Testing Error Correcting Codes  
by Multicanonical Sampling of Rare Events

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The idea of rare event sampling by Multicanonical Monte Carlo is applied to the estimation of the performance of error-correcting codes. The essence of the idea is importance sampling of the pattern of noises in the channel by Multicanonical Monte Carlo, which enables efficient estimation of tails of the distribution of bit errors. The proposed method is successfully tested with a convolutional code.

Dynamic Monte Carlo (Markov chain Monte Carlo, MCMC) algorithm is discovered in physics in 1950s and introduced to statistical data analysis in 1980s, now being recognized as an essential methodology in both fields. But the use of MCMC need not be restricted to these two fields. In fact, it is a general strategy for sampling from complicated distributions with unknown normalization constants, and there will be a number of potential applications in other fields.

In this note, we discuss a problem of estimating the distribution of bit errors of error-correcting codes for a given channel and a given input distribution. The essence of the proposed idea is importance sampling of the pattern of noises in the channel by MCMC. By using the proposed strategy, the tails of the error distribution are efficiently calculated. We also discuss that the use of Multicanonical Monte Carlo algorithm, a version of MCMC which utilizes iterative construction of the sampling weight, is ideally suited to the problem.

Note that efficient generation of unusual events is important not only for checking theories but also for practical purposes, because “rare events” become no more rare with some deviation from idealized models, such as unexpected correlation between noises.

Our proposal is on the use of MCMC for the evaluation of characteristics of codes and channels. Thus, it is essentially different from any idea on the use of MCMC for decoding messages. An example of the use of MCMC and related algorithm in this field is found in the references, which estimate channel capacity by Blahut-Arimoto algorithm.

The proposed approach is closely related to the idea of Hartmann, who studied large deviations of the output of a sequence alignment algorithm by a MCMC sampling. Similar approaches based on a MCMC sampling of quenched disorder is also used for estimating tails of the distributions of the ground state energy of physical systems and exploring finite temperature property of random magnets. The idea of using MCMC as a tool for sampling rare events that reduce the worst-case efficiency of an algorithm seems to have a wide range of applications.

Let us introduce a problem of estimating the performance of error-correcting codes. Fix a coding, a channel, and a decoding algorithm, and denote an input message and its distribution by \( x = \{x_i\} \) and \( P(x) \). The encoded message, an output of the noisy channel and its distribution is represented as \( z = \{z_i\}, y = \{y_i\} \) and \( P(y|x) \), respectively. The message decoded by the given method from \( y \) is given by \( T(y) \). Here any kind of decoding algorithm is allowed such as Viterbi-decoding, belief-propagation, and loopy-belief-propagation, as long as \( T(y) \) is stable and can be regarded as a deterministic decoder. Assuming the distance \( d(x, T(y)) \) between the input \( x \) and output \( T(y) \), we consider the problem of calculating the probability distribution

\[
P(d) = \sum_{x,y} \delta(d - d(x, T(y))) P(x) P(y|x)
\]

of the distance \( d \) between inputs and outputs. In most familiar cases, \( d \) is Hamming distance and \( P(d) \) is the distribution of bit errors with a given distribution \( P(x) \) of inputs.

A naive method for estimating \( P(d) \) is repeating the following procedures (1–3) independently until desired accuracy is attained: (1) A message \( x \) is generated from the distribution \( P(x) \). (2) The output \( y \) of the channel is sampled from \( P(y|x) \). (3) The distance \( d(x, T(y)) \) is calculated and recorded. This method is straightforward, but becomes highly computationally expensive when we are interested in the tails of the distribution of \( P(d) \), which correspond to rare events or large deviations under the assumption of the distribution \( P(x) \) of inputs.

The proposed method, which largely improves the efficiency of the estimation of the tails of \( P(d) \), is multicanonical sampling approximately proportional to \( P(d(x, T(y)))^{-1} P(x) P(y|x) \). Here \( P(d(x, T(y))) \) is defined by the expression where \( d \) of \( P(d) \) is substituted for \( d(x, T(y)) \). From the definition (1) of \( P(d) \), it is shown that the marginal distribution \( P^*(d) \) of bit errors \( d \) with this weight becomes nearly flat on the interval on which \( P(d) \neq 0 \), i.e.,

\[
P^*(d) \approx c \sum_{x,y} \delta(d - d(x, T(y))) P(d(x, T(y)))^{-1} P(x) P(y|x)
\]

\[
= c P(d)^{-1} \sum_{x,y} \delta(d - d(x, T(y))) P(x) P(y|x) = c
\]

where \( c \) is a constant. This enables both efficient sam-
sampling of the tails of the distribution \( P(d) \) and fast mixing of the Markov chain used for sampling. In this generic form, the method contains sampling of both \( x \) and \( y \) and looks somewhat complicated. But in the examples discussed below, it becomes simpler and reduces to the original idea of sampling pattern of the noise in the channel.

How can we realize such a sampling and fit the results of the sampling into the original problem? If we define the function \( P_M(d) \) by

\[
w_M(x, y) = P_M(d(x, T(y)))^{-1} P(x) P(x|y),
\]

the choice of weight \( w_M(x, y) \) reduces to the estimation of \( P_M(d) \) that realizes almost flat marginal distribution \( P^*(d) \). Then \( P_M(d) \) can be estimated by repeated preliminary runs of the simulation, just in the same way as the estimation of the weight in a conventional multicanonical algorithm. While any method in literature of multicanonical or Wang-Landau algorithm can be used for the estimation of the weight, in the following example we use a naive method with a histogram construction (entropic sampling). Once we obtain \( P_M(d) \) that gives a sufficiently flat distribution \( P^*(d) \), a reconstruction of the target distribution \( P(d) \) is given by \( P^*(d) P_M(d) \) with a suitable normalization constant.

Here we test the proposed method with a convolutional code, whose codewords are \( z_i^{(1)} = x_i x_{i+2} \) and \( z_i^{(2)} = x_i x_{i+3} x_{i+2} \). A binary symmetric channel (BSC) is assumed and a Viterbi decoder is used as \( T(y) \). In this example, the gauge invariance\(^3\) considerably simplifies the algorithm. In particular, we can fix the input \( x \) to an arbitrary bit sequence such as \( x_0 = 1111111 \cdots \) and the sampling of \( x \) and \( y \) reduces to the sampling of the pattern \( y \) of noises in the channel.\(^5\) Then the expression (2) becomes

\[
w_M(y) = P_M(d(x_0, T(y)))^{-1} P(x_0|y),
\]

but the proposed algorithm can be applied with some obvious modifications. In the following example, the length of original message and encoded message is 200 and \((200 - 2) \cdot 2 = 396\) respectively, and the probability of bit flip is set to 0.1. 20 iterations of preliminary runs are performed for the tuning of the weight and the final run is used for the calculation of the results.

Fig.1 gives an example of the convergence of algorithm, where the estimated bit error probabilities after the 4th, 12th, and 20th iteration are shown by solid circles (●), triangles (△), and open circles (○), respectively. The horizontal part of each curve indicates that no sample is obtained in the region.

In Fig.2, probabilities estimated in the 20th iteration are compared with the one by the naive method based on uniform random sampling. The symbol (●) corresponds to the result by the proposed method, where total of the preliminary and measurement runs requires about \( 8 \times 10^8 \) Viterbi decoding. The symbol (+) corresponds to the result by the naive method with \( 8.08 \times 10^8 \) Viterbi decoding. The results by the naive method are not shown in the region where the method dose not give a sample.

The proposed method gives the result in Fig.2 within a day of computation by a current personal computer and enables sampling from the right tail of the probability distribution \( P(d) \) where the naive method can hardly realize. On the other hand, the result by the proposed method (●) agrees with the one by the naive method (+) based on uniform random sampling in the range of higher probabilities, which supports the validity of the proposed method.

In the case of binary symmetric channel, the probability of bit flips is given and there is a small but finite possibility of flipping arbitrary number of bits up to the length of the encoded message. Then it is natural to expect that the right tail of the distribution \( P(d) \) corresponds to larger number of flipped bits in the channel. This tendency is illustrated in Fig.3, where the average of the numbers of flipped bits conditioned with a given value of bit errors \( d \) is plotted as a function of \( d \).
From this viewpoint, it will be more interesting to treat a channel of fixed number of flipped bits, instead of binary symmetric channel. It is equivalent to the sampling of the positions of flipped bits under the condition that the total number of flipped bits is given. An advantage of the proposed method is that we can easily adapt it to this kind of modification. In the present case, sampling from the channel of fixed number of flipped bits is simply realized by the introduction of Metropolis move of swapping the positions of a flipped bit and a conserved bit.

In Fig. 4, a result by the proposed method in the case of fixed number of flipped bits is shown. The length of the original message and encoded message is the same as the one of Fig. 1–Fig. 3, and the number of flipped bits is set to 40. The symbol $\bullet$ corresponds to the estimated probabilities by the proposed method. Preliminary runs with 9 iterations are performed for the tuning of the weight and the 10th run is used for the calculation of the results. Total of these runs requires about $7.3 \times 10^8$ Viterbi decoding. The symbol $\ast$ corresponds to the result by the naive method with $1.1 \times 10^9$ Viterbi decoding. The largest value 79 of the bit errors obtained by the proposed method seems to be the exact upper bound under the condition of Fig. 4, because it is stable against the increase of the weights for $d > 79$. In the tail region shown in Fig. 4, the probability is $\sim 10^{-20}$, which can never be estimated by the naive method.

In summary, we proposed an application of the idea of rare event sampling to the estimation of the performance of error-correcting codes. It is shown that a method based on multicanonical sampling of the pattern of noises gives an efficient way for sampling of the tails of the distribution of bit errors with given distributions of the input and noise.

A potential advantage of the proposed approach is that we can explicitly sample bit patterns of noises that give severe “damage” to encoded messages and cause large bit errors. It can be useful for the understanding of weak points of a given code. This idea of “weak point sampling” will be useful for a wide range of problems. Research in this direction as well as applications to realistic codes and channels are left for future studies.

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