Abstract. We study the structure of the symplectic invariant part $h_{g,1}^{Sp}$ of the Lie algebra $h_{g,1}$ consisting of symplectic derivations of the free Lie algebra generated by the rational homology group of a closed oriented surface $\Sigma$ of genus $g$.

First we describe the orthogonal direct sum decomposition of this space which is induced by the canonical metric on it and compute it explicitly up to degree 20. In this framework, we give a general constraint which is imposed on the $Sp$-invariant component of the bracket of two elements in $h_{g,1}$. Second we clarify the relations among $h_{g,1}$ and the other two related Lie algebras $h_{g,*}$ and $h_{g}$ which correspond to the cases of a closed surface $\Sigma$ with and without base point $* \in \Sigma$. In particular, based on a theorem of Labute, we formulate a method of determining these differences and describe them explicitly up to degree 20. Third, by giving a general method of constructing elements of $h_{g,1}^{Sp}$, we reveal a considerable difference between the two submodules of it, one is the $Sp$-invariant part of a certain ideal $h_{g,1}$ and the other is that of the Johnson image.

Finally we combine these results to determine the structure of $h_{g,1}$ completely up to degree 6 including the unstable cases where the genus 1 case has an independent meaning. In particular, we see a glimpse of the Galois obstructions explicitly from our point of view.

1. Introduction and Statements of the Main Results

Let $\Sigma_{g,1}$ be a compact oriented surface of genus $g \geq 1$ with one boundary component and we denote its first integral homology group $H_1(\Sigma_{g,1}; \mathbb{Z})$ simply by $H$ and let $H_Q = H \otimes \mathbb{Q}$. We denote by $L_{g,1}$ the free graded Lie algebra generated by $H_Q$ and let $h_{g,1}$ be the graded Lie algebra consisting of symplectic derivations of $L_{g,1}$. Let $h_{g,1}^+$ be the ideal consisting of derivations with positive degrees. This Lie algebra was introduced in the theory of Johnson homomorphisms (see [19]) and has been investigated extensively. We also consider closely related Lie algebras, denoted by $h_{g,*}$ and $h_{g}$ which correspond to the cases of a closed surface $\Sigma$ with and without base point $* \in \Sigma$.

Let $Sp(2g, \mathbb{Q})$ be the symplectic group which we sometimes denote simply by $Sp$. If we fix a symplectic basis of $H_Q$, then it can be considered as the standard representation of $Sp(2g, \mathbb{Q})$. Each piece $h_{g,1}(k), h_{g,*}(k), h_{g}(k)$, of the three graded Lie algebras, is naturally an $Sp$-module so that it has an irreducible decomposition. Let $h_{g,1}^{Sp}$ denote the Lie subalgebra of $h_{g,1}$ consisting of $Sp$-invariant elements. We denote by $h_{g,1}(2k)^{Sp}$ the degree $2k$ part of this Lie subalgebra. We use similar notations for the other two cases $h_{g,*}$ and $h_{g}$.
In [24] a canonical metric on \((H_Q^{\otimes 2k})^{Sp}\) is defined. We can consider \(h_{g,1}(2k)^{Sp}\) as a subspace of \((H_Q^{\otimes (2k+2)})^{Sp}\) so that it has the induced metric. To formulate our results, we use the following terminology. A Young diagram \(\lambda\) is denoted by \([\lambda_1 \lambda_2 \cdots \lambda_h]\) (\(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h\)) and the number of boxes in \(\lambda\), namely \(\lambda_1 + \cdots + \lambda_h\), is denoted by \(|\lambda|\). We also denote the number of rows of \(\lambda\), namely \(h\) in the above notation, by \(h(\lambda)\). For a given Young diagram \(\lambda\) as above, the symbol \(\lambda^d\) denotes another Young diagram \([\lambda_1 \lambda_1 \cdots \lambda_h(\lambda) \lambda_h(\lambda)\] which has multiple double floors. Also \(S_k\) denotes the symmetric group of order \(k\).

**Theorem 1.1.** With respect to the canonical metric on \(h_{g,1}(2k)^{Sp}\), there exists an orthogonal direct sum decomposition

\[
h_{g,1}(2k)^{Sp} \cong \bigoplus_{|\lambda|=k+1, \ h(\lambda) \leq g} H_\lambda
\]

in terms of certain subspaces \(H_\lambda\) which are indexed by Young diagrams \(\lambda = [\lambda_1 \cdots \lambda_h(\lambda)]\) with \((k+1)\) boxes. The dimension of \(H_\lambda\) is given by the following formula

\[
\dim H_\lambda = \frac{1}{(2k+2)!} \sum_{\gamma \in S_{2k+2}} \chi_{2k}(\gamma) \chi_{\lambda^d}(\gamma)
\]

where \(\chi_{2k}\) is the character

\[
\chi_{2k}(1^{k+2}) = (2k)! , \quad \chi_{2k}(1^1a^b) = (b-1)! \ a^{b-1} \mu(a) (\mu: \text{the Möbius function})
\]

\[
\chi_{2k}(a^b) = -(b-1)! \ a^{b-1} \mu(a) , \quad \chi_{2k}(\text{other conjugacy class}) = 0
\]

defined by Kontsevich and \(\chi_{\lambda^d}\) denotes the character of the irreducible representation of \(S_{2k+2}\) corresponding to the Young diagram \(\lambda^d = [\lambda_1 \lambda_1 \cdots \lambda_h(\lambda) \lambda_h(\lambda)]\) which has \((2k+2)\) boxes.

It follows that any \(Sp\)-invariant element \(\xi \in h_{g,1}(k)^{Sp}\) can be expressed as

\[
\xi = \sum_{|\lambda|=k+1, \ h(\lambda) \leq g} \xi_\lambda \quad (\xi_\lambda \in H_\lambda).
\]

We call \(\xi_\lambda\) the \(\lambda\)-coordinate of \(\xi\).

**Theorem 1.2.** The stable range with respect to the genus \(g\) of the spaces \(h_{g,1}(2k)^{Sp}\) is given by

\[
\dim h_{k,1}(2k)^{Sp} = \dim h_{k+1,1}(2k)^{Sp} = \cdots \quad (k \text{ odd})
\]

\[
\dim h_{k-1,1}(2k)^{Sp} = \dim h_{k,1}(2k)^{Sp} = \cdots \quad (k \text{ even}).
\]

Furthermore we have

\[
\dim h_{k,1}(2k)^{Sp} = \dim h_{k-1,1}(2k)^{Sp} + 1
\]

for any odd \(k \geq 3\).

By combining the above theorem with our earlier computation given in [25], we have determined the orthogonal decomposition of \(h_{g,1}(2k)^{Sp}\) explicitly for all \(2k \leq 20\) (see Table 2, Table 3 in Section 2 and Tables 9, 12 in Section 9).

It is a very important problem to determine whether a given element in \(h_{g,1}(k)^{Sp}\) can be expressed as a linear combination of brackets of elements of \(h_{g,1}\) with lower degrees and, in case it can be, we have a further problem of determining whether the difference
between various expressions give rise to non-trivial elements in $H_2(\mathfrak{h}_{g,1})$ or not. In this regard, we obtain the following result.

For each Young diagram $\lambda$ with $k$ boxes, let $V^k_\lambda$ denote the isotypical component of $H^\otimes_k$, considered as a GL-module, corresponding to $\lambda$, namely it is the sum of all the GL-submodules of $H^\otimes_k$ which are isomorphic to the GL-irreducible representation $\lambda_{GL}$. Then we can write

$$H^\otimes_k = \bigoplus_{|\lambda|=k} V^k_\lambda.$$  

We consider $\mathfrak{h}_{g,1}(k)$ to be a subspace of $H^\otimes_{k+2}$ and for each Young diagram $\lambda$ with $(k + 2)$ boxes, we set $\bar{H}_\lambda = V^k_{\lambda} \cap \mathfrak{h}_{g,1}(k)$. Then we can write

$$\mathfrak{h}_{g,1}(k) = \bigoplus_{|\lambda|=k+2} \bar{H}_\lambda$$

where $\bar{H}_\lambda$ is the totality of GL-irreducible summands of $\mathfrak{h}_{g,1}(k)$ which are isomorphic to $\lambda_{GL}$. Hence we can write

$$\bar{H}_\lambda \cong \lambda_{GL}^{\oplus m_\lambda}$$

where $m_\lambda$ denotes the multiplicity of $\lambda_{GL}$ in $\mathfrak{h}_{g,1}(k)$. We call $\bar{H}_\lambda$ the $\lambda$-isotypical component of $\mathfrak{h}_{g,1}(k)$. In relation to Theorem 1.1, we have the following. If $\lambda$ is a Young diagram with $(k + 1)$ boxes, then we have

$$H_\lambda \cong H^\text{Sp}_{\lambda} \quad \text{and} \quad \bar{H}_\lambda \cong H_\lambda \otimes \lambda_{GL}^k.$$  

**Theorem 1.3.** Let $\bar{H}_\lambda \subset \mathfrak{h}_{g,1}(k)$ be the $\lambda$-isotypical component of $\mathfrak{h}_{g,1}(k)$ and let $\bar{H}_\mu \subset \mathfrak{h}_{g,1}(\ell)$ be the $\mu$-isotypical component of $\mathfrak{h}_{g,1}(\ell)$ where $\lambda$ and $\mu$ denote Young diagrams with $(k + 2)$ boxes and $(\ell + 2)$ boxes respectively.

(i) The bracket $[\bar{H}_\lambda, \bar{H}_\mu]$ is included in a GL-submodule $B(\lambda, \mu)$ of $\mathfrak{h}_{g,1}(k + \ell)$ which is defined by

$$B(\lambda, \mu) = \bigoplus_{\nu \text{ satisfies } (C)} \bar{H}_\nu.$$  

Here the condition $(C)$ is given by

$(C): (\lambda_{GL} \otimes \mu_{GL})$ and $((\wedge^2 H_\text{Q} \otimes \nu_{GL})$ have a common GL-irreducible summand.

In particular, the height of such a $\nu$ must satisfy the following inequality

$$\max \{h(\lambda), h(\mu)\} - 2 \leq h(\nu) \leq h(\lambda) + h(\mu).$$

(ii) The $\text{Sp}$-invariant part $[\bar{H}_\lambda, \bar{H}_\mu]^\text{Sp}$ of $[\bar{H}_\lambda, \bar{H}_\mu]$ is included in a submodule $S(\lambda, \mu)$ of $\mathfrak{h}_{g,1}(k + \ell)^\text{Sp}$ which is defined by

$$S(\lambda, \mu) = \bigoplus_{\nu \text{ satisfies } (S)} H_\nu.$$  

Here the condition $(S)$ is given by

$(S): (\lambda_{GL} \otimes \mu_{GL})$ and $((\wedge^2 H_\text{Q} \otimes \nu_{GL})$ have a common GL-irreducible summand.

In particular, the $\nu$-coordinate of any element in $[\bar{H}_\lambda, \bar{H}_\mu]^\text{Sp}$ vanishes for all $\nu$ such that $2h(\nu) > h(\lambda) + h(\mu)$ or $2h(\nu) < \max \{h(\lambda), h(\mu)\} - 2$. 

Remark 1.4. The above result shows that the intersection matrix for the parings of the bracket operation
\[
\sum_{i+j=k} \mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(j) \rightarrow \mathfrak{h}_{g,1}(k)
\]
is, so to speak, “lower anti-triangular” with respect to the heights of Young diagrams which appear in the GL-irreducible decompositions of \(\mathfrak{h}_{g,1}(k)\)’s. Furthermore the non-vanishing area is a rather restricted one near the anti-diagonal.

Example 1.5. Asada-Nakamura [2] proved that there exists a unique copy \([2k+1,1]^2]_{GL} \subset \mathfrak{h}_{g,1}(2k+1)\) for any \(k \geq 1\). On the other hand, as an \(Sp\)-module, we have an \(Sp\)-irreducible decomposition
\[
[2k+1,1]^2]_{GL} = [2k+1,1]_{Sp} \oplus [2k,1]_{Sp} \oplus [2k+1]_{Sp}
\]
for all \(g \geq 3\). The second wedge product of each of these three irreducible components gives rise to an \(Sp\)-invariant element contained in \(\mathfrak{h}_{g,1}(4k+2)\). The above theorem implies that the \(\lambda\)-coordinate of this element vanishes for all \(\lambda\) with \(h(\lambda) > 3\). On the other hand, explicit computation for the case \(k = 1\) shows that the \([21,2]^{\lambda}\)-coordinate does not vanish so that the above theorem gives the best possible result.

Proofs of the above theorems are given in Section 2.

In Section 3, we compare the three Lie algebras \(\mathfrak{h}_{g,1}, \mathfrak{h}_{g,*}, \mathfrak{h}_g\). These Lie algebras are the rational forms of the corresponding Lie algebras \(\mathfrak{h}_{g,1}^Z, \mathfrak{h}_{g,*}^Z, \mathfrak{h}_g^Z\) which are defined over \(Z\). In the cases of the corresponding mapping class groups denoted by \(M_{g,1}, M_{g,*}, M_g\), the relations among them are described by the following two well-known extensions
\[
0 \rightarrow \mathbb{Z} \rightarrow M_{g,1} \rightarrow M_{g,*} \rightarrow 1,
1 \rightarrow \pi_1 \Sigma_g \rightarrow M_{g,*} \rightarrow M_g \rightarrow 1
\]
which hold for any \(g \geq 2\). In the cases of the above Lie algebras over \(Z\), the relations are described by the following extensions
\[
0 \rightarrow \mathbb{Z} \rightarrow \mathfrak{j}_{g,1}^Z \rightarrow \mathfrak{h}_{g,1}^Z \rightarrow \mathfrak{h}_{g,*}^Z \rightarrow 0,
0 \rightarrow \mathfrak{L}_g^Z \rightarrow \mathfrak{h}_{g,*}^Z \rightarrow \mathfrak{h}_g^Z \rightarrow 0
\]
where \(\mathfrak{j}_{g,1}^Z\) is a certain ideal of \(\mathfrak{h}_{g,1}^Z\) and \(\mathfrak{L}_g^Z\) denotes the Malcev Lie algebra, over \(Z\), of \(\pi_1 \Sigma_g\) (see [23] and Section 2 for more details).

In Section 4, we formulate a method of describing the \(Sp\)-decompositions of the two Lie algebras \(\mathfrak{h}_{g,*}, \mathfrak{h}_g\) which is based on a theorem of Labute [15].

Theorem 1.6. Over the rationals, we have a direct sum decomposition
\[
\mathfrak{h}_{g,1}(k) \cong \mathfrak{j}_{g,1}(k) \oplus \mathfrak{L}_g(k) \oplus \mathfrak{h}_g(k).
\]
Furthermore there exists an explicit method of determining the \(Sp\)-irreducible decompositions of the \(Sp\)-modules \(\mathfrak{j}_{g,1}(k)\) and \(\mathfrak{L}_g(k)\).

See Theorem 4.3 for the precise decompositions mentioned here. By applying this method and extending our results of [25], we obtain explicit \(Sp\)-decompositions of \(\mathfrak{j}_{g,1}(k), \mathfrak{L}_g(k)\) and \(\mathfrak{h}_g(k)\) for all \(k \leq 20\). See Section 4 for details.
If we apply various contractions to any GL-irreducible summand of GL-irreducible decomposition of $\mathfrak{h}_{g,1}(k)$, then we obtain various Sp-irreducible components. Of course non-isomorphic GL-irreducible summands may produce isomorphic Sp-irreducible components. Keeping this fact in mind, to analyze the structure of the Lie algebra $\mathfrak{h}_{g,1}$, we propose to take the process of contractions into consideration. We formulate this idea in Section 5 under the names of descendants and ancestors. See Definition 5.1 and Theorem 5.4.

In Section 6 we give a general method of constructing elements of $\mathfrak{h}_{g,1}^\text{Sp}$ and by using it, we reveal a considerable difference in property between the Sp-invariant parts of the two Lie subalgebras of $\mathfrak{h}_{g,1}$, one is the ideal $\mathfrak{j}_{g,1}$ and the other is the Johnson image $\text{Im} \tau_{g,1}$. See Theorem 6.8.

By making use of the above results, together with the Enomoto-Satoh map given in [7], we have extended known results considerably to obtain a complete description of the Lie algebra $\mathfrak{h}_{g,1}$ up to degree 6. It is summarized in the following theorem and we give more detailed structure theorem for $\mathfrak{h}_{g,1}(6)^\text{Sp}$ in Section 7 (see Theorem 7.5).

**Theorem 1.7.** The structure of the Lie algebra $\mathfrak{h}_{g,1}$ up to degree 6 is as in Table 7 where the symbol with double parentheses, e.g. $[3]$, means that it remains in the abelianization $H_1(\mathfrak{h}_{g,1}^\text{Sp})$.

| $k$ | $\mathfrak{h}_{g,1}(k)$ | $\mathfrak{j}_{g,1}(k)$ | $\mathcal{L}_{\tau}(k)$ | $\text{Im} \tau_g(k)$ | $\text{Coker} \tau_g(k)$ |
|-----|----------------|----------------|----------------|---------------------|---------------------|
| 1   | $1^{3}[1]$    | $0$          | $1^{2}$        | $1^{2}$            | $[3]$               |
| 2   | $2^{2}[1^{2}]$| $0$          | $2^{2}$        | $[3]$               |                     |
| 3   | $3^{2}[1]^{3}$| $0$          | $[21]$         | $3^{2}$            |                     |
| 4   | $4^{2}[1]^{3}[2^{4}]$ | $[2]$ | $31^{2}[1]^{2}$ | $2^{1}$            |                     |
| 5   | $5^{2}[1]^{3}[21]^{2}$ | $2[1]^{3}$ | $[21]^{2}[1]^{3}$ | $[5]$             |                     |
| 6   | $6^{2}[1]^{3}[21]^{3}[3]^{2}$ | $2[1]^{3}$ | $[21]^{2}[1]^{3}$ | $[5]$             |                     |

| $k$ | $\mathfrak{h}_{g,1}(k)$ | $\mathfrak{j}_{g,1}(k)$ | $\mathcal{L}_{\tau}(k)$ | $\text{Im} \tau_g(k)$ | $\text{Coker} \tau_g(k)$ |
|-----|----------------|----------------|----------------|---------------------|---------------------|
| 7   | $7^{2}[1]^{3}[21]^{3}[3]^{2}$ | $2[1]^{3}$ | $[21]^{2}[1]^{3}$ | $[5]$             |                     |

**Table 1.** List of $\mathfrak{h}_{g,1}(k)$, $\mathfrak{j}_{g,1}(k)$, $\mathcal{L}_{\tau}(k)$, $\text{Im} \tau_g(k)$, $\text{Coker} \tau_g(k)$.
In Table 1, the symbol $[1^3]$ denotes the $\text{Sp}$-irreducible representation corresponding to the Young diagram $[1^3] = [111]$. Also the symbol $\tau_g(k)$ denotes the Johnson homomorphism $\tau_g(k) : \mathcal{M}_g(k) \to \mathfrak{h}_g(k)$ for closed surface where $\{\mathcal{M}_g(k)\}_k$ denotes the Johnson filtration for the mapping class group $\mathcal{M}_g$. Details are given in Section 7.

In Section 8, we study the case of genus 1. This is motivated by the theory of universal mixed elliptic motives due to Hain and Matsumoto (cf. [11]). They show, among other things, that certain Galois obstructions appear in the $\text{Sp}$-invariant part $\mathfrak{h}^\text{Sp}_{g,1}$. We begin to study these elements from our point of view (see Theorem 8.5). Here the theory of Satoh [29] and Enomoto and Satoh [7] play an important role.

Finally we mention two reasons why we put an emphasis on the $\text{Sp}$-invariant part $\mathfrak{h}^\text{Sp}_{g,1}$ in our study of the whole Lie algebra $\mathfrak{h}_{g,1}$. One is that the Galois obstructions, predicted by Oda and proved by Nakamura [26] and Matsumoto [17] independently, appear in this $\text{Sp}$-invariant Lie subalgebra. See a recent survey article [18] of Matsumoto. The Galois obstructions also appear in the genus 1 symplectic derivation algebra as already mentioned above. The precise description of these obstructions is still a mystery and it should be a very important problem both in number theory and topology. See also Willwacher [30] for a related work.

The other concerns another important problem of deciding whether the composition

$$\mathfrak{h}^\text{Sp}_{g,1} \subset \mathfrak{h}_{g,1} \to \lim_{g \to \infty} H_1(\mathfrak{h}_{g,1})$$

is trivial. This is equivalent to the vanishing of the top homology group of $\text{Out} F_n$ with respect to its virtual cohomological dimension determined by Culler and Vogtmann [6] (see Conjecture 1.3. and Remark 1.5. in [25], and as for a recent result on $H_1(\mathfrak{h}_{g,1}^+)$, see [3]). This is also related to one more mystery in low dimensional topology because we can show that there exists a surjective homomorphism $H_1(\mathcal{H}_{g,1}; \mathbb{Q}) \to H_1(\mathfrak{h}_{g,1})$ where $\mathcal{H}_{g,1}$ denotes the group of homology cobordism classes of homology cylinders introduced by Garoufalidis and Levine [9]. It should be a very important problem to determine whether $H_1(\mathcal{H}_{g,1}; \mathbb{Q}) = 0$ or not. Recall here that Cha, Friedl and Kim [3] proved that $H_1(\mathcal{H}_{g,1}; \mathbb{Z}/2)$ contains $(\mathbb{Z}/2)\infty$ as a direct summand.

Acknowledgement The authors would like to thank Naoya Enomoto and Takao Satoh for enlightening discussion about the cokernel of the Johnson homomorphisms. Thanks are also due to Richard Hain, Makoto Matsumoto and Hiroaki Nakamura for helpful information about the arithmetic mapping class groups.

The authors were partially supported by KAKENHI (No. 24740040 and No. 24740035), Japan Society for the Promotion of Science, Japan.

2. ORTHOGONAL DECOMPOSITION OF $\mathfrak{h}^\text{Sp}_{g,1}$

In this section, we prove Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1 As already mentioned, the space $\mathfrak{h}_{g,1}(2k)^\text{Sp}$ can be considered to be a subspace of $\left( H_Q^{\otimes (2k+2)} \right)^\text{Sp}$. In [24], a canonical metric on $\left( H_Q^{\otimes (2k+2)} \right)^\text{Sp}$ is defined and
it is shown that there exists an orthogonal direct sum decomposition
\[
\left( H_Q^{\otimes(2k+2)} \right)^{Sp} \cong \bigoplus_{|\lambda|=k+1, \ h(\lambda) \leq g} U_\lambda
\]
in terms of certain subspaces \(U_\lambda\). As an \(S_{2k+2}\)-submodule of \(H_Q^{\otimes(2k+2)}\), \(U_\lambda\) is an irreducible \(S_{2k+2}\)-module corresponding to the Young diagram \(\lambda^\delta\). More precisely, \(GL\)-irreducible representation \(\lambda_{GL}^\delta\) appears in the \(GL\)-irreducible decomposition of \(H_Q^{\otimes(2k+2)}\) if and only if \(h(\lambda) \leq g\) and in this case its multiplicity is equal to the dimension of \(\lambda_{GL}^\delta\). On the other hand, each copy of \(\lambda_{GL}^\delta\) has a unique \(Sp\)-invariant element (up to scalars) and the totality of these \(Sp\)-invariant elements makes a basis of the subspace \(U_\lambda\).

Now it was shown in [23] (see also Proposition 5.6 below) that the space \(h_{g,1}(2k)\) is the image of certain projecting operator acting on \(H_Q^{\otimes(2k+2)}\) which is an element of \(Z[S_{2k+2}]\). More precisely, we have
\[
S_{2k+2} \circ (1 \otimes p_{2k+1})(H_Q^{\otimes(2k+2)}) = h_{g,1}(2k).
\]
By restricting to the \(Sp\)-invariant subspaces, we have
\[
S_{2k+2} \circ (1 \otimes p_{2k+1}) \left( H_Q^{\otimes(2k+2)} \right)^{Sp} = h_{g,1}(2k)^{Sp}.
\]
Since this operator is described in terms of the action of the symmetric group, it sends each subspace \(U_\lambda\) to itself. Hence if we define
\[
H_\lambda = S_{2k+2} \circ (1 \otimes p_{2k+1})(U_\lambda)
\]
we obtain the required orthogonal decomposition. Finally, the formula for the dimension of \(H_\lambda\) follows by applying Corollary 3.2 of [25] because \(\dim H_\lambda\) is equal to the multiplicity of \(\lambda_{GL}^\delta\) in \(h_{g,1}(2k)\). This completes the proof. \(\square\)

As was mentioned in our paper [25], we have determined the \(Sp\)-irreducible decompositions of \(h_{g,1}(k)\) for all \(k \leq 20\). By combining this with Theorem 1.1, we have determined the orthogonal decompositions of \(h_{g,1}(2k)^{Sp}\) for all \(2k \leq 20\). The results are given in Tables 2 and 3 below and further Tables 9-12 are given in Section 9. Here the symbol \(2[31]^\delta\) in Table 2 for example, means that there are two copies of the representation \([31]^\delta_{GL} = [3^21^2]_{GL}\) appear in \(h_{g,1}(6)\) and the unique \(Sp\)-invariant element in each copy contributes to \(h_{g,1}(6)^{Sp}\).

**Proof of Theorem 1.2** It is easy to see that the stable range of \((H_Q^{\otimes 2k})^{Sp}\) is \(g \geq k\). On the other hand, \(h_{g,1}(2k)^{Sp}\) is a submodule of \((H_Q^{\otimes(2k+2)})^{Sp}\) so that the stable range of

TABLE 2. Orthogonal decompositions of $h_{g,1}(2)^{Sp}, h_{g,1}(6)^{Sp}, h_{g,1}(8)^{Sp}$

| $g$  | $h_{g,1}(2)^{Sp}$ | $h_{g,1}(6)^{Sp}$ | $h_{g,1}(8)^{Sp}$ |
|------|-------------------|-------------------|-------------------|
| $= 1$ | 1 [2]$^a$ | 1 [4]$^b$ | 0 @ |
| $= 2$ | 4 [21]$^a$ [2]$^a$ | 2 [41]$^a$ [32]$^a$ | 3 [31]$^a$ |
| $\geq 3$ | 5 [21]$^a$ | 3 [31]$^a$ |

TABLE 3. Orthogonal decompositions of $h_{g,1}(10)^{Sp}, h_{g,1}(12)^{Sp}$

| $g$  | $h_{g,1}(10)^{Sp}$ | $h_{g,1}(12)^{Sp}$ |
|------|-------------------|-------------------|
| $= 1$ | 3 [6]$^a$ | 0 @ |
| $= 2$ | 51 15[51]$^a$6[26]$^a$4[24]$^a$7 [32]$^a$ | 190 3[61]$^a$103[52]$^a$56[43]$^a$ |
| $= 3$ | 97 19[41]$^a$4[24]$^a$3[24]$^a$4[23]$^a$ | 97 68[51]$^a$42[51]$^a$42[32]$^a$2[31]$^a$2[32]$^a$ |
| $= 4$ | 107 7[31]$^a$4[24]$^a$ | 643 36[41]$^a$60[32]$^a$2[31]$^a$2[32]$^a$ |
| $\geq 5$ | 108 [21]$^a$ | 650 [31]$^a$2[21]$^a$ |

$h_{g,1}(2k)^{Sp}$ is smaller than or equal to $g \geq k + 1$. Now, as was mentioned in [25], Proposition 4.1, for any Young diagram $\lambda$

$$\dim (\lambda_{GL})^{Sp} = \begin{cases} 1 & (\lambda: \text{multiple double floors, namely } \lambda = \mu^\delta \text{ for some } \mu) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $\lambda = [\lambda_1, \cdots, \lambda_h]$ be a Young diagram with $(2k + 2)$ boxes and $h$ rows and assume that it is with multiple double floors. Then it is easy to see the following:

$$h = 2k + 2 \Rightarrow \lambda = [1^{2k+2}],$$
$$h = 2k \Rightarrow \lambda = [2^21^{2k-2}],$$
$$h = 2k - 2 \Rightarrow \lambda = [3^21^{2k-4}] \text{ or } [2^41^{2k-6}].$$

Hence, to prove the result it is enough to show the following four facts:

(i) $[1^{2k+2}]_{GL}$ does not appear in the GL-irreducible decomposition of $h_{g,1}(2k)$,

(ii) $[2^21^{2k-2}]_{GL}$ appears in the GL-irreducible decomposition of $h_{g,1}(2k)$

\[\text{with multiplicity 1 for any odd } k \geq 1,\]

(iii) $[2^21^{2k-2}]_{GL}$ does not appear in the GL-irreducible decomposition of $h_{g,1}(2k)$

\[\text{for any even } k,\]

(iv) $[3^21^{2k-4}]_{GL}$ appears in the GL-irreducible decomposition of $h_{g,1}(2k)$

\[\text{for any even } k \geq 4 \text{ with non-zero multiplicity}.\]

(i) follows easily. Indeed $[1^{2k+2}]_{GL}$ is nothing other than the alternating product $\wedge^{2k+2}H_{Q}$ which is not invariant under the $\mathbb{Z}/(2k + 2)$-cyclic permutation of $H_{Q}^{\otimes(2k+2)}$ while any summand of $h_{g,1}(2k)$ must be so. (ii) and (iii) were proved by Enomoto and Satoh in [7]. Indeed, they proved that $[2^21^{2k-2}]_{GL}$ appears in $h_{g,1}(k)$ with multiplicity 1 for any $k$
such that \( k \equiv 1 \text{ or } 2 \mod 4 \) and does not appear otherwise. Finally (iv) follows from an explicit computation using the method of [25] but here we omit it.

To prove Theorem 1.3, we prepare the following.

**Lemma 2.1.** Let

\[ K_{ij} : H_Q^{\otimes (k+2)} \to H_Q^{\otimes k} \quad (1 \leq i < j \leq k+2) \]

be a linear mapping defined by

\[ K_{ij}(u_1 \otimes u_2 \otimes \cdots \otimes u_{k+2}) = (u_i \cdot u_j) u_1 \otimes \cdots \otimes \hat{u}_i \otimes \cdots \otimes \hat{u}_j \otimes \cdots \otimes u_{k+2} \quad (u_i \in H_Q) \]

where \( u_i \cdot u_j \) denotes the intersection number of \( u_i \) and \( u_j \), and the symbol \( \hat{u}_i \) means that we delete \( u_i \). Let \( V \subset H_Q^{\otimes (k+2)} \) be an irreducible \( GL \)-submodule which is isomorphic to \( \lambda_{GL} \) where \( \lambda \) is a Young diagram with \( (k+2) \) boxes. Consider the following condition \((C)\) posed on Young diagrams \( \nu \) with \( |\nu| = k \)

\[(C): (\wedge^2 H_Q \otimes \nu_{GL}) \text{ has a direct summand isomorphic to } \lambda_{GL}.\]

Then we have

\[ K_{ij}(V) \subset \bigoplus_{\nu \text{ satisfies } (C)} V^k_{\nu}. \]

**Proof.** Since the natural action of the symmetric group \( S_{k+2} \) on \( H_Q^{\otimes (k+2)} \) preserves the structure of a \( GL \)-module on it, it is enough to prove the case \( i = 1, j = 2 \). Now consider the following direct sum decomposition

\[ H_Q^{\otimes (k+2)} = (S^2 H_Q \otimes H_Q^{\otimes k}) \oplus (\wedge^2 H_Q \otimes H_Q^{\otimes k}). \]

Since the mapping \( K_{12} \) is trivial on the former summand, if we denote by \( q : H_Q^{\otimes (k+2)} \to \wedge^2 H_Q \otimes H_Q^{\otimes k} \) the natural projection onto the latter summand, then we have

\[ K_{12} = K_{12} \circ q. \]

It follows that \( K_{12}(V) = K_{12}(q(V)) \). Since \( q(V) \) is a \( GL \)-submodule of \( \wedge^2 H_Q \otimes H_Q^{\otimes k} \) isomorphic to \( \lambda_{GL} \), in view of the definition of the condition \((C)\), we can conclude that

\[ q(V) \subset \wedge^2 H_Q \otimes \left( \bigoplus_{\nu \text{ satisfies } (C)} V^k_{\nu} \right). \]

Now the mapping \( K_{12} \) on \( \wedge^2 H_Q \otimes H_Q^{\otimes k} \) is the contraction \( \wedge^2 H_Q \to \mathbb{Q} \) times the identity of \( H_Q^{\otimes k} \). We can now conclude that

\[ K_{12}(V) = K_{12}(q(V)) \subset \bigoplus_{\nu \text{ satisfies } (C)} V^k_{\nu} \]

as required.
Proof of Theorem 13. The bracket operation

\[ B : \mathfrak{h}_{g,1}(k) \otimes \mathfrak{h}_{g,1}(l) \to \mathfrak{h}_{g,1}(k + l) \]

is not a morphism of GL-modules so that we have to be careful here. First we observe that the above bracket can be extended to

\[ \tilde{B} : H_Q^{\otimes (k+2)} \otimes H_Q^{\otimes (l+2)} \to H_Q^{\otimes (k+l+2)} \]

which can be described as follows. Here we understand \( \mathfrak{h}_{g,1}(k) \) as a submodule of \( H_Q^{\otimes (k+2)} \) for any \( k \). We identify the domain of \( \tilde{B} \) with \( H_Q^{\otimes (k+l+4)} \) and define linear isomorphisms

\[ f^{(k,l)}_i : H_Q^{\otimes (k+l+4)} \to H_Q^{\otimes (k+l+4)} \quad (i = 2, 3, \ldots, l+2) \]

\[ b^{(k,l)}_i : H_Q^{\otimes (k+l+4)} \to H_Q^{\otimes (k+l+4)} \quad (i = 2, 3, \ldots, k+2) \]

by setting

\[ f^{(k,l)}_i(u_1 \otimes u_2 \otimes \cdots \otimes u_{k+2} \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{l+2}) = (v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_{k+2} \otimes v_{i+1} \otimes \cdots \otimes v_{l+2}) \]

\[ b^{(k,l)}_i(u_1 \otimes u_2 \otimes \cdots \otimes u_{k+2} \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{l+2}) = (u_1 \otimes u_2 \otimes \cdots \otimes u_i \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{i+2} \otimes u_{i+1} \otimes \cdots \otimes u_{k+2}) \]

Observe here that any of these linear isomorphisms corresponds to a certain element in the symmetric group \( S_{k+l+4} \) which acts on \( H_Q^{\otimes (k+l+4)} \) naturally. Also define a linear mapping

\[ K_i : H_Q^{\otimes (k+l+4)} \to H_Q^{\otimes (k+l+2)} \quad (i = 1, 2, \ldots, k+l+3) \]

by setting

\[ K_i(u_1 \otimes \cdots \otimes u_{k+l+4}) = (u_1 \cdot u_{i+1}) \otimes u_{i+2} \otimes \cdots \otimes u_{k+l+4} \]

Then we can write

\[ \tilde{B} = \sum_{i=2}^{k+2} K_i \circ b^{(k,l)}_i - \sum_{i=2}^{l+2} K_i \circ f^{(k,l)}_i. \]

It is easy to see that the restriction of \( \tilde{B} \) to \( \mathfrak{h}_{g,1}(k) \otimes \mathfrak{h}_{g,1}(l) \) is precisely the bracket operation described in [21].

Now we prove claim (i). Since \( \tilde{B} \) is equal to the bracket operation, for any two elements \( \xi \in \tilde{H}_\lambda \) and \( \eta \in \tilde{H}_\mu \), we have

\[ [\xi, \eta] = \tilde{B}(\xi \otimes \eta). \]

Since any of the mappings \( f^{(k,l)}_i, b^{(k,l)}_i \) corresponds to a certain element in the symmetric group \( S_{k+l+4} \) as already mentioned above, the image of \( \xi \otimes \eta \) by it is again contained in a GL-submodule of \( H_Q^{\otimes (k+l+4)} \) isomorphic to \( \lambda_{\text{GL}} \otimes \mu_{\text{GL}} \). Now we have to consider various \( K_i \) acting on these GL-submodules. Here we apply Lemma 2.1 with replacing \( H_Q^{\otimes (k+2)} \) and \( \lambda_{\text{GL}} \) by \( H_Q^{\otimes (k+l+4)} \) and any of the GL-irreducible component of \( \lambda_{\text{GL}} \otimes \mu_{\text{GL}} \), respectively. Then we can conclude that

\[ \tilde{B}(\xi \otimes \eta) \in B(\lambda, \mu) \]
as required.

Next we prove the condition on $h(\nu)$. The irreducible decomposition of the tensor product $\lambda_{GL} \otimes \mu_{GL}$ is given by the Littlewood-Richardson rule (see [8]) and we see that any of the irreducible summand in this decomposition is represented by a Young diagram whose number of rows, denoted by $h$, satisfies the inequality

$$\max \{h(\lambda), h(\mu)\} \leq h \leq h(\lambda) + h(\mu).$$

Since the height of the Young diagram $[1^2]$ corresponding to $\wedge^2 H_\mathbb{Q}$ is 2, we obtain the condition

$$\max \{h(\lambda), h(\mu)\} - 2 \leq h(\nu) \leq h(\lambda) + h(\mu)$$
on $h(\nu)$ as required.

The claim (ii) follows from (i) because we have the equality

$$\tilde{H}_{\lambda^s}^{Sp} = H_\lambda.$$

□

3. COMPARISON AMONG $h_{g,1}, h_{g,*}, h_g$ AND A DECOMPOSITION OF $h_{g,1}$

We recall the definitions of three kinds of symplectic derivation algebras, denoted by $h_{g,1}^Z, h_{g,*}^Z, h_g^Z$ from [23]. They correspond to three kinds of mapping class groups $M_{g,1}, M_{g,*}, M_g$ of $\Sigma_g$, relative to an embedded disk $D^2 \subset \Sigma_g$, relative to a base point $* \in \Sigma_g$ and without any decoration, respectively. They are Lie algebras defined over $\mathbb{Z}$ and to indicate this fact, we put superscript $Z$ on their symbols. The Lie algebras $h_{g,1}, h_{g,*}, h_g$ in the subtitle denote the rational forms of them and they are Lie algebras over $\mathbb{Q}$.

Let $\mathcal{L}_{g,1}^Z = \bigoplus_{k=1}^\infty \mathcal{L}_{g,1}^Z(k)$ be the free graded Lie algebra, over $\mathbb{Z}$, generated by $H$. Then the degree $k$ part of $h_{g,1}^Z$ can be written as

$$h_{g,1}^Z(k) = \text{Ker} \left( H \otimes \mathcal{L}_{g,1}^Z(k+1) \xrightarrow{[\cdot, \cdot]} \mathcal{L}_{g,1}^Z(k+2) \right).$$

Next let $\mathcal{L}_g^Z$ be the graded Lie algebra, over $\mathbb{Z}$, associated to the lower central series of $\pi_1 \Sigma_g$. Then the degree $k$ part of $h_{g,*}^Z$ can be written as

$$h_{g,*}^Z(k) = \text{Ker} \left( H \otimes \mathcal{L}_g^Z(k+1) \xrightarrow{[\cdot, \cdot]} \mathcal{L}_g^Z(k+2) \right).$$

Let $\omega_0 \in \mathcal{L}_{g,1}^Z(2)$ be the symplectic element and let $I_g = \bigoplus_{k=2}^\infty I_g(k)$ be the ideal of $\mathcal{L}_{g,1}$ generated by $\omega_0$. Then a theorem of Labute (see Theorem 4.2 below) says that there is a natural isomorphism $\mathcal{L}_g \cong \mathcal{L}_{g,1}/I_g$. Keeping this in mind, define an ideal $j_{g,1}^Z$ of $h_{g,1}^Z$ by setting

$$j_{g,1}^Z(k) = \text{Ker} \left( H \otimes I_g(k+1) \xrightarrow{[\cdot, \cdot]} I_g(k+2) \right).$$

Then it is easy to see that there is an isomorphism

$$h_{g,*}^Z \cong h_{g,1}^Z/j_{g,1}^Z.$$
Finally a result of Asada and Kaneko \textsuperscript{[1]} implies that the injective homomorphism 
\[ \pi, \Sigma_g \to \mathcal{M}_{g,*} \] 
induces an injection 
\[ \mathcal{L}_{g}^\mathbb{Z} \subset \mathfrak{h}_{g,*}^\mathbb{Z} \] 
of graded Lie algebras and the former is an ideal of the latter. Then we define \( \mathfrak{h}_g^\mathbb{Z} \) by setting 
\[ \mathfrak{h}_g^\mathbb{Z} = \mathfrak{h}_{g,*}^\mathbb{Z} / \mathcal{L}_{g}^\mathbb{Z}. \]

If we specify the genus \( g \), we write \( H_g \) instead of \( H \). Also fix a symplectic basis \( x_1, y_1, \ldots, x_g, y_g \) for \( H_g \) so that we have an injection \( i : H_g \subset H_{g+1} \) and a projection \( p : H_{g+1} \to H_g \) which is induced by setting \( p(x_{g+1}) = p(y_{g+1}) = 0. \)

**Lemma 3.1.** Under the natural inclusion \( i : H_g^{\otimes (k+2)} \subset H_{g+1}^{\otimes (k+2)} \), we have 
\[ i(\mathfrak{h}_{g,1}^\mathbb{Z}(k)) \subset \mathfrak{h}_{g+1,1}^\mathbb{Z}(k). \]

**Proof.** This is clear from the definition. \( \square \)

**Lemma 3.2.** Under the natural projection \( p : H_{g+1}^{\otimes (k+2)} \to H_g^{\otimes (k+2)} \), we have 
\[ p(\mathfrak{h}_{g+1,1}^\mathbb{Z}(k)) \subset \mathfrak{h}_{g,1}^\mathbb{Z}(k). \]

**Proof.** Any element \( f \in \mathfrak{h}_{g+1,1}(k) \) can be written as 
\[ f = \sum_{i=1}^{g} x_i \otimes \xi_i + x_{g+1} \otimes \xi_{g+1} + \sum_{i=1}^{g} y_i \otimes \eta_i + y_{g+1} \otimes \eta_{g+1} \]
where \( \xi_i, \xi_{g+1}, \eta_i, \eta_{g+1} \in \mathcal{L}_{g+1,1}(k+1) \) and 
\[ \sum_{i=1}^{g} [x_i, \xi_i] + [x_{g+1}, \xi_{g+1}] + \sum_{i=1}^{g} [y_i, \eta_i] + [y_{g+1}, \eta_{g+1}] = 0. \]

If we apply the projection \( p : \mathcal{L}_{g+1,1}(k+2) \to \mathcal{L}_{g,1}(k+2) \) to the above equality, we obtain 
\[ \sum_{i=1}^{g} [x_i, p(\xi_i)] + \sum_{i=1}^{g} [y_i, p(\eta_i)] = 0 \]
because \( p : \mathcal{L}_{g+1} \to \mathcal{L}_g \) is a Lie algebra homomorphism. On the other hand, we have 
\[ p(f) = \sum_{i=1}^{g} x_i \otimes p(\xi_i) + \sum_{i=1}^{g} y_i \otimes p(\eta_i). \]

We can now conclude that \( p(f) \in \mathfrak{h}_g^\mathbb{Z}(k) \) as required. \( \square \)

**Definition 3.3.** Suppose that, for any \( g \), we are given a graded submodule 
\[ m_g = \sum_{k=1}^{\infty} m_g(k) \subset \mathfrak{h}_{g,1}^\mathbb{Z} = \sum_{k=1}^{\infty} \mathfrak{h}_{g,1,1}^\mathbb{Z}(k). \]

(i) We call \( m_g \) \( i \)-stable (inclusion stable) if \( i(m_g(k)) \subset m_{g+1}(k) \) for any \( g \) and \( k \).

(ii) We call \( m_g \) \( p \)-stable (projection stable) if \( p(m_{g+1}(k)) \subset m_g(k) \) for any \( g \) and \( k \).

Even if the graded submodule \( m_g \) is non-trivial only for some fixed \( k \), namely \( m_g(l) = 0 \) for any \( l \neq k \) and \( g \), we still say that \( \{m_g(k)\}_g \) is \( i \)-stable or \( p \)-stable if it satisfies the above condition.

A similar definition can be applied to the rational form \( \mathfrak{h}_g,1 \) of \( \mathfrak{h}_{g,1}^\mathbb{Z} \).
**Proposition 3.4.** (i) The Johnson image $\text{Im } \tau_{g,1}^Z$ is $i$-stable.  
(ii) The ideal $i_{g,1}^Z$ is $p$-stable.  
(iii) The Sp-invariant Lie subalgebra $h_{g,1}^{\text{Sp}}$ is $p$-stable.

**Proof.** Claim (i) follows from the fact that the following diagram

$$
\begin{array}{ccc}
M_{g,1}(k) & \xrightarrow{\tau_{g,1}} & h_{g,1}(k) \\
\downarrow i & & \downarrow i \\
M_{g+1,1}(k) & \xrightarrow{\tau_{g+1,1}} & h_{g+1,1}^Z(k)
\end{array}
$$

is commutative.

Next we prove (ii). Since $p(\omega_0(g+1)) = \omega_0(g)$, the diagram

$$
\begin{array}{ccc}
I_{g+1}(k) & \xrightarrow{i} & \mathcal{L}_{g+1}^Z(k) \\
p & & \downarrow p \\
I_g(k) & \xrightarrow{i} & \mathcal{L}_g^Z(k)
\end{array}
$$

is commutative. It follows that the diagram

$$
\begin{array}{ccc}
H \otimes I_{g+1}(k+1) & \xrightarrow{\cdot 1} & I_{g+1}(k+1) \\
p & & \downarrow p \\
H \otimes I_g(k+1) & \xrightarrow{\cdot 1} & I_g(k+1)
\end{array}
$$

is also commutative. Therefore $p(j_{g+1,1}^Z(k)) = j_{g,1}^Z(k)$ as claimed. \hfill $\Box$

As for the comparison between $M_{g,1}(k)$ and $M_{g,*}(k)$, we have exact sequences

$$
0 \rightarrow \mathbb{Z} \rightarrow M_{g,1}(1) = \mathcal{I}_{g,1} \rightarrow M_{g,*}(1) = \mathcal{I}_{g,*} \rightarrow 1,
$$

$$
0 \rightarrow \mathbb{Z} \rightarrow M_{g,1}(2) = \mathcal{K}_{g,1} \rightarrow M_{g,*}(2) = \mathcal{K}_{g,*} \rightarrow 1
$$

for $k = 1, 2$ where the central subgroup $\mathbb{Z}$ is generated by the Dehn twist, which we denote by $\tau_0(g)$, along a simple closed curve which is parallel to the boundary curve of $\Sigma_{g,1}$. For $k \geq 3$, it is easy to see that the natural homomorphism $p : M_{g,1}(k) \rightarrow M_{g,*}(k)$ is injective so that we can consider $M_{g,1}(k)$ as a subgroup of $M_{g,*}(k)$. Here we have an important result of Hain [10] using the Hodge theory as follows. After tensoring with $\mathbb{Q}$, the graded modules associated with the following two filtrations of the Torelli group $\mathcal{I}_{g,*} = M_{g,*}(1)$ are isomorphic to each other. One is the filtration $\{M_{g,*}(k)\}_{k \geq 2}$ and the other is the filtration $\{p(M_{g,1}(k))\}_{k \geq 2}$ where $p : M_{g,1}(k) \rightarrow M_{g,*}(k)$ denotes the natural projection. From this result, we can deduce the following (this fact was already mentioned in [22]).

**Proposition 3.5.** (i) $M_{g,1}(k) \cong M_{g,*}(k)$ for any $k \geq 3$.  
(ii) $\text{Im } \tau_{g,1}(k) \cap j_{g,1}(k) = \{0\}$ and $\text{Im } \tau_{g,1}^Z(k) \cong \text{Im } \tau_{g,*}^Z(k)$ for any $k \neq 2$.

**Proof.** First of all, it is easy to deduce from the result of Hain mentioned above that, for any $k \geq 3$ the subgroup $M_{g,1}(k) \subset M_{g,*}(k)$ has finite index, because otherwise the associated graded modules of the above two filtrations would not be isomorphic.
The former part of the claim (ii) follows from this because if $\Im \tau_{g,1}(k) \cap j_{g,1}(k) \neq \{0\}$ for some $k \geq 3$, then it would imply that $M_{g,1}(k + 1)$ will have an infinite index in $M_{g,*,}(k + 1)$. Note here that $j_{g,1}(k)$ is a free $\mathbb{Z}$-module so that it has no torsion.

Next we prove that $M_{g,1}(3) = M_{g,*,}(3)$. For this, we show that $j_{g,1}(2)$ is isomorphic to $\mathbb{Z}$ and determine its generator. By definition, we have

$$j_{g,1}(2) = \text{Ker} \left( H \otimes I_g(3) \rightarrow I_g(4) \right).$$

Since $I_g(2) \cong \mathbb{Z}$ generated by the symplectic class $\omega_0 \in L_{g,1}(2)$, $I_g(3) = \{[u, \omega_0]; u \in H\} \cong H$. It follows that $H \otimes I_g(3) \cong H \otimes H$ and the bracket operation $H \otimes I_g(3) \rightarrow I_g(4)$ is given by $v \otimes [u, \omega_0] \mapsto [v, [u, \omega_0]]$. It can then be checked that the kernel of this bracket operation is isomorphic to $\mathbb{Z}$ generated by the element

$$\sum_{i=1}^{g} \{x_i \otimes [y_i, \omega_0] - y_i \otimes [x_i, \omega_0]\}$$

where $x_i, y_i$ ($i = 1, \ldots, g$) denotes a symplectic basis of $H$ as before. On the other hand, a result of [20] (Proposition 1.1) implies that $\tilde{\tau}_{g,1}^{\mathbb{Z}}(2)(\tau_0(g)) \in \mathfrak{h}_{g,1}(2)$ is precisely the above element (up to signs). It follows that

$$\tilde{\tau}_{g,1}^{\mathbb{Z}}(2)(\mathbb{Z}) = j_{g,1}(2).$$

Now let $\varphi \in M_{g,*}(3)$ be any element and choose any lift $\tilde{\varphi} \in M_{g,1}(2)$ of $\varphi$. Since $\tilde{\tau}_{g,1}^{\mathbb{Z}}(2)(\varphi) = 0$, we can write $\tilde{\tau}_{g,1}^{\mathbb{Z}}(2)(\tilde{\varphi}) = m \in j_{g,1}(2) \cong \mathbb{Z}$ for some $m$. Then we have $\tilde{\tau}_{g,1}^{\mathbb{Z}}(2)(\tilde{\varphi} \tau_0(g)^{-m}) = 0$ so that $\tilde{\varphi} \tau_0(g)^{-m} \in M_{g,1}(3)$. Since $\tilde{\varphi} \tau_0(g)^{-m}$ is also a lift of $\varphi$, we can conclude that $M_{g,1}(3) = M_{g,*}(3)$ as required. Next we prove $M_{g,1}(4) = M_{g,*}(4)$. Let $\psi \in M_{g,*}(4)$ be any element so that $\tilde{\tau}_{g,*}^{\mathbb{Z}}(3)(\psi) = 0$ and choose any lift $\tilde{\psi} \in M_{g,1}(3)$ of $\psi$. Since $\Im \tau_{g,1}(3) \cap j_{g,1}(3) = \{0\}$ as already proved above, we have $\tilde{\tau}_{g,1}(3)(\tilde{\psi}) = 0$. Therefore $\tilde{\psi} \in M_{g,1}(4)$ and we can conclude $M_{g,1}(4) = M_{g,*}(4)$. It is now clear that an inductive argument proves the claim (i). The latter part of (ii) follows from this completing the proof.

\textbf{Theorem 3.6.} Over the rationals, we have a direct sum decomposition

$$\mathfrak{h}_{g,1}(k) = j_{g,1}(k) \oplus L_{g}(k) \oplus \Im \tau_{g}(k) \oplus \text{Cok} \tau_{g}(k).$$

\textbf{Proof.} This follows from the following three exact sequences

$$0 \rightarrow j_{g,1} \rightarrow \mathfrak{h}_{g,1} \rightarrow \mathfrak{h}_{g,*} \rightarrow 0, \quad 0 \rightarrow L_{g} \rightarrow \mathfrak{h}_{g,*} \rightarrow \mathfrak{h}_{g} \rightarrow 0,$$

$$0 \rightarrow \Im \tau_{g} \rightarrow \mathfrak{h}_{g} \rightarrow \text{Cok} \tau_{g} \rightarrow 0.$$

\textbf{4. Determination of the ideal $j_{g,1}$}

In this section, we formulate a method of determining the Sp-irreducible decomposition of the ideal $j_{g,1}$. First we recall the following classical result (see e.g. [28]).
**Theorem 4.1.** Let $F_m$ be a free group of rank $m$ and let

$$L^\mathbb{Z}_{(m)} = \bigoplus_{k=1}^\infty L^\mathbb{Z}_{(m)}(k)$$

be the free graded Lie algebra generated by $H_1(F_m) \cong \mathbb{Z}^m$. Then the character of $L^\mathbb{Z}_{(m)}(k)$ as a $GL(m, \mathbb{Q})$-module is given by

$$\text{ch}(L^\mathbb{Z}_{(m)}(k)) = \frac{1}{k} \sum_{d|k} \mu(d)(x^d_1 + \cdots + x^d_m)^{k/d}.$$

In particular

$$\dim L^\mathbb{Z}_{(m)}(k) = \frac{1}{k} \sum_{d|k} \mu(d)m^{k/d}.$$

In our context, we have $L^\mathbb{Z}_{(2g)} = L^\mathbb{Z}_{g,1}$ and $L^\mathbb{Z}_{(2g)} = L_{g,1}$.

**Theorem 4.2 (Labute [15]).** Let $\Sigma_g$ be a closed oriented surface of genus $g$ and let $\Sigma_{g,1} = \Sigma_g \setminus \text{Int } D^2$. Let $L^\mathbb{Z}_g$ (resp. $L^\mathbb{Z}_{g,1}$) denote the graded Lie algebra associated to the lower central series of $\pi_1\Sigma_g$ (resp. $\pi_1\Sigma_{g,1}$). Then

$$L^\mathbb{Z}_g = L^\mathbb{Z}_{g,1}/(\omega_0)$$

where $(\omega_0)$ denotes the ideal generated by the symplectic class $\omega_0 \in L^\mathbb{Z}_{g,1}(2)$. Furthermore the $k$-th term $L^\mathbb{Z}_g(k)$ of the graded Lie algebra $L^\mathbb{Z}_g$ is a free $\mathbb{Z}$-module and its rank is given by

$$\text{rank } L^\mathbb{Z}_g(k) = \frac{1}{k} \sum_{d|k} \mu(k/d) \left[ \sum_{0 \leq i \leq [d/2]} (-1)^i \frac{d}{d-i} \binom{d-i}{i} (2g)^{d-2i} \right].$$

In the above formula, the part $i = 0$ corresponds to the case of $L^\mathbb{Z}_{g,1}(k)$.

Based on the above theorem of Labute, we can deduce the following method of determining the $\text{Sp}$-irreducible decomposition of $\Sigma_{g,1}(k)$. This is because his theorem gives not only the ranks of the relevant modules but the structure of them as $GL$-modules. Define

$$\tilde{I}(k) = -\frac{1}{k} \sum_{d|k} \mu(k/d) \left[ \sum_{1 \leq i \leq [d/2]} (-1)^i \frac{d}{d-i} \binom{d-i}{i} P_{k/d}^{(d-2i)} \right]$$

where $P_j$ denotes the $GL$-representation corresponding to the power sum

$$x_1^j + y_1^j + \cdots + x_g^j + y_g^j$$

so that $P_1 = H_\mathbb{Q}$ and if $d - 2i = 0$, then set $P_{k/d}^0 = [0]$ the $\text{Sp}$-trivial representation. Then we have the following result.

**Theorem 4.3.** (i) The $\text{Sp}$-irreducible decomposition of $\Sigma_{g,1}(k)$ is the same as that of the virtual $GL$-representation $\left( H_\mathbb{Q} \otimes \tilde{I}(k + 1) - \tilde{I}(k + 2) \right)$.

(ii) The $\text{Sp}$-irreducible decomposition of $L_g(k)$ is the same as that of the virtual $GL$-representation $\left( \frac{1}{k} \sum_{d|k} \mu(k/d) P_{k/d} - \tilde{I}(k) \right)$. 
By using these methods, we have computed the Sp-irreducible decompositions of the Sp-modules $j_{g,1}(k)$ and $L_g(k)$ explicitly for all $k \leq 20$. By combining with our earlier work in [25], we have now complete information about the Sp-irreducible decompositions of all the relevant Sp-modules up to degree 20. Here we only describe the Sp-invariant part. See Tables 4, 5, 6. The first table indicates the dimensions in the stable range whereas in the latter two tables (the cases $k = 6, 10$), we describe the unstable computations for later use.

**Table 4. Dimensions of $h_{g,1}(k)^{Sp}, j_{g,1}(k)^{Sp}, h_{g,\ast}(k)^{Sp}, L_g(k)^{Sp}, h_g(k)^{Sp}$**

| $k$ | $h_{g,1}(k)^{Sp}$ | $j_{g,1}(k)^{Sp}$ | $@ h_{g,\ast}(k)^{Sp}$ | $L_g(k)^{Sp}$ | $h_g(k)^{Sp}$ |
|-----|-------------------|-------------------|------------------------|----------------|----------------|
| 2   | 1                 | 1                 | @ 0                    | 0              | 0              |
| 4   | 0                 | 0                 | 0                      | 0              | 0              |
| 6   | 5                 | 2                 | 3                      | 1              | 2              |
| 8   | 3                 | 1                 | 2                      | 2              | 0              |
| 10  | 108               | 38                | 70                     | 34             | 36             |
| 12  | 650               | 210               | 440                    | 259            | 181            |
| 14  | 8795              | 2831              | 5964                   | 3215           | 2749           |
| 16  | 110610            | 34591             | 76019                  | 41858          | 34161          |
| 18  | 1710798           | 530466            | 1180332                | 644758         | 535574         |
| 20  | 29129790          | 8980269           | 20149521               | 11111008       | 9038513        |

**Table 5. Dimensions of $h_{g,1}(6)^{Sp}, j_{g,1}(6)^{Sp}, h_{g,\ast}(6)^{Sp}, L_g(6)^{Sp}, h_g(6)^{Sp}$**

| $g$ | $h_{g,1}(6)^{Sp}$ | $j_{g,1}(6)^{Sp}$ | $@ h_{g,\ast}(6)^{Sp}$ | $L_g(6)^{Sp}$ | $h_g(6)^{Sp}$ |
|-----|-------------------|-------------------|------------------------|----------------|----------------|
| 1   | 1                 | 1                 | 0                      | 0              | 0              |
| 2   | 4                 | 2                 | 2                      | 1              | 1              |
| $\geq 3$ | 5            | 2                 | 3                      | 1              | 2              |

**Table 6. Dimensions of $h_{g,1}(10)^{Sp}, j_{g,1}(10)^{Sp}, h_{g,\ast}(10)^{Sp}, L_g(10)^{Sp}, h_g(10)^{Sp}$**

| $g$ | $h_{g,1}(10)^{Sp}$ | $j_{g,1}(10)^{Sp}$ | $@ h_{g,\ast}(10)^{Sp}$ | $L_g(10)^{Sp}$ | $h_g(10)^{Sp}$ |
|-----|-------------------|-------------------|------------------------|----------------|----------------|
| 1   | 3                 | 3                 | 0                      | 0              | 0              |
| 2   | 51                | 27                | 24                     | 14             | 10             |
| 3   | 97                | 37                | 60                     | 31             | 29             |
| 4   | 107               | 38                | 69                     | 34             | 35             |
| $\geq 5$ | 108           | 38                | 70                     | 34             | 36             |

5. DESCENDANTS AND ANCESTORS

In this section, we propose a terminology which will hopefully be useful in analyzing the structure of the Lie algebra $h_{g,1}$. 
Remark 5.5. It is a very important problem to determine which $Z$ in $h$ non-trivial in $H$ \[2\] copy $Sp$ Im $\tau$ GL a given $Theorem 5.4.$ For any $h$ the corresponding $Sp$-descendants of each piece $h$ we can write $Let \[2\] Asada-Nakamura \[2\] proved that there exists a unique copy $V$ can be decomposed into a linear combination \[V = \bigoplus_{i=0}^{a} V_i\] of $Sp$-irreducible representations $V_i$. Here $V_0$ denotes the largest $Sp$-irreducible component of $V$ so that $V_0 \cong \lambda_{Sp}$ and any of the other $V_i (i \geq 1)$ is obtained from $V$ by applying various contractions successively. We say that $V_i$ is an $Sp$-descendant of $V$ of order $d$ if it is obtained by applying contractions $d$ times. In this case $V_i \cong \lambda_{Sp}^{(i)}$ where $\lambda^{(i)}$ is a Young diagram with $(k + 2 - 2d)$ boxes. We also call $Sp$-descendant of order 1 (resp. 2) an $Sp$-child (resp. an $Sp$-grandchild). Conversely we call $V$ the ancestor of $V_i$ for any $i$.

Example 5.2. Asada-Nakamura \[2\] proved that there exists a unique copy $[2k+1 1^{2}]_{GL} \subset h_{g,1}(2k+1)$ for any $k \geq 1$. The trace component $[2k+1]_{Sp} \subset h_{g,1}(2k+1)$ is an $Sp$-child of this unique copy. Enomoto-Satoh \[2\] proved that there exists a unique copy $[2^4k-1]_{GL} \subset h_{g,1}(4k+1)$ for any $k \geq 1$. Their anti-trace component $[1^{4k+1}]_{Sp} \subset h_{g,1}(4k+1)$ is an $Sp$-child of this unique copy.

Example 5.3. The $Sp$-invariant part given in Theorem \[1\] is, so to speak, the “last descendants” of each piece $h_{g,1}(2k)$.

Theorem 5.4. For any $k$ and any GL-irreducible component $V$ of $h_{g,1}(k)$ isomorphic to $\lambda_{GL}$, the corresponding $Sp$-irreducible component $V_0$ which is isomorphic to $\lambda_{Sp}$ is contained in $Im \tau_{g,1}(k)$ in a certain stable range (we can take $g \geq k + 3$).

Remark 5.5. It is a very important problem to determine which $Sp$-descendants of a given GL-irreducible component $V \subset h_{g,1}(k)$ belongs to $Im \tau_{g,1}(k)$ and/or remains non-trivial in $H_1(h_{g,1}^+)$.

To prove the above theorem, we prepare some terminology. Let $\sigma_i = (12 \cdots i) \in \mathfrak{S}_k$ be the cyclic permutation. Define two elements
\[p_k = (1 - \sigma_k)(1 - \sigma_{k-1}) \cdots (1 - \sigma_2)\]
\[S_k = \sum_{j=1}^{k} \sigma_j^i\]
in $\mathbb{Z}[^\mathfrak{S}_k]$ both of which act on $H_Q^{\otimes k}$ linearly. It is easy to see that $p_k(H_Q^{\otimes k}) = L_{g,1}(k)$.

Proposition 5.6 (see \[23\]). The subspace $H_Q \otimes L_{g,1}(k+1) \subset H_Q^{\otimes (k+2)}$ is invariant under the action of $S_{k+2}$ and
\[S_{k+2} (H_Q \otimes L_{g,1}(k+1)) = h_{g,1}(k)\].

It follows that
\[h_{g,1}(k) = S_{k+2} \circ (1 \otimes p_{k+1}) \left(H_Q^{\otimes (k+2)}\right)\].
Here we mention a relation with another method of expressing elements of $\mathfrak{h}_{g,1}(k)$, namely Lie spiders (see e.g. [16]). More precisely the following element

$$S_{k+2}(u_1 \otimes [u_2, [u_3, \cdots [u_{k+1}, u_{k+2}] \cdots]])$$

$$= u_1 \otimes [u_2, [u_3, \cdots [u_{k+1}, u_{k+2}] \cdots]] + u_2 \otimes ([u_3, [u_4, \cdots [u_{k+1}, u_{k+2}] \cdots], u_1] + u_3 \otimes ([u_4, [u_5, \cdots [u_{k+1}, u_{k+2}] \cdots], [u_1, u_2]] + \cdots + u_{k+2} \otimes \cdots [u_1, u_2, \cdots, u_k], u_{k+1}]$$

is represented by the Lie spider

$$\begin{array}{c}
\text{\quad} u_1 \\
\text{\quad} u_2 \\
\text{\quad} u_3 \\
\text{\quad} \vdots \\
\text{\quad} u_{k+1} \\
\text{\quad} u_{k+2}
\end{array}$$

where $u_i \in H_Q$.

**Proof of Theorem 5.4.** We use induction on $k$. It is a classical result of Johnson [12] that $\text{Im} \, \tau_{g,1}(1) = \mathfrak{h}_{g,1}(1) = \wedge^3 H_Q$. Hence the claim holds for $k = 1$.

It is easy to see that the bracket operation

$$H_Q \otimes \mathcal{L}_{g,1}(k) \rightarrow \mathcal{L}_{g,1}(k+1)$$

is surjective for any $k \geq 1$. It follows that $\mathfrak{h}_{g,1}(k)$ is generated by the elements of the form

$$S_{k+2}(u_1 \otimes [u_2, [u_3, \cdots [u_{k+1}, u_{k+2}] \cdots]]).$$

Let us denote the above element by $\ell(u_1, \ldots, u_{k+2})$. On the other hand, it is easy to see that the highest weight vector for $\lambda_{\text{Sp}}$ can be expressed as a linear combination of elements of the above form where any of $u_i$ ($i = 1, \ldots, k+2$) is equal to some $x_j$ with $j \leq k+2$. Here $x_1, \ldots, x_g, y_1, \ldots, y_g$ is a symplectic basis of $H$ for $g = k+3$. Hence it suffices to prove that $\ell(u_1, \ldots, u_{k+2})$ is contained in $\text{Im} \, \tau_{g,1}(k)$ for such $g$ whenever any $u_i$ is equal to $x_j$ ($j \leq k+2$). Now assume $k > 1$ and consider the following two elements

$$\ell(u_1, \ldots, u_k, x_{k+3}), \quad \ell(y_{k+3}, u_{k+1}, u_{k+2}).$$

Then the former element is contained in $\text{Im} \, \tau_{g',1}(k-1)$ with $g' = k+3$ by the induction assumption and the latter one belongs to $\mathfrak{h}_{g,1}(1) = \text{Im} \, \tau_{g,1}(1)$ for any $g \geq k+3$. Also it is easy to see that

$$[\ell(u_1, \ldots, u_k, x_{k+3}), \ell(y_{k+3}, u_{k+1}, u_{k+2})] = \ell(u_1, \ldots, u_{k+2}).$$

Therefore $\ell(u_1, \ldots, u_{k+2})$ is contained in $\text{Im} \, \tau_{g,1}(k)$ with $g = k+3$ as required. This completes the proof. \qed

**Proposition 5.7.** (i) For any $\lambda$ with $|\lambda| = k+2$, the isotypical component $\widetilde{H}_\lambda \subset \mathfrak{h}_{g,1}(k)$ as well as the corresponding $\text{Sp}$-isotypical component $(\tilde{H}_\lambda)_0$ is $i$-stable.

(ii) The $\text{Sp}$-invariant part $\mathfrak{h}_{g,1}^\text{Sp}$ is $p$-stable. Furthermore each component of the orthogonal direct sum decomposition of this space given by Theorem 1.1 is $p$-stable.

(iii) $j_{g,1}^\text{Sp}$ is $p$-stable.

(iv) The ideal $[\mathfrak{h}_{g,1}, \mathfrak{h}_{g,1}]$ is $i$-stable.
Proof. (i) follows from that the following three facts. The first is that the isotypical component $V^{k+2}_\lambda \subset (H_Q^{(k+2)})^{Sp}$ is $i$-stable which follows from the classical construction of this component. The second is that the submodule $h_{g,1}(k) \subset H_Q^{(k+2)}$ is $i$-stable. The third is that taking the contractions to any given “direction” is an $i$-stable operation. The rest of the assertions follow similarly using the facts that $(H_Q^{(k+2)})^{Sp} \subset H_Q^{(k+2)}$ is $p$-stable and the bracket operation of $h_{g,1}$ is $i$-stable in an obvious sense. □

Remark 5.8. In general, the $Sp$-invariant part of the Johnson image $(\text{Im} \tau_{g,1})^{Sp}$ is neither $i$-stable nor $p$-stable, and $h_{g,1}^{Sp}$ is not $i$-stable. These can be checked by direct computation. This is one of the reasons why the problem of determining (the $Sp$-invariant part of) the image of the Johnson homomorphism as well as the ideal $j_{g,1}$ is difficult. In a trial to overcome this difficulty, in Section 6 we introduce two kinds of basis for $h_{g,1}^{(2k)}$.  

6. Two kinds of bases for $h_{g,1}^{(2k)}$

In this section, we describe a general method of constructing elements of $h_{g,1}^{(2k)}$ and by using it we introduce two kinds of bases for it.

We begin by recalling a few facts from [23]. Let $D^\ell(2k)$ denote the set consisting of all the linear chord diagrams with $2k$ vertices. Here a linear chord diagram with $2k$ vertices is a partition of the set $\{1, 2, \ldots, 2k\}$ into $k$-tuple

$$C = \{(i_1, j_1), \ldots, (i_k, j_k)\}$$

of pairs $(i_l, j_l)$ ($l = 1, \ldots, k$) where we assume

$$i_1 < \cdots < i_k, \ i_l < j_l \quad (l = 1, \ldots, k).$$

We may also adopt a simpler notation $C = (i_1 j_1) \cdots (i_k j_k)$ than the above. This set $D^\ell(2k)$ has $(2k - 1)!!$ elements and let $QD^\ell(2k)$ be the vector space over $Q$ spanned by $D^\ell(2k)$. For each linear chord diagram $C \in D^\ell(2k)$, define

$$a_C \in (H_Q^{\otimes 2k})^{Sp}$$

by permuting the elements $(\omega_0)^{\otimes 2k}$ in such a way that the $s$-th part $(\omega_0)_s$ of this tensor product goes to $(H_Q)_{i_s} \otimes (H_Q)_{j_s}$, where $(H_Q)_i$ denotes the $i$-th component of $H_Q^{\otimes 2k}$, and then multiplied by the factor

$$\text{sgn } C = \text{sgn } \begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ i_1 & j_1 & \cdots & i_k & j_k \end{pmatrix}.$$
by setting $\Phi(C) = a_C$. Observe that the symmetric group $\mathfrak{S}_{2k}$ acts on both of $QD^f(2k)$ and $(H_Q^{\otimes 2k})^{Sp}$ naturally.

**Proposition 6.1** (see [23][24]). *The correspondence

$$\Phi : QD^f(2k) \to (H_Q^{\otimes 2k})^{Sp}$$

is surjective for any $g$ and bijective for any $g \geq k$. Furthermore this correspondence is “anti” $\mathfrak{S}_{2k}$-equivariant in the sense that

$$\Phi(\gamma(C)) = (\text{sgn} \gamma) \gamma(\Phi(C))$$

for any $C \in D^f(2k)$ and $\gamma \in \mathfrak{S}_{2k}$.

Let $\sigma_i = (12 \cdots i) \in \mathfrak{S}_k$ be the cyclic permutation as before. Define two elements

$$p'_k = (1 - (-1)^{k-1} \sigma_k)(1 - (-1)^{k-2} \sigma_{k-1}) \cdots (1 + \sigma_2),$$

$$S'_k = \sum_{j=1}^{k} (-1)^{j(k-1)} \sigma_j^j$$

in $\mathbb{Z}[\mathfrak{S}_k]$. Then by combining Proposition 5.6 with the above Proposition 6.1, we obtain the following result.

**Proposition 6.2.**

$\mathfrak{h}_{g,1}(2k)^{Sp} = \Phi \left( S'_{2k+2} \circ \sigma_{2k+2} \circ p'_{2k+1} \circ \sigma_{2k+2}^{-1}(QD^f(2k+2)) \right)$.

Thus we obtain a method of constructing elements of $\mathfrak{h}_{g,1}(2k)^{Sp}$. We remark here that computation by a computer is much easier in this context of $QD^f(2k+2)$ rather than that of $(H_Q^{\otimes (2k+2)})^{Sp}$ because in the latter case it becomes heavier and heavier according to the genus gets larger while in the former context the computation is independent of the genus. In particular, the explicit orthogonal decomposition of $(H_Q^{\otimes (2k+2)})^{Sp}$ can be obtained by applying various Young symmetrizers on $QD^f(2k+2)$ to obtain the corresponding decomposition of this space and then converting it to the space $(H_Q^{\otimes (2k+2)})^{Sp}$ by applying Proposition 6.1. To obtain the orthogonal decomposition of $\mathfrak{h}_{g,1}(2k)^{Sp}$, it is enough to apply further the operator $S'_{2k+2} \circ \sigma_{2k+2} \circ p'_{2k+1} \circ \sigma_{2k+2}^{-1}$ to the above decomposition of $QD^f(2k+2)$ and then apply the homomorphism $\Phi$.

More precisely, the explicit procedure goes as follows. In [24] a canonical metric on $QD^f(2k+2)$ is defined and it is shown that there exists an orthogonal direct sum decomposition

$$QD^f(2k+2) \cong \bigoplus_{|\lambda|=k+1} E_\lambda$$

in terms of certain subspaces $E_\lambda$. As an $\mathfrak{S}_{2k+2}$-submodule of $QD^f(2k+2)$, $E_\lambda$ is an irreducible $\mathfrak{S}_{2k+2}$-module corresponding to the Young diagram $2\lambda$. Since the operator $S'_{2k+2} \circ \sigma_{2k+2} \circ p'_{2k+1} \circ \sigma_{2k+2}^{-1}$ belongs to $\mathbb{Z}[\mathfrak{S}_{2k+2}]$, if we define

$$F_\lambda = (S'_{2k+2} \circ \sigma_{2k+2} \circ p'_{2k+1} \circ \sigma_{2k+2}^{-1})(E_\lambda),$$
then $F_\lambda$ is a subspace of $E_\lambda$ and
$$\Phi(F_\lambda) = H_{\lambda'} \subset \mathfrak{h}_{g,1}(2k)^{\text{Sp}}.$$  

Thus we obtain a method of constructing elements of $\mathfrak{h}_{g,1}(2k)^{\text{Sp}}$ which respects the orthogonal decomposition and is also independent of the genus $g$. Indeed, if we choose a basis $\{C_i^\lambda; i = 1, \ldots, \dim F_\lambda\}$ of $F_\lambda$ and set $v_i^\lambda = \Phi(C_i^\lambda)$. Then
$$(1) \{v_i^\lambda; |\lambda| = k + 1, i = 1, \ldots, \dim F_\lambda = \dim H_{\lambda'}\}$$
is a basis of $\mathfrak{h}_{g,1}(2k)^{\text{Sp}}$ in the stable range. We call this a $p$-stable basis because it is clearly $p$-stable in the obvious sense. This basis is suitable for describing $\tau_{g,1}^{\text{Sp}}$ which is $p$-stable. However, it is not $i$-stable (in fact no basis can be $i$-stable) and the description of $(\text{Im} \tau_{g,1})^{\text{Sp}}$ is rather complicated. The following definition is useful in analyzing this point.

**Definition 6.3.** For each Young diagram $\lambda$, let $\mu_\lambda$ be the number defined by the following formula
$$\mu_\lambda = \prod_{b: \text{ box of } \lambda} (2g - 2s_b + t_b)$$
where $s_b$ denotes the number of columns of $\lambda$ which are on the left of the column containing $b$ and $t_b$ denotes the number of rows which are above the row containing $b$ (see [24]). Then we define the eigenvalue of $H_\lambda$ to be $\mu_{\lambda'}$ where $\lambda'$ is the conjugate Young diagram of $\lambda$. When we specify the genus $g$, we write $\mu_{\lambda'}(g)$ for $\mu_{\lambda'}$ which is a polynomial in $g$ of degree $|\lambda|$.

**Example 6.4.**
$$\mu_{[4]}' = 2g(2g + 1)(2g + 2)(2g + 3), \quad \mu_{[1^4]}' = 2g(2g - 2)(2g - 4)(2g - 6).$$

Now define a linear mapping
$$\mathcal{K} : H_Q^{\otimes(2k+2)} \rightarrow \mathbb{Q}\mathcal{D}^\ell(2k + 2)$$
by setting
$$\mathcal{K}(\xi) = \sum_{C \in \mathcal{D}^\ell(2k+2)} \alpha_C(\xi) C.$$  

We use the same notation $\mathcal{K}$ for the restriction of the above mapping to the subspace $\mathfrak{h}_{g,1}(2k) \subset H_Q^{\otimes(2k+2)}$ as well as its further restriction to the subspace $\mathfrak{h}_{g,1}(2k)^{\text{Sp}} \subset \mathfrak{h}_{g,1}(2k)$.

**Proposition 6.5.** (i) The linear mapping $\mathcal{K}$ is $i$-stable in the sense that the following diagram is commutative
$$\begin{array}{c}
\mathfrak{h}_{g,1}(2k) \\
\downarrow i \\
\mathfrak{h}_{g+1,1}(2k)
\end{array} \xrightarrow{\mathcal{K}} \begin{array}{c}
\mathbb{Q}\mathcal{D}^\ell(2k + 2) \\
\downarrow \\
\mathbb{Q}\mathcal{D}^\ell(2k + 2)
\end{array}$$

(ii) The linear mapping
$$\mathcal{K} : \mathfrak{h}_{g,1}(2k)^{\text{Sp}} \rightarrow \mathbb{Q}\mathcal{D}^\ell(2k + 2)$$
is injective for any \( g \). Furthermore two subspaces \( K(H_\lambda) \) and \( K(H_\mu) \) (\( \lambda \neq \mu \)) are mutually orthogonal to each other with respect to the usual Euclidean metric on \( \mathbb{Q}D^f(2k + 2) \) which is induced by taking \( D^f(2k + 2) \) as an orthonormal basis.

(iii) For any element \( \xi \in F_\lambda \), we have the equality

\[
K(\Phi(\xi)) = \mu_\lambda \xi.
\]

Proof. (i) follows from the fact that the contraction is an \( i \)-stable operation. The former part of (ii) holds because any \( \text{Sp} \)-invariant tensor is detected by some iterated contractions. The latter part follows from a stronger statement proved in [24] that two subspaces \( K(U_\lambda) \) and \( K(U_\mu) \) (\( \lambda \neq \mu \)) are mutually orthogonal to each other with respect to the usual Euclidean metric. (iii) follows similarly because it is proved in the above cited paper that the equality \( K(\Phi(\xi)) = \mu_\lambda \xi \) holds for any \( \xi \in E_\lambda \).

Definition 6.6. Let \( V \subset \mathfrak{h}_{g,1}(2k)\text{Sp} \) be any specified subspace, e.g. \((\text{Im} \tau_{g,1}(2k))\text{Sp} \). We call a finite set \( D \subset D^f(2k + 2) \) a detector of \( V \) if the linear mapping

\[
K_D : \mathfrak{h}_{g,1}(2k)\text{Sp} \xrightarrow{K} \mathbb{Q}D^f(2k + 2) \xrightarrow{\text{proj}} \mathbb{Q}D
\]

is injective on \( V \).

Keeping the above Proposition 6.5 (iii) in mind, we make the following definition.

Definition 6.7. We modify the \( p \)-stable basis (1) by setting

\[
\bar{v}_\lambda^i = \frac{1}{\mu_X(g)} v_\lambda^i
\]

to obtain another basis

\[
\{ \bar{v}_\lambda^i ; |\lambda| = k + 1, i = 1, \ldots, \dim F_\lambda = \dim H_\lambda \}
\]

of \( \mathfrak{h}_{g,1}(2k)\text{Sp} \) in the stable range. We call this a normalized basis.

By combining the above results, we obtain the following theorem which shows that although there is a considerable difference between the two Lie subalgebras \( \mathfrak{j}_{g,1}\text{Sp} \) and \((\text{Im} \tau_{g,1}(2k))\text{Sp} \) it can be completely analyzed by rescaling each piece in the orthogonal decomposition by the corresponding eigenvalue.

Theorem 6.8. (i) The subspace \( \mathfrak{j}_{g,1}(2k)\text{Sp} \subset \mathfrak{h}_{g,1}(2k)\text{Sp} \) is \( p \)-stable so that the description of it in terms of a \( p \)-stable basis is constant with respect to \( g \).

(ii) In contrast with the case (i) above, the subspace \( (\text{Im} \tau_{g,1}(2k))\text{Sp} \subset \mathfrak{h}_{g,1}(2k)\text{Sp} \) is not \( p \)-stable. However, it is weighted stable in the following sense. Namely the description of it in terms of a normalized basis is constant with respect to \( g \).

Proof. It remains to prove the last claim of (ii). The set of values under \( K \) of \((\text{Im} \tau_{g,1}(2k))\text{Sp} \) is constant with respect to \( g \) in the stable range. On the other hand, Proposition 6.5 (iii) shows that the value under \( K \) of each member of a \( p \)-stable basis is the corresponding eigenvalue times a constant vector. Since the normalized basis cancels exactly this factor, the claim holds.
7. Description of $h_{g,1}(6)^{Sp}$

In this section, by applying the general results obtained in the preceding section, we give a complete description of the $Sp$-invariant part $h_{g,1}(6)^{Sp}$ of the weight 6 summand of $h_{g,1}$.

We begin by recalling the following decomposition.

**Proposition 7.1.** The GL-irreducible decomposition of $h_{g,1}(6)$ in a stable range is given by

$$h_{g,1}(6) = [62]_{GL} + [521]_{GL} + [51^3]_{GL} + [4^2]_{GL} + [431]_{GL} + 2[42^2]_{GL} + [421^2]_{GL} + [41^4]_{GL} + 2[3^2 1^2]_{GL} + [32^2 1]_{GL} + [32^1 3]_{GL} + [2^4]_{GL} + [2^2 1^4]_{GL}$$

By combining this with the results of the preceding sections, we can make Table 2 and Table 5 in Section 2. Next we construct an explicit basis of $h_{g,1}(6)^{Sp} \cong \mathbb{Q}^5$ as follows. The set $D'(8)$ of linear chord diagrams with 8 vertices has 105 elements and we enumerate them by the lexicographic order with respect to our notation of linear chord diagrams. We apply various Young symmetrizers and the projectors to the space $\mathbb{Q}D'(8)$. Then we obtain 5 elements

$$C_i \in \mathbb{Q}D'(8) \cong \mathbb{Q}^{105} \quad (i = 1, 2, 3, 4, 5)$$

explicitly described as

$$C_1 = (18, -2, -16, -2, 2, 0, -16, 11, 5, 0, 5, -5, 16, -3, -13, -2, -1, 3, 2, 3, -5, 0, 0, -2, -3, 5, 2, 1, -3, -16, 3, 13, 11, 0, -11, 5, -5, 0, -3, -8, 11, -2, -3, 5, 0, -5, 5, 5, 0, -5, -5, 0, 5, 5, -5, 0, 0, 0, 0, 16, -11, -5, -3, 0, 3, -13, 5, 8, 11, 0, -11, 2, 3, -5, 2, -2, 0, 1, -3, 2, -3, 0, 3, 5, 0, -5, -2, -1, 3, -18, 2, 16, 2, -2, 0, 16, -11, -5, 0, -5, 5, -16, 3, 13),$$

$$C_2 = (16, 4, -7, 4, -4, 0, -7, -8, -3, 0, -3, -4, 7, 3, 2, 4, -2, -3, -4, -6, 3, 0, 0, 4, 6, -3, -4, 2, 3, -7, -3, -2, -8, 0, 8, -3, 2, 0, 6, 8, -8, 4, 6, -3, 0, 3, 4, -3, 0, 3, -4, 0, -2, -3, 2, 0, 0, 0, 7, 8, 3, 3, 0, -6, 2, -2, -8, 0, 8, -4, -6, 3, -4, 4, 0, 2, 6, -4, 3, 0, -6, -3, 0, 3, 4, -2, -3, -16, -4, 7, -4, 4, 0, 7, 8, 3, 0, 3, 4, -7, -3, -2),$$

$$C_3 = (4, 0, -3, 0, 0, 0, -3, -2, -1, 0, -1, 2, 3, -1, 2, 0, 2, 1, 0, 0, 1, 0, 0, 0, 0, -1, 0, -2, -1, -3, 1, -2, -2, 0, 2, -1, 4, 0, 0, 4, -2, 0, 0, -1, 0, 1, -2, -1, 0, 1, 2, 0, -4, -1, 4, 0, 0, 0, 0, 3, 2, 1, -1, 0, 0, 2, -4, -4, -2, 0, 2, 0, 1, 0, 0, 0, -2, 0, 0, -1, 0, 0, -1, 0, 1, 0, 2, 1, -4, 0, 3, 0, 0, 0, 3, 2, 1, 0, 1, -2, -3, -1, -2),$$

$$C_4 = (0, 0, -1, 0, 1, -2, -1, 0, 0, 0, -1, 0, -2, -1, -3, 1, -2, -2, 0, 2, -1, 4, 0, 0, 4, -2, 0, 0, -1, 0, 1, -2, -1, 0, 1, 2, 0, -4, -1, 4, 0, 0, 0, 0, 3, 2, 1, -1, 0, 0, 2, -4, -4, -2, 0, 2, 0, 1, 0, 0, 0, -2, 0, 0, -1, 0, 0, -1, 0, 1, 0, 2, 1, -4, 0, 3, 0, 0, 0, 3, 2, 1, 0, 1, -2, -3, -1, -2),$$

$$C_5 = (1, 0, 1, 0, 2, 1, -4, 0, 3, 0, 0, 0, 3, 2, 1, 0, 1, -2, -3, -1, -2).$$
\[C_4 = (-2, -4, -2, -4, -2, 0, -2, 1, 1, 0, 1, 5, 2, 1, 5, -4, -3, -1, -2, -3, -1, 0, 0, 0, 2, 3, 1, 4, 3, 1, -2, -1, -5, 1, 0, -1, 1, -3, 0, 3, 0, 1, 2, 3, 1, 0, -1, -5, 1, 0, -1, 5, 0, 3, 1, -3, 0, 0, 0, 2, -1, -1, 1, 0, -3, 5, 3, 0, 1, 0, -1, -2, -3, -1, 4, 2, 0, 3, 3, -2, 1, 0, -3, 1, 0, -1, -4, -3, -1, 2, 4, 2, 2, 2, 0, 2, -1, -1, 0, -1, -5, -2, -1, -5),\]

\[C_5 = (-2, -1, 1, -1, -2, 0, 1, 1, 1, 0, 1, -1, -1, -2, -4, -1, 0, 2, -2, -3, -1, 0, 0, 0, 2, 3, 1, 1, 0, -2, 1, 2, 4, 1, 0, -1, 1, 3, 0, 3, 6, 1, 2, 3, 1, 0, -1, 1, 0, -1, -1, 0, -3, 1, 3, 0, 0, 0, -1, -1, -1, -2, 0, -3, -4, -3, -6, 1, 0, -1, -2, -3, -1, 1, 2, 0, 3, -2, -2, 0, -3, 1, 0, -1, -1, 0, 2, 2, 1, -1, 1, 2, 0, -1, -1, -1, 0, -1, 1, 1, 2, 4).\]

Finally we set

\[v_i = \Phi(C_i) \in \mathfrak{h}_{g,1}(6)^{Sp} \quad (i = 1, 2, 3, 4, 5).\]

Then these elements generate each component of the orthogonal decomposition of \(\mathfrak{h}_{g,1}(6)^{Sp}\) as depicted in Table 7 below.

**Table 7. Orthogonal decompositions of \((H_Q^{\otimes 8})^{Sp}\) and \(\mathfrak{h}_{g,1}(6)^{Sp}\)**

| \(\lambda\) | \(\mu_\lambda\) (eigen value of \(H_\lambda\)) | \(\dim U_\lambda\) | \(\dim H_\lambda\) | generators for \(H_\lambda\) |
|---|---|---|---|---|
| [4] | \((2g(2g + 1)(2g + 2)(2g + 3)\) | 14 | 1 | \(v_1\) |
| [31] | \((2g - 2)2g(2g + 1)(2g + 2)\) | 56 | 2 | \(v_2, v_3\) |
| [2^4] | \((2g - 2)(2g - 1)2g(2g + 1)\) | 14 | 1 | \(v_4, v_5\) |
| [21^2] | \((2g - 4)(2g - 2)2g(2g + 1)\) | 20 | 1 | \(v_5\) |
| [1^4] | \((2g - 6)(2g - 4)(2g - 2)2g\) | 105 | 5 | \(v_5\) |

Next we consider the dual elements \(\alpha_C\) \((C \in \mathcal{D}^e(8))\). It turns out that the set \(D\) of five elements

\[(12)(34)(56)(78), (12)(34)(57)(68), (12)(34)(58)(67), (12)(36)(47)(58), (12)(38)(46)(57)\]

which are the \((1, 2, 3, 8, 14)\)-th elements in the lexicographic order of \(\mathcal{D}^e(8)\), is a detector for \(\mathfrak{h}_{g,1}(6)^{Sp}\).

**Proposition 7.2.** *The evaluation homomorphism*

\[\mathcal{K}_D : \mathfrak{h}_{g,1}(6)^{Sp} \to \mathbb{Q}^5\]
Definition 7.3. In the stable range \( g \geq 3 \), we set

\[
\begin{align*}
\bar{v}_1 &= \frac{1}{2g(2g + 1)(2g + 2)(2g + 3)} v_1, \\
\bar{v}_2 &= \frac{1}{(2g - 2)2g(2g + 1)(2g + 2)} v_2, \\
\bar{v}_3 &= \frac{1}{(2g - 2)2g(2g + 1)(2g + 2)} v_3, \\
\bar{v}_4 &= \frac{1}{(2g - 2)(2g - 1)2g(2g + 1)} v_4, \\
\bar{v}_5 &= \frac{1}{(2g - 4)(2g - 2)2g(2g + 1)} v_5
\end{align*}
\]

so that \( \{\bar{v}_i\} \) is a normalized basis of \( \mathfrak{h}_{g,1}(6)^{Sp} \).

Here let us recall the Enomoto-Satoh map introduced in [7], which is a linear map \( ES_k : \mathfrak{h}_{g,1}(k) \to \mathfrak{a}_g(k - 2) \) where

\[
\mathfrak{a}_g = \bigoplus_{k=0}^{\infty} \mathfrak{a}_g(k)
\]
denotes the associative version of the derivation Lie algebras defined by Kontsevich [13][14]. They proved that \( \text{Im} \tau_{g,1}(k) \) is contained in \( \text{Ker} \ ES_k \) and it is a very important problem to study the quotient \( \text{Ker} \ ES_k/\text{Im} \tau_{g,1}(k) \). We know from Theorem 1.7 that this quotient is trivial for all \( k \leq 5 \).

Let us define the normalizer of the Johnson image \( \text{Im} \tau_{g,1} \) in \( \mathfrak{h}_{g,1} \) as \( \mathcal{N} = \bigoplus_{k} \mathcal{N}(k) \) where

\[
\mathcal{N}(k) = \{ \varphi \in \mathfrak{h}_{g,1}(k); [\varphi, \psi] \in \text{Im} \tau_{g,1}(k + l) \text{ for any } l \text{ and } \psi \in \text{Im} \tau_{g,1}(l) \}.
\]

This should be important in the study of the arithmetic mapping class group because the Galois obstructions appear in \( \mathfrak{h}_{g,1}^{Sp} \) and they normalize the image of the Johnson homomorphisms (cf. [18]).

Proposition 7.4. An element \( \varphi \in \mathfrak{h}_{g,1}(k) \) belongs to \( \mathcal{N}(k) \) if and only if \( [\varphi, \mathfrak{h}_{g,1}(1)] \subset \text{Im} \tau_{g,1}(k + 1) \).
Proof. Recall that Hain [10] proved that $\text{Im} \, \tau_{g,1}$ is generated by the degree 1 part. It follows that any element of $\text{Im} \, \tau_{g,1}(k)$ with $k \geq 2$ can be described as a linear combination of brackets of two elements in $\text{Im} \, \tau_{g,1}$ with lower degrees. Now consider the Jacobi identity

\[ [\varphi, [\psi, \chi]] + [\psi, [\chi, \varphi]] + [\chi, [\varphi, \psi]] = 0. \]

If we assume $\psi, \chi \in \text{Im} \, \tau_{g,1}$ and $\varphi$ normalizes both of $\psi$ and $\chi$, then the above identity implies that $\varphi$ also normalizes the bracket $[\psi, \chi]$. The claim follows from an easy inductive argument using this fact. A similar argument shows that the bracket mapping

\[ \text{Im} \, \tau_{g,1}(k) \otimes \text{Im} \, \tau_{g,1}(1) \xrightarrow{\text{Id}} \text{Im} \, \tau_{g,1}(k+1) \]

is surjective for any $k$. \hfill $\square$

**Theorem 7.5.** (i) $(\text{Im} \, \tau_{g,1}(6))^\text{Sp}$ is spanned by the following 2 elements

\[ \tau_1 = 3\bar{v}_1 - \bar{v}_2 + 8\bar{v}_3 + 5\bar{v}_5, \quad \tau_2 = 6\bar{v}_1 + 3\bar{v}_2 + 36\bar{v}_3 + 25\bar{v}_4 - 25\bar{v}_5. \]

(ii) $\text{Im} \, \tau_{g,1}(6)^\text{Sp}$ is spanned by the following 2 elements

\[ j_1 = v_2 - 5v_3 - 4v_4 + 2v_5, \quad j_2 = 3v_1 + 3v_2 + 3v_3 - v_4 - 2v_5. \]

(iii) $\dim (\text{Ker} \, ES_{6}/\text{Im} \, \tau_{g,1}(6))^{\text{Sp}} = 1$.

(iv) Modulo $(\text{Im} \, \tau_{g,1}(6))^\text{Sp}$, there exists a unique element in $h_{g,1}(6)^\text{Sp}$ which normalizes $\text{Im} \, \tau_{g,1}$. Namely

\[ \dim (\mathcal{N}(6)/\text{Im} \, \tau_{g,1}(6))^{\text{Sp}} = 1. \]

Proof. First we prove (i). Asada and Nakamura [2] proved that the leading term $[31^2]_{\text{Sp}} \subset h_{g,1}(3)$ is included in the image of the Johnson homomorphism. Let $\xi$ be the highest weight vector of this summand. We set $\eta = \iota(\xi)$ where $\iota$ denotes the symplectic automorphism given by $x_i \mapsto y_i, y_i \mapsto -x_i$ for all $i$. Then we have $[\xi, \eta] \in \text{Im} \, \tau_{g,1}(6)$ and explicit computation shows that

\[ \mathcal{K}_D([\xi, \eta]) = (0, 0, 0, -2, 2). \]

Next, by using the original work of Johnson [12] that $\text{Im} \, \tau_{g,1}(1) = h_{g,1}(1) \cong \wedge^3 H_Q$, we set

\[ \phi = [y_1 \wedge y_2 \wedge y_3, [y_1 \wedge y_2 \wedge y_3, \psi]] \]

where

\[ \psi = [[x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge y_5], [x_1 \wedge x_2 \wedge x_6, x_3 \wedge x_4 \wedge y_6]]. \]

Then clearly $\phi \in \text{Im} \, \tau_{g,1}(6)$ and explicit computation shows that

\[ \mathcal{K}_D(\phi) = (4, -1, -4, 4, -4). \]

It follows that $\mathcal{K}_D(\text{Im} \, \tau_{g,1}(6)^\text{Sp})$ is spanned by the 2 vectors

\[ t_1 = (0, 0, 0, -2, 2), \quad t_2 = (4, -1, -4, 4, -4) \]
Our task is then to find two linear combinations of \( v \) computation. Then explicit computation shows that the \( 2 \) cases we are done.

Also by computing the bracket mapping \( \tau \) choose the genus \( g \), we obtain the following two linear relations

\[
\begin{align*}
\mathcal{K}_D(9\bar{v}_1 - 8\bar{v}_2 + 4\bar{v}_3 - 25\bar{v}_4 + 50\bar{v}_5) + 90t_1 &= 0, \\
\mathcal{K}_D(-27\bar{v}_1 + 14\bar{v}_2 - 52\bar{v}_3 + 25\bar{v}_4 - 80\bar{v}_5) + 90t_2 &= 0.
\end{align*}
\]

It follows that \( \text{Im} \tau_{g,1}(6) \) is spanned by the following 2 elements

\[
9\bar{v}_1 - 8\bar{v}_2 + 4\bar{v}_3 - 25\bar{v}_4 + 50\bar{v}_5, \quad -27\bar{v}_1 + 14\bar{v}_2 - 52\bar{v}_3 + 25\bar{v}_4 - 80\bar{v}_5.
\]

The claimed 2 elements \( j_1, j_2 \) are obtained by a change of basis.

Next we prove (ii). We consider the following two elements

\[
p_1 = S_8(x_1 \otimes [\omega_0, [y_2, [x_2, [\omega_0, y_1]]]]), \quad p_2 = S_8(x_1 \otimes [y_2, [\omega_0, [x_2, [\omega_0, y_1]]]]).
\]

It is easy to see that both of these elements are contained in \( i_{g,1}(6) \) for any \( g \) because there are two copies of the symplectic form \( \omega_0 \) in the corresponding Lie spiders. Since we already know that \( i_{g,1}(6) \) is \( p \)-stable, to determine the required coefficients we may choose the genus \( g \) to be any one in the stable range \( g \geq 3 \). So we set \( g = 3 \) and begin computation. Then explicit computation shows that

\[
\mathcal{K}_D(p_1) = 4(98, -7, -81, 23, -10), \quad \mathcal{K}_D(p_2) = 24(0, -3, -3, 0, -1).
\]

Our task is then to find two linear combinations of \( v_i \)'s whose values under \( \mathcal{K}_D \) generate the 2 dimensional space spanned by the above two 5 dimensional vectors. Keeping \( g = 3 \), we obtain the following two linear relations

\[
\begin{align*}
\mathcal{K}_D(-36v_1 - 23v_2 - 101v_3 - 40v_4 + 50v_5) + 30240(98, -7, -81, 23, -10) &= 0, \\
\mathcal{K}_D(v_2 - 5v_3 - 4v_4 + 2v_5) + 6048(0, -3, -3, 0, -1) &= 0.
\end{align*}
\]

Since we have an identity

\[
-36v_1 - 23v_2 - 101v_3 - 40v_4 + 50v_5
\]

\[
= 13(v_2 - 5v_3 - 4v_4 + 2v_5) - 12(3v_1 + 3v_2 + 3v_3 - v_4 - 2v_5)
\]

\[
= 13j_1 - 12j_2,
\]

we are done.

To verify the accuracy of the above result, we made similar computations for the cases \( g = 4, 5, 6 \) and checked that the answers were the same as above.

Next we prove (iii). Explicit computation shows that \( a_g(4) \) and \( E_{6}(j_1), E_{6}(j_2) \in a_g(4) \) are linearly independent. Therefore \( \dim(\text{Ker} E_{6}) = 3 \). Since we already know that \( \dim(\text{Im} \tau_{g,1}(6)) = 2 \), the claim follows. We checked that \( E_{6}(\tau_1) = E_{6}(\tau_2) = 0 \) as it should be.

Finally we prove (iv). By combining the method of [25] and Theorem 4.3 we have

\[
\begin{align*}
h_{g,1}(7) &= 10[1]_{Sp} \oplus 15[3]_{Sp} \oplus \text{other terms}, \\
j_{g,1}(7) &= 3[1]_{Sp} \oplus 3[3]_{Sp} \oplus \text{other terms}.
\end{align*}
\]

Also by computing the bracket mapping

\[
h_{g,1}(1) \otimes \text{Im} \tau_{g,1}(6) \rightarrow h_{g,1}(7)
\]
explicitly, we find $6[1]_{Sp} \oplus 12[1^3]_{Sp}$ sitting inside $\text{Im} \tau_{g,1}(7)$. At this stage, we can conclude that the $[1^3]_{Sp}$-isotypical component of $\text{Im} \tau_{g,1}(7)$ is $12[1^3]_{Sp}$ and the multiplicity of the $[1]_{Sp}$-isotypical component of $\text{Im} \tau_{g,1}(7)$ is 6 or 7. Then we compute the Enomoto-Satoh map

$$ES_7 : h_{g,1}(7) \to a_g(5) = 15[1]_{Sp} \oplus 15[1^3]_{Sp} \oplus \text{other terms}$$

and find that it hits $3[1]_{Sp} \oplus 2[1^3]_{Sp} \subset a_g(5)$ while it detects only $2[1]_{Sp}$ out of $3[1]_{Sp} \subset i_{g,1}(7)$. We can now conclude that the $[1]_{Sp}$-isotypical component of $\text{Im} \tau_{g,1}(7)$ is $6[1]_{Sp}$ and we obtain the following exact sequence

$$0 \to (\text{Im} \tau_{g,1}(7))_1 \cong 6[1]_{Sp} \oplus 12[1^3]_{Sp} \to (\text{Ker} ES_7)_1 \to (r,s) [1]_{Sp} \oplus [1^3]_{Sp} \to 0$$

where $(\cdot)_1$ denotes the projection onto the $([1]_{Sp} \oplus [1^3]_{Sp})$-isotypical components and $r, s$ denote certain homomorphisms which we constructed explicitly. Next we consider the homomorphism

$$B_1 : h_{g,1}(6)_{Sp} \cong \mathbb{Q}^5 \to \text{Hom} (h_{g,1}(1), a_g(5))_{Sp}$$

defined by $B_1(v)(\xi) = ES_7([v, \xi]) \ (v \in h_{g,1}(6)_{Sp}, \xi \in h_{g,1}(1))$. Computation shows that $B_1$ hits only 1 dimension in the target so that $\dim \text{Ker} B_1 = 4$. Finally we consider the homomorphism

$$B_2 : \text{Ker} B_1 \subset h_{g,1}(6)_{Sp} \to \text{Hom} (h_{g,1}(1), [1]_{Sp} \oplus [1^3]_{Sp})_{Sp} \cong \mathbb{Q}^2$$

defined by $B_2(v)(\xi) = (r, s) ([v, \xi])$. It turns out that $B_2$ hits also only 1 dimension in the target so that $\dim \text{Ker} B_2 = 3$. Thus besides $(\text{Im} \tau_{g,1}(6))_{Sp}$, which is 2 dimensional, there exists one more dimension which normalizes $h_{g,1}(1)$. In view of Proposition 7.4 the claim is now proved.

**Remark 7.6.** It would be worthwhile to investigate whether the above summands $[1]_{Sp} \oplus [1^3]_{Sp}$ which the Enomoto-Satoh map does not detect, are detected by Conant’s new trace map given in [4] or not.

8. **Structure of the genus 1 case $h_{1,1}$**

In this section, we consider the case of genus 1 motivated by a recent work of Hain and Matsumoto mentioned in the introduction.

**Proposition 8.1.** (i) The irreducible decomposition of $[kl]_{GL}$ as an $Sp$-module is given by

$$[kl]_{GL} = [kl]_{Sp} + [k - 1, l - 1]_{Sp} + \cdots + [k - l + 1, 1]_{Sp} + [k - l]_{Sp}.$$

(ii) The restriction of $[kl]_{GL}$ to the subgroup $Sp(2, \mathbb{Q}) = SL(2, \mathbb{Q}) \subset GL(2g, \mathbb{Q})$ is given by

$$\text{Res}_{SL(2,\mathbb{Q})}^{GL(2g,\mathbb{Q})} [kl]_{GL} = [k - l]_{SL(2,\mathbb{Q})}.$$

**Proof.** (i) follows from the restriction law of the pair $Sp(2g, \mathbb{Q}) \subset GL(2g, \mathbb{Q})$ and (ii) follows from a further restriction to the subgroup $Sp(2, \mathbb{Q}) = SL(2, \mathbb{Q}) \subset GL(2g, \mathbb{Q})$ (cf. [8]).
Let $\mathfrak{h}_{g,1}(k)_{h \leq 2}$ denote the submodule of $\mathfrak{h}_{g,1}(k)$ consisting of $\text{GL}$-isotypical components of type $\lambda_{\text{GL}}$ with $h(\lambda) \leq 2$. Then as is well known, $\mathfrak{h}_{g,1}(k)_{h \leq 2}$ together with Proposition 8.1 completely determines the $\text{Sp}(2, \mathbb{Q})$-irreducible decomposition of $\mathfrak{h}_1(1-k)$. In this case, we can see that the converse is also true. Namely the $\text{Sp}(2, \mathbb{Q})$-irreducible decomposition of $\mathfrak{h}_{1,1}(k)$ determines $\mathfrak{h}_{g,1}(k)_{h \leq 2}$ completely. We indicate in Table 8 the explicit decompositions of the two modules for $k \leq 18$.

**Table 8.** Irreducible decompositions of $\mathfrak{h}_{1,1}(k)$ and $\mathfrak{h}_{g,1}(k)_{h \leq 2}$

| $k$  | Irreducible components of $\mathfrak{h}_{1,1}(k)$ | Irreducible components of $\mathfrak{h}_{g,1}(k)_{h \leq 2}$ |
|------|-----------------------------------------------|-------------------------------------------------|
| 1    | $\{0\}$                                       | $\{0\}$                                         |
| 2    | $[0]$                                         | $[2^2]$                                         |
| 3    | $[0]$                                         | $\{0\}$                                         |
| 4    | $[2]$                                         | $42$                                            |
| 5    | $\{0\}$                                       | $\{0\}$                                         |
| 6    | $[4,0]$                                       | $[62,44]$                                       |
| 7    | $[3]$                                         | $63$                                            |
| 8    | $[6,2^2]$                                     | $[82,2^4,64]$                                   |
| 9    | $[5,3,1]$                                     | $[83,74,65]$                                    |
| 10   | $[8,6,3,4,2,3,0]$                             | $[10,2,93,3,84,75,3,66]$                        |
| 11   | $[7,2,5,4,3,2,1]$                             | $[10,3,2,94,4,85,2,76]$                         |
| 12   | $[10,8,5,6,4,8,2]$                            | $[12,2,11,4,10,4,95,8,86]$                      |
| 13   | $[2,9,3,7,8,5,9,3,6,1]$                       | $[2,12,3,3,1,1,4,8,10,5,9,96,6,87]$             |
| 14   | $[12,10,7,8,9,6,18,4]$                        | $[14,2,13,3,7,12,4,9,11,5,18,10,6]$             |
|      | $[11,2,11,0]$                                 | $[11,9,7,11,8^2]$                             |
| 15   | $[2,11,5,9,14,7,21,5]$                        | $[2,14,3,5,13,4,14,12,5,21,11,6]$              |
|      | $[26,3,17,1]$                                 | $[26,10,7,17,98]$                              |
| 16   | $[14,2,12,9,10,16,8]$                         | $[16,2,2,15,3,9,14,4,16,13,5,38,12,6]$         |
|      | $[38,6,38,4,6,2,10,0]$                        | $[38,11,7,16,10,8,10,9^2]$                      |
| 17   | $[2,13,7,11,23,9,42,7]$                       | $[2,16,3,7,15,4,23,14,5,42,13,6]$              |
|      | $[68,5,72,3,48,1]$                            | $[68,12,7,72,11,8,48,10,9]$                    |
| 18   | $[16,2,14,12,12,26,10,67,8]$                  | $[18,2,17,3,12,16,4,26,15,5,67,14,6]$          |
|      | $[96,6,138,4,100,2,57,0]$                     | $[96,13,7,138,12,8,100,11,9,57,10^2]$          |

**Proposition 8.2.** (i) The leading term in the irreducible decomposition of $\mathfrak{h}_{1,1}(k)$ is given by

\[
\begin{align*}
[k - 2] & \quad (k \text{ is even}), \\
[k/6][k - 4] & \quad (k \text{ is odd } \geq 7), \quad \{0\} \quad \text{for } k = 1, 3, 5.
\end{align*}
\]

(ii) There exists a natural isomorphism

\[\mathfrak{h}_{1,1}(2k)^{\text{Sp}} \cong H_{[k+1]}\]

**Proof.** A proof of the former case of (i), which should be well known, can be given as follows. Asada and Nakamura [2] proved that the leading term of $\mathfrak{h}_{g,1}(2k)$ is $[2k,2]_{\text{GL}}$. 

By Proposition 8.1, this summand yields \([2k - 2]_{\text{Sp}}\) as its unique grandchild. Its further restriction to the genus 1 case, namely the restriction to the subgroup \(\text{Sp}(2, \mathbb{Q}) \subset \text{Sp}(2g, \mathbb{Q})\) is the required summand \([2k - 2]_{\text{Sp}(2, \mathbb{Q})}\). Again by Proposition 8.1, it is easy to see that no other summand in \(h_{g,1}(2k)\) can yield this component implying that the multiplicity of this component is 1.

Next we prove the latter part of (i). As an \(\text{Sp}(2, \mathbb{Q}) = \text{SL}(2, \mathbb{Q})\)-representation, the character of \(h_{1,1}(k)\) is

\[
\text{ch}(h_{1,1}(k)) = \text{ch}(H_{\mathbb{Q}})\text{ch}(L_{1,1}(k + 1)) - \text{ch}(L_{1,1}(k + 2))
= (x_1 + x_1^{-1})\text{ch}(L_{1,1}(k + 1)) - \text{ch}(L_{1,1}(k + 2)),
\]

where the character of \(L_{1,1}(l)\) is given by

\[
\text{ch}(L_{1,1}(l)) = \frac{1}{l} \sum_{d|l} \mu(d) (x_1^d + x_1^{-d})^{l/d} = \frac{1}{l} \sum_{d|l} \mu(d) \left\{ \sum_{i=0}^{l/d} \binom{l/d}{i} x_1^{l-2di} \right\}
\]

(cf. Theorem 4.1). Then the claim follows by checking the leading term with respect to \(x_1\) of this Laurent polynomial.

From the above formula, the degree of \(\text{ch}(L_{1,1}(l))\) is at most \(l\). However, the well-known formula \(\sum_{d|l} \mu(d) = 0\) for \(l \geq 2\) says that the coefficient of \(x_1^l\) is 0. The next term appears in degree \(l - 2\), because the degree of \(x_1\) in \(\text{ch}(L_{1,1}(l))\) decreases by two each time. Since we have \(d = i = 1\) in this case, the coefficient of \(x_1^{l-2}\) is 1.

To see the coefficient of \(x_1^{l-4}\), consider the terms with \(di = 2\). If \(l\) is odd, it suffices only to see the term with \(d = 1\) and \(i = 2\). The contribution of this term to the coefficient of \(x_1^{l-4}\) is \(\frac{\mu(1)}{l} \binom{l}{2} = \frac{l - 1}{2}\). If \(l\) is even, another term with \(d = 2\) and \(i = 1\) is involved. The contribution is given by \(\frac{\mu(2)}{l} \binom{l/2}{1} = -\frac{1}{2}\). Therefore the coefficient of \(x_1^{l-4}\) is \(\left\lfloor \frac{l - 1}{2} \right\rfloor\) for \(l\) of both cases.

To see the coefficient of \(x_1^{l-6}\), consider the terms with \(di = 3\). If \(l \not\equiv 0\) modulo 3, it is enough to see only the term with \(d = 1\) and \(i = 3\). The contribution of this term to the coefficient of \(x_1^{l-6}\) is \(\frac{\mu(1)}{l} \binom{l}{3} = \frac{(l - 1)(l - 2)}{6}\). Otherwise, we have another term with \(d = 3\) and \(i = 1\). The contribution is \(\frac{\mu(3)}{l} \binom{l/3}{1} = -\frac{1}{3}\). Therefore the coefficient of \(x_1^{l-6}\) is \(\left\lfloor \frac{(l - 1)(l - 2)}{6} \right\rfloor\).

Consequently, we have

\[
\text{ch}(L_{1,1}(l)) = x_1^{l-2} + \left\lfloor \frac{l - 1}{2} \right\rfloor x_1^{l-4} + \left\lfloor \frac{(l - 1)(l - 2)}{6} \right\rfloor x_1^{l-6} + (\text{lower degree terms})
\]
for $l \geq 2$. Hence
\[
\text{ch}(h_{1,1}(k)) = (x_1 + x_1^{-1}) \left( x_1^{k-1} + \left\lfloor \frac{k}{2} \right\rfloor x_1^{k-3} + \left\lfloor \frac{k(k-1)}{6} \right\rfloor x_1^{k-5} + \cdots \right)
- \left( x_1^k + \left\lfloor \frac{k+1}{2} \right\rfloor x_1^{k-2} + \left\lfloor \frac{(k+1)k}{6} \right\rfloor x_1^{k-4} + \cdots \right)
= \left( 1 + \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k+1}{2} \right\rfloor \right) x_1^{k-2} + \left( \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k(k-1)}{6} \right\rfloor - \left\lfloor \frac{(k+1)k}{6} \right\rfloor \right) x_1^{k-4} + \cdots .
\]
When $k$ is even, the coefficient of $x_1^{k-2}$ is 1, which checks the former part of (i) already proved. When $k$ is odd, the coefficient of $x_1^{k-2}$ is 0. However, it is easy to check that the coefficient of $x_1^{k-4}$ is given by $\left\lfloor \frac{k}{6} \right\rfloor$. If $k \geq 7$, this gives the leading term. In the cases where $k = 1, 3, 5$, we have $\text{ch}(h_{1,1}(k)) = 0$ since the non-negative degree part of the character $\text{ch}(h_{1,1}(k))$ vanishes and $\text{ch}(h_{1,1}(k))$ is invariant under the involution $x_1 \leftrightarrow x_1^{-1}$.

Finally we prove (ii). It is easy to see that the only isotypical component appearing in $h_{g,1}(k), k \leq 2$ which has multiple double floors is the one corresponding to the Young diagram $[k+1, k+1]$. Then the result follows from the definition of $H_{[k+1]}$.

By combining Proposition 8.1 with the latter part of Proposition 8.2 (i), we obtain the following.

**Corollary 8.3.** The leading term of the $GL$-irreducible decomposition of $h_{g,1}(2k + 1)$ is
\[
\left\lfloor \frac{k}{6} \right\rfloor [2k, 3]_{GL} \quad (k \geq 3).
\]

The summands $[2k - 2]_{Sp} \subset h_{1,1}(2k) \ (k = 1, 2, \ldots)$ play a central role in the theory of universal mixed elliptic motives due to Hain and Matsumoto (cf. [11] and [27]). More precisely, they consider the element
\[
\epsilon_{2k} \in [2k - 2]_{Sp} \subset h_{1,1}(2k) : \text{the highest weight vector}.
\]
Twice of this element is represented by the following Lie spider
\[
y_1 \quad x_1 \quad \cdots \quad x_1 \quad y_1
\]
while the highest weight vector of $[2k, 2]_{GL} \subset h_{g,1}(2k)$ is represented by the same Lie spider but we replace both $y_1$ by $x_2$.

They define
\[
u = \text{Lie subalgebra of } h_{1,1} \text{ generated by } \epsilon_{2k} \text{ for all } k = 0, 1, 2, \ldots
\]
where $\epsilon_0 \in h_{1,1}(0) \cong [2]_{Sp}$ is the highest weight vector, and they relate the structure of it with the theory of elliptic modular forms. Furthermore, they consider the action of
the absolute Galois group and, in particular, show the existence of a certain homomorphism
\[ \mathfrak{f} \to \mathfrak{h}_{1,1}^{\text{Sp}} \]
from the “fundamental Lie algebra”
\[ \mathfrak{f} = \text{free graded Lie algebra generated by } \sigma_3, \sigma_5, \ldots \]
to the \( \text{Sp} \)-invariant part of \( \mathfrak{h}_{1,1} \). It has the property that the image of \( \sigma_{2k+1} \), denoted by \( \tilde{\sigma}_{2k+1} \), lies in \( \mathfrak{h}_{1,1}(4k + 2)^{\text{Sp}} \) and moreover it should normalize \( u \).

**Proposition 8.4.** The kernel of the Enomoto-Satoh map \( ES : \mathfrak{h}_{g,1} \to a_g \) is an \( \text{Sp} \)-Lie subalgebra of \( \mathfrak{h}_{g,1} \).

**Proof.** Since \( ES \) is an \( \text{Sp} \)-equivariant mapping, \( \text{Ker} ES \) is an \( \text{Sp} \)-submodule of \( \mathfrak{h}_{g,1} \). On the other hand, a theorem of Satoh [29] implies that \( \text{Ker} ES \) is a Lie subalgebra of \( \mathfrak{h}_{g,1} \) in a certain stable range, namely for a sufficiently large \( g \). Furthermore, \( ES \) is an \( i \)-stable homomorphism in the sense that the following diagram is commutative:
\[
\begin{array}{ccc}
\mathfrak{h}_{g,1}(k) & \xrightarrow{ES} & a_g(k - 2) \\
\downarrow i & & \downarrow i \\
\mathfrak{h}_{g+1,1}(k) & \xrightarrow{ES} & a_{g+1}(k - 2).
\end{array}
\]
It follows that \( \text{Ker} ES \) is an \( \text{Sp} \)-Lie subalgebra of \( \mathfrak{h}_{g,1} \) without any condition on \( g \). \( \square \)

**Theorem 8.5.** (i) The summand \( [2k - 2]_{\text{Sp}} \subset \mathfrak{h}_{1,1}(2k) \) lies in the kernel of the Enomoto-Satoh map for any \( k \). It follows that \( ES(u) = 0 \).

(ii) The image of \( \sigma_5 \) in \( \mathfrak{h}_{1,1}(10)^{\text{Sp}} \) is characterized by the condition that \( ES([\epsilon_4, \sigma_5]) = 0 \). More precisely, we have
\[ \dim \text{Ker} \left( \mathfrak{h}_{1,1}(10)^{\text{Sp}} \xrightarrow{[\epsilon_4, \cdot]} \mathfrak{h}_{1,1}(14) \xrightarrow{ES} a_1(12) \right) = 1 \]
whereas \( \dim \mathfrak{h}_{1,1}(10)^{\text{Sp}} = 3 \).

**Proof.** First we prove (i). In view of Proposition 8.4, it suffices to show that \( ES(\epsilon_{2k}) = 0 \) for any \( k \geq 1 \) (the case \( k = 0 \) is trivial), where we regard \( \epsilon_{2k} \) as an element of \( \mathfrak{h}_{1,1}(2k) \subset \mathfrak{h}_{g,1}(2k) \) after stabilizations. The Enomoto-Satoh map \( ES \) is defined as the composition of group homomorphisms
\[
\mathfrak{h}_{g,1}(2k) \subset H_\mathbb{Q} \otimes L_{g,1}(2k + 1) \subset H_\mathbb{Q}^{(2k+2)} \\
\xrightarrow{K_{1,2}} H_\mathbb{Q}^{2k} \xrightarrow{\text{proj}} (H_\mathbb{Q}^{2k})_{\mathbb{Z}/2k} \cong (H_\mathbb{Q}^{2k})^{\mathbb{Z}/2k} = a_g(2k),
\]
where \( \text{proj} \) is the natural projection and the isomorphism \( (H_\mathbb{Q}^{2k})_{\mathbb{Z}/2k} \cong (H_\mathbb{Q}^{2k})^{\mathbb{Z}/2k} \) is given by
\[
u_1 \otimes \nu_2 \otimes \cdots \otimes \nu_{2k} \mapsto \sum_{i=1}^{2k} \sigma_{2k}^i (\nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_{2k})
\]
using the cyclic permutation \( \sigma_{2k} \) of entries of tensors. For \( l \geq 1 \), put
\[ X_l = [x_1, [x_1, \ldots, [x_1, y_1] \cdots]] = (-1)^l[[\cdots [y_1, x_1], x_1], \ldots, x_1] \in L_{g,1}(l + 1) \subset H_\mathbb{Q}^{(l+1)}. \]
As an element of \( H_Q \otimes L_{g,1}(2k + 1) \subset H_Q^{(2k+2)} \), we have
\[
\epsilon_{2k} = y_1 \otimes X_{2k} + x_1 \otimes [X_{2k-1}, y_1] + x_1 \otimes [X_{2k-2}, -X_1] + x_1 \otimes [X_{2k-3}, (-1)^2 X_2] + \cdots + x_1 \otimes [X_i, (-1)^{2k-2} X_{2k-2}] + x_1 \otimes [y_1, (-1)^{2k-1} X_{2k-1}] + y_1 \otimes ((-1)^2 X_{2k})
\]
\[
= 2y_1 \otimes X_{2k} + 2x_1 \otimes [X_{2k-1}, y_1] + \sum_{i=1}^{2k-2} x_1 \otimes [X_i, (-1)^{i+1} X_{2k-1-i}]
\]
\[
+ 2y_1 \otimes X_{2k} + 2(x_1 \otimes X_{2k-1} \otimes y_1 - x_1 \otimes y_1 \otimes X_{2k-1})
\]
\[
+ (-1)^{i+1} \sum_{i=1}^{2k-2} x_1 \otimes X_i \otimes X_{2k-1-i} + (-1)^i \sum_{i=1}^{2k-2} x_1 \otimes X_{2k-1-i} \otimes X_i.
\]

Note that the contraction \( K_{12} \) satisfies \( K_{12}(X \otimes Y) = K_{12}(X) \otimes Y \) for any \( X \in H_Q^{(i)} \) and \( Y \in H_Q^{(j)} \) with \( i \geq 2 \). Hence we have
\[
K_{12}(\epsilon_{2k}) = 2K_{12}(y_1 \otimes X_{2k}) + 2 \{K_{12}(x_1 \otimes X_{2k-1}) \otimes y_1 - K_{12}(x_1 \otimes y_1) \otimes X_{2k-1}\}
\]
\[
+ (-1)^{i+1} \sum_{i=1}^{2k-2} K_{12}(x_1 \otimes X_i) \otimes X_{2k-1-i} + (-1)^{i} \sum_{i=1}^{2k-2} K_{12}(x_1 \otimes X_{2k-1-i}) \otimes X_i.
\]

By induction, we can show that the equalities
\[
K_{12}(y_1 \otimes X_i) = \sum_{l=1}^{i-1} (-1)^i X_{i-l} \otimes x_1^{(i-l)} + (-1)^{l} y_1 \otimes x_1^{(i-l)}
\]
\[
K_{12}(x_1 \otimes X_i) = (-1)^l x_1^{(i-l)}
\]
hold. Since
\[
x_1^{(i)} \otimes X_1 \otimes x_1^{(2k-2-i)} = x_1^{(i)} \otimes x_1 \otimes y_1 \otimes x_1^{(2k-2-i)} - x_1^{(i)} \otimes y_1 \otimes x_1 \otimes x_1^{(2k-2-i)}
\]
\[
= 0 \in (H_Q^{(2k)})_{2k/2k},
\]
we have
\[
x_1^{(i-j)} \otimes X_{2k-1-j} \otimes x_1^{(j)} = 0 \in (H_Q^{(2k)})_{2k/2k}.
\]

Therefore
\[
proj \circ K_{12}(\epsilon_{2k}) = 2(-1)^{2k} y_1 \otimes x_1^{(2k-1)} + 2(-1)^{2k-1} x_1^{(2k-1)} \otimes y_1 = 0.
\]

This shows that \( ES(\epsilon_{2k}) = 0 \).

Next we prove (ii). We consider the following \((3+3)\) linear chord diagrams
\[
D_1 = (12)(34)(56)(78)(910)(1112), \quad U_1 = (12)(35)(46)(79)(810)(1112),
\]
\[
D_2 = (12)(34)(56)(79)(810)(1112), \quad U_2 = (12)(35)(47)(69)(810)(1112),
\]
\[
D_3 = (12)(34)(56)(710)(810)(912), \quad U_3 = (16)(29)(38)(411)(510)(712)
\]
in \( \mathcal{D}^{(12)} \). We define
\[
u_i = \Phi(S_{12} \circ \sigma_{12} \circ p_{11}' \circ \sigma_{12}^{-1}(U_i)) \in h_{1,1}(10)^{Sp} \quad (i = 1, 2, 3).
\]
It turns out that the set \( \{D_1, D_2, D_3\} \) can serve as a detector of \( h_{1,1}(10)^{Sp} \cong Q^3 \). In fact, the intersection matrix \((\alpha_{D_i}(u_j))\) is given by

\[
\begin{pmatrix}
46656 & 23328 & 3888 \\
3456 & 192 & -576 \\
-27648 & -14304 & -4824 \\
\end{pmatrix}
\]

which is non-singular. It follows that \( \{u_1, u_2, u_3\} \) is a basis of \( h_{1,1}(10)^{Sp} \). Next we compute the bracket \([\epsilon_4, u_i] \in h_{1,1}(14)\) and apply the Enomoto-Satoh mapping to it. Then we obtain 3 large tensors

\[
r_i = ES([\epsilon_4, u_i]) \in a_1(12) \quad (i = 1, 2, 3).
\]

Finally, we seek for a linear relation between \( r_1, r_2, r_3 \). It turns out that there exists a unique relation

\[
41r_1 - 51r_2 + 4r_3 = 0.
\]

We can now conclude that the element \( 41u_1 - 51u_2 + 4u_3 \in h_{1,1}(10)^{Sp} \) is the unique element (up to scalars) such that its bracket with \( \epsilon_4 \) is contained in \( u \), completing the proof.

\[\square\]

**Remark 8.6.** Pollack [27] determined the element \( \tilde{\sigma}_5 \in h_{1,1}(10)^{Sp} \cong Q^3 \) explicitly. We have checked that the unique element in \( h_{1,1}(10)^{Sp} \) (up to scalars) given above, coincides with his element. We are trying to extend our result to identify the element \( \tilde{\sigma}_7 \in h_{1,1}(14)^{Sp} \cong H_{[8]} \cong Q^{11} \).

9. **Tables of Orthogonal Decompositions of** \( h_{g,1}(2k)^{Sp} \)

In this section, we give Tables for the orthogonal decompositions of \( h_{g,1}(2k)^{Sp} \) for the cases \( 2k = 14, 16, 18, 20 \).

**Table 9.** Orthogonal decomposition of \( h_{g,1}(14)^{Sp} \)

| dim | eigenspaces |
|-----|-------------|
| 11 \((g = 1)\) | \(11) \^ 8 \^ 0\) |
| 1691 \((g = 2)\) | \(147) \^ 5 \^ 665) \^ 62) \^ 53) \^ 116) \^ 12 \^ 4 \^ 0\) |
| 6471 \((g = 3)\) | \(403) \^ 9 \^ 521) \^ 1436) \^ 431) \^ 665) \^ 42) \^ 3) \^ 232) \^ 3 \^ 2 \^ 0\) |
| 8505 \((g = 4)\) | \(337) \^ 1) \^ 1120) \^ 421) \^ 266) \^ 3 \^ 11) \^ 2) \^ 0\) |
| 8795 \((g = 5)\) | \(104) \^ 8 \^ 168) \^ 321) \^ 18) \^ 2) \^ 2) \^ 0\) |
| 8816 \((g = 6)\) | \(14) \^ 3) \^ 7) \^ 2) \^ 4) \^ 0\) |
| 8817 \((g \geq 7)\) | \(210) \^ 0\) |
### Table 10. Orthogonal decomposition of $\mathfrak{h}_{g,1}(16)^{Sp}$

| dim | eigenspaces |
|-----|-------------|
| $10 \ (g = 1)$ | $10|^{9}_{6}$ |
| $11842 \ (g = 2)$ | $440|^{8}_{7}|3028|^{7}_{6}|5860|^{6}_{5}|2504|^{5}_{4}$ |
| $69544 \ (g = 3)$ | $1776|^{7}_{6}|14616|^{6}_{5}|621|^{5}_{4}|21204|^{4}_{3}|531|^{3}_{2}|7664|^{2}_{1}$ |
| $3270|^{4}_{3}|8904|^{3}_{2}|432|^{2}_{1}$ |
| $268|^{3}_{2}|268|^{2}_{1}$ |
| $104190 \ (g = 4)$ | $2112|^{9}_{8}|12904|^{8}_{7}|521|^{7}_{6}|9744|^{6}_{5}|431|^{5}_{4}|6936|^{4}_{3}$ |
| $1776|^{3}_{2}|1776|^{2}_{1}$ |
| $2532|^{3}_{2}|2532|^{2}_{1}$ |
| $110610 \ (g = 5)$ | $960|^{9}_{8}|56280|^{8}_{7}|531|^{7}_{6}|3546|^{6}_{5}|1059|^{5}_{4}|239184|^{4}_{3}|78422$ |
| $11131 \ (g = 6)$ | $180|^{9}_{8}|312|^{8}_{7}|321|^{7}_{6}|97776|^{6}_{5}|199320|^{5}_{4}|531|^{4}_{3}|36780|^{3}_{2}|62826|^{2}_{1}$ |
| $111148 \ (g \geq 7)$ | $12|^{9}_{8}|5|^{8}_{7}|5|^{7}_{6}|5|^{6}_{5}|5|^{5}_{4}|5|^{4}_{3}|5|^{3}_{2}|5|^{2}_{1}|5|^{1}_{0}$ |

### Table 11. Orthogonal decomposition of $\mathfrak{h}_{g,1}(18)^{Sp}$

| dim | eigenspaces |
|-----|-------------|
| $57 \ (g = 1)$ | $57|^{10}_{8}$ |
| $100908 \ (g = 2)$ | $1710|^{9}_{8}|15053|^{8}_{7}|82|^{7}_{6}|42826|^{6}_{5}|73|^{5}_{4}|36780|^{4}_{3}|64|^{3}_{2}|4482|^{2}_{1}$ |
| $888099 \ (g = 3)$ | $8520|^{8}_{7}|97776|^{7}_{6}|721|^{6}_{5}|239184|^{5}_{4}|631|^{4}_{3}|78422|^{3}_{2}|62826|^{2}_{1}$ |
| $117024|^{4}_{3}|191095|^{3}_{2}|532|^{2}_{1}$ |
| $1548984 \ (g = 4)$ | $13584|^{11}_{10}|126540|^{10}_{9}|621|^{9}_{8}|199320|^{8}_{7}|531|^{7}_{6}|116340|^{6}_{5}|524|^{5}_{4}|32676|^{4}_{3}|145281$ |
| $15053|^{3}_{2}|15053|^{2}_{1}$ |
| $1710798 \ (g = 5)$ | $8842|^{10}_{9}|56280|^{9}_{8}|521|^{8}_{7}|44151|^{7}_{6}|35220|^{6}_{5}|126540|^{5}_{4}|62826|^{4}_{3}|145281|^{3}_{2}|145281|^{2}_{1}$ |
| $840|^{3}_{2}|13344|^{2}_{1}$ |
| $2600|^{9}_{8}|9619|^{8}_{7}|421|^{7}_{6}|2340|^{6}_{5}|3096|^{5}_{4}|321|^{4}_{3}|36780|^{3}_{2}|64|^{2}_{1}$ |
| $1728591 \ (g = 6)$ | $357|^{9}_{8}|605|^{8}_{7}|321|^{7}_{6}|67|^{6}_{5}|67|^{5}_{4}|67|^{4}_{3}|67|^{3}_{2}|67|^{2}_{1}$ |
| $1729620 \ (g = 7)$ | $12|^{10}_{9}|12|^{9}_{8}|12|^{8}_{7}|12|^{7}_{6}|12|^{6}_{5}|12|^{5}_{4}|12|^{4}_{3}|12|^{3}_{2}|12|^{2}_{1}$ |
| $1729657 \ (g \geq 9)$ | $21|^{9}_{8}$ |
Table 12. Orthogonal decomposition of $h_{g,1}(20)^{Sp}$

| dim    | eigenspaces                                                                 |
|--------|------------------------------------------------------------------------------|
| $108$  | $g = 1$                                                                       |
| $869798$ | $g = 2$                                                                       |
| $12057806$ | $g = 3$                                                                       |
| $25062360$ | $g = 4$                                                                       |
| $29129790$ | $g = 5$                                                                       |
| $29688027$ | $g = 6$                                                                       |
| $29728348$ | $g = 7$                                                                       |
| $29729957$ | $g = 8$                                                                       |
| $29729988$ | $g ≥ 9$                                                                       |

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Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
E-mail address: morita@ms.u-tokyo.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
E-mail address: sakasai@ms.u-tokyo.ac.jp

Department of Frontier Media Science, Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo, 164-8525, Japan
E-mail address: macky@fms.meiji.ac.jp