Superconvergence of both two and three dimensional rectangular Morley elements for biharmonic equations ∗

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Abstract

In the present paper, superconvergence of second order, after an appropriate postprocessing, is achieved for both the two and three dimensional first order rectangular Morley elements of biharmonic equations. The analysis is dependent on superconvergence of second order for the consistency error and a corrected canonical interpolation operator, which help to establish supercloseness of second order for the corrected canonical interpolation. Then the final superconvergence follows a standard postprocessing. For first order nonconforming finite element methods of both two and three dimensional fourth order elliptic problems, it is the first time that full superconvergence of second order is obtained without an extra boundary condition imposed on exact solutions. It is also the first time that superconvergence is established for nonconforming finite element methods of three dimensional fourth order elliptic problems. Numerical results are presented to demonstrate the theoretical results.

Keywords: Biharmonic equation; rectangular Morley element; superconvergence

1 Introduction

Because of significant applications in scientific and engineering computing, superconvergence analysis of finite element methods has become an active subject since 70’s last century. However, most of attentions have been paid on conforming and mixed finite element methods of second order problems, we refer interested readers to [3, 4, 12, 14] for more details. Since conforming finite element methods of fourth order problems are very complicated, most of popularly used elements in practice are nonconforming, for instance, [10, 19, 21, 25, 26, 27, 28]. However, for nonconforming finite elements, due to nonconformity of both trial and test functions, it becomes much more difficult to establish superconvergence properties and related asymptotic error expansions. For second order elliptic problems, there are a few superconvergence results on rectangular elements. In [5, 23], superconvergence of the gradient was obtained at the centers of elements for the Wilson element, which relies on the observation that the Wilson element space can be split into a conforming part and a nonconforming part. Due to superconvergence of consistency errors, superconvergence of the nonconforming rotated Q₁ element [20] and its variants was derived, see [8, 13, 18]. For the plate bending problem, there are only few superconvergence results for nonconforming finite elements. In [9], Chen first established the supercloseness of the corrected interpolation of the incomplete biquadratic element [29, 21] on uniform rectangular meshes. By using similar corrected interpolations as in [8], Mao et al. [17] first proved one and a half-order superconvergence for the Morley element [19] and the incomplete biquadratic nonconforming element on uniform rectangular

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meshes. In a recent paper [6], Hu and Ma proposed a new method by using equivalence between the Morley element and the first order Hellan-Herrmann-Johnson element and obtained one and a half-order superconvergence for the Morley element on uniform mesh. That half order superconvergence can be improved to one order superconvergence if the third order normal derivative of exact solutions vanishes on the boundary of the domain under consideration. Based on the equivalence to the Stokes equations and a superconvergence result of Ye [30] on the Crouzeix–Raivart element, Huang et al. [7] derived the superconvergence for the Morley element, which was postprocessed by projecting the finite element solution to another finite element space on a coarser mesh. See Lin and Lin [11] for superconvergence of the Ciarlet–Raviart scheme of the biharmonic equation. Note that all of those results are only for fourth order problems in two dimensions. Superconvergence of nonconforming finite element methods cannot be found for fourth order problems in three dimensions.

The purpose of the present paper is to analyze superconvergence of both the two-dimensional and three-dimensional rectangular Morley elements from [25]. Since both of them are nonconforming, one difficulty is to bound the consistency error. Another difficulty is from the canonical interpolation operator which does not admit supercloseness. To overcome the first difficulty, we use some special orthogonal property of the canonical interpolation operators of both the bilinear and trilinear elements when applied to the functions in the rectangular Morley element spaces. The other crucial observation is that the error between the (piecewise) gradient of functions in the discrete spaces and its mean is equal on two opposite edges (faces) of an element. In particular, this leads to superconvergence of second order for the consistency error. To deal with the second difficulty, we follow the idea from [3] to use a correction of the canonical interpolation. Together with the asymptotic expansion results from [9], this yields supercloseness of second order for such a corrected interpolation. Finally, based on the above superconvergence results, we follow the postprocessing idea from [14] to obtain a global superconvergent approximate solution, which converges at the second order convergence rate. It should be stressed that for first order nonconforming finite element methods of both two and three dimensional fourth order elliptic problems, it is the first time that full superconvergence of second order is obtained without an extra boundary condition imposed on exact solutions. It is also the first time that superconvergence is established for nonconforming finite element methods of three dimensional fourth order elliptic problems.

This paper is organized as follows. In the following section, we shall present the model problem and the rectangular Morley element. In section 3, we analyze the superconvergence property of the consistency error for the two-dimensional situation. In section 4, we make a correction of the canonical interpolation and obtain the superconvergence result after the postprocessing. In section 5, we establish the superconvergence result for the three-dimensional cubic Morley element. In the last section 6, we present some numerical results to demonstrate our theoretical results.

2 The model problem and the rectangular Morley element

2.1 The model problem

We consider the model fourth order elliptic problem: Given \( f \in L^2(\Omega) \), \( \Omega \subset \mathbb{R}^2 \) is a bounded Lipschitz domain,

\[
\begin{aligned}
\Delta^2 u &= f, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

(2.1)

The variational formula of problem (2.1) is to find \( u \in V := H^2_0(\Omega) \), such that

\[
a(u,v) := (\nabla^2 u, \nabla^2 v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}, \quad \text{for any } v \in V.
\]

(2.2)

where \( \nabla^2 u \) denotes the Hessian matrix of the function \( u \).

2.2 The two-dimensional rectangular Morley element

To consider the discretization of (2.2) by the rectangular Morley element method, let \( T_h \) be a regular uniform rectangular triangulation of the domain \( \Omega \). Given \( K \in T_h \), let \((x_{1,c}, x_{2,c})\) be the center of \( K \), the
Figure 1: degrees of freedom

meshsize $h$ and affine mapping:

$$\xi_1 = \frac{x_1 - x_{1,c}}{h}, \quad \xi_2 = \frac{x_2 - x_{2,c}}{h}, \quad \text{for any } (x_1, x_2) \in K. \quad (2.3)$$

On element $K$, the shape function space of the rectangular Morley element from [24] reads

$$P(K) := P_2(K) + \text{span}\{x_3, x_4\}, \quad (2.4)$$

here and throughout this paper, $P_l(K)$ denotes the space of polynomials of degree $\leq l$ over $K$. The nodal parameters are: for any $v \in C^1(K)$,

$$D(v) := \left(v(a_i), \frac{1}{|e_j|} \int_{e_j} \frac{\partial v}{\partial n_{e_j}} ds\right), \quad i, j = 1, 2, 3, 4, \quad (2.5)$$

where $a_i$ are vertices of $K$ and $e_j$ are edges with unit normal vectors $n_{e_j}$ of $K$, $|e_j|$ denote measure of edges $e_j$, see Figure 1. Let the reference element $\hat{K}$ be a square on $(\xi_1, \xi_2)$ plane, its vertices be $\hat{a}_1(1, -1), \hat{a}_2(1, 1), \hat{a}_3(-1, 1), \hat{a}_4(-1, -1)$, and its sides be $\hat{e}_1 = \hat{a}_1\hat{a}_2, \hat{e}_2 = \hat{a}_2\hat{a}_3, \hat{e}_3 = \hat{a}_3\hat{a}_4, \hat{e}_4 = \hat{a}_4\hat{a}_1$.

The nonconforming rectangular Morley element space is then defined by

$$V_h := \{v \in L^2(\Omega): v|_K \in P(K), \forall K \in T_h, v \text{ is continuous at all internal vertices and vanishes at all boundary vertices, and } \int_e \frac{\partial v}{\partial n_e} ds \text{ is continuous on internal edges } e \text{ and vanishes on boundary edges } e \text{ of } T_h\}. \quad (2.8)$$

The discrete problem of (2.2) reads: Find $u_h \in V_h$, such that

$$a_h(u_h, v_h) := (\nabla_h^2 u_h, \nabla_h^2 v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}, \quad \text{for any } v_h \in V_h. \quad (2.6)$$

where the operator $\nabla_h^2$ is the discrete counterpart of $\nabla^2$, which is defined element by element since the discrete space $V_h$ is nonconforming. Define a semi-norm over $V_h$ by

$$|u_h|^2_h := a_h(u_h, u_h), \quad \text{for any } u_h \in V_h. \quad (2.7)$$

Let $u$ and $u_h$ be the solutions of (2.2) and (2.6), respectively, by the second Strang Lemma (2.21,24), we have

$$|u - u_h|_h \leq C \left( \inf_{v_h \in V_h} |u - v_h|_h + \sup_{0 \neq w_h \in V_h} \frac{|a_h(u, w_h) - (f, w_h)|}{|w_h|_h} \right), \quad (2.8)$$

where the first term is the approximation error and the second one is the consistency error. Herein and throughout this paper, $C$ denotes a generic positive constant which is independent of the meshsize and may be different at different places.
3 Superconvergence of the rectangular Morley element in 2D

3.1 Superconvergence of the consistency error

Let \( I_h \) be piecewise bilinear interpolation operator on \( \Omega \), \( I_h : V_h \rightarrow B_h \),

\[
I_h v(P) = v(P), \quad \text{for any vertex } P \text{ of } T_h,
\]

where

\[
B_h = \{ v \in H^1(\Omega), \ v|_K \in Q_1(K), \ \forall K \in T_h \},
\]

and \( Q_l(K) \) denotes the space of all polynomials which are of degree \( \leq l \) with respect to each variable \( x_i \), over \( K \). Let the interpolation operator \( I_h \) be the counterpart of \( I_h \) on the reference element \( \hat{K} \). The bilinear interpolation operator \( I_h \) has the following error estimate:

\[
|v - I_h v|_{H^l(K)} \leq Ch^{2-l}|v|_{H^2(K)}, \quad l = 0, 1,
\]

for any \( v \in H^2(K) \). It is straightforward to see that \( I_h \) is well defined for any \( w_h \in V_h \). By Green’s formula,

\[
(f, I_h w_h) = (\Delta^2 u, I_h w_h) = -\int_{\Omega} \nabla \Delta u \cdot \nabla I_h w_h \, dx_1 dx_2.
\]

The integration by parts yields

\[
a_h(u, w_h) = -\sum_{K \in T_h} \int_K \nabla \Delta u \cdot \nabla w_h \, dx_1 dx_2 + \sum_{K \in T_h} \int_{\partial K} \partial^2 u \, \frac{\partial w_h}{\partial n} \, ds
\]

\[
+ \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial s \partial n} \, \frac{\partial w_h}{\partial s} \, ds,
\]

where \( \frac{\partial}{\partial n} \) and \( \frac{\partial}{\partial s} \) are tangential and normal derivatives along element boundaries, respectively. A combination of (3.4) and (3.5) yields

\[
a_h(u, w_h) - (f, w_h) = a_h(u, w_h) - (f, I_h w_h) + (f, I_h w_h - w_h)
\]

\[
= -\sum_{K \in T_h} \int_K \nabla \Delta u \cdot \nabla (w_h - I_h w_h) \, dx_1 dx_2 - \sum_{K \in T_h} \int_K f(w_h - I_h w_h) \, dx_1 dx_2
\]

\[
+ \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \, \frac{\partial w_h}{\partial n} \, ds + \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial s \partial n} \, \frac{\partial w_h}{\partial s} \, ds.
\]

The Cauchy-Schwarz inequality and the interpolation error estimate (3.3) lead to

\[
\left| \sum_{K \in T_h} \int_K f(w_h - I_h w_h) \, dx_1 dx_2 \right| \leq C h^2 ||f||_{L^2(\Omega)} |w_h|_{H^1},
\]

which indicates a superconvergence rate \( O(h^2) \).

In the following three lemmas, we will analyze superconvergence for the three remaining terms of (3.7).

**Lemma 3.1.** Suppose that \( u \in H^2(\Omega) \cap H^4(\Omega) \) and \( w_h \in V_h \). Then,

\[
\sum_{K \in T_h} \int_K \nabla \Delta u \cdot \nabla (w_h - I_h w_h) \, dx_1 dx_2 \leq C h^2 |u|_{H^4(\Omega)} |w_h|_{H^1}.
\]

**Proof.** On the reference element \( \hat{K} \), consider the following functional

\[
B_1(\hat{\phi}, \hat{w}_h) = \int_{\hat{K}} \hat{\phi} \frac{\partial^2 (\hat{w}_h - \hat{I}_K \hat{w}_h)}{\partial \xi_1^2} \, d\xi_1 d\xi_2,
\]

where \( \hat{\phi} \) is the counterpart of \( \phi \) on \( \hat{K} \), and \( \hat{I}_K \) is the counterpart of \( I_h \) on \( \hat{K} \).
The Bramble-Hilbert lemma gives
\[ B_1(\tilde{\phi}, \tilde{w}_h) \leq C \inf_{\tilde{\phi} \in P_h(\tilde{K})} ||\tilde{\phi} + \tilde{\phi}||_{L^2(\tilde{K})} ||\tilde{w}_h||_{H^2(\tilde{K})}, \]
\[ B_1(\tilde{\phi}, \tilde{w}_h) = 0, \forall \tilde{\phi} \in P_h(\tilde{K}), \ \forall \tilde{w}_h \in V_h. \]

Table 1: calculation of interpolation

| $\tilde{w}_h$ | 1 | $\xi_1$ | $\xi_2$ | $\xi_1 \xi_2$ | $\xi_1^2$ | $\xi_2^2$ | 1 | 1 | $\xi_1$ | $\xi_2$ |
|---------------|---|---------|---------|--------------|---------|---------|---|---|----------|----------|
| $I_0 \tilde{w}_h$ | 1 | $\xi_1$ | $\xi_2$ | $\xi_1 \xi_2$ | $\xi_1^2$ | $\xi_2^2$ | 1 | 1 | $\xi_1$ | $\xi_2$ |

A simple calculation leads to the interpolations, see Table 1.

It can be checked that
\[ \int \frac{\partial \Delta u}{\partial x_1} \frac{\partial (w_h - I_h w_h)}{\partial x_1} \, dx_1 \, dx_2 \leq Ch^2|u|_{H^4(K)}|w_h|_h \text{ for any } K \in T_h. \]  
\[ \int \frac{\partial \Delta u}{\partial x_2} \frac{\partial (w_h - I_h w_h)}{\partial x_2} \, dx_1 \, dx_2 \leq Ch^2|u|_{H^4(K)}|w_h|_h, \]
which completes the proof.

**Lemma 3.2.** Suppose that $u \in H^4_0(\Omega) \cap H^4(\Omega)$ and $w_h \in V_h$. Then,
\[ \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial n} \, ds \leq Ch^2|u|_{H^4(\Omega)}|w_h|_h. \]

**Proof.** Given $K \in T_h$, let $e_i, i = 1, \cdots, 4$ be its four edges. Define $\Pi_e^0 w = \frac{1}{|e_i|} \int_{e_i} w \, ds$ and $R_e^0 w = w - \Pi_e^0 w$, for any $w \in L^2(K)$, then we have
\[ \int_{e_i} R_e^0 w \, ds = 0. \]

Since $\int_{e_i} \frac{\partial w_h}{\partial n} \, ds$ is continuous on internal edges $e_i$ and vanishes on boundary edges of $T_h$, thus
\[
\sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial n} \, ds = \sum_{K \in T_h} \sum_{i=1}^{4} \int_{e_i} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial n} \, ds = \sum_{K \in T_h} \sum_{i=1}^{4} \int_{e_i} R_e^0 \frac{\partial w_h}{\partial n} \, ds = \sum_{K \in T_h} \sum_{i=1}^{4} J_i.
\]

We first analyze the following terms
\[ \sum_{K \in T_h} J_2 + J_4 = \sum_{K \in T_h} \left( \int_{e_2} \frac{\partial^2 u}{\partial n^2} R_{e_2}^0 \frac{\partial w_h}{\partial n} \, ds + \int_{e_4} \frac{\partial^2 u}{\partial n^2} R_{e_4}^0 \frac{\partial w_h}{\partial n} \, ds \right) = \sum_{K \in T_h} \left( \int_{e_2} \frac{\partial^2 u}{\partial x_1^2} R_{e_2}^0 \frac{\partial w_h}{\partial x_1} \, dx_2 + \int_{e_4} \frac{\partial^2 u}{\partial x_1^2} R_{e_4}^0 \frac{\partial w_h}{\partial x_1} \, dx_2 \right) = \sum_{K \in T_h} \int_{x_2,e_2,h} \left( \frac{\partial^2 u}{\partial x_1^2} R_{e_2}^0 \frac{\partial w_h}{\partial x_1} |_{e_2} - \frac{\partial^2 u}{\partial x_1^2} R_{e_4}^0 \frac{\partial w_h}{\partial x_1} |_{e_4} \right) \, dx_2. \]
For the rectangular Morley element, we have the following crucial property

\[ \mathcal{R}_{e_2}^0 \frac{\partial w_h}{\partial x_1} \bigg|_{e_2} = \mathcal{R}_{e_2}^0 \frac{\partial w_h}{\partial x_1} \bigg|_{e_3}, \quad \mathcal{R}_{e_1}^0 \frac{\partial w_h}{\partial x_1} \bigg|_{e_1} = \mathcal{R}_{e_1}^0 \frac{\partial w_h}{\partial x_2} \bigg|_{e_3}. \]

This implies

\[ \sum_{K \in \mathcal{T}_h} J_2 + J_4 = \sum_{K \in \mathcal{T}_h} \int_{x_{2,e-h}^{1}}^{x_{2,e+h}} \left( \frac{\partial^2 u}{\partial x_1^2} \bigg|_{e_2} - \frac{\partial^2 u}{\partial x_1^2} \bigg|_{e_1} \right) \mathcal{R}_{e_2}^0 \frac{\partial w_h}{\partial x_1} \bigg|_{e_2} \, dx_2 \]

\[ = \sum_{K \in \mathcal{T}_h} \int_{x_{2,e-h}^{1}}^{x_{2,e+h}} \mathcal{R}_{e_2}^0 \left( \frac{\partial^2 u}{\partial x_1^2} \bigg|_{e_2} - \frac{\partial^2 u}{\partial x_1^2} \bigg|_{e_1} \right) \mathcal{R}_{e_2}^0 \frac{\partial w_h}{\partial x_1} \bigg|_{e_2} \, dx_2. \]

The error estimate of the interpolation operators \( \Pi_{e_2}^0 \) yields

\[ \sum_{K \in \mathcal{T}_h} J_2 + J_4 \leq C h \sum_{K \in \mathcal{T}_h} \left\| \nabla \left( \frac{\partial^2 u}{\partial x_1^2} \bigg|_{e_2} - \frac{\partial^2 u}{\partial x_1^2} \bigg|_{e_1} \right) \right\|_{L^2(K)} |w_h|_h \]

\[ \leq C h^2 |u|_{H^4(\Omega)} |w_h|_h. \tag{3.15} \]

By the same argument, we can get

\[ \sum_{K \in \mathcal{T}_h} J_1 + J_3 \leq C h^2 |u|_{H^4(\Omega)} |w_h|_h. \tag{3.16} \]

Then, a combination of (3.15) and (3.16) completes the proof.

\[ \square \]

**Lemma 3.3.** Suppose that \( u \in H^4(\Omega) \cap H^4(\Omega) \) and \( w_h \in V_h \). Then,

\[ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial^2 u}{\partial s \partial n} \frac{\partial w_h}{\partial s} \, ds \leq C h^2 |u|_{H^4(\Omega)} |w_h|_h. \tag{3.17} \]

The proof of this lemma can follow the similar procedure of Lemma 3.2 and herein we omit it. According to above lemmas, we can obtain the following error estimate.

**Theorem 3.4.** Suppose that \( u \in H^4(\Omega) \), \( f \in L^2(\Omega) \) and for all \( w_h \in V_h \), then the consistency error can be estimated as

\[ a_h(u, w_h) - (f, w_h) \leq C h^2 (||f||_{L^2(\Omega)} + |u|_{H^4(\Omega)}) |w_h|_h. \]

### 3.2 Asymptotic expansion of the canonical interpolation

Given \( K \in \mathcal{T}_h \), we define the canonical interpolation operator \( \Pi_K : H^3(K) \to P(K) \) by, for any \( v \in H^3(K) \),

\[ \Pi_K v(P) = v(P) \text{ and } \int_{e} \frac{\partial \Pi_K v}{\partial n_e} \, ds = \int_{e} \frac{\partial v}{\partial n_e} \, ds, \tag{3.18} \]

for any vertex \( P \) of \( K \) and any edge \( e \) of \( K \). The interpolation operator \( \Pi_K \) has the following error estimates:

\[ |v - \Pi_K v|_{H^l(K)} \leq C h^{3-l} |v|_{H^3(K)}, \quad l = 0, 1, 2, 3, \tag{3.19} \]

provided that \( v \in H^3(K) \). Then the global version \( \Pi_h \) of the interpolation operator \( \Pi_K \) is defined as

\[ \Pi_h|_K = \Pi_K \text{ for any } K \in \mathcal{T}_h. \tag{3.20} \]

We need the following asymptotic expansion result from [9].
Lemma 3.5. Suppose that \( u \in H^4(\Omega) \), then for all \( v_h \in V_h \), we have

\[
a_h(u - \Pi_h u, v_h) \leq \sum_{K \in \mathcal{T}_h} \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \frac{\partial^3 v_h}{\partial x_1} \, dx_1 dx_2 + \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \frac{\partial^3 v_h}{\partial x_2} \, dx_1 dx_2 + \frac{1}{h} + \frac{1}{h^2} \|u\|_{H^4(\Omega)} |v_h|_h. \tag{3.21}
\]

It is straightforward from Lemma 3.5 to derive that by the inverse inequality

\[
a_h(u - \Pi_h u, v_h) \leq Ch \left( \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(\Omega)} + \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)} \right) |v_h|_h. \tag{3.22}
\]

Based on the above analysis and Theorem 3.4, Lemma 3.5, we can get the following error estimate of \( \|\Pi_h u - u_h\|_h \).

**Theorem 3.6.** Suppose that \( u \in H^4(\Omega) \), then we have

\[
\|\Pi_h u - u_h\|_h \leq Ch \left( \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(\Omega)} + \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)} \right). \tag{3.23}
\]

**Proof.** It follows from Theorem 3.4 and (3.22) that

\[
\|\Pi_h u - u_h\|_h^2 = a_h(\Pi_h u - u_h, \Pi_h u - u_h) = a_h(u - u_h, \Pi_h u - u_h) + a_h(\Pi_h u - u, \Pi_h u - u_h) \leq Ch \left( \|u\|_{H^4(\Omega)} + \|\Pi_h u - u_h\|_{\Omega} \right) \|\Pi_h u - u_h\|_h + Ch \left( \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(\Omega)} + \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)} \right) \|\Pi_h u - u_h\|_h,
\]

which completes the proof. \( \square \)

4 **Supercloseness of the correction interpolation**

In the view of Theorem 3.6, we cannot expect a higher order error estimate of \( \|\Pi_h u - u_h\|_h \). To overcome this difficulty, we follow the idea of [3] to make a correction of the interpolation \( \Pi_h u \). First of all, we define the correction term as follows

\[
R_K v = \sum_{j=5}^{8} a_j(v) \varphi_j, \quad v \in H^4(K), \tag{4.1}
\]

where \( a_j(v) \) read

\[
\begin{cases}
  a_j = -\frac{h}{6} \int_{\xi_j}^1 \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \, dx_2, & j = 5, 7, \\
  a_j = -\frac{h}{6} \int_{\xi_j}^1 \frac{\partial^3 u}{\partial x_1 \partial x_2} \, dx_1, & j = 6, 8, \tag{4.2}
\end{cases}
\]

and the basis functions \( \varphi_j \) read

\[
\begin{cases}
  \varphi_5 = \frac{h}{4} (\xi_1 + 1)^2 (\xi_1 - 1), \\
  \varphi_6 = \frac{h}{4} (\xi_2 + 1)^2 (\xi_2 - 1), \\
  \varphi_7 = \frac{h}{4} (\xi_1 + 1) (\xi_1 - 1)^2, \\
  \varphi_8 = \frac{h}{4} (\xi_2 + 1) (\xi_2 - 1)^2.
\end{cases}
\]

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Then the global version $R_h$ is defined as

$$R_h|_K = R_K, \text{ for any } K \in \mathcal{T}_h. \quad (4.3)$$

Define the correction interpolation $\Pi^*_K v$ as follows, for all $v \in H^4(K)$,

$$\Pi^*_K v = \Pi_K v - R_K v, \text{ } K \in \mathcal{T}_h, \quad (4.4)$$

Then the global version $\Pi^*_h$ is defined as

$$\Pi^*_h|_K = \Pi^*_K, \text{ for any } K \in \mathcal{T}_h. \quad (4.5)$$

Regarding the correction term $R_h$, we have the following lemma.

**Lemma 4.1.** Suppose that $u \in H^4(\Omega)$, then for all $v_h \in V_h$, we have

$$a_h(R_h u, v_h) = \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K (a_5 + a_7) \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K (a_5 + a_9) \frac{\partial^3 v_h}{\partial x_2^3} \, dx_1 \, dx_2. \quad (4.6)$$

**Proof.** Let $\xi_1$ and $\xi_2$ be defined as in (2.3). It follows from the definition of $P(K)$ that

$$\frac{\partial^2 v_h}{\partial x_1^2} = \frac{\partial^2 v_h}{\partial x_1^2} + h \frac{\partial^3 v_h}{\partial x_1^3} \xi_i, \quad i = 1, 2. \quad (4.7)$$

The definition of $a_h(\cdot, \cdot)$ yields

$$a_h(R_h u, v_h) = \sum_{K \in \mathcal{T}_h} \left( \int_K \frac{\partial^2 R_K u}{\partial x_1^2} \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 + 2 \int_K \frac{\partial^2 R_K u}{\partial x_1 \partial x_2} \frac{\partial^2 v_h}{\partial x_1 \partial x_2} \, dx_1 \, dx_2 ight. \right.$$  

\[ + \left. \int_K \frac{\partial^2 R_K u}{\partial x_2^2} \frac{\partial^2 v_h}{\partial x_2^2} \, dx_1 \, dx_2 \right). \quad (4.8)\]

We are in the position to calculate three terms on the right-hand side of (4.8). It follows the definition of $R_K$ that

\[ \int_K \frac{\partial^2 R_K u}{\partial x_1^2} \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 = \frac{h^{-1}}{4} \int_K \left[ (6\xi_1 + 2)a_5 + (6\xi_1 - 2)a_7 \right] \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 \]

\[ = \frac{h^{-1}}{4} \int_K \left[ 6\xi_1(a_5 + a_7) + 2(a_5 - a_7) \left( \frac{\partial^2 v_h}{\partial x_1^2} + h \frac{\partial^3 v_h}{\partial x_1^3} \xi_1 \right) \right] \, dx_1 \, dx_2 \]

\[ = \frac{h^{-1}}{4} \int_K 6\xi_1(a_5 + a_7) \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 + \frac{h^{-1}}{4} \int_K 6\xi_1(a_5 + a_7) h \frac{\partial^3 v_h}{\partial x_1^3} \, dx_1 \, dx_2 \]

\[ + \frac{h^{-1}}{4} \int_K 2(a_5 - a_7) \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 + \frac{h^{-1}}{4} \int_K 2(a_5 - a_7) h \frac{\partial^3 v_h}{\partial x_1^3} \, dx_1 \, dx_2. \]

Since coefficients like $\frac{\partial^2 v_h}{\partial x_1^2}$ and $\frac{\partial^3 v_h}{\partial x_1^3}$ are constants, we can get that by parity of function and symmetry of domains:

\[ \int_K 6\xi_1(a_5 + a_7) \, dx_1 \, dx_2 = 0, \quad \int_K 2(a_5 - a_7) h \xi_1 \, dx_1 \, dx_2 = 0. \]

Because of $a_5 = a_7$, hence, only one nonzero term is left, which reads

\[ \frac{h^{-1}}{4} \int_K 6\xi_1^2(a_5 + a_7) h \frac{\partial^3 v_h}{\partial x_1^3} \, dx_1 \, dx_2 = \frac{1}{2} \int_K (a_5 + a_7) \frac{\partial^3 v_h}{\partial x_1^3} \, dx_1 \, dx_2. \]

(4.9)

This yields

\[ \int_K \frac{\partial^2 R_K u}{\partial x_1^2} \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 \, dx_2 = \frac{1}{2} \int_K (a_5 + a_7) \frac{\partial^3 v_h}{\partial x_1^3} \, dx_1 \, dx_2. \]

(4.10)
A similar argument proves
\[
\int_K \frac{\partial^2 R_K u}{\partial x_1^2} \frac{\partial^2 v_h}{\partial x_1^2} \, dx_1 dx_2 = \frac{1}{2} \int_K (a_0 + a_8) \frac{\partial^3 v_h}{\partial x_1^2} \, dx_1 dx_2. \tag{4.11}
\]
Note that the basis functions \(\varphi_j (j = 5, 6, 7, 8)\) have no mixed terms, which leads to \(\frac{\partial^2 R_K}{\partial x_1 \partial x_2} = 0\). Thus
\[
\int_K \frac{\partial^2 R_K u}{\partial x_1 \partial x_2} \frac{\partial^2 v_h}{\partial x_1 \partial x_2} \, dx_1 dx_2 = 0, \tag{4.12}
\]
which completes the proof. ∎

Based on the above analysis, we can establish superclose results of the rectangular Morley element by the correction interpolation \(\Pi^*_h u\).

**Theorem 4.2.** Let \(u \in H^4(\Omega)\), \(u_h \in V_h\), be the solutions of \((4.20)\) and \((4.21)\), respectively, then we have
\[
||\Pi^*_h u - u_h||_h \leq C h^2 (||f||_{L^2(\Omega)} + ||u||_{H^4(\Omega)}). \tag{4.13}
\]

**Remark 4.3.** Comparing with the incomplete biquadratic plate element \([17]\), herein, the theorem does not require \(\frac{\partial^2 u}{\partial x_1 \partial x_2}\) to be zero on the boundary. Besides, the correction interpolation \(\Pi^*_h u\) still belongs to the space \(V_h\). Because of the boundary condition \(\frac{\partial u}{\partial n} \bigg|_{\partial \Omega} = 0\), it can be deduced that \(\frac{\partial u}{\partial x_1} \bigg|_{e_i} = 0\) and \(\frac{\partial u}{\partial x_2} \bigg|_{e_i} = 0\), \(e_i, e_j \in \partial \Omega, i = 1, 3, j = 2, 4\). Thus, \(\frac{\partial u}{\partial x_1^2} \bigg|_{e_i} = 0\) and \(\frac{\partial u}{\partial x_2^2} \bigg|_{e_i} = 0\), \(e_i, e_j \in \partial \Omega, i = 1, 3, j = 2, 4\).

**Proof.** On the reference element \(\hat{K}\), consider the functional
\[
B_2(\hat{u}, \hat{v}_h) = \frac{1}{3} \int_{\hat{K}} \frac{\partial^3 \hat{u}}{\partial \xi_1 \partial \xi_2^2} \frac{\partial^3 \hat{v}_h}{\partial \xi_1^2} \, d\xi_1 d\xi_2 + \frac{1}{3} \int_{\hat{K}} \frac{\partial^3 \hat{u}}{\partial \xi_1^2 \partial \xi_2} \frac{\partial^3 \hat{v}_h}{\partial \xi_2} \, d\xi_1 d\xi_2
\]
\[
- \frac{1}{6} \int_{\hat{K}} \left( \int_{\hat{e}_2} \frac{\partial^3 \hat{v}_h}{\partial \xi_1 \partial \xi_2^2} \, d\xi_2 \right) \frac{\partial^3 \hat{v}_h}{\partial \xi_1^2} \, d\xi_1 d\xi_2 - \frac{1}{6} \int_{\hat{K}} \left( \int_{\hat{e}_1} \frac{\partial^3 \hat{v}_h}{\partial \xi_2 \partial \xi_2^2} \, d\xi_2 \right) \frac{\partial^3 \hat{v}_h}{\partial \xi_2} \, d\xi_1 d\xi_2.
\]
It can be checked that
\[
\begin{cases}
B_2(\hat{u}, \hat{v}_h) \leq c||\hat{u}||_{H^4(\hat{K})}||\hat{v}_h||_{H^4(\hat{K})}, \\
B_2(\hat{u}, \hat{v}_h) = 0, \quad \forall \hat{u} \in P_3(\hat{K}), \quad \forall \hat{v}_h \in V_h.
\end{cases}
\]
Hence, the Bramble-Hilbert lemma gives
\[
B_2(\hat{u}, \hat{v}_h) \leq C||\hat{u}||_{H^4(\hat{K})}||\hat{v}_h||_{H^4(\hat{K})}. \tag{4.14}
\]
A scaling argument leads to
\[
\frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \frac{\partial^3 v_h}{\partial x_1^2} \, dx_1 dx_2 + \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \frac{\partial^3 v_h}{\partial x_2} \, dx_1 dx_2
\]
\[
+ \frac{1}{2} \int_K (a_5 + a_7) \frac{\partial^3 v_h}{\partial x_1^3} \, dx_1 dx_2 + \frac{1}{2} \int_K (a_6 + a_8) \frac{\partial^3 v_h}{\partial x_1^2 \partial x_2} \, dx_1 dx_2
\]
\[
\leq C h^2 (||u||_{H^4(\Omega)} ||v_h||_h).
\]
An application of Lemma \((4.5)\) and Lemma \((4.4)\) yields
\[
a_h(u - \Pi^*_h u, v_h) = a_h(u - \Pi^*_h u, v_h) + a_h(R_h u, v_h) \leq C h^2 ||u||_{H^4(\Omega)} ||v_h||_h. \tag{4.15}
\]
Then, together with Theorem \((4.4)\) and \((4.15)\), this gives
\[
||\Pi^*_h u - u_h||_h^2 = a_h(\Pi^*_h u - u_h, \Pi^*_h u - u_h)
\]
\[
= a_h(u - u_h, \Pi^*_h u - u_h) + a_h(\Pi^*_h u - u, \Pi^*_h u - u_h)
\]
\[
= [a_h(u, \Pi^*_h u - u_h) - (f, \Pi^*_h u - u_h)] + a_h(\Pi^*_h u - u, \Pi^*_h u - u_h)
\]
\[
\leq C h^2 (||f||_{L^2(\Omega)} + ||u||_{H^4(\Omega)}) ||\Pi^*_h u - u_h||_h,
\]
which completes the proof. ∎
Based on the superclose property, we can obtain the superconvergence result of the two-dimensional rectangular Morley element by a proper postprocessing technique. In order to attain the global superconvergence, we follow the idea of [14] to construct the postprocessing operator $\Pi_{3h}^3$ as follows.

We merge 9 adjacent elements into a macro element, $\tilde{K} = \bigcup_{i=1}^{9} K_i$, (see Figure 2), such that, in the macro element $\tilde{K}$,

$$\Pi_{3h}^3 w \in Q_3(\tilde{K}), \quad \forall w \in C(\tilde{K}).$$

(4.16)

We denote $Z_{ij}, i, j = 1, 2, 3, 4$ as the vertices of the 9 adjacent elements. Then, the operator $\Pi_{3h}^3$ satisfies

$$\Pi_{3h}^3 w(Z_{ij}) = w(Z_{ij}), \quad i, j = 1, 2, 3, 4.$$ (4.17)

Besides, the postprocessing operator $\Pi_{3h}^3$ has the following properties

$$\left\{ \begin{array}{l} \Pi_{3h}^3 (\Pi_h^3 u) = \Pi_{3h}^3 u, \quad \forall u \in H^4(\Omega), \\ |\Pi_{3h}^3 v_h|_h \leq C|v_h|_h, \quad \forall v_h \in V_h, \\ |u - \Pi_{3h}^3 u_h|_h \leq Ch^2|u|_{H^4(\Omega)}, \quad \forall u \in H^4(\Omega). \end{array} \right.$$ (4.18)

Then, we can get the following global superconvergent result.

**Theorem 4.4.** Let $u \in H^4(\Omega), u_h \in V_h$, be the solutions of (2.2) and (2.6), respectively, then we have

$$|u - \Pi_{3h}^3 u_h|_h \leq Ch^2(|f|_{L^2(\Omega)} + |u|_{H^4(\Omega)}).$$

(4.19)

**Proof.** It follows the properties (4.18) and Theorem 4.2 that

$$|u - \Pi_{3h}^3 u_h|_h \leq |u - \Pi_{3h}^3 \Pi_h^3 u_h|_h + |\Pi_{3h}^3 (\Pi_h^3 u - u_h)|_h \leq |u - \Pi_{3h}^3 u_h|_h + C||\Pi_h^3 u - u_h||_h \leq Ch^2(|f|_{L^2(\Omega)} + |u|_{H^4(\Omega)}),$$

(4.20)

which completes the proof.

5 Superconvergence of the cubic Morley element

In this section, we analyze the superconvergence property of the three-dimensional Morley element on cubic meshes with $\Omega \subset \mathbb{R}^3$. Let $T_h$ be a regular uniform cubic triangulation of the domain $\Omega \subset \mathbb{R}^3$. 
Given $K \in \mathcal{T}_h$, let $(x_1,c, x_2,c, x_3,c)$ be the center of $K$, the meshsize $h$ and affine mapping:

$$
\begin{align*}
\xi_1 &= \frac{x_1 - x_{1,c}}{h}, & \xi_2 &= \frac{x_2 - x_{2,c}}{h}, & \xi_3 &= \frac{x_3 - x_{3,c}}{h}, \quad \text{for any } (x_1, x_2, x_3) \in K.
\end{align*}
$$

(5.1)

On element $K$, the shape function space of the cubic Morley element reads

$$
P(K) := P_2(K) + \text{span}\{x_1^3, x_2^3, x_3^3, x_1 x_2 x_3\}.
$$

(5.2)

The nodal parameters are: for any $v \in C^1(K)$,

$$
D(v) = \left( v(a_i), \frac{1}{|F_j|} \int_{F_j} \frac{\partial v}{\partial n_{F_j}} \, ds \right), \quad i = 1, \ldots, 8, \quad j = 1, \ldots, 6,
$$

(5.3)

where $a_j$ are vertices of $K$ and $F_j$ are faces of $K$, see Figure 3.

5.1 Superconvergence of the consistency error

We also need the decomposition of the consistency error (3.6). For ease of reading, we recall the expression as follows

$$
a_h(u,w_h) - (f,w_h) = a_h(u,w_h) - (f,I_hw_h) + (f,I_hw_h - w_h)
$$

$$
= -\sum_{K \in \mathcal{T}_h} \int_K \nabla \Delta u \cdot \nabla (w_h - I_h w_h) \, dx_1 dx_2 dx_3
$$

$$
- \sum_{K \in \mathcal{T}_h} \int_K f(w_h - I_hw_h) \, dx_1 dx_2 dx_3 + \sum_{K \in \mathcal{T}_h} \sum_{1 \leq i \leq j \leq 3} \int_{\partial K} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial w_h}{\partial n_i} \, ds
$$

$$
+ \sum_{K \in \mathcal{T}_h} \sum_{1 \leq i \leq j \leq 3} \int_{\partial K} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial w_h}{\partial x_j} \, ds.
$$

(5.4)

A direct application of the interpolation error estimate (3.3) leads to

$$
\left| \sum_{K \in \mathcal{T}_h} \int_K f(w_h - I_hw_h) \, dx_1 dx_2 \right| \leq C h^2 \|f\|_{L^2(\Omega)} \|w_h\|_h.
$$

(5.5)

Lemma 5.1. Suppose that $u \in H^2_0(\Omega) \cap H^4(\Omega)$ and $w_h \in V_h$. Then,

$$
\sum_{K \in \mathcal{T}_h} \int_K \nabla \Delta u \cdot \nabla (w_h - I_hw_h) \, dx_1 dx_2 dx_3 \leq C h^2 \|u\|_{H^4(\Omega)} \|w_h\|_h.
$$
Proof. On the reference element $\hat{K}$, consider the following functional

$$B_3(\hat{\phi}, \hat{\omega}_h) = \int_{\hat{K}} \hat{\phi} \frac{\partial \hat{\omega}_h - \hat{I}_\hat{K}\hat{\omega}_h}{\partial \xi_1} \, d\xi_1 d\xi_2 d\xi_3,$$

(5.6)

A simple calculation leads to the interpolations, see Table 2.

| $\hat{I}_\hat{K}\hat{\omega}_h$ | $1$ | $\xi_1$ | $\xi_2$ | $\xi_3$ | $\xi_1 \xi_2$ | $\xi_1 \xi_3$ | $\xi_2 \xi_3$ | $\xi_1^2$ | $\xi_2^2$ | $\xi_3^2$ |
|-----------------------------|-----|-------|-------|-------|-----------|-----------|-----------|-------|-------|-------|

It follows that

$$\begin{cases}
B_3(\hat{\phi}, \hat{\omega}_h) \leq C \inf_{\hat{\phi} \in P_0(\hat{K})} || \hat{\phi} ||_{L^2(\hat{K})} || \hat{\omega}_h ||_{H^2(\hat{K})}, \\
B_3(\hat{\phi}, \hat{\omega}_h) = 0, \ \forall \hat{\phi} \in P_0(\hat{K}), \ \forall \hat{\omega}_h \in V_h.
\end{cases}$$

(5.7)

The Bramble-Hilbert lemma gives

$$B_3(\hat{\phi}, \hat{\omega}_h) \leq C \inf_{\hat{\phi} \in P_0(\hat{K})} || \hat{\phi} \hat{\omega}_h ||_{H^2(\hat{K})},$$

(5.8)

A substitution of $\phi = \frac{\partial \Delta u}{\partial x_1}$ into (5.7), plus a scaling argument yield

$$\int_{K} \frac{\partial \Delta u}{\partial x_1} \frac{\partial (w_h - I_h w_h)}{\partial x_1} \, dx_1 dx_2 dx_3 \leq Ch^2 |u|_{H^1(K)} |w_h|_h, \ \text{for any } K \in T_h.$$ 

(5.9)

A similar argument proves

$$\int_{K} \frac{\partial \Delta u}{\partial x_2} \frac{\partial (w_h - I_h w_h)}{\partial x_2} \, dx_1 dx_2 dx_3 \leq Ch^2 |u|_{H^1(K)} |w_h|_h, \ \text{for any } K \in T_h.$$ 

(5.10)

which complete the proof.

Next, we will analyze the last two terms of (5.7).

**Lemma 5.2.** Suppose that $u \in H_0^2(\Omega) \cap H^1(\Omega)$ and $w_h \in V_h$. Then,

$$\sum_{K \in T_h} \sum_{i=1}^{3} \int_{\partial K} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial w_h}{\partial n_i} \, ds \leq Ch^2 |u|_{H^1(\Omega)} |w_h|_h.$$ 

(5.11)

Proof. Given $K \in T_h$, let $F_i, i = 1, \cdots, 6$ be its faces. Define $\Pi^0_{F_i} w = \frac{1}{|F_i|} \int_{F_i} w \, ds$ and $R^0_{F_i} w = w - \Pi^0_{F_i} w$, for any $w \in L^2(K)$, then we have

$$\int_{F_i} R^0_{F_i} w \, ds = 0.$$ 

(5.12)

Since $\int_{F_i} \frac{\partial w_h}{\partial n_i} \, ds$ is continuous on internal faces $F_i$ and vanishes on boundary faces of $T_h$, thus

$$\sum_{K \in T_h} \sum_{i=1}^{3} \int_{\partial K} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial w_h}{\partial n_i} \, ds = \sum_{K \in T_h} \sum_{i=1}^{3} \int_{F_i} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial w_h}{\partial n_i} \, ds = \sum_{K \in T_h} \sum_{i=1}^{3} \int_{F_i} \frac{\partial^2 u}{\partial x_i^2} R^0_{F_i} \frac{\partial w_h}{\partial n_i} \, ds.$$ 

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Then we firstly analyze the following terms

\[
\sum_{K \in T_h} \sum_{i=1}^{3} (L_2 + L_3) n_i = \sum_{K \in T_h} \int_{F_3} \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} dx_1 dx_3 - \sum_{K \in T_h} \int_{F_3} \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} dx_1 dx_3
\]

For the cubic Morley element, we have the following crucial property

\[
\sum_{K \in T_h} \int_{x_1, c}^{x_3, c} \int_{x_3, c}^{x_3, c} \left( \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} - \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} \right) dx_1 dx_3.
\]

Then, a combination of (5.12), (5.13) and (5.14) completes the proof.

For the cubic Morley element, we have the following error estimate.

\[
R_0^0 \frac{\partial w_h}{\partial x_1} |_{F_1} = R_0^0 \frac{\partial w_h}{\partial x_1} |_{F_4}, \quad R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} = R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_3}, \quad R_0^0 \frac{\partial w_h}{\partial x_3} |_{F_3} = R_0^0 \frac{\partial w_h}{\partial x_3} |_{F_6}.
\]

This implies

\[
\sum_{K \in T_h} \sum_{i=1}^{3} (L_2 + L_3) n_i = \sum_{K \in T_h} \int_{x_1, c}^{x_3, c} \int_{x_3, c}^{x_3, c} \left( \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} - \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} \right) dx_1 dx_3
\]

The error estimate of the interpolation operators \( I_h^2 \) yields

\[
\sum_{K \in T_h} \sum_{i=1}^{3} (L_2 + L_3) n_i \leq Ch \sum_{K \in T_h} \left\| \nabla \left( \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} - \frac{\partial^2 u}{\partial x_1^2} R_0^0 \frac{\partial w_h}{\partial x_2} |_{F_2} \right) \right\|_{L^2(K)} |w_h|_h.
\]

A similar argument proves

\[
\sum_{K \in T_h} \sum_{i=1}^{3} (L_1 + L_4) n_i \leq Ch^2 |u|_{H^4(\Omega)} |w_h|_h,
\]

and

\[
\sum_{K \in T_h} \sum_{i=1}^{3} (L_3 + L_6) n_i \leq Ch^2 |u|_{H^4(\Omega)} |w_h|_h.
\]

Then, a combination of (5.12), (5.13) and (5.14) completes the proof.

**Lemma 5.3.** Suppose that \( u \in H^4(\Omega) \cap H^4(\Omega) \) and \( w_h \in V_h \). Then,

\[
\sum_{K \in T_h} \sum_{i=1}^{3} \int_{\partial K} \frac{\partial^2 u}{\partial x_1 \partial x_j} \frac{\partial w_h}{\partial x_j} n_i ds \leq Ch^2 |u|_{H^4(\Omega)} |w_h|_h.
\]

The proof of this lemma can follow the similar procedure of Lemma 5.2 and herein we omit it. According to above lemmas, we can obtain the following error estimate.

**Theorem 5.4.** Suppose that \( u \in H^4(\Omega) \), \( f \in L^2(\Omega) \). Then it holds that

\[
a_h(u, w_h) - (f, w_h) \leq Ch^2 (||f||_{L^2(\Omega)} + |u|_{H^4(\Omega)}) |w_h|_h, \text{ for any } w_h \in V_h.
\]

We need the following asymptotic expansion result of the canonical interpolation from [9].
Lemma 5.5. Suppose that $u \in H^4(\Omega), \Omega \subset \mathbb{R}^3$, then for all $v_h \in V_h$, we have

$$a_h(u - \Pi_h u, v_h) \leq \sum_{K \in T_h} \sum_{i \neq j} \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_i \partial x_j^2} \frac{\partial^3 v_h}{\partial x_i \partial x_j^2} \, dx_1 \, dx_2 \, dx_3 + C h^2 |u|_{H^4(\Omega)} |v_h|_h.$$ 

It is straightforward from Lemma 5.5 to derive that by the inverse inequality

$$a_h(u - \Pi_h u, v_h) \leq C h \sum_{i \neq j} \left\| \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right\|_{L^2(\Omega)} |v_h|_h.$$ 

(5.16)

Based on the analysis of the interpolation error and the consistency error, we can get the following error estimate of $|\Pi_h u - u_h|_h$.

Theorem 5.6. Suppose that $u \in H^4(\Omega)$, then we have

$$|\Pi_h u - u_h|_h \leq C h \sum_{i \neq j} \left\| \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right\|_{L^2(\Omega)}.$$ 

Proof. It follows from Theorem 5.4 and (5.16) that

$$|\Pi_h u - u_h|_h^2 = a_h(\Pi_h u - u_h, \Pi_h u - u_h)$$

$$= a_h(u - u_h, \Pi_h u - u_h) + a_h(\Pi_h u - u, \Pi_h u - u_h)$$

$$= [a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)] + a_h(\Pi_h u - u, \Pi_h u - u_h)$$

$$\leq C h^2 (\|f\|_{L^2(\Omega)} + |u|_{H^4(\Omega)}) |\Pi_h u - u_h|_h + C h \sum_{i \neq j} \left\| \frac{\partial^3 u}{\partial x_i \partial x_j^2} \right\|_{L^2(\Omega)} |\Pi_h u - u_h|_h.$$

which completes the proof.

5.2 Superconvergence of the correction interpolation

We can learn from Theorem 5.6 that the convergence of the error $|\Pi_h u - u_h|_h$ is only of order $O(h)$. Therefore, we follow the idea of [3] to make a correction of the interpolation to improve its convergence. The operator $\tilde{\Pi}_K$ is modified as

$$\tilde{\Pi}_K u = \Pi_K u - \tilde{R}_K u, \quad u \in H^4(K),$$

(5.17)

where $\tilde{R}_K u = \frac{14}{17} \sum_{j=9}^3 b_j(u) \tilde{\varphi}_j$ with

$$b_j = - \frac{1}{17} \int_{F_3} \left( \frac{\partial^3 v}{\partial x_1 \partial x_j^2} + \frac{\partial^3 v}{\partial x_2 \partial x_j^2} \right) \, dx_1 \, dx_2 \, dx_3, \quad j = 9, 12,$$

$$b_j = - \frac{1}{17} \int_{F_2} \left( \frac{\partial^3 v}{\partial x_2 \partial x_j^2} + \frac{\partial^3 v}{\partial x_3 \partial x_j^2} \right) \, dx_1 \, dx_3, \quad j = 10, 13,$$

$$b_j = - \frac{1}{17} \int_{F_3} \left( \frac{\partial^3 v}{\partial x_3 \partial x_j^2} + \frac{\partial^3 v}{\partial x_1 \partial x_j^2} \right) \, dx_1 \, dx_2, \quad j = 11, 14,$$

(5.18)

and the basis functions

$$\tilde{\varphi}_9 = \frac{4}{3} (\xi_1 + 1)^2 (\xi_1 - 1),$$

$$\tilde{\varphi}_{10} = \frac{4}{3} (\xi_2 + 1)^2 (\xi_2 - 1),$$

$$\tilde{\varphi}_{11} = \frac{4}{3} (\xi_3 + 1)^2 (\xi_3 - 1),$$

$$\tilde{\varphi}_{12} = \frac{4}{3} (\xi_1 + 1) (\xi_1 - 1)^2,$$

$$\tilde{\varphi}_{13} = \frac{4}{3} (\xi_2 + 1) (\xi_2 - 1)^2,$$

$$\tilde{\varphi}_{14} = \frac{4}{3} (\xi_3 + 1) (\xi_3 - 1)^2.$$  

(5.19)
\[ \xi_1 = \frac{x_1-x_1w}{h}, \quad \xi_2 = \frac{x_2-x_2w}{h}, \quad \xi_3 = \frac{x_3-x_3w}{h}, \quad \text{where we defined in (5.1)}. \]

Then the global version \( \tilde{\Pi}_h^3 \) and \( \tilde{R}_h \) are defined as

\[ \tilde{R}_h|_K = \tilde{R}_K, \quad \text{for any} \ K \in \mathcal{T}_h, \]
\[ \tilde{\Pi}_h^3|_K = \tilde{\Pi}_K, \quad \text{for any} \ K \in \mathcal{T}_h. \]

Thus, we can establish superclose results of the three-dimensional cubic Morley element by the correction interpolation \( \tilde{\Pi}_h^3u \).

**Theorem 5.7.** Let \( u \in H^4(\Omega), u_h \in V_h \), be the solutions of (2.2) and (2.6), respectively, then it holds

\[ ||\tilde{\Pi}_h^3u - u_h||_h \leq Ch^2(||f||_{L^2(\Omega)} + |u|_{H^4(\Omega)}). \quad (5.20) \]

**Proof.** On the reference element \( \tilde{K} \), consider the functional

\[ B_4(\tilde{u}, \tilde{v}_h) = \sum_{K \in \mathcal{T}_h} \sum_{i \neq j = 1}^3 \frac{1}{3} \int_R \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} \frac{\partial^3 \tilde{v}_h}{\partial \xi_i \partial \xi_j^2} \, d\xi_1 d\xi_2 d\xi_3 \]
\[ - \frac{1}{12} \int_R \left( \int_{\tilde{F}_1} \left( \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} + \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} \right) \, d\xi_2 d\xi_3 \right) \frac{\partial^3 \tilde{v}_h}{\partial \xi_i \partial \xi_j} \, d\xi_1 d\xi_2 d\xi_3 \]
\[ - \frac{1}{12} \int_R \left( \int_{\tilde{F}_2} \left( \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} + \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} \right) \, d\xi_1 d\xi_3 \right) \frac{\partial^3 \tilde{v}_h}{\partial \xi_i \partial \xi_j} \, d\xi_1 d\xi_2 d\xi_3 \]
\[ - \frac{1}{12} \int_R \left( \int_{\tilde{F}_3} \left( \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} + \frac{\partial^3 \tilde{u}}{\partial \xi_i \partial \xi_j^2} \right) \, d\xi_1 d\xi_3 \right) \frac{\partial^3 \tilde{v}_h}{\partial \xi_i \partial \xi_j} \, d\xi_1 d\xi_2 d\xi_3. \]

It can be checked that

\[ \left\{ \begin{array}{l}
B_4(\tilde{u}, \tilde{v}_h) \leq C||\tilde{u}||_{H^4(\tilde{K})} ||\tilde{v}_h||_{H^2(\tilde{K})}, \\
B_4(\tilde{u}, \tilde{v}_h) = 0, \quad \forall \tilde{u} \in P_3(\tilde{K}), \quad \forall \tilde{v}_h \in V_h.
\end{array} \right. \]

Hence, the Bramble-Hilbert lemma gives

\[ B_4(\tilde{u}, \tilde{v}_h) \leq C||\tilde{u}||_{H^4(\tilde{K})} ||\tilde{v}_h||_{H^2(\tilde{K})}. \quad (5.21) \]

A scaling argument leads to

\[ \sum_{K \in \mathcal{T}_h} \sum_{i \neq j = 1}^3 \frac{h^2}{3} \int_K \frac{\partial^3 u}{\partial x_i \partial x_j^2} \frac{\partial^3 v_h}{\partial x_i \partial x_j} \, dx \, dx_2 \, dx_3 + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K (b_0 + b_{12}) \frac{\partial^3 v_h}{\partial x_i} \, dx_1 \, dx_2 \, dx_3 \]
\[ + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K (b_{10} + b_{13}) \frac{\partial^3 v_h}{\partial x_i} \, dx_1 \, dx_2 \, dx_3 + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K (b_{11} + b_{14}) \frac{\partial^3 v_h}{\partial x_i} \, dx_1 \, dx_2 \, dx_3 \]
\[ \leq C h^2 |u|_{H^4(\Omega)} |v_h|_h. \]

An application of Lemma 5.5 yields

\[ a_h(u - \tilde{\Pi}_h^3u, v_h) = a_h(u - \Pi_h u, v_h) + a_h(\tilde{R}_h u, v_h) \leq Ch^2 |u|_{H^4(\Omega)} |v_h|_h. \quad (5.22) \]

Then, by Theorem 5.4 and (5.22), it yields

\[ ||\tilde{\Pi}_h^3u - u_h||_h^2 = a_h(\tilde{\Pi}_h^3u - u_h, \tilde{\Pi}_h^3u - u_h) \]
\[ = a_h(u - u_h, \tilde{\Pi}_h^3u - u_h) + a_h(\tilde{R}_h u - u_h, \tilde{\Pi}_h^3u - u_h) \]
\[ = [a_h(u - \tilde{\Pi}_h^3u - u_h) - (f, \tilde{\Pi}_h^3u - u_h)] + a_h(\tilde{\Pi}_h^3u - u_h, \tilde{\Pi}_h^3u - u_h) \]
\[ \leq Ch^2 (||f||_{L^2(\Omega)} + |u|_{H^4(\Omega)}) ||\tilde{\Pi}_h^3u - u_h||_h, \]

which completes the proof. \( \square \)
Furthermore, based on the superclose property, we can obtain the superconvergence result of the three-dimensional cubic Morley element by a proper postprocessing technique. In order to attain the global superconvergence, we follow the idea of [14] to construct the postprocessing operator $\Pi_3^{3h}$ as follows.

We merge 27 adjacent elements into a macro element, $\tilde{K} = \bigcup_{i=1}^{27} K_i$, (see Figure 4), such that, in the macro element $\tilde{K}$,

$$\Pi_3^{3h} w \in Q_3(\tilde{K}), \quad \forall w \in C(\tilde{K}).$$

(5.23)

We denote $Z_{ijk}, i, j, k = 1, 2, 3, 4$ as the vertices of the 27 adjacent elements. Then, the operator $\Pi_3^{3h}$ satisfies

$$\Pi_3^{3h} w(Z_{ijk}) = w(Z_{ijk}), \quad i, j, k = 1, 2, 3, 4.$$  

(5.24)

Besides, the postprocessing operator $\Pi_3^{3h}$ has the following properties

\[
\begin{cases}
\Pi_3^{3h}(\tilde{\Pi}_h^* u) = \Pi_3^{3h} u, & \forall u \in H^4(\Omega), \\
|\Pi_3^{3h} v_h|_h \leq C|v_h|_h, & \forall v_h \in V_h, \\
|u - \Pi_3^{3h} u_h|_h \leq Ch^2|u|_{H^4(\Omega)}, & \forall u \in H^4(\Omega).
\end{cases}
\]

(5.25)

Then, we can get the following global superconvergent result.

**Theorem 5.8.** Let $u \in H^4(\Omega), u_h \in V_h$, be the solutions of (2.2) and (2.6), respectively, then it holds

$$|u - \Pi_3^{3h} u_h|_h \leq Ch^2(||f||_{L^2(\Omega)} + |u|_{H^4(\Omega)}).$$

(5.26)

**Proof.** It follows the properties (5.23) and Theorem 5.7 that

\[
\begin{align*}
|u - \Pi_3^{3h} u_h|_h & \leq |u - \Pi_3^{3h} \tilde{\Pi}_h^* u_h|_h + |\Pi_3^{3h}(\tilde{\Pi}_h^* u - u_h)|_h \\
& \leq |u - \Pi_3^{3h} u|_h + C|\tilde{\Pi}_h^* u - u_h|_h \\
& \leq Ch^2(||f||_{L^2(\Omega)} + |u|_{H^4(\Omega)}),
\end{align*}
\]

(5.27)

which completes the proof.

\[\square\]

### 6 Numerical results

In this section, we present some numerical results of the two-dimensional rectangular Morley element and three-dimensional cubic Morley element to demonstrate our theoretical results. Herein, we denote $r$
as the rate of convergence. For the sake of simplicity, denote

$$\text{Err}_1 = |u - u_h|_h, \quad \text{Err}_2 = |\Pi_h u - u_h|_h, \quad \text{Err}_3 = |\Pi_h^* u - u_h|_h,$$
$$\text{Err}_4 = |u - \Pi^3_{3h} \Pi_h^* u|_h, \quad \text{Err}_5 = |\tilde{\Pi}^*_h u - u_h|_h, \quad \text{Err}_6 = |u - \Pi^3_{3h} \tilde{\Pi}_h^* u|_h.$$

In the two-dimensional case, we choose the square domain $\Omega_1 = [0, 1]^2$. We partition the domain $\Omega_1$ into the uniform squares with the meshsize $h = \frac{1}{N}$ for some integer $N$.

- In the first example, we use the function $u_1(x, y) = \sin^2(\pi x) \sin^2(\pi y)$ as the exact solution of problem (2.1).
- In the second example, we use the function $u_2(x, y) = x^2(1-x)^2 y^2(1-y)^2$ as the exact solution of problem (2.1).

The errors $\text{Err}_1, \text{Err}_2, \text{Err}_3, \text{Err}_4$ are computed on $\Omega_1$, the corresponding computational results of the two-dimensional rectangular Morley element are listed in Table 3 and Table 4, respectively. One can also refer to Figure 5 for logarithmic plot of the norms above-mentioned.

In the three-dimensional case, we choose the square domain $\Omega_2 = [0, 1]^3$. We partition the domain $\Omega_2$ into the uniform cubic meshes with the meshsize $h = \frac{1}{N}$ for some integer $N$.

- In the third example, we use the function $u_3(x, y, z) = \sin^2(\pi x) \sin^2(\pi y) \sin^2(\pi z)$ as the exact solution of problem (2.1).
- In the fourth example, we use the function $u_4(x, y, z) = x^2(1-x)^2 y^2(1-y)^2 z^2(1-z)^2$ as the exact solution of problem (2.1).

The errors $\text{Err}_1, \text{Err}_2, \text{Err}_5, \text{Err}_6$ are computed on $\Omega_2$, the corresponding computational results of the three-dimensional cubic Morley element are listed in Table 5 and Table 6, respectively. One can also refer to Figure 6 for logarithmic plot of the norms above-mentioned.

| Table 3: The errors of the 2-D rectangular Morley element for $u_1(x, y)$ |
|-----------------|------|------|------|------|
| $N$             | 6    | 12   | 24   | 48   |
| $\text{Err}_1$  | 3.801933642 | 1.848733847 | 0.916356489 | 0.457125924 |
| $r$             | —    | 1.040195809 | 1.012556679 | 1.00317321 |
| $\text{Err}_2$  | 1.97558386 | 1.04229950 | 0.526008399 | 0.263522765 |
| $r$             | —    | 0.922509198 | 0.986612148 | 0.997158239 |
| $\text{Err}_3$  | 1.614468104 | 0.436680422 | 0.111290839 | 0.028096575 |
| $r$             | —    | 1.886409182 | 1.97243010 | 1.985868661 |
| $\text{Err}_4$  | 3.503176938 | 1.427388728 | 0.359795304 | 0.09010039 |
| $r$             | —    | 1.295285573 | 1.988130023 | 1.997571101 |

From the tables and figures, we can see the superconvergent behaviors of the numerical solutions. Besides, in our examples, the exact solution $u_i(x, y), i = 1, 2$, or $u_i(x, y, z), i = 3, 4$, don’t satisfy the boundary condition $\frac{\partial^2 u_i}{\partial n^2} = 0, i = 1, \cdots, 4$, which are need for superconvergence of second order in two-dimensional case [6, 17]. However, our results still have the superconvergent property, which are coincide with our theoretical analysis.
Table 4: The errors of the 2-D rectangular Morley element for $u_2(x, y)$

| N  | 6  | 12 | 24 | 48 |
|----|----|----|----|----|
| Err1 | 0.014701829 | 0.007300451 | 0.003640499 | 0.001803668 |
| r   | —  | 1.009938149 | 1.003849379 | 1.013202406 |
| Err2 | 0.008054841 | 0.004151273 | 0.002093528 | 0.00100988 |
| r   | —  | 0.956302299 | 0.987617597 | 1.051752340 |
| Err3 | 0.006314902 | 0.001727848 | 0.000441448 | 0.00011096 |
| r   | —  | 1.869784037 | 1.968660896 | 1.992203814 |
| Err4 | 0.015833538 | 0.003874778 | 0.000631566 | 0.00024043 |
| r   | —  | 2.030798101 | 2.008272264 | 2.002152565 |

Table 5: The errors of the 3-D rectangular Morley element for $u_3(x, y, z)$

| N  | 6  | 12 | 24 | 48 |
|----|----|----|----|----|
| Err1 | 4.167950479 | 1.983259265 | 0.97524877 | 0.48523741 |
| r   | —  | 1.071464848 | 1.02403111 | 1.007079492 |
| Err2 | 2.569313835 | 1.299194244 | 0.64723551 | 0.32311494 |
| r   | —  | 0.983765976 | 1.00525448 | 1.002243303 |
| Err5 | 2.221462641 | 0.611765555 | 0.15663992 | 0.03948401 |
| r   | —  | 1.860459095 | 1.965526946 | 1.988115111 |
| Err6 | 3.862477845 | 1.283658784 | 0.319649973 | 0.07982083 |
| r   | —  | 1.589264895 | 2.005696886 | 2.001655786 |

Figure 5: The errors of the 2-D rectangular Morley element for $u_1(x, y)$ and $u_2(x, y)$
Table 6: The errors of the 3-D rectangular Morley element for $u_4(x, y, z)$

| N  | 6    | 12    | 24    | 48    |
|----|------|-------|-------|-------|
| Err1 | 0.001051488 | 0.000509571 | 0.00025198 | 0.00012562 |
| r   | 1.045077305 | 1.015973946 | 1.004243055 | 1.001368714 |
| Err2 | 0.000666808 | 0.000330208 | 0.00016451 | 0.000082177 |
| r   | 1.013896 | 1.00519979 | 1.001368714 | 1.00009933 |
| Err5 | 0.00055831 | 0.000154203 | 0.000039494 | 0.000009933 |
| r   | 1.856235564 | 1.965125435 | 1.991332077 | 2.003965450 |
| Err6 | 0.00096663 | 0.000232674 | 0.000057562 | 0.000014351 |
| r   | 2.054653764 | 2.015121382 | 2.003965450 | 2.003965450 |

Figure 6: The errors of the 3-D rectangular Morley element for $u_3(x, y, z)$ and $u_4(x, y, z)$

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