Hardy’s Identities and Inequalities on Cartan-Hadamard Manifolds

Joshua Flynn¹ · Nguyen Lam² · Guozhen Lu¹ · Saikat Mazumdar³

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Abstract
We study the Hardy identities and inequalities on Cartan-Hadamard manifolds using the notion of a Bessel pair. These Hardy identities offer significantly more information on the existence/nonexistence of the extremal functions of the Hardy inequalities. These Hardy inequalities are in the spirit of Brezis-Vázquez in the Euclidean spaces. As direct consequences, we establish several Hardy type inequalities that provide substantial improvements as well as simple understandings to many known Hardy inequalities and Hardy-Poincaré–Sobolev type inequalities on hyperbolic spaces in the literature.

Keywords Hardy’s identities, Hardy’s inequalities · Hardy-Poincaré–Sobolev inequalities · Cartan-Hadamard manifold · Hyperbolic space

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Joshua Flynn
joshua.flynn@uconn.edu
Nguyen Lam
nlam@grenfell.mun.ca
Guozhen Lu
guozhen.lu@uconn.edu
Saikat Mazumdar
saikat@math.iitb.ac.in; saikat.mazumdar@iitb.ac.in

¹ Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
² School of Science & Environment, Grenfell Campus, Memorial University of Newfoundland, Corner Brook, NL A2H5G4, Canada
³ Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India
1 Introduction

The main purpose of this article is to study the improvements of the $L^2$-Hardy type inequalities on Cartan-Hadamard manifold, i.e. a Riemannian manifold that is complete and simply connected and has everywhere nonpositive sectional curvature. We also sharpen Hardy inequalities on hyperbolic spaces in the literature.

We recall that on the Euclidean space $\mathbb{R}^N$, $N \geq 3$, the following celebrated Hardy inequality plays important roles in many areas such as analysis, probability and partial differential equations:

$$\int_{\mathbb{R}^N} |\nabla f|^2 \, dx \geq \left( \frac{N - 2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} \, dx, \quad f \in C_0^\infty \left( \mathbb{R}^N \right),$$

(1.1)

The constant $\left( \frac{N - 2}{2} \right)^2$ in the above Hardy inequality is optimal and is never achieved by nontrivial functions. Therefore, one may want to improve (1.1) by adding extra nonnegative terms to its right hand side. On the whole space $\mathbb{R}^N$, the operator $-\Delta - \left( \frac{N - 2}{2} \right)^2 \frac{1}{|x|^2}$ is known to be critical and there is no strictly positive $V \in V^1((0, \infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla f|^2 \, dx - \left( \frac{N - 2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} \, dx \geq \int_{\mathbb{R}^N} V(|x|) |f|^2 \, dx$$

holds for all $f \in C_0^\infty \left( \mathbb{R}^N \right)$ (see [33, Corollary 2.3.4], for e.g.). The situation is very different on bounded domains. In particular, it has been showed that extra nonnegative terms can be added to the Hardy inequality on bounded domains. For instance, let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$, then in order to investigate the stability of singular solutions of nonlinear elliptic equations, Brezis and Vázquez verified in [13] that for all $f \in W^{1,2}_0(\Omega)$:

$$\int_{\Omega} |\nabla f|^2 \, dx - \left( \frac{N - 2}{2} \right)^2 \int_{\Omega} \frac{|f|^2}{|x|^2} \, dx \geq z_0^2 \omega_N^2 |\Omega|^{-\frac{2}{N}} \int_{\Omega} |f|^2 \, dx$$

(1.2)

where $\omega_N$ is the volume of the unit ball and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$. We also mention that in [50], Vázquez and Zuazua established the following improved Hardy-Poincaré inequality: for any $1 \leq q < 2$, there exists a constant $C(q, \Omega) > 0$ such that for all $f \in W^{1,2}_0(\Omega)$:

$$\int_{\Omega} |\nabla f|^2 \, dx - \left( \frac{N - 2}{2} \right)^2 \int_{\Omega} \frac{|f|^2}{|x|^2} \, dx \geq C(q, \Omega) \|f\|_{W^{1,q}(\Omega)}^2.$$
It is interesting to note that the constant $z^2_0 \omega^2_N |\Omega|^{-\frac{n}{2}}$ in (1.2) is optimal when $\Omega$ is a ball and again is not attained in $W^{1,2}_0(\Omega)$. Therefore, Brezis and Vázquez also conjectured that $z^2_0 \omega^2_N |\Omega|^{-\frac{n}{2}} \int_{\Omega} |f|^2 \, dx$ is just the first term of an infinite series of extra terms that can be added to the right hand side of (1.2). This problem has attracted great attention and was investigated by many authors. See [1, 5, 11, 12, 19–21, 28, 30, 31], among others. We also refer the interested reader to [3, 33, 37, 38, 44, 48] which are excellent monographs on the topic. In particular, we note that in an attempt to improve, extend and unify several results in this direction, Ghoussoub and Moradifam [32] introduced the notion of a Bessel pair and studied its connections to Hardy inequalities. One of their results can be read as follows.

**Theorem A** Let $0 < R \leq \infty$, $B_R = B(0, R)$ be a ball centered at the origin with radius $R$, $V$ and $W$ be positive $C^1$-functions on $(0, R)$ such that $\int_0^R \frac{1}{r^{N-1} V(r)} \, dr = \infty$

and $\int_0^R r^{N-1} V(r) \, dr < \infty$. Then,

1. If $(r^{N-1} V, r^{N-1} W)$ is a Bessel pair on $(0, R)$, then for all $f \in C^\infty_0(B_R)$:

\[
\int_{B_R} V(|x|) |\nabla f|^2 \, dx \geq \int_{B_R} W(|x|) |f|^2 \, dx. \tag{1.3}
\]

2. If (1.3) holds for all $f \in C^\infty_0(B_R)$, then $(r^{N-1} V, r^{N-1} c W)$ is a Bessel pair on $(0, R)$ for some $c > 0$.

Here we say that a couple of $C^1$-functions $(V, W)$ is a Bessel pair on $(0, R)$ for some $0 < R \leq \infty$ if the ordinary differential equation

\[
(V y')' + W y = 0 \tag{1.4}
\]

has a positive solution $\varphi$ on the interval $(0, R)$.

Recently, Lam, Lu and Zhang further established in [39] and [40] Hardy’s identities which imply the Hardy inequalities and improved the results in Theorem A and other results of Hardy’s inequalities in the literature.

**Theorem B** Let $0 < R \leq \infty$, $A$ and $B$ be positive $C^1$–functions on $(0, R)$. Assume that for some $\alpha \in \mathbb{R}$, $\Delta d(x) - \frac{\alpha - 1}{d(x)}$ exists on $(0 < d(x) < R)$ in the sense of distributions and $(r^{\alpha-1} A, r^{\alpha-1} B)$ is a Bessel pair on $(0, R)$. Then for $u \in C^\infty_0([0 < d(x) < R])$:

\[ \tag{2} \]
\[
\int_{0<d(x)<R} A(d(x)) |\nabla u(x)|^2 \, dx - \int_{0<d(x)<R} B(d(x)) |u(x)|^2 \, dx
\]
\[
= \int_{0<d(x)<R} A(d(x)) \varphi^2(d(x)) \left| \nabla \left( \frac{u(x)}{\varphi(d(x))} \right) \right|^2 \, dx
\]
\[
- \int_{0<d(x)<R} A(d(x)) |u(x)|^2 \left[ \Delta d(x) - \frac{\alpha - 1}{d(x)} \varphi'(d(x)) \right] \varphi(d(x)) \, dx
\]

and
\[
\int_{0<d(x)<R} A(d(x)) |\nabla d(x) \cdot \nabla u(x)|^2 \, dx - \int_{0<d(x)<R} B(d(x)) |u(x)|^2 \, dx
\]
\[
= \int_{0<d(x)<R} A(d(x)) \varphi^2(d(x)) \left| \nabla d(x) \cdot \nabla \left( \frac{u(x)}{\varphi(d(x))} \right) \right|^2 \, dx
\]
\[
- \int_{0<d(x)<R} A(d(x)) |u(x)|^2 \left[ \Delta d(x) - \frac{\alpha - 1}{d(x)} \varphi'(d(x)) \right] \varphi(d(x)) \, dx.
\]

Here \( \varphi \) is the positive solution of
\[
\left( r^{\alpha-1} A(r) y'(r) \right)' + r^{\alpha-1} B(r) y(r) = 0
\]
on the interval \((0, R)\).

We note that the function \( d(x) \) above can be very general: it can be the distance from \( x \) to a point, to the boundary \( \partial \Omega \) or to a lower dimensional surfaces in \( \Omega \) with dimension \( \alpha \). Hardy type inequalities have been established earlier in the cases with general distance functions [4, 11] and in multipolar setting [10, 15, 16], etc. We also mention the paper [45] where Muckenhoupt pairs have been used to study the necessary and sufficient conditions for the validity of the Hardy inequality on one-dimensional space. More recently, Hardy identities and inequalities have also been established on Carnot groups in [24, 25] and for Dunkl operators in [52]. Furthermore, the notion of \( L^p \) Bessel pairs have also been introduced and \( L^p \) Hardy’s identities and inequalities have been established in [23] and [26]. Hardy’s identities and inequalities on more general Riemannian manifolds have also been proved in [27].

Hardy’s inequalities on Riemannian manifolds have also received much attention in the past decades. In particular, the following Hardy inequality has been first established on Riemannian manifold \((\mathbb{M}, g)\) by Carron in the paper [14]:
\[
\int_{\mathbb{M}} \rho^\alpha(x) |\nabla_g f|^2_g \, dV_g \geq \left( \frac{C + \alpha - 1}{2} \right)^2 \int_{\mathbb{M}} \rho^\alpha(x) \frac{|f|^2_g}{\rho^2(x)} \, dV_g \quad (1.5)
\]
where $\alpha \in \mathbb{R}$, $C + \alpha - 1 > 0$, $f \in C_0^\infty (M \setminus \rho^{-1} \{0\})$ and the weighted function $\rho$ satisfies the eikonal equation $|\nabla g \rho|_g = 1$ and $\Delta_g \rho \geq C \rho$ for some $C > 0$. Here $dV_g$, $\nabla_g$, $\Delta_g$ and $|\cdot|_g$ denote the volume element, gradient, Laplace–Beltrami operator and the length of a vector field with respect to the Riemannian metric $g$ on $M$, respectively. Further developments have been established in [6–9, 18, 36], for instance.

When $M$ is a $N$-dimensional Cartan–Hadamard manifold and $\rho = d(x, O)$ is the geodesic distance from a fixed point $O \in M$ and an arbitrary point $x \in M$, then $\rho$ satisfies all the aforementioned conditions. Moreover, it was showed in [14] that

$$\int_M |\nabla_g f|^2 dV_g \geq \left( \frac{N - 2}{2} \right)^2 \int_M |f|^2 \rho^2 (x) dV_g. \quad (1.6)$$

Moreover, the constant $\left( \frac{N - 2}{2} \right)^2$ was verified to be optimal in [51]. In particular, when $M$ is the hyperbolic space $\mathbb{H}^N$, we have

$$\int_{\mathbb{H}^N} |\nabla_H f|^2 dV_H \geq \left( \frac{N - 2}{2} \right)^2 \int_{\mathbb{H}^N} |f|^2 \rho^2 (x) dV_H \quad (1.7)$$

where $\rho (x)$ is the geodesic distance, $\nabla_H$ is the Riemannian gradient and $dV_H$ is the Riemannian volume form on $\mathbb{H}^N$. On the other hand, it is well-known that on $\mathbb{H}^N$, the $L^2$-spectrum is $\left( (\frac{N - 1}{2})^2, \infty \right)$. More precisely, we have the Poincaré–Sobolev inequality

$$\int_{\mathbb{H}^N} |\nabla_H f|^2 dV_H \geq \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |f|^2 dV_H \quad (1.8)$$

where $\left( \frac{N - 1}{2} \right)^2$ is sharp and is never attained by nontrivial functions in $W^{1,2} (\mathbb{H}^N)$. Let $\Delta_H$ be the Laplace-Beltrami operator on $\mathbb{H}^N$. In [2], the authors investigated the finiteness and infiniteness of the discrete spectrum of the Schrödinger operator $-\Delta_H + V$ and set up the following sharp improvements of the Poincaré–Sobolev inequality (1.8):

$$\int_{\mathbb{H}^N} |\nabla_H f|^2 dV_H - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |f|^2 dV_H$$

$$\geq \frac{1}{4} \int_{\mathbb{H}^N} |f|^2 \rho^2 (x) dV_H + \frac{(N - 1) (N - 3)}{4} \int_{\mathbb{H}^N} \frac{|f|^2}{\sinh^2 \rho (x)} dV_H. \quad (1.9)$$

Moreover, it is proved in [7] that the operator $-\Delta_H - (\frac{N - 1}{2})^2 - \frac{1}{4} \rho^2 (x) - \frac{(N - 1) (N - 3)}{4} \frac{1}{\sinh^2 \rho (x)}$ is critical in $\mathbb{H}^N \setminus \{0\}$ in the sense that for any $W > \frac{1}{4} \rho^2 + \cdots$
the inequality
\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 \, dV_{\mathbb{H}} - \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |f|^2 \, dV_{\mathbb{H}} \\
\geq \int_{\mathbb{H}^N} W |f|^2 \, dV_{\mathbb{H}} \quad \forall f \in C^\infty_0 \left(\mathbb{H}^N \setminus \{0\}\right)
\]
is not valid. Critical Hardy inequalities associated to the shifted Laplacian $-\Delta_{\mathbb{H}} - \lambda$ on the hyperbolic space $\mathbb{H}^n$ for $\lambda \leq \frac{(N-2)^2}{4\rho(x)^2}$ are also studied in [7] in the sense of [20]. This Hardy-Poincaré-Sobolev inequality has also been studied on larger classes of manifolds in [7]. Recently, there has been progress of establishing higher order Hardy-Sobolev-Maz’ya inequalities on hyperbolic spaces using Fourier analysis on hyperbolic spaces (see [41–43]). It is also worth mentioning that the problems of improving Hardy type inequalities as well as other functional and geometric inequalities using the effect of curvature have been studied intensively recently. We refer the interested reader to [9, 17, 22, 34, 35, 46, 47, 51], to name just a few.

Motivated by the aforementioned results, the main purpose of this article is to study the general Hardy type inequalities on Cartan–Hadamard manifolds. Moreover, we will set up some general Hardy identities that can be used to derive several substantial improvements of the Hardy inequality on Cartan–Hadamard manifolds. Our equalities not only provide straightforward understandings of several Hardy type inequalities, but also explain the existence and nonexistence of nontrivial optimizers.

Let $(\mathbb{M}, g)$ be a complete Riemannian manifold of dimension $N$. In a local coordinate system $\{x^i\}_{i=1}^N$, we can write

$$
g = \sum g_{ij} dx^i dx^j.
$$

The Laplace-Beltrami operator $\Delta_g$ with respect to the metric $g$ may then be written as

$$
\Delta_g = \sum \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial x^j}\right)
$$

where $(g^{ij}) = (g_{ij})^{-1}$. Denote by $\nabla_g$ the corresponding gradient. Then

$$
\langle \nabla_g f, \nabla_g h \rangle_g = \sum g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.
$$

We also denote

$$
|\nabla_g f|_g = \sqrt{\langle \nabla_g f, \nabla_g f \rangle_g}.
$$
Fix a point $O \in \mathbb{M}$ and denote by $\rho(x) = d(x, O)$ for all $x \in \mathbb{M}$, where $d$ denotes the geodesic distance on $\mathbb{M}$. Then $\rho(x)$ is Lipschitz continuous in $\mathbb{M}$.

For each point $O \in \mathbb{M}$, consider the exponential map $\exp_O : T_O \mathbb{M} \to \mathbb{M}$. For $X \in T_O \mathbb{M}$, let $\gamma(t)$ be the unique geodesic such that $\gamma(0) = O$ and $\gamma'(0) = X$. Then $\exp_O(tX) = \gamma(t)$ for $t > 0$. For small $t$, $\gamma$ is the unique minimal geodesic joining the points $O$ and $\exp_O(tX)$.

One can write

$$\mathbb{M} = \exp_O(U_O) \cup \text{Cut}(O),$$

where $\text{Cut}(O)$ denotes the cut locus of the point $O$ and $U_O$ is an open neighborhood of $O$ in $T_O \mathbb{M}$. Furthermore, $\exp_O : U_O \to \exp_O(U_O)$ is a diffeomorphism and $\text{Cut}(O) = \exp_O \partial U_O$. Also, the cut locus $\text{Cut}(O)$ has measure zero.

The distance function $\rho(x)$ is smooth on $\mathbb{M} \setminus (\text{Cut}(O) \cup \{O\})$ and it satisfies $|\nabla_g \rho(x)|_g = 1$ on $\mathbb{M} \setminus (\text{Cut}(O) \cup \{O\})$.

For a Cartan-Hadamard manifold $(\mathbb{M}, g)$, the exponential map $\exp_O : T_O \mathbb{M} \to \mathbb{M}$ is a diffeomorphism and $\text{Cut}(O) = \emptyset$, and then $\rho(x)$ is smooth in $\mathbb{M} \setminus \{O\}$ and $|\nabla_g \rho(x)|_g = 1$.

For any $R > 0$, denote by $B_R(O) = \{x \in \mathbb{M} : \rho(x) < \delta\}$ the geodesic ball in $\mathbb{M}$ with center at $O$ and radius $R$. Now, we choose an orthonormal basis $\{u, e_2, ..., e_N\}$ in $T_O \mathbb{M}$ and let $c(t) = \exp_O(tu)$ be a geodesic curve. Consider the Jacobi fields $\{Y_2(t), ..., Y_N(t)\}$ satisfying $Y_i(0) = 0$ and $Y_i'(0) = e_i$, so that the volume density function written in geodesic polar coordinates can be given by

$$J(u, t) = t^{-N+1} \sqrt{\det \langle [Y_i(t), Y_j(t)] \rangle}, \ t > 0.$$

We note that $J(u, t) \in C^{\infty}(T_O \mathbb{M} \setminus \{O\})$ and does not depend on $\{e_2, ..., e_N\}$. By the definition of the density function $J(u, t)$, we have the polar coordinates on $\mathbb{M}$:

$$\int_{\mathbb{M}} f(x) \, dV_g = \int \int_0^{\infty} f(\exp_O(tu)) J(u, t) t^{N-1} \, dr \, du.$$

Here $du$ denotes the canonical measure of the unit sphere of $T_O \mathbb{M}$.

For any function $f$ on $\mathbb{M}$, we also define the radial derivation $\partial_\rho = \frac{\partial}{\partial \rho}$ along the geodesic curve starting from $O$ by

$$\partial_\rho f(x) = \frac{d(f \circ \exp_O)}{dr} \left(\exp_O^{-1}(x)\right).$$

Here we denote $\frac{d}{dr}$ the radial derivation on $T_O \mathbb{M}$:

$$\frac{d}{dr} F(u) = \left\{ \frac{u}{|u|}, \nabla F(u) \right\}.$$
We note that by Gauss’s lemma, we have that \( \left| \partial_{\rho} f \right| \leq \left| \nabla_g f \right|_g \) for \( f \in C^1(\mathbb{M} \setminus \text{Cut}(O)) \).

The first main result of this article is the following Hardy type identities on the general complete Riemannian manifold \((\mathbb{M}, g)\):

**Theorem 1.1** Let \((\mathbb{M}, g)\) be a complete Riemannian manifold of dimension \(N\). Let \(O \in \mathbb{M}\) and take \(0 < R \leq d(O, \text{Cut}(O))\). Let \(V\) and \(W\) be positive \(C^1\) functions on \((0, R)\) such that \((r^{N-1}V, r^{N-1}W)\) is a Bessel pair on \((0, R)\). Then we have the following identities for all \(f \in C_0^\infty(\mathbb{M} \setminus (\text{Cut}(O) \cup \rho^{-1}\{0\}))\):

\[
\int_{BR(O)} V(\rho(x)) \left| \nabla_g f \right|^2_g dV_g - \int_{BR(O)} W(\rho(x)) |f|^2 dV_g \\
= \int_{BR(O)} V(\rho(x)) \varphi^2(\rho(x)) \left| \nabla_g \left( \frac{f}{\varphi(\rho(x))} \right) \right|_g^2 dV_g \\
- \int_{BR(O)} V(\rho(x)) |f|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} J'(u, \rho(x)) J(u, \rho(x)) dV_g
\]

and

\[
\int_{BR(O)} V(\rho(x)) \left| \partial_{\rho} f \right|^2 dV_g - \int_{BR(O)} W(\rho(x)) |f|^2 dV_g \\
= \int_{BR(O)} V(\rho(x)) \varphi^2(\rho(x)) \left| \partial_{\rho} \left( \frac{f}{\varphi(\rho(x))} \right) \right|_g^2 dV_g \\
- \int_{BR(O)} V(\rho(x)) |f|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} J'(u, \rho(x)) J(u, \rho(x)) dV_g.
\]

Here \(J'(u, t) = \frac{\partial J(u, t)}{\partial t}\), \(x = \exp_O(\rho u)\) and \(\varphi\) is the positive solution of

\[
\left( r^{N-1}V(r) \varphi'(r) \right)' + r^{N-1}W(r) \varphi(r) = 0.
\]

It is in fact possible to consider \(\varphi\) which have a zero at some \(r = R\), but are positive elsewhere and which satisfy the Bessel pair ODE on \((0, R) \cup (R, \infty)\). As the following theorem demonstrates, considering such \(\varphi\) allows one to establish global Hardy identities provided \(f\) is replaced by \(f - f(\exp)\) and provided \(f\) satisfies (1.10). We note that on finite interval \((0, R)\), this function \(\varphi\) satisfies the condition in Theorem 1.1. However, on the infinite interval \((0, \infty)\), this \(\varphi\) is allowed to be degenerate or singular at \(R\).
Theorem 1.2 Let \((\mathbb{M}, g)\) be a complete Riemannian manifold of dimension \(N\). Let 
\(O \in \mathbb{M}\) and take \(0 < R \leq d(O, \text{Cut}(O))\). Assume that \(V\) and \(W\) are positive \(C^1\)
functions on \((0, R) \cup (R, \infty)\) such that the ordinary differential equation

\[
\left( V(r) r^{N-1} \varphi'(r) \right)' + W(r) r^{N-1} \varphi(r) = 0
\]

has a positive solution \(\varphi\) on \((0, R) \cup (R, \infty)\).

Then for all \(f \in C^\infty_0(\mathbb{M} \setminus \text{Cut}(O) \cup \rho^{-1}\{0\})\) satisfying that for all \(u \in \mathbb{S}^{N-1}:\)

\[
\lim_{r \to R} V(r) \frac{\varphi'(r)}{\varphi(r)} \left| f \left( \exp_O (ru) \right) - f \left( \exp_O (Ru) \right) \right|^2 = 0, \quad (1.10)
\]

we have

\[
\int_{\mathbb{M}} V(\rho(x)) \left| \nabla_g \left( f - f \left( \exp_O (Ru) \right) \right) \right|^2 g \, dx - \int_{\mathbb{M}} W(\rho(x)) \left| f - f \left( \exp_O (Ru) \right) \right|^2 dV_g \\
= \int_{\mathbb{M}} V(\rho(x)) \varphi^2(\rho(x)) \left| \nabla_g \left( \frac{f - f \left( \exp_O (Ru) \right)}{\varphi(\rho(x))} \right) \right|^2 g \, dV_g \\
- \int_{\mathbb{M}} V(\rho(x)) \left| f - f \left( \exp_O (Ru) \right) \right|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} J'(u, \rho) J(u, \rho) \, dV_g
\]

and

\[
\int_{\mathbb{M}} V(\rho(x)) \left| \partial_\rho \left( f - f \left( \exp_O (Ru) \right) \right) \right|^2 g \, dx - \int_{\mathbb{M}} W(\rho(x)) \left| f - f \left( \exp_O (Ru) \right) \right|^2 dV_g \\
= \int_{\mathbb{M}} V(\rho(x)) \varphi^2(\rho(x)) \left| \partial_\rho \left( \frac{f - f \left( \exp_O (Ru) \right)}{\varphi(\rho(x))} \right) \right|^2 dV_g \\
- \int_{\mathbb{M}} V(\rho(x)) \left| f - f \left( \exp_O (Ru) \right) \right|^2 \frac{\varphi'(\rho(x))}{\varphi(\rho(x))} J'(u, \rho) dV_g.
\]

Here \(J'(u, t) = \frac{\partial J(u, t)}{\partial t}\) and \(x = \exp_O (\rho u)\).

By applying our main results to some explicit Bessel pairs on Cartan–Hadamard manifold, we obtain many interesting Hardy identities and inequalities. For instance, on the hyperbolic space, we obtain the following identities and inequalities that substantially improve (1.7) as consequences of our main results:
Theorem 1.3 For $f \in C^\infty_0(\mathbb{H}^N)$:

\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 dV_{\mathbb{H}} - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|f|^2}{\rho^2(x)} dV_{\mathbb{H}} \\
= \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} |\nabla_{\mathbb{H}} \left(\rho^{\frac{N-2}{2}}(x) f\right)|^2 dV_{\mathbb{H}} \\
+ \frac{(N-2)(N-1)}{2} \int_{\mathbb{H}^N} \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^2(x) \sinh \rho(x)} |f|^2 dV_{\mathbb{H}}
\]

and

\[
\int_{\mathbb{H}^N} |\partial_\rho f|^2 dV_{\mathbb{H}} - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|f|^2}{\rho^2(x)} dV_{\mathbb{H}} \\
= \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} |\partial_\rho \left(\rho^{\frac{N-2}{2}}(x) f\right)|^2 dV_{\mathbb{H}} \\
+ \frac{(N-2)(N-1)}{2} \int_{\mathbb{H}^N} \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^2(x) \sinh \rho(x)} |f|^2 dV_{\mathbb{H}}
\]

Obviously, our theorem gives the exact remainder and therefore provides the direct understanding for the Hardy inequality (1.7). Also, as a consequence of the above identities, we get that

\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 \geq \int_{\mathbb{H}^N} |\partial_\rho f|^2 dV_{\mathbb{H}} \\
\geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|f|^2}{\rho^2(x)} dV_{\mathbb{H}} \\
+ \frac{(N-2)(N-1)}{2} \int_{\mathbb{H}^N} \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^2(x) \sinh \rho(x)} |f|^2 dV_{\mathbb{H}}.
\]

Therefore, the operator $-\Delta_{\mathbb{H}} - \left(\frac{N-2}{2}\right)^2 \frac{1}{\rho^2(x)}$ is subcritical in $\mathbb{H}^N \setminus \{0\}$ as proved in [9], which is in contrast to the situation in the Euclidean setting [33, Corollary 2.3.4].

We also present the exact remainder for the Hardy-Poincaré–Sobolev inequality (1.9) and thus sharpen the inequality (1.8) and illustrate more precise understanding of (1.9):
Theorem 1.4 For \( f \in C^\infty_0 (\mathbb{H}^N) \):

\[
\int_{\mathbb{H}^N} |\nabla f|^2 \, dV_{\mathbb{H}} - \int_{\mathbb{H}^N} \left[ \frac{(N-1)^2}{4} + \frac{1}{4} \rho^2 (x) + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 \rho (x)} \right] |f|^2 \, dV_{\mathbb{H}}
\]

\[
= \int_{\mathbb{H}^N} \frac{\rho (x)}{\sinh^{N-1} \rho (x)} \left| \nabla \left( \frac{\sinh^{N-1} \rho (x) f}{\rho^2 (x)} \right) \right|^2 \, dV_{\mathbb{H}},
\]

and

\[
\int_{\mathbb{H}^N} |\partial_\rho f|^2 \, dV_{\mathbb{H}} - \int_{\mathbb{H}^N} \left[ \frac{(N-1)^2}{4} + \frac{1}{4} \rho^2 (x) + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 \rho (x)} \right] |f|^2 \, dV_{\mathbb{H}}
\]

\[
= \int_{\mathbb{H}^N} \frac{\rho (x)}{\sinh^{N-1} \rho (x)} \left| \partial_\rho \left( \frac{\sinh^{N-1} \rho (x) f}{\rho^2 (x)} \right) \right|^2 \, dV_{\mathbb{H}}.
\]

and

\[
\int_{\mathbb{H}^N} |\nabla f|^2 \, dV_{\mathbb{H}} \geq \int_{\mathbb{H}^N} |\partial_\rho f|^2 \, dV_{\mathbb{H}}
\]

\[
\geq \int_{\mathbb{H}^N} \left[ \frac{(N-1)^2}{4} + \frac{1}{4} \rho^2 (x) + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 \rho (x)} \right] |f|^2 \, dV_{\mathbb{H}}.
\]

We also obtain the following Hardy inequalities on hyperbolic spaces in the spirit of Brezis-Vázquez [13].

Theorem 1.5 Let \( 0 \leq \alpha \leq \frac{N-2}{2} \). For \( f \in C^\infty_0 (\mathbb{H}^N) \):

\[
\int_{0<\rho(x)<R} \left| \frac{\nabla f}{\rho} \right|^2 \, dV_{\mathbb{H}} - \left( \frac{(N-2)^2}{4} - \alpha^2 \right) \int_{0<\rho(x)<R} \frac{|f|^2}{\rho^2 (x)} \, dV_{\mathbb{H}}
\]

\[
= \frac{z^2_a}{R^2} \int_{0<\rho(x)<R} |f|^2 \, dV_{\mathbb{H}} + \int_{0<\rho(x)<R} \int_{\mathbb{H}^N} J_a \left( \frac{\partial_\rho f}{\rho} \right) \left( \frac{J_a \left( \frac{\partial_\rho x}{\rho} \right)}{\rho^{N-2} (x)} \right) \, dV_{\mathbb{H}}
\]

\[
- (N-1) \int_{0<\rho(x)<R} \left( \frac{2-N}{2} \frac{J_a}{\rho} \left( \frac{\partial_\rho x}{\rho} \right) \right) \left( \frac{\rho (x) \cosh \rho (x) - \sinh \rho (x)}{\rho (x) \sin \rho (x)} \right) |f|^2 \, dV_{\mathbb{H}}
\]

and

\[
\int_{0<\rho(x)<R} \left| \frac{\partial_\rho f}{\rho} \right|^2 \, dV_{\mathbb{H}} - \left( \frac{(N-2)^2}{4} - \alpha^2 \right) \int_{0<\rho(x)<R} \frac{|f|^2}{\rho^2 (x)} \, dV_{\mathbb{H}}
\]

\[
= \frac{z^2_a}{R^2} \int_{0<\rho(x)<R} |f|^2 \, dV_{\mathbb{H}} + \int_{0<\rho(x)<R} \int_{\mathbb{H}^N} J_a \left( \frac{\partial_\rho f}{\rho} \right) \left( \frac{J_a \left( \frac{\partial_\rho x}{\rho} \right)}{\rho^{N-2} (x)} \right) \, dV_{\mathbb{H}}
\]

\[
\left( \frac{\rho^2 - \rho^{2-\alpha}}{\rho} \left( \frac{\partial_\rho x}{\rho} \right) \right) \left( \frac{\rho (x) \cosh \rho (x) - \sinh \rho (x)}{\rho (x) \sin \rho (x)} \right) |f|^2 \, dV_{\mathbb{H}}
\]
− (N − 1) ∫_{0 < ρ(x) < R} \left( \frac{2 − N}{2ρ(x)} + \frac{z_α}{R} \frac{J′_α}{J_α} \left( \frac{κ}{R} ρ(x) \right) \right) \frac{ρ(x) \cosh ρ(x) − \sinh ρ(x)}{ρ(x) \sinh ρ(x)} |f|^2 dV_H

As a consequence of these identities, we get that

\[ \int_{0 < ρ(x) < R} |\nabla_{\mathbb{H}} f|^2 dV_H \]
\[ \geq \int_{0 < ρ(x) < R} |∂_ρ f|^2 dV_H \]
\[ \geq \left( \frac{(N − 2)^2}{4} − α^2 \right) \int_{0 < ρ(x) < R} \frac{|f|^2}{ρ^2(x)} dV_H + \frac{z_α^2}{R^2} \int_{0 < ρ(x) < R} |f|^2 dV_H \]
\[ − (N − 1) \int_{0 < ρ(x) < R} \left( \frac{2 − N}{2ρ(x)} + \frac{z_α}{R} \frac{J′_α}{J_α} \left( \frac{κ}{R} ρ(x) \right) \right) \frac{ρ(x) \cosh ρ(x) − \sinh ρ(x)}{ρ(x) \sinh ρ(x)} |f|^2 dV_H \]
\[ \geq \left( \frac{(N − 2)^2}{4} − α^2 \right) \int_{0 < ρ(x) < R} \frac{|f|^2}{ρ^2(x)} dV_H + \frac{z_α^2}{R^2} \int_{0 < ρ(x) < R} |f|^2 dV_H. \]

Here \( z_α \) is the first zero of the Bessel function of the first kind \( J_α(z) \).

Our paper is organized as follows: In Sect. 2, we use our main results on the Hardy identities to obtain several Hardy type inequalities and their improvements on Cartan-Hadamard manifolds. In Sect. 3, we will focus on deriving the Hardy identities and inequalities on hyperbolic spaces. We also provide the proofs of Theorems 1.3, 1.4 and 1.5 in this section. Proofs of main results (Theorems 1.1 and 1.2) will be presented in Sect. 4.

### 2 Hardy Inequalities on Cartan-Hadamard Manifolds

We note that if the sectional curvature \( K_M = −b \), then \( J(u, t) = J_b(t) \) does not depend on \( u \). Moreover

\[ J_b(t) = \begin{cases} 
1 & \text{if } b = 0 \\
\left( \frac{\sinh(\sqrt{b}t)}{\sqrt{b}t} \right)^{N − 1} & \text{if } b > 0.
\end{cases} \]

Also, if \( K_M \leq −b \leq 0 \), then by the Bishop-Gromov-Günther comparison theorem [29, p. 172], we have that

\[ \frac{J′(u, t)}{J(u, t)} \geq \frac{J′_b(t)}{J_b(t)} = \frac{N − 1}{t} D_b(t) \]
where $J'(u, t) = \frac{\partial J(u, t)}{\partial t}$,

$$D_b(t) = \begin{cases} 0 & \text{if } t = 0 \\ tc_t(t) - 1 & \text{if } t > 0 \end{cases}$$

and

$$c_t(t) = \begin{cases} \frac{1}{t} & \text{if } b = 0 \\ \sqrt{b} \coth \left( \sqrt{bt} \right) & \text{if } b > 0 \end{cases}.$$

Therefore, we obtain the following Hardy type inequality as a direct consequence of our Theorem 1.1.

**Theorem 2.1** Let $(\mathbb{M}, g)$ be a Cartan-Hadamard manifold of dimension $N$ and let $O \in \mathbb{M}$. Let $0 < R \leq \infty$, $V$ and $W$ be positive $C^1$-functions on $(0, R)$ such that $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair on $(0, R)$ with nonincreasing positive solution $\varphi$. Then for $f \in C^\infty_0(B_R(O) \setminus \rho^{-1}[0])$:

$$\int_{B_R(O)} V(\rho(x)) \left| \nabla_g f \right|^2_g dV_g - \int_{B_R(O)} W(\rho(x)) |f|^2 dV_g$$

$$\geq \int_{B_R(O)} V(\rho(x)) \left| \varphi^2(\rho(x)) \right| \left| \frac{\nabla_g \left( f \varphi(\rho(x)) \right)}{\varphi(\rho(x))} \right|^2_g dV_g$$

$$- \int_{B_R(O)} V(\rho(x)) |f|^2 \varphi'(\rho(x)) J'_b(\rho(x)) \frac{J_b(\rho(x))}{\varphi(\rho(x))} dV_g$$

$$\geq \int_{B_R(O)} V(\rho(x)) \left| \varphi^2(\rho(x)) \right| \left| \frac{\nabla_g \left( f \varphi(\rho(x)) \right)}{\varphi(\rho(x))} \right|^2_g dV_g$$

and

$$\int_{B_R(O)} V(\rho(x)) \left| \partial_\rho f \right|^2_g dV_g - \int_{B_R(O)} W(\rho(x)) |f|^2 dV_g$$

$$\geq \int_{B_R(O)} V(\rho(x)) \varphi^2(\rho(x)) \left| \partial_\rho \left( f \varphi(\rho(x)) \right) \right|^2_g dV_g$$

$$- \int_{B_R(O)} V(\rho(x)) |f|^2 \varphi'(\rho(x)) J'_b(\rho(x)) \frac{J_b(\rho(x))}{\varphi(\rho(x))} dV_g$$

$$\geq \int_{B_R(O)} V(\rho(x)) \left| \varphi(\rho(x)) \right| \left| \partial_\rho \left( f \varphi(\rho(x)) \right) \right|^2_g dV_g.$$

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Proof By the Bishop-Gromov-Günther comparison theorem [29, p. 172], we get
\[ \frac{J'(u,t)}{J(u,t)} \geq 0, \]
where \( J'(u,t) = \frac{\partial J(u,t)}{\partial t} \). Since \( \varphi \) is a nonincreasing function,
\[ -\varphi' (\rho (x)) J'(u,\rho) \geq -\varphi' (\rho (x)) \frac{j'_h(\rho)}{j_h(\rho)} \geq 0. \]
Hence, we can apply Theorem 1.1 to get the desired result.

By applying Theorem 2.1 to particular Bessel pairs, we obtain several Hardy type
inequalities with remainder terms on \( \mathbb{M} \). These results are listed as follows.

Corollary 2.1 for \( \lambda < N-2 \) and \( f \in C_0^\infty (\mathbb{M} \setminus \rho^{-1} \{0\}) : \)
\[
\int_M \frac{\left| \nabla_g f \right|^2}{\rho^\lambda (x)} dV_g - \left( \frac{N-\lambda-2}{2} \right)^2 \int_M \frac{|f|^2}{\rho^{\lambda+2} (x)} dV_g
\geq \int_M \frac{1}{\rho^{N-2} (x)} \left| \nabla_g \left( \rho^{\frac{N-\lambda-2}{2}} (x) f \right) \right|^2 g dV_g
\]
and
\[
\int_M \frac{\left| \partial_\rho f \right|^2}{\rho^\lambda (x)} dV_g - \left( \frac{N-\lambda-2}{2} \right)^2 \int_M \frac{|f|^2}{\rho^{\lambda+2} (x)} dV_g
\geq \int_M \frac{1}{\rho^{N-2} (x)} \left| \partial_\rho \left( \rho^{\frac{N-\lambda-2}{2}} (x) f \right) \right|^2 dV_g.
\] (2.1)

Proof We apply Theorem 2.1 to the Bessel pair \( (r^{N-1} r^{-\lambda}, r^{N-1} r^{-\lambda} (N-\lambda-2)^2 \frac{1}{r^2}) \) on
\( (0, \infty) \) with \( \varphi (r) = r^{\frac{2-N+\lambda}{2}} \) to get the desired results.

In the critical case \( \lambda = N-2 \), we have

Corollary 2.2 Let \( R > 0 \). We have for \( f \in C_0^\infty (B_R (O) \setminus \rho^{-1} \{0\}) : \)
\[
\int_{0<\rho(x)<R} \frac{\left| \nabla_g f \right|^2}{\rho^{N-2} (x)} dV_g - \frac{1}{4} \int_{0<\rho(x)<R} \frac{|f (x)|^2}{\rho^N (x) \left| \ln \frac{\rho(x)}{R} \right|^2} dV_g
\]
\geq \int_{0<\rho(x)<R} \frac{1}{\rho^{N-2} (x)} \left| \ln \frac{\rho(x)}{R} \right| \left| \nabla_g \left( \frac{f (x)}{\sqrt{\left| \ln \left| x \right| \right|}} \right) \right|^2 g dV_g
\]
and
\[
\int_{0<\rho(x)<R} \frac{\left| \partial_\rho f \right|^2}{\rho^{N-2} (x)} dV_g - \frac{1}{4} \int_{0<\rho(x)<R} \frac{|f (x)|^2}{\rho^N (x) \left| \ln \frac{\rho(x)}{R} \right|^2} dV_g
\]
\[
\geq \int_{0<\rho(x)<R} \frac{1}{\rho^{N-2}(x)} \left| \ln \frac{\rho(x)}{R} \right| \left| \frac{\partial_{\rho} \left( \frac{f(x)}{\sqrt{\ln |x|}} \right)}{R} \right|^2 dV_g.
\]

**Proof.** We apply Theorem 2.1 to the Bessel pair \( (r^{N-1} - \frac{1}{r^{N-2}}, r^{N-1} - \frac{1}{4r^N \ln \frac{r}{R}}) \) with \( \varphi = \sqrt{\ln \frac{r}{R}} \).

Actually, we can get the following version of the critical Hardy type inequalities on the whole space \( \mathbb{M} \) which is more general than Corollary 2.2.

**Corollary 2.3** Let \( R > 0 \). For any \( f \in C_0^\infty (\mathbb{M} \setminus \rho^{-1} \{0\}) \), we have

\[
\int_{\mathbb{M}} \frac{1}{\rho^{N-2}(x)} \left| \nabla_g \left( f(x) - f(\exp_O (Ru)) \right) \right|^2 dV_g
\]

\[
\geq \frac{1}{4} \int_{\mathbb{M}} \frac{\left| f(x) - f(\exp_O (Ru)) \right|^2}{\rho^N(x) \left| \ln \frac{\rho(x)}{R} \right|^2} dV_g
\]

\[
+ \int_{\mathbb{M}} \frac{1}{\rho^{N-2}(x)} \left| \ln \frac{\rho(x)}{R} \right| \left| \nabla \left( f(x) - f(\exp_O (Ru)) \right) \right|^2 dV_g.
\]

and

\[
\int_{\mathbb{M}} \frac{1}{\rho^{N-2}(x)} \left| \rho_{\rho} \left( f(x) - f(\exp_O (Ru)) \right) \right|^2 dV_g
\]

\[
\geq \frac{1}{4} \int_{\mathbb{M}} \frac{\left| f(x) - f(\exp_O (Ru)) \right|^2}{\rho^N(x) \left| \ln \frac{\rho(x)}{R} \right|^2} dV_g
\]

\[
+ \int_{\mathbb{M}} \frac{1}{\rho^{N-2}(x)} \left| \ln \frac{\rho(x)}{R} \right| \left| \rho_{\rho} \left( f(x) - f(\exp_O (Ru)) \right) \right|^2 dV_g.
\]

**Proof.** Apply the Theorem 1.2 to \( V(r) = \frac{1}{r^{N-2}}, W = \frac{1}{4r^N \ln \frac{r}{R}} \) and \( \varphi = \sqrt{\ln \frac{r}{R}} \).

Note that for any \( f \in C_0^\infty (\mathbb{M} \setminus \rho^{-1} \{0\}) \) and any \( u \in S^{N-1} : \)

\[
\lim_{r \to R} \left| V(r) \frac{\varphi'(r)}{\varphi(r)} \left| f(\exp_O (ru)) - f(\exp_O (Ru)) \right| \right|^2
\]
\[ = \lim_{r \to R} \frac{1}{r^{N-2}} \left| \left( \frac{\ln \frac{r}{R}}{\sqrt{\ln \frac{r}{R}}} \right)' \right| f \left( \exp_{O} (ru) \right) - f \left( \exp_{O} (Ru) \right) \right|^2 \]

\[ \lesssim \lim_{r \to R} \frac{1}{\ln \frac{r}{R}} (R - r)^2 = 0. \]

Now, by combining Corollaries 2.1 and 2.3, we obtain

**Corollary 2.4** Let \((M, g)\) be a Cartan-Hadamard manifold of dimension \(N\). For \(\lambda < N - 2\) and \(f \in C^\infty (M \setminus \rho^{-1} \{0\})\), we have

\[ \int_M \left| \frac{\partial_{\rho} f}{\rho^{\lambda/2} (x)} \right|^2 dV_g - \left( \frac{N - \lambda - 2}{2} \right)^2 \int_M \left| f \right|^2 \rho^{\lambda/2} (x) dV_g \geq \frac{1}{4} \sup_{R > 0} \int_M \left| f (x) - R^{\frac{N - \lambda - 2}{2}} f \left( \exp_{O} (Ru) \right) \rho_{\rho^{-1}}^{\frac{N - \lambda - 2}{2}} (x) \right|^2 \rho^{\lambda/2} (x) \left| \ln \frac{\rho(x)}{R} \right|^2 dV_g. \]  

**Proof** From Corollaries 2.1 and 2.3, we have

\[ \int_M \left| \frac{\partial_{\rho} f}{\rho^{\lambda/2} (x)} \right|^2 dV_g - \left( \frac{N - \lambda - 2}{2} \right)^2 \int_M \left| f \right|^2 \rho^{\lambda/2} (x) dV_g \geq \int_M \frac{1}{\rho^{N-2} (x)} \left| \partial_{\rho} \left( \rho^{\frac{N - \lambda - 2}{2}} (x) f \right) \right|^2 dV_g \]

\[ = \int_M \frac{1}{\rho^{N-2} (x)} \left| \partial_{\rho} \left( \rho^{\frac{N - \lambda - 2}{2}} (x) f - R^{\frac{N - \lambda - 2}{2}} f \left( \exp_{O} (Ru) \right) \right) \right|^2 dV_g \]

\[ \geq \frac{1}{4} \int_M \frac{\rho^{\frac{N - \lambda - 2}{2}} (x) f (x) - R^{\frac{N - \lambda - 2}{2}} f \left( \exp_{O} (Ru) \right) \rho^{N} (x) \left| \ln \frac{\rho(x)}{R} \right|^2}{\rho^{N} (x) \left| \ln \frac{\rho(x)}{R} \right|^2} dV_g \]

\[ + \int_M \frac{1}{\rho^{N-2} (x)} \left| \ln \frac{\rho(x)}{R} \right| \left| \partial_{\rho} \left( \rho^{\frac{N - \lambda - 2}{2}} f \left( \exp_{O} (Ru) \right) \rho^{N} \left( \rho \right) \left| \ln \frac{\rho(x)}{R} \right| \right)^2 \right| dV_g. \]

Note that the “virtual” optimizers of the weighted Hardy inequality (2.1) have the form \(\psi (\exp_{O} (u)) \rho^{\frac{N - \lambda - 2}{2}} (x)\) for some function \(\psi : \mathbb{S}^{N-1} \to \mathbb{R}\). These optimizers are virtual in the sense that, if equality were to hold in the Hardy inequality

\[ \int_M \left| \frac{\partial_{\rho} f}{\rho^{\lambda/2} (x)} \right|^2 dV_g \geq \left( \frac{N - \lambda - 2}{2} \right)^2 \int_M \left| f \right|^2 \rho^{\lambda/2} (x) dV_g \]
given by (2.1), then the remainder term would vanish, i.e., \( \partial_\rho \left( \rho^{\frac{N-\lambda-2}{2}} (x) f \right) = 0 \). Therefore, (2.2) can be read as a stability version of the weighted Hardy inequality (2.1).

We also obtain the following Hardy inequality in the spirit of Brezis and Vázquez [13]:

**Corollary 2.5** Let \((\mathbb{M}, g)\) be a Cartan-Hadamard manifold of dimension \(N\). For any \(R > 0\) and \(\lambda \leq N - 2\), we have for \(f \in C^\infty_0 (B_R (O) \setminus \rho^{-1} \{0\}) :\)

\[
\int_{B_R (O)} \frac{|\nabla_g f|_g^2}{\rho^\lambda (x)} dV_g - \left( \frac{N - \lambda - 2}{2} \right)^2 \int_{B_R (O)} \frac{|f|^2}{\rho^{\lambda+2} (x)} dV_g \geq \frac{z_0^2}{R^2} \int_{B_R (O)} \frac{|f|^2}{\rho^\lambda (x)} dV_g + \int_{B_R (O)} \left| \frac{J_0 \left( \frac{z_0}{R} \rho (x) \right)}{\rho (x)^{\frac{N-\lambda-2}{2}}} \right|^2 \left| \nabla_g \left( \frac{\rho (x)^{\frac{N-\lambda-2}{2}}}{J_0 \left( \frac{z_0}{R} \rho (x) \right) f} \right) \right|^2 dV_g
\]

and

\[
\int_{B_R (O)} \frac{\partial_\rho f}{\rho^\lambda (x)} dV_g - \left( \frac{N - \lambda - 2}{2} \right)^2 \int_{B_R (O)} \frac{|f|^2}{\rho^{\lambda+2} (x)} dV_g \geq \frac{z_0^2}{R^2} \int_{B_R (O)} \frac{|f|^2}{\rho^\lambda (x)} dV_g + \int_{B_R (O)} \left| \frac{J_0 \left( \frac{z_0}{R} \rho (x) \right)}{\rho (x)^{\frac{N-\lambda-2}{2}}} \right|^2 \left| \partial_\rho \left( \frac{\rho (x)^{\frac{N-\lambda-2}{2}}}{J_0 \left( \frac{z_0}{R} \rho (x) \right) f} \right) \right|^2 dV_g.
\]

**Proof** For any \(R > 0\), \(r^{N-1-N-\lambda} r^{N-1-N-\lambda} \left[ \frac{(N-\lambda-2)^2}{4} \right]^{\frac{1}{2}} + \frac{z_0^2}{R^2} \) is a Bessel pair on \((0, R)\) with \(\varphi (r) = r^{\frac{N-\lambda-2}{2}} J_0 \left( \frac{z_0}{R} \right) = r^{\frac{N-\lambda-2}{2}} J_0 (r)\). Here \(z_0 = 2.4048\ldots\) is the first zero of the Bessel function \(J_0 (z)\). Note that \(\varphi (r)\) is nonincreasing since \(N - \lambda - 2 \geq 0\).

### 3 Hardy Inequalities on Hyperbolic Spaces

In this section, we will investigate the Hardy identities and inequalities on the hyperbolic space \(\mathbb{H}^N\), which is the most important example of Cartan-Hadamard manifold. We use the Poincaré ball model of the hyperbolic space \(\mathbb{H}^N\). That is, the unit ball in \(\mathbb{R}^N\) centered at the origin and equipped with the metric

\[
ds^2 = \frac{4 \sum dx_i^2}{\left( 1 - |x|^2 \right)^2}.
\]
Also
\[ dV_{\mathbb{H}} = \frac{2^N}{(1 - |x|^2)^N} dx, \]
\[ \nabla_{\mathbb{H}} = \left( \frac{1 - |x|^2}{2} \right) \nabla, \]
where \( \nabla \) denotes the Euclidean gradient. Therefore
\[ \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}} = \int_{B} |\nabla u|^2 \frac{2^{N-2}}{(1 - |x|^2)^{N-2}} dx. \]

We also recall that the geodesic distance from \( x \) to 0 is \( \rho(x) = \ln \frac{1 + |x|}{1 - |x|} \). That is \( |x| = \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \). By using the Poincaré ball model and applying our Theorem 1.1 for the unit ball on the Euclidean space \( \mathbb{R}^N \), we get the following identity:

**Theorem 3.1** If \( \left( r^{N-1} \frac{1}{(1-r^2)^{N-2}} V, r^{N-1} \frac{4}{(1-r^2)^N} W \right) \) is a Bessel pair on \((0, 1)\), then we have for \( f \in C^\infty_0 (\mathbb{H}^N \setminus \{0\}) \):

\[
\int_{\mathbb{H}^N} V \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right) |\nabla_{\mathbb{H}} f|^2 dV_{\mathbb{H}} - \int_{\mathbb{H}^N} W \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right) |f|^2 dV_{\mathbb{H}} = \int_{\mathbb{H}^N} \phi^2 \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right) \left| \nabla_{\mathbb{H}} \left( \frac{f}{\phi \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right)} \right) \right|^2 dV_{\mathbb{H}}
\]

Here \( \phi \) is the positive solution of

\[
\left( \frac{r^{N-1}}{1 - r^2} \frac{1}{(1-r^2)^{N-2}} V (r) \phi' (r) \right)' + r^{N-1} \frac{4}{(1-r^2)^N} W (r) \phi (r) = 0
\]
on \((0, 1)\).

**Proof** Since \( \left( r^{N-1} \frac{1}{(1-r^2)^{N-2}} V, r^{N-1} \frac{4}{(1-r^2)^N} W \right) \) is a Bessel pair on \((0, 1)\) with solution \( \phi \), we get

\[
\int_{B(0,1)} \frac{1}{(1 - |x|^2)^{N-2}} V (|x|) |\nabla f|^2 dx - \int_{B(0,1)} \frac{4}{(1 - |x|^2)^N} W (|x|) |f|^2 dx = \int_{B(0,1)} \frac{1}{(1 - |x|^2)^{N-2}} V (|x|) \phi^2 (|x|) \left| \nabla \left( \frac{f}{\phi (|x|)} \right) \right|^2 dx.
\]
Equivalently,
\[
\int_{\mathbb{H}^N} V(|x|)|\nabla_{\mathbb{H}} f|^2 \, dV_{\mathbb{H}} - \int_{\mathbb{H}^N} W(|x|)|f|^2 \, dV_{\mathbb{H}} = \int_{\mathbb{H}^N} V(|x|)|\varphi^2(|x|)||\nabla_{\mathbb{H}} \left( \frac{f}{\varphi(|x|)} \right)|^2 \, dV_{\mathbb{H}}.
\]

Now, let
\[
F(r) = \frac{(1 - r^2)^{N-2}}{r^{N-1}},
\]
\[
G(r) = \int_r^1 F(t) \, dt,
\]
\[
V_2(r) = \frac{F^2(r)(1 - r^2)^2}{4(N-2)^2 G^2(r)}.
\]

Then we have

**Corollary 3.1** For \( f \in C_0^\infty(\mathbb{H}^N) \):
\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 \, dV_{\mathbb{H}} \geq \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{H}^N} V_2 \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right) |f|^2 \, dV_{\mathbb{H}}
\]
\[
= \int_{\mathbb{H}^N} \left| G \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right) \right|^2 \left| \nabla_{\mathbb{H}} \left( \frac{f}{\sqrt{G \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right)}} \right) \right|^2 \, dV_{\mathbb{H}}.
\]

As a consequence, we obtain the following Hardy type inequality that has been studied in [49]:
\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 \, dV_{\mathbb{H}} \geq \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{H}^N} V_2 \left( \frac{e^{\rho(x)} - 1}{e^{\rho(x)} + 1} \right) |f|^2 \, dV_{\mathbb{H}}.
\]

**Proof** We note that \( \frac{G}{N_0 y_{N-1}} \) is a fundamental solution of the hyperbolic Laplacian. We have that \( r^{N-1} \frac{1}{(1-r^2)^{N-2}}, r^{N-1} \frac{F^2(r)}{4G^2(r)} \frac{1}{(1-r^2)^{N-2}} \) is a Bessel pair on \((0,1)\) with
\[ \varphi = \sqrt{G(r)}. \] That is
\[
\left( r^{N-1} \frac{1}{(1-r^2)^{N-2}} \varphi' \right)' + r^{N-1} \frac{F^2(r)}{4G^2(r)} \frac{1}{(1-r^2)^{N-2}} \varphi = 0. \]
Indeed, a direct computation shows
\[
r^{N-1} \frac{1}{(1-r^2)^{N-2}} \varphi' = -\frac{1}{2\sqrt{G(r)}}
\]
Thus
\[
\left( r^{N-1} \frac{1}{(1-r^2)^{N-2}} \varphi' \right)' = -\left( \frac{1}{2\sqrt{G(r)}} \right)'
\]
\[
= \frac{1}{4} \frac{\sqrt{G(r)}G'(r)}{G^2(r)}
\]
\[
= -\frac{1}{2\sqrt{G(r)}} \frac{1}{4} \frac{\sqrt{G(r)}(1-r^2)^{N-2}}{r^{N-1}}
\]
\[
= -r^{N-1} \frac{F^2(r)}{4G^2(r)} \frac{1}{(1-r^2)^{N-2}} \varphi.
\]

Hence by Theorem 3.1, we obtain
\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 \, dV_{\mathbb{H}} = (N-2) \int_{\mathbb{H}^N} \left( \frac{N-2}{2} \right)^2 V_2 |f|^2 \, dV_{\mathbb{H}} + \int_{\mathbb{H}^N} |G(|x|)| \left| \nabla_{\mathbb{H}} \left( \frac{f}{\sqrt{G(|x|)}} \right) \right|^2 \, dV_{\mathbb{H}}.
\]

Now, we note that \( K_{\mathbb{H}^N} = -1 \). Therefore \( J(u, t) = J_1(t) \) does not depend on \( u \).
Moreover
\[
J_1(t) = \left( \frac{\sinh t}{t} \right)^{N-1}.
\]
and
\[
\frac{J'(u, t)}{J(u, t)} = \frac{J_1'(t)}{J_1(t)} = \frac{N-1}{t} (t \coth t - 1).
\]
Therefore, we can rewrite Theorem 1.1 as follows:

**Theorem 3.2** Let \( (r^{N-1} V, r^{N-1} W) \) be a Bessel pair on \( (0, R) \). Then we have the following identities
\[
\int_{B_R} V(\rho(x)) |\nabla_{\mathbb{H}} f|^2 \, dV_{\mathbb{H}} - \int_{B_R} W(\rho(x)) |f|^2 \, dV_{\mathbb{H}} \]
\[
\begin{align*}
= & \int_{B_R} V(\rho(x)) \left| \varphi^2(\rho(x)) \right| \left\| \nabla_{\mathbb{H}} \left( \frac{f}{\varphi(\rho(x))} \right) \right\|^2 dV_{\mathbb{H}} \\
& - (N - 1) \int_{B_R} V(\rho(x)) \frac{\varphi'(\rho(x)) \rho(x) \cosh(\rho(x)) - \sinh(\rho(x))}{\varphi(\rho(x)) \rho(x) \sinh(\rho(x))} |f|^2 dV_{\mathbb{H}}
\end{align*}
\]

and
\[
\begin{align*}
\int_{B_R} V(\rho(x)) \left| \partial_{\rho} f \right|^2 dV_{\mathbb{H}} & - \int_{B_R} W(\rho(x)) |f|^2 dV_{\mathbb{H}} \\
= & \int_{B_R} V(\rho(x)) \left| \varphi^2(\rho(x)) \right| \left\| \partial_{\rho} \left( \frac{f}{\varphi(\rho(x))} \right) \right\|^2 dV_{\mathbb{H}} \\
& - (N - 1) \int_{B_R} V(\rho(x)) \frac{\varphi'(\rho(x)) \rho(x) \cosh(\rho(x)) - \sinh(\rho(x))}{\varphi(\rho(x)) \rho(x) \sinh(\rho(x))} |f|^2 dV_{\mathbb{H}}.
\end{align*}
\]

By applying Theorem 3.2 to some explicit Bessel pairs, we obtain several improvements of the Hardy inequalities on hyperbolic spaces.

**Proof (Proof of Theorem 1.3)** We apply Theorem 3.2 to the Bessel pair \((r^{N-1}, r^{N-1} \left( \frac{N-2}{2} \right)^2 \frac{1}{r^2})\) on \((0, \infty)\). Note that in this case \(\varphi(r) = r^{-\frac{N-2}{2}}\). \qed

From Theorem 1.3, we can deduce the Hardy-Poincaré–Sobolev identities and inequalities that provide improved versions with exact remainder terms of the Hardy-Poincaré–Sobolev inequalities studied in [2, 7].

**Proof of Theorem 1.4** Let \(\Psi(r) = \frac{r}{\sinh r} \) and \(\Phi(r) = \Psi(r)^{\frac{N-1}{2}} = \left( \frac{r}{\sinh r} \right)^{\frac{N-1}{2}}\) and \(W(r) = -\frac{(r\Phi'(r))^t}{r^{N-1}\Phi} \). Then noting that
\[
\Psi'(r) = \frac{(1 - r \coth r)}{\sinh r}
\]
and
\[
\Phi'(r) = \frac{N - 1}{2} \Psi(r)^{\frac{N-3}{2}} \Psi'(r)
\]
\[
= \frac{N - 1}{2} \left( \frac{r}{\sinh r} \right)^{\frac{N-3}{2}} \frac{(1 - r \coth r)}{\sinh r}
\]
Hence
\[
\frac{\Phi'(r)}{\Phi(r)} = \frac{N - 1}{2} \frac{1 - r \coth r}{r}
\]
and

\[ r^{N-2} W(r) = \frac{N-1}{2} \left( r \left( \frac{r}{\sinh r} \right)^{\frac{N-2}{2}} \frac{1-r \coth r}{\sinh r} \right) \left( \frac{r}{\sinh r} \right)^{\frac{N-1}{2}} = \frac{N-1}{2} \left( \frac{1-r \coth r}{r^2} + \frac{N-3}{r^2} \frac{(1-r \coth r)^2}{r^2} \right) + \coth^2 r - 2 \frac{\coth r}{r} + \frac{1}{\sinh^2 r} \]

Note that \( (r^{N-1} \frac{1}{r^{N-2}} \Phi'(r))^' + r^{N-1} W(r) \Phi(r) = 0 \), we have by Theorem 3.2 that

\[ \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}} (x) \left| \nabla_{\mathbb{H}} \left( \rho^{\frac{N-2}{2}} (x) f \right) \right|^2 dV_{\mathbb{H}} \]

\[ = \int_{\mathbb{H}^N} W(\rho(x)) \left| \rho^{\frac{N-2}{2}} (x) f \right|^2 dV_{\mathbb{H}} \]

\[ + \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}} (x) \Phi^2(\rho(x)) \left| \nabla_{\mathbb{H}} \left( \frac{\rho^{\frac{N-2}{2}} (x) f}{\Phi(\rho(x))} \right) \right|^2 dV_{\mathbb{H}} \]

\[ + \frac{(N-1)^2}{2} \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}} (x) \left( \rho(x) \coth \rho(x) - \frac{1}{\rho(x)} \right)^2 \left| \rho^{\frac{N-2}{2}} (x) f \right|^2 dV_{\mathbb{H}}. \]

Therefore, from Theorem 1.3, we obtain

\[ \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 dV_{\mathbb{H}} \]

\[ = \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{H}^N} \frac{|f|^2}{\rho^2} dV_{\mathbb{H}} \]

\[ + \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}} (x) \left| \nabla_{\mathbb{H}} \left( \rho^{\frac{N-2}{2}} (x) f \right) \right|^2 dV_{\mathbb{H}} \]

\[ + \frac{(N-2)(N-1)}{2} \int_{\mathbb{H}^N} \rho(x) \coth \rho(x) - \frac{1}{\rho^2(x)} |f|^2 dV_{\mathbb{H}} \]

\[ = \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}} (x) \Phi^2(\rho(x)) \left| \nabla_{\mathbb{H}} \left( \frac{\rho^{\frac{N-2}{2}} (x) f}{\Phi(\rho(x))} \right) \right|^2 dV_{\mathbb{H}} \]

\[ + \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{H}^N} \frac{|f|^2}{\rho^2} dV_{\mathbb{H}} + \frac{(N-2)(N-1)}{2} \int_{\mathbb{H}^N} \rho(x) \coth \rho(x) - \frac{1}{\rho^2(x)} |f|^2 dV_{\mathbb{H}} \]

\[ + \frac{(N-1)^2}{2} \int_{\mathbb{H}^N} \left( \rho(x) \coth \rho(x) - \frac{1}{\rho(x)} \right)^2 |f|^2 + \int_{\mathbb{H}^N} \rho^{N-2} (x) W(\rho(x)) |f|^2 dV_{\mathbb{H}}. \]
We have

\[ \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} \left\| \nabla_{\mathbb{H}} \left( \rho^{\frac{N-2}{2}}(x) f \right) \right\|^2 dV_{\mathbb{H}} \]

\[ + \int_{\mathbb{H}^N} \left[ -\frac{(N-2)^2}{2} \frac{1}{\rho^2(x)} + \frac{(N-2)(N-1)}{4} \frac{1}{\rho^2(x)} \frac{\rho (x) \coth \rho (x) - 1}{\rho^2(x)} + \frac{(N-1)^2}{2} \left( \frac{\rho (x) \coth \rho (x) - 1}{\rho^2(x)} \right)^2 + \frac{1}{\sinh \rho} \right] |f|^2 dV_{\mathbb{H}} \]

\[ = \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} \Phi^2 (\rho (x)) \left\| \nabla_{\mathbb{H}} \left( \rho^{\frac{N-2}{2}}(x) f \right) \right\|^2 dV_{\mathbb{H}} \]

\[ + \int_{\mathbb{H}^N} \left[ \frac{(N-1)^2}{4} + \frac{1}{4} \frac{1}{\rho^2(x)} + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 \rho (x)} \right] |f|^2 dV_{\mathbb{H}} \]

In other words,

\[ \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 dV_{\mathbb{H}} - \int_{\mathbb{H}^N} \left[ \frac{(N-1)^2}{4} + \frac{1}{4} \frac{1}{\rho^2(x)} + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 \rho (x)} \right] |f|^2 dV_{\mathbb{H}} \]

\[ = \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} \Phi^2 (\rho (x)) \left\| \nabla_{\mathbb{H}} \left( \rho^{\frac{N-2}{2}}(x) f \right) \right\|^2 dV_{\mathbb{H}}. \]

Similarly, we also get

\[ \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} f|^2 dV_{\mathbb{H}} - \int_{\mathbb{H}^N} \left[ \frac{(N-1)^2}{4} + \frac{1}{4} \frac{1}{\rho^2(x)} + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 \rho (x)} \right] |f|^2 dV_{\mathbb{H}} \]

\[ = \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} \Phi^2 (\rho (x)) \left\| \partial_{\rho} \left( \rho^{\frac{N-2}{2}}(x) f \right) \right\|^2 dV_{\mathbb{H}}. \]

\[ \square \]

**Corollary 3.2** We have

\[ \int_{\mathbb{H}^N} \left( \frac{1}{\rho^{\lambda}(x)} \right) \left| \nabla_{\mathbb{H}} f \right|^2 dV_{\mathbb{H}} - \frac{(N - \lambda - 2)^2}{4} \int_{\mathbb{H}^N} \left| f \right|^2 dV_{\mathbb{H}} \]

\[ = \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} \left| \nabla_{\mathbb{H}} \left( \rho^{\frac{N-2}{2}}(x) f \right) \right|^2 dV_{\mathbb{H}} \]

\[ + \frac{(N - \lambda - 2)(N - 1)}{2} \int_{\mathbb{H}^N} \rho (x) \cosh \rho (x) - \sinh \rho (x) \right| f \right|^2 dV_{\mathbb{H}} \]

and

\[ \int_{\mathbb{H}^N} \left( \frac{1}{\rho^{\lambda}(x)} \right) \left| \partial_{\rho} f \right|^2 dV_{\mathbb{H}} - \frac{(N - \lambda - 2)^2}{4} \int_{\mathbb{H}^N} \left| f \right|^2 dV_{\mathbb{H}} \]
\[
\begin{align*}
&= \int_{\mathbb{H}^N} \frac{1}{\rho^{N-2}(x)} \left| \partial_\rho \left( \frac{\rho^{N-\lambda-2}}{2} f \right) \right|^2 dV_{\mathbb{H}} \\
&+ \frac{(N-\lambda-2)(N-1)}{2} \int_{\mathbb{H}^N} \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^{\lambda+2}(x) \sinh \rho(x)} \left| f \right|^2 dV_{\mathbb{H}}.
\end{align*}
\]

As a consequence of these identities, we get that for \( \lambda < N-2 \):

\[
\int_{\mathbb{H}^N} \frac{\left| \nabla_{\mathbb{H}} f \right|^2}{\rho^\lambda(x)} dV_{\mathbb{H}}
\geq \int_{\mathbb{H}^N} \frac{\left| \partial_\rho f \right|^2}{\rho^\lambda(x)} dV_{\mathbb{H}}
\geq \frac{(N-\lambda-2)^2}{4} \int_{\mathbb{H}^N} \frac{\left| f \right|^2}{\rho^{\lambda+2}(x)} dV_{\mathbb{H}} + \frac{(N-\lambda-2)(N-1)}{2} \int_{\mathbb{H}^N} \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^{\lambda+2}(x) \sinh \rho(x)} \left| f \right|^2 dV_{\mathbb{H}}
\geq \frac{(N-\lambda-2)^2}{4} \int_{\mathbb{H}^N} \frac{\left| f \right|^2}{\rho^{\lambda+2}(x)} dV_{\mathbb{H}}.
\]

**Proof** \( (\rho^{N-1-\lambda}, \rho^{N-1-\lambda} \frac{(N-\lambda-2)^2}{4} \frac{1}{r^2}) \) is a Bessel pair on \((0, \infty)\) with \( \varphi(r) = r^{\frac{2-N+\lambda}{2}} \).

**Corollary 3.3** We have

\[
\begin{align*}
\int_{0<\rho(x)<R} \frac{\left| \nabla_{\mathbb{H}} f \right|^2}{\rho^{N-2}(x)} dV_{\mathbb{H}} - \frac{z_0^2}{R^2} \int_{0<\rho(x)<R} \frac{\left| f \right|^2}{\rho^{N-2}(x)} dV_{\mathbb{H}}
\end{align*}
\]

\[
\begin{align*}
&= \int_{0<\rho(x)<R} \frac{J_0^2 \left( \frac{z_0}{R} \rho(x) \right)}{\rho^{N-2}(x)} \left| \nabla_{\mathbb{H}} \left( \frac{f}{J_0 \left( \frac{z_0}{R} \rho(x) \right)} \right) \right|^2 dV_{\mathbb{H}} \\
&- (N-1) \frac{z_0^2}{R} \int_{0<\rho(x)<R} \frac{J_0^2 \left( \frac{z_0}{R} \rho(x) \right)}{J_0 \left( \frac{z_0}{R} \rho(x) \right)} \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^{N-1}(x) \sinh \rho(x)} \left| f \right|^2 dV_{\mathbb{H}}
\end{align*}
\]

and

\[
\begin{align*}
\int_{0<\rho(x)<R} \frac{\left| \partial_\rho f \right|^2}{\rho^{N-2}(x)} dV_{\mathbb{H}} - \frac{z_0^2}{R^2} \int_{0<\rho(x)<R} \frac{\left| f \right|^2}{\rho^{N-2}(x)} dV_{\mathbb{H}}
\end{align*}
\]
\[
= \int_{0<\rho(x)<R} \frac{J_0^2 \left( \frac{z_0}{R} \rho(x) \right)}{\rho^{N-2}(x)} \left| \partial_\rho \left( \frac{f}{J_0 \left( \frac{z_0}{R} \rho(x) \right)} \right) \right|^2 dV_H
\]

\[
- (N-1) \frac{z_0}{R} \int_{0<\rho(x)<R} \frac{J_0' \left( \frac{z_0}{R} \rho(x) \right)}{J_0 \left( \frac{z_0}{R} \rho(x) \right)} \rho(x) \cosh \rho(x) - \sinh \rho(x) \frac{\rho^{N-1}(x) \sinh \rho(x)}{\rho^{N-2}(x)} |f|^2 dV_H.
\]

As a consequence of these identities, we get that
\[
\int_{0<\rho(x)<R} \frac{|\nabla_H f|^2}{\rho^{N-2}(x)} dV_H \geq \int_{0<\rho(x)<R} \frac{\left| \partial_\rho f \right|^2}{\rho^{N-2}(x)} dV_H \geq \frac{2}{R^2} \int_{0<\rho(x)<R} \frac{|f|^2}{\rho^{N-2}(x)} dV_H
\]

\[
- (N-1) \frac{z_0}{R} \int_{0<\rho(x)<R} \frac{J_0' \left( \frac{z_0}{R} \rho(x) \right)}{J_0 \left( \frac{z_0}{R} \rho(x) \right)} \rho(x) \cosh \rho(x) - \sinh \rho(x) \frac{\rho^{N-1}(x) \sinh \rho(x)}{\rho^{N-2}(x)} |f|^2 dV_H
\]

\[
\geq \frac{2}{R^2} \int_{0<\rho(x)<R} \frac{|f|^2}{\rho^{N-2}(x)} dV_H.
\]

**Proof** \( \left( r^{N-1,2-N}, r^{N-1,2-N} \frac{z_0}{R^2} \right) \) is a Bessel pair on \((0, R)\) with \( \varphi(r) = J_0 \left( \frac{z_0}{R} r \right) \) and \( \varphi'(r) = \frac{z_0}{R} J_0' \left( \frac{z_0}{R} r \right) \).

**Proof** (Proof of Theorem 1.5) We note that \( \left( r^{N-1,2-N}, r^{N-1,2-N} \frac{z_0}{R^2} \right) \) on \((0, R)\) with \( \varphi(r) = r^{\frac{z_0-N+1}{2}} J_\alpha \left( \frac{z_0}{R} r \right) \) \( 0 \leq \alpha \leq \frac{N-\lambda-2}{2} \). Here \( z_\alpha \) is the first zero of the Bessel function \( J_\alpha (z) \). Now, we can apply Theorem 3.2 to obtain the desired results.

**Corollary 3.4** We have
\[
\int_{0<\rho(x)<R} \frac{|\nabla_H f|^2}{\rho^{N-2}(x)} dV_H - \frac{1}{4} \int_{0<\rho(x)<R} \frac{|f|^2}{\rho^{N}(x)} \left| \ln \frac{\rho(x)}{R} \right|^2 dV_H
\]

\[
= \int_{0<\rho(x)<R} \frac{1}{\rho^{N-2}(x)} \left| \ln \frac{\rho(x)}{R} \right| \left| \nabla_H \left( \frac{f}{\sqrt{\ln \frac{|x|}{R}}} \right) \right|^2 dV_H
\]
\[ + \int_{0 < \rho(x) < R} \frac{1}{\rho^{N-2}(x)} \frac{1}{2} \left| \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^2(x) \sinh \rho(x)} \right| |f|^2 d\mathbb{V} \]

and

\[ \int_{0 < \rho(x) < R} \frac{1}{\rho^{N-2}(x)} \left| \frac{\partial \rho}{\partial f} \right|^2 d\mathbb{V} - \frac{1}{4} \int_{0 < \rho(x) < R} \frac{|f|^2}{\rho^N(x) \left| \ln \frac{\rho(x)}{R} \right|^2} d\mathbb{V} \]

\[ = \int_{0 < \rho(x) < R} \frac{1}{\rho^{N-2}(x)} \frac{1}{2} \left| \ln \frac{\rho(x)}{R} \left| \partial \rho \left( \frac{f}{\sqrt{\ln |x|}} \right) \right|^2 \right| d\mathbb{V} \]

\[ + \int_{0 < \rho(x) < R} \frac{1}{\rho^{N-2}(x)} \frac{1}{2} \left| \ln \frac{\rho(x)}{R} \left| \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^2(x) \sinh \rho(x)} \right| |f|^2 d\mathbb{V} \]

As a consequence of these identities, we get that

\[ \int_{0 < \rho(x) < R} \frac{|\nabla_{\mathbb{V}} f|^2}{\rho^{N-2}(x)} d\mathbb{V} \geq \int_{0 < \rho(x) < R} \frac{\left| \frac{\partial \rho}{\partial f} \right|^2}{\rho^{N-2}(x)} d\mathbb{V} \geq \frac{1}{4} \int_{0 < \rho(x) < R} \frac{|f|^2}{\rho^N(x) \left| \ln \frac{\rho(x)}{R} \right|^2} d\mathbb{V} \]

\[ + \int_{0 < \rho(x) < R} \frac{1}{\rho^{N-2}(x)} \frac{1}{2} \left| \ln \frac{\rho(x)}{R} \left| \frac{\rho(x) \cosh \rho(x) - \sinh \rho(x)}{\rho^2(x) \sinh \rho(x)} \right| |f|^2 d\mathbb{V} \geq \frac{1}{4} \int_{0 < \rho(x) < R} \frac{|f|^2}{\rho^N(x) \left| \ln \frac{\rho(x)}{R} \right|^2} d\mathbb{V}. \]

**Proof** \( \left( \rho^{N-1} \frac{1}{\rho^{N-2}}, \rho^{N-1} \frac{1}{4\rho^N |\ln \frac{\rho}{R}|^2} \right) \) is a Bessel pair on \((0, R)\) with \( \varphi(r) = \sqrt{|\ln \frac{r}{R}|} \) and \( \varphi'(r) = -\frac{1}{2r \sqrt{|\ln \frac{r}{R}|}}. \)
4 Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1**  Let \( f (x) = \varphi (\rho (x)) v (x) \), then

\[
\int_{B_R(O)} V (\rho (x)) |\nabla_g f|^2_g \, dV_g
\]

\[
= \int_{B_R(O)} V (\rho (x)) |\nabla_g (\varphi (\rho (x)) v (x))|^2_g \, dV_g
\]

\[
= \int_{B_R(O)} V (\rho (x)) \left| \varphi^2 (\rho (x)) |\nabla_g v|^2_g \, dV_g + \int_{B_R(O)} V (\rho (x)) \left| v^2 (x) |\nabla_g \varphi (\rho (x))|^2_g \, dV_g
\]

\[
+ \int_{B_R(O)} V (\rho (x)) \varphi (\rho (x)) \left( \nabla_g v^2, \nabla_g \varphi (\rho (x)) \right)_g \, dV_g
\]

\[
= \int_{B_R(O)} V (\rho (x)) \left| \varphi^2 (\rho (x)) |\nabla_g v|^2_g \, dV_g + \int_{B_R(O)} V (\rho (x)) \left| v^2 (x) |\varphi' (\rho (x))|^2_g \, dV_g
\]

\[
+ \int_{B_R(O)} V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \left( \nabla_g v^2, \nabla_g \rho (x) \right)_g \, dV_g
\]

Now, using the divergence theorem, we get

\[
\int_{B_R(O)} V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \left( \nabla_g v^2, \nabla_g \rho (x) \right)_g \, dV_g
\]

\[
= - \int_{B_R(O)} v^2 (x) \text{div} \left( V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \nabla_g \rho (x) \right)
\]

\[
= - \int_{B_R(O)} v^2 (x) V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \Delta_g \rho (x) \, dV_g
\]

\[
- \int_{B_R(O)} v^2 (x) V' (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \, dV_g
\]

\[
- \int_{B_R(O)} v^2 (x) V (\rho (x)) \varphi' (\rho (x)) \varphi' (\rho (x)) \, dV_g
\]

\[
- \int_{B_R(O)} v^2 (x) V (\rho (x)) \varphi (\rho (x)) \varphi'' (\rho (x)) \, dV_g
\]

Noting that (see [29, 4.B.2])

\[
\Delta_g \rho (x) = \frac{N - 1}{\rho (x)} + \frac{J' (u, \rho (x))}{J (u, \rho (x))}.
\]
Hence

\[ \int_{B_R(0)} V(\rho(x)) \left| \nabla_g \phi \right|^2 dV_g - \int_{B_R(0)} V(\rho(x)) \left| \nabla_g v \right|^2 dV_g \]

\[ = - \int_{B_R(0)} \varphi(\rho(x)) v^2(x) \left[ \frac{V(\rho(x)) \phi'(\rho(x)) \frac{N-1}{\rho(x)} + V'(\rho(x)) \phi'(\rho(x)) + V(\rho(x)) \phi''(\rho(x))}{J'(u, \rho(x))} \right] dV_g \]

\[ = \int_{B_R(0)} \varphi(\rho(x)) v^2(x) W(\rho(x)) - \int_{B_R(0)} v^2(x) V(\rho(x)) \varphi(\rho(x)) \phi'(\rho(x)) \frac{J'(u, \rho(x))}{J(u, \rho(x))} dV_g \]

\[ = \int_{B_R(0)} W(\rho(x)) |f|^2 dV_g - \int_{B_R(0)} v^2(x) V(\rho(x)) \varphi(\rho(x)) \phi'(\rho(x)) \frac{J'(u, \rho(x))}{J(u, \rho(x))} dV_g \]

Now, denote \( F(y) = f(\exp_O(y)), \Phi(y) = \varphi(\exp_O(y)) \) and \( \Psi(y) = v(\exp_O(y)) \). Then using the polar coordinate we get

\[ \int_{B_R(0)} W(\rho(x)) |f|^2 dV_g \]

\[ = \int_{\mathbb{S}^{N-1}} \int_0^R \rho^{N-1} W(\rho) \Phi(\rho) \Psi^2(\rho u) J(u, \rho) d\rho du \]

\[ = - \int_{\mathbb{S}^{N-1}} \int_0^R \partial_\rho \left( \rho^{N-1} V(\rho) \partial_\rho \Phi(\rho) \right) \Phi(\rho) \Psi^2(\rho u) J(u, \rho) d\rho du \]

\[ = \int_{\mathbb{S}^{N-1}} \int_0^R \rho^{N-1} V(\rho) \partial_\rho \Phi(\rho) \partial_\rho \left[ \Phi(\rho) \Psi^2(\rho u) J(u, \rho) \right] d\rho du \]

\[ = \int_{\mathbb{S}^{N-1}} \int_0^R \rho^{N-1} V(\rho) \left( \partial_\rho \Phi(\rho) \right)^2 \Psi^2(\rho u) J(u, \rho) d\rho du \]

\[ + 2 \int_{\mathbb{S}^{N-1}} \int_0^R \rho^{N-1} V(\rho) \partial_\rho \Phi(\rho) \Phi(\rho) \partial_\rho \Psi(\rho u) \Psi(\rho u) J(u, \rho) d\rho du \]

\[ + \int_{\mathbb{S}^{N-1}} \int_0^R \rho^{N-1} V(\rho) \partial_\rho \Phi(\rho) \Phi(\rho) \Psi^2(\rho u) \partial_\rho J(u, \rho) d\rho du. \]
Hence, we have
\[
\int_{B_R(O)} W(\rho(x)) |f|^2 \, dV_g
\]
\[
= \int_{S^{N-1}} \int_0^R \rho^{N-1} V(\rho) \left| \Psi(\rho u) \partial_\rho \Phi(\rho) + \Phi(\rho) \partial_\rho \Psi(\rho u) \right|^2 J(u, \rho) \, d\rho \, du
\]
\[
- \int_{S^{N-1}} \int_0^R \rho^{N-1} V(\rho) \left| \Phi(\rho) \partial_\rho \Psi(\rho u) \right|^2 J(u, \rho) \, d\rho \, du
\]
\[
+ \int_{S^{N-1}} \int_0^R \rho^{N-1} V(\rho) \partial_\rho \Phi(\rho) \partial_\rho J(u, \rho) \, d\rho \, du
\]
\[
= \int_{B_R(O)} V(\rho(x)) \left| \partial_\rho f \right|^2 \, dV_g - \int_{B_R(O)} V(\rho(x)) \left| \partial_\rho \left( \frac{f}{\varphi(\rho(x))} \right) \right|^2 \varphi^2(\rho(x)) \, dV_g
\]
\[
+ \int_{B_R(O)} \left( \frac{f}{\varphi(\rho(x))} \right)^2 \varphi(\rho(x)) \varphi'(\rho(x)) \frac{J'(u, \rho(x))}{J(u, \rho(x))} \, dV_g.
\]

**Proof of Theorem 1.2** Let \( f(x) - f(\exp_O(Ru)) = \varphi(\rho(x)) v(x) \). Then proceed as in the proof of Theorem 1.1, we get
\[
\int_{B_R(O)} V(\rho(x)) |\nabla_g (f - f(\exp_O(Ru)))|^2 \, dV_g
\]
\[
= \int_{B_R(O)} V(\rho(x)) |\nabla_g (\varphi(\rho(x)) v(x))|^2 \, dV_g
\]
\[
= \int_{B_R(O)} V(\rho(x)) |\varphi^2(\rho(x))| |\nabla_g v|^2 \, dV_g + \int_{B_R(O)} V(\rho(x)) \left| v^2 \right| |\nabla_g \varphi(\rho(x))|^2 \, dV_g
\]
\[
+ \int_{B_R(O)} V(\rho(x)) \varphi(\rho(x)) \left( |\nabla_g v|^2, |\nabla_g \varphi(\rho(x))|^2 \right) \, dV_g
\]
\[
= \int_{B_R(O)} V(\rho(x)) |\varphi^2(\rho(x))| |\nabla_g v|^2 \, dV_g + \int_{B_R(O)} V(\rho(x)) \left| v^2 \right| |\varphi'(\rho(x))|^2 \, dV_g
\]
\[
+ \int_{B_R(O)} V(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \left( |\nabla_g v|^2, |\nabla_g \rho(x)|^2 \right) \, dV_g.
\]

Now, using the divergence theorem, we get
\[
\int_{B_R(O)} V(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \left( |\nabla_g v|^2, |\nabla_g \rho(x)|^2 \right) \, dV_g
\]
\[
\begin{align*}
&= - \int_{BR(O)} v^2 \text{div} \left( V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \nabla_g \rho (x) \right) dV_g \\
&\quad - \int_{\partial BR(O)} v^2 V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \frac{\partial \rho}{\partial v} (x) dS_g.
\end{align*}
\]

By the assumption (C) on \(f\), we get
\[
\int_{\partial BR(O)} v^2 V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \frac{\partial \rho}{\partial v} (x) dS_g = 0.
\]

Hence
\[
\begin{align*}
&\int_{BR(O)} V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \left\langle \nabla_g v^2, \nabla_g \rho (x) \right\rangle_g dV_g \\
&\quad = - \int_{BR(O)} v^2 V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \Delta_g \rho (x) dV_g \\
&\quad \quad - \int_{BR(O)} v^2 V' (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) dV_g \\
&\quad \quad - \int_{BR(O)} v^2 V (\rho (x)) \varphi' (\rho (x)) \varphi' (\rho (x)) dV_g \\
&\quad \quad - \int_{BR(O)} v^2 V (\rho (x)) \varphi (\rho (x)) \varphi'' (\rho (x)) dV_g.
\end{align*}
\]

Again, using
\[
\Delta_g \rho (x) = \frac{N - 1}{\rho (x)} + \frac{J' (u, \rho)}{J (u, \rho)}
\]

we get
\[
\begin{align*}
&\int_{BR(O)} V (\rho (x)) \left\langle \nabla_g \left( f - f \left( \exp_{O} (Ru) \right) \right) \right\rangle_R^2 dV_g - \int_{BR(O)} V (\rho (x)) \left\langle \varphi^2 (\rho (x)) \right\rangle dV_g \\
&\quad = - \int_{BR(O)} \varphi (\rho (x)) v^2 \left[ V (\rho (x)) \varphi' (\rho (x)) \frac{N - 1}{\rho (x)} + V (\rho (x)) \varphi'' (\rho (x)) \right] dV_g \\
&\quad \quad - \int_{BR(O)} v^2 V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \frac{J' (u, \rho)}{J (u, \rho)} dV_g \\
&\quad \quad = \int_{BR(O)} \varphi^2 (\rho (x)) v^2 W (\rho (x)) - \int_{BR(O)} v^2 V (\rho (x)) \varphi (\rho (x)) \varphi' (\rho (x)) \frac{J' (u, \rho)}{J (u, \rho)} dV_g.
\end{align*}
\]
\[
\int_{B_R(O)} W(\rho(x)) \left| f - f\left(\exp_O (Ru)\right) \right|^2 \, dV_g \\
- \int_{B_R(O)} v^2 V(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \frac{J'(u, \rho)}{J(u, \rho)} \, dV_g.
\]

Similarly,
\[
\int_{\mathbb{M}\setminus B_R(O)} V(\rho(x)) \left| \nabla_g \left( f - f\left(\exp_O (Ru)\right) \right) \right|^2 \, dV_g \\
- \int_{\mathbb{M}\setminus B_R(O)} V(\rho(x)) \varphi^2(\rho(x)) \left| \nabla_g \left( \frac{f - f\left(\exp_O (Ru)\right)}{\varphi(\rho(x))} \right) \right|^2 \, dV_g \\
= \int_{\mathbb{M}\setminus B_R(O)} W(\rho(x)) \left| f - f\left(\exp_O (Ru)\right) \right|^2 \, dV_g \\
- \int_{\mathbb{M}\setminus B_R(O)} v^2 V(\rho(x)) \varphi(\rho(x)) \varphi'(\rho(x)) \frac{J'(u, \rho)}{J(u, \rho)} \, dV_g.
\]

Therefore
\[
\int_{\mathbb{M}} V(\rho(x)) \left| \nabla_g \left( f - f\left(\exp_O (Ru)\right) \right) \right|^2_g \, dx - \int_{\mathbb{M}} W(\rho(x)) \left| f - f\left(\exp_O (Ru)\right) \right|^2 \, dV_g \\
= \int_{\mathbb{M}} V(\rho(x)) \varphi^2(\rho(x)) \left| \nabla_g \left( \frac{f - f\left(\exp_O (Ru)\right)}{\varphi(\rho(x))} \right) \right|^2_g \, dx \\
- \int_{\mathbb{M}} V(\rho(x)) \left| f - f\left(\exp_O (Ru)\right) \right|^2 \varphi'(\rho(x)) \frac{J'(u, \rho)}{\varphi(\rho(x))} J(u, \rho) \, dV_g.
\]

Similarly,
\[
\int_{\mathbb{M}} V(\rho(x)) \left| \partial_\rho \left( f - f\left(\exp_O (Ru)\right) \right) \right|^2_g \, dx - \int_{\mathbb{M}} W(\rho(x)) \left| f - f\left(\exp_O (Ru)\right) \right|^2 \, dV_g \\
= \int_{\mathbb{M}} V(\rho(x)) \varphi^2(\rho(x)) \left| \partial_\rho \left( \frac{f - f\left(\exp_O (Ru)\right)}{\varphi(\rho(x))} \right) \right|^2_g \, dx \\
- \int_{\mathbb{M}} V(\rho(x)) \left| f - f\left(\exp_O (Ru)\right) \right|^2 \varphi'(\rho(x)) \frac{J'(u, \rho)}{\varphi(\rho(x))} J(u, \rho) \, dV_g.
\]

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