We generalize Laughlin’s flux insertion argument, originally discussed in the context of the quantum Hall effect, to topological phases protected by non-on-site unitary symmetries, in particular by parity symmetry or parity symmetry combined with an on-site unitary symmetry. As a model, we discuss fermionic or bosonic systems in two spatial dimensions with CP symmetry, which are, by the CPT theorem, related to time-reversal symmetric topological insulators (e.g., the quantum spin Hall effect). In particular, we develop the stability/instability (or “gappability”/“ingappablity”) criteria for non-chiral conformal field theories with parity symmetry that may emerge as an edge state of a symmetry-protected topological phase. A necessary ingredient, as it turns out, is to consider the edge conformal field theories on unoriented surfaces, such as the Klein bottle, which arises naturally from enforcing parity symmetry by a projection operation.

I. INTRODUCTION

One of the most fundamental and defining properties of the quantum Hall effect (QHE) is the charge pumping discussed in Laughlin’s thought experiment. This non-perturbative argument explains the extreme robustness of the QHE against disorder and interactions. In the language of quantum field theories, the charge pumping in Laughlin’s gauge argument is a manifestation of a quantum anomaly, i.e., the breakdown of a classical symmetry caused by quantum effects. This is an extreme case where quantum mechanical effects completely betray our expectations from classical physics. To be more precise, the quantum Hall state supports, in the presence of a boundary (an edge), a chiral edge state. If we focus on an edge, the total charge is not conserved within the edge, i.e., the U(1) symmetry associated with the particle number conservation is violated, as the charge leaks into the bulk precisely because of the QHE. This well-known bulk-boundary correspondence of the QHE relates the bulk topological properties and the gauge anomaly (non-conservation of charge) at the edge.

The connection to a quantum anomaly gives the conceptual backbone of the QHE. In fact, it is desirable to connect topological phases of any kind, not just the QHE, to a quantum anomaly for the following reasons. First, quantum anomalies often provide a way to detect a non-trivial topological nature of the state (e.g., charge pumping in the QHE described above), and hence gives an operational definition of a topological phase. Second, once a topological phase is characterized in terms of a quantum anomaly, it is most likely to be stable against interactions and disorder.

There are, however, a class of topological phases that
have emerged more recently and that are less well un-
derstood from this perspective. These are the so-called
symmetry-protected topological (SPT) phases. (For re-
cent works on SPT phases and in particular their con-
nection to anomalies, see, for example, Refs. 3–25.) In
typical situations, they do not have any topological prop-
erties in the absence of symmetry conditions such as time-
reversal. In particular, they are adiabatically connected to
a topologically trivial state if there is no symmetry.
In the presence of a certain set of symmetries, how-
ever, SPT phases are sharply distinct from a trivial state
and host a number of interesting topological properties.
Canonical examples of SPT phases include the Haldane
phase realized in one-dimensional quantum spin chains,
the quantum spin Hall effect (QSHE) in two dimensions,
and the three-dimensional time-reversal symmetric topo-
logical insulator.

One of our main goals in this paper is to provide a gen-
eral scheme that allows us to judge if a given phase is a
SPT phase or not, or, to be more precise, to diagnose un-
der which symmetry condition a given phase can possibly
be a SPT phase. While a fairly general topological clas-
sification of non-interacting fermion systems is possible,
judging if a given state can adiabatically be deformable
to a trivial state of matter or not is in general a non-
trivial question in the presence of strong interactions. In
fact, there are known examples where a would-be SPT
phase, for which one can define a topological invariant of
some sort at the level of single-particle wave functions,
can actually be adiabatically connected to a topologically
trivial phase once one includes interactions.

More specifically, we will focus on SPT/non-SPT
phases in (2+1) dimensions, and discuss “gappabil-
ity”/“ingappability” of their (1+1)-dimensional [(1+1)D]
edge states in the presence of symmetry conditions. We
consider a given (1+1)D gapless (conformal) field theory,
which may emerge as an edge state of a bulk theory, and
ask if its gapless nature can be protected by some symme-
try conditions. Once the ingappability of the edge state
is established, the corresponding bulk theory cannot adi-
abatically be connected to a topologically trivial state
that do not support a gapless edge theory – the state
in question is in a SPT phase protected by the symmet-
tries. On the other hand, if the edge theory turns out to
be gappable in the presence of the symmetries, the bulk
theory may be deformable to a topologically trivial state.

To diagnose gappability/ingappability of an edge the-
ory, a generalization of Laughlin’s gauge argument was
proposed in Ref. 20 in a way that can be applied to edge
states of SPT phases. The purpose of this paper is to ex-
tend the scheme proposed in Ref. 20 to study topolog-
ical phases protected by unitary non-on-site symmetries,
e.g., parity symmetries. (We will mainly be interested in
unitary symmetries, but anti-unitary symmetries such as
time-reversal symmetry are also relevant to our discus-
sion.)

A key ingredient of the strategy suggested in Ref. 20 is
the strict enforcement of symmetry conditions by a pro-
jection operation, or, to be more precise, an “orbifolding”
procedure in the edge conformal field theory (CFT).
An orbifold of a theory, which is invariant under a global
unitary on-site symmetry, is given by averaging the par-
tition function over boundary conditions twisted by a
group element in the symmetry group. Roughly speak-
ing, this procedure removes states that are not invariant
under the symmetry group. One can then study an adia-
batic evolution of the projected (“orbifolded”) partition
function. For example, if a U(1) symmetry is conserved,
as in Laughlin’s original argument, one can ask if the
orbifolded partition function is invariant or not under
a large U(1) gauge transformation. While an original
non-projected (non- orbifolded) theory may be anomaly
free, once orbifolded, the edge theory may fail to per-
form an anomaly-free adiabatic process. Non-invariance
of the orbifolded edge theory under a large U(1) gauge
transformation signals non-trivial topological properties
of the corresponding bulk state. One can also ask, per-
haps more fundamentally, the invariance/non-invariance
of the orbifolded system under large coordinate transfor-
mations, such as modular transformations on a space-
time manifold with non-zero genus, e.g., a torus.

This scheme is demonstrated to work for various ex-
amples. A similar projection or “gauging” procedure
is also employed in Ref. 22, where a criterion for the
ingappability/gappability of an edge theory is de-
erived using fractional statistics of the defects obtained
from gauging some unitary on-site global symmetry in a
gapped bulk theory.

In this paper, we extend the scheme proposed in Ref.
20 to SPT phases that are protected by unitary spatial
symmetries, in particular to parity symmetry (denoted by
P in the following). An interplay between spatial
(and, more generally, crystal) symmetries and topologi-
cal properties of electronic states have been studied ex-
tensively recently. Following the general strategy de-
scribed above, we consider the orbifolding or gauging pro-
cedure by a parity symmetry. Unlike orbifolding a uni-
tary on-site symmetry, orbifolding parity symmetry nat-
urally leads to a change of the topology of the spacetime
manifold of the edge theory. Once orbifolding parity sym-
metry, an edge theory is defined on an unoriented
(1+1)D spacetime surface, such as the Klein bottle instead
of a spacetime torus. We refer to these conformal field
theories as orientifold conformal field theories.

In the presence of yet another symmetry (represented
by a symmetry group G – the total symmetry group is
P × G) in addition to parity, there is a simple conse-
quence of the topology change from the torus to the Klein
bottle, which can be inferred by comparing their funda-
mental groups, i.e., the space of non-contractible loops on
these surfaces. On the torus, there are two independent
cycles and one can assign a group element to each cycle,
e.g., a U(1) phase factor; these two group elements (g1
and g2, say) represent boundary conditions along each
cycle. On the Klein bottle, on the other hand, because
of its unoriented nature, there is a certain restriction on
the group elements that one can assign for cycles; while one of the group elements, \( g_1 \), say, can be any element in \( G \), the other group element \( g_2 \) must satisfy \( g_2 = g_2^{-1} \) (see discussion near eq. (30) below for more details).

In this work, we focus on the cases where \( G \) is a U(1) symmetry (either charge U(1) or “spin” U(1) symmetry). One of our main observations is a crucial role played by symmetry (either charge U(1) or “spin” U(1) symmetry).

While our methodology is applicable to a wider class of systems with parity symmetry, in this paper, we choose to work with systems with parity combined with charge conjugation symmetry (CP symmetry). Specifically, we consider three examples: (i) fermionic systems with conserved charge U(1) symmetry and CP symmetry (CP symmetric topological insulators); (ii) bosonic systems with conserved charge U(1) and CP symmetries; and (iii) K-matrix theory with CP and U(1) symmetries. We discuss the topological classification of these systems by using the method of the generalized Laughlin argument.

The systems with CP symmetry are our canonical examples in the sense that they are closely related to SPT phases protected by time-reversal symmetry through the CPT theorem. For systems with Lorentz invariance, the CPT theorem tells us that any perturbation (mass terms and interactions) prohibited by T (CP) symmetry is also excluded by CP (T) symmetry. Thus, for Lorentz invariant systems, SPTs protected by time-reversal are automatically protected by CP symmetry as well.

For condensed matter systems, Lorentz invariance is not a prerequisite. However, for non-interacting fermions, it is known that the general topological classification can be obtained solely from the topological classification of Lorentz invariant Dirac Hamiltonians. When available, topological field theory descriptions of topological phases are also Lorentz invariant. In addition, it is known that, if one considers the entanglement spectrum as a tool to study SPT phases, CP symmetry of a physical Hamiltonian is translated to an effective time-reversal symmetry of the corresponding entanglement Hamiltonian if one bipartites the system into two subsystems that are related by CP symmetry. For these reasons, our method also provides a new insight into time-reversal symmetric topological systems, including the QSHE, by relating them to orientifold conformal field theories. Thus we provide a method for “twisting” or “gauging” time-reversal symmetry. (See recent discussion in Refs. 67 and 73.)

A. The outline and main results of the paper

The main results and outline of the paper can be summarized as follows: for the remainder of Sec. I, we will review gauge and chiral anomalies in (1+1) dimensions and their connection to topological phases in (2+1) dimensions. In particular, we rephrase the original Laughlin argument in a quantum field theory language, which we will use for our later discussion.

In Sec. II, we begin our discussion by introducing a free fermion model with CP and electromagnetic U(1) symmetries. We consider two kinds of CP symmetries, one which protects gapless edge states and hence leads to a non-trivial bulk symmetry-protected topological phase, and the other which does not lead to a topological insulator. An anomaly (“CP anomaly”) is identified in the non-chiral edge states of the CP-symmetric topological insulator. We then present, by using the CP-symmetric topological insulator as an example, a generalization of Laughlin’s argument to systems with parity symmetry (CP symmetry, in this case). By considering the partition function of the non-chiral edge theory with CP projection, it is shown that the distinction between the two cases shows up as the presence/absence of a \( \mathbb{Z}_2 \) flux on the Klein bottle (“\( g_2 \)” in the above notation). Under an adiabatic insertion of electromagnetic U(1) flux (“\( g_1 \)” in the above notation), the projected partition function is anomalous/anomaly-free when the edge theory is ingappable/gappable.

In Sec. III, the generalized Laughlin argument is applied to bosonic SPT phases with a single-component non-chiral boson edge theory with CP symmetry. The results are consistent with microscopic stability analysis of CP symmetric edge theories given in Ref. 53.

In Sec. IV, we consider a broader range of edge theories described by the K-matrix theory with CP symmetry. With the generalized Laughlin argument, we derive the stability criterion for the edge theories, which agrees with the stability criterion of the K-matrix theory with time-reversal symmetry.\( ^{34} \)\( ^{35} \) as expected from the CPT theorem.

We conclude in Sec. V. In Appendix A, we discuss the eigenvalue of the CP transformation of the ground state of edge CFTs, and in particular its evolution under an adiabatic evolution of the background flux. Once we choose to preserve the U(1) symmetry, the CP eigenvalue must be independent of the background flux, which we assume for the bulk of the paper. On the other hand, an alternative point of view is possible where we strictly enforce the U(1) symmetry. Once this is done, CP symmetry may be anomalous, and hence the ground state CP eigenvalue may be dependent on the background flux. This issue is discussed in Appendix A by making use of the state-operator correspondence of CFTs. Appendix B explains a technical detail that arises when diagnosing the stability of K-matrix edge theories in Sec. V.
B. The integer QHE and gauge anomaly

For chiral topological phases in two spatial dimensions, their edge states (which are chiral) are anomalous. When there is the electromagnetic U(1) symmetry, the chiral edge states are anomalous under infinitesimal as well as large U(1) gauge transformations. Even in the absence of the electromagnetic U(1) invariance, the edge states are still anomalous under infinitesimal as well as large diffeomorphisms.

For later use, let us review the anomaly under large gauge transformations at the edge of the integer QHE. (See, for example, Refs. [1] [2] for discussion on the edge theory of various quantum Hall states). (We will follow notations in Ref. [2]). The chiral edge mode of the integer QHE is described by the action

$$ S = \frac{1}{2\pi} \int dt dx \, i\psi_R^\dagger \partial_t \psi_R + i\psi_R^\dagger \partial_x \psi_R, $$

(1)

where $(t,x)$ is the spacetime coordinate of the edge theory, and chirality is chosen, say, to arrive at the right-moving fermions.

Following Laughlin’s thought experiment, we now insert magnetic flux into the system of a cylindrical shape. In terms of the fermion field in the edge theory, this amounts to imposing the following twisted boundary conditions both for space and time directions:

$$ \psi_R(t, x + 2\pi) = e^{2\pi i a} \psi_R(t, x), $$

$$ \psi_R(t + 2\pi \tau_2, x + 2\pi \tau_1) = e^{2\pi i b} \psi_R(t, x). $$

(2)

Here the edge theory is defined on a spatial circle of radius $2\pi$, and $\tau = \tau_1 + i\tau_2$ is the modular parameter of the spacetime torus. Under these boundary conditions, the right-moving partition function is computed to be

$$ Z_{[a,b]}(\tau) = q^{-\frac{1}{2} + \frac{1}{2}(a-1)^2} e^{-2\pi i (b-1/2)(a-1/2)} \times \prod_{n=0}^{\infty} \left[ 1 + e^{-2\pi i (b-1/2)} q^{n+a} \right] \times \prod_{n=-1}^{-\infty} \left[ 1 + e^{2\pi i (b-1/2)} q^{-n-a} \right] $$

$$ \times \prod_{n=0}^{\infty} \left[ 1 + e^{2\pi i (b-1/2)} q^{-n-a} \right] $$

$$ = \frac{1}{\eta(\tau)} q^{a-1/2} (-b+1/2) (0, \tau), $$

(3)

where $q = \exp(2\pi i \tau)$ and the theta function with characteristics is defined by

$$ \theta^{[\alpha \beta]}(\nu, \tau) \equiv \sum_{n \in \mathbb{Z}} e^{2\pi i (\nu + \beta)(n+\alpha)}. $$

(4)

While the classical theory, defined in terms of the action [1] together with the boundary condition [3], is invariant under large gauge transformations $a \to a + 1$ and $b \to b + 1$, the partition function violates this invariance:

$$ Z_{[a,b]} = Z_{[a+1,b]} = -e^{2\pi i a} Z_{[a,b+1]} = Z_{[-a,1-b]}, $$

(5)

and thus the edge theory is anomalous under this transformation [4].

C. The $S_z$ conserving QSHE and chiral anomaly

As yet another exercise, let us consider a bulk topological insulator characterized by non-zero spin Chern number. We require both charge U(1) and spin U(1) symmetries. The edge state, if it exists, also respects these symmetries, at least classically. However, either one of these U(1) symmetries must be spoiled by quantum mechanical effects. Let us now insist on the conservation of electromagnetic U(1) charge. One can then introduce an electromagnetic vector potential $A$. We then consider a non-chiral fermion coupled with the electromagnetic U(1) gauge field,

$$ S = \int d^2z \left( i\psi_R^\dagger D_z \psi_R + i\psi_L^\dagger D_z \psi_L \right), $$

(6)

where $D_z$ is a covariant derivative with the electromagnetic U(1) gauge field $A$. As is well known, the theory is not invariant under chiral gauge transformations, which in the present context are gauge transformations associated to the spin U(1) symmetry. This has to do with the presence of a non-trivial bulk topological phase protected by charge U(1) and spin U(1) symmetries.

The chiral anomaly comes about since the path integral measure is not invariant under chiral transformations. Let us consider the case where the Chern number associated to the vector potential $A$ is non-zero:

$$ Ch = \frac{i}{2\pi} \int \text{tr } F > 0, $$

(7)

where $F$ is the field strength of the external U(1) gauge potential $A$. Then, by the index theorem, the number of $\psi_L$ zero modes (= the number of $\psi_R^\dagger$ zero modes) is larger by $Ch$ than the number of $\psi_L^\dagger$ zero modes (= the number of $\psi_R$ zero modes). The path integral measure is given by

$$ \mathcal{D} [\psi^\dagger, \psi] = \prod_{\alpha=1}^{Ch} da_\alpha da^*_\alpha \prod_{n=1}^{\infty} db_n dc_\alpha db^*_n dc^*_\alpha, $$

(8)

where $\prod_{n=1}^{\infty} db_n dc_\alpha db^*_n dc^*_\alpha$ represents the “oscillator” part of the measure, whereas $\prod_{\alpha=1}^{Ch} da_\alpha da^*_\alpha$ represents the measure associated to the zero modes. While the measure $db_n dc_\alpha db^*_n dc^*_\alpha$ is invariant under both electromagnetic and spin U(1) global transformations, the zero mode part $da_\alpha da^*_\alpha$ has electromagnetic (vector) charge zero, but axial charge 2. Thus, the path integral measure is not invariant under the axial (spin) U(1) rotation. In fact, in the presence of nonzero flux with the Chern number $Ch$, the axial U(1) is broken down to its $\mathbb{Z}_{2Ch}$ subgroup.

To summarize, once we demand the electromagnetic U(1) symmetry to be preserved then, the chiral anomaly tells us that the spin U(1) [axial U(1)] must be broken at the edge – this is nothing but the QSHE, i.e., the spin quantum number is pumped by an adiabatic threading of the electromagnetic flux.
In fact, one has a choice — if one decides to preserve spin U(1) symmetry, instead of charge U(1), one could thread “spin flux” and consider the corresponding spin vector potential. Going through the above argument, one then concludes the charge is not conserved. This has to do with charge pumping by insertion of spin flux.

II. 2D FERMIonic TOPOLOGICAL PHASES PROTECTED BY CP SYMMETRY

In this section, we describe our methodology (a generalization of Laughlin’s argument) in terms of a simple two-dimensional fermionic system (although the method applies to a wider class of systems). The system of interest conserves the electromagnetic U(1) charge and respects a discrete symmetry, CP, that is a combination of parity, P: (x, y) → (−x, y) in two spatial dimensions, and charge conjugation, C, which is a unitary Z₂ on-site symmetry.

By the CPT theorem, the CP symmetric system (the CP symmetric topological insulator) is related to the time-reversal symmetric topological insulator. (In fact, they are equivalent when there is Lorentz invariance.) As two-dimensional insulators with time-reversal symmetry that squares to −1 are classified in terms of the Kane-Mele Z₂ topological invariant, so are CP symmetric insulators. The CP symmetric fermionic system can also be interpreted as a topological superconductor that conserves parity and the z-component of spin (this is an example of “T-duality”). See Ref. 53 for more details of superconducting systems equivalent (dual) to CP symmetric insulators.

A. CP symmetric insulators

a. The bulk tight-binding model A lattice model of the topological insulator with CP symmetry can be constructed on the two-dimensional square lattice by taking two copies of the above two-band Chern insulator with opposite chiralities. Consider the Hamiltonian in momentum space,

\[ H = \sum_{k \in \text{BZ}} \Psi^\dagger(k) \mathcal{H}(k) \Psi(k), \]

where \( \Psi(k) \) is a four-component fermion field with momentum \( k \), BZ represents the first Brillouin zone of the two-dimensional square lattice, and the single particle Hamiltonian in momentum space is given in terms of the \( 4 \times 4 \) matrix as,

\[ \mathcal{H}(k) = n_x(k) \tau_z \sigma_x + n_y(k) \tau_0 \sigma_y + n_z(k) \tau_0 \sigma_z, \]

where \( \sigma_\mu \) and \( \tau_\mu \) (\( \mu = 0, 1, 2, 3 \)) are two sets of Pauli matrices with \( \sigma_0 \) and \( \tau_0 \) being a \( 2 \times 2 \) unit matrix. The \( k \)-dependent three-component vector is given by

\[ \tilde{\eta}(k) = \begin{bmatrix} -\sin k_x \\ -\sin k_y \\ (\cos k_x + \cos k_y) + \mu \end{bmatrix}. \]

We will focus on the region \(-2 < \mu < 0 \) or \( 0 < \mu < +2 \). The Hamiltonian is invariant under the following two CP transformations

\[ (\mathcal{CP}) \Psi(\tau) (\mathcal{CP})^{-1} = U_{\text{CP}} \Psi^{\dagger}(\tilde{r}), \]

where \( r = (x, y) \) labels sites on the square lattice, \( \tilde{r} := (-x, y) \) and the \( 2 \times 2 \) unitary matrix \( U_{\text{CP}} \) is given by either of

\[ U_{\text{CP}} = \tau_z \sigma_x, \quad U_{\text{CP}}^T = +U_{\text{CP}}, \quad (\eta = +1), \]

\[ U_{\text{CP}} = \tau_y \sigma_x, \quad U_{\text{CP}}^T = -U_{\text{CP}}, \quad (\eta = -1). \]

To distinguish these two cases, we have introduced an index \( \eta; \eta = \pm 1 \) refers to the first/second case. We will also use the notation \( \eta = e^{2\pi i \epsilon} \) where \( \epsilon = 0, 1/2 \) for \( \eta = 1, -1 \), respectively. The distinction between these two CP symmetries can be summarized as

\[ (\mathcal{CP})^2 = e^{2\pi i N_f} \]

where CP acts on states with \( N_f \) fermions.

It turns out that imposing \( U_{\text{CP}} = \tau_z \sigma_x \) (\( \eta = 1 \)) leads to CP symmetric topological insulators. This can be seen by looking at the stability of the edge mode that can appear when we terminate the system in the \( y \)-direction (i.e., the edge is along the \( x \)-direction.) One can check, numerically, and also in terms of the continuum edge theory (see below), \( U_{\text{CP}} = \tau_z \sigma_x \) protects the edge state while \( U_{\text{CP}} = \tau_y \sigma_x \) does not in the presence of the electromagnetic U(1) symmetry.

b. The edge theory We now develop a continuum theory for the edge modes along the \( x \)-direction. The edge theory is described by, at low-energies, the free fermion Hamiltonian with relativistic dispersion:

\[ H = \frac{\psi}{2\pi} \int dx (\psi^\dagger_L \partial_x \psi_L - \psi^\dagger_R \partial_x \psi_R), \]

where the single-component complex fermion field operators \( \psi_L \) and \( \psi_R \) represent left-moving and right-moving electrons, and \( \psi \) is the Fermi velocity. The Hamiltonian conserves the U(1) charge

\[ F_V = \int dx \left[ \psi^\dagger_R \psi_R + \psi^\dagger_L \psi_L \right]. \]

The subscript “\( V \)” here represents the fact that this is a vector U(1) charge, as opposed to an axial U(1) charge,

\[ F_A = \int dx \left[ \psi^\dagger_R \psi_R - \psi^\dagger_L \psi_L \right]. \]

Since the edge runs along \( x \)-direction, the Hamiltonian with the edge preserves (is consistent with) CP symmetry, i.e., CP transformation is closed within the edge.
Corresponding to the bulk CP transformations, we consider the following two types of CP symmetry operations that act within the edge theory\footnote{We use $\tilde{\psi}$ to denote the complex conjugation.}

$$
(CP)\psi_L(x)(CP)^{-1} = \psi_R^\dagger(-x),
$$

$$
(CP)\psi_R(x)(CP)^{-1} = \eta \psi_L^\dagger(-x).
$$

The sign $\eta$ is $+/-$ respectively for topological/non-topological cases; there are two uniform fermion mass bilinears that are consistent with the charge $U(1)$ symmetry, $M_1 = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$, and $M_2 = -i(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L)$. These masses are odd under CP and prohibited when $\eta = +1$, whereas they are even under CP with $\eta = -1$. Thus,

$$
\eta = e^{2\pi i \epsilon} = \begin{cases} +1 & \text{"topological"} \\
-1 & \text{"trivial"} 
\end{cases}
$$

We conclude that the gapless edge theory is, at least at the quadratic level, stable (inapplicable); the stability-instability of the edge theory in the presence of interactions is one of our main focuses in the following sections.

Let us consider, in addition, quadratic but spatially inhomogeneous perturbations. The two uniform mass terms $M_1$ and $M_2$ are not allowed in the presence of CP symmetry with $\epsilon = 0$. However, one can still consider $\int dx M(x)$ where, $M(x) = a_1(x)M_1 + a_2(x)M_2$, as a perturbation to the edge theory. The perturbation is not allowed by CP symmetry if $a_1(x)$ is constant but allowed if $\tilde{a}(-x) = -\tilde{a}(x)$. This perturbation gaps out most of the edge theory, but not completely. At the point $x = 0$ which is invariant under CP symmetry, it leaves a zero energy mode. This is a type of zero energy mode akin to the zero energy bound state in a soliton in polyacetylene, and carries 1/2 charge. This is also similar to the mass domain wall in the helical edge mode of the QSHE discussed previously.\footnote{The difference, however, is that for the time-reversal symmetric quantum spin Hall effect, the mass domain wall breaks TRS; the only exception being the location of the kink. In the CP symmetric case, the mass domain wall, as a whole, preserves CP symmetry.}

### B. CP anomaly

Although the edge theory cannot be gapped at the quadratic level, whether or not it is gappable under arbitrary symmetry-preserving perturbations is not clear; a state that appears to be non-deformable to a trivial state may actually be deformable to a trivial state once one includes perturbations beyond the quadratic level. For the CP symmetric topological phase defined above, we now try to develop a generalized Laughlin argument. To put it differently, we will ask if there is a quantum anomaly or not that may guarantee the gapless nature of the edge theory.

Quite often, non-chiral theories are anomaly-free, and hence are not qualified as a topological phase without symmetry condition. However, there may be a tension between symmetry conditions and an attempt to make the theory self-consistent (anomaly-free). To be more precise, if one insists on the symmetry conditions, one might not be able to achieve anomaly-freedom. In the scheme proposed in Ref.\footnote{We note that the CP anomaly can be understood as a consequence of the chiral anomaly explained in Ref.\footnote{The argument goes as follows: as in the case of the chiral anomaly, we consider a background field with non-zero Chern number $Ch$. When non-zero and positive, there are zero modes in $\psi_L$ and $\psi_R^\dagger$, and the path-integral measure has a factor

$$
\prod_{\alpha=1}^{Ch} da_\alpha da^*_\alpha.
$$

Observe now that the Chern number $Ch$ flips its sign under parity, P. It also flips its sign under the charge conjugation, C. Thus, the Chern number remains invariant under the combination of CP. Let us first consider the case of $\eta = +1$, where CP transformation is given by $(CP)\psi_L(x)(CP)^{-1} = \psi_R^\dagger(-x)$, $(CP)\psi_R(x)(CP)^{-1} = \psi_L^\dagger(-x)$. Thus, by CP, the $(\psi_L, \psi_R^\dagger)$ zero modes in the background $A(x)$ are sent to the $(\psi_R^\dagger, \psi_L)$ zero modes in the background $-A(\tilde{x})$. Because of the Fermi statistics, the measure is transformed as

$$
\prod_{\alpha=1}^{Ch} da_\alpha da^*_\alpha \rightarrow (-1)^{Ch} \prod_{\alpha=1}^{Ch} da_\alpha da^*_\alpha.
$$

Since the field configurations $A(x)$ and $-A(\tilde{x})$ are smoothly connected, there is no way to define the measure so that it is invariant under CP. As we have seen, this case corresponds to topologically non-trivial bulk. On the other hand, when $\eta = -1$, $(CP)\psi_L(x)(CP)^{-1} = \psi_R^\dagger(-x)$, $(CP)\psi_R(x)(CP)^{-1} = -\psi_L^\dagger(-x)$, with the extra...}

B. CP anomaly
minus sign, the measure is invariant. In the next section, we make contact between these two cases (the case with and without parity anomaly) and topologically non-trivial and trivial insulators.

C. Generalized Laughlin’s argument

In the above considerations, we have (implicitly) assumed that the electromagnetic charge \( U(1) \) is strictly conserved. However, it would be possible to instead demand CP symmetry to be strictly conserved. Given a conflict between CP symmetry and charge conservation (when \( \epsilon = 0 \)) suggested by the above argument, it would not be possible to preserve electromagnetic \( U(1) \) charge symmetry once we demand CP symmetry. This suggests the following: let us twist the boundary condition by the conserved electromagnetic \( U(1) \) charge (denoted by \( a \) and \( b \) as before in our discussion in the QH edge). The partition function depends on these twisting angles. One can then enforce CP symmetry by performing a projection on to a space with definite CP eigenvalue. In the path integral picture, the enforcement of CP symmetry leads to a conformal field theory defined on an unoriented spacetime, i.e., the spacetime of the edge theory has the topology of the Klein bottle. Once we insist on CP symmetry, one may not be able to achieve gauge invariance under (large) \( U(1) \) gauge transformations. Equivalently, the partition function would not be invariant under \( a \to a+1 \) or \( b \to b+1 \). We view this conflict between the charge \( U(1) \) and CP symmetries as a signal for the existence of a bulk topological phase.

c. Twisted boundary conditions

Let us now canonically quantize the fermion theory in the presence of the following spatial boundary condition:

\[
\psi_L(x + \ell_1) = e^{2\pi i \nu_L} \psi_L(x), \quad \psi_R(x + \ell_1) = e^{2\pi i \nu_R} \psi_R(x).
\]

where the edge theory is put on a circle of circumference \( \ell_1 \). A discrete symmetry (CP, in our example) may be compatible/incompatible with the boundary condition. By acting with CP on the boundary condition \((22)\),

\[
(CP) \psi_L(x + \ell_1)(CP)^{-1} = e^{2\pi i \nu_L} (CP) \psi_L(x)(CP)^{-1} \Rightarrow \psi_L^\dagger(-x - \ell_1) = e^{2\pi i \nu_L} \psi_R^\dagger(-x) \Rightarrow e^{2\pi i \nu_L} \psi_R(x - \ell_1) = \psi_R(x),
\]

we conclude that CP symmetry is consistent with the twisted boundary condition when \( \nu_L = \nu_R \), i.e., only charge twist is allowed, \( \psi_L(x + \ell_1) = e^{2\pi i \nu_L} \psi_L(x) \), \( \psi_R(x + \ell_1) = e^{2\pi i \nu_L} \psi_R(x) \). By similar considerations, P symmetry is consistent with the twisted boundary condition only when \( \nu_L = 0, 1/2 \) and \( \nu_R = 0, 1/2 \).

d. The torus partition function

For the CP symmetric case, we thus consider the spatial boundary condition with \( \nu_L = \nu_R = a \). The corresponding partition function on the torus is

\[
Z(\tau, \bar{\tau}) = \text{Tr}_{a \otimes a} \left[ e^{-2\pi i \theta (b-1/2) F_{V_L} q_{R} L_{R} q_{L}} \right] = Z_{a,b}(\tau) Z_{a,b}(\bar{\tau}) (24)
\]

where \( \cdots \) denotes complex conjugation, and the Hamiltonian \( H = H_R + H_L \) is given in terms of the left- and right-moving parts as

\[
L_R = L_0 - \frac{c_R}{24}, \quad L_L = \bar{L}_0 - \frac{c_L}{24} \quad H_R = \frac{2 \pi v}{\ell_1} L_R, \quad H_L = \frac{2 \pi v}{\ell_1} L_L \quad (25)
\]

with \( c_L = c_R = 1 \). We have introduced the modular parameter through

\[
q = e^{2\pi i \tau}, \quad \tau = \tau_1 + i \tau_2, \quad \tau_2 = \frac{v \ell_2}{\ell_1}. \quad (26)
\]

Here, \( \ell_2 \) represents the inverse temperature and we have included, in addition to the (imaginary) time translation generated by the Hamiltonian, the space translation generated by the momentum with the corresponding periodicity \( \tau_1 \). (As we will see, \( \tau_1 \) will not play any role once we impose CP symmetry.)

Written as a product \( Z_{a,b}(\tau) Z_{a,b}(\bar{\tau}) \), where \( Z_{a,b}(\tau) \) is given by Eq. \( (22) \), the partition function is large gauge invariant under \( b \to b + 1 \) and \( a \to a + 1 \). One can also check that the partition function is modular invariant.

e. The Klein bottle partition function with CP symmetry

Let us now consider the partition function with CP projection:

\[
Z^{\text{Proj}}_{[a]}(\tau) = \text{Tr}_{a \otimes a} \left[ \frac{1 + CP}{2} e^{-2\pi i \theta (b-1/2) F_{V_L} q_{R} L_{R} q_{L}} \right] \quad (27)
\]

where we have inserted a projection operator, \( (1 + CP)/2 \). The first term in the projection gives nothing but the torus partition function. The second term can be interpreted as a path integral over the fermion fields on the Klein bottle (with twisted boundary condition in the time direction.) To develop this picture, we first perform the Wick rotation \( t = -i x_2 \). The insertion of CP operator into the trace has the effect that by translating a fermion field \( \psi_R \), say, once around the time direction, it comes back as \( (CP) \psi_R (CP)^{-1} \). Thus, the time direction boundary condition is \((\ell_2 = 2 \pi \tau_2)\)

\[
\psi_R(x_1, x_2) = -(CP) \psi_R(x_1, x_2 + \ell_2)(CP)^{-1}, \quad \psi_L(x_1, x_2) = -(CP) \psi_L(x_1, x_2 + \ell_2)(CP)^{-1}. \quad (28)
\]

where the factor \(-1\) comes from the antiperiodic boundary condition of the fermion fields (we have set \( b = 1/2 \) for simplicity). I.e.,

\[
\psi_R(x_1, x_2) = -\eta_R(x_1, x_2 + \ell_2), \quad \psi_L(x_1, x_2) = -\eta_L(x_1, x_2 + \ell_2). \quad (29)
\]
(Observe that \( \tau_1 \) is “projected out” by CP – see below.) The fermion fields are defined on the Klein bottle \((x_1, x_2) \equiv (x_1 + \ell_1, x_2) \equiv (-x_1, x_2 + \ell_2)\) with periodic boundary condition along \(x_1\) (possibly twisted by \(a\)), but with the CP-twisted boundary condition along \(x_2\) direction.

There is a simple consequence of the topology change from the torus to the Klein bottle, induced by CP projection. The partition function on a Riemann surface of genus \(g\) (denoted by \(\Sigma_g\)) is given as the sum over all possible monodromies on \(\Sigma_g\):

\[
Z^{\Sigma_g}(\tau) = \frac{1}{|\mathcal{G}|} \sum_{\alpha : \pi_1(\Sigma_g) \rightarrow \mathcal{G}} Z^{\Sigma_g}(\alpha; \tau),
\]

where \(\tau\) are the moduli of \(\Sigma_g\), \(\pi_1(\Sigma_g)\) is the fundamental group of \(\Sigma_g\), and \(Z^{\Sigma_g}(\alpha; m)\) denotes the partition function calculated with the particular set of monodromies \(\alpha\). For the torus, \(\pi_1(T^2) = \alpha, \beta\), \(\alpha \beta = \beta \alpha\), \(\alpha^2 = \beta^2 = 1\), \(\alpha \beta \beta^{-1} = 1\), \(\alpha \beta = \beta \alpha\). This means that, for any Abelian group \(\mathcal{G}\) by considering a correspondence \(\alpha \rightarrow g_1, \beta \rightarrow g_2\) where \(g_1, 2 \in \mathcal{G}\), the summation is \(\text{Hom}[\pi_1(T^2), \mathcal{G}] = (\mathcal{G})^2\). On the other hand, for the Klein bottle, the fundamental group is given by \(\pi_1(K) = (\alpha, \beta; \alpha \beta = \beta^{-1} \alpha)\). For an Abelian group, this means \(g_2 = (g_1)^{-1}\). As we will show below, the \(Z_2\) flux distinguishes topological and non-topological CP symmetric insulators.

Let us work out the effects of the projection explicitly. We first mode-expand the fermion fields as

\[
\psi_R(x) = \sqrt{\frac{2\pi}{\ell_1}} \sum_{r \in \mathbb{Z} + a} \psi_{R,r} e^{i \frac{2\pi}{\ell_1} r x},
\]

\[
\psi_L(x) = \sqrt{\frac{2\pi}{\ell_1}} \sum_{r \in \mathbb{Z} + a} \psi_{L,r} e^{i \frac{2\pi}{\ell_1} r x}.
\]

The CP transformation acts on the fermion modes as

\[
(CP) \psi_{L,r} (CP)^{-1} = \psi_{R,r},
\]

\[
(CP) \psi_{R,r} (CP)^{-1} = -\eta \psi_{L,r},
\]

For a given \(r > 0\), there are four states, \(|G\rangle_{Sa}, \psi_{R,r}^\dagger |G\rangle_{Sa}, \psi_{L,r} \psi_{R,r}^\dagger |G\rangle_{Sa}, \psi_{L,r} \psi_{R,r}^\dagger |G\rangle_{Sa}\), where \(|G\rangle_{Sa} \propto \psi_{L,r}^\dagger |0\rangle\) is the ground state for the boundary condition specified by \(a\). On these states, CP acts as, \(e.g.,\)

\[
(CP) |G\rangle_{Sa} = P_{[a]} |G\rangle_{Sa},
\]

\[
(CP) \psi_{L,r} \psi_{R,r}^\dagger |G\rangle_{Sa} = -\eta P_{[a]} \psi_{L,r} \psi_{R,r}^\dagger |G\rangle_{Sa}.
\]

Here, \(P_{[a]}\), the CP eigenvalue of the ground state, is, a priori, undetermined. We have demanded that the system is CP invariant, and hence the first equation follows. Since CP is unitary, the eigenvalue \(P_{[a]}\) should be a complex number of unit modulus.

For our discussion, it is crucial to know the \(a\)-dependence of the CP eigenvalue of the ground state. In particular, we need to compare the relative phase difference between \(P_{[a]}\) and \(P_{[a+1]}\). Under the assumption of the strict enforcement of CP symmetry, \(P_{[a]}\) should be independent of \(a\), and in particular, \(P_{[a]} = P_{[a+1]}\). (If \(P_{[a]}\) is dependent on \(a\), the projection operation in fact is ill-defined.)

It is also insightful to use an alternative but equivalent picture for the effects of the fluxes \(a\) and \(b\), where they are introduced as, instead of twisting angles for twisted boundary conditions, constant background gauge fields. In this picture, the Hamiltonian depends explicitly on \(a\) and is given by

\[
H(a) = \frac{v}{2\pi} \int dx [\psi_L^\dagger i \partial_x \psi_L - \psi_R^\dagger i \partial_x \psi_R] + \frac{a}{\ell_1} J_V,
\]

where \(J_V = vF_A\) is the current operator. The fermion fields obey boundary conditions that are independent of \(a\).

\[
\psi_L(x + \ell_1) = \psi_L(x), \quad \psi_R(x + \ell_1) = \psi_R(x).\]

Under an infinitesimal change in the flux \(a \rightarrow a + \delta a\), since the perturbation commutes with CP and hence does not mix states with different eigenvalues of CP, the CP eigenvalue \(P_{[a]}\) should be constant. For a similar discussion, see Refs. 72–74.

Finally, the projected partition function can be calculated in a straightforward fashion, leading to

\[
Z_{[a]}^{\text{Klein}} = \text{Tr}_{a \otimes a} \left[ (CP) e^{-2\pi i (b-1/2) F_{v}} q^{L} \bar{q}^{L} \right] = P_{[a]} e^{2\pi i (a-1/2)(\epsilon-1/2)} \frac{\eta(q\bar{q})}{\eta(q)\eta(\bar{q})} \theta \left[ \frac{a - 1/2}{(\epsilon - 1/2)} \right] (0, q\bar{q}),
\]

where we note \(q\bar{q} = e^{2\pi i \tau} e^{-2\pi i \tau} = e^{-4\pi \tau_2} = e^{-4\pi \tau^2 + \frac{\pi^2}{4}}\) and \(\eta(q\bar{q}) = \eta(2i \text{Im} \tau)\). Observe that the partition function is independent of \(\tau_1\), i.e., it is projected out by CP. Similarly, the chemical potential \(b\) also does not show up in the projected partition function.

With \(P_{[a]} = P_{[a+1]}\), which we enforce by CP symmetry, the partition function is invariant under \(a \rightarrow a + 1\) for the topologically trivial case (\(\epsilon = 1/2\)) whereas it is not for the topologically non-trivial case (\(\epsilon = 0\)),

\[
Z_{[a+1]}^{\text{Klein}} = e^{2\pi i (\epsilon-1/2)} Z_{[a]}^{\text{Klein}}.
\]

By comparison with the chiral partition function (\(\theta\)), we observe the distinction between topologically trivial (\(\epsilon = 1/2\)) and nontrivial (\(\epsilon = 0\)) cases shows up as a fictitious chemical potential (\(\pi\) flux in time direction). For the topological case the fermion effectively feels periodic boundary condition in time direction, whereas for the trivial case, the fermion effectively feels antiperiodic boundary condition. This anomaly vanishes if we consider two copies (more generally, an even number of copies) of the fermion theory, which suggests a \(Z_2\) classification of CP symmetric topological insulators.
III. 2D BOSONIC TOPOLOGICAL PHASES PROTECTED BY CP SYMMETRY

A. The edge theory

Armed with insights from the fermionic symmetry protected topological phases, we now discuss the bosonic topological phases. Below, we study the partition function of the edge of the bosonic CP symmetric topological insulator. We start from the single-component free boson theory on a ring of circumference $\ell$ defined by $Z = \int \mathcal{D}[\phi] \exp(iS)$ with the action

$$S = \frac{1}{4\pi\alpha' \alpha''} \int dt \int_0^\ell dx \left[ \frac{1}{v} (\partial_t \phi)^2 - v (\partial_x \phi)^2 \right],$$

where the $\phi$-field is compactified with the compactification radius $R$ as $\phi \equiv \phi + 2\pi R$; $\alpha'$ is the coupling constant of the boson theory. The canonical commutation relation is

$$[\phi(t, x), \partial_t \phi(t, x')] = 2\pi i \alpha' v \sum_{m \in \mathbb{Z}} \delta(x - x' - m\ell).$$

The theory can be quantized and decomposed into the left- and right-moving sectors. We introduce the chiral decomposition of the boson field $\phi$ as

$$\phi(t, x) = \varphi_L(x^+) + \varphi_R(x^-), \quad x^\pm := vt \pm x.$$  

and also the dual boson field as

$$\theta(t, x) = \varphi_L(x^+) - \varphi_R(x^-).$$

As in the fermionic CP symmetric topological insulator, we consider two kinds of CP symmetries specified by $\epsilon = 0, 1/2$ as follows:

$$(CP)\phi(t, x)(CP)^{-1} = -\phi(t, -x),$$

$$(CP)\theta(t, x)(CP)^{-1} = +\theta(t, -x) + 2\pi\epsilon\alpha'/R.$$  

The single-component boson model with these CP symmetries is studied in Ref. 53, and it was demonstrated, based on microscopic analysis of gapping potentials, that the case with $\epsilon = 0$ is gappable while the case with $\epsilon = 1/2$ is not. We will reproduce this result from the generalized Laughlin argument with CP symmetry.

f. Quantization  The mode expansions for the left- and right-moving boson fields is given by

$$\varphi_L(x^+) = x_L + \pi\alpha' p_L \frac{x^+}{\ell} + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \tilde{\alpha}_n \frac{\sin n x^+}{n},$$

$$\varphi_R(x^-) = x_R + \pi\alpha' p_R \frac{x^-}{\ell} + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \tilde{\alpha}_n \frac{\sin n x^-}{n},$$

where $[\alpha_m, \alpha_{-n}] = [\tilde{\alpha}_m, \tilde{\alpha}_{-n}] = m \delta_{m, n}, \quad (n, m > 0)$

$$[x_L, p_L] = [x_R, p_R] = i,$$

and all the other commutators vanish. With the periodic boundary condition

$$\phi(t, x + \ell) = \phi(t, x) + 2\pi R w, \quad w \in \mathbb{Z},$$

the allowed momentum values are

$$p_L = \frac{R}{\alpha'} w + \frac{k}{R}, \quad p_R = \frac{R}{\alpha'} w + \frac{k}{R},$$

$$p = \frac{1}{2} (p_L + p_R) = \frac{k}{R}, \quad \bar{p} = \frac{1}{2} (p_L - p_R) = \frac{\alpha'}{R},$$

where $k$ and $w$ are integers. Correspondingly, the chiral boson fields and the dual boson field obey

$$\varphi_L(x + \ell) - \varphi_L(x) = +\pi\alpha' p_L,$$

$$\varphi_R(x + \ell) - \varphi_R(x) = -\pi\alpha' p_R,$$

$$\phi(x + \ell) - \phi(x) = \pi\alpha' (p_L - p_R) = 2\pi R w,$$

$$\theta(x + \ell) - \theta(x) = \pi\alpha' (p_L + p_R) = 2\frac{\alpha'}{R} k.$$  

The set of bosonic exponents consistent with the boundary conditions are $\exp i \left[ \frac{k}{R} \theta + \frac{w R}{\alpha'} \right] = \exp i [p_L\varphi_L + p_R\varphi_R]$. The Hamiltonian is given by

$$H = H_L + H_R = \frac{2\pi v}{\ell} (L_L + L_R),$$

$$L_L = \frac{\alpha' p_L^2}{4} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \frac{1}{24},$$

$$\bar{L}_R = \alpha' p_R^2 \frac{2}{4} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24}.$$  

Observe that the spectrum depends only on $R/\sqrt{\alpha'}$ and is invariant under $R \rightarrow \alpha'/R$ as expected.

B. Twisted boundary conditions and twisted partition function

Two conserved U(1) charges, one for each left- and right-moving sector, can be introduced as follows:

$$N_{L,R} = \int_0^\ell dx \partial_x \varphi_{L,R} = \alpha' \pi p_{L,R},$$

which satisfy $[\varphi_L, N_L] = [\varphi_R, N_R] = \alpha' \pi i$. The operator $G(a_c, a_s) = \exp i [2\pi a_c R p + 2\pi a_s (\alpha'/R) \bar{p}]$. (49) generates translations in $\phi$ and $\theta$ as

$$\phi \rightarrow \phi + 2\pi a_c R, \quad \theta \rightarrow \theta + 2\pi a_s (\alpha'/R).$$

By using the $U(1) \times U(1)$ symmetry generators, it is possible to twist the spatial boundary condition as

$$\phi(x + \ell) = \phi(x) + 2\pi R a_c + 2\pi R w,$$

$$\theta(x + \ell) = \theta(x) + 2\frac{\alpha'}{R} a_s + 2\frac{\alpha'}{R} k.$$  

With the twisted boundary condition, the momenta are given by
\[ \alpha' \hat{p} = R(a_c + w), \quad \alpha' p = \frac{\alpha'}{R} (a_s + k). \] (52)

As compared to the original quantization conditions, in the presence of the twist, the quantization conditions on \( p \) and \( \hat{p} \) are “shifted” by \( a_c \) and \( a_s \). Below, we will focus on the twist by the diagonal \( U(1) \) symmetry, \( a_s = 0 \).

Let us now consider the partition function twisted both in time and space directions. It can be written as
\[ Z'_{[a_c, b_c]} = \text{Tr}_{a_c} \left[ \mathcal{G}(b_c) e^{2\pi i \theta (L_0 - c/24)} q^{(L_0 - c/24)} \right] \] (53)
where the trace is taken with the quantization conditions,
\[ \hat{p} = \Delta \hat{p} + \frac{R}{\alpha'} w, \quad \Delta \hat{p} = \frac{Ra_c}{\alpha'}, \]
\[ p = \Delta p + \frac{k}{R}, \quad \Delta p = \frac{(\alpha'/R)a_s}{\alpha'} \] (54).

On the other hand, the twist in time direction is implemented as an insertion of the operator \( \mathcal{G}(b_c) = \mathcal{G}(b_c, b_s = 0) \).

C. The CP projected partition function

Following the discussion for fermionic CP symmetric topological insulators, we now project with the CP operator. Inserting the projection operator in the partition function (53), we consider
\[ Z_{[a_c]}^{\text{proj}} = \text{Tr}_{a_c} \left[ \frac{1 + CP}{2} \mathcal{G}(b_c) q^{(L_0 - c/24)} q^{(L_0 - c/24)} \right]. \] (55)

We will consider an adiabatic process where we change \( a_c \) to \( a_c + 1 \) and ask if the CP projected partition function is invariant or not.

When evaluating the CP projected partition function, it is necessary to know the action of CP operators on the states in the Hilbert space. The symmetry transformation on \( \phi \) and \( \theta \) implies the action of CP on each mode in the mode expansion of \( \phi \) and \( \theta \):
\[ (CP) \alpha_n (CP)^{-1} = \bar{\alpha}_n, \]
\[ (CP) \phi_0 (CP)^{-1} = -\phi_0, \quad (CP) p (CP)^{-1} = -p, \]
\[ (CP) \theta_0 (CP)^{-1} = \theta_0 + 2\pi \epsilon (\alpha'/R), \quad (CP) \hat{p} (CP)^{-1} = \hat{p}. \] (56)

where \( \phi_0 = x_L + x_R \) and \( \theta_0 = x_L - x_R \). [Since CP flips the sign of \( p \), for generic value of \( a_c \), there is no state that is invariant under \( a_c \). For the purpose of the CP projection, we thus should set \( \Delta p \equiv 0 \Rightarrow a_s = 0 \).] See discussion near Eq. (25).

For later use, we need to know the action of CP on the states in the zero mode sector. We will use the momentum basis \( \{ | p, \hat{p} \rangle \} \), where \( | p, \hat{p} \rangle \) is the momentum eigen ket. Recall that due to the compactification condition, \( \phi_0 \equiv \phi_0 + 2\pi R \) and \( \theta_0 \equiv \theta_0 + 2\pi \alpha'/R \), the corresponding momenta lie in the BZ. Since CP transformation sends the momentum operators as \( p \to -p \) and \( \hat{p} \to \hat{p} \), the momentum eigen ket \( CP | p, \hat{p} \rangle \) must be equal to \( | -p, \hat{p} \rangle \) up to a phase, \( CP | p, \hat{p} \rangle = e^{i\pi x} | -p, \hat{p} \rangle \). Similarly, the ket \( CP | \phi_0, \theta_0 \rangle \) must be equivalent to \( | -\phi_0, \theta_0 + 2\pi \epsilon (\alpha'/R) \rangle \), \( CP | \phi_0, \theta_0 \rangle = e^{i\pi x} | -\phi_0, \theta_0 + 2\pi \epsilon (\alpha'/R) \rangle \). We can read off these phases from the Fourier representation of the basis ket:
\[ | p, \hat{p} \rangle = \int \text{d}\phi_0 \text{d}\theta_0 e^{i\pi (\phi_0 + \theta_0 + \pi R)} | \phi_0, \theta_0 \rangle, \]
as \[ CP | p, \hat{p} \rangle = e^{-i\pi \epsilon (w + a_s)} \int \text{d}\phi_0 \text{d}\theta_0 e^{-i\pi (\phi_0 + \theta_0 + \pi R)} e^{iB} | \phi_0, \theta_0 \rangle. \] (57)

In order to have \( CP | p, \hat{p} \rangle \propto | -p, \hat{p} \rangle \), we need to take \( B(\phi_0, \theta_0) = \text{const.} = B \), and we conclude
\[ CP | p, \hat{p} \rangle = e^{-i\pi \epsilon (w + a_s)} e^{iB} | -p, \hat{p} \rangle. \] (58)
i.e., \( A(p, \hat{p}) = B - 2\pi \epsilon \alpha'/R \). One can also check, from the inverse Fourier representation
\[ | \phi_0, \theta_0 \rangle = \sum_{p, \hat{p}} e^{-i\pi (\phi_0 + \theta_0 + \pi R)} | p, \hat{p} \rangle, \] (59)
that \( CP | \phi_0, \theta_0 \rangle = e^{iB} | -\phi_0, \theta_0 + 2\pi \epsilon \alpha'/R \rangle \).

Summarizing, acting with CP operator on the basis ket \( | p, \hat{p} \rangle \),
\[ CP | p, \hat{p} \rangle = P_{[a_c]} e^{-i\pi \epsilon \alpha'/R} | -p, \hat{p} \rangle, \] (60)
where \( P_{[a_c]} \) is independent of \( p \) and \( \hat{p} \), but may be dependent on the adiabatic parameters, \( a_c \) and \( b_c \). While the presence of the phase factor \( e^{-i\pi \alpha'/R} \) can directly be seen from the CP transformation laws on the zero mode operators, the phase factor \( P_{[a_c]} \) cannot be determined; this originates from our ignorance on the CP eigenvalue of the ground state, and, in particular, on its dependence on the adiabatic parameters. Based on our previous discussion, however, we enforce CP invariance for arbitrary value of the adiabatic parameters. This means that we demand that our CP operation does not depend on the inserted flux. While the Hilbert space changes adiabatically, we do not allow the phase factor to be dependent on \( a_c \). This is the same assumption we had before for the case of fermions. We could then choose \( B = \pi a_c \), and hence \( P_{[a_c]} = 1 \).

Having established the CP action on the zero-mode wave functions, we now calculate the projected partition function explicitly. Note also \( (CP) \mathcal{G}(b_c) (CP)^{-1} = \mathcal{G}(\bar{b}_c) \). This limits a reasonable value of \( 2\pi R b_c \) to be 0 and \( \pi \). Then, the Klein bottle partition function is
\[ Z_{[a_c]}^{\text{Klein}} = \text{Tr}_{a_c} \left[ (CP) \mathcal{G}(b_c) q^{(L_0 - c/24)} q^{(L_0 - c/24)} \right] \]
\[ = (qq)^{-\frac{1}{2R}} \prod_{n=1} (1 - (qq)^n)^{-1} \]
\[ \times \sum_{p, \hat{p}} \langle pp | (CP) e^{iR b_c \frac{1}{2}(pl + pr)} q^{\hat{p}} q^{\hat{p}} q^{\hat{p}} | pp \rangle. \] (61)
In order for \( \langle p \bar{p} | CP | p \bar{p} \rangle \) to be non-zero, \( p \) must be zero (\( k = 0 \)). Then, \( p_L = -p_R = \bar{p} \) and hence,

\[
Z_{[a_c]}^{\text{Klein}} = (q \bar{q})^{-\frac{1}{2}} \prod_{n=1}^{\infty} \left[ 1 - (q \bar{q})^n \right]^{-1} \times \sum_{w \in \mathbb{Z}} (q \bar{q})^\frac{1}{2} \left( \frac{\theta}{2} w^{2 + a_c} \right) e^{-i2\pi w}. \tag{62}
\]

When \( \epsilon = 1/2 \), the partition function is not invariant under \( a_c \rightarrow a_c + 1 \), as it picks up an overall minus sign, \( Z_{[a_c+1]}^{\text{Klein}} = -Z_{[a_c]}^{\text{Klein}} \). As in the fermionic CP symmetric topological insulator, the anomaly is of \( \mathbb{Z}_2 \) kind since it vanishes when we consider two copies (or any even number of copies) of the theory.

**IV. K-MATRIX THEORIES PROTECTED BY SYMMETRIES**

In this section, based upon the previous sections, we consider edge theories consisting of multiple free bosons that can describe, in addition to non-interacting topological insulators, interacting Abelian topologically ordered phases. We will develop a criterion for the stability of the edge theories in the presence of CP and U(1) symmetries.

**A. K-matrix theories**

Let us consider the K-matrix theory with \( N \) component compactified boson fields described by the Lagrangian

\[
\mathcal{L} = \frac{1}{4\pi} \left( K_{IJ} \partial_t \phi^I \partial_x \phi^J - V_{IJ} \partial_x \phi^I \partial_x \phi^J \right) + \frac{e}{2\pi} \epsilon^{\mu\nu} q_I \partial_\mu \phi^I A_\nu^J + \frac{s}{2\pi} \epsilon^{\mu\nu} S_I \partial_\mu \phi^I A_\nu^J, \tag{63}
\]

where \( K \) is an \( N \times N \) symmetric and invertible matrix with integer-valued matrix elements, \( V \) is an \( N \times N \) symmetric and positive definite matrix that accounts for the (screened) translation-invariant two-body interactions between electrons. The \( N \) component vector (“charge vector”) \( q_I \), together with the unit of electric charge \( e \), describe how the system couples to an external electromagnetic U(1) gauge potential, \( A_\mu^J \). Similarly, the \( N \) component vector (“spin vector”) \( S_I \), together with the unit of “spin” charge \( s \), describe how the system couples to an external “spin” U(1) gauge potential, \( A_\mu^J \) that couples to the spin-1/2 degrees of freedom along some quantization axis, \( z \)-axis, say.

The boson fields are compact variables, meaning field configurations \( \phi^I \) differ by an integer multiple of \( 2\pi \) are identified:

\[
\phi^I(t,x) \equiv \phi^I(t,x) + 2\pi n^I, \tag{64}
\]

with \( n^I \in \mathbb{Z} \) for all \( I = 1, \ldots, N \). The equal-time canonical commutation relations of the boson fields are given by

\[
[\phi^I(t,x), \partial_x \phi^J(t,x')] = -2\pi i (K^{-1})^{IJ} \delta(x-x'), \tag{65}
\]

or equivalently

\[
[\phi^I(t,x), \phi^J(t,x')] = -i\pi [(K^{-1})^{IJ} \text{sgn}(x-x') + \Theta^{IJ}], \tag{66}
\]

where the Klein factor

\[
\Theta^{IJ} := -(K^{-1})^{IK} \left[ \text{sgn}(K - L)(K_{KL} + Q_K Q_L) \right] (K^{-1})^{LJ}, \tag{67}
\]

is included to ensure that local excitations satisfy the proper commutation relations.

The goal of this section is to develop, in the presence of either charge or spin U(1) symmetry, together with a discrete symmetry (such as CP or parity symmetry), a stability (“ingappability”) criterion of the edge theory against interactions.

**g. The rotated basis** We start our discussion by quantizing the K-matrix theory with the (untwisted) compactification condition \( [2] \). We introduce, starting from the original boson fields \( \{ \phi^I \}_{I=1,\ldots,N} \), a new basis \( \{ \varphi^i \}_{i=1,\ldots,N} \) that is obtained by a rotation matrix \( e_i^J \) and its inverse \( e_i^{J*} \) as

\[
\varphi^i \equiv e_i^J \phi^J, \quad \varphi^J \equiv e_i^{J*} \varphi^i, \quad e_i^J e_i^{J*} = \delta_j^I, \quad e_i^J e_i^{J*} = \delta_j^I. \tag{68}
\]

The “vielbein” \( e_i^J \) and \( e_i^{J*} \) diagonalize the K-matrix as

\[
K_{IJ} = e_i^J \eta_{ij} e_j^I, \quad \eta_{ij} = e_i^J K_{IJ} e_j^J = \eta_{ij} \delta_{ij}. \tag{69}
\]

where \( \eta_{ij} = \eta_{ji} \delta_{ij} \) is a diagonal matrix. We also note

\[
(K^{-1})^{IJ} = e_i^{J*} (\eta^{-1})^{ij} e_i^J. \tag{70}
\]

In the following, by choosing \( e_i^J \) and \( e_i^{J*} \) properly, we assume that \( \eta_{ij} \)'s are \( \pm 1 \). In the rotated basis \( \varphi \), the Lagrangian can be written as

\[
\mathcal{L} = \frac{1}{4\pi} \left( \eta_{ij} \partial_t \varphi^i \partial_x \varphi^j - \nu \delta_{ij} \partial_x \varphi^i \partial_x \varphi^j \right) + \frac{e}{2\pi} \epsilon^{\mu\nu} \tilde{Q}_I \partial_\mu \varphi^I A_\nu^J + \frac{s}{2\pi} \epsilon^{\mu\nu} S_I \partial_\mu \varphi^I B_\nu^J, \tag{71}
\]

where we have introduced the charge and spin vectors in the rotated basis as

\[
\tilde{Q}_i \equiv e_i^{J*} Q_I, \quad \tilde{S}_i \equiv e_i^{J*} S_I, \tag{72}
\]

and assumed, for simplicity, \( e_i^{J*} V_{IJ} e_j^J = \nu_{ij} = \nu \delta_{ij} \). The compactification condition in the original basis \( \{ \phi^I \} \) is translated into, in the rotated basis,

\[
\varphi^i(t,x) \equiv \varphi^i(t,x) + 2\pi e_i^J n^I. \tag{73}
\]
Quantization without the background fields  As a warm up, we first quantize canonically the theory without the background fields on the spatial circle of radius $2\pi$:

$$\mathcal{L}_0 = \frac{1}{4\pi} \left( \eta_{ij} \partial_i \varphi^I \partial_j \varphi^I - v_{ij} \partial_i \varphi^I \partial_j \varphi^J \right).$$

(74)

The equal-time commutation relations are

$$[\varphi^i(t, x), \partial_x \varphi^j(t, x')] = -2\pi i (\eta^{-1})^{ij} \delta(x - x').$$

(75)

The mode expansion of $\varphi^i$ is given by

$$\varphi^i(t, x) = \varphi_0^i - p_j \left[ (\eta^{-1})^k v_{kl} (\eta^{-1})^l t + (\eta^{-1})^{ij} x \right] + \frac{i}{2} \sum_{n \neq 0} b_{ni} e^{-in[(\eta^{-1})^k v_{kl} (\eta^{-1})^l t + (\eta^{-1})^{ij} x]},$$

(76)

where

$$\delta_{ij} = \frac{1}{m} \delta_{ij} \delta_n.$$

(77)

Together with the commutation relation

$$[\varphi_0^i, p_j] = i \delta_{ij}, \quad [b_{ni}, b_{mj}] = \frac{1}{m} \delta_{ij} \delta_{n+m}.$$

(All other commutators vanish.)

As $p_i$ is conjugate to $\varphi^i(t, x)$, which obeys the compactification condition $\varphi^i(t, x) \simeq \varphi^i(t, x) + 2\pi e_j^i n_i$, the quantization condition of $p_i$ is given in terms of the reciprocal lattice vectors $e_i^I$ as

$$p_i = e_i^I m_I, \quad m_I \in \mathbb{Z}^N.$$

(78)

I.e., while the coordinates $\varphi^i$ are compactified on a lattice $\Gamma$ spanned by $\{e^I\}$, the momenta $p_i$ lie in the reciprocal (dual) lattice $\tilde{\Gamma}$ spanned by $\{e_i^I\}$. Observe also that, in a momentum eigenstate, the mode expansion (74) implies the boundary condition

$$\varphi^i(t, x + 2\pi) = \varphi^i(t, x) - 2\pi p_j (\eta^{-1})^{ij} = \varphi^i(t, x) + 2\pi e_j^I m_I (\eta^{-1})^{ij} = \varphi^i(t, x) + 2\pi e_j^I (K^{-1})^{IJ} m_J.$$  

(79)

For generic integral values of $m_J$, $(K^{-1})^{IJ} m_J$ are not integers and hence the boson fields obey twisted boundary conditions. The states corresponding to the momentum $p_j$ are represented by (by state-operator correspondence) the vertex operators

$$\hat{\exp} \hat{p}_i \varphi^i(t, x) \hat{=} \hat{\exp} \hat{m}_I \phi_I(t, x) \hat{=}$$

(80)

where $\hat{\cdot}$ represents normal-ordering.

Let us consider a subset of $\tilde{\Gamma}$ that is obtained by choosing $m_I = K_{IJ} \Lambda^J$ with $\Lambda^J \in \mathbb{Z}^N$. For this choice, the momentum is given by

$$p_i = \eta_{ij} e_j^I \Lambda^J.$$  

(81)

and the boson fields $\varphi^i$ obey untwisted boundary conditions. In the sector of the theory with this choice of momentum, all excitations are local (excitations consisting of exciting electron-like particles). The corresponding vertex operators are

$$\hat{\exp} \hat{i} \Theta(\Lambda) \hat{=} \hat{\exp} \hat{i} \Lambda^J K_{IJ} \phi_J(t, x) \hat{=},$$

(82)

To summarize, quantization of the K-matrix theory with the compactification conditions (63-64) gives rise to the spectrum of local (electrons) as well as non-local (quasiparticle) excitations, which are represented by untwisted and twisted boundary conditions, respectively. Once we specify the boundary condition by some integer vector $m$, we obtain the spectrum quantized within one sector (labeled by the equivalent class $|m\rangle$ with the relation $m \equiv m + K \Lambda$) of the total spectrum. There are $|\det K|$ sectors in this compactified K-matrix theory.

The Hamiltonian and total momentum are

$$H_0 = \frac{1}{4\pi} \int_0^{2\pi} dx \partial_x \varphi^i v_{ij} \partial_x \varphi^j$$

$$= \frac{1}{2} (\eta^{-1})^i v_{ij} (\eta^{-1})^j - \frac{1}{24} \text{Tr} (\eta^{-1} v_{ij})$$

$$+ \sum_{n=1}^{\infty} n^2 (\eta^{-1} b_n)^i v_{ij} (\eta^{-1} b_n)^j$$

(83)

and

$$P_0 = \frac{1}{4\pi} \int_0^{2\pi} dx \partial_x \varphi^i \eta_{ij} \partial_x \varphi^j$$

$$= \frac{1}{2} p_i (\eta^{-1})^{ij} p_j - \frac{1}{24} \text{Tr} (\eta^{-1}) + \sum_{n=1}^{\infty} n^2 b_{ni} (\eta^{-1})^{ij} b_{nj}.$$  

(84)

respectively. The eigenstates of $H_0$ and $P_0$ can be expressed as a direct product of their oscillator part (the Fock states generated by $b_{ni}$) and non-oscillator part (related to $\varphi_0^i$ and $p_i$). For the non-oscillator part, one can choose to use the momentum eigenvalues $\{p_i\}$, which have values $\{\eta_{ij} e_j^I \Lambda^J + e_i^I m_I\}$ as the boundary condition $\phi^I(t, x + 2\pi) = \phi^I(t, x) + 2\pi (K^{-1})^{IJ} m_J$ or $\varphi^i(t, x + 2\pi) = \varphi^i(t, x) + 2\pi e_j^I (K^{-1})^{IJ} m_J$ in the rotated basis, is specified, to label the eigenstates. We denote these eigenstates of sector $|m\rangle$ as $|A_m\rangle \equiv |A + K \Lambda m\rangle$.

The partition function for the sector $|m\rangle$ evaluated on a torus with modular parameter $\tau = \tau_1 + i\tau_2$ is given by

$$Z_m(\tau) = \text{Tr}_m \left[ e^{-2\pi i \tau} P_0 e^{-2\pi \tau H_0} \right].$$  

(85)

1. Twisted boundary conditions by U(1) symmetries

The K-matrix theory (13) has U(1)$^N$ symmetries. The corresponding conserved charges are given by

$$C^I \equiv \frac{1}{2\pi} \int_0^{2\pi} dx \partial_x \phi^I = -2\pi (K^{-1})^{IJ} e_j^I p_j.$$  

(86)

The global U(1) transformations associated to these charge degrees of freedom are generated by

$$G(\alpha) \equiv e^{-2\pi i \alpha^I C^I}$$

as

$$G(\alpha) \varphi^i(t, x) G(\alpha)^{-1}(t, x) = \varphi^i(t, x) + 2\pi (\eta^{-1})^{ij} e_j^I \alpha_J.$$  

(87)
where $\alpha$ is a vector consisting of twisting phases.

Now, starting from the original boundary condition for sector $[\mathbf{m}]$, we can generate a new twisted boundary condition by acting with $G$ as

$$\phi^j(t, x + 2\pi) = G(a)\phi^j(t, x)G(a)^{-1} + 2\pi(K^{-1})^Jm_J,$$

or, in the rotated basis,

$$\phi^j(t, x + 2\pi) = G(a)\phi^j(t, x)G(a)^{-1} + 2\pi(n^{-1})^j e^i_j m_J = \varphi(t, x) + 2\pi(n^{-1})^j e^i_j (m_J + a_J).$$

With this twisted boundary condition, the allowed values of the momenta $p$ are now shifted and given by

$$p_i = e^i_I (m_I + K_I\Lambda^J + a_J) \equiv e^i_I K_I\Lambda^J m_{a_i},$$

where

$$\Lambda^J_{m+a} := \Lambda^J + (K^{-1})^J (m_I + a_J), \quad \Lambda^J \in \mathbb{Z}^N.$$  \hspace{1cm} (91)

As in the untwisted case [in the absence of U(1) twisting phases], the eigenstates of the Hamiltonian and the total momentum can be expressed as a direct product of their oscillator part and non-oscillator part. One can choose to use the momentum eigenvalues, which are specified by a set of integers $\Lambda^J \in \mathbb{Z}^N$ to label non-oscillator part of the eigenstates. We denote these basis states as $|\Lambda^J_{m+a}\rangle$, which are given by the untwisted eigenstates with $\mathbf{m}$ shifted by $\mathbf{a}$.

2. Twisted partition function

The twisted partition function for sector $[\mathbf{m}]$ evaluated on a torus with modular parameter $\tau = \tau_1 + i\tau_2$ is given by

$$Z_{m[a, b]}(\tau) = \text{Tr}_{m+a} [G(b) e^{-2\pi i\tau_1 P_0} e^{-2\pi i\tau_2 H_0}],$$

where the trace is taken over the Hilbert space in the presence of the twisted boundary condition generated by $G(a)$. The operator insertion $G(b)$ generates, in the path-integral picture, twisted boundary condition in time direction. The partition function can be expressed as a product of the oscillator part and the zero-mode part as

$$Z_{m[a, b]}(\tau) = \xi(\tau) \sum_{\Lambda \in \mathbb{Z}^N} \zeta^\Lambda_{m[a, b]}(\tau) \langle \Lambda_{m+a} | \Lambda_{m+a} \rangle,$$

where

$$\zeta^\Lambda_{m[a, b]}(\tau) \equiv \exp \left( 2\pi i b^T \Lambda_{m+a} - \pi i \tau_1 \Lambda^T_{m+a} K \Lambda_{m+a} - \pi i \tau_2 \Lambda^T_{m+a} V \Lambda_{m+a} \right).$$

(93)

The oscillator part of the partition function $\xi(\tau)$ is independent of the twisting angles $\mathbf{a}$ and $\mathbf{b}$ and will not play any important role in the following discussion. The overlap $\langle \Lambda_{m+a} | \Lambda_{m+a} \rangle$ in Eq. (94) is simply $\langle \Lambda_{m+a} | \Lambda_{m+a} \rangle = 1$, but we displayed $\langle \Lambda_{m+a} | \Lambda_{m+a} \rangle$ in Eq. (94) for the later comparison.

3. Large gauge transformations

The large gauge transformations of U(1) symmetries are finite gauge transformations that preserve the spectrum of the theory. They are finite shifts of twisting phases $\mathbf{a}$ and $\mathbf{b}$ that preserve the U(1) operators $G$ [or more precisely, preserve the (twisted) boundary conditions] and can be deduced from the compactification condition of the K-matrix theory (94). For U(1)$^N$ symmetry, the large gauge transformations are given by

$$\mathbf{a} \rightarrow \mathbf{a} + K\delta, \quad \mathbf{b} \rightarrow \mathbf{b} + K\delta', \quad \forall \delta, \delta' \in \mathbb{Z}^N.$$  \hspace{1cm} (94)

To discuss the behavior of the twisted partition function under the large gauge transformation, let us consider $U(1) = U(1)_c \times U(1)_s$ symmetry: $\mathbf{a} = a_c Q + a_s S$ and $\mathbf{b} = b_c Q + b_s S$, where $Q$ and $S$ are charge and spin vectors, respectively. The minimal shifts are given by

$$\delta_c = 1/\beta_c, \quad \delta_s = 1/\beta_s,$$  \hspace{1cm} (95)

where $\beta_c \equiv \min_i |t^T K^{-1} Q|$ and $\beta_s \equiv \min_i |t^T K^{-1} S|$ represent the elementary charge and spin (the smallest fractional charge and spin of quasiparticle excitations) of the system, respectively. In other words, classically, the system is expected to be invariant under the following large gauge transformation:

$$a_c/s \rightarrow a_c/s + \delta_c/s, \quad b_c/s \rightarrow b_c/s + \delta_c/s.$$  \hspace{1cm} (96)

The invariance under the large gauge transformation may, however, be violated at the quantum level. From Eq. (93), we see, under large gauge transformations for the charge U(1)$_c$ symmetry,

$$Z_m(a_c + \delta_c, b_c, a_s, b_s) = Z_m(a_c, b_c, a_s, b_s),$$

$$Z_m(a_c, b_c + \delta_c, a_s, b_s) = e^{2\pi i\delta_c Q^T K^{-1}(a_c, Q+a_s, S) \cdot Z_m(a_c, b_c, a_s, b_s)}.$$  \hspace{1cm} (97)

Similarly, under large gauge transformations for the spin U(1)$_s$ symmetry,

$$Z_m(a_c, b_c, a_s + \delta_s, b_s) = Z_m(a_c, b_c, a_s, b_s),$$

$$Z_m(a_c, b_c, a_s, b_s + \delta_s) = e^{2\pi i\delta_s S^T K^{-1}(a_c, Q+a_s, S) \cdot Z_m(a_c, b_c, a_s, b_s)}.$$  \hspace{1cm} (98)

Observe that the way the partition function changes under the large gauge transformations does not depend on the sector $\mathbf{m}$ we specify.

In the cases where there is only the charge U(1)$_c$ symmetry, the above calculation tells us that $Z_{[a_c, b_c]}(\tau)$ is not invariant under the large-gauge transformations if

$$Q^T K^{-1} Q \neq 0.$$  \hspace{1cm} (99)

This large gauge anomaly is nothing but the quantum Hall effect.
B. Symmetry projected partition functions: generalities

Now let us move on to the situations of our main interest. We consider the $K$-matrix theory that preserves one of the U(1) symmetries, U(1)$_c$ or U(1)$_s$, but not both. We denote this U(1) symmetry by $\mathscr{G} = U(1)_{c,s}$. In addition, we assume the $K$-matrix theory is invariant under yet another global unitary symmetry; we call the corresponding symmetry group $\mathscr{G}'$. In our examples below, $\mathscr{G}'$ consists of a single discrete unitary symmetry transformation such as CP or P transformation. The total symmetry group is $\mathscr{G} \times \mathscr{G}'$.

Under the action of a symmetry generator $\mathcal{M} \in \mathscr{G}'$, the bosonic fields transform as:

$$M\phi(t, x)M^{-1} = U_M\phi(t, r_Mx) + \pi K^{-1} \chi_M, \quad (100)$$

where $U_M$ is an integral $N \times N$ matrix, $r_M$ is a real number, and $\chi_M$ is some $N$-component real vector. For on-site symmetry, $r_M = 1$. For non-on-site symmetry (below we consider parity, P, or some on-site symmetry combined with parity, such as CP), we have $r_M = -1$. Assuming the $K$-matrix theory is invariant under group $\mathscr{G}'$, $U_M$ and $r_M$ must satisfy

$$U_T^T K U_M = r_M K, \quad U_T^T U_M = r_M^2 V = V, \quad (101)$$

for any $\mathcal{M} \in \mathscr{G}'$. The invariance under $\mathscr{G}'$ also imposes constraints on the integer vector $Q$ or $S$ through the way the charge or spin current are transformed under $\mathscr{G}'$.

Following our discussion in the previous sections, our strategy to diagnose the stability of the edge theory is to enforce the invariance under $\mathscr{G}'$ by projection, and discuss the dependence of the projected partition function on the twisting phases. In order for this strategy to work, the twisted boundary conditions should be invariant under the symmetry $\mathscr{G}'$. Acting on the twisted boundary condition with a symmetry generator $\mathcal{M}$,

$$MK\phi(t, x + 2\pi)M^{-1} = MK\phi(t, x)M^{-1} + 2\pi (m + a)$$

$$\Rightarrow K\phi(t, r_M(x + 2\pi)) = K\phi(t, r_Mx) + 2\pi r_M U_T^T (m + a). \quad (102)$$

In order for the twisted boundary condition to be invariant under $\mathcal{M}$ for arbitrary value of $a_c$ ($a_s$), the charge vector $Q$ (the spin vector $S$) must satisfy

$$U_T^T m = m \quad \text{and} \quad U_T^T Q = Q \quad (U_T^T S = S), \quad (103)$$

respectively. In our discussion below, we assume, for given $\mathscr{G}$ and $\mathscr{G}'$, this condition is satisfied.

Finally, from the group structure of $\mathscr{G}'$ and the statistics of vertex operators, which represent local excitations, there are further constraints on the possible form of $U_M$ and $\chi_M$. This issue will be discussed in more details later with specific examples. In conclusion, a general $K$-matrix theory with symmetry group $\mathscr{G} \times \mathscr{G}'$ is described by the data $\{K, Q, S, \{U_M, \chi_M | M \in \mathscr{G}'\}\}$ that satisfies the conditions discussed above.

The symmetry projected partition function for the sector $|m\rangle$ is defined by

$$Z_{m[a, b]}^{\text{Proj}} \equiv \text{Tr}_{m+a} [P_{\mathscr{G}'} G(b) e^{-2\pi i r_1 P_0 e^{-2\pi r_2 H_0}}],$$

$$\text{where} \quad P_{\mathscr{G}'} = |\mathscr{G}'|^{-1} \sum_{\mathcal{M} \in \mathscr{G}'} \mathcal{M} \quad (104)$$

is the projection operator for the symmetry group $\mathscr{G}'$, satisfying $P_{\mathscr{G}'}^2 = P_{\mathscr{G}'}$. The trace in Eq. (104) is taken with respect to the Hilbert space in the presence of boundary conditions twisted by $G(a)$, and the insertion of the operator $G(b)$ inside the trace represents, in the path integral picture, the U(1) twisting phase in the temporal direction. As mentioned earlier, the twisting should be invariant under $\mathscr{G}'$, and hence typically only the charge twisting angles [$a_c, b_c$] or the spin twisting angles [$a_s, b_s$] is allowed. In this section, we discuss some general properties of $Z_{m[a, b]}^{\text{Proj}}$ keeping both charge and spin twisting angles. Once $\mathscr{G}'$ is given, and the invariance of the twisting boundary condition by $\mathscr{G}'$ is taken into account, it is easy to “switch off” either one of charge or spin angle.

The twisted partition function, that appears as a part of the projected partition function $Z_{m[a, b]}^{\text{Proj}}$, can be evaluated as

$$Z_{m[a, b]}^{M} (\tau) = \text{Tr}_{m+a} [\mathcal{M} G(b) e^{-2\pi i r_1 P_0 e^{-2\pi r_2 H_0}}]$$

$$= \xi^M(\tau) \sum_{\Lambda \in \mathbb{Z}^n} \zeta_{|m+a|}^\Lambda (\Lambda M_{m+a}), \quad (105)$$

where the oscillator part of the partition function $\xi^M(\tau)$ does not depend on the twisting phases $a$ and $b$.

The most crucial part of the calculations, as inferred from the previous examples of the Dirac fermions and the single-component boson, is the matrix element $\langle \Lambda_{m+a} | M | \Lambda_{m+a} \rangle$ in Eq. (105). As one can read off from Eq. (100), the transformation $\mathcal{M}$ maps the momentum eigenvalues $\Lambda_{m+a} \rightarrow r_M U_M \Lambda_{m+a}$, and hence $\mathcal{M} | \Lambda_{m+a} \rangle$ should be equal to $| r_M U_M \Lambda_{m+a} \rangle$ up to a phase factor.

To calculate this phase factor, in particular in the presence of the Klein factors, it is convenient to use the state-operator correspondence; according to the state-operator correspondence, for each sector of the Hilbert space constructed out of the zero-mode $|\Lambda_{m+a}\rangle$, we have a corresponding operator

$$\hat{\Lambda}^T K \phi \rangle \quad (106)$$

As a warm up, let us consider the untwisted ($a = 0$) counterpart when $m = 0$

$$\hat{\Lambda}^T K \phi \rangle \quad (107)$$
Now, using symmetry conditions (101) we have

\[ M \xi e^{i AT K \phi(t_0)} \equiv M^{-1} \]
\[ = e^{i \Delta \phi^\Lambda M} e^{i AT K (M \phi(t_0) M^{-1})} \]
\[ = e^{i \Delta \phi^\Lambda M} e^{i AT K (U_M \phi(t_0) + \pi K^{-1} \chi_M)} \]
\[ = e^{i \Delta \phi^\Lambda M} e^{i \pi A^T \chi_M} \]
\[ = e^{i \phi^\Lambda M} e^{i \pi A^T \chi_M} \] (108)

where \( e^{i \Delta \phi^\Lambda M} \) is the statistical phase factor of the vertex operator \( \xi \) under symmetry transformation \( M \), as explained in Appendix 3. For bosonic systems, such phase factor \( e^{i \Delta \phi^\Lambda M} \) equals to 1 because of the commutativity among bosons. For fermionic systems, however, we must take into account the anti-commutativity among fermions, which may lead to an additional phase factor for the transformation of the vertex operator.

In the presence of the twisting angles, the action of \( M \) on non-oscillator state \( | \Lambda_{m+a} \rangle \) may give rise to an additional phase factor which in principle depends on \( m + a \). Let us now take a close look at this. States \( | \Lambda_{m+a} \rangle \) labeled by shifted momentum \( \Lambda_{m+a} \) can be viewed as generated from a ground state \( GS_{m+a} \) by acting on some raising operator. Suppose the ground state \( GS_{m+a} \) is invariant under \( M \), then the eigenvalue of \( M \) for the ground state, defined by

\[ M | GS_{m+a} \rangle = P^M_{m+a} | GS_{m+a} \rangle, \] (109)

depends on the spatial twisting phases. The phase \( P^M_{m+a} \) together with \( e^{i \Delta \phi^\Lambda M + \pi A^T \chi_M} \) in Eq. (108), leads to

\[ M | \Lambda_{m+a} \rangle = P^M_{m+a} e^{i \Delta \phi^\Lambda M + \pi A^T \chi_M} | r_M U_M \Lambda_{m+a} \rangle. \] (110)

In other words, the dependence on the twisting angles comes only from \( P^M_{m+a} \) but not from \( e^{i \Delta \phi^\Lambda M + \pi A^T \chi_M} \). In fact, as we noted previously in the case of the Dirac fermion and the single-component boson, once we insist on invariance under \( \mathcal{G} \), the eigenvalues of the symmetry transformation would not change as we change \( a_{k,c} \). If so, the phase \( e^{i \Delta \phi^\Lambda M + \pi A^T \chi_M} \), which we get from the vertex operator of the untwisted theory, is the only phase factor that we need to keep track of.

Since \( M \) maps the momentum eigenvalues \( \Lambda_{m+a} \rightarrow r_M U_M \Lambda_{m+a} \), in the summation in Eq. (107), only those \( \Lambda \)s that satisfy \( \Lambda_{m+a} = r_M U_M \Lambda_{m+a} \) contribute. With the conditions (101) and (103), this means that the first term in \( \Lambda_{m+a} \) in Eq. (101) satisfies \( \Lambda = r_M U_M \Lambda \). Then the twisted partition function (104) is given by

\[ Z^M_{m,a,b}(\tau) = \xi^M(\tau) P^M_{m+a} \sum_{\Lambda \in \mathbb{Z}^N, \Lambda = r_M U_M \Lambda} e^{i \Delta \phi^\Lambda M + \pi A^T \chi_M} \xi^\Lambda_s | \Lambda_{m+a,b} \rangle(\tau). \] (111)

From Eq. (111) we observe that the symmetry projected partition function for the sector \( |m\rangle \) depends only on parameters \( m + a \) and \( b \). This means that the way the projected partition function changes under large gauge transformation does not depend on \( m \) (i.e., it is independent of sector). For compactness, we will drop the label \( m \) (or just set \( m = 0 \)) on partition functions in the following discussion.

C. \( \mathcal{G} \times \mathcal{G}' = U(1)_c \times Z^\text{CP}_2 \)

Now we consider the non-on-site CP symmetry. Let

\[ (CP) | \phi(t_0) \rangle (CP)^{-1} = U_{CP} | \phi(t, -x) \rangle + \pi K^{-1} \chi_{CP}, \] (112)

where \( U_{CP} \) is an integer \( N \times N \) matrix (the same as the dimension of \( K \)) and \( \chi_{CP} \) is some \( N \)-component real vector. In order for the system to be CP invariant, we require

\[ U_{CP}^T K U_{CP} = -K, \quad U_{CP}^T V U_{CP} = V, \quad U_{CP}^2 = I_N, \]
\[ U_{CP}^T Q = Q, \quad (I_N - U_{CP}^T) \chi_{CP} = 2\epsilon Q \pmod{2}, \] (113)

where \( I_N \) is the \( N \times N \) identity matrix and the value \( \epsilon = 0, 1/2 \) represents the sign of the CP operator squared for fermionic systems, with the relation

\[ (CP)^2 = e^{2\pi i N_f}, \] (114)

where \( N_f \) is the total fermion number operator.

In fact, these constraints on \( (K, Q, U_{CP}, \chi_{CP}) \) are identical to the corresponding data in \( K \)-matrix theories with time-reversal invariance. The most general gauge inequivalent solution (which exists for a non-chiral \( K \)-matrix theory; \( N \) must be even) is of the form

\[ K = \begin{pmatrix} 0 & A & B & B \\ A^T & 0 & C & -C \\ B^T & C^T & \Gamma & W \\ B^T & -C^T & W^T & -\Gamma \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q' \\ q \end{pmatrix}, \]
\[ U_{CP} = \begin{pmatrix} -I_M & 0 & 0 & 0 \\ 0 & I_M & 0 & 0 \\ 0 & 0 & 0 & I_N^{\frac{\pi}{2} - M} \\ 0 & 0 & I_N^{\frac{\pi}{2} - M} & 0 \end{pmatrix}, \]
\[ \chi_{CP} = \begin{pmatrix} x \\ 0 \\ (1 - 2\epsilon)x' \\ (1 - 2\epsilon)x^2 + 2\epsilon q \end{pmatrix}. \] (115)

Here, the matrix \( A \) is \( M \times M \), while the matrices \( B, C \) are \( M \times (N - M) \). The matrices \( \Gamma, W \) are both \( (\frac{\pi}{2} - M) \times (\frac{\pi}{2} - M) \). Similarly, \( q' \) is of dimension \( M \) and \( Q \) is of dimension \( (\frac{\pi}{2} - M) \). Finally, \( x \) is a \( M \)-dimensional vector consisting of 1’s and 0’s, while \( x' \) is a \( (\frac{\pi}{2} - M) \)-dimensional vector consisting of 1’s and 0’s. There are only a few constraints on \( (A, B, C, \Gamma, W, q, q', x, x') \). First, \( W \) must be antisymmetric: \( W = -W^T \). This
requirement follows from CP symmetry \cite{13}. Second, \( q' \) must be even-valued. This constraint comes from \( Q_I = K_{1I} \mod 2 \), which means the insulator is composed out of electrons. For the same reason, the parity of \( Q_I \) must match with that of \( K_{1I} \), but can be either even or odd. Finally, the greatest common factor of \( \{Q_I\} \) must be 1.

Once these data are given, we now calculate the CP symmetry projected partition function with charge U(1) symmetry \((a = a_eQ, b = b_cQ)\):

\[
Z^{\text{Proj}}_{[a_e, b_c]}(\tau) = \text{Tr}_{a_e} \left[ \mathcal{P}_{\text{CP}} \mathcal{G}(b_c)e^{-2\pi i \tau_1 P_0}e^{-2\pi \tau_2 H_0} \right],
\]

with \( \mathcal{P}_{\text{CP}} = \frac{1 + \mathcal{P}_{\text{CP}}}{2} \).

(116)

1. Bosonic systems

In the case where the system is composed of bosons, the most general data \((K, Q, U_{\text{CP}}, \chi_{\text{CP}})\) is given by \cite{13}, with an additional condition that the charge vector \(Q\) is even valued (and thus the diagonal elements of \(\Gamma\) are also even). In this CP invariant theory, the function \(\zeta^\Lambda_{[a_e, b_c]}(\tau)\) with the constraint \(\Lambda = -U_{\text{CP}}^T \Lambda\) is given by

\[
\zeta^\Lambda_{[a_e, b_c]}(\tau) = \exp \left( -\pi \tau_2 \Lambda^T_{a_e} V \Lambda_{a_e} \right),
\]

(117)

where \(\Lambda_{a_e} \equiv \Lambda + a_e K^{-1}Q\) and the fact \(Q^T \Lambda_{a_e} = \Lambda^T_{a_e} K \Lambda_{a_e} = 0\) (by CP symmetry) is used. Therefore, the partition function

\[
Z_{[a_e, b_c]}^{\text{CP}}(\tau) = \text{Tr}_{a_e} \left[ (\mathcal{CP}) \mathcal{G}(b_c)e^{-2\pi i \tau_1 P_0}e^{-2\pi \tau_2 H_0} \right]
\]

is calculated as [Eq. (113)]

\[
Z_{[a_e, b_c]}^{\text{CP}}(\tau) = \mathcal{P}_{[a_e]}^{\text{CP}}(\tau) \sum_{\Lambda \in \mathbb{Z}^N_{\Lambda = -U_{\text{CP}}^T \Lambda}} e^{-\pi \tau_2 \Lambda^T_{a_e} \Lambda_{a_e} + i\pi \Lambda^T \chi_{\text{CP}}},
\]

(119)

where \(\mathcal{P}_{[a_e]}^{\text{CP}}\) is the CP eigenvalue of the ground state. Observe that the charge U(1) transformation operator \(\mathcal{G}(b_c) = e^{-2\pi ib_c N_I}\) and the spatial translation operator (in space coordinate \(x\)) \(e^{-2\pi i \tau_1 P_0}\) in the partition function are both projected out, leading to the independence of \(a_e\) and \(\tau_1\) in \(Z^{\text{CP}}\). This can also be argued by the fact that the total charge \(J^0_c\) and momentum \(P_0\) are odd under CP, while the Hamiltonian \(H_0\) is even. For the same reason, the function \(\xi^{\text{CP}}\) just depends on \(\tau_2\). [See similar discussion near Eqs. (20) and (21)].

The bosonic CP symmetry projected partition function is given by

\[
Z^{\text{Proj}}_{[a_e, b_c]}(\tau) = \frac{1}{2} \left[ Z_{[a_e, b_c]}(\tau) + Z_{[a_e]}^{\text{CP}}(\tau_2) \right],
\]

(120)

with the form of \(Z_{[a_e, b_c]}(\tau)\) given by (105). Under a large gauge transformation \(a_e \rightarrow a_e + \delta e\) and \(b_c \rightarrow b_c + \delta c\), where \(\delta = (\min_1 |U^T K^{-1} Q|)^{-1}\), we have

\[
Z_{[a_e + \delta_e, b_c + \delta_c]}(\tau) = Z_{[a_e, b_c]}(\tau),
\]

(121)

\[
Z_{[a_e + \delta_e]}^{\text{CP}}(\tau_2) = \frac{P_{\text{CP}}^{\text{CP}}}{P_{\text{CP}}^{\text{CP}}} e^{-i\pi \Lambda^T \chi_{\text{CP}}} Z_{[a_e]}^{\text{CP}}(\tau_2),
\]

(122)

where \(\Lambda = \delta K^{-1}Q\) is an integer vector, and

\[
Z_{[a_e, b_c + \delta_c]}(\tau) = e^{2i \pi a_e Q^T K^{-1} Q \cdot Z_{[a_e]}(\tau_2)}.
\]

(123)

Since \(Q^T K^{-1} Q = 0\) by CP symmetry, the total projected partition function is invariant under \(b_c \rightarrow b_c + \delta_c\). The crucial part is the behavior of the partition function under \(a_e \rightarrow a_e + \delta e\). If we demand the CP eigenvalue be invariant under \(a_e \rightarrow a_e + \delta e\), i.e., \(P_{[a_e + \delta_e]}^{\text{CP}} = P_{[a_e]}^{\text{CP}}\), then the partition function is (not) large gauge invariant if the value of \(\Lambda^T \chi_{\text{CP}}\) is even (odd). Therefore, the quantity

\[
\Lambda^T \chi_{\text{CP}} = \delta K^T \chi_{\text{CP}} \equiv\delta
\]

(123)

gives the criterion: "\(\Lambda^T \chi_{\text{CP}} = \text{odd number}" corresponds to theory with anomaly (topological phase), while "\(\Lambda^T \chi_{\text{CP}} = \text{even number}" corresponds to theory without anomaly (trivial phase).

2. Fermionic systems

For fermionic systems, the most general data \((K, Q, U_{\text{CP}}, \chi_{\text{CP}})\) is given by

\[
\Lambda = \left( \begin{array}{cc} \Gamma & W \\ W^T & -\Gamma \end{array} \right), \quad Q = \left( \begin{array}{c} q \\ q \end{array} \right),
\]

\[
U_{\text{CP}} = \left( \begin{array}{cc} I_N & 0 \\ 0 & I_N \end{array} \right), \quad \chi_{\text{CP}} = \left( \begin{array}{cc} 2(1/2 - \epsilon) x' \\ 2(1/2 - \epsilon) x' + 2\epsilon q \end{array} \right).
\]

(124)

The calculation of the CP symmetry projected partition function in this theory can be done in the same way as the bosonic case, except the statistical phase factors that arise in:

\[
(CP)^{\frac{1}{4}} e^{i\Theta(\Lambda)} \stackrel{U_{\text{CP}}}{(CP)^{-\frac{1}{4}}} = e^{i\Delta\phi^\Lambda_{\text{CP}}} e^{i\pi \Lambda^T \chi_{\text{CP}}} \equiv e^{i\Theta(-U_{\text{CP}}^T \Lambda)^4},
\]

(125)

where

\[
i\Delta\phi^\Lambda_{\text{CP}} = i\pi \left( \sum_{l=1}^{N/2} \Lambda_l Q_l \right) \left( \sum_{j=N/2+1}^N \Lambda_j Q_j \right) \mod 2\pi i,
\]

(126)

is the statistical phase factor due to Fermi statistics derived in Appendix B. For CP invariant vectors \(\Lambda\) satisfying \(\Lambda = -U_{\text{CP}}^T \Lambda\), we can express \(\Lambda\) as \((\lambda, -\lambda)^T\), where \(\lambda\) is an \(N/2\) dimensional integer vector. Then the statistical phase can be expressed as

\[
i\Delta\phi^\Lambda_{\text{CP}} = -i\pi \sum_{l=1}^{N/2} \lambda_l q_l = -i\pi \lambda^T q \mod 2\pi i.
\]

(127)
On the other hand, 
\[ i\pi \Lambda^T \chi_{\text{CP}} = -i\epsilon \pi \Lambda^T q \mod 2\pi i \quad (128) \]
Writing \( \Lambda_{a_\epsilon}^T \Lambda_{a_\epsilon} = 2\Lambda_{a_\epsilon}^T \lambda_{a_\epsilon} \), where \( \lambda_{a_\epsilon} \equiv \lambda + \frac{e}{\tau_1} \lambda_c \) and \( \lambda_c \) is defined as 
\[ \delta_c K^{-1} Q = \Lambda_c \equiv \left( \begin{array}{c} \lambda_c \\ -\lambda_c \end{array} \right) \quad (129) \]
(remember that \( \Lambda_c = -U_{\text{CP}} \Lambda_c \), so \( \lambda_c \) is well-defined), then we have 
\[ Z_{[a_c, b_c]}^{\text{CP}}(\tau) = P_{[a_c]}^{\text{CP}}(\tau) \sum_{\Lambda \in \mathbb{Z}^N} e^{-\pi \tau_2 \Lambda_{a_\epsilon}^T \Lambda_{a_\epsilon} + i\pi \Lambda^T \chi_{\text{CP}} + i\Lambda \delta_c^\Lambda} \]
\[ = P_{[a_c]}^{\text{CP}}(\tau) \sum_{\Lambda \in \mathbb{Z}^{N/2}} e^{-2\pi \tau_2 \Lambda_{a_\epsilon}^T \Lambda_{a_\epsilon} - 2\pi i (\epsilon + 1/2) \lambda^T q}. \quad (130) \]
As in the case of the bosonic systems discussed previously, here \( Z_{[a_c, b_c]}^{\text{CP}}(\tau) \) depends only on \( a_c \) and \( \tau_2 \). The fermionic CP symmetry projected partition functions are given by 
\[ Z_{[a_c, b_c]}^{\text{Proj}}(\tau) = \frac{1}{2} \left[ Z_{[a_c, b_c]}^{\text{CP}}(\tau) + Z_{[a-c, b]}^{\text{CP}}(\tau) \right] \quad (131) \]
with \( Z_{[a_c, b_c]}^{\text{CP}}(\tau) \) given by Eq. (125). Under the large gauge transformation \( a_c \to a_c + \delta_c \) and \( b_c \to b_c + \delta_c \), 
\[ Z_{[a_c, b_c]}^{\text{CP}}(\tau) = Z_{[a_c, b_c]}^{\text{CP}}(\tau), \]
\[ Z_{[a_c, \delta_c b_c]}^{\text{CP}}(\tau) = Z_{[a_c, b_c]}^{\text{CP}}(\tau), \]
\[ Z_{[a_c + \delta_c b_c]}^{\text{CP}}(\tau) = \frac{P_{[a_c + \delta_c b_c]}^{\text{CP}}}{P_{[a_c]}^{\text{CP}}} \cdot e^{2\pi i (\epsilon - 1/2) \lambda^T q} \cdot Z_{[a_c]}^{\text{CP}}(\tau). \quad (132) \]
where \( Q^T K^{-1} Q = 0 \) (by CP symmetry) is used. Therefore, the fermionic theory is always anomaly-free if \( \epsilon = 1/2 \) \( ||(CP)^2 = (-1)^N \| \). For \( \epsilon = 0 \) \( ||(CP)^2 = 1 \| \), the quantity \( \lambda_{a_\epsilon}^T q \) gives the stability criterion: "\( \lambda_{a_\epsilon}^T q \) is odd number" corresponds to an anomalous theory (topological phase), while "\( \lambda_{a_\epsilon}^T q \) is even number" corresponds to theory without anomaly (trivial phase):
\[ \lambda_{a_\epsilon}^T q = \text{odd} \implies \text{stable edge ("topological")}, \]
\[ \lambda_{a_\epsilon}^T q = \text{even} \implies \text{unstable edge ("trivial")}. \quad (133) \]

D. Examples

1. The double Laughlin edge state

As an example, let us now consider the case of the doubled fermionic Laughlin state described by 
\[ K = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad U_{\text{CP}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[ Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \chi_{\text{CP}} = \begin{pmatrix} 0 \\ 2\epsilon \end{pmatrix}. \quad (134) \]
where \( \nu^{-1} \) is an odd integer and \( \epsilon \) can be either 0 or 1/2. The elementary charge in the system is \( \beta_e = \min_i |T K^{-1} Q| = \nu \), and the quantity \( \Lambda_c \) is given by \( \Lambda_c = K^{-1} Q / m_c = (1, -1)^T = (\lambda_c, -\lambda_c)^T \). From the previous discussion, the criterion for topological phases is: if \( \epsilon = 1/2 \), the system is in the trivial phase; if \( \epsilon = 0 \), since \( \lambda_{a_\epsilon}^T q = 1 \), the system is in the topological phase.

In this theory, we have \( \xi_{\text{CP}}(\tau) = \eta(2i\nu \tau_2)^{-1} \) and thus \( Z_{[a_c]}^{\text{CP}}(\tau) \) is given by Eq. (130). 
\[ Z_{[a_c]}^{\text{CP}}(\tau) = \frac{P_{[a_c]}^{\text{CP}}}{\eta(2i\nu \tau_2)} \sum_{\lambda \in \mathbb{Z}} e^{-2\pi \tau_2 (\lambda + i\nu a_c)^2 - i2\pi (\epsilon + 1/2) \lambda} \]
\[ = e^{2\pi i \nu a_c (\epsilon - 1/2)} \frac{P_{[a_c]}^{\text{CP}}}{\eta(2i\nu \tau_2)} \delta \left[ - (\epsilon - 1/2) \right] \left( 0, 2i\tau_2 \right). \quad (135) \]
The total CP symmetry projected partition function is given by Eq. (131). For \( \nu = 1 \), which corresponds to the "integer" CP symmetric system (without ground-state degeneracy), the results here agree exactly with the CP projected partition function obtained for the free fermion theory. Under a large gauge transformation \( a_c \to a_c + 1/\nu \), we have 
\[ Z_{[a_c + \frac{1}{\nu}]}^{\text{CP}}(\tau_2) = e^{2\pi i (\epsilon - 1/2)} \cdot \frac{P_{[a_c + \frac{1}{\nu}]}^{\text{CP}}}{P_{[a_c]}^{\text{CP}}} \cdot Z_{[a_c]}^{\text{CP}}(\tau_2). \quad (136) \]

Alternatively, the stability of the edge state of this theory can also be analyzed by enumerating potential (interaction) terms that can potentially gap the edge state without breaking CP and charge U(1) symmetries. They are given by 
\[ U(x) \cos [\Theta(\Lambda) - \alpha(x)] = U(x) \cos \left[ \frac{n}{\nu} (\phi_1 + \phi_2) - \alpha(x) \right], \quad (137) \]
where \( U(x) \) and \( \alpha(x) \) represent the strength and phase of the potential, respectively, which are allowed to be spatially inhomogeneous, and \( \Lambda^T = (n, -n), n \in \mathbb{Z} \) is a charge conserving vector. Under the CP transformation, 
\[ (CP) [U(x) \cos (\Theta(\Lambda) - \alpha(x))] (CP)^{-1} = U(x) \cos [\Theta(\Lambda) - 2(\epsilon + 1/2)n\pi - \alpha(x)] \],
where Eqs. (123) and (126) are used. For \( \epsilon = 1/2 \), the scattering term is CP invariant for any integer \( n \). Such perturbation can gap out the edge without breaking CP symmetry of the ground state \( \frac{1}{2} \langle \phi_1 + \phi_2 \rangle \). On the other hand, for \( \epsilon = 0 \) the scattering term is CP invariant just for even \( n \). In this case, however, the gapping perturbation also spontaneously breaks the CP symmetry of the ground state: \( \frac{1}{2} \langle \phi_1 + \phi_2 \rangle \to \frac{1}{2} \langle \phi_1 + \phi_2 \rangle - \pi \). The argument here agrees with our generalized Laughlin argument based on the CP projected partition function.
2. The fermionic $4 \times 4$ K-matrix theory

As yet another example, let us consider the fermionic K-matrix theory described by the following $4 \times 4$ K-matrix:

$$K = \begin{pmatrix} \Gamma & 0 & 0 & \Gamma \\ 0 & I_2 & 0 & I_2 \\ 0 & 0 & -b & -b \\ 0 & 0 & -b & -b \end{pmatrix},$$

$$Q = (1, 1, 1, 1)^T, \quad \chi_{CP} = (0, 0, 2\epsilon, 2\epsilon)^T,$$

(139)

where $\Gamma$ is a $2 \times 2$ matrix. It is convenient to parameterize the matrix as

$$K = \begin{pmatrix} b + us & b & 0 & 0 \\ b & b + vs & 0 & 0 \\ 0 & 0 & -b & -u \\ 0 & 0 & -b & -v \end{pmatrix},$$

(140)

where $b, u, v, s$ are integers and $u$ and $v$ have no common factor. In terms of these parameters, the elementary charge is

$$\beta_c = \min_i |l^T K^{-1} Q| = \frac{1}{(u + v)b + uv s},$$

(141)

and the quantities $\Lambda_c$ and $\lambda_c$ are given by

$$\Lambda_c = \frac{1}{m_c} K^{-1} Q = \begin{pmatrix} v \\ u \\ -v \\ -u \end{pmatrix} = \left( \begin{pmatrix} \lambda_c \\ -\lambda_c \end{pmatrix} \right).$$

(142)

From these, the criterion for the presence/absence of SPT phases is: if $\epsilon = 1/2$, the system is always in the trivial phase; if $\epsilon = 0$, since $\Lambda_c^T Q = u + v$, the parity of $u + v$ determines whether the phase is trivial ($u + v$ is even) or topological ($u + v$ is odd).

The CP twisted partition function $Z_{CP}$ is given by [Eq. (139)]

$$Z_{CP}^{\chi}[\tau] = \xi_{CP}^{\chi}[\tau] P_{CP}^{\chi} \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} e^{-2\pi \tau(2\lambda_1 m_c a_c v^2 + (\lambda_2 + m_c a_c u)^2)} \times e^{-2\pi i \epsilon (\epsilon + 1/2)(\lambda_1 + \lambda_2)}.$$  

(143)

Under a large gauge transformation $a_c \rightarrow a_c + 1/\beta_c$, the CP-twisted partition function transforms as

$$Z_{CP}^{\chi+1/\beta_c}[\tau_2] = e^{2\pi (\epsilon + 1/2)(u + v)} P_{CP}^{\chi+1/\beta_c} \cdot Z_{CP}^{\chi}[\tau_2].$$

(144)

We can also look for gapping potentials and see if we can gap the edge without breaking CP and charge U(1) symmetries. To gap out the 4 edge modes of theory [139], we need to find two linearly independent and charge conserving vectors $\Lambda_1$ and $\Lambda_2$ that satisfy Haldane’s null vector criterion:

$$\Lambda_1^T K \Lambda_1 = \Lambda_2^T K \Lambda_2 = \Lambda_1^T K \Lambda_2 = 0.$$  

(145)

Such $\Lambda_1$ and $\Lambda_2$ can be the following cases:

a. $\Lambda_1 = U_{CP}^T \Lambda_1$ and $\Lambda_2 = -U_{CP}^T \Lambda_2$: In this case, the charge conserving conditions $\Lambda_1^T Q = \Lambda_2^T Q = 0$ and Eq. (142) give that

$$\Lambda_1 = n_1 (1, -1, 1, 1)^T \equiv n_1 \Lambda_+,$$

$$\Lambda_2 = n_2 (v, u, -v, -u)^T \equiv n_2 \Lambda_+,$$

(146)

where $n_1, n_2 \in \mathbb{Z}$. Under CP the scattering term $\sum_{i=1}^2 U_i(x) \cos[\Theta(\Lambda_i) - \alpha_i(x)]$ transforms as

$$\langle \chi \rangle \left[ \sum_{i=1}^2 U_i(x) \cos[\Theta(\Lambda_i) - \alpha_i(x)] \right] \langle \chi \rangle^{-1}$$

$$= U_1(x) \cos[\Theta(\Lambda_1) - \alpha_1(x)] + U_2(x) \cos[\Theta(\Lambda_2)] - 2n_2 (\epsilon + 1/2)(u + v) \pi - \alpha_2(x).$$

(147)

By choosing $\alpha_1(x) = k\pi$, $k = 0, 1$, then, for $\epsilon = 1/2$, the scattering term is CP invariant for any integers $n_1$ and $n_2$. Such perturbation can gap out the edge without breaking CP symmetry of the ground state $\{\langle \chi \rangle, \langle \Theta(\Lambda_+) \rangle \}$. For $\epsilon = 0$, the scattering term is CP invariant if $n_2(u + v)$ is even. Under CP transformation the ground state transforms as

$$\{\langle \chi \rangle, \langle \Theta(\Lambda_+) \rangle \} \rightarrow \{\langle \chi \rangle, \langle \Theta(\Lambda_+) \rangle - (u + v)\pi\}.$$  

(148)

Therefore, the perturbation will gap out the edge with $(u + v$ is odd)/without $(u + v$ is even) breaking the CP symmetry of the ground state spontaneously.

b. $\Lambda_2 = -U_{CP}^T \Lambda_1$: In this case the CP invariant scattering term is

$$U(x) \left[ \cos[\Theta(\Lambda_1) - \alpha(x)] + \cos[\Theta(\Lambda_2)] + \pi \Lambda_1^T \chi + \Delta \phi_{CP} \right] - \alpha(x)] \left[ \chi \right]$$

(149)

where we used the fact that $\Lambda_1^T Q = 0$ (the charge neutrality condition) and $\Delta \phi_{CP} = \Delta \phi_{CP}^A$. Defining $\Lambda_+ = \Lambda_1 \pm \Lambda_2$, we can then find that $\Lambda_+ \in \mathbb{Z}$ is an integer multiple of $(v, u, -v, -u)$ and $\Lambda_\chi$ is an integer multiple of $(1, -1, 1, -1)$. From the analysis in (i), we know that $\langle \Theta(\chi, u, -v, -u) \rangle$ spontaneously breaks CP for odd $u + v$ ($\epsilon = 0$), so it is impossible that $\langle \Theta(\Lambda_+) \rangle$ and $\langle \Theta(\Lambda_2) \rangle$ (thus $\langle \Theta(\Lambda_1) + \Theta(\Lambda_2) \rangle$) can condensate without spontaneously breaking CP symmetry. The result for $\epsilon = 1/2$ is the same in (i): the perturbation can gap out the edge without breaking CP symmetry of the ground state. On the other hand, for even $u + v$ we can take

$$\Lambda_1^T = (1, -1, 1, -1), \quad \Lambda_+^T = (v, u, -v, -u),$$

(150)

so that

$$\Lambda_1^T = \frac{1}{2}(1 + v, -1 + u, 1 - v, -1 - u),$$

$$\Lambda_2^T = \frac{1}{2}(-1 + v, 1 + u, -1 - v, 1 - u).$$

(151)
Under CP the ground state transforms as
\[
\{\langle \Theta(\Lambda_1) \rangle, \langle \Theta(\Lambda_2) \rangle \} \\
\rightarrow \{\langle \Theta(\Lambda_2) \rangle - (u + v)(\epsilon + 1/2)\pi, \\
\langle \Theta(\Lambda_1) \rangle - (u + v)(\epsilon + 1/2)\pi\},
\]
which does not break CP spontaneously. Also, since there are no non-primitive linear combinations \(a_1\Lambda_1 + a_2\Lambda_2\) for any integers \(a_1\) and \(a_2\), the perturbation does not break CP for the whole family of condensations \(\langle \Theta(a_1\Lambda_1 + a_2\Lambda_2) \rangle\) (for both \(\epsilon = 0, 1\)). Therefore, the argument of the stability of the edge state by the microscopic analysis is consistent with the one by the large gauge invariance of the CP symmetry-projected partition function.

V. DISCUSSION

We have developed a general theoretical framework that allows us to determine under which conditions a given edge CFT is gappable/ingappable. While we have worked out particular examples with CP or P symmetry, our theoretical framework is applicable to other examples with local and non-local symmetries. For example, our methodology can be applicable to reflection symmetric fermionic SPT phases that are classified and tabulated in Refs. 44 and 45.

Our consideration in this paper is limited to an anomaly associated to global U(1) symmetries (once CP is enforced) and thus limited to systems with conserved U(1) symmetry (such as conservation of the particle number or z-component of SU(2) spin). We use a gauge flux of these U(1) symmetries as an adiabatic parameter in developing Laughlin’s gauge argument. For systems that do not have such continuous symmetry, one may need to consider an anomaly associated to gravitational degrees of freedom such as modular invariance. As one can see from the fact that the real part of the modular parameter \(\tau_3\) is projected out by orientifolding [recall discussion near Eq. [19]], the modular group of the torus \(\text{PSL}(2, \mathbb{Z})\) cannot be used to study conformal field theories on the Klein bottle. Nevertheless, an analogue of the S-modular transformation (which exchanges the space and time direction on the torus) still plays a role in orientifold conformal field theories. To be more precise, the “loop channel” calculations presented in this paper can be cast into an equivalent calculation in the “tree channel” by using crosscap states. We plan to visit these tree channel pictures in a forthcoming publication.

Related to this question, in this paper, we discussed partition functions of edge theories on the Klein bottle, but not the other spacetime manifolds such as annulus or Möbius strip. It may be interesting to ask if there is any role played by these other geometries. In oriented cases, the modular invariance on the torus is believed to be enough to define the conformal field theory on any (oriented) world sheet. For the unoriented cases, one may wonder if considering the consistency of the theory on the Klein bottle would be enough to define the conformal field theory on all (unoriented) worldsheets.

We will defer detailed studies of this issue for the future, but it may be worth pointing out the following: in string theory, conformal field theories on the Klein bottle appear in Type I superstring theory. There are, in addition to the Klein bottle, other worldsheet geometries such as a strip and the Möbius strip. Some properties are physically constrained by tadpole cancellation, and worldsheets with boundaries come hand-in-hand with D-branes. It would be interesting to explore what role these and other consistency conditions might play in a condensed matter setting.

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Appendix A: The CP eigenvalue of the ground state

In the bulk of the paper, we have enforced CP invariance at all steps of an adiabatic evolution (for all values of the flux \(a\)). In fact, the system (defined by Lagrangian with boundary conditions) is classically CP invariant, and hence one would assume this is so even at quantum level. What we discovered, under this assumption, is the violation of the electromagnetic U(1) symmetry. As we mentioned in Sec. 11, an alternative point of view is possible; if we were to enforce the electromagnetic U(1) symmetry, CP would then be violated (when the CP symmetry in question is “topological” – the one which leads topological insulators). Therefore, while the system preserves, at the classical level, both the electromagnetic U(1) and CP symmetries, there is a tension between these symmetries once we quantize the system. Once we demand the electromagnetic U(1) symmetry be strictly conserved, instead of enforcing CP, \(P_{a}\) may not be independent of \(a\). In the following, we determine \(P_{a}\) under the assumption of the U(1) conservation.

c. The IQHE As a warm up, let us start from an edge state of the (integer) quantum Hall system; it suffers from an anomaly; and hence cannot exist on its own. (We follow closely Ref. 5). The edge state of the IQHE is a chiral fermion \(\psi_R\). We consider an edge of circum-
ference $2\pi$, and impose the twisted boundary condition: 
\[ \psi_R(x + 2\pi) = e^{2\pi i} \psi_R(x). \]
From the state-operator correspondence, there is an operator associated to the ground state for a given $\nu$, which we call $A_\nu$. The operator can be determined from the following general principle: (i) any unitary on-site symmetry in field theories can be used to generate a twisting boundary condition; (ii) in CFT, Hilbert space with twisted boundary condition form an independent sector (Virasoro module); (iii) due to the state-operator correspondence, there is an operator that corresponds to a ground state of the twisted Hilbert space. The identification of the ground state operator can be done conveniently in terms of bosonization:
\[ \psi_R \simeq e^{i\varphi_R}. \] (A1)

One could then infer the operator corresponding to the ground state:
\[ A_\nu \equiv e^{i(-\nu+1/2)\varphi_R}, \] (A2)
where $\varphi_R$ is a chiral boson field. From this expression for the ground state operator one infers that the charge of the ground state is
\[ F_R = 1/2 - \nu. \] (A3)

Thus, we conclude that the ground state fermion number at the edge of the quantum Hall system changes as a function of twisting angle. Because of the spectral flow, as one changes $a \to a + 1$, the fermion number jumps by one (discontinuously). Had the charge been conserved (i.e., had there been no anomaly), the ground state fermion number should be independent of the twisting angle. The ground state charge \[ \pi \] is the origin of the factor $e^{-2\pi i(a-1/2)(a-1/2)}$ in the partition function [3, 4].

\[ \text{d. The QSHE with conserved } S_z \] Let us now consider the edge theory of a bulk quantum spin Hall system with conserving $S_z$. The edge state now consists of both left- and right-movers, $\psi_L$ and $\psi_R$. These fermion fields can be bosonized as
\[ \psi_L \sim e^{i\varphi_L}, \quad \psi_R \sim e^{-i\varphi_R}. \] (A4)
(Here, we do not include Klein factors while they are important in discussing CP symmetric topological insulators.) Following the same argument as in the case of the QHE, the operator corresponding the ground state of the left-moving sector is $e^{i(-\nu_L+1/2)\varphi_L}$ where $\nu_L$ is the twisting angle for the left movers. Similarly, the ground state for the right moving sector can be represented as $e^{i(-\nu_R+1/2)\varphi_R}$ By combining the left- and right-moving parts of the ground state properly, we have a ground state for the combined non-chiral system.

Below, we put more emphasis on U(1) charge conservation than $S_z$ conservation; we first give a priority to the U(1) charge conservation and see this necessary leads to violation of $S_z$ conservation – this is nothing but chiral anomaly. Since charge U(1) is conserved, it makes sense to twist boundary conditions by charge U(1) symmetry, $\nu_L = \nu_R$. We are thus lead to the ground state vertex operator
\[ e^{i(-\nu+1/2)\varphi_L} e^{+i(-\nu+1/2)\varphi_R} = e^{i(-\nu+1/2)\phi}. \] (A5)

Here, the non-chiral field $\phi = \varphi_L + \varphi_R$ is charge neutral. One could combine the left- and right-moving sectors differently to get $e^{i(-\nu+1/2)\theta}$ with $\theta = \varphi_L - \varphi_R$. This choice, however, is not consistent with charge U(1) conservation since $\theta$ is not charge neutral and the ground state fermion number $F_\nu = F_L + F_R$ changes as a function of $\nu$, $\nu = \nu + 1/2$.

While the ground state $e^{i(-\nu+1/2)\phi}$ is consistent with charge U(1) conservation, the price we paid is that the ground state is charged under spin $S_z$ conservation. This means that as one adiabatically inserts charge flux, the $S_z$ quantum number of the edge state changes – the spin is “pumped” from the edge in question to other edges, or vice versa.

\[ e. \text{ CP symmetric bosonic topological insulators} \] Let us now break the continuous the U(1) spin $S_z$ conservation and instead impose CP symmetry; we consider the case of CP symmetric topological insulators. The relevant symmetries are charge U(1) and CP. In particular, we focus on the bosonic version of the topological insulator that we discuss in Sec. [12]. The CP acts on the bosonic field as in Eq. [12]. Following above discussion, we consider the ground state that preserves the electromagnetic U(1) symmetry as a function of twisting angle $\nu$. The ground state is then given by
\[ e^{i\nu \theta R/\alpha'}. \] (A6)

The CP eigenvalue of the ground state is
\[ (CP) e^{i\nu \theta R/\alpha'} (CP)^{-1} = P_\nu e^{i\nu \theta R/\alpha'} \] where $P_\nu = e^{i2\pi \nu e}$. (A7)

Thus, for the topologically trivial case $\epsilon = 0$, the CP eigenvalue is independent of $\epsilon$, where as when $\epsilon = 1/2$ (topological), the ground state CP eigenvalue evolves as a function of $\nu$. This signals the conflict of the symmetry; once we choose to preserve the U(1), CP is necessarily broken.

Appendix B: Statistical phase factor of the chiral boson field under symmetry transformation

For any local quasiparticle excitation $\frac{\epsilon}{\pi} \exp i \Lambda^T K \phi$, where $\Lambda^T K \phi = \sum_I \Lambda_I (K \phi) = \sum_I \theta_I$, the symmetry

\[ e^{i(-\nu+1/2)\varphi_L} e^{+i(-\nu+1/2)\varphi_R} = e^{i(-\nu+1/2)\phi}. \] (A5)
transformation $\mathcal{G}$ acts as

$$ \mathcal{G} \mathfrak{e}^{i\Lambda^T K \phi} \mathcal{G}^{-1} = \mathcal{G} \mathfrak{e}^{\sum_I i\theta_I} \mathcal{G}^{-1} = \mathcal{G} \prod_I e^{i\theta_I} \mathcal{G}^{-1} = \mathcal{G} \prod_I e^{i\theta_I} \mathcal{G} \mathfrak{e}^{\frac{1}{2} \sum_{I<J} [i\theta_I, i\theta_J]} \mathcal{G}^{-1} = \mathcal{G} \prod_I e^{i\theta_I} \mathcal{G} \mathfrak{e}^{i \Delta \phi_{\mathcal{G}}} \mathcal{G}^{-1} = \mathcal{G} \prod_I e^{i\theta_I} \mathcal{G} \mathfrak{e}^{i \Delta \phi_{\mathcal{G}}} \mathcal{G}^{-1} \quad \text{(B1)}$$

where we have used the Baker-Campbell-Hausdorff formula (with the commutator $[i\theta_I, i\theta_J]$ being a $c$-number), the ordered-product $\prod_I$ is defined as an ordered product in the ascending order of indices, and

$$ i \Delta \phi_{\mathcal{G}} = \frac{1}{2} \sum_{I<J} ([G_i\theta_I, G_i\theta_J] - G[i\theta_I, i\theta_J] G^{-1}) \quad \text{(B2)} $$

Note that we keep the form $G[i\theta_I, i\theta_J] G^{-1}$ even if $[i\theta_I, i\theta_J]$ is a $c$-number, since in general $G$ can be an antunitary operator (e.g. T symmetry). On the other hand,

$$ G e^{i\Lambda^T K \phi} G^{-1} = e^{i\Phi} (\sum_I \Phi_I) G^{-1}, \quad \text{(B3)} $$

so we have

$$ \mathcal{G} \left( i\Lambda^T K \phi \right) G^{-1} = \sum_I G_i\theta_I G^{-1} + i \Delta \phi_{\mathcal{G}} \mod 2\pi i. \quad \text{(B4)} $$

This means the way that the operator $\mathcal{G}$ acts on the chiral boson field $\phi$ is not always linear, because some nontrivial phase factor $\Delta \phi_{\mathcal{G}}$ ($\neq 2\pi n$) might arise. In bosonic system, the phase factor is always the multiple of $2\pi i$, corresponding to Bose statistics, and thus we can ignore it (in this case $\mathcal{G}$ is linear in $\phi$). In fermionic systems, however, we must be careful with the phase factor, which might be nontrivial, because of the Fermi statistics.

In the following we take CP and T symmetries as examples.

\textbf{f. CP symmetry} From the canonical commutation relation \cite{[20]}, when $x \neq x'$ (but $x \rightarrow x'$ is taken when we consider the operator product expansion of vertex operators) and $I \neq J$, we have

$$ [(K\phi)_I(t, x), (K\phi)_J(t, x')] = -i\pi \text{sgn}(I - J) Q_I Q_J + 2\pi i N_{IJ}, \quad \text{(B5)} $$

where $N_{IJ}$ is the component of an integer matrix. Now for CP symmetry defined in Sec. \cite{[19]} the extra phase is given by

$$ i \Delta \phi_{\text{CP}} = -\frac{1}{2} \sum_{I<J} \Lambda_I \Lambda_J \left\{ [(U_{\text{CP}} K\phi)_I, (U_{\text{CP}} K\phi)_J] - [(K\phi)_I, (K\phi)_J] \right\}. \quad \text{(B6)} $$

Since $U_{\text{CP}}$ has the form \( \begin{pmatrix} 0 & 1_{N/2} \\ -1_{N/2} & 0 \end{pmatrix} \), where $N$ is an even integer, we have, for $1 \leq I < J \leq N$,

$$ \begin{aligned}
[(U_{\text{CP}} K\phi)_I, (U_{\text{CP}} K\phi)_J] &= -i\pi (U_{\text{CP}} Q)_I (U_{\text{CP}} Q)_J & \text{if } 1 \leq I < J \leq \frac{N}{2} \\
&= -i\pi (U_{\text{CP}} Q)_I (U_{\text{CP}} Q)_J & \text{if } 1 \leq I \leq \frac{N}{2} \text{ and } \frac{N}{2} + 1 \leq J \leq N \\
&= 0 & \text{otherwise.}
\end{aligned} \quad \text{(B7)} $$

Then

$$ i \Delta \phi_{\text{CP}} = -i\pi \sum_{1 \leq I < J \leq \frac{N}{2}} \Lambda_I \Lambda_J [(U_{\text{CP}} Q)_I (U_{\text{CP}} Q)_J - Q_I Q_J] $$

$$ + i\pi \sum_{\frac{N}{2} + 1 \leq I < J \leq N} \Lambda_I \Lambda_J [(U_{\text{CP}} Q)_I (U_{\text{CP}} Q)_J + Q_I Q_J] $$

$$ = i\pi \left( \sum_{I=1}^{N/2} \Lambda_I Q_I \right) \left( \sum_{J=\frac{N}{2}+1}^{N} \Lambda_J Q_J \right) \mod 2\pi i. \quad \text{(B8)} $$

where the second equality holds since the sum of the first two terms in the first equality vanishes. For CP invariant vector $\Lambda$, with $\Lambda = -U_{\text{CP}} \Lambda$, we can express $\Lambda$ as $(\lambda, -\lambda)^T$, where $\lambda$ is any $N/2$ dimensional integer vector. Then the statistical phase can be expressed as

$$ i \Delta \phi_{\text{CP}} = -i\pi \left( \sum_{I=1}^{N/2} \lambda_I q_I \right)^2 = -i\pi \sum_{I=1}^{N/2} \lambda_I q_I $$

$$ = -i\pi \lambda^T q \mod 2\pi i. \quad \text{(B9)} $$

\textbf{g. T symmetry} The set of data $\{K, Q, U_T, \chi_T\}$ for the T symmetric K-matrix theory is the same as the case of CP. The only difference is that T is an antunitary operator, which results in [from Eq. \text{(B4)}]

$$ T[(K\phi)_I, (K\phi)_J] T^{-1} = -[(K\phi)_I, (K\phi)_J], \quad \text{(B10)} $$
and hence reverses the sign in front of $Q_I Q_J$ in Eq. (B8)

$$i\Delta\phi_T^\Lambda = -\frac{i\pi}{2} \sum_{1 \leq i < J \leq \frac{N}{2}} \Lambda_I \Lambda_J [(U_T^\dagger Q_I)(U_T^\dagger Q_J) + Q_I Q_J]$$

$$- \frac{i\pi}{2} \sum_{\frac{N}{2} + 1 \leq I < J \leq N} \Lambda_I \Lambda_J [(U_T^\dagger Q_I)(U_T^\dagger Q_J) + Q_I Q_J]$$

$$+ \frac{i\pi}{2} \sum_{1 \leq I \leq \frac{N}{2}} \Lambda_I \Lambda_J [(U_T^\dagger Q_I)(U_T^\dagger Q_J) - Q_I Q_J]$$

$$- i\pi \left( \sum_{1 \leq I < J \leq \frac{N}{2}} + \sum_{\frac{N}{2} + 1 \leq I < J \leq N} \right) \Lambda_I \Lambda_J Q_I Q_J$$

mod $2\pi i$, (B11)

where the second equality holds since the third term in the first equality vanishes. For a $T$-invariant vector $\Lambda$ satisfying $\Lambda = -U_T^\dagger \Lambda$, we can express $\Lambda$ as $(\lambda, -\lambda)^T$, where $\lambda$ is any $N/2$ dimensional integer vector. Then the statistical phase is

$$i\Delta\phi_T^\Lambda = -2\pi i \sum_{1 \leq I < J \leq \frac{N}{2}} \lambda_I \lambda_J q_I q_J = 0 \mod 2\pi i.$$  (B12)

Therefore, in discussion of the K-matrix theory with $T$ symmetry, statistical phases are irrelevant and can safely be ignored, as pointed out in Ref. 59.

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As a clarifying remark, SPT phases are not topologically-ordered phases of matter in that they do not have defining properties of topological order such as non-trivial ground state degeneracy, fractional statistics, etc. Nevertheless, such phases are topologically non-trivial in the sense that they are not adiabatically connected to a trivial phase. On the other hand, there is a class of topologically-ordered phases that have some interesting properties in the presence of symmetries—they are called symmetry-enriched topological phases. The methodology proposed in this paper (a generalization of Laughlin’s argument) works both for SPT phases and symmetry-enriched topological phases.

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For a connection between orientifolds and time-reversal symmetric topological insulators and superconductors from the spacetime point of view (as opposed to the worldsheet point of view presented in this paper), see Refs. 23 and 24.

C.-T. Hsieh, T. Morimoto, and S. Ryu, unpublished.

The fermionic models with CP and charge U(1) symmetries can also be interpreted/realized as a BdG system with parity and spin U(1) symmetries (e.g., z-component of spin, $S_z$, is conserved).

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In the above calculations, the invariance is violated only for the temporal boundary condition. One in fact has a choice: by redefining $Z_{a,b} \rightarrow Z_{a,b}e^{-2i\alpha_{ab}}$, the partition function is now anomalous for spatial boundary condition. Such multiplication of the phase is related to (re-) assignment the U(1) charge to the ground state. See later discussion for more details.

In the presence of both U(1) and CP symmetries, by combining these two symmetries, one can generate a series of CP transformations:

$$U_a \psi_L(x) U_a^{-1} = e^{i\alpha} \psi_L(-x),$$

$$U_a \psi_R(x) U_a^{-1} = \eta e^{i\alpha} \psi_R(-x),$$

where $U_a = e^{i\alpha_FVP}$. While these transformations are each qualified to be called CP, the inclusion of such a U(1) phase factor does not play any essential role in our discussion.

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In addition to the Hamiltonian, there is a chemical potential which appears as an operator insertion $e^{-2\pi i(b-1/2)F_V}$ in the partition function. Viewing this operator as a part of the partition function, the system with the chemical potential is in general not invariant under CP since $(CP)F_V(CP)^{-1} = -F_V$. The only exceptions are the cases when $b = 0, 1/2$. When $b \neq 0, 1/2$, the system is not invariant under CP and so we cannot make a projection by CP symmetry. We therefore limit ourselves to $b = 0, 1/2$.

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The equivalence of the two pictures, one in terms of twisting boundary conditions, and the other in terms of background gauge fields, can be established by a gauge transformation that “unwinds” the boundary conditions, and vice versa. When the electromagnetic U(1) symmetry happens to be anomalous, care may be required in invoking such equivalence. (See, for example, Ref. 84.) In our approach, when an ambiguity such as the CP eigenvalue of the ground state arises, we follow what we expect in the
absence of anomalies. We test the consistency of such an assumption arising from enforcement of CP symmetry with the electromagnetic U(1) symmetry by inspecting the behavior of the partition function under the adiabatic process of flux insertion.

It is instructive to compare the CP projected partition function with the partition function with P projection. Parity transformation acts on fermion fields as

$$\mathcal{P}\psi_L(x)\mathcal{P}^{-1} = \eta e^{i\alpha}\psi_R(-x),$$

$$\mathcal{P}\psi_R(x)\mathcal{P}^{-1} = e^{i\alpha}\psi_L(-x).$$

In our fermionic edge theory, by analyzing mass terms, one can check that there is no topological phase protected by parity symmetry (of any kind) and the electromagnetic U(1) symmetry. The absence of topological phases can be seen from the fact that the Klein bottle partition function with parity projection is anomaly-free. First recall that P symmetry is consistent with the twisting boundary condition only when $\nu_L = -\nu_R$. As we require only the U(1) charge conservation, this means only $\nu_L = \nu_R = 0$ (periodic boundary condition) or $\nu_L = -\nu_R = 1/2$ (antiperiodic boundary condition) are allowed. With this in mind, the projection works, for a given $r > 0$, as

$$\text{Tr} \left[ \mathcal{P} e^{2\pi i (b-1/2)\mathcal{H}} q^\mathcal{H}_{LR} q^\mathcal{H}_{RL} \right] \propto \prod_r \left[ 1 + \eta e^{2\pi i (b-1/2)} e^{-2\pi i (\alpha)} \right].$$

Thus, the phase $\alpha$ as well as $\eta$ simply shifts the chemical potential $b$. In the case of P, we can freely change the time boundary condition $b$, but not the spatial boundary condition. Observe that this situation is opposite to what we had for CP symmetry. In the case of CP symmetry, we can freely change the space boundary condition, but not the time boundary condition. As before, we change $b \to b + 1$ and ask if the theory is invariant under this large gauge transformation or not. Depending on the spatial boundary conditions, (periodic/antiperiodic), the partition function may pick up an anomalous phase. However, observe that the chemical potential enters in the partition function as $e^{4\pi i (b-1/2)}$ not $e^{2\pi i (b-1/2)}$. Due to this doubling, there are no anomalous phases.

Here and in the following, the Dirac delta function $\delta(x-x')$ and $\text{sgn} (x - x')$ in the commutator should be interpreted as its periodic counter part, such as $\sum_{m \in \mathbb{Z}} \delta(x-x' - 2m\pi)$, when the system is put on a circle of circumference $2\pi$.

67 The linear combination $a_1A_1 + a_2A_2$ is non-primitive if there are some integer vector $A$ and integer $k > 1$ such that $a_1A_1 + a_2A_2 = kA$.

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80 While we have determined the ground state and its charge as above, we could take an alternative point of view. Let us assume that we actually do not know, a priori, that the U(1) symmetry is anomalous. We would like to test if this symmetry is anomalous or not. For this purpose, we pretend the charge U(1) is conserved. We do so since the charge U(1) is classically conserved, and if so, one would guess naively that the ground state fermion number does not change as we adiabatically change $a$ and $b$. Under this assumption, what one would discover is that the partition function is not invariant under $a \to a + 1$. Therefore, even though we started from the assumption that the U(1) is conserved, we run into the “inconsistency” in that the partition function is not invariant under $a \to a + 1$, in stead of $b \to b + 1$. We then conclude we cannot conserve the U(1) at the quantum level.

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83 S. Weinberg, *The quantum theory of fields* (Cambridge University Press, 1996).

84 K. Landsteiner, E. Megías, and F. Peña-Benitez, in *Lecture Notes in Physics, Berlin Springer Verlag*, Lecture Notes in Physics, Berlin Springer Verlag, Vol. 871, edited by D. Kharzeev, K. Landsteiner, A. Schmitt, and H.-U. Yee (2013) p. 433, [arXiv:1207.5808 [hep-th]].