NON-DESTRUCTIVE SAMPLE GENERATION FROM CONDITIONAL BELIEF FUNCTIONS

This paper presents a new approach to generate samples from conditional belief functions for a restricted but non trivial subset of conditional belief functions. It assumes the factorization (decomposition) of a belief function along a bayesian network structure. It applies general conditional belief functions.

1. THE PROBLEM

It is commonly acknowledged that we need to accept and handle uncertainty when reasoning with real world data. The most profoundly studied measure of uncertainty is the probability. There exist methods of so-called graphoidal representation of joint probability distribution - called Bayesian networks [7] - allowing for expression of qualitative independence, causality, efficient reasoning, explanation, learning from data and sample generation. However, the general feeling is that probability cannot express all types of uncertainty, including vagueness and incompleteness of knowledge. The Mathematical Theory of Evidence or the Dempster-Shafer Theory (DST) [8] has been intensely investigated in the past as a means of expressing incomplete knowledge. The interesting property in this context is that DST formally fits into the framework of graphoidal structures [9] which implies possibilities of efficient reasoning by local computations in large multivariate belief distributions given a factorization of the belief distribution into low dimensional component conditional belief

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* Institute of Computer Science, Polish Academy of Sciences, PL 01-237 Warsaw, ul. Ordona 21, klopotek@ipipan.waw.pl
functions. This in turn qualifies DST for usage in expert systems dealing with uncertainty as there exist efficient reasoning algorithms. But the concept of conditional belief functions is generally not usable for sample generation because composition of conditional belief functions is not granted to yield joint multivariate belief distribution, as some values of the belief distribution may turn out to be negative [2, 9]. Let us illustrate the problem with Bayesian networks in Fig.1a) and b). Table a) below gives marginal distribution of $X_1$ in Fig.1a,b), table b) - conditional distributions in Fig.1a), table c) - conditionals in Fig.1b).

In Fig.1a) $m_{X_1} \oplus m_{X_2|X_1} \oplus m_{X_3|X_2}$ and in Fig.1b) $m_{X_1} \oplus m_{X_2|X_1} \oplus m_{X_3|X_1} \oplus m_{X_4|X_3}$ are proper belief functions (with non-negative values of $m$). But in
Fig.1a) the function $m = m_{X_1} \oplus m_{X_2|X_1} \oplus m_{X_3|X_2} \oplus m_{X_4|X_3}$ is not a proper belief function, as visible in the table below:

|    | X1  | X2  | X3  | X4  | m              |
|----|-----|-----|-----|-----|----------------|
|    | ...... | ...... |     |     | 9.40444e-05    |
| {a} × {b} × {a} × {a} | {a} × {b} × {a} × {b} | {a} × {b} × {a} × {a,b} | {a} × {b} × {a,b} × {a} | {a} × {b} × {a,b} × {b} | -2.91556e-05 | -3.82222e-05 |

Also in the Fig.2b) the function $m = m_{X_1} \oplus m_{X_2|X_1} \oplus m_{X_3|X_2} \oplus m_{X_4|X_3} \oplus m_{X_5|X_1}$ is not a proper belief function as visible in the table below:

|    | X1  | X2  | X3  | X4  | X5  | m              |
|----|-----|-----|-----|-----|-----|----------------|
|    | ...... | ...... | ...... | ...... |     | -0.000107315   |
| {a} × {b} × {b} × {b} × {b} × {a,b} × {a} | {a} × {b} × {b} × {a,b} × {b} | 0.0022038 | -0.000107315 | ...... | ...... |

Hence, in DST, sample generation from a network and therefore the development of learning algorithms identifying graphoidal structure from data, understanding of causality and of mechanisms giving rise to belief distributions is hampered. E.g. beside [3], the known sample generation algorithms [1, 4, 5, 6, 11] do not use conditional belief functions and therefore (1) conditional independence between variables cannot be pre-specified for the sample and (2) a single generator pass may fail to generate a single sample element.

### 2. THE SOLUTION

In our solution to the problem of sample generation from conditional belief functions below we impose the restriction that in the bayesian network no two parents of a node are directly connected.

The fundamental idea behind the approach is to replace the conditional belief function with a specially defined conditional probability function while splitting some values of variables into subvalues. These subvalues take care of differences between belief function values between subsets and supersets of elementary values of variables. The proper generation of samples is run with these special conditional probability functions in a very traditional way, and after completion of sample generation the split values are again joined.
The main difficulties we encounter with handling conditional belief functions is that the conditional independence in DST is radically different from probabilistic independence and that the conditional mass functions \( m \) take negative values.

To overcome negativeness, we assume that the conditional belief functions are represented in terms of so-called \( K \) functions as introduced in [3]. Given that \( X \) is the set of all variables in the conditional belief function and \( q \) the set of conditioning variables, we have:

\[
K_q(A) = \sum_{B; A^q \subseteq B^q, A^{X-q} = B^{X-q}} m(B)
\]

For example, given \( m \) in table (a) below, we get \( K \) in table (b) below:

| \( X_1 \) | \( X_2 \) | \( m \) | \( X_1 \) | \( X_2 \) | \( K \) |
|---|---|---|---|---|---|
| \{a\} \times \{a\} | 0.166667 |
| \{a\} \times \{b\} | -0.0833333 |
| \{a\} \times \{a,b\} | -0.0833333 |
| \{b\} \times \{a\} | -0.0833333 |
| \{b\} \times \{b\} | 0.166667 |
| \{a,b\} \times \{a\} | 0.3 |
| \{a,b\} \times \{b\} | 0.3 |

\( K \)-function is nonnegative. For any level of conditioning variables the conditioned variables form a probability distribution.

Now we extend the set of values of every variable. If the set \( S \) is a set of values of an attribute, then we define the function \( MY() \) as \( MY(S) = S \) and \( SU() \) as \( SU(S) = \emptyset \). \( S \) is a V-expression. For any V-expression \( V \) for any proper non-empty subset \( s \subset MY(V) \) we define V-expressions \( s \odot V \) and \( s @ V \) and define functions \( MY(s \odot V) = MY(s @ V) = s, \) \( SU(s \odot V) = SU(s @ V) = V \). The only element of the set \( \{S\}^n \) is a V(n)-expression. \( MY(S^n) = S \) and \( SU(S^n) = \emptyset \). For any V-expression \( V \) for any proper non-empty subset \( s \subset MY(V) \), V(n)-expressions are elements of the set: \( V_n = \{s \odot V, s @ V\}^n \) and for every \( v_n \in V_n \) \( MY(v_n) = s, SU(v_n) = V \). Thus each V(n)-expression is a vector of \( n \) V-expressions.

Let \( X_j \) be a node in the belief network with \( n \) successors and let \( \pi(X_j) \) be the set of its predecessors in the network. Let \( K_{X_j|\pi(X_j)} \) be the \( K \)-function associated with this node. We transform it into a conditional probability function by replacing \( X_j \) with \( X'_j \) taking its values from the set of V(n)-expressions over
the set of values of \(X_j\), and every variable \(X_i \in \pi(X_j)\) is replaced with \(X_i^{\prime\prime}\) taking its values from the set of V-expressions over the set of values of \(X_j\). 

\(P(x_j' | x_i', \ldots, x_i')\) is calculated as follows:

1. If \(SU(x_i') = \ldots = SU(x_i') = \emptyset\) then for any subset of values \(s\) from the domain of \(X_j\): 
   \[
   \sum_{x_j': MY(x_j') = s} P(x_j' | x_i', \ldots, x_i') = K_{X_j | \pi(X_j)}(x_j' | x_i', \ldots, x_i').
   \]

2. If \(SU(x_j') \neq \emptyset\) then 
   \[
   P(x_j' | x_i', \ldots, x_i') = P(SU(x_j') | x_i', \ldots, x_i').
   \]

3. If \(x_i^{\prime\prime} = MY(x_i^{\prime\prime}) \cap SU(x_i^{\prime\prime})\) then
   \[
   P(x_j' | x_i', \ldots, x_i^{\prime\prime}, \ldots, x_i') = P(x_j' | x_i', \ldots, SU(x_i^{\prime\prime}), \ldots, x_i').
   \]

4. If \(x_i^{\prime\prime} = MY(x_i^{\prime\prime}) \cap SU(x_i^{\prime\prime})\) let \(x_i^*\) denote either \(x_i^{\prime\prime}\) or \(SU(x_i^{\prime\prime})\) and otherwise let \(x_i^*\) denote only \(x_i^{\prime\prime}\). If \(x_i^{\prime\prime} = MY(x_i^{\prime\prime}) \cap SU(x_i^{\prime\prime})\) let \(x_i^+\) denote either \(MY(x_i^{\prime\prime})\) and otherwise let \(x_i^+\) denote only \(x_i^{\prime\prime}\). Then
   \[
   P(x_j' | x_i^+, \ldots, x_i^+) = \text{average}_{x_i^+, \ldots, x_i^+}(P(x_j' | x_i^+, \ldots, x_i^+)).
   \]

Obviously \(P(x_j' | x_i', \ldots, x_i')\) has to be non-negative everywhere.

If \(X_j\) is a parent of another node in the network on the \(h\)th outgoing edge, then the respective \(x_j^{\prime\prime}\) acts as the \(h\)th element of the vector \(x_j'\).

With such a transformed probability distribution we generate the sample and then replace all the V- and V(n) expressions \(V\) with \(MY(V)\).

If \(X_2\) has a single successor and \(K_{X_2 | X_1}\) is of the form

| X1    | X2    | K   |
|-------|-------|-----|
| \{a\} | \{a\} | 0.516667 |
| \{a\} | \{b\} | 0.266667 |
| \{a\} | \{a,b\} | 0.216667 |
| \{b\} | \{a\} | 0.266667 |
| \{b\} | \{b\} | 0.516667 |

then the above rules lead to \(P(X_2' | X_1')\) of the form

| X1    | X2    | K   |
|-------|-------|-----|
| \{b\} | \{a,b\} | 0.216667 |
| \{a,b\} | \{a\} | 0.35 |
| \{a,b\} | \{b\} | 0.35 |
| \{a,b\} | \{a,b\} | 0.3 |
To verify the above sample generation algorithm, a program has been implemented allowing to generate the sample from conditional belief functions and to test DST conditional independence properties of the sample. The independence test is based on a previously elaborated layered independence test [2]. The PC algorithm of Spirtes/Glymour/Scheines [10] has been successfully tested for multivariate belief distributions for samples generated by our approach. Fig.1c) represents one of the networks recovered.

**REFERENCES**

[1] KAEMPKE T., *About assessing and evaluating uncertain information within the theory of evidence*. Decision Support Systems 4:433-439, 1988

[2] KLOPOTEK M.A., MATUSZEWSKI A., WIERZCHOŃ S.T., *Overcoming negative-valued conditional belief functions when adapting traditional knowledge acquisition tools to Dempster-Shafer Theory*. Proc. CESA’96 IMACS Multiconference (Conference on Expert System Applications), Lille-France, 9-12 July 1996, Vol.2, pp.948-953.

[3] KLOPOTEK M.A.: *Methods of Identification and Interpretations of Belief Distributions in the Dempster-Shafer Theory* (in Polish). Institute of Computer Science, Polish Academy of Sciences, Warsaw, Poland, 1998, ISBN 83-900820-8-x.

[4] KREINOVICH V. et al., *Monte-Carlo methods make Dempster-Shafer formalism feasible*. In: Advances in the Dempster-Shafer Theory of Evidence R.Yager, M. Fedrizzi, J. Kacprzyk: (eds) John Willey, New York, 175-191, 1994.

[5] MORAL S, WILSON N., *Markov-chain Monte-Carlo algorithm for the calculation of the Dempster-Shafer belief*. Proc. 12th Nt. Conf. On AI (AAAI-94) 269-274, 1994.
[6] MORAL S, WILSON N., Importance sampling Monte-Carlo algorithm for the calculation of Dempster-Shafer belief. Proc. IPMU’96, Granada 1-5.7.1996, Vol. III, 1337-1344

[7] PEARL J., Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. Morgan Kaufmann, San Mateo CA, 1988.

[8] SHAFER G., A Mathematical Theory of Evidence. Princeton University Press, 1976

[9] SHENOY P.P., Conditional Independence in Valuation-based Systems. International Journal of Approximate Reasoning, 10, 203-234, 1994.

[10] SPIRTES P., GLYMOUR C., SCHEINES R., Causation, Prediction and Search. Lecture Notes in Statistics 81, Springer-Verlag, 1993.

[11] WILSON N., A Monte Carlo algorithm for Dempster-Shafer belief. Proc. 7th Conf. On Uncertainty in AI, B.D’Ambrossio, P. Smets, AP. Bonisome (Eds.), Morgen-Kaufmann, 414-417, 1991