Stress and strain intensity factors within angle area of the boundary

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Abstract. Stress and strain state of the designs and structures within the area (zone) with angle cutouts, boundary cuts are characterized by emergence of the stress concentration areas and requires assessment of strength and reliability of facilities that is an actual task of the engineering practice. The theoretical study of stress and strain state within angle cutouts of the boundary area is reduced to research of singular solutions for the elastic problem with power singularity. Moreover, definition of stress or strain concentration in the non-regular point of the area boundary loses its significance. The stress and strain state within angle cutout top of the plane area boundary, which is recorded with use of intensity factor, is considered in the article. Two approaches to obtaining of formula for displacements and stresses within non-regular point of the plane area boundary with use of the stress and strain intensity factor are given. A difference in formula of stresses and displacement obtained for the limit values of stress and strain determine the practical significance of the work during experiments and determination of critical values for stress and strain.

1. Introduction
The stress-strain state of designs and structures with a geometrically non-linear border shape - cutouts, boundary cuts, is characterized by the emergence of the stress concentration areas, deformations with significant quantities and their gradients.

The solution of the physically linear problem of the theory of elasticity in the geometrically linearized formulation of the boundary conditions is determined by infinite stresses, strains and their gradients at the apex of an ideal cut or angular cut of the boundary of the area [1-7]. The concept of stress concentration at an irregular point of the area boundary loses its meaning.

The study of the stress-strain state of designs and structures in the area of an irregular point of the boundary is characterized by solving a homogeneous boundary value problem of the theory of elasticity [1-4, 8-14].

The power feature of the stress-strain state is due to the geometry of the shape of the boundary of the area — cutout, boundary cut, the order of which depends on the eigenvalues of the homogeneous elastic problem [1-2, 7-8, 15-16]. The eigenvalues of the homogeneous boundary-value problem depend on the shape of the boundary, the type of boundary conditions, the mechanical characteristics of the material of the area, and have many values [2, 7-10, 16].

The proper (singular) solutions of homogeneous boundary value problem are determined with an accuracy of unknown arbitrary constants.

Excluding the irregular point itself and its small neighborhood, it is possible to characterize the stress-strain state in a small area of the top of the cut as an asymptotic solution of an elastic homogeneous problem, depending on the unknown constants (intensity coefficients).

In fracture mechanics for an ideal mathematical section, singular stresses are characterized by stress intensity factors [4, 8, 9, 15].

In this work, the stress-strain state in the corner cut-out area is characterized by stress intensity factors and strain intensity factors that do not match.
The purpose of this research work is to develop a general approach for determining stress intensity factors and coefficients of intensity of deformations as stress value limits and deformations within an angular cut of the area boundary.

This research work considers two approaches to obtaining expressions for displacements and stresses in the vicinity of the apex of the cut-out of the flat area boundary using the stress intensity factors and the strain intensity factors. The expressions of stresses and displacements recorded with the help of stress intensity factors and strain intensity factors determine the practical significance of the study work when conducting experiments and determining critical values of stresses and strains.

The task of studying the intensity factors of stresses and strains is relevant for the study of the stress-strain state of designs and structures with geometric nonlinear shape of the boundaries.

2. Method of research

2.1. Problem Setting. Case of Plane Strain

The stress and strain state of composite structures within the areas of element mating under action of forced strains rupturing along the element contact line (surface) is characterized by the feature of power type. Stress and strain state within non-regular point of the area boundary under action of forced strains, particularly, temperature ones is determined by solving the uniform elastic boundary value problem. The concentrated efforts at the top of the cut are not considered. Solving of the uniform elastic problem within the area of non-regular point on special line is reduced [1, 2, 3] to solving the plane strain problem: 

\[ \sigma_x \neq 0, \sigma_y \neq 0, \tau_{xy} \neq 0, \tau_{sc} = \tau_{sc} = 0, \frac{\partial \sigma_{zz}}{\partial z} \neq 0, \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \]

and anti-plane strain problem: 

\[ \varepsilon_{xy} \neq 0; \varepsilon_{y} \neq 0; W \neq 0; n_z = 0, \varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{yy} = 0, U = V = 0. \]

Solving of the plane problems in the polar coordinate system is sought by variable separation method [3].

The uniform elastic boundary value problem for plane V-shaped area with symmetrical opening \(2\alpha\) within angle boundary point – top of the angle cutout with random opening is considered [1-5, 15-16]. Solving of the plane strain problem in displacements in polar coordinate system shall be sought as follows:

\[ u_r(r, \theta) = r^\alpha f(\theta), \quad u_\theta(r, \theta) = r^\alpha g(\theta) \]

where \( f(\theta), \ g(\theta) \) – unknown function of angle \( \theta \) to be determined, \( \lambda \) – unknown parameter. Substituting ratios (1) in Lame equation, formula for displacements and stresses within non-regular point of the area boundary are obtained [1-5, 7, 9, 15]:

\[ r^{-\lambda} u_r = Acos[(1 + \lambda)\theta] + Bsin[(1 + \lambda)\theta] + Ccos[(1 - \lambda)\theta] + Dsin[(1 - \lambda)\theta], \]

\[ r^{-\lambda} u_\theta = Bcos[(1 + \lambda)\theta] - A\sin[(1 + \lambda)\theta] + \nu_2 Dcos[(1 - \lambda)\theta] - \nu_2 C\sin[(1 - \lambda)\theta], \]

\[ \mu^{-1}r^{1-\lambda} \sigma_\theta = -2\lambda Acos[(1 + \lambda)\theta] - 2\lambda Bsin[(1 + \lambda)\theta] - (1 + \lambda)(1 - \nu_2) C\cos[(1 - \lambda)\theta] - (1 + \lambda)(1 - \nu_2) D\sin[(1 - \lambda)\theta], \]

\[ \mu^{-1}r^{1-\lambda} \tau_{\theta\phi} = -2\lambda A\sin[(1 + \lambda)\theta] + 2\lambda B\cos[(1 + \lambda)\theta] - (1 - \lambda)(1 + \nu_2) C\sin[(1 - \lambda)\theta] + (1 - \lambda)(1 - \nu_2) D\cos[(1 - \lambda)\theta], \]

\[ \mu^{-1}r^{1-\lambda} \sigma_r = 2\lambda[(1 + \lambda)\theta] + B\sin[(1 + \lambda)\theta] + \frac{3 - \lambda}{k - \lambda} C\cos[(1 - \lambda)\theta] + \frac{3 - \lambda}{k - \lambda} D\sin[(1 - \lambda)\theta], \]

\[ \sigma_\theta \neq 0; \sigma_y \neq 0; W \neq 0; n_z = 0, \varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{yy} = 0, U = V = 0. \]
where \( k = 3 - 4\nu \), \( \nu_2 = \frac{3 + \lambda - 4\nu}{3 - \lambda - 4\nu} \), factors \( A, B, C, D \) – random constants to be determined; \( \lambda \) – proper values of uniform boundary problem, generally, complex numbers, \( \nu, \mu \) – Poisson ratio and shear modulus accordingly.

Meeting the uniform boundary conditions: \( \sigma_{\theta} = \tau_{r\theta} = 0 \) at \( \theta = \pm \alpha \), two uniform linear equation systems are obtained in regard to unknown constants \( A, B, C, D \), which solving determines the ratios for the following factors:

\[
A = \frac{(1-\lambda)\sin[(1-\lambda)\alpha]}{(k-\lambda)\sin[(1+\lambda)\alpha]}
\]

\[
B = \frac{(1+\lambda)\sin[(1-\lambda)\alpha]}{(k-\lambda)\sin[(1+\lambda)\alpha]}
\]

\[
C = \frac{(k-\lambda)\sin[(1+\lambda)\alpha]}{2\lambda[(\lambda-1)\sin[(1-\lambda)\alpha] + (\lambda+1)\sin[(1+\lambda)\alpha]]}
\]

\[
D = \frac{(k-\lambda)\sin[(1+\lambda)\alpha]}{2\lambda[(\lambda-1)\sin[(1-\lambda)\alpha] + (\lambda+1)\sin[(1+\lambda)\alpha]]}
\]

The stress and strain state (2) is obtained with limit values of stresses within non-regular point of the boundary called stress intensity factors similar to stress intensity factors in mechanics of damage [3, 8, 12-13].

2.2. Area Stress Intensity Factors

To determine factors \( A \) and \( C \) in formula (2) for stresses \( \sigma_{\theta} \), go to the limit at \( \theta = 0 \):

\[
\lim_{r \to 0} \mu^{-1} r^{1-\lambda} \sigma_{\theta, \theta=0} = -2A\lambda - (1 + \lambda)(1 - \nu_2)C
\]

Specify

\[
K^\sigma_{\theta} = \lim_{r \to 0} \mu^{-1} r^{1-\lambda} \sigma_{\theta, \theta=0}
\]

Considering ratios (3) for \( A \) and \( C \), symbols (4), (5) we obtain unknown factors in formula (2) as follows:

\[
A = \frac{(1-\lambda)\sin[(1-\lambda)\alpha]}{(k-\lambda)\sin[(1+\lambda)\alpha]}
\]

\[
C = \frac{(k-\lambda)\sin[(1+\lambda)\alpha]}{2\lambda[(\lambda-1)\sin[(1-\lambda)\alpha] + (\lambda+1)\sin[(1+\lambda)\alpha]]}
\]

To determine factors \( B \) and \( D \) in formula (2) for stresses \( \tau_{r\theta} \), go to the limit at \( \theta = 0 \):

\[
\lim_{r \to 0} \mu^{-1} r^{1-\lambda} \tau_{r, \theta=0} = 2B\lambda + (1 + \lambda)(1 - \nu_2)D
\]

Specify

\[
K^\tau_{\theta} = \lim_{r \to 0} \mu^{-1} r^{1-\lambda} \tau_{r, \theta=0}
\]

Considering ratios (3) for \( B \) and \( D \), symbols (8), (9) we obtain unknown factors in formula (2) as follows:

\[
B = \frac{(1+\lambda)\sin[(1-\lambda)\alpha]}{2\lambda[(\lambda + 1)\sin[(1-\lambda)\alpha] - (1-\lambda)\sin[(1+\lambda)\alpha]]}
\]

\[
D = \frac{(k-\lambda)\sin[(1+\lambda)\alpha]}{2\lambda[(\lambda + 1)\sin[(1-\lambda)\alpha] - (1-\lambda)\sin[(1+\lambda)\alpha]]}
\]

Factors

\[
K^\sigma_{\theta} = \lim_{r \to 0} \mu^{-1} r^{1-\lambda} \sigma_{\theta, \theta=0}
\]

\[
K^\sigma_{\theta} = \lim_{r \to 0} \mu^{-1} r^{1-\lambda} \tau_{r, \theta=0}
\]

by analogy with symbols taken in the mechanics of damage [1, 2, 13–16] shall be determined as stress intensity factors, where
\[ \lambda = \min|\text{Re} \lambda |. \]

Considering the detected factors (6), (7), (10), (11) and symbols of limit stress values as stress intensity factors in form of (5), (9), initial stress and strain state within non-regular point of the area boundary (2) shall be recorded as a function of these factors.

2.3. Area Strain Intensity Factors

Specify formula for strains
\[ r^{1-\delta} \varepsilon_{\theta} = -\frac{(1 + \lambda - 4\nu)}{k - \nu} C \lambda \cos[(1 - \lambda)\theta] + \frac{(1 + \lambda - 4\nu)}{k - \nu} D \lambda \sin[(1 - \lambda)\theta], \] (12)

To determine factors A and C in formula (3) for strains \( \varepsilon_{r, \theta} \), go to the limit at \( \theta = 0 \):
\[ \lim_{\theta \to 0} r^{1-\delta} \varepsilon_{r, \theta} = \lambda [-A + \frac{(1 + \lambda - 4\nu)}{k - \nu} C]. \] (13)

Specify
\[ K_{\varepsilon}^{i} = \lim_{\theta \to 0} r^{1-\delta} \varepsilon_{r, \theta} = 0. \] (14)

Consideration ratios (3) for A and C, symbols (13), (14) we obtain unknown factors in formula (2) as follows:
\[ C = \frac{(k - \lambda)\sin[(1 + \lambda)\alpha]}{\lambda[(1 + \lambda - 4\nu)\sin[(1 + \lambda)\alpha] - (1 - \lambda)\sin[(1 - \lambda)\alpha]]} K_{\varepsilon}^{i}, \] (15)
\[ A = \frac{(1 - \lambda)\sin[(1 - \lambda)\alpha]}{\lambda[(1 + \lambda - 4\nu)\sin[(1 + \lambda)\alpha] - (1 - \lambda)\sin[(1 - \lambda)\alpha]]} K_{\varepsilon}^{i}. \] (16)

To determine factors B and D in the formula for strains \( \varepsilon_{r, \theta} \), go to the limit at \( \theta = 0 \):
\[ r^{1-\delta} \varepsilon_{r, \theta} = -2\lambda \sin[(1 + \lambda)\theta] + 2\lambda B \cos[(1 + \lambda)\theta] - (1 - \lambda)(1 - \nu) C \sin[(1 - \lambda)\theta] + (1 - \lambda)(1 - \nu) D \cos[(1 - \lambda)\theta]. \] (17)

Specify
\[ K_{\varepsilon}^{j} = \lim_{\theta \to 0} r^{1-\delta} \varepsilon_{r, \theta} = 0. \] (19)

Considering ratio
\[ \varepsilon_{r, \theta} = \mu^{-1} \tau_{r, \theta} \]
we obtain
\[ K_{\varepsilon}^{j} = \lim_{\theta \to 0} r^{1-\delta} \varepsilon_{r, \theta} = \lim_{\theta \to 0} \mu^{-1} r^{1-\delta} \tau_{r, \theta} = K_{\varepsilon}^{j}, \]
thus factors B and D have the same form (10), (11), i.e.
\[ B = \frac{(1 + \lambda)\sin[(1 - \lambda)\alpha]}{2\lambda[(\lambda + 1)\sin[(1 - \lambda)\alpha] - (1 - \lambda)\sin[(1 + \lambda)\alpha]]} K_{\varepsilon}^{j}, \] (20)
\[ D = \frac{(k - \lambda)\sin[(1 + \lambda)\alpha]}{2\lambda[(\lambda + 1)\sin[(1 - \lambda)\alpha] - (1 - \lambda)\sin[(1 + \lambda)\alpha]]} K_{\varepsilon}^{j}. \] (21)

Factors
\[ K^e_I = \lim_{r \to 0} r^{3/2} \varepsilon_r \theta, \theta = 0 \]
\[ K^\nu_{II} = \lim_{r \to 0} r^{3/2} \varepsilon_{r, \theta} \theta, \theta = 0 \]

by analogy with symbols taken in the mechanics of damage [3, 8, 13–15] shall be determined as strain intensity factors, where

\[ \lambda = \min |Re \hat{\lambda}|. \]

Considering the detected factors (15), (16), (20), (21) and symbols of limit strain values as strain intensity factors in form of (14), (19), initial stress and strain state within non-regular point of the area boundary (2) shall be recorded as a function of these factors.

For the anti-plane strain problem solving within the non-regular point of the area boundary shall be recorded similarly as follows:

\[ w = r^\lambda [ D_1 \sin \lambda \theta + D_2 \cos \lambda \theta], \]
\[ \mu^{-1} r^{1-\lambda} \tau_{xx} = \lambda_1 [ -D_1 \sin(1-\lambda_1) \theta + D_2 \cos(1-\lambda_1) \theta], \]
\[ \mu^{-1} r^{1-\lambda} \tau_{yy} = \lambda_1 [ D_1 \cos(1-\lambda_1) \theta + D_2 \sin(1-\lambda_1) \theta], \]

where \( D_1, D_2 \) – random constants to be determined, where \( \lambda_1 \) – proper values of uniform boundary problem.

Specifying

\[ K^1_{II1} = \lim_{r \to 0} \mu^{-1} r^{1-\lambda} \tau_{y, \theta}, \theta = 0, \quad K^2_{II1} = \lim_{r \to 0} \mu^{-1} r^{1-\lambda} \tau_{x, \theta}, \theta = 0, \]

Factors \( D_1 \) and \( D_2 \) are recorded, moreover, the stress and strain intensity factors numerically coincide.

3. Results

The procedure for determination of stress and strain state (2) within non-regular point of the plane area boundary is performed. The unknown factors are determined with use of limit stress values as stress intensity factors (5), (9) in the form of (6), (7), (10), (11) and with use of limit strain values as strain intensity factors (14), (19) in the form of (15), (16), (20), (21).

4. Discussion

Difference in stress and strain intensity factors for normal stresses and linear strains is detected. The intensity factors for shear stresses and angular strains coincide. In the main case of stress and strain state within the cut top of the area boundary one cannot speak about proportionality of stress and strain intensity factors that shall be considered during experimental determination of their values.

Conclusion

The novelty of the obtained results consists in the development of a general approach for determining stress intensity factors and strain intensity factors, which do not coincide in general case. The stress and strain state within the angle cut outs of the designs and structure boundary recorded in the form of limit values for stress and strain can be used in future for determination of stress and strain intensity factors that is an independent problem of mechanics of solids.

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