Holographic Principle in the Closed Universe:

a Resolution with Negative Pressure Matter

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ABSTRACT

A closed universe containing pressureless dust, more generally
perfect fluid matter with pressure-to-density ratio \( w \) in the range
\((\frac{1}{3}, -\frac{1}{3})\), violates holographic principle applied according to the
Fischler-Susskind proposal. We show, first for a class of two-fluid
solutions and then for the general multifluid case, that the closed
universe will obey the holographic principle if it also contains
matter with \( w < -\frac{1}{3} \), and if the present value of its total den-
sity is sufficiently close to the critical density. It is possible that
such matter can be realised by some form of ‘quintessence’, much
studied recently.

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1. The holographic principle implies that the degrees of freedom in a spatial region can all be encoded on its boundary, with a density not exceeding one degree of freedom per planck cell [1]. Accordingly, the entropy in a spatial region does not exceed its boundary area measured in planck units. Moreover, the physics of the bulk is describeable by the physics on the boundary. This has, indeed, been realised recently for some anti de Sitter spaces [2].

Fischler and Susskind (FS) have proposed [3] how to apply the holographic principle in cosmology, and showed that our universe, if flat or open, obeys this principle as long as its size is non zero - that is, non planckian. If closed, however, it violates this principle in the future even while its size is non zero. In fact, in some cases, the violation occurs while the universe is still expanding. This may indicate that closed universe is to be excluded as inconsistent, or some new behaviour must set in to accomodate the holographic principle [3].

The holographic principle has since been applied in the context of pre big bang scenario [4], singularity problem [5], and inflation [6, 7], mostly for flat universe. Recently, there have been two alternative proposals for the implementation of the holographic principle: by Easther and Lowe, based on second law of thermodynamics [7]; and, by Bak and Rey, using the 'cosmological apparent horizon' instead of particle horizon [8]. In both of these implementations, the closed universe also obeys the holographic principle naturally. Therefore, these proposals are perhaps the more natural ones than the FS proposal.

Nevertheless, it is of interest to study whether or not a closed universe is indeed to be excluded as inconsistent with the holographic principle, applied according to the FS proposal. We study this issue in this paper.

Throughout in the following we consider a closed universe, initially of zero size and expanding. It is assumed to contain more than one type of non interacting perfect fluid matter. The pressure $p_i$ and the density $\rho_i$ of the $i^{th}$ type of matter is related by the equation of state

$$p_i = w_i \rho_i \ , \quad -1 \leq w_i \leq 1 \ .$$

(1)

The parameter $w$ denotes the nature of the matter: $w = 0$ for pressureless dust, $w = \frac{1}{3}$ for radiation, etc. Furthermore, we assume that one of the $w$'s,
say $w_1$, lies in the range

$$-\frac{1}{3} < w_1 < \frac{1}{3} \quad (2)$$

so that if the corresponding matter were the only one present, then the universe violates the holographic principle in the future (see [3]).

We study the explicit solution for the two-fluid case with $w_1 + w_2 = -\frac{2}{3}$. We find that the closed universe obeys the holographic principle throughout its future if and only if the present value of the total density is sufficiently close to the critical density.

Using this solution, we show furthermore that the closed universe, containing at least one matter, with its $w$ lying in the range (2), obeys the holographic principle, applied according to the FS proposal, if it also contains at least one other matter with its $w$ satisfying $w < -\frac{1}{3}$, and if the present value of its total density is sufficiently close to the critical density. Thus, the closed universes need not be excluded as inconsistent, nor any new behaviour needs to set in; the above requirements will suffice.

If these conditions are also necessary then the holographic principle, applied according to the FS proposal, can be said to predict that if the total density at present of our universe exceeds the critical density, no matter by how small an amount, then it is closed and, hence, must also contain matter with $w < -\frac{1}{3}$.

We make a few remarks about matter with negative values of $w$. No known physical matter is of this type, except cosmological constant ($w = -1$). However, such matter, with $w \leq -\frac{1}{\sqrt{3}}$, was found necessary in [9] in avoiding big bang singularity within low energy string theory. Furthermore, Dirichlet - 0 and/or (-1) - branes [10] were envisaged as possible candidates for such matter. Also, the recent discovery, through the analyses of distant Supernovae, that the universe is accelerating at present [11] has sparked an enormous interest in the study of matter with $w < 0$. A realisation of such matter is the so called ‘quintessence’ - a time varying, spatially inhomogeneous component of energy density of the universe with negative pressure, much studied recently [12, 13]. Some of the references which study various candidates for matter with negative pressure and/or quintessence are given in [14]. It is possible that matter with $w < -\frac{1}{3}$, required here, can be realised by some one of the above candidates.

In the following, we first outline how the closed universe violates the holographic principle [3] in the future. We then present two-fluid solutions,
and obtain the conditions underwhich the holographic principle is obeyed. We then show that these conditions are also valid in general.

2. The line element for the homogeneous isotropic universe is given by

\[ ds^2 = -dt^2 + R^2(t) \left( \frac{dr^2}{1 - k r^2} + r^2 d\Omega_2^2 \right), \]

where \( R \) is the scale factor, \( d\Omega_2 \) is the standard line element on unit two dimensional sphere, and \( k = 0, -1, \) or \( +1 \) for flat, open, or closed universe respectively. The above line element can also be written as

\[ ds^2 = -dt^2 + R^2(t) (d\chi^2 + r^2 d\Omega_2^2), \]

where \( r = \chi, \sinh \chi, \) or \( \sin \chi \) for \( k = 0, -1, \) or \( +1 \) respectively. The coordinate size of the horizon is given by the parameter

\[ \chi = \int_0^t \frac{dt}{R}. \]

The holographic bound is given, upto numerical factors of \( \mathcal{O}(1) \), by \( S \leq A \) where \( S \) is the entropy in a given region and \( A \) is the area of its boundary in Planck units \([1]\). Applied to the closed universe according to FS proposal, it implies, upto numerical factors of \( \mathcal{O}(1) \), that

\[ \frac{S}{A} = \frac{\sigma(2\chi - \sin 2\chi)}{R^2 \sin^2 \chi} \leq 1, \tag{3} \]

where \( \sigma \) is the constant comoving entropy density \([3]\).

For a closed universe, initially of zero size and expanding, and containing only one type of matter with \( w > -\frac{1}{3} \), one obtains \([15]\), see below also,

\[ R^{1+3w} \propto \sin^2 \frac{1+3w}{2} \chi. \tag{4} \]

It can be seen that if \( w \geq \frac{1}{3} \), then the FS bound \((3)\) will be violated only when \( R \rightarrow 0 \), which is the Planckian regime. However, if \( -\frac{1}{3} < w < \frac{1}{3} \), then the FS bound \((3)\) will be violated as \( \chi \rightarrow \pi \). The violation occurs even while \( R \) is non zero. In fact, when \( w \leq 0 \), the violation occurs while the
universe is still expanding. This may indicate that such universes are to be excluded as inconsistent, or some new behaviour must set in to accommodate the holographic principle, applied according to the FS proposal [3].

3. The above conclusion is valid if the universe contains one type of matter only. In reality, however, more than one type of matter will be present in various, perhaps subdominant, quantities. It is then important to study such multifluid solutions before excluding closed universes as inconsistent.

The general multifluid solutions are difficult to obtain. In a few cases where exist [16], they typically involve elliptic functions and are often not in a useful form. However, for a class of models, we now present general solutions, in a form useful for our purposes.

Assume that the universe contains different types of non-interacting perfect fluid matter, with equations of state given as in (1). We assume, without loss of generality, that $w_i \neq -\frac{1}{3}$ since the effect of such matter is same as that of the $k$-term in equation (5) below. Also, define

$$\Omega_i \equiv \frac{\rho_{0i}}{\rho_c}, \quad \rho_c \equiv \frac{3H_0^2}{8\pi G}, \quad H_0 \equiv \left(\frac{\dot{R}}{R}\right)_0$$

where $\rho_{0i} (\neq 0)$, is the present value of $\rho_i$, $\rho_c$ is the critical density, $G$ is the Newton’s constant, and $H_0$ is the present value of the Hubble parameter. Einstein’s equations of motion can then be written as

$$\dot{R}^2 = -k + H_0^2 R_0^2 \sum_i \Omega_i \left(\frac{R}{R_0}\right)^{-3(1+w_i)} - (1 + 3w_i) \equiv f(R) \quad (5)$$

$$\rho_i = \rho_{0i} \left(\frac{R}{R_0}\right)^{-3(1+w_i)}, \quad (6)$$

where $R_0$ is the present value of $R$, and the upper dots denote the time derivatives. The present value of $\dot{R}$ gives the relation

$$k = H_0^2 R_0^2 \left(\sum_i \Omega_i - 1\right). \quad (7)$$

Throughout in the following, let $w_1$ lie in the range given in (4). Thus, if the corresponding matter were the only one present, then the universe
violates the holographic principle in the future (see [3]). We now define $y$ and $x$ as follows:

\[
\frac{R}{R_0} = \left( \frac{H_0^2 R_0^2 \Omega_1}{1 + 3 w_1} \right)^{\frac{1}{1 + 3 w_1}} y^{\frac{2 a}{1 + 3 w_1}} \quad (8)
\]

\[
\frac{dt}{R_0} = \frac{2 a}{1 + 3 w_1} \left( \frac{H_0^2 R_0^2 \Omega_1}{1 + 3 w_1} \right)^{\frac{1}{1 + 3 w_1}} y^{\frac{2 a}{1 + 3 w_1}} dx \quad , \quad (9)
\]

where $a$ and $q$ are positive constants to be chosen suitably, and we set $x = 0$ at $t = \chi = 0$. Clearly, the parameter $\chi$ is given by

\[
\chi = \int_0^t \frac{dt}{R} = \frac{2 a}{1 + 3 w_1} \int_0^x dx \ y^{\frac{2 a}{1 + 3 w_1}} . \quad (10)
\]

In terms of $y$ and $x$, equation (5) becomes

\[
\left( \frac{dy}{dx} \right)^2 = -ky^\alpha + \sum_i c_i y^{\alpha_i} \equiv g(y) , \quad (11)
\]

where the exponents $\alpha$ and $\alpha_i$, and the constants $c_i$, are given by

\[
\alpha = 2 + \frac{4 a (q - 1)}{1 + 3 w_1} \quad (12)
\]

\[
\alpha_i = \alpha - \frac{2 a (1 + 3 w_i)}{1 + 3 w_1} \quad (13)
\]

\[
c_i = H_0^2 R_0^2 \Omega_i \left( \frac{H_0^2 R_0^2 \Omega_1}{1 + 3 w_1} \right)^{\frac{1 + 3 w_1}{1 + 3 w_i}} . \quad (14)
\]

Note that $c_1 = 1$. The function $f(R)$ in (3), expressed in terms of $y$, becomes $f(R) = y^{-\alpha} g(y)$.

From now on, we set $k = +1$. If $q = a = 1$ then $(\alpha, \alpha_1) = (2, 0)$, and the solution (4) for the single fluid case (3) follows trivially. Consider two-fluid cases: $i = 1, 2$. The boundary conditions, corresponding to an universe, initially of zero size and expanding, are $R = y = 0$ and $\dot{R} > 0$ at $t = x = 0$.

**A:** Let $q = a = 1$, and $(1 + 3 w_1) = 2(1 + 3 w_2)$. To be definite, let $(w_1, w_2) = \left( \frac{1}{3}, 0 \right)$ and $(\Omega_1, \Omega_2) = (\Omega_r, \Omega_d)$ denoting radiation and pressureless dust respectively. Then $(\alpha, \alpha_1, \alpha_2) = (2, 0, 1)$, and equation (11) becomes

\[
\left( \frac{dy}{dx} \right)^2 = 1 + cy - y^2 , \quad c = \frac{H_0 R_0 \Omega_d}{\sqrt{\Omega_r}} .
\]
The solution for $R$ is given, after a straightforward algebra, by

\[ R = AR_0 \left( \sin(\chi - \alpha) + \sin \alpha \right), \quad t = \int_{0}^{\chi} d\chi \ R, \tag{15} \]

where the constants $A$ and $\alpha$ are given by

\[ 4A^2 = H_0^2 R_0^2 \left( 4\Omega_r + H_0^2 R_0^2 \Omega_d^2 \right), \quad \tan \alpha = \frac{c}{2}. \]

It follows from the above expressions that

\[ R|_{\chi=\pi} = 2AR_0 \sin \alpha = H_0^2 R_0^3 \Omega_d. \]

Hence, in a closed universe with both radiation and dust present, the FS bound (3) will be violated even while $R (= R|_{\chi=\pi})$ is non zero. This is true irrespective of the amount of radiation present, however large it may be. Only when $\Omega_d = 0$ exactly, will the FS bound (3) be obeyed all the way until $R = 0$, i.e. until the universe recollapses to zero size.

**B:** Let $2a = 1$, and $2(q - 1) = -(1 + 3w_1) = (1 + 3w_2)$. (For example, $(w_1, w_2) = (0, -\frac{2}{3})$.) Then $(\alpha, \alpha_1, \alpha_2) = (1, 0, 2)$, and equation (11) becomes

\[ \left( \frac{dy}{dx} \right)^2 = 1 - y + cy^2, \quad c = H_0^4 R_0^4 \Omega_1 \Omega_2. \]

Using equation (7), the constant $c$ can be written as

\[ c = \frac{\Omega_1 \Omega_2}{(\Omega_1 + \Omega_2 - 1)^2}, \tag{16} \]

where $\Omega_1 + \Omega_2 > 1$ since $k = +1$. The parameter $\chi$ and the time $t$ are given by

\[ \chi = \frac{1}{1 + 3w_1} \int_{0}^{x} \frac{dx}{\sqrt{y}}, \tag{17} \]

\[ t = \frac{R_0}{1 + 3w_1} \left( H_0^2 R_0^2 \Omega_1 \right)^{\frac{1}{1+3w_1}} \int_{0}^{x} \frac{dx}{\sqrt{y}} y^{\frac{1}{1+3w_1}}. \tag{18} \]

The details of the solution depend on whether or not $f_{\text{Min}} = \text{Min}(\frac{1}{y} - 1 + cy) = (2\sqrt{c} - 1)$ is negative or positive. Consider now each of these cases.
The solution for \( y \) is given, after a straightforward algebra, by

\[
y \sqrt{c} \sinh \alpha = \cosh \alpha - \cosh(x \sqrt{c} - \alpha), \quad \tanh \alpha \equiv 2 \sqrt{c}. \quad (19)
\]

Thus, \( y \) starts from zero at \( x = 0 \), expands to a maximum, given by \( y_{\text{max}} \sqrt{c} = \tanh \frac{\alpha}{2} \), at \( x \sqrt{c} = \alpha \), and then recollapses to zero at \( x \sqrt{c} = 2\alpha \). It can be seen from equation (18) that the recollapse occurs in a finite time.

The parameter \( \chi \) is given, after a straightforward algebra using equation (17) and the formula (2.464(32)) of [17], by

\[
\frac{1 + 3w_1}{2} \chi = \sqrt{\frac{2 \cosh \alpha}{1 + \cosh \alpha}} F(\phi, \beta), \quad (20)
\]

where

\[
F(\phi, \beta) = \int_0^\phi \frac{d\theta}{\sqrt{1 - \beta^2 \sin^2 \theta}} \quad (21)
\]

is the elliptic integral of the first kind. The parameters \( \phi \) and \( \beta \) are given by

\[
\sin^2 \phi = \frac{y}{y_{\text{max}}}, \quad \beta = \tanh \frac{\alpha}{2} < 1.
\]

Thus, as \( y \) expands from zero to \( y_{\text{max}} \) and then recollapses to zero, \( \phi \) increases monotonically from 0 to \( \frac{\pi}{2} \) to \( \pi \). Also, the parameter \( \chi \) increases monotonically with \( \phi \) and, since \( \beta^2 < 1 \), remains finite always.

It is now important to find out whether or not the value of \( \chi \) at the time of recollapse, \( \chi_* \equiv \chi|_{\phi=\pi} > \pi \). Clearly, if \( \chi_* > \pi \) then the FS bound (3) will be violated even while \( R \) is non zero. If \( \chi_* \leq \pi \) then the FS bound (3) will be obeyed all the way until \( R = 0 \), i.e., until the universe recollapses to zero size. Towards this end, note from equations (21) and (21) that

\[
\frac{1 + 3w_1}{2} \chi > \phi \quad \text{and, hence,} \quad \chi_* > \frac{2\pi}{1 + 3w_1}.
\]

Thus, for \( w_1 < \frac{1}{3} \), \( \chi_* > \pi \) and, therefore, the FS bound (3) will be violated even while \( R \) is non zero. In fact, for dust \( (w_1 = 0) \), the violation occurs while the universe is still expanding; that is, \( \chi = \pi \) even while \( \phi < \frac{\pi}{2} \).
The solution for \( y \) is given, after a straightforward algebra, by

\[ y\sqrt{c}\cosh \alpha = \sinh(x\sqrt{c} - \alpha) + \sinh \alpha , \quad \tanh \alpha \equiv \frac{1}{2\sqrt{c}}. \] (22)

Thus, \( y \) starts from zero at \( x = 0 \), expands to infinity as \( x \to \infty \). It can be seen from equation (18) that, for \( w_1 < \frac{1}{3} \), the required time \( t \) also \( \to \infty \).

The parameter \( \chi \) is given, after a straightforward algebra using equation (17) and the formula (2.464(16)) of [17], by

\[ (1 + 3w_1)\chi = c^{-\frac{1}{4}} F(\phi, \beta) , \] (23)

with \( F(\phi, \beta) \) as given in (21). The parameters \( \phi \) and \( \beta \) are now given by

\[ \cos \phi = \frac{1 - y\sqrt{c}}{1 + y\sqrt{c}} , \quad \beta^2 = \frac{1 + \tanh \alpha}{2} < 1 . \]

Thus, as \( y \) expands from 0 to infinity, \( \phi \) increases monotonically from 0 to \( \pi \). Also, \( \chi \) increases monotonically with \( \phi \) and, since \( \beta^2 < 1 \), remains finite always.

For the same reasons as given before, it is now important to find out whether or not the value of \( \chi \) as \( y \to \infty \), \( \chi_* \equiv \chi|_{\phi=\pi} > \pi \). Clearly if \( \chi_* > \pi \) then the FS bound (3) will be violated even while \( R \) is non zero and, in fact, increasing. If \( \chi_* < \pi \) then the FS bound (3) will be obeyed for all times \( t \). Towards this end, note from equation (23) that

\[ \chi_* = \frac{2c^{-\frac{1}{4}}}{1 + 3w_1} K(\beta) \] (24)

where we have used \( F(\pi, \beta) = 2F(\frac{\pi}{2}, \beta) \equiv 2K(\beta) \). Here, \( K(\beta) \) is the complete elliptic integral, and is finite since \( \beta^2 < 1 \). Thus, it follows from equation (24), the \( c \)-dependence of \( \beta \), and the properties of \( K(\beta) \), that \( \chi_* < \pi \) implies that \( c > c_* \), where \( c_* \) is the solution of equation (24) when \( \chi_* = \pi \). This, in turn, implies that

\[ f_{\text{Min}} > f_* \equiv 2\sqrt{c_*} - 1 \]

and also, from equation (16), that \( \Omega_2 < \Omega_{2*} \) for a given value of \( \Omega_1 \).
It also follows from equation (24), the $c$-dependence of $\beta$, and the properties of $K(\beta)$, that $c_*$ increases and, hence, $\Omega_2*$ decreases, as $w_1$ decreases. However, an explicit expression for $c_*$ is not available, although one can approximately determine $c_*$ using the tabulated values of $K(\beta)$. Then, $\Omega_2*$ can be determined for a given $\Omega_1$.

For example, if $w_1 = 0$ then $c_* \approx 2.684$, and

$$(\Omega_1, \Omega_2*) \approx (0.1, 1.103), (0.3, 1.041), (0.5, 0.912), \cdots$$

If $w_1 = \frac{1}{3}$, the highest value possible in the present solution, then $c_* \approx 0.416$, and

$$(\Omega_1, \Omega_2*) \approx (0.1, 1.501), (0.3, 1.857), (0.5, 2.082), \cdots$$

In all these cases, we have $\Omega_1 + \Omega_2* = 1 + \mathcal{O}(1)$. Thus, we see that $\chi_* < \pi$ and, hence, the FS bound (3) will be obeyed if matter, with $w < -\frac{1}{3}$, is present and if

$$0 < \Omega_1 + \Omega_2 - 1 \lesssim \mathcal{O}(1).$$

4. The above result is obtained for the two-fluid solutions where $(1 + 3w_1) = -(1 + 3w_2)$. However, this result is valid for general multifluid solutions also. Namely, the FS bound (3) will be obeyed if

(1) atleast one matter, with $w < -\frac{1}{3}$, is present, and

(2) the present value of the total density is sufficiently close to the critical density, *i.e.* $(\sum_1 \Omega_i - 1)$, which must be positive since $k = +1$, is sufficiently small.

This can be proved as follows. Let the multifluid system contain at least two types of matter, one with its $w \equiv w_1 > -\frac{1}{3}$, and another with its $w \equiv w_2 < -\frac{1}{3}$. It may now contain other types of matter also, with no further restrictions on $\{w_i\}$, $i = 1, 2, \cdots$. Then, the function

$$h(R) \equiv \sum_i \Omega_i \left(\frac{R}{R_0}\right)^{-(1+3w_i)} \quad (25)$$

has its non zero, and the only, minimum at a finite value of $R$. That is,

$$h(R) \geq h(R_m) > 0, \quad 0 < R_m < \infty.$$
We now consider an auxiliary two-fluid system, with the corresponding function given by
\[
\tilde{h}(R) \equiv \sum_{j=1,2} \tilde{\Omega}_j \left( \frac{R}{R_0} \right)^{-(1+3\tilde{w}_j)},
\]
where the tildes refer to the auxiliary system. By a simple, but slightly involved, analysis it can be shown that the parameters \( \tilde{w}_j \) and \( \tilde{\Omega}_j \), \( j = 1, 2 \), can be chosen\(^1\) such that \( (1 + 3\tilde{w}_1) = -(1 + 3\tilde{w}_2) > 0 \) and
\[
h(R) > \tilde{h}(R), \quad 0 \leq R \leq \infty.
\]
The parameter \( H_0 R_0 \), taken to be the same for both systems, is given by (see equation (7))
\[
H_0^2 R_0^2 \left( \sum_i \Omega_i - 1 \right) = 1.
\]
The solution for the auxiliary system is nothing but the one given in the previous section, where now the parameter \( \tilde{c} = H_0^4 R_0^4 \tilde{\Omega}_1 \tilde{\Omega}_2 \).

Note that
\[
f(R) = -1 + H_0^2 R_0^2 h(R) \quad \text{and} \quad \chi = \int_0^t \frac{dt}{R} = \int_0^R \frac{dR}{R \sqrt{f}},
\]
and similarly for \( \tilde{f}(R) \) and \( \tilde{\chi} \). Since \( h(R) > \tilde{h}(R) \) and \( H_0 R_0 \) is same for both the systems, it follows that
\[
f(R) > \tilde{f}(R) \quad \text{and, hence,} \quad \chi < \tilde{\chi}.
\]
However, it is shown in the previous section that \( \tilde{\chi} < \pi \) if \( \tilde{c} \) is sufficiently large. Therefore, it now follows that the parameter
\[
\chi < \pi
\]
and, hence, the FS bound (3) is obeyed in the future, if \( (\sum_i \Omega_i - 1) \), which must be positive, is sufficiently small; that is, if the present value of the total density is sufficiently close to the critical density.

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\(^1\)For example, choose \( \tilde{w}_1 \) such that \( \text{Min}\{|1 + 3\tilde{w}_i|\} > 1 + 3\tilde{w}_1 > 0 \), and \( \tilde{\Omega}_j \)'s such that \( \tilde{h}(R) \) also has its non zero, and the only, minimum at \( R = R_m \) and \( h(R_m) > \tilde{h}(R_m) > 0 \), where the inequalities are obeyed by sufficient margins.
We have thus shown that the closed universe, containing at least one matter, with its $w$ lying in the range $\left(2\right)$, obeys the holographic principle, applied according to the FS proposal, if it also contains at least one other matter with its $w$ satisfying $w < -\frac{1}{3}$, and if the present value of its total density is sufficiently close to the critical density. Thus, the closed universes need not be excluded as inconsistent, nor any new behaviour needs to set in; the above requirements will suffice.

If these conditions are also necessary, then they can be taken as predictions of holographic principle, applied to the closed universe according to the FS proposal. Thus, if the total density at present of our universe, which certainly contains pressureless dust, exceeds the critical density, no matter by how small an amount, then it is closed, $k = +1$. The holographic principle, applied according to the FS proposal, would then require that our universe must also contain matter with $w < -\frac{1}{3}$. It is possible that such matter can be realised by some form of quintessence, much studied recently.

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