Classical and Quantum Scattering of Maximally Charged Dilaton Black Holes

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Abstract

The classical and the quantum scattering of two maximally-charged dilaton black holes which have low velocities are studied. We find a critical value for the dilaton coupling, $a^2 = 1/3$. For $a^2 > 1/3$, two black holes are always scattered away and never coalesce together, regardless of the value of the impact parameter.

1 Introduction

The slow motion of solitons is described by a geodesic motion in a finite-dimensional moduli space if several forces between the static solitons are balanced with one another or are assumed to be very weak.

An example is found in the system consisting of BPS monopoles. This system has been extensively studied for a decade [1, 2, 3, 4, 5, 6]. The magnetic repulsive force and Higgs scalar force between monopoles at rest are cancelled by each other in this case. The slow motions of BPS monopoles are analyzed by using the metric on the moduli space [1, 2, 3, 4, 5, 6].

Another example is found in the multi-vortex system in the Abelian Higgs model with a critical relation between gauge and Higgs couplings. In this system, the electromagnetic force and scalar force are not only reduced to short-range forces but also cancelled by each other. The vortex moduli space is also investigated by several authors [7, 8, 9, 10].

In this paper, we examine a special kind of multi-black-hole system. Originally, dynamics of extreme Reissner–Nordstrom (RN) black holes at low energy was considered by Gibbons and Ruback [12]. In this system, the strength of the gravitational force is equal to that of the Coulomb force between the extreme RN black holes. The moduli space metric for the extreme RN black holes at arbitrary distances was shown by Ferrell and Eardley [13]. More recently, Traschen and Ferrell [14] investigated quantum-mechanical scattering of the extremal RN holes by quantizing the moduli.
The charged black holes in the system we study here are assumed to be coupled to dilaton fields [15, 16, 17, 18, 19, 20, 21, 22, 23]. Thus a (attractive) scalar force is involved in the system. The multi-centered static configuration of maximally charged dilaton black holes was shown by the present author recently [22]. In this case, three forces are balanced with one another in the static system described by the solution. The metric on the moduli space was also obtained in Ref. [23]. In this paper, the scattering of two maximally-charged dilaton black holes is more closely investigated by classical and quantum-mechanical methods.

The dynamics of the extremal holes may depend on the dilaton coupling. The existence of critical values for the coupling has recently been pointed out by several authors through the study of the classical and quantum nature of the charged dilaton black holes [15, 16, 17, 18, 19, 20, 21, 22, 23]. We will rediscover the critical behavior in the present study on classical and quantum treatments of the moduli space.

In Sec. 2, the static solution in the Einstein-Maxwell-dilaton system is reviewed and the derivation of the metric on moduli space is discussed. In Sec. 3, the classical two-body scattering is studied. We compare the result of a small-particle approximation with that of a geodesic approximation on the moduli space in the corresponding limit. The comparison is absolutely necessary to confirm the validity of the approximation, even if it does not provide much knowledge. The cross-section is found, especially for the case with the dilaton coupling of string theory.

In Sec. 4, the quantum-mechanical scattering is examined. Again, we compare the wave scattering by a black hole and the description by a quantization procedure on the collective motion of black holes. The cross-section for partial waves are discussed.

The conclusion is given in Sec. 5. Throughout this paper, we deal with a (1 + 3)-dimensional spacetime.

2 The Moduli Space Metric for the System Consisting of Maximally Charged Dilaton Black Holes

The Einstein-Maxwell-dilaton system contains a dilaton field $\phi$ coupled to a $U(1)$ gauge field $A_{\mu}$, besides the Einstein-Hilbert gravity.

The action for the fields with particle sources is

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[ R - 2(\nabla \phi)^2 - \epsilon^{-2a\phi} F_{\mu\nu} F^{\mu\nu} \right] - \sum_a \int ds_a \left[ m_a e^{a\phi} + e_a A_{\mu} \frac{dx_{\mu}}{ds_a} \right],$$

where $R$ is the scalar curvature and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The Newton constant has been set to be unity. The dilaton coupling to the Maxwell term is governed by a constant $a$. Here the coupling to the point
source has already been fixed for later use. \( s_a \) parametrizes each worldline of the particle.

The metric for the \( N \)-body system of maximally-charged dilaton black holes has been known as [22]

\[ ds^2 = -U^{-2}(x^k)dt^2 + U^2(x^k)\delta_{ij}dx^i dx^j, \]  

with

\[ U(x^k) = \left\{ F(x^k) \right\}^{1/(1+a^2)} \]  

and

\[ F(x^k) = 1 + \sum_{a=1}^{N} \frac{(1 + a^2)m_a}{|x - x_a|}. \]

Using these expressions, the vector one-form and dilaton configuration are written as

\[ A = \left( \frac{1}{1+a^2} \right)^{1/2} \left[ 1 - \left\{ F(x^k) \right\}^{-1} \right] dt \]  

and

\[ e^{-2a\phi} = \left\{ F(x^k) \right\}^{2a^2/(1+a^2)}. \]

In this solution, the asymptotic values of \( \phi \) and \( A \) are fixed to be zero.

The electric charge \( e_a \) and scalar charge \( \sigma_a \) of each black hole are associated with the corresponding mass \( m_a \) by

\[ e_a = (1 + a^2)^{1/2} m_a, \]  

\[ \sigma_a = am_a. \]  

One can explicitly verify cancellation of static forces between the black holes at large distances using the relations. If we set \( a = 0 \) in the solution, we find that the solution reduces to the Papapetrou-Majumdar solution [24]. The dimension of the parameter space of the static solution is \( 3N \), which is the number of coordinates describing the position of \( N \) black holes.

We anticipate that an addition of a small amount of kinetic energy to this static system can be treated by perturbation. This is justified on the assumption that radiation reactions can be ignored at low velocity.

Although we can obtain the Lienard-Wiechert potential at large distances in the system [12], we must apply the method of Ferrell and Eardley [13] to the system in order to get the information on the slow motion of the extremal black holes at arbitrary distances.

The perturbed metric and potential can be written in the form

\[ ds^2 = -U^{-2}dt^2 + 2N^i dx^i dt + U^2 dx^i dx^j, \]

\[ A = \left( \frac{1}{1+a^2} \right)^{1/2} \left[ 1 - \left\{ F(x^k) \right\}^{-1} \right] dt + A_i dx^i, \]

where \( U \) and \( F(x) \) are defined by (3) and (4). We need only to solve linearized equations with perturbed sources [where \( e_a = (1 + a^2)^{1/2} m_a \) is substituted] up
to $O(v)$ for $N_i$ and $A_i$ for our purpose. Each source plays the role of a maximally charged dilaton black hole.

The authors of Refs. [13] and [14] have emphasized the “smearing” of the source to avoid the violet divergences. It is known that divergences often appear in gravitating systems [25, 26]. The systematic regularization procedure is rather technical and yields no ambiguity.

Solving the Einstein and Maxwell equations and substituting the solutions, the perturbed dilaton field and sources to the field-theoretical action (1) with proper boundary terms, we get the effective Lagrangian up to $O(v^2)$ for the $N$-maximallycharged-dilaton-black-hole system: [23]

$$L = - \sum_a m_a + \frac{1}{2} \sum_a m_a v_a^2 = \frac{3-a^2}{8\pi} \int d^3x |F(x)|^2 \frac{1-(1+a^2)/(1+a^2)}{2|\mathbf{r}_a|^2|\mathbf{r}_b|^2} \sum_{a,b} (\mathbf{n}_a \cdot \mathbf{n}_b)(\mathbf{v}_a - \mathbf{v}_b)^2 m_a m_b$$

(11)

where $\mathbf{r}_a = \mathbf{x} - \mathbf{x}_a$ and $\mathbf{n}_a = \mathbf{r}_a/|\mathbf{r}_a|$. $F(x)$ is defined by (4).

Sending the value of dilaton coupling $a$ to zero in the above formal expression, we reproduce the result of Ref. [13]. Furthermore, in the large-distance limit [where we approximate $F(x)$ by 1], we recover exactly the same result obtained by the Lienard-Wiechert method [12].

Generally there exist many-body interactions. In general, we obtain infinite species of many-body interactions by expanding the function $F(x)$. The following are some special cases:

- $a^2 = 0$: The black holes are governed only by two-body, three-body and four-body interactions. [13]
- $a^2 = 1/3$: There are two-body and three-body interactions.
- $a^2 = 1$: There are only two-body interactions.
- $a^2 = 3$: There is no interaction.

The general expression (11) needs further regularization on performing the integration. If one wants to study the slow motion of the maximally-charged dilaton black holes in the low-velocity limit, one has to analyze a two-body system at the first step. We consider a two-black-hole system hereafter.

After regularization, the effective Lagrangian for a two-body system (consisting of black holes labelled with $a$ and $b$) can be rewritten as

$$L_{2B} = -M + \frac{1}{2} MV^2$$

\[+ \frac{1}{2} \mu v^2 \left\{ 1 - \frac{M}{\mu} - \frac{(3-a^2)M}{r} + \frac{M}{m_a} \left[ 1 + \frac{(1+a^2)m_a}{r} \right]^{(3-a^2)/(1+a^2)} \right.\]

\[+ \frac{M}{m_b} \left[ 1 + \frac{(1+a^2)m_a}{r} \right]^{(3-a^2)/(1+a^2)} \right\}, \tag{12}\]
where \( M = m_a + m_b, \mu = m_am_b/M, \mathbf{V} = (m_a\mathbf{v}_a + m_b\mathbf{v}_b)/M, \mathbf{v} = \mathbf{v}_a - \mathbf{v}_b \) and \( r = |\mathbf{x}_a - \mathbf{x}_b| \).

Since the motion of the center-of-mass is separable, we focus on the relative motion even if we proceed to the quantum-mechanical two-body problem. In terms of the metric on the moduli space, the separation of the motion means that the metric takes the block-diagonal form.

The metric for the relative motion is

\[
g_{ab} = \gamma(r)\delta_{ab},
\]

with

\[
\gamma(r) = 1 - \frac{M}{\mu} - \frac{(3-a^2)M}{r} + \frac{M}{m_a} \left[ 1 + \frac{(1+a^2)m_a}{r} \right]^{(3-a^2)/(1+a^2)}
\]

\[
+ \frac{M m_b}{r} \left[ 1 + \frac{(1+a^2)m_a}{r} \right]^{(3-a^2)/(1+a^2)}.
\]

In particular, when \( m_a = m_b = m \) (then \( M = 2m \) and \( \mu = m/2 \)), \( \gamma(r) \) becomes

\[
\gamma(r) = 1 - \frac{(3-a^2)M}{r} + 4 \left[ 1 + \frac{(1+a^2)M}{2r} \right]^{(3-a^2)/(1+a^2)} - 4.
\]

On the other hand, if \( m_b \ll m_a \) (then \( M \sim m_a \), i.e., \( \mu \to 0 \), we obtain

\[
\gamma(r) = \left[ 1 + \frac{(1+a^2)M}{2r} \right]^{(3-a^2)/(1+a^2)}.
\]

We examine the cases with some special values for \( a \).

For \( a = 0 \), we recover the result of Ref. [13]

\[
\gamma(r) = 1 + \frac{3M}{r} + \frac{3M^2}{r^2} + \frac{M^3}{r^3} \left( 1 - \frac{2\mu}{M} \right).
\]

(Ref. [14] includes misprints at this point.)

For \( a^2 = 1/3 \), we find

\[
\gamma(r) = \left( 1 + \frac{4M}{3r} \right)^2.
\]

For \( a^2 = 1 \), the value corresponding to the string theory, we find

\[
\gamma(r) = 1 + \frac{2M}{r}.
\]

Note that \( \gamma(r) \) does not depend on the reduced mass \( \mu \) for both cases \( a^2 = 1/3 \) and \( a^2 = 1 \).

For \( a^2 = 3 \), we get an obvious result

\[
\gamma(r) = 1.
\]

In the following section, we will describe the classical scattering of two maximally charged dilaton black holes using the metric on the moduli space.
3 The Classical Scattering of Two Maximally Charged Dilaton Black Holes

Once the metric is known, it is easy to compute geodesic motion on the moduli space. We pay attention to the two-body scattering problem. We assume that the scattering occurs on a scattering plane, where the distance \( r \) and the azimuthal angle \( \theta \) determine the configuration. Then the metric on the moduli space for the two-body system is reduced to

\[
ds_{MS}^2 = \gamma(r)(dr^2 + r^2 d\theta^2).
\]

(21)

To analyze the geodesic trajectory on the plane is straightforward. We find that the trajectory is described by the differential equation

\[
\left( \frac{dr}{d\theta} \right)^2 = r^4 \left[ \frac{\gamma(r)}{b^2} - \frac{1}{r^2} \right].
\]

(22)

where \( b \) denotes the impact parameter.

If the mass of one of the black holes, \( M \), overwhelms the other, the equation becomes

\[
\left( \frac{dr}{d\theta} \right)^2 = r^4 \left\{ \frac{1}{b^2} \left[ 1 + \frac{(1 + a^2)M}{r} \right]^{(3-a^2)/(1+a^2)} - \frac{1}{r^2} \right\}.
\]

(23)

Before solving the equation, we consider the point-particle equations of motion in the background field of a maximally charged dilaton black hole with mass \( M \). In this case, we can identify the coordinates of background geometry with the collective coordinates on the moduli space. The particle, playing the role of a maximally charged dilaton black hole, has mass \( m \) and charge \( e = (1 + a^2)^{1/2}m \). The action for the particle is

\[
\int ds \left[ me^{\alpha \phi} + eA_\mu \frac{dx^\mu}{ds} \right],
\]

(24)

[taking the same form as in (11)] and the background metric and fields are expressed by (2, 3, 5, 6) with

\[
F(r) = 1 + \frac{(1 + a^2)M}{r}.
\]

(25)

After some manipulation, we can show that the equation of motion in the background fields is once integrated to become

\[
\left( \frac{dr}{d\theta} \right)^2 = r^4 \left[ \left( \frac{E - m}{L} \right)^2 F^{4/(1+a^2)} + \frac{2m(E - m)}{L^2} F^{(3-a^2)/(1+a^2)} - \frac{1}{r^2} \right],
\]

(26)

where the constants \( E \) and \( L \) correspond respectively to energy and angular momentum of the incident particle at spatial infinity. The energy and angular
momentum at infinity can be written as $E = m(1 - v^2)^{-1/2}$ and $L = mvb(1 - v^2)^{-1/2}$, where $v$ is the asymptotic velocity of the incident particle. Then (26) becomes

$$
\frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 = \frac{1}{b^2} F^{(3-a^2)/(1+a^2)} - \frac{1}{r^2} + \frac{v^2}{4b^2} F^{4/(1+a^2)} + O(v^4). \tag{27}
$$

Apparently, in the low-velocity limit $v \to 0$, the result (27) coincides with the previous result (23) (for any $a^2$) which came from analyzing the moduli metric. Note that the moduli space metric in a general case contains more information; the geodesic approximation can quantitatively describe the motion in the system with arbitrary $m$ and $\mu$ in the low-velocity limit.

Now we study the two-body problem in the low-velocity limit. First, we examine the classical condition that the two extremal dilaton black holes coalesce. In the geodesic approximation, two black holes coalesce if the following algebraic equation for $r$ has no positive root:

$$
r^2 \gamma(r) = b^2. \tag{28}
$$

It is possible that the equation has a positive solution for any value of $b$. In such a case, black holes are always scattered away (provided that the low-velocity approximation is valid).

Next, we examine the deflection angle $\Theta$. $\Theta = \pi$ corresponds to the forward scattering, while $\Theta = 0$ corresponds to the backward scattering. Since the particle (black hole) is scattered away if (28) has a solution, the deflection angle is expressed in the integral form

$$
\Theta = \int_{r_0}^{\infty} \frac{2b dr}{r \sqrt{r^2 \gamma(r) - b^2}} - \pi, \tag{29}
$$

where $r_0$ is the maximum positive solution for Eq. (28).

Then, the classical differential cross-section (per solid angle) can be computed:

$$
d\sigma \over d\Omega = \left| \frac{\sin \Theta}{b} \frac{d\Theta}{db} \right|^{-1}. \tag{30}
$$

The above expressions are valid for general cases, i.e., the collective motion of maximally-charged dilaton black holes. For a while, we focus our attention on a particular case where the mass of the “target” black hole ($M$) is much larger than the “incident” black hole. In this case, the reduced mass $\mu$ becomes infinitesimal. Then, we must solve (23), or apply Eqs. (28)-(30) to the case with the moduli space metric (16).

Considering the coalescence condition, we distinguish the following cases.

- $a^2 \leq 1/3$: There is a critical value $b_{\text{coal}}$ of $b$, inside of which black holes coalesce [13]. $b_{\text{coal}}$ turns out to be

$$
b_{\text{coal}} = \begin{cases} 
1 - 3a^2 & (3 - a^2)/(2(1+a^2)] M \quad \text{for } a^2 < 1/3, \tag{31} \\
(4/3)M & \text{for } a^2 = 1/3. \tag{32}
\end{cases}
$$
• $a^2 > 1/3$: In this case, black holes never coalesce for any value of $b$ (provided that the slow-velocity approximation is used).

The scattering cross-section can be analytically obtained only for special values of $a^2$. The most impressive and simple result is found in the case with $a^2 = 1$, i.e., the coupling which appears in string theory. The deflection angle is given by

$$\Theta = 2 \tan^{-1} \frac{M}{b} \quad (a^2 = 1),$$

and then, the classical differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \frac{M^2}{\sin^4(\Theta/2)} \quad (a^2 = 1).$$

This has precisely the same $\Theta$-dependence as the Rutherford scattering! In this case, the deflection angle $\Theta$ approaches $\pi$ as $b/M$ goes to zero. This behavior can be attributed to the deficit angle near the origin on the moduli space for $a^2 = 1$ [23]. Note that, in this case with $a^2 = 1$, the result (33) and (34) holds for any masses of black holes provided that $M = m_a + m_b$.

We can relate the qualitative features of scattering (or coalescence) to the shape of the moduli space for the two-body system with arbitrary $a^2$ and masses. The metric on the (reduced) moduli space (21) can be rewritten by coordinate transformation as

$$ds_{MS}^2 = h(R)dR^2 + R^2 d\theta^2,$$

where $R = \gamma^{1/2}r$ and

$$h(R(r)) = \left(1 + \frac{r}{2\gamma} \frac{d\gamma}{dr}\right)^{-2}.$$

For $a^2 < 1/3$, $R$ increases as the separation $r$ goes to zero; thus the surface of the moduli space immersed into three dimensions seems to have a “throat” [13, 14]. If the trajectory goes across the throat, it represents the coalescence of black holes. For $a^2 = 1/3$, $R$ approaches a constant when $r$ is reduced to zero. The surface looks like a funnel. For $a^2 > 1/3$, both $R$ and $r$ approach zero at the same time. In the vicinity of the origin, $R = 0$, the metric is approximated by

$$ds_{MS}^2 = \left[ \frac{2(1 + a^2)}{3a^2 - 1} \right]^2 dR^2 + R^2 d\theta^2 \quad (R \sim 0, \text{for } a^2 > 1/3).$$

One can show that there is a deficit angle $\Delta \theta$ around the origin:

$$\Delta \theta = \frac{3 - a^2}{1 + a^2} \pi \quad (R \sim 0, \text{for } a^2 > 1/3).$$

This result is independent of the masses of black holes. Therefore, for $a^2 > 1/3$, the deflection angle $\Theta$ approaches $\Delta \theta$ in the limit of small $b$. 
4 The Quantum Scattering of Two Maximally Charged Dilaton Black Holes

In the previous section, we have seen that the classical low-energy scattering of two maximally charged dilaton black holes is well described by a geodesic motion on the (reduced) moduli space. We consider the quantum scattering in this section. The quantization of the moduli parameters has been discussed in Refs. [3, 4, 11, 14] for various systems.

We first review the quantization on the moduli space. Let us introduce a wave function \( \Psi \) on the moduli space, which obeys the Schrödinger equation
\[
\frac{i \hbar}{\partial t} \Psi = \left( -\frac{\hbar^2}{2\mu} \nabla^2 + \hbar^2 \xi R_{(MS)} \right) \Psi ,
\]
where \( \nabla^2 \) is the covariant Laplacian constructed from the moduli space metric and \( R_{(MS)} \) is the scalar curvature of the moduli space. We assume \( \xi = 0 \) in this paper though this term may be present in most general cases. Here, we have already removed the center-of-mass degrees of freedom, as usual.

The partial wave in a stationary state is
\[
\Psi = \psi_{ql}(r)Y_{lm}(\theta, \phi) \exp(-iEt/\hbar) ,
\]
where \( Y_{lm}(\theta, \phi) \) is the spherical harmonic function and \( E = \hbar^2 q^2/(2\mu) \).

Recalling the metric of the form (13), we can rewrite the equation for the partial wave function as
\[
\psi'' + \frac{2}{r} \psi' + \frac{\gamma'}{2\gamma} \psi' - \frac{l(l + 1)}{r^2} \psi + q^2 \gamma \psi = 0 ,
\]
where the prime denotes the derivative with respect to \( r \).

Now, we compare the Schrödinger equation with the scalar wave function, whose quantum has small mass \( m \) and charge \((1 + a^2)^{1/2}m\), in the background metric and fields of a maximally charged dilaton black hole with mass \( M \). The scalar field is obtained from the second quantization of the particle action (24). The wave function is given by
\[
\bar{\hbar}^2 (\nabla_\mu + iA_\mu) e^{-2ab\phi} (\nabla^\mu + iA^\mu) \psi - e^{-2ac\phi} m^2 \psi = 0 ,
\]
where the constants \( b \) and \( c \), which satisfy \( b - c = 1 \) [19], arise from an ambiguity in the procedure of the second quantization. The partial wave decomposition leads to
\[
\Psi = \psi_{ql}(r)Y_{lm}(\theta, \phi) \exp(-i\omega t) ,
\]
where \( \hbar \omega = (m^2 + \hbar^2 q^2)^{1/2} \).

Substituting (2), (3), (5) and (6) with (25) into (42), we obtain the differential equation for the partial wave:
\[
\psi'' + \frac{2}{r} \psi' + \frac{X'}{2X} \psi' - \frac{l(l + 1)}{r^2} \psi + F^{4/(1 + a^2)} \left[ (\hbar \omega - m)^2 + \frac{2m(\hbar \omega - m)}{F} \right] \psi = 0 ,
\]
with $X = e^{-4ab\phi}$.

In the nonrelativistic and low-energy limit, $\hbar q \ll m$, the wave equation takes the following form:

$$\psi'' + \frac{2}{r} \psi' + \frac{X'}{2X} \psi' - \frac{l(l+1)}{r^2} \psi = -\frac{F^{(3-a^2)/(1+a^2)}}{\hbar^2 q^2} \psi + O(q^4). \quad (45)$$

This coincides with Eq. (41), up to the term originating from an ambiguity of “operator ordering”. The difference coming from two approaches is expected to be unimportant provided that $\psi'$ is small near $r \sim 0$.

Let us analyze the scattering problem, adopting the wave equation on the moduli space (41). (Hereafter, we set $\hbar = 1$ unless otherwise stated, for simplicity.) In this paper, we do not intend to survey the scattering problem for all possible values of $a^2$. We pick up typical cases and use some typical approximation method to show peculiarities of charged dilaton black holes.

The differential equation can be written in a simple form by introducing a new coordinate $R = \int \sqrt{\gamma} dr$ and writing $\psi = \chi/(r\sqrt{\gamma})$. Then, one can find $\chi$

$$\chi'' R R + (q^2 - V) \chi = 0, \quad (46)$$

with

$$V = \frac{1}{2r\gamma} \left( \frac{r\gamma'}{\gamma} \right)' + \frac{l(l+1)}{r^2\gamma} = \frac{\gamma (r\gamma')' - r(\gamma')^2}{2 r^2 \gamma^3} + \frac{l(l+1)}{r^2\gamma}. \quad (47)$$

(Ref. [14] includes misprints at this point.)

As in the previous section, we treat the case with $\mu \to 0$, i.e., $m_b \ll m_a \sim M$. We have to remark that the results hold good for the cases with arbitrary values of $\mu$ if $a^2$ takes some special values such as 1 or $1/3$.

In the limit of $\mu \to 0$, $V$ takes the form

$$V(r(R)) = \frac{(3-a^2) \left[ \frac{r}{(1+a^2)M} \right]^{2(1-a^2)/(1+a^2)}}{2(1+a^2)^3 M^2 \left[ 1 + \frac{r}{(1+a^2)M} \right]^{(3a^2)/(1+a^2)}} + \frac{l(l+1)}{(1+a^2)^2 M^2 \left[ 1 + \frac{r}{(1+a^2)M} \right]^{(3-a^2)/(1+a^2)}}. \quad (48)$$

The range of the new coordinate $R$ depends on the dilaton coupling $a^2$. For $a^2 \leq 1/3$, $R$ varies from $-\infty$ to $+\infty$, as $r$ varies from 0 to $+\infty$. For $a^2 < 1/3$,

$$R \sim r + \left( \frac{3-a^2}{2} \right) M \ln \left[ \frac{r}{(1+a^2)M} \right] \quad \text{as } r \to \infty \quad \text{and}$$
For a totally region, we find

\[ V(\infty) > 0 \] to \( + \infty \) in finite barrier at \( R \) is realized when \( q \).

For \( a^2 > 1/3 \), the range of \( R \) is \([0, \infty]\). Asymptotically,

\[ R \sim r + \left( \frac{3 - a^2}{2} \right) M \ln \left( \frac{r}{(1 + a^2)M} \right) \quad \text{as } r \to \infty \]

and

\[ R \sim \frac{2(1 + a^2)^2}{3a^2 - 1} M \left( \frac{r}{(1 + a^2)M} \right)^{(3a^2 - 1)/(2(1 + a^2))} \quad \text{as } r \to 0. \]

Consequently, we can draw the shape of the potential \( V(R) \). In the asymptotic region, we find

\[ V(R) \sim \frac{(3 - a^2)M}{2R^3} + \frac{l(l + 1)}{R^2} \quad \text{for } R/(1 + a^2)M \gg 1 \quad (r \to \infty). \quad (49) \]

At the small separation length, on the other hand, \( V(R) \) looks like

\[ V(R) \sim \frac{3 - a^2}{2(1 + a^2)^3M^2} \left[ \frac{(3a^2 - 1)R}{2(1 + a^2)M} \right]^{-4(1 - a^2)/(1 - 3a^2)} + \frac{4(1 + a^2)^2 l(l + 1)}{(1 - 3a^2)^2 R^2}. \quad (50) \]

This should read as \( V(R \sim -\infty) \) for \( a^2 < 1/3 \), while as \( V(R \sim 0) \) for \( a^2 > 1/3 \).

For the critical value \( a^2 = 1/3 \), as \( R \to \infty \), the potential becomes

\[ V(R) = \frac{9}{16M^2} \exp \left( \frac{3R}{4M} \right) + \frac{9l(l + 1)}{16R^2} \exp \left( -\frac{3R}{2M} \right) \quad \text{for } R \to -\infty. \quad (51) \]

The potentials for some peculiar values of \( a^2 \) are plotted in Fig. 1(a-c) and Fig. 2. There are three cases which must be distinguished. The qualitative feature of the potential for \( a^2 < 1/3 \) is similar to that of the potential for \( a^2 = 0 \), which has been shown in Fig. 1 of Ref. [14]. The case for \( a^2 = 1/3 \) is a special case, where a barrier of an infinite height exists for \( l \neq 0 \).

For \( 1/3 < a^2 < 1 \), there is a similar barrier for \( l \neq 0 \), though \( R \) ranges from 0 to \( +\infty \). The case with \( a^2 = 1 \) is another critical case. For \( 1 < a^2 < 3 \), the infinite barrier at \( R = 0 \) exists for all \( l \).

The absorption of the wave into the origin is possible only if \( a^2 \leq 1/3 \). The capture may occur dominantly in \( l = 0 \) mode, as is seen from the figures. The qualitative feature of capture cross-section for the case with \( a^2 < 1/3 \) is similar to that for the case with \( a^2 = 0 \), which has been studied intensively by Traschen and Ferrell [14]. We thus only make a few remarks here.

In a classical picture, a particle can go beyond a potential barrier only if the amount of its energy is larger than the potential maximum. This situation is realized when \( q^2 > V_{\text{max}} \), where \( V_{\text{max}} \) is the maximum height of the potential \( V \). Semiclassical description is known to be valid for a large angular momentum, or equivalently, a large impact parameter. For large \( l \), we obtain

\[ V_{\text{max}} \sim \frac{4l(l + 1)}{(1 - 3a^2)^2M^2} \left( \frac{1 - 3a^2}{3 - a^2} \right)^{(3 - a^2)/(1 + a^2)} \quad (52) \]
On the other hand, the angular momentum is written in the semiclassical region as $\sim \hbar q b \sim \hbar \sqrt{l(l+1)}$. Therefore the capture by the black hole occurs when

$$b \sim \frac{\sqrt{l(l+1)}}{q} < \frac{1 - 3a^2}{2} \left( \frac{3 - a^2}{1 - 3a^2} \right)^{-(3-a^2)/(2(1+a^2))} M.$$  \hspace{1cm} (53)

This condition agrees with the classical result, (31). We thus find that the semiclassical treatment yields the same result obtained by the classical description.

For $a^2 > 1/3$, all waves are scattered back to infinity. This behavior corresponds to the classical picture. Since our analysis is originally based on the low-energy assumption, the partial-wave analysis on the scattering problem may be most appropriate.

In the case with $a^2 > 1/3$, the asymptotic behavior of the partial wave at infinity is expressed by using spherical Bessel functions:

$$\frac{1}{R} \chi_l(R) \sim a_l j_l(qR) + b_l n_l(qR)$$
$$\sim c_l \frac{1}{qR} \cos \left( qR - \frac{l + 1}{2} \pi + \delta_l \right) \text{ for } R \to \infty , \hspace{1cm} (54)$$

where $\delta_l$ is called a phase shift and $a_l = c_l \cos \delta_l$, $b_l = -c_l \sin \delta_l$.

The scattering amplitudes for partial waves are constructed from the phase shift, i.e., [27]

$$f_l(\Theta) = \frac{2l + 1}{2iq} (e^{2i\delta_l} - 1) P_l(\cos \Theta) , \hspace{1cm} (55)$$

where $P_l(x)$ is a Legendre polynomial. The differential cross-section $d\sigma/d\Omega$ is obtained from the amplitude

$$\frac{d\sigma}{d\Omega} = |f(\Theta)|^2 , \hspace{1cm} (56)$$

with

$$f(\Theta) = \sum_{i=0}^{\infty} f_i(\Theta) . \hspace{1cm} (57)$$

Returning to the wave equation, we can find an equation by using the constancy of the Wronskian. That is

$$\tan \delta_l = -q \int_0^\infty dR R \left[ V(R) - \frac{l(l+1)}{R^2} \right] j_l(qR) \frac{1}{a_l} \chi_l(qR) . \hspace{1cm} (58)$$

First we consider small $q$. Then, $\chi_l(qR)$ can be approximated by $a_l R j_l(qR)$. Consequently we obtain the result of the first Born approximation:

$$\tan \delta_l = -q \int_0^\infty dR R^2 \left[ V(R) - \frac{l(l+1)}{R^2} \right] |j_l(qR)|^2 . \hspace{1cm} (59)$$
The most dominant contribution comes from the $S$ wave. For example, we estimate the phase shift of the $S$ wave in the case of $a^2 = 1$. Since $V(0) = 1/(2r_+^2)$ and the half-width is roughly $r_+$ (as is seen from Fig. 2), we find

$$\delta_0 \sim -\frac{qr_+}{6} \quad (a^2 = 1),$$

where $r_+ = (1 + a^2)M$. For higher modes, $\delta_l \sim -(qr_+)^{2l+1}$. The total cross-section for the $S$ wave is roughly $4\pi r_+^2$, or rather small.

Next, we examine the case with large $q$. The low velocity and large $q$ can be realized if $\bar{\hbar} \ll 1$, because $v \sim \bar{\hbar}q/\mu$. Hence, we consider the semiclassical treatment to this end. The semiclassical approximation is valid for the scattering problem if the variation of the wavelength is smooth. We find the phase shift in this prescription: [27]

$$\delta_l = -qR_0 + \int_{R_0}^{\infty} dR \left\{ [q^2 - V(R) - 1/(4R^2)]^{1/2} - q \right\} + \frac{2l + 1}{4}\pi,$$

where $R_0$ satisfies $V(R_0) = q^2$.

For $a^2 > 1/3$, the angular-momentum barrier at $R = 0$ is dominant for $l \neq 0$. In this case, we can use (50) and obtain the following:

$$\delta_l = -\frac{\pi}{2} \left( \frac{3 - a^2}{3a^2 - 1} \right) \left[ l + \frac{1}{2} - \frac{5a^2 + 1}{16(1 + a^2)} \left( l + \frac{1}{2} \right)^{-1} \right],$$

which is valid for large $q$.

Again, we examine the $a^2 = 1$ case. Let us try to evaluate the total cross-section including higher wave modes in this case. The total cross-section is generally given by: [27]

$$\sigma = \frac{4\pi}{q^2} \sum_l (2l + 1) \sin^2 \delta_l.$$

Now, $\delta_l \simeq -(\pi/2)(l + 1/2)$, and then we get

$$\sigma = \frac{2\pi}{q^2} \sum_{l=0}^{l_{\text{max}}} (2l + 1) = \frac{2\pi}{q^2}(l_{\text{max}} + 1)^2 \quad (a^2 = 1).$$

If we regard $l_{\text{max}}$ as $\simeq q\bar{b}$, the total cross-section diverges owing to the contribution from $q\bar{b} \gg 1$, or equivalently, at the small deflection angle. This feature looks similar to the Coulomb scattering. This fact agrees well with the classical analysis for $a^2 = 1$.

5 Conclusion

In this paper, we have examined the classical and the quantum scattering problem of two maximally charged dilaton black holes. The system is governed by
a control parameter, \( a \), which determines the strength of coupling between a dilaton field and Maxwell field. We have found that the nature of the velocity-dependent force at the lowest order is sensitive to the dilaton coupling.

Therefore, the scattering of two dilaton black holes changes its manner as \( a^2 \) takes a different value. We have studied the two-body scattering by classical and quantum-mechanical analyses making use of the metric on the moduli space.

The small-particle (wave) approximation is found to reproduce the same result obtained by the analysis on the moduli space, both in classical and quantum cases. We have also found a critical value for \( a^2 \) in the scattering problem in this paper: that is \( a^2 = 1/3 \). Below this value, the coalescence (or absorption) of black holes occurs for a sufficiently small impact parameter. On the other hand, above the critical value, it is impossible for two black holes to merge into one. This behavior can be illustrated by both classical and quantum analyses on the moduli space.

We have mainly examined the case where one of the black holes is much more massive than the other (i.e., in the limit of infinitesimal reduced mass of the system). We must note, however, that the nature of the two-body problem at low energy is expected not to be sensitive to the reduced mass. For example, the moduli space metric is independent of the reduced mass of the system for the case with \( a^2 = 1/3 \) and \( a^2 = 1 \). Considering the geometry of the moduli space, we can safely say that the characteristics of the scattering are unchanged in quality by the finite reduced mass, especially when \( a^2 \) takes a finite value (but \( < 3 \)).

We have not yet understood all the aspects of many-body forces in the multi-maximally-charged-black-hole system. We must continue to make every effort to solve the few-body problem.

The analysis on the moduli space can be generalized to the many-body problems of black strings, black membranes, etc. [28]

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Figure 1: The potential $V(R)$ is plotted against $R$, for several cases. (a) $a^2 = 0$, (b) $a^2 = 1/3$, (c) $a^2 = 5/7$, (d) $a^2 = 1$ and (e) $a^2 = 7/5$. The $l = 0$ and $l = 1$ cases are shown in each figure. Here, $R$ is normalized by $r_+$ and $V(R)$ is normalized by $(r_+)^{-2}$, with $r_+ = (1 + a^2)M$. 
Figure 2: The potential $V(R)$ for $l = 0$ is plotted against $R$ in a graph. The normalization is the same as in Fig. 1.