NOETHERIAN RINGS OF LOW GLOBAL DIMENSION AND SYZYGETIC PRIME IDEALS

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Abstract. Let $R$ be a Noetherian ring. We prove that $R$ has global dimension at most two if, and only if, every prime ideal of $R$ is of linear type. Similarly, we show that $R$ has global dimension at most three if, and only if, every prime ideal of $R$ is syzygetic. As a consequence, one derives a characterization of these rings using the André-Quillen homology.

Let $R$ be a commutative ring. An ideal $I$ of $R$ is said to be of linear type if the graded surjective morphism $\alpha : S(I) \rightarrow R(I)$, from the symmetric algebra of $I$ to the Rees algebra of $I$, is an isomorphism; $I$ is said to be syzygetic if the second component $\alpha_2 : S_2(I) \rightarrow I^2$ is an isomorphism. It is known that $R$ has weak dimension at most one if, and only if, every ideal of $R$ is of linear type, and equivalently if, and only if, every ideal of $R$ is syzygetic. In particular, rings of weak dimension at most one are characterized in terms of the André-Quillen homology (see [12]).

Recall that the weak dimension of a ring $R$, denoted by $\text{w.dim}(R)$, is the supremum of the flat dimensions of all $R$-modules and that the global dimension of $R$, denoted by $\text{gl.dim}(R)$, is the supremum of the projective dimensions of all $R$-modules. Clearly $\text{w.dim}(R) \leq \text{gl.dim}(R)$, and when $R$ is Noetherian, they agree. Since $\text{gl.dim}(R) = \sup\{\text{gl.dim}(R_m) \mid m \in \text{Max}(R)\}$, then, for a Noetherian ring $R$, $\text{gl.dim}(R) \leq N$ is equivalent to $R_m$ being regular local with Krull dim $(R_m) \leq N$, for every maximal ideal $m$ of $R$ (see, e.g., [8]).

The purpose of this note is to extend these characterizations of rings of $\text{w.dim}(R) \leq 1$ to rings of global dimension at most two and three, but now in the Noetherian context. This is done in quite similar terms. Concretely, we prove the theorem below. Item (A), shown in general in [12], is included here just for the sake of completeness.

Theorem. Let $R$ be a Noetherian ring.

(A): $\text{gl.dim}(R) \leq 1 \iff$ every ideal of $R$ is of linear type $\iff$ every ideal of $R$ is syzygetic.

(B): $\text{gl.dim}(R) \leq 2 \iff$ every prime ideal of $R$ is of linear type.

(C): $\text{gl.dim}(R) \leq 3 \iff$ every prime ideal of $R$ is syzygetic.

Since the linear type and syzygetic conditions are clearly local, to prove this theorem, one can suppose that $(R, m, k)$ is a Noetherian local ring with maximal ideal $m$ and residue field $k$. Moreover, one can substitute the condition $\text{gl.dim}(R) \leq N$ by the condition “$R$ is regular local of Krull dim $(R) \leq N$”. If Krull dim $(R) \leq 1$, every nonzero proper ideal is generated by a nonzero divisor, hence of linear type and syzygetic (see, e.g., [6, Corollary 3.7]). Suppose that Krull dim $(R) \leq 2$ or 3. Since $R$ is regular local, then $m$ is generated by an $R$-regular sequence, so $m$ is of linear type (see, e.g., [6, Corollary 3.8]); moreover $R$ is a UFD, thus every height one prime ideal is principal (generated by a nonzero divisor), and so again of linear type; if Krull dim $(R) \leq 3$, then every height two prime ideal is perfect and generically a complete intersection, hence syzygetic (see, e.g., [6, Remark page 91]). This shows the “only if” implications.

Observe that the proof of [6, Corollary 3.8] shows that a Noetherian local ring with syzygetic maximal ideal is regular. Therefore, in order to prove the “if” implication in Theorem (A), it is
enough to display, in a two dimensional regular local ring, a non syzygetic ideal. Similarly, to prove prove the “if” implication in Theorem (B), it suffices to exhibit, in a three dimensional regular local ring, a height two prime ideal which is not of linear type. Finally, to prove the “if” implication in Theorem (C), we exhibit, in a four dimensional regular local ring, a height three prime ideal which is not syzygetic.

In this direction, we show the next result, a kind of rephrasing of [12] Lemma 3 but under the regular local hypothesis, and hence easier to prove it. We give a different alternative proof using [13] Corollary 4.11.

**Lemma 1.** Let \((R, m, k)\) be a regular local ring of Krull dimension 2. Let \(x, y\) be a regular system of parameters. Then \(m^2\) is a non syzygetic ideal.

**Proof.** Let \(I = m^2 = (x^2, xy, y^2)\) and \(J = (x^2, y^2)\). Using that \(x, y\) is an \(R\)-regular sequence, it is easy to check that \(xy \notin J\) (see [3] Theorem 9.2.2), to relate it to the simplest case of the Monomial Conjecture). Since \((xy)^2 \in JJ\), then \(J : xy \subseteq JJ : (xy)^2\). By [13] Corollary 4.11, we conclude that \(I\) is not syzygetic.

Next lemma displays, in a three dimensional regular local ring, a height two prime ideal which is not of linear type, thus generalizing [5] Corollary 2.7). There, in the context of the Shimoda Conjecture, one exhibits, in a three dimensional regular local ring, a non-complete intersection height two prime ideal. (Recall that complete intersection implies linear type.)

**Lemma 2.** Let \((R, m, k)\) be a regular local ring of Krull dimension 3. Let \(x, y, z\) be a regular system of parameters. Let \(I\) be the ideal of \(R\) generated by

\[
\begin{align*}
  f_1 &= y^3 - x^4 ,
  f_2 &= xyz - z^3 + x^4 - xy^3 ,
  f_3 &= x^2 y + y^2 z - x z^2 - x^3 y 
  & \text{and } f_4 = xy^2 - yz^2 - x^2 y^2 + x^3 z .
\end{align*}
\]

Then \(I\) is a height two prime ideal minimally generated by four elements. In particular, \(I\) is not of linear type.

Finally, the third lemma exhibits a non syzygetic height three prime ideal in a four dimensional regular local ring.

**Lemma 3.** Let \((R, m, k)\) be a regular local ring of dimension 4. Let \(x, y, z, t\) be a regular system of parameters. Let \(I\) be the ideal of \(R\) generated by

\[
\begin{align*}
  f_1 &= yz - xt ,
  f_2 &= z^3 - x^5 ,
  f_3 &= z^2 t - x^4 y ,
  f_4 &= z t^2 - x^3 y^2 ,
  f_5 &= z^3 t - x^2 y^3 ,
  f_6 &= y^4 - x^5 ,
  f_7 &= y^3 t - x^4 z ,
  f_8 &= y^2 t^2 - x^3 z^2 .
\end{align*}
\]

Then \(I\) is a height three prime ideal which is not a syzygetic ideal.

The general skeleton of the proofs of Lemmas 2 and 3 are similar to that of the proof of [5] Proposition 2.6]. Namely, once the candidate \(I\) is chosen, we show that \(I\) is perfect with the desired height, in particular, height unmixed. Then we pick an associated prime \(p\) to \(I\), which will be of the same height, and, by means of multiplicity theory, we show that \(xR + I\) and \(xR + p\) have the same colength, concluding, by Nakayama’s Lemma, that \(I\) and \(p\) are equal.

The ideal \(I\) displayed in Lemma 2 is a small variation of [7] Example 3.7]. Concretely, Huneke considers the height two prime ideal defined by the kernel of the homomorphism from the power series ring \(\mathbb{C}[X,Y,Z]\) to \(\mathbb{C}[t]\), sending \(X,Y,Z\) to \(t^6, t^7 + t^{10}, t^8\), respectively. He shows that this ideal is generated by the \(3 \times 3\) minors of a specified \(4 \times 3\) matrix \(L\), whose entries are either 0, or else among one of the monomial terms \(X, Y, Z, X^2, XY\) times a \(\pm 1, \pm 2\) integer coefficient. Our example consists in taking these \(3 \times 3\) minors, but substituting the variables \(X, Y, Z\) by the regular parameters \(x, y, z\) and, in order to avoid characteristic two problems, replacing in \(L\), \(\pm 2\) by \(\pm 1\) (surprisingly enough, it works).
As for the ideal $I$ considered in Lemma 3, we recover a particular case of a family of prime ideals with unbounded number of generators provided by Bresinsky in [2]. Concretely, we consider the kernel of the homomorphism from $\mathbb{K}[X,Y,Z,T]$, $\mathbb{K}$ any field, to $\mathbb{K}[t]$, sending $X,Y,Z,T$ to $t^{12}, t^{15}, t^{20}, t^{23}$ and then, as before, just substitute the variables by the regular parameters.

Before proceeding to prove Lemmas 2 and 3 we highlight the good behaviour of the syzygetic and linear type conditions through faithfully flat morphisms of rings. Indeed, this follows from [11 Corollaire 2.3] (see also [13 Theorem 2.4 and Example 2.3]), where one shows that these conditions are characterized in terms of the exactness of a complex of $R$-modules and noting that, if $R \to S$ is a faithfully flat morphism of rings, then $I \otimes_R S = IS$.

**Proof of Lemma 2** Since $(R, m)$ is a three dimensional regular local ring with maximal ideal $m$ generated by $x, y, z$, then its completion $(\hat{R}, \hat{m})$ is a three dimensional regular local ring with maximal ideal $\hat{m} = m\hat{R} = (x, y, z)\hat{R}$ generated by the regular system of parameters $x, y, z$. Let $I = (f_1, f_2, f_3, f_4)$ and $\hat{I} = I\hat{R} = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4)\hat{R}$. If we prove that $\hat{I}$ is prime and not of linear type, then $I = I\hat{R} \cap R$ is prime and not of linear type, because the completion morphism is faithfully flat (see, e.g., [10 § 8]). Therefore we can suppose that $R$ is complete.

First observe that $f_1, f_2, f_3, f_4$ are, up to a change of sign, the $3 \times 3$ minors of the matrix

$$\varphi_2 = \begin{pmatrix} x & xy & z \\ x & y & 0 \\ -z & -x^2 & -y \\ -y & -z & x \end{pmatrix}.$$  

In other words, $I = I_3(\varphi_2)$. Since $(f_1, f_2, x) = (x, y^3, z^3)$, then grade$(f_1, f_2, x) = 3$. By [3 Corollary 1.6.19], $f_1, f_2$ is an $R$-regular sequence in $I_2(\varphi_2)$ and so grade($I_3(\varphi_2)$) $\geq 2$. Let $\varphi_1$ be the $1 \times 4$ matrix defined as $(f_1, \ldots, f_4)$. By the Hilbert-Burch Theorem (e.g., [3 Theorem 1.4.16]),

$$0 \to F_2 = R^3 \xrightarrow{\varphi_2} F_1 = R^4 \xrightarrow{\varphi_1} F_0 = R \to R/I \to 0$$

is a free resolution of $R/I$. (It is minimal since $\varphi_2(R^3) \subseteq mR^4$ and $\varphi_1(R^4) = I \subseteq m$.) Therefore

$$2 \leq \text{grade}(I) = \min\{i \geq 0 \mid \text{Ext}^i_R(R/I, R) \neq 0\} \leq \text{proj dim}_R(R/I) \leq 2$$

and $I$ is a perfect ideal of grade 2 (see, e.g., [3 Theorem 1.2.5 and page 25]). In particular, $I$ is grade (and height) unmixed (see, e.g., [3 Proposition 1.4.15]) and so $m$ is not an associated prime to $I$.

Let $p$ be any associated prime of $I$ and set $D = R/p$. Thus $D$ is a one dimensional complete Noetherian local domain ([10 page 63]). Let $V$ be its integral closure in its quotient field $K$. Then $V$ is a one dimensional integrally closed Noetherian local domain, hence a DVR, a discrete valuation ring; note that $V$ is also complete (see, e.g., [15 Theorem 4.3.4]).

Let $\nu$ be the valuation on $K$ corresponding to $V$. Keep calling $x, y, z$ to the images of the regular system of parameters of $R$ in $V$. Set $\nu_x = \nu(x), \nu_y = \nu(y)$ and $\nu_z = \nu(z)$. In $V$, $f_1 = 0$. Applying $\nu$ to the equality $x^4 - y^3 = 0$, one gets $4\nu_x = 3\nu_y$. Thus $\nu_x = 3q$, for some integer $q \geq 1$. In fact, $q > 1$. Indeed, suppose that $q = 1, \nu_x = 3$ and $\nu_y = 4$. Since $f_2 = 0$ in $V$, then $z^3 = x(yz + x^3 - y^3)$. Applying $\nu$ to this equality, $3\nu_x \geq \min(12, 7 + \nu_z)$, which implies $\nu_z \geq 4$. Since $f_3 = 0$ in $V$, then $x^2 y = -y^2 z + xz^2 + x^3 y$. Applying $\nu$ to this equality, one gets $10 \geq \min(12, 11, 13)$, a contradiction. Therefore $\nu_x \geq 6$.

Observe that $xR + I = (x, y^3, y^2 z, yz^2, z^3)$. Set $S = R/xR$ and consider (by abuse of notation) $y, z$ a regular system of parameters of the regular local ring $(S, n)$, where $n = (y, z)$. Then $R/(xR + I) \cong S/n^3$. Since $xR = \text{Ann}_R(S)$, then $\text{length}_R(R/(xR + I)) = \text{length}_S(S/n^3)$. Since $y, z$ is a $S$-regular sequence, there exists a graded isomorphism $k[Y, Z] \cong G(n)$ of $k$-algebras, between the polynomial ring in two variables $Y, Z$ over the field $k = S/n$ and the associated graded ring of the ideal $n$. Using the two exact short sequences $0 \to n^i/n^{i+1} \to S/n^{i+1} \to S/n^i \to 0$, for $i = 1, 2$, one deduces that $\text{length}_S(S/n^3) = 6$. Therefore $\text{length}_R(R/(xR + I)) = 6$. 


On the other hand, since \( xR + I \subseteq xR + p \) and \( R/(xR + p) \cong (R/p)/(x \cdot R/p) = D/xD \), then

\[
6 = \text{length}_R(R/(xR + I)) \geq \text{length}_R(R/(xR + p)) = \text{length}_D(D/xD).
\]

Since \( f_1 = 0 \) and \( f_2 = 0 \) in \( D \), then \( y^3, z^3 \in xD \), and so \( xD \) is an \( m/p \)-primary ideal of the one dimensional Cohen-Macaulay local domain \( (D, m/p, k) \). Using [15, Proposition 11.1.10], we deduce that \( \text{length}_D(D/xD) = e_D(xD; D) \), the multiplicity of \( xD \) on \( D \). Clearly, \( V \) is a finitely generated Cohen-Macaulay \( D \)-module of rank \( 1 \) (see [6, Proposition 2.4]). Therefore, \( \text{length}_V(V/xV) \) is the degree of the extension of the residue fields of \( V \) and of \( D \). Since \( V \) is a DVR, \( \text{length}_V(V/xV) = \nu(x) = \nu_x \). Therefore, \( \text{length}_D(D/xD) = [k_V : k] \cdot \nu_x \). Summing up all together, we get

\[
6 = \text{length}_R(R/(xR + I)) \geq \text{length}_R(R/(xR + p)) = [k_V : k] \cdot \nu_x \geq 6.
\]

Therefore \( \text{length}_R(R/(xR + I)) = \text{length}_R(R/(xR + p)) \) and, by the additivity of the length with respect to exact short sequences, \( xR + I = xR + p \).

Note that \( x \not\in p \), otherwise \( p \supset xR + I \supset (x, y^3, z^3) \) and \( p = m \), a contradiction. Then \( p \cap xR = xp \). In particular, on tensoring \( 0 \to p/I \to R/I \to R/p \to 0 \) by \( R/xR \), one obtains the exact sequence \( 0 \to L/xL \to R/(xR + I) \to R/(xR + p) \to 0 \), where \( L = p/I \). Since \( xR + I = xR + p \), then \( L = xL \). By Nakayama’s Lemma, \( L = 0 \) and \( I = p \).

We conclude that \( I \) is a prime ideal of \( R \). Since the aforementioned resolution of \( R/I \) is minimal, \( I \) is minimally generated by \( 4 \) elements, which in particular implies that \( I \) is not of linear type, because the minimal number of generators of an ideal of linear type is bounded above by the dimension of the ring (see [6, Proposition 2.4]).

**Proof of Lemma** [3]. Since the proof of the present proposition is quite analogous to that of Lemma [2] we skip now some details and re-direct the reader to the former proof. For instance, as before, we can suppose that \( R \) is complete. Let \( \varphi_1 \) be the \( 1 \times 8 \) matrix defined as \((f_1, \ldots, f_8)\). Let \( \varphi_2 \) and \( \varphi_3 \) be the matrices defined as:

\[
\varphi_2 = \begin{pmatrix}
y^2t & y^3 & t^2 & zt & z^2 & x^3z & x^4 & -yt^2 & x^2y^2 & x^3y & x^4 & 0 \\
0 & 0 & 0 & 0 & -y & 0 & 0 & x^3 & 0 & 0 & -t & 0 \\
0 & 0 & -y & x & 0 & 0 & 0 & 0 & -t & z & x^3 & 0 \\
0 & 0 & x & 0 & 0 & 0 & 0 & -xy & z & 0 & 0 & -y^2 \\
0 & -z & 0 & 0 & 0 & 0 & -t & -x^3 & 0 & 0 & 0 & -x^2y \\
-z & x & 0 & 0 & 0 & -t & y & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & y & 0 & z & 0 & 0 & 0 & t & 0
\end{pmatrix}
\]

and

\[
\varphi_3 = \begin{pmatrix}
-t & 0 & 0 & -y & 0 & 0 & 0 & t & -x^2y \\
0 & 0 & -z & 0 & -yt & 0 & 0 & t & 0 \\
-x^3 & t & 0 & 0 & 0 & 0 & 0 & x & 0 \\
z & 0 & 0 & x & 0 & 0 & 0 & -z & x^3 \\
-y & 0 & 0 & 0 & -t & 0 & 0 & x & 0 \\
0 & x & -y & 0 & 0 & 0 & -x^3 & y^2 & 0 \\
0 & -y & 0 & 0 & -x^3 & 0 & 0 & x & 0 \\
x & 0 & 0 & 0 & z & 0 & 0 & 0 & t
\end{pmatrix}.
\]

Since \( \varphi_3 \cdot \varphi_2 = 0 \) and \( \varphi_2 \cdot \varphi_1 = 0 \), then

\[
0 \to F_3 = R^5 \xrightarrow{\varphi_3} F_2 = R^{12} \xrightarrow{\varphi_2} F_1 = R^8 \xrightarrow{\varphi_1} F_0 = R \to R/I \to 0
\]
is a complex of $R$-modules. To prove its exactness we will use the acyclicity criterion of Buchsbaum and Eisenbud (see, e.g., \[8\] Theorem 1.4.12)). Set $r_i = \sum_{j=i}^{3}(-1)^{j-i}\text{rank}F_j$, so that $r_1 = 1$, $r_2 = 7$ and $r_3 = 5$.

Note that, since $(f_2, f_5, f_6, x) = (x, y^4, z^3, t^3)$, then grade$(f_2, f_5, f_6, x) = 4$. By \[8\] Corollary 1.6.19, $f_2, f_5, f_6$ is an $R$-regular sequence in $I = I_1(\varphi_1)$. In particular, grade$(I) \geq 3$.

In order to prove grade$(I_{T}(\varphi_2)) \geq 2$, we look for minors of $\varphi_2$ with pure terms in one of the parameters. For instance, up to sign, the minor $g_1 := y^{10} - 2x^5y^6 + x^{10}y^2 \in I_{T}(\varphi_2)$, with pure term in $y$, is obtained from the $7 \times 7$ submatrix given by the rows 1, 2, 3, 4, 5, 7, 8 and the columns 2, 3, 4, 5, 6, 7, 12. Similarly, we get $g_2 := z^5 - 2x^5z^3 + x^{10}z^2 \in I_{T}(\varphi_2)$ from the $7 \times 7$ submatrix given by the rows 1, 3, 4, 5, 6, 7, 8 and the columns 1, 2, 5, 8, 9, 10, 11. Since $(g_1, g_2, x) = (x, y^{10}, z^5)$, then grade$(g_1, g_2, x) = 3$ and $g_1, g_2$ is an $R$-regular sequence in $I_{T}(\varphi_2)$ and grade$(I_{T}(\varphi_2)) \geq 2$.

As before, let us seek for minors of $\varphi_3$ with pure terms in one of the parameters. Thus $h_1 = y^6 - x^5y^2 \in I_5(\varphi_3)$ is obtained from the $5 \times 5$ submatrix given by the rows 1, 8, 9, 10, 11; $h_2 = z^5 - x^5z^2 \in I_5(\varphi_3)$ is obtained from the rows 3, 4, 6, 7, 12 and, finally, $h_3 = t^5 - x^2y^2 \in I_5(\varphi_3)$ is obtained from the rows 1, 2, 4, 5, 8. Since $(h_1, h_2, h_3, x) = (x, y^6, z^5, t^5)$, then $h_1, h_2, h_3$ is an $R$-regular sequence in $I_5(\varphi_3)$ and grade$(I_5(\varphi_3)) \geq 3$. We conclude that the complex above is a (minimal) free resolution of $R/I$.

Therefore $I$ is a perfect ideal of grade $3$. In particular, $I$ is height unmixed and so $\mathfrak{m}$ is not an associated prime to $I$.

Let $p$ be any associated prime of $I$ and set $D = R/p$. Thus $D$ is a one dimensional complete Noetherian local domain. Then $V$, the integral closure of $D$ in its quotient field $K$, is a DVR (see [13] Theorem 4.3.4]). Let $\nu$ be the valuation on $K$ corresponding to $V$. Set $\nu_x = \nu(x), \nu_y = \nu(y), \nu_z = \nu(z)$ and $\nu(t) = \nu_t$. In $V$, $f_2 = z^5 - x^5 = 0$, $f_5 = t^3 - x^2y^3 = 0$ and $f_6 = y^4 - x^5 = 0$. Applying $\nu$ to these equalities, one gets $3\nu_x = 5\nu_y, 3\nu_z = 2\nu_x + 3\nu_y$ and $4\nu_t = 5\nu_x$. The positive vector $(\nu_x, \nu_y, \nu_z, \nu_t) \in \mathbb{Z}^4$, with smallest $\nu_x \geq 1$, satisfying these three conditions is $(12, 15, 29, 23)$. (Clearly, this vector also satisfies all the other conditions arising from $f_i = 0$.) In particular, $\nu_x \geq 12$.

Let $(S, \mathfrak{n}, k)$ be the regular local ring with $S = R/xR$ and $\mathfrak{n} = m/xR = (y, z, t)$, by abuse of notation. One has $xR + I = (x, yz, z^3, z^2t, zt^2, t^3, y^4, y^3t, y^2t^2)$ and $R/(xR + I) \cong S/J$, where $J$ is the ideal of $S$ defined as $J = (yz, z^3, z^2t, zt^2, t^3, y^4, y^3t, y^2t^2)$. Since $xR = Ann_R(S)$, then length$_R(R/(xR + I)) = length_S(S/J)$, and so $I$ is a primary ideal of $S/J$. Moreover length$_S(S/J) = length_S(G/J^*)$. Hence, length$_S(G/J^*) = length_S(G/L)$. By the isomorphism $k[Y, Z, T] \cong G/\mathfrak{n}$, one deduces that $G/L$ is isomorphic to the $k$-vector space spanned by $1, Y, Y^2, T, Y^2T, Z, ZT, Z^2, T, T^2$.

As in Lemma 2 since $xR + I \subseteq xR + p$ and $R/(xR + p) \cong (R/p)/(x \cdot R/p) = D/xD$, then

$$12 = length_R(R/(xR + I)) \geq length_R(R/(xR + p)) = length_D(D/xD).$$

Since $f_6 = 0, f_2 = 0$ and $f_5 = 0$ in $D$, then $y^4, z^3, t^3 \subseteq xD$, and so $xD$ is an $\mathfrak{m}/\mathfrak{p}$-primary ideal of the one dimensional Cohen-Macaulay local domain $(D, \mathfrak{m}/\mathfrak{p}, k)$. Thus length$_D(D/xD) = e_D(xD; D)$. Since $V$ is a finitely generated Cohen-Macaulay $D$-module of rank $1$, length$_D(V/xV) = e_D(xD; D) \cdot rank_D(V) = e_D(xD; D)$. Moreover length$_D(V/xV) = [k_V:k] \cdot length_D(V/xV) = [k_V:k] \cdot \nu(x)$

Therefore, length$_D(D/xD) = [k_V:k] \cdot \nu_x$. Summing up all together,

$$12 = length_R(R/(xR + I)) \geq length_R(R/(xR + p)) = [k_V:k] \cdot \nu_x \geq 12.$$
of $R$. Set $H := (f_1, \ldots, f_7) \subset I$. Since the aforementioned resolution of $R/I$ is minimal, $f_8 \notin H$ and $H : f_8 \subsetneq R$. However, one can check that $f_8^2 = x^2yzf_1^2 - x^4f_1f_5 - x^2f_2f_7 + tf_5f_6 + x^2f_6f_7$. Thus $f_8^2 \in HI$ and $HI : f_8^2 = R$. Therefore, $H : f_8 \subsetneq HI : f_8^2$ and $I$ is not syzygetic (see [13 Lemma 4.2]).

In terms of the André-Quillen homology (see [1] and [14]; see also [9], for a new and recent treatment), and as a corollary of the Theorem, we state the following characterization of Noetherian rings of low global dimension. Again, just for the sake of completeness, we include item (A), shown in general in [12].

**Corollary.** Let $R$ be a Noetherian ring.

(A): $\mathrm{gl} \dim (R) \leq 1 \iff H_2(R, S, \cdot) = 0$, or $H_2(R, S, S) = 0$, for every quotient ring $S = R/I$.

(B): $\mathrm{gl} \dim (R) \leq 2 \iff H_2(R, S, \cdot) = 0$, for every quotient domain $S = R/I$.

(C): $\mathrm{gl} \dim (R) \leq 3 \iff H_2(R, S, S) = 0$, for every quotient domain $S = R/I$.

Note that unlike Theorem (B), Corollary (B) could be deduced directly from [5, Corollary 2.7], since the vanishing of the second André-Quillen homology $H_2(R, R/I, \cdot)$, in the Noetherian local case, is equivalent to $I$ being generated by an $R$-regular sequence. We give here a slightly different approach.

**Proof of the Corollary.** The equivalence between the first and the third condition in Corollary (A), follows immediately from the isomorphism $H_2(R, R/I, R/I) \cong \ker(\alpha_2)$ and the corresponding equivalence between the first and the third condition in Theorem (A) (see, e.g., [11 Corollaire 3.2]). Similarly, Corollary (C) follows immediately from this same isomorphism and Theorem (C).

It remains to prove the first equivalence of Corollary (A) and the equivalence of Corollary (B). To this end, recall that the vanishing of $H_2(R, R/I, \cdot)$ is also equivalent to $I$ being of linear type and $I/I^2$ being a flat $R/I$-module (see [11 Théorème 4.2]). Clearly, this characterization together the corresponding “if” implications in Theorem (A) and (B), show the “if” implications of Corollary (A) and (B), respectively. Finally, note that, as said before, if $\mathrm{gl} \dim (R) \leq 1$, then every nonzero ideal $I$ of $R$ is locally principal, hence its conormal module $I/I^2$ is $R/I$-flat. Similarly, if $\mathrm{gl} \dim (R) \leq 2$, any nonzero prime ideal $I$ of $R$ is either locally principal, or else maximal, hence in any case, its conormal module $I/I^2$ is again $R/I$-flat.

**Closing Remark.** If we omit the Noetherian assumption on the ring $R$, we know that Theorem (A) is true once we substitute $\mathrm{gl} \dim (R)$ for $\mathrm{w} \dim (R)$ (cf. [12]). Note that $\mathrm{w} \dim (R)$ can be strictly smaller than $\mathrm{gl} \dim (R)$, for instance, if $R$ is the ring of all algebraic integers (see, e.g., [16 1.3 Examples]). Therefore the “if” implication of Theorem (A), without the Noetherian hypothesis, is false. This suggests that one should also replace $\mathrm{gl} \dim (R)$ for $\mathrm{w} \dim (R)$ in the “if” implication of Theorem (B) and (C).

Just to have a flavour of the ins and outs of the non Noetherian setting, and to start with, we show the following simpler statement:

(1) *If $R$ is a commutative ring of $\mathrm{gl} \dim (R) \leq 2$, then every prime ideal of $R$ is of linear type.*

Indeed, since the linear type condition is local, we can suppose again that $(R, \mathfrak{m})$ is local. Then $R$ is either a regular local ring (of Krull dim $(R) \leq 2$), a valuation domain, or a so-called umbrella ring (see [16 2.2 Theorems], for the definitions and a proof). Concretely, it is shown that $R$ is a GCD domain with every prime ideal different from the maximal being flat, hence of linear type (see, e.g., [11 Remarque 2.6]). As for the maximal ideal $\mathfrak{m}$, it is shown that it is either principal, generated by two elements, or non finitely generated. In the first case, $\mathfrak{m}$ is of linear type whereas in the last case, it is shown that $R$ is a valuation domain, hence $\mathrm{w} \dim (R) \leq 1$, and so every ideal is of linear type, in particular, $\mathfrak{m}$ is also of linear type. Finally, if $\mathfrak{m}$ is generated by two elements, $a, b$, say, then there exists an exact sequence $0 \to R \xrightarrow{\varphi} R^2 \xrightarrow{\psi} \mathfrak{m} \to 0$, with $\varphi(1) = (\alpha, \beta)$, say, $\alpha, \beta \in R$, with $\gcd(\alpha, \beta) = 1$, and $\psi(u, v) = ua + vb$. Since $(b, -a) \in \ker(\psi)$, then there exists $\delta \in R$, such that
\[ a = -\delta \beta \text{ and } b = \delta \alpha. \] Note that \( \alpha, \beta \in m \), otherwise, if for instance \( \alpha \) is invertible, then \( \delta = \alpha^{-1}b \) and \( a = -\delta \beta = (-\alpha^{-1}b)\beta \) and \( m \) would be principal. Hence \( m = (a, b)R = \delta(\alpha, \beta)R \subseteq (\alpha, \beta)R \subseteq m \), and \( m = (\alpha, \beta) \) is generated by the regular sequence \( \alpha, \beta \), in particular, \( m \) is of linear type.

We do not know whether one can substitute \( \text{gl dim } (R) \leq 2 \) by \( \text{w.dim } (R) \leq 2 \) in (I); neither we know if the converse of (I) is true, even if we replace \( \text{gl dim } (R) \leq 2 \) by \( \text{w.dim } (R) \leq 2 \). This could be a line of enquiry in future work.

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