MINIMUM WIENER INDEX OF TRIANGULATIONS AND QUADRANGULATIONS

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Abstract. The Wiener index of a connected graph is the sum of the distances between all unordered pairs of vertices. We provide formulae for the minimum Wiener index of simple triangulations and quadrangulations with connectivity at least $c$, and provide the extremal structures, which attain those values. Our main tool is setting upper bounds for the maximum degree in highly connected triangulations and quadrangulations.

1. Definitions

Let $G$ be a connected graph. The Wiener index of $G$, denoted by $W(G)$, is the sum of the distances between all unordered pairs of vertices. In formula,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v),$$

where $d_G(u, v)$ denotes the number of edges on the shortest path between the two vertices $u$ and $v$. This index was introduced in 1947 [6] to predict the boiling point of alkanes. The Wiener index is perhaps the most frequently used graph parameter in the sciences.

Throughout this paper, every graph will be simple, finite and connected unless otherwise stated. For a graph $G$, the sets $V$ and $E$ represent the vertices and edges of $G$, respectively. The order of a graph is its number of vertices. The set of neighbors of the vertex $v$ is denoted by $N(v)$, and the degree of a vertex $v$ is denoted by $d(v) = |N(v)|$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of $G$, respectively. A cutset is a set of vertices, whose removal makes the graph disconnected. A non-complete graph $G$ of order $n \geq 3$ is $c$-connected for a positive integer $c$, if every cutset has size at least $c$; the connectivity $\kappa(G)$ of $G$ is the largest $c$ for which $G$ is $c$-connected. Clearly, if $G$ is not a complete graph, then $G$ has at least $\kappa(G) + 2$ vertices, the smallest cutset of $G$ has size $\kappa(G)$, and all degrees in $G$ are at least $\kappa(G)$. The notation $\simeq$ indicates the isomorphism of two graphs.

In this paper we will only be concerned with planar graphs. Those are the graphs that can be drawn in the plane (or equivalently, in the sphere), such that no edges cross. We will often rely on Euler’s formula, which states that for any finite, connected planar graph $G$ drawn in the plane,

$$n - e + f = 2,$$

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where \( n \) is the order, \( e \) is the number of edges, and \( f \) is the number of faces in \( G \). In this paper, **triangulations** and **quadrangulations** are simple graphs drawn in the sphere, in which every face is a triangle or every face is a quadrangle, respectively. Euler’s formula immediately implies that triangulations of order \( n \) have \( 3n - 6 \) edges and \( 2n - 4 \) faces, and quadrangulations with \( n \) vertices have \( 2n - 4 \) edges and \( n - 2 \) faces. It is well-known that triangulations are 3-connected but Euler’s formula does not allow them to be 6-connected, and quadrangulations are 2-connected but Euler’s formula does not allow them to be 4-connected. Whitney’s theorem [7] implies that all drawings of any 3-connected planar graph on the sphere are the same combinatorially, a conclusion that holds for all classes that we consider in this paper except general quadrangulations. We cite an elegant result of [5], although we do not use it explicitly: every 5-connected triangulation contains a spanning 3-connected quadrangulation. Comparison of Figures 5 and 9 incidentally gives an illustration for this result.

2. Results on Triangulations and Quadrangulations

Recently, there have been numerous results regarding the Wiener index on **triangulations** and **quadrangulations** of the sphere, which are edge maximal simple planar graphs, and edge maximal bipartite simple planar graphs, respectively. These recent results have mainly focused on upper bounds, see [1], [2], [3], [4]. Lower bounds for the Wiener index of such graphs were stated in [1], [2], without making extra assumptions on the connectivity. In this paper we complete the study of the minimum Wiener index of triangulations and quadrangulations, by determining the minimum Wiener index among \( c \)-connected simple triangulations and quadrangulations.

**Theorem 1** ([2]). Assume \( n \geq 6 \). The triangulation \( T_n^4 \) defined in Figure 1 minimizes the Wiener index among all triangulations of order \( n \). The triangulation \( T_n^4 \) is 4-connected. Consequently, the triangulation \( T_n^4 \) minimizes the Wiener index among all 4-connected \( n \)-vertex triangulations as well.

**Remark:** \( T_5^4 \) is the only triangulation of order 5, but it is not 4-connected. Gray vertices and dashed edges in the figures indicate the pattern to be repeated as \( n \) increases.

**Proof.** A triangulation contains \( 3n - 6 \) edges, thus there are exactly \( 3n - 6 \) pairs of vertices at distance 1 apart. If we can make sure that every remaining pair of vertices are at distance 2 apart, then we have a triangulation whose Wiener index is \( 2(\binom{n}{2} - (3n - 6)) + (3n - 6) = n^2 - 4n + 6 \), and this is clearly the minimum possible Wiener index. This is the case with \( T_n^4 \). Furthermore, it is easy to see that \( T_n^4 \) is 4-connected for all \( n \geq 6 \).

Triangulations and 4-connected triangulations fail to produce unique structures to minimize the Wiener index. With the aid of a computer, we evaluated the number of non-isomorphic triangulations of minimum Wiener index up to order 18, and the number of non-isomorphic 4-connected triangulations of minimum Wiener index up to order 22, see Table 1. Considering the large numbers in the Table 1, the classification of extremal structures seems hopeless. As will be shown, there are two 5-connected triangulations on 19
Figure 1. The triangulation $T_n^4$, which is the join of the cycle $C_{n-2}$ with the edgeless graph on two vertices, minimizes the Wiener index among all triangulations of order $n \geq 5$ and are 4-connected for $n \geq 6$.

Table 1. A summary of how many isomorphism classes, on $n$ vertices, attain the minimum Wiener index for general and 4-connected triangulations.

| Order | General Triangulation Count | 4-Connected Triangulation Count |
|-------|-----------------------------|---------------------------------|
| 4     | 1                           | 0                               |
| 5     | 1                           | 0                               |
| 6     | 2                           | 1                               |
| 7     | 5                           | 1                               |
| 8     | 12                          | 2                               |
| 9     | 36                          | 4                               |
| 10    | 99                          | 6                               |
| 11    | 255                         | 10                              |
| 12    | 614                         | 10                              |
| 13    | 1532                        | 14                              |
| 14    | 3908                        | 15                              |
| 15    | 10727                       | 19                              |
| 16    | 31242                       | 21                              |
| 17    | 96725                       | 25                              |
| 18    | 311735                      | 27                              |
| 19    |                              | 32                              |
| 20    |                              | 34                              |
| 21    |                              | 39                              |
| 22    |                              | 42                              |

vertices that minimize the Wiener index. All other graph classes studied in this paper will produce unique extremal graphs minimizing the Wiener index.

Theorem 2 ([1],[2]). Assume $n \geq 4$. The complete bipartite graph $K_{2,n-2}$ minimizes the Wiener index among all quadrangulations.

Proof. A quadrangulation contains $2n - 4$ edges, thus exactly $2n - 4$ pairs of vertices are at distance 1 apart. If we can make sure that every remaining pair of vertices are at distance 2 apart, we have a quadrangulation of Wiener index $2\left(\binom{n}{2} - (2n - 4)\right) + (2n - 4) = n^2 - 3n + 4$. 
This is the case with the quadrangulation $K_{2,n-2}$. Clearly this is the least possible Wiener index of a quadrangulation.

**Theorem 3.** Assume $n \geq 4$. Up to isomorphism, the graph $K_{2,n-2}$ is the unique minimizer of the Wiener index among all quadrangulations of order $n$.

**Proof.** Let $Q$ be a quadrangulation of order $n$ that has the same Wiener index as $K_{2,n-2}$, i.e. every non-adjacent pair of vertices are at distance 2. As quadrangulations are 2-connected, the minimum degree $\delta := \delta(Q) \geq 2$. Let $v$ be a vertex of $Q$ with $d(v) = \delta$, and let $u_1, \ldots, u_\delta$ be the neighbors of $v$. The remaining $n - \delta - 1$ vertices are at distance 2 from $v$. As quadrangulations are bipartite, these $n - \delta - 1$ vertices can only be adjacent to $u_1, \ldots, u_\delta$, and have degree at least $\delta$. Thus we get that $Q \simeq K_{\delta,n-\delta}$. Since $\delta$ is the minimum degree, $\delta \leq n - \delta$, therefore $Q$ contains $K_{\delta,\delta}$ as a subgraph. Since $Q$ is planar, we get $\delta = 2$ and $Q \simeq K_{2,n-2}$. □

**Figure 2.** The graph $K_{2,n-2}$ which minimizes the Wiener index among all quadrangulations of order $n$.

**Theorem 4.** Assume that $n \geq 8$, $n \neq 9$. The minimum Wiener index of 3-connected quadrangulations of order $n$ is

$$4 \left\lfloor \frac{n}{2} \right\rfloor^2 + \left( \left\lfloor \frac{n}{2} \right\rfloor + 21 \right) \left( \left\lfloor \frac{n}{2} \right\rfloor - 21 \right) - 5n + 449 = \begin{cases} \frac{5n^2}{4} - 5n + 8, & \text{if } n \text{ is even,} \\ \frac{5n^2}{4} - 3n - \frac{49}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The unique minimizer of the Wiener index among 3-connected quadrangulations of order $n$ is $Q_n^3$, defined in Figure 5.

The proof of this theorem is in Section 3, combining Lemma 11[e] and Theorems 12 and 13. No 3-connected quadrangulation exists of order $n \leq 7$ by Euler’s formula, and of order $n = 9$ by Lemma 7[c].

**Theorem 5.** Assume that $n \geq 12$, $n \neq 13$. The minimum Wiener index of 5-connected triangulations of order $n$ is

$$2n \left\lfloor \frac{n}{2} \right\rfloor + \left( \left\lfloor \frac{n}{2} \right\rfloor + 14 \right) \left( \left\lfloor \frac{n}{2} \right\rfloor - 14 \right) - 7n + 208 = \begin{cases} \frac{5n^2}{4} - 7n + 12, & \text{if } n \text{ is even,} \\ \frac{5n^2}{4} - 6n - \frac{9}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The unique minimizer of the Wiener index among 5-connected triangulations of order $n \neq 19$ is $T_n^5$, defined on Figure 9, while for $n = 19$, exactly two minimizers exist, namely $T_{19}^5$, and the 5-connected triangulation $X$ of order 19, defined on Figure 10.
The proof of this theorem is in Section 4 combining Lemma 18(e), Lemma 10(e) and Theorems 20 and 24. No 5-connected triangulation exists of order \( n \leq 11 \) by Euler’s formula, and of order \( n = 13 \) by Lemma 17.

3. Minimum Wiener Index of 3-Connected Quadrangulations

Note that Euler’s formula implies that there are no 3-connected quadrangulations on fewer than 8 vertices.

First, we define an auxiliary drawn graph, which we will use extensively in this section. Let \( v \) be a vertex of a 3-connected quadrangulation \( G \). We define the sunflower graph \( S_v \) around \( v \) (in the planar drawing of \( G \)), as \( v \) connected to its neighbors \( u_1, \ldots, u_d \) (listed in the cyclic order of the drawing, \( d = d(v) \)), and different vertices \( w_1, \ldots, w_d \) where \( w_i \) is connected to \( u_i \) and \( u_{i+1} \) (indices taken modulo \( d \), see Figure 3). We understand \( S_v \) as a part of the drawing of \( G \).

First we need to show that such a graph, with distinct vertices, exists in the drawing. We will also need some special properties of the sunflower graph, which will be shown in Lemma 6 below.

**Figure 3.** The sunflower graph \( S_v \) around \( v \) with \( d(v) = 8 \). The region \( R_v \) is shaded.

**Lemma 6.** Assume that \( Q \) is a drawing of a 3-connected quadrangulation. Then, for any vertex \( v \), \( Q \) contains a sunflower graph \( S_v \) with \( 2d(v) + 1 \) distinct vertices. Furthermore, the region \( R_v \) that contains \( v \) and is bounded by the cycle \( C_v = u_1w_1 \ldots u_{d(v)}w_{d(v)} \) contains no vertices or edges that are not in \( S_v \).

**Proof.** We know \( \delta(Q) \geq 3 \) by the 3-connectedness. Label the neighbors of \( v \) by \( u_1, \ldots, u_d \), in their planar cyclic order around \( v \). For each pair of successive neighbors \( u_i \) and \( u_{i+1} \) (indices taken modulo \( d \)), let \( w_i \neq v \) be their common neighbor that completes the face \( f_i \) that has \( u_i, v, u_{i+1} \) on its boundary. This means, in particular, that the interior of \( f_i \) has no vertices or edges. If \( y \) is a neighbor of \( u_i \) and \( y \notin \{v, w_{i-1}, w_i\} \) then \( y \) must lie between \( w_{i-1} \) and \( w_i \) in the planar cyclic order around \( u_i \), in particular, \( w_{i-1} \neq w_i \) as \( d(u_i) \geq 3 \). As \( Q \) is bipartite, \( u_i \neq w_j \) for all \( 1 \leq i, j \leq d \). We will show that each of the \( w_i \)'s must be
Assume that \( \text{Lemma 7.} \) distinct. As \( \mathcal{R}_v \) is the union of the faces \( f_i \), this finishes the proof. Assume that \( w_i = w_j \) for some \( j \neq i \). We already know that \( j \notin \{i - 1, i + 1\} \) and the vertices \( u_i, u_{i+1}, u_j, u_{j+1} \) are all different. We consider two regions of the planar drawing of \( Q \): \( \mathcal{R}_1 \) is bounded by the 4-cycle \( u_{i+1}vu_{i+1}w_i \) and does not contain the vertex \( u_i \), and \( \mathcal{R}_2 \) is bounded by the 4-cycle \( u_iwu_{j+1}w_j \) and does not contain the vertex \( u_{i+1} \). Thus the faces bounded by \( u_iwu_{i+1}w_i \) and \( u_jwu_{j+1}w_j \) are disjoint from \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). The neighbors of \( u_i \) that differ from \( v \) and \( w_i \) must lie in \( \mathcal{R}_2 \) and the neighbors of \( u_j \) that differ from \( v \) and \( w_i \) must lie in \( \mathcal{R}_1 \). Hence \( \{v, w_i\} \) separates \( u_i \) from \( u_j \) (See Figure 4), contradicting the fact that \( Q \) is 3-connected. \( \square \)

![Figure 4](image_url)

**Figure 4.** 2-element cutset appears in \( S_v \), when \( w_i = w_j \). The two faces \( vu_iwu_{i+1} \) and \( vu_jwu_{j+1} \) are shaded.

**Lemma 7.** Assume that \( G \) is a 3-connected quadrangulation with partite sets \( A, B \). Then

(a) \( \Delta(G) \leq \min\{|A| - 1, |B| - 1\} \);
(b) If \( |B| < |A| \), then for all \( x \in A \) we have \( d(x) \leq |B| - 2 \); and
(c) \( |V(G)| \neq 9 \), i.e., no 3-connected quadrangulations exist on 9 vertices.

**Proof.** (a) Let \( v \) be a vertex with degree \( \Delta = \Delta(G) \), we may assume \( v \in B \). As the sunflower \( S_v \) is a subgraph of the planar drawing of \( G \), \( \Delta \leq \min(|B| - 1, |A|) \). We are done unless \( |B| > |A| = \Delta \), so assume that is the case. As \( A = N(v) \), all neighbors of the vertices of \( B \) lie in \( N(v) \), in particular, every \( w_i \) has at least 3 neighbors in \( A \). For each \( i \) let \( k_i \) be the largest positive integer such that \( w_i \) has no neighbors in the set \( \{u_{i-t} : 1 \leq t \leq k_i - 1\} \cup \{u_{i+t} : 1 \leq t \leq k_i - 1\} \). Since for \( k = 1 \) the sets \( \{u_{i-t} : 1 \leq t \leq k - 1\} \) and \( \{u_{i+t} : 1 \leq t \leq k - 1\} \) are empty, such positive integers exist, they have an upper bound from the fact that \( w_i \) has at least 3 neighbors in \( A \), and for the largest such integer \( k_i \) we have that at least one of \( u_i, u_{i+1}, u_{i+1+k_i} \) is a neighbor of \( w_i \) that is different from \( u_i, u_{i+1} \). Choose \( i_0 \) such that \( k = k_{i_0} \) is minimal amongst the \( k_i \). By renumbering the \( u_i \) if necessary and changing the direction of the cyclic order we can assume that \( i_0 = 1 \) and \( w_1 \) is connected to \( u_1, u_2, u_{2+k} \) but none of \( u_{1-t}, u_{2+t} \) for all \( 1 \leq t \leq k - 1 \). Let \( \mathcal{R} \) be the region of the sphere bounded by the 4-cycle \( w_2vu_{2+k}w_1 \) that does not contain \( u_1 \). Consider \( w_2 \). By the definition of \( w_2 \) and the minimality of \( k \), \( w_2 \) lies in \( \mathcal{R} \) and it has at least one neighbor \( u_j \) that does not lie in \( \mathcal{R} \). The edge \( w_2u_j \) must cross the boundary of \( \mathcal{R} \), which contradicts the planarity of \( G \). Thus, we have \( \Delta(G) \leq \min\{|A| - 1, |B| - 1\} \), as claimed.
To prove the case (b), assume $|B| < |A|$, i.e. $n > 2|B|$. We already know that $\Delta(G) \leq |B| - 1$. Assume that $A$ contains a vertex $v$ of degree $|B| - 1$. Since $|A| = n - |B|$ and all other vertices of $A$ have degree at least 3, we have that $2n - 4 \geq (|B| - 1) + 3(n - |B| - 1) = 3n - 2|B| - 4$, so $n \leq 2|B|$, a contradiction.

To prove the case (c), assume to the contrary that $G$ has 9 vertices and partite sets $A, B$. We may assume $|B| < |A|$, and therefore $|B| \leq 4$. Then every vertex in $A$ has degree at most 2, a contradiction. □

Lemma 8. In a 3-connected quadrangulation $G$ of order $n$, the number of unordered pairs of vertices at distance 2 is at most
$$\frac{1}{2} \sum_v d^2(v) - 4(n - 2).$$
This estimate is exact precisely when $G$ has no non-facial 4-cycles.

Proof. Euler’s Formula gives us that any quadrangulation on $n$ vertices has $2(n - 2)$ edges and $n - 2$ faces. The number of 2-paths in $G$ is equal to $\sum_v \binom{d(v)}{2} = \frac{1}{2} \sum_v d^2(v) - 2(n - 2)$. This sum, however, overcounts the number of pairs of vertices distance 2 apart. In a 3-connected quadrangulation, two faces cannot share two consecutive edges from their boundaries. Thus, for each face, we are double counting the two pairs of vertices distance 2 apart, and so we may safely subtract $2(n - 2)$. There are pairs of vertices which we have double counted even after the substraction precisely when there are non-facial 4-cycles. □

Figure 5. The quadrangulation $Q^3_n$ of order $n = 2k \geq 8$ (left) and $n = 2k + 1 \geq 11$ (right), which minimizes the Wiener index among all 3-connected quadrangulations of order $n$. Gray vertices and dashed edges indicate the pattern to be repeated. The light gray regions are the sunflower graphs around a maximum degree vertex.

Lemma 9. Let $Q$ be a 3-connected quadrangulation of order $n$, with partite sets $A, B$. Then
$$W(Q) \geq 2n^2 + 2n - 8 - |A||B| - \sum_v d^2(v).$$
Equality holds in (1) precisely when the diameter of $Q$ is at most 4 and $Q$ has no non-facial 4-cycles.
Proof. Let $Q$ be an arbitrary 3-connected quadrangulation on $n$ vertices, with partite sets $A, B$. Let $D_i$ denote the number of unordered pairs of vertices at distance $i$ in $Q$. Clearly $W(Q) = \sum_i i \cdot D_i$. Observe that $D_1 = 2n - 4$, the number of edges; $D_2 \leq \frac{1}{2} \sum_v d^2(v) - 4(n - 2)$ by Lemma 8, $D_2 + D_4 + D_6 + D_8 + \cdots = \binom{|A|}{2} + \binom{|B|}{2}$, as pairs of vertices are at even distance precisely when they are from the same partite set; and finally, $D_1 + D_3 + D_5 + D_7 + \cdots = |A| \cdot |B|$, as pairs of vertices are at odd distance precisely when they are from different partite sets.

Combining all this information with the identity $|A| + |B| = n$, we obtain that

\[
W(Q) \geq (2n - 4) + 2D_2 + 3\left(|A| \cdot |B| - (2n - 4)\right) + 4\left(\binom{|A|}{2} + \binom{|B|}{2} \right) - D_2
\]

\[
= 2n^2 - 2n - |A||B| - 2(D_2 + 2n - 4)
\]

\[
\geq 2n^2 + 2n - 8 - |A||B| - \sum_v d^2(v).
\]

The first inequality in the displayed formula is an equality precisely when the diameter of $Q$ is at most 4, and the second inequality is an equality precisely when $Q$ has no non-facial 4-cycles. \hfill \square

The 3-connected quadrangulation $Q_n^3$ of order $n \geq 8$, $n \neq 9$ is defined in Figure 5. The following lemma is easy to verify, and we leave the details to the reader.

**Lemma 10.** Assume that $n \geq 8$, $n \neq 9$.

(a) $Q_n^3$ is a 3-connected quadrangulation.

(b) $Q_n^3$ has no non-facial 4-cycle.

(c) If $n$ is even, $Q_n^3$ has diameter 3 and degree sequence $\left\lfloor \frac{n}{2} \right\rfloor - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1, 3, \ldots, 3$.

(d) If $n$ is odd, $Q_n^3$ has diameter 4 and degree sequence $\left\lfloor \frac{n}{2} \right\rfloor - 1, \left\lfloor \frac{n}{2} \right\rfloor - 2, 4, 4, 3, \ldots, 3$.

(For $n = 11$, the terms in this sequence are not in decreasing order.)

(e) $W(Q_n^3) = \begin{cases} 
2n^2 - 5n + 8, & \text{if } n \text{ is even}, \\
2n^2 - 3n - \frac{49}{4}, & \text{if } n \text{ is odd}.
\end{cases}$

The following is obvious, and we make use of it frequently.

**Lemma 11.** Assume that $\sum_{i=1}^{n} x_i = a > 0$ is given, where the $x_i$’s are required to be integers from the interval $[b, c]$ with $0 \leq b$, and we have to maximize $\sum_{i=1}^{n} x_i^2$. As long as for some $(i \neq j)$ we have $b + 1 \leq x_j \leq x_i \leq c - 1$, we can increase the sum of squares while keeping the conditions by changing $x_i$ to $x_i + 1$ and $x_j$ to $x_j - 1$.

**Theorem 12.** Assume that the number $n \geq 8$ is even. The quadrangulation $Q_n^3$ defined in Figure 5 minimizes the Wiener index among all 3-connected quadrangulations of order $n$. Moreover, up to isomorphism, this minimizer is unique.

**Proof.** Let $Q$ be an arbitrary 3-connected quadrangulation on $n = 2k$ vertices, with partite sets $A, B$. Since $Q$ is 3-connected, for all $v$, we have $d(v) \geq 3$, and by Lemma 7, $d(v) \leq \Delta(Q) \leq \min(|A| - 1, |B| - 1) \leq \frac{n}{2} - 1$. By Lemma 11 and Lemma 10(c), $\sum_{v \in V(Q)} d^2(v) \leq$
\[ \sum_{v \in V(Q^3_n)} d^2(v) \] with equality precisely when \( Q \) has the same degree sequence as \( Q^3_n \). Also, 
\[ |A| \cdot |B| \leq \frac{|Q|^2}{4} \] with equality precisely when \( |A| = |B| = \frac{|Q|}{2} \). Lemma 9 gives that \( W(Q) \geq W(Q^3_n) \) with equality precisely when \( Q \) has the same degree sequence as \( Q^3_n \), \(|A| = |B| = \frac{|Q|}{2}\), \( Q \) has diameter at most 4 and no nonfacial 4-cycles. In particular, \( Q^3_n \) minimizes the Wiener index among \( n \)-vertex 3-connected quadrangulations.

We will show that the extremal quadrangulation is in fact unique. Assume that \( W(Q) = W(Q^3_n) \), so \( Q \) has the same degree sequence as \( Q^3_n \) and \(|A| = |B| = k \) vertices. Then in both \( A \) and \( B \) we have \( k - 1 \) vertices of degree 3, and the remaining one vertex must have degree \( k - 1 \) \((k - 1 \geq 3)\).

As before, let \( v \) be a vertex of maximum degree \( k - 1 \), and construct the sunflower graph \( S_v \) around \( v \). Since \( S_v \) has exactly \( n - 1 \) vertices, \( Q \) has one additional vertex \( v' \). This vertex \( v' \) is in the same partite class as the \( u_i \) vertices, and differs from \( v \). Each of the \( w_i \) has one edge not in \( S_v \) incident upon it, connecting them to either \( v' \) or one of the \( u_j \). If all vertices \( u_i \) have degree 3, then \( v' \) has degree \( k - 1 \geq 3 \), and it is adjacent to all \( w_i \) (in which case we have \( Q^3_n \)). Otherwise the degree of \( v' \) is 3 and exactly one of the \( u_i \) (say \( u_2 \) has degree \( k - 1 > 3 \) in \( Q \). Assume that the latter is the case. As \( w_1 \) and \( w_2 \) have an edge not in \( E(S_v) \) incident upon them, and \( w_1u_2, w_2u_2 \in E(S_v) \), both \( w_1 \) and \( w_2 \) are adjacent to \( v' \). Thus, \( v'w_1u_2w_2 \) bounds a facial region \( R \). As \( w_1u_2w_2 \), of which \( u_2 \) is an internal vertex, is the common boundary of \( R_v \) and \( R \), \( u_2 \) cannot have any edge outside of \( S_v \) incident upon it, a contradiction. \( \square \)

**Lemma 13.** Assume \( n = 2k + 1 \), and let \( Q \) be a 3-connected quadrangulation of order \( n \), with partite sets \( A, B \). If

\[ \sum_{v \in V(Q)} d^2(v) < 2k^2 + 12k + 10, \]

then \( W(Q) > W(Q^3_n) \). If \( \Delta(Q) \leq k - 2 \), then \( W(Q) > W(Q^3_n) \).

**Proof.** First note that by Lemma [III][d]

\[ \sum_{x \in V(Q^3_n)} d^2(x) = (k - 1)^2 + (k - 2)^2 + 2 \cdot 4^2 + 3^2(2k - 3) = 2k^2 + 12k + 10, \]

and also \(|A| \cdot |B| \leq k(k + 1) \) (note that \( k, k + 1 \) are the sizes of the partite classes in \( Q^3_n \)), so if (2) holds, then by Lemma 9 and Lemma 10[d] we have \( W(Q) > W(Q^3_n) \).

Since \( Q \) has odd number of vertices and minimum degree 3, the Handshaking Lemma implies \( \Delta(Q) \geq 4 \). Assume now that \( \Delta(Q) \leq k - 2 \), so \( k \geq 6 \). Let \( x_1, \ldots, x_{2k+1} \) be a sequence of integers that maximizes \( \sum x_i^2 \) subject to the conditions that that \( \sum x_i = 4(n - 2) \) and \( 3 \leq x_i \leq k - 2 \). If \( k = 6 \), the maximizing sequence is 4, 4, 4, 4, 4, 3, \ldots, 3 of length 13, and if \( k = 7 \) the maximizing sequence by Lemma [I] is 5, 5, 5, 4, 3, \ldots, 3 of length 15. In both of these cases, we have \( \sum x_i^2 < 2k^2 + 12k + 10 \). For \( k \geq 8 \), Lemma [I] gives that the maximizing sequence is \( k - 2, k - 2, 6, 3, 3, \ldots, 3 \), so

\[ \sum_{x \in V(Q)} d^2(x) \leq \sum_{i=1}^{2k+1} x_i^2 = 2(k - 2)^2 + 6^2 + 3^2(2k - 2) = 2k^2 + 10k + 26 \leq 2k^2 + 12k + 10. \]
Therefore \( W(Q) > W(Q_3^n) \) unless \( k = 8 \) and the degree sequence of \( Q \) is 6, 6, 6, 3, 3, \ldots, 3.

So for the rest of this proof \( k = 8 \), the degree sequence of \( Q \) is 6, 6, 6, 3, 3, \ldots, 3 and is of length 17. By Lemma \( 9 \) if \( W(Q) \leq W(Q_3^n) \), then \( W(Q) = W(Q_1^2) \), the diameter of \( Q \) is 4, \( Q \) has no nonfacial 4-cycles, \(|A| = 9\) and \(|B| = 8\). We will show that such a \( Q \) does not exist, which finishes the proof.

Because the sum of the degrees of the vertices in each partite class must be the same (in this case, 30), \( B \) contains exactly two of the degree 6 vertices. Let \( v \in B \) with degree 6, consider the sunflower \( S_v \), and label the 4 vertices outside \( S_v \) by \( x, y_1, y_2, y_3 \) such that \( B = \{v, w_1, \ldots, w_6, x\} \) and \( A = \{u_1, \ldots, u_6, y_1, y_2, y_3\} \). Since \( d(x) \in \{3, 6\} \), \( N(x) \subseteq A \) and at most one of the \( u_i \) has an edge not from \( S_v \) incident upon it, we have \( d(x) = 3 \) and without loss of generality \( y_1, y_2 \in N(x) \).

Assume first that \( N(x) = \{y_1, y_2, y_3\} \) and consider the sunflower \( S_x \). Let \( j_i \) be chosen such that \( w_{ji} \) is the common neighbor of \( y_i \) and \( y_{i+1} \) (indices taken modulo 3) in \( S_x \). Then each of the \( w_{ji} \) are different and have degree at least 4 in \( Q \), a contradiction.

So we can assume without loss of generality that \( N(x) = \{u_1, y_1, y_2\} \). Then the unique degree 6 vertex in \( A \) is \( u_1 \), so there are two different indices \( t_1 \) and \( t_2 \) such that \( w_{ti} \in N(u_1) \setminus \{w_1, w_6\} \). For \( i \in \{1, 2\} \) let \( z_i \) be the common neighbor of \( u_1 \) and \( y_i \) in the sunflower \( S_x \), and let \( z_3 \in \{w_1, \ldots, w_6\} \setminus \{z_1, z_2\} \). In particular, the degree of \( z_3 \) in \( Q \) is at least 4, therefore \( z_1 \) and \( z_2 \) must have degree 3 in \( Q \). If \( w_{ti} \in \{z_1, z_2\} \), then \( w_{ti} \) has degree at least 4 and consequently degree 6. This gives \( \{z_1, z_2\} \cap \{w_{ti}, w_{tj}\} = \emptyset \). Therefore without loss of generality \( w_1 = z_1 \), \( w_6 = z_2 \) and the \( u_1w_{ti} \) edges cannot run inside \( R_x \) or any of the faces bounded the 4-cycles \( u_1w_1y_1xu_1 \) and \( u_1w_6y_2xu_1 \), which leaves them no place to be, a contradiction.

**Theorem 14.** Assume that the number \( n \geq 11 \) is odd. The quadrangulation \( Q_3^n \) in Figure 5 minimizes the Wiener index among all 3-connected quadrangulations of order \( n \). Moreover, up to isomorphism, this minimizer is unique.

**Proof.** Let \( n = 2k + 1 \) and assume that \( Q \) is a 3-connected quadrangulation on \( n \) vertices of minimum Wiener index, and with partite sets \( A, B \), such that \(|A| > |B|\).

First we want to show that \(|A| = k + 1\), \(|B| = k\), and the degree sequence of \( Q \) restricted to the partite sets is the same as the degree sequence of \( Q_3^n \) restricted to its partite sets.

We have \(|A| \geq k + 1 \) and \(|B| \leq k\). Lemma \( 7[a] \) gives \( \Delta(Q) \leq |B| - 1 \leq k - 1 \). Lemma \( 13 \) gives \( \Delta(Q) = k - 1 \), which in turn shows \(|B| = k \) and \(|A| = k + 1 \). In addition, if \( d(v) = k - 1 \), Lemma \( 7[b] \) gives \( v \in B \).

As the degree sequence of quadrangulations is unique for \( n = 11 \) under the condition that every degree is 3 or 4, we may assume now that \( n \geq 13 \), i.e., \( k \geq 6 \). As the sum of the \( k \) degrees in \( B \) is the number of edges \( 2n - 4 = 4k - 2 \), and every degree is at least \( 3 \), only two degree sequences are possible for \( B \): \( (k - 1, 5, 3, 3, \ldots, 3) \) or \( (k - 1, 4, 4, 3, \ldots, 3) \). We claim that \( \Delta(A) \), the maximum degree of a vertex in \( A \) is \( k - 2 \). Lemma \( 7[b] \) showed \( \Delta(A) \leq |B| - 2 = k - 2 \).
Assume for contradiction that \( \Delta(A) \leq k - 3 \). Since the minimum degree is at least 3 and \( \sum_{x \in A} d(x) = 4k - 2 \), for \( k = 6 \) we get that \( 3 \cdot 7 = 4 \cdot 6 - 2 \), a contradiction. Therefore we have that \( k \geq 7 \),

\[
\sum_{x \in A} d^2(x) \leq (k - 3)^2 + 4^2 + 3^2(k - 1) = k^2 + 3k + 16,
\]

and

\[
\sum_{x \in V(Q)} d^2(x) \leq k^2 + 3k + 16 + (k - 1)^2 + 5^2 + 3^2(k - 2) = 2k^2 + 10k + 24.
\]

By Lemma \( [13] \) \( W(Q) > W(Q^3_n) \) when \( k \geq 8 \) so we may assume that \( k = 7 \). In particular, for \( k \geq 8 \) the degree sequence of \( A \) is \( (k - 2, 3, \ldots, 3) \).

If \( k = 7 \), the degree sequence of \( A \) is \( (4, 4, 3, 3, 3, 3, 3, 3) \) and the degree sequence of \( B \) is \( (6, 4, 4, 3, 3, 3, 3, 3) \) then \( \sum_{x \in V(Q)} d^2(x) = 190 < 192 = 2 \cdot 7^2 + 12 \cdot 7 + 10 \), and Lemma \( [13] \) contradicts the minimality of the Wiener index of \( Q \).

Hence the only case that remains to be checked is when \( k = 7 \), the degree sequence of \( A \) is \( (4, 4, 3, 3, 3, 3, 3) \) and the degree sequence of \( B \) is \( (6, 5, 3, 3, 3, 3, 3) \). In this case \( \sum_{x \in V(Q)} d^2(x) = 192 = \sum_{x \in V(Q_{10})} d^2(x) \), so by Lemma \( [9] \) and Lemma \( [10] \) \( (a) \) \( (d) \) the minimality of \( W(Q) \) implies that \( Q \) has no nonfacial 4-cycles. Let \( v \in B \), \( d(v) = 6 \) and consider the sunflower \( S_v \). Let \( x, y \) be the vertices outside \( S_v \). Then \( B = \{v, w_1, \ldots, w_6\} \) and \( A = \{u_1, \ldots, u_6, x, y\} \), without loss of generality \( d(w_1) = 5 \), and the rest of the \( w_i \) have degree 3. Therefore there is an \( i \in \{3, 4, 5, 6\} \) such that \( w_i \) is adjacent to \( u_i \). Since \( w_1 \) and \( u_i \) cut \( C_v \) into two paths, one contains \( w_2 \) and the other \( w_6 \), the vertices \( w_2 \) and \( w_6 \) lie inside two different regions bounded by the 4-cycle \( w_1u_jvu_1w_1 \). As this cycle is nonfacial, we have a contradiction.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sunflower_graph.png}
\caption{The quadrangulation \( Y \) of order 13 with Wiener index 164. The gray region shows the sunflower around a maximal degree vertex. The white vertices and the dotted edges form one of the non-facial 4-cycles.}
\end{figure}

So we have that the degree sequence of \( Q \) restricted to \( A \) is the same as the degree sequence of \( Q^3_n \) restricted to its \( A \), i.e. \( (k - 2, 3, \ldots, 3) \). We need to figure out what the degree sequence of \( Q \) restricted to \( B \) is. Assume \( k \geq 6 \), and \( B \) has degree sequence \( (k - 1, 5, 3, 3, \ldots, 3) \). Referring to the sunflower graph \( S_v \) at vertex \( v \), where \( d(v) = k - 1 \), we have \( B = \{v, w_1, \ldots, w_{k-1}\} \) and \( A = \{u_1, \ldots, u_{k-1}, x, y\} \). We can assume without loss of generality that \( w_1 \in B \) has degree 5, and for \( i : 2 \leq i \leq k - 1 \) set \( z_i \) be the unique
vertex in \( N(w_i) \setminus \{u_i, u_{i+1}\} \). Since \( w_1 \) is adjacent to 3 vertices of \( A \setminus \{u_i, w_2\} \), it is adjacent to at least one (and at most three) vertices in \( \{u_i : 3 \leq i \leq k - 1\} \), and consequently these vertices have degree at least 4. As \( A \) has a single vertex with degree more than 3, we conclude that there is a unique \( 3 \leq j \leq k - 1 \) that \( w_1 \) is adjacent to \( u_j \), \( d(u_j) = k - 2 \) and \( d(x) = d(y) = 3 \), and \( w_1 \) is adjacent to \( x \) and \( y \). In addition, for \( i : 2 \leq i \leq k - 1 \) we have that \( z_i \in \{x, y, u_j\} \); in particular \( z_{j-1}, z_j \in \{x, y\} \). We may assume without loss of generality that \( z_{j-1} = x \).

Let \( \mathcal{P} \) be the region bounded by \( C_v \) that is different from \( R_v \), and let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be the two subregions that the edge \( w_1 u_j \) cuts \( \mathcal{P} \) into; without loss of generality the boundary of \( \mathcal{R}_1 \) is the cycle \( w_1 u_2 w_2 u_3 \ldots u_j \). Now \( \mathcal{R}_1, \mathcal{R}_2 \) and \( R_v \) share only vertices on the boundary, and the common boundary of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) is the edge \( u_j w_1 \). \( \mathcal{R}_1 \) has \( j \geq 2 \geq 1 \) vertices \( w_2, \ldots, w_{j-1} \) from \( B \setminus \{w_1\} \) on its common boundary with \( R_v \), and for \( i : 2 \leq i \leq j - 1 \) the vertex \( z_i \) lies in \( \mathcal{R}_1 \) (inside or on the boundary). Since \( z_{j-1} = x \), \( x \) is inside \( \mathcal{R}_1 \). Let \( Q \) be the subregion of \( \mathcal{R}_1 \) bounded by the cycle \( w_1 u_2 w_2 \ldots u_{j-1} w_{j-1} x w_1 \). Then for \( i : 2 \leq i \leq j - 2 \) the vertex \( z_i \) lies in \( Q \), so \( z_i \in \{x, y\} \). \( \mathcal{R}_2 \) has \( k - j \geq 1 \) vertices \( w_j, w_{j+1}, \ldots, w_{k-1} \) from \( B \setminus \{w_1\} \) on its common boundary with \( R_v \), and for \( i : j \leq i \leq k - 1 \), \( z_i \) lies in \( \mathcal{R}_2 \) (inside or on the boundary). Since \( z_j \in \{x, y\} \) and \( x \) is inside \( \mathcal{R}_1 \), this implies that \( z_j = y \), \( y \) is inside \( \mathcal{R}_2 \), for all \( i : j \leq i \leq k - 1 \) we have \( z_i \in \{u_j, y\} \) and for all \( i : 2 \leq i \leq j - 1 \) we have \( z_i = x \). Similar logic as before gives that for all \( i : j \leq i \leq k - 1 \) we have \( z_i = y \) since \( d(x) = d(y) = 3 \), this means \( 3 = j - 1 = k - j + 1 \), so \( j = 4 \) and \( k = 6 \). We have \( Q \simeq Y \) (see Figure 6) and \( W(Q) = 164 > 160 = W(Q_{3n}^3) \), a contradiction.

For the rest of the proof we assume that \( n \geq 11 \), so \( k \geq 5 \). The integer sequence that maximizes the sum of squares, and satisfies the conditions we have for the degree sequence of \( G \) in \( A \) (respectively \( B \)) is \( k - 2, 3, \ldots, 3 \) (respectively \( k - 1, 4, 4, 3, \ldots, 3 \)), the degree sequence of \( Q_{3n}^3 \). Since \( Q_{3n}^3 \) is a 3-connected quadrangulation with diameter at most 4, this shows that \( W(Q_{3n}^3) \) is minimal, and the degree sequence of \( Q \) is the same as the degree sequence of \( Q_{3n}^3 \), and furthermore, the degree sequences of their respective partite sets are the same. Last, we need to show that \( Q \simeq Q_{3n}^3 \).

Let \( v \in Q \) with \( d(v) = k - 1 \), and consider the sunflower \( S_v \) around \( v \). Let \( x, y \) be the vertices of \( Q \) not in \( S_v \). Then \( B = \{v, u_1, w_2, \ldots, u_{k-1}\} \), \( A = \{u_1, \ldots, u_{k-1}, x, y\} \), and without loss of generality the two vertices of degree 4 in \( A \) are \( w_1 \) and \( w_j \).

If every \( u_i \) has degree 3 (this must happen in particular when \( k = 5 \) and vertices of \( A \) all have degree 3), then none of the \( u_i \) has a neighbor outside of \( S_v \). In this case \( w_1 \) and \( w_j \) must both be adjacent to \( x \) and \( y \). Without loss of generality the region \( R \) bounded by the cycle \( x w_1 u_2 w_2 \ldots u_j w_j x \) that does not contain \( v \) contains \( y \). (Otherwise we exchange the name of \( x \) and \( y \)). If \( j = k - 1 \), then \( x \) is on the interior of the 4-cycle \( y w_{k-1} u_1 w_1 y \) that does not contain \( v \), and the degree of \( x \) can only be 2, which is a contradiction. If \( j = 2 \), then \( R \) is bounded by a 4-cycle and \( y \) can have only degree 2, a contradiction. So \( 3 \leq j \leq k - 2 \), the \( k - 1 - j \geq 1 \) vertices \( w_{j+1}, w_{j+2}, \ldots, w_{k-1} \) must have \( x \) as their third neighbor, and the the \( j - 2 \geq 1 \) vertices \( w_2, w_3, \ldots, w_{j-1} \) must have \( y \) as their third neighbor. So \( \{d(x), d(y)\} = \{k + 1 - j, j\} = \{3, k - 2\} \), which gives \( j \in \{3, k - 2\} \). This is precisely the graph \( Q_{3n}^3 \).
Now let \( i \) be chosen so \( u_i \) have degree greater than 3. As all but one of the vertices of \( A \) have degree 3, for \( j \neq i \) we have \( d(u_j) = 3 \), \( u_j \) has the same neighbors in \( Q \) and \( S_v \), \( d(u_i) = k - 2 \), \( d(x) = d(y) = 3 \), and \( k \geq 6 \). This means that \( u_i \) is adjacent to precisely \( k - 5 \geq 1 \) of the vertices in \( \{w_s : s \notin \{i - 1, i\}, 1 \leq s \leq k - 1\} \), so it is adjacent to at least one \( w_{\ell} \) such that \( \ell \notin \{i - 1, i\} \). If \( i < \ell \leq k - 1 \) then \( u_i w_{\ell} w_{\ell + 1} u_i \) is a non-facial 4-cycle (as \( u_i w_{\ell} w_{\ell + 1} \ldots w_{\ell - 1} u_i \) lies in one of the regions bounded by this cycle while \( w_{\ell + 1} w_{\ell + 2} \ldots w_i u_i \) lies in the other region). If \( 1 \leq \ell < i - 1 \) then \( u_i w_{\ell + 1} w_{\ell + 1} u_i \) is a non-facial 4-cycle. Since \( Q \) can not have non-facial 4 cycles by Lemma 9, this is a contradiction.

4. Minimum Wiener Index of 5-Connected Triangulations

Euler’s formula shows that there are no 5-connected triangulations of order less than 12.

First we state some facts about triangulations of a simple \( n \)-gon not using additional vertices. Triangulations of an \( n \)-gon can be viewed as planar graphs, where the outer face is bounded by an \( n \) cycle and all other faces are bounded by a 3-cycle (we will refer to such faces as triangles).

**Lemma 15.** Let \( n \geq 4 \). Any triangulation of a simple \( n \)-gon uses \( n - 3 \) additional edges (i.e. edges which are not edges of the \( n \)-gon), and has at least 2 triangles with exactly two of their boundary edges on the \( n \)-gon.

**Proof.** The fact that the triangulation has \( n - 3 \) edges (and consequently \( n - 2 \) triangles) is easy to prove by induction on \( n \). When \( n \geq 4 \), all these triangles have at most 2 boundary edges on the \( n \)-gon. As there are \( n - 2 \) triangles inside and \( n \) edges on the \( n \)-gon itself, by the pigeonhole principle some two triangles must have two edges from edges of the \( n \)-gon. □

We need the following basic facts about 5-connected triangulations:

**Lemma 16.** Let \( T \) be a 5-connected triangulation of order \( n \). The following are true:

(a) Every 3-cycle is the boundary of a face and every 4-cycle is the boundary of a region whose interior does not contain vertices of the graph, and contains exactly one edge.

(b) Every edge lies on exactly two triangles. If \( abc \) and \( bcd \) are triangles of \( T \), then \( ad \) is not an edge of \( T \).

(c) For every edge \( xy \) of \( T \), there is precisely one 4-cycle in \( T \) that goes through its vertices, but does not use the \( xy \) edge; hence the number of 4-cycles in \( T \) is \( 3(n - 2) \).

(d) If \( x, y \) are non-adjacent vertices in \( T \), then there is at most one 4-cycle that contains them.

(e) Let \( D_i \) denote the number of unordered pairs of vertices at distance \( i \) in \( T \). We have \( D_1 = 3(n - 2) \) and

\[
D_2 = \frac{1}{2} \sum_{x \in V(T)} d^2(x) - 12(n - 2).
\]
\[ W(T) \geq 3 \binom{n}{2} + 6(n - 2) - \frac{1}{2} \sum_{x \in V(T)} d^2(x), \]

with equality if and only if \( T \) has diameter at most 3.

**Proof.**

\( \text{(a)} \) If \( C \) is a cycle that separates two regions that both contain vertices in their interior, then the vertices of \( C \) form a cutset, therefore \( C \) has at least 5 vertices.

\( \text{(b)} \) An edge bounds two faces that are triangles, and if there is a third 3-cycle using the edge, the other two edges of two of these 3-cycles give a 4-cycle that has vertices in both of its regions. If \( abc \) and \( bcd \) are triangles such that \( ad \) is an edge, then one of \( abd, acd \) would be a non-facial triangle unless \( n = 4 \). Both of these contradict \( \text{(a)} \) and \( \text{(b)} \) follows.

\( \text{(c)} \) Since every 4-cycle \( abcd \) bounds a region that has no vertices but has an edge (say \( ac \)), and if \( ac \) is an edge then \( bd \) cannot be an edge by \( \text{(b)} \) for every 4-cycle there is a unique edge that is not part of the cycle and connects two of its vertices. So we can map 4-cycles to edges by assigning this edge to each cycle. This map is injective. If two different 4-cycles would map to the same edge, this edge is part of three triangles, contradicting \( \text{(b)} \).

As each edge lies on two triangles which together form a 4 cycle, every edge is assigned to precisely one of these 4-cycles, so the map is surjective as well. Thus, the number of 4-cycles is the same as the number of edges, which is \( 3(n - 2) \) in any planar triangulation. \( \text{(c)} \) follows.

\( \text{(d)} \) Assume \( x, y \) are non-adjacent vertices that appear on two 4-cycles. As each 4-cycle containing \( x, y \) has two \( x-y \) paths of length 2, we have at least three \( x-y \) paths of length 2, say \( xa_1y, xa_2y, xa_3y \). By \( \text{(a)} \) for each \( i, j \in \{1, 2, 3\}, i \neq j \) the cycle \( C_{ij} = xaiya_jx \) bounds a region \( R_{ij} \) that contains no vertices in its interior. But the three regions \( R_{12}, R_{13}, R_{23} \) together with their boundaries cover the entire plane, so \( T \) has no other vertices besides \( x, y, a_1, a_2, a_3 \). This is a contradiction, as 5-connected triangulations must have at least 12 vertices.

\( \text{(e)} \) Observe that \( D_1 \) is exactly the number of edges of \( T, 3(n - 2) \). The formula \( \sum \binom{d(x)}{2} = \frac{1}{2} \sum d^2(x) - 3(n - 2) \) counts the number of paths of length 2 between unordered pairs of vertices. If an unordered pair of vertices has more than 1 such path, it appears on a 4-cycle, and by \( \text{(c)} \) and \( \text{(d)} \) this 4-cycle is unique. As each 4-cycle contains exactly two such unordered pairs of vertices, the number of unordered pairs of vertices that have a path of length 2 between them is \( \frac{1}{2} \sum d^2(x) - 9(n - 2) \) by \( \text{(e)} \). As every edge is contained in exactly one 4-cycle, this equals \( D_1 + D_2 \), proving \( \text{(e)} \).

\( \text{(f)} \) As \( \sum_i D_i = \binom{n}{2} \), we get

\[ W(T) = \sum_i iD_i \geq D_1 + 2D_2 + 3 \left( \binom{n}{2} - D_1 - D_2 \right) = 3 \binom{n}{2} - 2D_1 - D_2 \]

\[ = 3 \binom{n}{2} + 6(n - 2) - \frac{1}{2} \sum_{x \in V(T)} d^2(x), \]

and equality holds precisely when the diameter is at most 3. \( \square \)
Analogously to Section 3, we define an auxiliary drawn graph, which we will use extensively. Let $T$ be a 5-connected triangulation, and let $v \in V(T)$ have degree $d$. We define the mosaic graph $M_v$ at vertex $v$, together with its planar drawing, in the following way. $M_v$ contains the neighbors of $v$ in $G$, $u_1, u_2, \ldots, u_d$, with the edges $vu_i$, such that vertices $u_i$ are labeled according the clockwise cyclic order of the edges. We include the edges $u_iu_{i+1} \in E(T)$ for every $1 \leq i \leq d$ (indices are taken modulo $d$) in $M_v$, following the drawing of $T$. We also add a vertex $w_i \neq v$, which is a common neighbor of $u_i$ and $u_{i+1}$, together with edges joining them to $u_i$ and $u_{i+1}$ in $T$, following the drawing of $T$, for every $i$. We understand $M_v$ as a part of the drawing of $T$. We will show that $M_v$ has $2d + 1$ distinct vertices.

**Figure 7.** The mosaic graph $M_v$ around $v$, with $d = 8$. The grey region is $R_v$.

**Lemma 17.** If $T$ is a drawing of a 5-connected triangulation of order $n$, and $v$ is any vertex of $T$ with degree $d(v) = d$, the mosaic graph $M_v$ in $T$ has $2d + 1$ distinct vertices. Furthermore, the region $R_v$ that is bounded by the cycle $u_1w_1u_2w_2\ldots u_dw_d$ and contains the vertex $v$, contains edges and vertices from $T$ if and only if they are edges and vertices in the mosaic graph $M_v$. In addition, $T$ contains at least one vertex not in $M_v$, consequently $\Delta(T) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Moreover, $n \neq 13$.

**Proof.** Since $T$ is 5-connected, $\delta(T) \geq 5$. As before, label the neighbors of $v$ by $u_1, \ldots, u_d$, in their planar clockwise cyclic order around $v$. We get for free that $u_iu_{i+1}$ is an edge in $T$, since we have a triangulation. For each pair of successive neighbors $u_i$ and $u_{i+1}$ (indices taken modulo $d$), let $w_i \neq v$ be their common neighbor that completes the face that has $u_iu_{i+1}$ on its boundary, but not $v$. This means, in particular, that $R_v$ will satisfy the required property, so we just need to show that the vertices listed in $M_v$ are all distinct.

If $y$ is a neighbor of $u_i$ and $y \notin \{v, w_{i-1}, w_i, u_{i-1}, u_{i+1}\}$ then $y$ must lie between $w_{i-1}$ and $w_i$ in the planar cyclic order around $u_i$. In particular, as $d(u_i) \geq 5$, we have that $w_{i-1} \neq w_i$.

Also, $u_i \neq u_j$ for all $1 \leq i, j \leq d$. For $j \in \{i - 1, i\}$ this is obvious, and for other values of $j$ if $u_i = w_j$ then $v, u_i, u_j$ is a 3-element cutset.
Assume now that \( w_i = w_j \) for some \( j \neq i \). We already have that \( j \notin \{i - 1, i + 1\} \) and hence the vertices \( u_i, u_{i+1}, u_j, u_{j+1} \) are all distinct. We consider two regions of the planar drawing of \( T \): \( \mathcal{R}_1 \) is bounded by the 4-cycle \( u_{i+1}w_jw_i \) and does not contain the vertex \( u_i \), and \( \mathcal{R}_2 \) is bounded by the 4-cycle \( u_ivu_{j+1}w_i \) and does not contain the vertex \( u_{i+1} \).

The neighbors of \( u_i \) that differ from \( v, u_{i+1} \) and \( w_i \), lie in \( \mathcal{R}_2 \) and the neighbors of \( u_j \) that differ from \( v, u_{j+1} \) and \( w_i \), must lie in \( \mathcal{R}_1 \). Hence \( \{v, u_{i+1}, u_{j+1}, w_i\} \) separates \( u_i \) from \( u_j \) (see Figure 8), contradicting that \( T \) is 5-connected.

![Figure 8. 4-element cutset \( \{v, u_{i+1}, u_{j+1}, w_i\} \) in \( M_v \) when \( w_i = w_j \). The shaded regions are unions of faces, so they have no additional vertices.](image)

Now let \( v \) be a vertex of \( T \) with maximum degree, i.e., \( d(v) = \Delta(T) = \Delta \). The mosaic graph \( M_v \) around \( v \) contains \( 2\Delta + 1 \) vertices. If \( T \) contains a vertex that is not in \( M_v \), then \( 2\Delta + 2 < n \), and the claimed inequality follows.

If every vertex of \( T \) is in \( M_v \), then set of edges \( F \) not in \( M_v \) form a triangulation of the \( 2\Delta \)-cycle \( u_1w_1u_2w_2 \ldots u_{\Delta}w_{\Delta}u_1 \) on the region different from \( \mathcal{R}_v \); consequently \( |F| = 2\Delta - 3 \). Note that for any \( 1 \leq i < j \leq \Delta \), if \( u_iu_j \in F \), then the 3-cycle \( u_iu_jv \) separates the vertices \( w_i \) and \( w_j \), so \( u_i, u_j, v \) would be a cutset of size 3, a contradiction. If for any \( j \notin \{i, i-1\} \), \( u_iw_j \in F \), then the 4-cycle \( u_iw_ju_jv \) has the vertices \( w_i \) and \( w_{i-1} \) on its different sides, giving a cutset of size 4, which is also a contradiction. Therefore every edge in \( F \) connects two vertices of \( W = \{w_1, \ldots, w_{\Delta}\} \).

But then for every \( i \), the edges \( w_{i-1}u_i \) and \( u_iw_i \) lie on the boundary of the same face, giving \( w_{i-1}w_i \in F \). Hence \( w_1, \ldots, w_{\Delta} \) determines a \( \Delta \)-gon (all of the sides are in \( F \)), and this \( \Delta \)-gon is triangulated by the remaining edges of \( F \). Lemma 15 applies. Say, \( w_{i-1}, w_i, w_{i+1} \) is a triangle with two edges on the boundary of the \( \Delta \)-gon. Then \( d(w_i) = 4 \), contradicting the fact that \( T \) is 5-connected.

Finally, assume to the contrary that \( T \) has 13 vertices. Then \( \Delta(T) \leq 5 \), therefore \( T \) is 5-regular. The sum of degrees of \( T \) is odd, contradicting the Handshaking Lemma. □

For every \( n \geq 12, n \neq 13 \), the \( n \)-vertex triangulation \( T_n^5 \) is defined by Figure 9 (these will be our minimizers of the Wiener index). The following lemma is easy to verify and we leave the details to the reader.

**Lemma 18.** Assume that \( n \geq 12, n \neq 13 \).

(a) \( T_n^5 \) is a 5-connected triangulation.
Figure 9. The triangulation $T^5_n$, which minimizes the Wiener index among all 5-connected triangulations of order $n = 2k \geq 12$ (left) and of order $n = 2k + 1 \geq 15$ (right). Gray vertices and dashed edges indicate the pattern to be repeated. The shaded region shows the mosaic graph around a degree $\lfloor \frac{n}{2} \rfloor - 1$ vertex.

Figure 10. The 5-connected triangulation $X$. The two white vertices are at distance 4. The shaded region shows the mosaic graph around the degree 8 vertex.

(b) $T^5_n$ has diameter 3.
(c) For $n$ even, the degree sequence of $T^5_n$ is $\frac{n}{2} - 1, \frac{n}{2} - 1, 5, \ldots, 5$.
(d) For $n$ odd, the degree sequence of $T^5_n$ is $\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor - 2, 6, 6, 5, \ldots, 5$. (For $n = 15$, the terms in this sequence are not in decreasing order.)

(e)

$$W(T^5_n) = \begin{cases} \frac{5n^2}{4} - 7n + 12, & \text{if } n \text{ is even}, \\ \frac{5n^2}{4} - 6n - \frac{9}{4}, & \text{if } n \text{ is odd}. \end{cases}$$

The 5-connected triangulation $X$ of order 19, defined by Figure 10, has

(3)

$$W(X) = 335 = W(T^5_{19}).$$

We will also show that $X$ is the only 5-connected triangulation that is not isomorphic to any $T^5_n$ and achieves the minimum Wiener index for its order. Note that as $X$ is of diameter 4, the lower bound in Lemma 16 (f) cannot be used to compute $W(X)$. The different diameter, and also the different degree sequence, implies that $X \not\cong T^5_{19}$.

We define the extended mosaic graph $M^*_v$ by adding edges to $M_v$. Given a 5-connected triangulation $G$ and a vertex $v$ with mosaic graph $M_v$, we introduce the graph $M^*_v$, on the
vertex set of $M_v$, by setting

$$E(M_v^*) = E(M_v) \cup \{w_iw_{i+1} : 1 \leq i \leq d(v), w_iw_{i+1} \in E(G)\}.$$  

Note that $w_iw_{i+1} \in E(G)$ if and only if $d(u_{i+1}) = 5$. Let $R_v^*$ denote the extension of $R_v$ by adding to it the faces bounded by the 3-cycles $u_{i+1}w_iw_{i+1}$ for all edges $w_iw_{i+1} \in E(G)$; let $C_v$ denote the boundary cycle of $R_v^*$ and let $Q_v^*$ denote the other domain defined by the cycle $C_v$. Now all vertices of $G$ that are not vertices of $M_v$ and all edges of $E(G) \setminus E(M_v^*)$ lie in the region $Q_v^*$ of the drawing of $G$.

We will use the following notation in the rest of the section. Given a 5-connected triangulation $G$ and a vertex $v$, if $a, b$ are vertices of $C_v$, then $P_v(a, b)$ denotes the path on the cycle $C_v$ from $a$ to $b$ that follows the clockwise cyclic order. (So if $C_v = (w_1, w_2, \ldots, w_d)$ in clockwise cyclic order, then $P_v(w_1, w_2)$ is just the edge $w_1w_2$ with its endpoints, while $P_v(w_2, w_1)$ goes through all vertices of the cycle and misses only the edge $w_1w_2$.)

**Lemma 19.** Let $G$ be a 5-connected triangulation of order $n \geq 12$, and let $v$ be a vertex of $G$ with $d(v) = d$. Consider the extended mosaic graph $M_v^*$. The following are true:

(a) Every vertex $z \in V(C_v)$ has an edge of $E(G) \setminus E(M_v^*)$ incident upon it.

(b) If for some $z_1, z_2 \in V(C_v)$ we have $z_1z_2 \in E(G) \setminus E(M_v^*)$, then $z_1z_2$ cuts $Q_v^*$ into two subregions, each containing a vertex of $G$ in its interior, and $z_1, z_2 \in \{w_1, w_2, \ldots, w_d\}$.

(c) If $n$ is even and $d(v) = \frac{n}{2} - 1$, then $G \simeq T_n^3$.

**Proof.** Set $W = \{w_1, \ldots, w_d\}$ and $U = \{u_1, \ldots, u_d\}$.

(a) Observe that $W \subseteq V(C_v) \subseteq W \cup U$. Vertices in $W$ have degree at most 4 in $M_v^*$, and vertices of $U$ have degree 5 in $M_v^*$. If a vertex of $U$ has degree 5 in $G$, then it is not a vertex of $C_v$. [a]

(b) Let $z_1, z_2 \in V(C_v)$ where $z_1z_2 \in E(G) \setminus E(M_v^*)$. Assume first that $z_1z_2$ cuts $Q_v^*$ into two subregions, one of which (say $R$) contains no vertices in its interior. We will show that $R$ contains a (triangular) face $f$ such that the boundary of $f$ has two edges $e_1, e_2$ that are on the boundary of $R$ and $z_1z_2 \notin \{e_1, e_2\}$. This is obviously true when the boundary of $R$ is a 3-cycle. Otherwise the edges lying in the interior of $R$ are edges of $E(G) \setminus E(M_v^*)$ that form a triangulation of $R$, and by Lemma 15 this triangulation contains two faces with two boundary edges on the boundary of $R$. One of these faces, $f$, does not have the edge $z_1z_2$ on its boundary. Let $c$ be the common endvertex of the two edges $e_1, e_2$ on the boundary of $R$. Then $c \notin \{z_1, z_2\}$ and $c$ cannot have an edge from $E(G) \setminus E(M_v^*)$ incident upon it, contradicting [a]. So $z_1z_2$ cuts $Q_v^*$ into two subregions, both of which contains a vertex in its interior. Now assume to the contrary that $\{z_1, z_2\} \cap U \neq \emptyset$. We may assume that $z_1 = u_1$. Then $z_2 = u_{\ell}$ for some $3 \leq \ell \leq d - 1$ or $z_2 = w_j$ for some $3 \leq j \leq d - 2$ ($j \neq 2$ and $j \neq d - 1$, using Lemma 16(b) for edges $u_{\ell}w_2$ and $u_1u_d$). If $z_2 = u_{\ell}$, then $u_1u_\ell$ is a separating 3-cycle (as $w_2$ and $w_\ell$ are in different regions of this cycle), and if $z_2 = w_j$ then $u_1w_ju_{\ell}$ is a separating 4-cycle (as $w_2$ and $w_d$ are in different regions); both of which contradict the 5-connectedness of $G$.

(c) Assume now that $n$ is even and $d(v) = \frac{n}{2} - 1$. Lemma 17 gives $\Delta(G) = d(v)$. $G$ has exactly one vertex, say $x$, not in $M_v^*$, and hence in the region $Q_v^*$. Then the already
By Lemma 11 the integer sequence $y$ defined in Figure 9, is the unique minimizer of the Wiener index among all triangulations of order $n$. 

**Theorem 20.** Assume that $n \geq 12$ and $n$ is even. The triangulation $T_n^5$, which was defined in Figure 4, is the unique minimizer of the Wiener index among all 5-connected triangulations of order $n$.

**Proof.** Let $n \geq 12$ be even and assume $T$ is a 5-connected triangulation on $n = 2k$ vertices ($k \geq 6$). The degree sum of $T$ is $2(3n - 6) = 6n - 12$, and Lemma 17 gives $\Delta(T) \leq \frac{n}{2} - 1$. By Lemma 12 the integer sequence $y_1, \ldots, y_n$ that sums to $6n - 12$, satisfies $5 \leq y_i \leq \frac{n}{2} - 1$ and has the largest sum of squares is the sequence $\frac{n}{2} - 1, \frac{n}{2} - 1, 5, 5, \ldots, 5$, which is exactly the degree sequence of $T_n^5$ by Lemma 13(c). As $T_n^3$ has diameter 3, by Lemma 10(f) $T_n^5$ indeed has the minimum Wiener index among all 5-connected $n$-vertex triangulations. We know that the degree sequence of $T$ is the same as the degree sequence of $T_n^5$, so $T \simeq T_n^5$ follows from Lemma 19(e).

To characterize extremal triangulations of odd order, we need a bit more information about their structure.

**Lemma 21.** There are no 5-connected triangulations on 21 vertices with degree sequence $8, 8, 8, 5, \ldots, 5$.

**Proof.** Assume that $G$ is a 5-connected triangulation on 21 vertices with degree sequence $8, 8, 8, 5, \ldots, 5$. Let $v$ be a degree 8 vertex, and let $x_1, x_2, x_3, x_4$ be the vertices in $V(G) \setminus V(M_v^*)$. Set $X = \{x_1, x_2, x_3, x_4\}$, $U = \{u_i : 1 \leq i \leq 8\}$ and $W = \{w_i : 1 \leq i \leq 8\}$. Let $b$ be the number of connected components of the subgraph of $G$ induced by $X$ and $D_i$ be the component containing $x_i$, $c_i = |N(x_i) \cap X|$ and $\chi$ be the number of vertices of degree 8 in $X$.

Clearly $W \subseteq V(C_v)$, and by Lemma 19(a) and (b) all vertices of $V(C_v) \cap U$ have degree 8 (consequently $|V(C_v) \cap U| \leq 2$). Assume that $z \in V(C_v) \cap U$; then $|N(z) \cap X| = 3$. Let $\{x_i, x_j, x_k\} = N(z) \cap X$, then without loss of generality $x_i x_j x_k$ is a path in $G$ whose edges form faces with the edges $x_i z, x_j z, x_k z$. Moreover, if $e, f$ are the two edges on $C_v$ that are incident upon $z$, then the cyclic order of the edges that lie in or on the boundary of $Q_v^*$ around $z$ is $e, zx_i, zx_j, zx_k, f$ or $f, zx_i, zx_j, zx_k, e$; otherwise one of the triangles $zx_i x_j$ or $zx_j x_k$ is a separating triangle, which is a contradiction. Finally, $x_i x_k \notin E(G)$, as otherwise $zx_i x_k$ is a separating triangle.

Assume first that $U \cap V(C_v) \neq \emptyset$; without loss of generality $u_1 \in U \cap V(C_v)$, $u_1$ is adjacent to $x_2, x_3, x_4$ and $x_2 x_3 x_4$ is a path in $G$, consequently $x_2 x_4 \notin E(G)$. We have that either $b = 2$ and $D_1 = \{x_1\}$, or $b = 1$. Without loss of generality we may assume that the cyclic order of edges around $u_1$ in $Q_v^*$ is $u_1 w_1, u_1 x_2, u_1 x_3, u_1 x_4, u_1 w_8$. As $u_1 w_1, u_1 x_2$ (and also $u_1 x_3, u_1 w_8$) bound a common face, we have $w_1 x_2, w_8 x_4 \in E(G)$. Let $P^*$ be the region we get if we leave out from $Q_v^*$ the faces with $u_1$ on their boundary.

Consider the case when $|U \cap V(C_v)| = 2$, i.e. for some $j \neq 1$ the vertex $u_j$ also has degree 8. If $N(u_j) \cap X = N(u_1) \cap X = \{x_2, x_3, x_4\}$ (which must happen when $b = 2$), we
have that the path $x_2x_3x_4$ is induced in $D_2$. But then $u_1x_2u_3x_4$ is a separating 4-cycle, which is a contradiction. Therefore we must have $b = 1$, and without loss of generality for some $t \in \{2, 3\}$, $N(u_t) \cap X = \{x_1, x_t, x_{t+1}\}$ where $x_1x_t \in E(G)$ and $x_1x_{t+1} \notin E(G)$. This gives $|E(X)| \geq 3$, $|E(X, U)| = 6$, and (since the vertices of $X$ have degree 5) $|E(X, W)| = 20 - 6 - 2|E(X)| \leq 8$. On the other hand, since $b = 1$, Lemma 19(b) implies that the set of edges in $E(G) \setminus E(M^*_v)$ that are incident upon $W$ form the set $E(X, W)$, and (as $w_1$ and $w_8$ have degree 3 in $M^*_v$) consequently $|E(X, W)| \geq 10$, a contradiction. Therefore we must have $|U \cap V(C_v)| < 2$, i.e. $U \cap V(C_v) = \{u_1\}$.

Now we have that $U \cap V(C_v) = \{x_1\}$. Let $s$ and $t$ be the smallest and largest indices such that $w_s, w_t \in N(x_1)$; $1 \leq s < t + 1 \leq 8$. The path $w_sw_t$ cuts the set of edges in $M^*_v$. If $s \neq 1$ then $\{w_{s-1}, w_s, u_s, u_{s+1}, w_t, x_1\} \subseteq N(w_1)$, so $w_s$ must have degree 8. If $s = 1$, then $\{u_1, w_2, x_2, x_4, x_1\} \subseteq N(w_1)$, so $w_1$ has degree 8. Similar arguments imply that $w_t$ also has degree 8. But then $u_1, v, w_s, w_t$ all have degree 8, a contradiction. So we must have $b = 1$, i.e. $x_1$ is connected to at least one other vertex in $X$. If $x_1$ is connected to both $x_2$ and $x_4$, then (as the edges $x_1x_2$ and $x_1x_4$ are in $P^*$) $u_1x_2x_1x_4$ is a separating 4-cycle, which is a contradiction. We may assume $x_1x_2 \notin E(G)$, so $c_1 \in \{1, 2\}$. Then $|E(X, W)| = 13 + 3\chi - 2c_1$. Since the number of degree 8 vertices in $W$ is $1 - \chi$, $|E(X, W)| = 10 + 3(1 - \chi) = 13 - 3\chi$. This gives $2c_1 = 6\chi$, so $3$ divides $c_1$, which is a contradiction. Thus we must have $U \cap V(C_v) = \emptyset$.

Therefore $W = V(C_v)$ and every vertex in $W$ has degree 4 in $M^*_v$, so every vertex in $W$ has either one or 4 edges incident upon it from $E(G) \setminus E(M^*_v)$. Set $F = E(W) \setminus E(M^*_v)$ and $m_x = |E(X)|$. We have $2|F| + |E(W, X)| = \sum_{z \in W} (d(z) - 4) = 14 - 3\chi$ and $|E(X, W)| = (\sum_{x \in X} d(z)) - 2m_x = 20 + 3\chi - 2m_x$.

If $\chi = 2$, then all vertices of $W$ have at most one neighbor in $X$. Since the two vertices in $X$ that have degree 8 at least 5 neighbors in $W, this implies that $5 + 5 \leq |W| = 8$, a contradiction. Therefore $\chi \in \{0, 1\}$; at least one vertex in $W$ has degree 8.

Suppose $b = 1$. Then $3 \leq m_x \leq 5$, and $|E(X, W)| = 20 - 2m_x + 3\chi$. On the other hand by Lemma 19(b) $F = \emptyset$, so $|E(X, W)| = 14 - 3\chi$. This gives $m_x = 3 + 3\chi$, consequently $\chi = 0$, $m_x = 3$, and exactly two vertices (say $w_1, w_4$) in $W$ have degree 8. But then at least one of the 4-cycles of the form $w_1x_2w_4x_3x_1w_t$ is separating, which is a contradiction. Thus we must have $b > 1$.

Since $b > 1$, we must have either that the components spanned by $X$ are a $K_1$ and a $K_3$, or the subgraph generated by $X$ has exactly $4 - b$ edges (as all of its components are trees). In the former case $|E(X, W)| = 14 + 3\chi$, in the latter $|E(X, W)| = 12 + 2b + 3\chi \geq 16 + 3\chi > 14 - 3\chi \geq |E(X, W)|$, which is a contradiction. Therefore the components spanned by $X$ are a $K_1$ and a $K_3$. We get that $2|F| + 14 + 3\chi = 14 - 3\chi$, which gives $\chi = 0$, and $F = \emptyset$. So exactly two vertices (say $w_1, w_4$) in $W$ have degree 8, and $E(W, V(G)) \setminus E(M^*_v) = E(X, W)$. Without loss of generality the $K_3$ in $X$ is formed by the vertices $x_2, x_3, x_4$. But then $X \subseteq N(w_1)$, so the subgraph generated by $\{x_2, x_3, x_4, w_1\}$ is a
$K_4$. This is a contradiction, as one of the triangles \( w_1x_2x_3, w_1x_3x_4, w_1x_2x_4 \) is separating, contradicting the 5-connectedness of \( G \).

**Lemma 22.** Let \( n \geq 15 \) be odd, and let \( G \) be a 5-connected triangulation of order \( n \) with degree sequence \( d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n \). If \( W(G) \leq W(T_n^5) \), then \( d_1 = \left\lfloor \frac{n}{3} \right\rfloor - 1 \), and one of the following holds:

(a) \( n = 23 \) and the degree sequence of \( G \) is \( 10, 8, 8, 5, \ldots, 5 \)

(b) \( d_2 \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 \), \( d_3 + d_4 \leq 12 \), and consequently \( d_3 \leq 7 \), \( d_4 \leq 6 \) and \( d_5 = 5 \).

**Proof.** Set \( n = 2k + 1 \), then \( k = \left\lfloor \frac{n}{3} \right\rfloor \geq 7 \). As \( d_1 \leq k - 1 \), the only possible degree sequence for \( k = 7 \) is \( (6, 6, 6, 5, \ldots, 5) \), which satisfies the conclusion. Hence we may assume \( k \geq 8 \).

As \( T_n^5 \) has diameter 3, by Lemma [16](f) and Lemma [18](d) we must have

\[
\sum_{x \in V(G)} d^2(x) \geq \sum_{x \in V(T_n^5)} d^2(x) = (k - 1)^2 + (k - 2)^2 + 2 \cdot 6^2 + 5^2(2k - 3) = 2k^2 + 44k + 2.
\]

Assume first that \( d_1 \leq k - 2 \).

If \( k = 8 \), the only sequence possible is \( 6, 6, 6, 6, 5, \ldots, 5 \) whose sum of squares is \( 480 < 2 \cdot 8^2 + 44 \cdot 8 + 2 \), which is a contradiction.

If \( k = 9 \), then

\[
\sum_{x \in V(G)} d^2(x) \leq 3 \cdot 7^2 + 6^2 + 15 \cdot 5^2 = 558 < 560 = 2 \cdot 9^2 + 44 \cdot 9 + 2,
\]
a contradiction.

If \( k = 10 \), then By Lemma [21]

\[
\sum_{x \in V(G)} d^2(x) \leq 2 \cdot 8^2 + 7^2 + 6^2 + 17 \cdot 5^2 = 638 < 642 = 2 \cdot 10^2 + 44 \cdot 10 + 2,
\]
a contradiction.

If \( k \geq 11 \), then

\[
\sum_{x \in V(G)} d^2(x) \leq 2(k - 2)^2 + 8^2 + 5^2(2k - 2) = 2k^2 + 42k + 22 < 2k^2 + 44k + 2,
\]
a contradiction. So we proved that \( d_1 = k - 1 \). If \( k = 8 \), the only sequences possible are \( 7, 7, 6, 5, \ldots, 5 \) and \( 7, 6, 6, 6, 5 \ldots, 5 \), which satisfy the conclusion. Hence we may assume \( k \geq 9 \).

Assume next that \( d_2 \leq k - 3 \). If \( k = 9 \), the only sequence possible is \( 8, 6, 6, 6, 5, \ldots, 5 \) with degree square sum \( 558 < 2 \cdot 9^2 + 44 \cdot 9 + 2 \). If \( k = 10 \),

\[
\sum_{x \in V(G)} d^2(x) \leq 9^2 + 2 \cdot 7^2 + 6^2 + 17 \cdot 5^2 = 640 < 2 \cdot 10^2 + 44 \cdot 10 + 2.
\]

If \( k \geq 11 \), then

\[
\sum_{x \in V(G)} d^2(x) \leq (k - 1)^2 + (k - 3)^2 + 8^2 + 5^2(2k - 2) = 2k^2 + 42k + 24 \leq 2k^2 + 44k + 2.
\]
Lemma 23. Let $n \geq 15$ be odd, $k = \lfloor \frac{n}{2} \rfloor$, and let $G$ be a 5-connected triangulation of order $n$, with $W(G) \leq W(T^5_n)$. Let $v$ be a vertex with $d(v) = \Delta(G) =: \Delta$. Consider the extended mosaic graph $M^*_v$, and the sets $W = \{w_1, \ldots, w_\Delta\}$ and $U = \{u_1, \ldots, u_\Delta\}$. Let $x_1, x_2$ denote the two vertices not in $M^*_v$. The following statements are true:

(a) $\Delta = k - 1$, at most 4 degrees of $G$ are larger than 5, and for the largest 4 degrees $\Delta \geq d_2 \geq d_3 \geq d_4$ of $G$ we have either $d_2 \geq k - 2$, $d_3 \leq 7$, $d_4 \leq 6$ and $d_3 + d_4 \leq 12$, or $n = 23$, $d_2 = d_3 = 8 = k - 3$, $d_4 = 5$.
(b) For $i \in \{1, 2\}$, there are vertices $a_i, b_i \in V(C_v)$ such that $N(x_i) \setminus \{x_{3-i}\} = V(P_v(a_i, b_i))$. (We refer to these $a_i, b_i$ vertices in the forthcoming claims.) Furthermore, if $c \in V(P_v(a_1, b_1)) \cap V(P_v(a_2, b_2))$, then $c = a_1 = b_2$ or $c = a_2 = b_1$.
(c) $C_v = w_1w_2\ldots w_\Delta$, $d(x_i) \leq \Delta - 1$, and if $x_1x_2 \notin E(G)$ then $d(x_i) \leq \Delta - 2$ and $n \geq 19$.
(d) If $x_1x_2 \in E(G)$, then $G \simeq T^5_n$.
(e) If $x_1x_2 \notin E(G)$ then $a_1b_1a_2b_2 \in E(G) \setminus E(M^*_v)$. Moreover, for $i \in \{1, 2\}$, if $c_i \in V(P_v(b_i, a_{3-i}))$ and $z \in V(G) \setminus \{x_1, x_2\}$, such that $c_i, z \in E(G) \setminus E(M^*_v)$, then $z \in V(P_v(b_{3-i}, a_i))$, and the neighbors of $c_i$ in $P_v(b_{3-i}, a_i)$ form a consecutive sequence of vertices on this path.
(f) If $x_1x_2 \notin E(G)$, then $a_1 = b_2$ or $a_2 = b_1$.
(g) If $x_1x_2 \notin E(G)$, then $a_1 = b_2$ and $a_2 = b_1$.
(h) If $x_1x_2 \notin E(G)$, then $G \simeq X$ and $W(G) = W(T^5_n)$.

Proof. Note that $n \geq 15$, so $k \geq 7$. Let $G$, $v$, $x_1, x_2$ be as in the conditions. Lemma 22 yields (a).

Assume $i \in \{1, 2\}$. As $d(x_i) \geq 5$, $x_i$ has at least 4 neighbors on $C_v$, so there are vertices $a_i, b_i \in N(x_i) \cap V(C_v)$ such that all vertices in $N(x_i) \setminus \{x_{3-i}\}$ lie on the path $P_v(a_i, b_i)$ and $x_{3-i}$ does not lie in the interior of the subregion of $Q^*_v$ bounded by the cycle $x_1P_v(a_i, b_i)$. By Lemma 19(b) no two vertices in $P_v(a_i, b_i)$ can be joined by an edge that is not in $M^*_v$. As every vertex of $C_v$ has at least one edge incident upon it from $E(G) \setminus E(M^*_v)$, we must have $V(P_v(a_i, b_i)) \subseteq N(x_i)$, therefore $V(P_v(a_i, b_i)) = N(x_i) \setminus \{x_{3-i}\}$. As all edges incident upon $x_1$ or $x_2$ lie in the region $Q^*_v$ and do not cross, the two paths share all at most their endvertices.

Assume to the contrary that $c \in U \cap V(P_v(a_i, b_i))$. As $d(x_i) \geq 5$, $P_v(a_i, b_i)$ has at least 4 vertices. Therefore, there exists an internal vertex $c^*$ of the path $P_v(a_i, b_i)$, such that the edge $cc^*$ is an edge of this path. This means $c^* \in W$ (see Lemma 19(b)). $c^*$ has at most 3 edges incident upon it in $E(M^*_v)$, and the only edge in $E(G) \setminus E(M^*_v)$ incident upon $c^*$ is $c^*x_i$, contradicting $d(c^*) \geq 5$. Thus, $V(P_v(a_i, b_i)) \subseteq W$. By Lemma 19(a) the internal vertices of the paths $P(b_i, a_{3-i})$ each have at least one edge of $E(G) \setminus E(M^*_v)$ incident.
upon them. Thus, by the definition of \( a_i, b_i \) and \( M_i^* \), the internal vertices of the paths \( P(b_i, a_{3-i}) \) have to be adjacent to at least one other internal vertex of the paths \( P(b_i, a_{3-i}) \). Lemma \( 19(b) \) implies that \( V(P(b_i, a_{3-i})) \subseteq W \). So \( V(C_v) = W \). By part \( (b) \) we have

\[
|V(P_v(a_{3-i}, b_{3-i}))|, |V(P_v(a_i, b_i))| \geq 2.
\]

Hence \( \Delta = |W| \geq |V(P_v(a_i, b_i))| + |V(P_v(a_{3-i}, b_{3-i}))| \). As \( |N(x_i)| = |V(P_v(a_i, b_i))| \), this follows depending on whether \( x_1x_2 \) is non-edge or edge, the claimed upper bounds on \( d(x_i) \) follow. Assume \( x_1x_2 \notin E(G) \).

As we have \( 5 \leq d(x_i) \leq \Delta - 2, \) we have \( \Delta \geq 7, \) so \( n \geq 17 \). If \( n = 17 \), then we must have \( d(x_1) = d(x_2) = 5 \). As \( 8 = d(x_1) + d(x_2) - 2 \leq |W| = 7 \), this is a contradiction. \( (d) \) follows.

If \( x_1x_2 \in E(G) \). Assume that \( b_2 \neq a_1 \). Then either \( P_v(b_2, a_1) \) has an internal vertex \( c \) or \( b_2a_1 \in E(G) \). In the first case, as \( c \in C_v \), there is at least one edge in \( E(G) \setminus E(M_v^*) \) that has no vertices in its interior. Any edges between vertices of \( M_v^* \) incident upon \( c \) must lie in the subregion of \( Q_v^* \) bounded by the cycle \( P_v(b_2, a_1)x_1x_2 \). By Lemma \( 19(b) \) these edges must be of the form \( cx_1 \) or \( cx_2 \). But none of these are edges of \( G \), which is a contradiction.

In the second case \( b_2a_1 \in E(G) \), and the subregion of \( Q_v^* \) bounded by the 4-cycle \( b_2a_1x_1x_2 \) has no vertices in its interior, so we must have either \( x_1b_2 \in E(G) \) or \( x_2a_1 \in E(G) \), a contradiction. So \( a_1 = b_2 \) and \( a_2 = b_1 \), and \( V(C_v) = W \) by \( (c) \). All \( v \in W \) are incident to 4 edges of \( M_v^* \), but \( a_1, a_2 \in W \) are incident to 2 more edges, and vertices of \( W \setminus \{a_1, a_2\} \) are incident to no more, so \( d(a_1) = d(a_2) = 6 \), and \( (a) \) gives \( d_2 \geq k - 2 \). Since \( a_1, a_2 \) are vertices of degree greater than 5, and \( G \) has at most 4 vertices with degree greater than 5, we get \( \min(d(x_1), d(x_2)) = 5 \), which gives that \( G \simeq T_n^5 \) as claimed.

For the remaining cases assume that \( x_1 \) and \( x_2 \) are not adjacent, so \( n \geq 19 \) and \( \Delta \geq 8 \). This also implies that \( N(x_i) = V(P(a_i, b_i)) \), so the paths \( P(a_i, b_i) \) have at least 5 vertices.

In this case the edges \( x_ia_i, x_ib_i \) lie on the boundary of the same face, so \( a_1b_1 \in E(G) \), and \( a_1b_1x_i \) is a boundary of a face. Moreover, as \( |V(P(a_i, b_i))| \geq 5 \), \( a_ib_i \notin E(M_v^*) \). The rest of the statement is trivial if \( a_1 = b_2 \) and \( a_2 = b_1 \), so assume that is not the case. Consider the connected subregion \( \mathcal{R} \) of \( Q_v^* \) bounded by the cycle \( P(b_1, a_2) \), (that has length at least 3 by the assumption); it has no vertices in its interior. Any edges between vertices of the cycle \( P(b_1, a_2) \) are edges of this cycle or lie inside \( \mathcal{R} \). This finishes the proof unless \( a_1 \neq b_2 \) and \( a_2 \neq b_1 \), so consider that to be the case. Let \( c_i \in V(P(b_1, a_{3-i})) \) and \( z \in V(G) \setminus \{x_1, x_2\} \) such that \( c_i \not\in E(G) \setminus E(M_v^*) \). As \( c_i \) lies on the boundary of the connected subregion \( \mathcal{R} \), Lemma \( 19(b) \) gives that \( z \in V(P(b_3-i, a_i)) \), as claimed. Also, if \( z_1, z_2 \in V(P(b_3-i, a_i)) \) are different neighbors of \( c_i \) where \( z_1z_2 \) is not an edge of the path \( P_v(b_3-i, a_i) \), then by Lemma \( 19(b) \) any internal vertex \( z_3 \) of the \( z_1z_2 \) subpath of \( P(b_3-i, a_i) \) can only have the edge \( z_3c_i \) incident upon it from \( E(G) \setminus E(M_v^*) \). Since by Lemma \( 19(a) \) \( z_3 \) must have an edge from \( E(G) \setminus E(M_v^*) \) incident upon it, \( (e) \) follows.

Assume to the contrary that \( a_1 \neq b_2 \) and \( a_2 \neq b_1 \). By \( (e) \) we have \( a_1b_1, a_2b_2 \in E(G) \setminus E(M_v^*) \). Let \( \mathcal{R} \) be the connected subregion of \( Q_v^* \) bounded by the cycle \( P_v(b_2, a_1)P_v(b_1, a_2) \). \( G \) has at most 4 vertices with degree greater than 5. As \( d(v) = \Delta > 6 \), \( V(G) \setminus \{v\} \) has at most 3 vertices with degree greater than 5. In particular, \( C_v \) contains at most 3 vertices with degree greater than 5. As by \( (c) \) \( V(C_v) = W \), each of \( a_1, a_2, b_1, b_2 \) has at least 4 edges incident upon them in \( E(M_v^*) \), and two edges incident upon them from \( E(G) \setminus E(M_v^*) \) (the edges \( a_1b_1, a_2x_i, b_1x_i \)). This gives that \( a_1, b_1, a_2, b_2 \) have degree at least 6, a contradiction. \( (f) \) follows.
Figure 11. 5-connected triangulations of order $n = 21$, 23 and $n \geq 25$, which have the same degree sequence as $T_n^5$. The gray regions show the mosaic graphs around the vertex of degree $k - 1$. The gray vertices and dashed edges on the triangulation of order 25 indicate the pattern to be repeated to get the construction for higher odd order. The two white vertices are at distance 4.

(g) Assume to the contrary that $a_1 \neq b_2$ or $a_2 \neq b_1$. By (f), without loss of generality we have $a_1 = b_2$ and $a_2 \neq b_1$. By definition $a_1 x_1, a_1 x_2 \in E(G)$ and by (e) all vertices of $P_v(b_1, a_2)$ are neighbors of $a_1$. All other neighbors of $a_1$ are one of the 4 neighbors of $a_1$ in $M^*_v$. Consequently $d(a_1) = 6 + |V(P_v(b_1, a_2))| \geq 8$, so $d(a_1) = d_2 \in \{k - 1, k - 2, k - 3\}$, and if $d(a_1) = k - 3$, then $G$ contains no degree 6 vertices. As $V(C_v) = W$, we must have $d(a_2) = d(b_1) = 6$, as $a_2, b_1$ each have 4 neighbors in $M^*_v$, and both are joined to $a_1 = b_2$, and to a single $x_i$, and not joined to anything else. By (a), $d(a_1) \geq k - 2$, and, as $v, a_1, a_2, b_1$ are the 4 vertices of degree greater than 5, all other vertices (including $x_1$ and $x_2$) have degree 5. So every $w \in W \setminus \{a_1, a_2, b_1\}$ has 4 neighbors in $M^*_v$, and is joined by an edge in $E(G) \setminus E(M^*_v)$ to exactly one of the vertices $x_1, a_1, x_2$, and the paths $P(a_i, b_i)$ have 5 vertices each. As the sum of degrees is $6n - 12 = k - 1 + d(a_1) + 12 + 5(n - 4)$, we get $d(a_1) = k - 2$. As $d(a_1) \geq 8$, this gives $n \geq 21$. Figure 11 has the graph $G$ for all $n \geq 21$. Since $G$ has the same degree sequence as $T_n^5$ and $W(G) \leq W(T_n^5)$, by Lemma 16 (f) we must have $W(G) = W(T_n^5)$ and the diameter of $G$ is at most 3. However, $G$ has diameter at least 4, as demonstrated on Figure 11, a contradiction. (g) follows.

(h) By (g), $a_1 = b_2$ and $a_2 = b_1$. By (e), $a_1 a_2 \in E(G)$. By (e) $V(C_v) = W$, and any edge from $E(G) \setminus E(M^*_v)$ incident upon a vertex $w \in W \setminus \{a_1, a_2\}$ connects $w$ to exactly one of $x_1, x_2$. Each $a_i$ has 4 incident edges in $E(M^*_v)$, and in addition, it is joined to exactly 3 more vertices: $x_1, x_2, a_{3-i}$. So $d(a_1) = d(a_2) = 7 = d_2 = d_3$. By (a), $d_3 + d_4 \leq 12$. 
consequently all vertices of \( V(G) \setminus \{v, a_1, a_2\} \) (including \( x_1 \) and \( x_2 \)) have degree 5. As \( 6n - 12 = k - 1 + 14 + 5(n - 3) \), \( n = 19 \). We have that \( G \cong X \) and \( W(G) = W(T_{19}^5) \) (see Figure 10).

The following theorem now follows:

**Theorem 24.** Let \( n \geq 15 \) be odd. If \( n \neq 19 \), then the unique minimizer of the Wiener index among \( 5 \)-connected triangulations of order \( n \) is \( T_n^5 \). If \( n = 19 \), then there are precisely two minimizers, \( T_{19}^5 \) and \( X \).

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