THE SPACE OF GENERALIZED $G_2$-THETA FUNCTIONS OF LEVEL ONE

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Abstract. Let $C$ be a smooth projective complex curve of genus at least 2. For a simply-connected complex Lie group $G$ the vector space of global sections $H^0(\mathcal{M}(G), \mathcal{L}_G^\otimes l)$ of the $l$-th power of the ample generator $\mathcal{L}_G$ of the Picard group of the moduli stack of principal $G$-bundles over $C$ is commonly called the space of generalized $G$-theta functions or Verlinde space of level $l$. In the case $G = G_2$, the exceptional Lie group of automorphisms of the complex Cayley algebra, we study natural linear maps between the Verlinde space $H^0(\mathcal{M}(G_2), \mathcal{L}_G)$ of level one and some Verlinde spaces for $SL_2$ and $SL_3$. We deduce that the image of the monodromy representation of the WZW-connection for $G = G_2$ and $l = 1$ is infinite.

1. Introduction

Let $C$ be a smooth projective complex curve of genus $g \geq 2$. For a complex semi-simple Lie group $G$ we denote by $\mathcal{M}(G)$ the moduli stack of principal $G$-bundles over $C$. If $G$ is simply-connected, the Picard group of the stack $\mathcal{M}(G)$ is infinite cyclic and we denote by $\mathcal{L}$ its ample generator. The finite-dimensional vector spaces of global sections $H^0(\mathcal{M}(G), \mathcal{L}^\otimes l)$, the so-called spaces of generalized $G$-theta functions or Verlinde spaces of level $l$, have been intensively studied from different perspectives, e.g. gauge theory, mathematical theory of conformal blocks, and quantization. Note that much of the literature deals with the vector bundle case $G = SL_r$.

In this note we will study the Verlinde space $H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2})$ for the smallest exceptional Lie group $G_2$ and at level 1. The starting point of our investigation was the striking numerical relation between the dimensions of the Verlinde spaces for $G_2$ at level 1 and for $SL_2$ at level 3

(1) $\dim H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) = \frac{1}{2g} \dim H^0(\mathcal{M}(SL_2), \mathcal{L}_{SL_2}^\otimes 3) = \left(\frac{5 + \sqrt{5}}{2}\right)^{g-1} + \left(\frac{5 - \sqrt{5}}{2}\right)^{g-1}.$

These dimensions are computed by the Verlinde formula (see e.g. [B3] Corollary 9.8). It turns out that linear maps between these Verlinde spaces arise in a natural way by restricting to some distinguished substacks in $\mathcal{M}(G_2)$. The group $G_2$ contains the subgroups $SL_2$ and $SO_4$ as maximal reductive subgroups of maximal rank. These group inclusions induce maps

$$i : \mathcal{M}(SL_3) \rightarrow \mathcal{M}(G_2) \quad \text{and} \quad j : \mathcal{M}(SL_2) \times \mathcal{M}(SL_2) \rightarrow \mathcal{M}(G_2)$$

via the étale double cover $SL_2 \times SL_2 \rightarrow SO_4$.

Our main results are the following.

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**Theorem I.** For any smooth curve $C$ of genus $g \geq 2$ the linear map obtained by pull-back by the map $j$ of global sections of $\mathcal{L}_{G_2}$

$$j^* : H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) \longrightarrow \left[ H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2}^\otimes 3) \otimes H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2}) \right]_0$$

is an isomorphism.

**Theorem II.** For any smooth curve $C$ of genus $g \geq 2$ without vanishing theta-null the linear map obtained by pull-back by the map $i$ of global sections of $\mathcal{L}_{G_2}$

$$i^* : H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) \longrightarrow H^0(\mathcal{M}(\text{SL}_3), \mathcal{L}_{\text{SL}_3})_+$$

is surjective.

The subscripts 0 and + denote subspaces of invariant sections for the group of 2-torsion line bundles over $C$ and for the duality involution respectively.

The first example of isomorphism between Verlinde spaces was given in [B1] for the embedding $\mathbb{C}^* \subset \text{SL}_2$ at level 1. More recently, the rank-level dualities provide series of isomorphisms between Verlinde spaces (and their duals) for special pairs of structure groups. In this context Theorem I can be viewed as a new example.

Most of the constructions presented in this paper are valid for the coarse moduli spaces of semi-stable $G$-bundles over $C$. However, the generator $\mathcal{L}_{G_2}$ of the Picard group of the moduli stack $\mathcal{M}(G_2)$ does not descend [LS] to the moduli space $\mathcal{M}(G_2)$ because the Dynkin index of $G_2$ is 2. This forces us to use the moduli stack.

Theorem I has an application to the flat projective connection on the bundle of conformal blocks associated to the Lie algebra $\mathfrak{g}_2$ at level 1. Let $\pi : C \rightarrow S$ be a family of smooth projective curves and consider the vector bundle $\nabla^*_1(\mathfrak{g}_2)$ over $S$ whose fiber over the curve $C = \pi^{-1}(s)$ equals the conformal block $\nabla^*_1(\mathfrak{g}_2)$. Note that this conformal block is canonically (up to homothety) isomorphic to our space $H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2})$ by the general Verlinde isomorphism [LS]. By [U] the vector bundle $\nabla^*_1(\mathfrak{g}_2)$ is equipped with a flat projective connection, the so-called WZW-connection. Then we have the

**Corollary.** There exist families of smooth curves of any genus $g \geq 2$ for which the projective monodromy representation of the projective WZW-connection on $\nabla^*_1(\mathfrak{g}_2)$ has infinite image.

In section 2 we review the properties of the exceptional group $G_2$ and of its subgroups, as well as some results on the Verlinde spaces for $\text{SL}_2$ at low levels. In section 3 we give the proof of the main theorems.

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2. Moduli spaces and moduli stacks of principal $G_2$-bundles

In this section we review some results on the exceptional group $G_2$ and on the moduli of principal $G_2$-bundles over a smooth projective curve $C$.

2.1. The exceptional group $G_2$ and its rank-$2$ subgroups. The complex exceptional group $G_2$ is given by one of the following equivalent definitions:

- as the automorphism group $G_2 = \text{Aut}(\mathbb{O})$ of the complex 8-dimensional Cayley algebra or algebra of octonions $\mathbb{O}$ (see e.g. [Ba]).
- as the connected component of the stabilizer in $\text{GL}(V)$ of a non-degenerate alternating trilinear form $\omega : \Lambda^3 V \to \mathbb{C}$ on a complex 7-dimensional vector space $V$ (see e.g. [SK]).

We recall the following facts:

(a) For a generic trilinear form $\omega$ we have $\text{Stab}_{\text{GL}(V)}(\omega) = G_2 \times \mu_3$ and $\text{Stab}_{\text{SL}(V)}(\omega) = G_2$. Note that non-degenerate alternating forms form the unique dense $\text{GL}(V)$-orbit in $\Lambda^3 V^\ast$.

(b) Introducing $G_2$ as $\text{Aut}(\mathbb{O})$ there is a natural non-degenerate $G_2$-invariant trilinear form on the space of purely imaginary octonions $V = \text{Im}(\mathbb{O})$ given by $\omega(x, y, z) = \text{Re}(xyz)$, as well as a non-degenerate symmetric $G_2$-invariant bilinear form given by $q(x, y) = \text{Re}(xy)$. This shows that $G_2$ is a subgroup of $\text{SO}_7$.

(c) The complex Lie group $G_2$ is connected, simply-connected, has no center and is of dimension 14.

According to [BD] the group $G_2$ has, up to conjugation, two maximal Lie subgroups of maximal rank, i.e. of rank 2. These two subgroups are of type $A_2$ and $A_1 \times A_1$ respectively. As we could not find a reference in the literature, we will give for the reader’s convenience an explicit realization of these subgroups in $G_2$.

- $\text{SL}_3 \subset G_2$. We consider a non-degenerate alternating trilinear form $\omega \in \Lambda^3 V^\ast$ and define $G_2 = \text{Stab}_{\text{SL}(V)}(\omega)$. We associate to $\omega$ the quadratic form $q_\omega : \text{Sym}^2 V \to \mathbb{C}$, $q_\omega(x, y) = L_x \omega \wedge L_y \omega \wedge \omega \in \Lambda^7 V^\ast \cong \mathbb{C}$, where $L_x : \Lambda^3 V^\ast \to \Lambda^2 V^\ast$ denotes the contraction operator with the vector $x \in V$. Note that $\omega$ is non-degenerate iff $q_\omega$ is non-degenerate. We now choose a 3-dimensional subspace $W \subset V$ such that $W$ is isotropic for $q_\omega$ and such that the restriction $\omega_0 = \omega |_W \neq 0$. The following gives a description of $\text{SL}_3$ as a subgroup of $G_2$.

**Proposition 2.1.** With the above notation we have

$$\text{SL}_3 = \text{Stab}_{G_2}(W) = \{g \in G_2 \mid g(W) = W\}.$$  

More precisely, the subspace $W \subset V$ induces a natural decomposition

$$V = W \oplus \Lambda^2 W \oplus \mathbb{C},$$

which coincides with the decomposition of $V$ as $\text{SL}_3$-module.

**Proof.** We consider the composite map

$$\iota : \Lambda^2 W \hookrightarrow \Lambda^2 V \stackrel{L_\omega}{\longrightarrow} V^\ast,$$

where $L_\omega$ is contraction with $\omega \in \Lambda^3 V^\ast$. If we further compose with the projection $V^\ast \to W^\ast$, we obtain the isomorphism $\Lambda^2 W \xrightarrow{\sim} W^\ast$ induced by the non-zero restricted
form $\omega_0$. Hence $i$ is injective and we also denote by $\Lambda^2 W \subset V$ its image in $V$, which we identify with $V^*$ via the non-degenerate quadratic form $q_*$. Next we observe that $W \cap \Lambda^2 W = \{0\}$, since the composite map $W \to V^* \to W^*$ is zero — $W$ is isotropic. This shows that $W \oplus \Lambda^2 W$ is a hyperplane in $V$. Then we take the orthogonal complement to obtain the decomposition \((2)\). We observe that any $g \in \text{Stab}_{G_2}(W)$ also preserves the subspace $\Lambda^2 W \subset V$, hence the decomposition \((2)\). Moreover, since $g(\omega_0) = \omega_0$, we have $g \in \text{SL}_3 = \text{SL}(W)$. Hence $\text{Stab}_{G_2}(W) \subset \text{SL}_3$. On the other hand we consider the action of $G_2$ on the Grassmannian of isotropic subspaces $W \subset V$, which is of dimension 6. Hence $\dim \text{Stab}_{G_2}(W) \geq 8$, which leads to the equality $\text{Stab}_{G_2}(W) = \text{SL}_3$. \(\square\)

- $\text{SO}_4 \subset G_2$. We need to recall some basic facts on quaternions and octonions. We begin by recalling that the complex octonion algebra $\mathbb{O}$ is generated as $\mathbb{C}$-vector space by the 8 basis vectors $e_0 = 1, e_1, \ldots, e_7$ satisfying the relations given by the Fano plane (see e.g. \cite{Ba}). Then the algebra $\mathbb{O}$ contains as a subalgebra the complex quaternion algebra $\mathbb{H} = \mathbb{C}1 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ and we have a vector space decomposition

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e_4.$$  

We recall that the subgroup $U = \{p \in \mathbb{H} \mid p\overline{p} = 1\}$ of unit quaternions can be identified with the complex Lie group $\text{SL}_2$ and that there is a surjective group homomorphism $\varphi : U \times U \to \text{SO}(\mathbb{H}) = \text{SO}_4$, $\varphi(p,q) = [x \mapsto px\overline{q}]$ with kernel $\mathbb{Z}/2$ generated by $(-1,-1)$. Using the decomposition \((3)\) we consider the map $\psi : U \times U \to \text{SO}(\mathbb{O})$, $\psi(p,q) = (\varphi(p,p), \varphi(p,q))$. One easily checks that $\text{im}\ \psi \subset G_2$ and that $\text{ker}\ \psi = \text{ker}\ \varphi$. This gives a realization of $\text{SO}_4$ as subgroup of $G_2$. We also note that the center $Z(\text{SO}_4)$ is generated by $\varphi(-1,1) = -\text{Id}_\mathbb{H}$ and that $\text{SO}_4$ is the centralizer of the element $\psi(-1,1) = (\text{Id}_\mathbb{H}, -\text{Id}_\mathbb{H}) \in G_2$ of order 2 (see \cite{BD}).

2.2. The moduli space $M(G_2)$ and the moduli stack $\mathcal{M}(G_2)$. Because of the equality $\text{Stab}_{\text{SL}(V)}(\omega) = G_2$, a principal $G_2$-bundle $E_{G_2}$ is equivalent to a rank-7 vector bundle $V$ with trivial determinant equipped with a non-degenerate alternating trilinear form $\eta : \Lambda^3 V \to \mathcal{O}_C$. The correspondence is given by sending $E_{G_2}$ to $(V = E_{G_2}(V), \eta)$ via the embedding $G_2 \subset \text{SL}(V)$. Moreover, it is shown in \cite{S} that $E_{G_2}$ is semi-stable if and only if $V$ is semi-stable. We therefore obtain a map between coarse moduli spaces of semi-stable bundles $M(G_2) \to M(\text{SL}_7)$.

Although the embeddings of $\text{SL}_3$ and $\text{SO}_4$ in $G_2$ are defined only up to conjugation, the induced maps between coarse moduli spaces of semi-stable principal bundles

$$i : M(\text{SL}_3) \to M(G_2) \quad \text{and} \quad j : M(\text{SL}_2) \times M(\text{SL}_2) \to M(\text{SO}_4) \to M(G_2)$$

are well-defined. We find it more convenient to work with the simply-connected group $\text{SL}_2 \times \text{SL}_2$, which is a double cover of the subgroup $\text{SO}_4$. Abusing notation we also denote by $i$ and $j$ their composites with the map $M(G_2) \to M(\text{SL}_7)$. It follows from the description of the subgroups $\text{SL}_3$ and $\text{SO}_4$ in the previous section that

$$i(E) = E \oplus E^* \oplus \mathcal{O}_C \quad \text{and} \quad j(F,G) = \text{End}_0(F) \oplus F \otimes G.$$
Here $E$ is an $\text{SL}_3$-bundle and $F$, $G$ are $\text{SL}_2$-bundles. Note that $i(E)$ and $j(F,G)$ are semi-stable if $E$, $F$ and $G$ are semi-stable.

**Remark:** It is shown in \cite{G} that the singular locus of the moduli space $\text{M}(G_2)$ coincides with the union of the images $i(\text{M}(\text{SL}_3)) \cup j(\text{M}(\text{SO}_4))$.

We also denote by $i$ and $j$ the maps between the corresponding moduli stacks. Let $\mathcal{L}_G$ denote the ample generator of the Picard group $\text{Pic}(\mathcal{M}(G))$ when $G$ is a simply-connected group.

**Lemma 2.2.** With the above notation we have

$$i^* \mathcal{L}_{G_2} = \mathcal{L}_{\text{SL}_3} \quad \text{and} \quad j^* \mathcal{L}_{G_2} = \mathcal{L}_{\text{SL}_2}^3 \otimes \mathcal{L}_{\text{SL}_2}.$$

**Proof.** This follows straightforwardly from a Dynkin index computation using the tables in \cite{LS}.

We consider the involution $\sigma : \mathcal{M}(\text{SL}_3) \rightarrow \mathcal{M}(\text{SL}_3)$ given by taking the dual $\sigma(E) = E^*$. Then the line bundle $\mathcal{L}_{\text{SL}_3}$ is invariant under the involution $\sigma$. We consider the linearisation $\sigma^* \mathcal{L}_{\text{SL}_3} \overset{\sim}{\rightarrow} \mathcal{L}_{\text{SL}_3}$ which restricts to the identity over the fixed points of $\sigma$ and denote by $H^0(\mathcal{M}(\text{SL}_3), \mathcal{L}_{\text{SL}_3})_+$ the subspace of invariant sections.

The group of 2-torsion line bundles $\text{JC}[2]$ acts on $\mathcal{M}(\text{SL}_2)$ by tensor product and the Mumford group $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$, a central extension of $\text{JC}[2]$, acts linearly on $H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2})$ with level 1. The $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$-representation $H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2}^3) \otimes H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2})$ is of level 4 and therefore admits a linear $\text{JC}[2]$-action.

**Proposition 2.3.** The induced maps between Verlinde spaces

$$i^* : H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) \longrightarrow H^0(\mathcal{M}(\text{SL}_3), \mathcal{L}_{\text{SL}_3})_+$$

$$j^* : H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) \longrightarrow \left[H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2}^3) \otimes H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2})\right]_0$$

take values in the subspace invariant under the involution $\sigma$ and the $\text{JC}[2]$-action respectively.

**Proof.** First we show that the map $i : \mathcal{M}(\text{SL}_3) \rightarrow \mathcal{M}(G_2)$ is $\sigma$-invariant. There is a natural inclusion between Weyl groups $W(\text{SL}_3) \subset W(G_2)$. Consider an element $g \in G_2$ which lifts an element in $W(G_2) \setminus W(\text{SL}_3)$. Then $g \notin \text{SL}_3$. As the subalgebra $\mathfrak{sl}_3$ of $\mathfrak{g}_2$ corresponds to the long roots and as $W(G)$ preserves the Cartan-Killing form, the inner automorphism $C(g)$ of $G_2$ induced by $g$ preserves the subgroup $\text{SL}_3$. The restriction of $C(g)$ to $\text{SL}_3$ is an outer automorphism, which permutes its two fundamental representations. It thus induces the involution $\sigma$ on the moduli stack $\mathcal{M}(\text{SL}_3)$. Since any inner automorphism of $G_2$ induces the identity on the moduli stack $\mathcal{M}(G_2)$, we obtain that $i$ is $\sigma$-invariant.

Since $i^* \mathcal{L}_{G_2} = \mathcal{L}_{\text{SL}_3}$ and since $i$ is $\sigma$-invariant, the line bundle $\mathcal{L}_{\text{SL}_3}$ carries a natural $\sigma$-linearisation, namely the one which restricts to the identity over fixed points of $\sigma$. It is now clear that $\text{im}(i^*) \subset H^0(\mathcal{M}(\text{SL}_3), \mathcal{L}_{\text{SL}_3})_+$.

The second statement follows immediately from the fact that $j$ is invariant under the diagonal $\text{JC}[2]$-action on the moduli stack $\mathcal{M}(\text{SL}_2) \times \mathcal{M}(\text{SL}_2)$. \qed
2.3. A family of divisors in $\mathbb{P}H^0(\mathcal{M}G_2, \mathcal{L}_{G_2})$. Let $\theta(C)$ resp. $\theta^+(C)$ denote the set of theta-characteristics resp. even theta-characteristics over the curve $C$. The moduli stack $\mathcal{M}(\text{SO}_7)$ has two connected components $\mathcal{M}^+(\text{SO}_7)$ and $\mathcal{M}^-(\text{SO}_7)$ distinguished by the second Stiefel-Whitney class. Since $\mathcal{M}(G_2)$ is connected, the homomorphism $G_2 \subset \text{SO}_7$ induces a map

$$\rho : \mathcal{M}(G_2) \rightarrow \mathcal{M}^+(\text{SO}_7).$$

For each $\kappa \in \theta(C)$ we introduce the Pfaffian line bundle $\mathcal{P}_\kappa$ over $\mathcal{M}^+(\text{SO}_7)$ (see e.g. [BLS] section 5). We have

$$\rho^* \mathcal{P}_\kappa = \mathcal{L}_{G_2}.$$

Moreover, for $\kappa \in \theta^+(C)$, there exists a Cartier divisor $\Delta_\kappa \in \mathbb{P}H^0(\mathcal{M}^+(\text{SO}_7), \mathcal{P}_\kappa)$ with support

$$\text{supp}(\Delta_\kappa) = \{ E \in \mathcal{M}^+(\text{SO}_7) \mid \dim H^0(C, E(\mathbb{C}^7) \otimes \kappa) > 0 \},$$

where $E(\mathbb{C}^7)$ denotes the rank-7 vector bundle associated to $E$. We also denote by $\Delta_\kappa \in \mathbb{P}H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2})$ the pull-back $\rho^*(\Delta_\kappa)$ to $\mathcal{M}(G_2)$. We will show later (Corollary 3.2) that the family of divisors $\{\Delta_\kappa\}_{\kappa \in \theta^+(C)}$ spans the linear system $\mathbb{P}H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2})$. Abusing notation we also write $\Delta_\kappa$ for a section of $H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2})$ vanishing at the divisor $\Delta_\kappa$.

2.4. Verlinde spaces for $\text{SL}_2$ at level 1, 2 and 3. We denote $V_n = H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}_{\text{SL}_2}^\otimes n)$ for $n \geq 1$. We now review some results from [B2] describing special bases of the vector spaces $V_1 \otimes V_1$ and $V_2$.

First we recall that the Mumford group $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$ acts linearly on the space $V_n$ with level $n$, i.e., the center $\mathbb{C}^* \subset \mathcal{G}(\mathcal{L}_{\text{SL}_2})$ acts via $\lambda \mapsto \lambda^n$. For $n$ odd, there exists a unique (up to isomorphism) irreducible $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$-module $H_n$ of level $n$. Note that $\dim H_n = 2^n$. If $n$ is divisible by 4, any $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$-module $Z$ of level $n$ admits a linear $JC[2]$-action. We denote by $Z_0$ the $JC[2]$-invariant subspace of $Z$.

We now list the results needed in the proof of Theorem II.

**Lemma 2.4.** We have

$$\dim (V_1 \otimes V_3)_0 = \frac{1}{|JC[2]|} \dim V_1 \otimes V_3.$$ 

**Proof.** By the general representation theory of Heisenberg groups, the $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$-module $V_1 \otimes V_3$ decomposes into a direct sum of factors, which are all isomorphic to $H_1 \otimes H_3$. It is then straightforward to show that the space of $JC[2]$-invariants $(H_1 \otimes H_3)_0$ is 1-dimensional. \hfill \Box

**Proposition 2.5** ([B2]). The two $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$-modules $V_1 \otimes V_1$ and $V_2$ of level 2 decompose as direct sum of 1-dimensional character spaces for $\mathcal{G}(\mathcal{L}_{\text{SL}_2})$

$$V_1 \otimes V_1 = \bigoplus_{\kappa \in \theta(C)} \mathbb{C}\xi_\kappa, \quad V_2 = \bigoplus_{\kappa \in \theta^+(C)} \mathbb{C}d_\kappa.$$ 

The supports of the zero divisors $Z(d_\kappa)$ and $Z(\xi_\kappa)$ equal

$$\text{supp } Z(d_\kappa) = \{ E \in \mathcal{M}(\text{SL}_2) \mid \dim H^0(C, \text{End}_0(E) \otimes \kappa) > 0 \},$$

$$\text{supp } Z(\xi_\kappa) = \{ (E, F) \in \mathcal{M}(\text{SL}_2) \times \mathcal{M}(\text{SL}_2) \mid \dim H^0(C, E \otimes F \otimes \kappa) > 0 \}.$$
Moreover, if $C$ has no vanishing theta-null, $\xi_\kappa$ is mapped to $d_\kappa$ by the multiplication map $V_1 \otimes V_1 \rightarrow V_2$.

**Proposition 2.6 (A).** For a general curve the multiplication map of global sections

$$\mu : V_1 \otimes V_2 \rightarrow V_3$$

is surjective.

3. Proof of the Main Theorems

In this section we give the proof of the two theorems and of the corollary stated in the introduction.

3.1. **Theorem I.** The first step is to show that the two spaces appearing at both ends of the map $j^*$ have the same dimension. The dimension of the space on the RHS is computed by Lemma 2.4. The statement then follows from (1) and from the equalities $\dim V_1 = 2^g$ and $|JC[2]| = 2^{2g}$.

The next step is to show that $j^*$ is surjective for a general curve, which will imply by the first step that $j^*$ is an isomorphism for a general curve. We consider the following map

$$\alpha : V_1 \otimes V_1 \otimes V_2 \rightarrow V_1 \otimes V_3, \quad u \otimes v \otimes w \mapsto u \otimes \mu(v \otimes w),$$

where $\mu$ is the multiplication map introduced in Proposition 2.6. By Proposition 2.6 $\alpha$ is surjective for a general curve, hence its restriction to the subspace of $JC[2]$-invariant sections $\alpha_0 : (V_1 \otimes V_1 \otimes V_2)_0 \rightarrow (V_1 \otimes V_3)_0$ remains surjective. Moreover, one easily works out that the family of tensors $\{\xi_\kappa \otimes d_\kappa\}_{\kappa \in \theta^+(C)}$ forms a basis of $(V_1 \otimes V_1 \otimes V_2)_0$.

We will use the family of divisors $\{\Delta_\kappa\}_{\kappa \in \theta^+(C)}$ introduced in section 2.3.

**Lemma 3.1.** For all $\kappa \in \theta^+(C)$ we have the equality (up to a scalar)

$$j^*(\Delta_\kappa) = \alpha_0(\xi_\kappa \otimes d_\kappa).$$

**Proof.** Using the description of $j$ given in (1) and the description of the divisors $Z(d_\kappa)$ and $Z(\xi_\kappa)$ given in Proposition 2.5 we obtain the following decomposition as divisor in $\mathcal{M}(SL_2) \times \mathcal{M}(SL_2)$

$$j^*(\Delta_\kappa) = pr_1^*Z(d_\kappa) + Z(\xi_\kappa),$$

where $pr_1$ is the projection onto the first factor. This shows the lemma. \qed

Surjectivity (for a general curve) now follows as follows: since $\{\xi_\kappa \otimes d_\kappa\}_{\kappa \in \theta^+(C)}$ forms a basis of $(V_1 \otimes V_1 \otimes V_2)_0$ and since $\alpha_0$ is surjective, we see by Lemma 3.1 that the family $\{j^*(\Delta_\kappa)\}_{\kappa \in \theta^+(C)}$ generates $(V_1 \otimes V_3)_0$.

Finally, we complete the proof by showing that $j^*$ is an isomorphism for every smooth curve. We use the identification [LS] for any semi-simple, simply-connected complex Lie group $G$ of the Verlinde space $H^0(\mathcal{M}(G), L_0^{\otimes l})$ with the space of conformal blocks $V_1^*(\mathfrak{g})$ at level $l$, where $\mathfrak{g}$ is the Lie algebra of $G$, for the two cases $G = G_2$ and $G = SL_2 \times SL_2$. Then, [Be] Proposition 5.2 shows...
functoriality of the above isomorphism under group extensions. In our case \( SL_2 \times SL_2 \rightarrow G_2 \) the linear map \( j^* \) can therfore be identified with the natural map

\[
\beta_C : \mathcal{V}_{\mathbf{1}}^*(\mathfrak{g}_2) \longrightarrow \mathcal{V}_{\mathbf{3}}^*(\mathfrak{sl}_2) \otimes \mathcal{V}_{\mathbf{1}}^*(\mathfrak{sl}_2).
\]

We can define this linear map for a family of smooth curves \( \pi : C \rightarrow S \): by \cite{I1} there exist vector bundles of conformal blocks over the base \( S \) together with a homomorphism \( \beta \), which specializes over a point \( s \in S \) to the linear map \( \beta_{\pi^{-1}(s)} \). These vector bundles are equipped with flat projective connections, the so-called WZW connections.

We now observe that the Lie algebra embedding \( \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \subset \mathfrak{g}_2 \) is conformal — by direct computation. Then, by \cite{Be} Proposition 5.8, we obtain that the map \( \beta \) is projectively flat for the two WZW connections, hence its rank is constant in the family \( \pi : C \rightarrow S \). Since, by the previous step, \( \beta_C \) is injective for a general curve \( C \) (note that we do not take \( JC[2] \)-invariants on the conformal blocks), we conclude that \( \beta \) is injective for any smooth curve. Hence \( j^* \) is an isomorphism for any curve. This completes the proof.

From the above proof we immediately deduce the

**Corollary 3.2.** For a general curve the family \( \{ \Delta_\kappa \}_{\kappa \in \Theta^+(C)} \) linearly spans \( \mathbb{P}H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) \).

**Remark:** Note that Hitchin’s connection \cite{HI} is only defined on the vector bundle with fiber \( H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}^\mathbf{2}) \). We only get a connection for \( G_2 \) at level 1 via the isomorphism with the bundle of conformal blocks.

### 3.2. Theorem II.

We consider the family of divisors \( \{ \Delta_\kappa \}_{\kappa \in \Theta^+(C)} \) of \( \mathbb{P}H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2}) \) introduced in section 2.3. A straightforward computation shows that \( i^*(\Delta_\kappa) = H_\kappa \), where \( H_\kappa \in \mathbb{P}H^0(\mathcal{M}(SL_3), \mathcal{L})_+ \) is the divisor with support

\[
\text{supp}(H_\kappa) = \{ E \in \mathcal{M}(SL_3) \mid \dim H^0(C, E \otimes \kappa) > 0 \}.
\]

Therefore, in order to show surjectivity of \( i^* \), it is enough to show that the family \( \{ H_\kappa \}_{\kappa \in \Theta^+(C)} \) linearly spans \( \mathbb{P}H^0(\mathcal{M}(SL_3), \mathcal{L})_+ \). This is done as follows.

We introduce the Riemann Theta divisor \( \Theta = \{ L \in \text{Pic}^{g-1}(C) \mid \dim H^0(C, L) > 0 \} \) in the Picard variety \( \text{Pic}^{g-1}(C) \) parameterizing degree \( g - 1 \) line bundles over \( C \). We recall \cite{BNR} that there is a canonical isomorphism

\[
H^0(\text{Pic}^{g-1}(C), 3\Theta)^* \xrightarrow{\sim} H^0(\mathcal{M}(SL_3), \mathcal{L}),
\]

which is invariant for the two involutions, i.e. \( L \mapsto K_C \otimes L^{-1} \) on \( \text{Pic}^{g-1}(C) \) and \( \sigma \) on \( \mathcal{M}(SL_3) \) respectively. We thus obtain an isomorphism between subspaces of invariant divisors \( |3\Theta|^*_+ \cong \mathbb{P}H^0(\mathcal{M}(SL_3), \mathcal{L})_+ \). Is is easy to check that via this isomorphism \( H_\kappa = \varphi_{3\Theta}(\kappa) \), where

\[
\varphi_{3\Theta} : \text{Pic}^{g-1}(C) \longrightarrow |3\Theta|^*_+
\]

is the rational map given by the linear system \( |3\Theta|^*_+ \). In order to show that the family of points \( \{ \varphi_{3\Theta}(\kappa) \}_{\kappa \in \Theta^+(C)} \) linearly spans \( |3\Theta|^*_+ \), we factorize the map \( \varphi_{3\Theta} \) as

\[
\varphi_{4\Theta} : \text{Pic}^{g-1}(C) \longrightarrow |4\Theta|^*_+ \longrightarrow |3\Theta|^*_+,
\]
where the first map is the rational map given by the linear system $|4\Theta|^*_+$ and the second is the projection induced by the inclusion $H^0(3\Theta)_+ \to H^0(4\Theta)_+$. The result then follows from the main statement in [KPS] saying that $\{\varphi_{4\Theta}(\kappa)\}_{\kappa \in \theta^+(C)}$ is a projective basis of $|4\Theta|^*_+$ if $C$ has no vanishing theta-null.

**Remark:** For a curve of genus 2 we observe that both spaces have the same dimension, hence $i^*$ is an isomorphism in that case — note that any genus-2 curve is without vanishing theta-null.

3.3. **Corollary.** The statement of the corollary is proved in [LPS] for the conformal block $V^*_3(sl_2) = H^0(\mathcal{M}(SL_2), \mathcal{L}_{SL_2}^{23})$. Having already observed in the proof of Theorem I that the vector bundle map $i$ is projectively flat for the WZW-connections, it suffices to show the statement for the $JC[2]$-invariants of $V^*_3(sl_2) \otimes V^*_1(sl_2)$. This follows from [Be] Corollary 4.2.

4. **Some remarks**

Here we collect some additional computations.

4.1. **Verlinde formula for $l = 2$ and $g = 2$.** We just record the computation of the Verlinde number $\dim H^0(\mathcal{M}(G_2), \mathcal{L}^2) = 30$. Since the line bundle $\mathcal{L}^2$ descends to the coarse moduli space $\mathcal{M}(G_2)$ we obtain a rational $\theta$-map

$$\theta : \mathcal{M}(G_2) \longrightarrow |\mathcal{L}^2|^* = \mathbb{P}^{29}.$$ 

We refer to the paper [B4] for some results on the $\theta$-map on a genus-2 curve for vector bundles of small rank.

4.2. **Analogue for the exceptional group $F_4$.** There is a similar coincidence for the conformal embedding of Lie algebras $\mathfrak{sl}(2) \oplus \mathfrak{sp}(6) \subset \mathfrak{f}_4$. In fact, we observe that $\dim H^0(\mathcal{M}(F_4), \mathcal{L}_{F_4}) = \dim H^0(\mathcal{M}(G_2), \mathcal{L}_{G_2})$ and that $\dim H^0(\mathcal{M}(Sp_6), \mathcal{L}_{Sp_6}) = \dim H^0(\mathcal{M}(SL_2), \mathcal{L}_{SL_2}^{23})$ (symplectic strange duality). Moreover, $\ker(SL_2 \times Sp_6 \rightarrow F_4) = \mathbb{Z}/2$. These facts suggest a similar isomorphism for the Verlinde space $H^0(\mathcal{M}(F_4), \mathcal{L}_{F_4})$, but the method presented in this paper does to apply to that case.

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