Second-order Cosmological Perturbations Engendered by Point-like Masses

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Abstract

In the $\Lambda$CDM framework, presenting nonrelativistic matter inhomogeneities as discrete massive particles, we develop the second-order cosmological perturbation theory. Our approach relies on the weak gravitational field limit. The derived equations for the second-order scalar, vector, and tensor metric corrections are suitable at arbitrary distances, including regions with nonlinear contrasts of the matter density. We thoroughly verify fulfillment of all Einstein equations, as well as self-consistency of order assignments. In addition, we achieve logical positive results in the Minkowski background limit. Feasible investigations of the cosmological back-reaction manifestations by means of relativistic simulations are also outlined.

Key words: cosmological parameters – cosmology: theory – dark energy – dark matter – gravitation – large-scale structure of universe

1. Introduction

The conventional Lambda cold dark matter ($\Lambda$CDM) model conforms with the observational data (see, in particular, Ade et al. 2016) and embodies the mainstream of modern cosmology despite the distressing fact that the nature of dark ingredients of the universe still remains uncertain. The key assumption, being typical for this cosmological model as well as its numerous alternatives, is the existence of the homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) background, which is only slightly perturbed by inhomogeneities inherent in the distribution of the world’s filling material. The following quite natural question arises: can galaxies and their accumulations lead to considerable metric corrections and deeply affect the average cosmic expansion? The affirmative answer would bring us to general recognition of the so-called back-reaction (see the reviews by Clarkson et al. 2011; Räsänen 2011; Buchert & Räsänen 2012; Bolejko & Korzyński 2017, and references therein; also for a recent spirited debate about the magnitude of back-reaction effects see, in particular, Green & Wald 2013; Buchert et al. 2015, and references therein) and give at least a glimmer of hope to explain the apparent acceleration of expansion without mysterious dark energy.

Even if the answer is negative, it is extremely important to estimate deviations from the FLRW description and confront the theoretical predictions with the promising outcomes of such future space missions as Euclid (Scaramella et al. 2015; Amendola et al. 2016). For this purpose a reliable cosmological perturbation theory should be developed in the general relativity (GR) framework. Appropriateness at all cosmic scales and nonperturbative treatment of the matter density are those basic requirements that such a theory must meet. The first-order scheme complying with these reasonable demands and being suitable for relativistic N-body simulations has been successfully constructed for a system of discrete massive particles with nonrelativistic velocities by Eingorn (2016). Some predecessors and their drawbacks are reviewed in Eingorn (2016) as well.

The current paper is devoted to the generalization of the aforementioned first-order approach to the second order with respect to deviations of the metric coefficients from the corresponding background quantities. This generalization is particularly motivated by the fact that the first order is obviously insufficient for the comprehensive analysis of the possible back-reaction manifestations and the trustworthy prediction of their magnitude. The appeal to the second order with the purpose of revealing the corresponding observable features is also very promising in the era of precision cosmology (see Ben-Dayan et al. 2013; Bonvin et al. 2015a, 2015b, and references therein).

The narration is organized as follows: After reviewing the basic results of Eingorn (2016) in Section 2, we switch over to derivation and verification of equations for the second-order scalar, vector, and tensor perturbations in Section 3. Then, in Section 4, we focus attention on identification of the effective average energy density and pressure and propose the research program aimed at the perturbative computation of the cosmological back-reaction effects. Our achievements are laconically summarized in Section 5.

2. Discrete Picture of the First-order Cosmological Perturbations at All Scales

Let us start with an overview of the newly formulated first-order perturbation theory covering all cosmological spatial scales and permitting of nonlinear contrasts of the matter density (Eingorn 2016). We confine ourselves to the conventional $\Lambda$CDM model (with zero spatial curvature) and concentrate on those stages of the universe evolution when cold matter (dark and baryonic) and dark energy (being represented by the cosmological constant $\Lambda$) dominate while radiation or relativistic cosmic neutrinos make negligible contributions (see, however, Eingorn & Brilenkov 2015; Eingorn et al. 2016, for a broad generalization to the multicomponent case). Then the homogeneous and isotropic cosmological background is described by the unperturbed FLRW metric

$$dx^2 = a^2(d\tau^2 - \delta_{\alpha\beta}dx^\alpha dx^\beta), \quad \alpha, \beta = 1, 2, 3, \quad (1)$$
where $\eta$ and $x^a$, $\alpha = 1, 2, 3$, stand for the conformal time and comoving coordinates, respectively, and the corresponding Friedmann equations for the scale factor $a(\eta)$:

$$\frac{3H^2}{a^2} = \kappa \bar{\rho} + \Lambda, \quad \frac{2H' + \mathcal{H}^2}{a^2} = \Lambda. \quad (2)$$

Here $\mathcal{H}(\eta) \equiv a' / a$ (with prime denoting the derivative with respect to $\eta$) and $\kappa \equiv 8\pi G_N / c^4$ (with $c$ and $G_N$ representing the speed of light and Newtonian gravitational constant, respectively). Further, $\varepsilon$ is the energy density of the nonrelativistic pressureless matter, and the overline indicates averaging.

In the first-order approximation the real inhomogeneous universe is usually assumed to be described well by the perturbed metric (Bardeen 1980; Mukhanov 2005; Durrer 2008; Gorbunov & Rubakov 2011)

$$ds^2 = a^2[(1 + 2\Phi)d\eta^2 + 2B_{ij}dx^i dx^j] - (1 - 2\Phi)\delta_{ij}dx^i dx^j, \quad (3)$$

where $\Phi(\eta, r)$ is the scalar perturbation, while the spatial vector $\mathbf{B}(\eta, r) \equiv (B_1, B_2, B_3) \equiv (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ stands for the vector perturbation and satisfies the prevalent gauge condition

$$\nabla \mathbf{B} \equiv \delta^{(ij)} \partial_{\eta} B_{ij} \equiv 0. \quad (4)$$

Similarly to Clarkson & Umeh (2011) and Baumann et al. (2012), we have chosen the popular Poisson gauge but have not yet taken account of the tensor perturbations treating them as second-order quantities. As pointed out by Eingorn (2016), one can in principle allow for the first-order tensor perturbations associated with freely propagating gravitational waves (with no generator). They have not been explicitly included in Equation (3) since below we totally neglect their possible contributions to the sources of second-order metric corrections. At the same time, similarly to Eingorn (2016) but in contrast to Clarkson & Umeh (2011) and Baumann et al. (2012), the vector perturbation $\mathbf{B}$ has been already included in Equation (3) as a first-order quantity since it has a definite nonzero generator (see the right-hand side (rhs) of Equation (9) for $\mathbf{B}$ below).

In Eingorn (2016) the role of the inhomogeneous gravitational field source belongs to a system of separate nonrelativistic point-like particles with masses $m_n$, comoving radius vectors $r_n(\eta)$, and comoving peculiar velocities $\mathbf{v}_n(\eta) \equiv d\mathbf{r}_n / d\eta \equiv (\mathbf{v}_n^1, \mathbf{v}_n^2, \mathbf{v}_n^3)$, and the following expressions for $\Phi$, $\mathbf{B}$ are derived:

$$\Phi = \frac{1}{3} \sum_n \frac{m_n}{\vert \mathbf{r} - \mathbf{r}_n \vert} \exp(-q_n),$$

$$\frac{3\kappa c^2}{8\pi a} \sum_n \frac{m_n}{\vert \mathbf{r} - \mathbf{r}_n \vert} \mathbf{v}_n \cdot \mathbf{r}_n - \frac{3\kappa c^2}{8\pi a} \sum_n \frac{m_n [\mathbf{v}_n(\mathbf{r} - \mathbf{r}_n)]}{\vert \mathbf{r} - \mathbf{r}_n \vert} \cdot \frac{1 - (1 + q_n)\exp(-q_n)}{q_n^2},$$

$$B = \frac{\kappa c^2}{8\pi a} \sum_n \frac{m_n \mathbf{v}_n}{\vert \mathbf{r} - \mathbf{r}_n \vert} \exp(-2q_n / \sqrt{3}) [3 + 2\sqrt{3}q_n + 4q_n^2] - 3 + \frac{m_n [\mathbf{v}_n(\mathbf{r} - \mathbf{r}_n)]}{\vert \mathbf{r} - \mathbf{r}_n \vert} \cdot \frac{9 - (9 + 6\sqrt{3}q_n + 4q_n^2)\exp(-2q_n / \sqrt{3})}{q_n^2}, \quad (6)$$

Here $q_n(\eta, r) \equiv a |\mathbf{r} - \mathbf{r}_n| / \lambda$, with $\lambda(\eta) \equiv [2a^3 / (3\kappa \bar{\rho} c^2)]^{1/2} \sim a^{3/2}$ defining a finite range of Yukawa interaction. Being armed with the observed values of the Hubble constant $H_0 \approx 68$ km s$^{-1}$ Mpc$^{-1}$ and the parameter $\Omega_M \equiv \kappa \bar{\rho} c^4 / (3H_0^2 a_0^2) \approx 0.31$ (Ade et al. 2016), where $a_0$ denotes today’s scale factor, one can easily estimate the current value of the introduced characteristic cutoff scale: $\lambda_0 \approx 3.7$ Gpc (Eingorn 2016). An evident relationship $\bar{\rho} = \bar{\rho} c^2 / a^2$ establishes linkage between $\bar{\rho}(\eta) \sim a^{-3}$ and the constant average rest mass density $\bar{\rho}$. As regards the corresponding nonaveraged quantity, the rest mass density of the analyzed particle system in the comoving coordinates has the form

$$\rho(\eta, r) = \sum_n m_n \delta(\mathbf{r} - \mathbf{r}_n),$$

$$\rho_n(\eta, r) = m_n \delta(\mathbf{r} - \mathbf{r}_n). \quad (7)$$

The explicit analytical expressions (5) and (6) and their noteworthy features, along with the underlying perturbative approach and its numerous physical implications and advantages, are analyzed in detail by Eingorn (2016), and it makes no sense to manifest hair-splitting by repeating all deserving results here. Nevertheless, let us briefly enumerate those facts that are crucial for the clear purpose of the next section.

First of all, we stubbornly adhere to the following well-grounded argumentation (see also Baumann et al. 2012, for similar reasoning): the metric corrections and peculiar particle velocities are assumed to be small at arbitrary distances; however, the smallness of the density contrast $\delta \equiv \delta \rho / \bar{\rho}$ (where $\delta \rho \equiv \rho - \bar{\rho}$) is not demanded. For instance, $\rho \gg \bar{\rho}$ within galaxies, but even for huge density contrasts the metric is still approximated well by Equation (3). Therefore, one can linearize the Einstein equations in the first-order metric perturbations $\Phi$, $\mathbf{B}$ and velocities $\mathbf{v}_n$ without resorting to the unnecessary restrictive inequality $|\delta| \ll 1$. Thus, the nonlinear deviation of $\rho$ from its average value $\bar{\rho}$, actually occurring at sufficiently small scales, is absolutely unforbidden and fully taken into consideration, as opposed to the standard first-order relativistic perturbation theory (Mukhanov 2005; Durrer 2008; Gorbunov & Rubakov 2011) and its generalization to the second order (see, e.g., Bartolo et al. 2006).

Second, functions (5) and (6) are found as exact solutions of the Helmholtz equations (see Eingorn 2016, for the corresponding Fourier transforms):

$$\nabla^2 \Phi - \frac{3\kappa \bar{\rho} c^2}{2a} \Phi = \frac{\kappa c^2}{2a} \delta \rho - \frac{3\kappa c^2}{2a} \mathcal{H} \Xi, \quad (8)$$

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\[ \Delta B - \frac{2\kappa \rho c^2}{a} B = -\frac{2\kappa c^2}{a} \left( \sum_n \rho_n \bar{v}_n - \nabla \Xi \right), \quad (9) \]

where \( \Delta \equiv \delta \theta^2 \frac{\partial^2}{\delta v^2} \) is the Laplace operator in the comoving coordinates, and the auxiliary function \( \Xi(\eta, r) \) has been introduced:

\[
\Xi = \frac{1}{4\pi} \sum_n m_n \frac{(r - r_n) \bar{v}_n}{|r - r_n|^3}. \quad (10)
\]

Evidently, the presence of the term \( \sim \Phi \) in Equation (8) for the scalar perturbation \( \Phi \) is the reason for the Yukawa-type cutoff in solution (5). The aforementioned screening length \( \lambda \) is naturally related to the factor in this term: \( 3\kappa \rho c^2/(2a) \equiv a^2/\lambda^2 \). The contribution \( \sim B \) itself and the contribution \( \sim \Phi \) in Equation (9) arise in view of the fact that the energy-momentum fluctuations, generating the metric corrections, contain these corrections themselves (see Equations (2.13) and (2.14) in Eingorn 2016, as well as expressions (34)–(36) below).

The vector perturbation \( B \) (Equation (6)) obeys the gauge condition (Equation (4)). In addition, within the adopted accuracy perturbations (5) and (6) satisfy all remaining linearized Einstein equations, which are reduced to the triplet containing temporal derivatives:

\[
\Phi' + \mathcal{H} \Phi = -\frac{\kappa c^2}{2a} \Xi, \quad \Phi'' + 3\mathcal{H} \Phi' + (2\mathcal{H}' + \mathcal{H}^2) \Phi = 0, \quad B' + 2\mathcal{H} B = 0. \quad (11)
\]

Finally, auxiliary function (10) is found as the exact solution of the Poisson equation:

\[
\Delta \Xi = \nabla \sum_n \rho_n \bar{v}_n = -\sum_n \rho_n' \bar{v}_n. \quad (12)
\]

where the continuity equation \( \rho_n' + \nabla (\rho_n \bar{v}_n) = 0 \), which is satisfied for any \( n \)th particle identically, has been employed.

It presents no difficulty to show that the usual theory of hydrodynamical fluctuations emerges in the continuum limit. Indeed, let us momentarily regard \( \rho(\eta, r) \) as a continuous mass density field and \( \bar{v}(\eta, r) = \nabla \bar{v}^{(0)} + \bar{v}^{(1)} \) as a continuous velocity field (with \( \bar{v}^{(0)}(\eta, r) \) and \( \bar{v}^{(1)}(\eta, r) \) denoting the scalar and vector parts, respectively). Further, we replace \( \Xi \) by \( \bar{v}^{(0)} \) and \( \sum_n \rho_n \bar{v}_n - \nabla \Xi \) by \( \bar{v}^{(1)} \) in the case of linear density fluctuations. Then we make use of the relationship \( \varepsilon = \rho c^2/a^3 + (3\kappa \rho^2/a^3) \Phi \) in order to rewrite equations in terms of the energy density field \( \varepsilon(\eta, r) \) (instead of \( \rho \)). Finally, we introduce covariant spatial components of the 4-velocity \( u_{ij} = aB_{ij} - a\bar{v}_{,i} \beta = 1, 2, 3 \), and a spatial vector with these components \( u(\eta, r) \equiv (u_1, u_2, u_3) \) (instead of \( \bar{v} \)). We also single out the corresponding scalar and vector parts: \( u = \nabla u^{(0)} + u^{(1)} \), where \( u^{(0)}(\eta, r) = -a\bar{v}^{(0)} \) and \( u^{(1)}(\eta, r) = aB - a\bar{v}^{(1)} \). As a result, the rewritten Equations (8), (9), and (11) become equivalent to those describing hydrodynamical perturbations (see Mukhanov 2005, Section 7.3, in the pressureless matter case).

Since the first-order metric corrections inevitably contribute to the second-order ones, due to nonlinearity of the Einstein equations, we make the most out of the enumerated equations for \( \Phi \) and \( B \), as well as \( \Xi \), in the next section.

### 3. Splendors of the Second-order Theory

#### 3.1. Sources of Perturbations

Switching over to the second-order approximation, we present the metric as

\[
ds^2 = a^2(1 + 2\phi) d\eta^2 + 2B_{ij} dx^i dx^j - (1 - 2\phi) \delta_{ij} dx^i dx^j + \alpha^2 [2\Phi^2 d\eta^2 + 2B_{ij}^2 dx^i dx^j + (2\Phi \delta_{ij} + h_{ij}) d\eta dx^i dx^j]. \quad (13)
\]

where \( \Phi^{(2)}(\eta, r) \) and \( \Psi^{(2)}(\eta, r) \) are the second-order scalar perturbations, while the spatial vector \( B_{ij}^{(2)}(\eta, r) \equiv (B_{1j}^{(2)}, B_{2j}^{(2)}, B_{3j}^{(2)}) \equiv (B_1^{(2)}, B_2^{(2)}, B_3^{(2)}) \) stands for the second-order vector perturbation and satisfies the same gauge condition as Equation (4) for \( B \):

\[
\nabla B_{ij}^{(2)} = \delta_{ij} \frac{\partial h_{ij}^{(2)}}{\partial \phi} \equiv 0. \quad (14)
\]

Similarly to Clarkson & Umeh (2011) and Baumann et al. (2012), we have included the second-order tensor perturbations \( h_{ij}, \alpha, \beta = 1, 2, 3 \), in Equation (13). They obey the standard “transverse-traceless” gauge conditions:

\[
\delta_{ij} \frac{\partial h_{ij}}{\partial \phi} \equiv 0, \quad \delta^{ij} h_{ij} \equiv 0. \quad (15)
\]

It should be noted that we decompose the second-order metric corrections into scalar, vector, and tensor modes with respect to the unperturbed background. Thus, for example, the covariant divergence of the total vector \( B + B_{ij}^{(2)} \), defined via the perturbed spatial metric, does not vanish under the made gauge choice.

In order to elaborate on the Einstein equations

\[
G^k_i = \kappa T^k_i + \Lambda \delta^k_i, \quad i, k = 0, 1, 2, 3, \quad (16)
\]

for the sought-for functions \( \Phi^{(2)}, \Psi^{(2)}, B^{(2)}, \) and \( h_{ij} \), first of all it is necessary to calculate the mixed components of the Einstein tensor \( G^k_i \) and matter-energy-momentum tensor \( T^k_i \) up to the second order in metric corrections and their sources. Let us start with presenting the left-hand side (lhs) of Equation (16) as follows (see also Baumann et al. 2012, for a similar decomposition):

\[
G^k_i = (G^k_i)^{(0)} + (G^k_i)^{(1)} + (G^k_i)^{(2)} + (G^k_i)^{(3)}, \quad (17)
\]

Here \( (G^k_i)^{(0)} \) corresponds to the unperturbed cosmological background metric given by Equation (1). The expressions \( (G^k_i)^{(1)} \) (or, by analogy, \( (G^k_i)^{(2)} \)) are constructed from the terms being linear in the first-order (or second-order) perturbations \( \Phi \) and \( B \) (or \( \Phi^{(2)}, \Psi^{(2)}, B^{(2)}, \) and \( h_{ij} \)) and their spatiotemporal derivatives. Finally, \( (G^k_i)^{(3)} \) represent the second-order quantities containing products of \( \Phi, B \) and their derivatives. We express the temporal derivatives \( \dot{\Phi}, \dot{\Psi}, \dot{B}, \) and \( \dot{h}_{ij} \) in \( (G^k_i)^{(3)} \) by means of the functions \( \Phi \) and \( B \) themselves with the help of the triplet given by Equation (11). Thus, for example,

\[
G^0_\beta = (G^0_\beta)^{(0)} + (G^0_\beta)^{(1)} + (G^0_\beta)^{(2)} + (G^0_\beta)^{(3)}, \quad \beta = 1, 2, 3, \quad (18)
\]
where
\begin{align}
(G^{(0)}_j)^{(0)} &= 0, \quad (G^{(1)}_j)^{(0)} = \frac{1}{2a^2} \Delta B_{ij} + \frac{2}{a^2} \mathcal{H} \frac{\partial \Phi}{\partial \alpha} + \frac{2}{a^2} \frac{\partial \Phi'}{\partial \alpha}, \quad (G^{(2)}_j)^{(0)} = -\frac{1}{a^2} \Delta B_{ij} + \frac{2}{a^2} \mathcal{H} \frac{\partial \Phi}{\partial \alpha} + \frac{2}{a^2} \frac{\partial \Phi'}{\partial \alpha}, \quad (G^{(0)}_j)^{(1)} = \frac{1}{a^2} B_{ij} \Delta \Phi + \frac{1}{a^2} \mathcal{H} \frac{\partial \Phi}{\partial \alpha} - \frac{5}{a^2} \mathcal{H} \frac{\partial \Phi}{\partial \alpha}, \quad \delta_{ij}, \\
& \quad \quad - \frac{k^2}{a^2} \Delta \Phi - \frac{1}{a^2} \frac{\partial \Phi}{\partial \alpha} \frac{\partial \Phi}{\partial \alpha} + \frac{1}{a^2} \frac{\partial \Phi}{\partial \alpha} \frac{\partial \Phi}{\partial \alpha}.
\end{align}

Here $B^2 \equiv B_x^2 + B_y^2 + B_z^2$. The rest of the expressions $(G^{(k)}_j)^{(0)}$ and $(G^{(k)}_j)^{(1)}$ are well known (Mukhanov 2005; Durrer 2008; Gorbunov & Rubakov 2011):
\begin{align}
(G^{(0)}_j)^{(0)} &= \frac{3 \mathcal{H}^2}{a^2}, \quad (G^{(0)}_j)^{(1)} = \left(\frac{2 \mathcal{H}'}{a^2} + \frac{\mathcal{H}^2}{a^2}\right) \delta_{ij}, \quad (G^{(1)}_j)^{(0)} = \mathcal{H} \frac{\partial \Phi}{\partial \alpha} - \frac{6 \mathcal{H}^2}{a^2} - \frac{6 \mathcal{H} \Phi'}{a^2}, \quad (G^{(1)}_j)^{(1)} = -2 \left(\mathcal{H} \Phi + 3 \mathcal{H} \Phi + \mathcal{H} \Phi'\right) \delta_{ij}, \\
& \quad \quad - \frac{1}{a^2} \left[\mathcal{H} \left(\frac{\partial B_{ij}}{\partial \alpha} + \frac{\partial B_{ij}}{\partial \alpha}\right) + \frac{1}{a^2} \left(\frac{\partial B_{ij}}{\partial \alpha} + \frac{\partial B_{ij}}{\partial \alpha}\right)\right].
\end{align}

The expressions $(G^{(k)}_j)^{(2)}$ bear a strong resemblance to $(G^{(k)}_j)^{(1)}$ with due regard for the inequality $\Phi^{(2)} \approx \Psi^{(2)}$, as well as additional contributions from the tensor perturbations:
\begin{align}
(G^{(0)}_j)^{(2)} &= \frac{2}{a^2} \delta \Psi^{(2)} - \frac{6 \mathcal{H} \Phi^{(2)}}{a^2} - \frac{6 \mathcal{H} \Phi'}{a^2}, \\
(G^{(1)}_j)^{(2)} &= \frac{1}{2a^4} h^{(2)}_n + \frac{1}{a^2} \mathcal{H} h^{(2)}_n - \frac{1}{a^2} \Delta h_{ij}, \\
& \quad \quad - \frac{2}{a^4} \left[2 \mathcal{H} \left(\mathcal{H} \Phi^{(2)} + \mathcal{H} \Phi'\right) + \mathcal{H} \Phi'\right] \delta_{ij} - \frac{1}{2a^2} \left[\mathcal{H} \left(\frac{\partial B_{ij}^{(2)}}{\partial \alpha} + \frac{\partial B_{ij}^{(2)}}{\partial \alpha}\right) + \frac{1}{a^2} \left(\frac{\partial B_{ij}^{(2)}}{\partial \alpha} + \frac{\partial B_{ij}^{(2)}}{\partial \alpha}\right)\right].
\end{align}

Finally, the quantities $(G^{(0)}_j)^{(1)}$ and $(G^{(1)}_j)^{(1)}$ are quite cumbersome, and it makes no sense to reproduce them here.

As regards the mixed energy-momentum tensor components on the rhs of Equation (16), we resort to the well-known formulae for the analyzed system of point-like particles (Landau & Lifshitz 2000; Eingorn & Zhuk 2014):
\begin{align}
T_0^0 &= \frac{\rho c^2}{\sqrt{-g}} \frac{g_{00} + \sqrt{\gamma} g_{00} + g_{00} + g_{00} \nu m}{\sqrt{g_{00} + 2 g_{00} \nu + g_{00} \nu \nu}}, \\
T_0^0 &= \frac{\rho c^2}{\sqrt{-g}} \frac{g_{00} + \sqrt{\gamma} g_{00} + g_{00} + g_{00} \nu m}{\sqrt{g_{00} + 2 g_{00} \nu + g_{00} \nu \nu}}.
\end{align}

Consequently, with the same accuracy
\begin{align}
\frac{1}{\sqrt{-g}} &= \frac{1}{a^2} \left(\mathcal{H} \Phi + 3 \mathcal{H} \Phi + \mathcal{H} \Phi' + \mathcal{H} \Phi'\right) \delta_{ij}, \\
& \quad \quad + \frac{15}{2} \mathcal{H} \Phi' + \frac{1}{2} \mathcal{H} \Phi' + \frac{1}{2} \mathcal{H} \Phi' + \frac{1}{2} \mathcal{H} \Phi',
\end{align}

and Equations (28)–(30) are reduced to the formula
\begin{align}
T_0^0 &= \frac{\rho c^2}{a^2} \left(\mathcal{H} \Phi + 3 \mathcal{H} \Phi + \mathcal{H} \Phi' + \mathcal{H} \Phi'\right) \delta_{ij}, \\
& \quad \quad + \frac{15}{2} \mathcal{H} \Phi' + \frac{1}{2} \mathcal{H} \Phi' + \frac{1}{2} \mathcal{H} \Phi' + \frac{1}{2} \mathcal{H} \Phi',
\end{align}

respectively. Here $\nu m \equiv \delta_{ij} \Phi m \Phi$. By analogy with Eingorn (2016), we consider the quantities $\delta \rho$ and $\nu m$ as “importing” the first order of smallness in the Einstein equations. In other words, metric corrections, which are generated by these sources themselves, are assigned the first order. Indeed, in the above-mentioned Equations (8) and (9) the quantities $\delta \rho$ and $\nu m$ (as well as $\Xi$ containing $\nu m$) on the rhs play the role of sources generating the first-order metric corrections $\Phi$ and $B$ forming the lhs. Therefore, we have omitted such terms as, for example, $\sim \delta \rho \Delta \Phi$ in Equation (34) since this term is much smaller than the summand $\sim \delta \rho \Phi$ in the same parentheses at all cosmological scales and would import the third order of smallness (see also Chisari & Zaldarriaga 2011, for similar reasoning). The established thorough separation of the first- and second-order summands in the Einstein equations is strongly corroborated in Section 3.4 below.

Once again we employ a helpful decomposition
\begin{align}
T^0_0 &= (T^0_0)^{(0)} + (T^0_0)^{(1)} + (T^0_0)^{(2)} + (T^0_0)^{(3)},
\end{align}

The only nonzero component with the superscript “(0)” is $(T^0_0)^{(0)} = \rho c^2 / a^2$. The components $(T^0_0)^{(1)}$ (or, by analogy, $(T^0_0)^{(2)}$) are constructed from the terms being linear in the quantities $\delta \rho$, $\nu m$, $\Phi$, $B$ (or $\Psi$, $B$):
\begin{align}
(T^0_0)^{(1)} &= \frac{c^2}{a^2} \delta \rho + \frac{3 \rho c^2}{a^3} \Phi, \\
(T^0_0)^{(2)} &= \frac{c^2}{a^2} B - \frac{c^2}{a^2} \nu m \nu m, \quad (T^0_0)^{(3)} = 0.
\end{align}
\[ (T^0_0)^2 = \frac{3\kappa c^2}{a^3} \Psi^2, \quad (T^0_i)^2 = \frac{\bar{T}c^2}{a^3} B^{(2)i}, \quad (T^i_j)^2 = 0. \] (39)

Finally, the components with the superscript “(II)” contain products of \( \delta \rho, \bar{\nu}, \Phi, \) or \( B \):

\[ (T^0_0)^{II} = \frac{3\kappa c^2}{a^3} \delta \rho \Phi + \frac{c^2}{2a^3} \bar{\nu} \bar{\nu}^2 + \frac{15\kappa c^2}{2a^3} \Phi^2 - \frac{\bar{T}c^2}{2a^3} B^2, \]
\[ (T^0_i)^{II} = \frac{c^2}{a^3} \delta \rho B_0 + \frac{\bar{T}c^2}{a^3} \Phi B_0 + \frac{c^2}{a^3} \bar{\nu} \bar{\nu}^\delta \Phi, \]
\[ (T^i_j)^{II} = \frac{c^2}{a^3} \bar{\nu} \bar{\nu}^\delta B_0 - \frac{c^2}{a^3} \bar{\nu} \bar{\nu}^\delta \bar{\nu}^\beta. \] (40)

It is worth mentioning that owing to the zero value of \((T^0_0)^{(I)}\),
the anisotropic stress vanishes in the first-order approximation.

This is the cogent reason for using the same designation \( \Phi \equiv \Phi^{(1)} = \Psi^{(1)} \) for the equal first-order
combinations \( \Phi^{(1)} \) and \( \Psi^{(1)} \) from the very outset. Nevertheless, owing to the nonzero values of \((G^i_j)^{(II)}\)
and \((T^0_i)^{(II)}\) (or, more precisely, their combinations \(Q_{0\beta}\) introduced below),
the anisotropic stress does not vanish in the second-order approximation. Thus,
generally speaking, the second-order scalar perturbations \( \Phi^{(2)} \)
and \( \Psi^{(2)} \) are unequal: \( \Phi^{(2)} \neq \Psi^{(2)} \).

Let us conclude this subsection by introducing the promised helpful
combinations:

\[ Q_a = \kappa (T^0_0)^{(II)} - (G^i_i)^{(II)} \] (41)

and presenting their explicit expressions without concealing
anything, even despite the quite cumbersome form of some of them:

\[ Q_{00} = \frac{\kappa c^2}{2a^3} \bar{\nu} \bar{\nu}^2 - \frac{3\kappa^2 c^4}{4a^3} \bar{\nu}^2 + \frac{6\kappa c^2}{a^3} \mathcal{H} \Xi \Phi - \left( \frac{3\kappa \bar{T}c^2}{2a^3} + \frac{15\kappa c^2}{2a^3} \Phi^2 + \left( \frac{\kappa \bar{T}c^2}{2a^3} + \frac{3\mathcal{H}^2}{a^3} \right) B^2 \right) - \frac{2}{a^3} \Phi \Delta \Phi - \frac{3}{a^3} (\nabla \Phi)^2 + \frac{2}{a^3} \mathcal{H} B \nabla \Phi - \frac{1}{4a^2} B \Delta B \]
\[ + \frac{1}{8a^2} \nabla (B \nabla \mathcal{B}) - \frac{\kappa c^2}{a^3} \bar{\nu} \bar{\nu} B_i. \] (42)

\[ Q_{0\beta} = \frac{\kappa c^2}{a^3} \delta \rho B_0 + \frac{\kappa \bar{T}c^2}{a^3} \Phi B_0 + \frac{\kappa c^2}{a^3} \bar{\nu} \bar{\nu}^\delta \Phi - \frac{1}{a^2} B \Delta \Phi \]
\[ - \frac{1}{a^2} \mathcal{H} \frac{\partial (B^2)}{\partial x^0} + \frac{5}{a^2} \mathcal{H} \frac{\partial (\Phi^2)}{\partial x^0} + \frac{\kappa c^2}{a^3} \frac{\partial \Phi}{\partial x^0} - \frac{1}{a^2} \frac{\partial \Phi}{\partial x^0} B_i \frac{\partial B^2}{\partial x^0 \partial x^3}, \] (43)

\[ Q_{11} = -\frac{\kappa c^2}{a^3} \bar{\nu} \bar{\nu}^2 - \frac{\kappa^2 c^4}{4a^3} \bar{\nu}^2 + \frac{3\kappa c^2}{a^3} \mathcal{H} \Xi \Phi + \left( \frac{4\kappa \bar{T}c^2}{a^3} - \frac{5}{a^3} \mathcal{H}^2 \right) \Phi^2 - \frac{1}{a^3} \mathcal{H}^2 B^2 - \frac{4}{a^3} \Phi \Delta \Phi + \frac{4}{a^3} \frac{\partial (\Phi \Phi)}{\partial x^0} - 3 \frac{2}{a^3} \nabla (\nabla \Phi)^2 + \frac{2}{a^3} \frac{\partial (\Phi \Phi)}{\partial x^0} - \frac{2}{a^3} \mathcal{H} B_i \frac{\partial \Phi}{\partial x^0} \]
\[- \frac{3}{4a^2} B \Delta B + \frac{3}{2a^2} B_i \frac{\partial B_i}{\partial x^0} + \frac{1}{8a^2} \Delta (B \nabla \mathcal{B}) - \frac{1}{4a^2} \Delta (B \nabla \mathcal{B}) \]
\[- \frac{1}{a^3} \frac{\partial \Phi}{\partial x^0} - \frac{1}{a^3} \frac{\partial \Phi}{\partial x^0} B_i + \frac{2}{a^2} \frac{\partial (\Phi \Phi)}{\partial x^0} + \left[ x \rightarrow y \right], \] (45)

\[ Q_{22} \text{ and } Q_{33} \text{ are analogous,} \]

\[ Q_{12} = \frac{\kappa c^2}{2a^3} \bar{\nu} \bar{\nu} B_i - \frac{\kappa c^2}{2a^3} \bar{\nu} \bar{\nu} \bar{B}_i - \frac{1}{4a^2} B_i \frac{\partial \Phi}{\partial x^0} + \frac{1}{4a^2} B_i \frac{\partial (\Phi \Phi)}{\partial x^0} \]
\[ - \frac{1}{4a^2} B_i \frac{\partial (\Phi \Phi)}{\partial x^0} - \frac{1}{2a^2} B_i \frac{\partial \Phi}{\partial x^0} + \frac{1}{2a^2} B_i \frac{\partial \Phi}{\partial x^0} \]
\[ - \frac{1}{2a^2} B_i \frac{\partial \Phi}{\partial x^0} + \frac{1}{2a^2} B_i \frac{\partial \Phi}{\partial x^0} + \frac{1}{2a^2} B_i \frac{\partial \Phi}{\partial x^0} - \frac{1}{2a^2} B_i \frac{\partial \Phi}{\partial x^0} \]
\[ - \frac{1}{a^2} \mathcal{H} B_i \frac{\partial \Phi}{\partial x^0} + \frac{1}{a^2} \mathcal{H} B_i \frac{\partial \Phi}{\partial x^0} + \frac{2}{a^2} \frac{\partial (\Phi \Phi)}{\partial x^0} + \frac{\kappa c^2}{2a^2} \frac{\partial \Phi}{\partial x^0} + \frac{\kappa c^2}{2a^2} \frac{\partial \Phi}{\partial x^0} \]

where \([x \rightarrow y]\) stands for exactly the same terms with the
occurring-everywhere replacement of \( x \) by \( y \) and vice versa,
and \( Q_{13} \) and \( Q_{23} \) are analogous. The enumerated formulæ have been
derived, in particular, through the instrumentality of Equations (22) and (40) and equations from Section 2,
including the following direct consequence of Equation (2):

\[ \mathcal{H}^2 = \mathcal{H}' = \kappa a^2 c^2 / 2 = \kappa \bar{T}c^2 / (2a). \] From Equations (43) and (44) we get

\[ \frac{\partial Q_{0\beta}}{\partial x^0} = \frac{\partial Q_{01}}{\partial x} + \frac{\partial Q_{02}}{\partial y} + \frac{\partial Q_{03}}{\partial z}, \]

\[ = \Delta \left( \frac{\kappa c^2}{a^3} \Xi \Phi - \frac{1}{a^3} \mathcal{H} B^2 + \frac{5}{a^3} \mathcal{H} \Phi \Phi \right) \]
\[ - \frac{\kappa c^2}{a^3} \bar{\nu} \bar{\nu} (\nabla \Phi) + \frac{\kappa c^2}{a^3} \nabla \Phi \nabla \Xi + \frac{3\kappa c^2}{a^3} \mathcal{H} (B \nabla \Xi), \] (46)
respectively. Hereinafter summation over repeated Greek subscripts is implied without superfluousdecoding. It should also be noted that naturally $Q_{ik} = Q_{ki}$, in complete agreement with the symmetry inherent in the Einstein equations, which we write down in the very next subsection.

3.2. Scalar, Vector, and Tensor Sectors

Substituting Equations (17) and (37) into Equation (16) with due account taken of Equation (41), we immediately get the following:

1. 00-component:

$$
\frac{\partial Q_{00}}{\partial x^0} = 0,
$$

2. 0β-components:

$$
\left(\frac{1}{2\alpha^2} \partial_{\beta}\partial_{\alpha} + \frac{1}{\alpha^2} \partial_{\alpha} + \frac{1}{\alpha^2} \partial_{\beta}\right) = 0,
$$

3. 11-component:

$$
\frac{\partial Q_{11}}{\partial x^1} = 0,
$$

4. 12-component:

$$
\frac{\partial Q_{12}}{\partial x^2} = 0,
$$

All these equations clearly demonstrate the second-order scalar, vector, and tensor perturbations (represented by $\Phi^{(2)}$, $\Psi^{(2)}$, $B^{(2)}$, $h_{\alpha\beta}$, respectively) do not mix (Baumann et al. 2012) and are generated, in particular, by the quadratic combinations of the first-order scalar and tensor perturbations $\Phi$, $B$.

Now it is just the right time for the standard “scalar–vector–tensor” decomposition of $Q_{\alpha\beta}$:

$$
Q_{\alpha\beta} = Q^{(0)}\delta_{\alpha\beta} + \frac{\partial^2 Q^{(S)}}{\partial x^\alpha \partial x^\beta} + \frac{\partial Q^{(V)}}{\partial x^\alpha} + \frac{\partial Q^{(T)}}{\partial x^\alpha} + Q^{(T)}_{\alpha\beta},
$$

where $Q^{(0)}$ and $Q^{(S)}$ describe the scalar sector, while $Q^{(V)}$ and $Q^{(T)}_{\alpha\beta}$ describe the vector and tensor sectors, respectively, and satisfy the corresponding conditions:

$$
\frac{\partial Q^{(V)}}{\partial x^\alpha} = 0, \quad \frac{\partial Q^{(T)}}{\partial x^\alpha} = 0, \quad Q^{(T)}_{\alpha\beta} = 0.
$$

According to Equations (52) and (53), the introduced functions $Q^{(0)}$, $Q^{(S)}$, and $Q^{(V)}$ can be determined as solutions of the corresponding equations

$$
\Delta Q^{(0)} = \frac{1}{2} \Delta Q^{(0)}_{\alpha\beta} - \frac{1}{2} \frac{\partial^2 Q^{(0)}}{\partial x^\alpha \partial x^\beta},
$$

$$
\Delta Q^{(S)} = -\frac{1}{2} \Delta Q^{(S)}_{\alpha\beta} + \frac{3}{2} \frac{\partial^2 Q^{(S)}}{\partial x^\alpha \partial x^\beta},
$$

$$
\Delta Q^{(V)} = \Delta Q^{(V)}_{\alpha\beta} - \frac{1}{2} \frac{\partial^2 Q^{(V)}}{\partial x^\alpha \partial x^\beta},
$$

and then the remaining unknown functions $Q^{(T)}_{\alpha\beta}$ can be easily found from Equation (52). They act as the sole sources of gravitational waves (see the rhs of Equation (59) for $h_{\alpha\beta}$ below).

Let us synchronously perform the standard “scalar–vector” decomposition of $Q_{0\alpha}$:

$$
Q_{0\alpha} = \frac{\partial Q^{(0)}}{\partial x^\alpha} + Q^{(0)}_{\alpha\beta},
$$

where $Q^{(0)}$ and $Q^{(0)}_{\alpha\beta}$ denote the scalar and vector contributions, respectively. According to Equation (56), they can be determined as solutions of the corresponding equations

$$
\Delta Q^{(0)} = \frac{\partial Q_{0\alpha}}{\partial x^\alpha}, \quad \Delta Q^{(0)}_{\alpha\beta} = \frac{\partial^2 Q_{0\alpha}}{\partial x^\alpha \partial x^\beta},
$$

22- and 33-components are similar;
Finally, in order to abridge the notation, let us introduce the handy mixed-order quantities
\[ \Phi^{(12)} \equiv \Phi + \Phi^{(2)}, \quad \Psi^{(12)} \equiv \Phi + \Psi^{(2)}, \quad B^{(12)} \equiv B + B^{(2)}. \] (58)

Then the Einstein Equations (48)–(51) can be eventually rewritten as follows:

1. tensor sector:
\[ h^{(n)}_{\alpha \beta} + 2\mathcal{H}h^{(n)}_{\alpha \beta} - \Delta h^{(n)}_{\alpha \beta} = 2a^2 Q^{(1)}_{\alpha \beta}, \] (59)

2. vector sector:
\[ \Delta B^{(12)} - \frac{2\kappa \mathcal{P}^2}{a}B^{(12)} = -\frac{2\kappa \mathcal{P}^2}{a}(\rho \mathcal{P} - \nabla \Xi) + 2a^2 Q^{(1)}, \] (60)

\[ (B^{(12)})' + 2\mathcal{H}(B^{(12)}) = -2a^2 Q^{(V)}; \] (61)

3. scalar sector:
\[ \Phi^{(12)} - \Psi^{(12)} = a^2 Q^{(S)}, \] (62)

\[ \Delta \Psi^{(12)} - \frac{3\kappa \mathcal{P}^2}{2a}\Psi^{(12)} - 3\mathcal{H}[\Psi^{(12)} + \mathcal{H}(\Psi^{(12)})] = \frac{\kappa \mathcal{P}^2}{2a} \delta \rho + \frac{a^2}{2} Q_{00}, \] (63)

\[ (\Psi^{(12)})' + \mathcal{H}(\Psi^{(12)}) = -\frac{\kappa \mathcal{P}^2}{2a} \Xi + \frac{a^2}{2} Q^{(0)}; \] (64)

\[ \Delta \Psi^{(12)} - \Delta \Phi^{(12)} - 2(\Psi^{(12)})' - 4\mathcal{H}(\Phi^{(12)})' = -2\mathcal{H}(\Phi^{(12)})' - 2(2\mathcal{H}' + \mathcal{H}^2)\Phi^{(12)} = a^2 Q^{(0)}. \] (65)

Substituting Equation (64) into Equation (63), we get
\[ \Delta \Psi^{(12)} - \frac{3\kappa \mathcal{P}^2}{2a}\Psi^{(12)} = \frac{\kappa \mathcal{P}^2}{2a} \delta \rho - \frac{3\kappa \mathcal{P}^2}{2a} \mathcal{H} \Xi + \frac{a^2}{2} Q_{00} + \frac{3a^2}{2} \mathcal{H} Q^{(0)}. \] (66)

Recalling Equation (58) along with Equations (8) and (9), we reduce Equations (60), (62), and (66) to the equations
\[ \Delta B^{(2)} - \frac{2\kappa \mathcal{P}^2}{a}B^{(2)} = 2a^2 Q^{(1)}, \] (67)

\[ \Phi^{(2)} - \Psi^{(2)} = a^2 Q^{(S)}, \] (68)

\[ \Delta \Psi^{(2)} - \frac{3\kappa \mathcal{P}^2}{2a}\Psi^{(2)} = \frac{a^2}{2} Q_{00} + \frac{3a^2}{2} \mathcal{H} Q^{(0)}. \] (69)

Thus, we have derived the “master” Equations (59), (67), and (68) for the sought-for second-order cosmological perturbations. In the next subsection we show that the remaining “non-master” Equations (61), (64), and (65) are satisfied automatically provided that one takes advantage of the equations of motion governing the particle dynamics.

### 3.3. Verification of Equations

Without going into detail, let us outline the proof that the scalar sector Equations (64) and (65) containing temporal derivatives are really satisfied. In the first place, one finds a derivative of Equation (66) with respect to \( \eta \) and further expresses \((\Psi^{(12)})'\) from Equation (64). As a result,

\[ \frac{3\kappa \mathcal{P}^2}{2a} \mathcal{H} \Xi' = \mathcal{H}(\Delta \Phi^{(12)} - \frac{3\kappa \mathcal{P}^2}{2a} \mathcal{H} \Phi^{(12)}) \]

\[ - \frac{3\kappa \mathcal{P}^2}{2a} \mathcal{H} \Psi^{(12)} - \frac{\kappa \mathcal{P}^2}{2a} \mathcal{H} \delta \rho - \frac{a^2}{2} \Delta Q^{(1)} + \frac{3a^2}{2} (a^2 Q^{(0)})' + \frac{1}{2}(a^2 Q_{00})'. \] (69)

In the second place, one substitutes the expression for \((\Psi^{(12)})'\) from Equation (64) into Equation (65). As a result,

\[ \frac{3\kappa \mathcal{P}^2}{2a} \mathcal{H} \Xi' = \frac{3}{2} \mathcal{H}(\Delta \Phi^{(12)} - \frac{3}{2} \mathcal{H} \Delta \Psi^{(12)}) \]

\[ - 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}')\Phi^{(12)} - \frac{3\kappa \mathcal{P}^2}{2a} \mathcal{H} \Xi + 3a^2 \mathcal{H}^2 Q^{(1)} + \frac{3}{2} \mathcal{H}(a^2 Q^{(0)})' + \frac{3a^2}{2} \mathcal{H} Q^{(0)}. \] (70)

Therefore, it is enough to show that the rhs of Equation (69) is really equal to the rhs of Equation (70), since the lhs of these equations is the same, and then to prove either equation. We have successfully coped with both these onerous tasks. Equating the rhs of each, after lengthy calculation one eventually arrives at an identity. The following auxiliary formulae should be used on the way:

\[ \rho \mathcal{P}' \equiv \sum_n m_n \delta(r - r_n) \nu_n^2 = -\mathcal{H} \rho \mathcal{P} - \mathcal{P} \nabla \Phi - \mathcal{P} \mathcal{H} B \] (71)

in the first-order approximation (Eingorn 2016);

\[ (\rho \mathcal{P})' \equiv \left( \sum_n m_n \delta(r - r_n) \nu_n^2 \right)' = 2 \sum_n m_n \delta(r - r_n) \nu_n \nu_n' \]

\[ = -2\mathcal{H}(\rho \mathcal{P}^2 - 2\rho \Phi \nabla \Phi - \mathcal{H} \rho \mathcal{P} B \] (72)

within the adopted accuracy. The underlying equations of motion of the \( n \)-th particle have the form \( \mathcal{P} \nu_n = -\mathcal{H} \nu_n - \nabla \Phi - \mathcal{H} B \) (Eingorn 2016).

When the desired identity is achieved, it is enough to prove, for instance, the correctness of Equation (70). Now the accuracy of Equation (71) is insufficient, and it is necessary to take advantage of the spacetime interval for the \( n \)-th particle:

\[ ds_n = a (1 + 2\Phi + 2\Phi^{(2)} + 2(B_n + B_n^{(2)}) \nu_n) \]

\[ + \left[ (-1 + 2\Phi + 2\Psi^{(2)}) \delta_{\alpha \beta} + h_{\alpha \beta} \right] \nu_n^\alpha \nu_n^\beta 1/2 \, dt, \] (73)

where the metric corrections are computed at the point \( r = r_n \) and, as usual, do not include the divergent contributions from the considered particle itself. For the sake of simplicity we can confine ourselves to those terms in Equation (70) that are not quadratic in particle velocities, and then the Lagrange equations of motion have the form

\[ \mathcal{P} \nu_n = -\mathcal{H} \nu_n - \nabla \Phi + \mathcal{H} B + B' - 3\mathcal{H} \nu_n \Phi - \nabla(\Psi^2) - \nabla(\Phi^2) - \mathcal{H}(\Phi B + \mathcal{H} B^{(2)}) + (B^{(2)})'. \] (74)
Multiplication of Equation (74) by $\rho_n$ with subsequent summation over $n$ gives
\[
\rho^n = -\mathcal{H}\rho - \nabla\Phi - \delta\rho\nabla\Phi + \nabla^\gamma\nabla\Phi + \mathcal{H}\delta\rho B - \mathcal{H}\rho B + \mathcal{B}^2 - 3\mathcal{H}\rho\Phi + \nabla\Phi B + \mathcal{H}(\mathcal{B}^{(2)} + \mathcal{P}(\mathcal{B}^{(2)}))',
\]
where the last equation of triplet (11) has been used to replace the summand $\delta\rho B$ by $-2\mathcal{H}\delta\rho B$. We have also dropped all terms that would import the third order of smallness in the Einstein equations. If one additionally omits the terms importing the second order, then Equation (75) is reduced exactly to Equation (71).

Being armed with Equation (75), after exhausting calculation one turns Equation (70) into an identity. Thus, both initial non-master scalar sector Equations (64) and (65) are satisfied. The same applies to Equation (61). Indeed, suffice it to demonstrate that
\[
\Delta[(\mathcal{B}^{(12)})' + 2\mathcal{H}(\mathcal{B}^{(12)} + 2a^2Q^V)] - \frac{2\kappa\rho_0^2}{a}[(\mathcal{B}^{(12)})' + 2\mathcal{H}(\mathcal{B}^{(12)} + 2a^2Q^V)] = 0.
\]
Recalling Equation (60), one can reduce Equation (76) to the following equation:
\[
-\frac{2\kappa\rho_0^2}{a}\mathcal{H}(\mathcal{B}^{(12)}) - \frac{2\kappa^2}{a}\nabla\rho + \mathcal{H}(\mathcal{B}^{(12)})' = 0.
\]
In the framework of the above-mentioned simplification, which is without products of velocities, substitution of Equations (70) and (75) into Equation (77) eventually leads to the desired identity. Thus, the initial non-master vector sector Equation (61) is satisfied as well. Obviously, the same applies to gauge conditions (14) and (15) since exactly the same gauge conditions hold true for the corresponding rhs of Equations (67) and (59).

3.4. Self-consistent Separation of Summands

In the previous subsection we have demonstrated that the functions $\Phi^{(2)}$, $\Psi^{(2)}$, $\mathcal{B}^{(2)}$, $h_{00}$ determined as solutions of Equations (68), (67), and (59), respectively, satisfy all Einstein equations in the second-order approximation. The following relevant question arises: Is the undertaken separation of the first- and second-order terms well grounded and self-consistent? In other words, do we correctly and logically assign orders to different summands?

Of course, the answer is affirmative. As an illustrative example, let us single out two types of terms in Equation (63), namely, those that are either present or absent in the corresponding equation in the framework of the first-order approximation
\[
\Delta\Phi - \frac{3\kappa\rho_0^2}{2a}\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) + \mathcal{H}\nabla B = \frac{\kappa^2}{2a}\delta\rho,
\]
which is equivalent to Equation (8) in view of gauge condition (4) and the first equation of triplet (11). The vanishing last term on the lhs of Equation (78) is momentarily reinstated since it can be considered as being initially present in the corresponding 00-component of the Einstein equations as a part of $(G_0)^{(1)}(\mathcal{B})$ before applying gauge condition (4).

We designate the first derivatives with respect to each comoving spatial coordinate and conformal time as $1/L$ and $1/\tau$, respectively, as well as ascribe the orders of smallness $\epsilon$ and $\epsilon^2$ to the first- and second-order metric corrections. Then, for instance, $\delta\Phi \sim \epsilon/L^2$ while $(\Psi^{(2)})' \sim \epsilon^2/\tau$. As a result, taking into account the explicit expression (42) for $Q_{00}$, we have six terms of the first type (present in Equation (63) as well as in Equation (78)), namely,
\[
\frac{1}{L^2}\epsilon, \frac{\kappa\rho_0^2}{a}\epsilon, \frac{\mathcal{H}}{L}\epsilon, \frac{\mathcal{H}^2\epsilon}{a}, \frac{\kappa^2}{a}\delta\rho, \frac{\mathcal{H}}{L}\epsilon,
\]
and nine terms of the second type (present in Equation (63), but absent in Equation (78)):
\[
\frac{1}{L^2}\epsilon^2, \frac{\kappa\rho_0^2}{a}\epsilon^2, \frac{\mathcal{H}}{L}\epsilon^2, \frac{\mathcal{H}^2\epsilon^2}{a}, \frac{\kappa^2}{a}\epsilon^2, \frac{\mathcal{H}^2\epsilon^2}{a^2}.
\]
We distinguish between the coefficients $\kappa\rho_0^2/a$ and $\mathcal{H}^2$: they evolve synchronously during the matter-dominated stage of the universe evolution, but asynchronously during the $\Lambda$-dominated stage. The essence of the perturbative computation lies in the fact that for each term of the second type in Equation (80) there must exist a counterpart of the first type in Equation (79), such that their ratio is of the order of smallness $\epsilon$. This is what we intend to confirm right now.

It can be easily seen that the first five terms in Equation (80), divided by the corresponding first five terms in Equation (79), give precisely the order of smallness $\epsilon$. Obviously, the same applies to the sixth terms. Indeed, $\rho_0^2 \ll |\delta\rho|$ at arbitrary distances (Chisari & Zaldarriaga 2011), and the helpful estimate $\mathcal{H}^2$ by $\Phi$ by $\rho$ (see Baumann et al. 2012) holds true. Hence, $\rho_0^2/|\delta\rho| \sim \Phi$. Further, since $\kappa\rho_0^2/\epsilon \sim \Phi + \mathcal{H}\Phi'$ (Equation (11)) and $\kappa\rho_0^2/\rho^0 \sim \Delta B - 2\kappa\rho_0^2/\mathcal{B}$ (Equation (9)), the seventh term in Equation (80) is reduced to a combination of $(\mathcal{H}/\tau)^2\epsilon$ and $\mathcal{H}^2\epsilon^2$, while the eighth term is reduced to a combination of $\epsilon^2/L^2$ and $(\kappa\rho_0^2/\epsilon^2)\epsilon^2$. This quartet is already present in Equation (80); hence, the seventh and eighth summands add nothing new. Similarly, the last term $\kappa\rho_0^2/\epsilon^2 \sim \Phi$ (Equation (11)) and $\kappa\rho_0^2/\rho^0 \sim \Delta B - 2\kappa\rho_0^2/\mathcal{B}$ (Equation (2)); hence, in its turn, $\epsilon^2/\tau^2$ may be treated as a combination of $(\mathcal{H}/\tau)^2\epsilon^2$, $\mathcal{H}^2\epsilon^2$, and $(\kappa\rho_0^2/a)\epsilon^2$. Consequently, the last summand in Equation (80) also adds nothing new to those terms that are already available in the collection.

Thus, we have shown that the elaborated perturbative scheme is valid. This scheme elegantly resolves the formidable challenge briefly discussed in the introductory part of Clarkson & Umeh (2011): at any cosmological scale for each summand in the equations for the second-order metric corrections there exists a much larger counterpart in the corresponding equations for the first-order metric corrections. Therefore, in particular, the situation when magnitudes of $\Phi^{(2)}$ and $\Phi$ are comparable is really improbable. Quite the contrary, the inequality
\(|\Phi^{(2)}| \ll |\Phi|\) may be expected to occur everywhere, as it certainly should be in the framework of a self-consistent perturbation theory.

3.5. Minkowski Background Limit

In this subsection, again for the sake of simplicity, we momentarily ignore all terms being quadratic in particle velocities and concentrate on the Minkowski background limit: the scale factor \(a\) is now just a constant, \(H = 0, \dot{p} = 0\). Then, according to Eingorn (2016),

\[
\Phi = -\frac{\kappa c^2}{8\pi a} \sum_n m_n \left[ \frac{\nabla_n}{|r - r_n|} \right],
\]

(81)

\[
B = \frac{\kappa c^2}{4\pi a} \sum_n m_n \left[ \frac{\nabla_n \tilde{\nu}_n(r - r_n)}{|r - r_n|^2} \right].
\]

(82)

The sum of Newtonian potentials (81) is a solution of the standard Poisson equation

\[
\Delta \Phi = \frac{\kappa c^2}{2a} \rho = \frac{\kappa c^2}{2a} \sum_n m_n \delta(r - r_n).
\]

(83)

At the same time, from the second equation in Equation (68) and Equation (42) we get

\[
\Delta \Psi^{(2)} = -\Phi \Delta \Phi - \frac{3}{2} (\nabla \Phi)^2 = -\frac{3}{4} \Delta (\Phi^2) + \frac{\kappa c^2}{4a} \rho \Phi,
\]

(84)

where an evident relationship \(2(\nabla \Phi)^2 = \Delta (\Phi^2) - 2\Phi \Delta \Phi\) has been used along with Equation (83). Hence,

\[
\Psi^{(2)} = -\frac{3}{4} \Phi^2 - \frac{\kappa c^2}{16\pi a} \sum_n m_n \Phi|_{r = r_n}.
\]

(85)

After lengthy calculation, being based on Equations (44), (45), (47), and (54), one also finds

\[
Q^{(0)} = \frac{7}{4a^2} \Phi^2 - \frac{\kappa c^2}{16\pi a} \sum_n m_n \Phi|_{r = r_n}
\]

\[+ \frac{3\kappa c^2}{16\pi a^3} \sum_n m_n \left( \frac{r - r_n}{|r - r_n|} \right)(\nabla \Phi)|_{r = r_n}.
\]

(86)

Substitution of Equations (85) and (86) into the first line in Equation (68) gives

\[
\Phi^{(2)} = \frac{\kappa c^2}{8\pi a} \sum_n \left[ m_n \nabla_n \tilde{\nu}_n(r - r_n) \right]
\]

\[+ \frac{3\kappa c^2}{16\pi a^3} \sum_n m_n \left( \frac{r - r_n}{|r - r_n|} \right)(\nabla \Phi)|_{r = r_n}.
\]

(87)

As usual, the gravitational field produced by the \(n\)th particle is excluded from the factors \(\Phi|_{r = r_n}\) and \((\nabla \Phi)|_{r = r_n}\).

Let us compare solutions (82) and (87) with the corresponding adapted expressions

\[
B_{\text{LL}} = \frac{\kappa c^2}{16\pi a} \sum_n \left[ m_n \tilde{\nu}_n(r - r_n) \right],
\]

(88)

\[
\Phi^{(2)}_{\text{LL}} = \Phi^2 - \frac{\kappa c^2}{8\pi a} \sum_n \left[ m_n \Phi|_{r = r_n} \right],
\]

(89)

which are equivalent to those from the textbook by Landau & Lifshitz (2000, see Equations (106.15) and (106.13) therein). Here we ignore velocities squared as we arranged before. Of course, neither does Equation (82) coincide with Equation (88), nor does Equation (87) coincide with Equation (89). As pointed out by Eingorn (2016), the reason lies in the fact that our gauge conditions differ from those applied by Landau & Lifshitz (2000). Therefore, in order to reach agreement with this textbook, suffice it to find such a transformation of coordinates that would establish desired linkage. Apparently, it is enough to transform only the temporal coordinate: \(\eta \rightarrow \eta - A(\eta, r)\), and then \(\Phi^{(2)} \rightarrow \Phi^{(2)} + A'\) and \(B \rightarrow B + \nabla A\). Demanding that

\[
\Phi^{(2)} + A' = \Phi^{(2)}_{\text{LL}}, \quad B + \nabla A = B_{\text{LL}},
\]

(90)

with the help of Equations (82) and (87)-(89) we get

\[
A' = -\frac{3\kappa c^2}{16\pi a} \sum_n \left[ \frac{m_n}{|r - r_n|} \right], \quad \nabla A = \frac{3\kappa c^2}{16\pi a} \sum_n \left[ \frac{m_n \tilde{\nu}_n(r - r_n)}{|r - r_n|^3} \right].
\]

(91)

Action of \(\nabla\) on both sides of Equation (91) gives

\[
\nabla A' = -\frac{3\kappa c^2}{16\pi a} \sum_n \left[ \frac{m_n}{|r - r_n|} \right], \quad \nabla A = \frac{3\kappa c^2}{16\pi a} \sum_n \left[ \frac{m_n \tilde{\nu}_n(r - r_n)}{|r - r_n|^3} \right],
\]

(92)

and exactly the same result follows also from Equation (92). This incontestable fact ensures existence of the function \(A(\eta, r)\) and, consequently, of the above-mentioned coordinate transformation. Thus, agreement with Landau & Lifshitz (2000) has been reached.

4. Averaging Initiatives on the Eve of Cosmological Back-reaction Estimation

In view of the predictably zero average values of the first-order metric corrections (Eingorn 2016), the computation of the cosmological back-reaction effects should be based on the second-order perturbation theory. Without pretending to an exhaustive study, let us perform the Euclidean averaging, or smoothing (Clarkson et al. 2011), of the 00-component of Einstein equations (Equation (48)), multiplied by \(a^2/2\), and the sum of 11-, 22- and 33-components (see Equation (50)), multiplied by \((-a^2/6)\). We gather all terms containing \(\overline{\Psi^{(2)}}, \overline{\Phi^{(2)}}\) and their temporal derivatives on the lshs, while the other averaged contributions are gathered on the rhs:

\[-3\mathcal{H} \overline{\Psi^{(2)}} + 3\mathcal{H}^2 \overline{\Phi^{(2)}} - 3\frac{\kappa c^2}{2a} \overline{\Psi^{(2)}}\]

\[= \frac{1}{2} a^2 \overline{\mathcal{Q}_{00}} = \frac{1}{2} \kappa a^2 \overline{\mathcal{P}\overline{\Phi}},\]

(94)

\[\overline{\Psi^{(2)}} + \mathcal{H}(2 \overline{\Psi^{(2)}} + \overline{\Phi^{(2)}})' + (2\mathcal{H}' + \mathcal{H}^2) \overline{\Phi^{(2)}}\]

\[= -\frac{1}{6} a^2 \overline{\mathcal{Q}_{00}} = \frac{1}{2} \kappa a^2 \overline{\mathcal{P}\overline{\Phi}},\]

(95)

Here the overline indicates integrating over a comoving volume \(V\) and dividing by this volume in the limit of the infinite
integration domain ($V \rightarrow +\infty$). In addition, we have introduced the effective average energy density $\tilde{\varepsilon}^{(II)}(\eta)$ and pressure $\tilde{p}^{(II)}(\eta)$:

$$\kappa \tilde{\varepsilon}^{(II)} = \frac{\kappa c^2}{2a^2 \rho_b} - \frac{3 k^2 c^4}{4 a^4} \Phi^2 + \frac{6 \kappa c^2}{a^3} \Phi \tilde{\Phi}$$

$$- \left( \frac{3 \kappa c^2}{2 a^3} + \frac{15}{a^2} \Phi^2 \right) \Phi^2 + \left( \frac{12 \kappa c^2}{2 a^3} + \frac{3}{a^2} \Phi^2 \right) \Phi^2 - \frac{2 \kappa c^2}{a^3} \rho_b \Phi^2 - \frac{2}{a^2} \Phi \Delta \Phi - \frac{3}{a^2} (\nabla \Phi)^2 - \frac{1}{4 a^4} B \Delta B - \frac{\kappa c^2}{a^3} \rho_b B,$$

$$- \frac{2}{a^2} \Phi \Delta \Phi - \frac{3}{a^2} (\nabla \Phi)^2 - \frac{1}{4 a^4} B \Delta B - \frac{\kappa c^2}{a^3} \rho_b B,$$

(96)

$$\kappa \tilde{\varepsilon}^{(II)} \equiv \frac{\kappa c^2}{2a^2 \rho_b} + \frac{\kappa c^2}{2a^2 \rho_b} - \frac{3 k^2 c^4}{4 a^4} \Phi^2 + \frac{9 \kappa c^2}{2a^3} \Phi \tilde{\Phi}$$

$$+ \frac{15}{a^2} \Phi^2 \Phi^2 + \frac{3}{a^2} \Phi^2 \Phi^2 - \frac{\kappa c^2}{2a^3} \rho_b \Phi^2,$$

(98)

$$\kappa \tilde{p}^{(II)} = \frac{\kappa c^2}{3a^3 \rho_b} + \frac{\kappa c^2}{2a^2 \rho_b} + \frac{\kappa c^2}{2a^2 \rho_b} + \frac{3 k^2 c^4}{4 a^4} \Phi^2$$

$$- \frac{7 \kappa c^2}{a^3} \Phi \tilde{\Phi} - \frac{7 \kappa c^2}{a^3} \Phi \tilde{\Phi} - \frac{5}{a^2} \Phi^2 \Phi^2$$

$$+ \left( \frac{5 \kappa c^2}{a^3} + \frac{1}{a^2} \Phi^2 \right) \Phi^2 - \frac{\kappa c^2}{a^3} \rho_b B,$$

(99)

where explicit expressions (42) for $Q_{00}$ and (47) for $Q_{0a}$ have been used. Replacing $(\nabla \Phi)^2$ by $-\Phi \Delta \Phi$ and expressing $\Delta \Phi$ and $\Delta B$ from Equations (8) and (9), respectively, we rewrite Equations (96) and (97) in the more compact form:

$$\kappa \tilde{\varepsilon}^{(II)} = \frac{\kappa c^2}{2a^2 \rho_b} - \frac{3 k^2 c^4}{4 a^4} \Phi^2 + \frac{9 \kappa c^2}{2a^3} \Phi \tilde{\Phi}$$

$$- \frac{15}{a^2} \Phi^2 \Phi^2 + \frac{3}{a^2} \Phi^2 \Phi^2 - \frac{\kappa c^2}{2a^3} \rho_b \Phi^2,$$

(98)

$$\kappa \tilde{p}^{(II)} = \frac{\kappa c^2}{2a^2 \rho_b} + \frac{\kappa c^2}{2a^2 \rho_b} + \frac{\kappa c^2}{2a^2 \rho_b} + \frac{3 k^2 c^4}{4 a^4} \Phi^2$$

$$- \frac{7 \kappa c^2}{a^3} \Phi \tilde{\Phi} - \frac{7 \kappa c^2}{a^3} \Phi \tilde{\Phi} - \frac{5}{a^2} \Phi^2 \Phi^2$$

$$+ \left( \frac{5 \kappa c^2}{a^3} + \frac{1}{a^2} \Phi^2 \right) \Phi^2 - \frac{\kappa c^2}{a^3} \rho_b B.$$

(99)

Expressing $\tilde{\varepsilon}^{(II)}$ and $\tilde{p}^{(II)}$ from Equations (94) and (95), one can easily verify that these functions satisfy the standard conservation equation

$$(a^3 \tilde{\varepsilon}^{(II)})' + 3a^3 \tilde{p}^{(II)} = 0,$$

(100)

as it certainly should be. Hence, expressions (98) and (99) for the same functions must automatically satisfy this equation as well. This can be verified through the instrumentality of Equations (71) and (72), as well as equations from Section 2.

It is worth mentioning that if one keeps on the rhs of Equations (98) and (99) only the first two terms, which dominate at sufficiently small scales, and makes use of the relationship $\rho \Phi = -2 \rho_b^2$, which holds true for the virialized regions, then $\tilde{p}^{(II)} \rightarrow 0$ while $\tilde{\varepsilon}^{(II)} \rightarrow -[c^2/(2a^3)]\rho_b^2 \sim 1/a^3$. Thus, at virialized scales the effective pressure $\tilde{p}^{(II)}$ vanishes while the nonvanishing effective average energy density $\tilde{\varepsilon}^{(II)}$ brings us to a small time-independent renormalization of the corresponding background quantity $\tilde{\varepsilon}$, in full accord with Baumann et al. (2012) (see additionally Wetterich 2003, for earlier theoretical efforts and a “cosmic virial theorem”). The interpretation of the simulation outputs by Adamek et al. (2015) also suggests that “stable clustering” (implying virialized nonlinear structures) razes back-reaction from the cosmological battlefield. The underlying perturbative scheme advocated by Adamek et al. (2015) is compared with ours by Eingorn (2016). It is necessary to mention that this purely numerical scheme is characterized by the first-order accuracy for large enough distances and the second-order accuracy for sufficiently small distances, while the approach advocated in the current paper is characterized by the second-order accuracy everywhere and is fully analytical at least with respect to the first-order cosmological perturbations (5) and (6) and the sources (42)–(45) of the second-order ones.

It is noteworthy as well that the velocity-dependent summands can be easily distinguished from the velocity-independent ones in Equations (98) and (99), and there are only two types of contributions, which do not contain particle velocities: $\sim \rho_b \tilde{\varepsilon}^{(II)}$ and $\sim \rho_b \tilde{\varepsilon}^{(II)}$. Here $\Phi_0$ denotes the velocity-independent part of the first-order scalar perturbation $\Phi$ (Equation (5)), which is the sum of Yukawa potentials with the same interaction range $\lambda$ (up to an additive constant $1/3$):

$$\Phi_0 = \frac{1}{3} - \frac{\kappa c^2}{4 \pi a} \sum_n \frac{m_n}{|r - r_n|} \exp \left(-\frac{c a}{\rho_b} \frac{r - r_n}{\lambda} \right),$$

(101)

For illustration purposes, we compute both average quantities $\rho \Phi_0$ and $\rho \Phi_0^2$ analytically:

$$\rho \Phi_0 = \frac{1}{3} \rho_b \rho_\Phi = \frac{1}{3} \rho_b \rho \sum_n \frac{m_n m_k}{|r - r_n|} \exp \left(-\frac{c a}{\rho_b} \frac{r - r_n}{\lambda} \right),$$

(102)

$$\rho \Phi_0^2 = - \frac{1}{9} \rho_b \rho_\Phi \Phi_0 \rho_\Phi \Phi_0 = \frac{1}{9} \rho_b \rho \sum_n \frac{m_n m_k}{|r - r_n|} \exp \left(-\frac{c a}{\rho_b} \frac{r - r_n}{\lambda} \right).$$

(103)

It presents no difficulty to receive evidence that both these expressions tend to zero in the homogeneous mass distribution limit ($\sum m_n m_k \rightarrow p^2 \int d\rho_b d\eta$), as it certainly should be since $\Phi_0 = 0$ at any point in this test limit (Eingorn 2016). Reverting to Equations (98) and (99), we emphasize that collections of terms on the rhs of each may assist in the cosmological back-reaction estimation. We formulate the following quite feasible two-stage plan:

1. the launch of a new generation of cosmological N-body simulations based on the formalism developed by Eingorn (2016, see the equations of motion (3.6) therein);
2. the use of outputs of these simulations for the estimation of the effective average energy density $\tilde{\varepsilon}^{(II)}$ and pressure $\tilde{p}^{(II)}$, and the subsequent comparison with the background quantity $\bar{\varepsilon}$.

If the underlying inequalities $|\tilde{\varepsilon}^{(II)}| \ll \bar{\varepsilon}$ and $|\tilde{p}^{(II)}| \ll \bar{\varepsilon}$ become doubtful at any moment during the matter-dominated or $\Lambda$-dominated stages of the universe evolution, then this fact may serve as a sure sign of back-reaction significance and inappropriateness of the FLRW metric (1) and Friedmann Equations (2), with inevitable grave consequences. At the same time, if the inequalities being tested seem always unquestionable, this result by itself does not necessarily mean that back-reaction is insignificant, for the simple reason that we still rely on the initial assumption of the FLRW background existence and actually have no predictive power beyond. Nevertheless, if this key assumption is valid and $\bar{\varepsilon}$ is really much greater than...
perturbations have been derived. These equations are suitable at all spatial scales (naturally, except for the regions of strong gravitational fields in immediate proximity to such generators as black holes or neutron stars) and permit of nonlinear density contrasts.

(B) The Helmholtz Equations (67) and (68) incorporate exactly the same Yukawa interaction ranges as their corresponding counterparts (9) and (8) determining the first-order metric corrections.

(C) The constructed scheme passes three important corroborative tests:

1. we have verified that all Einstein equations are satisfied within the adopted accuracy along with the gauge conditions (Section 3.3);
2. we have confirmed the self-consistency of order assignments and the related expectation that the first-order metric corrections dominate over the second-order ones everywhere (Section 3.4);
3. in the Minkowski background limit the linkage with the textbook material has been established (Section 3.5).

(D) The highway to investigating the cosmological back-reaction effects beyond Newtonian gravitational physics has been outlined. The proposed relativistic simulations of the cosmic structure growth accompanied by the investigations of propagation of light and gravitational waves in the inhomogeneous universe would definitely assist in deepening and testing our knowledge of spacetime and the world’s filling material, including dark ingredients.

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