COUPLING AND DECOUPLING TO BOUND AN APPROXIMATING MARKOV CHAIN

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This simple note lays out a few observations which are well known in many ways but may not have been said in quite this way before. The basic idea is that when comparing two different Markov chains it is useful to couple them in such a way that they agree as often as possible. We construct such a coupling and analyze it by a simple dominating chain which registers if the two processes agree or disagree. We find that this imagery is useful when thinking about such problems. We are particularly interested in comparing the invariant measures and long time averages of the processes. However, since the paths agree for long runs, it also provides estimates on various stopping times such as hitting or exit times.

This work builds on the general ideas of Maximal couplings. See for instance \cite{1,10,16}. The analysis uses ideas from Poisson equations and Martingale methods which we find convenient. These ideas are quite standard (see for instance \cite{10,13}) but we are more directly inspired by \cite{6,8,12}. As this work was heading to completion, the authors became aware of the recent paper \cite{5} which also looks at coupling/decoupling of two different Markov process for an interesting, but different goal. Error bounds for approximations of uniformly mixing of Markov chains have been derived using other techniques by \cite{14}, among others.

In Section 1, we give our basic setup and quote some simple, classical convergence results. Proofs of some of the results are given in the Appendix for completeness. In Section 2, we introduce a nearby Markov chain and state the main approximation results of the note. In Section 3, we state the existence of a coupling leading to certain desirable estimates. In Section 3.1, we show how the results of Section 2 can be obtained using the coupling from Section 3. In Section 3.2, we build the coupling on which all results rest and introduce a two state change measure when the chains are coupled or decoupled. In Sections 3.3 and 3.4, we analyse the bounding chain. In Section 4, we show certain bounds are sharp. In Section 5, we provide a simple application to a Markov Chain Monte Carlo algorithm and show numerically that the results of the paper show a good level of approximation at considerable speed up by using an approximating chain rather than the original sampling chain.

Acknowledgements: This note grew out of various internal notes prompted by collaborations with Andrew Stuart, Mauro Maggioni, David Dunson, and Sayan Mukherjee. We thank them for encouragement and useful discussions. We are both indebted to MSRI where during the Fall of 2015 we had the time to largely write this version. We also thank the NSF for its support through grant DMS-1546130

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\textit{Date:} November 16, 2017.
1. Basic ergodic statements

Let \( P \) be a Markov transition kernel on a Polish space \( X \) with metric \(|\cdot|\). Given a function \( \phi: X \to \mathbb{R} \) and probability measure \( \nu \) on \( X \) we define:

\[
P \phi(x) = \int_X \phi(y) P(x, dy), \quad \nu P(dy) = \int_X P(x, dy) \nu(dx), \quad \nu \phi = \int_X \phi(y) \nu(dy).
\]

The following assumption is a version of a Doeblin Condition.

**Assumption 1.** There exists a constant \( a \in (0, 1) \) so that

\[
\|P(x, \cdot) - P(y, \cdot)\|_{TV} \leq 1 - a
\]

for all \( x, y \in X \).

A standard result of such a Doeblin Condition is the following.

**Theorem 1.** Under Assumption 1, there exists a unique stationary measure \( \mu \) for \( P \). Furthermore for any initial probability measures \( \nu_1 \) and \( \nu_2 \) one has

\[
\|\nu_1 P^n - \nu_2 P^n\|_{TV} \leq (1 - a)^n \|\nu_1 - \nu_2\|_{TV}
\]

Where for two probability measures \( \nu_1 \) and \( \nu_2 \), the Total Variation distance is defined by

\[
\|\nu_1 - \nu_2\|_{TV} = \frac{1}{2} \left( \sup_{|f|_\infty \leq 1} \nu_1 f - \nu_2 f \right) = \inf_{(X_1, X_2)} P(X_1 \neq X_2)
\]

where \(|f|_\infty = \sup_x |f(x)|\) and the infimum is over all couplings of \( \nu_1 \) and \( \nu_2 \). In other words, \( (X_1, X_2) \) are any random variables constructed on the same space with Law\((X_i) = \nu_i\). It is equally straightforward to prove the law of large numbers and concentration results. Defining

\[
|f|_* = \inf_{\lambda \in \mathbb{R}} \left( \sup_{x \in X} |f(x) - \lambda| \right)
\]

as in [7], we have the following results whose proofs for completeness are given in Appendix A.

**Theorem 2.** For any bounded \( f: \mathbb{R} \to X \), we have that

\[
\mathbb{E} \left( \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \mu f \right)^2 \leq \frac{4|f|_*^2}{a^2 n} \left( 2 + \frac{8}{n} \right)
\]

and for any \( \lambda > 0 \)

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \right| \geq \frac{4}{na} |f|_* + \frac{\lambda}{\sqrt{n}} |f|_* \right) \leq 2 \exp \left( - \frac{a^2 \lambda^2}{32} \right)
\]

It is worth noting that \(|f|_* \leq \min(|f|_\infty, |f - \mu f|_\infty)\) and hence either quantity on the right can replace \(|f|_*\) in the above estimates.

2. A nearby Markov chain

Now consider a second Markov chain \( \mathcal{P}_\epsilon \). As the notation suggests we are often interested in the setting when we have a collection of Markov kernels \( \{\mathcal{P}_\epsilon : \epsilon \in (0, \epsilon_0]\} \) for some constant \( \epsilon_0 \). We want to understand in what sense the long time dynamics of \( \mathcal{P}_\epsilon \) are close to those of \( \mathcal{P} \). We begin with the following simple assumption.
Assumption 2. There exists a constant $\epsilon > 0$ so that
\[
\|P(x, \cdot) - P(x, \cdot)\|_{TV} \leq \epsilon
\]
for all $x \in X$.

We have the following result

Proposition 3. Under Assumptions 1 and 2 any stationary distribution $\mu_\epsilon$ of $P_\epsilon$ satisfies
\[
\|\mu - \mu_\epsilon\|_{TV} \leq \frac{\epsilon}{a}
\]

Proof.
\[
\|\mu - \mu_\epsilon\|_{TV} \leq \|\mu P - \mu\|_{TV} + \|\mu P - \mu_\epsilon P\|_{TV} \leq (1 - a)\|\mu - \mu_\epsilon\|_{TV} + \epsilon
\]
The first inequality follows from the triangle inequality; the second used Assumption 1 for the first term and Assumption 2 for the second term. Rearranging the resulting inequality produces the quoted result. $\square$

Proposition 4. Let Assumptions 1 and 2 hold with $\epsilon \in (0, \frac{a}{2})$. Then Assumption 1 holds for the Markov operator $P_\epsilon$ with the constant “a” equal to $a - 2\epsilon$ which is less than 1 by construction. Hence for such $\epsilon$ the chain has a unique stationary distribution $\mu_\epsilon$ to which it converges exponentially.

We now consider couplings of the chains $X_n, X_n^\epsilon$ evolving according to the transition kernels $P$ and $P_\epsilon$ respectively with $X_0$ and $X_0^\epsilon$ as initial conditions.

Theorem 5. Assume that Assumption 3 and Assumption 1 hold. Then for any two probability measures $\nu_1$ and $\nu_2$ on $X$
\[
\left\|\frac{1}{n} \sum_{k=0}^{n-1} \nu_1 P^k - \frac{1}{n} \sum_{k=0}^{n-1} \nu_2 P_\epsilon^k\right\|_{TV} \leq \frac{\epsilon}{\alpha + \epsilon} + \frac{1 - (1 - \alpha - \epsilon)^n}{n(\alpha + \epsilon)} \left(\|\nu_1 - \nu_2\|_{TV} - \frac{\epsilon}{\alpha + \epsilon}\right).
\]
Furthermore, there exists a coupling of the process $(X_n, X_n^\epsilon)$ with Law$(X_0) = \nu_1$ and Law$(X_0^\epsilon) = \nu_2$ and a random constant $K = K(\alpha, \epsilon, X_0, X_0^\epsilon)$ so
\[
\frac{1}{2} \left|\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^\epsilon)\right| \leq \frac{\epsilon}{\alpha + \epsilon} + \frac{K}{n} + 2\sqrt{\frac{\log(n)}{n}}
\]
for all $n > 0$ and
\[
E\left(\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^\epsilon)\right)^2 \leq 4\|f\|_2^2 \left(\frac{c^2}{(\alpha + \epsilon)^2} + \frac{2}{n^2 (\alpha + \epsilon)^2} + \frac{2}{n} \frac{\alpha^2}{(\alpha + \epsilon)^2}\right)
\]
and for all $\lambda > 0$,
\[
P\left(\left|\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^\epsilon)\right| \geq 2\|f\|_\infty \left(\frac{\epsilon}{\alpha + \epsilon} + \frac{1}{n(\alpha + \epsilon)} + \frac{\lambda}{\sqrt{n}}\right)\right) \leq e^{-\frac{(\alpha + \epsilon)^2}{2}\lambda^2}.
\]

Remark 6. Taking $\nu_1$ equal to $\mu$, the invariant measure of $P$, will produce estimates involving $\mu$, $\mu f$ and related quantities more resembling Theorem 1. Alternative derivations of estimates with this form can be found at the end of the Appendix in Remark 12. The disadvantage of this alternative presentation is that it does not apply as directly to studying exit times and other more pathwise variables as covered by the next result, Theorem 1.
Let $g$ be a real valued function of a trajectory in $\mathbf{X}^\mathbb{N}$, which is the space of one-sided infinite sequences $\{X = (X_0, X_1, X_2, \ldots) : X_k \in \mathbf{X}, k \in \mathbb{N}\}$. We will write $X$ and $X^*$ for the entire trajectories.

**Theorem 7.** Assume that Assumption 2 holds and that $\tau$ is a stopping time adapted to the filtration $\mathcal{F}_n = \sigma(X_0, X_1, X_2, \ldots, X_n)$ with $\mathbb{E}\tau < \infty$. Let $g$ be a function of the path as described above. If $g(X)$ is measurable with respect to $\mathcal{F}_\tau$ then we have the following result:

$$\|\text{Law}(g(X)) - \text{Law}(g(X^*))\|_{TV} \leq \epsilon \mathbb{E}\tau$$

if $\text{Law}(X_0) = \text{Law}(X_0^*)$.

An interesting example of such a function is the hitting time of a set $A$. In this case, $g(X) = \inf\{n \geq 0 : X_n \in A\}$.

3. Understanding Through Coupling

The main point of this section is to give a path-wise perspective on results of this flavor. We will use the following assumption which in our setting is more natural than Assumption 1. It can be viewed as a “cross-Doeblin” condition.

**Assumption 3.** There exists a constant $\alpha \in (0, 1)$ and $\epsilon_0 \in (0, \alpha)$ so that

$$\|P_\epsilon(x, \cdot) - P(y, \cdot)\|_{TV} \leq 1 - \alpha$$

for all $x, y \in \mathbf{X}$ and $\epsilon \in (0, \epsilon_0]$.

**Proposition 8.** Assumptions 1 and 2 holding with parameters $\epsilon_0$ and $\alpha$ implies Assumptions 2 and 3 hold with the same $\epsilon_0$ and $\alpha = a - \epsilon$. Similarly assumptions 2 and 3 holding with parameters $\epsilon_0$ and $\alpha$ implies Assumptions 1 and 2 hold with the same $\epsilon_0$ and $\alpha = a - \epsilon$.

**Proof of Proposition 8.** If Assumptions 1 and 2 hold then

$$\|P_\epsilon(x, \cdot) - P(y, \cdot)\|_{TV} \leq \|P_\epsilon(x, \cdot) - P(x, \cdot)\|_{TV} + \|P(x, \cdot) - P(y, \cdot)\|_{TV} \leq 1 - a + \epsilon$$

which implies that Assumption 3 holds if $\epsilon < a$ with $\alpha = a - \epsilon$. In the other direction,

$$\|P(x, \cdot) - P(y, \cdot)\|_{TV} \leq \|P_\epsilon(x, \cdot) - P(y, \cdot)\|_{TV} + \|P_\epsilon(x, \cdot) - P(x, \cdot)\|_{TV} \leq 1 - \alpha + \epsilon$$

which completes the proof. □

The following Theorem 9 is one of the main results of this note. It gives the existence of a coupling with certain properties. With this result in hand, the proof of Theorem 5 follows in a fashion inspired by the Coupling Time inequality of Aldous used to bound mixing rates.

**Theorem 9.** Assume that Assumption 2 and Assumption 3 hold. Then for any pair of initial conditions $(X_0, X_0^*)$ there exists a coupling $(X_n, X_n^*)$ of the two chains so that

$$\frac{1}{n} \sum_{k=0}^{n-1} P(X_k \neq X_k^*) \leq \frac{\epsilon}{\alpha + \epsilon} + \frac{1 - (1 - \alpha - \epsilon)^n}{n(\alpha + \epsilon)} \left( P(X_0 \neq X_0^*) - \frac{\epsilon}{\alpha + \epsilon} \right).$$
Furthermore there exists a random constant $K = K(\alpha, \epsilon, X_0, X_0^\epsilon)$ such that with probability one

$$\frac{1}{n} \sum_{k=0}^{n-1} 1\{X_k \neq X_k^\epsilon\} \leq \frac{\epsilon}{\alpha + \epsilon} + \frac{K}{n} + 2\sqrt{\log(n) \over n}$$

for all $n \geq 0$. Finally we have the probabilistic bounds:

$$\mathbb{E}\left(\left|\frac{1}{n} \sum_{k=0}^{n-1} 1\{X_k \neq X_k^\epsilon\} - \frac{\epsilon}{\alpha + \epsilon}\right|^2\right) \leq \frac{2}{n^2} (\alpha + \epsilon)^2 + \frac{2}{n} (\alpha + \epsilon)^4$$

where $[x]^+ = \max(x, 0)$ and for any $\lambda > 0$

$$\mathbb{P}\left(\frac{1}{n} \sum_{k=0}^{n-1} 1\{X_k \neq X_k^\epsilon\} \geq \frac{\epsilon}{\alpha + \epsilon} + \frac{1}{n} (X_0 \neq X_0^\epsilon) + \frac{\lambda}{\sqrt{n}}\right) \leq e^{-\frac{(\alpha + \epsilon)^2}{2} \lambda^2}$$

The proof of the first part of this theorem will be given in Section 3.3.1. The second part with the almost sure estimates and probabilistic bounds is proved in Section 3.3.2.

We have a second result that speaks to the distributions of exit times and other path related quantities. The proof is given in Section 3.4.

**Theorem 10.** Assume that Assumption 2 holds. Let $\tau$ be a stoping time adapted to the filtration $\mathcal{F}_n = \sigma(X_0, X_1, X_2, \ldots, X_n)$ with $\mathbb{E}\tau < \infty$. Let $S_\epsilon = \inf\{n : X_n \neq X_n^\epsilon\}$ where $(X_n, X_n^\epsilon)$ is the coupled version of the process given in Theorem 9. If we assume that $X_0 = X_0^\epsilon$ then

$$\mathbb{P}(S_\epsilon \leq \tau) \leq \epsilon \mathbb{E}\tau$$

**3.1. Using the Coupling.** We now use the results of the previous section to prove Theorem 5 and Theorem 7. In all cases the idea is similar: use the fact that the two processes have been coupled to agree often. For the statements in Theorem 5, we use that they are equal for a controllable fraction of the time. For Theorem 7, we use that they are typically equal on a long interval of time if they agree initially.

**Proof of Theorem 5.** We begin by proving the first statment. Observe that

$$\frac{1}{2} \mathbb{E}\left(\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^\epsilon)\right) \leq ||f||_\infty \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(X_k \neq X_k^\epsilon)$$

$$\leq \frac{\epsilon}{\alpha + \epsilon} + \frac{1 - (1 - \alpha - \epsilon)^n}{n(\alpha + \epsilon)^2} (\alpha + \epsilon) \mathbb{P}\{X_0 \neq X_0^\epsilon\} - \epsilon$$

where the last estimate comes from Theorem 9. Since the above expression is true for any choice of couplings of the initial conditions $X_0$ and $X_0^\epsilon$, we are free to minimize over all such couplings. The Monge-Kantorovich Theorem states that $||\nu_1 - \nu_2||_{TV} = \inf \mathbb{P}\{X_0 \neq X_0^\epsilon\}$, where the infimum is taken over all couplings with marginals $\nu_1$ and $\nu_2$. This produces the right hand side of the bound, while taking the supremum over all $f$ with $||f||_\infty \leq 1$ produces the total variation norm on the left hand side. This completes the first statement. The third statement follows...
from the estimate

\[ \mathbb{E}\left( \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k') \right)^2 \leq \mathbb{E}\left( \frac{1}{n} \sum_{k=0}^{n-1} [f(X_k) - f(X_k')] 1_{\{X_k \neq X_k'\}} \right)^2 \]

\[ \leq 2 \|f\|_\infty^2 \mathbb{E}\left( \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{X_k \neq X_k'\}} \right)^2 \]

\[ \leq 4 \|f\|_\infty^2 \left[ \frac{\epsilon^2}{(\alpha + \epsilon)^2} + \mathbb{E}\left( \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{X_k \neq X_k'\}} - \frac{\epsilon}{\alpha + \epsilon} \right)^2 \right] \]

and from Theorem 9. The almost sure statement and exponential estimate follow similarly, also from Theorem 9.

The proof of Theorem 7 is very similar. □

Proof of Theorem 7 For any bounded function \( f : \mathbb{R} \to \mathbb{R} \), we have

\[ \mathbb{E}[f(g(X)) - f(g(X'))] = \mathbb{E}[f(g(X)) - f(g(X'))] 1_{\{\tau < S_\epsilon\}} + 1_{\{\tau \geq S_\epsilon\}} \]

\[ = \mathbb{E}[f(g(X)) - f(g(X'))] 1_{\{\tau < S_\epsilon\}} + \mathbb{E}[f(g(X)) - f(g(X'))] 1_{\{\tau \geq S_\epsilon\}} \leq 2 \|f\|_\infty \mathbb{P}(\tau \geq S_\epsilon) \]

Where \( S_\epsilon \) is the decoupling time defined in the statement of Theorem 10. The result now follows from Theorem 10 on the decoupling time \( S_\epsilon \). □

3.2. Construction of the Coupling. Given any two probability measures \( m_1 \) and \( m_2 \) on \( X \), one can always write them as a density relative to a common probability measure \( m \), namely \( m = \frac{1}{2}(m_1 + m_2) \). If \( m_i(dx) = f_i(x)m(dx) \) for \( i = 1, 2 \) for some \( f_1, f_2 \in L^1(X, m) \), we then define the measures \( m_1 \wedge m_2 \) and \( |m_1 - m_2| \) by \( (m_1 \wedge m_2)(dx) = (f_1 \wedge f_2)(x)m(dx) \) and \( |m_1 - m_2| (dx) = (f_1(x) - f_2(x))^+ m(dx) \) respectively. It is not hard to see that

\[ \|m_1 - m_\|_TV = 1 - (m_1 \wedge m_2)(X) = |m_1 - m_2|^+(X) = |m_2 - m_1|^+(X). \]

We define the following measures on \( X \) which will be used to construct our coupling. For any \( \xi = (\xi_1, \xi_2) \in X \times X \), we define

\[ Q_\epsilon(\xi, \cdot) = \frac{P_\epsilon(\xi_1, \cdot) \wedge P(\xi_2, \cdot)}{\rho_\epsilon(\xi)} \]

\[ R_\epsilon(\xi, \cdot) = \frac{[P_\epsilon(\xi_1, \cdot) - P(\xi_2, \cdot)]^+}{1 - \rho_\epsilon(\xi)} \]

\[ \tilde{R}_\epsilon(\xi, \cdot) = \frac{[P(\xi_2, \cdot) - P_\epsilon(\xi_1, \cdot)]^+}{1 - \rho_\epsilon(\xi)} \]

where \( \rho_\epsilon(\xi) = 1 - \|P_\epsilon(\xi_1, \cdot) - P(\xi_2, \cdot)\|_TV \). By the preceding observations these are all probability measures on \( X \) for fixed \( \xi \in X \times X \). Now define the following transition kernels in \( X \times X \) for \( \xi = (\xi_1, \xi_2) \) and \( x = (x_1, x_2) \) in \( X \times X \) by

\[ Q_\epsilon(\xi, dx) = \rho_\epsilon(\xi)Q_\epsilon(\xi, dx_1)\delta_{x_2}(dx_2) + (1 - \rho_\epsilon(\xi))(R_\epsilon(\xi, dx_1) \times \tilde{R}_\epsilon(\xi, dx_2)) \]

Observe that the marginals of \( Q_\epsilon(\xi, dx) \) are respectively \( P_\epsilon(\xi_1, \cdot) \) and \( P(\xi_2, \cdot) \).

Notice that under Assumptions 2 and 3, this construction has the following properties. If \( \chi = (\chi_1, \chi_2) \) is distributed according to \( Q_\epsilon(\xi, \cdot) \) then \( P(\chi_1 = \chi_2) = \rho_\epsilon(\xi) \) and

\[ \rho_\epsilon(\xi) \geq \begin{cases} 1 - \epsilon & \text{if } \xi_1 = \xi_2 \\ \alpha & \text{if } \xi_1 \neq \xi_2 \end{cases} \]
Letting $\chi_n = (\chi_n^{(1)}, \chi_n^{(2)})$ be the Markov chain on $\mathbf{X} \times \mathbf{X}$ defined by the transition density $Q_\epsilon$, we define the stochastic process $Z^*_n$ by

$$Z^*_n = \begin{cases} 0 & \text{if } \chi_n^{(1)} = \chi_n^{(2)} \\ 1 & \text{if } \chi_n^{(1)} \neq \chi_n^{(2)} \end{cases}$$

While $Z^*_n$ is not Markovian, we can define the random quantities $P(Z^*_{n+1} = k \mid Z^*_n = j)$ by $E(1\{Z^*_{n+1} = k\} \mid Z^*_n = j)$. Now observe that with probability one

$$P(Z^*_{n+1} = 0 \mid Z^*_n = 0) \geq 1 - \epsilon \quad \text{and} \quad P(Z^*_{n+1} = 0 \mid Z^*_n = 1) \geq \alpha.$$ 

Let $Y_n$ be the Markov chain on $\{0,1\}$ with the transition matrix

$$(2) \quad P_\epsilon = \begin{pmatrix} 1 - \epsilon & \epsilon \\ \alpha & 1 - \alpha \end{pmatrix}.$$ 

Assuming that $\epsilon < 1 - \alpha$, we have that with probability one

$$P(Z^*_{n+1} = 0 \mid Z^*_n = 0) \geq P(Y_{n+1} = 0 \mid Y_n = 0) = 1 - \epsilon$$

$$P(Z^*_{n+1} = 0 \mid Z^*_n = 1) \geq P(Y_{n+1} = 0 \mid Y_n = 1) = \alpha$$

$$P(Z^*_{n+1} = 0 \mid Z^*_n = 0) \geq P(Y_{n+1} = 0 \mid Y_n = 1) = \alpha.$$ 

Either directly from these estimates or from the fact that they imply that

$$P(Z^*_{n+1} \leq k \mid Z^*_n \leq Y_n) \geq P(Y_{n+1} \leq k \mid Z^*_n \leq Y_n)$$

for all $k \geq 0$ and $n \geq 0$, which is the assumption of classical stochastic dominance theorems, it is clear that one can construct a monotone coupling of the processes $Y_n$ and $Z^*_n$. That is, we can construct copies of $Y_n$ and $Z^*_n$ on the same probability space such that

$$P(Z^*_n \leq Y_n \text{ for all } n) = 1$$

provided $Z^*_0 \leq Y_0$. In particular, this implies that with probability one

$$(4) \quad \frac{1}{n} \sum_{k=0}^{n-1} 1\{X_k \neq X_k^*\} = \frac{1}{n} \sum_{k=0}^{n-1} 1\{Z_k^* = 1\} \leq \frac{1}{n} \sum_{k=0}^{n-1} 1\{Y_k = 1\} = \frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k).$$

Hence to control the fraction of time $X_n$ and $X_n^*$ disagree it is enough to bound the amount of time $Y_n = 1$. The analysis of this bounding chain is the topic of the next section.

3.3. Analysis of Bounding Chain. We now give the proofs of the estimates in Theorem 2. The basic idea is to use the fact that $Z^*_n$ is stochastically dominated by $Y_n$ in the sense of (3) and (4), to reduce all questions of interest to statements about the time that $Y_n$ spends in state 1. Since $Y_n$ is a simple two state Markov chain, the analysis is elementary and quite explicit.

3.3.1. Control in Expectation. The Markov transition matrix of the bounding chain is

$$P_\epsilon = \begin{pmatrix} 1 - \epsilon & \epsilon \\ \alpha & 1 - \alpha \end{pmatrix}.$$ 

It has generator $L_\epsilon = P_\epsilon - I$ and unique stationary measure $\mu_\epsilon$ given by

$$\mu_\epsilon = \begin{pmatrix} \frac{\alpha}{\alpha + \epsilon} & \frac{\epsilon}{\alpha + \epsilon} \end{pmatrix}$$

and satisfies by definition $\mu_\epsilon L_\epsilon = 0$ and $\mu_\epsilon P_\epsilon = \mu_\epsilon$. 
We define the following vectors

\[ \phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \phi_\epsilon = \mu_\epsilon \phi 1 = \begin{pmatrix} \frac{\alpha + \epsilon}{\alpha + \epsilon} \\ \frac{\alpha + \epsilon}{\alpha + \epsilon} \end{pmatrix} \]  
and further define \( \psi_\epsilon \) as the solution to the equation

\[ L_\epsilon \psi_\epsilon = -\bar{\phi}_\epsilon \]

It is straightforward to see that

\[ \psi_\epsilon = \sum_{k=0}^{\infty} \mathcal{P}_\epsilon^k \bar{\phi}_\epsilon \]

Observe that \( w_\epsilon \), defined by

\[ w_\epsilon = \left( \frac{-\epsilon}{\alpha} \right), \]

satisfies \( \mathcal{P}_\epsilon w_\epsilon = (1 - \epsilon - \alpha)w_\epsilon \) and hence \( w_\epsilon \) is right-eigenvector with eigenvalue \( 1 - \epsilon - \alpha \). Since \( \bar{\phi}_\epsilon = \frac{n}{\alpha + \epsilon} w_\epsilon \), we have that

\[ \psi_\epsilon = \left( \frac{\alpha}{\alpha + \epsilon} \right) \left( \sum_{k=0}^{\infty} (1 - \epsilon - \alpha)^k \right) w_\epsilon = \frac{\alpha}{(\alpha + \epsilon)^2} w_\epsilon \]

where we have again used the fact that \( \epsilon < 1 - \alpha \) so that \( 1 - \epsilon - \alpha \in (0, 1) \). Any initial distribution of \( (X_0, X_0^1) \) induced an initial distribution \( \nu \) for the \( Y_n \) chain by \( \nu(0) = \mathbf{P}(X_0 = X_0^1) \) and \( \nu(1) = \mathbf{P}(X_0 \neq X_0^1) \).

Combining the above properties we have that

\[ \nu \mathcal{P}_\epsilon^n \psi_\epsilon - \nu \psi_\epsilon = \sum_{k=0}^{n-1} \nu \mathcal{P}_\epsilon^k L_\epsilon \psi_\epsilon = \sum_{k=0}^{n-1} \nu \mathcal{P}_\epsilon^k \phi - n \nu \bar{\phi} \]

Rearranging this produces

\[ \frac{1}{n} \sum_{k=0}^{n-1} \nu \mathcal{P}_\epsilon^k \phi = \frac{\epsilon}{\alpha + \epsilon} + \frac{\nu \mathcal{P}_\epsilon^n \psi_\epsilon - \nu \psi_\epsilon}{n} = \frac{\epsilon}{\alpha + \epsilon} + \frac{\alpha}{n(\alpha + \epsilon)^2} (1 - (1 - \alpha - \epsilon)^n) \nu w_\epsilon \]

\[ = \frac{\epsilon}{\alpha + \epsilon} + \frac{1 - (1 - \alpha - \epsilon)^n}{n(\alpha + \epsilon)^2} \left( \alpha \mathbf{P}(X_0 \neq X_0^1) - \epsilon (1 - \mathbf{P}(X_0 \neq X_0^1)) \right) \]

Since \( \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}(X_k \neq X_k^1) \leq \frac{1}{n} \sum_{k=0}^{n-1} \nu \mathcal{P}_\epsilon^k \phi \), the preceding calculation proves Theorem \[\ref{thm:as_analysis_variance}\] after some algebra.

3.3.2. Almost Sure Analysis and Variance. Let \( (X_n, X_n^1) \) be the coupled versions of the chains constructed in the previous section and let \( Z_n \) and \( Y_n \) be the associated processes on \( \{0, 1\} \) also constructed in the previous sections. We now introduce slight abuse of notation by allowing \( \phi, \phi_\epsilon, \) and \( \psi_\epsilon \) to denote the associated real valued functions on \( \{0, 1\} \). For example

\[ \psi_\epsilon(y) = \begin{cases} 
\frac{\epsilon}{(\alpha + \epsilon)^2} & \text{if } y = 0 \\
\frac{\alpha}{(\alpha + \epsilon)^2} & \text{if } y = 1 
\end{cases} \]

Letting \( \mathcal{F}_n \) be the filtration generated by \( (Y_0, Y_1, \cdots, Y_n) \), define the Martingale increment

\[ I_n = \psi_\epsilon(Y_n) - \mathbb{E}(\psi_\epsilon(Y_n)|\mathcal{F}_{n-1}) \]
and the Martingale

\[ M_n = \sum_{k=1}^{n} I_k \]

with \( M_0 = 0 \).

Now since \( E(\psi_\epsilon(Y_n) - \psi_\epsilon(Y_{n-1}) | F_{n-1}) = (L\psi_\epsilon)(Y_{n-1}) \) we have

\[ \psi_\epsilon(Y_n) - \psi_\epsilon(Y_0) = \sum_{k=0}^{n-1} (L\psi_\epsilon)(Y_k) + M_n \]

and

\[ \frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k) - \frac{\epsilon}{\alpha + \epsilon} = \frac{1}{n} \sum_{k=0}^{n-1} (L\psi_\epsilon)(Y_k) = \frac{\psi_\epsilon(Y_0) - \psi_\epsilon(Y_n)}{n} - \frac{M_n}{n} \]

Next observe that

\[ -\frac{1}{\alpha + \epsilon} \{x_0 = x_0^c\} \leq \psi_\epsilon(Y_0) - \psi_\epsilon(Y_n) \leq \frac{1}{\alpha + \epsilon} \{x_0 \neq x_0^c\} . \]

Since |\( M_n - M_{n-1} \)| \( \leq \frac{1}{\alpha + \epsilon} \), we have by Azuma’s inequality

\[ P(\{|M_n| \geq \lambda \sqrt{n}\}) \leq 2e^{-\frac{(\alpha + \epsilon)^2 \lambda^2}{2}} . \]

Taking \( \lambda = 2\sqrt{\log(n)} \) and using the Borel-Cantelli lemma shows that there exists a random constant \( K \) so that

\[ |M_n| \leq K + 2\sqrt{n \log(n)} \]

for all \( n \geq 0 \). Combining these estimates produces the following result, which shows that under Assumptions \( \ref{ass1} \) and \( \ref{ass2} \) for any \( \epsilon \in (0, \alpha] \) and initial conditions \( X_0 \) and \( X_0^c \) there exists a random, positive constant \( K \), such that with probability one

\[ -\frac{1}{n(\alpha + \epsilon)} - \frac{K}{n} - 2\sqrt{\log(n)/n} \leq \frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k) - \frac{\alpha}{\alpha + \epsilon} - \frac{1}{n(\alpha + \epsilon)} + \frac{K}{n} + 2\sqrt{\log(n)/n} \]

for all \( n \geq 0 \). In addition for any \( \lambda > 0 \) and \( n > 0 \) one has

\[ P\left( \frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k) - \frac{\epsilon}{\alpha + \epsilon} \geq \frac{1}{n(\alpha + \epsilon)} + \frac{\lambda}{\sqrt{n}} \right) \leq e^{-\frac{(\alpha + \epsilon)^2 \lambda^2}{2}} . \]

Recalling that \( \frac{1}{n} \sum_{k=0}^{n-1} \{x_0 \neq x_0^c\} \leq \frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k), \) produces the first and last results given in Theorem \( \ref{thm1} \). To see the second result we return to (9). We use the fact that for \( \epsilon \in (0, \alpha] \)

\[ E|M_n|^2 = \sum_{k=1}^{n} E|I_k|^2 \leq n \frac{\alpha^2}{(\alpha + \epsilon)^4} \]

to see that if \( \epsilon \in (0, \alpha) \) then

\[ E \left| \frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k) - \frac{\epsilon}{\alpha + \epsilon} \right|^2 \leq 2E|\psi_\epsilon(Y_0) - \psi_\epsilon(Y_n)|^2 / n^2 + 2E|M_n|^2 / n^2 \leq 2 n^2 / (\alpha + \epsilon)^2 + 2 n (\alpha + \epsilon)^4 \]
With this, the proof of last estimate from Theorem \[9\] is completed by observing that
\[
\mathbb{E}\left(\frac{1}{n} \sum_{k=0}^{n-1} 1_{\{X_k \neq X_0\}} - \frac{\epsilon}{\alpha + \epsilon}\right)^2 \leq \mathbb{E}\left(\frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k) - \frac{\epsilon}{\alpha + \epsilon}\right)^2 \leq \mathbb{E}\left|\frac{1}{n} \sum_{k=0}^{n-1} \phi(Y_k) - \frac{\epsilon}{\alpha + \epsilon}\right|^2
\]

### 3.4. Analysis of Decoupling Time

Let \( S_\epsilon = \inf(n : X_n \neq X_n^\epsilon) \) and let \( \sigma_\epsilon = \inf\{n : Y_n = 1\} \) where \( Y_n \) is the 0-1 Markov process constructed in the previous section. Notice that the construction is still possible if \( \alpha = 0 \). In this case the state 1 is an absorbing state for the \( Y_n \) chain, but the stochastic ordering still holds.

Because of the stochastic ordering \( S_\epsilon \geq \sigma_\epsilon \). Hence for any stopping time \( \tau \),

\[
\mathbb{P}(S_\epsilon \leq \tau) \leq \mathbb{P}(\sigma_\epsilon \leq \tau)
\]

Now,

\[
\mathbb{P}(\sigma_\epsilon > \tau) = \sum_{k=0}^{\infty} \mathbb{P}(\sigma_\epsilon > k)\mathbb{P}(\tau = k) = \sum_{k=0}^{\infty} (1 - \epsilon)^k \mathbb{P}(\tau = k) = \mathbb{P}(1 - \epsilon)^\tau.
\]

Setting \( \Lambda(\epsilon) = \mathbb{E}(1 - \epsilon)^\tau \), if we temporarily assume that \( \tau \leq N \) almost surely for some constant \( N \) then \( \Lambda(\epsilon) \) is an everywhere differentiable function. Expanding around 0 for \( \epsilon > 0 \) and using the Lagrange remainder term produces

\[
\Lambda(\epsilon) = 1 - \epsilon \mathbb{E}\tau + \frac{1}{2} \epsilon^2 \mathbb{E}(\tau(\tau - 1)) \geq 1 - \epsilon \mathbb{E}\tau
\]

for some \( c \in [0, \epsilon] \). Since both the right and left hand side are well defined for any \( \epsilon \in (0, 1) \), when the stopping time only satisfies \( \mathbb{E}\tau < \infty \) (rather than the almost sure bound \( \tau < N \)), we conclude that

\[
\mathbb{P}(\sigma_\epsilon > \tau) \geq 1 - \epsilon \mathbb{E}\tau
\]

in general. Since \( \mathbb{P}(\sigma_\epsilon \leq \tau) = 1 - \mathbb{P}(\sigma_\epsilon > \tau) \) the result is proven.

### 4. Sharpness

Now we show that the total variation bound in Theorem \[5\] is tight by exhibiting a Markov chain satisfying the assumptions that achieves the bound. Let

\[
\mathbb{P} = \left(\begin{array}{cc}
1 - \beta & \beta \\
\beta & 1 - \beta
\end{array}\right)
\]

for \( \beta \leq 1/2 \). It is easy to verify by direct calculation that the invariant measure is \( \mu = \left(\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right) \) and \( \mathbb{P} \) satisfies the Doeblin condition with \( a = 2\beta \). \( \mathbb{P} \) has eigenvectors

\[
\phi_1 = \left(\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right), \quad \phi_2 = \left(\begin{array}{c}
-\frac{1}{2} \\
\frac{1}{2}
\end{array}\right)
\]

with eigenvalues 1 and \( 1 - 2\beta \), respectively. Any possible starting measure \( \nu \) can be expressed as \( \nu_\gamma = (\gamma, 1 - \gamma) \) for some \( \gamma \leq \frac{1}{2} \). Then \( \|\nu_\gamma - \mu\|_{TV} = \frac{1}{2} \left(\left|\frac{1}{2} - \gamma\right| + \left|\frac{1}{2} - (1 - \gamma)\right|\right) = \frac{1}{2} - \gamma \) when \( \gamma < \frac{1}{2} \); note if \( \gamma > \frac{1}{2} \), we get \( \gamma - \frac{1}{2} \).

Consider the perturbation

\[
\mathbb{P}_\epsilon = \left(\begin{array}{cc}
1 - (\beta - \epsilon) & \beta - \epsilon \\
\beta + \epsilon & 1 - (\beta + \epsilon)
\end{array}\right),
\]

which satisfies \( \sup_{x \in X} \|\mathbb{P}_\epsilon(x, \cdot) - \mathbb{P}(x, \cdot)\|_{TV} = \epsilon \) and \( \sup_{(x,y) \in X \times X} \|\mathbb{P}_\epsilon(x, \cdot) - \mathbb{P}(y, \cdot)\|_{TV} < 1 - (2\beta - \epsilon) = 1 - \alpha \). A diagonalization of \( \mathbb{P}_\epsilon \) is

\[
D_\epsilon = Q_\epsilon^{-1} \mathbb{P}_\epsilon Q_\epsilon = \frac{1}{2\alpha} \left(\begin{array}{cccc}
\beta + \epsilon & 1 & 0 & 1 - (\beta - \epsilon) \\
-1 & 1 & 0 & 1 - 2\beta \\
0 & 1 & 1 & \beta + \epsilon
\end{array}\right)
\]
where in the last step we took (see [9, pp 15-16] for example), so that
\[ n \text{ with } \Lambda \text{ a matrix inverse of the form } \]

Theorem 5, with large computational advantage. We consider approximation of a

\[ \nu \in \mathbb{R} \] one can achieve for Bayesian analysis of spatially indexed data. Gaussian processes are also com-

approach to achieve computational tractability for Gaussian process models, since

\[ \sum \| \nu \|_{TV} \] the accuracy in the total variation metric of approximations to Markov transition

We form \( \Lambda \) using a partial spectral decomposition. Algorithms for approximating

Therefore
\[ 1/\nu \sum_{k=0}^{n-1} \nu \mathcal{P}_k = \frac{1}{2\beta} \left( (\epsilon + \beta) + (\beta(2\gamma - 1) - \epsilon)(1 - 2\beta)^k \right) \]

so
\[ \frac{1}{\nu} \sum_{k=0}^{n-1} \nu \mathcal{P}_k = \frac{1}{2\beta} \left( (\epsilon + \beta) + (\beta(2\gamma - 1) - \epsilon)\frac{1 - (1 - 2\beta)^n}{2\beta n} \right) \]

= \left( \frac{\epsilon}{\alpha} + \frac{1}{2} \right) + \left( \| \mu - \nu \|_{TV} - \frac{\epsilon}{\alpha} \right) \frac{1 - (1 - a)^n}{an} \raisex{\frac{1}{2} \frac{1 - (1 - a)^n}{\alpha n}}

\[ \left\| \mu - \frac{1}{\nu} \sum_{k=0}^{n-1} \nu \mathcal{P}_k \right\|_{TV} \]

\[ = \frac{\epsilon}{\alpha + \epsilon} + \left( \| \mu - \nu \|_{TV} - \frac{\epsilon}{\alpha + \epsilon} \right) \frac{1 - (1 - a - \epsilon)^n}{(\alpha + \epsilon)n} \]

where \( \alpha \) is the constant appearing in Assumption \[ 2 \] For \( \alpha > \epsilon \), Corollary \[ 3 \] gives precisely this expression if one takes \( \nu_1 = \mu \) and \( \nu_2 = \nu \). Thus we conclude that the total variation bound in Theorem \[ 5 \] is sharp.

5. Application to MCMC for Gaussian processes

In this section we consider an application to a simple Markov chain Monte Carlo algorithm for sampling from the posterior distribution in a Gaussian process model

for Bayesian analysis of spatially indexed data. Gaussian processes are also commonly employed in nonparametric regression. Our aim is to exhibit a case where one can achieve \( \epsilon \ll \alpha + \epsilon \), and therefore an accurate approximation of \( \mathcal{P} \) by \( \mathcal{P}_\epsilon \) via

Theorem \[ 5 \] with large computational advantage. We consider approximation of a matrix inverse of the form

\[ (I_n + c\Sigma)^{-1} \approx (I_n + c\Lambda'\Lambda)^{-1} = I - \Lambda (c^{-1}I_n + \Lambda'\Lambda)^{-1}\Lambda' \]

with \( \Lambda \) a \( n \times q \) matrix with \( n \ll q \) and \( I_n \) is a \( k \times k \) identity matrix. This is a common approach to achieve computational tractability for Gaussian process models, since it replaces a non-parallel \( O(n^3) \) algorithm with a parallelizable \( O(n^2q) \) algorithm.

We form \( \Lambda \) using a partial spectral decomposition. Algorithms for approximating partial spectral decompositions without computing the full spectral decomposition, with applications to Gaussian process models, are given in \[ 2 \]. Our aim is to assess the accuracy in the total variation metric of approximations to Markov transition kernels \( \mathcal{P} \) that result from utilizing approximations of the form \[ 11 \] to generate \( \mathcal{P}_\epsilon \).

Consider a Gaussian process model with squared exponential (or “radial basis”) kernel

\[ z(w) = x_3 f(w) + \epsilon, \quad \epsilon \sim N(0, \sigma^2) \]

\[ \text{cov}(f(w_i), f(w_j)) = x_2 \exp(-x_1|w_i - w_j|^2/2) \]

The parameters of the model are \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), the positive orthant in \( \mathbb{R}^3 \). The points \( W = w_1, \ldots, w_n \) at which the process is sampled are treated as
fixed and known, and the observations of the process at these points are denoted \( z = (z(w_1), \ldots, z(w_n)) \). Bayesian inference on \( x \) requires choice of a prior distribution. A default choice is an inverse Gamma prior on \( x_3^2 \) with parameters \( a, b \) and density

\[
p(x_3^2 \mid a, b) = \frac{b^a}{\Gamma(a)} (x_3^2)^{-\frac{a}{2} - 1} e^{-\frac{b}{2} x_3^2}.
\]

For \( x_1 \), it is common (see e.g. [11]) to discretize the parameter space for \( x_1 \) to \( m \) points. We also do this for \( x_2 \), which allows us to numerically compute the transition matrix. Specifically, our prior has

\[
(x_1, x_2) \in X_1 \times X_2, \quad |X_1| = |X_2| = m,
\]

almost surely, where \(|X|\) is the cardinality of the finite set \( X \). We place prior mass \( m^{-2} \) on each atom, leading to the unnormalized posterior

\[
p(x) \propto |x_3^2(I_n + x_2 \Sigma(x_1, W))|^{-1/2} e^{-\frac{b}{2} z'(I + x_2 \Sigma(x_1, W))^{-1} z - \frac{a}{2} x_3^2}.
\]

where \( \Sigma(x_1, W) \) is a \( n \times n \) symmetric, positive-definite matrix with entries

\[
\{\Sigma(x_1, W)\}_{ij} = e^{-x_i \|w_i - w_j\|^2}.
\]

Integration over \( x_3^2 \) is available in closed form, leading to the likelihood for \( z \) marginal of \( x_3 \)

\[
L(z \mid x_1, x_2, W) \propto |I + x_2 \Sigma(x_1, W)|^{-\frac{1}{2}} \{b + z'(I + x_2 \Sigma(x_1, W))^{-1} z\}^{-\frac{a+n}{2}}.
\]

Because the priors on \( x_1, x_2 \) are discrete uniform on \( X_1, X_2 \), respectively, the posterior, which is the target distribution we want to sample, is proportional to (13) at the support points \( (x_1, x_2) \in X_1 \times X_2 \). We assess properties of the kernel \( \mathcal{P} \) defined by the update rule

\[
r(y_2, x_1) = \mathbf{P}(X_2 = y_2 \mid z, W, X_1 = x_1) = \frac{L(z \mid x_1, y_2, W)}{\sum_{y_2' \in X_2} L(z \mid x_1, y_2', W)}
\]

\[
s(y_1, x_2) = \mathbf{P}(X_1 = y_1 \mid z, W, X_2 = x_2) = \frac{L(z \mid y_1, x_2, W)}{\sum_{y_1' \in X_1} L(z \mid y_1', x_2, W)},
\]

which has invariant measure the posterior, and an approximating kernel \( \mathcal{P}_\epsilon \) that uses the same two-step update rule, but substitutes a low-rank approximation \( \Sigma_\epsilon(x_1, W) = \Lambda_\epsilon(x_1, W) \Lambda_\epsilon(x_1, W)' \) where \( \Lambda_\epsilon(x_1, W) \) is a \( n \times q_\epsilon \) matrix, with \( q_\epsilon \leq n \) as in (11). This is variously known as “predictive process” or “subset of regressors,” and is a common strategy for scaling computation in these models (see [2, 4] and references therein).

Observe that

\[
|\mathcal{P}((x_1, x_2), (y_1, y_2)) - \mathcal{P}_\epsilon((x_1^*, x_2^*), (y_1, y_2))| = |\mathcal{P}((x_1, \cdot), (y_1, y_2)) - \mathcal{P}_\epsilon((x_1^*, \cdot), (y_1, y_2))|
\]

\[
= |r(y_2, x_1)s(y_1, y_2) - r_\epsilon(y_2, x_1^*)s_\epsilon(y_1, y_2)| \equiv \alpha(x_1, x_1^*, y_1, y_2)
\]

so that

\[
\sup_{x_1, x_1^*} \|\mathcal{P}((x_1, \cdot), \cdot) - \mathcal{P}_\epsilon((x_1^*, \cdot), \cdot)\|_{TV} = \frac{1}{2} \sup_{x_1, x_1^*} \sum_{y_1} \sum_{y_2} \alpha(x_1, x_1^*, y_1, y_2),
\]

and the value of \( \alpha \) in Assumption 3 and \( \epsilon \) in Assumption 2 can be computed exactly.
To evaluate the practical usefulness of $\mathcal{P}_\epsilon$, we take $n = 1000$, $w_i = \frac{i}{n}$, then take 100 independent samples from (12) with $x_2 = 0.9$, $x_3^2 = 0.2$, $x_1 = -\log(0.01) / 0.45$.

Here, $x_1$ is chosen such that $\Sigma(x_1, W) = 0.01$ when $\|w_i - w_j\|^2 = 0.45 \max_{i,j} \|w_i - w_j\|^2$.

Setting parameters such that the spatial correlation decays to 0.01 (or, more generally, some small value) at distances equal to a specified fraction of the range of sampling points is typical in spatial statistics [3]. For construction of $\mathcal{P}_\epsilon$ and $\mathcal{P}$, we put $m = 10$ and

$$X_1 = \left\{ -\log(0.01), -\log(0.01), \ldots, -\log(0.01) \right\}$$
$$X_2 = \{0.5, 0.6, \ldots, 1.4\}.$$  

For each sample from (12), we compute $\alpha$ and $\epsilon$ for values of $q_\epsilon$ between 1 and $\min\{q_\epsilon : \epsilon < 10^{-10}\}$.

Figure 1 shows results. The vertical axis shows $\frac{\epsilon}{\alpha + \epsilon}$ as a function of $q_\epsilon$. Recall from Theorem 5, for example, that $\epsilon \ll \alpha + \epsilon$ results in high accuracy. The numerical simulation suggests one can achieve $\epsilon \ll \alpha + \epsilon$ with $q_\epsilon < n$, indicating that large computational gains from utilizing $\mathcal{P}_\epsilon$ are achievable. Indeed, $q_\epsilon = 30$ is enough to have $\frac{\epsilon}{\alpha + \epsilon} < 10^{-4}$ for all 100 replicate simulations, thus allowing inversion of a $1000 \times 1000$ matrix required to compute $\mathcal{P}$ to be replaced by inversion of a $30 \times 30$ matrix in computing $\mathcal{P}_\epsilon$ while achieving an accurate approximation. When $n$ is large, dimension reduction of this magnitude can have a very large computational benefit, suggesting the practical value of this commonly used strategy, and allowing the bounds in this paper to be immediately applied to MCMC for Gaussian process models when the prior on $(x_1, x_2)$ is discrete.

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Figure 1. The value of $\frac{\epsilon}{\alpha + \epsilon}$, with $\alpha$ defined as in Assumption 3, as a function of the number of columns in the matrix $\Lambda(x_1, W)$ ($q_\epsilon$) for 100 independent samples from the model in (12).

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**APPENDIX A. PROOFS OF BASIC RESULTS FOR $P$**

Given any $f: X \to \mathbb{R}$, consider the solution $\psi$ to the associated “Poisson” equation

$$L\psi = \mu f - f$$

where $\mu$ is the unique invariant measure of $P$, $\mu f = \int f d\mu$, and $L$ is the generator of the Markov Chain defined by $L = P - I$ where $I$ is the identity operator on $X$. Recalling the definition of $|f|_*$ from [1], we have the following result.

**Lemma 11.** If $f$ is bounded then there exits a unique solution $\psi$ of (14). Furthermore $|\psi|_\infty \leq 2|f|_*/\alpha$. 

Proof of Lemma 11. Define

\[ \psi_n = \sum_{k=0}^{n} P^k (f - \mu f) \]

We will see that desired \( \psi \) will be the limit of the \( \psi_n \). Since the definition of \( \psi \) does not change if \( f \) is replaced by \( f + \lambda \) for any constant \( \lambda \), we are free to assume that \( f \) is such that \( |f|_\infty = |f|_* \).

For any integers \( n > m > 0 \) we have

\[
|\psi_n - \psi_m|_\infty \leq \sum_{k=m+1}^{n} |P^k f - \mu f|_\infty \leq 2|f|_\infty \sum_{k=m+1}^{n} (1 - a)^k
\]

\[
= 2(1 - a)^{m+1}|f|_\infty \frac{1 - (1 - a)^{n-m}}{a}
\]

Hence \( \psi_n \) is a Cauchy sequence and the limit exists which we will call \( \psi \). Now observe that

\[
|\psi|_\infty \leq \sum_{k=0}^{\infty} |P^k f - \mu f|_\infty \leq 2|f|_\infty \sum_{k=0}^{\infty} (1 - a)^k \leq \frac{2|f|_\infty}{a}
\]

and hence \( \lim L\psi_n = L\psi \) since \( L \) is a bounded operator. Now observe that since \( L\mu f = 0 \)

\[
L\psi_n = \sum_{k=0}^{n} (P^{k+1} f - P^k f) = P^{n+1} f - f \longrightarrow \mu f - f \quad \text{as} \quad n \to \infty.
\]

Recalling that \( |f|_\infty = |f|_* \), the proof is concluded. \( \Box \)

Now

\[
\psi(X_n) - \psi(X_0) = \sum_{k=1}^{n} (\psi(X_k) - \psi(X_{k-1})) = \sum_{k=1}^{n} \mathbb{E}(\psi(X_k) - \psi(X_{k-1})|F_{k-1}) + M_n
\]

(15)

\[
= \sum_{k=0}^{n-1} (L\psi)(X_k) + M_n
\]

where \( F_k \) is the \( \sigma \)-algebra generated by the random variables \( (X_0, \ldots, X_k) \) and \( M_n \)

defined by

\[
M_n = \sum_{k=1}^{n} \psi(X_k) - \mathbb{E}(\psi(X_k)|F_{k-1})
\]

is a Martingale with respect to \( F_n \). Now since \( L\psi(X_k) = \mu f - f(X_k) \) rearranging

\[
\mu f - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \frac{\psi(X_n) - \psi(X_0)}{n} - \frac{M_n}{n}
\]

and we have

\[
\mathbb{E}M_n^2 = \sum_{k=1}^{n} \mathbb{E}\left(\psi(X_k) - \mathbb{E}(\psi(X_k)|F_{k-1})\right)^2.
\]
Letting $\mathcal{A}_{k-1} = \{ Y : Y \text{ is } \mathcal{F}_{k-1} \text{ measurable}, EY^2 < \infty \}$, observe that
\[
E \left( \psi(X_k) - E(\psi(X_k) | \mathcal{F}_{k-1}) \right)^2 = \inf_{Y \in A_{k-1}} E \left( \psi(X_k) - Y \right)^2 \leq E\psi(X_k)^2
\]
since the constant random variable $Y \equiv 0$ is in $A_k$ for all $k$, giving
\[
EM_{n}^2 = \sum_{k=1}^{n} E \left( \psi(X_k) - E(\psi(X_k) | \mathcal{F}_{k-1}) \right)^2 \leq \sum_{k=1}^{n} E\psi(X_k)^2 \leq \frac{4|f|_\infty^2}{a^2} n.
\]
So we have that
\[
E \left( \mu f - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \right)^2 \leq 2 E(\psi(X_n) - \psi(X_0))^2 + 2 EM_{n}^2 \leq \frac{4|f|_\infty^2}{a^2} \left( 2 + \frac{8}{n} \right)
\]
Again recalling that $|f|_\infty = |f|_*$, we get the first quoted result.

Now to see the second result, observe that $|M_{k+1} - M_k| \leq 4|f|_*/a$. Hence Azuma’s inequality implies that
\[
P(|M_n| \geq \lambda \sqrt{n|f|_\infty}) \leq 2 \exp \left( - \frac{a^2 \lambda^2}{32} \right)
\]
and returning to the above calculation
\[
P \left( \mu f - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \right) \geq \frac{4}{na|f|_\infty} + \frac{\lambda \sqrt{n|f|_\infty}}{2} \leq 2 \exp \left( - \frac{a^2 \lambda^2}{32} \right)
\]

**Remark 12** (Closeness by Perturbation Estimate). We now briefly give a version of the closeness result starting from Assumption \[7\] and Assumption \[8\], following the proof above.

Let $\psi$ be the solution to the Poisson equation \[14\] introduced previously. We begin by considering the analog of \[15\] for the epsilon chain.
\[
\psi(X_n^\epsilon) - \psi(X_0^\epsilon) = \sum_{k=0}^{n-1} (L_{\epsilon} \psi)(X_k^\epsilon) + M_n^\epsilon = \sum_{k=0}^{n-1} (L\psi)(X_k^\epsilon) + R_n^\epsilon + M_n^\epsilon
\]
where $L_{\epsilon} = \mathcal{P}_\epsilon - I$ is the generator associated to $\mathcal{P}_\epsilon$ and $M_n^\epsilon$ and $R_n^\epsilon$ are defined by
\[
M_n^\epsilon = \sum_{k=1}^{n-1} \psi(X_k^\epsilon) - E(\psi(X_k^\epsilon) | \mathcal{F}_{k-1}^\epsilon), \quad R_n^\epsilon = \sum_{k=0}^{n-1} ((\mathcal{P} - \mathcal{P}_\epsilon) \psi)(X_k^\epsilon)
\]
where $\mathcal{F}_{k}^\epsilon$ is the $\sigma$-algebra generated by $(X_0^\epsilon, \ldots, X_k^\epsilon)$. Notice that we have used the fact that $L - L_{\epsilon} = \mathcal{P} - \mathcal{P}_\epsilon$. Using \[14\], we obtain
\[
\mu f - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^\epsilon) = \frac{\psi(X_n^\epsilon) - \psi(X_0^\epsilon)}{n} + \frac{1}{n} R_n^\epsilon + \frac{1}{n} M_n^\epsilon
\]
Since the right hand side does not change if $f$ is replaced by $f + \lambda$, we can assume $|f|_\infty = |f|_*$. By Assumption \[3\] followed by Lemma \[11\], we see that
\[
\frac{1}{n} |R_n^\epsilon| \leq 2\epsilon |\psi|_\infty \leq \frac{4\epsilon}{a} |f|_\infty, \quad \frac{1}{n^2} E|M_n^\epsilon|^2 \leq \frac{4|f|_\infty^2}{a^2 n}, \quad |M_k^\epsilon - M_{k-1}^\epsilon| \leq \frac{4|f|_\infty}{a}
\]
From this we can quickly obtain estimates which are the analogue of those in Theorem 5. First observe that taking expectations of (16) and using these estimates produces

$$\mu f - \mathbb{E} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^e) \leq \frac{4}{an} |f|_\infty + \frac{4\epsilon}{a} |f|_\infty$$

Similarly we have,

$$\mathbb{E} \left( \mu f - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^e) \right)^2 \leq 3 \frac{4|\psi|_\infty^2}{n^2} + 3|R_n|^2 + 3\mathbb{E}|M_n^e|^2 \leq \frac{3}{a^2} (16\epsilon^2 + \frac{16}{n^2}) |f|_\infty^2 + \frac{12}{a^2 n} |f|_\infty^2$$

and as before using Azumà’s inequality

$$\mathbb{P} \left( |\mu f - \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^e)| \geq \frac{4}{a}(\epsilon + \frac{1}{n}) |f|_\infty + \frac{\lambda}{\sqrt{n}} |f|_\infty \right) \leq 2 \exp \left( - \frac{a^2 \lambda^2}{32} \right)$$