THE COLLATZ CONJECTURE & NON-ARCHIMEDEAN SPECTRAL THEORY - PART I.5 - HOW TO WRITE THE (WEAK) COLLATZ CONJECTURE AS A CONTOUR INTEGRAL

BY MAXWELL C. SIEGEL

Abstract. Let $q$ be an odd prime, and let $T_q : \mathbb{Z} \to \mathbb{Z}$ be the Shortened $qx + 1$ map, defined by $T_q(n) = n/2$ if $n$ is even and $T_q(n) = (qn + 1)/2$ if $n$ is odd. The study of the dynamics of these maps is infamous for its difficulty, with the characterization of the dynamics of $T_3$ being an alternative formulation of the famous Collatz Conjecture. This series of papers presents a new paradigm for studying such arithmetic dynamical systems by way of a neglected area of ultrametric analysis which we have termed $(p, q)$-adic analysis, the study of functions from the $p$-adics to the $q$-adics, where $p$ and $q$ are distinct primes. In this, the first-and-a-halfth paper of the series, as a first application, we show that the numen $\chi_q$ of $T_q$ can be used in conjunction with the Correspondence Principle (CP) and classic complex-analytic tools of analytic number theory to reformulate the study of periodic points of $T_q$ in terms of a contour integral via an application of Perron’s Formula to a Dirichlet series generated by $\chi_q$ and the function $M_q$ introduced in the first paper in this series, for which we establish functional equations, which we use to derive their meromorphic continuations to the left half-plane. The hypergeometric growth of the series as $\text{Re}(s) \to -\infty$ seems to preclude direct evaluation of the contour integrals via residues, but asymptotic results may still be achievable.

Notation & Preliminaries

All of the notation used in this paper builds upon the notation used in [31]. We recall that, for any real number $x$, we write $N_x$ to denote the set of all integers $\geq x$ (thus, $N_0 = \{0, 1, 2, \ldots\}$; $N_1 = \{1, 2, \ldots\}$). For any real number $x$, $\lfloor x \rfloor$ denotes the floor of $x$, the largest integer $\leq x$. Additionally, for any $n \in N_0$, we write $\#_1(n)$ to denote the number of 1s in the binary digits of $n$; $\lambda_2(n)$, meanwhile, denotes the total number of binary digits of $n$ (note that $\lambda_2(n) = \lfloor \log_2(n + 1) \rfloor$). As is traditional, we use $s = \sigma + it$ to denote the complex-variable input of our Dirichlet series. We adopt the standard convention of extending the binomial coefficient $\binom{n}{k}$ to complex values by way of the Gamma function:

\begin{equation}
\binom{s}{k} \overset{\text{def}}{=} \frac{s!}{k!(s-k)!} \overset{\text{def}}{=} \frac{\Gamma(s + 1)}{k! \Gamma(s-k+1)}
\end{equation}

Given a function $\lambda : N_0 \to \mathbb{C}$, we write $\zeta_\lambda$ to denote the Dirichlet series / Dirichlet generating function of $\lambda$:

\begin{equation}
\zeta_\lambda(s) \overset{\text{def}}{=} \sum_{n=1}^\infty \frac{\lambda(n)}{n^s}
\end{equation}
We will use both Big-O notation and the Vinogradov notation \( \ll \), with:

\[
\begin{align*}
  f(n) \ll g(n) \quad & \text{as } n \to \infty & (0.3) \\
  \updownarrow \\
  f(n) = O(g(n)) \quad & \text{as } n \to \infty & (0.4)
\end{align*}
\]

We also use subscripts on Big-O notation to indicate where the limiting variable is going; e.g.:

\[
\begin{align*}
  f(x) = O_{c}(g(x)) \quad & \text{as } x \to c & (0.5) \\
  \updownarrow \\
  f(x) = O(g(x)) \quad & \text{as } x \to c & (0.6)
\end{align*}
\]

We assume the reader has some familiarity with the basic theory of Dirichlet series and the Mellin transform. The papers of Flajolet et. al. on Mellin transforms—particularly [16, 17]—cover all the necessary information. However, for ease of reference, we state the most important results below.

**Preliminaries from Complex Analysis.**

**Definition 1 (Principal Value of an Integral).** Let \( c \in \mathbb{R} \), and let \( f(s) \) be a complex-valued function so that, for every real number \( T > 0 \), the integral:

\[
\int_{c-iT}^{c+iT} f(s) \, ds
\]

exists and is finite. (Note, we do not require sup\( T > 0 \) \( \left| \int_{c-iT}^{c+iT} f(s) \, ds \right| < \infty \)). Then, we write:

\[
\text{PV} \left\{ \int_{c-iT}^{c+iT} f(s) \, ds \right\} \overset{\text{def}}{=} \lim_{T \to \infty} \int_{c-iT}^{c+iT} f(s) \, ds
\]

provided that the limit on the right exists. We call \( \text{PV} \int_{c-i\infty}^{c+i\infty} f(s) \, ds \) the principal value of the expression \( \int_{c-i\infty}^{c+i\infty} f(s) \, ds \).

**Theorem 1 (Perron’s Formula[17]).** Let \( \lambda : \mathbb{N}_{1} \to \mathbb{C} \), let: \( c > 0 \) lie in the half-plane of absolute convergence of the Dirichlet Series \( \zeta_{\lambda} \). Then, for any \( m \geq 1 \) and any real \( x \geq 2 \):

\[
\frac{1}{m!} \sum_{k=1}^{\lfloor x \rfloor - 1} \lambda(k) \left( 1 - \frac{k}{n} \right)^{m} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s} \zeta_{\lambda}(s) \frac{ds}{s(s+1) \cdots (s+m)}
\]

When \( m = 0 \), we have:

\[
\sum_{k=1}^{\lfloor x \rfloor - 1} \lambda(k) = \text{PV} \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s} \zeta_{\lambda}(s) \frac{ds}{s} \right\}
\]

Perron’s Formula is a specific case of the following more general inversion formula:

**Theorem 2 (Mellin’s Summation Formula [17]).** Consider a function \( f : [0, \infty) \to \mathbb{C} \), and let:

\[
\mathcal{M} \{ f \} (s) \overset{\text{def}}{=} \int_{0}^{\infty} x^{s-1} f(x) \, dx
\]
denote the Mellin transform of $f$. Consider sequences $\{\lambda_k\}_{k \geq 1}$ and $\{\mu_k\}_{k \geq 1}$ of complex and positive-real numbers $\lambda_k$ and $\mu_k$, respectively. Then, for any $c \in \mathbb{R}$ so that both $\mathcal{M}\{f\}(s)$ and:

\begin{equation}
\sum_{k=1}^{\infty} \frac{\lambda_k}{\mu_k}
\end{equation}

converge absolutely on the strip $\text{Re}(s) = c$, we have:

\begin{equation}
\sum_{k=1}^{\infty} \lambda_k f(\mu_k x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \left( \sum_{k=1}^{\infty} \frac{\lambda_k}{\mu_k} \right) \mathcal{M}\{f\}(s) \, ds
\end{equation}

### 1. Introduction

In [31], we proved the Correspondence Principle (CP), which—among other things—showed that periodic points of $T_q$ in $\mathbb{Z}\{0\}$ were precisely those rational integer values which $T_q$’s numen, $\chi_q$, attained over the set of rational 2-adic integers which infinitely many digits. [31] gave several versions of this result. The one we shall use in this paper is:

**Corollary 1 (Correspondence Principle (Periodic Points), Ver. 2).**

I. Let $\Omega \subseteq \mathbb{Z}$ be any cycle of $T_q$. Then, viewing $\Omega$ as a subset of $\mathbb{Z}_q$, the intersection $\chi_q(\mathbb{Z}_2) \cap \Omega$ is non-empty. Moreover, for every $x \in \chi_q(\mathbb{Z}_2) \cap \Omega$, there is an $n \in \mathbb{N}_1$ so that:

\begin{equation}
x = \frac{\chi_q(n)}{1 - M_q(n)}
\end{equation}

II. Let $n \in \mathbb{N}_1$. If the quantity $x$ given by:

\begin{equation}
x = \frac{\chi_q(n)}{1 - M_q(n)}
\end{equation}

is a 2-adic integer, then $x$ is a periodic point of $T_q$ in $\mathbb{Z}_2$; if $x$ is in $\mathbb{Z}$, then $x$ is a periodic point of $T_q$ in $\mathbb{Z}$. Moreover, if $x \in \mathbb{Z}$, then $x$ is positive if and only if $M_q(n) < 1$, and $x$ is negative if and only if $M_q(n) > 1$.

**Remark 1.** When we speak of $T_q$ on $\mathbb{Z}_2$, we mean the standard extention of $T_q$ from a map on $\mathbb{Z}$ to a map on $\mathbb{Z}_2$ by way of the rule:

\begin{equation}
T_q(\hat{z}) \overset{\text{def}}{=} \begin{cases} \\
\frac{1}{2} & \text{if } \hat{z} \in 2\mathbb{Z}_2 \\
\frac{2\hat{z}+1}{2} & \text{if } \hat{z} \in 1 + 2\mathbb{Z}_2
\end{cases}
\end{equation}

**Remark 2.** Recall the identity:

\begin{equation}
\chi_q(B_2(n)) = \frac{\chi_q(n)}{1 - M_q(n)}, \quad \forall n \in \mathbb{N}_1
\end{equation}

where:

\begin{equation}
B_2(n) \overset{\text{def}}{=} \begin{cases} \\
0 & \text{if } n = 0 \\
\frac{n}{1 - 2^{A_2(n)}} & \text{if } n \geq 1
\end{cases}
\end{equation}
In this regard, Version 2 of the CP tells us that the behavior of the function \( n \mapsto \chi_q(B_2(n)) \) contains everything we could ever want to know about the periodic points of \( T_q \) in \( \mathbb{Z} \). This would suggest that we study \( \chi_q \circ B_2 : \mathbb{N}_0 \to \mathbb{Q} \) directly. While this can be done, it is not practical. If there is one unifying tool in the study of \( \chi_q \), it is the central importance of \( \chi_q \)'s functional equations:

\[
\chi_q(2z) = \frac{1}{2}\chi_q(z) \tag{1.6}
\]

\[
\chi_q(2z + 1) = \frac{q\chi_q(z) + 1}{2} \tag{1.7}
\]

Almost every useful result for \( \chi_q \) involves these identities in one form or another, and, to that end, \( \chi_q \circ B_2 \) is much more difficult to work with, simply because it does not satisfy functional equations of this affine-linear form. Indeed, a simple computation using the functional equations of \( \chi_q \) and \( M_q \) shows that \( \chi_q \circ B_2 \) satisfies the functional equation:

\[
(\chi_q \circ B_2)(2n) = \frac{1 - M_q(n)}{2 - M_q(n)} (\chi_q \circ B_2)(n) \tag{1.8}
\]

\[
(\chi_q \circ B_2)(2n + 1) = \frac{q(1 - M_q(n))(\chi_q \circ B_2)(n) + 1}{2 - qM_q(n)} \tag{1.9}
\]

Additionally, the \( 1 - M_q(n) \) denominator term makes \( \chi_q \circ B_2 \) behave erratically, blowing up whenever \( n \) makes the real number \( \frac{\lambda_2(n)}{\ln q - \lambda_2(n) \ln 2} \) small. While it is conceivable that \( \chi_q \circ B_2 \) could be profitably studied in its own right, doing so will likely be difficult. Fortunately, **Corollary 1** furnishes a much more manageable alternative. All we need to do is cancel the denominator:

\[
\frac{\chi_q(n)}{1 - M_q(n)} = x \tag{1.10}
\]

\[
(1 - M_q(n)) x - \chi_q(n) = 0
\]

By the CP, the periodic points of \( T_q \) are precisely those \( x \in \mathbb{Z}\setminus\{0\} \) for which \( (1 - M_q(n)) x - \chi_q(n) = 0 \) for some \( n \geq 1 \).

This is where Perron’s Formula comes into play.

**Definition 2.** Define the Dirichlet series:

\[
\zeta_M(q)(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{M_q(n)}{n^s} \tag{1.11}
\]

and:

\[
\zeta_{\chi_q}(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n^s} \tag{1.12}
\]

where \( s \) is a complex variable. Finally, for any integer \( x \), we define:

\[
F_q(s, x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(1 - M_q(n)) x - \chi_q(n)}{n^s} \tag{1.13}
\]
Remark 3. Observe the identity:

\[ F_q(s, x) = (\zeta(s) - \zeta_{M_q}(s)) x - \zeta_{X_q}(s) \]

where \( \zeta(s) \) is, of course, the Riemann Zeta Function. Moreover, as we will soon show, \( \text{Re}(s) > \sigma_q \) is the abscissa of absolute convergence for \( \zeta_{M_q}(s), \zeta_{X_q}(s), \) and \( F_q(s, x) \) (here, \( x \) is arbitrary), where:

\[ \sigma_q \overset{\text{def}}{=} \log_2 \left( \frac{q + 1}{2} \right) \]

Using Perron’s Formula (0.10), letting \( 0 < a < 1 \), and letting \( c > \sigma_q \), we can express the summatory function of \( (1 - M_q(n)) x - \chi_q(n) \) as:

\[ \sum_{k=1}^{n} ((1 - M_q(k)) x - \chi_q(k)) \overset{\text{PV}}{=} \int_{c-i\infty}^{c+i\infty} (n+a)^s - (n+a-1)^s \frac{ds}{s} F_q(s, x) ds, \forall n \geq 2 \]

Subtracting the \((n-1)\)th case from the \(n\)th case yields:

\[ (1 - M_q(n)) x - \chi_q(n) \overset{\text{PV}}{=} \int_{c-i\infty}^{c+i\infty} (n+a)^s - (n+a-1)^s \frac{ds}{s} F_q(s, x) ds, \forall n \geq 3 \]

Noting that:

\[ \chi_q(B_2(1)) = \frac{\chi_q(1)}{1 - M_q(1)} = \frac{1}{2} - q \]

\[ \chi_q(B_2(2)) = \frac{\chi_q(2)}{1 - M_q(2)} = \frac{1}{4} - q \]

upon applying Corollary 1, we obtain the paper’s main result, a reformulation of the Weak Collatz Conjecture (the assertion that 1, 2, 4 are the only periodic points of the Collatz map in the set of positive integers) as a contour integral:

**Theorem 3 (Contour Integral reformulation of the Weak Collatz Conjecture).** Let \( q \) be an odd prime, and let \( c > \sigma_q \). Then:

**I.** An integer \( x \in \mathbb{Z} \setminus \{-1, 0, 1\} \) is a periodic point of \( T_3 \) if and only if:

\[ \int_{c-i\infty}^{c+i\infty} (n+a)^s - (n+a-1)^s \frac{ds}{s} F_3(s, x) ds = 0 \]

for some integer \( n \geq 3 \) and some real number \( a \in (0, 1) \).

**II.** An integer \( x \in \mathbb{Z} \setminus \{-1, 0\} \) is a periodic point of \( T_5 \) if and only if:

\[ \int_{c-i\infty}^{c+i\infty} (n+a)^s - (n+a-1)^s \frac{ds}{s} F_5(s, x) ds = 0 \]

for some integer \( n \geq 3 \) and some real number \( a \in (0, 1) \).

**III.** If \( q \geq 7 \), an integer \( x \in \mathbb{Z} \setminus \{0\} \) is a periodic point of \( T_q \) if and only if:

\[ \int_{c-i\infty}^{c+i\infty} (n+a)^s - (n+a-1)^s \frac{ds}{s} F_q(s, x) ds = 0 \]
for some integer $n \geq 3$ and some real number $a \in (0, 1)$.

This Theorem is an excellent example of the power and versatility of the numen formalism presented in [31, 32]. By having reformulated the study of $T_q$’s dynamics in terms of the value-distribution of $\chi_q$—a fundamentally analytic problem—we have embedded our study in classical analysis and have thereby made it accessible to all the techniques and possibilities that analysis provides. Moreover, because of the broad applicability of the numen formalism to the dynamics of Hydra maps, we can achieve similar contour-integral reformulations of the question “is $x$ a periodic point?” to all other Hydra maps on $\mathbb{Z}, \mathbb{Z}^d$ for $d \geq 2$, and their isomorphic equivalents on the rings of integers of number fields.

In this paper, we shall analyze the Dirichlet series defined above, deriving functional equations for them and, from those, establishing their continuation to meromorphic functions on the complex plane. These functions will have a half-lattice of poles in the half-plane $\text{Re}(s) \leq \sigma_q$.

2. An Analysis of the Dirichlet Series of $\chi_q$ and $M_q$

As stated in the introduction, everything we will do here will end up coming back to the functional equations for $M_q$ and $\chi_q$. First, let us examine the abscissa of absolute convergence. To do that, we need to compute the summatory functions of $\chi_q$ and $M_q$.

**Proposition 1.** The summatory functions of $\chi_q$ and $M_q$ satisfy:

\begin{align}
\sum_{n=0}^{2N-1} \chi_q(n) &= \begin{cases} \frac{2^N}{3} 2^{N/3} & \text{if } q = 3 \\ \frac{2^N}{q-3} & \text{if } q \geq 5 \end{cases} \\
\sum_{n=0}^{2N-1} M_q(n) &= \frac{q}{q-1} \left( \frac{q+1}{2} \right)^N - \frac{1}{q-1}
\end{align}

**Proof:** We use an iterative-recursive method based on the functional equations of $\chi_q$ and $M_q$. This method is fundamental, and will be used in varying guises throughout this series of papers.

First, define:

\begin{equation}
S_q(N) \overset{\text{def}}{=} \sum_{n=0}^{2N-1} \chi_q(n)
\end{equation}

We then split the index of summation modulo 2 and apply $\chi_q$’s functional equations:

\begin{align*}
S_q(N) &= \sum_{n=0}^{2^{N-1}-1} (\chi_q(2n) + \chi_q(2n+1)) \\
&= \sum_{n=0}^{2^{N-1}-1} \left( \frac{\chi_q(n)}{2} + \frac{q\chi_q(n) + 1}{2} \right) \\
&= \sum_{n=0}^{2^{N-1}-1} \frac{1}{2} + \frac{q+1}{2} \sum_{n=0}^{2^{N-1}-1} \chi_q(n) \underbrace{S_q(N-1)}_{S_q(N-1)}
\end{align*}
This gives us the recursion relation:

\begin{equation}
S_q(N) = 2^{N-2} + \frac{q+1}{2}S_q(N-1)
\end{equation}

Nesting this gives:

\begin{align*}
S_q(N) &= 2^{N-2} + \frac{q+1}{2}S_q(N-1) \\
(\text{Use (2.4) for } S_q(N-1)) &= 2^{N-2} + 2^{N-3} \frac{q+1}{2} + \left( \frac{q+1}{2} \right)^2 S_q(N-2) \\
(\text{Use (2.4) for } S_q(N-2)) &= 2^{N-2} + 2^{N-3} \frac{q+1}{2} + 2^{N-4} \left( \frac{q+1}{2} \right)^2 + \left( \frac{q+1}{2} \right)^3 S_q(N-3) \\
&\vdots \\
&= \left( \frac{q+1}{2} \right)^N S_q(0) + \sum_{k=0}^{N-1} 2^{N-2-k} \left( \frac{q+1}{2} \right)^k
\end{align*}

Since:

\begin{equation}
S_q(0) = \sum_{n=0}^{2^a-1} \chi_q(n) = \chi_q(0) = 0
\end{equation}

we conclude that:

\begin{equation}
S_q(N) = \sum_{k=0}^{N-1} 2^{N-2-k} \left( \frac{q+1}{2} \right)^k = 2^{N-2} \sum_{k=0}^{N-1} \left( \frac{q+1}{4} \right)^k = \begin{cases} 
\frac{2^N}{q} & \text{if } q = 3 \\
\frac{2^N}{q-\frac{q+1}{2}} & \text{if } q \geq 5
\end{cases}
\end{equation}

Now, let:

\begin{equation}
S_q(N) \overset{\text{def}}{=} \sum_{n=0}^{2^a-1} M_q(n)
\end{equation}

Then, applying the same method as above, using the functional equations:

\begin{align*}
M_q(2n) &= \frac{M_q(n)}{2}, \text{ if } n \geq 1 \\
M_q(2n+1) &= \frac{qM_q(n)}{2}, \text{ if } n \geq 0
\end{align*}

we obtain:

\begin{align*}
S_q(N) &= 1 + \frac{1}{2} \sum_{n=1}^{2^a-1} M_q(n) + \frac{q}{2} \sum_{n=0}^{2^a-1} qM_q(n) \\
&= 1 + \frac{1}{2} \left( S_q(N-1) - M_q(0) \right) + \frac{q}{2} S_q(N-1) \\
&= \frac{1}{2} + \frac{q+1}{2} S_q(N-1) \\
(M_q(0) = 1) \Rightarrow \frac{1}{2} + \frac{q+1}{2} S_q(N-1)
\end{align*}

Nesting yields:

\begin{equation}
S_q(N) = \left( \frac{q+1}{2} \right)^N S_q(0) + \frac{1}{2} \sum_{n=0}^{N-1} \left( \frac{q+1}{2} \right)^n
\end{equation}
Here:

\[ S_q(0) = \sum_{n=0}^{2^q-1} M_q(n) = M_q(0) = 1 \]

and so, we get:

\[ S_q(N) = \left(\frac{q+1}{2}\right)^N + \frac{1}{2} \frac{\left(\frac{q+1}{2}\right)^N - 1}{\left(\frac{q+1}{2}\right)^N - 1} = \frac{q+1}{2} \frac{\left(\frac{q+1}{2}\right)^N - 1}{q-1} \]

Q.E.D.

For clarity’s sake, we record the following elementary estimate on partial sums of Dirichlet series in terms of the Big-O behavior of the coefficients’ summatory function.

**Proposition 2.** Let \( c \) be an integer \( \geq 2 \), let \( a(n) \) be a non-negative real-valued function, and suppose that:

\[ \sum_{n=0}^{c^{N-1}} a(n) \ll b^N \] as \( N \to \infty \)

for some real \( b \geq 1 \). Then:

\[ \left| \sum_{n=0}^{c^{N-1}} \frac{a(n)}{(n+1)^s} \right| \ll \left(\frac{b}{c^s}\right)^N \] as \( N \to \infty \)

where \( s = \sigma + it \). Consequently, the Dirichlet series converges absolutely for \( \text{Re}(s) > \log_c b \).

Proof: Using Abel’s summation formula, we have:

\[
\sum_{n=0}^{c^{N-1}} \frac{a(n)}{(n+1)^s} = \frac{1}{c^{Ns}} \sum_{n=0}^{c^{N-1}} a(n) + s \int_1^{c^N} \frac{\sum_{n=0}^{\lfloor x \rfloor-1} a(n)}{x^{s+1}} \, dx \\
(c^y = x; c^y \ln c = dx) = \frac{1}{c^{Ns}} \sum_{n=0}^{c^{N-1}} a(n) + s \ln c \int_0^N \frac{\sum_{n=0}^{[cy]-1} a(n)}{c^{sy}} \, dy
\]
Thus, for \( \text{Re}(s) = \sigma \):

\[
\left| \int_0^N \frac{\sum_{n=0}^{[c^y]-1} a(n)}{c^{\sigma y}} \, dy \right| \leq \int_0^N \frac{\sum_{n=0}^{[c^y]-1} a(n)}{c^\sigma y} \, dy \\
= \sum_{k=0}^{N-1} \int_k^{k+1} \frac{\sum_{n=0}^{[c^y]-1} a(n)}{c^{\sigma y}} \, dy \\
(\text{for } a(n) \geq 0) \leq \sum_{k=0}^{N-1} \int_k^{k+1} \frac{\sum_{n=0}^{[c^y]-1} a(n)}{c^{\sigma y}} \, dy \\
= \sum_{k=0}^{N-1} \left( \sum_{n=0}^{k+1} a(n) \right) \frac{c^{-\sigma k} - c^{-\sigma(k+1)}}{\sigma \ln c} \\
\leq \sum_{k=0}^{N-1} b^{k+1} c^{-\sigma k} - c^{-\sigma(k+1)} \frac{\sigma \ln c}{\sigma \ln c} \\
\leq \sum_{k=0}^{N-1} (b/c^\sigma)^k \\
\leq (b/c^\sigma)^N
\]

So:

\[
\left| \sum_{n=0}^{c^N-1} \frac{a(n)}{(n+1)} \right| \leq \frac{1}{c^{N\sigma}} \sum_{n=0}^{c^N-1} a(n) + \sigma \ln c \int_0^N \frac{\sum_{n=0}^{[c^y]-1} a(n)}{c^{\sigma y}} \, dy \\
\leq \frac{1}{c^{N\sigma}} b^N + (b/c^\sigma)^N \\
\leq (b/c^\sigma)^N
\]

Q.E.D.

**Proposition 3.** Let \( q \) be an odd prime, and let \( x \in \mathbb{Z} \). Then, \( \zeta_{\chi_q}(s) \) and \( \zeta_{M_q}(s) \), and \( F_q(s,x) \) converge absolutely for \( \text{Re}(s) > \sigma_q \).

Proof: **Proposition 1** shows that the summatory functions of \( \chi_q \) and \( M_q \) satisfy:

\[
\max \left\{ \sum_{n=0}^{2N-1} \chi_q(n), \sum_{n=0}^{2N-1} M_q(n) \right\} \leq \left( \frac{q+1}{2} \right)^N \text{ as } N \to \infty
\]

Applying **Proposition 2** yields the desired result for \( \zeta_{\chi_q}(s) \) and \( \zeta_{M_q}(s) \). Finally, since \( F_q(s,x) \) is a linear combination of \( \zeta_{\chi_q}(s) \), \( \zeta_{M_q}(s) \), and \( \zeta(s) \), its abscissa of convergence is \( \leq \) the minimum of the abscissa of convergence of those three Dirichlet series, which, for our choice of \( q \), is \( \text{Re}(s) > \sigma_q \).

Q.E.D.

Next, we establish an analytic continuation of \( F_q \) to a meromorphic function on \( \mathbb{C} \). To do this, we use the functional equations for \( M_q \) and \( \chi_q \) to establish functional equations for their ordinary generating functions (OGFs) (defined below). Exploiting the identity:
for all Re(s) greater than the abscissa of absolute convergence of the Dirichlet series on the left, we then convert the OGFs’ functional equations into functional equations for our Dirichlet series.

**Definition 3.** Define the OGFs \( g_{M_q} \) and \( g_{\chi_q} \) by:

\[
g_{M_q}(z) \overset{\text{def}}{=} \sum_{n=1}^{\infty} M_q(n) z^n
\]

\[
g_{\chi_q}(z) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \chi_q(n) z^n
\]

**Proposition 4.** The functional equations:

\[
g_{M_q}(z^2) = \frac{2}{qz + 1} g_{M_q}(z) - \frac{qz}{qz + 1}
\]

\[
g_{\chi_q}(z^2) = \frac{2}{qz + 1} g_{\chi_q}(z) - \frac{z}{qz + 1} \frac{1}{1 - z^2}
\]

hold for all complex \( z \) with \( |z| < 1 \).

Proof: 

I.

\[
g_{M_q}(z) = \sum_{n=1}^{\infty} M_q(n) z^n
\]

\[
= \sum_{n=0}^{\infty} M_q(2n + 1) z^{2n+1} + \sum_{n=1}^{\infty} M_q(2n) z^{2n}
\]

\[
= \frac{qz}{2} \sum_{n=0}^{\infty} M_q(n) z^{2n} + \frac{1}{2} \sum_{n=1}^{\infty} M_q(n) z^{2n}
\]

\[
= \frac{qz}{2} \left( 1 + g_{M_q}(z^2) \right) + \frac{1}{2} g_{M_q}(z^2)
\]

\[
= \frac{qz}{2} + \frac{qz + 1}{2} g_{M_q}(z^2)
\]

Hence:

\[
\frac{qz}{2} + \frac{qz + 1}{2} g_{M_q}(z^2) = g_{M_q}(z)
\]

\[
\Leftrightarrow
\]

\[
g_{M_q}(z^2) = \frac{2}{qz + 1} \left( g_{M_q}(z) - \frac{qz}{2} \right)
\]

II.
\[ g_{x_q}(z) = \sum_{n=1}^{\infty} \chi_q(n) z^n \]
\[ = \sum_{n=0}^{\infty} \chi_q(2n+1) z^{2n+1} + \sum_{n=1}^{\infty} \chi_q(2n) z^{2n} \]
\[ = \sum_{n=0}^{\infty} \frac{q \chi_q(n)}{2} z^{2n+1} + \frac{1}{2} \sum_{n=1}^{\infty} \chi_q(n) z^{2n} \]
\[ = \frac{q z}{2} \sum_{n=0}^{\infty} \chi_q(n) z^{2n} + \frac{1}{2} \sum_{n=0}^{\infty} z^{2n+1} + \frac{1}{2} g_{x_q}(z^2) \]
\[ (\chi_q(0) = 0) ; = \frac{q z}{2} g_{x_q}(z^2) + \frac{1}{2} \frac{z}{2 (1 - z^2)} + + \frac{1}{2} g_{x_q}(z^2) \]
\[ = \frac{1}{2} \frac{z}{2 (1 - z^2)} + \frac{q z + 1}{2} g_{x_q}(z^2) \]

Hence:

\[ \frac{1}{2} \frac{z}{2 (1 - z^2)} + \frac{q z + 1}{2} g_{x_q}(z^2) = g_{x_q}(z) \]

\[ \Downarrow \]

\[ (2.23) \quad g_{x_q}(z^2) = \frac{2}{q z + 1} \left( g_{x_q}(z) - \frac{1}{2} \frac{z}{2 (1 - z^2)} \right) \]

Q.E.D.

To simplify matters, we now define an OGF for the sequence \((1 - M_q(n)) x - \chi_q(n)\).

**Definition 4.** For each \(x \in \mathbb{Z}\), define the function \(z \mapsto f_q(z, x)\) by:

\[ f_q(z, x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} ((1 - M_q(n)) x - \chi_q(n)) z^n = \left( \frac{z}{1 - z} - g_{M_q}(z) \right) x - g_{x_q}(z) \]

The functional equations for the \(g\)s give us a functional equation for \(f_q\):

**Proposition 5.** We have:

\[ f_q(z, x) = -\frac{(q - 1) x}{2} \frac{z}{1 - z} - \frac{1}{2} \frac{z}{2 (1 - z^2)} + \frac{q z + 1}{2} f_q(z^2, x), \ \forall \ |z| < 1 \]
Proof: Using Proposition 4, we have:

\[ f_q(z^2, x) = \left( \frac{z}{1-z} - g_{M_q}(z^2) \right) x - g_{\chi_q}(z^2) \]

\[ = \left( \frac{z}{1-z} - \left( \frac{2}{qz+1} g_{M_q}(z) - \frac{qz}{qz+1} \right) \right) x - \left( \frac{2}{qz+1} g_{\chi_q}(z) - \frac{z}{qz+1} \right) \]

\[ = \left( \frac{z}{1-z} + \frac{qz}{qz+1} \right) x + \frac{z}{qz+1} \frac{1}{1-z^2} + \frac{2}{qz+1} \left( \frac{-xg_{M_q}(z) - g_{\chi_q}(z)}{1-z^2} \right) \]

\[ = \frac{xz}{1-z} + \frac{qzx}{qz+1} + \frac{z}{qz+1} \frac{1}{1-z^2} + \frac{2}{qz+1} \left( \frac{-xz}{1-z} + \left( \frac{z}{1-z} - g_{M_q}(z) \right) x - g_{\chi_q}(z) \right) \]

\[ = \frac{xz}{1-z} + \frac{qxz}{qz+1} + \frac{z}{qz+1} \frac{1}{1-z^2} - \frac{2}{qz+1} \frac{xz}{qz+1} \frac{1}{1-z} + \frac{2}{qz+1} f_q(z, x) \]

\[ = \frac{x}{1-z} - \frac{xz}{qz+1} + \frac{z}{qz+1} \frac{1}{1-z^2} - \frac{xz}{qz+1} \frac{1}{1-z} + \frac{2}{qz+1} f_q(z, x) \]

\[ = \frac{(q-1)xz(1+z) + z + 2(1-z^2)f_q(z, x)}{(qz+1)(1-z^2)} \]

and so:

\[ f_q(z^2, x) = \frac{(q-1)xz(1+z) + z + 2(1-z^2)f_q(z, x)}{(qz+1)(1-z^2)} \]

\[ \iff \]

\[ f_q(z, x) = \frac{(qz+1)(1-z^2)f_q(z^2, x) - (q-1)xz(1+z)}{2(1-z^2)} - \frac{z}{2(1-z^2)} \]

\[ = \frac{qz+1}{2} f_q(z^2, x) - \frac{(q-1)xz}{2} \frac{z}{1-z} - \frac{z}{2(1-z^2)} \]

Q.E.D.

Using \( f_q(z, x) \)'s functional equation in conjunction with the Mellin transform, we can obtain a functional equation for \( F_q(s, x) \).

**Theorem 4 (Functional Equation & Analytic Continuation of \( F_q \)).** \( F_q(s, x) \) satisfies the functional equation:

\[ F_q(s, x) = -\frac{1}{2} \frac{2^s ((q-1)x+1) - 1}{2^s - 2^{q}} \zeta(s) + \frac{q/2}{2^s - 2^{q}} \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^n \binom{s+n-1}{n} F_q(s+n, x) \]

where:

\[ \binom{s+n-1}{s-1} = \frac{1}{n!} \frac{\Gamma(s+n)}{\Gamma(s)} \]
Proof: Using the identity: which holds for all \( s \) with \( \text{Re}(s) > \sigma_q \), we have:

\[
(2.28) \quad F_q(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} f_q \left( e^{-y}, x \right) dy
\]

Replacing \( x \) with \( e^{-y} \) in (2.25) and applying (2.15) yields

\[
(2.29) \quad F_q(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} g \frac{e^{-y} + 1}{2} f_q \left( e^{-2y}, x \right) dy - \frac{(q-1)x}{2} \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} e^{-y} dy
\]

\[
- \frac{1}{2} \frac{1}{\Gamma(s)} \int_0^\infty \frac{y^{s-1} e^{-y}}{1 - e^{-2y}} dy
\]

(I) becomes:

\[
(2.30) \quad \frac{1}{2} \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} f_q \left( e^{-2y}, x \right) dy + \frac{q}{2} \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} e^{-y} f_q \left( e^{-2y}, x \right) dy
\]

which is:

\[
(2.31) \quad \frac{1}{2s+1} \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} f_q \left( e^{-u}, x \right) du + \frac{q}{2s+1} \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-u/2} f_q \left( e^{-u}, x \right) du
\]

For the integral on the left, we expand \( e^{-u/2} \) as a power series and integrate term by term:

\[
\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-u/2} f_q \left( e^{-u}, x \right) du = \sum_{n=0}^{\infty} \left( \frac{-1/2)^n}{n!} \right) \frac{1}{\Gamma(s)} \int_0^\infty u^{s+n-1} f_q \left( e^{-u}, x \right) du
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{-1/2)^n}{n! n} \right) \frac{\Gamma(s+n)}{\Gamma(s)} \frac{1}{\Gamma(s+n)} \int_0^\infty u^{s+n-1} f_q \left( e^{-u}, x \right) du
\]

\[
\left( \frac{\Gamma(s+n)}{n! \Gamma(s)} \right) = \left( s+n-1 \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{-1/2)^n}{n! n} \right) \left( s+n-1 \right) F_q(s+n,x)
\]

So:

\[
\frac{1}{\Gamma(s)} \int_0^\infty g \frac{e^{-y} + 1}{2} f_q \left( e^{-2y}, x \right) dy = \frac{F_q(s,x)}{2s+1} + \frac{q}{2s+1} \sum_{n=0}^{\infty} \left( \frac{-1/2)^n}{n! n} \right) F_q(s+n,x)
\]

\[
= \frac{q+1}{2s+1} F_q(s,x) + \frac{q}{2s+1} \sum_{n=1}^{\infty} \left( \frac{-1/2)^n}{n! n} \right) F_q(s+n,x)
\]

Meanwhile, (II) is:

\[
(2.32) \quad \frac{1}{\Gamma(s)} \int_0^\infty \frac{y^{s-1} e^{-y}}{1 - e^{-y}} dy = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} e^{-ny} dy = \zeta(s)
\]
while (III) is:

\[
\frac{1}{\Gamma(s)} \int_0^\infty \frac{y^{s-1}e^{-y}}{1-e^{-2y}} dy = \sum_{n=0}^{\infty} \frac{1}{\Gamma(s)} \frac{1}{1/(2n+1)^s} \int_0^\infty y^{s-1}e^{-(2n+1)y} dy
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}
\]

Putting everything together, (2.29) becomes:

\[
(2.33) \quad F_q(s, x) = \frac{q+1}{2s+1} F_q(s, x) - \frac{(q-1)x}{2} \zeta(s) - \frac{1}{2} \left(1 - 2^{-s}\right) \zeta(s)
\]

\[
+ \frac{q}{2s+1} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n \binom{s+n-1}{n} F_q(s+n, x)
\]

and so:

\[
F_q(s, x) = \frac{-(q-1)x}{2} \zeta(s) - \frac{1}{2} \left(1 - 2^{-s}\right) \zeta(s) + \frac{q}{2} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n \binom{s+n-1}{n} F_q(s+n, x)
\]

\[
= \frac{-1}{2} 2^s ((q-1)x+1) - \frac{1}{2} \zeta(s) + \frac{q/2}{2^s-2^{\sigma_q}} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n \binom{s+n-1}{n} F_q(s+n, x)
\]

Q.E.D.

Using (2.26), we can analytically continue \( F_q \) as a function of \( s \). The next few results chronicle \( F_q \)'s singularities. First, however, a notation:

**Definition 5.** We define the function \( S_q(s, x) \) by:

\[
(2.34) \quad S_q(s, x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n \binom{s+n-1}{n} F_q(s+n, x)
\]

Note that, for any \( x \) and any \( \delta > 0 \), \( S_q(s, x) \) converges uniformly to a holomorphic function on the half plane \( \text{Re}(s) \geq \sigma_q - 1 + \delta \).

**Remark 4.** With this notation, (2.26) can be written as:

\[
(2.35) \quad F_q(s, x) = \frac{1}{2^s-2^{\sigma_q}} \left(1 - 2^s ((q-1)x+1) \zeta(s) + \frac{q}{2} S_q(s, x)\right)
\]

**Remark 5.** For \( n \geq 1 \):

\[
(2.36) \quad \binom{s+n-1}{n} = \frac{1}{n!} \prod_{k=0}^{n-1} (s+k)
\]
is a degree $n$ polynomial in $s$ with positive rational coefficients and a constant term of 0, we have that its absolute value is bounded by a constant multiple of $|s|^n$, where the constant is the value of the polynomial at $s = 1$; that is:

\[
\left| \frac{s + n - 1}{n} \right| \leq \left( \frac{1}{n!} \prod_{k=0}^{n-1} (1 + k) \right) |s|^n = \frac{n!}{n} |s|^n = |s|^n
\]

and so:

\[
|S_q(s, x)| \leq \sum_{n=1}^{\infty} \left( -\frac{s}{2} \right)^n |F_q(s + n, x)|
\]

We conclude by computing the residues of $F_3$. The general case of $F_q$ is more complicated.

**Corollary 2 (Singularities of $F_3$).** I. For $k \in \mathbb{Z} \setminus \{0\}$, $F_3(s, x)$ has a simple pole at $s = 1 + \frac{2k\pi i}{\ln 2}$, the residue of which is:

\[
\frac{3}{\ln 16} S_3 \left( 1 + \frac{2k\pi i}{\ln 2}, x \right) - \frac{4x + 1}{\ln 16} \zeta \left( 1 + \frac{2k\pi i}{\ln 2} \right)
\]

II. $F_3(s, x)$ has a double pole at $s = 1$, the residue of which is:

\[
\frac{3S_3(1, x)}{\ln 16} - \left( \left( \frac{1}{2} + \frac{\gamma}{\ln 2} \right) x + \left( \frac{3}{8} + \frac{\gamma}{4 \ln 2} \right) \right)
\]

III. $F_3(s, x)$ has a simple pole at $s = 0$, the residue of which is:

\[
-\frac{3(4x + 1)}{16 \ln 2}
\]

More generally, $F_3(s, x)$ has a simple pole at $s = -n$ for all integers $n \geq 0$. Letting $R_{3,n}$ denote the residue of $F_3(s, x)$ at $s = -n$, we then have that the $R_{3,n}$'s satisfy the recurrence relation:

\[
R_{3,n+1} = \frac{3(4x + 1)}{1 - 2^n} - \frac{3/2}{1 - 2^n} \sum_{k=0}^{n} 2^k \left( \begin{array}{c} n + 1 \\ k \end{array} \right) R_{3,k}, \forall n \geq 0
\]

**Proof:** For convenience, here is (2.35) again:

\[
F_q(s, x) = \frac{1}{2^x - 2^{\sigma_q}} \left( \frac{1 - 2^s ((q - 1)x + 1)}{2} \zeta(s) + \frac{q}{2} S_q(s, x) \right)
\]

Fix $x \in \mathbb{Z}$. Note that $s = \sigma_q$ is the largest positive real number at which $F_q(s, x)$ has a singularity, and that $S_q(s, x)$ is holomorphic at this value of $s$. Using (2.35), we see that the singularity of $F_q(s, x)$ at $s = \sigma_q$ must come from either:

\[
\frac{1 - 2^s ((q - 1)x + 1)}{2} \zeta(s)
\]

or

\[
\frac{1}{2^x - 2^{\sigma_q}}
\]

Since $\zeta(s)$ has a simple pole at $s = 1$, the limit:

\[
\lim_{s \to 1} \frac{1 - 2^s ((q - 1)x + 1)}{2} = -(q - 1)x - \frac{1}{2}
\]
and the fact that \( x \in \mathbb{Z} \) tells us the only singularity of \( \frac{1-2^x((q-1)x+1)}{2} \zeta(s) \) is a simple pole at \( s = 1 \) of residue \( -(q-1)x - \frac{1}{2} \). Meanwhile, \( (2^s - 2^{\sigma_q})^{-1} \) has simple poles at \( s = \sigma_q + \frac{2k\pi i}{\ln 2} \)
for all \( k \in \mathbb{Z} \).

So, suppose \( q = 3 \), so that \( \sigma_q = 1 \). Then (2.35) becomes:

\[
F_3(s, x) = \left( \frac{1}{2} - 2^{s-1}(2x+1) \right) \frac{\zeta(s)}{2^s - 2} + \frac{3}{2} S_3(s, x) \]

where \( S_3(s, x) \) is holomorphic for \( \Re(s) > \sigma_q - 1 = 0 \).

• Here:

\[
\left( \frac{1}{2} - 2^{s-1}(2x+1) \right) \frac{\zeta(s)}{2^s - 2}
\]

has a double pole at \( s = 1 \), while \( \frac{3}{2} S_3(s, x) \) has a simple pole there; in total, \( F_3(s, x) \) has a double pole at \( s = 1 \).

To compute the residue of \( F_3(s, x) \) at \( s = 1 \), we use the series expansions:

\[
\zeta(s) = \frac{1}{s-1} + \gamma + O((s-1)) \text{ as } s \to 1
\]

\[
\frac{s-1}{2^s - 2} = \frac{1}{\ln 4} - \frac{s-1}{4} + O((s-1)^2) \text{ as } s \to 1
\]

which gives:

\[
(s-1)^2 F_3(s, x) = \frac{1 - 2^s(2x+1)}{\ln 16} + \left( \frac{\gamma}{\ln 16} - \frac{1}{8} \right) (1 - 2^s(2x+1)) + \frac{3S_3(s, x)}{\ln 16} (s-1)
\]

\[+ O((s-1)^2) \]

Since:

\[
\text{Res} [F_3(s, x) : 1] = \lim_{s \to 1} \frac{d}{ds} \left\{ (s-1)^2 F_3(s, x) \right\}
\]

differentiating the previous equation with respect to \( s \) and taking the limit as \( s \to 1 \) yields the residue:

\[
\text{Res} [F_3(s, x) ; 1] = \frac{3S_3(1, x)}{\ln 16} - \left( \frac{1}{2} + \frac{\gamma}{\ln 2} \right) x + \left( \frac{3}{8} + \frac{\gamma}{4\ln 2} \right)
\]

• Since \( S_3(s, x) \) and \( \zeta(s) \) are holomorphic at:

\[
s = \sigma_3 + \frac{2k\pi i}{\ln 2} = 1 + \frac{2k\pi i}{\ln 2}, \forall k \in \mathbb{Z} \setminus \{0\}
\]

and since:

\[
\lim_{s \to 1 + \frac{2k\pi i}{\ln 2}} \left( \frac{1}{2} - 2^{s-1}(2x+1) \right) = -2x - \frac{1}{2}
\]

which is never zero (since \( x \in \mathbb{Z} \)) we have that the \( (2^s - 2)^{-1} \) term is the only singular part of \( F_3(s, x) \) at \( s = 1 + \frac{2k\pi i}{\ln 2} \) for non-zero \( k \), and so, \( F_3(s, x) \) has simple poles at \( s = 1 + \frac{2k\pi i}{\ln 2} \)
for $k \in \mathbb{Z} \setminus \{0\}$. The residues of these poles are:

$$
\text{Res} \left[ F_3(s, x) : 1 + \frac{2k\pi i}{\ln 2} \right] = \lim_{s \to 1 + \frac{2k\pi i}{\ln 2}} \left( s - 1 - \frac{2k\pi i}{\ln 2} \right) \left( \frac{1}{2} - 2^{s-1} (2x + 1) \right) \frac{\zeta(s)}{2^s - 2} + 3 S_3(s, x)
$$

$$
= \lim_{s \to 0} \frac{s}{2^s - 1} \frac{(1 - 2^s (4x + 2)) \zeta(s + 1 + \frac{2k\pi i}{\ln 2}) + 3 S_3(s + 1 + \frac{2k\pi i}{\ln 2}, x)}{1/\ln 2}
$$

$$
= \frac{3}{\ln 16} S_3 \left( 1 + \frac{2k\pi i}{\ln 2}, x \right) - \frac{4x + 1}{\ln 16} \zeta \left( 1 + \frac{2k\pi i}{\ln 2} \right)
$$

\[\bullet\] As $s \to 0$, (2.35) tends to:

$$
F_3(0, x) = \left( \frac{1}{2} - \frac{2x + 1}{2} \right) \frac{\zeta(0)}{1 - 2} + \frac{3}{2} \frac{S_3(0, x)}{1 - 2} = -\frac{x}{2} - \frac{3}{2} S_3(0, x)
$$

Now:

$$
S_3(s, x) = -\frac{s}{2} F_3(s + 1, x) + \frac{s(s + 1)}{8} F_3(s + 2, x) - \frac{s(s + 1) (s + 2)}{2^3 \cdot 3!} F_3(s + 3, x) + \cdots
$$

Since $F_3(s + n, x)$ is holomorphic at $s = 0$ uniformly with respect to $n \geq 2$, we have:

$$
S_3(s, x) = -\frac{1}{2} s F_3(s + 1, x) + O_0(s) \text{ as } s \to 0
$$

Since $F_3(s, x)$ has a double pole at $s = 1$, $s F_3(s + 1, x)$ has a simple pole at $s = 0$, and thus, $S_3(0, x)$ is a simple pole. Consequently, $F_3(s, x)$ has a simple pole at $s = 0$, the residue of which is given by:

$$
\lim_{s \to 0} s F_3(s, x) = \lim_{s \to 0} s \left( -\frac{x}{2} - \frac{3}{2} S_3(s, x) \right)
$$

$$
= -\frac{1}{2} \lim_{s \to 0} s S_3(s, x)
$$

$$
\left( S_3(s, x) = -\frac{1}{2} s F_3(s + 1, x) + O_0(s) \right); = \frac{3}{4} \lim_{s \to 0} s^2 F_3(s + 1, x)
$$

$$
= \frac{3}{4} \lim_{s \to 1} (s - 1)^2 F_3(s, x)
$$

(Use (2.41)); = \frac{3}{4} \lim_{s \to 1} (s - 1)^2 \left( \frac{1}{2} - 2^{s-1} (2x + 1) \right) \frac{\zeta(s)}{2^s - 2}

$$
+ \frac{3}{4} \lim_{s \to 1} (s - 1)^2 \frac{3 S_3(s, x)}{2^s - 2}
$$

Here, $S_3(s, x)$ is holomorphic at $s = 1$. Since $1/ (2^s - 2)$ has a simple pole at $s = 1$, we have:

$$
\lim_{s \to 1} (s - 1)^2 \frac{3 S_3(s, x)}{2^s - 2} = 0
$$
On the other hand, \( \zeta(s) / (2^s - 2) \) has a double pole at \( s = 1 \), so multiplying it by \((s - 1)^2\) and letting \( s \to 1 \) will give us a constant. Consequently:

\[
\text{Res} \left[ F_3(s, x) : 0 \right] = \lim_{s \to 0} s F_3(s, x) \\
= -\frac{3}{2} \lim_{s \to 0} s S_3(s, x) \\
= \frac{3}{4} \lim_{s \to 1} (s - 1)^2 F_3(s, x) \\
= \frac{3}{4} \lim_{s \to 1} (s - 1)^2 \left( \frac{1}{2} - 2^{s-1}(2x + 1) \right) \frac{\zeta(s)}{2s - 2} \\
= -\frac{3}{4} \left( -2x - \frac{1}{2} \right) \times \lim_{s \to 1} \frac{(s - 1)^2 \zeta(s)}{2^s - 2} \\
= -\frac{3}{8} (4x + 1) \times \lim_{s \to 1} (s - 1) \zeta(s) \times \lim_{z \to 1} \frac{z - 1}{2^z - 2} \\
= -\frac{3}{16 \ln 2} (4x + 1)
\]

Note that this also shows that:

\[
(2.42) \quad \lim_{s \to 1} (s - 1)^2 F_3(s, x) = -\frac{4x + 1}{4 \ln 2}
\]

More generally, by induction, (2.35) shows that \( F_3(s, x) \) has a simple pole at \( s = -k \) for \( k \geq 0 \), the residue of which is:

\[
\lim_{s \to -k} (s + k) F_3(s, x) = \lim_{s \to -k} (s + k) \frac{\left( \frac{1}{2} - 2^{s-1}(2x + 1) \right) \zeta(s) + \frac{3}{2} S_3(s, x)}{2^s - 2} \\
= \frac{3/2}{2^{-k} - 2} \lim_{s \to -k} (s + k) S_3(s, x) \\
= \frac{3/2}{2^{-k} - 2} \lim_{s \to 0} s S_3(s - k, x)
\]

Here:

\[
S_3(s - k, x) = \sum_{n=1}^{k} \left( -\frac{1}{2} \right)^n \binom{s + n - k - 1}{n} F_3(s + n - k, x) \\
+ \left( -\frac{1}{2} \right)^k \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^n \binom{s + n - 1}{n + k} F_3(s + n, x) \\
= \sum_{n=0}^{k-1} \left( -\frac{1}{2} \right)^{k-n} \binom{s - n - 1}{k - n} F_3(s - n, x) \\
+ \left( -\frac{1}{2} \right)^k \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^n \binom{s + n - 1}{n + k} F_3(s + n, x)
\]
As \( s \to 0 \), note that \( F_3(s - n, x) \) tends to the simple pole \( F_3(-n, x) \) for all \( n \in \{0, \ldots, k - 1\} \).

Since:

\[
\begin{align*}
\binom{s - n - 1}{k - n} &= \frac{1}{(k - n)!} \prod_{j=0}^{k-n-1} (s - n - 1 - j) \\
(\ell = j + n + 1) &= \frac{1}{(k - n)!} \prod_{\ell=n+1}^{k} (s - \ell) \\
(\text{let } s \to 0) &= \frac{1}{(k - n)!} \prod_{\ell=n+1}^{k} (-\ell) \\
&= (-1)^{k-n} \prod_{\ell=1}^{k} \frac{\ell}{(k - n)!} \\
&= (-1)^{k-n} \binom{k}{n} \\
\end{align*}
\]

we have that:

\[
\lim_{s \to 0} s \sum_{n=0}^{k-1} \left( -\frac{1}{2} \right)^{k-n} \binom{s - n - 1}{k - n} F_3(s - n, x) = \sum_{n=0}^{k-1} \left( -\frac{1}{2} \right)^{k-n} (-1)^{k-n} \binom{k}{n} \text{Res } \left[ F_3(s, x) : -n \right] \\
= \sum_{n=0}^{k-1} \left( \frac{1}{2} \right)^{k-n} \binom{k}{n} \text{Res } \left[ F_3(s, x) : -n \right]
\]

Meanwhile, since \( F_3(s + n, x) \) is holomorphic at \( s = 0 \) uniformly with respect to \( n \geq 2 \), we have:

\[
\lim_{s \to 0} s \left( -\frac{1}{2} \right)^{k} \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^{n} \binom{s + n - 1}{n + k} F_3(s + n, x) = \left( -\frac{1}{2} \right)^{k+1} \lim_{s \to 0} \binom{s}{k+1} F_3(s + 1, x) \\
= \left( -\frac{1}{2} \right)^{k+1} \prod_{j=1}^{k} (-j) \lim_{s \to 0} s^2 F_3(s + 1, x) \\
= -\left( \frac{1}{2} \right)^{k+1} \lim_{s \to 1} (s - 1)^2 F_3(s, x) \\
(2.42) = -\left( \frac{1}{2} \right)^{k+1} \frac{4x + 1}{k+1} - \frac{4 \ln 2}{k+1} \\
= \frac{4x + 1}{4 \ln 2} \left( \frac{1}{2} \right)^{k+1}
\]

Thus:

\[
\lim_{s \to 0} s S_3(s - k, x) = \frac{4x + 1}{4 \ln 2} \left( \frac{1}{2} \right)^{k+1} + \sum_{n=0}^{k-1} \left( \frac{1}{2} \right)^{k-n} \binom{k}{n} \text{Res } \left[ F_3(s, x) : -n \right]
\]
and so:

\[
\text{Res} \left[ F_3(s, x) : -k \right] = \lim_{s \to -k} (s + k) F_3(s, x) = \frac{3/2}{2 - k - 2} \lim_{s \to 0} s S_3(s - k, x) = \frac{3/2}{2 - k - 2} \left( \frac{4x + 1}{4 \ln 2} \frac{1}{k + 1} \left( \frac{1}{2} \right)^{k+1} + \sum_{n=0}^{k-1} \left( \frac{1}{2} \right)^{k-n} \binom{k}{n} \text{Res} \left[ F_3(s, x) : -n \right] \right)
\]

Thus, letting:

\[
R_{3,n} \overset{\text{def}}{=} \text{Res} \left[ F_3(s, x) : -n \right]
\]

we have:

\[
R_{3,k} = \frac{3/(4x+1)}{1 - 2^{k+1}} \frac{1}{k + 1} + \frac{3/2}{1 - 2^{k+1}} \sum_{n=0}^{k-1} 2^n \binom{k}{n} R_{3,n}
\]

re-indexing (swapping \(k\) and \(n\)) gives:

\[
R_{3,n} = \frac{3/(4x+1)}{1 - 2^{n+1}} \frac{1}{n + 1} + \frac{3/2}{1 - 2^{n+1}} \sum_{k=0}^{n-1} 2^k \binom{n}{k} R_{3,k}
\]

which is the desired recursive formula.

Q.E.D.

In conjunction with Perron’s Formula, these residues can be used to asymptotically analyze equation (1.17), however, the functional equation (2.26) can be used to show that \(F_q(s, x)\) grows hyperexponentially as \(\text{Re}(s) \to -\infty\), which makes it impossible to evaluate the integral in (1.17) exactly by shifting the contour of integration to left infinity. Nevertheless, it is conceivable that clever exploitation of the properties of Perron-type formulae and the growth of the Riemann Zeta Function and to get something interesting out of this. Finally, since the set-up used in this paper can be applied to any Hydra map for which the methods of [31, 32], it may be of interest to explore if there are any instances—particularly degenerate ones—where the Mellin transforms methods of this paper might bear useful fruit.

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