Graph Quantum Groups

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Abstract

We define a new mathematical structure (graph quantum group) which combines the tower of algebras associated with a graph $\mathcal{G}$ and the structure of a Hopf algebra $\mathcal{A}$. In this structure Ocneanu’s string operators become quantum group intertwiners. We present some examples of graph quantum groups.
1 Introduction

There are two basic mathematical structures underlying the integrability of two dimensional lattice models, namely quantum groups [1] and Temperley-Lieb-Jones algebras [2, 3], which are associated respectively to vertex models and IRF or RSOS models [4]. The lattice variables for vertex models are in correspondence with irreps of a Hopf algebra $A$ while for IRF models they are labelled by the nodes of a given graph $G$. The TLJ algebra is defined using the graph $G$ by the Jones fundamental construction [3]. When the graph $G$ is defined by means of the decomposition rules of the tensor products of irreps of $A$ and the nodes are associated to these irreps, then a vertex-IRF map can be defined relating these two types of models [2].

It is well known that solvable integrable models can usually be interpreted as factorizable scattering theories [5]. In the case of vertex models one finds a scattering of particles whose internal quantum numbers are governed by irreps of the Hopf algebra $A$, which plays the role of a symmetry acting on asymptotic states. On the other hand the IRF models are more suitable for a description of scattering of kinks interpolating between asymptotic vacua configurations. Under this interpretation the nodes of the graph $G$ are the vacua and the links are the kinks [7].

The study of $N = 2$ integrable massive field theories has led to the construction of a more general class of factorizable scattering theories which combine in a non trivial way the two classes mentioned above. In these theories the solitons are kinks with extra internal quantum numbers associated to a Hopf algebra $A$ and such that the scattering $S$-matrices depend in general, in a rather complicated way, on both the Hopf algebra $A$ and the graph $G$ which characterizes the kinks.

This physical problem has motivated us to generalize the definition of quantum groups in order to include in a non trivial way the graph $G$. We call this new structure graph quantum group (GQG) [8].

2 Graph Quantum Groups: definition

In order to define a graph quantum group, $(A, G, w)$, we will use:

i) A Hopf algebra $A$

ii) A connected graph $G$

iii) A map $w$:
which associates to each link of the graph \( \Gamma \) an irreducible finite dimensional representation of \( A \).

We will reduce our construction to graphs whose incidence matrix \( \Lambda_{a,b} \) (\( a, b \) nodes of \( \Gamma \)) takes only the values 0 or 1, i.e., there exist at most one link between two nodes.

Given two arbitrary nodes \( a, b \) of \( \Gamma \) we define the space \( \Omega_{a,b}^{(n)} \) of paths with \( n \) links starting at the node \( a \) and finishing at the node \( b \):

\[
\Omega_{a,b}^{(n)} = \{ \xi = (a_0, a_1, \ldots, a_{n-1}, a_n) \mid a_0 = a, a_n = b ; \Lambda_{a_i,a_{i+1}} = 1 \}
\]

We shall use the notation \( \xi(i) = a_i \) to characterize the value of the path \( \xi \) at level \( i \).

Using the map \( w \) we can associate to each path \( \xi \) in \( \Omega_{a,b}^{(n)} \) a \( n \)th tensor product of irreps of \( A \):

\[
w : \Omega_{a,b}^{(n)} \to \otimes^n \text{Rep } A
\]

\[
\xi \to w(\xi) = w(a_0, a_1) \otimes \cdots \otimes w(a_{n-1}, a_n)
\]

We will call a plaquette of \( \Gamma \) any couple of paths \( (\gamma^+, \gamma^-) \) of length two starting and finishing at the same points in \( \Gamma \), i.e., \( \gamma^+ = (a, d, c), \gamma^- = (a, b, c) \) such that \( \Lambda_{a,b} = \Lambda_{b,c} = \Lambda_{a,d} = \Lambda_{d,c} = 1 \). Sometimes we will use the notation \( \begin{pmatrix} a & d \\ b & c \end{pmatrix} \) for representing a plaquette.

**Definition 1.** For any plaquette \( (\gamma^+, \gamma^-) \) of \( \Gamma \) we define the plaquette operators \( T_i(\gamma^+, \gamma^-), i = 1, \ldots, n-1 \), acting on the space of paths as follows:

\[
\xi \to \xi' = T_i(\gamma^+, \gamma^-)\xi
\]

\[
\xi'(j) = \prod_{k=0}^{2} \delta_{\xi(i-1+k), \gamma^-(k)} \times \begin{cases} \xi(j) & j \neq i \\ \gamma^+(1) & j = i \end{cases}
\]

The vanishing of the Kronecker’s delta symbol in eq.\( (4) \) will mean that the result of the action of \( T_i(\gamma^+, \gamma^-) \) on \( \xi \) produces an ”empty” path \( \phi \).

The plaquette operators can be composed in a path way. They for instance satisfy:

\[
T_i(\gamma^+, \gamma^-) T_i(\eta^+, \eta^-) = \delta_{\gamma^-, \eta^+} T_i(\gamma^+, \eta^-) \\
T_i(\gamma^+, \gamma^-) T_j(\eta^+, \eta^-) = T_j(\eta^+, \eta^-) T_i(\gamma^+, \gamma^-), \ |i - j| \geq 2
\]
By the map $w$ we can lift the plaquette operators $T_i(\gamma^+, \gamma^-)$ to operators $R_i(\gamma^+, \gamma^-)$ acting on tensor products of irreps:

$$
\begin{align*}
\xi & \xrightarrow{w} w(\xi) \\
T_i(\gamma^+, \gamma^-) & \downarrow \quad \downarrow \\
\xi' & \xrightarrow{w} w(\xi')
\end{align*}
$$

(6)

So far we have considered $A$ to be a non affine Hopf algebra, however if $A$ is affine then we can associate to each link $(a, b)$ an affine irrep:

$$(a, b) \in \text{Links}(G) \rightarrow (w(\xi), \theta)$$

(7)

where $\theta$ is the affine parameter (see section 3 for explicit examples). Interpreting now a plaquette operator $(\gamma^+, \gamma^-)$ as a two particle elastic scattering process, we associate with it an affine parameter $\theta_{12}$ representing the relative rapidity, namely for a plaquette

$$
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
$$

we use the following map (7):

$$
\begin{align*}
(a, b) & \rightarrow (w(a, b), \theta_1) \\
(b, c) & \rightarrow (w(b, c), \theta_2) \\
(a, d) & \rightarrow (w(a, d), \theta_2) \\
(d, c) & \rightarrow (w(d, c), \theta_1)
\end{align*}
$$

(8)

**Definition 2.-** Associated with the map $w$ we define the sets:

$$
g_{a,b}^{(n,w)} = \{ (\xi, w(\xi)) \mid \xi \in \Omega_{a,b}^{(n)} \}$$

(9)

and the vector spaces $C[g_{a,b}^{(n,w)}]$, which consist of the linear combinations:

$$
v = \sum_{(a_0, a_1, \ldots, a_{n-1}) \in \Omega_{a,b}^{(n)}} \sum_{m_1, \ldots, m_n} v_{m_1, \ldots, m_n}^{a_0, \ldots, a_n} e_{m_1}^{(a_0, a_1)} \otimes \cdots \otimes e_{m_n}^{(a_{n-1}, a_n)}
$$

(10)

where $e_{m_i}^{(a_{i-1}, a_i)}(m_i = 1, \ldots, \dim V_{w(a_{i-1}, a_i)})$ is a basis of the vector space $V_{w(a_{i-1}, a_i)}$.

**Definition 3.-** On $C[g_{a,b}^{(n,w)}]$ we define the "Yang-Baxter" operators $S_i(\theta)$ as follows:
the representation
operators $S$
Yang-Baxter relation [4]:
for any element $g$
a graphic representation of this equation) :

to the graph-Yang-Baxter equation (see figure 1 for
This is equivalent to say that for any plaquette
Definition 4.- A triplet $(A, G, w)$ is a graph quantum group if there exist plaquette operators $S_i(\gamma^+, \gamma^-)$ such that:

i) $A$ -covariance:

\[ S_i(\gamma^+, \gamma^-)(\theta) \rho_{w(\gamma^-)}[\Delta(g)] = \rho_{w(\gamma^+)}[\Delta(g)]S_i(\gamma^+, \gamma^-)(\theta) \] (12)

for any element $g$ of $A$. By $\rho_{w(\gamma^+)}[\Delta(g)]$ we mean the action of $\Delta(g)$ particularized to the representation $w(\gamma^+)$. 

ii) graph-Yang-Baxter equation (gYB): The operators $S_i(\theta)$ must satisfy the Yang-Baxter relation [4]:

\[ S_i(\theta) S_{i+1}(\theta + \theta') S_i(\theta') = S_{i+1}(\theta') S_i(\theta + \theta') S_{i+1}(\theta) \]
\[ S_i(\theta) S_j(\theta') = S_j(\theta') S_i(\theta) \quad |i - j| \geq 2 \] (13)

Condition i) means that the plaquette operators are intertwiners of the action of $A$. This is equivalent to say that for any plaquette $(\gamma^+, \gamma^-)$ the representations $w(\gamma^+)$ and $w(\gamma^-)$ are equivalent. Using now the matrix representation (11) of the $S_1$ operators we deduce that the eq.(13) is equivalent to the graph-Yang-Baxter equation (see figure 1 for a graphic representation of this equation):

\[ \sum_{d, \{m', m''\}} S_{m_1 m_2}^{m_2 m_3} \left( a_1 \ a_2 \ (\theta) \right) S_{m_2 m_3}^{m_3 m_4} \left( \frac{d}{a_3} \ a_4 \right) \left( \theta + \theta' \right) S_{m_3 m_4}^{m_4 m_5} \left( a_1 \ a_2' \ \frac{d}{d_3} \ a_3' \right) \left( \theta' \right) \] (14)

Given two paths $(\eta^+, \eta^-)$ of length $m$ bigger than two, starting and finishing at the same nodes, i.e. $\eta^+(0) = \eta^-(0), \eta^+(m) = \eta^-(m)$, we can construct different tilings of the
Figure 1: graph-Yang-Baxter equation.

region encircled by the path \( \eta^+ \cup \eta^- \) using the plaquettes defined above. Each tiling give rise to an operator \( S(\eta^+, \eta^-) \) (obtained by composing the corresponding plaquette operators). The gYB equation implies that the operator \( S(\eta^+, \eta^-) \) is tiling independent. By analogy with Ocneanu’s path model of subfactors we shall call \( S(\eta^+, \eta^-) \) string operators \([9]\). These operators satisfy discrete Migdal-Makeenko type equations:

\[
S(\eta^+, \eta^-) = \sum_{\tilde{\eta}} S(\eta^+, \tilde{\eta}) S(\tilde{\eta}, \eta^-) \quad (15)
\]

Using elementary plaquette operators we can also define discrete path derivatives. The collection of all string operators of length \( m \) define an algebra \( S_m \). Our construction defines a tower of algebras \( \cdots \subset S_m \subset S_{m+1} \subset \cdots \), which acts on tensor products of irreps of the Hopf algebra \( A \). It would be interesting to know whether this tower defines a subfactor.

Comments:
1) The lift by the map \( w \) of the plaquette operator, which according to condition i) of the definition 4 is an intertwiner of the algebra \( A \), is not in general an element of the centralizer of \( A \) and therefore cannot be derived from a universal \( R \)-matrix.

2) In the definition of GQG’s we may restrict ourselves to a certain family of plaquettes. This restriction should not mean that the reduced set of plaquettes can be defined by any subgraph of \( G \). We shall call this structure restricted graph quantum group (rGQG).
3 Examples

Example 1: N=2 supersymmetric massive theories.

All integrable $N=2$ supersymmetric massive theories \cite{10, 11, 12} define graph quantum groups \cite{8}. The lift of the plaquette operator represents the scattering $S$ matrix for their solitonic spectrum. In many cases the $S$-matrix operator factorizes into the $R$ matrix of the $N=2$ Hopf algebra and a solution to the IRF Yang Baxter equation for a given graph \cite{13, 15}. Next we define the graph quantum group associated to any integrable $N=2$ massive theory.

i) The Hopf algebra $\mathcal{A}$ is the $N=2$ algebra generated by the susy charges $Q^\pm, \bar{Q}^\pm$, the topological central charge $W$ and the fermion number $\mathcal{F}$, satisfying the following (anti)commutation relations:

\begin{align*}
(Q^\pm)^2 &= (\bar{Q}^\pm)^2 = \{Q^+, Q^-\} = \{Q^-, Q^+\} = 0, \\
\{Q^+, Q^-\} &= P, \quad \{Q^+, \bar{Q}^-\} = \bar{P}, \\
\{Q^+, \bar{Q}^+\} &= W, \quad \{Q^-, \bar{Q}^-\} = \bar{W}, \\
[\mathcal{F}, Q^\pm] &= \pm Q^\pm, \\
[\mathcal{F}, \bar{Q}^\pm] &= \pm \bar{Q}^\pm
\end{align*}

and comultiplication rules:

\begin{align*}
\Delta Q^\pm &= Q^\pm \otimes 1 + e^{\pm i\pi \mathcal{F}} \otimes Q^\pm \\
\Delta \bar{Q}^\pm &= \bar{Q}^\pm \otimes 1 + e^{\mp i\pi \mathcal{F}} \otimes \bar{Q}^\pm \\
\Delta W &= W \otimes 1 + 1 \otimes W \\
\Delta P &= P \otimes 1 + 1 \otimes P
\end{align*}

(17)

ii) The graph $\mathcal{G}$ is defined by the spectrum as follows \cite{13, 14}: The nodes and links of $\mathcal{G}$ represent respectively the different vacua configurations and the Bogomolnyi solitons.

iii) The map $w$ is defined associating to each link $(a, b)$ a two dimensional irrep of $\mathcal{A}$, labelled by the casimirs, $m_{a,b}$ which gives the mass, $\Delta_{a,b}$ which is the eigenvalue of the topological central charge, and the fermion number $f_{a,b}$, as follows:

\begin{align*}
\pi_{a,b}(\theta)(Q^-) &= \begin{pmatrix} 0 & 0 \\ \sqrt{m_{a,b} e^{\theta/2}} & 0 \end{pmatrix}, \quad \pi_{a,b}(\theta)(Q^+) = \begin{pmatrix} 0 \sqrt{m_{a,b} e^{\theta/2}} \\ 0 & 0 \end{pmatrix}
\end{align*}
\[\pi_{a,b}(\theta) (\bar{Q}^+) = \begin{pmatrix} 0 & 0 \\ \omega_{a,b} \sqrt{m_{a,b}} e^{-\theta/2} & 0 \end{pmatrix}, \quad \pi_{a,b}(\theta) (\bar{Q}^-) = \begin{pmatrix} 0 & \omega_{a,b}^* \sqrt{m_{a,b}} e^{-\theta/2} \\ 0 & 0 \end{pmatrix} \] (18)

\[\pi_{a,b}(\theta) (\mathcal{F}) = \begin{pmatrix} f_{a,b} & 0 \\ 0 & f_{a,b} - 1 \end{pmatrix}\]

where \(\omega_{a,b} = \frac{\Delta_{a,b}}{\Delta_{a,b}}\). If the \(N = 2\) massive theory admits a lagrangian representation then the map \(w\) and the graph \(\mathcal{G}\) can be defined in terms of a non degenerate Landau-Ginzburg superpotential \(W\) \[11\]. In this case the nodes of the graph are the critical points of \(W\), the topological charge \(\Delta_{a,b}\) is given by \(W_b - W_a\), where \(W_a\) is the critical value of \(W\) at the point \(a\), and the fermion numbers are computed from the index theorem formula:

\[\exp(2\pi i f_{a,b}) = \text{phase} \left( \frac{\det H(b)}{\det H(a)} \right)\] (19)

iv) To define the plaquette operators we consider elastic processes satisfying equal mass and equal fermion number for opposite sides.

In table 1 we list the GQG intertwiners, solutions to conditions i) and ii) of the definition 4, for a reduced set of \(N = 2\) massive Landau-Ginzburg theories \(\text{see reference } [8] \text{ for details)}.

**Example 2: Quantum Groups with continuous Spec**

Generically we can equip a given Hopf algebra with a GQG structure whenever there exist couples of equivalent representations of the type:

\[\rho_1 \otimes \rho_2 \simeq \rho_3 \otimes \rho_4\] (20)

where the irreps \(\rho_3\) and \(\rho_4\) differ from \(\rho_1\) or \(\rho_2\). This situation can be expected to occur when the Hopf algebra possesses finite dimensional irreps parametrized by continuous casimirs, as it is the case of Hopf algebras at root of unit \([17]\). To illustrate how this may happen we shall considered the Hopf algebra \(\tilde{U}_q(A_1)\) \([18]\) with generators \(E, F, K, U\). If the deformation parameter \(q\) is a \(\ell\) root of unit, i.e. \(q^\ell = 1\), then \(\tilde{U}_q(A_1)\) has a large center generated, in addition to the usual quadratic casimir \(C\), by \(X = E^{\ell'}, Y = F^{\ell'}, Z = K^{\ell'}\) \((\ell' = \ell\) if \(\ell\) is odd and \(\ell' = \ell/2\) if \(\ell\) is even \) and the central element \(U\). These four casimirs form a central Hopf subalgebra with comultiplications:

\[\Delta X = X \otimes 1 + Z U^{\ell'} \otimes X\]
\[\Delta Y = Y \otimes Z^{-1} + U^{\ell'} \otimes Y\] (21)
Table 1: All these examples are perturbations of ADE Landau-Ginzburg models. The superpotential $T_{k+2}(x)$ of the model $A_{k+1}$ is the Chebyshev polynomial $T_n(2 \cos \theta) = 2 \cos n\theta$.

\[ \Delta Z = Z \otimes Z \]
\[ \Delta U = U \otimes U \]

The conditions to have the equivalence shown in equation (20) is that $X, Y, Z, U$ take the same values on both representations:

\[
X(\rho_1 \otimes \rho_2) = X(\rho_3 \otimes \rho_4) \Rightarrow x_1 + z_1 u_1^{\ell^2} x_2 = x_3 + z_3 u_3^{\ell^2} x_4
\]
\[
Y(\rho_1 \otimes \rho_2) = Y(\rho_3 \otimes \rho_4) \Rightarrow y_1 z_2^{-1} + u_1^{-\ell^2} y_2 = y_3 z_4^{-1} + u_3^{-\ell^2} y_4
\]
\[
Z(\rho_1 \otimes \rho_2) = Z(\rho_3 \otimes \rho_4) \Rightarrow z_1 z_2 = z_3 z_4
\]
\[
U(\rho_1 \otimes \rho_2) = U(\rho_3 \otimes \rho_4) \Rightarrow u_1 u_2 = u_3 u_4
\]  

For each solution of these equations one is able to find a $\ell^2 \times \ell^2$ matrix $R \left( \begin{array}{c} \rho_3 \rho_4 \\ \rho_1 \rho_2 \end{array} \right)$, which is an intertwiner realizing the equivalence (21).

To look for solutions of the graph-Yang-Baxter equation we need to consider equivalences of the form:

\[
\rho_1 \otimes \rho_2 \otimes \rho_3 \simeq \rho_4 \otimes \rho_5 \otimes \rho_6
\]  

Model & Superpotential $W$ & $S$-matrices & \\
--- & --- & --- & \\
$A_{k+1}(t_1)$ & $\frac{x^{k+2}}{k+2} - x$ & Intertwiners of nilpotent irreps of $U_q(A^{(1)}_1)(q^4 = 1)$ & \\
$A_{k+1}(t_2)$ & $\frac{x^{k+2}}{k+2} - \frac{x^2}{2}$ & Susy generalization of chiral Potts Boltzmann weights & \\
$A_{k+1}(t_k)$ & $\frac{T_{k+2}(x)}{k+2}$ & Intertwiners of spin 1/2 irrep of $U_q(A^{(1)}_1)(q^4 = 1)$ × Andrews-Baxter-Forrester Boltzmann weights & \\
$D_{k+3}(\tau)$ & $\frac{x^{k+2}}{2(k+2)} + \frac{xy^2}{2} - y$ & Intertwiners of nilpotent irreps of $\tilde{U}_q(A^{(1)}_1)(q^4 = 1)$ & \\
$D_{k+2}(t_2)$ & $\frac{x^{k+1}}{2(k+1)} + \frac{xy}{2} - x$ & Susy generalization of chiral Potts Boltzmann weights & \\
$E_6(t_7)$ & $\frac{2}{3} + \frac{2}{3} - xy$ & Susy generalization of chiral Potts Boltzmann weights & \\
$E_8(t_{16})$ & $\frac{x^3}{3} + \frac{y^3}{3} - xy$ & Susy generalization of chiral Potts Boltzmann weights &
which are guaranteed provided:

\[ X(\rho_1 \otimes \rho_2 \otimes \rho_3) = X(\rho_4 \otimes \rho_5 \otimes \rho_6), \text{ etc} \quad (24) \]

To have a gYB one should be able to factorize eq.\((23)\) in the usual Yang-Baxter sequences:

\[ \rho_7 \otimes \rho_8 \otimes \rho_3 \rightarrow \rho_7 \otimes \rho_9 \otimes \rho_6 \]
\[ \rho_1 \otimes \rho_2 \otimes \rho_3 \quad \rho_4 \otimes \rho_5 \otimes \rho_6 \]
\[ \rho_1 \otimes \rho_{10} \otimes \rho_{11} \rightarrow \rho_4 \otimes \rho_{12} \otimes \rho_{11} \quad (25) \]

for some intermediate irreps \(\rho_7, \ldots, \rho_{12}\). Each step in eq.\((25)\) involves a set of eqs. of the type \((22)\), i.e.:

\[ X(\rho_1 \otimes \rho_2) = X(\rho_7 \otimes \rho_8), X(\rho_2 \otimes \rho_3) = X(\rho_{10} \otimes \rho_{11}) \]
\[ X(\rho_1 \otimes \rho_2) = X(\rho_7 \otimes \rho_8), X(\rho_2 \otimes \rho_3) = X(\rho_{10} \otimes \rho_{11}) \]
\[ X(\rho_1 \otimes \rho_2) = X(\rho_7 \otimes \rho_8), X(\rho_2 \otimes \rho_3) = X(\rho_{10} \otimes \rho_{11}) \]
\[ X(\rho_7 \otimes \rho_9) = X(\rho_4 \otimes \rho_5), X(\rho_{12} \otimes \rho_{11}) = X(\rho_5 \otimes \rho_6) \quad (26) \]

and similarly for \(Y, Z, U\). It is easy to see that equations \((26)\) with the constraint \((24)\) determine the casimirs \(X, Y, Z, U\) of the six intermediate irreps but one, which can be choosen at will. This freedom is the origin of the sum on IRF labels in the graph-Yang-Baxter equation.

We shall finally make some further comments. Based on the formal connection between Ocneanu’s path model \([9]\) and Witten’s string vertex \([19]\), the construction that we have developed suggest a possible way to define discrete strings possessing internal degrees of freedom governed by quantum groups. Each string configuration, i.e. a path \(\xi\) on the Bratteli diagram of the graph \(\mathcal{G}\), can be associated with a ket \(|\xi\rangle\) in the vector space \(\mathbb{C}[\xi, w(\xi)]\) (see definition 2). The string operators act now on these string states as prescribed by equation \((11)\). The interest of this structure is the non-trivial interplay between the geometry of the strings, characterized by the graph \(\mathcal{G}\), and the internal degrees of freedom which we associate with a quantum algebra \(\mathcal{A}\). Another possible application is the construction of new integrable lattice models which mix vertex and IRF degrees of freedom.
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