Equivalence of approximation by convolutional neural networks and fully-connected networks

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Abstract

Convolutional neural networks are the most widely used type of neural networks in applications. In mathematical analysis, however, mostly fully-connected networks are studied. In this paper, we establish a connection between both network architectures. Using this connection, we show that all upper and lower bounds concerning approximation rates of fully-connected neural networks for functions $f \in \mathcal{C}$—for an arbitrary function class $\mathcal{C}$—translate to essentially the same bounds on approximation rates of convolutional neural networks for functions $f \in \mathcal{C}^{\text{equi}}$, with the class $\mathcal{C}^{\text{equi}}$ consisting of all translation equivariant functions whose first coordinate belongs to $\mathcal{C}$.

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1 Introduction

The recent overwhelming success of machine learning techniques such as deep learning [5, 7, 16] has prompted many theoretical works trying to provide a mathematical explanation for this extraordinary performance. One line of research focuses on analysing the underlying computational architecture that is given by a neural network. In the context of approximation theory, it is possible to describe the capabilities of this architecture meaningfully. First and foremost, the universal approximation theorem (see [4, 6, 9]) shows that any continuous function on a compact domain can be approximated arbitrarily well by neural networks. Besides, more refined approximation results relate the size (in terms of number of neurons or number of free parameters) of an approximating neural network to its approximation fidelity; see for instance [1, 12, 13, 14, 15, 17, 18]. These results include upper bounds on the sufficient size of a network, but also prescribe lower bounds on the necessary size of a network, required for certain approximation task.

While the results described above offer valuable insight into the functionality and capability of neural networks, their practical relevance is limited. Indeed, all mentioned results consider so-called fully-connected neural networks (FNNs). In most applications, however, so-called convolutional neural networks (CNNs), [8], are employed.

Of course, also the approximation theory of CNNs has been studied, albeit to a much lesser extent. Notable examples include [3, 20] and [19]. In fact, in [20] a universal approximation theorem was introduced for CNNs that are based on a convolution after zero padding. Additionally, [19] demonstrates a universal approximation theorem for CNNs where the convolution is based on a group structure. We are not aware, however, of any results that describe the relation between the approximation capabilities of a CNN and the number of its free parameters.

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In this work, we study to what extent the approximation properties of CNNs and FNNs are related. Specifically, we study CNNs as defined in [19] and intuitively described in [3, Chapter 9.5, Equation (9.7)]. These CNNs are translation equivariant functions, which means that a shift in coordinates in the input results in a corresponding shift of the output. For these networks, we prove (a generalisation of) the following result: For every FNN with a certain number of free parameters \( W \) representing a function \( g : \mathbb{R}^N \rightarrow \mathbb{R} \) there exists a CNN with \( O(W) \) parameters, representing a translation equivariant function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) such that \( g_1 = f \), where \( g_1 : \mathbb{R}^N \rightarrow \mathbb{R} \) denotes the first coordinate of \( g \). Conversely, for every CNN with \( W \) free parameters, representing a function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \), there exists an FNN with \( O(W) \) parameters, representing a function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) such that \( g_1 = f \). As a consequence, any upper or lower approximation bounds of FNNs for a given function class are equivalent to approximation bounds of CNNs for the associated function class of translation equivariant functions. As a by-product, our results also give an elementary proof of the universality of CNNs for the class of continuous translation equivariant functions.

The convolutional networks used in practice often employ a form of pooling after each layer (see [3, Section 9.3]); besides, the convolutions are sometimes zero-padded (see [5, Section 9.5]) instead of periodic. However, both of these techniques destroy the translation equivariance. For this reason—and for the sake of mathematical simplicity—we restrict ourselves to the case of periodic convolutions without pooling in this short note.

The paper is structured as follows: We begin by introducing FNNs and CNNs in Sections 2 and 3, respectively. The aforementioned equivalence of approximation rates is then demonstrated in Section 4.

2 Fully-connected neural networks

Let \( \mathcal{G} \) be a finite group of cardinality \( |\mathcal{G}| \in \mathbb{N} \). For a finite set \( I \), we denote the set of real-valued sequences with index set \( I \) by \( \mathbb{R}^I = \{ (x_i)_{i\in I} \mid x_i \in \mathbb{R} \ \forall i \in I \} \). In this note, we consider neural network functions the input of which are elements of \( \mathbb{R}^{|M|} \times \mathcal{G} \), where \( |M| := \{1, \ldots, M\} \). For such networks, we will compare the expressivity of FNNs with that of CNNs. Even though the group structure of \( \mathcal{G} \) will not be used in the present section, it will be essential for the definition of CNNs in the next section.

The following definition of FNNs is standard in the mathematical literature on neural networks; only the restriction to inputs in \( \mathbb{R}^{|C_0|} \times \mathcal{G} \) is slightly unusual.

**Definition 2.1.** Let \( \mathcal{G} \) be a finite group, let \( C_0, L \in \mathbb{N} \), and \( N_1, \ldots, N_L \in \mathbb{N} \). A fully-connected neural network \( \Phi \) is a sequence of affine-linear maps \( \Phi = (V_1, \ldots, V_L) \), where \( V_1 : \mathbb{R}^{C_0} \times \mathcal{G} \rightarrow \mathbb{R}^{N_1} \) and \( V_\ell : \mathbb{R}^{N_{\ell-1}} \rightarrow \mathbb{R}^{N_\ell} \) for \( 1 \leq \ell \leq L \). The architecture \( A(\Phi) \) is given by \( A(\Phi) := (C_0 \cdot |\mathcal{G}|, N_1, \ldots, N_L) \). For an arbitrary function \( \varrho : \mathbb{R} \rightarrow \mathbb{R} \) (called the activation function), we define the \( \varrho \)-realisation of the network \( \Phi = (V_1, \ldots, V_L) \) as

\[
R_{\varrho}(\Phi) : \mathbb{R}^{|C_0|} \times \mathcal{G} \rightarrow \mathbb{R}^{N_L}, \quad x \mapsto x_L,
\]

where \( x_0 := x \), \( x_{\ell+1} := \varrho(V_{\ell+1}(x_\ell)) \) for \( 0 \leq \ell \leq L - 2 \), and \( x_L := V_L(x_{L-1}) \).

In the expression above, \( \varrho \) is applied component-wise, that is, \( \varrho((x_1, \ldots, x_n)) = (\varrho(x_1), \ldots, \varrho(x_n)) \).

For an affine-linear map \( V : \mathbb{R}^I \rightarrow \mathbb{R}^J \), there is a uniquely determined vector \( b \in \mathbb{R}^J \) and a linear map \( A : \mathbb{R}^I \rightarrow \mathbb{R}^J \) such that \( V(x) = Ax + b \) for all \( x \in \mathbb{R}^I \). We then set \( \|V\|_0 := \|b\|_0 + \|A\|_0 \), where \( \|b\|_0 := \{|j \in J \mid b_j \neq 0\} \) denotes the number of non-zero entries of \( b \), and \( \|A\|_0 := \sum_{i \in I} \|A_{\delta_i}\|_0 \), with \( (\delta_i)_{i \in I} \) denoting the standard basis of \( \mathbb{R}^I \). With this notation, we define the number of weights \( W(\Phi) \) and the number of neurons \( N(\Phi) \) as

\[
W(\Phi) := \sum_{\ell=1}^L \|V_\ell\|_0 \quad \text{and} \quad N(\Phi) := C_0 \cdot |\mathcal{G}| + \sum_{\ell=1}^L N_\ell.
\]
3 Convolutional neural networks

For a finite group $G$ and functions $a, b : G \to \mathbb{R}$, we denote by $a * b$ the convolution of $a$ and $b$, defined by

$$a * b : G \to \mathbb{R}, \quad g \mapsto \sum_{h \in G} a(h) b(h^{-1}g).$$

The first step in computing the output of a CNN is to convolve the input with different convolution kernels. This leads to different channels, each of which has the same dimension. Each layer of the network thus has a spatial dimension (the number of elements $|G|$ of the group $G$) and a channel dimension (the number of channels). After the convolution step, the different channels are combined in an affine-linear fashion, but only along fixed spatial coordinates. Finally, the activation function is applied component-wise, and the whole procedure is repeated on the next layer, with input given by the output of the present layer.

We shall now turn this informal description into a formal definition. The definitions might appear to be overly technical, but these technicalities will be important later to estimate the number of parameters of a CNN.

Before stating that definition, we introduce a few notations which will allow for more succinct expressions. First, for $x = (x_{i,g})_{i \in [M], g \in G} \in \mathbb{R}^{[M] \times G}$, we write $x_i = (x_{i,g})_{g \in G}$ for $i \in [M]$. Likewise, we will identify a family $(x_i)_{i \in [M]}$, where $x_i \in \mathbb{R}^G$, with the family $(x_i(g))_{i \in [M], g \in G} \in \mathbb{R}^{[M] \times G}$.

Finally, if $I, J$ are sets, and if $F : \mathbb{R}^I \to \mathbb{R}^J$, then we define the lifting of $F$ as the map $F^† : \mathbb{R}^I \times G \to \mathbb{R}^J \times G$ that results from applying $F$ along fixed spatial coordinates. Formally, this means

$$F^† : \mathbb{R}^I \times G \to \mathbb{R}^J \times G, \quad (x_{j,g})_{j \in J, g \in G} \mapsto ((F((x_{j,g})_{j \in I})_i)_{i \in I, g \in G}.$$  

It is not hard to verify $(F \circ G)^† = F^† \circ G^†$ for $F : \mathbb{R}^I \to \mathbb{R}^J$ and $G : \mathbb{R}^K \to \mathbb{R}^I$.

Given these notations, we can state two fundamental definitions. We start by defining the maps that perform the convolutional steps in a CNN.

**Definition 3.1.** Let $G$ be a finite group. Given $k, C_1 \in \mathbb{N}$, we say that a map $B : \mathbb{R}^{[C_1] \times G} \to \mathbb{R}^{[k] \times [C_1] \times G}$ is filtering, with $k$ filters, if there are $a_1, \ldots, a_k \in \mathbb{R}^G$ such that

$$B((x_i)_{i \in [C_1]}) = (x_j * a_r)_{(r,j) \in [k] \times [C_1]} \quad \forall (x_i)_{i \in [C_1]} \in \mathbb{R}^{[C_1] \times G}. \quad (3)$$

In this case, we write $B \in \text{filter}(G, k, C_1)$, and set $\|B\|_{\ell_0 \text{filter}} := \sum_{r=1}^k \|a_r\|_{\ell^0}$. This is well-defined, since the filters $a_1, \ldots, a_k$ are uniquely determined by $B$.

As already outlined before, a CNN performs convolution steps and then affine-linear transformations along fixed spatial coordinates. This operation is formalised in the following definition.

**Definition 3.2.** Given $k, C_1, C_2 \in \mathbb{N}$, we say that a map $T : \mathbb{R}^{[C_1] \times G} \to \mathbb{R}^{[C_2] \times G}$ is a spatially-convolutional, semi-connected map with $k$ filters, if $T$ can be written as $T = A^† \circ B$ for a map $B \in \text{filter}(G, k, C_1)$ and an affine-linear map $A : \mathbb{R}^{[k] \times [C_1]} \to \mathbb{R}^{[C_2]}$. In this case, we write $T \in \text{Conv}(G, k, C_1, C_2)$, and

$$\|T\|_{\ell_0 \text{conv}} := \min \left\{ \|A\|_{\ell^0} + \|B\|_{\ell_0 \text{filter}}, \quad A : \mathbb{R}^{[k] \times [C_1]} \to \mathbb{R}^{[C_2]} \text{ affine-linear}, \quad B \in \text{filter}(G, k, C_1), \quad T = A^† \circ B \right\}. \quad (4)$$

**Remark 3.3.** Note that every spatially-convolutional, semi-connected map $T \in \text{Conv}(G, k, C_1, C_2)$ is a special affine-linear map. Furthermore, the number of weights of $T$ as an “ordinary” affine-linear map can be estimated up to a multiplicative constant by $\|T\|_{\ell_0 \text{conv}}^2$; in fact, we have

$$\|T\|_{\ell^0} \leq |G|^2 \cdot \|T\|_{\ell_0 \text{conv}}^2. \quad (4)$$

To see this, choose an affine-linear map $A : \mathbb{R}^{[k] \times [C_1]} \to \mathbb{R}^{[C_2]}$ and a filtering map $B \in \text{filter}(G, k, C_1)$ such that $T = A^† \circ B$ and $\|T\|_{\ell_0 \text{conv}} = \|A\|_{\ell^0} + \|B\|_{\ell_0 \text{filter}}$. Furthermore, choose $a_1, \ldots, a_k \in \mathbb{R}^G$ such that $B$ satisfies Equation (3), and let $A_{\text{lin}} : \mathbb{R}^{[k] \times [C_1]} \to \mathbb{R}^{[C_2]}$ be linear and $b \in \mathbb{R}^{[C_2]}$ such that $A(\cdot) = b + A_{\text{lin}}(\cdot)$. 

\[3\]
Now, define \( b^i := (b_j)_{j \in [C_1], g \in \mathcal{G}} \in \mathbb{R}^{[C_2] \times \mathcal{G}} \). It is then not hard to see that \( T = b^i + A_{\text{lin}}^i \circ B \), where the map \( A_{\text{lin}}^i \circ B : \mathbb{R}^{[C_1] \times \mathcal{G}} \to \mathbb{R}^{[C_2] \times \mathcal{G}} \) is linear. Furthermore, for arbitrary \( i_0 \in [C_1], j \in [C_2] \) and \( h_0, g \in \mathcal{G} \), we have

\[
\left( A_{\text{lin}}^i \circ B \right)_{j, g} \cdot \delta_{i_0, h_0} = \left( A_{\text{lin}} \left( \delta_{i_0, i} \cdot \delta_{h_0, g} \right) \right)_{r \in [k], i \in [C_1]} = \sum_{r=1}^{k} (A_{\text{lin}})_{j, (r, i_0)} \cdot a_r (h_0^{-1} g),
\]

where we identified the linear map \( A_{\text{lin}} \) with the matrix associated to it (by virtue of the standard basis). The above identity implies that

\[
\| T \|_{\ell^0} = \| b^i \|_{\ell^0} + \| A_{\text{lin}}^i \circ B \|_{\ell^0} \leq \sum_{j=1}^{C_2} \sum_{g \in \mathcal{G}} \mathbb{1}_{b_j \neq 0} + \sum_{j=1}^{C_2} \sum_{i=1}^{C_1} \sum_{h_0, g \in \mathcal{G}} \sum_{r=1}^{k} \mathbb{1}_{(A_{\text{lin}})_{j, (r, i_0)} \neq 0} \cdot \| a_r (h_0^{-1} g) \|_{\ell^0}
\]

\[
\leq |\mathcal{G}| \cdot b_{\ell} + |\mathcal{G}| \cdot \left( \max_{r=1, \ldots, k} \| a_r \|_{\ell^0} \right) \cdot \sum_{j=1}^{C_2} \sum_{i=1}^{C_1} \sum_{r=1}^{k} \mathbb{1}_{(A_{\text{lin}})_{j, (r, i_0)} \neq 0}
\]

\[
\leq |\mathcal{G}| \cdot b_{\ell} + |\mathcal{G}| \cdot \left( \max_{r=1, \ldots, k} \| a_r \|_{\ell^0} \right) \cdot \| A_{\text{lin}} \|_{\ell^0} \leq |\mathcal{G}|^2 \cdot \| T \|_{\ell^0_{\text{conv}}},
\]

Here, we used the change of variables \( h = h_0^{-1} g \) to see \( \sum_{h_0, g \in \mathcal{G}} \mathbb{1}_{a_r (h_0^{-1} g) \neq 0} = \sum_{h_0 \in \mathcal{G}} \| a_r \|_{\ell^0} = |\mathcal{G}| \cdot \| a_r \|_{\ell^0} \). Furthermore, we used in the last step that \( \| b \|_{\ell^0} + \| A_{\text{lin}} \|_{\ell^0} = \| A \|_{\ell^0} \leq \| T \|_{\ell^0_{\text{conv}}} \), and that \( \| a_r \|_{\ell^0} \leq |\mathcal{G}| \), since \( a_r \in \mathbb{R}^{\mathcal{G}} \).

We now define CNNs similarly to FNNs, with the modification that the affine-linear maps in the definition of the network are required to be spatially-convolutional, semi-connected.

**Definition 3.4.** Let \( L \in \mathbb{N} \), let \( \mathcal{G} \) be a finite group, let \( C_0, C_1, \ldots, C_L \in \mathbb{N} \), and let \( k_1, \ldots, k_L \in \mathbb{N} \). A convolutional neural network \( \Phi \) with \( L \) layers, channel counts \( (C_0, C_1, \ldots, C_L) \), and filter counts \( (k_1, \ldots, k_L) \) is a sequence of spatially-convolutional, semi-connected maps \( \Phi = (T_1, \ldots, T_L) \) such that \( T_{L} \in \text{Conv}(\mathcal{G}, k_{L}, C_{L-1}, C_{L}) \) for \( 1 \leq L \leq L \).

For a convolutional neural network \( \Phi = (T_1, \ldots, T_L) \) and an activation function \( \varphi : \mathbb{R} \to \mathbb{R} \), we define the \( \varphi \)-realisation of \( \Phi \) as

\[
R_{\varphi}(\Phi) : \mathbb{R}^{[C_0] \times \mathcal{G}} \to \mathbb{R}^{[C_L] \times \mathcal{G}}, \quad x \mapsto x_L,
\]

where \( x_0 := x \), \( x_{\ell+1} := \varphi (T_{\ell+1}(x_\ell)) \) for \( 0 \leq \ell \leq L - 2 \), and \( x_L := T_L(x_{L-1}) \).

In the construction above, we again apply \( \varphi \) component-wise.

We call \( C(\Phi) := \sum_{\ell=0}^{L} C_{\ell} \) the number of channels of \( \Phi \); the number \( C_0 = C_0(\Phi) \in \mathbb{N} \) is called the number of input channels of \( \Phi \), while \( C_L = C_L(\Phi) \in \mathbb{N} \) is the number of output channels. The number of weights is \( W_{\text{conv}}(\Phi) := \sum_{\ell=1}^{L} \| T_{\ell} \|_{\ell^0_{\text{conv}}} \).

**Remark.** It could be more natural to call \( W_{\text{conv}}(\Phi) \) the number of free parameters, instead of “the number of weights”. We chose the present terminology primarily to be consistent with the established terminology for FNNs.

**Remark 3.5.** With the identification \( \mathbb{R}^{[C_1] \times \mathcal{G}} \cong \mathbb{R}^{[C_1 \cdot |\mathcal{G}|]} \), each CNN \( \Phi = (T_1, \ldots, T_L) \) is also an FNN, simply because each of the maps \( T_{\ell} \in \text{Conv}(\mathcal{G}, k_{\ell}, C_{\ell-1}, C_{\ell}) \) is an affine-linear map \( T_{\ell} : \mathbb{R}^{[C_{\ell-1}] \times \mathcal{G}} \to \mathbb{R}^{[C_{\ell}] \times \mathcal{G}} \).

When interpreting \( \Phi \) as an FNN, it has architecture \( A(\Phi) = (C_0 \cdot |\mathcal{G}|, C_1 \cdot |\mathcal{G}|, \ldots, C_L \cdot |\mathcal{G}|) \), and thus \( N(\Phi) \leq |\mathcal{G}| \cdot C(\Phi) \). Furthermore, as a consequence of Remark 3.3, we see

\[
W(\Phi) = \sum_{\ell=1}^{L} \| T_{\ell} \|_{\ell^0} \leq |\mathcal{G}|^2 \cdot \sum_{\ell=1}^{L} \| T_{\ell} \|_{\ell^0_{\text{conv}}} = |\mathcal{G}|^2 \cdot W_{\text{conv}}(\Phi).
\]
As we have just seen, CNNs are special FNNs. Hence, it is natural to ask to what extent these networks can achieve the same approximation properties as FNNs. It turns out that the restriction to CNNs is significant, since CNNs can only approximate so-called translation equivariant functions. To make the concept of translation equivariance more precise, and, in particular, meaningful for functions with different input and output dimensions, we first introduce the notion of vectorisation: For a set $I$ and a function $H : \mathbb{R}^G \to \mathbb{R}^G$ we define the $I$-vectorisation $H^{vec, I}$ of $H$ as

$$H^{vec, I} : \mathbb{R}^I \times G \to \mathbb{R}^I \times G, \quad (x_i)_{i \in I} \mapsto (H(x_i))_{i \in I}.$$  

(6)

Next, we define the concept of translation equivariance.

**Definition 3.6.** Let $I, J$ be index sets, and $G$ a finite group. A function $F : \mathbb{R}^I \times G \to \mathbb{R}^J \times G$ is called translation equivariant, if $F \circ S^g_{vec, I} = S^g_{vec, J} \circ F$ for all $g \in G$, where $S^g_{vec, I}$ denotes the $I$-vectorisation of the shift operator $S_g : \mathbb{R}^G \to \mathbb{R}^G, (x_h)_{h \in G} \mapsto (x_{g^{-1}h})_{h \in G}$, and analogously for $S^g_{vec, J}$.

As previously announced, every realisation of a CNN is translation equivariant, as the following proposition demonstrates.

**Proposition 3.7.** Let $G$ be a finite group, let $g : \mathbb{R} \to \mathbb{R}$ be any function, and let $\Phi$ be a CNN. Then the $g$-realisation $R_g(\Phi)$ is translation equivariant.

**Proof.** Recall that convolutions are translation equivariant, in the sense that $S_g(x * a) = (S_g x) * a$ for $a, x \in \mathbb{R}^G$ and $g \in G$; this can be seen directly from the definition of the convolution in Equation (1). Therefore, a filtering map $B$ as in Equation (3) satisfies $S^g_{vec, [k] \times [C_1]} \circ B = B \circ S^g_{vec, [C_1]}$ for all $g \in G$.

Now, for a permutation $\pi$ of $G$, let us write $C_\pi : \mathbb{R}^G \to \mathbb{R}^G, (x_h)_{h \in G} \mapsto (x_{\pi(h)})_{h \in G}$. A direct computation shows for any function $F : \mathbb{R}^I \to \mathbb{R}^J$ that $F^\top \circ C^\top_{\pi} = C^\top_{\pi} \circ F^\top$. Clearly, $S_g = C_\pi$ for a suitable permutation $\pi = \pi_g$. Overall, we thus see for any $T = A^\top \circ B \in \text{Conv}(G, k, C_1, C_2)$ that

$$T \circ S^g_{vec, [C_1]} = A^\top \circ S^g_{vec, [k] \times [C_1]} \circ B = S^g_{vec, [C_2]} \circ A^\top \circ B = S^g_{vec, [C_2]} \circ T \quad \forall g \in G.$$  

Since the activation function $g$ is applied component-wise, this ensures $g(T(S^g_{vec, [C_1]} x)) = S^g_{vec, [C_2]} (g(T x))$ for all $x \in \mathbb{R}^{[C_1] \times G}$ and $T \in \text{Conv}(G, k, C_1, C_2)$. By iterating this observation, we get the claim. \qed

The proposition shows that all realisations of CNNs are translation equivariant. The approximation theory of CNNs was studied before, for instance in the works [3, 20]. These works, however, only consider a restricted class of convolutions. In [19, Theorem 3.1], the universality of CNNs for the class of translation equivariant functions was established. Nevertheless, until now it was not known what kind of approximation rates CNNs yield. In the next section, we will see that there is in fact a fundamental connection between the approximation capabilities of CNNs and FNNs.

4 Approximation rates of convolutional neural networks for translation equivariant functions

We start by demonstrating in Subsection 4.1 how every CNN can be interpreted as an FNN, and—more importantly—how to each FNN one can associate a CNN, such that the first coordinate of the realisation of the CNN is identical to the realisation of the FNN. Afterwards, we demonstrate how this yields an equivalence between the approximation rates of CNNs and FNNs. We close with a concrete example showing how our results can be used to translate approximation results for FNNs into approximation results for CNNs.
4.1 The transference principle

We will measure approximation rates with respect to \(L^p\) norms of vector-valued functions. For these (quasi)-norms, we use the following convention: For \(d \in \mathbb{N}\), for any finite index set \(J\), any measurable subset \(\Omega \subset \mathbb{R}^d\), and any measurable function \(f : \Omega \to \mathbb{R}^J\), we define

\[
\|f\|_{L^p(\Omega, \mathbb{R}^J)} := \|x \mapsto \|f(x)\|_p\|_{L^p(\Omega)}.
\]

Note that this implies for \(F : \Omega \to [G]^{J \times G}\) that

\[
\|F\|_{L^p(\Omega, [G]^{J \times G})}^p = \sum_{g \in G} \|F_g\|_{L^p(\Omega, \mathbb{R}^J)}^p,
\]

where \(F_g := (F)_{g} : \Omega \to \mathbb{R}^J\) denotes the \(g\)-th component of \(F\). Here, the function \(\pi_g^J\) is the projection onto the \(g\)-th component, given by

\[
\pi_g^J : [G]^{J \times G} \to \mathbb{R}^J, (x_{j,h})_{j \in J, h \in G} \mapsto (x_{j,g})_{j \in J}.
\]

**Remark.** One could equally well define \(\|F\|_{L^p(\Omega, [G]^{J \times G})}\) as \(\|\|F\|_{L^p(\Omega)}\) where \(|F(x)| = |F(x)|\) denotes the euclidean norm of \(F(x)\). It is not hard to see that both (quasi)-norms are equivalent since \(J\) and \(G\) are finite; furthermore, the constant of the norm equivalence only depends on \(|J|\) and \(|G|\).

We denote the identity element of \(G\) by 1 and we observe that if \(F, G : \mathbb{R}^{I \times G} \to \mathbb{R}^{I \times G}\) are translation equivariant and \((F)_1 = (G)_1\) then \(F = G\); indeed, it suffices to show for all \(g \in G\) that \((F)_g = (G)_g\). This holds since we have

\[
(F)_g = \pi_1^I \circ S_{g-1}^{vec, I} \circ F = \pi_1^I \circ F \circ S_{g-1}^{vec, I} = (F)_1 \circ S_{g-1}^{vec, I}
\]

for every translation equivariant function \(F : \mathbb{R}^{I \times G} \to \mathbb{R}^{J \times G}\).

Given a finite index set \(I \neq \emptyset\), we say that a subset \(\Omega \subset \mathbb{R}^{I \times G}\) is \(G\)-invariant, if \(S_{g}^{vec, I}(\Omega) \subset \Omega\) for all \(g \in G\). An example of such a set is \(\prod_{i \in I} \Omega_i^G\), where the sets \(\Omega_i \subset \mathbb{R}\) for \(i \in I\) can be chosen arbitrarily. Since \(S_{g}^{vec, I}(x) = P_g x\) for all \(x \in \mathbb{R}^{I \times G}\) and a suitable permutation matrix \(P_g\), Equations (7) and (9) show for any \(p \in (0, \infty)\), any measurable \(G\)-invariant set \(\Omega \subset \mathbb{R}^{I \times G}\) and any two translation equivariant functions \(F, G : \mathbb{R}^{I \times G} \to \mathbb{R}^{J \times G}\) that

\[
\|F - G\|_{L^p(\Omega, [G]^{J \times G})} = \left(\sum_{g \in G} \|(F)_g - (G)_g\|_{L^p(\Omega, \mathbb{R}^J)}^p\right)^{1/p} = |G|^{1/p} \cdot \|F\|_{L^p(\Omega, [G]^{J \times G})} - \|G\|^{1/p} \cdot \|F\|_{L^p(\Omega, [G]^{J \times G})},
\]

and this clearly remains true for \(p = \infty\).

We can now state the transference principle between FNNs and CNNs.

**Theorem 4.1.** Let \(G\) be a finite group, let \(\varepsilon \in [0, \infty)\), \(p \in (0, \infty]\), and \(C_0, N \in \mathbb{N}\). Let \(\Omega \subset \mathbb{R}^{[C_0] \times G}\) be \(G\)-invariant and measurable.

Let \(F : \mathbb{R}^{[C_0] \times G} \to \mathbb{R}^{[N] \times G}\) be translation equivariant, let \(q : \mathbb{R} \to \mathbb{R}\) be an activation function, and let \(\Phi\) be an FNN with architecture \(A(\Phi) = (C_0 : |G|, N_1, \ldots, N_{L-1}, N)\) satisfying \(\|\Phi\|_{L^p(\Omega, [G]^{J \times G})} \leq \varepsilon\).

Then there is a CNN \(\Psi\) with channel counts \((C_0, N_1, \ldots, N_{L-1}, N)\), with filter counts \((N_1 \cdot C_0, 1, \ldots, 1)\), and with \(W_{conv}(\Psi) \leq 2 \cdot W(\Phi)\) and \(\|F - R(\Psi)\|_{L^p(\Omega, [G]^{J \times G})} \leq |G|^{1/p} \cdot \varepsilon\). Here, we use the convention \(|G|^{1/\infty} = 1\).

**Remark.** 1) In fact, the proof shows that the network \(\Psi\) can be chosen independently of the activation function \(q\), unless \(\Phi = (V_1, \ldots, V_L)\) with \(V_\ell = 0\) for some \(\ell \in \{1, \ldots, L\}\).

2) Since we can choose \(\varepsilon = 0\) and \(\Omega = \mathbb{R}^{[C_0] \times G}\), the theorem shows in particular that if \((F)_1 = R(\Phi)\) for an FNN \(\Phi\) of architecture \((C_0 : |G|, N_1, \ldots, N_{L-1}, N)\), then \(F = R(\Psi)\) for a CNN \(\Psi\) with channel counts \((C_0, N_1, \ldots, N_{L-1}, N)\), with filter counts \((N_1 \cdot C_0, 1, \ldots, 1)\), and with \(W_{conv}(\Psi) \leq 2 \cdot W(\Phi)\).
3) In addition to the number of layers and weights, the complexity of the individual weights can also be relevant. Given a subset \( \Lambda \subset \mathbb{R} \), we say that an affine-linear map \( V : \mathbb{R}^J \to \mathbb{R}^J \) has weights in \( \Lambda \) if \( V(x) = b + A x \) for all \( x \in \mathbb{R}^l \) and certain \( b \in \Lambda^0 \) and \( A \in \Lambda^{J \times 1} \). Likewise, we say that an FNN \( \Phi = (V_1, \ldots, V_L) \) has weights in \( \Lambda \) if all \( V_\ell \) have weights in \( \Lambda \).

The proof of the theorem shows that if the FNN \( \Phi = (V_1, \ldots, V_L) \) has weights in \( \Lambda \), and if \( V_\ell \neq 0 \) for some \( \ell \), then the CNN \( \Psi = (T_1, \ldots, T_L) \) constructed in the theorem satisfies \( T_\ell = A_\ell^\top \circ B_\ell \) where all \( A_\ell, B_\ell \) have weights in \( \Lambda \cup \{0,1\} \).

Proof. In view of Equation (10), it suffices to show that there is a CNN \( \Psi \) with the asserted channel counts, filter counts, and number of weights, such that \( (R_\varnothing(\Psi))_1 = R_\varnothing(\Phi) \).

Let \( \Phi = (V_1, \ldots, V_L) \). For brevity, set \( N_0 := C_0 \) and \( N_L := N \), and furthermore \( k_1 := N_1 \cdot C_0 \) and \( k_\ell := 1 \) for \( \ell \in \{2, \ldots, L\} \).

We first handle a few special cases, in order to avoid tedious case distinctions later on. First, if \( \|V_\ell\|_{\varnothing} \neq 0 \), then \( R_\varnothing(\Phi) \equiv 0 \). Then, let \( \Psi = (T_1, \ldots, T_L) \), where \( T_\ell = A_\ell^\top \circ B_\ell \) with \( A_\ell : \mathbb{R}^{[k_\ell] \times [N_{\ell-1}]} \to \mathbb{R}^{[N_\ell]} \), \( x \to 0 \), and with \( B_\ell : \mathbb{R}^{[N_{\ell-1}] \times \mathcal{G}} \to \mathbb{R}^{[k_\ell] \times [N_{\ell-1}] \times \mathcal{G}} \), \( (x_i)_{i \in [N_{\ell-1}]} \mapsto (x_i \ast 0)_{i \in [k_\ell], i \in [N_{\ell-1}]} \) for all \( \ell \in \{1, \ldots, L\} \). It is then trivial to verify that \( \Psi \) has the desired number of filters and channels, that \( (R_\varnothing(\Psi))_1 = 0 \), and that \( \|T_\ell\|_{\varnothing, \text{conv}} = 0 \) for all \( \ell = 1, \ldots, L \), so that \( W(\varnothing) = 0 \leq 2 \cdot W(\Phi) \).

Next, if \( \|V_\ell\|_{\varnothing} > 0 \), but \( \|V_\ell\|_{\varnothing} = 0 \) for some \( \ell \in \{1, \ldots, L - 1\} \), then there is some \( c \in \mathbb{R}^N \) such that \( \|c\|_{\varnothing, \text{conv}} \leq W(\Phi) \) and \( \|c\|_{\varnothing} \equiv c \). Indeed, \( c = V_L \circ c_0 \) for some \( c_0 \in \mathbb{R}^{N_{L-1}} \). Besides, for any \( A \in \mathbb{R}^{n \times k}, \ b \in \mathbb{R}^n \) and \( x \in \mathbb{R}^k \), we have \( (A x + b)_j = b_j + \sum_{\ell=1}^k A_{\ell,j} x_\ell \), which shows that \( 1_{(A x + b)_j \neq 0} \leq 1_{b_j \neq 0} + \sum_{\ell=1}^k 1_{A_{\ell,j} \neq 0} \), and hence
\[
\|A x + b\|_{\varnothing} = \sum_{j=1}^n 1_{(A x + b)_j \neq 0} \leq \sum_{j=1}^n 1_{b_j \neq 0} + \sum_{\ell=1}^k \sum_{j=1}^n 1_{A_{\ell,j} \neq 0} = \|A(\cdot) + b\|_{\varnothing}. \]
Therefore, \( \|c\|_{\varnothing} = \|V_L c_0\|_{\varnothing} \leq \|V_L\|_{\varnothing} \leq W(\Phi) \).

Given such a vector \( c \in \mathbb{R}^N \) with \( R_\varnothing(\Phi) \equiv c \) and \( \|c\|_{\varnothing} \leq W(\Phi) \), we define \( \Psi = (T_1, \ldots, T_L) \), where \( T_\ell = A_\ell^\top \circ B_\ell \), and where \( B_1, \ldots, B_{L-1} \) and \( A_1, \ldots, A_{L-1} \) are defined just as in the previous case, and where finally
\[
A_L : \mathbb{R}^{[k_L] \times [N_{L-1}]} \to \mathbb{R}^{[N_L]}, \ x \mapsto c. \]
This is well-defined, since \( N_L = N \), whence \( c \in \mathbb{R}^{N} \equiv \mathbb{R}^{[N_L]} \). It is not hard to see that \( (R_\varnothing(\Psi))_1 = c = R_\varnothing(\Phi) \), that \( \Psi \) has the right number of filters and channels, and that \( W(\Psi) = \|A_L\|_{\varnothing} = \|c\|_{\varnothing} \leq W(\Phi) \).

In the following, we can thus assume \( \|V_\ell\|_{\varnothing} > 0 \) for all \( \ell \in \{1, \ldots, L\} \). Below, we will repeatedly make use of the following fact: If \( v \in \mathbb{R}^G \), and if we define \( v^* \in \mathbb{R}^G \) by \( v^*_g := v_{g-1} \) for \( g \in \mathcal{G} \), then
\[
(x \ast v^*)_1 = \sum_{h \in \mathcal{G}} x_h v^*_h = \sum_{h \in \mathcal{G}} x_h v_h = (x, v)_{\mathbb{R}^G} \quad \forall x \in \mathbb{R}^G. \tag{11} \]

Furthermore, \( x \ast \delta_1 = x \) for all \( x \in \mathbb{R}^G \), where \( (\delta_1)_1 = 1 \) and \( (\delta_1)_g = 0 \) for \( g \in \mathcal{G} \setminus \{1\} \).

Recall that \( \Phi = (V_1, \ldots, V_L) \). Since \( V_1 : \mathbb{R}^{[C_0] \times \mathcal{G}} \to \mathbb{R}^{[C_0]} \) is affine-linear, there are \( v_j^0 \in \mathbb{R}^G \) and \( b_j \in \mathbb{R} \) (for \( j \in [N_1] \) and \( i \in [C_0] \)) such that \( V_1(\cdot) = b + V_1^{\text{lin}}(\cdot) \), where \( b = (b_j)_{j \in [N_1]} \) and
\[
V_1^{\text{lin}} : \mathbb{R}^{[C_0] \times \mathcal{G}} \to \mathbb{R}^{[N_1]}, \quad (x_i)_{i \in [C_0]} \mapsto \left( \sum_{i=1}^{C_0} (x_i, v_j^0)_{\mathbb{R}^G} \right)_{j \in [N_1]}.
\]

We now define \( T_1 := A_1^\top \circ B_1 \in \text{Conv}(\mathcal{G}, C_0 \cdot N_1, C_0, N_1) \), where
\[
B_1 : \mathbb{R}^{[C_0] \times \mathcal{G}} \to \mathbb{R}^{[N_1] \times [C_0] \times [C_0] \times \mathcal{G}}, \quad (x_i)_{i \in [C_0]} \mapsto (x_i \ast (v_j^0)^*)_{j \in [N_1], i \in [C_0]}
\]
and $A_1 : \mathbb{R}^{[N_1] \times [C_0] \times [C_0]} \to \mathbb{R}^{[N_1]}$, $y \mapsto b + A_1^{\text{lin}} y$, where

$$A_1^{\text{lin}} : \mathbb{R}^{[N_1] \times [C_0] \times [C_0]} \to \mathbb{R}^{[N_1]}, \quad (y_{j,i,i}, y_{i,i})_{j \in [N_1], i \in [C_0]} \mapsto \left( \sum_{i=1}^{C_0} \mathbf{1}_{v \neq 0} \cdot y_{i,i} \right)_\ell \in [N_1].$$

As a consequence of these definitions, we see for arbitrary $x = (x_{j,i})_{j \in [C_0], i \in [G]} \in \mathbb{R}^{[C_0] \times [G]}$ and $\ell \in [N_1]$ that

$$((A_1 \circ B_1) x)_\ell = A_1 \left( (B_1 x)_{j,i,\ell} \right) = A_1 \left( (x_{j} * (v_j^\ell)^*)_{j,i} \right)_{(j,i) \in [N_1] \times [C_0] \times [C_0]}.$$

(Def. of $A_1$ and Eq. (10))

$$b_\ell + \sum_{i=1}^{C_0} (x_i, v_i^\ell)_{\ell} \cdot \mathbf{1}_{v_i \neq 0} = b_\ell + \sum_{i=1}^{C_0} (x_i, v_i^\ell)_{\ell} = (V_1 x)_\ell.$$

In other words, with the projection map $\pi_1^{[N_1]}$ defined in Equation (5), we have

$$\pi_1^{[N_1]} \circ T_1 = \pi_1^{[N_1]} \circ A_1 \circ B_1 = V_1. \quad (12)$$

Furthermore, we see directly from the definition of $A_1^{\text{lin}}$ that

$$\left( A_1^{\text{lin}} \delta_{j,i} \right)_\ell = \sum_{n=1}^{N_1} \left( \mathbf{1}_{v \neq 0} \cdot \delta_{j,i} \right)_{n,n} \in \delta_{j,i} \cdot \mathbf{1}_{v \neq 0} = \delta_{j,\ell} \quad \forall j, \ell \in [N_1] \text{ and } i \in [C_0].$$

Therefore,

$$\|A_1^{\text{lin}}\|_{\ell^0} = \sum_{j=1}^{N_1} \sum_{i=1}^{C_0} \sum_{\ell=1}^{N_1} \mathbf{1}_{(A_1^{\text{lin}} \delta_{j,i})_{\ell} \neq 0} = \sum_{j=1}^{N_1} \sum_{i=1}^{C_0} \mathbf{1}_{v \neq 0} \leq \sum_{j=1}^{N_1} \sum_{i=1}^{C_0} \|v_j\|_{\ell^0} = \|V_1^{\text{lin}}\|_{\ell^0},$$

and hence $\|A_1\|_{\ell^0} = \|b\|_{\ell^0} + \|A_1^{\text{lin}}\|_{\ell^0} \leq \|b\|_{\ell^0} + \|V_1^{\text{lin}}\|_{\ell^0} = \|V_1\|_{\ell^0}$.

Next, note

$$\|B_1\|_{\ell^0} = \sum_{j=1}^{N_1} \sum_{i=1}^{C_0} \|v_j\|_{\ell^0} = \|V_1^{\text{lin}}\|_{\ell^0} \leq \|V_1\|_{\ell^0},$$

which finally implies

$$\|T_1\|_{\ell^0} \leq \|A_1\|_{\ell^0} + \|B_1\|_{\ell^0} \leq 2 \cdot \|V_1\|_{\ell^0}.$$  

Next, for $\ell \in \{2, \ldots, L\}$, we define $T_\ell := V_\ell^\dagger \circ B_\ell \in \text{Conv}(G, 1, N_{\ell-1}, N_{\ell})$, where

$$B_\ell : \mathbb{R}^{[N_{\ell-1}] \times [G]} \to \mathbb{R}^{[N_{\ell-1}] \times [G]}, \quad x = (x_i)_{i \in [N_{\ell-1}]} \mapsto (x_i + \delta_1)_{i \in [N_{\ell-1}]} = x.$$

Note because of $B_\ell x = x$ that $T_\ell = V_\ell^\dagger$. Furthermore, note $\|B_\ell\|_{\ell^0} = \|\delta_1\|_{\ell^0} = 1 \leq \|V_\ell\|_{\ell^0}$, since we excluded the case $\|V_\ell\|_{\ell^0} = 0$ at the beginning of the proof.

$$\|T_\ell\|_{\ell^0} \leq \|V_\ell\|_{\ell^0} + \|B_\ell\|_{\ell^0} \leq 2 \cdot \|V_\ell\|_{\ell^0}.$$  

We now define the CNN $\Psi := (T_1, \ldots, T_L)$, noting that this network indeed has the required number of channels and filters, and that $W_{\text{conv}}(\Psi) = \sum_{\ell=1}^{L} \|T_\ell\|_{\ell^0} \leq 2 \sum_{\ell=1}^{L} \|V_\ell\|_{\ell^0} = 2 \cdot W(\Phi)$. By Proposition 3.7 we see that $R_\Phi(\Psi)$ is translation equivariant. Since $F$ is also translation equivariant, Equations (9) and (10) show that we only need to verify $[R_\Phi(\Psi)]_1 = R_\Phi(\Phi)$. But this is easy to see: We saw above that $T_\ell = V_\ell^\dagger$ for $\ell \in \{2, \ldots, L\}$, which easily implies $T_1^{[N_1]} \circ T_\ell = V_\ell \circ \pi_1^{[N_{\ell-1}]}$. Since furthermore the activation function $\phi$ is applied component-wise, we see

$$[R_\phi(\Psi)]_1 = T_1 \circ \pi_1^{[N_1]} \circ \pi_1^{[N_{\ell-1}]} \circ \cdots \circ \phi \circ V_2 \circ \pi_1^{[N_1]} \circ \cdots \circ \phi \circ T_1,$$

(Eq. (22) and $\pi_1^{[N_{\ell-1}]}$, $\phi \circ \pi_1^{[N_{\ell-1}]}$ since $\phi$ acts component-wise).

$$= (V_L \circ \phi \circ V_{L-1} \circ \cdots \circ \phi \circ V_2) \circ \phi \circ V_1 \circ \cdots \circ \phi \circ V_2 \circ \phi \circ V_1,$$

$$= R_\phi(V_1, \ldots, V_L) = R_\phi(\Phi),$$

as desired.
Remark 4.2. Let $\Psi = (T_1, \ldots, T_L)$ be a CNN with channel counts $(C_0, C_1, \ldots, C_L)$ and filter counts $(k_1, \ldots, k_L)$. The FNN $\Phi^\Psi$ associated to $\Psi$ is $\Phi^\Psi := (T_1, \ldots, T_{L-1}, T'_L)$, where $T'_L := \pi_{C_L} \circ T_L$.

The properties of the network $\Phi^\Psi$ are closely related to those of $\Psi$; in particular, the following holds:

- $R_\varepsilon(\Phi^\Psi) = [R_\varepsilon](\Psi))$;
- If $\|F - R_\varepsilon(\Phi^\Psi)\|_{L^p(\Omega, \mathbb{R}^{C_{L+1} \times \mathbb{G}})} \leq \varepsilon$ for some function $F : \mathbb{R}^{C_0 \times \mathbb{G}} \to \mathbb{R}^{C_L \times \mathbb{G}}$ and some $p \in (0, \infty]$, then $\|F - R_\varepsilon(\Phi^\Psi)\|_{L^p(\Omega, \mathbb{R}^{C_L})} \leq \varepsilon$;
- $\Phi^\Psi$ has architecture $A(\Phi^\Psi) = ([|\mathbb{G}| \cdot C_0, |\mathbb{G}| \cdot C_1, \ldots, |\mathbb{G}| \cdot C_{L-1}, C_L)$, and hence $N(\Phi^\Psi) \leq |\mathbb{G}| \cdot C(\Psi)$; and
- $W_{\text{conv}}(\Phi^\Psi) \leq |\mathbb{G}|^2 \cdot W_{\text{conv}}(\Psi)$.

The very last property is a consequence of Equation (4.5), combined with the estimate $\|\pi|\mathbb{G}| \circ T_L\|_{L^p} \leq \|T_L\|_{L^p}$.

4.2 Equivalence of approximation rates

For $C_0, N \in \mathbb{N}$ and any function class $C \subset \{F : \mathbb{R}^{C_0 \times \mathbb{G}} \to \mathbb{R}^{N}\}$, we call

$$C_{\text{equiv}} := \left\{ G : \mathbb{R}^{C_0 \times \mathbb{G}} \to \mathbb{R}^{N \times \mathbb{G}} \mid \text{G translation equivariant and } (G)_1 \in C \right\}$$

the equivariant function class associated to $C$.

The properties of the network $\Phi^\Psi$ imply that for any function class $C \subset \{F : \mathbb{R}^{C_0 \times \mathbb{G}} \to \mathbb{R}^{N}\}$, the approximation rate of FNNs in terms of the number of neurons [or number of weights] is equivalent (up to multiplicative constants that depend only on $|\mathbb{G}|$) to the approximation rate of CNNs in terms of the number of channels [or number of weights] for the associated equivariant function class $C_{\text{equiv}}$.

As a result, all upper and lower approximation bounds established for FNNs (such as for instance [1] [2] [4] [10] [11] [12] [14] [18]) directly imply the same bounds for CNNs for the corresponding translation equivariant function classes. As concrete examples of this, we now state the approximation theorem for CNNs that corresponds to [14, Theorem 3.1]. In the following result, the activation function $\rho$ that is used is the so-called rectified linear unit (ReLU) given by $\rho(x) = \max\{0, x\}$.

Proposition 4.3. Let $\mathcal{G}$ be a finite group and let $d \in \mathbb{N}$ and $\beta, p \in (0, \infty)$. Then there is a constant $c = c(d, |\mathcal{G}|, \beta, p) > 0$ such that for any $\varepsilon < (0, 1/2)$ and any translation equivariant function $F : \mathbb{R}^{d \times \mathcal{G}} \to \mathbb{R}^\mathcal{G}$ satisfying $\|F\|_{C^{0}([-1/2, 1/2]^{d} \times \mathcal{G})} \leq 1$, there is a CNN $\Psi^F_{\varepsilon}$ with at most $(2 + \lfloor \log_2 \beta \rfloor) \cdot (11 + \beta / (d |\mathcal{G}|))$ layers and such that

$$W_{\text{conv}}(\Psi^F_{\varepsilon}) \leq c \cdot \varepsilon^{-d |\mathcal{G}| / \beta} \quad \text{and} \quad \|R_\varepsilon(\Psi^F_{\varepsilon}) - F\|_{L^p\left([-1/2, 1/2]^{d} \times \mathbb{G}^\mathcal{G}\right)} \leq \varepsilon.$$

Here, $\rho : \mathbb{R} \to \mathbb{R}, x \mapsto \max\{0, x\}$ is the ReLU.

Remark. 1) For the precise definition of $\|f\|_{C^0}$, we refer to [14, Section 3.1].

2) Under a certain encodability assumption (see [14, Section 4]) on the weights of the approximating networks, one can show using Remark 4.2 that the approximation rate from above is optimal up to a factor that is logarithmic in $1/\varepsilon$. However, since this encodability condition is quite technical, we do not state this result in detail.

Proof. Let $D := d \cdot |\mathcal{G}|$ and $L := (2 + \lfloor \log_2 \beta \rfloor) \cdot (11 + \beta / D)$, as well as $f := (F)_1 : \mathbb{R}^{d \times \mathcal{G}} \to \mathbb{R}$. With the constant $c_0 = c_0(D, \beta) > 0$ provided by [14, Theorem 3.1], we see that there is an FNN $\Phi^F_{\varepsilon}$ with at most $L$ layers, with $W(\Phi^F_{\varepsilon}) \leq c_0 \cdot (|\mathcal{G}|^{-1/p} \cdot \varepsilon)^{-D/\beta}$, and with $\|R_\varepsilon(\Phi^F_{\varepsilon}) - f\|_{L^p([-1/2, 1/2]^{d} \times \mathcal{G})} \leq |\mathcal{G}|^{-1/p} \cdot \varepsilon$. Thus, setting $c := 2c_0 \cdot |\mathcal{G}|^{D/(\beta p)}$, Theorem 4.1 yields a CNN $\Psi^F_{\varepsilon}$ satisfying all the stated properties. \qed
For the sake of brevity, we refrain from explicitly stating the CNN versions of the results in [1, 2, 4, 10, 11, 12, 18].

Finally, we remark that Theorem 4.1 yields a new, simplified proof of the universal approximation theorem for CNNs that was originally derived by Yarotsky [19]. This theorem states that if $\Omega \subset \mathbb{R}^{|C_0|} \times \mathcal{G}$ is $\mathcal{G}$-invariant and compact, and if $F : \mathbb{R}^{|C_0|} \times \mathcal{G} \rightarrow \mathbb{R}^{|N|} \times \mathcal{G}$ is continuous and translation equivariant, then $F$ can be uniformly approximated on $\Omega$ by $\varphi$-realisations of CNNs of any fixed depth $L \geq 2$, as long as $\varphi$ is not a polynomial.

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