BER and Outage Probability Approximations for LMMSE Detectors on Correlated MIMO Channels

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Abstract—This paper is devoted to the study of the performance of the linear minimum mean-square error (LMMSE) receiver for (receive) correlated multiple-input multiple-output (MIMO) systems. By the random matrix theory, it is well known that the signal-to-noise ratio (SNR) at the output of this receiver behaves asymptotically like a Gaussian random variable as the number of receive and transmit antennas converge to $+\infty$ at the same rate. However, this approximation being inaccurate for the estimation of some performance metrics such as the bit error rate (BER) and the outage probability, especially for small system dimensions, Li et al. proposed convincingly to assume that the SNR follows a generalized gamma distribution which parameters are tuned by computing the first three asymptotic moments of the SNR. In this paper, this technique is generalized to (receive) correlated channels, and closed-form expressions for the first three asymptotic moments of the SNR are provided. To obtain these results, a random matrix theory technique adapted to matrices with Gaussian elements is used. This technique is believed to be simple, efficient, and of broad interest in wireless communications. Simulations are provided, and show that the proposed technique yields in general a good accuracy, even for small system dimensions.

Index Terms—Bit-error rate (BER), correlated channels, gamma approximation, large random matrices, minimum mean-square error (MMSE), multiple-input multiple-output (MIMO), outage probability, signal-to-noise ratio (SNR).

I. INTRODUCTION

SINCE the mid-1990s, digital communications over multiple-input multiple-output (MIMO) wireless channels have aroused an intense research effort. It is indeed well known since Telatar’s work [1] that antenna diversity increases significantly the Shannon mutual information of a wireless link; in rich scattering environments, this mutual information increases linearly with the minimum number of transmit and receive antennas. Since the findings of [1], a major effort has been devoted to analyze the statistics of the mutual information. Such an analysis has strong practical impacts: For instance, it can provide information about the gain obtained from scheduling strategies [2]; it can be used as a performance metric to optimally select the active transmit antennas [3]; etc.

The early results on MIMO channels mutual information concerned channels with centered independent and identically distributed (i.i.d.) entries. It is of interest to study the statistics of this mutual information for more practical (correlated) MIMO channels. In this course, many works established the asymptotic normality of the mutual information in the large-dimension regime for the so-called Kronecker correlated channels [4], [5], for general spatially correlated channels [6], and for general variance profile channels [7].

Another performance index of clear interest is the signal-to-noise ratio (SNR) at the output of a given receiver. In this paper, we focus on one of the most popular receivers, namely the linear Wiener receiver, also called linear minimum mean-squared error (LMMSE) receiver. In this context, an outage event occurs when the SNR at the LMMSE output lies beneath a given threshold. One purpose of this paper is to approximate the associated outage probability for an important class of MIMO channel models. Another performance index associated with the SNR is the bit-error rate (BER) which will be also studied herein.

Outage probability approximations have been provided in recent works for various channels, under very specific technical conditions (in the case where the moment generating function (MGF) [8] or the probability density function (pdf) [9] have closed-form expressions; when a first-order expansion of the pdf can be derived [10]; in the more general case, where the moment generating function can be approximated by using Padé approximations [11], etc.). All these results deal with specific situations where the statistics of the SNR could be derived for finite system dimensions.

Alternatively, by making use of large random matrix theory, one can study the behavior of the SNR in the asymptotic regime where the channel matrix dimensions grow to infinity. For fairly general channel statistical models, it is then possible to prove the convergence of the SNR to deterministic values and even establish its asymptotic normality (see, for instance, [12] and [13]). However, this Gaussian approximation is not accurate when the channel dimensions are small. This is confirmed in, e.g., [14], where it is shown that the asymptotic BER based on the sole Gaussian approximation is significantly smaller than the empirical estimate. A more precise approximation of the BER or the outage probability is expected if one chooses to approximate the SNR probability distribution with a distribution: 1) which is supported by $\mathbb{R}_+$ (indeed, a Gaussian random variable takes...
Let \( y = \sqrt{\frac{v K}{\rho}} \mathbf{H}^\dagger \mathbf{w}_0 \), then it is well known that the LMMSE estimator is given by
\[
\mathbf{g} = (\Sigma \Sigma^* + \rho \mathbf{I}_N)^{-1} y.
\]
Writing the received vector \( \mathbf{r} = \mathbf{s}_0 y + \mathbf{r}_n \), where \( s_0 y \) is the relevant term and \( \mathbf{r}_n \) represents the interference-plus-noise term, the SNR at the output of the LMMSE estimator is given by
\[
\beta_K = \left| s_0^* y \right|^2 / \left| s_0^* \mathbf{r}_n \right|^2.
\]
Plugging the expression of \( \mathbf{g} \) given above into this expression, one can show that the SNR \( \beta_K \) is given by
\[
\beta_K = y^* \left( \frac{1}{K} \mathbf{H}^\dagger \mathbf{W} \mathbf{P} \mathbf{W}^* \mathbf{H}^\dagger + \rho \mathbf{I}_N \right)^{-1} y
\]
with \( \mathbf{P} = \text{diag}(p_1, \ldots, p_K) \) and \( \mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_K] \). Let \( \Psi = \mathbf{U} \mathbf{D} \mathbf{U}^* \) be a spectral decomposition of \( \Psi \). Then, \( \beta_K \) is written as
\[
\beta_K = \frac{p_0}{K} \mathbf{w}_0^* \mathbf{D}^\dagger \left( \frac{1}{K} \mathbf{D}^\dagger \mathbf{U}^* \mathbf{W} \mathbf{P} \mathbf{W}^* \mathbf{U}^* \mathbf{D} + \rho \mathbf{I}_N \right)^{-1} \mathbf{D}^\dagger \mathbf{U}^* \mathbf{w}_0
\]
where \( \mathbf{z} = \mathbf{U}^* \mathbf{w}_0 \) (resp., \( \mathbf{Z} = \mathbf{U}^* \mathbf{W}^\dagger \)) is an \( N \times 1 \) vector with complex independent standard Gaussian entries (resp., an \( N \times K \) matrix with independent Gaussian entries).

Under appropriate assumptions, it can be proved that \( \beta_K \) admits a deterministic approximation as \( K, N \to \infty \), the ratio being bounded below by a positive constant and above by a finite constant. Furthermore, its fluctuations can be precisely described under the same asymptotic regime (for a full and rigorous computation based on random matrix theory, see [13]). As it will appear shortly, a deterministic approximation of the third centered moment of \( \beta_K \) is needed and will be computed in the sequel.

III. BER AND OUTAGE PROBABILITY APPROXIMATIONS

A. A Quick Reminder of the Generalized Gamma Distribution

Recall that if a random variable \( X \) follows a generalized gamma distribution \( \Gamma(\alpha, b, \xi) \), where \( \alpha \) and \( b \) are, respectively, referred to as the shape and scale parameters, then
\[
\mathbb{E}X = \alpha b, \quad \text{var}(X) = \alpha b^2, \quad \text{and} \quad \mathbb{E}(X - \mathbb{E}X)^3 = (\xi + 1)\alpha b^3.
\]
The pdf of the generalized gamma distribution with parameters \( (\alpha, b, \xi) \) does not have a closed form expression but its MGF \( M(s) \) is given as [17]:
\[
M(s) = \begin{cases} 
\exp \left( \frac{\alpha}{\xi} \left( 1 - (1 - b \xi s)^{-\frac{1}{\xi}} \right) \right), & \text{if } \xi > 1 \\
\left( 1 - s b \right)^{-\alpha}, & \text{if } \xi = 1 \\
\exp \left( \frac{\alpha}{\xi} \left( 1 - b \xi s \right)^{-\frac{1}{\xi}} - 1 \right), & \text{if } \xi > 1.
\end{cases}
\]
B. BER Approximation

Using quadrature phase-shift keying (QPSK) constellations with Gray encoding [18], and assuming that the noise at the LMMSE output is Gaussian, the BER $\eta_{\text{Gauss}}$ is given by

$$\eta_{\text{Gauss}} = EQ(\sqrt{\beta_K})$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$ and the expectation is taken over the distribution of the SNR $\beta_K$. Based on the asymptotic normality of the SNR, [19] and [20] proposed to use the approximation $\eta_{\text{asy. moment}}$ of the BER given by

$$\eta_{\text{asy. moment}} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dx.$$

where $\beta_K$ denotes an asymptotic deterministic approximation of the first moment of $\beta_K$. It was shown however in [15] that this expression is inaccurate since a Gaussian random variable allows negative values and has a zero third moment while the output SNR is always positive and has a nonzero third moment for finite system dimensions. To overcome these difficulties, Li et al. [15] approximate the BER by considering first that the SNR follows a gamma distribution with scale $\alpha$ and shape $b$, these parameters being tuned by equating the first two moments of the gamma distribution with the first two asymptotic moments of the SNR. However, the third asymptotic moment was shown to be different from the third moment of the gamma distribution which only depends on the scale $\alpha$ and shape $b$. In light of this consideration, Li et al. [15] refine this approximation and consider that the SNR follows a generalized gamma distribution which is adjusted by assuming that its first three moments equate the first three asymptotic moments of the SNR. As expected, this approximation has proved to be more accurate than the gamma approximation, and so will be the one considered in this paper. Next, we briefly review this technique, which we will rely on to provide accurate approximations for the BER and outage probability.

Let $E_{\infty}(\beta_K)$, $\text{var}_{\infty}(\beta_K)$, and $S_{\infty}(\beta_K)$ denote, respectively, the deterministic approximations of the asymptotic central moments of $\beta_K$. Then, the parameters $\xi$, $\alpha$, and $b$ are determined by solving

$$E_{\infty}(\beta_K) = \alpha b, \quad \text{var}_{\infty}(\beta_K) = \alpha b^2 \text{ and } S_{\infty}(\beta_K) = (\xi + 1)\alpha b^3.$$ 

thus giving the following values:

$$\alpha = \frac{(E_{\infty}(\beta_K))^2}{\text{var}_{\infty}(\beta_K)}, \quad \beta = \frac{\text{var}_{\infty}(\beta_K)}{E_{\infty}(\beta_K)}, \quad \text{and } \xi = \frac{S_{\infty}(\beta_K)E_{\infty}(\beta_K)}{(\text{var}_{\infty}(\beta_K))^2} - 1.$$ 

Using the MGF, one can use the following approximation $\eta$ of the BER by using the following relation that holds for QPSK constellation [21]:

$$\eta = \frac{1}{\pi} \int_0^{\pi/2} M \left( -\frac{1}{2 \sin^2 \theta} \phi \right) d\phi,$$

Note that similar expressions for the BER exist for other constellations and can be derived by plugging the following identity involving the function $Q(x)$ [21]:

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp \left( -\frac{x^2}{2 \sin^2 \theta} \right) d\phi$$

into the BER expression.

C. Outage Probability Approximation

Only the MGF has a closed-form expression. Knowing the MGF, one can compute numerically the cumulative distribution function by applying the saddle-point approximation technique [22]. Denote by $K(y) = \log(M(y))$ the cumulant generating function, by $y$ the threshold SNR and by $t_y$ the solution of

$$K'(t_y) = y.$$ 

Let $w_0$ and $u_0$ be given by

$$w_0 = \text{sign}(t_y)\sqrt{2(t_y y - K(t_y))}$$

and $u_0 = t_y \sqrt{K''(t_y)}$. The saddle-point approximate of the outage probability is given by

$$P_{\text{out}} = \Phi(w_0) + \phi(u_0) \left( \frac{1}{w_0} - \frac{1}{u_0} \right),$$

where $\Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ and $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ denote, respectively, the standard normal cumulative distribution function and probability distribution function.

So far, we have presented the technique that will be used in simulations for the evaluation of the BER and outage probability. This technique is heavily based on the computation of the three first asymptotic moments of the SNR $\beta_K$, an issue that is dealt with in the next section.

IV. ASYMPTOTIC MOMENTS

A. Assumptions

Recall from Section II the various definitions $K, N, D, \hat{D}$. In the following, we assume that both $K$ and $N$ go to $+\infty$, their ratio being bounded below and above as follows:

$$0 < \ell^- = \liminf \frac{K}{N} \leq \ell^+ = \limsup \frac{K}{N} < +\infty.$$ 

In the sequel, $K \to \infty$ will refer to this asymptotic regime. We will frequently write $D_K$ and $\hat{D}_K$ to emphasize the dependence in $K$, but may drop the subscript $K$ as well. Assume the following mild conditions.

Assumption A1: There exist real numbers $d_{\text{max}} < \infty$ and $\tilde{d}_{\text{max}} < \infty$ such that

$$\sup_K \|D_K\| \leq d_{\text{max}} \text{ and } \sup_K \|\hat{D}_K\| \leq \tilde{d}_{\text{max}}$$

where $\|D_K\|$ and $\|\hat{D}_K\|$ are the spectral norms of $D_K$ and $\hat{D}_K$. 

Assumption A2: The normalized traces of $D_K$ and $\hat{D}_K$ satisfy

$$\inf_K \frac{1}{K} \text{Tr}(D_K) > 0 \text{ and } \inf_K \frac{1}{K} \text{Tr}(\hat{D}_K) > 0.$$
B. Asymptotic Moments Computation

In this subsection, we provide closed-form expressions for the first three asymptotic moments. We shall first introduce some deterministic quantities that are used for the computation of the first, second, and third asymptotic moments.

Proposition 1: (cf. [4]) For every integer $K$ and any $t > 0$, the system of equations in $(\delta_{K}, \hat{\delta}_{K})$

\[
\begin{align*}
\delta_{K} &= \frac{1}{K} \text{Tr}(D_{K}(I + t D_{K}D_{K})^{-1}), \\
\hat{\delta}_{K} &= \frac{1}{K} \text{Tr}(\hat{D}_{K}\left(I + t \hat{D}_{K}\hat{D}_{K}\right)^{-1})
\end{align*}
\]

admits a unique solution $(\delta_{K}(t), \hat{\delta}_{K}(t))$ satisfying $\delta_{K}(t) > 0$, $\hat{\delta}_{K}(t) > 0$. Let $T$ and $\hat{T}$ be the $N \times N$ and $K \times K$ diagonal matrices defined by

\[
T = (I + t \delta_{K}D)^{-1} \quad \text{and} \quad \hat{T} = (I + t \hat{\delta}_{K}\hat{D})^{-1}.
\]

Note that in particular: $\delta = \frac{1}{K} \text{Tr}(DT)$ and $\hat{\delta} = \frac{1}{K} \text{Tr}(\hat{D}\hat{T})$. Define also $\gamma$ and $\hat{\gamma}$ as $\gamma = \frac{1}{K} \text{Tr}(D^2T^2)$ and $\hat{\gamma} = \frac{1}{K} \text{Tr}(\hat{D}^2\hat{T}^2)$. Finally, replace $t$ by $\frac{1}{\rho}$ and introduce the following deterministic quantities:

\[
\Omega_{K}^{2} = \frac{\gamma_{\hat{\gamma}}}{\rho^{2}} \left(\frac{\gamma_{\hat{\gamma}}}{\rho^{2} - \gamma_{\hat{\gamma}}} + 1\right), \quad \nu_{K} = \frac{2\rho^{3}}{K (\rho^{2} - \gamma_{\hat{\gamma}})^{3}} \left[\text{Tr}(D^3T^3) - \frac{\gamma_{\hat{\gamma}}^{3}}{\rho^{3}} \text{Tr}(\hat{D}^3\hat{T}^3)\right].
\]

As usual, $\alpha_{K} = O(\beta_{K})$ means that $\alpha_{K}(\beta_{K})^{-1}$ is uniformly bounded as $K \rightarrow \infty$. Then, the first three asymptotic moments are given by the following theorem.

Theorem 1: Assuming that the matrices $D$ and $\hat{D}$ satisfy the conditions stated in A1 and A2, then the following convergences hold true.

1) First asymptotic moment [12], [13]

\[
\frac{\delta_{K}}{\rho} = O(1) \quad \text{and} \quad \frac{\delta_{K}}{\rho} \quad K \rightarrow \infty \quad \rightarrow 0.
\]

2) Second asymptotic moment [12], [13]

\[
\Omega_{K} = O(1) \quad \text{and} \quad KE\left(\frac{\beta_{K}}{\rho_{0}} - E\left(\frac{\beta_{K}}{\rho_{0}}\right)\right)^{2} - \Omega_{K}^{2} \quad K \rightarrow \infty \quad \rightarrow 0.
\]

3) Third asymptotic moment

\[
\nu_{K} = O(1) \quad \text{and} \quad KE\left(\frac{\beta_{K}}{\rho_{0}} - E\left(\frac{\beta_{K}}{\rho_{0}}\right)\right)^{3} - \nu_{K} \quad K \rightarrow \infty \quad \rightarrow 0.
\]

The two first items of the theorem are proved in [13] (beware that the notations used in this article are the same as those in [4] and slightly differ from those used in [13]). Proof of the third item of the theorem is postponed to the Appendix.

Remark 1: One can note that the third asymptotic moment is of order $O(K^{-2})$. This is in accordance with the asymptotic normality of the SNR, where the third moment of $\sqrt{K}(\beta_{K} - E(\beta_{K}))$ will eventually vanish, as this quantity becomes closer to a Gaussian random variable. However, its value remains significant for small-dimension systems.

V. SIMULATION RESULTS

In our simulations, we consider a MIMO system in the uplink direction. The base station is equipped with $N$ receiving antennas and detects the symbols transmitted by a particular user in the presence of $K$ interfering users. We assume that the correlation matrix $\Psi$ is given by $\Psi(i,j) = \frac{1}{\sqrt{N}} e^{i\pi ij}$ with $0 \leq a < 1$. Recall that $\hat{P}$ is the matrix of the interfering users’ powers. We set $\hat{P}$ (up to a permutation of its diagonal elements) to

\[
\hat{P} = \begin{cases} 
\text{diag}(\{4P, 5P\}), & \text{if } K = 2 \\
\text{diag}(\{P, P, 2P, 4P\}), & \text{if } K = 4
\end{cases}
\]

where $P$ is the power of the user of interest. For $K = 2^{p}$ with $3 \leq p \leq 5$, we assume that the powers of the interfering sources are arranged into five classes as in Table I. We investigate the impact of the correlation coefficient $a$ on the accuracy of the asymptotic moments when the input SNR is set to 15 dB for $N = K$ (Fig. 1) and $N = 2K$ (Fig. 2). In these figures, the relative error on the estimated first three moments $\frac{||\hat{\mu} - \mu||}{\mu}$ and $\mu$ denote, respectively, the asymptotic and empirical moments is depicted with respect to the correlation coefficient $a$. These simulations show that when the number of antennas is small, the asymptotic approximation of the second and third moments degrades for large correlation coefficients ($a$ close to one). Despite these discrepancies for $a$ close to 1, simulations show that the BER and the outage probability are well approximated even for small system dimensions. Indeed, Fig. 3 shows the evolution of the empirical BER and the theoretical BER predicted by (1) versus the input SNR for different values of $\beta_{K}$ and $N$. In Fig. 4, the saddle point-approximate of the outage probability given by (2) is compared with the empirical one. In both Figs. 3 and 4, 2000 channel realizations have been considered, and in Fig. 4, the input SNR has been set to 15 dB. These figures show that even for small system dimensions, the BER is well approximated for a wide range of SNR values. For high SNR values, the proposed approximation tends to underestimate the BER. This tends to show that one should go beyond the first three moments and take into account higher order moments to estimate more accurately the BER at high SNR. The outage probability is also well approximated except for small values of the SNR threshold that are likely to be in the tail of the asymptotic distribution.

APPENDIX I
PROOF OF THEOREM 1

In the sequel, we shall heavily rely on the results and techniques developed in [4]. In the sequel, $D$ and $\hat{D}$ are,
are i.i.d. standard complex Gaussian, $\mathbf{X}$ is an $N \times K$ matrix defined by
\[ \mathbf{X} = \mathbf{D}^{1/2} \mathbf{Z} \mathbf{D}^{1/2}. \]
We shall often write $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_K]$ where the $\mathbf{x}_j$’s are the columns of $\mathbf{X}$. We recall hereafter the mathematical tools that will be of constant use in the sequel.

A. Notations
Define the resolvent matrix $\mathbf{H}$ by
\[ \mathbf{H} = \left( \frac{t}{K} \mathbf{D}^{1/2} \mathbf{Z} \mathbf{D}^{1/2} + \mathbf{I}_N \right)^{-1} = \left( \frac{t}{K} \mathbf{X} \mathbf{X}^\ast + \mathbf{I}_N \right)^{-1}. \]
We introduce the following intermediate quantities:
\[ \beta(t) = \frac{1}{K} \text{Tr}(\mathbf{DH}), \quad \alpha(t) = \frac{1}{K} \text{Tr}(\mathbf{DEH}), \quad \text{and} \quad \beta = \beta - \alpha. \]
Matrix
\[ \hat{\mathbf{R}}(t) = \text{diag}(r_1, \ldots, r_K) \]
is a $K \times K$ diagonal matrix defined by
\[ \hat{\mathbf{R}}(t) = (\mathbf{I} + t\hat{\alpha}(t) \hat{\mathbf{D}}_K)^{-1}. \]
Let $\hat{\alpha} = \frac{1}{t} \text{Tr}(\hat{\mathbf{D}} \hat{\mathbf{R}})$. Then, matrix $\mathbf{R}(t) = \text{diag}(r_1, \ldots, r_N)$ is an $N \times N$ matrix defined by
\[ \mathbf{R}(t) = (\mathbf{I} + t\hat{\alpha}(t) \mathbf{D})^{-1}. \]

B. Mathematical Tools
The results below, of constant use in the proof of Theorem 1, can be found in [4].
1) Differentiation Formulas:
\[ \frac{\partial \mathbf{H}_{pq}}{\partial X_{ij}} = -\frac{t}{K} [X^\ast \mathbf{H}]_{pq} H_{pq} = -\frac{t}{K} [\mathbf{x}_j^\ast \mathbf{H}]_{pq} H_{pq}, \]
\[ \frac{\partial \mathbf{H}_{pq}}{\partial X_{ij}} = -\frac{t}{K} [\mathbf{H} \mathbf{x}_j]_{pq} H_{iq} = -\frac{t}{K} [\mathbf{H} \mathbf{x}_j]_{pq} H_{iq} \]
2) Integration by Parts Formula for Gaussian Functionals: Let $\Phi$ be a $\mathcal{C}^1$ complex function polynomially bounded together with its derivatives, then
\[ E[\mathbf{X}_{ij} \Phi(\mathbf{X})] = d_d d_j E \left[ \frac{\partial \Phi(\mathbf{X})}{\partial X_{ij}} \right]. \]
3) Poincaré–Nash Inequality: Let $\mathbf{X}$ and $\Phi$ be as above, then
\[ \text{Var}(\Phi(\mathbf{X})) \leq \sum_{i=1}^{N} \sum_{j=1}^{K} d_i d_j E \left[ \left| \frac{\partial \Phi(\mathbf{X})}{\partial X_{ij}} \right|^2 + \left| \frac{\partial \Phi(\mathbf{X})}{\partial X_{ij}} \right|^2 \right]. \]
whose spectral norm are uniformly bounded in $K$, then the following hold true:
\[
\frac{1}{K} \text{Tr}(AR) = \frac{1}{K} \text{Tr}(AT) + O(K^{-2})
\]
\[
\frac{1}{K} \text{Tr}(BR) = \frac{1}{K} \text{Tr}(BT) + O(K^{-2}).
\]

Proposition 3: Let $(A_K), (B_K)$, and $(C_K)$ be three sequences of $N \times N, K \times K$, and $N \times N$ diagonal deterministic matrices whose spectral norm are uniformly bounded in $K$. Consider the following functions:
\[
\Phi(X) = \frac{1}{K} \text{Tr} \left( A H \frac{X B X^*}{K} \right)
\]
\[
\Psi(X) = \frac{1}{K} \text{Tr} \left( A H D \frac{X B X^*}{K} \right).
\]
Then,
1) the following estimations hold true:
\[
\text{var} \Phi(X), \text{var} \Psi(X), \text{var}(\beta), \text{and} \quad \text{var} \left( \frac{1}{K} \text{Tr} AHCH \right) \quad \text{are} \quad O(K^{-2});
\]
2) the following approximations hold true:
\[
E[\Phi(X)] = \frac{1}{K} \text{Tr} \left( \bar{D} \bar{T} B \right) \frac{1}{K} \text{Tr} \left( ADT \right) + O(K^{-2}) \quad (7)
\]
\[
E[\Psi(X)] = \frac{1}{1 - \bar{\sigma}^2 \gamma^2} \left( \frac{1}{K^2} \text{Tr} \left( \bar{D} \bar{T} B \right) \text{Tr} (AD^2T^2) \right. \\
\left. - \frac{1}{K^2} \text{Tr} \left( \bar{D} \bar{T}^2 B \right) \text{Tr} (ADT) \right) + O(K^{-2}) \quad (8)
\]
\[
E \frac{1}{K} \text{Tr}[AHDH] = \frac{1}{1 - \bar{\sigma}^2 \gamma^2} \frac{1}{K} \text{Tr}(ADT^2) + O(K^{-2}). \quad (9)
\]

Proofs of Propositions 2 and 3 are essentially provided in [4]. In the same vein, the following proposition will be needed.

Proposition 4: Let $(A_K), (B_K)$, and $(C_K)$ be three sequences of $N \times N, K \times K$, and $N \times N$ diagonal deterministic matrices whose spectral norm are uniformly bounded in $K$. Consider the following function:
\[
\varphi(X) = \frac{1}{K} \text{Tr} \left[ A H A H A H \frac{X B X^*}{K} \right].
\]
Then $\text{var} \varphi(X) = O(K^{-2})$ and $\text{var} \left( \frac{1}{K} \text{Tr} A H A H A H \right) = O(K^{-2})$. 

Proof of Proposition 4 is essentially the same as the proof of Proposition 3 part 1). It is provided for completeness and postponed to Appendix II.

C. End of Proof of Theorem 1

We are now in position to complete the proof of Theorem 1. Using the notations of [4], the SNR can be written as
\[
\beta_K = \frac{t p_0}{K} z^T D^2 H(t) D^2 z
\]

Fig. 2. Absolute value of the relative error when $N = 2K$. (a) First moment of the SNR, (b) Second moment of the SNR, (c) Third moment of the SNR.

4) Deterministic Approximations and Various Estimations:

Proposition 2: Let $(A_K)$ and $(B_K)$ be two sequences of, respectively, $N \times N$ and $K \times K$ diagonal deterministic matrices.
Hence, the third moment is given by

\[
\begin{align*}
E(\beta_K - E(\beta_K))^3 &= (\frac{t\tau_0}{K^3})^3 E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^3, \\
&= (\frac{t\tau_0}{K^3})^3 E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} + \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^3 \\
&= (\frac{t\tau_0}{K^3})^3 \left[ E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^3 \\
&+ 3E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right) \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^2 \\
&+ 3E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right) \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^2 \\
&+ 3E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^2 \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right) \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^2 \\
&+ E \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^3 \right] \\
&= (\frac{t\tau_0}{K^3})^3 \left[ E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^3 \\
&+ 3E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right) \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^2 \\
&+ 3E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right) \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^2 \\
&+ 3E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^2 \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right) \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^2 \\
&+ E \left( \text{Tr}D\text{HD} - \text{ETr}D\text{HD} \right)^3 \right].
\end{align*}
\]

Fig. 3. BER versus input SNR. (a) \( N = K = 4 \) and \( \alpha = 0 \). (b) \( N = K = 4 \) and \( \alpha = 0.9 \). (c) \( N = 2K = 4 \) and \( \alpha = 0 \). (d) \( N = 2K = 4 \) and \( \alpha = 0.9 \).

where \( t = \frac{1}{\beta} \). Hence,

\[
E(\beta_K - E(\beta_K))^3 = \frac{(t\tau_0)^3}{K^3} E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^3
\]

In order to deal with the first term of the right-hand side of (10), notice that if \( M \) is a deterministic matrix and \( \mathbf{x} \) is a standard Gaussian vector, then

\[
E(\mathbf{x}^* M \mathbf{x} - \text{Tr}M)^3 = \text{Tr}(M^3)E(|x_1|^2 - 1)^3
\]

(such an identity can be easily proved by considering the spectral decomposition of \( M \)). Hence

\[
E \left( z^* D^{1/2} H D^{1/2} z - \text{Tr}D\text{HD} \right)^3 = E \text{Tr}(D)^3 E \left( |Z_{11}|^2 - 1 \right)^3
\]

\[
= 2E \text{Tr}(DHDHHD).
\]
Indeed and according to Proposition 3. It remains to deal with concentration results for the spectral measure of random matrices [23] (see also [15, eqs. (86)–(87)], where details are provided). Consequently, we end up with the following approximation:

\[ K^2 \mathbb{E} (\beta_K - \mathbb{E}(\beta_K))^3 \]

which is \( \mathcal{O}(1) \) according to Proposition 3. It remains to deal with \( \mathbb{E}(\text{Tr}DH - \text{Tr}DH)^3 \), which can be proved to be uniformly bounded in \( K \) using concentration results for the spectral measure of random matrices [23] (see also [15, eqs. (86)–(87)], where details are provided). Consequently, we end up with the following approximation:

\[ \frac{(t\rho_0)^3}{K} \mathbb{E} \left( |Z_{11}|^2 - 1 \right)^3 \mathbb{E}\text{Tr}DH^2DH^2DH + \mathcal{O}(K^{-1}) \]

which is deterministic but still depends on the distribution of the entries via the expectation operator \( \mathbb{E} \). The rest of the proof is devoted to providing a deterministic approximation of \( \mathbb{E}\text{Tr}DH^2DH^2DH \) depending on \( \gamma, \bar{z}, T \) and \( \bar{T} \). Note that \( H = I - \frac{1}{K}HXX^* \), thus

\[
[\text{HDHD}]_{pp} = [\text{HDHD}]_{pp} - t \left[ \text{HDHD} \frac{XX^*}{K} \right]_{pp}
\]

\[
= [\text{HDHD}]_{pp} - \frac{t}{K} \sum_{j=1}^{K} [\text{HDHD}Hx_j]_{pp} X_{pj}, \quad (11)
\]
Let us deal with the second term of (11). We have

\[
\frac{1}{K} \mathbb{E} \left[ \text{HD} \mathbf{d} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} = \frac{1}{K} \sum_{k=1}^{N} \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{d} \mathbf{H} \mathbf{x}_k | p \right] \mathbb{X}_{kj} \mathbb{X}_{pj}.
\]

Using the integration by part formula (5), we get

\[
\mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} = \sum_{k=1}^{N} d_k \mathbb{d}_j \delta(p - k) \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{d} \mathbf{H} \mathbf{x}_k \right] + \sum_{k=1}^{N} d_k \mathbb{d}_j
\]

\[
\times \mathbb{E} \left[ \mathbb{X}_{pj} \sum_{l,m=1}^{N} \frac{\partial [H_{pk} d_{lm} H_{km} H_{mk}]}{\partial x_{kj}} \right]
\]

\[
= d_p \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] - \frac{t}{K} \sum_{k,l,m=1}^{N} d_k \mathbb{d}_j d_{lm} d_{k,l,m} \mathbb{E} \left[ \mathbb{X}_{pj} H_{pk} \mathbb{H} \mathbf{x}_j | p \right] H_{km} H_{mk} \]

\[
- \frac{t}{K} \sum_{k,l,m=1}^{N} d_k \mathbb{d}_j d_{lm} d_{k,l,m} \mathbb{E} \left[ H_{pk} H_{km} \mathbb{H} \mathbf{x}_j | p \right] H_{kk} \]

\[
= d_p \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] - \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right) \mathbb{T} \left( \mathbf{D} \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
- \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
- \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

Multiplying the right- and the left-hand sides by \( \tilde{r}_j = \frac{1}{1 + t \mathbb{d}_j} \), we get

\[
\mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} = \tilde{r}_j \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} - \frac{t}{K} \tilde{r}_j
\]

\[
\times \mathbb{E} \left[ \mathbb{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{d}_j \mathbb{T} \left( \mathbf{D} \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right) \mathbb{T} \left( \mathbf{D} \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
- \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
- \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right).
\]

Plugging (12) into (11), we obtain

\[
\mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} = \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} - \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
+ \frac{t}{K} \sum_{j=1}^{K} \tilde{r}_j \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
+ \frac{t}{K} \sum_{j=1}^{K} \tilde{r}_j \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
+ \frac{t}{K} \mathbb{d}_j \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

Hence

\[
(1 + t \mathbb{d}_j) \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} = \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} + \frac{t^2}{K^2} \mathbb{d}_j
\]

\[
\times \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
+ \frac{t^2}{K^2} \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right)
\]

\[
+ \frac{t^2}{K^2} \mathbb{E} \left[ \mathbb{H} \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right] \mathbb{X}_{pj} \mathbb{T} \left( \mathbf{D} \mathbf{H} \mathbf{D} \mathbf{H} \mathbf{x}_j | p \right).
\]
Multiplying by \(d_p\), summing over \(p\), and dividing by \(K\), we obtain

\[
E \left( \frac{1}{K} \text{Tr}[\mathbf{DHDHG}] \right) = E \left( \frac{1}{K} \sum_{p=1}^{K} d_p [\text{HDHDG}]_{pp} \right) = \frac{1}{K} \sum_{p=1}^{K} r_p d_p E[\text{HDHG}]_{pp} + \frac{\ell^2}{K^3} \times E[\text{Tr}(\mathbf{DHDHG}) \text{Tr}(\mathbf{DRXHDX})] + \frac{\ell^2}{K^2} E \beta \text{Tr}(\mathbf{DRDHDHGDX})
\]

\[\triangleq \chi_1 + \chi_2 + \chi_3 + \chi_4 \tag{14}\]

where

\[
\chi_1 = \frac{1}{K} E \text{Tr}(\mathbf{DRHDHG})
\]

\[
\chi_2 = \frac{\ell^2}{K} E \text{Tr}(\mathbf{DHDHG}) \frac{1}{K} \text{Tr} \left( \frac{\mathbf{DRH} \mathbf{HDX}}{K} \right)
\]

\[
\chi_3 = \frac{\ell^2}{K} E \text{Tr}(\mathbf{DHDHG}) \frac{1}{K} \text{Tr} \left( \frac{\mathbf{DRHDHGDX}}{K} \right)
\]

\[
\chi_4 = \frac{\ell^2}{K} E \beta \text{Tr} \left( \frac{\mathbf{DRDHDHGDX}}{K} \right)
\]

According to Proposition 3, \(\text{var} \frac{1}{K} \text{Tr}(\mathbf{DRHDXHDX})\) is of order \(O(K^{-2})\). Similarly, \(\text{var}(\beta) = O(K^{-2})\). Hence, using Cauchy–Schwarz inequality, we get the estimation \(\chi_4 = O(K^{-2})\). It remains to work out the expressions involved in \(\chi_1, \chi_2\) and \(\chi_3\) by removing the terms with expectation and replacing them with deterministic equivalents.

Since \(\text{var} \frac{1}{K} \text{Tr}(\mathbf{DRHDHG}) = O(K^{-2})\) by Proposition 3 and \(\text{var} \frac{1}{K} \text{Tr}(\mathbf{DHDHG}) = O(K^{-2})\) by Proposition 4, we have

\[
\chi_2 = \frac{\ell^2}{K} E \text{Tr}(\mathbf{DHDHG}) E \left( \frac{1}{K} \text{Tr} \left[ \frac{\mathbf{DRH} \mathbf{HDX}}{K} \right] \right) + O(K^{-2})
\]

\[
\chi_3 = \frac{\ell^2}{K} E \text{Tr}(\mathbf{DHDHG}) \frac{1}{K} \text{Tr}(\mathbf{DRDHT}) + O(K^{-2})
\]

where (a) follows from Proposition 3 part 2) and (b), from Proposition 2. Similar arguments yield

\[
\chi_1 = \frac{1}{1 - \ell^2 \gamma} \frac{1}{K} \text{Tr}(\mathbf{D^2 RDT}) + O(K^{-2})
\]

\[
\chi_1 = \frac{1}{1 - \ell^2 \gamma} \frac{1}{K} \text{Tr}(\mathbf{D^3 T}) + O(K^{-2})
\]

Plugging (16), (15), and (17) into (14), we obtain

\[
\frac{1}{K} E \text{Tr}(\mathbf{DHDHG}) = \left( \frac{\beta \frac{1}{K} - E[\beta \frac{1}{K}]}{p_0} \right)^3
\]

\[
K^2 \left( \frac{\beta \frac{1}{K} - E[\beta \frac{1}{K}]}{p_0} \right)^3 = \frac{1}{K} (\ell^2 \gamma)^3 \text{Tr}(\mathbf{D^3 T}) - \frac{\ell^3 \gamma^3}{K} \text{Tr}(\mathbf{D^2 T}) + O(K^{-2})
\]

Hence

\[
\nu_K = \frac{2p^3}{K(p^2 - \gamma^2)^3} \left[ \text{Tr}(\mathbf{D^3 T}) - \frac{\gamma^3}{p^3} \text{Tr}(\mathbf{D^3 T}) \right] + O \left( \frac{1}{K} \right)
\]

The fact that

\[
\text{var}(\varphi(X)) \leq \sum_{i=1}^{N} \sum_{j=1}^{K} d_i \tilde{d}_j \left| \frac{\partial \varphi}{\partial X_{ij}} \right|^2 + \sum_{i=1}^{N} \sum_{j=1}^{K} d_i \tilde{d}_j \left| \frac{\partial \varphi}{\partial X_{ij}} \right|^2
\]

is of order \(O(1)\) is straightforward and its proof is omitted. Proof of Theorem 1 is completed.

**APPENDIX II**

**PROOF OF PROPOSITION 4**

The proof mainly relies on Poincaré–Nash inequality. Using the Poincaré–Nash inequality, we have

\[
\text{var}(\varphi(X)) \leq \sum_{i=1}^{N} \sum_{j=1}^{K} d_i \tilde{d}_j \left| \frac{\partial \varphi}{\partial X_{ij}} \right|^2 + \sum_{i=1}^{N} \sum_{j=1}^{K} d_i \tilde{d}_j \left| \frac{\partial \varphi}{\partial X_{ij}} \right|^2
\]
We only deal with the first term of the last inequality (the second term can be handled similarly). We have
\[ \varphi(X) = \frac{1}{K^2} \sum_{p,r,s,t} \sum_{i=1}^K c_{pp} H_{pr} A_{rt} H_{rs} A_{st} X_{tu} B_{uv} X_{pu}^*. \]
After straightforward calculations using the differentiation formula (3), we get
\[ \frac{\partial \varphi}{\partial X_{ij}} = \phi_{ij}^{(1)} + \phi_{ij}^{(2)} + \phi_{ij}^{(3)} + \phi_{ij}^{(4)} \]
where
\[ \phi_{ij}^{(1)} = -\frac{t}{K^3} [X^* H A H A X B^* C H]^* \]
\[ \phi_{ij}^{(2)} = -\frac{t}{K^3} [X^* H A H X B^* C A H]^* \]
\[ \phi_{ij}^{(3)} = -\frac{t}{K^3} [X^* H X B^* C A H]^* \]
\[ \phi_{ij}^{(4)} = \frac{1}{K^2} [B X^* C A H]^* \]
Hence
\[ \left| \frac{\partial \varphi}{\partial X_{ij}} \right|^2 \leq 4 \left( |\phi_{ij}^{(1)}|^2 + |\phi_{ij}^{(2)}|^2 + |\phi_{ij}^{(3)}|^2 + |\phi_{ij}^{(4)}|^2 \right) \]
and we get the inequality given at the top of the page. We only prove that the first term of the right-hand side is of order \( K^{-2} \); the other terms being handled similarly. Using Cauchy-Schwarz inequality, we get
\[
4 \sum_{i=1}^N \sum_{j=1}^K d_d d_j \mathbb{E} \left[ |\phi_{ij}^{(1)}|^2 \right] \\
\leq \frac{4t^2}{K^6} \text{detr} ||H||^2 ||C||^2 \text{ETr} \\
	imes ((HA)^2 H X D X^* H (AH)^2 (XB^*)^2) \\
\leq \frac{4t^2}{K^6} \text{detr} ||H||^2 ||C||^2 \left( \text{ETr}(HA)^2 H X D X^* H \\
	imes (AH)^2 (HA)^2 H X D X^* H (AH)^2 \right)^{\frac{1}{2}} \\
\times \left( \text{ETr} (XB^*)^4 \right)^{\frac{1}{2}} \\
\leq \frac{4t^2}{K^2} \text{detr} ||H||^2 ||C||^2 \\
\times ||A||^4 \sqrt{\frac{1}{K} \left( \frac{XD^*}{K} \right)^2} \sqrt{\frac{1}{K} \left( \frac{XB^*}{K} \right)^4} \]
where the first inequality follows by using the fact that \( \text{Tr}(AB) \leq ||B|| \text{Tr}(A) \), \( A \) being Hermitian nonnegative matrix and the second follows by applying twice Cauchy–Schwarz inequalities
\[ \text{Tr}(AB) \leq \sqrt{\text{Tr}(A^* A)} \sqrt{\text{Tr}(BB^*)} \]
and
\[ \mathbb{E} X Y \leq \sqrt{\mathbb{E} X^2 \mathbb{E} Y^2}. \]
We end the proof of the first statement by using the fact that \( \frac{1}{K} \text{Tr}(X B^* X)^n \) is uniformly bounded in \( K \) whenever \( B \) is a sequence of diagonal matrices with uniformly bounded spectral norm and \( n \) is a given integer.

The second statement follows from the resolvent identity
\[ \frac{1}{K} \text{Tr}(A H A H A H \ldots) = \frac{1}{K} \text{Tr}(A H A H A H X X \ldots). \]
According to the first part of the proposition
\[ \text{var} \left( \frac{1}{K} \text{Tr}(A H A H A H X X \ldots) \right) = \mathcal{O}(K^{-2}). \]
Now, \( \text{Tr}(A H A H) = \text{Tr} A^2 H A H \) and \( \text{var} \frac{1}{K} \text{Tr} A^2 H A H = \mathcal{O}(K^{-2}) \) by Proposition 3 part 1). Hence, applying inequality
\[ \text{var}(X + Y) \leq \text{var}(X) + \text{var}(Y) + 2 \sqrt{\text{var}(X) \text{var}(Y)} \]
yields the desired result. Proof of Proposition 4 is completed.

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