Phases in two dimensional \( p_x + ip_y \) superconducting systems with next-nearest-neighbor interactions

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A chiral \( p_x + ip_y \) superconducting system with nearest and next-nearest hopping and pairing terms is considered. Gap closures, as various parameters of the system are varied, are found analytically and used to identify the topological phases. The phases are characterized by Chern numbers (ranging from \(-3\) to \(3\)), and (numerically) by response to introduction of weak disorder, edges, and magnetic fields in an extreme type-II limit, focusing on the low-energy modes (which presumably become zero-energy Majorana modes for large lattices and separations). Several phases are found, including a “hybrid” phase with Chern number 3 that cannot be thought of in terms of a single range of interaction (and which supports no additional defect modes), and an easy to understand phase with Chern number 2 that may host an additional, disorder resistant, Majorana mode. The energies of the vortex quasiparticle modes were found to oscillate as vortex position varied. The spatial length scale of these oscillations was found for various points in the Chern number 3 phase and increased as criticality was approached.

I. INTRODUCTION

Recently, there has been much interest in topological features of various condensed matter systems, in particular Majorana fermions\(^{11-13}\). We focus on chiral \( p_x + ip_y \) superconductors, which can develop a nonzero Chern number and consequential topological features, including zero-energy Majorana fermions around defects.\(^{13,14}\) Majorana modes are also expected in the lattice version of chiral superconductors: the gap function in the lattice looks roughly like \( \sin(nk_x) + i\sin(nk_y) \), so it reduces to \( n\partial_x + in\partial_y \) in the continuum limit where we expect guaranteed 0-energy Majorana modes.

In contrast to the continuum case where the range of interaction simply rescales the gap function, the range plays a more interesting role on the lattice. Previous work suggested that including longer-ranged interactions leads to novel phases. The motivation is as seen in FIG. 1 if all interactions are of range \( n \), a number of noninteracting sublattices form, each with their own properties, e.g. Chern number. In the more realistic case when interactions of different ranges are included, intermediate behavior is expected. Additionally, the pairing terms at different ranges may, in principle, have different chirality, i.e., \( p_x \pm ip_y \). Our system of interest has a combination of nearest-neighbor (NN) and next-nearest neighbor (NNN) hopping (respectively, \( t_1 \) and \( t_2 \)), and \( p + ip \) pairing (respectively, \( \Delta_1 \) and \( \Delta_2 \)) terms, at chemical potential \( \mu \).

The five parameters \((t_1, t_2, \Delta_1, \Delta_2, \text{and } \mu)\) constitute a parameter space rich enough to include the well-known BEC and BCS superconducting systems, as well as their two-sublattice versions. Noticing that the BEC-BCS transition is topological in nature, we search for the surfaces in parameter space where the band gap collapses and topological phase transitions occur. We posit that the regions bounded by these surfaces will represent the different phases of the system. The phase diagram has features that make it easier to visualize than a general five-dimensional phase diagram: the phase transitions depend only on the three ratios of parameters \( \alpha = \frac{\Delta_2}{\Delta_1} \) and the scaled hopping terms \( t_1/\mu, t_2/\mu \); for fixed values of \( \alpha \), all phase transitions are lines in the \( t_1-t_2 \) plane; finally, depending on the chirality of the NNN pairing terms (i.e., if it is the same or opposite chirality as the NN term), there are respectively four or five such lines, three of which are independent of \( \alpha \). An analytical calculation finds all Chern numbers possible with NN and NNN terms; they range from \(-3\) to \(+3\).

The simplest way to visualize the phase diagram for a given value of \( \alpha \) is to plot all the phase transition lines in the \( t_1-t_2 \) plane; FIG. 2 is the phase diagram for the same chirality case.
and $\alpha = 1/2$. The $\alpha$ dependence of the phase transition lines can be presented by specifying the points of closest approach to the $t_1$-$t_2$ origin. FIG. 3 parametrically plots these points of closest approach as a function of $\alpha$. In both cases, the phase transition line or lines remain fixed for $\alpha^2 \geq 2$; i.e., for $|\Delta_2|^2 \leq \frac{1}{2} |\Delta_1|^2$, no change in the topology of the system can occur by modifying the pairing terms. For the same chirality case, only one $\alpha$-dependent phase transition line occurs; when $\alpha$ is turned on from zero, the line moves in from the far right or left, for respectively positive or negative values of $\alpha$. For the opposite chirality case, two lines appear symmetrically in $t_1/\mu$; their placement is symmetric as a function of $\alpha$.

In the purely NN case, the change in Chern number from 0 to 1 signals the BEC-BCS phase transition, and we therefore anticipate an analogous difference in the behavior for other Chern number changes. Going beyond a defect-free analysis, a numerical diagonalization around zero energy (on lattices with size $\sim 150$ square) is performed to investigate the low-energy quasiparticle response of the system to three kinds of defects: edges, on-site disorder, and magnetic fields in an extreme type-II limit with vortices in $\Delta$. Edges are introduced by adding terms of the form $O_i c_i^\dagger c_i$; $O_i$ is set to be very large outside of some defined region, so that the low energy spectrum does not access states involving sites outside of the defined region. On-site disorder is added in a similar way: $O_i$ inside the region of interest takes on a value of $E_D$ with probability $\frac{1}{2}$ and $-E_D$ with the same probability (and 0 otherwise). For the magnetic field, we assume a very long magnetic screening length so that the magnetic field is constant, but still a finite superconducting coherence length $\xi$ so that vortices appear in the superconducting order parameter $\Delta$.

We now summarize our results. Localized defects produce low-energy quasiparticle excitations localized around the defect; at higher energies, increasingly less localized states form. The details of the edge modes, including their energies, depend strongly on the details of the edge. Conversely, vortex core modes localized far from the edge behave independently of choice of details about the edge, e.g., square or circular. The vortex core modes interact with each other, but

FIG. 2: Phase diagram with $\alpha = \frac{\Delta_1}{\Delta_2} = \frac{1}{2}$. The large numbers indicate the Chern number for various values of $t_1$, $t_2$, separated by lines where the gap collapses. The dots indicate values of $t_1$, $t_2$, $\mu$ investigated; especially $(t_1, t_2) = (-2, 1)$ with $\mu \geq 0$.

FIG. 3: Phase diagram for same (top) and opposite (bottom) chirality. The three $\alpha$-independent phase transition lines are plotted directly, while the one or two $\alpha$-dependent lines are presented instead as parametric plots of the points of closest approach to the origin; particular values of $\alpha$ are marked. Crossing the $\alpha$-dependent lines causes the Chern number to jump by $+4$ or $-2$, respectively, for the same or opposite chirality cases (the contribution occurs for the phase on the side opposite the line as the origin, relative to the large plotted numbers). Compare with FIG. 2.
increasingly weakly with distance: the energies of the lowest quasiparticle vortex core exhibit exponentially damped oscillations as the vortices are separated. The length scales of the edge-vortex and vortex-vortex interactions were different; edge-vortex hybridization occurs over a much longer distance. Furthermore, as the phase transition is approached, the quasiparticle excitations tend to be less localized; all length scales appear to grow.

A system with \( \alpha = \frac{1}{2}, (t_1, t_2) = (-2, 1) \) and \( \mu \) varying, i.e. on the path shown in FIG. 2, crosses several phase transitions. As long as the system is far enough away from the phase transition that the edge-vortex hybridization can be neglected, the energy of the lowest quasiparticle was found to oscillate. Indeed, while in the Chern number 3 portion of the phase diagram, we find a spatial period of oscillation that appears to increase linearly with the chemical potential, \( \Delta \) field superconducting order parameter \( \Delta \), and Hamiltonian

\[
H = \sum_{ij} h_{ij} c_i^\dagger c_j + \frac{1}{2} \sum_{ij} \Delta_{ij} c_i^\dagger c_j + \text{h.c.}
\]

\[
= \frac{1}{2} \sum_{ij} \left[ c_i^\dagger \begin{bmatrix} h_{ij} & \Delta_{ij} \end{bmatrix} c_j \right] c_i^\dagger c_j
\]

\[
= \frac{1}{2} \sum_{ij} \left[ c_i^\dagger c_j \right] \mathcal{H}_{ij} c_i^\dagger c_j = E_g + \sum_n E_n \Psi_n \Psi_n (1)
\]

The Bogoliubov-de Gennes Hamiltonian is diagonalized in the last step in terms of the ground state energy \( E_g \), quasi-
Indeed, the system takes on Chern numbers \(-Z\) with topological classification putting the system in Altland-Zirnbauer chirality possibility, for now we consider both cases. Although we will later discard the opposite where the chiralities are the same, and one where the chiralities are opposite. Once the NN chirality is fixed, however, the two chiralities can be related by inverting any one axis (i.e., turning the system over). Without loss of generality, we assume that the NN pairing has counterclockwise chirality; the two chiralities can be plotted implicitly, giving an exact solution for the location in the Brillouin zone for each zero; see FIG. 5. Fortunately, the explicit solution for the locations of the zeros are unexpectedly not needed, as we will see momentarily.

\begin{equation}
\cot(k_x)^2 = \frac{\alpha^2}{2 - \alpha^2} \left[ 1 \pm \sqrt{\frac{4}{4 + \alpha^4}} \right]
\end{equation}

for both the same and opposite chirality cases. Because the location of each zero depends on the value of \(\alpha\), we call these zeros “unpinned,” (or \(\alpha\)-dependent) in contrast to the “pinned” ones that are always at the high-symmetry points.

\begin{equation}
\Psi_n = \sum_i \left[ u_i^{(n)} c_i + v_i^{(n)} c_i^\dagger \right]
\end{equation}

As illustrated in FIG. 5, \(h\) includes nearest and next-nearest neighbor hopping \((t_1\) and \(t_2\) terms), and \(\Delta\) includes analogous pairing \((\Delta_1\) and \(\Delta_2\) terms). For both the NN and NNN pairing terms in the figure, the phase of \(\Delta\) advances under counterclockwise rotation, creating the chirality of the order parameter. Without loss of generality, we assume that the NN pairing has counterclockwise chirality; the two chiralities can be related by inverting any one axis (i.e., turning the system over). Once the NN chirality is fixed, however, the two chiralities of the NNN pairing represent physically distinct cases: one where the chiralities are the same, and one where the chiralities are opposite. Although we will later discard the opposite chirality possibility, for now we consider both cases.

The pairing terms explicitly break time-reversal symmetry, putting the system in Altland-Zirnbauer symmetry class D, with topological classification \(\mathbb{Z}\) given by the Chern number. Indeed, the system takes on Chern numbers \(-3\) through 3.

\textbf{B. Gap Closing Momenta and Symmetries}

The gap closes for momenta \(k = (k_x, k_y)\) such that

\begin{equation}
0 = \frac{1}{2\Delta_2} \Delta(k) = \alpha \left[ \sin(k_x) + i \sin(k_y) \right] + \sin(k_x \mp k_y) + i \sin(k_x \pm k_y)
\end{equation}

(by the top sign corresponds to the same chirality, and the bottom to the opposite chirality case), and

\begin{equation}
0 = \frac{1}{\mu} h(k) = -1 + 2t_1 \left[ \cos(k_x) + \cos(k_y) \right] + 4t_2 \cos(k_x) \cos(k_y)
\end{equation}

These conditions require an inversion symmetry, \(k \rightarrow -k\). For the chiralities to be the same, a \(\pi\) rotation symmetry of the zeros is required, \((k_x, k_y) \rightarrow (-k_x, k_y)\). For the chiralities to be opposite, a more subtle symmetry of the zeros of the pairing order is required:

\begin{equation}
k \rightarrow k' = \left[ \frac{\pi - k_y}{\pi + k_x} \right]
\end{equation}

\textbf{C. Zeros of} \(\Delta\)

There are the four “pinned” (or \(\alpha\)-independent) zeros of \(\Delta\) at the symmetry points (i.e., where \(\sin k_x = \sin k_y = 0\)). Assuming that the sine terms do not vanish, the remaining zeros of \(\Delta\) can be shown to satisfy

\begin{equation}
\cot(k_x)^2 = \frac{\alpha^2}{2 - \alpha^2} \left[ 1 \pm \sqrt{\frac{4}{4 + \alpha^4}} \right]
\end{equation}

\textbf{D.} \(h\) at the zeros of \(\Delta\)

For \(k\) such that \(\Delta(k) = 0\), the band gap closes if \(h(k) = 0\). Furthermore, by looking at the sign of \(h(k)\) as well as the phase winding of \(\Delta\) around the zero, the Chern number can be directly found (see Appendix 3C for details). Indeed, at the pinned zeros, the high-symmetry points \(\Gamma, X, Y,\) and \(M\),

\begin{equation}
\begin{pmatrix}
\begin{array}{c}
h(0,0) \\
\h(\pm \pi, 0) \\
\h(0, \pm \pi) \\
\h(\pm \pi, \pm \pi)
\end{array}
\end{pmatrix}
= \frac{1}{4\mu} \begin{pmatrix}
t_2 + t_1 - \frac{1}{4} \\
-t_2 - \frac{3}{4} \\
t_2 - t_1 - \frac{3}{4} \\
t_2 - t_1 - \frac{1}{4}
\end{pmatrix}
\end{equation}

These four equations, one of which is redundant, imply three \(\alpha\)-independent phase transition lines in the \(t_1-t_2\) plane. The
Chern number changes by 1 when crossing each line (except for the $t_1 = -\frac{1}{4}$ double line, where the change is 2).

For the $\alpha$-dependent zeros, given Equation (5), $h$ can be evaluated at the momenta of the unpinned zeros of $\Delta$ (it is a tedious but straightforward considerations of cases). For the same chirality case, $h$ at all four unpinned zeros is,

$$\mu h(k) = -1 - t_1\alpha(2 - \alpha^2) - t_2\alpha^4$$  \hspace{1cm} (6)

In the opposite chirality case, $h$ takes on two different values in an inversion-symmetry guaranteed way:

$$\mu h(k) = -1 \pm t_1\alpha\sqrt{4 + \alpha^2} + t_2\alpha^4$$  \hspace{1cm} (7)

The condition that unpinned zeros are included or excluded by the Fermi surface (i.e., $h(k) \leq 0$) can be recast in a more transparent way; define

$$\beta = \left[ -\alpha(2 - \alpha^2) -\alpha^4 \right] \quad \text{or} \quad \left[ \pm\alpha\sqrt{4 + \alpha^2} \right]$$  \hspace{1cm} (8)

and the above condition can be restated as

$$0 \leq \mu h(k) = -1 + t \cdot \beta \quad \text{or} \quad \frac{1}{\beta} \leq t \cdot \hat{\beta}$$  \hspace{1cm} (9)

I.e., there is a phase transition line, with closest approach to the $t_1$-$t_2$ origin given by $Z = \frac{\beta}{\hat{\beta}}$. This allows for a compact description of the phase diagram in the $t_1$-$t_2$ plane, by parametrically plotting $Z$ as a function of $\alpha$ (FIG. 3). If the projection is greater (i.e., you are on the opposite side of the line as the origin), the unpinned zeros contribute. A concrete example is provided for the same-chirality case with $\alpha = \frac{1}{2}$ in FIG. 6. In the same chirality case, all four zeros are either included or excluded, while in the opposite chirality case the zeros come in two pairs (corresponding to the two branches of $\beta$ and curves in FIG. 3).

E. Physical Cases

In the preceding, we considered a rather large class of possible Hamiltonians, not all necessarily valid mean-field approximations of any real systems. Indeed, it is not immediately obvious what experimental signatures would be present in a real material with superconducting order of opposite chiralities at different interaction lengths, or if such order is ever energetically favorable. We therefore defer discussion of these more exotic Hamiltonians, focusing instead on the same chirality case, with $t_1 < 0$ and $t_2 > 0$, which is not unlike the superconducting band is the strontium ruthenates. In particular, $\alpha = 1/2$ and $(t_1, t_2) = (-2, 1)$ with various values of $\mu \geq 0$; see FIG. 7. We are motivated to choose these values of parameters because they cut through several nontrivial phases with distinct Chern numbers.
FIG. 7: (color online) Plot of a “bulk” mode on a 150 by 150 lattice, \((t_1, t_2) = (-2, 1), \mu = 1\). The plots are of the magnitudes of the \([u, v]\) parts of the Bogoliubov-de Gennes wavefunction. The normalization is chosen such that the average is 1. The Hamiltonian includes two vortices separated by 13.2 in the \(x\)-direction (position shown on the plots, with red circles of radius \(\xi = 1.6\) around them). However, as the plot shows, this particular quasiparticle mode does not inhabit the vortex significantly; instead, the probability tends to be spread throughout the bulk of the material. The energy is out of the gap: \(\approx 2.5\).

III. DEFECTS AND MAGNETIC FIELDS

A. Magnetic Fields: Flux Tubes and Vortices

When an external magnetic field is imposed on a superconductor, currents develop in an effort to eject the magnetic field from the bulk of the superconductor, minimizing the associated free energy. As a result, magnetic fields attenuate on a length scale \(\lambda\), the London penetration depth. As the magnetic fields grow more powerful, the system can enter the type-II regime, where it becomes energetically favorable for magnetic field to penetrate the superconductor in localized regions, “flux tubes,” in which the superconducting order parameter locally vanishes. The flux tubes carry integer multiples of the magnetic flux quantum because the order parameter’s phase must be well-defined upon rotation by a full \(2\pi\) radians. The superconducting coherence length \(\xi\) determines the scale on which the order parameter changes; it is distinct from \(\lambda\). The magnetic flux tubes create real-space vortices in the magnetic field and superconducting order parameter.

In the two dimensional case at hand, the associated currents are also essentially two dimensional, limiting the ability of these currents to create response magnetic fields. The natural simplifying limit is to take \(\lambda \to \infty\), neglecting the magnetic field of the response supercurrent, and to assume a constant, unaffected, external magnetic field. Notwithstanding, we still assume the presence of vortices in the superconducting order parameter; \(\xi\) remains finite. These vortices are therefore localized regions of vanishing superconducting order parameter \(\Delta\), without associated magnetic inhomogeneity.

We are guided by the semiclassical relation

\[
v_s = \frac{1}{m^*} \left( \nabla \phi - \frac{e^*}{c} A \right)
\]

where \(v_s\) is the superfluid velocity, \(\phi\) is the phase of the superconducting order parameter, and \(A\) is the vector potential, in London gauge. When far away from a vortex \((r \gg \lambda)\), we expect \(v_s = 0\) (we assume this) and \(B = 0\). Integrating the semiclassical relation around the vortex yields

\[
2\pi n = \oint \nabla \phi \cdot dl = \frac{e^*}{c} \Phi_m
\]

(i.e., the well-known fact that an integer multiple of magnetic flux quanta penetrates through a flux tube). When the path is reduced, as long as the order parameter does not vanish, the change of \(\phi\) around the loop must remain the same integer multiple of \(2\pi\). Thus, even for \(r < \lambda\), the order parameter winds an integer multiple of \(2\pi\) around each vortex.

With a qualitative description of the behavior of the order parameter (the magnitude falls off near vortices, and the phase winds an integer multiple of \(2\pi\) around each vortex), a quantitative model to perform a numerical simulation must now be established. We use the model

\[
\Delta_{jk} = \Delta^{(0)}_{|k-j|} D(j, k)e^{i\phi_{jk}}
\]

The phase of the order parameter \(\theta_{jk}\) is a geometric mean of the expected phases at \(i\) and \(j\):

\[
e^{i\theta_{jk}} = \frac{e^{i\phi_k} + e^{i\phi_j}}{|e^{i\phi_k} + e^{i\phi_j}|}
\]
FIG. 8: (color online) Plot of “vortex mode;” compare with FIG. 7. The energy of this mode is in-gap: \( \approx 5.3 \times 10^{-3} \).

FIG. 9: (color online) Plot of “edge mode;” compare with FIG. 7. The energy of this mode is in-gap: \( \approx 3.1 \times 10^{-3} \).

\[ D(j, k) = \frac{d_{\text{eff}}(j, k)}{\sqrt{d_{\text{eff}}(j, k)^2 + \xi^2}} \]  \hspace{1cm} (13)

\( \xi \) is the superconducting coherence length, and

\[ d_{\text{eff}}^{-1}(j, k) = \sum_n \left( \min_{x \text{ between } j \text{ and } k} |x - v_n| \right)^{-1} \]  \hspace{1cm} (14)

\( x \) lies on the line connecting \( j \) and \( k \). The hopping terms \( h_{ij} \) acquire a Peierls phase due to the magnetic vector potential

\[ h_{jk} = h_{k-j}^{(0)} e^{i\phi} \int_0^l A \cdot d\ell \]

The semiclassical relation (10) relates \( A \) in the London gauge: \( \nabla \cdot A = 0 \) and the normal component of \( A \cdot \hat{n} \) becomes the physically meaningful boundary supercurrent. Putting the zero of \( A \) at the center of the sample, the simple form \( A \propto \rho \phi \) implies a constant magnetic field; additionally for this choice of \( A \), the boundary current happily vanishes for circular geometry. For non-circular geometries, this approximation will remain valid provided that the edge (and associated currents) are far from the features of interest.
B. Edges

The simplest edges are produced by omitting certain terms in the Hamiltonian, i.e., setting all terms of the form $h_{ij}$ and $\Delta_{ij}$ to zero for $ij$ which cross an edge. While natural and simple, this procedure has a drawback: if two edges are introduced, an artificially “sharp” corner is produced and leads to the development of “corner modes,” quasiparticle defect states with the vast majority of statistical weight of the wavefunction near these artificial corners. One might be concerned that such an unphysical feature might poison the simulation.

If the edge were smoother the problem would be naturally eliminated. To create a more general shape of edge, on-site terms $O_i c_i^\dagger c_i$ are added with $O_i$ increasingly large near and beyond the edges of the system. For our purposes, the edge is made very steep and circular, i.e., it goes from 0 inside a circular region of the lattice, to a very large number outside it. The lattice sites with large on-site energies must play no role in the low energy spectrum of the Hamiltonian.

C. Disorder

By adjusting the $O_i$ terms, on site disorder is produced, representing quenched chemical impurities in the lattice. The model is simply:

$$O_i = \begin{cases} 
0, & \text{with probability } 1 - p \\
-E_i, & \text{with probability } p/2 \\
E_d, & \text{with probability } p/2
\end{cases}$$

In cases where the locations of vortices were moved, the same disorder realization was used.

IV. NUMERICAL RESULTS

Now, we discuss the results of diagonalizing the Bogoliubov de-Gennes Hamiltonian \[ \] for eigenvalues near 0, with the defects and magnetic field discussed in the previous section.

A. Vortex Core Mode Oscillations

When two magnetic vortices tubes are placed at various locations throughout the lattice, the quasiparticle energy spectrum changes. As the vortices are separated, the energies of the lowest quasiparticles exhibit damped oscillation, as seen in FIG. 10 and FIG. 11. The dominant Fourier component of these oscillations is found (and inverted) to give a spatial period. FIG. 12 plots these spatial periods versus $\mu$ for the particular value of $(t_1, t_2) = (-2, 1)$, $\alpha = 1/2$. Only cases where the vortex core mode did not hybridize with the edges were included here; even small distortions to the oscillations disturb the spatial period significantly. We therefore could only examine systems far away from criticality; only points in the Chern number 3 phase were far enough from criticality.

FIG. 10: Plot of several lowest quasiparticle energies vs separation of two vortices in the $x$ direction on a 150 by 150 circular lattice, $\alpha = 1/2$, $\mu = 1$, $\xi = 1.6$, and $(t_1, t_2) = (-2, 0, 1, 0)$. Negative energies correspond to a negative spatial-inversion symmetry. Similar oscillations of low-energy modes were noticed by Mizushima and Machida.\[\] There is only one guaranteed zero-energy Majorana mode, in contrast to the two seen in FIG. 11.

Interestingly, as plotted in the Figure, there appears to be a linear dependence of the spatial period on $\mu$, with slope close to 0.8. Mizushima and Machida\[\] observe a similar oscillation with vortex separation, related to $k_F$.

B. Majorana Mode Count

For Chern number 3, only one vortex mode exists; see FIG. 10. However, for Chern number ±2, we find two 0-energy modes when two vortices are introduced. FIG. 11 examines modes in the $-2$ region near $(t_1, t_2) = (0, 2)$. The adjacent $(t_1, t_2) = (-1.5, 2)$ region with Chern number +2 probably also supports an additional Chern mode, but the issue there is complicated by the fact that the system is usually quite close to criticality. I.e., the zeros of $\Delta(k)$ occur where the $h_0$ is relatively small, leading to a divergence of correlation lengths. Effective analysis requires the edge/vortex hybridization be suppressed; for $\alpha = 1/2$, much larger systems have to be simulated.

C. Disorder

One may be concerned that these additional Majorana modes are destroyed by disorder, motivating the prescription for adding on-site disorder to $O_i$ detailed previously. The same simulations with disorder added are shown in FIG. 13.

\[\]
FIG. 11: Compare with FIG. 10: circular lattice, $\alpha = \frac{1}{2}$, $\mu = 1$, and $\xi = 1.6$, with various hopping strengths. Notice the two oscillating low energy excitations, possibly with an exponentially damped envelope. Presumably, in the limit of large separation of the vortices, these vortex core states become zero-energy Majorana modes. We suspect that a significant portion of the Chern number $-2$ phase enjoys these multiple Majorana modes, though proving they are topologically protected is of course impossible with a purely numerical approach. Resistance to weak disorder is discussed later FIG. 13. For weak disorder, there are no meaningful differences from the disorder free case; pairs of vortex modes that existed before the disorder persist after turning on the weak disorder. In reality, vortices would become pinned to disorder sites. A more detailed calculation would not install vortices at prespecified locations. Despite these caveats, we believe that these additional modes warrant further analytical investigation.

V. CONCLUSION

Chiral $p$-wave superconductors on a lattice support additional, interesting phases beyond the two well-known (topologically trivial) BEC and (Chern number 1) BCS phases. Some of these new phases (e.g., Chern number 2) have an intuitive description in terms of sublattices, which seem to survive variation of parameters as well as the addition of weak disorder; both zero energy modes present in the two-sublattice case appear to survive. The intrinsically “hybrid” Chern number 3 phase, on the other hand, does not support any additional modes, perhaps related to the fact that it cannot be related to any simple model with clean sublattice separation.

FIG. 12: Plot of (inverse) dominant Fourier component, or spatial period, of oscillations of lowest quasiparticle energy in two-vortex systems separation. All systems $(t_1, t_2) = (-2, 1)$ while $\mu$ varies; the Chern number was always 3. A linear fit was performed, and included. Edge-vortex hybridization plagued numerical trials at larger values of $\mu$; it rapidly becomes unclear which quasiparticles are the “vortex” modes. Even relatively weak hybridization of the edge with the vortex modes can subtly deform the oscillating form, causing dramatic changes in the slope of the fit line.

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Appendix A: Spatial Inversion Symmetry

In the absence of added disorder, the model systems we consider have a spatial inversion symmetry which can be exploited to enhance the clarity of some plots (especially to highlight oscillating behavior). The hopping terms straightforwardly satisfy $h_{-i,-j} = h_{ij}$, while the $p$-wave symmetry of the superconducting order parameter implies $\Delta_{-i,-j} = -\Delta_{ij}$. Let $I$ realize the inversion symmetry in position space (i.e., $(Ih)_{ij} = h_{-i,-j}$) and put $\mathcal{F} = \tau_3 \otimes I$ ($\tau_3$ acts on the space of Nambu spinors). Clearly, $\mathcal{F}^2 = 1$, and $\mathcal{F}^\dagger = \mathcal{F}$, so
the eigenvalues of $\mathcal{J}$ are $\pm 1$. The quick calculation

$$\mathcal{J}^\dagger \mathcal{H}_{ij} \mathcal{J} = \mathcal{I}^\dagger \tau_3 \mathcal{H}_{ij} \tau_3 \mathcal{I}$$

$$\mathcal{I}^\dagger \begin{bmatrix} h_{ij} - \Delta_{ij} & -\Delta_{ji} \\ -\Delta_{ji} & -h_{ij} \end{bmatrix} \mathcal{I} = \begin{bmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ji} & -h_{ij} \end{bmatrix} = \mathcal{H}_{ij} \quad (A1)$$

shows that the inversion symmetry $\mathcal{J}$ relates quasiparticles of the form $(u_i, v_i)$ to $(u_{-i}, -v_{-i})$. For appropriate energy eigenstates, $\psi = \pm \mathcal{J} \psi$ and the subparts $u$ and $v$ therefore have separate (and opposite) inversion symmetries given by $\mathcal{I}u = \pm u$ and $\mathcal{I}v = \mp v$. These eigenvalues can be changed using the Bogoliubov-de Gennes particle-hole symmetry (i.e., $\Xi = \tau_3 \otimes K$, where $K$ is complex conjugation); the symmetry-related negative-energy pair has opposite $\mathcal{J}$ eigenvalue:

$$\Xi \mathcal{J} = \tau_3 K \tau_3 \mathcal{J} = -i \tau_2 \mathcal{J} = -\mathcal{J} \Xi \quad (A2)$$

By acting with $\Xi$, we can restrict our attention to quasiparticles with positive $\mathcal{J}$ eigenvalue at the expense of considering “negative” energies (which would have otherwise been discarded). Or, more simply, “the sign of $E$ encodes the eigenvalue of $\mathcal{J}$.” Plotting eigenvalues as such often provides a more clear view of their behavior, for example energy oscillations. See, for example, FIG. 11 for such a plot.

In the case of disorder, the symmetry $\mathcal{J}$ is clearly broken by the additional terms. Nonetheless, the overlap $\langle \psi | \mathcal{J} | \psi \rangle$ is still meaningful: if positive, we can still identify $\psi$ as “symmetric”-like or otherwise. In FIG. 13 the sign of the overlap is plotted as the shape of the symbol.

**Appendix B: Hamiltonian Details**

Here, the hopping and pairing terms used in Hamiltonian (1) are given explicitly:

$$\Delta_{ij} = \sum_n A_n (\delta_{i-j,R_n} - \delta_{j-i,R_n}) \quad (B1a)$$

$$h_{ij} = (O_i - \mu) \delta_{ij} + \sum_n B_n (\delta_{i-j,R_n} + \delta_{j-i,R_n}) \quad (B1b)$$

where $R_n$ are the real space vectors identifying the range and direction of the interactions (which are always lattice vectors), and $A$ and $B$ give the associated strengths (and phases). The on-site component $h_{ii} = -\mu + O_i$ matrix has been singled out, and separated into chemical potential $\mu$ and the $O_i$, which include disorder and edge-creating terms.

All pairing terms are $p + ip$, so the $A$ terms come in pairs: one real, and another in the perpendicular direction, purely imaginary. To control the explosion of terms, we make a bookkeeping decision and make $R_{\pm n}$ rotations of each other, and have $A_{\pm n}$ differ in phase by $\pi/2$.

As in FIG. 5 the NN terms have the form

$$R_{1,-1} = \hat{x}, \hat{y} \quad \text{with} \quad B_{\pm 1} = t_1 \quad \text{and} \quad A_{1,-1} = -i \Delta_1, \Delta_1.$$
Δ₁ and t₁ are assumed to be real. The NNN terms are similarly, 
\[ R_{k2} = \hat{x} \mp \hat{y} \quad \text{with} \quad B_{\pm 2} = t_2 \]
and
\[ A_{2,+2} = -i\Delta_2, \Delta_2 \quad \text{and} \quad A_{2,-2} = \Delta_2, -i\Delta_2. \]
In the + branch, the next-nearest pairing has the same chirality, and in the − branch it has opposite chirality.

Appendix C: Chern Number Review

Here, we review Chern number calculation in the defect free case, as in section [I] for nearest and next-nearest interactions. Put the Fourier transform of the Hamiltonian (1)
\[ \mathcal{H}_{ij} \to \mathcal{H}_k = n_k \cdot \tau, \]
where \( \tau \) is a vector of Pauli matrices and
\[ n_k = \begin{bmatrix} \Re \Delta_k \\ -\Im \Delta_k \end{bmatrix}. \] (C1)
The Chern number is obtained by integrating the Berry curvature
\[ \frac{1}{2\pi} \nabla_k \times \langle | \Psi_k \rangle \nabla_k | \Psi_k \rangle = \frac{1}{4\pi} (\nabla_k \Psi_k) \times (\nabla_k n_k) \]
\[ = \frac{1}{4\pi} \hat{n} \cdot \frac{\partial \hat{n}}{\partial k_x} \times \frac{\partial \hat{n}}{\partial k_y} \] (C2)
over the Brillouin zone. (Both equalities are due to straightforward calculation, though the second is tedious to show.) The vector-valued function \( n \) maps momentum space to \( \mathbb{R}^3 \), and characterizes the Cooper pairing (and corresponding quasiparticles) at a given momentum. The vanishing of \( n \) corresponds precisely to nodes in the band structure. Therefore, in the fully-gapped regime, the unit vector \( n \) maps \( T^2 \) to \( S^2 \), and the above integral is just the degree of the map \( n \), an integer.\(^\text{23}\)
According to the Hopf classification, the degree characterizes the mapping \( n \) topologically, i.e. up to homotopy. We emphasize here that we have so far said nothing about the presence of zero-energy modes or sublattices: only the topologically invariant Chern number.

One can do slightly better than this. By smoothly deforming \( n \) so that \( n = \mp \hat{z} \) except when \( h_k \) vanishes, the Chern number is seen to depend only on the winding of the phase of the superconducting order parameter around the Fermi surface (of the parent state, i.e., where \( h_k = 0 \)). Because such a smooth deformation will not close the band gap, the topological invariant is unchanged. The integral over the Brillouin zone therefore becomes a line integral over this \( h_k = 0 \) surface, which is sensitive only to the winding of the superconducting order parameter’s phase \( \phi \).

The winding of \( \phi \) can only occur around zeros of \( \Delta \), and always in multiples of \( 2\pi \). Neglecting higher-order zeros of \( \Delta \), one simply counts the number of zeros enclosed by the Fermi surface, and note whether their winding is clockwise or counterclockwise to get the Chern number. To get the sign of this answer correct, “enclosed” is taken to mean the particle-like side of the Fermi surface. We emphasize at this point that we are dealing with a quadratic, single-band Hamiltonian. Analogous results for multi-band Hamiltonians would be more complicated.

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