SOME RESULTS FOR \( q \)-POLY-BERNOULLI POLYNOMIALS WITH A PARAMETER

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Abstract. The main object of this paper is to investigate a new class of the generalized \( q \)-poly-Bernoulli numbers and polynomials with a parameter. We give explicit formulas and a recursive method for the calculation of the \( q \)-poly-Bernoulli numbers and polynomials. As a consequence, we derive a method for the calculation of the special values at negative integral points of the Arakawa–Kaneko zeta function also known as generalized Hurwitz zeta function.

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1. INTRODUCTION

Let \( q \) be an indeterminate with \( 0 \leq q < 1 \). The \( q \)-analogue of \( x \) is defined by

\[
[x]_q = \frac{1 - q^x}{1 - q}
\]

with \([0]_q = 0\) and \( \lim_{q \to 1} [x]_q = x \). Recently Komatsu in [12] introduced and studied a new family of polynomials, called \( q \)-poly-Bernoulli polynomials \( B_{n,k,q}(z) \) with a real parameter \( \rho \) which are defined by the following generating function:

\[
F_{q,\rho}(t; z) := \frac{\rho}{1 - e^{-\rho t}} \mathrm{Li}_{k,q} \left( \frac{1 - e^{-\rho t}}{\rho} \right) e^{-t z} = \sum_{n=0}^{\infty} B_{n,k,q}(z) \frac{t^n}{n!},
\]

\((n \geq 0; k \in \mathbb{Z}; \rho \neq 0)\)

where \( \mathrm{Li}_{k,q}(z) \) is the \( q \)-polylogarithm function [11] defined by

\[
\mathrm{Li}_{k,q}(z) = \sum_{n=1}^{\infty} \frac{z^n}{[n]_q}.
\]

Clearly, we have

\[
\lim_{q \to 1} B_{n,k,q}(z) = B_{n,k}(z).
\]
which is the poly-Bernoulli polynomial with a \( \rho \) parameter [7], and

\[
\lim_{q \to 1} \text{Li}_{k,q}(z) = \text{Li}_k(z).
\]

which is the ordinary polylogarithm function, defined by

\[
\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}.
\] (1.2)

In addition, when \( z = 0 \), \( B^{(k)}_{n,\rho}(0) = B^{(k)}_{n,\rho} \) is the poly-Bernoulli number with a \( \rho \) parameter. When \( z = 0 \) and \( \rho = 1 \), \( B^{(k)}_{n,1}(0) = B^{(k)}_n \) is the poly-Bernoulli number [1–3, 10] defined by

\[
\frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B^{(k)}_n \frac{t^n}{n!},
\] (1.3)

In this paper, we propose to investigate a new class of the generalized \( q \)-poly-Bernoulli numbers and polynomials with a parameter which we call \( (m;q) \)-poly-Bernoulli polynomials with a parameter \( m \). We establish several properties of these polynomials. The study of \( (m,q) \)-poly-Bernoulli polynomials with a parameter yields an interesting algorithm for calculating \( B^{(k)}_{n,m}(z;\rho;q) \). As an application, we derive a recursive method for the calculation of the special values at negative integral points of the Arakawa-Kaneko zeta function.

We first recall some basic definitions and some results [8, 16] that will be useful in the rest of the paper. The (signed) Stirling numbers \( s(n;i) \) of the first kind are the coefficients in the following expansion:

\[
x (x - 1) \cdots (x - n + 1) = \sum_{i=0}^{n} s(n;i) x^i, \quad n \geq 1
\]

and satisfy the recurrence relation given by

\[
s(n + 1,i) = s(n,i - 1) - ns(n,i) \quad (1 \leq i \leq n).
\] (1.4)

The Stirling numbers of the second kind, denoted \( S(n,i) \) are the coefficients in the expansion

\[
x^n = \sum_{i=0}^{n} S(n,i) x (x - 1) \cdots (x - i + 1), \quad n \geq 1.
\]

These numbers count the number of ways to partition a set of \( n \) elements into exactly \( i \) nonempty subsets.

The exponential generating functions for \( s(n,i) \) and \( S(n,i) \) are given by

\[
\sum_{n=i}^{\infty} s(n,i) \frac{z^n}{n!} = \frac{1}{i!} [\ln(1 + z)]^i
\]
and
\[
\sum_{n=1}^{\infty} S(n, i) \frac{z^n}{n!} = \frac{1}{i!} (e^z - 1)^i,
\]
respectively.

The weighted Stirling numbers \( S_n^i(x) \) of the second kind are defined by (see [5,6])
\[
S_n^i(x) = \frac{1}{i!} \Delta^i x^n = \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} (x+j)^n,
\]
where \( \Delta \) denotes the forward difference operator. The exponential generating function of \( S_n^k(x) \) is given by
\[
\sum_{n=1}^{\infty} S_n^i(x) \frac{z^n}{n!} = \frac{1}{i!} e^{xz} (e^z - 1)^i
\]
and weighted Stirling numbers \( S_n^i(x) \) satisfy the following recurrence relation:
\[
S_{n+1}^i(x) = S_n^{i-1}(x) + (x+i) S_n^i(x) \quad (1 \leq i \leq n).
\]

In particular, we have for nonnegative integer \( r \)
\[
S_n^i(0) = S(n, i) \quad \text{and} \quad S_n^i(r) = \binom{n+r}{i+r}. \]

where \( \binom{n}{i} \) denotes the \( r \)-Stirling numbers of the second kind [4].

2. \textit{The \((m,q)\)-Poly-Bernoulli Numbers with a Parameter} \( \rho \)

In order to compute \( B_{n,\rho,q}(k) := B_{n,\rho,q}(0) \), we define \((m,q)\)-poly-Bernoulli numbers \( B_{n,m}(\rho,q) \) with a parameter \( \rho \) in terms of \( m \)-Stirling numbers of the second kind by:
\[
B_{n,m}(\rho,q) = \frac{(-\rho)^n (m+1)_q^k}{m!} \sum_{i=0}^{n} \frac{(m+i)! S_n^i(m)}{(-\rho)^i (m+i+1)_q^i}, \quad m \geq 0 \quad (2.1)
\]
with \( B_{0,m}(\rho,q) = 1 \) and \( B_{n,0}(\rho,q) = B_{n,\rho,q} \).

By direct computation from (2.1), we find
\[
B_{0,m}(\rho,q) = 1, \quad B_{1,m}(\rho,q) = (m+1) \left( \frac{q^{m+1} - 1}{q^{m+2} - 1} \right)^k - m.
\]
The following theorem gives us a relation between the \((m,q)\)-poly-Bernoulli numbers \(B_{n,m}(\rho,q)\) and \(q\)-poly-Bernoulli numbers \(B_n^{(k)}(\rho,q) := B_n^{(k)}\).

**Theorem 1.** For \(m \geq 0\), we have

\[
B_{n,m}^{(k)}(\rho,q) = \frac{(-\rho)^m [m+1]_q ^k}{m!} \sum_{i=0}^{m} s(m,i) \frac{B_{n+i}^{(k)}(\rho,q)}{(-\rho)^i}.
\]  

**(2.2)**

**Proof.** The explicit formula (2.2) can be derived from a known result in [14, p. 681, Corollary 1] for the Stirling transform upon specializing the initial sequence

\[a_{0,m} = \frac{m!}{(-\rho)^m [m+1]_q ^k}.
\]

The next theorem contains the exponential generating function for \((m,q)\)-poly-Bernoulli numbers with a parameter \(\rho\).

**Theorem 2.** The exponential generating function for \(B_{n,m}^{(k)}(\rho,q)\) is given by

\[
\sum_{n=0}^{\infty} B_{n,m}^{(k)}(\rho,q) \frac{t^n}{n!} = \frac{(-\rho)^m [m+1]_q ^k}{m!} \frac{e^{mt}}{m!} \left( e^{-t} \frac{d}{dt} \right)^m E_{q,0}(t;z).
\]

**Proof.** We have

\[
\sum_{n=0}^{\infty} B_{n,m}^{(k)}(\rho,q) \frac{t^n}{n!} = \frac{(-\rho)^m [m+1]_q ^k}{m!} \sum_{i=0}^{m} s(m,i) \sum_{n=0}^{\infty} \frac{B_{n+i}^{(k)}(\rho,q) t^n}{(-\rho)^i} \frac{1}{n!}
\]

\[
= \frac{(-\rho)^{m+n} [m+1]_q ^k}{m!} \sum_{i=0}^{m} s(m,i) \frac{d^i}{dt^i} \left( \frac{\rho}{1-e^t} \text{Li}_k(q \left( \frac{1-e^t}{\rho} \right)) \right).
\]

Since [13]

\[
\sum_{i=0}^{m} s(m,i) \left( \frac{d}{dt} \right)^i = e^{mt} \left( e^{-t} \frac{d}{dt} \right)^m,
\]

we get the desired result. 

Next, we propose an algorithm, which is based on a three-term recurrence relation, for calculating the \((m,q)\)-poly-Bernoulli numbers \(B_{n,m}^{(k)}(\rho,q)\) with a parameter \(\rho\).
Theorem 3. For every integer \( k \), the \( B_{n,m}^{(k)}(\rho,q) \) satisfies the following three-term recurrence relation:

\[
B_{n+1,m}^{(k)}(\rho,q) = (m + 1) \left( \frac{q^{m+1} - 1}{q^{m+2} - 1} \right)^k B_{n,m+1}^{(k)}(\rho,q) - \rho m B_{n,m}^{(k)}(\rho,q) \tag{2.3}
\]

with the initial sequence given by \( B_{0,m}^{(k)}(\rho,q) = 1 \).

Proof. From (2.2) and (1.4), we have

\[
B_{n,m+1}^{(k)}(\rho,q) = \frac{(-\rho)^{m+1} [m + 2]_q^k}{(m + 1)!} \sum_{i=0}^{m+1} (s(m,i) - ms(m,i)) \frac{B_{n+i}^{(k)}(\rho,q)}{(-\rho)^i}.
\]

After some simplifications, we find that

\[
B_{n,m+1}^{(k)}(\rho,q) = \frac{1}{[m + 1]_q^k} \frac{(-\rho)^{m+1} [m + 2]_q^k}{m + 1} \left( \frac{1}{(-\rho)} B_{n+1,m}^{(k)}(\rho,q) - m B_{n,m}^{(k)}(\rho,q) \right).
\]

This evidently equivalent to (2.3). \( \Box \)

Remark 1. If we set \( \rho = 1, k = 1 \) and \( q \to 1 \), in (2.3), we get

\[
B_{n+1,m} = \frac{(m + 1)^2}{(m + 2)} B_{n,m+1} - m B_{n,m}, \tag{2.4}
\]

an algorithm for the classical Bernoulli numbers with \( B_1 = \frac{1}{2} \). See [15] for the case \( B_1 = -\frac{1}{2} \).

3. The \((m, q)\)-Poly-Bernoulli Polynomials with a Parameter \( \rho \)

For \( m \geq 0 \), let us consider the \((m, q)\)-poly-Bernoulli polynomials with a parameter \( B_{n,m}^{(k)}(\rho,q) \) as follows:

\[
B_{n,m}^{(k)}(\rho,q) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} B_{i,m}^{(k)}(\rho,q) z^{n-i}. \tag{3.1}
\]

It is easy to show that the generating function of \( B_{n,m}^{(k)}(\rho,q) \) is given by

\[
\sum_{n \geq 0} B_{n,m}^{(k)}(\rho,q) \frac{z^n}{n!} = e^{-zt} \sum_{n \geq 0} B_{n,m}^{(k)}(\rho,q) \frac{t^n}{n!} = \frac{1}{m!} (-\rho)^{m+n} [m + 1]_q^k e^{(m-z)t} \left( e^{-t \frac{d}{dt}} \right)^m F_{q,\rho}(t; z).
\]

Next, we show an explicit formula about \( B_{n,m}^{(k)}(\rho,q) \).
Theorem 4. The following formula holds true

\[ B_{n;m}^{(k)}(z;\rho) = \frac{m+1}{m!} \sum_{i=0}^{n} (-\rho)^{n-i} \binom{n}{i} \frac{(m+i)!}{[m+i+1]_q} \mathcal{S}_n^i \left( \frac{z}{\rho} + m \right) . \]

Proof. From (3.1), we have

\[ B_{n,m}^{(k)}(z;\rho) = \frac{m+1}{m!} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \sum_{j=0}^{i} \frac{(-\rho)^{i-j} (m+j)! \mathcal{S}_j^i (m)}{[m+j+1]_q} z^{n-i} \]

By using the relation

\[ \mathcal{S}_n^i (x+y) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{S}_l^i (x) y^{n-l}, \]

we obtain

\[ B_{n,m}^{(k)}(z;\rho) = \frac{m+1}{m!} \sum_{j=0}^{n} (-\rho)^{n-j} \binom{n}{j} \sum_{i=0}^{n} \binom{n}{i} \mathcal{S}_i^j (m) \left( \frac{z}{\rho} + m \right)^{n-i} . \]

Theorem 5. The polynomials \( B_{n,m}^{(k)}(z;\rho) \) satisfy the following three-term recurrence relation:

\[ B_{n+1,m}^{(k)}(z;\rho) = (m+1) \left( \frac{q^m - 1}{q^{m+1} - 1} \right)^k B_{n,m+1}^{(k)}(z;\rho) \]

\[ + (z-\rho m) B_{n,m}^{(k)}(z;\rho), \quad (3.2) \]

with the initial sequence given by

\[ B_{0,m}^{(k)}(z;\rho) = 1. \]

Proof. From (3.1), we get

\[ \frac{d}{dz} B_{n,m}^{(k)}(z;\rho) = \sum_{i=0}^{n} (n-i) (-1)^{n-i} \binom{n}{i} B_{i,m}^{(k)}(\rho) z^{n-i-1} \]

\[ = n \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} B_{i,m}^{(k)}(\rho) z^{n-i-1} \]

\[ - n \sum_{i=1}^{n} (-1)^{n-i} \binom{n-1}{i-1} B_{i,m}^{(k)}(\rho) z^{n-i-1} . \]
Then
\[
\frac{d}{dz} B^{(k)}_{n,m}(z; \rho, q) = n B^{(k)}_{n,m}(z; \rho, q)
\]
\[
- n \sum_{i=0}^{n-1} (-1)^{n-i-1} \binom{n-1}{i} B^{(k)}_{i+1,m}(\rho, q) z^{n-i-1}.
\]

Now, using (2.3), we have
\[
\frac{d}{dz} B^{(k)}_{n,m}(z; \rho, q) = n B^{(k)}_{n,m}(z; \rho, q) + n \rho m \sum_{i=0}^{n-1} \binom{n-1}{i} B^{(k)}_{i,m}(\rho, q) (-z)^{n-i-1}
\]
\[
- n (m + 1) \left( \frac{q^m - 1}{q^{m+2} - 1} \right)^k \sum_{i=0}^{n-1} \binom{n-1}{i} B^{(k)}_{i,m+1}(\rho, q) (-z)^{n-i-1},
\]
which, after simplification, yields
\[
\frac{d}{dz} B^{(k)}_{n-1,m}(z; \rho, q) = B^{(k)}_{n,m}(z; \rho, q) - (m + 1) \left( \frac{q^m - 1}{q^{m+2} - 1} \right)^k B^{(k)}_{n-1,m+1}(z; \rho, q)
\]
\[
+ \rho m B^{(k)}_{n-1,m}(z; \rho, q),
\]
which is obviously equivalent to (3.2) and the proof is complete. \(\square\)

As a consequence of Theorem 5, one can deduce a three-term recurrence relation for \((m, q)\)-poly-Bernoulli polynomials with a parameter \(\rho\) and negative upper indices \(B^{(-k)}_{n,m}(z; \rho, q)\).

**Corollary 1.** The \(B^{(-k)}_{n,m}(z; \rho, q)\) satisfies the following three-term recurrence relation:

\[
B^{(-k)}_{n+1,m}(z; \rho, q) = (m + 1) \left( \frac{q^{m+2} - 1}{q^{m+1} - 1} \right)^k B^{(-k)}_{n,m+1}(z; \rho, q)
\]
\[
+ (z - \rho m) B^{(-k)}_{n,m}(z; \rho, q),
\]
with the initial sequence given by
\[
B^{(-k)}_{0,m}(z; \rho, q) = 1.
\]

The next result gives a method for the calculation of the special values at negative integral points of the Arakawa–Kaneko zeta function. Recall that the Arakawa-Kaneko zeta function \(\xi_k(s, x)\), for \(s \in \mathbb{C}, x > 0, k \in \mathbb{Z}\), is defined by [9]

\[
\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{-xt} dt
\]
It is well-known that the special values at negative integral points are given in terms of poly-Bernoulli polynomials

$$\xi_k(-n, x) = (-1)^n B_n^{(k)}(x), n \geq 0.$$ 

We now present the following algorithm for $\xi_k(-n, x)$. We start with the sequence $K_{0,m} = 1$ as the first row of the matrix $(K_{n,m})_{n,m \geq 0}$. Each entry is determined recursively by

$$K_{n+1,m}(k, x) = \frac{(m + 1)^{k+1}}{(m + 2)^k} K_{n,m+1}(k, x) + (x - m) K_{n,m}(k, x).$$

Then

$$\xi_k(-n, x) = (-1)^n K_{n,0}(k, x)$$

where $K_{n,0}(k, x)$ are the first column of the matrix $(K_{n,m})_{n,m \geq 0}$.

4. Conclusion

In our present research, we have investigated a new class of the generalized $q$-poly-Bernoulli numbers and polynomials with a parameter. As a consequence, we derive a method for the calculation of the special values at negative integral points of the Arakawa-Kaneko zeta function. We have also given a recursive method for the calculation of $q$-poly-Bernoulli numbers and polynomials with parameter.

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