Queues with heterogeneous servers and uninformed customers: who works the most?

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Abstract

In this paper, we consider systems that can be modelled by $M | M | n$ queues with heterogeneous servers and non informed customers. Considering any two servers: we show that the probability that the fastest server is busy is smaller than the probability that the slowest server is busy. Moreover, we show that the effective rate of service done by the fastest server is larger than effective rate of service done by the slowest server.

Keywords: queues; multiserver queues; heterogeneous servers; Markovian processes; slow server.

1 Introduction

In this work, we consider queues with heterogeneous server, we focus on the uninformed customers case [1, 2]. A practical motivation for this is that there are plenty of systems with servers which operate in parallel and are heterogeneous in their capabilities, but without the customer being aware of their service rates differences. For instance: tellers in a bank, cashiers in a supermarket, agents in airport check-in, and may others.

Thus, in order to determine who works the most for uninformed customers, we model the system by a $M | M | n$ queue [3] with heterogeneous servers and uninformed customers. The relevant probability distribution obtained through the use of the balance equations was derived in [2].

Considering any two servers $l$ and $m$, and using the probability distribution: we compute the difference between the probability that server $l$ is busy and the probability that server $m$ is busy:

- we compute the difference between the probability that server $l$ is busy and the probability that server $m$ is busy $\mathcal{P}_l^n - \mathcal{P}_m^n$;
- we compute the difference between the effective rates of service done by servers $l$ and $m$ is $\mu_l \mathcal{P}_l^n - \mu_m \mathcal{P}_m^n$.

For a given $0 < \lambda < \mu^n$, and $\mu_l > \mu_m$, the first difference is negative and second difference is positive, which leads to the desired result.

The third expression given in the theorem only combines these two results in an upper bound -lower bound form.
2.1 Notation

In order to study the steady state behavior of the Markov process, we utilize the notation defined in table [1]
2.2 Balance Equations

2.2.1 Preliminary Considerations

As in [2], let us consider a state \( i \) for a system with \( n \) servers. At state \( i \), the system has \( |S^n_i| \) busy servers. The states that are neighbors of state \( i \) are given by the set \( D^n_i \). There are transitions to state \( i \), from states in which the system has \( |S^n_i| - 1 \) busy servers and from states in which the system has \( |S^n_i| + 1 \) busy servers.

The set of states that are neighbor of state \( i \) and in which the system has \( |S^n_i| + 1 \) busy servers is given by \( U^n_{|S^n_i|+1} \cap D^n_i \). The transitions from a state \( j \) in this set to state \( i \) happen when server \( g^n(i, j) \) finishes serving its customer.

The set of states that are neighbor of state \( i \) and in which the system has \( |S^n_i| - 1 \) busy servers is given by \( U^n_{|S^n_i|−1} \cap D^n_i \). The transitions from a state \( j \) in this set to state \( i \) happen when a customer arrives at the system in state \( j \). This means that one of the \( n - (|S^n_i| - 1) \) idle servers will become busy. There is no preference among these servers, which implies that the transition rate is given by the arrival rate divided by the number of idle servers.

2.2.2 Balance Equations

The steady state probability \( p^n_i \), probability that the system is in state \( i \), is given by the solution to the following linear system:

\[
(\lambda + \sum_{k \in S^n_i} \mu_k) p^n_i - \sum_{j \in U^n_{|S^n_i|−1} \cap D^n_i} \frac{\lambda}{n - (|S^n_i| - 1)} p^n_j - \sum_{j \in U^n_{|S^n_i|+1} \cap D^n_i} \mu g^n(i,j) p^n_j = 0,
\]

\[
0 \leq i \leq 2^n - 1;
\]

\[
\lambda p^n_{i-1} = \mu^n p^n_i, \ i \geq 2^n;
\]

\[
\sum_{i=0}^{\infty} p^n_i = 1.
\]

2.2.3 Main Results

**Theorem 2.1** Let us assume that the Markov process with which we are dealing is time homogeneous, irreducible, and remains in each state for a positive length of time and is incapable of passing through an infinite number of states in a finite time [4]. Then, for a given \( 0 < \lambda < \mu^n \), and \( \mu_l > \mu_m \), we have that

- \( \mathcal{P}_l^n < \mathcal{P}_m^n \);
- \( \mu_l \mathcal{P}_l^n > \mu_m \mathcal{P}_m^n \);
- \( \frac{\mu_m}{\mu_l} \mathcal{P}_m^n < \mathcal{P}_l^n < \mathcal{P}_m^n \).
Proof.

From [2], we have that

\[ p^n_i = \frac{(n - |S^n_i|)!}{n!} \lambda^{|S^n_i|} \prod_{j \in S^n_i} \frac{\mu_j^n}{\mu_j^{n_0}}; \quad 0 \leq i \leq 2^n - 1. \]

For a state \( i, \quad 0 \leq i \leq 2^n - 1, \) such that server \( l \) is busy, server \( m \) is idle, and there are other \( k - 1 \) servers busy, we have that

\[ p^n_i = \lambda \left( \frac{n - k}{n!} \right) \lambda^{k-1} P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 1)p^n_0. \]

The probability that server \( l \) is busy and server \( m \) is idle in the system with \( n \) servers is then given by

\[ p^n_{l,m} = \frac{\lambda}{\mu_l} \sum_{k=1}^{n-1} \left( \frac{n - k}{n!} \right) \lambda^{k-1} P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 1)p^n_0. \]

The difference between the probability that server \( l \) is busy and the probability that server \( m \) is busy is then given by

\[ p^n_{l,m} - p^n_{m,l} = \left( \frac{\lambda}{\mu_l} - \frac{\lambda}{\mu_m} \right) \sum_{k=1}^{n-1} \left( \frac{n - k}{n!} \right) \lambda^{k-1} P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 1)p^n_0 + \mu_l P^n. \]

The effective rate of service done by server \( l \) is given by \( \mu_l P^n_l \).

Thus, the difference between the effective rates of service done by servers \( l \) and \( m \) is given by \( \mu_l P^n_l - \mu_m P^n_m \).

Noticing that

\[ P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 1) = \frac{1}{\mu_m} P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 2) + P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 1). \]

We have that

\[ \mu_l P^n_l - \mu_m P^n_m = \left( \frac{1}{\mu_m} - \frac{1}{\mu_l} \right) \sum_{k=1}^{n-1} \left( \frac{n - k}{n!} \right) \lambda^k P(S_i^n - (\mu_l^{-1}, \mu_m^{-1}), k - 2) + (\mu_l - \mu_m) P. \]

\[ \blacksquare \]
3 Applications

There are several systems with servers working in parallel. Moreover in many of these systems, the servers are heterogeneous in their capabilities, without the customers being aware of who is fast and who is slow. As examples of systems of this kind, we could cite: tellers in a bank, cashiers in a supermarket, agents in an airport check-in, etc [2].

4 Concluding remarks

In order to answer the question: who works the most? We considered systems that could be modelled by $M \mid M \mid n$ queues with heterogeneous servers and non informed customers. Through the use of the probability distribution previously derived, we were able to show that the fastest server works less in the sense that the probability that the fastest server is busy is smaller than the probability that the slowest server is busy. On the other hand, we showed that effective rate of service of the fastest server is larger than that of the slowest server. These results were also combined in an upper bound - lower bound form.

References

[1] Rubinovitch, M. The Slow Server Problem. J. Appl. Prob. 22,205-213. (1985)

[2] Cabral, F. B. The Slow Server Problem for Uninformed Customers Queueing Systems. 50(4);353-370 (2005),

[3] Kleinrock, L. Queueing Systems. New York, John Wiley, 1975-76, 2v.

[4] Kelly, F. P. Reversibility and Stochastic Networks. New York, John Wiley, 1979.