Decidability of the Elementary Theory of a Torsion-Free Hyperbolic Group

Olga Kharlampovich, Alexei Myasnikov

January 24, 2014

Abstract

Let $\Gamma$ be a torsion free hyperbolic group. We prove that the elementary theory of $\Gamma$ is decidable and admits an effective quantifier elimination to boolean combination of $\forall\exists$-formulas. The existence of such quantifier elimination was previously proved in [37].

1 Introduction

It was proved in [37] that every first order formula in the theory of a torsion free hyperbolic group is equivalent to a boolean combination of $\forall\exists$-formulas. We will prove the following result.

Theorem 1. Let $\Gamma$ be a torsion free hyperbolic group. There exists an algorithm given a first-order formula $\phi$ to find a boolean combination of $\forall\exists$-formulas that define the same set as $\phi$ over $\Gamma$.

This theorem will be proved in Section 6. We will also prove the following result in Section 4.

Theorem 2. The $\forall\exists$-theory of a torsion-free hyperbolic group is decidable.

These results imply

Theorem 3. The elementary theory of a torsion-free hyperbolic group is decidable.

Notice, that an algorithm to solve systems of equations in torsion free hyperbolic groups was constructed in [35]. The problem of solving equations in such groups was reduced to the problem of solving equations in a free group, and Makanin algorithm was used for solving equations in a free group [39]. Decidability of the existential theory of a free group was shown in [30], and decidability of the existential theory of a torsion free hyperbolic group was proved in [37], [8], and, later, in [22]. It was shown in [29] that the elementary theory of a free group is decidable (solution of an old problem of Tarski). In [21] we proved the statement of Theorem 1 for a free group.
The techniques that we are using can be applied in some other important classes of groups, for example, partially commutative groups (right-angled Artin groups). It is known that the compatibility problem for systems of equations over partially commutative groups is decidable, see [10]. Moreover, the universal (existential) and positive theories of partially commutative groups are also decidable, see [11] and [6]. An effective description of the solution set of systems of equations over a partially commutative group was given in [5] (using an analogue of Makanin-Razborov diagrams), and this description can be applied to study elementary theories of these groups.

2 Preliminary facts

2.1 Toral relatively hyperbolic groups

A finitely generated group $G$ that is hyperbolic relative to a collection $\{P_1, \ldots, P_k\}$ of subgroups is called toral if $P_1, \ldots, P_k$ are all abelian and $G$ is torsion-free.

Many algorithmic problems in (toral) relatively hyperbolic groups are decidable, and in particular we take note of the following for later use.

**Lemma 1.** In every toral relatively hyperbolic group $G$, the following hold.

1. The conjugacy problem in $G$, and hence the word problem, is decidable.
2. If $g \in G$ is a hyperbolic element (i.e. not conjugate to any element of any $H_i$), then the centralizer $C(g)$ of $g$ is an infinite cyclic group. Further, a generator for $C(g)$ can be effectively constructed.

**Proof.** The word problem was solved in [13] and the conjugacy problem in [2]. For the second statement, let $G = \langle A \rangle$ and let $g \in G$ be a hyperbolic element. Theorem 4.3 of [33] shows that the subgroup

$$E(g) = \{ h \in G \mid \exists n \in \mathbb{N} : h^{-1}g^n h = g^n \}$$

has a cyclic subgroup of finite index. Since $G$ is torsion-free, $E(g)$ must be infinite cyclic (see for example the proof of Proposition 12 of [31]). Clearly $C(g) \leq E(g)$, hence $C(g)$ is infinite cyclic.

To construct a generator for $C(g)$, consider the following results of D. Osin (see the proof of Theorem 5.17 and Lemma 5.16 in [32]):

1. there exists a constant $N$, which depends on $G$ and the word length $|g|$ and can be computed, such that if $g = f^n$ for some $f \in G$ and with $n$ positive then $n \leq N$;
2. there is a computable function $\beta : \mathbb{N} \to \mathbb{N}$ such that if $f$ is an element of $G$ with $f^n = g$ for some positive $n$, then $f$ is conjugate to some element $f_0$ satisfying $|f_0| \leq \beta(|g|)$.
We proceed as follows. Let \( \mathcal{F} \) be the set of all \( f \in G \) such that \( |f| \leq \beta(|g|) \) and \( h^{-1}f^n h = g \) for some \( h \in G \) and \( 1 \leq n \leq N \). It is finite, non-empty (since \( g \) is an element), and can be computed (since conjugacy is decidable). Let \( f \) be an element of \( \mathcal{F} \) such that the exponent \( n \) is maximum amongst elements of \( \mathcal{F} \) and find an element \( h \in G \) such that \( h^{-1}f^n h = g \) (we may find \( h \) by enumeration).

We claim that if \( \bar{g} \) is a generator of \( C(g) \) then either \( h^{-1}fh = \bar{g} \) or \( h^{-1}fh = \bar{g}^{-1} \). Indeed, \( h^{-1}fh \in C(g) \) since it commutes with \( g = (h^{-1}fh)^n \), hence \( h^{-1}fh = \bar{g}^k \) for some \( k \) and so

\[
g = (h^{-1}fh)^n = \bar{g}^k.
\]

Suppose \( k > 0 \). Since \( \bar{g}^kn = g \), it implies that \( \bar{g} \) is conjugate to some element \( g_0 \) with \( |g_0| \leq \beta(|g|) \). Then \( g_0\bar{g}^n \) is conjugate to \( g \), so by \( \beta \) \( kn \leq N \) hence \( g_0 \in \mathcal{F} \). By maximality of the exponent in the choice of \( f \), \( k \) must be 1 and \( h^{-1}fh = \bar{g} \). If \( k < 0 \), a similar argument shows that \( h^{-1}fh = \bar{g}^{-1} \).

**Definition 1.** Let \( G \) be a group generated by a finite set \( X \), \( \{P_1, \ldots, P_m\} \) be a collection of subgroups of \( G \). A subgroup \( R \) of \( G \) is called relatively quasi-convex with respect to \( \{P_1, \ldots, P_m\} \) (or simply relatively quasi-convex when the collection \( \{P_1, \ldots, P_m\} \) is fixed) if there exists a constant \( \sigma > 0 \) such that the following condition holds. Let \( f, g \) be two elements of \( R \), and \( p \) an arbitrary geodesic path from \( f \) to \( g \) in \( \text{Cayley}(G, X \cup \mathcal{P}) \), where \( \mathcal{P} \) is the union of all subgroups in \( \{P_1, \ldots, P_m\} \). Then for any vertex \( v \in p \), there exists a vertex \( w \in R \) such that

\[
dist_X(u, w) \leq \sigma.
\]

Note that, without loss of generality, we may assume one of the elements \( f, g \) to be equal to the identity since both the metrics \( \text{dist}_X \) and \( \text{dist}_{X \cup \mathcal{P}} \) are invariant under the left action of \( G \) on itself.

It is easy to see that, in general, this definition depends on \( X \). However in case of relatively hyperbolic groups it does not depend on \( X \) [32].

**Proposition 1.** [33] There exists an algorithm which given a finite presentation of a toral relatively hyperbolic group \( G \) and finitely generated relatively quasi-convex subgroups \( H \) and \( R \) of \( G \) (given by finite generating sets)

1) finds a finite family \( \mathcal{J} \) of non-trivial intersections \( J = H^g \cap R \neq 1 \) such that any non-trivial intersection \( H^{g^r} \cap R \) has form \( J^r \) for some \( r \in R \) and \( J \in \mathcal{J} \). One can effectively find the generators of the subgroups from \( \mathcal{J} \).

2) solves the membership problem for \( H \).

**Corollary 1.** Let \( H, R \) be finitely generated relatively quasi-convex subgroups of a toral relatively hyperbolic group \( G \). Then one can effectively verify whether or not \( R \) is conjugate into \( H \), and if it is, then find a conjugator.

**Proof.** Using Proposition 1 we can find all the non-trivial intersections \( J = R \cap g^{-1}Hg \). For each such subgroup \( J \) we check if all the generators of \( R \) belong to \( J \). If such \( J \) exists, then \( J \) is \( R \) and \( R \) is conjugate into \( H \). If such \( J \) does not exist, then \( R \) is not conjugate into \( H \).
2.2 JSJ decomposition of toral relatively hyperbolic groups

Definition 2. A splitting of a group \( G \) is a graph of groups decomposition. The splitting is called abelian if all of the edge groups are abelian. An elementary splitting is a graph of groups decomposition for which the underlying graph contains one edge. A splitting is reduced if it admits no edges carrying an amalgamation of the form \( A *_C C \).

Let \( G \) be a toral relatively hyperbolic group. A reduced splitting of \( G \) is called essential if

(1) all edge groups are abelian; and

(2) if \( E \) is an edge group and \( x^k \in E \) for some \( k > 0 \) then \( x \in E \). A reduced splitting of \( G \) is called primary if it is essential and all noncyclic abelian subgroups of \( G \) are elliptic (that is conjugate into vertex subgroups of the splitting).

Proposition 2. [9] There is an algorithm which takes a finite presentation for a freely indecomposable toral relatively hyperbolic group, \( \Gamma \) say, as input and outputs a graph of groups which is a primary JSJ decomposition for \( \Gamma \).

Proposition 3. ([9], Theorem 3.41) Suppose that \( G \) is toral relatively hyperbolic relative to \( \mathcal{P} = \{P_1, \ldots, P_n\} \), that \( \Lambda \) is a graph of groups decomposition of \( G \) with finitely presented vertex groups and finitely presented edge groups which are either word-hyperbolic or else contained in a conjugate of some parabolic subgroup. Suppose furthermore that each \( P_i \in \mathcal{P} \) is conjugate into some vertex group of \( \Lambda \). Let \( V \) be a vertex group of \( \Lambda \). Define \( \mathcal{P}_V \) to be a collection of representatives of conjugacy classes of intersections of \( V \) with conjugates of the \( P_i \). Then \( V \) is hyperbolic relative to \( \mathcal{P}_V \).

Proposition 4. ([7], Theorem 0.1 and Corollary 4.1 and [19]) In the proposition above the QH vertex groups embed as fully quasi-convex subgroups in \( G \), the other vertex groups embed as relatively quasi convex subgroups.

Proposition 5. There is an algorithm which takes a finite presentation for a freely indecomposable toral relatively hyperbolic group \( G \), and finitely generated subgroups \( H_1, \ldots, H_n \) of \( G \) as input, and outputs a graph of groups which is a primary abelian JSJ decomposition for \( G \) relative to \( H_1, \ldots, H_n \).

Proof. We consider the case of \( n = 1 \) because the general case can be proved similarly. We first construct a primary JSJ decomposition \( D \) of \( G \) relative to the set of generators of \( H \). This can be done using [9]. We will need the following lemma.

Lemma 2. Given a primary elementary abelian splitting of a toral relatively hyperbolic group \( G \) and its finitely generated subgroup \( H \), there is an algorithm to decide if \( H \) is elliptic in this splitting and if it is not, to produce an element from \( H \) which is hyperbolic.

Proof. Suppose \( G = A *_{C} B \) is an elementary abelian splitting. By the CSA property of toral relatively hyperbolic groups (see [15], Lemma 2.5), and [14], \( C \) must be maximal abelian either in \( A \) or in \( B \). Each generator of \( H \) can
be written in an amalgamated product normal form as $h = a_1 b_1 \ldots a_k b_k$ with $a_i \in A, b_i \in B$. Notice that every element that is conjugate to a cyclically reduced element in $G$ can be obtained from this element by a cyclic permutation post composed with conjugation by an element from $C$ (see, for example, [27]). Therefore if an element is conjugate into $A$, then the reduced form of some of its cyclic permutations belongs to $A$ (there is an algorithm to decide this because we only have to be able to solve the membership problem in $C$ for $A$ and $B$). If none of the generators is conjugate into $A$ or $B$, then the lemma is proved. Otherwise, we can conjugate $H$ and suppose that the first (nontrivial) generator of $H$ belongs, say, to $A$. If the first generator belongs to $A$, then $H$ is conjugate to $A$ if and only if all the other generators belong to $A$.

HNN extensions can be considered similarly, this proves the lemma.

Corollary 2. Given a primary abelian splitting of a toral relatively hyperbolic group $G$ and its finitely generated subgroup $H$, there is an algorithm to decide if $H$ is elliptic in this splitting and if it is not, to produce an element from $H$ which is hyperbolic.

We can now finish the proof of the proposition. If $H$ is not elliptic with respect to an elementary splitting corresponding to some edge connecting a rigid subgroup and a QH vertex group $Q$ (with corresponding surface $S_Q$) of $D$, then we can construct a hyperbolic element $h \in H$. We construct a relative JSJ decomposition $D_h$ adding the maximal cyclic subgroup containing $h$ to the set of peripheral subgroups. All elementary splittings of $G$ corresponding to $D_h$ are also elementary splittings corresponding to $D$, but since $<h>$ is elliptic in $D_h$ and hyperbolic in $D$, there are strictly less elementary splittings corresponding $D_h$ than to $D$. Moreover, not all the elementary splittings corresponding to simple closed curves on $Q$ and edges from $Q$ are splittings of $D_h$. Indeed, the application of some of the canonical Dehn twists of $S_Q$ would change $h$, therefore the group of canonical automorphisms of $D_h$ (see [20]) does not contain these Dehn twists. Since generators of the Mapping class group of $S_Q$ are Dehn twists along particular non-separating simple closed curves on $S_Q$, (see [28], [18]), the size of the QH subgroup in $D_h$ that is a subgroup of $Q$ (if any) is strictly less than $\text{size}(Q)$. Then the regular size $\text{size}(D_h)$ of $D_h$, namely the tuple $\text{size}(D_h) = (\text{size}(Q_1), \ldots, \text{size}(Q_r))$ of sizes of the MQH subgroups in decreasing order (defined in Section 2.3, [20]) is strictly less than $\text{size}(D)$.

We now check is $H$ is elliptic in the quadratic decomposition corresponding to $D_h$ (in this decomposition we collapse all the edges between non-QH subgroups of $D_h$). If $H$ is not elliptic, we find a hyperbolic element $h_1$ and construct a decomposition $D_{h,h_1}$ adding the maximal cyclic subgroup containing $h_1$ to the set of peripheral subgroups. $\text{size}(D_{h,h_1}) < \text{size}(D_h)$. Since the regular size cannot decrease infinitely, eventually we will obtain a splitting $D_Q$ such that MQH subgroups of this splitting are exactly MQH subgroups of the JSJ decomposition relative to $H$.

We can now check if $H$ is elliptic in the decomposition obtained from $D_Q$ by collapsing all the edges between non-abelian subgroups. We then increase if
necessary the edge groups between rigid and abelian subgroups. This operation decreases the \( ab(D) \) (the sum of ranks of the abelian vertex groups minus the sum of ranks of the edge groups between abelian and rigid subgroups). Again, we cannot decrease the abelian rank infinitely, so eventually we get a decomposition \( D_{Q,A} \) of \( G \) such that \( H \) is elliptic to all \( QH \) and abelian subgroups of this decomposition.

We now collapse those edges between rigid subgroups of \( D_{Q,A} \) that correspond to elementary splittings of \( G \) for which \( H \) is not elliptic.

3 Effective description of homomorphisms to \( \Gamma \)

In this section, following [22] and [20] we describe an algorithm that takes as input a finite system of equations \( S \) over \( \Gamma \) and produces a tree diagram \( T \) that encodes the set \( \text{Hom}_\Gamma(\Gamma_{R(S)}, \Gamma) \). When \( S \) is a system without coefficients, we interpret \( S \) as relators for a finitely presented group \( G = \langle Z \mid S \rangle \) and the diagram \( T \) encodes instead the set \( \text{Hom}(G, \Gamma) \).

There are two ingredients in this construction: first, the reduction of the system \( S \) over \( \Gamma \) to finitely many systems of equations over free groups, and second, the construction of Hom-diagrams (Makanin-Razborov diagrams) for systems of equations over free groups.

Fix \( \Gamma = \langle A \mid R \rangle \) a finitely presented torsion-free hyperbolic group, \( F \) the free group on \( A \), and \( \pi : F \to \Gamma \) the canonical epimorphism.

The map \( \pi \) induces an epimorphism \( F[X] \to \Gamma[X] \), also denoted \( \pi \), by fixing each \( x \in X \). For a system of equations \( S \subset F[X] \), we study the corresponding system \( S^\pi \subset \Gamma[X] \) which we denote again by \( S \). The radical of \( S \) over \( \Gamma \) (the normal subgroup that consists of all elements of \( \Gamma[X] \) that are sent to the identity by all solutions of \( S \)) will be denoted \( R_\Gamma(S) \).

The coordinate group is defined as \( \Gamma_{R(S)} = \Gamma[X]/R(S) \), where \( X \) is precisely the set of variables appearing in \( S \).

Notice that the relators \( R \) of \( \Gamma \) are in the radical \( R_\Gamma(S) \) for every system of equations \( S \), hence

\[
F_{R_\Gamma(S)} = \Gamma_{R(S)}.
\]

Let \( \overline{\cdot} \) denote the canonical epimorphism \( F(X, A) \to \Gamma_{R(S)} \). For a homomorphism \( \phi : F(X, A) \to K \) we define \( \overline{\phi} : \Gamma_{R(S)} \to K \) by

\[
\overline{\phi}(\overline{w}) = \phi(w),
\]

where any preimage \( w \) of \( \overline{w} \) may be used. We will always ensure that \( \overline{\phi} \) is a well-defined homomorphism.

3.1 Reduction to systems of equations over free groups

In [35], the problem of deciding whether or not a system of equations \( S \) over a torsion-free hyperbolic group \( \Gamma \) has a solution was solved by constructing canonical representatives for certain elements of \( \Gamma \). This construction reduced
the problem to deciding the existence of solutions in finitely many systems of equations over free groups, which had been previously solved. The reduction may also be used to find all solutions to $S$ over $\Gamma$, as described below.

**Lemma 3.** Let $\Gamma = \langle A \mid R \rangle$ be a torsion-free $\delta$-hyperbolic group and $\pi : F(A) \to \Gamma$ the canonical epimorphism. There is an algorithm that, given a system $S(Z,A) = 1$ of equations over $\Gamma$, produces finitely many systems of equations

$$S_1(X_1,A) = 1, \ldots, S_n(X_n,A) = 1$$

over $F$, constants $\lambda, \mu > 0$, and homomorphisms $\rho_i : F(Z,A) \to F_{R(S_i)}$ for $i = 1, \ldots, n$ such that

(i) for every $F$-homomorphism $\phi : F_{R(S_i)} \to F$, the map $\rho_i \phi \pi : \Gamma_{R(S)} \to \Gamma$ is a $\Gamma$-homomorphism, and

(ii) for every $\Gamma$-homomorphism $\psi : \Gamma_{R(S)} \to \Gamma$ there is an integer $i$ and an $F$-homomorphism $\phi : F_{R(S_i)} \to F(A)$ such that $\rho_i \phi \pi = \psi$. Moreover, for any $z \in \Gamma$, the word $z^\rho_i \phi$ labels a $(\lambda, \mu)$-quasigeodesic path for $z^\psi$ in $\Gamma$.

Further, if $S(Z) = 1$ is a system without coefficients, the above holds with $G = \langle Z \mid S \rangle$ in place of $\Gamma_{R(S)}$ and ‘homomorphism’ in place of ‘$\Gamma$-homomorphism’.

**Proof.** The result is an easy corollary of Theorem 4.5 of [35], but we will provide a few details.

We may assume that the system $S(Z,A)$, in variables $z_1, \ldots, z_l$, consists of $m$ constant equations and $q - m$ triangular equations, i.e.

$$S(Z,A) = \left\{ \begin{array}{c}
  z_{\sigma(j,1)} z_{\sigma(j,2)} z_{\sigma(j,3)} = 1 \quad j = 1, \ldots, q - m \\
  z_s = a_s
\end{array} \right. \quad s = l - m + 1, \ldots, l$$

where $\sigma(j,k) \in \{ 1, \ldots, l \}$ and $a_i \in \Gamma$. A construction is described in [35] which, for every $m \in \mathbb{N}$, assigns to each element $g \in \Gamma$ a word $\theta_m(g) \in F$ satisfying

$$\theta_m(g) = g$$

called its *canonical representative*. The representatives $\theta_m(g)$ are not ‘global canonical representatives’, but do satisfy useful properties for certain $m$ and certain finite subsets of $\Gamma$, as follows.

Let $L = q \cdot 2^{5050(\delta + 1)^6(2|A|)^{2\delta}}$. Suppose $\psi : F(Z,A) \to \Gamma$ is a solution of $S(Z,A)$ and denote

$$\psi(z_{\sigma(j,k)}) = g_{\sigma(j,k)}.$$ 

Then there exist $h_k^{(j)}, c_k^{(j)} \in F(A)$ (for $j = 1, q - m$ and $k = 1, 2, 3$) such that

\[ \text{[The constant of hyperbolicity $\delta$ may be computed from a presentation of $\Gamma$ using the results of [35].] } \]
(i) each $c_k^{(j)}$ has length less than $L$ (as a word in $F$),

(ii) $c_1^{(j)} c_2^{(j)} c_3^{(j)} = 1$ in $\Gamma$,

(iii) there exists $m \leq L$ such that the canonical representatives satisfy the following equations in $F$:

\[
\begin{align*}
\theta_m(g_{\sigma(j,1)}) &= h_1^{(j)} c_1^{(j)} \left(h_2^{(j)}\right)^{-1} \quad (2) \\
\theta_m(g_{\sigma(j,2)}) &= h_2^{(j)} c_2^{(j)} \left(h_3^{(j)}\right)^{-1} \quad (3) \\
\theta_m(g_{\sigma(j,3)}) &= h_3^{(j)} c_3^{(j)} \left(h_1^{(j)}\right)^{-1}. \quad (4)
\end{align*}
\]

In particular, when $\sigma(j,k) = \sigma(j',k')$ (which corresponds to two occurrences in $S$ of the variable $z_{\sigma(j,k)}$) we have

\[
\begin{equation}
\begin{split}
h_k^{(j)} c_k^{(j)} \left(h_k^{(j)}\right)^{-1} &= h_k^{(j')} c_k^{(j')} \left(h_k^{(j')}\right)^{-1}. \quad (5)
\end{split}
\end{equation}
\]

Moreover, $\theta_m(g_{\sigma(j,i)})$ labels a $(\lambda, \mu)$-quasigeodesic. Consequently, we construct the systems $S(X_i, A)$ as follows. For every positive integer $m \leq L$ and every choice of $3(q-m)$ elements $c_1^{(j)}, c_2^{(j)}, c_3^{(j)} \in F$ ($j = 1, \ldots, q-m$) satisfying (i) and (ii), we build a system $S(X_i, A)$ consisting of the equations

\[
\begin{align*}
X^{(j)} c_k^{(j)} \left(x_k^{(j)}\right)^{-1} &= x_k^{(j')} c_k^{(j')} \left(x_k^{(j')}\right)^{-1} \quad (6) \\
X^{(j)} c_k^{(j)} \left(x_{k+1}^{(j)}\right)^{-1} &= \theta_m(a_s) \quad (7)
\end{align*}
\]

where an equation of type (6) is included whenever $\sigma(j,k) = \sigma(j',k')$ and an equation of type (7) is included whenever $\sigma(j,k) = s \in \{l-m+1, \ldots, l\}$. To define $\rho_i$, set

\[
\rho_i(z_s) = \begin{cases} 
X^{(j)} c_k^{(j)} \left(x_k^{(j)}\right)^{-1}, & 1 \leq s \leq l-m \text{ and } s = \sigma(j,k) \\
\theta_m(a_s), & l-m+1 \leq s \leq l
\end{cases}
\]

where for $1 \leq s \leq l-m$ any $j,k$ with $\sigma(j,k) = s$ may be used.

If $\psi : F(Z) \to \Gamma$ is any solution to $S(Z, A) = 1$, there is a system $S(X_i, A)$ such that $\theta_m(g_{\sigma(j,k)})$ satisfy (2)-(4). Then the required solution $\phi$ is given by

\[
\phi(x_j^{(k)}) = h_j^{(k)}.
\]

The bound of $L$ here, and below, is extremely loose. Somewhat tighter, and more intuitive, bounds are given in [34].

The word problem in hyperbolic groups is decidable.
Indeed, (iii) implies that \( \phi \) is a solution to \( S(X_i, A) = 1 \). For \( s = \sigma(j, k) \in \{1, \ldots, l - m\} \),
\[
\sigma_j \phi = h_j^{(j)} c_k (h_{k+1}^{(j)})^{-1} = \theta_\phi (g_\sigma(j, k))
\]
and similarly for \( s \in \{l - m + 1, \ldots, l\} \), hence \( \psi = \rho_i \phi \pi \).

Conversely, for any solution \( \phi(x^{(k)}_j) = h_j^{(k)} \) of \( S(X_i) = 1 \) one sees that by \( \square \),
\[
\sigma_j \phi h_j^{(j)} c_1 c_2 c_3 (h^{(j)}_1)^{-1}
\]
which maps to 1 under \( \pi \) by (ii), hence \( \rho_i \phi \pi \) induces a homomorphism.

### 3.2 Encoding solutions with the tree \( T \)

An algorithm is described in Section 7 of \cite{20} which constructs, for a given system of equations \( S(X, A) \) over the free group \( F \), a canonical diagram encoding the set of solutions of \( S \). The diagram consists of a directed finite rooted tree \( T \) that has, in particular, the following properties. Let \( G = F_{R(S)} \).

(i) Each vertex \( v \) of \( T \) is labelled by a pair \((G_v, Q_v)\) where \( G_v \) is an \( F \)-quotient of \( G \) and \( Q_v \) the subgroup of canonical automorphisms in \( \text{Aut}_F(G_v) \) (see \cite{20} for the definition). The root \( v_0 \) is labelled by \((G, 1)\) and every leaf is labelled by \((F(Y) * F, 1)\) where \( Y \) is some finite set (called free variables).

Each \( G_v \), except possibly \( G_{v_0} \), is fully residually \( F \).

(ii) Every (directed) edge \( v \to v' \) is labelled by a proper surjective \( F \)-homomorphism \( \pi(v, v') : G_v \to G_{v'} \).

(iii) For every \( \phi \in \text{Hom}_F(G, F) \) there is a path \( p = v_0 v_1 \ldots v_k \) where \( v_k \) is a leaf labelled by \((F(Y) * F, 1)\), elements \( \sigma_i \in Q_{v_i} \), and a \( F \)-homomorphism \( \phi_0 : F(Y) * F \to F \) such that
\[
\phi = \pi(v_0, v_1) \pi(v_1, v_2) \sigma_2 \ldots \pi(v_{k-2}, v_{k-1}) \sigma_{k-1} \pi(v_{k-1}, v_k) \phi_0. \tag{8}
\]

(iv) The canonical splitting of each fully residually free group \( G_v \) is its Grushko decomposition followed by the abelian JSJ decompositions of the factors.

The algorithm gives for each \( G_v \) a finite presentation \( \langle A_v \mid R_v \rangle \), and for each \( Q_v \) a finite list of generators in the form of functions \( A_v \to (A_v \cup A_v^{-1})^\ast \). Note that the choices for \( \phi_0 \) are exactly parametrized by the set of functions from \( Y \) to \( F \).

Let \( S(Z, A) = 1 \) be a finite system of equations over \( \Gamma \). We will construct a diagram \( T \) to encode the set of solutions of \( S(Z, A) = 1 \). Namely, we will construct a tree of fundamental sequences encoding all solutions of a system \( S(Z, A) = 1 \) of equations over \( \Gamma \) using the tree of fundamental sequences for “covering” systems of equations over \( F \) constructed in Lemma 3.

We apply Lemma 3 and construct the systems \( S_1(X_1, A) = 1 \ldots S_n(X_n, A) = 1 \). Create a root vertex \( v_0 \) labelled by \( F(Z, A) \). For each of the systems \( S_i(X_i, A) \), let \( T_i \) be the tree constructed above. Build an edge from \( v_0 \) to
the root of $T_i$ labelled by the homomorphism $\rho_i$. For each leaf $v$ of $T_i$, labelled by $F(Y) \ast F$, build a new vertex $w$ labelled by $F(Y) \ast \Gamma$ and an edge $v \rightarrow w$ labelled by the homomorphism $\pi : F(Y) \ast F \rightarrow F(Y) \ast \Gamma$ which is induced from $\pi : F \rightarrow \Gamma$ by acting as the identity on $F(Y)$.

Define a branch $b$ of $\mathcal{T}$ to be a path $b = v_0v_1 \ldots v_k$ from the root $v_0$ to a leaf $v_k$. Let $v_1$ be labelled by $F_{R(S_i)}$ and $v_k$ by $F(Y) \ast \Gamma$. We associate to $b$ the set $\Phi_b$ consisting of all homomorphisms $F(Z) \rightarrow \Gamma$ of the form

$$\rho_i\pi(v_1, v_2)\sigma_2 \cdot \cdot \cdot \pi(v_{k-2}, v_{k-1})\sigma_{k-1}\pi(v_{k-1}, v_k)\pi_Y \phi$$ (9)

where $\sigma_j \in Q_{v_j}$ and $\phi \in \text{Hom}_\Gamma(F(Y) \ast \Gamma, \Gamma)$. Since $\text{Hom}_\Gamma(F(Y) \ast \Gamma, \Gamma)$ is in bijective correspondence with the set of functions $\Gamma^Y$, all elements of $\Phi_b$ can be effectively constructed. We have obtained the following theorem.

**Proposition 6. [22]** There is an algorithm that, given a system $S(Z, A) = 1$ of equations over $\Gamma$, produces a diagram encoding its set of solutions. Specifically,

$$\text{Hom}(\Gamma_{R(S)}, \Gamma) = \{ \phi \mid \phi \in \Phi_b, \ b \text{ is a branch of } \mathcal{T} \}$$

where $\mathcal{T}$ is the diagram described above. When the system is coefficient-free, then the diagram encodes $\text{Hom}(G, \Gamma)$ where $G = \langle Z \mid S \rangle$.

Note that in the diagram $\mathcal{T}$, the groups $G_v$ appearing at vertices are not quotients of coordinate group $\Gamma_{R(S)}$ and that to obtain a homomorphism from $\Gamma_{R(S)}$ to $\Gamma$ one must compose maps along a complete path ending at a leaf of $\mathcal{T}$. In [16] it is shown that for any toral relatively hyperbolic group there exist $\text{Hom}$-diagrams with the property that every group $G_v$ is a quotient of $\Gamma_{R(S)}$ and that every edge map $\pi(v, v')$ is a proper surjective homomorphism.

**Proposition 7. [22]** There is an algorithm to replace each fundamental sequence constructed in Proposition 6 with a fundamental sequence over $\Gamma$ with the following properties:

1. The image of each non-abelian vertex group is non-abelian,
2. all rigid subgroups, edge groups, and subgroups of abelian vertex groups generated by the edge groups are mapped to the next level monomorphically.
3. Let $H$ be a subgroup generated by a rigid subgroup $R$ together with subgroups of the abelian vertex groups which are connected to $R$, generated by all their edge groups. Each such $H$ is mapped to the next level monomorphically.

**Remark 1.** In Sela’s terminology [36] fundamental sequences are called resolutions and fundamental sequences with these properties are called strict resolutions. In [24] we used fundamental sequences with properties 1-2, but we should have used “strict” fundamental sequences. We will borrow the term “strict” for fundamental sequences with the above properties.

10
We will recall the proof of this proposition.

Proof. The terminal group of each fundamental sequence is $G_v = F(Y) \ast F$. We replace it with $F(Y) \ast \Gamma$. Then we change the next from the bottom level of the fundamental sequence, the group $G_{v_{k-1}}$ and the epimorphism $\pi(v_{k-1}, v_k)$. We consider all possible collapses of the abelian JSJ decomposition $D_{k-1}$ of $G_{v_{k-1}}$:

1. an edge group in $D_{k-1}$ is mapped by $\pi(v_{k-1}, v_k)\phi$ to the trivial group,
   in particular, some of the boundary elements of a MQH subgroup are mapped to a trivial group,

2. some MQH subgroup is mapped to an abelian group,

3. some non-abelian vertex group is mapped to an abelian group.

If there are no collapses, then we construct the two-levels canonical NTQ group with bottom level $F(Y) \ast \Gamma$ as in [20]. Suppose there are collapses. First we collapse all the edge groups between two non-QH vertex groups that are mapped to the trivial element. We collapse those abelian and MQH subgroups that are mapped to the trivial element, replace those non-abelian subgroups that are mapped to the abelian subgroup with the maximal free abelian quotient. We remove then all the edges that are connected to MQH vertex groups and are mapped into the identity. If we remove an edge corresponding to a stable letter, then we add the cyclic subgroup generated by a stable letter as a free factor. Suppose $Q$ is an MQH subgroup that is mapped to a non-abelian group. We replace by the identity all the boundary elements $p_i$ that are mapped to the identity. This is equivalent to filling with disks the corresponding boundary components of the surface $S_Q$. If the obtained subgroup is not a QH subgroup anymore we consider it as a rigid subgroup (in case it is given by the relations $x^2y^2 = 1, x^2c_1c_2 = 1$ or $c_1^2c_2^3c_3^2 = 1$) or as an HNN extension of a rigid subgroup (in case $x^2y^2c^2 = 1$). For each connected component of the obtained graph of groups that does not contain $\Gamma$ we add a new letter so that the fundamental group of that connected component is mapped into the conjugate of $F(Y) \ast \Gamma$ by this new letter. With the obtained decomposition we associate the NTQ group. At the bottom we have a free product of a free group, group $\Gamma$ and some conjugates of $\Gamma$. This group is toral relatively hyperbolic. We can continue this way and change consecutively all the higher levels of the fundamental sequence. On each level for each connected component of the obtained graph of groups that does not contain $\Gamma$ we add a new generator so that the fundamental group of this connected component is mapped into $\Gamma \ast F(Y)$ conjugated by the new generator. This completes the construction of fundamental sequences and corresponding NTQ groups over $\Gamma$.

We denote by $T(S, \Gamma)$ the constructed tree of strict fundamental sequences over $\Gamma$.

Similarly one can prove the following result.
Proposition 8. There is an algorithm, given a finitely presented residually \( \Gamma \) group \( G = \langle X \mid S \rangle \) and a finite number of finitely generated subgroups \( H_1, \ldots, H_k \) of \( G \) to construct a finite number of strict fundamental sequences over \( \Gamma \) ending with groups with no sufficient splitting modulo \( H_1, \ldots, H_k \) (see [20]) such that all the homomorphisms from \( G \) to \( \Gamma \) factor through one of them.

Remark 2. In Sela’s terminology rigid and solid limit groups are similar objects to groups with no sufficient splitting.

Proposition 9. If \( G \) is a toral relatively hyperbolic group, and \( S(X, A) = 1 \) a system of equations having a solution in \( G \), then there exists an algorithm to construct a finite number of strict fundamental sequences that encode all solutions of \( S(X, A) = 1 \) in \( G \).

Proof. The proof is similar to the proof of Proposition 7. We assign to each solution of \( S(X) = 1 \) canonical representatives in the free product \( \tilde{G} \) of a free group and parabolic subgroups using \( [8] \) and a disjunction of systems of equations over \( \tilde{G} \). By \( [4] \), the solution set of a system of equations in \( \tilde{G} \) is described by a finite number of fundamental sequences over \( \tilde{G} \). We will have a statement similar to Proposition 6. Then we replace each fundamental sequence with a fundamental sequence over \( G \) as we did in the proof of Proposition 7.

3.3 Splittings of \( \Gamma \) limit groups

In the previous section we constructed the tree of strict fundamental sequences encoding all solutions of a finite system \( S(Z, A) = 1 \) of equations over \( \Gamma \) using the tree of fundamental sequences for “covering” systems of equations over \( F \). In this section we show how to construct splittings of \( \Gamma \)-limit groups (not necessary finitely presented) using splittings of the “covering” \( F \)-limit groups.

Proposition 10. Let \( H \) be the image of the group \( \Gamma_{R(S)} \), where \( S = S(Z, A) \), in the NTQ group \( N \) corresponding to a strict fundamental sequence in \( T(S, \Gamma) \). Suppose \( H \) is freely indecomposable. Then there is an algorithm to find the generators (in \( N \)) of the rigid subgroups in the primary abelian JSJ decomposition of \( H \) and to find the JSJ decomposition (with rigid subgroups given by their generators in \( N \)).

Proof. The notion of a generic family of solutions of an NTQ system is defined in [20] (Definition 22). Let \( H \) be the image of the group \( \Gamma_{R(S)} \), where \( S = S(Z, A) \), in the NTQ group \( N \) corresponding to a strict fundamental sequence in \( T(S, \Gamma) \). A generic family of homomorphisms from \( H \) to \( \Gamma \) is a generic family for the NTQ system corresponding to a (canonical) embedding of \( H \) into an NTQ group over \( \Gamma \) as in (Section 7, [24]), or [15]. The notion of a positive unbounded family of homomorphisms is given in (Section 7.14, [20]). Such a family corresponds, in particular, to positive unbounded powers of Dehn twists (and generalized Dehn twists) for the JSJ decomposition of \( H \), and is generic. We first will prove the following lemma.
Lemma 4. Let $H$ be the image of the group $\Gamma_{R(S)}$, where $S = S(Z,A)$, in the NTQ group $N$ corresponding to a strict fundamental sequence in $T(S,\Gamma)$. Suppose $H$ is freely indecomposable. If a generic family of homomorphisms from $H$ to $\Gamma$, that corresponds to positive unbounded powers of Dehn twists (and generalized Dehn twists) for the JSJ decomposition of $H$, factors through $N$, then there is an algorithm to find the generators (in $N$) of the rigid subgroups in the primary abelian JSJ decomposition of $H$.

Proof. Suppose that $N$ is obtained from the fundamental sequence corresponding to branch $b \in T$ for the system $S_1(X_1,A) = 1$ over a free group (recall that $T$ is the tree of canonical fundamental sequences for $S_1(X_1,A) = 1$ over $F$). If $S_1(X_1,A) = 1$ is irreducible, the root vertex of $b$ corresponds to $F_{R(S_1)}$. If it is reducible, then the second from the root vertex $v$ corresponds to one of the fully residually free quotients of $F_{R(S_1)}$. Assume, wlog, that $F_{R(S_1)}$ is itself fully residually free. Then the root of the tree $T$ (top level) corresponds to a primary JSJ decomposition of $F_{R(S_1)}$.

We take the decomposition of $H$ induced from the decomposition of the top level of $N$. Since a generic family of solutions of $S(Z,A) = 1$ factors through $N$, then rigid subgroups in this decomposition are exactly rigid subgroups of $H$.

Now we have to find them. The group $N$ is obtained from $F_{R(S_1)}$. Let $K$ be the subgroup generated by $\rho_1(Z)$ in $F_{R(S_1)}$. We first take intersections of $K$ with rigid subgroups in the JSJ-decomposition of $F_{R(S_1)}$. Denote by $R_1,\ldots,R_t$ the obtained subgroups. We claim that the images of $R_1,\ldots,R_t$ in $N$ with some exceptions will be rigid subgroups in a primary JSJ decomposition of $H$. Denote these images by $\bar{R}_1,\ldots,\bar{R}_t$. There are the following exceptions:

1) Some $\bar{R}_i$ may be mapped onto an abelian group. If $\bar{R}_i$ is connected by an edge to some non-abelian $\bar{R}_j$, then the image in $N$ of $< \bar{R}_i \cup \bar{R}_j >$ generates a rigid subgroup. If all the edges between $\bar{R}_i$ and $\bar{R}_j$ are mapped into the identity, then $\bar{R}_i$ becomes an abelian vertex group.

2) The image of some QH subgroup may become a rigid subgroup, we will see this when changing the fundamental sequences in the proof of Proposition [7]

Let us prove this. If we take a generic family of solutions for $N$ that is obtained from solutions of $S_1(X_1,A) = 1$ in $F$, this family restricted on $H$ is a generic family for $H$, therefore it contains a generic sub-family of solutions $\{\phi_i\}$ such that the images $\phi_i(h)$ of the generators $h$ of abelian subgroups, QH-subgroups and stable letters of the JSJ decomposition of $H$ grow much faster with $i \to \infty$ than the images of the elements in a fixed rigid subgroup. The construction of solutions in the generic family involves products of big powers. They label $(\lambda,\mu)$-quasigeodesics in $\Gamma$, and by Theorem 3.17, [1], we can assume that $\lambda,\mu$ are the same for all the family. Hence the length of the canonical representatives of the images $\phi_i(h)$ of the generators $h$ of abelian subgroups, QH-subgroups and stable letters of the JSJ decomposition of $H$ grow much faster with $i \to \infty$ than the length of the canonical representatives of the images of the elements in a fixed rigid subgroup of $H$. Basically, every abelian splitting of $H$ lifts to a splitting of its pre-image $K$ induced by a splitting of $F_{R(S_1)}$. $\square$
Definition 3. We call a subgroup of \( \Gamma \) almost immutable if it is elliptic in all the hierarchical decompositions of \( \Gamma \), namely, in the JSJ decomposition of \( \Gamma \), JSJ decomposition of the corresponding vertex group, and in the JSJ decompositions of all the further vertex groups.

Lemma 5. Let \( H \) be the image of the group \( \Gamma_R(S) \), where \( S = S(Z, A) \), in the NTQ group \( N \) corresponding to a strict fundamental sequence in \( T(S, \Gamma) \). There is an algorithm to construct a presentation of \( H \) as a series of amalgamated products and \( HNN \)-extensions with abelian (or trivial) edge groups beginning with cyclic groups, \( \Gamma \), and a finite number of almost immutable subgroups of \( \Gamma \) given by finite generating sets.

Moreover, if \( g_1, \ldots, g_k \) are generators of this presentation, and \( h_1, \ldots, h_s \) are images of the generators of \( \Gamma_R(S) \) in \( N \) (they are also generators of \( H \)), then there is an algorithm to express \( g_1, \ldots, g_k \) in terms of \( h_1, \ldots, h_s \) and vice versa.

Indeed, we construct some decomposition of \( H \) even if there is no generic family of homomorphisms from \( H \) to \( \Gamma \) factoring through \( N \), and then continue inductively.

We can now finish the proof of the Proposition 10. There is some branch of the tree \( T(S, \Gamma) \) such that a generic family of homomorphisms for \( H \) factors through this branch. Using Lemma 5 we can effectively present each image of \( \Gamma \) in variables \( v_1(\ldots, v_k, w_1, \ldots, w_m) = 1 \) are relations for the presentation of \( H \) as in Lemma 6. Denote the system obtained by substituting expressions for \( g_1, \ldots, g_k, w_1, \ldots, w_m \) in variables \( h_1, \ldots, h_s \) by \( Q(h_1, \ldots, h_s, A) = 1 \). Suppose \( \tilde{g}_i = \tilde{v}_i(h_1, \ldots, h_s), i = 1, \ldots, k \) and \( \tilde{w}_i = \tilde{v}_i(h_1, \ldots, h_s), i = k+1, \ldots, m \), where \( \tilde{w}_1, \ldots, \tilde{w}_m \in \Gamma \), and \( R(g_1, \ldots, g_k, w_1, \ldots, w_m) = 1 \) are relations for the presentation of the other image of \( F_R(S) \), say \( \tilde{H} \) in some other NTQ group. Denote the system obtained by substituting expressions for \( \tilde{g}_1, \ldots, \tilde{g}_k, \tilde{w}_1, \ldots, \tilde{w}_m \) in variables \( h_1, \ldots, h_s \) by \( \tilde{Q}(h_1, \ldots, h_s, A) = 1 \). The universal formula

\[
\forall h_1, \ldots, h_s(Q(h_1, \ldots, h_s, A) = 1 \rightarrow \tilde{Q}(h_1, \ldots, h_s, A) = 1)
\]

is true in \( \Gamma \) if and only if \( \tilde{H} \) is a quotient of \( H \). And \( \tilde{H} \) is isomorphic to \( H \) if and only if they are quotients of each other.

Therefore, we can decide if the image of \( \Gamma_R(S) \) in one NTQ group is isomorphic to the image in the other NTQ group and find the isomorphism. If two such images are isomorphic we can compare the obtained decompositions. Since we know that one of the decompositions is a JSJ decomposition we can decide which one. The proposition is proved.

We will use the following result without further references.

\[ \Box \]
Proposition 11. Let $\Gamma$ be a non-elementary torsion-free hyperbolic group, and let $L$ be a non-abelian, freely indecomposable, strict $\Gamma$-limit group. Then $L$ admits an essential cyclic splitting. If, furthermore, $L$ is a restricted $\Gamma$-limit group then the splitting may be chosen so that the coefficient subgroup is elliptic.

3.4 Quasi-convex closure

Let $\Gamma$ be a torsion free non-elementary hyperbolic group, $H$ a subgroup of $\Gamma$ given by generators. We will describe a certain procedure that either finds a splitting of $H$ or constructs a group $K$, $H \leq K \leq \Gamma$ such that $K$ is quasi-convex in $\Gamma$ and rigid. We will call this $K$ a *quasi-convex closure of $H$*. Take a JSJ decomposition $D_1$ of $\Gamma$. If $H$ is not elliptic, there is induced splitting of $H$, otherwise, let $K_1$ be a rigid subgroup such that $H$ is conjugate into $K_1$. Such a subgroup $K_1$ is quasi-convex and hyperbolic as a vertex group in $D_1$ by Proposition 3, and the quasi-convexity constants can be found effectively. Then take the JSJ decomposition $D_2$ of $K_1$ and find again a vertex subgroup containing a conjugate of $H$ or find a splitting of $H$. We continue this way until a subgroup $K_n$ is rigid. Denote $K = K_n$. Hierarchical accessibility for hyperbolic groups is proved in [29].

Let now $H$ be the image of the group $\Gamma_{R(S)}$, where $S = S(Z,A)$, in the NTQ group $N$ corresponding to a strict fundamental sequence in $T(S,\Gamma)$. Using Lemma 4 we can construct a presentation of $H$ as a series of amalgamated products and HNN-extensions with abelian (or trivial) edge groups beginning with cyclic groups, $\Gamma$ and a finite number of almost immutable subgroups of $\Gamma$ given by finite generating sets. We replace each almost immutable subgroup $I$ by its quasi-convex closure (denote it $I_q$). Consider the group generated by $H$ and all subgroups $I_q$. Denote it by $H_q$ which we call a *quasi-convex closure of $H$*. Notice that $H_q$ is relatively quasi-convex in $N$.

Notice that in [3] some algorithmic results were obtained for $\Gamma$-limit groups for locally quasi convex torsion free hyperbolic group $\Gamma$.

4 Decision algorithm for $\forall \exists$-sentences

In this section we will prove Theorem 2. Notice that the analog of the implicit function theorem and parametrization theorem ([29], Theorem 12) for a torsion free hyperbolic group was stated by Sela in [37], Section 2 (the construction of a completion and a closure of a well structured resolution is equivalent to the statement of the Parametrization theorem). In the rest of the paper we will only consider fundamental sequences satisfying the first and second restrictions from [20], Sections 7.8, 7.9. To make this paper self-contained we recall these sections here.
4.1 First restriction on fundamental sequences

Let $S(Z, A) = 1$ be a system of equations over $\Gamma$. We can assume that it is irreducible. We construct fundamental sequences for $S = 1$ as in Proposition 7.

Let $$\Gamma_{R(\mathcal{U})} * F(t_1, \ldots, t_k) = P_1 * \ldots * P_q * <t_1> * \ldots * <t_k> \quad (10)$$

be a reduced free decomposition of a maximal shortening quotient $\Gamma_{R(\mathcal{U})} * F(t_1, \ldots, t_k)$ modulo $F_{R(S)}$ (this shortening quotient is exactly the group corresponding to the second from the top level of the corresponding fundamental sequence), and $\pi : \Gamma_{R(S)} \to \Gamma_{R(\mathcal{U})} * F(t_1, \ldots, t_{\beta_i})$ the epimorphism.

Let $P$ be the subgroup generated by variables $X$ and standard coefficients $C$ of a regular quadratic equation $Q_i = 1$ corresponding to some fixed MQH subgroup in the JSJ decomposition of $F_{R(S)}$. Consider a free decomposition $\pi(P) = K_1 * \ldots * K_p * <t_{k_{ij}} > * \ldots * <t_{k_{ij}} >$ inherited from the free decomposition $(10)$ such that each standard coefficient is conjugated into some $K_j$, and each $K_j$ has a conjugate of some coefficient. Then there is a canonical automorphism that transforms $X$ into variables $X_1$ with the following properties:

1) the family $X_1$ can be represented as a disjoint union of sets of variables $X_{11}, \ldots, X_{1r}$;

2) every solution of $S = 1$ can be transformed by a canonical automorphism corresponding to $Q_i = 1$ into a solution of the system obtained from $S = 1$ by replacing $Q_i = 1$ by a system of several quadratic equations $Q_{ij1}(X_{11}, C) = 1, \ldots, Q_{ijr}(X_{1r}, C) = 1$ with standard coefficients from $C$;

3) each quadratic equation $Q_{ij} = 1$ either is coefficient-free or has coefficients from $C$ which are conjugated into some $K_r$;

4) $X_{ij}^r$ is a solution of the system $Q_{ij1}(X_{11}, C^{\pi_1}) = 1, \ldots, Q_{ijr}(X_{1r}, C^{\pi_1}) = 1$;

5) if $Q_{ij} = 1$ is coefficient-free, then $X_{ij}^r$ is a solution of maximal possible dimension or the corresponding surface group is a subgroup of $\Gamma$;

6) if $Q_{ij} = 1$ is not coefficient-free, then $Q_{ij} = 1$ cannot be transformed by a canonical automorphism corresponding to $Q_{ij} = 1$ into an equation

$$Q_{ij1}(X_{ij1})Q_{ij2}(X_{ij2}) = 1$$

such that $Q_{ij1}(X_{ij1}) = 1$ is coefficient-free and $Q_{ij2}(X_{ij2}) = 1$ has non-trivial coefficients from $C$ which are conjugated into some $K_r$ and such that $X_{ij}^r$ is a solution of the system $Q_{ij1}(X_{ij1}) = 1$, $Q_{ij2}(X_{ij1}, C^{\pi}) = 1$.

Suppose $Q_{ip} = 1$ is some equation in variables $X_{ip}$ in this family which has coefficients from $C$. Each homomorphism in a fundamental sequence of homomorphisms from $\Gamma_{R(U)}$ to $\Gamma$ is a composition $\sigma_1 \pi \phi$, where $\sigma_1$ is a canonical automorphism of $\Gamma_{R(S)}$, and $\phi$ is a homomorphism from $\Gamma_{R(\mathcal{U})} * F(t_1, \ldots, t_k)$ to $\Gamma$. We will include into the fundamental sequence only the compositions $\sigma_1 \pi \phi$ for which $\{ \phi \}$ is non-special and satisfies the following property: for each $j$, $Q_{ij} = 1$ is not split into a system of two quadratic equations $Q_{ij1}(X_{ij1}) = 1$ and $Q_{ij2}(X_{ij2}) = 1$ with disjoint sets of variables such that $Q_{ij1} = 1$ is coefficient-free and $Q_{ij2} = 1$ has coefficients from $C$ which are conjugated into some $K_r$ and such that $X_{ij1}^{\pi} \phi$ is a solution of $Q_{ij1} = 1$ and $Q_{ij2}(X_{ij2})^{\pi} \phi = 1$. 

16
4.2 Second restriction on fundamental sequences

Suppose the family of homomorphisms $\sigma_1 \pi_1 \ldots \sigma_n \pi_n \tau$ is a strict fundamental sequence, corresponding to the NTQ system $Q(X_1, \ldots, X_n) = 1$:

$$Q_1(X_1, \ldots, X_n) = 1, \ldots, Q_n(X_n) = 1$$

adjoint with free variables $t_1, \ldots, t_k$. Here the restriction of $\sigma_i$ on $\Gamma_{R(Q_i, \ldots, Q_n)}$ is the canonical automorphism on $\Gamma_{R(Q_i, \ldots, Q_n)}$, identical on variables from $X_{i+1}, \ldots, X_n$ and on all free variables from the higher levels, $\pi_i : \Gamma_{R(Q_i, \ldots, Q_n)} \ast F(t_1, \ldots, t_{k_i-1}) \to \Gamma_{R(Q_{i+1}, \ldots, Q_n)} \ast F(t_1, \ldots, t_{k_i})$.

We only consider strict fundamental sequences. Recall, that in particular, this means that we include in the fundamental sequence only such homomorphisms that give nonabelian images of the regular subsystems of $Q_i = 1$ on all levels (the rest can be included into a finite number of fundamental sequences), and the images of the edge groups of the JSJ decompositions on all levels are nontrivial. We also only consider homomorphisms that do not factor through a fundamental sequence $\text{Var}_{\text{fund}}(Q(b_1))$ such that the tuple of dimensions $(k_1, \ldots, k_n)$ (where the numbers $k_1, \ldots, k_n$ are dimensions of the free factors from the previous paragraph) for $b_1$ is greater than the corresponding tuple for $b$ in the lexicographic order.

We can suppose that all fundamental sequences that we consider satisfy the following properties. Let $\Gamma_{R(Q_i, \ldots, Q_n)}$ be a free product of some factors. Then

1) the images of abelian factors under $\pi_i$ are different factors of $F(t_{k_{i-1}+1}, \ldots, t_{k_i})$;

2) the images under $\pi_i$ of factors which are surface groups are different factors of $F(t_{k_{i-1}+1}, \ldots, t_{k_i})$ or of $\Gamma$;

3) if some quadratic equation in $Q_i = 1$ has free variables in this fundamental sequence, then these variables correspond to some variables among $t_{k_{i-1}+1}, \ldots, t_{k_i}$, the images under $\pi_i$ of coefficients of quadratic equations cannot be conjugated into $F(t_{k_{i-1}+1}, \ldots, t_{k_i})$;

4) different factors in the free decomposition of $\Gamma_{R(Q_i, \ldots, Q_n)}$ are sent into different factors in the free decomposition of $\Gamma_{R(Q_{i+1}, \ldots, Q_n)} \ast F(t_{k_{i-1}+1}, \ldots, t_{k_i})$.

**Proposition 12.** For a system of equations $S$ over $\Gamma$ one can effectively construct a finite set of strict fundamental sequences that satisfy the first and second restriction above, such that every solution of $S$ factors through one of these fundamental sequences.

**Proof.** Using Proposition 7 we can construct a finite number of strict fundamental sequences over $\Gamma$ such that each solution of $S$ factors through one of them. In addition, if the corresponding fundamental sequence over a free group satisfies first and second restrictions, then the fundamental sequence obtained
in Proposition 7 can be also constructed satisfying these restrictions. To do this algorithmically one only needs to solve the word problem in $\Gamma$-limit groups which are given as subgroups of NTQ groups. And the word problem in NTQ groups is solvable because they are toral relatively hyperbolic. 

**Definition 4.** We call fundamental sequences satisfying the first and second restrictions well aligned fundamental sequences.

### 4.3 Induced NTQ systems and fundamental sequences

In this subsection we modify the construction of [20], Section 7.12 for a torsion free hyperbolic group $\Gamma$. We will construct, given an NTQ system $S = 1$ over $\Gamma$, corresponding NTQ group $\Gamma_{R(S)}$, and the well aligned fundamental sequence of solutions, the induced NTQ group, NTQ system, and the well aligned fundamental sequence of solutions for a subgroup $K$ of $\Gamma_{R(S)}$.

Let $S = 1$ be an NTQ system over $\Gamma$:

$$S_1(X_1, X_2, \ldots, X_n, A) = 1,$$
$$S_2(X_2, \ldots, X_n, A) = 1,$$
$$\ldots$$
$$S_n(X_n, A) = 1$$

and $\pi_i: \Gamma_i \to \Gamma_{i+1}$ a fixed $\Gamma_{i+1}$-homomorphism (a solution of $S_i(X_1, \ldots, X_n) = 1$ in $\Gamma_{i+1} = \Gamma_{R(S_{i+1}, \ldots, S_n)}$, $\Gamma_{n+1} = G$, which is a free product of $\Gamma$, free group and some subgroups of $\Gamma$). Let $K$ be a finitely generated subgroup (or $\Gamma$-subgroup) of $\Gamma_{R(S)}$. Then there exists a system $W(Y) = 1$ such that $K = \Gamma_{R(W)}$. We will describe here how to embed $K$ more economically into a NTQ group $\Gamma_{R(Q)}$ such that $\Gamma_{R(Q)} \leq \text{quasi isom.} \Gamma_{R(S)}$ and assign to $Q = 1$ a fundamental sequence that includes all the solutions of $W = 1$ relative to $S = 1$.

Canonical automorphisms on different levels for $Q = 1$ will be induced by canonical automorphisms for $S = 1$, mappings between different levels for $Q = 1$ will be induced by mappings for $S = 1$.

Without loss of generality we can suppose that $\Gamma_{R(S)}$ is freely indecomposable modulo $\Gamma$. The top quadratic system of equations $S_1(X_1, \ldots, X_n) = 1$ corresponds to a splitting $D$ of $\Gamma_{R(S)}$. The non-QH non-abelian subgroups of $D$ are factors in a free decomposition of $<X_2, \ldots, X_n>$. Consider the induced splitting of the quasi convex closure $K_q$ denoted by $D_K$. This splitting may give a free factorization $K_q = K_1 \ast \ldots \ast K_k$, where $\Gamma \leq K_1$. Consider each factor separately. Consider $K_1$. Each edge $e$ in the decomposition of $K_1$ that connects two rigid vertex groups is composed from a two edges $e_1$ and $e_2$ that are adjacent and both in the orbit of the same edge $\bar{e}$ in the Bass-Serre tree corresponding to the graph of groups $D$. Moreover, $\bar{e}$ connects a rigid vertex group to an abelian vertex group in $D$, because if it connects a rigid vertex group to a QH-subgroup, then a QH subgroup would appear between two rigid vertex groups instead of edge $e$. Increasing $K_1$ by a finite number of suitable elements
from abelian vertex groups of $\Gamma_{R(S)}$ we join together non-QH non-abelian subgroups of $D_K$ which are conjugated into the same non-QH non-abelian subgroup of $D$ by elements from abelian vertex groups in $\Gamma_{R(S)}$. There is an algorithm to do this by Proposition 1. Moreover, we can take these elements in abelian subgroups such a way that their images on the next level are trivial. In this way we obtain a group $\bar{K}_1$ such that $D_{\bar{K}_1}$ does not have edges between non-QH non-abelian subgroups, and generators of edge groups connecting non-QH non-abelian subgroups to abelian subgroups not having roots in $<X_2, \ldots, X_N>$. We then add conjugating elements so that all rigid subgroups that are mapped to the same free factor on the next level are conjugate into one subgroup. The images of these elements in $\Gamma_{R(S)}$ are mapped to the identity by $\pi_1$. Denote the group that we obtained by $\hat{K}_1$. Therefore, the relations for $\hat{K}_1$ give the radical of a quadratic system solvable in $<X_2, \ldots, X_n>$. Moreover, $\pi_1(\Gamma_1) = \pi_1(\hat{K}_1)$ because we added elements that are sent by $\pi_1$ into the identity. Since we will be considering only well aligned fundamental sequences, we fix the family of elements in the QH subgroups of $\hat{K}_1$ that are mapped to the identity by $\pi_1$ so that corresponding quadratic equations split into systems satisfying the first restriction.

Now consider separately each factor of $\pi_1(\Gamma_1) \cap <X_2, \ldots, X_n>$ and enlarge it the same way. Working similarly with each $\Gamma_i$ we consider all the levels of $S = 1$ from the top to the bottom. At the bottom, $G$ may be a free product. We replace the subgroups of $\Gamma$ in the bottom level of the image of $K_q$ that are conjugate into subgroups of $\Gamma$ in the free product $G$ by corresponding subgroups of $\Gamma$ in $G$ and obtain an intermediate group that we denote by $H_1$. Then we repeat the whole construction for $H_1$ in place of $K_q$, obtain $H_2$ and repeat the construction again. We will eventually stop, namely obtain that $H_i = H_{i+1}$, because every time when we repeat the construction if $H_i \neq H_{i+1}$ then there is some level $j$ such that on all the levels higher than $j$ the decompositions are the same as in the previous step and on level $j$ one of the following characteristics decreases:

1. the number of free factors in the free decomposition on level $j$ of $H_i$,
2. if the number of free factors does not decrease, then the number of edges and vertices of the induced decomposition on level $j$ of $H_i$ decreases,
3. if the number of free factors in the free decomposition on level $j$ of $H_i$, and the number of edges and vertices does not decrease, then the size of the decomposition of level $j$ of $H_i$ is decreased.

We end up with an NTQ system $Q = 1$ such that $K \leq \Gamma_{R(Q)} \leq_{\text{quasi isom.}} \Gamma_{R(S)}$, and the image of the top $j$ levels of $\Gamma_{R(Q)}$ on the level $j + 1$ is contained in the image of $K$ on this level. Each QH subgroup of the induced system is a finite index subgroup in some QH subgroup of $S = 1$. This NTQ system $Q = 1$ is called induced NTQ system and the corresponding well aligned fundamental sequence is called induced fundamental sequence.
Similarly if we have a fundamental sequence (and NTQ system) modulo a subgroup we can define induced fundamental sequence (and NTQ system) modulo a subgroup.

There is an algorithm for the construction because we begin with $K_q$ that is relatively quasi convex in the relatively hyperbolic group $\Gamma_{R(S)}$, and, therefore, can use Proposition 11 to find different intersections of $K_q$ with conjugates of vertex groups in decompositions of $\Gamma_{R(S)}$.

4.4 First step

We will now describe the algorithm for construction of the $\forall\exists$-tree. Consider the sentence

$$\Phi = \forall X \exists Y (U(X,Y) = 1 \wedge V(X,Y) \neq 1) \quad (11)$$

If the sentence is true then $\forall X \exists Y U(X,Y) = 1$. By Proposition 11 we can effectively find all formula solutions $Y = f(X)$ of $U(X,Y) = 1$ in $F(Y) \ast \Gamma$ that are collected in a finite number of strict fundamental sequences. To check whether the sentence (11) is true we now have to check it only for those values of $X$ for which for any formula solution $y = f(X)$ we have $V(X,f(X)) = 1$. Since this should happen for all formula solutions in each fundamental sequence of formula solutions, we apply the implicit function theorem again and get a system of equations on the terminal groups of these fundamental sequences. Therefore we can effectively construct a system of equations $U_0(X) = 1$ on the values of $X$ for which the sentence still has to be verified. Let $G = \Gamma_{R(U_0)}$, where $U_0 = 1$ is such a system. We define now a tree $T_{E_A}(\Phi) = T_{E_A}(G)$ oriented from the root, and assign to each vertex of $T_{E_A}(G)$ some set of homomorphisms in $G \to \Gamma$. We assign all set of homomorphisms $G \to \Gamma$ to the initial vertex $v_\rightarrow$. We can construct a finite number of NTQ systems corresponding to branches $b$ of the process described in Propositions 11, 12, $S(b)$, their correcting extensions: $S_{corr}(b) = 1$ ($S_{corr}(b) : S_1(X_1, \ldots, X_n) = 1, \ldots, S_n(X_n) = 1$), and corresponding well aligned fundamental sequences $Var_{fund}(S(b))$. For each such fundamental sequence we assign a vertex $v_{\text{fin}}$ to the tree $T_{E_A}(\Phi)$. We draw an edge from vertex $v_{\rightarrow}$ to each vertex corresponding to $Var_{fund}(S(b))$.

Let $Var_{fund}(S(b))$ be assigned to $v_0$. Since every branch of the tree will be constructed independently of the others we will now describe the construction for $Var_{fund}(S(b))$. By the analog of the Parametrization theorem (20), Theorem 12) for $\Gamma$ (that is also proved by Sela [37]) and by proposition 9 we can collect all values of $Y$ given by formulas $Y = f(X_1, \ldots, X_n)$ in variables $X_1, \ldots, X_n$ of $F_{R(S(b))}$. Sentence (11) is now verified for all values of $X$ except those for which for all $Y = f(X_1, \ldots, X_n)$ we have $V(X_1, \ldots, X_n, f(X_1, \ldots, X_n)) = 1$. As on the previous step this gives the equation $U_1(X_1, \ldots, X_n) = 1$ which is equivalent to a system of all equations $V(X_1, \ldots, X_n, f(X_1, \ldots, X_n)) = 1$. We will only consider values of $X_1, \ldots, X_n$ satisfying this system that are minimal in $Var_{fund}(S(b))$ with respect to canonical automorphisms on all the levels modulo the image of $G$. This is enough because if for a specialization $X^\phi$ there exists a minimal specialization of $X_1^\phi, \ldots, X_n^\phi$ that does not satisfy this system,
then the sentence is true for $X^\theta$.

Let $V_{\text{fund}}(U_1)$ be the subset of homomorphisms from the set $V_{\text{fund}}(S(b))$ going through the corrective extension $S_{\text{corr}}(b) = 1$, minimal with respect to the canonical automorphisms modulo the image of $G$ on all levels, and satisfying the additional equation $U_1(X_1, \ldots, X_n) = 1$. We introduce a vertex $\hat{v}_{1i}$ and draw an edge connecting $\hat{v}_{ni}$ to $\hat{v}_{1i}$.

Let $G$ be a finitely generated group. Recall that any family of homomorphisms $\Psi = \{\psi_i : G \to \Gamma\}$ factors through a finite set of maximal fully residually $\Gamma$ groups $H_1, \ldots, H_k (= \Gamma$-limit groups) that all are quotients of $G$. We first take a quotient $G_1$ of $G$ by the intersection of the kernels of all homomorphisms from $\Psi$, and then construct maximal fully residually $\Gamma$ quotients $H_1, \ldots, H_k$ of $G_1$. We say that $\Psi$ discriminates groups $H_1, \ldots, H_k$, and that each $H_i$ is a fully residually $\Gamma$ group discriminated by $\Psi$.

Let $G_1$ be a fully residually $\Gamma$ group discriminated by the set of homomorphisms $V_{\text{fund}}(U_1)$ (we do not need its relations). Consider the family of fundamental sequences for $G_1$ modulo the images $R_1, \ldots, R_s$ of the rigid subgroups in the abelian decompositions of the factors of the free decomposition of the subgroup $H_1 = \langle X_2, \ldots, X_n \rangle$. By Proposition 10 we know the generators of $R_1, \ldots, R_s$ and by Proposition 4 can effectively construct these fundamental sequences. If the dimension of such a sequence is greater than $k_1$, (that can be found effectively) then the corresponding homomorphisms are contained in the fundamental sequence for $U = 1$ with the number $k_1$ greater than that for $V_{\text{fund}}(S(b))$, and we do not consider this sequence here. So we only consider well aligned canonical fundamental sequences $c$ for $G_1$ modulo $R_1, \ldots, R_s$ (corresponding to coefficients of quadratic equations $S_1 = 1$ of the top level for $S_{\text{corr}}(b) = 1$) with, in particular, the following properties: (1) they have dimension less or equal than $k_1$, (2) Rigid subgroups of $G$ are elliptic in the splitting of the top level of the block-NTQ group discriminated by the fundamental sequence, (3) $c$ is consistent with the decompositions of surfaces corresponding to quadratic equations of $S_1$, by a collection of simple closed curves mapped to the identity. Namely, if we refine the JSJ decomposition of $G$ by adding splittings corresponding to the simple closed curves that are mapped to the identity when $G$ is mapped to free product in Subsection 4.1, then the standard coefficients on all the levels of $c$ are images of elliptic elements in this decomposition.

Suppose a fundamental sequence $c$ has the top dimension component $k_1$. If the system corresponding to the top level of the sequence $c$ is the same as $S_1 = 1$, we extend the fundamental sequences modulo $R_1, \ldots, R_s$ by canonical fundamental sequences for $H_1$ modulo rigid subgroups in the factors in the free decomposition of the subgroup $\langle X_2, \ldots, X_n \rangle$. If such a sequence has dimension greater than or equal to $k_2$, then the corresponding solution can be factored through a fundamental sequence for $U = 1$ of the greater dimension.

So we only consider such sequences of dimension less than or equal to $k_2$. If the sum of first two dimensions is strictly smaller than $k_1 + k_2$, we do the same as in case when the first dimension is smaller than $k_1$ (see below). We continue this way to construct fundamental sequences $Var_{\text{fund}}(S_1(b))$. We draw edges from the vertex corresponding to $Var_{\text{fund}}(U_1)$ to the vertices $Var_{\text{fund}}(S_1(b))$. 21
Suppose now that the fundamental sequence $c$ for $G_1$ modulo $R_1, \ldots, R_s$ has dimension strictly less than $k_1$ or has dimension $k_1$, but the system corresponding to the top level of $c$ is not the same as $S_1 = 1$. Suppose also that $G \neq \Gamma$. Then we use the following lemma (in which we suppose that $R_1, \ldots, R_s$ are non-trivial).

**Lemma 6.** The image $G_t$ of $G$ in the group $H_t$ appearing on the terminal level $t$ of sequence $c$ is a proper quotient of $G$ unless each of the homomorphisms that factors through $H_t$ is an embedding of each of freely indecomposable factors of $H_t$ into $\Gamma$.

**Proof.** Consider the terminal group of $c$; denote it $H_t$. Suppose $G_t$ is isomorphic to $G$. Denote the abelian JSJ decomposition of $H_t$ by $D_t$. Then there is an abelian decomposition of $G$ induced by $D_t$. Therefore rigid (non-abelian and non-QH) subgroups and edge groups $R_1, \ldots, R_s$ of $G$ are elliptic in this decomposition.

There exists a decomposition of some free factor $P_i$ which is induced from $D_t$. But this is impossible because this means that the homomorphisms we are considering can be shortened by applying canonical automorphisms of $P_i$ modulo those subgroups from $\{R_1, \ldots, R_s\}$ which are conjugated into $P_i$.

We can effectively write a presentation for $G_t$ as a chain of amalgamated products and HNN extensions over abelian subgroups beginning with almost immutable subgroups of $\Gamma$, cyclic groups, and $\Gamma$ itself (using Lemma 5). We do not continue if we obtained proper quotients of subgroups of $\Gamma$ (corresponding homomorphisms factor through other fundamental sequences assigned to other vertices $\hat{v}_{0,i}$ of the $T_{EA}$). Construct a complete set $F$ of fundamental sequences that encode all the homomorphisms $G_t \to \Gamma$. One can extract from $c$ modulo the terminal level the induced well aligned fundamental sequence for $G$. Denote this induced fundamental sequence by $c_2$. Consider the block-NTQ group $\bar{G}$ generated by the top $p$ levels of the NTQ group corresponding to the fundamental sequence $c$, and the correcting extension of the NTQ group $\Gamma_{R(S_2(b_2))}$ corresponding to some fundamental sequence from $\mathcal{F}$ and amalgamated along the top level of $\Gamma_{R(S_2(b_2))}$ (containing $G_t$). Consider a fundamental sequence $c_3$ that consists of homomorphisms obtained by the composition of a homomorphism from $c_2$ and from a fundamental sequence corresponding to $b_2$. One can apply the parametrization theorem to the NTQ group corresponding to $c_3$ and get formula solutions of $V(X,Y) = 1$ in the correct extension of this group (as in [20], Section 7.5), and, therefore, in the corresponding extension of $\bar{G}$. Assign a vertex $\hat{v}_{2ik}$ of the tree $T_{EA}(\Phi)$ to sequence $c_3$. We draw an edge from the vertex of $T_{EA}(G)$ corresponding to $V_{\text{fund}}(U_1)$ to $\hat{v}_{2ik}$. By the parametrization theorem we can collect all formula solutions of $U(X,Y) = 1$ over this extension of $\bar{G}$. Those formula solutions for which $V(X,Y) = 1$ (if exist) will give an additional equation $U_2 = 1$ for generators of this extension of $\bar{G}$. 

22
4.5 Second step

We will describe the next step in the construction of $T_{EA}(\Phi)$ which basically is general. Let $c_3 = V_{\text{fund}}(S^{(1)}(d))$ be a fundamental sequence corresponding to some vertex $\hat{v}_3$ of $T_{EA}(G)$, let $c$ be, as before, the corresponding canonical fundamental sequence for $G_1$ modulo $R_1, \ldots, R_s$. Consider the set of those minimal homomorphisms from $\bar{G}$ to $\Gamma$ which are going through the fundamental sequence $c$, which factor through a corrective extension $\bar{V}_{\text{corr}}(S^{(1)}(d))$, and satisfy the additional equation $U_2 = 1$. Let $G_2$ be one of the fully residually $\Gamma$ groups discriminated by this set (we do not need to know its presentation at this step, but we will find it later after constructing a complete set of fundamental sequences encoding all homomorphisms going through $c$ which factor through a corrective extension $\bar{V}_{\text{corr}}(S^{(1)}(d))$, and satisfy the equation $U_2 = 1$).

The case when the natural image $G^{(2)}$ of $G$ in $G_2$ is a proper quotient of $G$ is the first "easy" case. There is an algorithm to see this. The group $G$ is represented as a subgroup of some NTQ group $N$ as a series of amalgamated products over abelian subgroups and HNN’s from $\Gamma$, almost immutable subgroups of $\Gamma$ (denote then $I_1, \ldots, I_t$) and cyclic groups. We can assume that $G$ is freely indecomposable. Adding a system of equations on $N$ we can construct another family of NTQ groups, and represent $G^{(2)}$ the same way by Lemma 5. Therefore we can assign a finite system of equations with coefficients to $G$ and another system to $G^{(2)}$ and decide if $G^{(2)}$ is isomorphic to $G$ using decidability of the universal theory of $\Gamma$.

If $G^{(2)}$ is a proper quotient of $G$ (we can check this using Lemma 6), we begin the construction with $G$ replaced by $G^{(2)}$.

In this case we assign to vertices $\hat{v}_4$ the fundamental sequences corresponding to branches of $T_{CE}(G^{(2)})$ and draw edges from $\hat{v}_2$ to all these vertices. In all the other considerations below we suppose that $G^{(2)}$ is isomorphic to $G$.

Suppose the JSJ decomposition for the top level of $c$ corresponds to the equation $S_{11}(X_{11}, X_{12}, \ldots) = 1$; some of the variables $X_{11}$ are quadratic, the others correspond to extensions of centralizers. Construct a fundamental sequence $c^{(2)}$ as in Proposition 7 for $G_2$ modulo the rigid subgroups of the top level of $c$ in the factors in the free decomposition of the subgroup generated by $X_{12}, \ldots$.

Denote by $N_0^1$ the image of the subgroup generated by $X_1, \ldots, X_n$ in the group discriminated by $c$. Denote $N_0^2 = X_1, \ldots, X_n > c$. Denote by $N_0^2 = X_1, \ldots, X_n > c^{(2)}$ the image of $< X_1, \ldots, X_n > c^{(2)}$. If $N_0^2$ is a proper quotient of $N_0^1$ (we have an algorithm to check this) and $c$ is not the same as the top level of $S(b)$, we have another "easy" case. In this case we do what we did at the previous step taking $N_0^2$ instead of $G_2$ and we do not consider vertices corresponding to NTQ systems with the same top level as $S^{(1)}(d)$.

In all the cases below we suppose that $N_0^1$ is isomorphic to $N_0^2$.

**Case 1.** The top levels of $c$ and $c^{(2)}$ are the same, then we go to the second level of $c$ and consider it the same way as the first level.

**Case 2.** The top levels of $c$ and $S_1$ are the same (therefore $c$ has only one level). We work with $c^{(2)}$ the same way as we did for $c$. Then the image
of $G$ on some level $k$ of $c^{(2)}$ is a proper quotient of $G$ by Lemma 6. If at some point the sum of dimensions for $c^{(2)}$ is not maximal, we amalgamate the fundamental sequence induced by the top part of $c^{(2)}$ above level $k$ for $G$, and each fundamental sequence for this quotient, then take the family of correcting extensions. Then we construct the block-NTQ group as we did on the first step, denote it by $N_2$.

**Case 3.** The top levels of $c$ and $S_1$ are not the same and the top levels of $c$ and $c^{(2)}$ are not the same. If $N_{0}^{2}$ is a proper quotient of $N_0^{1}$, then we apply step 1 to $N_{0}^{2}$ instead of $G$. Doing this, we do not have to consider those fundamental sequences $c$ that have only one level.

If $N_{0}^{2}$ is isomorphic to $N_0^{1}$ then there is a level $k$ of $c^{(2)}$ such that we can suppose that the image of either $G$ or $N_{0}^{1}$ on this level is a proper quotient. To verify if the image $N_{0}^{1}$ of $N_0^{1}$ is a proper quotient we replace $N_0^{1}$ by its quasi convex closure $N_{0,q}^{1}$ and verify if the image of $N_{0,q}^{1}$ is a proper quotient.

Consider fundamental sequences for $N_{0}^{1}$ modulo the images of subgroups $R_1, \ldots, R_s$, and apply to them step 1. Denote the obtained fundamental sequences by $f_i$. Construct fundamental sequences for the subgroup generated by the images of $X_1, \ldots, X_n$ with the top part being induced from the top part of $c^{(2)}$ (above level $k$) and bottom part being some $f_i$, but not the sequence with the same top part as $c$. We construct a block-NTQ group amalgamating the top $k − 1$ levels of $c^{(2)}$ and the block-NTQ group constructed for $f_i$ as on the first step. There exists a formula solution over this group. (Notice, that constructing the NTQ group induced by $N_{0}^{2}$ from the top part of $c^{(2)}$ we simultaneously deciding if $N_{0}^{2}$ is a proper quotient of $N_0^{1}$.)

If all the levels of $c$ and $c^{(2)}$ are the same, so we never have cases 2 and 3, then the block fundamental sequence consists of a sequence of induced fundamental sequences for $G$ and its images. For each level of each induced fundamental sequence there is an abelian decomposition. Consider instead of $c_2$ the fundamental sequence $c_4$ induced from $c$ for the group $G_{corr}$. There exists some level $k$ such that the abelian decompositions for $c_4$ will coincide with abelian decompositions for $c_2$ for levels above $k$, and on level $k$ either the number of free factors in the free decomposition for $c_4$ is less than for $c_2$ or the number of factors is the same, but the regular size of the decompositions (lexicographically ordered tuple $(sizeQ_1, \ldots, sizeQ_m)$ of sizes of MQH subgroups) for $c_4$ is smaller than for $c_2$, or the regular sizes are the same but the abelian size $ab$ of the decompositions for $c_4$ is smaller than for $c_2$. Here if $R$ is an abelian decomposition, by $ab(R)$ (the abelian size) we denote the sum of the ranks of abelian vertex groups in $R$ minus the sum of the ranks of the edge groups for the edges from them.

### 4.6 General step

We now describe the $n$’th step of the construction. Denote by $N_i$ the block-NTQ group constructed on the $i$’th step, and by $N_{j}^{i}$, $j > i$ the image of it on the $j$’th step. Let $\{j_k, k = 1, \ldots, s\}$ be all the indices for which the top level of $N_{j_k}^{i+1}$ is different from the top level of $N_{j_k}$. 


If $G$ or some of the groups $N^{jk}_{n-1}$ is not embedded into $N^n_{n-1}$ (we can check this the same way as we checked whether the image of $N^n_0$ is a proper quotient in Step 2, Case 3). We replace the first such group by its proper quotient in $N^n_{n-1}$ and consider only the fundamental sequences that have the top level different from $N^n_{n-1}$. In all other cases we can suppose that the groups $G$ and all $N^{jk+1}_{n-1}$ are embedded into $N^n_{n-1}$.

**Case 1.** The top levels of $c^{(n)}$ and $c^{(n-1)}$ are the same. In this case we go to the second level and consider it the same way as the first level.

If going from the top to the bottom of the block-NTQ system, we do not obtain the case considered above or Cases 2, 3 and all the levels of the top block of $N_{n-1}$ and $N_n$ are the same, we consider the group $G_{cor}$ which was constructed for the induced fundamental sequence corresponding to the homomorphisms from $G$ going through $N_n$, and the fundamental sequence induced by $c^{(n)}$ for this group as we did on the second step when cases 2 and 3 were not applicable on all the levels of $c^{(2)}$.

**Case 2.** The top levels of $c^{n-1}$ and $c^{n-2}, \ldots, c^{n-i}$ are the same, and the top levels of $c^{n-1}$ and $c^n$ are not the same. Then on some level $p$ of $c^{(n)}$ we can suppose that the image of $N^{n-i}_{n-i-1}$ is a proper quotient (or the fundamental sequence goes through another branch constructed on the previous step). Consider fundamental sequences $f_i$ for this quotient modulo the rigid subgroups and apply to them the procedure described on step 1. Consider only sequences with the top level different from $c^{(n-i)}$. Construct $N_n$ as a block-NTQ group with the top part being $c^{(n)}$ above level $p$ and the bottom part being the block-NTQ group corresponding to $f_i$.

**Case 3.** The top levels of $c^{(n-2)}$ and $c^{(n-1)}$ are not the same and the top levels of $c^{(n-1)}$ and $c^{(n)}$ are not the same. Then on some level $p$ of $c^{(n)}$ the image of $N^{n-1}_{n-2}$ is a proper quotient. Construct a block-NTQ group as in the previous case.

In this way we continue the construction of the tree $T_{EA}(\Phi)$.

### 4.7 The $\forall \exists$ tree is finite

It is convenient to define as in [37], Definition 4.2, the notion of complexity of the fundamental sequence $(Cmplx(Var_{fund}))$ at follows:

$$Cmplx(Var_{fund}) = (dimVar_{fund} + factorsVar_{fund}, (sizeQ_1, \ldots, sizeQ_m), ab(V_{fund}(Q)))$$

where $factorsVar_{fund}$ is the number of freely indecomposable, non-cyclic terminal factors embedded into $\Gamma$ (Sela calls $dimVar_{fund} + factorsVar_{fund}$ the Kurosh rank of the resolution), and $(sizeQ_1, \ldots, sizeQ_m)$ is the regular size of the system. The complexity is a tuple of numbers which we compare lexicographically from the left.

In this subsection we will prove the following.

**Theorem 4.** The tree $T_{EA}(\Phi)$ is finite.

Proof. We begin with the characterization of the case when the fundamental sequence $c$ has only one level.
Let an NTQ system $Q(X_1, \ldots, X_n) = 1$ have the form

$$Q_1(X_1, \ldots, X_n) = 1,$$

$$\ldots$$

$$Q_n(X_n) = 1,$$

where $Q_1 = 1$ corresponds to the top level of JSJ decomposition for $\Gamma_{R(Q)}$, variables from $X_1$ are either quadratic or correspond to extensions of centralizers. Consider this system together with a fundamental sequence $V_{\text{fund}}(Q)$ defining it. Let $V_{\text{fund}}(U_1)$ be the subset of $V_{\text{fund}}(Q)$ satisfying some additional equation $U_1 = 1$, and $G_1$ a group discriminated by this subset. Consider the family of those canonical fundamental sequences for $G_1$ modulo the images of rigid subgroups of $G$, $R_1, \ldots, R_s$ in the factors $P_1, \ldots, P_s$ in the free decomposition of the subgroup $< X_2, \ldots, X_m >$, which have the same Kurosh rank modulo them as $Q_1 = 1$. Denote this free decomposition by $H_1^*$. In constructing this fundamental sequence we take into consideration only those homomorphisms of the quotient $\Gamma$-limit group that embed the images of the terminal non-cyclic freely indecomposable factors into $\Gamma$. Denote such a fundamental sequence by $c$, and corresponding NTQ system $S = 1 (mod H_1)$, where $S = 1$ has form

$$S_1(X_{11}, \ldots, X_{1m}) = 1$$

$$\ldots$$

$$S_m(X_{1m}) = 1.$$

Denote by $D_Q$ a canonical decomposition corresponding to the group $\Gamma_{R(Q)}$. Non-QH, non-abelian subgroups in this decomposition are $P_1, \ldots, P_s$. Abelian and QH subgroups correspond to the system $Q_1(X_1, \ldots, X_n) = 1$. For each $i$ there exists a canonical homomorphism

$$\eta_i : \Gamma_{R(Q)} \rightarrow \Gamma_{R(S_i, \ldots, S_m)}$$

such that $P_1, \ldots, P_s$ are mapped into rigid subgroups in the canonical decomposition of $\eta_i(\Gamma_{R(Q)})$.

Each QH subgroup in the decomposition of $\Gamma_{R(S_i, \ldots, S_m)}$ as an NTQ group is a QH subgroup of $\eta_i(\Gamma_{R(Q)})$. By [35], Lemma 2.7, for each QH subgroup $Q_1$ of $\eta_i(\Gamma_{R(Q)})$ there exists a QH subgroup of $\Gamma_{R(Q)}$ that is mapped into a subgroup of finite index in $Q_1$. The size of this QH subgroup is, obviously, greater or equal to the size of $Q_1$. Those QH subgroups of $\Gamma_{R(Q)}$ that are mapped into QH subgroups of the same size by some $\eta_i$ are called stable.

**Lemma 7.** In the conditions above there are the following possibilities:

(i) The set of homomorphisms going through $c$ is generic for each regular quadratic equation in $Q_1 = 1$ and $ab(c) = ab(V_{\text{fund}}(Q))$ (in this case $c$ has only one level identical to $Q_1$); 

(ii) It is possible to reconstruct system $S = 1$ such a way that size $(S) < size (Q_1)$; 

(iii) size $(S) = size (Q_1)$, $ab(c) < ab(V_{\text{fund}}(Q))$. 

26
Proof. The fundamental sequence \( c \) modulo the decomposition \( H_1 \ast \) has the same dimension as \( Q_1 = 1 \). The Kurosh rank of \( Q_1 = 1 \) is the sum of the following numbers:

1) the dimension of a free factor \( F_1 = F(t_0, \ldots, t_{k_0}) \) in the free decomposition of \( F_{R(Q)} \) corresponding to an empty equation in \( Q_1 = 1 \);

2) the number of abelian factors;

3) the sum of dimensions of surface groups factors;

4) the number of free variables of quadratic equations with coefficients in \( Q_1 = 1 \) corresponding to the fundamental sequence \( V_{\text{fund}}(Q) \),

5) \( f\text{actors}\text{Var}_{\text{fund}} \).

Because \( c \) has the same Kurosh rank, the free factor \( F_1 \) is unchanged. By 1) and 2) in Section 4.2, abelian and surface factors are sent into different free factors.

Let \( Q_{1i} = 1 \) be one of the standard quadratic equations in the system \( Q_1 = 1 \). If the set of solutions of \( Q_{1i} = 1 \) over \( F_{R(Q_2 \ldots Q_n)} \) that factor through the system \( S = 1 \) is a generic set for \( Q_{1i} = 1 \), then by the analog of (20, Theorem 9) we conclude that \( S = 1 \) can be reconstructed so that it contains only one quadratic equation as a part of the system \( S_m = 1 \). Indeed, suppose a QH subgroup \( \overline{Q}_{1i} \), corresponding to \( Q_{1i} = 1 \) mapped on some level \( s \) of \( S = 1 \) onto a subgroup of the same size. Then it is stable. Suppose also that a QH subgroup of \( F_{R(Q)} \) that is a subgroup of \( \overline{Q}_{1i} \) is projected on some level \( k \) above \( s \) into a QH subgroup \( \overline{Q}_k \). Then this projection is a monomorphism. On all the levels above \( s \) we can join the image of a subgroup of \( \overline{Q}_{1i} \) to a non-QH subgroup adjacent to it (and not count it in the size). We can join the image of it on level \( m \) by the isomorphic copy of \( \overline{Q}_{1i} \).

Is all QH subgroups corresponding to \( Q_1 = 1 \) are stable, then the regular size of \( S = 1 \) is the same as the regular size of \( Q_1 = 1 \) and if \( ab(c) = ab( V_{\text{fund}}(Q)) \), then reconstructed \( S = 1 \) has only one level.

The lemma is proved.

To finish the proof of Theorem 4, notice that by Lemma 7, every time we apply the transformation of Case 3 (we refer to the cases from Section 4.6) in the construction of \( AE \)-tree we either (i) decrease the dimension in the top block, therefore decrease the Kurosh rank, or (ii) replace the NTQ system in the top block by another NTQ system of the same dimension but with smaller size, or (iii) \( ab(c) \) decreases. Hence the complexity defined in the beginning of this section decreases. Hence Case 3 cannot be applied infinitely many times to the top block. If we apply Case 2, we consider the second block for proper quotients of a finite number of groups. Hence, starting from some step, we come to a situation, when the fundamental sequences factor through the same block-NTQ system, and the image of \( G \) in the last level of these systems is a proper quotient of \( G \). Case 1 cannot appear infinitely many times because every time
the induced fundamental sequence has the same decomposition on levels above some level $k$ and has a decrease in the complexity of the decomposition on level $k$. Theorem 4 is proved.

The effectiveness of the construction of the finite $\forall\exists$-tree for sentence $\Phi$ implies that the $\forall\exists$-theory of the group $\Gamma$ is decidable. This proves Theorem 2.
5 Effectiveness of the global bound in finiteness results

In this section we will give a proof of the effectiveness of the global bound in Theorem 11 [20] and show how to generalize the proof for the hyperbolic group case. In Section 5.4 of [20] we defined the notion of a sufficient splitting of a group $K$ modulo a class of subgroups $\mathcal{K}$. Let $\Gamma$ be a non-elementary torsion-free hyperbolic group with generators $A, P = A \cup \{p_1, \ldots, p_k\}, H = \langle P \rangle$. Let $\mathcal{K}$ consist of one subgroup $\mathcal{K} = \{H\}$. Let $K = \langle X, P | S(X, P) \rangle$, and suppose that $K$ does not have a sufficient splitting modulo $H$. Let $D$ be an abelian JSJ decomposition of $K$ modulo $H$.

We give the notion of algebraic solutions. Let $K_1$ be a fully residually $\Gamma$ quotient of the group $K$, $\kappa: K \to K_1$ the canonical epimorphism, and $H_1 = \kappa(H)$ the canonical image of $H$ in $K_1$. An elementary abelian splitting of $K_1$ modulo $H_1$ which does not lift into $K$ is called a new splitting.

**Definition 5.** (Definition 20 [20]) In the notation above the quotient $K_1$ is called reducing if one of the following holds:

1. $K_1$ has a non-trivial free decomposition modulo $H_1$;
2. $K_1$ has a new elementary abelian splitting modulo $H_1$.

We say that a homomorphism $\phi: K \to K_1$ is special if $\phi$ either maps an edge group of $D$ to the identity or maps a non-abelian vertex group of $D$ to an abelian subgroup.

Let $\mathcal{R} = \{K/R(r_1), \ldots, K/R(r_s)\}$ be a complete reducing system for $K$ (the existence of such system in for a free group is proved in [20], this can be similarly proved for a torsion free hyperbolic group). Now we define algebraic and reducing solutions of $S = 1$ in $F$ with respect to $\mathcal{R}$. Let $\phi: H \to F$ be a fixed $F$-homomorphism and $\text{Sol}_\phi$ the set of all homomorphisms from $K$ onto $F$ which extend $\phi$. A solution $\psi \in \text{Sol}_\phi$ is called reducing if there exists a solution $\psi' \in \text{Sol}_\phi$ in the $\sim_{\text{MAX}}$-equivalence class of $\psi$ which satisfies one of the equations $r_1 = 1, \ldots, r_k = 1$. All non-reducing non-special solutions from $\text{Sol}_\phi$ are called $K$-algebraic (modulo $H$ and $\phi$).

**Theorem 5.** (cf. [24], Theorem 6) Let $H \leq K$ be as above. The fact that for parameters $P$ there are exactly $N$ non-equivalent Max-classes of $K$-algebraic solutions of the equation $S(X, P) = 1$ modulo $H$ can be written algorithmically as a boolean combination of conjunctive $\exists \forall$-formulas (these are formulas of type [24]).

**Proof.** The generating set $X$ of $K$ corresponding to the decomposition $D$ can be partitioned as $X = X_1 \cup X_2$ such that $G = \langle X_2 \cup P \rangle$ is the fundamental group of the graph of groups obtained from $D$ by removing all QH-subgroups. If $c_e$ is a given generator of an edge group of $D$, then we know how a generalized fractional Dehn twist (AE-transformation or extended automorphism in the terminology of [20], [26]) $\sigma$ associated with edge $e$ acts on the generators from
the set $X$. Namely, if $x \in X$ is a generator of a vertex group, then either $x^\tau = x$
 or $x^\sigma = c^{-m}xc^m$, where $c$ is a root of the image of $c_e$ in $F$, or in case $e$ is
 an edge between abelian and rigid vertex groups and $x$ belongs to the abelian
 vertex group, $x^\sigma = xc^m$. Similarly, if $x$ is a stable letter then either $x^\sigma = x$
 or $x^\sigma = xc^m$.

One can write elements $c_e$ as words in generators $X_2$, $c_e = c_e(X_2)$. Denote
 $T = \{t_i, \ i = 1, \ldots, m\}$. Consider the formula

$$\exists X_1 \exists X_2 \forall Y \forall T \forall Z \left(S(X_1, X_2, P) = 1 \land \bigwedge_{i=1}^m [t_i, c_i(X_2)] = 1 \land Z = X_2^{\sigma^T} \land S(Y, X_2, P) = 1 \land V(Y, Z, P) = 1 \right).$$

It says that there exists a solution of the equation $S(X_1, X_2, P) = 1$ that is
 not Max-equivalent to a solution $Y, Z, P$ that satisfies $V(Y, Z, P) = 1$. If now
 $V(Y, Z, P) = 1$ is a disjunction of equations defining maximal reducing quo-
tients, then this formula states that for parameters $P$ there exists at least one
 Max-class of algebraic solutions of $S(X, P) = 1$ with respect to $H$.

Denote

$$\tau(T, X_2, Y, Z) = \left(\bigwedge_{i=1}^m [t_i, c_i(X_2)] = 1 \land Z = X_2^{\sigma^T} \land S(Y, X_2, P) = 1 \land V(Y, Z, P) = 1 \right).$$

The following formula states that for parameters $P$ there exists at least two
 non-equivalent Max-classes of algebraic solutions of $S(X, P) = 1$ with respect
to $H$.

$$\theta_2(P) = \exists X_1, X_3 \exists X_2, X_4 \forall Y, Y' \forall T, T', T'' \forall Z, Z' \left(S(X_1, X_2, P) = 1 \land S(X_3, X_4, P) = 1 \land \bigwedge_{i=1}^m [t_i, c_i(X_2)] = 1 \land X_2^{\sigma^{T''}} = X_4 \right).$$

Similarly one can write a formula $\theta_N(P)$ that states for parameters $P$ there
 exist at least $N$ non-equivalent Max-classes of algebraic solutions of $S(X, P) = 1$
 with respect to $H$.

Then $\theta_N(P) \land \neg \theta_{N+1}(P)$ states that there are exactly $N$ non-equivalent
 Max-classes. The theorem is proved.

**Theorem 6.** (for the case when $\Gamma$ is a free group this is [20], Theorem 11)

Let $H, K$ be finitely generated fully residually $\Gamma$ groups such that $\Gamma \leq H \leq K$
 and $K$ does not have a sufficient splitting modulo $H$. Let $D$ be an abelian JSJ
decomposition of $K$ modulo $H$ (which may be trivial). There exists a constant
$N = N(K, H)$ such that for each $\Gamma$-homomorphism $\phi : H \to \Gamma$ there are at most
$N$ algebraic pair-wise non-equivalent with respect to $\sim_{AEQ}$ and, therefore, with
respect to $\sim_{MAX}$, homomorphisms from $K$ to $\Gamma$ that extend $\phi$.

Moreover, the constant $N$ for the number of $\sim_{MAX}$-non-equivalent homomorphisms can be found effectively.
Proof. The statement about the existence of such constant $N$ is Theorem 3.5 [37] (although there is no proof of Theorem 3.5 there). We will show how to find this constant effectively. To make presentation easier, we consider first the case when the group $K$ from the formulation of Theorem 11 does not have a splitting modulo $H$. (in the terminology of [37] it is a rigid limit group). We consider the formula

$$
\exists P \exists Y_1, \ldots, Y_m (\wedge_{i=1}^m S(P, Y_i) = 1 \wedge Y_i \neq Y_j (i \neq j) \wedge_{i=1}^k \wedge_{i=1}^m r_i (P, Y_i) \neq 1).
$$

We know from Theorem 11 that possible number $m$ of algebraic solutions is bounded. Therefore for some positive integer $m$ such a formula will be false. The minimal such $m$ can be found because the existential theory of $\Gamma$ is decidable. Therefore $N = m - 1$.

Now we consider the case when the group $K$ has a splitting modulo $H$ but not a sufficient splitting ($K$ is solid in terminology of [37]). This case is more complicated because we have to write that solutions corresponding to tuples $Y_1, \ldots, Y_m$ are not reducing and not in the same $\sim_{\text{MAX}}$ equivalence classes for $i \neq j$. This means that there exist no elements representing QH subgroups and no elements commuting with edge groups of the JSJ decomposition of $K$ modulo $H$ such that application of generalized fractional Dehn twists corresponding to these elements take some of these solutions to reducing solutions or take one solution to the other. This fact can be expressed in terms of $\exists \forall$-sentence that is true if and only if there exists a homomorphism $H \to F$ that can be extended to $m$ algebraic and not $\sim_{\text{MAX}}$ equivalent homomorphisms $K \to F$. The decidability of $\exists \forall$-theory of $\Gamma$ was proved in the previous section. Then the bound on $m$ can be found effectively because we can find for which $m$ the sentence is false and therefore such homomorphism $H \to F$ does not exist. \qed
6 Quantifier elimination algorithm

In this section we will prove Theorem 1. Consider the following formula

$$\Theta(P) = \exists Z \forall X \exists Y (U(A, P, Z, X, Y) = 1 \land V(A, P, Z, X, Y) \neq 1), \quad (12)$$

where $A$ is a generating set of $\Gamma$. This formula $\Theta(P)$ is the negation of the formula $\Phi$ considered in [20].

The existence of quantifier elimination to boolean combinations of $\exists \forall$-formulas for $\Gamma$ was proved in [37]. Earlier it was proved in [39], [20] that every formula in the theory of a free group $F$ is equivalent to a boolean combination of $\exists \forall$-formulas. The general scheme of the proofs in [39] and in [20] is quite similar: to use the so-called implicit function theorem (= existence of formula solutions in the covering closure of a limit group) and to approximate any definable set and get its stratification using certain verification process (based on the implicit function theorem) that stops after finite number of steps. But all the necessary technical results are proved differently (using actions on $\mathbb{R}$-trees in [39], and using elimination process and free action of fully residually free groups on $\mathbb{Z}^n$-trees, which is equivalent to the existence of free length functions in $\mathbb{Z}^n$, in [20]). The proof in [20] is also algorithmic. It will be more convenient for us to follow our proof in [20] and to use our terminology but refer to [37] for necessary technical results.

To obtain effective quantifier elimination to boolean combinations of $\exists \forall$-formulas it is enough to give an algorithm to find such a boolean combination that defines the set defined by $\Theta(P)$.

The procedure is similar for a free group and for a torsion free hyperbolic group $\Gamma$. We recall how the procedure goes. For every tuple of elements $P$ for which $\Theta(P)$ is true, there exists some $Z$ and (by the Merzljakov theorem (Proposition 2.1, [37])) a solution $Y = f(A, P, Z, X)$ of $U = 1 \land V \neq 1$ in $F(X) \ast \Gamma$. All such solutions for all possible values of $P$ belong to a finite number of fundamental sequences with terminal groups $\Gamma_R(U_{1,i}) \ast F(X)$, where $U_{1,i} = U_{1,i}(A, P, Z, Z^{(1)})$ and $\Gamma_R(U_{1,i})$ is a group with no sufficient splitting modulo $<A, P, Z>$ (see Section 12.2, [20]). These groups can be effectively found by Proposition 8.

We now consider each of these fundamental sequences separately. Below we will not write the constants $A$ in the equations but assume that equations may contain constants. Those values $P, Z$ for which there exist a value of $X$ such that the equation

$$V(P, Z, X, f(Z, Z^{(1)}, P)) = 1$$

is satisfied for any function $f$ give a system of equations on $\Gamma_R(U_{1,i}) \ast F(X)$. This system is equivalent to a finite subsystem (to one equation in the case when we consider formulas with constants). Let $G$ be the coordinate group of this system and $G_{i, i} \in J$ be the corresponding fully residually $\Gamma$ groups.

We introduced in Section 12.2, [20], the tree $T_X(G)$ which is constructed the same way as $T_{EA}(\Phi)$ with $X, Y$ considered as variables and $P, Z, Z^{(1)}$ as
parameters. To each group $G_i$ we assign fundamental sequences modulo $<P, Z, Z^{(1)}>$. Their terminal groups are groups $\Gamma_{R(V_{2,i})}$, where

$$V_{2,i} = V_{2,i}(P, Z, Z^{(1)}, Z_1^{(2)})$$

that do not have a sufficient splitting modulo $<P, Z, Z^{(1)}>$. Then we find all formula solutions $Y$ of the conjunction

$$U(P, Z, X, Y) = 1 \land V(P, Z, X, Y) \neq 1$$

in the corrective normalizing extensions of the NTQ groups corresponding to these fundamental sequences for $X$ (see [20], Theorem 12). These formula solutions $Y$ are described by a finite number of fundamental sequences with terminal groups $\Gamma_{R(U_{2,i})}$, where $U_{2,i} = U_{2,i}(P, Z, Z^{(1)}, Z_1^{(2)}, Z^{(2)})$. Then again we investigate the values of $X$ that make the word $V(P, Z, X, Y)$ equal to the identity for all these formula solutions $Y$. And we continue the construction of $T_X(G)$. We can prove that this tree is finite exactly the same way as we proved the finiteness of the $\exists \forall$-tree. We will call $T_X(\Theta)$ the parametric $\exists \forall$-tree for the formula $\Theta(P)$.

For each branch of the tree $T_X$ we assign a sequence of groups $\Gamma_{R(U_{1,i})}, \Gamma_{R(V_{2,i})}, \ldots, \Gamma_{R(U_{r,i})}$ as in [20], Section 12.2. Corresponding irreducible systems of equations are:

$$U_{1,i} = U_{1,i}(P, Z, Z^{(1)}),$$

$$U_{m,i} = U_{m,i}(P, Z, Z^{(1)}, Z_1^{(m)}, Z^{(m)}), \quad m = 2, \ldots, r,$$

which correspond to the terminal groups of fundamental sequences describing $Y$ of level $(m, m - 1)$, and

$$V_{m,i} = V_{m,i}(P, Z, Z^{(1)}, Z_1^{(m)}), \quad m = 2, \ldots, r$$

which correspond to the terminal groups of fundamental sequences describing $X$ of level $(m, m)$. They correspond to vertices of $T_X$ that have distance $m$ to the root.

For each $m$ the group $\Gamma_{R(U_{m,i})}$ does not have a sufficient splitting modulo the subgroup $<P, Z, Z^{(1)}, Z_1^{(m)}>$. and the group $\Gamma_{R(V_{m,i})}$ does not have a sufficient splitting modulo the subgroup $<P, Z, Z^{(1)}>$. On each step we consider terminal groups of all levels. Below we will sometimes skip index $i$ and write $U_m, V_m$ instead of $U_{m,i}, V_{m,i}$.

**Proposition 13.** A complete system of reducing quotients of a group with no sufficient splitting modulo a subgroup can be found effectively.

**Proof.** Suppose first that a group $K$ does not have a splitting modulo a subgroup $H$. For each reducing quotient $K_i$ of $K$ there is a discriminating family $\Psi$ of homomorphisms such that for every homomorphism $\psi \in \Psi$ the restriction $\psi_H$
can be extended by infinitely many ways to homomorphisms from $K_1$ to $\Gamma$. Using canonical representatives we can construct a system of equations $S = 1$ in the free group $F$ that is a natural pre-image of $\Gamma$ ($\pi : F \to \Gamma$) such that each solution of $S = 1$ in $F$ corresponds to some homomorphism $K \to \Gamma$ and every homomorphism $K \to \Gamma$ corresponds to some solution of $S = 1$. We can run the Elimination process for $S = 1$ modulo the pre-image of $H$. Each homomorphism from the pre-image of $H$ to $F$ can be extended by infinitely many ways to a solution of $S = 1$ corresponding to a homomorphism from $K_1$ to $\Gamma$. Modifying the obtained quotients of $F_{R(S)}$ into quotients of $K$, we will obtain, different quotients of $K$ but, in particular, we will obtain all the maximal reducing quotients of $K$ because the new splittings for quotients of $K$ will be seen in the Elimination process over a free group. Then we have to compare the reducing quotients that we obtain and take the maximal ones. We can decide if one reducing quotient $R_1$ is a quotient of the other $R_2$ because we know the images of the generators of $R_2$ in $R_1$. (Notice that these groups are universally equivalent to $\Gamma$ and the universal theory of $\Gamma$ is decidable.)

Suppose now that $K$ has a (non-sufficient) splitting modulo $H$. For each reducing quotient $K_1$ of $K$ there is a discriminating family $\Psi$ of homomorphisms such that for every homomorphism $\psi \in \Psi$ the restriction $\psi_H$ can be extended by infinitely many ways to homomorphisms from $K_1$ to $\Gamma$ which are minimal in their AEQ classes for $K$ (see [20] for definition of AEQ class). We again can take a system of equations $S = 1$ in the free group $F$ (the pre-image of $\Gamma$, $\pi : F \to \Gamma$) such that each solution of $S = 1$ in $F$ corresponds to some homomorphism $K \to \Gamma$ and every homomorphism $K \to \Gamma$ corresponds to some solution of $S = 1$. We can run the Elimination process for $S = 1$ modulo the pre-image of $H$. Each homomorphism from the pre-image of $H$ to $F$ can be extended by infinitely many ways to a solution of $S = 1$ corresponding to a homomorphism from $K_1$ to $\Gamma$ that is AEQ-minimal as a homomorphism from $K$ to $\Gamma$. Modifying the obtained quotients of $F_{R(S)}$ into quotients of $K$, we will obtain reducing quotients of $K$ and, in particular, all the maximal reducing quotients of $K$. Then we have to compare the reducing quotients that we obtain and take the maximal ones.

\[ \Box \]

6.1 Algorithm for the construction of the tree $T_X$.

**Proposition 14.** There is an algorithm to construct the following:

1) the finite parametric $\exists \forall$-tree $T_X$,

2) for each branch of the tree $T_X$ the finite family of groups

\[ \Gamma_{R(U_{1,i})}, \Gamma_{R(V_{2,i})}, \ldots, \Gamma_{R(V_{r,i})}, \Gamma_{R(U_{r,i})}, \]

3) for each vertex of the tree, a fundamental sequence describing either $X$ (if the associated group is $\Gamma_{R(U_{j,i})}$) or $Y$ (if the associated group is $\Gamma_{R(V_{j,i})}$).
Proof. This uses the algorithm from Proposition 13 to construct a complete system of reducing quotients and Proposition 8 that states that we can construct fundamental sequences modulo a finite set of finitely generated subgroups algorithmically.

The tree $T_X$ is finite, as in [20] we have schemes of levels $(1,0), (1,1), (2,1), (2,2)$ etc up to some number $(m,m)$.

### 6.2 Configuration groups

We will concentrate on level $(2,1)$ now. In Definition 27 and Definition 28, [20] we define initial fundamental sequences of levels $(2,1)$ and $(2,2)$ and width $i$ (the possible width is bounded) modulo $P$. Since we are now considering the formula $\Theta$ such that $\Theta = \neg \Phi$ for the formula $\Phi$ considered in [20], we will slightly change the definition here. It will be more convenient to replace condition (6) from Definition 28 of [20] by its negation and add this negation on level $(2,1)$.

**Definition 6.** Let $\Gamma_{R(V_2,1)}, \ldots, \Gamma_{R(V_2,t)}$ be the whole family of groups on level $(1,1)$. To construct the initial fundamental sequences of level $(2,1)$ and width $i = i_1 + \ldots + i_s$, we consider the fundamental sequences modulo the subgroup $\langle P \rangle$ for the groups discriminated by $i$ solutions of the systems

$$U_{2,m_s}(P, Z, Z^{(1)}, Z_1^{(2,i,s)}, Z^{(2,i,s)}) = 1, \ j = 1, \ldots, i_s, \ s = 1, \ldots, t,$$

with properties:

1. $Z^{(1)}, Z_1^{(2,j,s)}, Z^{(2,j,s)}$ are algebraic;
2. $Z_1^{(2,j,s)}$ are not MAX-equivalent to $Z_1^{(2,p,s)}, p \neq j, p, j = 1, \ldots, i_s, s = 1, \ldots, t$;
3. for any of the finite number of values of $Z_1^{(2)}$ the fundamental sequences for $V_{2,s}(P, Z, Z^{(1)}, Z_1^{(2)}) = 1$ are contained in the union of the fundamental sequences for $U_{2,m_s}(P, Z, Z^{(1)}, Z_1^{(2,j)}, Z^{(2,j)}) = 1$ for different values of $Z^{(2,j,s)}$;
4. there is no non-equivalent $Z_1^{(2,i,s+1)}$, algebraic, solving $V_{2,s} = 1, s = 1, \ldots, t$.
5. the solution $P, Z, Z^{(1)}$ does not satisfy a proper equation which implies $V = 1$ for any value of $X$.
6. for any $s$, the solution $P, Z, Z^{(1)}, Z_1^{(2,i,s)}, Z^{(2,i,s)}$ can not be extended to a solution of some

$$V_{3,s}(P, Z, Z^{(1)}, Z_1^{(2,i,s)}, Z^{(2,i,s)}, Z_1^{(3,i,s)}) = 1.$$

We call this group a configuration group. We also call a tuple

$$Z, Z^{(1)}, Z_1^{(2,i,s)}, Z^{(2,i,s)}, j = 1, \ldots, i_s, \ s = 1, \ldots, t$$

satisfying the conditions above a certificate for $\Theta$ for $P$ (of level $(2,1)$ and width $i$). We add to the generators of the configuration group additional variables $Q$.
for the primitive roots of a fixed set of elements for each certificate (these are primitive roots of the images of the edge groups and abelian vertex groups in the relative JSJ decompositions of the groups $\Gamma_{R(V_2)}$).

Identically to the proof of ([20], Lemma 27) one can show that for each value of parameters $P$ for which there exists a certificate, either there exists a generic family of certificates corresponding to the fundamental sequence, or any certificate in this fundamental sequence can be extended by $Z_1^{(2,i_s+1,s)}$ such that the whole tuple factors through one of the groups $W_{\text{surplus}}$ (defined before Lemma 27) and going through one of the fundamental sequences for which $Z_1^{(2,i_s+1,s)}$ is either reducing or Max-equivalent to one of $Z_1^{(2,j,s)}, j = 1, \ldots, i_s$. In the former case we say that the fundamental sequence has depth 1, it the later case we will consider fundamental sequences of depth 2.

Notice that we do not know a system of equations defining a configuration group. We, therefore, need the following result.

**Proposition 15.** Let $H = \Gamma_{R(W)}$ be one of the configuration groups with generators

$$P, Z, Z^{(1)}, Z_1^{(2,j,s)}, Z_1^{(2,j,s)}, Q, j = 1, \ldots, i_s, s = 1, \ldots, t.$$

Then there is an algorithm to find each terminal group of each fundamental sequence for $H$ modulo $P$.

**Proof.** We will first prove the statement of the proposition for the case when $\Gamma$ is a free group. Let $\Gamma = F$. As in the proof in [20], Theorem 11, we extensively use the technique of generalized equations described in [20], Subsection 4.3 and Section 5. The reader has to be familiar with these sections of [20]. In the proof of Theorem 11, [20] we show how to construct given a group $K$ that does not have a sufficient splitting modulo a subgroup $H$ a finite system of cut equations $\Pi$ (see [20], Section 7.7) for a minimal in its Max-class solution such that the intervals of $\Pi$ are labeled by values of the generators of $H$. For each system

$$U_{2,m_s}(P, Z, Z^{(1)}, Z_1^{(2,j,s)}, Z_1^{(2,j,s)}) = 1$$

we construct a cut equation modulo the parametric subgroup $<P, Z, Z^{(1)}, Z_1^{(2,j,s)}>$. The intervals of this cut equation are labeled by $P, Z, Z^{(1)}, Z_1^{(2,j,s)}$. For the intervals labeled by $P, Z, Z^{(1)}$ we add a cut equation for the system

$$V_{2,m}(P, Z, Z^{(1)}, Z_1^{(2,j,s)}) = 1$$

modulo the parametric subgroup $<P, Z, Z^{(1)}>$. For the intervals labeled by $P, Z$ we add cut equations for the system

$$U_{1,m}(P, Z, Z^{(1)}) = 1$$

modulo the parametric subgroup $<P, Z>$. The intervals labeled by $P$ will be the same for all these cut equations. Similarly we identify all the intervals labeled by the same variables that occurs in different cut equations. We now add
to the obtained cut equation, which can be also considered as a generalized equation, the inequalities that guarantee that conditions (1)-(6) are satisfied. These inequalities are just indicating that specializations of variables corresponding to some sub-intervals of the generalized equation must not be identities. But this is a standard requirement for a solution of a generalized equation. For example, we write an equation $r_1(Z_1^{(2,j)}) = \lambda_1$ and set that $\lambda_1$ is a base of the generalized equation. Then the condition $\lambda_1 \neq 1$ must be automatically satisfied for a solution of a generalized equation. So we can construct a finite number of generalized equations such that each certificate corresponding to minimal in their Max-classes specializations is a solution of one of these generalized equations $GE$.

We now construct fundamental sequences of solutions of the equations $GE$ modulo $<P>$. Notice that not all solutions from the fundamental sequence satisfy the necessary inequalities, but if we restrict the sets of automorphisms on all the levels to those whose application preserves corresponding generalized equations, we will have solutions of inequalities too. Therefore, a generic family of solutions does satisfy the inequalities. Using our standard procedure we construct fundamental sequences induced by the subgroup with generators

$$P, Z, Z^{(1)}, Z_1^{(2,j,s)}, Z^{(2,j,s)}, Q, j = 1, \ldots, i_s, s = 1, \ldots, t.$$ 

The subgroups generated by the images of these generators in the terminal groups of these fundamental sequences are precisely the terminal groups of the fundamental sequences for $H$ modulo $P$.

Let now $\Gamma$ be a torsion free hyperbolic group. For each system over $\Gamma$ we construct a system of equations over $F$ using canonical representatives and a cut equation modulo parameters subgroup $<P, Z, Z^{(1)}, Z_1^{(2,j,s)}>$. For this system. For the intervals labeled by $P, Z$ we add equations in a free group constructed using canonical representatives for the system $U_{1,m}(P, Z, Z^{(1)}) = 1$ over $\Gamma$ and add cut equations modulo the parameters subgroup $<P, Z >$. The inequalities over $\Gamma$ will correspond to inequalities over $F$ which, again, indicate that specializations of some variables of the generalized equation must be non-trivial. This is a standard requirement for a solution of a generalized equation over $F$. We construct fundamental sequences modulo $<P>$. Then we transform these fundamental sequences into fundamental sequences over $\Gamma$ as we did in the proof of Proposition 7. Then we construct fundamental sequences induced by the subgroup with generators

$$P, Z, Z^{(1)}, Z_1^{(2,j,s)}, Z^{(2,j,s)}, Q, j = 1, \ldots, i_s, s = 1, \ldots, t.$$ 

The subgroups generated by the images of these generators in the terminal groups of these fundamental sequences are the groups we are looking for. □
This implies the following result.

**Corollary 3.** There is an algorithm to construct the initial fundamental sequences for $Z$ of level (2,1) and width $i$ related to $\Gamma_{R(V)}$.

Lemmas 27, [20], states that the set of parameters $P$ for which there exists a fundamental sequence of level (2,1) and width $i$ and a certificate, consists of those $P$ for which there exists a generic family of certificates (generic certificate) and those for which all the certificates factor through a proper projective image of this fundamental sequence. Actually, Lemma 27 deals with certificates satisfying only properties (1)-(5), but the proof does not change if we add property (6) to the definition of a certificate.

For a given value of $P$ the formula $\Theta$ can be proved on level (2,1) and depth 1 if and only if the following conditions are satisfied.

(a) There exist algebraic solutions for some $U_{i,coeff} = 1$ corresponding to the terminal group of a fundamental sequence $V_{i,fund}$ for a configuration group modulo $P$.

(b) These solutions do not factor through the fundamental sequences that describe solutions from $V_{i,fund}$ that do not satisfy one of the properties (1)-(6). There is a finite number of such fundamental sequences.

(c) These solutions do not factor through the terminal groups of fundamental sequences of level (2,1) and greater depth derived from $V_{i,fund}$.

(d) $(P, Z, Z^{(1)})$ cannot be extended to a solution of $V = 1$ by arbitrary $X$ ($X$ of level 0) and $Y$ of level (1,0).

In this case there is a generic certificate of level (2,1) width $i$ and depth 1. These conditions can be described by a boolean combination of conjunctive $\exists \forall$-formulas of type $\Theta$. Similarly we consider fundamental sequences of level (2,1) width $i$ and depth 2 and deeper fundamental sequences of level (2,1) width $i$. We construct the projective tree (see [20], Section 11) to construct these deeper sequences.

We now need another algorithmic result that states that the main technical tool of the procedure of constructing the projective tree, tight enveloping NTQ groups and fundamental sequences, can be effectively constructed.

### 6.3 Tight Enveloping NTQ Groups

We now have to modify the definition of a **tight enveloping NTQ group and fundamental sequence**. This is done as follows. Given a fully residually $\Gamma$ group $G = \Gamma_{R(U)}$, the NTQ system $W = 1$ corresponding to a fundamental sequence for $U = 1$ (with the quadratic system $S_1(X_1, \ldots, X_n) = 1$ corresponding to the top level), a system of equations $P = 1$ with coefficients in $\Gamma_{R(W)}$ having a solution in some extension of $\Gamma_{R(W)}$ we first construct fundamental sequences (satisfying first and second restrictions) for $P = 1$ modulo the rigid subgroups
of $F_{R(U)}$ (that are subgroups of the second level of $\Gamma_{R(W)}$). Consider one of these fundamental sequences and construct the NTQ group for it. Denote it $\Gamma_{R(L_1)}$. Suppose $G$ is embedded into $\Gamma_{R(L_1)}$.

(a) We take the NTQ group induced by the image of $G$ in $\Gamma_{R(L_1)}$ from $\Gamma_{R(U)}$ as described in Subsection 4.3. This does not increase the Kurosh rank, because we add only elements from abelian subgroups. Denote this group by $\text{Ind}(\Gamma_{R(U)})$. Then we do the following.

(b) The terminal level of $\text{Ind}(F_{R(U)})$ admits a free decomposition induced from the free decomposition of the terminal group of the NTQ group $\Gamma_{R(L_1)}$, $M = M_1 \ast \ldots \ast M_s \ast F$, where $F$ is a free group (possibly trivial) and $M_1, \ldots, M_s$ are embedded into conjugates of the non-cyclic freely indecomposable factors in the free decomposition of the terminal level of $\Gamma_{R(L_1)}$. We replace each $M_i$ by the factor that contains it.

(c) We add to $\text{Ind}(F_{R(U)})$ all the QH subgroups of $\Gamma_{R(L_1)}$ that have non-trivial intersection with $\text{Ind}(\Gamma_{R(U)})$, do not have free variables, the corresponding level of $\text{Ind}(\Gamma_{R(U)})$ intersects non-trivially some of their adjacent vertex groups, and their addition decreases the Kurosh rank.

(d) We add also those QH subgroups $Q$ of the group $\Gamma_{R(L_1)}$ that intersect $\text{Ind}(\Gamma_{R(U)})$ in a subgroup of finite index (in $Q$) and have less free variables than the subgroup in the intersection.

(e) We add all the elements that conjugate different QH subgroups (abelian vertex groups) of $\text{Ind}(\Gamma_{R(U)})$ into the same QH subgroup of $L_1$ if this decreases the Kurosh rank.

(f) We add edge groups of abelian subgroups of the enveloping group that have non-trivial intersection with $\text{Ind}(\Gamma_{R(U)})$ if this does not increase the Kurosh rank.

We make these steps (which we call adjustment) iteratively by levels from the top to the bottom (considering on level $i + 1$ the image of the group extended on level $i$) of the system corresponding to $\text{Ind}(\Gamma_{R(U)})$ and denote the obtained group by $\text{Adj}(\Gamma_{R(U)})$.

We repeat the adjustment iteratively as many times as possible. Then we add once more those QH subgroups of different levels of the enveloping group that contain QH subgroups of the tight enveloping NTQ group as subgroups of finite index and whose addition decreases the Kurosh rank. We call the constructed NTQ group the tight enveloping NTQ group. We will also call the corresponding system (fundamental sequence) the tight enveloping system (fundamental sequence) and use the notation $(TEnv(S_1))$. As a size of a QH subgroup $Q$ in the tight enveloping NTQ group we consider the size of the QH subgroup in the enveloping group containing $Q$ as a subgroup of a finite index.

The Kurosh rank of the tight enveloping fundamental sequence $(TEnv(S_1))$ is less than or equal to the Kurosh rank of $S_1$ modulo free factors of $\langle X_2, \ldots, X_n \rangle$. 

39
(because we consider only fundamental sequences compatible with the free factorization of the subgroup \(\langle X_2, \ldots, X_n \rangle\) and compatible with the splitting of quadratic equations as discussed in Subsections 4.1, 4.2). If the Kurosh ranks are the same, we can always reorganize the levels of the enveloping system \(L_1\) into another \(Env(S_1)\) so that they have the same fundamental solutions and \(\text{size}(TEnv(S_1)) \leq \text{size}(S_1)\). If all the parameters (Kurosh rank, size, ab) are the same, then \(TEnv(S_1)\) has one level the same as \(S_1\). (Notice, that the Kurosh rank of the tight enveloping NTQ fundamental sequence \((TEnv(S_1))\) is the same as the maximal Kurosh rank of the corresponding subgroup in the terminal group in the enveloping fundamental sequence modulo free factors of \(\langle X_2, \ldots, X_n \rangle\).

**Proposition 16.** 1) Given a fully residually \(\Gamma\) group \(G = \Gamma_{R(U)}\), the canonical NTQ system \(W = 1\) corresponding to a branch of the canonical embedding tree \(T_{CE}(\Gamma_{R(U)})\) of the system \(U = 1\), a system of equations \(\mathcal{P} = 1\) with coefficients in \(\Gamma_{R(W)}\) having a solution in some extension of \(\Gamma_{R(W)}\), there is an algorithm for the construction of tight enveloping NTQ groups and fundamental sequences.

2) The bound in Lemma 28 from [20] can be found effectively.

**Proof.** 1) The first algorithm can be constructed using Proposition 1, because the construction begins with the induced NTQ group \(\text{Ind}(F_{R(U)})\), and this group is relatively hyperbolic as well as \(\Gamma_{R(L_1)}\).

2) The bound in Lemma 28 from [20] can be found effectively as in Theorem 6.

We also consider similarly fundamental sequences of all levels \((m, m-1)\). We now can make all the steps of the quantifier elimination procedure (to boolean combination of formulas \(\mathcal{P}\)) algorithmically.

This proves Theorem 1.
References

[1] M. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Springer, 1999.

[2] I. Bumagin, The conjugacy problem for relatively hyperbolic groups, Algebraic and Geometric Topology, 4(2004), 1013-1040.

[3] I. Bumagin, J. Macdonald, Groups discriminated by a locally quasi-convex hyperbolic group, arXiv:1307.6783 [math.GR].

[4] M. Casals-Ruiz, I. Kazachkov, On systems of equations over free products of groups. J. Algebra 333 (2011), 368-426.

[5] M. Casals-Ruiz, I. Kazachkov, On systems of equations over free partially commutative groups, Memoirs of the American Mathematical Society 2011; 153 pp., V. 212.

[6] M. Casals-Ruiz, I. Kazachkov, Elements of Algebraic Geometry and the Positive Theory of Partially Commutative Groups, Canad. J. Math. 62 (2010), pp. 481-519.

[7] F. Dahmani, Combination of convergence groups, Geometry & Topology Volume 7 (2003) 933-963.

[8] F. Dahmani, Existential questions in (relatively) hyperbolic groups, Israel Journal of Mathematics, 173 (2009), 91-124.

[9] F. Dahmani, D. Groves. The isomorphism problem for toral relatively hyperbolic groups. Publ. Math. Inst. Hautes Etudes Sci. No. 107 (2008), 211290.

[10] V. Diekert, A. Muscholl, Solvability of Equations in Graph Groups is Decidable, Internat. J. Algebra Comput. 16 (2006), pp. 1047-1069.

[11] V. Diekert, M. Lohrey, Existential and Positive Theories of Equations in Graph Products, Theory Comput. Syst. 37 (2004), pp. 133-156.

[12] D. Epstein and D. Holt. Computation in word-hyperbolic groups. Internat. J. Algebra Comput., 11(4):467–487, 2001.

[13] B. Farb. Relatively hyperbolic groups. Geom. Funct. Anal., 8(5):810–840, 1998.

[14] D. Gildenhuys, O. Kharlampovich, A. Myasnikov, CSA groups and separated free constructions, Bull. Austral. Math. Soc., Vol. 52 (1995), 63–84.

[15] D. Groves, Limit groups for relatively hyperbolic groups, I. The basic tools. Algebr. Geom. Topol., 9(3) (2009), 1423-1466.

[16] D. Groves. Limit groups for relatively hyperbolic groups. II. Makanin-Razborov diagrams. Geom. Topol., 9:2319–2358, 2005.
[17] D. Groves, H. Wilton, \textit{Conjugacy classes of solutions to equations and inequalities over hyperbolic groups}, J Topology (2010) 3 (2): 311-332.

[18] S. P. Humphries. Generators for the mapping class group. In Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), volume 722 of Lecture Notes in Math., pages 4447. Springer, Berlin, 1979.

[19] G.C. Hruska, \textit{Relative hyperbolicity and relative quasi-convexity for countable groups}, http://arxiv.org/pdf/0801.4596.pdf

[20] O.Kharlampovich, A. Myasnikov, \textit{Elementary theory of free non-abelian groups}. Journal of Algebra, 2006, Volume 302, Issue 2, p. 451-552.

[21] O. Kharlampovich, A. Myasnikov, \textit{Effective JSJ decompositions}. Group Theory: Algorithms, Languages, Logic, Contemp. Math., AMS (Borovik, editor), CONM/378, 87-212, 2005.

[22] O. Kharlampovich, J. Macdonald, \textit{Effective embedding of residually hyperbolic groups into direct products of extensions of centralizers}, Accepted to J. of Group Theory, February 2013. Preprint at arXiv:1202.3835v1.

[23] O. Kharlampovich and A. Myasnikov, \textit{Definable sets in a hyperbolic group}, IJAC, 2013, no 1, 91-110.

[24] O. Kharlampovich and A. Myasnikov, \textit{Quantifier elimination algorithm to boolean combination of $\exists\forall$-formulas in the theory of a free group}, arXiv:1207.1900.

[25] O. Kharlampovich, A. Myasnikov, P. Weil, \textit{Intersection of relatively quasi-convex subgroups in toral relatively hyperbolic groups}, preprint.

[26] O. Kharlampovich and A. Myasnikov. \textit{Implicit function theorem over free groups}, Journal of Algebra, vol 290/1, pp. 1–203, 2005.

[27] R.C.Lyndon, P.E.Schupp, \textit{Combinatorial group theory}. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York, 1977.

[28] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. Proc. Cambridge Philos. Soc., 60:769778, 1964.

[29] L. Louder, N. Touikan, Strong accessibility for finitely presented groups, arXiv:1302.5451.

[30] G. Makanin, \textit{Decidability of universal and positive theories of a free group}, (English translation) Mathematics of the USSR-Izvestiya 25 (1985), 7588; original: Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya 48 (1984), 735749.
[31] A. Myasnikov and V. Remeslennikov. *Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups.* Internat. J. Algebra Comput., 6(6):687–711, 1996.

[32] D. Osin, *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems,* Mem. Amer. Math. Soc. 179 (2006), no. 843.

[33] D. Osin. *Elementary subgroups of relatively hyperbolic groups and bounded generation.* Internat. J. Algebra Comput., 16(1):99–118, 2006.

[34] P. Papasoglu, An algorithm detecting hyperbolicity. Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), 193200, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 25, Amer. Math. Soc., Providence, RI, 1996.

[35] E. Rips and Z. Sela. *Canonical representatives and equations in hyperbolic groups.* Invent. Math., 120(3):489–512, 1995.

[36] Z. Sela. *Diophantine geometry over groups I: Makanin-Razborov diagrams,* Publications Mathematiques de l’IHES 93(2001), 31-105.

[37] Z. Sela. *Diophantine geometry over groups VIII: The elementary theory of a hyperbolic group,* Proceedings of the LMS, 99(2009), 217-273.

[38] Z. Sela. *Diophantine geometry over groups IV: an iterative procedure for validation of a sentence,* Israel Journal of Mathematics, 143 (2004), 1-130.

[39] Z. Sela. *Diophantine geometry over groups V: Quantifier elimination,* Israel Jour. of Math. 150(2005), 1-197.