Global wellposedness and scattering for the defocusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions $d \geq 9$

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Abstract

We consider the defocusing energy-critical nonlinear Schrödinger equation of fourth order $iu_t + \Delta^2 u = -|u|^8_d u - 4 u$. We prove that any finite energy solution is global and scatters both forward and backward in time in dimensions $d \geq 9$.

Key Words: Defocusing, Energy-critical, Fourth order Schrödinger equations, Global well-posedness, Scattering.

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1 Introduction

In this paper, we will investigate the defocusing energy-critical Schrödinger equation of fourth order, namely,

$$\begin{cases}
    iu_t + \Delta^2 u = -|u|^8_d u, & \text{in } \mathbb{R}^d \times \mathbb{R}, \\
    u(0) = u_0(x), & \text{in } \mathbb{R}^d.
\end{cases} \tag{1}$$

The name ‘energy-critical’ refers to the fact that the scaling symmetry

$$u(t, x) \mapsto u_\lambda(t, x) := \lambda^{\frac{4}{d-4}} u(\lambda^4 t, \lambda x)$$

leaves both the equation and the energy invariant. The energy of a solution is defined by

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\Delta u(t, x)|^2 dx + \frac{d-4}{2d} \int_{\mathbb{R}^d} |u(t, x)|^{\frac{2d}{d-4}} dx$$

and is conserved under the flow. We refer to the Laplacian term in the formula above as the kinetic energy and to the second term as the potential energy.
Definition 1.1 (Solutions.). A function \( u : I \times \mathbb{R}^d \to \mathbb{C} \) on a non-empty time interval \( t_0 \in I \subset \mathbb{R} \) is a solution to (1) if it lies in the class \( \mathcal{C}_t^0 \tilde{H}_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{\frac{2(d+4)}{d-4}}(K \times \mathbb{R}^d) \) for all compact \( K \subset I \), and obeys the Duhamel formula

\[
    u(t) = e^{i(t-t_0)\Delta^2}u(t_0) + i \int_{t_0}^t e^{i(t-\tau)\Delta^2} F(u(\tau)) d\tau
\]

for all \( t \in I \), where \( F(u) = |u|^{\frac{d}{d-4}}u \). We refer to \( I \) as the lifespan of \( u \). We say that \( u \) is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that \( u \) is a global solution if \( I = \mathbb{R} \).

Conjecture 1.1. Let \( d \geq 5 \) and let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a solution to (1) with finite energy \( E \), then \( I = \mathbb{R} \) and

\[
    \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+4)}{d-4}} dxdt \leq C(E) < \infty.
\]

This conjecture has been verified for radial data by B. Pausader [22]. In this paper, we will verify this conjecture for general data in dimensions \( d \geq 9 \). In fact, we establish the following theorem:

Theorem 1.1. Let \( d \geq 9 \) and let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a solution to (1) with finite energy \( E \), then \( I = \mathbb{R} \) and

\[
    \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+4)}{d-4}} dxdt \leq C(E) < \infty.
\]

The ideas and techniques for fourth order nonlinear Schrödinger equations come from the study of classical nonlinear Schrödinger equations. For the energy critical nonlinear Schrödinger equations

\[
    \begin{array}{l}
    iu_t + \Delta u = \lambda |u|^{\frac{d}{2-4}}u, \quad \text{in} \ \mathbb{R}^d \times \mathbb{R}, \\
    u(0) = u_0(x), \quad \text{in} \ \mathbb{R}^d,
    \end{array}
\]

(2)

the local well-posedness and global well-posedness for small data were established by T. Cazenave and F. B. Weissler [4] regardless of the sign of \( \lambda \). The global well-posedness and scattering for large data have been extensively studied.

For the defocusing case \( \lambda = +1 \), J. Bourgain proved global well-posedness and scattering for radial solution in dimensions three and four in [3], with the “induction on energy” strategy he invented. Subsequently, G. Grillakis [9] gave a different argument which recovered part of [3], namely, global existence from smooth, radial, finite energy data. Later on, T. Tao [26] generalized the results of Bourgain to any dimension \( d \geq 3 \) and got bounds on various spacetime norms of the solution which are exponential type in the energy, which improved Bourgain’s tower type bounds. J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao [6] established global well-posedness and scattering for solutions in energy space in dimension three. The method is similar in spirit to the induction on energy strategy of Bourgain, but they performed the induction analysis
in both frequency space and physical space simultaneously, and replaced the Morawetz inequality by an interaction Morawetz estimate. The principle advantage of the interaction Morawetz estimate is that it is not localized in spatial origin and so is better able to handle nonradial solutions. E. Ryckman and M. Visan extended this results to dimensions four and higher in [24], [29].

A new and efficient approach to the energy-critical nonlinear Schrödinger equations was introduced by C. E. Kenig and F. Merle [10], where they obtained global well-posedness and scattering for radial data with energy and kinetic energy less than those of ground state in the focusing case in dimensions $3 \leq d \leq 5$. Their arguments work equally well for the defocusing case. They employed a (concentration) compactness technique in place of previous localization arguments. They reduced matters to a rigidity theorem using a concentration compactness argument, with the aid of localized Virial identity. The radiality enters only at one point in the proof of the rigidity theorem because of the difficulty in controlling the motion of spatial translation of global solutions. Earlier steps in this direction include [1], [2], [13], [14] and [19]. R. Killip and M. Visan [16] improved this results to general solutions in $d \geq 5$. The method is to reduce minimal kinetic energy blow up solutions to almost periodic solutions modulo symmetries, which match one of the three scenarios: finite time blowup, low-to-high cascade and soliton. Then the aim is to eliminate such solutions. The finite time blowup solutions can be precluded using the method in [10]. For the other two types of solutions, R. Killip and M. Visan proved that they admit additional regularities, namely, they belong to $\dot{H}^{-\epsilon}_x$ for some $\epsilon > 0$. In particular, they are in $L^2_x$. Similar ideas have appeared in [15], [16] and [17] when dealing with mass-critical nonlinear Schrödinger equations. But a remarkable difficulty comes from the minimal kinetic energy blowup solution because the kinetic energy, unlike the energy, is not conserved. Related arguments (for the cubic NLS in dimension three) appear in [12]. The low-to-high cascade can be precluded by negative regularity and the conservation of mass. It remains to preclude the soliton. In this case, one need to control the motion of spatial center function of the soliton solution, which can be obtained by using the method from [8] and [11] and the negative regularity. The fist step is to note that a minimal kinetic energy blowup solution with finite mass must have zero momentum. A second ingredient needed to control the motion of spatial center function is a compactness property of the orbit of $\{u(t)\}$ in $L^2_x$. The argument from [8] gives that the spatial translation is $o(t)$ instead of $O(t)$ given by simple argument as $t \to \infty$. Finally the soliton-like solution is precluded by using a truncated Virial identity. However, the negative regularity in [16] cannot be obtained in dimensions three and four because the dispersion is too weak.

**Definition 1.2 (Symmetry group).** For any phase $\theta \in \mathbb{R}/2\pi \mathbb{Z}$, position $x_0 \in \mathbb{R}^d$ and scaling parameter $\lambda > 0$, we define the unitary transformation $g_{\theta,x_0,\lambda} : \dot{H}^2(\mathbb{R}^d) \to \dot{H}^2(\mathbb{R}^d)$ by the formula

$$[g_{\theta,x_0,\lambda}f](x) := \lambda^{-\frac{d+4}{2}} e^{i\theta} f(\lambda^{-1}(x-x_0)).$$

We let $G$ be the collection of such transformations. If $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a function, we define $T_{g_{\theta,x_0,\lambda}}u : \lambda^4 I \times \mathbb{R}^d \to \mathbb{C}$ where $\lambda^4 I := \{\lambda^4 t : t \in I\}$ by the formula

$$[T_{g_{\theta,x_0,\lambda}}u](t, x) := \lambda^{-\frac{d+4}{2}} e^{i\theta} u(\lambda^{-4}t, \lambda^{-1}(x-x_0)).$$
Definition 1.3 (Almost periodic solutions). Let $d \geq 5$. A solution $u$ to (1) with lifespan $I$ is said to be almost periodic modulo $G$ if there exist functions $N : I \rightarrow \mathbb{R}^+$, $x : I \rightarrow \mathbb{R}^d$ and $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $t \in I$, and $\eta > 0$,

$$\int_{|x-x(t)| \geq C(\eta)/N(t)} |\Delta u(t, x)|^2 dx \leq \eta$$

(3)

and

$$\int_{|\xi| \geq C(\eta)N(t)} |\hat{\xi}^4| \hat{u}(t, \xi)|^2 d\xi \leq \eta.$$

(4)

We refer to the function $N$ as the frequency scale function for the solution $u$, $x$ the spatial center function, and to $C$ as the compactness modulus function.

By the Ascoli-Arzela theorem, a family of functions is precompact in $\dot{H}^2_x$ if and only if it is norm-bounded and there exists a compactness modulus function $C$ so that

$$\int_{|x| \geq C(\eta)} |\Delta f(x)|^2 dx + \int_{|\xi| \geq C(\eta)} |\xi|^4 |\hat{f}(\xi)|^2 d\xi \leq \eta$$

for all functions $f$ in the family. By Sobolev embedding, any solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ that is almost periodic modulo $G$ must also satisfy

$$\int_{|x-x(t)| \geq C(\eta)/N(t)} |u(t, x)|^{2d} dx \leq \eta.$$  

(5)

Remark 1.1. By Ascoli-Arzela theorem, the above definition is equivalent to either of the following two statements:

1. The quotient orbit $\{Gu(t) : t \in I\}$ is a precompact set of $G \setminus \dot{H}^2$, where $G \setminus \dot{H}^2$ is the moduli space of $G$-orbits $G\hat{f} := \{gf : g \in G\}$ of $\dot{H}^2(\mathbb{R}^d)$.

2. There exists a compact subset $K$ of $\dot{H}^2$ such that $u(t) \in GK$ for all $t \in I$; equivalently there exists a group function $g : I \rightarrow G$ and a compact subset $K$ such that $g^{-1}(t)u(t) \in K$ for any $t \in I$.

Remark 1.2. A further consequence of compactness modulo $G$ is the existence of a function $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ so that

$$\int_{|x-x(t)| \leq c(\eta)/N(t)} |\Delta u(t, x)|^2 dx + \int_{|\xi| \leq c(\eta)N(t)} |\xi|^4 |\hat{u}(t, \xi)|^2 d\xi \leq \eta$$

(6)

for all $t \in I$ and $\eta > 0$.

In fact, since $K$ is compact in $\dot{H}^2(\mathbb{R}^d)$, there exists $c(\eta)$ such that

$$\sup_{f \in K} \int_{|x| < c(\eta)} |\Delta f|^2 dx < \eta.$$
Thus
\[
\int_{|x-x(t)| \leq c(\eta)/N(t)} |\Delta u(t, x)|^2 dx = \int_{|x| < c(\eta)} |\Delta g^{-1}(t)u(t)|^2 dx < \sup_{f \in K} \int_{|x| < c(\eta)} |\Delta f|^2 dx < \eta.
\]

We can prove similarly that there exists \(c(\eta)\) such that
\[
\int_{|\xi| \leq c(\eta)N(t)} |\xi|^4 |\hat{u}(t, \xi)|^2 d\xi \leq \eta.
\]

In [20], we have made a lot of preparations including the following two theorems:

**Theorem 1.2** (Reduction to almost periodic solutions, [20]). Suppose \(d \geq 5\) is such that Conjecture [7] failed. Then there exists a maximal-lifespan solution \(u : \mathbb{I} \times \mathbb{R}^d \to \mathbb{C}\) to (1) such that \(E(u) < \infty\), \(u\) is almost periodic modulo \(G\), and \(u\) blows up both forward and backward in time. Moreover, \(u\) has minimal kinetic energy among all blowup solutions, that is
\[
\sup_{t \in \mathbb{I}} \|\Delta u(t)\|_{L^2} < \sup_{t \in J} \|\Delta v(t)\|_{L^2}
\]
for all maximal-lifespan solutions \(v : J \times \mathbb{R}^d \to \mathbb{C}\) that blowup at least one time direction.

**Theorem 1.3** (Three special scenarios for blowup, [20]). Fix \(d \geq 5\) and suppose that Conjecture [7] fails for this choice of \(d\). Then there exists a minimal kinetic energy, maximal-lifespan solution \(u : \mathbb{I} \times \mathbb{R}^d \to \mathbb{C}\), which is almost periodic modulo symmetries, and obeys
\[
S_I(u) = \int_{\mathbb{I}} \int_{\mathbb{R}^d} |u(t, x)|^{2(d+4)/(d-4)} dx dt = \infty, \quad E(u) < \infty.
\]

We can also ensure that the lifespan \(\mathbb{I}\) and the frequency scale function \(N : \mathbb{I} \to \mathbb{R}^+\) match one of the following three scenarios:

I. (Finite time blowup.) We have that either \(\inf I| < \infty\) or \(\sup I < \infty\).

II. (Soliton-like solution.) We have \(\mathbb{I} = \mathbb{R}\) and
\[
N(t) = 1, \quad \text{for all } t \in \mathbb{R}.
\]

III. (Low-to-high frequency cascade.) We have \(\mathbb{I} = \mathbb{R}\) and
\[
\inf_{t \in \mathbb{R}} N(t) \geq 1, \quad \text{and} \quad \limsup_{t \to \infty} N(t) = \infty.
\]

This paper is devoted to precluding the existence of solutions that satisfy the criteria in Theorem 1.3. The argument here is a direct “fourth order” analogue of that in [16]. The key step in all three scenarios above is to prove additional regularity, that is, the solution \(u\) lies in \(L^2\) or better. The finite time blow up can be precluded using the method of C. E. Kenig, F. Merle [10], that is, we prove that the \(L^2\) norm of \(u(t)\) converges to
zero as $t$ approaches the finite endpoint. Since mass is conserved, this implies that $u$ is identically zero. To preclude the the other two types, we will prove that they have negative regularities. This is achieved in two stages. First, we prove that the solution belongs to $L^\infty_t L^p_x$ for certain values of $p$ less than $2d/(d-4)$. The second step is to upgrade the decay proved in the first step to $L^2_x$-based spaces. Thus we can preclude the low-to-high frequency cascade by negative regularity and the conservation of mass.

To preclude the soliton-like solutions, we adapt a different argument from [16] because no Galilean type transformation is known for the nonlinear Schrödinger equations of fourth order. We first prove that the $L^p_x (1 < p < \infty)$ norm of soliton solution is bounded from below. In fact, we can see from the proof that this is true for any almost periodic solutions. Next, using the negative regularity for the soliton solution, we derived an interaction Morawetz estimate. The interaction Morawetz estimate holds only for soliton (and low-to-high cascade) instead of all actual solutions here. Moreover, we needn’t localize the soliton solution in either physics or frequency space as in [9] because it belongs to $L^\infty_t H^2_x$. Finally we prove that some spacetime norm of the soliton is infinity, which contradicts the spacetime bound obtained from the interaction Morawetz estimate. In addition, this argument can be applied to other defocusing Schrödinger-type equations to preclude the soliton-like solution once one prove that such solution admits sufficient regularity.

At last, we will mention that the defocusing assumption is only used in precluding the soliton. So the negative regularity for low-to-high cascade and soliton remains true in focusing case. If one has the Galilean type transformation, then the global well-posedness and scattering for focusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions $d \geq 9$ can probably be solved using the method in [16]. The dimension restriction appears in the proof of the negative regularity because the dispersion is not strong enough to perform the double Duhamel trick.

After the paper was finished, we learned that B. Pausader [23] has obtained independently similar result in dimension $d = 8$ and the high dimensional results can also be obtained using his method.

The rest of the paper is organized as follows: In Section 2, we introduce some notations and preliminaries. Section 3 is devoted to deriving a very important property of almost periodic solutions: double Duhamel formula. In Section 4, we preclude the finite time blow up solutions. In Section 5, we prove the negative regularity for low-to-high cascade and soliton. In Section 6, we preclude the low-to-high cascade and in Section 7, we kill the soliton.

2 Notations and preliminaries

We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \lesssim CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity $X$ such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. The fact that these constants depend upon the dimension $d$ will be
suppressed. If $C$ depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some $C_u$ depending on $u$; similarly for $X \sim_u Y$, $X = O_u(Y)$, etc. We denote by $X_\pm$ any quantity of the form $X \pm \varepsilon$ for any $\varepsilon > 0$. Throughout this paper, we denote $\frac{2d}{d-4}$ by $2^\#$.

For any spacetime slab $I \times \mathbb{R}^d$, we use $L^q_t L^r_x(I \times \mathbb{R}^d)$ to denote the Banach space of functions $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ whose norm is

$$\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} := \left( \int_I \|u(t)\|_{L^r_x}^q \right)^{\frac{1}{q}} < \infty,$$

with the usual modifications when $q$ or $r$ are equal to infinity. When $q = r$ we abbreviate $L^q_t L^q_x$ as $L^q_{t,x}$.

We define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

For $s \in \mathbb{R}$, we define the fractional differentiation/integral operator

$$|\nabla|^s f(\xi) := |\xi|^s \hat{f}(\xi),$$

which in turn defines the homogeneous Sobolev norm

$$\|f\|_{H^s(\mathbb{R}^d)} := |||\nabla|^s f|||_{L^2(\mathbb{R}^d)}.$$

We recall some basic facts about Littlewood-Paley theory. Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\widehat{P_{\leq N}} f(\xi) := \varphi(\xi/N) \hat{f}(\xi),$$
$$\widehat{P_{> N}} f(\xi) := (1 - \varphi(\xi/N)) \hat{f}(\xi),$$
$$\widehat{P_N} f(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \hat{f}(\xi)$$

and similarly $P_{< N}$ and $P_{\geq N}$. We also define

$$P_{M < \cdot \leq N} := P_{\leq N} - P_{\leq M} = \sum_{M < N' \leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when $M$ and $N$ are dyadic numbers; in particular, all summations over $N$ or $M$ are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow $M$ and $N$ to not be a power of 2. Note that $P_N$ is not truly a projection; to get around this, we will occasionally need to use the fattened Littlewood-Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \quad (7)$$
They obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

As all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta^2}$, as well as with the differential operators such as $i\partial_t + \Delta^2$. We will use the basic properties of these operators many times, including

**Lemma 2.1** (Bernstein estimates). For $1 \leq p \leq q \leq \infty$,

\[
\|\nabla|^{\pm s} P_N f\|_{L_p^q(\mathbb{R}^d)} \sim N^{\pm s} \|P_N f\|_{L_p^q(\mathbb{R}^d)},
\]

\[
\|P_{\leq N} f\|_{L_p^q(\mathbb{R}^d)} \lesssim N^{d/p - d/q} \|P_{\leq N} f\|_{L_p^q(\mathbb{R}^d)},
\]

\[
\|P_N f\|_{L_p^q(\mathbb{R}^d)} \lesssim N^{d/p - d/q} \|P_N f\|_{L_p^q(\mathbb{R}^d)}.
\]

We also need the following fractional chain rule [5]:

**Lemma 2.2** (Fractional chain rule, [5]). Suppose $G \in C^1(\mathbb{C})$, $s \in (0,1]$ and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then,

\[
\|\nabla|^{s} G(u)\|_p \lesssim \|G'(u)\|_{p_1} \|\nabla|^{s} u\|_{p_2}.
\]

Another tool we will use is a form of Gronwall’s inequality that involves both the past and the future, ‘causal’ in the terminology of [27].

**Lemma 2.3** (A Gronwall inequality, [16]). Given $\gamma > 0$, $0 < \eta < \frac{1}{2}(1 - 2^{-\gamma})$, and $\{b_k\} \in \ell^\infty(\mathbb{Z}^+)$, let $x_k \in \ell^\infty(\mathbb{Z}^+)$ be a non-negative sequence obeying

\[
x_k \leq b_k + \eta \sum_{l=0}^{\infty} 2^{-\gamma |k-l|} x_l \text{ for all } k \geq 0.
\]

Then

\[
x_k \lesssim \sum_{l=0}^{k} r^{k-l} b_l \text{ for all } k \geq 0
\]

for some $r = r(\eta) \in (2^{-\gamma}, 1)$. Moreover, $r \downarrow 2^{-\gamma}$ as $\eta \downarrow 0$.

### 3 Double Duhamel formula

In this section, we prove the Double Duhamel formula. Similar formula has appeared in [28]. For completeness, we give the proof, see also [28].

**Lemma 3.1** (Double Duhamel formula). Let $u$ be an almost periodic solution to (1) on its maximal-lifespan $I$. Then, for all $t \in I$,

\[
u(t) = \lim_{T \sup I} \int_{t}^{T} e^{i(t-t')\Delta^2} F(u(t')) dt' = -\lim_{T \inf I} \int_{t}^{T} e^{i(t-t')\Delta^2} F(u(t')) dt',
\]

as weak limits in $H^2$. 

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Proof. Suppose $u$ be an almost periodic solution to (1) on its maximal-lifespan $I$, then we claim that $e^{-it\Delta^2} u(t)$ is weakly convergent in $\dot{H}^2(\mathbb{R}^d)$ to zero as $t \to \sup I$ or $t \to \inf I$.

We just prove the claim as $t \to \sup I$, as the other case is similar. By almost periodicity, we have a compact subset $K \in \dot{H}^2(\mathbb{R}^d)$ and group elements $g_{\theta(t),x_0(t),\lambda(t)} \in G$ for each $t \in I$ such that

$$g_{\theta(t),x_0(t),\lambda(t)}^{-1} u(t) \in K.$$  

Suppose first that $\sup I$ is finite, and thus $u$ exhibits forward blowup in finite time. By Corollary 4.10 in [20], we conclude that this forces $\lambda(t)$ to go to zero as $t \to \sup I$. Thus the operator $g_{\theta(t),x_0(t),\lambda(t)}$ are weakly convergent to zero. By the compactness of $K$, this implies

$$\lim_{t \to \sup I} \sup_{f \in K} |\langle \Delta g_{\theta(t),x_0(t),\lambda(t)} f, \Delta \phi \rangle|_{L^2(\mathbb{R}^d)} = 0$$

for all $\phi \in \dot{H}^2(\mathbb{R}^d)$. From this and (8), we see that $u(t)$ converges weakly to zero as $t \to \sup I$. Since $\sup I$ is finite and the propagator curve $t \mapsto e^{-it\Delta^2}$ is continuous in the strong operator topology, we see that $e^{-it\Delta^2} u(t)$ converges weakly to zero, as desired.

Now suppose instead that $\sup I$ is infinite. It will suffice to show that

$$\lim_{t \to +\infty} \langle \Delta e^{-it\Delta^2} u(t), \phi \rangle_{L^2_2(\mathbb{R}^d)} = 0$$

for all test functions $\phi \in C_0^\infty(\mathbb{R}^d)$. Applying (8) and duality, it suffices to show that

$$\lim_{t \to +\infty} \sup_{f \in K} |\langle \Delta g_{\theta(t),x_0(t),\lambda(t)} f, e^{it\Delta^2} \phi \rangle|_{L^2_2(\mathbb{R}^d)} = 0.$$  

By the compactness of $K$, it therefore suffices to show that

$$\lim_{t \to +\infty} |\langle \Delta g_{\theta(t),x_0(t),\lambda(t)} f, e^{it\Delta^2} \phi \rangle|_{L^2_2(\mathbb{R}^d)} = 0$$

for each $f \in \dot{H}^2$. But the claim follows from the stationary phase expansion of $e^{it\Delta^2} \phi$, the point being that $e^{it\Delta^2} \phi$ acquires a quartic phase oscillation as $t \to \infty$ which cannot be renormalized by any of the symmetries.

Now recall the Duhamel formula

$$u(t) = e^{it\Delta^2} e^{-it_+\Delta^2} u(t_+) + i \int_t^{t_+} e^{i(t-t')\Delta^2} F(u(t')) dt'$$

for any $t, t_+ \in I$. Letting $t_+$ converge to $\sup I$, then we conclude the backward Duhamel formula

$$u(t) = i \int_t^{\sup I} e^{i(t-t')\Delta^2} F(u(t')) dt',$$

where the improper integral is interpreted in a conditionally convergent sense in the weak topology, that is

$$\langle u(t), f \rangle = \lim_{t_+ \to \sup I} \langle \Delta i \int_t^{t_+} e^{i(t-t')\Delta^2} F(u(t')) dt', f \rangle_{L^2_2(\mathbb{R}^d)}$$
for all \( f \in L^2_x(\mathbb{R}^d) \). Similarly, we have the forward Duhamel formula

\[
  u(t) = -i \int_{\inf I}^{t} e^{i(t-t')\Delta} F(u(t')) dt'.
\]

\[\square\]

4 Finite time blow up

In this section we preclude scenario I in Theorem 1.3. The argument is essentially taken from [10], see also [16], [20].

**Theorem 4.1** (No finite-time blowup). Let \( d \geq 5 \). Then there are no maximal-lifespan solutions \( u : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) to (1) that are almost periodic modulo \( G \), obey

\[
  S_I(u) = \infty
\]

and are such that either \( |\inf I| < \infty \) or \( \sup I < \infty \).

**Proof.** Suppose for a contradiction that there existed such a solution \( u \). Without loss of generality, we may assume \( \sup I < \infty \). Then by Corollary 4.10 in [20],

\[
  \lim inf_{t \rightarrow \sup I} N(t) = \infty. \tag{10}
\]

We now show that this implies that

\[
  \lim sup_{t \rightarrow \sup I} \int_{|x| \leq R} |u(t,x)|^2 dx = 0 \text{ for all } R > 0. \tag{11}
\]

Indeed, let \( 0 < \eta < 1 \) and \( t \in I \). By Hölder, Sobolev embedding and energy conservation,

\[
  \int_{|x| \leq R} u(t,x)^2 dx \lesssim \int_{|x-x(t)| \leq \eta R} |u(t,x)|^2 dx + \int_{|x| \leq R \atop |x-x(t)| > \eta R} |u(t,x)|^2 dx \lesssim (\eta R)^4 \|u\|_{L^2(\mathbb{R}^d)}^2 + R^4 \left( \int_{|x-x(t)| > \eta R} |u(t,x)|^{2d/(d-4)} dx \right)^{\frac{d-4}{d}} \lesssim (\eta R)^4 E(u) + R^4 \left( \int_{|x-x(t)| > \eta R} |u(t,x)|^{2d/(d-4)} dx \right)^{\frac{d-4}{d}}.
\]

Letting \( \eta \rightarrow 0 \), we can make the first term on the right-hand side of the inequality above as small as we wish. On the other hand, by [10], almost periodicity and Remark 1.2, we see that

\[
  \lim sup_{t \rightarrow \sup I} \int_{|x-x(t)| > \eta R} |u(t,x)|^{\frac{2d}{d-4}} dx < \lim sup_{t \rightarrow \sup I} \int_{|x-x(t)| > C(\epsilon)/N(t)} |u(t,x)|^{\frac{2d}{d-4}} dx < \epsilon,
\]

for any \( \epsilon > 0 \). Thus

\[
  \lim sup_{t \rightarrow \sup I} \int_{|x-x(t)| > \eta R} |u(t,x)|^{\frac{2d}{d-4}} dx = 0.
\]
by the arbitrary of $\epsilon$.

For $t \in I$, define

$$M_R(t) := \int_{\mathbb{R}^d} \phi\left(\frac{|x|}{R}\right) |u(t, x)|^2 dx,$$

where $\phi$ is a smooth, radial function such that $\phi(r) = 1$ for $r \leq 1$ and $\phi = 0$ for $r \geq 2$. By (11),

$$\limsup_{t \nearrow \sup I} M_R(t) = 0 \quad \text{for all } R > 0. \quad (12)$$

On the other hand,

$$\partial_t M_R(t) = -2 \text{Im} \int \Delta \left(\phi\left(\frac{|x|}{R}\right)\right) \bar{u} \Delta u dx - 2 \text{Im} \int \nabla \left(\phi\left(\frac{|x|}{R}\right)\right) \cdot \nabla \bar{u} \Delta u dx.$$

So by Hölder and Hardy’s inequality, we have

$$|\partial_t M_R(t)| \lesssim \int_{|x| \sim R} \frac{|u||\Delta u|}{R^2} dx + \int_{|x| \sim R} \frac{||\nabla u||\Delta u|}{R} dx \lesssim \|u\|_2 ||\Delta u||_2 + \|\nabla u\|_2 ||\Delta u||_2 \lesssim E(u).$$

Thus,

$$M_R(t_1) = M_R(t_2) + \int_{t_2}^{t_1} \partial_t M_R(t) dt \lesssim M_R(t_2) + |t_1 - t_2| E(u)$$

for all $t_1, t_2 \in I$ and $R > 0$. Let $t_2 \nearrow \sup I$ and invoking (12), we have

$$M_R(t_1) \lesssim |\sup I - t_1| E(u).$$

Now letting $R \to \infty$ and using the conservation of mass, we obtain $u_0 \in L^2_\infty(\mathbb{R}^d)$. Finally, letting $t_1 \nearrow \sup I$, we deduce $u_0 = 0$. Thus $u \equiv 0$, which contradicts (9).

5 Negative regularity

**Theorem 5.1** (Negative regularity in global case). Let $d \geq 9$ and let $u$ be a global solution to (1) that is almost periodic modulo $G$. Suppose also that $E(u) < \infty$ and

$$\inf_{t \in \mathbb{R}} N(t) \geq 1. \quad (13)$$

Then $u \in L^\infty_t \dot{H}^{-\epsilon}(\mathbb{R} \times \mathbb{R}^d)$ for some $\epsilon = \epsilon(d) > 0$. In particular, $u \in L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^d)$.

Let $u$ be a solution to (1) that obeys the hypothesis of Theorem 5.1. Let $\eta > 0$ be a small constant to be chosen later. Then by Remark 1.2 combined with (13), there exists $N_0 = N_0(\eta)$ such that

$$||\Delta u\|_{L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^d)} \leq \eta. \quad (14)$$
We define
\[
A(N) = \begin{cases} 
N^{\frac{4}{d-4}} \sup_{t \in \mathbb{R}} \|u_N(t)\|_{L_x^{2d-8d+8}(\mathbb{R}^d)} & \text{for } d \geq 12 \\
N^{\frac{4}{d-4}} \sup_{t \in \mathbb{R}} \|u_N(t)\|_{L_x^{2d-8d+8}(\mathbb{R}^d)} & \text{for } 9 \leq d < 12
\end{cases}
\]
for frequencies \(N < 10N_0\).

We next prove a recurrence formula for \(A(N)\).

**Lemma 5.1.** For all \(N < 10N_0\),
\[
A(N) \lesssim_u \left( \frac{N}{N_0} \right)^\alpha + \eta^\frac{8}{d-4} \sum_{\frac{N_0}{10} \leq N_1 \leq N_0} \left( \frac{N}{N_1} \right)^\alpha A(N_1) + \eta^\frac{8}{d-4} \sum_{N_1 < \frac{N_0}{10}} \left( \frac{N_1}{N} \right)^\alpha A(N_1),
\]
where \(\alpha = \min\left\{ \frac{4}{d-4}, \frac{1}{2} \right\}\).

**Proof.** We first give the proof in dimensions \(d \geq 12\). Fix \(N \leq 10N_0\), by time translation symmetry, it suffices to prove
\[
N^{-\frac{4}{d-4}} \|u_N(0)\|_{L_x^{2d-8d+8}(\mathbb{R}^d)} \lesssim_u \left( \frac{N}{N_0} \right)^\alpha + \eta^\frac{8}{d-4} \sum_{\frac{N_0}{10} \leq N_1 \leq N_0} \left( \frac{N}{N_1} \right)^\alpha A(N_1) + \eta^\frac{8}{d-4} \sum_{N_1 < \frac{N_0}{10}} \left( \frac{N_1}{N} \right)^\alpha A(N_1).
\]

Using Lemma 3.1 into the future followed by the triangle inequality, Bernstein and the dispersive estimate, that is (3.7) in [22], we estimate
\[
N^{-\frac{4}{d-4}} \|u_N(0)\|_{L_x^{2d-8d+8}(\mathbb{R}^d)} \lesssim N^{-\frac{4}{d-4}} \left( \int_0^1 e^{-it\Delta^2} P_N F(u(t)) dt \right)_{L_x^{2d-8d+8}(\mathbb{R}^d)} + N^{-\frac{4}{d-4}} \left( \int_0^\infty e^{-it\Delta^2} P_N F(u(t)) dt \right)_{L_x^{2d-8d+8}(\mathbb{R}^d)}
\]
\[
\lesssim N^2 \left( \int_0^1 e^{-it\Delta^2} P_N F(u(t)) dt \right)_{L_x^2(\mathbb{R}^d)} + N^{-\frac{4}{d-4}} \int_0^\infty t^{-\frac{4}{d-4}} dt \|P_N F(u]\|_{L_x^{2d-8}(\mathbb{R}^d)}
\]
\[
\lesssim N^{-2} \|P_N F(u)\|_{L_x^1 L_x^{2d-8}(\mathbb{R}^d)} + N^\frac{2d-8d+8}{d-4} \|P_N F(u)\|_{L_x^{\infty} L_x^{2d-8}(\mathbb{R}^d)}
\]
\[
\lesssim N^\frac{4}{d-4} \|P_N F(u)\|_{L_x^2 L_x^{2d-8}(\mathbb{R}^d)}.
\]

Using the Fundamental Theorem of Calculus, we decompose
\[
F(u) = O\left( \|u > N_0\| \right) + O\left( \|u > N_0\| \right) + F\left( \|u_N \| \right)
\]
\[
+ u^{\frac{8}{d-4}} \int_0^1 F\left( u^{\frac{8}{d-4}} \right) dt + u^{\frac{4}{d-4}} \int_0^1 F\left( u^{\frac{4}{d-4}} \right) dt.
\]

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The contribution to the right-hand side of (15) coming from terms that contain at least one copy of \( u_{\geq N_0} \) can be estimated in the following manner: Using Hölder, Bernstein and the energy conservation, we have
\[
N^{\frac{4}{d-4}} \left\| P_N O(\|u_{> N_0}\|; \|u_{\leq N_0}\|^{\frac{8}{d-4}}) \right\|_{L^\infty_t L^2_x} \lesssim \|u\|_{L^\infty_t L^{\frac{2d(d-4)}{d-8}}_x}  \quad (\mathbb{R} \times \mathbb{R}^d)
\]
Next we turn to the contribution to the right-hand side of (15) coming from the last two terms in (16). It suffices to consider the first of them since similar arguments can be used to deal with the second.

First we note that \( \Delta u \in L^\infty_t L^2_x \), we have
\[
F_z(u) \in \mathcal{A}_{\frac{d(d-4)}{8}}^{\frac{8}{d-4}}.
\]
Furthermore, as \( P_{\geq N} F_z(u) \) is restricted to high frequencies, the Besov characterization of the homogeneous Hölder continuous functions (see [25], VI. 7.8) yields
\[
\| P_{\geq N} F_z(u) \|_{L^\infty_t L^2_x} \lesssim N^{-\frac{8}{d-4}} \| \Delta u \|_{L^\infty_t L^2_x}^{\frac{8}{d-4}}.
\]
In fact,
\[
\| P_{\geq N} F_z(u) \|_{L^\infty_t L^2_x} \lesssim \sum_{M > N} \| P_M F_z(u) \|_{L^\infty_t L^2_x} \lesssim \sum_{M > N} M^{-\frac{8}{d-4}} \| \Delta u \|_{L^\infty_t L^2_x}^{\frac{8}{d-4}}.
\]
Thus, by Hölder and (14),
\[
N^{\frac{4}{d-4}} \left\| P_N (u_{< N} \int_0^1 \sum_{\frac{N}{10}}^1 F_z(u_{\frac{N}{10} \leq \leq N_0} + \theta u_{< \frac{N}{10}}) d\theta) \right\|_{L^\infty_t L^2_x} \lesssim \| P_{\geq N} (\int_0^1 \sum_{\frac{N}{10}}^1 F_z(u_{\frac{N}{10} \leq \leq N_0} + \theta u_{< \frac{N}{10}}) d\theta) \|_{L^\infty_t L^2_x} \lesssim \sum_{N_1 < \frac{N}{10}} \left( \frac{N_1}{N} \right)^{\frac{4}{d-4}} A(N_1).
\]
Hence, the contribution coming from the last two terms in (15) is acceptable.

We are left to estimate the contribution of $F(u_{\frac{N}{10}} \leq u \leq N_0)$. We need only show

$$
\| F(u_{\frac{N}{10}} \leq u \leq N_0) \|_{L_t^\infty L_x^{d-8}} \leq \eta^{d-4} \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} A(N_1).
$$
(17)

As $d \geq 12$, we have $\frac{8}{d-4} \leq 1$. Using the triangle inequality, Bernstein, (14) and Hölder, we estimate

$$
\begin{align*}
\| F(u_{\frac{N}{10}} \leq u \leq N_0) \|_{L_t^\infty L_x^{d-8}} & \lesssim \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \\
& \lesssim \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \\
& \lesssim \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \\
& \lesssim \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \\
& \lesssim \eta^{d-4} N_2^{-\frac{8}{d-4}} \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \\
& \lesssim \eta^{d-4} N_2^{-\frac{8}{d-4}} \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \\
& \lesssim \eta^{d-4} N_2^{-\frac{8}{d-4}} \sum \frac{N_i^{-\frac{4}{d-4}}}{N_1} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}} \| u_{N_1} \|_{L_t^\infty L_x^{d-8}} \| u_{N_2} \|_{L_t^\infty L_x^{d-8}}.
\end{align*}
$$

This proves (17) and so completes the proof of the lemma in dimensions $d \geq 12$. 

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Consider now \( 9 \leq d < 12 \). Arguing as for (15), we have
\[
N^{-\frac{1}{2}} \| u_N(0) \|_{L^2_t L^2_x (\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}} \| P_N F(u) \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)},
\]
which we estimate by decomposing the nonlinearity as in (16). First we have
\[
N^{-\frac{1}{2}} \| P_N O(|u| \leq N_0) \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}} \| u \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \lesssim \left( \frac{N}{N_0} \right)^{\frac{1}{2}}.
\]
Next using Bernstein and Lemma 2.2 together with (14), we have
\[
N^{-\frac{1}{2}} \| P_N u \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}} \| u \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}} \| \nabla F(u) \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}} \| \Delta u \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)} \lesssim \eta^{\frac{1}{2 \theta}} \sum_{N_1 < \frac{N}{N_0}} (\frac{N_1}{N})^{\frac{1}{2}} A(N_1).
\]
Finally we estimate
\[
\| F(u_x) \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)}.
\]
We denote the maximal integer less than or equal to \( \frac{d+4}{d-4} \) by \( k(d) \), then
\[
\| F(u_x) \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \lesssim \sum_{\frac{N}{10} \leq N_1, \ldots, N_{k(d)}, M \leq N_0} \| u_{N_1} u_{N_2} \cdots u_{N_{k(d)}} u_M \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \| u_M \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)} \lesssim \sum_{\frac{N}{10} \leq N_1, \ldots, N_{k(d)}, M \leq N_0} \| u_{N_1} \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \prod_{j=2}^{k(d)} \| u_{N_j} \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)} \| u_M \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)}
+ \sum_{\frac{N}{10} \leq N_1, \ldots, N_{k(d)}, M \leq N_0} \| u_M \|_{L^\infty_t L^{2d \theta} x (\mathbb{R}^d)} \prod_{j=2}^{k(d)} \| u_{N_j} \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)} \times \prod_{j=2}^{k(d)} \| u_{N_j} \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)} \times \prod_{j=2}^{k(d)} \| u_{N_j} \|_{L^\infty_t L^{d \theta} x (\mathbb{R}^d)}
\]
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\[ \lesssim \eta^{\frac{8}{d-4}} \sum_{\frac{N_0}{10} \leq N_1 \leq N_0} \sum_{\frac{N_0}{10} \leq M_1 \leq N_1 \leq N_0} \left( \frac{N_1}{M_1} \right)^{k(d)} \left( M_1^{\frac{8}{d-4}} \right)^{\frac{d+4}{d-4} \cdot k(d)} \times \left( \frac{N_1^{\frac{1}{2}}}{M_1^{\frac{1}{2}}} \right)^{1+k(d)} \left( \frac{N_1^{\frac{1}{2}}}{M_1^{\frac{1}{2}}} \right)^{\frac{d+4}{d-4} \cdot k(d)} \lesssim \eta^{\frac{8}{d-4}} \sum_{\frac{N_0}{10} \leq N_1 \leq N_0} N_1^{\frac{1}{2}} A(N_1). \]

Putting everything together completes the proof of the lemma in the case of \( 9 \leq d < 12 \).

**Proposition 5.1** (\( L^p \) breach of scaling). Let \( u \) be as in Theorem 5.1 Then

\[ u \in L_t^\infty L_x^p \text{ for } \frac{2d(d+4)}{d^2-8} \leq p < \frac{2d}{d-4}. \]

In particular,

\[ F(u) \in L_t^\infty A_2^r \text{ for } \frac{2d(d+4)(d-4)}{d^3+8d^2-16d-64} \leq r < \frac{2d}{d+8}. \]  \( (18) \)

**Proof.** We only present the details for \( d \geq 12 \). The treatment of \( 9 \leq d < 12 \) is completely analogous. Combining Lemma 5.1 and Lemma 2.3 we deduce

\[ \| u_N \|_{L_t^\infty L_x^{2d(d-4)}(\mathbb{R} \times \mathbb{R}^d)} \lesssim_u N^{\frac{s}{2d+8}} \text{ for } N \leq 10N_0 \]  \( (19) \)

In applying Lemma 2.3 we set \( N = 10 \cdot 2^{-k} N_0 \), \( x_k = A(10 \cdot 2^{-k} N_0) \) and take \( \eta \) sufficiently small.

By interpolation followed by (19), Bernstein and energy conservation,

\[ \| u_N \|_{L_t^\infty L_x^{2d(d-4)}} \lesssim_u \| u_N \|_{L_t^\infty L_x^{2d(d-4)}}^{\frac{(d-4)(\frac{d+2}{d-4} - \frac{1}{p})}{2d(d-4)}} \| u_N \|_{L_t^\infty L_x^{2d(d-4)}} \lesssim N^{8(\frac{d+2}{2d} - \frac{1}{p}) - N^{-1+(d-4)(\frac{d+2}{2d} - \frac{1}{p})}} \lesssim_u N^{\frac{d}{2} - \frac{4}{d} \cdot \frac{d+4}{d-4}} \]

for all \( N \leq 10N_0 \). Then using Bernstein, we have

\[ \| u \|_{L_t^\infty L_x^p} \lesssim \| u_{\leq N_0} \|_{L_t^\infty L_x^p} + \| u_{> N_0} \|_{L_t^\infty L_x^p} \lesssim_u \sum_{N \leq N_0} N^{\frac{d}{2} - \frac{4}{d} \cdot \frac{d+4}{d-4}} + \sum_{N > N_0} N^{\frac{d}{2} - \frac{4}{d} \cdot \frac{d+4}{d-4}} \lesssim_u 1. \]

(18) follows by paraproduct and Hölder inequality.
**Proposition 5.2** (Some negative regularity). Let \( d \geq 9 \) and let \( u \) be as in Theorem 5.1. Assume further that \( F(u) \in L^\infty_t \dot{H}^{s-\sigma}_x \) for some \( \frac{2d(d+1)(d-4)}{d^2+8d^2-16d-16} \leq r < \frac{2d}{d+8} \) and some \( 0 \leq s \leq 2 \). Then there exists \( s_0 = s_0(r,d) > 0 \) such that \( u \in L^\infty_t \dot{H}^{s-s_0}_x \).

**Proof.** It suffices to prove that

\[
\| \nabla |^s u_N \|_{L^\infty_t L^2_x} \lesssim u \ N^{s_0} \quad \text{for all } N > 0 \quad \text{and} \quad s_0 = \frac{d}{r} - \frac{d}{2} - 4 > 0.
\] (20)

Indeed, by Bernstein combined with energy conservation,

\[
\| \nabla |^s u_N \|_{L^\infty_t L^2_x} \leq \| \nabla |^s u^0 \|_{L^\infty_t L^2_x} + \| \nabla |^s u^1 \|_{L^\infty_t L^2_x} \lesssim u \sum_{N \leq 1} N^{s_0} + \sum_{N > 1} N^{(s-s_0)+2} \lesssim u^1.
\]

We are left to prove (20). By time-translation symmetry, it suffices to prove

\[
\| \nabla |^s u_N(0) \|_{L^2_x} \lesssim u \ N^{s_0} \quad \text{for all } N > 0 \quad \text{and} \quad s_0 = \frac{d}{r} - \frac{d+8}{2} > 0.
\]

Using the Duhamel formula (5.11) both in the future and in the past, we write

\[
\| \nabla |^s u_N(0) \|_{L^2_x}^2 = \lim_{T \to -\infty} \lim_{T' \to +\infty} \left| \int_0^T e^{-it\Delta^2} P_N |\nabla|^s F(u(t))dt, -i \int_{T'}^0 e^{-i\tau\Delta^2} P_N |\nabla|^s F(u(\tau))d\tau \right|
\]

\[
\leq \int_{-\infty}^{+\infty} \int_0^0 \left| \langle P_N |\nabla|^s F(u(t)), e^{i(t-\tau)\Delta^2} P_N |\nabla|^s F(u(\tau)) \right| dt d\tau.
\]

We estimate the term inside the integrals in two ways. On one hand, using Hölder and the dispersive estimate,

\[
\left| \langle P_N |\nabla|^s F(u(t)), e^{i(t-\tau)\Delta^2} P_N |\nabla|^s F(u(\tau)) \right| \lesssim \| P_N |\nabla|^s F(u(t)) \|_{L^2_x} \| e^{i(t-\tau)\Delta^2} P_N |\nabla|^s F(u(\tau)) \|_{L^2_x} \lesssim |t-\tau|^{-\frac{d}{2}+\frac{d}{2}} \| F(u(t)) \|_{\dot{H}^s_x}^2.
\]

On the other hand, using Bernstein,

\[
\left| \langle P_N |\nabla|^s F(u(t)), e^{i(t-\tau)\Delta^2} P_N |\nabla|^s F(u(\tau)) \right| \lesssim \| P_N |\nabla|^s F(u(t)) \|_{L^2_x} \| e^{i(t-\tau)\Delta^2} P_N |\nabla|^s F(u(\tau)) \|_{L^2_x} \lesssim N^{2d(\frac{d}{2}-\frac{1}{2})} \| F(u(t)) \|_{\dot{H}^s_x}^2.
\]

Thus,

\[
\| \nabla |^s u_N(0) \|_{L^2_x}^2 \lesssim \| F(u(t)) \|_{\dot{H}^s_x}^2 \int_{-\infty}^{+\infty} \int_0^0 \min\{ |t-\tau|^{-\frac{d}{2}+\frac{d}{2}}, N^{-2d(\frac{d}{2}-\frac{1}{2})} \} dt d\tau \lesssim N^{2s_0} \| F(u(t)) \|_{\dot{H}^s_x}^2,
\]

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where we use the fact that \( r < \frac{2d}{d+8} \). It’s here that the dimension restriction is imposed.

\[ \text{Proof of Theorem 5.1} \]

Proposition 5.1 allows us to apply Proposition 5.2 with \( s = 2 \). We conclude that \( u \in L_t^{\infty} H^{2-s_0} \) for some \( s_0 = s_0(r, d) > 0 \). Thus we deduce that \( F(u) \in L_t^{\infty} \dot{A}_{2-s_0}^{\infty} \) for some \( \frac{2d(d+4)(d-4)}{d + 8d^2 - 16d - 64} \leq r < \frac{2d}{d+8} \). We are thus in the position to apply Proposition 5.2 again and obtain \( u \in L_t^{\infty} H^{2-2s_0} \). Iterating this procedure finitely many times, we derive \( u \in L_t^{\infty} H^{-\varepsilon} \) for some \( 0 < \varepsilon < s_0 \).

This completes the proof of Theorem 5.1.

6 The low-to-high frequency cascade

In this section, we use the negative regularity provided by Theorem 5.1 to preclude low-to-high frequency cascade solutions.

\[ \text{Theorem 6.1 (Absence of cascades). Let } d \geq 9. \text{ There are no global solutions to (1) that are low-to-high frequency cascades in the sense of Theorem 1.3} \]

\[ \text{Proof.} \]

Suppose for a contradiction that there existed such a solution \( u \). Then by Theorem 5.1, \( u \in L_t^{\infty} L_x^2 \). Thus by the conservation of mass,

\[ 0 \leq M(u) = M(u(t)) = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx < \infty \quad \text{for all } t \in \mathbb{R}. \]

Fix \( t \in \mathbb{R} \) and let \( \eta > 0 \) be a small constant. By compactness,

\[ \int_{|\xi| \leq c(\eta) N(t)} |\xi|^4 |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta. \]

On the other hand, as \( u \in L_t^{\infty} \dot{H}^{-\varepsilon} \) for some \( \varepsilon > 0 \),

\[ \int_{|\xi| \leq c(\eta) N(t)} |\xi|^{-2\varepsilon} |\hat{u}(t, \xi)|^2 \, d\xi \lesssim u \]

Hence, by H"{o}lder,

\[ \int_{|\xi| \leq c(\eta) N(t)} |\hat{u}(t, \xi)|^2 \, d\xi \lesssim u \eta^{\frac{2}{1+\varepsilon}}. \]

Meanwhile,

\[ \int_{|\xi| \geq c(\eta) N(t)} |\hat{u}(t, \xi)|^2 \, d\xi \leq [c(\eta) N(t)]^{-4} \int_{\mathbb{R}^d} |\xi|^4 |\hat{u}(t, \xi)|^2 \, d\xi \]

\[ \leq [c(\eta) N(t)]^{-4} E(u). \]

Therefore, we obtain

\[ 0 \leq M(u) \lesssim u \eta^{\frac{2}{1+\varepsilon}} \]

for all \( t \in \mathbb{R} \). As \( u \) is a low-to-high cascade, there is a sequence of times \( t_n \to \infty \) so that \( N(t_n) \to \infty \). As \( \eta > 0 \) is arbitrary, we may conclude \( M(u) = 0 \) and \( u \equiv 0 \). This concludes the fact that \( S_I(u) = \infty \), thus settling Theorem 6.1. \( \square \)
7 Soliton-like solutions

In this section, we preclude the soliton-like solutions, namely, we prove

**Theorem 7.1** (Absence of solitons). Let \( d \geq 9 \). There are no global solutions to \( \square u = 0 \) that are solitons in the sense of Theorem 1.3.

First we prove that the potential cannot be very small.

**Proposition 7.1** (Potential energy bounded from below). Let \( u \) be the soliton-like solutions in the sense of Theorem 1.3, then we have

\[
\inf_{t \in \mathbb{R}} \| u(t, x) \|_{L^\frac{2d}{d-4}_x} > 0.
\]

**Proof.** Suppose for a contradiction that \( \inf_{t \in \mathbb{R}} \| u(t, x) \|_{L^\frac{2d}{d-4}_x} = 0 \). Then there exists a sequence \( \{t_n\} \) such that

\[
\lim_{n \to \infty} \| u(t_n, x) \|_{L^\frac{2d}{d-4}_x} = 0,
\]

where \( t_n \to 0 \) (up to time translation) or \( t_n \to \pm \infty \).

Since \( u \in C^0_t(\mathbb{R}, \mathcal{H}^2_x(\mathbb{R}^d)) \), for any \( \varepsilon > 0 \), there exists an interval \( \tilde{I} = (a, +\infty) \) if \( t_n \to +\infty \); \( \tilde{I} = (-\infty, b) \) if \( t_n \to -\infty \), such that

\[
\| u(t, x) \|_{L^\infty_t L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)} < \varepsilon.
\]

Using Lemma 3.1 and Strichartz estimates, we have

\[
\| u \|_{L^\frac{2d}{d-4}_t L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)} \lesssim \int_t^{+\infty} e^{i(t-t')\Delta^2} \left( \| u \|_{L^\frac{8}{d-4}_x} \right)(t') dt' \| u \|_{L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)}
\]

\[
\lesssim \| u \|_{L^\frac{2d}{d-4}_t L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)}
\]

\[
\lesssim \| u \|_{L^\frac{2d}{d-4}_t L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)} \| u \|_{L^\frac{8}{d-4}_t L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)}
\]

\[
\lesssim \varepsilon \| u \|_{L^\frac{2d}{d-4}_t L^\frac{2d}{d-4}_x(\tilde{I} \times \mathbb{R}^d)}.
\]

Thus we get that \( u \equiv 0 \) on \( \tilde{I} \). By energy conservation, \( u \equiv 0 \) on \( \mathbb{R} \), which contradicts \( S_R(u) = \infty \). \( \square \)

**Proposition 7.2** (Concentration of \( L^p \) norm). Let \( u \) be the soliton-like solution in the sense of Theorem 1.3 then for every \( 1 < p < +\infty \), we have

\[
\inf_{t \in \mathbb{R}} \| u(t, x) \|_{L^p_x} > 0.
\]
Proof. By Theorem 5.2, \( u \in L^2_x(\mathbb{R}^d) \). If \( p > 2^\# \), then interpolation

\[
\| u(t) \|_{L^p_t} \lesssim \| u(t) \|_{L^{\frac{4p}{4p-(d-4)p}}_t} \| u(t) \|_{L^{\frac{4p}{2d-(d-4)p}}_t},
\]

combined with Proposition 7.1, yields that

\[
\inf_{t \in \mathbb{R}} \| u(t) \|_{L^p_t} > 0.
\]

If \( 1 < p < 2^\# \), by interpolation

\[
\| u(t) \|_{L^p_t} \lesssim \| u(t) \|_{L^{\frac{4p}{2d-(d-4)p}}_t} \| u(t) \|_{L^{\frac{4p}{d}}_t},
\]

and mass conservation, we have

\[
\inf_{t \in \mathbb{R}} \| u(t) \|_{L^p_t} > 0.
\]

Finally we consider the case of \( 2 < p < 2^\# \). If \( \inf_{t \in \mathbb{R}} \| u(t, x) \|_{L^p_x} = 0 \), then there exists \( \{ t_n \} \) such that \( \lim_{n \to \infty} \| u(t_n, x) \|_{L^p_x} = 0 \). On the other hand, by [1], [6] and Proposition 7.1, we have

\[
\lim_{n \to \infty} \| P_{c(\eta)<c(\eta)} u(t_n) \|_{L^\#_p} \gtrsim 1,
\]

as long as \( \eta \) is chosen sufficiently small. By Sobolev embedding,

\[
\| P_{c(\eta)<c(\eta)} u(t_n) \|_{L^\#_p} \lesssim \| \nabla \|^{\frac{1}{p}} \| P_{c(\eta)<c(\eta)} u(t_n) \|_{L^p_x} \lesssim c(\eta) \| u(t_n) \|_{L^p_x} \to 0, \quad \text{as} \ n \to \infty.
\]

This contradicts (23), hence completes the proof. 

To kill the soliton, we need the interaction Morawetz estimate. The Interaction Morawetz estimates was obtained in [23] in dimension \( d \geq 7 \) and then was extended to dimensions \( d \geq 5 \) in [21].

Proposition 7.3 (Interaction Morawetz estimates, [21]). Let \( u \in C^0_t(I, H^2_x(\mathbb{R}^d)) \) be the solution to

\[
\begin{cases}
  iu_t + \Delta^2 u = \lambda |u|^{p-1} u, & \text{in} \ \mathbb{R} \times \mathbb{R}^d, \\
  u(0) = u_0(x), & \text{in} \ \mathbb{R}^d,
\end{cases}
\]

where \( 1 < p \leq 2^\# - 1 \). Then if \( d > 5 \), we have

\[
\int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^5} dx dy dt + \int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, y)|^2 |u(t, x)|^{\frac{2d}{d-4}}}{|x-y|^\frac{d-4}{4}} dx dy dt \lesssim u_1.
\]  

(24)

If \( d = 5 \), we have

\[
\int_I \int_{\mathbb{R}^5} |u(t, x)|^4 dx dt + \int_I \int_{\mathbb{R}^5 \times \mathbb{R}^5} \frac{|u(t, y)|^2 |u(t, x)|^{\frac{10}{2}}}{|x-y|} dx dy dt \lesssim u_1.
\]  

(25)
Proposition 7.3 and Theorem 5.1 yield Corollary 7.1. Fix $d \geq 9$. Suppose $u$ is the soliton-like solution to (1) in the sense of Theorem 1.3, then we have
\begin{equation}
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \frac{u(t,x)}{|x-y|^5} \right|^2 \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \frac{u(t,y)}{|x-y|} \right|^\frac{2d}{d-4} dx dy dt \lesssim u_1.
\end{equation}

(26)

Now we can kill the soliton thus complete the proof the Theorem 1.1.

Proof of Theorem 7.1. Fix $d \geq 9$. Let $u$ be the soliton-like solution as in Theorem 1.3. Then by Corollary 7.1, we have
\begin{equation}
\left\| \left\| \nabla \right\|^{-\frac{d-5}{2}} \frac{|u|^2}{L^2_{t,x}} \right\|^2_{L^\infty_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \lesssim u_1.
\end{equation}

On the other hand, by Sobolev embedding and energy conservation, we have
\begin{equation}
\left\| \nabla |u|^2 \right\|^2_{L^\infty_{t} L^d_{x} (\mathbb{R} \times \mathbb{R}^d)} \leq C \left\| \nabla u \right\|^2_{L^\infty_{t} L^\frac{2d}{d-4}_{x} (\mathbb{R} \times \mathbb{R}^d)} \left\| u \right\|^2_{L^\infty_{t} L^\frac{2d}{d-4}_{x} (\mathbb{R} \times \mathbb{R}^d)} \lesssim u_1.
\end{equation}

Therefore, by interpolation, we have
\begin{equation}
\left\| |u|^2 \right\|^2_{L^{2(d-3)}_{t} L^{\frac{d(d-3)}{d^2-7d+15}}_{x} (\mathbb{R} \times \mathbb{R}^d)} \leq C \left\| \left\| \nabla \right\|^{-\frac{d-5}{2}} \frac{|u|^2}{L^2_{t,x}} \right\|^2_{L^\infty_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \left\| \nabla |u|^2 \right\|^\frac{d-3}{d}_{L^\infty_{t} L^d_{x} (\mathbb{R} \times \mathbb{R}^d)},
\end{equation}

hence
\begin{equation}
\left\| u \right\|^\frac{2d(d-3)}{d^2-7d+15}_{L^{2(d-3)}_{t} L^{\frac{2d(d-3)}{d^2-7d+15}}_{x} (\mathbb{R} \times \mathbb{R}^d)} \lesssim u_1.
\end{equation}

(27)

However, by Proposition 7.2
\begin{equation}
\left\| u \right\|^\frac{2d(d-3)}{d^2-7d+15}_{L^{2(d-3)}_{t} L^{\frac{2d(d-3)}{d^2-7d+15}}_{x} (\mathbb{R} \times \mathbb{R}^d)} \gtrsim u_1.
\end{equation}

So
\begin{equation}
\left\| u \right\|^\frac{2d(d-3)}{d^2-7d+15}_{L^{2(d-3)}_{t} L^{\frac{2d(d-3)}{d^2-7d+15}}_{x} (\mathbb{R} \times \mathbb{R}^d)} = +\infty,
\end{equation}

which contradicts (27). This completes the proof of Theorem 7.1.

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