Diophantine approximation by special primes

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Abstract

We show that whenever $\delta > 0$, $\eta$ is real and constants $\lambda_i$ satisfy some necessary conditions, there are infinitely many prime triples $p_1$, $p_2$, $p_3$ satisfying the inequality $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/12+\delta}$ and such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most 28 prime factors.

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1 Introduction and statements of the result.

In 1973 Vaughan [13] proved that whenever $\delta > 0$, $\eta$ is real and constants $\lambda_i$ satisfy some necessary conditions, there are infinitely many prime triples $p_1$, $p_2$, $p_3$ such that

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi - \delta}$$

(1)

for $\xi = 1/10$. Latter the upper bound for $\xi$ was improved by Baker and Harman [1] to $\xi = 1/6$, by Harman [4] to $\xi = 1/5$ and the best result up to now is due to K. Matomäki [9] with $\xi = 2/9$.

On the other hand a famous and still unsolved problem in Number Theory is the prime-twins conjecture, which states that there exist infinitely many prime numbers $p$ such that $p + 2$ is also a prime.

Up to now many hybrid theorems were proved. One of the best result belongs to K. Matomäki and Shao [8]. They proved that every sufficiently large odd integer $n$ such that $n \equiv 3 \pmod{6}$ can be represented as a sum

$$n = p_1 + p_2 + p_3$$
of primes $p_1, p_2, p_3$ such that

$$p_1 + 2 = P'_2, \quad p_2 + 2 = P''_2, \quad p_3 + 2 = P'''_2,$$

where $P_l$ is a number with at most $l$ prime factors.

In the present paper we consider (1) with primes of the form specified above. We prove the following theorem.

**Theorem 1.** Suppose that $\lambda_1, \lambda_2, \lambda_3$ are non-zero real numbers, not all of the same sign, that $\eta$ is real, and that $\lambda_1/\lambda_2$ is irrational. Let $\xi = 1/12$ and $\delta > 0$. Then there are infinitely many ordered triples of primes $p_1, p_2, p_3$ for which

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi + \delta}$$

and

$$p_1 + 2 = P'_2, \quad p_2 + 2 = P''_2, \quad p_3 + 2 = P'''_2.$$

By choosing the parameters in a different way we may obtain other similar results, for example $\xi = -9/350, p_i + 2 = P_20, i = 1, 2, 3$.

Result of this type were obtained by Dimitrov and Todorova [3]. Combining the circle and sieve methods and using the Bombieri – Vinogradov prime number theorem they proved (2) with right-hand side $[\log(\max p_j)]^{-A}, A > 1$ and primes $p_1, p_2, p_3$ such that $p_i + 2 = P_8, i = 1, 2, 3$. In this paper we improve the right-hand side of [3]. Obviously this is at the expense of the number of the prime factors of $p_i + 2$.

## 2 Notations and some lemmas.

For positive $A$ and $B$ we write $A \asymp B$ instead of $A \ll B \ll A$. As usual $\varphi(n)$ and $\mu(n)$ denote Euler’s function and Möbius’ function. Let $(m_1, m_2)$ and $[m_1, m_2]$ be the greatest common divisor and the least common multiple of $m_1, m_2$ respectively. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. As usual, $[y]$ denotes the integer part of $y$, $e(y) = e^{2\pi iy}$. The letter $\varepsilon$ denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write $x^\varepsilon \log x \ll x^\varepsilon$. Since $\lambda_1/\lambda_2$ is irrational, there are infinitely many different convergents $a_0/q_0$ to its continued fraction, with

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}$$

where $(a_0, q_0) = 1, q_0 \geq 1$ and $a_0 \neq 0$. We choose $q_0$ to be large in terms of $\lambda_1, \lambda_2, \lambda_3$ and $\eta$, and make the following definitions.
\( X = q_0^{12/5} ; \)  
\( \tau = X^{-5/6 \log X} ; \)  
\( \vartheta = X^{-1/12 + \delta}, \ \delta > 0 ; \)  
\( H = \frac{\log^2 X}{\vartheta} ; \)  
\( z = X^\beta, \ 0 < \beta < 1/30 ; \)  
\( D = X^{47/450 - \varepsilon_0}, \ \varepsilon_0 = 0.001 ; \)  
\( P(z) = \prod_{2 < p \leq z} \ p, \ \text{p\ -prime number} ; \)  
\( I(\alpha) = \int_{\lambda_0 X}^{X} e(\alpha y) dy . \)  

The value of \( \beta \) will be specified latter.

Let \( \lambda^\pm(d) \) be the lower and upper bounds Rosser’s weights of level \( D \), hence

\[ |\lambda^\pm(d)| \leq 1, \ \lambda^\pm(d) = 0 \text{ if } d \geq D \text{ or } \mu(d) = 0 . \tag{12} \]

For further properties of Rosser’s weights we refer to [5], [6].

**Lemma 1.** Let \( \vartheta \in \mathbb{R} \) and \( k \in \mathbb{N} \). There exists a function \( \theta(y) \) which is \( k \) times continuously differentiable and such that

\[ \theta(y) = 1 \quad \text{for} \quad |y| \leq 3\vartheta/4 ; \]

\[ 0 \leq \theta(y) < 1 \quad \text{for} \quad 3\vartheta/4 < |y| < \vartheta ; \]

\[ \theta(y) = 0 \quad \text{for} \quad |y| \geq \vartheta . \]

and its Fourier transform

\[ \Theta(x) = \int_{-\infty}^{\infty} \theta(y)e(-xy)dy \]

satisfies the inequality

\[ |\Theta(x)| \leq \min \left( \frac{7\vartheta}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{k}{2\pi|x|\vartheta/8} \right)^k \right) . \]

**Proof.** See [10].

**Lemma 2.** Let \( X \geq 2, k \geq 2 \). We have

\[ \sum_{n \leq X} \frac{1}{\varphi(n)} < \log X . \]
3 Outline of the proof.

Consider the sum

\[ \Gamma(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X, |\lambda_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \theta} \log p_1 \log p_2 \log p_3. \]  

(13)

Any non-trivial estimate from below of \( \Gamma(X) \) implies solvability of \( |\lambda_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \theta \) in primes such that \( p_i + 2 = P_h, h = [\beta^{-1}] \).

We have

\[ \Gamma(X) \geq \tilde{\Gamma}(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X, \theta(p_i, z) = 1, i = 1, 2, 3} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \log p_1 \log p_2 \log p_3. \]  

(14)

On the other hand

\[ \tilde{\Gamma}(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \Lambda_1 \Lambda_2 \Lambda_3 \log p_1 \log p_2 \log p_3, \]  

(15)

where

\[ \Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d), i = 1, 2, 3 \]

(16)

We denote

\[ \Lambda_i^\pm = \sum_{d|(p_i+2, P(z))} \lambda^\pm(d), i = 1, 2, 3 \]

From the linear sieve we know that \( \Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+ \) (see [2], Lemma 10). Then we have a simple inequality

\[ \Lambda_1 \Lambda_2 \Lambda_3 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \]  

(17)

analogous to this one in [2], Lemma 13).

Using (15) and (17) we obtain

\[ \tilde{\Gamma}(X) \geq \Gamma_0(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \times (\Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+) \log p_1 \log p_2 \log p_3. \]  

(18)

Let

\[ \Gamma_0(X) = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X) - 2\Gamma_4(X), \]  

(19)
where for example
\[ \Gamma_1(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \log p_1 \log p_2 \log p_3 \] (20)
and so on. We shall consider the sum \( \Gamma_1(X) \). The rest can be considered in the same way.

From (16) and (20) we have
\[ \Gamma_1(X) = \sum_{d_1 \mid P(z)} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \log p_1 \log p_2 \log p_3. \]

Using the inverse Fourier transform for the function \( \theta(x) \) we get
\[
\Gamma_1(X) = \sum_{d_1 \mid P(z)} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \\
\times \int_{-\infty}^{\infty} \Theta(t) e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t) dt \\
= \int_{-\infty}^{\infty} \Theta(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt,
\]
where
\[ L^\pm (t, X) = \sum_{d_1 \mid P(z)} \lambda^\pm (d) \sum_{\lambda_0 X < p \leq X} e(pt) \log p. \] (21)

We divide \( \Gamma_1(X) \) into three parts
\[ \Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X), \] (22)
where
\[ \Gamma_1^{(1)}(X) = \int_{|t| \leq \tau} \Theta(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt, \] (23)
\[ \Gamma_1^{(2)}(X) = \int_{\tau < |t| < H} \Theta(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt, \] (24)
\[ \Gamma_1^{(3)}(X) = \int_{|t| \geq H} \Theta(t) e(\eta t) L^- (\lambda_1 t, X) L^+ (\lambda_2 t, X) L^+ (\lambda_3 t, X) dt. \] (25)

We shall estimate \( \Gamma_1^{(3)}(X), \Gamma_1^{(1)}(X), \Gamma_1^{(2)}(X) \) respectively in the sections 4, 5, 6. In section 7 we shall complete the proof of the Theorem.
4 Upper bound for $\Gamma_1^{(3)}(X)$.

Lemma 3. For the integral $\Gamma_1^{(3)}(X)$, defined by (25), we have

$$\Gamma_1^{(3)}(X) \ll 1.$$  \hfill (26)

Proof. See [3], Lemma 2.

5 Asymptotic formula for $\Gamma_1^{(1)}(X)$.

The first lemma we need in this section is the following.

Lemma 4. Let $\lambda \neq 0$. Using the definitions (11) and (21) we have

(i) \( \tau \int_{-\tau}^{\tau} |L^\pm(\lambda\alpha, X)|^2 d\alpha \ll X \log X \),

(ii) \( \tau \int_{-\tau}^{\tau} |I(\lambda\alpha)|^2 d\alpha \ll X \log X \).

Proof. We only prove (i). The inequality (ii) can be proved likewise.

Having in mind (5), (12) and (21) we get

\[
\int_{-\tau}^{\tau} |L^\pm(\lambda\alpha, X)|^2 d\alpha = \sum_{d_i \mid P(\tau)} \lambda^\pm(d_1) \lambda^\pm(d_2) \\
\times \sum_{\lambda \neq p_1, p_2 \leq X} \log p_1 \log p_2 \int_{-\tau}^{\tau} e(\lambda(p_1 - p_2)\alpha) d\alpha \\
\ll \sum_{d_i \leq D} \sum_{\lambda \neq p_1, p_2 \leq X} \log p_1 \log p_2 \min\left(\tau, \frac{1}{|p_1 - p_2|}\right) \\
\ll \left(\log X\right)^2 \sum_{d_i \leq D} (U\tau + V),
\]

\hfill (27)
where

\[ U = \sum_{\lambda_0 X < n_1, n_2 \leq X \atop n_1 + 2 \equiv 0 (d_1)} \frac{1}{|n_1 - n_2|}, \quad V = \sum_{\lambda_0 X < n_1, n_2 \leq X \atop n_2 + 2 \equiv 0 (d_2)} \frac{1}{|n_1 - n_2|}. \]

We have

\[ U \ll \sum_{\lambda_0 X < n_1 \leq X \atop n_1 + 2 \equiv 0 (d_1)} \sum_{\lambda_0 X < n_2 \leq X \atop n_2 + 2 \equiv 0 (d_2)} \frac{1}{|n_1 - n_2|} \ll \sum_{\lambda_0 X < n_1 \leq \frac{X}{n_1 + 2 \equiv 0 (d_1)}} \frac{1}{\tau d_2} \sum_{\lambda_0 X < n_1 \leq X \atop n_1 + 2 \equiv 0 (d_1)} \frac{X}{\tau d_1 d_2}. \quad (28) \]

Obviously \( V \leq \sum_l V_l \) where

\[ V_l = \sum_{\lambda_0 X < n_1, n_2 \leq X \atop l < |n_1 - n_2| \leq 2l} \frac{1}{|n_1 - n_2|} \]

and \( l \) takes the values \( 2^d / \tau, d = 0, 1, 2, \ldots \), with \( l \leq X \).

We have

\[ V_l \ll \frac{1}{l} \sum_{\lambda_0 X < n_1 \leq X \atop l < |n_1 - n_2| \leq 2l} \sum_{\lambda_0 X < n_2 \leq X \atop n_2 + 2 \equiv 0 (d_2)} \frac{1}{l} \sum_{\lambda_0 X < n_1 \leq X \atop n_1 + 2 \equiv 0 (d_1)} \frac{l}{d_2} \ll \frac{1}{d_2} \sum_{\lambda_0 X < n_1 \leq X \atop n_1 + 2 \equiv 0 (d_1)} \frac{X}{d_1 d_2}. \quad (30) \]

The assertion in (i) follows from (5), (27) – (30).

The next lemma gives us asymptotic formula for the sums \( L^\pm (\alpha, X) \) denoted by (21).

**Lemma 5.** Let \( D \) is defined by (2), and \( \lambda(d) \) be complex numbers defined for \( d \leq D \) such that

\[ |\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d) = 0. \quad (31) \]

If

\[ L(\alpha, X) = \sum_{d \leq D} \lambda(d) \sum_{\lambda_0 X < p \leq X \atop p + 2 \equiv 0(d)} e(p\alpha) \log p \]
then for $|\alpha| \leq \tau$ we have

$$L(\alpha, X) = I(\alpha) \sum_{d \leq D} \lambda(d) \varphi(d) + \mathcal{O}\left(\frac{X}{(\log X)^A}\right),$$  \hspace{1cm} (32)$$

where $A > 0$ is an arbitrary large constant.

Proof. This lemma is very similar to results of Tolev [12]. Inspecting the arguments presented in [12, Lemma 10], the reader will easily see that the proof of Lemma 5 can be obtained by the same manner. \hfill \Box

Let

$$L_i^\pm = L^\pm(\lambda_i t, X),$$
$$\mathcal{M}_i^\pm = \mathcal{M}^\pm(\lambda_i t, X) = I(\lambda_i t) \sum_{d \leq D} \frac{\lambda^\pm(d)}{\varphi(d)}. \hspace{1cm} (33)$$

We use the identity

$$L_i^\pm L_j^\pm L_k^\pm = \mathcal{M}_i^\pm \mathcal{M}_j^\pm \mathcal{M}_k^\pm + (L_i^\pm - \mathcal{M}_i^\pm) \mathcal{M}_j^\pm \mathcal{M}_k^\pm + L_i^\pm (L_j^\pm - \mathcal{M}_j^\pm) \mathcal{M}_k^\pm + L_i^\pm L_j^\pm (L_k^\pm - \mathcal{M}_k^\pm). \hspace{1cm} (34)$$

Replace

$$J_1 = \int_{|t| \leq \tau} \Theta(t)e(\eta t)\mathcal{M}^-(\lambda_1 t, X)\mathcal{M}^+(\lambda_2 t, X)\mathcal{M}^+(\lambda_3 t, X)dt. \hspace{1cm} (35)$$
Then from (21), (23), (33), (34), (35), Lemma 1 and Lemma 5 we obtain
\[
\Gamma_1^{(1)}(X) - J_1 = \int_{|t| \leq \tau} \Theta(t) e(\eta t) \left( L^-(\lambda_1 t, X) - M^-(\lambda_1 t, X) \right) M^+(\lambda_2 t, X) M^+(\lambda_3 t, X) dt
\]
\[
+ \int_{|t| \leq \tau} \Theta(t) e(\eta t) L^-(\lambda_1 t, X) \left( L^-(\lambda_2 t, X) - M^-(\lambda_2 t, X) \right) M^+(\lambda_3 t, X) dt
\]
\[
+ \int_{|t| \leq \tau} \Theta(t) e(\eta t) L^-(\lambda_1 t, X) L^+(\lambda_2 t, X) \left( L^+(\lambda_3 t, X) - M^+(\lambda_3 t, X) \right) dt
\]
\[
\ll \vartheta \frac{X}{(\log X)^A} \left( \int_{|t| \leq \tau} |M^+(\lambda_2 t, X) M^+(\lambda_3 t, X)| dt \right)
\]
\[
+ \int_{|t| \leq \tau} |L^-(\lambda_1 t, X) M^+(\lambda_3 t, X)| dt + \int_{|t| \leq \tau} |L^-(\lambda_1 t, X) L^+(\lambda_2 t, X)| dt
\]
\[
\ll \vartheta \frac{X}{(\log X)^A} \left( \int_{|t| \leq \tau} |M^+(\lambda_2 t, X)|^2 dt + \int_{|t| < \tau} |M^+(\lambda_3 t, X)|^2 dt \right)
\]
\[
+ \int_{|t| \leq \tau} |L^-(\lambda_1 t, X)|^2 dt + \int_{|t| \leq \tau} |L^+(\lambda_2 t, X)|^2 dt . \quad (36)
\]

On the other hand (33) and Lemma 2 give us
\[
|M^\pm(\lambda t, X)| \ll |I(\lambda t)| \log X . \quad (37)
\]

Bearing in mind (36), (37) and Lemma 4 we find
\[
\Gamma_1^{(1)}(X) - J_1 \ll \vartheta \frac{X^2}{(\log X)^{A-5}} . \quad (38)
\]

Arguing as in [3] for the integral defined by (35) we get
\[
J_1 = B(X) \left( \sum_{d \mid P(z)} \frac{\lambda^-(d)}{\varphi(d)} \right) \left( \sum_{d \mid P(z)} \frac{\lambda^+(d)}{\varphi(d)} \right)^2 + O(\vartheta \tau^{-2} \log^3 X) , \quad (39)
\]

where
\[
B(X) = \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \theta(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) dy_1 dy_2 dy_3 .
\]
According to ([3], Lemma 4) we have

\[ B(X) \gg \vartheta X^2. \]  

(40)

Usung (5), (6), (38) and (39) we obtain

\[ \Gamma_1^{(1)}(X) = B(X) \left( \sum_{d \mid P(z)} \frac{\lambda^{-}(d)}{\varphi(d)} \right) \left( \sum_{d \mid P(z)} \frac{\lambda^{+}(d)}{\varphi(d)} \right)^2 + \mathcal{O} \left( \vartheta \frac{X^2}{(\log X)^{4-\varepsilon}} \right). \]  

(41)

Let

\[ G^\pm = \sum_{d \mid P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}. \]  

(42)

Thus from (41) and (42) it follows

\[ \Gamma_1^{(1)}(X) = B(X) G^-(G^+)^2 + \mathcal{O} \left( \vartheta \frac{X^2}{(\log X)^{4-\varepsilon}} \right). \]  

(43)

6 Upper bound for \( \Gamma_1^{(2)}(X) \).

The treatment of the intermediate region depends on the following lemma.

**Lemma 6.** Suppose \( \alpha \in \mathbb{R} \) with a rational approximation \( \frac{a}{q} \) satisfying \( \left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2} \); where \( (a, q) = 1, q \geq 1, a \neq 0 \). Let \( D \) is defined by (9), and \( \omega(d) \) be complex numbers defined for \( d \leq D \) and let \( \omega(d) \ll 1 \). If

\[ \mathcal{L}(X) = \sum_{d \leq D} \omega(d) \sum_{\substack{X/2 < p \leq X \\text{p is prime}}} e(p\alpha \log p) \]  

(44)

then

\[ \mathcal{L}(X) \ll X^\varepsilon \left( X^{11/12} + \frac{X}{q^{1/2}} + X^{1/2}q^{1/2} + q \right), \]

where \( \varepsilon \) is an arbitrary small positive number.

**Proof.** See [11], Lemma 1. \( \square \)

Let us consider any sum \( L^\pm(\alpha, X) \) denoted by (21). We represent it as sum of finite number sums of the type

\[ L(\alpha, Y) = \sum_{d \leq D} \omega(d) \sum_{\substack{Y/2 < p \leq Y \\text{p is prime}}} e(p\alpha \log p), \]
where
\[ \omega(d) = \begin{cases} \lambda(d), & \text{if } d \mid P(z), \\ 0, & \text{otherwise.} \end{cases} \]

We have
\[ L^\pm(\alpha, X) \ll \max_{\lambda_0 X \leq Y \leq X} |L(\alpha, Y)|. \]

If
\[ q \in \left[ X^{1/6}, X^{5/6} \right], \]
then from Lemma 6 for the sums \( L(\alpha, Y) \) we get
\[ L(\alpha, Y) \ll Y^{11/12+\varepsilon}. \]

Therefore
\[ L^\pm(\alpha, X) \ll \max_{\lambda_0 X \leq Y \leq X} Y^{11/12+\varepsilon} \ll X^{11/12+\varepsilon}. \] (46)

Let
\[ V(t, X) = \min \{|L^\pm(\lambda_1 t, X)|, |L^\pm(\lambda_2 t, X)|\}. \] (47)

We shall prove the following

**Lemma 7.** Let \( t, X, \lambda_1, \lambda_2 \in \mathbb{R}, \)
\[ |t| \in (\tau, H), \]
where \( \tau \) and \( H \) are denoted by (4) and (7), \( \lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q} \) and \( V(t, X) \) is defined by (47). Then there exists a sequence of real numbers \( X_1, X_2, \ldots \to \infty \) such that
\[ V(t, X_j) \ll X_j^{11/12+\varepsilon}, \quad j = 1, 2, \ldots. \] (49)

**Proof.** Our aim is to prove that there exists a sequence \( X_1, X_2, \ldots \to \infty \) such that for each \( j = 1, 2, \ldots \) at least one of the numbers \( \lambda_1 t \) and \( \lambda_2 t \) with \( t \), subject to (48) can be approximated by rational numbers with denominators, satisfying (45). Then the proof follows from (46) and (47).

Let \( q_0 \) be sufficiently large and \( X \) be such that \( X = q_0^{12/5} \) (see (41)). Let us notice that there exist \( a_1, q_1 \in \mathbb{Z} \), such that
\[ \left| \lambda_1 t - \frac{a_1}{q_1} \right| < \frac{1}{q_1 q_0^2}, \quad (a_1, q_1) = 1, \quad 1 \leq q_1 \leq q_0^2, \quad a_1 \neq 0. \] (50)

From Dirichlet’s Theorem ([7], p.158) it follows the existence of integers \( a_1 \) and \( q_1 \), satisfying the first three conditions. If \( a_1 = 0 \) then \( |\lambda_1 t| < \frac{1}{q_1 q_0^2} \) and from (48) it follows
\[ \lambda_1 \tau < \lambda_1 |t| < \frac{1}{q_0^2}, \quad q_0^2 < \frac{1}{\lambda_1 \tau}. \]
From the last inequality, (41) and (50) we obtain

\[ X^{5/6} < \frac{X^{5/6}}{\lambda_1 \log X}, \]

which is impossible for large \( q_0 \), respectively, for a large \( X \). So \( a_1 \neq 0 \). By analogy there exist \( a_2, q_2 \in \mathbb{Z} \), such that

\[ \left| \lambda_2 t - \frac{a_2}{q_2} \right| < \frac{1}{q_2 q_0}, \quad (a_2, q_2) = 1, \quad 1 \leq q_2 \leq q_0^2, \quad a_2 \neq 0. \quad (51) \]

If \( q_i \in \left[ X^{1/6}, \frac{X^{5/6}}{2} \right] \) for \( i = 1 \) or \( i = 2 \), then the proof is completed. From (41), (50) and (51) we have

\[ q_i \leq X^{5/6} = q_0^2, \quad i = 1, 2. \]

Thus it remains to prove that the case

\[ q_i < X^{1/6}, \quad i = 1, 2 \quad (52) \]

is impossible. Let \( q_i < X^{1/6}, i = 1, 2 \). From (7), (18), (50) – (52) it follows

\[ 1 \leq |a_i| < \frac{1}{q_0^2} + q_i \lambda_i |t| < \frac{1}{q_0^2} + q_i \lambda_i H, \]

\[ 1 \leq |a_i| < \frac{1}{q_0^2} + \lambda_i X^{1/4-\delta} \log^2 X, \quad i = 1, 2. \quad (53) \]

We have

\[ \lambda_1 \lambda_2 = \frac{\lambda_1 t}{\lambda_2 t} = \frac{a_1}{q_1} + \left( \lambda_1 t - \frac{a_1}{q_1} \right) = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \Xi_1}{1 + \Xi_2}, \quad (54) \]

where \( \Xi_i = \frac{q_i}{a_i} \left( \lambda_i t - \frac{a_i}{q_i} \right), \quad i = 1, 2 \). According to (50), (51) and (51) we obtain

\[ |\Xi_i| < \frac{q_i}{|a_i|} \frac{1}{|q_0^2 q_i^2|} < \frac{1}{q_0^2}, \quad i = 1, 2, \]

\[ \lambda_1 \lambda_2 = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathcal{O} \left( \frac{1}{q_0^2} \right)}{1 + \mathcal{O} \left( \frac{1}{q_0^2} \right)} = \frac{a_1 q_2}{a_2 q_1} \left( 1 + \mathcal{O} \left( \frac{1}{q_0^2} \right) \right). \]

Thus \( \frac{a_1 q_2}{a_2 q_1} = \mathcal{O}(1) \) and

\[ \frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + \mathcal{O} \left( \frac{1}{q_0^2} \right). \quad (55) \]
Therefore, both fractions $\frac{a_0}{q_0}$ and $\frac{a_1q_2}{a_2q_1}$ approximate $\frac{\lambda_1}{\lambda_2}$. Using (50), (52) and inequality (53) with $i = 2$ we obtain

$$|a_2|q_1 < 1 + \lambda_i X^{5/12 - \delta} \log^2 X < \frac{q_0}{\log X}$$  \hspace{1cm} (56)

so $|a_2|q_1 \neq q_0$ and the fractions $\frac{a_0}{q_0}$ and $\frac{a_1q_2}{a_2q_1}$ are different. Then using (56) it follows

$$\left| \frac{a_0 - a_1q_2}{q_0 - a_2q_1} \right| = \frac{|a_0a_2q_1 - a_1q_2q_0|}{|a_2|q_1q_0} \geq \frac{1}{|a_2|q_1q_0} > \frac{\log X}{q_0^2}. \hspace{1cm} (57)$$

On the other hand, from (3) and (55) we have

$$\left| \frac{a_0 - a_1q_2}{q_0 - a_2q_1} \right| \leq \left| \frac{a_0}{q_0} - \frac{\lambda_1}{\lambda_2} \right| + \left| \frac{\lambda_1}{\lambda_2} - \frac{a_1q_2}{a_2q_1} \right| \ll \frac{1}{q_0},$$

which contradicts (57). This rejects the assumption (52). Let $q_0^{(1)}, q_0^{(2)}, \ldots$ be an infinite sequence of values of $q_0$, satisfying (3). Then using (4) one gets an infinite sequence $X_1, X_2, \ldots$ of values of $X$, such that at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ can be approximated by rational numbers with denominators, satisfying (45). Hence, the proof is completed.

Let us estimate the integral $\Gamma_1^{(2)}(X_j)$, denoted by (24). Using (47), (49) and Lemma 1 we find

$$\Gamma_1^{(2)}(X_j) \ll \vartheta \int_{\tau < |t| < H} V(t, X_j) \left( |L^- (\lambda_1 t, X_j) L^+ (\lambda_3 t, X_j)| + |L^+ (\lambda_2 t, X_j) L^+ (\lambda_3 t, X_j)| \right) dt$$

$$\ll \vartheta \int_{\tau < |t| < H} V(t, X_j) \left( |L^- (\lambda_1 t, X_j)|^2 + |L^+ (\lambda_2 t, X_j)|^2 + |L^+ (\lambda_3 t, X_j)|^2 \right) dt$$

$$\ll \vartheta X_j^{11/12 + \varepsilon} \max_{1 \leq k \leq 3} \mathcal{I}_k,$$  \hspace{1cm} (58)

where

$$\mathcal{I}_k = \int_{\tau}^H |L^\pm (\lambda_k t, X_j)|^2 \, dt.$$  

Arguing as in [3] we obtain

$$\mathcal{I}_k \ll X_j^{13/12 - \delta} (\log X_j)^7.$$  \hspace{1cm} (59)

Using (58), (59) and choosing $\varepsilon < \delta$ we get

$$\Gamma_1^{(2)}(X_j) \ll \vartheta X_j^{11/12 + \varepsilon} X_j^{13/12 - \delta} (\log X_j)^7 \ll \vartheta \frac{X_j^2}{(\log X_j)^{A-5}}.$$  \hspace{1cm} (60)
Summarizing (22), (26), (43) and (60) we find
\[ \Gamma_1(X_j) = B(X_j)G^-(G^+)^2 + \mathcal{O}\left(\vartheta \frac{X_j^3}{(\log X_j)^{4-5}}\right). \]  

(61)

7 Proof of the Theorem.

Since the sums \(\Gamma_2(X_j), \Gamma_3(X_j)\) and \(\Gamma_4(X_j)\) are estimated in the same way then from (13), (14), (18), (19) and (61) we obtain
\[ \Gamma(X_j) \geq B(X_j)W(X_j) + \mathcal{O}\left(\vartheta \frac{X_j^2}{(\log X_j)^{4-5}}\right), \]  

(62)

where
\[ W(X_j) = 3 \left(G^+\right)^2 \left(G^- - \frac{2}{3}G^+\right) \]  

(63)

and \(G^\pm\) are defined by (42).

Let \(f(s)\) and \(F(s)\) are the lower and the upper functions of the linear sieve. We know that if
\[ s = \frac{\log D}{\log z}, \quad 2 \leq s \leq 3 \]  

(64)

then
\[ f(s) = \frac{2e^\gamma \log(s-1)}{s}, \quad F(s) = \frac{2e^\gamma}{s} \]  

(65)

where \(\gamma = 0.577...\) is the Euler constant (see Lemma 10, [2]). Using (42) and Lemma 10 [1] we get
\[ F(z) \left(f(s) + \mathcal{O}\left((\log X)^{-1/3}\right)\right) \leq G^- \leq F(z) \leq G^+ \leq F(z) \left(F(s) + \mathcal{O}\left((\log X)^{-1/3}\right)\right). \]  

(66)

Here
\[ F(z) = \prod_{2 < p \leq z} \left(1 - \frac{1}{p-1}\right) \asymp \frac{1}{\log X}. \]  

(67)

To estimate \(W(X_j)\) from below we shall use the inequalities (see (66)):
\[ G^- - \frac{2}{3}G^+ \geq F(z) \left(f(s) - \frac{2}{3}F(s) + \mathcal{O}\left((\log X)^{-1/3}\right)\right) \]  

(68)
Let $X = X_j$. Then from (63) and (68) it follows

$$W(X_j) \geq 3F^3(z) \left( f(s) - \frac{2}{3} F(s) + O((\log X)^{-1/3}) \right)$$

(69)

Choose $s = 2.948$.

Then by (8), (9) and (64) we find

$$\beta = 0.035089.$$  

It is not difficult to compute that for sufficiently large $X$ we have

$$f(s) - \frac{2}{3} F(s) > 10^{-5}.$$  

(70)

Choose $A \geq 10$.

Then by (3), (10), (62), (67), (69) and (70) we obtain:

$$\Gamma(X_j) \gg \frac{X_j^{23/12+\delta}}{((\log X_j)^3}.$$  

(71)

The last inequality implies that $\Gamma(X_j) \to \infty$ as $X_j \to \infty$.

By the definition (13) of $\Gamma(X)$ and the inequality (71) we conclude that for some constant $c_0 > 0$ there are at least $c_0 X_j^{23/12+\delta}$ triples of primes $p_1, p_2, p_3 \leq X_j$, $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \delta$ such that for every prime factor $p$ of $p_j + 2$, $j = 1, 2, 3$ we have $p \geq X_0.035089$.

The proof of the Theorem is complete.

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