Abstract: Identities involving cyclic sums of terms composed from Jacobi elliptic functions evaluated at $p$ equally shifted points were recently found. The purpose of this paper is to re-express these cyclic identities in terms of ratios of Jacobi theta functions, since many physicists prefer using Jacobi theta functions rather than Jacobi elliptic functions.
1 Introduction.

Recently, we have discovered many new cyclic identities involving the Jacobi elliptic functions \( \text{sn}(x, m) \), \( \text{cn}(x, m) \), \( \text{dn}(x, m) \), where \( m \) is the elliptic modulus parameter \((0 \leq m \leq 1)\). These mathematical identities are described in references [1, 2, 3], henceforth referred to as I, II and IIa respectively. The functions \( \text{sn}(x, m) \), \( \text{cn}(x, m) \), \( \text{dn}(x, m) \) are doubly periodic functions with periods \((4K(m), i2K'(m))\), \((4K(m), 2K(m) + i2K'(m))\), \((2K(m), i4K'(m))\), respectively [4]. Here, \( K(m) \) denotes the complete elliptic integral of the first kind, and \( K'(m) = K(1 - m) \). The \( m = 0 \) limit gives \( K(0) = \pi/2 \) and trigonometric functions: \( \text{sn}(x, 0) = \sin x \), \( \text{cn}(x, 0) = \cos x \), \( \text{dn}(x, 0) = 1 \). The \( m \to 1 \) limit gives \( K(1) \to \infty \) and hyperbolic functions: \( \text{sn}(x, 1) \to \tanh x \), \( \text{cn}(x, 1) \to \text{sech} x \), \( \text{dn}(x, 1) \to \text{sech} x \). For simplicity, from now on we will not explicitly display the modulus parameter \( m \) as an argument of the Jacobi elliptic functions.

The cyclic identities discussed in I and II play an important role in showing that a kind of linear superposition is valid for many nonlinear differential equations of physical interest [5, 6]. In all identities, the arguments of the Jacobi functions in successive terms are separated by either \( 2K(m)/p \) or \( 4K(m)/p \), where \( p \) is an integer. Each \( p \)-point identity of rank \( r \) involves a cyclic homogeneous polynomial of degree \( r \) (in Jacobi elliptic functions with \( p \) equally spaced arguments) related to other cyclic homogeneous polynomials of degree \( r - 2 \) or smaller.

In II, it was shown that all our identities follow from four master identities. It was also shown that corresponding to every such identity, one can obtain new identities corresponding to pure imaginary shifts by multiples of \( i2K'(m)/p \) or \( i4K'(m)/p \), as well as identities corresponding to complex shifts by multiples of \( 2[K(m) + iK'(m)])/p \) or \( 4[K(m) + iK'(m)])/p \). Identities involving the nine secondary Jacobi elliptic functions [cd \((x, m)\), ns \((x, m)\), ds \((x, m)\), etc.] were also discussed. Furthermore, in II we gave results for several identities involving Weierstrass elliptic functions and ratios of Jacobi theta functions, both of which are intimately related with Jacobi elliptic functions [4].

In particular, in II we showed that given any identity for the Jacobi elliptic functions, we can immediately write down the corresponding identity for the ratio of Jacobi theta functions, since the ratio of any two Jacobi theta functions is also doubly periodic. In discussions with several physicists, it became apparent that they are more comfortable thinking in terms of Jacobi theta functions rather than Jacobi elliptic functions. In order for them to fully appreciate the power of various cyclic identities previously obtained in I, II and IIa, we decided to re-express these identities in terms of the ratios of
Jacobi theta functions.

In this context it is worth noting that the connection between the four Jacobi theta functions
\( \theta_1(z, \tau), \theta_2(z, \tau), \theta_3(z, \tau), \theta_4(z, \tau) \) and the Jacobi elliptic functions is given by [7]

\[
\begin{align*}
\text{sn}(u, m) &= \frac{1}{m^{1/4}} \frac{\theta_1(z, \tau)}{\theta_4(z, \tau)}, \\
\text{cn}(u, m) &= \frac{(1 - m)^{1/4}}{m^{1/4}} \frac{\theta_2(z, \tau)}{\theta_4(z, \tau)}, \\
\text{dn}(u, m) &= (1 - m)^{1/4} \frac{\theta_3(z, \tau)}{\theta_4(z, \tau)},
\end{align*}
\]

(1)

where \( z \equiv \frac{u \pi}{2K(m)} \) and \( \tau = iK'(m)/K(m) \). Therefore, any of our cyclic identities for real, imaginary or complex shifts can also be re-written as identities for the ratios of Jacobi theta functions for shifts in units of \( \pi/p, \pi \tau/p \) or \( \pi(1 + \tau)/p \) respectively if the ratios of theta functions involved are of period \( \pi \) (or twice these values if the period is \( 2\pi \)). In particular note that while \( \theta_{3,4}(z) \) are of period \( \pi \), \( \theta_{1,2}(z) \) are of period \( 2\pi \).

For simplicity, from now on we will not explicitly display \( \tau \) as an argument of the Jacobi theta functions. We also point out that the constants \( m, 1 - m \) and \( K(m) \) can also be entirely re-expressed in terms of theta functions:

\[
m^{1/4} = \frac{\theta_2(0)}{\theta_3(0)}, \quad (1 - m)^{1/4} = \frac{\theta_4(0)}{\theta_3(0)}, \quad \frac{2K(m)}{\pi} = \theta_3^2(0).
\]

(2)

Further, the Jacobi zeta function \( Z(u) \) and the complete elliptic integral of the second kind \( E \), which also appear in some of the identities, can also be expressed in terms of Jacobi theta functions:

\[
Z(u) = \frac{1}{\theta_3^2(0)} \frac{\theta_4'(z)}{\theta_4(z)}, \quad E = [1 - \frac{\theta_4''(0)}{\theta_3'(0)\theta_4(0)}]K = [1 - \frac{\theta_4''(0)}{\theta_3'(0)\theta_4(0)}]\pi \theta_3^2(0).
\]

(3)

Finally, let us note a remarkable fact about the cyclic identities: [8]

\[
\sum_{j=1}^{p} g(z_j)[h(z_{j+1}) \pm h(z_{j-1})] = \pm \sum_{j=1}^{p} h(z_j)[g(z_{j+1}) \pm g(z_{j-1})],
\]

(4)

\[
\sum_{j=1}^{p} (-1)^j g(z_j)[h(z_{j+1}) \pm h(z_{j-1})] = \mp \sum_{j=1}^{p} (-1)^j h(z_j)[g(z_{j+1}) \pm g(z_{j-1})],
\]

(5)

where \( h(z) \) and \( g(z) \) are combinations of ratios of Jacobi theta functions. Hence we shall only mention one of the two cyclic identities in order to avoid duplication.

The plan of this paper is as follows. We first consider the cyclic identities in I and II following from master identities MI-I to MI-IV. In Sec. 2 to Sec. 5, we give a list of corresponding cyclic identities in terms of the ratios of Jacobi theta functions. Finally, in Sec. 6 we summarize the results obtained and also indicate how to generalize these results to shifts in units of \( T \tau/p \) as well as \( T(1 + \tau)/p \) (where the
period $T$ is $\pi$ or $2\pi$). Further, we also indicate how to obtain identities for other ratios of $\theta$ functions as well as for their products.

Throughout this paper, we use the notation $1 \leq r < p$ and $(r,p) = 1$, i.e. they are co-prime. It may be noted that for MI-I and MI-II identities the period for the ratios of $\theta$ functions is $\pi$ while it is $2\pi$ for MI-III and MI-IV identities. Further, whereas the MI-I and MI-II identities are valid for odd as well as even $p$, for MI-III and MI-IV cases, nontrivial identities are obtained only when $p$ is an odd integer. It is worth keeping in mind that the identities given in this paper are not exhaustive but are meant to be representative identities of low rank.

## 2 Identities following from MI-I.

As shown in IIa, one of the simplest MI-I identities is given by (see Eq. (22) of IIa)

$$
\sum_{j=1}^{p} \text{dn}(x_j)\text{dn}(x_{j+1})\text{dn}(x_{j+2}) = \left[ \text{cs}^2(2K/p) - 2\text{cs}(2K/p)\text{cs}(4K/p) \right] \sum_{j=1}^{p} \text{dn}(x_j),
$$

(6)

where

$$
x_j = u + (j - 1)T/p,
$$

(7)

with $T$ being $2K$ or $4K$ depending on whether the identity is of type MI-I,II or type MI-III,IV respectively. On using the relation between Jacobi theta and Jacobi elliptic functions [Eqs. (1) and (2)], the corresponding identity in terms of theta functions is

$$
\sum_{j=1}^{p} \frac{\theta_3(z_j)\theta_3(z_{j+1})\theta_3(z_{j+2})}{\theta_4(z_j)\theta_4(z_{j+1})\theta_4(z_{j+2})} = \frac{\theta_2(\pi/p)}{\theta_1(\pi/p)} \left[ \frac{\theta_2(\pi/p)}{\theta_1(\pi/p)} - 2\frac{\theta_2(2\pi/p)}{\theta_1(2\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
$$

(8)

Here $z_j \equiv z + (j - 1)\pi/p$ with $z = u\pi/2K = u/\theta_3^2(0)$.

Proceeding in the same way we now rewrite the various MI-I identities obtained in IIa in terms of the ratios of Jacobi theta functions. For example identities (83) to (111) [except identities (105) and (109)] of IIa take the forms given below:

$$
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 0.
$$

(9)

$$
\sum_{j=1}^{p} \frac{\theta_3(z_j)\theta_3(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} \frac{\theta_3(z_{j+(l-1)r})}{\theta_4(z_{j+(l-1)r})} = \left[ \prod_{k=1}^{\frac{l-1}{2}} \frac{\theta_3^2(k\pi r/p)}{\theta_1^2(k\pi r/p)} \right] \frac{2(-1)^{\frac{l-1}{2}}}{\theta_3^2(0)} \left[ \prod_{n \neq k, n=1}^{l} \frac{\theta_2([n-k]r\pi/p)}{\theta_1([n-k]r\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} ,
$$

(10)
where \( l \) is any odd integer (\( \geq 3 \)). In the special case when \( p \) is also odd and \( l = p \), this identity takes the elegant form

\[
P_j^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} = \frac{(p-1)\theta_2^2(n\pi/p)}{\theta_1^2(n\pi/p)} \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]  

(11)

\[
\sum_{j=1}^{p} \frac{\theta_2^2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \left[ \frac{\theta_2^2(0)}{\theta_3(0)\theta_4(0)} - \frac{\theta_2^2(r\pi/p)}{\theta_1^2(r\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

(12)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_3(z_{j+r})}{\theta_4(z_{j+r})^2} + \frac{\theta_2(z_{j-r})\theta_3(z_{j-r})}{\theta_4(z_{j-r})^2} \right] = -2 \frac{\theta_2(z_{0})}{\theta_2(0)\theta_1(r\pi/p)} \left[ \frac{\theta_3(r\pi/p)}{\theta_1(r\pi/p)} - \frac{\theta_3(0)\theta_4(r\pi/p)}{\theta_4(0)\theta_1(r\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

(13)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_3(z_{j+r})}{\theta_4(z_{j+r})^2} + \frac{\theta_1(z_{j-r})\theta_3(z_{j-r})}{\theta_4(z_{j-r})^2} \right] = -2 \frac{\theta_2^2(0)}{\theta_2(0)\theta_3(0)} \frac{\theta_2(r\pi/p)}{\theta_1(r\pi/p)} \left[ \frac{\theta_3(r\pi/p)}{\theta_1(r\pi/p)} - \frac{\theta_3(0)\theta_4(r\pi/p)}{\theta_4(0)\theta_1(r\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

(14)

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_3(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2 \frac{\theta_2(r\pi/p)\theta_2(s\pi/p)}{\theta_3(0)\theta_1(s\pi/p)} \left[ \frac{\theta_2(r\pi/p)}{\theta_1(s\pi/p)} - \frac{\theta_2(s\pi/p)}{\theta_1(s\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

(15)

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_2(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_2(z_{j-r})\theta_2(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2 \frac{\theta_2(r\pi/p)\theta_3(s\pi/p)}{\theta_1(s\pi/p)} \left[ \frac{\theta_2(r\pi/p)}{\theta_1(s\pi/p)} - \frac{\theta_2(s\pi/p)}{\theta_1(s\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

(16)

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_1(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_1(z_{j-r})\theta_1(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = 2 \frac{\theta_4(r\pi/p)\theta_4(s\pi/p)}{\theta_1(s\pi/p)} \left[ \frac{\theta_2(r\pi/p)\theta_2(s\pi/p)}{\theta_1(s\pi/p)} - \frac{\theta_2(s\pi/p)}{\theta_1(s\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

(17)
\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_{j+r}) \theta_3(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r}) \theta_2(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} = 2 \left[ \frac{\theta_3(r\pi/p) \theta_3(0)}{\theta_4(r\pi/p) \theta_2(0)} \right] \sum_{j=1}^{p} \frac{\delta_3(z_j)}{\delta_4(z_j)}.
\]

(18)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r}) \theta_3(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r}) \theta_3(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = 2 \left[ \frac{\theta_4(r\pi/p) \theta_4(0)}{\theta_1(r\pi/p) \theta_2(0)} \right] \sum_{j=1}^{p} \frac{\delta_3(z_j)}{\delta_4(z_j)}.
\]

(19)

\[
\sum_{j=1}^{p} \frac{\delta_2^2(z_j)}{\delta_4^2(z_j)} \left[ \frac{\theta_2(z_{j+r}) \theta_3(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_2(z_{j-r}) \theta_3(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right]
\times \left( \frac{\theta_2(r\pi/p) \theta_2(t\pi/p) \theta_3(s\pi/p) \theta_4(t\pi/p)}{\theta_1(s\pi/p) \theta_4(t\pi/p) \theta_4(r\pi/p) \theta_1(t\pi/p)} \right)
\times \left[ \frac{\theta_3(r\pi/p) \theta_3(s\pi/p) \theta_4(t\pi/p) \theta_4(s\pi/p)}{\theta_1(r\pi/p) \theta_4(t\pi/p) \theta_4(s\pi/p) \theta_1(r\pi/p)} \right]
\times \left[ \frac{\theta_3(r\pi/p) \theta_3(s\pi/p) \theta_4(t\pi/p) \theta_4(s\pi/p)}{\theta_1(r\pi/p) \theta_4(t\pi/p) \theta_4(s\pi/p) \theta_1(r\pi/p)} \right]
\times \sum_{j=1}^{p} \frac{\theta_4(z_j)}{\theta_4(z_j)}.
\]

(20)

\[
\sum_{j=1}^{p} \frac{\delta_1(z_j)^2 \theta_4(z_j)}{\delta_4(z_j)^2} \left[ \frac{\theta_2(z_{j+r}) \theta_1(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_2(z_{j-r}) \theta_1(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = -2 \frac{\delta_2^2(r\pi/p)}{\delta_1^2(r\pi/p)} \left[ 1 + \frac{\delta_3(z_j) \theta_2(z_j)}{\delta_4(z_j) \theta_4(z_j)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j)}{\delta_4^2(z_j)}.
\]

(21)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)^2} \left[ \frac{\theta_3(z_{j+r}) \theta_4(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r}) \theta_4(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = 2 \frac{\delta_1^2(z_j) \theta_4(z_{j+r}) \theta_4(z_{j+s})}{\delta_4^2(z_{j}) \theta_4(z_{j+r})} \frac{\theta_3(0) \theta_3(0) \theta_4(0) \theta_4(0)}{\theta_3(0) \theta_3(0) \theta_4(0) \theta_4(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j)}{\theta_4(z_j)^2}.
\]

(22)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)^2} \left[ \frac{\theta_2(z_{j+r}) \theta_3(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_2(z_{j-r}) \theta_3(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = -2 \frac{\delta_2(z_j) \theta_2(r\pi/p)}{\delta_1^2(r\pi/p)} \left[ \frac{\theta_4(r\pi/p)}{\theta_4(0)} + \frac{\theta_3(r\pi/p)}{\theta_3(0)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j)}{\theta_4(z_j)^2}.
\]

(23)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)^2 \theta_4(z_j)}{\theta_1(z_j)^2} \left[ \frac{\theta_1(z_{j+r}) \theta_4(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_1(z_{j-r}) \theta_4(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = -2 \frac{\delta_4(z_j) \theta_2(r\pi/p) \theta_4(r\pi/p)}{\theta_2(0) \theta_4(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j)}{\theta_4(z_j)^2}.
\]

(24)
\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^3(z_{j+r})} + \frac{\theta_3^2(z_{j-r})}{\theta_4^2(z_{j-r})} \right] = 2 \frac{\theta_3(0)\theta_2(r\pi/p)\theta_3(r\pi/p)}{\theta_2(0)\theta_1^2(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)}.
\]

(25)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_3(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2 \frac{\theta_2(r\pi/p)\theta_2(s\pi/p)}{\theta_1(r\pi/p)\theta_1(s\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)}.
\]

(26)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_1(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_1(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = 2 \frac{\theta_1(r\pi/p)\theta_4(s\pi/p)}{\theta_1(r\pi/p)\theta_1(s\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)}.
\]

(27)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_3(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2 \frac{\theta_2(0)\theta_4(r\pi/p)\theta_2(s\pi/p)}{\theta_2(0)\theta_1(r\pi/p)\theta_1(s\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)}.
\]

(29)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_3(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2 \frac{\theta_2(0)\theta_3(r\pi/p)\theta_2(s\pi/p)}{\theta_3(0)\theta_1(r\pi/p)\theta_1(s\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)}.
\]

(30)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^3(z_{j+r})} + \frac{\theta_3^2(z_{j-r})}{\theta_4^2(z_{j-r})} \right] = -2 \frac{\theta_3^2(0)\theta_2^2(r\pi/p)}{\theta_3^2(0)\theta_4^2(0)} \left[ \frac{\theta_3^2(r\pi/p)}{\theta_4^2(0)} + \frac{\theta_3^2(r\pi/p)}{\theta_4^2(0)} + 3 \frac{\theta_3(r\pi/p)\theta_4(r\pi/p)}{\theta_3(0)\theta_4(0)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)}.
\]

(31)
is necessarily odd.

As emphasized in II, such identities are only valid when $p$ is even and hence $r$ (being co-prime to $p$) is necessarily odd.

For example, the identities (196) to (201), (207), (208) and (215) of IIa take the form given below:

$$\sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_3^2(z_{j+r}) + \theta_3(z_{j-r})}{\theta_4^3(z_{j+r}) + \theta_4(z_{j-r})} \right] = \frac{2 \theta_3^2(0) \theta_3(r \pi/p) \theta_4(r \pi/p)}{\theta_3(0) \theta_4(0) \theta_4^2(r \pi/p)} \sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^3(z_j)}$$

$$2 \frac{\theta_3^2(r \pi/p)}{\theta_4^2(r \pi/p)} \left[ 1 - \frac{\theta_3^2(0) \theta_3(r \pi/p) \theta_4(r \pi/p)}{\theta_3(0) \theta_4(0) \theta_4^2(r \pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} . \quad (32)$$

$$\sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_3^2(z_{j+r}) + \theta_3(z_{j-r})}{\theta_4^3(z_{j+r}) + \theta_4(z_{j-r})} \right] = \frac{-2 \theta_3^2(0) \theta_3(r \pi/p) \theta_4^2(r \pi/p)}{\theta_3(0) \theta_4(0) \theta_4^2(r \pi/p)} \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}$$

$$\frac{\theta_3^2(0) \theta_3(r \pi/p) \theta_4^2(r \pi/p)}{\theta_3(0) \theta_4(0) \theta_4^2(r \pi/p)} \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} . \quad (33)$$

$$\sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_2(z_{j+s}) + \theta_1(z_{j-r}) \theta_2(z_{j-s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s}) + \theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = \frac{-2 \theta_3^2(0) \theta_3(r \pi/p) \theta_3(s \pi/p)}{\theta_3(0) \theta_4(0) \theta_4(r \pi/p)} \sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)}$$

$$\frac{\theta_3(s \pi/p) \theta_4(r \pi/p)}{\theta_4(s \pi/p)} \sum_{j=1}^{p} \frac{\theta_3(z_j) \theta_2(z_j)}{\theta_4(z_j) \theta_4(r \pi/p)} . \quad (34)$$

2.1 MI-I Identities with alternating signs.

Let us now write a few MI-I identities with alternate signs in terms of the ratios of theta functions. As emphasized in II, such identities are only valid when $p$ is even and hence $r$ (being co-prime to $p$) is necessarily odd.

For example, the identities (196) to (201), (207), (208) and (215) of IIa take the form given below:

$$\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+1}) + \theta_2(z_{j-1})}{\theta_4(z_{j+1}) + \theta_4(z_{j-1})} \right] = 0 . \quad (36)$$
\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)\theta_4(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)}.
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)\theta_3(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} = -\left[ \frac{\theta_2(r\pi/p)\theta_2(2r\pi/p)}{\theta_1(r\pi/p)\theta_1(2r\pi/p)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)\theta_3(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} \theta_3(z_{j+a}) = -\left[ \frac{\theta_2(r\pi/p)\theta_2(s\pi/p)}{\theta_1(r\pi/p)\theta_1(s\pi/p)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2\frac{\theta_2^2(0)}{\theta_4^2(r\pi/p)} \left[ \frac{\theta_3(r\pi/p)\theta_4(r\pi/p)}{\theta_3(0)\theta_4(0)} + \frac{\theta_2^2(r\pi/p)}{\theta_2^2(0)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r})\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = -2\frac{\theta_2^2(0)\theta_2(r\pi/p)}{\theta_2(0)\theta_4^2(r\pi/p)} \left[ \frac{\theta_2(r\pi/p)}{\theta_3(0)} + \frac{\theta_4(r\pi/p)}{\theta_4(0)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2\frac{\theta_2^2(0)\theta_2(r\pi/p)}{\theta_2(0)\theta_4^2(r\pi/p)} \left[ \frac{\theta_3(r\pi/p)}{\theta_3(0)} + \frac{\theta_4(r\pi/p)}{\theta_4(0)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})\theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2\frac{\theta_2^2(0)}{\theta_4^2(r\pi/p)} \left[ \frac{\theta_3^2(r\pi/p)}{\theta_3(0)\theta_4(0)} - \frac{\theta_2^2(r\pi/p)}{\theta_2^2(0)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)\theta_3(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} = -\left[ \frac{\theta_2(r\pi/p)\theta_2(s\pi/p)}{\theta_1(r\pi/p)\theta_1(s\pi/p)} \right] \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)}{\theta_4(z_j)} .
\]
\[ \sum_{j=1}^{n} (-1)^{j-1} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+1})}{\theta_4(z_{j+1})} + \frac{\theta_3(z_{j-1})}{\theta_4(z_{j-1})} \right] = 2\frac{\theta_2^2(0) \theta_3(\pi/p) \theta_4(\pi/p)}{\theta_3(0) \theta_4(0) \theta_4^2(\pi/p)} \sum_{j=1}^{n} (-1)^{j-1} \frac{\theta_1(z_j) \theta_2(z_j)}{\theta_4^2(z_j)}. \tag{44} \]

### 3 Identities following from MI-II.

We shall now write down some of the MI-II identities from IIa in terms of the ratios of theta functions. As in the previous section, in this section too \( z_j \equiv z + (j-1)\pi/p \) with \( z = u\pi/2K = u/\theta_3^2(0) \). Identities (112) to (141) [except identities (120), (121), (126), (130) to (133) and (136)] of IIa, when expressed in terms of theta functions, are given by

\[ \sum_{j=1}^{p} \frac{\theta_3(z_j) \theta_3(z_{j+r})}{\theta_4(z_j) \theta_4(z_{j+r})} = \frac{p \theta_3(0) \theta_3(r\pi/p)}{\theta_4(0) \theta_4(r\pi/p)} \left[ 1 - \frac{\theta_3'(r\pi/p) \theta_3(r\pi/p)}{\theta_3^2(0) \theta_3(r\pi/p) \theta_1(r\pi/p)} \right]. \tag{45} \]

\[ \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_1(z_{j+r})}{\theta_4(z_j) \theta_4(z_{j+r})} = \frac{p \theta_1'(r\pi/p) \theta_2(r\pi/p)}{\theta_2(0) \theta_3(r\pi/p) \theta_2(r\pi/p)} \tag{46} \]

\[ \sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_2(z_{j+r})}{\theta_4(z_j) \theta_4(z_{j+r})} = \frac{p \theta_2(0) \theta_2(r\pi/p)}{\theta_4(0) \theta_4(r\pi/p)} \left[ 1 - \frac{\theta_3(r\pi/p) \theta_4'(r\pi/p)}{\theta_3^2(0) \theta_1(r\pi/p) \theta_2(r\pi/p)} \right]. \tag{47} \]

\[ \sum_{j=1}^{p} \frac{\theta_3(z_j) \theta_3(z_{j+1}) \cdots \theta_3(z_{j+(l-1)})}{\theta_4(z_j) \theta_4(z_{j+1}) \cdots \theta_4(z_{j+(l-1)})} = \frac{p}{\pi} \int_0^\pi dz \frac{\theta_3(z) \theta_3(z + \pi/p) \cdots \theta_3(z + [l-1]\pi/p)}{\theta_4(z) \theta_4(z + \pi/p) \cdots \theta_4(z + [l-1]\pi/p)}, (l\text{ even}). \tag{48} \]

\[ \sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})}{\theta_4(z_{j-r})} \right] = 0. \tag{49} \]

\[ \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 0. \tag{50} \]

\[ \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 0. \tag{51} \]

\[ \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+s}) \theta_3(z_{j+r})}{\theta_4(z_{j+s}) \theta_4(z_{j+r})} + \frac{\theta_1(z_{j-s}) \theta_3(z_{j-r})}{\theta_4(z_{j-s}) \theta_4(z_{j-r})} \right] = 0. \tag{52} \]

We now write down results for those cyclic identities in IIa in which the right hand side contained a definite integral which we could not then evaluate. However, subsequently, we derived local identities using which we were able to evaluate all these integrals. We are therefore giving answers using results in [...].

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\[
\sum_{j=1}^{p} \frac{\theta_3^2(z_j)\theta_3^2(z_{j+r})}{\theta_4^2(z_j)\theta_4^2(z_{j+r})} = -2\frac{\theta_3^2(r\pi/p)}{\theta_4^2(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_3^2(z_j)}{\theta_4^2(z_j)} \\
+ p \left[ \frac{\theta_3^2(0)\theta_3^2(r\pi/p)}{\theta_4^2(0)\theta_4^2(r\pi/p)} + \frac{\theta_3^4(0)\theta_3^2(r\pi/p)}{\theta_4^2(0)\theta_4^2(0)\theta_4^2(r\pi/p)} \right] \frac{2\theta_3^2(0)\theta_2(r\pi/p)\theta_3(r\pi/p)\theta_4'(r\pi/p)}{\theta_4^2(0)\theta_4^2(0)\theta_4^2(r\pi/p)} \right]. 
\] (53)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_2(z_{j+r})}{\theta_4^2(z_{j+r})} + \frac{\theta_1(z_{j-r})\theta_2(z_{j-r})}{\theta_4^2(z_{j-r})} \right] \\
= 4\frac{\theta_3(0)\theta_4(0)\theta_3(r\pi/p)\theta_4(r\pi/p)}{\theta_2(0)\theta_4^2(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_3^2(z_j)}{\theta_4^2(z_j)} \\
- 2p \frac{\theta_3^4(0)\theta_4(0)\theta_3(r\pi/p)}{\theta_2(0)\theta_4^2(0)\theta_4^2(r\pi/p)} \left[ \theta_3(r\pi/p) - \theta_2(r\pi/p)\theta_4'(r\pi/p) \right]. 
\] (54)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_3(z_{j+r})}{\theta_4^2(z_{j+r})} + \frac{\theta_2(z_{j-r})\theta_3(z_{j-r})}{\theta_4^2(z_{j-r})} \right] \\
= -4\frac{\theta_3(0)\theta_2(r\pi/p)\theta_3(r\pi/p)}{\theta_2(0)\theta_4^2(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_3^2(z_j)}{\theta_4^2(z_j)} \\
+ 2p \frac{\theta_3^4(0)}{\theta_4^2(0)\theta_4^2(r\pi/p)} \left[ 2\theta_3(r\pi/p)\theta_2(r\pi/p) - \theta_4'(r\pi/p) \right] \frac{\theta_3^2(r\pi/p)}{\theta_2(0)\theta_3(0)} \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \\
+ \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} - \frac{\theta_4'(r\pi/p)\theta_3^2(r\pi/p)}{\theta_2(0)\theta_3(0)\theta_4^2(0)} \theta_3(r\pi/p) \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} + \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \right]. 
\] (55)

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})}{\theta_4^2(z_{j+r})} + \frac{\theta_3(z_{j-r})}{\theta_4^2(z_{j-r})} \right] \\
= 4\frac{\theta_3(0)\theta_4(r\pi/p)\theta_4(r\pi/p)}{\theta_2(0)\theta_4^2(r\pi/p)} \sum_{j=1}^{p} \theta_3^2(z_j) \\
- 2p \frac{\theta_3^4(0)\theta_2(0)\theta_2(r\pi/p)}{\theta_3(0)\theta_4^2(0)\theta_4^2(r\pi/p)} \left[ \theta_3^2(r\pi/p) \right] \\
+ \frac{\theta_3^2(r\pi/p)}{\theta_4^2(0)} \right] \right] - \frac{\theta_4'(r\pi/p)\theta_3^2(r\pi/p)}{\theta_2(0)\theta_3(0)\theta_4^2(0)} \theta_3(r\pi/p) \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} + \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \left]. 
\] (56)

\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^2(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^2(z_{j-r})} \right] \\
= 2\frac{\theta_3^2(0)\theta_3(r\pi/p)\theta_4(r\pi/p)}{\theta_3(0)\theta_4(0)\theta_4^2(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_3^2(z_j)}{\theta_4^2(z_j)} \\
- 2p \frac{\theta_3^4(0)\theta_2(0)\theta_3(r\pi/p)}{\theta_3(0)\theta_4(0)\theta_4^2(r\pi/p)} \left[ \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \right] \\
- \frac{\theta_4'(r\pi/p)\theta_3^2(r\pi/p)}{\theta_2(0)\theta_3(0)\theta_4^2(0)} \frac{\theta_3(r\pi/p)}{\theta_2(0)} \left]. 
\] (57)

\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^2(z_{j+r})} + \frac{\theta_3^2(z_{j-r})}{\theta_4^2(z_{j-r})} \right] \\
= 2\frac{\theta_3^2(0)\theta_2(r\pi/p)\theta_3(r\pi/p)}{\theta_2(0)\theta_4^2(0)\theta_4^2(r\pi/p)} \sum_{j=1}^{p} \frac{\theta_3^2(z_j)}{\theta_4^2(z_j)} \\
- 2p \frac{\theta_3^4(0)\theta_2^2(0)\theta_3(r\pi/p)}{\theta_3(0)\theta_4^2(0)\theta_4^2(r\pi/p)} \left[ \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \frac{\theta_3^2(r\pi/p)}{\theta_2(0)} \right] \\
- \frac{\theta_4'(r\pi/p)\theta_3^2(r\pi/p)}{\theta_2(0)\theta_3(0)\theta_4^2(0)} \frac{\theta_3(r\pi/p)}{\theta_2(0)} \left]. 
\] (58)
\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \frac{\theta_3^3(0)}{\theta_4^4(0)} \theta_2(r\pi/p) \theta_4(r\pi/p) \sum_{j=1}^{p} \frac{\theta_3^3(z_j)}{\theta_4^4(z_j)} \\
+ 2p \frac{\theta_2(0)}{\theta_4(0)} \theta_2(r\pi/p) \left[ 1 - \frac{\theta_1^1(0)}{\theta_2^2(0)} \theta_2^2(r\pi/p) \theta_4^4(0) \theta_4^4(r\pi/p) \right].
\]

(59)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^4(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^4(z_{j-r})} \right] = -2 \frac{\theta_3^3(0)}{\theta_4^4(0)} \theta_2(r\pi/p) \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}.
\]

(60)

\[
\sum_{j=1}^{p} \frac{\theta_2^2(z_j) \theta_1(z_j) \theta_3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \frac{\theta_2^2(0)}{\theta_4^4(0)} \theta_2(r\pi/p) \theta_4(r\pi/p) \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}.
\]

(61)

\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j) \theta_1(z_j) \theta_3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^4(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^4(z_{j-r})} \right] = 2 \frac{\theta_3^3(0)}{\theta_4^4(0)} \theta_2(r\pi/p) \theta_4(r\pi/p) \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}.
\]

(62)

\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j) \theta_1(z_j) \theta_2(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^4(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^4(z_{j-r})} \right] = -4 \frac{\theta_2^2(0)}{\theta_3(0) \theta_4(0)} \frac{\theta_3(0) \theta_4(0)}{\theta_4^4(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}
\]

(63)

\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j) \theta_1(z_j) \theta_3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^4(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^4(z_{j-r})} \right] = -4 \frac{\theta_3^3(0)}{\theta_2(0) \theta_3(0) \theta_4(0)} \frac{\theta_2(0) \theta_3(0) \theta_4(0)}{\theta_4^4(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}
\]

(64)

\[
\sum_{j=1}^{p} \frac{\theta_3^3(z_j) \theta_1(z_j) \theta_3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^4(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^4(z_{j-r})} \right] = -4 \frac{\theta_3^3(0)}{\theta_2(0) \theta_3(0) \theta_4(0)} \frac{\theta_2(0) \theta_3(0) \theta_4(0)}{\theta_4^4(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}
\]

(65)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)} \left[ \frac{\theta_3^3(z_{j+r})}{\theta_4^4(z_{j+r})} + \frac{\theta_3^3(z_{j-r})}{\theta_4^4(z_{j-r})} \right] = 2 \frac{\theta_3^3(0)}{\theta_4^4(0)} \frac{\theta_3^3(0) \theta_3(0) \theta_4(0)}{\theta_4^4(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_4^4(z_j)}
\]

(66)
3.1 MI-II Identities with alternating signs.

Let us now write the MI-II identities with alternating signs as given by Eqs. (219) to (239) [except identities (231), (232), (233) and (237)] of II in terms of Jacobi theta functions. It should be noted here that in this case \( p \) is necessarily even and that \( r, s \) are therefore odd integers co-prime to \( p \).

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)\theta_3(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} = \frac{2}{\theta_3(0)\theta_4(0)} \frac{\theta_2(r\pi/p)}{\theta_1(r\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_1'(z_j)}{\theta_4(z_j)}.
\]

(67)

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_1(z_j)}{\theta_4(z_j)} \frac{\theta_4(z_{j+r})}{\theta_2(z_j)\theta_4(z_{j+r})} = \frac{2}{\theta_2(0)\theta_3(0)} \frac{\theta_4(r\pi/p)}{\theta_1(r\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)}.
\]

(68)

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_2(z_j)\theta_2(z_{j+r})}{\theta_4(z_j)\theta_4(z_{j+r})} = -\frac{2}{\theta_2(0)\theta_4(0)} \frac{\theta_3(r\pi/p)\theta_3(2r\pi/p)}{\theta_1(r\pi/p)\theta_1(3r\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)}.
\]

(69)

This generalizes for any even number \( l < p \) to:

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z_j)\theta_3(z_{j+r})...\theta_3(z_{j+(l-1)r})}{\theta_4(z_j)\theta_4(z_{j+r})...\theta_4(z_{j+(l-1)r})} = (-1)^{l/2} \frac{2}{\theta_3^2(0)} \left( \sum_{k=1}^{l/2} (-1)^{k-1} \prod_{n=1, n \neq k}^{l} \frac{\theta_2((n-k)r\pi/p)}{\theta_1((n-k)r\pi/p)} \right) \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)}.
\]

(70)

Similarly for any even integer \( l \leq p, \theta_1 \) and \( \theta_2 \) functions satisfy the identities \( p \geq 4 \)

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_1(z_j)\theta_1(z_{j+r})...\theta_1(z_{j+(l-1)r})}{\theta_4(z_j)\theta_4(z_{j+r})...\theta_4(z_{j+(l-1)r})} = \frac{2}{\theta_3^2(0)} \left( \sum_{k=1}^{l/2} (-1)^{k-1} \prod_{n=1, n \neq k}^{l} \frac{\theta_4((n-k)r\pi/p)}{\theta_1((n-k)r\pi/p)} \right) \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)}.
\]

(71)

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_2(z_j)\theta_2(z_{j+r})...\theta_2(z_{j+(l-1)r})}{\theta_4(z_j)\theta_4(z_{j+r})...\theta_4(z_{j+(l-1)r})} = (-1)^{l/2} \frac{2}{\theta_3^2(0)} \left( \theta_3(0)/\theta_2(0) \right)^2 \left( \sum_{k=1}^{l/2} (-1)^{k-1} \prod_{n=1, n \neq k}^{l} \frac{\theta_3((n-k)r\pi/p)}{\theta_1((n-k)r\pi/p)} \right) \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)}.
\]

(72)
When \( l = p \ (p \geq 4) \), the last two identities reduce to

\[
\Pi_{j=1}^{\infty} \frac{\theta_4(z_j)}{\theta_4'(z_j)} = \frac{1}{\theta_2'^2(0)} \left( \Pi_{n=1}^{(p-2)} \frac{\theta_3^3(n\pi/p)}{\theta_1^4(n\pi/p)} \right) \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{74}
\]

\[
\Pi_{j=1}^{\infty} \frac{\theta_2(z_j)}{\theta_4(z_j)} = (-1)^{p/2} \frac{1}{\theta_2'^2(0)} \left( \Pi_{n=1}^{(p-2)} \frac{\theta_3^3(n\pi/p)}{\theta_1^4(n\pi/p)} \right) \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{75}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4(z_j)}{\theta_4'(z_j)} \left[ \frac{\theta_1(z_j)}{\theta_4'(z_j)} + \frac{\theta_2(z_j)\theta_2(z_j)}{\theta_4'(z_j)} \right] = -\frac{4}{\theta_3(0)\theta_4(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{76}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_1(z_j)}{\theta_4'(z_j)} \left[ \frac{\theta_2(z_j)}{\theta_4'(z_j)} + \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4'(z_j)} \right] = -\frac{4}{\theta_2(0)\theta_3(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{77}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_2(z_j)}{\theta_4'(z_j)} \left[ \frac{\theta_1(z_j)}{\theta_4'(z_j)} + \frac{\theta_3(z_j)\theta_3(z_j)}{\theta_4'(z_j)} \right] = -\frac{4}{\theta_2(0)\theta_3(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{78}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z)}{\theta_4'(z_j)} \left[ \frac{\theta_3(z)}{\theta_4'(z_j)} + \frac{\theta_3(z)\theta_3(z)}{\theta_4'(z_j)} \right] = 2 \frac{\theta_3^3(0)\theta_3(0)\theta_3(\pi/p)\theta_4(\pi/p)}{\theta_2(0)\theta_4(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{79}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z)}{\theta_4'(z_j)} \left[ \frac{\theta_2(z)}{\theta_4'(z_j)} + \frac{\theta_2(z)\theta_3(z)}{\theta_4'(z_j)} \right] = \frac{2\theta_4^3(0)\theta_2(0)\theta_3(\pi/p)\theta_4(\pi/p)}{\theta_2(0)\theta_4(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{80}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z)}{\theta_4'(z_j)} \left[ \frac{\theta_1(z)}{\theta_4'(z_j)} + \frac{\theta_1(z)\theta_3(z)}{\theta_4'(z_j)} \right] = \frac{2\theta_4^3(0)\theta_2(0)\theta_3(\pi/p)\theta_4(\pi/p)}{\theta_2(0)\theta_4(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{81}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z)}{\theta_4'(z_j)} \left[ \frac{\theta_1(z)}{\theta_4'(z_j)} + \frac{\theta_1(z)\theta_3(z)}{\theta_4'(z_j)} \right] = \frac{-2\theta_3^3(0)\theta_3(\pi/p)\theta_4(\pi/p)}{\theta_3(0)\theta_4(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{82}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_3(z)}{\theta_4'(z_j)} \left[ \frac{\theta_2(z)}{\theta_4'(z_j)} + \frac{\theta_2(z)\theta_3(z)}{\theta_4'(z_j)} \right] = \frac{2\theta_4^3(0)\theta_2(\pi/p)\theta_4(\pi/p)}{\theta_2(0)\theta_4(0)\theta_1^2(\pi/p)} \sum_{j=1}^{p} (-1)^{j-1} \frac{\theta_4'(z_j)}{\theta_4(z_j)} . \tag{83}
\]
4 Identities following from MI-III.

We shall now rewrite the identities following from master identity MI-III in terms of theta functions. It is worth recalling that unlike the previous two sections, the period for these identities, as well as those in the next section is $2\pi$. Further, $p$ is necessarily an odd integer. The identities (144) to (170) [except identities (152),(154),(161),(163),(164) and (168)] of Ha, when re-expressed in terms of Jacobi theta functions are given below.

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 0. \tag{84}
\]

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} \frac{\theta_1(z_{j+(l-1)r})}{\theta_4(z_{j+(l-1)r})} = \left[ (-1)^{l-1/2} \Pi_{k=1}^{r} \frac{\theta_2^2(2r_{k}\pi/p)}{\theta_4^2(2r_{k}\pi/p)} \right]^{l-1/2} \prod_{n\neq k, n=1}^{l} \frac{\theta_4(2[n-k]r_{n}\pi/p)}{\theta_4(z_{n\pi/p})} \prod_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)}, \tag{85}
\]

where $l$ is any odd integer ($3 \leq l \leq p$). Note that unlike the last two sections, here $z_j \equiv z + \frac{2(j-1)\pi}{\lambda}$ where as before $z \equiv \frac{u\pi}{2\lambda} = \frac{u\pi}{2\theta_4(0)}$. In the special case when $l = p$, this identity takes the elegant form

\[
\Pi_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} = (-1)^{\frac{p-1}{2}} \Pi_{n=1}^{p} \frac{\theta_2^2(2n\pi/p)}{\theta_4^2(2n\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)}. \tag{86}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2^2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})}{\theta_4(z_{j-r})} \right] = -2 \frac{\theta_2^2(0)}{\theta_4^2(2r\pi/p)} \left[ \frac{\theta_2(2r\pi/p)}{\theta_4(0)} \frac{\theta_3(2r\pi/p)}{\theta_4(0)} - \frac{\theta_3(2r\pi/p)}{\theta_4(0)} \right] \prod_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)}. \tag{87}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \frac{\theta_2^3(0)}{\theta_4(0)\theta_4(2r\pi/p)} \left[ \frac{\theta_2(2r\pi/p)}{\theta_4(0)} - \frac{\theta_3(2r\pi/p)}{\theta_4(0)} \right] \prod_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)}. \tag{88}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \frac{\theta_2^3(0)}{\theta_4(0)\theta_4(2r\pi/p)} \left[ \frac{\theta_3(2r\pi/p)}{\theta_4(0)} - \frac{\theta_2(2r\pi/p)}{\theta_2(0)} \right] \prod_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)}. \tag{89}
\]
\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+r}) \theta_2(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_2(z_{j+r}) \theta_2(z_{j-s})}{\theta_4(z_{j+r}) \theta_4(z_{j-s})} \right] = -2 \left[ \frac{\theta_3(2r\pi/p) \theta_3(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} .
\]

(90)

\[
\sum_{j=1}^{p} \theta_1(z_j) \left[ \frac{\theta_3(z_{j+r}) \theta_3(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r}) \theta_3(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = -2 \left[ \frac{\theta_2(2r\pi/p) \theta_2(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} .
\]

(91)

\[
\sum_{j=1}^{p} \theta_3(z_j) \left[ \frac{\theta_3(z_{j+r}) \theta_1(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r}) \theta_1(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = -2 \left[ \frac{\theta_2(2r\pi/p) \theta_2(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \right] \frac{\theta_2(2r\pi/p) \theta_2(0)}{\theta_1(2r\pi/p) \theta_3(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} .
\]

(92)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_2^2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2^2(z_{j-r})}{\theta_4(z_{j-r})} \right] = \frac{2 \theta_2^2(2r\pi/p)}{\theta_4^2(2r\pi/p)} \left[ 1 + \frac{\theta_2(0) \theta_2(2r\pi/p) \theta_3(2r\pi/p)}{\theta_2(0) \theta_3(0) \theta_4^2(2r\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(93)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_2(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r}) \theta_2(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \left[ \frac{\theta_2(2r\pi/p)}{\theta_2(0)} + \frac{\theta_3(2r\pi/p)}{\theta_3(0)} \right] \sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(94)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_1(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_1(z_{j-r}) \theta_1(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = 2 \left[ \frac{\theta_4(2r\pi/p) \theta_4(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)} .
\]

(95)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r}) \theta_2(z_{j+s})}{\theta_4(z_{j+r}) \theta_4(z_{j+s})} + \frac{\theta_2(z_{j-r}) \theta_2(z_{j-s})}{\theta_4(z_{j-r}) \theta_4(z_{j-s})} \right] = -\frac{2 \theta_2^2(0) \theta_3(2r\pi/p) \theta_3(2s\pi/p)}{\theta_4^2(0) \theta_1(2r\pi/p) \theta_1(2s\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(96)
\[
\begin{align*}
\sum_{j=1}^{p} \frac{\theta_3(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_3(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] \\
= -2 \frac{\theta_2(2r\pi/p)\theta_2(2s\pi/p)}{\theta_1(2r\pi/p)\theta_1(2s\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)}. \quad (97)
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_1(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_3(z_{j-r})\theta_1(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] \\
= 2 \frac{\theta_3(0)\theta_3(2r\pi/p)\theta_4(2s\pi/p)}{\theta_2(0)\theta_1(2r\pi/p)\theta_1(2s\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \cdot \quad (98)
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)\theta_3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})}{\theta_4(z_{j-r})} \right] \\
= -2 \frac{\theta_2(0)\theta_3(2r\pi/p)\theta_3(2r\pi/p)}{\theta_1(2r\pi/p)\theta_2(0)\theta_3(0)\theta_4^2(0)} \sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} + 3 \frac{\theta_2(2r\pi/p)\theta_3(2r\pi/p)}{\theta_2(0)\theta_3(0)} \sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)}. \quad (100)
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)\theta_3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})}{\theta_4(z_{j-r})} \right] \\
+ 2 \frac{\theta_2^2(2r\pi/p)\theta_4(0)}{\theta_1(2r\pi/p)} \left[ \frac{\theta_2^2(2r\pi/p)}{\theta_4^2(0)} - \frac{\theta_2(2r\pi/p)\theta_3(2r\pi/p)}{\theta_2(0)\theta_3(0)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \cdot \quad (101)
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)\theta_3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})}{\theta_4(z_{j-r})} \right] \\
+ 2 \frac{\theta_2^2(2r\pi/p)\theta_4(0)}{\theta_1(2r\pi/p)} \left[ \frac{\theta_2^2(2r\pi/p)}{\theta_4^2(0)} + \frac{\theta_2^2(2r\pi/p)}{\theta_4^2(0)} \right] + 3 \frac{\theta_2(2r\pi/p)\theta_3(2r\pi/p)}{\theta_2(0)\theta_3(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \cdot \quad (102)
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)\theta_3(z_j)}{\theta_4^3(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_3(z_{j+s})}{\theta_4(z_{j+r})\theta_4(z_{j+s})} + \frac{\theta_2(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j-r})\theta_4(z_{j-s})} \right] \\
= 2 \frac{\theta_2^2(0)\theta_2(2r\pi/p)\theta_3(2r\pi/p)}{\theta_2(0)\theta_3(0)\theta_4^2(2r\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} + 2 \frac{\theta_2^2(2r\pi/p)\theta_2^2(2r\pi/p)}{\theta_4^2(0)} \sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \cdot \quad (103)
\end{align*}
\]
5 Identities following from MI-IV.

We shall now rewrite the identities following from master identity MI-IV in terms of theta functions. As in the last section, the period for these identities is 2π. Further, p is necessarily an odd integer. The identities (171) to (194) [except identities (181), (183), (186) and (192)] of IIa, when re-expressed in terms of Jacobi theta functions are given below. As in the previous section, in this section too, 

\[ z_j \equiv z + 2(j-1)\pi/p \]

with \( z = u\pi/2K = u/\theta_3^2(0) \).

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_2(z_j) \theta_3(z_j)}{\theta_3(z_j)} \left[ \frac{\theta_2(z_{j+r}) \theta_3(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_2(z_{j-r}) \theta_3(z_{j-r})}{\theta_4(z_{j-r})} \right] = 2 \frac{\theta_4(0) \theta_2(2\pi/p) \theta_3(2\pi/p)}{\theta_3^2(2\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_3(z_j)}{\theta_3(z_j)} \ . \tag{105}
\]

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r})}{\theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r})}{\theta_4(z_{j-r})} \right] = 0 \ . \tag{106}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_2(z_{j+r}) \theta_2(z_{j+4r}) \ldots \theta_2(z_{j+(l-1)r})}{\theta_4(z_j) \theta_4(z_{j+r}) \theta_4(z_{j+4r}) \ldots \theta_4(z_{j+(l-1)r})} = \left[ \prod_{k=1}^{l-1} \frac{\theta_3^2(2kr\pi/p)}{\theta_4^2(2kr\pi/p)} \right] \prod_{n \neq k, n=1}^{l} \frac{\theta_3(2[n-k]r\pi/p)}{\theta_4(2[n-k]r\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \ , \tag{107}
\]

where \( l \) is any odd integer (3 \( \leq l \leq p \)). In the special case when \( l = p \), this identity takes the form

\[
\Pi_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} = \Pi_{n=1}^{p-1} \frac{\theta_3^2(2n\pi/p)}{\theta_4^2(2n\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \ . \tag{108}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j) \theta_2(z_{j+r}) \theta_2(z_{j+4r}) \ldots \theta_2(z_{j+(l-1)r})}{\theta_4(z_j) \theta_4(z_{j+r}) \theta_4(z_{j+4r}) \ldots \theta_4(z_{j+(l-1)r})} = 2 \frac{\theta_4(0) \theta_2(2\pi/p) \theta_4(2\pi/p)}{\theta_3^2(2\pi/p)} \left[ \frac{\theta_2(2\pi/p)}{\theta_2(0)} - \frac{\theta_4(0) \theta_3^2(2\pi/p)}{\theta_3^2(0) \theta_4(2\pi/p)} \right] \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \ . \tag{109}
\]
\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+r}) \theta_3(z_{j+r})}{\theta_4^2(z_{j+r})} + \frac{\theta_2(z_{j-r}) \theta_3(z_{j-r})}{\theta_4^2(z_{j-r})} \right] \\
= -2 \frac{\theta_2^3(0) \theta_3(2r\pi/p)}{\theta_3(0) \theta_4^2(2r\pi/p)} \left[ \frac{\theta_2(2r\pi/p)}{\theta_4(0)} - \frac{\theta_4(2r\pi/p)}{\theta_4(0)} \right] \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)}. \tag{110}
\]

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_2(z_{j+r})}{\theta_4(z_{j+r}) \theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r}) \theta_2(z_{j-r})}{\theta_4(z_{j-r}) \theta_4(z_{j-r})} \right] \\
= -2 \frac{\theta_2^3(0) \theta_3(2r\pi/p)}{\theta_3(0) \theta_4^2(2r\pi/p)} \left[ \frac{\theta_2(2r\pi/p)}{\theta_4(0)} - \frac{\theta_4(2r\pi/p)}{\theta_4(0)} \right] \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)}. \tag{111}
\]

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_2(z_{j+s}) \theta_3(z_{j+s})}{\theta_4(z_{j+s}) \theta_4(z_{j+s})} + \frac{\theta_2(z_{j-s}) \theta_3(z_{j-s})}{\theta_4(z_{j-s}) \theta_4(z_{j-s})} \right] \\
= -2 \frac{\theta_4(2[r-s]\pi/p) \theta_2(2r\pi/p)}{\theta_1(2[r-s]\pi/p) \theta_1(2r\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)}. \tag{112}
\]

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_2(z_{j+r})}{\theta_4(z_{j+r}) \theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r}) \theta_2(z_{j-r})}{\theta_4(z_{j-r}) \theta_4(z_{j-r})} \right] \\
= 2 \frac{\theta_4(2r\pi/p) \theta_4(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)}. \tag{113}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_1(z_{j+r})}{\theta_4(z_{j+r}) \theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r}) \theta_1(z_{j-r})}{\theta_4(z_{j-r}) \theta_4(z_{j-r})} \right] \\
= 2 \frac{\theta_4(2r\pi/p) \theta_4(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)}. \tag{114}
\]

\[
\sum_{j=1}^{p} \frac{\theta_3(z_j)}{\theta_4(z_j)} \left[ \frac{\theta_3(z_{j+s}) \theta_3(z_{j+s})}{\theta_4(z_{j+s}) \theta_4(z_{j+s})} + \frac{\theta_3(z_{j-s}) \theta_3(z_{j-s})}{\theta_4(z_{j-s}) \theta_4(z_{j-s})} \right] \\
= -2 \frac{\theta_2(2r\pi/p) \theta_2(2s\pi/p)}{\theta_1(2r\pi/p) \theta_1(2s\pi/p)} \sum_{j=1}^{p} \frac{\theta_2(z_j)}{\theta_4(z_j)}. \tag{115}
\]

\[
\sum_{j=1}^{p} \frac{\theta_2^3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r}) \theta_3(z_{j+r})}{\theta_4(z_{j+r}) \theta_4(z_{j+r})} + \frac{\theta_1(z_{j-r}) \theta_3(z_{j-r})}{\theta_4(z_{j-r}) \theta_4(z_{j-r})} \right] \\
= -2 \frac{\theta_2^3(0) \theta_2(2r\pi/p)}{\theta_3(0) \theta_4^2(2r\pi/p)} \left[ \frac{\theta_4(2r\pi/p)}{\theta_4(0)} + \frac{\theta_2(2r\pi/p)}{\theta_4(0)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j) \theta_3(z_j)}{\theta_4(z_j)}. \tag{116}
\]
\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_3(z_{j+r})}{\theta_2^2(z_{j+r})} + \frac{\theta_2(z_{j-r})\theta_3(z_{j-r})}{\theta_2^2(z_{j-r})} \right] = -2\frac{\theta_2^2(0)\theta_3(2\pi/p)}{\theta_3(0)\theta_4^2(2\pi/p)} \left[ \frac{\theta_4(2\pi/p)}{\theta_4(0)} + \frac{\theta_2(2\pi/p)}{\theta_2(0)} \right] \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(117)

\[
\sum_{j=1}^{p} \frac{\theta_2^2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r}) + \theta_1(z_{j-r})}{\theta_4(z_{j+r}) + \theta_4(z_{j-r})} \right] = 2\frac{\theta_2^3(0)\theta_2(2\pi/p)\theta_3(2\pi/p)}{\theta_3^2(0)\theta_4^2(2\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(118)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_4^3(z_{j+r}) + \theta_4^3(z_{j-r})}{\theta_4^2(z_{j+r}) + \theta_4^2(z_{j-r})} \right] = 2\frac{\theta_2^3(0)\theta_4^2(2\pi/p)}{\theta_4^2(0)\theta_4^2(2\pi/p)} \left[ \frac{\theta_4^2(2\pi/p)}{\theta_4^2(0)} + \frac{\theta_2^2(2\pi/p)}{\theta_2^2(0)} + \frac{\theta_2^2(2\pi/p)}{\theta_2^2(0)} \right] + 3\frac{\theta_2^2(0)\theta_4(2\pi/p)\theta_4^2(2\pi/p)}{\theta_5^2(0)\theta_2(2\pi/p)\theta_4(0)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(119)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_2(z_{j+r})\theta_4(z_{j+s}) + \theta_2(z_{j-r})\theta_4(z_{j-s})}{\theta_4(z_{j+r})\theta_4(z_{j+s}) + \theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2\frac{\theta_1(0)\theta_4(2\pi/p)}{\theta_4(0)\theta_4(2\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(120)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_3(z_{j+r})\theta_3(z_{j+s}) + \theta_3(z_{j-r})\theta_3(z_{j-s})}{\theta_4(z_{j+r})\theta_4(z_{j+s}) + \theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2\frac{\theta_2^2(0)\theta_2(2\pi/p)\theta_2(2\pi/p)}{\theta_2^2(0)\theta_2(2\pi/p)\theta_2(2\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(121)

\[
\sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_1(z_{j+s}) + \theta_1(z_{j-r})\theta_1(z_{j-s})}{\theta_4(z_{j+r})\theta_4(z_{j+s}) + \theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = 2\frac{\theta_2^3(0)\theta_4(2\pi/p)\theta_4(2\pi/p)}{\theta_3^2(0)\theta_1(2\pi/p)\theta_1(2\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(122)

\[
\sum_{j=1}^{p} \frac{\theta_2(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} \left[ \frac{\theta_1(z_{j+r})\theta_2(z_{j+s}) + \theta_1(z_{j-r})\theta_2(z_{j-s})}{\theta_4(z_{j+r})\theta_4(z_{j+s}) + \theta_4(z_{j-r})\theta_4(z_{j-s})} \right] = -2\frac{\theta_2(0)\theta_4(2\pi/p)\theta_3(2\pi/p)}{\theta_3^2(0)\theta_1(2\pi/p)\theta_1(2\pi/p)} \sum_{j=1}^{p} \frac{\theta_1(z_j)\theta_3(z_j)}{\theta_4^2(z_j)} .
\]

(123)
increasing as well as basic ratios
ratios like say \( \theta \)
\( T \frac{\tau}{p} \)
We shall now give some general comments about extending the above results in several directions.

6 Comments and discussion.

We shall now give some general comments about extending the above results in several directions.

(i) **Identities for auxiliary functions:** Until now we have discussed identities for the three basic ratios \( \theta_1(z)/\theta_4(z) \), \( \theta_2(z)/\theta_4(z) \), \( \theta_3(z)/\theta_4(z) \). What about the identities for the remaining nine ratios like say \( \theta_1(z)/\theta_2(z) \)? These are readily obtained by making use of the identities coming from increasing \( z \) by the half periods \( \pi/2 \), \( \pi \tau/2 \) and \( \pi(1+\tau)/2 \). For example, using \[ \theta_1(z) \theta_2(z) = -i \theta_3(z + \frac{\pi}{2}[1+\tau]) \theta_4(z + \frac{\pi}{2}[1+\tau]), \] we immediately obtain identities for the ratio \( \theta_1(z)/\theta_2(z) \).

(ii) **Identities for shifts in units of \( \pi \tau/p \) or \( \pi(1-\tau)/p \):** So far we have focused our attention on identities involving ratios of Jacobi theta functions evaluated at points separated by gaps of \( T/p \) with \( T \) real (and equal to \( \pi \) or \( 2\pi \)). But since the ratios of theta functions are also periodic with period \( T\tau/p \) as well as \( T(1-\tau)/p \), we can convert each of our identity to another one involving points separated either by gaps of \( T\tau/p \) or \( T(1-\tau)/p \). In fact this is easily done using the modular transformations \[
\frac{\theta_1(z, \tau_1)}{\theta_4(z, \tau_1)} = -i \frac{\theta_1(\tau z, \tau)}{\theta_3(\tau z, \tau)} \quad \frac{\theta_2(z, \tau_1)}{\theta_4(z, \tau_1)} = \frac{\theta_2(\tau z, \tau)}{\theta_2(\tau z, \tau)} \quad \frac{\theta_3(z, \tau_1)}{\theta_4(z, \tau_1)} = -i \frac{\theta_3(\tau z, \tau)}{\theta_2(\tau z, \tau)},
\]
\[
\frac{\theta_1(z, \tau_3)}{\theta_4(z, \tau_3)} = \theta_1([1-\tau]z, \tau) \quad \frac{\theta_2(z, \tau_3)}{\theta_4(z, \tau_3)} = \frac{\theta_3([1-\tau]z, \tau)}{(i)^{1/2} \theta_4([1-\tau]z, \tau)} \quad \frac{\theta_3(z, \tau_3)}{\theta_4(z, \tau_3)} = \frac{\theta_4([1-\tau]z, \tau)}{(i)^{1/2} \theta_4([1-\tau]z, \tau)},
\]
where \( \tau_1 = -1/\tau \) while \( \tau_3 = \tau/(1 - \tau) \) which correspond to changing the modular parameter \( m \) to \( 1 - m \) and \( 1/m \) respectively.

(iii) **Identities for products of ratios of Jacobi theta functions:** In previous discussions, we have evaluated Jacobi theta functions at points separated by gaps of \( T/p, T\tau/p \) or \( T(1 - \tau)/p \). An obvious question is if we can also obtain identities for products of ratios of theta functions like say \( \theta_1 \theta_2 / \theta_3 \theta_4 \). The answer is yes and in fact these can be easily obtained from the identities derived in this paper by noting the relations

\[
\frac{\theta_2(z)\theta_3(z)}{\theta_1(z)\theta_4(z)} = \frac{\theta_3(z)\theta_3(2z + \pi\tau/2) + \theta_2(z)\theta_2(2z + \pi\tau/2)}{\theta_4(z)\theta_4(2z + \pi\tau/2)}.
\]  
(129)

\[
\frac{\theta_1(z)\theta_3(z)}{\theta_2(z)\theta_4(z)} = \frac{\theta_2(z)\theta_1(2z + \pi\tau/2) - i\theta_4(z)\theta_3(2z + \pi\tau/2)}{\theta_3(z)\theta_4(z)}.
\]
(130)

\[
\frac{\theta_1(z)\theta_2(z)}{\theta_3(z)\theta_4(z)} = \frac{\theta_3(z)\theta_1(2z + \pi\tau/2) - i\theta_4(z)\theta_2(2z + \pi\tau/2)}{\theta_2(z)\theta_4(z)}.
\]
(131)

\[
\frac{\theta_2(z)\theta_4(z)}{\theta_1(z)\theta_3(z)} = \frac{\theta_2(z)\theta_1(2z + \pi\tau/2) + i\theta_4(z)\theta_3(2z + \pi\tau/2)}{\theta_3(z)\theta_4(z)}.
\]
(132)

\[
\frac{\theta_1(z)\theta_4(z)}{\theta_2(z)\theta_3(z)} = \frac{\theta_2(z)\theta_1(2z + \pi\tau/2) - \theta_3(z)\theta_2(2z + \pi\tau/2)}{\theta_3(z)\theta_4(z)}.
\]
(133)

\[
\frac{\theta_3(z)\theta_4(z)}{\theta_1(z)\theta_2(z)} = \frac{\theta_3(z)\theta_1(2z + \pi\tau/2) + i\theta_4(z)\theta_2(2z + \pi\tau/2)}{\theta_2(z)\theta_4(z)}.
\]
(134)

Another interesting question to which we do not know the answer is, just as the ratios of Jacobi theta functions (which usually occur with genus one in string theory) satisfy various identities, do the higher genus theta functions also satisfy similar identities?

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