String-Loop Corrections Versus Non-Extremality

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Abstract

We discuss a magnetic black-hole solution to the equations of motion of the string-loop-corrected effective action. At the string-tree level, this solution is the extremal magnetic black hole described by the "chiral null model." In the extremal case, the string-loop correction is constant, and this fact is used to analytically solve the loop-corrected equations of motion. In distinction to the tree-level solution, the resulting loop-corrected solution has the horizon at a finite distance from the origin; its location is a function of the loop correction. The loop-corrected configuration is compared with a string-tree-level non-extremal magnetic black hole solution which also has the horizon at a finite distance from the origin. We find that for an appropriate choice of free parameters of solutions, the loop-corrected magnetic black hole can be approximated by a tree-level non-extremal solution. We compare the thermodynamic properties of the loop-corrected and non-extremal solutions.

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At present, string theory is considered the best candidate for a fundamental theory that would provide a consistent quantum theory of gravity unified with the other interactions [1,2]. In particular, string theory provides a powerful approach to the physics of black holes (review and further refs. in [3,4]). In this setting a fundamental problem is that of understanding how the intrinsically stringy effects modify Einstein gravity. In this paper we discuss two of these effects: presence of scalar fields such as the dilaton and the moduli fields and higher-genus contributions modifying the tree-level effective action. We focus on higher-genus corrections, because string theory, being a theory formulated on a world sheet, intrinsically contains string-loop corrections from higher topologies of the world sheet (these vanish only for higher supersymmetries $N \geq 4$), while $\alpha'$ corrections can vanish in certain constructions based on conformal field theories and for a large class of backgrounds.

We consider the one-string-loop (torus topology) corrections to a special class of backgrounds: magnetic 4D black holes provided by the ”chiral null models” embedded in (compactified) heterotic string theory [5,6]. String-loop effects modify the gauge couplings in the effective action. In the general case, these corrections are rather complicated functions of the moduli (fields that describe compact dimensions) [7–9]. However, for the extremal magnetic black-hole solution, the loop correction $\Delta$ is constant. Assuming that the loop correction $\Delta$ is numerically small, we obtain an analytic solution to the loop-corrected equations of motion. The dilaton of the string-tree-level solution increases at small distances. Thus, at small distances, the effective gauge coupling $e^{-\phi} + \Delta$ is sensitive to string-loop correction. This produces an important physical effect: the horizon of the loop-corrected magnetic black hole is shifted from the point $r = 0$ to a finite distance determined by the loop correction.

The loop-corrected solution is compared with a non-extremal string-tree-level magnetic black hole solution. Since for small loop correction $\Delta$ the effective string coupling $e^{-\phi}$ at large distances is much larger than the loop correction, adjusting free parameters of the non-extremal solution, the large-$r$ asymptotic of the loop-corrected metric can be made close to that of the non-extremal solution. Requiring that the locations of the horizons of both solutions are of the same order, it is possible to fix the free parameters of both solutions and to approximate string-loop-corrected black hole solution by a tree-level non-extremal configuration.

In section 2 we review the structure of the 4D magnetic black hole in heterotic string theory provided by the chiral null model.

In section 3 we solve the loop-corrected equations of motion. Starting from the extremal magnetic black-hole solution at the string-tree level, we obtain an analytic solution to the loop-corrected equations of motion.

In section 4 we find the position of the horizon of the loop-corrected magnetic black hole as a function of the loop correction $\Delta$ and the free parameters of solution. We show that by adjusting the free parameters of both solutions the loop-corrected solution can be approximated by a non-extremal string-tree-level magnetic solution.

In section 5 we discuss some thermodynamic properties of the loop-corrected magnetic black hole. We calculate the Hawking temperature and geometric entropy of the black hole and compare them with the corresponding quantities of the non-extremal solution. Finally, we comment on possible microscopic derivation of the entropy.
II. DYONIC 4D BLACK HOLE IN TOROIDALLY COMPACTIFIED HETEROTIC STRING THEORY

We begin with a brief review of the chiral null models \[5,6\]. The chiral null model is a nonlinear 2D $\sigma$-model interpreted as a string world-sheet Lagrangian with nontrivial backgrounds:

$$L = F(x)\partial_u\left[\bar{\partial}v + K(u,x)\bar{\partial}u + 2A_i(u,x)\bar{\partial}x^i\right] + (G_{ij} + B_{ij})(x)\partial x^i\bar{\partial}x^j + \mathcal{R}\Phi(x). \tag{1}$$

An important special case of the chiral null models (1) are the chiral null models with curved transverse part of the form

$$L = F(x)\partial_u\left[\bar{\partial}v + K(x)\bar{\partial}u\right] + f(x)k(x)[\partial x^4 + a_s(x)\partial x^s][\bar{\partial}x^4 + a_s(x)\bar{\partial}x^s]$$

$$+ f(x)k^{-1}(x)\partial x^s\bar{\partial}x^s + b_s(x)(\partial x^4\bar{\partial}x^s - \bar{\partial}x^4\partial x^s) + \mathcal{R}\Phi(x), \tag{2}$$

$$\mathcal{R} \equiv \frac{1}{4} \alpha' \sqrt{g^{(2)}} R^{(2)}.$$  

Here $x^s = (x^1, x^2, x^3)$ and $v = 2t$ are non-compact space-time coordinates, $u = y_2$ and $x^4 = y_1$ are compact toroidal coordinates. Written in “4D form”, the chiral null model is

$$L = (G'_{\mu\nu} + B'_{\mu\nu})(x)\partial x^\mu\bar{\partial}x^\nu + G_{mn}(x)[\partial y^m + A^{(1)m}_\mu(x)\partial x^\mu][\bar{\partial}y^n + A^{(1)n}_\nu(x)\bar{\partial}x^\nu]$$

$$+ A^{(2)}_{\mu\nu}(x)(\partial y^n\bar{\partial}x^\mu - \bar{\partial}y^n\partial x^\mu) + \mathcal{R}\phi(x). \tag{3}$$

The 4D string-frame background is related to the fields in the Lagrangian (2) as \[11\]

$$G'_{\mu\nu} = G_{\mu\nu} - G_{mn}A^{(1)m}_\mu A^{(1)n}_\nu, \quad B'_{\mu\nu} = B_{\mu\nu}, \tag{4}$$

$$A^{(1)n}_\mu \equiv A^n_\mu = G^{nm}G_{n\mu}, \quad A^{(2)}_{\mu\nu} = B_{\mu\nu}, \quad \phi = \Phi - \frac{1}{4} \ln \Delta, \quad \Delta \equiv \det G_{mn}, \tag{5}$$

and their explicit form can be read off from (4):

$$A^1_\mu = (0, a_s), \quad A^2_\mu = (K^{-1}, 0), \quad B_{1\mu} = (0, b_s), \quad B_{2\mu} = (F', 0).$$

Following \[10\] we consider a special class of chiral null models where the functions $a_s$ and $b_s$ satisfy the equations

$$2\partial[a_q] = -\epsilon_{pq}\partial^sf, \quad 2\partial[a_q] = -\epsilon_{pq}\partial^sk^{-1}.$$  

Here $f$ and $k^{-1}$ are three-dimensional harmonic functions

$$\partial_s\partial^sf = 0, \quad \partial_s\partial^sk^{-1} = 0$$

and the dilaton $\phi$ is

$$\phi = \frac{1}{2} \ln f.$$  

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The four-dimensional Einstein-frame metric is

\[ g_{\mu \nu} = e^{-2\phi} G'_{\mu \nu}. \] (6)

From the sigma-model perspective, the 4D backgrounds of the above functional form are solutions to the equations ensuring conformal invariance of the theory. These backgrounds have also the space-time interpretation as black holes that can be obtained by solving the equations of motion of the 4D low energy effective action which (in the Einstein frame) is

\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial_\mu \phi)^2 - (\partial_\mu \sigma)^2 - (\partial_\mu \gamma)^2 \right. 
- \frac{1}{4} e^{-\phi + 2\gamma} F_{(1)}^2 - \frac{1}{4} e^{-\phi + 2\sigma} F_{(2)}^2 
\left. \right. 
- \frac{1}{4} e^{-\phi - 2\gamma} F_{(3)}^2 - \frac{1}{4} e^{-\phi - 2\sigma} F_{(4)}^2 \right). \] (7)

Here \( g_{\mu \nu} \) is the (Einstein frame) metric, \( \sigma \) and \( \gamma \) are the moduli related to the radii of the \( T^2 \). The action (7) is a particular case of the general 4D effective action which is obtained by dimensional reduction of 10D supergravity action [11] by compactifying on the product of the tori and by truncating to the relevant set of fields. We shall consider the following solution to the equations of motion [6,12]:

\[ ds^2 = -\Lambda(r) dt^2 + \Lambda^{-1} (dr^2 + r^2 d\Omega_2^2), \]
\[ \Lambda^2(r) = FK^{-1} k f^{-1}, \]
\[ 2\phi = \ln FK^{-1} k f^{-1}, \]
\[ e^{2\sigma} = FK, \]
\[ e^{2\gamma} = f k, \] (8)

where \( F^{-1}, K, f \) and \( k^{-1} \) are harmonic functions. \( F_{(1)} \) and \( F_{(3)} \) correspond to electric fields, \( F_{(2)} \) and \( F_{(4)} \) are magnetic field strengths.

Following the general discussion [7–9], the one-loop-corrected 4D action has the following form:

\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial_\mu \phi)^2 - (\partial_\mu \sigma)^2 - (\partial_\mu \gamma)^2 \right. 
- \frac{1}{4} e^{2\gamma}(e^{-\phi} + \Delta) F_{(1)}^2 - \frac{1}{4} e^{2\sigma}(e^{-\phi} + \Delta) F_{(2)}^2 
\left. \right. 
- \frac{1}{4} e^{-2\gamma}(e^{-\phi} + \Delta) F_{(3)}^2 - \frac{1}{4} e^{-2\sigma}(e^{-\phi} + \Delta) F_{(4)}^2 \right). \] (9)

Here the loop correction \( \Delta \) is a function of \( G_{11} = f k \) and \( G_{22} = FK \) and in the general case has a rather complicated form.

Let us consider purely magnetic \((F^{-1} = K = 1)\) extremal black holes \((f^{-1} = k)\). In this case in [8] \( \sigma = \gamma = 0 \) and backgrounds of the chiral null model are expressed by a single function (we consider one-center solution) \( f_0 \):
\[ f_0(r) = 1 + \frac{P}{r}, \]
\[ \Lambda = f = k^{-1} = f_0, \]
\[ a_\varphi = b_\varphi = P(1 - \cos \vartheta), \]
\[ \phi = \ln f_0 \quad \gamma = 0, \]

where \( a_\varphi \) and \( b_\varphi \) are the nonzero components of potentials in spherical coordinates. The magnetic field strengths are \( F_{(1)ij} = F_{(3)ij} = -\varepsilon_{ijk} \partial^k f_0 \), and in spherical coordinates have a single nonzero component \( F_{\vartheta\varphi} = -P \sin \vartheta \). For the magnetic extremal solution, the components of the internal metric \( G_{11} \) and \( G_{22} \) are constants, and hence perturbatively \( \Delta \) is also a constant which we assume to be numerically small. In this case the effective action (10) takes the form

\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial_\mu \phi)^2 - (\partial_\mu \gamma)^2 \right. \]
\[ - \frac{1}{4} e^{2\gamma} (e^{-\phi} + \Delta) F_{(1)}^2 - \frac{1}{4} e^{-2\gamma} (e^{-\phi} + \Delta) F_{(3)}^2 \bigg). \]

**III. SOLUTION OF THE EQUATIONS OF MOTION**

We look for a static spherically-symmetric solution of the field equations of the loop-corrected action (11). The general ansatz for the metric is

\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + e^\mu d\Omega^2. \]

The field strength \( F_{ij} \) is assumed to have the same functional form as in the tree-level case: \( F_{ij} = -\varepsilon_{ijk} \partial^k f(r) \), where the function \( f(r) \) is to be determined from the field equations. In spherical coordinates this ansatz yields the only nonzero component

\[ F_{\vartheta\varphi} = f'(r) r^2 \sin \vartheta. \]

The equations of motion resulting from the action (11) are

\[ D^2 \phi + \frac{1}{4} e^{-\phi} (e^{2\gamma} F_{(1)}^2 + e^{-2\gamma} F_{(3)}^2) = 0, \]

\[ \partial_\mu \left( \sqrt{-g} (e^{-\phi} + \Delta) e^{\pm 2\gamma} g^{\mu \nu} g^{\nu \nu'} F_{(1,3)\mu \nu'} \right) = 0, \]

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \frac{1}{2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu \nu} (\partial \phi)^2 \right) - \left( \partial_\mu \gamma \partial_\nu \gamma - \frac{1}{2} g_{\mu \nu} (\partial \gamma)^2 \right) \]
\[ - \frac{1}{4} (e^{-\phi} + \Delta) \left[ e^{2\gamma} \left( 2(F_{(1)}^2)_{\mu \nu} - \frac{1}{2} g_{\mu \nu} F_{(1)}^2 \right) + e^{-2\gamma} \left( 2(F_{(3)}^2)_{\mu \nu} - \frac{1}{2} g_{\mu \nu} F_{(3)}^2 \right) \right] = 0. \]

\(^1\)One-loop corrections are calculated with tree-level expressions.
and
\[ D^2 \gamma - \frac{1}{4} (e^{-\phi} + \Delta)(e^{2\gamma} F^2_{(1)} - e^{-2\gamma} F^2_{(3)}) = 0. \] (17)

At the tree level $\gamma = 0$ (see (10)) and $F^2_{(1)} = F^2_{(3)}$. In the first order in $\Delta$, we have $\gamma = \Delta \gamma_1$. Expanding all the expressions in (17) to the first order in $\Delta$, we obtain
\[ D^2 \gamma_1 - \frac{1}{2} \left[ e^{-\phi} (F^2_{(1)} + F^2_{(3)}) \right] (0) \gamma_1 = 0. \]

Here the subscript $(0)$ stands for the tree-level expressions. From this equation it follows that $\gamma_1 = 0$ is a solution. Multiplying this equation by $\gamma_1$ and integrating over the space-time, we obtain
\[ -\int d^4x \sqrt{g} (D \gamma_1)^2 = \int d^4x \sqrt{g} \left( e^{-\phi} F^2_{(1)} \right) (0) \gamma_1^2. \]

Since $F^2_{(1,3)} > 0$ (see below), this equation can be satisfied only if $\gamma_1 = 0$.

Thus, in the first order in $\Delta$, we have $\gamma = 0$ also, and henceforth we set $\gamma = 0$.

The field strengths have the following nonzero components
\[ (F^2)_{\phi\phi} = (f'(r)r^2 \sin \vartheta)^2 e^{-\mu}, \quad (F^2)_{\phi\vartheta} = (f'(r)r^2)^2 e^{-\mu}, \]
\[ F^2 = 2(f'(r)r^2)^2 e^{-2\mu}. \] (18)

Eq. (14) takes the form
\[ \phi'' - \left( \frac{\lambda'}{2} - \frac{\nu'}{2} - \mu' \right) \phi' + \frac{e^{\lambda-\phi}}{2} F^2 = 0. \] (19)

Away from the origin, Eq. (15) reduces to
\[ \partial_\vartheta \left( \sqrt{-g} (e^{-\phi} + \Delta) g^{\vartheta\vartheta} g^{\phi\phi} F_{\vartheta\phi} \right) = 0. \] (20)

Since $\sqrt{-g} \sim \sin \vartheta$, $g^{\vartheta\vartheta} \sim \sin^{-2} \vartheta$ and $F_{\vartheta\phi} \sim \sin \vartheta$, Eq. (20) is satisfied identically. To account for a singularity at the origin (point-like source of magnetic charge), we calculate the flux of magnetic field through an arbitrary sphere enclosing the origin
\[ \int_{S^2} F = \int d\varphi d\vartheta \sin \vartheta F_{\vartheta\phi} = \text{Const}. \]

The Einstein equations (16) (with one index lifted) take the form (13):
\[ e^{-\lambda} \left( \mu'' + \frac{3}{4} \mu'^2 - \frac{\mu' \lambda'}{2} \right) - e^{-\mu} + \frac{1}{4} e^{-\lambda} \phi'^2 + \frac{1}{4} (e^{-\phi} + \Delta) F^2 = 0, \] (21)
\[ e^{-\lambda} \left( \frac{\mu'^2}{2} + \mu' \nu' \right) - 2e^{-\mu} - \frac{1}{2} e^{-\lambda} \phi'^2 + \frac{1}{2} (e^{-\phi} + \Delta) F^2 = 0, \] (22)
\[ e^{-\lambda} (2\mu'' + 2\nu'' + \mu'^2 + \nu'^2 - \mu' \lambda' - \nu' \lambda' + \mu' \nu') + e^{-\lambda} \phi'^2 - (e^{-\phi} + \Delta) F^2 = 0. \] (23)
Our next aim is to solve the equations of motion to the first order in $\Delta$. In the leading order ($\Delta = 0$) we have

\begin{align*}
\nu^{(0)} &= -\ln f_0, \\
\lambda^{(0)} &= \ln f_0, \\
\mu^{(0)} &= \ln f_0 + 2\ln r, \\
\phi^{(0)} &= \ln f_0, \\
F_{\varphi\vartheta}^{(0)} &= 2q'^2, \\
q' &= \frac{f''}{f_0},
\end{align*}

in the first order in $\Delta$ we look for a solution in the form

\begin{align*}
\nu &= -\ln f_0 + \Delta n, \\
\lambda &= \ln f_0 + \Delta l, \\
\mu &= \ln f_0 + 2\ln r + \Delta m, \\
\phi &= \ln f_0 + \Delta \varphi, \\
F_{\varphi\vartheta} &= P(1 + \Delta \tau) \sin \vartheta.
\end{align*}

Here $n, m, l$ and $\phi$ are unknown functions, $p$ is a number. One also has

\begin{align*}
F^2 &= 2q'^2(1 + \Delta \tau),
\end{align*}

where $\tau = 2p - 2m$. In Appendix A we present the detailed solution of the equations of motion assuming that $m = l = -n$ \footnote{There are four equations \eqref{eq:21} and \eqref{eq:22}-\eqref{eq:23} for four unknown functions $m$, $n$, $l$ and $\phi$. Our choice corresponds to the requirement that in the first order in $\Delta$, as in the leading order, $\mu + \nu = 2\ln r$.} Although there are more equations than unknown functions, it appears that the ansatz is consistent. We obtain the following solution:

\begin{align*}
n &= -m = -l = A_{-1} \frac{1}{f_0} + A_0 + A_1 f_0 + \frac{1}{2} f_0 \ln f_0 \quad (26)
\end{align*}

and

\begin{align*}
\varphi &= -A_{-1} \frac{1}{f_0} + A_0 + 2p + (A_1 + \frac{1}{2}) f_0 + \frac{1}{2} f_0 \ln f_0. \quad (27)
\end{align*}

Here $A_i$ and $p$ are arbitrary constants. The constants in the above solution are partially constrained by imposing the conditions that at large $r$ the metric is asymptotic to the Minkowski metric and that the asymptotic value of the dilaton is unity:

\begin{align*}
A_{-1} + A_1 + A_0 &= 0, \\
-A_{-1} + A_1 + A_0 + \frac{1}{2} + 2p &= 0. \quad (28)
\end{align*}

From the relations \eqref{eq:28} it follows that

\begin{align*}
A_{-1} &= p + \frac{1}{4}, \quad (29)
\end{align*}
IV. HORIZON AND MATCHING OF SOLUTIONS AT SMALL AND LARGE DISTANCES

Let us discuss our solution. The expressions (26)-(27) were obtained by making two expansions: (i) in $\Delta$, assuming that corrections are smaller than the leading-order expressions: $|\ln f_0| > |\Delta n|$, $|\ln f_0| > |\Delta l|$, etc., and (ii) by expanding the exponents $e^{\Delta n}$, $e^{\Delta l}$, etc. to the first order in $\Delta$ assuming that $1 > |\Delta n|$, $|\Delta l|$, ... The bounds that define the range of validity of our solution result both in constraints on the free parameters $A_i$ and on the domain of variation of $r$.

Let us consider the $g_{00}$ component of the metric. Introducing new variable

$$x = f_0(r)^{-1} = \frac{r}{r + P},$$

we rewrite the expression for the metric component $g_{00}$ as

$$e^{\nu} = x \left[ 1 + \Delta \left( A_{-1} (x - 1) + A_1 \left( \frac{1}{x} - 1 \right) + \frac{1}{2x} \ln \frac{1}{x} \right) \right].$$

(30)

Here we used relations (28) to eliminate $A_0$.

The range of validity of the expression (30) is determined by the inequalities

$$1 > \left| \Delta \left( A_{-1} (x - 1) + A_1 \left( \frac{1}{x} - 1 \right) + \frac{1}{2x} \ln \frac{1}{x} \right) \right|$$

(31)

and

$$\ln \frac{1}{x} > \left| \Delta \left( A_{-1} (x - 1) + A_1 \left( \frac{1}{x} - 1 \right) + \frac{1}{2x} \ln \frac{1}{x} \right) \right|.$$  

(32)

Sufficient conditions for validity of the the inequalities (31) and (32) are their validity for each separate term in (31) and (32). Solving the inequalities, we obtain

$$1 > |\Delta A_{-1}|, \quad 1 \gg |\Delta A_1|$$

$$x > |\Delta A_1|, \quad x > |\Delta| \ln \frac{1}{|\Delta|}.$$  

(33)

Note that we assume that $|\Delta| \ll 1$.

Let us extrapolate the expression (30) to the region $x \sim |\Delta| \ln \frac{1}{|\Delta|}$ and look for a zero (a would-be horizon) of the function $g_{00}$ at small $x$. For small $x$ we obtain the equation

\[ \text{It is remarkable that solution (26)-(27) is obtained in an analytic form for all } r \text{ for which we can use perturbation expansion in } \Delta. \text{ Usually it is possible to solve such problem only for small and large } r \text{ and to sew the asymptotic solutions.} \]

\[ \text{There is another range } 1 > A_1 > (e - 1)^{-1} \text{ in which there exists a solution of the inequalities. However, for } A_1 \text{ in this interval, the range of small } x \text{ which we are interested in is excluded.} \]
\[ \frac{1}{\Delta} + A_{-1} = -\left( \frac{A_1}{x} + \frac{1}{2x} \ln \frac{1}{x} \right). \]  

(34) 

This equation has a solution (zero of the time component of the metric \( g_{00} \)) provided \( \Delta < 0 \). Introducing new variable \( y \) by the relation \( x = e^{A_1 - y} \), we transform Eq. (34) to the form \( ye^{y} = A \), where \( A = 2 \left( \frac{1}{|\Delta|} - A_{-1} \right) e^{2A_1} \). For \( A \gg 1 \) the asymptotic expansion of solution of this equation is [14]: 

\[ y = \ln A - \ln \ln A + \frac{\ln \ln A}{\ln A} + O \left( \frac{\ln \ln A}{\ln A} \right)^2. \]

An approximate solution to Eq. (34) is 

\[ x_0 \approx \frac{A_1 + \ln \left( \frac{1}{|\Delta|} - A_{-1} \right)}{\frac{1}{|\Delta|} - A_{-1}} = O \left( |\Delta| \ln \frac{1}{|\Delta|} \right). \]  

(35) 

Let us consider the dilaton. In Eqs. (14)-(16) the parameter \( \Delta \) enters through the combination \( \Delta + e^{-\phi} \), where 

\[ e^{-\phi} = x \left[ 1 - \Delta \left( A_{-1}(1-x) + (A_1 + \frac{1}{2})(\frac{1}{x} - 1) + \frac{1}{2x} \ln \frac{1}{x} \right) \right]. \]  

(36) 

Near the horizon, at \( r_0 \ll r \), the function \( e^{-\phi} \) is of order \( |\Delta| \ln \frac{1}{|\Delta|} \). In this region, \( \Delta + e^{-\phi} \) and \( e^{-\phi} \) are of the same order. At large \( r \), \( e^{-\phi} \) approaches unity and \( e^{-\phi} \gg \Delta \). This suggests that we can look for a solution of the field equations in the whole range \( r_0 < r < \infty \), neglecting \( \Delta \) as compared to \( e^{-\phi} \), which has a simpler form than the loop-corrected solution and at the same time is sufficiently close to the loop-corrected solution, in particular, has the horizon shifted from the origin to \( r > 0 \).

Let us consider a non-extremal solution of the field equations (14)-(16) at \( \Delta = 0 \). The metric is [13] [4]: 

\[ ds^2 = - \left( 1 + \frac{k \sinh^2 \mu}{r} \right)^{-1} \left( 1 - \frac{k}{r} \right) dt^2 + \left( 1 + \frac{k \sinh^2 \mu}{r} \right) \left[ \left( 1 - \frac{k}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \right], \]

(37) 

the dilaton and magnetic field strength are  

5It is important that we considered only smooth decreasing functions \( g_{00} \) and \( e^\phi \). The component of the metric \( g_{rr} = e^\lambda = e^{-\nu} \) in the near-horizon region is a large rapidly varying function.  

6In ref. [15], as well as in related papers cited therein, the action was obtained by dimensional reduction from higher dimensions. The 4D vector fields and scalars appear as metric components with a 4D and internal indices and with both internal indices respectively. In the case at hand, the action (11) contains two scalar fields \( \phi \) and \( \gamma \) and two vector fields. Although one vector field comes from the metric and the other from antisymmetric tensor, it can be verified explicitly that the action (11) meets all the requirements of [15].
\[ e^\phi = \left(1 + \frac{k \sinh^2 \mu}{r}\right); \quad F_{(1)} = F_{(3)} = \frac{k \sinh 2\mu}{2r^2} \varepsilon_2, \]  

(38)

where \(\varepsilon_2\) is the volume of the unit 2-sphere, \(k\) is the position of horizon. The extremal limit is \(k \to 0\) and \(\mu \to \infty\) with \(ke^{2\mu}\) fixed. The \(g_{00}\) component of the metric interpolates smoothly between unity at \(r = \infty\) and zero at the horizon. Next, we explore a possibility to approximate the loop-corrected solution by a more simple tree-level non-extremal magnetic black hole.

To find a non-extremal field configuration which is sufficiently close to the loop-corrected one, it is natural to demand that (i) both solutions have equal magnetic charges, (ii) both solutions have identical asymptotic behavior, i.e. at large \(r\) the leading \(O(\frac{1}{r})\) terms of the metrics and dilatons of both solutions are equal, and (iii) locations of the horizons of both solutions are of the same order. Let us discuss these requirements successively.

Equating the magnetic charges of both solutions, we have

\[
\frac{k}{2} \sinh 2\mu = P(1 - |\Delta|p). \tag{39}
\]

The asymptotics of the component \(g_{00}\) of the loop-corrected metric (other components differ from this expression by the sign of the \(O(\frac{1}{r})\) term) is

\[
g^{(1)}_{00} = 1 - \frac{P}{r} \left[1 + |\Delta| (A_1 - p + \frac{1}{4}) \right] + O\left(\frac{1}{r^2}\right),
\]

where we used the relations (28) and (29). The asymptotics of the corresponding component of the metric of non-extremal solution is

\[
g^{(2)}_{00} = 1 - \frac{P}{r} \left[1 + b + \frac{k}{p}\right] + O\left(\frac{1}{r^2}\right),
\]

where

\[ b = \frac{k \sinh^2 \mu}{P} - 1. \]

Assuming that the non-extremal solution is close to extremality, from Eq. (39) we obtain

\[
b \simeq -\frac{k}{2P} - |\Delta|p. \tag{40}
\]

Equating the asymptotics of both metrics and substituting (40), we have

\[
A_1 = \frac{k}{2P|\Delta|} - \frac{1}{4}. \tag{41}
\]

Asymptotics of the functions \(e^{-\phi}\) are

\[
e^{-\phi^{(1)}} = 1 - \frac{P}{r} \left[1 - |\Delta| (A_1 + p + \frac{5}{4}) \right] + O\left(\frac{1}{r^2}\right)
\]

and

\[
e^{-\phi^{(2)}} = 1 - \frac{P}{r} (1 + b) + O\left(\frac{1}{r^2}\right).
\]

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Using (40), we have
\[ e^{-\phi(1)} = 1 - \frac{P}{r}[1 - |\Delta|\left(\frac{k}{2P|\Delta|} + p + 1\right)] \]
and
\[ e^{-\phi(2)} = 1 - \frac{P}{r}[1 - |\Delta|\left(\frac{k}{2P|\Delta|} + p\right)]. \]

We see that although it is impossible to obtain strict asymptotic equality \( e^{-\phi(1)} = e^{-\phi(2)} \),
these expressions are very close if \( \left(\frac{k}{2P|\Delta|} + p\right) \gg 1 \).

Further specification requires additional assumptions. Assuming that locations of horizons of both solutions are of the same order, i.e. \( k \sim P|\Delta|\ln \frac{1}{|\Delta|} \), we have
\[ A_1 \sim \ln \frac{1}{|\Delta|}, \tag{42} \]
and
\[ \sinh^2 \mu \sim \frac{1}{|\Delta|} \ln \frac{1}{|\Delta|}. \]

For small \( |\Delta| \ll 1 \) we have \( \sinh^2 \mu \gg 1 \), i.e. the non-extremal configuration is close to extremality. Taking the free parameter \( p = O(1) \) and noting that \( \ln \frac{1}{|\Delta|} \gg 1 \), we find that both asymptotics of the metrics and dilatons are very close to each other.

Assuming validity of the strict equality, \( k = r_0 \), and taking \( p \sim 1 \), from the equation for the horizon (35) we obtain
\[ k \approx P\frac{A_1 + \frac{1}{|\Delta|}}{\frac{1}{|\Delta|}}, \]
where we neglected the constants of order \( O(1) \) as compared to \( \frac{1}{|\Delta|} \). Substituting for \( A_1 \) the expression (41), we have \( k = 2P|\Delta|\ln \frac{1}{|\Delta|} \).

V. THERMODYNAMIC PROPERTIES

Let us discuss some thermodynamic properties of the loop-corrected configuration. The loop-corrected ADM mass calculated from the asymptotics of the the metric is
\[ M = P \left[ 1 + |\Delta|\left(-A_{-1} + A_1 + \frac{1}{2}\right) \right] \tag{43} \]

To calculate the Hawking temperature we consider the near-horizon region where we take \( r = r_0 + \rho^2 \) and, assuming that \( r_0 \gg \rho^2 \), expand the metric in powers of \( \rho^2 \). The Hawking temperature is determined from the requirement that the resulting metric has no conical singularity. We obtain
\[ T_{bh} = \frac{1}{4\pi \left(1 + \frac{|\Delta|}{r_0}\right)}. \tag{44} \]

The geometric entropy defined as one fourth of the horizon area is

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\[ S_{bh} = \pi x_0^2 \left( 1 + \frac{|\Delta|}{r_0} \right). \] (45)

Assuming the validity of the second law of thermodynamics, i.e. that the thermodynamic relation \( \delta E_{bh} = T\delta S \) is approximately satisfied with \( \delta S = S_{bh} \) and \( T = T_{bh} \), we obtain \( \delta E_{bh} \sim r_0 \sim P|\Delta|\ln \frac{1}{|\Delta|} \). On the other hand, taking

\[ \delta E_{bh} \sim \delta M = P|\Delta|(-A_{-1} + A_1 + \frac{1}{2}), \] (46)

we obtain that

\[ (-A_{-1} + A_1) \sim \ln \frac{1}{|\Delta|} \]

which is consistent with the estimate (41).

Let us discuss how the thermodynamic properties of the loop-corrected solution match those of the tree-level one. The energy, charge and geometric entropy of the non-extremal magnetic black hole (37)-(38) are

\[ E \sim k(3 + \cosh 2\mu), \]
\[ Q \sim k \sinh 2\mu, \]
\[ S_{bh} = \pi k^2 \cosh^2 \mu. \] (47)

Assuming that the non-extremal solution is close to extremality, i.e. \( \mu \gg 1 \), the excess of energy above the extremal limit is

\[ \delta E \sim k. \] (48)

Supposing that the non-extremal solution is close to the loop-corrected configuration in the sense discussed in the preceding section, and substituting in the expression (44) for \( \delta M \) the expressions for \( A_1 \) and \( A_{-1} \) from (40), we obtain again that \( \delta M \sim k \sim P|\Delta|\ln \frac{1}{|\Delta|} \). Finally, we can note that the Hawking temperature \( T_{bh} = \frac{1}{4\pi k \cosh \mu} \) and the geometric entropy \( S_{bh} = \pi k^2 \cosh^2 \mu \) of the non-extremal solution are close to the corresponding loop-corrected expressions.

The entropy of an extremal dyonic black hole with nonzero electric and magnetic charges is equal to \( S_{bh} = \frac{\pi}{\sqrt{Q_1 Q_2 P_1 P_2}} \). In the non-extremal case the same formula for the geometric entropy is valid with the charges \( P_i = k \sinh \gamma_i \cosh \gamma_i \) and \( Q_i = k \sinh \delta_i \cosh \delta_i \), \( i = 1, 2 \), where \( \gamma_i \) and \( \delta_i \) are boost parameters. [14] Extremal dyonic black hole is obtained in the limit \( k \to 0, \gamma_i \to \infty, \delta_i \to \infty \), with \( P_i \) and \( Q_i \) held fixed. The solution (37)-(38) can be considered as a non-extremal counterpart of the extremal magnetic solution of the chiral null model with two magnetic charges. [11] The entropy of the non-extremal solution can be obtained by the same substitution of charges as discussed above. Setting in the expression

\[ ^7 \text{Note that at the string-tree level, at } \Delta = 0, \text{ the entropy of magnetic black hole vanishes.} \]
for the entropy of the extremal dyonic black hole $Q_1 = Q_2 = k$ and $P_1 = P_2 = k \cosh^2 \mu$ we obtain for the entropy of the non-extremal magnetic black hole the following expression:

$$S_{bh} = \pi k^2 \cosh^2 \mu,$$

which is the same as above (47).

Statistical entropy of the non-extremal 4D dyonic black hole can be calculated using the D-brane technique in the near-extremal limit where all the charges are large [17]. In the case at hand, for non-extremal magnetic black hole, we cannot repeat this calculation literally. If, however, in the near-extremal limit, $k \to 0, \gamma \to \infty$ magnetic black hole can be substituted with dyonic black hole with the electric charges $Q_1 = Q_2 = k$ and the D-brane counting could be extended to this case, it would provide statistical origin of the entropy of the non-extremal magnetic black hole. Anyway, the above remarks can be of relevance in other approaches to counting of the statistical entropy [18].

VI. DISCUSSION

In this paper we have discussed a solution to the equations of motion of the string-loop-corrected effective action in the simplest setting: when the moduli fields and therefore the string-loop correction to the gauge couplings are constant. As the tree-level solution we chose the magnetic black hole given by the chiral null model.

The main result of our study is that the horizon of the loop-corrected solution is moved away from the origin and that the loop-corrected solution can be approximated by a non-extremal black hole. It was found that the requirements that both the loop-corrected and the non-extremal solutions have the same large-$r$ asymptotics and positions of their horizons are of the same order can be fulfilled if the free parameters of both solutions are connected in a special way.

The string-tree-level chiral null model provides a solution to the low energy effective action which in a special renormalization scheme receives no $\alpha'$ corrections [8]. The solution of the one-loop-corrected effective action we considered (9) is no longer expressed in terms of harmonic functions and the $\alpha'$-corrections are present. However, because now the horizon is shifted away from the origin, it is possible that the $\alpha'$ corrections are small and can be treated perturbatively.

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APPENDIX A: SOLUTION OF FIELD EQUATIONS

In the first order in parameter $\Delta$ the Einstein equations (21)-(23) are

$$m'' + m'(q' + \frac{3}{r}) - l'(\frac{1}{2} q' + \frac{1}{r}) + \varphi'\frac{q'}{2} - \frac{l - m}{r^2} + \frac{1}{2} q'^2 s = -\frac{1}{2} f_0 q'^2,$$

\[(A1)\]
\[ m'\frac{2}{r} + n'(q' + \frac{2}{r}) - \varphi'q' - 2\frac{l - m}{r^2} + q'^2s = -f_0q'^2, \quad (A2) \]

\[ m'' + n'' + m'\frac{2}{r} - l'\frac{1}{r} + n'(-q' + \frac{1}{r}) + \varphi'q' - q'^2s = f_0q'^2. \quad (A3) \]

Here

\[ s = l - \varphi + \tau = l - \varphi - 2m + 2p \quad (A4) \]

We also need the equation for the dilaton \((\varphi')\) in the \(O(\Delta)\) order:

\[ \varphi'' + 2r\varphi' + \frac{1}{2}(2m' + n' - l')q' + q'^2s = 0. \quad (A5) \]

We look for a solution such that \(m = l = -n\), because in this case, as at the tree level, the components of the metric satisfy the relation \(g_{tt} = g^{-1}_{rr}\). Substituting this ansatz in equations \((A1)-(A5)\), we have

\[ m'' + m'\left(\frac{1}{2}q' + \frac{2}{r}\right) + \varphi'q' + \frac{1}{2}q'^2s = -\frac{1}{2}f_0q'^2, \quad (A6) \]

\[ m'q' + \varphi'q' - q'^2s = f_0q'^2, \quad (A7) \]

and

\[ \varphi'' + 2r\varphi' + q'^2s = 0, \quad (A8) \]

where

\[ s = 2p - m - \varphi \]

Eqs. \((A2)\) and \((A3)\) reduce to the same Eq. \((A7)\). Combining the equations as \((A6)-(A7)+(A2)\), substituting the expression for \(s\), and introducing new variable \(u = m + \varphi\), we obtain

\[ u'' + u'(q' + \frac{2}{r}) - q'^2(u - 2p) = 0 \quad (A9) \]

\[ u' + q'(u - 2p) = f_0q' \quad (A10) \]

and

\[ \varphi'' + \frac{2}{r}\varphi' - q'^2(u - 2p) = 0 \quad (A11) \]

Eqs. \((A9)\) and \((A10)\) are not independent: the first one can be obtained from the second one as

\[ (A10)' + \frac{2}{r}(A10) = (A9) \]
Thus, finally we obtain the following system

$$u' + q'u = (f_0 + 2p)q'$$  \hspace{1cm} (A12)

and

$$\varphi'' + \frac{2}{r}\varphi' + (2p - u)q'^2 = 0$$  \hspace{1cm} (A13)

Eq. \hspace{0.1cm} (A12) can be integrated yielding

$$u = \frac{C'}{f_0} + \frac{f_0}{2} + 2p,$$

where \(C\) is an arbitrary constant. Using this expression we integrate (A13):

$$\varphi = C_2 + C_1 f_0 + \frac{C}{2f_0} + \frac{1}{2} f_0 \ln f_0$$  \hspace{1cm} (A14)

Finally, for \(m = -\varphi + u\) we obtain

$$m = -C_2 + 2p + (C_1 - \frac{1}{2})f_0 + \frac{C}{2f_0} - \frac{1}{2} f_0 \ln f_0.$$  \hspace{1cm} (A15)

The relation between the constants used here and those in the main test is: \(A_0 = C_2 - 2p;\)
\(A_1 = C_1 - \frac{1}{2}\); and \(A_{-1} = -C/2\).
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