Quasiperiodic tilings in hyperbolic space

Reinhard Lück
Weilstetter Weg 16, D-70567 Stuttgart, Germany
r.v.lueck@web.de

Abstract. The standard methods to derive quasiperiodic patterns in Euclidean space were tested for hyperbolic space, which is represented by the Poincaré disc. A novel method to transform the icosahedral tiling of rhombohedra into the Poincaré disc is discussed.

1. Introduction
Hyperbolic tilings are well-known to result in rotational symmetries which are in Euclidean space only possible with quasiperiodicity. However, the question arises if quasiperiodicity is also possible in hyperbolic space. Dolbilin and Frettlöh [1] described 'Böröczky tilings in high dimensional hyperbolic space' and Ch. Goodman-Strauss [2, 3] discussed a 'strongly aperiodic set of tiles in the hyperbolic plane'. Publications of both presented examples in the so-called half-plane, these examples can be considered as limit periodic [4].

The Poincaré disc is generally used for representation of hyperbolic space. Regular tilings of the Poincaré disc are described following [5] as $p$-$q$ tilings ($p\times q$), where $p$ denotes a regular polygon with $p$-fold symmetry and $q$ is meaning the number of polygons sharing a vertex. The dual of $(p\times q)$ is $(q\times p)$. In the present paper, standard methods for derivation of quasiperiodic patterns in Euclidean space will be discussed for hyperbolic space represented by the Poincaré disc. A novel method to transform icosahedral tilings into hyperbolic space is added.

2. Substitution method
The substitution of tiles in quasiperiodic patterns results generally in tilings composed of similar tiles at a lower scale [4]. This procedure may be applied repeatedly in Euclidean space. Selfsimilarity is an important property of these tilings. The substitution can be applied in some regular hyperbolic tilings; the result is another tiling which is not similar. A repeated application of substitution is not possible. Therefore, the substitution method does not work for the derivation of quasiperiodic patterns in the Poincaré disc.

3. Dualisation of multigrids
De Bruijn [6] has introduced the method of dualisation of multigrids in order to derive quasiperiodic patterns; these patterns are mainly composed of rhombi. There is a little chance to apply this method in hyperbolic space. The grids in this method are equidistant. The distances in Ammann bar grids [6] differ and are related to eigenvalues of substitution matrices. Ammann bar multigrids have also been applied to derive quasiperiodic patterns for several symmetries. Its application to the Poincaré disc seems to be impossible.
4. Dualisation of nets

Stampfli [8, 9] superimposed two hexagon nets which were rotated to each other by $\pi/6$ to derive by dualisation a novel twelvefold quasiperiodic pattern. Gäbler and Stampfli [10] described this method in more detail. They showed that also two triangle nets and two square nets produce quasiperiodic patterns. Superimposing of tilings is possible in hyperbolic space as well. A tiling with $n$-fold rotational symmetry at the centre of the Poincaré disc may be copied, and the copy is rotated by a rotation angle around the centre of the Poincaré disc. For the present investigation, it was checked: Can dualisation of two superimposed hyperbolic tilings produce aperiodic hyperbolic tilings?

To characterize such a procedure a set of parameters is needed. These parameters are the tiling, described by $p$ and $q$, the rotational symmetry of the centre and the rotation angle (Table 1). This is the method to investigate coincidence site lattices (CSL) in hyperbolic tilings [11]. In Euclidean geometry, the coincidence density is described by $\Sigma$. $1/\Sigma$ is the fraction of coinciding lattice points, which may be atoms or vertices of tilings in periodic lattices. In quasilattices it is fairly the same. $\Sigma$ is related to $k$ which is the index for colour symmetry. Superimposing of two $p$-$q$ tilings rotated to each other can result in coinciding vertices, c.f. (i) in Table 1. It is also found that only half of the coinciding vertices are related to colour symmetry, c.f. (ii) in Table 1. This kind of colour symmetry shows only symmetry with respect to rotation. The coincidence is resulting in an enantiomorphic pair of coloured tilings [12]. The third possibility is, (iii) in Table 1, that no coincidence is observed, neither for vertices nor for tile centres. Dualisation of such superimposed tilings should result in an aperiodic tiling. Crossing lines form rhombi by dualisation. The fourth case (iv) is that some vertices are coinciding, these coincidences are singular, no colour index can be determined. The coincidence can be at the singular boundary (v). A few examples are listed in Table 1, two of them are depicted in Figures 1 and 2. Figure 1 is corresponding to the formation of an aperiodic hyperbolic tiling.

Table 1. Examples of results after superimposing hyperbolic tilings with rotation to each other

| Tiling (pq) | (73) | (54) | (38) | (57) | (37) | (83) | (54) | (38) | (57) | (37) |
|------------|------|------|------|------|------|------|------|------|------|------|
| Rot.sym.   | 7    | 5    | 3    | 7    | 5    | 3    | 4    | 4    | 8    | 8    |
| Rot.angle | $\pi$ | $\pi$ | $\pi$ | $\pi$ | $\pi/4$ | $\pi/4$ | $\pi/8$ | $\pi/8$ | $\pi$ | $\pi/2$ |
| c.f. text  | (i)  | (i)  | (ii) | (ii) | (iii) | (iii) | (iii) | (iv) | (iv) | (v)  |
| $k$        | 8    | 6    | 10   | 8    | 6    | 10   | –    | –    | –    | –    |

It should be added that some hyperbolic tilings exhibit only continuous lines without any break. This holds for $(p^q)$ when $q$ is even. It is analogous to multigrids resulting from triangle $(3^6)$ and square tilings $(4^4)$ in Euclidean space (see Section 3).

![Figure 1](image1.png) Superimposed $(4^3)$ tilings rotated by $\pi/4$. The dual tiling $(5^4)$ is included.

![Figure 2](image2.png) Superimposed $(5^3)$ tilings rotated by $\pi$. The dual tiling $(4^3)$ is included. Coincidence results in the shown colour symmetry with 6 colours, which has only rotational symmetry.
5. Matching rules
There may exist matching rules for decoration of tiles which force aperiodic tilings. This would be analogous to the decoration of the 16 Wang tiles. Searching for it is a challenge and out of scope of the present investigation. It is simple to decorate tiles such that a consisting periodic tiling is prevented.

6. Cut and projection
A projection in hyperbolic space should be performed in a higher dimensional ball of Poincaré type [13]. The projection should be positioned perpendicular to the Poincaré disc. The width of the strip to be projected may be a problem.

7. Transformation of ‘open’ surfaces
Periodic surfaces in three-dimensional space – as ‘open’ Fermi surfaces, periodic minimal surfaces etc. – can be mapped onto two-dimensional tilings in hyperbolic space. In the present paper, there is no space for an example of a periodic surface in three dimensions transformed to the Poincaré disc. The ‘generic primitive icosahedral tiling’ [14, 15] composed of obtuse and acute rhombohedra should be transformed to an ‘open’ surface. The rhombi will be replaced with connecting prisms of different adjusted heights.

For a dual tiling, however, the rhombohedra will be replaced by truncated versions of the rhombohedra. The rectangular facets of the prisms or the facets of the truncated rhombohedra will form the ‘open’ surface. The surfaces divide the space into an inner and an outer area. These open surfaces will be transformed into hyperbolic space presented by a Poincaré disc.

7.1. An icosahedral tiling with ‘connecting prisms’
As a non-conformal approximation we used the tiling \((4\text{6})\), this means that 6 squares are sharing a vertex. For this approximation, the rectangular facets of the prisms are replaced with squares. Four squares are forming the connecting prisms. There are sharp and obtuse angles. As an example six orientations of the facets of prisms corresponding to a rhombohedron are shown in Figure 3. A similar representation of a 20-star which is composed of 20 acute rhombohedra would require 30 different orientations of facets.

Figure 3. Facets of 6 prisms which belong to a single rhombohedron are shown. There are 6 orientations of rhombi, each rhombus is connected to a prism, which are composed of four facets in two different orientations. The three orientations of prisms are marked by the colours red, green and blue. The different orientations are distinguished by dark and light colours. One should keep in mind that a facet appears in the Poincaré disc several times (strictly spoken infinitely often).

7.2. An icosahedral tiling composed of truncated rhombohedra
A rhombohedron is composed of 8 corners and the two rhombohedra have four types of corners. Two corners of the obtuse and acute rhombohedra have three-fold rotational symmetry, six corners have only reflection symmetry. In order to design the new polyhedra, the corners are truncated in such a manner that the new facets form a continuous surface. The Golden rhombi are reduced to ‘Golden rectangles’; these rectangles are not part of the continuous surface and will therefore be omitted. The
facets formed by truncation correspond to four different types of hexagons with the edge lengths 1 and \( \tau \) [\( \tau = (\sqrt{5} + 1)/2 \)]. The 24 different vertices of this icosahedral tiling [14, 15] are surrounded by hexagons in the same manner as the centres of the original rhombohedra. Relative frequencies of polyhedra and of vertices are equal.

**Figure 4.** Two types of truncated rhombohedra, each one is composed of 2 regular hexagons and 6 distorted hexagons. Facets of rectangular form are omitted to build up a continuous open surface. The interior of the truncated polyhedra belong to an inner area and whereas the outer area belongs to the 24 vertices of the ‘generic primitive icosahedral tiling’ [14, 15].

| Table 2. Properties of tiles formed by truncation of rhombohedra |
|---------------------|-----------------|-----------------|-----------------|
| Shape of tile       | Large hexagon   | Small hexagon   | Large distorted hexagon | Small distorted hexagon |
| Corresponding solid angle in ico rhombohedron tiling | \( \pi / 5 \) | \( \pi / 5 \) | \( 3\pi / 5 \) | \( \pi / 5 \) |
| Orientation perp axis | five-fold | five-fold | three-fold | three-fold |
| Edge lengths of tiles | \( \tau, \tau, \tau, \tau, \tau \) | \( 1, 1, 1, 1, 1 \) | \( 1, \tau, \tau, 1, \tau, \tau \) | \( \tau, 1, 1, \tau, 1, \tau \) |
| Angles of tiles      | \( 2\pi / 3 \) | \( 2\pi / 3 \) | \( 4\pi / 5, 3\pi / 5, 3\pi / 5 \) | \( 2\pi / 5, 4\pi / 5, 4\pi / 5 \) |
| Angles in conformal tiling in Poincaré disc | \( \pi / 2 - \varepsilon \) | \( \pi / 2 + \varepsilon' \) | \( \pi / 2 - \varepsilon' \) | \( \pi / 2 + \varepsilon \) |
| Relative frequency   | \( 1 / 4\tau^2 \) | \( 1 / 4\tau \) | \( 3 / 4\tau \) | \( 3 / 4\tau^2 \) |

The surface produced in this way is composed of four types of hexagons and different vertex configurations. There are four vertex configurations with reflection symmetry and three enantiomorphic pairs. Its tiling is quasiperiodic and can be mapped onto a Poincaré disc. As a first approximation, the hexagons may be replaced with regular decorated hexagons in a (6\(^6\)) tiling; this approximation is not conformal but topologically equivalent. In the exact conformal transformation, the angles at the vertices should be altered. In the small hexagons, the angle should be larger than \( \pi / 2 \), however, in the large hexagons the angle should be smaller than \( \pi / 2 \). The angles in the distorted hexagons should be corresponding. A consequence is that the lines will not be further straight at vertices.

**Figure 5.** Truncated ‘Golden’ rhombohedra, acute (blue/green) and obtuse (red/yellow). The Poincaré discs show a representation of the truncated rhombohedra in a non-conformal approximation with the corresponding colouring. The uncoloured areas represent space for adjacent tiles.
To understand the aperiodic hyperbolic tiling we will presently focus on non-conformal transformation into hyperbolic space. That means the distorted hexagons will be approximated by regular hexagons and the small and the large regular hexagons are approximated by hexagons of identical size. The applied hyperbolic tiling is \((6^4)\), and the different hexagons are distinguished by colourings according to figure 5.

**Figure 6.** Vertex configuration of the so-called 20-star represented by 20 regular hexagons of a ‘Buckyball’; the 12 regular pentagons are open to guarantee a surface which can be transformed into hyperbolic space. At the right, these hexagons are mapped onto a Poincaré disc. 10 of the hexagons can be easily found, the other 10 are close to the boundary of the Poincaré disc.

The vertex configuration of the highest symmetry in icosahedral tiling is that of the so-called 20-star. This is composed of 20 acute rhombohedra. For our purpose, the acute rhombohedra have to be replaced with 20 truncated rhombohedra. Figure 6 shows the vertex configuration in form of a Buckyball with omitted pentagons and a representation in Poincaré disc. The remaining facets of the truncated rhombohedra are added in Figure 7. The vertex configuration with the smallest number of rhombohedra is composed of two acute and two obtuse rhombohedra. In the Poincaré disc it is more complex (Figure 8).

**Figure 7.** Representation of the so-called 20-star together with an enlarged part. The uncoloured areas represent space for adjacent tiles.
It can be imagined that any three-dimensional icosahedral tiling composed of rhombohedra can be transformed to the Poincaré disc in the manner described. The disadvantage is that the resolution of tiling is very low at the boundary of the Poincaré disc. The resolution can be improved by zooming on parts. Apart from this, the tiling can be moved within the Poincaré disc to focus on other regimes.

![Figure 8. Representation of the simplest vertex configuration in the icosahedral rhombohedron tiling. It consists of two acute and two obtuse truncated rhombohedra (cf. Figure 3). The white areas represent adjacent tiles, which should be added at the 12 golden rectangles of the rhombic dodecahedron (also called Bilinski polyhedron) formed by the original rhombohedra.](image)

8. Acknowledgement

The original hyperbolic tilings were made available by Dirk Frettlöh.

References

[1] Dolbilin N and Frettlöh D 2010 Properties of Böröczky tilings in high dimensional hyperbolic space *European J. Combin.* 31 1181 - 1195

[2] Goodman-Strauss Ch 2005 A strongly aperiodic set of tiles in the hyperbolic plane *Inventiones Mathematicae* 159 119 - 132

[3] Goodman-Strauss Ch 2010 A hierarchical strongly aperiodic set of tiles in the hyperbolic plane *Theoretical Computer Science* 411 1085 - 1093

[4] Baake M and Grimm U 2013 Aperiodic order 1 A mathematical invitation Cambridge University Press

[5] Grünbaum B and Shephard J M 1987 Tilings and patterns Freeman, New York

[6] de Bruijn N G 1981 Algebraic theory of Penrose’s non-periodic tilings of the plane *Kon. Nederl. Akad. Wetensch. Proc. Ser. A* 84 39 - 52 and 53 - 66

[7] Lück R 1993 Basic ideas of Ammann bars grids *Int. J. Mod. Phys. B* 6-7 1437 - 1453

[8] Stampfli P 1986 *Helv. Phys. Acta* 59 1260.

[9] Stampfli P 1990 New quasiperiodic lattices from the grid method *Quasicrystals networks and molecules of fivefold symmetry* Ed Hagittai I (VHC Publishers New York) 201-221

[10] Gähler F and Stampfli P 1993 The dualisation method revisited: Dualisation of product Laguerre complexes as a unifying framework *Int. J. Mod. Phys. B* 6-7 1333 - 1349

[11] Lück R and Frettlöh D 2008 Ten colours in quasiperiodic and regular hyperbolic tilings *Z. Krist.* 223 782-784

[12] Frettlöh D 2008. Counting perfect colourings of plane regular tilings *Z. Krist.* 223 773 - 776

[13] Lück R and Frettlöh D 2014 Hyperbolic icosahedral tilings by buckyballs *Acta Physica Polonica A* 126 524 -526

[14] Henley C L 1986 Sphere packings and local environments in Penrose tilings *Phys. Rev. B* 34 797 - 816

[15] Baake M, Ben-Abraham S I, Klitzing R, Kramer P, Schlottmann M 1994 Classification of local configurations in quasicrystals *Z. Krist.* A 50 553 - 566