Dynamics of Potentials in Bianchi Type Scalar-Tensor Cosmology

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Abstract

The present study investigates the nature of the field potential via new technique known as reconstruction method for the scalar field potentials. The key point of this technique is the assumption that Hubble parameter is dependent on the scalar field. We consider Bianchi type I universe in the gravitational framework of scalar-tensor gravity and explore the general form of the scalar field potential. In particular, this field potential is investigated for the matter contents like barotropic fluid, the cosmological constant and Chaplygin gas. It is concluded that for a given value of Hubble parameter, one can reconstruct the scalar potentials which can generate the cosmology motivated by these matter contents.

Keywords: Scalar-tensor theory; Scalar field; Field potentials.
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1 Introduction

The reality of cryptic dominant component of the universe distribution labeled as dark energy (DE) and its resulting phenomena of cosmic acceleration...
has become a center of interest for the researchers. The existence of this unusual sort of DE is supported by the observational results of many astronomical experiments like Supernova (Ia) [1, 2], Wilkinson Microwave Anisotropy Probe (WMAP) [3] and Sloan Digital Sky Survey (SDSS) [4], galactic cluster emission of X-rays [5], large scale-structure [6] and weak lensing [7]. These experiments reveal the present day cosmic acceleration by evaluating the luminosity distance relation of some type of objects known as standard candles. They also lead to the conclusion that our universe is nearly flat.

In order to resolve these issues, numerous attempts are made which can be categorized on the basis of the used technique. Basically, two approaches have been reported in this context: the modification in the matter configuration of the Lagrangian density and the modification in the whole gravitational framework described by the action. The Chaplygin gas [8] and its modified forms [9], cosmological constant [10], tachyon fields [11], quintessence [12], viscosity effects [13] and k-essence [14] etc. are some DE candidates belonging to the first category. The second approach includes examples of modified theories like \( f(R) \) gravity [15], Gauss-Bonnet gravity [16], \( f(T) \) theory [17], \( f(R,T) \) gravity [18] and scalar-tensor theories [19]. The study of scalar-tensor theories in the subject of cosmology has a great worth due to its vast applications and success [20].

The complete history of the universe from the early inflationary epoch to the final era of cosmic expansion can successfully be discussed by using scalar field as DE candidate [21]. Basically, the alternating gravitational theories are proposed by the inclusion of some functions or terms as a possible modification of Einstein gravity that cannot be derived from the fundamental theory. This raises a question about the appropriate choice of these functions by checking their cosmological viability. However, the process of reconstruction provides a way for having a cosmologically viable choice of these functions. Such a procedure has been adopted by many researchers [22]-[31]. The reconstruction procedure is not a new technique as it has a long history for the reconstruction of DE models. In order to have a better understanding of this technique, we may refer the readers to study some interesting earlier papers [32]. Basically, this technique enables one to find the form of the scalar field potential as well as scalar field for a particular value of the Hubble parameter in terms of scale factor or cosmic time.

It is worth investigating the nature of scalar field potential in the context of scalar-tensor theories. Using reconstruction approach, the nature of the field potential for a minimally coupled scalar-tensor theory has been discussed.
The scalar potentials for tachyon field [23] as well as for solutions involving two scalar fields [24] have been reconstructed through this technique. This is also extended to the modified gravitational frameworks including non-minimal coupled scalar-tensor theories [25], Gauss-Bonnet gravity [26], $F(T)$ theory [27] and the non-local gravity model [28]. Kamenshchik et al. [29] used this technique to reconstruct the scalar field potential for FRW universe in the induced gravity and discussed it for some types of matter distribution which can reproduce cosmic evolution. The same authors [30] used superpotential approach to reconstruct the field potential for FRW model in a non-minimally coupled scalar-tensor gravity and explored its nature for different cases like de Sitter and barotropic solutions describing the cosmic evolution.

In this paper, we discuss the nature of the field potential using the reconstruction procedure for locally rotationally symmetric (LRS) Bianchi type I (BI) universe model. The paper is organized as follows. In the next section, we provide a general discussion of this technique and explore the form of scalar field potential. Section 3 is devoted to study the field potentials using the barotropic fluid, the cosmological constant and the Chaplygin gas as matter contents. In the last section, we discuss and conclude the results.

2 General Formulation of the Field Potential

The scalar-tensor gravity is generally determined by the action [31]

$$S = \int \sqrt{-g} \left[ U(\phi) R - \frac{\omega(\phi)}{2} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} + V(\phi) \right] d^4x; \quad \mu, \nu = 0, 1, 2, 3, \quad (1)$$

where $U$ is the coupling of geometry and the scalar field, $V$ is the self-interacting potential, $R$ is the Ricci scalar and $\omega$ is the interaction function. We can discuss different cases of scalar-tensor theories by taking different values of $U(\phi)$. When both $U$, $\omega$ are constants, the above action yields the Einstein-Hilbert action with quintessence scalar field, for $U = \phi$ with $\omega = \omega_0$, $\omega(\phi)$, it corresponds to simple Brans-Dicke (BD) and the generalized BD gravity with scalar potential, respectively. For $U(\phi) = \frac{1}{2} \gamma \phi^2$, where $\gamma$ is any non-zero constant and constant $\omega$, it leads to the action of the induced gravity. Anisotropic and spatially homogeneous extension of flat FRW model, BI universe with the expansion factors $A$ and $B$ is given by the metric.
\[ ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)(dy^2 + dz^2) \]  

and the respective Ricci scalar is

\[ R = -2\left[\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB}\right]. \]

The average scale factor \( a(t) \), the universe volume \( V \), the directional Hubble parameters (\( H_1 \) along \( x \) direction while \( H_2 \) along \( y \) and \( z \) directions) and the mean Hubble parameter are given by

\[
\begin{align*}
a(t) &= (AB^2)^{1/3}, \\
V &= a^3(t) = AB^2, \\
H_1 &= \frac{\dot{A}}{A}, \\
H_2 &= H_3 = \frac{\dot{B}}{B}, \\
H(t) &= \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right).
\end{align*}
\]

In order to deal with highly non-linear equations, we take a physical assumption for the scale factors, i.e., \( A = B^m; \ m \neq 0,1 \) \([34]\). This condition is originated from the fact that in a spatially homogeneous model, the normal congruence to homogeneous expansion corresponds to the proportionality of the shear scalar \( \sigma \) and the expansion scalar \( \theta \), in other words, the ratio of these quantities \( \frac{\sigma}{\theta} \) is constant. This condition has been used by many researchers for the discussion of exact solutions \([35]\). The above condition further yields the relations \( \frac{\dot{A}}{A} = m\frac{\ddot{B}}{B} \) and \( \frac{\dot{A}}{A} = m\frac{\ddot{B}}{B} + m(m - 1)\frac{\dot{B}^2}{B^2} \), consequently the Ricci scalar takes the form

\[ R = -2\left[ (m + 2)\frac{\ddot{B}}{B} + (m^2 + m + 1)\frac{\dot{B}^2}{B^2} \right] \]  

For BI universe model, we have \( \sqrt{-g} = B^{(m+2)} \) and the respective point-like Lagrangian density constructed by partial integration \([36]\) of the above action (when \( \omega = \omega_0 \), where \( \omega_0 \) is an arbitrary constant) is given by

\[
\begin{align*}
L(B, \phi, \dot{B}, \dot{\phi}) &= 2(m + 2)B^{(m+1)}\frac{dU}{d\phi}\dot{B}\dot{\phi} + 2B^m\dot{B}^2(1 + 2m)U(\phi) \\
&- \frac{\omega_0}{2}B^{m+2}\dot{\phi}^2 + V(\phi)B^{m+2},
\end{align*}
\]
where we have neglected the boundary terms. In order to formulate the corresponding field equations, we use the Euler-Lagrange equations

$$\frac{\partial L}{\partial \dot{B}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 0, \quad \frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \phi} \right) = 0,$$

which describe the dependent field equation for the BI model and the evolution equation of scalar field. Thus we have

$$2(m + 2) \frac{d^2 U}{d\phi^2} \dot{\phi}^2 - 2(m + 2) \frac{dU}{d\phi} \ddot{\phi} - 4(1 + 2m) \frac{dU}{d\phi} \frac{\dot{B}}{B} \dot{\phi}$$

$$- 4(1 + 2m) U(\phi) \frac{\dot{B}}{B} = 0,$$

(5)

$$\omega_0 \ddot{\phi} + \omega_0 (m + 2) \dot{\phi} \frac{\dot{B}}{B} + 2(1 + 2m) \frac{dU}{d\phi} \frac{\dot{B}^2}{B^2} - 2(m + 2)(m + 1) \frac{dU}{d\phi} \frac{dV}{d\phi} \frac{\dot{B}^2}{B^2}$$

$$- 2(m + 2) \frac{\dot{B}}{B} \frac{dU}{d\phi} = 0.$$

(6)

The energy relation (conserved quantity) for the Lagrangian density can be written as $E_L = \dot{B} \frac{\partial L}{\partial \dot{B}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L$ that yields the independent field equation for BI universe (when substituted equal to zero)

$$2(1 + 2m) U(\phi) \frac{\dot{B}^2}{B^2} + 2(m + 2) \frac{dU}{d\phi} \frac{\dot{B}}{B} \dot{\phi} - \frac{\omega_0}{2} \dot{\phi}^2 - V(\phi) = 0.$$

(7)

When $m = 1$, these equations reduce to the case of FRW universe.

For the special choice of $U$, we evaluate the scalar potential in terms of scale factor, directional Hubble parameter and scalar field. We consider the directional Hubble parameter as a function of scale factor or cosmic time by taking different cases of matter contents. The scalar field is found as a function of scale factor or cosmic time and then the scale factor as a function of scalar field by inverting the obtained expression. Finally, we evaluate the Hubble parameter in terms of scalar field and hence the form of scalar potential. We shall explore the nature of the potential that can generate the cosmic evolution described by these matter contents. Equation (7) yields

$$V(\phi) = 2(1 + 2m) U(\phi) \frac{\dot{B}^2}{B^2} + 2(m + 2) \frac{dU}{d\phi} \frac{\dot{B}}{B} \dot{\phi} - \frac{\omega_0}{2} \dot{\phi}^2$$
or equivalently,

\[ V(\phi) = [2(1 + 2m)U(\phi) + 2(m + 2)\phi_B B \frac{dU}{d\phi} - \frac{\omega_0}{2} \phi_B B^2] H_2^2, \quad (8) \]

which provides

\[
\frac{dV}{d\phi} = 2(1 + 2m)H_2^2 \frac{dU}{d\phi} + 4(1 + 2m)U(\phi) \frac{H_2 \dot{H}_2}{\phi} + 2(m + 2)H_2 \phi \frac{d^2U}{d\phi^2} + 2(m + 2)\dot{H}_2 \frac{dU}{d\phi} - \omega_0 \ddot{\phi}.
\]

Using this equation in Eq. (8), it follows that

\[
\omega_0 (m + 2) \phi^2 - 2(m^2 + 2) \frac{dU}{d\phi} \dot{\phi} \dot{H}_2 + 4(1 + 2m)U \dot{H}_2 + 2(m + 2) \dot{H}_2 \frac{dU}{d\phi} - \omega_0 \ddot{\phi} = 0.
\]

We investigate two cases for the coupling function \( U \), i.e., when \( U = U_0 \), where \( U_0 \) is a non-zero constant and \( U \equiv U(\phi) \). In the first case, Eq. (9) becomes

\[
\phi^2 + \left( \frac{4(1 + 2m)U_0}{\omega_0(m + 2)} \right) \dot{H}_2 = 0; \quad \omega_0 \neq 0, \quad m \neq -2
\]

For the scalar field in terms of scale factor \( B \), we have

\[
\phi^2 + \left[ \frac{A(1 + 2m)U_0}{\omega_0(m + 2)} \right] \frac{H'_2}{H_2 B} = 0, \quad (10)
\]

where prime indicates derivative with respect to scale factor, yielding solution

\[
\phi(B) = \int \left( \pm \sqrt{-H_2(B)B \frac{A(1 + 2m)U_0}{\omega_0(m + 2)} \frac{dH_2}{dB} } \right) dB + c_1,
\]

where \( c_1 \) is a constant of integration. One can solve this integral for particular values of the Hubble parameter. In the second case, we consider \( U \equiv U(\phi) \) (a non-minimal coupling of geometry and scalar field). Equation (10) can
be written for scalar field in terms of scale factor and directional Hubble parameter as

\[ \phi'' + \phi' \left( \frac{H'_2}{H_2} \right) + \phi'^2 \left[ \omega_0/2 + \frac{dU}{d\phi} \right] + \frac{2(1 + 2m)U H'_2}{(m + 2)B H_2} + \frac{m(1 - m)\phi'}{(m + 2)B} = 0. \] (11)

This equation is discussed for two particular choices of \( U \).

When \( U = \phi \), i.e., the simple BD gravity, it follows that

\[ \phi'' + \phi' \left( \frac{H'_2}{H_2} \right) + \omega_0/2 \phi'^2 + \frac{2(1 + 2m)\phi H'_2}{(m + 2)B H_2} + \frac{m(1 - m)\phi'}{(m + 2)B} = 0. \] (12)

For the case of induced gravity described by \( U(\phi) = \gamma \omega_0/2 \), Eq.(11) yields

\[ \phi'' + \phi' \left( \frac{H'_2}{H_2} \right) + \phi'^2 \left[ \frac{\omega_0/2 + \gamma}{\gamma \phi} \right] + (1 + 2m)\gamma \phi \frac{H'_2}{(m + 2)B H_2} + m(1 - m)\phi' = 0. \] (13)

These two equations are difficult to solve analytically unless the function \( H_2(B) \) is given. For the sake of simplicity, we introduce a new variable \( x \equiv \frac{\phi'}{\phi} \) which yields \( \frac{\phi''}{\phi} = x' + x^2 \) and hence Eq.(13) turns out to be

\[ x' + x^2 \left( \frac{2\gamma + \omega_0}{\gamma} \right) + x \left( \frac{H'_2}{H_2} \right) + \frac{(1 + 2m) H'_2}{(m + 2)B H_2} + \frac{m(1 - m)x}{(m + 2)B} = 0. \] (14)

Further, we assume \( x \equiv \frac{2\gamma + \omega_0 + 4\gamma}{\omega_0 + 4\gamma f} \), where \( f \) is an arbitrary function of the scale factor \( B \). Also, \( x = \frac{\phi'}{\phi} \) thus integration leads to \( \phi = f^{2\gamma/(\omega_0 + 4\gamma)} \). Using this value of \( x \) in Eq.(14), we obtain

\[ f'' + f' \left( \frac{H'_2}{H_2} \right) + \omega_0 + 4\gamma \left( \frac{1 + 2m}{m + 2} f \right) \frac{H'_2}{B H_2} + \frac{m(1 - m)f'}{(m + 2)B} = 0. \] (15)

We see that Eq.(12) is difficult to transform in \( x \) by the above transformation. If we consider the scalar field as a constant then Eq.(8) yields the scalar potential \( V = 2(1 + m)UH^2_{2,0} \), where \( H^2_{2,0} \) is constant directional Hubble parameter. Multiplying the Klein-Gordon equation (6) both sides with this value of \( V \), we obtain the scalar potential

\[ V = V_0 U \frac{m^2 + 2m + 3}{1 + m} = V_0 \frac{\gamma}{2} \frac{2m^2 + 2m + 3}{1 + m}, \]

which is obviously a constant (as \( V_0 \) and \( \phi \) are constants).
When \( \omega \equiv \omega(\phi) \), the field equations (5) and (7) remain the same except that the constant \( \omega_0 \) is replaced by \( \omega(\phi) \) while Eqs. (11) becomes

\[
\omega(\phi) \ddot{\phi} + \omega(\phi)(m + 2)\frac{\dot{B}}{B} + \frac{\dot{\phi}^2}{2} \frac{d\omega}{d\phi} + 2(1 + 2m) \frac{dU \dot{B}^2}{dB \frac{d\phi}{B^2}} - 2(m + 2)(m + 1) \frac{dU}{d\phi} \frac{dV}{d\phi} \frac{\dot{B}^2}{B^2} - 2(m + 2) \frac{\dot{B}}{B} \frac{dU}{d\phi} = 0. \tag{16}
\]

Solving the field equations (5), (7) and (16), we have the same expressions as Eqs. (10), (12) and (15) except \( \omega_0 \) is replaced by \( \omega(\phi) \). In the following, we discuss Eqs. (10), (12) and (15) separately to construct potential.

## 3 Potential Construction

Now we discuss the scalar field potential by taking three different matter contents.

### 3.1 Barotropic Fluid

First we consider the barotropic fluid (a particular case of the perfect fluid) with equation of state (EoS), \( p = k\rho, \ 0 < k < 1 \), where \( p \) and \( \rho \) are pressure and density, while \( k \) is the EoS parameter. In order to find the evolution of Hubble parameter due to barotropic fluid, we consider the Einstein field equations for BI universe model as

\[
(1 + 2m)H^2_2 = \rho, \quad \left( \frac{m + 3}{2} \right) \dot{H}_2 + \left( \frac{m^2 + m + 4}{2} \right) H^2_2 = -p, \tag{17}
\]

where we have used the condition \( A = B^m \) and also combined the two dependent field equations. The integration of the energy conservation equation yields \( \rho = \rho_0 B^{-(1+k)(m+2)} \), where \( \rho_0 \) is an integration constant. Consequently, the directional Hubble parameters are found to be

\[
H_2(B) = \frac{H_1(B)}{m} = \left[ \frac{m^2 + m + 4}{1 + 2m} + 2k \right] \frac{2\rho_0}{(1 + k)(m + 2)(m + 3)}^{1/2} \times \frac{B^{-(1+k)(m+2)}}{2}, \tag{18}
\]

where the integration constant is taken to be zero. The evolution of Hubble parameter is \( \frac{H'_1(B)}{H_2(B)} = -\frac{(1+k)(m+2)}{2B} \). The corresponding deceleration parameter
turns out to be positive, i.e., $q = -1 + \frac{3(k+1)}{2}$ which is consistent with the barotropic fluid. Using these values in Eq. (11), we obtain

$$\phi(B) = \ln(\phi_0 B^{\pm \sqrt{\frac{2(1+k)(1+2m)U_0}{\omega_0}}})$$

where $\phi_0$ is a non-zero integration constant. This shows that the constant coupling of geometry and scalar field, i.e., $U = U_0$ for the barotropic fluid leads to the logarithmic form of scalar field which further corresponds to expanding or contracting scalar field versus scale factor $B$ on the basis of sign. Consequently, the scale factors turn out to be

$$A(\phi) = \exp\left(\frac{4m}{2n^2 + 8n + 8} c_2^{-1} \phi^{n+2}\right), \quad B(\phi) = \exp\left(\frac{4}{2n^2 + 8n + 8} c_2^{-1} \phi^{n+2}\right).$$

We see that the scale factors are of exponential form which indicate rapid cosmic expansion for the expanding scalar field. The corresponding field potential is

$$V(B) = [2(1+2m)U_0 - (1+k)(1+2m)U_0](m^2 + m + 4) (1 + 2m) + 2k)$$

$$\times \frac{2\rho_0}{(1+k)(m+2)(m+3)} B^{-(1+k)(m+2)}. \quad (19)$$

This is of power law nature and indicates inverse power law behavior for $m > 0$ as $0 < k < 1$.

For the variable $\omega(\phi)$, we consider the ansatz $\omega(\phi) = \omega_0 \phi^n; \ n > 0$ so that the scalar field takes the following form

$$\phi(B) = [c_2 \ln(B)^2 n^2 + 4 \ln(B)^2 n - 2 \ln(B)n^2 c_1 - 8 \ln(B) nc_1 + 4 \ln(B)^2$$

$$- 8 \ln(B)c_1 + c_1^2 n^2 + 4n c_1^2 + 4c_1^2]^{1/(n+2)} (2^{\frac{1}{n+2}})^{-2},$$

where $c_1$ is an integration constant and $c_2 = \frac{2(1+2m)(1+k)U_0}{\omega_0}$. For the sake of simplicity, we take $c_1 = 0$ and hence the scalar field becomes

$$\phi(B) = \frac{c_2^{1/(n+2)} (\ln(B^{2n^2+8n+8}))^{1/(n+2)}}{2^{(1/(n+2))^2}}.$$ 

Thus the scale factors in exponential form are

$$A(\phi) = \exp\left(\frac{4m}{2n^2 + 8n + 8} c_2^{-1} \phi^{n+2}\right), \quad B(\phi) = \exp\left(\frac{4}{2n^2 + 8n + 8} c_2^{-1} \phi^{n+2}\right).$$
Consequently, the potential turns out to be

\[
V(B) = [2(1 + 2m)U_0 - \frac{\omega_0}{2} \left( \frac{c_2^{1/(n+2)} \ln(B^{2n^2+8n+8})}{2(1/(n+2))^2} \right)^n c_2^{2/(n+2)} \left( 2^{1/(n+2)} - 4 \right) \times \left( \frac{2n^2 + 8n + 8}{(n + 2)^2} \right) \ln(B^{2n^2+8n+8})^{-2(1+n)/n+2} \right) \left( \frac{m^2 + m + 4}{1 + 2m} + 2k \right) \times \frac{2\rho_0}{(1 + k)(m + 2)(m + 3)} B^{-(1+k)(m+2)},
\]

which contains the product of inverse power law and logarithmic functions of the scale factor.

For \( U = \phi \), Eq.(12) takes the form

\[
\phi'' + \left( \frac{m(1 - m)}{m + 2} - \frac{(1 + k)(m + 2)}{2} \right) \frac{\phi'}{B} + \frac{\omega^2}{2} \phi'^2 - (1 + 2m)(1 + k) \frac{\phi}{B^2} = 0.
\]

When \( \omega = \omega_0 \) or \( \omega(\phi) = \omega_0 \phi^n \), the solution to this differential equation is quite complicated and cannot provide much insights. However, if we take \( m = -1/2 \) and \( \omega = \omega_0 \), then this leads to

\[
\phi(B) = \frac{2}{\omega_0} \ln\left[ \frac{\frac{6}{\omega_0} \left( 4c_3 B^{3/4(3+k)} + 9c_4 + 3c_4 k \right)}{3 + k} \right],
\]

where \( c_3 \) and \( c_4 \) are integration constants and \( \omega_0 \neq 0 \). The respective scale factors are

\[
A(\phi) = \left[ \frac{1}{4c_3} \left( \frac{6}{\omega_0} \exp\left( \frac{\omega_0}{2} \phi \right) - 9c_4 - 3c_4 k \right) \right]^{m/(3/4(3+k))},
\]

\[
B(\phi) = \left[ \frac{1}{4c_3} \left( \frac{6}{\omega_0} \exp\left( \frac{\omega_0}{2} \phi \right) - 9c_4 - 3c_4 k \right) \right]^{1/(3/4(3+k))}
\]

and the corresponding scalar field potential turns out to be

\[
V(B) = \left( 2m + 2 \right) \left( \frac{3c_3(3 + k) B^{3/4(3+k)}}{\frac{\omega_0}{2} \left( 4c_3 B^{3/4(3+k)} + 9c_4 + 3c_4 k \right)} \right) - \frac{\omega_0}{2} \left( \frac{3c_3(3 + k) B^{3/4(3+k)}}{\frac{\omega_0}{2} \left( 4c_3 B^{3/4(3+k)} + 9c_4 + 3c_4 k \right)} \right)^2.
\]

We can conclude that the scalar field is described by logarithmic function and the scale factors are of exponential nature which yields expansion for increasing scalar field while the potential turns out to be of power law nature.
Now we discuss the induced gravity case and evaluate the function $f$ by using the Hubble parameter and its evolution in Eq.\((15)\) which leads to

$$f'' + \left[ \frac{m(1-m)}{m+2} - \frac{(1+k)(m+2)}{2} \right] f' - \frac{\omega_0 + 4\gamma}{4\gamma} (1+2m)(1+k) \frac{f}{B^2} = 0$$

whose solution is

$$f(B) = c_5 B^{r_1} + c_6 B^{r_2}; \quad r_{1,2} = \frac{1-c_7}{2} \pm \frac{1}{2} \sqrt{c_7^2 + 1 - 2c_7 - 4c_8},$$

where $c_5$ and $c_6$ are arbitrary constants while $c_7$ and $c_8$ are given by

$$c_7 = -\frac{2m + (3+k)m^2 + 4 + 4(1+m)k}{2(m+2)}, \quad c_8 = -(1+2m)(1+k) \frac{\omega_0 + 4\gamma}{4\gamma}.$$

The corresponding scalar field is $\phi(B) = (c_5 B^{r_1} + c_6 B^{r_2})^{\frac{\omega_0 + 4\gamma}{4\gamma}}$ which is clearly of power law nature. Since it is difficult to invert this expression for the scale factor $B$ in terms of $\phi$, so we take either $c_5 = 0$ or $c_6 = 0$, which leads to either

$$A(\phi) = \frac{1}{c_5^{m}} \phi^{\frac{m(\omega_0 + 4\gamma)}{4\gamma}}, \quad B(\phi) = \frac{1}{c_5} \phi^{\frac{\omega_0 + 4\gamma}{4\gamma}},$$

or

$$A(\phi) = \frac{1}{c_6^{m}} \phi^{\frac{m(\omega_0 + 4\gamma)}{4\gamma}}, \quad B(\phi) = \frac{1}{c_6} \phi^{\frac{\omega_0 + 4\gamma}{4\gamma}}.$$

We see that the scale factors are also of power law nature and show expanding or contracting behavior depending upon the values of the involved parameters. The scalar field potential (13) then turns out to be

$$V(B) = \frac{(1+2m)\gamma + 4\gamma^2(m+2)r_{1,2}}{\omega_0 + 4\gamma} - \frac{2\omega_0 \gamma^2 r_{1,2}^2}{(\omega_0 + 4\gamma)^2} \left( \frac{m^2 + m + 4}{1 + 2m} + 2k \right)$$

$$\times \left( \rho_0 c_{5,6}^{\frac{\omega_0 + 4\gamma}{4\gamma}} B^{\frac{4\gamma r_{1,2}}{\omega_0 + 4\gamma} - (1+k)(m+2)} \right).$$

This may be of positive or inverse power law nature depending upon the values of parameters.

For variable $\omega$, the analytical solution of Eq.\((15)\) is not possible. However, the corresponding numerical solution can be found by using the initial
Figure 1: Plots show the field potential versus scale factor $B$. Plots (a), (b), (c) and (d) correspond to the field potentials given by Eqs. (19), (20), (22) and (24), respectively. Here $m = 2$, $\rho_0 = 1$, $U_0 = 3$, $k = 0.5$ and $\omega_0 = 0.9$ in all plots except for the plot (c), where $m = -0.5$. 
conditions $f(1) = 0.67$ and $f'(1) = 1.95$ and is given by the polynomial interpolation

$$f(B) = 0.014B^8 - 0.4615B^7 + 6.3599B^6 - 47.5667B^5 + 206.9123B^4$$
$$-529.2141B^3 + 772.0721B^2 - 587.8872B + 180.4408,$$

(25)

where we have taken $m = 2$, $\gamma = 0.25$, $k = 0.5$ and $\omega = 0.9\phi^2$. The corresponding scalar field is $\phi(B) = (f(B))^{\frac{1}{2\gamma}}$, yielding the form of the field potential in polynomial form which represents positive power law nature. Here the scalar field is in polynomial form which cannot be inverted for scale factor $B$.

We have plotted the potentials given by Eqs. (19), (20), (22) and (24) versus scale factor $B$ as shown in Figure 1. It is found that in all cases, the scalar field potentials are positive decreasing functions except for the plot (c) which has a signature flip from positive to negative with the increase in scale factor (this graph corresponds to the negative value of $m$). We can conclude that for a positive behavior of the field potential (which is physically acceptable), we should take positive range of $m$.

### 3.2 Cosmological Constant

In this case, we take $p = -\rho$ and hence the energy density becomes a constant, i.e., $\rho = \rho_0$. The corresponding directional Hubble parameters and its evolution are given by

$$\frac{H_1(B)}{m} = H_2(B) = \sqrt{\frac{4\rho_0}{m + 3}(1 - \frac{m^2 + m + 4}{1 + 2m})B}, \quad \frac{H'_2(B)}{H_2(B)} = \frac{1}{2B\ln(B)},$$

(26)

The deceleration parameter turns out to be a dynamical quantity $q = -(1 + \frac{1}{2(m+2)\ln(B)})$. It is interesting to mention here that in our case, the directional Hubble parameters are dependent on the scale factor $B$ (due to anisotropy) whereas in the case of FRW universe, the Hubble parameter is independent of the scale factor, i.e., it turns out to be constant. We use these values in the previously discussed three cases, i.e., $U = U_0$, $\phi$ and $U = \frac{1}{2}\gamma\phi^2$. Equation (10) provides $(\phi')^2 = \frac{(1+2m)U_0}{\omega_0(m+2)}\frac{1}{B^2\ln(B)}$ whose integration leads to

$$\phi(B) = \pm \sqrt{-2\ln(B)c_{10} + c_9},$$

where $c_9$ is an integration constant while $c_{10} = \frac{2(1+2m)U_0}{\omega_0(m+2)}$. This leads to the scale factor as an exponential function of the
scalar field $B(\phi) = \exp(-1/2c_{10}(\phi - c_9))$. Likewise, for $\omega = \omega_0\phi^n$, the scalar field is found to be

$$
\phi(B) = (2^{-2/(n-2)})^2 \left[ \frac{\pm \sqrt{-2\ln(B) + c_{11}}}{(n-2)(2\ln(B)c_{10} + c_{11}^2)} \right],
$$

where $c_{11}$ is an integration constant while $c_{10}$ is the same as above. Using these values in Eq.(8), the field potential can be determined which would include the product terms of scale factor and logarithmic function.

In the case of simple BD gravity, Eq.(12) is not easy to solve for both cases $\omega = \omega_0$ and $\omega = \omega_0\phi^n$. However, the corresponding numerical solutions can be constructed in a similar way as we have discussed in the previous case. The scalar field as well as the potentials constructed, in this way, would be of polynomial nature. For $m = -1/2$, it leads to $\phi'' + \frac{\omega_0}{\phi} \phi' = 0$ and hence

$$
\phi(B) = \frac{2\ln\left(\frac{\omega_0 + \frac{1}{2}c_{13}\omega_0}{\omega_0}\right)}{\omega_0}, \quad B(\phi) = \frac{2}{c_{12}\omega_0}(\exp(\omega_0\phi/2) - \frac{c_{13}\omega_0}{2}),
$$

where $c_{12}$ and $c_{13}$ are integration constants. The field potential corresponding to these values can be obtained from Eq.(8) which would be of power law nature. For the case of induced gravity, Eq.(15) provides

$$
f'' + \frac{f'}{2B\ln B} + \frac{m(1-m)f'}{m+2}B + \frac{(\omega_0 + 4\gamma)(1+2m)}{4\gamma (m+2)} \frac{f}{B}(\frac{1}{2B\ln B}) = 0.
$$

Solving this equation, we have the solution in terms of Kummer functions

$$
f(B) = c_{14}KummerM\left(\frac{1}{4}(-m(1-m)\gamma + (2\omega_0 + 9\gamma)m + 6\gamma + \omega_0(m+2)
+ 3(m+2)\gamma(m+2-m(1-m))((m+2)\gamma(m+2-m(1-m)))^{-1}, 3/2,
+ (-m - 2 + m(1-m))(m+2)\ln(B)\sqrt{\ln(B)}B^{1/2(m+2)(m+2-m(1-m))})
+ c_{15}KummerU\left(\frac{1}{4}(-m(1-m)\gamma + (2\omega_0 + 9\gamma)m + 6\gamma + \omega_0(m+2)
+ 3(m+2)\gamma(m+2-m(1-m))((m+2)\gamma(m+2-m(1-m)))^{-1}, 3/2,
+ (-m - 2 + m(1-m))(m+2)\ln(B)\sqrt{\ln(B)}B^{1/2(m+2)(m+2-m(1-m))})
\right),
$$

where $c_{14}$ and $c_{15}$ are integration constants. Since $\phi = f^{2\gamma/(\omega_0 + 4\gamma)}$, consequently the scalar field potential can be determined (it would be a lengthy
expression in Kummer function). For $\omega_0 = -4\gamma$, the solution is

$$f(B) = c_{16} + \left( \int \frac{B^{-m(1-m)/m+2}}{\sqrt{\ln(B)}} dB \right) c_{17}, \quad (30)$$

where $c_{16}$ and $c_{17}$ are integration constants. The corresponding potential can be determined by using the value of the scalar field $\phi = f_{-\omega_0+4\gamma}$ in Eq.(8). It would include the integral term and hence cannot be categorized as power law, exponential or logarithmic form.

### 3.3 Chaplygin Gas

Finally, we consider the Chaplygin gas EoS as DE candidate which is defined by $p = -\frac{C}{\rho}$, where $C$ is some positive constant. In order to discuss the potential, we use the above EoS parameter in the energy conservation equation and then integration leads to $\rho(B) = (C + c_{18}B^{-2(m+2)})^{1/2}$, where $c_{18}$ is an integration constant. Using this value in Eq.(17), it follows that

$$H_2^2(B) = \frac{4C^{1/2}}{m+3} (1 - \frac{m^2 + m + 4}{1 + 2m} \ln(B) + \frac{4c_{18}}{(m+3)^{1/2}}) (1 + \frac{m^2 + m + 4}{2(1 + 2m)}B^{-2(m+2)}), \quad (31)$$

whose evolution yields

$$\frac{H'_2}{H_2} = \frac{p_1 - 2(m + 2)p_2 B^{-2(m+2)}}{2B(p_1 \ln(B) + p_2 B^{-2(m+2)})}, \quad (32)$$

where $p_1 = \frac{4C^{1/2}}{m+3} (1 - \frac{m^2 + m + 4}{1 + 2m})$ and $p_2 = \frac{c_{18}}{(m+3)^{1/2}} (1 + \frac{m^2 + m + 4}{1 + 2m})$. For the constant coupling of scalar field and geometry ($U = U_0$) with $\omega = \omega_0$, we have

$$\phi(B) = \int \pm \sqrt{2} (\omega_0 (m + 2)(B^{-2(1+m)}p_2 + B^2 p_1 \ln(B)) U_0 (1 + 2m)(-p_1 + 2mp_2 B^{-2(m+2)} + 4p_2 B^{-2(m+2)} ((B^{-2(m+2)}p_2 + B^2 p_1 \ln(B))) \times \omega_0 (m + 2))^{-1/2}. \quad (31)$$

Thus we can determine the field potential that can generate the cosmic evolution of Chaplygin gas matter (it would be in integral form). For $\omega = \omega_0 \phi^n$, 15
the scalar field is
\[
\frac{2\phi(B)^{(n+2)/2}}{n + 2} + \int \left[ (\omega_0(m + 2)(p_2 + \ln(B')B'^{2m+4}p_1))^{-1}(\phi(B)^{n/2}B'^{2(1+m)}
\times (-2U_0\omega_0(2m^2 + 5m + 2)\phi(B)^{-n}(-2B' - 4mp_2^2m - 2p_1\ln(B')mp_2B'^{4-2m}
+B^8p_1^2\ln(B') - 4p_1\ln(B')p_2B'^{4-2m} + B'^{4-2m}p_1p_2 - 4B'^{4-m}p_2^2))^{1/2})B'^{-6} \right] = 0.
\]

Clearly, it is not possible to have an explicit expression for scalar field in terms of scale factor \(B\) and hence the form of the respective field potential cannot be determined. For simple BD gravity with \(\omega = \omega_0\) and \(\omega = \omega_0\phi^n\), we could not find analytical solutions but numerical solutions can be constructed in a similar pattern as we have discussed earlier. For induced gravity, analytical solution is only possible if we take \(p_2 = 0\), which further implies the same cases as we have found in the cosmological constant case (as \(\frac{H_2'}{H_2} = \frac{1}{2\ln(B')}\)).

## 4 Summary and Discussion

This paper investigates scalar field potentials by a new technique known as the reconstruction technique for the field potentials. We have applied this technique to BI universe model in the context of general scalar-tensor theory. The general form of the field potential without assigning any values of \(U\), \(V\) and \(H_2\) has been explored. We have also discussed two particular cases of \(U\), i.e., when it is a constant and \(U = U(\phi)\). In both cases, the field potential depends upon the scale factor \(B\), the scalar field and the directional Hubble parameter \(H_2\). Further, we have taken two cases for \(\omega\), i.e., \(\omega = \omega_0\) and \(\omega = \omega_0\phi^n\). It is found that an explicit form of the field potential cannot be found in terms of scale factor unless we choose some particular value of the Hubble parameter. For this purpose, we have taken the evolution of Hubble parameter motivated by the barotropic fluid, the cosmological constant and the Chaplygin gas matter contents. In literature [33, 38], four types of scalar field potentials have usually been discussed, i.e., the positive and inverse power laws, the exponential and the logarithmic potentials while other forms are multiple of these four types.

For the barotropic fluid, the potential can be found but it is not possible for the simple BD gravity. We have also observed that for constant \(U\), the scalar fields are logarithmic functions for both \(\omega = \omega_0\) and \(\omega = \omega_0\phi^n\), while the scale factors are of exponential nature. Also, for simple BD gravity with
For $m = -0.5$ and $\omega = \omega_0$, the scale factors are exponential functions while for the induced gravity, they turn out to be of power law form. In order to examine their behavior, we have plotted the field potentials versus scale factor $B$ as shown in Figure 1. It is concluded that the field potentials are positive and decrease to zero except for the case of simple BD gravity where we have taken negative value of $m$. We may conclude that for positive field potential, we should impose the condition $m > 0$. We have also discussed a numerical approach (polynomial interpolation) for the cases where no analytical solution exists. Likewise, for the cosmological constant candidate of DE with constant coupling function $U$, we can determine the form of the field potential without taking any condition for both $\omega$, however in other cases, we have to impose some certain conditions.

In the case of Chaplygin gas matter contents, the scalar field potential can be discussed only for $\omega = \omega_0$ with $U = U_0$. However, in other cases, either the explicit analytical solution is not possible or we have the same expression of the field potential as in the case of cosmological constant. It would be worthwhile to investigate the form of the field potential for the exponential form of coupling function of scalar field and geometry. This procedure may lead to some interesting results when the chameleon mechanism is taken into account in the framework of scalar-tensor gravity.

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