On a Conjecture of Thomassen

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Abstract

In 1989, Thomassen asked whether there is an integer-valued function \( f(k) \) such that every \( f(k) \)-connected graph admits a spanning, bipartite \( k \)-connected subgraph. In this paper we take a first, humble approach, showing the conjecture is true up to a \( \log n \) factor.

1 Introduction

Erdős noticed [4] that any graph \( G \) with minimum degree \( \delta(G) \) at least \( 2k - 1 \) contains a spanning, bipartite subgraph \( H \) with \( \delta(H) \geq k \). The proof for this fact is obtained by taking a maximal edge-cut, a partition of \( V(G) \) into two sets \( A \) and \( B \), such that the number of edges with one endpoint in \( A \) and one in \( B \), denoted \( |E(A,B)| \), is maximal. Observe that if some vertex \( v \) in \( A \) does not have degree at least \( k \) in \( G[B] \), then by moving \( v \) to \( B \), one would increase \( |E(A,B)| \), contrary to maximality. The same argument holds for vertices in \( B \). In fact this proves that for each vertex \( v \in V(G) \), by taking such a subgraph \( H \), the degree of \( v \) in \( H \), denoted \( d_H(v) \), is at least \( d_G(v)/2 \). This will be used throughout the paper.

Thomassen observed that the same proof shows the following stronger statement. Given a graph \( G \) which is at least \( (2k - 1) \) edge-connected (that is one must remove at least \( 2k - 1 \) edges in order to disconnect the graph), then \( G \) contains a bipartite subgraph \( H \) for which \( H \) is \( k \) edge-connected. In fact, each edge-cut keeps at least half of its edges. This observation led Thomassen to conjecture that a similar phenomena also holds for vertex-connectivity.

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Before proceeding to the statement of Thomassen’s conjecture, we remind the reader that a graph $G$ is said to be $k$ vertex-connected or $k$-connected if one must remove at least $k$ vertices from $V(G)$ in order to disconnect the graph (or to remain with one single vertex). We also let $\kappa(G)$ denote the minimum integer $k$ for which $G$ is $k$-connected. Roughly speaking, Thomassen conjectured that any graph with high enough connectivity also should contain a $k$-connected spanning, bipartite subgraph. The following appears as Conjecture 7 in [3].

**Conjecture 1.** For all $k$, there exists a function $f(k)$ such that for all graphs $G$, if $\kappa(G) \geq f(k)$, then there exists a spanning, bipartite $H \subseteq G$ such that $\kappa(H) \geq k$.

In this paper we prove that Conjecture 1 holds up to a log $n$ factor by showing the following:

**Theorem 1.** For all $k$ and $n$, and for every graph $G$ on $n$ vertices the following holds. If $\kappa(G) \geq 10^{10}k^3 \log n$, then there exists a spanning, bipartite subgraph $H \subseteq G$ such that $\kappa(H) \geq k$.

Because of the log $n$ factor, we did not try to optimize the dependency on $k$ in Theorem 1. However, it looks like our proof could be modified to give slightly better bounds.

## 2 Preliminary Tools

In this section, we introduce a number of preliminary results.

### 2.1 Mader’s Theorem

The first tool is the following useful theorem due to Mader [2].

**Theorem 2.** Every graph of average degree at least $4\ell$ has an $\ell$-connected subgraph.

Because we are interested in finding bipartite subgraphs with high connectivity, the following corollary will be helpful.

**Corollary 1.** Every graph $G$ with average degree at least $8\ell$ contains a (not necessarily spanning) bipartite subgraph $H$ which is at least $\ell$-connected.

**Proof.** Let $G$ be such a graph and let $V(G) = A \cup B$ be a partition of $V(G)$ such that $|E(A, B)|$ is maximal. Observe that $|E(A, B)| \geq |E(G)|/2$, and therefore, the bipartite graph $G'$ with parts $A$ and $B$ has average degree at least $4\ell$. Now, by applying Theorem 2 to $G'$ we obtain the desired subgraph $H$.  

\[ \square \]
2.2 Merging k-connected Graphs

We will also make use of the following easy expansion lemma.

**Lemma 1.** Let $H_1$ and $H_2$ be two vertex-disjoint graphs, each of which is k-connected. Let $H$ be a graph obtained by adding $k$ independent edges between these two graphs. Then, $\kappa(H) \geq k$.

**Proof.** Note first that by construction, one cannot remove all the edges between $H_1$ and $H_2$ by deleting fewer than $k$ vertices. Moreover, because $H_1$ and $H_2$ are both $k$-connected, each will remain connected after deleting less than $k$ vertices. From here, the proof follows easily. \qed

Next we will show how to merge a collection of a few $k$-connected components and single vertices into one $k$-connected component. Before stating the next lemma formally, we will need to introduce some notation. Let $G_1, \ldots, G_t$ be $t$ vertex-disjoint $k$-connected graphs, let $U = \{u_{t+1}, \ldots, u_{t+s}\}$ be a set consisting of $s$ vertices which are disjoint to $V(G_i)$ for $1 \leq i \leq t$, and let $R$ be a $k$-connected graph on the vertex set $\{1, \ldots, t + s\}$. Also let $X = (G_1, \ldots, G_t, u_{t+1}, \ldots, u_{t+s})$ be a $(t + s)$-tuple and $X_i$ denote the $i$th element of $X$. Finally, let $\mathcal{F}_R := \mathcal{F}_R(X)$ denote the family consisting of all graphs $G$ which satisfy the following:

(i) the disjoint union of the elements of $X$ is a spanning subgraph of $G$, and

(ii) for every distinct $i, j \in V(R)$ if $ij \in E(R)$, then there exists an edge in $G$ between $X_i$ and $X_j$, and

(iii) for every $1 \leq i \leq t$, there is a set of $k$ independent edges between $V(G_i)$ and $k$ distinct vertex sets $\{V(X_{ji}), \ldots, V(X_{ji})\}$, where $V(u_i) = \{u_i\}$.

**Lemma 2.** Let $G_1, \ldots, G_t$ be $t$ vertex-disjoint graphs, each of which is $k$-connected, and let $U = \{u_{t+1}, \ldots, u_{t+s}\}$ be a set of $s$ vertices for which $U \cap V(G_i) = \emptyset$ for every $1 \leq i \leq t$. Let $R$ be a $k$-connected graph on the vertex-set $\{1, \ldots, t + s\}$, and let $X = \{G_1, \ldots, G_t, u_{t+1}, \ldots, u_{t+s}\}$. Then, any graph $G \in \mathcal{F}_R(X)$ is $k$-connected.

**Proof.** Let $G \in \mathcal{F}_R(X)$, and let $S \subseteq V(G)$ be a subset of size at most $k - 1$. We wish to show that the graph $G' := G \setminus S$ is still connected. Let $x, y \in V(G')$ be two distinct vertices in $G'$; we shall show that there exists a path in $G'$ connecting $x$ to $y$. Towards this end, we first note that if both $x$ and $y$ are in the same $G_i$, then because each $G_i$ is $k$-connected, there is nothing to prove. Moreover, if both $x$ and $y$ are in distinct elements of $X$ which are also disjoint from $S$, then we are also finished, as follows. Because $R$ is $k$-connected, if we delete all of the vertices in $R$ corresponding to elements of $X$ which intersect $S$, the resulting graph is still connected. Therefore, one can easily find a path between the elements containing $x$ and $y$ which goes only through “untouched” elements of $X$, and hence, there exists a path connecting $x$ and $y$.

The remaining case to deal with is when $x$ and $y$ are in different elements of $X$, and at least one of them is not disjoint with $S$. Assume $x$ is in some such $X_i$ (y will be
treated similarly). Using Property (iii) of $\mathcal{F}_R$, there is at least one edge between $X_i$ and an untouched $X_j$. Therefore one can find a path between $x$ and some vertex $x'$ in an untouched $X_j$. This takes us back to the previous case.

2.3 Main Technical Lemma

A directed graph or digraph is a set of vertices and a collection of directed edges; note that bidirectional edges are allowed. For a directed graph $D$ and a vertex $v \in V(D)$ we let $d^+_D(v)$ denote the out-degree of $v$. We let $U(D)$ denote the underlying graph of $D$, that is the graph obtained by ignoring the directions in $D$ and merging multiple edges. In order to find the desired spanning, bipartite $k$-connected subgraph in Theorem 1, we look at sub-digraphs in an auxiliary digraph.

The following is our main technical lemma and is the main reason why we have a log $n$ factor.

**Lemma 3.** If $D$ is a finite digraph on at most $n$ vertices with minimum out-degree

$$\delta^+(D) > (k - 1) \lceil \log n \rceil,$$

then there exists a sub-digraph $D' \subseteq D$ such that

1. For all $v \in V(D')$ we have $d^+_{D'}(v) \geq d^+_D(v) - (k - 1) \lceil \log n \rceil$, and
2. $\kappa(U(D')) \geq k$.

**Proof.** If $\kappa(U(D)) \geq k$, then there clearly is nothing to prove. So we may assume that $\kappa(U(D)) \leq k - 1$. Delete a separating set of size at most $k - 1$. The smallest component, say $C_1$, has size at most $n/2$ and for any $v \in V(C_1)$, every out-neighbor of $v$ is either in $V(C_1)$ or in the separating set that we removed, and so

$$d^+_{C_1}(v) \geq d^+_D(v) - (k - 1).$$

We continue by repeatedly applying this step, and note that this process must terminate. Otherwise, after at most log $n$ steps we are left with a component which consists of one single vertex and yet contains at least one edge, a contradiction.

3 Highly Connected Graphs

With the preliminaries out of the way, we are now ready to prove Theorem 1.

**Proof.** Let $G$ be a finite graph on $n$ vertices with

$$\kappa(G) \geq 10^{10} k^3 \log n.$$ 

In order to find the desired subgraph, we first initiate $G_1 := G$ and start the following process.
As long as \( G_i \) contains a bipartite subgraph which is at least \( k \)-connected on at least \( 10^3 k^2 \log n \) vertices, let \( H_i = (S_i \cup T_i, E_i) \) be such a subgraph of maximum size, and let \( G_{i+1} := G_i \setminus V(H_i) \). Note that \( H_1 \) must exist as

\[
\delta(G_1) \geq 10^{10} k^3 \log n - 2k \geq 8000 k^2 \log n,
\]

and so by Corollary 1, \( G_1 \) must contain a \( k \)-connected bipartite subgraph of size at least \( 10^3 k^2 \log n \).

Let \( H_1, \ldots, H_t \) be the sequence obtained in this manner, and note that all the \( H_i \)'s are vertex disjoint with \( \kappa(H_i) \geq k \) and \( |V(H_i)| \geq 10^3 k^2 \log n \). Observe that if \( H_1 \) is spanning, then there is nothing to prove. Therefore, suppose for a contradiction that \( H_1 \) is not spanning. Let \( V_0 := V(G_{t+1}) = \{v_1, \ldots, v_s\} \) be the subset of \( V(G) \) remaining after this process; note that it might be the case that \( V_0 = \emptyset \). Because each \( H_i \) is a bipartite, \( k \)-connected subgraph of \( G_i \) of maximum size and \( G \) is \( 10^{10} k^3 \log n \) connected, we show that the following are true:

(a) For every \( 1 \leq i < j \leq t \), there are less than \( 4k \) independent edges between \( H_i \) and \( H_j \), and

(b) for every \( j > i \) and \( v \in V(G_j) \), the number of edges in \( G \) between \( v \) and \( H_i \), denoted by \( d_G(v, V(H_i)) \), is less than \( 2k \), and

(c) for every \( 1 \leq i \leq t \), there exists a set \( M_i \) consisting of exactly \( 10^3 k^2 \log n \) independent edges, each of which has exactly one endpoint in \( H_i \).

Indeed, for showing (a), note that if there are at least \( 4k \) independent edges between \( H_i \) to \( H_j \), by pigeonhole principle, at least \( k \) of them are between the same part of \( H_i \) (say \( S_i \)) and the same part of \( H_j \) (say \( S_j \)). Therefore, the graph obtained by joining \( H_i \) to \( H_j \) with this set of at least \( k \) edges is a \( k \)-connected (by Lemma 1), bipartite graph and is larger than \( H_i \), contrary to the maximality of \( H_i \).

For showing (b), note that if there are at least \( 2k \) between \( v \) and \( H_i \), then there are at least \( k \) edges incident with \( v \) touch the same part of \( H_i \), and let \( F \) be a set of \( k \) such edges. Second, we mention that joining a vertex of degree at least \( k \) to a \( k \)-connected graph trivially yields a \( k \)-connected graph. Next, since all the edges in \( F \) are touching the same part, the graph obtained by adding \( v \) to \( V(H_i) \) and \( F \) to \( E(H_i) \), will also be bipartite. This contradicts the maximality of \( H_i \).

For (c), note first that since \( H_1 \) is not spanning, using (b) we conclude that in the construction of the bipartite subgraphs \( H_1, \ldots, H_t \) in the process above,

\[
\delta(G_2) \geq 10^{10} k^3 \log n - 2k \geq 8000 k^2 \log n.
\]

Therefore, using Corollary 1, it follows that \( G_2 \) contains a bipartite subgraph of size at least \( 10^3 k^2 \log n \) which is also \( k \)-connected.

Therefore, the process does not terminate at this point, and \( H_k \) exists (that is, \( t \geq 2 \)). It also follows that for each \( 1 \leq i \leq t \) we have \( |V(G) \setminus V(H_i)| \geq 10^3 k^2 \log n \). Next, note that \( G \) is \( 10^{10} k^3 \log n \) connected, and that each \( H_i \) is of size at least \( 10^3 k^2 \log n \). For each
\[ i, \text{ consider the bipartite graph with parts } V(H_i) \text{ and } V(G) \setminus V(H_i) \text{ and with the edge-set consisting of all the edges of } G \text{ which touch both of these parts. Using König’s Theorem (see [5], p. 112), it follows that if there is no such } M_i \text{ of size } 10^4k^2\log n, \text{ then there exists a set of strictly fewer than } 10^3k^2\log n \text{ vertices that touch all the edges in this bipartite graph (a vertex cover). By deleting these vertices, one can separate what is left from } H_i \text{ and its complement, contrary to the fact that } G \text{ is } 10^{10}k^3\log n \text{ connected.}

In order to complete the proof, we wish to reach a contradiction by showing that one can either merge few members of \( \{H_1, \ldots, H_t\} \) with vertices of \( V_0 \) into a \( k \)-connected component or find a \( k \)-connected component of size at least \( 10^4k^2\log n \) which is contained in \( V_0 \). In order to do so, we define an auxiliary digraph, using a special subgraph \( G' \subseteq G \), and use Lemmas 3 and 2 to achieve the desired contradiction. We first describe how to find \( G' \).

First, we partition \( V_0 \) into two sets, say \( A \) and \( B \), where

\[
A = \left\{ v \in V_0 : d_G \left( v, \bigcup_{i=1}^t V(H_i) \right) \geq 10^4k^3\log n \right\},
\]

and observe that, using (b), since \( A \subseteq V_0 \), any vertex \( a \in A \) must send edges to more than

\[
10^4k^3\log n/(2k) = 5000k^2\log n
\]

distinct elements in \( X := \{H_1, \ldots, H_t, v_1, \ldots, v_s\} \). For each \( 1 \leq i \leq t \), let \( M_i \) be a set as described in (c). Observe that, using (b), each such \( M_i \) touches more than

\[
10^3k^2\log n/(4k) = 250k \log n
\]
distinct elements of \( X \setminus \{H_i\} \). Let \( M'_i \subseteq M_i \) be a subset of size exactly \( 250k \log n \) such that each pair of edges in \( M'_i \) touches two distinct elements of \( X \setminus \{H_i\} \), which of course are distinct from \( G_i \). Recall that \( H_i = (S_i \cup T_i, E_i) \) for every \( 1 \leq i \leq t \).

For \( Y := \{S_1, \ldots, S_t, T_1, \ldots, T_t, v_1, \ldots, v_s\} \), let

\[
\Phi : Y \to \{L, R\}
\]

be a mapping, generated according to the following random process:

Let \( X_1, \ldots, X_t, Y_1, \ldots, Y_s \sim \text{Bernoulli}(1/2) \) be mutually independent random variables. For each \( 1 \leq i \leq t \), if \( X_i = 1 \), then let \( \Phi(S_i) = L \) and \( \Phi(T_i) = R \). Otherwise, let \( \Phi(S_i) = R \) and \( \Phi(T_i) = L \). For every \( 1 \leq j \leq s \), if \( Y_j = 1 \), then let \( \Phi(v_j) = L \), and otherwise \( \Phi(v_j) = R \). Now, delete all of the edges between two distinct elements of \( Y \) which receive the same label according to \( \Phi \).

Finally, define \( G' \) as the spanning bipartite graph of \( G \) obtained by deleting all of the edges within \( A \) and for distinct \( i \) and \( j \), the edges between \( H_i \) and \( H_j \) which are not contained in \( M'_i \cup M'_j \).

Recall by construction, using \( \Phi \) we generated labels at random; therefore, by using Chernoff bounds (for instance see [1]), one can easily check that with high probability the following hold:
(i) For every $1 \leq i \leq t$, each set $M'_i \cap E(G')$ touches at least (say) $120k \log n$ other elements of $X$, and

(ii) for each $b \in B$, the degree of $b$ into $A \cup B$ is at least (say) $d_{G'}(b, A \cup B) \geq 10^5k^3 \log n$, and

(iii) for each vertex $a \in A$, there exist edges between $a$ and $\cup_{i=1}^t V(H_i)$ that touch at least (say) $2000k^2 \log n$ distinct members of $\{H_1, \ldots, H_t\}$.

Note that here we relied on the luxury of losing the $\log n$ factor for using Chernoff bounds, but it seems like we could easily handle this “cleaning process” completely by hand.

Now we are ready to define our auxiliary digraph $D$. To this end, we first orient edges (again, bidirectional edges are allowed, and un-oriented edges are considered as bidirectional) of $G'$ in the following way:

For every $1 \leq i \leq t$, we orient all of the edges in $E(G') \cap M'_i$ out of $H_i$. We orient all of the edges between $A$ and $\cup_{i=1}^t V(H_i)$ out of $A$. We orient edges between $B$ and $\cup_{i=1}^t V(H_i)$ arbitrarily, and we orient the remaining edges within $A \cup B$ in both directions.

Now, we define $D$ to be the digraph with vertex set $V(D) = X$, and $xy \in E(D)$ if and only if there exists an edge between $x$ and $y$ in $G'$ which is oriented from $x$ to $y$.

In order to complete the proof, we first note that with high probability $D$ is a digraph on at most $n$ vertices with out-degree $\delta^+(D) > (k - 1)[\log n]$. This follows immediately from Properties (i)-(iii) as well as the way we oriented the edges. Therefore, one can apply Lemma 3 to find a sub-digraph $D' \subseteq D$ such that

1. For all $v \in V(D')$ we have $d^+_D(v) \geq d^+_D(v) - (k - 1)[\log n]$, and

2. $\kappa(U(D')) \geq k$.

In fact, with high probability, $\delta^+(D) \geq 120k \log n \geq k + (k - 1)[\log n]$ . Note that by construction, every pair of edges which are oriented out of some $H_i$ must be independent and go to different components. Using Property 1. above combined with the fact that $\delta^+(D') \geq \delta^+(D) - (k - 1)[\log n] \geq k$, we may conclude that the subgraph $G'' \subseteq G'$ induced by the union of all the components in $V(D')$ satisfies $G'' \in \mathcal{F}_{U(D')}(V(D'))$. Applying Lemma 2 with $X = V(D')$ and $R = U(D')$, it follows that $\kappa(G'') \geq k$.

In order to obtain the desired contradiction, we consider the following two cases:

Case 1: $V(G'')$ contains $V(H_i)$ for some $i$. We note that this case is actually impossible because it would contradict the maximality of $H_i$ for the minimal index $i$ such that $V(H_i) \subseteq V(G'')$.

Case 2: $V(G'') \subseteq A \cup B$. We note that in this case, there must be at least one vertex $b \in B \cap V(G'')$. Indeed, $G''$ is $k$-connected, and there are no edges within $A$. Now, it follows from Properties 1. and (ii) above that

$$d^+_D(b) \geq d^+_D(b) - (k - 1)[\log n] \geq 10^4k^3 \log n.$$
Thus, it follows that $|V(G'')| \geq 10^4 k^3 \log n$. Combining this observation with the facts that $G''$ is $k$-connected and $V(G'') \subseteq A \cup B$, we obtain a contradiction. This case can not arise because $G''$ should have been included as one of the bipartite subgraphs $\{H_1, \ldots, H_t\}$.

This completes the proof. □

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