Dynamics of Multidimensional Secesson

Arne Soulier and Tim Halpin-Healy
Physics Department, Barnard College, Columbia University, NY NY 10027-6598
(September 1, 2002)

We explore a generalized Seceder Model with variable size selection groups and higher dimensional genotypes, uncovering its well-defined mean-field limiting behavior. Mapping to a discrete, deterministic version, we pin down the upper critical size of the multiplet selection group, characterize all relevant dynamically stable fixed points, and provide a complete analytical description of its self-similar hierarchy of multiple branch solutions.

Dynamical phenomena in which an initially homogeneous population of weakly interacting individual agents can disperse, aggregate and form clusters arises in many different physical, biological, and sociological contexts. Condensation and droplet formation is, of course, a well-known example in physics; galaxy formation and clustering is another. In traffic patterns, the real-space jams that plague highway driving are, for some, a daily reminder of such intrinsic tendencies in correlated systems far-from-equilibrium. The formation of swarms and herds in zoology, or the flocking of birds, provide additional illustrations. In these cases, particularly, joining the group yields advantages over standing out alone, be it by better exploration of food resources, protection from predators, or easing the aerodynamic flow in flight. Nevertheless, sometimes, as in fashion trends and similar social (or even financial) settings, standing apart from the crowd can also be a seed for the formation of new groups, splitting off the mainstream, though maybe becoming the mainstream themselves later on. In these instances, steady-state multiple groups can be the norm. Such matters are manifest in recent, though now classic implementations of Arthur’s variant of the El Farol Bar problem, as for example, discussed by Zhang and Challet, where a multitude of competing agents, armed with limited memory strategies, compete via statistical Sisyphian dynamics to be in the minority group. Interestingly, with stochasticity introduced to the decision-making process, Johnson and coworkers uncovered a tendency towards self-organized segregation within such evolutionary minority games. Subsequently, Hod & Nakar discovered a dynamical phase transition in this setting, between 2-group segregation and single group clustering, driven by the economic cost-benefit ratio defined in the model. In biological systems, clustering can appear on multiple scales, with aggregation a consequence of dynamic correlations, whether they be hidden or explicit. Even so, the complexity that arises, for example, in statistical models of evolution, resulting in the formation of species is not, per se, self-evident via the direct interplay of mutation and selection- the system can dynamically bring itself to a critical state. The Seceder Model was introduced, initially, to demonstrate that an interactive mechanism favoring individuality cannot only create distinct groups, but also yields a rich diversity of cluster-forming dynamics. The essential tack was to give a small advantage to individuals that distinguish themselves from others. This is not unnatural, since in epidemics, for example, genetic differences can enhance long-term survival probabilities. Likewise, for players in a minority game, distinctness may be the advantageous property. In this Letter, we consider the Seceder Model in its broadest sense. Its description is simple enough: within a population of individuals, each described by a genotype variable, choose a subset from the population and calculate its average. From within this selection multiplet, the individual most distant from the mean is the parent to be. Create an offspring by taking this parent’s value plus a small uniform deviate. Finally, replace a randomly chosen member of the population by this new offspring. The process is then iterated through many generational time-steps.

Despite the complexity of its segregative dynamics, the Seceder Model may be amenable to traditional methodological approaches. For example, the nontrivial scaling exhibited by the Seceder envelope, as well as the interesting time evolution of the group number could be thought of as fluctuation-dominated non-classical behavior. In this spirit, it is natural to consider the dimensionality dependences inherent in the model, with the expectation of finding a simpler, mean-field or classical nonequilibrium dynamics within some sector of this larger parameter space. Some wisdom in this regard may be had from the Bak-Sneppen model of punctuated evolution, wherein a similar, innocuously trim update algorithm engenders an extraordinarily rich spatiotemporal dynamics. Even so, a mean-field limit of this model was subsequently engineered and further explored, simplified scaling retrieved by introducing system-wide correlations to the interactions. Here, for the Seceder Model, we’re motivated by similar goals. Clearly, the number of parameters is restricted- we have the population size N, which is understood to diverge in the thermodynamic limit. There is also the size m of the multiplet selection group; finally, the dimensionality d of the genotype variable, which determines the nature of the base-space through which the population groups mark their trajectories, provides an additional degree of freedom.

We begin by enlarging the selection group from which the parent is chosen. Naively, we’d expect the limit m → N, which introduces increasing cross-correlation within the society, to elicit eventually, a mean-field type of behavior, if only in the extreme case when m = N, when we’re averaging over the entire population using
the societal mean to determine the most distant, reproduced individual. Indeed, this is the case. The surprise, however, comes with the abruptness of the transition. There is, already, a marked change of behavior as we switch from a triplet \(m = 3\) to a quartet \(m = 4\) selection group. In Figure 1, we show single runs of the \(d = 1\) Seceder Model using multiplet groups \(m = 3 - 8\), within an essentially infinite population, \(N = 512\). For \(m = 3\), we have trademark Seceder demeanor, with self-similar branching characterized by three dominant, but fluctuating arms, centered about the origin, with ample small-scale stochastic structure associated with the transient appearance of variously short-lived subbranches. Rather than a gradual transition, we find for \(m = 4\) that the typical stable configuration suddenly involves two groups, not three. In addition, these two branches exhibit only the most modest sorts of fluctuations, as is evident from the figure. Increasing the selection group to \(m = 5\) further diminishes the fluctuations, but hardly affects the tilt of what seems to be the nearly linear divergence of the two groups. Next, for \(m = 6\&7\), there is, strangely, a discrete jump to an altogether different, but closely allied pair of trajectories. With \(m = 8\&9\), another jump, and so it goes with each successive even-odd pair of multiplet selection groups. As \(m \rightarrow N\), the trajectories form an extremely well-defined V-shaped wedge, with little fluctuation at all, as we’d expect of a mean-field limit. Thus we see that the dominant dynamic of secession involves, for \(m \geq 4\), segregation into two evenly populated opposing groups with a free interchange of individuals over the course of time. Similar self-organized segregation was reported recently by Johnson et al., within the context of an evolutionary minority game. Ensemble averaging over many realizations, we have systematically studied the growth of the population diameter over time. Only for triplet selection, \(m = 3\), do we find a fractional power-law dependence, the diameter asymptotically scaling with an exponent very close to \(3/4\), our measured value being \(0.74±0.01\) for this one dimensional case.

With the Seceder Model defined as above, the genotype space is a continuum. Clearly, discretizing the model alters no essential features. Indeed, much can be gleaned by considering this discrete Seceder Model in its deterministic limit, wherein the most distinct individual is reproduced exactly, rather than yielding a merely approximate next of kin. One is lead to a set of nonlinear coupled ODEs, first-order rate equations for the concentration simplex \((x_1, x_2, ..., x_B)\), describing the evolution of the discrete set of \(B\) genotypes possible within the population: \(\dot{x}_j = \sum_{i_1, ..., i_m = 1}^B \alpha_{i_1}^{i_2}...i_m x_{i_1} x_{i_2} ... x_{i_m} - x_j\), for \(j \in \{1, ..., B\}\). These equations transform the Seceder Model into an evolving chemical reaction system whose dynamics are dictated by the law of mass kinetics and the constraints of unit dilution flux, possessing some features reminiscent of earlier efforts on the hypercycle model, and generalized replicator equations. Here, one is looking at the stability of an \(B\)–branch solution generated by \(m\)–multiplet selection group dynamics. The coefficients are zero unless the genotype/individual is the distant outlier- either in isolation, in which case \(\alpha = 1\), or as happens occasionally, sharing that distinction with \(p\) other individuals, with \(\alpha = 1/p\). As a practical matter, the coefficients, combinatoric in origin, can be generated systematically via a multinomial expansion \((x_1 + x_2 + ... + x_B)^m\) and then carefully dividing numerical prefactors in appropriate proportions amongst relevant rate variables \(\dot{x}_j\). Probability conservation demands this connection to the multinomial expansion, but it’s the parceling out of terms that guarantees the complexity of the model. With this set of ODEs in hand, essentially providing a coarse-grained real-space renormalization-group [RSRG] prescription of the original Seceder Model, we follow the flow equations for the concentration variables, characterizing all relevant fixed points. Within this broader \(mB\) space, the \(d = 1\) Seceder Model exhibits its full richness. As an indication of the wealth of this geometric pattern formation, consider for the moment triplet selection dynamic, \(m = 3\), where a self-similar hierarchy of multibranch fixed points emerges. We examine, to illustrate, the case \(B = 4\), for which:

\[\dot{x}_1 = x_1^3 + 3x_1(x_2^2 + x_3^2 + x_4^2) + 3x_1x_2x_3 + 6x_1x_3x_4 - x_1\]

\[\dot{x}_2 = x_2^3 + 3x_2(x_1^2 + x_3^2 + x_4^2) + 3x_2x_3x_4 - x_2\]

Because of branch symmetry about the central axis, the flow equations for the remaining variables are easily obtained via the interchange \(x_1 \leftrightarrow x_4\) and \(x_2 \leftrightarrow x_3\) and, indeed, the globally stable fixed point (FP) must lie within this reduced subspace, with mirror variables identified. For the case at hand, invoking the constraint \(x_2 = 1/2 - x_1\) and demanding \(\dot{x}_1 = 0\) leads to the cubic equation \(7x_1^3 - 6x_1^2 + 5/4x_1 = 0\), yielding \((x_1, x_2, x_3, x_4) = (\frac{1}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14})\), as well as the less stable 2–branch solutions \((\frac{1}{3}, 0, 0, \frac{1}{3})\) and \((0, \frac{1}{3}, \frac{1}{3}, 0)\). Of course, if we simply numerically integrate the coupled ODEs and follow the trajectories from a randomly generated initial condition, we find with 100\% probability to our unique superstable FP. The situation for \(B = 5\) is slightly different- insisting upon symmetry \(x_5 = x_1\) and \(x_4 = x_2\), in addition to the
normalization constraint $x_3 = 1 - 2x_2 - 2x_1$, we have the recurring 2-branch solutions with $x_1 = 0$ and $x_2 = 1/2$ and vice versa, leaving us with two coupled bilinear equations in the variables $x_1$ and $x_2$, represented graphically as a rotated, displaced ellipse and hyperbola within the unit square. There are two intersection points: one full-fledged superstable 5-branch solution, $(\frac{2}{17}, \frac{5}{17}, \frac{9}{17}, \frac{9}{17}, \frac{1}{17})$, the other, an unstable lower-dimensional 4-branch solution $(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$. Note that, in the latter instance, $x_1 + x_2 = 1/2 = x_4 + x_5$, so that this unstable solution can, thanks to the gap $x_3 = 0$, be understood, via coarse-graining, as literally self-similar to its two-branch cousin $(\frac{1}{2}, 0, \frac{1}{2})$. This sort of hierarchical connection manifests itself regularly whenever we uncover a lower-dimensional fixed point; i.e., vanishing $x_i$ in the branch structure. The most compelling instance of this phenomenon appears when we search for a superstable 8-branch FP. In fact, there is none. The ODE flows converge on a peculiar 6-branch solution, $(\frac{1}{8}, \frac{3}{8}, 0, \frac{3}{8}, \frac{5}{8}, 0, \frac{3}{8}, \frac{5}{8})$, which is exactly self-similar to the strongly attractive 3-branch fixed point $(\frac{1}{3}, 0, \frac{1}{3}, 0)$. Interestingly, this lesser 8-branch solution is distinct from the straight out 6-branch, roughly $(0.30, 0.12, 0.08, 0.08, 0.12, 0.30)$, easily shown to be irrational, as is the 7-branch and all those beyond 8. The 9, 10 and 12-branch FPs show no zeros, but such behavior becomes increasingly rare. The 11-branch solution has two gaps, $x_4 = x_9 = 0$, but with $x_1 = x_1' \approx 0.271, x_2 = x_1' \approx 0.089, x_3 = x_3' \approx 0.044$ and $x_5 = x_5' \approx 0.047, x_6 \approx 0.100$, can be coarse-grained to a broad 3-branch again, though in this case only approximately, self-similar to our dominant FP $(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. Likewise, the 15-branch, although 19 and 21-branch FPs show three gaps and a self-similarity to the 4-branch $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

An additional payoff of this RSRG treatment of the Seceder Model is an explanation of the relative stability of 2 and 3-branch solutions for triplet ($m = 3$) and higher multiplet ($m \geq 4$) selection groups. Recall Figure 1, which made it clear that for large $m$, the dynamics of the $d = 1$ stochastically Seceder Model are controlled entirely by the strongly attractive fixed point of the 2-branch pattern. For triplet selection, however, a hierarchy of multi-branch solutions is manifest, which meant the frequent appearance of a 3-branches, and somewhat occasionally 4 and 5-branches, but a complete absence of 2-branch behavior. The essential dichotomy can be understood graphically by following the flows for 3-branch dynamics in the deterministic Seceder Model, assuming triplet selection group $m = 3$, illustrated in Figure 2a, where we show the [111] plane $x_1 + x_2 + x_3 = 1$. We find our superstable fixed point, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, within the unit triangle. We note, in particular, that the outmost 2-branch solution $(\frac{1}{2}, 0, \frac{1}{2})$ is unstable to small perturbations off the edge. In turn, the simple branch FPs at the triangle vertices are entirely unstable. As $m \to 4$, however, this interior FP merges with that at the midpoint on the triangle’s lower edge, reversing the flow and stabilizing the 2-branch dynamics. From this vantage point, it is clear that quartet, rather than triplet, selection is the marginal case, a fact quickly confirmed by a stability analysis of the $(\frac{1}{2}, 0, \frac{1}{2})$;

![FIG. 2. a) RSRG flows in $x_1:x_2:x_3$—space for the $d = 1$ deterministic Seceder Model. A superstable 3-branch FP exists within the equilateral triangle for $m < 4$ only.]
Indeed, that is precisely the characteristic behavior of the triplet Seceder Model, where 3-branch dynamics are typically seen, with occasional 4, 5 or 6-branch runs.

We should stress, in this regard, that the 3-branch solution is an extraordinarily robust feature of this model, becoming even more so in higher dimensions, where the genotype is specified by an $d$-component vector rather than a single real number, the case we’ve focussed on thus far. For example, in $d=2$, an initially homogeneous, or highly polarized population for that matter- see Figure 3, will eventually segregate into three distinct groups heading off along the symmetry axes of a triangle, each cluster equidistant from the other two. In $d=3$ dimensions, we might expect four groups, perhaps, localized at the corners of an expanding tetrahedron, preserving the notion of equal distance. Interestingly, however, this does not happen at all. Again, we observe the formation of just three groups- note Figure 4; the effect is stronger still for $d \geq 4$. Apparently, asymptotic higher-dimensional secession involves segregational collapse to a greatly reduced, two-dimensional, subspace- the hyperplane defined by three fuzzy points of an expanding equilateral triangle whose angular orientation may vary from one realization to the next, but whose essential geometry does not. Interestingly, this dimensional reduction can be understood within the context of the deterministic model- one considers the stability of $d+1$ equally separated groups in $d$ dimensions to perturbations (ultimately, statistical in nature) that bring one group closer to the rest [19]. For $d \geq 3$, we find that the minority group is unstable and will go extinct, whereas for $d < 3$, the zero FP associated with this vanishing group reverses stability, yielding three separatist clusters whose relative population sizes are set by the degree of symmetry breaking.

In sum, we have revealed the mean-field limit of the multidimensional Seceder Model. For selection multiplet sizes $m \geq m_c = 4$, the nonequilibrium dynamics produce a steady-state with two opposing groups, independent of $d$. In the extreme societal Seceder limit ($m \to N$), the noisy dynamics dies away, leaving two tightly knit groups. For $m = 3$, multiple groups are typical, with three the norm. Higher dimensional genotypes/strategies produce, surprisingly, no further fragmentation. Using a coarse-grained RG prescription, which discretizes and renders deterministic the model, we analytically uncover a self-similar hierarchy of multiple branch FPs in a gapped spectrum. Additional work, concerning the intermittent extinction dynamics of individual groups, early-time transient behaviors, as well as kinetic symmetry-breaking phenomena, will be reported elsewhere [20].

Financial support for Tim HH has been provided by NSF DMR-0083204, Condensed Matter Theory.

[1] J. A. Blackman, et al., Phys. Rev. Lett. 84, 4409 (2000).
[2] M. Drinkwater, Science 287, 1217 (2000).
[3] F. Schweitzer, et al., Phys. Rev. Lett. 80, 5044 (1998).
[4] J. Parrish, et al., Science 284, 99 (1999).
[5] J. Toner and Y. Tu, Phys. Rev. Lett. 75, 4326 (1995).
[6] E.V. Albano, Phys. Rev. Lett. 77, 2129 (1996).
[7] W.B. Arthur, Am. Econ. Assoc. Proc. 84, 406 (1994).
[8] D. Challet and Y.-C. Zhang, Physica A 246, 407 (1997).
[9] N.F. Johnson, et al., Phys. Rev. Lett. 82, 3360 (1999).
[10] S. Hod and E. Nakar, Phys. Rev. Lett. 88, 238702 (2002).
[11] B. Drosel, Phys. Rev. Lett. 82, 5144 (1999).
[12] S. Kauffman, The Origins of Order (Oxford Press, 1993).
[13] P. Dittrich, et al., Phys. Rev. Lett. 84, 3205 (2000).
[14] P. Bak and K. Sneppen, Phys. Rev. Lett. 71, 4083 (1993).
[15] H. Flyvbjerg, et al., Phys. Rev. Lett. 71, 4087 (1993).
[16] B. Derrida et al., Phys. Rev. Lett. 73, 906 (1994).
[17] M. Marsili et al., Phys. Rev. Lett. 80, 1457 (1998).
[18] see, e.g., J. Hofbauer and K. Sigmund, Evolutionary Games & Population Dynamics (Cambridge Press, 1998).
[19] we break the symmetry by hand for the odd group, writing its ODE coefficient $\alpha = (1 - K)/d$. Stability analysis reveals that critical symmetry-breaking parameter $K_c = 2/(d - 1)$ decreases with dimensionality; statistical fluctuations thereby induce cascadal reduction to $d = 2$. 
[20] A. Soulier, N. Arkus and T. Halpin-Healy, in preparation.