$K \rightarrow \pi\pi\gamma$ decays: a search for novel couplings in kaon decays

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Abstract

We analyze $K \rightarrow \pi\pi\gamma$ decays in the framework of Chiral Perturbation Theory. We study the different Dalitz plot distributions, trying to find regions where $O(p^6)$ contributions could be more easily detected. To fulfill this program we compute all the the $O(p^4)$ loop and counterterm contributions, finding a substantial agreement with the existing calculations and adding some small missing terms in $K_S \rightarrow \pi^+\pi^-\gamma$. 

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1. Introduction

The amplitudes for $K \rightarrow \pi\pi\gamma$ decays can be generally decomposed as the sum of two terms: the internal bremsstrahlung ($A_{IB}$) and the direct emission ($A_{DE}$) [1-4]. The first one, which represents only the bremsstrahlung contribution of the external charged particles, is completely predicted by QED in terms of the $K \rightarrow \pi\pi$ amplitude [5]. The second one, which is obtained by subtracting $A_{IB}$ from the total amplitude, depends on the direct $K \rightarrow \pi\pi\gamma$ coupling and furnishes a test for mesonic interaction models.

The pole for the photon energy going to zero would tend to enhance $A_{IB}$ compared to $A_{DE}$. Nevertheless in $K_L \rightarrow \pi^+\pi^-\gamma$ and $K^\pm \rightarrow \pi^\pm\pi^0\gamma$ the inner bremsstrahlung is respectively suppressed by the CP conservation and the $\Delta I = 1/2$ rule. As a result in these channels it is easier to study the direct emission, testing low energy interaction models.

Chiral Perturbation Theory (ChPT) is supposed to be a reliable framework to study $K \rightarrow \pi\pi\gamma$ decays [6, 7]. In this effective quantum field theory, based on symmetry principles, the physical amplitudes are expanded in powers of meson masses and momenta. At the lowest non-trivial order ($\pi^2$), in this effective quantum field theory, based on symmetry principles, the physical amplitudes are expanded in powers of meson masses and momenta. At the lowest non-trivial order ($\pi^2$), the internal bremsstrahlung appears, while the direct emission, which requires more derivatives, starts at $\pi^4$. Experiments seem to indicate that in $K_L \rightarrow \pi^+\pi^-\gamma$ and possibly in $K^\pm \rightarrow \pi^\pm\pi^0\gamma$, not only $\pi^4$ but also $\pi^6$ magnetic contributions might be important [8, 4]. More in general, we can say that the question of the size of $\pi^6$ contribution in radiative non-leptonic kaon decays is very interesting, as it has been emphasized in the case of $K_L \rightarrow \pi^0\pi^0\gamma$ [9, 10].

In this paper we have analyzed $K \rightarrow \pi\pi\gamma$ decays, looking for kinematical regions where order $\pi^6$ electric contributions could be relevant and possibly detected. To this aim we have computed all the one loop amplitudes and the corresponding $\pi^6$ counterterms contributions. We substantially agree with the existing calculation for $K_L \rightarrow \pi^+\pi^-\gamma$, $K^\pm \rightarrow \pi^\pm\pi^0\gamma$ [4] and $K_S \rightarrow \pi^+\pi^-$ [11]. In addition we have calculated the small missing terms due $\pi K$ and $K\eta$ loops in $K_S \rightarrow \pi^+\pi^-\gamma$. Using these results we find some kinematical region, in particular for $K_L \rightarrow \pi^+\pi^-\gamma$ and possibly for $K^\pm \rightarrow \pi^\pm\pi^0\gamma$ decays, where order $\pi^6$ electric contributions could be relevant.

The paper is organized as follows: in section 2 we give a short discussion on kinematics, Low theorem and ChPT. In section 3, 4, 5 and 6 we analyze respectively $K_L \rightarrow \pi^+\pi^-\gamma$, $K_S \rightarrow \pi^+\pi^-\gamma$, $K^\pm \rightarrow \pi^\pm\pi^0\gamma$ and $K_{L,S} \rightarrow \pi^0\pi^0\gamma$ decays. Section 7 contains some concluding remarks and finally in the appendices we discuss the loop functions and some variable transformations.

2. Kinematics, Low theorem and ChPT

Due to Lorentz and gauge invariance we can define two invariant amplitudes for the processes $K(p_K) \rightarrow \pi_1(p_1)\pi_2(p_2)\gamma(q, \bar{e})$: the electric and the magnetic one. Using the dimensionless amplitudes $E$ and $M$, defined in Ref.[4], we can write:

$$A(K \rightarrow \pi\pi\gamma) = \tilde{E}(q) [E(z_1)(p_1q_{2\mu} - p_2q_{1\mu}) + M(z_2)\epsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\rho q_\sigma] / m_K^2,$$  \hspace{1cm} (1)

with

$$z_1 = \frac{p_1q}{m_K}, \hspace{1cm} z_3 = \frac{p_3q}{m_K}, \hspace{1cm} z_3 = z_1 + z_2.$$  \hspace{1cm} (2)

Summing over photon helicities there is no interference among electric and magnetic terms:

$$\frac{\partial^2 \Gamma}{\partial z_1 \partial z_2} = \frac{m_K}{(4\pi)^3} [((E(z_1))^2 + |M(z_1)|^2) [z_1z_2(1 - 2z_3 - r_1^2 - r_2^2) - r_1^2 z_2^2 - r_2^2 z_1^2]].$$  \hspace{1cm} (3)

As we have already said in the introduction, generally one prefers also to decompose the total amplitude as a sum of inner bremsstrahlung and direct emission:

$$A(K \rightarrow \pi_1\pi_2\gamma) = A_{IB} + A_{DE}.$$  \hspace{1cm} (4)
$A_{IB}$, which in the classic limit would correspond to the charged particle radiation, in QED is completely predicted by the Low theorem [5], which relates radiative and non-radiative amplitudes in the limit of photon energy going to zero. For $K \rightarrow \pi \pi \gamma$ transitions the theorem reads:

$$A_{IB}(K \rightarrow \pi \pi \gamma) = e \left( \frac{\vec{q}_b}{q_b} - \frac{\vec{q}_a}{q_a} \right) A(K \rightarrow \pi \pi),$$  

(5)

where $(p_a, p_b) \equiv (p_+, p_-)$ for the neutral kaon decays and $(p_a, p_b) \equiv (p_+, p_K)$ or $(p_a, p_b) \equiv (p_K, p_-)$ for the charged ones. Thus in the $E(z_i)$ amplitude the two contributions $A_{IB}$ and $A_{DE}^{Electric}$ can interfere, differently from the $M(z_i)$ amplitude, where only a direct emission contribution appears:

$$|A(K \rightarrow \pi_1 \pi_2 \gamma)|^2 = |A_{IB}|^2 + 2 \cdot \text{Re} \{ A_{IB} A_{DE}^{Electric} \} + |A_{DE}^{Electric}|^2 + |A_{DE}^{Magnetic}|^2.$$  

(6)

By opportune kinematic integration, eq.(3) can be transformed in a photon energy distribution (see appendix B). In the limit of the photon energy ($E^*_\gamma$ in the kaon rest frame) going to zero one can write:

$$\frac{d\Gamma}{dE^*_\gamma} = \frac{\alpha}{E^*_\gamma} + \beta + \gamma \cdot E^*_\gamma + ...$$  

(7)

The content of Low theorem is a prediction for $\alpha$ and $\beta$, from pure QED, in terms of the physical amplitude $A(K \rightarrow \pi \pi)$.

We will now proceed analyzing $K \rightarrow \pi \pi \gamma$ decays in the framework of ChPT. In this effective quantum field theory the octet of the pseudoscalar fields$^1$,

$$\phi = \begin{bmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} \\ \pi^- \\ K^- \end{bmatrix} \begin{bmatrix} \pi^+ \\ -\pi^0/\sqrt{2} + \eta/\sqrt{6} \\ K^0 \end{bmatrix} \begin{bmatrix} K^+ \\ -\eta/\sqrt{2} \\ -\eta \sqrt{3/2} \end{bmatrix},$$  

(8)

is identified with the octet of pseudo Goldstone bosons coming from the spontaneous symmetry breaking of $G = SU(3)_L \times SU(3)_R$ into $H = SU(3)_L \times SU(3)_R$. The lagrangian, expressed in terms of $\phi$, is expanded in powers of meson masses and momenta, and is dictated only by the transformation law of the interactions.

At the lowest order $p^2$, the strong part of the lagrangian is completely fixed and is given by:

$$L_S^{(2)} = \frac{F_0^2}{4} \left\{ \text{tr}(D_\mu U^\dagger D^\mu U) + \text{tr}(\chi U^\dagger + U \chi^\dagger) \right\},$$  

(9)

where $U = e^{i\sqrt{2}\phi/F_0}$ is a unitary $3 \times 3$ matrix which transforms linearly under $G$ and $\chi$ is proportional to a scalar field, transforming like $U$, which must acquire a non-vanishing expectation value in order to reproduce the correct values of the meson masses [7]. At this order we can assume $F_0 = F_\pi = F_K \simeq 93$MeV. The covariant derivative $D_\mu U$ is given by:

$$D_\mu U = \partial_\mu U - ir_\mu U + iU l_\mu,$$  

(10)

where $l_\mu$ and $r_\mu$ are the external gauge fields of $SU(3)_L \times SU(3)_R$. The electromagnetic field can be introduced setting $l_\mu = r_\mu = eQA_\mu$, where $Q = \text{diag}(2/3, -1/3, -1/3)$.

The $|\Delta S| = 1$ weak lagrangian transforms under $G$ as an $(8_L, 1_R)$ or a $(27_L, 1_R)$. At the lowest order $p^2$ can be written as:

$$L_{\Delta S}^{(2)} = G_S F_\pi^4 \text{tr} \left( \lambda D_\mu U^\dagger D^\mu U \right) + G_{27} F_\pi^4 \left[ (U^\dagger D_\mu U)_{23}(D_\mu U^\dagger)_{11} + \frac{2}{3}(U^\dagger D_\mu U)_{21}(D_\mu U^\dagger)_{13} \right] + \text{h.c.},$$  

(11)

$^1$ The chosen phase convention does not satisfy the Condon-Shortley-De Swart one [12].
where $\lambda = (\lambda_6 - i\lambda_7)/2$ and $\lambda_{6,7}$ are the usual Gell-Mann matrices. Experimentally, from $K \to \pi\pi$ decays, the coupling constant $G_8$ and $G_{27}$ are fixed to be:

$$|G_8| \approx 9 \cdot 10^{-6} \text{ GeV}^{-2}, \quad \frac{G_{27}}{G_8} \approx \frac{1}{18}. \quad (12)$$

At order $p^2$, in $K \to \pi\pi\gamma$ decays, since there are not enough powers of momenta, only an inner bremsstrahlung amplitude will appear. In agreement with the Low theorem we found:

$$A_{IB}(K \to \pi\pi\gamma)^{\mathcal{O}(p^2)} = e \left( \frac{\tilde{q}_p}{q_p} - \frac{\tilde{q}_{pa}}{q_{pa}} \right) A(K \to \pi\pi)^{\mathcal{O}(p^2)}. \quad (13)$$

Actually we will take eq.(13) as a definition of the inner bremsstrahlung amplitudes, meaning that this relation must hold order by order in ChPT. We recall also the $\mathcal{O}(p^2)$ results for the CP conserving $K \to \pi\pi$ amplitudes:

$$A(K_S \to \pi^+\pi^-)^{\mathcal{O}(p^2)} = + 2(G_8 + \frac{2}{3} G_{27}) F_{\pi}(m_K^2 - m^2_{\pi}), \quad (14)$$

$$A(K^+ \to \pi^+\pi^0)^{\mathcal{O}(p^2)} = + \frac{5}{3} G_{27} F_{\pi}(m_K^2 - m^2_{\pi}).$$

ChPT is not a renormalizable theory but the requirement of unitarity implies that meson loops have to be considered. The loop contributions, which start at order $p^4$, introduce divergences which must be re-absorbed, order by order, by corresponding counterterms. While the physical amplitudes (loops + counterterms) are scale independent, the coefficients of the counterterms depend on the renormalization scale of the loops. Gasser and Leutwyler [7] have classified the $p^4$ operators for the strong and electromagnetic lagrangian and have determined their coefficients by comparison with the experiments. The authors of Ref.[13] have shown that these coefficients are well reproduced by vector and axial vector meson exchange choosing the renormalization scale $\mu$ around the $p$-mass.

The effective Lagrangian at next to leading order, both in the strong and weak sector, has anomalous and non-anomalous operators [14]. At this order the anomalous operators contribute only to the magnetic amplitudes $M(z_i)$, which have been extensively studied in Ref.[3, 4] and will be not re-analyzed here. The non-anomalous octet part of the $p^4$ $|\Delta S| = 1$ Lagrangian, relevant for $K \to \pi\pi\gamma$ decays, can be written as:

$$\mathcal{L}_{|\Delta S| = 1}^{(4)} = G_8 F_{\pi} [N_{14}W_{14} + N_{15}W_{15} + N_{16}W_{16} + N_{17}W_{17}] + \text{h.c.,} \quad (15)$$

where

$$W_{14} = \text{itr} \left( \lambda u^\dagger \left\{ f_{\mu\nu}^{\mu\nu} , u_\mu u_\nu \right\} u \right), \quad W_{15} = \text{itr} \left( \lambda u^\dagger u_\mu f_{\mu\nu}^{\mu\nu} u_\nu u \right), \quad (16)$$

$$W_{16} = \text{itr} \left( \lambda u^\dagger \left\{ f_{\mu\nu}^{\mu\nu} , u_\mu u_\nu \right\} u \right), \quad W_{17} = \text{itr} \left( \lambda u^\dagger u_\mu f_{\mu\nu}^{\mu\nu} u_\nu u \right),$$

$$\lambda = (\lambda_6 - i\lambda_7)/2, \quad u^2 = U, \quad u_\mu = i u^\dagger D_\mu U u = u^\dagger, \quad f_{\mu\nu}^{\mu\nu} = u F_{L}^{\mu\nu} u^\dagger \pm u^\dagger F_{R}^{\mu\nu} u$$

and

$$F_{L}^{\mu\nu} = \partial^\mu t^\nu - \partial^\nu t^\mu + i[l^\mu, t^\nu], \quad F_{R}^{\mu\nu} = \partial^\mu t^\nu - \partial^\nu t^\mu + i[l^\mu, t^\nu]. \quad (17)$$

In the electric transitions of $K \to \pi\pi\gamma$ decays only the following counterterm combination appears:

$$N_{14} - N_{15} - N_{16} - N_{17}. \quad (18)$$

Chiral symmetry alone does not predict the values of the constants $N_i$, these however can be estimated using various Vector Meson Dominance (VMD) models. Following Ref.[15], the combination in eq.(19) can be expressed in terms of the parameter $k_f$:

$$N_{E_1}^{(4)} = (4\pi)^2 [N_{14} - N_{15} - N_{16} - N_{17}] = - k_f \frac{8\pi^2 F_{\pi}^2}{M_\pi^2} \sim o(1). \quad (20)$$
At $o(p^6)$ obviously the number of E-type counterterms increases, normalizing $\mathcal{L}^{(6)}$ as $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ we can write in general:

$$\mathcal{L}^{(6)}_{|\Delta S|=1} = G_8 \sum_i N_i^{(6)} W_i^{(6)} + \text{h.c.}$$

We will not try to classify all the $W_i^{(6)}$ but we will consider only two typical operators:

$$W_E^{(6)} = \frac{i}{(4\pi)^4} \text{tr} \left( u\lambda u^\dagger \left( u\gamma^\mu S_{\mu\nu}^u + u_{\nu} S_{\mu\nu}^u u^\dagger \gamma^\mu \right) \right),$$

$$W_M^{(6)} = \frac{1}{(4\pi)^4} \text{tr} \left( u\lambda u^\dagger \left( J_{\mu\nu}^u, \gamma^\rho, \nabla^\rho u_{\mu}, u_{\nu} \right) \right),$$

($\nabla^\rho$ is the "covariant" derivative defined in ref.[13], which for our purposes acts like $\partial^\rho$). The first one, similar to the one considered in Ref.[16], generates the same kinematical dependence of the $o(p^4)$ operators, i.e. give a contribution to the electric amplitudes of $K \rightarrow \pi\pi\gamma$ decays proportional to the one of the $o(p^4)$ counterterms. The second one, with more derivatives, generates a substantially different kinematical dependence. The size of the coefficients of these operator is an open question. With the normalization in eqs.(22)-(23), naive chiral power counting would suggest $N_E^{(6)} \sim N_M^{(6)} \sim o(1)$ nevertheless, for different arguments, sensibly larger estimates for $N_E^{(6)}$ and $N_M^{(6)}$ were made respectively in Ref.[16] and [17]. Waiting for further theoretical insight, we will not commit ourself to any particular model to predict the size of these coefficients, but we will investigate the possibility to bound or even to measure them.

3. $K_L \rightarrow \pi^+\pi^-\gamma$

In this decay the electric and magnetic contribution of eq.(3) are competing and they have been both measured [8]. The inner bremsstrahlung violates CP through the $K_L \rightarrow \pi^+\pi^-$ amplitude and, neglecting the $\Delta I = 3/2$ suppressed terms, can be written as:

$$E_{IB}^o(z_i) = + \frac{e e A(K_S \rightarrow \pi^+\pi^-)}{m_K(z_+z_-)}, \quad (p_1 = p_+, \ p_2 = p_-),$$

which at the lowest order in ChPT is:

$$E_{IB}^o(p^2)(z_i) = \frac{2eeG_8F_{\pi}(m_K^2-m_\pi^2)}{m_K(z_+z_-)}.$$
For the IB both the spectrum and the rate are in agreement with eq.(24). The dipole magnetic contribution can be written in first approximation as

$$M_1(z_3) \simeq M_0 \cdot (1 + \hat{c} z_3),$$

(28)

where $M_0$ and $\hat{c}$ are fixed by the rate and the spectrum respectively. The large experimental vale for the slope: $\hat{c}_{ex} = (-1.7 \pm 0.5)$ [21], which vanishes at $o(p^4)$ in ChPT, carries important dynamical informations. It is not only a pure $o(p^6)$ contribution, but it is also substantially larger of a typical vector meson exchange effect [4], showing that, at least in the weak anomalous sector, local $p^6$ contributions are sizable and can play a role in discriminating different models.

Regarding the electric multipoles, at $o(p^4)$ there are no local CP conserving counterterms and consequently the loop contributions are finite [22, 23]. We have computed these amplitudes confirming the earlier results of Ref.[4]. The potential contributions are given by $\pi \pi$ [6] $p$ [7] $\gamma$ [8] and $K \eta$ $KK$ [9] intermediate states (see fig.1). Nevertheless, due to CP conservation, only the small $\pi K$ and $K \eta$ loops are non-vanishing. Their contributions can be parametrized using the functions $h_{\pi K}$ and $h_{K \eta}$ defined in the appendix A:

$$E^{(4)}_{\text{loop}}(z_i) = \frac{eG_S m_K (m^2_K - m^2_\pi)}{8\pi^2 F_\pi} \left[ h_{\pi K}(z_-) + h_{K \eta}(z_-) - h_{\pi K}(z_+) - h_{K \eta}(z_+) \right].$$

(29)

The functions $h_{ij}$ depend only on one kinematical variable. Considering the Taylor expansion of these functions ($h_{ij}(z_i) = \sum_n h_{ij}^0 z_i^n$), with the coefficients reported table 1, we found:

$$E^{(4)}_{\text{loop}}(z_i) \simeq \frac{eG_S m_K (m^2_K - m^2_\pi)}{8\pi^2 F_\pi} \left[ 0.05 (z_+ - z_-) \right].$$

(30)

Consistently with the CP conservation only an $E_2$ term survives but it is suppressed by several factors compared to $E_{IB}$ in eq.(25): i) the angular momentum barrier, ii) the absence of the photon energy pole, iii) the chiral suppression factor $(m_{3/2}/F_\pi)^2$ and iv) the small coefficient 0.05 generated by the chiral loops.

Other $E_2$ contribution can arise at $o(p^6)$. The operator $W^{(6)}_{E_2}$ is forbidden by CP conservation, like the $o(p^4)$ counterterms, on the other hand the operator $W^{(6)}_{E_2}$ in eq.(23) gives:

$$E^{(6)}_{E_2}(z_i) = \frac{eG_S m_K^5}{48\pi^4 F_\pi^2} N^{(6)}_{E_2}(z_+ - z_-).$$

(31)

Since $N^{(6)}_{E_2} \sim o(1)$ the $E_2$ multipole should be dominated by the local $o(p^6)$ counterterms instead of the $o(p^4)$ loop contributions. This is one of the main message of this paper. At higher order chiral loops will generate a small rescattering phase $\delta_{E_2}$ to the term in eq.(31).

Now the question is whether the $E_2$ multipole can be detected. The electric contribution to eq.(3) is

$$|E_{IB} + E_{DE}|^2 \simeq \left[ |E_{IB}|^2 + 2 Re(E_{IB} E_{E_2}) \right]$$

$$= |e A(K_S \rightarrow \pi^+ \pi^-)|^2 \left[ 1 + C^L_{int} z_+ z_- (z_+ - z_-) \right],$$

(32)

where

$$C^L_{int} \simeq \frac{m^4_K \cos(\phi_0 + \delta_0 - \delta_{E_2})}{48|\epsilon|^4 F_\pi^4} N^{(6)}_{E_2},$$

(33)

$\delta_0 \simeq (39 \pm 5)^\circ$ [19] is the the $\pi^+ \pi^-$ $I=0$ phase shift, $\phi_0 \simeq (43.67 \pm 0.14)^\circ$[24] is the phase of $\epsilon$ and $\delta_{E_2}$ is expected to be small. Unfortunately the argument of the cosine in eq.(33) tends to be about 80°, smearing somewhat the effect. Nevertheless we can try to study the interference term $C^{L}_{int}$, and thus $N^{(6)}_{E_2}$, measuring the (indirect) CP violating asymmetry under pion exchange, i.e. the observable:

$$\Delta^L = \frac{N(K_L \rightarrow \pi^+ \pi^- ; \cos(\theta) > 0) - N(K_L \rightarrow \pi^+ \pi^- ; \cos(\theta) < 0)}{N(K_L \rightarrow \pi^+ \pi^- ; \cos(\theta) > 0) + N(K_L \rightarrow \pi^+ \pi^- ; \cos(\theta) < 0)}.$$

(34)
where $\theta$ is the angle between $\gamma$ and $\pi^+$ momenta in the di-pion rest frame (see appendix B). For $E^*_{\gamma} > 20$ MeV

$$|\Delta^L_{(E^*_{\gamma} > 20 \text{ MeV})}| = 1.2 \cdot 10^{-4}|C^L_{\text{int}}| \sim 10^{-3}|N^{(6)}_{E_2}|,$$

thus at least $10^{11} K_L$ are necessary in order to observe a non vanishing effect.

We finally note that $\Delta^L$ could receive a contribution also from the interference between $E_2$ and a direct CP violating multipole $E_1$. Nevertheless this contribution it is not enhanced by the interference with the inner bremsstrahlung and therefore should be suppressed respect to the previous one.

4. $K_S \rightarrow \pi^+ \pi^- \gamma$

In this channel we will consider only the electric amplitudes since the dipole magnetic contribution is CP violating while higher multipoles are phase space suppressed and do not interfere with electric amplitudes for unpolarized photons. The CP conserving bremsstrahlung term

$$E_{IB}(z_i) = \frac{eA(K_S \rightarrow \pi^+ \pi^-)}{m_K(z_+ z_-)}, \quad (p_1 = p_+, p_2 = p_-),$$

dominates giving a contribution to the branching ratio

$$\text{BR}(K_S \rightarrow \pi^+ \pi^- \gamma)_{IB,E^*_{\gamma} > 20 \text{ MeV}} \simeq 4.80 \cdot 10^{-3}$$

and a Dalitz Plot distribution, reported for instance in Ref.[25, 2], which agree very nicely with the data [8].

At the lowest order in ChPT only the IB is present, the electric dipole component appear at the next order. The two gauge invariant contributions $E_{IB}$ and $E_{DE}$ are separated according to the ChPT definition of IB (eq.(13)). At order $o(p^4)$ there are loops and counterterms. Since the counterterm contribution to $A(K_S \rightarrow \pi^+ \pi^- \gamma)_{DE}$

$$E^{(4)}_{CT}(z_i) = \frac{4eG_Fm_K^3}{4\pi^2 F_\pi^2} [N_{14} - N_{15} - N_{16} - N_{17}] = \frac{eG_Fm_K^3}{4\pi^2 F_\pi} N^{(4)}_{E_1}$$

is scale independent [22, 23], the direct emission loop contribution has to be finite [11]. As in the $K_L$ case the possible intermediate states appearing in the meson loop are $\pi\pi$, $\pi K$, $K\eta$ and $KK$ (see fig.1). In Ref.[11] only the $\pi\pi$ loop was computed, since this is the only one which has an absorptive part. Interestingly a large dispersive contribution was found, even larger than the absorptive one in the allowed kinematical regions. This of course urged a calculation of the other dispersive contributions and a confirmation of the previous $\pi\pi$ loop calculation. Analogously to eq.(29), our result for the loop contribution can be written as:

$$E^{(4)}_{loop}(z_i) = -\frac{eG_Fm_K^3}{8\pi^2 F_\pi^2} \{4h_{\pi\pi}(-z_3) + h_{\pi K}(z_+ z_{\pi K}) + h_{K\eta}(z_+ z_{K\eta}) + (z_+ \leftrightarrow z_-)\},$$

where $h_{\pi\pi}$, $h_{\pi K}$ and $h_{K\eta}$ are defined in the appendix A (the $KK$ contribution vanishes in the limit $m_{K^+} = m_{K^0}$). Regarding the $\pi\pi$ loop we confirm the interesting result that the dispersive contribution is larger than the absorptive one, however we disagree on the analytic structure. We find instead

$$E^{(4)}_{\pi\pi}(E^*_{\gamma}) = -\frac{eG_Fm_K^3}{8\pi^2 F_\pi^2 E^*_{\gamma}^2} \left\{ (m_K^2 - m_{\pi^+}^2)(m_K^2 - 2E^*_{\gamma} m_K) \left[ \beta \ln \left( \frac{1 + \beta}{\beta - 1} \right) - \beta_0 \ln \left( \frac{1 + \beta_0}{\beta_0 - 1} \right) \right] \right\}$$

$$-E^*_{\gamma}m_K(2m_{\pi^+}^2 - m_K^2) + m_{\pi^+}^2(m_K^2 - m_{\pi^+}^2) \left[ \ln^2 \left( \frac{1 + \beta_0}{\beta_0 - 1} \right) - \ln^2 \left( \frac{1 + \beta}{\beta - 1} \right) \right],$$

while the authors of Ref.[11] for the polynomial term had $(3m_{\pi^+}^2 - 2m_K^2)$. Though the difference is numerically insignificant has some deep meaning: i) the full amplitude is proportional to $(m_K^2 - m_{\pi^+}^2)$, i.e. to the weak
vertex of fig.1 with the pions on-shell, ii) the $E^{(4)}_{\text{loop}}(E^*_\gamma)$ amplitude vanishes in the limit $E^*_\gamma \to 0$, as required by Low theorem\(^\dagger\). Of course with the help of these considerations one understands better why the $KK$ loop contribution is zero. Furthermore it is also confirmed by our calculation that the $\pi\pi$ loop contribution is dominant as suggested in Ref.[11].

Another interesting point of our calculation is the kinematical dependence of the loop contributions. Using the Taylor expansion of the function $h_{ij}$ in eq.(39) we obtain:

$$E^{(4)}_{\text{loop}}(z_i) \simeq -\frac{eGsm_K(m^2_K - m^2_\pi)}{4\pi^2F_\pi} \left[ 1.3 + 1.1z_3 - i(0.6 + 0.9z_3) \right].$$  \hfill (41)

Since the kinematical dependence of eq.(41) is mild, in principle it could be possible to distinguish particular $o(p^6)$ contributions, like the one of the operator $W^{(6)}_{E_2}$:

$$E^{(6)}_{E_2}(z_i) = -\frac{eGsm_K^2}{24\pi^3F_\pi^2}N^{(6)}_{E_2} \left[ m^2_K - m^2_\pi - \frac{3}{2}m^2_\pi z_3 \right].$$  \hfill (42)

On the other hand it is impossible to distinguish among the flat contributions of the $o(p^4)$ counterterms and the operator $W^{(6)}_{E_1}$:

$$E^{(6)}_{E_1}(z_i) = -\frac{eGsm_K^2}{32\pi^3F_\pi}N^{(6)}_{E_1}.$$  \hfill (43)

Of course all these contributions give a very small interferential branching ratio compared to the IB in eq.(37). As explained in the discussion regarding the bremsstrahlung of $K_L \to \pi^\pm\pi^\mp\gamma$, for $A(K_L \to \pi^\pm\pi^\mp)$ we will use the experimental value of $|A(K_L \to \pi^\pm\pi^\mp)|$ and $\delta^0_0 \simeq (39 \pm 5)^\circ$ [20, 19]. Neglecting the $p^6$ contribution, using the central value of $\delta^0_0$ and varying $N^{(4)}_{E_1}$ we obtain the results reported in table 2.

Differently from the $K_L$ case, due to the large IB, in $K_L \to \pi^\pm\pi^\mp\gamma$ decays it is practically impossible to measure a CP violating interference between an even and an odd multipole.

5. $K^\pm \to \pi^\pm\pi^0\gamma$

Since the initial and the final states of $K^\pm \to \pi^\pm\pi^0\gamma$ decays are not CP eigenstates, in these channels the inner bremsstrahlung amplitude together with the lowest electric and magnetic transitions are present, also in the limit of CP conservation. Actually $E_{1B}$ is suppressed by the $\Delta I = 1/2$ rule:

$$E_{1B}(z_i) = \pm\frac{eA(K^\pm \to \pi^\pm\pi^0)}{m_K(z_\pm z_3)}, \quad (p_1 = p_\pm, \ p_2 = p_0).$$  \hfill (44)

Experimentalists generally choose one kinematical variable as the kinetic energy of the charged pion ($T_\pi^* \text{in the CMS}$), which is not affected by the problem of identifying the $\pi^0$ photons. In the range $55$ MeV $\leq T_\pi^* \leq 90$ MeV the theoretical prediction for internal bremsstrahlung branching ratio is:

$$BR(K^\pm \to \pi^\pm\pi^0\gamma)_{1B, 55\text{MeV} \leq T_\pi^* \leq 90\text{MeV}} = 2.61 \cdot 10^{-4}. \hfill (45)$$

After the subtraction of this contribution from the experimental branching ratio there is a clear evidence of a direct emission component [18]:

$$BR(K^\pm \to \pi^\pm\pi^0\gamma)_{DE, 55\text{MeV} \leq T_\pi^* \leq 90\text{MeV}} = (1.8 \pm 0.4) \cdot 10^{-5}. \hfill (46)$$

It is only one order of magnitude less than the IB in eq.(45), instead of the typical $10^{-2} \sim 10^{-3}$ suppression factor of the interferential contribution, because both the electric and magnetic amplitudes are not suppressed by the $\Delta I = 1/2$ rule.

\(^\dagger\) The author of Ref.[11] have recently checked their calculation confirming our result.
From the analysis of the Dalitz Plot distribution, the DE component seems more likely due to a magnetic transition [26], but the present data are not conclusive about this point. The theoretical discussions of the magnetic amplitudes has been done for instance in Ref.[3, 4]. Here we will concentrate on the electric ones in the framework of ChPT. At the lowest order, as usual, there is only the IB contribution:

\[ E_{IB}^{o(p^4)}(z_i) = \pm \frac{5eG_2F_\pi(m_K^2 - m_\pi^2)}{3m_K(z_\pm z_3)}. \]  (47)

At next order there are loops and counterterms. The \( o(p^4) \) counterterm combination which appears is the same finite one of eq.(38):

\[ E_{CT}^{(4)}(z_i) = \frac{eGsm_K^3}{8\pi^2F_\pi}N_{E_1}^{(4)}, \]  (48)

thus the loop contributions are finite. In the octet limit \( (G_{27} = 0) \) only the small \( \pi K \) and \( K\eta \) loops are non-vanishing. We have computed them finding a little discrepancy with the earlier computation of Ref.[4]. Our result is:

\[ E_{\text{loop}}^{(4)}(z_i) = \pm \frac{eGsm_K(m_K^2 - m_\pi^2)}{8\pi^2F_\pi} \left[ h_{\pi K}(z_+) + h_{K\eta}(z_+) \right] \]
\[ \approx \pm \frac{eGsm_K(m_K^2 - m_\pi^2)}{8\pi^2F_\pi} \left[ .16 + .05z_+ \right], \]  (49)

weather in Ref.[4] the \( h_{K\eta} \) function was multiplied by a factor 2/3\(^1\). Due to the smallness of the loop contributions this discrepancy is numerically irrelevant.

As in the \( K_L \to \pi^+\pi^-\gamma \) case, in \( K^\pm \to \pi^\pm\pi^0\gamma \) decays it turns out that the loop contributions are very small and thus the \( o(p^6) \) local operators may be relevant. However in this case the \( o(p^4) \) counterterms are not vanishing and should represent the dominant effect. To disentangle the \( o(p^4) \) contributions from the \( o(p^6) \) ones it is necessary to look for particular kinematical dependencies. As in the previous cases the operator \( W_{E_1}^{(6)} \) gives only a flat contribution which renormalizes the one of the \( o(p^4) \) counterterms:

\[ N_{E_1}^{(4)} \to N_{E_1}^{(4)} - \frac{m_K^2}{8\pi^2F_\pi^2}N_{E_1}^{(6)}. \]  (50)

On the other hand the operator \( W_{E_2}^{(6)} \) gives a non-flat contribution:

\[ E_{E_2}^{(6)}(z_i) = \pm \frac{eGsm_K^3}{24\pi^2F_\pi^2}N_{E_2}^{(6)} \left[ m_K^2 - m_\pi^2 - \frac{3}{2}m_K^2z_3 \right], \]  (51)

which could be detected studying the \( T_\pi^\star \) dependence of the Dalitz Plot distribution (see appendix B).

6. \( K_{L,S} \to \pi^0\pi^0\gamma \)

Since no charged particles are involved, \( K_{L,S} \to \pi^0\pi^0\gamma \) decays are completely different from those considered before. The absence of charged particles implies the absence of the inner bremsstrahlung amplitude and thus a very small branching ratio. Furthermore these decays are suppressed by Bose statistics, which requires both electric and magnetic multipoles to be even. As a consequence none of them has been observed yet.

In the limit of CP conservation \( K_L \to \pi^0\pi^0\gamma \) is a pure electric transition and \( K_S \to \pi^0\pi^0\gamma \) is a pure magnetic one. We will consider only the former. As it has been shown in Ref.[27], in \( K_L \to \pi^0\pi^0\gamma \) not only the \( o(p^4) \) local contributions are forbidden, but also the one-loop amplitude is vanishing. Therefore this decay can receive contributions at least of order \( p^6 \) in ChPT and could be very useful in order to fix, or even

\(^1\) The author of Ref.[4] have recently checked their calculation confirming our result.
to bound, these new couplings. Unfortunately it is very difficult to observe, due to the large background of $K_L \to 3\pi^0$.

A typical operator which gives a non vanishing contribution to $K_L \to \pi^0\pi^0\gamma$ is $W_E^{(6)}$:

$$E^{(6)}_{E_2}(z_i) = \frac{e G m_K^5}{48 \pi^4 F_\pi^2} N_{E_2}^{(6)} (z_1 - z_2).$$

Using eq.(52) we can parametrize the branching ratio of $K_L \to \pi^0\pi^0\gamma$ in terms of $N_{E_2}^{(6)}$:

$$BR(K_L \to \pi^0\pi^0\gamma) \simeq 10^{-9} |N_{E_2}^{(6)}|^2.$$  

In Ref.[17] the optimistic guess of $BR(K_L \to \pi^0\pi^0\gamma) \simeq 10^{-8}$, together with $BR(K_S \to \pi^0\pi^0\gamma) \simeq 10^{-11}$, was made. This prediction would suggest $N_{E_2}^{(6)} \sim 3$, which is certainly bigger than the naive power counting estimate but cannot be excluded a priori.

7. Conclusions

The analysis presented here completes the previous $o(p^4)$ calculation on $K \to \pi\pi\gamma$ decays and explores the effects of $o(p^6)$ local operators in the electric transitions.

Regarding the $o(p^4)$ DE loop contributions, we find substantial agreement with the existing calculations. In Ref.[11] it was shown that the dispersive pion loop contribution is larger than the absorptive one. This has urged us to calculate also the $\pi K$ and $\eta K$ loops, which give only a dispersive contribution. We find that the naive expectation of Ref.[11], that these contribution are small, is confirmed. Furthermore our calculation has clarified some erroneous considerations of Ref.[11].

We have also analyzed the contributions to $K \to \pi\pi\gamma$ decays of two typical $o(p^6)$ operators (eqs.(22)-(23)), which have been studied for different purposes in the literature. Interestingly we find that in $K_L \to \pi^+\pi^-\gamma$ the local $o(p^6)$ contributions can be substantially larger than the $o(p^4)$ ones.

Acknowledgements

We would like to thank G. Ecker and H. Neufeld for very useful discussions.
Appendix A

Following Ref.[4], in order to write the loop functions \( h_{ij}(z) \) we first introduce the expansion of the three point integrals with \( q^2 = 0 \):

\[
\int \frac{d^4l}{(2\pi)^4} \frac{\mu \nu}{[(l^2 - m_i^2)(l + q)^2 - m_j^2][(l - p)^2 - m_j^2]} = \mu \nu C_{20}(p^2, (p + q)^2, m_i^2, m_j^2) + "p^\mu, q^\nu" \text{ terms."} \tag{A.1}
\]

With this definition, the finite functions \( h_{ij}(z) \) are given by:

\[
h_{ij}(z) = \frac{4\pi^2}{pq} \frac{m_K^2}{m_K^2} \left[ C_{20}(p^2, (p + q)^2, m_i^2, m_j^2) - C_{20}(p^2, p^2, m_i^2, m_j^2) \right], \tag{A.2}
\]

where \( z = pq/m_K^2 \) and \( p^2 = m_K^2 \) (for \( h_{\pi K} \) and \( h_{K\eta} \)) or \( p^2 = m_\pi^2 \) (for \( h_{\pi\pi} \)).

These functions can be explicitly written in terms of one dimensional integrals. Defining:

\[
f_1(z) = -\frac{m_i^2}{2zm_K^2} \int_0^1 \frac{dx}{x} \ln \left( \frac{m_i^2(1 - x) + zm_j^2 - x(1 - x)(p^2 + 2zm_K^2)}{m_i^2(1 - x) + zm_j^2 - x(1 - x)p^2} \right),
\]

\[
f_2(z) = \frac{p^2 + m_i^2 - m_j^2 + 2zm_K^2}{2zm_K^2} \int_0^1 \frac{dx}{x} \ln \left( \frac{m_i^2(1 - x) + zm_j^2 - x(1 - x)(p^2 + 2zm_K^2)}{m_i^2(1 - x) + zm_j^2 - x(1 - x)p^2} \right),
\]

\[
f_c = \frac{m_i^2}{p^2} \int_0^1 \frac{dx}{x} \ln \left( \frac{m_i^2(1 - x) + zm_j^2 - x(1 - x)p^2}{m_i^2} \right) + \frac{m_i^2}{p^2} - 1,
\]

we can write:

\[
h_{ij}(z) = \frac{1}{4\pi} \left[ f_1(z) + f_2(z) + f_c \right]. \tag{A.4}
\]

As expected by the Low theorem, \( (f_1(z) + f_2(z))^\frac{m_i^2}{m_j^2} - f_c \), so that \( h_{ij}(z) \) has no pole for \( z \to 0 \).

In the case \( m_i = m_j \) the integrals \( f_1, f_2 \) and \( f_c \) can be done explicitly. For the \( \pi\pi \) function \( (m_i = m_j = m_\pi, p^2 = m_K^2) \) we found:

\[
h_{\pi\pi}(z) = \frac{1}{8z^2} \left\{ (1 + 2z) \left[ \beta \ln \left( \frac{1 + \beta}{\beta - 1} \right) - \beta_0 \ln \left( \frac{1 + \beta_0}{\beta_0 - 1} \right) \right] \right. + \frac{m_i^2}{m_K^2} \left[ \ln^2 \left( \frac{1 + \beta_0}{\beta_0 - 1} \right) - \ln^2 \left( \frac{1 + \beta}{\beta - 1} \right) \right] - 2z \}, \tag{A.5}
\]

where

\[
\beta_0 = \sqrt{1 - \frac{4m_i^2}{m_K^2}} \quad \text{and} \quad \beta = \sqrt{1 - \frac{4m_i^2}{m_K^2(1 + 2z)}}. \tag{A.6}
\]

It is interesting to note that the function \( h_{\pi\pi}(z) \) is simply related to the function \( H(z) \), defined in Ref.[28], which appears in \( K \to \gamma\gamma^* \) decays:\footnote{We thank G. Ecker for clarifying us this point}

\[
h_{\pi\pi}(z) = -H(1 + 2z). \tag{A.6}
\]
Appendix B

In $K_{L,S} \to \pi^+\pi^-\gamma$ decays, the variables which can be more easily studied from the experimental point of view are: i) the photon energy in the kaon rest frame ($E^*_\gamma$), ii) the angle between $\gamma$ and $\pi^+$ momenta in the di-pion rest frame ($\theta$). The relations between ($E^*_\gamma$, $\theta$) and the $z_i$ are:

\[ z_3 = \frac{E^*_\gamma}{m_K} \quad z_\pm = \frac{E^*_\gamma}{2m_K} (1 \mp \beta \cos(\theta)), \quad (B.1) \]

where $\beta = \sqrt{1 - 4m_\pi^2/(m_K^2 - 2m_\pi E^*_\gamma)}$. The kinematical limits on $E^*_\gamma$ and $\theta$ are given by:

\[ 0 < E^*_\gamma < \frac{m_K^2 - 4m_\pi^2}{2m_K}, \quad -1 \leq \cos(\theta) \leq 1. \quad (B.2) \]

Finally the differential rate in terms of these variables is:

\[ \frac{\partial^2 \Gamma}{\partial E^*_\gamma \partial \cos(\theta)} = \frac{E^*_\gamma \beta}{2m_K} \frac{\partial^2 \Gamma}{\partial z_1 \partial z_2} = \left[ |E|^2 + |B|^2 \right] \frac{E^*_\gamma \beta^3}{512\pi^3 m_K^3} \left( 1 - \frac{2E^*_\gamma}{m_K} \right) \sin^2(\theta). \quad (B.3) \]

In $K^\pm \to \pi^\pm\pi^0\gamma$ decays the situation is different, it is more useful to study the differential rate as a function of [26]: i) the charged pion kinetic energy in the kaon rest frame ($T^*_c$), ii) the adimensional variable $W^2 = (qp_K)(qp_\pm)/(m_\pi^2 m_K^2)$. These variables are related to the $z_i$ by:

\[ z_\pm = \frac{1}{2m_K^2} \left[ m_K^2 + m_\pi^2 - m_\pi^0 - 2m_K m_\pi - 2m_K^2 T^*_c \right], \quad z_3 z_\pm = \frac{m_\pi^2}{m_K^2} W^2. \quad (B.4) \]

While for $T^*_c$ it is easy to write the limits:

\[ 0 < T^*_c < \frac{(m_K - m_\pi^+)^2 - m_\pi^0}{2m_K}, \quad (B.5) \]

for $W$ the expressions are cumbersome and we refer to the figure 3.3 of Ref.[2]. The advantage in using these variables lies in the fact that, through the $W^2$-dependence, it is easier to disentangle the different contributions of inner bremsstrahlung, direct emission and interference [29]:

\[ \frac{\partial^2 \Gamma}{\partial T^*_c \partial W^2} = \frac{\partial^2 \Gamma_{IB}}{\partial T^*_c \partial W^2} \left\{ 1 + \frac{m_\pi^2}{m_K^2} 2 \text{Re} \left( \frac{E_{DE}}{eA} \right) W^2 + \frac{m_\pi^2}{m_K^2} \left( \frac{|E_{DE}|}{eA} \right)^2 + \left( \frac{|M_{DE}|}{eA} \right)^2 \right\} W^4 \right\}, \quad (B.6) \]

where $A = A(K^\pm \to \pi^\pm\pi^0)$. 

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Tables and Figures

Table 1

Coefficients of the Taylor expansion of the functions \( h_{ij} \), for \( ij = \pi \pi, \pi K \) and \( K \eta \).

|       | \( h^0_{ij} \)       | \( h^1_{ij} \)       | \( h^2_{ij} \)       |
|-------|----------------------|----------------------|----------------------|
| \( \pi \pi \) | +.36 - .15i          | -.30 + .22i          | +.37 - .42i          |
| \( \pi K \)    | -.12                 | -.045                | < 10^-2              |
| \( K \eta \)  | -.04                 | -.005                | < 10^-2              |

Table 2

Inner bremsstrahlung and direct emission contributions to the branching ratios of \( K_S \rightarrow \pi^+ \pi^- \gamma \) decay, for different values of the cut in the photon energy. Different values of the \( o(p^4) \) counterterms are chosen.

\[ E_\gamma^* > 20 MeV \quad E_\gamma^* > 50 MeV \quad E_\gamma^* > 100 MeV \]

|       | \( 10^3 \cdot IB \) | \( 10^6 \cdot DE (k_f = 0) \) | \( 10^6 \cdot DE (k_f = +.5) \) | \( 10^6 \cdot DE (k_f = +1) \) | \( 10^6 \cdot DE (k_f = -.5) \) | \( 10^6 \cdot DE (k_f = -1) \) |
|-------|----------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|       | 4.80                 | -6.2                        | -10.5                       | -14.8                       | -1.9                        | +2.4                        |
|       | 1.75                 | -5.0                        | -8.3                        | -11.7                       | -1.6                        | +1.8                        |
|       | .31                  | -2.0                        | -3.3                        | -4.7                        | -.6                         | +.7                         |

Figure 1

\( o(p^4) \) loop diagrams for \( K \rightarrow \pi \pi \gamma \) direct emission amplitudes. The symbols \( \circ \) and \( \bullet \) represent the strong and the weak vertex, respectively. The photon has to be attached to any charged line or to any vertex.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9408219v1