Propagating Property of a Second-order Noncanonical Optical Vortex Beam in a Strongly Focusing System

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Abstract. The noncanonical optical vortex is an optical vortex with a nonconstant phase gradient around its center, i.e. the phase is not a linear function of the azimuthal angle. The expression of the strongly focused field of a (conventional) second order noncanonical vortex beam has been derived analytically and the field distribution is discussed. It has been found that the intensity distribution at the focal plane can exhibit rich patterns, which is more complicated than that of the first-order. The transverse focal shift phenomenon in current case also exits and is changed with the semi-aperture angle $\alpha$ in a different way. It also can be seen that the intensity maxima can be one to four on the focal plane. Our research may provide a new method for controlling of the structured optical field, and will give theoretical supports for the study of higher-order noncanonical optical vortices.

1. Introduction

An optical vortex beam refers to a type of beam with orbital angular momentum (OAM). In the center of the beam, there exists a phase singularity and the optical intensity is zero [1-3]. Since they can carry any mode of OAM theoretically and have shown their unique properties, optical vortices have attracted a lot of attention from researchers, and have been applied in a wide range of research areas, such as in optical microscope [4], optical micromanipulation [5-7], quantum Communication [8-10] and other aspects. However, most researches focus on canonical vortices [11-21](i.e. the canonical vortices are commonly called as optical vortices), and there are a few of researches on noncanonical vortices. The noncanonical optical vortices are also the optical vortices with a helical wavefront, but the phase gradient is not a constant, which is different from the canonical vortices [22-24]. The propagation properties of optical vortex in a high numerical aperture system are always interesting in many researches, for their important role in exploring new optical structures and behaviours, thus in this article we will look at the the propagation properties of a noncanonical vortex beam in a strongly focusing system.

The traditional ‘focal shift’ is due to the low Fresnel coefficient in the focusing system, which causes the maximum field intensity to appear near the lens and deviate from the geometrical focus. This is mainly refers to the longitudinal focal shift [25]. The transverse focal shift (TFS) phenomenon refers to the transverse shift of the intensity peak on the focal plane, which can be defined as the distance between the maximum intensity point of the field and the geometrical focus [26-29]. Recently, Li Jinhong and others have studied the strongly focusing properties of the first-order noncanonical vortex beams and have observed the TFS in that field. In this article, the second-order noncanonical vortex will...
be analysed and as we will see the TFS also can be observed and the interesting propagation properties of this higher order noncanonical vortex beam will be shown.

2. Theoretical Derivation

Firstly, let us consider a second-order noncanonical optical vortex, with its complex amplitude at the transverse plane can be expressed as

\[ V_\lambda(x, y) = \exp\left[-\left(x^2 + y^2\right)/w_0^2\right](x + icy)^2, \]  

(1)

where \((x, y)\) are the Cartesian coordinates, \(w_0\) is the beam waist, \(c\) is the phase distribution factor and determines the noncanonical intensity of the optical vortex, which is usually called the ‘anisotropy parameter’. When \(c\) is a real number, this expression is a second-order noncanonical optical vortex. If \(c = 1\), this vortex degenerates into a second-order canonical vortex, i.e. the traditional optical vortex. From the analysis of Eq. (1), if \(|c| < 1\), the phase changes slowly near the \(x\)-axis, and the intensity maximum is also obtained on the \(x\)-axis. If \(|c| > 1\), the phase of the wave changes slowly near the \(y\)-axis, and the intensity maximum is also obtained on the \(y\)-axis. Figure 1 shows the optical field intensity and phase distribution of the second-order noncanonical optical vortex on the transverse plane with different anisotropy parameter.

Assuming that this second-order noncanonical optical vortex is linearly polarized at the \(x\) direction, and is incident on a strongly focusing system with a focal length of \(f\) and a semi-aperture angle of \(\alpha\) (as shown in figure 2), where the geometrical focus is taken at the coordinate origin. According to the Richards-Wolf vectorial diffraction theory, the electric field \(E(\rho, \phi, z)\) at an observation point in the focal field can be expressed as

\[
E(\rho, \phi, z) = \sum_{s} E_s = -\frac{ik}{2\pi} \int_0^\alpha \int_0^{2\pi} fV_\lambda(\theta, \phi)\sqrt{\cos\theta}\sin\theta \times \left[ \cos^2\theta + \sin^2\theta(1-\cos\theta) \right] \exp(-kz, \cos\theta) \exp\left[ik\rho, \sin\theta \cos(\phi - \phi')\right] \mathrm{d}\phi \mathrm{d}\theta.
\]

(2)

\(\alpha = 30^\circ\).
Figure 2. Illustration of a strongly focusing system.

Where \( (\rho_s, z_s, \phi_s) \) is the cylindrical coordinates in the focusing region, \( k \) is the wave number and \( k = 2\pi / \lambda \) (\( \lambda \) is the wavelength of the free space). By the Abbe sine condition, namely \( r = f \sin \theta \), the equation (1) can be transformed into

\[
V_\lambda (r, \phi) = \exp \left[ -\left( f \sin \theta \right)^2 / \alpha_0^2 \right] (r \cos \phi + i \epsilon r \sin \phi)^2, \tag{3}
\]

Substituting (3) into (2) and integrating \( \phi \) in the equation (2), we can get

\[
e_s(\rho_s, z_s, \phi_s) = -ik \int_0^\alpha P(\theta_s) I_s(\theta_s, \rho_s, \phi_s) \exp(ikz_s \cos \theta) d\theta, \tag{4}
\]

\[
e_x(\rho_s, z_s, \phi_s) = -ik \int_0^\alpha P(\theta_s) I_x(\theta_s, \rho_s, \phi_s) \exp(ikz_s \cos \theta) d\theta, \tag{5}
\]

\[
e_y(\rho_s, z_s, \phi_s) = -ik \int_0^\alpha P(\theta_s) I_y(\theta_s, \rho_s, \phi_s) \exp(ikz_s cos \theta) d\theta, \tag{6}
\]

\[
P(\theta) = (f \sin \theta)^3 \exp \left[ -\left( f \sin \theta \right)^2 / \alpha_0^2 \right] \sqrt{\cos \theta}, \tag{7}
\]

\[
I_x(\theta, \rho_s, \phi_s) = I_{x0}(\theta, \rho_s, \phi_s) + I_{x2}(\theta, \rho_s, \phi_s) + I_{x4}(\theta, \rho_s, \phi_s), \tag{8}
\]

\[
I_x(\theta, \rho_s, \phi_s) = I_{x0}(\theta, \rho_s, \phi_s) + I_{x2}(\theta, \rho_s, \phi_s) + I_{x4}(\theta, \rho_s, \phi_s), \tag{9}
\]

\[
I_y(\theta, \rho_s, \phi_s) = I_{y0}(\theta, \rho_s, \phi_s) + I_{y2}(\theta, \rho_s, \phi_s) + I_{y4}(\theta, \rho_s, \phi_s), \tag{10}
\]

In the equation (7) - (10):

\[
I_{x0}(\theta, \rho_s, \phi_s) = \frac{1}{8} \left[ (3 - c^2) \cos \theta + 1 - 3c^2 \right] J_0 (k \rho_s \sin \theta), \tag{11}
\]

\[
I_{x2}(\theta, \rho_s, \phi_s) = -\frac{1}{4} \left[ (\cos \theta + c)(1 + c) \exp(i2\phi_s) + (c - 1)(c - \cos \theta) \exp(-i2\phi_s) \right] J_2 (k \rho_s \sin \theta), \tag{12}
\]

\[
I_{x4}(\theta, \rho_s, \phi_s) = -\frac{1}{16} \left[ (1 - \cos \theta) \left[ (1 + c)^2 \exp(i4\phi_s) + (c - 1)^2 \exp(-i4\phi_s) \right] J_4 (k \rho_s \sin \theta), \tag{13}
\]

\[
I_{y0}(\theta, \rho_s, \phi_s) = -\frac{1}{4} ic(1 - \cos \theta) J_0 (k \rho_s \sin \theta), \tag{14}
\]

\[
I_{y2}(\theta, \rho_s, \phi_s) = -\frac{1}{4} (c^2 - 1)(1 - \cos \theta) \sin 2\phi_s J_2 (k \rho_s \sin \theta), \tag{15}
\]
\[ I_{x4}(\theta; \rho_0, \phi_0) = \frac{1}{16} i (1 - \cos \theta) \left[ (1 + c)^2 \exp(i4\phi_0) - (c - 1)^2 \exp(-i4\phi_0) \right] J_4(k\rho_0 \sin \theta), \quad (16) \]
\[ I_{y4}(\theta; \rho_0, \phi_0) = -\frac{1}{8} i \sin \theta \left[ (3 - c^2 + 2c) \exp(i\phi_0) + (3 - c^2 - 2c) \exp(-i\phi_0) \right] J_4(k\rho_0 \sin \theta), \quad (17) \]
\[ I_{z4}(\theta; \rho_0, \phi_0) = \frac{1}{8} i \sin \theta \left[ (c + 1)^2 \exp(i3\phi_0) + (c - 1)^2 \exp(-i3\phi_0) \right] J_4(k\rho_0 \sin \theta), \quad (18) \]

where \( J_n \) is the first kind Bessel function of order \( n \). From equations (4)~(18), one can see that the three field components of the electric field in the focal region generated by the second-order noncanonical optical vortex are not only affected by the semi-aperture angle \( \alpha \) of the system, but also by the anisotropy parameter \( c \).

3. Results and Discussion

Based on the equations derived above, the properties of the second-order noncanonical optical vortex in the strongly focusing system are discussed in this section.

Firstly, we observe the distribution of the optical field along the propagation axis. From equations (4)~(18), we can see that when \( \rho_0 = 0 \), only \( I_{x0} \) and \( I_{y0} \) are not zero, that means the optical field has no longitudinal component along the propagation axis, that is, there is no \( z \) component, which is the same as the case when the canonical second-order optical vortex is strongly focused, i.e., the optical field along the propagation axis only has the transverse field components.

Secondly, let us discuss the total optical field intensity distribution on the focal plane. Figure 3 shows the total optical field intensity and the phase distribution of the \( e_x \) component of the focal plane when \( \alpha = 30^\circ \) with different anisotropy parameter \( c \). From this figure, one can see that the intensity distribution of the second-order noncanonical vortex wave with linear polarization is more complicated than that of the first-order. Through observing the phase plots of the \( e_x \) component in Figure 3, we can see that there are two main vortex points (et. singularities) in the field component. When \( c < 1 \), these two singularities are symmetrically distributed on the \( x \)-axis; when \( y < 1 \), the two singularities are symmetrically distributed on the \( y \)-axis, which has a strong influence on the optical intensity distribution. At the same time, it can be observed that with the change of \( c \), the field on the focal plane can have a transverse focal shift, and the number of the intensity maxima can be one, two, three and four on the focal plane, which is quite different from the intensity distribution when the incident wave is first-
order noncanonical vortex (one or two maximum points for the first order), that will be analyzed in detail below.

![Figure 4](image)

**Figure 4.** Optical field intensity distribution for different parameters ($\alpha = 30^\circ, 80^\circ$) on the focal plane.

Here the influence of the semi-aperture angle $\alpha$ and the anisotropy parameter $c$ on the field intensity distribution on the focal plane are discussed. Figure 4 shows the total field intensity distribution on the focal plane for selected value of the semi-aperture angle, here $\alpha$ is chosen as $30^\circ$ and $80^\circ$, respectively. From the figure 4, we can see that those different values of $\alpha$ will significantly affect the distribution of the optical field intensity in the focal plane. Firstly, as $\alpha$ increases, the maximum intensity point will be closer to the central axis; secondly, when $\alpha$ and $c$ are small, the intensity maximum is first concentrated at the focal point, and as $c$ increases, the intensity maxima will spread to the $x_y$-axis, and then the maximum points will rotate to the $y_x$-axis, and finally return to the focus, that is the TFS phenomenon of the second-order vortex wave in the strongly focusing field. While for large value of $\alpha$ (such as $\alpha = 80^\circ$), this rule is just the opposite. In order to observe this phenomenon more clearly, Figure 5 shows the variation of the position of the intensity maxima on the focal plane for different values of $c$ and $\alpha$ (here since the symmetry of the intensity maxima, only one of the maxima points at one transverse plane is shown).

![Figure 5](image)

**Figure 5.** The variation of the intensity maximum points of the optical field on the focal plane with $c$ ($\alpha = 30^\circ, 60^\circ, 80^\circ$).

Due to the complexity of the second-order noncanonical vortex, the variation of the intensity maxima on the focal plane with $c$ is different from that in the first-order case. Note that at some values of $c$ in figure 5 there is no value, that is because at these points there are three or four intensity maxima. Figure 5 shows that when $\alpha$ is small, the intensity maxima follow the rule: first at the origin, then spread to the...
After that rotate to the \( y \)-axis, and finally return to the origin, which is the same as in the strongly focused first-order noncanonical beam. But as \( \alpha \) becomes larger, the difference appears. From the figure 5, it can be seen that when \( \alpha = 60^\circ \), the intensity maxima do not spread from the focus to the \( x \)-axis, but directly spread to the \( y \)-axis and then return to the origin; while when \( \alpha = 80^\circ \), the intensity maxima first spread from the origin to the \( y \)-axis, then rotate to the \( x \)-axis and return to the origin.

From figure 4 and figure 5, the following conclusions can be drawn:

1) The TFS will occur for a second-order noncanonical vortex beam in a high numerical aperture system. It is essentially caused by the anisotropy parameter of the noncanonical vortices.

2) The semi-aperture angle \( \alpha \) and the anisotropy parameter \( c \) affect the TFS in different ways, and by comparing with the first-order noncanonical vortex wave, the second-order vortex wave has more complicated TFS properties.

![Figure 6. Distribution of total field intensity on the focal plane (\( \alpha = 30^\circ, 45^\circ \)).](image)

![Figure 7. Distribution of total field intensity along the propagation direction (\( \alpha = 30^\circ, 45^\circ, 60^\circ \)).](image)

From figure 6, we can see that the number of intensity maxima for the second-order noncanonical vortex wave on the focal plane can be one, two, three, or even four. This is another characteristic of the second-order noncanonical vortex wave by strongly focusing, which is different from the case of first-order noncanonical vortex wave. For the first-order case, the field intensity can have one or two maximum point/points on the focal plane. This difference is mainly caused by the non-generic property of the higher-order noncanonical vortex and the effect of phase singularities on the intensity distribution.

Finally, we will look at the optical field distribution along the propagation direction. Figure 7 shows the intensity distribution of the total optical field along the propagation direction for the second-order noncanonical optical vortex in the focal region, here \( c = 0.6, \alpha = 30^\circ, 40^\circ, 80^\circ \). From this, we can observe
that regardless of the value of the semi-aperture angle $\alpha$, the optical field intensity distribution along the propagation direction will show obvious rotations. When $\alpha = 30^\circ$, the two maxima are symmetrically distributed on the $x$, -axis on the focal plane, and on the transverse planes with $z_c = \pm \lambda, \pm 2\lambda$, the two maxima deviate and are no longer distributed on the $x$, -axis, but rotate with the propagation; when $\alpha = 45^\circ$, there are four maxima appear on the focal plane, which are symmetrically distributed along the $x$, -axis and $y$, -axis respectively, while as $z_c$ changes, the off-axis rotation phenomenon also appears; when $\alpha = 60^\circ$, the two intensity maxima are symmetrically distributed on the $y$, -axis. Similarly, for $z_c = \pm \lambda, \pm 2\lambda$, the two maxima are no longer distributed on the $y$, -axis, but rotate off-axis with the beam propagation.

4. Conclusion
This article mainly studies the propagation properties of the second-order, linearly polarized noncanonical vortex beam in a high numerical aperture system, and discusses the effects of the semi-aperture angle $\alpha$, the anisotropy parameter $c$ on the optical field intensity distribution. The main results of this article can be summarized as follows: 1) When the incident wave is a second-order noncanonical vortex wave with linear polarization, the optical field in the focal region has no longitudinal component along the propagation axis, which is the same as that for the corresponding canonical second-order vortex wave. 2) The TFS also can be observed for the second-order noncanonical vortex wave, which can be controlled the semi-aperture angle $\alpha$ and the anisotropy parameter $c$. The behaviours of the TFS for the second-order noncanonical vortex is quite different from that of the first-order noncanonical vortex. 3) The number of the intensity maxima on the focal plane can be not only one or two, but also three or four, which is distinguishing characteristic of the focused field for the second-order noncanonical vortex. 4) The intensity pattern of the focused optical field for the second-order noncanonical vortex will rotate as the beam propagates. When the anisotropy parameter $c$ is fixed, the semi-aperture angle $\alpha$ can affect the field distribution in a nontrivial way. The results of this article will provide new methods for controlling of the three-dimensional optical field, and may give theoretical support for the application of high-order noncanonical vortices and the exploration of other types of noncanonical vortices.

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