Analytic solutions of fractional Integro-Differential Equations of Volterra Type

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Abstract. In this paper, we propose a new method for constructing a solution of the integro-differential equations of Volterra type. The particular solutions of the homogeneous and of the inhomogeneous equation will be constructed and the Cauchy type problems will be investigated. Note that this method is based on construction of normalized systems functions with respect to the differential operator’s fractional order.

1. Introduction
Let $\alpha > 0, \nu \geq 0, \beta \in R, \lambda \neq 0$. On the domain $0 < t \leq d < \infty$ consider an integral-differential equation of the following form:

$$D^\alpha y(t) = \lambda t^\beta I^\nu y(t) + f(t), \quad 0 < t \leq d < \infty,$$

(1)

where for any $\delta > 0 : I^\delta y(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t - \tau)^{\delta-1} y(\tau)d\tau$, and $D^\alpha$ is the derivative of $\alpha$ order in the Riemann-Liouville sense, i.e. $D^\alpha y(t) = \frac{d^m}{dt^m}I^{\alpha-m}y(t)$, $m = [\alpha] + 1$.

Questions related to theorems about existence and uniqueness of solutions of Cauchy type and Dirichlet type problems for linear and nonlinear fractional order differential equations have been developed in sufficient detail (see [1] for the main results, review of papers, and references). In [2], equation (1) was studied in the case $\beta = 0$. In a more general case, an equation of the type (1) and a Cauchy-type problem for them were studied in [3, 4]. Theorems on existence and uniqueness of a solution of Cauchy-type problem have been proved. We note that explicit solutions have been constructed only for certain types of linear differential equations of fractional order. Solutions of certain elementary homogeneous and inhomogeneous equations, obtained by the selection method or by expansion of the desired solution into a quasi-power series, are known. Moreover, explicit solutions of a Cauchy-type problem for certain differential equations of fractional order were found in [1] by the method of reduction to an equivalent Volterra integral equation. Further, in [5], using the properties of Mittag-Leffler type functions:

$$E_{\alpha,m,l}(z) = \sum_{i=0}^{\infty} c_i z^i, c_0 = 1, c_i = \prod_{k=0}^{i-1} \frac{\Gamma[\alpha(km + l) + 1]}{\Gamma[\alpha(km + l + 1) + 1]}, i \geq 1,$$

(2)
an algorithm for constructing a solution of the differential equation (1) in the case \( \nu = 0 \) was proposed. Moreover, the case, when \( f(t) = 0 \) and \( f(t) \) is a quasi-polynomial, was considered. In this paper we propose a new method for constructing an explicit solution of integral-differential equations of fractional order. This method is based on construction of normalized systems. Let \( f \) be defined in a domain \( \Omega \) and \( \lambda \in X \). Moreover, the case, when \( f(t) \) is called just normalized. The main properties of the \( f \) - normalized systems of functions with respect to the operators \( (L_1, L_2) \) on \( \Omega \) have been described in [8]. Let us consider the main property of the \( f \) - normalized systems.

**Proposition 1.** If a system of functions \( \{f_i(x)\}_{i=0}^{\infty} \) is \( f \) - normalized with respect to \( (L_1, L_2) \) on \( \Omega \), then the functional series \( y(x) = \sum_{i=0}^{\infty} f_i(x), x \in \Omega \), is a formal solution of the equation:

\[
(L_1 - L_2) y(x) = f(x), x \in \Omega.  
\]

The next proposition allows to construct a \( f \) - normalized system with respect to a pair of operators \( (L_1, L_2) \).

**Proposition 2.** If for \( L_1 \) there exists a right inverse operator \( L_1^{-1} \), i.e. \( L_1 \cdot L_1^{-1} = E \), where \( E \) is a unit operator and \( L_1 f_0(x) = f(x) \), then a system of the functions

\[
f_i(x) = \left( L_1^{-1} \cdot L_2 \right)^i f_0(x), i \geq 1,  
\]

is \( f \) - normalized with respect to a pair of the operators \( (L_1, L_2) \) on \( \Omega \).

**Proof.** Since \( L_1 \cdot L_1^{-1} = E \) is the unit operator, then for all \( i = 1, 2, ..., \), we have

\[
L_1 f_i(x) = L_1 \left( L_1^{-1} \cdot L_2 \right) \left( L_1^{-1} \cdot L_2 \right)^{i-1} f(x) = L_2 \left( L_1^{-1} \cdot L_2 \right)^{i-1} f(x) = L_2 f_{i-1}(x).  
\]

Consequently, \( L_1 f_i(x) = L_2 f_{i-1}(x) \) and by assumption of the theorem \( L_1 f_0(x) = f(x) \) i.e. the system \( f_i(x) = \left( L_1^{-1} \cdot L_2 \right)^i f_0(x), i \geq 0, \) is \( f \) - normalized with respect to the pair of operators \( (L_1, L_2) \).
3. Properties of operators $I^\alpha$ and $D^\alpha$

We denote $C_{\delta}[0,d] = \{f(t) : \exists \delta \in [0, 1), t^\delta f(t) \in C[0,d]\}$.

The following statements are known [1].

**Lemma 1.** Let $\alpha > 0$. Then for all $f(t) \in C_{\delta}[0,d]$ the equality

$$D^\alpha [I^\alpha[f]](t) = f(t)$$

holds for all $t \in (0,d)$. If $f(t) \in C[0,d]$, then (5) holds for all $t \in [0,d]$.

**Lemma 2.** Let $\alpha > 0$, $m = [\alpha] + 1$ $s \in R$. Then the following equalities hold:

$$I^{\alpha t^s} = \frac{\Gamma(s + 1)}{\Gamma(s + 1 + \alpha)} t^{s+\alpha}, s > -1,$$

$$D^{\alpha t^s} = \frac{\Gamma(s + 1)}{\Gamma(s + 1 - \alpha)} t^{\alpha-s}, s > \alpha-1, D^{\alpha t^s} = 0, s = \alpha - j, j = 1, 2, ..., m.$$  

**Lemma 3.** Let $\alpha > 0$, $m = [\alpha] + 1, 0 \leq \delta < 1$ and $f(t) \in C_{\delta}[a,b]$. Then

1) if $\alpha < \delta$, then $I^{\alpha f(t)} \in C_{\delta-a}[a,b]$ and

$$||I^{\alpha f}||_{C_{\delta-a}[a,b]} \leq M||f||_{C_{\delta}[a,b]}, M = \frac{\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)};$$

2) if $\alpha \geq \delta$, then $I^{\alpha f(t)} \in C[a,b]$ and

$$||I^{\alpha f}||_{C[a,b]} \leq M||f||_{C_{\delta}[a,b]}, M = \frac{(b-a)^{\alpha-\delta}\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)}.$$  

4. Construction of $f-$ normalized systems

In this section we construct normalized systems with respect to the pair of the operators

$$\left(D^\alpha, \lambda t^\beta I^\nu\right).$$

To do it from Proposition 2 it follows that it is necessary to find all solutions of the equation $D^\alpha y(t) = 0$ and a right inverse for the operator $D^\alpha$. According to statement of Lemma 1, the right inverse of the the operator $D^\alpha$ is the operator $I^\alpha$, and due to (6) linear independent solutions of the equation $D^\alpha y(t) = 0$ are functions $t^{\alpha s}, s_j = \alpha - j, j = 1, 2, ..., m$. Hereinafter, we denote $L_1 = D^\alpha$ and $L_2 = \lambda t^\beta I^\nu$. Then the equation (1) is represented as (3).

For real numbers $\alpha > 0, \nu \geq 0, \delta > 0, s \in R$ we introduce the following coefficients:

$$C_{\alpha,\nu}(\delta, s, i) = \prod_{k=0}^{i-1} \frac{\Gamma(\delta k + s + 1)}{\Gamma(\delta k + s + 1 + \nu)} \cdot \frac{\Gamma[\delta(k + 1) + s + 1 - \alpha]}{\Gamma[\delta(k + 1) + s + 1]}, i \geq 1, C_{\alpha,\nu}(\delta, s, 0) = 1, s \in R.$$  

We assume that $C_{\alpha,\nu}(\delta, s, i) \neq 0$. Let $s_j = \alpha - j, j = 1, 2, ..., m$ and $f_{0,s_j}(t) = t^{s_j}$, then due to (7): $L_1 f_{0,s_j}(t) = 0, j = 1, 2, ..., m$. We consider the system of functions:

$$f_{i,j}(t) = \left(I^\alpha \cdot \lambda t^\beta I^\nu\right)^i f_{0,s_j}(t), i = 1, 2, ..., m.$$  

(9)

Since $(D^\alpha)^{-1} = I^\alpha$ and $D^\alpha f_{0,s_j}(t) = 0$, then Proposition 2 implies that the system (9) is 0–normalized with respect to the pair of the operators $(D^\alpha, \lambda t^\beta I^\nu)$. We find explicit form of the system of functions $f_i(t)$. Hereinafter, everywhere we will assume that $\alpha > 0, m = [\alpha] + 1, \nu \geq 0, \beta > -\{\alpha\} - \nu$. The following proposition is valid.

**Lemma 4.** Let $s \geq \alpha - m$. Then for the functions $g_i(t) = \left(I^\alpha \cdot t^\beta I^\nu\right)^i t^s, i \geq 1$, the following equality holds:

$$g_i(t) = C_{\alpha,\nu}(\alpha + \beta + \nu, s, i) t^{(\alpha + \beta + \nu)i+s}, i = 1, 2, ...$$

(10)
From the lemma in the cases $j = \alpha - j, j = 1, 2, ..., m$, we obtain

$$C_{\alpha,\nu}(\delta, \alpha - j, i) = \prod_{k=0}^{i-1} \frac{\Gamma[\nu(k+1)\delta + 1 + \alpha - j]}{\Gamma[\nu(k+1)\delta + 1 + \alpha + j]} \cdot \frac{\Gamma[(k+1)\delta + 1 - j]}{\Gamma[(k+1)\delta + 1 + \alpha - j]}, i \geq 1, \delta = \alpha + \beta + \nu. \quad (11)$$

Consider the function

$$u_j(z) = \sum_{i=0}^{\infty} C_{\alpha,\nu}(\alpha + \beta + \nu, s_j, i) z^i,$$  

(12)

where $z$ is a complex number.

In [9] it is shown that for the coefficients of the function (2) the following asymptotic estimate holds:

$$\frac{c_i}{c_{i+1}} = \frac{\Gamma[\alpha(\nu + 1) + 1]}{\Gamma[\alpha(\nu + 1)]} \sim (\alpha \nu)^i \alpha \nu \to \infty$$

Thus, the function (2) is entire. Denote $\delta = \alpha + \beta + \nu$ and rewrite the coefficients $C_{\alpha,\nu}(\delta, \alpha, j, i)$ as follows:

$$C_{\alpha,\nu}(\delta, \alpha - j, i) = \prod_{k=0}^{i-1} \frac{\nu(k+1)\delta + 1 + \alpha - j}{\nu(k+1)\delta + 1 + \alpha + j} \cdot \frac{\Gamma[\alpha(\nu + 1) + 1]}{\Gamma[\alpha(\nu + 1)]}, i \geq 1, \nu > 0.$$

Further, the asymptotic estimate

$$\frac{C_{\alpha,\nu}(\delta, \alpha - j, i)}{C_{\alpha,\nu}(\delta, \alpha, j, i+1)} \sim (\delta i)^{\nu + \alpha} \to \infty (i \to \infty)$$

yields that $u_j(z), j = 1, 2, ..., m$, from (12) are also entire functions. Lemma 4 and Proposition 2 implies the following

**Lemma 5.** Let $s_j = \alpha - j, j = 1, 2, ..., m$. Then at all values $j = 1, 2, ..., m$ the system of functions

$$f_{i,j}(t) = \lambda^i C_{\alpha,\nu}(\alpha + \beta + \nu, s_j, i) t^{(\alpha + \beta + \nu)i + s_j}, i = 0, 1, ...$$

is 0-normalized with respect to the pair of operators $(D^\alpha, \lambda^i t^\nu)$ on the domain $t > 0$.

Using the main property of normalized systems we get the following proposition:

**Theorem 1.** Let $s_j = \alpha - j, j = 1, 2, ..., m$. Then at all values $j = 1, 2, ..., m$ the functions

$$y_j(t) = \sum_{i=0}^{\infty} f_{i,j}(t) = t^{s_j} \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}(\alpha + \beta + \nu, s_j, i) t^{(\alpha + \beta + \nu)i} \quad (13)$$

are linearly independent solutions of the homogenous equation (1). Moreover, for all $j = 1, 2, ..., m - 1, y_j(t) \in C[0, d]$ and $y_m(t) \in C_{m-\alpha}[0, d]$.

Now we turn to construction of a solution of inhomogenous equation. Let $f(t) \in C[0, d]$. Then by the statement of Lemma 1 for the function $f_0(t) = I^\alpha f(t)$ the following equality is true:

$$L_1 f_0(t) = D^\alpha I^\alpha f(t) = f(t).$$

Consider the system:

$$f_i(t) = \left( I^\alpha \lambda^i t^\nu \right)^i f_0(t) \equiv \lambda^i \left( I^\alpha t^\nu \right)^i f_0(t), i = 1, 2, ... \quad (14)$$
Lemma 6. Let \( f(t) \in C[0,d], d < \infty \). Then the system of functions (14) is \( f(t) \) - normalized with respect to the pair of operators \( (D^\alpha, \lambda t^\beta I^\nu) \) on the domain \( t > 0 \).

Theorem 2. Let \( f(t) \in C[0,d] \) and functions \( f_i(t), i \geq 0 \) be defined by (14). Then the function

\[
y_f(x) = \sum_{i=0}^{\infty} f_i(t)
\]

is a particular solution of the equation (1) from the class \( C[0,d] \).

Proof. Estimate the series (15). We have

\[
|y_f| \leq \sum_{i=0}^{\infty} |f_i(t)| \leq \frac{||f||_{C[0,d]}}{(\Gamma(\alpha + 1))} \left[ 1 + \sum_{i=1}^{\infty} |\lambda|^i \alpha, \nu \right] t^{i(\alpha + \beta)}.
\]

Since the last series is uniformly convergent on the domain \( 0 \leq t \leq d \), then sum of the series, and hence the function \( y_f(t) \) belong to the class \( C[0,d] \). Theorem is proved.

Now we study representation of the functions (14) for certain particular cases of functions \( f(t) \).

Lemma 7. Let \( f(t) = t^\mu, \mu > -1 \). Then a particular solution of the equation (1) has the following form:

\[
y_f(t) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + \alpha)} \sum_{k=0}^{\infty} \lambda^k C_{\alpha,\nu}(\alpha + \beta + \nu, \mu + \alpha, k) t^{k(\alpha + \beta + \nu)}.
\]

Proof of the lemma follows from (10).

Theorem 3. Let \( f(t) = \sum_{j=1}^{p} \lambda_j t^{\mu_j}, \mu_j > -1 \). Then a particular solution of the equation (1) has the following form:

\[
y_f(t) = \sum_{j=1}^{p} \lambda_j \frac{\Gamma(\mu_j + 1)}{\Gamma(\mu_j + 1 + \alpha)} \sum_{k=0}^{\infty} \lambda^k C_{\alpha,\nu}(\alpha + \beta + \nu, \mu_j + \alpha, k) t^{k(\alpha + \beta + \nu)}.
\] (16)

In the case \( \nu = 0 \) the representation (16) of a particular solution of (1) coincides with the result of [5] (see Theorem 2, formula (27)).

Now we give an algorithm for constructing particular solutions of the inhomogeneous equation (1) in the case when \( f(t) \) is an analytic function.

Theorem 4. Let \( f(t) \) be an analytic function. Then a particular solution of the equation (1) has the form

\[
y_f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(\alpha + k + 1)} \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}(\alpha + \beta, k + \alpha, i) t^{i(\alpha + \beta)}.
\] (17)

5. Solution of Cauchy type problem for homogenous equation

Consider the following Cauchy type equation:

\[
D^\alpha y(t) = \lambda t^\beta I^\nu y(t) + \sum_{j=1}^{p} \lambda_j t^{\mu_j}, 0 < t \leq d < \infty,
\] (18)

\[
D^\alpha y(t)|_{t=0} = b_k, k = 1, 2, ..., m - 1,
\] (19)

where \( D^{\alpha-m} = I^{m-\alpha}, b_k \) are real numbers. First we study the homogenous equation (18) - (19).
Theorem 4. Let \( \lambda_j = 0, j = 1, 2, ..., p \). Then solution of the Cauchy equation (18) - (19) exists, unique and can be represented in the form:

\[
y(t) = \sum_{j=1}^{m} \frac{b_j}{\Gamma(\alpha - j + 1)} I^{\alpha-j} \sum_{i=0}^{\infty} \lambda_i^j C_{\alpha,\nu}(\alpha + \beta + \nu, \alpha - j, i) t^{(\alpha + \beta + \nu) i}. \tag{20}
\]

Proof. Let \( \lambda_j = 0, j = 1, 2, ..., p \). According to Theorem 1 the function \( y(t) \) in (20) is solution of the equation (18). Let us show that \( y(t) \) satisfies initial conditions (19). Due to (5) - (6), we have:

\[
I^{\alpha - k} t^{\alpha - j} = 0, j > k, D^{\alpha - k} t^{\alpha - j} = \Gamma[\alpha + \beta + \nu] t^{\alpha + \beta + \nu - k - j}, j \leq k.
\]

Thus, for the functions \( f_{i,j}(t), j = 1, 2, ..., m \), we get:

\[
D^{\alpha - k} f_{i,j}(t) \bigg|_{t=0} = \lim_{t \to 0} D^{\alpha - k} f_{i,j}(t) = \begin{cases} \Gamma(\alpha + \beta + \nu - k - j), i = 0, k = j \\ 0, k \neq j, i \geq 0 \end{cases}.
\]

Then

\[
D^{\alpha - j} y_j(0) = \lim_{t \to 0} D^{\alpha - j} y_j(t) = \Gamma(\alpha - j + 1), D^{\alpha - k} y_j(0) = \lim_{t \to 0} D^{\alpha - k} y_j(t) = 0, k \neq j.
\]

Consequently,

\[
D^{\alpha - k} y(t) \bigg|_{t=0} = \lim_{t \to 0} D^{\alpha - k} y(t) = \frac{b_k}{\Gamma(\alpha - k + 1)} \Gamma(\alpha - k + 1) = b_k.
\]

Theorem is proved.

From the theorems 3 and 4 we get the following statement.

Theorem 5. If \( \mu_j > -1, j = 1, 2, ..., p \), then solution of the Cauchy problem (18) - (19) exists, unique and can be represented in the form:

\[
y(t) = \sum_{j=1}^{m} \frac{b_j I^{\alpha - j}}{\Gamma(\alpha - j + 1)} \sum_{i=0}^{\infty} \lambda_i^j C_{\alpha,\nu}(\alpha + \beta + \nu, \alpha - j, i) t^{(\alpha + \beta + \nu) i} + \sum_{j=1}^{p} \frac{\lambda_j \Gamma(\mu_j + 1) t^{\mu_j}}{\Gamma(\mu_j + 1 + \alpha)} \sum_{k=0}^{\infty} \lambda_k^j C_{\alpha,\nu}(\alpha + \beta + \nu, \mu_j + \alpha, k) t^{k(\alpha + \beta + \nu)}.
\]

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