A note on zero-one laws in metrical Diophantine approximation

Victor Beresnevich*           Sanju Velani†
York                        York

Dedicated to Wolfgang Schmidt on the occasion of his 75th birthday

1 Introduction

Given \( \psi : \mathbb{N} \to [0, +\infty) \), let \( \mathcal{A}(\psi) \) denote the set of \( x \in [0, 1] \) such that

\[
|qx + p| < \psi(q)
\]

holds for infinitely many \((p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \). In 1924, Khintchine [15] established a beautiful and strikingly simply criterion for the ‘size’ of \( \mathcal{A}(\psi) \) expressed in terms of Lebesgue measure. Under the condition that \( \psi \) is monotonic, Khintchine’s theorem states that the measure of \( \mathcal{A}(\psi) \) is one (respectively, zero) if the sum \( \sum_q \psi(q) \) diverges (respectively, converges). The monotonicity condition is only required in the divergence case and moreover it is absolutely crucial. Duffin and Schaeffer [10] constructed a non-monotonic function \( \psi \) for which \( \sum_q \psi(q) \) diverges but \( \mathcal{A}(\psi) \) is of zero measure. In other words, without the monotonicity assumption, Khintchine’s theorem is false and the famous Duffin-Schaeffer conjecture provides the appropriate statement. The key difference is that in (1), we impose coprimality on the integers \( p \) and \( q \). Let \( \mathcal{A}'(\psi) \) denote the resulting subset of \( \mathcal{A}(\psi) \). The Duffin-Schaeffer conjecture states that the measure of \( \mathcal{A}'(\psi) \) is one (respectively, zero) if the sum \( \sum_q \varphi(r) \psi(q) q^{-1} \) diverges (respectively, converges). Although various partial results have been obtained, the full conjecture represents a key unsolved problem in metric number theory – see [4, 14] for details. Returning to the raw set \( \mathcal{A}(\psi) \), without monotonicity and coprimality the appropriate analogue of Khintchine’s theorem has been formulated by Catlin [9]. The Catlin conjecture also remains open.

The upshot of the above discussion is that currently we are unable to prove analogues of Khintchine’s theorem for either of the fundamental sets \( \mathcal{A}(\psi) \) and \( \mathcal{A}'(\psi) \). However, it is known that the Lebesgue measure of \( \mathcal{A}(\psi) \) and \( \mathcal{A}'(\psi) \) is either 0 or 1. In the case of \( \mathcal{A}(\psi) \)

*EPSRC Advanced Research Fellow, grant EP/C54076X/1
†Research supported by EPSRC grant EP/E061613/1 and INTAS grant 03-51-5070
this zero-one law is due to Cassels [8] and in the case of $A'(\psi)$ it is due to Gallagher [11]. The goal of this note is to establish the higher dimensional analogues of these classical zero-one laws. For a discussion concerning the higher dimensional analogues of the conjectures of Duffin-Schaeffer and Catlin see [4].

Throughout, $m \geq 1$ and $n \geq 1$ are integers. Given $\Psi : \mathbb{Z}^m \to [0, +\infty)$, let $A_{n,m}(\Psi)$ be the set of $X \in [0, 1]^{nm}$ such that

$$|qX + p| < \Psi(q)$$

holds for infinitely many $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$. Here $| \cdot |$ denotes the supremum norm in $\mathbb{R}^m$, $X$ is regarded as an $n \times m$ matrix and $q$ is regarded as a row. Thus, $qX \in \mathbb{R}^m$ represents a system of $m$ real linear forms in $n$ variables. In higher dimensions the set $A'(\psi)$ has two natural generalizations:

$$A'_{n,m}(\Psi) = \{ X \in [0, 1]^{nm} : (2) \text{ holds for } \text{i.m. } (p, q) \text{ with } \gcd(p, q) = 1 \}$$

$$A''_{n,m}(\Psi) = \{ X \in [0, 1]^{nm} : (2) \text{ holds for } \text{i.m. } (p, q) \text{ with } \gcd(p_i, q_j) = 1 \text{ for } j \leq m \} .$$

Here 'i.m.' stands for 'infinitely many' and $\gcd(p, q)$ denotes the greatest common divisor of all the components of $p$ and $q$. If $\gcd(p, q) = 1$ then we say that $p$ and $q$ are coprime.

Before we state our main result, let us agree on the following notation: $A_{n,m}(\Psi)$ will denote any of the fundamental sets $A_{n,m}(\Psi)$, $A'_{n,m}(\Psi)$ and $A''_{n,m}(\Psi)$. Thus, a statement for $A_{n,m}(\Psi)$ is valid for $A_{n,m}(\Psi)$, $A'_{n,m}(\Psi)$ and $A''_{n,m}(\Psi)$. Also, $|X|$ will denote the $k$-dimensional Lebesgue measure of the set $X \subset \mathbb{R}^k$.

**Theorem 1** For any $n, m$ and $\Psi$ we have that $|A_{n,m}(\Psi)| \in \{0, 1\}$.

## 2 Auxiliary results

In this section we group together various self contained statements that we appeal to during the course of establishing Theorem 1. Most are higher dimensional analogues of well known one-dimensional statements. Indeed, the one-dimensional version of our first result can be found in [8].

**Lemma 1** Let $\{B_i\}$ be a sequence of balls in $\mathbb{R}^k$ with $|B_i| \to 0$ as $i \to \infty$. Let $\{U_i\}$ be a sequence of Lebesgue measurable sets such that $U_i \subset B_i$ for all $i$. Assume that for some $c > 0$,

$$|U_i| \geq c|B_i| \quad \text{for all } i .$$

Then the sets

$$U = \limsup_{i \to \infty} U_i := \bigcap_{i \geq 1} \bigcup_{j \geq i} U_i \quad \text{and} \quad B = \limsup_{i \to \infty} B_i := \bigcap_{j \geq 1} \bigcup_{i \geq j} B_i$$

have the same Lebesgue measure.
Proof. Let $U_i := \bigcup_{j \geq i} U_j$ and $D_j := B \setminus U_j$. Then, $D := B \setminus U = \bigcup_j D_j$ and Lemma 1 states that $D$ has measure zero. Equivalently, every $D_j$ must have zero measure. Assume the contrary. Then, there is an $l \in \mathbb{N}$ such that $|D_l| > 0$ and therefore there is a density point $x_0$ of $D_l$. Since $x_0 \in B$, we have that $x_0 \in B_{j_i}$ for a sequence $j_i$. Since $|B_{j_i}| \to 0$, we have that $|D_l \cap B_{j_i}| \sim |B_{j_i}|$ as $i \to \infty$. Since $D_j \supset D_l$ for all $j \geq l$, it follows that

$$|D_{j_i} \cap B_{j_i}| \sim |B_{j_i}| \quad \text{as} \quad i \to \infty.$$  \hfill (4)

On the other hand, by construction $D_{j_i} \cap U_{j_i} = \emptyset$. Thus, in view of (3) we have that

$$|B_{j_i}| \geq |U_{j_i}| + |D_{j_i} \cap B_{j_i}| \geq c|B_{j_i}| + |D_{j_i} \cap B_{j_i}|,$$

i.e. $|D_{j_i} \cap B_{j_i}| \leq (1 - c) |B_{j_i}|$ for all sufficiently large $i$. This contradicts (4). \hfill \Box

The following lemma is the higher dimensional analogue of the well know one-dimensional ‘ergodic’ property of rational transformations – see for example [11], Lemma 3, [14], Lemma 2.2 or [23], Lemma 7.

**Lemma 2** For any integer $l \geq 2$ and $s \in \mathbb{Z}^k$ consider the transformation of the unit cube $[0,1]^k$ into itself given by

$$T : x \mapsto l \cdot x + \frac{1}{l} \cdot s \quad \text{(mod 1)}.$$  

Let $A$ be a subset of $[0,1]^k$ such that $T(A) \subseteq A$. Then $A$ is of Lebesgue measure 0 or 1.

**Proof.** Let $A$ be as in the statement. Then $T^\nu(A) \subseteq A$, where $T^\nu : x \mapsto l^\nu x + \frac{s}{l}$ (mod 1) is the $\nu$-th iterate of $T$. Let $\chi_A$ be the characteristic function of $A$. It follows that

$$\chi_A(x) \leq \chi_A\left(l^\nu x + \frac{s}{l}\right).  \quad (5)$$

Suppose that $|A| > 0$. Then there is a density point $x_0$ of $A$. Let $C_\nu$ be the cube in $[0,1]^k$ centred at $x_0$ of sidelength $l^{-\nu}$. Then

$$|A \cap C_\nu| = \int_{C_\nu} \chi_A(x) dx \leq \int_{C_\nu} \chi_A\left(l^\nu x + \frac{s}{l}\right) dx = \frac{1}{|B|} \int_{[0,1]^k} \chi_A(x) dx = |C_\nu| \cdot |A|.$$  

Since $x_0$ is a density point of $A$ and $\text{diam} C_\nu \to 0$ as $\nu \to \infty$, the left hand side of the above equality is asymptotically $|C_\nu|$. Therefore, $|A| = 1$. \hfill \Box

Given a ball $B = B(x,r)$ and a real number $c > 0$, we denote by $cB$ the ‘scaled’ ball $B(x,cr)$. The next lemma is a basic covering result from geometric measure theory usually referred to as the $5r$-lemma. For further details and proof the reader is referred to [19].
Lemma 3 Every collection $\mathcal{C}$ of balls of uniformly bounded diameter in a metric space $\Omega$ contains a disjoint subcollection $\mathcal{G}$ such that

$$\bigcup_{B \in \mathcal{C}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$ 

We immediately make use of the covering lemma to show that the Lebesgue measure of a reasonably general lim sup set is unchanged with respect to ‘scaling’ by a constant factor.

Lemma 4 Let $\{S_i\}_{i \in \mathbb{N}}$ be a sequence of subsets in $[0, 1]^k$, $\{\delta_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers such that $\delta_i \to 0$ as $i \to \infty$ and let

$$\Delta(S_i, \delta_i) := \{x \in [0, 1]^k : \text{dist}(S_i, x) < \delta_i\}.$$ 

Then for any real number $C > 1$, the sets

$$A := \limsup_{i \to \infty} \Delta(S_i, \delta_i) \quad \text{and} \quad B := \limsup_{i \to \infty} \Delta(S_i, C\delta_i)$$

have the same Lebesgue measure.

Proof. First of all notice that the sets $\Delta(S_i, \delta_i)$ are open and therefore Lebesgue measurable. Since $C > 1$ we have that $A \subset B$. For each $i \in \mathbb{N}$, let $\mathcal{B}_i$ denote the collection of balls $\{B(x, \delta_i) : x \in S_i\}$. Thus, we have that $\Delta(S_i, \delta_i) = \bigcup_{B \in \mathcal{B}_i} B$. By Lemma 3 there is a disjoint subcollection $\mathcal{G}_i$ of $\mathcal{B}_i$ such that

$$\bigcup_{B \in \mathcal{G}_i} B \subset \Delta(S_i, \delta_i) \subset \bigcup_{B \in \mathcal{B}_i} B \subset \bigcup_{B \in \mathcal{G}_i} 5B. \quad (6)$$

Since $S_i \subset [0, 1]^k$ we have that every ball $B \in \mathcal{G}_i$ is contained in the cube $[-\delta_i, 1 + \delta_i]^k$. It follows that $\mathcal{G}_i$ is a finite disjoint collection of balls.

If $z \in \Delta(S_i, C\delta_i)$, then there is a $y \in S_i$ such that $|z - y| < C\delta_i$. Furthermore, by (6) there exists a ball $B = B(x, \delta_i) \in \mathcal{G}_i$ such that $y \in 5B$. Therefore, $|z - x| \leq |z - y| + |y - x| < (5 + C)\delta_i$. Thus we have shown that

$$\Delta(S_i, C\delta_i) \subset \bigcup_{B \in \mathcal{G}_i} (5 + C)B. \quad (7)$$

Now given a constant $\lambda > 0$, let $\mathcal{C}(\lambda) := \limsup_{i \to \infty} \bigcup_{B \in \mathcal{G}_i} \lambda B$. This is the set of $x$ such that $x \in \lambda B$ for some $B \in \mathcal{G}_i$ for infinitely many $i$. Then, (6) and (7) imply that

$$\mathcal{C}(1) \subset A \subset B \subset \mathcal{C}(5 + C). \quad (8)$$

By Lemma 1, the sets $\mathcal{C}(\lambda)$ with $\lambda > 0$ have the same Lebesgue measure irrespective of $\lambda$. Therefore, in view of (8) the sets $A$ and $B$ must have the same Lebesgue measure. \qed
3 Proof of Theorem 1

On following the arguments of [11], it is easily verified that $A_{n,m}^o(\Psi) = [0, 1]^{nm}$ if the following condition
\[ \Psi(q)/|q| \to 0 \text{ as } |q| \to \infty \] is violated. Therefore, without loss of generality we assume that (9) is satisfied.

When considering $A_{n,m}^o(\Psi)$, the error of approximation is rigidly determined by the function $\Psi$. In proving Theorem 1, it is extremely useful to introduce a certain degree of flexibility within the error of approximation. Given $A_{n,m}^o(\Psi)$, let
\[ F_{n,m}(\Psi) = \bigcup_{k=1}^{\infty} A_{n,m}^o(k\Psi) . \]
Clearly, $F_{n,m}(\Psi) \supset A_{n,m}^o(\Psi)$. However, as a consequence of Lemma 4 we have that
\[ |F_{n,m}(\Psi)| = |A_{n,m}^o(\Psi)| . \] (10)
Clearly, Theorem 1 follows on establishing the analogous statement for $F_{n,m}(\Psi)$.

**Theorem 2** For any $n, m$ and $\Psi$ we have that $|F_{n,m}(\Psi)| \in \{0, 1\}$.

3.1 Proof of Theorem 2

We establish the theorem by considering the sets $F_{n,m}(\Psi)$, $F'_{n,m}(\Psi)$ and $F''_{n,m}(\Psi)$ separately.

**The set $F_{n,m}(\Psi)$:** Clearly, $F_{n,m}(\Psi)$ is invariant under the translation $T : X \mapsto 2X \pmod{1}$. Thus, the desired statement for the set $F_{n,m}(\Psi)$ is a trivial consequence of Lemma 2.

**The set $F'_{n,m}(\Psi)$:** By definition, $F'_{n,m}(\Psi)$ consists of points $X \in [0, 1]^{nm}$ for which there exists a constant $C = C(X) > 0$ such that
\[ |qX + p| < C \Psi(q) \quad \text{and} \quad \gcd(p, q) = 1 \] (11)
holds for infinitely many $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$. Now, for each prime $l$ consider the following subsets of $F''_{n,m}(\Psi)$:
\[ S_0(l) = \{ X \in [0, 1]^{nm} : \exists C > 0 \text{ so that (11) holds for i.m. } (p, q) \text{ with } l \nmid d = \gcd(q) \} , \]
\[ S_1(l) = \{ X \in [0, 1]^{nm} : \exists C > 0 \text{ so that (11) holds for i.m. } (p, q) \text{ with } l \parallel d = \gcd(q) \} , \]
\[ S_2(l) = \{ X \in [0, 1]^{nm} : \exists C > 0 \text{ so that (11) holds for i.m. } (p, q) \text{ with } l^2 \mid d = \gcd(q) \} . \]
Here \( l \parallel d \) means that \( l \) divides \( d \) but \( l^2 \) does not divide \( d \). Note that

\[
F'_{n,m}(\Psi) = S_0(l) \cup S_1(l) \cup S_2(l).
\]  

(12)

Suppose \( X \in S_0(l) \). Then (11) is satisfied for infinitely many \((p, q)\) with \( l \nmid d = \gcd(q) \). On setting \( q' := q \) and \( p' := l \), we have that

\[
|q'(lX) + p'| < lC\Psi(q')
\]

holds for infinitely many \((p', q')\) \( \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\} \) with

\[
\gcd(p', q') = 1.
\]

(13)

The coprimality condition is readily verified by making use of the fact that \( l \nmid \gcd(q) \). Thus, if \( X \in S_0(l) \) then \( lX \in S_0(l) \). Therefore the set \( S_0(l) \) is invariant under the transformation \( T : X \mapsto lX \pmod{1} \) and Lemma 2 implies that \(|S_0(l)|\) is 0 or 1.

For \( j \in \{1, \ldots, n\} \), let \( S_{1,j}(l) \) denote the set of \( X \in [0,1]^{nm} \) such that (11) is satisfied for infinitely many \((p, q)\) with \( l\parallel q_j \). Recall, that \( q_j \) is the \( j \)'th coordinate of \( q = (q_1, \ldots, q_n) \). Clearly,

\[
S_1(l) = \bigcup_{j=1}^n S_{1,j}(l).
\]

Suppose \( X \in S_{1,j}(l) \) for some \( j \in \{1, \ldots, n\} \). Let \( S_j \in \mathbb{Z}^{nm} \) denote the integer matrix with zero entries everywhere except in the \( j \)-th row where every entry is 1. Then \( qS_j = (q_j, \ldots, q_j) \in \mathbb{Z}^m \). By definition, (11) is satisfied for infinitely many \((p, q)\) with \( l\parallel q_j \). On setting \( q' := q \) and \( p' := lp - \frac{1}{l}qS_j \), we have that

\[
\left| q'(lX + \frac{1}{l}S_j) + p' \right| < lC\Psi(q')
\]

(14)

holds for infinitely many \((p', q')\) \( \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\} \) satisfying (13). Thus, if \( X \in S_{1,j}(l) \) then \( lX + \frac{1}{l}S_j \in S_{1,j}(l) \). Therefore the set \( S_{1,j}(l) \) is invariant under the transformation

\[
T : X \mapsto lX + \frac{1}{l}S_j \pmod{1}
\]

and Lemma 2 implies that \(|S_{1,j}(l)|\) is 0 or 1. Thus, \( S_1(l) \) is a finite union of sets with measure 0 or 1 and so \(|S_1(l)|\) is also 0 or 1.

In view of (12), the upshot of the above results for \(|S_0(l)|\) and \(|S_1(l)|\) is that if there exists a prime \( l \) such that \( S_0(l) \) or \( S_1(l) \) is of positive measure then \(|F'_{n,m}(\Psi)| = 1\). Thus, without loss of generality we can assume that such a prime does not exist and so by (12) we have that

\[
|S_2(l)| = |F'_{n,m}(\Psi)| \quad \text{for every prime } l.
\]

(15)
Suppose $X \in S_2(l)$ and fix any $S \in \mathbb{Z}^{nm}$. Then \((11)\) is satisfied for infinitely many $(p, q)$ with $l^2|d = \gcd(q)$. On setting $q' := q$ and $p' := p - \frac{1}{l} q S$, we have that

$$\left| q' \left( X + \frac{1}{l} S \right) + p' \right| < C \Psi(q)$$

(16)

holds for infinitely many $(p', q') \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$ satisfying \((13)\). Thus, if $X \in S_2(l)$ then $X + \frac{1}{l} S \in S_2(l)$ for any $S \in \mathbb{Z}^{nm}$. Therefore the set $S_2(l)$ is invariant under any transformation $X \mapsto X + \frac{1}{l} S \pmod{1}$ with $S \in \mathbb{Z}^{nm}$. In other words, $S_2(l)$ is $\frac{1}{l}$-periodic in every coordinate. Thus, for any cube $C_l$ in $[0, 1]^{nm}$ of sidelength $\frac{1}{l}$ we have that

$$|S_2(l) \cap C_l| = |S_2(l)| \cdot |C_l|.$$

In view of \((15)\), it follows that

$$|F'_{n,m}(\Psi) \cap C_l| = |F'_{n,m}(\Psi)| \cdot |C_l|$$

(17)

for any prime $l$. Now suppose that $|F'_{n,m}(\Psi)| > 0$. Then there is a density point $X_0$ of $F'_{n,m}(\Psi)$. For each prime $l$, let $C_l$ denote the cube in $[0, 1]^{nm}$ centred at $X_0$ of sidelength $\frac{1}{l}$. Then,

$$|F'_{n,m}(\Psi) \cap C_l| \sim |C_l| \quad \text{as} \quad l \to \infty.$$

This together with \((17)\) implies that $|F'_{n,m}(\Psi)| = 1$ and thereby completes the proof of Theorem 1 for the set $F'_{n,m}(\Psi)$.  

\[\square\]

**The set** $F''_{n,m}(\Psi)$ : To establish the desired zero-one statement for the set $F''_{n,m}(\Psi)$, we modify in the obvious manner the argument given above for the set $F'_{n,m}(\Psi)$. Naturally, “gcd($p_j, q$) = 1 for all $j = 1, \ldots, m$” will replace “gcd($p, q$) = 1” appearing in \((11)\). Similarly, the condition that “gcd($p'_j, q'$) = 1 for all $j = 1, \ldots, m$” will replace the coprimality condition \((13)\). The rest remains pretty much unchanged.  

\[\square\]

**4 Further results and questions**

**Ψ-well approximable points.** The various sets of $Ψ$-well approximable points are defined by requiring that the constant $C > 0$ appearing in \((11)\) can be made arbitrarily small. More precisely,

$$W_{n,m}^\circ(\Psi) := \bigcap_{k=1}^\infty A_{n,m}^\circ(k^{-1}\Psi).$$

Lemma 4 readily implies the following statement.

**Theorem 3** For any $n, m$ and $Ψ$ we have that $|W_{n,m}^\circ(\Psi)| = |A_{n,m}^\circ(Ψ)|$.  

7
Theorem 3 combined with (10) and Theorem 2 trivially implies the zero-one law for \( \Psi \)-well approximable sets.

**Corollary 1** For any \( n, m \) and \( \Psi \) we have that \(|\mathcal{W}_{n,m}^\circ(\Psi)| \in \{0, 1\}\).

**\( \Psi \)-badly approximable points.** Naturally, the various sets of \( \Psi \)-badly approximable points can be thought of as being complementary to the \( \Psi \)-well approximable sets. More precisely,

\[
\mathcal{B}_{n,m}^\circ(\Psi) := \mathcal{A}_{n,m}^\circ(\Psi) \setminus \mathcal{W}_{n,m}^\circ(\Psi).
\]

An immediate consequence of Theorem 3 is the following result.

**Corollary 2** For any \( n, m \) and \( \Psi \) we have that \(|\mathcal{B}_{n,m}^\circ(\Psi)| = 0\).

The classical set of badly approximable real numbers \( \text{Bad} := \mathcal{B}_{1,1}(q \mapsto q^{-1}) \) is known to have full Hausdorff dimension; i.e. \( \dim \text{Bad} = 1 \). For a general function \( \psi : \mathbb{N} \to [0, +\infty) \) with various mild growth conditions, Bugeaud [7], answering a question posed in [5], has shown that \( \mathcal{B}_{1,1}(\psi) \) has full Hausdorff dimension; i.e. \( \dim \mathcal{B}_{1,1}(\psi) = \dim \mathcal{A}_{1,1}(\psi) \). It view of this and Corollary 2 it is reasonable to ask the following question.

**Question 1** Is it true that the set \( \mathcal{B}_{n,m}(\Psi) \) has full Hausdorff dimension; i.e.

\[
\dim \mathcal{B}_{n,m}(\Psi) = \dim \mathcal{A}_{n,m}(\Psi) ?
\]

A weaker form of this question in which \( \mathcal{B}_{n,m}(\Psi) \) is replaced by \( \mathcal{A}_{n,m}(\Psi) \setminus \mathcal{A}_{n,m}(\Psi') \) with \( \Psi'(q) = o(\Psi(q)) \) as \( |q| \to \infty \), can be found in [5]. Note that if answer to the above question is yes, then we automatically have that \( \dim \mathcal{B}_{n,m}^\circ(\Psi) = \dim \mathcal{A}_{n,m}^\circ(\Psi) \).

**Multi-error approximation.** Observe that the inequality given by (2) can be rewritten as a system of \( m \) inequalities; namely

\[
|qX^{(j)} + p_j| < \Psi(q) \quad j = 1, \ldots, m,
\]

where \( X^{(j)} \) is the \( j \)-th column of \( X \). Thus, the error of approximation associated with each linear form is determined by \( \Psi \) and is the same. More generally, we consider the system

\[
|qX^{(j)} + p_j| < \Psi_j(q) \quad j = 1, \ldots, m,
\]

(18)

with \( \Psi_j : \mathbb{Z}^n \to [0, +\infty) \) and so the error of approximation is allowed to differ from one linear form to the next. Let \( \mathcal{A}_{n,m}^\circ(\Psi_1, \ldots, \Psi_m) \) denote the ‘multi-error’ analogue of \( \mathcal{A}_{n,m}^\circ(\Psi) \) – obtained by replacing (2) with (18) in the definition of \( \mathcal{A}_{n,m}^\circ(\Psi) \). Naturally, this enables us to define the multi-error analogues of \( \mathcal{F}_{n,m}^\circ(\Psi) \), \( \mathcal{W}_{n,m}^\circ(\Psi) \) and \( \mathcal{B}_{n,m}^\circ(\Psi) \).

Without much effort, it is possible to establish the multi-error analogue of Theorem 2 – the proof is practically unchanged.
Theorem 4 For any \( n, m \) and \( \Psi_1, \ldots, \Psi_m \) we have that \( |F_{n,m}(\Psi_1, \ldots, \Psi_m)| \in \{0, 1\} \).

If the statement of Lemma 4 can be generalized to the multi-error framework the above theorem would answer the following question and thereby yield the analogue of Theorem 1.

Question 2 Is it true that \( |F_{n,m}(\Psi_1, \ldots, \Psi_m)| = |W_{n,m}(\Psi_1, \ldots, \Psi_m)| \)?

Note that if the answer to Question 2 is yes, then so is the answer to our next question.

Question 3 Is it true that \( |B_{n,m}(\Psi_1, \ldots, \Psi_m)| = 0 \)?

Multiplicative approximation. Given \( \Psi : \mathbb{Z}^n \to [0, +\infty) \), let \( A_{n,m}^\times(\Psi) \) be the set of \( X \in [0, 1]^{nm} \) such that
\[
\prod_{i=1}^{m} \|qX^{(j)}\| < \Psi(q)
\]
holds for infinitely many \( q \in \mathbb{Z}^n \). Here \( \| \cdot \| \) denotes the distance to the nearest integer. Naturally, this enables us to define the associated multiplicative sets \( F_{n,m}^\times(\Psi), W_{n,m}^\times(\Psi) \) (multiplicatively \( \Psi \)-well approximable points) and \( B_{n,m}^\times(\Psi) \) (multiplicatively \( \Psi \)-badly approximable points). Clearly, if \( \Psi := \Psi_1 \cdots \Psi_m \) then
\[
A(\Psi_1, \ldots, \Psi_m) \subset A_{n,m}^\times(\Psi), \\
F(\Psi_1, \ldots, \Psi_m) \subset F_{n,m}^\times(\Psi), \\
W(\Psi_1, \ldots, \Psi_m) \subset W_{n,m}^\times(\Psi).
\]

However, it is easily seen that
\[
B(\Psi_1, \ldots, \Psi_m) \not\subset B_{n,m}^\times(\Psi).
\]

Question 4 Is it true that \( A_{n,m}^\times(\Psi), F_{n,m}^\times(\Psi) \) and \( W_{n,m}^\times(\Psi) \) are of measure 0 or 1?

Question 5 Is it true that \( |B_{n,m}^\times(\Psi)| = 0 \)?

Note that when \( n = 1, m = 2 \) and \( \Psi(q) := q^{-1} \), the answer to Question 5 is for obvious reasons yes. Indeed, it is conjectured that
\[
B_{1,2}^\times(q \mapsto q^{-1}) = \emptyset.
\]

This is Littlewood’s famous conjecture in the theory of Diophantine approximation.
Approximation by rational planes. The inequality given by (2) takes on two ‘extreme’ forms of rational approximation depending on whether $n = 1$ or $m = 1$. When $m = 1$, it corresponds to approximating arbitrary points by $(n - 1)$-dimensional rational planes (i.e. rational hyperplanes) and gives rise to the dual theory of Diophantine approximation. When $n = 1$, it corresponds to approximating arbitrary points by 0-dimensional rational planes (i.e. rational points) and gives rise to the simultaneous theory of Diophantine approximation. For $d \in \{0, \ldots, n - 1\}$, it is natural to consider the Diophantine approximation theory in which points in $\mathbb{R}^n$ are approximated by $d$-dimensional rational planes – the dual and simultaneous theories just represent the extreme. The foundations have been developed in some depth by W.M. Schmidt [22] in the sixties and more recently by M. Laurent [17]. However, apart from the extreme cases, there appears to be no analogue of Theorem 1 within the theory of approximation by $d$-dimensional rational planes.

Approximation by algebraic numbers. Sprindzuk’s [23] celebrated proof of Mahler’s conjecture [18] led to Baker [1] making the following stronger conjecture that was eventually established by Bernik [6]. Let $n \in \mathbb{N}$ and $\psi : \mathbb{N} \to (0, +\infty)$ be monotonic. Then for almost every real $x$ the inequality

$$|P(x)| < H(P)^{-n+1}\psi(H(P)) \quad (20)$$

holds for finitely many $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ if

$$\sum_{r=1}^{\infty} \psi(r) < \infty. \quad (21)$$

Here $H(P)$ is the height of $P$; i.e. the maximum of the absolute values of the coefficients of $P$. The case $\psi(h) := h^{-1-\varepsilon}$ corresponds to Mahler’s conjecture. In [2] it has been shown that if the sum in (21) diverges and $\psi$ is monotonic, then for almost every real $x$ inequality (20) holds infinitely often. More recently [3], the monotonicity assumption in Bernik’s convergence result has been removed. However, removing the monotonicity assumption from the divergence result remains an open problem akin to the Duffin-Schaeffer conjecture. In the first instance, it would be natural and desirable to ask for a zero-one law.

**Question 6** Is it true that the set of $x \in [0, 1]$ such that (20) holds for infinitely many $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ is of measure 0 or 1?

The following is a related question concerning explicit approximation by algebraic numbers.

**Question 7** Is it true that the set of real $x \in [0, 1]$ such that

$$|x - \alpha| < H(\alpha)^{-n}\psi(H(\alpha)) \quad (22)$$

holds for infinitely many real algebraic $\alpha$ of $\deg \alpha \leq n$ is of measure 0 or 1?
Here $H(\alpha)$ stands for the height of the minimal defining polynomial of $\alpha$. If (21) is satisfied, a simple application of the Borel-Cantelli lemma shows that the set under consideration is of measure zero. On the other hand, if the sum in (21) diverges and $\psi$ is monotonic the set under consideration is known to have measure one – see [2]. The upshot is that Question 7 only needs to be considered when $\psi$ is non-monotonic and the sum in (21) diverges.

Acknowledgements. We would like to thank Wolfgang Schmidt for his many wonderful theorems that have greatly influenced our research. Also, thank you for your generous support over the years. Finally and most importantly – happy number seventy five Wolfgang!

References

[1] A. Baker, On a theorem of Sprindžuk, Proc. Royal Soc. Series A, 292 (1966), pp. 92–104.

[2] V. Beresnevich, On approximation of real numbers by real algebraic numbers, Acta Arith., 90 (1999), pp. 97–112.

[3] V. Beresnevich, On a theorem of V. Bernik in the metric theory of Diophantine approximation, Acta Arith., 117 (2005), pp. 71–80.

[4] V. Beresnevich, V. Bernik, M. Dodson, and S. Velani, Classical metric diophantine approximation revisited. Preprint, 2007, arXiv:0803.2351.

[5] V. Beresnevich, D. Dickinson, and S. Velani, Sets of exact ‘logarithmic’ order in the theory of Diophantine approximation, Math. Ann., 321 (2001), pp. 253–273.

[6] V. Bernik, On the exact order of approximation of zero by values of integral polynomials, Acta Arithmetica, 53 (1989), pp. 17–28. (In Russian).

[7] Y. Bugeaud, Sets of exact approximation order by rational numbers, Math. Ann., 327 (2003), pp. 171–190.

[8] J. W. S. Cassels, Some metrical theorems in Diophantine approximation. I, Proc. Cambridge Philos. Soc., 46 (1950), pp. 209–218.

[9] P. A. Catlin, Two problems in metric Diophantine approximation. I, J. Number Theory, 8 (1976), pp. 282–288.

[10] R. J. Duffin and A. C. Schaeffer, Khintchine’s problem in metric Diophantine approximation, Duke Math. J., 8 (1941), pp. 243–255.

[11] P. X. Gallagher, Approximation by reduced fractions, J. Math. Soc. Japan, 13 (1961), pp. 342–345.
[12] P. X. Gallagher, *Metric simultaneous diophantine approximation. II*, Mathematika, 12 (1965), pp. 123–127.

[13] A. Groshev, *A theorem on a system of linear forms*, Dokl. Akad. Nauk SSSR, 19 (1938), pp. 151–152. (In Russian).

[14] G. Harman, *Metric number theory*, vol. 18 of LMS Monographs New Series, Clarendon Press, 1998.

[15] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann., 92 (1924), pp. 115–125.

[16] A. Khintchine, *Zur metrischen Theorie der diophantischen Approximationen*, Math. Zeitschr., 24 (1926), pp. 706–714.

[17] M. Laurent, *On transfer inequalities in Diophantine approximation*. March 2007, arXiv:math/0703146.

[18] K. Mahler, *über das Maß der Menge aller S-Zahlen*, Math. Ann., 106 (1932), pp. 131–139.

[19] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995.

[20] A. Pollington and R. C. Vaughan, *The k-dimensional Duffin and Schaeffer conjecture*, Mathematika, 37 (1990), pp. 190–200.

[21] W. M. Schmidt, *Metrical theorem on the fractional parts of sequences*, Trans. Amer. Math. Soc., 110 (1964), pp. 493–518.

[22] W. M. Schmidt, *On heights of algebraic subspaces and diophantine approximations*, Ann. of Math. (2), 85 (1967), pp. 430–472.

[23] V. Sprindžuk, *Metric theory of Diophantine approximation*, John Wiley & Sons, New York-Toronto-London, 1979. (English transl.).

Victor V. Beresnevich: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.

   e-mail: vb8@york.ac.uk

Sanju L. Velani: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.

   e-mail: slv3@york.ac.uk