Curvature estimates for minimal hypersurfaces via generalized longitude functions

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Abstract On some specified convex supporting sets of spheres, we find a generalized longitude function $\theta$ whose level sets are totally geodesic. Given a (weakly) harmonic map $u$ into $S^n$, the composition of $\theta$ and $u$ satisfies an elliptic equation of divergence type. With the aid of Harnack’s inequalities, we establish the image shrinking property and then regularity results. Applying such results to study the Gauss image of minimal hypersurfaces in Euclidean spaces, we obtain curvature estimates and corresponding Bernstein type theorems.

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1 Introduction

Let $(N, h)$ be a Riemannian manifold and $V$ be an open subset of $N$. If every compact subset $K$ of $V$ supports a strictly convex function, then we say $V$ is a convex supporting set of $N$. This concept is introduced by Gordon in [6], which is a natural generalization of the geodesic convex set. It is shown by Jost, Xin and the author in [9] that the complement of a half-equator, i.e. $S^n \setminus S^{n-1}_+$, is a maximal convex supporting set of $S^n$, since if we add even a single point to $S^n \setminus S^{n-1}_+$, it will contain a closed geodesic. Combining the construction of convex functions and some analytic technologies, one can show the smoothness of an weakly harmonic map $u : M \to S^n$ whose image is contained in a compact subset of $S^n \setminus S^{n-1}_+$. Moreover, a Bernstein type theorem follows from the above regularity result and the well-known Ruh–Vilms theorem [13]:
Theorem 1.1 [9] Let $M^m \subset \mathbb{R}^{m+1}$ be an embedded complete minimal hypersurface with Euclidean volume growth. Assume the following Neumann–Poincaré inequality

$$\int_{B_R(y)} |u - \bar{u}_{B_R(y)}|^2 \leq CR^2 \int_{B_R(y)} |\nabla u|^2 \quad \forall u \in C^\infty(B_R(y))$$

holds for every $y \in M$ and $R > 0$ with a positive constant $C$ not depending on $R$ and $y$, where $B_R(y)$ denotes the extrinsic ball centered at $y$ and of radius $R$ and $\bar{u}_{B_R(y)}$ is the average value of $v$ on $B_R(y)$. If the Gauss image of $M$ is contained in a compact subset of $S^m \setminus S^m_{+}^{-1}$, then $M$ has to be an affine linear subspace.

This is an improvement of Moser’s theorem [11], which says that an entire minimal graph $M = \{(x, f(x)) : x \in \mathbb{R}^n\}$ has to be affine linear provided that the slope of the function $f$ is uniformly bounded. From the viewpoint of Gauss maps, boundedness of $|Df|$ means that the Gauss image of $M$ is contained in a closed subset of an open hemisphere. Please note that $S^m \setminus S^m_{+}^{-1}$ contains the upper hemisphere and the lower hemisphere.

It is natural for us to raise the following two questions.

Firstly, is $S^n \setminus S^n_{+}^{-1}$ the unique maximal convex supporting set of $S^n$? If not, given a convex supporting set $\mathcal{V} \subset S^n$ which is not contained in $S^n \setminus S^n_{+}^{-1}$, can we derive regularity results (or Bernstein type results) when the image under harmonic map (or Gauss map, respectively) is contained in a compact subset of $\mathcal{V}$?

Recall that Moser’s theorem was also improved by Ecker–Huisken in [5], which says that any entire minimal graph has to be affine linear whenever $|Df| = o(\sqrt{|x|^2 + f^2})$. In other words, one can obtain a Bernstein type theorem for a minimal hypersurface $M$ whose Gauss image lies in an open hemisphere; when $y \in M$ approaches infinity, the image of $y$ under the Gauss map is allowed to approach the equator (the boundary of hemisphere) in a controlled manner. Similarly, under the fundamental assumption that $\gamma(M) \subset S^n \setminus S^n_{+}^{-1}$, where $\gamma$ denotes the Gauss map, can we derive Bernstein type results by imposing an additional condition on the rate of convergence of $\{\gamma(y_k) : k \in \mathbb{Z}^+\}$ to the boundary of $S^n \setminus S^n_{+}^{-1}$ for any sequence $\{y_k \in M : k \in \mathbb{Z}^+\}$ tending to infinity? This is our second question.

We partially answer above two questions in the present paper. But our technique is a bit different from [9]. The function $\theta$ on spheres, so-called generalized longitude function, play a crucial role in our statement.

Let $\pi$ be the natural projection from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^2$, then $\pi$ maps $S^n$ onto $\mathbb{D}$, the two-dimensional closed unit disk. It is easily-seen that the preimage of $(0, 0)$ under $\pi$ is a sub-sphere of codimension 2, which is denoted by $S^{n-2}$. Once $\mathcal{V}$ is a simply-connected subset of $S^n \setminus S^n_{+}^{-2}$, the composition of $\pi$ and the angular coordinate of $\mathbb{D} \setminus \{(0, 0)\}$ yields a real-valued function on $\mathcal{V}$, denoted by $\theta$. When $n = 2$, $\theta$ becomes the longitude function, so $\theta$ is called a generalized longitude function. Each level set of $\theta$ is contained in a hemisphere of codimension 1, which is totally geodesic. It is not hard for us to calculate $\text{Hess} \theta$ and moreover, prove $\mathcal{V}$ is a convex-supporting set (see Proposition 2.1). This means $S^n \setminus S^n_{+}^{-1}$ is not the unique maximal convex supporting set. But it is still unknown what is the sufficient and necessary condition for a subset of $S^n$ be a convex supporting set.

Using the composition formula, we can deduce a partial differential equation (see 2.8) that $\theta \circ u$ satisfies whenever $u$ is a (weakly) harmonic map into $\mathcal{V}$. We note that the equation can also be derived in the framework of warped product structure (see [17]). Following the idea of Moser [11] and Bombieri–Giusti [2], one can derive Harnack’s inequalities for $\theta \circ u$ when $M$ satisfies the so-called local DSVP-condition with respect to a fixed point $y_0 \in M$. 

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Here ‘D’ represents the existence of a distance function $d$, the metric topology induced by which coincides with the initial topology; ‘V’ denotes the condition on the volume growth of metric balls centered at $y_0$ as a function of their radius; ‘S’ and ‘P’ are respectively Sobolev type inequalities and Neumann–Poincaré inequalities for functions defined on metric balls centered at $y_0$ with uniform constants. It is easy to show the local DSVP-condition is weaker than the DVP-condition in [9]. Harnack’s inequalities implies the so-called image shrinking property, and moreover a regularity theorem of weakly harmonic maps into spheres with image restrictions (see Theorem 4.2), which not only generalizes but also improves the regularity theorem in [9]. Furthermore, if we replace the Gauss image restriction on $M$ in Theorem 1.1 by a more general condition that $\gamma(M)$ is contained in a closed, simply-connected subset of $S^m \setminus S^{m-2}$, then again based on the image shrinking property one can get the corresponding Bernstein type result, and the first question is partially answered.

Finally, in conjunction with the Gauss image shrinking property and Ecker–Huisken’s [5] curvature estimates for minimal graphs, we deduce curvature estimates for minimal hypersurfaces with Gauss image restrictions, which implies a Bernstein type theorem as follows.

**Theorem 1.2** Let $M^m \subset \mathbb{R}^{m+1}$ be an embedded complete minimal hypersurface with Euclidean volume growth. Assume that there is $y_0 \in M$, such that the following Neumann–Poincaré inequality

$$\int_{B_R(y_0)} |v - \bar{v}_R|^2 \leq C R^2 \int_{B_R(y_0)} |\nabla v|^2 \quad \forall v \in C^\infty(B_R(y_0))$$

holds with a positive constant $C$ not depending on $R$, where $\bar{v}_R$ is the average value of $v$ on $B_R(y_0)$. If the Gauss image of $M$ is contained in $S^m \setminus S^{m-2}$, and

$$\sup_{B_R(y_0)} d(\cdot, S^{m-2})^{-1} \circ \gamma = o(\log \log R)$$

then $M$ has to be an affine linear subspace.

The above theorem is comparable with Theorem 1.1.

Firstly, Theorem 1.1 requires the Neumann–Poincaré inequality holds for every extrinsic ball centered at each point of $M$ and of arbitrary radius with a uniform constant; so the assumption of Theorem 1.2 on $M$ is weaker.

Please note that $\partial(S^m \setminus S^{m-1}) = S^{m-2} \cup A$ with $A$ the preimage of the interval $(0, 1]$ under $\pi$. According to the assumption of Theorem 1.1, $\{\gamma(y_k)\}$ cannot converge to the boundary of $S^m \setminus S^{m-1}$ for any sequence $\{y_k : k \in \mathbb{Z}^+\}$ in $M$ tending to infinity. In contrast, $\{\gamma(y_k)\}$ is allowed to converge to an arbitrary point in $A$ at arbitrary speed, or any point of $S^{m-2}$ in a controlled manner. Hence Theorem 1.2 partially answers the second question that we have raised.

As shown in [2,10], area-minimizing hypersurfaces satisfy the local DSVP-condition, hence Theorem 5.3 and Corollary 5.1 immediately follow from Theorem 1.2. Unfortunately we do not know whether the above Bernstein type results are optimal.

### 2 Generalized longitude functions on spheres

There is a covering map $\chi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \to S^2 \setminus \{N, S\}$,

$$(\varphi, \theta) \mapsto (\cos \varphi \cos \theta, \cos \varphi \sin \theta, \sin \varphi)$$
where \( N \) and \( S \) are the north pole and the south pole, \( \varphi \) and \( \theta \) are latitude and longitude, respectively. \( \{ \varphi, \theta \} \) is called the geographic coordinate of \( S^2 \). Each level set of \( \theta \) is a meridian, i.e., a half of a great circle connecting the north pole and the south pole. Although \( \chi \) is not one-to-one, the restriction of \( \chi \) on \( (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi) \) is a bijection onto an open domain \( \mathbb{V} \) that is obtained by deleting the International Date Line from \( S^2 \). It has been shown in [6, 9] that \( \mathbb{V} \) is a maximal convex supporting set of \( S^2 \), i.e. any compact set \( K \subset \mathbb{V} \) supports a strictly convex function.

The longitude function \( \theta \) and open domain \( \mathbb{V} \) can be generalized to higher dimensional cases.

Let \( \pi \) be the natural projection from \( \mathbb{R}^{n+1} \) onto \( \mathbb{R}^2 \), which maps \( (x_1, \ldots, x_{n+1}) \) to \( (x_1, x_2) \), then it is easily-seen that \( \pi(S^n) = \mathbb{D} \), the two-dimensional closed unit disk. On it we shall use the polar coordinate system. The radial coordinate and the angular coordinate are respectively denoted by \( r \) and \( \theta \). In other words, there exists a covering mapping \( \chi : (0, 1] \times \mathbb{R} \rightarrow \mathbb{D}\setminus\{(0, 0)\} \)

\[
(r, \theta) \mapsto (r \cos \theta, r \sin \theta).
\]

Assume \( \mathbb{V} \) is a simply connected subset of \( S^n \setminus S^{n-2} = \pi^{-1}(\mathbb{D}\setminus\{(0, 0)\}) \), then the lifting properties for covering spaces enable us to find a smooth mapping \( \Psi : \mathbb{V} \rightarrow (0, 1] \times \mathbb{R} \)

\[
x \mapsto \Psi(x) = (r(x), \theta(x))
\]
such that the following commutative diagram holds

\[
\begin{array}{ccc}
\mathbb{V} & \xrightarrow{\Psi} & (0, 1] \times \mathbb{R} \\
\downarrow{\text{Id}} & & \downarrow{\chi} \\
\mathbb{V} & \xrightarrow{\pi} & \mathbb{D}\setminus\{(0, 0)\}
\end{array}
\]

In other words, \( (x_1, x_2) = \pi(x) = \chi \circ \Psi(x) = r(x) \left( \cos \theta(x), \sin \theta(x) \right) \) \( \forall x \in \mathbb{V} \). (2.1)

One can proceed as in [9, Sect. 2.2] to get the following equations

\[
\text{Hess } r = -r g_s + r d\theta \otimes d\theta \\
\text{Hess } \theta = -r^{-1} (dr \otimes d\theta + d\theta \otimes dr)
\]

where \( g_s \) denotes the standard metric on \( S^n \).

For an arbitrary compact subset \( K \subset \mathbb{V} \), there is a constant \( c \in (0, 1) \), such that \( r > c \) on \( K \). Therefore, the function

\[
\phi := \theta + \arcsin(cr^{-1})
\]

is well-defined on \( K \). A direct calculation same as in [9, Sect. 2.2] shows Hess \( \phi(X, X) > 0 \) for every \( X \in TK \) satisfying \( |X| = 1 \) and \( d\phi(X) = 0 \). Thereby Lemma 2.1 in [9] enable us to get \( \lambda \) large enough, such that

\[
F := \lambda^{-1} \exp(\lambda \phi)
\]

is strictly convex on \( K \). Hence we have

**Proposition 2.1** Any simply connected subset of \( S^n \setminus S^{n-2} \) is a convex supporting set of \( S^n \).

Let

\[
\overline{S^n} := \{ (x_1, \ldots, x_{n+1}) \in S^n : x_1 \geq 0, x_2 = 0 \}.
\]
Then \( S^n \setminus \overline{S}^{n-1}_+ \) is obviously a simply connected subset of \( S^n \setminus S^{n-2} \), and the above proposition tells us \( S^n \setminus \overline{S}^{n-1}_+ \) is a convex supporting set. Thereby Proposition 2.1 generalizes the conclusion of Theorem 2.1 in [9]. But it is still unsolved what is the sufficient and necessary condition for a subset of \( S^n \) to be a convex supporting set. To see the relationship between convex supporting sets and closed geodesics, please have a look at Appendix.

Now we assume \( M \) is an \( m \)-dimensional Riemannian manifold, \( V \) is a simply connected subset of \( S^n \setminus S^{n-2} \). If \( u : M \to V \) is a harmonic map, then \( \theta \circ u \) is a smooth function on \( M \). Using the composition formula, we have

\[
\Delta(\theta \circ u) = \text{Hess} \, \theta (u_\ast e_\alpha, u_\ast e_\alpha) + d\theta (\tau (u))
= -2(r^{-1} \circ u) d\theta (u_\ast e_\alpha)
= -2(r^{-1} \circ u) \left[ \nabla (r \circ u), \nabla (\theta \circ u) \right]
\]

(2.7)

where \( \tau \) denotes the tensor field of \( u \), which is identically zero since \( u \) is harmonic, \( \langle , \rangle \) is the Riemannian metric on \( M \) and \( \nabla \) denotes the Levi–Civita connection on \( M \). Here and in the sequel we denote by \( \{ e_1, \ldots, e_m \} \) a local orthonormal frame field on \( M \). We use the summation convention and assume the range of indices

\[ 1 \leq \alpha \leq m. \]

(2.7) is equivalent to

\[
\Delta(\theta \circ u) + 2(r^{-1} \circ u) \left[ \nabla (r \circ u), \nabla (\theta \circ u) \right] = 0
\]

and multiplying both sides by \( r^2 \circ u \) yields

\[
\text{div} ((r^2 \circ u) \nabla (\theta \circ u)) = 0.
\]

(2.8)

Here \( \text{div} \) is the divergence with respect to the metric on \( M \).

Please note that if we weaken the condition on \( u \) by just assuming it is a weakly harmonic map, then a direct computation similar to [8, Sect. 8.5] and [9] shows that \( \theta \circ u \) is a weak solution to the partial differential equation (2.8). More precisely, for an arbitrary smooth function \( \phi \) with compact support in \( M \),

\[
\int_M (r^2 \circ u) \langle \nabla \phi, \nabla (\theta \circ u) \rangle \, d\mu = 0.
\]

(2.9)

(2.8) and (2.9) shall play an important part in the next sections.

**Remark 2.1** Solomon [17] showed that \( S^n \setminus S^{n-2} \) has a so-called warped product structure. More precisely, if we denote

\[
S^n_{+} := \{(y_1, \ldots, y_n) \in S^{n-1} : y_1 > 0\},
\]

then there is a diffeomorphism \( F \) from \( S^n_{+} \times S^1 \) onto \( S^n \setminus S^{n-2} \)

\[
F((y_1, \ldots, y_n), \varphi) = (y_1 \cos \varphi, y_1 \sin \varphi, y_2, \ldots, y_n).
\]

From the viewpoint, \( \varphi \) can be regarded as a smooth \( S^1 \)-valued function on \( S^n \setminus S^{n-2} \), and the level sets of \( \varphi \) are all totally geodesic and orthogonal to \( \frac{\partial}{\partial \varphi} \).

Please note that the function \( \theta \) we have defined can be seen as the lift of the restriction of \( \varphi \) on any simply connected subset of \( S^n \setminus S^{n-2} \). Hence (2.8) can also be derived from Lemma 1 in [17].
3 Harnack’s inequalities for elliptic differential equations

Let \((M, g)\) be a Riemannian manifold, \(A\) be a section of \(T^* M \otimes TM\), such that for every \(y \in M\) and nonzero \(X, Y \in T_y M\),

\[
\langle X, A(Y) \rangle = \langle Y, A(X) \rangle \tag{3.1}
\]

and

\[
\langle X, A(X) \rangle > 0. \tag{3.2}
\]

Then we say \(A\) is symmetric and positive definite, and

\[
\text{div}(A(\nabla f)) = 0 \tag{3.3}
\]

is a partial differential equation of elliptic type.

At first, we assume there is a distance function \(d\) on \(M\), and the metric topology induced by \(d\) is equivalent to the initial topology of \(M\). Moreover, for each \(y_1, y_2 \in M\), \(d(y_1, y_2) \leq \rho(y_1, y_2)\), where \(\rho(\cdot, \cdot)\) is the intrinsic distance induced by the Riemannian metric on \(M\). Obviously, \(\rho\) is one of these functions.

Now we fix \(y_0 \in M\), and let \(B_R\) denote the ball centered at \(y_0\) of radius \(R\) given by the distance function \(d\). We assume every function in \(B_R\) of \(H^1_0\) type is also a \(L^2\)-function with \(v > 1\), and there is a positive constant \(K_1\), such that for every \(r \in [\frac{R}{2}, R]\) and \(v \in H^1_0(B_R)\),

\[
\left( \int_{B_r} |v|^2 \right)^{\frac{1}{2}} \leq K_1 r \left( \int_{B_r} |\nabla v|^2 \right)^{\frac{1}{2}}. \tag{3.4}
\]

This is a Sobolev type inequality. Here \(\int_{\Omega} v\) denotes the average value of \(v\) on an arbitrary domain \(\Omega \subset M\), i.e.

\[
\int_{\Omega} v := \frac{\int_{\Omega} v \ast 1}{\text{Vol}(\Omega)}. \]

Let \(K_2, K_3\) be positive constants, such that

\[
\text{Vol}(B_R) \leq K_2 \text{Vol} \left( B_{\frac{R}{2}} \right), \tag{3.5}
\]

\[
\sup_{v \neq 0, \int_{B_{3R} \setminus \frac{R}{2}} v^2 = 0} \frac{\int_{B_{3R} \setminus \frac{R}{2}} |\nabla v|^2 \ast 1}{\int_{B_{3R} \setminus \frac{R}{2}} |v|^2 \ast 1} \leq K_3 R^2. \tag{3.6}
\]

And

\[
\lambda_1 := \inf_{X \in TB_R, X \neq 0} \frac{\langle X, A(X) \rangle}{\langle X, X \rangle}, \tag{3.7}
\]

\[
\lambda_2 := \sup_{X \in TB_R, X \neq 0} \frac{\langle X, A(X) \rangle}{\langle X, X \rangle}, \tag{3.8}
\]

\[
L := \frac{\lambda_2}{\lambda_1}. \tag{3.9}
\]

By classical spectrum theory of harmonic operators, if we denote by \(\mu_2\) the second eigenvalue of \(\Delta v + \mu v = 0\) in \(B_{3R} \setminus \frac{R}{2}\), where \(v\) has a vanishing normal derivative on the boundary of \(B_{3R} \setminus \frac{R}{2}\).
then the left hand side of (3.6) equals $\mu^{-1}_2$. For this reason, an equivalent form of (3.6)

$$\int_{B_{3R}} |v - \tilde{v}|^2 * 1 \leq K_3 R^2 \int_{B_{3R}} |\nabla v|^2 * 1$$

(3.10)

is called a Neumann–Poincaré inequality in some references, e.g. [4].

Following the idea of [11] and [2], it is not hard for us to derive a version of Harnack’s inequality as follows.

**Proposition 3.1** Let $(M, g)$ be a Riemannian manifold equipped with a distance function $d$ as above. If $f$ is a positive (weak) solution to (3.3) on the metric ball centered at $y_0$ and of radius $R$, then there exists a positive constant $C_0$, only depending on $K_1, K_2, K_3$ and $\nu$, but not on $f, L$ or $R$, such that

$$\log f_+, \frac{R}{2} - \log f_-, \frac{R}{2} \leq C_0 L^{\frac{1}{2}}.$$  

(3.11)

Here and in the sequel,

$$f_+, R := \sup_{B_R} f, \quad f_-, R := \inf_{B_R} f.$$  

(3.12)

**Remark 3.1** For divergence form elliptic partial differential equations in an open domain $\Omega \subset \mathbb{R}^n$, one can deduce Harnack’s inequality (see [11])

$$\sup_{\Omega'} f \leq C(\Omega', \Omega)^{\frac{1}{2}} \inf_{\Omega'} f$$

(3.13)

for any $\Omega' \subset \subset \Omega$. In addition when $\Omega$ is convex one can take

$$C(\Omega', \Omega) = \left( \frac{\text{diam } \Omega}{\text{dist}(\Omega', \partial \Omega)} \right)^\beta$$

(3.14)

with a positive constant $\beta$. Bombieri–Giusti [2] proved a version of Harnack’s inequality for solutions of divergence form elliptic partial differential equations on area-minimizing hypersurfaces in Euclidean spaces. The above proposition is a further generalization. Please note that the example

$$\frac{\partial^2 f}{\partial x^2} + L \frac{\partial^2 f}{\partial y^2} = 0$$

with solution

$$f = \exp \left( L^{\frac{1}{2}} x \right) \cos y$$

shows that the dependence on $L$ in (3.13) cannot be improved.

Now we let $f$ be an arbitrary (weak) $L^\infty$-solution of (3.3) on $B_R$ (not necessarily positive), then $f - f_-, R + \epsilon$ is obviously a positive (weak) $L^\infty$-solution for each $\epsilon > 0$. Applying Proposition 3.1 to $f - f_-, R + \epsilon$ yields

$$\log(f_+, \frac{R}{2} - f_-, R + \epsilon) - \log(f_-, \frac{R}{2} - f_-, R + \epsilon) \leq C_0 L^{\frac{1}{2}},$$

i.e.

$$f_+, \frac{R}{2} - f_-, R + \epsilon \leq \exp(C_0 L^{\frac{1}{2}})(f_-, \frac{R}{2} - f_-, R + \epsilon).$$
Letting $\varepsilon \to 0$ implies
\[ f_+ \frac{\varepsilon}{2} - f_- R \leq \exp(C_0 L^\frac{1}{2}) (f_- \frac{\varepsilon}{2} - f_- R), \]
then
\[ f_+ \frac{\varepsilon}{2} - f_- \frac{\varepsilon}{2} = (f_+ \frac{\varepsilon}{2} - f_- R) - (f_- \frac{\varepsilon}{2} - f_- R) \leq (1 - \exp(-C_0 L^\frac{1}{2})) (f_+ R - f_- R). \]
Thereby we get an estimate for the oscillation of $f$ as follows.

**Corollary 3.1** Our assumption on $M$ is same as in Proposition 3.1. If $f$ is a (weak) $L^\infty$-solution of (3.3) in $B_R$, then the oscillation of $f$ on $B_R$ could be estimated by
\[ \text{osc}_{B_R} f \leq (1 - \exp(-C_0 L^\frac{1}{2})) \text{osc}_{B_R} f \] with a positive constant $C_0$ depending on $K_1, K_2, K_3$ and $\nu$, but not on $L$ and $R$.

**4 Image shrinking property of harmonic maps**

Our notations and assumptions are same as in Sects. 2 and 3. From (2.8) and (3.3), it is easily seen that $\theta \circ u$ satisfies a divergence form elliptic partial differential equation with $A = (r^2 \circ u) \text{Id}$, where $\text{Id}$ denotes the smooth section of $T^*M \otimes TM$ satisfying $\text{Id}(X) = X$ for every $X \in TM$.

Since $r$ is a $(0, 1]$-valued function,
\[ L = \frac{\sup_{B_R} (r^2 \circ u)}{\inf_{B_R} (r^2 \circ u)} \leq \sup_{B_R} (r^{-2} \circ u). \] (4.1)

Now we make an additional assumption on $M$: there exists $R_0 \in (0, +\infty)$, and one can find uniform constants $K_1, K_2, K_3$ and $\nu$ which are all independent of $R \in (0, R_0]$, such that the estimates (3.4), (3.5) and (3.6) hold. As a matter of convenience, we say $M$ satisfies the **local DSVP-condition** with respect to $y_0$ in the sequel.

**Remark 4.1** The local DSVP-condition is comparable with the DVP-condition defined in [9]. From the work of Saloff–Coste [14] and Biroli–Mosco [1], if the doubling property (3.5) holds for arbitrary $y \in M, R \leq R_0$, and the Neumann–Poincaré inequality (3.10) holds for arbitrary $B_{\frac{3R}{2}}(y) \subset M$ with uniform constants $K_2$ and $K_3$, then the Sobolev inequality (3.4) is satisfied whenever $B_{2R}(y) \subset M$ and $R \leq \frac{R_0}{2}$, with constants $\nu$ and $K_1$ depending only on $K_2$ and $K_3$. This means that the DVP-condition implies the local DSVP-condition with respect to every $y \in M$.

**Corollary 3.1** gives
\[ \text{osc}_{B_R} (\theta \circ u) \leq (1 - \exp(-C_0 \sup_{B_R} (r^{-1} \circ u))) \text{osc}_{B_R} (\theta \circ u) \] (4.2)
with a constant $C_0$ independent of $R \leq R_0$. As a matter of convenience, we shall use abbreviations as follows
\[ \Theta = \theta \circ u, \quad M(R) := \sup_{B_R} (r^{-1} \circ u) \ (R \in (0, R_0]) \] (4.3)
in the sequel. Taking logarithms of both sides of (4.2) gives
\[
\log \text{osc}_{B_{\frac{r}{t}}} \Theta - \log \text{osc}_{B_R} \Theta \leq \log \left(1 - \exp(-C_0M(R))\right)
\]  
(4.4)
for arbitrary \( R \leq R_0 \). After iteration we arrive at
\[
\log \text{osc}_{B_{2^{-k_0}}} \Theta - \log \text{osc}_{B_{R_0}} \Theta \leq \sum_{j=0}^{k-1} \log \left(1 - \exp(-C_0M(2^{-j}R_0))\right)
\]  
(4.5)
for every \( k \in \mathbb{Z}^+ \). By additionally defining \( M(R) := M(R_0) \) when \( R \in [R_0, 2R_0] \), \( M(R) \) can be regarded as an increasing function on \((0, 2R_0] \). The right hand of above inequality could be estimated by
\[
\sum_{j=0}^{k-1} \log \left(1 - \exp(-C_0M(2^{-j}R_0))\right)
\leq \int_{-1}^{2R_0} \log \left(1 - \exp(-C_0M(2^{-t}R_0))\right)dt
\leq (\log 2)^{-1} \int_{2^{-k+1}R_0}^{2R_0} R^{-1} \log \left(1 - \exp(-C_0M(R))\right)dR. 
\]  
(4.6)

For arbitrary \( R \leq \frac{R_0}{2} \), there exists \( k \in \mathbb{Z}^+ \), such that \( 2^{-k-1}R_0 < R \leq 2^{-k}R_0 \). Hence one can get an estimate of the oscillation by combining (4.5) and (4.6):
\[
\log \text{osc}_{B_{\frac{r}{t}}} \Theta - \log \text{osc}_{B_{R_0}} \Theta \leq (\log 2)^{-1} \int_{4R}^{2R_0} R^{-1} \log \left(1 - \exp(-C_0M(R))\right)dR. 
\]  
(4.7)

Consider the function \( t \in (0, \exp(-C_0)]) \mapsto \frac{-\log(1-t)}{t} \). Since
\[
\lim_{t \to 0^+} -\frac{\log(1-t)}{t} = \lim_{t \to 0^+} -\frac{[\log(1-t)]'}{t'} = 1,
\]
there is a positive constant \( c_1 \), depending only on \( C_0 \), such that
\[
-\frac{\log(1-t)}{t} \geq c_1 \quad \text{i.e.} \quad \log(1-t) \leq -c_1 t
\]  
(4.8)
for all \( t \in (0, \exp(-C_0)] \). Particularly
\[
\log \left(1 - \exp(-C_0M(R))\right) \leq -c_1 \exp(-C_0M(R)). 
\]  
(4.9)
Substituting (4.9) into (4.7) gives
\[
\log \text{osc}_{B_{\frac{r}{t}}} \Theta - \log \text{osc}_{B_{R_0}} \Theta \leq -(\log 2)^{-1}c_1 \int_{4R}^{2R_0} R^{-1} \exp(-C_0M(R))dR. 
\]  
(4.10)
Again using the monotonicity of $M(R)$ implies
\[
\log \text{osc}_{B_R} \Theta - \log \text{osc}_{B_{R_0}} \Theta \leq - (\log 2)^{-1} c_1 \exp(-C_0 M(R_0)) \int_{4R}^{2R_0} R^{-1} dR
\]
\[
= - (\log 2)^{-1} c_1 \exp(-C_0 M(R_0)) \log \left( \frac{R_0}{2R} \right). \tag{4.11}
\]

From the estimates we can get the so-called image shrinking property of (weakly) harmonic maps.

**Theorem 4.1** Let $(M, g)$ be a Riemannian manifold satisfying the local DSVP-condition with respect to $y_0 \in M$ with constants $R_0 > 0$, $\nu > 1$ and $K_1, K_2, K_3 > 0$, $\forall$ be a simply connected subset of $S^n \setminus S^{n-2}$. If $u : M \to S^n$ is a (weakly) harmonic map, $u(B_{R_0}) \subset K$, where $K$ is a compact subset of $\mathbb{V}$, then there exists positive constants $C_0$ and $C_1$, depending only on $\nu, K_1, K_2, K_3$ and $K$, such that the image of $B_{R_1}$ under $u$ is contained in a closed geodesic ball of radius $\arccos \left( \frac{1}{2} M(R_1)^{-1} \right) < \frac{\pi}{2}$ in $S^n$, where
\[
R_1 := \frac{1}{2} \exp \left( - C_1 \exp(C_0 M(R_0)) \right) R_0. \tag{4.12}
\]

In particular, if $u(B_{R_0}) \subset S^n \setminus S_+^{n-1}$, then our conclusion holds true when we just assume
\[
M(R_0) := \sup_{B_{R_0}} (r^{-1} \circ u) < +\infty.
\]

**Remark 4.2** Theorem 4.1 is comparable with Theorem 5.1 in [9]. Firstly, as shown in Remark 4.1, the local DSVP-condition can be derived from the DVP-condition, so our assumption on $M$ is weaker than that in [9]. Secondly, in [9], the image shrinking property of $u$ requires that $u(B_{R_0})$ is contained in a compact set $K \subset S^n \setminus S_+^{n-1}$, which is a simply connected subset of $S^n \setminus S^{n-2}$, therefore the first statement generalizes the corresponding conclusion in [9]. Finally, if $u(B_{R_0}) \subset S^n \setminus S_+^{n-1}$, then the image shrinking property still holds when just assuming the composition of $r^{-1}$ and $u$ is bounded; on the other hand, $\theta \circ u(y)$ is allowed to approach 0 or 2\pi as $y$ approaches the boundary of $B_{R_0}$. This is an improvement.

**Proof** Since $K$ is a closed subset of $\mathbb{V}$, $\Theta = \theta \circ u$ is a bounded function on $B_{R_0}$. More precisely, there is a positive constant $c_2$ depending only on $K$, such that
\[
\text{osc}_{B_{R_0}} \Theta \leq c_2. \tag{4.13}
\]

The constants $C_0$ and $c_1$ have been given above. Now we put
\[
C_1 := \log 2 \log \left( \frac{3c_2}{2\pi} \right) c_1^{-1} \tag{4.14}
\]
then by (4.12),
\[
R_1 = \frac{1}{2} \exp \left( - \log 2 \log \left( \frac{3c_2}{2\pi} \right) c_1^{-1} \exp(C_0 M(R_0)) \right) R_0 \tag{4.15}
\]
Substituting (4.15) into (4.10) and (4.11) yields
\[
\log \text{osc}_{B_{R_1}} \Theta - \log \text{osc}_{B_{R_0}} \Theta \leq - \log \left( \frac{3c_2}{2\pi} \right).
\]
In conjunction with (4.13) we have \( \text{osc}_{B_{R_1}} \Theta \leq \frac{2\pi}{3} \). It enables us to find \( \theta_0 \in \mathbb{R} \), such that
\[
(\theta \circ u)|_{B_{R_1}} \leq \left[ \theta_0 - \frac{\pi}{3}, \theta_0 + \frac{\pi}{3} \right].
\]
Let \( x_0 := (\cos \theta_0, \sin \theta_0, 0, \ldots, 0) \in S^n \), then for arbitrary \( y \in B_{R_1} \),
\[
(u(y), x_0) = r \circ u(y)(\cos(\theta - \theta_0)) \geq \frac{1}{2} r \circ u(y) \geq \frac{1}{2} M(R_1)^{-1}
\]
which implies \( u(B_{R_1}) \) is contained in the closed geodesic ball centered at \( x_0 \) and of radius \( \arccos \left( \frac{1}{2} M(R_1)^{-1} \right) \).

Noting that condition \( u(B_{R_0}) \subset S^n \setminus S^{n-1}_+ \) implies \( \text{osc}_{B_{R_0}} \theta \leq 2\pi \), we can derive the second statement in the same way. □

Furthermore a regularity theorem for weakly harmonic maps easily follows.

**Theorem 4.2** Let \((M, g)\) be an arbitrary Riemannian manifold, \( u : M \to S^n \) be a weakly harmonic map, and \( V \) be a simply connected subset of \( S^n \setminus S^{n-2} \). Given \( y_0 \in M \), if there is a neighborhood \( U \) of \( y_0 \), such that \( u(U) \subset K \) with \( K \) a compact subset of \( V \), then \( u \) is smooth on a neighborhood of \( y_0 \). In particular, if the image of \( U \) under \( u \) is contained in \( S^n \setminus S^{n-1}_+ \), and
\[
\sup_U (r^{-1} \circ u) < +\infty,
\]
then \( u \) is smooth near \( y_0 \).

**Remark 4.3** The above conclusion is not only an improvement, but also a generalization of the regularity theorem in [9], which says a weakly harmonic map \( u \) into \( S^n \) is smooth near \( y_0 \) if the image of a neighborhood of \( y_0 \) is contained in a compact subset of \( S^n \setminus S^{n-1}_+ \).

**Proof** We just give the proof of the first statement here, because the proof of the second one is quite similar.

By the definition of Riemannian manifolds, each point has a coordinate patch with induced metric. Hence without loss of generality we can assume \( U \) is a Euclidean ball centered at \( y_0 = 0 \) and of radius \( R_0 \) equipped with metric \( g = g^{\alpha \beta} dy^\alpha dy^\beta \), where \((y^1, \ldots, y^m)\) denotes Euclidean coordinate, and there exists two positive constants \( \lambda \) and \( \mu \), such that
\[
\lambda^2 |\xi|^2 \leq g_{\alpha \beta}(y)\xi^\alpha \xi^\beta \leq \mu^2 |\xi|^2 \tag{4.16}
\]
for arbitrary \( y \in U \) and \( \xi \in \mathbb{R}^m \). By a standard scaling argument, we can assume \( \lambda = 1 \) without loss of generality.

Let \( d \) be the canonical Euclidean distance function, i.e.
\[
d : (y_1, y_2) \in U \times U \mapsto |y_1 - y_2|,
\]
then obviously \( d(\cdot, \cdot) \leq \rho(\cdot, \cdot) \), the intrinsic distance induced by \( g \).

Denote \( dy = dy^1 \wedge \cdots \wedge dy^m \), then \( *1 = \sqrt{\det(g_{\alpha \beta})} dy \) with
\[
1 \leq \sqrt{\det(g_{\alpha \beta})} \leq \mu^m. \tag{4.16}
\]
It is well-known that \( \nabla v = g^{\alpha \beta} D^\alpha v D^\beta v \) where \((g^{\alpha \beta})\) is the inverse matrix of \((g_{\alpha \beta})\), hence
\[
|\nabla v|^2 = g^{\alpha \beta} D^\alpha v D^\beta v \geq \mu^{-2} |Dv|^2. \tag{4.17}
\]
Recall that the classical Sobolev inequality says
\[
\left( \int_{B_R} v^{\frac{m}{m-1}} \, dy \right)^{\frac{m-1}{m}} \leq C(m) \int_{B_R} |Dv| \, dy
\]
(4.18)
for arbitrary nonnegative \( C^1 \)-function \( v \) whose supporting set is contained in \( B_R \). Then it could be derived from (4.16), (4.17) and (4.18) that
\[
\left( \int_{B_R} v^{\frac{m}{m-1} \ast 1} \right)^{\frac{m-1}{m}} \leq \left( \mu^m \int_{B_R} v^{\frac{m}{m-1}} \, dy \right)^{\frac{m-1}{m}} \leq \mu^{m-1} C(m) \int_{B_R} |Dv| \, dy \leq \mu^m C(m) \int_{B_R} |\nabla v| \ast 1.
\]

By Hölder’s inequality, it is easily-seen that for arbitrary \( q \geq \frac{m}{m-1} \),
\[
\left( \int_{B_R} v^q \ast 1 \right)^{\frac{1}{q}} \leq \frac{q(m-1)}{m} \mu^m C(m) \left( \int_{B_R} |\nabla v|^{1+\frac{1}{q}} \ast 1 \right)^{\frac{1}{m+\frac{1}{q}}}
\]
and moreover
\[
\left( \int_{B_R} v^q \right)^{\frac{1}{q}} \leq \frac{q(m-1)}{m} \mu^m C(m) V(R) \frac{1}{\pi} \left( \int_{B_R} |\nabla v|^{1+\frac{1}{q}} \ast 1 \right)^{\frac{1}{m+\frac{1}{q}}}
\]
(4.19)
(4.20)
It directly follows from (4.16) that \( V(R) \leq \mu^m \omega_m R^m \), with \( \omega_m \) the volume of \( m \)-dimensional Euclidean disk equipped with the canonical metric. Hence (4.20) enable us to choose
\[
v := \begin{cases} 
4 & \text{if } m = 2 \\
\frac{m}{m-2} & \text{if } m \geq 3
\end{cases}
\]
and
\[
K_1 := \frac{2v(m-1)}{m} \mu^{m+1} \omega_m C(m)
\]
to ensure (3.4) holds true.

By a straightforward calculation similar to [9, Sect. 6.1], one can make sure (3.5) and (3.10) hold true by putting \( K_2 := (2\mu)^m \) and \( K_3 := \frac{9}{4} \pi^{-2} \mu^{m+2} \). Hence Theorem 4.1 enable us to find two constant \( C_0 \) and \( C_1 \), depending only on \( m \), \( \mu \) and \( K \), such that \( u(B_{R_1}) \) is contained in a closed geodesic ball of radius \( < \frac{\pi}{2} \), where
\[
R_1 := \frac{1}{2} \exp \left( -C_1 \exp(C_0 M(R_0)) \right) R_0 \quad \text{with} \quad M(R_0) := \sup_{U} (r^{-1} \circ u).
\]

Now we can proceed as in [7] and [9] to obtain estimates of the oscillation of \( u \) and moreover the Hölder estimates for \( u \), which implies \( u \) is Hölder continuous in a neighborhood of \( y_0 \). Finally \( u \) has to be smooth near \( y_0 \) by the higher regularity results for harmonic maps. \( \square \)
5 Curvature estimates for minimal hypersurfaces

Let $M^m$ be an orientable minimal hypersurface (not necessarily complete) embedded in $\mathbb{R}^{m+1}$. Denote the restriction of Euclidean distance function on $M$ by $d$:

$$(y_1, y_2) \in M \times M \mapsto |y_1 - y_2|.$$ 

Then it is easily-seen that $d(y_1, y_2) \leq \rho(y_1, y_2)$, which are the extrinsic and intrinsic distance, respectively. Since the inclusion map $i : M \rightarrow \mathbb{R}^{m+1}$ is injective, the metric topology induced by $d$ coincides with initial topology of $M$.

Fix $y_0 \in M$, denote by $B_R$ the intersection of $M$ and the Euclidean ball centered at $y_0$ and of radius $R$, which is also a metric ball given by $d$. As shown in [10], for every nonnegative function $v$ of $C^1$-type which vanishes outside a compact subset of $B_R$, the following Sobolev inequality

$$\left( \int_{B_R} v \frac{m}{m-1} \right)^{\frac{m-1}{m}} \leq C(m) \int_{B_R} |\nabla v| \frac{1}{m}$$

holds. Then as in the proof of Theorem 4.2, one can arrive at

$$\left( \int_{B_R} v^{\frac{m}{m-1}} \right)^{\frac{1}{m}} \leq \frac{q(m-1)}{m} C(m) V(R) \left( \int_{B_R} |\nabla v|^{\frac{1}{m-1}} \right)^{\frac{1}{m-1}} + \frac{1}{m-1} + \frac{1}{q}$$

for arbitrary $q \geq \frac{m}{m-1}$. Here and in the sequel, $V(R) := \text{Vol}(B_R)$.

Given $y \in M$ and $R > 0$, the volume density is defined by

$$D(y, R) := \frac{V(y, R)}{\omega_m R^m}$$

The well-known monotonicity theorem tells us $D(y, R)$ is nondecreasing in $R$ and $\lim_{R \to 0^+} D(y, R) = 1$. Thus for an arbitrary constant $R_0 > 0$,

$$\omega_m R^m \leq V(R) \leq D(R_0) \omega_m R^m$$

for all $R \in (0, R_0]$, where $D(R_0)$ is the abbreviation of $D(y_0, R_0)$.

By (5.2) and (5.4), if we take

$$v := \begin{cases} 4 & \text{if } m = 2 \\ \frac{m}{m-2} & \text{if } m \geq 3 \end{cases}$$

and

$$K_1 := \frac{2v(m-1)}{m} D(R_0) \frac{1}{\omega_m} C(m),$$

then (3.4) holds for every $v \in H^1(B_R)$ with $R \in (0, R_0]$. (5.4) also implies the so called doubling property that

$$V(R) \leq 2^m D(R_0) V\left( \frac{R}{2} \right)$$

for all $R \in (0, R_0]$. In other words, we can choose $K_2 := 2^m D(R_0)$ so that (3.5) holds.
Denote by $\mu_2(R)$ the second eigenvalue of $\Delta v + \mu v = 0$ in $B_R$, where the normal derivative of $v$ vanishes on the boundary of $B_R$. Then $\mu_2(R)$ is obviously continuous in $R$. We claim
\[
\sup_{R \in [0, R_0]} R^{-2} \mu_2(R)^{-1} < +\infty. \tag{5.6}
\]
To prove it, it is sufficient to show
\[
\limsup_{R \to 0^+} R^{-2} \mu_2(R)^{-1} < +\infty. \tag{5.7}
\]
After choosing a suitable coordinate, we can assume $y_0 = 0$ and $T_{y_0} M$ is orthogonal to the $(m + 1)$-th coordinate vector without loss of generality. The classical implicit function theorem implies the existence of a sufficiently small number $R^- \leq R_0$, such that $B_{R^-} = \{ y = (z, f(z)) : z \in \Omega \}$

with a star-like domain $\Omega$ in $\mathbb{R}^m$ and a function $f : \Omega \to \mathbb{R}$ satisfying $f(0) = 0, |Df|(0) = 0, |f(z)| < 1$ and $|Df(z)| < 1$ for every $z \in \Omega$. Therefore $B_{R^-}$ is diffeomorphic to a Euclidean ball of radius $R^-$, and the diffeomorphism is given by
\[
\chi : y = (z, f(z)) \mapsto \left| \frac{y}{|z|} \right| z,
\]
which maps $B_R$ onto a Euclidean ball of radius $R$ for each $R \leq R^-$. Hence the canonical Neumann–Poincaré inequality on Euclidean spaces implies
\[
\int_{B_R} |v - \bar{v}|^2 \ast 1 \leq CR^2 \int_{B_R} |\nabla v|^2 \ast 1 \tag{5.8}
\]
for each $R \leq R^-$ and every function $v$ on $B_R$ of $H^{1,2}$-type, where $C$ is a positive constant which depends only on $R^-, m$ and $M$. Choosing $v$ to be an eigenfunction corresponding to the eigenvalue $\mu_2(R)$ implies (5.7), and our claim (5.6) is proved. Now we denote
\[
\Lambda(R_0) := \sup_{R \in (0, R_0]} R^{-2} \mu_2(R)^{-1}
\]
and take $K_3 := \frac{9}{16} \Lambda(R_0)$, then (3.6) holds for each $R \leq R_0$.

Since $M$ is orientable, there exists a smooth unit normal vector field $v$ on $M$. The Gauss map $\gamma : M \to S^m$ is defined by
\[
\gamma(y) = v(y) \tag{5.10}
\]
via parallel translation in $\mathbb{R}^m$. Ruh–Vilms [13] proved that $M$ has constant mean curvature if and only if $\gamma$ is a harmonic map. Suppose the image of $B_{R_0}$ under the Gauss map is contained in $S^m \setminus S^m_+$, and
\[
M(R_0) := \sup_{B_{R_0}} (r^{-1} \circ \gamma) < +\infty. \tag{5.11}
\]
The image shrinking property of harmonic maps (Theorem 4.1) allows us to find $x_0 \in S^m$ and two positive constants $C_2$ and $C_3$ depending only on $m, D(R_0)$ and $\Lambda(R_0)$, such that
\[
(\gamma(y), x_0) \geq \frac{1}{2} M(R_1)^{-1} \tag{5.12}
\]
for every $y \in B_{R_1}$, where
\[
R_1 := \frac{1}{2} \exp \left( - C_3 \exp(C_2 M(R_0)) \right) R_0. \tag{5.13}
\]
Denote
\[ f = (\gamma(\cdot), x_0) \] (5.14)
then \( f \) is a positive function on \( B_{R_1} \). Since \( \gamma \) is a harmonic map into a Euclidean sphere,
\[ \Delta f = -f|d\gamma|^2. \] (5.15)
Denote by \( B \) the second fundamental form of \( M \) in \( \mathbb{R}^{m+1} \). As shown in [18, Sect. 3.1], the energy density of the Gauss map satisfies
\[ E(\gamma) = \frac{1}{2}|d\gamma|^2 = \frac{1}{2}|B|^2. \] (5.16)
Substituting (5.16) into (5.15) yields
\[ \Delta f = -|B|^2 f. \] (5.17)
Let
\[ h := f^{-1}, \] (5.18)
then from (5.17) we arrive at
\[ \Delta h = |B|^2 h + 2h^{-1}|\nabla h|^2. \] (5.19)
The following Simons’ identity [16] is well-known
\[ \Delta |B|^2 = -2|B|^4 + 2|\nabla B|^2. \] (5.20)
With the aid of Codazzi equations, Schoen–Simon–Yau [15] get a Kato-type inequality as follows
\[ |\nabla B|^2 \geq \left(1 + \frac{2}{m}\right)|\nabla |B||^2. \] (5.21)
And it follows from (5.20) and (5.21) that
\[ \Delta |B|^2 \geq -2|B|^4 + 2 \left(1 + \frac{2}{m}\right)|\nabla |B||^2. \] (5.22)
Based on (5.19) and (5.22), \( \Delta(|B|^p h^q) \) can be easily estimated for arbitrary \( p, q > 0 \); by choosing suitable \( p, q \), one can proceeded as in [5] to get
\[ \Delta(|B|^p h^q) \geq 0 \] (5.23)
for any \( p \geq \frac{m-2}{2} \). The mean value inequality on minimal submanifolds (see [3,12]) can be applied to get
\[ |B|^p h^q(y_0) \leq C(m)V(R)^{-\frac{1}{2}} \left(\int_{B_R} |B|^{2p} h^{2p} \right)^{\frac{1}{2}} \] (5.24)
for every \( R \leq R_1 \). Again using the inequality for \( \Delta(|B|^p h^q) \), one can get the following estimate as in [5]:
\[ \int_{B_{R_1}} |B|^{2p} h^{2p} \eta^{2p} \ast 1 \leq C(p) \int_{B_{R_1}} h^{2p} |\nabla \eta|^{2p} \ast 1. \] (5.25)
Here $p \geq \max\{3, m - 1\}$ and $\eta$ can be taken to be any smooth function which vanishes outside a compact subset of $B_{R_1}$. Now we choose $\eta$ to be standard cut-off function satisfying $\text{supp } \eta \subset B_{R_1}$, $\eta \equiv 1$ on $B_{\frac{R_1}{2}}$ and $|\nabla \eta| \leq c_0 R_1^{-1}$, and we get

$$\int_{B_{\frac{R_1}{2}}} |B|^{2p} h^{2p} * 1 \leq C(p) c_0^p R_1^{-2p} V(R_1) \sup_{B_{R_1}} h^{2p}. \quad (5.26)$$

Substituting (5.26) into (5.24) implies

$$|B|^p h^p(y_0) \leq C(m, p) \left( \frac{V(R_1)}{V(R_2)} \right)^{\frac{1}{2}} R_1^{-p} \sup_{B_{R_1}} h^p. \quad (5.27)$$

Combining with (5.27), (5.5), (5.13) and (5.12), we obtain an a priori curvature estimate as follows:

**Theorem 5.1** Let $M^m$ be an orientable minimal hypersurface embedded in $\mathbb{R}^{m+1}$, $y_0 \in M$ and

$$B_R := \{ y \in M : |y - y_0| < R \}.$$

If there is a positive number $R_0$, such that the Gauss image of $B_{R_0}$ is contained in $S^m \setminus S^m_{+}$, and

$$\sup_{B_{R_0}} (r^{-1} \circ \gamma) < +\infty,$$

then

$$|B|(y_0) \leq C_4 R_0^{-1} \exp \left( C_3 \exp(C_2 \sup_{B_{R_0}} (r^{-1} \circ \gamma)) \right). \quad (5.28)$$

Here $C_2, C_3, C_4$ are positive constants only depending on $m$, $D(R_0) := \frac{V(R_0)}{\omega_m R_0^m}$ and $\Lambda(R_0) := \sup_{R \in (0, R_0]} R^{-2} \mu_2(R)^{-1}$.

**Remark 5.1** If the Gauss image of $B_{R_0}$ is contained in $K \subset \mathbb{S}$, where $\mathbb{S}$ is a simply connected subset of $S^m \setminus S^m_{+}$, then $\sup_{B_{R_0}} (r^{-1} \circ \gamma) < +\infty$. From the image shrinking property, we can proceed as above and get

$$|B|(y_0) \leq C_5 R_0^{-1}. \quad (5.29)$$

with a positive constant $C_5$ depending only on $m, D(R_0), \Lambda(R_0)$ and $K$.

Now we additionally assume $M$ is complete and the Gauss image of $M$ is contained in $S^m \setminus S^m_{+}$. If there exists $y_1 \in M$ and a positive constant $C$ such that

$$D(y_1, R) \leq C$$

for every $R < +\infty$, then we say $M$ has Euclidean volume growth. For arbitrary $y \in M$, if we denote $d := d(y, y_1)$, then

$$V(y, R) \leq V(y_1, R + d) \leq C \omega_m (R + d)^m$$

and moreover

$$D(y, R) \leq C \left( \frac{R + d}{R} \right)^m.$$
Letting $R \to +\infty$ implies $\lim_{R \to +\infty} D(y, R) \leq C$, then the monotonicity theorem tells us
\begin{equation}
D(y, R) \leq C
\end{equation}
for every $y \in M$ and $R < +\infty$. Using the image shrinking property and above curvature estimates, one can get a Bernstein type theorem as follows.

**Theorem 5.2** Let $M^m$ be an orientable complete minimal hypersurface embedded in $\mathbb{R}^{m+1}$. Assume $M$ has Euclidean volume growth, and the Gauss image of $M$ is contained in $S^m \setminus S_{+}^{m-1}$. If there is $y_0 \in M$, such that $\lim_{R \to +\infty} \Lambda(R) < +\infty$, and
\begin{equation}
\sup_{B_R}(r^{-1} \circ \gamma) = O(\log \log R).
\end{equation}
Then $M$ has to be an affine linear subspace.

**Proof** Denote $\Theta := \theta \circ \gamma$ and $M(R) := \sup_{B_R}(r^{-1} \circ \gamma)$. The Ruh–Vilms theorem implies $\gamma$ is a harmonic map. For every $R \in (0, +\infty)$, Theorem 4.1 enable us to find $\theta_0(R) \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ and two positive constants $C_0$ and $C_1$ depending only on $m$, $\lim_{R \to +\infty} D(R)$ and $\lim_{R \to +\infty} \Lambda(R)$, such that
\begin{equation}
|\Theta(y) - \theta_0(R)| \leq \frac{\pi}{3}
\end{equation}
for every $y \in B_{R'}$, where
\begin{equation}
R' := \frac{1}{2} \exp (-C_1 \exp(C_0 M(R))) R.
\end{equation}

By the compactness of $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$, there is an monotonically increasing sequence $\{R_j : j \in \mathbb{Z}^+\}$ satisfying $\lim_{j \to \infty} R_j = +\infty$ and $\lim_{j \to \infty} \theta_0(R_j) = \theta_\infty \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$. Denote
\begin{equation}
R'_j := \frac{1}{2} \exp (-C_1 \exp(C_0 M(R_j))) R_j.
\end{equation}
(5.32) implies for arbitrary $\varepsilon > 0$, there exists a positive integer $k$, such that for every $j \geq k$, $M(R_j) \leq \varepsilon \log \log R_j$, hence
\begin{equation}
R'_j \geq \frac{R_j}{2 \exp \left(C_1 (\log R_j)^{C_{0}\varepsilon}\right)}.
\end{equation}

When $\varepsilon$ is sufficiently small and $R_j$ is sufficiently large, one have $C_1 (\log R_j)^{C_{0}\varepsilon} \leq \frac{1}{2} \log R_j$, which implies
\begin{equation}
R'_j \geq \frac{1}{2} R_j^{\frac{1}{2}}
\end{equation}
and hence $\lim_{j \to \infty} R'_j = +\infty$.

Hence for arbitrary $y \in M$, we can find $l \in \mathbb{Z}^+$, such that $d(y_0, y) \leq R'_j$ whenever $j \geq l$. (5.33) tells us
\begin{equation}
|\Theta(y) - \theta_0(R_j)| \leq \frac{\pi}{3}.
\end{equation}
Letting $j \to \infty$ in above inequality we arrive at
\begin{equation}
|\Theta(y) - \theta_\infty| \leq \frac{\pi}{3}
\end{equation}
\end{document}
for every \( y \in M \). This implies the Gauss image of \( M \) is contained in an open hemisphere centered at \( x_0 := (\cos \theta_\infty, \sin \theta_\infty, 0, \ldots, 0) \in S^m \).

Let \( h := (\gamma(\cdot), x_0)^{-1} \), then for arbitrary \( y \in M \), similarly to above we can arrive at the following estimate

\[
|B|^p h^p(y) \leq C(m, p) \left( \frac{V(R)}{V(R_\frac{1}{2})} \right)^{\frac{1}{p}} R^{-p} \sup_{B_R(y)} h^p
\]

\[
\leq c_3 R^{-p} \sup_{B_{R+\delta}(y)} (r^{-1} \circ \gamma)^p;
\]

i.e.

\[
|B|(y) \leq c_3 R^{-1} \sup_{B_{R+d}(y)} (r^{-1} \circ \gamma) \tag{5.37}
\]

with \( d := d(y, y_0) \) and \( c_3 \) a positive constant depending only on \( m, p \) and \( \lim_{R \to +\infty} D(R) \).

Letting \( R \to +\infty \) forces \( |B|(y) = 0 \). Therefore \( M \) has to be flat.

**Remark 5.2** \( \lim_{R \to +\infty} \Lambda(R) < +\infty \) is equivalent to saying that the Neumann–Poincaré inequality

\[
\int_{B_R(y_0)} |v - \tilde{v}_R|^2 * 1 \leq C R^2 \int_{B_R(y)} |\nabla v|^2 * 1 \quad \forall v \in C^\infty(B_R(y_0)) \tag{5.38}
\]

holds for any \( R \in (0, \infty) \) with a positive constant \( C \).

**Remark 5.3** In Theorem 5.2, if we replace the condition on Gauss image of \( M \) by the assumption that \( \gamma(M) \subset K \subset \nabla \) with \( \nabla \) a simply connected subset of \( S^m \setminus S^{m-2} \). Then again based on image shrinking property we can get the corresponding Bernstein type result.

Especially if \( M \) is area-minimizing, the Neumann–Poincaré inequality (5.38) holds for every \( y_0 \in M \) and \( R > 0 \) with a uniform positive constant \( C \) only depending on \( m \) (see [2]). Moreover, for the volume of extrinsic balls we have (see [2])

\[
\text{Vol}(B_R(y)) \leq \frac{m + 1}{2} \omega_{m+1} R^m \quad \forall y \in M. \tag{5.39}
\]

Therefore \( M \) satisfies local DSVP-condition with respect to every \( y \in M \) with constants \( K_1, K_2, K_3 \) and \( v \) which all depend only on \( m \) and furthermore (4.10) holds with positive constants \( C_0 \) and \( c_1 \) only depending on \( m \). Starting from (4.10) one can derive another Bernstein type theorem as follows.

**Theorem 5.3** There exists a positive constant \( \varepsilon \) depending only on \( m \), such that the following result holds: Let \( M^m \) be a complete area-minimizing hypersurface embedded in \( \mathbb{R}^{m+1} \), whose Gauss image is contained in \( S^m \setminus S^{m-1} \), and

\[
\sup_{B_R(y_0)} (r^{-1} \circ \gamma) \leq \varepsilon(m) \log \log R \tag{5.40}
\]

for every \( R \) no less than a positive constant \( R_- \), then \( M \) has to be an affine linear space.

**Proof** The meanings of \( C_0 \) and \( c_1 \) are as above. Let

\[
\varepsilon = C_0^{-1}, \tag{5.41}
\]

\( \square \)
then
\[
\int_{R_-}^{+\infty} R^{-1} \exp(-C_0 M(R))dR \geq \int_{R_-}^{+\infty} R^{-1} (\log R)^{-C_0 \varepsilon} dR
\]
\[
= \int_{R_-}^{+\infty} (\log R)^{-1} d\log R
\]
\[
= \log \log R|_{R_-}^{+\infty} = +\infty. \tag{5.42}
\]
By (4.10) and (5.42), for arbitrary \( R \geq R_- \), one can take \( R_0 \) large enough, such that
\[
\log \text{osc}_{B_R}(\Theta) - \log \text{osc}_{B_{R_0}}(\Theta) \leq -\log 3.
\]
In conjunction with \( \text{osc}_{B_{R_0}}(\Theta) \leq 2\pi \), we can find \( \theta_0(R) \in \left[ \frac{\pi}{3}, \frac{5\pi}{3} \right] \), such that
\[
\Theta|_{B_R} \in \left[ \theta_0(R) - \frac{\pi}{3}, \theta_0(R) + \frac{\pi}{3} \right].
\]
The compactness of \( \left[ \frac{\pi}{3}, \frac{5\pi}{3} \right] \) enable us to find a strictly increasing sequence \( \{ R_j : j \in \mathbb{Z}^+ \} \) tending to \(+\infty\) and satisfying \( \lim_{j \to \infty} \theta_0(R_j) = \theta_{\infty} \in \left[ \frac{\pi}{3}, \frac{5\pi}{3} \right] \). Similarly to above we can derive \( |\Theta(y) - \theta_{\infty}| \leq \frac{\pi}{3} \) for every \( y \in M \). It follows that the Gauss image of \( M \) is contained in a closed subset of open hemisphere. Finally Ecker–Huisken’s estimates [5] implies \( M \) has to be affine linear.

Remark 5.4 Similarly, if the Gauss image of \( M \) is contained in \( K \subset \mathbb{V} \) with \( \mathbb{V} \) a simply connected subset of \( S^m \setminus S^{m-2} \), then our conclusion still holds true. Thereby we not only improve, but also generalize the results of Theorem 6.6 in [9].

Corollary 5.1 There is a positive constant \( \delta \) depending only on \( m \), such that the following result holds: Let \( f \) be an entire solution of the minimal surface equation
\[
\sum_{i=1}^{m} D^i \left( \frac{D^i f}{\sqrt{1 + |Df|^2}} \right) = 0 \tag{5.43}
\]
in \( \mathbb{R}^m \) with \( f(0) = 0 \). if
\[
\left( \sum_{i=1}^{m-1} (D^i f)^2 \right)^{\frac{1}{2}} \leq \delta \log \log \left( f^2 + x^2 \right)^{\frac{1}{2}} \tag{5.44}
\]
holds for every \( x \in \mathbb{R}^m \) satisfying \( |x| \geq R_- \), where \( R_- \) is a positive constant, then \( f \) has be to be affine linear.

Proof Under the assumptions, \( M := \{(x, f(x)) : x \in \mathbb{R}^m\} \) is an entire minimal graph, which is an area-minimizing hypersurface embedded in \( \mathbb{R}^{m+1} \) (see [19, Sect. 6.2] ). For every \( x \in \mathbb{R}^m \),
\[
\gamma(x, f(x)) = (1 + |Df|^2)^{-\frac{1}{2}}(-D^1 f, -D^2 f, \ldots, -D^m f, 1)
\]
with \( \gamma \) the Gauss map on \( M \). Thus the Gauss image of \( M \) is contained in the upper hemisphere. Now we define \( \pi : S^m \to \mathbb{B} \) by
\[
(x_1, \ldots, x_{m+1}) \to (x_m, x_{m+1}).
\]
Then
\[ r^{-2} \circ \gamma = \frac{1 + |Df|^2}{1 + (Dm f)^2} = 1 + \frac{\sum_{i=1}^{n-1} (D^i f)^2}{1 + (Dm f)^2} \leq 1 + \sum_{i=1}^{n-1} (D^i f)^2. \quad (5.45) \]

\( f(0) = 0 \) implies \( 0 \in M \). Take \( y_0 := 0 \), then for any \( y := (x, f(x)) \in M \), \( y \in B_R(y_0) \) if and only if \( (f^2 + x^2)^{\frac{1}{2}} < R \). Now we put \( \delta(m) := \frac{1}{\epsilon(m)} \), where the definition of \( \epsilon(m) \) is same as in Theorem 5.3, then (5.45) and (5.44) imply (5.40) when \( R \) is large enough. And the conclusion immediately follows from Theorem 5.3. \( \square \)

**Remark 5.5** By Corollary 5.1, if \( D^1 f, \ldots, D^m f \) are uniformly bounded in \( \mathbb{R}^m \), then \( f \) is affine linear. So Corollary 5.1 is an improvement of Theorem 8 in [2]. It is also comparable with Ecker–Huisken’s Bernstein type theorem [5].

### 6 Appendix

As shown in [6], any harmonic map from a compact Riemannian manifold into a convex supporting set \( V \) has to be constant. Especially, an arbitrary closed geodesic in \( V \) can be viewed as the image of a harmonic map from \( S^1 \) into \( V \), hence every convex supporting set cannot contain any closed geodesic.

Conversely, if a subset \( V \) of a Riemannian manifold \((M, g)\) contains no closed geodesic, does \( V \) have to be convex supporting? Unfortunately the answer is ‘no’. The following is a counterexample. Let \( M =: S^2, S^1 \) be the equator and

\[ A := \left\{ (\cos \theta, \sin \theta, 0) \in S^1 : \theta \in \left[ 0, \frac{\pi}{3} \right] \cup \left[ \frac{2\pi}{3}, \pi \right] \cup \left[ \frac{4\pi}{3}, \frac{5\pi}{3} \right] \right\}. \]

Noting that an arbitrary great circle \( C \) intersects \( S^1 \) at least two antipodal points, we have \( C \cap A \neq \emptyset \) and hence \( V := S^2 \setminus A \) contains no closed geodesic. On the other hand, we show \( V \) is not a convex supporting set. Let \( R_{\frac{\pi}{2}} \) be \( \frac{\pi}{2} \)-rotation around \( x_3 \)-axis. Choose a compact subset \( K \), such that \( N, S \in K \) and \( R_{\frac{\pi}{2}}(K) = K \) (if not, replace \( K \) instead of \( K \cup R_{\frac{\pi}{2}}(K) \cup R_{\frac{3\pi}{2}}(K) \)). Assume \( f \) is a \( C^2 \)-convex function on \( K \), then \( f \circ R_{\frac{\pi}{2}} \) and \( f \circ R_{\frac{3\pi}{2}} \) are also convex, hence \( h := f + f \circ R_{\frac{\pi}{2}} + f \circ R_{\frac{3\pi}{2}} \) is a convex function on \( K \) which is invariant under \( R_{\frac{\pi}{2}} \). Since \( N \) and \( S \) are fixed points under \( R_{\frac{\pi}{2}} \), we have

\[ (R_{\frac{\pi}{2}})_* (\nabla h) = \nabla h \]

at \( N \) and \( S \). Thus \( \nabla h(N) = \nabla h(S) = 0 \), which means \( h \) has two critical points in \( K \), and causes a contradiction to the convexity of \( h \). Therefore \( V \) cannot be a convex supporting set.

Moreover, \( S^n \setminus S^{n-1}_+ \) is the unique maximal convex supporting set of \( S^n \) that contains the upper hemisphere and the lower hemisphere.

**Proposition 6.1** Let \( V \) be an open and connected convex supporting set of \( S^n \), if \( S^n_+ \cup S^n_- \subset V \), then \( V \subset S^n \setminus S^{n-1}_+ \).

**Proof** Let \( \{e_1, \ldots, e_{n+1}\} \) be an orthonormal basis of \( \mathbb{R}^{n+1} \), such that

\[ S^n_+ = \{ x \in S^n : (x, e_1) > 0 \}, \quad S^n_- = \{ x \in S^n : (x, e_1) < 0 \}. \]
By the definition of convex supporting sets, for an arbitrary compact subset $K \subset \mathbb{V}$, we can find a strictly convex function $f$ on $K$. Now we choose a family of compact sets $\{K_i \subset \mathbb{V}: i = 1, 2, \ldots\}$, such that $K_i \subset K_j$ for arbitrary $i < j$, $\mathbb{V} = \bigcup_{i=1}^{\infty} K_i$, and each $K_i$ satisfies the following two conditions: (I) $K_i$ is invariant under the reflection with respect to the hyperplane $(\cdot, e_1) = 0$; (II) for any $x \in K_i$ satisfying $(x, e_1) = 0$, the geodesic from $e_1$ to $-e_1$ which goes through $x$ is contained in $K_i$. We denote by $f_i$ the convex function on $K_i$.

Now we denote by $\psi$ the reflection with respect to the hyperplane $(\cdot, e_1) = 0$, then obviously $\psi$ is an isometry and hence $f_i \circ \psi$ is also strictly convex. Let

$$h_i := \frac{1}{2} (f_i + f_i \circ \psi),$$

then $h_i$ is a strictly convex function which is invariant under $\psi$, in particular $h_i(e_1) = h_i(-e_1)$.

If $\nabla h_i = 0$ at $e_1$, then $h_i \circ \psi = h_i$ implies $\nabla h_i = 0$ at $-e_1$, which means that $h_i$ has 2 critical points in $K$ and causes a contradiction to the convexity of $h$. Hence $\nabla h_i \neq 0$. Denote $v_i := \nabla h_i$. Now we claim

$$H_i := \{ x \in S^n : (x, e_1) = 0, (x, v_i) > 0 \}$$

satisfies $H_i \cap K_i = \emptyset$. We prove it by Reductio ad absurdum. Assume $x \in H_i \cap K_i$, then by Condition (II) there is a geodesic $\gamma$ lying in $K$ which connects $e_1$ and $-e_1$ and goes through $x \in K_i \cap H_i$. Hence $(\dot{\gamma}, v_i) > 0$ and moreover $\frac{d}{dt} \big|_{t=0} (h_i \circ \gamma) > 0$. The convexity of $h_i$ implies $h_i \circ \gamma$ is a strictly increasing function, which contradict to $h_i(e_1) = h_i(-e_1)$.

The compactness of $\{v \in T_{e_1} S^n : |v| = 1\}$ enable us to find a subsequence of $\{v_i : i \in \mathbb{Z}^+\}$ converging to a unit vector in $T_{e_1} S^n$. Without loss of generality one can assume

$$\lim_{i \to \infty} v_i = e_2.$$

Denote

$$S_{n-1}^\pm := \{ x \in S^n : (x, e_1) = 0, (x, e_2) > 0 \}.$$

Then for every $x \in S_{n-1}^\pm$, we can find $k \in \mathbb{Z}^+$, such that $x \in H_i$ for every $i \geq k$. Therefore $x \not\in K_i$ and furthermore $x \not\in \bigcup_{i=k}^{\infty} K_i = \mathbb{V}$. Thus $S_{n-1}^\pm \cap \mathbb{V} = \emptyset$, and the conclusion immediately follows. \qed

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