Riesz bases for $L^2(\partial \Omega)$ and regularity for the Laplace equation in Lipschitz domains

Abdellatif CHAÏRA, Soumia TOUHAMI

Université Moulay Ismail, Faculté des Sciences, Laboratoire de Mathématiques et leurs Applications, Équipe EDP et Calcul Scientifique, BP 11201 Zitoune, 50070 Meknès, Maroc.

Abstract. In a paper from 1996, D. Jerison and C. Kenig among other results provided a $H^{1/2}$ regularity result for the Dirichlet problem for the Laplace equation in Lipschitz domains. In this article, we adopt a Hilbertian approach to construct two Riesz bases for $L^2(\partial \Omega)$, which will allow to find in a different way some of the results of D. Jerison and C. Kenig, and G. Savaré (1998) about the regularity issue of the Laplace equation.

Keywords: Dirichlet problem, Laplace equation, Lipschitz domain, Hilbertian method, Riesz basis

1 Introduction and results

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \geq 2$, with boundary $\partial \Omega$, let us consider for $g$ defined on $\partial \Omega$, the Dirichlet problem for the Laplace equation

$$\begin{cases} 
\Delta v = 0 & (\Omega) \\
v = g & (\partial \Omega).
\end{cases}$$

We briefly recall the history of the problem (1). If $g$ is continuous on $\partial \Omega$ it is well known that $\Omega$
being regular for the Laplacian $\Delta$, the problem (1) has a unique solution given by

$$v(x) = \int_{\partial \Omega} g(y) \, d\omega^x(y),$$

where $\omega^x$ is the harmonic measure for $\Omega$ with pole $x \in \Omega$. For $g \in H^{1/2}(\partial \Omega)$, the problem (1) is variational and has a unique solution according to the Hilbertian theory of Sobolev spaces [8].

The problem (1) when the data consisted either of functions in $L^2(\partial \Omega)$ or of functions with first derivatives in $L^2(\partial \Omega)$, had attracted significant research attention. This began with the work of J. Nečas [19]. Using Rellich Identity:

$$\int_{\Omega} (m, \nabla u) dx + \frac{1}{2} \int_{\Omega} \text{div}(m)|\nabla u|^2 dx + \frac{1}{2} \int_{\partial \Omega} (m, \nu) d\sigma,$$

where $m \in (C^\infty(\mathbb{R}^d))^d$ is a vector field, $\partial_\nu$ is the normal derivative operator associated to $\Omega$ and $(.,.)_{\mathbb{R}^d}$ denotes the inner product on $\mathbb{R}^d$, J. Nečas proved the following result:

- **Rellich-Nečas lemma.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a constant $c_\Omega > 0$ depending on the geometry of $\Omega$ such that for all $u \in H^1_\Delta(\Omega) \cap H^1_0(\Omega)$

$$\|\partial_\nu u\|_{L^2(\partial \Omega)} \leq c_\Omega \|\Delta u\|_{L^2(\Omega)},$$

where $H^1_\Delta(\Omega) = \{ u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega) \}.$

The Rellich-Nečas lemma allows to define the very weak solution of the Dirichlet problem for the Laplace equation (1). Indeed, we say that $v \in L^2(\Omega)$ is a **very weak solution** of the problem (1) if for all $u \in H^1_\Delta(\Omega) \cap H^1_0(\Omega)$ we have

$$-\int_{\Omega} v \Delta u \, dx + \int_{\partial \Omega} g \, \partial_\nu u \, d\sigma = 0. \quad (2)$$

The last formulation makes sens according to Rellich-Nečas lemma and means that

$$A'v = \mu,$$

where $A'$ from $L^2(\Omega)$ to the dual space of $H^1_\Delta(\Omega) \cap H^1_0(\Omega)$ denoted $(H^1_\Delta(\Omega) \cap H^1_0(\Omega))'$, is the transpose of the Laplacian with Dirichlet conditions

$$\varphi \rightarrow A\varphi = -\Delta \varphi$$

defined from $H^1_\Delta(\Omega) \cap H^1_0(\Omega)$ to $L^2(\Omega)$, and where $\mu$ is the linear form given by

$$\varphi \rightarrow -\int_{\partial \Omega} g \, \partial_\nu \varphi \, d\sigma,$$

which is continuous according to Rellich-Nečas lemma. Since $A$ is an isomorphism from $H^1_\Delta(\Omega) \cap H^1_0(\Omega)$ into $L^2(\Omega)$, it follows that $A'$ is also an isomorphism from $L^2(\Omega)$ to $(H^1_\Delta(\Omega) \cap H^1_0(\Omega))'$, and this proves the existence and the uniqueness of $v \in L^2(\Omega)$, solution of (1). Dahlberg in [7], established that the harmonic measure and the surface measure associated to $\Omega$ are mutually absolutely
continuous, furthermore, the Random-Nikodym derivative of harmonic measure with respect to surface measure satisfies a reverse Hölder inequality which allows to solve the problem (1) with data in $L^2(\partial \Omega)$. In [12], D. Jerison and C. Kenig provided another proof of Dahlberg’s results using an integral identity due to Rellich, and after that in [13], they gave optimal estimates for the Dirichlet problem when the data has one derivative in $L^2(\partial \Omega)$, where they combined Rellich formulas with Dahlberg’s results. D. Verchota in [23], following the works of Coifman-McIntosch and Meyer [4], had been interested to the invertibility of classical layer potentials for Laplace equation on the boundaries of bounded Lipschitz domains and the applications to the Dirichlet and Neumann problems. In [14], D. Jerison and C. Kenig studied the inhomogenous Dirichlet problem for the Laplacian in Lipschitz domains with data in trace spaces, where they used the strategy of reduction to the homogenous problem. The two important tools in their paper were the investigation of traces of Sobolev spaces on the boundary and the characterization of Sobolev and Besov spaces of harmonic functions. Savaré in [21], developed a variational argument based on the usual Niremberg’s difference quotient technique to deal with the regularity of the solutions of Dirichlet and Neumann problems for some linear and quasilinear elliptic equations in Lipschitz domains.

The main purpose of this paper is to construct two Riesz bases for $L^2(\partial \Omega)$ (see §4) and show how it will be possible to give another proof of the $H^{1/2}$ regularity results about the Dirichlet problem for the Laplacian previously established by Jerison and Kenig in [13] and by Savaré in [21] (see §5). In the following we give a first description of the approach that we will follow in this paper and which will be detailed in the next sections.

Consider the solution operator of the problem (1)$$K : L^2(\partial \Omega) \rightarrow L^2(\Omega), \quad g \mapsto v,$$where $v$ is the very weak solution of (1) and consider its adjoint operator $K^*$, which takes each $f \in L^2(\partial \Omega)$ to $-\partial_\nu u^0$ into $L^2(\partial \Omega)$, where $u^0$ is the solution of the Dirichlet problem for the following Poisson equation
\begin{align*}
\begin{cases}
-\Delta u^0 = f & (\Omega) \\
\Gamma u^0 = 0 & (\partial \Omega),
\end{cases}
\end{align*}
(3)
where $\Gamma$ is the trace operator from $H^1(\Omega)$ to $L^2(\partial \Omega)$. Consider also the embedding operator from $H^1(\Omega)$ into $L^2(\Omega)$ denoted $E$. For $f \in L^2(\Omega)$, the adjoint operator $E^*$ is the solution operator of Robin problem for the following Poisson equation
\begin{align*}
\begin{cases}
-\Delta u = f & (\Omega) \\
\partial_\nu u + \Gamma u = 0 & (\partial \Omega).
\end{cases}
\end{align*}
(4)
By setting $E_1^* = E^* - E_0^*$ and $u^1 = E_1^* f$, where $E_0^*$ denotes the solution operator of (3), it follows that $u^1$ is a solution of the following Dirichlet problem for the Laplace equation
\begin{align*}
\begin{cases}
-\Delta u^1 = 0 & (\Omega) \\
\Gamma u^1 = \Gamma u & (\partial \Omega),
\end{cases}
\end{align*}
(5)
where $u$ is the solution of (4). Let us set $\Gamma_\delta^* = F_\delta^*(I + F_1 F_\delta^*)^{-1/2} \Gamma^*$, where $F_1$ is the Moore-Penrose inverse of the adjoint operator $E_1 = (E_1^*)^*$, and $\Gamma^*$ is the adjoint of the trace operator $\Gamma$. We will
show in section 3 of this paper that
\[ \Gamma_0^* K^* = (I + F_1^* F_1)^{-1/2} P_{\mathcal{H}(\Omega)} \]
is compact and self-adjoint, where \( P_{\mathcal{H}(\Omega)} \) is the orthogonal projection onto the space of harmonic square-integrable functions which is called the Bergman space and denoted in this text by \( \mathcal{H}(\Omega) \). Consequently, there exists a sequence of couples \( (\kappa_n, \phi_n)_{n \geq 1} \in \mathbb{R}_+^* \times \mathcal{H}(\Omega) \) associated with \( \Gamma_0^* K^* \) such that
\[ \Gamma_0^* K^* \phi_n = \kappa_n^2 \phi_n. \]
Moreover, \( (\phi_n)_{n \geq 1} \) is an orthonormal basis for \( \mathcal{H}(\Omega) \). By setting for all \( n \geq 1 \),
\[ \Gamma_0^* \phi_n = \kappa_n y_n \text{ and } K^* \phi_n = \kappa_n g_n, \]
the main purpose of the present work is to prove the following result.

**Theorem 1.1** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain. Then, the sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) defined above, are Riesz bases for \( L^2(\partial \Omega) \).

One of the main consequences of Theorem 1.1 is the following classical regularity result.

**Theorem 1.2** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain. Then, for \( g \in L^2(\partial \Omega) \), the very weak solution of the Dirichlet problem for the Laplace equation (1) lies in \( H^{1/2}(\Omega) \) and there exist two positive constants \( c_1, c_2 \) depending on the geometry of \( \Omega \) such that
\[ c_1 \|g\|_{L^2(\partial \Omega)} \leq \|v\|_{H^{1/2}(\Omega)} \leq c_2 \|g\|_{L^2(\partial \Omega)}. \]
Moreover, the solution operator \( K \) is compact and injective.

The plan of the paper is the following: the next section contains some known and new facts about the Moore-Penrose inverse and a brief recall of some preliminary results for Riesz bases and related sequences, and also some basic results for Sobolev spaces in Lipschitz domains. In section 3, we present the main key tools to deal with Theorem 1.1 and Theorem 1.2. Section 4 will be devoted to study the sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) and to present the remaining arguments to conclude our main result (Theorem 1.1). In section 5, a regularity result for the Dirichlet problem for the Laplace equation (1) will be derived (Theorem 1.2).

## 2 Preliminaries and basic results

Let \((\mathcal{H}_1, (\cdot, \cdot)_1)\) and \((\mathcal{H}_2, (\cdot, \cdot)_2)\) be two Hilbert spaces with the associated inner products \((\cdot, \cdot)_1, (\cdot, \cdot)_2\) and the induced norms \(\|\cdot\|_1, \|\cdot\|_2\), and throughout this article, unless otherwise mentioned, they will be simply denoted \(\mathcal{H}_1\) and \(\mathcal{H}_2\). A linear operator from \(\mathcal{H}_1\) to \(\mathcal{H}_2\) is a pair consisting of a subspace \(\mathcal{D}(A)\) of \(\mathcal{H}_1\) together with a linear map \(A : \mathcal{D}(A) \to \mathcal{H}_2\). We call \(\mathcal{D}(A)\) the domain of the operator \(A\) and write \((A, \mathcal{D}(A)) = A\). \(\mathcal{N}(A)\) denotes its null space, \(\mathcal{R}(A)\) its range space and \(\mathcal{G}(A)\) its graph. In the case \((A, \mathcal{D}(A))\) is bounded, we write simply \(A\). The set of all bounded operators from \(\mathcal{H}_1\) into \(\mathcal{H}_2\) is denoted by \(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)\), and if \(\mathcal{H}_1 = \mathcal{H}_2\), \(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)\) is denoted \(\mathcal{B}(\mathcal{H})\). For two linear operators \((A, \mathcal{D}(A))\) and \((B, \mathcal{D}(B))\) from \(\mathcal{H}_1\) into \(\mathcal{H}_2\), \((B, \mathcal{D}(B))\) is called an extension of \((A, \mathcal{D}(A))\) if
\[ \mathcal{D}(A) \subset \mathcal{D}(B) \quad \text{and} \quad \forall x \in \mathcal{D}(A), Ax = Bx, \]
and this fact is denoted by $A \subset B$.

For a linear operator $(A, \mathcal{D}(A))$ on a Hilbert space $\mathcal{H}$, there are several ways of defining the notion of positivity, in this paper, this corresponds to the following $$(Ax, x) \geq 0 \quad \forall x \in \mathcal{D}(A),$$

in such case we write $A \geq 0$ and say that $(A, \mathcal{D}(A))$ is positive. $(A, \mathcal{D}(A))$ is said to be densely defined if $\mathcal{D}(A)$ is dense in $\mathcal{H}$, i.e., $\overline{\mathcal{D}(A)} = \mathcal{H}$, where $\overline{\mathcal{D}(A)}$ denotes the closure of $\mathcal{D}(A)$. $(A, \mathcal{D}(A))$ is said to be closed if its graph is closed in $\mathcal{H}_1 \times \mathcal{H}_2$, where the inner product in $\mathcal{H}_1 \times \mathcal{H}_2$ is defined for all $x, u \in \mathcal{H}_1$ and $y, v \in \mathcal{H}_2$ by $$(x, y, (u, v)) = (x, u)_1 + (y, v)_2.$$ The set of all closed densely defined operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ is denoted by $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. The adjoint of a densely defined operator from $\mathcal{H}_1$ into $\mathcal{H}_2$ is denoted $(A^*, \mathcal{D}(A^*))$ where $\mathcal{D}(A^*)$ is defined to be the set of all $y \in \mathcal{H}_2$ for which there exists $z \in \mathcal{H}_1$ such that $$(Ax, y)_2 = (x, z)_1 \quad \forall x \in \mathcal{D}(A).$$

Since $\mathcal{D}(A)$ is dense, it follows that $z$ is unique. We put $A^*y = z$, then we have: $$(x, A^*y)_1 = (Ax, y)_2, \forall x \in \mathcal{D}(A), y \in \mathcal{D}(A^*),$$

and $A^*$ is closed. Moreover, if $A$ is closed, $A^*$ is densely defined.

**Lemma 2.1** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $(A, \mathcal{D}(A)), (B, \mathcal{D}(B))$ be two linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ such that $A \subset B$. Then if $\mathcal{D}(A)$ is dense, we have $B^* \subset A^*$.

A linear operator $(A, \mathcal{D}(A))$ on a Hilbert space $\mathcal{H}$ is said to be self-adjoint if $A^* = A$ which means that $\mathcal{D}(A^*) = \mathcal{D}(A)$ and that $A^*x = Ax$ for all $x \in \mathcal{D}(A)$. Many of the operators which we shall study in this paper are positive self-adjoint and the condition of self-adjointness is of profound importance to define the powers of any fractional order of $(A, \mathcal{D}(A))$. A bounded linear operator $A$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ is said to be compact if for any bounded sequence $(f_n)_{n \geq 1}$ of elements of $\mathcal{H}_1$, the sequence $(Af_n)_{n \geq 1}$ has a norm convergent subsequence. The following theorem is stated in [5, Theorem 3.4].

**Theorem 2.1 (Schauder’s Theorem)** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $A \in B(\mathcal{H}_1, \mathcal{H}_2)$. Then, $A$ is compact if and only if its adjoint $A^*$ is compact.

For further lectures, see [6] and [15].

When an operator is not invertible in the strict sense, one can define its Moore-Penrose inverse. The next subsection is devoted to provide some known and new facts about this concept that will play a key role in this text.
2.1 The Moore-Penrose Inverse

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces, $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ a closed densely defined operator and $(A^*, \mathcal{D}(A^*))$ its adjoint. The Moore-Penrose inverse of $(A, \mathcal{D}(A))$ denoted $(A^\dagger, \mathcal{D}(A^\dagger))$ is defined as the unique linear operator in $\mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$ such that

$$\mathcal{D}(A^\dagger) = \mathcal{R}(A) \oplus \mathcal{N}(A^\dagger), \quad \mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$$

and satisfying the followings

$$\begin{align*}
AA^\dagger A &= A \\
A^\dagger AA^\dagger &= A^\dagger
\end{align*}$$

where $P_{\mathcal{R}(A)}$ and $P_{\mathcal{R}(A^\dagger)}$ denote the orthogonal projections onto $\mathcal{R}(A)$ and $\mathcal{R}(A^\dagger)$ respectively. Moreover, $(A, \mathcal{D}(A))$ is the Moore-Penrose inverse of $(A^\dagger, \mathcal{D}(A^\dagger))$ and $\mathcal{R}(A)$ is closed if and only if $(A^\dagger, \mathcal{D}(A^\dagger))$ is bounded. According to a fundamental result of Von Neumann (see [10] and [16]), for $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, the operators $(I + AA^*)^{-1}$ and $A^*(I + AA^*)^{-1}$ are everywhere defined and bounded. Moreover, $(I + AA^*)^{-1}$ is self-adjoint. Similarly, the operators $(I + A^*A)^{-1}$ and $A(I + A^*A)^{-1}$ are everywhere defined and bounded, and $(I + A^*A)^{-1}$ is self-adjoint. Moreover, we have the following

$$(I + AA^*)^{-1}A \subset A(I + A^*A)^{-1}$$

and

$$(I + A^*A)^{-1}A^* \subset A^*(I + AA^*)^{-1}.$$ (see [10] and [16]).

In the following, we state some identities that go back to Labrousse [16]:

**Proposition 2.1** Let $A \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$ such that $B = A^\dagger$, then

1. $A(I + A^*A)^{-1} = B^*(I + BB^*)^{-1}$;
2. $(I + A^*A)^{-1} + (I + BB^*)^{-1} = I + P_{\mathcal{N}(B^*)}$;
3. $A^*(I + AA^*)^{-1} = B(I + B^*B)^{-1}$;
4. $(I + AA^*)^{-1} + (I + B^*B)^{-1} = I + P_{\mathcal{N}(A^*)}$;
5. $(I + AA^*)^{-1} + (I + B^*B)^{-1} = I$ (if $A^*$ is injective);
6. $\mathcal{N}(A^*(I + AA^*)^{-1/2}) = \mathcal{N}(A^*) = \mathcal{N}(B)$.

Some of the results we will present in the rest of this subsection about the Moore-Penrose inverse, are stated for the first time and will prove useful throughout the rest of this paper.

**Proposition 2.2** Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces, $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B$ its Moore-Penrose inverse, then for all $x \in \mathcal{H}_1$ one has

$$\|x\|^2 = \|B^*(I + BB^*)^{-1/2}x\|^2 + \|(I + BB^*)^{-1/2}x\|^2.$$

Moreover, if $x \in \mathcal{R}(B)$ then

$$\|x\|^2 = \|(I + BB^*)^{-1/2}x\|^2 + \|(I + A^*A)^{-1/2}x\|^2.$$
Proof. The first part of the proposition was proved in [17]. Now, for \( x \in \mathcal{R}(B) = \mathcal{N}(B^*)^\perp \) where \( \mathcal{N}(B^*)^\perp \) denotes the orthogonal complement of \( \mathcal{N}(B^*) \), we have according to the fourth item of Proposition 2.1 that

\[
(I + A^*A)^{-1}x + (I + BB^*)^{-1}x = x,
\]

which implies that

\[
\|x\|_1^2 = (x, x)_1 = ((I + A^*A)^{-1}x + (I + BB^*)^{-1}x, x)_1
\]

\[
= \|(I + A^*A)^{-1}x\|_1^2 + \|(I + BB^*)^{-1}x\|_1^2. \quad \square
\]

We will extensively make use of the following result:

**Proposition 2.3** Let \( \mathcal{H}_1, \mathcal{H}_2 \) be two Hilbert spaces, \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) and \( B \) its Moore-Penrose inverse, then the operator \( B^*(I + BB^*)^{-1/2} \) is bounded, has a closed range and its Moore-Penrose inverse is given by

\[
T_B = B(I + B^*B)^{-1/2} + A^*(I + B^*B)^{-1/2}.
\]

Moreover, the adjoint operator of \( T_B \) is \( T_B^* \), where

\[
T_B^* = B^*(I + BB^*)^{-1/2} + A(I + BB^*)^{-1/2}.
\]

**Proof.** For \( x \in \mathcal{H}_1 \), we have according to Proposition 2.2 that

\[
\|x\|_1^2 = \|B^*(I + BB^*)^{-1/2}x\|_2^2 + \|(I + BB^*)^{-1/2}x\|_1^2,
\]

and if \( x \in \mathcal{R}(B) \), then

\[
\|x\|_1^2 = \|(I + BB^*)^{-1/2}x\|_1^2 + \|(I + A^*A)^{-1/2}x\|_1^2,
\]

which implies that for all \( x \in \mathcal{R}(B) = \mathcal{N}(B^*)^\perp \), we have

\[
\|B^*(I + BB^*)^{-1/2}x\|_2 = \|(I + A^*A)^{-1/2}x\|_1.
\]

Since \( A \) is bounded, it follows that \((I + A^*A)^{-1/2}\) is bounded, invertible and has a bounded inverse. Moreover, there exists a positive constant \( c \) such that for all \( x \in \mathcal{H}_1 \)

\[
c \|x\|_1 \leq \|(I + A^*A)^{-1/2}x\|_1 \leq \|x\|_1
\]

and if \( x \in \mathcal{R}(B) \),

\[
c \|x\|_1 \leq \|B^*(I + BB^*)^{-1/2}x\|_2 \leq \|x\|_1.
\]

We therefore deduce that \( B^*(I + BB^*)^{-1/2} \) has a bounded Moore-Penrose inverse, and a direct verification leads to

\[
T_BB^*(I + BB^*)^{-1/2}T_B = T_B
\]

and that

\[
B^*(I + BB^*)^{-1/2}T_BB^*(I + BB^*)^{-1/2} = B^*(I + BB^*)^{-1/2}.
\]
Moreover, we have
\[ T_B B^*(I + B B^*)^{-1/2} = B(I + B^* B)^{-1/2} T_{B^*} = P_{R(B)}, \]
and
\[ T_{B^*} B(I + B^* B)^{-1/2} = B^*(I + B B^*)^{-1/2} T_B = P_{R(B^*)}. \]
Therefore, \( T_B \) is the Moore-Penrose inverse of \( B^*(I + B B^*)^{-1/2} \). On the other hand, since
\[ (B(I + B^* B)^{-1/2})^* = B^*(I + B B^*)^{-1/2} \]
and
\[ (A^*(I + B^* B)^{-1/2})^* = A(I + B B^*)^{-1/2}, \]
we obtain that
\[ (T_B)^* = (B(I + B^* B)^{-1/2})^* + (A^*(I + B^* B)^{-1/2})^* = B^*(I + B B^*)^{-1/2} + A(I + B B^*)^{-1/2} = T_{B^*}. \]
Hence,
\[ (T_B)^* = T_{B^*}. \]
\[ \square \]

**Corollary 2.1** The operator \( B^*(I + B B^*)^{-1/2} \) is an isomorphism from \( N(B^*)^\perp \) to \( R(B^*) \).

**Corollary 2.2** The operator \( T_B \) is an isomorphism from \( R(B^*) \) to \( N(B^*)^\perp \).

The next result provides a decomposition for an arbitrary bounded operator in terms of its Moore-Penrose inverse.

**Proposition 2.4** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two Hilbert spaces, \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) and \( (B, D(B)) \) its Moore-Penrose inverse. Then, we have the following decomposition
\[ A = (I + B^* B)^{-1/2} T_{B^*}, \]
where \( T_{B^*} = B^*(I + B B^*)^{-1/2} + A(I + B B^*)^{-1/2} \).

**Proof.** A direct verification leads to
\[ (I + B^* B)^{-1/2} T_{B^*} = (I + B^* B)^{-1/2} \left( B^*(I + B B^*)^{-1/2} + A(I + B B^*)^{-1/2} \right). \]
Moreover, since
\[ (I + B^* B)^{-1/2} B^* \subset B^*(I + B B^*)^{-1/2}, \]
and that
\[ B^*(I + B B^*)^{-1} = A(I + A^* A)^{-1} \]
from the third item of Proposition 2.1, it follows that
\[
(I + B^*B)^{-1/2}T_{B^*} = B^*(I + BB^*)^{-1} + A(I + BB^*)^{-1} = A(I + A^*A)^{-1} + A(I + BB^*)^{-1} = A\left( (I + A^*A)^{-1} + (I + BB^*)^{-1} \right).
\]

Moreover, for \( x \in \mathcal{N}(A) \), we have:
\[
(I + B^*B)^{-1/2}T_{B^*}x = A\left( (I + A^*A)^{-1} + (I + BB^*)^{-1} \right)x = A(2x) = 2Ax = 0.
\]

For \( x \in \mathcal{R}(B) \), it follows according to Proposition 2.1 that
\[
(I + B^*B)^{-1/2}T_{B^*}x = Ax.
\]

Hence, for all \( x \in \mathcal{H}_1 = \mathcal{N}(A) \oplus \mathcal{R}(B) \),
\[
Ax = (I + B^*B)^{-1/2}T_{B^*}x. \quad \square
\]

Further detailed results concerning the Moore-Penrose inverse concept could be found in \([10, 16] \) and \([17] \). Another important theoretical background in Functional Analysis that will be useful in this paper is Riesz bases concept and related sequences, and most of the basic results that we will remind here are stated in \([3, 11] \) and \([24] \).

### 2.2 Riesz bases and related sequences

A sequence \((x_k)_{k \geq 1}\) in a Hilbert space \( \mathcal{H} \) is said to be complete if
\[
\text{span}(x_k)_{k \geq 1} = \mathcal{H},
\]
and minimal if each element of the sequence lies outside the closed linear span of the others, i.e.,
\[
x_j \notin \text{span}(x_k)_{k \neq j}, \forall j \in \mathbb{N}.
\]

We say that \((x_k)_{k \geq 1}\) has a biorthogonal if there exists a sequence \((z_k)_{k \geq 1}\) in \( \mathcal{H} \) such that
\[
(x_m, z_n) = \delta_{mn} \quad \text{(Kronecker's } \delta \text{ symbol)},
\]
and in this case we say that \((x_k)_{k \geq 1}\) and \((z_k)_{k \geq 1}\) are biorthogonal.

The next lemma is stated in \([3, \text{ Lemma 3.3.1}] \).

**Lemma 2.2** Let \((x_k)_{k \geq 1}\) be a sequence in a Hilbert space \( \mathcal{H} \). Then

1. \((x_k)_{k \geq 1}\) has a biorthogonal \((z_k)_{k \geq 1}\) if and only if \((x_k)_{k \geq 1}\) is minimal.
2. If a biorthogonal sequence for \((x_k)_{k \geq 1}\) exists, then it is uniquely determined if and only if \((x_k)_{k \geq 1}\) is complete in \( \mathcal{H} \).
A sequence \((x_k)_{k \geq 1}\) is called a Bessel sequence if there exists a constant \(b > 0\) such that

\[
\sum_{k=1}^{\infty} |(x, x_k)_\mathcal{H}|^2 \leq b \|x\|^2, \quad \forall x \in \mathcal{H}.
\]

The constant \(b\) is called a Bessel bound or an upper bound for \((x_k)_{k \geq 1}\), and the smallest upper bound \(b\) for \((x_k)_{k \geq 1}\), will be denoted \(b_X\). The following lemma stated in [11, Theorem 7.4], characterizes all Bessel sequences for \(\mathcal{H}\) starting with one orthonormal basis.

**Lemma 2.3** Let \((w_k)_{k \geq 1}\) be an orthonormal basis for \(\mathcal{H}\). Then the Bessel sequences for \(\mathcal{H}\) are precisely the sequences \((Uw_k)_{k \geq 1}\), where \(U\) is a bounded linear operator on \(\mathcal{H}\).

A sequence \((x_k)_{k \geq 1}\) in a Hilbert space \(\mathcal{H}\) is said to be a Riesz basis for \(\mathcal{H}\) if there exists an orthonormal basis \((w_k)_{k \geq 1}\) for \(\mathcal{H}\) and an isomorphism \(U\) on \(\mathcal{H}\) such that

\[
\forall k \geq 1, \quad x_k = Uw_k.
\]

The next theorem stated in [3, Theorem 3.6.6], gives equivalent conditions for \((x_k)_{k \geq 1}\) being a Riesz basis.

**Theorem 2.2** For a sequence \((x_k)_{k \geq 1}\) in a Hilbert space \(\mathcal{H}\), the following statements are equivalent:

1. \((x_k)_{k \geq 1}\) is a Riesz basis for \(\mathcal{H}\).

2. \((x_k)_{k \geq 1}\) is complete in \(\mathcal{H}\) and there exist \(a, b > 0\) such that for all finite scalar sequence \((c_k)_{k \geq 1}\)

\[
a \sum_{k=1}^{\infty} |c_k|^2 \leq \sum_{k=1}^{\infty} \|c_kx_k\|^2 \leq b \sum_{k=1}^{\infty} |c_k|^2.
\]

3. \((x_k)_{k \geq 1}\) is a complete Bessel sequence, and has a complete biorthogonal sequence \((y_k)_{k \geq 1}\) which is also a Bessel sequence.

For a given sequence \(X = (x_k)_{k \geq 1}\) in \(\mathcal{H}\), let us introduce some related operators. The synthesis operator associated with \(X = (x_k)_{k \geq 1}\) is defined as follows:

\[
\mathcal{D}(S_X) = \{(c_k)_{k \geq 1} \in \ell^2(\mathbb{N}^*) / \sum_k c_k x_k \text{ converges}\},
\]

and for \((c_k)_{k \geq 1} \in \mathcal{D}(S_X),

\[
S_X(c_k)_{k \geq 1} = \sum_{k=1}^{\infty} c_k x_k.
\]

Since the finite sequences are dense in \(\ell^2(\mathbb{N}^*)\) and contained in \(\mathcal{D}(S_X)\), the synthesis operator \(S_X\) is densely defined. The analysis operator associated with the sequence \(X = (x_k)_{k \geq 1}\) is defined by

\[
\mathcal{D}(A_X) = \{x \in \mathcal{H} / ((x, x_k)_\mathcal{H})_{k \geq 1} \in \ell^2(\mathbb{N}^*)\},
\]

and for \(x \in \mathcal{D}(A_X),

\[
A_X x = ((x, x_k)_\mathcal{H})_{k \geq 1}.
\]

The following lemma is stated in [11, Theorem 7.4].
Lemma 2.4 Let \((x_k)_{k \geq 1}\) be a sequence in \(\mathcal{H}\). Then, \((x_k)_{k \geq 1}\) is a Bessel sequence if and only if the associated synthesis operator is bounded.

The next lemma is stated in [2, Lemma 3.1] and [3, Lemma 8.4.2].

Lemma 2.5 Let \(\mathcal{H}\) be a Hilbert space and \(X = (x_n)_{n \geq 1}\) an arbitrary sequence in \(\mathcal{H}\). Then, the following hold

1. The analysis operator \(A_X\) is closed.

2. If the analysis operator \(A_X\) is densely defined, then the adjoint operator \(A_X^*\) is an extension of the synthesis operator \(S_X\), i.e., \(S_X \subset A_X^*\).

Note that if \((x_k)_{k \geq 1}\) is an orthonormal basis for \(\mathcal{H}\), the associated analysis operator \(A_X\) is a unitary isomorphism. In the case \((x_k)_{k \geq 1}\) is a Bessel sequence, \(A_X\) is bounded and

\[
S_X = A_X^*,
\]

where \(S_X\) is the synthesis operator associated with \((x_k)_{k \geq 1}\).

The rest of this section concerns some basic results for Sobolev spaces in Lipschitz domains.

2.3 Sobolev spaces in Lipschitz domains

Throughout this section, \(\Omega\) is an open subset of \(\mathbb{R}^d\), \(d = 1, 2, 3, \ldots\), \(\partial\Omega\) its boundary and \(\overline{\Omega}\) its closure. \(C^k(\Omega)\) denotes the space of functions mapping \(\Omega\) into \(\mathbb{C}\) such that all partial derivatives up to order \(k\) are continuous, where \(k \in \mathbb{Z}_+\) and we denote by \(C^k(Q)\), for \(Q\) a closed subset of \(\mathbb{R}^d\), the space of restrictions to \(Q\) of all functions in \(C^k(\mathbb{R}^d)\).

Consider the multi-index \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d\). We define \(|\alpha| = \sum_{k=1}^d \alpha_k\). For \(f \in C^m(\Omega)\) and \(|\alpha| \leq m\), we define

\[
\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f.
\]

If \(Q \subset \mathbb{R}^d\) is compact, we may equip \(C^k(Q)\) with the norm

\[
\|\varphi\|_{C^k(Q)} = \sup_{x \in Q, |\alpha| \leq k} |(\partial^\alpha \varphi)(x)|.
\]

We denote by \(C^\infty(\Omega)\) for closed \(Q \subset \mathbb{R}^d\), the intersection of all \(C^k(Q)\), for \(k \in \mathbb{Z}_+\). The closure of the set \(\{x \in \Omega \mid f(x) \neq 0\}\) where \(f \in C(\Omega)\), is called the support of \(f\) and denoted \(\text{supp} f\). A function \(f \in C^\infty(\Omega)\) is said to be a test function if \(\text{supp} f\) is a compact subset of \(\Omega\) and the set of all test functions on \(\Omega\) is denoted by \(C^\infty_c(\Omega)\). For a sequence \((\varphi_n)_{n \geq 1}\) in \(C^\infty_c(\Omega)\) and \(\varphi \in C^\infty_c(\Omega)\), we say that \((\varphi_n)_{n \geq 1}\) converges to \(\varphi\) in \(C^\infty_c(\Omega)\) if there exists a compact \(Q \subset \Omega\) such that for all \(n \geq 1\) \(\text{supp}(\varphi_n) \subset Q\) and for all multi-index \(\alpha \in \mathbb{Z}_+^d\), the sequence \((\partial^\alpha \varphi_n)_{n \geq 1}\) converges uniformly to \(\partial^\alpha \varphi\). The space \(C^\infty_c(\Omega)\) induced by this convergence is denoted \(\mathcal{D}(\Omega)\). Moreover, the action of a linear map \(u : \mathcal{D}(\Omega) \rightarrow \mathbb{C}\) on the test function \(\varphi\) is denoted by \(\langle u, \varphi \rangle\).

A distribution on \(\Omega\) is a linear map \(u : \mathcal{D}(\Omega) \rightarrow \mathbb{C}\) such that for all compact \(Q \subset \Omega\), there exists \(m \in \mathbb{Z}_+\) and \(c \geq 0\) such that

\[
|\langle u, \varphi \rangle| \leq c \|\varphi\|_{C^m(Q)} \quad \forall \varphi \in \mathcal{D}(\Omega),
\]
where \( m \) and \( c \) may depend on \( Q \). We denote by \( \mathcal{D}'(\Omega) \) the vector space of distributions on \( \Omega \). For \( u \in \mathcal{D}'(\Omega) \) a distribution, one can define its partial derivative with respect to \( x_i \) to be the distribution \( \frac{\partial u}{\partial x_j} \), specified by

\[
< \frac{\partial u}{\partial x_j}, \varphi > = - < u, \frac{\partial \varphi}{\partial x_j} > \quad \forall \varphi \in \mathcal{D}(\Omega).
\]

Once the derivative has been defined, it will be easy to define recursively higher derivatives by induction, i.e.,

\[
\frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_j} \right).
\]

We denote by \( H^k(\Omega) \) the Sobolev space of all distributions \( u \) defined on \( \Omega \) such that all partial derivatives of order at most \( k \) lie in \( L^2(\Omega) \), i.e.,

\[
\partial^{\alpha}u \in L^2(\Omega), \quad \forall |\alpha| \leq k.
\]

\( H^k(\Omega) \) equipped with the norm

\[
\|u\|_{k,\Omega} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |\partial^{\alpha}u|^2 \, dx \right)^{1/2},
\]

associated with the inner product

\[
(u,v)_{k,\Omega} = \sum_{|\alpha| \leq k} \int_{\Omega} \partial^{\alpha}u \, \overline{\partial^{\alpha}v} \, dx, \quad \forall u, v \in H^k(\Omega)
\]

is a Hilbert space, where \( \overline{\partial^{\alpha}v} \) is the conjugate of \( \partial^{\alpha}v \). Sobolev spaces \( H^s(\Omega) \) for non-integer \( s \) are defined by the real interpolation method (see [1], [18] and [22]).

**Definition 2.1** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with boundary \( \partial \Omega \) and closure \( \overline{\Omega} \). We say that \( \partial \Omega \) is Lipschitz continuous if for every \( x \in \partial \Omega \) there exists a coordinate system \((\widehat{y}, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}\), a neighborhood \( Q_{\delta,\delta'}(x) \) of \( x \) and a Lipschitz function \( \gamma_x : \widehat{Q}_\delta \rightarrow \mathbb{R} \) with the following properties:

1. \( \Omega \cap Q_{\delta,\delta'}(x) = \{(\widehat{y}, y_d) \in Q_{\delta,\delta'}(x) / \gamma_x(\widehat{x}) < y_d\} \);
2. \( \partial \Omega \cap Q_{\delta,\delta'}(x) = \{(\widehat{y}, y_d) \in Q_{\delta,\delta'}(x) / \gamma_x(\widehat{x}) = y_d\} \);

where

\[
Q_{\delta,\delta'}(x) = \{(\widehat{y}, y_d) \in \mathbb{R}^d / \|\widehat{y} - \widehat{x}\|_{\mathbb{R}^{d-1}} < \delta \text{ and } |y_d - x_d| < \delta' \}
\]

and

\[
\widehat{Q}_\delta(x) = \{\widehat{y} \in \mathbb{R}^{d-1} / \|\widehat{y} - \widehat{x}\|_{\mathbb{R}^{d-1}} < \delta\}
\]

for \( \delta, \delta' > 0 \). An open connected subset \( \Omega \subset \mathbb{R}^d \) whose boundary is Lipschitz continuous is called a Lipschitz domain.

If \( \Omega \) is a Lipschitz hypograph, then according to Mclean [18], we can construct Sobolev spaces on its boundary \( \partial \Omega \) in terms of Sobolev spaces on \( \mathbb{R}^{d-1} \), as follows. For \( g \in L^2(\partial \Omega) \), we define

\[
g_x(\widehat{x}) = g(\widehat{x}, \gamma(\widehat{x})) \text{ for } \widehat{x} \in \mathbb{R}^{d-1},
\]
put
\[ H^s(\partial \Omega) = \{ g \in L^2(\partial \Omega) \mid g_\gamma \in H^s(\mathbb{R}^{d-1}) \text{ for } 0 \leq s \leq 1 \}, \]
and equip this space with the inner product
\[ (g, y)_{s, \partial \Omega} = (g_\gamma, y_\gamma)_{s, \mathbb{R}^{d-1}}. \]
where
\[ (u, v)_{s, \mathbb{R}^{d-1}} = \int_{\mathbb{R}^{d-1}} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi. \]
Recalling that any Lipschitz function is almost everywhere differentiable so, any Lipschitz hypograph \( \Omega \) has a surface measure \( \sigma \), and an outward unit normal \( \nu \) that exists \( \sigma \)-almost everywhere on \( \partial \Omega \). If \( \Omega \) is a Lipschitz hypograph then
\[ d\sigma(x) = \sqrt{1 + \|\nabla \gamma(\hat{x})\|^2_{\mathbb{R}^{d-1}}} \, d\hat{x} \]
and
\[ \nu(x) = \frac{(-\nabla \gamma(\hat{x}), 1)}{\sqrt{1 + \|\nabla \gamma(\hat{x})\|^2_{\mathbb{R}^{d-1}}}} \]
for almost every \( x \in \partial \Omega \).
Suppose now that \( \Omega \) is a Lipschitz domain. Since \( \partial \Omega \subset \bigcup_{x \in \partial \Omega} Q_{\delta, \delta'}(x) \) and that \( \partial \Omega \) is compact, there exist then \( x^1, x^2, \ldots, x^n \in \partial \Omega \) such that
\[ \partial \Omega \subset \bigcup_{j=1}^n Q_{\delta, \delta'}(x^j). \]
It follows that the family \( (W_j) = (Q_{\delta, \delta'}(x^j)) \) is a finite open cover of \( \partial \Omega \), i.e., each \( W_j \) is an open subset of \( \mathbb{R}^d \), and \( \partial \Omega \subset \bigcup_j W_j \).
Let \( (\varphi_j) \) be a partition of unity subordinate to the open cover \( (W_j) \) of \( \partial \Omega \), i.e.,
\[ \varphi_j \in \mathcal{D}(W_j) \quad \text{and} \quad \sum_j \varphi_j(x) = 1 \quad \text{for all } x \in \partial \Omega. \]
The inner product in \( H^s(\partial \Omega) \) is then defined by
\[ (u, v)_{H^s(\partial \Omega)} = \sum_j (\varphi_j u, \varphi_j v)_{H^s(\partial \Omega_j)}, \]
where \( \Omega_j \) can be transformed to a Lipschitz hypograph by a rigid motion, i.e., by a rotation plus a translation and satisfies
\[ W_j \cap \Omega = W_j \cap \Omega_j \text{ for each } j. \]
It is interesting to mention that a different choice of \( (W_j), (\Omega_j) \) and \( (\varphi_j) \) would yield the same space \( H^s(\partial \Omega) \) with an equivalent norm, for \( 0 \leq s \leq 1 \). For further lectures see ([11] and [18]).

The following lemmas are stated in ([9] and [19]).

**Lemma 2.6** For a bounded Lipschitz domain \( \Omega \) with boundary \( \partial \Omega \), the space \( H^{1/2}(\partial \Omega) \) is dense in \( L^2(\partial \Omega) \).
Lemma 2.7 Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Then, the space $H^s(\Omega)$ is compactly imbedded in $H^{s'}(\Omega)$ for all $s' < s$ in $\mathbb{R}$.

Definition 2.2 For a bounded Lipschitz domain with boundary $\partial\Omega$, the space $H^{-1/2}(\partial\Omega)$ is the dual space of $H^{1/2}(\partial\Omega)$.

Several mathematicians contributed to the study of the trace spaces in Lipschitz domains, most notably Gagliardo on $W^{1,p}(\Omega)$ for $1 \leq p \leq +\infty$ (see [8]) and Costabel on $H^s(\Omega)$ for $\frac{1}{2} < s < \frac{3}{2}$ (see [6]).

Throughout the rest of this paper, $\Omega$ denotes a bounded Lipschitz domain of $\mathbb{R}^d$.

3 The main key ingredients

Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^d$, $d \geq 2$. The trace map takes each continuous function $u$ on $\overline{\Omega}$ to its restriction on $\partial\Omega$. Under the condition $\Omega$ is a bounded Lipschitz domain, this trace map may be extended to be a continuous surjective operator denoted $\Gamma_s$ from $H^s(\Omega)$ to $H^{s-1/2}(\partial\Omega)$, for $\frac{1}{2} < s < \frac{3}{2}$ (see [1], [6], [18] and [19]). The range space and the null space of $\Gamma_s$ are respectively given by

$$R(\Gamma_s) = H^{s-1/2}(\partial\Omega) \text{ and } N(\Gamma_s) = H^s_0(\Omega),$$

where $H^s_0(\Omega)$ is the closure in $H^s(\Omega)$ of infinitely differentiable functions compactly supported in $\Omega$.

Let us set $\Gamma = T_1\Gamma_1$, where $\Gamma_1$ is the trace operator from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$ and $T_1$ is the embedding operator from $H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$. According to Gagliardo (see [8]), it follows that $R(\Gamma) = H^{1/2}(\partial\Omega)$ and $N(\Gamma) = H^1_0(\Omega)$. Since $\Gamma_1$ is bounded and $T_1$ is compact (see [19]), $\Gamma$ is compact. Moreover, since $R(\Gamma)$ is dense in $L^2(\partial\Omega)$, we have the following lemma:

Lemma 3.1 Let $\Gamma$ be the trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$. Then, the adjoint operator $\Gamma^*$ is injective and compact.

Now, we induce $H^1(\Omega)$ by the following inner product

$$(u,v)_{\partial\Omega} = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\partial\Omega} \Gamma u \Gamma v \, d\sigma \quad \forall \ u, v \in H^1(\Omega). \quad (6)$$

The associated norm $\|\cdot\|_{\partial\Omega}$ is given by

$$\|u\|_{\partial\Omega} = \left( \|
abla u\|^2_{L^2(\Omega)} + \|\Gamma u\|^2_{L^2(\partial\Omega)} \right)^{1/2}, \quad (7)$$

and $H^1(\Omega)$ induced with the inner product $(.,.)_{\partial\Omega}$ will be denoted $H^1_\partial(\Omega)$. For $v \in C^1(\overline{\Omega})$ the normal derivative map $\partial_\nu, v$ maps each $v$ to $\partial_\nu v = \nu \cdot (\nabla v)|_{\partial\Omega}$ onto $L^2(\partial\Omega)$. Moreover, under the condition $\Omega$ is a bounded Lipschitz domain, $\partial_\nu, v$ may be extended to be a bounded linear operator denoted $\hat{\partial}_\nu$ from $H^1_\Delta(\Omega)$ to $H^{-1/2}(\partial\Omega)$ (see [9]). In the following, we recall Green’s formula (see [9] and [19]).
Proposition 3.1 "Green’s formula" Let \( \Omega \) be a bounded Lipschitz domain, then for all \( u \in H^1_\Delta(\Omega) \) and \( v \in H^1(\Omega) \) one has

\[
\int_\Omega \nabla u \nabla v dx = - \int_\Omega \Delta u \, Ey \, dx + \langle \hat{\partial}_\nu u, \Gamma_1 v \rangle,
\]

where \( E \) is the embedding operator from \( H^1(\Omega) \) into \( L^2(\Omega) \) and \( \langle ., . \rangle \) is the duality pairing between \( H^{-1/2}(\partial \Omega) \) and \( H^{1/2}(\partial \Omega) \).

The following proposition characterizes \( \Gamma^* \).

Proposition 3.2 For all \( g \in L^2(\partial \Omega) \), \( \Gamma^* \) is the solution operator of Robin problem for the following Laplace equation

\[
\begin{cases}
\Delta z = 0 & (\Omega) \\
\partial_\nu z + \Gamma z = g & (\partial \Omega),
\end{cases}
\]

where \( \partial_\nu \) is the normal derivative operator considered as non-bounded from \( H^1_\Delta(\Omega) \) to \( L^2(\partial \Omega) \).

Proof. Let \( g \in L^2(\partial \Omega) \) and \( z = \Gamma^* g \). We have:

\[
(\Gamma^* g, v)_{\partial \Omega} = \int_\Omega \nabla z \nabla v dx + \int_{\partial \Omega} \Gamma z \Gamma v d\sigma \quad (*)
\]

so that if \( v \in H^1_0(\Omega) = N(\Gamma) \), then we obtain

\[
\int_\Omega \nabla z \nabla v \ dx = 0.
\]

Since the previous equality characterizes the \( H^1 \)-harmonic functions, then we may write:

\[
\Delta z = 0 \text{ in } \mathcal{D}'(\Omega).
\]

Applying Green’s formula to (*), we obtain that

\[
\int_\Omega \nabla z \nabla v dx + \int_{\partial \Omega} \Gamma z \Gamma v d\sigma = \langle \hat{\partial}_\nu z, \Gamma_1 v \rangle + \int_{\partial \Omega} \Gamma z \Gamma v d\sigma
\]

\[
= \int_{\partial \Omega} g \Gamma v d\sigma,
\]

which leads to the following duality pairing on \( H^{1/2}(\partial \Omega) \times H^{-1/2}(\partial \Omega) \)

\[
\langle \hat{\partial}_\nu z + \hat{\Gamma} z, \Gamma_1 v \rangle = \langle \hat{g}, \Gamma_1 v \rangle,
\]

where \( \hat{y} \) denotes the embedding of an element \( y \in L^2(\partial \Omega) \) in \( H^{-1/2}(\partial \Omega) \).

Viewing \( \mathcal{R}(\Gamma_1) = H^{1/2}(\partial \Omega) \), it follows that

\[
\hat{\partial}_\nu z + \hat{\Gamma} z = \hat{g},
\]

15
\[ \partial_\nu z = g - \Gamma z. \]

Consequently, \( \partial_\nu z \) belongs to the range of the embedding operator from \( L^2(\partial\Omega) \) into \( H^{-1/2}(\partial\Omega) \), which means that \( \partial_\nu z \in L^2(\partial\Omega) \) and that
\[ \partial_\nu z + \Gamma z = g. \]

The trace operator \( \Gamma \) being bounded, one considers its Moore-Penrose inverse which we denote by \( \Lambda = \Gamma^\dagger \in \mathcal{C}(L^2(\partial\Omega), H^1_0(\Omega)) \), such that
\[ \mathcal{D}(\Lambda) = \mathcal{R}(\Gamma) = H^{1/2}(\partial\Omega) \text{ and } \mathcal{N}(\Lambda^*) = \mathcal{N}(\Gamma) = H^1_0(\Omega). \]

Moreover, \( \Lambda \) is characterized by the following.

**Proposition 3.3** Let \( \Gamma \) be the trace operator from \( H^1_0(\Omega) \) into \( L^2(\partial\Omega) \) and \( \Lambda \) its Moore-Penrose inverse. Then, \( \Lambda \) is the solution operator of the Dirichlet problem for the Laplace equation with data in \( H^{1/2}(\partial\Omega) \). Moreover, we have:
\[ \mathcal{R}(\Lambda) = \mathcal{H}^1(\Omega), \]
where \( \mathcal{H}^1(\Omega) = \{ v \in H^1(\Omega) \mid \Delta v = 0 \text{ in } \mathcal{D}'(\Omega) \} \).

**Proof.** Since \( \Gamma \) is bounded, it follows that its Moore-Penrose inverse \( \Lambda \) is closed and densely defined with closed range. Moreover, from Lemma 3.1, \( \Gamma^* \) is injective, which implies that \( \mathcal{D}(\Lambda) = \mathcal{R}(\Gamma) \). Also, for \( g \in \mathcal{D}(\Lambda) \) let \( v = \Lambda g \). For \( w \in \mathcal{D}(\Lambda^*) \), we have
\[ (v, w)_{\mathcal{D}, \Omega} = \int_\Omega \nabla v \nabla w dx + \int_{\partial\Omega} \Gamma v \Gamma w d\sigma = (\Lambda g, w)_{\mathcal{D}, \Omega} = \int_{\partial\Omega} g \Lambda^* w d\sigma, \]
so that if \( w \in \mathcal{N}(\Lambda^*) = \mathcal{N}(\Gamma) = H^1_0(\Omega) \), the following holds
\[ \int_\Omega \nabla v \nabla w dx = 0. \]

Since the previous equality holds for all \( w \in H^1_0(\Omega) \) and characterizes the \( H^1 \)–harmonic functions, it follows that
\[ \begin{cases} \Delta v = 0 & (\Omega) \\ \Gamma v = g & (\partial\Omega). \end{cases} \]

We now consider the embedding operator:
\[ E : H^1_0(\Omega) \rightarrow L^2(\Omega) \]
\[ v \mapsto Ev \]
which maps each \( v \in H^1(\Omega) \) to itself into \( L^2(\Omega) \) but obviously with different topologies. \( H^1(\Omega) \) is induced with the inner product \( (.,.)_{\mathcal{D}, \Omega} \) and \( L^2(\Omega) \) with its usual inner product \( (.,.)_{0, \Omega} \). The following theorem characterizes \( E^* \).
Theorem 3.1 Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^d$ and $E$ the embedding operator from $H^1_0(\Omega)$ into $L^2(\Omega)$. Then for $f \in L^2(\Omega)$, the adjoint operator $E^*$ is the solution operator of Robin problem for the following Poisson equation

$$\begin{cases}
-\Delta u = f & (\Omega) \\
\partial_\nu u + \Gamma u = 0 & (\partial\Omega).
\end{cases}$$

(8)

Proof. Let $f \in L^2(\Omega)$ and $v \in H^1(\Omega)$. Putting $u = E^* f$, one has

$$\int_\Omega fEvdx = (E^* f, v)_{\partial\Omega} = \int_\Omega \nabla u\nabla v \, dx + \int_{\partial\Omega} \Gamma u \Gamma v \, d\sigma.$$  

(9)

Now, if $v \in C^\infty_c(\Omega)$ then,

$$(E^* f, v)_{\partial\Omega} = \int_\Omega fEvdx = \int_\Omega \nabla v\nabla u \, dx = \langle -\Delta u, v \rangle_{\mathscr{D}'(\Omega), \mathscr{D}(\Omega)}.$$  

Therefore,

$$f = -\Delta u \quad \text{in} \quad \mathscr{D}'(\Omega).$$

Applying Green’s formula to (9), one has

$$\int_\Omega fEvdx = -\int_\Omega Ev \Delta u \, dx + \langle \partial_\nu u, \Gamma_1 v \rangle + \int_{\partial\Omega} \Gamma v \Gamma u \, d\sigma$$  

$$= \int_\Omega fEvdx + \langle \partial_\nu u + \Gamma u, \Gamma_1 v \rangle,$$  

so far,

$$\langle \partial_\nu u + \Gamma u, \Gamma_1 v \rangle = 0 \quad \forall v \in H^1(\Omega).$$

Moreover, since $\mathcal{R}(\Gamma_1) = H^{1/2}(\partial\Omega)$, it follows that

$$\partial_\nu u + \Gamma u = 0 \quad \text{in} \quad H^{-1/2}(\partial\Omega).$$

Consequently, $\partial_\nu u$ belongs to the range of the embedding operator acting from $L^2(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ and $\partial_\nu u \in L^2(\partial\Omega)$, which implies that

$$\partial_\nu u + \Gamma u = 0 \quad \text{in} \quad L^2(\partial\Omega). \quad \square$$

Now, for $f \in L^2(\Omega)$, let us consider $E_0^*$ the solution operator of the Dirichlet problem for the following Poisson equation

$$\begin{cases}
-\Delta u^0 = f & (\Omega) \\
\Gamma u^0 = 0 & (\partial\Omega)
\end{cases}.$$
By setting $E_1^* = E^* - E_0^*$ and $u^1 = E_1^* f$, it follows that $u^1$ is a solution of the Dirichlet problem for the following Laplace equation

$$
\begin{cases}
-\Delta u^1 = 0 & (\Omega) \\
\Gamma u^1 = \Gamma u & (\partial \Omega),
\end{cases}
$$

where $u$ is the solution of (8). Furthermore, using Rellich-Nečas Lemma, we can prove the following crucial theorem:

**Theorem 3.2** Let $\Gamma$ be the trace operator from $H^1_\partial(\Omega)$ into $L^2(\partial\Omega)$ and $E_1^*$ defined as above. Then, we have $E_1^* = \Gamma^* K^*$, where $K$ is the solution operator of the Dirichlet problem for the Laplace equation (1).

**Proof.** Putting $u^1 = E_1^* f$, we have $u^1 = u - u^0$, where $u^0$ and $u$ are solutions of the followings problems

$$
\begin{cases}
-\Delta u^0 = f & (\Omega) \\
\Gamma u^0 = 0 & (\partial\Omega)
\end{cases}
$$

and

$$
\begin{cases}
-\Delta u = f & (\Omega) \\
\partial_\nu u + \Gamma u = 0 & (\partial\Omega)
\end{cases}
$$

respectively. Since $u^1, u^0 \in H^1_\Delta(\Omega)$, it follows that

$$
\widehat{\partial_\nu u^1}, \widehat{\partial_\nu u^0} \in H^{-1/2}(\partial\Omega),
$$

therefore,

$$
\partial_\nu u = \widehat{\partial_\nu u^0} + \widehat{\partial_\nu u^1} \in H^{-1/2}(\partial\Omega).
$$

On the one hand, $\partial_\nu u = -\Gamma u$ implies that $\partial_\nu u \in L^2(\partial\Omega)$, and by Rellich-Nečas Lemma, we have $\partial_\nu u^0 \in L^2(\partial\Omega)$. Which implies that $\partial_\nu u^1 \in L^2(\partial\Omega)$. On the other hand, since

$$
\partial_\nu u^1 + \Gamma u^1 = -\partial_\nu u^0 - \partial_\nu u = -\partial_\nu u^0,
$$

and that the adjoint operator $K^*$ takes each $f \in L^2(\partial\Omega)$ to $-\partial_\nu u^0$ onto $L^2(\partial\Omega)$, where $u^0$ is the solution of the Dirichlet problem for the Poisson equation (3), it follows that

$$
\partial_\nu u^1 + \Gamma u^1 = K^* f,
$$

and that $u^1$ is the unique solution of

$$
\begin{cases}
-\Delta u^1 = f & (\Omega) \\
\partial_\nu u^1 + \Gamma u^1 = -\partial_\nu u^0 & (\partial\Omega).
\end{cases}
$$

Therefore,

$$
E_1^* = \Gamma^* K^*. \quad \square
$$
Now, the operator $E_1$ being bounded, one considers its Moore-Penrose inverse which we denote by $F_1$ such that
\[
\mathcal{D}(F_1) = \mathcal{R}(E_1) \oplus \mathcal{N}(E_1^*)
\]
and
\[
\mathcal{R}(F_1) = \mathcal{H}^1(\Omega) \text{ and } \mathcal{N}(F_1) = \mathcal{N}(E_1^*) = H_0^1(\Omega).
\]
According to Proposition 2.3, the operator $F_1^*(I + F_1F_1^*)^{-1/2}$ acting from $H^1(\Omega)$ into $L^2(\Omega)$ is bounded with closed range, i.e.,
\[
\mathcal{R}(F_1^*(I + F_1F_1^*)^{-1/2}) = \mathcal{R}(F_1^*) = \mathcal{H}(\Omega)
\]
and
\[
\mathcal{N}(F_1^*(I + F_1F_1^*)^{-1/2}) = \mathcal{N}(F_1^*) = H_0^1(\Omega),
\]
and from Corollary 2.1, we have the following lemma.

**Lemma 3.2** The operator $F_1^*(I + F_1F_1^*)^{-1/2}$ is an isomorphism from $\mathcal{H}^1(\Omega)$ into $\mathcal{H}(\Omega)$.

Let us now set
\[
\Gamma_0^* = F_1^*(I + F_1F_1^*)^{-1/2}\Gamma^*.
\]

The following lemma characterizes $\Gamma_0^*$.

**Lemma 3.3** The operator $\Gamma_0^*$ defined above is compact and injective.

**Proof.** Knowing that $\mathcal{R}(F_1^*(I + F_1F_1^*)^{-1/2}) = \mathcal{H}(\Omega)$, we have $\mathcal{R}(\Gamma_0^*) \subset \mathcal{H}(\Omega)$. Moreover, $\Gamma$ being compact (see §3), it follows by Schauder’s theorem (Theorem 2.1) that $\Gamma^*$ is compact as well. Therefore, the boundedness of $F_1^*(I + F_1F_1^*)^{-1/2}$ implies that $\Gamma_0^*$ is compact. The injectivity of $\Gamma_0^*$ holds for the reason that $\Gamma^*$ is injective and that $\mathcal{R}(\Gamma^*) \subset \mathcal{N}(F_1^*(I + F_1F_1^*)^{-1/2})^\perp$. \[\square\]

Let us now return to the Dirichlet problem for the Laplace equation (1), where we have considered its solution operator $K$ and its adjoint $K^*$. Composing $\Gamma_0^*$ by $K^*$, we obtain
\[
\Gamma_0^*K^* = F_1^*(I + F_1F_1^*)^{-1/2}\Gamma^* K^*,
\]
and in view of Theorem 3.2, we have $E_1^* = \Gamma^* K^*$, which leads to
\[
\Gamma_0^*K^* = F_1^*(I + F_1F_1^*)^{-1/2}E_1^*.
\]

On the other hand, since $(I + F_1^*F_1)^{-1/2}F_1^* \subset F_1^*(I + F_1F_1^*)^{-1/2}$ and that $\mathcal{R}(E_1^*) \subset \mathcal{D}(F_1^*)$, one has
\[
\Gamma_0^*K^* = (I + F_1^*F_1)^{-1/2}F_1^*E_1^*,
\]
and viewing $F_1^*E_1^* = P_{\mathcal{H}(\Omega)}$, it follows that
\[
\Gamma_0^*K^* = (I + F_1^*)^{-1/2}P_{\mathcal{H}(\Omega)}.
\]

Therefore,
\[
\mathcal{R}(\Gamma_0^*K^*) = \mathcal{R}((I + F_1^*)^{-1/2}P_{\mathcal{H}(\Omega)}).
\]

Viewing $\Gamma_0^*$ is compact, $K \in \mathcal{B}(L^2(\partial\Omega), L^2(\Omega))$ and that $(I + F_1^*F_1)^{-1/2}P_{\mathcal{H}(\Omega)}$ is self-adjoint, it follows that $\Gamma_0^*K^*$ is compact and self-adjoint. Therefore, there exists a sequence $((\kappa_n, \phi_n))_{n \geq 1}$ in $\mathbb{R}^+_* \times \mathcal{H}(\Omega)$ such that for all $n \geq 1$,
\[
\Gamma_0^*K^*\phi_n = \kappa_n^2\phi_n.
\]

Moreover, the sequence $(\phi_n)_{n \geq 1}$ is an orthonormal basis for the Bergman space $\mathcal{H}(\Omega)$.

The aim of the next section is to prove the main result of this paper (Theorem 1.1).
4 The sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\)

As we stated in the first section of this paper, the sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are defined for all \(n \geq 1\), by

\[
\Gamma_0^* \phi_n = \kappa_n y_n \text{ and } K^* \phi_n = \kappa_n g_n,
\]

where \(((\kappa_n, \phi_n))_{n \geq 1}\) in \(\mathbb{R}^*_+ \times \mathcal{H}(\Omega)\) is a sequence of couples associated to \(\Gamma_0^* K^*\). A first remark is that \(\Gamma_0^*\) and \(K\) satisfy the following

\[
\Gamma_0^* g_n = \kappa_n \phi_n = K y_n.
\]

Denote by \(G(\partial \Omega)\) and \(Y(\partial \Omega)\) the closures of \(\text{span}(g_n)_{n \geq 1}\) and \(\text{span}(y_n)_{n \geq 1}\) respectively. The principal objective of this section will be to prove that the sequences \((g_n)_{n}\) and \((y_n)_{n}\) are Riesz bases. This will be the key ingredient to prove the \(H^{1/2}\) regularity result for the problem (1), stated in Theorem 1.2.

**Lemma 4.1** The sequences \((y_n)_{n \geq 1}\) and \((g_n)_{n \geq 1}\) are biorthogonal and \((y_n)_{n \geq 1}\) is complete.

**Proof.** Let \(m, n \geq 1\). We have

\[
(g_n, y_m)_{0, \partial \Omega} = \frac{1}{\kappa_n \kappa_m} (\kappa_n g_n, \kappa_m y_m)_{0, \partial \Omega}
\]

\[
= \frac{1}{\kappa_n \kappa_m} (K^* \phi_n, \Gamma_0 \phi_m)_{0, \partial \Omega}
\]

\[
= \frac{1}{\kappa_n \kappa_m} (\Gamma_0^* K^* \phi_n, \phi_m)_{0, \Omega}
\]

\[
= \frac{\kappa_n^2}{\kappa_n \kappa_m} (\phi_n, \phi_m)_{0, \Omega}.
\]

Since \((\phi_n)_{n \geq 1}\) is an orthonormal basis for \(\mathcal{H}(\Omega)\), it follows that \((g_n, y_m)_{0, \partial \Omega} = \delta_{nm}\), therefore \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are biorthogonal. To prove that \((y_n)_{n \geq 1}\) is complete, one standard way is to consider an element \(g \in L^2(\partial \Omega)\) such that for all \(n \geq 1\),

\[
(g, y_n)_{0, \partial \Omega} = 0
\]

and prove that \(g = 0\). Multiplying (10) by \(\kappa_n\), it follows that

\[
0 = \kappa_n (g, y_n)_{0, \partial \Omega} = (g, \kappa_n y_n)_{0, \partial \Omega} = (g, \Gamma_0 \phi_n)_{0, \partial \Omega} = (\Gamma_0^* g, \phi_n)_{0, \Omega}
\]

therefore, \((\Gamma_0^* g, \phi_n)_{0, \Omega} = 0\) for all \(n \geq 1\), and since \((\phi_n)_{n \geq 1}\) is an orthonormal basis for \(\mathcal{H}(\Omega)\) and that \(\Gamma_0^* g \in \mathcal{H}(\Omega)\), we obtain that \(\Gamma_0^* g = 0\) which implies that \(g = 0\) according to the injectivity of \(\Gamma_0^*\) from Lemma 3.3. □

The following corollary is a consequence of Lemma 4.1 and Lemma 2.2.

**Corollary 4.1** The sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are minimal.

**Lemma 4.2** \(\mathcal{R}(K^*)\) and \(\mathcal{R}(\Gamma_0)\) are densely imbedded in \(G(\partial \Omega)\) and \(Y(\partial \Omega)\) respectively. Moreover,

\[
\mathcal{Y}(\partial \Omega) = L^2(\partial \Omega).
\]
Proof. First, let us prove that \( \mathcal{R}(K^*) \subset \mathcal{G} (\partial \Omega) \). Since \( (\phi_n)_{n \geq 1} \) is an orthonormal basis for \( \mathcal{H}(\Omega) \), we have for all \( v \in \mathcal{H}(\Omega) \)

\[
v = \sum_{n=1}^{\infty} (v, \phi_n)_{0, \Omega} \phi_n = \lim_{n \to +\infty} \sum_{j=1}^{n} (v, \phi_j)_{0, \Omega} \phi_j,
\]

and since \( K^* \) is bounded, it follows that

\[
K^* v = \lim_{n \to +\infty} \sum_{j=1}^{n} (v, \phi_j)_{0, \Omega} K^* \phi_j = \lim_{n \to +\infty} \sum_{j=1}^{n} (v, \phi_j)_{0, \Omega} \kappa_j g_j.
\]

Therefore, \( K^* v \) is the limit of a linear combination sequence of elements of \( (g_n)_{n \geq 1} \). Hence \( K^* v \in \mathcal{G} (\partial \Omega) \), then \( \mathcal{R}(K^*) \subset \mathcal{G} (\partial \Omega) \) and since \( \text{span}(g_n)_{n \geq 1} \subset \mathcal{R}(K^*) \), it follows that \( \mathcal{R}(K^*) \) is densely imbedded in \( \mathcal{G} (\partial \Omega) \). In a similar way, we obtain that \( \mathcal{R}(\Gamma_0) \subset \mathcal{Y}(\partial \Omega) \). Moreover, here is another way to obtain the completeness of the sequence \( (g_n)_{n \geq 1} \) : since the operator \( \Gamma_0^* \) is injective according to Lemma 3.3, \( \mathcal{R}(\Gamma_0) \) is then dense in \( L^2(\partial \Omega) \). Therefore, we obtain that \( \mathcal{Y}(\partial \Omega) = L^2(\partial \Omega) \). □

Lemma 4.3 The sequences \( (\kappa_n g_n)_{n \geq 1} \) and \( (\kappa_n y_n)_{n \geq 1} \) are Bessel sequences.

Proof. Viewing \( (\phi_n)_n \) is an orthonormal basis for \( \mathcal{H}(\Omega) \) and that the operators \( K^* \) and \( \Gamma_0^* \) are bounded, it follows by Lemma 2.3 that \( (\kappa_n g_n)_{n \geq 1} \) and \( (\kappa_n y_n)_{n \geq 1} \) are Bessel sequences. □

Corollary 4.2 The synthesis operators associated with the sequences \( (\kappa_n g_n)_{n \geq 1} \) and \( (\kappa_n y_n)_{n \geq 1} \) are bounded.

Corollary 4.3 The analysis operators associated with the sequences \( (\kappa_n g_n)_{n \geq 1} \) and \( (\kappa_n y_n)_{n \geq 1} \) are bounded.

In the rest of this paper, we denote by \( (A_G, \mathcal{D}(A_G)) \) and \( (A_Y, \mathcal{D}(A_Y)) \) the analysis operators associated with the sequences \( (g_n)_{n \geq 1} \), \( (y_n)_{n \geq 1} \), and by \( (S_G, \mathcal{D}(S_G)) \) and \( (S_Y, \mathcal{D}(S_Y)) \) their associated synthesis operators respectively. Denote also by \( A_\kappa \) the analysis operator associated with the orthonormal basis \( (\phi_n)_{n \geq 1} \) and by \( M_\kappa \) the multiplication operator on \( \ell^2(\mathbb{N}^*) \) by the sequence \( \kappa = (\kappa_n)_{n \geq 1} \) such that

\[
M_\kappa : \ell^2(\mathbb{N}^*) \longrightarrow \ell^2(\mathbb{N}^*),
\]

such that for all scalar sequence \( (x_n)_{n \geq 1} \in \ell^2(\mathbb{N}^*) \),

\[
M_\kappa(x_n)_{n \geq 1} = (\kappa_n x_n)_{n \geq 1}.
\]

In particular, for the sequences \( G = (g_n)_{n \geq 1} \) and \( Y = (y_n)_{n \geq 1} \), we adopt the following notation:

\[
M_\kappa G = (\kappa_n g_n)_{n \geq 1} = \kappa G \quad \text{and} \quad M_\kappa Y = (\kappa_n y_n)_{n \geq 1} = \kappa Y.
\]

The next lemma will prove to be crucial.
Lemma 4.4 We have
\[ A_Y K^* = M_\kappa A_\Phi = A_G \Gamma_0. \]

Proof. For \( v \in \mathcal{H}(\Omega) \), we have
\[
M_\kappa A_\Phi v = (\kappa_n(v, \phi_n)_{0, \Omega})_{n \geq 1}.
\]

In a similar way, we obtain
\[
M_\kappa A_\Phi v = (\kappa_n(v, \phi_n)_{0, \Omega})_{n \geq 1}.
\]

Corollary 4.4 The analysis operators \((A_G, \mathcal{D}(A_G))\) associated with the sequence \((g_n)_{n \geq 1}\) is closed and densely defined.

Proof. According to Lemma 2.5, \((A_G, \mathcal{D}(A_G))\) is closed. On the other hand, we have shown in Lemma 4.4, that \(\mathcal{R}(\Gamma_0) \subset \mathcal{D}(A_G)\), and since \(\mathcal{R}(\Gamma_0)\) is dense in \(L^2(\partial \Omega)\), it follows then that the analysis operators \((A_G, \mathcal{D}(A_G))\) is densely defined.

We will extensively make use of the following lemma:

Lemma 4.5 The following hold
1. \(M_\kappa A_G \subset A_G K = A_\kappa G\)
2. \(M_\kappa A_Y \subset A_G \Gamma_0^* = A_\kappa Y\)
3. \(K^* A_\Phi = A_G^* M_\kappa\)
4. \(\Gamma_0 A_\Phi^* = A_\gamma G\)

Proof. Let \( g \in \mathcal{D}(A_G) \). We have
\[
M_\kappa A_G g = (\kappa_n(g, g_n)_{0, \partial \Omega})_{n \geq 1}.
\]
Since for all \( n \geq 1 \),
\[
\kappa_n(g, g_n)_{0, \partial \Omega} = (g, \kappa_n g_n)_{0, \partial \Omega} = (g, K^* \phi_n)_{0, \partial \Omega} = (K g, \phi_n)_{0, \partial \Omega},
\]

it follows that for all \( g \in \mathcal{D}(A_G) \),
\[
M_\kappa A_G g = A_\Phi K g = A_\kappa G g
\]
and the first inclusion holds. In a similar way, one can prove (2). Having established in Corollary 4.4 that the analysis operator \((A_G, \mathcal{D}(A_G))\) is densely defined and that \(A_\kappa Y\) is bounded in Corollary 4.3, one considers their adjoint operators \(A_\kappa^*, A_\kappa^* Y\) respectively. Therefore, the items (3) and (4) hold by considering the adjoints in (1) and (2). □

The following result will prove useful in the rest of this text.

**Lemma 4.6** The operator \(A_\kappa Y\) is injective and has a dense range.

**Proof.** For \( g \in L^2(\partial \Omega) \), the equality \( A_\kappa Y g = 0 \) implies that for all \( n \geq 1, (g, \kappa_n y_n)_{\partial \Omega} = 0 \). Moreover, viewing \( \kappa_n \neq 0 \), it follows that \( (g, y_n)_{\partial \Omega} = 0 \). Since the sequence \( Y = (y_n)_{n \geq 1} \) is complete, we get \( g = 0 \). Therefore, \( A_\kappa Y \) is injective. On the other hand, from the second item of Lemma 4.5, we have \( A_\kappa Y = A_\Phi \Gamma_0^\ast \), and since \( A_\Phi \) is a unitary isomorphism and that \( \mathcal{R}(\Gamma_0^\ast) \) is dense in \( \mathcal{H}(\Omega) \), we deduce that \( \mathcal{R}(A_\kappa Y) \) is dense in \( \ell^2(\mathbb{N}^*) \). □

**Corollary 4.5** The operator \(A_\kappa^* Y\) is bounded, injective and has a dense range.

**Lemma 4.7** The operator \((M_\frac{1}{\kappa} A_G, \mathcal{D}(M_\frac{1}{\kappa} A_G))\) is the inverse of \(A_\kappa^* Y\). Moreover, it is surjective and has a surjective adjoint.

**Proof.** From Lemma 4.5, we have \( \Gamma_0 A_\Phi^* = A_\kappa^* Y, \) which implies that for all \( (c_n)_{n \geq 1} \in \ell^2(\mathbb{N}^*), \)
\[
M_\frac{1}{\kappa} A_G A_\kappa^* (c_n)_n = M_\frac{1}{\kappa} A_G \Gamma_0 A_\Phi^* (c_n)_n.
\]
In view of Lemma 4.4, we have \( A_G \Gamma_0 = M_\kappa A_\Phi \), which leads to
\[
M_\frac{1}{\kappa} A_G A_\kappa^* (c_n)_n = M_\frac{1}{\kappa} M_\kappa A_\Phi A_\Phi^* (c_n)_n.
\]
Moreover, since \( A_\Phi \) is a unitary isomorphism, we have
\[
A_\Phi A_\Phi^* = I_{\ell^2(\mathbb{N}^*)},
\]
therefore,
\[
M_\frac{1}{\kappa} A_G A_\kappa^* (c_n)_n = (c_n)_n.
\]
Hence, we obtain that
\[
\mathcal{R}(A_\kappa^*) \subset \mathcal{D}(M_\frac{1}{\kappa} A_G)
\]
and that
\[
M_\frac{1}{\kappa} A_G A_\kappa^* = I_{\ell^2(\mathbb{N}^*)}.
\]
Having previously established in Corollary 4.5 that \(A_\kappa^* Y\) is bounded injective with dense range, it follows that \((M_\frac{1}{\kappa} A_G, \mathcal{D}(M_\frac{1}{\kappa} A_G))\) is its unique inverse. Moreover, it is surjective and has a surjective adjoint. □

**Corollary 4.6** The operator \((A_\kappa^* M_\frac{1}{\kappa}, \mathcal{D}(A_\kappa^* M_\frac{1}{\kappa}))\) is the inverse of \(A_\kappa Y\).
Corollary 4.7 The operators \((A^*_G, \mathcal{D}(A^*_G))\) and \((A^*_Y, \mathcal{D}(A^*_Y))\) are surjective. Moreover, the sequence \((g_n)_{n\geq 1}\) is complete.

**Proof.** Since the operator \((A^*_G M_1, \mathcal{D}(A^*_G M_1))\) is surjective, clearly \((A^*_G, \mathcal{D}(A^*_G))\) is also surjective as well, then \(\mathcal{R}(A^*_G) = L^2(\partial \Omega)\), and this implies that the sequence \((g_n)_{n\geq 1}\) is complete, i.e,

\[ \mathcal{G}(\partial \Omega) = L^2(\partial \Omega). \]

On the other hand, we have

\[ \text{span}(g_n)_{n\geq 1} \subset \mathcal{R}(K^*), \]

and according to Lemma 4.2, \(\mathcal{R}(K^*)\) is densely imbedded in \(\mathcal{G}(\partial \Omega)\). Therefore, we deduce that \(\mathcal{R}(K^*)\) is dense in \(L^2(\partial \Omega)\). Moreover, by Lemma 4.4, we have

\[ \mathcal{R}(K^*) \subset \mathcal{D}(A_Y). \]

Therefore, the operator \((A_Y, \mathcal{D}(A_Y))\) is densely defined. In analogue with the proof of Lemma 4.7, instead of \(A^*_Y = A^*_Y M_\kappa\), we consider the operator \(A^*_G M_\kappa = A^*_G\), then one can prove that \(M_\kappa A_Y\) is the inverse of \(A^*_G M_\kappa\) and this establishes that \((A^*_Y, \mathcal{D}(A^*_Y))\) is surjective. □

Corollary 4.8 The solution operator of the Dirichlet problem for the Laplace equation (1) is injective.

The next proposition is an essential step towards the main result of this paper.

**Proposition 4.1** The following hold

\[ \Gamma_0 = S_Y A_Y K^* = S_Y A_G \Gamma_0, \]

and

\[ K^* = S_G A_G \Gamma_0 = S_G A_Y K^*. \]

**Proof.** Consider the orthonormal basis \((\phi_n)_{n\geq 1}\) for \(\mathcal{H}(\Omega)\). For \(v \in \mathcal{H}(\Omega)\), we have the following representation

\[ v = \sum_{k=1}^{\infty} (v, \phi_k)_{0,\Omega} \phi_k \]

and since \(\Gamma_0\) is bounded, we obtain that

\[ \Gamma_0 v = \sum_{k=1}^{\infty} (v, \phi_k)_{0,\Omega} \Gamma_0 \phi_k \]

\[ = \sum_{k=1}^{\infty} (v, \phi_k)_{0,\Omega} \kappa_k y_k \]

\[ = \sum_{k=1}^{\infty} (v, \kappa_k \phi_k)_{0,\Omega} y_k \]

\[ = \sum_{k=1}^{\infty} (v, K y_k)_{0,\Omega} y_k \]

\[ = S_Y A_Y K^* v, \]
and that

\[ \Gamma_0 v = \sum_{k=1}^{\infty} (v, \phi_k)_{0,\Omega} \Gamma_0 \phi_k \]
\[ = \sum_{k=1}^{\infty} (v, \phi_k)_{0,\Omega} \kappa_k y_k \]
\[ = \sum_{k=1}^{\infty} (v, \kappa_k \phi_k)_{0,\Omega} y_k \]
\[ = \sum_{k=1}^{\infty} (v, \Gamma^*_0 y_k)_{0,\Omega} y_k \]
\[ = S_Y A_G \Gamma_0 v. \]

Hence,

\[ \Gamma_0 = S_Y A_Y K^* = S_Y A_G \Gamma_0. \]

In a similar way, one can establish that

\[ K^* = S_G A_G \Gamma_0 = S_G A_Y K^*. \]

An interesting consequence of Proposition 4.1 is the following:

**Corollary 4.9** *The synthesis and analysis operators associated with the sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) satisfy*

\[ S_Y A_G \subset I_{L^2(\partial\Omega)} \]

and

\[ S_G A_Y \subset I_{L^3(\partial\Omega)}. \]

**Proof.** From Proposition 4.1, we have

\[ \Gamma_0 = S_Y A_Y K^* = S_Y A_G \Gamma_0. \]

Moreover, we showed in Lemma 4.2 that \( \mathcal{R}(\Gamma_0) \) is dense in \( L^2(\partial\Omega) \). We therefore have

\[ S_Y A_G \subset I_{L^2(\partial\Omega)}. \]

Similarly, we have according to Proposition 4.1 that

\[ K^* = S_G A_G \Gamma_0 = S_G A_Y K^*, \]

and by Corollary 4.8 that the operator \( K \) is injective, which implies that \( \mathcal{R}(K^*) \) is dense in \( L^2(\partial\Omega) \). Hence,

\[ S_G A_Y \subset I_{L^2(\partial\Omega)}. \]

**Corollary 4.10** *The synthesis operators \( S_G \) and \( S_Y \) are bounded.*
Proof. In Corollary 4.7, we showed that \((A_G, D(A_G))\) and \((A_Y, D(A_Y))\) are surjective, it follows that \(\mathcal{R}(A_G)\) and \(\mathcal{R}(A_Y)\) are closed. Moreover, from Corollary 4.9 the synthesis operators \((S_G, D(S_G))\) and \((S_Y, D(S_Y))\) are the inverse of the analysis operators \((A_Y, D(A_Y))\) and \((A_G, D(A_G))\) respectively. Consequently, \(S_G\) and \(S_Y\) are bounded. □

Now, we can deduce the principal result of this paper.

**Corollary 4.11** The sequences \((y_n)_{n \geq 1}\) and \((g_n)_{n \geq 1}\) are Riesz bases for \(L^2(\partial \Omega)\).

**Proof.** Having shown in Corollary 4.10 that the synthesis operators \(S_G\) and \(S_Y\) associated with the sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) respectively are bounded. It follows according to Lemma 2.4 that the sequences \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are Bessel sequences. On the other hand, \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are biorthogonal and complete according to Lemma 4.1 and Corollary 4.7. Consequently, \((g_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are Riesz Bases according to Theorem 2.2. □

5 Regularity result for the Dirichlet problem

Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^d\). For \(0 \leq s \leq 1\), we denote by \(\mathcal{H}^s(\Omega)\) the space of real harmonic functions on the usual Sobolev space \(H^s(\Omega)\), i.e.,

\[
\mathcal{H}^s(\Omega) = \{ v \in H^s(\Omega) \mid \Delta v = 0 \text{ in } \Omega \}
\]

and by \(\mathcal{H}_s(\Omega)\) the following:

\[
\mathcal{H}_s(\Omega) = \{ \ (I + F_1^* F_1)^{-s/2} v \mid v \in \mathcal{H}(\Omega) \},
\]

where \(F_1\) is the Moore-Penrose inverse of the embedding operator \(E_1\). A first characterization of \(\mathcal{H}^*(\Omega)\) is given in the following proposition:

**Proposition 5.1** Let \(\Omega \subset \mathbb{R}^d\) be a bounded Lipschitz domain. Then, \(\mathcal{H}^1(\Omega) = \mathcal{H}_1(\Omega)\) with an equivalence of norms.

**Proof.** According to Proposition 2.4, we have

\[
E_1 = (I + F_1^* F_1)^{-1/2} T_{F_1^*}
\]

where

\[
T_{F_1^*} = F_1^* (I + F_1 F_1^*)^{-1/2} + E_1 (I + F_1 F_1^*)^{-1/2}.
\]

Moreover, since the algebraic equality \(\mathcal{R}(E_1) = \mathcal{H}^1(\Omega)\) holds and that

\[
\mathcal{H}_1(\Omega) = \{ \ (I + F_1^* F_1)^{-1/2} v \mid v \in \mathcal{H}(\Omega) \},
\]

we obtain the algebraic equality between \(\mathcal{H}^1(\Omega)\) and \(\mathcal{H}_1(\Omega)\), and all what is needed to prove is the equivalence of norms. To this end, for \(u \in \mathcal{H}_1(\Omega)\) consider the graph norm

\[
\|u\|_{\mathcal{H}_1(\Omega)} = \|(I + F_1^* F_1)^{1/2} u\|_{0,\Omega}.
\]

For \(v \in \mathcal{H}^1(\Omega),\ E_1 v \in \mathcal{H}_1(\Omega)\) and

\[
\|(I + F_1^* F_1)^{1/2} E_1 v\|_{0,\Omega} = \|T_{F_1^*} v\|_{0,\Omega}.
\]
Viewing $T_{F_1}$ is an isomorphism from $\mathcal{H}^1(\Omega)$ into $\mathcal{H}(\Omega)$ by Corollary 2.2. This assures the existence of two positive constants $c_1'$ and $c_2'$ not depending on $v$ such that

$$c_1' \|v\|_{0,\Omega} \leq \|(I + F_1^* F_1)^{1/2} E_1 v\|_{0,\Omega} \leq c_2' \|v\|_{0,\Omega}.$$ 

Therefore, we deduce that the norms $\|\cdot\|_{\mathcal{H}^1(\Omega)}$ and $\|\cdot\|_{\mathcal{H}(\Omega)}$ are equivalent. □

The following corollary could be deduced using the real interpolation method (see [1], [18] and [22]).

**Corollary 5.1** Assume $0 \leq s \leq 1$, then $\mathcal{H}^s(\Omega)$ form an interpolatory family. Moreover, we have

$$\mathcal{H}^s(\Omega) = \mathcal{H}_s(\Omega)$$

with equivalence of norms.

**Proposition 5.2** Let $((\kappa_n, \phi_n))_{n \geq 1}$ be the sequence of couple in $\mathbb{R}_+^* \times \mathcal{H}(\Omega)$ associated with $\Gamma_0 K^*$ such that $\Gamma_0 K^* \phi_n = \kappa_n^2 \phi_n$. Then the following statements are equivalent

1. $v \in \mathcal{H}^s(\Omega)$.
2. $\sum_{n=1}^{+\infty} \frac{1}{\kappa_n^{2s}} |(v, \phi_n)_{0,\Omega}|^2$ converges.

**Proof.** Let $v \in \mathcal{H}_s(\Omega)$. There exists then $\phi \in \mathcal{H}(\Omega)$ such that

$$v = (I + F_1^* F_1)^{-s/2} \phi,$$

which implies that

$$(v, \phi_n)_{0,\Omega} = ((I + F_1^* F_1)^{-s/2} \phi, \phi_n)_{0,\Omega} = (\phi, (I + F_1^* F_1)^{-s/2} \phi_n)_{0,\Omega}.$$

According to the Spectral Theorem [6, Theorem 5.1], it follows that

$$(v, \phi_n)_{0,\Omega} = (\phi, (I + F_1^* F_1)^{-s/2} \phi_n)_{0,\Omega} = (\phi, \kappa_n^{2s} \phi_n),$$

which implies that

$$\frac{1}{\kappa_n^{2s}} (v, \phi_n)_{0,\Omega} = (\phi, \phi_n)_{0,\Omega},$$

we therefore obtain that

$$\sum_{n=1}^{+\infty} \frac{1}{\kappa_n^{2s}} |(v, \phi_n)_{0,\Omega}|^2 = \sum_{n=1}^{+\infty} |(\phi, \phi_n)_{0,\Omega}|^2.$$

Since $\phi \in \mathcal{H}(\Omega)$ and $\sum_{n=1}^{+\infty} |(\phi, \phi_n)_{0,\Omega}|^2 < +\infty$, it follows then that

$$\sum_{n=1}^{+\infty} \frac{1}{\kappa_n^{2s}} |(v, \phi_n)_{0,\Omega}|^2 < +\infty, \quad \forall v \in \mathcal{H}_s(\Omega),$$

therefore according to Corollary 5.1, we conclude that for $v \in \mathcal{H}^s(\Omega)$, $\sum_{n} \frac{1}{\kappa_n^{2s}} |(v, \phi_n)_{0,\Omega}|^2$ converges. □

From Corollary 5.1 and Proposition 5.2, we deduce:
Corollary 5.2 We have
\[ \mathcal{R}(K) \subset \mathcal{R}((I + F_i^* F_i)^{-1/4} P_{\mathcal{H}(\Omega)}) \]
and
\[ \mathcal{R}(\Gamma_0^* g) \subset \mathcal{R}((I + F_i^* F_i)^{-1/4} P_{\mathcal{H}(\Omega)}). \]

Proof. Let \( g \in L^2(\Omega) \). For all \( n \geq 1 \), we have
\[ (Kg, \phi_n)_{0,\Omega} = (g, K^* \phi_n)_{0,\Omega} = (g, \kappa_n g_n)_{0,\partial \Omega}, \]
which leads to
\[ \left| \frac{1}{\kappa_n} (Kg, \phi_n)_{0,\Omega} \right| = \left| (g, g_n)_{0,\partial \Omega} \right| \]
and since \( (g_n)_{n \geq 1} \) is a Bessel sequence, \( \sum_n \left| (g, g_n)_{0,\partial \Omega} \right|^2 \) converges, and this implies that \( \sum_n \left| \frac{1}{\kappa_n} (Kg, \phi_n)_{0,\Omega} \right|^2 \) converges, therefore \( Kg \in H^{1/2}(\Omega) \). Similarly, for all \( n \geq 1 \),
\[ (\Gamma_0^* g, \phi_n)_{0,\Omega} = (g, \Gamma_0 \phi_n)_{0,\Omega} = (g, \kappa_n y_n)_{0,\partial \Omega} \]
so
\[ \left| \frac{1}{\kappa_n} (\Gamma_0^* g, \phi_n)_{0,\Omega} \right| = \left| (g, y_n)_{0,\partial \Omega} \right|. \]
Similarly, viewing \( (y_n)_{n \geq 1} \) is a Bessel sequence, it follows that \( \sum_n \left| (g, y_n)_{0,\partial \Omega} \right|^2 \) converges and therefore \( \sum_n \left| \frac{1}{\kappa_n} (\Gamma_0^* g, \phi_n)_{0,\Omega} \right|^2 \) converges, thus \( \Gamma_0^* g \in H^{1/2}(\Omega) \).

It turns out that the very weak solution of the Dirichlet problem for the Laplace equation (1), which is of the form \( v = Kg \) lies in \( H^{1/2}(\Omega) \). □

Now, we can state the main result of this section.

Theorem 5.1 Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain. For \( g \in L^2(\partial \Omega) \), the very weak solution of the Dirichlet problem for the Laplace equation (1) lies in \( H^{1/2}(\Omega) \) and there exist two positive constants \( c_\Omega \) and \( c'_\Omega \) depending on the geometry of \( \Omega \) such that:
\[ c'_\Omega \| g \|_{L^2(\partial \Omega)} \leq \| v \|_{H^{1/2}(\Omega)} \leq c_\Omega \| g \|_{L^2(\partial \Omega)}, \]

Moreover, the solution operator \( K \) is compact and injective.

Proof. According to Corollary 5.2,
\[ \mathcal{R}(K) \subset \mathcal{R}((I + F_i^* F_i)^{-1/4} P_{\mathcal{H}(\Omega)}) \]
which implies that
\[ \mathcal{R}(K) \subset H^{1/2}(\Omega). \]
On the other hand, according to Lemma 2.7, \( H^{1/2}(\Omega) \) is compactly imbedded in \( L^2(\Omega) \), therefore \( K \) is compact. The injectivity of \( K \) holds from Corollary 4.8. On the other hand, since
\[ \mathcal{H}_{1/2}(\Omega) = H^{1/2}(\Omega), \]
and that
\[ \|Kg\|_{\mathcal{H}_{1/2}(\Omega)} = \|(I + F_1^* F_1)^{1/4}Kg\|_{0,\Omega}, \]
we have
\[ ((I + F_1^* F_1)^{1/4}Kg, \phi_n)_{0,\Omega} = (Kg, (I + F_1^* F_1)^{1/4}\phi_n)_{0,\Omega} = (g, \kappa_n^{-1}\phi_n)_{0,\Omega} = (g, g_n)_{0,\partial\Omega}. \]

Therefore, we obtain that
\[ \|Kg\|_{\mathcal{H}_{1/2}(\Omega)}^2 = \sum_{n=1}^{\infty} |(g, g_n)_{0,\partial\Omega}|^2. \]

Moreover, since \((g_n)_{n\geq 1}\) is a Riesz basis for \(L^2(\partial\Omega)\), there exist two constants \(b_G\) and \(a_G\) such that
\[ a_G \|g\|_{0,\partial\Omega}^2 \leq \|Kg\|_{\mathcal{H}_{1/2}(\Omega)}^2 \leq b_G \|g\|_{0,\partial\Omega}^2, \]
where \(b_G\) is the smallest upper bound of the sequence \((g_n)_{n\geq 1}\) and \(a_G\) is the greatest lower bound of the sequence \((g_n)_{n\geq 1}\), therefore, we obtain that
\[ \sqrt{a_G} \|g\|_{0,\partial\Omega} \leq \|v\|_{\mathcal{H}_{1/2}(\Omega)} \leq \sqrt{b_G} \|g\|_{0,\partial\Omega}. \]

Moreover, in view of Corollary 5.1, we have the equivalence of the norms \(\|\cdot\|_{\mathcal{H}_{1/2}(\Omega)}\) and \(\|\cdot\|_{\mathcal{H}_{1/2}(\Omega)}\), which implies that there exist two positive constants \(c_{\Omega}\) and \(c'_{\Omega}\) depending on the geometry of \(\Omega\) such that
\[ c_{\Omega} \|g\|_{0,\partial\Omega} \leq \|v\|_{\mathcal{H}_{1/2}(\Omega)} \leq c'_{\Omega} \|g\|_{0,\partial\Omega}. \]

**Remark:** This work is part of the second author’s ongoing Ph.D. research, which is carried out at Moulay Ismail University, Meknes-Morocco.

**References**

[1] Adams R. A., and Fournier J. F., Sobolev spaces. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] Casazza P.G., Christensen O., Li S. and Linder A., Riesz-Fischer sequences and Lower Frame Bounds, A. Anal. Anwend. 21, 305-314, 2002.

[3] Christensen O., An introduction to frames and Riesz bases. Birkhauser, 2016.

[4] Coifman R.R., McIntosh A. and Meyer Y., L’intégrale de Cauchy définit un opérateur borné sur \(L^2\) pour les courbes Lipschitziennes, Ann. of Math. 116(1982), 361-387.

[5] Conway J.B., A course in functional analysis, second edition, Springer-Verlag, New York, 1985.
[6] Costabel M., Boundary integral operators on Lipschitz domains: elementary results, SIAM J. Math. Anal. 19 (1988), 613-626. MR 89h:35090

[7] Dahlberg B., Estimates of harmonic measure, Arch. Rational Mech. Anal. 65(1977), no. 3, 275-288.

[8] Gagliardo E., Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in $n$ variabili, Rend. Sem. Mat. Univ. Padova 27, (1957) pp. 284-305.

[9] Grisvard Pierre, Elliptic problems in nonsmooth domains, Monographs and Studies in Mathematics, 24. Pitman Advanced Publishing Program. Boston-London-Melbourne: Pitman Publishing (1985).

[10] Groetsch C.W., Inclusions and identities for the Moore-Penrose inverse of a closed linear operator, Mathematische Nachrichten, 171, 157-164 (1995).

[11] Heil C., A Basis Theory Primes: Expanded Edition, Applied and Numerical Analysis, Springer (1998).

[12] Jerison D., and Kenig C., An identity with applications to harmonic measure Bull. Amer. Math.Soc (1980), 447-451.

[13] Jerison D., and Kenig C., Boundary value problems in Lipschitz domains, MAA Studies in Math., Studies in Partial Differential Equations, W. Littman, ed.23 (1982), 1-68.

[14] Jerison D., and Kenig C., The inhomogenous Dirichlet problem in Lipschitz domains, J. Funct. Anal., 130(1): 161-219, 1995.

[15] Kato T., Perturbation Theory for Linear Operators, Reprint of the corr. print. of the 2nd ed. 1980, Classics in Mathematics. Berlin: Springer-Verlag.

[16] Labrousse Jean-Philippe, Inverses généralisés d’opérateurs non bornés, Proc. Amer. Math Soc., Vol. 115, N° 1, May (1992) pp; 125-1209.

[17] Labrousse Jean-Philippe et Mbekhta Mostapha, les opérateurs points de continuité pour la conorme et l’inverse de Moore-Penrose, Houston J. of Math., 18, 7-23, 1992.

[18] McLean, W., Strongly elliptic systems and boundary integral equations. Cambridge University Press, Cambridge, 2000.

[19] Nečas J., Les Méthodes Directes en Théorie des Équations Elliptiques, Masson, Paris, (1967).

[20] Rellich F., Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral, Math. Z. 46,(1940), pp. 635-636.

[21] Savaré G., Regularity Results for Elliptic Equations in Lipschitz Domains, J. Funct. Anal., 152 (1998), no. 1, 176-201.

[22] Tartar Luc, An introduction to Sobolev spaces and interpolation spaces. Lecture Notes of the Unione Matematica Italiana, 3. Springer, Berlin; UMI, Bologna, 2007.
[23] Verchota G., Layer Potentials and Regularity for the Dirichlet Problem for Laplace’s Equation in Lipschitz Domains, J. Funct. Anal., 59, 572-611 (1984).

[24] Young R., An introduction to nonharmonic Fourier series. Academic Press New York, 1980.