A Remarkable Property of the Dynamic Optimization Extremals*

Delfim F. M. Torres
delfim@mat.ua.pt

R&D Unit Mathematics and Applications
Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal

Abstract

We give conditions under which a function $F(t, x, u, \psi_0, \psi)$ satisfies the relation $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x}$ along the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of an optimal control problem, where $H$ is the corresponding Hamiltonian. The relation generalizes the well known fact that the equality $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ holds along the extremals of the problem, and that in the autonomous case $H \equiv constant$. As applications of the new relation, methods for obtaining conserved quantities along the Pontryagin extremals and for characterizing problems possessing given constants of the motion are obtained.

Keywords: dynamic optimization, optimal control, Pontryagin extremals, constants of the motion.

1 Introduction

A dynamic optimization continuous problem poses the question of what is the optimal magnitude of the choice variables, at each point of time, in a given interval. To tackle such problems, three major approaches are available: dynamic programming; the calculus of variations; and the powerful

*Presented at the contributed session Optimal Control and Calculus of Variations of the 4th International Optimization Conference in Portugal, Optimization 2001, Aveiro, July 23–25, 2001. Accepted for publication in the journal Investigação Operacional, Vol. 22, Nr. 2, 2002, pp. 253–263.
and insightful optimal control. The calculus of variations is a classical subject, born in 1696 with the brachistochrone problem, whose field of applicability is broadened with optimal control theory. Dynamic programming is based on the solution of a partial differential equation, known as the Hamilton-Jacobi-Bellman equation, in order to compute a value function. Dynamic programming is well designed to deal with optimization problems in discrete time. All these techniques are well known in the literature of operations research (see e.g. [3, 4, 31]), systems theory (see e.g. [13]), economics (see e.g. [8, 19] and [22, Capítulo 14]) and management sciences (see e.g. [12])). Here, we are concerned with the methods and procedures of optimal control. This approach allows the effective study of many optimization problems arising in such fields as engineering, astronautics, mathematics, physics, economics, business management and operations research, due to its ability to deal with restrictions on the variables and nonsmooth functions (see e.g. [12, 17, 20, 27]).

At the core of optimal control theory is the Pontryagin maximum principle – the celebrated first order necessary optimality condition – whose solutions are called (Pontryagin) extremals and which are obtained through a function $H$ called Hamiltonian, akin to the Lagrangian function used in ordinary calculus optimization problems (see e.g. [21, 27]). For autonomous problems of optimal control, i.e. when the Hamiltonian $H$ does not depend explicitly on time $t$, a basic property of the Pontryagin extremals is the remarkable feature that the corresponding Hamiltonian is constant along the extremals (see e.g. [23, 16]). In classical mechanics this property corresponds to energy conservation (see e.g. [18, 24]), while in the calculus of variations it corresponds to the second Erdmann necessary optimality condition (see e.g. [1]). For problems of optimal control that depend upon time $t$ explicitly (non-autonomous problems), the property amounts to the fact that the total derivative with respect to time of the corresponding Hamiltonian equals the partial derivative of the Hamiltonian with respect to time:

$$\frac{dH}{dt}(t, x(t), u(t), \psi_0, \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi_0, \psi(t))$$

for almost all $t$ (see e.g. [23, 2, 14]). This corresponds to the DuBois-Reymond necessary condition of the calculus of variations (see e.g. [4]). Recent applications, in many different contexts of the calculus of variations and optimal control, show the fundamental nature of the property (1). It has been used in [11, 4, 23] to establish Lipschitzian regularity of minimizers; in [10] to establish some existence results; and in [23, 30] to prove some generalizations of first Noether’s theorem. The techniques used in the proof of
the relation are also very useful, and have been applied in contexts far away from dynamic optimization (see e.g. [15]). In this note we give conditions under which a function $F(t, x, u, \psi_0, \psi)$ satisfies the equality
\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x},
\]
almost everywhere, along the Pontryagin extremals. For $F = H$ equality (2) reduces to (1). As a corollary, we obtain a necessary and sufficient condition for $F(t, x, u, \psi_0, \psi)$ to be a constant of the motion. From it, one is able to find constants of the motion that depend on the control and that are not momentum maps, that is, one can find preserved quantities $F(t, x(t), u(t), \psi_0, \psi(t))$ along the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem, which are not of the form $\psi(t) \cdot C(x(t))$. This is in contrast with the results obtained in [5], where the conserved quantities are always of the form $\psi(t) \cdot C(x(t))$. Our condition provides also a method for the characterization of optimal control problems with given constants of the motion. All these possibilities are illustrated with examples.

2 Preliminaries

Without loss of generality (see e.g. [2]), we will be considering the optimal control problems in Lagrange form with fixed initial time $a$ and fixed terminal time $b$ ($a < b$).

2.1 Formulation of the Optimal Control Problem

The problem consists of minimize a cost functional of the form
\[
J[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) \, dt,
\]
called the performance index, among all the solutions of the vector differential equation
\[
\dot{x}(t) = \varphi(t, x(t), u(t)) \quad \text{for a.a. } t \in [a, b].
\]
The state trajectory $x(\cdot)$ is a $n$-vector absolutely continuous function
\[
x(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n);
\]
and the control $u(\cdot)$ is a $r$-vector measurable and bounded function satisfying the control constraint $u(t) \in \Omega$,
\[
u(\cdot) \in L_\infty([a, b]; \Omega).
\]
The set $\Omega \subseteq \mathbb{R}^r$ is called the control set. In general, the problem may include some boundary conditions and state constrains, but they are not relevant for the present study: the results obtained are independent of those restrictions. We assume the functions $L : [a, b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ and $\varphi : [a, b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ to be continuous on $[a, b] \times \mathbb{R}^n \times \Omega$ and to have continuous derivatives with respect to $t$ and $x$.

2.2 The Pontryagin Maximum Principle

We shall now formulate the celebrated Pontryagin maximum principle [23], which is a first-order necessary optimality condition. The maximum principle provides a generalization of the classical calculus of variations first-order necessary optimality conditions and can treat problems in which upper and lower bounds are imposed on the control variables – a possibility of considerable interest in operations research (see [12]).

**Theorem 1 (Pontryagin maximum principle).** Let $(x(\cdot), u(\cdot))$ be a minimizer of the optimal control problem. Then, there exists a nonzero pair $(\psi_0, \psi(\cdot))$, where $\psi_0 \leq 0$ is a constant and $\psi(\cdot)$ a $n$-vector absolutely continuous function with domain $[a, b]$, such that the following hold for almost all $t$ on the interval $[a, b]$:

(i) the Hamiltonian system

\[
\begin{align*}
    \dot{x}(t) &= \frac{\partial H(t, x(t), u(t), \psi_0, \psi(t))}{\partial \psi}, \\
    \dot{\psi}(t) &= -\frac{\partial H(t, x(t), u(t), \psi_0, \psi(t))}{\partial x};
\end{align*}
\]

(ii) the maximality condition

\[H(t, x(t), u(t), \psi_0, \psi(t)) = \max_{v \in \Omega} H(t, x(t), v, \psi_0, \psi(t));\]

with the Hamiltonian $H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u)$.

**Definition 1.** A quadruple $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ satisfying the Hamiltonian system and the maximality condition is called a (Pontryagin) extremal.

**Remark 1.** Different terminology for the function $H$ can be found in the literature. The Hamiltonian $H$ is sometimes called “unmaximized Hamiltonian”, “pseudo-Hamiltonian” or “Pontryagin function”.

4
Remark 2. Transversality conditions may also appear in the Pontryagin maximum principle. These conditions depend on the specific boundary conditions under consideration. Our methods do not require the use of such transversality conditions and the results obtained are, as already mentioned, valid for arbitrary boundary conditions.

Remark 3. The maximality condition is a static optimization problem. The method of solving the optimal control problem (3)–(4) via the maximum principle consists of finding the solutions of the Hamiltonian system by the elimination of the control with the aid of the maximality condition. The required optimal solutions are found among these extremals.

The proof of the following theorem can be found, for example, in [23, 2].

Theorem 2. If \((x(\cdot), u(\cdot), \psi_0, \psi(\cdot))\) is a Pontryagin extremal, then the function \(H(t, x(t), u(t), \psi_0, \psi(t))\) is an absolutely continuous function of \(t\) and satisfies the equality (4), where on the left-hand side we have the total derivative with respect to \(t\), and on the right-hand side the partial derivative of the Hamiltonian with respect to \(t\).

As a particular case of Theorem 2, when the Hamiltonian does not depend explicitly on \(t\), that is when the optimal control problem is autonomous – functions \(L\) and \(\varphi\) do not depend on \(t\) – then the value of the Hamiltonian evaluated along an arbitrary Pontryagin extremal \((x(\cdot), u(\cdot), \psi_0, \psi(\cdot))\) of the problem turns out to be constant:

\[
H(x(t), u(t), \psi_0, \psi(t)) \equiv \text{const}, \quad t \in [a, b] .
\]

We remark that Theorem 2 is a consequence of the Pontryagin maximum principle. We shall generalize Theorem 2 in Section 3. Before, we review some facts from functional analysis needed in the proof of our result.

2.3 Facts from Functional Analysis

First we introduce the concept of an absolutely continuous function in \(t\) uniformly with respect to \(s\).

Definition 2. Let \(\phi(s, t)\) be a real valued function defined on \([a, b] \times [a, b]\). The function \(\phi(s, t)\) is said to be an absolutely continuous function in \(t\) uniformly with respect to \(s\) if, given \(\varepsilon > 0\), there exists \(\delta > 0\), independent of \(s\), such that for every finite collection of disjoint intervals \((a_j, b_j) \subseteq [a, b]\)

\[
\sum_j (b_j - a_j) \leq \delta \Rightarrow \sum_j |\phi(s, b_j) - \phi(s, a_j)| \leq \varepsilon \quad (s \in [a, b]) .
\]
The proof of the following two propositions can be found in [14, p. 74].

**Proposition 3.** Let \( F(t,x,u,\psi_0,\psi) \), \( F : [a,b] \times \mathbb{R}^n \times \Omega \times \mathbb{R}_0^- \times \mathbb{R}^n \to \mathbb{R} \), be continuously differentiable with respect to \( t, x, \psi \) for \( u \) fixed, and assume that there exists a function \( G(\cdot) \in L_1([a,b]; \mathbb{R}) \) such that

\[
\left\| \nabla_{(t,x,\psi)} F(t,x(t),u(s),\psi_0,\psi(t)) \right\| \leq G(t) \quad (s,t \in [a,b]).
\]

Then \( \phi(s,t) = F(t,x(t),u(s),\psi_0,\psi(t)) \) is absolutely continuous in \( t \) uniformly with respect to \( s \) on \([a,b]\).

**Proposition 4.** Let \( \phi(s,t), \phi : [a,b] \times [a,b] \to \mathbb{R} \), be an absolutely continuous function in \( t \) uniformly with respect to \( s \) satisfying

\[
\phi(t,t) = \max_{s \in [a,b]} \phi(s,t)
\]

in a set dense in \([a,b]\). Then the function \( \phi(t,t) \) can be uniquely extended to a function \( m(t) \) absolutely continuous on \([a,b]\).

### 3 Main Result

Our result is a generalization of the Theorem 3.

**Theorem 5.** If \( F(t,x,u,\psi_0,\psi) \) is a real valued function as in Proposition 3 and besides satisfies

\[
F(t,x(t),u(t),\psi_0,\psi(t)) = \max_{v \in \Omega} F(t,x(t),v,\psi_0,\psi(t)) \quad (5)
\]

a.e. in \( t \in [a,b] \) along any Pontryagin extremal \((x(\cdot),u(\cdot),\psi_0,\psi(\cdot))\) of the optimal control problem, then \( t \to F(t,x(t),u(t),\psi_0,\psi(t)) \) is absolutely continuous and the equality

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} \quad (6)
\]

holds along the extremals.

**Proof.** Our proof is an extension of the standard proof of Theorem 3. Let \((x(\cdot),u(\cdot),\psi_0,\psi(\cdot))\) be a Pontryagin extremal of the problem. Setting \( v = u(s) \) in (3), we obtain that \( \phi(s,t) = F(t,x(t),u(s),\psi_0,\psi(t)) \) satisfies

\[
\phi(t,t) \geq \phi(s,t), \quad s \in [a,b],
\]

(7)
for $t$ in a set of full measure on $[a, b]$. Proposition 4 then implies that $m(t) = \phi(t, t) = F(t, x(t), u(t), \psi_0, \psi(t))$ is an absolutely continuous function on $[a, b]$. It remains to prove that

$$m(t) = \phi(t, t) = F(t, x(t), u(t), \psi_0, \psi(t))$$

is an absolutely continuous function on $[a, b]$. It remains to prove that

$$\dot{m}(t) = \frac{\partial F}{\partial t}(\pi(t)) + \frac{\partial F}{\partial x}(\pi(t)) \cdot \dot{x}(t) + \frac{\partial F}{\partial \psi}(\pi(t)) \cdot \dot{\psi}(t),$$

where $\pi(t) = (t, x(t), u(t), \psi_0, \psi(t))$. Since

$$m(t + h) - m(t) = \phi(t + h, t + h) - \phi(t + h, t + h) - \phi(t, t + h),$$

and by the hypotheses the left-hand side and the second term on the right-hand side have a limit as $h \to 0$, one concludes that the first term on the right must have a limit as well. From (7)

$$\phi(t + h, t + h) \geq \phi(t, t + h)$$

and it follows that $\frac{\phi(t + h, t + h) - \phi(t, t + h)}{h}$ is nonnegative when $h > 0$ and nonpositive when $h < 0$; thus, its limit must be zero when $h \to 0$. In this way we obtain

$$\dot{m}(t) = \lim_{h \to 0} \frac{F(t + h, x(t + h), u(t), \psi_0, \psi(t)) - F(t, x(t), u(t), \psi_0, \psi(t))}{h}$$

and the conclusion follows from the Hamiltonian system.

**Corollary 6.** Let $F(t, x, u, \psi_0, \psi) : [a, b] \times \mathbb{R}^n \times \Omega \times \mathbb{R}_-^+ \times \mathbb{R}^n \to \mathbb{R}$, be continuously differentiable with respect to $t$, $x$, $\psi$ for $u$ fixed; and $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ be an extremal. If

(i) $F(t, x(t), u(t), \psi_0, \psi(t))$ is absolutely continuous in $t$;

(ii) $F(t, x(t), u(t), \psi_0, \psi(t)) = \max_{v \in \Omega} F(t, x(t), v, \psi_0, \psi(t))$ a.e. in $a \leq t \leq b$;

then the equality (8) holds along the extremal.

Possible applications of Theorem 5 follow in the next section.

## 4 Applications of the Main Result

Solving the Hamiltonian system by the elimination of the control with the aid of the maximality condition is typically a difficult task. Therefore, it is worthwhile to look for circumstances which make the solution easier. This is the case when the extremals don’t change the value of a given function. Indeed, the existence of such a function, called constant of the motion, may...
be used for reducing the dimension of the Hamiltonian system (see e.g. Módulo 5)). In extreme cases, with a sufficiently large number of (independent) constants of the motion, one can solve the problem completely.

4.1 Constants of the Motion

From Theorem 5, one immediately obtains a necessary and sufficient condition for a function to be a constant of the motion.

**Definition 3.** A quantity $F(t, x, u, \psi_0, \psi)$ which is constant along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem, is called a constant of the motion.

**Corollary 7.** Under the conditions of Theorem 5, $F(t, x, u, \psi_0, \psi)$ is a constant of the motion if and only if
\[
\frac{\partial F}{\partial t} + \sum_{i=1}^{4} \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial \psi_i} - \sum_{i=1}^{4} \frac{\partial F}{\partial \psi_i} \frac{\partial H}{\partial x_i} = 0
\]
holds, almost everywhere, along the Pontryagin extremals of the optimal control problem.

**Example 1.** ($n = 4$, $r = 2$, $\Omega = \mathbb{R}^2$) Let us consider the problem
\[
\int_a^b \left((u_1(t))^2 + (u_2(t))^2\right) dt \rightarrow \min,
\]
\[
\begin{align*}
\dot{x}_1(t) &= x_3(t) \\
\dot{x}_2(t) &= x_4(t) \\
\dot{x}_3(t) &= -x_1(t) \left((x_1(t))^2 + (x_2(t))^2\right) + u_1(t) \\
\dot{x}_4(t) &= -x_2(t) \left((x_1(t))^2 + (x_2(t))^2\right) + u_2(t).
\end{align*}
\]
The corresponding Hamiltonian function is
\[
H(x_1, x_2, x_3, x_4, u_1, u_2, \psi_0, \psi_1, \psi_2, \psi_3, \psi_4) = \psi_0 \left(u_1^2 + u_2^2\right) + \psi_1 x_3 + \psi_2 x_4 - \psi_3 x_1 \left(x_1^2 + x_2^2\right) + \psi_3 u_1 - \psi_4 x_2 \left(x_1^2 + x_2^2\right) + \psi_4 u_2.
\]
We claim that
\[
F = -\psi_1 x_2 + \psi_2 x_1 - \psi_3 x_4 + \psi_4 x_3
\]
is a constant of the motion for the problem. Direct calculations show that
\[
\frac{\partial F}{\partial t} + \sum_{i=1}^{4} \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial \psi_i} - \sum_{i=1}^{4} \frac{\partial F}{\partial \psi_i} \frac{\partial H}{\partial x_i} = \psi_4 u_1 - \psi_3 u_2.
\]
From the maximality condition it follows that $\frac{\partial H}{\partial u_1} = 0$ and $\frac{\partial H}{\partial u_2} = 0$, that is, $2\psi_0 u_1 + \psi_3 = 0$ and $2\psi_0 u_2 + \psi_4 = 0$. Using these last two identities in (10) one concludes from Corollary 7 that (9) is a constant of the motion.
4.2 Characterization of Optimal Control Problems

We shall endeavor here to find a method to synthesize optimal control problems with given constants of the motion. If a function $F$ is fixed \textit{a priori}, we can regard equality (8) as a partial differential equation in the unknown Hamiltonian $H$. Obviously, if this differential equation admits a solution, then an optimal control problem can be constructed with the constant of the motion $F$. We shall illustrate the general idea in special situations.

\textbf{Example 2.} The Hamiltonian $H$ is a constant of the motion if and only if $\frac{\partial H}{\partial t} = 0$. Condition is trivially satisfied for autonomous problems.

\textbf{Example 3.} Function $\psi x + Ht$ is a constant of the motion if and only if $H = \frac{\partial H}{\partial x} x - \frac{\partial H}{\partial \psi} \psi - \frac{\partial H}{\partial t} t$. Condition is satisfied, for example, for problems of the form $(0 < a < b)$

$$\int_a^b L(tx(t), u(t)) \frac{dt}{t} \rightarrow \min,$$

$$\dot{x}(t) = \frac{\varphi(tx(t), u(t))}{t^2}.$$

\textbf{Example 4.} We conclude from Corollary 7 that a necessary and sufficient condition for $H\psi x$ to be a constant of the motion is

$$\psi x \frac{\partial H}{\partial t} + \psi H \frac{\partial H}{\partial \psi} - Hx \frac{\partial H}{\partial x} = 0.$$

A simple problem with constant of the motion $H\psi x$ is therefore

$$\int_a^b L(u(t)) \ dt \rightarrow \min,$$

$$\dot{x}(t) = \varphi(u(t)) x(t).$$

\textbf{Example 5.} The following optimization problem is important in the study of cubic polynomials on Riemannian manifolds (see [3, p. 39] and [26]). Here we consider the particular case when one has 2-dimensional state and $n$ controls:

$$\int_0^T \left( (u_1(t))^2 + \cdots + (u_n(t))^2 \right) dt \rightarrow \min,$$

$$\begin{cases}
    \dot{x}_1(t) = x_2(t), \\
    \dot{x}_2(t) = X_1(x_1(t)) u_1(t) + \cdots + X_n(x_1(t)) u_n(t).
\end{cases} \tag{11}$$

Functions $X_i(\cdot)$, $i = 1, \ldots, n$, are assumed smooth. The Hamiltonian for the problem is

$$H = \psi_0 (u_1^2 + \cdots + u_n^2) + \psi_1 x_2 + \psi_2 (X_1(x_1)u_1 + \cdots + X_n(x_1)u_n).$$
As far as the problem is autonomous, the Hamiltonian is a constant of the motion. We are interested in finding a new constant of the motion for the problem. We will look for one of the form

\[ F = k_1 \psi_1 x_1 + k_2 \psi_2 x_2, \]

where \( k_1 \) and \( k_2 \) are constants. This is a typical constant of the motion, known in the literature by momentum map (see [5]). First we note that

\[ \frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial x_1} = k_1 \psi_1, \quad \frac{\partial F}{\partial x_2} = k_2 \psi_2, \quad \frac{\partial F}{\partial \psi_1} = k_1 x_1, \quad \frac{\partial F}{\partial \psi_2} = k_2 x_2, \]

and

\[ \frac{\partial H}{\partial x_1} = \psi_2 \left( X'_1(x_1) u_1 + \cdots + X'_n(x_1) u_n \right), \quad \frac{\partial H}{\partial x_2} = \psi_1, \]
\[ \frac{\partial H}{\partial \psi_1} = x_2, \quad \frac{\partial H}{\partial \psi_2} = X_1(x_1) u_1 + \cdots + X_n(x_1) u_n. \]

Substituting these quantities into (8) we obtain that

\[ k_1 \psi_1 x_2 + k_2 \psi_2 \left( X_1(x_1) u_1 + \cdots + X_n(x_1) u_n \right) - k_1 x_1 \psi_2 \left( X'_1(x_1) u_1 + \cdots + X'_n(x_1) u_n \right) - k_2 x_2 \psi_1 = 0. \]

The equality is trivially satisfied if \( k_1 = k_2 \) and \( X'_i(x_1) x_1 = X_i(x_1), \ i = 1, \ldots, n \). We have just proved the following proposition.

**Proposition 8.** If the homogeneity condition \( X_i(\lambda x_1) = \lambda X_i(x_1) \ (i = 1, \ldots, n), \ \forall \lambda > 0, \) holds, then \( \psi_1(t)x_1(t) + \psi_2(t)x_2(t) \) is constant in \( t \in [0, T] \) along the extremals of the problem (11).

**Acknowledgments**

The author is in debt to A. V. Sarychev for the many useful advises, comments and suggestions. The research was supported by the program PRODEP III 5.3/C/200.009/2000.

**References**

[1] Ambrosio L., Ascenzi O., Buttazzo G. Lipschitz Regularity for Minimizers of Integral Functionals with Highly Discontinuous Integrands. J. Math. Anal. Appl. 142, 1989, pp. 301–316.
[2] Berkovitz L. D. Optimal Control Theory. Applied Mathematical Sciences 12, Springer-Verlag, New York, 1974.

[3] Bertsekas D. P. Dynamic Programming and Optimal Control, Vol. I (2nd ed.). Athena Scientific, Belmont, Massachusetts, 2000.

[4] Bertsekas D. P. Dynamic Programming and Optimal Control, Vol. II. Athena Scientific, Belmont, Massachusetts, 1995.

[5] Blankenstein G., van der Schaft A. Optimal control and implicit Hamiltonian systems. In: Isidori A., Lamnabhi-Lagarrigue F., Respondek W. (eds). Nonlinear control in the year 2000, vol. 1 (Paris). Springer, London. 2001, pp. 185–205.

[6] Camarinha M. A Geometria dos Polinómios Cúbicos em Variedades Riemannianas. Ph.D. thesis, Departamento de Matemática, Universidade de Coimbra, Coimbra, 1996.

[7] Cesari L. Optimization—Theory and Applications. Springer-Verlag, New York, 1983.

[8] Chiang A. C. Elements of Dynamic Optimization. McGraw-Hill Inc, 1992.

[9] Clarke F. H. Optimization and Nonsmooth Analysis. John Wiley & Sons Inc., New York, 1983.

[10] Clarke F. H. An Indirect Method in the Calculus of Variations. Trans. Amer. Math. Soc. 336, 1993, pp. 655–673.

[11] Clarke F. H., Vinter R. B. Regularity Properties of Solutions to the Basic Problem in the Calculus of Variations. Trans. Amer. Math. Soc. 289, 1985, pp. 73–98.

[12] Connors M. M., Teichroew D. Optimal Control of Dynamic Operations Research Models. International Textbook Company, Scranton, Pennsylvania, 1967.

[13] Elgerd O. I. Control Systems Theory. McGraw-Hill Inc, 1967.

[14] Fattorini H. O. Infinite Dimensional Optimization and Control Theory. Encyclopedia of Mathematics and Its Applications 62, Cambridge University Press, Cambridge, 1999.
[15] Freiling G., Jank G., Sarychev A. Non-blow-up Conditions for Riccati-
type Matrix Differential and Difference Equations. Results Math. 37,
2000, pp. 84–103.

[16] Gamkrelidze R. V. Principles of Optimal Control Theory. Mathematical
Concepts and Methods in Science and Engineering 7, Plenum Press,
New York, 1978.

[17] Isaacs R. Differential Games – A Mathematical Theory with Applica-
tions to Warfare and Pursuit, Control and Optimization. Dover Publi-
cations Inc., Mineola, New York, 1999.

[18] Lauwerier H. A. Calculus of Variations in Mathematical Physics. Math-
ematical Centre Tracts 14, Mathematisch Centrum, Amsterdam, 1966.

[19] Léonard D., Van Long N. Optimal Control Theory and Static Opti-
mization in Economics. Cambridge University Press, Cambridge, 1992.

[20] Pereira F. L. Control Design for Autonomous Vehicles: A Dynamic Op-
timization Perspective. European Journal of Control 7, 2001, pp. 178–
202.

[21] Pinch E. R. Optimal Control and the Calculus of Variations. Oxford
University Press, Oxford, 1995.

[22] Pires C. Cálculo para Economistas. McGraw-Hill de Portugal Lda.,
2001.

[23] Pontryagin L. S., Boltyanskii V. G., Gamkrelidze R. V., Mischenko
E. F. The Mathematical Theory of Optimal Processes. John Wiley,
New York, 1962.

[24] Rund H. The Hamilton–Jacobi Theory in the Calculus of Variations, Its
Role in Mathematics and Physics. D. Van Nostrand Co., Ltd., London–
Toronto, Ont.–New York, 1966.

[25] Sarychev A. V., Torres D. F. M. Lipschitzian Regularity of Minimizers
for Optimal Control Problems with Control-Affine Dynamics. Applied
Mathematics and Optimization, 41, 2000, pp. 237–254.

[26] Silva Leite F., Camarinha M., Crouch P. Elastic Curves as Solutions
of Riemannian and Sub-Riemannian Control Problems. Math. Control
Signals Systems 13, 2000, pp. 140–155.
[27] Smith D. R. Variational Methods in Optimization. Dover Publications Inc., Mineola, New York, 1998.

[28] Staicu V. Equações Diferenciais. Relatório da disciplina de Equações Diferenciais, Provas de Agregação em Matemática, Universidade de Aveiro, 2000.

[29] Torres D. F. M. Conservation Laws in Optimal Control. Dynamics, Bifurcations and Control, Lecture Notes in Control and Information Sciences 273, Springer-Verlag, Berlin, Heidelberg, 2002, pp. 287–296.

[30] Torres D. F. M. On the Noether Theorem for Optimal Control. European Journal of Control, 8(1) 2002, pp. 56–63.

[31] Valadares Tavares L., Nunes Correia F. Optimização Linear e Não Linear – Conceitos, Métodos e Algoritmos. Fundação Calouste Gulbenkian, Lisboa, 1986.