LIOUVILLE TYPE THEOREMS FOR SINGULAR INTEGRAL EQUATIONS AND INTEGRAL SYSTEMS

XIAOHUI YU

The Center for China's Overseas Interests, Shenzhen University
Shenzhen Guangdong, 518060, P.R. China

(Communicated by Wenxiong Chen)

Abstract. In this paper, we establish some Liouville type theorems for positive solutions of some integral equations and integral systems in $\mathbb{R}^N$. The main technique we use is the method of moving planes in an integral form.

1. Introduction. In this paper, we study the nonexistence of positive solutions for the following integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{t} |x-y|^{N-\alpha}} f(u(y)) \, dy \text{ in } \mathbb{R}^N$$

and integral system

$$\begin{cases}
u(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{t} |x-y|^{N-\alpha}} f(u(y), v(y)) \, dy \text{ in } \mathbb{R}^N, \\
v(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{t} |x-y|^{N-\alpha}} g(u(y), v(y)) \, dy \text{ in } \mathbb{R}^N,
\end{cases}$$

where $N \geq 2$, $0 < \alpha < N$ and $0 \leq t < \alpha$. The solutions of (1) and (2) are closely related to the following partial differential equation and partial differential system in $\mathbb{R}^N$

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{1}{|y|^t} f(u(x)) \text{ in } \mathbb{R}^N$$

and

$$\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{1}{|y|^t} f(u(x), v(x)) \text{ in } \mathbb{R}^N, \\
(-\Delta)^{\frac{\alpha}{2}} v(x) = \frac{1}{|y|^t} g(u(x), v(x)) \text{ in } \mathbb{R}^N.
\end{cases}$$

In fact, every positive smooth solution of (1) or (2) multiplied by a constant satisfies (3) or (4) respectively. Here $(-\Delta)^{\frac{\alpha}{2}}$ is defined by

$$(-\Delta)^{\frac{\alpha}{2}} u = (|\xi|^\alpha u^\wedge)^\vee,$$

where $\wedge$ is the Fourier transformation and $\vee$ is its inverse.

In this paper, we are interested in the nonexistence results of (1) and (2). When $\alpha = 2$ and $t = 0$, there are a great number of results on the nonexistence results of (3) and (4), see [1, 3, 4, 14, 15, 16, 29]. The first result is [15], in which the
authors proved, among other things, that if \( f(u) = u^p \), then problem (3) has no positive \( C^2 \) solutions provided \( 0 < p < \frac{N+2}{N-2} \). This result is optimal in the sense that for any \( p \geq \frac{N+2}{N-2} \), there are infinitely many positive solutions to (3). Thus the Sobolev exponent \( \frac{N+2}{N-2} \) is the dividing exponent between existence and nonexistence of positive solutions. It is natural to ask whether similar results hold for system of

\[
\begin{align*}
-\Delta u &= v^p \quad \text{in } \mathbb{R}^N, \\
-\Delta v &= u^q \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

(5)

It has been conjectured, see for example [13], that the hyperbola

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, p > 0, q > 0
\]

is the dividing curve between existence and nonexistence for problem (5). This conjecture is supported by the results that there are no radial positive solutions to problem (3) provided that \( p,q \) satisfy \( \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} \), see Mitidieri [28] for \( p > 1, q > 1 \) and Serrin and Zou [30] for \( p > 0, q > 0 \). Moreover, it is proved in [32] that there are indeed infinitely many radial solutions provided that \( \frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N} \). This solves the conjecture in the sense of radial case. As for the non-radial case, this conjecture has not been solved completely, we refer the readers to [13, 28, 30, 31, 33]. We will not give further comments on these results. We note that all these results concerning the particular nonlinear term \( f(u) = u^p \). A natural question is whether problem (3) possesses a positive solution for general nonlinearity \( f(u) \). As for \( \alpha = 2 \), this question has been solved by L.Damascelli and F.Gladiali in [11]. They proved that problem (3) does not possess positive solutions when \( f \) is subcritical. The main tool they used is the method of moving planes. We note that \( f \) is only required to be continuous in this paper. So we can not conclude that the weak solutions of problem (3) are of \( C^2 \) class. Moreover, since \( f \) is not assumed to be Lipschitz continuous, the usual maximum principle does not work. In developing the method of moving plane, the authors in [11] used the technique based on integral inequalities, an idea originally due to S.Terracini’s work [34] and [35]. After the work of [11], Y.Guo and J.Liu obtained the Liouville type results of the elliptic system, i.e., problem (4) with \( \alpha = 2 \) and \( t = 0 \) in [17]. We note that all the results listed above are concerning the case \( \alpha = 2 \) and \( t = 0 \). In a recent paper [36], we established Liouville type theorems for integral equation (1) and integral system (2) with \( 0 < \alpha < N \) and \( t = 0 \). In this paper, we want to know whether similar results hold for \( 0 < \alpha < N \) and \( 0 \leq t < \alpha \). Up to now, there are a lot of works on this type of integral equations and integral systems. For example, in [23], the authors studied integral system (2) with \( f(u, v) = v^p, g(u, v) = u^q \). They proved that if \( \frac{N}{N-\alpha} < p, q \leq \frac{N+\alpha}{N-\alpha} \) but not both equal to \( \frac{N+\alpha}{N-\alpha} \), then \( u = v \equiv 0 \) under some integral conditions on \( u, v \). In the same spirit of this, W.Chen and C.Li studied the Lane-Emden conjecture in [5]. Recently, Y.Fang and W.Chen studied the Liouville type theorem for poly-harmonic equation in \( \mathbb{R}^N_+ \) in [12]. The main ingredient in these papers is the moving plane method based on the maximum principle of integral forms, which was first introduced by W.Chen, C.Li and B.Ou in [9] and [10] and then used widely in [6, 8, 18, 19, 20, 24, 25, 26, 27]. For more details, please see also [7] for a survey. At the same time, moving spheres based on the maximum principle of integral forms was also introduced by Y.Li in [21]. Inspired by these results, we study the Liouville type results for problems (1)
and (2), i.e., the general nonlinear problems. Our first result concerns the Liouville type result of problem (1). Our first result is the following

**Theorem 1.1.** Let \( u \in C^0(\mathbb{R}^N) \) be a positive solution of problem (1), where \( N \geq 2, 0 < \alpha < N, 0 \leq t < \alpha \) and \( f : [0, +\infty) \to \mathbb{R} \) is a continuous function with the properties

(i) \( f(s) \) is nondecreasing in \( (0, +\infty) \).

(ii) \( h(s) = \frac{f(s)}{s^{N-\alpha}} \) is nonincreasing in \( (0, +\infty) \) and \( h(s) \) is not a constant. Then \( u \equiv 0 \).

Next, we study the integral system (2). We first study a simpler

\[
\begin{align*}
\begin{cases}
    u(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^t |x-y|^{N-\alpha}} f(v(y)) \, dy & \text{in } \mathbb{R}^N, \\
    v(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^t |x-y|^{N-\alpha}} g(u(y)) \, dy & \text{in } \mathbb{R}^N,
\end{cases}
\end{align*}
\]

(7)

that is, we assume \( f \) depends only on \( v \) and \( g \) depends only on \( u \). Our second result is the following

**Theorem 1.2.** Let \( (u, v) \in C^0(\mathbb{R}^N) \times C^0(\mathbb{R}^N) \) be a positive solution of problem (7). Suppose that \( f, g : [0, +\infty) \to \mathbb{R} \) are continuous and satisfy

(i) \( f(s), g(s) \) are nondecreasing in \( (0, +\infty) \);

(ii) \( h(s) = \frac{f(s)}{s^{N-\alpha}}, k(s) = \frac{g(s)}{s^{N-\alpha}} \) are nonincreasing in \( (0, +\infty) \);

(iii) either \( h \) or \( k \) is not a constant.

Then \( (u, v) \equiv (0, 0) \).

**Theorem 1.2** can be extended to more general case, in which both \( f \) and \( g \) depend on \( (u, v) \). We have the following

**Theorem 1.3.** Let \( (u, v) \in C^0(\mathbb{R}^N) \times C^0(\mathbb{R}^N) \) be a positive solution of problem (2). Suppose that \( f, g : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) are continuous functions satisfying

(i) \( f(s, l), g(s, l) \) are nondecreasing in \( l \) for fixed \( s \) and \( f(s, l), g(s, l) \) are nondecreasing in \( s \) for fixed \( l \);

(ii) there exist \( p_1, q_1 \geq 0, p_1 + q_1 = \frac{N+\alpha-2t}{N-\alpha} \) such that \( h(s, l) = \frac{f(s, l)}{s^{q_1} l^{p_1}} \) is nonincreasing in \( l \) for fixed \( s \) and \( h(s, l) \) is nonincreasing in \( s \) for fixed \( l \);

(iii) there exist \( p_2, q_2 \geq 0, p_2 + q_2 = \frac{N+\alpha-2t}{N-\alpha} \) such that \( k(s, l) = \frac{g(s, l)}{s^{q_2} l^{p_2}} \) is nonincreasing in \( l \) for fixed \( s \) and \( k(s, l) \) is nonincreasing in \( s \) for fixed \( l \);

(iv) either \( h \) or \( k \) is not a constant.

Then \( (u, v) \equiv (0, 0) \).

Using the same technique we can consider the more general system

\[
\begin{align*}
\begin{cases}
    u(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} \left( \sum_{i=1}^{n} f_i(u(y), v(y)) \right) \, dy & \text{in } \mathbb{R}^N, \\
    v(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} \left( \sum_{i=1}^{n} g_i(u(y), v(y)) \right) \, dy & \text{in } \mathbb{R}^N.
\end{cases}
\end{align*}
\]

(8)

We have the similar result as Theorem 1.3.

**Theorem 1.4.** Let \( (u, v) \in C^0(\mathbb{R}^N) \times C^0(\mathbb{R}^N) \) be a positive solution of problem (8). Suppose that \( f = \sum_{i=1}^{n} f_i, g = \sum_{i=1}^{n} g_i, f_i, g_i : [0, +\infty) \times [0, +\infty) \to \mathbb{R}, i = 1, 2, \ldots, n \) are continuous functions satisfying
Corollary 1. Let \( g(s,l) \) be a positive solution of
\[
\int_{\mathbb{R}^N} \frac{u(y)^p}{|y|^{2N-2\alpha}} \, dy \text{ in } \mathbb{R}^N
\]
for some \( 0 < p < \frac{N+\alpha-2t}{N-\alpha} \), then \( u \equiv 0 \).

Corollary 2. Let \( (u,v) \in C^0(\mathbb{R}^N) \times C^0(\mathbb{R}^N) \) be a positive solution of
\[
\begin{aligned}
\int_{\mathbb{R}^N} \frac{u(y)^p}{|y|^{2N-2\alpha}} \, dy & \quad \text{ in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} \frac{v(y)^q}{|y|^{2N-2\alpha}} \, dy & \quad \text{ in } \mathbb{R}^N.
\end{aligned}
\]
for some \( 0 < p, q < \frac{N+\alpha-2t}{N-\alpha} \) and \( p, q \) not equal to \( \frac{N+\alpha-2t}{N-\alpha} \) at the same time, then \( u = v \equiv 0 \).

Corollary 3. Let \( (u,v) \in C^0(\mathbb{R}^N) \times C^0(\mathbb{R}^N) \) be a positive solution of
\[
\begin{aligned}
\int_{\mathbb{R}^N} \frac{u(y)^{p_1}v(y)^{q_1}}{|y|^{2N-2\alpha}} \, dy & \quad \text{ in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} \frac{u(y)^{p_2}v(y)^{q_2}}{|y|^{2N-2\alpha}} \, dy & \quad \text{ in } \mathbb{R}^N.
\end{aligned}
\]
for some \( 0 < p_1, p_2, q_1, q_2 < \frac{N+\alpha-2t}{N-\alpha} \), \( p_i + q_i < \frac{N+\alpha-2t}{N-\alpha} \), \( i = 1, 2 \) and \( p_1 + q_1, p_2 + q_2 \) not equal to \( \frac{N+\alpha-2t}{N-\alpha} \) at the same time, then \( u = v \equiv 0 \).

According to Theorem 1.1 to Theorem 1.4, we have the following corollaries.

This paper is organized as follows. We prove Theorem 1.1 and Theorem 1.2 in Section 2 and Section 3 respectively. Theorem 1.3 and Theorem 1.4 are proved in Section 4. In the following, we denote by \( C \) a positive constant which may vary from line to line.

2. Proof of Theorem 1.1. Let us study the positive solutions to the integral equation
\[
u(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{2N-2\alpha}} f(u(y)) \, dy \text{ in } \mathbb{R}^N.
\]
Since we don’t know the behaviors of \( u \) at infinity, we introduce the Kelvin’s transform of \( u \) as \( v = u(x \sqrt{\frac{x^2}{|x|^{N-\alpha}}}) \), then a direct calculation shows that \( v(x) \) solves
\[
v(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N-\alpha}} f(|y|^{N-\alpha} v(y)) \, dy \text{ in } \mathbb{R}^N.
\]
By the definition of \( h(s) = \frac{f(s)}{s^{N+\alpha-2t}} \), we deduce that \( v(x) \) satisfies
\[
v(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N-\alpha}} h(|y|^{N-\alpha} v(y)) v(y) \frac{N+\alpha-2t}{N-\alpha} \, dy.
\]
Since \( u \) is continuous in \( \mathbb{R}^N \), we conclude that \( v \) is continuous and strictly positive in \( \mathbb{R}^N \setminus \{0\} \) with possible singularity at the origin. Moreover, \( v \) decays at infinity as \( u(0)|x|^{\alpha-N} \). If we denote by \( \tau = \frac{N+\alpha}{N-\alpha} \), then we infer from the decay speed of \( v \) that
\[
v(x) \in L^{\tau+1} \cap L^{\infty}(\mathbb{R}^N \setminus B_r(0)) \quad (9)
\]
for all \( r > 0 \).
Now we use the moving plane method to prove our result. For a given real number \( \lambda > 0 \), define
\[
\Sigma_\lambda = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_1 \geq \lambda \}, \quad T_\lambda = \{ x \in \mathbb{R}^N | x_1 = \lambda \}
\]
and let \( x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n) \) and \( u_\lambda(x) = u(x^\lambda) \).

**Lemma 2.1.**
\[
v(x) - v_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^\lambda-y|^{N-\alpha}} \right) \frac{h(|y|^{N-\alpha} v(y)) v(y) \frac{N+\alpha-2t}{N-\alpha}}{|y|^t} \, dy
\]
\[
- \frac{h(|y^\lambda|^{N-\alpha} v(y^\lambda)) v(y^\lambda) \frac{N+\alpha-2t}{N-\alpha}}{|y^\lambda|^t} \, dy \quad (10)
\]

**Proof.** A direct calculation implies
\[
v(x) = \int_{\Sigma_\lambda} \frac{1}{|x-y|^{N-\alpha}} h(|y|^{N-\alpha} v(y)) v(y) \frac{N+\alpha-2t}{N-\alpha} \, dy
\]
\[
+ \int_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{N-\alpha}} h(|y|^{N-\alpha} v(y)) v(y) \frac{N+\alpha-2t}{N-\alpha} \, dy
\]
and
\[
v_\lambda(x) = \int_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{N-\alpha}} h(|y^\lambda|^{N-\alpha} v(y^\lambda)) v(y^\lambda) \frac{N+\alpha-2t}{N-\alpha} \, dy
\]
\[
+ \int_{\Sigma_\lambda} \frac{1}{|x^\lambda-y^\lambda|^{N-\alpha}} h(|y^\lambda|^{N-\alpha} v(y^\lambda)) v(y^\lambda) \frac{N+\alpha-2t}{N-\alpha} \, dy.
\]

Since \( |x-y| = |x^\lambda - y^\lambda| \) and \( |x^\lambda - y^\lambda| = |x^\lambda - y| \), then the assertion of this lemma holds easily.

**Lemma 2.2.** Under the assumptions of Theorem 1.1, there exists \( \lambda_0 > 0 \) such that for all \( \lambda \geq \lambda_0 \), we have \( v_\lambda(x) \geq v(x) \) for all \( x \in \Sigma_\lambda \).

**Proof.** First we note that for any \( \lambda > 0 \), we have \( |y| > |y^\lambda| \) for \( \forall y \in \Sigma_\lambda \). If we denote by \( \Sigma^*_\lambda = \{ y \in \Sigma_\lambda | v(y) > v_\lambda(y) \} \), then for any \( y \in \Sigma^*_\lambda \), we have \( |y|^{N-\alpha} v(y) > |y|^\lambda |N-\alpha| v(y^\lambda) \). We infer from the monotonicity of \( h(s) \)
\[
h(|y|^{N-\alpha} v(y)) \leq h(|y^\lambda|^{N-\alpha} v(y^\lambda)),
\]
and hence we have
\[
\frac{h(|y|^{N-\alpha} v(y)) v(y) \frac{N+\alpha-2t}{N-\alpha}}{|y|^t} \leq \frac{h(|y^\lambda|^{N-\alpha} v(y^\lambda)) v(y^\lambda) \frac{N+\alpha-2t}{N-\alpha}}{|y^\lambda|^t}.
\]
As for \( y \in \Sigma_\lambda \setminus \Sigma^0_\lambda \), we have
\[
\frac{h(|y|^{N-\alpha}v(y))v(y) \frac{N+\alpha-2t}{N-\alpha}}{|y|^t} = \frac{f(|y|^{N-\alpha}v(y))}{|y|^{N+\alpha-t}} \leq \frac{f(|y|^{N-\alpha}v_\lambda(y))}{|y|^t|y|^{N+\alpha-2t}} = \frac{f(|y|^{N-\alpha}v_\lambda(y))}{|y|^t|y|^{N+\alpha-2t}} v_\lambda(y) \frac{N+\alpha-2t}{N-\alpha} \\
\leq \frac{f(|y|^{N-\alpha}v_\lambda(y))}{|y|^t|y|^{N+\alpha-2t}} v_\lambda(y) \frac{N+\alpha-2t}{N-\alpha} = \frac{h(|y|^{N-\alpha}v(y))v(y) \frac{N+\alpha-2t}{N-\alpha}}{|y|^t}.
\]

Then by Lemma 2.1 we have
\[
v(x) - v_\lambda(x) \leq \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}} \right) h(|y|^{N-\alpha}v(y))[v(y) \frac{N+\alpha-2t}{N-\alpha} - v(y) \frac{N+\alpha-2t}{N-\alpha}]\,dy \\
\leq C \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}} \right) h(|y|^{N-\alpha}v(y))v(y)^{\alpha^*(t)-2}(v(y) - v(y^\lambda))\,dy,
\]

where \( \alpha^*(t) = \frac{2(N-t)}{N-\alpha} \). On the other hand, since \( |y|^{N-\alpha}v(y) \equiv u(|y|) \) is bounded for \( \lambda > 1 \) and \( y \in \Sigma_\lambda \), we conclude that \( h(|y|^{N-\alpha}v(y)) \) is bounded for \( y \in \Sigma_\lambda \). Hence we deduce that
\[
v(x) - v_\lambda(x) \leq C \int_{\Sigma_\lambda} \frac{1}{|x-y|^{N-\alpha}} v(y)^{\alpha^*(t)-2}(v(y) - v(y^\lambda))\,dy. \quad (11)
\]

By the Hardy-Littlewood-Sobolev inequality, see for example [22], it follows that for any \( q > \frac{N}{N-\alpha} \),
\[
\|v(x) - v(x^\lambda)\|_{L^q(\Sigma^-_\lambda)} \leq C(\int_{\Sigma_\lambda} v(y)^{\alpha^*(t)}\,dy)^{\frac{N-\alpha}{2N}} \|v(y) - v(y^\lambda)\|_{L^q(\Sigma^-_\lambda)}.
\]

Since \( v(x) \in L^{t+1}(\mathbb{R}_N \setminus B_r(0)) \), we can choose \( \lambda_0 \) large enough such that
\[
C(\int_{\Sigma_\lambda} v(y)^{t+1}\,dy)^{\frac{N-\alpha}{2N}} \leq \frac{1}{2} \text{ for all } \lambda \geq \lambda_0.
\]

Then we conclude \( \|v(x) - v(x^\lambda)\|_{L^q(\Sigma^-_\lambda)} = 0 \) for all \( \lambda \geq \lambda_0 \). Therefore \( \Sigma^-_\lambda \) must be measure 0 and hence empty. \( \square \)

We now move the plane \( x_1 = \lambda_0 \) to the left as long as \( v(x) \leq v_\lambda(x) \) holds for \( \forall x \in \Sigma_\lambda \). Suppose that this procedure stops at some \( \lambda = \lambda_1 \), then we have \( v(x) \leq v_\lambda(x) \). Moreover, we have the following result.

**Lemma 2.3.** \( \lambda_1 \leq 0 \).

**Proof.** We prove the conclusion by contradiction. We will prove that if \( \lambda_1 > 0 \), then the plane can be moved further to the left. More precisely, there exists an \( \epsilon > 0 \), such that \( v(x) \leq v_\lambda(x) \), \( x \in \Sigma_\lambda \) for all \( \lambda \in [\lambda_1 - \epsilon, \lambda_1] \). This contradicts the choice of \( \lambda_1 \).

Since \( v(x) \leq v_\lambda(x) \), then we can infer from Lemma 2.1 that \( v < v_\lambda \) in the interior of \( \Sigma_\lambda \). Let \( \overline{\Sigma_\lambda} = \{x \in \Sigma_\lambda, v(x) \geq v_\lambda(x)\} \), then obviously \( \overline{\Sigma_\lambda} \) has
measure 0 and \( \lim_{\lambda \to \lambda_1} \Sigma_{\lambda} = \Sigma_{\lambda_1} \). We infer from equation (11) that
\[
\|v(x) - v(x^\lambda)\|_{L^q(\Sigma^{-}_\lambda)} \leq C(\int_{\Sigma^{-}_\lambda} v(y)^{q \epsilon} \, dy)^{\frac{q-1}{q}} \|v(y) - v(y^\lambda)\|_{L^q(\Sigma^{-}_\lambda)}.
\]
Condition (9) ensures that one can choose \( \varepsilon > 0 \) sufficiently small so that for all \( \lambda \in [\lambda_1 - \varepsilon, \lambda_1] \),
\[
C(\int_{\Sigma^{-}_\lambda} v(y)^{q \epsilon} \, dy)^{\frac{q-1}{q}} \leq \frac{1}{2}.
\]
Now by (12), we have \( \|v(x) - v(x^\lambda)\|_{L^q(\Sigma^{-}_\lambda)} = 0 \) and hence \( \Sigma^{-}_\lambda \) must be empty. \( \square \)

**Proof of Theorem 1.1.** Let \( v \) be the Kelvin’s transformation of \( u \) at point \( p \in \mathbb{R}^N \), we first prove that \( v \) is radially symmetric. We use the method of moving plane and prove the symmetry in every direction. Without loss of generality, we choose the \( x^1 \) direction and prove that \( v \) is symmetric in the \( x^1 \) direction. We can carry out the procedure as the above and conclude that \( \lambda_1 \leq 0 \). Then we conclude by continuity that \( v(x) \leq v_0(x) \) for all \( x \in \Sigma_0 \). We can also perform the moving plane procedure from the left and find a corresponding \( \lambda'_1 \geq 0 \). Then we get \( v_0(x) \leq v(x) \) for \( x \in \Sigma_0 \). This fact and the above inequality imply that \( v(x) \) is symmetric with respect to \( T_0 \). We perform the above procedure in every direction, then \( v \) and hence \( u \) are radially symmetric around the point of the Kevin’s transform. Since \( p \) is arbitrary, we conclude that \( u \) is constant. Finally, we infer from equation (1) that \( u \equiv 0 \). \( \square \)

3. **Proof of Theorem 1.2.** The spirit of the proof of Theorem 1.2 is similar to the proof of Theorem 1.1. We denote by \( \Sigma_\lambda, T_\lambda, x^\lambda \) and \( u_\lambda(x) \) as above. We consider the Kelvin’s transform \( w, z \) of \( u, v \) defined by
\[
w(x) = \frac{1}{|x|^{N-\alpha}} u\left(\frac{x}{|x|^{2}}\right), \quad z(x) = \frac{1}{|x|^{N-\alpha}} v\left(\frac{x}{|x|^{2}}\right).
\]
Then a direct calculation implies that \( w, z \) satisfy
\[
\begin{align*}
&\left\{ \begin{array}{l}
w(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} f(|y|^{N-\alpha} z(y)) \frac{1}{|y|^{N+\alpha-t}} \, dy, \\
z(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} g(|y|^{N-\alpha} w(y)) \frac{1}{|y|^{N+\alpha-t}} \, dy.
\end{array} \right.
\end{align*}
\]
By the definitions of \( h(s) = \frac{f(s)}{s^{N-\alpha}} \), \( k(s) = \frac{g(s)}{s^{N-\alpha}} \), we have
\[
\begin{align*}
&\left\{ \begin{array}{l}
w(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N-\alpha}} h(|y|^{N-\alpha} z(y)) z(y) \frac{y^{N+2\tau}}{s^{N-\alpha}} \, dy, \\
z(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N-\alpha}} k(|y|^{N-\alpha} w(y)) w(y) \frac{y^{N+2\tau}}{s^{N-\alpha}} \, dy.
\end{array} \right.
\end{align*}
\]
It is easy to see that \( w, z \) are continuous and strictly positive in \( \mathbb{R}^N \setminus \{0\} \) with possible singularity at the origin. Moreover, they decay at infinity as \( u(0) \frac{1}{|x|^{N-\alpha}} \) and \( v(0) \frac{1}{|x|^{N-\alpha}} \) respectively, so we have \( w, z \in L^{\tau+1} \cap L^\infty(\mathbb{R}^N \setminus B_r(0)) \) for any \( r > 0 \) with \( \tau = \frac{N+2\alpha}{N-\alpha} \). In the spirit of Lemma 2.1, we have the following result.
Lemma 3.1.

\[
w(x) - w_{\lambda}(x) = \int_{\Sigma_{\lambda}} \left( \frac{1}{|x - y|^{N-\alpha}} - \frac{1}{|x_{\lambda} - y|^{N-\alpha}} \right) \left[ h(|y|^{N-\beta}z(y))z(y) \right] \left( \frac{N+\alpha-2t}{|y|^{t}} \right) dy,
\]

\[
z(x) - z_{\lambda}(x) = \int_{\Sigma_{\lambda}} \left( \frac{1}{|x - y|^{N-\alpha}} - \frac{1}{|x_{\lambda} - y|^{N-\alpha}} \right) \left[ k(|y|^{N-\beta}w(y))w(y) \right] \left( \frac{N+\alpha-2t}{|y|^{t}} \right) dy.
\]

Proof. The proof of this lemma is similar to the proof of Lemma 2.1, we omit it. \(\square\)

Lemma 3.2. Under the assumptions of Theorem 1.2, there exists \(\lambda_0 > 0\) such that for all \(\lambda \leq \lambda_0\), we have \(w_{\lambda}(x) \geq w(x)\) and \(z_{\lambda}(x) \geq z(x)\) for all \(x \in \Sigma_{\lambda}\).

Proof. Since \(\lambda > 0\), it follows that \(|y| > |y'|\) for all \(y \in \Sigma_{\lambda}\). If we denote \(\Sigma_{\lambda}^w = \{y \in \Sigma_{\lambda}|w(y) > w_{\lambda}(y)\}\) and \(\Sigma_{\lambda}^z = \{y \in \Sigma_{\lambda}|z(y) > z_{\lambda}(y)\}\), then for any \(y \in \Sigma_{\lambda}^w\), we have

\[
k(|y|^{N-\beta}w(y)) \leq k(|y|^{N-\beta}w(y'))
\]

for all \(y \in \Sigma_{\lambda} \setminus \Sigma_{\lambda}^w\), we have

\[
k(|y|^{N-\beta}w(y))w(y) \frac{N+\alpha-2t}{|y|^{t}} = g(|y|^{N-\beta}w(y)) \frac{N+\alpha-t}{|y|^{t}} \leq g(|y|^{N-\beta}w_{\lambda}(y)) \frac{N+\alpha-2t}{|y|^{t}} \leq g(|y|^{N-\beta}w_{\lambda}(y)) \frac{N+\alpha-2t}{|y|^{t}} w_{\lambda}(y) \frac{N+\alpha-2t}{|y|^{t}} \leq k(|y|^{N-\beta}w_{\lambda}(y))w(y) \frac{N+\alpha-2t}{|y|^{t}} \frac{N+\alpha-2t}{|y|^{t}}.
\]

Hence we have

\[
z(x) - z_{\lambda}(x) \leq C \int_{\Sigma_{\lambda}} \frac{1}{|x - y|^{N-\alpha}} w(y)^{a^*(t) - 2} (w(y) - w(y')) dy
\]

by the same process of proving Lemma 2.2. Similarly we have

\[
w(x) - w_{\lambda}(x) \leq C \int_{\Sigma_{\lambda}} \frac{1}{|x - y|^{N-\alpha}} z(y)^{a^*(t) - 2} (z(y) - z(y')) dy.
\]

We infer from the Hardy-Littlewood-Sobolev inequality that, for any \(q > \frac{N}{N-\alpha}\),

\[
\|z(x) - z(x')\|_{L^1(\Sigma_{\lambda})} \leq C(\int_{\Sigma_{\lambda}} w(y)^{q+1} dy)^{\frac{N-\alpha}{N}} \|w(y) - w(y')\|_{L^q(\Sigma_{\lambda})}.
\]
and
\[ \|w(x) - w(x^\lambda)\|_{L^r(\Sigma_\lambda^w)} \leq C(\int_{\Sigma_\lambda} z(y)^{r+1} dy)^{\frac{1}{r+1}} \|z(y) - z(y^\lambda)\|_{L^s(\Sigma_\lambda^w)}. \]

The above two equations imply
\[ \|z(x) - z(x^\lambda)\|_{L^r(\Sigma_\lambda^w)} \leq C(\int_{\Sigma_\lambda} w(y)^{r+1} dy)^{\frac{1}{r+1}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma_\lambda^w)} \]
and
\[ \|w(x) - w(x^\lambda)\|_{L^s(\Sigma_\lambda^w)} \leq C(\int_{\Sigma_\lambda} w(y)^{r+1} dy)^{\frac{1}{r+1}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma_\lambda^w)}. \]

Since \( w(x), z(x) \in L^{r+1}(\mathbb{R}^N \setminus B_r(0)) \) for any \( r > 0 \), we can choose \( \lambda_0 \) large enough such that \( C(\int_{\Sigma_\lambda} w(y)^{r+1} dy)^{\frac{1}{r+1}} \leq \frac{1}{2} \) for all \( \lambda \geq \lambda_0 \). Then we conclude \( \|z(y) - z(y^\lambda)\|_{L^s(\Sigma_\lambda^w)} = \|w(y) - w(y^\lambda)\|_{L^s(\Sigma_\lambda^w)} = 0 \) for all \( \lambda \geq \lambda_0 \). Therefore both \( \Sigma_\lambda^w \) and \( \Sigma_\lambda^z \) must be measure 0 and hence empty.

We now move the plane \( x_1 = \lambda_0 \) to the left as long as \( w(x) \leq w_\lambda(x) \) and \( z(x) \leq z_\lambda(x) \) hold for \( \forall x \in \Sigma_\lambda \). Suppose that this procedure stops at some \( \lambda = \lambda_1 \), then we have \( w(x) \leq w_{\lambda_1}(x) \) and \( z(x) \leq z_{\lambda_1}(x) \) for \( \forall x \in \Sigma_{\lambda_1} \). Moreover, we have the following result.

**Lemma 3.3.** \( \lambda_1 \leq 0 \).

**Proof.** We prove the conclusion by contradiction. We will prove that if \( \lambda_1 > 0 \), then the plane can be moved further to the left. More precisely, there exists an \( \varepsilon > 0 \), such that \( w(x) \leq w_{\lambda_1}(x) \) and \( z(x) \leq z_{\lambda_1}(x) \), then we can infer from Lemma 3.1 that \( z < z_{\lambda_1} \) and \( w < w_{\lambda_1} \) in the interior of \( \Sigma_{\lambda_1} \). Let \( \Sigma_{\lambda_1}^w = \{ x \in \Sigma_{\lambda_1} \mid w(x) \geq w_{\lambda_1}(x) \} \) and \( \Sigma_{\lambda_1}^z = \{ x \in \Sigma_{\lambda_1} \mid z(x) \geq z_{\lambda_1}(x) \} \), then obviously both \( \Sigma_{\lambda_1}^w \) and \( \Sigma_{\lambda_1}^z \) have measure 0 and \( \lim_{\lambda \to \lambda_1} \Sigma_{\lambda_1}^w = \Sigma_{\lambda_1}^w \lim_{\lambda \to \lambda_1} \Sigma_{\lambda_1}^z = \Sigma_{\lambda_1}^z \). We infer from equations (13) and (14) that, for any \( q > \frac{N}{N-r} \),
\[ \|w(x) - w(x^\lambda)\|_{L^q(\Sigma_\lambda^w)} \leq C(\int_{\Sigma_\lambda^w} z(y)^{r+1} dy)^{\frac{1}{r+1}} \|z(y) - z(y^\lambda)\|_{L^s(\Sigma_\lambda^w)} \] (15)
and
\[ \|z(x) - z(x^\lambda)\|_{L^q(\Sigma_\lambda^z)} \leq C(\int_{\Sigma_\lambda^z} w(y)^{r+1} dy)^{\frac{1}{r+1}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma_\lambda^z)}. \] (16)
That is
\[ \|w(x) - w(x^\lambda)\|_{L^q(\Sigma_\lambda^w)} \leq C(\int_{\Sigma_\lambda^w} z(y)^{r+1} dy)^{\frac{1}{r+1}} \cdot \left( \int_{\Sigma_\lambda^w} w(y)^{r+1} dy \right)^{\frac{1}{r+1}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma_\lambda^w)}. \]
By the definitions of $h$ continuous and strictly positive in $\Sigma^t_\lambda$ defined by the moving plane method, but the calculation is more complicated. We only prove for problem (2), i.e., Theorem 1.3 and Theorem 1.4. The spirit of the proofs is still the $x$ and prove the symmetry in every direction. Without loss of generality, we choose let $\lambda u$ and prove that $w,z$ are radially symmetric. We still use the method of moving plane procedure from the left and find a corresponding $\lambda x$ that is, $w(z) = 0$ and hence $\Sigma^{w}_\lambda, \Sigma^{z}_\lambda$ must be empty.

Proof of Theorem 1.2. Let $w, z$ be the Kelvin’s transformation of $u, v$ at $p$, we first prove that $w, z$ are radially symmetric. We still use the method of moving plane and prove the symmetry in every direction. Without loss of generality, we choose the $x^1$ direction and prove that $w, z$ are symmetric in the $x^1$ direction. We can carry out the procedure as the above and conclude that $\lambda_1 \leq 0$, then we conclude by continuity that $w(x) \leq w_0(x)$ and $z(x) \leq z_0(x)$ for all $x \in \Sigma_0$. We can also perform the moving plane procedure from the left and find a corresponding $\lambda_1^* \geq 0$, that is, $w_0(x) \leq w^*_0(x)$ and $z_0(x) \leq z^*_0(x)$ for $x \in \Sigma_0$. This fact and the above inequality imply that $w(x), z(x)$ are symmetric with respect to $T_0$. We perform the above procedure in all directions and for all $p \in \mathbb{R}^N$, then $w, z$ and hence $u, v$ are radially symmetric around the point of the Kevin’s transform. Since $p$ is arbitrary, we conclude that $u, v$ are constants. Since either $h$ or $k$ is not a constant, it follows that $u = v \equiv 0$.

4. Proofs of Theorem 1.3 and Theorem 1.4. Now we prove the general results for problem (2), i.e., Theorem 1.3 and Theorem 1.4. The spirit of the proofs is still the moving plane method, but the calculation is more complicated. We only prove Theorem 1.3, the proof of Theorem 4 is similar and we omit it. We still denote by $\Sigma_\lambda, T_\lambda, \alpha_\lambda$ and $\alpha_\lambda(x)$ as above. We also consider the Kelvin’s transform $w, z$ of $u, v$ defined by

$$ w(x) = \frac{1}{|x|^{N-\alpha}} u\left(\frac{x}{|x|^2}\right), \quad z(x) = \frac{1}{|x|^{N-\alpha}} v\left(\frac{x}{|x|^2}\right). $$

Since $(u, v)$ solves (2), then a direct calculation implies that $w, z$ satisfy

$$ \begin{cases} 
  w(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} f(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) \frac{1}{|y|^{N+\alpha-\delta}} dy, \\
  z(x) = \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} g(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) \frac{1}{|y|^{N+\alpha-\delta}} dy.
\end{cases} $$

By the definitions of $h(s, l) = \frac{f(s, l)}{s^{\alpha}}$, $k(s, l) = \frac{g(s, l)}{s^{\alpha}}$, we have

$$ \begin{cases} 
  w(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N-\alpha}} h(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) w(y)^{p_1} z(y)^{q_1} dy, \\
  z(x) = \int_{\mathbb{R}^N} \frac{1}{|y|^{N-\alpha}} k(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) w(y)^{p_2} z(y)^{q_2} dy,
\end{cases} $$

where $p_1 + q_1 = p_2 + q_2 = \frac{N+\alpha-2\delta}{N-\alpha}$. For the same reason we have that $w, z$ are continuous and strictly positive in $\mathbb{R}^N \setminus \{0\}$ with possible singularity at the origin. Moreover, they decay at infinity as $u(0) \frac{1}{|x|^{N-\alpha}}$ and $v(0) \frac{1}{|x|^{N-\alpha}}$ respectively, so we
have $w, z \in L^{r+1} \cap L^\infty(\mathbb{R}^N \setminus B_r(0))$ for any $r > 0$ with $\tau = \frac{N+\alpha}{N-\alpha}$. In the spirit of Lemma 2.1 and Lemma 3.1, we have the following result.

**Lemma 4.1.**

$$w(x) - w_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}} \right) \left[ f(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) - h(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) \right] \frac{1}{|y|^{N+\alpha-\tau}} \, dy,$$

$$z(x) - z_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}} \right) \left[ g(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) - k(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) \right] \frac{1}{|y|^{N+\alpha-\tau}} \, dy.$$

By the definitions of $h(s, l)$ and $k(s, l)$, the above two equations can be written as

$$w(x) - w_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}} \right) \left[ h(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) w(y)^{p_1} z(y)^{q_1} \right] \frac{1}{|y|^{\alpha}} \, dy,$$

$$z(x) - z_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}} \right) \left[ k(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) w(y)^{p_2} z(y)^{q_2} \right] \frac{1}{|y|^{\alpha}} \, dy.$$

**Proof.** The proof is a direct calculation, see the proofs of Lemma 2.1 and Lemma 3.1.

**Lemma 4.2.** Under the assumptions of Theorem 1.3, there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, we have $w_\lambda(x) \geq w(x)$ and $z_\lambda(x) \geq z(x)$ for all $x \in \Sigma_\lambda$.

**Proof.** Since $\lambda > 0$, it follows that $|y| > |y|^{\lambda}$ for all $y \in \Sigma_\lambda$. We still denote by $\Sigma^w_\lambda = \{ y \in \Sigma_\lambda | w(y) > w_\lambda(y) \}$ and $\Sigma^z_\lambda = \{ y \in \Sigma_\lambda | z(y) > z_\lambda(y) \}$. In the spirit of Lemma 2.2 and Lemma 3.2, we need to estimate $\|w(x) - w(x^{\lambda})\|_{L^p(\Sigma^w_\lambda)}$ and $\|z(x) - z(x^{\lambda})\|_{L^p(\Sigma^z_\lambda)}$. We first estimate $\|w(x) - w(x^{\lambda})\|_{L^p(\Sigma^w_\lambda)}$.

(i) If $w(y) \geq w_\lambda(y)$ and $z(y) \leq z(y^{\lambda})$, then by the assumptions in Theorem 1.3, we have

$$f(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y)) \geq f(|y|^{N-\alpha} w(y^{\lambda}), |y|^{N-\alpha} z(y^{\lambda})) \frac{w_\lambda(y)}{w(y)}.$$

Since

$$\frac{f(|y|^{N-\alpha} w(y), |y|^{N-\alpha} z(y))}{|y|^{N-\alpha} w(y)} \frac{w_\lambda(y)}{w(y)} \geq \frac{f(|y|^{N-\alpha} w(y^{\lambda}), |y|^{N-\alpha} z(y^{\lambda}))}{|y|^{N-\alpha} w(y^{\lambda})} \frac{w_\lambda(y)}{w(y)} \frac{w(y)}{w(y^{\lambda})}.$$
we deduce that
\[
|y|^{N+\alpha-t} f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y)) \\
\geq |y|^{N+\alpha-2t} \frac{w(y)}{w(y)} \frac{z^\alpha}{N-\alpha},
\]
which further implies that
\[
\frac{1}{|y|^{N+\alpha-t}} f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y)) - \frac{1}{|y|^{N+\alpha-t}} f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))
\leq \frac{1}{|y|^{N+\alpha-t}} f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))[1 - \left(\frac{w(y)}{w(y)}\right)^{N+\alpha-2t}] \frac{N+\alpha-2t}{N-\alpha} (1 - \frac{w(y)}{w(y)})
\leq \frac{C}{|y|^{N+\alpha-t}} f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))(w(y) - w(y^\alpha))
\leq \frac{1}{|y|^t} \frac{C}{|y|^{2(\alpha-t)}} (w(y) - w(y^\alpha)).
\]

(ii) If \(w(y) \geq w_\alpha(y)\) and \(z(y) > z(y^\alpha)\), then we have
\[
\frac{f(|y|^{-\alpha}w(y^\alpha), |y|^{-\alpha}z(y^\alpha))}{|y|^{N+\alpha-2t}} \frac{w(y^\alpha)}{w(y^\alpha)^\alpha q_1} \geq \frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))}{|y|^{N+\alpha-2t}} \frac{w(y)}{w(y)^\alpha q_1},
\]
that is
\[
\frac{f(|y|^{-\alpha}w(y^\alpha), |y|^{-\alpha}z(y^\alpha))}{|y|^{N+\alpha-t}} \geq \frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))}{|y|^{N+\alpha-t}} \frac{w(y)}{w(y)^\alpha q_1} (\frac{z(y^\alpha)}{z(y)})^q_1.
\]
So we have
\[
\frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))}{|y|^{N+\alpha-t}} - \frac{f(|y|^{-\alpha}w(y^\alpha), |y|^{-\alpha}z(y^\alpha))}{|y|^{N+\alpha-t}}
\leq \frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))}{|y|^{N+\alpha-t}} [1 - \left(\frac{w(y^\alpha)}{w(y)}\right)^\alpha q_1 (\frac{z(y^\alpha)}{z(y)})^q_1]
\leq \frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))}{|y|^{N+\alpha-t}} [1 - \left(\frac{w(y^\alpha)}{w(y)}\right)^{N+\alpha-2t} (\frac{z(y^\alpha)}{z(y)})^{N+\alpha-2t}]\frac{N+\alpha-2t}{N-\alpha} \left[1 - \frac{w(y^\alpha)}{w(y)} + (1 - \frac{z(y^\alpha)}{z(y)})\right]
\leq \frac{N+\alpha-2t}{N-\alpha} \frac{1}{|y|^{N+\alpha-t}} \frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y))}{w(y)} (w(y) - w(y^\alpha))
\leq \frac{f(|y|^{-\alpha}w(y), |y|^{-\alpha}z(y)) (z(y) - z(y^\alpha))}{z(y)}
\leq \frac{1}{|y|^t} \frac{C}{|y|^{2(\alpha-t)}} [(w(y) - w(y^\alpha)) + (z(y) - z(y^\alpha))]
(iii) If \( w(y) < w_\lambda(y) \) and \( z(y) \geq z(y^\lambda) \), then we can change the roles of \( w, z \) in case (i) and obtain

\[
\frac{1}{|y|^{N+\alpha-t}} f(|y|^{-\alpha} w(y), |y|^{-\alpha} z(y)) - \frac{1}{|y|^{\lambda N+\alpha-t}} f(|y^\lambda|^{-\alpha} w(y^\lambda), |y^\lambda|^{-\alpha} z(y^\lambda)) \\
\leq \frac{1}{|y|^t |y|^{2(\alpha-t)}} (z(y) - z(y^\lambda)).
\]

(iv) Finally, if \( w(y) < w_\lambda(y) \) and \( z(y) < z(y^\lambda) \), then

\[
f(|y|^{-\alpha} w(y), |y|^{-\alpha} z(y)) \\
\leq f(|y|^{-\alpha} w(y^\lambda), |y|^{-\alpha} z(y^\lambda)) \\
= \frac{1}{|y|^t |y|^{N+\alpha-t}} f(|y|^{-\alpha} w(y^\lambda), |y|^{-\alpha} z(y^\lambda)) w(y^\lambda)^{p_1} z(y^\lambda)^{p_2} \\
\leq \frac{1}{|y^\lambda|^t |y^\lambda|^{N+\alpha-t}} f(|y^\lambda|^{-\alpha} w(y^\lambda), |y^\lambda|^{-\alpha} z(y^\lambda)) w(y^\lambda)^{p_1} z(y^\lambda)^{p_2} \\
\leq f(|y^\lambda|^{-\alpha} w(y^\lambda), |y^\lambda|^{-\alpha} z(y^\lambda)).
\]

Hence for every \( x \in \Sigma^\alpha_\lambda \), we deduce from equations (19)(21)(22)(23) that

\[
w(x) - w_\lambda(x) \\
\leq \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^t |y-x|^{N-\alpha}} |y|^{2(\alpha-t)} \left[ (w(y) - w(y^\lambda)) + (z(y) - z(y^\lambda)) \right] dy \\
\leq \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^t |y-x|^{N-\alpha}} |y|^{2(\alpha-t)} (w(y) - w(y^\lambda)) dy \\
+ \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^t |y-x|^{N-\alpha}} |y|^{2(\alpha-t)} (z(y) - z(y^\lambda)) dy.\]

Similarly, we have

\[
z(x) - z_\lambda(x) \\
\leq \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^t |y-x|^{N-\alpha}} |y|^{2(\alpha-t)} \left[ (w(y) - w(y^\lambda)) + (z(y) - z(y^\lambda)) \right] dy \\
\leq \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^t |y-x|^{N-\alpha}} |y|^{2(\alpha-t)} (w(y) - w(y^\lambda)) dy \\
+ \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^t |y-x|^{N-\alpha}} |y|^{2(\alpha-t)} (z(y) - z(y^\lambda)) dy.\]

It follows from the Hardy-Littlewood-Sobolev inequality that, for any \( q > \frac{N}{N-\alpha} \),

\[
\|w(x) - w(x^\lambda)\|_{L^q(\Sigma^\alpha_\lambda)} \\
\leq C \left( \int_{\Sigma^\alpha_\lambda} \frac{1}{|y|^{2N}} dy \right)^{\alpha-t} \|w(y) - w(y^\lambda)\|_{L^q(\Sigma^\alpha_\lambda)} + \|z(x) - z(x^\lambda)\|_{L^q(\Sigma^\alpha_\lambda)}
\]
and
\[
\|z(x) - z(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} \\
\leq C\left(\int_{\Sigma_{x^\lambda}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} + \|z(x) - z(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})}.
\]

Choose \(\lambda_0\) large enough, such that \(C\left(\int_{\Sigma_{x^\lambda}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \leq \frac{1}{4}\) for all \(\lambda \geq \lambda_0\), then we deduce that
\[
\|w(y) - w(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} = \|z(x) - z(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} = 0.
\]
That is, \(\Sigma^w_{x^\lambda}\) and \(\Sigma^\lambda_{x^\lambda}\) have measure 0 and hence empty.

We now move the plane \(x_1 = \lambda_0\) to the left as long as \(w(x) \leq w_\lambda(x)\) and \(z(x) \leq z_\lambda(x)\) hold for all \(\lambda \in [\lambda_1 - \varepsilon, \lambda_1]\). Suppose that this procedure stops at some \(\lambda = \lambda_1\), then we have \(w(x) \leq w_{\lambda_1}(x)\) and \(z(x) \leq z_{\lambda_1}(x)\). Moreover, we have the following lemma.

**Lemma 4.3.** \(\lambda_1 \leq 0\).

**Proof.** We prove the conclusion by contradiction. We will prove that if \(\lambda_1 > 0\), then the plane can be moved further to the left. More precisely, there exists \(\varepsilon > 0\), such that \(w(x) \leq w_{\lambda_1}(x)\) and \(z(x) \leq z_{\lambda_1}(x)\), \(x \in \Sigma_{\lambda_1}\) for all \(\lambda \in [\lambda_1 - \varepsilon, \lambda_1]\). Then this contradicts the choice of \(\lambda_1\).

Since \(w(x) \leq w_{\lambda_1}(x)\) and \(z(x) \leq z_{\lambda_1}(x)\), then we can infer from Lemma 4.1 that \(z < z_{\lambda_1}\) and \(w < w_{\lambda_1}\) in the interior of \(\Sigma_{\lambda_1}\). Let \(\Sigma^w_{\lambda_1} = \{x \in \Sigma_{\lambda_1} | w(x) \geq w_{\lambda_1}(x)\}\) and \(\Sigma^z_{\lambda_1} = \{x \in \Sigma_{\lambda_1} | z(x) \geq z_{\lambda_1}(x)\}\), then obviously both \(\Sigma^w_{\lambda_1}\) and \(\Sigma^z_{\lambda_1}\) have measure 0 and \(\lim_{\lambda \to \lambda_1} \Sigma^w_{\lambda} = \Sigma^w_{\lambda_1}, \lim_{\lambda \to \lambda_1} \Sigma^z_{\lambda} = \Sigma^z_{\lambda_1}\). We infer from equations (24) and (25) that, for any \(q > \frac{\alpha}{N-\alpha},\)
\[
\|w(x) - w(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} \leq C\left(\int_{\Sigma^w_{\lambda_1}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \|z(y) - z(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} \\
+ \left(\int_{\Sigma^z_{\lambda_1}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})}. (26)
\]

Similarly, we have
\[
\|z(x) - z(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} \leq C\left(\int_{\Sigma^w_{\lambda_1}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \|z(y) - z(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} \\
+ \left(\int_{\Sigma^z_{\lambda_1}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \|w(y) - w(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})}. (27)
\]

We infer from equations (26) and (27) that
\[
\|w(x) - w(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} + \|z(x) - z(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} \\
\leq C\left(\int_{\Sigma^w_{\lambda_1} \cup \Sigma^z_{\lambda_1}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \|z(y) - z(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} + \|w(y) - w(y^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})}. (28)
\]

Choose \(\varepsilon > 0\) small enough, such that for all \(\lambda \in [\lambda_1 - \varepsilon, \lambda_1]\),
\[
C\left(\int_{\Sigma^w_{\lambda_1} \cup \Sigma^z_{\lambda_1}} \frac{1}{|y|^{2N}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \leq \frac{1}{2}.
\]
Now by equation (28), we have \(\|w(x) - w(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} = \|z(x) - z(x^\lambda)\|_{L^s(\Sigma^\lambda_{x^\lambda})} = 0\) and hence \(\Sigma^w_{x^\lambda}, \Sigma^z_{x^\lambda}\) must be empty.
Proof of Theorem 1.3. Let \( w, z \) be the Kelvin’s transformation of \( u, v \) at \( p \), we first prove that \( w, z \) are radially symmetric. We still use the method of moving plane and prove the symmetry in every direction. Without loss of generality, we choose the \( x_1 \) direction and prove that \( w, z \) are symmetric in the \( x_1 \) direction. We can carry out the procedure as the above and conclude that \( \lambda_1 \leq 0 \), then we conclude by continuity that \( w(x) \leq w_0(x) \) and \( z(x) \leq z_0(x) \) for all \( x \in \Sigma_0 \). We can also perform the moving plane procedure from the left and find a corresponding \( \lambda_1' \geq 0 \), and then we get \( w_0(x) \leq w(x) \) and \( z_0(x) \leq z(x) \) for \( x \in \Sigma_0 \). This fact and the above inequality imply that \( w(x), z(x) \) are symmetric with respect to \( T_0 \), we perform the above procedure in every direction, then it follows that \( w, z \) and hence \( u, v \) are radially symmetric around the point of the Kelvin’s transform. Since \( p \) is arbitrary, we conclude that \( u, v \) are constants. By the assumption that either \( h(t) \) or \( k(t) \) is not a constant, we deduce that \( u = v \equiv 0 \). \[\square\]

Acknowledgements. The author would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] H. Berestycki, L. A. Caffarelli and L. Nireberg, Further qualitative properties for elliptic equations in unbounded domains, Ann. Norm. Sup. Pisa. Cl. Sci., 25 (1997), 69–94.
[2] G. Bianchi, Nonexistence of positive solutions to semilinear elliptic equations on \( R^N \) or \( R_+^N \) through the method of moving planes, Comm. P.D.E., 22 (1997), 1671–1690.
[3] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, Comm. Pure App. Math., XLII (1989), 271–297.
[4] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J., 63 (1991), 615–622.
[5] W. Chen and C. Li, An integral system and the Lane-Emden conjecture, Disc. Cont. Dyn. Sys., 24 (2009), 1167–1184.
[6] W. Chen and C. Li, Classification of positive solutions for nonlinear differential and integral systems with critical exponents, Acta Mathematica Scientia, 29 (2009), 949–960.
[7] W. Chen and C. Li, Methods on Nonlinear Elliptic Equations, AIMS Book Series, vol. 4, 2010.
[8] W. Chen and C. Li, Radial symmetry of solutions for some integral systems of Wolff type, Discrete Contin. Dyn. Syst., 30 (2011), 1083–1093.
[9] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure and Appl. Math., LIX (2006), 330–343.
[10] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, Comm. P.D.E., 30 (2005), 59–65.
[11] L. Damascelli and F. Gladiali, Some nonexistence results for positive solutions of elliptic equations in unbounded domains, Rev. Mat. Iberoamericana, 20 (2004), 67–86.
[12] Y. Fang and W. Chen, A Liouville type theorem for poly-harmonic Dirichlet problems in a half space, Advances in Mathematics, 229 (2012), 2835–2867.
[13] D. G. De Figueiredo and P. L. Felmer, A liouville type theorem for elliptic systems, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 21 (1994), 387–397.
[14] D. G. De Figueiredo and B. Sirakov, Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems, Math. Ann., 333 (2005), 231–260.
[15] B. Gidas and J. Spruk, A priori bounds of positive solutions of nonlinear elliptic equations, Comm. P.D.E., 6 (1981), 883–901.
[16] B. Gidas, W. Ni and L. Nirenberg, Symmetry and related properties via maximum principle, Commun. Math. Phys., 68 (1979), 525–598.
[17] Y. Guo and J. Liu, Liouville type theorems for positive solutions of elliptic system in \( R^N \), Comm. P.D.E., 33 (2008), 263–284.
[18] F. Hang, X. Wang and X. Yan, An integral equation in conformal geometry, Ann. Inst. H. Poincare Anal. Non Lineaire, 26 (2009), 1–21.
[19] C. Jin and C. Li, Symmetry of solutions to some systems of integral equations, *Proc. Amer. Math. Soc.*, 134 (2006), 1661–1670.

[20] C. Li and L. Ma, Uniqueness of positive bound states to Shrodinger systems with critical exponents, *SIAM J. Math. Anal.*, 40 (2008), 1049–1057.

[21] Y. Li, Remarks on some conformally invariant integral equations: the method of moving spheres, *J. Eur. Math. Soc.*, 6 (2004), 153–180.

[22] E. Lieb, Sharp Constants in the Hardy-Littlewood-Sobolev inequalities, *Ann. of Math.*, 118 (1983), 349–374.

[23] L. Ma and D. Chen, A Liouville type theorem for an integral system, *Comm. Pure and Appl. Anal.*, 5 (2006), 855–859.

[24] L. Ma and D. Chen, Radial symmetry and monotonicity results for an integral equation, *J. Math. Anal. Appl.*, 2 (2008), 943–949.

[25] L. Ma and D. Chen, Radial symmetry and uniqueness of non-negative solutions to an integral system, *Math. Comput. Modelling*, 49 (2009), 379–385.

[26] C. Ma, W. Chen and C. Li, Regularity of solutions for an integral system of Wolff type, *Advances in Mathematics*, 226 (2011), 2676–2699.

[27] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.*, 195 (2010), 455–467.

[28] E. Mitidieri, Non-existence of positive solutions of semilinear systems in $\mathbb{R}^N$, *Diff. Int. Eq.*, 9 (1996), 465–479.

[29] W. M. Ni and J. Serrin, Non-existence theorems for quasilinear partial differential equations, *Rend. Circ. Mat. Palermo, Suppl.*, 8 (1985), 171–185.

[30] J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden system, *Diff. Int. Eq.*, 9 (1996), 635–653.

[31] J. Serrin and H. Zou, Existence of positive solutions of Lane-Emden system, *Atti Sem. Mat. Fis. Univ. Modena. Sippl.*, 46 (1998), 369–380.

[32] J. Serrin and H. Zou, The existence of positive entire solutions of elliptic Hamiltonian system, *Comm. P.D.E.*, 23 (1998), 577–599.

[33] P. Souplet, The proof of the Lane-Emden conjecture in four space dimensions, *Advances in Mathematics*, 221 (2009), 1409–1437.

[34] S. Terracini, Symmetry properties of positives solutions to some elliptic equations with nonlinear boundary conditions, *Diff. Int. Eq.*, 8 (1995), 1911–1922.

[35] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, *Adv. Diff. Eq.*, 1 (1996), 241–264.

[36] X. Yu, Liouville type theorems for integral equations and integral systems, *Calculus of Variations and Partial Differential Equations*, 46 (2013), 75–95.

Received October 2015; revised February 2016.

E-mail address: yuxiao_211@163.com