Global solutions of a free boundary problem for selfgravitating scalar fields.

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Abstract

The weak cosmic censorship hypothesis can be understood as a statement that there exists a global Cauchy evolution of a selfgravitating system outside an event horizon. The resulting Cauchy problem has a free null-like inner boundary. We study a selfgravitating spherically symmetric nonlinear scalar field. We show the global existence of a spacetime with a null inner boundary that initially is located outside the Schwarzschild radius or, more generally, outside an apparent horizon. The global existence of a patch of a spacetime that is exterior to an event horizon is obtained as a limiting case.

1. Introduction

The cosmic censorship hypothesis [1] can be informally stated as singularities are hidden inside black holes. Various attempts to formalize that statement have led to a collection of results that can be assembled into two categories, one of which may be called ”geometric” and the other ”dynamic”.

The first approach, reviewed for instance in [2] and [3], seems to be strongly influenced by the post-Leray [4] notion of the global hyperbolicity. The latter requires (in addition to the standard hyperbolicity condition as stated for instance in [5]) stringent smoothness properties of coefficients of a hyperbolic operator as well as a causality condition. In a
related stream of research a considerable effort is put into the examination of geometric quantities in various models, in order to define a notion of a singularity and then, to identify those singularities that must be enclosed by event horizons. A singularity can be understood as a point of space-time having divergent curvatures; a cosmic censorship hypothesis would require that all geodesics originating at a singularity remain inside a black hole. The post-Leray apparatus of ideas makes it plausible, however, to associate singularities with the existence of geodesics having a finite length [6]. On the other hand, an increasing number of counterexamples to resulting versions of the cosmic censorship hypothesis has given inspiration to various notions of truly ”singular” singularities, of which I mention a subclass of the so-called strong singularities [7] that is associated with the existence of apparent horizons [8].

The dynamic approach has been initiated, to my knowledge, by Eardley and Moncrief [8]. It bases on the notion of a Cauchy problem as formulated, for instance, in [5]. A weak version of the cosmic censorship hypothesis can be formalized as follows, in the case of asymptotically flat spacetimes:

*Given ”reasonable” asymptotically flat initial data of Einstein-matter field equations and assuming ”reasonable” energy conditions, there exists a Cauchy evolution that is global in the sense, that a solution does exist outside black holes for arbitrarily large values of the time of an asymptotic observer.*

It is possible to formulate a ”dynamic” version of the cosmic censorship in cosmological models [8], where the geometric approach runs into a trouble because there exist serious conceptual problems in defining a notion of a black hole. Most of the existing literature is concerned with the validity of that version of the cosmic censorship in various cosmological models [8].

An analysis of spherically symmetric systems suggests that the two approaches to the cosmic censorship can converge. Trapped surfaces, that are inherent to the study of the cosmic censorship conjecture by Królak [7], have been investigated in spherically symmetric geometries [10], [11]. Their appearance is always connected with a large concentration of
matter. If a sphere $S$ that is centered around a point of symmetry is trapped, then it contains amount of matter $M(V) = \int_{V(S)} \rho dV$ of the order $L(S)$ \cite{[11]}, where $V(S)$ is the volume inside $S$ and $L(S)$ is the geodesic radius of $S$. And conversely, a large amount of matter, $M(V) > L(V)$, leads to the formation of an apparent horizon. The quantity $\int_{V(S)} \rho dV$ can be bounded from above and from below by a suitable Sobolev norm (usually $H_1$) of a matter field. Sobolev norms are, in turn, natural objects in the dynamic approach; an evolving system does exist so long as its Sobolev norm is finite. Singularities can be defined as those regions of space-time that give an infinite contribution to a Sobolev norm.

Take an initial configuration of a compact support of a hyperbolic system of matter fields coupled to Einstein equations. Hyperbolicity (I always use the Petrovsky meaning of that term; see Section 3) means that the support of matter remains finite during a finite evolution. Assume that a Cauchy evolution can be analyzed in the Sobolev class $H_1$. If an evolution breaks down then there exists a sphere $S$ such that a contribution to the Sobolev norm $H_1$ coming from the interior of $S$ becomes big in comparison to $L(S)$ - hence there must exist a trapped surface and an event horizon enclosing $S$. That means that, in the spherically symmetric case at least (and under the assumption that there exists an evolution in $H_1$ - that fact is not obvious), the notion of strong singularity is related to the notion of dynamic singularity.

The "dynamic" version of the weak cosmic censorship hypothesis leads to an external Cauchy problem with a free inner boundary being a null cone, as is explained below. Take a four-dimensional (asymptotically flat) Lorentzian manifold $M$ and define its space-like foliation by hypersurfaces $\Sigma_t$, where $t$ is the asymptotic time, $t \leq \infty$. A black hole $H$ can be defined as the largest piece of $M$ that still can be enclosed by a null cone $\delta H$ such that the area of the intersection $\delta H \cap \Sigma_t$ remains uniformly bounded. (That definition is motivated by the fact that an area of a spherical black hole is bounded from above by $16\pi m^2$, where $m$ is the asymptotic mass \cite{12}.) The weak cosmic censorship hypothesis reads in this context as follows: for "reasonable" initial data and "reasonable" matter, there should exist a global Cauchy evolution of a region exterior to $H$. The inner boundary (event horizon) $\delta H$ of that
region is a null cone so that we arrive naturally to an external Cauchy problem with an event horizon as a free null-like inner boundary.

In this paper I will study a more general case of a Cauchy evolution with a free null-like boundary that does not necessarily coincides with an event horizon. A selfgravitating nonlinear scalar field is used as a matter model. The main results of the paper are contained in Theorems 1 and 2 of Section 4. Their formulation would require a number of preliminary definitions, so instead let me just discuss the most important points.

The fact that I find the most interesting is that even the local version of the Cauchy problem requires the positivity of the potential (selfinteraction) term $W(\phi) \geq 0$. Using a technical jargon, negative selfinteraction $W(\phi)$ means that scalar field equations (see 28, 29) would lose their strict hyperbolicity. (23) implies that the loss of hyperbolicity is particularly plausible inside a region enclosed by the Schwarzschild radius $2m$.

The smoothness of an evolving solution is locally preserved, that is its differentiability properties are kept intact for a small enough interval of time. The global existence is also proven, through standard apriori estimates, in the differentiability class $\phi \in H_k$, ($k \geq 3$). Having the global existence outside a null cone $\delta H$, one is given a geometry of $\delta H$ and a scalar field on $\delta H$; they in turn can be regarded as initial values of the characteristic Cauchy problem. The present work is supplementary to the investigation of Christodoulou [13], who investigated the existence of a solution of the characteristic Cauchy problem for (massless) scalar fields.

Although I deal only with scalar fields, my impression is that the formalism presented below is capable to include other forms of nontachyonic matter, that is assuming that matter fields are described by hyperbolic equations and their energy-momentum tensor satisfies some energy conditions. Spherically symmetric Yang - Mills and SU(2) nonlinear sigma model do in fact satisfy the needed conditions. Einstein - Vlasov system does not belong to that category that can be analyzed with techniques presented here, but let me mention that a recent work of Rein, Rendall and Schaeffer shows that there exists a global solution for Einstein - Vlasov system [14], if matter does not form a singularity at the symmetry center.
The content of the rest of the paper is following. Section 2 presents the Einstein equations coupled to a spherically symmetric field equations in (1+3) splitting. They can be reduced to a system of integro-differential equations with matter fields being dynamical variables. The reader may find it useful that one can explicitly express components of the spherically symmetric metric as some functionals of matter-related terms - see formulae (14 - 18) - in any splitting of the space-time, that is for any value of the trace \( trK \) of the extrinsic curvature.

Section 3 describes the Einstein - scalar field equations in the so-called polar gauge. That way of foliating a space-time is particularly convenient when discussing the external Cauchy problem with a free inner boundary. In most gauges one is confronted with the need to impose an additional boundary condition at the inner boundary; in contrast with that the polar gauge does not require any extra conditions.

Section 4 contains the main result, Theorems 1 and 2. The proof of the local Cauchy solvability bases on the standard compactness method. The main part of the local part of the proof of Theorem 1 is relegated to Section 5 while the global part is based on important \( L_p \) and \( H_l \) global estimates that are proven in Section 6. In Section 6 a change of dependent variables leads to a set of first order equations that are ”almost” linear; that allows one to obtain a set of apriori global estimates. Section 7 discusses a generalization of the proof of Theorem 1 that finally leads to the proof of Theorem 2.

2. Equations.

The most general metric of a spherically symmetric spacetime is given by the expression

\[
\mathrm{d}s^2 = -N^2(r, t)\mathrm{d}t^2 + a(r, t)\mathrm{d}r^2 + R^2(r, t)d\Omega^2,
\]

where \( t \) is a time coordinate, \( r \) is a radial coordinate, \( R \) is the areal radius and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the standard line element on the unit sphere with with the angle variables \( 0 \leq \phi < 2\pi \) and \( 0 \leq \theta \leq \pi \). I assume that at spatial infinity \( N = 1 \) (hence \( t \) coincides with
the proper time of an observer who is far away from the center of symmetry) and $a = 1$.

Initial data of Einstein-scalar fields equations on an exterior part $\Sigma^\text{out}_t$ of a Cauchy slice $\Sigma_t$ consist of $(g_{ij}, K_{ij}, \rho, j_i)$ where $g_{ij}$ is the intrinsic metric, $K_{ij} = \frac{\partial g_{ij}}{2N}$ is the extrinsic curvature, $\rho = -T^0_0$ is the matter energy density and $j_i = NT^i_0$ is the matter current density. $T_{\mu\nu}$ is the energy - momentum tensor. Initial data satisfy the constraints

\begin{align}
(3) \ R - K_{ij}K^{ij} + (\text{tr} K)^2 &= 16\pi \rho \\
(4) \ \nabla_i K^{ij} - \nabla^j \text{tr} K &= -8\pi j^i
\end{align}

where $(3) \ R$ is the scalar curvature of the intrinsic metric. Above (and elsewhere in the paper) I assume the Einstein summation rule with the exception of indices $r, \theta$ and $\phi$ whose repetition is supposed never to mean summation. Latin indices change from 1 to 3 while Greek indices range from 0 to 4.

In the spherically symmetric case it is convenient to formulate the whole set of Einstein equations in terms of the extrinsic curvature $K_{ij}$ (which describes the embedding of the three-dimensional hypersurface $\Sigma$ into a spacetime) and the mean curvature $p$ (which is a trace of the two-dimensional extrinsic curvature that describes the embedding of a two-dimensional sphere centered around the symmetry center into $\Sigma_t$). The interior metric of $\Sigma_t$ is

\[ ds^2_{(3)} = adr^2 + R^2 d\Omega^2. \]

The mean curvature of a centered two-sphere as embedded in an initial three dimensional hypersurface is

\[ p = \frac{2\partial_r R}{\sqrt{aR}}. \]

If the trace $\text{tr} K$ of $K_{ij}$ is fixed then there is only one independent component of the extrinsic curvature, say the radial-radial component $K = K^r_r$, and the remaining two components are each equal to $\frac{\text{tr} K - K}{2}$. 


\[ trK - K = 2K^\phi_{\phi} = 2K^\phi_{\phi}. \] (6)

In terms of \( K \) and \( p \) the constraints can be written as
\[
\frac{\partial_r (pR)}{\sqrt{a}} = -8\pi R\rho - \frac{3R}{4} (K)^2 + \frac{R}{4} (trK)^2 + \frac{R}{2} KtrK - \frac{Rp^2}{4} + \frac{1}{R} \] (7)

and
\[
\frac{\partial_r (R^3(K - trK))}{\sqrt{a}} = -8\pi R^3 \frac{j_r}{\sqrt{a}} - ptrKR^3 \] (8)

In the spherically symmetric case the full set of the Einstein equations consists of the two preceding ones, the evolution equation
\[
\partial_0 (K - trK) = \frac{3N}{2} (K)^2 + \frac{N}{2} (trK)^2 - 2NKtrK - \frac{p^2 R}{\sqrt{a}} \partial_r \frac{N}{pR} + 8\pi N(T^r_r + \rho) \] (9)

and the lapse equation
\[
\nabla_\gamma \partial^\gamma N = N \left( \frac{3}{2} (K)^2 + (trK)^2 / 2 - KtrK + 4\pi (\rho + T^i_i) \right) + \partial_0 trK. \] (10)

The Einstein equations and Bianchi identities yield the energy-momentum conservation equations, which in the case of spherical symmetry reduce to
\[
\partial_0 \frac{j_r}{\sqrt{a}} + N(K + trK) \frac{j_r}{\sqrt{a}} + \frac{N}{\sqrt{a}} \partial_r T^r_r + \frac{\partial_r N}{\sqrt{a}} (T^r_r + \rho) + Np(T^r_r - T^\phi_{\phi}) = 0 \] (11)

\[- \partial_0 \rho - \frac{N}{\sqrt{a}} j_r - \frac{N}{\sqrt{a}} \partial_r j_r + \frac{2\partial_r N}{a} j_r - NK(T^r_r - T^\phi_{\phi}) - NtrK(\rho + T^\phi_{\phi}) = 0. \] (12)

Using the above equations, one can express metric coefficients in terms of \( T_{\mu\nu} \) and \( trK \).

From the momentum constraint one finds
\[
RK(R) - RtrK(R) = \frac{C + 8\pi \int_R^\infty j_r r^3 dr}{R^2} + 2 \int_R^\infty trK r^2 dr/R^2. \] (13)

The parameter \( C \) is constant on a particular Cauchy slice and it must be set to 0 on slices including the symmetry center. \( C \) is arbitrary, however, on slicings that do not include the world line \( R = 0 \).

The integration of the Hamiltonian constraints leads, after some algebra, to the expression
\[ pR = 2 \sqrt{1 - \frac{2m}{R} + \frac{2m(R)}{R} + \frac{R^2}{4}(trK - K)^2}, \]  

(14)

where \( m \) is the asymptotic (ADM) mass. The function \( m(R) \) can be interpreted as a local energy energy density (it is easy to notice that \( m(0) = m \)) and it is given by the equation

\[ m(R) = 4\pi \int_{R}^{\infty} dr r^2 (\rho + rj_r (trK - K)). \]

(15)

The lapse \( N \) can be determined from (9),

\[ N = \frac{pR}{2} \left( 1 + 4 \int_{R}^{\infty} \frac{\partial_t (K_r^3 - trK_r^3) dr}{\beta(r)p^3r^5} \right) \beta(R) \]  

(16)

where

\[ \beta(r) = e^{\int_{r_0}^{r} \left( 16\pi (-T_r^r - \rho) + 2KtrK + 2(trK)^2 \right) \frac{1}{p^2} ds}. \]

(17)

The lapse \( N \) satisfies (10), which can be shown by using the conservation equation (11).

The line element can be written directly in terms of \( p, K, trK, R \) and \( N \):

\[ ds^2 = dt^2(-N^2 + \frac{N^2(trK^R - KR^2)}{(pR)^2}) - 2N trK - \frac{K}{p} dtdR + \frac{4}{(pR)^2} dR^2 + R^2 d\Omega^2. \]

(18)

Equations (13 - 18) demonstrate that all metric functions can be expressed as certain functionals of matter-related terms. That is simply a manifestation of the well known fact that spherically symmetric gravitation does not carry degrees of freedom independent of matter.

The scalar field equation can be cast into one of two equivalent forms

\[ D_D^2 \phi = -\frac{\theta'}{2} \frac{\theta'}{2} \frac{D^2 \phi}{-\frac{\partial_t N}{p^3r^5} \phi} - trK D\phi + W', \]

(19)

\[ D_D^2 \phi = -\frac{\theta}{2} \frac{\theta}{2} \frac{D^2 \phi}{-\frac{\partial_t N}{p^3r^5} \phi} + trK D\phi + W'. \]

(20)

Here the differential operators \( D, D \) are defined as follows

\[ D = \frac{1}{N} \partial_t + \frac{1}{\sqrt{a}} \partial_r \quad D = \frac{-1}{N} \partial_t + \frac{1}{\sqrt{a}} \partial_r. \]

(21)

\( \theta' = p + K - trK, \quad \theta = p - K + trK \) are the optical scalars and \( W' = \partial_W W(\phi). \) \( W(\phi) \) is the scalar field selfinteraction potential.
The nonzero components of the energy-momentum tensor of the scalar field are

\[ T^0_0 = -\frac{1}{4}((D\phi)^2 + (D\phi)^2 + 4W(\phi)), \quad T^r_r = \frac{1}{4}((D\phi)^2 + (D\phi)^2 - 4W(\phi)) \]
\[ T^0_r \sqrt{a} = -\frac{1}{4N}((D\phi)^2 - (D\phi)^2), \quad T^\phi_\phi = T^\theta_\theta = -\frac{1}{2}(D\phi D\phi - 2W(\phi)). \]

(22)

3. Definition of the external Cauchy problem.

We will deal with an external Cauchy problem with a free inner boundary. Define \( \Sigma^\text{out}_0 \) as an open end extending outside a sphere of a coordinate radius \( R_0 \).

Initial data entirely determine the geometry of the initial slice and, given the conservation equation of the energy momentum tensor, the lapse function \( N \); see equations (13 - 20). That specifies also null cones attached at \( \Sigma_0 \). Take now an outgoing null cone \( H \) originating at a radius \( R_0 \). The open end \( \Sigma^\text{out}_0 \) shall give rise to a foliation defined by open Cauchy ends \( \Sigma^\text{out}_t \); notice, however, that the foliation is nonunique in general. That is because on all future \( (t > 0) \) open ends \( \Sigma^\text{out}_t \) the extrinsic curvature \( K \) (see (13)) depends on a parameter \( C \) that is arbitrary. One would have to impose a new condition on the free inner boundary (in addition to the fixing of the trace of extrinsic curvature \( trK \) of \( \Sigma^\text{out}_t \)) that should guarantee that the lapse function \( N \) is strictly positive in \( \Sigma^\text{out}_t \). This is in order to guarantee that the proper time \( \tau = \int N dt \) runs forward in the outer region. This demand is quite restrictive, in general. ¿From formula (16) follows that the lapse can vanish on a centered sphere \( S \) if the mean curvature \( p(S) = 0 \), but it might vanish also elsewhere, if the second factor of (16) equals 0. The second possibility is more difficult to deal with, since the factor in question depends on the acceleration of matter (see (11) and (13)). Fortunately, the arising difficulty can be avoided by imposing the polar gauge condition \( trK = K \); that condition removes the arbitrariness that is present in other gauges so that there is no free data at the inner boundary. Indeed, in such a case equations (14 - 18) become
\[ pR = 2\sqrt{1 - \frac{2m}{R} + \frac{2m(R)}{R}}, \quad (23) \]

\[ m(R) = 4\pi \int_R^\infty drr^2 \rho, \quad (24) \]

\[ N = \frac{pR}{2} \beta(R), \quad (25) \]

\[ \beta(r) = e^{16\pi \int_r^\infty (-\nabla_T - \rho) \frac{1}{pR} ds}. \quad (26) \]

and

\[ ds^2 = -dt^2 N^2 + \frac{4}{(pR)^2} dR^2 + R^2 d\Omega^2. \quad (27) \]

The parameter \( m \) appearing above is a free parameter that will be assumed to satisfy the inequality \( m \geq m(R_0) \); the equality might take place only if the energy density \( \rho \) vanishes at the inner boundary (exact conditions are given in Theorems 1 and 2). (23) - (26) imply that \( N \) is positive provided that \( p > 0 \). The momentum constraint (8) can be used in order to replace \( trK \) by \( \frac{8\pi}{p} j_r/\sqrt{a} \) and the equation (9) allows one to eliminate the gradient \( \partial_r N \).

With all that equations (19) and (20) lead to

\[ (\partial_0 + \frac{NpR}{2} \partial_R)V = \frac{8\pi N}{p} V(j - T) - \frac{Np}{2} (U + V/2) - \frac{NV}{pR^2} + NW'(\phi), \quad (28) \]

\[ (\partial_0 - \frac{NpR}{2} \partial_R)U = \frac{8\pi N}{p} U(j + T) + \frac{Np}{2} (V + U/2) + \frac{NU}{pR^2} - NW'(\phi); \quad (29) \]

above we set

\[ V = D\phi \]

\[ U = D\phi \]

\[ j = \frac{j_r}{\sqrt{a}} \]

\[ T = T_r. \quad (30) \]
We will refer to (28) and (29) as to "reduced Einstein-scalar field equations". The whole dynamics of the selfgravitating scalar field is given by the scalar field equations (28) and (29). The scalar field $\phi$ can be written as an integral quantity $\phi(R, t) = \phi(R, 0) + \int_0^t d\tau N(U - V)/2$ so that functions $U$ and $V$ can be used as dynamical variables. The initial data of the whole system of Einstein - scalar field equations consist of $U$ and $V$ supplemented by a value of $\phi$ at a single point (say $\phi(t = 0, \infty) = 0$).

Equations (28) and (29) are strictly hyperbolic in the sense of Petrovsky provided that $NpR$ is strictly positive. One can show that $NpR$ can vanish only if $pR = 0$. On the other hand, $\partial_r \beta$ (and hence some of coefficients of the reduced equations) can become singular at those points, where the mean curvature $p$ vanishes. Thus in our case strict hyperbolicity is a prerequisite for the existence of a local causal evolution. That means, from the inspection of (23) and (25), that close to the sphere with $R = 2m$ the contribution $m(R)$ has to be positive, that is guaranteed if $\rho \geq 0$; the latter condition is satisfied if the selfinteraction $W(\phi)$ of the scalar field is nonnegative. Thus, even the local version of the Cauchy problem might require the positivity of the potential term $W(\phi) \geq 0$.

As pointed above, the system of equations (28) and (29) is strictly hyperbolic in a region that does not contain minimal surfaces. From the inspection of formulae for $p$ and $N$ it is obvious that both quantities are strongly positive outside a sphere of the Schwarzschild radius $R = 2m$, so that outside that region the information propagates causally and time runs forward. That would mean that the sort of the external Cauchy problem that is described in the beginning of this Section is legitimate, at least for initial open ends $\Sigma_0^{out}$ originating out of the sphere $R = 2m$. We will show that to be true even in a more general situation. We shall stress, however, that the coordinates that we use do not allow for the investigation of regions inside apparent horizons because in the polar gauge apparent horizons coincide with minimal surfaces, where our system of coordinates breaks down.
4. The Cauchy solution.

The reduced equations (28, 29) are integro-differential and hyperbolic; notice that they are nonlocal. I did not find any mathematical result that can yield directly the existence of a local in time solution for equations of that type. For that reason we have to prove the existence of a local evolution starting from first principles [16].

There are several methods to prove the existence of a local Cauchy solution. We will use an approach that bases on results of Petrovsky [3] and on properties of Sobolev spaces. The main result is formulated in Theorem 1. Its proof consists of following main points. Firstly, a sequence of functions will be generated iteratively for data given on an extended initial hypersurface. Secondly, Lemma 3, proven in the next Section, shows that the sequence is uniformly bounded for a short period of time outside a ”rigid” cone. Then, in Lemma 4, standard compactness theorems of functional analysis ensure the existence of a convergent subsequence, that is the sought local in time solution of the reduced equations. Finally, Lemma 7 shows the existence of a global Cauchy evolution.

Let us recall that Sobolev spaces $H^k(V)$ can be defined as a completion of $C^k$-functions in the norm $||f||_{H^k(V)} = \int_V dV \Sigma_{i=0}^k (D^i f)^2$ where $D^i f = \Sigma_{k_1,k_2,...,k_i} \partial_{k_1}...\partial_{k_i} f$ and $n$ is the dimension of a riemannian manifold $V$. In the case of spherical symmetry the norm reads $||f|| = \int drr^2 \Sigma_{i=0}^k (\partial^i r f)^2$. The main result is the following one.

**Theorem 1.** Let the initial data of the reduced equations (28) and (29) be $U, V \in H_k$ and $\phi \in H_{k+1} \cap L_{n_0}$, $k=2, 3, ...$. Assume that $W(x) > 0$ for $x \neq 0$ and $W(0) = 0$ and

$$|\partial_x^l W(x)| \leq A(l, n_l)|x|^{n_l}$$

for some $0 \leq n_l < \infty$ and a constant $A$ depending only on $n_l$ and $l = 0, ..., k$. Let $\Sigma_0^{out}$ be an open end, $R_0 = \inf||x|: x \in \Sigma_0^{out}| > 2m + \eta, \eta > 0$ and $m \geq m(R_0)$ be an asymptotic mass of the configuration (with strict inequality $m > m(R_0)$ if $\partial_x^i f \neq 0, f = U, V$, at least for one value $i = 0, ...k - 1$).

Then

i) there exists a local Cauchy evolution of $\Sigma_0^{out}$, i. e. a foliation $\Sigma_t^{out}$ for some $\eta'(...
\[
0 \leq t < \eta', \quad \text{with} \quad U, V \in H_k \quad \text{and} \quad \phi \in H_{k+1}. \quad \text{In addition,} \quad U, V \in C^{k-1}_-(\cup_{0 \leq t < \eta'} \Sigma_{t}^{\text{out}}); \]

ii) solutions are unique;

iii) solutions are global in \( H_k \) and the null boundary \( \delta H \) of \( \cup_{0 \leq t < \eta'} \Sigma_t^{\text{out}} \) escapes to spatial infinity, i.e., the area of the inner boundary \( \Sigma_t^{\text{out}} \) goes to infinity as \( t \to \infty \).

Actually, one can prove a slightly stronger version:

\textbf{Theorem 2.} Let the initial data of the reduced equations (28) be \( U, V \in H_k \) and \( \phi \in H_{k+1} \cap L_{n_0}, \quad k=2, 3, ... \). Assume that \( W(x) > 0 \) for \( x \neq 0 \) and \( W(0) = 0 \) and

\[
|\partial_x^{l} W(x)| \leq A(l, n_l)|x|^{n_l}
\]

for some \( 0 \leq n_l < \infty \) and a constant \( A \) depending only on \( n_l \) and \( l = 0, \ldots, k \). Let \( \Sigma_0^{\text{out}} \) be an open end, \( R_0 = \inf[|x| : x \in \Sigma_0^{\text{out}}] \) and \( m \geq m(R_0) \) be an asymptotic mass of the configuration (with strict inequality \( m > m(R_0) \)) if \( \partial_i x f \neq 0, \ f = U, V, \) at least for one value \( i = 0, \ldots, k - 1 \).

Assume that minimal surfaces are absent in \( \Sigma_0^{\text{out}} \). Then

i) there exists a local Cauchy evolution of \( \Sigma_0^{\text{out}} \), i.e., a foliation \( \Sigma_t^{\text{out}} \) for some \( \eta'(0 \leq t < \eta') \), with \( U, V \in H_k \) and \( \phi \in H_{k+1} \). In addition, \( U, V \in C^{k-1}_-(\cup_{0 \leq t < \eta'} \Sigma_t^{\text{out}}); \)

ii) solutions are unique;

iii) solutions are global in \( H_k \) and the null boundary \( \delta H \) of \( \cup_{0 \leq t < \eta'} \Sigma_t^{\text{out}} \) either

a) escapes to spatial infinity, i.e., the area of the inner boundary \( \Sigma_t^{\text{out}} \) goes to infinity as \( t \to \infty \),

or

b) stabilizes at the radius \( R = 2m_B \), where \( m_B \) is the Bondi mass of a black hole.

\textbf{Comments.} The condition that \( m \) may be equal to \( m(R_0) \) only if a sufficient number of derivatives of \( U \) and \( V \) vanishes, is always satisfied if initial data on \( \Sigma_0^{\text{out}} \) are obtained by restriction from initial data on a whole slice (including the origin) \( \Sigma_0 \). Conditions \( U, V \in H_k(\Sigma_0) \) mean (through Sobolev embeddings theorems) that \( k - 1 \) derivatives of \( U \) and \( V \) are continuous. If \( m = m(R_0) \) then \( U(R) \) and \( V(R) \) must identically vanish for \( R \leq R_0 \), and therefore also their derivatives up to the order \( k - 1 \). The symbol \( C^k \) means a class of functions which are almost \( C^k \), that is they are of Hölder class \( C^{k-1+\kappa} \) for any \( 0 < \kappa < 1 \). Let us point
out that the smoothness of $U$ and $V$ can be improved to $C^{k-1/2}$, at least for initial data of compact support, following an argument of [15].

In what follows we will prove Theorem 1 in four steps and five lemmae; the proof of two of them is placed in separate sections.

Theorem 2 can be obtained by improving one of the elements in the reasoning.

**Proof.** Step I of Theorem 1.

Define a $H_k$ extension $\tilde{\Sigma}_0^{out} = \{x \in \Sigma_0: 2m + \eta - \delta \leq |x| < \infty\}$ of $\Sigma_0^{out}$ by assuming following initial data in the interval $[2m + \eta - \delta, 2m + \eta]$

$$\partial_r^k f(R) = 0, \quad \partial_r^{k-1} f(R) = \partial_r^{k-1} f(R_0), \quad \partial_r^{k-i} f(R) = \partial_r^{k-i} f(R_0) - \int_R^{R_0} dr \partial_r^{k-i+1} f(r),$$

where $f = U, V$ and $i = 2, \ldots, k$. The initial values of the scalar field in the extension can be obtained from integrating $U$ and $V$, $\phi(R) = \phi(R_0) - \int_R^{R_0} dr \frac{1}{Npr}(U + V)$. One can check (see Appendix) that $H_k$ norms of $U$ and $V$ over the extended manifold $\tilde{\Sigma}_0^{out}$ differ from $H_k$ norms of $U$ and $V$ over $\Sigma_0^{out}$ by a term bounded above by $\delta CH_k(\Sigma_0^{out})$, where $C$ is a constant. Similarly, the mass function $m(R) \leq m(R_0) + \delta CH_1(\Sigma_0^{out})$ for $R \in [2m + \eta - \delta, 2m + \eta]$. Thus for $\delta$ sufficiently small the extended manifold can have the same asymptotic mass $m$ and lie outside the Schwarzschild radius $2m$.

Let us define following quantities

$$p_0(R, t) = p(R, t = 0)$$
$$N_0(R, t) = N(R, t = 0)$$
$$U_0(R, t) = U(R, t = 0),$$
$$V_0(R, t) = V(R, t = 0),$$
$$\phi_0(R, t) = \phi(R, t = 0)$$

as being identically equal to the initial data. Let $\rho_0$, $T_0$ and $j_0$ be the initial values of the energy density $\rho$, $T$ and $j$, respectively. Define also, for $n \geq 1$, functions

$$\alpha_n(R, t) = \sqrt{1 - \frac{2m - 8\pi \int_R^\infty dr r^2 \rho_n(r, t)}{R}}$$

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\begin{align*}
\beta_n(R, t) &= e^{-4\pi \int_R^\infty dr \frac{(2\pi \rho_n + \rho_n)}{\alpha_n^2}} \\
\rho_n &= \frac{1}{4}((U_n)^2 + (V_n)^2 + 4W(\phi_n)), \quad T_n = \frac{1}{4}((U_n)^2 + (V_n)^2 - 4W(\phi_n)) \\
\alpha_n^2 &\equiv \frac{1}{4}(2\pi)^2 (\rho_n^2 - (U_n)^2), \\
\phi_n(t, R) &= \phi(R, 0) + \int_0^t d\tau \alpha_{n-1}(\tau, R) \beta_{n-1}(\tau, R) \left(U_{n-1}(\tau, R) - V_{n-1}(\tau, R)\right) / 2. \quad (32)
\end{align*}

All formulae above are expressed in terms of $U_n$ and $V_n$ which (for $n > 0$) are defined as solutions of following evolution equations

\begin{align*}
(\partial_0 + \alpha_{n-1}^2 \beta_{n-1} \partial_R) V_n &= 4\pi \beta_{n-1} R V_n (j_n - T_n) - \frac{\alpha_{n-1}^2 \beta_{n-1}}{R} (U_n + V_n/2) - \frac{\beta_{n-1} V_n}{2R} + \alpha_{n-1} \beta_{n-1} W'(\phi_n), \quad (33) \\
(\partial_0 - \alpha_{n-1}^2 \beta_{n-1} \partial_R) U_n &= 4\pi \beta_{n-1} R U_n (j_n + T_n) + \frac{\alpha_{n-1}^2 \beta_{n-1}}{R} (V_n + U_n/2) + \frac{\beta_{n-1} U_n}{2R} - \alpha_{n-1} \beta_{n-1} W'(\phi_n), \quad (34)
\end{align*}

with initial conditions $U_n(R, 0) = U_0(R)$ and $V_n(R, 0) = V_0(R)$.

Define $\Omega_t = [(t', x) : 0 \leq t' \leq t, |x| \geq 2m + \eta - \delta + 2t']$ and $\tilde{\Sigma}_t^{out} = [x \in \Omega_t : x^0 = t]$. Define

$$E_{kn}(\tilde{\Sigma}_t^{out}) = \int_{\tilde{\Sigma}_t^{out}} \left( |\partial^k_x U_n|^2 + |\partial^k_x V_n|^2 \right). \quad (35)$$

The essential part of step I is following.

**Lemma 3.** Let the initial data of the reduced equations (28) and (29) be of compact support and $U_0, V_0 \in H_k(\tilde{\Sigma}_0^{out})$, $k \geq 2$, for $k=2, 3, \ldots$. Assume that

$$W(x) \geq 0, \quad |\partial^l_x W(x)| \leq A(l, n_l)|x|^{n_l}$$

for some $0 \leq n_l < \infty$ and a constant $A$ depending only on $n_l$ and $l$. Then, for a small enough time $t$:

i) for each $n > 0$ there exists a solution $(U_n, V_n)$ of equations (33, 34) in $\Omega_t$;  

ii)Sobolev norms $H_k(\tilde{\Sigma}_t^{out})$ of $U_n$ and $V_n$ are uniformly bounded for $t' < t$.
\[ E_{ln}(\Sigma_{t}^{out}) \leq \frac{1}{(E_{ln}^0 - l - 2 - C_{l}t)^{1/(l+2)}}, \]  

where \( l = 0, 1, \ldots, k \), \( E_{ln}^l = E_{ln}(\tilde{\Sigma}_{0}^{out}) \) is the initial value of the norm \( E_{ln} \) and \( C_{l}'s \) are some constants;

iii) coefficients \( \alpha_{n}, \beta_{n}, \phi_{n} \) of (33, 34) are at least \( C^{k} \) as functions of \((t, R)\) with a norm \( |f|_{C^{h}} = \sup_{\sum_{k_{1}+k_{2}=0}} |\partial_{k_{1}}^{l} \partial_{R}^{k_{2}} f| \) (where \( f = \alpha_{n}, \beta_{n} \)) uniformly bounded by (up to a constant coefficient depending only on initial data)

\[ (E_{kn}(\Sigma_{t}^{out}))^{2k} \leq \frac{1}{(E_{ln}^{k} - k - 2 - C_{l}t)^{2k/(k+2)}} \]  

and, in particular, \( \alpha^{2} \beta_{n} \leq 2; \)

iv) \( L_{2}(\Sigma_{t}^{out}) \) norms of \( \partial_{k}^{l} \partial_{R}^{m} f \) (\( f = U_{1}, V_{1} \) and \( l > 0, l + m \leq k \)) are uniformly bounded;

v) \( |\partial_{R}^{l} \phi_{n}(t, R)| \leq C_{3} + C_{4}t \), where \( C_{3} \) and \( C_{4} \) are constants depending only on initial data and \( 0 < l \leq k \).

We postpone the proof of Lemma 3 to the next Section. Assuming that it holds true one shows the following

**Lemma 4.** Assume conditions of Theorem 1 and, in addition, that initial data are of compact support and \( k \geq 2 \). There exists a solution \((U, V) \in H_{k}(\Omega_{t}) \) and \((U, V) \in C^{k-1}(\Omega_{t}) \) of the reduced equations (28, 29) in \( \Omega_{t} \).

Proof of Lemma 4. Notice that there exists a compact set \( \Omega_{t} \) that contains supports of all functions \((U_{n}, V_{n})\). From iii) and iv) of Lemma 3, \( U_{n}, V_{n} \) constitute a bounded subset of \( H_{k}(\Omega_{t}) \). Therefore there exists a subsequence \( U_{n_{k}}, V_{n_{k}} \) that is weakly convergent in \( H_{k}(\Omega_{t}) \) and strongly convergent in \( H_{k-1}(\Omega_{t}) \) to a limit \((U, V)\). \((U, V)\) is Hölder \( C^{\mu} \) continuous, hence \((U, V)\) is a limit also in the \( L_{\infty} \) norm. \((U, V)\) solves, by the construction of the iteration, the reduced equations (28, 29) in \( \Omega_{t} \). The solution belongs to \( H_{k}(\Omega_{t}) \) and, since our problem is essentially two-dimensional, it is \( C^{k-1+\kappa}(\Omega_{t}) \), for any \( 0 < \kappa < 1 \), by the Sobolev embedding theorem. That ends the proof of Lemma 4.
Lemma 5. Under conditions of Theorem 1 and for $k \geq 2$, a solution $(U, V)$ is unique in $\Omega_t$.

The proof goes in a completely standard way [5] and we omit it.

Step II of the proof of Theorem 1.

We can remove the compactness condition of the support of initial data.

Lemma 6. Under conditions of Theorem 1 a solution $(U, V)$ with (noncompact) initial data that are of class $H_k$, $k= 2, 3,\ldots$, exists in $\Omega_t$.

Sketch of the proof of Lemma 6.

We shall discuss the case with $k = 2$ only. Take a $C^\infty$ sequence of functions of compact support $U_0\eta, V_0\eta$ that approximates $(U_0, V_0)$ in $H_2(\tilde{\Sigma}_t^\text{out})$. Each pair of the sequence $U_0\eta, V_0\eta$ gives rise to a solution $U_{t\eta}, V_{t\eta}$ of the reduced equations (28, 29) in $\Omega_t$. From the reduced equations one derives, after a lengthy but simple algebra,

$$
(\partial_t + 2\partial_R)
\left(
|U_{tn}|^2_{H^2(\tilde{\Sigma}_t^\text{out})} + |V_{tn}|^2_{H^2(\tilde{\Sigma}_t^\text{out})}
\right) \leq
C
\left(
|U_{tn}|^2_{H^2(\tilde{\Sigma}_t^\text{out})} + |V_{tn}|^2_{H^2(\tilde{\Sigma}_t^\text{out})}
\right)^3;
$$

Thus the sequence of solutions is uniformly bounded in $H_2(\Omega_t)$, for sufficiently small but nonzero $t > 0$. Therefore there exists a subsequence $(U_{tn_k}, V_{tn_k})$ that is weakly convergent to $(U, V)$ in $H_2(\Omega_t)$ and strongly convergent in $H^1(\Omega_t^M)$, where $\Omega_t^M = [(R, t)\in\Omega_t : R < M]$. The last fact and the Hölder continuity of $(U, V)$ and $(U_{tn_k}, V_{tn_k})$ imply pointwise convergence of the subsequence in $\Omega_t^M$. A careful analysis of the reduced equation shows that for $M > 1$ the reduced equations are nearly linear. A perturbation analysis shall prove the existence of a solution in $\Omega_t \setminus \Omega_t^M$ that matches to a solution $(U, V)$. That concludes the proof of Lemma 6.

Step III of the proof of Theorem 1.

The inner boundary $\delta\Omega_t$ of $\Omega_t$ consists of pairs $(t, 2m + \eta - \delta + 2t)$. The outgoing null rays in $\Omega_t$ are determined from the differential equation $\frac{dR}{dt} = \frac{(|pR|)^2}{4} - \beta$. From (23 - 26) we
conclude that $\frac{dR}{dt} < 1$ at $\delta\Omega_t$. From the smoothness condition we infer that $\frac{(pR)^2}{4} \beta^{1/2}$ is at least a bounded function along a null ray. Thus $\delta\Omega_t$ is a space-like three-surface. Therefore there exists a null outgoing geodesic joining each point $(t, R)$ (for small enough $t$) of $\delta\Omega_t$ with a point $P$ on the initial open end $\Sigma^\text{out}_0$. Let us choose $t$ such that the point $P$ lies on the boundary of $\Sigma^\text{out}_0$; that is possible since by definition $\Sigma^\text{out}_0 \subset \tilde{\Sigma}^\text{out}_0$. Then there exists a time development $\Sigma^\text{out}_t \subset \tilde{\Sigma}^\text{out}_t$, defined by the solution $(U, V)$ of the reduced Einstein-scalar field equations, of the initial open end $\Sigma^\text{out}_0$, i.e. $\Sigma^\text{out}_0$ is future null geodesically complete. That accomplishes the local proof of Theorem 1.

Step IV of the proof of Theorem 1.

**Lemma 7.** Define $g = \frac{V}{pR}$ and $h = \frac{U}{pR}$. Under conditions of Theorem 1,

Proposition 7.1. Banach norms $L_{2j}$ of $g$ and $h$ are equivalent to norms $L_{2j}$ of $U$ and $V$, respectively;

Proposition 7.2. Sobolev norms $H_l$ and integral norms $L_{2j}$ of $g$ and $h$ are uniformly bounded by $Ce^{Ct}$ for any integer $j$, $l \leq k$ and for some constant $C$;

Proposition 7.3. The scalar field $\phi$ is uniformly bounded by a constant $C$ depending only on an asymptotic (ADM) mass of the configuration and on the areal radius $R_0$ of the initial slice $\Sigma^\text{out}_0$,

$$|\phi(R)| \leq C.$$ (39)

The proof of Lemma 7 is highly technical and we postpone it to Section 6. Using the above Propositions we infer that, under conditions of Theorem 1, there exists a uniform bound on $L_{2j}$ norms of $U$ and $V$. We show that $H_k$ norms of $U$ and $V$ remain bounded during a finite evolution. Indeed,

$$||\partial_r U||^2_{L^2(\Sigma_t)} = \int_{\Sigma_t} dV \left( \partial_r (p_r h) \right)^2 \leq$$

$$2 \int_{\Sigma_t} dV \left( (\partial_r (p_r h))^2 + (p_r \partial_r h)^2 \right) \leq$$

$$C \int_{\Sigma_t} dV \left( (U^2 + V^2)^2 h^2 + (\partial_r h)^2 \right),$$ (40)
for some $C$. The first inequality is just a trivial application of $2ab \leq a^2 + b^2$ and in the second one we use $0 < pr \leq 2$, expression $[23]$ (with $\rho = \frac{1}{4}(U^2 + V^2) + W(\phi)$) and Proposition 7.3. The last expression of $[30]$ is globally bounded by applying the Schwartz inequality and then using Propositions 7.1 and 7.2. Thus the norm $||\partial_r U||_{L^2(\Sigma)}^2$ is globally bounded.

Similarly one shows the global boundedness of $||\partial_r V||_{L^2(\Sigma)}^2$ and, iteratively, of higher Sobolev norms of functions $U$ and $V$.

One can show that the area of the inner boundary of $\Sigma_t$ goes to infinity as $t \to \infty$. Indeed, the inner boundary $\bigcup_t \Sigma_t$ is null and its areal radius $R$ satisfies the differential equation $\frac{dR}{dt} = \frac{(pR)^2}{4} - \beta$. Initially we have $R_0 > 2m + \eta$, with $\eta > 0$; thus from $[23]$, $[26]$ and the equation we infer that $(pR)^2(t) > 4(1 - \frac{2m}{2m + \eta})$ and $\beta(R_t) > e^{2(1 - \frac{2m}{2m + \eta})} \int_R^t drr^2(p + \eta) \geq e^{2(1 - \frac{2m}{2m + \eta})}$. Thus $\frac{dR}{dt}$ is strictly positive nad $R_t$ goes to infinity. That completes the global part of Theorem 1.

5. Proof of Lemma 3.

The approximating equations $[33]$ and $[34]$ are nonlinear in $U_n, V_n$, with polynomial nonlinearities in each order $n$, so that their coefficients are of class $C^\infty$ as functions of unknown variables $U_n, V_n$. A long but simple calculation yields following equations

$$
\left(\partial_0 + 2\partial_R\right) \int_R^\infty dr \frac{r^2}{2} \left((\partial^k_R U_n)^2 + (\partial^k_R V_n)^2\right) =
\frac{R^2}{2} \left((\partial^k_R U_n)^2(-2 - \alpha_{n-1}\beta_{n-1}) + (\partial^k_R V_n)^2(-2 + \alpha_{n-1}\beta_{n-1})\right) |_R + \\
\int_R^\infty drr^2 \left[\Gamma_n - \frac{1}{2r^2} \partial_r(\alpha_{n-1}\beta_{n-1}) + k \partial_r(\alpha_{n-1}\beta_{n-1})\right] \left((\partial^k_r U_n)^2 - (\partial^k_r V_n)^2\right) + \\
8\pi r \beta_{n-1} j_n(\partial^k_r V_n)^2 + \partial^k_r(\alpha_{n-1}\beta_{n-1} V_n)(\partial^k_r V_n - \partial^k_r U_n)\right] + \\
\int_R^\infty drr^2 \sum_{i=1}^k \left[\left(\partial^k_r \Gamma_n \frac{(k)}{i} + (1 - \delta_{ki}) \partial^{i+1}_r(\alpha_{n-1}\beta_{n-1}) \left(\frac{(k)}{i+1}\right)\right) \left(\partial^k_r U_n \partial^{k-i}_r U_n - \partial^k_r V_n \partial^{k-i}_r V_n\right) + \\
8\pi \partial^k_r(\beta_{n-1} V_n)^2 \left(\frac{(k)}{i} \partial^k_r V_n \partial^{k-i}_r V_n + \left(\frac{(k)}{i}\right) \partial^k_r \left(\frac{\alpha_{n-1}\beta_{n-1}}{r}\right) \left(\partial^k_r V_n \partial^k_r U_n - \partial^k_r U_n \partial^k_r V_n\right)\right],
$$

where
\[ \Gamma_n = 4\pi\beta_{n-1}R(j_n + T_n) + \frac{\alpha_{n-1}^2\beta_{n-1}}{2R} + \frac{\beta_{n-1}}{2R}, \quad \binom{k}{i} = \frac{k!}{i!(k-i)!}. \]  

(42)

In particular one obtains

\[ \partial_0 \int_R^\infty r^2 dr \rho_n \leq \frac{1}{4} (V_n^2 - U_n^2) \alpha_{n-1}^2 \beta_{n-1}R^2 + \]
\[ \frac{1}{2} \int_R^\infty dr r^2 \beta_{n-1} \left[ W'(\phi_n) \left( V_n - V_{n-1} - U_n + U_{n-1} \right) + 16\pi j_n \left( W(\phi_n) - W(\phi_{n-1}) \right) \right]. \]

(43)

Let us recall an obvious inequality

\[ |f(R)| \leq \frac{1}{R^{1/2}} \left( \int_R^\infty (\partial_R f)^2 r^2 dr \right)^{1/2} \]

(44)

that holds true for any function of compact support that belongs to \( H_1. \)

One gets from (41), applying the Schwartz inequality and pointwise estimates (44), the inequality

\[ \frac{dE_{kn}}{dt} \leq C \left[ E_{kn} \left( \sup |\Gamma_n| + \sup \rho_{n-1} + \sup |j_n| \right) + E_{kn}^{1/2} \left| \partial_r^k (\alpha_{n-1} \beta_{n-1} V') \right|_{L^2} + \right. \]
\[ \left. \sum_{i=1}^{k-1} E_{k-i,n}^{1/2} E_{kn}^{1/2} \left( \sup |\partial_r^i \Gamma_n| + \sup |\partial_r^{i+1} (\alpha_{n-1} \beta_{n-1})| \right) + \right. \]
\[ \left. E_{kn}^{1/2} E_{1n}^{1/2} \left( ||\partial_r^k \Gamma_n||_{L^2} + ||\partial_r^{k+1} (\alpha_{n-1} \beta_{n-1})||_{L^2} \right) \right]. \]

(45)

Here and below constants may change from a line to line, but they do depend only on initial data and \( k. \) Using (44) and various Hölder estimates in order to bound the sup - terms on the left hand side of (45) one derives the differential inequality

\[ (\partial_t + 2\partial_R) E_{kn}(\Sigma_{\text{out}}^n) \leq C E_{kn} \left( E_{1n}^2 + E_{1,n-1} + E_{k-1,n-1}^k (E_{k-1,n} + E_{kn-1}) \right). \]

(46)

The interesting fact is that for \( k > 1 \) the inequality (46) is linear in the term \( E_{kn}. \)

I will use the method of exact induction in order to prove Lemma 3. For \( n = 1 \) the coefficients of (33) and (34) are time-independent. Sobolev embeddings theorem (or, strictly saying, (44)) implies that \( U_0 \) and \( V_0 \) are of Hölder class \( C^{k-1/2}. \) Therefore the coefficients of (32) are of class \( C^{k+1/2} \) for \( R \geq R_0 \) as functions of \( R \) and (being time-independent) they are bounded and continuous in the class \( C^{k}, k \geq 2 \) as functions of \( t, R. \) The eigenvalue \( \alpha_0\beta_0 \) of the hyperbolic operator is bounded from above by 1 and it is strictly nonzero so that in the
case \( n = 1 \) the approximating equation is hyperbolic. A result of Petrovsky \( \text{[5]} \) ensures the existence of \( U_1, V_1 \in C^{k-1}(\Omega_t) \), confirming part i) of lemma 3.

(46) becomes now

\[
(\partial_t + 2\partial_R)E_{k1}(\Sigma_{t}^{\text{out}}) \leq C_2 E_{k1} \left( E_{11}^2 + E_{1,0} + E_{k-10}^k (E_{k-11} + E_{k0}) \right),
\]

where \( E_{k0} = E_{kn}(\Sigma_0^{\text{out}}) \) is the initial value of the norm \( E_{kn} \) and \( C_1, C_2 \) are some constants depending only on initial data. The \( C^{k-1} \) smoothness of the solution \( U_1, V_1 \) guarantees that \( E_{l1} \) is a continuous function of time \( t \) for \( l < k \). Now, let \( E_{l1}(t) > E_0 \) for \( t' \geq t \geq 0 \). Then (47) implies, after elementary calculations,

\[
E_{l1}(\Sigma_{t}^{\text{out}}) \leq \frac{1}{\left( E_{l0}^{-1/2} - C_l t \right)^{1/(l+2)}},
\]

where we used the initial value condition \( E_{l1} = E_{l0} \). The above trick does not work for \( E_{k1} \) since the latter need not be continuous. In order to show a corresponding estimation for \( E_{k1} \) one has to insert estimations (18) into (47) and integrate the latter. The linearity of (47) in \( E_{k1} \) allows one to get an estimation as in (48). That proves the point ii) of Lemma 3.

In order to show iii) let us notice that \( \partial_R^k \alpha_n \) or \( \partial_R^k \beta_n \) contain derivatives of \( U_n \) and \( V_n \) of the order \( k-1 \) at most; that is functions \( \alpha_1 \) and \( \beta_1 \) are of class \( C^k \) for fixed \( t \). That can be shown quite generally, in any order. A direct calculation shows that

\[
\partial_0 \alpha_n^2 = \frac{8\pi}{R} \partial_0 \int_R^\infty dr r^2 \rho_n
\]

(49) and (13) show that the order of differentiability of \( \partial_0 \alpha_n \) is equal to the order of differentiability of \( U_n \) and \( V_n \), if coefficients of the preceding order, \( \beta_{n-1}, \alpha_{n-1} \) are of class \( C^k \). Therefore \( \partial_0 \alpha_1 \) is \( C^{k-1/2} \) in \( \Sigma_t^{\text{out}} \) and \( C^{k-1} \) in \( \Omega_t \) because \( U_1, V_1 \in C^{k-1/2} \) and they are \( C^{k-1} \) in \( \Omega_t \). One shows that

\[
\partial_0 \beta_n = -4\pi \beta_n \int_R^\infty dr r^2 \left[ \frac{2\rho_n^2}{r \alpha_n^2} \partial_0 \alpha_n^2 + \right.
\]

\[
2j_n \alpha_n^2 \beta_n \beta_{n-1} \partial_r \frac{1}{r \alpha_n^2} + \frac{\beta_{n-1}}{r \alpha_n^2} \left[ W'(\phi_n) (V_n - U_n) + 16\pi r j_n (W(\phi_n) - W(\phi_{n-1})) \right]
\]

\[\ + \]
\[-8\pi \beta_n j_n R \alpha_n^2 \alpha_{n-1} \beta_{n-1};\]

obviously \(\partial_0 \beta_1\) is \(C^{k-1/2}\) in \(\Sigma_t^{out}\). Thus both \(\alpha_1\) and \(\beta_1\) are at least \(C^k\) in \(\Omega_t\). That accomplishes the first part of the statement iii). \(C^{k-1}\) - differentiability of \(U_0\) and \(V_0\) implies that also \(\phi_1(t, R) = \phi_0(R) + \int_0^t ds\alpha_0 \beta_0 \frac{U_n - V_n}{2}\) is (at least) \(C^k\). The estimate v) is trivially true for \(\phi_1\). The second part of iii), the estimate (37) and that of \(\alpha_1^2 \beta_1\), can be obtained from \(C^k\) estimates of \(\alpha_1\) and \(\beta_1\), expressing them in terms of initial data (through (48)).

Differentiation of the approximating equations (33, 34) with respect \(t\) and \(R\), integration over \(\Sigma_t^{out}\) and the use of ii) and iii) allows one to express those norms of functions \(U_1\) and \(V_1\) that are specified in point iv) in terms of initial data. That finishes the proof of the first step of the induction hypothesis.

Now, let it be true for some \(n\). Using the induction hypothesis and the same reasoning as above, one shows that the coefficients of the \(n + 1\) equation (33, 34) are at least \(C^k\), so that there exists a solution \(U_{n+1}\) and \(V_{n+1}\), by a result of Pietrovski (49). The second power of its Sobolev norm, \(E_{l,n+1}\) (where \(l < k\)), is either less than \(E_{l,n}\) for some \(t\) close to \(t = 0\) or greater than \(E_{l,n}\). In the former case it satisfies the bound of (48) by the induction hypothesis while in the latter case we have, from (46)

\[ (\partial_t + 2\partial_R)E_{l,n+1}(\Sigma_t^{out}) \leq \left(E_{l,n+1}(\Sigma_t^{out})\right)^{l+1}, \] (50)

which again yields

\[ E_{l,n+1}(\Sigma_t^{out}) \leq \frac{1}{(E_{l0} - Clt)^{1/(l+2)}}. \] (51)

The estimation of \(E_{kn+1}\) can now be obtained in the same token as that of \(E_{k1}\). Hence i) and ii) are proven. The remaining steps, iii) and iv) are shown identically as in the case \(n = 1\). We will check the validity of v). Notice that

\[ \phi_{n+1} = \phi_0(R) + \int_0^t ds\alpha_n \beta_n \frac{U_n - V_n}{2}; \]

the right hand side of that equation is bounded from above by \(\sup |\phi_0(R)| + C \int_0^t dsE_{1n}^{1/2}\), using (44). The induction hypothesis ii) and direct integration immediately yield v). That
ends the proof of Lemma 3. Let us point out that the uniform bounds proven above show the existence of all approximating functions in a strictly positive time $t$; from (51) follows that $t$ can be bounded from below by a number that depends only on initial data and does not depend on $n$.

6. Proof of Lemma 7.

Let $R_t = \inf_{\Sigma^0_{textout}}(r)$; we have

$$R_t = R_0 + \int_0^t ds (\partial_t + \frac{NpR}{2} \partial_R)R = R_0 + \int_0^t ds \frac{NpR}{2} \geq R_0 > 2m$$

(52)

(these inequalities follow from the assumption made in Theorem 1 that $\Sigma^0_{textout}$ is placed outside a centered sphere of the Schwarzschild radius $2m$). Using (14) and (52), we conclude that

$$Rp(t, R) \geq 2\sqrt{1 - \frac{2m}{R_0}} > 0$$

(53)

on all future slices $\Sigma^0_t$. Since (14) implies also $Rp \leq 2$, we infer that there exist two nonzero and finite numbers $\alpha, \beta$ such that

$$\alpha g \leq \frac{V}{pR} \leq \beta g, \quad \alpha h \leq \frac{U}{pR} \leq \beta h;$$

that fact trivially implies the equivalence of $L_p$ norms of pairs $(g, V)$ and $(h, U)$, for any $p$, and hence also for even values of $p$. That proves Proposition 7.1.

One can show that

$$\partial_0(pR) = 8\pi NR_j$$

(54)

The reduced equations (28) and (29) lead now to following system of equation for functions $g = \frac{V}{pR}$ and $h = \frac{U}{pR}$

$$(\partial_0 + \frac{NpR}{2} \partial_R)g = 16\pi \frac{Ng}{p} W(\phi) - \frac{2N}{pR^2}g - \frac{Nph}{2} + \frac{N}{pR} W'(\phi),$$

(55)

$$(\partial_0 - \frac{NpR}{2} \partial_R)h = -16\pi \frac{Nh}{p} W(\phi) + \frac{2N}{pR^2}h + \frac{Npg}{2} - \frac{N}{pR} W'(\phi).$$

(56)
The interesting fact is that these equations are almost linear in unknowns; although coefficients are some functionals of $g$ and $h$, they are pointwise bounded by constants that are independent of both functions.

Embeddings theorems imply that initial data $(U, V)$ that are of Sobolev class $H_k$ ($k > 1$) must belong to $L_p(\Sigma^\text{out}_t)$ for any $p > 2$; that implies, taking into account the definition of $g, h$ that initial data of the latter are also of class $L_p$. Let $n$ be an even number. Multiply (55) and (56) by $g^{n-1}$ and $h^{n-1}$, respectively and integrate over $\Sigma^\text{out}_t$. Integrating by parts and using (61) (see below) one arrives at

\[
\left(\partial_0 + \frac{N p R}{2} \partial_R \right) \int_R^\infty drr^2 \left( \frac{1}{n} (h^n + g^n) \right) = - \frac{N p R^3}{n} h^n + \\
\int_R^\infty drr^2 \left[ \frac{2N}{pr} (g^n - h^n) \left( \frac{(pr)^2}{4nr} + \frac{1-n}{nr} + \frac{8\pi (n-1)}{n} r W(\phi) \right) \right] \\
g h \frac{N p}{2} (h^{n-2} - g^{n-2}) + \frac{NW'(\phi)}{pr} (g^{n-1} - h^{n-1}),
\]

(57)

The assumptions of Lemma 7 $W(x) > 0$ for $x \neq 0$ and $W(0) = 0$ imply that for initial data of compact support the scalar field $\phi$ vanishes at spatial infinity; then one can use (14) and replace $\partial_r \phi$ by $\frac{1}{pr}(U + V)$ (see (30)), which leads finally to the estimation

\[
\phi(R, t) \leq C(||U||_{L^2(\Sigma^\text{out}_t)} + ||V||_{L^2(\Sigma^\text{out}_t)}) \leq C(||U||_{L^2(\Sigma^\text{out}_0)} + ||V||_{L^2(\Sigma^\text{out}_0)});
\]

(58)

the last inequality is valid only if the selfinteraction term $W(\phi)$ is nonnegative and it follows essentially from the conservation of asymptotic mass $m$. The constant $C$ can be found explicitly and the right hand side of (58) can be found to be bounded by $\sqrt{\frac{2}{1-\frac{2m}{R_0}}} \sqrt{\frac{m}{\pi R_0}}$. With this we prove Proposition 7.3.

With the inequality (58) and having hitherto known $L_\infty$ estimate $p R \leq 2$, we get from (51)

\[
\left(\partial_0 + \frac{N p R}{2} \partial_R \right) \int_R^\infty drr^2 \left( \frac{1}{n} (h^n + g^n) \right) \leq C \int_R^\infty drr^2 \left( \frac{1}{n} (h^n + g^n) \right),
\]

(59)

where $C$ is a constant depending only on $n$ and initial data. Therefore $L_n$ norms of $g$ and $h$ are bounded in a global time evolution. That proves the $L_p$ part of Proposition 7.2.
Differentiating equations (55) and (56) with respect \( r \), multiplying them by \( \partial_r g \) and \( \partial_r h \), respectively, and integrating over \( \Sigma^\text{out} \) one arrives at
\[
\frac{d}{dt} \bigg|_{\text{out}} \int_R^\infty dr r^2 \left( \frac{1}{2} (\partial_r g)^2 + (\partial_r h)^2 \right) = \\
-\frac{N p R^3}{2} (\partial_r h)^2 + \int_R^\infty dr r^2 \left( \left( (\partial_r g)^2 - (\partial_r h)^2 \right) \left( -\frac{1}{4} \partial_r (N p r) + 16 \pi \frac{N W}{p} - \frac{2 N}{p r^2} + \frac{N p r^2}{2} \right) + \\
\partial_r \left( \frac{N}{p r} W'(\phi) \right) (\partial_r g - \partial_r h) + (g \partial_r g - h \partial_r h) \partial_r (16 \pi \frac{N W}{p} - \frac{2 N}{p r^2}) + \frac{1}{2} (g \partial_r h - h \partial_r g) \partial_r (N p) \right) \tag{60}
\]

Notice that coefficients \( \frac{1}{4} \partial_r (N p r) \), \( \frac{16 \pi N W}{p} + \frac{N}{p r^2} \) in front of \( (\partial_r h)^2 \) and \( (\partial_r g)^2 \) are pointwise bounded on \( \Sigma^\text{out} \). That follows from obvious estimates \( 0 < N, p R/2 \leq 1, R > 2m \), from the equation
\[
-\frac{r^3}{4} \partial_r (N p r^{-1}) = -\frac{N}{p r} + \frac{3 N p r}{4} - 4 \pi N p^{-1} r (T - \rho) = \\
-\frac{N}{p r} + \frac{3 N p r}{4} + 8 \pi N r W(\phi)/p \tag{61}
\]

and from Proposition 7.3. Using the Schwartz inequality, Proposition 7.3 and the already proven part of Proposition 7.2 one obtains from (60) the inequality
\[
\frac{d}{dt} \bigg|_{\text{out}} \int_R^\infty dr r^2 \left( \frac{1}{2} (\partial_r g)^2 + (\partial_r h)^2 \right) \leq C \int_R^\infty dr r^2 \left( (\partial_r g)^2 + (\partial_r h)^2 \right) \tag{62}
\]
with a constant \( C \) depending smoothly on various norms \( L_{2j} \) and on the estimation of the scalar field. From (62) we infer that the integral \( \int_R^\infty dr r^2 \left( (\partial_r g)^2 + (\partial_r h)^2 \right) \) remains bounded during a finite evolution. Therefore \( H_1 \) norms of \( U \) and \( V \) remain bounded.

Now, let there exists a global estimate for the Sobolev norm \( H_{k-1} \) of \( g \) and \( h \). In one dimension (to which our problem essentially reduces) the boundedness of \( H_k \) implies the existence of \( L_p \) estimates of \( \partial^l f \), \( f = V, U \), for \( l < k \) and for any integer \( p > 2 \). Differentiating reduced equations (55) and (56) \( k \) times with respect \( r \), integrating the resulting equations, using various Hölder inequalities and the induction hypothesis, one obtains an inequality analogous to (62) for the \( H_k \) norm. That leads to the conclusion that \( H_k \) norm is also globally bounded. That ends the proof of Proposition 7.2.
7. Proof of Theorem 2.

Now I will discuss shortly the proof of Theorem 2. It proceeds in a way analogous to the proof of Theorem 1. In particular, as in the former case, one defines the approximation procedure of Step 1 of Theorem 1. The only difference between the two cases is due to the fact that previously we used an open initial end \( \Sigma_0^{\text{out}} \) with \(|x| > 2m\) while now it penetrates a region inside the Schwarzschild radius \(|x| = 2m\). The difference is important, since the former choice guarantees strict hyperbolicity of all approximating equations (33, 34) - all coefficients \(\alpha_n^2, \beta_n\) are uniformly bounded away from 0 - while now we have to show that.

Let \(R_0 = 2m\) be the areal radius of the inner boundary of \(\Sigma^{\text{out}}_0\). Define \(R_1 = R_0 - \tau > 0\) and \(\tilde{\Sigma}^{\text{out}}_0 = \{x \in \Sigma_0 : R_1 \leq |x| < \infty\}\) (an open end containing \(\Sigma^{\text{out}}_0\)). Let \(S^{\text{out}}_r = \{x \in \Sigma_0 : R_0 \leq r \leq |x| < \infty\}\) be an open end contained in \(\Sigma^{\text{out}}_0\). Define a four-manifold \(\Omega_t = \{(t', x') : 0 \leq t' \leq t, |x| \geq R_1 + 2t'\}\) foliated by hypersurfaces \(\tilde{\Sigma}^{\text{out}}_{t'} = \{x \in \Omega_t : x^0 = t\}\) and another four-manifold \(\Omega_{S,t} = \{(t', x) : 0 \leq t' \leq t, |x| \geq r - 2t'\}\) with leaves \(S^{\text{out}}_{rt} = \{x \in \Omega_{rt} : x^0 = t\}\). Obviously, \(S^{\text{out}}_{r0} = S^{\text{out}}_r\).

As in Section 4 we define energy norms of solutions of the approximating equations (33), (34) for each family of foliations,

\[
E^\Sigma_{kn}(\tilde{\Sigma}^{\text{out}}_t) = \left( |\partial_t^k U_n|_{L_2(\tilde{\Sigma}^{\text{out}}_t)}^2 + |\partial_r^k V_n|_{L_2(\tilde{\Sigma}^{\text{out}}_t)}^2 \right),
\]

(63)

\[
E^S_{kr}(\tilde{\Sigma}^{\text{out}}_{rt}) = \left( |\partial_t^k U_n|_{L_2(\tilde{\Sigma}^{\text{out}}_{rt})}^2 + |\partial_r^k V_n|_{L_2(\tilde{\Sigma}^{\text{out}}_{rt})}^2 \right).
\]

(64)

All iterative solutions satisfy the initial data, \(U_n(t, R) = U_0(R), V_n(t, R) = V_0(R)\); norms of initial data are denoted simply as \(E_{k0}\) and \(E_{k0}^S\) respectively, for each of the two families of foliations.

The main technical result of Section 4, Lemma 3, has to be replaced by

**Lemma 3’**. Let the initial data of the approximating equations (28) and (29) be of compact support and let there exists a nonzero \(\tau\) such that

\[
\inf_{R_0 \leq t \leq \infty} \left[ E_0^S - (2m - r + 2\tau) \right] > C_6 \tau,
\]

(65)
where $C_6$ is a constant defined below that depends on initial data. Let $U_0, V_0 \epsilon H_k(\Sigma^\text{out}_0)$, $k \geq 2$), for $k=2, 3,...$ Assume that $W(x) > 0$ if $x \neq 0$ and $V(0) = 0$ and

$$\left| \partial_x^l W(x) \right| \leq A(l, n_t) |x|^{|n_t|}$$

for some $0 \leq n_t < \infty$ and a constant $A$ depending only on $n_t$ and $l$. Then for $t < \tau$

O) equations (33) and (34) are strictly hyperbolic in $\Omega$;

i) for each $n > 0$ there exists a solution $(U_n, V_n)$ of equations (33, 34) in $\Omega_1$;

ii) Sobolev norms $H_k(\Sigma^\text{out}_t)$ of $U_n$ and $V_n$ are uniformly bounded,

$$E_{kn}(\Sigma^\text{out}_t) \leq \frac{1}{(E_{k0} - l/2 - C_1 t)^{1/(l+2)}},$$

where $l = 0, 1, ..., k$ and $C_1$'s are some constants;

iii) coefficients $\alpha_n, \beta_n$ of (33, 34) are at least $C^k$ as functions of $(t, R)$ with a norm $|f|_{C^k} = \sup\Sigma^k_{k_1+k_2=0} |\partial_0^{k_1} \partial_R^{k_2} f|$ (where $f = \alpha_n, \beta_n$) uniformly bounded by (up to a constant coefficient depending only on initial data)

$$\left( E_{kn}(\Sigma^\text{out}_t) \right)^{2k} \leq \frac{1}{(E_{k0} - k/2 - C_1 t)^{2k/(k+2)}},$$

iv) $L_2(\Sigma^\text{out}_t)$ norms of $\partial_0^{l} \partial_R^{m} f$ ($f = U_n, V_n$ and $l > 0, l + m \leq k$) are uniformly bounded;

v) $|\partial_R^{l} \phi_n(t, R)| \leq C_3 + C_4 t$, where $C_3$ and $C_4$ are constants depending only on $l$ and initial data and $0 < l \leq k$.

**Proof.** The only new element in comparison with Lemma 3 is the strict hyperbolicity assertion, point O above. We have to show that the product $\alpha_n^2 \beta_n$ is strongly positive for sufficiently small values of $\tau$. That will be achieved if we show that $\alpha_n$ itself is positive, that is (from (32))

$$R + 2t - 2m + 8\pi \int_{R+t}^{\infty} drr^2 \rho_n(r, t) > 0$$

for $R \geq R_1$. That is obviously true for the $n = 1$ equation, since all functions with a suffix "0" (including $\alpha_0$) are time-independent and initially $\alpha$ is positive. Now let the $n-th$ step
of the induction hypothesis be true. We shall show that the \((n+1)\)th equations are strictly hyperbolic. Manipulating with equations (33) and (34) one finds the equation

\[
(\partial_t - 2\partial_R)2 \int_R^\infty dr r^2 \rho_n(r, t) = (\partial_t - 2\partial_R) \int_R^\infty dr \frac{r^2}{2} ((U_n)^2 + (V_n)^2 + 2W(\phi_n)) = \\
\frac{R^2}{2} \left( (U_n)^2(2 - \alpha_{n-1}^2\beta_{n-1}) + (V_n)^2(2 + \alpha_{n-1}^2\beta_{n-1}) \right) + \\
\int_R^\infty dr r^2 \left[ 8\pi \beta_{n-1} r j_n(\rho_n - \rho_{n-1} + T_{n-1} - T_n) + V'\alpha_{n-1}\beta_{n-1}(U_{n-1} - U_n + V_n - V_{n-1}) \right]
\]

that describes the evolution of the \(n-th\) order external energy (modulo a coefficient \(4\pi\)) \(\int_R^\infty dr r^2 \rho_n(r, t)\) along the foliation formed by leaves \(S_{\mathcal{R}t}^{\text{out}}\). From that we obtain (dropping out the positive boundary term and using pointwise estimates in order to bound the last integral of (69))

\[
(\partial_t - 2\partial_R)2 \int_R^\infty dr r^2 \rho_n(r, t) \geq \\
\int_R^\infty dr r^2 \left[ 8\pi \beta_{n-1} r j_n(\rho_n - \rho_{n-1} + T_{n-1} - T_n) + V'\alpha_{n-1}\beta_{n-1}(U_{n-1} - U_n + V_n - V_{n-1}) \right] \geq \\
-C \left( \sup(|j_n| + |V_n| + |U_n| + |V_{n-1}| + |U_{n-1}|) \right) \int_R^\infty dr r^2 \rho_n(r, t),
\]

where \(C\) is a constant. Using now (44) and (35) we observe that \(\sup(|j_n| + |V_n| + |U_n| + |V_{n-1}| + |U_{n-1}|)\) can be bounded from above by \(\tilde{C}_1E_{1n}^\Sigma(\tilde{\Sigma}_{\mathcal{R}t}^{\text{out}}) + \tilde{C}_2 \left( E_{1n}^\Sigma(\tilde{\Sigma}_{\mathcal{R}t}^{\text{out}}) \right)^{1/2},\) where \(\tilde{C}_1\) and \(\tilde{C}_2\) are constants. Using now the induction assumption (36) and using the obvious inequality \(\sqrt{x} + x \leq 1 + 2x\), we arrive at

\[
(\partial_t - 2\partial_R)2 \int_R^\infty dr r^2 \rho_n(r, t) \geq \\
-C_4 \left( 1 + \frac{1}{(E_{1n}^{-3} - C_1t)^{1/3}} \right) \int_R^\infty dr r^2 \rho_n(r, t),
\]

which yields

\[
\int_{R-2t}^\infty dr r^2 \rho_n(r, t) \geq e^{-C_5t} \int_{R}^\infty dr r^2 \rho(r, t = 0) \geq \\
\int_{R}^\infty dr r^2 \rho(r, t = 0) - C_6 t.
\]

All constants above depend only on initial values and the degree of nonlinearity of the potential term \(W(\phi_n)\). The last constant enters the condition (35) stated in Lemma 3'. Using (72) we can bound the left hand side of (68) as follows
\[ R + 2t - 2m + 8\pi \int_{R+2t}^{\infty} drr^2 \rho_n(r,t) \geq R - 2m + 8\pi \int_{R+4t}^{\infty} drr^2 \rho(r,t = 0) + t(2 - C_6). \] (73)

Take now \( \tau \) of (65) and notice that the condition (65) is satisfied by all values \( 0 \leq t \leq \tau \).

Notice that
\[ E_0^{SR+4t} = 8\pi \int_{R+4t}^{\infty} drr^2 \rho(r,t = 0); \]
inserting the condition \( E_0^{SR+4t} - (2m - (R + 4t) + 2\tau) > C_6 t > 0 \) of (65) into (13) we obtain \( (\tau - t)(2 + C_6) \), that is the last expression of (73) is positive for \( t < \tau \). Thus we arrive at the desired result
\[ \alpha_2(x, t) = R + 2t - 2m + 8\pi \int_{R+2t}^{\infty} drr^2 \rho_n(r,t) > 0. \]

The approximating equations (33) and (34) are strictly hyperbolic in \( \Omega_t \).

The rest of the proof of Lemma 3’ is similar to that of Lemma 3. One can prove also results corresponding to those of lemmae 4-7, thus accomplishing the proof of a local part of Theorem 2 and the first half of its global part. I omit details. The proof of the remaining global statements of the point iii) of Theorem 2 is given elsewhere [15]. Below I will sketch the main idea.

Assume that we have a complete initial Cauchy hypersurface with an apparent horizon, that is (using the polar gauge) with a minimal 2-surface on the initial slice. As a side remark let us point out that the area of an outermost apparent horizon cannot decrease (see, e. g., a proof outlined in [12]); in fact it has to increase whenever matter crosses through the horizon, which moves acausally outwards. Asymptotically the spherically symmetric apparent horizont becomes null and its areal radius stabilizes at \( 2m_B \leq 2m \), where \( m_B \) is the Bondi mass. Thus all null geodesics starting outward from \( R \geq 2m_B \) at \( t = 0 \) are complete. The biggest set \( H \) that is still future null complete can be constructed as follows. Take a part \( \Sigma_r \) of the initial hypersurface that does not include minimal surfaces. Then data on \( \Sigma_r \) give rise to a local evolution, according to the local part of Theorem 2. The global evolution prolongs until the free inner boundary ”freezes” near a minimal surface; the lapse collapses to values close to 0 and the area of the inner boundary is practically constant. In such a case one can take a slightly smaller initial open end \( \Sigma_{r'} \subset \Sigma_r \); that evolves to a spacetime that freezes at a later time than the previous one. Continuing that procedure
ad infinitum one finds finally an open end such that a corresponding spacetime $H$ exists globally and the area $dH$ of its null inner boundary stabilizes at a finite value $4\pi R_B^2$ for $t \to \infty$. That limiting inner boundary $dH$ is an event horizon and half of $R_B$ is the Bondi mass. It follows from the construction that all null geodesics originating in $H$ and directed outward are complete.

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Appendix

Lemma.

Assume notation of the main text. Let $U, V \in H_2(\Sigma_0^{out})$. Define a $H_k$ extension $\tilde{\Sigma}_0^{out} = \{x \in \Sigma_0 : 2m + \eta - \delta \leq |x| < \infty\}$ of $\Sigma_0^{out}$ by assuming following initial data in the interval $[2m + \eta - \delta, 2m + \eta]$ 

$$\partial_r^k f(R) = 0, \quad \partial_r^{k-1} f(R) = \partial_r^{k-1} f(R_0), \quad \partial_r^{k-i} f(R) = \partial_r^{k-i} f(R_0) - \int_{R}^{R_0} dr \partial_r^{k-i+1} f(r),$$

where $f = U, V$ and $i = 2, \ldots, k$. Then

$$||f||_{H_k(\tilde{\Sigma}_0^{out})} \leq (1 + C\delta) ||f||_{H_k(\Sigma_0^{out})},$$

for sufficiently small $\delta$ and some constant $C$.

Proof. The two norms in question differ by a sum of terms of the form

$$\int_{R_0-\delta}^{R_0} (\partial_r^{k-1} f(R))^2 R^2 dR;$$

that sum is a polynomial in $\delta$ that can be bounded from above by $C\delta \sum_{i=1}^{k-1} (\partial_r^{i} f(R_0))^2$. By one of Sobolev inequalities (see the proof of Proposition 7.3), $|\partial_r^{i} f(R_0)| \leq C||\partial_r^{i+1} f||_{\Sigma_0^{out}}$; combining all information one arrives at the desired inequality.
The initial values of the scalar field in the extension can be obtained from integrating $U$ and $V$, $\phi(R) = \phi(R_0) - \int_R^{R_0} dr \frac{1}{pr}(U + V)$. That would lead to an estimation of the contribution of the potential term to the mass $m$ that is due to the enlargement of the initial open end. Together with the preceding Lemma, that would give $m(R_0 - \delta) \leq m$ for sufficiently small $\delta$. 
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