OUT OF EQUILIBRIUM PHASE TRANSITIONS
AND A TOY MODEL FOR
DISORIENTED CHIRAL CONDENSATES

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Abstract

We study the dynamics of a second order phase transition in a situation that mimics a sudden quench to a temperature below the critical temperature in a model with dynamical symmetry breaking. In particular we show that the domains of correlated values of the condensate grow as $\sqrt{t}$ and that this result seems to be largely model independent.
I. Introduction

Most of the studies in finite temperature field theory deal with problems in thermal equilibrium. There are systems though, where conditions change so fast that to take the equilibrium formalism, even as a first approximation, would be inappropriate. One of the situations where non equilibrium phenomena are interesting is the dynamics of phase transitions. They have been considered by a number of authors, mostly in connection with problems in cosmology. More recently the interest in phase transitions occurring in relativistic heavy ion collisions has grown due to the possibility of producing in the laboratory the deconfined phase of strong interactions. Clear signals that the quark gluon plasma was indeed produced and the study of its properties is made difficult by the fact that in the laboratory this phase is produced in a small region and is very short lived. Non equilibrium phenomena are quite normal occurrence in this kind of experiment and this provides another motivation for considering the dynamical aspects of phase transitions in relativistic field theories.

One very interesting proposal on how to decide whether the quark gluon plasma was created or not during a heavy ion collision was suggested many years ago and since then has been studied by many authors [1]. It is based on the fact that besides the deconfining phase transition, we expect in QCD another phase transition at about the same temperature, namely, the chiral symmetry restoration. The picture goes as follows. The value of the order parameter $\bar{\psi}\psi$ becomes zero at high temperatures reached in the plasma formed by the colliding nuclei and chiral symmetry is restored. With the expansion of the plasma though, this temperature decreases and eventually chiral symmetry is broken again. The direction of symmetry breaking in isospin space, however, does not necessarily have to be the same as it originally was before the collision took place. Therefore, if the value of the order parameter is, say, in the $\pi^0$ direction, after hadronization this plasma will be converted into neutral pions only, and not charged ones. This could lead to spectacular events with large number of pions correlated in isospin space coming out of one collision
and signaling clearly that chiral symmetry restoration was achieved inside the plasma. Events of this kind might have already been seen in cosmic rays (the so called Centauro events [2]) as it has been suggested [3]. The difficulty with this picture is that the direction in which chiral symmetry is broken does not have to be the same all over the plasma. More likely there will be domains in which the order parameter is the same, but different domains will have different symmetry breaking directions. This domain structure is a very general phenomenon in condensed matter, cosmology, etc. If the typical domains are small and only a few pions can be made with energy contained in each of them, then the correlation in isospin space can hardly be distinguished from that of a random production. In equilibrium there are indications that this is indeed the case. Recently Rajagopal and Wilczek [3] have suggested that if the rapid cooling of the plasma is modelled as a sudden quench below the critical temperature the size of correlated domains can be much larger than the size in equilibrium, growing with time as some power of time. This conclusion has been supported by analogies with condensed matter systems obeying classical dissipative stochastic first order equations. However, as has been stressed by Boyanovsky et al. [4] the situation in a relativistic field theory, where the order parameter obeys a quantized conservative second order equation does not necessarily have to be the same. The same problem of determining the rate of growth of domain structures out of equilibrium is also of importance in cosmology, since the size of the domains determine the number of different topological defects produced during the phase transition. In this paper, we report on an attempt to study this phenomenon systematically within the context of a model that exhibits dynamical symmetry breaking.

The method we will use to analyze this problem is the Closed Time Path formalism invented in the sixties by Schwinger and Keldysh [5]. Since chiral symmetry breaking is not well understood in QCD even at zero temperature, we will take the Nambu-Jona-Lasinio [6] model as an effective theory having the essential physics of chiral symmetry breaking present in QCD. In this model the critical temperature is a function of the coupling
constant $g$ and, instead of coupling our system to some heat bath to generate the quenching, we will mimic the drop in temperature by suddenly changing $g$, and consequently the critical temperature. Ideally one should study such a phenomenon starting from an initial temperature $T$ (and coupling constant $g$ such that $T > T_c(g)$ ) and then suddenly increasing $g$ to $g'$ at some instant $t_0$ such that $T < T_c(g')$. The system, in this case, will find itself in the wrong phase and the formation of domains in the broken phase will take place. Our aim is to calculate the value of $\langle \bar{\psi}(t, \vec{r})\psi(t, \vec{r})\bar{\psi}(t, 0)\psi(t, 0) \rangle$ which gives a measure of the distance scale over which isospin is correlated. However, such a calculation is quite complicated and to bring out the qualitative features we choose an alternate but parallel model where the interaction $(g)$ is turned on suddenly at $t = 0$ such that the initial temperature $T < T_c(g)$. We find that the domain size grows with time (for large times) as $\sqrt{t}$ in a model independent manner.

**II. Closed Time Path formalism**

In this section we briefly review the Closed Time Path formalism. All essential details can be found in the references [5,7,8]. Suppose we are interested in the expectation value of some observable $A$ at instant $t$ in a given ensemble. It is given, in general by

$$<A>(t) = \text{Tr}_\rho(t)A = \text{Tr}_\rho(0)U(0, t)AU(t, 0), \quad (1)$$

where $\rho$ is the density matrix that describes the (mixed) state of the system and $U$ is the evolution operator (in the Schrödinger picture) corresponding to a hamiltonian $H$ that is, in general, time dependent. The density matrix does not necessarily have to commute with the hamiltonian, in which case it describes a non equilibrium state. For $t \leq 0$ we assume that the system is in equilibrium so that at $t = 0$, $\rho$ can be written as

$$\rho(0) = \frac{e^{-\beta H_i}}{Tr e^{-\beta H_i}}, \quad (2)$$

for some initial hamiltonian $H_i$. For positive times we take $H$ as the hamiltonian which governs the dynamics of the system. We can write now

$$\rho(0) = \frac{e^{-\beta H_i}}{Tr e^{-\beta H_i}} = \frac{U(T - i\beta, T)}{Tr U(T - i\beta, T)}, \quad (3)$$

4
where \( T \) is any time \( T < 0 \), since \( U(T - i\beta, T) \) involves only the hamiltonian \( H_i \). We can now go back to equation (1). After substituting equation (3), inserting the identity operators \( 1 = U(0, T)U(T, 0), 1 = U(t, T')U(T', t) \) and rearranging the product inside the trace we obtain

\[
\langle A \rangle (t) = \frac{\text{Tr} U(T - i\beta, T)U(T, T')U(T', t)AU(t, T)}{\text{Tr} U(T - i\beta, T)U(T, T')U(T', T)},
\]

(4)

where \( T' \) is some time larger than \( t \). Equation (4) can be pictured as describing the evolution of the system from \( T \) to \( T' \) (with an insertion of the operator \( A \) at \( t \)), and then backwards in time to \( T \) and finally along the imaginary time axis to \( T - i\beta \). We can now take the limits \( T \to -\infty \) and \( T' \to \infty \). This suggests the definition of a generating functional

\[
Z_c[J_c] = \text{Tr} U_J(-\infty - i\beta, -\infty)U_J(-\infty, \infty)U_J(\infty, -\infty),
\]

(5)

where the subscript \( c \) is to remind us that the quantity is defined in the contour in the complex \( t \)-plane described above. \( U_J \) stands for the evolution operator under the influence of the external sources \( J_c \). Clearly, if \( H \) equals \( H_i \) and is time independent, and if the source \( J_c \) is the same along the entire time contour, \( Z_c \) is the partition function of the system and this formalism reduces to the well known imaginary time formalism. We are, however, dealing with a nonequilibrium phenomenon where \( H(t) \) is different along the time contour and, consequently we choose the external sources to be different along the two branches of the contour to allow us to obtain Green’s functions by taking derivatives with respect to the sources. \( Z_c \) has a path integral representation

\[
Z_c[J_c] = \int_{(anti)periodic} D\phi \ e^{i\int_c (\mathcal{L} + J\phi) d^4x}.
\]

(6)

with \( \phi \) denoting a generic field of the theory. As usual we can separate the quadratic part of the lagrangian and expand the interaction part in a power series to obtain a perturbative expansion. It turns out that the free propagators vanish if they connect a point in the segment \((-\infty, -\infty - i\beta)\) to a point on the real axis. Thus the contribution of this segment
effectively decouples and contributes only to an overall normalization [8]. Therefore, we are effectively left with an integral on the real axis (both ways). It is convenient to rewrite this as a normal integral over the real line along one branch, but the propagator then acquires a $2 \times 2$ matrix structure

$$\Delta_{ab} = \begin{vmatrix} \Delta_{++} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{--} \end{vmatrix}$$  

where

$$\Delta_{++}(x - y) = <T(\phi(x)\phi(y))>_\beta$$
$$\Delta_{--}(x - y) = <T^*(\phi(x)\phi(y))>_\beta$$
$$\Delta_{+-}(x - y) = <\phi(y)\phi(x)>_\beta$$
$$\Delta_{-+}(x - y) = <\phi(x)\phi(y)>_\beta.$$  

Here $T$ and $T^*$ stand for time ordering and antitime ordering respectively. The matrix structure and the different boundary conditions satisfied by the functions $\Delta_{ab}$ can be understood remembering that in the part of the contour that runs in the negative direction (we call it ”−” as opposed to the other one that we call ”+”) the time ordering gets reversed, and that any point in the ”+” branch is earlier than any point in the ”−” branch. Besides the matrix structure of the propagator the other difference in relation to the usual Feynman rules at $T = 0$ is that we have the interaction term in both branches of the contour and consequently there are two kinds of vertices, one connecting fields living on the ”+” branch and the other (with opposite sign) connecting fields defined on the ”−” branch.

III. The Model

The model we would like to consider is the Nambu-Jona-Lasinio model in four dimensions, with two flavors and $N$ colors.

$$\mathcal{L} = i\bar{\psi}_\alpha \gamma^\mu \partial_\mu \psi_\alpha + \frac{g^2}{4NA^2}[(\bar{\psi}_\alpha \psi_\alpha)^2 - (\bar{\psi}_\alpha \gamma_5 \tau^i \psi_\alpha)^2],$$  

where $\alpha = 1, ..., N$, $\tau^i$’s are the generators of flavour $SU(2)$, and the flavor indices are suppressed for simplicity. This model is nonrenormalizable in four dimensions. However,
we treat this, for our discussions, as an effective low energy theory for energy scales below the cut off $\Lambda$. The theory has the chiral symmetry
\[
\psi_\alpha \rightarrow e^{i\theta\gamma_5}\psi_\alpha
\]
\[
\bar{\psi}_\alpha \rightarrow \bar{\psi}_\alpha e^{i\theta\gamma_5}.
\] (10)
As usual we can introduce auxiliary fields $\sigma$ and $\pi^i$ to write the Lagrangian (9) in an equivalent form
\[
\mathcal{L} = i\bar{\psi}_\alpha \gamma^\mu \partial_\mu \psi_\alpha - \frac{N\Lambda^2}{2} \sigma^2 - \frac{N\Lambda^2}{2} \pi^i \pi^i + \frac{g}{\sqrt{2}} \sigma \bar{\psi}_\alpha \psi_\alpha + i \frac{g}{\sqrt{2}} \pi^i \bar{\psi}_\alpha \gamma_5 \tau^i \psi_\alpha.
\] (11)
As is well known, the chiral symmetry (10) of the theory is dynamically broken if $g^2 > 2\pi^2$ and restored at $T_c = \text{const.}(1 - \frac{2\pi^2}{g^2})^{\frac{1}{2}}\Lambda$. In our discussion we will take the coupling constant $g$ to be time dependent with the simple form
\[
g(t) = \theta(t)g,
\] (12)
and the initial temperature $T$ smaller than the critical temperature
\[
T_c = \text{const.}(1 - \frac{2\pi^2}{g^2})^{\frac{1}{2}}\Lambda.
\] (13)
Thus, according to the model we have chosen, for $t < 0$ we have a gas of massless free fermions in equilibrium at a temperature $T$ and the state of the system is invariant under the chiral transformation of equation (10). At $t = 0$ we turn on the interaction such that the system suddenly finds itself at a temperature much below the critical temperature $T_c$ given in (13). This guarantees that the effect of the sudden change will make the chiral symmetric initial state of the system unstable, almost in parallel to the case of an expanding plasma as discussed in the introduction. An important point to note, however, is that, since both the dynamics and the initial state are chirally symmetric, $\sigma$ will have a vanishing vacuum expectation value in the ensemble. However, in each element of the ensemble, different regions of the space will have different directions of symmetry breaking and, although the
expectation value of $\sigma$ (averaged over the ensemble) remains zero, correlations of the $\sigma$ (and $\pi$) field will grow.

The complete $\sigma$ (or $\pi$) propagator is obtained from

$$D_{ab}^{-1}(x, y) = D_{ab}^0 -1(x, y) + \Sigma_{ab}(x, y),$$  \hspace{1cm} (14)

where $a, b = \pm$, $D^0$ the zeroth order propagator and $\Sigma$ is the self-energy. The discussion of domain growth, however, is best carried out in terms of the physical propagators (namely, the retarded, advanced and correlated ones) and, therefore, we transform from the $\pm$ basis to the physical basis through the transformation $[5,7],$

$$D' = VDV^{-1}, \quad \Sigma' = V\Sigma V^{-1},$$ \hspace{1cm} (15)

with

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. $$ \hspace{1cm} (16)

Thus,

$$D' = \begin{pmatrix} 0 & -D_A \\ D_R & D_c \end{pmatrix},$$ \hspace{1cm} (17)

$$\Sigma' = \begin{pmatrix} \Sigma_c & \Sigma_R \\ -\Sigma_A & 0 \end{pmatrix}. $$ \hspace{1cm} (18)

with

$$D_R = D_{++} - D_{+-}$$
$$D_A = -D_{++} + D_{-+}$$
$$D_c = D_{++} + D_{--}$$
$$\Sigma_R = \Sigma_{++} + \Sigma_{+-}$$
$$\Sigma_A = -\Sigma_{++} - \Sigma_{-+}$$
$$\Sigma_c = \Sigma_{++} + \Sigma_{--}. $$ \hspace{1cm} (19)

The functions $D_{R(A)}$ are the usual retarded (advanced) functions defined as

$$D_R(x, y) = \theta(x^0 - y^0) < [\phi(x), \phi(y)] >_{\beta}$$
$$D_A(x, y) = \theta(y^0 - x^0) < [\phi(x), \phi(y)] >_{\beta},$$ \hspace{1cm} (20)
and $D_c$, which gives the fermionic correlation function we are interested in is

$$D_c(x, y) = <[\phi(x), \phi(y)]_+ >_\beta.$$  

(21)

In case of Fermi fields we would exchange commutators by anticommutators and vice versa.

The difference in sign in the definition of $\Sigma_{R(A)}$ and $D_{R(A)}$ is due to the fact that $\Sigma_{+-}$ and $\Sigma_{-+}$ include a ”−” sign coming from a ”−” type vertex. Using equation (14) we find the relation between the propagator and the self energy to be

$$D_R = D_R^0 \frac{1}{1 + \Sigma_R D_R^0},$$  

(22)

$$D_A = D_A^0 \frac{1}{1 + \Sigma_A D_A^0},$$  

(23)

and

$$D_c = D_R \Sigma_c D_A + \frac{1}{1 + D_R^0 \Sigma_R} D_c^0 \frac{1}{1 + \Sigma_A D_A^0}.$$  

(24)

Note that in a non equilibrium situation these functions will depend on each time argument separately, and not only on their difference. In other words, in a non equilibrium situation there is no time translation invariance. Thus, we can not diagonalise them by taking a Fourier transform and the order of the functions in (22, 23, 24) is important. Consequently, unlike the equilibrium case, it is in general difficult to invert expressions such as $1 + \Sigma_R D_R^0$.

IV. The Calculation

The functions $D_{R(A)}$ carry information about the dynamics of the model and at zeroth order are independent of the initial state (they are expectation values of c-numbers which vanish at equal times by microcausality). It is the function $D_c$ that contains the information about the state of the system, like particle number distributions, temperature, etc.

For instance, for free fermions we have (We use the metric $\eta_{\mu\nu} = (+−−−)$)

$$S_{R(A)}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik.x} (k^\mu \gamma_\mu + m) \frac{1}{k^2 - m^2 \pm i\epsilon k_0},$$

$$S_c(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik.x} (k^\mu \gamma_\mu + m) 2\pi i(2n(k_0) - 1)\delta(k^2 - m^2),$$  

(25)
where
\[
n(k_0) = \frac{1}{e^{\beta |k_0|} + 1}
\] (26)
is the Fermi distribution function. We want to calculate \( D_c \) for the \( \sigma \) (and \( \pi \)) propagators at equal times. At tree level these are not real (on shell) propagating particles, consequently they do not thermalise. Another way of saying this is that they are too heavy (their masses are of the order of the cut-off) and they are Boltzmann suppressed at any temperature much smaller than the cut-off. We have then
\[
\begin{align*}
D^0_R &= \frac{1}{iN\Lambda^2}, \\
D^0_A &= -\frac{1}{iN\Lambda^2}, \\
D^0_c &= 0.
\end{align*}
\] (27)

We will calculate the self-energy in the leading order in \( \frac{1}{N} \) so that the main contribution comes from the fermionic one loop bubble diagram. But we note first that since \( g(t) = \theta(t)g_0, \Sigma_{R(A)}(x, y) \) and \( \Sigma_c(x, y) \) vanish if either \( x^0 < 0 \) or \( y^0 < 0 \). Consequently, they have the generic form
\[
\Sigma(x, y) = \theta(x^0)\bar{\Sigma}(x, y)\theta(y^0).
\] (28)
where the quantity with an overbar is calculated with a constant \( g \) much the same way as in an equilibrium calculation. As a result, the inversion of quantities such as \( (1 + \Sigma_R D^0_R)^{-1} \) becomes trivial for positive times. For example, \( (1 + \Sigma_R D^0_R)^{-1} \) satisfies the equation
\[
(1 + \Sigma_R D^0_R)^{-1}(1 + \Sigma_R D^0_R) = 1.
\] (29)

Written out explicitly, this has the form
\[
\int d^4z \, (1 + \Sigma_R D^0_R)^{-1}(x, z)[\delta^4(z - y) + \frac{1}{iN\Lambda^2} \theta(z^0)\bar{\Sigma}_R(z - y)\theta(y^0)] = \delta^4(x - y).
\] (30)

By definition \( \bar{\Sigma}_R \) vanishes for \( z^0 < y^0 \) and consequently, if we restrict to \( y^0 > 0 \), the two step functions in the second term can be ignored and the inversion can be done much the
same way as for a constant coupling, equilibrium case. The knowledge of \((1 + \Sigma_{R}D_{R}^{0})^{-1}\) for positive times is enough to calculate the retarded propagator for positive times \((x^{0}, y^{0} > 0)\)

\[
D_{R}(x, y) = \int dz \ D_{R}^{0}(x, z)(1 + \Sigma_{R}D_{R}^{0})^{-1}(z, y) = \tilde{D}_{R}(x - y).
\]

(31)

The same argument goes through for the advanced function as well but this is not always the case. For the Feynman Greens function, for example, the analogue of equation (30) is a complicated integral equation. This comes about because \(\Sigma_{++}\) describes propagation in both, forward and backward time direction so, even after restricting the final points \(x^{0}\) and \(y^{0}\) to be positive, \(z^{0}\), the intermediate time coordinate, can be negative. Since the dynamics for negative times is different from that at positive times the inversion is not straightforward. Since our non equilibrium retarded and advanced functions are the same as the ones in equilibrium (always for \(x^{0}, y^{0} > 0\) only) we have (It will become clear shortly that this is all we need for our discussion.)

\[
\tilde{D}_{R}(x, y) = \int \frac{d^{4}k}{(2\pi)^{4}} e^{-ik(x-y)} \frac{1}{iN\Lambda^{2} + \Sigma_{R}(k)},
\]

(32)

\[
\tilde{D}_{A}(x, y) = \int \frac{d^{4}k}{(2\pi)^{4}} e^{-ik(x-y)} \frac{1}{-iN\Lambda^{2} + \Sigma_{A}(k)}.
\]

(33)

From equations (24) and (27) then, we obtain for \(x^{0}, y^{0} > 0\)

\[
D_{c}(x, y) = \int d^{4}zd^{4}z' \ D_{R}(x, z)\Sigma_{c}(z, z')D_{A}(z', y)
= \int d^{4}zd^{4}z' \ D_{R}(x - z)\theta(z^{0})\Sigma_{c}(z - z')\theta(z^{'0})D_{A}(z' - y).
\]

(34)

Note that to evaluate \(D_{c}\) we need \(D_{R(A)}\) only for positive times as stated earlier. To find \(\Sigma_{R}, \Sigma_{A}\) and \(\Sigma_{c}\) we first calculate \(\bar{\Sigma}_{ab}\), \(a, b = \pm\) using the diagrammatic methods and then use the definitions in (19). \(\bar{\Sigma}_{ab}\) is simply obtained from the Feynman rules discussed above to be

\[
\bar{\Sigma}_{ab} = -ab \ g^{2}N \ Tr \int \frac{d^{4}k}{(2\pi)^{4}} iS_{ab}(k)iS_{ba}(p + k),
\]

(35)
where $S_{ab}$ are the finite temperature, real time propagators at the initial temperature $T$

\[
S_{++}(p) = (\gamma^\mu p_\mu + m) \left( \frac{1}{p^2 - m^2 + i\epsilon} + 2\pi i n(p_0)\delta(p^2 - m^2) \right)
\]
\[
S_{--}(p) = (\gamma^\mu p_\mu + m) \left( -\frac{1}{p^2 - m^2 - i\epsilon} + 2\pi i n(p_0)\delta(p^2 - m^2) \right)
\]
\[
S_{+-}(p) = (\gamma^\mu p_\mu + m)2\pi i(n(p_0) - \theta(-p_0))\delta(p^2 - m^2)
\]
\[
S_{-+}(p) = (\gamma^\mu p_\mu + m)2\pi i(n(p_0) - \theta(p_0))\delta(p^2 - m^2).
\]

and "ab" is a sign factor coming from the fact that the "-" vertices carry a negative sign.

Note that in our theory $m = 0$ and using these definitions the calculation of $\Sigma_{ab}$ is straightforward. We only give the results here:

\[
\Sigma_{++} = i4g^2N \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ \frac{p_0\omega_k + \vec{p},\vec{k}}{(p_0 - \omega_k)^2 - \omega_{p+k}^2} - \frac{i\pi}{\omega_k} (p_0\omega_k - \vec{p},\vec{k}) \right\}
\]
\[
\delta((p_0 + \omega_k)^2 - \omega_{p+k}^2)(\frac{1}{2}\text{sgn}(p_0) + n(\omega_k) - n(\omega_k)n(\omega_{p+k}))
\]
\[
+ p_0 \rightarrow -p_0
\]
\[
\Sigma_{--} = \Sigma_{++}^*(p^*)
\]
\[
\Sigma_{+-} = 4\pi g^2N \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \left[ (p_0\omega_k - \vec{p},\vec{k})\delta((p_0 + \omega_k)^2 - \omega_{p+k}^2) \right]
\]
\[
(n(\omega_k) - 1)(n(p_0 + \omega_k) - \theta(-p_0 - \omega_k))
\]
\[
+ (p_0^2 - \vec{p},\vec{k})\delta((p_0 - \omega_k)^2 - \omega_{p+k}^2)\]
\[
n(\omega_k)(n(p_0 - \omega_k) - \theta(-p_0 + \omega))
\]
\[
\Sigma_{-+} = 4\pi g^2N \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \left[ (p_0\omega_k - \vec{p},\vec{k})\delta((p_0 + \omega_k)^2 - \omega_{p+k}^2) \right]
\]
\[
n(\omega_k)(n(p_0 - \omega_k) - \theta(-p_0 + \omega_k))
\]
\[
+ (p_0^2 - \vec{p},\vec{k})\delta((p_0 - \omega_k)^2 - \omega_{p+k}^2)
\]
\[
(n(\omega_k) - 1)(n(p_0 + \omega_k) - \theta(-p_0 - \omega_k))
\].

We can now construct $\Sigma_R$, $\Sigma_A$ and $\Sigma_c$. They are given, after some rearrangement and with the help of the change of variables $k \rightarrow -k - p$ in various places (these are finite
integrals since the momenta are cut off at $\Lambda$), by

$$\bar{\Sigma}_R(p) = i4g^2N\int \frac{d^3k}{(2\pi)^3}\left(\frac{1 - 2n(\omega_k)}{2\omega_k}\right)[p_0\omega_k + \vec{p}\vec{k}](p_0 - \omega_k + i\epsilon)^2 - \omega_{p+k}^2 + p_0, \epsilon \rightarrow -p_0, -\epsilon]$$

$$\Sigma_A(p) = \bar{\Sigma}_R(p^*)$$

and

$$\bar{\Sigma}_c(p) = -4g^2\pi N(p_0^2 - \vec{p}^2)\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\left[\delta((p_0 + \omega_k)^2 - \omega_{p+k}^2)(\frac{1}{2}sgn(p_0) + n(\omega_k) - n(\omega_k)n(p_0 + \omega_k)) + p_0 \rightarrow p_0\right].$$

Consequently we can also obtain $\bar{D}_R(A)$ in a straightforward manner. We will need to know the position of the pole(s) $\epsilon_p(\epsilon_p^*)$ of $\bar{D}_R(A)(p_0, \vec{p})$ in order to evaluate $D_c$. We determine the position of the pole in the following way. At the pole of $\bar{D}_R$ we have

$$iN\Lambda^2 + \bar{\Sigma}_R(p_0, \vec{p}) = 0. \quad (40)$$

If we set $\vec{p} = 0$ in the above expression, we obtain

$$\Lambda^2 \left(1 - \frac{g^2}{2\pi^2}\right) + \frac{g^2}{8\pi^2p_0^2}\ln\left(-\frac{4\Lambda^2}{p_0^2}\right) - \frac{16g^2}{\pi^2} \int^\Lambda_0 dk \frac{1}{e^{\beta k} + 1} \frac{k^3}{p_0^2 - 4k^2} = 0. \quad (41)$$

A simple graphical analysis shows that, for $g^2 > 2\pi^2$ (the symmetry breaking case) and temperatures smaller than $T_c$, there is always a solution of equation (41) with $p_0^2$ negative. Let us call this solution $-M^2$. To find the next term in the series expansion we substitute $p_0^2 \rightarrow -M^2 + bp^2$ and keep terms up to second order in $\vec{p}$. After some tedious algebra we obtain

$$\left(\frac{7}{3} - b\right)I_3 + \left(\frac{5}{3} - b\right)I_5 = 0, \quad (42)$$

where

$$I_3 = 4M^4 \int^\Lambda_0 dk \tgh(\beta k/2)\frac{k^3}{(M^2 + 4k^2)^3} \quad (43)$$

and

$$I_5 = 16M^2 \int^\Lambda_0 dk \tgh(\beta k/2)\frac{k^5}{(M^2 + 4k^2)^3}. \quad (44)$$
The solution to equation (42) can be easily seen to lie within the range
\[ \frac{5}{3} < b < \frac{7}{3}. \] (45)

The position of the pole up to this order is given by
\[ p^2_0 = \epsilon_p^2 = -M^2 + b\vec{p}^2, \]
with \( b \) determined as above. This way we have
\[ \epsilon_p \sim \pm i(M - \frac{b\vec{p}^2}{2M}). \] (46)

The wrong sign of the mass \((-M^2)\) is a signal of instability of the initial state. One could go on to higher orders but this will suffice for our discussion.

Using our previous results we have now, with \( x = (t, \vec{r}) \) and \( y = (t, \vec{0}) \)
\[ D_c(x, y) = \int d^4zd^4z'\theta(z^0)\theta(z'^0) \int \frac{d^4k}{(2\pi)^4} \frac{dp}{(2\pi)^4} \frac{dq}{(2\pi)^4} e^{-ik.(x-z)-ip.(z-z')-iq.(z'-y)} \]
\[ \times \bar{D}_R(k)\Sigma_c(p)\bar{D}_A(q) \]
\[ = \int \frac{d^4k}{(2\pi)^4} \frac{dp_0 dq_0}{2\pi 2\pi} e^{i\vec{k}.\vec{r}} e^{i(k_0-q_0)t} \frac{i}{k_0-p_0-i\epsilon} \frac{i}{p_0-q_0+i\epsilon} \]
\[ \times \bar{D}_R(k_0, \vec{k})\Sigma_c(p_0, \vec{k})\bar{D}_A(q_0, \vec{k}). \] (47)

Note that it is the limited range of integrations on \( z^0, z'^0 \) that produces the denominators shown above instead of the usual \( \delta \) functions that assure conservation of energy. As the system is unstable, \( \bar{D}_R \) grows exponentially with time so that one should be careful in defining its Fourier transform. It is well known [9] that if \( \bar{D}_R \sim e^{\alpha t} \) for large \( t \), the Fourier transform \( \bar{D}_R(\omega) \) will be analytic only above the line \( \text{Im} \omega = \alpha \) and the inverse Fourier transform is given by an integral along a line just above \( \text{Im} \omega = \alpha \). The opposite happens for the advanced function, namely, the region of analyticity is \( \text{Im} \omega < \alpha \). With this in mind we can perform the integrations over \( k_0 \) and \( q_0 \) to obtain
\[ \int \frac{dk_0}{2\pi} e^{-ik_0t} \bar{D}_R(k_0, \vec{k}) \frac{1}{k_0-p_0+i\epsilon} = -i\bar{D}_R(\epsilon_k, \vec{k}) \frac{e^{-i\epsilon_k t}}{\epsilon_k - p_0} - i\epsilon^{-ip_0t} \bar{D}_R(p_0, \vec{k}) \]
\[ \int \frac{dq_0}{2\pi} e^{iq_0t} \bar{D}_A(q_0, \vec{k}) \frac{1}{q_0-p_0-i\epsilon} = i\bar{D}_A(\epsilon^*_k, \vec{k}) \frac{e^{i\epsilon_k t}}{\epsilon^* - p_0} + i\epsilon^{ip_0t} \bar{D}_A(p_0, \vec{k}), \] (48)
where $\bar{d}_{R(A)}$ are the residues of $\bar{D}_{R(A)}$ at the pole $\epsilon_k(\epsilon_k^*)$. As is clear from the earlier discussion, $\epsilon_k$ is complex leading to both exponentially growing as well as damped behaviour. For large times, it is the exponentially growing solution which will dominate. Furthermore, the terms not involving $\epsilon_k$ do not grow with time and are the ones that would be there had $g$ been kept constant and the range of integration in $z^0, \bar{z}^0$ was $(-\infty, \infty)$ (namely, for the case of equilibrium). We will omit this term from now on since the exponential growth gives the dominant contribution. Putting this back in (47) we have

$$D_c(t, \vec{r}) = \int \frac{dp_0}{2\pi} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.\vec{r}} |\bar{d}_R(\epsilon_k, \vec{k})|^2 e^{-i(\epsilon_k - \epsilon_k^*)t} \frac{1}{|p_0 - \epsilon_k|^2} \Sigma_c(p_0, \vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.\vec{r}} e^{-i(\epsilon_k - \epsilon_k^*)t} f(\vec{k}^2, T), \quad (49)$$

where

$$f(\vec{k}^2, T) = g^2 N \int \frac{d^3p}{(2\pi)^3} \frac{|\bar{d}_R(\epsilon_k, \vec{k})|^2}{|\omega_p + \omega_{p+k} + \epsilon_k|^2} \left[ \frac{(\omega_k + \omega_{p+k})^2 - \vec{k}^2}{|\omega_p + \omega_{p+k} + \epsilon_k|^2} \left( \frac{1}{2} - n(\omega_p) + n(\omega_p)n(\omega_{p+k}) \right) 
+ \frac{(\omega_k - \omega_{p+k})^2 - \vec{k}^2}{|\omega_p - \omega_{p+k} - \epsilon_k|^2} \left(\frac{1}{2} \text{sgn}(\omega_p - \omega_{p+k}) - n(\omega_p) + n(\omega_p)n(\omega_{p+k})\right) 
+ \frac{(\omega_{p+k} - \omega_p - \epsilon_k)^2 - \vec{k}^2}{|\omega_p + \omega_{p+k} - \epsilon_k|^2} \left(\frac{1}{2} \text{sgn}(\omega_p - \omega_{p+k}) - n(\omega_p) + n(\omega_p)n(\omega_{p+k})\right) 
+ \frac{(\omega_p + \omega_{p+k})^2 - \vec{k}^2}{|\omega_p + \omega_{p+k} - \epsilon_k|^2} \left(-\frac{1}{2} - n(\omega_p) + n(\omega_p)n(\omega_{p+k})\right) \right].$$

$$\quad (50)$$

A little analysis shows that the function $f$ vanishes for $\vec{k}^2 \to 0$ so, for small $\vec{k}^2$ we can write ($k = |\vec{k}|$)

$$f(k^2, T) \to T^3 g(k/T), \quad (51)$$

with $g(x) \to 0$ as $x \to 0$ as a power, e.g. $g(x) \sim x^\alpha, \alpha > 0$. We evaluate $D_c$ in equation (49) for large times using the saddle point approximation. The saddle point is given by

$$k_s = \frac{i2Mr}{bt}. \quad (52)$$

Therefore, $k \to 0$ as $t \to \infty$ and we are justified in disregarding terms of order $O(p^4)$ in
the expansion of $\epsilon_p$ in equation (46). This gives for large times

$$D_c(t, r) = C(T) \left( \frac{r}{t} \right)^\alpha \exp{\frac{-M}{2t} e^{-\frac{\epsilon^2}{L^2(t)}}}$$

where $C(T)$ is a function independent of $(t, \vec{r})$ and $L(t)$ is the typical size of the domains given by

$$L(t) = \sqrt{\frac{b}{2M} t}.$$  \hspace{1cm} (54)

This is the same scaling as obtained by Boyanovsky et al. [4] for the scalar $\phi^4$ theory with a mass square that suddenly changes sign. The size of the domains, of course, depends on the details of the theory since it is determined by $b$ and $M$ but the power law dependence on $t$ in equation (54) seems to depend only on the fact that the first non constant term in the expansion of $\epsilon_p$ for small $\vec{p}$ is quadratic in $\vec{p}$. We see from (53) that the correlation grows indefinitely with time but this is an artifact of our approximation. This growth will actually stop when inside a domain the value of $\sigma$ is that of the equilibrium and the phase transition will be over. This however would involve a rearrangement of the chiralities of the fermions that can not be achieved in leading order in $\frac{1}{N}$ since in this case the fermions do not scatter.

V. Conclusion

We have shown how to use the Closed Time Path formalism to describe the dynamics of a phase transition. In particular we analysed the dynamics of domain growth following quenching in a simple model exhibiting chiral symmetry breaking. We found that the domains with correlated isospin grow as $\sqrt{t}$ and we have argued that this result is not dependent on details of our model.

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