THE $\omega$-LIMIT SET OF A FLOW WITH ARBITRARILY MANY SINGULAR POINTS ON A SURFACE AND SURGERIES TO ADD SINGULAR POINTS

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Abstract. Area-preserving flows on surfaces are one of the fundamental dynamical systems and are studied from a physical point of view via the connection for such flows with solid-state physics and pseudo-periodic topology. On the other hand, the limit behaviors of orbits of flows on surfaces are captured by the Poincaré-Bendixson theorem using the $\omega$-limit sets. In this paper, we consider what kinds of the $\omega$-limit sets do appear in the area-preserving (or, more general, non-wandering) flows on compact surfaces, and show that the $\omega$-limit set of any non-closed orbit of such a flow with arbitrarily many singular points on a compact surface is either a subset of singular points or a locally dense $\mathbb{Q}$-set. To show this, we demonstrate the dependence between the $\omega$-limit sets and the $\alpha$-limit sets of points. Moreover, we show the wildness of surgeries to add totally disconnected singular points and the tameness of those to add finitely many singular points for flows on surfaces.

1. Introduction

The Poincaré-Bendixson theorem is one of the most fundamental tools to capture the limit behaviors of orbits of flows and was applied to various phenomena (e.g. \cite{6,13,23,26,39,41,43}). In \cite{7}, Birkhoff introduced the concepts of $\omega$-limit set and $\alpha$-limit set of a point. Using these concepts, one can describe the limit behaviors of orbits stated in the works of Poincaré and Bendixson in detail. In fact, the Poincaré-Bendixson theorem was generalized for flows on surfaces in various ways \cite{3,4,10,13,15,16,20,22,29,31,35,44,47,49,51}, and also for foliations \cite{30,38}, translation lines on the sphere \cite{26}, geodesics for a meromorphic connection on Riemann surfaces \cite{1,12}, group actions \cite{24}, and semidynamical systems \cite{8}.

Area-preserving flows on compact surfaces are one of the basic and classic examples of dynamical systems, also known as locally Hamiltonian flows or equivalently multi-valued Hamiltonian flows. The measurable properties of such flows are studied from various aspects \cite{9,12,17,19,25,28,40,46}. For instance, the study of area-preserving flows for their connection with solid-state physics and pseudo-periodic topology was initiated by Novikov \cite{27}. The orbits of such flows also arise in pseudo-periodic topology, as hyperplane sections of periodic manifolds (cf. \cite{3,54}).

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1.1. Statements of main results. In this paper, we consider the following classification problem and the wildness and tameness of surgeries to add singular points of flows on surfaces.

1.1.1. Classifications of the \( \omega \)-limit sets of points in the non-wandering and Hamiltonian flows on surfaces. We consider the following question.

**Question 1.** What kinds of the \( \omega \)-limit sets of points do appear in the non-wandering, divergence-free, and Hamiltonian flows with arbitrarily many singular points on compact surfaces, respectively?

We answer that only nowhere dense subsets of singular points and locally dense \( Q \)-sets appear in such cases. Similar results hold for locally dense orbits of flows and non-closed orbits of gradient flows (see §3.3 for details).

To describe precise statements of the results, we recall some concepts as follows. An orbit is **closed** if it is singular or periodic, and it is **locally dense** if its closure has a nonempty interior. An orbit \( O \) is **recurrent** if \( O \subseteq \omega(O) \cup \alpha(O) \). A \( Q \)-set is the closure of a non-closed recurrent orbit. We have the following dichotomy for any non-closed orbit of a non-wandering flow.

**Theorem A.** The \( \omega \)-limit set of any non-closed orbit for a non-wandering flow with arbitrarily many singular points on a compact surface is either a nowhere dense subset of singular points or a locally dense \( Q \)-set.

We have the following observation from the previous theorem.

**Corollary B.** The \( \omega \)-limit set of any non-closed orbit for a Hamiltonian flow with arbitrarily many singular points on a compact surface consists of singular points.

To demonstrate Theorem A (see §3.2 for details), we show the following dichotomy for any locally dense orbit, which supplements a generalization of the Poincaré-Bendixson theorem [51, Theorem A].

**Theorem C.** The \( \omega \)-limit set of any locally dense orbit for a flow with arbitrarily many singular points on a compact surface is either a nowhere dense subset of singular points or a locally dense \( Q \)-set.

The previous theorem says that the \( \omega \)-limit set and the \( \alpha \)-limit set of any point are not independent in general. For instance, the \( \omega \)-limit set of a point whose \( \alpha \)-limit set is a limit circuit is not a locally dense \( Q \)-set.

1.2. Wildness and tameness of surgeries to add singular points. We consider the following question for invariance of the Hamiltonian property by surgeries to add singular points.

**Question 2.** Is the Hamiltonian property for flows on surfaces invariant under multiplying a bump function to a Hamiltonian vector field?

We answer that the Hamiltonian property for flows on compact surfaces is invariant when only finitely many singular points are added, but that the property is not invariant even if totally disconnected singular points are added. More precisely, every non-zero vector field can be deformed into a vector field with wandering domains using a bump function with totally disconnected critical points as follows.
Theorem D. For any non-zero vector field $X$ on a compact surface $S$, there is a smooth function $f: S \to \mathbb{R}$ such that

1. The vector field $fX$ is not non-wandering.
2. Every orbit of $X$ is a union of orbits of $fX$.
3. The critical point set $f^{-1}(0)$ is totally disconnected.
4. If the singular point set $\text{Sing}(X)$ is totally disconnected, then so is $\text{Sing}(fX)$.

On the other hand, the Hamiltonian property of vector fields on compact surfaces is invariant under adding finitely many singular points. To state more precisely, we recall some concepts and a fact as follows. A nonempty subset $A$ of a topological space $X$ is a level set of a function $f: X \to \mathbb{R}$ if there is a value $c \in \mathbb{R}$ such that $A = f^{-1}(c)$. Recall that, for any Hamiltonian vector field $X$ on a surface $S$ with the Hamiltonian $h: S \to \mathbb{R}$, the set of connected components of level sets of the restriction $h|_{S - \text{Sing}(X)}$ is the set of orbits of the restriction $X|_{S - \text{Sing}(X)}$ and is a codimension one foliation on the surface $S - \text{Sing}(X)$. Therefore we introduce a pre-Hamiltonian flow as follows.

Definition 1. A flow $v$ on an orientable surface is pre-Hamiltonian if there is a continuous function $H: S \to \mathbb{R}$ such that the set of connected components of level sets of the restriction $H|_{S - \text{Sing}(v)}$ is the set of orbits of the restriction $v|_{S - \text{Sing}(v)}$ and is a codimension one foliation on the surface $S - \text{Sing}(v)$.

Then $H$ is called the pre-Hamiltonian of $v$. We have the following equivalence under finiteness of singular points.

Theorem E. The following are equivalent for a flow $v$ with finitely many singular points on a compact surface $S$:

1. The flow $v$ is Hamiltonian.
2. The flow $v$ is pre-Hamiltonian.

This implies that Hamiltonian property of flows on compact surfaces is invariant under adding finitely many singular points.

The present paper consists of five sections. In the next section, as preliminaries, we introduce fundamental concepts. In § 3, classifications of the $\omega$-limit sets of points in the non-wandering and Hamiltonian flows on surfaces are demonstrated, and a remark is stated. In § 4, the tameness of surgeries to add finitely many singular points is described. The final section demonstrates the wildness of surgeries to add totally disconnected singular points.

2. Preliminaries

2.1. Topological notion. Denote by $\overline{A}$ the closure of a subset $A$ of a topological space and by $\partial A := \overline{A} - \text{int}A$ the boundary of $A$, where $B - C$ is used instead of the set difference $B \setminus C$ when $B \subseteq C$. A subset of a compact surface $S$ is essential if it is not null homotopic in $S^*$, where $S^*$ is the resulting closed surface from $S$ by collapsing all boundary components into singletons.

A curve is a continuous mapping $C : I \to X$ where $I$ is a non-degenerate connected subset of a circle $S^1$. A curve is simple if it is injective. We also denote by $C$ the image of a curve $C$. Denote by $\partial C := C(\partial I)$ the boundary of a curve $C$, where $\partial I$ is the boundary of $I \subset S^1$. A simple curve is a simple closed curve if its domain is $S^1$ (i.e., $I = S^1$). A simple closed curve is also called a loop. An arc is a simple curve whose domain is an interval. An orbit arc is an arc contained in an orbit.
2.2. Notion of dynamical systems. By a surface, we mean a paracompact two-dimensional manifold, that does not need to be orientable. A flow is a continuous $\mathbb{R}$-action on a manifold. From now on, we suppose that flows are on surfaces unless otherwise stated. Let $\nu : \mathbb{R} \times S \to S$ be a flow on a surface $S$. For $t \in \mathbb{R}$, define $\nu_t : S \to S$ by $\nu_t := \nu(t, \cdot)$. For a point $x$ of $S$, we denote by $O(x)$ the orbit of $x$, $O^+(x)$ the positive orbit (i.e. $O^+(x) := \{ \nu_t(x) \mid t > 0 \}$). A subset is invariant if it is a union of orbits. A point $x$ of $S$ is singular if $x = \nu_t(x)$ for any $t \in \mathbb{R}$ and is periodic if there is a positive number $T > 0$ such that $x = \nu_T(x)$ and $x \neq \nu_t(x)$ for any $t \in (0, T)$. A point is closed if it is singular or periodic. An orbit is singular (resp. periodic, closed) if it contains a singular (resp. periodic, closed) point. Denote by $\text{Sing}(\nu)$ the set of singular points and by $\text{Per}(\nu)$ (resp. $\text{Cl}(\nu)$) the union of periodic (resp. closed) orbits.

A point is wandering if there are its neighborhood $U$ and a positive number $N$ such that $\nu_t(U) \cap U = \emptyset$ for any $t > N$. A point is non-wandering if it is not wandering (i.e. for any its neighborhood $U$ and for any positive number $N$, there is a number $t \in \mathbb{R}$ with $|t| > N$ such that $\nu_t(U) \cap U \neq \emptyset$). Denote by $\Omega(\nu)$ the set of non-wandering points, called the non-wandering set. A flow is non-wandering if any points are non-wandering.

The $\omega$-limit (resp. $\alpha$-limit) set of a point $x$ is $\omega(x) := \bigcap_{n \in \mathbb{N}} \{ x \mid \nu_t(x) \mid t < n \}$ (resp. $\alpha(x) := \bigcap_{n \in \mathbb{N}} \{ x \mid \nu_t(x) \mid t > n \}$). For an orbit $O$, define $\omega(O) := \omega(x)$ and $\alpha(O) := \alpha(x)$ for some point $x \in O$. Note that an $\omega$-limit (resp. $\alpha$-limit) set of an orbit is independent of the choice of point in the orbit. A separatrix is a non-singular orbit whose $\alpha$-limit or $\omega$-limit set is a singular point.

A point $x$ is recurrent if $x \in \omega(x) \cup \alpha(x)$. Denote by $R(\nu)$ the set of non-closed recurrent points. An orbit is recurrent if it contains a recurrent point. Recall that a Q-set is the closure of a non-closed recurrent orbit.

An orbit is proper if it is embedded, locally dense if its closure has a nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense) if its orbit is proper (resp. locally dense). Denote by $\text{LD}(\nu)$ (resp. $\text{E}(\nu), P(\nu)$) the union of locally dense orbits (resp. exceptional orbits, non-closed proper orbits). Recall that a non-wandering flow on a compact surface has no exceptional orbits (i.e. $E(\nu) = \emptyset$) because of [50] Lemma 2.3. We have the following observations (see [52] §2.2.1 for details). The union $\text{P}(\nu)$ of non-closed proper orbits is the set of non-recurrent points, and that $R(\nu) = \text{LD}(\nu) \cup \text{E}(\nu)$, where $\cup$ denotes a disjoint union. Moreover, we have a decomposition $S = \text{Sing}(\nu) \cup \text{Per}(\nu) \cup \text{P}(\nu) \cup R(\nu)$.

2.2.1. Quasi-circuits and quasi-Q-sets. A closed connected invariant subset is a non-trivial quasi-circuit if it is a boundary component of an open annulus, contains a non-recurrent orbit, and consists of non-recurrent orbits and singular points. A non-trivial quasi-circuit $\gamma$ is a quasi-semi-attracting quasi-circuit if there is a point $x \in A$ with $O^+(x) \subset A$ such that $\omega(x) = \gamma$. Moreover, a quasi-circuit is not a circuit in general.

The transversality for a continuous flow can be defined using tangential spaces of surfaces, because each flow on a compact surface is topologically equivalent to a $C^1$-flow by Gutierrez’s smoothing theorem [21]. An $\omega$-limit (resp. $\alpha$-limit) set of a point is a quasi-Q-set if it intersects an essential closed transversal infinitely many times. A quasi-Q-set is not Q-set in general, but a Q-set is a quasi-Q-set [51] Lemma 3.8.
2.2. Types of singular points. A point $x$ is a center if, for any its neighborhood $U$, there is an invariant open neighborhood $V \subset U$ of $x$ such that $U - \{x\}$ is an open annulus that consists of periodic orbits, as in the left on Figure 1. A $\partial$-k-saddle (resp. $k$-saddle) is an isolated singular point on (resp. outside of) $\partial S$ with exactly $(2k + 2)$-separatrices, counted with multiplicity as in Figure 1. A multi-saddle is a $k$-saddle or a $\partial$-(k/2)-saddle for some $k \in \mathbb{Z}_{\geq 0}$. A 1-saddle is topologically an ordinary saddle, and a $\partial$-(1/2)-saddle is topologically a $\partial$-saddle. The union of multi-saddles and their separatrices is called the multi-saddle connection diagram. Any connected components of the multi-saddle connection diagram are called multi-saddle connections.

3. PROOFS OF MAIN RESULTS AND A REMARK ON GRADIENT CASES

We demonstrate the main results for the $\omega$-limit sets as follows.

3.1. Proof of Theorem C Let $v$ be a flow with arbitrarily many singular points on a compact surface $S$ and $x \in LD(v)$. Then the orbit closure $\overline{O(x)}$ is a locally dense Q-set and so a neighborhood of $O(x)$. Moreover, the point $x$ is non-wandering. Proposition 2.2 and Lemma 2.3] implies that $\overline{O(x)} \cap (\text{Per}(v) \cup E(v)) = \emptyset$ and so that $\overline{O(x)} \subseteq \text{Sing}(v) \cup P(v) \cup LD(v)$. Since any point whose $\omega$-limit set is a limit cycle is wandering, the $\omega$-limit set $\omega(x)$ is not a limit cycle. Lemma 3.5] implies that $\omega(x)$ is not a quasi-semi-attracting limit quasi-circuit. By Theorem A, the $\omega$-limit set $\omega(x)$ is either a nowhere dense subset of singular points, a locally dense Q-set, or a quasi-Q-set in $\text{Sing}(v) \cup P(v)$. We claim that $\omega(x)$ is not a quasi-Q-set in $\text{Sing}(v) \cup P(v)$. Indeed, assume that $\omega(x)$ is a quasi-Q-set in $\text{Sing}(v) \cup P(v)$. By definition of quasi-Q-set, there is a closed transversal $\gamma$ intersecting $\omega(x)$ infinitely many times. Since the loop $\gamma$ is compact, there is an accumulation point $y \in \omega(x) \subseteq \overline{O(x)}$. There is a closed sub-arc $I$ of $\gamma$ contained in $\overline{O(x)}$ intersecting $\omega(x)$ infinitely many times. Fix an orbit arc $C$ with $\partial C = \partial I$. By the waterfall construction to the loop $C \cup I$, we obtain a closed transversal $\mu \subset \overline{O(x)} \cap (P(v) \cup LD(v))$ intersecting $\omega(x)$ infinitely many times. If the intersection $O^+(x) \cap \mu$ has an accumulation point, then $O^+(x) \cap \mu$ is negative recurrent and so $\omega(x) = \overline{O(x)}$ is a locally dense Q-set, which contradicts that $\omega(x)$ is a quasi-Q-set in $\text{Sing}(v) \cup P(v)$. Thus the intersection $O^+(x) \cap \mu$ has no accumulation points. Therefore there is a sub-arc $J: [-1,1] \to \mu$ with $J(-1) \in O^+(x)$ as in Lemma 3.2]. Applying Lemma 3.2] to the sub-arc $J$, the $\omega$-limit set $\omega(x)$ is a quasi-circuit, which contradicts that $\omega(x)$ is a quasi-Q-set that consists of singular points and non-recurrent points.

Thus the $\omega$-limit set $\omega(x)$ is either a nowhere dense subset of singular points or a locally dense Q-set.
3.2. Proof of Theorem A. By \[50\] Lemma 2.4, we have that \( S = \text{Cl}(v) \cup \text{P}(v) \cup \text{LD}(v) = \text{Sing}(v) \cup \text{Per}(v) \cup \text{LD}(v) \). From \[50\] Proposition 2.6, we have \( \text{Sing}(v) \cup \text{P}(v) = \{ y \in S \mid \omega(y) \cup \alpha(y) \subseteq \text{Sing}(v) \} \). Fix a point \( x \in S - \text{Cl}(v) \). By Theorem A, we may assume that \( x \in S - (\text{Cl}(v) \cup \text{LD}(v)) = \text{P}(v) \). Since \( \text{Sing}(v) \cup \text{P}(v) = \{ y \in S \mid \omega(y) \cup \alpha(y) \subseteq \text{Sing}(v) \} \), the \( \omega \)-limit set \( \omega(x) \) consists of singular points. \[51\] Theorem A implies that the \( \omega \)-limit set \( \omega(x) \) is a nowhere dense subset of singular points.

We have the following nonexistence of no non-closed recurrent orbits for Hamiltonian flows on surfaces.

Lemma 3.1. Every Hamiltonian flow with arbitrarily many singular points on any surface has no non-closed recurrent orbits.

Proof. Let \( v \) be a Hamiltonian flow with arbitrarily many singular points on a surface \( S \). Fix a Hamiltonian \( h \) generating \( v \). Assume that there is a non-closed recurrent orbit \( O \). Fix a point \( x \in O \). By the flow box theorem for a continuous flow on a compact surface (cf. Theorem 1.1, p.45 \[1\]), there is a closed disk \( U \) which can be identified with \( [-1,1] \times [-1,1] \) such that \( x \) corresponds to the origin \((0,0)\) and the arc \( C_t := [-1,1] \times \{ t \} \) for any \( t \in [-1,1] \) is contained in some orbit. By the definition of Hamiltonian vector field, for any \( t \in [-1,1] \), there is a number \( r_t \in \mathbb{R} \) with \( h(C_t) = \{ r_t \} \) and the function \( r : [-1,1] \to \mathbb{R} \) defined by \( r(t) := r_t \) is strictly increasing or decreasing. Moreover, we have \( h(O) = \{ 0 \} \). On the other hand, since \( O \) is non-closed recurrent, the orbit \( O \) intersects \( U - C_0 \). Set an arc \( C_t' \subset U \cap (U - C_0) \). This means that \( t' \neq 0 \in h(O) \), which contradicts \( h(O) = \{ 0 \} \). Thus there are no non-closed recurrent orbits.

The previous lemma and Theorem A imply Corollary A.

3.3. A remark on an analogous statement for gradient flows. Notice that a similar statement holds for a gradient flow with arbitrarily many singular points on a (possibly non-compact) surface.

Proposition 3.2. The \( \omega \)-limit set of any non-closed orbit for a gradient flow with arbitrarily many singular points on a surface consists of singular points.

Proof. Let \( v \) be a gradient flow with arbitrarily many singular points on a compact surface \( S \). There is the height function \( h : S \to \mathbb{R} \) such that the flow generated by the gradient vector field \( X_h := \text{grad}(h) \) is topologically equivalent to \( v \).

We claim that there is no non-closed recurrent orbit. Indeed, assume that there is a non-closed recurrent orbit \( O \). Fix a point \( x \in O \) with \( Q = \omega(x) \) or \( Q = \alpha(x) \). By time reversion if necessary, we may assume that \( Q = \omega(x) \). Then there is a closed transversal \( \gamma \subset \text{P}(v) \) intersecting \( Q \) infinitely many times. Therefore \( \gamma \) intersects \( O^+(x) \) infinitely many times. Then there is a point \( y \in \gamma \cap \omega(x) \subset \text{P}(v) \). By the existence of the height function \( h : S \to \mathbb{R} \) of \( v \), the value of \( y \) is finite. Since \( \gamma \) is compact, the norms of the gradients of \( h \) on \( \gamma \) are separated from zero. This means that \( \lim_{t \to \infty} h(v_t(x)) = -\infty \). From \( y \in \gamma \cap \omega(x) \subset \text{P}(v) \), we have \( h(y) = \lim_{t \to \infty} h(v_t(x)) = -\infty \), which contradicts \( h(y) \in \mathbb{R} \).

Then \( S = \text{Sing}(v) \cup \text{P}(v) \). Fix a point \( x \in S - \text{Sing}(v) = \text{P}(v) \). We claim that \( \omega(x) \subseteq \text{Sing}(v) \). Indeed, fix a height function \( h : S \to \mathbb{R} \) generating the gradient flow \( v \) up to topological equivalence. Assume that there is a non-recurrent point \( y \in \omega(x) \). Then there is a closed transverse arc \( I \) with \( y \in \partial I \cap (I \cap \omega(x)) \). By the existence of the height function \( h : S \to \mathbb{R} \) of \( v \), the value of \( y \) is finite. Since
I is compact, the norm of the vector field generated grad $h$ by $h$ on $I$ is separated from zero. This means that $\lim_{t \to \infty} h(v_t(x)) = -\infty$. From $y \in I \cap \omega(x) \subset P(v)$, we have $h(y) = \lim_{t \to \infty} h(v_t(x)) = -\infty$, which contradicts $h(y) \in \mathbb{R}$. Thus $\omega(x) \subset S - P(v) = \text{Sing}(v)$. □

4. ON PRE-HAMILTONIAN FLOWS

We have the following statements.

**Lemma 4.1.** Any Hamiltonian flow on an orientable compact surface is pre-Hamiltonian.

**Proof.** Let $v$ be a Hamiltonian flow on an orientable compact surface $S$. By definition of Hamiltonian flow, there is a Hamiltonian $h : S \to \mathbb{R}$ whose Hamiltonian vector field generates a flow $w$ which is topologically equivalent to $v$. By Corollary 14 and definition of Hamiltonian vector field, any orbits of the restriction flow $w|_{S - \text{Sing}(w)}$ are closed in the surface $S - \text{Sing}(w)$ and are connected components of level sets of the restriction $h|_{S - \text{Sing}(w)}$. From the flow box theorem for a continuous flow on a compact surface, the set of orbits of $v|_{S - \text{Sing}(w)}$ is a codimension one foliation on the surface $S - \text{Sing}(w)$. Using the topological conjugacy $k : S \to S$ from $v$ to $w$, the composition $H := h \circ k$ is a desired continuous function $H : S \to \mathbb{R}$. □

**Lemma 4.2.** Let $v$ be a pre-Hamiltonian flow on an orientable compact surface $S$. If the image of $\text{Sing}(v)$ by the pre-Hamiltonian is totally disconnected, then $v$ is non-wandering and $S = \text{Sing}(v) \cup \text{Per}(v)$.

**Proof.** Let $v$ be a flow on an orientable compact surface $S$ and $H : S \to \mathbb{R}$ the pre-Hamiltonian of $v$. Since $\text{Sing}(v)$ is closed and so compact, the image $H(\text{Sing}(v))$ is compact and so closed. Fix any point $x \in S$ with $H(x) \notin H(\text{Sing}(v))$. Then the orbit $O(x)$ is a connected component of $H^{-1}(H(x))$ and so is a closed subset. Since $x$ is not a singular point, the orbit $O(x)$ is a periodic orbit. This means that $S - H^{-1}(H(\text{Sing}(v)))$ consists of periodic orbits. Fix any point $y \in H^{-1}(H(\text{Sing}(v))) - \text{Sing}(v)$. Then the orbit $O(y)$ is a connected component of $H^{-1}(H(y)) \setminus \text{Sing}(v)$, and any neighborhood of $O(y)$ in $S - \text{Sing}(v)$ is not contained in $H^{-1}(H(y)) \setminus \text{Sing}(v)$. By the continuity of $H$ and the totally disconnectivity of $H(\text{Sing}(v))$, any neighborhood of $O(y)$ in $S - \text{Sing}(v)$ intersects $H^{-1}(\mathbb{R} - H(\text{Sing}(v))) = S - H^{-1}(H(\text{Sing}(v))) \subseteq \text{Per}(v)$. This means that $H^{-1}(H(\text{Sing}(v))) - \text{Sing}(v) \subset \text{Per}(v)$ and so that $S = \text{Sing}(v) \cup \text{Per}(v)$. Therefore $v$ is non-wandering. □

The totally disconnectivity of the image $H(\text{Sing}(v))$ of the singular point set by the pre-Hamiltonian is necessary and can not be replaced by one of the singular point set $\text{Sing}(v)$ in the previous lemma (see Corollary 5.2). The extended orbit space $S/\nu_{\text{ex}}$ is a quotient space $S/\sim$ defined by $x \sim y$ if either $x$ and $y$ are contained in a multi-saddle connection or there is an orbit that contains $x$ and $y$ but is not contained in any multi-saddle connections. We prove Theorem 17 as follows.

**Proof of Theorem 17** By Lemma 4.1 the assertion (1) implies the assertion (2). Let $v$ be a pre-Hamiltonian flow on an orientable compact surface $S$, $H : S \to \mathbb{R}$ the pre-Hamiltonian, and $\mathcal{F}$ the foliation on $S - \text{Sing}(v)$ induced by $H|_{S - \text{Sing}(v)}$. By the Baire category theorem, any leaves of $\mathcal{F}$ have the empty interior. From Lemma 14.2 the flow $v$ is non-wandering and $S = \text{Sing}(v) \cup \text{Per}(v)$. By [11 Theorem 3], any singular points of $v$ are either centers or multi-saddles.
We claim that there are no locally dense orbits. Indeed, assume that there is a locally dense orbit $O$. Then the closure $\overline{O}$ has a nonempty interior. Put $c := H(O) \in \mathbb{R}$. Since $\overline{O} \subseteq H^{-1}(H(O)) = H^{-1}(c)$, there is a connected component $L$ of the inverse image $H^{-1}(c)$ has a nonempty interior. By definition of pre-Hamiltonian, the connected component $L$ is a leaf of the codimension one foliation $\mathcal{F}$, which contradicts that any leaf has the empty interior.

From [53 Lemma 3.1], the multi-saddle connection diagram is the complement of the union of centers and $\text{Per}(v)$ and the extended orbit space $S/v_{\text{ex}}$ is a finite directed topological graph. Therefore any multi-saddle connections are connected components of $S - \text{Per}(v)$. Since any level sets are closed, each multi-saddle connection is contained in some level set of $H$. This means that the quotient map $\text{pr}_{\text{ex}} : S \to S/v_{\text{ex}}$ can be obtained by collapsing connected components of level sets of $H$ into singletons. Therefore there is an order-preserving continuous mapping $h : S/v_{\text{ex}} \to \mathbb{R}$ with $H = h \circ \text{pr}_{\text{ex}}$. This implies that the directed graph $S/v_{\text{ex}}$ has no directed cycles. From [53 Theorem B], the flow $v$ is Hamiltonian. \hfill $\Box$

5. Wildness of totally disconnected singular point sets

We construct a non-trivial flow box with totally disconnected singular points that interrupts a flow.

5.1. Non-trivial flow box with totally disconnected singular points. Recall that the Minkowski sum $A + B$ is defined by $A + B := \{a + b \mid a \in A, b \in B\}$. Steinhaus shown that $C + C = [0, 2]$, where $C$ is the Cantor ternary set [45]. Identify $\frac{1}{2}C \subset [0, 1/2] \subset \mathbb{R}/\mathbb{Z} = : T$. Using the Whitney theorem, we construct the following non-trivial flow box with totally disconnected singular points.

**Lemma 5.1.** Consider a vector field $X = (0, 1)$ on a closed square $D := [-1, 2]^2$ and let $D^2 := [0, 1]^2 \subset D$. Then there is a smooth function $f : D \to \mathbb{R}$ with $D - [-1/2, 3/2]^2 \subset f^{-1}(1)$ satisfying the following conditions:

1. The flow box $D^2$ contains a wandering domain with respect to $fX$.
2. For any point $x \in D^2$, we have either $O^+_f(x) \subset D^2$ or $O^-_f(x) \subset D^2$.
3. The intersection $\text{Sing}(fX)$ is totally disconnected.

**Proof.** Set $\mathcal{M} := \{(x, x + y) \mid x, y \in \frac{1}{2}C\} = \bigcup_{x \in \frac{1}{2}C} \{x\} \times (\{x\} + \frac{1}{2}C) \subset [0, 1/2] \times [0, 1] \subset D^2$. By construction of $\mathcal{M}$, the subset $\mathcal{M}$ is a closed subset. By Whitney theorem [48] (cf. [27] Theorem 1.1.4), there is a $C^\infty$ function $f_0 : D \to [0, 1]$ with $f_0^{-1}(0) = \mathcal{M}$. Using a bump function $\varphi : D \to [0, 1]$ with $[0, 1]^2 \subset \varphi^{-1}(0)$ and $[-1, 2]^2 - [-1/2, 3/2]^2 \subset \varphi^{-1}(1)$, a $C^\infty$ function $f := f_0(1 - \varphi) + \varphi : D \to [0, 1]$ satisfies $f^{-1}(0) = \mathcal{M}$ and $D - [-1/2, 3/2]^2 \subset f^{-1}(1)$. Then the vector field $fX$ is desired. \hfill $\Box$

Using such a non-trivial flow box interrupting a flow, we show the deformation to create wandering domains.

**Proof of Theorem 7.** By Lemma 5.1, we can replace a trivial flow box with the non-trivial one as in Lemma 5.1 by multiplying a smooth function $f : S \to \mathbb{R}$ such that $f = 1$ outside of the trivial flow box, and that the restriction of $f$ to the trivial flow box is as in Lemma 5.1 in the trivial flow box. Therefore the assertion holds. \hfill $\Box$

We have the following statement.
Corollary 5.2. There is a pre-Hamiltonian flow with wandering domains and totally disconnected singular points.

Proof. Consider a Hamiltonian \( h: \mathbb{S}^2 \to \mathbb{R} \) on the unit sphere \( \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \) defined by \( h(x, y, z) = z \). Replacing a trivial flow box for the Hamiltonian vector field \( X \), by Theorem D, there is a smooth function \( f: \mathbb{S}^2 \to \mathbb{R} \) such that the flow \( v \) generated by the vector field \( fX \) is not no-wandering and the singular point set \( \text{Sing}(v) \) is totally disconnected. By construction, the function \( h \) is the pre-Hamiltonian of \( v \). \( \square \)

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