Effective Action and Schwinger Pair Production in Strong QED

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Abstract. Some field theoretical aspects, such as the effective action and Schwinger pair production, are critically reviewed in strong QED. The difference of the boundary conditions on the solutions of the field equation is discussed to result in the effective action both in the Coulomb and time-dependent gauge. Finally, the apparent spin-statistics inversion is also discussed, where the WKB action for bosons (fermions) works well for fermion (boson) pair-production rate.

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INTRODUCTION

The development of QED has laid a cornerstone for quantum field theories, whose renormalization scheme has been regarded as a paradigm for quantum field theories. However, in spite of enormous successes of QED, such as prediction and measurement of the anomalous magnetic moment, it has been studied and tested only in the weak-field limit. QED processes are classified into weak and strong QED according to the strength of electromagnetic field involved. The critical strength is $E_c = m^2 c^3 / \hbar q (1.3 \times 10^{16} \text{ V/cm})$ for the electric field (electron) and $B_c = m^2 c^2 / \hbar q (4.4 \times 10^{13} \text{ G})$ for the magnetic field. With an external field applied, the free Maxwell theory, $\mathcal{L} = -\mathcal{F} = -F_{\mu \nu} F^{\mu \nu} / 4$, gets quantum corrections due to interactions with virtual charged pairs from the vacuum, which lead to the effective action $\mathcal{L}_{\text{eff}} = -\mathcal{F} + \delta \mathcal{L} (\mathcal{F}, \mathcal{G})$, where $\mathcal{G} = F_{\mu \nu} F^{\mu \nu} / 4$. The prominent consequence of strong QED is that the imaginary part of $\delta \mathcal{L}$ for a strong electric field leads to the vacuum decay and the real part plays the role of a media for light propagation.

Historically, Sauter, Heisenberg and Euler, and Weisskopf studied as early as thirties the effective action of strong electromagnetic fields [1] and Schwinger used the proper-time method to derive the exact one-loop effective action in a gauge invariant form in a constant electromagnetic field [2]. Though the original motivation was a theoretical interest in quantum field theory, the past two decades have witnessed enormous attractions in strong QED. It is partly because the X-ray free electron laser from Linac Coherent Light Source (LCLS) at SLAC [3] and Extreme Light Infrastructure (ELI) [4] will produce an electric field near the critical strength for electron-position pair production. These terrestrial experiments are expected to provide many interesting physics. More astonishing physics comes from astrophysical objects, neutron stars. Neutron stars have magnetic fields ranging from $10^8 \text{ G}$ to $10^{15} \text{ G}$ and more than one-tenth of them are believed to have magnetic fields stronger than $10^{14} \text{ G}$, the so-called magnetars [5], at least
one order greater than the critical strength.

To find the effective action or to compute the pair-production rate is a nontrivial task for a generic profile of electromagnetic field, and only certain field configurations have been worked out exactly (for review, see Ref. [6]). The Sauter-type electric field \( E(t) = E_0 \text{sech}^2(t/T) \), or is confined to a local region, \( E(x) = E_0 \text{sech}^2(x/L) \), has been used to calculate the pair-production rate by employing approximate schemes [8, 9, 10, 11, 12, 13, 14], whose results are compared with the exact result [15]. The resolvent technique [16] and the evolution operator method together with a gamma function renormalization scheme [17, 18] have been used to find the effective action for the Sauter-type electric field, \( E(t) = E_0 \text{sech}^2(t/T) \). The purpose of this proceedings is to clarify some of issues of the effective action and Schwinger pair production. First, the boundary conditions on the solutions of field equation will be discussed for the time-dependent and the Coulomb gauges. The Coulomb gauge, which is a convenient choice for an electric field distributed in space, requires a different boundary condition for the effective action from the time-dependent gauge. Second, an interesting observation is made that there seems to be an apparent spin-statistic inversion for the WKB action for the pair-production rate, where the WKB action for bosons works well for fermion pair production and vice versa.

**EFFECTIVE ACTION IN COULOMB GAUGE**

One may find the pair-production rate either from the solutions of the Klein-Gordon and Dirac equation or the effective action. At the one-loop level, the solutions of the field equation carry all the information about the effective action and thereby the pair-production rate. The physical mechanism for Schwinger pair production by an electric field in the time-dependent gauge is quite similar to particle production by a time-changing spacetime: the ingoing wave at one asymptotic region is scattered by either the electric field or the spacetime metric and splits into a mixture of positive and negative frequency waves at the other asymptotic region [19]. So, to find the effective action, one has to impose the boundary condition that initially there is only an ingoing wave which is scattered by interactions. With this boundary condition imposed, the effective action is determined by the Bogoliubov coefficient for the outgoing waves. Using the evolution operator and regularizing gamma function, the exact one-loop effective action was obtained for a constant electric field and the Sauter-type electric field in the time-dependent gauge [17, 18].

On the other hand, the field equation for a spatially localized electric field in the Coulomb gauge becomes a tunneling problem. To simplify the physics, let us consider an electric field along the \( x \)-direction in the Coulomb gauge. The Klein-Gordon equation for scalar QED and the Dirac equation for spinor QED have the (spin diagonal) Fourier component [in natural units with \( \hbar = c = 1 \)]

\[
\left[ \frac{\partial^2}{\partial t^2} + (\omega - qA_0(x))^2 - (m^2 + k_\perp^2 + 2i\sigma qE(x)) \right] \phi_{0k,\sigma}(x) = 0, \tag{1}
\]

where \( \sigma = 0 \) for spin-0 bosons and \( \sigma = \pm 1/2 \) for spin-1/2 fermions. The equation describes the tunneling problem under the inverted harmonic potential. The incident
wave function corresponds to the ingoing solution and the reflected wave function to the outgoing solution [8]:

$$\phi_{\omega k \sigma}(x) = \alpha_{\omega k \sigma} \varphi_{\omega k \sigma}(x) + \beta_{\omega k \sigma}^* \varphi_{\omega k \sigma}(x).$$  \hspace{1cm} (2)

Here, $$\varphi_{\omega k \sigma}$$ is the ingoing wave function. The tunneling wave function

$$\phi_{\omega k \sigma}(x) = \gamma_{\omega k \sigma} \varphi_{\omega k \sigma}(x)$$  \hspace{1cm} (3)

corresponds to pair production.

From the causality with respect to the group velocity, the flux conservation law reads

$$|\alpha_{\omega k \sigma}|^2 = |\beta_{\omega k \sigma}|^2 - (-1)^{2|\sigma|+1} |\gamma_{\omega k \sigma}|^2,$$  \hspace{1cm} (4)

whose relative ratios lead to the Bogoliubov relation [8, 20]

$$\left|\frac{\alpha_{\omega k \sigma}}{\beta_{\omega k \sigma}}\right|^2 + (-1)^{2|\sigma|+1} |\frac{\gamma_{\omega k \sigma}}{\beta_{\omega k \sigma}}|^2 = 1.$$  \hspace{1cm} (5)

As the tunneling probability is related with pair production and the reflection probability with vacuum-to-vacuum transition [8], one may identify the Bogoliubov coefficients as

$$\mu_{\omega k \sigma} = \frac{\alpha_{\omega k \sigma}}{\beta_{\omega k \sigma}}, \quad \nu_{\omega k \sigma} = \frac{\gamma_{\omega k \sigma}}{\beta_{\omega k \sigma}}.$$  \hspace{1cm} (6)

Now, the Bogoliubov transformation may take the form [21]

$$\hat{a}_{\omega k \sigma}^{\text{out}} = \mu_{\omega k \sigma} \hat{a}_{\omega k \sigma} + \nu_{\omega k \sigma}^* \hat{b}_{\omega k \sigma}^{\text{in}},$$  \hspace{1cm} (7)

where $$\hat{a}$$ and $$\hat{b}$$ denote particle and anti-particle operators, respectively. Then, the outgoing vacuum is

$$|0,\text{out}\rangle = \prod_{\omega k \sigma} U_{\omega k \sigma} |0,\text{in}\rangle,$$  \hspace{1cm} (8)

where $$U_{\omega k \sigma}$$ is the evolution operator in the Coulomb gauge given by

$$U_{\omega k \sigma} = e^{\xi_{\omega k \sigma} \sigma \hat{a}_{\omega k \sigma}^\dagger \hat{a}_{\omega k \sigma} + \theta_{\omega k \sigma} \sigma (\hat{a}_{\omega k \sigma}^\dagger \hat{a}_{\omega k \sigma} + \hat{b}_{\omega k \sigma}^\dagger \hat{b}_{\omega k \sigma}) - (-1)^{2|\sigma|+1} e^{-\xi_{\omega k \sigma} \sigma \hat{a}_{\omega k \sigma}^\dagger \hat{a}_{\omega k \sigma} + \theta_{\omega k \sigma} \sigma (\hat{a}_{\omega k \sigma}^\dagger \hat{a}_{\omega k \sigma} + \hat{b}_{\omega k \sigma}^\dagger \hat{b}_{\omega k \sigma})}.$$  \hspace{1cm} (9)

Here, the parameters $$\xi_{\omega k \sigma}$$ and $$\theta_{\omega k \sigma}$$ are determined by the Bogoliubov coefficients only. The last factor of Eq. (9) annihilates pairs and the middle factor gives only a phase factor but the prefactor creates pairs. Thus, the outgoing vacuum always contains pairs of particles and anti-particles with the opposite momenta, implying pair production. Then, it follows that mean number of produced pairs for a given quantum number is

$$N_{\omega k \sigma} = |\nu_{\omega k \sigma}|^2 = \left|\frac{\gamma_{\omega k \sigma}}{\alpha_{\omega k \sigma}}\right|^2.$$  \hspace{1cm} (10)
The one-loop effective action given by \[17, 18\]
\[
\mathcal{L}_{\text{eff}} = -\left( -1 \right)^{2|\sigma|+1} i \sum_{\omega_{\mathbf{k} \perp \sigma}} \ln(\mu^{*}_{\omega_{\mathbf{k} \perp \sigma}}),
\]  
leads to the general relation between the imaginary part of the effective action and the mean number of produced pairs:
\[
2(\text{Im} \mathcal{L}_{\text{eff}}) = -\left( -1 \right)^{2|\sigma|+1} \sum_{\omega_{\mathbf{k} \perp \sigma}} \ln[1 - (1)^{2|\sigma|+1} N_{\omega_{\mathbf{k} \perp \sigma}}].
\]  

For the sake of simplicity, let us consider a constant electric field in the Coulomb gauge, \(A_0 = -Ex\). Now, the tunneling solution at the asymptotic region at \(x = -\infty\) is
\[
D_{p}(\zeta) = \sqrt{\frac{2\pi}{\Gamma(-p)}} e^{-ip(1+1/2)} D_{p-1}(i\zeta) + e^{-ip\pi} D_{p}(-\zeta),
\]  
while at the asymptotic region \(x = +\infty\) it is
\[
\phi_{\omega_{\mathbf{k} \perp \sigma}}(x) = D_{p}(\zeta),
\]
where \(D_{p}\) is the parabolic cylinder function and
\[
\zeta = \sqrt{\frac{2}{qE}} e^{i\pi/4} (\omega + qEx), \quad p^{*} = -\frac{1}{2} - i\frac{m^2 + k_{\perp}^2 + 2i\sigma qE}{2(qE)}.
\]

Thus, the Bogoliubov coefficient is found to be
\[
\mu_{\mathbf{k}} = \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{i(p-1)\pi/2}.
\]  

There is a caveat in finding the effective action from Eq. (11). Now, the argument of the gamma function is the complex conjugate of that of the time-dependent gauge in Refs. [17, 18]. So, the contour integral for the proper-time integral using the gamma function regularization in Refs. [17, 18] should be taken in the fourth quadrant instead of the first quadrant. The renormalized exact one-loop effective action is then given by
\[
\mathcal{L}_{\text{eff}}^\text{eff} = \left( -1 \right)^{2|\sigma|+1} \frac{(2|\sigma|+1)}{16\pi^2} \int_{0}^{\infty} \frac{ds}{s^3} e^{-m^2s} [(qEs) f(s) - g(s)]
+ i\frac{(2|\sigma|+1)(qE)^2}{16\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{2|\sigma|+1}(n+1)}{n^2} e^{-\frac{mq^2}{2n}},
\]
where \(f(s) = \text{cosec}(qEs)\) and \(g(s) = 1 + (qEs)^2/6\) for bosons, and \(f(s) = \cot(qEs)\) and \(g(s) = 1/s - (qEs)^3/3\) for fermions. Note that the effective action is independent of the gauges used as expected. The vacuum persistence (twice the imaginary part) can be written as
\[
2\text{Im}(\mathcal{L}_{\text{eff}}^\text{eff}) = -\left( -1 \right)^{2|\sigma|+1} \frac{(2|\sigma|+1)qE}{(2\pi)} \int \frac{d^2k_{\perp}}{(2\pi)^2} \ln[1 - \left( -1 \right)^{2|\sigma|+1} N_{k}],
\]  

where $N_k$ is the mean number of the produced pairs:

$$N_k = e^{-\frac{\pi (m^2 + k^2_\perp)}{qE}}.$$  \hspace{1cm} (19)

**APPARENT SPIN-STATISTICS INVERSION FOR THE PAIR-PRODUCTION RATE**

In Ref. [10] the pair-production rate by a spatially localized electric field in the Coulomb gauge, $E(x) = -dA_0(x)/dx$, is given by

$$N_{\omega k_\perp \sigma} = e^{-\text{Re}(\mathcal{S}_{\omega k_\perp \sigma})},$$  \hspace{1cm} (20)

where the WKB approximation of the action $\mathcal{S}$ is

$$\mathcal{S}^{(0)}_{\omega k_\perp \sigma} = \int \sqrt{-Q_{\omega k_\perp \sigma}}(x) dx, \quad Q_{\omega k_\perp \sigma} = m^2 + k^2_\perp + 2i\sigma qE(x) - (\omega - qA_0(x))^2.$$  \hspace{1cm} (21)

Here, the contour integral is taken outside of the branch cut connecting two roots of $Q_{\omega k_\perp \sigma}(x)$ in a complex plane $x$. For the Sauter-type electric field, $E(x) = E_0 \text{sech}^2(x/L)$, the pair-production rate (20) turns out to be a good approximation scheme [10], whose total mean number up to quadratic terms of momenta is equivalent to the worldline instanton plus the prefactor [12, 13].

An interesting point is observed that the pair-production rate for fermions and bosons is better approximated by the WKB action for bosons and fermions, respectively. This apparent inversion of spin-statistics can be shown for $E(x) = E_0 \text{sech}^2(x/L)$ and $E(t) = E_0 \text{sech}^2(t/T)$ by including the next-to-leading order correction to the WKB action,

$$\mathcal{S}^{(2)}_{\omega k_\perp \sigma} = -i \int \left( -\sqrt{-Q_{\omega k_\perp \sigma}} \left[ 5(Q_{\omega k_\perp \sigma}')^2 - 4Q_{\omega k_\perp \sigma}Q_{\omega k_\perp \sigma}' ight] \right) dx.$$  \hspace{1cm} (22)

In fact, the following relations hold

$$\text{Re}(\mathcal{S}^{(0)}_{\text{sp}} + \mathcal{S}^{(2)}_{\text{sc}}) \approx \mathcal{S}^{(0)}_{\text{sc}},$$  \hspace{1cm} (23)

and

$$\mathcal{S}^{(0)}_{\text{sc}} + \mathcal{S}^{(2)}_{\text{sc}} \approx \text{Re}(\mathcal{S}^{(0)}_{\text{sp}}).$$  \hspace{1cm} (24)

The above relation is numerically confirmed for the profile $E(t) = E_0 \text{sech}^2(t/T)$, which can be shown by comparing Figs. 1 and 3, and Figs. 2 and 4, respectively, of Ref. [22] and for the profile $E(t) = E_0 \cos(\omega t + \phi)e^{-t^2/(2\tau^2)}$ [23]. The physical reasoning for this apparent inversion of spin-statistics is not known. The inversion of spin-statistics is also known for the power spectrum of the vacuum noise seen by a uniformly accelerated observer in odd dimensions [24]. The connection of Unruh effect with Schwinger mechanism has been studied in Ref. [25]. However, the WKB action for Schwinger mechanism depends on the dimensionality only through the transverse momenta, while Unruh effect is approximately the Boltzmann distribution and is independent of spin. To exploit the underlying physics for this apparent inversion of spin-statistics requires a further study.
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