ON THE ANALYTICITY OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS WITH FRACTIONAL DISSIPATION

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Abstract. By using a new bilinear estimate, a pointwise estimate of the generalized Oseen kernel and an idea of fractional bootstrap, we show in this note that solutions to the Navier-Stokes equations with fractional dissipation are analytic in space variables.

1. Introduction

We are interested in the initial value problem of $d$ dimensional generalized Navier-Stokes equations with fractional dissipation

$$u_t + u \nabla u + (-\Delta)^{\gamma/2} u + \nabla p = 0, \quad \text{div} u = 0,$$

$$u(0, x) = u_0(x) \quad x \in \mathbb{R}^d,$$  \tag{1.1-1.2}

where $\gamma \in (1, 2]$ is a fixed parameter and the initial data $u_0$ is in some Banach space to be specified later.

In a well-known paper [9] Kato proved that for $\gamma = 2$ the problem is locally well-posed for $u_0 \in L^d$. Kato’s method is based on perturbation theory of the Stokes kernel and is different from the energy methods used in the seminal paper [12] by Leray. The so called mild solutions are constructed via a fixed point argument by considering the corresponding integral equations. Kato’s results have been generalized by many authors in various function spaces. (See, for example, [4], [7], [10], [18], [20]). With minor modifications, this method can also be applied to show the local well-posedness of the generalized Navier-Stokes equation (1.1)-(1.2) with initial data $u_0 \in L^d$ and the global well-posedness for small data (see Proposition 2.1).

In [15], Masuda initiated the study the spatial analyticity of solutions to the Navier-Stokes equations. The temporal analyticity was proved

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by Foias and Temam in an important paper [2]. The study of analyticity of the Navier-Stokes equations was continued by many authors. (See, for example, [5], [6], [11], [13], [20] and [17].) In particular, in a very recent paper [17], Miura and Sawada showed that the solutions by Koch and Tataru [10] are spatial analytic. A similar smoothness result is also obtained in a recent preprint [3] by Germain, Pavlović and Staffilani for both the $L^\infty$ and the Carleson norms.

Usually, spatial analyticity of solutions to the Navier-Stokes equations is obtained by using either the fixed point argument (see, for example, [20], [17] and [3]) or a variation of Foias and Temam’s method (see, for example, [14]). For both methods, one always needs some kind of smallness assumption on either the initial data or the solution itself. The main result (Theorem 2.2) of this paper is that the spatial analyticity is an intrinsic property of the solutions to the Navier-Stokes equations. The philosophy is that the existence of the solution in certain path spaces implies its analyticity without any smallness assumption. Our method is based on new estimates of the kernels, a semi-group property of the mild solutions and a so called fractional bootstrap argument.

The remaining part of the article is organized as follows: the main results are given in the following section. Section 3 is devoted to a new pointwise estimate of the generalized Oseen kernel and a corresponding estimate of the generalized heat kernel. In Section 4 we prove Theorem 2.2 mainly by using the aforementioned fractional bootstrap argument. Finally, the proof of Corollary 2.6 is given in the last section.

2. The main results

Define $G(t, x) = G_\gamma(t, x)$ by its Fourier transform $\hat{G}_\gamma(t, \xi) = e^{-t|\xi|\gamma}$ for $t > 0$. Then $G_\gamma(t, x)$ is the fundamental solution of the linear operator $\partial_t + (-\Delta)^{\gamma/2}$ and it has the scaling property

$$G_\gamma(t, x) = t^{-d/\gamma}G_\gamma(1, t^{-\frac{1}{\gamma}}x).$$

(2.1)

It is well-known that (1.1)-(1.2) can be rewritten into an integral equation

$$u(t) = G(t) * u_0 - \int_0^t G(t-r, \cdot) * P(u\nabla u)(r, \cdot) \, dr,$$

$$= G(t) * u_0 - \int_0^t K(t-r, \cdot) * (u\nabla u)(r, \cdot) \, dr,$$

(2.2)
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where $P$ is the Helmholtz projection, and $K = PG$ is the Oseen kernel (see Section 3 and Proposition 3.1).

For $q \in (\frac{d}{\gamma - 1}, \infty], T \in (0, \infty]$, introduce the Banach spaces

$$X_{q,T} = BC([0,T), L^\frac{d}{d-1}_x) \cap \{u \mid t^\alpha u \in BC((0,T), L^q_x)\}, \quad \alpha = 1 - \frac{1}{\gamma} - \frac{d}{q \gamma},$$

with norm

$$\|u\|_{X_{q,T}} = \max \{\|u\|_{L^\frac{d}{d-1}_x L^\infty([0,T))}, \|t^\alpha u\|_{L^q_x L^\infty((0,T))}\},$$

The classical Kato’s method easily gives the following local well-posedness result.

**Proposition 2.1.** Assume $u_0$ is in the closure of $\{u \in C^\infty_0(\Omega) ; \text{div} u = 0\}$ in the scaling invariant Lebesgue space $L^{\frac{d}{d-1}}(\mathbb{R}^d)$. Then for any $q \in (\frac{d}{\gamma - 1}, \infty]$, (2.2) has a unique solution in $X_{q,T}$ for some $T \in (0, \infty]$.

Here we state our main results of this note.

**Theorem 2.2.** Suppose $u$ is a solution to (2.2) in $\mathbb{R}^d \times (0, T)$ for some $T \in (0, \infty]$, and satisfies $\|t^\alpha u\|_{L^q_x L^\infty((0,T))} < \infty$ for some $q \in (\frac{d}{\gamma - 1}, \infty]$. Then for any $t \in (0, T)$ and $q' \in [q, \infty]$, we have

$$\|D^k u(t, \cdot)\|_{L^{q'}_x} \leq C^{k+1} t^{\frac{k}{\gamma} - \alpha' k},$$

where $\alpha' = 1 - \frac{1}{\gamma} - \frac{d}{q \gamma}$ and $C$ is independent of $k$ and $q'$. Consequently, $u(t, \cdot)$ is spatial analytic.

**Remark 2.3.** If $T = \infty$, estimate (2.3) implies the decay in time of higher order Sobolev norms. Furthermore, the radius of convergence of Taylor’s expansion of $u(t, \cdot)$ increases with time at a rate proportional to $t^{1/\gamma}$.

**Remark 2.4.** In Theorem 2.2 we don’t assume any condition on the initial data $u_0$. The philosophy here is that the mere existence of the solution implies its analyticity.

**Remark 2.5.** In the case when $\gamma = 2$, a similar result is obtained in a recent interesting paper [5] by Giga and Sawada. Our proof is more direct, and essentially different from theirs. Moreover, for general $\gamma \in (1, 2]$, their method only gives a less satisfactory estimate

$$\|D^k u(t, \cdot)\|_{L^3_x} \leq C^{k+1} t^{\frac{k}{\gamma} - \alpha \frac{2k}{\gamma}},$$

which doesn’t imply the spatial analyticity of $u$ if $\gamma < 2$.

The next corollary is a simple consequence of Proposition 2.1 and Theorem 2.2.
Corollary 2.6. Suppose $u$ is a solution to (2.2) in $\mathbb{R}^d \times (0, T)$ for some $T \in (0, \infty]$ and satisfies $u \in C([0, T), L_x^{d/(\gamma - 1)})$. Then there exists a countable subset $\Omega$ of $(0, T)$ such that $u(t, \cdot)$ is spatial analytic for any $t \in (0, T) \setminus \Omega$.

3. Pointwise estimates of the generalized Oseen kernel

The generalized Oseen kernel

We will need the following pointwise estimates of higher derivatives of the generalized Oseen kernel with an explicit control of constants. The proof of a similar result but with no control of constants can be found in [14], Proposition 11.1. We will not use Proposition 3.1 in its full generality. However, the estimate itself is of independent interest and we are not able to find it in the literature.

Proposition 3.1. Assume $d \geq 3$ and $\gamma \in (0, \infty)$. For $1 \leq j, m < d$ and $t > 0$, the operator $O_{j,m,t} = \frac{1}{\Delta} \partial_j \partial_m e^{-t\Lambda^\gamma}$ is a convolution operator whose kernel is given by

$$K_{j,m,t}(x) = \frac{1}{t^{d/\gamma}} K_{j,m}(\frac{x}{t^{1/\gamma}}),$$

where $K_{j,m}$ is a smooth function. There exists a constant $C = C(d, \alpha, \gamma)$ such that, for any integer $k \geq 0$ and $-1 < \alpha \leq 1$,

$$|(1 + |x|)^{d+k+\alpha} \partial^k \Lambda^\alpha K_{j,m}(x)| \leq C^{k+1} k^k$$

for all $x \in \mathbb{R}^d$.

Proof. Let $G_\gamma(t, y)$ be the generalized heat kernel which satisfies the scaling property (2.1). Consider first the case $0 \leq |x| \leq 1$. We have

$$\max_{0 \leq |x| \leq 1} \left| D^{k+2} \Lambda^{\alpha-2} G_\gamma(1, y) \right|$$

$$\leq \left\| D^{k+2} \Lambda^{\alpha-2} G_\gamma(1, y) \right\|_{L_x^\infty} \leq \int_{\mathbb{R}^d} |\xi|^{k+\alpha} e^{-|\xi|^\gamma} d\xi \leq C^{k+1} k^k.$$
To this end, write \( x = t^{-1/\gamma} \hat{n}, \hat{n} \in S^d \), \( y = t^{-1/\gamma} z \), \( 0 < t \leq 1 \), and we have

\[
|x|^{k+\alpha+d} D_x^{k+2} \Lambda^\alpha \int \frac{1}{|x-y|^{d-2}} G_\gamma(1, y) dy
= C_1 |x|^{k+\alpha+d} \int \frac{1}{|x-y|^{d-2+\alpha}} D_y^{k+2} G_\gamma(1, y) dy
= C_1 \int \frac{1}{|\hat{n} - z|^{d-2+\alpha}} D_z^{k+2} G_\gamma(t, z) dz.
\]

where \( C_1 = C_1(d, \gamma, \alpha) \) is another constant.

Now note that as \( t \to 0 \), \( G_\gamma(t, x) \to \delta_0(x) \) where \( \delta_0 \) is the Dirac distribution on \( \mathbb{R}^d \). Therefore it is easy to see that the right-hand side of the above converges to

\[
D_z^{k+2} \left( \frac{1}{|\hat{n} - z|^{d-2+\alpha}} \right) \bigg|_{z=0}.
\]

Clearly this is bounded by \( C^{k+1} \cdot k^k \) for some constant \( C > 0 \). Remark that this heuristic argument suggests why the optimal bound on the constants is of the form \( C^{k+1} \cdot k^k \).

To complete our argument, we write

\[
\int \frac{1}{|\hat{n} - z|^{d-2+\alpha}} D_z^{k+2} G_\gamma(t, z) dz
= \int_{|\hat{n} - z| \leq 1/2} \frac{1}{|\hat{n} - z|^{d-2+\alpha}} D_z^{k+2} G_\gamma(t, z) dz
+ \int_{|\hat{n} - z| > 1/2} \frac{1}{|\hat{n} - z|^{d-2+\alpha}} D_z^{k+2} G_\gamma(t, z) dz
= I + II.
\]

To estimate I, we use the representation of \( G_\gamma(t, z) \) through the heat kernel [19]:

\[
G_\gamma(t, z) = \int_0^\infty \frac{1}{(\pi^{\frac{d}{2}} s^{\frac{1}{2}} t^\gamma)^d} \exp \left\{ - \left| \frac{z}{s^{\frac{1}{2}} t^\gamma} \right|^2 \right\} dF(s), \quad (3.1)
\]
where \( dF(\cdot) \) is a probability measure. This gives us

\[
D_k^{k+2} G_\gamma(t, z) = \int_0^\infty \frac{1}{(\pi^{\frac{d}{2}} s^{\frac{d}{2}} t^{\frac{d}{2}})^d} D_k^{k+2} \exp \left\{ - \left| \frac{z}{s^{\frac{d}{2}} t^{\frac{d}{2}}} \right|^2 \right\} dF(s).
\]

where \( H_{k+2}(z) \) is the \( d \)-dimensional Hermite polynomial of degree \( k+2 \). We now use the following pointwise estimate of Hermite polynomials [8]:

\[
|H_n(x)| \leq (2^n n!)^\frac{1}{2} e^{x^2/2}.
\] (3.2)

We then have

\[
|D_k^{k+2} G_\gamma(t, z)| \\
\leq \int_0^\infty \left( t^{\frac{1}{2}} s^{\frac{1}{2}} \right)^{(k+d+2)} \cdot 2^{k+1} \cdot ((k + 2)!)^\frac{1}{2} \cdot \exp \left\{ - \frac{1}{2} \left| \frac{z}{s^{\frac{d}{2}} t^{\frac{d}{2}}} \right|^2 \right\} dF(s).
\]

Since in case I, \( |\tilde{n} - z| \leq \frac{1}{2} \) and therefore \( |z| \geq \frac{1}{2} \), we have

\[
|D_k^{k+2} G_\gamma(t, z)| \\
\leq C^{k+1} \cdot ((k + 2)!)^{\frac{1}{2}} \int_0^\infty \left( t^{\frac{1}{2}} s^{\frac{1}{2}} \right)^{(k+d+2)} \exp \left\{ - \frac{1}{8} \left| \frac{z}{s^{\frac{d}{2}} t^{\frac{d}{2}}} \right|^2 \right\} dF(s) \\
\leq C^{k+1} k^k.
\]

where the last inequality follows from the fact that \( dF(s) \) is a probability measure and the elementary inequality

\[
\sup_{x > 0} x^{k+d+2} e^{-x^2/2} \leq C^{k+1} \cdot k^k.
\] (3.3)
The estimate of II is similar. By integration by parts and (3.1) we have,

\[ |II| \leq \sum_{j=0}^{k+1} \int_0^\infty (s^{\frac{1}{2}} t^\frac{1}{2})^{-d} \int_{|z-\hat{n}|=\frac{1}{2}} D^j_z \left( \frac{1}{|z-\hat{n}|^{d-2+\alpha}} \right) \cdot \left| D^{k-j+1} \exp \left\{ -\left| \frac{z}{s^{\frac{1}{2}} t^\frac{1}{2}} \right|^2 \right\} \right| ds \sigma(z) dF(s) + \]

\[ + \int_0^\infty (s^{\frac{1}{2}} t^\frac{1}{2})^{-d} \int_{|z-\hat{n}|>\frac{1}{2}} D^{k+2} \left( \frac{1}{|z-\hat{n}|^{d-2+\alpha}} \right) \exp \left\{ -\left| \frac{z}{s^{\frac{1}{2}} t^\frac{1}{2}} \right|^2 \right\} d\sigma(z) dF(s). \]

On \( |z-\hat{n}| = \frac{1}{2} \), we have \( |z| \geq \frac{1}{2} \) and therefore by (3.2)

\[ D^{k-j+1} \exp \left\{ -\left| \frac{z}{s^{\frac{1}{2}} t^\frac{1}{2}} \right|^2 \right\} = \left( s^{\frac{1}{2}} t^\frac{1}{2} \right)^{-(k-j+1)} \cdot H \alpha_{k-j+1} \left( \frac{z}{s^{\frac{1}{2}} t^\frac{1}{2}} \right) \cdot \exp \left\{ -\left| \frac{z}{s^{\frac{1}{2}} t^\frac{1}{2}} \right|^2 \right\} \]

\[ \leq \left( s^{\frac{1}{2}} t^\frac{1}{2} \right)^{-(k-j+1)} \cdot 2^{\frac{k-j+1}{2}} (k-j+1)! \cdot \frac{1}{8} \left( \frac{1}{s^{\frac{1}{2}} t^\frac{1}{2}} \right)^2 \cdot \exp \left\{ -\left| \frac{z}{s^{\frac{1}{2}} t^\frac{1}{2}} \right|^2 \right\} . \]

Note also that for \( |z-\hat{n}| > \frac{1}{2} \), we have \( \left| D^{k+2} \left( \frac{1}{|z-\hat{n}|^{d-2+\alpha}} \right) \right| \leq C^{k+1} k! \).

These estimates together with the elementary inequality (3.3) immediately give us

\[ |II| \leq \sum_{j=0}^{k+1} C^{k+1} j! (k-j+1)! + C^{k+1} k^k \leq C^{k+1} k^k. \]

The proposition is now proved. \( \square \)

We also need the following lemma.

**Lemma 3.2.** Let \( \gamma \in (0, \infty) \), \( p \in [1, \infty] \). Let \( k \geq 0 \) be an integer and \( \varepsilon \in (0, 1] \). Then for some constant \( C = C(\gamma, \varepsilon) > 0 \), we have

\[ \left\| D^k \Lambda^\alpha G(t, \cdot) \right\|_{L^p} \leq C^{k+1} k^\gamma t^{-\frac{k+\alpha}{p} - \frac{d(1-\frac{1}{p})}{\gamma}}, \quad (3.4) \]

for any \( \alpha \in [\varepsilon - 1, 1] \) such that \( k + \alpha \geq \varepsilon \) or \( k = \alpha = 0 \). Note that the constant \( C \) can be taken to be independent of \( p \).

**Proof.** This follows from a similar pointwise estimate as in Proposition 3.1. We omit the details. \( \square \)
4. Proof of Theorem 2.2

Since $u$ is divergence free, we have $u \nabla u = \nabla \cdot (u \otimes u)$. Therefore, by using integration by parts, the integral equation (2.2) is equivalent to
\[
 u(t) = G(t) \ast u_0 - B(u, u),
\]
(4.1)
where
\[
 B(u, u) := \int_0^t \nabla K(t - s, \cdot) \ast (u \otimes u)(s, \cdot) \, ds
\]
is a bilinear term. The following lemma is probably known. We provide a sketched proof for the sake of completeness.

**Lemma 4.1.** Under the assumptions of Theorem 2.2, for any $t \in (0, T)$ and $q' \in [q, \infty]$, we have
\[
 \|u(t, \cdot)\|_{L^{q'}_x} \leq C t^{-\alpha'},
\]
(4.2)
where $\alpha' = 1 - \frac{1}{q} - \frac{d}{q\gamma}$ and the constant $C$ is independent of $q'$.

**Proof.** We shall use a bootstrap argument. Assume for some positive constant $C_0$,
\[
 \|t^\alpha u(t, \cdot)\|_{L^q_x L^\infty(0, T)} \leq C_0.
\]
We fix a $t \in (0, T)$ and choose $s \in (t/3, 2t/3)$ such that
\[
 s^\alpha \|u(s, \cdot)\|_{L^q_x} \leq C_0.
\]
From (4.1) and the semigroup property of $G$, it holds that
\[
 u(t, \cdot) = G(t - s) \ast u(s, \cdot) - \int_s^t \nabla K(t - r, \cdot) \ast (u \otimes u)(r, \cdot) \, dr.
\]
Taking the $L^{q'}_x$ norm on both sides and using Minkowski’s inequality, Young’s inequality, Hölder’s inequality, Proposition 3.1 and Lemma 3.2, we get
\[
 \|u(t, \cdot)\|_{L^{q'}_x} \leq \|G(t - s) \ast u(s, \cdot)\|_{L^{q'}_x} + \int_s^t \|\nabla K(t - r, \cdot) \ast (u \otimes u)(r, \cdot)\|_{L^{q'}_x} \, dr
\]
\[
 \leq \|G(t - s)\|_{L^1} \|u(s, \cdot)\|_{L^q} + \int_s^t \|\nabla K(t - r, \cdot)\|_{L^2} \|u(r, \cdot)\|_{L^2}^2 \, dr
\]
\[
 \leq C(t - s)^{-\frac{d}{q'}(1 - \frac{1}{r_1})} s^{-\alpha} + C \int_s^t (t - r)^{-\frac{d}{q} - \frac{1}{q'}(1 - \frac{1}{r_2})} r^{-2\alpha} \, dr,
\]
where $r_1$ and $r_2$ satisfy
\[
 1 + \frac{1}{q'} = \frac{1}{r_1} + \frac{1}{q}, \quad 1 + \frac{1}{q'} = \frac{1}{r_2} + \frac{2}{q}.
\]
Because $s \in (t/3, 2t/3)$,

$$\|u(t, \cdot)\|_{L^{q'}} \leq C t^{-\alpha'} + C t^{-\alpha'} \int_{1/3}^{1} (1 - r)^{-\frac{1}{\gamma} - \frac{d}{\gamma} \frac{1}{2}} \gamma^{-2\alpha} \, dr.$$ 

Since $q > \frac{d}{\gamma - 1}$, the last integral is finite if

$$0 \leq \frac{1}{q} - \frac{1}{q'} \leq \frac{1}{2d} (\gamma - 1 - \frac{d}{q}). \tag{4.3}$$

Hence, (4.2) is proved for $q'$ satisfying (4.3). A finite iteration of this argument gives (4.3) for any $q' \in [q, \infty]$. The lemma is proved. □

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2** In this proof we shall denote by $C_1$ constants which may vary from line to line but does not depend on $k$ or $q'$. Let $N \geq 1$ be an integer sufficiently large such that

$$1 + \frac{1}{N} + \frac{2d}{Nq} + \frac{d}{q} < \gamma.$$ 

Define a finite sequence of numbers $q_0 \geq q$ such that

$$\frac{1}{q_n} = \frac{1}{q} - \frac{n}{Nq}, \quad 0 \leq n \leq N.$$ 

Also define for $l = 0, 1, \cdots, N - 1$, $q' \in [q, \infty]$,

$$A(k, l, q') = \left\| t^{\alpha' + \frac{k+l}{\gamma}} D_x^{k} \Lambda^{N} u \right\|_{L^{q'}_{x} L^{\infty}_{t}(0,T)},$$

and

$$A(k, l) = \sup_{q \leq q' \leq \infty} A(k, l, q').$$

We shall derive a set of recurrent inequalities for $A(k, l)$. To this end, by using the semigroup proper of $G$, write

$$u(t, \cdot) = G\left(\frac{t}{k+2}\right) * u\left(\frac{k+1}{k+2} t\right) - \int_{\frac{k+1}{k+2} t}^{t} \nabla K(t - r, \cdot) * (u \otimes u)(r, \cdot) \, dr.$$ 

Call the first term in the above sum linear term and the other bilinear term. We have four cases.

**Case 1**: estimate of the linear term, $k \geq 0$, $1 \leq l \leq N - 1$. By Lemma
we have

\[
\left\| t^{\alpha + \frac{k+1}{N}} D_x^{k+N} \left( G\left( \frac{t}{k+2} \right) \ast u\left( \frac{k+1}{k+2} t \right) \right) \right\|_{L^p_x L^q_t}
\leq \left\| \frac{1}{N} \Lambda_\frac{k}{N} G\left( \frac{t}{k+2} \right) \right\|_{L^p_x L^q_t} \left\| t^{\alpha + \frac{k+1}{N}} D_x^{k+N} u\left( \frac{k+1}{k+2} t \right) \right\|_{L^p_x L^q_t}
\leq C_\gamma (k+2)^{\frac{k}{N\gamma}} \cdot \left( \frac{k+2}{k+1} \right)^{\alpha + \frac{k+1}{N\gamma}} A(k, l - 1)
\leq C_1 (k+1)^{\frac{k}{N\gamma}} A(k, l - 1).
\]

**Case 2:** estimate of the linear term, \( l = 0, k \geq 1 \). This case is similar to Case 1 and we have

\[
\left\| t^{\alpha + \frac{k}{N}} D_x^k \left( G\left( \frac{t}{k+2} \right) \ast u\left( \frac{k+1}{k+2} t \right) \right) \right\|_{L^p_x L^q_t}
= \left\| t^{\alpha + \frac{k}{N}} D_x A^{1/N} D_x \Lambda^{-1} G\left( \frac{t}{k+2} \right) \ast D_x^{k-1} \Lambda^{\frac{k-1}{N}} u\left( \frac{k+1}{k+2} t \right) \right\|_{L^p_x L^q_t}
\leq C_1 \cdot (k+1)^{\frac{k}{N\gamma}} A(k-1, N-1).
\]

**Case 3:** estimate of the nonlinear term for \( k \geq 0, 1 \leq l \leq N - 1 \). Consider \( q' \in [q, \infty] \), obviously \( \frac{1}{q'} \in [\frac{1}{q_{n+1}}, \frac{1}{q_n}] \) for some \( 0 \leq n \leq N - 1 \). For any two functions \( f, g \), and \( 0 \leq \varepsilon < 1 \), \( 2 < p < \infty \), the following fractional Leibniz inequality is well known:

\[
\| \Lambda^\varepsilon (f g) \|_{L^{p/2}_x} \leq C_p (\| \Lambda^\varepsilon f \|_{L^p_x} \| g \|_{L^p_x} + \| \Lambda^\varepsilon g \|_{L^p_x} \| f \|_{L^p_x}),
\]

where the constant \( C_p \) depends on \( p \). In what follows, we shall only apply the fractional Leibniz inequality when \( p = q_n, 0 \leq n \leq N - 1 \). In this way the constants will not depend on \( q' \). Now by Lemma 3.2, Proposition 3.1, Young’s inequality and fractional Leibniz inequality,
we have
\[
\left\| t^{\alpha'} \frac{d^{k+1}}{dt^{k+1}} \int_{t^{k+1}}^{t} \nabla K(t-r, \cdot) \ast (u \otimes u)(r) \, dr \right\|_{L^q_x L^{\infty}_t} \\
= \left\| t^{\alpha'} \frac{d^{k+1}}{dt^{k+1}} \int_{t^{k+1}}^{t} \Lambda^{\frac{\gamma}{q}} \nabla K(t-r, \cdot) \ast \left( D^k_x \Lambda^{\frac{1}{q}} (u \otimes u)(r) \right) \, dr \right\|_{L^q_x L^{\infty}_t} \\
\leq C_1 \left( \int_{1-\frac{1}{k+2}t}^{1} (1-r)^{-\frac{k+1}{q} - \frac{d}{q} \left(q_n - 1 \right)} \, dr \right) \sum_{j=0}^{k} \binom{k}{j} \cdot \left( 1 - \frac{1}{k + 2} \right)^{-\frac{k+1}{q} - \frac{d}{q} \left(q_n - 1 \right)} \cdot A(j, l-1, q_n) A(k-j, 0, q_n) \\
\leq C_1 \sum_{j=0}^{k} \binom{k}{j} A(j, l-1) A(k-j, 0).
\]

The last integral converges since we have
\[
\frac{1 + \frac{1}{N}}{\gamma} + \frac{d}{\gamma} \left(q_n - 1 \right) \leq \frac{1 + \frac{1}{N}}{\gamma} + \frac{d}{\gamma q} + \frac{d}{N q \gamma} < 1.
\]

**Case 4:** estimate of the nonlinear term for \( k \geq 1 \) and \( l = 0 \). This case is similar to Case 3 but slightly trickier. The trick is to write
\[
D^k_x \int_{t^{k+1}}^{t} \nabla K(t-r, \cdot) \ast (u \otimes u)(r) \, dr \\
= \int_{t^{k+1}}^{t} D^k_x \Lambda_{-\frac{N-1}{q}} \nabla K(t-r, \cdot) \ast D^{k-1}_x \Lambda_{-\frac{N-1}{q}} (u \otimes u)(r) \, dr.
\]

Now the rest of the proof follows essentially the same line as in Case 3. We have
\[
\left\| t^{\alpha'} \frac{d^{k}}{dt^{k}} \int_{t^{k+1}}^{t} \nabla K(t-r, \cdot) \ast (u \otimes u)(r) \, dr \right\|_{L^q_x L^{\infty}_t} \\
\leq C_2 \sum_{j=0}^{k-1} \binom{k-1}{j} A(j, N-1) A(k-1-j, 0).
\]

Concluding from the above four cases, and by Lemma 4.1, we have the following recurrent inequalities for \( A(k, l) \):

For \( k = 0, l = 0 \),
\[
A(0, 0) \leq C.
\]
For $k \geq 0$, $1 \leq l \leq N$, denote $A(k, N) = A(k + 1, 0)$, and we have

$$A(k, l) \leq C_1(k + 1) n^k A(k, l - 1) + C_1 \sum_{j=0}^{k} \binom{k}{j} A(j, l - 1) A(k - j, 0).$$

Here $C$ and $C_1$ are constants greater than 1. For $k \geq 0$, $1 \leq l \leq N$, $0 \leq j \leq k$, denote $n_1 = Nj + l - 1$, $n_2 = N(k - j)$ and $n = Nk + l$. The following inequality is easy to prove by using Stirling's formula:

$$\binom{k}{j} \leq C_1 \left( \frac{n^n}{n_1^{n_1} n_2^{n_2}} \right)^{\frac{1}{n}}.$$

Then it is not difficult to see that $A(k, l) \leq F(Nk + l)$, where $F(n)$ is a sequence of numbers satisfying

$$F(0) \leq C,$$

and for $n \geq 1$,

$$F(n) = C_1 n^{n+1} F(n - 1) + C_1 \sum_{n_1=0}^{n-1} \frac{n^n}{n_1^{n_1} (n - 1 - n_1)^{(n-1-n_1)/N}} F(n_1) F(n - 1 - n_1).$$

Clearly $F(n) \leq (C_1 C)^{n+1} n^{n/N} G(n)$, where $G(0) = 1$ and

$$G(n) = 2 \sum_{n_1=0}^{n-1} G(n_1) G(n - 1 - n_1).$$

By the method of formal power series, it is easy to show that for some constant $C > 0$,

$$G(n) \leq C^{n+1}.$$

This immediately yields that

$$A(k, l) \leq C^{k+1} k^k.$$

Our theorem is proved.

5. Proof of Corollary 2.6

This section is devoted to the proof of Corollary 2.6. For any $t \in [0, T)$, by Proposition 2.1 (2.2) has a unique solution $\bar{u} \in X_{q, \varepsilon_t}$ with initial data $\bar{u}_0 = u(t, \cdot)$ for some $q \in \left(\frac{d}{\gamma - 1}, \infty\right]$ and $\varepsilon_t \in (0, T - t)$. By Theorem 2.2 $\bar{u}(s, \cdot)$ is spatial analytic for $s \in (0, \varepsilon_t)$. The same is true for $u(t + s, \cdot)$ because of the following uniqueness result of the mild solution to (1.1)-(1.2).
Lemma 5.1. The mild solution to (2.2) in $C([0, T_1], L^d_{x}(\gamma^{-1}))$ is unique for any $T_1 \in (0, \infty]$.

Indeed, for $\gamma = 2$ this lemma is proved in [16]. The proof there can be easily modified to cover the case $\gamma \in (1, 2]$. We leave the details to interested readers.

Denote
\[ \Omega = \{ t \in (0, T) \mid u(t, \cdot) \text{ is not spatial analytic} \}. \]

For any $t \in \Omega$, we choose a rational number $q_t$ in $(t, t + \varepsilon_t)$. It is easy to see that for different such $t_1$ and $t_2$, the corresponding $q_{t_1}$ and $q_{t_2}$ can not coincide with each other. Therefore, $\Omega$ is at most a countable set, and the corollary is proved.

Concluding remark with some minor modifications our method also applies to the periodic boundary condition case. We leave the details to interested readers.

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