Eigenvalues of a $H$-generalized join graph operation constrained by vertex subsets

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Abstract

Considering a graph $H$ of order $p$, a generalized $H$-join operation of a family of graphs $G_1, \ldots, G_p$, constrained by a family of vertex subsets $S_i \subseteq V(G_i)$, $i = 1, \ldots, p$, is introduced. When each vertex subset $S_i$ is $(k_i, \tau_i)$-regular, it is deduced that all non-main adjacency eigenvalues of $G_i$, different from $k_i - \tau_i$, for $i = 1, \ldots, p$, remain as eigenvalues of the graph $G$ obtained by the above mentioned operation. Furthermore, if each graph $G_i$ of the family is $k_i$-regular, for $i = 1, \ldots, p$, and all the vertex subsets are such that $S_i = V(G_i)$, the $H$-generalized join operation constrained by these vertex subsets coincides with the $H$-generalized join operation. Some applications on the spread of graphs are presented. Namely, new lower and upper bounds are deduced and an infinity family of non regular graphs of order $n$ with spread equals $n$ is introduced.

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1 Notation and main concepts

We deal with undirected simple graphs herein simply called graphs. For each graph $G$, the vertex set is denoted by $V(G)$ and its edge set by $E(G)$. Usually, we consider that the graph $G$ has order $n$, that is $V(G) = \{1, \ldots, n\}$. An edge with end vertices $i$ and $j$ is denoted by $ij$ and then we say that the vertices $i$ and $j$ are adjacent or neighbors. The number of neighbors of a vertex $i$ is the degree of $i$ and the neighborhood of $i$ is the set of its neighbors, $N_G(i) = \{j \in V(G) : ij \in E(G)\}$. The maximum and minimum degree of the vertices of $G$ is denoted by $\Delta(G)$ and $\delta(G)$, respectively. The complement of $G$, denoted by $\overline{G}$ is such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{ij : ij \notin E(G)\}$. A path of length $p-1$, $P_p$, in $G$ is a sequence of vertices $i_1, \ldots, i_p$ all distinct except, eventually the first and the last) and such that $i_ji_{j+1} \in E(G)$, for $j = 1, \ldots, p-1$. If $i_1 = i_p$, then it is a closed path usually called cycle of length $p$ and denoted $C_p$.

A graph $G$ is connected if there is a path between each pair of distinct vertices. A complete graph of order $n$, where each pair of distinct vertices is connected by an edge, is denoted by $K_n$. The complement of $K_n$, $\overline{K_n}$, is called the null graph. A graph $G$ is bipartite if $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end vertex in $V_1$ and the other one in $V_2$. This graph $G$ is called complete bipartite and it is denoted $K_{p,q}$, if $|V_1| = p$, $|V_2| = q$ and each vertex of $V_1$ is connected with every vertex of $V_2$.

The adjacency matrix of a graph $G$, $A(G) = (a_{i,j})$, is the $n \times n$ matrix

$$a_{i,j} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix $A(G)$ is a nonnegative symmetric with entries which are 0 and 1 and then all of its eigenvalues are real. Furthermore, since all its diagonal entries are equal to 0, the trace of $A(G)$ is zero. If $G$ has at least one edge, then $A(G)$ has a negative eigenvalue not greater than $-1$ and a positive eigenvalue not less than the average degree of the vertices of $G$.

Considering any matrix $M$ we denote its spectrum (the multiset of the eigenvalues of $M$) by $\sigma(M)$. The spectrum of the adjacency matrix of a graph $G$, $\sigma(A(G))$, is simply denoted by $\sigma(G)$ and the eigenvalues of $A(G)$ are also called the eigenvalues of $G$. An eigenvalue $\lambda$ of a graph $G$ is called non-main if its associated eigenspace, denoted $\varepsilon_G(\lambda)$, is orthogonal to the all one vector, otherwise is called main.

Usually, the multiplicities of the eigenvalues are represented in the multiset $\sigma(G)$ as powers in square brackets. For instance, $\sigma(G) = \{\lambda_1^{[m_1]}, \ldots, \lambda_q^{[m_q]}\}$ denotes that $\lambda_1$ has multiplicity $m_1$, $\lambda_2$ has multiplicity $m_2$, and so on. Throughout the paper, the eigenvalues of a graph $G$ with $n$ vertices, $\lambda_1(G), \ldots, \lambda_n(G)$, are ordered as follows: $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. If $\lambda$ is an eigenvalue of the graph $G$ and $u$ is an associated eigenvector, the pair $(\lambda, u)$ is called an eigenpair of $G$.

Considering a graph $G$ of order $n$ and a vertex subset $S \subseteq V(G)$, the characteristic vector of $S$ is the vector $x_S \in \{0, 1\}^n$ such that $(x_S)_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise.} \end{cases}$

A vertex subset $S$ is $(k, \tau)$-regular if $S$ induces a $k$-regular graph in $G$ and every
vertex out of \( S \) has \( \tau \) neighbors in \( S \), that is,
\[
|N_G(i) \cap S| = \begin{cases} 
  k & \text{if } i \in S \\
  \tau & \text{otherwise.}
\end{cases}
\]

When the graph \( G \) is \( k \)-regular, for convenience, \( S = V(G) \) is considered \( (k, 0) \)-regular. There are several properties of graphs related with \( (k, \tau) \)-regular sets (see [2, 3]). For instance, we may refer the following properties:

- A graph \( G \) has a perfect matching if and only if its line graph has a \((0, 2)\)-regular set.
- A graph \( G \) is Hamiltonian if and only if its line graph has a \((2, 4)\)-regular set inducing a connected graph.
- A graph \( G \) of order \( n \) is strongly regular with parameters \((n, p, a, b)\) if and only if \( \forall v \in V(G) \) the vertex subset \( S = N_G(v) \) is \((a, b)\)-regular in \( G - v \) (where \( G - v \) is the graph obtained from \( G \) deleting the vertex \( v \)).

\section{Generalized join graph operation with vertex subset constraints}

Considering two vertex disjoint graphs \( G_1 \) and \( G_2 \), the join of \( G_1 \) and \( G_2 \) is the graph \( G_1 \lor G_2 \) such that \( V(G_1 \lor G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \lor G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1) \land y \in V(G_2)\} \). A generalization of the join operation was first introduced in [10] under the designation of generalized composition and more recently in [1] with the designation of \( H \)-join, defined as follows:

Consider a family of \( p \) graphs, \( \mathcal{F} = \{G_1, \ldots, G_p\} \), where each graph \( G_j \) has order \( n_j \), for \( j = 1, \ldots, p \), and a graph \( H \) such that \( V(H) = \{1, \ldots, p\} \). Each vertex \( j \in V(H) \) is assigned to the graph \( G_j \in \mathcal{F} \). The \( H \)-join (generalized composition) of \( G_1, \ldots, G_p \) is the graph \( G = \bigvee_{H} \{G_j : j \in V(H)\} \) \( (H[G_1, \ldots, G_p]) \) such that \( V(G) = \bigcup_{j=1}^{p} V(G_j) \) and

\[
E(G) = \left( \bigcup_{j=1}^{p} E(G_j) \right) \cup \left( \bigcup_{rs \in E(H)} \{uv : u \in V(G_r), v \in V(G_s)\} \right).
\]

Now, we generalize the above \( H \)-join operation according to the next definition.

\textbf{Definition 1} Consider a graph \( H \) of order \( p \) and a family of \( p \) graphs \( \mathcal{F} = \{G_1, \ldots, G_p\} \). Consider also a family of vertex subsets \( S = \{S_1, \ldots, S_p\} \), such that \( S_i \subseteq V(G_i) \) for \( i = 1, \ldots, p \). The \( H \)-generalized join operation of the family of graphs \( \mathcal{F} \) constrained by the family of vertex subsets \( S \), denoted by \( \bigvee_{(H,S)} \mathcal{F} \),
produces a graph such that

$$V\left(\bigvee_{(H,S)}\mathcal{F}\right) = \bigcup_{i=1}^{p} V(G_i),$$

$$E\left(\bigvee_{(H,S)}\mathcal{F}\right) = \left(\bigcup_{i=1}^{p} E(G_i)\right) \cup \{xy : x \in S_i, y \in S_j, ij \in E(H)\}.$$  

Notice that the particular case of the $H$-generalized join operation of the family of graphs $\mathcal{F} = \{G_1, \ldots, G_p\}$ constrained by the family of vertex subsets $S = \{V(G_1), \ldots, V(G_p)\}$, coincides with the above described $H$-generalized join operation.

**Example 1** The Figure 1 depicts an example of a $H$-generalized join operation, with $H = P_3$, of a family of graphs $\mathcal{F} = \{G_1, G_2, G_3\}$ constrained by the family of vertex subsets $S = \{S_1, S_2, S_3\}$, where $S_1 = \{a, b\}$, $S_2 = \{d, f\}$, and $S_3 = \{g, i, j\}$. 

![Figure 1: The $H$-generalized join operation of the family of graphs $\mathcal{F} = \{G_1, G_2, G_3\}$, constrained by the family of vertex subsets $S = \{S_1, S_2, S_3\}$, where $S_1 = \{a, b\} \subset V(G_1), S_2 = \{d, f\} \subset V(G_2)$ and $S_3 = \{g, i, j\} \subset V(G_3)$.](image)

Now it is worth to recall the following result.

**Lemma 1** [2] Let $G$ be a graph with a $(\kappa, \tau)$-regular set $S$, where $\tau > 0$, and $\lambda \in \sigma(A(G))$. Then, denoting the characteristic vector of $S$ by $x_S$, $\lambda$ is non-main if and only if

$$\lambda = \kappa - \tau \quad \text{or} \quad x_S \in (\mathcal{E}_G(\lambda))^\perp,$$
where \((E_G(\lambda))^{-1}\) denotes the vector space orthogonal to the eigenspace associated to the eigenvalue \(\lambda\).

From now on, given a graph \(H\), we denote
\[
\delta_{i,j}(H) = \begin{cases} 
1 & \text{if } ij \in E(H) \\
0 & \text{otherwise.}
\end{cases}
\]

**Theorem 1** Consider a graph \(H\) of order \(p\) and a family of \(p\) graphs \(\mathcal{F} = \{G_1, \ldots, G_p\}\) such that \(|V(G_i)| = n_i, i = 1, \ldots, p\). Consider also the family of vertex subsets \(S = \{S_1, \ldots, S_p\}\), where \(S_i \in \{S_i^\prime \subseteq V(G_i) : \text{either } S_i^\prime \text{ or } V(G_i) \setminus S_i^\prime \text{ is } (k_i, \tau_i)-\text{regular for some integers } k_i, \tau_i\}\), for \(i = 1, \ldots, p\). Let \(G = \bigvee_{(H,S)} \mathcal{F}\). If \(\lambda \in \sigma(G_i) \setminus \{k_i - \tau_i\} \text{ for some } i \in \{1, \ldots, p\}\) is non-main, then \(\lambda \in \sigma(G)\).

**Proof.** Denoting \(\delta_{i,j} = \delta_{i,j}(H)\), then \(\delta_{i,j} x_{S_i} x_{S_j}^T\), where \(x_{S_i}\) and \(x_{S_j}\) are the characteristic vectors of \(S_i\) and \(S_j\), respectively, is an \(n_i \times n_j\) matrix whose entries are zero if \(ij \notin E(H)\), otherwise
\[
\left(\delta_{i,j} x_{S_i} x_{S_j}^T\right)_{q,r} = \begin{cases} 
1 & \text{if } q \in S_i \land r \in S_j \\
0 & \text{otherwise.}
\end{cases}
\]

Then the adjacency matrix of \(G\) has the form
\[
A(G) = \begin{pmatrix}
A(G_1) & \delta_{1,2} x_{S_1} x_{S_2}^T & \cdots & \delta_{1,p-1} x_{S_1} x_{S_{p-1}}^T & \delta_{1,p} x_{S_1} x_{S_p}^T \\
\delta_{2,1} x_{S_2} x_{S_1}^T & A(G_2) & \cdots & \delta_{2,p-1} x_{S_2} x_{S_{p-1}}^T & \delta_{2,p} x_{S_2} x_{S_p}^T \\
\delta_{3,1} x_{S_3} x_{S_1}^T & \delta_{3,2} x_{S_3} x_{S_2}^T & \cdots & \delta_{3,p-1} x_{S_3} x_{S_{p-1}}^T & \delta_{3,p} x_{S_3} x_{S_p}^T \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_{p-1,1} x_{S_{p-1}} x_{S_1}^T & \delta_{p-1,2} x_{S_{p-1}} x_{S_2}^T & \cdots & A(G_{p-1}) & \delta_{p-1,p} x_{S_{p-1}} x_{S_p}^T \\
\delta_{p,1} x_{S_p} x_{S_1}^T & \delta_{p,2} x_{S_p} x_{S_2}^T & \cdots & \delta_{p,p-1} x_{S_p} x_{S_{p-1}}^T & A(G_p)
\end{pmatrix}
\]

Let \(u_i\) be an eigenvector of \(A(G_i)\) associated to the non-main eigenvalue \(\lambda_i \neq k_i - \tau_i\), with \(1 \leq i \leq p\). Then,
\[
A(G) = \begin{pmatrix}
0 \\
\vdots \\
0 \\
u_i \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
\delta_{i,1} \left(x_{S_1}^T u_i\right) x_{S_1} \\
\vdots \\
\delta_{i-1,1} \left(x_{S_{i-1}}^T u_i\right) x_{S_{i-1}} \\
\delta_{i,1} \left(x_{S_i}^T u_i\right) x_{S_i} \\
\vdots \\
\delta_{p,1} \left(x_{S_p}^T u_i\right) x_{S_p}
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0 \\
\lambda_i u_i \\
\vdots \\
0
\end{pmatrix},
\]

since \(x_{S_i}\) is the characteristic vector of the vertex subset \(S_i\) and \(S_i \text{ or } V(G_i) \setminus S_i\) is \((k_i, \tau_i)-\text{regular} \) (take into account that \(\lambda_i\) is non-main and then we may apply Lemma \([1]\)).

From the proof of Theorem \([1]\) we may conclude the following corollary.
Corollary 1 Consider a graph $H$ of order $p$ and a family of $p$ graphs $\mathcal{F} = \{G_1, \ldots, G_p\}$ such that $|V(G_i)| = n_i, i = 1, \ldots, p$. Consider also the family of vertex subsets $\mathcal{S} = \{V(G_1), \ldots, V(G_p)\}$. Let $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$. If $\lambda \in \sigma(G_i)$ for some $i \in \{1, \ldots, p\}$ is non-main, then $\lambda \in \sigma(G)$.

Proof. Consider an eigenpair $(\lambda, u)$ of a graph $G_i$, for some $i \in \{1, \ldots, p\}$, where $\lambda$ is non-main. Then, taking into account the equations (11) where, in this case, $x_{S_i}$ is the all one vector, the result follows. \(\blacksquare\)

Notice that in the above corollary, $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$ coincides with the $H$-join operation of the family of graphs $\mathcal{F}$ (generalized composition $H[G_1, \ldots, G_p]$ in the terminology of [10]).

Example 2 Consider the Example [4] where $V(G_1) = \{a, b, c\}$ and $S_1 = \{a, b\}$ is $(1, 1)$-regular, $V(G_2) = \{d, e, f\}$ and $S_2 = \{d, f\}$ is $(0, 2)$-regular, $V(G_3) = \{g, h, i, j\}$ and $S_3 = \{g, i\}$ is $(2, 2)$-regular:

- The eigenpairs of $A(G_1)$ are \(\left(\sqrt{2}, \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)\) and \(\left(-\sqrt{2}, \begin{bmatrix} -\sqrt{2} \\ 1 \\ 1 \end{bmatrix}\right)\).

- The eigenpairs of $A(G_2)$ are \(\left(\sqrt{2}, \begin{bmatrix} \sqrt{2} \\ 2 \\ 2 \end{bmatrix}\right), \left(0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)\) and \(\left(-\sqrt{2}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)\).

- The eigenpairs of $A(G_3)$ are \(\left(\frac{\sqrt{17}}{2}, \begin{bmatrix} \frac{\sqrt{17}}{2} \\ \frac{1 + \sqrt{17}}{2} \\ \frac{1 + \sqrt{17}}{2} \end{bmatrix}\right), \left(0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)\) and \(\left(\frac{\sqrt{17}}{2}, \begin{bmatrix} \frac{1 - \sqrt{17}}{2} \\ \frac{1 - \sqrt{17}}{2} \\ \frac{1 - \sqrt{17}}{2} \end{bmatrix}\right)\).

Let $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$, where $H = P_3$. Then, denoting $\delta_{i,j} = \delta_{i,j}(H)$ and defining the characteristic vectors of the vertex subsets $S_1$, $S_2$ and $S_3$ considering their elements by alphabetic order, we obtain:

\[
A(G) = \begin{pmatrix}
A(G_1) & \delta_{1,2}x_{S_1}x_{S_2}^T & \delta_{1,3}x_{S_1}x_{S_3}^T \\
\delta_{2,1}x_{S_2}x_{S_1}^T & A(G_2) & \delta_{2,3}x_{S_2}x_{S_3}^T \\
\delta_{3,1}x_{S_3}x_{S_1}^T & \delta_{3,2}x_{S_3}x_{S_2}^T & A(G_3)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
A(G_1) & [1 \\ 1 \\ 1] & \text{0}_{3 \times 4} \\
[1 \\ 0 \\ 1] & A(G_2) & [1 \\ 1 \\ 1] \\
\text{0}_{4 \times 3} & [1 \\ 1 \\ 1] & A(G_3)
\end{pmatrix}
\]

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According to Theorem 1, \( \{0, -1\} \subset \sigma(A(G)) \). Notice that \( S_1 \subseteq V(G_1) \) is \((1,1)\)-regular and thus we are not able to get a conclusion about if the eigenvalue 0 of \( A(G_1) \) is or not an eigenvalue of \( A(G) \). On the other hand \( S_2 \subseteq V(G_2) \) is \((0,2)\)-regular and \( S_3 \subseteq V(G_3) \) is \((2,2)\)-regular. In fact,

\[
\sigma(A(G)) = \{4.44999, 1.86239, 0, 0, -1, -1.3822, -1.51442, -3.02546\}.
\]

Consider a graph \( H \) of order \( p \), a family of graphs \( \mathcal{F} = \{G_1, \ldots, G_p\} \), where each graph \( G_i \) has order \( n_i \), and a family of vertex subsets \( \mathcal{S} = \{S_1, \ldots, S_p\} \), where for each \( i \in \{1, \ldots, p\} \), \( S_i \subseteq V(G_i) \). If \( G = \bigcup_{(H,S)} \mathcal{F} \) and \((\lambda, \hat{u})\) is an eigenpair of \( A(G) \), decomposing \( \hat{u} \) such that \( \hat{u} = \left( \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_p \end{array} \right) \), where each \( u_i \) is a subvector of \( \hat{u} \) with \( n_i \) components, then \( \lambda \hat{u} = A(G)\hat{u} \), that is,

\[
\lambda \left( \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_p \end{array} \right) = \left( \begin{array}{c} A(G_1)u_1 + \left( \sum_{j \neq 1} \delta_{1,j} x_{S_i}^T u_j \right) x_{S_1} \\ A(G_2)u_2 + \left( \sum_{j \neq 2} \delta_{2,j} x_{S_2}^T u_j \right) x_{S_2} \\ \vdots \\ A(G_p)u_p + \left( \sum_{j \neq p} \delta_{p,j} x_{S_p}^T u_j \right) x_{S_p} \end{array} \right),
\]

(2)

where \( \delta_{i,j} = \delta_{i,j}(H) \).

Furthermore, if we assume that \( G_i \) is \( d_i \)-regular and \( S_i \) or its complement is \((k_i, \tau_i)\)-regular, for \( i = 1, \ldots, p \), respectively, according to Theorem 1,

\[
\bigcup_{i=1}^{p} (\sigma(G_i) \setminus \{d_i, k_i - \tau_i\}) \subset \sigma(G),
\]

since by one hand, as it is well known, all the eigenvalues of each graph \( G_i \) are non-main but \( d_i \), on the other hand, if a regular graph has a \((k, \tau)\)-regular vertex subset, then \( k - \tau \) is a non-main eigenvalue 2.

Additionally, assuming that \( S_i = V(G_i) \), for \( i = 1, \ldots, p \), the remaining eigenvalues of \( G \) can be computed as follows: let us define \( \tilde{u} \), setting each of its subvectors \( u_i = \theta_i e_{n_i} \), for \( i = 1, \ldots, p \), where each \( e_{n_i} \) is an all one vector with \( n_i \) componentes and \( \theta_1, \ldots, \theta_p \) are scalars. Then the system (2) becomes

\[
\lambda \left( \begin{array}{c} \theta_1 e_{n_1} \\ \theta_2 e_{n_2} \\ \vdots \\ \theta_p e_{n_p} \end{array} \right) = \left( \begin{array}{c} \left( d_1 \theta_1 + \sum_{j \neq 1} \delta_{1,j} \theta_j n_j \right) e_{n_1} \\ \left( d_2 \theta_2 + \sum_{j \neq 2} \delta_{2,j} \theta_j n_j \right) e_{n_2} \\ \vdots \\ \left( d_p \theta_p + \sum_{j \neq p} \delta_{p,j} \theta_j n_j \right) e_{n_p} \end{array} \right).
\]
Therefore, \((\lambda, \hat{u})\) is an eigenpair for \(A(G)\) if and only if \((\lambda, \hat{\theta})\), where \(\hat{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)^T\), is an eigenpair of the matrix

\[
M = \begin{pmatrix}
d_1 & \delta_{1,2}n_2 & \ldots & \delta_{1,p}n_p \\
\delta_{2,1}n_2 & d_2 & \ldots & \delta_{2,p}n_p \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{p,1}n_2 & \delta_{p,2}n_2 & \ldots & d_p
\end{pmatrix}, \tag{3}
\]

that is, \(M\hat{\theta} = \lambda\hat{\theta}\).

Setting \(D = \text{diag}(d_1, \ldots, d_p)\) and \(N = \text{diag}(n_1, \ldots, n_p)\), then \(M = A(H)N + D\) is similar to the symmetric matrix

\[
M' = \begin{pmatrix}
d_1 & \delta_{1,2}\sqrt{n_1n_2} & \ldots & \delta_{1,p}\sqrt{n_1n_p} \\
\delta_{2,1}\sqrt{n_1n_2} & d_2 & \ldots & \delta_{2,p}\sqrt{n_2n_p} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{p,1}\sqrt{n_1n_p} & \delta_{p,2}\sqrt{n_2n_p} & \ldots & d_p
\end{pmatrix}, \tag{4}
\]

since \(M' = KMK^{-1}\) with \(K = \text{diag}(\sqrt{n_1}, \ldots, \sqrt{n_p})\). Therefore, \(M' = D + KA(H)K\) and \(\sigma(M) = \sigma(M')\).

Based on the above analysis, we are able to deduce the following result.

**Theorem 2** Consider a graph \(H\) of order \(p\) and a family of regular graphs \(F = \{G_1, \ldots, G_p\}\), where each regular graph \(G_i\) has degree \(d_i\) and order \(n_i\). Consider the family of vertex subsets \(S = \{S_1, \ldots, S_p\}\), where

\[
S_i \in \{S_i' \subseteq V(G_i) : S_i' \text{ or } V(G_i) \setminus S_i' \text{ is } (k_i, \tau_i) \text{- regular, for some } k_i, \tau_i\},
\]

for \(i = 1, \ldots, p\). Assume that \(G = \bigvee_{(H,S)} F\) and \(M'\) is the matrix defined in (3). If \(S_i = V(G_i)\), for \(i = 1, \ldots, p\), then

\[
\sigma(G) = \left(\bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i\}\right) \cup \sigma(M'),
\]

otherwise \(\sigma(G) \supseteq \bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i, k_i - \tau_i\}\).

**Proof.** The conclusions are direct consequence of the above analysis, taking into account that if \(S_i = V(G_i)\), for \(i = 1, \ldots, p\), then each \(S_i\) is \((k_i, \tau_i)\)-regular, with \(k_i = d_i\) and \(\tau_i = 0\).

3 Some applications on the spread of graphs

3.1 Definitions and basic results

Given a \(n \times n\) complex matrix \(M\), the spread of \(M\), \(s(M)\), is defined as \(\max_{i,j} |\lambda_i(M) - \lambda_j(M)|\), where the maximum is taken over all pairs of eigenvalues of \(M\). Then

\[
s(M) = \max_{x,y} (x^* M x - y^* M y) = \max_{i,j} \sum_{i,j} m_{i,j} (\bar{x}_i x_j - \bar{y}_i y_j),
\]

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where $z^*$ is the conjugate transpose of $z$ and the maximum is taken over all pairs of unit vectors in $\mathbb{C}^n$.

**Theorem 3** \[8\] \[s(M) \leq \left(2 \sum_{i,j} |m_{i,j}|^2 - \frac{2}{n} \sum_i |m_{i,i}|^2\right)^{1/2}, \text{ with equality if and only if } M \text{ is a normal matrix (that is, such that } M^*M = MM^*\text{), with } n-2 \text{ of its eigenvalues all equal to the average of the remaining two.} \]

Several results on the spread of normal and Hermitian matrices were presented in \[6, 9\].

In this paper, we consider only the spread of adjacency matrices of simple graphs and we define the spread of a graph $G$ as the spread $s(A(G))$, which is simply denoted by $s(G)$. Therefore,

$$s(G) = \max_{i,j} \{|\lambda_i(G) - \lambda_j(G)|\},$$

where the maximum is taken over all pairs of eigenvalues of the adjacency matrix of $G$. If the graph $G$ has order $n$, then $s(G) = \lambda_1(G) - \lambda_n(G)$ and replacing the matrix $M$ of Theorem 3 by $A(G)$, it follows that

$$s(G) = \lambda_1(G) - \lambda_n(G) \leq \sqrt{4|E(G)|}, \quad (5)$$

Denoting the average degree of the vertices of $G$ by $\overline{d}(G)$, from (5) it follows that

$$s(G) \leq \sqrt{2n\overline{d}(G)} < \sqrt{2n(n-1)} \quad (6)$$

if $n > 2$, since $\overline{d}(G) \leq n-1$ and $\overline{d}(G) = n-1$ if and only if $G = K_n$. Notice that $\sigma(K_n) = \{n-1, (-1)^{n-1}\}$ and then $s(K_n) = n$. Furthermore,

$$\overline{d}(G) \leq \frac{n}{2} \Rightarrow s(G) \leq n. \quad (7)$$

In \[4\] the following upper bounds on the spread of a graph were obtained.

**Theorem 4** \[4\] If $G$ is a graph of order $n$, then

$$s(G) \leq \lambda_1(G) + \sqrt{2|E(G)| - \lambda_2^2(G)} \leq 2\sqrt{|E(G)|}. \quad (8)$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $|E(G)| = 0$ or $G = K_{p,q}$, for some $p$ and $q$.

**Theorem 5** \[4\] If $G$ is a regular graph of order $n$, then $s(G) \leq n$. Equality holds if and only if the complement of $G, \overline{G}$, is disconnected.

Additional results on the spread of graphs can be found in \[4, 7\].
3.2 The spread of the join of two graphs

Now it is worth to recall the join of two vertex disjoint graphs $G_1$ and $G_2$ which is the graph $G_1 \lor G_2$ obtained from their union connecting each vertex of $G_1$ to each vertex of $G_2$. Considering this graph operation, as direct consequence of Theorem\[2\] we have the following corollaries. Notice that Corollary\[2\] is well known (see, for instance, [10]).

**Corollary 2** If $G_i$ is a $d_i$-regular graph of order $n_i$, for $i = 1, 2$, then

$$\sigma(G_1 \lor G_2) = \bigcup_{i=1}^{2} (\sigma(A(G_i)) \setminus \{d_i\}) \cup \{\beta_1, \beta_2\}.$$  

where $\beta_1$ and $\beta_2$ are eigenvalues of the matrix $M' = \begin{pmatrix} \frac{d_1}{\sqrt{n_1 n_2}} & \sqrt{n_1 n_2} \\ \frac{d_2}{\sqrt{n_1 n_2}} & \frac{d_2}{\sqrt{n_1 n_2}} \end{pmatrix}$, that is,

$$\beta_1 = \frac{d_1 + d_2 + \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2}, \quad (9)$$

$$\beta_2 = \frac{d_1 + d_2 - \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2}. \quad (10)$$

**Corollary 3** Consider a $d_i$-regular graph of order $n_i$, for $i = 1, 2$, and the graph $G = G_1 \lor G_2$ of order $n = n_1 + n_2$. Then

$$s(G) = \begin{cases} \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}, & \text{if } \lambda_n(G) = \beta_2 \\ \frac{d_2 - d_1 + \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2} + s(G_1), & \text{if } \lambda_n(G) = \lambda_{n_1}(G_1) \\ \frac{d_1 - d_2 + \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2} + s(G_2), & \text{if } \lambda_n(G) = \lambda_{n_2}(G_2). \end{cases} \quad (11)$$

Furthermore, setting $R = \sqrt{(d_1 - d_2)^2 + 4n_1 n_2},$

$$s(G) = \max\{R, \frac{d_2 - d_1 + R}{2} + s(G_1), \frac{d_1 - d_2 + R}{2} + s(G_2)\}.$$  

**Proof.** According to Corollary\[2\], $\sigma(A(G)) = \bigcup_{i=1}^{2} (\sigma(A(G_i)) \setminus \{d_i\}) \cup \{\beta_1, \beta_2\}$, where $\beta_1$ and $\beta_2$ have the values (9) and (10), respectively. On the other hand, $\lambda_{n_i}(G_i) = d_i - s(G_i)$, for $i = 1, 2$. Therefore, the equalities in (11) follows, as well as the second part. \qed

**Corollary 4** Let $G_i$ be a $d_i$-regular graph of order $n_i$, for $i = 1, 2$, and $G = G_1 \lor G_2$. If $|d_1 - d_2| > |n_1 - n_2|$, then $s(G) > n = n_1 + n_2$.

**Proof.** By construction, it is immediate that the order of $G$ is $n = n_1 + n_2$. Taking into account that $\beta_1$ and $\beta_2$, in (9) and (10) of Corollary\[2\], respectively, are eigenvalues of the adjacency matrix of $G = G_1 \lor G_2$, then

$$s(G) \geq \beta_1 - \beta_2 = \sqrt{(d_1 - d_2)^2 + 4n_1 n_2} > n_1 + n_2 = n.$$
Notice that $\sqrt{(d_1 - d_2)^2 + 4n_1n_2} > n_1 + n_2 \iff (d_1 - d_2)^2 + 4n_1n_2 > n_1^2 + n_2^2 + 2n_1n_2 \iff (d_1 - d_2)^2 > (n_1 - n_2)^2$. 

Considering the complete graph $K_k$, for which $\sigma(K_k) = \{-1\}^{k-1}, k-1\}$, and the null graph $K_{n-k}$, for which $\sigma(K_{n-k}) = \{0\}^{n-k}$, and denote the join of these graphs by $G(n,k)$ (that is $G(n,k) = K_k \vee K_{n-k}$), according to Corollary 2, $\sigma(G(n,k)) = \{-1\}^{k-1}, \{0\}^{n-k-1}, \beta_1, \beta_2\}$, with

$$\beta_1 = \frac{k - 1 + \sqrt{(k-1)^2 + 4k(n-k)}}{2}$$
$$\beta_2 = \frac{k - 1 - \sqrt{(k-1)^2 + 4k(n-k)}}{2}.$$ 

Therefore, $s(G(n,k)) = \beta_1 - \beta_2 = \sqrt{(k-1)^2 + 4k(n-k)}$. Furthermore, when $\frac{n+1}{2} < k < n-1$, the hypothesis of Corollary 3 hold for these graphs and then $s(G(n,k)) > n$.

**Theorem 6** [4] Among the family of graphs $G(n,k) = K_k \vee K_{n-k}$, with $1 \leq k \leq n-1$, the maximum of $s(G(n,k))$ is attained when $k = \lfloor 2n/3 \rfloor$.

In [4] the following conjecture was checked by computer for graphs of order $n \leq 9$.

**Conjecture 1** [4] The maximum spread $s(n)$ of the graphs of order $n$ is attained only by $G(n, \lfloor 2n/3 \rfloor)$, that is, $s(n) = \lfloor (4/3)(n^2 - n + 1) \rfloor^{\frac{1}{2}}$ and so $\frac{1}{\sqrt{3}}(2n-1) < s(n) < \frac{1}{\sqrt{3}}(2n-1) + \frac{\sqrt{3}}{n-2}$.

### 3.3 The spread of the generalized join of graphs

Throughout this subsection we consider a graph $H$ of order $p$ and a family of regular graphs $\mathcal{F} = \{G_1, \ldots, G_p\}$, where each regular graph $G_i$ has degree $d_i$ and order $n_i$. We consider also $M = A(H)N + D$, where $N = \text{diag}(n_1, \ldots, n_p)$ and $D = \text{diag}(d_1, \ldots, d_p)$, and we define $d_i - s(G_i) = \min\{d_i - s(G_i) : i = 1, \ldots, p\}$ and the matrix

$$P = \begin{pmatrix}
0 & \sqrt{n_1n_2} & \cdots & \sqrt{n_1n_p} \\
\sqrt{n_2n_2} & 0 & \cdots & \sqrt{n_2n_p} \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{n_1n_p} & \sqrt{n_2n_p} & \cdots & 0
\end{pmatrix}.$$

Using the above notation, with the following theorems, we state upper and lower bounds on the spread of $G = \bigvee_H \mathcal{F}$.

**Theorem 7** If $G = \bigvee_H \mathcal{F}$, then

$$s(G) = s(M) + \max_{1 \leq i \leq p} \{\lambda_p(M) + s(G_i) - d_i, 0\}. \quad (12)$$
Furthermore,

\[ s(G) \geq n_\downarrow \left( s(H) - (\bar{d}_\uparrow - \bar{d}_\downarrow) \right) - (n_\uparrow - n_\downarrow) \left( \lambda_p(H) + \bar{d}_\uparrow \right) \quad (13) \]

where \( \bar{d}_\uparrow = \max_{1 \leq i \leq p} \frac{d_i}{n_i} \) (\( \bar{d}_\downarrow = \min_{1 \leq i \leq p} \frac{d_i}{n_i} \)) and \( n_\uparrow = \max_{1 \leq i \leq p} n_i \) (\( n_\downarrow = \min_{1 \leq i \leq p} n_i \)).

**Proof.** According to Theorem 2, \( \sigma(G) = (\bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i\}) \cup \sigma(M) \). Then \( \forall i \in \{1, \ldots, p\} \lambda_{n_i}(G_i) = d_i - s(G_i) \in \sigma(G) \) and hence

\[ \lambda_n(G) \in \{d_i - s(G_i), i = 1, \ldots, p\} \cup \{\lambda_n(M)\}. \]

Since \( \lambda_1(G) = \lambda_1(M) \) (notice that \( \lambda_1(G) \geq d_i \forall i \in \{1, \ldots, p\} \)), the equality \( (12) \) holds.

Now, we prove the inequality \( (13) \). Consider the symmetric matrix \( M' = KA(H)K + D \) in \( (4) \), where \( K = \text{diag}(\sqrt{n_1}, \ldots, \sqrt{n_p}) \), which is similar to the matrix \( M \). Let \( (\lambda, x) \) be an eigenpair of \( H \), where \( x \) is such that \( \sum_{i=1}^p x_i^2 = 1 \). Setting \( y = K^{-1}x \), then

\[
\begin{align*}
\lambda_n(G) &\leq \min \sigma(M) = \min \sigma(M') \\
&\leq \frac{y^T (KA(H)K + D) y}{y^T y} \\
&= \frac{x^T A(H)x + x^T K^{-1}DK^{-1}x}{x^T K^{-2}x} \\
&= \frac{\lambda x^T x + x^T D\sum_{i=1}^p \frac{d_i}{n_i}x_i^2}{x^T K^{-2}x} \\
&\leq \frac{\lambda + \sum_{i=1}^p \frac{d_i}{n_i}x_i^2}{\sum_{i=1}^p \frac{x_i^2}{n_i}} \leq \lambda_1(M') \leq \lambda_1(G).
\end{align*}
\]

Taking into account that \( \bar{d}_\uparrow = \max_{1 \leq i \leq p} \frac{d_i}{n_i} \) (\( \bar{d}_\downarrow = \min_{1 \leq i \leq p} \frac{d_i}{n_i} \)) and \( n_\uparrow = \max_{1 \leq i \leq p} n_i \) (\( n_\downarrow = \min_{1 \leq i \leq p} n_i \)), we may conclude the following.

- If \( \lambda = \lambda_p(H) \), then \( \lambda_n(G) \leq \frac{\lambda_p(H) + \bar{d}_\downarrow}{\frac{n_\downarrow}{n_\uparrow}} = n_\uparrow \left( \lambda_p(H) + \bar{d}_\downarrow \right) \).
- If \( \lambda = \lambda_1(H) \), then \( \lambda_1(G) \geq \frac{\lambda_1(H) + \bar{d}_\downarrow}{\frac{n_\downarrow}{n_\uparrow}} = n_\uparrow \left( \lambda_1(H) + \bar{d}_\downarrow \right) \).

Therefore, \( s(G) \geq n_\downarrow \left( \lambda_1(H) + \bar{d}_\downarrow \right) - n_\uparrow \left( \lambda_p(H) + \bar{d}_\uparrow \right) = n_\downarrow \left( \lambda_1(H) + \bar{d}_\downarrow \right) - n_\uparrow \left( \lambda_p(H) + \bar{d}_\uparrow \right) - (n_\uparrow - n_\downarrow) \left( \lambda_p(H) + \bar{d}_\uparrow \right) \). \[ \Box \]

As an immediate consequence of Theorem 7, we have the following corollary.

**Corollary 5**  If the graph \( H \) has at least one edge and \( G = \bigvee_H \mathcal{F} \), then

\[ s(G) \geq n_\downarrow \left( s(H) - (\bar{d}_\uparrow - \bar{d}_\downarrow) \right). \]

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Proof. From \([\ref{13}]\), it follows
\[
\begin{align*}
s(G) & \geq n_{1} \left( s(H) - \langle d_{\dagger} - \tilde{d}_{\dagger} \rangle \right) - (n_{\dagger} - n_{1}) \left( \lambda_{p}(H) + \tilde{d}_{\dagger} \right) \\
& \geq n_{1} \left( s(H) - \langle d_{\dagger} - \tilde{d}_{\dagger} \rangle \right) 
\end{align*}
\] (14)
The inequality (14) is obtained taking into account that \(\tilde{d}_{\dagger} \leq 1\) and, since \(H\) has at least one edge, \(\lambda_{p}(H) \leq -1\) and therefore, \((n_{\dagger} - n_{1}) \left( \lambda_{p}(H) + \tilde{d}_{\dagger} \right) \leq 0\).

Using this corollary, and taking into account that \(\tilde{d}_{\dagger} \) and \(\tilde{d}_{\dagger} \) are both in the interval \((0, 1)\), it follows that \(s(G) \geq n_{1}(s(H) - 1)\).

Theorem 8 If \(G = \bigvee_{H} F\), then
\[s(G) \leq \max_{1 \leq i \leq p} d_{i} + \lambda_{1}(H)\lambda_{1}(P) - \min\{d_{i}, -s(G_{i}), \lambda_{p}(M)\}.\]

Proof. By Theorem\([\ref{2}]\) \(\sigma(G) = (\bigcup_{i=1}^{p} \sigma(G_{i}) \setminus \{d_{i}\}) \cup \sigma(M)\), where \(M = D + A(H) \circ P\), with \(D = \text{diag}(d_{1}, \ldots, d_{p})\), and \(\circ\) denotes the Hadamard product (see, for instance, \([\ref{3}]\)). Since when we have two symmetric nonnegative matrices of order \(p\), \(A\) and \(B\), \(\lambda_{1}(A + B) \leq \lambda_{1}(A) + \lambda_{1}(B)\) and \(\lambda_{1}(A \circ B) \leq \lambda_{1}(A \otimes B) = \lambda_{1}(A)\lambda_{1}(B)\), where \(\otimes\) is the Kronecker product, we may conclude that \(\lambda_{1}(M) \leq \lambda_{1}(D) + \lambda_{1}(A)\lambda_{1}(P) \leq \lambda_{1}(D) + \lambda_{1}(H)\lambda_{1}(P) = \max_{1 \leq i \leq p} d_{i} + \lambda_{1}(H)\lambda_{1}(P)\).

Since \(\lambda_{n}(G) = \min\{d_{i}, -s(G_{i}), \lambda_{p}(M)\}\), it follows that,
\[s(G) \leq \max_{1 \leq i \leq p} d_{i} + \lambda_{1}(H)\lambda_{1}(P) - \min\{d_{i}, -s(G_{i}), \lambda_{p}(M)\}.
\]

Theorem 9 If the graph \(H\) has at least one edge and \(G = \bigvee_{H} F\), then
\[s(M) \leq s(G) < s(M) + \max_{1 \leq i \leq p} \{d_{i}\}.
\]

Proof. By Theorem\([\ref{2}]\) \(s(G) = s(M) + \max_{1 \leq i \leq p} \{\lambda_{p}(M) - \lambda_{n_{i}}(G_{i}), 0\}\).

1. If \(\max_{1 \leq i \leq p} \{\lambda_{p}(M) - \lambda_{n_{i}}(G_{i}), 0\} = 0\), then the left inequality holds as equality and the right inequality is strict.

2. Otherwise, assume that \(\exists i^{*} \in \{1, \ldots, p\}\) such that \(\max_{1 \leq i \leq p} \{\lambda_{p}(M) - \lambda_{n_{i}}(G_{i}), 0\} = \lambda_{p}(M) - \lambda_{n_{i^{*}}}(G_{i^{*}})\). Since,
\[\lambda_{p}(M) - \lambda_{n_{i^{*}}}(G_{i^{*}}) < -\lambda_{n_{i^{*}}}(G_{i^{*}}) \leq d_{i^{*}} \leq \max_{1 \leq i \leq p} \{d_{i}\},\]
then the right inequality holds. Notice that, when \(H\) has at least one edge, \(\lambda_{p}(M) < 0\). In fact, if \(ij \in E(H)\), the matrix \(B_{ij} = \begin{pmatrix} d_{i} & \sqrt{\lambda_{n_{i}}(M)} \sqrt{n_{i}n_{j}} \\ \sqrt{n_{i}n_{j}} & d_{j} \end{pmatrix}^{T}\)

is a principal submatrix of \(P_{ij}M^{T}_{ij}\), where \(P_{ij}\) is permutation matrix. Therefore, \(\lambda_{p}(M) = \lambda_{p}(P_{ij}M^{T}_{ij}) \leq \lambda_{2}(B_{ij}) < 0\). The left inequality follows from the fact that the eigenvalues of \(M\) are also eigenvalues of \(G\).
3.4 An infinite family of non regular graphs of order \( n \) with spread equal to \( n \).

**Theorem 10** Consider the positive integers \( p, q \geq 3 \) and \( n \in \mathbb{N} \) such that \( n \geq p + q + 3 \). Let \( H = P_3 \) and let \( F = \{G_1, G_2, G_3\} \) be a family of graphs, where \( G_1 = C_p, \ G_2 = C_q \) and \( G_3 = C_{n-p-q} \). If \( S = \{S_1, S_2, S_3\} \) is such that \( S_i = V(G_i) \) for \( i = 1, 2, 3 \), then the graph

\[
G = \bigvee_{(H,S)} F.
\]

is non regular and is such that \( s(G) \leq n \). Furthermore, \( s(G) = n \) if and only if \( q = \frac{n}{2} \).

**Proof.** By definition of generalized join, it is immediate that \( G \) is non regular. By Theorem 2

\[
\sigma(G) = \bigcup_{i=1}^{3} (\sigma(G_i) \setminus \{2\}) \cup \{\beta_1, \beta_2, \beta_3\},
\]

where \( \beta_i, \) with \( i \in \{1, 2, 3\} \), are the roots of the characteristic polynomial of the matrix

\[
M = \begin{pmatrix}
2 & q & 0 \\
p & 2 & n-p-q \\
0 & q & 2
\end{pmatrix}.
\]

Then \( \beta_1 = 2, \beta_2 = 2 + \sqrt{q(n-q)} \), and \( \beta_3 = 2 - \sqrt{q(n-q)} \). Notice that the largest eigenvalue of \( M \) is \( \beta_2 \) and \( \lambda_{\min}(G) = \beta_3 = 2 - \sqrt{q(n-q)} < -2 \) (taking into account the values of \( p, q \) and \( n \) and since \( \lambda_{\min}(G_i) \geq -2 \), for \( i = 1, 2, 3 \)). Therefore,

\[
s(G) = 2\sqrt{q(n-q)}.
\]

Since \( q(n-q) \leq \frac{n^2}{4} \) and \( q(n-q) = \frac{n^2}{4} \) if and only if \( q = \frac{n}{2} \), the result follows.

As immediate consequence of Theorem 10 if \( n \) is an even positive number not less than 12, \( q = \frac{n}{2} \) and \( 3 \leq p \leq \frac{n-p}{2} \) then the graph \( G \) defined in (15) is such that \( s(G) = n \).

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