of mass 1. Indeed $\tilde{\Omega}_{p+1}^{E}$ is by definition $\tilde{\Omega}_{p+1}^{E}(0, 0, [u'])$. So

$$
\Pi_* (\tilde{\Omega}_{p+1}^{E}) = \sigma_F^*(\tilde{\Omega}_{p+1}^{E}),
$$

since the equality is true outside $\mathbb{P}(F_2)$ and both sides of the equality give no mass to $\mathbb{P}(F_2)$. Similarly:

$$
\Pi_*((\omega_{FS,k-1}^{P-1})_{|E}) = \sigma_F^*(\omega_{FS,k-1}^{P-1}).
$$

And our normalization of $\tilde{\Omega}$ implies that $\Pi_*(\tilde{\Omega}_{p+1}^{E})$ is of mass 1 so $\Pi_*(L_{|E}|)$ is of mass 1. Then $\nu(T_{LS}, 0)$ can be interpreted as the mass of $\Pi(E)$ for the current $\Pi_*(\tilde{T}_{|E})$ which is also the mass of $(p_2)_*(T)$ on $\Pi(E)$. \hfill \Box

End of the proof of Theorem 3.1. The Lelong number $\nu(T, 0)$ is the same than the Lelong number $\nu(\sigma_2(T), (0, 0))$ (more generally, the Lelong numbers of the pull-back of a current by a submersion are preserved by Proposition (2.3) in [Méo98]). This and Lemma 3.4 applied to $\tilde{T} := (p_2)_*(\tilde{T}) = \lim_n p_1^*(T_n)$ imply that $\nu(T, 0)$ is the mass of $\Pi(E)$ for the current $(p_2)_*(\tilde{T})$ (we use that $(p_2)_*(p_2)^* = \text{id}$). Proposition 3.5 implies the result. \hfill \Box

Several remarks are in order here.

Remark 3.9 The transform $L_p$ is compatible with the cohomology, that is if $T$ and $T'$ are cohomolouges on $H^{p, p}(X, \mathbb{C})$ so are $L_p(T)$ and $L_p(T')$ by Lemma 5.3. Indeed, if $T = T' + dd^c \alpha$ with $\alpha$ of bidegree $(k - p - 1, k - p - 1)$ then since the $\tilde{\pi}_i$ are submersions and $\tilde{\Omega}$ is closed we have:

$$
L_p(T) - L_p(T') = dd^c (\tilde{\pi}_1^* (\tilde{\pi}_2^* (\alpha) \wedge \tilde{\Omega}_{p+1}^{E})).
$$

Remark 3.10 The choice $\tilde{\Omega}_{p+1}^{E}$ can be replaced by a strongly positive closed smooth form $\Theta$ on $\tilde{X} \times X$ of bidegree $(p + 1, p + 1)$ such that the mass of $\Theta \wedge [\pi_2^{-1}(y)]$ is equal to 1 for any $y$ (this mass is a constant for cohomological reasons, so we just have to normalize it).

Remark 3.11 The same method allows us to prove that: "for any current $T$ of bideimension $(p, p)$ $(p < k - 1)$ and any $p < q \leq k - 1$, there exists a positive closed current $T_q$ of bideimension $(q, q)$ depending continuously and linearly of $T$ which has the same Lelong number as $T$ at every point".

4 Generalized Demailly’s inequality

Using Theorem 3.1 we follow the argument of Méo ([Méo98]) to prove Theorem 1.1. We use the notations of the introduction. Demailly proved the following regularization result:
Theorem 4.1 ([Dem92b]) Let $S$ be a positive closed current of bidegree $(1, 1)$ on a compact Kähler manifold $X$. Let $\varphi$ be a quasi-psh function such that $S = \alpha + \ddc \varphi$ where $\alpha$ is a smooth $(1, 1)$ form. Then for all $c > 0$, there exists a decreasing sequence $(\varphi_{c,l})_{l \geq 1}$ of functions converging to $\varphi$ such that:

- $\varphi_{c,l}$ is smooth outside $X \setminus E_c$;
- $\ddc \varphi_{c,l} + \alpha + A \|S\| \omega \geq 0$ where $A$ is a constant that depends only on $X$ and $\omega$;
- for all $x \in X$, $\nu(\varphi_{c,l}, x) = (\nu(\varphi, x) - c)_+ := \max(\nu(\varphi, x) - c, 0)$.

We apply this result for $S = T_{LS}$. The current:

$$T \wedge (\ddc \varphi_{c_{p-1},l} + \alpha + A \|S\| \omega) \wedge \cdots \wedge (\ddc \varphi_{c_q,l} + \alpha + A \|S\| \omega)$$

is well defined by the theory of intersection of currents (see [Dem93] and [FS95]) because for $c_j > b_j$, the set of points at which $\varphi_{c_j,l}$ is not bounded is contained in $E_{c_j,l}$ which is of dimension less or equal to $j$. Using [Dem97, Corollary (7.9) p. 194], its Lelong number at $x$ is greater or equal than:

$$\nu(T, x)(\nu(T, x) - c_{p-1}) + \cdots (\nu(T, x) - c_q)_+.$$ 

Siu’s theorem ([Siu74]) implies that this current is greater than:

$$\sum_n \nu_{q,n}(\nu_{q,n} - c_{p-1}) + \cdots (\nu_{q,n} - c_q)_+ [Z_{q,n}].$$

Observe that the mass of the current $\ddc \varphi_{c,l} + \alpha + A \|S\| \omega \geq 0$ is equal to $(1 + A) \|T\|$. So taking the masses gives:

$$\sum_n \nu_{q,n}(\nu_{q,n} - c_{p-1}) + \cdots (\nu_{q,k} - c_q)_+ [Z_{q,n}] \leq C \|T\|^{p-q+1}.$$ 

Theorem 4.1 follows from letting the $c_j$ go to $b_j$.

5 Transformation of pluriharmonic currents

We want to generalize the results of Sections 2 and 3 to the case of pluriharmonic currents. Once again we write "linear operator" on a convex cone instead of an affine operator on a convex cone which extends to a linear operator on the vector space it spans.

Theorem 5.1 The Lelong-Skoda transform $\mathcal{L}_p$ is a well defined linear continuous operator from the cone of positive pluriharmonic currents of bideimension $(p, p)$ ($p < k-1$) to the cone of positive pluriharmonic currents of bidimension $(1, 1)$. The transform preserves Lelong numbers.

Recall that a positive current $T$ of bideimension $(p, p)$ is said to be pluriharmonic if $\ddc T = 0$ in the sense of currents (see [FS95]). For such current, Skoda proved that the Lelong number is well defined (Proposition 1 in [Sko82]). More precisely, let $T$ be a positive pluriharmonic current of bideimension $(p, p)$ in an open set $U$ of $\mathbb{C}^k$. Then,
for \( x \) in \( U \), we have that the positive measure \( T(y) \wedge (d_y d_y^* \log |x-y|)^p \) is well defined on \( U \setminus \{x\} \) since \( \log |x-y| \) is smooth here. And for \( 0 < r_1 < r_2 \) we have the identity:

\[
\frac{1}{r_2^p} \int_{|x-y|<r_2} T(y) \wedge (\frac{1}{2} dd^c |y|^2)^p - \frac{1}{r_1^p} \int_{|x-y|<r_1} T(y) \wedge (\frac{1}{2} dd^c |y|^2)^p = \int_{r_1 < |x-y|<r_2} T(y) \wedge (d_y d_y^* \log |x-y|)^p. \tag{4}
\]

In particular, the non negative quantity \( \frac{1}{r^p} \int_{|x-y|<r} T(y) \wedge (\frac{1}{2} dd^c |y|^2)^p \) decreases with \( r \) to a number \( \nu(T, x) \) called the Lelong number of \( T \) at \( x \). Let \( (T_n) \) be a sequence of smooth positive pluriharmonic currents converging to \( T \) in the sense of currents in a neighborhood of \( x \). Then since \( \lim_{n \to \infty} T_n(y) \wedge (\frac{1}{2} dd^c |y|^2)^p = T(y) \wedge (\frac{1}{2} dd^c |y|^2)^p \) in the sense of measures, we have for \( r \) generic that:

\[
\lim_{n \to \infty} \frac{1}{r^p} \int_{|x-y|<r} T_n(y) \wedge (\frac{1}{2} dd^c |y|^2)^p \to \frac{1}{r^p} \int_{|x-y|<r} T(y) \wedge (\frac{1}{2} dd^c |y|^2)^p.
\]

For \( T_n \) smooth, we can let \( r_1 \) goes to \( 0 \) in Skoda’s formula (1) and we see that:

\[
\frac{1}{r^p} \int_{|x-y|<r} T_n(y) \wedge (\frac{1}{2} dd^c |y|^2)^p = \int_{|x-y|<r} T_n(y) \wedge (d_y d_y^* \log |x-y|)^p.
\]

Combining the last two equalities gives:

\[
\nu(T, x) = \lim_{r \to 0} \lim_{n \to \infty} \int_{|x-y|<r} T_n(y) \wedge (d_y d_y^* \log |x-y|)^p. \tag{5}
\]

In the case of pluriharmonic currents, we have the following integration by parts formula (see [Dom93]) :

**Lemma 5.2** Let \( T \) be a pluriharmonic current of bidimension \( (p,p) \) on \( U \subset X \) and \( f \) be a smooth form of bidegree \( (p-1, p-1) \) equal to zero near \( \partial U \), then:

\[
\int_U T \wedge dd^c f = 0.
\]

Once again, the Lelong number at a point does not depend on the choice of coordinates (see [AB96]), so we can speak of Lelong numbers on a manifold. Lemma 2.1 still applies for positive pluriharmonic currents. That is: if \( \hat{X} \to X \) is the blow-up of \( X \) at \( x \) and \( \hat{T} \) is a cluster value of the sequence \( v^*(T_n) \), then the mass of \( \hat{T} \) on the exceptional divisor is equal to the Lelong number \( \nu(T, x) \) (the proof is exactly the same using formula (5)).

Now we define in the same way \( L_p(T) \) which is a well defined positive pluriharmonic current of bidegree (1, 1). Observe that the arguments of Section 3 remain valid so we can conclude in the same way.

Let us mention the points where the argument need some modifications. In [DS04b], the authors also prove Theorem 4.3 for positive pluriharmonic currents. In Lemma 4.7 we use the previous argument instead of Stokes formula: we use formula (4) applied to \( L_p(T_n) \) instead of formula (2) \( (L_p(T_n) \) is continuous and not smooth but that is enough here). To prove that the measures \( T|_E \wedge L|_E \) and \( T|_E \wedge \omega_{FS,2k-1}^{k+p} \) have the same mass, observe that \( \omega_{FS,2k-1}^{k+p} \) and \( L|_E \) are cohomologous since they have the same mass.
so $\omega^{k+p}_{F,SE_{2k-1}} - L_{1E} = dd^cf$ where $f$ is a form with coefficient in $L^1$. And we conclude using the previous integration by part formula.

In an open set $U$ of $\mathbb{C}^k$, it is not true that the set $E_c = \{ z \in U, \nu(T, y) > c \}$ is analytic for a pluriharmonic current $T$. Indeed, consider in $\mathbb{C}^2$ the current $T = h(z_1)[z_2 = 0]$ where $h$ is a non-constant non-negative harmonic function in $\mathbb{C}$ and $[z_2 = 0]$ is the current of integration on $\{ z_2 = 0 \}$. In a compact manifold, any pluriharmonic function is constant. That raises the question:

Open problem (Dinh): Let $T$ be a positive pluriharmonic current on a compact Kähler manifold $(X, \omega)$. Does Siu’s theorem still hold for $T$, i.e. is $E_c$ analytic for $c > 0$?

A relative result was proved in [DL03] in the case of rectifiable currents. The previous theorem simplifies the question to the case of bidegree $(1, 1)$.

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