On a class of generalized Takagi functions with linear pathwise quadratic variation

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Abstract
We consider a class \( \mathcal{X} \) of continuous functions on \([0,1]\) that is of interest from two different perspectives. First, it is closely related to sets of functions that have been studied as generalizations of the Takagi function. Second, each function in \( \mathcal{X} \) admits a linear pathwise quadratic variation and can thus serve as an integrator in Föllmer’s pathwise Itô calculus. We derive several uniform properties of the class \( \mathcal{X} \). For instance, we compute the overall pointwise maximum, the uniform maximal oscillation, and the exact uniform modulus of continuity for all functions in \( \mathcal{X} \). Furthermore, we give an example of a pair \( x, y \in \mathcal{X} \) such that the quadratic variation of the sum \( x + y \) does not exist.

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1 Introduction
In this note, we study a class \( \mathcal{X} \) of continuous functions on \([0,1]\) that is of interest from several different perspectives. On the one hand, just as typical Brownian sample paths, each function \( x \in \mathcal{X} \) admits the linear pathwise quadratic variation \( \langle x \rangle_t = t \) in the sense of Föllmer [13] and therefore can serve as an integrator in Föllmer’s pathwise Itô calculus. On the other hand, \( \mathcal{X} \) is a subset, or has a nonempty intersection, with classes of functions that have been studied as generalizations of Takagi’s celebrated example [29] of a nowhere differentiable continuous function. We will now explain the connections of our results with these two separate strands of literature.

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1.1 Contributions to Föllmer’s pathwise Itô calculus

In 1981, Föllmer [13] proposed a pathwise version of Itô’s formula, which, as a consequence, yields a strictly pathwise definition of the Itô integral as a limit of Riemann sums. Some recent developments have led to a renewed interest in this pathwise approach. Perhaps the most important of these developments is the conception of functional pathwise Itô calculus by Dupire [10] and Cont and Fournié [6, 7], which for instance is crucial in defining partial differential equations on path space [11]. Another source for the renewed interest in pathwise Itô calculus stems from the growing awareness of model ambiguity in mathematical finance and the resulting desire to reduce the reliance on probabilistic models; see, e.g., [15] for a recent survey and [3, 4, 8, 14, 26, 27] for case studies with successful applications of pathwise Itô calculus to financial problems. A systematic introduction to pathwise Itô calculus, including an English translation of [13], is provided in [28].

A function \( x \in C[0,1] \) can serve as an integrator in Föllmer’s pathwise Itô calculus if it admits a continuous pathwise quadratic variation \( t \mapsto \langle x \rangle_t \) along a given refining sequence of partitions of \([0,1]\). This condition is satisfied whenever \( x \) is a sample path of a continuous semimartingale, such as Brownian motion, and does not belong to a certain nullset. This nullset, however, is generally not known explicitly, and so it is not possible to tell whether a specific realization \( x \) of Brownian motion does indeed admit a continuous pathwise quadratic variation. The first purpose of this note is to provide a rich class \( \mathcal{X} \) of continuous functions that can be constructed in a straightforward manner and that do admit the nontrivial pathwise quadratic variation \( \langle x \rangle_t = t \) for all \( x \in \mathcal{X} \). The functions in \( \mathcal{X} \) can thus be used as a class of test integrators in pathwise Itô calculus. Our corresponding result, Proposition 2.5, extends a previous result by Gantert [18, 19], from which it follows that \( \langle x \rangle_1 = 1 \) for all \( x \in \mathcal{X} \).

Still within this context, a second purpose of this note is to investigate whether the existence of \( \langle x \rangle \) and \( \langle y \rangle \) implies the existence of \( \langle x + y \rangle \) (or, equivalently, the existence of the pathwise quadratic covariation \( \langle x,y \rangle \)). For typical sample paths of a continuous semimartingale this implication is always true. In the literature on pathwise Itô calculus, however, it has been taken for granted that the existence of \( \langle x + y \rangle \) cannot be deduced from the existence of \( \langle x \rangle \) and \( \langle y \rangle \). In Proposition 2.6 we will now give an example of two functions \( x, y \in \mathcal{X} \) for which \( \langle x + y \rangle \) does indeed not exist. To the knowledge of the author, such an example has so far been missing from the literature.

1.2 Contributions to the theory of generalized Takagi functions

In 1903, Takagi [29] proposed an example of a continuous function on \([0,1]\) that is nowhere differentiable. This function has since been rediscovered several times and its properties have been studied extensively; see the recent surveys by Allaart and Kawamura [2] and Lagarias [24]. While the original Takagi function itself does not belong to our class \( \mathcal{X} \), there are at least two classes of functions whose study was motivated by the Takagi function and that are intimately connected with \( \mathcal{X} \). One family of functions is the “Takagi class” introduced in 1984 by Hata and Yamaguti [20]. Similar but more restrictive function classes were introduced
earlier by Faber [12] or Kahane [21]. The Takagi class has a nonempty intersection with \( \mathcal{X} \) but neither one is included in the other. More recently, Allaart [1] extended the Takagi class to a more flexible class of functions. This family now contains \( \mathcal{X} \). By extending arguments given by Kono [23] for the Takagi class, Allaart [1] studies in particular the moduli of continuity of certain functions in his class.

In contrast to these previous studies, the focus of this paper is not so much on the individual features of functions \( x \in \mathcal{X} \) but rather on uniform properties of the entire class \( \mathcal{X} \). Here we compute the overall pointwise maximum, the uniform maximal oscillation, and the exact uniform modulus of continuity for all functions in \( \mathcal{X} \). In these computations we cannot rely on previous methods that were conceived for the analysis of the Takagi functions and its generalizations. For instance, neither the result and arguments from Kono [23] nor the ones from Allaart [1] apply to the modulus of continuity of functions in \( \mathcal{X} \), and a new approach must be taken. It exploits the self-similar structure of \( \mathcal{X} \) and its members.

A special role in our analysis will be played by the function \( \hat{x} \), defined in (2.2) below. It has previously appeared in the work of Ledrappier [25], who studied the Hausdorff dimension of its graph. Here we will determine its global maximum and its exact modulus of continuity. In particular the results on the global maximum of \( \hat{x} \) will be needed in our analysis of the uniform properties of \( \mathcal{X} \), but these results are also interesting in their own right.

This paper is organized as follows. In the subsequent Section 2 we first introduce our class \( \mathcal{X} \) and then discuss its uniform properties in Theorems 2.2 and 2.3 and Corollary 2.4. We then recall Föllmer’s [13] notions of pathwise quadratic variation and covariation and state our corresponding results. All proofs are given in Section 3.

2 Statement of results

Recall that the \textit{Faber–Schauder functions} are defined as

\[
e_{\emptyset}(t) := t, \quad e_{0,0}(t) := (\min\{t, 1-t\})^+, \quad e_{n,k}(t) := 2^{-n/2}e_{0,0}(2^n(t-k))
\]

for \( t \in \mathbb{R}, n = 1, 2, \ldots, \) and \( k \in \mathbb{Z} \). The graph of \( e_{n,k} \) looks like a wedge with height \( 2^{-n/2} \), width \( 2^{-n} \), and center at \( t = (k + \frac{1}{2})2^{-n} \). In particular, the functions \( e_{n,k} \) have disjoint support for distinct \( k \) and fixed \( n \). Now let coefficients \( \theta_{n,k} \in \{-1, +1\} \) be given and define for \( N \in \mathbb{N} \) the continuous functions

\[
x^N(t) := \sum_{n=0}^{N-1} \sum_{k=0}^{2^n-1} \theta_{n,k}e_{n,k}(t), \quad 0 \leq t \leq 1.
\]  

(2.1)

It is well known (see, e.g., [1]) and easy to see that, due the uniform boundedness of the coefficients \( \theta_{n,k} \), the functions \( x^N(t) \) converges uniformly in \( t \) to a continuous function \( x(t) \) as \( N \uparrow \infty \). Let us denote by

\[
\mathcal{X} := \{ x \in C[0,1] \mid x = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n,k}e_{n,k} \text{ for coefficients } \theta_{n,k} \in \{-1, +1\} \}
\]
the class of limiting functions arising in this way. A function \( x \in \mathcal{X} \) belongs to the “Takagi class” introduced by Hata and Yamaguti [20] if and only if the coefficients \( \theta_{n,k} \) in (2.1) are independent of \( k \). Moreover, \( \mathcal{X} \) is a subset of the class of generalized Takagi functions studied by Allaart [1]. The original Takagi function, however, is obtained by taking \( \theta_{n,k} = 2^{-n/2} \) and therefore does not belong to \( \mathcal{X} \).

Remark 2.1 (On similarities with Brownian sample paths). The functions in \( \mathcal{X} \) can exhibit interesting fractal structures; see Figure 1 on the other hand, displays some similarities with the sample paths of a Brownian bridge. This similarity is not surprising since the well-known Lévy–Ciesielski construction of the Brownian bridge consists in replacing the coefficients \( \theta_{n,k} \in \{-1, +1\} \) with independent standard normal random variables (see, e.g., [22]). As a matter of fact, using ideas of de Rham [9] and Billingsley [5], it was shown in Theorem 3.1 (iii) that functions in \( \mathcal{X} \) share with Brownian sample paths the property of being nowhere differentiable. Moreover, Ledrappier [25] showed that the Hausdorff dimension of the graph of the function

\[
\hat{x} := \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} e_{n,k}
\]

is the same as that of the graphs of typical Brownian trajectories, namely \(3/2\). In Proposition 2.5 we will see, moreover, that the functions in \( \mathcal{X} \) have the same pathwise quadratic variation as Brownian sample paths.

Our first result is concerned with (uniform) maxima and oscillations of the functions in \( \mathcal{X} \). It will also be concerned with the function \( \hat{x} \) defined in (2.2), a function that will play a special role throughout our analysis. The maximum of the original Takagi function was computed by Kahane [21], but his method does not apply in our case, and more complex arguments are needed here.

Theorem 2.2 (Uniform maximum and oscillations). The class \( \mathcal{X} \) has the following uniform properties.

(a) The uniform maximum of functions in \( \mathcal{X} \) is attained by \( \hat{x} \) and given by

\[
\max_{x \in \mathcal{X}} \max_{t \in [0,1]} |x(t)| = \max_{t \in [0,1]} \hat{x}(t) = \frac{1}{3} (2 + \sqrt{2}).
\]

Moreover, the maximum of \( \hat{x}(t) \) is attained in \( t = \frac{1}{3} \) and \( t = \frac{2}{3} \).

(b) The maximal uniform oscillation of functions in \( \mathcal{X} \) is

\[
\max_{x \in \mathcal{X}} \max_{s,t \in [0,1]} |x(t) - x(s)| = \frac{1}{6} (5 + 4\sqrt{2}),
\]

where the respective maxima are attained by \( s = 1/3 \), \( t = 5/6 \), and

\[
x^* := e_{0,0} + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{2^n-1} e_{n,k} - \sum_{\ell=2^{n-1}}^{2^n-1} e_{n,\ell} \right).
\]

(2.3)
In our next result, we will investigate the modulus of continuity of $\hat{x}$ and the uniform modulus of continuity of the class $\mathcal{X}$. Kono [23] analyzed the moduli of continuity for some functions in the Takagi class of Hata and Yamaguti [20], and Allaart [1] later extended this result. However, neither the result and arguments from [23] nor the ones from [1] apply to the functions in $\mathcal{X}$ because the sequence $a_n := 2^{n/2}$ is not bounded. To state our results, let us denote by

$$\nu(h) := \lfloor -\log_2 h \rfloor, \quad h > 0,$$

(2.4)

the integer part of $\log_2 \frac{1}{h}$, and define

$$\hat{\omega}(h) := \left(1 + \frac{1}{\sqrt{2}}\right) h 2^{\nu(h)/2} + \frac{1}{3} (\sqrt{8} + 2) 2^{-\nu(h)/2},$$

$$\omega_{\text{max}}(h) := (1 + \sqrt{2}) h 2^{\nu(h)/2} + \frac{1}{3} (4 + \sqrt{8}) 2^{-\nu(h)/2}.$$
Figure 2: Plots of $x \in \mathcal{X}$ when the coefficients $\theta_{n,k}$ form an independent and identically distributed $\{-1, +1\}$-valued random sequence such that $\theta_{n,k} = +1$ with respective probabilities $\frac{1}{2}$ (left) and $\frac{1}{3}$ (right). The dashed line corresponds to the approximation $\langle x \rangle^8$ of the quadratic variation along the $8$th dyadic partition $T_8$.

Note that both $\widehat{\omega}(h)$ and $\omega_{\text{max}}(h)$ are of the order $O(\sqrt{h})$ as $h \downarrow 0$. More precisely,

\[
\liminf_{h \downarrow 0} \frac{\widehat{\omega}(h)}{\sqrt{h}} = 2\sqrt{\frac{4}{3} + \sqrt{2}}, \quad \limsup_{h \downarrow 0} \frac{\widehat{\omega}(h)}{\sqrt{h}} = \frac{1}{6}(11 + 7\sqrt{2}),
\]

\[
\liminf_{h \downarrow 0} \frac{\omega_{\text{max}}(h)}{\sqrt{h}} = 2\sqrt{\frac{8}{3} + 2\sqrt{2}}, \quad \limsup_{h \downarrow 0} \frac{\omega_{\text{max}}(h)}{\sqrt{h}} = \frac{1}{6}(14 + 11\sqrt{2}).
\]

**Theorem 2.3 (Moduli of continuity).**

(a) The function $\widehat{x}$ has $\widehat{\omega}$ as its modulus of continuity. More precisely,

\[
\limsup_{h \downarrow 0} \max_{0 \leq t \leq 1-h} \frac{|\widehat{x}(t + h) - \widehat{x}(t)|}{\omega(h)} = 1.
\]

(b) An exact uniform modulus of continuity for functions in $\mathcal{X}$ is given by $\omega_{\text{max}}$. That is,

\[
\limsup_{h \downarrow 0} \sup_{x \in \mathcal{X}} \max_{0 \leq t \leq 1-h} \frac{|x(t + h) - x(t)|}{\omega_{\text{max}}(h)} = 1.
\]

We have the following corollary to Theorems 2.2 and 2.3.

**Corollary 2.4.** $\mathcal{X}$ is a compact subset of $C[0,1]$ with respect to the topology of uniform convergence.

Theorem 2.3 implies moreover that each $x \in \mathcal{X}$ is Hölder continuous with exponent $\frac{1}{2}$ and hence admits a finite 2-variation in the sense that

\[
\sup_{T} \sum_{t \in T} (x(t') - x(t))^2 < \infty, \quad (2.5)
\]
where the supremum is taken over all partitions \( T \) of \([0, 1]\) and \( t' \) denotes the successor of \( t \) in \( T \), i.e.,
\[
t' = \begin{cases} 
\min\{u \in T \mid u > t\} & \text{if } t < 1, \\
1 & \text{if } t = 1.
\end{cases}
\]
Each \( x \in \mathcal{D} \) can therefore serve as an integrator in the pathwise integration theory of rough paths; see, e.g., Friz and Hairer [17]. A different pathwise integration theory was proposed earlier by Föllmer [13]. It is based on the following notion of pathwise quadratic variation. Instead of considering the supremum over all partitions as in (2.5), one fixes an increasing sequence of partitions \( T_1 \subset T_2 \subset \cdots \) of \([0, 1]\) such that the mesh of \( T_N \) tends to zero. Here we will henceforth concentrate on the sequence of dyadic partitions:
\[
T_N := \{k2^{-N} \mid k = 0, \ldots, 2^N\}, \quad N = 1, 2, \ldots
\] (2.6)
For \( x \in C[0, 1] \) one then defines the sequence
\[
\langle x \rangle^N_t := \sum_{s \in T_N, s \leq t} (x(s') - x(s))^2.
\] (2.7)
The function \( x \in C[0, 1] \) is said to admit the continuous quadratic variation \( \langle x \rangle \) along the sequence \( (T_N) \) if for all \( t \in [0, 1] \) the limit
\[
\langle x \rangle_t := \lim_{N \to \infty} \langle x \rangle^N_t
\] (2.8)
exists, and if \( t \mapsto \langle x \rangle_t \) is a continuous function. Föllmer’s pathwise Itô calculus uses this class of functions \( x \) as integrators.

For given \( x \in C[0, 1] \), the approximations \( \langle x \rangle^N_t \) are typically not monotone in \( N \), and so it is not clear \textit{a priori} whether the limit in (2.8) exists. Moreover, even if the limit exists, it may depend strongly on the particular choice of the underlying sequence of partitions. For instance, it is known that for any \( x \in C[0, 1] \) there exists a refining sequence \((S_N)_{N \in \mathbb{N}}\) of partitions such that the quadratic variation of \( x \) along \((S_N)_{N \in \mathbb{N}}\) vanishes identically; see [16, p. 47]. It is also not difficult to construct \( x \in C[0, 1] \) for which the limit in (2.8) exists but satisfies \( \langle x \rangle_t = \frac{1}{2} \) \((1/2, 1) \) and is hence discontinuous.

So the first question that arises in this context is how one can obtain functions that do admit a continuous quadratic variation along the sequence \((T_N)_{N \in \mathbb{N}}\)? Of course one can take all sample paths of a Brownian motion (or, more generally, of a continuous semimartingale) that are not contained in a certain null set \( A \). But \( A \) is generally not given explicitly, and so it is not possible to tell whether a specific realization \( x \) of Brownian motion does indeed admit the quadratic variation \( \langle x \rangle_t = t \) along \((T_N)_{N \in \mathbb{N}}\). Moreover, this selection principle for functions \( x \) lets a probabilistic model enter through the backdoor, although the initial purpose of pathwise Itô calculus was to get rid of probabilistic models altogether.

\[\text{Pathwise Itô calculus also works for càdlàg functions } x, \text{ but this requires that the continuous part of } x \text{ admits a continuous quadratic variation along } (T_N); \text{ see } [13] \text{ and } [6]. \text{ For this reason we will concentrate here on the case of continuous functions } x.\]
In the following proposition, we show that each \( x \in \mathcal{X} \) admits the linear pathwise quadratic variation \( \langle x \rangle_t = t \) for \( t \in [0,1] \). This extends a result by Gantert [13, 19], from which it follows that \( \langle x \rangle_1 = 1 \) for all \( x \in \mathcal{X} \). Our proposition implies that each \( x \in \mathcal{X} \) can serve as an integrator in Föllmer’s pathwise Itô calculus.

**Proposition 2.5.** Every \( x \in \mathcal{X} \) admits the quadratic variation \( \langle x \rangle_t = t \) along \( (T_N)_N \in \mathbb{N} \).

The second question that we will address in this context is concerned with a standard assumption that is made in pathwise Itô calculus whenever the covariation of two functions \( x, y \in C[0,1] \) is needed. Let

\[
\langle x, y \rangle_t^N := \sum_{s \in T_N, s \leq t} (x(s') - x(s))(y(s') - y(s)) \tag{2.9}
\]

and observe that

\[
\langle x, y \rangle_t^N = \frac{1}{2} \left( \langle x + y \rangle_t^N - \langle x \rangle_t^N - \langle y \rangle_t^N \right). \tag{2.10}
\]

If \( x \) and \( y \) admit the continuous quadratic variations \( \langle x \rangle \) and \( \langle y \rangle \) along \( (T_N)_{N \in \mathbb{N}} \), then it follows from (2.10) that the covariation of \( x \) and \( y \),

\[
\langle x, y \rangle_t := \lim_{N \uparrow \infty} \langle x, y \rangle_t^N, \tag{2.11}
\]

exists along \( (T_N)_{N \in \mathbb{N}} \) and is continuous in \( t \) if and only if \( x + y \) admits a continuous quadratic variation along \( (T_N)_{N \in \mathbb{N}} \). When \( x \) and \( y \) are sample paths of Brownian motion or, more generally, of a continuous semimartingale, the quadratic variation \( \langle x + y \rangle \), and hence \( \langle x, y \rangle \), will always exist almost surely. But for arbitrary functions \( x, y \in C[0,1] \) it has so far not been possible to reduce the existence of the limit in (2.11) to the existence of \( \langle x \rangle \) and \( \langle y \rangle \). In our next proposition we will provide an example of two functions \( x, y \in \mathcal{X} \) for which the limit in (2.11) does not exist, even though \( \langle x \rangle_t = t = \langle y \rangle_t \). This shows that the existence of \( \langle x, y \rangle \) and \( \langle x + y \rangle \) is not implied by the existence of \( \langle x \rangle \) and \( \langle y \rangle \). It follows in particular that the class of functions that admit a continuous quadratic variation along \( (T_N)_{N \in \mathbb{N}} \) is not a vector space.

To the knowledge of the author, a corresponding example has so far been missing from the literature.

**Proposition 2.6.** Consider the functions

\[
\hat{x} = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} e_{n,k} \quad \text{and} \quad y = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} (-1)^n e_{n,k},
\]

which belong to \( \mathcal{X} \) and hence admit the quadratic variation \( \langle \hat{x} \rangle_t = t = \langle y \rangle_t \) along \( (T_N) \). Then

\[
\lim_{N \uparrow \infty} \langle \hat{x} + y \rangle_t^{2N} = \frac{4}{3} \quad \text{and} \quad \lim_{N \uparrow \infty} \langle \hat{x} + y \rangle_t^{2N+1} = \frac{8}{3} \tag{2.12}
\]
and
\[
\lim_{N \to \infty} (\hat{x}, y)^{2N} = \frac{-1}{3} \quad \text{and} \quad \lim_{N \to \infty} (\hat{x}, y)^{2N+1} = \frac{1}{3}
\] (2.13)

In particular, the limits of \((\hat{x} + y)^{N}\) and \((\hat{x}, y)^{N}\) do not exist as \(N \to \infty\).

See the two upper panels in Figure 1 for plots of the two functions \(\hat{x}\) and \(y\) occurring in Proposition 2.6. See Figure 3 for a plot of
\[
\hat{x} + y = \sum_{n=0}^{\infty} \sum_{k=0}^{2^{2n} - 1} e_{2n,k}.
\] (2.14)

Figure 3: Plot of the function \(\hat{x} + y\) defined in (2.14).

3 Proofs

3.1 Proof of Theorem 2.2

We start with the following lemma, which computes the maximum of
\[
\hat{x}^{N} := \sum_{n=0}^{N-1} \sum_{k=0}^{2^{2n} - 1} e_{n,k}.
\]

Lemma 3.1. Let
\[
J_N := \frac{1}{3}(2^N - (-1)^N), \quad N = 0, 1, 2, \ldots
\]
be the sequence of Jacobsthal numbers, and let
\[
m_N := \frac{1}{3} \left(2 + \sqrt{2} + (-1)^{N+1}2^{-N}(\sqrt{2} - 1)\right) - 2^{-N/2}.
\]
Then $\hat{x}^N$ has exactly two maximal points, which are given by

$$t^-_N := 2^{-N}J_N \quad \text{and} \quad t^+_N := 2^{-N}J_{N+1},$$

and the global maximum of $\hat{x}^N$ is given by

$$\max_{t \in [0,1]} \hat{x}^N(t) = \hat{x}^N_{t^-_N} = \hat{x}^N_{t^+_N} = m_N.$$

**Proof.** We first note that by the symmetry of $\hat{x}^N$ with respect to $t = 1/2$ and the fact that $\frac{1}{2} - t_N = t_N - \frac{1}{2}$ it is sufficient to prove the result for the restriction of $\hat{x}^N$ to $[0, \frac{1}{2}]$. We will therefore simply write $t_N$ in place of $t^-_N$. Let

$$s^-_N := \frac{\hat{x}_{t^-_N} - \hat{x}_{t^-_N - 2^{-N}}}{2^{-N}} \quad \text{and} \quad s^+_N := \frac{\hat{x}_{t^+_N + 2^{-N}} - \hat{x}_{t^+_N}}{2^{-N}}$$

denote the left- and right-hand slopes of $\hat{x}^N$ at $t_N$. We use induction on $N$ to prove the assertion together with the additional auxiliary claim that

$$s^-_N = 2^{N+1} + (1 + (-1)^N 2^{N+1})(\sqrt{2} - 1), \quad s^+_N = -2^{N+1} + (1 + (-1)^N 2^{N+1})(\sqrt{2} - 1). \quad (3.1)$$

See Figure 4 for an illustration of the proof.

For $N = 1$, we have $\hat{x}^N(t) = e_{0,0}(t) = \min\{t, 1 - t\}$, which is clearly maximized at $t = \frac{1}{2} = 2^{-1}J_1 = 2^{-1}J_2$ and has maximal value $\hat{x}^1_{1/2} = \frac{1}{2} = m_1$. We also have $s^-_1 = 1$ and $s^+_1 = -1$, which is consistent with (3.1).

Now we assume that the result is correct for $N$. It is clear that the new maximum of $\hat{x}^{N+1}$ must be attained either in $t_N - 2^{-(N+1)}$ or in $t_N + 2^{-(N+1)}$. By induction hypothesis and since the slopes of Schauder functions $e_{N,k}$ are equal to $\pm 2^N$, the slope of $\hat{x}^{N+1}$ to the left of $t_N$ is

$$\hat{s}^-_N := s^-_N - 2^{N/2} = \begin{cases} \sqrt{2} - 1 & \text{if } N \text{ is even}, \\ (1 - 2^{N+1})(\sqrt{2} - 1) & \text{if } N \text{ is odd}. \end{cases} \quad (3.2)$$

The slope of $\hat{x}^{N+1}$ to the right of $t_N$ is

$$\hat{s}^+_N := s^+_N + 2^{N/2} = \begin{cases} (1 + 2^{N+1})(\sqrt{2} - 1) & \text{if } N \text{ is even}, \\ \sqrt{2} - 1 & \text{if } N \text{ is odd}. \end{cases} \quad (3.3)$$

For $N$ even, we clearly have $\hat{s}^+_N > -\hat{s}^-_N$, and so the new maximum of $\hat{x}^{N+1}$ must be attained in

$$t_{N+1} = t_N + 2^{-(N+1)} = \frac{2^{-N}}{3}(2^N - 1) + 2^{-(N+1)} = \frac{2^{-(N+1)}}{3}(2^{N+1} + 1) = 2^{-(N+1)}J_{N+1}.$$

Moreover, it follows immediately that

$$s^-_{N+1} = \hat{s}^+_N = 2^N + (1 - 2^N)(\sqrt{2} - 1).$$
It is also clear that
\[ s^+_{N+1} = s^+_N - 2^{N/2} = -2^{N/2} + (1 - 2^{N/2})(\sqrt{2} - 1), \]
which coincides with our formula (3.1). Since \( \hat{x}_{tN}^{N+1} = \hat{x}_{tN}^N \), we get furthermore from our induction hypothesis that
\[ \hat{x}_{tN+1}^{N+1} = \hat{x}_{tN}^N + s^-_{N+1}2^{-(N+1)} = \frac{1}{3}(2 + \sqrt{2} + 2^{-(N+1)}(\sqrt{2} - 1)) - 2^{-\frac{N+1}{2}} = m_{N+1}, \]
where the second identity requires a computation. This completes the induction in case \( N \) is even.

For \( N \) odd, we have \(-\hat{s}_N \geq \hat{s}_N^+ \) (with equality iff \( N = 1 \)). Therefore, the maximum of \( \hat{x}_{tN+1}^{N+1} \) must be attained in \( t_{N+1} = t_N - 2^{-(N+1)}, \) which is again found equal to \( 2^{-(N+1)}J_{N+1} \). It follows that \( s^+_{N+1} = -\hat{s}_N \), and one easily establishes our formulas for \( s^-_{N+1} \) and \( \hat{x}_{tN+1}^{N+1} \) as in the previous case.

\[ \text{Figure 4: Illustration of the proof of Lemma 3.1. The functions } \hat{x}^N \text{ are plotted over the interval } [0, 1/2] \text{ for various values of } N, \text{ together with the corresponding values for } t_N \text{ and } m_N. \]

Proof of Theorem 2.2 (a). For \( t \) fixed, \( x(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n,k} e_{n,k}(t) \) is maximized by taking all coefficients \( \theta_{n,k} \) equal to +1, i.e., \( |x(t)| \leq |\hat{x}(t)| \) for all \( x \in \mathcal{X}^* \) and \( t \in [0,1] \). The statement on maximum and maximizers of \( \hat{x} \) follows immediately from Lemma 3.1 by noting that \( m_N \to \frac{1}{3}(2 + \sqrt{2}) \), \( t_N \to \frac{1}{3} \), and \( t_N^+ \to \frac{3}{5} \) as \( N \uparrow \infty \).

Before proving Theorem 2.2 (b), we need the following lemma.

Lemma 3.2. We have
\[ M := \sup_{x \in \mathcal{X}^*} \max_{s,t \in [0,1]} |x(t) - x(s)| = \sup_{x \in \mathcal{X}^*} \max_{s \in [0,1/2]} \max_{t \in [1/2,1]} |x(t) - x(s)|. \]
Proof. We prove the assertion by showing that it would be suboptimal to take $s$ and $t$ both from either $[0, 1/2]$ or $[1/2, 1]$. By symmetry we can concentrate on $[0, 1/2]$, and so we assume by way of contradiction that for any given $\varepsilon$ there exists $x^\varepsilon \in \mathcal{X}$ such that
\[
\max_{s, t \in [0, 1/2]} |x^\varepsilon(t) - x^\varepsilon(s)| \geq M - \varepsilon.
\]

Note that the Faber–Schauder functions have the following scaling property:
\[
\sqrt{2} e_{n,k}(t) = e_{n-1,k}(2t) \quad \text{for } t \geq 0, n \geq 1, \text{ and } k \geq 0.
\]

So if $\theta_{n,k}$ are the coefficients in the Faber–Schauder development of $x^\varepsilon$, then we have that
\[
x^\varepsilon(t) = e_{0,0}(t) + \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n,k} e_{n,k}(t) = t + \frac{1}{\sqrt{2}} y(2t), \quad t \in [0, 1/2],
\]
where $y = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n+1,k} e_{n,k}$ belongs to $\mathcal{X}$. Hence,
\[
\max_{s, t \in [0, 1/2]} |x^\varepsilon(t) - x^\varepsilon(s)| \leq \max_{s, t \in [0, 1/2]} |t - s| + \frac{1}{\sqrt{2}} \max_{u, v \in [0, 1]} |y(u) - y(v)|.
\]
This gives
\[
M - \varepsilon \leq \frac{1}{2} + \frac{1}{\sqrt{2}} M
\]
and in turn that
\[
M - \frac{2\varepsilon}{2 - \sqrt{2}} \leq \frac{1}{2 - \sqrt{2}} < 1.708 < 1.776 < \frac{5}{6} + \frac{\sqrt{8}}{3} = \max_{s, t \in [0, 1]} |x^*(t) - x^*(s)| \leq M.
\]

By taking $\varepsilon$ sufficiently small we obtain the desired contradiction.

Proof of Theorem 2.2 (b). One easily sees from Lemma 3.2 that $\max_{x \in \mathcal{X}} \max_{s, t \in [0, 1]} |x(t) - x(s)|$ is attained by $x^*$. We have $x^* = \hat{x}$ on $[0, 1/2]$, and so the maximum of $x^*$ is attained in $s = 1/3$ with value $1/3(2 + \sqrt{2})$, due to the first part of this theorem. Moreover, on $[1/2, 1]$ we have that $-x^*(t) = \hat{x}_{t-1/2} - 1/2$. It therefore follows from part (a) of the theorem that the minimum of $x^*$ is attained at $t = 5/6$ with minimal value $1/2 - 1/3(2 + \sqrt{2})$. Therefore,
\[
\max_{s, t \in [0, 1]} |x^*(t) - x^*(s)| = \frac{2}{3} (2 + \sqrt{2}) - \frac{1}{2} = \frac{5}{6} + \frac{\sqrt{8}}{3}.
\]

3.2 Proofs of Theorem 2.3 and of Corollary 2.4

Faber–Schauder functions $e_{n,k}$ can be linear with slope $\pm 2^{n/2}$ on intervals of length $2^{-(n+1)}$. Using this fact for $n \leq N - 2$ and for the interval $[t, t + 2^{-N}]$, we get
\[
|x(t + h) - x(t)| \leq \sum_{n=0}^{N-2} h 2^{n/2} + \sum_{n=N-1}^{\infty} 2^{n-1} \sum_{k=0}^{\infty} \theta_{n,k} (e_{n,k}(t + h) - e_{n,k}(t)), \tag{3.6}
\]
where \( \theta_{n,k} \) are the coefficients in the Faber–Schauder development of \( x \). The left-hand sum is equal to \( h(1 + \sqrt{2})(2^{(N-1)/2} - 1) \). This observation suffices in the situation of [20] and \([\star]\) to determine the corresponding moduli of continuity, because in these papers the Faber–Schauder coefficients are such that the sum on the right-hand side can be asymptotically neglected.

In our case, the sum on the right-hand side of (3.6) cannot be neglected. To deal with it, we note that the Faber–Schauder functions have the following scaling properties:

\[
\sqrt{2}e_{n,k}(t) = e_{n-1,k}(2t) \quad \text{and} \quad e_{n,k}(t - 2^{-n}) = e_{n,k+\ell}(t)
\]

(3.7)

for \( t \in \mathbb{R}, n \geq 0, \) and \( k, \ell \in \mathbb{Z} \) (see also (3.5)). For \( t \in [0, 1 - 2^{-N}] \) given, let \( \ell \in \mathbb{N}_0 \) be such that \( \ell 2^{-N} \leq t < (\ell + 1)2^{-N} \) and define \( s := t - \ell 2^{-N} \). Then both \( 2^{N-1}s \) and \( 2^{N-1}(s + h) \) belong to \( [0, 1] \). Then the scaling properties (3.7) imply that for \( n \geq N \),

\[
e_{n,k}(t + h) - e_{n,k}(t) = 2^{1-N} \left( e_{n-(N-1),k-\ell 2^n-N}(2^{N-1}(s + h)) - e_{n-(N-1),k-\ell 2^n-N}(2^{N-1}s) \right) \quad \text{(3.8)}
\]

The case \( n = N - 1 \) needs additional care, depending on whether \( \ell \) is even or odd.

**Case 1: \( \ell \) is even.** In this case, \([\ell 2^{-N}, (\ell + 2)2^{-N}] \) contains both \( t \) and \( t + h \) and is equal to the support of \( e_{N-1,\ell/2} \). Therefore, the identity (3.8) extends to \( n = N - 1 \) and we arrive at

\[
\sum_{n=N-1}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n,k} \left( e_{n,k}(t + h) - e_{n,k}(t) \right) = 2^{1-N} \left( y(2^{N-1}(s + h)) - y(2^{N-1}s) \right)
\]

(3.9)

for

\[
y := \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n+N-1,k+\ell 2^n-N} e_{n,k} \in \mathcal{X}^*.
\]

(3.10)

**Case 2: \( \ell \) is odd.** In this case, the interval \([\ell 2^{-N}, (\ell + 1)2^{-N}] \) contains \( t \) and belongs to the support of \( e_{N-1,\ell/2} \), while the interval \([((\ell + 1)2^{-N}, (\ell + 2)2^{-N}] \) belongs to the support of \( e_{N-1,(\ell+1)/2} \). It may or may not contain \( t + h \). We need to distinguish two further cases, depending on whether the coefficients \( \theta_{N-1,\ell/2} \) and \( \theta_{N-1,(\ell+1)/2} \) are identical or different.

**Case 2a: \( \theta_{N-1,\ell/2} = \theta_{N-1,(\ell+1)/2} \).** We have

\[
e_{N-1,\ell/2}(r) + e_{N-1,(\ell+1)/2}(r) = 2^{\frac{N-1}{2}} \left( 1 - e_{0,0}(2^{\frac{N-1}{2}} r - \ell/2) \right) \quad \text{for} \quad r \in [\ell 2^{-N}, (\ell 21)2^{-N}].
\]

We thus arrive at again at (3.9), but this time for

\[
y := -\theta_{N-1,\ell/2} e_{0,0} + \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n+N-1,k+\ell 2^n-N} e_{n,k} \in \mathcal{X}^*.
\]

**Case 2b: \( \theta_{N-1,\ell/2} = -\theta_{N-1,(\ell+1)/2} \).** Since \( e_{N-1,\ell/2}(r) - e_{N-1,(\ell+1)/2}(r) \) is linear on \([\ell 2^{-N}, (\ell + 2)2^{-N}] \) with slope \(-2^{\frac{N-1}{2}}\), we get

\[
|x(t + h) - x(t)| \leq \sum_{n=0}^{N-1} 2^{n/2} + \sum_{n=N}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n,k} \left( e_{n,k}(t + h) - e_{n,k}(t) \right) + 2^{\frac{N-1}{2}} \left( y(2^N(s + h)) - y(2^N s) \right),
\]
where
\[ y(r) = \begin{cases} y_0(r) & \text{if } 0 \leq r \leq 1, \\ y_1(r - 1) & \text{if } 1 \leq r \leq 2, \end{cases} \] (3.11)
and \( y_0, y_1 \in \mathcal{X}^* \) are given by
\[ y_0 := \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_n + N, k + \ell 2^n \epsilon_{n,k}, \quad y_1 := \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_n + N, k + (\ell + 1) 2^n \epsilon_{n,k} \] (3.12)

**Proof of Theorem 2.3 (a).** Since \( \theta_{n,k} = 1 \) for all \( n, k \), (3.6), (3.9), (3.10), and Theorem 2.2 (a) imply that in Case 1
\[ |\hat{x}(t + h) - \hat{x}(t)| \leq h(1 + \sqrt{2})(2^{(N - 1)/2} - 1) + 2^{1-N} \hat{x}(2^{N-1}(s + h))) - \hat{x}(2^{N-1}s)| \] (3.13)
\[ \leq h(1 + \sqrt{2})2^{(N - 1)/2} + \frac{2^{1-N}}{3}(2 + \sqrt{2}) = \hat{\omega}(h). \]

In Case 2a we get a similar estimate, but on the right-hand side of (3.13), \( \hat{x} \) needs to be replaced by
\[ \hat{y} := -\epsilon_{0,0} + \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \epsilon_{n,k}. \]
But note that \( \hat{y}(t) = \hat{x}(\varphi(t)) - 1/2 \), where \( \varphi(t) = 1/2 - t \) for \( t \leq 1/2 \) and \( \varphi(t) = t - 1/2 \) for \( t \geq 1/2 \). Therefore, \( |\hat{x}(t + h) - \hat{x}(t)| \leq \hat{\omega}(h) \) also holds in Case 2a. Finally, Case 2b cannot occur. This concludes the proof of “\( \leq \)”. To prove “\( \geq \)”, we let \( t := 0 \) and \( h_N := \frac{2}{3}2^{-N} \). Then we are in Case 1, and
\[ |\hat{x}(t + h_N) - \hat{x}(t)| = (1 + \sqrt{2})(2^{(N - 1)/2} - 1)h_N + 2^{(1-N)/2} \hat{x}(2^{N-1}h_N) = \hat{\omega}(h_N) - (1 + \sqrt{2})h_N, \]
by Theorem 2.2 (a). Since \( \hat{\omega}(h_N) = O(\sqrt{h_N}) \), we get “\( \geq \)”. \( \square \)

**Proof of Theorem 2.3 (b).** In Case 1, (3.6), (3.9), (3.10), and Theorem 2.2 (b) imply that for each \( x \in \mathcal{X} \)
\[ |x(t + h) - x(t)| \leq h(1 + \sqrt{2})(2^{(N - 1)/2} - 1) + 2^{1-N} \sup_{y \in \mathcal{X}} |y(2^{N-1}(s + h))) - y(2^{N-1}s)| \]
\[ \leq h(1 + \sqrt{2})2^{(N - 1)/2} + \frac{2^{1-N}}{6}(5 + 4\sqrt{2}), \]
which is easily seen to be dominated by \( \omega_{\text{max}}(h) \).

The same inequality as in Case 1 holds in Case 2a. In Case 2b, we obtain
\[ |x(t + h) - x(t)| \leq h(1 + \sqrt{2})(2^{N/2} - 1) + 2^{-N/2} \sup_{h \leq 2^{-N}} \sup_{0 \leq s \leq 1 - h} |y(2^N(s + h))) - y(2^Ns)|, \]

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where \( y \) is as in (3.11) for \( y_0, y_1 \in \mathcal{X} \). Clearly, the supremum on the right-hand side is attained for \( y_0 = y_1 = \hat{x} \) and then equal to \( \frac{2}{3} (2 + \sqrt{2}) \) according to Theorem 2.2 (a). Therefore,
\[
|x(t + h) - x(t)| \leq h(1 + \sqrt{2}) \frac{2^{N/2} + 2^{-N/2}}{3}(2 + \sqrt{2}) = \omega_{\text{max}}(h).
\]
This concludes the proof of “\( \leq \)”. To prove “\( \geq \)”, we define \( x \in \mathcal{X} \) through
\[
x_N := \sum_{n \geq 0, n \neq N-1} \sum_{k=0}^{2^{n-1}-1} e_{n,k} + \sum_{k=0}^{2^{N-1}-1} (-1)^k e_{N-1,k}.
\]
Then we let \( t_N := \frac{5}{3} 2^{-N} \) and \( h_N := \frac{2}{3} 2^{-N} \), so that we are in Case 2b with \( \ell = 1 \) and \( s = \frac{2}{3} 2^{-N} \). We consequently get from Theorem 2.2 (a) that
\[
|x_N(t_N + h_N) - x_N(t_N)| = h_N(1 + \sqrt{2}) \left( \frac{2^{N/2} - 1}{2} - 2^{-N/2} \right) \left| \hat{x}(1/3) - \hat{x}(2/3) \right| = \omega_{\text{max}}(h_N) - h_N(1 + \sqrt{2}).
\]
This concludes the proof of “\( \geq \)”.

**Proof of Corollary 2.4.** Theorem 2.2 implies that the family \( \mathcal{X} \) is uniformly bounded and Theorem 2.3 (b) yields that \( \mathcal{X} \) is equicontinuous. Therefore it only remains to show that \( \mathcal{X} \) is closed in \( C[0,1] \). Following Hata and Yamaguti [20], let \((x_m)_{m \in \mathbb{N}}\) be a sequence in \( \mathcal{X} \) that converges uniformly to some \( x \in C[0,1] \). We clearly have \( x(0) = x(1) = 0 \). It is well known that any such function can be uniquely represented as a uniformly convergent series of Faber–Schauder functions,
\[
x = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} a_{n,k}(x) e_{n,k},
\]
where the coefficients \( a_{n,k} \) are given as
\[
a_{n,k}(x) = 2^{n/2} \left( 2x \left( \frac{2k + 1}{2^{n+1}} \right) - x \left( \frac{k + 1}{2^{n}} \right) - x \left( \frac{k}{2^{n}} \right) \right).
\]
Clearly, we have \( a_{n,k}(x_m) \to a_{n,k}(x) \) for all pairs \( n, k \) as \( m \to \infty \). But since each \( x_m \in \mathcal{X} \) and the representation (3.14) is unique, we must have \( a_{n,k}(x_m) \in \{-1, +1\} \), which implies that also \( a_{n,k}(x) \in \{-1, +1\} \) and in turn that \( x \in \mathcal{X} \).

### 3.3 Proofs of Propositions 2.5 and 2.6

For \( x \in \mathcal{X} \) with Faber–Schauder expansion
\[
x = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \theta_{n,k} e_{n,k}
\]
and \(N \in \mathbb{N}\) we define
\[
x^N = \sum_{n=0}^{N-1} \sum_{k=0}^{2^n-1} \theta_{n,k} e_{n,k}.
\]

**Proof of Proposition 2.5.** Note that \(x(s) = x^N(s)\) for \(s \in \mathbb{T}_N\). Hence,
\[
\langle x \rangle^N_t = \sum_{s \in \mathbb{T}_N, s \leq t} (x(s') - x(s))^2 = \sum_{s \in \mathbb{T}_N, s \leq t} (x^N(s') - x^N(s))^2. \tag{3.15}
\]

We will now compute \(\langle x \rangle^N_t\) recursively. To this end, let \(s \in \mathbb{T}_N\) be such that \(s \leq t\). We have \(x^{N+1}(s) = x^N(s)\), \(x^{N+1}(s') = x^N(s')\), and, due to the linearity of \(e_{n,k}\) on \([s, s']\) for \(n \leq N\),
\[
x^{N+1}\left(\frac{s+s'}{2}\right) = x^N\left(\frac{s+s'}{2}\right) + \sigma \Delta_{N+1} = \frac{1}{2} (x^N(s) + x^N(s')) + \sigma \Delta_{N+1},
\]
where \(\sigma \in \{-1, 1\}\) and \(\Delta_{N+1} = 2^{-(N+2)/2}\) is the maximal amplitude of a Faber–Schauder function \(e_{N,t}\). Therefore, when passing from \(N\) to \(N + 1\), the term \((x^N(s') - x^N(s))^2\) in (3.15) will be replaced by
\[
\left(\frac{x^{N+1}(s') - x^{N+1}(s)}{2}\right)^2 + \left(\frac{x^{N+1}(s) - x^{N+1}(s')}{2}\right)^2
= \left(x^N(s') - \frac{1}{2} x^N(s) - x^N(s')\right)^2 + \left(x^N(s) - \frac{1}{2} x^N(s) - x^N(s')\right)^2
= \frac{1}{2} (x^N(s') - x^N(s))^2 + 2 \sigma^2 \Delta_{N+1}
= \frac{1}{2} (x^N(s') - x^N(s))^2 + 2^{-N-1}
\]
provided that \(\frac{s+s'}{2} \leq t\). When \(t \in \mathbb{T}_N\), the case in which \(s \leq t\) and \(\frac{s+s'}{2} > t\) can only happen for \(s = t\). So in this case the term \((x^N(s') - x^N(s))^2 = (x^N(t') - x^N(t))^2\) in (3.15) will be replaced by
\[
\left(\frac{x^{N+1}(t') - x^{N+1}(t)}{2}\right)^2 = \left(\frac{1}{2} (x^N(t) + x^N(t')) - \sigma \Delta_{N+1}\right)^2
= \frac{1}{4} (x(t + 2^{-N}) - x(t))^2 - \sigma 2^{-(N+2)/2} (x(t + 2^{-N}) - x(t)) + 2^{-N-2}.
\]
For \(t \in \mathbb{T}_N\) we thus obtain
\[
\langle x \rangle^{N+1}_t = \frac{1}{2} \langle x \rangle^N_t + \sum_{s \in \mathbb{T}_N, s \leq t} 2^{-N-1} + \varepsilon_{N+1} = \frac{1}{2} \langle x \rangle^N_t + \frac{1}{2} t + \varepsilon_{N+1},
\]
where
\[
\varepsilon_{N+1} = -\frac{1}{4} (x(t + 2^{-N}) - x(t))^2 - \sigma 2^{-(N+2)/2} (x(t + 2^{-N}) - x(t)) + 2^{-N-2}.
\]
By iterating this argument, we get for \(t \in \mathbb{T}_N\) and \(K \in \mathbb{N}\) that
\[
\langle x \rangle^{N+K}_t = 2^{-K} \langle x \rangle^N_t + t \sum_{i=1}^{K} 2^{-i} + \sum_{i=1}^{K} 2^{-K} \varepsilon_{N+i}.
\]

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Therefore, when passing from

\[ \lim_{K \uparrow \infty} (x)_t^{N+K} = t \sum_{i=1}^{\infty} 2^{-i} = t. \]

This proves that \( \lim_N (x)_t^N = t \) for all \( t \in \bigcup_N T_N \). But any given \( t \in [0,1] \) can be approximated arbitrarily close by \( s, u \in \bigcup_N T_N \) such that \( s \leq t \leq u \), and we have \( (x)^N_s \leq (x)_t^N \leq (x)_u^N \) for all \( N \). Thus, we must also have \( \lim_N (x)_t^N = t \) for all \( t \in [0,1] \).

**Proof of Proposition 2.6** The claim (2.12) follows immediately from (2.13) and (2.10). As in the proof of Proposition 2.5 we have that

\[
\langle \hat{x}, y \rangle_1^N := \sum_{s \in E_N} (\hat{x}(s') - \hat{x}(s))(y(s') - y(s)) = \sum_{s \in T_N} (\hat{x}^N(s') - \hat{x}^N(s))(y^N(s') - y^N(s)),
\]

that \( \hat{x}^{N+1}(s') = \hat{x}^N(s') \), \( \hat{x}^{N+1}(s') = \hat{x}^N(s') \), and that

\[
\hat{x}^{N+1}(\frac{s + s'}{2}) = \hat{x}^N(\frac{s + s'}{2}) + \Delta_{N+1} = \frac{1}{2}(\hat{x}^N(s) + \hat{x}^N(s')) + \Delta_{N+1},
\]

where \( \Delta_{N+1} = 2^{-(N+2)/2} \) is the maximal amplitude of a Faber–Schauder function \( e_{N,t} \). For \( y \) we have \( y^{N+1}(s') = y^N(s') \), \( y^{N+1}(s') = y^N(s') \), and

\[
y^{N+1}(\frac{s + s'}{2}) = y^N(\frac{s + s'}{2}) + (-1)^{N+1} \Delta_{N+1} = \frac{1}{2}(y^N(s') + y^N(s')) + (-1)^{N+1} \Delta_{N+1}.
\]

Therefore, when passing from \( N \) to \( N+1 \), each term \( (\hat{x}^N(s') - \hat{x}^N(s))(y^N(s') - y^N(s)) \) in (3.16) will be replaced by

\[
\left( \hat{x}^{N+1}(s') - \hat{x}^{N+1}(\frac{s + s'}{2}) \right) \left( y^{N+1}(s') - y^{N+1}(\frac{s + s'}{2}) \right) + \left( \hat{x}^{N+1}(\frac{s + s'}{2}) - \hat{x}^{N+1}(s) \right) \left( y^{N+1}(\frac{s + s'}{2}) - y^{N+1}(s) \right)\]

\[
= \left( \frac{1}{2}(\hat{x}^N(s') - \hat{x}^N(s)) - \Delta_{N+1} \right) \left( \frac{1}{2}(y^N(s') - y^N(s)) - (-1)^{N+1} \Delta_{N+1} \right) + \left( \frac{1}{2}(\hat{x}^N(s') - \hat{x}^N(s)) + \Delta_{N+1} \right) \left( \frac{1}{2}(y^N(s') - y^N(s)) + (-1)^{N+1} \Delta_{N+1} \right)\]

\[
= \frac{1}{2}(\hat{x}^N(s') - \hat{x}^N(s))(y^N(s') - y^N(s)) + 2(-1)^{N+1} \Delta_{N+1}^2 + (-1)^{N+1} 2^{-N-1}\]

So

\[
\langle \hat{x}, y \rangle_1^{N+1} = \frac{1}{2}(\hat{x}, y)_1^N + (-1)^{N+1} \sum_{s \in T_N} 2^{-N-1} = \frac{1}{2}(\hat{x}, y)_1^N + (-1)^{N+1}
\]

and in turn

\[
\langle \hat{x}, y \rangle_1^{2N+1} = \frac{1}{4}(\hat{x}, y)_1^{2N-1} + \frac{1}{4} \quad \text{and} \quad \langle \hat{x}, y \rangle_1^{2N+2} = \frac{1}{4}(\hat{x}, y)_1^{2N} - \frac{1}{4}.
\]

From here, (2.13) follows easily. \(\square\)
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