The issue of Dark Energy in String Theory

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Recent astrophysical observations, pertaining to either high-redshift supernovae or cosmic microwave background temperature fluctuations, as those measured recently by the WMAP satellite, provide us with data of unprecedented accuracy, pointing towards two (related) facts: (i) our Universe is accelerated at present, and (ii) more than 70% of its energy content consists of an unknown substance, termed dark energy, which is believed responsible for its current acceleration. Both of these facts are a challenge to String theory. In this review I outline briefly the challenges, the problems and possible avenues for research towards a resolution of the Dark Energy issue in string theory.

1 Introduction

Recent Astrophysical Data, from either studies of distant supernovae type Ia [1], or precision measurements of temperature fluctuations in the cosmic microwave background radiation from the WMAP satellite [2], point towards a current-era acceleration of our Universe, as well as a very peculiar energy budget for it. 70% of the energy density of which consists of an unknown energy substance, termed dark Energy. In fact, global best-fit models of a compilation of all the available data at present are provided by simple Einstein-Friedman Universes with a (four space-time dimensional) positive cosmological constant $\Lambda$, whose value saturate the Newtonian upper limit obtained from galactic dynamics, namely in order of magnitude

$$\Lambda \sim 10^{-122} M_P \quad (M_P = 10^{19} \text{ GeV}).$$

(1)

Although, as a classical (general relativistic) field theory, such a model is fairly simple, from a quantum theory view point it appears to be the less understood at present. The reason is simple: Since in cosmology [3] the radiation and matter energy densities scale with inverse powers of the scale factor, $a^{-4}$ and $a^{-3}$ respectively, in a Universe with positive cosmological constant $\Lambda$,
the vacuum energy density remains constant and positive, and eventually dominates the energy budget. The asymptotic (in time) Universe becomes a \textit{de Sitter} one, and in such a Universe the scale factor will increase exponentially, 

\[ a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}} t}, \]  

thereby implying that the Universe will eventually enter an inflationary phase again, and in fact it will accelerate eternally, since \( \ddot{a} > 0 \), where the over-dot denotes derivative with respect to the Robertson-Walker cosmic time, \( t \), defined by:

\[ ds^2_{\text{RW}} = -dt^2 + a^2(t) ds^2_{\text{spatial}} \]  

In such de Sitter Universes there is unfortunately a \textit{cosmic horizon}

\[ \delta \propto \int_{t_0}^{t_{\text{End}}} \frac{cdt}{a(t)} < \infty \]  

where \( t_{\text{End}} \) indicates the end of time. For a closed Universe \( t_{\text{End}} < \infty \), but for an open or flat Universe \( t_{\text{End}} \to \infty \). The Cosmic Microwave (CMB) data of WMAP and other experiments at present indicate that our Universe is spatially flat, and hence \( t_{\text{End}} \to \infty \).

The presence of a cosmic horizon implies that it is not possible to define pure state vectors of quantum asymptotic (in time) states. Therefore, the entire concept of a well-defined and gauge invariant Scattering matrix \( S \) breaks down in quantum field theories defined on such de Sitter space time backgrounds. For string theory this is bad news, because precisely by construction [4], perturbative string theory is based on the well-defined nature of scattering amplitudes of various excitations, and hence on a well-defined S-matrix [5]. This is a challenge for string theory, and certainly one of the most important issues I would like to discuss in this brief review.

A straightforward way out, would be \textit{quintessence}-like scenaria for dark energy [6], according to which the latter is due to a potential of a time dependent scalar field, which has not yet reached its equilibrium point. If, then, the asymptotic value of the dark energy vanishes in such a way so as not to have a cosmic horizon, then the model could be accommodated within string-inspired effective field theories, and could thus characterise the low-energy limit of strings, given that an asymptotic S-matrix could be defined in such a case.

However, this does not mean that de Sitter Universes \textit{per se} cannot be accommodated somehow into a (possibly non perturbative) string theory framework. Their anti-de-Sitter (AdS, negative cosmological constant) counterparts certainly do, and in fact there have been important development towards a holographic property of quantum field theories in such Universes, due to the celebrated Maldacena conjecture [7], concerning quantum properties of (supersymmetric) conformal field theories on the boundary of AdS space time. As we shall discuss in the next section, similar conjectures [8] may characterise
their de Sitter counterparts, and this may be a way forward to accommodate such a space time into string theory.

Finally, a more straightforward (perturbative) approach to discuss de Sitter and inflationary scenarios in string theories, will be to use the so-called non-critical (or Liouville) string framework [9], dealing with a mathematically consistent way of discussing strings propagating in non-conformal backgrounds, such as the de Sitter space time. This theory, however, at least as far as computation of the pertinent correlation functions are concerned, has not been developed to the same level of mathematical understanding as the critical strings. A crucial ingredient in this approach is the identification of the Liouville mode with the target time [10], which allows for some non-conformal backgrounds in string theory, including de Sitter space times and accelerated Universes, to be accommodated in a mathematically consistent manner.

We should stress at this point, that the above considerations, regarding S-matrix amplitudes in de Sitter Universes, refer to pure perturbative string theories. In the modern approach to string theory, where membrane (D-brane) structures [11] also appear as mathematically consistent entities, the presence of a dark energy on the string theory on the brane is unavoidable, unless extreme conditions on unbroken supersymmetry and static nature of brane worlds are imposed. However, in brane cosmology one needs moving branes, in order to obtain a cosmological space time [12], and in this case, target space time supersymmetry breaks down, due to the brane motion, resulting in non-trivial vacuum energy contributions on the brane [13].

The structure of the article will be the following: in section 2, I will deal with mathematical properties of de Sitter space times: after reviewing briefly basic features of this geometry, I will describe modern approaches to the issue of placing a quantum field theory in de Sitter space times, by discussing briefly a holographic conjecture, put forward by Strominger [8], according to which a quantum field theory on the single boundary of de Sitter space can be related to a classical theory in the bulk, in a way not dissimilar to the celebrated Maldacena conjecture [7] for anti-de Sitter spaces (negative cosmological constant space times). In section 3, I will discuss the issue of cosmic horizons in perturbative string theory, and give further arguments that consistent perturbative strings cannot be characterized by such horizons. In section 4, I will discuss quintessence scenario in strings, where the dilaton behaves as the quintessence field, responsible for the current acceleration of the Universe. I will discuss two opposite examples, a pre Big-Bang scenario [14], in which the string coupling increases at late times, with string loop corrections playing a dominant rôle, and another scenario [10, 15], in which the string coupling becomes more and more perturbative as the time passes, leading asymptotically to a vanishing dark energy, in such a way that S-matrix states can be defined. In this second scenario the current-era acceleration parameter turns out to be proportional to the square of the string coupling, which at present enjoys perturbative values compatible with particle physics phenomenology. I will briefly discuss predictions of such models in the context of recent data,
but also unresolved problems. I will not discuss the issue of dark energy in
brane cosmologies in this article, as this is a topic covered by other lecturers
in the school [13]. Conclusions and directions for future research in the issue
of Dark Energy in Strings will be presented in section 5.

2 De Sitter (dS) Universes from a Modern Perspective

In this section we shall give a very brief overview of the most important
properties of de Sitter space, relevant for our discussion. For more details we
refer the reader to [8], and references therein, where a concise exposition of
the most important properties of classical and quantum theories of de Sitter
space is given.

2.1 Classical Properties

The classical Geometrical picture of a de Sitter space time is that of a single-
sheet hyperboloid, depicted in figure 1. This hypersurface can be constructed
from the flat (d+1)-dimensional Minkowski space time, with coordinates
$(X^0, X^i), i = 1, \ldots d$, by means of the equation:

$$-(X^0)^2 + (X^1)^2 + \ldots + (X^d)^2 = \ell^2$$

(5)

where the parameter $\ell$ has units of length, and is called the de Sitter radius.

The classical Einstein equations, which yield as a solution this space time,
involve a positive cosmological constant.
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There are various coordinates one can use for the description of such space times, whose detailed description can be found in [8]. The most useful one, and most relevant for our purposes, which helps us understand the causal properties of the de Sitter space time, are the conformal coordinates, \((T, \theta_i)\), \(i = 1, \ldots, d\), in terms of which the metric element reads:

\[
ds^2 = \frac{1}{\cos^2 T} \left( -dT^2 + d\Omega^2_{d-1} \right)
\]

where \(\Omega\) is the usual angular part, expressed in terms of \(\theta_i\)’s.

In terms of these coordinates, one arrives easily at the Penrose diagram for the de Sitter space, depicted in fig. 2(a), which contains all the information about the causal structure.

A peculiar feature of this space, but quite important for the development of a consistent quantum gravity theory, is the fact that no single observer can access the entire space time (see figs. 2(b),(c)). As we see from the figure, the causal past and future regions of an observer sitting, say, at the south pole will only be the portions \(O^-\) and \(O^+\), respectively. Their intersection (called

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + A g_{\mu\nu} = 0, \quad A = \frac{(d-2)(d-1)}{2\ell^2}
\]
causal diamond) is the only region of the de Sitter space that is fully accessible to an observer at the south pole.

2.2 Quantum Field Theory on dS: thermodynamical properties

Due to the aforementioned problems of inaccessibility of the entire region of dS space to a classical observer, a consistent formulation of quantum field theory in such space times is still an open issue, and certainly it is expected to be rather different from the corresponding one in Minkowskian space times, where such inaccessibility problems are absent.

The situation somewhat resembles that of a Black Hole (BH). In that problem, there is an horizon, for an asymptotic observer, who lies far away from it and makes his/her measurements locally. The entropy \( S \) associated with an event horizon in the BH case is given by the Bekenstein-Hawking area law [16, 17]

\[
S = \frac{1}{4G_N} A
\]

where \( G_N \) is the gravitational (Newton) constant, and \( A \) is the area of the horizon. This is a macroscopic formula, which essentially describes how properties of event horizons in General Relativity change as their parameters are varied.

The quantum BH Hawking radiate, and from this type of particle creation one comes to the conclusion that there is a temperature \( T_H \) characterising the exterior space time (‘Hawking’ temperature), measured at infinity. For the case of a Schwarzschild BH

\[
T_H = \frac{\hbar}{8\pi M}
\]

where \( M \) is the mass (energy) of the BH. In the BH case this temperature is found by integrating the thermodynamic relation \( dS_{BH}/dM = 1/T_H \).

From a quantum field theory point of view, we can understand such formulae from the fact that they describe some effective “loss” of information, associated with modes that go beyond the horizon, and hence are lost for ever for the classical asymptotic observer.

Indeed, if one considers the exterior portion of space time of the BH as an open system, and the interior as constituting the “environment”, with which the physical world interacts, then the Bekenstein-area law formula may be derived simply even in flat Minkowski space times with a boundary of area \( A \). For instance, Srednicki [18] has demonstrated that by tracing the density matrix of a massless scalar field (taken as a toy, but illustrative, example) over degrees of freedom residing inside an imaginary sphere, embedded in a flat Minkowski space time, the result leads to an entropy for the scalar field which is proportional to the area, and not the volume of the sphere.

In view of this analogy, one would therefore expect that in all cases, where there is a region in space time inaccessible to an asymptotic observer of a
quantum field theory on such geometries, there should be an entropy associated with the area of the region. This should also be expected in the dS case, in view of the existence of a de Sitter horizon. If one postulates some thermodynamic properties associated with the entropy, one arrives also at an effective temperature concept.

The deep issue in the black hole case is to understand the precise coefficient $1/4G_N$ in Eq. (8), in other words develop a sufficiently correct quantum theory for such space times, in which it will be possible to count the microstates of the BH exactly. If the latter are associated to a Von Neumann entropy, $S_{VN} = -k_B \text{Tr} \ln \rho$, where $\rho$ is the density matrix of the system under consideration (the quantum BH in our case), and the Tr is taken over all microstates, then one should show that $S_{VN} = S$ given by (8).

At present this is one of the most important issues in theoretical physics. A precise counting of microstates, however, leading to a relation of the form (8) has become possible for certain highly supersymmetric black hole backgrounds in string theory, saturating the so-called Bogomolnyi-Prasad-Sommerfeld (BPS) bound [19]. It is, though, still unproven for general, non supersymmetric black holes, which are the likely types to be encountered in our physical world.

We now come to the dS case. Indeed, as one would expect from the above generic arguments, there should be an entropy associated with the horizon. In fact it is, and there is also a temperature (‘Gibbons-Hawking temperature’) [20], in complete analogy with the BH case. In fact, the Temperature is given in terms of the de Sitter radius by:

$$T_{GH} = \frac{1}{2\pi \ell}$$

(10)

and the entropy, associated with the de Sitter horizon of area $A$, is given exactly by the formula (8).

These properties can be proven by considering a quantum field on the dS background and evaluating its Green functions. Such an analysis shows that, in the case of massive quantum fields, an observer, moving along a time-like geodesic of dS space, observes a thermal bath of particles, when the massive field is in its vacuum state $|0\rangle$. It turns out that the correct type of Green functions to be used in this case are the thermal ones. For details we refer the reader to the lectures by Strominger [8], and references therein. Such an analysis allows for the computation of the effective temperature the dS space is associated with.

The entropy of the de Sitter space $S_{dS}$, then, is found following the argument suggested by Gibbons and Hawking [20], according to which

$$\frac{dS_{dS}}{d(-E_{dS})} = \frac{1}{T_{GH}}$$

(11)

where $E_{dS}$ is the energy of the dS space. Notice the minus sign in front of $E_{dS}$. This stems from the fact that what we call energy in dS space is not as simple...
as the mass of the BH case. To understand qualitatively what might happen in the dS case, we should first start from the principle outlined above, that the entropy of the space is associated with “stuff” behind the horizon. We do not, at present, have any idea what the “microstates” of the dS vacuum are, but let us suppose for the sake of the argument, that an entropy is associated with them (this assumption is probably correct).

In general relativity energy is defined as an integral of a total derivative over a space-time volume, which therefore reduces to a surface integral on the boundary of the surface, and hence vanishes for a closed surface. Because of this vanishing result, if we consider a closed surface on de Sitter space, and we put, say, positive energy on the south pole, then there must be necessarily some negative energy at the north pole to compensate, and yield a zero result.

Therefore, the singularity at the north pole, behind the dS horizon, will correspond to negative energy. From the BH analogy, it is therefore more sensible to vary with respect to this negative energy, and this explains the relative minus sign in (11) yielding the correct expression for the area law in the dS case.

The important point to notice, however, is that, despite the formal similarity of the dS with the BH, in the former case no one understands, at present, the precise microscopic origin of the entropy and temperature. It is not clear what precisely are the microstates behind the horizon, which constitute the “environment” with which the quantum field theory interacts.

This question acquires much bigger importance in cosmologies with a positive cosmological constant, which are currently favoured by the astrophysical data [1,2]. Indeed in such cases, the asymptotic (in time) Universe will enter a pure de-Sitter-space phase, since all the matter energy density will be diluted, scaling with the scale factor as $a^{-3}$, thereby leaving us only with the constant vacuum energy contribution $\Lambda$. As discussed in the beginning of the lecture, the cosmological horizon will be given by (4), and in this case the dS radius $\ell$, in terms of which the entropy and temperature are expressed, is associated with $\Lambda$ by (6), essentially its square root.

### 2.3 Lack of Scattering Matrix and intrinsic CPT Violation in dS?

The important question, therefore, from a quantum-field-theory viewpoint on such cosmologies and in general dS-like space times, concerns the kind of quantum field theories one can define consistently in such a situation. In this respect, the situation is dual to the BH case in the following sense: in a BH, there is an horizon which defines a space time boundary for an asymptotic observer who lies far away from it. In a full quantum theory the BH evaporates due to Hawking radiation. Although the above thermodynamics arguments are valid for large semi-classical BH, one expects the Hawking evaporation process to continue until the BH acquires a size comparable to the characteristic scale of quantum gravity (QG), the Planck length $\ell_P = 1/M_P$, with $M_P \sim 10^{19}$ GeV. Such microscopic BH may either evaporate completely, leaving behind a
naked singularity, or, better -thus satisfying the cosmic censorship hypothesis, according to which there are no unshielded space-time singularities in the physical world - disappear in a space-time “foam”, namely in a QG ground state, consisting of dynamical “flashing on and off” microscopic BH. In such a case, an initially pure quantum state will in principle be observed as mixed by the asymptotic observer, given that “part of the state quantum numbers” will be kept inside the foamy black holes (“effective information loss”), and hence these will constitute degrees of freedom inaccessible to the observer.

Barring the importing concept of holographic properties, which may indeed characterise such singular space times in QG, to which we shall come later on, a situation like this will imply an effective non-unitary evolution of quantum states of matter in such backgrounds, and hence gravitational decoherence.

A similar situation will characterise the dS space, which is dual to the BH analogue, in the sense that the observer is inside the (cosmological) horizon, in contrast to the BH where he/she was lying outside. However the situation concerning the inability to define asymptotically pure state vectors for the quantum state of matter fields remains in this case.

The lack of a proper definition of pure “out” state asymptotic vectors in both situations, implies that a gauge invariant scattering matrix is also ill defined in the dS case. By a theorem due to Wald then [21], one cannot define in such quantum field theories a quantum mechanical CPT operator. This leads to quantum decoherence of matter propagating in such de Sitter space times. For more details I refer the interested reader in [22], where possible phenomenological consequences of such decoherence are discussed in detail.

2.4 Holographic properties of dS? Towards a quantum gravity theory

A final, but important aspect, that might characterise a quantum theory in de Sitter space times, is the aforementioned property of holography. If this happens, then the above-mentioned information loss paradox will not occur, and a mathematically consistent quantum mechanical picture of gravity in the presence of space-time boundaries will be in place.

I must stress, at this point, an important issue for which there is often confusion in the literature. If quantum gravity turns out to lead to open-system quantum mechanics for matter theories, this is not necessarily a mathematical inconsistency. It simply means that there is information carried out by the quantum-gravitational degrees of freedom, which however may not be easy to retrieve in a perturbative treatment. Of course, even in such situations, the complete system, gravity plus matter, is mathematically a closed quantum system. On the other hand, if holography is valid, then one simply does not have to worry about any effective loss of information due to the space time boundary, and hence the situation becomes much cleaner.

Holographic properties of anti-de-Sitter (AdS) spaces (negative cosmological constant) are encoded in the celebrated Maldacena conjecture [7], accord-
ning to which the quantum correlators of a conformal quantum field theory on the boundary of the AdS space can be evaluated by means of classical gravity in the bulk of this space. This conjecture, known with the abbreviation AdS/CFT correspondence, has been verified to a number of highly supersymmetric backgrounds in string theory, but of course it may not be valid in (realistic) non conformal, non supersymmetric cases. The issue for such cases is still open.

A similar conjecture in de Sitter space times has been put forward by Strominger [8]. The conjecture, which is not proven at present, can be formulated as follows:

Consider an operator $\phi(x_i)$ of quantum gravity in a de Sitter space, inserted at points $x_i$ on the hypersurfaces $I^-$ or $I^+$. The dS/CFT conjecture states that correlation functions of this operator at the points $x_i$ can be generated by an appropriate Euclidean conformal field theory

$$\langle \phi(x_1) \ldots \phi(x_i) \rangle_{\text{dS}^{d+1}} \leftrightarrow \langle \mathcal{O}_\phi(x_1) \ldots |\text{calO}_\phi(x_i)\rangle_{S^d}$$

(12)

where $\mathcal{O}_\phi(x)$ is an operator of the CFT associated with the operator $\phi$.

For the simple, but quite instructive case, of a three dimensional dS$_3$ space, a proof of this correspondence has been given in [8], making appropriate use of properties of the asymptotic symmetry group of gravity for dS$_3$. We refer the interested reader to that work, and references therein, for more details.

Before closing this section, we would like to stress that the dS/CFT conjecture may not be valid in realistic cosmologies, in which the quantum field theories of relevance are certainly not conformal. If, however, this conjecture is valid, then this is a very big step towards a CPT invariant, non-perturbative, construction of a quantum theory of gravity.

The holographic principle [23] will basically allow for any possible information loss associated with the presence of the cosmological horizon to decay with the cosmic time, in such a way that an asymptotic observer will not eventually loose any information. This will allow for a consistent CPT operator to be defined, then, according to the above-mentioned theorem of Wald [21].

If true for the dS case, one expects a similar holographic property to be valid for the BH case as well. In fact recently, Hawking argued [24] this to be the case in a BH quantum theory of gravity, but in my opinion his arguments are not supported by any rigorous calculation. Hawking’s argument is based on the fact that any consistent theory of gravity should involve an appropriate sum over topologies, including the Minkowskian one (trivial). In Hawking’s argument, then, the Euclidean path integrals over the non-trivial topologies, that would give non-unitary contributions, and hence information loss, lead to expressions in scattering amplitudes that decay exponentially with time, thereby leaving only the trivial topology contributions, which are unitary. As we said, however, there is no rigorous computation involved to support this argument, at least at present, not withstanding the fact that the Euclidean formalism seems crucial to the result (although, arguably we know of no other way of performing a proper quantum gravity path integral). Hence, the issue of
unitarity in effective low-energy theories of quantum gravity is still wide open in my opinion, and constitutes a challenge for both theory and phenomenology of quantum gravity [22].

3 No Horizons in Perturbative (Critical) String Theory

As discussed above, if holography is valid, there should, in principle, be no issue regarding string theory and CPT would be a good symmetry of the theory, as seems desirable from a modern M-theory point of view [25].

If, however, holography is not valid for realistic non-supersymmetric, non-conformal theories, then such a situation is most problematic in string theory, which, as mentioned in the beginning, at least in its perturbative treatment is based on a formalism with well-defined scattering amplitudes [5].

Apart from the scattering-matrix and CPT-based issues, there are other arguments that exclude the existence of horizons in perturbative string theory [26]. These arguments derive from considerations of the shape of the potentials arising from supersymmetry breaking scenarios in perturbative string theory, whose coupling (before compactification) is defined by the exponential of the dilaton field \( g_s = e^\phi \).

The situation becomes cleanest if we consider, for simplicity and definiteness, the case of a single scalar, canonically normalised, field \( \phi \), playing the rôle of the quintessence field in a Robertson-Walker space time with scale factor \( a(t) \), with \( t \) the cosmic time. Such a field could be the dilaton, or other modulus field from the string multiplet [4].

Consider the lowest order Friedmann equation, as well as the equation of motion of the field \( \phi \) in \( D + 1 \) dimensions (the overdot denotes cosmic-time derivative), which are (formally) derived from the \( \sigma \)-model \( \beta \)-functions of a perturbative string theory

\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right) = \frac{2\kappa^2}{D(D-1)} E = \frac{(\dot{\phi})^2}{2} + V(\phi) ,
\]

\[
\ddot{\phi} + DH \dot{\phi} + V'(\phi) = 0 ,
\]

(13)

with \( E \) the total energy of the scalar field, \( V \) its potential, and a prime indicating variations with respect to the field \( \phi \). We obtain the following expressions for the scale factor \( a(t) \) and the cosmic horizon \( \delta \):

\[
a(t) = \exp \left( \int d\phi \sqrt{\frac{E}{D(D-1)(E-V)}} \right) ,
\]

\[
\delta = \int_{-\infty}^{\infty} \frac{dt}{a} = \int d\phi \frac{1}{a} = \int d\phi \frac{1}{a \sqrt{2(E-V)}} .
\]

(14)

The condition for the existence of a cosmic horizon is of course the convergence of the integral on the right-hand-side of the expression for \( \delta \). This depends
on the asymptotic behaviour of the potential $V$ as compared to the total energy $E$. This behaviour can be studied in a generic perturbative string theory, based on the form of low energy potentials of possible quintessence candidates, such as dilaton, moduli etc. Because realistic string theories involve at a certain stage supersymmetry in target space, which is broken as we go down to the four dimensional world after compactification, or as we lower the energy from the string (Planck) scale, such arguments depend on the form of the potential, dictated by supersymmetry-breaking considerations. The form is such that $\delta \to \infty$ in (14), and hence there are no horizons. I will not repeat these arguments here, because the above-mentioned CPT/scattering-matrix based argument is more general, and encompasses such cases, and is the most fundamental reason for incompatibility of perturbative strings with space-time backgrounds with horizons. I refer the interested reader to the literature [26].

I would like to stress, however, that these arguments refer to the traditional critical strings, without branes, where a low-energy field theory derives from conformal invariance conditions. From this latter point of view it is straightforward to understand the problem of incorporating cosmologies with horizons, such as inflation or in general de Sitter space times, in perturbative strings. A tree-world-sheet $\sigma$-model on, say, graviton backgrounds, whose conformal invariance conditions would normally yield the target-space geometry, reads to order $\alpha'$ ($\alpha'$ denotes henceforth the Regge slope) [4]:

$$\beta_{\mu\nu} = R_{\mu\nu} + \ldots$$

(15)

where the $\ldots$ indicate contributions from other background fields, such as dilaton $\ldots$.

Ignoring the other fields, conformal invariance of the perturbative stringy $\sigma$-model would require a Ricci flat $R_{\mu\nu} = 0$ background, which is not the case of a dS space, for which (c.f. (6))

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

(16)

To generate such corrections in the early days of string theory, Fischler and Susskind [27] had to invoke renormalization-group corrections to the above-tree level $\beta$-function (15), induced by higher string loops, i.e. higher topologies of the $\sigma$-model world sheet. Tadpoles $\mathcal{J}$ of dilatons at one string loop order (torus topologies) yielded a dS (or AdS depending on the sign of $\mathcal{J}$) type contribution to the graviton $\beta$-function, $\mathcal{J} g_{\mu\nu}$. The basic idea behind this approach is to accept that world-sheet surfaces of higher topologies with handles whose size is smaller than the short-distance cutoff of the world-sheet theory, will not be ‘seen’ as higher- topologies but appear ‘effectively’ as tree level ones. They will, therefore, lead to loop corrections to the traditional tree-level $\beta$-functions of the various background fields, which cannot be discovered at tree level. Conformal invariance implies of course that tori with such small handles are equivalent to world-sheet spheres but with a long thin tube connected to them. For more details on this I refer the interested reader in my lectures in the first Aegean School [3].
Nevertheless, this approach does not solve the problem, despite its formal simplicity and elegance. The reason is two fold: first, string-loop perturbation theory is not Borel-resummable, and as such, the expansion in powers of genus of closed Riemann surfaces with handles (and holes if open strings are included), does not converge mathematically, hence it cannot give sensible answers for strong or intermediate string couplings. It is indeed, expected, that the dark energy is a property of a full theory of quantum gravity, and as such, an explanation of it should not be restricted only to perturbative string theory. Second, a string propagating in a space-time with a loop-induced cosmological constant will not be characterised by a well-defined scattering matrix, which by definition, as already mentioned, is a ‘must’ for perturbative string theory.

Thus, the issue remains as to what kind of dark energy one is likely to encounter in string theory.

4 Dilaton quintessence and String Theory

4.1 An expanding Universe in String Theory

One of the simplest, and most natural quintessence fields, to generate a dynamical dark energy component for the string Universe is the dilaton, \( \Phi \), a scalar field that appears in the basic gravitational multiplet of any (super)string theory [4]. Dilaton cosmology has been originated by Antoniadis, Bachas, Ellis and Nanopoulos in [28], where the basic steps for a correct formulation of an expanding Robertson-Walker Universe in string theory have been taken, consistent with conformal invariance conditions \(^1\). The crucial rôle of a time dependent dilaton field had been emphasized.

In [28] a time-dependent dilaton background, with a linear dependence on time in the so-called \( \sigma \)-model frame was assumed. Such backgrounds, even when the \( \sigma \)-model metric is flat, lead to exact solutions (to all orders in \( \alpha' \)) of the conformal invariance conditions of the pertinent stringy \( \sigma \)-model, and so are acceptable solutions from a perturbative viewpoint. It was argued in [28] that such backgrounds describe linearly-expanding Robertson-Walker Universes, which were shown to be exact conformal-invariant solutions, corresponding to Wess-Zumino models on appropriate group manifolds.

The pertinent \( \sigma \)-model action in a background with graviton \( G \), antisymmetric tensor \( B \) and dilaton \( \Phi \) reads [4]:

\[
S_\sigma = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \left[ \sqrt{-g} G_{\mu\nu} \partial_\mu X^\alpha \partial_\nu X^\beta + i \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \sqrt{-\gamma} R^{(2)} \Phi \right],
\]

\(^1\) In fact that work was actually the first work on Liouville supercritical strings [9], with the Liouville mode identified with the target time, although this had not been recognised in the original work, but later [10].
where $\Sigma$ denotes the world-sheet, with metric $\gamma$ and the topology of a sphere, $\alpha$ are world-sheet indices, and $\mu, \nu$ are target-space-time indices. The important point of [28] was the rôle of target time $t$ as a specific dilaton background, linear in that coordinate, of the form
\[ \Phi = \text{const} - \frac{1}{2} Q t, \tag{18} \]
where $Q$ is a constant and $Q^2 > 0$ is the $\sigma$-model central-charge deficit, allowing this supercritical string theory to be formulated in some number of dimensions different from the critical number. Consistency of the underlying world-sheet conformal field theory, as well as modular invariance of the string scattering amplitudes, required discrete values of $Q^2$, when expressed in units of the string length $M_s$ [28]. This was the first example of a non-critical string cosmology, with the spatial target-space coordinates $X_i$, $i = 1, \ldots, D - 1$, playing the rôle of $\sigma$-model fields. This non-critical string was not conformally invariant, and hence required Liouville dressing [9]. The Liouville field had time-like signature in target space, since the central charge deficit $Q^2 > 0$ in the model of [28], and its zero mode played the rôle of target time.

As a result of the non-trivial dilaton field, the Einstein term in the effective $D$-dimensional low-energy field theory action is conformally rescaled by $e^{-2\Phi}$. This requires a redefinition of the $\sigma$-model-frame space-time metric $g_{\sigma\mu\nu}$ to the 'physical' Einstein metric $g^E_{\mu\nu}$:
\[ g^E_{\mu\nu} = e^{-\Phi/D(2D-2)} G_{\mu\nu} . \tag{19} \]
Target time must also be rescaled, so that the metric acquires the standard Robertson-Walker (RW) form in the normalized Einstein frame for the effective action:
\[ ds^2_E = -dt^2_E + a_E^2(t_E) \left( dr^2 + r^2 d\Omega^2 \right) , \tag{20} \]
where we show the example of a spatially-flat RW metric for definiteness, and $a_E(t_E)$ is an appropriate scale factor, which is a function of $t_E$ alone in the homogeneous cosmological backgrounds we assume throughout.

The Einstein-frame time is related to the time in the $\sigma$-model frame [28] by:
\[ dt_E = e^{-2\Phi/(D-2)} dt \quad \rightarrow \quad t_E = \int t e^{-2\Phi(t')/(D-2)} dt' . \tag{21} \]
The linear dilaton background (18) yields the following relation between the Einstein and $\sigma$-model frame times:
\[ t_E = c_1 + D - 2 \frac{Q}{Q^2} e^{\Phi/(D-2)} , \tag{22} \]
where $c_1$ is an appropriate (positive) constant. Thus, a dilaton background (18) that is linear in the $\sigma$-model time scales logarithmically with the Einstein time (Robertson-Walker cosmic time) $t_E$: 
\begin{equation}
\Phi(t_E) = (\text{const}.) - \frac{D-2}{2} \ln\left( \frac{Q}{D-2} t_E \right). \tag{23}
\end{equation}

In this regime, the string coupling [4]:
\begin{equation}
g_s = \exp(\Phi(t)) \tag{24}
\end{equation}
varies with the cosmic time \( t_E \) as \( g_s^2(t_E) \equiv e^{2\Phi} \propto \frac{1}{t_E^D} \), thereby implying a vanishing effective string coupling asymptotically in cosmic time. In the linear dilaton background of [28], the asymptotic space-time metric in the Einstein frame reads:
\begin{equation}
ds^2 = -dt_E^2 + a_0^2 t_E^2 (dr^2 + r^2 d\Omega^2) \tag{25}
\end{equation}
where \( a_0 \) a constant. Clearly, there is no acceleration in the expansion of the Universe (25).

The effective low-energy action on the four-dimensional brane world for the gravitational multiplet of the string in the Einstein frame reads [28]:
\begin{equation}
S_{\text{brane}}^{\text{eff}} = \int d^4 x \sqrt{-g} \left\{ R - 2(\partial_\mu \Phi)^2 - \frac{1}{2} e^{4\Phi} (\partial_\mu b)^2 - \frac{2}{3} e^{2\Phi} \delta c \right\}, \tag{26}
\end{equation}
where \( b \) is the four-dimensional axion field associated with a four-dimensional representation of the antisymmetric tensor, and \( \delta c = C_{\text{int}} - c^* \), where \( C_{\text{int}} \) is the central charge of the conformal world-sheet theory corresponding to the transverse (internal) string dimensions, and \( c^* = 22(6) \) is the critical value of this internal central charge of the (super)string theory for flat four-dimensional space-times. The linear dilaton configuration (18) corresponds, in this language, to a background charge \( Q \) of the conformal theory, which contributes a term \(-3Q^2\) (in our normalization) to the total central charge. The latter includes the contributions from the four uncompactified dimensions of our world. In the case of a flat four-dimensional Minkowski space-time, one has \( C_{\text{total}} = 4 - 3Q^2 + C_{\text{int}} = 4 - 3Q^2 + c^* + \delta c \), which should equal 26 (10). This implies that \( C_{\text{int}} = 22 + 3Q^2 (6 + 3Q^2) \) for bosonic (supersymmetric) strings.

An important result in [28] was the discovery of an exact conformal field theory corresponding to the dilaton background (23) and a constant-curvature (Milne) static metric in the \( \sigma \)-model frame (or, equivalently, a linearly-expanding Robertson-Walker Universe in the Einstein frame). The conformal field theory corresponds to a Wess-Zumino-Witten two-dimensional world-sheet model on a group manifold \( O(3) \) with appropriate constant curvature, whose coordinates correspond to the spatial components of the four-dimensional metric and antisymmetric tensor fields, together with a free world-sheet field corresponding to the target time coordinate. The total central charge in this more general case reads \( C_{\text{total}} = 4 - 3Q^2 - \frac{k}{6} \delta c + C_{\text{int}}, \) where \( k \) is a positive integer corresponding to the level of the Kac-Moody algebra associated with the WZW model on the group manifold. The value of \( Q \) is chosen in such a way that the overall central charge \( c = 26 \) and the theory is conformally invariant. Since such unitary conformal field theories have
discrete values of their central charges, which accumulate to integers or half-integers from below, it follows that the values of the central charge deficit $\delta c$ are discrete and finite in number. From a physical point of view, this implies that the linear-dilaton Universe may either stay in such a state for ever, for a given $\delta c$, or tunnel between the various discrete levels before relaxing to a critical $\delta c = 0$ theory. It was argued in [28] that, due to the above-mentioned finiteness of the set of allowed discrete values of the central charge deficit $\delta c$, the Universe could reach flat four-dimensional Minkowski space-time, and thus exit from the expanding phase, after a finite number of phase transitions.

The analysis in [28] also showed that there are tachyonic mass shifts of order $-Q^2$ in the bosonic string excitations, but not in the fermionic ones. This implies the appearance of tachyonic instabilities and the breaking of target-space supersymmetry in such backgrounds, as far as the excitation spectrum is concerned. The instabilities could trigger the cosmological phase transitions, since they correspond to relevant renormalization-group world-sheet operators, and hence initiate the flow of the internal unitary conformal field theory towards minimization of its central charge, in accordance with the Zamolodchikov $c$-theorem [29]. In semi-realistic cosmological models [15] such tachyons decouple from the spectrum relatively quickly. On the other hand, as a result of the form of the dilaton in the Einstein frame (23), we observe that the dark-energy density for this (four-dimensional) Universe, $\Lambda \equiv e^{2\Phi} \delta c$, is relaxing to zero with a $1/t^{(D-2)}_E$ dependence on the Einstein-frame time for each of the equilibrium values of $\delta c$. Therefore, the breaking of supersymmetry induced by the linear dilaton is only an obstruction [30], rather than a spontaneous breaking, in the sense that it appears only temporarily in the boson-fermion mass splittings between the excitations, whilst the vacuum energy of the asymptotic equilibrium theory vanishes.

4.2 Pre Big Bang Scenarios

After the work of [28], dilaton cosmology has been discussed in a plethora of interesting works, most of them associated with the so-called ‘pre-Big-Bang’ (pBB) cosmologies [14], suggested by Veneziano, and pursued further by Gasperini, Veneziano and collaborators. For the interested reader, this type of cosmology has been reviewed by the author in the first Aegean School [3], where I refer the interested reader for more details.

The basic feature behind the approach, is the fact that the dilaton has such time dependence in these models that, as the cosmic time elapses, the string coupling $g_s = e^\Phi$ grows stronger at late stages of the Universe. The dilaton potential in the pre Big-Bang approach, which may be generated by higher string loops, has the generic form depicted in fig. 3 [14]. The situation is opposite that of [28], where as we have seen the string coupling becomes weaker with the cosmic time, and perturbative strings are sufficient for a description of the Universe at late epochs.
Fig. 3. The dilaton potential in the pre Big-Bang scenario of string cosmology. The string coupling grows strong at late times, and hence current-era is described by strongly-coupled strings, where higher string loop corrections matter.

I will not discuss in detail the pBB theories, since there are excellent reviews on the subject [14], where I refer the interested reader for more details. For our purposes here, I would like to emphasize the basic predictions of this model regarding the rôle of dilaton as a quintessence field, responsible for late-time acceleration.

The starting point is the string-frame, low-energy, string-inspired effective action with graviton and dilaton backgrounds [4], to lowest order in the $\alpha' \exp$ expansion, but including dilaton-dependent loop (and non-perturbative) corrections, which are essential given that at late epochs the dilaton grows strong in pBB scenarios. Such corrections are encoded in a few “form factors” [14] $\psi(\phi)$, $Z(\phi)$, $\alpha(\phi)$, ..., and in an effective dilaton potential $V(\phi)$. The effective action reads:

$$S = -\frac{M_P^2}{2} \int d^4 x \sqrt{-g} \left[ e^{-\psi(\phi)} \bar{R} + Z(\phi) \left( \nabla_\phi \right)^2 + \frac{2}{M_P^2} V(\phi) \right]$$

$$- \frac{1}{16\pi} \int d^4 x \sqrt{-g} \frac{\alpha(\phi)}{F_{\mu\nu}^2 + \Gamma_m(\phi, g, \text{matter})}$$

where we follow the conventions of [14].

The four dimensional action above is the result of compactification. It is also assumed that the the corresponding moduli have been frozen at the string scale. In the approach of [14] it is assumed that the form factors $\psi(\phi)$, $Z(\phi)$, $\alpha(\phi)$ approach a finite limit as $\phi \rightarrow +\infty$ while, in the same limit, $V \rightarrow 0$. The fields appearing in the matter action $\Gamma_m$ are in general non-minimally and non-universally coupled to the dilaton (also because of the loop corrections).

In the Einstein frame the action (27) becomes

$$S = -\frac{M_P^2}{2} \int d^4 x \sqrt{-g} \left[ R - \frac{k(\phi)^2}{2} (\nabla \phi)^2 + \frac{2}{M_P^2} \hat{V}(\phi) \right]$$
\[-\frac{1}{16\pi} \int d^4 x \sqrt{-g} F_{\mu\nu}^2 + \Gamma_m(\phi, c_1^2 g_{\mu\nu} e^\psi, \text{matter}) , \tag{28}\]

where
\[k^2(\phi) = 3\psi'^2 - 2e^\psi Z, \quad \hat{V} = c_1^2 e^{2\psi} V. \tag{29}\]

The pertinent equations of motion for the graviton field read:
\[6H^2 = \rho + \rho_\phi, \tag{30}\]
\[4\dot{H} + 6H^2 = -p - p_\phi, \tag{31}\]

while the dilaton equation is:
\[k^2(\phi) \left( \ddot{\phi} + 3H \dot{\phi} \right) + k(\phi) k'(\phi) \dot{\phi}^2 + \hat{V}'(\phi) + \frac{1}{2} [\psi'(\phi)(\rho - 3p) + \sigma] = 0. \tag{32}\]

In the above equations \(H = \dot{a}/a\), a dot denotes differentiation with respect to the Einstein cosmic time, and we have used the definitions:
\[\rho_\phi = \frac{1}{2} k^2(\phi) \dot{\phi}^2 + \hat{V}(\phi), \quad \rho_\phi = \frac{1}{2} k^2(\phi) \dot{\phi}^2 - \hat{V}(\phi). \tag{33}\]

After some manipulations the pertinent equations of motion, describing the dynamics of the system, read:
\[2H^2 k^2 \frac{d^2 \phi}{d\chi^2} + k^2 \left( \frac{1}{2} \rho_m + \frac{1}{3} \rho_r + \hat{V} \right) \frac{d\phi}{d\chi} + 2H^2 k k' \left( \frac{d\phi}{d\chi} \right)^2 + 2\hat{V}' + \psi' \rho_m + \sigma = 0 , \tag{34}\]

where \(\chi = \ln a\), with \(a\) the scale factor in units of the present day scale.

The matter evolution equation, on the other hand, can be split into the various components (radiation (r), baryonic (d)):
\[\frac{d\rho_r}{d\chi} + 6\rho_r - \frac{\sigma_r \dot{\phi}}{2} = 0 , \tag{35}\]
\[\frac{d\rho_r}{d\chi} + 3\rho_b - \frac{1}{2} (\psi' \rho_b + \sigma_b) \frac{d\phi}{d\chi} = 0 , \tag{35}\]
\[\frac{d\rho_d}{d\chi} + 3\rho_d - \frac{1}{2} (\psi' \rho_d + \sigma_d) \frac{d\phi}{d\chi} = 0. \tag{35}\]

And for the dilaton-energy density \(\rho_\phi\) one can obtain the equation
\[\frac{d\rho_\phi}{d\chi} + 6\rho_\phi - 6\hat{V}(\phi) + \frac{1}{2} (\psi' \rho_m + \sigma) \frac{d\phi}{d\chi} = 0. \tag{36}\]
Fig. 4. Time evolution of $\rho_\phi$ for $q = 0$ (dash-dotted curve), $q = 0.01$ (dashed curve) and $q = 0.1$ (dotted curve). The initial scale is $a_i = 10^{-20}a_{eq}$, and the epoch of matter-radiation equality corresponds to $\chi \simeq 46$. Left panel: the dilaton energy density is compared with the radiation (thin solid curve) and matter (bold solid curve) energy density. Right panel: the dilaton energy density (in critical units) is compared with the analytical estimates for the focusing and dragging phases.

The analysis of [14], based on these equations, leads to predictions regarding the behaviour of the various cosmological parameters of the pBB dilaton cosmology.

Under various approximations and assumptions, which I will not go through, but I would stress that they are due to the fact that the various form factors and the dilaton couplings to matter are not known in this approach due to the (uncontrolled) loop corrections, one can obtain the asymptotic evolution of the Hubble factor and of the dominant energy density in this approach,

$$H \sim a^{-3/(2+q)}, \quad \rho \sim a^{-6/(2+q)},$$

where $q = \mathcal{O}(1)$ and is expressed in terms of the various energy densities in the model $q = q = 0.1 + \Delta \phi - \Delta \phi^* - \Delta \phi^2$.

The evolution of the various cosmological parameters in a typical of such pBB models is given in figures 4 and 5, taken from the second ref. in [14].

4.3 Non Critical Strings and Dark Energy

Pre Big Bang scenarios, as we have discussed, involve strong string couplings, and hence the various form factors appearing in the effective actions are unknown.

An alternative approach, is to invoke the weak coupling late-era dilaton cosmology of [28], which has the advantage that at late eras perturbative $\sigma$-model calculations are reliable, and hence one can perform concrete computations and predictions. The analysis of [28] however has to be generalised to include inflationary and other backgrounds with horizons, if the dark matter issue and accelerating Universes are to be tackled. This cannot be achieved with the simple linear dilaton backgrounds of [28].
In [10] we went one step beyond the analysis in [28], and considered more complicated σ-model metric backgrounds that did not satisfy the σ-model conformal-invariance conditions, and therefore needed Liouville dressing [9] to restore conformal invariance. Such backgrounds could even be time-dependent, living in \((d+1)\)-dimensional target space-times. Various mathematically-consistent forms of non-criticality can be considered, for instance cosmic catastrophes such as the collision of brane worlds [31,32]. Such models lead to supercriticality of the associated σ models describing stringy excitations on the brane worlds. The Liouville dressing of such non-critical models results in \((d+2)\)-dimensional target spaces with two time directions. An important point in [10] was the identification of the (world-sheet zero mode of the) Liouville field with the target time, thereby restricting the Liouville-dressed σ model to a \((d+1)\)-dimensional hypersurface of the \((d+2)\)-dimensional target space, thus maintaining the initial target space-time dimensionality. We stress that this identification is possible only in cases where the initial σ model is supercritical, so that the Liouville mode has time-like signature [9,28]. In certain models [31,32], such an identification was proven to be energetically preferable from a target-space viewpoint, since it minimized certain effective potentials in the low-energy field theory corresponding to the string theory at hand.

All such cosmologies require some physical reason for the initial departure from the conformal invariance of the underlying σ model that describes string excitations in such Universes. The reason could be an initial quantum fluctuation, or, in brane models, a catastrophic cosmic event such as the collision of two or more brane worlds. Such non-critical σ models relax asymptotically to conformal σ models, which may be viewed as equilibrium points in string theory space, as illustrated in Fig. 6. In some interesting cases of relevance to...
Fig. 6. A schematic view of string theory space, which is an infinite-dimensional manifold endowed with a (Zamolodchikov) metric. The dots denote conformal string backgrounds. A non-conformal string flows (in a two-dimensional renormalization-group sense) from one fixed point to another, either of which could be a hypersurface in theory space. The direction of the flow is irreversible, and is directed towards the fixed point with a lesser value of the central charge, for unitary theories, or, for general theories, towards minimization of the degrees of freedom of the system.

cosmology [15], which are particularly generic, the asymptotic conformal field theory is that of [28] with a linear dilaton and a flat Minkowski target-space metric in the $\sigma$-model frame. In others, the asymptotic theory is characterized by a constant dilaton and a Minkowskian space-time [31]. Since, as we discussed in [10] and review briefly below, the evolution of the central-charge deficit of such a non-critical $\sigma$ model, $Q^2(t)$, plays a crucial rôle in inducing the various phases of the Universe, including an inflationary phase, graceful exit from it, thermalization and a contemporary phase of accelerating expansion, we term such Liouville-string-based cosmologies $Q$-Cosmologies.

The use of Liouville strings to describe the evolution of our Universe has a broad motivation, since non-critical strings are associated with non-equilibrium situations, as are likely to have occurred in the early Universe. The space of non-critical string theories is much larger than that of critical strings. It is therefore remarkable that the departure from criticality may enhance the predictability of string theory to the extent that a purely stringy quantity such as the string coupling $g_s$ may become accessible to experiment via its relation to the present-era cosmic acceleration parameter: $g_s^2 = -q^0$ [33]. Another example arises in a non-critical string approach to inflation, if the Big Bang is identified with the collision of two D-branes [32]. In such a scenario, astrophysical observations may place important bounds on the recoil velocity of the brane worlds after the collision, and lead to an estimate of the separation of the branes at the end of the inflationary period.

In such a framework, the identification of target time with a world-sheet renormalization-group scale, the zero mode of the Liouville field [10], provides a novel way of selecting the ground state of the string theory. This is not necessarily associated with minimization of energy, but could simply be a re-
sult of cosmic chance. It may be a random global event that the initial state of our cosmos corresponds to a certain Gaussian fixed point in the space of string theories, which is then perturbed into a Big Bang by some relevant (in a world-sheet sense) deformation, which makes the theory non-critical, and hence out of equilibrium from a target space-time viewpoint. The theory then flows, as indicated in Fig. 6, along some specific renormalization-group trajectory, heading asymptotically to some ground state that is a local extremum corresponding to an infrared fixed point of this perturbed world-sheet σ-model theory. This approach allows for many ‘parallel universes’ to be implemented, and our world might be just one of these. Each Universe may flow between different fixed points, its trajectory following a perturbation by a different operator. It seems to us that this scenario is more attractive and specific than the landscape scenario [27], which has recently been advocated as a framework for parametrizing our ignorance of the true nature of string/M theory.

Let us briefly review the basic formalism. We consider a σ-model action deformed by a family of vertex operators $V_i$, corresponding to ‘couplings’ $g_i$, which represent non-conformal background space-time fields from the massless string multiplet, such as gravitons, $G_{\mu \nu}$, antisymmetric tensors, $B_{\mu \nu}$, dilatons $\Phi$, their supersymmetric partners, etc.:

$$S = S_0(X) + \sum_i g_i \int d^2 z V_i(X) ,$$

(38)

where $S_0$ represents a conformal σ model describing an equilibrium situation. The non-conformality of the background means that the pertinent $\beta^i$ function $\beta^i \equiv d g^i / d \ln \mu \neq 0$, where $\mu$ is a world-sheet renormalization scale. Conformal invariance would imply restrictions on the background fields/σ-model couplings, $g^i$, corresponding to the constraints $\beta^i = 0$, which are equivalent to equations of motion derived from a target-space effective action for the corresponding fields $g^i$. The entire low-energy phenomenology and model building of critical string theory is based on such restrictions [4].

In the non-conformal case $\beta^i \neq 0$, the theory is in need of dressing by the Liouville field $\phi$ in order to restore conformal symmetry [9]. The field $\phi$ acquires dynamics through the integration over world-sheet covariant metrics in the path integral, and may be viewed as a local dynamical scale on the world sheet [10]. If the central charge of the (supersymmetric) matter theory is $c_m > 25(9)$ (i.e., supercritical), the signature of the kinetic term of the Liouville coordinate in the dressed σ-model is opposite to that of the σ-model fields corresponding to the other target-space coordinates. As mentioned previously, this opens the way to the important step of interpreting the Liouville field physically by identifying its world-sheet zero mode $\phi_0$ with the target time in supercritical theories [10]. Such an identification emerges naturally from the dynamics of the target-space low-energy effective theory by minimizing the effective potential [31].

In terms of the Liouville renormalization-group scale, one has the following equation relating Liouville-dressed couplings $g^i$ and $\beta$ functions in the non-
critical string case:

\[ \ddot{g}^i + Q \dot{g}^i = \mp \beta^i(g_j), \tag{39} \]

where the - (+) sign in front of the \( \beta \)-functions on the right-hand-side applies to super(sub)critical strings, the overdot denotes differentiation with respect to the Liouville zero mode, \( \beta^i \) is the world-sheet renormalization-group \( \beta \) function (but with the renormalized couplings replaced by the Liouville-dressed ones as defined by the procedure in [9]), and the minus sign on the right-hand side (r.h.s.) of (39) is due to the time-like signature of the Liouville field. Formally, the \( \beta^i \) of the r.h.s. of (39) may be viewed as power series in the (weak) couplings \( g^i \). The covariant (in theory space) \( G_{ij} \beta^j \) function, with \( G_{ij} \) the (Zamolodchikov) metric, may be expanded as:

\[ G_{ij} \beta^j = \sum_{i_n} \langle V_i V_i \ldots V_i \rangle_{\phi} g^i \ldots g^i, \tag{40} \]

where \( V_i \) indicates Liouville dressing as in [9] \( \langle \ldots \rangle_{\phi} = \int d\phi dr \exp(-S(\phi, r, g^i)) \) denotes a functional average including Liouville integration, and \( S(\phi, r, g^i) \) is the Liouville-dressed \( \sigma \)-model action, including the Liouville action [9].

In the case of stringy \( \sigma \) models, the diffeomorphism invariance of the target space results in the replacement of (39) by:

\[ \ddot{g}^i + Q(t) \dot{g}^i = \mp \tilde{\beta}^i, \tag{41} \]

where the \( \tilde{\beta}^i \) are the Weyl anomaly coefficients of the stringy \( \sigma \) model in the background \( \{g^i\} \), which differ from the ordinary world-sheet renormalization-group \( \beta^i \) functions by terms of the form:

\[ \tilde{\beta}^i = \beta^i + \delta g^i \tag{42} \]

where \( \delta g^i \) denote transformations of the background field \( g^i \) under infinitesimal general coordinate transformations, e.g., for gravitons [4] \( \beta^G_{\mu\nu} = \beta^G_{\mu\nu} + \nabla_{(\mu} W_{\nu)} \), with \( W_{\mu} = \nabla_{\mu} \Phi \), and \( \beta^G_{\mu\nu} = R_{\mu\nu} \) to order \( \alpha' \) (one \( \sigma \)-model loop).

The set of equations (39),(41) defines the \textit{generalized conformal invariance conditions}, expressing the restoration of conformal invariance by the Liouville mode. The solution of these equations, upon the identification of the Liouville zero mode with the original target time, leads to constraints in the space-time backgrounds [10, 31], in much the same way as the conformal invariance conditions \( \beta^i = 0 \) define consistent space-time backgrounds for critical strings [4]. It is important to remark [10] that the equations (41) can be derived from the \textit{variation of an off shell} action. This follows from general properties of the Liouville renormalization group, which guarantee that the appropriate Helmholtz conditions in the string-theory space \( \{g^i\} \) for the Liouville-flow dynamics to be derivable from an action principle are satisfied.

When applied to dilaton cosmologies, with dilaton and graviton backgrounds, this approach yields interesting results, including a modified asymptotic scaling of the dark matter energy density, \( a^{-2} \) with the scale factor, as
well as an expression of the current-era acceleration parameter of the Universe roughly proportional to the square of the string coupling, $q_0 \propto -(g_0^s)^2$, $g_0^s = e^{2\Phi}$, with $\Phi$ the current era dilaton (this proportionality relation becomes exact at late eras, when the matter contributions become negligible due to cosmic dilution). The current-era dark energy in this framework relaxes to zero with the Einstein cosmic time as $1/t^2$, and this scaling law follows from the generalised conformal invariance conditions (41), characterising the Liouville theory, as well as the identification of time with the Liouville mode [10].

To be specific, after this identification, the relevant Liouville equations (41) for dilaton and graviton cosmological backgrounds, in the Einstein frame [28], read [33]:

$$3 H^2 - \tilde{\varrho}_m - \varrho_\phi = \frac{e^{2\phi}}{2} \tilde{\varrho}_\phi$$

$$2 \dot{H} + \dot{\varrho}_m + \varrho_\phi + \dot{p}_m + p_\phi = \frac{\tilde{\varrho}_{ii}}{a^2}$$

$$\dot{\phi} + 3H\dot{\phi} + \frac{1}{4} \frac{\partial \tilde{V}_{\text{all}}}{\partial \phi} + \frac{1}{2} (\dot{\varrho}_m - 3\dot{p}_m) = -\frac{3}{2} \frac{\tilde{\varrho}_{ii}}{a^2} - \frac{e^{2\phi}}{2} \tilde{\varrho}_\phi .$$  \hspace{1cm} (43)

where $\dot{\varrho}_m$ and $\dot{p}_m$ denote the matter energy density and pressure respectively, including dark matter contributions. As usual, the overdot denotes derivatives with respect to the Einstein time, and $H$ is the Hubble parameter of the Robertson-Walker Universe. The r.h.s of the above equations denotes the non-critical string off-shell terms, due to the non-equilibrium nature of the pertinent cosmology. The latter could be due to an initial cosmically catastrophic event, such as the collision of two brane worlds:

$$\tilde{\varrho}_\phi = e^{-2\phi} (\dot{\phi} - \dot{\phi}^2 + Q e^\phi \dot{\phi})$$

$$\tilde{\varrho}_{ii} = 2 a^2 (\dot{\phi} + 3H\dot{\phi} + \dot{\phi}^2 + (1 - q)H^2 + Q e^\phi (\dot{\phi} + H)) .$$  \hspace{1cm} (44)

Notice the dissipative terms proportional to $Q \dot{\phi}$, which are responsible for the terminology “Dissipative Cosmology” used alternatively for Q-cosmology [33]. In these equations, $q$ is the deceleration $q \equiv -\ddot{a}/\dot{a}^2$. The potential appearing in (43) is defined by $\tilde{V}_{\text{all}} = 2Q^2 \exp (2\phi) + V$ where, for the sake of generality, we have allowed for an additional potential term in the string action $-\sqrt{-G} V$.

A brief summary of the results of our analysis for a model-case Q-cosmology, are presented in figs. 7, 8, 9 and 10. The model is discussed in some detail in ref. [33]. Notice the late-era presence of exotic $a^{-2}$-scaling of matter species, attributed to dark matter, denoted by $\rho_\text{c}$ in the figures.

The reader is invited to compare these results with the ones of critical-string dilaton cosmologies in pre-Big-bang scenaria presented above (c.f. figs. 4 and 5), in particular with respect to the effects of the non-critical, off-shell terms “$G$”, which appear significant at the current era [33].

An important result of the analysis of [33] is the fact that the conventional Boltzmann equation, controlling the evolution of species densities, needs to
The issue of Dark Energy in String Theory

The dilaton \( \phi \), the (square root of the) central charge deficit \( Q \) and the ratio \( a/a_0 \) of the cosmic scale factor as functions of the Einstein time \( t_{\text{Einstein}} \). The present time is located where \( a/a_0 = 1 \) and in the figure shown corresponds to \( t_{\text{today}} \approx 1.07 \). The input values for the densities are \( \rho_b = 0.238, \rho_e = 0.0 \) and \( w_e \) is 0.5. The dilaton value today is taken \( \phi = 0.0 \). Right panel: The values of \( \Omega_i \equiv \rho_i/\rho_c \) for the various species as functions of \( t_{\text{Einstein}} \).

be modified in Q-cosmology, in order to incorporate consistently the effects of the dilaton dissipative pressure \( \sim \dot{\phi} \) and the non-critical (relaxation) terms, \( "G" \):

\[
\frac{dn}{dt} = -3 \, H \, n - <\sigma v> (n^2 - n_{eq}^2) + \dot{\phi} \, n + "G/m_X" . \tag{45}
\]

The respective relic density of the species \( X \), with mass \( m_X \), is then obtained from \( \Omega_X \, h_0^2 = n \, m_X \, h_0^2 \), after solving this modified equation. This may have important phenomenological consequences, in particular when obtaining constraints on supersymmetric particle-physics models from astrophysical data.
Fig. 9. Left panel: The deceleration $q$ and the dimensionless Hubble expansion rate $\dot{H} \equiv \frac{\dot{H}}{\sqrt{3} H_0}$ as functions of $t_{Einstein}$. Right panel: The derivative of the dilaton and its ratio to the dimensionless expansion rate.

Fig. 10. Left panel: The ratio $|q|/g_s^2$ as function of the redshift for $z$ ranging from $z = 0.2$ to future values $z = -0.6$, for the inputs discussed in the main text. The rapid change near $z \approx 0.16$ signals the passage from deceleration to the acceleration period. Right panel: The values of the string coupling constant plotted versus the redshift in the range $z = 0.0 - 1.0$.

We shall not discuss these issues further here, due to lack of space. For more details we refer the interested reader to the literature [10, 33].

5 Conclusions

In this work we have reviewed various issues related to the consistent incorporation of Dark Energy in string theory. We have discussed only traditional string theory and did not cover the modern extension, including membranes. This topic has been covered by other lecturers in the School.

One of the most important issues concerns de Sitter space, and in general space-times with horizons in string theory. We have studied general properties, including holographic scenaria, which may be the key to an inclusion of such
space times in the set of consistent (possibly non-perturbative) ground states of strings.

We have also seen that perturbative strings are incompatible with space times with horizons, mainly due to the lack of a scattering matrix. However, non-critical strings may evade this constraint, and we have discussed briefly how accelerating universes can be incorporated in non-critical (Liouville) strings. The use of Liouville strings to describe the evolution of our Universe is natural, since non-critical strings are associated with non-equilibrium situations which undoubtedly occurred in the early Universe.

The dilaton played an important rôle in string cosmology, and we have seen how it can act as a quintessence field, responsible for the current-era acceleration of the Universe.

There are many phenomenological tests of this class of cosmologies that can be performed, which the generic analysis presented here is not sufficient to encapsulate. Tensor perturbations in the cosmic microwave background radiation is one of them. The emission of gravitational degrees of freedom from the hot brane to the cold bulk, during the inflationary and post-inflationary phases in models involving brane worlds is something to be investigated in detail. A detailed knowledge of the dependence of the equation of state on the redshift is something that needs to be looked at in the context of specific models. Moreover, issues regarding the delicate balance of the expansion of the Universe and nucleosynthesis, which requires a very low vacuum energy, must be resolved in specific, phenomenologically semi-realistic models, after proper compactification to three spatial dimensions, in order that the conjectured cosmological evolution has a chance of success.

Finally, the compactification issue per se is a most important part of a realistic stringy cosmology. In our discussion above, we have assumed that a consistent compactification takes place, leading to effective four-dimensional string-inspired equations of motion. In realistic scenarios, however, details of how the extra dimensions are compactified play a key rôle in issues like supersymmetry breaking.

In this review I did not discuss higher-curvature modifications of the low-energy Einstein action, which characterise all string-inspired models, including brane worlds scenarios. Such terms may play an important rôle in Early Universe cosmology. For instance, they may imply initial singularity-free string cosmologies [34], or non-trivial black hole solutions with (secondary) dilaton hair [35], which can play a rôle in the Early universe sphaleron transitions. So, before closing the lecture, I will devote a few words on their form.

In ordinary string theory, which is the subject of the present lecture, such higher-order terms possess ambiguous coefficients in the effective action. This is a result of local field redefinitions, which leave the (low-energy) string scattering amplitudes invariant, and hence cannot be determined by low energy considerations. In ordinary string theory [4], with no space-time boundaries in (the low-energy) target space time, such ambiguities imply that the so-called ghost-free Gauss-Bonnet combination $\frac{1}{g_s} (R_{\mu\nu\rho\sigma}^2 - 4 R_{\mu\nu}^2 + R^2)$, with $g_s = e^\Phi$
the string coupling and $\phi$ the dilaton field, can always be achieved for the quadratic curvature terms in the string-inspired low-energy effective action, which constitutes the first non trivial order corrections to Einstein term in bosonic and heterotic string effective actions.

However, in the case of brane worlds, with closed strings propagating in the bulk, things are not so simple. As discussed in [36], field redefinition ambiguities for the bulk low-energy graviton and dilaton fields, that would otherwise leave bulk string scattering amplitudes invariant, induce brane (boundary) curvature and cosmological constant terms, with the unavoidable result of ambiguities in the terms defining the Einstein and cosmological constant terms on the brane. This results in (perturbative in $\alpha'$) ambiguities in the cross-over scale of four-dimensional brane gravity, as well as the brane vacuum energy. It is not clear to me, however, whether these ambiguities are actually present in low-energy brane world scenarios. I believe that these bulk-string ambiguities can be eliminated once the brane effective theory is properly defined, given that closed and open strings also propagate on the brane world hypersurfaces, and thus are characterised by their own scattering amplitudes. Matching these two sets of scattering amplitudes properly, for instance by looking at the conformal theory describing the splitting of a closed-string bulk state, crossing a brane boundary, into two open string excitations on the brane, may lead to unambiguous brane cross-over and cosmological constant scales, expressed in terms of the bulk string scale and coupling. These are issues that I believe deserve further investigation, since they affect early Universe cosmologies, where such higher-curvature terms are important. I will not, however, discuss them further here.

I would like to close this lecture with one more remark on the non-equilibrium Liouville approach to cosmology advocated in [10, 33], and discussed last in this article. This approach is based exclusively on the treatment of target time as an irreversible dynamical renormalization-group scale on the world sheet of the Liouville string (the zero mode of the Liouville field itself). This irreversibility is associated with fundamental properties of the world-sheet renormalization group, which lead in turn to the loss of information carried by two-dimensional degrees of freedom with world-sheet momenta beyond the ultraviolet cutoff [29] of the world-sheet theory. This fundamental microscopic time irreversibility may have other important consequences, associated with fundamental violations of CPT invariance [22] in both the early Universe and the laboratory, providing other tests of these ideas.

Acknowledgements
It is my pleasure to thank the organisers, and especially E. Papantonopoulos, for the invitation to lecture in this very interesting school and workshop. This work is partially supported by funds made available by the European Social Fund (75%) and National (Greek) Resources (25%) - (EPEAEK II) - PYTHAGORAS.
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