NORM ESTIMATES AND ASYMPTOTIC FAITHFULNESS OF THE QUANTUM SU(n) REPRESENTATIONS OF THE MAPPING CLASS GROUPS

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Abstract. We give a direct proof for the asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups using peak sections in Kodaira embedding. We give also estimates on the norm of the parallel transport of the projective connection on the Verlinde bundle. The faithfulness has been proved earlier in [1] using Toeplitz operators of compact Kähler manifolds and in [10] using skein theory.

1. Introduction

Let Σ be a closed oriented surface of genus g ≥ 2 and p ∈ Σ. We consider the moduli space M of flat SU(n)-connections P on Σ \ {p} with fixed holonomy a center element d ∈ Z/nZ ∼= Z_{SU(n)} of SU(n). We assume that n and d are coprime, in the case of g = 2 we also allow (n, d) = (2, 0), namely the SU(2)-connections with trivial holonomy.

There is a canonical symplectic form ω on M obtained by integrating wedge product of Lie algebra su(n)-valued connection forms. The natural action of the mapping class group Γ of Σ on (M, ω) is symplectic. Let L be the Hermitian line bundle over M and ∇ the compatible connection in L constructed by Freed [7]. By [7, Proposition 5.27], the curvature of ∇ is \sqrt{-1} \frac{i}{2\pi} \omega. Given any element σ in the Teichmüller space T the symplectic manifold M can be equipped with a Kähler structure so that L becomes a holomorphic ample line bundle L_σ. The Verlinde bundle V_k is defined by

\[ V_k = H^0(M_\sigma, L_\sigma^k). \]

It is known by the works of Axelrod-Della Pietra-Witten [5] and Hitchin [8] that the projective bundle P(V_k) is equipped with a natural flat connection. Since there is an action of the mapping class group Γ of Σ on V_k covering its action on T, which preserves the flat connection in P(V_k), we get for each k, a finite dimensional projective representation. This sequence of projective representations π_{k,n,d}, k ∈ N_+, is the quantum SU(n) representation of the mapping class group Γ.

V. Turaev [14] conjectured that there should be no nontrivial element of the mapping class group acting trivially under π_{k,n,d} for all k, keeping (n, d) fixed.
This property is called \textit{asymptotic faithfulness} of the quantum SU\((n)\) representations \(\pi_{k}^{n,d}\). In \cite[Theorem 1]{1}, J. E. Andersen proved Turaev’s conjecture, namely the following

\textbf{Theorem 1.1} \cite[Theorem 1]{1}). Let \(\pi_{k}^{n,d}\) be the representation of the mapping class group. Assume that \(n\) and \(d\) are coprime or that \((n, d) = (2, 0)\) when \(g = 2\), then

\begin{equation}
\bigcap_{k=1}^{\infty} \text{Ker}(\pi_{k}^{n,d}) = \begin{cases} \{1, H\}, & g = 2, \quad (n, d) = (2, 0) \\ \{1\}, & \text{otherwise}, \end{cases}
\end{equation}

where \(H\) is the hyperelliptic involution on genus \(g = 2\) surfaces.

This theorem is proved in \cite{1} by considering the action of the mapping class group on functions on \(M\) as symbols of Toeplitz operators on holomorphic sections of \(\mathcal{L}_{\sigma}^{k}\) for large \(k\). A different proof using skein theorem is given in \cite{10}; see also \cite{2, 3, 4} and references therein for further developments. The existing proofs seem rather involved. We shall give a somewhat more direct and elementary proof using peak sections in the Kodaira embedding.

We describe briefly our approach to Theorem 1.1. Write \(\pi_{k}^{n,d}\) as \(\pi_{k}\) throughout the rest of the paper.

For any element \(\phi\) of the mapping class group \(\Gamma\) and any point \(\sigma \in \mathcal{T}\), let \(\sigma(t) : [0, 1] \to \mathcal{T}\) be a smooth curve connecting \(\phi(\sigma)\) and \(\sigma\). Denote by \(P_{\phi(\sigma), \sigma(t)}\) the parallel transport from \(\phi(\sigma)\) to \(\sigma(t)\) with respect to the projective flat connection \((2.1)\) below. For any \(s \in H^{0}(M_{\sigma}, \mathcal{L}_{\sigma}^{k})\), set

\[ s(t) := P_{\phi(\sigma), \sigma(t)} \circ \phi^{*}(s) \in H^{0}(M_{\sigma(t)}, \mathcal{L}_{\sigma(t)}^{k}). \]

Here \(\phi^{*}\) is the induced action of \(\phi\) on the total space of the Verlinde bundle. For any smooth function \(\rho : M \to (0, 1]\) define a rescaled Hermitian structure on \(\mathcal{H}_{k}\) by

\[ \langle s_{1}, s_{2} \rangle_{\rho} = \int_{M} \rho(s_{1}, s_{2}) \frac{\omega^{m}}{m!}, \quad \|s\|_{\rho}^{2} = \langle s, s \rangle_{\rho}. \]

We shall study the variation of \(\|s(t)\|_{\rho}^{2}\) and obtain

\begin{equation}
\tag{1.2} e^{-\frac{C_{e+kC}}{\kappa+n}} \|s\|_{\rho \circ \phi = 1}^{2} \leq \|P_{\phi(\sigma), \sigma} \phi^{*}(s)\|_{\rho}^{2} \leq e^{\frac{C_{e+kC}}{\kappa+n}} \|s\|_{\rho \circ \phi = 1}^{2},
\end{equation}

(see Lemma 3.2). Here \(C_{e}\) and \(C\) are positive constants independent of \(k\). We prove that if \(\phi \in \bigcap_{k=1}^{\infty} \text{Ker}\pi_{k}\) then the induced action of \(\phi\) on \(M\) is the identity. First of all it follows that the representation \(\phi \to P_{\phi(\sigma), \sigma} \circ \phi^{*}\) is projectively trivial on the space \(H^{0}(M_{\sigma}, \mathcal{L}_{\sigma}^{k})\),

\[ P_{\phi(\sigma), \sigma} \circ \phi^{*} = \pi_{k}(\phi) = c_{k} \text{Id} \]

for some constant \(c_{k} = c_{k}(\phi) \neq 0\). By taking \(\rho = 1\) and using \((1.2)\), we get a lower bound of \(c_{k}^{2}\), i.e. \(c_{k}^{2} \geq e^{-\frac{C_{e+kC}}{\kappa+n}}\), which converges to \(e^{-C}\) as \(k \to \infty\), so \(c_{k}^{2} > c\) for some constant \(c > 0\). If \(\phi\) on \(M\) is not the identity, say \(\phi(p) \neq p\) we can construct appropriate weight function \(\rho\) and peak section \(s\) at \(p\) so that
the right hand side $e^{-k/n} ||s||^2_{\rho,\phi^{-1}}$ is arbitrarily smaller than $e^{-C}$ while as $\|P_{\phi(\sigma)} \circ \phi^*(s)\|^2_{\rho} = c_k^2 ||s||^2_{\rho}$ has a uniform lower bound $e^{-C}$, a contradiction to (1.2). Thus $\phi$ acts as identity on $M$, and it follows further by standard arguments that $\phi$ itself is the identity element in $\Gamma$ under the assumption on $\{g, n, d\}$ or a hyperelliptic involution for genus $g = 2$ surfaces.

This article is organized as follows. In Section 2 we fix notation and recall some basic facts on the Verlinde bundle, the projective flat connection and the peak section. Theorem 1.1. is proved in Section 3.

We would like to thank Jorgen Andersen for some informative explanation of his results.

2. Preliminaries

The results in this section can be found in [1, 5, 8, 9, 13] and references therein.

Let $\Sigma$ be a closed oriented surface of genus $g \geq 2$ and $p \in \Sigma$. Let $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)} = \{cI, c^n = 1\}$, the center of $SU(n)$. We assume that $n$ and $d$ are coprime, in the case of $g = 2$ we also allow $(n, d) = (2, 0)$. Let $M$ be the moduli space of flat $SU(n)$-connections $P$ on the $\Sigma \setminus \{p\}$ with fixed holonomy $d$ around $p$. $M$ is then a compact smooth manifold of dimension $m = (n^2 - 1)(g - 1)$ with tangent vectors given by the Lie algebra $su(n)$-valued connection 1-forms.

There is a canonical symplectic form $\omega$ on $M$ by taking the trace of the integration of products of 1-forms, the natural action of the mapping class group $\Gamma$ on $M$ is symplectic. Let $L$ be the Hermitian line bundle over $M$ and $\nabla$ the compatible connection in $L$ with curvature $\sqrt{-1}/2\pi \omega$; see [7, Proposition 5.27]. The induced connection in $L^k$ will also be denoted by $\nabla$.

Let $T$ be the Teichmüller space of $\Sigma$ parametrizing all marked complex structures on $\Sigma$. By a classical result of Narasimhan and Seshadri [11], each $\sigma \in T$ induces a Kähler structure on $M$ and thus a Kähler manifold $M_\sigma$. By using the $(0, 1)$-part of $\nabla$, the bundle $L$ is then equipped with a holomorphic structure, which we denote by $L_\sigma$. Thus the manifold $T$ also parameterizes Kähler structures $I_\sigma$, $\sigma \in T$ on $(M, \omega)$ and the holomorphic line bundles $L_\sigma$. For any positive integer $k$, we have inside the trivial bundle $H^0(M, L_{\sigma}^k)$ the finite dimensional subbundle, Verlinde bundle $V_k$, given by

$$V_k(\sigma) = H^0(M_\sigma, L_{\sigma}^k)$$

for $\sigma \in T$. By Axelrod, Della Pietra, Witten [5] and Hitchin [8], there is a projective flat connection in $V_k$ given by

$$\hat{\nabla}_v = \hat{\nabla}_v^t - u(v), \quad v \in T(T),$$

(2.1)
where $\nabla^T$ is the trivial connection in $\mathcal{H}_k$. The expression of $u(v)$ can be found in [1, Formula (7)], and is given by

\begin{equation}
(2.2) \quad u(v) = \frac{1}{2(k + n)} \left( \sum_{r=1}^{R} \nabla_{X_r(v)} \nabla_{Y_r(v)} + \nabla_{Z(v)} + nv[F] \right) - \frac{1}{2} v[F],
\end{equation}

where $F : T \rightarrow C^\infty(M)$ is a smooth function such that $F(\sigma)$ is real-valued on $M$ for all $\sigma \in T$. \{X_r(v), Y_r(v), Z(v)\} $\subset C^\infty(M \sigma, T)$ are a finite set of vector fields of $M \sigma$ taking value in the holomorphic tangent space $T$.

Since $\mathcal{L}_\sigma$ is an ample line bundle over $M \sigma$, one may take a large $k$ such that $\mathcal{L}_\sigma^k$ is a very ample line bundle. Then the Kodaira embedding is given by

$$
\Phi_\sigma : M \rightarrow \mathbb{P}(H^0(M \sigma, \mathcal{L}_\sigma^k)^*), \quad p \mapsto \Phi_\sigma^k(p) = \{s \in H^0(M \sigma, \mathcal{L}_\sigma^k), s(p) = 0\}.
$$

A peak section $s^k_p \in H^0(M \sigma, \mathcal{L}_\sigma^k)$ of $\mathcal{L}_\sigma^k$ at a point $p \in M$ is a unit norm generator of the orthogonal complement of $\Phi_\sigma^k(p)$ such that

$$
|s^k_p(p)|^2 = \sum_{i=1}^{N_k} |s_i(p)|^2,
$$

where $N_k = \dim H^0(M \sigma, \mathcal{L}_\sigma^k)$ and $\{s_i\}_{1 \leq i \leq N_k}$ is an orthonormal basis of $H^0(M \sigma, \mathcal{L}_\sigma^k)$ with respect to the standard $L^2$-metric; see [9, Definition 5.1.7]. The existence of peak sections is well-known, and for any sequence $\{r_k\}$ with $r_k \rightarrow 0$ and $r_k \sqrt{k} \rightarrow \infty$, one has

\begin{equation}
(2.3) \quad \int_{B(p, r_k)} |s^k_p(x)|^2 \omega^n \frac{1}{n!} = 1 - o(1), \quad \text{for } k \rightarrow \infty;
\end{equation}

see e. g. [9, Formula (5.1.25)] and [13, Lemma 1.2].

### 3. A DIRECT APPROACH TO THE ASYMPTOTIC FAITHFULNESS

In this section, we will present an elementary proof of Theorem 1.1 using peak sections.

Fix $\sigma \in T$. For any $\phi \in \Gamma$, the mapping class group of $\Sigma$, let $\sigma(t) : [0, 1] \rightarrow T$ be a smooth curve with $\sigma(0) = \phi(\sigma), \sigma(1) = \sigma$. For any $s \in H^0(M \sigma, \mathcal{L}_\sigma^k)$, set

\begin{equation}
(3.1) \quad s(t) := P_{\phi(\sigma), \sigma(t)} \circ \phi^*(s) \in H^0(M_{\sigma(t)}, \mathcal{L}_{\sigma(t)}^k),
\end{equation}

where $P_{\phi(\sigma), \sigma(t)}$ is the parallel transport from $\phi(\sigma)$ to $\sigma(t)$ with respect to the projective flat connection $\mathcal{L}_\sigma^k$. For any smooth function $\rho : M \rightarrow (0, 1]$ we define a Hermitian structure on $\mathcal{H}_k$ by

\begin{equation}
(3.2) \quad \langle s_1, s_2 \rangle = \int_M \rho(s_1, s_2) \frac{\omega^n}{m!},
\end{equation}

and denote $\|s\|^2 = \langle s, s \rangle$. (We note that the question of projectiveness of the norm (3.2) with respect to the connection (2.1) is systematically studied in [12].)
Lemma 3.1. We have the following estimate for the differential operator $u$ along $\sigma(t)$,

$$\left| \langle u(\sigma'(t))s(t), s(t) \rangle \right| \leq \frac{C_\rho + kC}{2(k + n)} \| s(t) \|_\rho^2,$$

where the constant $C = \max_{[0,1] \times M} \left| \frac{\partial F(\sigma(t))}{\partial t} \right|$ and

$$C_\rho = \max_{[0,1] \times M} | \Lambda_t \bar{\partial}_k (Z(\sigma'(t))^\ast \rho)^{-1} \sum_{r=1}^{R} \max_{[0,1] \times M} | \Lambda_t \bar{\partial}_k (Y_r(\sigma'(t))^\ast \Lambda_t \bar{\partial}_k (X_r(\sigma'(t))^\ast \rho)) \rho^{-1} |$$

are independent of $k$. Here $X^*$ denotes the dual 1-form of $X$ such that $X^*(X) = |X|^2_\omega$.

Proof. By (2.2) and (3.2) we have

$$\left| \langle u(\sigma'(t))s(t), s(t) \rangle \right| = \left| \int_M (\rho u(\sigma'(t))s(t), s(t)) \frac{\omega^m}{m!} \right|$$

$$\leq \frac{1}{2(k + n)} \left| \int_M (\nabla_{Z(\sigma'(t))} s(t), \rho s(t)) \frac{\omega^m}{m!} \right|$$

$$+ \frac{1}{2(k + n)} \left| \int_M \left( \sum_{r=1}^{R} \nabla_{X_r(\sigma'(t))} \nabla_{Y_r(\sigma'(t))} s(t), \rho s(t) \right) \frac{\omega^m}{m!} \right|$$

$$+ \frac{k}{2(k + n)} \left| \int_M \rho \left( \frac{\partial F(\sigma(t))}{\partial t} \right) s(t), s(t) \right) \frac{\omega^m}{m!} \right|. \tag{3.3}$$

The tangent vectors $X, Y, Z$ are $(1,0)$-vectors and $\nabla$ above can all be replaced by $\nabla^{(1,0)}$, which we still denote by $\nabla$. By [6, Chapter VII, Theorem (1.1)], the adjoint of $\nabla$ on $M_{\sigma(t)}$ is $\nabla^{(1,0),\ast} = \sqrt{-1} [\Lambda_t, \bar{\partial}_k]$ where $\Lambda_t$ is the adjoint of multiplication operator $\omega \wedge \cdot$ by the Kähler metric.

The first term in the RHS of (3.3) can be estimated as

$$\left| \int_M (\nabla_{Z(\sigma'(t))} s(t), \rho s(t)) \frac{\omega^m}{m!} \right| = \left| \langle \nabla_{Z(\sigma'(t))} s(t), \rho s(t) \rangle \right|$$

$$= \left| \langle s(t), \nabla^\ast (Z(\sigma'(t))^\ast \rho s(t)) \rangle \right|$$

$$= \left| \langle s(t), \sqrt{-1} \Lambda_t \bar{\partial}_k (Z(\sigma'(t))^\ast \rho)^{-1} \cdot \rho s(t) \rangle \right|$$

$$\leq \max_{[0,1] \times M} | \Lambda_t \bar{\partial}_k (Z(\sigma'(t))^\ast \rho)^{-1} | \cdot \| s(t) \|_\rho^2, \tag{3.4}$$
where the third equality holds since \( s(t) \) is a holomorphic section of \( \mathcal{L}^k_{\sigma(t)} \), i.e. \( \bar{\partial}_t s(t) = 0 \). Similarly the second term is bounded by

\[
\left| \int_M \left( \sum_{r=1}^R \nabla_{x_r(\sigma'(t))} \nabla_{y_r(\sigma'(t))} s(t), \rho s(t) \right) \frac{\omega^m}{m!} \right| \\
\leq \sum_{r=1}^R \left| \left( \nabla_{x_r(\sigma'(t))} \nabla_{y_r(\sigma'(t))} s(t), \rho s(t) \right) \right| \\
\leq \sum_{r=1}^R \left| \left( s(t), -\Lambda_t \bar{\partial}_t \left( Y_r(\sigma'(t))^* \Lambda_t \bar{\partial}_t (X_r(\sigma'(t))^* \rho) \right) \rho^{-1} \cdot \rho s(t) \right) \right| \\
\leq \sum_{r=1}^R \max_{[0,1] \times M} \left| \Lambda_t \bar{\partial}_t \left( Y_r(\sigma'(t))^* \Lambda_t \bar{\partial}_t (X_r(\sigma'(t))^* \rho) \right) \rho^{-1} \right| \cdot \| s(t) \|_\rho^2.
\]

For the last term in the RHS of (3.3), we have

\[
\left| \int_M \rho \left( \frac{\partial F(\sigma(t))}{\partial t} s(t), s(t) \right) \frac{\omega^m}{m!} \right| \leq \max_{[0,1] \times M} \left| \frac{\partial F(\sigma(t))}{\partial t} \right| \cdot \| s(t) \|_\rho^2.
\]

Substituting (3.4), (3.5) and (3.6) into (3.3), we obtain

\[
\frac{1}{2(k+n)} \max_{[0,1] \times M} \left| \Lambda_t \bar{\partial}_t (Z(\sigma'(t))^* \rho) \rho^{-1} \right| \cdot \| s(t) \|_\rho^2 \\
+ \frac{1}{2(k+n)} \sum_{r=1}^R \max_{[0,1] \times M} \left| \Lambda_t \bar{\partial}_t \left( Y_r(\sigma'(t))^* \Lambda_t \bar{\partial}_t (X_r(\sigma'(t))^* \rho) \right) \rho^{-1} \right| \cdot \| s(t) \|_\rho^2 \\
+ \frac{k}{2(k+n)} \max_{[0,1] \times M} \left| \frac{\partial F(\sigma(t))}{\partial t} \right| \cdot \| s(t) \|_\rho^2 = C_{\rho} + kC \cdot \| s(t) \|_\rho^2.
\]

completing the proof.

\[\square\]

**Proposition 3.2.** We have the following estimate for the norm of the parallel transport \( P_{\phi(\sigma),\sigma} \),

\[
e^{-\frac{C_{\rho} + kC}{k+n}} \| s \|^2_{\rho_\phi^{-1}} \leq \| P_{\phi(\sigma),\sigma} \phi^*(s) \|^2_\rho \leq e^{-\frac{C_{\rho} + kC}{k+n}} \| s \|^2_{\rho_\phi^{-1}},
\]

for all \( s \in H^0(M_\sigma, \mathcal{L}_\sigma^k) \).

**Proof.** Using the definition of \( s(t) \) in (3.1) we have

\[
\nabla_{\sigma'(t)} s(t) = 0.
\]
By (2.1) and (3.8) we deduce that
\[
\frac{d}{dt} \|s(t)\|_\rho^2 = \langle \nabla^t_{\sigma'(t)}s(t), s(t) \rangle_\rho + \langle s(t), \nabla^t_{\sigma'(t)}s(t) \rangle_\rho
\]
\[
= \int_M (\rho u(\sigma'(t))) s(t), s(t) + \langle s(t), \rho u(\sigma'(t)) s(t) \rangle_\rho \omega^m_m!
\]
\[
= 2 \Re(a(u(\sigma'(t)) s(t), s(t))_\rho.
\]
This is treated in Lemma 3.1 and we find
\[
- \frac{C_\rho + kC}{k + n} \|s(0)\|_\rho^2 \leq \frac{d}{dt} \|s(t)\|_\rho^2 \leq \frac{C_\rho + kC}{k + n} \|s(t)\|_\rho^2.
\]
Hence
\[
e^{-\frac{C_\rho + kC}{k + n} \|s(0)\|_\rho^2} \leq \|s(1)\|_\rho^2 \leq e^{\frac{C_\rho + kC}{k + n} \|s(0)\|_\rho^2}.
\]
Now \(\sigma(t)\) is a curve from \(\phi(\sigma)\) to \(\sigma\), \(P_{\phi(\sigma), \sigma(0)} = P_{\phi(\sigma), \phi(\sigma)} = \text{Id}, \sigma(1) = \sigma\), and
\[
s(0) = \phi^*(s), \quad s(1) = P_{\phi(\sigma), \sigma} \phi^*(s).
\]
The norm of \(s(0)\) is given by
\[
\|s(0)\|_\rho^2 = \|\phi^*s\|_\rho^2 = \int_M \rho|\phi^*s|^2 \omega^m_m! = \int_M \rho|s|^{2 \omega^m_m!}
\]
\[
= \int_M \rho \circ \phi^{-1}|s|^2 \omega^m_m! = \|s\|^2 \rho_{\phi^{-1}}.
\]
Here we have used the fact that \(\phi\) induces a symplectomorphism of \(M\), i.e. \(\phi^*\omega = \omega\).

Combining (3.10) and (3.9) we have
\[
e^{-\frac{C_\rho + kC}{k + n} \|s\|^2 \rho_{\phi^{-1}}} \leq \|\phi(\sigma), \sigma) \phi^*(s)\|^2_\rho \leq e^{\frac{C_\rho + kC}{k + n} \|s\|^2 \rho_{\phi^{-1}}}.
\]

We prove now Theorem 1.1.

The proof of Theorem 1.1: We consider first the case of \(g \geq 3\), \(n\) and \(d\) are coprime. Suppose \(\phi \in \bigcap_{k=1}^\infty \ker \pi_k\). We prove that \(\phi\) is the identity element.

The projective representation of the mapping class group \(\Gamma\) is defined via the flat connection, in particular \(\Gamma\) acts on the space of covariant constant sections over Teichmüller space,
\[
P_{\phi(\sigma), \sigma} \circ \phi^* = \pi_k(\phi) = c_k \text{Id}
\]
when acting on the element of \(H^0(M_\sigma, L^k_\sigma)\) for some constant \(c_k \neq 0\).

By taking \(\rho = 1\) and using Lemma 3.2, we get
\[
e^{-\frac{C_1 + kC}{k + n} \|s\|^2 \rho_{\phi^{-1}}} \leq c_k^2 \leq e^{\frac{C_1 + kC}{k + n} \|s\|^2 \rho_{\phi^{-1}}}
\]
We prove first $\phi$ acts on $M$ as identity. Otherwise suppose $\phi \neq \text{Id}$. Then there exists a point $p \in M$ such that $p \neq \phi^{-1}(p)$. Let $V_p, U_p \subset M$ be two small neighborhoods of $p$ with

$$p \in V_p \Subset U_p, \quad \phi^{-1}(V_p) \subset M - U_p.$$  

Let $\rho : M \to (0, 1]$ be a smooth function on $M$ satisfying

$$\rho(x) = \begin{cases} 
1, & x \in V_p, \\
\frac{1}{e^{2C} + 1}, & x \in M - U_p.
\end{cases}$$  

(3.15)

For each large $k$ we take the initial section $s$ to be the peak section $s^k_p$ of the point $p$. By (3.12), (3.13), (3.15) and (2.3), we find

$$\|P_{\phi^i, \sigma} \circ \phi^i(s^k_p)\|_\rho = c_k^2 \int_M \rho|s^k_p|^2 \frac{\omega^m}{m!} \geq e^{-\frac{C_1 + kC}{k + n}} \int_{V_p} |s^k_p|^2 \frac{\omega^m}{m!} \geq e^{-\frac{C_1 + kC}{k + n}} (1 - o(1)).$$  

(3.16)

On the other hand, by (3.14), (3.15) and (2.3), we have also

$$\|s^k_p\|_{\rho \circ \phi^{-1}}^2 = \int_M \rho \circ \phi^{-1}|s^k_p|^2 \frac{\omega^m}{m!} \leq \frac{1}{e^{2C} + 1} \int_{V_p} |s^k_p|^2 \frac{\omega^m}{m!} + \int_{M - V_p} |s^k_p|^2 \frac{\omega^m}{m!} \leq \frac{1}{e^{2C} + 1} + o(1).$$  

(3.17)

Substituting (3.16) and (3.17) into (3.7) we obtain

$$e^{-\frac{C_1 + kC}{k + n}} (1 - o(1)) \leq e^{-\frac{C_2 + kC}{k + n}} \left( \frac{1}{e^{2C} + 1} + o(1) \right).$$

As $k \to \infty$ it gives

$$e^{-C} \leq c_k \cdot \frac{1}{e^{2C} + 1} = \frac{e^{-C}}{1 + e^{-2C}} < e^{-C},$$

which is a contradiction. So $\phi$ acts on $M$ as the identity. It follows then from the standard argument [1] that $\phi$ itself is the identity element in $\Gamma$.

Now in the case $g = 2, (n, d) = (2, 0)$, the same proof above concludes that if $\phi \in \bigcap_{k=1}^\infty \text{Ker}(\pi_k^{2,0})$ then it acts trivially on $M$. It is then either the identity or the hyper-elliptic involution $H$; see [1]. On the other hand $H$ indeed acts trivially under all $\pi_k^{2,0}$ by its definition. Thus $\bigcap_{k=1}^\infty \text{Ker}(\pi_k^{2,0}) = \{1, H\}$. \qed
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