Estimates for norms of two-weighted summation operators on trees for $1 < p < q < \infty$

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Abstract

In this paper, estimates for norms of weighted summation operators (discrete Hardy-type operators) on a tree are obtained for $1 < p < q < \infty$ and for arbitrary weights and trees.

1 Introduction

First we give some notation.

Throughout this paper we consider graphs $\mathcal{G}$ with finite or countable vertex set, which will be denoted by $V(\mathcal{G})$. Also we suppose that the graphs have neither multiple edges nor loops. Given a function $f : V(\mathcal{G}) \to \mathbb{R}$ and a number $1 < p < \infty$, we set

$$\|f\|_{l^p(\mathcal{G})} = \left( \sum_{\xi \in V(\mathcal{G})} |f(\xi)|^p \right)^{1/p}.$$ 

Denote by $l^p(\mathcal{G})$ the space of functions $f : V(\mathcal{G}) \to \mathbb{R}$ with finite norm $\|f\|_{l^p(\mathcal{G})}$.

Let $\mathcal{T} = (\mathcal{T}, \xi_0)$ be a tree rooted at $\xi_0$. We introduce a partial order on $V(\mathcal{T})$ as follows: we say that $\xi' > \xi$ if there exists a simple path $(\xi_0, \xi_1, \ldots, \xi_n, \xi')$ such that $\xi = \xi_k$ for some $k \in \{0, \ldots, n\}$; by the distance between $\xi$ and $\xi'$ we mean the quantity $\rho(\xi, \xi') = \rho(\xi', \xi) = n + 1 - k$. In addition, we set $\rho(\xi, \xi) = 0$. If $\xi' > \xi$ or $\xi' = \xi$, we write $\xi' \geq \xi$. For $j \in \mathbb{Z}_+$ and $\xi \in V(\mathcal{T})$, let

$$V_j(\xi) := \{ \xi' \geq \xi : \rho(\xi, \xi') = j \}.$$ 

Given $\xi \in V(\mathcal{T})$, we denote by $\mathcal{T}_\xi = (\mathcal{T}_\xi, \xi)$ the subtree in $\mathcal{T}$ with the vertex set

$$V(\mathcal{T}_\xi) = \{ \xi' \in V(\mathcal{T}) : \xi' \geq \xi \}.$$ 

Let $W \subset V(\mathcal{T})$. We say that $\mathcal{G} \subset \mathcal{T}$ is a maximal subgraph on the vertex set $W$ if $V(\mathcal{G}) = W$ and if any two vertices $\xi', \xi'' \in W$ that are adjacent in $\mathcal{T}$ are also adjacent in $\mathcal{G}$. Given $\xi, \xi' \in V(\mathcal{T})$, $\xi \leq \xi'$, we denote by $[\xi, \xi']$ the maximal subgraph on the vertex set $\{ \eta \in V(\mathcal{T}) : \xi \leq \eta \leq \xi' \}$. 

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Let \( \mathcal{G} \) be a subgraph in \((T, \xi_0)\). Denote by \( V_{\text{max}}(\mathcal{G}) \) and \( V_{\text{min}}(\mathcal{G}) \) the set of maximal and minimal vertices in \( \mathcal{G} \), respectively.

Let \((T, \xi_0)\) be a tree, and let \( u, w : V(T) \to [0, \infty) \) be weight functions. Define the summation operator \( S_{u,w,T} \) by

\[
S_{u,w,T}f(x) = w(x) \sum_{\xi' \leq \xi} u(\xi') f(\xi'), \quad x \in V(T), \quad f : V(T) \to \mathbb{R}.
\]

Given \( 1 < p < q < \infty \), by \( \mathcal{G}_{T,u,w}^{p,q} \) we denote the operator norm of \( S_{u,w,T} : l_p(T) \to l_q(T) \), which is the minimal constant \( C \) in the inequality

\[
\left( \sum_{\xi \in V(T)} w^q(\xi) \left| \sum_{\xi' \leq \xi} u(\xi') f(\xi') \right|^q \right)^{1/q} \leq C \left( \sum_{\xi \in V(T)} \|f(\xi)\|^p \right)^{1/p}, \quad f : V(T) \to \mathbb{R}.
\]

Let \( X, Y \) be arbitrary sets, \( f_1, f_2 : X \times Y \to \mathbb{R}_+ \). We write \( f_1(x, y) \lesssim f_2(x, y) \) (or \( f_2(x, y) \gtrsim f_1(x, y) \)) if, for any \( y \in Y \), there exists \( c(y) > 0 \) such that \( f_1(x, y) \leq c(y) f_2(x, y) \) for each \( x \in X \); \( f_1(x, y) \asymp f_2(x, y) \) if \( f_1(x, y) \lesssim f_2(x, y) \) and \( f_2(x, y) \lesssim f_1(x, y) \).

**Theorem 1.** Let \((\mathcal{A}, \xi_0)\) be a tree, and let \( u, w : V(\mathcal{A}) \to \mathbb{R}_+ \). Suppose that \( 1 < p < q < \infty \). Then

\[
\mathcal{G}_{\mathcal{A},u,w}^{p,q} \asymp \sup_{p,q} \|u\|_{l_p([\xi_0, \xi])} \|w\|_{l_q(\mathcal{A})}.
\]

In [1] this result was proved under some restrictions on weights (see Theorems 1.2 and 3.6).

Given \( f : V(\mathcal{A}) \to \mathbb{R} \), we set

\[
\|f\|_{l_q(l_p(\mathcal{A}))} = \left( \sum_{j=0}^{\infty} \left( \sum_{\xi \in V_j^A(\xi_0)} |f(\xi)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.
\]

By \( \mathcal{G}_{\mathcal{A},u,w}^{p,q} \) we denote the operator norm of \( S_{u,w,A} : l_q(l_p(\mathcal{A})) \to l_q(\mathcal{A}) \). If \( p \leq q \), then \( \|f\|_{l_p(\mathcal{A})} \geq \|f\|_{l_q(l_p(\mathcal{A}))} \) and

\[
\mathcal{G}_{\mathcal{A},u,w}^{p,q} \geq \mathcal{G}_{\mathcal{A},u,w}^{p,q}.
\]

Let

\[
u(\xi) = u_j, \quad w(\xi) = w_j, \quad \xi \in V_j^A(\xi_0).
\]

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In addition, we suppose that there exist a number $C_* \geq 1$ and a function $S : \mathbb{Z}_+ \rightarrow (0, \infty)$ satisfying the following conditions:

$$C_*^{-1} \frac{S(j')}{S(j)} \leq \text{card } V_{j-j'}^A(\xi) \leq C_* \frac{S(j')}{S(j)}, \quad \xi \in V_j^A(\xi_0), \quad j' \geq j;$$

$$S(0) = 1;$$

there exist $R_0 \geq R > 1$ such that

$$R_0 \geq \frac{S(j + 1)}{S(j)} \geq R.$$

**Theorem 2.** Suppose that $1 < p < q < \infty$ and conditions (2), (3), (4), (5) hold. Then

$$\hat{\mathcal{G}}_{p,q,A,u,w} = \sup_{j \in \mathbb{Z}_+} u_j \left( \sum_{i \geq j} w_i \frac{S(i)}{S(j)} \right)^{\frac{1}{q}} = \sup_{\xi \in V(A)} u(\xi) \|w\|_{l_q(A_{\xi})}.$$ 

2 **Proof of Theorem 1**

The lower estimate for $\mathcal{G}_{p,q,A,u,w}$ was obtained in [1, Lemma 3.3]. In addition, the following result was proved (see [1, Lemma 3.1]).

**Lemma 1.** Let $1 < p < q < \infty$. Then there exists $\sigma = \sigma(p, q) \in (0, \frac{1}{8})$ with the following property: if $(\mathcal{A}, \xi_0)$ is a tree with finite vertex set, $u, w : V(\mathcal{A}) \rightarrow (0, \infty)$,

$$\frac{\|w\|_{l_q(A_0)}}{\|w\|_{l_q(A_{\xi})}} \leq \sigma \quad \text{for any} \quad \xi \in V(\mathcal{A}), \quad \eta \in V_1^A(\xi),$$

then $\hat{\mathcal{G}}_{p,q,A,u,w} = \sup_{\xi \in V(\mathcal{A})} u(\xi) \|w\|_{l_q(A_{\xi})}$.

Lemma 1 was proved by induction; here the discrete analogue of Evans – Harris – Pick theorem [2] was applied.

Let $\{T_j\}_{j \in \mathbb{N}}$ be a family of subtrees in $\mathcal{T}$ such that $V(T_j) \cap V(T_{j'}) = \emptyset$ for $j \neq j'$ and $\cup_{j \in \mathbb{N}} V(T_j) = V(\mathcal{T})$. Then $\{T_j\}_{j \in \mathbb{N}}$ is called a partition of the tree $\mathcal{T}$. Let $\xi_j$ be the minimal vertex of $T_j$. We say that the tree $T_s$ succeeds the tree $T_j$ (or $T_j$ precedes the tree $T_s$) if $\xi_j < \xi_s$ and

$$\{\xi \in \mathcal{T} : \xi_j \leq \xi < \xi_s\} \subset V(T_j).$$

**Proof of Theorem 1**. By B. Levi’s theorem, without loss of generality we may assume that the tree $\mathcal{A}$ has a finite vertex set. In addition, we may consider only strictly positive weight functions $u, w$.  

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Let \( \sigma = \sigma(p, q) \in (0, \frac{1}{8}) \) be as defined in Lemma 11. Given \( \xi \in V(A) \), we denote
\[
V_{\xi, \sigma} = \{ \eta \geq \xi : \|w\|_{l_q(A_\eta)} \geq \sigma \|w\|_{l_q(A_\xi)} \}. 
\] (6)
Then \( \xi \in V_{\xi, \sigma} \). Denote by \( A_{\xi, \sigma} \) the maximal subgraph in \( A \) on the vertex set \( V_{\xi, \sigma} \). Notice that \( A_{\xi, \sigma} \) is a tree.

We claim that
\[
\|u\|_{l_{p'}(A_{\xi, \sigma})} \|w\|_{l_q(A_{\xi, \sigma})} \leq \sup_{p, q, \eta \in V(A)} \|u\|_{l_{p'}(V_{\xi, \eta})} \|w\|_{l_q(A_{\eta})}. 
\] (7)
Indeed, let \( \eta \in V(A_{\xi, \sigma}) \). Then there exists a vertex \( \zeta_{\eta} \in V_{\max}(A_{\xi, \sigma}) \) such that \( \eta \in [\xi, \zeta_{\eta}] \). Indeed, otherwise for any vertex \( \zeta \in V(A_{\xi, \sigma}) \cap V(A_\eta) \), the set \( V(A_{\xi, \sigma}) \cap V_1^A(\zeta) \) is nonempty. Hence, we can construct an infinite chain \( \zeta_1 < \zeta_2 < \zeta_3 < \ldots \), \( \zeta_j \in V(A_{\xi, \sigma}) \). This contradicts with the assumption that the set \( V(A) \) is finite.

Thus, \( V(A_{\xi, \sigma}) = \cup_{\xi \in V_{\max}(A_{\xi, \sigma})}[\xi, \zeta] \). Therefore,
\[
\|u\|_{l_{p'}(A_{\xi, \sigma})} \|w\|_{l_q(A_{\xi, \sigma})} \leq \sum_{\xi \in V_{\max}(A_{\xi, \sigma})} \sum_{\xi \leq \eta \leq \zeta} u^{q'}(\eta). 
\] (8)
Let us prove that
\[
\text{card} V_{\max}(A_{\xi, \sigma}) \leq 1. 
\] (9)
Indeed, if \( \zeta, \zeta' \in V_{\max}(A_{\xi, \sigma}) \), then these vertices are incomparable and \( V(A_\zeta) \cap V(A_{\zeta'}) = \emptyset \). Hence,
\[
\|w\|_{l_q(A_\zeta)} \geq \sum_{\xi \in V_{\max}(A_{\xi, \sigma})} \|w\|_{l_q(A_\xi)} \geq \text{card} V_{\max}(A_{\xi, \sigma}) \sigma^q \|w\|_{l_q(A_\xi)}; 
\]
i.e., \( \text{card} V_{\max}(A_{\xi, \sigma}) \leq \sigma^{-q} \). From (8) and (9) it follows that there exists a vertex \( \zeta_* \in V_{\max}(A_{\xi, \sigma}) \) such that
\[
\|w\|_{l_q(A_\zeta)} \|u\|_{l_{p'}(A_{\xi, \sigma})} \leq \|w\|_{l_q(A_{\zeta_*})} \sum_{\xi \leq \eta \leq \zeta_*} u^{p'}(\eta) \leq \sigma^{-p'} \|w\|_{l_q(A_{\zeta_*})} \sum_{\xi \leq \eta \leq \zeta_*} u^{p'}(\eta). 
\] (6)
This implies (7).

Let us construct a partition of the tree \( A \) into subtrees \( (A_m, \hat{\xi}_m), 1 \leq m \leq m_* \), such that
\[
A_m = A_{\hat{\xi}_m, \sigma}, \quad 1 \leq m \leq m_*.
\] (10)
To this end we construct a family of partitions of the tree \( A = \hat{A}_k \cup G_k \), \( 0 \leq k \leq k_* \); here \( A_k \) is a subtree in \( A \) rooted at \( \xi_0 \), \( V(A_k) = \sqcup_{m=1}^{m_k} V(A_m) \), and \( G_k \) is a disjoint union of trees \( A_{m_k, j}, 1 \leq j \leq j_k \). In addition, \( m_{k+1} > m_k \).
1. We set $V(A_0) = \emptyset$, $G_0 = A$, $m_0 = 0$.

2. Suppose that a partition $A = A_k \sqcup G_k$ is constructed and $V(G_k) \neq \emptyset$. Denote by $A_{k+1}$ the maximal subgraph on the vertex set $V(A_k) \cup (\cup_{1 \leq j \leq k} V(A_{nk_j, \sigma}))$, and by $G_{k+1}$, the maximal subgraph on the vertex set $G_k \setminus (\cup_{1 \leq j \leq k} A_{nk_j, \sigma})$. Let $\{A_m\}_{1 \leq m \leq m_{k+1}} = \{A_m\}_{1 \leq m \leq m_k} \cup \{A_{nk_j, \sigma}\}_{1 \leq j \leq k}$. If $V(G_{k+1}) = \emptyset$, then we stop.

From (6) and (10) it follows that
\[
\|w\|_{l^q(A_{k+1})} < \sigma \quad \text{if} \quad A_k \text{ succeeds } A_m.
\] (11)

We define the tree $D$ with the vertex set $\{\hat{\xi}_m\}_{1 \leq m \leq m_*}$ as follows: we write $\hat{\xi}_l \in V_D(\hat{\xi}_m)$ if and only if the tree $A_l$ succeeds the tree $A_m$. Let $\hat{u}, \hat{w} : V(D) \to (0, \infty)$,
\[
\hat{u}(\hat{\xi}_m) = \|u\|_{l^p(A_{\xi_m, \sigma})}, \quad \hat{w}(\hat{\xi}_m) = \|w\|_{l^q(A_{\xi_m, \sigma})}.
\] (12)

Then
\[
\mathcal{S}_{p,q}^{A,\hat{u},\hat{w}} \lesssim \mathcal{S}_{p,q}^{D,\hat{u},\hat{w}}
\] (13)

(it can be proved similarly as Lemma 3.4 in [1]).

We have
\[
\|\hat{w}\|_{l^q(D_{\xi_m})} = \|w\|_{l^q(A_{\xi_m})}.
\] (14)

Hence,
\[
\|\hat{w}\|_{l^q(D_{\xi})} \lesssim \sigma, \quad \xi \in V(D), \quad \xi' \in V(D_{\xi}).
\] (15)

Applying Lemma 1 we get that
\[
\mathcal{S}_{D,\hat{u},\hat{w}}^{p,q} \lesssim \sup_{p,q} \|u\|_{l^p(A_{\xi_m, \sigma})} \|w\|_{l^q(A_{\xi_m})} \leq \sup_{1 \leq m \leq m_*} \|u\|_{l^p(A_{\xi_m, \sigma})} \|w\|_{l^q(A_{\xi_m})} \leq \sup_{\xi \in V(A)} \|u\|_{l^p(A_{\xi, \sigma})} \|w\|_{l^q(A_{\xi})} \lesssim \sup_{\xi, \eta \in V(A)} \|u\|_{l^p([\xi_0, \eta])} \|w\|_{l^q(\xi)}.
\] (16)

This together with (13) completes the proof.

3 Proof of Theorem 2

The following result was proved by G. Bennett [3].

\[\]
Theorem A. (see [3]). Let $1 < p \leq q < \infty$, and let $\hat{u} = \{\hat{u}_n\}_{n \in \mathbb{Z}^+}$, $\hat{w} = \{\hat{w}_n\}_{n \in \mathbb{Z}^+}$ be non-negative sequences such that

$$M_{\hat{u}, \hat{w}} := \sup_{m \in \mathbb{Z}^+} \left( \sum_{n=m}^{\infty} \hat{u}_n^q \right)^{\frac{1}{q}} \left( \sum_{n=0}^{m} \hat{u}_n^{p'} \right)^{\frac{1}{p'}} < \infty.$$ 

Let $\mathcal{S}_{p,q}^{\hat{u}, \hat{w}}$ be the minimal constant $C$ in the inequality

$$\left( \sum_{n=0}^{\infty} \hat{w}_n \sum_{k=0}^{n} \hat{u}_k f_k \right)^{\frac{1}{q}} \leq C \left( \sum_{n \in \mathbb{Z}^+} |f_n|^p \right)^{\frac{1}{p}}, \quad \{f_n\}_{n \in \mathbb{Z}^+} \in l_p.$$ 

Then $\mathcal{S}_{p,q}^{\hat{u}, \hat{w}} \simeq M_{\hat{u}, \hat{w}}$.

Proof of Theorem 2. By (1), (2), (3) and Theorem 1, it is sufficient to prove that

$$\mathcal{S}_{p,q}^{\hat{u}, \hat{w}} \simeq \sup_{p,q,C,R,R_0} u_j \left( \sum_{i \geq j} w_i^q S(i) \right)^{\frac{1}{q}}.$$ 

Denote by $\{\eta_{i,j}\}_{i \in I_j}$ the set of vertices $V_{A,j}(\xi_0)$.

Let $h : V(A) \to \mathbb{R}^+$, $\|h\|_{l^p(A)} = 1$, $f(\xi) = \sum_{\eta \in \xi} u(\eta)h(\eta)$. We estimate from above the magnitude

$$\sum_{\xi \in V(A)} w^q(\xi) \left( \sum_{\eta \in \xi} u(\eta)h(\eta) \right)^q = \sum_{\xi \in V(A)} w^q(\xi) f^q(\xi).$$ 

Let $n \in \mathbb{Z}^+$. Denote by $X_n$ the disjoint union of intervals $\Delta_{n,i}$ ($i \in I_n$) of unit length. Let $\mu_n$ be a measure on $X_n$ such that $\mu_n(\Delta_{n,i}) = 1$ and the restriction of $\mu_n$ on $\Delta_{n,i}$ is the Lebesgue measure. Then

$$\mu_n(X_n) = \text{card } V_{A,n}^{\hat{u}}(\xi_0) \simeq \frac{3}{C} S(n). \quad (15)$$ 

We define the function $\varphi : X_n \to \mathbb{R}$ by

$$\varphi|_{\Delta_{n,i}} = f(\eta_{n,i}). \quad (16)$$

Given $0 \leq k \leq n$, $s \in I_k$, we set $Q_{k,s} = \cup_{\eta_{n,i} \geq \eta_{k,s}} \Delta_{n,i}$. Then

$$\mu_n(Q_{k,s}) = \text{card } \{i \in I_n : \eta_{n,i} \geq \eta_{k,s}\} \simeq \frac{S(n)}{C} S(k). \quad (17)$$

Let

$$P_k \varphi|_{Q_{k,s}} = \sum_{\eta < \eta_{k,s}} u(\eta)h(\eta), \quad E_k \varphi = \varphi - P_k \varphi. \quad (18)$$
In addition, we set
\[ P_{n+1}\varphi := \varphi, \quad E_{n+1}\varphi := 0. \quad (19) \]

Let us estimate from above the value
\[ \sum_{i \in I_n} |f(\eta_{n,i})|^q \overset{(16)}{=} \|\varphi\|_{L^q(X_n)}. \quad (20) \]

To this end, we argue similarly as in [1, Theorem 4.1], [5].

Denote by \( \varphi^* : [0, \mu_n(X_n)] \to \mathbb{R}_+ \) the non-increasing rearrangement of the function \( |\varphi| \). Then for any \( t \in [0, \mu_n(X_n)] \) there exists a set \( Y_t \subset X_n \) such that
\[ \mu_n(Y_t) = t, \quad \forall x \in Y_t \quad |\varphi(x)| \geq \varphi^*(t). \quad (21) \]

Then
\[ \varphi^*(t) \leq \frac{\|\varphi\|_{L^p(Y_t)}}{[\mu_n(Y_t)]^{1/p}}. \quad (22) \]

Let \( k \in \mathbb{Z}_+ \),
\[ \frac{\mu_n(X_n)}{S(k+1)} < t \leq \frac{\mu_n(X_n)}{S(k)}. \quad (23) \]

We set \( k_n = \min\{k, n+1\} \). Then
\[ E_k\varphi \|_{L^p(Y_t)} \overset{(18), (19)}{=} \|E_k\varphi\|_{L^p(Y_t)} + \sup_{x \in X_n} |P_k\varphi(x)|[\mu_n(Y_t)]^{1/p}. \]

This together with (22) yields that
\[ \varphi^*(t) \leq \|E_k\varphi\|_{L^p(Y_t)}[\mu_n(Y_t)]^{-1/p} + \sup_{x \in X_n} |P_k\varphi(x)|. \]

From (5), (21) and (23) we obtain
\[ \varphi^*(t) \leq \|E_k\varphi\|_{L^p(Y_t)}[\mu_n(X_n)]^{-1/p}[S(k)]^{1/p} + \sup_{x \in X_n} |P_k\varphi(x)|. \quad (24) \]

Let \( k_n \geq 1 \) (otherwise, \( P_k\varphi = 0 \)), and let \( x \in Q_{k_n-1,s} \). For any \( 0 \leq j \leq k_n - 1 \) there exists the unique \( l_j \in I_j \) such that \( \eta_{j,l_j} \leq \eta_{k_n-1,s} \). Then
\[ (P_{j+1}\varphi - P_j\varphi)|_{Q_{j,l_j}} \overset{(18), (19)}{=} \text{const}, \quad 0 \leq j \leq k_n - 1. \quad (25) \]

Observe that \( P_0\varphi = 0 \). Hence,
\[ |P_k\varphi(x)| \leq \sum_{j=0}^{k_n-1} |P_{j+1}\varphi(x) - P_j\varphi(x)| \leq \sum_{j=0}^{k_n-1} \|P_{j+1}\varphi - P_j\varphi\|_{C(Q_{j,l_j})}. \overset{(25)}{=} \]
= \sum_{j=0}^{k_n-1} \| P_{j+1} \varphi - P_j \varphi \|_{L^p(Q_{j+1})} \left[ \mu_n(Q_{j+1}) \right]^{-1/p} \lesssim_{\rho, \check{C}_*} 1 \tag{17, 18, 19}
\lesssim \sum_{j=0}^{k_n-1} \left[ \frac{S(j)}{S(n)} \right]^{1/p} \left( \| E_j \varphi \|_{L^p(Q_{j+1})} + \| E_{j+1} \varphi \|_{L^p(Q_{j+1})} \right) \lesssim_{\rho, \check{C}_*, R_0} 1 \tag{5}
\lesssim \sum_{j=0}^{k_n} \left[ \frac{S(j)}{S(n)} \right]^{1/p} \| E_j \varphi \|_{L^p(X_n)} \leq \min_{\rho, \check{C}_*, R_0} \sum_{j=0}^{\min\{k_n, n\}} \left[ \frac{S(j)}{S(n)} \right]^{1/p} \| E_j \varphi \|_{L^p(X_n)}.$

This together with (15) and (24) implies that
\[
\varphi^*(t) \lesssim_{\rho, \check{C}_*, R, R_0} \sum_{j=0}^{\min\{k_n, n\}} \left[ \frac{S(j)}{S(n)} \right]^{1/p} \| E_j \varphi \|_{L^p(X_n)} \quad \text{if} \quad \frac{\mu_n(X_n)}{S(k + 1)} < t \leq \frac{\mu_n(X_n)}{S(k)}. \tag{26}
\]

Therefore,
\[
\int_{X_n} |\varphi(x)|^q \, dx = \int_0^{\mu_n(X_n)} |\varphi^*(t)|^q \, dt \lesssim_{\rho, \check{C}_*, R, R_0} 1 \tag{14, 15, 26}
\lesssim \sum_{k=0}^{n} \frac{S(n)}{S(k)} \left( \sum_{j=0}^{k} \left[ \frac{S(j)}{S(n)} \right]^{1/p} \| E_j \varphi \|_{L^p(X_n)} \right) =: A. \tag{27}
\]

We claim that
\[
A \lesssim_{\rho, \check{C}_*, R, R_0} \sum_{j=0}^{n} \left( \frac{S(j)}{S(n)} \right)^{\frac{1}{p} - \frac{q}{q'}} \| E_j \varphi \|_{L^p(X_n)}. \tag{28}
\]

Indeed, let \( \psi_j = \left( \frac{S(j)}{S(n)} \right)^{\frac{1}{p} - \frac{q}{q'}} \| E_j \varphi \|_{L^p(X_n)}. \) Then (28) follows from Theorem A and the estimate
\[
\sup_{0 \leq k \leq n} \left( \sum_{j=0}^{k} \left[ \frac{S(j)}{S(n)} \right]^{q'/q} \right)^{\frac{1}{q'}} \left( \sum_{j=k}^{n} \frac{S(n)}{S(j)} \right)^{\frac{1}{q'}} \lesssim_{\rho, \check{C}_*, R, R_0} 1.
\]

We have
\[
\| E_j \varphi \|_{L^p(X_n)} \leq \left\| \sum_{i=j+1}^{n+1} (P_i \varphi - P_{i-1} \varphi) \right\|_{L^p(X_n)} \leq \sum_{i=j+1}^{n+1} \| P_i \varphi - P_{i-1} \varphi \|_{L^p(X_n)}.
\]

From (27) and (28) we get that
\[
\| \varphi \|_{L^q(X_n)} \lesssim_{\rho, \check{C}_*, R, R_0} \sum_{j=0}^{n+1} \left( \frac{S(j)}{S(n)} \right)^{\frac{1}{p} - \frac{q}{q'}} \left( \sum_{i=j+1}^{n+1} \| P_i \varphi - P_{i-1} \varphi \|_{L^p(X_n)} \right)^q \lesssim_{\rho, \check{C}_*, R, R_0} \sum_{j=1}^{n+1} \left( \frac{S(j)}{S(n)} \right)^{\frac{1}{p} - 1} \| P_j \varphi - P_{j-1} \varphi \|_{L^p(X_n)}^q.
\]
The last inequality follows from Theorem A and (5) since
\[
\sup_{1 \leq j \leq n+1} \left( \sum_{i=1}^{j} \left( \frac{S(i)}{S(n)} \right)^{q/(p+1)} \right)^{\frac{1}{q'}} \left( \sum_{i=j}^{n+1} \left( \frac{S(n)}{S(i)} \right)^{q'(1-\frac{1}{q})} \right)^{\frac{1}{q'}} \lesssim_{p,q,R,R_0} 1.
\]

Further,
\[
\|P_j \varphi - P_{j-1} \varphi \|_{L^p(X_n)}^p \leq \sum_{s \in I_{j-1}} \int_{Q_j, s} \left| \sum_{\eta \in \eta_{j-1,s}} u(\eta) h(\eta) - \sum_{\eta < \eta_{j-1,s}} u(\eta) h(\eta) \right|^p d\mu_n \lesssim_{p,q,R,R_0} \sum_{s \in I_{j-1}} |h(\eta_{j-1,s})|^p.
\]

This together with (5) and (29) implies that
\[
\|\varphi\|_{L^q(X_n)}^q \lesssim \sum_{k=0}^{n} \frac{S(n)}{S(k)} u_k \left( \sum_{s \in I_k} |h(\eta_{k,s})|^p \right)^{\frac{2}{p}}.
\]

Hence,
\[
\sum_{n=0}^{\infty} w_n^q \sum_{i \in I_n} |f(\eta_{n,i})|^q \lesssim_{p,q,R,R_0} \sum_{n=0}^{\infty} w_n^q \sum_{k=0}^{n} \frac{S(n)}{S(k)} u_k \left( \sum_{s \in I_k} |h(\eta_{k,s})|^p \right)^{\frac{2}{p}} =
\]
\[
\leq \sum_{k=0}^{\infty} \left( \sum_{s \in I_k} |h(\eta_{k,s})|^p \right)^{\frac{2}{p}} u_k \sum_{n=k}^{\infty} w_n^q \frac{S(n)}{S(k)} \leq \left[ \sup_{0 \leq k < \infty} u_k \sum_{n=k}^{\infty} w_n^q \frac{S(n)}{S(k)} \right] \left( \sum_{k=0}^{\infty} \left( \sum_{s \in I_k} |h(\eta_{k,s})|^p \right)^{\frac{2}{p}} \right)^{\frac{2}{p}}.
\]

This completes the proof.

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