Normal coideal subalgebras of semisimple Hopf algebras

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Abstract. The restriction functor from the category of representations of a semisimple Hopf algebra to the category of representations of a normal coideal subalgebra is studied. It is shown that this functor has a similar behavior to the restriction functor to the category of representations of a normal Hopf subalgebra. Commutator subalgebras as normal left (right) coideal subalgebras are also studied.

1. Introduction
Semisimple Hopf algebras were intensively studied in the last ten years especially due to their connections with other fields in mathematics and physics such as quantum field theory, fusion categories and topological invariants of knots and manifolds.

In principle the structure of a Hopf algebra can be recovered from the structure of a normal Hopf subalgebra, its corresponding quotient Hopf algebra and some additional cohomological data (see [1]). For this reason, normal Hopf subalgebras of semisimple Hopf algebras are an important tool in the study of semisimple Hopf algebras, especially in their classification.

Recently it was proven in [2] that Hopf subalgebras are normal if and only if they are depth two subalgebras. This result was then extended by the author to coideal subalgebras in [3]. Moreover it was shown that in this situation, depth two and normality in the sense defined by Rieffel in [10] also coincide.

In this paper we study normal left coideal subalgebras $L$ of semisimple Hopf algebras $H$. We show that the restriction functor from left $H$-modules to left $L$-modules has a similar behavior to the restriction functor to normal Hopf subalgebras. Using Rieffel’s equivalence classes, formulae for this restriction functor are given in Section 4. Similar results for the induction functor are also proven in the same section.

The proof of these results is using the notion of double coset for semisimple finite dimensional Hopf algebras that was introduced in [4].

The left (right) commutator subalgebra of a semisimple Hopf algebra is defined as the smallest normal left (right) coideal subalgebra with the property that the corresponding quotient is a commutative Hopf algebra. We show that a semisimple Hopf algebra over an algebraically closed
field of characteristic zero is of Frobenius type provided that its left (right) commutator algebra is of Frobenius type.

Recall that a finite dimensional algebra $A$ over a field $k$ is called of Frobenius type if the dimension of each irreducible $A$-module divides the dimension of $A$. One of Kaplansky’s conjectures states that any semisimple Hopf algebra over an algebraically closed field is of Frobenius type.

The paper is organized as follows. In the second section we recall the basics on semisimple Hopf algebras that are needed for the rest of the paper. Section 3 proves some properties of the induction and restriction functors determined by a left coideal subalgebra of a semisimple Hopf algebra. Using the coset decomposition from [4] we prove in Section 4 the above mentioned formulae for induction and restriction. In Section 5 we define conjugate modules for normal coideal subalgebras, similar to the conjugate modules for normal subgroups. It is shown in Theorem 5.9 that if an irreducible module is a constituent of a restricted module then all the conjugate modules are also constituents of the restricted module. The left (right) commutator subalgebra of a semisimple Hopf algebra are defined in the last section. In Theorem 6.4 we show that if the commutator algebra of a semisimple Hopf algebra is of Frobenius type then the Hopf algebra itself is of Frobenius type.

We work over an algebraically closed field $k$ of characteristic zero. For a $k$-vector space $V$ we denote by $|V|$ the dimension $\dim_k(V)$ of $V$ as vector space over $k$. All Hopf algebraic notations are those from [8] except the fact that we drop the sigma symbol in Sweedler’s comultiplication.

2. Preliminaries
Throughout of this paper $H$ will be a finite dimensional semisimple Hopf algebra over an algebraically closed field $k$ of characteristic zero. Then $H$ is also cosemisimple and $S^2 = \text{Id}$ [7]. The set of irreducible characters of $H$ is denoted by $\text{Irr}(H)$. The Grothendieck group $\mathcal{G}(H)$ of the category of finite dimensional left $H$-modules is a ring under the tensor product of modules. Then $C(H) = \mathcal{G}(H) \otimes_Z k$ is a semisimple subalgebra of $H^*$ [9] and it has a basis given by the characters of the irreducible $H$-modules. By duality $C(H^*)$ is a semisimple subalgebra of $H$ and coincides with the set of cocommutative elements of $H$.

Since $H$ is also cosemisimple [7] the set of simple subcoalgebras of $H$ is in bijection with the set of irreducible characters of $H^*$ (see [6] for this correspondence).

For a vector subspace $S$ of a Hopf algebra $H$ we denote by $\omega(S)$ the vector subspace of $H$ generated by $s - \epsilon(s)1$ for all $s \in S$.

2.1. Coideal subalgebras
Recall that a left coideal subalgebra $S$ of $H$ is a subalgebra $S$ of $H$ with $\Delta(S) \subset H \otimes S$. Then $S$ is also a semisimple Hopf algebra by Lemma 4.0.2 of [3].

A left coideal subalgebra of $H$ is called normal if $L$ is closed under the left adjoint action of $H$ on itself, i.e $a_1LS(a_2) \subset L$ for all $a \in H$. If $H$ is involuntary this is equivalent to $a_2LS(a_1) \subset L$ for all $a \in H$.

It follows from [3] that $S$ is also a semisimple algebra if $H$ is a semisimple Hopf algebra. Denote by $C(S)$ the linear span of the trace functionals on $S$. It is the vector space with basis given by the set $\text{Irr}(S)$ of irreducible characters of $S$.

2.2. The correspondence between Hopf ideals and normal left coideal subalgebras
Let $H$ be a finite dimensional Hopf algebra. Based on the structure of relative Hopf modules, Takeuchi described in [13] a one to one correspondence between the Hopf ideals of $H$ and the normal left coideal subalgebras of $H$. This correspondence can be summarized as follows. If $I$ is a Hopf ideal of $H$ then define $L(I) := H^{\omega \pi}$ where $\pi$ is the usual Hopf map $\pi : H \rightarrow H/I$. Then it is easy to check that $L(I)$ is a normal left coideal subalgebra of $H$ and $I = \omega(L(I))H$. 

Conversely if \( L \) is a normal coideal subalgebra of \( H \) then \( \omega(L)H \) is a Hopf ideal and \( L(\omega(L)H) = L \). Denote by \( H//L \) the quotient Hopf algebra \( H/\omega(L)H \) and by \( \pi_L : H \to H//L \).

2.3. Coset decomposition

Let \( B \) be a Hopf subalgebra \( H \). By Corollary 2.5 of [4] there is a coset decomposition for \( H \\
H = \oplus_{C/\sim} CB. \\

where \( \sim \) is an equivalence relation on the set of simple subcoalgebras of \( H \) given by \( C \sim C' \) if and only if \( CB = C'B. \) In [4] this equivalence relation is denoted by \( \pi^H_{k,n}. \)

In terms of characters it follows from Remark 2.9 of [4] that \( d \sim d' \) if and only if
\[
\frac{d \Lambda_B}{\epsilon(d)} = \frac{d' \Lambda_B}{\epsilon(d')}
\]

where \( \Lambda_B \in B \) is an integral of \( B \).

2.4. Subcoalgebra associated to a comodule

Let \( W \) be a right \( H^* \)-module. Then \( W \) is a right \( H \)-comodule and one can associate to it a subcoalgebra of \( H \) denoted by \( C_{\omega} [6] \). This is the minimal subcoalgebra of \( H \) with the property that \( \rho(W) \subseteq C_{\omega} \otimes W \). If \( W \) is a simple comodule and \( q = |W| \) then \( |C_{\omega}| = q^2 \) and the associated coalgebra \( C_{\omega} \) is a co-matrix coalgebra. It has a basis \( \{x_{ij}\}_{1 \leq i,j \leq q} \) such that \( \Delta(x_{ij}) = \sum_{l=0}^{q} x_{il} \otimes x_{lj} \) for all \( 1 \leq i,j \leq q \). Moreover \( W \cong k < x_{11} > 1 \leq i \leq q \) as right \( H \)-comodules where \( \rho(x_{11}) = \Delta(x_{11}) = \sum_{l=0}^{q} x_{il} \otimes x_{1l} \) for all \( 1 \leq i \leq q \). The character of \( W \) as right \( H^* \)-module is \( d \in C(H^*) \subseteq H \) and it is given by \( d = \sum_{i=1}^{q} x_{i1} \). Then \( \epsilon(d) = q \) and the simple subcoalgebra \( C_{\omega} \) is sometimes denoted by \( C_d \).

3. Restriction to normal coideal subalgebras

In this section we prove some formulae for restriction of modules to arbitrary coideal subalgebras.

Let \( S \) be a left coideal subalgebra of a given Hopf algebra \( H \). Then for any \( H \)-module \( M \) and any \( S \)-module \( V \) the comodule structure \( \rho : S \to H \otimes S \) defines via pullback, an \( S \)-module structure on \( M \otimes V \). Denote this module structure by \( M \otimes V \).

This makes the category of \( S \)-modules a left module category over the tensor category \( H \)-modules [14]. The following Proposition also appears in [3]. For the sake of completeness we sketch its proof here.

**Proposition 3.1.** Let \( S \subseteq H \) be a right \( H \)-comodule subalgebra of \( H \). Then \( M \otimes V \uparrow^H_S \cong (M \downarrow^H_S \otimes V) \uparrow^H_S \) for any \( S \)-module \( V \) and any \( H \)-module \( M \).

**Proof.** The map \( T : H \otimes_S (M \otimes V) \to M \otimes (H \otimes_S V) \) given by \( a \otimes_S (m \otimes v) \mapsto a_1 m \otimes (a_2 \otimes_S v) \) is well defined map and a morphism of \( H \)-modules with inverse given by \( m \otimes (a \otimes_S v) \mapsto a_2 \otimes_S (S^{-1}(a_1)m \otimes v) \).

**Corollary 3.2.** Let \( S \subseteq H \) be a left coideal subalgebra of \( H \). Then
\[
M \otimes \epsilon_S \uparrow^H_S = M \downarrow^H_S \uparrow^H_S
\]

for any \( H \)-module \( S \).

**Proof.** Put \( V = k \), the trivial \( S \)-module in the above Proposition. Note that \( M \otimes k = M \downarrow^H_S \).

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The following Proposition is a straightforward computation. It shows that the restriction functor \( \text{res} : H \to \text{mod} \to S \to \text{mod} \) is a morphism of \( H \to \text{mod} \) module categories.

**Proposition 3.4.** Let \( S \) be a left coideal subalgebra of \( H \) and \( M, N \) be two \( H \)-modules. Then

\[
(M \otimes N) \downarrow^H_S = M \otimes N \downarrow^H_S
\]

**Corollary 3.5.** Let \( S \) be a left coideal subalgebra of \( H \) and \( M, N \) be two \( H \) modules such that \( N \) is a trivial \( S \)-module. Then

\[
M \otimes N \cong M^{[N]}
\]
as \( S \)-modules.

**Proof.** One has that \( s(m \otimes n) = s_1 m \otimes s_2 n = s_1 m \otimes \epsilon(s_2) n = sm \otimes n \) for all \( m \in M, n \in N \) and \( s \in S \).

### 3.1. On two endofunctors on \( H \to \text{mod} \) and respectively \( S \to \text{mod} \)

Define the endofunctors:

\[
\mathcal{T} : S \to \text{mod} \to S \to \text{mod} \text{ given by } \mathcal{T}(V) = V \uparrow^H_S \downarrow^H_S
\]

for any \( S \)-module \( V \). Also define

\[
\mathcal{V} : H \to \text{mod} \to H \to \text{mod} \text{ given by } \mathcal{V}(M) = M \downarrow^H_S \uparrow^H_S
\]

for any \( H \)-module \( M \).

The following formulae for composition powers of these endofunctors were proven by induction on \( n \) in \([3]\):

**Lemma 3.8** (Lemma 2.5.7 of \([3]\)). For any \( S \)-module \( V \) and any \( H \)-module \( M \) it follows that:

(i) \[
\mathcal{V}^n(M) = M \otimes (\epsilon_S \uparrow^H_S)^n
\]

(ii) \[
\mathcal{T}^{n+1}(V) = V \uparrow^H_S \otimes \mathcal{T}^n(\epsilon_S)
\]

for all \( n \geq 1 \).

### 3.2. Rieffel’s equivalence relation for a coideal subalgebra

Let \( S \subseteq H \) be a left coideal subalgebra of the semisimple Hopf algebra \( H \). Then as above \( S \) is also semisimple algebra. For any \( V \) and \( W \) two \( S \)-modules one defines \( V \sim W \) if and only if there is a simple \( H \)-module \( M \) such that \( V \) and \( W \) are both constituents of \( M \downarrow^H_S \). The relation \( \sim \) is reflexive and symmetric but not transitive in general. Its transitive closure is denoted by \( \approx \). Thus we say that \( V \approx W \) if and only if there is \( m \geq 1 \) and a sequence \( V_{01}, V_{11}, \ldots, V_{m-11}, V_{mm} \) of simple \( S \)-modules such that \( V = V_{01} \sim V_{11} \sim V_{21} \sim \cdots \sim V_{m-11} \sim V_{mm} = W \). As explained in \([5]\) it follows that \( V \approx W \) if and only if there is \( n \geq 0 \) such that \( V \) is a constituent to \( T^n(W) \). This is also equivalent to \( W \) to be a constituent of \( T^n(V) \). In the rest of the paper the above equivalence relation \( \approx \) is denoted by \( a^H_S \). This equivalence relation is considered in \([10]\) in the context of extensions of semisimple rings.

Similarly one can define an equivalence relation \( u^H_S \) on the set of irreducible \( H \)-modules. For two irreducible \( H \)-modules \( M \) and \( N \) we say that \( M \sim N \) if and only if their restriction to \( S \) have a common constituent. Then \( u^H_S \) is the transitive closure of \( \sim \). Similarly, it can be shown that \( M \) and \( N \) are equivalent if and only if there is \( n \geq 0 \) such that \( M \) is a constituent to \( V^n(N) \). This is also equivalent to the fact that \( N \) is a constituent of \( V^n(M) \).
3.3. The formula for the induced trivial character
Let \( L \) be a normal left coideal subalgebra of \( H \) and \( \epsilon_L := \epsilon_H|_L \) be the trivial character of \( L \).

**Proposition 3.11.** Let \( L \) be a normal left coideal subalgebra of a finite dimensional semisimple Hopf algebra \( H \) and \( A = H//L \). If \( t_A \in A^* \) is the integral on \( A \) with \( t_A(1) = |A| \) then

\[
\epsilon_L \uparrow^H_L = t_A.
\]

Moreover \( t_A \downarrow^H_L = \frac{|H|}{|L|} \epsilon_L \).

**Proof.** One has from [7] that

\[
t_A = \sum_{\chi \in \text{Rep}(A//L)} \chi(1) \chi.
\]

It is easy to see that \( \chi \in \text{Rep}(A//L) \) if and only if \( L \) acts trivially on the representation associated to \( \chi \). Therefore in this situation \( \chi \downarrow_L = \chi(1) \epsilon_L \). By Frobenius reciprocity it follows that \( \epsilon_L \) is a constituent of \( \epsilon_L \uparrow^H_L \) with multiplicity \( \chi(1) \). Thus \( \epsilon_L \uparrow^H_L = t_A \). The second statement follows from Corollary 2.4.3 of [3]. \( \square \)

4. A dual relation

Let \( L \) be a normal left coideal subalgebra of \( H \) and \( A := H//L \). Then the natural projection \( \pi : H \to A \) is a surjective Hopf map and then \( \pi^* : A^* \to H^* \) is an injective Hopf map. We identify \( A^* \) with its image \( \pi^*(A^*) \) in \( H^* \). This is a Hopf subalgebra of \( H^* \). In this section we will study the equivalence relation \( r^H_{k, A^*} \) on \( \text{Irr}(H^*) = \text{Irr}(H) \). Recall the definition of \( r^H_{k, A^*} \) from subsection 2.3.

**Proposition 4.1.** Let \( L \) be a normal left coideal subalgebra of a semisimple Hopf algebra \( H \) and \( A = H//L \). Consider the equivalence relation \( r^H_{k, A^*} \) on \( \text{Irr}(H) \). Then \( \chi \sim \mu \) if and only if their restrictions to \( L \) have a common constituent.

**Proof.** With the notations of subsection 2.3 note that \( \Lambda_{A^*} = t_A \). Using equation 2.1 the equivalence relation \( r^H_{k, A^*} \) on \( \text{Irr}(H) \) becomes \( \chi \sim \mu \) if and only if \( m_H(\chi, \mu t_A) > 0 \). On the other hand, applying the previous Proposition it follows that:

\[
m_H(\chi, \mu t_A) = m_H(\chi, \mu \epsilon_L \uparrow^H_L) = m_H(\chi, \mu \downarrow^H_L \uparrow^H_L) = m_L(\chi \downarrow^H_L, \mu \downarrow^H_L).
\]

In the second equality we have applied formula 3.3. Thus \( \chi \sim \mu \) if and only if their restriction to \( L \) have a common constituent. \( \square \)

**Corollary 4.2.** Let \( L \) be a normal left coideal subalgebra of a semisimple Hopf algebra \( H \) and \( A = H//L \). With the above notations it follows that \( u^A_L = r_{k, H^*} \).

**Theorem 4.3.** Let \( L \) be a normal left coideal subalgebra of a semisimple Hopf algebra \( H \) and \( A = H//L \). Consider the equivalence relation \( r^H_{k, A^*} \) on \( \text{Irr}(H) \). Then \( \chi \sim \mu \) if and only if

\[
\frac{\chi|_L}{\chi(1)} = \frac{\mu|_L}{\mu(1)}.
\]

**Proof.** Let \( C_1, C_2, \ldots, C_s \) be the equivalence classes of \( r^H_{k, A^*} \) on \( \text{Irr}(H) \) and let

\[
a_i := \sum_{\chi \in C_i} \chi(1) \chi \quad (4.4)
\]
for $1 \leq i \leq l$. If $\mathcal{C}_i$ is the equivalence class of the trivial character $\epsilon_A$ of $A$ then the definition of $r^H_{k,A^*}$ implies that $a_1 = t_A$. Formula 2.1 becomes

$$\frac{\chi}{\chi(1) |A|} = \frac{a_i}{a_i(1)}$$

for any irreducible character $\chi \in \mathcal{C}_i$.

By Proposition 3.11 one has that $t_A \uparrow^H_L = |A| \epsilon_L$. Thus by restricting to $L$ both terms of the relation 4.5 and applying Lemma 3.5 one obtains that:

$$\frac{\chi}{\chi(1)} = \frac{a_i}{a_i(1)}$$

Therefore if $\chi \sim \mu$ then $\frac{\chi}{\chi(1)} = \frac{\mu}{\mu(1)}$. The converse follows from Proposition 4.1. □

**Corollary 4.7.** With the above notations one has that $u^H_L = r^H_{k,(u_H/L)}$ as equivalence relations on $\text{Irr}(H)$. \[4.1\]

**4.1. Formulae for the restriction and induction functor**

Let $L$ be a normal left coideal subalgebra of $A$. Consider $\mathcal{L}_0, \ldots, \mathcal{L}_s$ the equivalence classes of $d^H_L$. By Proposition 4.3.1 of [3] one may assume $\mathcal{L}_0 = \{\epsilon_L\}$. Define

$$l_i := \sum_{\alpha \in \mathcal{L}_i} \alpha(1)\alpha,$$

for all $0 \leq i \leq s$. Also as before let $\mathcal{C}_0, \ldots, \mathcal{C}_s$ be the corresponding equivalence classes of $u^H_L$ and let

$$a_i := \sum_{\chi \in \mathcal{C}_i} \chi(1)\chi.$$

The previous Theorem implies the following Corollary:

**Corollary 4.8.** With the above notations, if $\chi \in \mathcal{C}_i$ and $\alpha \in \mathcal{L}_i$ then one has the following formulae:

$$\chi \downarrow^L = \frac{\chi(1) |H|}{a_i(1) |L|} l_i.$$  \[4.9\]

and

$$\alpha \uparrow^H_L = \frac{\alpha(1) |H|}{a_i(1) |L|} a_i.$$

**Proof.** It follows form the previous Theorem that the restriction of two irreducible $H$-characters to $L$ either have the same common constituents or they have no common constituents. Let $t_H$ be the integral on $H$ with $t_H(1) = |H|$. One has that $t_H = \sum_{i=1}^{s} a_i$ as $t_H$ is the regular character of $H$. Since $H$ is free as left $L$-module (see [12]) it follows that the restriction of $t_H$ to $L$ is the regular character $r_L$ of $L$ multiplied by $|H|/|L|$. Thus $t_H \downarrow^H_L = |H|/|L|r_L$. But $r_L = \sum_{\alpha \in \text{Irr}(L)} \alpha(1)\alpha$. Since all the irreducible constituents of $a_i$ are inside $L_i$ it follows from Theorem 4.3 that

$$a_i \downarrow^L = \frac{|H|}{|L|} l_i$$

for any $i$ with $0 \leq i \leq s$. \[4.10\]
Then if $\chi \in \mathcal{C}_i$ formula 4.6 implies that

$$\chi \downarrow_L = \frac{\chi(1)}{a_i(1)} \frac{|H|}{|L|} l_i. \quad (4.11)$$

Evaluating at 1 the above equality one gets $a_i(1) = \frac{|H|}{|L|} l_i(1)$. By Frobenius reciprocity the above restriction formula implies that if $\alpha \in \mathcal{L}_i$ then

$$\alpha \uparrow^H_L = \frac{\alpha(1)}{a_i(1)} \frac{|H|}{|L|} a_i. \quad (4.12)$$

5. The restriction of modules to normal left coideal subalgebras

Let $G$ be a finite group and $H$ a normal subgroup of $G$. If $M$ is an irreducible $H$-module then

$$M \uparrow^G_H = \bigoplus_{i=1}^s \frac{g_i}{g} M$$

where $\frac{g}{g}$ is a conjugate module of $M$ and $\{g_i\}_{i=1}^s$ is a set of representatives for the left cosets of $H$ in $G$. For $g \in G$ the $H$-module $\frac{g}{g} M$ has the same underlying vector space as $M$ and the multiplication with $h \in H$ is given by $h \cdot m = (ghg^{-1})m$ for all $m \in M$. It is easy to see that $\frac{g}{g} M \cong \frac{g'}{g'} M$ if $gH = g'H$.

Let $L$ be a normal left coideal subalgebra of $H$ and $M$ be an irreducible $L$-module. In this section we will define the notion of a conjugate module similar to the one introduced in [4] for normal Hopf subalgebras. It was also previously considered in [11] in the cocommutative case.

If $d \in \text{Irr}(H^*)$ we define a conjugate module $\frac{d}{d} M$. We will show that as in the group situation, the irreducible constituents of $M \uparrow^H_L$ and $\bigoplus_{d \in \text{Irr}(H^*)} \frac{d}{d} M$ are the same for any irreducible $L$-module $M$.

5.1. The conjugate modules

Let $M$ be an irreducible $L$-module with character $\alpha \in C(L)$. We consider the following notion of conjugate module similar to the one introduced in [4] for normal Hopf subalgebras. It was also previously considered in [11] in the cocommutative case.

If $W$ is a right $H^*$-module then $W \otimes M$ becomes an $L$-module with the following structure:

$$l(w \otimes m) = w_0 \otimes (Sw_1lw_2)m \quad (5.1)$$

for all $l \in L$, $w \in W$ and $m \in M$. Note that $l(w \otimes m) \in W \otimes M$ since $H$ is involutory. Here we used that any right $H^*$-module $W$ is a left $H$-comodule with the structure map $\rho(w) = w_2 \otimes w_0$.

In order to check that $W \otimes M$ is an $L$-module one has that

$$l(l', (w \otimes m)) = l(w_0 \otimes (Sw_1lw_2)m) = w_0 \otimes (Sw_1lw_2)(Sw_3lw_4)m = w_0 \otimes (Sw_1lw_3lw_4)m = (l')(w \otimes m)$$

for all $l, l' \in L$, $w \in W$ and $m \in M$.

It can be easily checked that if $W \cong W'$ as right $H^*$-modules then $W \otimes M \cong W' \otimes M$ as $L$-modules. Thus for any irreducible character $d \in \text{Irr}(H^*)$ associated to a simple $H$-comodule $W$ one can define the $L$-module $\frac{d}{d} M \cong W \otimes M$. Its character will be denoted with $\frac{d}{d}$ where $\alpha$ is the character of $M$. Clearly $W \otimes (M \oplus N) = (W \otimes M) \oplus (W \otimes N)$ for any two $L$-modules $M$ and $N$ of $L$. 

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5.2. Formula for the conjugate character

**Proposition 5.2.** Let $L$ be a normal left coideal subalgebra of $H$ and $M$ be an irreducible $L$-module with character $\alpha \in C(L)$. Suppose that $W$ is an irreducible right $H^*$-module with character $d \in \text{Irr}(H^*)$. Then the character $d\alpha$ of the $L$-module $dM$ is given by the following formula:

$$d\alpha(x) = \alpha(S(d_2 x d_1))$$  \hspace{1cm} (5.3)

for all $x \in L$.

*Proof.* Indeed as in subsection 2.4 one may suppose that $W \cong k < x_{i1} \mid 1 \leq i \leq q >$ where $C_d = k < x_{ij} \mid 1 \leq i, j \leq q >$ is the coalgebra associated to $W$ and $q = \varepsilon(d) = |W|$. Then formula 5.1 becomes $l(x_{i1} \otimes w) = \sum_{r=1}^{q} x_{r1} \otimes (S(x_{rj})lx_{ij})m$. Since $d = \sum_{i=1}^{q} x_{ii}$ one gets the formula for the character $d\alpha$. $\square$

**Proposition 5.4.** With the above notations one has that $d^d\alpha = d' (d'\alpha)$ for all $d, d' \in \text{Irr}(H^*)$ and $\alpha \in C(L)$. This shows that $C(L)$ is a left $C(H^*)$-module.

*Proof.* Clearly $1\alpha = \alpha$. The above formula can be checked directly using the formula for the character $d\alpha$ from the previous Proposition. Indeed for any $x \in L$ one has that $d^d\alpha(x) = \alpha(S(d_2 d_2 x d_1 d_1')) = (d'\alpha)(Sd_2 x d_1) = (d' (d'\alpha))(x)$. $\square$

5.3. Compatibility with restriction

Let $N$ be a left $H$-module and $W$ a right $H^*$-module. Then $W \otimes N$ becomes an $H$-module with the following structure:

$$h(w \otimes m) = w_0 \otimes (S(w_{-1})hw_{-2})m$$  \hspace{1cm} (5.5)

It can be checked that $W \otimes N \cong N^{[W]}$ as $H$-modules. Indeed the map $\phi : W \otimes N \to w_0 \otimes w_{-1}n$ is an isomorphism of $H$-modules where $\varepsilon W$ is considered left $H$-module with the trivial action. Its inverse is given by $w \otimes n \mapsto w_0 \otimes S(w_{-1})n$. To check that $\phi$ is an $H$-module map one has that

$$\phi(h(w \otimes n)) = \phi(w_0 \otimes (S(w_{-1})hw_{-2})n) = w_0 \otimes w_{-1}(S(w_{-2}hw_{-3})n) = w_0 \otimes hw_{-1}n = h.(w_0 \otimes w_{-1}n) = h\phi(w \otimes n)$$

for all $w \in W$, $m \in M$ and $h \in H$.

**Proposition 5.6.** Let $L$ be a normal left coideal subalgebra of $H$ and $M$ be an irreducible $L$-module with character $\alpha \in C(L)$. If $d \in \text{Irr}(H^*)$ then

$$\frac{1}{\varepsilon(d)} d\alpha \uparrow^H_L = \alpha \uparrow^H_L$$

*Proof.* Using the notations from subsection 4.1 let $L_i$ be the subset of $\text{Irr}(L)$ which contains $\alpha$. It is enough to show that the irreducible constituents of $d\alpha$ are contained in this set and then the induction formula 4.12 from the same subsection can be applied for each of these constituents.

For this, suppose $N$ is an irreducible $H$-module and

$$N \downarrow^L_L = \oplus_{i=1}^r N_i$$  \hspace{1cm} (5.7)
where \( N_i \) are irreducible \( L \)-modules. The above result implies that \( W \otimes N \cong N_i^{[W]} \) as \( H \)-modules. Therefore \( (W \otimes N) \downarrow_L = (N \downarrow_L)^{[W]} \) as \( L \)-modules. But \( (W \otimes N) \downarrow_L = \bigoplus_{i=1}^s (W \otimes N_i) \) where each \( W \otimes N_i \) is an \( L \)-module by 5.1. Thus

\[
\bigoplus_{i=1}^s N_i^{[W]} = \bigoplus_{i=1}^s (W \otimes N_i) \tag{5.8}
\]

This shows that if \( N_i \) is a constituent of \( N \downarrow_L \) then \( W \otimes N_i \) has all the irreducible \( L \)-constituents among those of \( N \downarrow_L \).

**Proposition 5.9.** Let \( L \) be a left normal coideal subalgebra of \( H \) and \( M \) be an irreducible \( L \)-module. Then \( M \uparrow^H_L \otimes_L \) and \( \bigoplus_{d \in \Irr(H^*)} d M \) have the same irreducible constituents.

**Proof.** Let \( S \) be the set of simple subcoalgebras of \( H \). Then \( H = \bigoplus_{C \in S} C \). It follows that the induced module \( M \uparrow^H_L \) can be written as the sum:

\[
M \uparrow^H_L = H \otimes_L M = \bigoplus_{C \in S} C L \otimes_L M.
\]

Note that each \( C L \) is a free \( L \)-module by Theorem 6.1 of \([12]\). On the other hand each \( C L \otimes_L M \) is an \( L \)-module by left multiplication with elements of \( L \) since

\[
l.(c l' \otimes_L m) = c_3(Cc_2l_1)l' \otimes_L m = c_3 \otimes_L (Cc_2l_1)l'm
\]

for all \( l, l' \in L, c \in C \) and \( m \in M \). Thus \( M \uparrow^H_L \) restricted to \( L \) is the sum of the \( L \)-modules \( C L \otimes_L M \) with \( C \in S \). On the other hand it can be checked directly that the composition of the canonical maps \( C \otimes M \rightarrow C L \otimes M \rightarrow C L \otimes_L M \) is a surjective morphism of \( L \)-modules. Let \( d \) be the character associated to \( C \). Since \( C \otimes M \cong \epsilon(d) M \) as \( L \)-modules this implies that \( C L \otimes_L M \) is a homeomorphic image of \( \epsilon(d) \) copies of \( d M \). Therefore the irreducible constituents of \( M \uparrow^H_L \) are among those of \( \bigoplus_{d \in \Irr(H^*)} d M \). In the proof of the previous Proposition we have shown the other inclusion. Thus \( M \uparrow^H_L \otimes_L \) and \( \bigoplus_{d \in \Irr(H^*)} d M \) have the same irreducible constituents.

**Remark 5.10.** It follows that the equivalence class of a character \( \alpha \in \Irr(L) \) under the relation \( d^H_L \) is given by all the irreducible constituents of \( d^H_L \) as \( d \) runs through all irreducible characters of \( H^* \).

6. On the commutator subalgebra and the dimension of an irreducible \( A \)-module

Let \( S \) be the set of one dimensional modules of a semisimple Hopf algebra \( H \) and

\[
I := \bigcap_{M \in S \Ann_H(M)}.
\]

Since \( S \) is closed under tensor product it follows from \([15]\) that \( I \) is a Hopf ideal of \( H \). Let \( \pi : H \rightarrow H/I \) be the canonical Hopf projection.

**Definition 6.1.** The normal left coideal subalgebra \( L := H^{\omega \pi} \) of \( A \) is called the left commutator subalgebra of \( A \).

It follows that \( H/\!\!/L \) is a commutative Hopf algebra since \( I = \omega(L)H \) and all the irreducible representations of \( H/I \) are one dimensional. Similarly \( R := \omega \pi^{\omega \pi} H \) is called the right commutator subalgebra of \( A \).

**Lemma 6.2.** Let \( L \) and \( L' \) be two normal left coideal subalgebras. If \( \omega(L)H \subset \omega(L')H \) then \( L \subset L' \).
Proof. Let $\phi : H//L \rightarrow H//L'$ be given by $h \mapsto \hat{h}$. Then it is easy to check that $\phi$ is a well defined Hopf map and $\phi \circ \pi_L = \pi_{L'}$ where $\pi_L$ and $\pi_{L'}$ are the Hopf projections of $H$ onto $H//L$ respectively $H//L'$. Thus $L = H^\text{co}\pi_L \subset H^\text{co}\pi_{L'} = L'$.

Next proposition shows that $L$ is the smallest normal left coideal subalgebra of $H$ with $H//L$ a commutative Hopf algebra.

**Proposition 6.3.** Let $H$ be a semisimple Hopf algebra, $L$ its left commutator subalgebra and $L'$ an arbitrary normal left coideal subalgebra of $H$. If $H//L'$ is a commutative Hopf algebra then $L \subset L'$.

**Proof.** Since all the representations of $H//L'$ are one dimensional it follows that $\text{Irr}(H//L') \subset S$. Thus by taking annihilators one has $\omega(L')H \supset I = \omega(L)H$ and therefore $L' \supset L$ by Lemma 6.2.

Recall that a finite dimensional algebra $A$ is said to be of Frobenius type if the dimension of any irreducible representation of $A$ divides the dimension of $A$.

**Theorem 6.4.** Suppose that the left commutator algebra of a semisimple Hopf algebra $H$ is of Frobenius type. Then $H$ is also of Frobenius type.

**Proof.** Let as above $S$ be the set of one dimensional modules of $H$ and $L$ its left commutator algebra. Then $\text{Irr}(H//L) = S$. Let $M$ be an irreducible $H$-module and let $V$ be an irreducible constituent of $M \downarrow_H L$. Then by Equation 4.5 it follows all the other constituents of $H \otimes_L V$ are of the type $M \otimes X$ with $X \in S$. Thus all the irreducible constituents of $H \otimes_L V$ have the same dimension, namely the dimension $|M|$ of $M$. This implies that $|M|$ divides $|H|\frac{|V|}{|L|}$. But since $L$ is of Frobenius type it follows that $|V|$ divides dimension $|L|$ of $L$. Therefore $|M|$ divides the dimension $|H|$ of $H$ and $H$ is also of Frobenius type.

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