Nonexistence of solutions for elliptic equations with supercritical nonlinearity in nearly nontrivial domains

February 7, 2019

Riccardo MOLLE\textsuperscript{a}, Donato PASSASEO\textsuperscript{b}

\textsuperscript{a}Dipartimento di Matematica, Universit\`a di Roma “Tor Vergata”,
Via della Ricerca Scientifica n. 1, 00133 Roma, Italy.

\textsuperscript{b}Dipartimento di Matematica “E. De Giorgi”, Universit\`a di Lecce,
P.O. Box 193, 73100 Lecce, Italy.

Abstract. - We deal with nonlinear elliptic Dirichlet problems of the form
\[
\text{div}(|Dv|^{p-2}Dv) + f(u) = 0 \quad \text{in } \Omega, \quad u \in H^{1,p}_0(\Omega)
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $p > 1$ and $f$ has supercritical growth from the viewpoint of Sobolev embedding.

Our aim is to show that there exist bounded contractible non star-shaped domains $\Omega$, arbitrarily close to domains with nontrivial topology, such that the problem does not have nontrivial solutions. For example, we prove that if $n = 2$, $1 < p < 2$, $f(u) = |u|^{q-2}u$ with $q > \frac{2p}{2-p}$ and $\Omega = \{(\rho \cos \theta, \rho \sin \theta) : |\theta| < \alpha, |\rho - 1| < s\}$ with $0 < \alpha < \pi$ and $0 < s < 1$, then for all $q > \frac{2p}{2-p}$ there exists $\bar{s} > 0$ such that the problem has only the trivial solution $u \equiv 0$ for all $\alpha \in (0, \pi)$ and $s \in (0, \bar{s})$.

MSC: 35J20; 35J60; 35J65.

Keywords: Supercritical Dirichlet problems, contractible domains, nonexistence of solutions.

E-mail address: molle@mat.uniroma2.it (R. Molle).
1 Introduction

Let us consider the Dirichlet problem

\[ \text{div}( |Du|^{p-2} Du) + f(u) = 0 \quad \text{in } \Omega, \quad u \in H^{1,p}_0(\Omega) \quad (1.1) \]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \), \( n \geq 2 \) and \( p > 1 \).

It is well known that if the function \( f : \mathbb{R} \to \mathbb{R} \) has critical or supercritical growth from the viewpoint of the Sobolev embedding \( H^{1,p}_0 \hookrightarrow L^q(\Omega) \), the usual methods to find solutions of this problem do not work (see for instance [3]).

For example, if \( 1 < p < n \) and \( f(t) = |t|^{q-2}t \) with \( q \geq \frac{np}{n-p} \) (the critical Sobolev exponent), then the existence of nontrivial solutions to problem

\[ \text{div}( |Du|^{p-2} Du) + |u|^{q-2}u = 0 \quad \text{in } \Omega, \quad u \in H^{1,p}_0(\Omega) \quad (1.2) \]

is strictly related to the shape of \( \Omega \). If \( \Omega \) is star-shaped, problem (1.2) has only the trivial solution \( u \equiv 0 \), as a consequence of a Pohozaev type identity (see [26]). On the other hand, if \( \Omega \) is an annulus, one can easily find infinitely many radial solutions (as pointed out by Kazdan and Werner in [9]). Hence, many researches have been devoted to study the effect of the domain shape on the existence of nontrivial solutions to problem (1.2), following some stimulating questions posed by Brezis, Nirenberg, Rabinowitz, etc. . . . (see [2]). In particular, the case where \( p = 2, n \geq 3, q \geq \frac{2n}{n-2} \) has been considered in many papers.

Answering a question of Nirenberg, Bahri and Coron proved in [1] the existence of a positive solution when \( p = 2, n \geq 3, q = \frac{2n}{n-2} \) and \( \Omega \) has nontrivial topology, in the sense that some homology group is nontrivial (see also [5, 28], concerning the case of domains with small holes).

Notice that for \( q > \frac{2n}{n-2} \) the condition that \( \Omega \) has nontrivial topology is neither sufficient nor necessary to guarantee the existence of nontrivial solution. In fact (answering a question posed by Rabinowitz) the second author proved in [19, 22] that there exist exponents \( q > \frac{2n}{n-2} \) and nontrivial domains \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \) such that the problem

\[ \Delta u + |u|^{q-2}u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \quad (1.3) \]

has only the trivial solution \( u \equiv 0 \).

Moreover, for all \( q \geq \frac{2n}{n-2} \) there exist contractible domains \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \) such that problem (1.3) has positive and sign-changing solutions (see [6, 8, 10, 11, 13, 14, 16–18, 20, 21, 24, 25] and the references therein).

More precisely, for all \( \alpha \in (0, \pi) \) and \( s \in (0, 1) \), let us consider for example the piecewise smooth contractible domain \( \Omega \) of the form

\[ \Omega^{\alpha,s}_n = \{ (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : (x, |y|) \in S^{\alpha,s} \} \quad (1.4) \]
where
\[ S^{\alpha,s} = \{(\rho \cos \theta, \rho \sin \theta) \in \mathbb{R}^2 : 0 \leq \theta < \alpha, \ |\rho - 1| < s \}. \] (1.5)

Then, the following assertions hold for problem (1.3) with \( \Omega = \Omega^{\alpha,s}_n \) and \( n \geq 3 \):

- for all \( q \geq \frac{2n}{n-2} \) there exists \( \bar{\varepsilon}_q > 0 \) such that, if \( \pi - \bar{\varepsilon}_q < \alpha < \pi \), then problem (1.3) has positive and sign changing solutions; moreover, for \( q > \frac{2n}{n-2} \), the number of solutions tends to infinity as \( \alpha \to \pi \) (see [12–14, 16, 18, 21, 24, 25], etc. . . . );
- for all \( \alpha > \frac{\pi}{2} \) there exists \( \bar{\alpha}_q \geq \frac{2n}{n-2} \) such that problem (1.3) with \( q \geq \bar{\alpha}_q \) has at least one positive solution (see [13]);
- for all \( \alpha > \frac{\pi}{2} \) there exists \( \bar{\varepsilon}_\alpha > 0 \) such that problem (1.3) with \( \frac{2n}{n-2} < q < \frac{2n}{n-2} + \bar{\varepsilon}_\alpha \) has positive solutions (see [10, 12], etc. . . . ).

These results (that have been stimulated by an interesting question posed by Brezis in [2]) show that, even if the Pohozaev nonexistence result can be extended to non star-shaped domains (see [4, 7] and also [11, 23, 27] for related phenomena), it cannot be extended to all contractible domains when \( p = 2 \) and \( n \geq 3 \).

The nonexistence result obtained in the present paper, on the contrary, suggests that the situation is quite different if \( n = 2 \) and \( 1 < p < 2 \). In fact, as a direct consequence of Theorem 2.4, we have the following proposition.

**Proposition 1.1** Assume \( n = 2 \) and \( 1 < p < 2 \). Then, for all \( q > \frac{2n}{2-p} \) there exists \( \bar{s} \in (0, 1) \) such that problem (1.2) with \( \Omega = \Omega^{\alpha,s}_2 \) has only the trivial solution \( u \equiv 0 \) for all the pairs \( (\alpha, s) \) such that \( s \in (0, \bar{s}) \) and \( \alpha \in (0, \pi) \).

Since, for all \( s \in (0, \bar{s}) \), the domain \( \Omega^{\alpha,s}_2 \) is contractible for all \( \alpha \in (0, \pi) \), is star-shaped for \( \alpha \) small enough and is close to a domain with nontrivial topology when \( \alpha \) is close to \( \pi \), Proposition 1.1 suggests the following natural question (analogous to the well known one posed by Brezis in [2]): if \( n = 2 \) and \( 1 < p < 2 \), can one extend Pohozaev’s nonexistence result for star-shaped domains to all the contractible domains of \( \mathbb{R}^2 \)?

The nonexistence result presented in this paper suggests that this question might have a positive answer.

## 2 Integral identity and nonexistence result

The following lemma generalizes Pohozaev identity.

**Lemma 2.1** Let \( \Omega \) be a piecewise smooth bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \) and \( p > 1 \). Assume that \( u \in H^{1,p}_0(\Omega) \) is a solution of the equation
\[
\text{div}(|Du|^{p-2}Du) + f(u) = 0 \quad \text{in} \ \Omega,
\] (2.1)
where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function. Then, for all \( v = (v_1, \ldots, v_n) \in C^1(\overline{\Omega}, \mathbb{R}^n) \), the function \( u \) satisfies the integral identity

\[
\left(1 - \frac{1}{p}\right) \int_{\partial \Omega} |Du|^{p-2} v \cdot \nu d\sigma = \int_{\Omega} |Du|^{p-2} (dv[Du] \cdot Du) dx + \int_{\Omega} \text{div} \left( F(u) - \frac{1}{p} |Du|^p \right) dx,
\]

(2.2)

where \( \nu \) denotes the outward normal to \( \partial \Omega \), \( dv[\xi] = \sum_{i=1}^n (D_i v) \xi_i \) \( \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and \( F(t) = \int_0^t f(\tau) d\tau \forall t \in \mathbb{R} \).

**Proof** In order to prove (2.2) it suffices to apply the Gauss-Green formula to the function \( (v \cdot Du)|Du|^{p-2}Du \).

Thus, we obtain

\[
\int_{\partial \Omega} (v \cdot Du)|Du|^{p-2}(Du \cdot \nu)d\sigma = \int_{\Omega} (v \cdot Du)|Du|^{p-2}D_i u dx
\]

\[
= \int_{\Omega} \sum_{i=1}^n D_i \left( \sum_{j=1}^n v_j D_j u \cdot |Du|^{p-2}D_i u \right) dx \tag{2.3}
\]

\[
= \int_{\Omega} \sum_{i,j=1}^n \left( D_i v_j D_j u |Du|^{p-2}D_i u + v_j D_{i,j} u |Du|^{p-2}D_i u \right.
\]

\[
\left. + v_j D_j u D_i (|Du|^{p-2}D_i u) \right) dx.
\]

Since \( u \equiv 0 \) on \( \partial \Omega \), we have \( Du = (Du \cdot \nu) \nu \) and, as a consequence,

\[
\int_{\partial \Omega} (v \cdot Du)|Du|^{p-2}(Du \cdot \nu)d\sigma = \int_{\partial \Omega} |Du|^p(v \cdot \nu)d\sigma. \tag{2.4}
\]

Notice that

\[
\int_{\Omega} \sum_{i,j=1}^n v_j D_{i,j} u |Du|^{p-2}D_i u dx = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^n v_j |Du|^{p-2}D_j |D_i u|^2 dx
\]

\[
= \frac{1}{p} \int_{\Omega} \sum_{j=1}^n v_j D_j |Du|^p dx \tag{2.5}
\]

\[
= \frac{1}{p} \int_{\partial \Omega} |Du|^p v \cdot \nu d\sigma - \frac{1}{p} \int_{\Omega} \text{div} v |Du|^p dx.
\]

Moreover, since \( u \) solves equation (2.1),

\[
\int_{\Omega} \sum_{i,j=1}^n v_j D_j u D_i (|Du|^{p-2}D_i u) dx = - \int_{\Omega} \sum_{j=1}^n v_j D_j u f(u) dx
\]

\[
= - \int_{\Omega} \sum_{j=1}^n v_j D_j F(u) dx = \int_{\Omega} \text{div} v \cdot F(u). \tag{2.6}
\]
Then, (2.2) follows easily from (2.3), (2.4), (2.5), (2.6).

q.e.d.

Lemma 2.2 On the piecewise smooth domain $\Omega_{2}^{\alpha,s} = \{(\rho \cos \theta, \rho \sin \theta) \in \mathbb{R}^2 : |\theta| < \alpha, |\rho - 1| < s\}$ let us consider the vector field $v \in C^1(\Omega_{2}^{\alpha,s}, \mathbb{R}^2)$ defined by

$$v(\rho \cos \theta, \rho \sin \theta) = (\rho - 1)(\cos \theta, \sin \theta) + \rho \theta(- \sin \theta, \cos \theta).$$  

(2.7)

Then,

a) $v \cdot \nu > 0$ on $\partial \Omega_{2}^{\alpha,s} \forall \alpha \in (0, \pi), \forall s \in (0, 1)$;

b) $\text{div} v(\rho \cos \theta, \rho \sin \theta) = 3 - \frac{1}{\rho} \forall (\rho \cos \theta, \rho \sin \theta) \in \Omega_{2}^{\alpha,s}$;

c) $\text{div}(\rho \cos \theta, \rho \sin \theta)[\xi] \cdot \xi = \xi_N^2 + \left(2 - \frac{1}{\rho}\right)\xi_T^2 \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$,

where

$$\xi_N = \xi_1 \cos \theta + \xi_2 \sin \theta \quad \text{and} \quad \xi_T = -\xi_1 \sin \theta + \xi_2 \cos \theta.$$  

(2.8)

Proof Property (a) is a simple consequence of the definition of $\Omega_{2}^{\alpha,s}$ and $v$. In order to prove (b) and (c) it suffices to notice that

$$\text{div}(\rho \cos \theta, \rho \sin \theta)[\xi] = \xi_N(\cos \theta, \sin \theta) + \xi_N(- \sin \theta, \cos \theta)$$

$$-\xi_T(\cos \theta, \sin \theta) + \xi_T\left(2 - \frac{1}{\rho}\right)(- \sin \theta, \cos \theta),$$

(2.9)

as one can verify by direct computation.

q.e.d.

Corollary 2.3 Let $\Omega = \Omega_{2}^{\alpha,s}$ and $v \in C^1(\Omega_{2}^{\alpha,s}, \mathbb{R}^2)$ be as in Lemma 2.2. Let $f$ and $F$ be as in Lemma 2.1. Then every solution of the Dirichlet problem

$$\text{div}(|Du|^{p-2}Du) + f(u) = 0 \quad \text{in} \Omega_{2}^{\alpha,s}, \quad u \in H^1_0(\Omega_{2}^{\alpha,s})$$  

(2.10)

satisfies the inequality

$$0 \leq \left[1 - \frac{2}{p} + \left(1 + \frac{1}{p}\right)\frac{s}{1-s}\right] \int_{\Omega_{2}^{\alpha,s}} |Du|^p dx + \int_{\Omega_{2}^{\alpha,s}} \text{div} v \cdot F(u) dx.$$  

(2.11)

The proof follows directly from Lemmas 2.1 and 2.2 (taking into account that $\left|1 - \frac{1}{\rho}\right| \leq \frac{s}{1-s} \forall \rho \in (1-s, 1+s)$).

Now, we can prove a nonexistence result for nontrivial solutions in the domain $\Omega_{2}^{\alpha,s}$. 
Theorem 2.4 Let $\Omega = \Omega_{2,q}$ be as in Lemma 2.2, $f$ and $F$ be as in Lemma 2.1 and assume that $1 < p < 2$ and there exists $q > \frac{2p}{2-p}$ such that

$$tf(t) \geq qF(t) \geq 0 \quad \forall t \in \mathbb{R}.$$  \hfill (2.12)

Then, there exists $\bar{s} \in (0,1)$ such that the Dirichlet problem (2.10) has only the solution $u \equiv 0$ for every pair $(\alpha, s)$ such that $s \in (0, \bar{s})$ and $\alpha \in (0, \pi)$.

Proof Notice that $u \equiv 0$ is obviously a solution of Problem (2.10) because the assumption (2.12) clearly implies $f(0) = 0$. Let us prove that it is the unique solution. Since $0 \leq F(u) \leq \frac{1}{q}uf(u)$, from Lemma 2.2 and Corollary 2.3 we obtain that every solution $u$ of the Dirichlet problem (2.10) must satisfy

$$0 \leq \left[1 - \frac{2}{p} + \left(1 + \frac{1}{p}\right) \frac{s}{1-s}\right] \int_{\Omega_{2,q}} |Du|^p dx + \left[2 + \frac{s}{1-s}\right] \frac{1}{q} \int_{\Omega_{2,q}} uf(u) dx.$$  \hfill (2.13)

Notice that

$$\int_{\Omega_{2,q}} uf(u) dx = \int_{\Omega_{2,q}} |Du|^p dx$$  \hfill (2.14)

as $u$ solves the Dirichlet problem (2.10). Therefore we obtain,

$$0 \leq \left[1 - \frac{2}{p} + \frac{2}{q} + \left(1 + \frac{1}{p} + \frac{1}{q}\right) \frac{s}{1-s}\right] \int_{\Omega_{2,q}} |Du|^p dx.$$  \hfill (2.15)

Since $1 - \frac{2}{p} + \frac{2}{q} < 0$ for $q > \frac{2p}{2-p}$, there exists $\bar{s} \in (0,1)$ such that $1 - \frac{2}{p} + \frac{2}{q} + \left(1 + \frac{1}{p} + \frac{1}{q}\right) \frac{s}{1-s} < 0 \forall s \in (0, \bar{s})$. Therefore, if $s \in (0, \bar{s})$ and $u$ solves the Dirichlet problem (2.10), we must have

$$\int_{\Omega_{2,q}} |Du|^p dx = 0,$$  \hfill (2.16)

so the proof is complete.

q.e.d.

Finally, notice that we obtain in particular Proposition 1.1 when in Theorem 2.4 we choose $f(u) = |u|^{q-2}u$ (which obviously satisfies condition (2.12)).

Acknowledgement. The authors have been supported by the “Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)” of the Istituto Nazionale di Alta Matematica (INdAM) - Project: Equazioni di Schrodinger nonlineari: soluzioni con indice di Morse alto o infinito.

The second author acknowledges also the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006
References

[1] A. Bahri - J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. 41 (1988), 253-294.

[2] H. Brézis, *Elliptic equations with limiting Sobolev exponents – the impact of topology*, Comm. Pure Appl. Math. 39 (suppl.) (1986), S17-S39.

[3] H. Brézis - L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*. Comm. Pure Appl. Math. 36 (1983), no. 4, 437–477.

[4] A. Carpio Rodríguez - M. Comte - R. Lewandowski, *A nonexistence result for a nonlinear equation involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992), no. 3, 243–261.

[5] J.M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 7, 209-212.

[6] E.N. Dancer, *A note on an equation with critical exponent*, Bull. London Math. Soc. 20 (1988), no. 6, 600–602.

[7] E.N. Dancer - K. Zhang, *Uniqueness of solutions for some elliptic equations and systems in nearly star-shaped domains*, Nonlinear Anal. 41 (2000), no. 5-6, Ser. A: Theory Methods, 745–761

[8] W.Y. Ding, *Positive solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains*, J. Partial Differential Equations 2 (1989), no. 4, 83–88.

[9] J. Kazdan and F. W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. 28 (1975), no. 5, 567-597.

[10] R. Molle - D. Passaseo, *Positive solutions for slightly super-critical elliptic equations in contractible domains*, C. R. Math. Acad. Sci. Paris 335 (2002), no. 5, 459–462.

[11] R. Molle - D. Passaseo, *Nonlinear elliptic equations with critical Sobolev exponent in nearly starshaped domains*, C. R. Math. Acad. Sci. Paris 335 (2002), no. 12, 1029–1032.

[12] R. Molle - D. Passaseo, *Positive solutions of slightly supercritical elliptic equations in symmetric domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 5, 639–656.

[13] R. Molle - D. Passaseo, *Nonlinear elliptic equations with large supercritical exponents*, Calc. Var. Partial Differential Equations 26 (2006), no. 2, 201–225.

[14] R. Molle - D. Passaseo, *Multiple solutions of supercritical elliptic problems in perturbed domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 3, 389–405.
[15] L. Moschini - S.I. Pohozaev - A. Tesei, Existence and nonexistence of solutions of nonlinear Dirichlet problems with first order terms, J. Funct. Anal. 177 (2000), no. 2, 365–382.

[16] D. Passaseo, Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains, Manuscripta Math. 65 (1989), no. 2, 147–165.

[17] D. Passaseo, On some sequences of positive solutions of elliptic problems with critical Sobolev exponent, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 3 (1992), no. 1, 15–21.

[18] D. Passaseo, Existence and multiplicity of positive solutions for elliptic equations with supercritical nonlinearity in contractible domains, Rend. Accad. Naz. Sci. XL Mem. Mat. (5) 16 (1992), 77–98.

[19] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains, J. Funct. Anal. 114 (1993), no. 1, 97–105.

[20] D. Passaseo, Multiplicity of positive solutions for the equation \( \Delta u + \lambda u + u^{2^*-1} = 0 \) in noncontractible domains, Topol. Methods Nonlinear Anal. 2 (1993), no. 2, 343-366.

[21] D. Passaseo, The effect of the domain shape on the existence of positive solutions of the equation \( \Delta u + u^{2^*-1} = 0 \), Topol. Methods Nonlinear Anal. 3 (1994), no. 1, 27-54.

[22] D. Passaseo, New nonexistence results for elliptic equations with supercritical nonlinearity, Differential Integral Equations 8 (1995), no. 3, 577–586.

[23] D. Passaseo, Some concentration phenomena in degenerate semilinear elliptic problems, Nonlinear Anal. 24 (1995), no. 7, 1011–1025.

[24] D. Passaseo, Multiplicity of nodal solutions for elliptic equations with supercritical exponent in contractible domains, Topol. Methods Nonlinear Anal. 8 (1996), no. 2, 245–262 (1997).

[25] D. Passaseo, Nontrivial solutions of elliptic equations with supercritical exponent in contractible domains, Duke Math. J., 92 (1998), no. 2, 429–457.

[26] S.I. Pohozaev, On the eigenfunctions of the equation \( \Delta u + \lambda f(u) = 0 \), Soviet. Math. Dokl. 6 (1965), 1408-1411.

[27] S.I. Pohozaev - A. Tesei, Existence and nonexistence of solutions of nonlinear Neumann problems, SIAM J. Math. Anal. 31 (1999), no. 1, 119–133.

[28] O. Rey, The role of the Green’s function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal. 89 (1990), no. 1, 1-52.