SUBDIVISIONAL SPACES AND GRAPH BRAID GROUPS

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Abstract. We study the problem of computing the homology of the configuration spaces of a finite cell complex $X$. We proceed by viewing $X$, together with its subdivisions, as a subdivisional space—a kind of diagram object in a category of cell complexes. After developing a version of Morse theory for subdivisional spaces, we decompose $X$ and show that the homology of the configuration spaces of $X$ is computed by the derived tensor product of the Morse complexes of the pieces of the decomposition, an analogue of the monoidal excision property of factorization homology.

Applying this theory to the configuration spaces of a graph, we recover a cellular chain model due to Świątkowski. Our method of deriving this model enhances it with various convenient functorialities, exact sequences, and module structures, which we exploit in numerous computations, old and new.

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1. Introduction

Consider the following problem: given a cell structure on a space $X$, compute the homology of the configuration space

$$B_k(X) := \{(x_1, \ldots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\}/\Sigma_k.$$ 

In this work, we provide new tools to address this problem by combining ideas from homotopy theory and robotics. We apply these tools to study the homology of configuration spaces of graphs.
1.1. Configuration spaces and gluing. First introduced in [FN62], configuration spaces of manifolds have a long and rich history in algebraic topology—for a taste, we direct the reader to [Arn69], [Seg73], [CLM76], [Tot96], and [LS05]. One powerful idea in this area has been the idea that configuration spaces behave predictably under collar gluings; that is, if we decompose a manifold $M$ into two open submanifolds $M_0$ and $M_1$ glued along the thickening of an embedded codimension one submanifold $N$, then the configuration spaces of $M$ are in some sense determined by the configuration spaces of these smaller manifolds. This decomposition technique has borne considerable fruit—see [McD75], [Böd87], [BC88], [BCT89], [FT00], and [Knu17].

One articulation of this gluing property comes from the theory of factorization homology, which provides a quasi-isomorphism

$$C_\ast(B(M)) \simeq C_\ast(B(M_0)) \bigotimes_{C_\ast(B(N \times \mathbb{R}))} C_\ast(B(M_1)),$$

where $B(M) := \bigsqcup_{k \geq 0} B_k(M)$ [AF15]. Unfortunately, these chain complexes are only algebras and modules up to structured homotopy, so this quasi-isomorphism may be difficult to use in computations. To address this issue, we will combine this gluing property with ideas drawn from a different school of thought.

1.2. Configuration spaces and cell structures. Since their introduction in [GK98] and [Ghr02], configuration spaces of graphs have been studied intensively—see [Abr00], [Far03], and [Far05] and the references therein. According to a theorem of [Ghr02], the configuration spaces of a graph $\Gamma$ are all classifying spaces for their fundamental groups, the so-called graph braid groups of $\Gamma$.

The key to approaching these spaces is to notice that the cell structure of a graph $\Gamma$ yields a cellular approximation $B_\square^k(\Gamma) \to B_k(\Gamma)$, which becomes a homotopy equivalence after finite subdivision [Abr00]. Thus, one may apply the Morse theory for cell complexes developed in [For98]. This approach has led to many computations—see [FS05], [Far06], [FS08], [FS12], [KKP12], [Sab09], and [KP12].

There are two immediate obstacles to extending the success of cellular methods in the case of a graph to higher dimensional complexes. First, the geometry in higher dimensions may be too difficult to handle directly. To address this problem, we will use the cut-and-paste idea employed in the manifold case, working with locally defined Morse data. Second, the approximation $B_\square^k$ may fail to capture the homotopy type of the configuration space in higher dimensions, even after finite subdivision.

1.3. Subdivisional spaces and decomposition. Fortunately, the cellular approximation does improve with subdivision, and, by applying the method of [Abr00], we prove in Theorem 2.8 that, for any suitably convergent set $\mathcal{P}$ of subdivisions of $X$, the map

$$\text{colim}_{X' \in \mathcal{P}} B_\square^k(X') \to B_k(X)$$

is a weak homotopy equivalence. Thus, in the presence of such a convergent subdivisional structure $\mathcal{P}$, the study of the configuration spaces of $X$ is equivalent to the study of the diagram of configuration complexes determined by $\mathcal{P}$.
These diagrammatics are formalized within the framework of subdividal spaces, which behave rigidly in some ways and like continuous objects in others. In particular, cellular chains and an abstract form of Morse theory lift effortlessly to the subdividal context, whereas the subdividal configuration space \( B^{SD}(X) \) captures the homotopy type of \( B(X) \). Our first main result is a gluing theorem in this context.

**Decomposition Theorem** (Theorem 3.19). Given a decomposition \( X \cong X_0 \amalg A \times I \)
\( X_1 \) as subdividal spaces, together with suitable locally-defined Morse data, the homology of \( B(X) \) is computed by the derived tensor product of Morse complexes

\[
I(B^{SD}(X_0)) \otimes_{I(B^{SD}(A \times I))} I(B^{SD}(X_1)).
\]

1.4. Świątkowski complexes. In the second half of the paper, we apply this theory to the semi-classical case of graphs. What results is a family of chain complexes computing the homology of graph braid groups, which first appeared in the work of Świątkowski [Świ01]. Let \( \Gamma \) be a graph with vertices \( V \) and edges \( E \). We set

\[
S(\Gamma) := \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S(v),
\]

where \( S(v) \) is the free Abelian group generated by \( \{ \emptyset, v \} \amalg H(v) \). Here \( H(v) \) is the set of half-edges at \( v \). The Świątkowski complex of \( \Gamma \) is the (bigraded differential) \( \mathbb{Z}[E] \)-module \( S(\Gamma) \)—see §4.2 for details. Our second main result is the following:

**Comparison Theorem** (Theorem 4.5). There is a natural isomorphism of bigraded Abelian groups

\[
H_*(B(\Gamma)) \cong H_*(S(\Gamma)).
\]

To derive Theorem 4.5 from the decomposition theorem, we fragment \( \Gamma \) completely. We take \( \Gamma_0 \) to be a disjoint union over \( v \in V \) of the star graphs \( S_{d(v)} \), where \( d(v) \) is the number of half-edges incident on \( v \). We take \( \Gamma_1 \) to be a disjoint union of intervals, one for each edge in \( E \). We obtain \( \Gamma \) by gluing these pieces along \( 2|E| \) disjoint intervals. We define local Morse data by putting the vertex of each star graph “at the top,” so that configurations flow down the legs (see §4.3)—and the resulting Morse complex is a reduced version of the Świątkowski complex for the star graph. The decomposition theorem gives the isomorphism

\[
H_*(B(\Gamma)) \cong H_* \left( \bigotimes_{v \in V} \mathbb{Z}[H(v)] \otimes S(v) \otimes \mathbb{Z}[E], \partial \right),
\]

and the righthand side is isomorphic to \( S(\Gamma) \) by inspection.

1.5. Homology computations. The Świątkowski complex has many desirable features. It is finite dimensional in each bidegree and finitely generated as a \( \mathbb{Z}[E] \)-module. It connects configuration spaces of different cardinalities by the action of \( \mathbb{Z}[E] \). It depends only on intrinsic graph theoretic data and requires no choice of subdivision. It decomposes geometrically, assigning a short exact sequence to the removal of a vertex. It is functorial for embeddings among graphs, so relations at the level of atomic subgraphs impose global constraints. Some of these properties are evident or implicit already in [Świ01]; several are new. These features amount to a robust computational toolkit, which we exploit extensively in §5 and Appendix C.
1.6. **How to read this paper.** The reader concerned mainly with graph braid groups may wish to start with just enough of §4 to see our conventions on graphs and the definition of the Świątkowski complex before skipping directly to the computations of §5, returning to the theory later. Starting from the beginning is recommended for the reader interested in configuration spaces in general, higher dimensional applications, or variations on the ideas of factorization homology.

1.7. **Relation to previous work.** This paper grew out of the desire to combine the local-to-global approach to configuration spaces of graphs promised by the stratified factorization homology developed in [AFT17] with the combinatorial character and computational ease of the discrete Morse theoretic model of [FS05], following [Abr00]. Although we do not directly employ the results of any of this work, its ideas permeate the theory developed in §2–4.

The Świątkowski complex first appeared in [Świ01] (see also [Lüt14]). There Świątkowski constructed a cubical complex lying inside $B_k(\Gamma)$ as a deformation retract to study the fundamental group. The cellular chain complex of this cubical complex is isomorphic to the weight $k$ subcomplex of $S(\Gamma)$. This observation implies a weaker version of Theorem 4.5 which contains more direct geometric content.

A similar edge stabilization mechanism, in a different complex and for trees only, was studied by Ramos [Ram16].

1.8. **Future directions.** We defer pursuit of the following ideas to future work.

1. **Edge stabilization.** In the sequel to this paper, we show that the $\mathbb{Z}[E]$-action is geometric, arising from a new family of stabilization maps at the level of the configuration spaces themselves, and we carry out a detailed investigation of its properties.

2. **Destabilization.** Dual to the process of adding points is that of splitting configurations apart, which may be phrased as a cocommutative coalgebra structure for which $\mathbb{Z}[E]$ acts by coderivations.

3. **Higher dimensions.** Little research has been done on configuration spaces of higher dimensional cell complexes in general—see [Gal01] and [AP17] for rare examples. Replicating the computational success of the Świątkowski complex in higher dimensions amounts to identifying tractable local Morse data.

4. **Cup products.** The diagonal is not a cellular embedding, but it is an embedding of subdivisional spaces, so our methods may shed light on the cohomology rings of configuration spaces. This is already very interesting for graphs—see [Sab09].

5. **Ordered configurations.** Our program translates with minor modifications to the context of ordered configurations, and we expect to recover an enhanced version of the cellular chain complex of the cubical model constructed in [Lüt14].

1.9. **Conventions.** Graded objects are concentrated in non-negative degrees. This restriction is only used in Proposition A.7. We write $\text{Ch}_2$ for the category of chain complexes.

Bigradings of modules are by degree and weight. The braiding isomorphism for a tensor product of modules has a sign which depends on degree and not on weight: if $x$ and $y$ have degree $i$ and $j$, the braiding isomorphism takes $x \otimes y$ to $(-1)^{ij} y \otimes x$. We write $[m]$ for the degree shift functor by $m$ and $\{n\}$ for the weight shift functor.
by \( n \) so that the degree \( i \) and weight \( j \) component of \( M[m]\{n\} \) is the degree \( i - m \) and weight \( j - n \) component of \( M \).

Symmetric monoidal functors are strong monoidal. We use the phrases “(natural) weak equivalence” to refer to a (natural) isomorphism in the relevant homotopy category and the phrases “(natural) quasi-isomorphism” to refer to a (natural) chain map which induces an isomorphism on homology groups.

We write \( C^{\text{sing}}(X) \) for the singular chain complex of the topological space \( X \). If \( X \) is a CW complex, we denote the cellular chain complex of \( X \) by \( C(X) \).

1.10. **Acknowledgments.** The second author thanks Felix Boes for sharing computational tools. The third author thanks Daniel Lütgehetmann for noticing the connection to the work of Świątkowski [Świ01].

2. **Subdivisional spaces**

Following the ideas of [Abr00], we approximate the configuration spaces of a cell complex by cell complexes, and we show, in Theorem 2.8, that the homotopy types of these approximations often “converge” to the homotopy type of the true configuration space under transfinite subdivision. We then introduce the framework of subdivisional spaces, which sets a complex \( X \) equipped with a set of subdivisions \( \mathcal{P} \) on equal footing with the corresponding collection of configuration complexes \( \{B^k_n(X')\}_{X' \in \mathcal{P}} \). We identify a natural theory of homology for these objects, the complex of subdivisional chains, and we show that it is homotopically well-behaved.

2.1. **Complexes and subdivision.** If \( X \) is a CW complex and \( c \subseteq X \) is an \( n \)-cell, we write \( \partial c \) for the image of \( \partial D^n \) under a characteristic map for \( c \), and we set \( \hat{c} := c \setminus \partial c \). The \( n \)-skeleton of \( X \), which is to say the union of the cells of \( X \) of dimension at most \( n \), is denoted \( \text{sk}_n(X) \). A cellular map is a map preserving skeleta.

From now on, a complex will be a finite CW complex. We choose to restrict our attention to the finite case for the sake of convenience only.

**Definition 2.1.** Let \( f : X \to Y \) be a cellular map between complexes. We say that \( f \) is

1. **regular** if \( f \) preserves both closed and open cells;
2. **an isomorphism** if \( f \) is regular and bijective;
3. **an embedding** if \( f \) is regular and injective;
4. **a subdivision** if \( f \) is bijective and preserves subcomplexes;
5. **a subdivisional embedding** if \( f \) is injective and preserves subcomplexes.

Thus, a subdivisional embedding factors via its image into a subdivision followed by an embedding.

Given a complex \( X \), we write \( \text{SD}(X) \) for the category whose objects are subdivisions of \( X \) and whose morphisms are commuting triangles of subdivisions. Note that, since a subdivision is in particular a homeomorphism, there can be at most one morphism in \( \text{SD}(X) \) with fixed source and target.

**Remark 2.2.** Some authors consider a subdivision to be the inverse to what we have defined to be a subdivision. We choose this convention because it matches the direction of the functoriality that will arise naturally in the examples of interest. Modulo this issue of direction, our notion of subdivision is equivalent to that of [LW69, Def. II.6.2].
2.2. Configuration complexes. We now introduce the main object of study.

**Definition 2.3.** Let $X$ be a topological space.

1. The $k$th ordered configuration space of $X$ is $\text{Conf}_k(X) = \{(x_1, \ldots, x_k) : x_i \neq x_j \text{ if } i \neq j\}$.

2. The $k$th unordered configuration space of $X$ is the quotient $B_k(X) = \text{Conf}_k(X)/\Sigma_k$.

Unfortunately, a cell structure on $X$ does not induce an obvious cell structure on $\text{Conf}_k(X)$; however, following [Abr00], there is a cellular approximation.

**Definition 2.4.** Let $X$ be a complex.

1. The $k$th ordered configuration complex of $X$ is the largest subcomplex $\text{Conf}_k(X) \subseteq X^k$ contained in $\text{Conf}_k(X)$.

2. The $k$th unordered configuration complex of $X$ is the quotient $B_k(X) = \text{Conf}_k(X)/\Sigma_k$.

In other words, a cell $(c_1, \ldots, c_k)$ of $X^k$ lies in $\text{Conf}_k(X)$ if and only if $c_i \cap c_j = \emptyset = \emptyset$ for $i \neq j$.

The quotient $B_k(X)$ is again a complex, which we view as an approximation to $B_k(X)$. These approximations enjoy a certain functoriality.

**Lemma 2.5.** Let $s : X \to X'$ be a subdivision. The restriction of $s^k$ to $\text{Conf}^\square_k(X) \subseteq X^k$ contained in $\text{Conf}^\square_k(X)$ factors $\Sigma_k$-equivariantly through $\text{Conf}^\square_k(X')$ as a subdivisional embedding.

**Theorem 2.8** (Convergence theorem). For any complex $X$, any convergent $\mathcal{P} \subseteq \text{SD}(X)$, and any $k \geq 0$, the natural map

$$\text{colim}_{X' \in \mathcal{P}} \text{Conf}^\square_k(X') \to \text{Conf}_k(X)$$

is a weak homotopy equivalence.

**Proof.** Since spheres are compact, any map of spheres or homotopy of maps of spheres lies in $\text{Conf}^\square_k(X')$ for some subdivision $X'$ in $\mathcal{P}$. \qed

**Remark 2.9.** The content of the convergence theorem is that configuration complexes are useful in the study of configuration spaces whenever $X$ admits a convergent subdivisional structure. How large is this class of complexes?
Remark 2.10. It is natural to wonder whether finite subdivision suffices to recover the correct homotopy type. We do not address this question here, noting only that the method of proof completely breaks down; indeed, the homotopy groups of Conf$_2(D^3) \simeq S^2$ are all non-zero above degree one [IMW16].

We close by noting that the configuration complexes interact predictably with disjoint unions.

Lemma 2.11. There is a natural commuting diagram
\[
\prod_{i+j=k} B_i^\square(X) \times B_j^\square(Y) \xrightarrow{\simeq} B_k^\square(X \amalg Y)
\]
\[
\prod_{i+j=k} B_i(X) \times B_j(Y) \xrightarrow{\simeq} B_k(X \amalg Y)
\]
in which the bottom arrow is a homeomorphism and the top an isomorphism.

2.3. Subdivisional spaces. Complexes and subdivisional embeddings form a category which we denote by $\mathcal{C}x^{SD}$.

Definition 2.12. A subdivisional space is a functor $\mathcal{X} : \mathcal{P} \to \mathcal{C}x^{SD}$ with $\mathcal{P}$ a filtered category.

We write $\mathcal{E}mb^{SD}$ for the category whose objects are subdivisional spaces and whose morphisms are given by
\[
\text{Hom}_{\mathcal{E}mb^{SD}}(\mathcal{X}, \mathcal{Y}) = \lim_{p \in \mathcal{P}} \text{colim}_{q \in \mathcal{Q}} \text{Hom}_{\mathcal{C}x^{SD}}(\mathcal{X}(p), \mathcal{Y}(q)).
\]

In other words, $\mathcal{E}mb^{SD}$ is the category of ind-objects $\text{Ind}(\mathcal{C}x^{SD})$. We shall make very little use of the general theory of ind-objects, but the reader looking for further information on the subject may consult [KS06, Ch. 6].

Remark 2.13. The formula for hom sets in $\text{Ind}(\mathcal{C})$ is derived from the following intuitions:

1. an object of $\mathcal{C}$ should determine an ind-object of $\mathcal{C}$;
2. a general ind-object of $\mathcal{C}$ should be a filtered colimit of objects of $\mathcal{C}$; and
3. objects of $\mathcal{C}$ should be compact as ind-objects.

We say that $\mathcal{X} : \mathcal{P} \to \mathcal{C}x^{SD}$ is indexed on $\mathcal{P}$. Subdivisional spaces indexed on different categories may be isomorphic.

Example 2.14. A subdivisional structure $\mathcal{P} \subseteq \mathcal{SD}(X)$ determines a subdivisional space $\mathcal{X} : \mathcal{P} \to \mathcal{C}x^{SD}$ sending $X \to X'$ to $X'$.

Most complexes admit many subdivisional structures and many non-isomorphic realizations as subdivisional spaces. Roughly, we imagine that a subdivisional structure $\mathcal{P}$ forces $X$ to be isomorphic to each of the subdivisions contained in $\mathcal{P}$.

Example 2.15. Since the product of filtered categories is filtered, the levelwise Cartesian product and disjoint union of two subdivisional spaces is again a subdivisional space. Note that the former is not the categorical product, nor is the latter the categorical coproduct.
Definition 2.16. The spatial realization functor $| - |$ is the composite
\[ \mathcal{E}mb^{SD} = \text{Ind}(\mathcal{C}x^{SD}) \rightarrow \text{Ind}(\mathcal{T}op) \xrightarrow{\text{colim}} \mathcal{T}op. \]

Example 2.17. If $\mathcal{X}$ is any of the subdivisional spaces of Example 2.14, there is a canonical homeomorphism $|\mathcal{X}| \cong X$.

According to Lemma 2.5, configuration complexes are functorial for subdivisional embeddings, so we may make the following definition.

Definition 2.18. Let $\mathcal{X} : \mathcal{P} \rightarrow \mathcal{C}x^{SD}$ be a subdivisional space. The $k$th ordered subdivisional configuration space of $\mathcal{X}$ is the subdivisional space
\[ \mathcal{P} \xrightarrow{\mathcal{X}} \mathcal{C}x^{SD} \xrightarrow{\text{Conf}} \mathcal{C}x^{SD}. \]

Similarly, we have the $k$th unordered subdivisional configuration space $B_{SD}^k(\mathcal{X})$. We will be particularly interested in this construction when $\mathcal{X}$ comes from a complex $X$ with a convergent subdivisional structure, for in this case Theorem 2.8 gives a weak homotopy equivalence
\[ |B_{SD}^k(\mathcal{X})| \sim B_k(X). \]

It will often be convenient to consider configuration spaces of all finite cardinalities simultaneously, and we set
\[ B^{SD}(\mathcal{X}) := \coprod_{k \geq 0} B_{SD}^k(\mathcal{X}) \]
(note that the indicated disjoint union does in fact exist in $\mathcal{E}mb^{SD}$, since $\mathcal{E}mb^{SD}$ admits filtered colimits). Thus, we have a functor
\[ B^{SD} : \mathcal{E}mb^{SD} \rightarrow \mathcal{E}mb^{SD}. \]

Using Lemma 2.11, we see that $B^{SD}$ naturally carries the structure of a symmetric monoidal functor, where $\mathcal{E}mb^{SD}$ is symmetric monoidal under disjoint union in the domain and Cartesian product in the codomain.

2.4. Subdivisional chains. Since the complex of cellular chains is functorial for cellular maps between complexes, and, in particular, for subdivisional embeddings, we may make the following definition.

Definition 2.19. The functor $C^{SD}$ of subdivisional chains is the composite
\[ \mathcal{E}mb^{SD} = \text{Ind}(\mathcal{C}x^{SD}) \xrightarrow{\text{Ind}(C)} \text{Ind}(\mathcal{C}h_Z) \xrightarrow{\text{colim}} \mathcal{C}h_Z. \]

Viewing $\mathcal{E}mb^{SD}$ and $\mathcal{C}h_Z$ as symmetric monoidal under Cartesian product and tensor product, respectively, $C^{SD}$ naturally carries the structure of a symmetric monoidal functor (here we use that cellular chains sends products to tensor products and that the tensor product distributes over colimits).

Remark 2.20. In contrast to the cellular chain complex, $C^{SD}(\mathcal{X})$ is typically a very large object. For example, if $\mathcal{X}$ is obtained by equipping the interval $I$ with the subdivisional structure $SD(I)$, then $C^{SD}(\mathcal{X})$ is uncountably generated.

The fundamental fact about the functor of subdivisional chains is the following.
Proposition 2.21. There is a natural weak equivalence
\[ C^{\text{SD}}(-) \simeq C^{\text{sing}}(|-|) \]
of functors from \( \text{Emb}^{\text{SD}} \) to chain complexes.

In the proof, we use the following intermediary.

Construction 2.22. Let \( X \) be a CW complex. We define \( S^{\text{cell}}(X) \) to be the simplicial set given in simplicial degree \( n \) by the set of cellular maps \( \sigma : \Delta^n \to X \), where \( S \) is the standard functor of singular simplices, and this map is a weak homotopy equivalence of simplicial sets by the cellular approximation theorem. Since both the induced map on geometric realizations and the composite \( |S^{\text{cell}}(X)| \to |S(X)| \to X \) are cellular by construction, we obtain the zig-zag
\[ C(X) \leftarrow C(|S^{\text{cell}}(X)|) \rightarrow C(|S(X)|) \cong C^{\text{sing}}(|X|) \]

Proof of Proposition 2.21. Consider the following diagram:

\[
\begin{array}{ccc}
\text{Ind}(C^{\text{SD}}) & \xrightarrow{\text{colim}} & \text{Ind(Top)} \\
\text{Ind}(\text{C}^{\text{sing}}) & \xrightarrow{\text{colim}} & \text{Ind(Top)} \\
\end{array}
\]

The composition along the top and right is \( C^{\text{sing}}(|-|) \). The composition along the bottom is \( C^{\text{SD}} \). The right square commutes up to natural isomorphism because the colimit is filtered. The triangle and bigon on the left commute up to natural objectwise quasi-isomorphism. Since filtered colimits preserve quasi-isomorphisms, the conclusion follows.

Thus there is a homotopically well-behaved natural isomorphism \( H_*(C^{\text{SD}}(X)) \cong H_*(|X|) \). To make this statement precise, we use homotopy colimits for diagrams of chain complexes—reminders and references are in Appendix A. The following corollary will be a key ingredient in our proof of Theorem 3.19 below.

Corollary 2.23. Let \( X \) be a subdivisional space and \( F : \mathcal{D} \to \text{Emb}^{\text{SD}} \) a functor equipped with a natural transformation to the constant functor at \( X \). If the induced map \( \text{hocolim}_{\mathcal{D}} |F| \to |X| \) is a weak homotopy equivalence, then the induced map
\[ \text{hocolim}_{\mathcal{D}} C^{\text{SD}}(F) \to C^{\text{SD}}(X) \]
is a quasi-isomorphism.

Proof. Applying homology to the natural weak equivalence of Proposition 2.21, and using the fact that a levelwise weak homotopy equivalence of functors induces a weak homotopy equivalence on homotopy colimits (Lemma A.4), we obtain the isomorphisms in the following commutative square:

\[
\begin{array}{ccc}
H_*(\text{hocolim}_{\mathcal{D}} C^{\text{sing}}(|F|)) & \xrightarrow{i} & H_*(C^{\text{sing}}(|X|)) \\
\downarrow & & \downarrow \\
H_*(\text{hocolim}_{\mathcal{D}} C^{\text{SD}}(F)) & \xrightarrow{i} & H_*(C^{\text{SD}}(X)).
\end{array}
\]

From our assumption and Proposition A.8, the top map is an isomorphism, and the claim follows.  \( \square \)
3. Decomposition

We prove our first main result, the decomposition theorem, stated below as Theorem 3.19. A careful formulation of this result requires that we supply a certain amount of definitional groundwork, and this task will occupy our attention in §3.1–3.3. The theorem and its proof appear in §3.4.

The proof of the decomposition theorem is premised on various manipulations of homotopy colimits. For the convenience of the reader less familiar with categorical homotopy theory, we have included a brief review of the relevant terminology and results in Appendix A, as well as a number of references.

3.1. Decompositions and gaps. We first make precise the data involved in the type of decomposition that we wish to consider. Before doing so, we remind the reader of two operations on subdivisions. First, by restricting in the source and target, a subdivision yields a subdivision on any subcomplex. Second, the Cartesian product of two subdivisions is a subdivision of the Cartesian product. Both of these constructions respect further subdivision, so a subdivisional structure on a complex yields a subdivisional structure on any subcomplex by restriction, and any subdivisional structures on two complexes yield a subdivisional structure on their product.

Definition 3.1. An \((r\text{-fold})\) decomposition of the complex \(X\) is the data of

1. a collection of complexes \(\{\tilde{X}_0, \ldots, \tilde{X}_r, A_1, \ldots, A_r\}\);
2. for each \(0 \leq j \leq r\), a pair of embeddings \(A_j \to \tilde{X}_j \leftarrow A_{j+1}\) with disjoint images, where \(A_0 = A_{r+1} = \emptyset\) by convention;
3. an isomorphism \(X \cong \tilde{X}_0 \coprod_{A_1 \times \{0\}} (A_1 \times I) \coprod_{A_2 \times \{0\}} (A_2 \times I) \cdots \coprod_{A_r \times \{0\}} (A_r \times I) \coprod_{A_r \times \{1\}} \tilde{X}_r\);

4. a subdivisional structure \(\mathcal{P}_X \subseteq SD(X)\) restricting to a product of subdivisional structures \(\mathcal{P}_{A_j \times SD(I)}\) on \(A_j \times I\) for each \(1 \leq j \leq r\).

We say that the decomposition is convergent if \(\mathcal{P}_X\) is so.

Given a decomposition, we set

\[X_j := (A_j \times I) \coprod_{A_j \times \{1\}} \tilde{X}_j \coprod_{A_{j+1} \times \{0\}} (A_{j+1} \times I)\]

These complexes are called the components of the decomposition, and the complexes \(A_j \times I\) are the bridges.

Remark 3.2. Essentially all of our results hold if \(SD(I)\) is replaced with a convergent subdivisional structure on \(I\).

We typically abbreviate the data of a decomposition to the letter \(\mathcal{E}\). Note that, by restriction, a decomposition determines subdivisional structures on all of the complexes involved, and every inclusion between two such lifts to a morphism of subdivisional spaces.

Definition 3.3. Let \(\mathcal{E}\) and \(\mathcal{F}\) be \(r\)-fold decompositions of \(X\) and \(Y\), respectively. A map of decompositions from \(\mathcal{E}\) to \(\mathcal{F}\) is a map \(f : X \to Y\) of subdivisional spaces
whose restrictions fit into a commuting diagram of subdivisional spaces

\[
\begin{array}{cccccc}
\tilde{X}_0 & \rightarrow & A_1 \times I & \rightarrow & \tilde{X}_1 & \rightarrow & \cdots & \rightarrow & \tilde{X}_j & \rightarrow & \cdots & \rightarrow & \tilde{X}_r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{Y}_0 & \rightarrow & B_1 \times I & \rightarrow & \tilde{Y}_1 & \rightarrow & \cdots & \rightarrow & \tilde{Y}_j & \rightarrow & \cdots & \rightarrow & \tilde{Y}_r,
\end{array}
\]

and such that \( f|_{A_j \times I} = f|_{A_j} \times \text{id}_I \) for each \( 1 \leq j \leq r \).

A decomposition gives rise to a combinatorial relationship to a certain poset.

**Definition 3.4.** The category of \((r\text{-fold})\) **gaps** is the partially ordered set \( \mathcal{G}_r \) of nonempty open subsets \( A \subseteq [0, r] \) such that

1. the complement of \( A \) is a (possibly empty) finite union of closed intervals of positive length;
2. if \( i \in \{0, \ldots, r\} \) lies in the closure of \( A \), then \( i \in A \); and
3. for every \( 1 \leq j \leq r \), \( A \cap [j-1, j] \neq \emptyset \).

**Construction 3.5.** Given a decomposition \( \mathcal{E} \) of \( X \), there is a continuous map \( \pi_\mathcal{E} : X \rightarrow [0, r] \) specified by requiring that

1. \( \pi_\mathcal{E}(\tilde{X}_j) = \{j\} \), and
2. \( \pi_\mathcal{E}|_{A_j \times I} \) is the projection onto \( I \cong [j-1, j] \).

If \( A \subseteq [0, r] \) is a gap, condition (4) of Definition 3.1 provides the inverse image of \([0, r] \setminus A\) with a canonical subdivisional structure, and we obtain this way a functor

\[ \gamma_\mathcal{E} : \mathcal{G}_r^{op} \rightarrow \text{Emb}^{SD}. \]

If \( f : \mathcal{E} \rightarrow \mathcal{F} \) is a map of decompositions, then \( f(\pi_\mathcal{E}^{-1}([0, r] \setminus A)) \subseteq \pi_\mathcal{F}^{-1}([0, r] \setminus A) \) for every \( A \in \mathcal{G}_r \), so we may interpret \( f \) as a natural transformation from \( \gamma_\mathcal{E} \) to \( \gamma_\mathcal{F} \).

Each \( \gamma_\mathcal{E}(A) \) is a (possibly empty) union of some number of components of the form \((A_j \times [a, 1]) \cup \tilde{X}_j \cup (A_{j+1} \times [0, b])\) with \( 0 < a, b < 1 \) or \( U = A_j \times [c, d] \) with
0 < c < d < 1. We refer to such a component as a *basic*. We say that the former type of basic is of *component type* and the latter of *bridge type*. The term “basic” is borrowed from [AFT17], whose ideas heavily influence our approach to Theorem 3.19.

Note that $X$ itself typically does not lie in the image of $\gamma_\mathcal{E}$, since $\emptyset \notin \mathcal{S}_r$.

### 3.2. Local invariants

In this paper, our main interest in the decomposition theorem stated below will be as a tool to study configuration spaces, but the proof of the theorem will only make use of a few key features of these spaces.

**Definition 3.6.** Let $\mathcal{E}$ be a decomposition of $X$. An $\mathcal{E}$-*local invariant* is a symmetric monoidal functor $F : (\text{Emb}^{\text{SD}}, \Pi) \to (\text{Emb}^{\text{SD}}, \times)$ such that the natural map

$$\text{hocolim}_{\mathcal{S}_r} |F(\gamma_\mathcal{E})| \to |F(X)|$$

is a weak equivalence. A map of local invariants (possibly for different decompositions) is a symmetric monoidal natural transformation.

**Remark 3.7.** At the cost of greater verbal overhead, it is possible to work with invariants that are only defined locally relative to a given $\mathcal{E}$. All of our results carry over into this more general context.

We now check that this condition is satisfied in the example of greatest interest to us.

**Proposition 3.8.** The symmetric monoidal functor $B^{\text{SD}}$ is an $\mathcal{E}$-local invariant for any convergent decomposition $\mathcal{E}$.

**Proof.** The claim will follow by two-out-of-three after verifying that each of the numbered arrows in the commuting diagram

$$\text{hocolim}_{A \in \mathcal{S}_r} B\left(\pi^{-1}_\mathcal{E}\left([0, r] \setminus \mathcal{A}\right)\right) \overset{(1)}{\longrightarrow} B(X)$$

$$\downarrow \quad (2) \quad \downarrow$$

$$\text{hocolim}_{A \in \mathcal{S}_r} B\left(\left|\gamma_\mathcal{E}(A)\right|\right) \overset{(3)}{\longrightarrow} B(X)$$

$$\uparrow \quad (4) \quad \uparrow (5)$$

$$\text{hocolim}_{\mathcal{S}_r} |B^{\text{SD}}(\gamma_\mathcal{E})| \longrightarrow |B^{\text{SD}}(X)|,$$

is a weak homotopy equivalence, where $X$ is the subdivisional space determined by $X$ and the subdivisional structure $\mathcal{P}_X \subseteq \text{SD}(X)$ of $\mathcal{E}$. The second equivalence follows from the fact that $B$ sends homotopy equivalences through injective maps to weak homotopy equivalences, and the third follows from (1) and (2) by two-out-of-three. Theorem 2.8 gives the fifth and (together with Lemma A.4) the fourth, using our assumption on $\mathcal{E}$ (see the discussion following Definition 2.18). Thus, it remains to verify the first equivalence.

Consider the collection $\mathcal{U} := \{B(\pi^{-1}_\mathcal{E}\left([0, r] \setminus \mathcal{A}\right)) : A \in \mathcal{S}_r\}$, which is an open cover of $B(X)$. In fact, $\mathcal{U}$ is a *complete* cover in the sense of Definition A.5; to see
this, we note that for $S$ finite,
\[
\bigcap_s B(\pi_\mathcal{E}^{-1}([0,r] \setminus \mathcal{A}_s)) = B\left(\bigcap_s \pi_\mathcal{E}^{-1}([0,r] \setminus \mathcal{A}_s)\right) \\
= B\left(\pi_\mathcal{E}^{-1}\left(\bigcap_s [0,r] \setminus \mathcal{A}_s\right)\right) \\
= B\left(\pi_\mathcal{E}^{-1}\left([0,r] \setminus \bigcup_s \mathcal{A}_s\right)\right),
\]
and that a finite union of closures of gaps is again the closure of a gap. With this observation, the desired equivalence follows from Theorem A.6. □

Remark 3.9. The local invariant $B^{SD}$, from its definition as a disjoint union over finite cardinalities, is naturally a graded subdivisional space. Furthermore, every map induced by an inclusion $U \subseteq V$ automatically preserves this grading, as do the isomorphisms $B^{SD}(U_1 \cup U_2) \cong B^{SD}(U_1) \times B^{SD}(U_2)$. It is very useful to keep track of this grading in studying configuration spaces (see Theorem 4.5, for example), and we will often do so implicitly.

3.3. Flows and bimodules. The decomposition theorem allows us to study configuration spaces and other local invariants in terms of local information. In order to get a combinatorial handle on the local information, we axiomatize what it means to simplify a local invariant coherently.

Definition 3.10. The category of abstract flows, $\mathcal{Fl}_Z$, has objects $(C, \pi)$, where $C$ is a chain complex and $\pi : C \xrightarrow{\sim} C$ is an idempotent quasi-isomorphism. A morphism $(C, \pi) \to (C', \pi')$ is a chain map $f$ such that $\pi' \circ f = \pi' \circ f \circ \pi$. Such a map is called flow compatible.

We will use two functors $\mathcal{Fl}_Z \to \text{Ch}_Z$. The forgetful functor takes $(C, \pi)$ to $C$ and is the identity on morphisms. The Morse complex $I$ takes $(C, \pi)$ to $\pi(C)$ and $f$ to $\pi \circ f$. The subcategory of objects of $\mathcal{Fl}_Z$ with underlying chain complexes flat in each degree inherits a symmetric monoidal structure for which each of these functors is symmetric monoidal.

We think of $\pi$ as the limit of a flow on $C$ and the elements of the associated Morse complex as the critical points of that flow.

Remark 3.11. A discrete flow in the sense of [For98] gives rise to an abstract flow; indeed, our definitions of abstract flow and flow compatible map were motivated by the desire to work functorially with discrete Morse data.

Definition 3.12. Let $X$ be a subdivisional space indexed on $\mathcal{P}$.

1. A subdivisional flow on $X$ is the data of the dashed lift in the diagram

2. A map $f$ of subdivisional spaces equipped with subdivisional flows is flow compatible if $C(f)$ lies in the image of the forgetful functor $\text{Ind}(\mathcal{Fl}_Z) \to \text{Ind}(\text{Ch}_Z)$.

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Definition 3.13. Let $\mathcal{X} : \mathcal{P} \to \mathcal{C}^{\text{SD}}$ be a subdivisional space equipped with a subdivisional flow $g = \{g_p : C(\mathcal{X}(p)) \to C(\mathcal{X}(p))\}_{p \in \mathcal{P}}$. The associated Morse complex is the chain complex

$$I(C^{\text{SD}}(\mathcal{X}), g) = \colim_P I(C(\mathcal{X}(p)), g_p).$$

The Morse complex is functorial for flow compatible maps and comes equipped with a natural quasi-isomorphism $C^{\text{SD}}(\mathcal{X}) \sim \to I(C^{\text{SD}}(\mathcal{X}), g)$, since a filtered colimit of quasi-isomorphisms is a quasi-isomorphism. In particular, the Morse complex computes the homology of $|\mathcal{X}|$. Typically, when the choice of subdivisional flow $g$ is clear from context, we abbreviate the Morse complex to $I(C^{\text{SD}}(\mathcal{X}))$ or simply $I(\mathcal{X})$.

If $\mathcal{X}$ and $\mathcal{Y}$ are equipped with subdivisional flows, then, since the functor of cellular chains is symmetric monoidal and takes values in levelwise flat chain complexes, the product $\mathcal{X} \times \mathcal{Y}$ inherits a canonical subdivisional flow, which we refer to as the product flow. Note that, in this case, there are canonical isomorphisms $I(\mathcal{X} \times \mathcal{Y}) \cong I(\mathcal{X}) \otimes I(\mathcal{Y})$ satisfying obvious associativity and commutativity relations.

Definition 3.14. Let $\mathcal{E}$ be a decomposition of $X$ and $F$ an $\mathcal{E}$-local invariant. A local flow on $F$ is the data of a subdivisional flow on $F(\gamma_{\mathcal{E}}(A))$ for each $A \in \mathcal{G}_r$, subject to the following conditions:

1. for $A \subseteq B$, the induced map $F(B) \to F(A)$ is flow compatible; and
2. the isomorphism $F(\gamma_{\mathcal{E}}(A)) \times F(\gamma_{\mathcal{E}}(B)) \cong F(\gamma_{\mathcal{E}}(A) \amalg \gamma_{\mathcal{E}}(B))$ and its inverse are each flow compatible, where the lefthand side carries the product flow, whenever $\gamma_{\mathcal{E}}(A)$ and $\gamma_{\mathcal{E}}(B)$ are disjoint in $X$.

A map between local invariants equipped with local flows is flow compatible if each of its components is so.

A local flow in particular determines the dashed lift in the diagram

$$\mathcal{G}_r^{op} \xrightarrow{\Emb^{SD}} \mathcal{E}mb^{SD} \xrightarrow{F} \mathcal{E}mb^{SD} \xrightarrow{C^{SD}} \mathcal{C}h_{\mathbb{Z}}.$$
of these “Morse complexes” is canonically isomorphic to the Morse complex of any corresponding basic.

In a more homotopical context, it would be sensible to require a weaker condition. The strictness of isotopy invariance is motivated by the computational nature of our goals, and it has the following important consequence, which is drawn from the theory of factorization algebras—see [Gin13] and the references therein.

**Construction 3.16.** Let $\mathcal{E}$ be a decomposition of $X$ and $F$ an $\mathcal{E}$-local invariant equipped with an isotopy invariant local flow. For each $1 \leq j \leq r$, the Morse complex $I(F(A_j \times I))$ of the bridge carries a natural associative algebra structure, for which the Morse complexes $I(F(X_{j-1}))$ and $I(F(X_j))$ of the relevant components are right and left modules, respectively.

We indicate how the algebra structure arises (the module structures are similar). Set $R_j := I(F(A_j \times I))$; then the unit map $\eta_j : \mathbb{Z} \to R_j$ is induced by the inclusion $\emptyset \to A_j \times I$. As for the multiplication, we take any two disjoint subbasics $U_1$ and $U_2$ of bridge type contained in $A_j \times I$ and define $\mu_j$ to be the composite

$$
R_j \otimes R_j \xrightarrow{\cong} I(F(U_1)) \otimes I(F(U_2)) \xrightarrow{\cong} I(F(U_1 \times F(U_2)) \xrightarrow{\cong} I(F(U_1) \amalg F(U_2)) \xrightarrow{\cong} I(F(A_j \times I)) = R_j,
$$

where in the second line we have used that the Morse complex of a product subdivisional flow is the tensor product of the Morse complexes.

Any two choices for $U_1$ and $U_2$ may be connected by a zig-zag of inclusions of disjoint pairs of basics of bridge type, so, by naturality of the maps in question, $\mu_j$ is independent of this choice. Tracing through the construction shows that $\eta_j$ is a unit for $\mu_j$. For associativity, we consider a configuration of five bridge type basics with containments as in Figure 2. We use $U_1$ and $U_{23}$ to define the outer multiplication and $U_2$ and $U_3$ for the inner multiplication in the expression $\mu_j \circ (\text{id} \otimes \mu_j)$, and we use $U_{12}$ and $U_3$ to define the outer multiplication and $U_1$ and $U_2$ to define the inner multiplication in the expression $\mu_j \circ (\mu_j \otimes \text{id})$. Then both expressions are given in terms of maps $I(F(U_1)) \otimes I(F(U_2)) \otimes I(F(U_3)) \to R_j$, which coincide because of the associativity of the structure morphisms for local invariants.

It will be convenient to have terminology for this emergent algebraic structure.

**Definition 3.17.** An $r$-fold bimodule is

1. a collection $(R_1, \ldots, R_r, M_0, \ldots, M_r)$ of chain complexes,
2. the structure of a differential graded unital associative algebra on $R_j$ for each $1 \leq j \leq r$, and
3. the structure of an $(R_j, R_{j+1})$-bimodule on $M_j$ for each $0 \leq j \leq r$, where $R_0 = R_{r+1} = \mathbb{Z}$ by convention.

A map of $r$-fold bimodules $(R_1, \ldots, R_r, M_0, \ldots, M_r) \to (R'_1, \ldots, R'_r, M'_0, \ldots, M'_r)$ consists of a collection of maps of algebras $R_j \to R'_j$ together with a map of $(R_j, R_{j+1})$-bimodules from $M_j$ to $M'_j$ for each $j$, where $M'_j$ carries the bimodule structure induced by restriction along the maps $R_j \to R'_j$ and $R_{j+1} \to R'_{j+1}$.
It is clear from Construction 3.16 and the definition of a local flow on a local invariant that a flow compatible map between local invariants intertwines the various algebra and module structures.

We summarize the discussion so far in categorical terms. There is a category $\mathcal{I}_{\text{inv}}$ whose objects are triples $(X, E, F)$ consisting of a complex $X$, an $r$-fold decomposition $E$, and an $E$-local invariant $F$ equipped with an isotopy invariant local flow (suppressed in the notation), and whose morphisms are pairs of a map of decompositions and a flow compatible map of local invariants. There is a second category $\mathcal{B}_{\text{bimod}}$ whose objects are $r$-fold bimodules and whose morphisms are maps of such. What we have constructed so far is a canonical lift of the Morse complex to a functor

$$I : \mathcal{I}_{\text{inv}} \rightarrow \mathcal{B}_{\text{bimod}}.$$ 

3.4. The decomposition theorem. Our goal is to use the algebraic structures of Construction 3.16 to express the global value of a local invariant with an isotopy invariant local flow in terms of its values on the pieces of the decomposition. In order to make a precise statement, we first need to spell out how the various pieces of an $r$-fold bimodule may be assembled to give a corresponding “global value.”

**Definition 3.18.** Let $(R_1, \ldots, R_r, M_0, \ldots, M_r)$ be an $r$-fold bimodule.

1. The simplicial bar construction on these data is the $r$-fold simplicial chain complex given in degree $(n_1, \ldots, n_r)$ by

$$\text{Bar}_\Delta(M_0, R_1, \ldots, R_r, M_r)(n_1, \ldots, n_r) = M_0 \otimes R_1^{\otimes n_1} \otimes \cdots \otimes R_r^{\otimes n_r} \otimes M_r.$$ 

The face maps are defined by the respective module action and algebra multiplication maps, and the degeneracies are defined by the respective units.

2. The bar construction or bar complex on these data is the total chain complex of the multicomplex obtained from the simplicial bar construction by taking the respective alternating sums of the face maps in each simplicial direction.

The bar complex of the $r$-fold bimodule is a functor in a straightforward manner and computes the homology of the derived tensor product

$$M_0 \otimes_{R_1}^L \cdots \otimes_{R_r}^L M_r.$$
Indeed, depending on one’s point of view, this prescription may even be taken as the definition of the derived tensor product. For details on these matters, the reader may consult [Smi67], for example.

We are now equipped to state our main result.

**Theorem 3.19** (Decomposition theorem). There is a natural weak equivalence connecting the two composites in the diagram

\[
\begin{array}{ccc}
\mathcal{N}_{\text{iso}}^{(X,\mathcal{E},F)\to F(X)} & \xrightarrow{\mathcal{E} \text{mb}^{\text{SD}}} & \mathcal{C}^{\text{SD}} \\
\downarrow I & & \downarrow C^{\text{SD}} \\
\mathcal{B}\text{imod}_r & \xrightarrow{\text{Bar}} & \mathcal{C}h_\mathbb{Z}.
\end{array}
\]

In particular, there is a weak equivalence

\[
C^{\text{SD}}(F(X)) \simeq I(F(X_0)) \mathcal{L} I(F(A_1 \times I)) \mathcal{L} \cdots \mathcal{L} I(F(X_r)).
\]

We turn now to the proof of Theorem 3.19. For the sake of brevity, when considering the \(r\)-fold bimodule arising from an \(\mathcal{E}\)-local invariant \(F\), we use the somewhat abusive notation \(\text{Bar}(I(\mathcal{E}))\) for the corresponding bar complex. Since \(\text{Bar}(I(\mathcal{E}))\) arises from the multi-simplicial object \(\text{Bar}_\Delta(I(\mathcal{E}))\), and since \(C^{\text{SD}}(F(X))\) may be recovered as a homotopy colimit over \(\mathcal{G}^{\text{op}}_r\), our strategy in relating these two objects will be to relate the categories \(\Delta^r\) and \(\mathcal{G}_r\).

**Definition 3.20.** Let \(A \subseteq [0,r]\) be a gap. The \(j\)th trace of \(A\) is the ordered set

\[
\tau_j(A) = \pi_0(A \cap [j-1,j]),
\]

with the ordering induced by the standard orientation of \(\mathbb{R}\).

It follows from the definitions that the set \(\tau_j(U)\) is always non-empty, so the various traces extend to a functor

\[
\tau : \mathcal{G}_r \to \Delta^r.
\]

Using the trace, we may relate the Morse complex to the bar construction.

**Construction 3.21.** We define a natural transformation

\[
\psi : I(F(\gamma_\mathcal{E})) \to \text{Bar}_\Delta(I(\mathcal{E})) \circ \tau^{\text{op}}
\]

of functors from \(\mathcal{G}^{\text{op}}_r\) to chain complexes. For a gap \(A\), we have

\[
\text{Bar}_\Delta(I(\mathcal{E}))(\tau^{\text{op}}(U)) \cong I(F(X_0)) \otimes I(F(A_1 \times I))^{\otimes \tau_1(A)} \otimes \cdots \\
\cdots \otimes I(F(A_r \times I))^{\otimes \tau_r(A)} \otimes I(F(X_r))
\]

and we define the component \(\psi_A\) by expressing \(\gamma_\mathcal{E}(A)\) as a disjoint union of basics and tensoring together the maps induced by the inclusions of each basic into the corresponding component or bridge. Naturality follows from flow compatibility of the structure maps involved, together with the fact that these Morse complexes arise from product flows, both of which are guaranteed by Definition 3.14.

As a matter of terminology, we say that an \(r\)-fold gap \(A \in \mathcal{G}_r\) is separated if \(A \cap \{0, \ldots, r\} = \emptyset\). Write \(\mathcal{G}_r \subseteq \mathcal{G}_r\) for the full subcategory of separated gaps.
Lemma 3.22. Let $E$ be a decomposition of $X$ and $F$ an $E$-local invariant with an isotopy-invariant local flow. The canonical map

$$hocolim_{\mathcal{S}_D} I(F(\gamma_E)) \xrightarrow{\sim} hocolim_{\mathcal{S}_D} \text{Bar}_\Delta(I(E)) \circ \tau^{op}$$

induced by $\psi$ is a quasi-isomorphism.

Proof. A gap $A$ is separated if and only if $\gamma_E(A)$ intersects each $\widetilde{X}_j$ non-vacuously, so, by isotopy invariance, $\psi_A$ is a quasi-isomorphism for separated $A$. Thus, by Lemma A.4 and Proposition A.12, it suffices to note that the inclusion $\mathcal{S}_r \subseteq \mathcal{S}_r$ is homotopy initial (so that the inclusion of opposite categories is homotopy final). Indeed, all of the undercategories in question are filtered and hence contractible. □

Theorem 3.19 relies on the following fact about the functor $\tau$.

Lemma 3.23. For any object $S \in \Delta^r$, the inclusion $\iota : \tau^{-1}(S) \rightarrow (S \downarrow \tau)$ is homotopy initial.

In one form or another, this fact is certainly well-known to experts. In the name of a self-contained narrative, we nevertheless include a proof, which is deferred to §3.5 below. For now, we draw the following consequence (see Appendix A for notation).

Corollary 3.24. Let $V : (\Delta^{op})^r \rightarrow \mathcal{C}_\mathbb{Z}$ be a multi-simplicial chain complex. There is a natural weak equivalence $\text{hoLan}_{\tau^{op}}(V \circ \tau^{op}) \simeq V$.

Proof. We have that

$$\text{hoLan}_{\tau^{op}}(V \circ \tau^{op})(S) = \text{hocolim}_{(\tau^{op})^{-1}(S)} (V \circ \tau^{op} \circ \text{forget}) (A.9)$$

$$\simeq \text{hocolim}_{(\tau^{op})^{-1}(S)} (V \circ \iota \circ \text{forget} \circ \iota) (3.23, A.12)$$

$$= \text{hocolim}_{(\tau^{op})^{-1}(S)} (V \circ S)$$

$$\simeq V(S) (A.13)$$

where in the last step we have used that the category $\tau^{-1}(S)$ is contractible. To see why this is so, we note that $(S \downarrow \tau)$ is contractible, having a final object (since $\mathcal{S}_r$ has the final object $[0, r]$), and invoke Corollary A.14 and Lemma 3.23 a second time. □

Proof of Theorem 3.19. We have the following column of quasi-isomorphisms:

$$C_{SD}(F(X)) \simeq \text{hocolim}_{\mathcal{S}_D} C_{SD}(F(\gamma_E)) (2.23)$$

$$\simeq \text{hocolim}_{\mathcal{S}_D} I(F(\gamma_E)) (A.4)$$

$$\simeq \text{hocolim}_{\mathcal{S}_D} \tau^* \text{Bar}_\Delta(I(E)) (3.22, A.4)$$

$$\cong \text{hoLan}_*(\text{Bar}_\Delta(I(E)) \circ \tau^{op})(*) (A.10)$$

$$\simeq \text{hoLan}_*(\text{hoLan}_*(\text{Bar}_\Delta(I(E)) \circ \tau^{op})(*) (\text{left Kan extensions compose})$$

$$\cong \text{hoLan}_{(\Delta^{op})^r} \text{hoLan}_*(\text{Bar}_\Delta(I(E)) \circ \tau^{op})(*) (A.10)$$

$$\simeq \text{hoLan}_{(\Delta^{op})^r} \text{Bar}_\Delta(I(E)) (3.24, A.4)$$

$$\simeq \text{Bar}(I(E)) (A.7).$$
Naturality follows from flow compatibility and inspection of Construction 3.21.

3.5. Proof of Lemma 3.23. The fundamental observation in the proof is the following. Let $\tilde{G}_{1,k} \subseteq \tilde{G}_1$ denote the (non-full) subcategory with objects the separated gaps with exactly $k$ components and morphisms the $\pi_0$-bijective inclusions.

**Lemma 3.25.** For every $k > 0$, the category $\tilde{G}_{1,k}$ is contractible.

**Proof.** We define a functor $\chi : \tilde{G}_1 \to \text{Top}$ by letting $\chi(A) \subseteq B_k(A)$ be the subspace of configurations that intersect each connected component of $A$ non-trivially. The proof will be complete upon establishing the chain of weak homotopy equivalences

$$|N\tilde{G}_{1,k}| \simeq \text{hocolim}_{\tilde{G}_{1,k}} \simeq \text{hocolim}_{\tilde{G}_{1,k}} \chi \simeq \text{pt},$$

where $\text{pt}$ is the constant functor with value a singleton. The first equivalence is immediate from Definition A.3, and the second follows from Lemma A.4 and the fact that $\chi(A)$ is homeomorphic to the product of the connected components of $A$ and hence contractible. To establish the third equivalence, we note that the collection $\{\chi(A) : A \in \tilde{G}_{1,k}\}$ of open subsets of $B_k((0,1))$ is a basis for its topology (and thus a complete cover). Since this configuration space is contractible, the desired equivalence now follows from Theorem A.6. □

Now, a separated $r$-fold gap is nothing more or less than an $r$-tuple of separated 1-fold gaps. In other words, there is a commuting diagram of functors

$$\tilde{G}_r \xrightarrow{\iota} G_r \xleftarrow{\iota} \tilde{G}_1 \xrightarrow{\tau} \Delta^r.$$ 

Writing $\tilde{\tau} : \tilde{G}_r \to \Delta^r$ for either composite and fixing an object $S = (S_1, \ldots, S_r) \in \Delta^r$, we have another commuting diagram of functors of the form

$$\tau^{-1}(S) \xrightarrow{\iota} (S \downarrow \tau) \xleftarrow{\iota} \tilde{\tau}^{-1}(S) \xrightarrow{\iota} \Pi_{j=1}^r \tilde{\tau}^{-1}(S_j) \xrightarrow{\iota} \Pi_{j=1}^r (S_j \downarrow \tilde{\tau}).$$

The strategy will be to understand $\iota$ by understanding each of the other arrows in the diagram (a direct comparison is possible but somewhat more involved), beginning with the upper vertical arrows.

**Lemma 3.26.** The inclusions of $\tilde{\tau}^{-1}(S)$ and $(S \downarrow \tilde{\tau})$ into $\tau^{-1}(S)$ and $(S \downarrow \tau)$, respectively, are each homotopy initial.

**Proof.** We give the proof for the former inclusion only, the latter differing only in requiring more notation. Fixing $A \in \tau^{-1}(S)$, we must check the contractibility of the category of $r$-fold gaps $B$ such that

1. $B \subseteq A$,
2. $\tau(B \subseteq A) = \text{id}_S$, and
Proof of Lemma 3.23. so contractibility follows from Lemma 3.25. □

First, we note that this category is non-empty; indeed, we may obtain such a $B$ from $A$ by removing a sufficiently small neighborhood of each $j \in \{0, \ldots, r\}$ from $A$. Moreover, any $B$ satisfying these three conditions is contained one of this form. Since this subcollection is clearly filtered, the claim follows by Example A.2. □

Next, we consider the middle horizontal arrow.

Lemma 3.27. The functor $\tilde{1}$ is homotopy initial.

Proof. The property of being homotopy initial is preserved by products and equivalences of categories, so it suffices to consider the case $r = 1$. We establish some notation.

(1) An object of $(S \downarrow \tilde{\tau})$ is a pair $(A, f)$, where $f : S \to T$ is a map of ordered sets and $A$ is a union of open subintervals of $(0, 1)$ whose components have disjoint closures and such that $\pi_0(A) = T$ as ordered sets.

(2) A morphism is a union of open subintervals of $(0, 1)$ whose components have disjoint closures and such that $\pi_0(A) = T$ as ordered sets.

(3) The functor $\tilde{1}$ is defined by sending $A \in \tau^{-1}(S)$ to the pair $(A, \text{id}_S)$. We wish to prove the contractibility, for each $(A, f)$, of the category $(\tilde{1} \downarrow (A, f))$, which is nothing other than the category of gaps $B \subseteq A$ with $\pi_0(B) = S$ and $\pi_0(B \subseteq A) = f$. By inspection, we have the isomorphism

$$(\tilde{1} \downarrow (A, f)) \cong \prod_{t \in \text{im}(f)} \tilde{S}_{1, |f^{-1}(t)|},$$

so contractibility follows from Lemma 3.25. □

Proof of Lemma 3.23. It is a fact that homotopy initial functors satisfy a partial two-out-of-three property; that is, if $T_1$ is homotopy initial, then $T_2 \circ T_1$ is homotopy initial if and only if $T_2$ is so (see [Lur09, Prop. 4.1.1.3(2)], for example). Applying this fact to the composite $\tilde{\tau}^{-1}(S) \to \tau^{-1}(S) \to (S \downarrow \tau)$ and invoking Lemma 3.26 reduces the lemma to verifying that the composite is homotopy initial. This composite coincides with the composite $\tilde{\tau}^{-1}(S) \to (S \downarrow \tilde{\tau}) \to (S \downarrow \tau)$, so this claim follows from Lemmas 3.26 and 3.27, since homotopy initial functors compose. □

4. Application to graphs

We use the theory developed above to study the configuration space of a graph. We establish conventions, introduce the Świątkowski complex, and use the technology developed in the previous section to prove Theorem 4.5, which asserts that this complex computes the homology of the configuration spaces of a graph functorially.

4.1. Conventions on graphs. A graph is a finite 1-dimensional CW complex $\Gamma$. Its 0-cells and 1-cells are its vertices and edges $V(\Gamma)$ and $E(\Gamma)$, or simply $V$ and $E$. The vertices of an edge $V(e)$ are the vertices contained in the closure of that edge in $\Gamma$. We write $E(v)$ for the set of edges incident to the vertex $v$. A half-edge is an end of an edge. The set of half-edges of $\Gamma$ is $H(\Gamma)$ or simply $H$. We write $H(v)$ for the set of half-edges incident to $v$ and $H(e)$ for the set of half-edges contained in $e$. For $h$ in $H$, we write $v(h)$ and $e(h)$ for the corresponding vertex and edge.

Any sufficiently small neighborhood of a vertex $v$ is homeomorphic to a cone on finitely many points; this finite number is the valence of $v$, denoted $d(v)$. The vertex $v$ is isolated if $d(v) = 0$ and essential if $d(v) \geq 3$. An edge with a 1-valent
A vertex is a tail. A self-loop at a vertex is an edge whose entire boundary is attached at that vertex. A graph is simple if it has no self-loops and no pair of edges with the same vertices.

**Example 4.1.** The cone on \( \{1, \ldots, n\} \) is a graph \( S_n \) with \( n+1 \) vertices. These graphs are called star graphs and the cone point the star vertex.

The interval \( I \) is a graph with two vertices \( 0 \) and \( 1 \) and one edge between them. It is isomorphic to \( S_1 \) and homeomorphic to \( S_2 \) but it will be convenient to have alternate notation for it.

![Figure 3. Star graphs](image)

**Definition 4.2.** Let \( f : \Gamma_1 \to \Gamma_2 \) be a continuous map between graphs. We say that \( f \) is a graph morphism if

1. the inverse image \( f^{-1}(V(\Gamma_2)) \) is contained in \( V(\Gamma_1) \) and
2. the map \( f \) is injective.

We call a graph morphism a smoothing if it is a homeomorphism and a graph embedding if it preserves vertices. A graph morphism can be factored into a graph embedding followed by a smoothing. The composite of graph morphisms is a graph morphism, and we obtain in this way a category \( \text{Grph} \). Although the objects of \( \text{Grph} \) are simply finite 1-dimensional CW complexes, not all morphisms are cellular. A subgraph is the image of a graph embedding. A graph morphism \( f : \Gamma_1 \to \Gamma_2 \) induces a map \( E(f) : E(\Gamma_1) \to E(\Gamma_2) \), a partially defined map \( V(f) : f^{-1}(V(\Gamma_2)) \to V(\Gamma_2) \), and a map \( H(v)(f) : H(v) \to H(f(v)) \) for each \( v \in f^{-1}(V(\Gamma_2)) \).

![Figure 4. There is a graph morphism (in fact a smoothing) from left to right but not from right to left.](image)

Since graph morphisms are injective, they induce maps at the level of configuration spaces. Thus, it is natural to view \( H_*(B(-)) \) as a functor from the category \( \text{Grph} \) to bigraded Abelian groups (with a weight grading for cardinality).

### 4.2. The Świątkowski complex

We now introduce our main tool in the study of \( H_*(B(\Gamma)) \).

**Construction 4.3 (Świątkowski complex).** Let \( \Gamma \) be a graph. For each vertex \( v \in V \), we set \( S(v) = \mathbb{Z}\langle \emptyset, v, h \in H(v) \rangle \) and regard this Abelian group as bigraded with \( |\emptyset| = (0,0), |v| = (0,1) \), and \( |h| = (1,1) \).
The Świątkowski complex of $\Gamma$ is the differential bigraded $\mathbb{Z}[E]$-module
\[ S(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S(v), \]
where $|e| = (0, 1)$ for $e \in E$, and differential determined by setting
\[ \partial(h) = e(h) - v(h). \]
Since $\partial$ is $\mathbb{Z}[E]$-linear, the module structure descends to homology.

A graph morphism $f : \Gamma_1 \to \Gamma_2$ determines a map $S(f) : S(\Gamma_1) \to S(\Gamma_2)$. This takes edges to their images under $\Gamma$. If $f(v)$ is a vertex of $\Gamma_2$, then the induced map takes $S(v)$ to $S(f(v))$ using $f$. If $f(v)$ is in the edge $e$ in $\Gamma_2$, the map factors through $S(v) \to \mathbb{Z}[e]$ where $\emptyset$ goes to 1, $v$ to $e$, and $h \in H(v)$ to 0. By inspection, $S(f)$ respects the bigrading, differential, and module structures.

Remark 4.4. The generators of $S(\Gamma)$ describe “states” in the configuration spaces of $\Gamma$. The module $S(v)$ records the local states allowed at the vertex $v$, with $\emptyset$ corresponding to the absence of a particle at $v$, the element $v$ to a stationary particle at $v$, and the element $h \in H(v)$ to a path in which a particle moves infinitesimally along the edge containing $h$. A general state is obtained by prescribing a local state at each vertex and a number of particles on each edge, and the differential is the cellular differential taking a path to its endpoints.

We view $S$ as a functor from $\mathcal{G}ph$ to the category of bigraded chain complexes and the action of a (weight-graded) ring. A morphism is a weight-graded morphism of rings and a compatible morphism of differential bigraded modules. We denote the degree $i$ and weight $k$ component of $S(\Gamma)$ by $S_i(\Gamma)_k$. Our main result concerning the Świątkowski complex is the following:

**Theorem 4.5** (Comparison theorem). There is an isomorphism
\[ H_*(B(\Gamma)) \cong H_*(S(\Gamma)) \]
of functors from $\mathcal{G}ph$ to bigraded Abelian groups.

Remark 4.6. The weight $k$ subcomplex $S_i(\Gamma)_k$ is isomorphic to the cellular chains of the cubical complex exhibited in [Świ01], if every vertex is essential. This implies Theorem 4.5 at the level of objects.

Here we present the $\mathbb{Z}[E]$-module structure on the Świątkowski complex algebraically. In work in preparation, we show that this structure arises from an $E$-indexed family of maps of topological spaces $B_k(\Gamma) \to B_{k+1}(\Gamma)$ that increase the number of points on an edge. Such stabilization maps were known to exist for tails [AP17] and for trees at the level of Morse complexes [Ram16], but stabilization at arbitrary edges is new, and the sequel will be devoted the study of its properties.

It is often useful to consider a smaller variation on the Świątkowski complex.

**Definition 4.7.** Let $\Gamma$ be a graph and $U$ a subset of $V(\Gamma)$. For each $v \in U$, let $\tilde{S}(v) \subseteq S(v)$ be the subspace spanned by $\emptyset$ and the differences $h_{ij} := h_i - h_j$ of half-edges. The reduced Świątkowski complex (relative to $U$), is
\[ S^U(\Gamma) := \mathbb{Z}[E] \otimes \bigotimes_{v \in V \setminus U} S(v) \otimes \bigotimes_{v \in U} \tilde{S}(v), \]
considered as a subcomplex and submodule of $S(\Gamma)$. To be explicit, the differential is determined by $\partial(h_{ij}) = e(h_i) - e(h_j)$.
Note 4.8. When $U = V$ is the full set of vertices, we write $\tilde{S}(\Gamma) := S^V(\Gamma)$. When $U$ is the set of 1-valent vertices, we write $S^U(\Gamma) := S^U(\Gamma)$. Both $\tilde{S}(-)$ and $S^U(-)$ are functorial for graph morphisms.

When $\Gamma$ is a disjoint union of star graphs and intervals, we write $S^\partial (\Gamma)$ (an abuse of notation) for the reduced Świątkowski complex of $S_n$ relative to the set of non-star vertices. Since $S_1$ and the interval are isomorphic, this requires a specification of which such components are star graphs (and which vertex is the star vertex). The construction $S^\partial (-)$ is functorial only for graph morphisms $f : \Gamma_1 \to \Gamma_2$ such that $f^{-1}(v)$ is a star vertex whenever $v$ is a star vertex. Thus, for example, the smoothing $S_2 \to I$ is allowed, but the inclusion $I \to S_n$ of a leg is not.

Proposition 4.9. For any graph $\Gamma$ and any $U \subseteq V(\Gamma)$ containing no isolated vertices, the inclusion $\iota : S^U(\Gamma) \to S(\Gamma)$ is a quasi-isomorphism.

We omit a detailed proof, which can be seen, e.g., by a spectral sequence argument, filtering by polynomial degree in vertices and half-edges intersecting $U$.

Remark 4.10. The full Świątkowski complex has a canonical basis, while the reduced version lacks one in general. The reduction at (some subset of) the 1-valent vertices of a graph retains a canonical basis: if $v$ is 1-valent, then $S(v)$ is spanned by $\{ \emptyset \}$. The corresponding reduced complex is the Świątkowski complex of a “graph” in which the 1-valent vertex has been deleted, leaving a half-open tail. All of our constructions and results can be made rigorous for such non-compact “graphs.”

Our strategy in proving Theorem 4.5 will be to apply the tools developed earlier in the paper, especially Theorem 3.19. In order to do so, we must produce a decomposition, introduce Morse theory on the subdivisinal configuration spaces of the pieces, and analyze the resulting Morse complexes.

Construction 4.11 (Canonical decomposition). Let $\Gamma$ be a graph. Subdivide $\Gamma$ by adding four vertices to each edge; denote the resulting graph by $\Gamma_\#$. There is a canonical graph morphism $\Gamma_\# \to \Gamma$ inducing a homeomorphism on configuration spaces.

Removing from each edge of $\Gamma$ the open interval defined by the outer pair of added vertices produces a graph $\tilde{\Gamma}_0 \cong \coprod_{v \in V} S_d(v)$. On the other hand, for each edge, we have the closed interval defined by the inner pair of added vertices, and we let $\Gamma_1 \cong E \times I$ be the union of these closed intervals. Clearly, we have

$$\Gamma_\# \cong \tilde{\Gamma}_0 \coprod_{A_1 \times \{0\}} (A_1 \times I) \coprod_{A_1 \times \{1\}} \tilde{\Gamma}_1,$$

where $A_1$ is a finite set with $|A_1| = 2|E|$.

We define the canonical decomposition of $\Gamma_\#$ (abusively, of $\Gamma$) by choosing the full poset of subdivisions $SD(\Gamma_\#)$, which is filtered and convergent.

The main ingredient in the proof of Theorem 4.5 is the following.

Proposition 4.12. Let $\Gamma$ be a graph, and regard $B^{SD}$ as a local invariant on the canonical decomposition of $\Gamma$. There is an isotopy invariant local flow on $B^{SD}$ such
that

\[ I(A_1 \times I) \cong \mathbb{Z}[E] \otimes \mathbb{Z}[E] \]
\[ I(\Gamma_0) \cong \bigotimes_{v \in V} S^\alpha(S_{d(v)}) \]
\[ I(\Gamma_1) \cong \mathbb{Z}[E] \]

as associative algebras and modules, respectively.

The proof of this result amounts to constructing a couple abstract flows, calculating their Morse complexes, and checking various compatibilities. These tasks will occupy our attention in §4.3–4.4 below, but the idea behind the constructions is very simple. Intuitively, we define a flow on a star by pulling the cone point up, allowing points to flow down the legs, and we define a flow on an interval by allowing points to flow according to some fixed orientation. The reader who finds this heuristic sufficiently convincing may skip ahead to the computations.

For now, we deduce the following:

**Proof of Theorem 4.5, construction of isomorphism.** Applying Theorem 3.19 with the local flow of Proposition 4.12 produces an isomorphism

\[ H_*(B(\Gamma)) \cong H_* \left( \bigotimes_{v \in V} S^\alpha(S_{d(v)}) \bigotimes_{\mathbb{Z}[E] \otimes \mathbb{Z}[E]} \mathbb{Z}[E] \right). \]

Since \( \bigotimes_{v \in V} S^\alpha(S_{d(v)}) \) is a free \( \mathbb{Z}[E] \otimes \mathbb{Z}[E] \)-module, the derived tensor product is computed by the ordinary tensor product, and the proof is complete upon noting the canonical isomorphism

\[ \bigotimes_{v \in V} S^\alpha(S_{d(v)}) \bigotimes_{\mathbb{Z}[E] \otimes \mathbb{Z}[E]} \mathbb{Z}[E] \cong S(\Gamma). \]

4.3. **Interval and star flows.** In this section, we construct the abstract flows (in the sense of §3.3) that form the building blocks of the local flows alluded to in Proposition 4.12. These flows are inspired by the discrete flows of [FS05] and could be constructed in the same manner, but, in working locally, we are able to avoid the machinery of discrete Morse theory.
**Definition 4.13** (Interval flow). Let $I \to I'$ be a subdivision. We fix an orientation of $I'$ and define an order on the vertices by declaring the negative direction to be the direction of decrease. Define an endomorphism $\pi$ of $C(B_k^\square(I'))$ by declaring that $\pi$

1. takes any (positively signed) generator which is a set of $k$ many 0-cells to the (positively signed) set of the $k$ least 0-cells and
2. takes any generator which is a set containing a 1-cell to zero.

Using that the complex $B_k^\square(I')$ is either contractible or empty, depending on whether $k$ is greater than the number of 0-cells, the following result is easily verified.

**Lemma 4.14.** The interval flow is an abstract flow on $C(B_k^\square(I'))$.

**Definition 4.15** (Star flow). Let $S_n \to S'_n$ be a subdivision. For convenience, order the vertices in the $i$th leg by declaring the direction away from the star vertex $v$ to be the direction of decrease; call them $v_{i,0}, \ldots, v_{i,N_i}$, with $v_0$ the least vertex and $v_{N_i} = v$ the star vertex. Let $e_{i,j}$ be the 1-cell containing $v_{i,j-1}$ and $v_{i,j}$. Given a tuple $(r_1, \ldots, r_n)$ with $0 \leq r_i \leq N_i$, write $V_r$ for the set of vertices containing the $r_i$ least vertices in the $i$th leg. Define an endomorphism $\pi$ of $C(B_k^\square(S'_n))$ by declaring that $\pi$

1. takes any (positively signed) set of 0-cells not including $v_r$ with $r_i$ 0-cells in the $i$th leg to the (positively signed) set $V_r$,
2. takes any (positively signed) set of 0-cells including $v$ and $r_i$ additional 0-cells in the $i$th leg to the (positively signed) union of $V_r$ and $\{v\}$,
3. takes any set containing a 1-cell not of the form $e_{j,N_j}$ to zero, and
4. takes any set containing $e_{j,N_j}$ and $r_i$ additional (positively signed) 0-cells in the $i$th leg (necessarily not including $v$) to the sum

$$\sum_{r=r_j+1}^{N_j} V_r \cup \{e_{j,r}\}.$$

See Figure 6.

![Figure 6. Examples of two of the cases for star flow (Definition 4.15)](image)

**Lemma 4.16.** The star flow is an abstract flow on $C(B_k^\square(S'_n))$.

**Proof.** Idempotence is immediate except in the last case. There the image of the cell containing $e_{j,N_j}$ and $r_i$ vertices in the $i$th leg is the sum of the cell $V_r \cup \{e_{j,N_j}\}$ with several cells in the kernel of $\pi$. The map $\pi$ is a chain map by inspection (it suffices to check 2-cell and 1-cell generators).

Let us see that $\pi$ is a quasi-isomorphism. For brevity, we write $C$ for $C(B_k^\square(S'_n))$ and $I$ for its Morse complex. Consider the subcomplex $A$ of $C$ spanned by generators (i.e., sets of cells of $B_k^\square(S'_n)$) which do not contain any 1-cell intersecting $v$. 

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Since $\pi(A) \subseteq A$, there is a commuting diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & C/A & \longrightarrow & 0 \\
0 & \longrightarrow & A \cap I & \longrightarrow & I & \longrightarrow & I/(A \cap I) & \longrightarrow & 0
\end{array}
$$

of exact sequences. The quotient complex $C/A$ is linearly isomorphic to the subspace $B \subseteq C$ spanned by generating cells containing an edge intersecting $v$, but, in the quotient, the differential ignores the special edge. Similarly, $I/(A \cap I) \cong (\pi(B), 0)$. It follows that $A \xrightarrow{\pi} (A \cap I)$ and $C/A \xrightarrow{\pi} I/(A \cap I)$ are both quasi-isomorphisms, since the domains are either contractible or empty in every weight, while the codomains have no differentials. The claim follows by the five lemma. □

**Lemma 4.17.** There is a canonical chain map $I(B_k^\square(S'_n)) \to S^\square(S_n)_k$ that is an isomorphism if $S'_n$ has at least $k + 1$ vertices in each leg.

**Proof.** A basis for the weight $k$ subcomplex of $S^\square(S_n)$ is given by the set of elements of the form $e_1^{r_1} \cdots e_n^{r_n} \otimes x$, where $x \in \{\emptyset, v, h_1, \ldots, h_n\}$, the $r_j$ are non-negative integers, and $\sum r_j + \text{wt}(x) = k$.

The coimage of $\pi$ in the star flow is a quotient of the set of cells containing either the 0-cell $v$, the 1-cell $e_{j,N}$, or no cell containing $v$, along with $r_j$ additional 0-cells in the $j$th leg for each $j$. The relation identifies two such configurations with the same central configuration and the same $r_j$ for each $j$ but different choices of 0-cells away from the center vertex. Then we send such a configuration to the element $e_1^{r_1} \cdots e_n^{r_n} \otimes x$ where $x$ is:

1. the element $v$ if $c$ contains the star vertex $v$,
2. the element $h_i$ if $c$ contains the edge $e_{j,N}$, and
3. the element $\emptyset$ otherwise.

This map is prima facie injective, and it is surjective if $S'_n$ has the hypothesized number of vertices in each leg. It is straightforward to verify that this map is also a chain map. □

**4.4. Compatibilities.** In this section, we check that the abstract flows constructed in the previous section are compatible with inclusion and subdivision, completing the proof of Proposition 4.12 and thereby of the isomorphism statement of Theorem 4.5.

Let $\Gamma \to \Gamma'$ be a subdivision of graphs, with $\Gamma$ either the interval graph $I$ or the star graph $S_n$, and let $\Xi' \subseteq \Gamma'$ be a subgraph that is a subdivision of a disjoint union $\Xi$ of intervals and stars; thus, $\Xi \to \Gamma$ is a graph morphism. We equip $B_k^\square(\Gamma')$ with either the interval flow or the star flow, depending on the case in consideration. Through the isomorphism of Lemma 2.11, the configuration complex $B_k^\square(\Xi')$ also inherits an abstract flow, which depends on choices of star points, since subdivided intervals, 1-stars, and 2-stars are intrinsically indistinguishable, and on choices of orientation for the interval components.

**Lemma 4.18.** The inclusion $i : B_k^\square(\Xi') \to B_k^\square(\Gamma')$ is flow compatible provided that any component of $\Xi'$ containing a star vertex of $\Gamma'$ carries the star flow at that
Moreover, in this case, the following diagram commutes:

\[
\begin{array}{ccc}
I(B_k^\Box(\Xi')) & \xrightarrow{I(B_k^\Box(i))} & I(B_k^\Box(\Gamma')) \\
\downarrow & & \downarrow \\
S^\lambda(\Xi)_k & \xrightarrow{S^\lambda(\Xi\to\Gamma)} & S^\lambda(\Gamma)_k.
\end{array}
\]

**Proof.** We prove only the case when \( \Gamma \) is a star graph and \( \Gamma' \) is equipped with the star flow (the interval case is easier), focusing on the verification of flow compatibility. Write \( \pi_\Xi \) for the abstract flow on \( B_k^\Box(\Xi') \) and \( \pi_\Gamma \) for the abstract flow on \( B_k^\Box(\Gamma') \).

Let \( c \) be a cell of \( B_k^\Box(\Xi') \), written as a symmetric product of cells of \( \Xi' \).

If \( c \) contains an edge that is not adjacent to the star vertex of \( \Gamma' \), then so does \( i(c) \). Then by Definitions 4.13 and 4.15, \( \pi_\Gamma(i(c)) = 0 = \pi_\Xi(i(\pi_\Xi(c))) \). On the other hand, if \( c \) contains no such edge, then by the same definitions and inspection, \( \pi_\Xi(c) \) is the sum of a cell \( c' \), which is obtained by moving the vertices of \( c \) into their minimal positions in the relevant components of \( \Xi' \), with a linear combination of cells, each containing an edge not adjacent to the star vertex of \( \Gamma' \). The image of these latter cells are all in the kernel of \( \pi_\Gamma \), so \( \pi_\Gamma(i(\pi_\Xi(c))) = \pi_\Gamma(i(c)) \). But the same characterization also shows that \( \pi_\Gamma(i(c')) = \pi_\Gamma(i(c)) \), since both are obtained by moving the same number of vertices in each leg of \( \Gamma' \) into their minimal positions.

The commutativity claim is essentially immediate from what has already been said and the description of the isomorphism in Lemma 4.17. \( \square \)

Similar considerations apply in the case of a subdivision:

**Lemma 4.19.** For subdivisions \( V' \to V'' \) and \( S'_n \to S''_n \) of the interval and the star graph \( S_n \) respectively, the induced subdivisional embeddings \( B_k^\Box(V') \to B_k^\Box(V'') \) and \( B_k^\Box(S_n) \to B_k^\Box(S'_n) \) are flow compatible, and the following diagrams commute:

\[
\begin{array}{ccc}
I(B_k^\Box(V')) & \longrightarrow & I(B_k^\Box(V'')) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
S^\lambda(S_n)_k & \longrightarrow & S^\lambda(S'_n)_k.
\end{array}
\]

**Proof of Proposition 4.12.** By Lemma 4.19, the interval and star flows determine subdivisional flows on any disjoint union of intervals and stars equipped with any subdivisional structure. By the same result and Lemma 4.18, any embedding among such is flow compatible if the preimage of every star point is a star point. We use the star flow on component type basics in \( \Gamma_0 \) with the vertices of \( \Gamma \) as star vertices and the interval flow for all other basics. Since these subdivisional flows respect the isomorphisms \( B^{SD}(U \amalg V) \cong B^{SD}(U) \times B^{SD}(V) \), we obtain a local flow on the local invariant \( B^{SD} \), isotopy invariant by Lemma 4.18. By inspection in the interval case and Lemma 4.17, there is an identification of the Morse complex \( I(B^{SD}(U)) \cong S^\lambda(U) \) which is natural for inclusions among disjoint unions of basics by Lemmas 4.18 and 4.19. Isotopy invariance and the identification of the algebra and module structures follow. \( \square \)

### 4.5. Naturality for graph morphisms.

In this section, we show that the isomorphism of Theorem 4.5 is functorial. To use the naturality clause of Theorem 3.19 we will show that every graph morphism lifts to a flow compatible map of local invariants. It suffices to do so for graph embeddings and smoothings separately.
If $f : \Gamma_1 \to \Gamma_2$ is a graph embedding, then every edge of $\Gamma_1$ is mapped homeomorphically to an edge of $\Gamma_2$, so we may choose auxiliary vertices so that there is a commuting diagram of graph morphisms:

$$
\begin{array}{c}
\Gamma_1 \xrightarrow{f} \Gamma_2 \\
\uparrow \quad \quad \uparrow \\
(\Gamma_1)_# \quad \to \quad (\Gamma_2)_# 
\end{array}
$$

Then from the definition of the canonical decomposition, we have a commuting diagram of graph embeddings, and, in particular, of subdivisional spaces, which is to say a map of decompositions:

$$
\begin{array}{c}
\coprod_{v \in V(\Gamma_1)} S_{d(v)} \xleftarrow{\coprod_{e \in E(\Gamma_1)} pt \sqcup pt} \coprod_{e \in E(\Gamma_1)} I \\
\downarrow \quad \downarrow \\
\coprod_{v \in V(\Gamma_2)} S_{d(v)} \xleftarrow{\coprod_{e \in E(\Gamma_2)} pt \sqcup pt} \coprod_{e \in E(\Gamma_2)} I.
\end{array}
$$

It follows from Lemmas 4.18 and 4.19 that the identity map on $B^{SD}$ is flow compatible when regarded as a map between the induced local invariants on these two decompositions. Since every basic is a disjoint union of intervals and stars, the same lemmas show that the induced map at the level of Morse complexes coincides with the induced map on Świątkowski complexes. Thus, naturality holds for graph embeddings.

The case of a smoothing reduces to the following scenario. Fix an edge $e_0 \in \Gamma$, and let $\Gamma'$ be the graph obtained from $\Gamma$ by adding a bivalent vertex $v_0$ to $e_0$, subdividing it into $e_1$ and $e_2$. There is a smoothing $f : \Gamma' \to \Gamma$, and every smoothing is a composite of isomorphisms and smoothings of this kind. This smoothing does not respect the canonical decompositions of $\Gamma'$ and $\Gamma$, but it does respect a different pair of decompositions $\mathcal{E}'$ and $\mathcal{E}$, depicted in the following diagram and Figure 7:

$$
\begin{array}{c}
\coprod_{v_0 \neq v \in V'} S_{d(v)} \xleftarrow{\coprod_{e \in E} pt \sqcup pt} \coprod_{e \in E'} I \xleftarrow{pt \sqcup pt} S_2 \\
\downarrow \quad \downarrow \\
\coprod_{v \in V} S_{d(v)} \xleftarrow{\coprod_{e \in E} pt \sqcup pt} \coprod_{e \in E'} I \xleftarrow{pt \sqcup pt} I.
\end{array}
$$

As before, Lemma 4.18 guarantees that corresponding map of local invariants is flow compatible; moreover, Theorem 3.19 implies that the right-hand cell commutes in the diagram below. The left-hand cell commutes by inspection. Since the commuting of the outer square is what is to be proven, it will then suffice to verify that
Figure 7. The canonical decomposition (labelled $\tilde{\epsilon}'_{\text{fld}}$) and the decomposition $\tilde{\epsilon}'$ of the theta graph $\Theta_3$ with an additional bivalent vertex are depicted in the first row. The canonical decomposition (labelled $\tilde{\epsilon}_{\text{par}}$) and the decomposition $\tilde{\epsilon}$ of the theta graph $\Theta_3$ itself are depicted in the second row. The notation used is that of Appendix B; compare Figure 11.

the two remaining triangles also commute.

\[
\begin{align*}
H_*(S(\Gamma')) & \cong H_*(S(f)) \cong H_*(S(\Gamma)) \\
& \cong H_*(S(f)) \\
& \cong H_*(B(f)) \cong H_*(B(\Gamma))
\end{align*}
\]
This question of comparing different decompositions of a fixed complex is the subject of Appendix B. The main tool there, Proposition B.5, is used twice. In §B.2, it is used to establish a compatibility which is applicable to $\mathcal{E}$ and the canonical decomposition of $\Gamma$. This first compatibility guarantees the commutativity of the lower triangle provided that every inclusion among disjoint unions of basics for the two decompositions is flow compatible. Then, in §B.3, the proposition is used to establish a compatibility applicable to $\mathcal{E}'$ and the canonical decomposition of $\Gamma'$. This second compatibility guarantees the commutativity of the upper triangle provided the corresponding inclusions are flow compatible. As before, these compatibilities are a direct consequence of Lemma 4.18.

5. Homology of graph braid groups

We begin our computational study of the homology of configuration spaces of graphs in earnest. Highlights include Proposition 5.13, an exact sequence associated to the removal of a vertex; Proposition 5.14, which asserts that multiplication by any fixed edge is injective; Proposition 5.17, which identifies the homology in top degree in the case of a trivalent graph; Proposition 5.22, a full computation in the case of the complete graph on four vertices; and Appendix C, which gives a streamlined derivation of the characterization of the first homology due to [KP12]. Except for this last, these results appear to be new.

5.1. Graphs, star and loop classes, and relations. Since a graph embedding induces a map at the level of Świątkowski complexes, the homology of the configuration spaces of a graph is constrained by relations originating in its subgraphs. In this section, we acquaint ourselves with a few useful relations of this kind.

Recall that the star graph $S_n$ is the cone on the set $\{1, \ldots, n\}$. This graph has $n+1$ vertices: a central vertex $v_0$ of valence $n$ and $n$ vertices $\{v_1, \ldots, v_n\}$ of valence 1. Its half-edges $E$ are $\{e_1, \ldots, e_n\}$, and it has $2n$ corresponding half-edges $H$, of which $n$ of these, $h_1, \ldots, h_n$, are at $v_0$ and one, $h'_i$, is at $v_i$ for $i \neq 0$.

We will also consider several other graphs in this section.

**Definition 5.1.** The cycle graph $C_n$ is a topological circle equipped with $n$ bivalent vertices and $n$ edges. The lollipop graph $L_n$ is obtained by attaching an extra edge $e_0$ to the cycle graph $C_n$ at one half-edge. The theta graph $\Theta_n$ is the topological suspension (double cone) on $n$ points, with vertices the cone points $v_1$ and $v_2$.

![Cycle graphs, lollipop graphs, and theta graphs](image)

The reduced Świątkowski complex of the star graph $\tilde{S}(S_n)$ is concentrated in degrees 0 and 1 with a single differential

$$\mathbb{Z}[E][h_1, \ldots, h_n] \xrightarrow{\partial} \mathbb{Z}[E][\emptyset].$$
Lemma 5.2.  

(1) The homology group \( H_1(B(S_3)) \) is freely generated as a \( \mathbb{Z}[E] \)-module by a single class \( \alpha \) in \( H_1(B_2(S_3)) \).

(2) The homology group \( H_1(B(C_n)) \) is generated as a \( \mathbb{Z}[E] \)-module by a single class \( \gamma \) in \( H_1(B_1(C_1)) \) subject to the relations \( e_i \gamma = e_j \gamma \).

Proof. For \( S_3 \), it is straightforward to see that \( H_0(B(S_3)) \) is rank one in each weight. By observation, the chain \( a := a_{123} = e_1 h_{23} + e_2 h_{31} + e_3 h_{12} \) is closed. Checking the Euler characteristic shows that \( \mathbb{Z}[E](\alpha) \) has the correct rank and thus is the entirety of the kernel of \( \partial \).

For \( C_1 \), the complex \( S(C_1) \) has no differential and its degree one subspace is isomorphic to the module described. There is a (non-unique) smoothing from \( C_n \) to \( C_1 \); the induced map on Świątkowski complexes identifies every edge of \( C_n \) with the unique edge of \( C_1 \). \( \square \)

The sign of the generating class \( \alpha \) in \( H_1(B_2(S_3)) \) depends on the choice of ordering of the half-edges. We will use the notation \( \alpha_{123} = \alpha_{231} = \alpha_{312} \) for \( \alpha \) with the convention employed here and \( \alpha_{132} = \alpha_{321} = \alpha_{213} \) for the generator with the opposite sign.

Definition 5.3. Let \( S_3 \rightarrow \Gamma \) be a graph morphism. We call the image of the class \( \alpha \) in \( H_1(B_2(\Gamma)) \) a star class and a representing cycle of the form above a star cycle.

Let \( C_n \rightarrow \Gamma \) be a graph morphism. We call the image of the class \( \gamma \) in \( H_1(B(\Gamma)) \) a loop class and a representing cycle a loop cycle.

A star class depends only on the isotopy class of the graph morphism which induces it. Note also that we have such a morphism whenever there are three distinct half-edges at a vertex; it is not necessary that the corresponding edges be distinct.

Lemma 5.4. Star and loop classes are non-trivial.

Proof. There is a natural homomorphism \( \sigma_\Gamma : H_1(B_2(\Gamma)) \rightarrow \mathbb{Z}/2\mathbb{Z} \), which is induced on Abelianizations by the homomorphism from the braid group \( \pi_1(B_2(\Gamma)) \) recording the permutation of the endpoints of a braid. By naturality, evaluating \( \sigma_\Gamma \) on a star class gives the same answer as evaluating \( \sigma_{S_3} \) on a generator of \( H_1(B_2(S_3)) \). Since any cycle representing such a generator interchanges the two points of the configuration, \( \sigma_\Gamma \) takes the value 1 on this generator.

Loop classes live in \( H_1(B_1(\Gamma)) \) which is naturally isomorphic to \( H_1(\Gamma) \); a loop in \( \Gamma \) is never trivial. \( \square \)

Star classes play a pivotal role in the remainder of the paper. For example, we have the following result.

Lemma 5.5. The \( \mathbb{Z}[E] \)-module \( H_1(B_k(S_n)) \) is generated by star classes.

Proof. We proceed by induction on \( n \). Since \( S_n \) is topologically an interval for \( n \in \{1, 2\} \), the claim holds trivially in these cases, and the case \( n = 3 \) follows from Lemma 5.2, so we may assume that \( n \geq 4 \).

Any degree 1 element \( a \in S^0(S_n) \) can be written in the form \( a = \sum_{i=1}^n p_i e_i \) with \( p_i \in \mathbb{Z}[E] \), and imposing the condition \( \partial a = 0 \) yields the two equations

\[
\partial a = \sum_{i=1}^n p_i (e_i - v) = 0 \iff \begin{cases} 
\sum_{i=1}^n p_i = 0; \\
\sum_{i=1}^n e_i p_i = 0.
\end{cases}
\]
For each \( i \leq n - 2 \), we write \( p_i = (e_i - e_{i-1})p_i' + r_i \), where \( r_i \) does not involve the variable \( e_n \). Now we can rewrite \( a \) partially in terms of the star cycles \( a_{i,n-1,n} \) as follows:

\[
a = \sum_{i=1}^{n-2} (p_i'a_{i,n-1,n} + r_i h_i) + q_{n-1}h_{n-1} + q_n h_n,
\]

where

\[
q_{n-1} := p_{n-1} - \sum_{i=1}^{n-2} (e_i - e_{n})p_i', \quad q_n := p_n - \sum_{i=1}^{n-2} (e_{n-1} - e_i)p_i'.
\]

Write \( q_{n-1} \) and \( q_n \) as

\[
q_{n-1} = e_n q_{n-1}' + r_{n-1}, \quad q_n = e_n q_n' + r_n,
\]

where \( r_{n-1} \) and \( r_n \) do not involve the variable \( e_n \). Then considering terms involving \( e_n \) in the equations (1) using the fact that \( \partial a_{i,n-1,n} = 0 \), we have

\[
\begin{cases}
e_n(q_{n-1}' + q_n') = 0; \\
e_{n-1}q_{n-1}' + e_n q_n' + r_n = 0
\end{cases} \iff (e_n - e_{n-1})q_n' + r_n = 0.
\]

Since \( r_n \) does not involve the variable \( e_n \), it follows that \( q_n' = r_n = 0 \), whence \( q_n = 0 \). We further conclude that \( q_{n-1}' = 0 \), and so \( q_{n-1} \) does not involve the variable \( e_n \); therefore, \( a - \sum_{i=1}^{n-2} a_{i,n-1,n} \) does not involve \( e_n \) or \( h_n \) and so must lie in the image of the map \( S^q(-) \) induced by the inclusion \( S_{n-1} \to S_n \) that misses the \( n \)th leg. The inductive hypothesis now completes the proof. \( \square \)

**Proposition 5.6.** If \( \Gamma \) is connected, then \( H_1(B(\Gamma)) \) is generated as a \( \mathbb{Z}[E] \)-module by star classes and loop classes.

**Proof.** Assume first that \( \Gamma \) is a tree. In this case the claim follows from Proposition 5.15, Lemma 5.5, and induction on the number of essential vertices of \( \Gamma \). In the general case, subdivide each edge of \( \Gamma \) twice, calling the resulting graph \( \Gamma' \). Each edge of \( \Gamma \) corresponds to three edges of \( \Gamma' \); let \( E_{mid} \) be the set of edges of \( \Gamma' \) none of whose vertices are vertices of \( \Gamma \). Let \( E_0 \) be a subset of \( E_{mid} \) whose complement is a spanning tree \( \mathcal{T} \) of \( \Gamma' \). For \( e \in E_0 \), write \( \Gamma^{[e]} \) for the disjoint union of the unique cycle subgraph \( C_e \) contained in \( e \cup \mathcal{T} \) with a disjoint edge for each edge of \( E_{mid} \setminus E(C_e) \). There are canonical graph embeddings of \( \mathcal{T} \) and \( C_e \) into \( \Gamma' \) which induce graph morphisms from \( \mathcal{T} \) and \( C_e \) into \( \Gamma \). Write \( \tilde{S}(\Gamma) := \tilde{S}(\mathcal{T}) \oplus \bigoplus_{e \in E_0} \tilde{S}(\Gamma^{[e]}) \); then we have a map of differential graded \( \mathbb{Z}[E] \)-modules

\[
\phi : \tilde{S}(\Gamma) \to \tilde{S}(\Gamma)
\]

induced by these graph morphisms. It is clear that \( H_1(B(\Gamma^{[e]})) \) is generated as a \( \mathbb{Z}[E] \)-module by its unique loop class for each \( e \in E_0 \), and we have already shown that \( H_1(B(\Gamma_{\mathcal{T}})) \) is generated by star classes; therefore, it will suffice to show that \( \phi \) induces a surjection on \( H_1 \).

For \( e \in E_0 \), write \( e' \) and \( e'' \) for the unique pair of edges of \( \mathcal{T} \) with \( \phi(e) = \phi(e') = \phi(e'') \). At the chain level, \( \phi|_{\tilde{S}_1(\mathcal{T})} \) is surjective with kernel the \( \mathbb{Z}[E] \)-span of the set \( \{e' - e'' : e \in E_0\} \). Let \( c_e \in \tilde{S}(C_e) \) be a cycle representing a loop class and choose a degree 1 element \( b_e \in \tilde{S}(\mathcal{T}) \) with \( \phi(b_e) = \phi(c_e) \) and \( \partial b_e = \pm(e' - e'') \). Then \( \phi(b_e - c_e) = 0 \) and \( \partial(b_e - c_e) = \pm(e' - e'') \in \tilde{S}(\mathcal{T}) \).
Now, suppose we are given \( b \in \tilde{S}(T) \) with \( \phi(T) \) closed. Then

\[
\partial b = \sum_{e \in E_0} p_i(e' - e''),
\]

and we set

\[
b' := b + \sum_{e \in E_0} \mp p_i(b_e - c_e) \in \tilde{S}(\Gamma).
\]

Then \( \phi(b') = \phi(b) \) and \( b' \) is closed, as desired. \( \square \)

Now we turn to relations.

**Lemma 5.7.**

(1) The homology group \( H_0(B(S_2)) \) is generated by the class of the empty configuration \( \emptyset \) subject to the relation \( e_1 \emptyset = e_2 \emptyset \).

(2) In \( H_1(B(S_4)) \), the star classes satisfy the relation

\[
e_1 \alpha_{234} - e_2 \alpha_{341} + e_3 \alpha_{412} - e_4 \alpha_{123} = 0.
\]

(3) In \( H_1(B(L_n)) \), the loop and star classes satisfy the relation

\[
(e - e_0)\gamma = \alpha
\]

where \( e \) is an edge of the cycle subgraph.

(4) Let the edges of \( \Theta_3 \) be numbered from 1 to 3 and likewise for the half-edges at \( v_1 \) and at \( v_2 \). Then the star class \( \alpha_{123} \) at \( v_1 \) and the star class \( \alpha_{321} \) at \( v_2 \) are equal.

**Proof.**

(1) The cokernel of the differential is generated freely by \( (e_1 - e_2) \).

(2) The claim follows by expansion of the star cycles \( a_{ijk} \).

(3) It suffices to verify the claim for \( L_1 \) where it is already true for the (unique) chain level representatives using the reduced Świątkowski complex.

(4) The chain \( h_{12} \otimes h'_{13} - h_{13} \otimes h'_{12} \) in \( S_2(\Theta_3) \) bounds the difference between the corresponding star cycles.

\( \square \)

**Definition 5.8.** Let \( \Gamma \) be a graph.

(1) Let \( S_2 \to \Gamma \) be a graph morphism. We call the induced relation in \( H_0(B(\Gamma)) \) an \textit{I}-relation.

(2) Let \( S_4 \to \Gamma \) be a graph morphism. We call the induced relation on star classes in \( H_1(B(\Gamma)) \) an \textit{X}-relation.

(3) Let \( L_n \to \Gamma \) be a graph morphism. We call the induced relation on loop and star classes in \( H_1(B(\Gamma)) \) a \textit{Q}-relation.

(4) Let \( \Theta' \to \Theta_3 \) be a smoothing and \( \iota : \Theta' \to \Gamma \) a graph embedding. Denoting the preimage of \( v_i \) in \( \Theta' \) by \( v'_i \), we call the relation induced on the star classes at \( \iota(v'_1) \) and \( \iota(v'_2) \) in \( H_1(B(\Gamma)) \) a \textit{Θ}-relation.

(5) Let \( C_n \to \Gamma \) be a graph morphism. We call the induced relation (from Lemma 5.2) on loop classes in \( H_1(B(\Gamma)) \) an \textit{O}-relation.

Repeated use of the \textit{I}-relation implies the well-known fact that \( H_0(B_k(\Gamma)) \) is one-dimensional for \( \Gamma \) connected.

These atomic classes combine naturally.

**Definition 5.9.** Consider a graph \( \Gamma_0 \) written as the disjoint union of \( n_1 \) cycle graphs and \( n_2 \) copies of \( S_3 \). Since the configuration spaces of the components all
have torsion-free homology, there is a class $\beta$ in $H_{n_1 + n_2}(B_{n_1 + 2n_2}(\Gamma_0))$ corresponding to the tensor product of the loop classes in each cycle graph and the star classes in each star graph. If $\Gamma_0 \to \Gamma$ is a graph morphism, we call the image of $\beta$ in $H_{n_1 + n_2}(B_{n_1 + 2n_2}(\Gamma))$ the external product of the corresponding loop and star classes. We will use juxtaposition to indicate the external product—writing $\alpha_1 \alpha_2$ for the external product of classes $\alpha_1$ and $\alpha_2$—and we caution the reader that this construction is only partially defined and so does not define a product on homology.

5.2. Euler characteristic. One calculation that requires no further tools is that of the Euler characteristic. This result has been known at least since [Gal01].

**Corollary 5.10.** The Euler characteristic of $B_k(\Gamma)$ is given by

$$\chi(B_k(\Gamma)) = \sum_{U \subseteq V} (-1)^{|U|} \left( k - |U| + |E| - 1 \right) \prod_{v \in U} (d(v) - 1).$$

Defining the Euler–Poincaré series of $\Gamma$ to be the formal power series

$$P_\chi(\Gamma)(t) = \sum_{k=0}^{\infty} \chi(B_k(\Gamma)) t^k,$$

we have the following reformulation.

**Corollary 5.11.** Let $\Gamma$ be a graph. Then

$$P_\chi(\Gamma)(t) = \prod_{v} \frac{1 + t(1 - d(v))}{(1 - t)^{d(v)}}.$$

**Proof.** For $n \geq 0$, the formula holds for the graph $G_{2n}$ that is the wedge of $n$ circles. It also holds, for $m \leq n$ non-negative, for the graph $G_{2m+1,2n+1}$, which is given by connecting the vertices of $G_{2m}$ and $G_{2n}$ by a single edge. Since the formula of Corollary 5.10 does not depend on which pairs of vertices share an edge, the claim follows. \qed

5.3. Exact sequences. One particularly nice tool afforded by the Świątkowski complex is an exact sequence which we use to reduce computations of $H_*(B(\Gamma))$ to computations for simpler graphs.

**Definition 5.12.** Let $v$ be a vertex of the graph $\Gamma$. Then we write $\Gamma_v$ for the vertex explosion of $\Gamma$ at $v$, that is, the graph obtained by

1. replacing the vertex $v$ with $\{v\} \times H(v)$ and
2. modifying the attaching map for half-edges attached at $v$ by letting such a half-edge $h$ be attached at $v \times h$.

There is a graph morphism from $\Gamma_v$ to $\Gamma$ which takes each edge to itself, takes the vertex $v \times h$ to $e(h)$, and takes each other vertex to itself. Defining this morphism requires choices (of precisely where in $e(h)$ to send $v \times h$) but the isotopy class of this graph morphism is unique. See Figure 9.

**Proposition 5.13.** Fix a half-edge $h_0 \in H(v)$. There is a short exact sequence of differential bigraded $\mathbb{Z}[E]$-modules

$$0 \to \tilde{S}(\Gamma_v) \xrightarrow{\phi} \tilde{S}(\Gamma) \xrightarrow{\psi} \bigoplus_{h \in H(v) \setminus \{h_0\}} \tilde{S}(\Gamma_v)[1] \{1\} \to 0.$$
Figure 9. A local picture of vertex explosion along with an intermediate graph which admits a graph morphism from $\Gamma_v$ and a smoothing to $\Gamma$.

Proof. Polarize $b$ in bidegree $(i, k)$ in $\bar{S}(\Gamma)$ by its component in $\bar{S}(v)$, i.e., write

$$b = b_0 + \sum_{h \in H(v) \setminus \{h_0\}} (h - h_0)b_h$$

where $|b_0| = (i, k)$, $|b_h| = (i - 1, k - 1)$, and each of the tensors $b_0$, and $b_h$ contains no elements of $H(v)$. The subspace of such elements is precisely $\bar{S}(\Gamma_v)$, and, since $\partial$ does not create half-edges, both the inclusion of $\bar{S}(\Gamma_v)$ as $b_0$ and the projection to $\bigoplus b_h$ are chain maps. It is immediate that both respect the action of $\mathbb{Z}[E]$.

In the corresponding long exact sequence, the connecting homomorphism $\delta$ from $\bigoplus H_*(\bar{S}(\Gamma_v)) \to H_*(\bar{S}(\Gamma_v))\{-1\}$ is explicitly given by the formula

$$\delta\beta_h = (e(h) - e(h_0))\beta_h.$$  

5.4. Edge injectivity and bridge decompositions. We will make repeated use of the following result as a technical tool, but it is interesting in itself, and we think of it as a first step in an investigation of the $\mathbb{Z}[E]$-module structure enjoyed by the homology of the configuration spaces of $\Gamma$, in the tradition of the study of homological stability. This study will be continued in detail in future work in preparation—see also [Ram16].

Proposition 5.14. For any $e \in E$, multiplication by $e$ is injective on $H_*(B(\Gamma))$.

Proof. Any monomial $p(E, V)$ in the edges and vertices of $\Gamma$ can be rewritten as a polynomial in variables $e$, $e - e'$, and $e - v$. Write $U$ for the collection of generators other than $e$, which is in bijection with $E \setminus \{e\} \cup V$. Then an arbitrary chain in $\bar{S}(\Gamma)$ can be written in the form

$$b = \sum_{i=0}^{M} e^i b_i(U, H)$$

for some (graded commutative) polynomials $b_i$ (not all polynomials are possible).

Let $c$ be a cycle in $S(\Gamma)_k$, and suppose that $ec = \partial b$. The proof will be complete once we are assured that $c$ is itself a boundary. To see this, we note that the differential, acting on $b_i(U, H)$, cannot introduce a factor of $e$ in this basis; that is,

$$\partial b = \sum_{i=0}^{M} e^i b'_i(U, H),$$
where $b_i' = \partial b_i$. Then we have
\[
\sum_{i=0}^{M} e_i c_{i-1}(U, H) = \sum_{i=0}^{M} e_i b_i'(U, H).
\]
Thus, $\partial b_0 = b_0' = 0$, so $e c = \partial b = \partial (b - b_0)$. Since $b - b_0$ is divisible by $e$, we have $c = \partial (\frac{b - b_0}{e})$, as desired. □

With Proposition 5.14 in hand, we can describe the effect on $H_*(B(\Gamma))$ of cutting $\Gamma$ into two disconnected components at a vertex.

**Proposition 5.15.** Let $\Gamma$ be a connected graph and $v$ a bivalent vertex whose removal disconnects $\Gamma$, and write $e_1$ and $e_2$ for the edges at $v$. There is an isomorphism
\[
H_*(B(\Gamma)) \cong H_*(B(\Gamma_v))/e_1 \sim e_2
\]
of $\mathbb{Z}[E]$-modules. In particular, for any field $\mathbb{K}$, we have
\[
H_*(B(\Gamma), \mathbb{K}) \cong H_*(B(\Gamma_v), \mathbb{K}) \mathbb{K}[e_1] \times H_*(B(\Gamma_v''), \mathbb{K}),
\]
where $\Gamma_v'$ and $\Gamma_v''$ are the connected components of $\Gamma_v$.

**Proof.** Applying Proposition 5.13 at the vertex $v$, we obtain the exact sequence
\[
0 \rightarrow \tilde{S}(\Gamma_v) \xrightarrow{\delta} \tilde{S}(\Gamma) \xrightarrow{\psi} \tilde{S}(\Gamma_v)[1]\{1\} \rightarrow 0.
\]
We claim that the connecting homomorphism in the corresponding long exact sequence is injective. Indeed, suppose that $\delta \beta = 0$ for some $\beta \in H_i(B_k(\Gamma_v))[1]\{1\}$; then, applying the formula of Proposition 5.13, we have $(e_1 - e_2)\beta = 0$. Since $\Gamma_v$ is disconnected, the homology of $B_k(\Gamma_v)$ is naturally graded by the number of points lying in $\Gamma_v$; therefore, writing $\beta = \sum_{j=0}^k \beta_j$, we obtain the system of equations
\[
e_1 \beta_k = e_2 \beta_0 = 0
\]
\[
e_1 \beta_j = e_2 \beta_{j+1} \quad (0 \leq j < k).
\]
Applying Proposition 5.14 repeatedly now implies that $\beta_j = 0$ for every $0 \leq j \leq k$.

The statement over $\mathbb{Z}$ now follows by exactness and the formula for $\delta$, and the statement over $\mathbb{K}$ follows from statement over $\mathbb{Z}$ and the Künneth theorem. □

### 5.5. High degree homology for graphs of maximal valence three.
Here we apply the exact sequence of Proposition 5.13 to study $H_*(B(\Gamma))$ when every vertex of $\Gamma$ has valence 3 or 1 and $i$ is close to the number of trivalent vertices of $\Gamma$. According to our conventions, a graph may have self-loops (at most one self-loop per vertex), and multiple edges connecting a given pair of vertices. In this subsection, unless otherwise specified, $\Gamma$ is a graph all of whose vertices are of valence 3 or 1, and $N$ denotes the number of essential vertices of $\Gamma$.

**Construction 5.16.** Suppose $\Gamma$ has $r$ self-loops. To each essential vertex $v$ we associate a homology class $\beta_v \in H_1(B(\Gamma))$, well-defined up to sign: if $v$ has a self-loop, then $\beta_v$ is the corresponding loop class; otherwise, $\beta_v$ is the star class at $v$. Taking the external product over the essential vertices of the $\beta_v$, we obtain a homology class $\beta \in H_N(B_{2N-1}(\Gamma))$, well-defined up to sign, called the canonical class of $\Gamma$. 

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Proposition 5.17. We have an isomorphism

\[ H_N(B(\Gamma)) \cong \mathbb{Z}[E]\beta. \]

Proof. Without loss of generality, we may assume \( \Gamma \) is connected. We proceed by induction on \( N \), the base cases of a lollipop or a star having been covered in §5.1. Assume that \( N \geq 2 \), fix an essential vertex \( v \), and assume that the claim is known for \( \Gamma_v \).

Applying Proposition 5.13 at \( v \), we obtain the exact sequence

\[ \cdots \to H_N(\tilde{S}(\Gamma_v)) \to H_N(\tilde{S}(\Gamma)) \to \bigoplus_{h \in H(v) \setminus \{h_0\}} H_{N-1}(\tilde{S}(\Gamma_v)) \{1\} \xrightarrow{\delta} \cdots \]

Since \( \Gamma_v \) has \( N-1 \) essential vertices, the first term vanishes, so the homology group of interest is the kernel of \( \delta \). The inductive hypothesis identifies the domain of \( \delta \) as

\[ \left( \bigoplus_{h \in H(v) \setminus \{h_0\}} \mathbb{Z}[E(v)] \right) \otimes \mathbb{Z}[E \setminus E(v)]\beta', \]

where \( \beta' \) is the canonical class of \( \Gamma_v \) and \( \delta \) acts on the first factor. Denoting by \( \Gamma^{(v)} \) the subgraph containing \( v \) and all of its edges (either a star or a lollipop), this first factor is naturally identified with the degree 1 component of \( \tilde{S}(\Gamma^{(v)}) \). Under this identification, it is immediate from the formula of Proposition 5.13 that the condition of lying in the kernel of \( \delta \) is precisely the condition of lying in the kernel of \( \partial \) from degree one to degree zero of \( \tilde{S}(\Gamma^{(v)}) \). Since \( \Gamma^{(v)} \) has only one vertex,

\[ \ker \partial \cong H_1(B(\Gamma^{(v)})) \cong \mathbb{Z}[E(v)]\beta_v, \]

and the claim follows. \( \square \)

Remark 5.18. For a general graph (with no restriction on valence) it is straightforward to show that \( H_N(B_k(\Gamma)) = 0 \) for \( k < N \). Similar methods to those used in Proposition 5.17 show, for instance, that if \( \Gamma \) has no bivalent vertices or self-loops and at least one vertex, that \( H_N(B_N(\Gamma)) = 0 \) as well. Any further sharpening of this result must take into account, e.g., that \( H_2(B_3(\Theta_4)) \) is non-zero.

Similar inductive arguments using Proposition 5.13 may be used to demonstrate the following two results.

Proposition 5.19. Suppose \( \Gamma \) is simple. There is an isomorphism

\[ H_{N-1}(B(\Gamma)) \cong \bigoplus_{d(v)=3} \mathbb{Z}[E] \hat{\beta}_v/(e\hat{\beta}_v - e'\hat{\beta}_v : e, e' \in E(v)) \]

of \( \mathbb{Z}[E] \)-modules, where \( \hat{\beta}_v \in H_{N-1}(B_{2N-2}(\Gamma)) \) is the external product of \( \beta_w \) for \( w \neq v \).

Proposition 5.20. If \( \Gamma \) is simple, then \( H_{N-2}(B(\Gamma)) \) is torsion-free.

Sketch of proofs of Proposition 5.19 and 5.20. In both cases, by Proposition 5.13 at a fixed essential vertex \( v_0 \), the desired homology group \( A \) fits into a short exact sequence

\[ 0 \to \text{coker} \delta_M \to A \to \ker \delta_{M-1} \to 0 \]

for some \( M \).

In the case of Proposition 5.19, we can use Proposition 5.17 and the explicit formula for the connecting homomorphism to show that the cokernel entry of the
short exact sequence consists of the terms indexed by vertices other than $v_0$. Then by induction and an explicit calculation, the kernel entry splits and yields the term indexed by $v_0$.

In the case of Proposition 5.20, the kernel entry is torsion-free by induction so any torsion would have to come from the cokernel term. By explicitly examining two cases using the formula for the connecting homomorphism and Proposition 5.19 (the cases corresponding to whether the vertex indexing the summand in Proposition 5.19 is adjacent to $v_0$ or not), we conclude that the cokernel term is itself torsion-free. □

5.6 Case study: the complete graph $K_4$. We apply our results to give a complete description of $H_*(B(K_4))$ as a $\mathbb{Z}[E]$-module, where $K_4$ is the complete graph on four vertices. As an intermediary result, we also give the corresponding computation for the net graph $N$.

![Figure 10. The complete graph $K_4$ and the net graph $N$](image)

Notation 5.21. We number the vertices of $K_4$ and write $e_{ab}$ for the edge connecting vertices $a$ and $b$; thus, $e_{ab} = e_{ba}$ and $e_{aa}$ is undefined. Write $h_b^{(a)}$ for the half-edge at $a$ with edge $e_{ab}$. The action of the symmetric group $\Sigma_4$ on the vertices extends to an action on $S(K_4)$ by bigraded chain maps intertwining the $\mathbb{Z}[E]$-action; indeed, a choice of parametrization defines an action of $\Sigma_4$ on $K_4$ by graph isomorphisms, and the Świątkowski complex is functorial. We write $\gamma_a$ for the loop class avoiding vertex $a$ and $\alpha_a$ for the star class at vertex $a$, with signs fixed as follows:

- $\gamma_4 = [h_2^{(1)} - h_3^{(1)} + h_3^{(2)} - h_1^{(2)} + h_1^{(3)} - h_2^{(3)}]$
- $\alpha_4 = \alpha_{123} = [(h_2^{(1)} - h_3^{(1)})e_{34} + (h_3^{(4)} - h_1^{(4)})e_{24} + (h_2^{(4)} - h_3^{(4)})e_{14}]$
- $\gamma_a = (-1)^{\text{sgn}(\sigma)}\sigma(\gamma_4)$, and similarly for $\alpha_a$, where $\sigma$ is the cyclic permutation taking 4 to $a$.

We extend this notation to the net graph $N$ by naming all 1-valent vertices 4; this is not ambiguous because we will never refer to the half-edges at these vertices and because every vertex is adjacent to at most one such vertex.

Proposition 5.22. The $\mathbb{Z}[E]$-module structure of $H_*(B(K_4))$ is given in terms of generators and relations as follows: Let $a, b, c, d \in \{1, 2, 3, 4\}$ be distinct.

- $H_0(B(K_4))$ is generated by $\emptyset$ subject to relations identifying all edges.
- $H_1(B(K_4))$ is generated by $\gamma_a$ and $\alpha_a$ subject to the following relations:

  $$\sum_{a=1}^4 \gamma_a = 0 \quad \gamma_a(e_{bc} - e_{cd}) = 0 \quad \gamma_a(e_{ab} - e_{ac}) = 0$$

  $$\alpha_a = \alpha_b \quad \alpha_a(e - e') = 0 \quad \gamma_a(e_{bc} - e_{ab}) = \alpha_b$$

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\[ H_2(B(K_4)) \] is generated by \( \gamma_a \alpha_a \) and \( \alpha_a \alpha_b \) subject to the following relations:
\[
\sum_{a=1}^4 \gamma_a \alpha_a = 0 \\
\alpha_a \alpha_b(e - e') = 0 \text{ if } e, e' \neq e_a \beta \in E \\
\gamma_a \alpha_a(e_{bc} - e_{cd}) = 0 \\
\gamma_a \alpha_a(e_{bc} - e_{ab}) = \alpha_a \alpha_a
\]

\[ H_3(B(K_4)) \] is generated by \( \alpha_a \alpha_b \alpha_c \) subject to the relations \( \alpha_a \alpha_b \alpha_c(e_{ad} - e_{bd}) = 0 \).

\[ H_4(B(K_4)) \] is freely generated by \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \).

We will prove this result by relating \( K_4 \) to the net graph \( N \), which is obtained by exploding a vertex. The corresponding result for \( N \) is the following.

**Lemma 5.23.** The \( \mathbb{Z}[E] \)-module structure of \( H_*(B(N)) \) is given in terms of generators and relations as follows: Let \( a, b, c \in \{1, 2, 3\} \) be distinct.

- \( H_0(B(N)) \) is generated by \( \emptyset \) subject to relations identifying all edges.
- \( H_1(B(N)) \) is generated by \( \gamma_4 \) and \( \alpha_a \) subject to the following relations:
  \[
  \alpha_a(e - e') = 0 \text{ if } e, e' \neq e_4 \\
  \gamma_4(e_{ab} - e_{ac}) = 0 \\
  \gamma_4(e_{ac} - e_{a4}) = \alpha_a
  \]
- \( H_2(B(N)) \) is generated by \( \alpha_a \alpha_b \) subject to the relation \( \alpha_a \alpha_b(e - e') = 0 \) if \( e, e' \notin \{e_4, e_4, e_4\} \).
- \( H_3(B(K_4)) \) is freely generated by \( \alpha_1 \alpha_2 \alpha_3 \).

**Proof.** This follows from Propositions 5.17, 5.19, and 5.6, along with the \( O, Q, \) and \( I \)-relations and a rank-counting argument (using Corollary 5.10, say).

**Proof of Proposition 5.22.** The statement for \( H_0 \) holds for any connected graph, the statement for \( H_3 \) follows from Proposition 5.19, and the statement for \( H_4 \) follows from Proposition 5.17. For \( H_1 \), generation follows from Proposition 5.6 (along with the identification of loop classes containing four edges as the sum of two loop classes of the given form). The relation involving only loop classes follows from a relation in \( H_1(K_4) \) itself. The other stated relations follow directly from the \( \Theta, I, Q, \) and \( O \)-relations. In fact, \( H_1(B_n(G)) \) is completely known in general [KP12]. The relevant part of that computation, which we reprove as Lemma C.9, shows by inspection of the relations that our presentation is complete. Thus, it remains to prove the statement for \( H_2 \). We may assume that \( k \geq 2 \), since \( H_2 \) vanishes in lower weight.

The outline of the proof is as follows. Using the long exact sequence of Proposition 5.13, we obtain an explicit \( \mathbb{Z} \)-linear description of \( H_2 \). Then, denoting by \( M \) the \( \mathbb{Z}[E] \)-module presented by generators and relations in the statement of the proposition, we exhibit a \( \mathbb{Z}[E] \)-linear map from \( M \) to \( H_2 \), which our integral description shows to be surjective. Finally, we verify that the two modules have the same integral rank in each weight.

The portion of the long exact sequence relevant to our purpose is
\[
\cdots \to H_2(B_{k-1}(N)) \oplus_2 \delta_2 \to H_2(B_k(N)) \to H_2(B_k(K_4)) \to H_1(B_{k-1}(N)) \oplus_2 \delta_1 \to H_1(B_k(N)) \to \cdots
\]

The kernel of \( \delta_1 \) is free over \( \mathbb{Z} \), so this portion of the sequence splits integrally, and we have \( H_2(B_k(K_4)) \cong \text{coker } \delta_2 \oplus \text{ker } \delta_1 \) as \( \mathbb{Z} \)-modules. Passing to the cokernel of \( \delta_2 \) merely identifies \( e_{a4} \) and \( e_{b4} \) in \( H_2(B_k(N)) \) and thus contributes the space of external products \( \alpha_a \alpha_b \beta(e) \) for \( a, b \in \{1, 2, 3\} \), subject to the final relation \( \alpha_a \alpha_b(e - e') = 0 \)
if \( e, e' \neq e_{ab} \). On the other hand, \( \ker \delta_1 \) consists of the isomorphic image of the classes \( e_m^a \gamma^k \alpha_a \in H_2(B_4(K)) \) for \( a \in \{1, 2, 3\} \) and \( 0 \leq m \leq k - 3 \). Verifying this fact is a tedious calculation.

As for relations, the second, third, and fourth follow from the \( O \)-relation, the \( I \)-relation, and the \( Q \)-relation, respectively. For the first relation, counting ranks shows that \( \sum_{a=1}^4 r_a \gamma^a \alpha_a = 0 \) with at least one \( r_a \) non-zero, and symmetry under cyclic permutations implies that \( r_a \) is constant in \( a \). The absence of torsion in \( H_2(B_4(K)) \) forces \( r_a \) to be a unit.

For the rank calculation, we note that the \( \ker \delta_2 \) is the direct sum of three copies of the space of degree \( k - 4 \) polynomials in two variables, which has integral rank \( 3(k - 3) \), while \( \ker \delta_1 \) is the direct sum of three copies of the space of degree \( k - 3 \) polynomials in two variables, which has integral rank \( 3(k - 2) \). It follows that the integral rank of \( H_2(B_4(K)) \) is \( 6k - 15 \), which is also a lower bound on the rank of \( M \) by surjectivity. Thus, it remains to show that the integral rank of \( M \) is at most \( 6k - 15 \). To see why this is the case, we note that the relations shown imply that \( M \) is spanned integrally in weight \( k \) by the classes \( e_{ab}^k \gamma^m \alpha_a \), where \( 1 \leq a, b \leq 4 \), \( 0 \leq m \leq k - 4 \), and \( e \neq e_{ab} \) is fixed, together with the classes \( e_{bc}^k \gamma^m \alpha_a \), where \( a, b, c \neq 4 \). There are \( 6(k - 3) \) generators of the former type and \( 3 \) of the latter type, so the rank is at most \( 6k - 15 \), as claimed. \( \square \)

APPENDIX A. REMINDERS ON HOMOTOPY COLIMITS

In this appendix, we present a summary of some relevant facts and definitions concerning homotopy colimits and related matters. For the general theory and full details, [Dug17], [GJ09] and [Hir02] are good references.

The objects of the simplicial indexing category \( \Delta \), being finite ordered sets, may naturally be regarded as categories, and the arrows of \( \Delta \) determine functors among these categories.

**Definition A.1.** Let \( D \) be a category. The nerve of \( D \) is the simplicial set

\[
N\Delta_\bullet := \text{Fun}(\Delta^\bullet, D).
\]

An \( p \)-simplex \( \sigma \in N\Delta_p \) is a string \( \sigma(p) \to \cdots \to \sigma(0) \) of composable morphisms in \( D \). We say that \( D \) is contractible if the geometric realization \( |N\Delta| \) is so.

**Example A.2.** A filtered or cofiltered category is contractible provided it is not empty. In particular, any category with an initial or terminal object is contractible.

**Definition A.3.** Let \( F : D \to \text{Top} \) be a functor. The homotopy colimit of \( F \), denoted \( \text{hocolim}_D F \), is the geometric realization of the simplicial space given in simplicial degree \( p \) by

\[
\prod_{\sigma \in N\Delta_p} F(\sigma(p))
\]

with face and degeneracy maps induced by those of \( N\Delta \).

---

1Lemma A.4 implies that the homotopy colimit induces a functor from the homotopy category of functors from \( D \) to \( \text{Top} \) to the homotopy category of \( \text{Top} \). It is typical to either define the homotopy colimit as this latter functor or to define a homotopy colimit as any functor with similar properties to that in our definition which induces this same functor at the level of homotopy categories. We will not need this level of generality.
The homotopy colimit, thus defined, is functorial on the functor category, so that a natural transformation of functors \( F \rightarrow G \) induces a map on homotopy colimits. A fundamental property of homotopy colimits is that they are homotopy invariant.

**Lemma A.4.** The homotopy colimit of a natural weak homotopy equivalence between functors \( F \) and \( G \) is a weak homotopy equivalence.

**Definition A.5.** Let \( X \) be a topological space and \( \mathcal{U} \) an open cover of \( X \). We say \( \mathcal{U} \) is complete if it is possible to write any finite intersection of elements of \( \mathcal{U} \) as a union of elements of \( \mathcal{U} \).

We view the open cover \( \mathcal{U} \) as partially ordered under inclusion and thereby as a category. There is a tautological functor \( \Gamma : \mathcal{U} \rightarrow \text{Top} \) taking \( U \in \mathcal{U} \) to the topological space \( U \). Our main use of homotopy colimits is via the following result.

**Theorem A.6 ([DI04, Prop. 4.6]).** If \( \mathcal{U} \) is a complete cover of \( X \), then the natural map

\[
\text{hocolim}_{\mathcal{U}} \Gamma \rightarrow X
\]

is a weak homotopy equivalence.

One can also define homotopy colimits for functors valued in chain complexes. For simplicity we only define a special case; more detail is available in the references at the beginning of the appendix.

Given a simplicial chain complex \( V : \Delta \rightarrow \mathbb{C}h_{\mathbb{Z}} \), we may construct a bicomplex \( \text{Alt}(V) \) by taking the alternating sum of the face maps.

**Proposition A.7 ([Dug17, Prop. 19.9]).** Let \( V : \Delta \rightarrow \mathbb{C}h_{\mathbb{Z}} \) be a simplicial chain complex concentrated in non-negative degrees. There is a natural weak equivalence

\[
\text{hocolim}_{\Delta} V \simeq \text{Tot}(\text{Alt}(V)),
\]

where \( \text{Tot} \) denotes the total complex.

The following standard result asserts that the notions of homotopy colimit valued in topological spaces and chain complexes are compatible.

**Proposition A.8.** Let \( F : \mathcal{D} \rightarrow \text{Top} \) be a functor. There is a natural quasi-isomorphism

\[
\text{hocolim}_{\mathcal{D}} \text{C}^{\text{sing}}(F) \simeq \text{C}^{\text{sing}}(\text{hocolim}_{\mathcal{D}} F).
\]

We shall also make use of a relative version of the homotopy colimit construction. Recall that, given a functor \( T : \mathcal{D}_1 \rightarrow \mathcal{D}_2 \) and the existence of enough colimits in \( \mathcal{C} \), the restriction functor \( T^* : \text{Fun}(\mathcal{D}_2, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{D}_1, \mathcal{C}) \) admits a left adjoint \( \text{Lan} \), the left Kan extension functor. The homotopical version of this construction is the following.\(^2\)

**Definition A.9.** Let \( \mathcal{D}_2 \xleftarrow{T} \mathcal{D}_1 \xrightarrow{F} \mathcal{C} \) be functors with \( \mathcal{C} = \text{Top} \) or \( \mathcal{C} = \mathbb{C}h_{\mathbb{Z}} \). The homotopy left Kan extension of \( F \) along \( T \) is the functor from \( \mathcal{D}_2 \) to \( \mathcal{C} \) given (on objects) by

\[
\text{hoLan}_T F(d) = \text{hocolim}_{(T \downarrow d)} (F \circ \text{forget}).
\]

\(^2\)As with the homotopy colimit, the homotopy left Kan extension may be defined more invariantly in terms of homotopy categories. With this setup, our definition is a proposition—see [Dug17, Prop. 10.2], for example.
Here, for an object \( d \in \mathcal{D}_2 \), the overcategory \((T \downarrow d)\) has as objects the pairs \((d', f)\) with \( d' \in \mathcal{D}_1 \) and \( f : T(d') \to d \) a morphism in \( \mathcal{D}_2 \). A morphism from \( f : T(d') \to d \) to \( g : T(d'') \to d \) is a morphism \( h : d' \to d'' \) such that \( g \circ T(h) = f \). The forgetful functor to \( \mathcal{D}_1 \) takes \((d', f)\) to \( d' \). The construction \((T \downarrow d)\) is functorial in \( d \), and \( \text{hoLan}_T F \) extends to a functor using this functoriality and the functoriality of the homotopy colimit.

**Example A.10.** When \( \mathcal{D}_2 \) is the trivial category \( 1 \) with one object \( * \) (so that there is a unique functor \( * : \mathcal{D}_1 \to 1 \)), this construction recovers the homotopy colimit:

\[
\text{hoLan}_* F(*) \cong \text{hocolim} \mathcal{D}_1 F
\]

Dually, the objects of the undercategory \((d \downarrow T)\) has as objects the pairs \((f, d')\) with \( d' \in \mathcal{D}_1 \) and \( f : d \to T(d') \) a morphism in \( \mathcal{D}_2 \), and as morphisms morphisms \( h \) in \( \mathcal{D}_1 \) satisfying a dual condition. Overcategories and undercategories are very useful in the calculation of homotopy colimits. In order to say how this is so, we require a preliminary definition.

**Definition A.11.** Let \( T : \mathcal{D}_1 \to \mathcal{D}_2 \) be a functor. We say that \( T \) is

1. **homotopy final** if \((d \downarrow T)\) is contractible for every \( d \in \mathcal{D}_2 \), or
2. **homotopy initial** if \((T \downarrow d)\) is contractible for every \( d \in \mathcal{D}_2 \).

That is, \( T \) is homotopy initial just in case \( T^{\text{op}} : \mathcal{D}_1^{\text{op}} \to \mathcal{D}_2^{\text{op}} \) is homotopy final.

**Proposition A.12** ([Rie14, Thm. 8.5.6]). Let \( \mathcal{D}_1 \xrightarrow{T} \mathcal{D}_2 \xrightarrow{F} \mathcal{C} \) be functors with \( \mathcal{C} = \text{Top} \) (or \( \mathcal{C} = \text{Ch}_{\mathbb{Z}} \)). If \( T \) is homotopy final, then the natural map

\[
\text{hocolim} \mathcal{D}_1 T^* F \to \text{hocolim} \mathcal{D}_2 F
\]

is a weak homotopy equivalence (quasi-isomorphism).

We will make use of the following immediate consequences of this result.

**Corollary A.13.** Let \( \mathcal{D} \) be a category and \( \mathcal{C} : \mathcal{D} \to \mathcal{C} \) the constant functor at \( c \in \mathcal{C} \), where \( \mathcal{C} = \text{Top} \) (or \( \mathcal{C} = \text{Ch}_{\mathbb{Z}} \)). If \( \mathcal{D} \) is contractible, then the natural map

\[
\text{hocolim} \mathcal{D} \mathcal{C} \to c
\]

is a weak homotopy equivalence (quasi-isomorphism).

**Corollary A.14.** Let \( T : \mathcal{D}_1 \to \mathcal{D}_2 \) be any functor. If \( T \) is homotopy final, then the induced map \( \text{ND}_1 \to \text{ND}_2 \) is a weak homotopy equivalence.

**Appendix B. Comparing decompositions**

In this appendix, we tighten the combinatorial side of the correspondence between decompositions and relative tensor products arising from Theorem 3.19. We present a general conceptual framework in which comparison questions may be formulated, and we provide one uniform answer to such questions in the form of Proposition B.5. We then apply this comparison result in two examples used in the main text. We hope that these examples will make manifest that decompositions and relative tensor products may be manipulated in an essentially identical way.
B.1. Abstract setup. Throughout this section, we work exclusively with decompositions $E$ and $F$ of a fixed complex $X$. We assume that $F$ is a symmetric monoidal functor determining both an $E$-local invariant and an $F$-local invariant. With more notational baggage, one could work in a more general setting.

Definition B.1. An $(r,s)$-comparison scheme is a functor $\alpha : \Delta^r \to \Delta^s$.

We think of a comparison scheme as specifying the pattern of an operation on iterated bimodules, as reflected in the simplicial bar construction, or, equally, the pattern of an operation on decompositions, as reflected in the trace.

Notation B.2. Given an $r$-fold decomposition $E$, an $s$-fold decomposition $F$, and an $(r,s)$-comparison scheme $\alpha$, we write $G_\alpha$ for the category of pairs $(A,B)$ with $A \in G_r$, $B \in G_s$, $\gamma_E(A) \subseteq \gamma_E(B)$, and $\alpha(\tau(A)) = \tau(B)$. By design, this category fits into a commuting diagram

$$
\begin{array}{ccc}
G_r & \xrightarrow{p} & G_\alpha & \xrightarrow{q} & G_s \\
\tau \downarrow & & \alpha \downarrow & & \tau \\
\Delta^r & \to & \Delta^s.
\end{array}
$$

Definition B.3. We say that $E$ and $F$ are $\alpha$-comparable if $p : G_\alpha \to G_r$ is homotopy initial.

Roughly, $\alpha$-comparability asserts that every subspace in the image of $\gamma_E$ is contained in a contractible collection of subspaces in the image of $\gamma_F$ with compatible patterns of connected components.

We now add local flows into the mix.

Definition B.4. Suppose that $F$ is equipped with isotopy invariant local flows on both $E$ and $F$.

1. We say that a comparison scheme $\alpha$ is flow compatible if, for every $(A,B) \in G_\alpha$, the inclusion $\gamma_E(A) \subseteq \gamma_F(B)$ is flow compatible.

2. Fix a flow compatible comparison scheme $\alpha$. A comparison datum is a map $\text{Bar}_\Delta(I(E)) \to \text{Bar}_\Delta(I(F)) \circ \alpha^{op}$ in the homotopy category of $r$-fold simplicial chain complexes, which fits into the commuting diagram

$$
\begin{array}{ccc}
p^*I(F(\gamma_E)) & \xrightarrow{q*} & q^*I(F(\gamma_F)) \\
p^*\psi \downarrow & & q^*(\text{Bar}_\Delta(I(F)) \circ \tau^{op}) \\
p^*(\text{Bar}_\Delta(I(E)) \circ \tau^{op}) & \xrightarrow{q^*} & p^*(\text{Bar}_\Delta(I(F)) \circ \alpha^{op} \circ \tau^{op}).
\end{array}
$$

The upper horizontal arrow exists by flow compatibility. We call the map $\text{Bar}(I(E)) \to \text{Bar}(I(F))$ induced by a comparison datum the “comparison map.”

We these definitions in hand, the following result is essentially formal.

Proposition B.5. If $E$ and $F$ are $\alpha$-comparable, then the composite

$$
\text{Bar}(I(E)) \to \text{Bar}(I(F)) \simeq C^{SD}(F(X))
$$

of the comparison map followed by the weak equivalence of Theorem 3.19 applied to $F$ coincides with the weak equivalence of Theorem 3.19 applied to $E$. In particular, the comparison map is a quasi-isomorphism.
B.2. Parenthesization. For our first sample application, we begin by noting that the derived tensor product computed by the bar construction on an iterated bimodule enjoys a great deal of symmetry, remaining invariant under any parenthesization of the factors. To give one example, there is a canonical quasi-isomorphism

\[ M_0 \otimes_{R_1} M_1 \otimes_{R_2} M_2 \simeq M_0 \otimes_{R_1} (M_1 \otimes_{R_2} M_2) \]

The corresponding comparison in the context of the decomposition theorem is between, on the one hand, the decomposition \( E \) with components \( X_0, X_1, \) and \( X_2 \) and bridges \( A_1 \times I \) and \( A_2 \times I \); and, on the other hand, the decomposition \( E_{par} \) with components \( X_0 \) and \( X_1 \Pi_{A_2 \times I} X_2 \) and bridge \( A_1 \times I \) (see Figure 11). In order to apply Proposition B.5, we proceed as follows.

1. We choose our \((2,1)\)-comparison scheme \( \alpha : \Delta^2 \to \Delta \) to be the projection onto the first factor.
2. With this choice, \( E \) and \( E_{par} \) are \( \alpha \)-comparable. Indeed, the undercategory in question is filtered.
3. We assume we are given isotopy invariant local flows for which \( \alpha \) is flow compatible. This point will depend on the specifics of the situation.
4. In the manner of Construction 3.16, we build a collection of maps

\[ I(F(X_1)) \otimes I(F(A_2)) \otimes \cdots \otimes I(F(A_2)) \otimes I(F(X_2)) \to I(F(X_1 \Pi_{A_2 \times I} X_2)) \]

using isotopy invariance and flow compatibility of the structure maps of \( F \) and the comparison scheme. These maps respect the simplicial structure maps, and the induced map of simplicial chain complexes is a comparison datum.

Invoking Proposition B.5, we see that the quasi-isomorphism

\[ \text{Bar}(I(E)) \xrightarrow{\sim} \text{Bar}(I(E_{par})) \]

is compatible with those supplied by Theorem 3.19.

Remark B.6. It should be clear that an arbitrary parenthesization may be treated in an identical fashion.

B.3. Folding. Our second example concerns the canonical quasi-isomorphism

\[ M_0 \otimes_{R_1} M_1 \otimes_{R_2} M_2 \simeq (M_0 \otimes M_2) \otimes_{R_1 \otimes R_2^{op}} M_1 \]

The corresponding comparison in the context of the decomposition theorem is between the decomposition \( E \) already introduced and the decomposition \( E_{fld} \) with components \( X_0 \Pi X_2 \) and \( X_1 \) and bridge \( (A_1 \Pi A_2) \times I \) obtained by “folding the end over” (see Figure 11). The data in the this case is the following.

1. We choose our \((1,2)\)-comparison scheme \( \alpha : \Delta \to \Delta^2 \) to be the identity in the first factor and the “flip” automorphism in the second.
2. With this choice, \( E_{fld} \) and \( E \) are \( \alpha \)-comparable. Indeed, the undercategory in question has a final object.
3. We assume we are given isotopy invariant local flows for which \( \alpha \) is flow compatible. This point will depend on the specifics of the situation.
4. Observe that the simplicial chain complexes \( \text{Bar}_\Delta(E_{fld}) \) and \( \text{Bar}_{\Delta_0}(E) \circ \alpha^{op} \) are identical in each simplicial degree, except that the former uses the local flow for \( E_{fld} \), while the latter uses the local flow for \( E \). The comparison datum comes from matching terms and using the flow compatibility of \( \alpha \).
Invoking Proposition B.5, we see that the quasi-isomorphism
\[
\text{Bar}(I(\mathcal{E}_{\text{fld}})) \xrightarrow{\sim} \text{Bar}(I(\mathcal{E}))
\]
is compatible with those supplied by Theorem 3.19.

Appendix C. Degree one homology of graph braid groups

The homology group \( H_1(B_k(\Gamma)) \) for a general connected graph \( \Gamma \) is completely understood; indeed, according to a theorem of Ko–Park [KP12, Thm. 3.16], this group may be identified solely in terms of connectivity and planarity data from \( \Gamma \). Their argument proceeds through an intricate combinatorial and linear algebraic analysis of the discrete Morse data constructed in [FS05], which quickly becomes very technical.

In this appendix, we use the Świątkowski complex to simplify the calculation of \( H_1 \) by giving streamlined proofs of four key lemmas of [KP12], which appear in §C.3 below, cross-referenced with the corresponding results in the original text. These four results directly imply the calculation of \( H_1 \), which we outline below but do not state in full. We make no claim of originality in this appendix, our purpose being only to demonstrate the efficiency of the Świątkowski complex in applications.

C.1. Cuts and connectivity. In order to proceed, we will require some terminology from graph theory. Note that these invariants should only be used as defined on simple graphs and may behave unexpectedly on general graphs.

Definition C.1. Let \( \Gamma \) be a simple graph. A \textit{k-cut} is a set of \( k \) vertices whose removal topologically separates at least two vertices of \( \Gamma \). A simple graph is \textit{k-connected} if it has at least \( k + 1 \) vertices and no \((k - 1)\)-cuts.
Given a 1-cut \( v \) in \( \Gamma \), a \( v \)-component of \( \Gamma \) is the closure in \( \Gamma \) of a connected component of the complement of \( v \) in \( \Gamma \).

The importance of connectivity for our purposes arises from the following classical result, called Menger’s theorem.

**Theorem C.2 ([Men27]).** Let \( k > 0 \). A simple graph is \( k \)-connected if and only if, for distinct vertices \( x \) and \( y \) in \( \Gamma \), there exist \( k \) paths from \( x \) to \( y \) in \( \Gamma \), disjoint except at endpoints.

In other words, a simple graph is \( k \)-connected (for positive \( k \)) if and only if any pair of vertices \( x \) and \( y \) there is a graph embedding of a subdivision of the theta graph \( \Theta_k \) into \( \Gamma \) taking the two vertices of the theta graph to \( x \) and \( y \).

As the following result shows, high connectivity places strong constraints on the behavior of the first homology.

**Proposition C.3.** If \( \Gamma \) is simple and 3-connected, then any two star classes in \( H_*(B(\Gamma)) \) coincide up to sign.

**Proof.** By 3-connectivity, any two vertices of \( \Gamma \) are joined by three distinct paths, so a star class at one vertex coincides with some star class at any other vertex by the \( \Theta \)-relation. Thus, it suffices to verify that two star classes \( \alpha \) and \( \alpha' \) at a fixed vertex \( v \) coincide up to sign. We may assume further that \( \alpha \) and \( \alpha' \) are induced by inclusions of \( S_3 \) differing only at a single half-edge, so that we have an inclusion of \( S_4 \) into \( \Gamma \) (with endpoint vertices \( v_1, \ldots, v_4 \) and central vertex \( v \) all distinct by 3-connectedness). We will show that the four star classes in \( \Gamma \) obtained in this way agree up to sign.

A corollary of Menger’s theorem is the fact that in a \( k \)-connected graph, any two sets of vertices, each of size \( k \), can be joined by \( k \) disjoint paths [BM08, Prop. 9.4]. Since \( \Gamma_v \) is 2-connected, two disjoint paths join \( \{v_1, v_4\} \) and \( \{v_2, v_3\} \). By relabelling suppose one joins \( v_1 \) to \( v_2 \) and the other joins \( v_3 \) to \( v_4 \). Similarly, two disjoint paths join \( \{v_1, v_2\} \) to \( \{v_3, v_4\} \). By switching the labels on \( v_1 \) and \( v_2 \) if need be, we may assume one joins \( v_1 \) to \( v_3 \) and the other joins \( v_2 \) to \( v_4 \). We now have two extensions of the inclusion \( S_4 \to \Gamma \) to an inclusion of a subdivision of the figure-eight graph \( 8 \) as in Figure 12, and the \( Q \)-relation and \( I \)-relation imply that \( \alpha_{123} = \alpha_{124} \), that \( \alpha_{142} = \alpha_{143} \), and that \( \alpha_{123} = \alpha_{423} \). □

![Figure 12. The figure-eight graph 8 and two inclusions of subdivisions of it at a vertex of valence at least 4 in a triconnected graph](image)

C.2. **Minors and planarity.** Recall that a graph is said to be planar if it can be embedded in \( \mathbb{R}^2 \). The goal of this section is to establish the following connection between planarity and configuration spaces.
Lemma C.4. Let $\Gamma$ be a 3-connected simple graph. If $\Gamma$ is non-planar, then any star class of $H_1(B_2(\Gamma))$ is 2-torsion.

As before, the proof will proceed by reduction to atomic cases. In order to see how this reduction will proceed, we require an auxiliary notion.

**Definition C.5.** Let $\Gamma$ and $\Gamma'$ be graphs. We say that $\Gamma$ is a **minor** of $\Gamma'$ if $\Gamma$ may be obtained up to isomorphism from $\Gamma$ by repeated application of the following operations:

1. Contract an edge;
2. Remove an edge;
3. Remove an isolated vertex.

The relevance of minors for our purposes is the following classical result (see Figures 13 and 14 for terminology).

**Theorem C.6 ([Wag37]).** A graph is non-planar if and only if it admits $K_5$ or $K_{3,3}$ as a minor.

In order to apply this criterion, we must first clarify the relationship between the Świątkowski complex of a graph and that of its minors. We begin by noting that, if $\Gamma$ is a minor of $\Gamma'$, then $E$ is naturally identified with a subset of $E'$. 

**Lemma C.7.** Let $\Gamma$ be a minor of $\Gamma'$. There is a map $f : S(\Gamma) \rightarrow S(\Gamma')$ of differential graded $\mathbb{Z}[E]$-modules such that $f_*(\alpha)$ is a star class whenever $\alpha$ is.

**Proof.** We may assume that $\Gamma$ is obtained from $\Gamma'$ by a single application of one of the operations of Definition C.5. In the case of operation (2) or (3), $\Gamma$ is a subgraph of $\Gamma'$, and the claim is immediate from the standard functoriality of the Świątkowski complex. Since the same holds for the contraction of a tail or a self-loop, we may assume that $\Gamma$ is obtained from $\Gamma'$ by contracting an edge with two distinct vertices $v_1$ and $v_2$. Denote the corresponding half-edges by $h_1$ and $h_2$, and let $v_0$ be the vertex that is the image of the closure of $e$ under the quotient map $\Gamma' \rightarrow \Gamma$.

In order to define $f$, we note that, under the quotient map, each vertex $v \neq v_0$ of $\Gamma$ is canonically identified with a vertex $\check{v}$ of $\Gamma'$, and each half-edge $h$ of $\Gamma$ is canonically identified with a half-edge $\check{h}$ of $\Gamma'$. With this in mind, we set

$$f(v) = \begin{cases} \check{v} & v \neq v_0 \\ e & v = v_0 \end{cases}$$

and

$$f(h) = \begin{cases} \check{h} & v(h) \neq v_0 \\ \check{h} - h_j & v(h) = v_j. \end{cases}$$

By inspection, $f$ respects the differential and $\mathbb{Z}[E]$-action, and the claim regarding star classes is a direct calculation with star cycles. \qed

**Proof of Lemma C.4.** Since $\Gamma$ is non-planar, it admits $K_5$ or $K_{3,3}$ as a minor, and we may push a star class of the minor forward using Lemma C.7 to obtain a star class in $\Gamma$. Therefore, it suffices to prove the claim for $K_5$ and $K_{3,3}$.

In the case of $K_5$, we can find an embedded copy of the theta graph $\Theta_3$ as in Figure 13. This shows that the star class $\alpha_{123}$ at $v$ is the negative of the star class $\alpha_{12'3'} = \alpha_{12'2'}$ at $v'$. The picture depicted and concomitant $\Theta$-relation can be rotated by $\frac{2\pi}{5}$; applying such relations 5 times shows that the star class $\alpha_{123}$ is equal to $(\alpha_{123})^5 = (-1)^5\alpha_{123}$.

In the case of $K_{3,3}$, we have two embedded subdivision of $\Theta_3$, as in Figure 14. Applying the $\Theta$-relation twice, we see that both a star class at the upper left vertex and its additive inverse are equal to the same star class at the upper right vertex. \qed
C.3. Four lemmas of Ko–Park. We are now equipped to prove the results of [KP12] alluded to above, which together easily yield a complete calculation of $H_1(B_k(\Gamma))$ for an arbitrary connected graph $\Gamma$. We content ourselves with an outline of this computation, directing the interested reader to [KP12] for details.

Since any graph has a simple subdivision, we may assume that $\Gamma$ is simple. In this case, there are classical decomposition theorems from graph theory associating to a 1-connected simple graph a set of 2-connected simple graphs, obtained by repeated 1-cuts, and to each of these a set of 3-connected simple graphs, obtained by repeated 2-cuts. With this tool in hand, together with the observation that it suffices to assume that $\Gamma$ is 1-connected, the argument proceeds as follows.

1. In characterizing the effect of a 1-cut, Lemma C.10 reduces the computation for $\Gamma$ to the computation for each associated 2-connected graph.
2. Assuming now that $\Gamma$ is 2-connected, Lemma C.8 grants that $H_1(B_k(\Gamma))$ is independent of $k$ for $k \geq 2$.
3. Lemma C.13 describes the effect of a 2-cut on $H_1(B_2(\Gamma))$ for 2-connected $\Gamma$, which reduces to the computation for each associated 3-connected graph.
4. Finally, Lemma C.9 computes $H_1(B_2(\Gamma))$ for 3-connected $\Gamma$.

With careful bookkeeping of cuts and components, one arrives at an explicit answer, which we do not repeat here.

The first result is a sharpened version of the degree 1 component of Proposition 5.14 in the presence of sufficient connectivity.

Lemma C.8 ([KP12, Lem. 3.12]). Suppose that $\Gamma$ is simple and 2-connected. For any $e \in E$ and $k \geq 2$, multiplication by $e^{k-2}$ induces an isomorphism

\[ H_1(B_2(\Gamma)) \cong H_1(B_k(\Gamma)). \]

Proof. By Propositions 5.14 and 5.6, it will suffice to show that $p(E)\alpha$ and $p(E)\gamma$ are divisible by $e$ in the expected range, where $p(E)$ is a monomial in $E$, $\alpha$ is a star class at the vertex $v$, and $\gamma$ is a loop class.

In the case of a star class $\alpha$, there is nothing to show provided $p(E) \in \mathbb{Z}[e]$, so we may write $p(E)\alpha = [q(E)e']a$, where $e' \neq e$ and $[a] = \alpha$. Since $\Gamma$ is 2-connected,
both \(e'\) and \(e\) have vertices distinct from \(v\), and there is an injective path in \(\Gamma \setminus \{v\}\) between them. Letting \(h\) denote the alternating sum of the half-edges involved in this path, we have \(\partial(q(E)\alpha) = q(E)\alpha - q(E)e'a\). An easy induction on the degree of \(q\) completes the argument in this case.

In the case of a loop class \(\gamma\), we write \(p(E)\gamma = [q(E)e'c]\), where \(e' \neq e\) and \([c] = \gamma\). Since \(\Gamma\) is connected, there is a path connecting \(e'\) to \(e\). If this path contains none of the edges or vertices involved in \(c\), then the same argument as before shows that \(q(E)ec \sim q(E)e'c\); on the other hand, if the path is contained entirely in \(c\), then the same conclusion follows by the \(O\)-relation. Thus, we may assume that \(e\) lies in the complement of \(c\), \(e'\) lies in \(c\), and that the two share a vertex. By the \(Q\)-relation, we have \(q(E)ec \sim \pm q(E)e'c \pm q(E)a\), where \([a]\) is a star class. Since the case of a star class is known, this completes the proof.

The second result is the base calculation, the 3-connected case.

**Lemma C.9** ([KP12, Lem. 3.15]). If \(\Gamma\) is 3-connected, then \(H_1(B_2(\Gamma)) \cong \mathbb{Z}^{\beta_1(\Gamma)} \oplus K\), where

\[
K = \begin{cases} 
\mathbb{Z} & \text{if } \Gamma \text{ is planar} \\
\mathbb{Z}/2 & \text{else.}
\end{cases}
\]

Here \(K\) is generated by a star class and \(\mathbb{Z}^{\beta_1(\Gamma)}\) by the inclusion of cycle graphs into \(\Gamma\).

**Proof.** Choose a set of loops \(c_i\) in \(\Gamma\) representing a basis for \(H_1(\Gamma)\) and a star class \(\alpha\). Define a map

\[
\left(\bigoplus_{i=1}^{\beta_1(\Gamma)} H_1(B_2(c_i))\right) \oplus K \cong \mathbb{Z}^{\beta_1(\Gamma)} \oplus K \xrightarrow{\psi} H_1(B_2(\Gamma))
\]

using the inclusions of \(c_i\) in \(\Gamma\) and taking the generator of \(K\) to \(\alpha\). This map is well-defined by Lemma C.4 and surjective by Propositions 5.6 and C.3.

Next, we “thicken” \(\Gamma\), replacing vertices with disks and edges with strips, obtaining a surface \(\Sigma\) of genus \(g\) with \(b\) boundary components equipped with an embedding \(\iota: \Gamma \to \Sigma\) that is also a homotopy equivalence (see, [Kon92, p. 4-5], for example, and the references therein for more on this classical tactic). Note that \(g > 0\) if \(\Gamma\) is non-planar, while we may take \(g = 0\) if \(\Gamma\) is planar. Moreover, by comparing Euler characteristics, we find that

\[
\beta_1(\Gamma) = 2g + b - 1.
\]

Now, from the explicit presentation for \(\pi_1(B_2(\Sigma))\) given in [Bel04], it is easy to see that \(\iota\) induces a surjection \(\pi_1(B_2(\Gamma)) \to \pi_1(B_2(\Sigma))\) on fundamental groups, and thus also on first homology. By direct computation, the same presentation gives

\[
H_1(B_2(\Sigma)) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}^{b-1} & g = 0 \\
\mathbb{Z}/2 \oplus \mathbb{Z}^{2g+b-1} & g \geq 1
\end{cases}
\cong \mathbb{Z}^{\beta_1(\Gamma)} \oplus K,
\]

Thus, \(H_1(B_2(\Gamma))\) both admits a surjection from and a surjection onto \(\mathbb{Z}^{\beta_1(\Gamma)} \oplus K\), which implies the claim by finite dimensionality.

The third result describes the effect on \(H_1\) of a 1-cut.
Lemma C.10 ([KP12, Lem. 3.11]). Let $\Gamma$ be a graph, $v$ a 1-cut of valence $\nu$ in $\Gamma$, and $\{\Gamma^{(i)}\}_{i=1}^{\nu}$ the set of $v$-components of $\Gamma_v$. There is an isomorphism

$$H_1(B_k(\Gamma)) \cong \left( \bigoplus_{i=1}^{\mu} H_1(B_k(\Gamma^{(i)})) \right) \oplus \mathbb{Z}^{N(k,\Gamma,v)}$$

where

$$N(k,\Gamma,v) = (\nu - 2) \left( \frac{k + \mu - 2}{k - 1} \right) - \left( \frac{k + \mu - 2}{k} \right) - (\nu - \mu - 1).$$

Proof. Applying Proposition 5.13 at $v$, we obtain the exact sequence

$$\cdots \to \bigoplus_{|H(v)|=1}^{H(v)} H_1(B_{k-1}(\Gamma_v)) \xrightarrow{\delta_1} H_1(B_k(\Gamma_v)) \to H_1(B_k(\Gamma)) \to \bigoplus_{|H(v)|=1}^{H(v)} H_0(B_{k-1}(\Gamma_v)) \xrightarrow{\delta_0} H_0(B_k(\Gamma_v)) \to \cdots$$

Since zeroth homology is free, $\ker \delta_0$ is as well, so $H_1(B_k(\Gamma)) \cong \coker \delta_1 \oplus \ker \delta_0$.

Assume $\Gamma$ is connected for simplicity; then $\pi_0(B_k(\Gamma_v))$ is in bijection with the set of partitions of $k$ into $\mu$ distinguished blocks and we conclude by exactness that

$$\text{rk} \ker \delta_0 = (\nu - 1) \left( \frac{k + \mu - 2}{k - 1} \right) - \left( \frac{k + \mu - 1}{k} \right) + 1.$$ 

A similar argument shows $H_1(B_k(\Gamma^{(i)})) \cong \coker \delta_1^{(i)} \oplus \ker \delta_1^{(i)}$ and $\bigoplus_i \ker \delta_1^{(i)} \cong \mathbb{Z}^{(\nu - \mu)}$. Then, after a little combinatorial rearrangement, all that remains is to show that $\coker \delta_1 \cong \bigoplus_{i=1}^{\mu} \coker \delta_1^{(i)}$. The maps $H_1(B_k(\Gamma^{(i)})) \to H_1(B_k(\Gamma_v))$ arising from the various inclusions induce a map $\bigoplus_{i=1}^{\mu} \coker \delta_1^{(i)} \to \coker \delta_1$. We will show that this map is an isomorphism.

By the Künneth formula (again using the fact that zeroth homology is free) the homology $H_1(B_k(\Gamma_v))$ splits as a direct sum over $i$. The $i$th summand consists of homology classes in $H_1(B_k(\Gamma^{(i)}))$ equipped with an ordered partition of $k - k'$ into $\mu - 1$ blocks for some $k' \leq k$. Passing to the quotient $\coker \delta_1$, every equivalence class in the $i$th summand has a representative homology class with $k' = k$. Since this representative is in the image of the map in question, surjectivity follows.

For injectivity, we note that the map $H_1(B_k(\Gamma^{(i)})) \to H_1(B_k(\Gamma_v))$ lands in the $i$th summand of the direct sum decomposition, and $\delta_1$ respects this decomposition. But a naïve choice of indexing set for the direct sum in the domain of $\delta_1$ reveals that $\coker \delta_1^{(i)} \to \coker \delta$ is injective for fixed $i$. \hfill \Box

The final result describes the effect on $H_1$ of a 2-cut. We will establish a result for a certain type of decomposition.

Notation C.11. Let $\{x, y\}$ be a 2-cut in a 2-connected simple graph $\Gamma$. An $\{x, y\}$-decomposition consists of a (redundant) collection of subgraphs of $\Gamma$, namely $(G_1, G_{-1}, P_1, P_{-1}, \Gamma_1, \Gamma_{-1}, \mathcal{C})$, where

1. the subgraph $G_1$ contains $x$ and $y$ and $G_1 \setminus \{x, y\}$ is a connected component of $\Gamma \setminus \{x, y\}$,
2. the subgraph $G_{-1}$ is $\Gamma \setminus G_1 \cup \{x, y\}$,
3. the subgraph $P_i$ is a simple path in $G_i$ between $x$ and $y$,
4. the subgraph $\Gamma_i$ is the union of $G_i$ and $P_{-i}$,
Figure 15. An example of an \{x, y\}-decomposition. Every diamond is a push-out in \( \mathcal{T}_{\text{op}} \).

(5) the subgraph \( C \) is the union of \( P_1 \) and \( P_{-1} \).

See Figure 15 for an example. This notation is not symmetric in its indices because of the connectivity requirement on \( G_1 \setminus \{x, y\} \).

Lemma C.12. Let \( \Gamma \) be a 2-connected simple graph with a 2-cut \( \{x, y\} \) and an \{x, y\}-decomposition. The natural map of Świątkowski complexes \( S(P_i) \to S(G_i) \) admits a retraction \( S(G_i) \to S(P_i) \) with the property that generators of \( S(G_i) \) which avoid \( x \) and its half-edges (respectively \( y \) and its half-edges) are taken to sums of generators of \( S(P_i) \) which avoid \( x \) and its half-edge (respectively \( y \) and its half-edge).

Proof. Suppose we are given a retraction \( f : S_{\leq m}(G_i) + S(P_i) \to S(P_i) \) with the desired property. The problem of extending this map to a generator \( b \) of degree \( m + 1 \) is that of choosing a nullhomotopy in \( S(P_i) \) for \( f(\partial(b)) \) having the desired property. Since \( S(P_i) \) and its reductions at \( x \), at \( y \), and at \( \{x, y\} \) are all contractible in each weight, this extension is unobstructed as long as \( m > 0 \). Thus, by induction on \( m \), it will suffice to produce a retraction \( f : S_{\leq 1}(G_i) + S(P_i) \to S(P_i) \) with the desired property. For this, we extend the identity on \( S_{\leq 1}(P_i) \) by

1. sending each edge or vertex not lying in \( P_i \) to a fixed arbitrarily chosen edge \( e \) of \( P_i \),
2. sending each half-edge whose vertex is not in \( P_i \) to zero, and
3. sending each half-edge whose vertex \( v \) is in \( P_i \) to a chain in \( S(P_i) \) (avoiding \( x \) and \( y \) except potentially at \( v \)) whose boundary is the difference between \( v \) and \( e \). This is possible because \( P_i \) is topologically an interval with endpoints \( x \) and \( y \). \( \square \)
Lemma C.13 ([KP12, Lem. 3.13]). Let $\Gamma$ be a 2-connected simple graph with a 2-cut $\{x, y\}$ and an $\{x, y\}$-decomposition. Then the sequence

$$0 \rightarrow H_1(B_2(\mathbb{C})) \rightarrow H_1(B_2(\Gamma_1)) \oplus H_1(B_2(\Gamma_{-1})) \rightarrow H_1(B_2(\Gamma)) \rightarrow 0$$

is split exact, where the maps are induced by the respective inclusions, with the map $H_1(B_2(\Gamma_{-1})) \rightarrow H_1(B_2(\Gamma))$ twisted by an overall sign.

Proof. The composite of the middle two maps is the sum of the map induced by the inclusion of $\mathbb{C}$ into $\Gamma$ with its negative, hence zero. For injectivity of the second map, we appeal to the argument of Lemma C.9, which shows that the loop class corresponding to $\mathbb{C}$ is non-zero. For surjectivity of the third map, by 2-connectedness of $\Gamma$, any graph morphism of $S_3$ to $\Gamma$ can be extended to an embedding of some lollipop graph. By Proposition 5.6 and the $Q$-relation it suffices to show that classes of the form $e\gamma$ lie in the image, where $e$ is an edge and $\gamma$ a loop class.

If $e$ does not lie in the representing loop of $\gamma$, then by 2-connectedness of $\Gamma$ we may write $e\gamma = e'\gamma + e''\gamma$ where $e'$ lies within the representing loops of both $\gamma'$ and $\gamma''$. By the $O$-relation, any two choices for $e$ within $\gamma'$ represent the same class. This reduces us to classes $e\gamma$ where $\gamma$ passes through both $G_1$ and $G_{-1}$ and $e$ lies in $G_1$. Then $e\gamma$ can be rewritten as $e\gamma_1 + e\gamma_{-1}$, where $\gamma_i$ is a loop class in $\Gamma_i$ for $i \in \{-1, 1\}$. The first of these lies in the image of $H_1(B_2(\Gamma_1))$, and the second lies in the image of $H_1(B_2(\Gamma_{-1}))$ by the connectivity property of $G_1$.

It remains to show exactness and splitting, for which we will use the retracts of Lemma C.12. A generator of $S(\Gamma)$ is a product of edges, half-edges, and vertices of $G_1$ and $G_{-1}$ (which intersect only at the vertices $x$ and $y$), and so, by applying these retracts on each of these two pieces, we obtain retracts $\pi_i : S(\Gamma) \rightarrow S(\Gamma_i)$ and $\rho_i : S(\Gamma_i) \rightarrow S(\mathbb{C})$ of the maps induced by the respective inclusions. These retracts are well-defined because of the “boundary conditions” on generators involving $x$ and $y$ in the retracts of Lemma C.12. Moreover, we have the commuting diagram

$$
\begin{array}{ccc}
S(\Gamma_i) & \xrightarrow{\rho_i} & S(\mathbb{C}) \\
\downarrow & & \downarrow \\
S(\Gamma) & \xrightarrow{\pi_{-1}} & S(\Gamma_{-1}).
\end{array}
$$

The map $\rho_1$ splits the sequence. For exactness, consider the composite

$$H_1(B_2(\Gamma_1)) \oplus H_1(B_2(\Gamma_{-1})) \rightarrow H_1(B_2(\Gamma)) \xrightarrow{(\pi_1, -\pi_{-1})} H_1(B_2(\Gamma_1)) \oplus H_1(B_2(\Gamma_{-1})).$$

Any map in the kernel of the first map is a fortiori in the kernel of the composition. Now writing $\iota_*$ for any map on homology induced by an inclusion, and using the commuting diagram above, this composition is given by

$$(\beta_1, \beta_{-1}) \mapsto (\beta_1 - \iota_*(\rho_{-1})_*\beta_{-1}, \beta_{-1} - \iota_*(\rho_1)_*(\beta_1)).$$

This expression vanishes if and only if $\beta_1 = \iota_*(\rho_1)_*(\beta_1)$, in which case $(\beta_1, \beta_{-1})$ is the image of $(\rho_1)_*(\beta_1)$, showing exactness. □

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