Discrete Symmetry and GUT Breaking

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Abstract

We study the supersymmetric GUT models where the supersymmetry and GUT gauge symmetry can be broken by the discrete symmetry. First, with the ansatz that there exist discrete symmetries in the branes’ neighborhoods, we discuss the general reflection $Z_2$ symmetries and GUT breaking on $M^4 \times M^1$ and $M^4 \times M^1 \times M^1$. In those models, the extra dimensions can be large and the KK states can be set arbitrarily heavy. Second, considering the extra space manifold is the annulus $A^2$ or disc $D^2$, we can define any $Z_n$ symmetry and break any 6-dimensional $N = 2$ supersymmetric $SU(M)$ models down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{M-4}$ models for the zero modes. In particular, there might exist the interesting scenario on $M^4 \times A^2$ where just a few KK states are light, while the others are relatively heavy. Third, we discuss the complete global discrete symmetries on $M^4 \times T^2$ and study the GUT breaking.

PACS: 11.10.Kk; 11.25.Mj; 04.65.+e; 11.30.Pb

Keywords: Grand Unified Theory; Symmetry Breaking; Extra Dimensions

October 2001

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1 Introduction

Grand Unified Theory (GUT) gives us an simple and elegant understanding of the quantum numbers of the quarks and leptons, and the success of gauge coupling unification in the Minimal Supersymmetric Standard Model strongly support this idea. The Grand Unified Theory at high energy scale has been widely accepted now, there are some problems in GUT: the grand unified gauge symmetry breaking mechanism, the doublet-triplet splitting problem, and the proton decay, etc.

As we know, one obvious approach to break GUT gauge symmetry is Higgs mechanism [1], which is discussed extensively in phenomenology. Another approach is spin connection embedding, which is used in the weakly coupled heterotic string $E_8 \times E_8$, and M-theory on $S^1/Z_2$ [2]. Because the Calabi-Yau manifold has $SU(3)$ holonomy, the observable $E_8$ gauge group can be broken down to $E_6$ by spin connection embedding. In addition, the GUT gauge symmetry can be broken down to a low energy subgroup by means of Wilson lines, provided that the fundamental group of the extra space manifold or orbifold is non-trivial [3].

Recently, a new scenario to explain above questions in GUT has been suggested by Kawamura [4, 5, 6], and further discussed by a lot of papers [7, 8]. The key point is that the GUT gauge symmetry exists in 5 or higher dimensions and is broken down to the 4-dimensional $N = 1$ supersymmetric Standard Model like gauge symmetry for the zero modes due to the discrete symmetries in the branes’ neighborhoods, which become the non-trivial orbifold projections on the multiplets and gauge generators in GUT. So, we would like to call it the discrete symmetry approach. The attractive models have been constructed explicitly, where the supersymmetric 5-dimensional and 6-dimensional GUT models are broken down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ model, where $n$ is the rank of GUT group, through the compactification on various orbifolds. The GUT gauge symmetry breaking and doublet-triplet splitting problems have been solved neatly by the orbifold projections, and other interesting phenomenology, like $\mu$ problems, gauge coupling unifications, non-supersymmetric GUT, gauge-Higgs unification, proton decay, etc, have also been discussed [4, 8]. By the way, it seems to us that this approach is similar to the Wilson line approach, but not the same, for example, the fundamental group of extra space manifold can be trivial in the discrete symmetry approach, and we may not break the supersymmetry by Wilson line approach.

On the other hand, large extra dimension scenarios with branes have been an very interesting subject for the past few years, where the gauge hierarchy problem can be solved because the physical volume of extra dimensions may be very large and the higher dimensional Planck scale might be low [9], or the metric for the extra dimensions has warp factor [10]. Naively, one might think the masses of KK states are $\sqrt{\sum_i n_i^2/R_i^2}$, where $R_i$ is the radius of the $i$-th extra dimension. However, it is shown that this is not true if one considered the shape moduli [11] or the local discrete symmetry in the brane neighborhood [12], and it may be possible to maintain the ratio (hierarchy) between the higher dimensional Planck scale and 4-dimensional...
Planck scale while simultaneously making the KK states arbitrarily heavy. So, a lot of experimental bounds on the theories with large extra dimensions are relaxed. Moreover, the gauge symmetry and supersymmetry can be broken if we consider the local discrete symmetry \[12\].

In this paper, we study the supersymmetric GUT models where the supersymmetry and GUT gauge symmetry can be broken by the discrete symmetries in the branes’ neighborhoods or on the extra space manifold. We require that for the zero modes in the bulk, the supersymmetric GUT models are broken down to the 4-dimensional \(N = 1\) supersymmetric \(SU(3) \times SU(2) \times U(1)^{n-3}\) model, and above the GUT scale or including the zero modes and KK modes, the bulk should preserve the original GUT gauge symmetry and supersymmetries, i. e., we can not project out all the zero modes and KK modes of the fields in the theories. In addition, we define two discrete symmetries \(Z_n\) and \(Z'_n\) are equivalent if: (1) \(n = n'\); (2) in order to satisfy our requirement, the representation for the generator of \(Z_n\) in the adjoint representation of GUT group \(G\) must be the same as that for the generator of \(Z'_n\).

First, we would like to explore the general scenarios where the GUT gauge symmetry and supersymmetry can be broken by the discrete symmetries in the brane neighborhood, and the masses of KK states can be set arbitrarily heavy. Our ansatz is that there exist the discrete symmetries (local or global) in the special branes’ neighborhoods, which become the additional constraints on the KK states. The KK states, which satisfy the discrete symmetries, remain in the theories, while the KK states, which do not satisfy the discrete symmetries, are projected out. Therefore, we can construct the theories with only zero modes for all the KK modes are projected out, or the theories which have large extra dimensions and arbitrarily heavy KK modes because there is no simple relation between the mass scales of extra dimensions and the masses of KK states. In addition, the bulk gauge symmetry and supersymmetry can be broken on the special branes for the zero and KK modes, and in the bulk for the zero modes by local and global discrete symmetries.

(I) We generalize our previous models \[12\] to the models on the space-time \(M^4 \times S^1\) and \(M^4 \times I^1\), where the \(M^4\) is the 4-dimensional Minkowski space-time. And we point out that the general models on \(M^4 \times I^1\) can be obtained from the general models on \(M^4 \times S^1\) by moduloin the \((Z_2)^{k_0}\) symmetry in which \(k_0\) is the positive integer. Moreover, we find that, to satisfy our ansatz and requirement, there are at most two non-equivalent \(Z_2\) symmetries, which can be local or global. Therefore, we can only discuss the 5-dimensional \(N = 1\) supersymmetric \(SU(5)\) model. In this scenario, the bulk 4-dimensional \(N = 2\) supersymmetry and \(SU(5)\) gauge symmetry are broken down to the 4-dimensional \(N = 1\) supersymmetry and \(SU(3) \times SU(2) \times U(1)\) gauge symmetry on the special brane with GUT breaking \(Z_2\) symmetry for all the modes, and in the bulk for the zero modes. And the 3-branes preserve half of the bulk supersymmetry. Moreover, the masses of KK states can be set arbitrarily heavy although the physical size of the fifth dimension can be large, even at millimeter range. By the way, one can also discuss the non-supersymmetric \(SU(6)\) and \(SO(10)\) breaking, however, there are zero modes for \(A^\tilde{a}\) where \(\tilde{a}\) is the index related to the broken gauge generators under two non-equivalent \(Z_2\) projections.
We study the models on the space-time $M^4 \times M^1 \times M^1$ where $M^1$ can be $S^1$, $S^1/Z_2$, and $I^1$. Because the extra space manifold is the product of two 1-dimensional manifold and the discussions are similar, as representatives, we discuss the models on the space-times $M^4 \times S^1 \times S^1$ where there are parallel 4-branes with $Z_2$ symmetry along the fifth and sixth dimensions, and there are at most four non-equivalent $Z_2$ symmetries. We also discuss the models on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$ where there are only four 4-branes at boundaries and some 3-branes in the bulk, and there are three non-equivalent $Z_2$ symmetries. In those models, the extra dimensions can be large and the masses of KK states can be set arbitrarily heavy. The 6-dimensional non-supersymmetric GUT models and $N = 1$ supersymmetric GUT models can be considered as special cases of $N = 2$ supersymmetric GUT models, so, we discuss the 6-dimensional $N = 2$ supersymmetric GUT models. Because $N = 2$ 6-dimensional supersymmetric theory has 16 real supercharges, which corresponds to $N = 4$ 4-dimensional supersymmetric theory, we can not have hypermultiplets in the bulk, and then, we have to put the Standard Model fermions on the brane. As representatives, we discuss the 6-dimensional $N = 2$ supersymmetric $SU(6)$ and $SO(10)$ models on the space-time $M^4 \times S^1 \times S^1$, and the 6-dimensional $N = 2$ supersymmetric $SU(6)$ models with gauge-Higgs unification on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$. For the zero modes, the bulk 4-dimensional $N = 4$ supersymmetry and $SU(6)$ or $SO(10)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry.

Second, we discuss the models where the extra space manifold is the disc $D^2$ or annulus $A^2$. In this kind of scenarios, we can naturally use the complex coordinates and introduce global $Z_n$ symmetry for any positive integer $n$, so, we can break any $SU(M)$ gauge symmetry for $M \geq 5$ down to the $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry. Similar to above, we only study the 6-dimensional $N = 2$ supersymmetric GUT models with the Standard Model fermions on the boundary 4-branes or on the 3-brane at origin if the extra space manifold is disc $D^2$. There are 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry in the bulk and on the 4-branes for the zero modes, and on the 3-brane at origin in the disc $D^2$ scenario. Including all the KK states, we will have 4-dimensional $N = 4$ supersymmetry and $SU(M)$ gauge symmetry in the bulk, and on the 4-branes. By the way, if we put the Standard Model fermions on the 3-brane at origin, the extra dimensions can be large and the gauge hierarchy problem can be solved for there does not exist the proton decay problem at all. As an example, we discuss the 6-dimensional $N = 2$ supersymmetric $SU(6)$ model on $M^4 \times A^2$ or $M^4 \times D^2$ with $Z_9$ symmetry. Moreover, if the extra space manifold is annulus $A^2$, for suitable choices of the inner radius and outer radius, we might construct the models where only a few KK states are light, and the other KK states are relatively heavy due to the boundary condition on the inner and outer boundaries, so, we might produce the light KK states of gauge fields at future colliders, which is very interesting in collider physics.

In addition, if the extra space manifold is an sector of $D^2$ or an segment of $A^2$, we point out that the masses of KK states can be set arbitrarily heavy if the range of angle is small enough. However, we can not define the discrete symmetry $Z_n$ for
n > 2 on the sector of \( D^2 \) or segment of \( A^2 \), so, it is not interesting for us to discuss the supersymmetric GUT breaking in this case.

Third, we discuss the complete global discrete symmetry on the space-time \( M^4 \times T^2 \). We prove that the possible global discrete symmetries on the torus is \( Z_2, Z_3, Z_4, \) and \( Z_6 \). We also discuss the 6-dimensional \( N = 2 \) supersymmetric \( SU(5) \) models on the space-time \( M^4 \times T^2 \) with \( Z_6 \) symmetry, where the Standard Model fermions on the observable 3-brane at one of the fixed points. There are 4-dimensional \( N = 1 \) supersymmetry and the Standard Model gauge symmetry in the bulk for the zero modes, and on the 3-brane at \( Z_6 \) fixed point for all the modes. Including the KK states, we will have the 4-dimensional \( N = 4 \) supersymmetry and \( SU(5) \) gauge symmetry in the bulk, the 4-dimensional \( N = 1 \) supersymmetry and \( SU(5) \) gauge symmetry on the 3-branes at \( Z_3 \) fixed points, and the 4-dimensional \( N = 4 \) supersymmetry and \( SU(3) \times SU(2) \times U(1) \) gauge symmetry on the 3-branes at \( Z_2 \) fixed points. The Standard Model fermions and Higgs fields can be on any 3-brane at one of the fixed points. In particular, if we put the Standard Model fermions and Higgs fields on the 3-brane at \( Z_6 \) fixed point, the extra dimensions can be large and the gauge hierarchy problem can be solved because there is no proton decay problem at all.

This paper is organized as follows: in section 2, we discuss the discrete symmetry in the brane neighborhood in general. We study the discrete symmetry on the space-time \( M^4 \times M^1, M^4 \times M^1 \times M^1, M^4 \times A^2, M^4 \times T^2 \) in sections 3, 5, 7, 9, respectively. And we discuss the supersymmetric GUT breaking on the space-time \( M^4 \times M^1, M^4 \times M^1 \times M^1, M^4 \times A^2, M^4 \times T^2 \) in sections 4, 6, 8, 10 respectively. Our discussion and conclusion are given in section 11.

### 2 Discrete Symmetry in the Brane Neighborhood

We assume that in a \((4+n)\)-dimensional space-time manifold \( M^4 \times M^n \) where \( M^4 \) is the 4-dimensional Minkowski space-time and \( M^n \) is the manifold for extra space dimensions, there exist some topological defects, or we call them branes for simplicity. The special branes, which we are interested in, have co-dimension one or more than one. Assuming we have \( K \) special branes and using the \( I-th \) special brane as a representative, our ansatz is that in the open neighborhood \( M^4 \times U_I \) \((U_I \subset M^n)\) of the \( I-th \) special brane, there is a global or local discrete symmetry\(^2\) which forms a discrete group \( \Gamma_I \), where \( I = 1, 2, ..., K \). And the Lagrangian is invariant under the discrete symmetries. In addition, we require that above the GUT scale or including all the KK states, the bulk should preserve the original GUT gauge symmetries and supersymmetries, i.e., we can not project out all the KK states of the fields in the theories, and the supersymmetric GUT models are broken down to the 4-dimensional

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\(^2\)Global discrete symmetry is a “special” case of local discrete symmetry. The key difference is that, the space-time manifold can modulo the global discrete symmetry and become a quotient space-time manifold or orbifold.
$N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ model in the bulk for the zero modes.

Assume the local coordinates for extra dimensions in the $I$ – th special brane neighborhood are $y_I^1, y_I^2, \ldots, y_I^n$, the action of any element $\gamma_i^I \subset \Gamma_I$ on $U_I$ can be expressed as

$$\gamma_i^I : (y_I^1, y_I^2, \ldots, y_I^n) \subset U_I \rightarrow (\gamma_i^1 y_I^1, \gamma_i^2 y_I^2, \ldots, \gamma_i^n y_I^n) \subset U_I ,$$

where the $I$ – th special brane position is the only fixed point, line, or hypersurface for the whole group $\Gamma_I$ as long as the neighborhood is small enough.

The Lagrangian is invariant under the discrete symmetry in the neighborhood $M^4 \times U_I$ of the $I$ – th special brane, i. e., for any element $\gamma_i^I \subset \Gamma_I$

$$\mathcal{L}(x^\mu, \gamma_i^1 y_I^1, \gamma_i^2 y_I^2, \ldots, \gamma_i^n y_I^n) = \mathcal{L}(x^\mu, y_I^1, y_I^2, \ldots, y_I^n) ,$$

where $(y_I^1, y_I^2, \ldots, y_I^n) \subset U_I$. So, for a generic bulk multiplet $\Phi$ which fills a representation of bulk gauge group $G$, we have

$$\Phi(x^\mu, \gamma_i^1 y_I^1, \gamma_i^2 y_I^2, \ldots, \gamma_i^n y_I^n) = \eta_i^\Phi (R_{\gamma_i^I})^{l_\Phi} \Phi(x^\mu, y_I^1, y_I^2, \ldots, y_I^n)(R_{\gamma_i^I}^{-1})^{m_\Phi} ,$$

where $\eta_i^\Phi$ is an element of the discrete symmetry and can be determined from the Lagrangian (up to an element in $\Gamma_I$ for the matter fields), $l_\Phi$ and $m_\Phi$ are the non-negative integers determined by the representation of $\Phi$ under the gauge group $G$. Moreover, $R_{\gamma_i^I}$ is an element in $G$, and $R_{\Gamma_I}$ is a discrete subgroup of $G$. We will choose $R_{\gamma_i^I}$ as the matrix representation for $\gamma_i^I$ in the adjoint representation of gauge group $G$. The consistent condition for $R_{\gamma_i^I}$ is

$$R_{\gamma_i^I} R_{\gamma_j^I} = R_{\gamma_i^I \gamma_j^I} , \forall \gamma_i^I, \gamma_j^I \subset \Gamma_I .$$

Mathematical speaking, the map $R : \Gamma_I \rightarrow R_{\Gamma_I} \subset G$ is a homomorphism. Because the special branes are fixed under the discrete symmetry transformations, the gauge group on the $I$ – th special brane is the subgroup of $G$ which commutes with $R_{\Gamma_I}$, and we denote the subgroup as $G/R_{\Gamma_I}$. For the zero modes, the bulk gauge group is broken down to the subgroup of $G$ which commutes with all $R_{\Gamma_I}$, i. e., $R_{\Gamma_1}, R_{\Gamma_2}, \ldots, R_{\Gamma_K}$, and we denote the subgroup as $G/\{R_{\Gamma_1}, R_{\Gamma_2}, \ldots, R_{\Gamma_K}\}$. In addition, if the theory is supersymmetric, the special branes will preserve part of the bulk supersymmetry, and the zero modes in the bulk also preserve part of the supersymmetry, in other words, the supersymmetry can be broken on the special branes for all the modes, and in the bulk for the zero modes.

In addition, we only have the KK states which satisfy the local and global discrete symmetries in the theories because the KK modes, which do not satisfy the local and global discrete symmetries, are projected out under our ansatz. Therefore, we can construct the theories with only zero modes because all the KK modes are projected out, or the theories which have large extra dimensions and arbitrarily heavy KK states for there is no simple relation between the mass scales of extra dimensions and the masses of KK states. By the way, we are only interested in the second kind of scenarios.
3 Discrete Symmetry on the Space-Time $M^4 \times M^1$

We would like to generalize our previous models to the models on the space-time $M^4 \times S^1$, and $M^4 \times I^1$. We obtain that the general models on $M^4 \times I^1$ can be obtained from the general models on $M^4 \times S^1$ by moduling the $(Z_2)^{k_0}$ symmetry in which $k_0$ is the positive integer. And the models on $M^4 \times S^1/(Z_2 \times Z_2')$ is an special case for $k_0=2$ and no 3-branes in the bulk.

We assume the corresponding coordinates for the space-time are $x^\mu$, ($\mu = 0, 1, 2, 3$), $y \equiv x^5$, the radius for the circle $S^1$ is $R$, and the length for the interval $I^1$ is $\pi R$. We also assume that there are some special 3-branes along the fifth dimension, and there is a $Z_2$ reflection symmetry in each 3-brane neighborhood.

3.1 Discrete Symmetry on $M^4 \times S^1$

Assuming we have $n+1$ parallel 3-branes along the $S^1$, and their fifth coordinates are: $y_0 = 0 < y_1 < y_2 < ... < y_n < 2\pi R$. We define the local fifth coordinate for the $i$-th brane as $y_i' \equiv y - y_i$, and then, $y_0' \equiv y$. In addition, the equivalent class for the reflection $Z_2$ symmetry in the $i-th$ brane neighborhood is $y_i' \sim -y_i'$. And for that $Z_2$ symmetry, we define the corresponding $Z_2$ operator $P_i$ for $i=0, 1, 2, ..., n$, whose eigenvalue is $\pm 1$, i. e., for a generic field or function, we have

$$P_i \phi(x^\mu, y_i') = \pm \phi(x^\mu, y_i').$$

(5)

By the way, $P_i^2 = 1$, so, $\{1, P_i\}$ forms a $Z_2$ group.

Because if $y_i/(2\pi R)$ is an irrational number, we will project out all the KK states, which can not satisfy our requirement. So, we assume

$$y_i = \frac{p_i}{q_i}2\pi R, \quad \text{for} \quad i = 1, 2, ..., n,$$

(6)

where $p_i$ and $q_i$ are relative prime positive integers.

Assume $L$ is the least common multiple for all $p_i + q_i$, i. e.,

$$L \equiv [p_1 + q_1, p_2 + q_2, ..., p_n + q_n].$$

(7)

By Unique Prime Factorization theorem, we obtain that

$$L = 2^{k_0}s_1^{k_1}s_2^{k_2}...s_j^{k_j},$$

(8)

where $2 < s_1 < s_2 < ... < s_j$, $k_0$ is the non-negative integer, $s_i$ is the prime number and $k_i$ is the positive integer for $i = 1, 2, ..., j$. Moreover, we define the effective physical radius $r$ as

$$r = \begin{cases} 
4R/L & \text{for} \ k_0 \geq 2, \\
2R/L & \text{for} \ k_0 = 1, \\
R/L & \text{for} \ k_0 = 0.
\end{cases}$$

(9)
For a generic bulk field $\phi$, we obtain the KK modes expansions

$$\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{n+1} \pi R}} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{r},$$

(10)$$\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{r},$$

(11)$$\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n+1)y}{r},$$

(12)$$\phi_{--}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n+2)y}{r},$$

(13)

where $n$ is a non-negative integer. The 4-dimensional fields $\phi_{++}^{(2n)}$, $\phi_{+-}^{(2n+1)}$, $\phi_{-+}^{(2n+1)}$ and $\phi_{--}^{(2n+2)}$ acquire masses $2n/r$, $(2n+1)/r$, $(2n+1)/r$ and $(2n+2)/r$ upon the compactification. Zero modes are contained only in $\phi_{++}$ fields, thus, the matter content of massless sector is smaller than that of the full 5-dimensional multiplet. Moreover, because $0 < r \leq R$, the masses of KK states $(n/r)$ can be set arbitrarily heavy if $L$ is large enough, i.e., we choose suitable $p_i$ and $q_i$ for some $i$, for example, if $1/R$ is about TeV, $p_i = 10^{13} - 1$ and $q_i = 10^{13} + 1$, we obtain that $1/r$ is at least about $10^{16}$ GeV, which is the usual GUT scale. Therefore, there is no simple relation between the physical size of the fifth dimension and the mass scales of KK modes.

For $k_0 = 0$ and $k_0 = 1$, we obtain that

$$P_i \phi_{++}(x^\mu, y) = \phi_{++}(x^\mu, y),$$

(14)$$P_i \phi_{+-}(x^\mu, y) = -\phi_{+-}(x^\mu, y),$$

(15)

for all $i = 0, 1, 2, ..., n$. So, we only have one non-equivalent $Z_2$ symmetry. And for $i = 0$, we always have above equations for $P_0$.

And for $k_0 \geq 2$, if $(p_i + q_i)$ is a multiple of $2^{k_0}$, i.e., $2^{k_0} | (p_i + q_i)$, we obtain that

$$P_i \phi_{++}(x^\mu, y) = \phi_{++}(x^\mu, y),$$

(16)$$P_i \phi_{+-}(x^\mu, y) = -\phi_{+-}(x^\mu, y),$$

(17)and if $(p_i + q_i)$ is not a multiple of $2^{k_0}$, i.e., $2^{k_0} \nmid (p_i + q_i)$, we obtain that

$$P_i \phi_{++}(x^\mu, y) = \phi_{++}(x^\mu, y),$$

(18)
\[ P_i \phi_{\pm}(x^\mu, y) = -\phi_{\mp}(x^\mu, y). \]  

(19)

So, we have two non-equivalent \( Z_2 \) symmetries.

Because we need discrete symmetry to break the bulk gauge symmetry and supersymmetry, we will concentrate on the scenario with \( k_0 \geq 2 \). To be explicit, we would like to give two examples: (I) \( n = 1 \) and \( 4 | (p_1 + q_1) \), in this simple case, we can have two local \( Z_2 \) symmetries; (II) Suppose \( n = 3 \), \( 4 | (p_1 + q_1) \), \( p_2 = q_2 = 1 \), \( p_3 = q_1 \), and \( q_3 = p_1 \), we have one global \( Z_2 \) symmetry and one local \( Z_2 \) symmetry. And the local \( Z_2 \) symmetry will become global if \( p_1 = 1 \) and \( q_1 = 3 \). This is the scenario discussed in Ref. [12] where the global \( Z_2 \) symmetry has been modulod from the manifold.

Furthermore, if we require that the models have one global \( Z_2 \) symmetry, then, modulo this global \( Z_2 \) symmetry, we obtain the models with discrete symmetry on the space-times \( M^4 \times S^1 / Z_2 \). And the two \( Z_2 \) symmetries in the two boundary 3-branes’ neighborhoods are equivalent. Let us explain this in detail: suppose we have \( 2n + 2 \) special 3-branes, we require that \( y_0 = 0 \), \( y_{n+1} = \pi R \), \( p_i = q_{2n+2-i} \) and \( q_i = p_{2n+2-i} \) where \( i = 1, 2, ..., n \), we will have one global \( Z_2 \) symmetry in which the equivalent class is \( y \sim -y \). Moduloing this equivalent class, we obtain the models on \( M^4 \times S^1 / Z_2 \).

In general, if we require that the models have global \( (Z_2)^{k_0} \) symmetry for \( k_0 > 1 \), then, modulo the global \( (Z_2)^{k_0} \) symmetry, we obtain the models with discrete symmetry on the space-times \( M^4 \times S^1 / (Z_2)^{k_0} \). And the two \( Z_2 \) symmetries in the two boundary 3-branes’ neighborhoods are not equivalent. As an example, we discuss the models with \( k_0 = 2 \). Suppose we have \( 4n + 4 \) 3-branes, \( y_{n+1} = \pi R / 2 \), \( y_{2n+2} = \pi R \), \( y_{3n+3} = 3\pi R / 2 \). And for \( i = 1, 2, ..., n \), we have \( p_i = q_{4n+4-i} \), \( q_i = p_{4n+4-i} \), \( p_{2n+2-i} = p_i' \), \( q_{2n+2-i} = q_i' \), \( p_{2n+2+i} = p_i' \), \( q_{2n+2+i} = q_i' \), where \( p_i' \) and \( q_i' \) are relative prime positive integer and satisfy the equation

\[ \frac{p_i'}{q_i'} = \frac{q_i - p_i}{2(p_i + q_i)}. \]  

(20)

If \( n = 0 \), we obtain the models on \( M^4 \times S^1 / (Z_2 \times Z_2') \) [13].

### 3.2 Discrete Symmetry on \( M^4 \times I^1 \)

In this subsection, we would like to consider the discrete symmetry on the space-time \( M^4 \times I^1 \). Assume on two boundary 3-branes, the fields should satisfy the Dirichlet or Neumann boundary condition, we show that the general models on the space-time \( M^4 \times I^1 \), contain the models on \( M^4 \times S^1 / Z_2 \) and the models on \( M^4 \times S^1 / (Z_2)^{k_0} \) for \( k_0 > 1 \). Assuming we have \( n + 2 \) parallel 3-branes along the \( I^1 \), and their fifth coordinates are: \( y_0 = 0 < y_1 < y_2 < ... < y_{n+1} = \pi R \). We define the local fifth coordinate for the \( i \)-th brane as \( y_i = y-y_i \). And the equivalent class for the reflection \( Z_2 \) symmetry in the \( i-th \) brane neighborhood is \( y_i' \sim -y_i' \). Moreover, for that \( Z_2 \) symmetry, we define the corresponding \( Z_2 \) operator \( P_i \) for \( i=0, 1, 2, ..., n \), whose eigenvalue is \( \pm 1 \), i.e., for a generic field or function, we have

\[ P_i \phi(x^\mu, y_i') = \pm \phi(x^\mu, y_i'). \]  

(21)
Because if \( y_i / (\pi R) \) is an irrational number, we will project out all the KK states, which can not satisfy our requirement. So, we assume

\[
y_i = \frac{p_i}{q_i} \pi R, \quad \text{for } i = 1, 2, ..., n ,
\]

(22)

where \( p_i \) and \( q_i \) are relative prime positive integers.

Assume \( L \) is the least common multiple for all \( p_i + q_i \), i.e.,

\[
L \equiv \left[ p_1 + q_1, p_2 + q_2, ..., p_n + q_n \right] .
\]

(23)

By Unique Prime Factorization theorem, we obtain that

\[
L = 2^{l_0} s_1^{l_1} s_2^{l_2} ... s_j^{l_j} ,
\]

(24)

where \( 2 < s_1 < s_2 < ... < s_j \), \( l_0 \) is the non-negative integer, \( s_i \) is the prime number and \( l_i \) is the positive integer for \( i = 1, 2, ..., j \). For the models on the space-time \( M^4 \times S^1 / \mathbb{Z}_2 \), we define the effective physical radius \( r \) as

\[
r = \begin{cases} 
2R / L & \text{for } l_0 \geq 1 , \\
R / L & \text{for } l_0 = 0 .
\end{cases}
\]

(25)

For \( l_0 = 0 \), we can still define the effective radius as

\[
r = \frac{2R}{L} .
\]

(26)

This kind of the models can not be obtained from the models in the last subsection by moduloing one global \( \mathbb{Z}_2 \) symmetry, however, they can be obtained from the models in the last subsection by moduloing global \( (\mathbb{Z}_2)^{k_0} \) symmetries for \( k_0 > 1 \).

For a generic bulk field \( \phi \), we obtain the KK modes expansions

\[
\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{n+2} \pi R}} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{r} ,
\]

(27)

\[
\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n + 1)y}{r} ,
\]

(28)

\[
\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n + 1)y}{r} ,
\]

(29)

\[
\phi_{--}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n + 2)y}{r} ,
\]

(30)
where $n$ is a non-negative integer. The 4-dimensional fields $\phi^{(2n)}_{++}$, $\phi^{(2n+1)}_{+-}$, and $\phi^{(2n+2)}_{--}$ acquire masses $2n/r$, $(2n + 1)/r$, $(2n + 1)/r$ and $(2n + 2)/r$ upon the compactification. And the zero modes are contained only in $\phi_{++}$ fields. Moreover, because $0 < r \leq R$, the masses of KK states $(n/r)$ can be set arbitrarily heavy if $L$ is large enough, i.e., we choose suitable $p_i$ and $q_i$, for some $i$. So, there is no simple relation between the physical size of the fifth dimension and the mass scales of KK states.

(I) First, we discuss the models on the space-times $M^4 \times S^1/Z_2$. We should keep in mind that there is one global $Z_2$ symmetry has been moduloed from $S^1$, which we can call it as $P_0$ or $P_{n+1}$.

For $l_0 = 0$, we obtain that

\begin{align}
P_i \phi_{++}(x^\mu, y) &= \phi_{++}(x^\mu, y), \\
P_i \phi_{+-}(x^\mu, y) &= -\phi_{+-}(x^\mu, y),
\end{align}

for all $i = 1, 2, ..., n$. Under our assumption, those $Z_2$ symmetry are equivalent to the global $Z_2$ symmetry, so, we just have one independent $Z_2$ symmetry.

And for $l_0 \geq 1$, if $(p_i + q_i)$ is a multiple of $2^{l_0}$, i.e., $2^{l_0}|(p_i + q_i)$, we obtain that

\begin{align}
P_i \phi_{++}(x^\mu, y) &= \phi_{++}(x^\mu, y), \\
P_i \phi_{+-}(x^\mu, y) &= -\phi_{+-}(x^\mu, y),
\end{align}

this $Z_2$ symmetry is not equivalent to the global symmetry. And if $(p_i + q_i)$ is not a multiple of $2^{l_0}$, i.e., $2^{l_0} \nmid (p_i + q_i)$, we obtain that

\begin{align}
P_i \phi_{++}(x^\mu, y) &= \phi_{++}(x^\mu, y), \\
P_i \phi_{+-}(x^\mu, y) &= -\phi_{+-}(x^\mu, y),
\end{align}

this $Z_2$ symmetry is equivalent to the global $Z_2$ symmetry. In short, we have two independent $Z_2$ symmetries, in which one can be considered as global $Z_2$ symmetry.

Because we need discrete symmetry to break the bulk gauge symmetry and supersymmetry, we will concentrate on the scenario with $l_0 \geq 1$. To be explicit, we would like to give one examples: $n = 1$ and $2|(p_1 + q_1)$. In this simple case, we can have one local $Z_2$ symmetries and one global $Z_2$ symmetry. And that local $Z_2$ symmetry becomes global if $p_1 = q_1 = 1$.

(II) If $l_0 = 0$ and $r = 2R/L$, this kind of models can be obtained from the models in the last subsection by moduloing the global $(Z_2)^{k_0}$ symmetries for $k_0 > 1$, so, $P_0$ and $P_{n+1}$ are two non-equivalent $Z_2$ symmetries. If at start point, the original extra space manifold is $I^1$, we can consider that there are no global $Z_2$ symmetries for $P_0$ and $P_{n+1}$.
Because $l_0 = 0$, there are two cases: $p_i$ is odd and $q_i$ is even, or $p_i$ is even and $q_i$ is odd. If $p_i$ is even,

$$P_i \phi_{\pm}(x^\mu, y) = \phi_{\pm}(x^\mu, y),$$  \hspace{1cm} (37)

$$P_i \phi_{\mp}(x^\mu, y) = -\phi_{\mp}(x^\mu, y),$$  \hspace{1cm} (38)

and if $p_i$ is odd

$$P_i \phi_{\pm}(x^\mu, y) = \phi_{\mp}(x^\mu, y),$$  \hspace{1cm} (39)

$$P_i \phi_{\mp}(x^\mu, y) = -\phi_{\mp}(x^\mu, y).$$  \hspace{1cm} (40)

Therefore, we can have two local $Z_2$ symmetries, which can be thought as global symmetries if we consider the original manifold for the extra dimension is $S^1$.

4 GUT Breaking on the Space-Time $M^4 \times M^1$

In this section, we would like to discuss the supersymmetric $SU(5)$ model on the space-time $M^4 \times M^1$ with two discrete $Z_2$ symmetries. We assume that the $SU(5)$ gauge fields and two 5-plet Higgs hypermultiplets in the bulk, and the Standard Model fermions can be on the 3-brane or in the bulk.

As we know, the $N = 1$ supersymmetric theory in 5-dimension have 8 real supercharges, corresponding to $N = 2$ supersymmetry in 4-dimension. The vector multiplet physically contains a vector boson $A_M$ where $M = 0, 1, 2, 3, 5$, two Weyl gauginos $\lambda_{1,2}$, and a real scalar $\sigma$. In terms of 4-dimensional $N = 1$ language, it contains a vector multiplet $V(A_\mu, \lambda_1)$ and a chiral multiplet $\Sigma((\sigma + i A_5)/\sqrt{2}, \lambda_2)$ which transform in the adjoint representation of $SU(5)$. And the 5-dimensional hypermultiplet physically has two complex scalars $\phi$ and $\phi^c$, a Dirac fermion $\Psi$, and can be decomposed into two 4-dimensional chiral multiplet $\Phi(\phi, \psi \equiv \Psi_R)$ and $\Phi^c(\phi^c, \psi^c \equiv \Psi_L)$, which transform as conjugate representations of each other under the gauge group. For instance, we have two Higgs chiral multiplets $H_u$ and $H_d$, which transform as 5 and $\overline{5}$ under $SU(5)$ gauge symmetry, and their mirror $H^c_u$ and $H^c_d$, which transform as $\overline{5}$ and 5 under $SU(5)$ gauge symmetry.

The general action for the $SU(5)$ gauge fields and their couplings to the bulk hypermultiplet $\Phi$ is [14]

$$S = \int d^5x \frac{1}{k g^2} \operatorname{Tr} \left[ \frac{1}{4} \int d^2 \theta \left( W^\alpha W_\alpha + \text{H.C.} \right) \right. $$

$$+ \int d^4 \theta \left( (\sqrt{2} \partial_5 + \Sigma) e^{-V} (\sqrt{2} \partial_5 + \Sigma) + \partial_5 e^{-V} \partial_5 e^V \right) \right]$$

$$+ \int d^5x \left[ \int d^4 \theta \left( \Phi^c e^V \Phi + \Phi e^{-V} \Phi^c \right) \right. $$

$$+ \int d^2 \theta \left( \Phi^c (\partial_5 - \frac{1}{\sqrt{2}} \Sigma) \Phi + \text{H.C.} \right) \right].$$  \hspace{1cm} (41)
Because the action is invariant under the parity $P$, we obtain that under the parity operator $P_i$, the vector multiplet transforms as

$$V(x^\mu, y'_i) \rightarrow V(x^\mu, -y'_i) = P_i V(x^\mu, y'_i) P_i^{-1},$$  \hspace{1cm} (42)

$$\Sigma(x^\mu, y'_i) \rightarrow \Sigma(x^\mu, -y'_i) = -P_i \Sigma(x^\mu, y'_i) P_i^{-1},$$  \hspace{1cm} (43)

if the hypermultiplet $\Phi$ is a 5 or $\bar{5}$ $SU(5)$ multiplet, we have

$$\Phi(x^\mu, y'_i) \rightarrow \Phi(x^\mu, -y'_i) = \eta_\Phi P_i \Phi(x^\mu, y'_i),$$  \hspace{1cm} (44)

$$\Phi^c(x^\mu, y'_i) \rightarrow \Phi^c(x^\mu, -y'_i) = -\eta_\Phi P_i \Phi^c(x^\mu, y'_i),$$  \hspace{1cm} (45)

and if the hypermultiplet $\Phi$ is a 10 or $\bar{10}$ $SU(5)$ multiplet, we have

$$\Phi(x^\mu, y'_i) \rightarrow \Phi(x^\mu, -y'_i) = \eta_\Phi P_i \Phi(x^\mu, y'_i) P_i^{-1},$$  \hspace{1cm} (46)

$$\Phi^c(x^\mu, y'_i) \rightarrow \Phi^c(x^\mu, -y'_i) = -\eta_\Phi P_i \Phi^c(x^\mu, y'_i) P_i^{-1},$$  \hspace{1cm} (47)

where $\eta_\Phi = \pm 1$.

For simplicity, let us denote the two non-equivalent $Z_2$ symmetries as $P$ and $P'$. We choose the following matrix representations for the parities $P$ and $P'$ which are expressed in the adjoint representation of $SU(5)$

$$P = \text{diag}(+1, +1, +1, +1, +1), \quad P' = \text{diag}(-1, -1, -1, +1, +1).$$  \hspace{1cm} (48)

So, upon the parity $P'$, the gauge generators $T^A$ where $A = 1, 2, ..., 24$ for $SU(5)$ are separated into two sets: $T^a$ are the gauge generators for the Standard Model gauge group, and $T^\hat{a}$ are the other broken gauge generators

$$P T^a P^{-1} = T^a, \quad P T^{\hat{a}} P^{-1} = T^{\hat{a}},$$  \hspace{1cm} (49)

$$P' T^a P'^{-1} = T^a, \quad P' T^{\hat{a}} P'^{-1} = -T^{\hat{a}}.$$  \hspace{1cm} (50)

Choosing $\eta_{H_u} = +1$ and $\eta_{H_d} = +1$, we obtain the particle spectra, which are given in Table 1. The bulk 4-dimensional $N = 2$ supersymmetry and $SU(5)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry in the bulk for the zero modes, and on the special 3-branes which preserve $Z_2$ symmetry $P'$ for all the modes. Including the KK states, the gauge symmetry on the special 3-branes, which preserve $Z_2$ symmetry $P$, is $SU(5)$. In addition, the 4-dimensional supersymmetry on the 3-branes is 1/2 of the bulk 4-dimensional supersymmetry or $N = 1$ due to the $Z_2$ symmetry in the brane neighborhood. Moreover, the Standard Model fermions can be in the bulk or on the 3-brane, and the discussion are similar to those in Ref. [4-8], so, we will not repeat them here.

By the way, one can also discuss the non-supersymmetric $SU(6)$ and $SO(10)$ breaking, however, there are zero modes for $A^\hat{a}_5$ where $\hat{a}$ is the index related to the broken gauge generators under two $Z_2$ symmetries.
Table 1: Parity assignment and masses \((n \geq 0)\) of the fields in the SU(5) gauge and Higgs multiplets. The indices \(F, T\) are for doublet and triplet, respectively.

| \((P, P')\) | field | mass |
|------------|-------|------|
| (+, +)     | \(V_\mu^a, H_u^F, H_d^F\) | \(\frac{2n}{r}\) |
| (+, −)     | \(V_\mu^a, H_u^T, H_d^T\) | \(\frac{2n+1}{r}\) |
| (−, +)     | \(\Sigma^a, H_u^cT, H_d^cT\) | \(\frac{2n+1}{r}\) |
| (−, −)     | \(\Sigma^a, H_u^cF, H_d^cF\) | \(\frac{2n+2}{r}\) |

5 Discrete Symmetry on the Space-Time \(M^4 \times M^1 \times M^1\)

We would like to discuss the models where there are some parallel 4-branes with \(\mathbb{Z}_2\) reflection symmetry along the fifth and sixth dimensions on the space-time \(M^4 \times M^1 \times M^1\), in which \(M^1\) can be \(S^1, S^1/\mathbb{Z}_2,\) and \(I^1\). Because the extra space manifold is the product of two 1-dimensional manifold, we only discuss the models on the space-times \(M^4 \times S^1 \times S^1\) as a representative because the discussions of the models with other combinations are similar. In addition, we discuss the models on the space-time \(M^4 \times S^1/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2\) where there are some 3-branes with \(\mathbb{Z}_2\) symmetry in the bulk.

The corresponding coordinates for the space-time are \(x^\mu, (\mu = 0, 1, 2, 3, y \equiv x^5, z \equiv x^6),\) and the radii for the \(y\) and \(z\) directions are \(R_1\) and \(R_2\), respectively.

5.1 Discrete Symmetry on \(M^4 \times S^1 \times S^1\)

First, we would like to discuss the discrete symmetry on the space-time \(M^4 \times S^1 \times S^1\). Assume that along the \(y\) and \(z\) directions, we have \(n + 1\) and \(m + 1\) parallel 4-branes with \(\mathbb{Z}_2\) reflection symmetry, respectively. The 5-th coordinates for the parallel 4-branes along the \(y\) direction are \(y_0 = 0 < y_1 < y_2 < ... < y_n < 2\pi R_1\), and the 6-th coordinates for the parallel 4-branes along the \(z\) direction are \(z_0 = 0 < z_1 < z_2 < ... < z_m < 2\pi R_2\).

We denote the local coordinate for the \(i\)-th 4-brane along the \(y\) direction as \(y_i' \equiv y - y_i\). In addition, for the \(\mathbb{Z}_2\) symmetry in the \(i-th\) 4-brane neighborhood, we define a \(\mathbb{Z}_2\) operator \(P_i^y\) for \(i=0, 1, 2, ..., n\), whose eigenvalue is \(\pm 1\), i. e., for a generic field or function, we have

\[
P_i^y \phi(x^\mu, y_i', z) = \pm \phi(x^\mu, y_i', z) .
\] (51)

Similarly, we denote the local coordinate for the \(i\)-th 4-brane along the \(z\) direction as \(z_i' \equiv z - z_i\). And for the \(\mathbb{Z}_2\) symmetry in the \(i-th\) 4-brane neighborhood,
we define a $\mathbb{Z}_2$ operator $P_z^i$ for $i=0,1,2,\ldots,m$

$$P_z^i \phi(x^\mu, y, z'_i) = \pm \phi(x^\mu, y, z'_i).$$

(52)

Because if $y_i/(2\pi R_1)$ or $z_i/(2\pi R_1)$ is an irrational number, we will project out all the KK states, which can not satisfy our requirement. So, we assume

$$y_i = \frac{p^y_i}{q^y_i} 2\pi R_1, \quad \text{for } i = 1, 2, \ldots, n ,$$

(53)

$$z_i = \frac{p^z_i}{q^z_i} 2\pi R_2, \quad \text{for } i = 1, 2, \ldots, m ,$$

(54)

where $p^y_i$ and $q^y_i$ are relative prime positive integers, and $p^z_i$ and $q^z_i$ are relative prime positive integers.

Assume $L_y$ is the least common multiple for all $p^y_i + q^y_i$, and $L_z$ is the least common multiple for all $p^z_i + q^z_i$, i.e.,

$$L_y \equiv [p^y_1 + q^y_1, p^y_2 + q^y_2, \ldots, p^y_n + q^y_n] ,$$

(55)

$$L_z \equiv [p^z_1 + q^z_1, p^z_2 + q^z_2, \ldots, p^z_m + q^z_m] .$$

(56)

By Unique Prime Factorization theorem, we obtain that

$$L_y = 2^{k_0} s_1^{k_1} s_2^{k_2} \ldots s_u^{k_u} ,$$

(57)

$$L_z = 2^{l_0} t_1^{l_1} t_2^{l_2} \ldots t_v^{l_v} ,$$

(58)

where $2 < s_1 < s_2 < \ldots < s_u$, $2 < t_1 < t_2 < \ldots < t_v$, $k_0$ and $l_0$ are the non-negative integers, $s_i$ and $t_j$ are the prime numbers, and $k_i$ and $l_j$ are the positive integers for $i = 1, 2, \ldots, u$ and $j = 1, 2, \ldots, v$. And we define the effective physical radii $r_1$ and $r_2$ as

$$r_1 = \begin{cases} 
4R_1/L_y & \text{for } k_0 \geq 2 , \\
2R_1/L_y & \text{for } k_0 = 1 , \\
R_1/L_y & \text{for } k_0 = 0 ,
\end{cases}$$

(59)

$$r_2 = \begin{cases} 
4R_1/L_z & \text{for } l_0 \geq 2 , \\
2R_1/L_z & \text{for } l_0 = 1 , \\
R_1/L_z & \text{for } l_0 = 0 .
\end{cases}$$

(60)
For a generic bulk field $\phi$, we obtain the KK modes expansions

$$
\phi_{+++-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+++-}^{(2n,2m)}(x^\mu) A_{++}^{2n}(y, r_1) A_{+-}^{2m}(z, r_2),
$$

(61)

$$
\phi_{++--}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{++--}^{(2n,2m+1)}(x^\mu) A_{++}^{2n}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(62)

$$
\phi_{+-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n,2m+1)}(x^\mu) A_{++}^{2n}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(63)

$$
\phi_{++--}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n,2m+2)}(x^\mu) A_{++}^{2n}(y, r_1) A_{+-}^{2m+2}(z, r_2),
$$

(64)

$$
\phi_{++--}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+1)}(x^\mu) A_{++}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(65)

$$
\phi_{+-++}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(66)

$$
\phi_{--+--}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+2)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m+2}(z, r_2),
$$

(67)

$$
\phi_{+-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m}(z, r_2),
$$

(68)

$$
\phi_{+-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(69)

$$
\phi_{+-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(70)

$$
\phi_{+-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2),
$$

(71)
\[
\phi_{---}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi^{(2n+1,2m+2)}(x^\mu) A^{2n+1}_{-+}(y, r_1) A^{2m+2}_{-+}(z, r_2), \tag{72}
\]

\[
\phi_{---}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi^{(2n+2,2m)}(x^\mu) A^{2n+2}_{--}(y, r_1) A^{2m}_{--}(z, r_2), \tag{73}
\]

\[
\phi_{---}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi^{(2n+2,2m+1)}(x^\mu) A^{2n+2}_{--}(y, r_1) A^{2m+1}_{--}(z, r_2), \tag{74}
\]

\[
\phi_{---}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi^{(2n+2,2m+2)}(x^\mu) A^{2n+2}_{--}(y, r_1) A^{2m+2}_{--}(z, r_2), \tag{75}
\]

\[
\phi_{---}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi^{(2n+1,2m+1)}(x^\mu) A^{2n+1}_{-+}(y, r_1) A^{2m+1}_{-+}(z, r_2), \tag{76}
\]

where

\[
A^{2n+1}_{++}(y, r_1) = \frac{1}{\sqrt{2^{2n+1} \pi R_1}} \cos \frac{2ny}{r_1}, \tag{77}
\]

\[
A^{2n+1}_{+-}(y, r_1) = \frac{1}{\sqrt{\pi R_1}} \cos \frac{(2n + 1)y}{r_1}, \tag{78}
\]

\[
A^{2n+1}_{-+}(y, r_1) = \frac{1}{\sqrt{\pi R_1}} \sin \frac{(2n + 1)y}{r_1}, \tag{79}
\]

\[
A^{2n+2}_{-+}(y, r_1) = \frac{1}{\sqrt{\pi R_1}} \sin \frac{(2n + 2)y}{r_1}. \tag{80}
\]

Similarly, we define \(A^{2n+1}_{++}(z, r_2), A^{2n+1}_{+-}(z, r_2), A^{2n+1}_{-+}(z, r_2), A^{2n+2}_{-+}(z, r_2)\).

The 4-dimensional fields \(\phi^{(n,m)}(x^\mu)\) acquire masses \(\sqrt{n^2/r_1^2 + m^2/r_2^2}\) upon the compactification. And the zero modes are contained only in \(\phi_{++++}\) fields. Moreover, because \(0 < r_1 \leq R_1\) and \(0 < r_2 \leq R_2\), the masses of KK states \((\sqrt{n^2/r_1^2 + m^2/r_2^2})\) can be set arbitrarily heavy if \(L_y\) and \(L_z\) are large enough, i.e., we choose suitable \((p^i_y, q^i_y)\) for some \(i\), and \((p^j_z, q^j_z)\) for some \(j\). So, there is no simple relation between the physical size of the extra dimensions and the mass scales of KK modes.

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For $k_0 = 0$ and $k_0 = 1$, we obtain that

\[ P_y^0 \phi_{++-+}(x^\mu, y, z) = \phi_{++-+}(x^\mu, y, z) , \tag{81} \]

\[ P_y^0 \phi_{-++-}(x^\mu, y, z) = -\phi_{-++-}(x^\mu, y, z) , \tag{82} \]

for all $i = 0, 1, 2, ..., n$. So, we only have one independent $Z_2$ symmetry. And for $i = 0$, we always have above equations for $P_0^y$.

Moreover, for $k_0 \geq 2$, if $(p^y_i + q^y_i)$ is a multiple of $2^{k_0}$, i. e., $2^{k_0} | (p^y_i + q^y_i)$, we obtain that

\[ P_y^i \phi_{+++-}(x^\mu, y, z) = \phi_{+++-}(x^\mu, y, z) , \tag{83} \]

\[ P_y^i \phi_{+-++}(x^\mu, y, z) = -\phi_{+-++}(x^\mu, y, z) , \tag{84} \]

and if $(p^y_i + q^y_i)$ is not a multiple of $2^{k_0}$, i. e., $2^{k_0} \nmid (p^y_i + q^y_i)$, we obtain that

\[ P_y^i \phi_{++-+}(x^\mu, y, z) = \phi_{++-+}(x^\mu, y, z) , \tag{85} \]

\[ P_y^i \phi_{-++-}(x^\mu, y, z) = -\phi_{-++-}(x^\mu, y, z) . \tag{86} \]

So, we have two non-equivalent $Z_2$ symmetries along the $y$ direction.

Similarly, for $l_0 = 0$ and $l_0 = 1$, we obtain that

\[ P_z^0 \phi_{++-+}(x^\mu, y, z) = \phi_{++-+}(x^\mu, y, z) , \tag{87} \]

\[ P_z^0 \phi_{-++-}(x^\mu, y, z) = -\phi_{-++-}(x^\mu, y, z) , \tag{88} \]

for all $i = 0, 1, 2, ..., m$. So, we only have one independent $Z_2$ symmetry. And for $i = 0$, we always have above equations for $P_0^z$.

For $l_0 \geq 2$, if $(p^z_i + q^z_i)$ is a multiple of $2^{l_0}$, i. e., $2^{l_0} | (p^z_i + q^z_i)$, we obtain that

\[ P_z^i \phi_{++-+}(x^\mu, y, z) = \phi_{++-+}(x^\mu, y, z) , \tag{89} \]

\[ P_z^i \phi_{-++-}(x^\mu, y, z) = -\phi_{-++-}(x^\mu, y, z) , \tag{90} \]

and if $(p^z_i + q^z_i)$ is not a multiple of $2^{l_0}$, i. e., $2^{l_0} \nmid (p^z_i + q^z_i)$ we obtain that

\[ P_z^i \phi_{++-+}(x^\mu, y, z) = \phi_{++-+}(x^\mu, y, z) , \tag{91} \]

\[ P_z^i \phi_{-++-}(x^\mu, y, z) = -\phi_{-++-}(x^\mu, y, z) . \tag{92} \]

So, we have two non-equivalent $Z_2$ symmetries along the $z$ direction.
Therefore, we can have at most four non-equivalent $Z_2$ symmetries. Because we need discrete symmetry to break the bulk gauge symmetry and supersymmetry, we will concentrate on the scenario with $k_0 \geq 2$ and $l_0 \geq 2$. To be explicit, we would like to give two examples: (I) $n = 1$, $m = 1$, $4|(p_1^y + q_1^y)$, $4|(p_2^z + q_2^z)$, in this simple case, we can have four local $Z_2$ symmetries. (II) Suppose $n = 3$, $m = 3$, $4|(p_1^y + q_1^y)$, $4|(p_2^y + q_2^y)$, $p_2^y = q_2^y = 1$, $p_2^z = q_2^z = 1$, $p_3^y = q_3^y = p_3^z = q_3^z = q_1^z$, and $q_3^z = p_1^z$, we will have two global $Z_2$ symmetries and two local $Z_2$ symmetries. The local $Z_2$ symmetry will become global if $p_1^y = 1$ and $q_1^y = 3$, or $p_1^z = 1$ and $q_1^z = 3$. The two global $Z_2$ symmetries can be modulated from the manifold. In general, we may have global $(Z_2)^{k_0}$ and $(Z_2)^{l_0}$ symmetries, the extra space orbifold will be $S^1/(Z_2)^{k_0} \times S^1/(Z_2)^{l_0}$ if we modulo those global $Z_2$ symmetries.

5.2 Discrete Symmetry on $M^4 \times S^1/Z_2 \times S^1/Z_2$

In this subsection, we would like to discuss the discrete symmetry on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$, where there are some 3-branes with $Z_2$ symmetry in the bulk. And we denote two global $Z_2$ symmetries $y \sim -y$ and $z \sim -z$ as $P^y$ and $P^z$. For simplicity, we assume that there are only four 4-branes which are the boundary branes on $S^1/Z_2 \times S^1/Z_2$, and the 3-branes are only in the bulk.

Suppose we have $n$ 3-branes in the bulk, and their coordinates are $(y_i, z_i)$ where $0 < y_i < \pi R_1$ and $0 < z_i < \pi R_2$. We denote the local coordinates for the $i$-th 3-brane as $y'_i \equiv y - y_i$, and $z'_i \equiv z - z_i$. And then, the equivalent class for the reflection $Z_2$ symmetry in the $i$-th 3-brane neighborhood is $(y'_i, z'_i) \sim (-y'_i, -z'_i)$. For that $Z_2$ symmetry, we define the corresponding $Z_2$ operator $P_i$ for $i = 1, 2, ..., n$, whose eigenvalue is $\pm 1$, i.e., for a generic field or function, we have

$$P_i \phi(x^\mu, -y'_i, -z'_i) = \pm \phi(x^\mu, y'_i, z'_i) .$$

Because if $y_i/(2\pi R_1)$ or $z_i/(2\pi R_1)$ is an irrational number, we will project out all the KK states, which can not satisfy our requirement. So, we assume

$$y_i = \frac{p_i^y}{q_i^y} \pi R_1, \quad \text{for } i = 1, 2, ..., n ,$$

$$z_i = \frac{p_i^z}{q_i^z} \pi R_2, \quad \text{for } i = 1, 2, ..., m ,$$

where $p_i^y$ and $q_i^y$ are relative prime positive integers, and $p_i^z$ and $q_i^z$ are relative prime positive integers.

Assume $L_y$ is the least common multiple for all $p_i^y + q_i^y$, and $L_z$ is the least common multiple for all $p_i^z + q_i^z$, i.e.,

$$L_y \equiv [p_1^y + q_1^y, p_2^y + q_2^y, ..., p_n^y + q_n^y] ,$$

$$L_z \equiv [p_1^z + q_1^z, p_2^z + q_2^z, ..., p_m^z + q_m^z] .$$
By Unique Prime Factorization theorem, we obtain that

\[ L_y = 2^{k_0} s_1^{k_1} s_2^{k_2} \ldots s_u^{k_u}, \quad (98) \]

\[ L_z = 2^{l_0} t_1^{l_1} t_2^{l_2} \ldots t_v^{l_v}, \quad (99) \]

where \( 2 < s_1 < s_2 < \ldots < s_u \), \( 2 < t_1 < t_2 < \ldots < t_v \), \( k_0 \) and \( l_0 \) are the non-negative integers, \( s_i \) and \( t_j \) are the prime numbers, and \( k_i \) and \( l_j \) are the positive integers for \( i = 1, 2, \ldots, u \) and \( j = 1, 2, \ldots, v \). And we define the effective physical radii \( r_1 \) and \( r_2 \) as

\[
r_1 = \begin{cases} 
2R_1/L_y & \text{for } k_0 \geq 1, \\
R_1/L_y & \text{for } k_0 = 0,
\end{cases} \quad (100)
\]

\[
r_2 = \begin{cases} 
2R_2/L_z & \text{for } l_0 \geq 1, \\
R_2/L_z & \text{for } l_0 = 0.
\end{cases} \quad (101)
\]

Because we require that every fields have zero modes or KK modes, we can only have one additional non-equivalent \( Z_2 \) symmetry for all the 3-branes, which is not equivalent to two global \( Z_2 \) symmetries \( P^y \) and \( P^z \). And in this scenario, \( k_0 \geq 1, l_0 \geq 1; 2^{k_0}(p_1^{y} + q_1^{y}) \) for \( i = 1, 2, \ldots, n \), \( 2^{l_0}(p_2^{z} + q_2^{z}) \) for \( j = 1, 2, \ldots, m \).

Because all the \( Z_2 \) symmetries for the 3-branes are equivalent, we write them as \( P^{y'z'} \). Denoting the field with \( (P^y, P^z, P^{y'z'})=(\pm, \pm, \pm) \) by \( \phi_{\pm\pm\pm} \), we obtain the following KK mode expansions

\[
\phi_{+++}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \phi_{+++}^{(2n,2m)}(x^\mu)A_{++}^{2n}(y, r_1)A_{+++}^{2m}(z, r_2) + \phi_{+++}^{(2n+1,2m+1)}(x^\mu)A_{++}^{2n+1}(y, r_1)A_{+++}^{2m+1}(z, r_2) \right), \quad (102)
\]

\[
\phi_{+-+}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \phi_{+-+}^{(2n,2m+1)}(x^\mu)A_{++}^{2n}(y, r_1)A_{++-}^{2m+1}(z, r_2) + \phi_{+-+}^{(2n+1,2m+2)}(x^\mu)A_{++}^{2n+1}(y, r_1)A_{++-}^{2m+2}(z, r_2) \right), \quad (103)
\]

\[
\phi_{-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \phi_{-+-}^{(2n,2m+1)}(x^\mu)A_{++}^{2n}(y, r_1)A_{-+-}^{2m+1}(z, r_2) + \phi_{-+-}^{(2n+1,2m+2)}(x^\mu)A_{++}^{2n+1}(y, r_1)A_{-+-}^{2m+2}(z, r_2) \right), \quad (104)
\]

\[
\phi_{--+}(x^\mu, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \phi_{--+}^{(2n,2m+2)}(x^\mu)A_{++}^{2n}(y, r_1)A_{--+}^{2m+2}(z, r_2) + \phi_{--+}^{(2n+1,2m+3)}(x^\mu)A_{++}^{2n+1}(y, r_1)A_{--+}^{2m+3}(z, r_2) \right), \quad (105)
\]
In this section, we would like to discuss the GUT breaking on the space-time $M$. 

To be explicit, we would like to give an example: $n = 1$, $2|(p_i^x + q_i^x)$, and $2|(p_j^x + q_j^x)$. In this simple example, the local $Z_2$ symmetry for the 3-brane can become global if $p_i^x = q_i^x = 1$ and $p_j^x = q_j^x = 1$, and this global symmetry can be moduloid, then, the space-time is $M^4 \times (S^1/Z_2 \times S^1/Z_2)/Z_2$.

6 GUT Breaking on the Space-Time $M^4 \times M^1 \times M^1$

In this section, we would like to discuss the GUT breaking on the space-time $M^4 \times M^1 \times M^1$. As we know, the 6-dimensional $N = 1$ supersymmetric theory is chiral, where the gaugino (and gravitino) has positive chirality and the matters (hypermultiplets) have negative chirality, so, it often has anomaly except that we put the Standard Model fermions on the brane and add a multiplet in the adjoint representation of gauge group or some other matter contents in the bulk to cancel the gauge anomaly. The 6-dimensional non-supersymmetric GUT models and $N = 1$ supersymmetric GUT models can be considered as special cases of $N = 2$ supersymmetric
GUT models, so, we only discuss the 6-dimensional $N = 2$ supersymmetric GUT models. $N = 2$ supersymmetric $SU(5)$, $SU(6)$, $SU(7)$, $SO(10)$, and $SO(12)$ models on the space-time $M^4 \times T^2/(Z_2)^3$ and $M^4 \times T^2/(Z_2)^4$ have been studied completely in Ref. [8], and those discussions can be extended to the complete discussions of GUT breaking on the space-time $M^4 \times M^1 \times M^1$ where there are three and four $Z_2$ symmetries. Because the discussions for GUT breaking are similar, we will not make the complete discussions for $SU(M)$ and $SO(2M)$ models in this paper. To explain the idea, we will discuss the $N = 2$ supersymmetric $SU(6)$ and $SO(10)$ models on $M^4 \times S^1 \times S^1$ where there are four $Z_2$ symmetries, and the $N = 2$ supersymmetric $SU(6)$ model with gauge-Higgs unification on $M^4 \times S^1 / Z_2 \times S^1 / Z_2$ where there are three $Z_2$ symmetries.

Let us explain the 6-dimensional gauge theory with $N = 2$ supersymmetry. $N = 2$ supersymmetric theory in 6-dimension has 16 real supercharges, corresponding to $N = 4$ supersymmetry in 4-dimension. So, only the vector multiplet can be introduced in the bulk, and we have to put the Standard Model fermions on the 4-branes, 3-branes or 4-brane intersections. In terms of the 4-dimensional $N = 1$ language, it contains a vector multiplet $V(A_\mu, \lambda_1)$, and three chiral multiplets $\Sigma_5$, $\Sigma_6$, and $\Phi$. All of them are in the adjoint representation of the gauge group. In addition, the $\Sigma_5$ and $\Sigma_6$ chiral multiplets contain the gauge fields $A_5$ and $A_6$ in their lowest components, respectively.

In the Wess-Zumino gauge and 4-dimensional $N = 1$ language, the bulk action is [4]

$$
S = \int d^6x \left\{ \text{Tr} \left[ \int d^2 \theta \left( \frac{1}{4kg^2} W^\alpha W_\alpha + \frac{1}{kg^2} \left( \Phi \partial_5 \Sigma_6 - \Phi \partial_6 \Sigma_5 - \frac{1}{\sqrt{2}} \Phi [\Sigma_5, \Sigma_6] \right) \right) + H.C. \right] + \int d^4 \theta \frac{1}{kg^2} \text{Tr} \left[ \sum_{i=5}^{6} \left( (\sqrt{2} \partial_i + \Sigma_5^i) e^{-V} (\sqrt{2} \partial_i + \Sigma_i) e^{V} + \partial_i e^{-V} \partial_i e^{V} \right) + \Phi^i e^{-V} \Phi e^{V} \right] \right\} .
$$

(110)

And the gauge transformation is given by

$$
e^V \rightarrow e^\Lambda e^V e^{\Lambda^t},
$$

(111)

$$
\Sigma_i \rightarrow e^\Lambda (\Sigma_i - \sqrt{2} \partial_i) e^{-\Lambda},
$$

(112)

$$
\Phi \rightarrow e^\Lambda \Phi e^{-\Lambda},
$$

(113)

where $i = 5, 6$.

From the action, we obtain the vector multiplet transformations under the $Z_2$ operators $P^y_1$, $P^z_j$, $P^{y'z'}_i$

$$
V(x^\mu, -y'_i, z) = P^y_i V(x^\mu, y'_i, z)(P^y)^{-1}
$$

(114)

$$
\Sigma_5(x^\mu, -y'_i, z) = -P^y_i \Sigma_5(x^\mu, y'_i, z)(P^y)^{-1}
$$

(115)
\[ \Sigma_6(x^\mu, -y_i', z) = P^y_i \Sigma_6(x^\mu, y_i', z)(P^y_i)^{-1}, \]  
(116)

\[ \Phi(x^\mu, -y_i', z) = -P^y_i \Phi(x^\mu, y_i', z)(P^y_i)^{-1}, \]  
(117)

\[ V(x^\mu, y, -z'_j) = P^z_j V(x^\mu, y, z'_j)(P^z_j)^{-1}, \]  
(118)

\[ \Sigma_6(x^\mu, y, -z'_j) = P^z_j \Sigma_6(x^\mu, y, z'_j)(P^z_j)^{-1}, \]  
(119)

\[ \Phi(x^\mu, y, -z'_j) = -P^z_j \Phi(x^\mu, y, z'_j)(P^z_j)^{-1}, \]  
(120)

\[ V(x^\mu, -y_i', -z'_i) = P^{y'z'} V(x^\mu, y_i', z'_i)(P^{y'z'})^{-1}, \]  
(122)

\[ \Sigma_5(x^\mu, -y_i', -z'_i) = -P^{y'z'} \Sigma_5(x^\mu, y_i', z'_i)(P^{y'z'})^{-1}, \]  
(123)

\[ \Sigma_6(x^\mu, -y_i', -z'_i) = -P^{y'z'} \Sigma_6(x^\mu, y_i', z'_i)(P^{y'z'})^{-1}, \]  
(124)

\[ \Phi(x^\mu, -y_i', -z'_i) = P^{y'z'} \Phi(x^\mu, y_i', z'_i)(P^{y'z'})^{-1}. \]  
(125)

### 6.1 $SU(6)$ and $SO(10)$ breaking on $M^4 \times S^1 \times S^1$

In this subsection, we would like to discuss the $SU(6)$ and $SO(10)$ models on $M^4 \times S^1 \times S^1$. We require that: (1) there are no zero modes for the chiral multiplets $\Sigma_5$, $\Sigma_6$ and $\Phi$; (2) for the zero modes, we only have 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^2$ model.

For simplicity, we assume that there are four 4-branes, where two along the $y$ direction and two along the $z$ direction, $4|(p^y_i + q^y_i)$, and $4|(p^z_i + q^z_i)$. So, we will have four local $Z_2$ symmetries: $P^y_0$, $P^y_1$, $P^z_0$ and $P^z_1$.

We will choose the unit matrix representations for $P^y_0$ and $P^z_0$ in the adjoint representation of GUT gauge group. So, considering the zero modes, under $P^y_0$ projection, we can break the 4-dimensional $N = 4$ supersymmetry to $N = 2$ supersymmetry with $(V, \Sigma_6)$ forming a vector multiplet and $(\Sigma_5, \Phi)$ forming a hypermultiplet, and we can break the 4-dimensional $N = 2$ supersymmetry to $N = 1$ supersymmetry further by $P^z_0$ projection.

For a generic bulk field $\phi(x^\mu, y, z)$, we can define four parity operators $P^y_0$, $P^y_1$, $P^z_0$ and $P^z_1$, respectively. Denoting the field with $(P^y_0, P^y_1, P^z_0, P^z_1)=\pm, \pm, \pm, \pm, \pm, \pm$ by $\phi_{+++}$, we obtain the KK mode expansions, which are those given in Eq.s (61-76).
(I) SU(6) Model. We need to choose the matrix representations for parity operators $P_0^y$, $P_1^y$, $P_0^z$ and $P_1^z$, which are expressed in the adjoint representation of SU(6). Because $SU(6) \supset SU(5) \times U(1)$; $SU(4) \times SU(2) \times U(1)$; $SU(3) \times SU(3) \times U(1)$, we obtain that, in general, $P_0^y$ and $P_1^y$ just need to be any two different representations from these three representations: diag(+1, +1, +1, +1, +1, −1), diag(−1, −1, −1, +1, +1, −1), and diag(−1, −1, −1, +1, +1, +1). So, the matrix representations for $P_0^y$, $P_0^z$, $P_1^y$ and $P_1^z$ are

\[
P_0^y = \text{diag}(+1, +1, +1, +1, +1), \quad P_0^z = \text{diag}(+1, +1, +1, +1, +1), \quad (126)
\]

\[
P_1^y = \text{diag}(+1, +1, +1, +1, −1), \quad P_1^z = \text{diag}(−1, −1, +1, +1, −1), \quad (127)
\]

or

\[
P_1^y = \text{diag}(+1, +1, +1, +1, −1), \quad P_1^z = \text{diag}(−1, −1, +1, +1, +1), \quad (128)
\]

or

\[
P_1^y = \text{diag}(−1, −1, −1, +1, +1), \quad P_1^z = \text{diag}(−1, −1, −1, +1, +1). \quad (129)
\]

And we would like to point out that

\[
SU(6)/\{\text{diag}(+1, +1, +1, +1, +1, −1)\} \approx SU(5) \times U(1), \quad (130)
\]

\[
SU(6)/\{\text{diag}(−1, −1, −1, +1, +1, −1)\} \approx SU(4) \times SU(2) \times U(1), \quad (131)
\]

\[
SU(6)/\{\text{diag}(−1, −1, −1, +1, +1, +1)\} \approx SU(3) \times SU(3) \times U(1). \quad (132)
\]

(II) SO(10) Model. We choose the following matrix representations for the parity operators $P_0^y$, $P_0^z$, $P_1^y$, and $P_1^z$, which are expressed in the adjoint representation of SO(10)

\[
P_0^y = \text{diag}(+\sigma_0, +\sigma_0, +\sigma_0, +\sigma_0) , \quad (133)
\]

\[
P_0^z = \text{diag}(+\sigma_0, +\sigma_0, +\sigma_0, +\sigma_0) , \quad (134)
\]

\[
P_1^y = \text{diag}(\sigma_2, \sigma_2, \sigma_2, \sigma_2) , \quad (135)
\]

\[
P_1^z = \text{diag}(−\sigma_0, −\sigma_0, −\sigma_0, +\sigma_0) , \quad (136)
\]

where $\sigma_0$ is the 2 × 2 unit matrix and $\sigma_2$ is the Pauli matrix.

And we would like to point out that

\[
SO(10)/P_0^y \approx SU(5) \times U(1), \quad (137)
\]

\[
SO(10)/P_1^y \approx SU(4) \times SU(2) \times SU(2) . \quad (138)
\]

Now, we discuss the GUT breaking for $SU(6)$ and $SO(10)$ together. Assume $G = SU(6)$ or $G = SO(10)$. Under $P_1^y$ and $P_1^z$ parities, the gauge generators $T^A$, \footnote{For $SU(6)$ model and $SO(10)$ model, one can interchange the matrix representations $P_1^y$ and $P_1^z$, i.e., $P_1^y \leftrightarrow P_1^z$, and the discussions are similar.}
Table 2: Parity assignment and masses \((n \geq 0, m \geq 0)\) for the vector multiplet in the \(SU(6)\) or \(SO(10)\) models on \(M^4 \times S^1 \times S^1\).

| \((P_0^y, P_1^y, P_0^z, P_1^z)\) | field | mass |
|--------------------------------|-------|------|
| \((+, +, +, +)\) | \(V^a_{\mu} \) | \(\sqrt{(2n)^2/r_1^2 + (2m)^2/r_2^2}\) |
| \((+, +, +, -)\) | \(V^a_{\mu} \) | \(\sqrt{(2n)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((+, -, +, +)\) | \(V^a_{\mu} \) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m)^2/r_2^2}\) |
| \((+, -, +, -)\) | \(V^a_{\mu} \) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((- , - , +, +)\) | \(\Sigma^a \) | \(\sqrt{(2n + 2)^2/r_1^2 + (2m)^2/r_2^2}\) |
| \((- , - , +, -)\) | \(\Sigma^a \) | \(\sqrt{(2n + 2)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((- , + , +, +)\) | \(\Sigma^a \) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m)^2/r_2^2}\) |
| \((- , + , +, -)\) | \(\Sigma^a \) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((- , - , +, +)\) | \(\Sigma^a \) | \(\sqrt{(2n)^2/r_1^2 + (2m + 2)^2/r_2^2}\) |
| \((- , - , +, -)\) | \(\Phi^a \) | \(\sqrt{(2n + 2)^2/r_1^2 + (2m + 2)^2/r_2^2}\) |
| \((- , - , +)\) | \(\Phi^a \) | \(\sqrt{(2n + 2)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((- , - , -)\) | \(\Phi^a \) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 2)^2/r_2^2}\) |
| \((- , - , +)\) | \(\Phi^a \) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |

where \(A=1, 2, \ldots, 35\) for \(SU(6)\) and \(45\) for \(SO(10)\) are separated into four sets: \(T^a, b\) are the gauge generators for \(SU(3) \times SU(2) \times U(1) \times U(1)\) gauge symmetry, \(T^a, b, \hat{T}^a, b\), and \(T^a, \hat{b}\) are the other broken gauge generators which belong to \(\{G/P^y \cap \{\text{coset } G/P^y\}\}, \{\text{coset } G/P^y \cap \text{coset } G/P^z\},\) and \(\{\text{coset } G/P^y \cap \{\text{coset } G/P^z\}\}\), respectively. Therefore, under \(P_0^y, P_0^z, P_1^y\) and \(P_1^z\), the gauge generators transform as

\[
P_0^y T^{A,B} (P_0^y)^{-1} = T^{A,B}, \quad P_0^z T^{A,B} (P_0^z)^{-1} = T^{A,B},
\]

\[
P_1^y T^{a,b} (P_1^y)^{-1} = T^{a,b}, \quad P_1^y T^{a,b} (P_1^y)^{-1} = -T^{a,b},
\]

\[
P_1^z T^{A,b} (P_1^z)^{-1} = T^{A,b}, \quad P_1^z T^{A,b} (P_1^z)^{-1} = -T^{A,b}.
\]
Table 3: For the model $G = SU(6)$ or $G = SO(10)$ on $M^4 \times S^1 \times S^1$, the gauge superfields, the number of 4-dimensional supersymmetry and gauge symmetry on the intersections of 4-branes, which are located at, $(y = 0, z = 0)$, $(y = 0, z = z_1)$, $(y = y_1, z = 0)$, and $(y = y_1, z = z_1)$, or on the 4-branes which are located at $y = 0$, $z = 0, y = y_1, z = z_1$.

| Brane Position | fields | SUSY | Gauge Symmetry |
|---------------|--------|------|----------------|
| $(0, 0)$      | $V^{A,B}_\mu$ | $N = 1$ | $G$ |
| $(0, z_1)$   | $V^{A,b}_\mu, \Sigma^A_6$ | $N=1$ | $G/P^z_1$ |
| $(y_1, 0)$   | $V^{a,B}_\mu, \Sigma^a_5$ | $N=1$ | $G/P^y_1$ |
| $(y_1, z_1)$ | $V^{a,b}_\mu, \Sigma^a_6, \Sigma^b_6, \Phi^{a,b}$ | $N=1$ | $SU(3) \times SU(2) \times U(1) \times U(1)$ |
| $y = 0$      | $V^{A,B}_\mu, \Sigma^A_6$ | $N=2$ | $G$ |
| $z = 0$      | $V^{A,B}_\mu, \Sigma^A_5$ | $N=2$ | $G$ |
| $y = y_1$    | $V^{a,B}_\mu, \Sigma^a_5, \Sigma^b_6, \Phi^{a,b}$ | $N=2$ | $G/P^y_1$ |
| $z = z_1$    | $V^{a,b}_\mu, \Sigma^a_5, \Sigma^b_6, \Phi^{a,b}$ | $N=2$ | $G/P^z_1$ |

The particle spectra are given in Table 2, and the gauge superfields, the number of 4-dimensional supersymmetry and the gauge group on the 4-branes or 4-brane intersections are given in Table 3. For the zero modes, we only have 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1) \times U(1)$ model in the bulk. And from Table 3, we obtain that including the KK modes, the 3-brane (the intersection of 4-branes) and 4-brane preserve $N = 1$ and $N = 2$ supersymmetry, respectively. The gauge group on the 3-brane can be $G$, or $G/P^z_1$, or $G/P^y_1$, or $SU(3) \times SU(2) \times U(1) \times U(1)$. And the gauge group on the 4-brane can be $G$, or $G/P^y_1$ or $G/P^z_1$. The phenomenology discussions are similar to those in Ref. [3].

6.2 $SU(6)$ Breaking on $M^4 \times S^1/Z_2 \times S^1/Z_2$

In this subsection, we would like to discuss the $SU(6)$ model on $M^4 \times S^1/Z_2 \times S^1/Z_2$ where there are two global $Z_2$ symmetries and one local $Z_2$ symmetry. Although we can not project out all the zero modes for the chiral multiplets $\Sigma_5, \Sigma_6$ and $\Phi$, we require that the extra zero modes are only from $\Phi$ and form two Higgs doublets, which is called gauge-Higgs unifications. Our basic requirements are: (1) there are no zero modes for the chiral multiplets $\Sigma_5$ and $\Sigma_6$; (2) considering the zero modes, there are only one pair of Higgs doublets because if we had two pairs of Higgs doublets, we may have flavour changing neutral current problem.
For simplicity, we assume that there are only four 4-branes, which are the boundaries for $S^1/Z_2 \times S^1/Z_2$, there is only one 3-brane in the bulk, and $2|\rho^+_i + q^+_i \rangle$, $2|\rho^-_i + q^-_i \rangle$. So, we will have two global $Z_2$ symmetries $P^y$ and $P^z$, and one local $Z_2$ symmetry $P^{y'z'}$. In order to project out all the zero modes of $\Sigma_5$ and $\Sigma_6$, we would like to choose the matrix representation of $P^y$ is equal to that of $P^z$.

For a generic bulk field $\phi(x^\mu, y, z)$, we can define three parity operators $P^y$, $P^z$, $P^{y'z'}$, respectively. Denoting the field with $(P^y, P^z, P^{y'z'})=(\pm, \pm, \pm)$ by $\phi_{\pm\pm\pm}$, we obtain the KK mode expansions which are given in Eq.s (102-109).

There are two scenarios which have gauge-Higgs unification,

(1) We choose the matrix representations for $P^y$, $P^z$ and $P^{y'z'}$ as following

$$ P^y = P^z = \text{diag}(+1, +1, +1, +1, +1, -1) \ , $$

$$ P^{y'z'} = \text{diag}(-1, -1, -1, +1, +1, +1) \ . $$

(2) We choose the matrix representations for $P^y$, $P^z$, and $P^{y'z'}$ as following

$$ P^y = P^z = \text{diag}(-1, -1, -1, +1, +1, -1) \ , $$

$$ P^{y'z'} = \text{diag}(-1, -1, -1, +1, +1, +1) \ . $$

Under $P^y$ (or $P^z$) and $P^{y'z'}$ parities, the gauge generators $T^A$, where $A=1, 2, ..., 35$ for $SU(6)$ are separated into four sets: $T^{a,b}$ are the gauge generators for $SU(3) \times SU(2) \times U(1) \times U(1)$ gauge symmetry, $T^{a,b}$, $T^{a\bar{b}}$, and $T^{\bar{a}b}$ are the other broken gauge generators which belong to $\{G/P^y \cap \{\text{coset } G/P^{y'z'}\}\}$, $\{\{\text{coset } G/P^y\} \cap G/P^{y'z'}\}$, and $\{\{\text{coset } G/P^y\} \cap \{\text{coset } G/P^{y'z'}\}\}$, respectively. Therefore, under $P^y$, $P^z$, and $P^{y'z'}$, the gauge generators transform as

$$ P^y T^{a,b} (P^y)^{-1} = T^{a,b} \ , \quad P^y T^{\bar{a},\bar{b}} (P^y)^{-1} = -T^{\bar{a},\bar{b}} \ , $$

$$ P^z T^{a,b} (P^z)^{-1} = T^{a,b} \ , \quad P^z T^{\bar{a},\bar{b}} (P^z)^{-1} = -T^{\bar{a},\bar{b}} \ , $$

$$ P^{y'z'} T^{A,b} (P^{y'z'})^{-1} = T^{A\bar{b}} \ , \quad P^{y'z'} T^{\bar{A},\bar{b}} (P^{y'z'})^{-1} = -T^{\bar{A}\bar{b}} \ . $$

We present the particle spectra in Table 4, and the number of 4-dimensional supersymmetry and gauge symmetry on the 3-brane, 4-brane intersections and 4-branes in Table 5. Because the 3-brane is the fixed point under $P^{y'z'}$ symmetry, it preserves 4-dimensional $N=2$ supersymmetry. And the 4-branes are the fixed lines under one global symmetry, the 4-brane intersections and 4-branes preserve 4-dimensional $N=1$ and $N=2$ supersymmetry, respectively. The phenomenology discussions are also similar to those in Ref.

7 Discrete Symmetry on the Space-Time $M^4 \times A^2$

As we know, in each point of the 2-dimensional real manifold, there is an open neighborhood homeomorphic to $R^2$ in the real coordinates or $C^1$ in the complex coordinates, and its rotation group is locally $SO(2)$ or $U(1)$. So, We may define the $Z_n$ discrete symmetry on the 2-dimensional manifold in which $n$ is any positive integer and
Table 4: Parity assignment and masses \((n \geq 0, m \geq 0)\) for the vector multiplet in the \(SU(6)\) model on \(M^4 \times S^1/Z_2 \times S^1/Z_2\).

| \((P^y, P^z, P^{y'})\) | field \(V^a_{\mu}, \Phi^a_{\hat{\mu}}\) | mass |
|-----------------|-----------------|------|
| \((+, +, +)\)   | \(V^{ab}_{\mu}, \Phi^{\hat{ab}}_{\hat{\mu}}\) | \(\sqrt{(2n)^2/r_1^2 + (2m)^2/r_2^2}\)\ or \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((+, +, -)\)   | \(V^{ab}_{\mu}, \Phi^{\hat{ab}}_{\hat{\mu}}\) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m)^2/r_2^2}\)\ or \(\sqrt{(2n)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((+, +, +)\)   | \(\Sigma^{\hat{a}}_5, \Sigma^a_6\) | \(\sqrt{(2n)^2/r_1^2 + (2m + 1)^2/r_2^2}\)\ or \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 2)^2/r_2^2}\) |
| \((+, -, -)\)   | \(\Sigma^{\hat{a}}_5, \Sigma^a_6\) | \(\sqrt{(2n)^2/r_1^2 + (2m + 2)^2/r_2^2}\)\ or \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((-+, +)\)     | \(\Sigma^{\hat{a}}_5, \Sigma^a_6\) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m)^2/r_2^2}\)\ or \(\sqrt{(2n + 2)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((-+, -)\)     | \(\Sigma^{\hat{a}}_5, \Sigma^a_6\) | \(\sqrt{(2n + 2)^2/r_1^2 + (2m)^2/r_2^2}\)\ or \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((-+, +)\)     | \(V^{ab}_{\mu}, \Phi^{\hat{ab}}_{\hat{\mu}}\) | \(\sqrt{(2n + 1)^2/r_1^2 + (2m)^2/r_2^2}\)\ or \(\sqrt{(2n + 2)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |
| \((-+, -)\)     | \(V^{ab}_{\mu}, \Phi^{\hat{ab}}_{\hat{\mu}}\) | \(\sqrt{(2n + 2)^2/r_1^2 + (2m)^2/r_2^2}\)\ or \(\sqrt{(2n + 1)^2/r_1^2 + (2m + 1)^2/r_2^2}\) |

we can break any supersymmetric \(SU(M)\) GUT models down to the 4-dimensional \(N = 1\) supersymmetric \(SU(3) \times SU(2) \times U(1)^{M-4}\) models for the zero modes. One obvious candidate for the extra space manifold is the disc \(D^2\) where there is one 3-brane at origin and one 4-brane at outer boundary. To be general, we can consider that the extra space manifold is the annulus where there are two boundary 4-branes. For simplicity, we denote the annulus as \(A^2\).

Furthermore, we discuss the KK modes expansions on the space-time \(M^4 \times (\text{an Segment of } A^2)\), and find that the masses of KK states might be set arbitrarily heavy if the range of the angle is small enough.
Table 5: For the $G = SU(6)$ model with gauge-Higgs unification on $M^4 \times S^1/Z_2 \times S^1/Z_2$. The gauge superfield $V_\mu$, the number of 4-dimensional supersymmetry and gauge symmetry on the 4-brane intersections or 3-brane, which are located at the $(y = 0, z = 0)$, $(y = 0, z = \pi R_2)$, $(y = \pi R_1, z = 0)$, $(y = \pi R_1, z = \pi R_2)$, and $(y = y_1, z = z_1)$, and on the 4-branes which are located at the fixed lines $y = 0$, $y = \pi R_1, z = 0, z = \pi R_2$.

| Brane Position | Fields | SUSY | Gauge Symmetry |
|----------------|--------|------|----------------|
| $(0, 0), (0, \pi R_2), (\pi R_1, 0), (\pi R_1, \pi R_2)$, $V_{\mu}^{a,b}$ | $N = 1$ | $G/P^y$ or $G/P^z$ |
| $(y_1, z_1)$ | $V_{\mu}^{A,b}$ | $N=2$ | $G/P^y$ or $G/P^z$ |
| $y = 0, y = \pi R_1$, $z = 0, z = \pi R_2$ | $V_{\mu}^{a,b}$ | $N=2$ | $G/P^y$ or $G/P^z$ |

7.1 Discrete Symmetry on $M^4 \times A^2$

We consider that the extra space manifold is annulus $A^2$. The convenient coordinates for the annulus $A^2$ is circular coordinate $(r, \theta)$, and it is easy to change it to the complex coordinates by $z = re^{i\theta}$. We assume that the inner radius of the annulus is $R_1$, and the outer radius of the annulus is $R_2$. Taking $R_1 = 0$, we obtain the Disc $D^2$. Considering $Z_n$ symmetry on $A^2$, we define

$$\omega = e^{i\frac{2\pi}{n}} ,$$

(149)

and we define the generator for $Z_n$ as $\Omega$ which satisfies $\Omega^n = 1$.

For a generic bulk multiplet $\Phi$ which fills a representation of the bulk gauge group $G$, we have

$$\Omega \Phi(x^\mu, z, \bar{z}) = \Phi(x^\mu, \omega z, \omega^{n-1} \bar{z}) = \eta_\Phi(R_{\Omega}) \Phi(x^\mu, z, \bar{z})(R_{\Omega}^{-1})^{m}\Phi ,$$

(150)

where $\eta_\Phi \subset Z_n$, and $R_{\Omega}$ is an element in the adjoint representation of $G$ which satisfies $R_{\Omega}^n = 1$.

For a generic field $\phi(x^\mu, z, \bar{z})$ with eigenvalue $\omega^l$ under the operator $\Omega$, we write it as $\phi_{\omega^l}(x^\mu, z, \bar{z})$, i. e.,

$$\Omega \phi_{\omega^l}(x^\mu, z, \bar{z}) = \omega^l \phi_{\omega^l}(x^\mu, z, \bar{z}) .$$

(151)

And the KK modes expansions for $\phi_{\omega^l}(x^\mu, z, \bar{z})$ are

$$\phi_{\omega^l}(x^\mu, z, \bar{z}) = \sum_{j=\infty}^{\infty} \sum_{k=1}^{\infty} \phi_{\omega^l}^{(jk)}(x^\mu) f_{jk}(z, \bar{z}) ,$$

(152)
where \( l = 0, 1, \ldots, n - 1 \), and

\[
f^{\omega l}_{jk}(z, \bar{z}) = \sum_{s=0}^{n-1} \omega^{(n-s)} f_{jk}(\omega^s z, \omega^{n-s} \bar{z}) . \tag{153}
\]

And the function \( f_{jk}(z, \bar{z}) \) are defined by

\[
f_{jk}(z, \bar{z}) = J_j(\lambda_{jk} r) e^{i j \theta} , \tag{154}
\]

or

\[
f_{jk}(z, \bar{z}) = J_j(\lambda_{jk} |z|)(z/|\bar{z}|)^j , \tag{155}
\]

where \( J_j(\lambda_{jk} r) \) is the first order Bessel function, and it satisfies the Dirichlet or Neumann boundary condition at \( r = R_1 \) and \( r = R_2 \)

\[
J_j(\lambda_{jk} r) = 0 \text{ or } \frac{d J_j(\lambda_{jk} r)}{d \lambda_{jk} r} = 0 , \quad \text{for } r = R_1 \text{ and } R_2 . \tag{156}
\]

The zero modes are contained only in \( \phi_{\omega 0} \) fields, i. e. \( l = 0 \).

First, we consider that the extra space manifold is a disc \( D^2 \), i. e., \( R_1 = 0 \), so, we only have boundary condition at \( r = R_2 \), and then we can have all the KK states. There is also one fixed point under \( Z_n \) symmetry, which is the center of the disc. One might wonder whether there is an singularity for \( f_{jk}(z, \bar{z}) \) at origin \( r = 0 \) when \( j \neq 0 \), however, there is no singularity and we only have zero modes at origin because \( J_0(0) = 1 \) and \( J_s(0) = 0 \) for \( s \geq 1 \). By the way, the global \( Z_n \) symmetry can be modulated from \( D^2 \), and the corresponding orbifold is \( D^2/Z_n \).

Second, we consider that the extra space manifold is an annulus \( A^2 \). The \( Z_n \) symmetry act free on the annulus \( A^2 \), so, we can obtain the quotient manifold \( A^2/Z_n \) by moduloin the \( Z_n \) symmetry. We will also have much less KK states, i. e., a lot of KK states in the summation might be absent because the boundary conditions at \( r = R_1 \) and \( r = R_2 \) must be satisfied simultaneously. The interesting phenomenology is that, we might have the scenario in which only a few KK states are light and the other KK states are relatively heavy, so, we may produce the light KK states of the gauge fields at future colliders.

### 7.2 KK modes on \( M^4 \times (\text{an Segment of } A^2) \)

If the extra space-manifold is an segment of \( A^2 \), we would like to discuss the KK modes. We assume that the inner radius of the annulus is \( R_1 \), the outer radius of the annulus is \( R_2 \), and the angle \( \theta \) is

\[
0 \leq \theta \leq \alpha 2\pi , \tag{157}
\]

where \( 0 < \alpha < 1 \). The KK modes expansion for a generic field \( \phi \) is

\[
\phi(x^\mu, z, \bar{z}) = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \phi^{(jk)}(x^\mu) f_{jk}(z, \bar{z}) , \tag{158}
\]
where
\[ f_{jk}(z, \bar{z}) = J_{j/(4\alpha)}(\lambda_{jk}|z|)e^{ij\theta/(4\alpha)}, \] (159)
or
\[ f_{jk}(z, \bar{z}) = J_{j/(4\alpha)}(\lambda_{jk}|z|)(z/|\bar{z}|)^{j/(4\alpha)}. \] (160)

And at \( r = R_1 \) and \( r = R_2 \), the function \( J_{j/(4\alpha)}(\lambda_{jk}r) \) should satisfy the Dirichlet boundary condition or Neumann condition,
\[ J_j(\lambda_{jk}r) = 0 \quad \text{or} \quad \frac{dJ_j(\lambda_{jk}r)}{d\lambda_{jk}r} = 0. \] (161)

In short, if \( \alpha \) is very small, then, \( j/(4\alpha) \) will be very large, and the KK states may be set arbitrarily heavy, which is similar to that in Ref. [11].

We can define the \( Z_2 \) reflection symmetry on the sector of \( D^2 \) or the segment of \( A^2 \). However, we cannot define the discrete symmetry \( Z_n \) for \( n > 2 \) on the sector of \( D^2 \) or segment of \( A^2 \), so, it is not interesting for us to discuss the supersymmetric GUT breaking in this case.

### 8 GUT breaking on the Space-Time \( M^4 \times A^2 \)

In this section, we would like to discuss the 6-dimensional \( N = 2 \) supersymmetric \( SU(M) \) GUT models on the space-time \( M^4 \times A^2 \) or \( M^4 \times D^2 \).

In principle, we can break any \( SU(M) \) gauge group on the space-time \( M^4 \times A^2 \) or \( M^4 \times D^2 \) because we can choose \( Z_n \) symmetry in which \( n \) is very large. As example, we will discuss the \( SU(6) \) models on the space-time \( M^4 \times A^2 \) or \( M^4 \times D^2 \) with \( Z_9 \) symmetry, or one can consider the space-time as \( M^4 \times A^2/Z_9 \) or \( M^4 \times D^2/Z_9 \).

The \( N = 2 \) supersymmetry in 6-dimension corresponds to \( N = 4 \) supersymmetry in 4-dimension, thus, only the gauge multiplet can be introduced in the bulk. This multiplet can be decomposed under the 4-dimensional \( N = 1 \) supersymmetry into a vector multiplet \( V \) and three chiral multiplets \( \Sigma, \Phi, \Phi^c \) in the adjoint representation, with the fifth and sixth components of the gauge field, \( A_5 \) and \( A_6 \), contained in the lowest component of \( \Sigma \). The Standard Model fermions are on the boundary 4-brane at \( r = R_1 \) or \( r = R_2 \) for the annulus \( A^2 \), and on the 3-brane at origin or on the boundary 4-brane at \( r = R_2 \) for the disc \( D^2 \).

In the Wess-Zumino gauge and 4-dimensional \( N = 1 \) language, the bulk action is [14]
\[
S = \int d^6x \left\{ \text{Tr} \left[ \int d^2\theta \left( \frac{1}{4kg^2} \mathcal{W}^a \mathcal{W}_a + \frac{1}{kg^2} \left( \Phi^c \partial \Phi - \frac{1}{\sqrt{2}} \Sigma[\Phi, \Phi^c] \right) \right) \right] + \text{h.c.} \right. \\
+ \int d^4\theta \frac{1}{kg^2} \text{Tr} \left[ (-\sqrt{2} \partial \bar{\Sigma} + \Sigma^t)e^{-V}(-\sqrt{2} \partial + \Sigma)e^V \right] \\
+ \int d^4\theta \frac{1}{kg^2} \text{Tr} \left[ \Phi^c e^{-V}\Phi^c e^V + \Phi^t e^{-V}\Phi e^V \right], \] (162)
Notice that we consider $Z_9$ symmetry, then, $\omega = e^{i2\pi/9}$. From above action, we obtain the transformations of gauge multiplet under $\Omega$ as

\[
V(\omega z, \omega^8 \bar{z}) = R_\Omega V(z, \bar{z}) R_\Omega^{-1},
\]

\[
\Sigma(\omega z, \omega^8 \bar{z}) = \omega^8 R_\Omega \Sigma(z, \bar{z}) R_\Omega^{-1},
\]

\[
\Phi(\omega z, \omega^8 \bar{z}) = \omega^m R_\Omega \Phi(z, \bar{z}) R_\Omega^{-1},
\]

\[
\Phi^c(\omega z, \omega^8 \bar{z}) = \omega^{10-m} R_\Omega \Phi^c(z, \bar{z}) R_\Omega^{-1},
\]

where $0 \leq m \leq 8$, $R_\Omega$ is an element in the adjoint representation of the GUT gauge group and satisfies the equation $R_\Omega^9 = 1$. To be compatible with our previous discussions in section 6, we choose $m = 8$, and then,

\[
\Phi(\omega z, \omega^8 \bar{z}) = \omega^8 R_\Omega \Phi(z, \bar{z}) R_\Omega^{-1},
\]

\[
\Phi^c(\omega z, \omega^8 \bar{z}) = \omega^{2} R_\Omega \Phi^c(z, \bar{z}) R_\Omega^{-1}.
\]

Now, we would like to discuss the supersymmetric $SU(6)$ model. We choose the following matrix representations for the $Z_9$ operator $\Omega$, $R_\Omega$, which are expressed in the adjoint representaion of $SU(6)$

\[
R_\Omega = \text{diag}(\omega^2, \omega^2, \omega^2, \omega^8, \omega^8, \omega^5).
\]

So, upon the $Z_9$ operator $\Omega$, the gauge generators $T^A$ where $A=1, 2, \ldots, 35$ for $SU(6)$ are separated into two sets: $T^a$ are the gauge generators for the $SU(3) \times SU(2) \times U(1)^2$ gauge group, and $T^\hat{a}$ are the other broken gauge generators

\[
R_\Omega T^a R_\Omega^{-1} = T^a, \quad R_\Omega T^\hat{a} R_\Omega^{-1} = -T^\hat{a}.
\]

First, we consider that the extra space manifold is the annulus $A^2$. For the zero modes, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry in the bulk and on the 4-branes at $r = R_1$ and $r = R_2$. Including the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(6)$ gauge symmetry in the bulk, and on the 4-branes at $r = R_1$ and $r = R_2$.

Second, we consider that the extra space manifold is the disc $D^2$. For the zero modes, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry in the bulk and on the 4-brane at $r = R_2$. Including all the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(6)$ gauge symmetry in the bulk, and on the 4-brane at $r = R_2$. In addition, we always have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry on the 3-brane at origin in which only the zero modes exist. So, if we put the Standard Model fermions

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on the 3-brane at origin, the extra dimensions can be large and the gauge hierarchy problem can be solved for there does not exist the proton decay problem at all.

In order to break the extra $U(1)$ symmetry, we have to introduce the extra chiral multiplets which are singlets under the Standard Model gauge symmetry, and use Higgs mechanism. And if we considered the chiral model on the observable brane, we will have to introduce the exotic particles due to the anomaly cancellation.

In short, we can introduce $Z_n$ symmetry to break any supersymmetric $SU(M)$ GUT models as long as $n$ is large enough. There are 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry in the bulk and on the 4-branes for the zero modes, and on the 3-brane at origin in the disc $D^2$ scenario. Including all the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(M)$ gauge symmetry in the bulk, and on the 4-branes. In addition, the Standard Model fermions are on the boundary 4-brane at $r = R_1$ or $r = R_2$ if the extra space manifold is annulus $A^2$, and on the 3-brane at origin or on the boundary 4-brane at $r = R_2$ if the extra space manifold is disc $D^2$.

## 9 Discrete Symmetry on the Space-Time $M^4 \times T^2$

In this section, we would like to discuss the discrete symmetry on the space-time $M^4 \times T^2$. Because for any point in $T^2$, there is an open neighborhood which is homeomorphic to $R^2$, naively, one might think that one can introduce any $Z_n$ symmetry on $T^2$. However, this is not true. In fact, we can prove that the only discrete symmetries on the torus from the rotation group $SO(2) \text{ or } U(1)$ are $Z_2$, $Z_3$, $Z_4$, and $Z_6$.

The proof is the following. In the complex coordinates, the torus $T^2$ can be defined by $C^1$ moduled the equivalent classes: $z \sim z + 2\pi R_1$ and $z \sim z + 2\pi R_2\omega$ where $\omega = e^{i\theta}$. So, we need to discuss the $Z_2$, $Z_3$ and $Z_n$ symmetries for $n > 3$ separately.

First, we consider $Z_2$ symmetry, the equivalent class is $z \sim -z$, and the two fixed points are $z = 0$ and $z = \pi R_1 + \pi R_2 \omega$. So, the global $Z_2$ symmetry can be defined on the general torus $T^2$. And the 3-branes can be located at the fixed points.

Second, we consider $Z_3$ symmetry. We have to choose $R_1 = R_2 = R$. Define $\theta = 2\pi/3$, we obtain that the equivalent class $z \sim z + 2\pi R \omega$ is equivalent to the equivalent class $z \sim z + 2\pi R e^{i\pi/3}$. And there are three fixed points: $z = 0$, $z = 2\pi R e^{i\pi/6}/\sqrt{3}$, and $z = 4\pi R e^{i\pi/6}/\sqrt{3}$. The 3-branes can be located at the fixed points.

Third, in order to define the $Z_n$ symmetry for $n > 3$, we have to choose $\theta = 2\pi/n$ and $R_1 = R_2 = R$. In addition, $\omega$ should satisfy the following equations

\begin{align}
2\pi R \omega &= l'2\pi R + k'2\pi R \omega, \\
2\pi R \omega \omega &= l2\pi R + k2\pi R \omega,
\end{align}

where $l$, $k$, $l'$ and $k'$ are integers. The first equation is satisfied by choosing $l' = 0$ and $k' = 1$. Moreover, from the second equation, we obtain

\[ \omega = \frac{l \pm \sqrt{l^2 + 4k}}{2}. \]
The complete solutions are \( l = 0 \) and \( k = \pm 1 \), \( l = 1 \) and \( k = -1 \). And then, all the possible \( \omega \) are \( \pm 1, \pm i, e^{i2\pi/3}, e^{i2\pi/6} \). Therefore, the discrete symmetries \( Z_n \) for \( n > 3 \) on the torus from the rotation group \( SO(2) \) or \( U(1) \) are \( Z_4 \) and \( Z_6 \). Moreover, for \( Z_4 \) discrete symmetry, there are two \( Z_4 \) fixed points: \( z = 0 \) and \( z = \sqrt{2\pi}Re^{i\pi/4} \), two \( Z_2 \) fixed points: \( z = \pi R \) and \( z = \pi Re^{i\pi/2} \). For \( Z_6 \) discrete symmetry, there is one \( Z_6 \) fixed point \( z = 0 \), and there are two \( Z_3 \) fixed points: \( z = 2\pi Re^{i\pi/6}/\sqrt{3} \) and \( z = 4\pi Re^{i\pi/6}/\sqrt{3} \), three \( Z_2 \) fixed points: \( z = \sqrt{3}\pi Re^{i\pi/6}, z = \pi R \) and \( z = \pi Re^{i\pi/3} \). The 3-branes can be located at those fixed points.

For the general \( T^2 \) defined by the equivalent classes: \( z \sim z + 2\pi R_1 \) and \( z \sim z + 2\pi R_2 e^{i\theta} \), the KK modes expansion for a generic field \( \phi \) is

\[
\phi(x^\mu, z, \bar{z}) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \phi^{(jk)}(x^\mu) f_{jk}(z, \bar{z}) ,
\]

where

\[
f_{jk}(z, \bar{z}) = \exp\{i[(a - ib)z + (a + ib)\bar{z}]\} ,
\]

where

\[
a = \frac{j}{2R_1} ,
\]

\[
b = \frac{1}{\sin \theta} \left( \frac{k}{2R_2} - \frac{j}{2R_1} \cos \theta \right) .
\]

The mass for \( \phi^{jk} \) is

\[
M_{\phi^{jk}}^2 = \frac{1}{\sin \theta^2} \left[ \frac{j^2}{4R_1^2} + \frac{k^2}{4R_2^2} - \frac{2jk}{4R_1R_2} \cos \theta \right] .
\]

The zero modes are contained in \( \phi^{(00)} \) sector, i.e., \( j = 0 \) and \( k = 0 \).

The fundamental group for the torus is \( \mathbb{Z} \oplus \mathbb{Z} \), so, one might think we can break the gauge symmetry by Wilson line in the mean time. The key question is how to define the suitable KK modes’ expansions for the bulk fields in the Wilson line approach, because we require that, the fields without zero modes, should have KK modes’ excitations.

10 GUT Breaking on the Space-Time \( M^4 \times T^2 \)

We would like to consider the 6-dimensional \( N = 2 \) supersymmetric \( SU(5) \) model on \( M^4 \times T^2 \) with \( Z_6 \) symmetry.

Notice that we consider \( Z_6 \) symmetry, we define \( \omega = e^{i2\pi/6} \). From the action in Eq. (162), we obtain the transformations of gauge multiplet under \( \Omega \) as

\[
V(\omega z, \omega^5 \bar{z}) = R_\Omega V(z, \bar{z}) R^{-1}_\Omega ,
\]
\[ \Sigma(\omega z, \omega^5 \bar{z}) = \omega^5 R_\Omega \Sigma(z, \bar{z}) R_\Omega^{-1}, \quad (180) \]

\[ \Phi(\omega z, \omega^5 \bar{z}) = \omega^5 R_\Omega \Phi(z, \bar{z}) R_\Omega^{-1}, \quad (181) \]

\[ \Phi^c(\omega z, \omega^5 \bar{z}) = \omega^2 R_\Omega \Phi^c(z, \bar{z}) R_\Omega^{-1}, \quad (182) \]

where \( R_\Omega \) is an element in the adjoint representation of \( SU(5) \) and satisfies the equation \( R_\Omega^6 = 1 \).

We choose the following matrix representations for the \( Z_6 \) operator \( \Omega \), \( R_\Omega \), which are expressed in the adjoint representation of \( SU(5) \):

\[ R_\Omega = \text{diag}(1, 1, 1, -1, -1). \quad (183) \]

So, upon the \( Z_6 \) operator \( \Omega \), the gauge generators \( T^A \) where \( A = 1, 2, ..., 24 \) for \( SU(5) \) are separated into two sets: \( T^a \) are the gauge generators for the Standard Model gauge group, and \( T^{\hat{a}} \) are the other broken gauge generators

\[ R_\Omega \ T^a \ R_\Omega^{-1} = T^a, \quad R_\Omega \ T^{\hat{a}} \ R_\Omega^{-1} = -T^{\hat{a}}. \quad (184) \]

In addition, the representation of the generator for \( Z_2 \) symmetry is

\[ R_\Omega^3 = \text{diag}(1, 1, 1, -1, -1), \quad (185) \]

and the representation of the generator for \( Z_3 \) symmetry is

\[ R_\Omega^2 = \text{diag}(1, 1, 1, 1). \quad (186) \]

Therefore, the gauge symmetries on the 3-branes at \( Z_2 \) fixed points and on the 3-branes at \( Z_3 \) fixed points are

\[ SU(5)/R_\Omega^3 \approx SU(3) \times SU(2) \times U(1), \quad (187) \]

\[ SU(5)/R_\Omega^2 \approx SU(5). \quad (188) \]

The gauge multiplet, the number of 4-dimensional supersymmetry and gauge symmetry on the 3-branes are given in Table 6. In short, we have 4-dimensional \( N = 1 \) supersymmetry and the Standard Model gauge symmetry in the bulk for the zero modes, and on the 3-brane at \( Z_6 \) fixed point for all the modes. Including the KK states, we will have the 4-dimensional \( N = 4 \) supersymmetry and \( SU(5) \) gauge symmetry in the bulk, the 4-dimensional \( N = 1 \) supersymmetry and \( SU(5) \) gauge symmetry on the 3-branes at \( Z_3 \) fixed points, and the 4-dimensional \( N = 4 \) supersymmetry and \( SU(3) \times SU(2) \times U(1) \) gauge symmetry on the 3-branes at \( Z_2 \) fixed points. The Standard Model fermions and Higgs fields can be on any 3-brane at one of the fixed points. In particular, if we put the Standard Model fermions and Higgs fields on the 3-brane at \( Z_6 \) fixed point, the extra dimensions can be large and the gauge hierarchy problem can be solved because there is no proton decay problem at all.
Table 6: For the $G = SU(5)$ model on $M^4 \times T^2$ with $Z_6$ symmetry. The gauge multiplet, the number of 4-dimensional supersymmetry and gauge symmetry on the 3-branes, which are located at the $Z_6$ fixed point $z = 0$; the $Z_3$ fixed points: $z = 2\pi \text{Re}^{\pi/6}/\sqrt{3}$, and $z = 4\pi \text{Re}^{\pi/6}/\sqrt{3}$; and the $Z_2$ fixed points: $z = \sqrt{3}\pi \text{Re}^{\pi/6}$, $z = \pi R$, and $z = \pi \text{Re}^{\pi/3}$.

| Brane Position | Fields | SUSY | Gauge Symmetry |
|----------------|--------|------|----------------|
| $z = 0$        | $V^a$  | $N = 1$ | $SU(3) \times SU(2) \times U(1)$ |
| $z = 2\pi \text{Re}^{\pi/6}/\sqrt{3}$, $z = 4\pi \text{Re}^{\pi/6}/\sqrt{3}$ | $V^A, \Sigma^a, \Phi^a, (\Phi^c)^{\hat{a}}$ | $N = 1$ | $SU(5)$ |
| $z = \sqrt{3}\pi \text{Re}^{\pi/6}$, $z = \pi R, z = \pi \text{Re}^{\pi/3}$ | $V^a, \Sigma^A, \Phi^A, (\Phi^c)^A$ | $N = 4$ | $SU(3) \times SU(2) \times U(1)$ |

11 Discussion and Conclusion

With the ansatz that there exist local or global discrete symmetries in the special branes’ neighborhoods, we discuss the general reflection $Z_2$ symmetries on the space-time $M^4 \times M^1$ and $M^4 \times M^1 \times M^1$. We obtain that we can have at most two $Z_2$ symmetries on $M^4 \times M^1$ and four $Z_2$ symmetries on $M^4 \times M^1 \times M^1$. As representatives, we discuss the $N = 1$ supersymmetric $SU(5)$ model on the space-time $M^4 \times M^1$ where the Standard Model fermions can be in the bulk or on the 3-brane, the $N = 2$ supersymmetric $SU(6)$ and $SO(10)$ models on the space-time $M^4 \times S^1 \times S^1$ and the $N = 2$ supersymmetric $SU(6)$ model with gauge-Higgs unification on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$, where the Standard Model fermions must be on the 4-brane, or 3-brane, or 4-brane intersection. For the zero modes, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{n-3}$ gauge symmetry in which $n$ is the rank of GUT gauge group. The gauge symmetry and supersymmetry may be broken on the 3-branes, or 4-branes, or 4-brane intersections. In particular, in those models, the extra dimensions can be large and the masses of KK states can be set arbitrarily heavy.

In addition, we discuss the discrete $Z_n$ symmetry on the space-time $M^4 \times A^2$ and $M^4 \times D^2$ in which $n$ is any positive integer. In this kind of scenarios, we can break any $SU(M)$ gauge symmetry for $M \geq 5$ down to the $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry by introducing the global $Z_n$ symmetry as long as $n$ is large enough. In general, considering the 6-dimensional $N = 2$ supersymmetry, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry in the bulk and on the 4-branes for the zero modes, and on the 3-brane at origin where only the zero modes exist in the disc $D^2$ scenario. Including all the KK states, we will have
4-dimensional $N = 4$ supersymmetry and $SU(M)$ gauge symmetry in the bulk, and
on the 4-branes. The Standard Model fermions should be on the boundary 4-brane or
3-brane at origin. By the way, if we put the Standard Model fermions on the 3-brane
at origin, the extra dimensions can be large and the gauge hierarchy problem can
be solved for there does not exist the proton decay problem at all. Moreover, if the
extra space manifold is annulus $A^2$, for suitable choices of the inner radius and outer
radius, we might construct the models where only a few KK states are light and the
other KK states are relatively heavy due to the boundary conditions on the inner and
outer boundaries, so, we might produce the light KK states of gauge fields at future
colliders, which is very interesting in collider physics.

And if the extra space manifold is an sector of $D^2$ or an segment of $A^2$, we
point out that the masses of KK states can be set arbitrarily heavy if the range of
angle is small enough.

Furthermore, we discuss the complete global discrete symmetry on the space-
time $M^4 \times T^2$. We prove that the possible global discrete symmetries on the torus is
$Z_2$, $Z_3$, $Z_4$, and $Z_6$. We also discuss the 6-dimensional $N = 2$ supersymmetric $SU(5)$
models on the space-time $M^4 \times T^2$ with $Z_6$ symmetry. There are 4-dimensional
$N = 1$ supersymmetry and the Standard Model gauge symmetry in the bulk for
the zero modes, and on the 3-brane at $Z_6$ fixed point for all the modes. Including
the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(5)$
gauge symmetry in the bulk, the 4-dimensional $N = 1$ supersymmetry and $SU(5)$
gauge symmetry on the 3-branes at $Z_3$ fixed points, and the 4-dimensional $N = 4$
supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry on the 3-branes at $Z_2$
fixed points. The Standard Model fermions and Higgs fields can be on any 3-brane
at one of the fixed points. In particular, if we put the Standard Model fermions and
Higgs fields on the 3-brane at $Z_6$ fixed point, the extra dimensions can be large and
the gauge hierarchy problem can be solved because there is no proton decay problem
at all.

The phenomenology in those scenarios deserve further study.

Acknowledgments

This research was supported in part by the U.S. Department of Energy under Grant
No. DOE-EY-76-02-3071.

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