WEIGHT CONJECTURES FOR $\ell$-COMPACT GROUPS AND SPETSES

RADHA KESSAR, GUNTER MALLE, AND JASON SEMERARO

Abstract. Fundamental conjectures in modular representation theory of finite groups, more precisely, Alperin’s weight conjecture and Robinson’s ordinary weight conjecture, can be expressed in terms of fusion systems. We use fusion systems to connect the modular representation theory of finite groups of Lie type to the theory of $\ell$-compact groups. Under some mild conditions we prove that the fusion systems associated to homotopy fixed points of $\ell$-compact groups satisfy an equation which for finite groups of Lie type is equivalent to Alperin’s weight conjecture.

For finite reductive groups, Robinson’s Ordinary weight conjecture is closely related to Lusztig’s Jordan decomposition of characters and the corresponding results for Brauer $\ell$-blocks. Motivated by this, we define the principal block of a spets attached to a spetsial $\mathbb{Z}_\ell$-reflection group, using the fusion system related to it via $\ell$-compact groups, and formulate an analogue of Robinson’s conjecture for this block. We prove this formulation for an infinite family of cases as well as for some groups of exceptional type.

Our results not only provide further strong evidence for the validity of the weight conjectures, but also point toward some yet unknown structural explanation for them purely in the framework of fusion systems.

1. Introduction and statement of results

In this paper we connect fusion systems originating from homotopy fixed points on $\ell$-compact groups with the representation theory of so-called spet ses by means of a natural generalisation of Alperin’s weight conjecture from modular representation theory of finite groups.

Let $\ell$ be a prime. An $\ell$-compact group is a topological object that may loosely be regarded as a homotopical analogue of a compact Lie group, built from an $\ell$-adic reflection group as Weyl group. It has been shown to give rise to fusion systems on certain finite $\ell$-groups. In representation theory, a spets is a yet rather mysterious analogue of a finite reductive group with a complex reflection group as Weyl group, for which some shadow of Deligne–Lusztig character theory can be formulated. Apparent similarities between...
these two classes of objects have led a number of authors to expect that there should be a more formal connection between them. With this goal in mind, the third author observed various numerical consistencies between the exotic 2-fusion systems associated to a 2-compact group with Weyl group the exceptional complex reflection group $G_{24}$ and an ad hoc defined set of irreducible characters associated to the spets with the same Weyl group, in the spirit of a generalisation of Alperin’s weight conjecture. This famous local-global conjecture from 1986 is an attempt to relate the character theory of a finite group to the fusion system on its Sylow subgroups.

In the present paper, we provide a formal theoretical context to these observations. Thus the goals of the present paper are to:

- associate various global invariants to spetses (such as the principal $\ell$-block);
- formulate analogues for spetses and $\ell$-compact groups of various local-global counting conjectures for groups;
- prove these conjectures (in some cases);
- highlight/explain techniques from algebraic topology relevant to the study of group representations.

In order to motivate and explain our constructions, conjectures and results, we begin by discussing global and local data associated to finite groups of Lie type.

A finite group of Lie type is the group of fixed points $G^F$ of a connected reductive linear algebraic group $G$ under a Steinberg endomorphism $F$ with respect to an $\mathbb{F}_q$-structure for some prime power $q$ coprime to $\ell$. The ordinary irreducible characters of $G^F$ are constructed from the $\ell$-adic cohomology groups of so-called Deligne–Lusztig varieties on which $G^F$ acts. Lusztig has given a combinatorial parametrisation of these characters by Lusztig series $\mathcal{E}(G, s)$ indexed by (classes of) semisimple elements $s$ in the Langlands dual group, which is purely in terms of the Weyl group $W$ of $G$. Furthermore, for any prime $\ell$ different from the defining characteristic of $G$, the distribution of these characters into Brauer $\ell$-blocks has also been shown to be controlled by $W$. This is described by $e$-Harish-Chandra theory, where $e$ denotes the order of $q$ modulo $\ell$. For example, when $\ell$ is very good for $G$ then the unipotent characters of $G^F$ lying in the principal $\ell$-block $B_0$ are those in the principal $e$-Harish-Chandra series. Moreover, the latter is in bijection with the irreducible characters of the corresponding relative Weyl group denoted here $W_\ell$, a complex reflection group provided by Lehrer–Springer theory.

The local data we consider for $G^F$ is most succinctly described in terms of fusion systems. Recall that a fusion system $\mathcal{F}$ on a finite $\ell$-group $S$ is a category with objects the subgroups of $S$ and morphisms certain injective group homomorphisms satisfying some weak axioms. It is called saturated if it satisfies two additional “Sylow axioms”, derived from the motivating example of the fusion system $\mathcal{F}_\ell(G)$ induced by a finite group $G$ on a Sylow $\ell$-subgroup. A subgroup $P \leq S$ is $\mathcal{F}$-centric if $C_S(Q) = Z(Q)$ for all $\mathcal{F}$-conjugates $Q$ of $P$ and $\mathcal{F}$-radical if $\text{Out}_\mathcal{F}(P)$ has no non-trivial normal $\ell$-subgroup. The class $\mathcal{F}^c$ of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups plays an important role in the local-global conjectures we consider. If $k$ is an algebraically closed field of characteristic $\ell$, $G$ is a finite group, $B_0$ is the principal $\ell$-block of $G$ and $\text{IBr}(B_0)$ the set of irreducible $\ell$-Brauer characters of $G$ lying in $B_0$, Alperin’s weight conjecture (AW conjecture) claims the equality
\(|\text{IBr}(B_0)| = w(\mathcal{F}_\ell(G))\) where, for a saturated fusion system \(\mathcal{F}\),

\[
w(\mathcal{F}) := \sum_{P \in \mathcal{F}^\text{cent}/\mathcal{F}} z(k\text{Out}_\mathcal{F}(P)),
\]

and the sum runs over \(\mathcal{F}\)-conjugacy class representatives of \(\mathcal{F}\)-centric, \(\mathcal{F}\)-radical subgroups. Here, for a finite group \(H\), \(z(kH)\) denotes the number of isomorphism classes of projective simple \(kH\) modules.

In the case \(G^F\) is a finite group of Lie type with Weyl group \(W\) then assuming \(\ell\) is very good for \(G\), and \(q\) of order \(e\) modulo \(\ell\), then \(|\text{IBr}(B_0)| = |\text{Irr}(W_e)|\) (see Proposition 4.1). Hence, the above discussion implies that the AW conjecture for the principal \(\ell\)-block is equivalent to the assertion

(1) \[w(\mathcal{F}_\ell(G^F)) = |\text{Irr}(W_e)|.\]

We wish to argue that the left hand side of this equality is also “generic” in the Weyl group \(W\). For this, we require some algebraic topology, specifically the theory of \(\ell\)-compact groups.

To any \(\ell\)-compact group \(X\) is associated a reflection group \(W\) on a \(\mathbb{Z}_\ell\)-lattice \(L\). If \(X\) is connected, then any prime power \(q\) coprime to \(\ell\) determines a self-equivalence \(\psi^q\) of \(X\), unique up to homotopy, called an unstable Adams operator. If \(X\) is moreover simply connected and \(\ell\) is odd, then Broto–Møller have shown in [12] that for any self-equivalence \(\tau\) of \(X\) whose image in the outer automorphism group of \(X\) is of \(\ell'\)-order, the space of homotopy fixed points under \(\tau \psi^q\) is the classifying space of an \(\ell\)-local finite group, and in particular the pair \((q, \tau)\) determines a saturated fusion system \(\mathcal{F}(\tau X(q))\) on a finite \(\ell\)-group (see Theorem 3.2). In Theorem 3.6 we describe the underlying \(\ell\)-group of this fusion system. Moreover \((X, \tau, q)\) determines a certain relative Weyl group \(W_e\) where \(e\) is the order of \(q\) modulo \(\ell\) described in Theorem 3.4. If \(W\) is rational and \(G^F\) is a group of Lie type associated to \(W\) as above, results of Friedlander–Mislin [28] and Quillen [49] imply that for suitable \(X\) and \(\tau\), \(\mathcal{F}_\ell(G^F) = \mathcal{F}(\tau X(q))\) (see Remark 3.3), and in this case the equality (1) is equivalent to

\[w(\mathcal{F}(\tau X(q))) = |\text{Irr}(W_e)|.\]

In view of this, it becomes tempting to ask whether the same equality is true without the assumption that \(W\) is rational, so that \(\mathcal{F}(\tau X(q))\) is a potentially exotic fusion system. Our first local-global result is the following generalisation of the AW conjecture (see Theorem 4.2); here, the new notion of very good prime for a \(\mathbb{Z}_\ell\)-reflection group is defined in terms of its maximal rank subgroups, see Definition 2.4.

**Theorem 1 (AW conjecture for \(\ell\)-compact groups).** Let \(\ell > 2\), \(X\) be a simply connected \(\ell\)-compact group with Weyl group \((W, L)\), \(q\) a prime power prime to \(\ell\) and \(\tau\) an automorphism of \(X\) whose image in the outer automorphism group of \(X\) has finite order prime to \(\ell\). If \(\ell\) is very good for \((W, L)\), then

\[w(\mathcal{F}(\tau X(q))) = |\text{Irr}(W_e)|,\]

where \(e\) is the order of \(q\) modulo \(\ell\).

From this, we recover the validity of the AW conjecture for the principal \(\ell\)-blocks of simply connected groups of Lie type, in particular the previously unknown cases of types
$E_6$, $E_7$, and $E_8$, for all very good primes $\ell > 2$. For the infinite series, we prove Theorem 1 by developing an equivariant version of the Alperin–Fong proof of the AW conjecture for general linear groups $\mathbb{Q}$. We also have results for cases when $\ell$ is not very good, viz. for the Aguadé groups (see Theorem 4.4).

Now let us return to the above situation where $G$ is a connected reductive group. If $\ell$ is very good for $G$ then the ordinary irreducible characters $\text{Irr}(B_0)$ in the principal block are a union of $e$-Harish-Chandra series associated to $\ell$-elements of $G^F$. We seek to describe this union purely in terms of the Weyl group $W$ of $G$.

To this end, we require a notion of $\ell$-subgroups and their centralisers in $\ell$-compact groups. Fortunately, owing to historical interest in homotopy decompositions of classifying spaces, such a theory is available to us. In particular, centralisers of $\ell$-compact groups are again $\ell$-compact. In Theorem 5.2 we prove the following, extending [6, Thm. 1.9] which deals with the case $|Q| = \ell$.

**Theorem 2.** Suppose that $X$ is a connected $\ell$-compact group with torsion-free fundamental group, $Q$ is a finite cyclic $\ell$-group and $f : BQ \to X$ is a morphism. Then the centraliser $C_X(Q, f)$ is connected.

We provide a conceptual argument when the Weyl group has order prime to $\ell$; the general case requires a case-by-case approach as in the case $|Q| = \ell$. Jesper Grodal has communicated another proof to us relying on properties of Lannes $T$-functor.

Now let $q$ be a prime power as above and $\mathcal{F} = \mathcal{F}(\tau(X(q)))$ be the fusion system associated to $X$, $q$ and $\tau$ as above and let $S$ be the underlying $\ell$-group of $\mathcal{F}$. Each $s \in S$ gives rise to a centraliser space $C_X(s)$ defined in terms of a fixed embedding of the classifying space $BS$ of $S$, with Weyl group denoted $W(s)$. If $W$ is spetsial, and $G$ denotes the $\mathbb{Z}_\ell$-spets associated to $W$ and the automorphism of $W$ induced by $\tau$ (see Section 6), then we invoke Theorem 2 to construct a $\mathbb{Z}_\ell$-spets $C_G(s)$. Now attached to any $\mathbb{Z}_\ell$-spets is a collection of unipotent characters whose degrees are polynomials over $\mathbb{Q}_\ell$. By analogy with Lusztig’s theory, we define a set of characters $\mathcal{E}(G(q), s)$ in bijection with the unipotent characters $\text{Uch}(C_G(s))$. Motivated by the situation for finite reductive groups, if $\ell$ is very good for $(W, L)$ we define the principal $\ell$-block

$$\text{Irr}(B_0) := \coprod_{s \in S/\mathcal{F}} \mathcal{E}(G(q), s)_1$$

of $G$, where $\mathcal{E}(G(q), s)_1$ denotes the subset of $\mathcal{E}(G(q), s)$ in bijection with the principal $e$-Harish-Chandra series of $\text{Uch}(C_G(s))$, and the union runs over $\mathcal{F}$-conjugacy class representatives of elements in $S$. We also provide a degree formula for such characters generalising Lusztig’s Jordan decomposition formula for characters of rational spetses. In further analogy with finite group theory, we call $\mathcal{F}$ the fusion system of $G(q)$ and denote it by $\mathcal{F}(G(q))$.

If $G$ is a finite group and $\chi \in \text{Irr}(G)$, the $\ell$-adic valuation $\nu_\ell([G]/\chi(1))$ is called the defect of $\chi$. Let $\text{Irr}^d(G)$ be the subset of irreducible characters of defect $d$. For the principal $\ell$-block $B_0$ of $G$, Robinson’s ordinary weight conjecture (OW conjecture) (see [50], and also [8, 5.49]) claims the equality $|\text{Irr}^d(B_0)| := |\text{Irr}^d(G) \cap \text{Irr}(B_0)| = m(\mathcal{F}_\ell(G), d)$.
where, for a saturated fusion system $\mathcal{F}$,$$
m(\mathcal{F},d) := \sum_{P \in \mathcal{F}_c} w_P(\mathcal{F},d),$$the sum runs over representatives of classes of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups and $w_P(\mathcal{F},d)$ is a certain alternating count of projective simple modules associated to stabilisers of elements in $\text{Irr}^d(P)$. We propose the following analogue of the OW conjecture for spet ses:

**Conjecture 1 (OW conjecture for spet ses).** Let $G$ be a simply connected $\mathbb{Z}_\ell$-spets for which $\ell > 2$ is very good, let $q$ be a power of a prime different from $\ell$, $B_0$ be the principal block of $G(q)$ and $\mathcal{F}(G(q))$ be the associated fusion system. Then$$|\text{Irr}^d(B_0)| = m(\mathcal{F}(G(q)),d) \quad \text{for all } d \geq 0,$$where $\text{Irr}^d(B_0)$ is the subset of characters in $\text{Irr}(B_0)$ of $\ell$-defect $d$.

We show that the defects of characters in $\text{Irr}(B_0)$ are locally controlled (see Proposition 6.5), so $|\text{Irr}^d(B_0)|$ can be re-expressed in terms of characters of the groups $W(s)$ defined above. Thus, Conjecture 1 can be further generalised to a statement about arbitrary simply connected $\ell$-compact groups (see Conjecture 6.12). We give evidence for this by verifying it when $|W|$ is prime to $\ell$ and for the smallest non-trivial infinite family of reflection groups. This also supports the general positivity conjecture ([35, Conj. 2.5]) that for any saturated fusion system $\mathcal{F}$, $m(\mathcal{F},d) \geq 0$ for all $d \geq 0$. Our Conjecture 1 also implies the ordinary weight conjecture for the principal $\ell$-block of $G^F$ for all semisimple algebraic groups $G$ with Frobenius map $F$ whenever $\ell > 2$ is very good for $G$ and $F$ induces an $\ell'$-automorphism of $W$.

We close the introduction with some remarks concerning the case $\ell = 2$, the restriction on the order of $\tau$ and the simply connected hypothesis in Theorem 2 and Conjecture 1. The connected 2-compact groups have been classified by Andersen–Grodal [5]. They show that the exotic 2-compact group DI(4) associated to $G_{24}$, constructed by Dwyer–Wilkerson, is the only simple 2-compact group not arising as the 2-completion of a compact connected Lie group. In [37], Lynd and the third author show the conclusion of Theorem 1 holds in this case. The paper [52] performs the computations necessary to verify the conclusion of Conjecture 1 holds for a suitably adapted definition of the principal block. Note that the prime 2 is bad for $G_{24}$ by Proposition 2.6. The restriction on the order of $\tau$ as well as the simply connected hypothesis come from our being in the setting of [12, Thm A] (see Theorem 3.2). In many situations the assumptions on [12, Thm A] are either automatically satisfied (see Remark 3.5) or are not necessary (see Remark 3.3). It would be desirable to know to what extent these restrictions may be relaxed.

**Structure of the paper:** In Section 2 we collect background material and prove some new results on reflection groups which are required for later sections. Among other things, this section includes a fundamental property of stabilisers (Proposition 2.3) needed in the proof of Theorem 2. Section 3 recalls the description of the fusion system $\mathcal{F}(X(q))$ constructed in [12]. We also derive, in Section 4.3, a description of its underlying $\ell$-group. In Section 4, we restate Theorem 1 as Theorem 4.2, outline its proof and explain how the AW conjecture for principal blocks of finite groups of Lie type for very good primes follows from it (Corollary 4.3). In Section 5 we investigate homotopy fixed-point centralisers of
ℓ-compact groups and prove Theorem 2 as part of Theorem 5.2. These results allow us to define in Section 6 the centralisers of ℓ-elements and also the principal block of a Zℓ-spets. Sections 7 and 8 complete the proof of Theorem 1 in the cases when X is of generalised Grassmannian and exceptional type respectively.

Acknowledgement: The authors would like to thank Bob Oliver and Albert Ruiz for providing us with references for Section 7, Carles Broto and Jesper Møller for clarifying some points in [12] and Bob Oliver and Jesper Grodal for helpful discussions.

Contents

1. Introduction and statement of results 1
2. Zℓ-Reflection groups 6
3. From Zℓ-reflection groups to fusion systems via ℓ-compact groups 12
4. Alperin’s weight conjecture for homotopy fixed point fusion systems 15
5. Centralisers 17
6. Weight conjectures for Spetses 22
7. Generalised Grassmannians 30
8. Exceptional and Aguadé groups 45
9. Appendix: Weights for wreath products 48
References 49

2. Zℓ-Reflection groups

In this section we recall some properties of finite reflection groups, set our notation, show a crucial result on stabilisers and define the notion of good and very good primes for Zℓ-reflection groups.

Let R be a principal ideal domain of characteristic zero with field of fractions K. An R-reflection group is a pair (W, L) where L is a finitely generated free R-module and W ≤ AutR(L) is a finite group generated by reflections, i.e., non-trivial elements fixing point-wise an R-submodule of corank 1. We say that (W, L) is irreducible if the KW-module K ⊗R L is irreducible. If (W, L) is an R-reflection group then so is (W, L′) := (W, R′ ⊗R L) for any principal ideal domain R′ containing R. In this case we say (W, L′) is obtained from (W, L) by extension of scalars. Two R-reflection groups (W, L) and (W′, L′) are isomorphic if there exists an R-linear isomorphism φ : L → L′ such that φ′φ−1 = W′.

2.1. Classification of reflection groups. Let ℓ be a prime. A Zℓ-reflection group is exotic if it is irreducible with character field strictly containing Q. Every Zℓ-reflection group (W, L) decomposes as a direct product

(W, L) = (W1 × W2, L1 ⊕ L2),

where (W1, L1) is an extension of scalars of a Z-reflection group and (W2, L2) is a direct product of exotic Zℓ-reflection groups which are uniquely determined up to permutation of factors (see, e.g., [6 Thm. 11.1]).
Recall that every \( \mathbb{Z} \)-reflection group \((W, L)\) is isomorphic to \((W_G, L_G)\) for some (not uniquely determined) compact connected Lie group \(G\) with Weyl group \(W_G\) and cocharacter lattice \(L_G\). By the Shephard–Todd classification theorem (see e.g. [31, Tab. 1]) the irreducible complex reflection groups fall into an infinite series of monomial groups \(G(e, r, n)\) (with \(n \geq 1\), \(e \geq 2\), \(r|e\)), the symmetric groups \(S_n\) and 34 exceptional groups (the rank of \(L\) is at most 8 in the exceptional cases).

The irreducible \(\mathbb{Z}_\ell\)-reflection groups are also known, see e.g., [6]; for this paper we find it convenient to subdivide them into the following six categories:

- the \(\mathbb{Z}_\ell\)-reflection groups of order prime to \(\ell\) (also called Clark–Ewing groups);
- the symmetric groups \(S_n\) for \(n \geq \ell\);
- the imprimitive groups \(G(e, r, n)\) with \(n \geq \ell\), \(e \geq 2\) and \(r|e|(\ell - 1)\) (with associated \(\ell\)-compact groups called generalised Grassmannians in [12]);
- the five rational Weyl groups of exceptional types \(G_2, F_4, E_6, E_7\) and \(E_8\) for \(\ell \in \{2,3\}, \{2,3\}, \{2,3,5\}, \{2,3,5,7\}, \{2,3,5,7\}\) respectively;
- the four exceptional reflection groups \(G_{12}, G_{29}, G_{31}\) and \(G_{34}\) at \(\ell = 3,5,5\) and 7 respectively, studied by Aguadé [1]; and
- the exceptional reflection group \(G_{24}\) at \(\ell = 2\).

The Clark–Ewing groups comprise the symmetric groups \(S_n\) for \(n < \ell\), the imprimitive groups \(G(e, r, n)\) with \(n < \ell\), \(e \geq 2\), \(r|e|(\ell - 1)\), or \(5 \leq e = r|\ell + 1|\) when \(n = 2\) (the dihedral groups), as well as those exceptional \(\mathbb{Z}_\ell\)-reflection groups of order prime to \(\ell\).

2.2. Relative Weyl groups. Let \((W, L)\) be a \(\mathbb{Z}_\ell\)-reflection group, \(V := \mathbb{Q}_\ell \otimes \mathbb{Z}_\ell L\) and let \(\phi \in N_{GL(V)}(W)\). Recall that there exists a fundamental system of homogeneous invariants \(f_1, \ldots, f_r\), \(r = \text{rk}(L)\), of \(W\) in the symmetric algebra on \(V\) which are eigenvectors for \(\phi\), with eigenvalues \(\varepsilon_i\) say. The uniquely determined multiset \(\{(d_i, \varepsilon_i) \mid 1 \leq i \leq r\}\), where \(d_i := \deg f_i\), are the generalised degrees of \(W\phi\). The order polynomial of the coset \(W\phi\) is defined as

\[
O_\varepsilon(W\phi) := \prod_i (x^{d_i} - \varepsilon_i) \in \mathbb{Q}_\ell[x].
\]

Note that \(O_\varepsilon(W\phi) \in \mathbb{Z}_\ell[x]\) if \(\phi \in N_{GL(L)}(W)\).

**Theorem 2.1** (Lehrer–Springer). Let \(g \in W\phi\) have fixed space \(V(g,1) := \ker_V(g-1)\) of maximal possible dimension among elements in \(W\phi\). Then:

(a) \(W_\phi := N_W(V(g,1))/C_W(V(g,1))\) acts faithfully as a reflection group on \(V(g,1)\), where \(C_W(V(g,1))\) is the point-wise stabiliser of \(V(g,1)\) in \(W\).

(b) The group \(W_\phi\) is uniquely determined up to the conjugation action of \(W\) on the set of its subquotients.

(c) We have \(O_\varepsilon(W\phi) = \prod_{i, \varepsilon_i=1} (x^{d_i} - 1)\), where \(\{(d_i, \varepsilon_i)\}\) are the generalised degrees of \(W\phi\) on \(\mathbb{Q}_\ell \otimes \mathbb{Z}_\ell L\).

(d) Assume \(\phi \in N_{GL(L)}(W)\). Then \(W_\phi\) is a \(\mathbb{Z}_\ell\)-reflection group on the fixed space \(L_\phi = \ker_L(g-1)\) of \(g\) in \(L\).

**Proof.** Parts (a) and (c) are [36, Thm. 1.1]. By [36, Lemma 3.1] the maximal eigenspaces \(V(g,1)\) for \(g\) running over \(W\phi\) are all \(W\)-conjugate, whence (b). Part (d) is clear. \(\square\)

We call \(W_\phi\) the relative Weyl group of a Sylow 1-torus of \(W\phi\) by analogy with the case of finite reductive groups and spetses. If \(V(g,1)\) contains an eigenvector of \(g\) not lying in
any of the reflecting hyperplanes of \( W \) then \( g \) is called \( 1\)-regular. In this case, Springer showed that \( W_\phi \cong C_W(g) \).

**Example 2.2.** We describe the relative Weyl groups in imprimitive groups when \( \phi \) is a scalar. Let \( W = G(m, r, n) \) with \( m/l - 1 \) and let \( \phi = \zeta \in \mathbb{Z}_l^* \) be a primitive \( \ell \)th root of unity. Then

\[
W_\zeta = \begin{cases} 
G(me', r, \frac{\zeta}{e}) & \text{if } re \mid mn, \\
G(me', 1, \frac{\zeta}{e} - 1) & \text{if } re \nmid mn \text{ but } e \mid mn, \\
G(me', 1, \lfloor \frac{\zeta}{e} \rfloor) & \text{otherwise},
\end{cases}
\]

where \( e' = e/\gcd(e, m) \) (see [39, (2.6) and (5.4)] and [12, A.10]).

### 2.3. Stabilisers.

The following general fact will be needed in Section 5 to prove a Steinberg-type result on centralisers in \( \ell \)-compact groups; it might be of independent interest. Recall that parabolic subgroups of a \( \mathbb{Q}_l \)-reflection group \( W \) are by definition the point-wise stabilisers of subspaces. By a theorem of Steinberg [53, 1.20] these are reflection subgroups.

**Proposition 2.3.** Let \( (W, L) \) be a \( \mathbb{Z}_l \)-reflection group, \( \bar{T} = \mathbb{Z}/\ell^\alpha \otimes_{\mathbb{Z}_l} L \) the corresponding discrete torus. Let \( v \in \bar{T} \) and set \( H = C_W(v) \). Then \( H/H_{\text{ref}} \) is of \( \ell \)-power order, where \( H_{\text{ref}} \) denotes the normal subgroup generated by the reflections in \( H \). Moreover, if \( W \) is exotic then \( H/H_{\text{ref}} \) is generated by reflections, and if \( W \) is an \( \ell' \)-group then \( H = H_{\text{ref}} \) is a parabolic subgroup.

**Proof.** We provide a conceptual argument for the first assertion; the proof of the second statement will be completed by a case-by-case argument.

First assume that \( |H| \) is prime to \( \ell \). Let \( a \in \mathbb{N} \). By the Maschke argument the canonical surjection \( L \to L/\ell^a L \) restricts to a surjection \( L^H \to (L/\ell^a L)^H \). Indeed, let \( v \in (L/\ell^a L)^H \) and let \( y \in L \) be any lift of \( v \). Then \( \frac{1}{|H|} \text{Tr}^H_y(y) \in L^H \) has image \( \frac{1}{|H|} \text{Tr}^H_y(v) = v \in L/\ell^a L \).

Now let \( v \in \bar{T} \) be an element of order \( \ell^a \). By the above, there exists \( x \in L \) such that \( v \in x + \ell^a L \) and \( C_W(v) \leq C_W(x) \). Since \( C_W(x) \leq C_W(v) \) we have that \( C_W(x) = C_W(v) \). Now the stabiliser of any \( x \in L \) is the same as the stabiliser of its image in \( \mathbb{Q}_l \otimes_{\mathbb{Z}_l} L \), which is a reflection representation over a field of characteristic 0. But there, by the result of Steinberg [53, 1.20] stabilisers of vectors are reflection subgroups and in fact parabolic subgroups.

In the general case, we may apply the above to a collection of subgroups of \( H \) of order coprime to \( \ell \) generating \( O^\ell(H) \) to see that \( H_{\text{ref}} \geq O^\ell(H) \), giving the first claim.

Now assume that \( W \) is exotic. By the first part we may assume that \( \ell \) divides \( |W| \). Here we show the claim by a case-by-case argument. If \( v \) has order \( \ell \), then the result was shown in [6, Thm. 1.9].

Let us first discuss the four Aguadé groups. Here, Sylow \( \ell \)-subgroups are cyclic of order \( \ell \). By the first part we only need to consider stabilisers \( C_W(v) \) containing an element of order \( \ell \). A direct computation shows that the fixed space in \( \Omega_1(\bar{T}) \) of such an element is 1-dimensional and moreover the centraliser of this subspace is a symmetric group \( H = \mathfrak{S}_\ell \) (see also the last part of the proof of [6, Thm. 7.1]). Thus, \( H \) acts on \( \Omega_1(\bar{T}) \) as the Weyl group of the simply connected group \( \text{SL}_\ell \). Here, the claim follows by [54, 2.14 and 2.16]. Similarly, for the exotic \( \mathbb{Z}_2 \)-reflection group \( G_{24} \), which acts as \( \text{GL}_3(2) \)
on $\hat{L} := \mathbb{F}_2 \otimes_{\mathbb{Z}} L$, the stabilisers of elements in $\hat{L}$ are Weyl groups of simply connected type $B_3$ and we conclude as before.

It remains to consider the case $W = G(e, r, n)$, with $e \geq 2$ and $r | c | (\ell - 1)$. Here we may argue directly following [44] Lemma 7.11 (which describes centralisers of elementary abelian subgroups of $\hat{T}$). We identify $\hat{T}$ with $\mathbb{Z}/\ell^\infty \otimes_{\mathbb{Z}} L = (\mathbb{Z}/\ell^\infty)^n$ (written additively). Identify $W$ with $A(e, r, n) \rtimes S_n$, where $A(e, r, n)$ is the subgroup of diagonal $n \times n$-matrices over $\mathbb{Z}_\ell$ such that all diagonal entries are $e$-th roots of unity and the determinant is an $e/r$-th root of unity [39]. Let $t = (\lambda_1, \ldots, \lambda_n) \in \hat{T}$. We declare indices $i$ and $j$, $1 \leq i, j \leq n$, to be equivalent if $\lambda_i = \zeta\lambda_j$ for some $e$-th root of unity $\zeta \in \mathbb{Z}_\ell$. Denote by $n(t)$ the partition of $\{1, \ldots, n\}$ into equivalence classes. Suppose that there are $u_0$ indices $i$ such that $\lambda_i = 0$ and $s$ further equivalences classes containing $u_1, \ldots, u_s$ elements respectively.

Now for $a = (\zeta_1, \ldots, \zeta_n) \in A(e, r, n)$ and $\sigma \in S_n$, we have

$$(a, \sigma)_t = (\zeta_1\lambda_{\sigma(1)}, \ldots, \zeta_n\lambda_{\sigma(n)}).$$

So, $(a, \sigma) \in C_W(t)$ if and only if $\lambda_i = \zeta_i\lambda_{\sigma(i)}$ for all $i$, and then $\sigma$ preserves the partition $n(t)$. Now, the group of $e$-th roots of unity in $\mathbb{Z}_\ell^\times$ acts fixed point freely on $\mathbb{Z}/\ell^\infty \setminus \{0\}$. Hence if $(a, \sigma) \in C_W(t)$ is such that $\sigma = \tau\sigma_1 \cdots \sigma_s$, where $\tau \in S_{u_0}$ and $\sigma_j \in S_{u_j}$, $1 \leq j \leq s$, then for all $i$ in the $j$-th equivalence class, $\zeta_i$ is uniquely determined by $\lambda_i = \zeta_i\lambda_{\sigma_j(i)}$. In particular, all entries of $a$ other than those in the 0th equivalence class are determined by $\sigma_1 \cdots \sigma_s$. Conversely, if $a$ and $\sigma = \tau\sigma_1 \cdots \sigma_s$ are such that for all $1 \leq j \leq s$, and all $i$ in the $j$th class $\lambda_i = \zeta_i\lambda_{\sigma_j(i)}$, then $\prod_{i' \sim i} \lambda_{i'} = 1$ and hence $(a, \sigma) \in C_W(t)$. Thus the map

$$G(e, r, u_0) \times \prod_{1 \leq j \leq e} S_{u_j} \to C_W(t), \quad (a_0, \tau) \times \prod_{1 \leq j \leq e} \sigma_j \mapsto (a, \tau\sigma_1 \cdots \sigma_s),$$

where the entries of the components of $a$ in the 0th class are the entries of $a_0$ and the entries of the components corresponding to the $j$th class are determined by the equations $\lambda_i = \zeta_i\lambda_{\sigma_j(i)}$, is an isomorphism. Now the image of a transposition $\sigma_j \in S_{u_j}$ under the above map is a reflection. Since the image of $G(e, r, u_0)$ is also generated by reflections, $C_W(t)$ is indeed generated by reflections. The proof is complete. \hfill $\square$

2.4. Good, bad and very good primes. We extend the notion of bad primes to $\mathbb{Z}_\ell$-reflection groups. For Weyl groups these are defined in terms of torsion of the root system modulo closed subsystems, or of coefficients of the highest root (see e.g. [42] §B.5). Since there is no general concept of root systems for $\mathbb{Z}_\ell$-reflection groups (yet), we propose a different characterisation which turns out to agree with the usual one for Weyl groups.

We say that a reflection subgroup of a $\mathbb{Q}_\ell$-reflection group $W$ has maximal rank if its fixed space agrees with that of $W$. For reflection groups in positive characteristic, there can exist subspace stabilisers which are not parabolic; for example, a maximal rank subgroup could have invariant vectors. It is this phenomenon that underlies our definition of bad primes.

**Definition 2.4.** Let $(W, L)$ be a $\mathbb{Z}_\ell$-reflection group. We say that $\ell$ is **good** for $W$ if all maximal rank reflection subgroups of $W$ (on $L \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$) have the same number of trivial composition factors on $\mathbb{F}_\ell \otimes_{\mathbb{Z}_\ell} L$. Otherwise we call $\ell$ a **bad prime** for $W$. We say that $\ell$ is **very good** for $(W, L)$ if it is good for $W$, and $L_W$ and $(L^*)_W$ are both torsion-free.
Here, we write $L_W = L/(w-1)L \mid w \in W$ for the coinvariants of $(W, L)$. Let us point out that all exotic $\mathbb{Z}_\ell$-reflection groups have $L_W = 0$, that is, they are simply connected, see [46, Prop. 1.6]. So for exotic reflection groups, all good primes are very good.

Observe that the notion of being very good may depend on the chosen $\mathbb{Z}_\ell$-lattice for $W$, while being good only depends on $W$. We have the following properties:

**Lemma 2.5.** Assume $(W, L)$ is a $\mathbb{Z}_\ell$-reflection group.

(a) Then $\ell$ is good for $W$ if and only if it is good for all irreducible factors of $W$.

(b) If $(W, L) = (W_1 \times W_2, L_1 \oplus L_2)$ is a direct decomposition, then $\ell$ is very good for $(W, L)$ if and only if it is so for $(W_1, L_1)$ and $(W_2, L_2)$.

(c) If $\ell$ is good for $W$, then it is so for all of its reflection subgroups.

(d) If $W$ is a Weyl group, the notion of bad primes agrees with the classical one.

**Proof.** Parts (c) and (d) will be a consequence of Proposition 2.6 in conjunction with (a). The other claims are obvious. □

**Proposition 2.6.** Let $(W, L)$ be an irreducible $\mathbb{Z}_\ell$-reflection group.

(a) The bad primes $\ell > 2$ are as given in Table 1, and $\ell = 2$ is bad unless $W \cong S_n$.

(b) The only groups with good primes that are not very good are $W = S_n$ with $\ell | n$.

| $\ell$ | $G(6, 6, 2)$ | $G_{12}$ | $G_{28}$ | $G_{29}$ | $G_{31}$ | $G_{34}$ | $G_{35}$ | $G_{36}$ | $G_{37}$ |
|-------|-------------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\ell$ | 3           | 3       | 3       | 5       | 7       | 3       | 3       | 3, 5    |

Note that $G(6, 6, 2), G_{28}, G_{35}, G_{36}, G_{37}$ are the Weyl groups of types $G_2, F_4, E_6, E_7$ and $E_8$, respectively, and $G_{12}, G_{29}, G_{31}$ and $G_{34}$ are the Aguadé groups.

**Proof.** If $W$ is a Clark–Ewing group, that is, $W$ has order prime to $\ell$, then $\bar{L} := \mathbb{F}_\ell \otimes_{\mathbb{Z}_\ell} L$ is semisimple and moreover, any subgroup of $W$ has a trivial composition factor on $\bar{L}$ if and only if it has one on $L$. This shows that $\ell$ is very good for $W$.

For Weyl groups, if $\ell$ appears in Table 1 then there is a maximal rank subgroup occurring as the Weyl group of the centraliser of an isolated $\ell$-element (see e.g. [9, §§4,5]) of a corresponding linear algebraic group in characteristic different from $\ell$, so $\ell$ is bad according to our definition. On the other hand, the description of maximal rank subgroups via Borel–de Siebenthal [42, 13.2] shows that only the listed cases occur. For example, for $W = S_n$ there is no proper maximal rank subgroup at all and so all primes are good.

For $G(e, r, n)$ with $e \geq 2$ dividing $\ell - 1$, the description in the proof of Proposition 2.3 shows that there are no proper maximal rank stabilisers. Among the exotic exceptional $\mathbb{Z}_\ell$-reflection groups, only the four Aguadé groups and $G_{24}$ have order divisible by $\ell$. As already observed in the proof of Proposition 2.3, for the former groups there exists an element $0 \neq v \in \bar{L}$ with stabiliser a proper maximal rank subgroup, viz. a symmetric group $\mathfrak{S}_\ell$. The group $G_{24}$ has a maximal rank reflection subgroup $A_1^\ell$. Clearly that has trivial composition factors in characteristic 2, so 2 is bad for $G_{24}$.

Now assume that $\ell$ is good for $(W, L)$. If the reflection character of $W$ on $L$ does not lie in the same $\ell$-block as the trivial character, then the condition for $\ell$ being very
good is obviously satisfied. This is the case, for example, if \( W \) has a non-trivial normal \( \ell' \)-subgroup. This deals with the groups \( G(e, r, n) \) with \( e \geq 2 \), as well as with the exceptional Weyl groups. Thus only \( W = \mathfrak{S}_n \) remains to be considered. Here, the reflection character lies in the principal \( \ell' \)-block only if \( \ell \mid n \).

**Remark 2.7.** For so-called well-generated complex reflection groups \( W \), Broué–Corran–Michel \[13\] §8 define the notion of bad prime ideals and a bad ideal when \( W \) is moreover spetsial (see Section 6.1). These are certain ideals in the ring of integers of the field of definition of \( W \). In case \( W \) is also a \( \mathbb{Z}_\ell \)-reflection group, it can be checked from the tables in [15] that \( \ell \) is bad in our sense if and only if it divides the norm of their “bad ideal”.

**Remark 2.8.** Let \( \ell > 2 \) and \((W, L_0)\) be the Weyl group of a connected reductive group \( G \) in characteristic different from \( \ell \), and set \( L := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} L_0 \). Then \( L_W = 0 \) if and only if \( G \) is semisimple with fundamental group of \( \ell' \)-order, and \((L')_W = 0 \) if and only if \( G \) is semisimple with \(|Z(G)/Z^0(G)| \) of \( \ell' \)-order. Furthermore, \( L_W \) is torsion-free if and only if \( G \) has derived subgroup with fundamental group an \( \ell' \)-group.

Thus, for example, if \( G \) is simple of classical type \( B, C \) or \( D \), then all primes \( \ell > 2 \) are very good for the corresponding \((W, L)\).

The following will be relevant for the descent to relative Weyl groups:

**Proposition 2.9.** Let \((W, L)\) be a \( \mathbb{Z}_\ell \)-reflection group with \( L_W = 0 \) and \( \phi \in N_{\text{GL}(L)}(W) \). If \( \ell > 2 \) is very good for \((W, L)\), then it is so for \((W_\phi, L_\phi)\).

**Proof.** Assume \( W = W_1 \times W_2 \) is a direct decomposition into \( \phi \)-stable reflection subgroups. Then \( L_i := \langle (w - 1)L \mid w \in W_i \rangle \) is \( \phi \)-stable for \( i = 1, 2 \), and since \( L_W = 0 \) we have \( L = L_1 \oplus L_2 \). Furthermore, \( L_\phi = (L_1)_\phi \oplus (L_2)_\phi \) and \((L_i)_W = 0 \). Thus we may assume that \((W, L) = (W' \times \cdots \times W', L' \oplus \cdots \oplus L')\) where \((W', L')\) is irreducible and \( \phi \) permutes the \( m \) factors transitively. Set \( \phi' := \phi^m |_{L'} \), then

\[
L_\phi = \{(x, \phi(x), \ldots, \phi^{m-1}(x)) \mid x \in L_{\phi'}\} \cong L_{\phi'}.
\]

Thus by Lemma 2.5 we are reduced to the case of irreducible \((W, L)\).

By inspection from Proposition 2.2 if \( \ell \) is good for \((W, L)\) then it is so for all of its relative Weyl groups. Any \( 1 \neq w \in Z(W) \) acts by a root of unity \( 1 \neq \xi \in \mathbb{Z}_\ell^\times \) on \( L \) and thus stabilises \( L_\phi \). Since \( \xi - 1 \in Z^\times_\ell \), we obtain \((L_\phi)_0 = 0 = (L_\phi)_0\) and so \( \ell \) is very good for \((W_\phi, L_\phi)\). By [33] Tab. 1 and 6] the only \( \mathbb{Z}_\ell \)-reflection groups with trivial centre are \( G(e, r, n) \) with \( e \geq 3 \) and \( \gcd(e, ne/r) = 1 \), \( \mathfrak{S}_n \) and \( W(D_n) \) with \( n \) odd.

In the first case, proper relative Weyl groups again have type \( G(f, s, m) \) with \( f > 2 \) dividing \( \ell - 1 \) (see Example 2.2), so \( \ell \) is very good for them. In the other two cases, \( \phi \) acts on \( L \) like an element \( w \in W \) times a scalar \( \xi \), and so \( w \in N_W(L_\phi) \) acts as the scalar \( \xi^{-1} \) on \( L_\phi \). Hence we are done unless \( \xi = 1 \). Finally, for \( \xi = 1 \) we have \( W_\phi = W \).

The conclusion fails to hold for \( \ell = 2 \) since \( 2 \) is very good for the \( \mathbb{Z}_2 \)-reflection group attached to \( \text{SL}_n \) with \( n \) odd, but its relative Weyl group for \( \phi = -\text{Id} \) is of type \( B_{(n-1)/2} \), for which \( 2 \) is bad.

We do not know whether \( \ell > 2 \) being very good for \((W, L)\) descends to any relative Weyl group \((W_\phi, L_\phi)\) if \( L_W \neq 0 \). According to [33] Prop. C.4 combined with [6] Thm. 12.2], if \( \phi \) is of \( \ell' \)-order then this is at least true for the property \( L_W \) being torsion-free.
3. From $\mathbb{Z}_\ell$-reflection groups to fusion systems via $\ell$-compact groups

We explain how to obtain a fusion system starting from a $\mathbb{Z}_\ell$-reflection group and a prime power $q$ via homotopy fixed points on the associated $\ell$-compact group, and we develop some properties of the fusion system thus obtained.

3.1. $\ell$-compact groups. We follow the exposition of [6], [24] and [12]. Let $\ell$ be a prime. Recall that the Bousfield–Kan $\ell$-completion, denoted $X \mapsto X^\wedge$, is a functor from the category of spaces (that is, simplicial sets) to itself, along with a natural transformation $\text{Id} \to (-)^\wedge$. A space $X$ is said to be $\ell$-complete if $X \to X^{\wedge}$ is a homotopy equivalence.

An $\ell$-compact group is a pointed, connected and $\ell$-complete space $X$ such that the cohomology $H^*(\Omega X; \mathbb{F}_\ell)$ is finite dimensional over $\mathbb{F}_\ell$. Here $\Omega X$ refers to the space of based loops in $X$. A morphism between $\ell$-compact groups is a map of pointed spaces. Two morphisms $\varphi, \varphi': X \to Y$ between $\ell$-compact groups are conjugate if $\varphi$ and $\varphi'$ are freely homotopic and $\varphi$ is an isomorphism (or equivalence) if it is a homotopy equivalence. Further, $\varphi : X \to Y$ is a monomorphism if the homotopy fibre of $\varphi$ has finite $\mathbb{F}_\ell$-homology.

We note that a morphism between $\ell$-compact groups may also be thought of as a based homotopy class of based maps, but we will use the former point of view.

We choose this setup which we find more intuitive than the conventional one in which an $\ell$-compact group $X$ is presented as a triple $(X, B\pi_1X, e)$ where $B\pi_1X = X$, $X$ is a space homotopy equivalent to $\Omega X$ and $e : X \to \Omega X$ is a chosen homotopy equivalence.

An $\ell$-compact group $X$ is connected if $\Omega X$ is connected. The reader should beware of the slight notational ambiguity of this terminology as the space $X$ itself is always connected. We also note that if $Y$ is a connected $\ell$-compact group, then two morphisms $\varphi, \varphi' : X \to Y$ are conjugate if and only if there is a based homotopy between $\varphi$ and $\varphi'$.

If $G$ is a compact Lie group whose component group is a finite $\ell$-group, then $(BG)^\wedge$ is an $\ell$-compact group, where for any topological group $G$, $BG$ refers to the classifying space of $G$. The $\ell$-completed classifying space $T = (BU(1)^r)^\wedge$ of a compact torus $U(1)^r$ is called an $\ell$-compact torus of rank $r$. If $X$ is an $\ell$-compact group, then a maximal torus of $X$ is a monomorphism $i : T \to X$ from an $\ell$-compact torus $T$ to $X$ such that the homotopy fibre of $i$ has non-zero Euler characteristic. Here the Euler characteristic of a space is taken to be the alternating sum of the dimensions of its $\mathbb{F}_\ell$-homology groups.

By the fundamental results of Dwyer and Wilkerson [24] Prop. 8.11, Thm. 8.13], every $\ell$-compact group has a maximal torus $i : T \to X$ which is unique up to conjugation in the sense that if $i' : T' \to X$ is any maximal torus, then there exists an isomorphism $\varphi : T' \to T$ such that $i' = i \circ \varphi$ are conjugate.

Fix an $\ell$-compact group $X$ with maximal torus $i : T \to X$ and set $L_X := \pi_2(T)$, a free $\mathbb{Z}_\ell$-module of the same rank as $T$. Let $W_X(T)$ be the topological monoid of self-maps of $T$ over $X$ (defined after replacing $i$ by an equivalent fibration) [24] Def. 9.6] and set $W_X = \pi_0(W_X(T))$. By [24] Prop. 9.5, Thm. 9.7, Thm. 9.17, $W_X$ is a finite group and if $X$ is connected, then $W_X$ is identified with the set of conjugacy classes of self-equivalences $\varphi$ of $T$ such that $i$ and $i \circ \varphi$ are conjugate. This sets up (for connected $X$) a faithful action of $W_X$ on $L_X$ as a finite $\mathbb{Z}_\ell$-reflection group. The pair $(W_X, L_X)$ is the Weyl group of $X$. Occasionally and when there is no risk of ambiguity, we will just use the first component $W_X$ to denote the Weyl group of $X$. Note that $(W_X, L_X)$ is independent of the choice of $T$ up to isomorphism.
Suppose that $X$ is connected. Given any homotopy self-equivalence $\alpha : X \to X$, there exists a self-equivalence $\alpha_T : T \to T$ such that $\alpha \circ i$ is conjugate to $i \circ \alpha_T$. Further, the conjugacy class of $\alpha_T$ is uniquely determined up to the action of $W_X$ (see [13, Thm. 1.2]). This induces a group homomorphism $\Phi_X : \text{Out}(X) \to N_{\text{GL}(L_X)}(W_X)/W_X$ where $\text{Out}(X)$ is the outer automorphism group of $X$, that is, the group of conjugacy (i.e., homotopy) classes of homotopy self-equivalences of $X$. For odd $\ell$, we have the following classification result.

**Theorem 3.1.** [6, Thm. 1.1] Suppose that $\ell$ is an odd prime. The assignment

$$X \leadsto (W_X, L_X)$$

induces a bijective correspondence between isomorphism classes of connected $\ell$-compact groups and isomorphism classes of $\mathbb{Z}_\ell$-reflection groups. Here, the map

$$\Phi_X : \text{Out}(X) \to N_{\text{GL}(L_X)}(W_X)/W_X$$

is an isomorphism.

### 3.2. Fusion systems from homotopy fixed points

Let $X$ be a connected $\ell$-complete group, with Weyl group $(W, L)$. For $q \in \mathbb{Z}_\ell^*$, we denote by $\psi^q$ a self-equivalence of $X$ whose class in $\text{Out}(X)$ is the image under $\Phi_X^{-1}$ of the coset $\lambda_q W \in N_{\text{GL}(L)}(W)/W$ where $\lambda_q := q\text{id} \in \text{GL}(L)$; $\psi^q$ is called an unstable Adams operator corresponding to $q$. For an equivalence $\tau$ of $X$ whose class in $\text{Out}(X)$ has finite order, let $\tau X(q)$ denote the space of homotopy fixed points $X^{h\tau\psi^q}$. Thus $\tau X(q)$ is the space of all paths $\gamma : [0, 1] \to X$ such that $\gamma(1) = \tau\psi^q(\gamma(0))$ (with the compact-open topology). Note that $\tau X(q)$ may also be described by the homotopy pullback diagram:

$$\begin{array}{ccc}
\tau X(q) & \xrightarrow{i} & X \\
\downarrow & & \downarrow \Delta \\
X & \xrightarrow{(1, \tau\psi^q)} & X \times X
\end{array}$$

where $\Delta : X \to X \times X$ is the diagonal embedding, and that up to homotopy equivalence $\tau X(q)$ only depends on the free homotopy class of $\tau\psi^q$ and hence only on the class of $\tau\psi^q$ in $\text{Out}(X)$ (see [13, Rem. 2.3, Lemma B.1]).

Suppose that $X$ is a space and $S$ is a finite $\ell$-group for which there is a continuous map $f : BS \to X$. By the functoriality of the classifying space construction to each injective group homomorphism $\varphi \in \text{Inj}(P, Q)$ is associated a map of spaces $B\varphi : BP \to BQ$, and we may form a category $\mathcal{F}_{S,f}(X)$ whose objects are the subgroups of $S$ with morphisms given by

$$\text{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q) := \{ \varphi \in \text{Inj}(P, Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi \}$$

for each $P, Q \leq S$ [8, Def. III.3.3]. To such an $f$ is also associated a certain “centric linking category” $\mathcal{L}^c_{S,f}(X)$ (see [8, Def. III.3.3]). An $\ell$-complete space $X$ is said to be a classifying space of a saturated fusion system if there exists a finite $\ell$-group $S$ and a morphism $f : BS \to X$ such that $\mathcal{F}_{S,f}(X)$ is saturated and $(S, \mathcal{F}_{S,f}(X), \mathcal{L}^c_{S,f}(X))$ is an $\ell$-local finite group (see [8, Def. I.4.4]) with classifying space $X$, i.e., $X \simeq [\mathcal{L}^c_{S,f}(X)]^\ell$. By [11, Thm. 7.4] (see also [8, Thm. III.4.25]), the homotopy type of $X$ determines the triple $(S, \mathcal{F}_{S,f}(X), \mathcal{L}^c_{S,f}(X))$ up to isomorphism, and in particular an $\ell$-complete space can be the classifying space of at most one (isomorphism class of) saturated fusion system.
An ℓ-compact group $X$ is called simply connected if $\Omega X$ is simply connected. For odd $\ell$, this is equivalent to the underlying $\mathbb{Z}_\ell$-reflection group $(W, L)$ having $L_W = 0$ (see [6 Thm. 1.7]). The following is shown in [12 Thm. A]:

**Theorem 3.2.** Let $\ell$ be an odd prime and $X$ be a simply connected ℓ-compact group. Let $q$ be a power of a prime $p \neq \ell$ and $\tau$ be a self-equivalence of $X$ whose class in $\text{Out}(X)$ has finite order prime to $\ell$. Then $^\tau X(q)$ is a classifying space of a saturated fusion system.

**Remark 3.3.** (a) Let $q$ be a prime power, prime to $\ell$, $G$ a connected reductive algebraic group over $\mathbb{F}_q$ with Weyl group $W$ and cocharacter lattice $L_0$, and $F : G \to G$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure corresponding to $\phi \in N_{GL(L_0)}(W)$. Let $X$ be a connected ℓ-compact group with Weyl group $(W, \mathbb{Z}_\ell \otimes \mathbb{Z} L_0)$ and let $\tau$ be a homotopy self-equivalence of $X$ whose class in $\text{Out}(X)$ corresponds to $\Phi_X^{-1}(W\phi)$. Then by a theorem of Friedlander $(BG^F)^\zeta \simeq ^\tau X(q)$ (see [13 Thm. 3.1]). We note that here there is no assumption on $\ell$ nor on simple connectivity of $X$.

(b) By the existence and uniqueness of centric linking systems established by Chermak [23] given any saturated fusion system $\mathcal{F}$ on a finite $\ell$-group there is a unique (up to homotopy equivalence) space $X$ such that $X$ is a classifying space of $\mathcal{F}$. Thus, in the above theorem we may also speak of $^\tau X(q)$ being the classifying space of a saturated fusion system.

(c) Theorem A of [12] is stated under the a priori stronger assumption that $\tau$ is of finite $\ell'$-order but as shown in [12 Thm. B] the two conditions are in fact equivalent.

In the situation of Theorem 3.2 we denote by $X(q)$ the space $^\tau X(q)$ with $\tau$ equal to the identity map, that is, the homotopy fixed points in $X$ under $\psi^q$. We then have the following comparison and recognition theorem. The action of the group $\Gamma$ on $X$ below is in the sense of homotopical actions (see [12 Section 6], [33 Section C.1]).

**Theorem 3.4.** Let $\ell$ be an odd prime, $X$ a connected ℓ-compact group with Weyl group $(W, L)$ and $\tau$ a self-equivalence of $X$ whose class in $\text{Out}(X)$ has finite order prime to $\ell$, with $W\phi \in N_{GL(L)}(W)/W$ associated to $\tau$ via Theorem 3.7. Let $q$ be a power of a prime $p \neq \ell$ and write $q = q_0 \zeta$ where $\zeta \in \mathbb{Z}_\ell^*$ is a primitive $e$-th root of unity and $q_0 \equiv 1 \pmod{\ell}$. Let $\Gamma$ be the subgroup of $\text{Out}(X)$ generated by the class of $\tau \psi^q$.

(a) There is a canonical action of $\Gamma$ on $X$ such that the homotopy fixed point space $X^{h\Gamma}$ is a connected ℓ-compact group with Weyl group $(W_{\phi\zeta}, L_{\phi\zeta})$, and

$$^\tau X(q) \simeq X^{h\Gamma}(q_0).$$

If $X$ is simply connected, then so is $X^{h\Gamma}$.

(b) If $q^e \in \mathbb{Z}_\ell^* \setminus \{1\}$ is such that $q \equiv q' \pmod{\ell}$ and $(q^e - 1)_e = (q'^e - 1)_e$, then

$$X(q) \simeq X(q').$$

**Proof.** The first and last assertion of (a) is [12 Thm. B], the homotopy equivalence in (a) for simply connected $X$ is given in the proof of [12 Thm. A] and (b) for simply connected $X$ is [12 Thm. E(2)]. The general case of (a) is given in [33 Thm. 1.2], the Weyl group is identified in [33 Thm. C.3] using Lehrer–Springer theory (Theorem 2.1), and (b) is in [33 Cor. C.9].
Remark 3.5. In Theorems 3.2 and 3.3 the assumption on the order of the class of \( \tau \) is unnecessary if the underlying \( \mathbb{Z}_p \)-reflection group \((W, L)\) of \( X \) is irreducible. Indeed, by [18, Prop. 3.13], the group \( N_{GL(L)}(W)/W \) has order prime to \( \ell \) except when \( W \) is of type \( D_4 \) and \( \ell = 3 \). But for the latter situation, the restriction on \( \tau \) is not necessary since Friedlander’s result (see Remark 3.3(a)) covers this case.

Thus, if \( W \) is reducible, outer automorphisms of order divisible by \( \ell \) can only arise from wreath product situations. It would seem interesting to extend the above theorems to cover this case as well, in the spirit of Remark 6.9.

3.3. The Sylow \( \ell \)-subgroups. We identify the \( \ell \)-subgroup \( S \) underlying the saturated fusion system determined by the homotopy fixed points via Theorem 3.2.

Theorem 3.6. Let \( \ell \) be an odd prime, and \( X \) a simply connected \( \ell \)-compact group with Weyl group \((W, L)\) and a self-equivalence \( \tau \) whose class in \( \text{Out}(X) \) is of finite order prime to \( \ell \), with \( W \phi \) the corresponding element of \( N_{GL(L)}(W)/W \). Let \( q \) be a prime power and \( \zeta \in \mathbb{Z}_\ell^* \) the primitive \( e \)-th root of unity such that \( q \equiv \zeta \pmod{\ell} \). Then the saturated fusion system \( F \) associated to \( \tau X(q) \) via Theorem 3.2 satisfies:

(a) The \( \ell \)-group \( S \) underlying \( F \) is a semidirect product \( S = (\mathbb{Z}/\ell^n \mathbb{Z})^* \rtimes (W_\phi)_\ell \), where \( \ell^n = (q^e - 1)/\ell \), \( r = \text{rk}(L_\phi) \) and \( (W_\phi)_\ell \) denotes a Sylow \( \ell \)-subgroup of \( W_\phi \).

(b) We have the Steinberg-type order formula

\[
|S| = \prod_{i: \zeta^{d_i} \epsilon_i = 1} (q^{d_i} - 1)/\ell = \left( \prod_{i} (q^{d_i} - \epsilon_i^{-1}) \right)/\ell = O(q(W_\phi^{-1})/\ell)
\]

where \( (d_1, \epsilon_1), \ldots, (d_r, \epsilon_r) \) are the generalised degrees of \( W \phi \) on \( Q_\ell \otimes \mathbb{Z}_\ell L \).

Proof. Write \( q = \zeta q_0 \), so \( q_0 \equiv 1 \pmod{\ell} \). By Theorem 3.4(a) we have \( \tau X(q) \simeq X^{\text{ht}}(q_0) \) where \( \Gamma \) is the subgroup of \( \text{Out}(X) \) generated by the class of \( \tau \psi^q \). Now \( q_0 \approx q^e \), so

\[
\nu(q^e - 1) = \nu((q_0^e - 1)) = \nu(q_0 - 1),
\]

where \( \nu \) denotes the \( \ell \)-adic valuation on \( Q_\ell \). Hence \( X^{\text{ht}}(q_0) \simeq X^{\text{ht}}(q^e) \) have isomorphic fusion systems by Theorem 3.3(b). So we may instead consider the fusion system determined by \( X^{\text{ht}}(q^e) \). Here, as \( q^e \equiv 1 \pmod{\ell} \) and as by Theorem 3.4(a), \( W_\phi \) is the Weyl group of \( X^{\text{ht}} \), the assertion in (a) follows from [12, Prop. 7.6]

By Theorem 2.1(b), \( W_\phi \) is a reflection group with multiset of degrees given by \( D := \{ d_i | \zeta^{d_i} \epsilon_i = 1 \} \), so a well-known formula shows \( |W_\phi| = \prod_{d \in D} d \). On the other hand, as \( e \) is the order of \( q \) modulo \( \ell \) it is easily seen that \( \nu((q^{de} - 1)/(q^e - 1)) = \nu((q^e - 1)) \) for any \( d \geq 1 \), whence

\[
\nu((q^{de} - 1)/(q^e - 1)) = \nu(|W_\phi|).
\]

The formula stated in (b) now follows from (a) and from the definition of the order polynomial (see Section 2.2). \( \square \)

4. Alperin’s weight conjecture for homotopy fixed point fusion systems

Let \( k \) be an algebraically closed field of characteristic \( \ell \) and let \( F \) be a saturated fusion system on a finite \( \ell \)-group. Recall the function \( w(F) \) defined in the introduction. If \( B_0 \) is the principal block of a finite group with fusion system \( F \) then \( w(F) \) counts the number
of weights associated to $B_0$. Alperin’s weight conjecture (AW conjecture) \cite{2} predicts that $w(\mathcal{F}) = l(B_0)$ where $l(B_0)$ is the number of isomorphism classes of simple $B_0$-modules. When the underlying group is a finite group of Lie type, then $\epsilon$-Harish-Chandra theory combined with the theory of basic sets and block theory gives a highly non-trivial formula for $l(B_0)$ in terms of relative Weyl groups, which leads to:

**Proposition 4.1.** Let $G$ be a connected reductive algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism $F : G \to G$ acting as $\phi$ on the Weyl group $W$ of $G$. If $\ell$ is a very good prime for $G$, and $q$ has order $\epsilon$ modulo $\ell$, then

$$B_0(G^F) \text{ satisfies AW conjecture } \iff w(\mathcal{F}_0(G^F)) = |\text{Irr}(W_{\phi\ell})|,$$

where $\zeta$ is a primitive $\epsilon$th root of unity.

Since $W$ is rational, the isomorphism type of $(W_{\phi\ell}, L_{\phi\ell})$ is independent of the choice of $\zeta$.

**Proof.** Set $B_0 = B_0(G^F)$. It suffices to show that $l(B_0) = |\text{Irr}(W_{\phi\ell})|$. Now $l(B_0) = |\text{Irr}(B_0) \cap \mathcal{E}(G^F, 1)|$ is the number of unipotent characters of $G^F$ in $B_0$, by \cite[Thm. A]{29} and Remark \cite{28} By \cite[Thm. 5.24]{17}, $\text{Irr}(B_0) \cap \mathcal{E}(G^F, 1)$ is the principal $\epsilon$-Harish-Chandra series of $G^F$, which in turn by \cite[Thm. 3.2]{17} is in bijection with $\text{Irr}(W_{\phi\ell})$. \hfill $\square$

In light of the above, our first main result, Theorem 1 from the introduction and which combined with the theory of basic sets and block theory gives a highly non-trivial formula for $l(B_0)$ in terms of relative Weyl groups, which leads to:

**Theorem 4.2.** Let $\ell > 2$, $X$ a simply connected $\ell$-compact group with Weyl group $(W, L)$, $\tau$ an automorphism of $X$ whose class in $\text{Out}(X)$ has finite order prime to $\ell$. Let $q$ be a power of a prime different from $\ell$ and let $\mathcal{F}(\tau X(q))$ be the saturated fusion system associated to $\tau X(q)$. If $\ell$ is very good for $(W, L)$, then

$$w(\mathcal{F}(\tau X(q))) = |\text{Irr}(W_{\phi\ell})|,$$

where $W\phi = \Phi_X(\tau) \in N_{GL(L)}(W)/W$ and $\zeta \in \mathbb{Z}_\ell^*$ is the root of unity with $q \equiv \zeta \pmod{\ell}$.

**Proof.** First note that by Proposition \cite{29} $\ell$ is very good for $(W_{\phi\ell}, L_{\phi\ell})$. Hence, by Theorem \cite{3} we may assume that $q \equiv 1 \pmod{\ell}$ and $\tau = \text{Id}$, so $W_{\phi\ell} = W$. Now any simply connected $\mathbb{Z}_\ell$-reflection group is the direct product of irreducible simply connected $\mathbb{Z}_\ell$-reflection groups by \cite[11.1]{6}. Analogously, if $\mathcal{F}$ is the product of saturated fusion systems $\mathcal{F}_1, \mathcal{F}_2$, then $\mathcal{F}^{\text{cr}} = \mathcal{F}_1^{\text{cr}} \times \mathcal{F}_2^{\text{cr}}$ by \cite[Lemma 3.1]{7}, and thus $w(\mathcal{F}) = w(\mathcal{F}_1)w(\mathcal{F}_2)$. Hence, it suffices to consider the irreducible cases. By Proposition \cite{2} this means that either $W$ is a Clark–Ewing group, a group $G(e, r, n)$ with $e \geq 2$ and $r|e|\ell - 1$, a Weyl group of type $E_6$, $E_7$ or $E_8$ with $\ell = 5$, $\ell \in \{5, 7\}$, $\ell = 7$ respectively, or $G_n$ with $\ell | n$. In the latter case, the assertion holds by the main result of \cite{27} combined with Proposition \cite{11}.

For Clark–Ewing groups $W$ the order is prime to $\ell$. Then by Theorem \cite{3, Thm. III.5.10}, $\mathcal{F}(\tau X(q)) = \mathcal{F}_S(N)$, where $N = S \rtimes W$. Thus, $S$ is the only centric radical subgroup, $W$ is its outer automiser and $|\text{Irr}(W)| = z(kW) = w(\mathcal{F}(\tau X(q)))$ since $|W|$ is coprime to $\ell$.

The other cases will be considered in Sections \cite{7, Thm. III.5.10} and \cite{8}. \hfill $\square$
Note that by Remark 3.3(a) the assumption on \( \tau \) having order prime to \( \ell \) is not necessary if \((W, L)\) arises from a \(\mathbb{Z}\)-reflection group. Thus, combined with Proposition 4.1, Theorem 4.2 yields the following.

**Corollary 4.3.** Alperin’s weight conjecture holds for the principal \( \ell \)-blocks of finite reductive groups \( G^F \), whenever \( G \) is semisimple of simply connected type and \( \ell > 2 \) is a very good prime for \( G^F \).

It is known that the conclusion of Theorem 4.2 does not extend to all primes \( \ell \) even for rational Weyl groups, since both the description of blocks by \( e \)-Harish-Chandra series as well as unipotent characters being basic sets can fail to hold. Examples for this can be found in [30, §1.2] and [26]. It is thus not surprising that for bad primes we obtain a different result (this will be shown in Section 8.1):

**Theorem 4.4.** If \( W \) is one of the four Aguadé exceptional \( \mathbb{Z}_\ell \)-reflection groups, and \( q \equiv 1 \pmod{\ell} \) then

\[
w(\mathcal{F}(X(q))) = |\text{Irr}(W)| + 1.
\]

**Remark 4.5.** (a) It would be interesting to compute \( w(\mathcal{F}) \) for the fusion systems \( \mathcal{F} \) of the principal \( \ell \)-blocks of \( F_4, E_6, E_7, E_8 \) for bad primes \( \ell \). To our knowledge, the weights have not been determined in these cases. The validity of the AW conjecture for \( G_2(q) \) at the bad prime \( \ell = 3 \) has been shown by An [4, (3B)]; here again the number of weights is \(|\text{Irr}(W)| + 1\).

(b) Note that in all of the above results the number of weights is independent of the power of \( \ell \) dividing \( q^e - 1 \).

5. **Centralisers**

An important ingredient for our generalisation of the ordinary weight conjecture will be the existence of centralisers, which we now discuss.

5.1. **Centralisers in \( \ell \)-compact groups.** For \( Q \) a discrete group, we consider \( BQ \) as a pointed space with a chosen base point. Extending the terminology from Section 3.1 by a morphism from \( BQ \) to an \( \ell \)-compact group \( X \), we will mean a continuous, pointed map \( f : BQ \to X \). Two morphisms \( f, f' : BQ \to X \) are called conjugate if \( f \) and \( f' \) are freely homotopic.

For spaces \( Y, Z \), \( \text{Map}(Y, Z) \) denotes the function space of maps from \( Y \) to \( Z \) and the component containing a particular map or homotopy class \( f \) is denoted \( \text{Map}(Y, Z)_f \). For \( Q \) a discrete group, and a morphism \( f \) from \( BQ \) to an \( \ell \)-compact group \( X \), we set \( C_X(Q, f) := \text{Map}(BQ, X)_f \). If \( R \leq Q \), and \( g : BR \to X \) is a continuous map we say that \( f \) is an extension of \( g \) if \( g \simeq f \circ Bi \), where \( i : R \to Q \) is the inclusion map, and we denote by \( f : BR \to X \) also the composition \( f \circ Bi \).

From now on, and for the rest of this section, let \( X \) be a connected \( \ell \)-compact group with maximal torus \( i : T \to X \) and Weyl group \( (W, L) \). Then \( W \) identifies with the set of conjugacy classes of self-equivalences \( \varphi \) of \( T \) such that \( i \circ \varphi \) and \( i \) are conjugate (see Section 3.1) and for \( w \in W \), we denote by \( w : T \to T \) any representative of the conjugacy class of self-equivalences indexed by \( w \).

We say that a morphism \( f : BQ \to X \) *factors through* \( T \) if there exists a morphism \( j : BQ \to T \) such that \( i \circ j \) is conjugate to \( f \), and in this case we call \( j \) a *factorisation*
Lemma 5.1. Let $Q$ be a finite cyclic $\ell$-group. Then any morphism $f : BQ \to X$ factors through $T$. If $j, j'$ are two factorisations of $f$, then $j'$ is conjugate to $w \circ j$ for some $w \in W$.

Proof. For the first assertion, see for instance [45, Thm. 2.6]. The second assertion is [45, Prop. 4.1].

The $\ell$-compact torus $T$ has a discrete approximation $B\hat{T} \to T$ as in [24, Sec. 6] (see also [25, Sec. 6]) with $\hat{T} = \mathbb{Z}/\ell^\infty \otimes \mathbb{Z}_2 \cong (\mathbb{Z}/\ell^r)^\vee$ where $r$ is the rank of $T$. If $Q$ is a finite $\ell$-group or a discrete torus, then to every morphism $j : BQ \to BT$ is associated a group homomorphism $\overline{j} : Q \to \hat{T}$, uniquely determined by $j$ such that $BJ : BQ \to B\hat{T}$ is a lift of $j$, i.e., such that composition of $BJ$ with $B\hat{T} \to T$ is conjugate to $j$ [25, Prop. 3.2] (here note that if $Q$ is a finite $\ell$-group, $Q$ is a discrete approximation of $BQ$). We call $\overline{j}$ the discrete approximation to $j$. For a subset $A \subset \hat{T}$ we denote by $C_W(A)$ the subgroup of elements $w \in W$ which fix $A$ point-wise.

By [24, Prop. 5.1] for any morphism $f : BQ \to X$, the centraliser $C_X(Q, f)$ is an $\ell$-compact group. If $f$ factors through $T$, then the Weyl group of $C_X(Q, f)$ may be described as follows. Let $j : BQ \to T$ be a factorisation of $f$ and $\overline{j}$ its discrete approximation. Then $i' : T \to C_X(Q, f)$ is a maximal torus of $C_X(Q, f)$ for some morphism $i'$ whose composition with the evaluation map $C_X(Q, f) \to X$ at the base point of $X$ is conjugate to $i$. Since $f$ is conjugate to $i \circ j$, we have $C_X(Q, f) = C_X(Q, i \circ j)$. By [25, Prop. 4.4] and [25, Thm. 7.6], the Weyl group of $C_X(Q)$ with respect to $i'$ identifies with $(C_W(\overline{j}(Q), L))$.

We now prove Theorem 2. The case $|Q| = \ell$ was shown in [6, Thm. 1.9].

Theorem 5.2. Suppose that $X$ is a connected $\ell$-compact group with torsion-free fundamental group and Weyl group $(W, L)$. Then for any morphism $f : BQ \to X$ with $Q$ a finite cyclic $\ell$-group, the $\ell$-compact group $C_X(Q, f)$ is connected.

Proof. By [6, Thm. 11.1] and [5, Thm. 1.1] there is a direct decomposition $(W, L) = (W_G \times W_Y, (Z_\ell \otimes Z_\ell) \oplus L_Y)$, where $(W_G, L_G)$ is the Weyl group of a compact Lie group whose fundamental group has $\ell$-torsion and $W_Y$ is a direct product of exotic $\mathbb{Z}_\ell$-reflection groups. Since this induces a corresponding direct decomposition of the pair $(W, T)$ and thus of stabilisers, we may assume that $(W, L)$ either comes from a compact Lie group whose fundamental group has $\ell$-torsion, or is an exotic $\mathbb{Z}_\ell$-reflection group. In the former case, the assertion follows from Steinberg’s result [51, Cor. 2.16].

Now let $(W, L)$ be exotic and let $(C_W(\overline{j}(s)), L)$ be the Weyl group of $C_X(Q, f)$ as described above the theorem and denote by $W_1$ the Weyl group of the connected component of $C_X(Q, f)$. By [25, Thm. 7.6 and Rem. 7.7], $W_1 \leq C_W(\overline{j}(s))$ is the subgroup generated by those $w \in C_W(\overline{j}(s))$ whose image in $W$ is a reflection. Here note that if $\ell = 2$, then by the classification of $2$-compact groups in [5], $W = G_{24}$ and therefore for any reflection in $W$, of which there is just one class, the subgroups $\sigma(s)$ and $F(s)$ from [25, Def. 7.3] agree. Now by [25, Thm. 4.7], $C_X(Q, f)$ is connected if and only if $W_1 = C_W(\overline{j}(s))$. Hence
$C_X(Q,f)$ is connected if and only if $C_W(\tilde{j}(s))$ is a reflection subgroup of $W$. Since $W$ is exotic, the latter holds by Proposition 2.3. \hfill $\Box$

Let $\alpha$ be a self-equivalence of $X$ and let $\alpha_T : T \to T$ be such that $i \circ \alpha_T$ is conjugate to $\alpha \circ i$ (see the discussion before Theorem 3.1); such an $\alpha_T$ is called a lift of $\alpha$. The following is a rephrasing of [12, Lemma 7.3].

**Lemma 5.3.** Let $\alpha$ be a self-equivalence of $X$, $\alpha_T : T \to T$ a lift of $\alpha$ and $\tilde{\alpha}_T : \tilde{T} \to \tilde{T}$ its discrete approximation. Let $Q$ be a finite cyclic $\ell$-group, $f : BQ \to X$ a morphism, $j$ a factorisation of $f$ and $\tilde{j}$ the discrete approximation of $j$. Let $i' : T \to C_X(Q,f)$ be a maximal torus of $C_X(Q,f)$ where $i'$ is a morphism whose composition with the evaluation map $C_X(Q,f) \to X$ at the base point of $X$ is conjugate to $i$.

(a) $f$ is conjugate to $\alpha \circ f$ if and only if there exists $w \in W$ such that $\alpha_T \circ j$ is conjugate to $w \circ j$ (equivalently $\tilde{\alpha}_T \circ \tilde{j} = w \circ \tilde{j}$). If it exists then $w$ is unique up to elements of $C_W(\tilde{j}(Q))$.

(b) For any $w \in W$ such that $\alpha_T \circ j$ is conjugate to $w \circ j$, the induced homotopy equivalence $\alpha_\#$ of $C_X(Q,f)$ has a lift to $T$ of the form $w^{-1} \alpha_T$, where $\alpha_\#$ denotes composition with $\alpha$.

**Proof.** The hypotheses imply that $i \circ \alpha_T \circ j$ is conjugate to $i \circ j$, i.e., that $\alpha_T \circ j$ is a factorisation of $f$. Hence by Lemma 5.1 $\alpha_T \circ j$ is conjugate to $w \circ j$ for some $w \in W$. Moving to discrete approximations yields $\tilde{\alpha}_T \circ \tilde{j} = w \circ \tilde{j}$. The uniqueness up to $C_W(\tilde{j}(Q))$ is clear. For the final assertion see [12, Lemma 7.2]. \hfill $\Box$

**Remark 5.4.** Suppose that $\alpha \circ f = f$ in the above. If $j' : BQ \to T$ is another factorisation of $f$, and $w' \in W$ with $\tilde{\alpha}_T \circ \tilde{j}' = w' \circ \tilde{j}'$, then by uniqueness of factorisations $\tilde{j}' = x \circ \tilde{j}$ for some $x \in W$. Setting $C = wC_W(\tilde{j}(Q))$ and $C' = w'C_W(\tilde{j}'(Q))$ we have that $C' = x^{-1}C \tilde{\alpha}_T x$, where by $\tilde{\alpha}_T x$ we mean $\tilde{\alpha}_T \circ x \circ \tilde{\alpha}_T^{-1} \in \text{Aut}(\tilde{T})$. A similar remark holds about the choice of restriction $\alpha_T$.

### 5.2. Homotopy fixed point centralisers.

We continue with the notation of this section. So $X$ is a connected $\ell$-compact group with torsion-free fundamental group, maximal torus $i : T \to X$ and Weyl group $(W,L)$. Let $\tau$ be a self equivalence whose order in $\text{Out}(X)$ is finite. Let $S$ be a finite $\ell$-group and, for some $q \in \mathbb{Z}_\ell^*$, let $f : BS \to \tau X(q)$ be a morphism of loop spaces. For $s \in S$, define the centralisers

$$C_X(s) := C_X((s),\iota \circ f) \quad \text{and} \quad C_{X(q)}(s) := C_{X(q)}((s),f),$$

where $\iota$ is an in $(\ast)$ in Section 3.2. Set

$$W(s) := C_W(\tilde{j}(s))$$

where $j : B(s) \to T$ is a factorisation of (the restriction to $(s)$ of) $\iota \circ f$ and $\tilde{j} : (s) \to \tilde{T}$ the discrete approximation of $j$. Recall from the previous section that $(W(s),L)$ is the Weyl group of the connected $\ell$-compact group $C_X(s)$.

**Proposition 5.5.** With $X$, $S$ and $f : BS \to \tau X(q)$ as above, for any $s \in S$ we have:

(a) With $\iota_\#$ denoting composition with $\iota$ and $\tau\psi_\#$ denoting composition with $\tau\psi_\#$,

$$C_{X(q)}(s) \simeq C_X(s)^{\iota\psi_\#}.$$
(b) If \( ^\ast X(q) \) is the classifying space of a saturated fusion system \( \mathcal{F} \) on \( S \) via \( f \) then for any fully \( \mathcal{F} \)-centralised \( s \in S \), \( C_{X(q)}(s) \) is the classifying space of \( C^e_f(s) \).

Note that by Theorem 3.2 the assumption in (b) is satisfied if \( X \) is in addition simply connected and \( \tau \) has \( \ell' \) order.

**Proof.** In (a), it suffices to prove that

\[
\begin{array}{ccc}
C_{X(q)}(s) & \xrightarrow{\iota^\#} & C_X(s) \\
\downarrow & & \downarrow \Delta \\
C_X(s) & \xrightarrow{(1,\tau^q)} & C_X(s) \times C_X(s)
\end{array}
\]

is a homotopy pullback diagram (see [13] Rem. 2.3). Now \( C_X(s) \) is connected by Theorem 3.2. Let \( \beta = \iota \circ f \) and \( \alpha = \tau^q \). By definition, \( \Delta \circ \iota \simeq (1, \alpha) \). Composing with \( f \) on the right and with projection onto the second component on the left, we obtain \( \beta \simeq \alpha \circ \beta \). Now (a) follows by applying [12] Lemma 7.2 with \( f \) in place of \( g \), and (b) follows from [11] Thm. 6.3 and Prop. 2.5(c)]. \( \square \)

Now assume \( \ell > 2 \) and let \( \phi \in N_{\text{GL}(L)}(W) \) be the element (uniquely determined up to multiplication by \( W \)) such that \( \phi \) corresponds to the conjugacy class of \( \tau \) in \( \text{Out}(X) \), i.e., \( \hat{W} \phi \) is the image under \( \Phi_X \) of the homotopy class of \( \tau^q \). Now let \( s \in S \) and let \( \tau^q \) be as in Proposition 5.5. Since \( C_X(s) \) is connected by Theorem 5.2 there exists \( \phi_s \in N_{\text{GL}(L)}(W(s)) \) (uniquely determined up to multiplication by \( W(s) \)) such that \( \phi_s \lambda_q \) corresponds to the conjugacy class of \( \tau^q \) via Theorem 3.1, where \( \lambda_q = q \text{Id} \in \text{GL}(L) \), i.e., \( W(s) \phi_s \lambda_q \) is the image under \( \Phi_{C_X(s)} \) of the homotopy class of \( \tau^q \).

Lemma 5.3 may be used to identify \( W(s) \phi_s \). Let \( \tau^q_T \) be a lift of \( \tau \) to \( T \) and let \( \psi^q_T \) be a lift of \( \psi^q \) such that \( \psi^q_T \) is the map \( x \mapsto x^q \), \( x \in \hat{T} \). As explained in the proof of Proposition 5.5, \( \iota \circ f \) is conjugate to \( \tau^q \circ \iota \circ f \). Hence, by Lemma 5.3 applied with \( \alpha = \tau^q \), \( Q = (s) \) and \( \iota \circ f \) replacing \( f \), there exists \( w \in W \) such that \( \tau^q_T \psi^q_T \circ j \) is conjugate to \( w \circ j \) and for any such \( w \), \( w^{-1} \tau^q_T \psi^q_T \) is a lift of \( \tau^q \) to \( T \). Moving over to discrete approximations, it follows that \( w^{-1} \tau^q_T \psi^q_T \) centralises \( j(s) \) and \( w^{-1} \tau^q_T \) normalises \( W(s) \). Thus we may take for \( \phi_s \) the element \( w^{-1} \tau^q_T \).

We conjecture the following order formula for centralisers (compare with Theorem 3.6(b)).

**Conjecture 5.6.** In the situation of Proposition 5.5 assume that \( \ell > 2 \), \( X \) is simply connected and \( \tau \) has order prime to \( \ell \). Then for any fully \( \mathcal{F}(\tau^q) \)-centralised \( s \in S \),

\[
\nu_\ell(|C_S(s)|) = \nu_\ell(O_q(W(s)\phi_s^{-1}))
\]

In Proposition 5.7 below we show that the conclusion of the conjecture holds whenever the data \((W, L, \tau, q)\) corresponds to a finite group of Lie type and in Lemma 7.18 we show that it holds for generalised Grassmannians \( X(e, r, n)(q) \) when \( \tau = 1 \) and \( q \equiv 1 \pmod{\ell} \).

### 5.3. Finite groups of Lie type.

We record here the translation of the constructions above for finite groups of Lie type. The discussion below is an enhancement of Remark 5.3 and is a combination of results of Quillen, Friedlander, and Mislin.
Let $p \neq \ell$ be a prime, $G$ a connected reductive group over $\mathbb{F}_p$ with maximal torus $T$, Weyl group $W$ and cocharacter lattice $L_0$. Let $X = B\tilde{G}_\ell^\wedge$ where $\tilde{G}$ is a connected reductive group over $\mathbb{C}$ with the same root datum as $G$. Then $X$ is a connected $\ell$-compact group with Weyl group $\left(W, Z_{\ell} \otimes Z L_0\right)$ and underlying maximal torus $i: T \to X$. Let $B\tilde{T} \to T$ be a discrete approximation with $\tilde{T} = \mathbb{Z}/\ell^\infty \otimes L_0$. We identify the subgroup of $T$ consisting of all $\ell$-power elements with $\tilde{T}$ via the restriction of the $W$-equivariant isomorphism $T \cong \mathbb{F}_{\ell}^\times \otimes_{\mathbb{Z}} L_0$. Let $F_0: G \to G$ be a standard Frobenius morphism such that $T$ is $F_0$-stable and $F_0(x) = x^p$ for $x \in T$. For each $d \geq 0$ there exist maps $\iota_d: (BGF_0^d)^\wedge_\ell \to X$ such that:

(a) The restriction of $B\tilde{T} \to T$ to $B(\tilde{T} \cap G F_0^d)$ is a factorisation of the restriction of $\iota_d$ to $B(\tilde{T} \cap G F_0^d)$ where we identify $B(\tilde{T} \cap G F_0^d)$ to its $\ell$-completion.

(b) For all $d, d'$ such that $d \mid d'$, we have a homotopy commutative diagram

\[
\begin{array}{ccc}
(BGF_0^d)^\wedge_\ell & \xrightarrow{\iota_d} & X \\
\text{inc} & & 1 \\
(BGF_0^{d'})^\wedge_\ell & \xrightarrow{\iota_{d'}} & X
\end{array}
\]

where the left map is inclusion and the right hand side map is the identity (see [28, Corollary 1.3]).

Let $q$ be a power of $p$ and $F: G \to G$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure such that $T$ is $F$-stable. Then there exists a self-equivalence $\tau$ of $X$ (induced from a finite order automorphism of $\tilde{G}$), a map $\iota: (BGF)^\wedge_\ell \to X$ and a positive integer $r$ with $F^r = F_0^r$ such that

\[
\begin{array}{ccc}
(BGF)^\wedge_\ell & \xrightarrow{\iota} & X \\
\text{inc} & & 1 \\
(BGF_0)^\wedge_\ell & \xrightarrow{\iota_r} & X
\end{array}
\]

is homotopy commutative and

\[
\begin{array}{ccc}
(BGF)^\wedge_\ell & \xrightarrow{\iota} & X \\
& & 1 \\
X & \xrightarrow{(1, \tau \psi)} & X \times X
\end{array}
\]

is a homotopy pullback. In particular, $(BGF)^\wedge_\ell \simeq X^{hr \psi}$ and consequently $F^r(X(q)) = F_S(GF)$ for a Sylow $\ell$-subgroup $S$ of $GF$ (see [13, Thm. 3.1]). Moreover, setting $L = \mathbb{Z}_\ell \otimes_{\mathbb{Z}} L_0$ and $\phi := 1 \otimes F \in GL(L)$ we have that $W\phi = \Phi_X(\tau)$, where $\Phi_X$ is as in Theorem 3.1.

Let $s \in GF$ be an $\ell$-element. We explain how to identify the coset $W(s)\phi_s$ of Conjecture 5.6. Since $s = F(s)$, $s$ is contained in an $F$-stable maximal torus $T_w := gTg^{-1}$ of
\( \mathbf{G} \), with \( g^{-1}F(g) = w \in N_{\mathbf{G}}(T) \) (see e.g. [42 §25.1]). We may assume that \( g \in G^{F_d} \) for some multiple \( d \) of \( r \). So, \( g^{-1}sg \in \hat{T} \cap G^{F_d} \). Let \( Bc_{g^{-1}} : \langle s \rangle \to \hat{T} \cap G^{F_d} \) be induced from \( c_{g^{-1}} : \langle s \rangle \to \hat{T} \cap G^{F_d} \). By (a) and (b) above we obtain the homotopy commutative diagram:

\[
\begin{array}{ccc}
B\langle s \rangle & \xrightarrow{\text{inc}} & (BG)^{\wedge}_\ell \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
B\langle s \rangle & \xrightarrow{\text{inc}} & (BG^{F_d}^{\wedge})_\ell \\
\end{array}
\]

Thus \( c_{g^{-1}} : \langle s \rangle \to \hat{T} \) is the discrete approximation of a factorisation of \( \iota \circ \text{inc} : B\langle s \rangle \to X \). It follows that \( W(s) = C_W(t) \), where \( t = g^{-1}sg \in \hat{T} \). Moreover, the equation \( F(s) = s \) translates to

\[
w^{\tau_F}(t^q) = wF(t) = t.
\]

By the recipe for \( \phi_s \) given at the end of the previous subsection, we have

\[
\phi_s = w\phi.
\]

**Proposition 5.7.** With the notation above, suppose that \( \ell \) is odd and \( s \in S \) is fully \( \mathcal{F}(^X(q)) \)-centralised. Then

\[
\nu_t([C_S(s)]) = \nu_t(O_q(W(s)\phi_s^{-1})).
\]

**Proof.** We have \( F \circ c_g = c_g \circ wF \) as maps from \( T \) to \( T_w \). So by transport of structure \( gC_W(t)g^{-1} \) is the Weyl group of \( C_G(s) \) with respect to the reference torus \( T_w \), and \( F \) acts on it as \( w\phi \). By the Steinberg order formula for finite groups of Lie type, \( |C_G(s)^F|_\ell = (O_q(W(s)\phi_s^{-1}))_\ell \). Since \( \mathcal{F}(^X(q)) = \mathcal{F}_S(G^F) \), the fully centralised assumption on \( s \) means that \( C_S(s) \) is a Sylow \( \ell \)-subgroup of \( C_G(s)^F \). The result follows. \( \square \)

The next result will be used in section 7

**Lemma 5.8.** With the above notation, let \( s \in G^{F_d} \) be an \( \ell \)-element. If \( j : B\langle s \rangle \to T \) is a factorisation of \( \iota_d \circ \text{inc} : B\langle s \rangle \to X \) then \( j(s) \) and \( s \) are \( G \)-conjugate.

**Proof.** As above, there exist \( d \geq 0 \) and \( g \in G^{F_d} \) such that \( gsg^{-1} \in \hat{T} \), so \( Bc_g : B\langle s \rangle \to T \) is a factorisation of \( \iota_d \circ \text{inc} : B\langle s \rangle \to X \) with discrete approximation \( c_g \). If \( j : B\langle s \rangle \to T \) is another factorisation, then by Lemma 5.1 \( j \) is conjugate to \( w \circ Bc_g \) for some \( w \in W \). By uniqueness of discrete approximations, \( j = w \circ c_g \) and the result follows. \( \square \)

6. **Weight conjectures for Spetses**

So far, we have stayed on the “local” side of fusion systems. Now as for finite groups we want to compare this to some “global” side, for which we need to introduce suitable global objects. This will be the spets, and as a special case, a finite reductive group.
6.1. Spetses and their unipotent degrees. The term “spetses” denotes a hypothetical mathematical object resembling a connected reductive linear algebraic group but whose Weyl group is a non-rational, finite complex reflection group. Attached to a spets is a collection of data that behave like combinatorial data arising in the representation theory of finite reductive groups, more precisely, analogues of unipotent characters, subdivided into Harish-Chandra series, with Fourier matrices attached to the m, and the like. Such data have been constructed for the so-called spetsial complex reflection groups: a proper subset of all complex reflection groups defined via certain integrality or rationality properties of their associated Hecke algebras, see [40 §3], encompassing and much larger than the collection of all rational and even all real reflection groups.

The spetsial complex reflection groups are the direct products of irreducible complex reflection groups of the following types:

- a real irreducible reflection group;
- $G(e, 1, n)$ or $G(e, e, n)$ for some $n \geq 1$, $e \geq 3$; or
- $G_n$, with $n \in \{4, 6, 8, 14, 24, 25, 26, 27, 29, 32, 33, 34\}$

(see [41 §8]). We say that a $\mathbb{Z}_\ell$-reflection group $(W, L)$ is spetsial if its extension to $\mathbb{C}$ is spetsial.

Definition 6.1. A $\mathbb{Z}_\ell$-spets $G = (W, L)$ is a spetsial $\mathbb{Z}_\ell$-reflection group $(W, L)$ together with an element $\phi \in N_{\text{GL}(L)}(W)$. The order (polynomial) of $G$ is

$$|G| := x^N \prod_{i=1}^{r} (\varepsilon_i - 1)^{-2} \Omega_{\mathbb{Q}_\ell}(W, \phi) = x^N \prod_{i=1}^{r} (x^{d_i} - \varepsilon_i) \in \mathbb{Z}_\ell[x],$$

where $\{(d_i, \varepsilon_i) \mid 1 \leq i \leq r\}$ are the generalised degrees of $W\phi$ on $\overline{\mathbb{Q}_\ell} \otimes_{\mathbb{Z}_\ell} L$ as in Section 2.2 and $N$ is the number of reflections in $W$ (see [19 Def. 1.44]).

This is a subclass of spetses constructed from arbitrary complex spetsial reflection groups (see [13]), augmented by also fixing a $\mathbb{Z}_\ell$-lattice for $W$. We will need the following observation:

Proposition 6.2. Let $(W, L)$ be a spetsial $\mathbb{Z}_\ell$-reflection group with $L_W$ torsion-free. Then for any $s \in \hat{T} = \mathbb{Z}/\ell^\infty \otimes_{\mathbb{Z}_\ell} L$, the $\mathbb{Z}_\ell$-reflection group $C_W(s)$ is spetsial.

Proof. Arguing as in the proof of Theorem [52] we see that $(C_W(s), L)$ is a reflection subgroup of $(W, L)$. For Weyl groups, all reflection subgroups are again Weyl groups. So by [6 Thm. 11.1] we may assume that $W$ is exotic. Our argument here is case-by-case. For Clark–Ewing groups, centralisers are parabolic subgroups, and parabolic subgroups of spetsial groups are spetsial by [41 §8]. Next assume that $W = G(e, r, n)$ is in the infinite series, with $3 \leq e|(\ell - 1)$, and $r \in \{1, e\}$ as $W$ is spetsial. Then Proposition 2.3 shows the claim. So we are left to consider $W = G_{29}$ and $W = G_{34}$, at $\ell = 5, 7$ respectively. As mentioned in the proof of Proposition 2.3, here the centralisers of non-trivial elements $s$ are either symmetric groups $\mathfrak{S}_\ell$, hence spetsial, or of order prime to $\ell$ and thus parabolic. □

Attached to a $(\mathbb{Z}_\ell)$-spets $G = (W, \phi, L)$ is a set $\text{Uch}(G)$ of unipotent characters [39,19]. Each $\gamma \in \text{Uch}(G)$ has a degree $\gamma(1)$ which is a polynomial with coefficients in the character field of $W$ on $L$. Thus, in particular if $W$ is a $\mathbb{Z}_\ell$-reflection group then $\gamma(1) \in \mathbb{Q}_\ell[x]$. One unipotent character that always exists is the trivial character $1_G$, with degree $1_G(1) = 1$. 


The unipotent degrees are not necessarily contained in \( \mathbb{Z}_l[x] \). Recall our Definition 2.3 of bad primes. One observes the following by a case-by-case check (see the formulas in [39] and the lists in [19], respectively, and also compare with [15, Con. 8.3] about ‘bad ideals’):

**Proposition 6.3.** Let \( G = (W\phi, L) \) be a \( \mathbb{Z}_l \)-spets. Then \( \ell \) is good for \( W \) if and only if the degrees of all unipotent characters of \( G \) have \( \ell \)-adically integral coefficients.

Let \( G = (W\phi, L) \) be a \( \mathbb{Z}_l \)-spets. Then, for any root of unity \( \zeta \in \mathbb{Z}_l^* \) the set of unipotent characters \( \text{Uch}(G) \) is naturally partitioned into so-called \( \zeta \)-Harish-Chandra series, and one among them, the principal \( \zeta \)-Harish-Chandra series \( \mathcal{E}(G, 1, \zeta) \) of \( \text{Uch}(G) \) containing \( 1_G \), is in bijection with \( \text{Irr}(W_{\phi^{-1}}) \) (see [39, Folg. 3.16 and 6.11] and [19, 4.31]), where \( W_{\phi^{-1}} \) is the corresponding relative Weyl group, see Section 2.2.

**Example 6.4.** Let \( q \) be a prime power. If \( W \) is a Weyl group, then letting \( G \) be a connected reductive algebraic group defined over \( \mathbb{F}_q \) with Weyl group \( W \), and \( F : G \to G \) the corresponding Frobenius endomorphism, acting as \( \phi \) on \( W \) as in Remark 2.3(a), then \( |G^F| = |G|_{x=q} \), where \( G = (W\phi^{-1}, L) \) and \( \text{Uch}(G) \) can be identified with the set of unipotent characters (in the sense of Lusztig, see e.g. [32 Def. 2.3.8]) of the finite reductive group \( G^F \), such that the degree of \( \gamma \in \text{Uch}(G) \) is precisely the polynomial \( \gamma(1) \in \mathbb{Q}[x] \) for which \( \gamma(1)|_{x=q} \) is the degree of the corresponding unipotent character of \( G^F \). Further, if \( q \) has order \( e \) modulo \( \ell \), then for any primitive \( e \)-root of unity \( \zeta \in \mathbb{Z}_l \), the principal \( e \)-Harish-Chandra series of \( G^F \) is in bijection with the principal \( \zeta \)-Harish-Chandra series of \( G \).

There is the following important “local control” of heights for unipotent characters of spetses (the Weyl group case of which was used in the proof of [20, Thm. 4.7]):

**Proposition 6.5.** Let \( G = (W\phi, L) \) be a \( \mathbb{Z}_l \)-spets, \( q \) a prime power and \( \zeta \in \mathbb{Z}_l^* \), the root of unity with \( q \equiv \zeta \pmod{\ell} \) (respectively \( q \equiv \zeta \pmod{4} \) when \( \ell = 2 \)). Then there is a bijection \( \Psi : \mathcal{E}(G, 1, \zeta) \xrightarrow{1:\rightarrow} \text{Irr}(W_{\phi^{-1}}) \) with

\[
\nu_e(\gamma(1)|_{x=q}) = \nu_e(\Psi(\gamma)(1)) \quad \text{for all } \gamma \in \mathcal{E}(G, 1, \zeta).
\]

In particular, the distribution of \( \ell \)-heights in the principal \( \zeta \)-Harish-Chandra series depends solely on the relative Weyl group \( W_{\phi^{-1}} \).

**Proof.** By [39, Folg. 3.16 and 6.11] and [19, 4.31] there is a bijection \( \Psi : \mathcal{E}(G, 1, \zeta) \to \text{Irr}(W_{\phi^{-1}}) \) such that \( \gamma(1)|_{x=\zeta} = \pm \Psi(\gamma)(1) \), for suitable signs depending on \( \gamma \). Furthermore, \( \gamma(1) \) is a constant times a product of cyclotomic polynomials over \( \mathbb{Z}_\ell \); by the afore-mentioned specialisation property, none of these factors is \( x - \zeta \).

Writing \( \tilde{\gamma}(1) := \gamma(1)(\zeta^{-1}x) \) and \( \tilde{q} := \zeta^{-1}q \) we now claim that \( \nu_\ell(\gamma(1)|_{x=q}) = \nu_\ell(\tilde{\gamma}(1)|_{x=\tilde{q}}) = \nu_\ell(\Psi(\gamma)(1)) \), where no factor of \( \tilde{\gamma}(1) \) equals \( x - 1 \). Since \( \tilde{q} \equiv 1 \pmod{\ell} \), a polynomial \( f \in \mathbb{Z}_\ell[x] \) has \( f(\tilde{q}) \equiv 0 \pmod{\ell} \) if and only if \( f(1) \equiv 0 \pmod{\ell} \). Now any cyclotomic polynomial over \( \mathbb{Q} \) is a product of factors \( (x^m - 1)/(x - 1) \) and their inverses, and as \( \Phi_m(\tilde{q}) \) and \( \Phi_m(1) \) are divisible by \( \ell \) at most once for any \( m > 1 \), any cyclotomic polynomial over \( \mathbb{Q} \) different from \( x - 1 \) has at most one factor \( f \) over \( \mathbb{Z}_\ell \) with \( f(\tilde{q}) \) (and \( f(1) \)) divisible by \( \ell \). Since for
any $m \geq 1$, $(\bar{q}^m - 1)/(\bar{q} - 1)$ and $((x^m - 1)/(x - 1))|_{x=1}$ are divisible by the same power of \ell (by an application of L'Hospital's rule), our claim now follows.

\section*{6.2. The fusion system and principal block of a $\mathbb{Z}_\ell$-spets.}

For the rest of this section, let $\ell > 2$ be a prime. Let $G = (W \phi^{-1}, L)$ be a simply connected $\mathbb{Z}_\ell$-spets (that is, $(W, L)$ is simply connected) with $\phi$ of $\ell$-order. For $q$ a prime power not divisible by $\ell$ set $G(q) := (W \phi^{-1}, L, q)$. Let $X$ be a connected $\ell$-compact group with Weyl group $(W, L)$, and let $\tau$ be a homotopy self-equivalence of $X$ with $\Phi_X(\tau) = W \phi$ (see Theorem 3.1). By Theorem 3.2, $\tau X(q)$ is the classifying space of a saturated fusion system, $\mathcal{F}(\tau X(q))$ and we set $\mathcal{F}(G(q)) := \mathcal{F}(\tau X(q))$. By Theorem 3.6(b) and Definition 6.1 if $S$ is the underlying $\ell$-group of $\mathcal{F}(G(q))$, then $|S| = (|G|)_{x=q} \ell$.

We let $f : BS \to \tau X(q)$ and $\iota : \tau X(q) \to X$ be as defined in Section 5.2. For $s \in S$ let $j : B\langle s \rangle \to T$ be a factorisation of (the restriction to $\langle s \rangle$ of) $\iota \circ f$ and $j : \langle s \rangle \to \tilde{T}$ the discrete approximation of $j$ and let $W(s) = C_W(\tilde{j}(s))$ and $\phi_s$ be as defined after Proposition 5.3. Recall that $\phi_s \in W \phi$ normalises $W(s)$. By Proposition 6.2, the $\mathbb{Z}_\ell$-reflection group $W(s)$ is spetsial. Let $G(s) := (W(s)\phi_s^{-1}, L)$ be the associated $\mathbb{Z}_\ell$-spets, with unipotent characters $\text{Uch}(C_G(s))$. By Lemma 5.3 and Remark 5.4, the isomorphism class of $C_G(s)$ is independent of the choice of $j$. Moreover, we have:

\textbf{Lemma 6.6.} The order polynomial $|C_G(s)|$ divides $|G|$ in $\mathbb{Z}_\ell[x]$.

\textbf{Proof.} Let $\xi \in \overline{\mathbb{Q}_\ell}^\times$ be a zero of $|G|$. The Sylow theorems [16, Thm. 3.4] show that the multiplicity of $x - \xi$ as a factor of $|G|$ equals the maximal dimension of a $\xi$-eigenspace of elements $g \in W \phi^{-1}$, and similarly for $|C_G(s)|$. As $W(s)\phi_s^{-1} \subseteq W \phi^{-1}$, the claim follows.

We let

$$\mathcal{E}(G(q), s) = \{\gamma_{q, \lambda}^s \mid \lambda \in \text{Uch}(C_G(s))\}$$

denote a set in bijection with $\text{Uch}(C_G(s))$ and call it the \emph{characters of $G(q)$ in the series $s$}. The sets $\text{Uch}(C_G(s))$ are in canonical bijection for conjugate elements $s$. The degree of $\gamma_{q, \lambda}^s$ is defined as

$$\gamma_{q, \lambda}^s(1) := (|G : C_G(s)|_{x^q} \lambda(1))|_{x=q} \in \mathbb{Q}_\ell.$$ 

Here, $|G : C_G(s)|_{x^q}$ means the prime-to-$x$ part of the polynomial $|G|/|C_G(s)| \in \mathbb{Z}_\ell[x]$.

This is inspired by (and specialised to) Lusztig's formula for Jordan decomposition of characters (see e.g. [32, Thm. 2.6.4]) in the case of rational spetses, i.e., finite reductive groups. Note that our notion of characters of $G(q)$ in the series $s$ is different from that in finite reductive groups where $s$ is taken to be an element in the dual group. However as explained in the proof of Proposition 6.8 our assumptions on $\ell$ allow for this change.

When $s = 1$ we just write $\lambda^q$ for $\gamma_{q, \lambda}^1$, where now $\lambda^q(1) = \lambda(1)|_{x=q}$, and call $\mathcal{E}(G(q), 1)$ the unipotent characters of $G(q)$. The $\ell$-\textit{defect} of $\gamma_{q, \lambda}^s \in \mathcal{E}(G(q), s)$ is defined to be the $\ell$-adic valuation

$$\nu_\ell(|G|_{x=q}) - \nu_\ell(\gamma_{q, \lambda}^s(1)) = \nu_\ell(|C_G(s)|_{x=q}) - \nu_\ell(\lambda(1)_{x=q}).$$

Further, for $\zeta \in \mathbb{Z}_\ell^\times$ the root of unity with $q \equiv \zeta \pmod{\ell}$ we denote by $\mathcal{E}(G(q), s)_\zeta$ the subset of $\mathcal{E}(G(q), s)$ in bijection with the principal $\zeta$-Harish-Chandra series of $\text{Uch}(C_G(s))$. 

\newpage
By Proposition 6.5 \(\mathcal{E}(G(q), s)_1\) is in bijection with \(\mathrm{Irr}(W(s), q^{-1})\). The following definition is inspired by the results of Cabanes–Enguehard [21, Thm.] on unipotent \(\ell\)-blocks of finite reductive groups; indeed, if \(G\) is a rational spets for which \(\ell\) is very good, then what we define are exactly the characters in the principal \(\ell\)-block of the corresponding finite group of Lie type (see Proposition 6.8 below):

**Definition 6.7.** Let \(G = (W\phi^{-1}, L)\) be a simply connected \(\mathbb{Z}_\ell\)-spets such that \(\ell\) is very good for \(G\). Then define the *characters in the principal block* \(B_0\) of \(G(q)\) as

\[
\mathrm{Irr}(B_0) := \bigcap_{s \in S/F} \mathcal{E}(G(q), s)_1,
\]

where the union runs over a set \(S/F\) of fully centralised representatives \(s\) of \(\mathcal{F}(G(q))\)-conjugacy classes in \(S\). For a non-negative integer \(d\) we let

\[
\mathrm{Irr}^d(B_0) := \{\chi \in \mathrm{Irr}(B_0) \mid d = \nu_\ell(|G|_{x=q}) - \nu_\ell(1)\},
\]

the set of irreducible characters in \(B_0\) of \(\ell\)-defect \(d\).

**Proposition 6.8.** Let \(G\) be a semisimple algebraic group defined over \(\mathbb{F}_q\) with respect to a Frobenius endomorphism \(F : G \to G\). Suppose that \(F\) has \(\ell\)-order in its action on the Weyl group and that \(\ell > 2\) is a very good prime for \(G\). Let \(B_0(G)\) be the principal \(\ell\)-block of \(G := G^F\) and let \(B_0\) be the principal block of the associated spets \(G(q)\). Then

\[
|\mathrm{Irr}^d(B_0(G))| = |\mathrm{Irr}^d(B_0)| \quad \text{for all } d \geq 0.
\]

**Proof.** Let \((G^*, F)\) be dual to \((G, F)\) and let \(S^*\) be a Sylow \(\ell\)-subgroup of \(G^* := G^{*F}\). Then

\[
\mathrm{Irr}(B_0(G)) = \bigcap_s (\mathrm{Irr}(B_0(G)) \cap \mathcal{E}(G(s), s))
\]

where the union runs over \(G^*\)-classes of elements \(s \in S^*\) and \(\mathcal{E}(G, s) \subseteq \mathrm{Irr}(G)\) is the Lusztig series corresponding to the \(G^*\)-class of \(s\). Let \(e\) be the order of \(q\) modulo \(\ell\). Then, according to the description of characters in the principal \(\ell\)-block of \(G\) for very good primes in [21, Thm.], \(\mathrm{Irr}(B_0(G)) \cap \mathcal{E}(G, s)\) is in defect preserving bijection with the principal \(e\)-Harish-Chandra series, \(\mathcal{E}(L(s)^F, 1)_1\) of \((L(s)^F, 1)_1\) where \(L(s)\) is an \(F\)-stable Levi subgroup of \(G\) in duality with the Levi subgroup \(C_{G^*}(s)\) of \(G^*\).

Now let \(S\) be a Sylow \(\ell\)-subgroup of \(G\). By [30] Prop. 4.2 under our assumption on \(\ell\) there is a bijection between \(G^*\)-classes in \(S\) and \(G^*\)-classes in \(S^*\) preserving centralisers through duality. Hence \(\mathrm{Irr}(B_0(G))\) is in defect preserving bijection with \(\bigcap_s \mathcal{E}(C_{G^*}(s), 1)_1\) where now the union runs over \(G^*\)-classes of \(\ell\)-elements in \(S\) or alternatively (see Section 5.3) over fully centralised representatives of \(\mathcal{F}(G(q)) = \mathcal{F}_{G_1^*}(G^*)\)-classes of \(S\). By the discussion above Proposition 5.7 \((C_{G^*}(s), F)\) corresponds to the spets \(C_{G^*}(s)(q)\). Now the claim follows from Example 4.1. 

**Remark 6.9.** Let us comment on the restrictive assumption on the order of \(\phi\). We can associate a saturated fusion system to any spets as follows. For irreducible \((W, L)\) we proceed as above using Remark 3.5. Now assume that \((W, L) = (\prod_{i=1}^r W_i, \bigoplus_{i=1}^r L_i)\) with irreducible factors \((W_i, L_i)\) permuted transitively by \(\phi\). Then the unipotent characters of \(G\) are in bijection with those of \(G_1 = (W_1 \phi^{-1}, L_1)\), and we may set \(\mathcal{F}(G(q)) := \mathcal{F}(G_1(q^*))\). Finally, if \((W, L) = (W_1 \times W_2, L_1 \oplus L_2)\) is a \(\phi\)-stable direct decomposition, then by [19] 4.1
the unipotent characters of $G$ are the Cartesian product of those of $G_i = (W_i \phi^{-1}|_{L_i}, L_i)$, $i = 1, 2$, and their degrees multiply. We then set $\mathcal{F}(G(q)) := \mathcal{F}(G_1(q)) \times \mathcal{F}(G_2(q))$. In this way, we have related a fusion system to any $G$ and $q$, but this is not always known to arise from the known homotopy fixed point construction on $\ell$-compact groups.

6.3. The ordinary weight conjecture for spetses. If $\mathcal{F}$ is a saturated fusion system on a finite $\ell$-group $S$, we now recall functions $k(\mathcal{F}), m(\mathcal{F}, d)$ for $d$ a non-negative integer, and $m(\mathcal{F}) = \sum_{d \geq 0} m(\mathcal{F}, d)$ associated to $\mathcal{F}$ defined in Sections 1 and 2 of [35], where $k(\mathcal{F}) := k(\mathcal{F}, \alpha)$ etc. when the associated Külshammer–Puig family of classes $\alpha$ is zero. These invariants appear in conjectures concerning the number of characters in the principal $\ell$-block of a finite group with fusion system $\mathcal{F}$.

The integer $m(\mathcal{F}, d)$ is an alternating sum count of projective simple modules associated to stabilisers of certain pairs $(\sigma, \mu)$, where $\sigma$ is a chain of $\ell$-subgroups in the (outer) $\mathcal{F}$-automorphism group of some $\mathcal{F}$-centric subgroup $Q \leq S$ and $\mu$ is an irreducible character of $Q$ of defect $d$. In fact, by [35] Lemma 7.5 only radical centric subgroups $Q \in \mathcal{F}^c$ will contribute to the sum.

More precisely, recall from [35] §2 that

$$m(\mathcal{F}, d) = \sum_{Q \in \mathcal{F}^c / \mathcal{F}} w_Q(\mathcal{F}, d),$$

where $w_Q(\mathcal{F}, d)$ is the contribution coming from the set $N_Q$ of non-empty normal chains of $\ell$-subgroups of $\text{Out}_\mathcal{F}(Q)$ of the form $(1 = X_0 < X_1 < \cdots < X_m)$.

If $B_0$ is the principal block of a finite group with fusion system $\mathcal{F}$ then Robinson’s ordinary weight conjecture (OW conjecture) is the assertion that $m(\mathcal{F}, d)$ counts the number of ordinary irreducible characters of defect $d$ in $B_0$. This leads us to the following analogue for spetses (Conjecture 6.10 from the introduction). By Proposition 6.8 if $G$ is associated to a finite group $G^F$ as in Example 6.4 then the assertion of the conjecture is equivalent to the assertion that the OW conjecture holds for the principal block of $G^F$.

**Conjecture 6.10** (OW conjecture for spetses). Let $G$ be a simply connected $\mathbb{Z}_\ell$-spets such that $\ell > 2$ is very good for $G$, and $q$ be a power of a prime different from $\ell$. Let $B_0$ be the principal block of $G(q)$. Then

$$|\text{Irr}^d(B_0)| = m(\mathcal{F}(G(q)), d) \quad \text{for all } d \geq 0.$$

From the local control of heights in Proposition 6.5 we obtain the following expression for the left hand side in Conjecture 6.10.

**Proposition 6.11.** Let $G = (W \phi^{-1}, L)$ be a simply connected $\mathbb{Z}_\ell$-spets such that $\ell > 2$ is very good for $G$. Let $q$ be a power of a prime different from $\ell$, $\mathcal{F} = \mathcal{F}(G(q))$ the associated saturated fusion system on $S$ and $B_0$ the principal block. Then

$$|\text{Irr}^d(B_0)| = \sum_{s \in S / \mathcal{F}} |\text{Irr}^{d-u_s}(W(s \phi^{-1} \zeta^{-1})|$$

where $\zeta \in \mathbb{Z}_\ell^\times$ is the root of unity with $q \equiv \zeta \pmod{\ell}$, and $u_s = \nu_\ell(|C_G(s)|_{x=q}) - \nu_\ell(|W(s \phi^{-1} \zeta^{-1})|).$
Proof. Suppose \( \chi \in \text{Irr}^d(B_0) \), so \( \chi = \gamma^q_{s,\lambda} \) for some \( \lambda \in \text{Uch}(C_G(s)) \), and \( s \in S \) is fully \( \mathcal{F} \)-centralised. By definition \( d = \nu_{\ell}(|C_G(s)|_{x=q}) - \nu_{\ell}(|\lambda(1)_{x=q}|) \). Now the result follows since \( \nu_{\ell}(\lambda(1)_{x=q}) = \nu_{\ell}(\Psi(\lambda)(1)) \) by Proposition 6.5. \( \square \)

This leads us to the following reformulation of the OW conjecture which is new even for principal blocks of finite groups of Lie type and which on the other hand makes sense even in the non-spetsial case. It shows in particular that the OW conjecture for finite groups of Lie type is a purely local statement.

**Conjecture 6.12** (OW conjecture for \( \ell \)-compact groups). Let \( X \) be a simply connected \( \ell \)-compact group with Weyl group \((W, L)\) for which \( \ell > 2 \) is very good, and \( \tau \) an automorphism of \( X \) of finite order prime to \( \ell \). Let \( W_{\phi} = \Phi_X(\tau) \), \( q \) be a power of a prime different from \( \ell \), and \( F = \mathcal{F}(\tau) \) the associated fusion system on \( S \). Then

\[
\text{m}(\mathcal{F}, d) = \sum_{s \in S/F} |\text{Irr}^{d-v_{\ell}(W(s)_{\phi,\zeta})}| \quad \text{for all } d \geq 0,
\]

where \( (W(s), L) \) is the \( \mathbb{Z}_\ell \)-reflection group underlying \( C_X(s) \), \( \zeta \in \mathbb{Z}_\ell^\times \) is the root of unity with \( q \equiv \zeta \pmod{\ell} \), and \( v_s = \nu_{\ell}(O_q(W(s)_{\phi,\zeta}^{-1})) - \nu_{\ell}(|W(s)_{\phi,\zeta}|) \).

Note that since \( W_{\phi^{-1}\zeta^{-1}} = W_{\phi,\zeta} \), and since the order formulas for spetses and reflection cosets agree at \( \ell \), by Proposition 6.11 Conjecture 6.12 is a generalisation of Conjecture 6.10.

There is a coarser version of Conjecture 6.10 obtained by summing over all \( d \). Under the hypotheses of Conjecture 6.10 this asserts that

\[
|\text{Irr}(B_0)| = \text{m}(\mathcal{F}(G(q))).
\]

Passing through localisation afforded by Proposition 6.11 we obtain under the hypothesis of Conjecture 6.12 the conjectural equation

\[
\text{m}(\mathcal{F}) = \sum_{s \in S/F} |\text{Irr}(W(s)_{\phi,\zeta})| \quad \text{for } \ell \text{-compact groups which we call } \text{“Summed OW conjecture” or SOW conjecture for short.}
\]

For \( \mathcal{F} \) a saturated fusion system on a finite group \( S \) we set

\[
\text{k}(\mathcal{F}) := \sum_{s \in S/F} \text{w}(C_F(s)).
\]

If \( B_0 \) is the principal block of a finite group with fusion system \( \mathcal{F} \) then, by [35, Prop. 4.5] assuming the AW conjecture holds for all principal blocks, \( \text{k}(\mathcal{F}) = \text{k}(B_0) \) is the number of ordinary characters in \( B_0 \). In [35] it has been conjectured that \( \text{m}(\mathcal{F}) = \text{k}(\mathcal{F}) \) for an arbitrary saturated fusion system \( \mathcal{F} \). This yields the following consequence of Theorem 4.2:

**Corollary 6.13.** If the AW conjecture holds for all principal \( \ell \)-blocks of all finite groups then the SOW conjecture holds for \( X(e, r, n)(q) \) with \( r|e|(|\ell - 1|) \) and \( e \geq 2 \), for all prime powers \( q \equiv 1 \pmod{\ell} \).
Proof. Let $X = X(e, r, u)$ and let $\mathcal{F}$ be the fusion system corresponding to $X(q)$ via Theorem 5.2. Then, the required equality is

$$m(\mathcal{F}) = \sum_{s \in S/\mathcal{F}} |\text{Irr}(W(s)_{\phi_s})|.$$  

By our assumption on the validity of the AW conjecture, by [35, Cor. 1.3] we have $m(\mathcal{F}) = k(\mathcal{F})$. Note that [35, Cor. 1.3] assumes the AW conjecture for all blocks of all finite groups but it can be easily seen from the proof that in the trivial 2-cocycle case, it suffices to assume the AW conjecture only for principal blocks. Let $s \in S$ be fully $\mathcal{F}$-centralised. By Proposition 5.5 and Theorem 7.17, $C_\mathcal{F}(s)$ is a direct product of fusion systems $\mathcal{F}_i$ where $\mathcal{F}_i$ corresponds to some $X_i(q^{d_i})$ with $X_i$ an $\ell$-compact group with Weyl group $G(e_i, r_i, n_i)$. Moreover, $W(s)_{\phi(s)} = \prod_i G(e_i, r_i, n_i)$ by Lemma 7.18. Applying the weight count for generalised Grassmannians to each component $X_i$ (noting that $q^{d_i} \equiv 1 \pmod{\ell}$) and passing to direct products gives $w(C_\mathcal{F}(s)) = |\text{Irr}(W(s)_{\phi_s})|$. Hence,

$$m(\mathcal{F}) = k(\mathcal{F}) = \sum_{s \in S/\mathcal{F}} w(C_\mathcal{F}(s)) = \sum_{s \in S/\mathcal{F}} |\text{Irr}(W(s)_{\phi_s})|$$

where the second equality is by definition. \hfill \square

A very similar argument shows the following, where we don’t need to assume the AW conjecture by using [35, Rem. 7.8]:

**Proposition 6.14.** Conjecture 6.12 holds whenever $W$ has order prime to $\ell$ and $q \equiv 1 \pmod{\ell}$.

When $q \equiv 1 \pmod{\ell}$, we prove Conjecture 6.12 for the ‘smallest cases’ in the infinite series not covered by the preceding result, viz. the groups $W = G(e, r, \ell)$, in Section 7.3.

As discussed in Section 4, the $\ell$-blocks of finite reductive groups for not very good primes behave somewhat differently, so Conjecture 6.12 does not extend to arbitrary $\ell$. This is exemplified by the following, which will be shown in Section 8.2; see also Remark 6.10.

**Proposition 6.15.** The conclusion of Conjecture 6.12 holds for the four Aguadé exotic $\mathbb{Z}_\ell$-reflection groups, except for defect $d = 2$.

In the excluded case, when $d = 2$ and $(W, \nu_2(q - 1)) \neq (G_{12}, 1)$ we have instead

$$m(\mathcal{F}, d) = \sum_{s \in S/\mathcal{F}} |\text{Irr}^{d-v_2}(W(s)_{\phi_s})| + \ell.$$  

Similarly, if $W = G_{24}$ and $\ell = 2$ then [52, Thm. 2.4] shows that the conclusion of Conjecture 6.12 holds except when $d = 4$ or $d = \nu_2(q^2 - 1) + 4$ where it predicts $m(\mathcal{F}, d) = 0$. In fact $m(\mathcal{F}, d)$ is respectively 2 and 4 in these cases (see [52, Table 1]).

**Remark 6.16.** A natural way to account for the $\ell$ additional characters of defect 2 in the principal $\ell$-blocks of Aguadé spetses $G_\ell(q)$ would be to expand the set $\text{Irr}(B_n)$ to include the $\ell$ characters in $\text{Uch}(G_\ell(q))$ of defect 2. This is well-justified since for the groups of Lie type $^2F_4(q^2)$ and $E_6(q)$, in which $G_{12}$ and $G_{13}$ occur as relative Weyl groups, the principal $\ell$-blocks for $\ell = 3, 5$ respectively, also contain unipotent characters outside the principal series: For $^2F_4(q^2)$, $q^2 \equiv -1 \pmod{3}$, according to [35] the principal 3-block contains 11...
unipotent characters, 8 of which lie in the principal 2-series (for \( q^2 \)) and are in bijection with \( \text{Irr}(G_{12}) \). Here by Himstedt [34, Tab. C5] we have \( l(B_0) = 9 = |\text{Irr}(G_{12})| + 1 \).

For \( E_8(q) \) with \( q \equiv \pm 2 \pmod{5} \), so \( e = e_5(q) = 4 \), according to Enguehard [26, Tab. I] the principal 5-block contains the unipotent characters in the principal 4-series (in bijection with \( \text{Irr}(G_{31}) \)) plus five further unipotent characters of positive height.

7. Generalised Grassmannians

In this section we provide proofs of our main results for the infinite series of imprimitive irreducible \( \mathbb{Z}_\ell \)-reflection groups \( W = G(e, r, n) \), \( r | \ell - 1 \), \( e \geq 2 \), where throughout, \( \ell \) denotes an odd prime. For uniformity of notation and because these groups occur naturally in our discussion, we include the case \( e = 1 \). So, \( G(1, 1, n) \) denotes the (non-irreducible) \( \mathbb{Z}_\ell \)-reflection group \( (\mathfrak{S}_n, L) \) where \( L = \mathbb{Z}_\ell^n \) is the permutation module of \( \mathfrak{S}_n \).

We denote by \( X(e, r, n) \) the connected \( \ell \)-compact group corresponding to \( G(e, r, n) \) via Theorem 5.1. For any prime power \( q \) prime to \( \ell \), \( X(1, 1, n)^{b, c, q} \cong (BGL_n(q))_{\ell}^\circ \) and if \( q' \) is a prime power such that \( q' \equiv 1 \pmod{\ell} \) and \( q \) has order \( e \) modulo \( \ell \) with \( \nu_q(q^e - 1) = \nu_q(q' - 1) \), then by a theorem of Ruiz [51] described in detail in §7.1.3 \( F(X(e, r, n)(q')) \) is a subsystem of the \( \ell \)-fusion system of \( GL_{en}(q) \). Thus weight calculations for the system \( F(X(e, r, n)(q')) \) can be done via descent from the fusion system of \( GL_{en}(q) \).

7.1. Proof of Theorem 4.2 for generalised Grassmannians. Here we show the validity of the AW conjecture for the fusion systems attached to \( G(e, r, n) \), generalising the results of Alperin and Fong [3] for the general linear groups.

7.1.1. The set up. Let \( q \) be a prime power prime to \( \ell \) and let \( e \) be the order of \( q \) modulo \( \ell \). Set \( k = \mathbb{F}_q \) and \( K = \mathbb{F}_{q^e} \). Let \( V' \) be an \( n \)-dimensional \( k \)-vector space and set \( V = K \otimes_k V' \).

Then we may view \( V \) both as an \( n \)-dimensional \( K \)-vector space and as an \( ne \)-dimensional \( k \)-vector space. We identify \( GL_K(V) \leq GL_k(V') \) with \( GL_n(q^e) \leq GL_{en}(q) \) and \( GL_n(q) = GL_k(V') \) with a subgroup of \( GL_K(V) \) via the map which sends \( \phi \in GL_k(V') \) to its unique \( K \)-linear extension to \( V' \).

Fix a direct sum decomposition

\[
V' = V'_1 \oplus \cdots \oplus V'_n
\]

of \( V' \) into one-dimensional \( k \)-spaces and for each \( i \), choose a basis element \( v_i \) of \( V'_i \). Set \( V_i = K \otimes_k V'_i \) and let \( \mathfrak{S}_n \leq GL_k(V') \) be the symmetric group on \( \{ v_1, \ldots, v_n \} \). We identify the \( n \)-fold direct product \( K^\times \times \cdots \times K^\times \) with its image in \( GL_K(V) \) via the map which sends \( (\lambda_1, \ldots, \lambda_n) \) to the \( K \)-linear map

\[
\sum_{1 \leq j \leq n} x_j \otimes v_j \mapsto \sum_{1 \leq j \leq n} \lambda_j x_j \otimes v_j \quad \text{for} \quad x_j \in K.
\]

Let \( \Delta \cong C_{\ell^n} \) be the Sylow \( \ell \)-subgroup of \( K^\times \) and set

\[
D = \Delta \times \cdots \times \Delta \leq GL_K(V),
\]

the Sylow \( \ell \)-subgroup of \( K^\times \times \cdots \times K^\times \) (\( n \) factors). Let

\[
n = a_0 + a_1 \ell + \cdots + a_m \ell^m \quad \text{with} \quad 0 \leq a_i < \ell,
\]
Lemma 7.1. Let $D_k$ with its image in $GL_k$ has an irreducible character of zero $\ell$.

$\pi$ is a Sylow $\ell$-subgroup of $GL_k$ contained in the above Young subgroup. Then,

$$E \cong \prod_{0 \leq i \leq m} \prod_{1 \leq j \leq a_i} C_\ell \cdot \cdots \cdot C_\ell,$$

$D \cap E = 1$ and

$$S := DE \cong \prod_{0 \leq i \leq m} (C_{\ell^i} \cdot C_\ell \cdot \cdots \cdot C_\ell)^{a_i}$$

is a Sylow $\ell$-subgroup of $GL_k(V)$ contained in $GL_K(V)$. Let $F$ denote the fusion system of $GL_k(V)$ on $S$.

Let $\Phi$ be the generator of $Gal(K/k) \cong C_\ell$ defined by $\Phi(x) = x^i$, $x \in K$. Identify $\langle \Phi \rangle^n$ with its image in $GL_k(V)$ via the map which sends $(\Phi^{i_1}, \ldots, \Phi^{i_n})$ to the $k$-linear map $V \to V$ defined by

$$\sum_{1 \leq j \leq n} x_j \otimes v_j \mapsto \sum_{1 \leq j \leq n} x_j^{i_j} \otimes v_j \text{ for } x_j \in K.$$

Then

$$C_{GL_k(V)}(D) = (K^\times)^n, \quad N_{GL_k(V)}(D) = (K^\times)^n \langle \Phi \rangle^n \mathcal{S}_n, \quad (K^\times)^n \cap \langle \Phi \rangle^n \mathcal{S}_n = 1, \quad \langle \Phi \rangle^n \cap \mathcal{S}_n = 1$$

and $\mathcal{S}_n$ normalises $\langle \Phi \rangle^n$. Thus the natural map $N_{GL_k(V)}(D) \to Aut(D)$ induces an isomorphism

$$Out_F(D) \cong \langle \Phi \rangle^n \rtimes \mathcal{S}_n \cong C_\ell \cdot \cdots \cdot C_\ell.$$

We will identify $Aut_F(D)$ with $\langle \Phi \rangle^n \rtimes \mathcal{S}_n$ via the above. Set $H = \langle \Phi \rangle^n$ the base subgroup of $Aut_F(D)$ and set $H_0 = [H, \mathcal{S}_n]$. Then

$$H_0 = \{ (\Phi^{i_1}, \ldots, \Phi^{i_n}) \mid i_1 + \cdots + i_n \equiv 0 \pmod{\ell} \}.$$

We define $\pi : Aut_F(D) \to \mathbb{Z}/\ell\mathbb{Z}$ by

$$\tag{2} (\Phi^{i_1}, \ldots, \Phi^{i_n}) \sigma \mapsto i_1 + \cdots + i_n \pmod{\ell}, \text{ for } (\Phi^{i_1}, \ldots, \Phi^{i_n}) \in H, \sigma \in \mathcal{S}_n,$$

a surjective group homomorphism with kernel $H_0 \mathcal{S}_n$.

7.1.2. On the Alperin–Fong description of centric, radical and weight contributing subgroups. We recall some of the key results of [3] on counting weights in principal blocks of finite general linear groups. For a saturated fusion system $F$ on a finite $\ell$-group $S$ and a subgroup $R \leq S$, we say that $R$ is $F$-weight contributing if $R$ is $F$-centric and $Out_F(R)$ has an irreducible character of zero $\ell$-defect. We note some elementary facts.

Lemma 7.1. Let $F$ be a saturated fusion system on a finite $\ell$-group $S$ and let $R \leq S$.

(a) If $R$ is weight contributing, then $R$ is $F$-centric and $F$-radical.
(b) Suppose that $F = F_S(G)$ for some finite group $G$ with $S$ as Sylow $\ell$-subgroup. Then $R$ is $F$-centric if and only if $Z(R)$ is a Sylow $\ell$-subgroup of $C_G(R)$. If $R$ is both $F$-centric and $F$-radical, then $R$ is $G$-radical, i.e., $O_{\ell}(G/R) = 1$. In particular, if $R$ is $F$-weight contributing then $R$ is $G$-radical.

Recall that $\ell \neq 2$. Denote by $C$ the set of finite sequences $c = (c_1, \ldots, c_t)$ of strictly positive integers including the empty sequence $. For $c \in C$, write $|c| := c_1 + \cdots + c_t$. For $c \in C$ and non-negative integers $m, \gamma, \alpha$, let $R_{m,\alpha,\gamma,c}$ denote a corresponding basic subgroup of $GL_k(V)$ as defined in [3, Sec. 4]. We will assume that the extra-special component of $R_{m,\alpha,\gamma,c}$ is of exponent $\ell$.

**Proposition 7.2.** Let $R$ be a radical $\ell$-subgroup of $GL_k(V)$. There exist decompositions

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s, \quad R = R_0 \times R_1 \times \cdots \times R_s,$$

of $k$-vector spaces and of groups such that $R_0$ is the trivial subgroup of $GL_k(V_0)$ and for each $i \geq 1$, $R_i = R_{m_i,\alpha_i,\gamma_i,c_i}$ is a basic subgroup of $GL_k(V_i)$ with $\dim_k(V_i) = e^{m_i+\alpha_i+\gamma_i+|c_i|}$.

(a) $Z(R)$ is a Sylow $\ell$-subgroup of $C_G(R)$ if and only if $V_0 = \{0\}$ and $m_i = 0$ for all $i \geq 2$.

(b) The $GL_k(V)$-conjugacy classes of radical $\ell$-subgroups of $GL_k(V)$ are in bijection with the set of assignments

$$f : \mathbb{N}_+ \times \mathbb{N} \times \mathbb{N} \times C \to \mathbb{N} \text{ such that } \sum_{(m,\alpha,\gamma,c)} \ell^{m+\alpha+\gamma+|c|}f(m,\alpha,\gamma,c) \leq n.$$ 

Under this bijection, $f$ corresponds to the class of groups $R = \prod_{(m,\alpha,\gamma,c)} R_{m,\alpha,\gamma,c}^{f(m,\alpha,\gamma,c)}$.

(c) The $GL_k(V)$-classes of radical $\ell$-subgroups $R$ such that $Z(R)$ is a Sylow $\ell$-subgroup of $C_G(R)$ correspond to functions $f$ as in (b) satisfying $f(m,\alpha,\gamma,c) = 0$ for $m \geq 2$ and

$$\sum_{(m,\alpha,\gamma,c)} \ell^{m+\alpha+\gamma+|c|}f(m,\alpha,\gamma,c) = n.$$

**Proof.** All statements follow from Section 4 of [3]. The first statement and (b) are given in [3 (4A)]. For (a), suppose first that $V_0$ is non-trivial. Since the $k$-dimension of $V_i$ for $i \geq 1$ is divisible by $e$ and $\dim_k V = en$, $e$ divides $\dim_k(V_i)$ and it follows that $GL_k(V_0)$ has a non-trivial Sylow $\ell$-subgroup, say $E$. Since clearly $E \leq C_{GL_k(V)}(R)$ and $E \cap R = 1$, it follows that $R$ is not a Sylow $\ell$-subgroup of $RC_{GL_k(V)}(R)$, equivalently that $Z(R)$ is not a Sylow $\ell$-subgroup of $C_{GL_k(V)}(R)$.

Hence we may assume from now on that $V_0 = \{0\}$. So, $Z(R)$ is a Sylow $\ell$-subgroup of $C_{GL_k(V)}(R)$ if and only if for all $i \geq 1$, $Z(R_i)$ is a Sylow $\ell$-subgroup of $C_{GL_k(V)}(R_i)$. Hence we may assume that $i = 1$ and $R = R_{m,\alpha,\gamma,c}$. By construction $R = R_{m,\alpha,\gamma,c}A_e$ where $R_{m,\alpha,\gamma} := R_{m,\alpha,\gamma,c}$, $Z(R)$ is isomorphic to the diagonally embedded copy of $Z(R_{m,\alpha,\gamma})$ and $C_{GL_k(V)}(R)$ is isomorphic to the diagonally embedded copy of $C_{GL_k(W)}(R_{m,\alpha,\gamma})$ where $W$ is the subspace underlying $R_{m,\alpha,\gamma}$. Hence we may assume that $c = ()$. In this case, as explained in [3 Sec. 4], $Z(R) \cong K^\times$ where $K$ is the extension of $k$ of degree $e^{\alpha}$ (hence the extension of $k$ of degree $e^{\alpha}$ and $C_{GL_k(V)}(R)$ is isomorphic to $GL_m(K^\times)$. It follows that $Z(R)$ is a Sylow subgroup of $C_{GL_k(V)}(R)$ if and only if $m = 1$. This proves (a).

Finally, (c) is a consequence of the preceding statements.
From now on, we write $R_{\alpha,\gamma,\xi} := R_{1,\alpha,\gamma,\xi}$. We next identify the outer automorphism groups of $GL_k(V)$-radical and centric subgroups (see Section 4 of [3]):

**Proposition 7.3.** Let $R = \prod_{(1,\alpha,\gamma,\xi)} R_{\alpha,\gamma,\xi}^{f(\alpha,\gamma,\xi)} \leq S$ be an $\mathcal{F}$-centric and $GL_k(V)$-radical subgroup with corresponding decomposition

$$V = \bigoplus_{(1,\alpha,\gamma,\xi)} V_{\alpha,\gamma,\xi}^{f(\alpha,\gamma,\xi)}$$

from Proposition 7.2. We have:

(a) $\text{Out}_\mathcal{F}(R) \cong N_{GL_k(V)}(R)/RC_{GL_k(V)}(R) \cong \prod_{\alpha,\gamma,\xi} N_{\alpha,\gamma,\xi} \lhd \mathcal{S}_{f(\alpha,\gamma,\xi)}$

where $N_{\alpha,\gamma,\xi} := N_{GL_k(V_{\alpha,\gamma,\xi})}(R_{\alpha,\gamma,\xi})/R_{\alpha,\gamma,\xi}C_{GL_k(V_{\alpha,\gamma,\xi})}(R_{\alpha,\gamma,\xi})$.

(b) Suppose that $R = R_{1,\alpha,\gamma,\xi}$. Then

$$N_{GL_k(V)}(R)/RC_{GL_k(V)}(R) \cong (\text{Sp}_{2\gamma}(\ell) \rtimes C_{\ell,0}) \times \text{GL}_{c_1}(\ell) \times \ldots \times \text{GL}_{c_\ell}(\ell),$$

where we set $\text{Sp}_{2\gamma}(\ell) := 1$ if $\gamma = 0$.

**Lemma 7.4.** Let $R = \prod_{(1,\alpha,\gamma,\xi)} R_{\alpha,\gamma,\xi}^{f(\alpha,\gamma,\xi)} \leq S$ be $\mathcal{F}$-centric. If $R$ is $\mathcal{F}$-weight contributing, then $f(\alpha,\gamma,\xi) = 0$ for all $\alpha \geq 1$.

**Proof.** Suppose that $R$ is $\mathcal{F}$-weight contributing and let $R_i = R_{1,\alpha,\gamma,\xi}$ be one of its factors. By Proposition 7.3, $\text{Out}_\mathcal{F}(R_i)$ contains a normal subgroup isomorphic to $\text{Sp}_{2\gamma}(\ell) \rtimes C_{\ell,0}$. Hence $\text{Sp}_{2\gamma}(\ell) \rtimes C_{\ell,0}$ has an irreducible character say $\chi$ of $\ell$-defect zero and if $\chi_0$ is an irreducible character of $\text{Sp}_{2\gamma}(\ell)$ lying below $\chi$, then $\chi_0$ is also of $\ell$-defect zero. Now the Steinberg character of $\text{Sp}_{2\gamma}(\ell)$ is the unique character of $\ell$-defect zero (if $\gamma = 0$, then we take the trivial character as Steinberg character). It follows that $\chi_0$ is $\text{Sp}_{2\gamma}(\ell) \rtimes C_{\ell,0}$-stable and hence since $C_{\ell,0}$ is cyclic, $\chi$ is an extension of $\chi_0$ and in particular, $\chi$ and $\chi_0$ have the same degree. Thus, $\alpha = 0$. □

7.1.3. *Descent to $r > 1$.* Recall that for a saturated fusion system $\mathcal{F}$ on a finite $\ell$-group $S$, there is a group $\Gamma_{\ell'}(\mathcal{F})$ whose lattice of subgroups is in one-to-one correspondence with the lattice of saturated fusion subsystems of $\mathcal{F}$ of index coprime to $\ell$ (see [8] Thm. I.7.7) and let $\theta : \text{Mor}(\mathcal{F}) \to \Gamma_{\ell'}(\mathcal{F})$ be the canonical map as defined before [8] Thm I.7.7. We need the following fact:

**Lemma 7.5.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$ and suppose $T \leq S$ is abelian and weakly $\mathcal{F}$-closed. If $P, Q \leq T$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ then there exists $\hat{\varphi} \in \text{Aut}_{\mathcal{F}}(T)$ with $\hat{\varphi}|_P = \varphi$.

**Proof.** Choose $R \leq S$ and $\psi \in \text{Hom}_{\mathcal{F}}(Q, R)$ with $R$ fully $\mathcal{F}$-normalised. Notice that $R \leq T$ since $T$ is weakly $\mathcal{F}$-closed. Since $\mathcal{F}$ is saturated, there exist morphisms $\beta_1 \in \text{Hom}_{\mathcal{F}}(N_S(P), S)$ and $\beta_2 \in \text{Hom}_{\mathcal{F}}(N_S(Q), S)$ extending $\psi \circ \varphi$ and $\psi$ respectively. Since $T$ is abelian $T \leq N_S(P), N_S(Q)$, so $\beta_1$ and $\beta_2$ both restrict to $\mathcal{F}$-automorphisms of $T$ (where we also use the fact that $T$ is weakly $\mathcal{F}$-closed.) Therefore $\hat{\varphi} := (\beta_2|_T)^{-1} \circ \beta_1|_T \in \text{Aut}_{\mathcal{F}}(T)$ extends $\varphi$, as needed. □
Proposition 7.6. Suppose that $\ell$ is an odd prime. Then
\[
\Gamma_{\ell}^e(F) \cong \begin{cases} 
\mathbb{Z}/e\mathbb{Z} & \text{if } n \geq \ell, \\
\mathbb{Z}/e\mathbb{Z} \rtimes S_n & \text{if } n < \ell.
\end{cases}
\]

Proof. This is Theorem 5.10 of [51] (see also [47]). \hfill $\square$

We will use a precise description of $\theta$ given in [47].

Lemma 7.7. Let $A = \{g \in S \mid o(g) = \ell, \ker(g - \text{Id}_V) = 0\}$. Then $A \subseteq D$.

Proof. We follow an argument in the proof of [48] Ex. 6.2. Note that the set $A$ defined here is stable under $F$-conjugation and therefore contained in the set denoted $X$ in [47] Lemma 3.3. Let $g = (\lambda_1, \ldots, \lambda_n) \in A$ with $(\lambda_1, \ldots, \lambda_n) \in D$ and $g \in E$. By assumption $g^\ell = 1$. Hence, $\ell^\ell = 1$ and $g(\sigma g^{-1}) \cdots (\sigma^\ell g^{-\ell}(\ell - 1)) = 1$. The first condition implies that every non-trivial cycle of $\sigma$ is an $\ell$-cycle and the second condition implies that if $(i_1, \ldots, i_\ell)$ is an $\ell$-cycle in $\sigma$, then the product $\lambda_{i_1} \cdots \lambda_{i_\ell} = 1$. Letting $u$ be the number of $\langle \sigma \rangle$-orbits of size $\ell$ in $\{1, \ldots, n\}$, it follows that the characteristic polynomial of $g$ viewed as a $K$-linear transformation of $V$ is divisible by $(x^\ell - 1)^u$. Thus, if $u \neq 0$, then $1$ is an eigenvalue of $g$ in its action on $V$. \hfill $\square$

Let $\pi : \text{Aut}_F(D) \to \mathbb{Z}/e\mathbb{Z}$ be the homomorphism defined in (2) and let $Z$ be the unique subgroup of order $\ell$ of $Z(\text{GL}_K(V))$.

Proposition 7.8. There exists a surjective homomorphism $\Pi : \Gamma_{\ell}^e(F) \to \mathbb{Z}/e\mathbb{Z}$ with the following property: for any pair of $F$-centric subgroups $P, Q \leq S$ and any morphism $\alpha \in \text{Hom}_F(P, Q)$, there exists $\beta \in \text{Aut}_F(D)$ such that $\beta|_Z = \alpha|_Z$ and for each such $\beta$, $\Pi(\theta(\alpha)) = \pi(\beta)$. If $n \geq \ell$, then $\Pi$ is an isomorphism.

Proof. For this we note by Lemma 7.7 that $\mathcal{F}^\Pi \leq D$. Further, we check that $H_0 S_\pi$ is the normal closure of $\text{Aut}_{\mathcal{F}(Z)}(D)$ in $\text{Aut}_F(D)$. The first assertion follows by [47] Prop. 3.3(a)]. The second is immediate from Proposition 7.6. \hfill $\square$

Let $\mathcal{F}' := \mathcal{F}_S(\text{GL}_K(V))$ be the fusion system of $\text{GL}_K(V)$ on $S$.

Proposition 7.9. Let $P, Q \leq S$ be $F$-centric subgroups. Then $\Pi \circ \theta(\varphi) = 1$ for all $\varphi \in \text{Hom}_F(P, Q)$.

Proof. Let $P, Q \leq S$ be $F$-centric and let $\varphi \in \text{Hom}_F(P, Q)$. Since $D$ is also weakly $\mathcal{F}$-closed (see [14] Prop. 5.13(b)), $\varphi|_{\mathcal{F}(X)}$ extends to an $\mathcal{F}$-automorphism $\hat{\varphi}$ of $D$ by Lemma 7.5. But since $\text{Aut}_F(D) = S_n$, $\pi(\hat{\varphi}) = 1$. \hfill $\square$

For $r|e$, we denote by $\mathcal{F}^{(r)}$ the $\ell'$-index fusion subsystem of $\mathcal{F}$ corresponding to the subgroup $\Pi^{-1}(r\mathbb{Z}/e\mathbb{Z})$ of $\Gamma_{\ell}^e(F)$ as described in [3] Thm. I.7.7.

Proposition 7.10. Let $r|e$ and let $q'$ be a prime power such that $q' \equiv 1 \pmod{\ell}$ and $v_\ell(q^e - 1) = v_\ell(q' - 1)$. Then $\mathcal{F}^{(r)}$ is isomorphic to $\mathcal{F}(X(e, r, n)(q'))$.

Proof. By Theorem 5.11(b), $X(e, r, n)(q') \simeq X(e, r, n)(q^e)$ have isomorphic associated fusion systems. Now the result follows from [51] Thm. 6.3]. \hfill $\square$

Proposition 7.11. Let $r|e$ and let $P \leq S$.
(a) $P$ is $\text{GL}_K(V)$-radical if and only if $P$ is $\text{GL}_k(V)$-radical. If $Q \leq S$ is $\text{GL}_k(V)$-radical, then $P$ and $Q$ are $\mathcal{F}$-conjugate if and only if $P$ and $Q$ are $\mathcal{F}'$-conjugate.

(b) The $\mathcal{F}$-classes of $\mathcal{F}$-centric, $\text{GL}_k(V)$-radical subgroups of $S$ coincide with the $\mathcal{F}'$-classes of $\mathcal{F}'$-centric, $\text{GL}_k(V)$-radical subgroups of $S$.

(c) $P$ is $\mathcal{F}$-centric if and only if $P$ is $\mathcal{F}(r)$-centric and $P$ is $\mathcal{F}$-radical if and only if $P$ is $\mathcal{F}(r)$-centric and $\mathcal{F}(r)$-radical.

(d) The $\mathcal{F}$-classes of $\mathcal{F}$-centric, $\text{GL}_k(V)$-radical subgroups of $S$ coincide with the $\mathcal{F}(r)$-classes of $\mathcal{F}(r)$-centric, $\text{GL}_k(V)$-radical subgroups of $S$.

Proof. (a) Applying Proposition 7.2(b) with $k$ replaced by $K$ yields the same description of the $\text{GL}_K(V)$-conjugacy classes of radical subgroups of $\text{GL}_K(V)$ as that of the $\text{GL}_k(V)$-conjugacy classes of radical subgroups of $\text{GL}_k(V)$. Since every $K$-decomposition of the type of Proposition 7.2 is also a $k$-decomposition, every $\text{GL}_K(V)$-conjugacy class of $\text{GL}_K(V)$-radical subgroups is contained in a unique $\text{GL}_k(V)$-conjugacy class of $\text{GL}_k(V)$-radical subgroups and every $\text{GL}_k(V)$-conjugacy class of $\text{GL}_k(V)$-radical subgroups contains a unique $\text{GL}_K(V)$-conjugacy class of $\text{GL}_K(V)$-radical subgroups. Now (a) follows since $S \leq \text{GL}_K(V)$.

(b) It suffices to prove that the bijection between radical subgroups given in (a) preserves the centric and radical property. But this is immediate from Proposition 7.2.

(c) For the first assertion see [8, Lemma I.7.6]. Now suppose that $P$ is $\mathcal{F}$-centric. Since $\Gamma_e(\mathcal{F})$ is cyclic, by [8, Thm. I.7.7], $\text{Out}_{\mathcal{F}(r)}(P)$ is a normal subgroup of $\text{Aut}_{\mathcal{F}}(P)$ of $\ell'$-index, and consequently $\text{Out}_{\mathcal{F}(r)}(P)$ is a normal subgroup of $\text{Out}_{\mathcal{F}}(P)$ of $\ell'$-index. This proves the second assertion.

(d) By (c), it suffices to prove that any two $\mathcal{F}$-conjugate $\mathcal{F}$-centric $\text{GL}_k(V)$-radical subgroups $P, Q \leq S$ are $\mathcal{F}(r)$-conjugate. By part (a), $P$ and $Q$ are $\mathcal{F}'$-conjugate. Let $\alpha : P \to Q$ be an isomorphism in $\mathcal{F}'$. By Lemma 7.9, $\Pi \circ \theta(\alpha) = 1$ hence by [8, Thm. I.7.7], $\alpha \in \mathcal{F}(r)$, proving (d). \hfill $\Box$

Lemma 7.12. For any $r | e$ a subgroup $R \leq S$ is $\mathcal{F}$-weight contributing if and only if it is $\mathcal{F}(r)$-weight contributing.

Proof. As described above, if $R$ is $\mathcal{F}$-weight contributing, then $R$ is both $\mathcal{F}$-centric and $\mathcal{F}$-radical and similarly for $\mathcal{F}(r)$. By Proposition 7.11 the set of $\mathcal{F}(r)$-centric, $\mathcal{F}(r)$-radical subgroups of $S$ coincides with the set of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of $S$. Let $R \leq S$ be $\mathcal{F}$-centric and $\mathcal{F}$-radical. As explained in the proof of Proposition 7.11(c), $\text{Out}_{\mathcal{F}(r)}(R)$ is a normal subgroup of $\text{Out}_{\mathcal{F}}(R)$ of $\ell'$-index. Thus an irreducible character $\chi$ of $\text{Out}_{\mathcal{F}}(R)$ is of $\ell$-defect zero if and only if the irreducible characters of $\text{Out}_{\mathcal{F}(r)}(R)$ covered by $\chi$ are of $\ell'$-defect zero. \hfill $\Box$

7.1.4 Counting weights. By the above results, in order to count $\mathcal{F}$- and $\mathcal{F}(r)$-weights we need only consider those $\mathcal{F}$-centric, $\text{GL}_k(V)$-radical subgroups of $S$ whose basic components are all of type $R_{1,0,\gamma,\ell}$. We will concentrate on such subgroups from now on.

We denote

$$\mathfrak{F} := \{ f : \mathbb{N} \times C \to \mathbb{N} \mid \sum_{(\gamma, \ell)} \ell^{r+|k|} f(\gamma, \ell) = n \}.$$
For $f \in \mathcal{F}$ choose a direct sum decomposition into $k$-subspaces
\begin{equation}
V' = \bigoplus_{(\gamma, c) \in \mathbb{N} \times C} \bigoplus_{1 \leq i \leq f(\gamma, c)} V'_{\gamma, c, i}
\end{equation}
such that
\[
\dim_k (V'_{\gamma, c, i}) = \ell^{\gamma_i |c|}
\]
for $(\gamma, c) \in \mathbb{N} \times C$, $1 \leq i \leq f(\gamma, c)$, and such that for each $(\gamma, c, i)$, there is a subset $Y$ of $\{1, \ldots, n\}$ and an index $(i', j')$ of the direct sum decomposition of $V'$ underlying $S$ such that
\[
V'_{i', j'} = \sum_{y \in Y} V'_y \quad \text{we have that } V'_{\gamma, c, i} = \sum_{y \in Y} V'_y \quad \text{and } Y \subseteq Y'.
\]
Such a decomposition is always possible given the arithmetic constraints on $f$. Consider the induced decomposition
\[
V = \bigoplus_{(\gamma, c) \in \mathbb{N} \times C} \bigoplus_{1 \leq i \leq f(\gamma, c)} V_{\gamma, c, i}
\]
where $V_{\gamma, c, i} = K \otimes_k V'_{\gamma, c, i}$. Set
\[
R_f = \prod_{(\gamma, c) \in \mathbb{N} \times C} \prod_{1 \leq i \leq f(\gamma, c)} R_{\gamma, c, i}^{(i)}
\]
where for each $i$, $R_{\gamma, c, i}^{(i)} \leq \text{GL}_k(V_{\gamma, c, i}) \leq \text{GL}_k(V'_{\gamma, c, i})$ is a basic subgroup of type $R_{0, 1, \gamma, c}$.

By Propositions 7.2 and 7.11, $f$ determines the groups $R_f$ up to $F$-conjugacy and the $F^{(r)}$-class of $R_f$ determines $f$. We may and will assume that the groups $R_f$ are subgroups of $S$.

For $i \geq 0$, let $c_i$ be the sequence $(1, \ldots, 1)$ of length $i$ and define $f_S, f_D : \mathbb{N} \times C \to \mathbb{N}$ by
\[
f_S(\gamma, c) = \begin{cases} a_i & \text{if } (\gamma, c) = (0, c_i), \\ 0 & \text{otherwise,} \end{cases}, \quad f_D(\gamma, c) = \begin{cases} n & \text{if } (\gamma, c) = (0, ()), \\ 0 & \text{otherwise.} \end{cases}
\]
Then $f_S, f_D \in \mathcal{F}$ and $S = R_{f_S}$ and $D = R_{f_D}$.

**Proposition 7.13.** Let $R = R_f \leq S$ be as above. We have:

(a) \[
\text{Out}_F(R) = \prod_{(\gamma, c)} N_{\gamma, c} \cdot \mathcal{G}_{f(\gamma, c)},
\]
where $N_{\gamma, c} = (\text{Sp}_{2\gamma}(c) \times C_{\gamma, c}) \times \text{GL}_{c_1}(\ell) \times \cdots \times \text{GL}_{c_{\ell}}(\ell)$ and $C_{\gamma, c}$ is cyclic of order $e$.

(b) For $(\gamma, c)$ with $f(\gamma, c) \neq 0$, let $\alpha_{\gamma, c} \in \text{Aut}_F(R)$ be a lift of a generator of $C_{\gamma, c}$. Then $\Pi \circ \theta((\alpha_{\gamma, c})) = \mathbb{Z}/e\mathbb{Z}$.

(c) We have $\Pi \circ \theta(\text{Aut}_F(R)) = r\mathbb{Z}/e\mathbb{Z}$, and $\ker(\Pi \circ \theta)$ contains all $\mathcal{G}_{f(\gamma, c)} \text{Inn}(R)$.

(\gamma, c) \in \mathbb{N} \times C.

**Proof.** The assertion in (a) is proved in Proposition 7.3. For (b), we may assume that $R = R_f$, where $f$ is the characteristic function of some $(\gamma, c) \in \mathbb{N} \times C$ and $n = \ell^{\gamma_i |c|}$. So $R$ is of type $R_{1, 0, \gamma, c}$. Let us first consider the case that $c = (0)$. Set $N := N_{\text{GL}_k(V)}(R)$, $C = C_{\text{GL}_k(V)}(R)$ and let $N^0 := C_N(Z(R))$. By [3, Sec. 4], $N^0 = LCR$, where $L \cong \text{Sp}_{2\gamma}(\ell)$, $[C, L] = 1$ and $L \cap RC = 1$. Also, $N = N^0 \rtimes (\sigma)$ with $o(\sigma) = e$.

Now consider the conjugation action of $(\sigma)$ on $Z(R)$. By definition of $N^0$, this action is faithful. By definition of $R$, $Z(R)$ is the Sylow $\ell$-subgroup of $Z(\text{GL}_k(V)) \cong K^\times$. Note that $Z$ is a subgroup of order $\ell$ of $Z(R)$. Let $Z(R) = \langle g \rangle$ and suppose that $\sigma(g) = g'$. Then...
Since $\sigma(g)$ has the same eigenvalues as $g$, we see that $\sigma(g) = g^q$ for some $0 \leq j \leq e - 1$. Since the action of $\sigma$ on $Z(R)$ is faithful, by replacing $\sigma$ by a suitable power we may assume that $\sigma(g) = g^q$. Thus, letting $c_\sigma$ denote the image of $\sigma$ in $\text{Aut}_F(R)$, we see that restriction of $c_\sigma$ to $Z(R)$ coincides with the restriction of $\Phi_{\{1, \ldots, n\}}$ to $Z(R)$, where $\Phi_{\{1, \ldots, n\}} \in \text{Aut}_F(D)$ is the automorphism of $D$ acting as $\Phi$ on all components. Thus by Proposition 7.13
\[
\Pi \circ \theta(c_\sigma) = \pi(\Phi_{\{1, \ldots, n\}}) \equiv n \pmod{e}.
\]
Since $n$ is a power of $\ell$, $n$ and $e$ are relatively prime and surjectivity follows. The case of general $\sigma$ follows from the case above by [3, Eq. (4.1)].

By [8, Thm. I.7.7], $\mathcal{F}^{(r)} = \langle (\Pi \circ \theta)^{-1}(r\mathbb{Z}/e\mathbb{Z}) \rangle$ and $\text{Aut}_{\mathcal{F}^{(r)}}(R) = \text{Aut}_F(R) \cap (\Pi \circ \theta)^{-1}(r\mathbb{Z}/e\mathbb{Z})$. Hence the first statement of (c) follows from (b). By construction of the isomorphism in (a), each $\mathcal{S}_\gamma,\ell$ is the image of some subgroup of $\mathcal{S}_n \cap \mathcal{N}_\mathcal{G}_{\mathcal{L}(V)}(R)$ under the canonical map $\mathcal{N}_\mathcal{G}_{\mathcal{L}(V)}(R) \to \text{Out}_F(R)$ and conjugation by an element of $\mathcal{S}_n$ goes to the identity under $\Pi \circ \theta$. This proves the second assertion of (c).

For $c \in \mathcal{C}$, denote by $\mathcal{A}(c)$ the set of irreducible characters of $\mathcal{G}_{c_1}(\ell) \times \cdots \times \mathcal{G}_{c_\ell}(\ell)$ of the form $\chi_1 \cdot \cdots \cdot \chi_\ell$, where each $\chi_j$ is a Steinberg character of $\mathcal{G}_{c_j}(\ell)$. Note that $|\mathcal{A}(c)| = (\ell - 1)^\ell$.

**Proposition 7.14.** (a) The set of $\mathcal{F}$-weights (up to conjugation) is in bijection with the set $\mathcal{W}$ of assignments
\[
w : \mathbb{N} \times \mathbb{Z}/e\mathbb{Z} \times \bigcup_{c \in \mathcal{C}} \mathcal{A}(c) \to \{\ell\text{-cores}\}
\]
such that
\[
\sum_{c} \sum_{(\gamma,x,\varphi) \in \mathbb{N} \times \mathbb{Z}/e\mathbb{Z} \times \mathcal{A}(c)} \ell^{r+|c|} |w(\gamma, x, \varphi)| = n.
\]
(b) Let $\mathbb{Z}/r\mathbb{Z}$ act on $\mathcal{W}$ via $y.w(\gamma, x, \varphi) := w(\gamma, y + x, \varphi)$ for $y \in \mathbb{Z}/r\mathbb{Z}$, $w \in \mathcal{W}$ and $(\gamma, x, \varphi) \in \mathbb{N} \times \mathbb{Z}/e\mathbb{Z} \times \mathcal{A}$. Then the $\mathcal{F}^{(r)}$-weights (up to conjugation) are indexed by the $\mathbb{Z}/r\mathbb{Z}$-orbits of $\mathcal{W}$ with each orbit contributing as many weights as the order of the stabiliser of a point of the orbit.

**Proof.** By Lemma 7.3 and Lemma 7.12 any $\mathcal{F}$- (respectively $\mathcal{F}^{(r)}$)-weight-conjugating subgroup is conjugate to an $R_f$ for some $f \in \mathcal{F}$. By Propositions 7.2 and 7.11 distinct choices of $f$ give rise to distinct $\mathcal{F}$- (respectively $\mathcal{F}^{(r)}$)-conjugacy classes of subgroups of $S$. Thus, we are reduced to the case when $R = R_f$. Then, (a) is a consequence of Proposition 7.3(a) and Proposition 7.2 applied with $G = \text{Out}_F(R)$.

By Proposition 7.13(b),(c), $\text{Out}_F(R)/\text{Out}_{\mathcal{F}^{(r)}}(R) \cong \mathbb{Z}/e\mathbb{Z}$ and for any $(\gamma, c)$ with $f(\gamma, c) \neq 0$, the image of $\alpha_{\gamma, c}$ is a generator of $\text{Out}_F(R)/\text{Out}_{\mathcal{F}^{(r)}}(R) \cong \mathbb{Z}/e\mathbb{Z}$. This means that the action of $\text{Irr}(\text{Out}_F(R)/\text{Out}_{\mathcal{F}^{(r)}}(R))$ on $\text{Irr}^0(\mathcal{N}_{\gamma,\ell})$ given by
\[
\lambda \cdot \varphi = \lambda_0 \cdot \varphi \quad \text{for} \lambda \in \text{Irr}(\text{Out}_F(R)/\text{Out}_{\mathcal{F}^{(r)}}(R)), \varphi \in \text{Irr}^0(\mathcal{N}_{\gamma,\ell}),
\]
where $\lambda_0$ is the $(\gamma, c, i)$-component of the restriction of $\lambda$ to $\mathcal{N}_{\gamma,\ell}$ for some (and hence all) $1 \leq i \leq f_{\gamma,\ell}$, is a regular action. Thus the $\mathbb{Z}/e\mathbb{Z}$-set $\text{Irr}^0(\mathcal{N}_{\gamma,\ell})$ can be identified with the $\mathbb{Z}/e\mathbb{Z}$-set $\mathbb{Z}/e\mathbb{Z} \times \mathcal{A}(c)$ (acting through the left regular action on the first component).

Now (b) follows by Proposition 7.13 and by Proposition 7.2 applied with $G = \text{Out}_F(R)$ and $M = \text{Out}_{\mathcal{F}^{(r)}}(R)$. \qed
We now give an equivariant form of (1A) of \[3\].

**Lemma 7.15.** Let \(e, \ell, r \mid e\) and \(n\) be as above. For each \(d \geq 0\), let

\[ I_d = \mathbb{Z}/e\mathbb{Z} \times \{(d, j) \mid 0 \leq j < \ell^d\}. \]

View \(I_d\) as a \(\mathbb{Z}/r\mathbb{Z}\)-set via the left regular action on the first component and let \(I = \bigcup_d I_d\).

Let \(\mathcal{L}\) be the set of \(e\)-tuples of partitions of \(n\) viewed as a \(\mathbb{Z}/r\mathbb{Z}\)-set via the left regular action on indices and

\[ \mathcal{W} := \left\{ w : I \to \{\ell\text{-cores} \mid \sum_{i,j,d} \ell^d w(i, d, j) = n\right\} \]

viewed as a \(\mathbb{Z}/r\mathbb{Z}\)-set via \(i'.w(i, d, j) = w(i + i', d, j)\).

Then \(\mathcal{W}\) and \(\mathcal{L}\) are isomorphic \(\mathbb{Z}/r\mathbb{Z}\)-sets.

This assertion follows by inspection of the proof of (1A) of \[3\], which we recall here for the convenience of the reader:

**Proof.** Let \((\lambda_1, \lambda_2, \ldots, \lambda_e) \in \mathcal{L}\). Fix \(i\) with \(1 \leq i \leq e\). For each \(d \geq 0\) and \(0 \leq j < \ell^d\) we define a family of \(\ell\)-cores \(\kappa_{i,j}^d\) inductively as follows. Set \(\lambda_i^0 := \lambda_i\). Then if \(d \geq 0\) and \(\lambda_i^d\) is defined for \(0 \leq j < \ell^d\), we let \(\kappa_{i,j}^d\) be the \(\ell\)-core of \(\lambda_i^d\) and \((\lambda_i^{d+1}, \ldots, \lambda_i^{d+j+\ell-1})\) be its \(\ell\)-quotient. By construction \(\sum_{d \geq 0} \sum_{0 \leq j < \ell^d} \ell^d |\kappa_{i,j}^d| = |\lambda_i|\) and each \(\lambda_i\) is uniquely determined by the \(\kappa_{i,j}^d\). Now for each \(1 \leq i \leq e\) define \(w(i, d, j) := \kappa_{i,j}^d\) and observe that \(w \in \mathcal{W}\) since \(\sum_{i=1}^e |\lambda_i| = n\) by assumption. Moreover it is clear by construction that the resulting map \(\mathcal{L} \to \mathcal{W}\) is both a bijection and \(\mathbb{Z}/r\mathbb{Z}\)-equivariant. \(\square\)

**Completion of the proof of Theorem 4.2.** By Theorem 3.4 and Proposition 7.10, it suffices to show that \(w(\mathcal{F}^{(r)}) = |\text{Irr}(G(e, r, n))|\). For each \(d > 0\), let \((\mathbb{N} \times \mathcal{C})_d^\ell\) be the subset of \(\mathbb{N} \times \mathcal{C}\) consisting of pairs \((\gamma, \zeta)\) such that \(\gamma + |\zeta| = d\). By the proof of (4C) of \[3\], we have that

\[ \sum_{(\gamma, \zeta) \in (\mathbb{N} \times \mathcal{C})_d^\ell} |\mathcal{A}(\zeta)| = \ell^d. \]

Let

\[ I_d = \mathbb{Z}/e\mathbb{Z} \times \bigcup_{(\gamma, \zeta) \in (\mathbb{N} \times \mathcal{C})_d^\ell} \mathcal{A}(\zeta). \]

Then \(\bigcup_{\zeta} (\mathbb{N} \times \mathbb{Z}/e\mathbb{Z} \times \mathcal{A}(\zeta))\) may be naturally identified with \(\bigcup_d I_d\). Thus by Lemma 7.15 and by Proposition 7.14 the \(\mathcal{F}^{(r)}\)-weights are indexed by the \(\mathbb{Z}/r\mathbb{Z}\)-orbits of \(\mathcal{L}\) with each orbit contributing as many weights as the order of the stabiliser of a point of the orbit. Here \(\mathcal{L}\) is as defined in Lemma 7.15. Since the irreducible characters of \(G(e, r, n)\) are also indexed by the \(\mathbb{Z}/r\mathbb{Z}\)-orbits of \(\mathcal{L}\) in the same way, the proof is complete. \(\square\)

### 7.2. Centralisers

Let \(e, r, n\) be positive integers with \(r \mid e \mid (\ell - 1)\). Let \(q\) and \(q'\) be prime powers such that such that \(q' \equiv 1 \pmod{\ell}\), \(q\) has order \(e\) modulo \(\ell\) and \(\nu_r(q' - 1) = \nu_q(q' - 1)\). We describe centralisers of \(\ell\)-elements in \(X(e, r, n)(q')\) as defined in Section 5 via the identification of \(X(e, 1, n)(q')\) with \(X(1, 1, en)(q)\) from Theorem 3.4 and Example 2.2.

Let us first record some relevant information for the general linear groups. For \(i \geq 0\) we let \(\mathcal{F}_i \subset \mathbb{F}_q[x]\) denote the collection of all irreducible polynomials of degree \(e\ell^i\) all of whose roots have \(\ell\)-power order. The following is easy and well-known:
Lemma 7.16. Let $G = \text{GL}_n(q)$. Then, via their characteristic polynomials, $G$-conjugacy classes of $\ell$-elements of $G$ are in bijection with polynomials

$$(x - 1)^{a_1} \prod_{i \geq 0} \prod_{f \in \mathfrak{E}_i} f^{a_f} \in \mathbb{F}_q[x]$$

of degree $n$; thus $a_1, a_f \geq 0$ satisfy $a_1 + e \sum_{i \geq 0} \ell^i \sum_{f \in \mathfrak{E}_i} a_f = n$.

The centraliser of an $\ell$-element $s \in G$ with this characteristic polynomial is

$$C_G(s) \cong \text{GL}_{a_1}(q) \times \prod_{i \geq 0} \prod_{f \in \mathfrak{E}_i} \text{GL}_{a_f}(q^{\ell^i}).$$

Let $\zeta \in \mathbb{Z}_q^*$ be the root of unity with $q \equiv \zeta \pmod{\ell}$. By Theorem 3.4 and Example 2.2, $X(e, 1, n) \cong X(1, 1, en)_W$ where $\Gamma$ is the subgroup of $\text{Out}(X(1, 1, en))$ generated by the class of $\psi^\gamma$. There is a homotopy commutative diagram

$$\begin{array}{ccc}
X(1, 1, en)(q) & \xrightarrow{\iota} & X(1, 1, en) \\
\downarrow{s} & & \uparrow{\gamma} \\
X(e, 1, n)(q') & \xrightarrow{\iota_1} & X(e, 1, n)
\end{array}$$

where $\iota$ and $\iota_1$ fit into homotopy pullback squares as in (*) in Section 3.2.

Let $S \leq \text{GL}_{en}(q)$ be a fixed Sylow $\ell$-subgroup. By [51, Thm. 6.3] and its proof, for each $r|e$, there is a map $\beta_r : BS \to X(e, r, n)(q')$ and a homotopy monomorphism $X(e, r, n)(q') \to X(e, 1, n)(q')$ whose composition $BS \to X(e, r, n)(q') \to X(e, 1, n)(q')$ factorises as the inclusion $BS \to B\text{GL}_{en}(q)^\wedge \cong X(1, 1, en)(q)$ followed by $g$ and such that $\mathcal{F}_{S,\beta_r}(X(e, r, n)(q'))$ is the saturated subsystem $\mathcal{F}^{(r)}$ of $\mathcal{F}_{\text{GL}_{en}(q)}(S)$ as in Proposition 7.10. For each $s \in S$ set

$$C_{X(e, r, n)}(s) := C_{X(e, r, n)}(s, \tau_r \circ \beta_r), \quad C_{X(e, r, n)(q')}(s) := C_{X(e, r, n)(q')}(s, \beta_r)$$

where again $\tau_r : X(e, r, n)(q') \to X(e, r, n)$ fits into a homotopy pullback square as in (*) in Section 3.2.

Theorem 7.17. With the notation above, for any $\ell$-element $s \in S$ with characteristic polynomial

$$(x - 1)^{a_1} \prod_{i \geq 0} \prod_{f \in \mathfrak{E}_i} f^{a_f} \quad \text{with } a_1 + e \sum_{i \geq 0} \ell^i \sum_{f \in \mathfrak{E}_i} a_f = en$$

we have

$$C_{X(e, r, n)}(s) \cong X(e, r, a_1/e) \times \prod_{i \geq 0} \prod_{f \in \mathfrak{E}_i} X(1, 1, a_f)^{\ell^i}.$$ 

If moreover $s$ is fully centralised in the fusion system $\mathcal{F}^{(r)}$, then

$$C_{X(e, r, n)(q')}(s) \cong X(e, r, a_1/e)(q') \times \prod_{i \geq 0} \prod_{f \in \mathfrak{E}_i} X(1, 1, a_f)(q)^{\ell^i}.$$ 

Here, the products are understood to run over those $f$ with $a_f \neq 0$.

Proof. For the first assertion, by Theorem 3.1 it suffices to show that the Weyl group of $C_X(s)$ is isomorphic to the Weyl group of the right hand side. For this, we first identify the discrete tori of $X(e, r, n)$ and $X(e, 1, n)$ such that the Weyl group $W^{(r)} := G(e, r, n)$
of the former acts as a subgroup of the Weyl group $W = G(e, 1, n)$ of the latter. It follows from the proof of Proposition \ref{prop:factorisation} that $C_{W^{(r)}}(s) = G(e, r, u_0) \times \prod_j \mathfrak{S}_{u_j}$ if and only if $C_W(s) = G(e, 1, u_0) \times \prod_j \mathfrak{S}_{u_j}$. It hence suffices to discuss the case $r = 1$.

Set $X' = X(1, 1, en) \cong B \mathsf{GL}_{en}(\mathbb{C})$. Let $T'$ be the subgroup of diagonal matrices of $G := \mathsf{GL}_{en}(\mathbb{F})$, $W' = N_G(T')/T' \cong \mathfrak{S}_{en}$ and $L'$ the cocharacter lattice of $T'$. Then as explained in Section \ref{sec:factorisation} (W', L') is the Weyl group of $X'$ and by Theorem \ref{thm:factorisation}(a) the Weyl group $(W, L)$ of $X$ is $(W'_e, L'_e)$ (see the discussion above the statement of the theorem).

Further,

$$W'_e = C_{W'}(w_0) = \langle (1, \ldots, e) \times \cdots \times (e(n-1) + 1, \ldots, en) \rangle \rtimes H,$$

where

$$w_0 := (1, \ldots, e) \cdots (e(n-1) + 1, \ldots, en) \in \mathfrak{S}_{en}$$

is a product of $n$ disjoint cycles of length $e$, $H \cong \mathfrak{S}_n$ acts by wreathing the base subgroup and $L'_e$ is the 1-eigenspace of $w_0\zeta$ (see Section \ref{sec:cocharacter}). Consequently the subgroup $\hat{T}$ of $T'$ of $\ell$-power order elements consists of diagonal matrices of the form

$$\text{diag}(x_1, \ldots, x_{en}) \text{ with } x_{ue+i+1} = x^\zeta_{ue+i} \text{ for all } 0 \leq u \leq n-1, 1 \leq i \leq e-1.$$  

Here we are writing $\hat{T}$ multiplicatively and for $x \in \mathbb{Z}/\ell\mathbb{Z}$ by $x^\zeta$ we mean the image of $x$ under the action of $\zeta \in \mathbb{Z}_\ell^\times \cong \text{Aut}(\mathbb{Z}/\ell\mathbb{Z})$. In particular if $x$ has order $\ell^d$, then $x^\zeta$ is $x^n$, for any $n \in \mathbb{Z}$ congruent to $\zeta$ modulo $\ell^d$.

Let $t = \text{diag}(x_1, \ldots, x_{en}) \in \hat{T}$. Let \{1, \ldots, en\} = $J_0 \cup \ldots \cup J_m$ be the partition induced by the relation $x_i = x_j$, where $J_0$ consists of the indices $i$ with $x_i = 1$ (by abuse of notation we allow the possibility $J_0 = \emptyset$). Now $J_u \rightarrow J'_u := \{x^\zeta \mid x \in J_u\}$ defines a free action of $\langle \zeta \rangle$ on the set $\{J_u \mid 1 \leq u \leq m\}$. By suitable re-indexing we may assume that $\{J_1^1, \ldots, J_m^1\}$ is a set of orbit representatives for this action, where $m' = m/e$. For each $1 \leq v \leq m'$, let $J'_v = \bigcup_{0 \leq b \leq e-1} J_v^b$. Then \{1, \ldots, en\} = $J_0 \cup J_1 \cup \ldots \cup J_{m'}$ is a partition,

$$C_{W'}(t) \cong \mathfrak{S}_{[J_0]} \times \prod_{v=1}^{m'} \mathfrak{S}_{[J_v]}.$$  

and

$$C_W(t) = C_{C_{W'}(w_0)}(t) \cong G(e, 1, |J_0|/e) \times \prod_{v=1}^{m'} \mathfrak{S}_{[J_v]}.$$  

Suppose that $t \in \hat{T}$ has characteristic polynomial

$$(x - 1)^{a_1} \prod_{i \geq 0} \prod_{f \in \delta} f^{a_f} \in \mathbb{F}_q[x]$$

with $a_1 + e \sum_{i \geq 0} \ell^i \sum_{f \in \delta} a_f = en$.

If $\lambda$ is an eigenvalue of $t$, then $\lambda^\zeta = \lambda^{f^d}$ when $\lambda$ has degree $\ell^d$ over $\mathbb{F}_q$. Thus,

$$C_W(t) = C_{C_{W'}(w_0)}(t) \cong G(e, 1, a_1/e) \times \prod_{i \geq 0} \prod_{f \in \delta} G(1, 1, a_f)^{\ell^d}.$$  

Let $s \in S$ and suppose that $t \in \hat{T}$ is the image of $s$ under a discrete approximation of a factorisation of the restriction of $\beta_1$ to $B\langle s \rangle$. Then $t$ is also the image of $s$ under a discrete approximation of $\gamma \circ \iota_1 \circ \beta_1$ where $\gamma, \iota_1$ are as in the homotopy commutative diagram at
the beginning of the section. But \( \gamma \circ \iota_1 \circ \beta_1 \) is also the composition of \( \text{inc} : BS \to X'(q) \) with \( X'(q) \to X' \). By Lemma 5.3 applied to \( G = \text{GL}_{en}(F) \), we have that \( t \) and \( s \) have the same characteristic polynomial. This proves the first assertion.

Before moving on to the second assertion, let us record a fact which will be used in the proof of Lemma 7.18. Let \( t \in \hat{T} \) have characteristic polynomial as above. Since \( \nu_t(q^b - 1) = \nu_t((q^e)^b - 1) \) for all non-negative integers \( b \) and \( e \), the order of \( q \) modulo \( \ell \), there exists \( w \in H \) which is a product of cycles of length \( \ell^r \) for each \( f \) of degree \( \ell^r \) (counting multiplicities) such that \( w.t^f = t \), with \( H \cong S_n \) as above. In particular, note that \( w \in G(e, e, n) \).

For the second part, we identify the centraliser via its fusion system. If \( r = 1 \), by Lemma 7.10 and Proposition 7.10, the fusion system of \( C_{\text{GL}_{en}(q)}(s) \) is the fusion system on the right hand side, so we conclude with Proposition 5.3(b). For \( r > 1 \) we argue by induction on the order of \( s \). If \( s^{\ell^r} = 1 \) with \( a = \nu_t(q^e - 1) \), the claim follows by [12, Prop. 11.2].

Now let \( s \in S \) be fully centralised and suppose that \( s \) has order \( \ell^d \), \( d > a \). Set \( u = s^{\ell^d-1} \). We may replace \( s \) by a suitable \( F^{(r)} \)-conjugate such that both \( s \) and \( u \) are fully \( F^{(r)} \)-centralised and hence by [11, Prop. 3.8(c)] both \( s \) and \( u \) are fully \( F \)-centralised. Write \( \text{GL}_{en}(q) = \text{GL}_k(V) \) and let \( W_0 \subseteq V \) be the 1-eigenspace of \( u \). Since \( u \) is semisimple, there is a decomposition \( V = W_0 \oplus W_1 \) into \( \langle u \rangle \)-invariant subspaces. By Lemma 7.10,

\[
C_{\text{GL}_k(V)}(u) = \text{GL}_k(W_0) \times C_{\text{GL}_k(W_1)}(u).
\]

Setting \( S_j := C_S(u) \cap \text{GL}_k(W_j) \) we have \( C_S(u) = S_0 \times S_1 \). Since \( u \) is fully \( F \)-centralised, \( S_j \) is a Sylow \( \ell \)-subgroup of \( \text{GL}_k(W_j) \) and \( S_1 \) is a Sylow \( \ell \)-subgroup of \( C_{\text{GL}_k(W_1)}(u) \).

Set \( G_0 = F_{S_0}(\text{GL}_k(W_0)) \) and \( G_1 = F_{S_1}(C_{\text{GL}_k(W_1)}(u)) \). As explained in the proof of [51, Lemma 6.2], the decomposition of \( C_{X(q^e)}(u) \) given in the first part of our statement translates to a corresponding decomposition for fusion systems

\[
C_{F^{(r)}}(u) = G_0^{(r)} \times G_1.
\]

Now \( s \) preserves the decomposition \( V = W_0 \oplus W_1 \). Write \( s = (s_0, s_1) \) where \( s_j \in S_j \). Then the characteristic polynomial of \( s \) on \( V \) decomposes as \( h_0(x)h_1(x) \) where \( h_j(x) \) is the characteristic polynomial of \( s_j \) on \( W_j \), \( j = 0, 1 \). Moreover, \( h_0(x) \) and \( h_1(x) \) are relatively prime. In other words,

\[
h_0(x) = (x - 1)^{s_1} \prod_{i \geq 0} \prod_{f \in \mathfrak{F}_i} f^{b_f} \quad \text{and} \quad h_1(x) = \prod_{i \geq 0} \prod_{f \in \mathfrak{F}_i} f^{c_f},
\]

where for each \( i \geq 0 \) and each \( f \in \mathfrak{F}_i \), either \( b_f = 0 \), \( c_f = a_f \) or vice versa.

We have

\[
C_{F^{(r)}}(s) = C_{C_{F^{(r)}}(u)}(s) = C_{G_0^{(r)}}(s_0) \times C_{G_1}(s_1).
\]

Since \( C_{G_1}(s_1) = F_{C_{S_1}(s_1)}(C_{\text{GL}_k(W_1)}(s_1)) \), and since up to \( \ell \)-completion \( X(1, 1, c_f)(q^{\ell^r}) \) is homotopy equivalent to \( B \text{GL}_{ec_f}(q^{\ell^r}) \) we obtain from Lemma 7.10 that \( C_{G_1}(s_1) \) is the fusion system associated to

\[
\prod_{i \geq 0} \prod_{f \in \mathfrak{F}_i} X(1, 1, c_f)(q^{\ell^r}).
\]
Lemma 7.19. \[ C_{e,r,s}(a_0) \] is the fusion system associated to \( C_{X(e,r,a_{01}/e)(q')} (s_0) \). Since \( s_0^{q'-1} = u|_{W_0} = 1 \), the order of \( s_0 \) is strictly less than the order of \( s \). Hence,

\[
C_{X(e,r,a_{01}/e)(q')} (s_0) \cong X(e, r, b_1) \times \prod_{i \geq 0} \prod_{f \in \mathcal{F}_i} X(1, 1, b_f)(q'^{e_i})
\]

by induction. The result follows as by Proposition 5.5(b), \( C_{s_i}(s) \) is the fusion system associated to \( C_{X(q'_i)} (s) \).

\[
\Box
\]

Lemma 7.18. In the notation of Theorem 7.17, let \( s \in S \) have characteristic polynomial \((x - 1)^{a_1} \prod_{i \geq 0} \prod_{f \in \mathcal{F}_i} (x - 1)^{a_i} \). Let \((W(s) \phi_s, q')\) be as in Conjecture 5.6. Then we may choose

\[
\phi_s = \prod_{i \in \mathcal{F}_i} \sigma_{i,f} \in \mathcal{G}_n \leq G(e, r, n)
\]

such that for each \( i \) and \( f \), \( \sigma_{i,f} \) is a product of \( a_f \) disjoint cycles of length \( \ell_i \), and

\[
W(s) \phi_s \cong G(e, r, a_{11}/e) \times \prod_{i \geq 0} \prod_{f \in \mathcal{F}_i} G(1, 1, a_f).
\]

In particular, Conjecture 5.6 holds in this case.

Proof. Let \((W, L)\) be identified with \((C_{W*}(w_0), L_{\zeta})\) as in the proof of Theorem 7.17 and let \( t \in T \) with \( \tilde{\gamma}(s) = t \). Then by the recipe for \( \phi_s \), we may take for \( \phi_s \) the element \( w \in H \) as described in the proof of Theorem 7.17. One checks that \( w \) has maximal possible \( \zeta \)-eigenspace on \( L \) amongst the elements of \( C_W(t)w \) and further that \( C_W(t) \) acts faithfully on this \( \zeta \)-eigenspace. The first assertion follows. The description of \( \phi_s \) given above and of \( C_{X(q')} (s) \) in Theorem 7.17 yields Conjecture 5.6. \( \Box \)

7.3. Proof of Conjecture 6.12 for \( G(e, r, \ell) \) and \( q \equiv 1 \pmod{\ell} \). Assume \( r|e|(\ell-1) \) and let \( X \) be the connected \( \ell \)-compact group with Weyl group \((W, L)\) where \( W = G(e, r, \ell) \), \( e > 1 \). Let \( \ell \) be very good for \( W \) and \( \tau \) be an automorphism of \( X \) of finite order prime to \( \ell \). Since \( \ell \) is odd, the element \( \phi \in N_{G_W(L)}(W) \) to which \( \tau \) corresponds via Theorem 5.1 is a scalar (see [18, Prop. 3.13]). It therefore suffices to consider the case \( \tau = 1 \). Let \( q \equiv 1 \pmod{\ell} \) and recall that by Proposition 7.10 the fusion system for \( X(q) \) may be realised as a fusion system inside a general linear group. Precisely, we choose \( q_0 \) to have order \( e \) modulo \( \ell \) and set \( a := \nu_\ell(q_0 - 1) \) so that \( F = F(X(q)) \) is a subsystem of the \( \ell \)-fusion system of \( GL_{e}(q_0) \) on a Sylow \( \ell \)-subgroup \( S \).

Recall the sets \( \mathcal{F}_i \subset \mathcal{F}_{q_0}[\ell] \) defined before Lemma 7.10. An easy counting argument shows that

\[
|\mathcal{F}_i| = \begin{cases} \frac{(\ell^a - 1)}{e} & \text{if } i = 0, \\ \frac{(\ell^{a+1} - \ell^{a-1})}{e} & \text{otherwise.} \end{cases}
\]

We begin by identifying the \( F \)-centric radical subgroups.

Lemma 7.19. Let \( W = G(e, r, \ell) \). Then the \( F \)-centric radical subgroups \( R \) are:

1. \( R = S \cong C_{e \ell} \times C_\ell \) with \( \text{Out}_F(R) \cong C_{\ell-1} \times C_{e/r} \);
2. \( R = D \cong (C_{e \ell})^\ell \) with \( \text{Out}_F(R) \cong W \); and
3. \( R = E := Z_2(S)(\sigma) \cong C_{e \ell} \times \ell^{a+1}/2 \) (a central product) where \( \sigma \) is any element satisfying \( S = D(\sigma) \), and \( \text{Out}_F(E) \cong \text{SL}_2(\ell) \times C_{e/r} \).
For $d \geq 0$ and $R$ an $F$-centric radical subgroup, $w_R(F, d)$ is given in Table 2 where "−" indicates that there are no characters of that defect. Moreover

$$m(F, a\ell) = \alpha + \beta = \sum_{\chi \in Irr^E(D)/\text{W}} z(kI_W(\chi)).$$

Table 2. $w_R(F, d)$ for $R \in F^{cr}$

| $d/R$ | $S$ | $D$ | $E$ |
|-------|-----|-----|-----|
| $a\ell$ | $\alpha$ | $\beta$ | $-$ |
| $a\ell + 1$ | $\ell(r_0r + e/r)$ | $-$ | $-$ |
| $a + 1$ | $-$ | $-$ | $r_1r$ |
| $a + 2$ | $-$ | $-$ | $0$ |

Proof. For $W = G(e, r, \ell)$ the set $\mathcal{F}$ defined after Lemma 7.12 consists of the two functions $f_S$ and $f_D$ defined before Proposition 7.13 and one other function given by

$$f(\gamma, \epsilon) = \begin{cases} 1 & \text{if } \gamma = 1, \epsilon = (0), \\ 0 & \text{otherwise}. \end{cases}$$

Set $E := R_{0,1,1,0}$. Then $E$ is a basic subgroup in the sense of [3, Sect. 4]. From this reference, and using the notation of Section 7.1, we see that $E = ZE_0$, where $Z$ is the Sylow $\ell$-subgroup of $Z(GL_K(V))$ and $E_0$ is extra-special of order $\ell^3$ and exponent $\ell$. Moreover, $N_{GL_K(V)}(E)/E_{C_{GL_K(V)}(V)(E)} \cong SL_2(\ell)$ and $N_{GL_K(V)}(E)$ is an extension of $N_{GL_K(V)}(E)/E_{C_{GL_K(V)}(V)(E)}$ by a cyclic group of order $e/r$ (here $C_{GL_K(V)}(E) = C_{GL_K(V)}(E) = Z(GL_K(V)) \cong K^* \times K^*$). We deduce that (3) holds. The structure of the $F$-outer automorphism groups in cases (1) and (2) can be read off from Proposition 7.13.

To compute the weights, suppose first that $R = E$. Here we see that every character of $E$ is either of defect $a + 1$ or $a + 2$. An argument to show that $w_E(F, a + 2) = 0$ appears in [3], Lemma 8.7. Now every non-linear irreducible character of $E$ is induced by a character of $Z_2(S)$ which does not contain $Z(E) \cong C_{\ell^1}$ in its kernel. Counting, we obtain $(\ell^{a+1} - \ell^a)/\ell$ characters this way. The action of $Out_F(E)$ on $\text{Irr}^{a+1}(E)$ partitions this set into $\ell^a - (\ell - 1)/\ell$ equally sized orbits (with trivial stabiliser). We have $|N_E| = 2$ where $N_E$ is as defined in Section 6.3, and contributions to $w_E(F, a + 1)$ from the trivial and non-trivial chains are easily calculated to be $\ell^{a-1}(\ell - 1)/\ell = r_1r$ and 0 respectively.

Next suppose that $R = S$ and $d = a\ell + 1$. We identify $Out_F(S) \cong C_{\ell-1} \times C_{e/r}$ with the subgroup of $GL_2(\ell)$ generated by the matrices $\text{diag}(1, \omega)$ and $\text{diag}(\omega^{(\ell-1)r/e}, 1)$, where $\omega$ is a generator of $F_\ell^*$. Under this identification, the action of $Out_F(S)$ on $\text{Irr}^{\ell+1}(S)$ is the action on column vectors $(u, v) \in \mathbb{Z}/\ell^a \oplus \mathbb{Z}/\ell$. A complete set of orbit representatives is given by $\{(0, 0), (u, 0), (0, 1), (u, 1)\}$ where there are exactly $r_0r$ choices for the non-zero element $u$. The stabilisers of these representatives have orders $(\ell - 1)e/r, \ell - 1, e/r$ and 1 respectively, so

$$w_S(F, a\ell + 1) = (\ell - 1)e/r + (\ell - 1)r_0r + e/r + r_0r = \ell(r_0r + e/r).$$

It remains to prove the last statement. Set $d := a\ell$ and under the identification of $W$ with $\text{Aut}_F(D)$, set $N := N_W(\text{Aut}_S(D))$. By Clifford theory, $\text{Ind}^S_D$ is a surjective map.
calculating we get Lemma 7.16, and in general there are $r$ from $44$ RADHA KESSAR, GUNTER MALLE, AND JASON SEMERARO

Proof. We discuss the occurring defects from which the result follows.

This shows that induction induces a stabiliser-preserving bijection between $N$-orbits on \{ $\chi \in \text{Irr}(D) \mid Z(S) \nsubseteq \ker(\chi)$ \} and $\text{Out}_F(S)$-orbits on $\text{Irr}^d(S)$. Since $z(k(I_N(\chi))) = 0$ whenever $Z(S) \leq \ker(\chi)$, for $\chi \in \text{Irr}(D)$, we obtain

$$\alpha + \beta = w_D(F, d) + w_S(F, d)$$

as needed. \qed

We need the following combinatorial facts:

**Lemma 7.20.** The following hold:

(a) $|\text{Irr}(\mathfrak{G}_\ell)| - z(k\mathfrak{G}_\ell) = \ell$.

(b) $|\text{Irr}(W)| - z(kW) = e\ell/r$ for $W = G(e, r, \ell)$ with $r|e|(\ell - 1)$.

**Proof.** Part (a) counts the partitions of $\ell$ with an $\ell$-hook, and these are precisely the hooks. For (b) we use the parametrisation of $\text{Irr}(G(e, 1, \ell))$ by $e$-tuples of partitions $\lambda$ of $\ell$. The characters not of $\ell$-defect zero are then precisely those for which one of the parts $\lambda_i$ of $\lambda$ has an $\ell$-hook. This can only happen when $|\lambda_i| = \ell$, and then by (a) there are exactly $\ell$ such. This shows (b) in the case $r = 1$. For general $r$ the linear characters of $G(e, 1, \ell)/G(e, r, \ell) \cong C_r$ act on the parametrising $e$-tuples of partitions by cyclic shift. A character with a non-trivial stabiliser thus corresponds to an $e$-tuple invariant under a non-trivial cyclic shift, of order $1 < d|e$. But then $d$ divides $\ell$, which is not possible as $e|(|\ell - 1)$. So by Clifford theory, we have

$$|\text{Irr}(G(e, r, \ell))| = \frac{1}{r}[\text{Irr}(G(e, 1, \ell))] \quad \text{and} \quad z(kG(e, r, \ell)) = \frac{1}{r}z(kG(e, r, \ell)),$$

from which the result follows. \qed

**Proposition 7.21.** Conjecture 6.12 holds for $W = G(e, r, \ell)$, with $r|e|(\ell - 1)$ in the case $q \equiv 1 \pmod{\ell}$.

**Proof.** We discuss the occurring defects $d$ in turn, and repeatedly use Lemma 7.18 to identify the groups $W(s)\hat{\phi}_s$ in Conjecture 6.12. Suppose first that $d = a\ell + 1$. If $s = 1$ then $W(s)\hat{\phi}_s = W$ and $|\text{Irr}^1(W)| = \ell e/r$ by Lemma 7.20(b). Next assume $1 \neq s \in Z(S)$ so that $W(s)\hat{\phi}_s = \mathfrak{G}_s$. When $r = 1$ there are exactly $r_0$ classes of such elements by Lemma 7.16, and in general there are $r_0 r$ classes by Theorem 7.17. Now $|\text{Irr}^1(W(s)\hat{\phi}_s)| = \ell$
by Lemma 7.20(a), and we obtain
\[ \sum_{s \in Z(S)/F} |\text{Irr}^1(W(s)_{\phi_s})| = \ell e/r + \ell r_0 r, \]
in accordance with Table 2.

If \( d = a + 1 \) then by Lemma 7.16 it suffices to consider classes of elements \( s \) for which \( a_f = 1 \) for some \( f \in \mathcal{S}_1 \) (so that \( \nu_f(|C_S(s)|) = a + 1 \)). By Theorem 7.17 such classes split into \( r \) distinct \( F^{(r)} \)-classes and since here \( W(s)_{\phi_s} = W(s) = 1 \), we obtain \( r_1 r \) characters altogether, as needed by Table 2.

The only occurring defects are \( a + 1, a\ell + 1 \) and \( a\ell \), so to complete the proof it suffices to show that
\[ m(F) = \sum_{s \in S/F} |\text{Irr}(W(s)_{\phi_s})|. \]

Applying Lemmas 7.19 and 7.20 and using the fact that \( \ell \nmid |W(s)_{\phi_s}| \) whenever \( s \in D \setminus Z(S) \), the right hand side of (4) is equal to
\[
|\text{Irr}(W)| + \sum_{1 \neq s \in Z(S)/F} |\text{Irr}(W(s)_{\phi_s})| + \sum_{s \in S/D \setminus F} |\text{Irr}(W(s)_{\phi_s})| + \sum_{s \in S/D \setminus F} |\text{Irr}(W(s)_{\phi_s})|
\]
\[
= |\text{Irr}(W)| + r_0 r |\text{Irr}(|G_e|) + m(F, a\ell) - \sum_{s \in Z(S)/F} z(kI\text{Irr}(W(s))) + r_1 r
\]
\[
= |\text{Irr}(W)| - z(kW) + r_0 r |\text{Irr}(|G_e|) - r_0 r z(kG_e) + m(F, a\ell) + r_1 r
\]
\[
= \ell e/r + r_0 r e + m(F, a\ell) + r_1 r = m(F),
\]
as needed. \( \square \)

Since Conjecture 5.6 holds for \( W = G(e, r, \ell) \) by Lemma 7.18 Proposition 7.21 implies Conjecture 6.10 holds for the \( \mathbb{Z}_e \)-spets associated to \( W \) in the case \( q \equiv 1 \pmod{\ell} \) (see the remark immediately after the statement of Conjecture 6.12).

8. Exceptional and Aguadé groups

In this section we prove our results from Sections 4 and 5 for the exceptional Weyl groups and the four Aguadé exotic \( \mathbb{Z}_e \)-reflection groups.

8.1. Proof of Theorem 4.2 for exceptional and Aguadé groups. Let \((W, L)\) be a \( \mathbb{Z}_e \)-reflection group of type \( E_n, n \in \{6, 7, 8\} \) with associated \( \ell \)-compact group \( X, \ell \) a good prime dividing \( |W| \), and \( q \) a prime power. By the reductions made after its statement, in order to prove Theorem 4.2 for \( X(q) \) we may assume \( q \equiv 1 \pmod{\ell} \) and \( \tau = 1 \).

Lemma 8.1. Let \((W, L), X, \ell \) and \( q \) be as above. Then the associated fusion system \( F := F(X(q)) \) satisfies:
(a) \( F = F_\ell(E_n(q)) \) is the \( \ell \)-fusion system of \( E_n(q) \) on one of its Sylow \( \ell \)-subgroups \( S \).
(b) For some \( \sigma \in S \) we have \( S = T(\sigma) \) for \( T \leq S \) a homocyclic subgroup of index \( \ell \) and rank \( n \).
(c) \( F^{(\tau)} = \{S, T, E\} \) where \( E = Z_2(S)(\sigma) \). The \( F \)-automisers and number of weights are as given in Table 3.
(d) Theorem 4.2 holds for \( F \).
Theorem 4.2 for $F$ in [12, p. 1821]. The $q$-completed classifying spaces of compact Lie groups of type $^2E_6$ and $E_8$, while the other cases are exotic (see [12, Sec. 1]). Let $q \equiv 1 \pmod{\ell}$ be a prime power; we also assume that $a := \nu_q(q - 1) > 1$ when $n = 12$. For $n \in \{12, 29, 31, 34\}$ let $X_n$ denote the $\ell$-completed classifying group associated to $G_n$ and $G_n = G_n(q) = F(X_n(q))$ denote the corresponding fusion system with $\ell$ as indicated.

Let $G_n'$ be the fusion system on a Sylow $\ell$-subgroup $S_n$ of $\text{SL}_\ell(q)$. So $|S_n : n| = \ell^n$, where $S_n = \hat{T}_n(\sigma) \cong C_{\ell^n} \rtimes C_{\ell}$ is a Sylow $\ell$-subgroup of $\text{GL}_\ell(q)$ and $\hat{T}_n = \hat{S}_n \cap \text{GL}_1(q)^\ell$. Note, in particular, that $S_n$ is a group of maximal class.

Lemma 8.2. Let $(W, L)$, $\ell$ and $q$ be as above. Then for $n \in \{12, 29, 31, 34\}$, the fusion system $G_n$ associated to $X_n$ satisfies:

(a) $G_n$ is a simple fusion system containing $G_n'$ as a subsystem (in fact the centraliser of the centre of $S_n$).
(b) We have $S_n = T_n(\sigma)$ where $T_n := S_n \cap \hat{T}_n \cong C_{\ell^n}^{\ell-1}$.
(c) $G_n'/G_n = \{S_n, T_n, D_n, E_n\}$, where $D_n \cong E_n = Z_2(S_n(\sigma)) \cong \ell_+^{n+2}$. The $G_n$-automisers and numbers of weights are as given in Table 4.
(d) Theorem 4.2 holds for $G_n$.

### Table 3. Out$_F(P)$ for $P \in F^{cr}$, and $z(k\text{Out}_F(P))$ (in bold)

| $(n, \ell)$ | $S$ | $z$ | $T$ | $E$ | $\ell$ | $|\text{Irr}(W)|$ |
|-------------|-----|-----|-----|-----|-------|----------------|
| $(6, 5)$    | $C_4 \times C_2$ | 8   | W 15 | $\text{SL}_2(5).2$ | 2 | 25 |
| $(7, 5)$    | $C_4 \times C_2 \times \mathfrak{S}_3$ | 24  | W 30 | $\text{SL}_2(5).2 \times \mathfrak{S}_3$ | 6 | 60 |
| $(7, 7)$    | $C_6 \times C_2$ | 12  | W 46 | $\text{SL}_2(7).2$ | 2 | 60 |
| $(8, 7)$    | $C_6 \times (C_2)^2$ | 24  | W 84 | $\text{SL}_2(7).2 \times C_2$ | 4 | 112 |

Proof. Part (a) may be deduced from the description of the “sporadic cases” described in [12, p. 1821]. The $F$-centric radical subgroups are described in [22, Table 2.2], and their structure and $F$-automisers can be determined from the results in [22, Sec. 2]. Theorem 4.2 for $F$ now follows from Table 3.

Now, let $(W, L)$ be an exotic $\mathbb{Z}_\ell$-reflection group of order divisible by $\ell > 2$. Then $W$ is one of $G_{12}$, $G_{29}$, $G_{31}$ or $G_{34}$, with $\ell = 3, 5, 7$ or 7 respectively. The associated $\ell$-compact groups were constructed by Aguadé [1]. In the first and third cases, these arise as $\ell$-completed classifying spaces of compact Lie groups of type $^2E_6$ and $E_8$, while the other cases are exotic (see [12, Sec. 1]). Let $q \equiv 1 \pmod{\ell}$ be a prime power; we also assume that $a := \nu_q(q - 1) > 1$ when $n = 12$. For $n \in \{12, 29, 31, 34\}$ let $X_n$ denote the $\ell$-completed classifying group associated to $G_n$ and $G_n = G_n(q) = F(X_n(q))$ denote the corresponding fusion system with $\ell$ as indicated.

Let $G_n'$ be the fusion system on a Sylow $\ell$-subgroup $S_n$ of $\text{SL}_\ell(q)$. So $|S_n : n| = \ell^n$, where $S_n = \hat{T}_n(\sigma) \cong C_{\ell^n} \rtimes C_{\ell}$ is a Sylow $\ell$-subgroup of $\text{GL}_\ell(q)$ and $\hat{T}_n = \hat{S}_n \cap \text{GL}_1(q)^\ell$. Note, in particular, that $S_n$ is a group of maximal class.

Lemma 8.2. Let $(W, L)$, $\ell$ and $q$ be as above. Then for $n \in \{12, 29, 31, 34\}$, the fusion system $G_n$ associated to $X_n$ satisfies:

(a) $G_n$ is a simple fusion system containing $G_n'$ as a subsystem (in fact the centraliser of the centre of $S_n$).
(b) We have $S_n = T_n(\sigma)$ where $T_n := S_n \cap \hat{T}_n \cong C_{\ell^n}^{\ell-1}$.
(c) $G_n'/G_n = \{S_n, T_n, D_n, E_n\}$, where $D_n \cong E_n = Z_2(S_n(\sigma)) \cong \ell_+^{n+2}$. The $G_n$-automisers and numbers of weights are as given in Table 4.
(d) Theorem 4.2 holds for $G_n$.

### Table 4. Out$_{G_n}(R)$ and $z(k\text{Out}_{G_n}(R))$ for $R \in G_n^{cr}$

| $\ell$ | $S_n$ | $T_n$ | $D_n$ | $E_n$ | $|\text{Irr}(G_n)|$ |
|-------|-------|-------|-------|-------|----------------|
| 12    | $C_{\ell-1} \times C_{\ell-1}$ | $G_{\ell-1}$ | $\text{SL}_2(\ell)$ | $\text{GL}_2(\ell)$ | 8 |
| 29    | 16    | 17    | 1     | 4     | 37 |
| 31    | 16    | 39    | 1     | 4     | 59 |
| 34    | 36    | 127   | 1     | 6     | 169 |

Proof. See [12, Sec. 10] for (a)–(c). Part (d) then follows from Table 4.
In the case \( a = 1 \) for \( W = G_{12} \), apart from \( \{ S \cong 3_1^{1+2} \} \) there are two \( G_{12} \)-classes of centric radical subgroups represented by \( V_1, V_2 \) with \( V_i \cong C_3^2 \) and \( \text{Out}_{G_{12}}(V_i) \cong \text{GL}_3(2) \). Since \( \text{Out}_{G_{12}}(S) \cong D_8 \), we find
\[
w(G_{12}) = z(k\text{Out}_{G_{12}}(S)) + z(k\text{Out}_{G_{12}}(V_1)) + z(k\text{Out}_{G_{12}}(V_2)) = 5 + 2 + 2 = |\text{Irr}(W)| + 1.
\]

8.2. The ordinary weight conjecture for Aguadé groups. Here we show Proposition 6.15 for the Aguadé groups. For \( n \in \{12, 29, 31, 32\} \) we set \( G_n := G_n(q) = \mathcal{F}(G_n(q)) \), where \( q \equiv 1 \pmod{\ell} \).

**Lemma 8.3.** For \( d \geq 0 \) and \( R \) a \( G_n \)-centric radical subgroup, \( w_R(G_n, d) \) is given in Table 5 where “-” indicates that there are no characters of that defect. Moreover
\[
m(G_n, a(\ell - 1)) = \alpha + \beta = \sum_{\chi \in \text{Irr}(T_n)/G_n} z(kI_{G_n}(\chi)).
\]

| \( \ell - 1 \) | \( S_n \) | \( T_n \) | \( D_n \) | \( E_n \) |
|---|---|---|---|---|
| \( (\ell - 1)a \) | \( \alpha \) | \( \beta \) | \( - \) | \( - \) |
| \( (\ell - 1)a + 1 \) | \( \ell^2 \) | \( - \) | \( - \) | \( - \) |
| 2 | \( - \) | \( - \) | \( \ell - 1 \) | 1 |
| 3 | \( - \) | \( - \) | 0 | 0 |

**Proof.** This is analogous to the argument given in the proof of Lemma 7.19. In particular if \( R \in \{D_n, E_n\} \) we apply that argument in the case \( R = E \) and \( a = 1 \) to conclude that
\[
w_R(G_n, 3) = 0, \quad \text{and} \quad w_R(G_n, 2) = \begin{cases} \ell - 1 & \text{if } R = D_n, \\ 1 & \text{if } R = E_n. \end{cases}
\]

If \( R = S_n \) then since \( R \) has maximal class, it has \( \ell^2 \) linear characters and the action of \( \text{Out}_{G_n}(S_n) \) on \( \text{Irr}^{a(\ell - 1)+1}(S_n) \) is the natural action of the subgroup of \( \text{GL}_2(\ell) \) generated by the matrices \( \text{diag}(1, \omega) \) and \( \text{diag}(\omega, 1) \) where \( \omega \) is a generator of \( F_\ell^\times \). The column vectors \( \{(0, 0), (1, 0), (0, 1), (1, 1)\} \) form a complete set of orbit representatives for this action, so
\[
m(G_n, a(\ell - 1) + 1) = (\ell - 1)^2 + 2(\ell - 1) + 1 = \ell^2.
\]

Finally, the argument to prove the last statement is analogous to that given in Lemma 7.19 so we omit this. \( \square \)

**Proof of Proposition 6.15.** For a \( G_n \)-class represented by \( 1 \neq s \in S_n \), let \( G_{n,s} := (G_n)(s) \) be the centraliser Weyl group from Proposition 5.5. These are described in [12, Prop. 10.1]: if \( s \notin Z(S_n), \left| G_{n,s} \right| \) is prime to \( \ell \), and otherwise \( G_{n,s} = \mathcal{G}_\ell \). If \( d = (a - 1)\ell + 1 \) then by Lemmas 7.20 and 8.3
\[
\sum_{s \in Z(S)/F} |\text{Irr}^1(G_{n,s})| = |\text{Irr}^1(G_n)| + |\text{Irr}^1(\mathcal{G}_\ell)| = \ell(\ell - 1) + \ell = m(G_n, d),
\]
as expected. Now, again by Lemmas 7.20 and 8.3
\[
\sum_{s \in S/G_n} |\text{Irr}(G_{n,s})| = |\text{Irr}(G_n)| + \sum_{1 \neq s \in Z(S_n)/G_n} |\text{Irr}(G_{n,s})| + \sum_{s \in Z(S_n)/G_n} |\text{Irr}(G_{n,s})| = |\text{Irr}(G_n)| + |\text{Irr}(\mathfrak{S}_n)| + m(G_n, a(\ell - 1)) - \sum_{s \in Z(S_n)/G_n} z(kG_{n,s}) = |\text{Irr}(G_n)| - z(kG_n) + |\text{Irr}(\mathfrak{S}_n)| - z(k\mathfrak{S}_n) + m(G_n, a(\ell - 1)) = \ell(\ell - 1) + \ell + m(G_n, a(\ell - 1)) = m(G_n) - m(G_n, 2).
\]
Finally, note that when \( n = 12 \) and \( a = 1 \), we have \( S_{12} \cong 3_+^{1+2} \) and \( m(G_{12}, 3) = 9 = |\text{Irr}^1(G_{12})| + |\text{Irr}^1(\mathfrak{S}_3)\) (see \([35\text{ Tab. 2}]\).)

9. Appendix: Weights for wreath products

Here we prove a result which was used in Section 7.2.

Lemma 9.1. Let \( G = N \wr \mathfrak{S}_n \) be a finite group. There is a natural bijection
\[
\Psi : \text{Irr}^0(G) \to \mathcal{W}
\]
between the set \( \text{Irr}^0(G) \) of \( \ell \)-defect zero characters of \( G \) and the set of functions
\[
\mathcal{W} := \left\{ w : \text{Irr}^0(N) \to \{\ell\text{-cores}\} \mid \sum_{\varphi \in \text{Irr}^0(N)} |w(\varphi)| = n \right\}.
\]

See \([3\text{ Sec. 4}]\). The map \( \Psi \) is defined as follows: Let \( \chi \in \text{Irr}^0(G) \), \( \tau \in \text{Irr}(B) \) with \( \chi \in \text{Irr}(G|\tau) \), where \( B = N^n \) is the base subgroup of \( G \). Let \( H \) be the stabiliser of \( \tau \) in \( G \) and let \( \chi' \in \text{Irr}(H|\tau) \) such that \( \chi = \text{Ind}^G_H(\chi') \). Then \( \tau \in \text{Irr}^0(B) \) and \( \chi' \in \text{Irr}^0(H) \). Let \( \tau = \tau_1 \otimes \cdots \otimes \tau_n \) with \( \tau_j \in \text{Irr}^0(N) \) and for each \( \varphi \in \text{Irr}^0(N) \), denote by \( n_{\chi,\varphi} = n_\varphi \) the number of \( j \) such that \( \varphi \cong \tau_j \). Then
\[
H \cong \prod_{\varphi \in \text{Irr}^0(N)} N \wr \mathfrak{S}_{n_x},
\]
and we have an identification
\[
\text{Irr}^0(H|\tau) = \prod_{\varphi \in \text{Irr}^0(N)} \text{Irr}^0(N \wr \mathfrak{S}_{n_x}|\varphi \otimes \cdots \otimes \varphi).
\]

For \( \varphi \in \text{Irr}^0(N) \), let \( \hat{\varphi} \) be the canonical extension to \( N \wr \mathfrak{S}_{n_x} \) of \( \varphi^\otimes n \), i.e., \( \hat{\varphi} \) is the tensor induced character \( \varphi^\otimes n \). Tensoring with \( \hat{\varphi} \) induces a bijection \( \ell_{\varphi} : \text{Irr}^0(\mathfrak{S}_{n_x}) \to \text{Irr}^0(N \wr \mathfrak{S}_{n_x}|\varphi \otimes \cdots \otimes \varphi) \).

Suppose that \( \chi' \) corresponds to \( \prod_{\varphi} \chi'_{\varphi} \) via \((5)\). Then \( \Psi(\chi)(\varphi) = \kappa_{\varphi} \), where \( \kappa_{\varphi} \) is the \( \ell \)-core partition of \( n_{\varphi} \) labelling \( \ell^{-1}_{\varphi}(\chi'_{\varphi}) \).

Now let \( \lambda \) be a linear character of \( G' := G/([G,G]\mathfrak{S}_n) \) and let \( \chi, \tau, H \) and \( \chi' \) as above. Since \( \mathfrak{S}_n \) is in the kernel of \( \lambda \), \( \lambda|_B = \lambda_0^\otimes n \) for some \( \lambda_0 \in \text{Irr}(N) \). The character \( \lambda \chi \)
covers $\lambda|_{B_0}\tau$, $H = \text{Stab}_G(\lambda|_{B_0}\tau)$ and for all $\varphi \in \text{Irr}^0(N)$, we have $n_{\lambda,\chi,\lambda_0,\varphi} = n_{\chi,\varphi} = n_{\varphi}$. Further, $\lambda_0\varphi = \lambda|_{N_i}\varphi$, hence for any $\kappa \in \text{Irr}(\mathcal{S}_n)$, 

$$i_{\lambda_0,\varphi}(\kappa) = \lambda_0\varphi \otimes \kappa = \lambda|_{N_i}\varphi_i(\kappa).$$

Since 

$$\lambda \cdot \chi = \lambda \cdot \text{Ind}_H^G(\chi') = \text{Ind}_H^G(\lambda|_H \cdot \chi') = \prod_{\varphi \in \text{Irr}^0(N)} \lambda|_{N_i}\varphi \cdot \chi_{\varphi}$$

we obtain 

(6) 

$$\Psi(\lambda \cdot \chi)(\lambda_0 \cdot \varphi) = \Psi(\chi).$$

Let $\text{Irr}(G')$ act on $\text{Irr}^0(G)$ via $\chi \mapsto \lambda \cdot \chi$ (where $\chi \in \text{Irr}^0(G)$, $\lambda \in \text{Irr}(G')$), and on $\mathcal{W}$ via 

$$\lambda.w(\varphi) := w(\lambda_0^{-1} \cdot \varphi), \quad \text{where } w \in \mathcal{W}, \ \lambda \in \text{Irr}(G'), \ \varphi \in \text{Irr}^0(N).$$

By Equation (6), $\Psi$ is an isomorphism of $\text{Irr}(G')$-sets.

The above discussion carries over to direct products $G = \prod_{i \in I} N_i \wr \mathcal{S}_n_i$. Set 

$$\mathcal{W} := \left\{ w : \bigcup_{i \in I} \text{Irr}^0(N_i) \to \{\ell\text{-cores}\} \text{ with } \sum_{\varphi \in \text{Irr}^0(N_i)} |w(\varphi)| = n_i \right\}$$

and let $M$ be a subgroup of $G$ containing $[G, G]$ and all $\mathcal{S}_n_i$, $i \in I$. If $\lambda \in \text{Irr}(G/M)$, the restriction of $\lambda$ to $\prod_{i \in I} N_i^{n_i}$ is of the form $\prod_{i \in I} \lambda_i^{n_i}$ for some $\lambda_i \in \text{Irr}(N_i)$. The group $\text{Irr}(G/M)$ acts on $\mathcal{W}$ via 

$$\lambda.w_i(\varphi) = w(\lambda_i^{-1} \cdot \varphi) \quad \text{for } \lambda \in \text{Irr}(G/M), \ w \in \mathcal{W}, \ i \in I, \ \varphi \in \text{Irr}^0(N_i).$$

Then the $\text{Irr}(G/M)$-sets $\text{Irr}^0(G)$ and $\mathcal{W}$ are isomorphic where as before the $\text{Irr}(G/M)$-action on $\text{Irr}^0(G)$ is via multiplication. The Clifford theory of cyclic extensions gives the following immediate consequence of the above:

**Proposition 9.2.** With the above notation, suppose that $G/M$ is a cyclic $\ell'$-group. Then $\text{Irr}^0(M)$ is indexed by the set of $D := \text{Irr}(G/M)$-orbits of $\mathcal{W}$ in such a way that the $D$-orbit of $w \in \mathcal{W}$ corresponds to $|\text{Stab}_D(w)|$ elements of $\text{Irr}^0(M)$.

**References**

[1] J. Aguadé, Constructing modular classifying spaces. Israel J. Math. 66 (1989), 23–40.

[2] J. Alperin, Weights for finite groups. The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 369–379, Proc. Sympos. Pure Math., 47, Amer. Math. Soc., Providence, RI, 1987.

[3] J. Alperin and P. Fong, Weights for symmetric and general linear groups. J. Algebra 131 (1990), 2–22.

[4] J. An, Weights for the Chevalley groups $G_2(q)$. Proc. Lond. Math. Soc. 69 (1994) 2–46.

[5] K. Andersen and J. Grodal, The classification of 2-compact groups. J. Amer. Math. Soc. 22 (2009), 387–436.

[6] K.K.S. Andersen, G. Grodal, J.M. Møller, and A. Viruel, The classification of $p$-compact groups for $p$ odd. Ann. of Math. (2) 167 (2008), 95–210.

[7] K. Andersen, B. Oliver, and J. Ventura, Reduced, tame and exotic fusion systems. Proc. Lond. Math. Soc. 105 (2012), 87–152.

[8] M. Aschbacher, R. Kessar, and B. Oliver, Fusion Systems in Algebra and Topology. London Mathematical Society Lecture Note Series, 391, Cambridge University Press, Cambridge, 2011.

[9] C. Bonnafé, Quasi-isolated elements in reductive groups. Comm. Algebra 33 (2005), 2315–2337.
[10] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver, Extensions of p-local finite groups. *Trans. Amer. Math. Soc.* **359** (2007), 3791–3858.

[11] C. Broto, R. Levi, and B. Oliver, The homotopy theory of fusion systems. *J. Amer. Math. Soc.* **16** (2003), 779–856.

[12] C. Broto and J. Möller, Chevalley p-local finite groups. *Algebr. Geom. Topol.* **7** (2007), 1809–1919.

[13] C. Broto, J. Möller, and B. Oliver, Equivalences between fusion systems of finite groups of Lie type. *J. Amer. Math. Soc.* **25** (2012), 1–20.

[14] C. Broto, J. Möller, and B. Oliver, Automorphisms of fusion systems of finite simple groups of Lie type. *Memoirs Amer. Math. Soc.* **1267** (2019), 1–117.

[15] M. Broué, R. Corran, and J. Michel, Cyclotomic root systems and bad primes. *Adv. Math.* **325** (2018), 375–458.

[16] M. Broué and G. Malle, Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis. *Math. Ann.* **292** (1992), 241–262.

[17] M. Broué, G. Malle, and J. Michel, Generic blocks of finite reductive groups. *Asterisque* No. 212 (1993), 7–92.

[18] M. Broué, G. Malle, and J. Michel, Towards spetses I. *Transform. Groups* **4** (1999), 157–218.

[19] M. Broué, G. Malle, and J. Michel, Split spetses for primitive reflection groups. *Asterisque* No. 359 (2014).

[20] O. Brunat and G. Malle, Characters of positive height in blocks of finite quasi-simple groups. *Int. Math. Res. Not. IMRN* **17** (2015), 7763–7786.

[21] M. Cabanes and M. Enguehard, On unipotent blocks and their ordinary characters. *Invent. Math.* **117** (1994), 149–164.

[22] D. Craven, B. Oliver, and J. Semeraro, Reduced fusion systems over p-groups with abelian subgroup of index p: II. *Adv. Math.* **322** (2017), 201–268.

[23] A. Chermak, Fusion systems and localities. *Acta Math.* **211** (2013), 47–139.

[24] W. G. Dwyer and C. W. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math.* (2) **139** (1994), 395–442.

[25] W. G. Dwyer and C. W. Wilkerson, The center of a p-compact group. *The Čech Centennial* (Boston, MA, 1993), Contemp. Math. **181**, 119–157, Amer. Math. Soc., Providence, RI, 1995.

[26] M. Enguehard, Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais. *J. Algebra* **230** (2000), 334–377.

[27] Z. Feng, The blocks and weights of finite special linear and unitary groups. *J. Algebra* **523** (2019), 53–92.

[28] E. M. Friedlander and G. Mislin, Cohomology of classifying spaces of complex Lie groups and related discrete groups. *Comment. Math. Helv.* **59** (1984), 347–361.

[29] M. Geck, Basic sets of Brauer characters of finite groups of Lie type II. *J. Lond. Math. Soc.* **47** (1993), 225–268.

[30] M. Geck and G. Hiss, Basic sets of Brauer characters of finite groups of Lie type. *J. Reine Angew. Math.* **418** (1991), 173–188.

[31] M. Geck and G. Malle, Reflection groups. *Handbook of Algebra. Vol. 4*, 337–383. Elsevier/North-Holland, Amsterdam, 2006.

[32] M. Geck and G. Malle, *The Character Theory of Finite Groups of Lie Type: A Guided Tour*. Cambridge Studies in Advanced Mathematics, 187, Cambridge University Press, Cambridge, 2020.

[33] J. Grodal and A. Lahtinen, String topology of finite groups of Lie type. Preprint. arXiv:2003.07852, 2020.

[34] F. Himstedt, On the decomposition numbers of the Ree groups $^2 F_4(q^2)$ in non-defining characteristic. *J. Algebra* **325** (2011), 364–403.

[35] R. Kessar, M. Linckelmann, J. Lynd, and J. Semeraro, Weight conjectures for fusion systems. *Adv. Math.* **357** (2019), 106825.

[36] G. I. Lehrer and T. A. Springer, Reflection subquotients of unitary reflection groups. *Canad. J. Math.* **51** (1999), 1175–1193.
[37] J. Lynd and J. Semeraro, Weights in a Benson–Solomon block. Preprint. arXiv:1712.02826, 2017.
[38] G. Malle, Die unipotenten Charaktere von $^2F_4(q^2)$. Comm. Algebra 18 (1990), 2361–2381.
[39] G. Malle, Unipotente Grade imprimitiver komplexer Spiegelungsgruppen. J. Algebra 177 (1995), 768–826.
[40] G. Malle, Spetses. Doc. Math. Extra Vol. ICM II (1998), 87–96.
[41] G. Malle, On the generic degrees of cyclotomic algebras. Represent. Theory 4 (2000), 342–369.
[42] G. Malle and D. Testerman, Linear Algebraic Groups and Finite Groups of Lie Type. Cambridge Studies in Advanced Mathematics, 133. Cambridge University Press, Cambridge, 2011.
[43] J. Møller, Rational isomorphisms of $p$-compact groups. Topology 35 (1996), 201–225.
[44] J. Møller, $N$-determined $p$-compact groups. Fund. Math. 173 (2002), 201–300.
[45] J. Møller and D. Notbohm, Connected finite loop spaces with maximal tori. Trans. Amer. Math. Soc. 350 (1998), 3483–3504.
[46] D. Notbohm, For which pseudo-reflection groups are the $p$-adic polynomial invariants again a polynomial algebra? J. Algebra 214 (1999), 553–570.
[47] B. Oliver and A. Ruiz, Simplicity of fusion systems of finite simple groups. Trans. Amer. Math. Soc. 374 (2021), 7743–7777.
[48] R. Oliver and J. Ventura, Extensions of linking systems with $p$-group kernel. Math. Ann. 338 (2007), 983–1043.
[49] D. Quillen, On the cohomology and $K$-theory of the general linear groups over a finite field. Ann. of Math. (2) 96 (1972), 552–586.
[50] G. Robinson, Local structure, vertices and Alperin’s conjecture. Proc. London Math. Soc. 72 (1996), 312–330.
[51] A. Ruiz, Exotic normal fusion subsystems of general linear groups. J. London Math. Soc. (2) 76 (2007), 181–196.
[52] J. Semeraro, A 2-compact group as a spets. Exp. Math. 32 (2023), 140–155.
[53] R. Steinberg, Endomorphisms of Linear Algebraic Groups. Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.
[54] R. Steinberg, Torsion in reductive groups. Adv. Math. 15 (1975), 63–92.