A note on Todorov surfaces
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Abstract

Let $S$ be a Todorov surface, i.e., a minimal smooth surface of general type with $q = 0$ and $p_g = 1$ having an involution $i$ such that $S/i$ is birational to a $K3$ surface and such that the bicanonical map of $S$ is composed with $i$.

The main result of this paper is that, if $P$ is the minimal smooth model of $S/i$, then $P$ is the minimal desingularization of a double cover of $P^2$ ramified over two cubics. Furthermore it is also shown that, given a Todorov surface $S$, it is possible to construct Todorov surfaces $S_j$ with $K^2_{S_j} = 1, \ldots, K^2_{S_j} - 1$ and such that $P$ is also the smooth minimal model of $S_j/i_j$, where $i_j$ is the involution of $S_j$. Some examples are also given, namely an example different from the examples presented by Todorov in [To2].

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1 Introduction

An involution of a surface $S$ is an automorphism of $S$ of order 2. We say that a map is composed with an involution $i$ of $S$ if it factors through the double cover $S \rightarrow S/i$. Involutions appear in many contexts in the study of algebraic surfaces. For instance in most cases the bicanonical map of a surface of general type is non-birational only if it is composed with an involution.

Assume that $S$ is a smooth minimal surface of general type with $q = 0$ and $p_g \neq 0$ having bicanonical map $\phi_2$ composed with an involution $i$ of $S$ such that $S/i$ is non-ruled. Then, according to [Xi] Theorem 3, $p_g(S) = 1$, $K^2_S \leq 8$ and $S/i$ is birational to a $K3$ surface (Theorem 3 of [Xi] contains the assumption $\deg(\phi_2) = 2$, but the result is still valid assuming only that $\phi_2$ is composed with an involution).

Todorov ([To2]) was the first to give examples of such surfaces. His construction is as follows. Consider a Kummer surface $Q$ in $P^3$, i.e., a quartic having as only singularities 16 nodes $a_i$. The double cover of $Q$ ramified over the intersection of $Q$ with a general quadric and over the 16 nodes of $Q$ is a surface of general type with $q = 0$, $p_g = 1$ and $K^2 = 8$. Then, choose $a_1, \ldots, a_6$ in general position and let $G$ be the intersection of $Q$ with a general quadric through $j$ of the nodes $a_1, \ldots, a_6$. The double cover of $Q$ ramified over $Q \cap G$ and over the remaining $16 - j$ nodes of $Q$ is a surface of general type with $q = 0$, $p_g = 1$ and $K^2 = 8 - j$.

Imposing the passage of the branch curve by a 7-th node, one can obtain a surface with $K^2 = p_g = 1$ and $q = 0$. This is the so-called Kunev surface. Todorov ([To1]) has shown that the Kunev surface is a bidouble cover of $P^2$ ramified over two cubics and a line.
I refer to [Mo] for an explicit description of the moduli spaces of Todorov surfaces.

We call Todorov surfaces smooth surfaces \( S \) of general type with \( p_g = 1 \) and \( q = 0 \) having bicanonical map composed with an involution \( i \) of \( S \) such that \( S/i \) is birational to a K3 surface.

In this paper we prove the following:

**Theorem 1** Let \( S \) be a Todorov surface with involution \( i \) and \( P \) be the smooth minimal model of \( S/i \). Then:

a) there exists a generically finite degree 2 morphism \( P \to \mathbb{P}^2 \) ramified over two cubics;

b) for each \( j \in \{1, \ldots, K_2^2 - 1\} \), there is a Todorov surface \( S_j \), with involution \( i_j \), such that \( K_2^2 S_j = j \) and \( P \) is the smooth minimal model of \( S_j/i_j \).

The idea of the proof is the following. First we verify that the evenness of the branch locus \( B' + \sum A_i \subset P \) implies that each nodal curve \( A_i \) can only be contained in a Dynkin graph \( G \) of type \( A_{2n+1} \) or \( D_n \). Then we use a Saint-Donat result to show that \( A_i \) can be chosen such that the linear system \( |B' - G| \) is free. This implies b). Finally we conclude that there is a free linear system \( |B'_0| \) with \( B'_0^2 = 2 \), which gives a).

**Notation and conventions**

We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface \( S \), the canonical class is denoted by \( K \), the geometric genus by \( p_g := h^0(S, \mathcal{O}_S(K)) \), the irregularity by \( q := h^1(S, \mathcal{O}_S(K)) \) and the Euler characteristic by \( \chi = \chi(\mathcal{O}_S) = 1 + p_g - q \).

A \((-2)\)-curve or nodal curve on a surface is a curve isomorphic to \( \mathbb{P}^1 \) such that \( C^2 = -2 \). We say that a curve singularity is negligible if it is either a double point or a triple point which resolves to at most a double point after one blow-up.

The rest of the notation is standard in algebraic geometry.

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## 2 Preliminaries

The next result follows from [Sti] (4.1), Theorem 5.2, Propositions 5.6 and 5.7.

**Theorem 2** ([Sti]) Let \( |D| \) be a complete linear system on a smooth K3 surface \( F \), without fixed components and such that \( D^2 \geq 4 \). Denote by \( \varphi_D \) the map given by \( |D| \). If \( \varphi_D \) is non-birational and the surface \( \varphi_D(F) \) is singular then there exists an elliptic pencil \( |E| \) such that \( ED = 2 \) and one of these cases occur:
(i) $D = O_F(4E + 2\Gamma)$ where $\Gamma$ is a smooth rational irreducible curve such that $\Gamma E = 1$. In this case $\varphi_D(F)$ is a cone over a rational normal twisted quartic in $\mathbb{P}^4$; 

(ii) $D = O_F(3E + 2\Gamma_0 + \Gamma_1)$, where $\Gamma_0$ and $\Gamma_1$ are smooth rational irreducible curves such that $\Gamma_0 E = 1$, $\Gamma_1 E = 0$ and $\Gamma_0 \Gamma_1 = 1$. In this case $\varphi_D(F)$ is a cone over a rational normal twisted cubic in $\mathbb{P}^3$; 

(iii) a) $D = O_F(2E + \Gamma_0 + \Gamma_1)$, where $\Gamma_0$ and $\Gamma_1$ are smooth rational irreducible curves such that $\Gamma_0 E = \Gamma_1 E = 1$ and $\Gamma_0 \Gamma_1 = 0$; 

b) $D = O_F(2E + \Delta)$, with $\Delta = 2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$ ($N \geq 0$), where the curves $\Gamma_i$ are irreducible rational curves as in Figure 1.

In both cases $\varphi_D(F)$ is a quadric cone in $\mathbb{P}^3$. 

Moreover in all the cases above the pencil $|E|$ corresponds under the map $\varphi_D$ to the system of generatrices of $\varphi_D(F)$.

3 Proof of Theorem 1

We say that a curve $D$ is nef and big if $DC \geq 0$ for every curve $C$ and $D^2 > 0$. In order to prove Theorem 1 we show the following:

Proposition 3 Let $P$ be a smooth K3 surface with a reduced curve $B$ satisfying:

(i) $B = B' + \sum_{t \in \{9, \ldots, 16\}} A_t$, where $B'$ is a nef and big curve with at most negligible singularities, the curves $A_t$ are disjoint (-2)-curves also disjoint from $B'$ and $B \equiv 2L$, $L^2 = -4$, for some $L \in \text{Pic}(P)$.

Then:

a) Let $\pi : V \to P$ be a double cover with branch locus $B$ and $S$ be the smooth minimal model of $V$. Then $q(S) = 0$, $p_g(S) = 1$, $K_S^2 = t - 8$ and the bicanonical map of $S$ is composed with the involution $i$ of $S$ induced by $\pi$;

b) If $t \geq 10$, then $P$ contains a smooth curve $B_0'$ and (-2)-curves $A'_1, \ldots, A'_{t-1}$ such that $B_0'^2 = B'^2 - 2$ and $B_0 := B'_0 + \sum_{i=1}^{t-1} A'_i$ also satisfies condition (i).
Proof:

a) Let \( L \equiv \frac{1}{2}B \) be the line bundle which determines \( \pi \). From the double cover formulas (see e.g. \[BPV\]) and the Riemann Roch theorem,
\[
q(S) = h^1(P, \mathcal{O}_P(L)),
p_g(S) = 1 + h^0(P, \mathcal{O}_P(L)),
\]
\[
h^0(P, \mathcal{O}_P(L)) + h^0(P, \mathcal{O}_P(-L)) = h^1(P, \mathcal{O}_P(L)).
\]
Since \( 2L - \sum A_i \) is nef and big, the Kawamata-Viehweg’s vanishing Theorem (see e.g. \[EV\] Corollary 5.12, c)) implies \( h^1(P, \mathcal{O}_P(-L)) = 0 \).
Hence
\[
h^1(P, \mathcal{O}_P(L)) = h^1(P, \mathcal{O}_P(K_P - L)) = h^1(P, \mathcal{O}_P(-L)) = 0
\]
and then \( q(S) = 0 \) and \( p_g(S) = 1 \).
As
\[
h^0(P, \mathcal{O}_P(2K_P + L)) = h^0(P, \mathcal{O}_P(L)) = 0,
\]
the bicanonical map of \( S \) is composed with \( i \) (see \[CM\] Proposition 6.1]).
The \((-2)\)-curves \( A_1, \ldots, A_t \) give rise to \((-1)\)-curves in \( V \), therefore
\[
K^2_S = K^2_V + t = 2(K_P + L)^2 + t = 2L^2 + t = t - 8.
\]

b) Denote by \( \xi \subset P \) the set of irreducible curves which do not intersect \( B' \) and denote by \( \xi_i, \ i \geq 1 \), the connected components of \( \xi \). Since \( B'^2 \geq 2 \), the Hodge-index Theorem implies that the intersection matrix of the components of \( \xi \) is negative definite. Therefore, following \[BPV\] Lemma I.2.12, the \( \xi_i \)'s have one of the five configurations: the support of \( A_n, D_n, E_6, E_7 \) or \( E_8 \) (see e.g. \[BPV\] III.3 for the description of these graphs).

Claim 1: Each nodal curve \( A_i \) can only be contained in a graph of type \( A_{2n+1} \) or \( D_n \).

Proof: Suppose that there exists an \( A_i \) which is contained in a graph of type \( E_6 \). Denote the components of \( E_6 \) as in Figure 2

![Figure 2: E6](image)

If \( A_i = a_3 \) or \( A_i = a_6 \), then \( a_6 B = a_6 a_3 = 1 \) or \( a_3 B = 1 \), contradicting \( B \equiv 2L \).
If \( A_i = a_1 \) or \( A_i = a_2 \), then \( a_2 B = 1 \) or \( a_1 B = 1 \), the same contradiction. By the same reason, \( A_i \neq a_4 \) and \( A_i \neq a_5 \).
Analogously one can verify that each $A_i$ can not be in a graph of type $A_{2n}$, $E_7$ or $E_8$. ♦

The possible configurations for the curves $A_i$ in the graphs are shown in Figure 3. Fix one of the curves $A_i$ and denote by $G$ the graph containing it.

**Figure 3:** The numbers represent the multiplicity and the dotted curve represents a general element $B'_0$ in $|B' - G|$.

**Claim 2:** We can choose $A_i$ such that the linear system $|B' - G|$ has no fixed components (and thus no base points, from [SU, Theorem 3.1]).

**Proof:** Denote by $\varphi_{|B'|}$ the map given by the linear system $|B'|$. We know that $\varphi_{|B'|}$ is birational or it is of degree 2 (see [SU, Section 4]). If $\varphi_{|B'|}$ is birational or the point $\varphi_{|B'|}(G)$ is a smooth point of $\varphi_{|B'|}(P)$, the result is clear, since $|B' - G|$ is the pullback of the linear system of the hyperplanes containing $\varphi_{|B'|}(G)$ and $\varphi_{|B'|}(\varphi_{|B'|}(G)) = G$ (see [BPV] Theorems III 7.1 and 7.3).

Suppose now that $\varphi_{|B'|}$ is non-birational and that $\varphi_{|B'|}(G)$ is a singular point of $\varphi_{|B'|}(P)$. Then $B'$ is linearly equivalent to a curve with one of the configurations described in Theorem 2. Except for the last configuration, $G$ contains at most two $(-2)$-curves. But $t \geq 9$, thus in these cases there exists another graph $G'$ containing a curve $A_j$ such that $\varphi_{|B'|}(G')$ is a non-singular point of $\varphi_{|B'|}(P)$ (notice that Theorem 2 implies that $\varphi_{|B'|}(P)$ contains only one singular point).

So we can suppose that $B'$ is equivalent to a curve with a configuration as in Theorem 2 (iii, b). None of the curves $\Gamma_0, \ldots, \Gamma_n$ can be one of the curves $A_j$. For this note that: if $\Gamma_0 = A_j$, then $EB = E(B' + \sum A_i) = 2 + 3 = 3 \not\equiv 0$ (mod 2); if $\Gamma_1 = A_j$, then $\Gamma_0B = \Gamma_0\Gamma_1 = 1 \not\equiv 0$ (mod 2); etc. Again this configuration can contain at most two curves $A_j$, the components $\Gamma_{N+1}, \Gamma_{N+2}$. ♦
Let $B'_0$ be a smooth curve in $|B' - G|$. If $G$ is an $A_{2n+1}$ graph, then, using the notation of Figure 3,

$$B'_0 + \sum_{i=1}^{n} E_i + \sum_{i=1}^{t} A_i \equiv \left( B' - \sum_{i=1}^{n+1} A_i \right) + \sum_{i=1}^{t} A_i \equiv \equiv B' + \sum_{i=1}^{t} A_i - 2 \sum_{i=1}^{n+1} A_i \equiv 0 \pmod{2}.$$

Therefore the curve $B_0 := B'_0 + \sum_{i=1}^{n} E_i + \sum_{i=1}^{t} A_i$ satisfies condition (i).

The case where $G$ is a $D_m$ graph is analogous. \hfill \Box

**Proof of Theorem 1** : Let $V \to S$ be the blow-up at the isolated fixed points of the involution $i$ and $W$ be the minimal resolution of $S/i$. We have a commutative diagram

$$\begin{array}{ccc}
V & \longrightarrow & S \\
\downarrow \pi & & \downarrow \\
W & \longrightarrow & S/i.
\end{array}$$

The branch locus of $\pi$ is a smooth curve $B = B' + \sum_{i} A_i$, where the curves $A_i$ are $(−2)$-curves which contract to the nodes of $S/i$. Let $P$ be the minimal model of $W$ and $B \subset P$ be the projection of $B$. Let $L \equiv \frac{1}{2}B$ be the line bundle which determines $\pi$.

First we verify that $B$ satisfies condition (i) of Proposition 3: from [CM, Proposition 6.1], $\chi(O_W) - \chi(O_S) = K_W(K_W + L)$, hence $K_W(K_W + L) = 0$, which implies that $B$ has at most negligible singularities; now from [Mo, Theorem 5.2] we get $K_S^2 = \frac{1}{2}B^2 + 1 = p_g(S) = \frac{1}{2}(K_S^2 - t) + 3$, thus $t = K_S^2 + 8$ and $B^2 = B^2 - 2t = 2K_S^2 - 2t = -16$, which gives $(B/2)^2 = -4$ and $B^2 \geq 2$; finally $B$ is nef because, on a $K3$ surface, an irreducible curve with negative self intersection must be a $(−2)$-curve.

Now using Proposition 3, b) and a) we obtain statement b). In particular we get also that $P$ contains a curve $B'_0$ and $(−2)$-curves $A'_i$, $i = 1, \ldots, 9$, such that $B_0 := B'_0 + \sum_{i} A'_i$ is smooth and divisible by 2 in the Picard group. Moreover, the complete linear system $|B'_0|$ has no fixed component nor base points and $B'_0^2 = 2$. Therefore, from [St], $|B'_0|$ defines a generically finite degree 2 morphism

$$\varphi := \varphi|_{B'_0} : P \to \mathbb{P}^2.$$

Since $g(B'_0) = 2$, this map is ramified over a sextic curve $\beta$. The singularities of $\beta$ are negligible because $P$ is a $K3$ surface.

We claim that $\beta$ is the union of two cubics. Let $p_i \in \beta$ be the singular point corresponding to $A'_i$, $i = 1, \ldots, 9$. Notice that the $p_i$'s are possibly infinitely near.
Let $C \subset \mathbb{P}^2$ be a cubic curve passing through $p_i$, $i = 1, \ldots, 9$. As $C + \varphi_*(B'_0)$ is a plane quartic, we have

$$\left(\varphi^*(C) - \sum_{i=1}^{9} A'_i\right) + B'_0 + \sum_{i=1}^{9} A'_i \equiv \varphi^*(C + \varphi_*(B'_0)) \equiv 0 \pmod{2},$$

hence also $\varphi^*(C) - \sum_{i=1}^{9} A'_i \equiv 0 \pmod{2}$, i.e. there exists a divisor $J$ such that

$$2J \equiv \varphi^*(C) - \sum_{i=1}^{9} A'_i.$$

Since $P$ is a $K_3$ surface, the Riemann Roch theorem implies that $J$ is effective. From $J A'_i = 1$, $i = 1, \ldots, 9$, we obtain that the plane curve $\varphi_*(J)$ passes with multiplicity 1 through the nine singular points $p_i$ of $\beta$. This immediately implies that $\varphi_*(J)$ is not a line nor a conic, because $\beta$ is a reduced sextic. Therefore $\varphi_*(J)$ is a reduced cubic. So $\varphi_*(J) \equiv C$ and then

$$\varphi^*(\varphi_*(J)) \equiv 2J + \sum_{i=1}^{9} A'_i.$$

This implies that $\varphi_*(J)$ is contained in the branch locus $\beta$, which finishes the proof of a).

4 Examples

Todorov gave examples of surfaces $S$ with bicanonical image $\phi_2(S)$ birational to a Kummer surface having only ordinary double points as singularities. The next sections contain an example with $\phi_2(S)$ non-birational to a Kummer surface and an example with $\phi_2(S)$ having an $A_{17}$ double point.

4.1 $S/i$ non-birational to a Kummer surface

Here we construct smooth surfaces $S$ of general type with $K^2 = 2,3$, $p_g = 1$ and $q = 0$ having bicanonical map of degree 2 onto a $K_3$ surface which is not birational to a Kummer surface.

It is known since [Hu] that there exist special sets of 6 nodes, called Weber hexads, in the Kummer surface $Q \in \mathbb{P}^3$ such that the surface which is the blow-up of $Q$ at these nodes can be embedded in $\mathbb{P}^3$ as a quartic with 10 nodes. This quartic is the Hessian of a smooth cubic surface.

The space of all smooth cubic surfaces has dimension 4 while the space of Kummer surfaces has dimension 3. Thus it is natural to ask if there exist Hessian "non-Kummer" surfaces, i.e. which are not the embedding of a Kummer surface blown-up at 6 points. This is studied in [Ro], where the existence of "non-Kummer" quartic Hessians $H$ in $\mathbb{P}^3$ is shown. These are surfaces with 10 nodes $a_i$ such that the projection from one node $a_1$ to $\mathbb{P}^2$ is a generically 2 : 1 cover of $\mathbb{P}^2$ with branch locus $a_1 + a_2$ satisfying: $a_1, a_2$ are smooth cubics.
tangent to a nondegenerate conic $C$ at 3 distinct points. We use this in the following construction.

Let $\alpha_1$, $\alpha_2$ and $C$ be as above. Take the morphism $\pi : W \to \mathbb{P}^2$ given by the canonical resolution of the double cover of $\mathbb{P}^2$ with branch locus $\alpha_1 + \alpha_2$. The strict transform of $C$ gives rise to the union of two disjoint $(-2)$-curves $A_1, A_2 \subset W$ (one of these correspond to the node $\alpha_1$ from which we have projected).

Let $T \in \mathbb{P}^2$ be a general line. Let $A_3, \ldots, A_{11} \subset W$ be the disjoint $(-2)$-curves contained in $\pi^*(\alpha_1 + \alpha_2)$. We have $\pi^*(T + \alpha_1) \equiv 0 \pmod{2}$, hence, since $\alpha_1$ is in the branch locus, also

$$\pi^*(T) + \sum_{i=3}^{11} A_i \equiv 0 \pmod{2}.$$ 

The linear systems $|\pi^*(T) + A_2|$ and $|\pi^*(T) + A_1 + A_2|$ have no fixed components nor base points (see [St, (2.7.3) and Corollary 3.2]). The surface $S$ is the minimal model of the double cover of $W$ ramified over a general element in

$$|\pi^*(T) + A_2| + \sum_{i=2}^{11} A_i \quad \text{or} \quad |\pi^*(T) + A_1 + A_2| + \sum_{i=1}^{11} A_i.$$

4.2 $\phi_2(S)$ with $A_{17}$ and $A_1$ singularities

This section contains a brief description of a construction of a surface $S$ of general type having bicanonical image $\phi_2(S) \subset \mathbb{P}^3$ a quartic $K3$ surface with $A_{17}$ and $A_1$ singularities. I omit the details, which were verified using the Computational Algebra System Magma.

Let $C_1$ be a nodal cubic, $p$ an inflection point of $C_1$ and $T$ the tangent line to $C_1$ at $p$. The pencil generated by $C_1$ and $3T$ contains another nodal cubic $C_2$, smooth at $p$. The curves $C_1$ and $C_2$ intersect at $p$ with multiplicity 9.

Let $\rho : X \to \mathbb{P}^2$ be the resolution of $C_1 + C_2$ and $\pi : W \to X$ be the double cover with branch locus the strict transform of $C_1 + C_2$. Denote by $\tilde{T}$ the line containing the nodes of $C_1$ and $C_2$ and by $l \subset W$ the pullback of the strict transform of $T$. The map given by $(\rho \circ \pi)^* l$ is birational onto a quartic $Q$ in $\mathbb{P}^3$ with an $A_1$ and $A_{17}$ singularities (notice that $l$ is a $(-2)$-curve and $((\rho \circ \pi)^* l + l = 0$).

Let $B' \in |(\rho \circ \pi)^* l + l|$ be a smooth element and $A_1, \ldots, A_9$ be the disjoint $(-2)$-curves contained in $(\rho \circ \pi)^* p$. Let $S$ be the minimal model of the double cover of $W$ with branch locus $B' + \sum_{i=1}^{9} A_i + l$. The surface $Q$ is the image of the bicanonical map of $S$ and $p_g(S) = 1$, $q(S) = 0$, $K_S^2 = 2$.

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