COMBINATORIAL FORMULAE FOR FINITE-TYPE INVARIANTS VIA PARITIES

MICAH CHRISMAN AND VASSILY OLEGOVICH MANTUROV

Abstract. The celebrated theorem of Goussarov states that all finite-type (Vassiliev-Goussarov) invariants of classical knots can be expressed in terms of Polyak-Viro combinatorial formulae. These formulae intrinsically use non-realizable Gauss diagrams and virtual knots.

Some of these formulae can be naturally extended to virtual knots; however, the class of finite-type invariants of virtual knots obtained by using these formulae (so-called Goussarov-Polyak-Viro finite-type invariants) is very small. Kauffman gave a more natural notion of finite-type invariants, which, however, turned out to be quite complicated: even invariants of order zero form an infinite-dimensional space.

Recently, the second named author introduced the notion of parity which turned out to be extremely useful for many purposes in virtual knot theory and low-dimensional topology; in particular, they turned out to be useful for constructing invariants of free knots, the latter being very close to the notion of order 0 invariants.

In the present paper we use the concept of parity to enlarge the notion of Goussarov-Polyak-Viro combinatorial formulae and provide explicit formulae for these invariants. Not all of the new invariants are of GPV finite-type, but they all are of Kauffman finite-type. Also, we establish some relations with the with the standard GPV formulae.

Keywords: knot, virtual knot, parity, finite-type invariant, combinatorial formula

Dedication: To Louis H. Kauffman, in celebration of his 65th birthday.

1. Introduction

1.1. Overview. Combinatorial formulae for finite-type invariants of classical knots were introduced by Polyak and Viro in [PV]. In [GPV], it was shown that every finite-type invariant of classical knots has a combinatorial formula. The formulae themselves are in terms of some linear combinations of arrow diagrams. The arrow diagrams correspond to certain subdiagrams of knots. The diagrams need not represent a classical knot, but all represent some virtual knot. However, the combinatorial formula for the classical knot invariant may not be an invariant of virtual knots.

On the other hand, the same idea can be used to find invariants of virtual knots. All such formulae represent finite-type invariants of virtual knots. In fact, they exhibit two different notions of finite-type. The first class of invariants, called Kauffman finite-type invariants, is parallel to the classical knot case. The second class of invariants, called Goussarov-Polyak-Viro finite-type invariants, are all Kauffman finite-type invariants. However, not all Kauffman finite-type invariants are of GPV finite-type. In fact, not even the Birman-Lin coefficients of the Jones-Kauffman polynomial are of GPV finite-type (see [Ka1], [C1], [C2]).

In this paper, a method is described which extends the notion of combinatorial formulae to include many more Kauffman finite-type invariants. All of the new formulae are of Kauffman finite-type, but surprisingly not all of them are of GPV finite-type. Also, some of the new formulae are invariant under the virtualization move. This is a property which no GPV finite-type invariant exhibits [C2].

The method of extension uses the second author’s recent discovery of parities in knot theory [Ma1]. A parity is a certain function on the Gauss diagram of a knot which behaves nicely under the Reidemeister moves. Parities have been used to find invariants of free knots and links [Ma1]. They have also been used to extend many other invariants of virtual knots [Ma].

The main idea behind the extension is as follows. Given a Gauss diagram, one marks every arrow with either a 0 or a 1, according to a given parity. Next one looks at the sum of all subdiagrams of this embellished diagram. Then we apply various kinds of projections. In one case, we destroy all diagrams having an even arrow. In another case, we destroy all diagrams having any odd arrows. Also, there are invariants that arise from projections which fall somewhere between these two extremes. The projections are not necessarily nicely behaved under Reidemeister moves, but they turn out to be useful for finding combinatorial formulae.

The most important projection for parities is the one which destroys all odd arrows. This has been used to improve many invariants. It was first proved in [Ma1] that such a projection is well defined (i.e. it maps equivalent diagrams to equivalent ones). This projection is related to the definition of the parity hierarchy.
which gives an infinite sequence of distinct parities (see Section 2.3). Another immediate consequence from this is the existence of an even counterpart and an odd counterpart of every combinatorial formula defined via the parity of bunches (see Sections 2.5, 3.2, and 19).

For each of the various projections, we construct an analogue of the Polyak algebra. For a very large class of examples of parities, the corresponding invariants turn out to be of Kauffman finite-type. Moreover, the underlying groups can be computed explicitly in many useful cases. In the case that all subdiagrams having an odd arrow are projected to zero, there is an isomorphism with the usual Polyak algebra (although, the resulting invariants are very different). In the case that all diagrams having an even arrow are projected to zero, we show that there is a basis for the space of invariants which depends only on the symbol of the invariant (i.e. on the degree n part). In fact, the basis can be computed explicitly for every degree. The main technical difficulty in this paper relies upon the solution of a system of equations that rectifies this symbol with the second Reidemeister move.

Parities can also be used to clarify a different kind of extension question. The theories of classical knots and long classical knots are identical. However, the theories of virtual knots and long virtual knots are very different (see, e.g. [Ma4]). This phenomenon is even observable at the level of combinatorial formulae. For example, there is a combinatorial formula for the order two invariant of compact classical knots [PV]. The formula does not extend to the virtual case. All such formulae vanish [GPV]. However, there are two linearly independent combinatorial formulae for long virtual knots. One might wonder why these formulae fail to provide invariants of compact virtual knots. In this paper, we show that the order two invariants of long virtual knots pull back to invariants of zero index knot diagrams. They are invariants up to Reidemeister moves involving only zero index diagrams. Moreover, the two invariants of long virtual knots coincide identically on this set.

Throughout the paper, we will look at invariants for compact virtual knots and for long virtual knots. The definition of the invariants is essentially the same but the resulting groups of invariants will turn out to be somewhat different. However, we will show that for one of our groups of invariants, every combinatorial \( P \)-formula for compact virtual knots arises from an identical system of equations as a combinatorial \( P \)-formulae for long virtual knots (see Section 3.8). In fact we will see that the two types of invariants take on essentially the same form.

The layout of the paper is as follows. In the remainder of Section 1.2 we review the definitions and relevant theorems concerning virtual knots and finite-type invariants.

In Section 2 we give an introduction to parities with numerous examples. Examples for parities on long virtual knots come from an index of arrows in a chord diagram. The index and its properties are also investigated in Section 2. In Theorems 13 and 14 we discuss the extension problem for the order two invariants of long virtual knots.

Section 3 begins with a description of the parity enhanced Polyak algebra. For a given \( n \), there are \( n \) distinct groups of Kauffman finite-type invariants having order \( \leq n \). The corresponding invariants are collectively referred to as combinatorial \( P \)-formulae. Examples for small \( n \) can be computed very easily by hand (see Section 3.1). In Section 3.4 we present a sufficient condition (called switch symmetry) on a parity \( \mathcal{P} \) so that all combinatorial \( \mathcal{P} \)-formulas are of Kauffman finite-type. All of our examples of parities turn out to satisfy the sufficient condition. Also in Section 3.6 we establish the surprising fact that there are combinatorial formulae that are not of GPV finite-type and that there are combinatorial formulae that are virtualization invariant.

For some of our projections, it is possible to construct combinatorial \( \mathcal{P} \)-formulae using known GPV formulae. In fact, we prove in Theorem 19 that every homogeneous GPV formula of order exactly \( n \) (i.e. one in which each term has \( n \) arrows) can be decomposed into an even part and an odd part. The even and odd parts are each given by combinatorial \( \mathcal{P} \)-formulae.

The last and most technical part of Section 3 deals with the projection where all subdiagrams having an even arrow are mapped to zero. The dimension of these invariants is computed exactly in Theorem 23. We describe an explicit generating set for them in terms of certain polynomials in Sections 3.5.1, 3.7.1, and 3.7.5. Here the structures for the Wilson line and Wilson loop case diverge. The Wilson loop case is considered separately in Section 3.8.

In Section 4.1 we present some computational results on the dimensions of the other combinatorial \( \mathcal{P} \)-formulae on the Wilson line. Also, we present 11 rationally linearly independent combinatorial \( \mathcal{P} \)-formulae of order 2 on the Wilson line. These invariants correspond to a projection which is not extremal.
1.2. Knots Diagrams, Virtual Knots, and Gauss Diagrams. Let $K: S^1 \to \mathbb{R}^2$ be a knot diagram. An orientation of a knot diagram is a choice of one of the two possible ways to traverse $S^1$: clockwise or counter-clockwise. Given an orientation of a knot, a crossing in a knot diagram is endowed with a local orientation (or local writhe). A $\oplus$ crossing is given by the right hand rule whereas a $\ominus$ crossing is given by the left hand rule (see Figure 1). In general, the crossing configuration and the local orientations are the combinatorial data that is counted by combinatorial knot invariants.

![Figure 1. The local orientation of a crossing](image)

Knot diagrams are equivalent up to a sequence of Reidemeister moves (see Figure 2) and planar isotopies. If $K$ and $K'$ are knot diagrams such that $K$ and $K'$ are the left hand side and right hand side of the same Reidemeister move respectively, we will write $K \leftrightarrow K'$. The same notation will be used for Gauss diagrams.

![Figure 2. The Reidemeister moves](image)

The Gauss diagram of a knot diagram $K: S^1 \to \mathbb{R}^2$ is defined as follows. On a copy of $S^1$ (this copy of $S^1$ is called the Wilson loop), mark all pairs of points $x, y \in S^1$ such that $K(x) = k(y)$. The points $x$ and $y$ are connected by a chord. Each chord is endowed with a direction according to the relation of the incident arcs in the knot diagram. The resulting arrow points from the overcrossing arc to the undercrossing arc. In addition, each arrow is marked $\oplus$ or $\ominus$ according to the local orientation of the crossing. Two Gauss diagrams are equivalent if there is an orientation preserving diffeomorphism of the Wilson loop mapping one to the other and preserving both the direction and sign of each arrow.

![Figure 3. The Kishino knot and its Gauss diagram](image)

There is also a natural notion of Gauss diagrams for long knots. In this case, the points are marked on a copy of $\mathbb{R}^1$ (this copy is called the Wilson line). Otherwise the construction is the same.

Every knot and long knot diagram has an associated Gauss diagram, but not every collection of signed arrows on the Wilson loop or Wilson line corresponds to a knot or long knot. However, there is correspondence between Gauss diagrams and virtual knots/long virtual knots.

A virtual knot diagram $V$ (see [Ka1] and [GPV]) is an immersion $V: S^1 \to \mathbb{R}^2$ such that all points where the immersion fails to be one-to-one, the intersection is transversal. In addition, every such intersection is endowed with either an overcrossing, an undercrossing or a virtual crossing. Virtual crossings are denoted by a small circle surrounding a transversal intersection. All crossings which are not virtual are called classical. Long virtual knot diagrams are defined in an analogous way.

Two virtual knot diagrams (or long virtual knot diagrams) are said to be equivalent if they are obtained from one other by a finite sequence of planar isotopies, the moves $\Omega_1, \Omega_2, \Omega_3$ and the four virtual moves
Figure 4. The Virtual Moves

(see Figure 3). An equivalence class of virtual knot diagrams under this notion of equivalence is called a virtual knot (or long virtual knot).

A Gauss diagram of a virtual knot (or long virtual knot) is found by connecting points $x$ and $y$ on the Wilson loop (resp. the Wilson line) by a chord if $V(x) = V(y)$ and the crossing is classical. All virtual crossings are ignored in the formation of the Gauss diagram. The chord is directed towards $x$ if $y$ is on the overcrossing arc and directed towards $y$ if $x$ is on the overcrossing arc. As usual, each arrow is marked with the local orientation of the crossing when $V$ is oriented. An example is given in Figure 3.

A surprising and useful fact (see [GPV] for a proof) is that if two virtual knots or long virtual knots have equivalent Gauss diagrams, then the virtual knots are themselves equivalent via a sequence of moves taken only from the set of virtual moves. Moreover, since every Gauss diagram represents some virtual knot, it follows that it is sufficient to consider only Gauss diagrams up to some equivalence corresponding to the moves $\Omega_1, \Omega_2, \Omega_3$ (Note: Two diagrams equivalent by planar isotopies will certainly have equivalent Gauss diagrams). It follows from a theorem of Östlund [Ost] that all Reidemeister moves may be obtained from the following transformations on Gauss diagrams.

Figure 5. Sufficient set of Reidemeister moves in Gauss diagram notation

The information in a Gauss diagram may be reduced using an intersection graph. Intersection graphs were originally defined by Chmutov, Duzhin and Lando. They appeared originally in the study of chord diagrams for Vassiliev invariants of classical knots (see [CDL]). For Gauss diagrams, two arrows $a$ and $b$ are said to intersect (or to be linked) if their endpoints alternate on the Wilson loop or line. We write $(a, b) = (b, a) = 1$ if $a$ and $b$ intersect and $(a, b) = 0$ otherwise. The intersection graph is the graph with a vertex for each arrow of the diagram and an edge between two vertices $a$ and $b$ exactly when $(a, b) = 1$ (see Figure 6). In this paper, the intersection graph plays a starring role; it is used to determine the parity of arrows in a Gauss diagram.

Figure 6. A long virtual knot and its intersection graph

The second named author and D.P. Ilyutko showed in [IM1] and [IM2] that intersection graphs can be used to create entirely new knot theories. In particular, one may write out Reidemeister relations on arbitrary signed graphs. Some of these graph-links do not correspond to the intersection graph of a Gauss diagram at all. There is a parallel here with the relation between Gauss diagrams of knots and virtual knots.
Also as in virtual knot theory, many invariants of classical knots extend to graph-links. The lesson for us is that the intersection graph encodes a lot of information about the knottedness of a virtual knot.

It was shown by Goussarov (see [GPV]) that if two classical knot diagrams are equivalent by a sequence of virtual and classical moves, then they are knot diagrams of equivalent knots. The following well-known condition gives an efficient method to show that a Gauss diagram must correspond to a virtual knot diagram (although it may be equivalent to a classical knot).

**Proposition 1.** If $K$ is a classical knot diagram, then the degree of every vertex of the intersection graph of $K$ is even.

### 1.3. Finite-Type Invariants

In this subsection, we review the two different notions of finite-type invariants for virtual knots and some related results. It should be noted that there are other generalizations of Vassiliev finite-type invariants and combinatorial formulæ. For example, there is Fiedler’s work on Gauss diagram formulæ for invariants of knots in thickened surfaces (see [Fd]). Also, Andersen and Mattes showed in [AM] there is the universal invariant for such finite-type invariants which arise from configuration space integrals. More recent work in this area has been done by Grishanov and Vassiliev (see [GrVa]).

Another interesting approach to combinatorial formulæ for knot invariants comes from considering the cohomology of knot spaces (the initial idea which led Vassiliev to his definition of finite type invariants). Vassiliev used this in [Va] to find combinatorial formulæ which differ from the Gauss diagram approach.

Our focus is entirely upon Kauffman’s generalization of finite-type invariants to virtual knots and the GPV notion of combinatorial formulæ.

#### 1.3.1. Kauffman Finite-Type

In [Ka1], Kauffman introduced the notion of graphical finite-type invariants. This notion of finite-type invariant is the one which is most similar to the well-known diagrammatic formulation of finite-type for classical knots. In Kauffman’s version, singular knots are replaced with 4-valent graphs. Later on, by a finite-type invariant of virtual knots we mean a Kauffman finite-type invariant, unless otherwise specified.

Let $K_\bullet: S^1 \to \mathbb{R}^2$ (or $\mathbb{R}^1 \to \mathbb{R}^2$ for long knots) be an immersion such that at each point where the map fails to be one-to-one the intersection is transversal. Moreover, it is required that each such self-intersection is embellished with one of three possible crossing types: over/under crossing, virtual crossing, or a graphical vertex. In addition to planar isotopies, the Reidemeister moves, and virtual moves, one adds the rigid vertex isotopy moves:

![Rigid vertex isotopy moves](image)

The same happens when one considers knots in a thickened surfaces $S_g \times I$, so, we adopt the same set of rigid vertex isotopy moves for the case of virtual knots as well.

Two knotted 4-valent graphs are said to be equivalent if they are obtained from one another by a finite sequence of planar isotopies, Reidemeister moves, and virtual moves, one adds the rigid vertex isotopy moves:

Two virtual knot invariant $v$ can be extended to an invariant of knotted 4-valent graphs by successive application of the rule:

$$v\left(\begin{array}{c}
\end{array}\right) = v\left(\begin{array}{c}
\end{array}\right) - v\left(\begin{array}{c}
\end{array}\right)$$

The fact that the extension indeed defines an invariant of knotted 4-valent graphs is an easy consequence of the definition and the moves $\Omega_2$, $\Omega_3$, $\text{Vr}2$, $\text{Vr}3$, and $\text{Vr}4$. This extension to knotted 4-valent graphs will also be denoted $v$.

A virtual knot invariant $v$ is said to be of Kauffman finite-type of order $\leq n$ if $v(K_\bullet) = 0$ for all knotted 4-valent graphs $K_\bullet$ with more than $n$ graphical vertices.

There are numerous examples of Kauffman finite-type invariants. The coefficient of $x^n$ in the power series expansion of the Birman-Lin substitution $A \to e^x$ of the Jones-Kauffman polynomial (for virtual knots or long knots) is a rational valued Kauffman finite-type invariant of order $\leq n$ (see [Ka1] for proof). There is an even more discriminating generalization of the Jones-Kauffman polynomial due to Manturov [Mau]. The Birman-Lin coefficients of the generalization are also of Kauffman finite-type. More examples of Kauffman finite-type invariants are given in the next section.
1.3.2. Goussarov-Polyak-Viro Finite-Type. This notion of finite-type invariants ultimately arises from the construction of combinatorial formula for Vassiliev invariants in [PV]. The formal construction was carried out in [GPV].

As usual, we consider immersions $K : S^1 \to \mathbb{R}^2$ (or $\mathbb{R}^1 \to \mathbb{R}^2$) where each place at which the map fails to be one-to-one we have a transversal intersection. Moreover, each such self-intersection is embellished with either an over/under crossing, a virtual crossing, or an over/under semi-virtual crossing:

\begin{center}
\includegraphics[width=0.7\textwidth]{crossings.png}
\end{center}

Note that the definition of these crossing types depends also the orientation of the crossing. For brevity we have drawn the crossings but omitted the orientation of the strands.

At the moment, virtual knot diagrams are considered equivalent only up to planar isotopies. No other relations are factored out at this time.

Virtual knot or long virtual knot invariants are extended to these semi-virtual diagrams by using the relation:

$$v\left(\begin{array}{c}
\includegraphics[width=0.2\textwidth]{over-under.png}
\end{array}\right) = v\left(\begin{array}{c}
\includegraphics[width=0.2\textwidth]{over-under.png}
\end{array}\right) - v\left(\begin{array}{c}
\includegraphics[width=0.2\textwidth]{over-under.png}
\end{array}\right)$$

The extension of a virtual knot invariant is also denoted $v$. A virtual knot invariant is said to be of Goussarov-Polyak-Viro finite-type of order $\leq n$ if $v(K) = 0$ for all semi-virtual knots $K$ with more than $n$ semi-virtual crossings. For brevity, they are often called GPV finite-type invariants.

In [GPV] it was shown why every Goussarov-Polyak-Viro finite-type invariant is of Kauffman finite type. Indeed, it follows from the relation:

$$v\left(\begin{array}{c}
\includegraphics[width=0.2\textwidth]{over-under.png}
\end{array}\right) = v\left(\begin{array}{c}
\includegraphics[width=0.2\textwidth]{over-under.png}
\end{array}\right) - v\left(\begin{array}{c}
\includegraphics[width=0.2\textwidth]{over-under.png}
\end{array}\right)$$

The advantage of the GPV finite-type invariants is that they admit a purely algebraic universal description in terms of Gauss diagrams. This is the perspective which is the main subject of this paper. One of objectives of our paper is to extend this algebraic set-up by adding parity considerations.

Let $A$ denote set of dashed arrow diagrams. These are just like Gauss diagrams except that every arrow is drawn dashed. Let $D$ denote the collection of Gauss diagrams, up to equivalence. The map $i : Z[D] \to Z[A]$ just makes every arrow of a Gauss diagram dashed. Define $I_{GPV} : Z[D] \to Z[A]$ on generators by:

$$I_{GPV}(D) = \sum_{D' \subset D} i(D')$$

Here the sum is taken over all subdiagrams of $D$: those diagrams whose arrows are taken from a subset of the arrows of $D$ (signs included). This map has a satisfying interpretation in terms of Gauss diagrams. A dashed arrow represents a semi-virtual crossing. Indeed:

\begin{center}
\includegraphics[width=0.5\textwidth]{dashed_arrow.png}
\end{center}

Rearranging this gives a schematic definition of $I_{GPV}$. This can be made precise. Let $\mathcal{K}$ denote the collection of equivalence classes of knots and $\Delta \Omega$ the submodule of $Z[D]$ generated by the relations in Figure 5. The previous discussion shows that $Z[\mathcal{K}] \cong Z[D]/\Delta \Omega$. Let $\Delta P$ denote the Polyak relations in Figure 7. Define $\bar{A} = Z[A]/\Delta P$.

The Polyak relations can be interpreted as the image of the moves $\Omega_1$, $\Omega_2$, and $\Omega_3$ under the map $I_{GPV}$. The terms in the image of a move are grouped together so that they differ only as in the drawn arcs. Factoring out by the resulting relations (i.e. the ones in Figure 7) gives a sufficient condition that the $\Omega$ moves be satisfied. In fact, the condition is also necessary.

**Theorem 2** (Goussarov, Polyak, Viro [GPV]). The map $I_{GPV} : Z[D] \to Z[A]$ is an isomorphism. The inverse can be defined explicitly:

$$I_{GPV}^{-1}(A) = \sum_{A' \subset A} (-1)^{|A-A'|} i^{-1}(A)$$
The main interest in GPV finite-type invariants comes from the following remarkable theorem.

**Theorem 3** (Goussarov, Polyak, Viro, [GPV]). The map $(I_{GPV})_n : \mathbb{Z}[\mathcal{X}] \to \bar{A} \to \bar{A}_n$ is universal in the sense that if $G$ is any abelian group, and $v$ is a GPV finite-type invariant of order $\leq n$, then there is a map $v' : \bar{A}_n \to G$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{Z}[\mathcal{X}] & \xrightarrow{v} & G \\
\downarrow^{I_{GPV}} & & \downarrow^{v'} \\
\bar{A} & \xrightarrow{v'} & \bar{A}_n
\end{array}
$$

In particular, the vector space of rational valued GPV finite-type invariants of order $\leq n$ is finite dimensional and can be identified with $\text{Hom}_{\mathbb{Q}}(\bar{A}_n, \mathbb{Q})$.

The groups $\bar{A}_n$ fall into a natural sequence of surjections:

$$
\cdots \to \bar{A}_n \to \bar{A}_{n-1} \to \cdots \to \bar{A}_3 \to \bar{A}_2 \to \bar{A}_1 \to \bar{A}_0
$$

Also, they satisfy the following useful short exact sequence:

$$
0 \to \frac{\bar{A}_n}{\bar{A}_{n-1} + \Delta P} \to \bar{A}_n \to \bar{A}_{n-1} \to 0
$$

There is a pairing $\langle \cdot, \cdot \rangle : \mathbb{Z}[\mathcal{A}] \times \mathbb{Z}[\mathcal{A}] \to \mathbb{Z}$ defined on generators by $\langle D_1, D_2 \rangle = 1$ if $D_1 = D_2$ and $\langle D_1, D_2 \rangle = 0$ if $D_1 \neq D_2$.

A GPV combinatorial formula of type $\leq n$ is an element $F \in \mathbb{Z}[\mathcal{A}]$ such that if $D \in \mathbb{D}$ with $\text{coeff}(D, F) \neq 0$, then $D$ has $\leq n$ arrows. Moreover, it is required that $\langle F, r \rangle = 0$ for every $r \in \Delta P$. By Theorem 3, a combinatorial formula generates an integer valued virtual knot or long virtual knot invariant:

$$
\langle F, I_{GPV}(\cdot) \rangle : \mathbb{Z}[\mathcal{D}] \to \mathbb{Z}
$$

The main interest in GPV finite-type invariants comes from the following remarkable theorem.

**Theorem 4** (Goussarov [GPV]). If $v$ is an integer valued finite-type invariant of order $\leq n$ of classical knots, then there is a combinatorial formula $F \in \mathbb{Z}[\mathcal{A}]$ (on the Wilson line) such that every summand of $F$ has at most $n$ dashed arrows and for all classical knots $K$,

$$
v(K) = \langle F, I_{GPV}(K) \rangle
$$
It is important to note that the theorem does not assert that $F$ is a combinatorial formula for a virtual knot/long knot invariant as defined in the previous paragraph.

1.3.3. Finite-Type Invariants and the Virtualization Move. In [C1], the first named author extended Eisermann's twist lattices [E] to Kauffman and GPV finite-type invariants of virtual knots (also see [E] for related references). This provides an elementary tool by which to distinguish the two flavors of finite-type. An example of a twist sequence for Kauffman finite-type invariants is:

| all arrows signed $\odot$ | all arrows signed $\oplus$ |
|--------------------------|--------------------------|
| $\ldots$                 | $\ldots$                 |
| $k = -2, k = 0$          | $k = -1, k = 0$          |
| $k = 1, k = 2, k = 3, \ldots$ | $k = 1, k = 2, k = 3, \ldots$ |

An example of a twist sequence for GPV finite-type invariants is (called a fractional twist sequence):

| all arrows signed $\odot$ | all arrows signed $\oplus$ |
|--------------------------|--------------------------|
| $\ldots$                 | $\ldots$                 |
| $k = -2, k = 0$          | $k = -1, k = 0$          |
| $k = 1, k = 2, k = 3, \ldots$ | $k = 1, k = 2, k = 3, \ldots$ |

A twist lattice (fractional twist lattice) is a function $\Phi : \mathbb{Z}^m \to \mathbb{D}$ such that each of the $m$ standard inclusions $\mathbb{Z} \to \mathbb{Z}^m \to \mathbb{D}$ is a twist sequence (resp. fractional twist sequence). We have the following generalization of Eisermann’s theorem.

**Theorem 5** (Chrisman [C1]). A virtual knot or virtual long knot invariant $v : \mathbb{Z}[\mathbb{D}] \to \mathbb{Q}$ is of Kauffman finite-type (GPV finite-type) of order $\leq n$ if and only if for every twist lattice (resp. fractional twist lattice) $\Phi : \mathbb{Z}^m \to \mathbb{D}$, the composition $v \circ \Phi : \mathbb{Z}^m \to \mathbb{Q}$ is a polynomial of degree $\leq n$.

Kauffman pointed out in [Ka1] that there are graphical finite-type invariants of order $\leq 2$ which are not of GPV finite-type of order $\leq 2$. In [C1], a twist sequence argument was used to extend this result to all orders. In particular, Birman-Lin’s coefficients of the Jones-Kauffman polynomial are of Kauffman finite-type of order $\leq n$ but not of GPV finite-type of order $\leq m$ for any $m$. In Section 3.6, a fractional twist sequence argument will be used to show that all of the new invariants are of Kauffman finite-type but not all of them are of GPV finite-type.

The obstruction for the Jones-Kauffman polynomial is its invariance under the so-called virtualization move (see [FKM] and [Ma3] for further discussion):

$$\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \rangle = \langle \begin{array}{c} \nearrow \\ \nwarrow \nearrow \searrow \nwarrow \end{array} \rangle$$

**Figure 8. The Virtualization Move**

The virtualization move has a very simple diagrammatic description: in a Gauss diagram of a virtual knot, it changes the direction of some arrow without changing the sign.

For classical knots, it usually suffices to know the signs of chords of a Gauss diagram to restore the arrow directions (if we require a diagram to have no virtual crossings). For virtual knots, this will not work. A different choice of an arrow direction may yield two inequivalent virtual knots.

A natural question is whether one needs arrow directions for Gauss diagrams of classical knots in the first place. To eliminate this piece of combinatorial data, one needs to prove the following virtualization conjecture, first stated in [FKM]: if two classical knot diagrams are connected by a sequence of Reidemeister moves, virtual moves, and virtualizations, then the knots are isotopic.

In particular, it is very important to find examples of invariants supporting this conjecture (i.e., invariants of classical knots extended to the case of virtual knots such that the resulting extension does not detect any virtualization moves). There are many such invariants known. For example, the Jones polynomial [Ka1], some of its generalizations [Ma5], and Khovanov homology [Ma4] all share this property.

It was shown by the first named author in [C2] that every GPV finite-type invariant must detect the virtualization move on some pair of virtual knots/virtual long knots. In Sections 3.6 and 3.8 it will be shown that some of the new combinatorial formulae are invariant under the virtualization move.
1.4. **Flat Knots, Free Knots, and Invariants of Order Zero and One.** It is well known that any classical knot may be turned into the unknot by some sequence of crossing switches. The implication for finite-type invariants is that there are no nonconstant order zero invariants for classical knots.

For virtual knots and virtual long knots, all GPV finite-type invariants of order zero are constants. This follows easily from the fact that there is only one diagram having no arrows.

The story for Kauffman finite-type invariants of order zero is much more complicated. These invariants are related to flat virtual knots [Ka1]. A flat virtual knot is an equivalence class of virtual knot diagrams. Two virtual knot diagrams are in the same flat equivalence class if they may be obtained from one another by a sequence of crossing changes, virtual moves, or planar isotopies.

Flat virtual knots are represented in the plane as virtual knot diagrams where the over/under crossing information has been forgotten. The nonvirtual crossings are indicated just as usual intersection points of two lines. For each such diagram, there corresponds a signed chord diagram. There is a virtual knot having either choice of sign in the flat equivalence class. However, there is only one choice of the direction of the arrow so that the chord diagram corresponds with the representation of the flat equivalence class. Changing both direction and sign of an arrow corresponds to switching the crossing from over to under or vice versa.

Diagrams of flat virtual knots are considered equivalent up to planar isotopies, virtual moves and the moves below:

\[
\begin{align*}
\text{→} & \quad \text{→} & \quad \leftarrow & \quad \leftarrow \\
\text{→} & \quad \text{→} & \quad \leftarrow & \quad \leftarrow \\
\text{→} & \quad \text{→} & \quad \leftarrow & \quad \leftarrow \\
\text{→} & \quad \text{→} & \quad \leftarrow & \quad \leftarrow \\
\end{align*}
\]

Let \( \mathcal{F} \) denote the resulting collection of equivalence classes of all these moves. The relations above imply that there is a natural projection \( f : \mathcal{K} \rightarrow \mathcal{F} \).

**Proposition 6.** If \( v : \mathcal{K} \rightarrow \mathbb{Q} \) is a Kauffman finite-type invariant of order zero, then there is an invariant of flat knots \( \bar{v} : \mathcal{F} \rightarrow \mathbb{Q} \) so that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{v} & \mathbb{Q} \\
\downarrow f & & \downarrow \bar{v} \\
\mathcal{F} & & \\
\end{array}
\]

On the other hand, if \( \bar{v} : \mathcal{F} \rightarrow \mathbb{Q} \) is a flat virtual knot invariant then \( \bar{v} \circ f : \mathcal{K} \rightarrow \mathbb{Q} \) is a Kauffman finite-type invariant of order 0.

**Proof.** This follows immediately from the definitions and the various notions of equivalence. \( \square \)

It is well-known (see [Ma1] and [Ka1]) that the flat projection of the Kishino knot (see Figure 3) is not flat equivalent to the unknot. Since there are nontrivial flat virtual knots, it follows that there are nontrivial Kauffman finite-type invariants of order zero.

Flat virtual knots may themselves be simplified to free knots (see [Ma1]). Free knots are equivalence classes of virtual knot diagrams modulo both crossing changes and virtualization moves. A virtualization move changes the classical crossing of a virtual knot from over to under or vice versa, but the sign of the crossing remains unchanged (see Figure 8). Gauss diagrams of free knots therefore correspond to unsigned chord diagrams where chords represent any of the possible classical crossings. Indeed, any set of choices for both direction and sign of the arrows in an unsigned diagram will correspond to some virtual knot in the free equivalence class. As usual, virtual crossings are not accounted for in the chord diagram of a free knot.

Alternatively, one may define free knots as immersions of 4-valent graphs. The immersions have a unique unicursal component and a specified opposite edge structure. The opposite edge structure in turn specifies some Euler circuits of the graph. This is how one traverses the free knot in one of two possible directions.

Free knots are considered equivalent up to planar isotopies, virtual moves, and the following moves, drawn here in chord diagram notation:

\[
\begin{align*}
\text{→} & \quad \text{→} & \quad \leftarrow & \quad \leftarrow \\
\text{→} & \quad \text{→} & \quad \leftarrow & \quad \leftarrow \\
\end{align*}
\]

Denote the collection of the resulting equivalence classes of free knots by \( \mathcal{G} \). There is a natural projection \( g : \mathcal{K} \rightarrow \mathcal{G} \) that factors through \( \mathcal{F} \). The following proposition follows from the various definitions.
Proposition 7. If \( v : K \to \mathbb{Q} \) is a Kauffman finite-type invariant of order zero that is invariant under the virtualization move, then there is a \( \bar{v} : \mathcal{S} \to \mathbb{Q} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
K & \xrightarrow{v} & \mathbb{Q} \\
\downarrow{g} & & \downarrow{\bar{v}} \\
\mathcal{S} & &
\end{array}
\]

On the other hand, if \( \bar{v} : \mathcal{S} \to \mathbb{Q} \) is an invariant of free knots, then \( \bar{v} \circ g : K \to \mathbb{Q} \) is a virtualization invariant Kauffman finite-type invariant of order zero.

It was shown by the second named author in [Ma1] that there are nontrivial free knots (it was independently reproved by A.Gibson [Gib]). The above proposition implies that there are nontrivial virtualization invariant Kauffman finite-type invariants of order zero. Another strong invariant of free knots is given in [MM]. However, note that the free projection of the Kishino knot is trivial: its chord diagram can be trivialized by two application of the free version of the Reidemeister two move (see [Ma1]). Hence, not all Kauffman finite-type invariants of order zero are virtualization invariant.

There are many nice and easily computable invariants of free knots. Many of them rely upon parity considerations. In fact, it was for the analysis of free knots that this concept was developed (see [Ma1]).

The new results of this paper do not produce any new invariants of order zero.

The invariants of order one have an essentially parallel tale to those of order zero. For classical knots, all finite-type invariants of order one are constant. The relation P1 also kills nonconstant order one GPV invariants. However, there are many Kauffman finite-type invariants of order one. In fact, the new invariants presented in Section 3.5 form an infinite number of them (as every parity defines two linearly independent formulae of order one).

2. Parities and the Index

The use of parity was motivated by the very first example of parity of chords on a Gauss diagrams: a chord of a Gauss diagram is even whenever the number of chords it intersects (it is linked with) is even; otherwise this chord is odd. However, the first fundamental work which treats general notion of parity is [Ma1]: one can notices that the notion of parity can be axiomatized by taking some key properties, and there are lots of other parities except the Gauss one. For a list of parities and many of their uses, see the paper of the second named author [Ma1]. In this section, we give the definitions and examples that will be used throughout the paper. We also discuss the index and its application to the extension problem for order two invariants of long virtual knots.

2.1. Definition of Parity. Let \( \mathcal{D}^{(1,0)} \) denote the collection of Gauss diagrams (on either the Wilson loop or Wilson line) which are embellished with a 1 or a 0 at every arrow. For \( D \in \mathcal{D} \), let \( C(D) \) denote the set of arrows (or chords) of \( D \). Parities satisfy conditions imposed on Gauss diagrams. It is important to note that in a given condition or relation, all participating diagrams are identical outside the drawn intervals. A parity is a couple of functions \( P = (P, p_D) \), where \( P : \mathcal{D} \to \mathcal{D}^{(1,0)} \), \( p_D : C(D) \to \mathbb{Z}_2 \) and \( P \) is the function which assigns a 1 or a 0 at every arrow according to \( p_D \). Moreover, we require that \( P \) satisfies the following conditions:

1. (a) For all \( D \in \mathcal{D} \), if \( D = \begin{array}{c} i \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \), then \( p_D(i) = 0 \).
   (b) If \( D = \begin{array}{c} i \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \) and \( D' = \begin{array}{c} i \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \), then for all \( j \in C(D) \cap C(D') \), \( p_D(j) = p_{D'}(j) \).

2. (a) For all \( D \in \mathcal{D} \), if \( D = \begin{array}{c} \odot \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \), then \( p_D(i) = p_D(j) \).
   (b) If \( D = \begin{array}{c} \odot \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \) and \( D' = \begin{array}{c} \odot \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \), then for all \( k \in C(D) \cap C(D') \), \( p_D(k) = p_{D'}(k) \).

3. Suppose that \( D \) and \( D' \) are given as below.

\[
D = \begin{array}{c} \odot \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} \quad D' = \begin{array}{c} \odot \\
\uparrow \ \uparrow \ \uparrow \\
\downarrow \ \downarrow \ \downarrow \\
\odot \ \odot \ \odot \\
\end{array} 
\]
It is important to note that these defining conditions on parities imply parity conditions on all other types of Reidemeister moves. This is a consequence of Ostlund’s theorem 2.1.

2.2. The functorial map $f$. It turns out that the parity axioms listed above lead to a simple and powerful map on the set of free knots and, more generally, on the set of virtual knots.

Let $\mathcal{P}$ be any parity and $D$ be a Gauss diagram. Let $f(D)$ be a diagram obtained from $D$ by making all odd crossings virtual. In other words, we remove all odd arrows from the Gauss diagram.

The following theorem follows from definitions.

**Theorem 8.** Let $\mathcal{P}$ be an parity. The map $f$ is a well-defined map on the set of all virtual knots. For a Gauss diagram $D$, $f(D) = D$ if and only if all arrows of $D$ are even with respect to $\mathcal{P}$ (i.e. marked with a 0). Otherwise, the number of arrows of $f(D)$ is strictly less than the number of arrows of $D$.

The map $f$ plays a role in defining the parity hierarchy (see Section 2.3), establishing the even/odd decomposition of homogenous GPV formulae (see Section 2.2), and proving that long virtual knots having zero index at every arrow are closed under Reidemeister moves (see Theorem 12 and preceding definitions).

2.3. The Index and Examples of Parities. For $D \in \mathcal{D}$, let $G_D$ denote the intersection graph of $D$ (as defined by Chmutov and Duzhin [CDL]). An arrow of $D$ corresponds to a vertex in $G_D$. If the degree of the vertex is odd, the corresponding chord is decorated with a 1. If the vertex is even, the chord is decorated with a 0. It was shown in [Ma1], [Ma2] that this defines a parity $\mathcal{P}$. We will refer to this as the Gaussian parity (or intersection parity) and denote it by $\mathcal{P}_0$.

This gives rise to an infinite sequence of parities $\mathcal{P}_n$, called the parity hierarchy, that was first constructed in [Ma2]. Let $D$ be a Gauss diagram on the Wilson line. For an arrow $x$ of $D$ and arrow $y$ we say that an endpoint $E$ of $y$ is between the endpoints $E_1, E_2$ of $x$ (with $E_1 < E < E_2$) if $E_1 < E < E_2$. We define an index $I_D(x)$ on each arrow $x$ of a Gauss diagram $D$ by using the formula:

$$I_D(x) = \sum_{y \in \mathcal{D}, y \neq x \neq z} \delta(x, y) \sigma(x) \sigma(y)$$

where $\delta(x, y) = 1$, if the arrowhead of $y$ lies between the endpoints of $x$ and $-1$ otherwise and $\sigma(z)$ is the sign of the arrow $z$. The symbol $(x, y)$ is as defined in Section 2.2.

**Lemma 9.** The index satisfies the following properties.

1. If $D$ and $D'$ are as in (1) of Section 2.2 then $I_D(i) = 0$ and $I_D(x) = I_D'(x)$ for all $x \neq i$.
2. If $D, D'$ are as in (2) of Section 2.2 then $I_D(x) = I_D'(x)$ for all $x \neq i, j$ and $I_D(i) = -I_D(j)$.
3. If $D$ and $D'$ are as in (3) of Section 2.2 then $I_D(x) = I_D'(x)$ for all $x \neq i, j, k$ and $I_D(l) = I_D'(l')$ for $l \in \{i, j, k\}$.

Define a parity $\mathcal{P}_1 = (P_1, p_1)$ as follows. Let $D$ be a Gauss diagram and $x \in C(D)$. If $(P_0, p_0)(x) = 1$ for some $x \in C(D)$, define $(p_1)D(y) = (p_0)D(y)$ for all $y \in C(D)$. Otherwise, all arrows are even. In this case, either $I_D(x) \equiv 0 \pmod{4}$ or $I_D(x) \equiv 2 \pmod{4}$. In the first case, set $(p_1)D(x) = 0$ (as $I_D(x)$ is an even multiple of 2). In the second case, $(p_1)D(x) = 1$ (as $I_D(x)$ is an odd multiple of 2).

The same idea is used to define a parity $\mathcal{P}_n = (P_n, p_n)$ inductively. Let a Gauss diagram be decorated as in $\mathcal{P}_{n-1}$. If a diagram has an odd arrow, define $(p_n)D(y) = (p_{n-1})D(y)$ for all $y \in C(D)$. Otherwise, $I_D(y) \equiv 0 \pmod{2^n}$ for all $y \in C(D)$. Hence it follows that either $I_D(x) \equiv 0 \pmod{2^{n+1}}$ or $I_D(x) \equiv 2^n \pmod{2^{n+1}}$. In the second case, $(p_n)D(x) = 0$ (as the index is an even multiple of $2^n$) and in the second case, $(p_n)D(x) = 1$ (as the index is an odd multiple of $2^n$).

A second way to think of the parity of hierarchies is through the functorial map $f$ of Section 2.2. For a Gauss diagram $D$ embellished according to the Gaussian parity, if $f(D) \neq D$, then the labels of $D$ are constant throughout the hierarchy. Otherwise, $f(D) = D$ and the labels of $D$ are altered according to the congruence of the index modulo 4, as above.

Note that when the map $f$ is applied to a diagram $D$, the embellishment of $f(D)$ according to some parity $\mathcal{P}$ may not correspond to the embellishment of the corresponding arrow in $D$. In this sense, the map $f$ does not preserve the parity of an arrow although it preserves equivalence of virtual knots.
Theorem 10. \( \mathcal{P}_n \) is a parity for every \( n \in \mathbb{N} \cup \{0\} \). Moreover, if \( n \neq m \), then \( \mathcal{P}_n \) and \( \mathcal{P}_m \) do not coincide on all diagrams and hence are distinct.

Proof. The proof is by induction, starting with \( n = 1 \). By Lemma 9 the Reidemeister moves preserve the index. It is thus sufficient to show that everything works modulo four. Condition (1) of Section 2.1 is satisfied since \( I_D(i) = 0 \).

For condition (2) of Section 2.1, first note that for all \( x \neq i, j \), the contribution from arrows \( i \) and \( j \) to \( I_D(x) \) adds to zero. Also \( I_D(i) = -I_D(j) \). If \( D \) is a diagram such that all arrows have even index, it follows that \( I_D(i) \equiv I_D(j) \pmod{4} \).

The only tricky thing to check is the \( \Omega^3 \) move. In \( \mathcal{P}_0 \), we have that there is an even number of odd arrows in an \( \Omega^3 \) move. The odd arrows remain odd in \( \mathcal{P}_1 \). So an \( \Omega^3 \) move involving diagrams with an odd arrow is automatically satisfied.

Consider then the case of an \( \Omega^3 \) move involving diagrams all of whose arrows are even. It is easier, and in no way damaging to the proof, to think of the \( \Omega^3 \) move on the Wilson loop. Since the index is preserved by the move, it is sufficient to consider only one side of it. Hence, consider one of the cases for RHS given below.

Consider the three grey boomerangs drawn above. They represent the initial and terminal sides of certain arrows which intersect pairs of arrows taken from \( \{i, j, k\} \). If the boomerang crosses arrows \( p \) and \( q \) and the contribution to \( I_D(p) \) is \( r \), then the contribution to \( I_D(q) \) is \(-r \). It follows that:

\[
I_D(i)/2 + I_D(j)/2 + I_D(k)/2 = 0
\]

Therefore, an even number \( I_D(i)/2, I_D(j)/2, I_D(k)/2 \) is odd. So an even number of \( I_D(i), I_D(j), \) and \( I_D(k) \) is congruent to 2 (mod 4). The other cases follow similarly. This completes the proof that \( \mathcal{P}_1 \) is a parity. The proof that the induction step is satisfied is virtually identical and is therefore omitted.

For the claim that the parities are distinct, it is sufficient to show that there is a Gauss diagram \( D_n \) such \( (p_k)_{D_n}(x) = \) 0 for all \( x \in C(D_n) \) and \( k < n \) and \( (p_n)_{D_n}(y) = 1 \) for some \( y \in C(D_n) \). It is easy to check that the following diagrams \( D_n \) satisfy this property.

2.4. Parities of Two Component Links. As another example we have a parity for two-component links. Let \( \mathcal{L} \) denote the set of 2-component links (classical or virtual). A crossing of a diagram \( L \in \mathcal{L} \) is said to be even if it is formed by one component of the link \( L \), and odd if it is formed by two components. It can be easily checked that this parity satisfies all parity axioms.

2.5. Parities of Bunches. Let \( \mathcal{P} \) be any parity and let \( n \in \mathbb{N} \). Let \( \mathcal{P}^n \) denote the parity of \( n \)-bunches (i.e. sets of arrows with exactly \( n \) elements) which is defined as follows. Set \( \mathcal{P}^1 = \mathcal{P} \). For \( n > 1 \), suppose that \( D \in \mathcal{D} \) has at least \( n \) arrows and that \( a_1, \ldots, a_n \in C(D) \), \( a_i \neq a_j \). The bunch \( \{a_1, \ldots, a_n\} \) is said to be even if \( p_D(a_i) = 0 \) for all \( i, 1 \leq i < n \). Otherwise, the bunch \( \{a_1, \ldots, a_n\} \) is said to be odd.

We will use parities of \( n \)-bunches to obtain an interpretation of the invariants on \( \mathcal{O}_n \) in Section 2.3. This ultimately leads to the decomposition formula of Section 2.3.

2.6. Zero Index Diagrams and the GPV Invariants of Order Two. The order two Vassiliev invariant for compact classical knots was shown by Polyak and Viro in [PV] to be given by the combinatorial formula
The action of $k \in \mathbb{Z}_{2n}$ is defined by $k$ iterations of the action of 1 and $-1$ is identified with $2n-1$ applications of 1.

**Lemma 11.** $\mathbb{Z}_{2n} \cdot D = \{\overline{D}\}$

**Proof.** This is obvious from the definitions. \hfill $\square$

Let $\mathcal{Z}$ denote those Gauss diagrams of zero index on the Wilson line. By Lemma [11] there is a well-defined map $\Pi$ from $\mathcal{Z}$ to the Gauss diagrams on the Wilson loop, $\Pi(D) = \overline{D}$. Let $\overline{Z}$ denote the image of $\Pi$.

**Definitions:** Let $\mathcal{S}$ be a set of Gauss diagrams which are either all on the Wilson loop or all on the Wilson line. A Reidemeister move in $\mathcal{S}$ is a Reidemeister move $D \leftrightarrow D'$, where $D, D' \in \mathcal{S}$ (see Figure 5). Two diagrams $D, D' \in \mathcal{S}$ are Reidemeister equivalent in $\mathcal{S}$ if there is a sequence $D = D_0 \leftrightarrow D_1 \leftrightarrow D_1 \leftrightarrow \cdots \leftrightarrow D_{m-1} \leftrightarrow D_m = D'$ where $D_i \leftrightarrow D_{i+1}$ are Reidemeister moves in $\mathcal{S}$ for all $i$, $0 \leq i \leq m - 1$. A function $v : \mathcal{S} \to \mathbb{Q}$ is Reidemeister invariant in $\mathcal{S}$ if whenever $D, D' \in \mathcal{S}$ are Reidemeister equivalent in $\mathcal{S}$, then $v(D) = v(D')$.

**Theorem 12.** If $D, D' \in \mathcal{Z}$ are Reidemeister equivalent, then they are Reidemeister equivalent in $\mathcal{Z}$.

**Proof.** Suppose that $D, D' \in \mathcal{Z}$ are Reidemeister equivalent. Then there is a sequence $S$ of Reidemeister moves $D = D_0 \leftrightarrow D_1 \leftrightarrow \cdots \leftrightarrow D_p = D'$. If all of the $D_i$ are in $\mathcal{Z}$, then there is nothing to prove. Otherwise, it must be true that there are some arrows amongst the $D_i$ which have nonzero index.

Define an arrow to be odd if it has nonzero index and even if it has a zero index. By Lemma [11] this almost defines a parity on the collection of Gauss diagrams. The deviant condition is (3a) of Section 2.2 in an O3 move, it is possible that all three arrows have nonzero index. Therefore, it is possible that all three arrows are marked odd. However, the map $f$ of Section 2.2 deletes such arrows from both diagrams and hence preserves the equivalence of such a deviant pair of diagrams.

Therefore the conclusion of Theorem [11] also holds for this near parity. Apply the functorial map $f$ to the entire sequence of moves. Note that $f$ fixes the endpoints of this path (i.e. $f(D) = D$ and $f(D) = D'$). Since $f$ is functorial, the new sequence is also a sequence of Reidemeister moves and identities of the form...
E ↔ E. Moreover, the total number of arrows in the new sequence \( f(S) \) is less than the total number of arrows of \( S \).

The new sequence may have arrows which have nonzero index. Therefore, we apply the map \( f \) again. Since \( f \) decreases the total number of arrows in the sequence, there is a natural number \( N \) such that if \( a, b \geq N \), then \( f^n(S) = f^N(S) \). By Theorem 8 all the terms of the sequence \( S \) are even (note that some of the diagrams may be empty diagrams). For the near parity defined in this proof, this means that all of the arrows have zero index. Since \( D \) and \( D' \) are fixed by \( f \), we have that \( D \) and \( D' \) are Reidemeister equivalent in \( \mathbb{Z} \).

**Theorem 13.** Let \( v \) denote \( v_{21} \) or \( v_{22} \). Suppose \( D \in \mathbb{Z} \) has \( n \) arrows. Then for any \( \zeta \in \mathbb{Z}_{2n} \), \( v(\zeta \cdot D) = v(D) \). Moreover, \( v \) may be considered as a function \( \bar{v} \) on \( \mathbb{Z} \) which is Reidemeister invariant in \( \mathbb{Z} \).

**Proof.** First note that for all \( x \in C(D) \) (which coincides identically with \( C(\zeta \cdot D) \)), we have that \( I_D(x) = I_{\zeta \cdot D}(x) \). Let \( x \) be the arrow of \( D \) whose rightmost endpoint is rightmost of all arrow endpoints on the Wilson line. Assume that \( x \) points right. We will prove the result for \( \zeta = 1 \). There are two ways that arrows might intersect \( x \). These correspond to the only two ways that arrows may intersect \( x \) in \( 1 \cdot D \).

Let \( M^x_{i,j} \), \( 1 \leq i, j \leq 2 \), denote the number of configurations in the \((i,j)\) position in the boxed array immediately above, where the count is weighted by the signs of arrows. Since the index of \( x \) is zero in both \( D \) and \( 1 \cdot D \), it follows that \( M^x_{1,1} = M^x_{2,1} \) and \( M^x_{1,2} = M^x_{2,2} \). Moreover, since there is a one-to-one correspondence between the two diagrams in each row, it must be that \( M^x_{1,1} = M^x_{1,2} = M^x_{2,1} = M^x_{2,2} \). A similar statement can be made when \( x \) points left.

We will prove the theorem only in the case of the combinatorial formula \( \zeta(\cdot) \). Intersecting arrow pairs not involving \( x \) have the same contribution to the value of this invariant on \( D \) and \( 1 \cdot D \). Since \( M^x_{1,1} = M^x_{1,2} \), they also have the same contribution on arrows involving \( x \). Hence:

\[
\left\langle \, \begin{array}{c}
\alpha \\
\beta
\end{array} \, , \, I_{GPV}(D) \right\rangle = \left\langle \, \begin{array}{c}
\alpha \\
\beta
\end{array} \, , \, I_{GPV}(1 \cdot D) \right\rangle
\]

For the final claim, note that for \( \bar{v} \in \mathbb{Z} \), \( v \) has the same value on every element of \( \Pi^{-1}(\bar{v}) \). Hence, \( v \) may be considered as a function \( \bar{v} \) on \( \mathbb{Z} \).

Suppose then that \( D, D' \in \mathbb{Z} \) and \( D \leftrightarrow D' \) is a Reidemeister move. Select a point \( \theta \) on the Wilson loop of \( D \) and \( D' \) so that \( \theta \) is not in the interval of affected arrows of the move. Cutting \( D \) and \( D' \) at this point gives two diagrams \( E \) and \( E' \) where \( E \leftrightarrow E' \) is a Reidemeister move on the Wilson line. Then \( v(E) = v(E') \). By the previously established first claim of this theorem, we have that \( \bar{v}(D) = \bar{v}(D') \). Thus, \( \bar{v} \) is Reidemeister invariant in \( \mathbb{Z} \).

**Theorem 14.** For all \( D \in \mathbb{Z} \), \( v_{21}(D) = v_{22}(D) \).

**Proof.** The proof has three parts. First it is proved for all diagrams \( D \) having \( \oplus \) signs at every arrow. Then it is proved for all diagrams \( D \) having exactly one arrow signed \( \oplus \). The proof is concluded with an induction on the number of arrows signed \( \oplus \).

Suppose then that all arrows of \( D \) are signed \( \oplus \). Two arrows \( x \) and \( y \) of \( D \) intersect in one of four ways:

Since the index at each arrow is zero and the signs are all \( \oplus \), we have:

\[
0 = \sum_{x \in C(D)} I(x) = \sum_{(x,y) \in C(D) \times C(D) \setminus \Delta C(D)} \delta(x,y)
\]

Also note that diagrams of the second and third kind satisfy \( \delta(x,y) + \delta(y,x) = 0 \). On the other hand, diagrams of the first kind satisfy \( \delta(x,y) + \delta(y,x) = -2 \) while diagrams of the fourth kind satisfy \( \delta(x,y) + \delta(y,x) = 2 \).
Therefore, the number of +2 contributions to the above sum must be identically equal to the number of -2 contributions to the above sum. It follows that \(v_{21}(D) = v_{22}(D)\).

Suppose then that \(D\) has exactly one arrow \(x\) which is signed \(\ominus\). Assume that \(x\) points right. There are four ways which arrows may intersect \(x\):

1. \(\ominus\)
2. \(\ominus\)
3. \(\ominus\)
4. \(\ominus\)

The idea is to switch both the direction and sign of \(x\). This creates a diagram \(D'\) of zero index such that the sign of every arrow is \(\ominus\). By the previous case, we have \(v_{21}(D') = v_{22}(D')\). Let \(M_1^+\) denote the number of distinct intersecting arrow pairs of the form: \(\ominus\), where neither of the two arrows is \(x\). Let \(M_2^-\) denote the number of arrow pairs of \(D\) of the form: \(\ominus\), where neither of the two arrows is \(x\). Let \(N_1, N_2, N_3, N_4\) denote the number of arrow pairs as in (1)-(4) above. Since \(I(x) = 0\), we have the following relation:

\[N_1 - N_3 - N_2 + N_4 = 0\]

Since \(v_{21}(D) = v_{22}(D')\), we have that \(M_1^+ + N_2 = M_2^- + N_1\). Combining these relations together gives \(M_1^+ - N_3 = M_2^- - N_4\). But this just says that \(v_{21}(D) = v_{22}(D)\).

To complete the proof, we proceed by induction on the number \(k\) of arrows signed \(\ominus\). The case \(k = 1\) is the case considered in the preceding paragraphs. Suppose then that if the number of \(\ominus\) signs in \(D\) is \(k\), then \(v_{21}(D) = v_{22}(D)\). Let \(D\) be a diagram with \(k + 1\) arrows signed \(\ominus\). Choose an arbitrary arrow \(x\) signed \(\ominus\). Let \(D'\) be the diagram obtained by switching both the arrow direction and sign of \(x\). By the induction hypothesis, \(v_{21}(D') = v_{22}(D')\).

Let \(M_1\) (respectively \(M_2\)) denote the contribution of diagrams of the form: \(\ominus\) (respectively \(\ominus\)), weighted by the product of the arrow signs, where neither of the two intersecting arrows is \(x\). Also, let \(N_1, N_2, N_3, N_4\) denote the number of configurations (1)-(4) as above where the sign of the arrow \(y\) intersecting \(x\) may be anything, the sign of \(x\) is \(\ominus\) and each contributing configuration is weighted by the product of the signs. Since \(I(x) = 0\), we again have the relation:

\[N_1 - N_2 = N_3 - N_4\]

Applying the induction hypothesis to \(D'\) gives \(M_1 + N_2 = M_2 + N_1\). Together, these two relations say that \(M_1 - N_3 = M_2 - N_4\). Hence, \(v_{21}(D) = v_{22}(D)\). This completes the proof.

3. COMBINATORIAL FORMULAE AND PARITIES

3.1. The Algebraic Construction of the New Formulae. The construction is the same for the Wilson line or the Wilson loop. Consider \(A^{[1,0]}\), the collection of Gauss diagrams with dashed arrows having a 0 or a 1 at every arrow (the empty Gauss diagram is by decree also in \(A^{[1,0]}\)). Let \(A^{[1]}\) denote those diagrams having a 1 at every arrow, \(A^{[0]}\) those diagrams having a 0 at every arrow, \(A_{\{1\}}\) denote those diagrams having a 1 at some arrow, and \(A_{\{0\}}\) those diagrams having a 0 at some arrow. Here we are not considering any parity. So for a given Gauss diagram there are \(2^k\) ways to attach a 0 or a 1 to the arrows of some Gauss diagram. We define some relations in \(\mathbb{Z}[A^{[1,0]}]\) as follows.

\[Q1: \quad 0 \quad \ominus = 0, \quad Q2: \quad \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} = 0\]

\[Q3: \quad \begin{align*}
\frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} + \frac{\delta \ominus}{\delta \ominus} =
\end{align*}\]

In Q3, we include all possibilities where \(i = i', j = j', k = k'\), and \(i + j + k = i' + j' + k' = 0\). Note also that the \(\delta\) in the Q2 relations refers to a value of 1 or 0.

Define \(\Delta Q = \langle Q1, Q2, Q3\rangle\). Let \(k, n \in \mathbb{N}, k \leq n\). Let \(E_k\) denote the free abelian group generated by those diagrams having \(\geq k\) arrows marked 0. Let \(A_n\) denote the free abelian group generated by those
diagrams having more than \( n \) arrows. Let \( A^k \) denote the free abelian group generated by those diagrams having fewer than \( k \) arrows.

We define the following quotients:
\[
\mathcal{O}_{\infty,k} = \frac{\mathbb{Z}[A^{(1,0)}]}{(\Delta Q, E_k, A^k)}, \quad \mathcal{O}_{n,k} = \frac{\mathbb{Z}[A^{(1,0)}]}{(\Delta Q, E_k, A^k, A_n)}, \quad \mathcal{O}_n := \mathcal{O}_{n,n}
\]

Following [GPV], we define a map \( i : \mathbb{Z}[\mathcal{D}^{(1,0)}] \to \mathbb{Z}[A^{(1,0)}] \) on generators by \( i(D) = D \) with all arrows drawn dashed and all sign and \( \mathbb{Z}_2 \) markings unchanged. Let \( \mathcal{P} \) denote any parity. Define \( I^\mathcal{P} : \mathbb{Z}[\mathcal{D}] \to \mathbb{Z}[A^{(1,0)}] \) on generators by:
\[
I^\mathcal{P}(D) = \sum_{D' \subseteq P(D)} i(D')
\]

where the sum is taken over all subdiagrams \( D' \) of \( P(D) \), with the corresponding arrows in \( D' \) embellished exactly as in \( P(D) \). Also, we have the natural projections onto the quotients \( \mathcal{O}_{\infty,k}, \mathcal{O}_{n,k} \) and \( \mathcal{O}_n \):
\[
\pi^\mathcal{P}_{\infty,k} : \mathbb{Z}[A^{(1,0)}] \to \mathcal{O}_{\infty,k}, \\
\pi^\mathcal{P}_{n,k} : \mathbb{Z}[A^{(1,0)}] \to \mathcal{O}_{n,k}, \\
\pi^\mathcal{P}_n : \mathbb{Z}[A^{(1,0)}] \to \mathcal{O}_n
\]

The crucial point in the construction of the new invariants are these projections. Diagrams with a certain number of arrows marked zero are mapped to zero. To obtain virtual knot invariants, we simply take the composition of these projections with \( I^\mathcal{P} \). Hence, we define:
\[
I^\mathcal{P}_{\infty,k} = \pi^\mathcal{P}_{\infty,k} \circ I^\mathcal{P}, \quad I^\mathcal{P}_{n,k} = \pi^\mathcal{P}_{n,k} \circ I^\mathcal{P}, \quad I^\mathcal{P}_n = \pi^\mathcal{P}_n \circ I^\mathcal{P}
\]

It is important to note at this point that the group \( \mathcal{O}_{n,1} \) corresponds to the projection where all subdiagrams having an even arrow are mapped to zero. This is one of the extremal examples mentioned in the introduction.

**Theorem 15.** Let \( \mathcal{P} \) be any parity and \( 1 \leq k \leq n \). If \( v \in \text{Hom}_{\mathcal{Z}}(\mathcal{O}_{n,k}, \mathbb{Q}) \), then \( v \circ I^\mathcal{P}_{n,k} \) is a virtual knot (or virtual long knot) invariant. The statement also holds when \( n = \infty \) and \( k < \infty \).

**Proof.** Given any Reidemeister relation, apply the map \( I^\mathcal{P} \). The distribution of zeros and ones in the \( \Delta Q \) relations coincides with the parity definition in Section 2.1. Collect similar terms in the image. By similar terms, we mean all terms which are identical outside the affected arrows. The grouped terms all lie in \( \Delta Q \). The fact that these all vanish in \( \mathcal{O}_{n,k} \) is sufficient to guarantee that \( v \circ I^\mathcal{P}_{n,k} \) is an invariant. \( \square \)

Just as in the GPV case, we have for each \( k \) a sequence of surjections:
\[
\cdots \to \mathcal{O}_{n,k} \to \mathcal{O}_{n-1,k} \to \cdots \to \mathcal{O}_{k+1,k} \to \mathcal{O}_{k,k}
\]

There is also the following short exact sequence which will be used in the proof of Theorem 20:
\[
0 \to \frac{\mathcal{O}_{n,k}}{(A^k, E_k, A_{n-1}, \Delta Q)} \to \mathcal{O}_{n,k} \to \mathcal{O}_{n-1,k} \to 0
\]

Combinatorial formulae may be defined in direct analogy to those in [GPV]. There is the pairing \( \langle \cdot, \cdot \rangle : \mathbb{Z}[A^{(1,0)}] \times \mathbb{Z}[A^{(1,0)}] \to \mathbb{Z} \) defined on generators by \( \langle D_1, D_2 \rangle = 1 \) if \( D_1 = D_2 \) and \( \langle D_1, D_2 \rangle = 0 \) if \( D_1 \neq D_2 \). For a given pair \((n,k)\), \( 1 \leq k \leq n \), a combinatorial formula is an element \( F \in \mathbb{Z}[A^{(1,0)}] \) such that:
1. every \( D \in A^{(1,0)} \) with \( \text{coeff}(D, F) \neq 0 \) has number of arrows between \( k \) and \( n \) (inclusive),
2. every \( D \in A^{(1,0)} \) with \( \text{coeff}(D, F) \neq 0 \) has less than \( k \) arrows marked 0, and
3. for every \( r \in \Delta Q \), \( (F, r) = 0 \).

For a given parity \( \mathcal{P} \) and a pair \((n,k)\), a combinatorial formula \( F \) defines a virtual knot or long virtual knot invariant by the rule:
\[
\langle F, I^\mathcal{P}\rangle
\]

Moreover, it follows from the definitions that \( \langle F, \cdot \rangle \in \text{Hom}_{\mathcal{Z}}(\mathcal{O}_{n,k}, \mathbb{Q}) \). The collection of all such invariants for a parity \( \mathcal{P} \) are referred to as combinatorial \( \mathcal{P} \)-formula.
In addition to the odd groups $\mathcal{O}_{n,k}$, there are also the even groups $\mathcal{E}_n$. They are defined by the quotients:

$$\mathcal{E}_\infty = \frac{\mathbb{Z}[A(1,0)]}{\langle \Delta Q, A(1) \rangle}, \quad \mathcal{E}_n = \frac{\mathbb{Z}[A(1,0)]}{\langle \Delta Q, A(1), A_n \rangle}$$

Note that $\mathcal{E}_\infty$ and $\mathcal{E}_n$ both contain the empty diagram. The even case exhibits the natural projections $\pi_\infty : \mathbb{Z}[A(1,0)] \to \mathcal{E}_\infty$ and $\pi_n : \mathbb{Z}[A(1,0)] \to \mathcal{E}_n$. Of course there is the analogue of $I_{GPV}$ for a given parity $\mathcal{P}$, $I_n^\mathcal{P} = \pi_n \circ I_{\mathcal{P}} : \mathbb{Z}[D] \to \mathcal{E}_n$.

Note that this is the projection where all diagrams having an odd arrow are sent to zero. This is to be compared with the functorial map $f$ of Section 2.2 which deletes all arrows from a diagram having an odd arrow. Indeed the projection defining the group $\mathcal{E}_\infty$ has the same effect as the map $I_{GPV} \circ f$.

**Theorem 16.** Let $\mathcal{P}$ be any parity. If $v \in \text{Hom}_\mathbb{Z}(\mathcal{E}_n, \mathbb{Q})$, then $v \circ I_n^\mathcal{P}$ is a virtual knot (or virtual long knot) invariant. The statement also holds when $n = \infty$.

Combinatorial formulae for the even groups can be defined in a similar way to the case of the odd groups. This will be discussed in more detail in Section 3.3.

Now that the groups have been constructed abstractly, we proceed to show that they are generally nontrivial and possess some interesting structures. Our exposition starts with the least technically difficult results and proceeds to those which are most technically difficult.

- We show in Section 3.2 that $\mathcal{O}_n$ is generally nontrivial by showing that homogeneous GPV formulae generate invariants in $\text{Hom}_\mathbb{Z}(\mathcal{O}_n, \mathbb{Q})$.
- We show in Section 3.3 that $\mathcal{E}_n$ is nontrivial by showing that that $\mathcal{E}_n$ and $\tilde{A}_n$ are isomorphic. Together with Section 3.2 this yields an even/odd decomposition of homogeneous GPV finite-type invariants.
- We proceed in Section 3.4 to look at the case of $\mathcal{O}_{n,1}$. We show that there is a more efficient set of relations with which one can compute. In Section 3.5 we use the relations to compute a generating set for all invariants on $\mathcal{O}_{n,1}$ with $n \leq 3$. A more detailed analysis of these groups is withhold until Section 3.7.
- In Section 3.6 we give a sufficient condition for all invariants to be of Kauffman finite-type. We discuss some combinatorial $\mathcal{P}$-formula which are virtualization invariant and some which are not of GPV finite-type.
- In Section 3.7 we compute the dimension of $\text{Hom}_\mathbb{Z}(\mathcal{O}_{n,1}, \mathbb{Q})$ by explicitly constructing a basis of combinatorial $\mathcal{P}$-formulae.

3.2. $\mathcal{O}_n$ and Homogeneous GPV Formulae. The easiest way to establish the existence of combinatorial $\mathcal{P}$-formulae is to use known GPV formulae. To this end we use homogeneous GPV formulae. A homogeneous GPV combinatorial formula of order $n$ is a GPV combinatorial formula $F$ having exactly $n$ arrows in every summand. The easiest examples of these are the invariants $v_{21}$ and $v_{22}$. Also, there is the well-known Casson invariant. For order three, there is the following example, which was found by a *Mathematica* program.

$$1 \quad + \quad + \quad -1 \quad + \quad + \quad -1 \quad + \quad +$$

$$+1 \quad + \quad + \quad + \quad + \quad -1 \quad + \quad + \quad -1 \quad + \quad + \quad +$$

If $F = \sum_{D \in \mathcal{D}} \text{coeff}(D, F) \cdot D$ is a GPV combinatorial formulae that is homogeneous of order $n$, define $F^n$ as follows. If $\text{coeff}(D, F) \neq 0$, the contribution to $F^n$ is the sum over all labellings of the arrows of $D$ with zeros and ones so that the result is an odd $n$-bunch. In particular, a labelled diagram contributes whenever
it has at least one odd arrow. Hence there are a total of \(2^n - 1\) diagrams in \(F^o\) arising from the contribution of \(D\). This sum will be denoted by:

\[
\sum_{D \in D} D^{(1)}
\]

where the sum is over all ways that the \(n\) arrows form an odd \(n\)-bunch. Define

\[
F^o = \sum_{D \in D} \sum_{D^{(1)} \subseteq D} \text{coeff}(D, F) \cdot D^{(1)}
\]

If \(n = 0\) and \(F\) is the empty diagram, we define \(F^o = 0\).

**Theorem 17.** If \(F\) is a GPV combinatorial formula which is homogeneous of order \(n\), then:

\[
\langle F^o, \cdot \rangle \in \text{Hom}_\mathbb{Z}(\mathcal{O}_n, \mathbb{Q}).
\]

Hence, if \(\mathcal{P}\) is any parity, \(\langle F^o, I[\mathcal{P}](\cdot) \rangle\) is an invariant of virtual knots (or virtual long knots).

**Proof.** The formula vanishes on all diagrams having more than \(n\) arrows, less than \(n\) arrows, or \(n\) even arrows. The only things that needs to be checked are the relations in \(\Delta Q\).

**Q1:** An isolated arrow, i.e. one in which the head and the tail are adjacent to one another, is necessarily even in \(\mathcal{P}\). If it is part of an odd bunch, at least one of the other arrows is odd in \(\mathcal{P}\). Any summand \(D\) in \(F\) having an isolated arrow must have \(\text{coeff}(D, F) = 0\) since \(F\) is a GPV invariant. It follows that all \(Q1\) relations are satisfied for \(F^o\).

**Q2:** We have that \(\langle F, \cdot \rangle\) satisfies all \(P2\) relations (see Figure 7). Since \(F\) is homogeneous of order \(n\), the \(P2\) relations may be divided into two types:

\[
\begin{align*}
\begin{array}{c}
\vdots \vdots \\
\bullet \circ
\end{array}
\end{align*} = 0,
\begin{align*}
\begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} + 
\begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} = 0
\end{align*}
\]

The same division occurs in the \(Q2\) relations for \(F^o\). Now, the two arrows involved in the \(Q2\) relation are either both marked zero or both marked one. If they are both marked one, any \(n\)-bunch containing them will be odd. In this case the two types of relations are satisfied because \(F\) is a GPV combinatorial formula.

If the arrows are both marked zero, then one of the arrows will be in an odd \(n\)-bunch only when there is an arrow outside the drawn interval which is marked one. In this case, it occurs in all three diagrams in the \(P11\) relation. Hence the fact that the relation is satisfied for \(F^o\) follows from the fact that it is satisfied for \(F\) in the GPV case.

**Q3:** Since \(F\) is homogeneous of order \(n\), the \(P3\) relations also separate into two types:

\[
\begin{align*}
A : & \quad \begin{array}{c}
\vdots \vdots \\
\bullet \circ
\end{array} + \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} + \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} = \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} + \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} + \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array}
\end{align*}
\]

\[
B : \quad \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array} = \begin{array}{c}
\vdots \vdots \\
\circ \bullet
\end{array}
\]

The same division occurs for \(F^o\) in \(Q3\).

The sets \(\{i, j, k\}\) and \(\{i', j', k'\}\) contain either three arrows are marked zero or two arrows marked one.

Suppose that all three are marked zero. Consider first the six term relation \(A\). The only way that any two amongst the arrows may be contained in an odd \(n\)-bunch is if there is an arrow outside the drawn intervals which is marked zero. This arrow appears in all six diagrams, whence the relation is satisfied by \(F^o\) in the same way it is satisfied for \(F\). Similarly, the relation \(B\) is satisfied.

Suppose that there are exactly two arrows in each of \(\{i, j, k\}\) and \(\{i', j', k'\}\) which are marked one. We know that \(pD(l) = pD(l')\) for \(l = i, j, k\). Moreover, any choice of two arrows from the sets \(\{i, j, k\}\) and \(\{i', j', k'\}\) contains at least one arrow which is marked one. Therefore, each diagram in the relation \(A\) is in an odd bunch. The diagrams in \(B\) are also in an odd bunch. Once again, the fact that \(F\) satisfies \(A\) and
In this case, either both arrows are marked zero or both arrows are marked 1. If both arrows are marked 0, the relation is trivially satisfied.

3.3. Even Arrow Groups and Even/Odd Decomposition. In this section, we establish the isomorphism between \( \mathcal{E}_n \) and the Polyak groups \( \tilde{\mathcal{A}}_n \). Also, we prove a decomposition theorem for homogeneous GPV finite-type invariants.

Recall that the projection defining \( \mathcal{E}_n \) is the one which maps all subdiagrams having an odd arrow to zero, or equivalently the map which deletes all odd arrows and sums over all subdiagrams of the resulting diagram.

**Theorem 18.** The Polyak group \( \tilde{\mathcal{A}}_n \) is isomorphic to the even group \( \mathcal{E}_n \) for all \( n \in \mathbb{N} \cup \{ \infty \} \).

**Proof.** Define \( f_e : \mathbb{Z} [\tilde{\mathcal{A}}] \to \mathbb{Z} [A^{(1,0)}] \) to be the map which assigns a zero to every arrow of every diagram. The empty diagram is mapped to itself (recall that the empty diagram is in \( A^{(1,0)} \) by decree). The isomorphism of interest is established via the projections (and also the second Noether isomorphism theorem):

\[
\mathbb{Z} [A] \xrightarrow{f_e} \mathbb{Z} [A^{(1,0)}] \xrightarrow{\text{coeff} (\cdot)} \mathbb{Z} [A^{(1,0)}] / (\Delta Q, A^{(1,0)})
\]

The set of \( \Delta Q \) relations in the final quotient is taken modulo the set of relations \( \Delta Q \cap \langle A^{(1)} \rangle \).

A Q1 relation appears in \( \Delta Q \cap \langle A^{(1)} \rangle \) only if some arrow other than the affected one is marked 1.

In a Q2 relation, the markings of affected arrows must be the same and hence the only Q2 relations in \( \Delta Q \cap \langle A^{(1)} \rangle \) are those in which some arrow other than the affected arrows is marked 1.

In a Q3 relation, either (1) all affected arrows are marked 0, or (2) only one is marked 0. If (2), then any choice of two of the three arrows contains an odd arrow. Therefore, every configuration of type (2) is in \( \Delta Q \cap \langle A^{(1)} \rangle \). A type (1) relation appears in \( \Delta Q \cap \langle A^{(1)} \rangle \) only if some other arrow is marked 1.

It follows that the only nontrivial relations in the final quotient are those relations Q1,Q2,Q3 where each constituent diagram has a zero at every arrow. But this is the image under \( f_e \) of (P1,P2,P3). Hence, \( \tilde{\mathcal{A}}_\infty \cong \mathcal{E}_\infty \). Since the grading is preserved by \( f_e \), we also have \( \tilde{\mathcal{A}}_n \cong \mathcal{E}_n \).

If \( f_e : \mathbb{Z} [A] \to \mathbb{Z} [A^{(1,0)}] \) is as in Theorem 18 and \( F \in \mathbb{Z} [A] \), define \( F^e = f_e (F) \).

**Theorem 19.** Let \( F \) be a GPV combinatorial formula which is homogeneous of order \( n \). Then for any parity \( \mathcal{P} \), there is decomposition of \( F \) into its even and odd parts:

\[
\langle F, I_{GPV} (\cdot) \rangle = \langle F^o, I_{[\mathcal{P}]} (\cdot) \rangle + \langle F^e, I_{[\mathcal{P}]} (\cdot) \rangle
\]

**Proof.** Suppose that \( n \geq 1 \). Let \( D \in \mathcal{D} \) and \( D' \subset P(D) \) be a subdiagram with corresponding arrows marked as prescribed by the parity. If \( D' \) does not have exactly \( n \) arrows, then LHS and RHS of both equations are zero regardless of any parity considerations. Assume then that \( D' \) has precisely \( n \) arrows.

Suppose first that all the arrows of \( D' \) are marked with a zero. Then \( \langle F^o, i (D') \rangle = 0 \) and \( \langle F, i (D') \rangle = \langle F^e, i (D') \rangle \).

On the other hand, suppose that at least one of the arrows in \( D' \) is marked 1. Then \( \langle F^o, i (D') \rangle = 0 \). If \( D' \) (forgetting all numerical markings) satisfies coeff\((D', F) = 0\), then the desired decomposition is trivially true. If coeff\((D', F) \neq 0\) then we have by definition of \( F^o \) that:

\[
\langle F, i (D') \rangle = \text{coeff} (D', F) = \langle F^o, i (D') \rangle
\]

Thus, the formula holds. If \( n = 0 \), then \( F \) is the empty diagram. Since, \( F^e \) is the empty diagram and \( F^o = 0 \), the formula also holds in this case.

3.4. A Special Construction for \( \mathcal{O}_{n,1} \). In this group, the projection forces any subdiagram having a 0 at some arrow to vanish. Therefore we may assume that all arrows are marked with a 1. In other words, all arrows are considered to be odd. This allows for a substantial simplification of the relations in \( \Delta Q \). We consider each one in turn.

Q1: All Q1 relations are automatically satisfied since there is at least one arrow marked with a 0.

Q2: In this case, either both arrows are marked zero or both arrows are marked 1. If both arrows are marked 0, the relation is trivially satisfied.
If both arrows are marked 1, the relation cannot be simplified further. So the only Q2 relations needed are those with $\delta = 1$ and all other arrows marked 1. Relations Q2 where all arrows (including those not in the drawn intervals) are marked one will be denoted $Q2(1)$.

**Q3:** Consider then the markings of arrows involved in a Q3 relation.

By the parity condition (3a), either all three of the arrows are marked 0 or only one is marked 0. If all arrows are 0, then the condition is trivially satisfied. Otherwise, there are three cases: of the triple, either only the arrows labeled $(i, i')$ are even or only the arrows labeled $(j, j')$ are even or only the arrows labeled $(k, k')$ are even. In each case, all but two of the labeled diagrams vanish. The resulting three relations are given below.

- $Q3a: = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\oplus$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\oplus$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\oplus$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\oplus$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\oplus$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\oplus$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array}$

- $Q3b: = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\odot$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\odot$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\odot$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\odot$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\odot$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\odot$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array}$

- $Q3c: = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\odot$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\odot$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\odot$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\odot$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\odot$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\odot$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array}$

We will refer to the collection $\{Q3a, Q3b, Q3c\}$ as the **commutativity relations**. These observations are easily made into the following theorem. When all arrows in the diagrams are marked one, they will be denoted $\{Q3a(1), Q3b(1), Q3c(1)\}$. Also, let $A^{(1)}_n$ denote the set of dashed arrow diagrams having more than $n$ arrows, all of which are marked 1.

**Theorem 20.** For every $n \geq 1$, there is an isomorphism of groups:

$$\mathbb{O}_{n,1} \cong \mathbb{Z}[A^{(1)}_n] \langle A^{(1)}_n, Q2(1), Q3a(1), Q3b(1), Q3c(1) \rangle$$

3.5. **Computations of $\text{Hom}_{\mathbb{Z}}(\mathbb{O}_{n,1}, \mathbb{Q})$ for $n \leq 3$.** We look at the Wilson line case first and then turn to the Wilson loop case. By Theorem 20 we may assume that all arrows are marked 1, so we will not write out the markings in this case. The combinatorial $\mathcal{P}$-formula of order 1 are generated by the following:

$$F^3 = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\oplus$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\oplus$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\oplus$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array} \ominus \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\odot$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\odot$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\odot$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array} , \quad F'' = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\oplus$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\oplus$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\oplus$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array} \ominus \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) node[below]{$\odot$} -- (1,0) node[below]{$\ominus$} -- (2,0) node[below]{$\odot$};
\draw (0,1) node[above]{$\ominus$} -- (1,1) node[above]{$\odot$} -- (2,1) node[above]{$\ominus$};
\end{tikzpicture}
\end{array}$$

It is clear that these both vanish on all relations in Theorem 20. Now, for invariants of higher order, it is convenient to introduce a new notation. Consider the set of symbols $\{L_\oplus, L_\ominus, R_\oplus, R_\ominus\}$, where $L$ is interpreted as an arrow directed left, $R$ is interpreted as an an arrow pointing right and the subscript signifies the local writhe number of the arrow. For $D \in \mathcal{D}^{(1,0)}$ having $n$ arrows, denote by:

$$L_\oplus^a L_\ominus^b R_\oplus^c R_\ominus^d (D)$$

the number of subdiagrams of $D$ consisting of $a$ arrows marked 1 pointing left and signed $\oplus$, $b$ arrows marked 1 pointing left and signed $\ominus$, $c$ arrows marked 1 pointing right and signed $\oplus$, and $d$ arrows marked 1 pointing right and signed $\ominus$. Note that some of $a, b, c, d$ might be zero. The term only counts subdiagrams whose odd arrows match the nonzero exponents.
The usefulness of this notation is immediately clear when one considers the commutativity relations. These relations say that two arrows having the same sign and having adjacent endpoints can have those endpoints rearranged in any way.

On the other hand, if a \( F \) combinatorial \( \mathcal{P} \)-formula of order \( n \), then \( \langle F, D + D' \rangle = 0 \), if \( D \) and \( D' \) are diagrams having \( n \) arrows that are identical except in the sign of a single arrow (this is a consequence of the Q2 relation). Therefore, if \( D \) is a summand of \( D \) having \( n \) arrows, we must also have any diagram which is obtained from \( D \) by switching any subset of the signs of \( D \) (and weighted with an appropriate coefficient).

In particular, every such combinatorial \( \mathcal{P} \)-formula must also contain the choice of all \( \oplus \) signs for \( D \).

Hence, any subset of arrows may be arranged on the Wilson line in any way as long as the number of arrows pointing left and the number of arrows pointing right is preserved. We see that in general the combinatorial \( \mathcal{P} \)-formulae will be expressible as linear combinations of symbols of the form \( L_{a}^{i} L_{b}^{j} R_{c}^{k} R_{d}^{l} (D) \).

It is now a simple matter to construct the combinatorial \( \mathcal{P} \)-formulae of order 2. Starting with an arrow diagram having two arrows signed positive, we include all appropriate rearrangements and sign changes. In doing so, we may introduce a diagram which participates in a Q2 relation. Accounting for this boils down to solving a small system of equations. For example, we see that there is a \( F^{rr} \in \mathbb{Z}[A^{(1)}] \) such that for all \( D \in \mathcal{D} \):

\[
\langle F^{rr}, I[\mathcal{P}] (D) \rangle = L_{a} R_{b} (D) - L_{a} R_{b} (D) - L_{a} R_{b} (D) + L_{a} R_{b} (D)
\]

Since this does not count any subdiagrams of odd arrows having two arrows pointing in the same direction and vanishes on subdiagrams having 1 arrow, we conclude that this satisfies all Q2 relations having two or fewer arrows. By the above discussion it satisfies all Q2 relations for diagrams having more than two arrows and all commutativity relations. It is therefore a combinatorial \( \mathcal{P} \)-formula.

The previous examples have been shown to be invariants by doing little more than choosing a decent notation. Generally the hardest part is to make sure that the Q2 relations are satisfied. We will construct a formula \( F \in \mathcal{A}^{(1)} \) of order \( n \) whose order \( n \) part contains only terms of the form \( L_{a}^{i} L_{b}^{j} \), where \( i + j = n \). Some of the summands of \( F \) will have two arrows grouped as in the Q2 relation. A sufficient condition to guarantee that the Q2 relations is that the coefficients \( c_{a,b} \) of terms of the form \( L_{a}^{i} L_{b}^{j} \) satisfy:

\[
c_{a,b} + c_{a-1,b} + c_{a,b-1}
\]

where \( a, b \geq 1 \). For \( n = 2 \), it follows that there is an \( F^{ll} \in \mathcal{A}^{(1)} \) and an \( F^{rr} \in \mathcal{A}^{(1)} \) such that:

\[
\langle F^{ll}, I[\mathcal{P}] (D) \rangle = 2 \cdot L_{a}^{2} (D) - 2 \cdot L_{a} L_{a} (D) + 2 \cdot L_{a} (D) + L_{a} (D)\]
\[
\langle F^{rr}, I[\mathcal{P}] (D) \rangle = 2 \cdot R_{b}^{2} (D) - 2 \cdot R_{b} R_{b} (D) + 2 \cdot R_{b} (D) + R_{b} (D)
\]

A similar computation can be made for invariants of order exactly 3. They are generated by the four solutions \( F^{rrr}, F^{lll}, F^{lrl}, \) and \( F^{lll} \) which are given below:

\[
\langle F^{rrr}, I[\mathcal{P}] (D) \rangle = 2 \cdot L_{a}^{2} (D) - 2 \cdot L_{a} R_{b}^{2} (D)\]
\[
\langle F^{lrl}, I[\mathcal{P}] (D) \rangle = 2 \cdot L_{a}^{2} R_{b}^{2} (D) - 2 \cdot L_{a} L_{a} R_{b}^{2} (D) - 2 \cdot L_{a} R_{b} R_{b} (D) + 2 \cdot L_{a} L_{a} R_{b} R_{b} (D)
\]

\[
\langle F^{lll}, I[\mathcal{P}] (D) \rangle = L_{a}^{3} (D) - L_{a}^{2} L_{a} (D) + L_{a}^{2} L_{b} (D) - L_{a}^{2} (D) - L_{a}^{2} R_{b} (D) + L_{a}^{2} R_{b} (D) - L_{a}^{2} R_{b} (D)
\]

Now consider the case of the Wilson loop. We define terms of the form:

\[
N_{a}^{b} N_{b}^{a} (D)
\]

which stand for the number of subdiagrams of \( D \) consisting of exactly \( a \) arrows marked with a 1 and signed \( \oplus \) and \( b \) arrows marked with a 1 and signed \( \ominus \).

All of the new combinatorial formulae on the Wilson loop may be written as linear combinations of these terms. Indeed, consider a combinatorial formula \( F \) on the Wilson loop of order \( n \). Suppose that \( F \) has a term \( D \) consisting of \( n \) arrows signed \( \oplus \). By the commutativity relations, it is easy to see that \( F \) has the same value on any dashed diagram having \( n \) dashed arrows signed \( \oplus \). Moreover, if \( D' \) is obtained from \( D \)
by changing the sign of an arrow to \( \circ \), then \( \langle F, D \rangle = - \langle F, D' \rangle \). If follows that every dashed diagram with \( n \) arrows must be a summand of \( F \) weighted with some appropriate coefficient.

The Q2 relations can be satisfied in a similar manner to that of the Wilson line case. Some summands \( D \) of \( F \) possess two arrows as in a Q2 relation. It follows that a sufficient condition for a formulae to exist is that the coefficients \( c_{a,b} \) of the terms counted by \( N^a_{\circ} N^b_{\circ}(D) \) satisfy:

\[
c_{a,b} + c_{a-1,b} + c_{a,b-1} = 0
\]

where \( a, b \geq 1 \). As this is the exact same system of equations that we solved for the \( L^n \) and \( R^n \) case for \( n \leq 3 \), we get the following combinatorial formulae for the Wilson loop for \( n \leq 3 \):

\[
\begin{align*}
\langle F^n, I_\circ^3(D) \rangle &= N^3_{\circ}(D) - N^3_{\circ}(D) \\
\langle F^{mn}, I_\circ^3(D) \rangle &= 2 \cdot N^3_{\circ}(D) - 2 \cdot N^2_{\circ} N^1_{\circ}(D) + 2 \cdot N^2_{\circ}(D) + N^1_{\circ}(D) + N^1_{\circ}(D) \\
\langle F^{mn}, I_\circ^3(D) \rangle &= N^3_{\circ}(D) - N^2_{\circ} N^1_{\circ}(D) + N^2_{\circ} N^1_{\circ}(D) - N^2_{\circ}(D) + N^2_{\circ}(D) - N^2_{\circ}(D)
\end{align*}
\]

In Section 3.7, we will show how to generate all invariants in \( \text{Hom}_2(\mathbb{O}_n, \mathbb{Q}) \) as rational linear combinations of combinatorial \( \mathcal{P} \)-formulae.

3.6. Combinatorial \( \mathcal{P} \)-formulae and Finite-Type Invariants. In this section we provide a sufficient condition for a parity to yield Kauffman finite-type invariants. We also show that some of the new formulae are not of GPV finite-type and some are even virtualization invariant.

**Definition:** A parity \( \mathcal{P} = (P, p_D) \) is said to be **switch symmetric** if for all \( D, D' \) as in Figure 9, \( p_D(i) = p_{D'}(i') \) and for all \( j \in C(D) \cap C(D') \), \( p_D(j) = p_{D'}(j) \). Note that in this picture, the lettered embellishment is the label of the arrow, not its parity.

\[
D = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}, \quad D' = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
\]

**Figure 9.** Definition of Switch Symmetry

**Theorem 21.** Let \( \mathcal{P}_n \) for \( n \in \mathbb{N} \cup \{0\} \) denote the Gaussian parity and its hierarchy of parities. Then \( \mathcal{P}_n \) is switch symmetric for every \( n \).

**Proof.** The Gaussian parity is clearly switch symmetric. For \( \mathcal{P}_n \), note that if \( D \) and \( D' \) are obtained from one another by changing both the direction and sign of the arrow labelled \( x \), we have \( I_{D,y}(y) = I_{D',y}(y) \) for all \( y \in C(D) \). This proves the theorem. \( \square \)

**Theorem 22.** If \( \mathcal{P} \) is a switch symmetric parity and \( v \in \text{Hom}_2(\mathbb{O}_{n,k}, \mathbb{Q}) \) (or \( \text{Hom}_2(\mathbb{E}_{n,k}, \mathbb{Q}) \)), then \( v \circ I_{n,\mathcal{P}} \) (respectively, \( v \circ I_{n,\mathcal{P}}^r \)) is a Kauffman finite-type invariant of order \( \leq n \). The result holds on both the Wilson line and the Wilson loop.

**Proof.** Suppose that \( K \) is rigid 4-valent graph with one graphical vertex. The resolution of this crossing may be expressed in terms of Gauss diagrams as \( D \rightarrow D' \), where \( D \) and \( D' \) are as in Figure 9. Now, every arrow of \( D \) corresponds to an arrow of \( D' \) having the same parity. We apply \( I_\mathcal{P} \) to obtain:

\[
I_\mathcal{P} : \begin{array}{c}
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ
\end{array} + \begin{array}{c}
\circ \\
\circ
\end{array}
\]

\[
I_\mathcal{P}^r : \begin{array}{c}
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ
\end{array} + \begin{array}{c}
\circ \\
\circ
\end{array}
\]

The second terms on the RHS of these equations are identical because \( \mathcal{P} \) is switch symmetric. Therefore, every term in \( I_\mathcal{P}(D \rightarrow D') \) contains at least one arrow. In general, a long singular virtual knot with \( n+1 \) singular crossings will yield at least \( n+1 \) arrows in every nonzero term. Hence, \( v \circ I_{n,\mathcal{P}} \) (or \( v \circ I_{n,\mathcal{P}}^r \)) vanishes on all 4-valent graphs with more than \( n \) graphical vertices. This proves the theorem. \( \square \)
Now, consider the Gaussian parity \( \mathcal{P}_0 \). In the Wilson line case, we look at the combinatorial \( \mathcal{P}_0 \)-formulae of order \( \leq 3 \) from Section 3.5. It is certainly the case that the invariants of order one are not of GPV finite-type as there are no GPV finite-type invariants of order one at all. For \( 2 \leq n \leq 3 \), we apply Theorem 5. The following is a fractional twist sequence of type FSR [C1]:

Consider the combinatorial \( \mathcal{P} \)-formula \( F^{rr} \). The formula vanishes on every even term of the sequence and is nonzero on every odd term. Hence, the long virtual knot invariant defined by \( F^{rr} \) evaluates on this sequence to a nonzero function with infinitely many zeros. It is therefore not a polynomial. Thus it is not of GPV finite-type of any order. On the other hand, if one looks at the twist sequence of type FSL obtained from the above sequence by reversing the orientation of the Wilson line, then a similar argument shows that the combinatorial \( \mathcal{P} \)-formula defined by \( F^{ll} \) is not of GPV finite-type. A similar argument can be applied to \( F^{rrr} \) and \( F^{lll} \) are not of GPV finite-type. In fact, one may even use the same fractional twist sequences.

Now we turn to the invariants defined by \( F^{rl} \), \( F^{rr} \) and \( F^{rl} \). Consider the following twist sequence of type FSL.

All three of the invariants defined by \( F^{rl} \), \( F^{rr} \) and \( F^{rl} \) vanish on the even terms of this sequence and are nonvanishing on the odd terms. So once again, these are not polynomials and hence not of GPV finite-type.

Another way to show that something is not of GPV finite-type of any order is to show that it is invariant under the virtualization move [C2]. Recall that the virtualization move is as given in Figure 8. To show that a combinatorial \( \mathcal{P} \)-formula \( F \) is invariant under the virtualization move, it is sufficient to show that if \( D \) is a summand of \( F \) and \( D' \) is a dashed arrow diagram obtained by changing the direction of some subset of arrows (but not the signs) then \( D' \) is also a summand of \( F \) with the same coefficient.

**Theorem 23.** For the Gaussian parity, the following combinatorial \( \mathcal{P} \)-formulae on the Wilson line are invariant under the virtualization move:

\[
F^r + F^l
\]

\[
F^{ll} + 2 \cdot F^{rl} + F^{rr}
\]

\[
2 \cdot F^{rrr} + F^{rl} + F^{rr} + 2 \cdot F^{lll}
\]

**Proof.** The formulae are given explicitly in Section 3.5. Adding them up as indicated shows that the sufficient condition described in the previous paragraph holds true. \( \square \)

For the Wilson loop case, it turns out that all combinatorial \( \mathcal{P} \)-formulae are virtualization invariant. The proof requires a computation of the groups \( \text{Hom}_2(\mathcal{O}_{n,1}, \mathbb{Q}) \). Therefore, we postpone further comments until Section 3.8.

### 3.7. The Dimension of \( \text{Hom}_2(\mathcal{O}_{n,1}, \mathbb{Q}) \)

The aim of this section is to prove the following theorem:

**Theorem 24.**

1. **On the Wilson line**, there are \( n + 1 \) linearly independent (over \( \mathbb{Q} \)) combinatorial \( \mathcal{P} \)-formula of order exactly \( n \) for every \( n \). Hence, \( \text{dim}_2(\text{Hom}_2(\mathcal{O}_{n,1}, \mathbb{Q})) = n(n + 3)/2 \).

2. **On the Wilson loop**, there is exactly one combinatorial \( \mathcal{P} \)-formula of order exactly \( n \) for every \( n \). Hence, \( \text{dim}_2(\text{Hom}_2(\mathcal{O}_{n,1}, \mathbb{Q})) = n \).

The proof of the theorem has several steps. As the proof is quite long, we give an outline of our plan of attack. The Wilson line case is considered first. Then the Wilson loop case follows as an easy corollary. The plan for the Wilson line is as follows:

1. First we generalize the approach of Section 3.5. We look for formulae that are polynomials whose terms are of the form \( L^a L^b R^c R^d \). Such terms will automatically satisfy the commutativity relations. The hard part is satisfying the Q2 relations. We set up a system of equations (see Equations 4) and proceed to show that a solution to this system will yield a combinatorial \( \mathcal{P} \)-formula.

2. Next we find an upper bound for the dimension by using a short exact sequence argument (see Section 3.6).
(3) Then we show in Section 3.7.3 that if there is a combinatorial $\mathcal{P}$-formula generated by $L^n$ and $R^n$ for every $n$ (see Theorem 24), then the upper bound on the dimension is realized by simple products of the $L^n$ and $R^n$ (see Lemma 25). These products turn out to be linearly independent.

(4) Finally, one shows that there are combinatorial $\mathcal{P}$-formulae generated by $L^n$ and $R^n$ for every $n$.

This requires solving a system of equations that uses only integer arithmetic. The case that $n$ is even (see Section 3.7.4) is treated separately from the case that $n$ is odd (see Section 3.7.5).

After all of these steps are completed for the Wilson line, we proceed with the case of the Wilson loop (see in Section 3.5).

3.7.1 General Considerations for the Wilson line. Let $F$ be a combinatorial $\mathcal{P}$-formula of order $n$ on the Wilson line (i.e. it is of order $\leq n$ but not of order $\leq m$ for any $m < n$). Let $D \in \mathcal{A}^{(1)}$. Since $F \in \mathbb{Z}[\mathcal{A}^{(1)}]$, we have an integer which is the coefficient of $D$ as a term of $F$. Hence we may write $\text{coeff}(D, F)$ unambiguously.

Suppose that $D, D' \in \mathcal{A}^{(1)}$ have $n$ arrows. The following necessary conditions on $F$ are clear from the relations in Theorem 20.

(1) If $D, D'$ have $n$ arrows and are obtained from one another by changing the sign of $k$ arrows, then $\text{coeff}(D', F) = (-1)^k \text{coeff}(D, F)$.

(2) Suppose that all the arrows of $D$ and $D'$ have the same sign (i.e. all $\oplus$ or all $\ominus$). If $D$ and $D'$ have the same number of arrows pointing left and the same number of arrows pointing right, then $\text{coeff}(D, F) = \text{coeff}(D', F)$.

(3) Therefore, if $\text{coeff}(D, F) \neq 0$, then every $D'$ which has the same number of arrows pointing left as $D$ and the same number of arrows pointing right as $D$, then $\text{coeff}(D', F) = \pm \text{coeff}(D, F)$.

We will use the same notation as in Section 3.5 for $L_\oplus, L_\ominus, R_\oplus, R_\ominus$. To ease reading, we will write:

$$w := R_\oplus, x := R_\ominus, y := L_\ominus, z := L_\oplus$$

We will show that every odd arrow diagram $D$ having $n$ arrows signed $\oplus$ generates a combinatorial $\mathcal{P}$-formula $F$ of order $n$ such that the only terms having exactly $n$ arrows are related to $D$ as in the above necessary conditions. Therefore, $\langle F, I[\mathcal{P}] \rangle$ must always contain the following terms, if $D$ has $n$ arrows pointing right and $n_2$ arrows pointing left:

$$\sum_{i+j=n_1} \sum_{k+l=n_2} (-1)^{j+l} w^i x^j y^k z^l$$

Let $c_{ij}^{kl}$ denote the coefficient of $w^i x^j y^k z^l$ where $i + j, k + l \leq n_1 + n_2 - 1$. Then we search for $F$ so that $\langle F, I[\mathcal{P}] \rangle$ may be written as:

$$c_0 \sum_{i+j=n_1} \sum_{k+l=n_2} (-1)^{j+l} w^i x^j y^k z^l + \sum_{m_1=1}^{n_1-1} \sum_{m_2=1}^{n_2-1} \sum_{i+j=1}^{m_1} \sum_{k+l=1}^{m_2} c_{ij}^{kl} w^i x^j y^k z^l$$

where $c_0$ and the $c_{ij}^{kl}$ satisfy the system of equations below:

$$0 = (-1)^{j+l} c_0 + c_{i,j-1,j}^{k,l} + c_{i,j,j-1}^{k,l}, \quad k + l = n_2, 1 \leq j \leq n_1 - 1$$

$$0 = (-1)^{j+l} c_0 + c_{i,j-1,j}^{n_2-1-l,l} + c_{i,j,j-1}^{n_2-1-l,l}, \quad i + j = n_1, 1 \leq l \leq n_2 - 1$$

$$0 = c_{i,j}^{k,l} + c_{i-1,j}^{k,l} + c_{i,j-1}^{k,l}, \quad 2 \leq i + j \leq n_1 - 1, 0 \leq k + l \leq n_2$$

$$0 = c_{i,j}^{k-1,l} + c_{i,j}^{k,l-1}, \quad 0 \leq k + l \leq n_2 - 1, 0 \leq i + j \leq n_1$$

$$0 = c_{i,j}^{2-l,1} + c_{i,j}^{1,l-1}, \quad n_2 \geq 1, k + l \leq n_2 - 1, i + j = n_1 + 1$$

$$0 = c_{i-1,j}^{1-1,k-1} + c_{i,j}^{l-1,k-1}, \quad n_1 \geq 1, i + j \leq n_1 - 1, k + l = n_2 + 1$$

As will be shown in Theorem 25, these equations all correspond to the $Q_2^{(1)}$ relations.

For a variable $c_{ij}^{kl}$, we define $\text{deg}(c_{ij}^{kl}) = i + j + k + l$. For an equation $r$ in the list above, we define $\text{deg}(r)$ to be the maximum of the degrees of the variables in $r$ if $c_0$ is not a variable in $r$ and $\text{deg}(r) = n$ if $c_0$ is a variable in $r$.

**Theorem 25.** If \{\$c_0, c_{ij}^{kl}$\} is an integer solution to the system \{\$c_0, c_{ij}^{kl}$\}, then there is a combinatorial formula $F \in \mathbb{Z}[\mathcal{A}^{(1)}]$ such that

$$\langle F, I[\mathcal{P}] \rangle = c_0 \sum_{i+j=n_1} \sum_{k+l=n_2} (-1)^{j+l} w^i x^j y^k z^l + \sum_{m_1=1}^{n_1-1} \sum_{m_2=1}^{n_2-1} \sum_{i+j=1}^{m_1} \sum_{k+l=1}^{m_2} c_{ij}^{kl} w^i x^j y^k z^l$$
where every diagram in \( F \) having \( n \) arrows has \( n_1 \) arrows pointing right and \( n_2 \) arrows pointing left.

**Proof.** For every \( D \in A^{(1)} \), define \( \text{coeff}(D, F) \) as follows. If \( D \) has \( n_1 \) arrows pointing right and \( n_2 \) arrows pointing left, then \( \text{coeff}(D, F) = (-1)^{j+l} c_0 \), where \( j = \) the number of right arrows of \( D \) signed \( \ominus \) and \( k = \) the number of left arrows of \( D \) signed \( \ominus \). If \( D \) has \( > n_1 \) right arrows or \( > n_2 \) left arrows, \( \text{coeff}(D, F) = 0 \). If \( D \) has \( < n_1 \) right arrows and \( \leq n_2 \) left arrows or has \( \leq n_1 \) right arrows and \( < n_2 \) left arrows, set \( \text{coeff}(D, F) = c_{ij}^{k} \), where:

\[
\begin{align*}
i & = \# \text{ of right arrows signed } \oplus \\
j & = \# \text{ of right arrows signed } \ominus \\
k & = \# \text{ of left arrows signed } \oplus \\
l & = \# \text{ of left arrows signed } \ominus 
\end{align*}
\]

We define the potential combinatorial formula \( F \) by:

\[
F = \sum_{D \in A^{(1)}} \text{coeff}(D, F) \cdot D
\]

Note that the only nonzero summands of \( F \) having \( n \) arrows have \( n_1 \) arrows pointing right and \( n_2 \) arrows pointing left. To show that \( F \) is a combinatorial formula, it needs to be checked that \( \langle F, r \rangle = 0 \) for all \( r \) satisfying the relations of Theorem 20. For \( r \in \{Q3a^{(1)}, Q3b^{(1)}, Q3c^{(1)}\} \), the result follows from the fact that these moves change neither the direction nor the sign of any arrow.

The only relations that need to be checked are the \( Q2 \) relations. Write a \( Q2 \) relation as \( \langle A, F \rangle \) as follows. If \( \langle A, F \rangle = 0 \), then \( \langle A, F \rangle = 0 \) for all \( A \). However, we have \( \langle A, F \rangle = 0 \) if \( A \) has \( |A| = n \) arrows, \( n = n_1 + n_2 \), \( n_1 \) arrows pointing right and \( n_2 \) arrows pointing left, then the result is also clear. The only other case with \( A \) having \( n \) arrows is the case when there are \( n_1 \) right arrows and \( n_2 \) left arrows. In this case we have:

\[
\langle F, A \rangle = (-1)^{i+l} c_0 = (-1)^{i+l+1} c_0 = -\langle F, A \rangle
\]

In the case that \( A \) has \( n = n_1 + n_2 \) arrows, there are several possibilities. Let \( i_A \) be the number of arrows of \( A \) pointing right and signed \( \oplus \), \( j_A \) the number of arrows of \( A \) pointing right and signed \( \ominus \), \( k_A \) the number of arrows of \( A \) pointing left and signed \( \oplus \), \( l_A \) be the number of arrows of \( A \) pointing left and signed \( \ominus \).

Suppose that \( i_A + j_A \geq n_1 + 2 \). In a \( Q2 \) relation involving rightward arrows we have \( \text{coeff}(A, F) = \text{coeff}(A, F) = \text{coeff}(A, F) = 0 \). The same thing happens if the \( Q2 \) relation involves leftward arrows. Thus, all \( Q2 \) relations of this form are satisfied. Now suppose that \( k_A + l_A \geq n_2 + 2 \). By symmetry with previous case, it follows that all \( Q2 \) relations involving diagrams of this form are satisfied.

Now suppose that \( i_A + j_A = n_1 + 1 \). If \( r \) involves rightward arrows, then \( \text{coeff}(A, F) = \text{coeff}(A, F) = \text{coeff}(A, F) = 0 \). If \( r \) involves leftward arrows, we still have \( \text{coeff}(A, F) = 0 \). However, we have \( \text{coeff}(A, F) = c_{i_A,j_A-1}^{k_A,l_A} \) and \( \text{coeff}(A, F) = c_{i_A-1,j_A}^{k_A,l_A} \). By relations (2), we have that:

\[
\begin{align*}
c_{i_A,j_A-1}^{k_A,l_A} + c_{i_A-1,j_A}^{k_A,l_A} & = 0
\end{align*}
\]

Hence all \( Q2 \) relations of this form are satisfied. A similar argument works in the case that \( k_A + l_A = n_2 + 1 \).

Another case is that \( i_A + j_A = n_1 \) and \( k_A + l_A = n_2 \). Assume without loss of generality that \( r \) involves rightward arrows. Then:

\[
\begin{align*}
\langle F, r \rangle & = \langle F, A \rangle + \langle F, A \rangle + \langle F, A \rangle \\
& = \text{coeff}(A, F) + \text{coeff}(A, F) + \text{coeff}(A, F) \\
& = (-1)^{j_A+k_A} c_0 + c_{i_A,j_A-1}^{k_A,l_A} + c_{i_A-1,j_A}^{k_A,l_A} \\
& = 0
\end{align*}
\]
Finally suppose that \( i_A + j_A + k_A + l_A \leq n_1 + n_2 - 1 \), \( i_A + j_A \leq n_1, j_A + k_A \leq n_2 \). We may assume without loss of generality that \( r \) involves only rightward arrows. Then:

\[
\langle F, r \rangle = \langle F, A_\pm \rangle + \langle F, A_+ \rangle + \langle F, A_- \rangle
\]

\[
= \text{coeff}(A_+, F) + \text{coeff}(A_-, F)
\]

\[
= c_{A_A A_A} + c_{A_A A_A} + c_{A_A -1 A_A}
\]

\[
= 0
\]

Since \( \langle F, r \rangle = 0 \) for all relations \( r \), it follows that \( F \) is a combinatorial \( \mathcal{P} \)-formula of order \( \leq n \). \( \square \)

We will say that there is a **combinatorial formula generated by \( R^{n+1}L^{n_2} \)** if there is an integer solution \( \{c_0, c_i^j\} \) to the system \( \mathbf{3.7.4} \). It will be shown in Sections \( \mathbf{3.7.3} \) and \( \mathbf{3.7.5} \) that there are combinatorial formula generated by \( R^n \) and \( L^n \). It will then follow from Section \( \mathbf{3.7.2} \) that there are combinatorial formula generated by \( R^{n+1}L^{n_2} \) for every choice of \( n_1 \) and \( n_2 \).

### 3.7.2. An Upper Bound on the Dimension

We proceed with the plan as outlined in Section \( \mathbf{3.7} \). The upper bound is established by using a short exact sequence argument. Later, we show that this upper bound is achieved by combinatorial \( \mathcal{P} \)-formulae.

**Lemma 26.** An upper bound for the dimension is given by \( \dim_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(\mathbb{O}_{n,1}, \mathbb{Q})) \leq n(n+3)/2 \).

**Proof.** Theorem \( \mathbf{3.7} \) implies the following short exact sequence:

\[
0 \rightarrow A^{(1)}_{t,1} + \Delta Q^{(1)} \rightarrow A^{(1)}_t + \Delta Q^{(1)} \rightarrow \mathbb{O}_{t,1} \rightarrow \mathbb{O}_{t-1,1} \rightarrow 0
\]

where \( \Delta Q^{(1)} = \langle Q2^{(1)}, Q3a^{(1)}, Q3b^{(1)}, Q3c^{(1)} \rangle \). Let \( \mathbb{Z}_t \) denote the leftmost nonzero module in the sequence. It follows by induction that we may identify \( \text{Hom}_{\mathbb{Z}}(\mathbb{O}_{n,1}, \mathbb{Q}) \) as a subgroup of:

\[
\bigoplus_{t=1}^n \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_t, \mathbb{Q})
\]

It is thus sufficient to show that \( \dim_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_t, \mathbb{Q})) \leq t + 1 \) for \( 1 \leq t \leq n \).

We proceed as in [Pol]. Let \( A^{(1)}_t \) denote the free abelian group generated by those signed odd arrow diagrams having \( t \) arrows. It follows from Noether’s isomorphism theorem that:

\[
\frac{A^{(1)}_{t,1} + \Delta Q^{(1)}}{A^{(1)}_t + \Delta Q^{(1)}} \cong \frac{A^{(1)}_t}{A^{(1)}_t \cap (A^{(1)}_t + \Delta Q^{(1)})}
\]

Let \( Q3_t \) denote those \( Q3a^{(1)}, Q3b^{(1)} \) and \( Q3c^{(1)} \) relations having \( t \) arrows in every diagram. We define a relation on \( A^{(1)}_t \):

\[
\text{NS}_t : \quad \begin{array}{c}
\oplus \\
\end{array} + \begin{array}{c}
\oplus \\
\end{array}
\]

It is easy to see that \( \langle \text{NS}_t, Q3_t \rangle \subset A^{(1)}_t \cap (A^{(1)}_t + \Delta Q^{(1)}) \). Therefore, we have the exact sequence below:

\[
\frac{A^{(1)}_t}{\langle Q3_t, \text{NS}_t \rangle} \rightarrow \frac{A^{(1)}_t}{A^{(1)}_t \cap (A^{(1)}_t + \Delta Q^{(1)})} \rightarrow 0
\]

Finally, let \( B^{(1)}_t \) denote the free abelian group generated by unsigned odd arrow diagrams. Let \( R3_t \) denote the collection of all unsigned \( Q3_t \) relations. We define a map:

\[
\Xi_t : \frac{A^{(1)}_t}{\langle Q3_t, \text{NS}_t \rangle} \rightarrow \frac{B^{(1)}_t}{\langle R3_t \rangle}
\]

\[
\Xi_t(D) = (-1)^{|D|}
\]

where \( D \) has \( k \) arrows signed \( \oplus \) and \( |D| \) denotes \( D \) with all its signs forgotten. \( \Xi_t \) is a one-to-one and onto homomorphism. Hence:

\[
\frac{A^{(1)}_t}{\langle Q3_t, \text{NS}_t \rangle} \cong \frac{B^{(1)}_t}{\langle R3_t \rangle}
\]
The relations $R_3$, imply that any two diagrams having the same number of arrows pointing left and the same number of arrows pointing right are equivalent. Since there are between $0$ and $t$ arrows pointing right we have:

$$\dim\left(\text{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_2}{(R_3)}, \mathbb{Q}\right)\right) = t + 1$$

If we now chase around the diagrams long enough, we see that $\dim\left(\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_t, \mathbb{Q})\right) \leq t + 1$. This proves the lemma.

3.7.3. **Assuming Formulae for $L^n$ and $R^n$, Theorem 24 (1) is Proved.** In this section, we show that there is a combinatorial $F$-formula generated by $R^{n_1}L^{n_2}$ for every $n_1$ and $n_2$. The following theorem will be proved later:

**Theorem 27.** There is a combinatorial formula generated by $R^n$ and $L^n$ for every choice of $n$.

*Proof.* This is proved in Section 3.7.4 when $n$ is even and Section 3.7.5 when $n$ is odd.

Let $n_1, n_2 \in \mathbb{N}$ be given. Let $E^r$ denote the combinatorial formula generated by $R^{n_1}$ and $E^l$ denote the combinatorial formula generated by $L^{n_2}$. By Theorem 27 and Lemma 25 we have polynomials $f_{n_1}(w, x)$ and $f_{n_2}(y, z)$ such that:

$$\langle E^r, I[F]\rangle = f_{n_1}(w, x)$$
$$\langle E^l, I[F]\rangle = f_{n_2}(y, z)$$

The coefficients of $f_{n_1}$ and $f_{n_2}$ are determined by solutions $\{c_0, c_{ij} : i + j \leq n_1, i, j \geq 0\}$ and $\{d_0, d_{kl} : k + l \leq n_2, k, l \geq 0\}$ to the system (2) (here unnecessary indices have been deleted). Let $n = n_1 + n_2$. Define:

$$f_n(w, x, y, z) = f_{n_1}(w, x) \cdot f_{n_2}(y, z)$$

**Lemma 28.** The set of coefficients in $f_n$ is a solution to (3) and hence, by Theorem 24 there is an $F \in \mathbb{Z}[A^{(1)}]$ such that:

$$\langle F, I[F]\rangle = f_n(w, x, y, z)$$

where every summand in $F$ having $n$ arrows has exactly $n_1$ arrows pointing right and $n_2$ arrows pointing left.

*Proof.* It is only necessary to prove that the coefficients in $f_n$ satisfy the system of equations in (2). For the top two types on the list, we have:

$$(-1)^{i+l} c_0 d^l + (-1)^i d^l c_{i-1, j} + (-l) d^l c_{i, j-1} = \begin{cases} (-1)^l d^l((-1)^l c_0 + c_{i-1, j} + c_{i, j-1}) \\ 0 \end{cases}$$

$$(-1)^{i+l} c_0 d^l + (-1)^i d^l c_{i-1, j} + (-l) d^l c_{i, j-1} = \begin{cases} (-1)^l c_0((-1)^l d_0 + d^{k-1,l} + d^{k,l-1}) \\ 0 \end{cases}$$

The second two types of equations may be written as follows:

$$c_{ij} d^{k,l} + c_{i-1,j} d^{k,l} + c_{i,j-1} d^{k,l} = d^{k,l}(c_{ij} + c_{i-1,j} + c_{i,j-1})$$

$$c_{ij} d^{k,l} + c_{i,j,l} d^{k,l} + c_{ij} d^{k,l-1} = c_{ij} \left( d^{k,l} + d^{k-1,l} + d^{k,l-1} \right)$$

$$c_{ij} d^{k,l} + c_{i,j,l} d^{k,l} + c_{ij} d^{k,l-1} = 0$$

The last two types of equations may be written as:

$$(-1)^i c_0 d^{k,l} + (-1)^{i-1} c_0 d^{k,l} = 0$$

$$(-1)^i d^l c_{ij} + (-1)^{i-1} d^l c_{ij} = 0$$

This completes the proof of the lemma.

As an exercise, the reader might enjoy factoring the formulae in Section 3.5. These were computed by explicitly solving a system of equations, but they factor into formulae of smaller degree as described here.
Proof of Theorem 24 (1). Let \( n \in \mathbb{N} \) be given. The previous work has shown that there is a combinatorial \( \mathcal{P} \)-formula generated by \( R^n \), \( R^{n-1}L \), \( R^{n-2}L^2 \), \ldots , \( RL_{n-1} \) and \( L^n \).

Now, it is clear from the arrow diagrams in \( R^mL^{n_2} \) having \( n_1 + n_2 \) arrows that the formula generated by \( R^mL^{n_2} \) cannot be written as a linear combination of the formulae \( R^mL^{n_j} \), where \( n_i + n_j = n \) and \( n_i \neq n_1, n_j \neq n_2 \). Hence the set of formulae generated by \( R^mL^{n_j} \) are linearly independent for every \( n \). Hence, there are \( n+1 \) rationally linearly independent combinatorial \( \mathcal{P} \)-formula of order exactly \( n \) for every \( n \). This implies that a lower bound for the dimension is \( n(n+3)/2 \). This, along with Lemma 26 proves the theorem.

3.7.4. Formulae for \( L^n \) and \( R^n \), \( n \) even. By symmetry, it suffices just to look at one of the two, say \( R^n \). We simplify the equations in this case and remove any unnecessary indices. In particular, we are looking for expressions of the form:

\[
c_0 \sum_{i=0}^{n} (-1)^i w^{n-i} x^i + \sum_{k=1}^{n-1} \sum_{i+j=k} c_{ij} w^i x^j
\]

where \( c_0 \) and the \( c_{ij} \) satisfy the equations:

\[
\text{TOP} = \{ (-1)^i c_0 + c_{n-i-1,i} + c_{n-i,i-1} = 0 : 1 \leq i \leq n-1 \} \\
\text{BOT} = \{ c_{ij} + c_{i-1,j} + c_{i,j-1} = 0 : 2 \leq i+j \leq n-1, i,j \geq 1 \}
\]

Here, TOP refers to those equations having degree \( n \) (which are of highest degree and hence on the top) and BOT refers to those equations having degree less than \( n \) (i.e. on the bottom).

For the case that \( n \) is even, add the following symmetry conditions to TOP and BOT:

\[
c_{ij} = c_{ji} \forall i,j
\]

Applying this to TOP gives:

\[
(-1)^i c_0 + c_{n-i-1,i} + c_{n-i,i-1} = (-1)^i c_0 + c_{i,n-i-1} + c_{i-1,n-i} = (-1)^{n-i} c_0 + c_{i-1,n-i} + c_{i,n-i-1} = (-1)^{n-i} c_0 + c_{n-(n-i)-1,n-i} + c_{n-(n-i),n-i-1}
\]

Hence it is sufficient to consider the system \( \text{TOP}' \) defined by:

\[
\text{TOP}' = \{ (-1)^i c_0 + c_{n-i-1,i} + c_{n-i,i-1} = 0, (-1)^{n/2} c_0 + 2c_{n/2-1,n/2} = 0 : 1 \leq i \leq n/2-1 \}
\]

We set \( c_{n/2-1,n/2} = (-1)^{n/2-1} \) and add it to \( \text{TOP}' \). We may label the variables so that \( \text{TOP}' \) is given by the following augmented matrix:

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & -1 \\
0 & 1 & 1 & \ldots & 1 & 0 \\
0 & 0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & (-1)^{n/2} \\
0 & 0 & 0 & \ldots & 1 & 0 \\
\end{pmatrix}
\]

\[
\begin{array}{c|c}
\hline
0 & (-1)^{n/2-1} \\
\hline
\end{array}
\]

Lemma 29. The system TOP has a solution which is given by:

\[
c_0 = 2 \\
c_{i,n-i-1} = (-1)^{i+1}(2i+1-n), \ n/2 \leq i \leq n-1, \\
c_{ij} = c_{ji}
\]

Proof. Let \( m = n/2 + 1 \). Consider a solution to the above augmented matrix for \( \text{TOP}' \) as a column vector \( (a_1, \ldots, a_m) \). One immediately obtains that \( a_{m-1} = (-1)^{n/2-1} \) and \( a_m = 2 \). The other equations may be rewritten as:

\[
a_k + a_{k+1} + (-1)^{k+1} \cdot 2
\]

The solution to this recursion (as can be immediately checked by substitution) is:

\[
a_k = (-1)^{k+1}(2(m-k) - 1)
\]

Finally, if we set \( a_k = c_{n-k,n-k-1} \) for \( 1 \leq k \leq m-1 \) and \( c_0 = a_m \), we have the result for \( \text{TOP}' \) after a change of variables. The result for TOP follows by adding the symmetry conditions. \( \square \)
For \( k < n \), denote by \( \text{BOT}_k \) the system of equations \( \text{BOT} \) modulo the symmetry \( c_{ij} = c_{ji} \). Note that \( \text{BOT}_{2k}^+ \) and \( \text{BOT}_{2k+1}^+ \) both have \( k \) distinct equations. They are given by:

\[
\begin{align*}
\text{BOT}_{2k+1}^+ &= \{ c_{\alpha,\beta} + c_{\alpha-1,\beta} + c_{\alpha,\beta-1} = 0 \mid \alpha + \beta = 2k + 1, \alpha > \beta, 1 \leq \beta \leq k \} \\
\text{BOT}_{2k}^+ &= \{ c_{\alpha,\beta} + c_{\alpha-1,\beta} + c_{\alpha,\beta-1} = 0 \mid \alpha + \beta = 2k, \alpha > \beta, 1 \leq \beta \leq k - 1 \} \\
&\quad \cup \{ c_{k,k} + 2 \cdot c_{k,k-1} = 0 \} 
\end{align*}
\]

The trick of the proof is to use the equations in \( \text{BOT}_{n-1}^+ \), \( \text{BOT}_{n-2}^+ \), and the solution to \( \text{TOP}' \), to make a system of equations which is nearly square. Define:

\[
B_{n/2-1} = \text{BOT}_{n-1}^+ \cup \text{BOT}_{n-2}^+ \cup \{ c_{n-i-1,i} = (-1)^{i+1}(2t + 1 - n) \text{ for } n/2 \leq i \leq n - 2 \} \cup \{ c_{(n-2)/(n-2)/2-1} = 1 \}
\]

This gives an \((3n/2 - 2) \times (3n/2 - 2)\) system \( A_{n/2-1} \cdot \vec{C}_{n/2-1} = \vec{b}_{n/2-1} \), where \( \vec{b}_{n/2-1} \) is a nonzero vector determined by \( B_{n/2-1} \). We will show that this system has a unique solution. In particular, it gives values to the variables of degree \( n - 3 \). This process is then continued on a similar grouping of the remainder of the equations in \( \text{BOT}' \).

Specifically, the equations of order \( n - 3 \) or less in \( \text{BOT}' \) are grouped as follows. For \( 1 \leq k \leq n/2 - 2 \) we look at the systems \( \text{BOT}_{2k+1}^+ \) and \( \text{BOT}_{2k}^+ \). Some equations are added in to ensure that the system is square. In fact, add in the equations \( c_{2k+1-s,s} = W_s^k \) for \( 1 \leq s \leq k \) where the \( W_s^k \) stand for some constants yet to be determined. We also add the equation \( c_{k,k-1} = 1 \). The result is a system of equations:

\[
B_k = \text{BOT}_{2k+1}^+ \cup \text{BOT}_{2k}^+ \cup \{ c_{2k+1-s,s} = W_s^k : 1 \leq s \leq k \} \cup \{ c_{k,k-1} = 1 \}
\]

This system is a \((3k + 1) \times (3k + 1)\) system of the form \( M_k \cdot \vec{C}_k = \vec{b}_k \), where \( \vec{b}_k \) is a nonzero vector containing zeros, the \( W_s^k \) and a 1.

We complete the proof of Theorem 27 for every even \( n \) in three steps:

1. It is shown that the homogeneous system \( M_k \cdot \vec{c} = \vec{0} \) has a unique solution by showing that \( M_k \) can be row reduced, using only integer arithmetic, to an upper triangular matrix with all ones along the diagonal.

2. The solution for \( M_k \cdot \vec{c} = \vec{b}_k \) determines the arbitrary constants of the \( W_s^{k-1} \) in \( M_{k-1} \cdot \vec{c} = \vec{b}_{k-1} \).

3. Since the vector \( \vec{b}_{n/2-1} \) has no arbitrary constants (in fact, its values come from Lemma 20), we can solve the systems in the order \( k = n/2 - 1, k = n/2 - 2, \ldots, k = 1 \). The result turns out to be an integer solution \( \{ c_0, c_{ij} \} \) of (2) with \( c_0 \neq 0 \). By Theorem 25 this is a combinatorial formula.

Steps (2) and (3) are clear. It remains to show that \( M_k \) is invertible. We construct a matrix for \( M_k \) as follows. Let \( Z_k \) be the \( k \times k \) zero matrix, \( I_k \) the \( k \times k \) identity matrix. We define a \( k \times (k + 1) \) matrix \( S_k \) and a \( k \times k \) matrix \( T_k \) by:

\[
S_k = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1
\end{bmatrix},
T_k = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 2
\end{bmatrix}.
\]

In addition, we will employ the \((k + 1) \times k \) matrix \( E_k \), the \((k + 1) \times k \) matrix \( J_k \), and the \( k \times (k + 1) \) matrix \( L_k \) defined below:

\[
E_k = \begin{bmatrix}
Z_k \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix},
J_k = \begin{bmatrix}
I_k \\
0 & \cdots & 0
\end{bmatrix},
L_k = \begin{bmatrix}
0 & \vdots & 0 \\
& & I_k
\end{bmatrix}.
\]
We will show that the following \((3k + 1) \times (3k + 1)\) matrix has rank \((3k + 1)\) and is hence invertible.

\[
M_k = \begin{bmatrix}
S_k & Z_k & I_k \\
L_k & T_k & Z_k \\
Z_{k+1} & E_k & J_k
\end{bmatrix}
\]

Note that columns 1 to \(k\) correspond to those variables in \(B_k\) of degree 2\(k\), columns \(k + 2\) to 2\(k + 1\) those variables in \(B_k\) of degree 2\(k - 1\), and columns 2\(k + 2\) to 3\(k + 1\) to those variables in \(B_k\) of degree 2\(k + 1\). Also note that rows 1 to \(k\) correspond to those equations in \(\text{BOT}_{2k+1}^2\), rows \(k + 1\) to 2\(k\) correspond to those equations in \(\text{BOT}_{2k}^2\), rows 2\(k + 1\) to 3\(k + 1\) correspond to those equations involving the \(W_k\), and row 3\(k + 1\) corresponds to the equation \(c_{k,k-1} = 1\). We have also specified that column 2\(k + 1\) corresponds to the variable \(c_{k,k-1}\). In what follows, all row operations use only integer arithmetic.

**Lemma 30.** There is a sequence of row operations on \(M_k\) so that \(M_k = M'_k\) where:

\[
M'_k = \begin{bmatrix}
S'_k & Z_k & I'_k \\
L_k & T_k & Z_k \\
Z_{k+1} & E_k & J_k
\end{bmatrix}, \quad \text{and} \quad S'_k = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & (-1)^{k-1} \\
0 & 1 & 0 & \cdots & 0 & (-1)^{k-2} \\
& & \vdots & & & \vdots \\
& & \vdots & & & \vdots \\
0 & 0 & 0 & \cdots & 1 & (-1)^3
\end{bmatrix}
\]

Now we take the 3\(k + 1\) row (i.e. the last row) and use eliminate the 2 in \(T_k\). This gives the lemma:

**Lemma 31.** There is a sequence of row operations so that \(M'_k = M''_k\), where:

\[
M''_k = \begin{bmatrix}
S''_k & Z_k & I''_k \\
L_k & T_k & Z_k \\
Z_{k+1} & E_k & J_k
\end{bmatrix}, \quad \text{and} \quad T'_k = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
& & \vdots & & & \vdots \\
& & \vdots & & & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

Now using the 2\(k\)-th row, we may eliminate the last column in \(S'_k\), giving the following lemma:

**Lemma 32.** There is a sequence of row operations so that \(M''_k = M'''_k\), where:

\[
M'''_k = \begin{bmatrix}
S'''_k & Z_k & I'''_k \\
L_k & T_k & Z_k \\
Z_{k+1} & E_k & J_k
\end{bmatrix}, \quad \text{and} \quad S'''_k = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
& & \vdots & & & \vdots \\
& & \vdots & & & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]

Now use the second to \(k\)-th rows of \(M'''_k\) to eliminate the ones in the first through \((k - 1)\)-th rows of \(L_k\). This gives the lemma:

**Lemma 33.** There is a sequence of row operations such that \(M'''_k = M''''_k\), where:

\[
M''''_k = \begin{bmatrix}
S''''_k & Z_k & I''''_k \\
L_k & T_k & Z_k \\
Z_{k+1} & E_k & J_k
\end{bmatrix}, \quad \text{and} \quad L'_k = \begin{bmatrix}
Z_{k-1} & \vdots \\
\vdots & \vdots \\
0 & 0
\end{bmatrix}
\]

If the reader has been paying attention, she can see that we are very nearly done. First permute rows 2\(k + 1\) to 3\(k + 1\) by using the cycle \((3k + 1)(3k) . . . (3k + 2)(2k + 1) 2k\). The result is a matrix \(M''''_k\) having ones along the diagonal in columns 2\(k + 1\) to 3\(k + 1\) and only zeros below the diagonal in the same columns. Finally, we can permute the rows \(k + 1\) to 2\(k\) of \(M''''_k\) by using the cycle \((2k)(2k - 1) . . . (k + 2 k + 1).\) The result is a matrix having all ones along the diagonal and only zero’s below the diagonal. Thus we have proved:

**Lemma 34.** The matrix \(M_k\) can be row reduced by using only integer arithmetic to an upper triangular matrix having only ones along the diagonal. Hence \(M_k\) has rank 3\(k + 1\).

**Lemma 35.** There are combinatorial \(P\)-formulae generated by \(R^n\) and \(L^n\) for every \(n \in 2\mathbb{N}\).

*Proof.* The matrix \(M_k\) is a square matrix which is row reduced to an upper triangular matrix with all ones along the diagonal. Since the elements of the vectors \(L_k\) are always integers, solving from the bottom to the top necessarily yields integer solutions. Applying steps (2)-(3) above yields an integral solution \(\{c_0, c_1\}\) to the system (2) with \(c_0 ne 0\). Hence, there is a corresponding combinatorial \(P\) formula. \(\square\)
3.7.5. Formulae for $L^n$ and $R^n$, $n$ odd. This case is similar to the case $n$ even, so we only sketch the proof. Let $\text{TOP}'$ and $\text{BOT}'$ be the system of equations obtained from $\text{TOP}$ and $\text{BOT}$ (as defined in Section 3.7.5) by considering the skew symmetries:

\[
\begin{align*}
\text{TOP}' & \quad : \\
& \quad \text{BOT}'
\end{align*}
\]

In this case, $\text{TOP}'$ may be written as below. Note that the variable $c_{qq}$ does not appear in any equations in $\text{TOP}'$:

\[
\text{TOP}' = \begin{cases}
(-1)^j c_0 + c_{i,j-1} + c_{i-1,j} = 0 & i + j = n, 1 \leq j \leq (n-3)/2 \\
(-1)^{(n-1)/2} c_0 + c_{(n+1)/2,(n-3)/2} = 0
\end{cases}
\]

Adding the equation $c_{(n+1)/2,(n-3)/2} = (-1)^{(n+1)/2}$ to $\text{TOP}'$ gives the augmented matrix:

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & -1 \\
0 & 1 & 1 & \ldots & 0 & 1 \\
0 & 0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & (-1)^{(n-1)/2} \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

Lemma 36. The system of equations $\text{TOP}$ has a solution given by:

\[
\begin{align*}
c_0 & = 1 \\
c_{(n-1)/2+k,(n-1)/2-k} & = (-1)^{(n+1)/2-k}, 1 \leq k \leq \frac{n-1}{2} \\
c_{ij} & = -c_{ij}, \quad i + j = n - 1, j < i \\
c_{(n-1)/2,(n-1)/2} & = 0
\end{align*}
\]

Proof. The proof is similar to the $n$ even case. □

Let $\text{BOT}'_j$ denote those equations in $\text{BOT}'$ of degree $k$. In particular, we have:

\[
\begin{align*}
\text{BOT}'_2j & = \{ c_{a,b} + c_{a-1,b} + c_{a,b-1} = 0 & \alpha + \beta = 2j, \alpha > \beta, 1 \leq \beta \leq j - 1 \} \\
\text{BOT}'_{2j-1} & = \{ c_{a,b} + c_{a-1,b} + c_{a,b-1} = 0 & \alpha + \beta = 2j - 1, \alpha > \beta, 1 \leq \beta \leq j - 2 \} \\
& \cup \{ c_{j,j-1} + c_{j,j-2} = 0 \}
\end{align*}
\]

Consider the system of equations $B_j$ define by:

\[
B_j = \text{BOT}'_2j \\
\cup \text{BOT}'_{2j-1} \\
\cup \{ c_{j,j-1} = (-1)^{j-1} \} \\
\cup \{ c_{j+s,j-s} = W_j^s | 1 \leq s \leq j - 1 \}
\]

Here the $W_j^s$ are constants that are yet to be determined that corresponding to values of the variables of degree $2j$. Note that the system $B_j$ has $3j - 2$ variables and $3j - 2$ equations. Let $S_k$, $Z_k$, $L_k$, and $J_k$ be as before. Define $H_k$ to be the $k \times k$ matrix with a $k$ in the $(k,k)$ position and 0 everywhere else. Also, define $Y_k$ to be the $k \times (k-1)$ matrix of all zeros. In addition, we need the following $k \times jk$ matrix $T_k$:

\[
T_k = \begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

For the system $B_j$, we arrange the variables into a matrix $M_j$ so that the columns 1 to $j$ correspond to the variables of degree $2j - 1$, the columns $j + 1$ to $2j - 1$ correspond to the variables of degree $2j - 2$, etc.
and the columns $2j$ to $3j - 2$ correspond to those variables of degree $2j$. Moreover, the equations are arranged so that rows 1 to $j - 1$ correspond to $\text{BOT}_{2j}$, rows $j$ to $2j - 2$ correspond to $\text{BOT}_{2j-1}$, rows $2j - 1$ to $3j - 3$, correspond to the equations containing the $W_j^\prime$, and row $3j - 2$ corresponds to those equation $c_{j,j-1} = (-1)^{j-1}$. In the matrix below, the $j$-th column corresponds to the variable $c_{j,j-1}$.

\[ M_j = \begin{bmatrix} S_{j-1} & Z_{j-1} & I_{j-1} \\ L_{j-1} & T_{j-1} & Z_{j-1} \\ H_j & Y_j & J_{j-1} \end{bmatrix} \]

Therefore, the system of equations $B_j$ corresponds to the matrix equation $M_j \cdot \bar{C}_j = \bar{b}_j$, where $\bar{b}_j$ contains some zeros, the $W_j^\prime$, and $(-1)^{j-1}$.

**Lemma 37.** The $(3j - 2) \times (3j - 2)$ matrix $M_j$ can be row reduced, by using integer only arithmetic, to an upper triangular matrix with all ones along the diagonal. Hence the $M_j$ has rank $3j - 2$.

**Proof.** This is similar to Section 3.7.4. □

**Lemma 38.** There is a combinatorial $P$-formula of order $R^n$ and $L^n$ for every odd $n \in \mathbb{N}$.

**Proof.** We proceed as in the case where $n$ is even. We have a solution to the system of equations for $\text{TOP}'$ giving unique values for the variables of the degree $n - 1$. These determine the value of the constants $W_j' (n-1)/2$. We solve the systems $M_j \bar{C}_j = \bar{b}_j$ in the order $j = (n-1)/2$, $j = (n-3)/2$, ..., $j = 2$. The $W_j$ in the vector $\bar{b}_j$ are determined by the solution to the system $M_{j+1} \bar{C}_{j+1} = \bar{b}_{j+1}$. This follows because the variables of lowest order in $B_{j+1}$ have degree $2j$ and the $W_j$ correspond to values of the variables of order $2j$. These are known because these are the ones for which we just solved! In each instance, Lemma 37 gives an integer solution $\bar{C}_j$. The last equation to solve is:

\[ c_{1,1} + c_{1,0} + c_{0,1} = 0 \]

If $n = 1$, this is actually the only equation. In this case we set $c_{11} = 0$, $c_{10} = 1$, and $c_{01} = -1$. If $n \neq 1$, this is solved simply by setting $c_{10} = c_{01} = c_{11} = 0$. This clearly abides by the symmetry condition. The $\bar{C}_j$, the solution to $\text{TOP}'$, and this last equation give a solution $\{c_0, c_{ij}\}$ to the system (2). □

This completes the proof of Theorem 27.

### 3.8. The Case of $\text{Hom}_\mathbb{Z}(\mathbb{O}, \mathbb{Q})$ on the Wilson loop.

Define an unsigned dashed arrow diagram on the Wilson loop as follows:

\[ \xi_n = \]

Denote by $\mathbb{Z}[\xi_n]$ the free abelian group generated by $\xi_n$. Consider a dashed arrow diagram on the Wilson loop having all arrows signed $\oplus$. Modulo the commutativity relations, this diagram is equivalent to any other diagram having all arrows signed $\oplus$. Using a short exact sequence argument which is nearly identical to the one given in the proof of Theorem 20 on may show that $\text{Hom}_\mathbb{Z}(\mathbb{O}, \mathbb{Q})$ is identified as a subgroup of:

\[ \bigoplus_{r=1}^{n} \text{Hom}_\mathbb{Z}(\mathbb{Z}[\xi], \mathbb{Q}) \]

The following lemma follows immediately from this observation.

**Lemma 39.** An upper bound for the dimension of $\text{Hom}_\mathbb{Z}(\mathbb{O}, \mathbb{Q})$ on the Wilson loop is $n$.

We show that this upper bound is realized by producing a combinatorial $P$ formula of order exactly $n$ for every $n$. Following Section 3.3 we look for formulae that are linear combinations of terms of the form $N^0_{\oplus} N^0_{\ominus}$. If there is such a formula $F$, it satisfies some obvious conditions.

1. If $D, D'$ have $n$ arrows and are obtained from one another by changing the sign of $k$ arrows, then $\text{coeff}(D', F) = (-1)^k \text{coeff}(D, F)$. 
(2) Since $F$ is of order exactly $n$, $F$ must contain all diagrams having $n$ arrows signed $\oplus$ (i.e. those matching $N^n_{\oplus}$).

(3) Therefore, if $D,D'$ are dashed arrow diagrams having $n$ arrows, then $\text{coeff}(D,F) = \pm \text{coeff}(D',F)$. Let $w := N_{\oplus}$ and $x := N_{\oplus}$. The desired formula is of the form:

$$c_0 \sum_{i=0}^{n} (-1)^i w^{n-i} x^i + \sum_{k=1}^{n-1} \sum_{i+j=k} c_{ij} w^i x^j.$$

where $c_0, c_{ij} \in \mathbb{Z}, 0 \leq i,j \leq n$. It remains only to show that the coefficients may be chosen so that all Q2 relations are satisfied. Note that all the Q2 relations look locally the same on the Wilson loop as they do on the Wilson line. So the system of equations is also the same. In particular, we may write:

$$\text{TOP} = \{ (-1)^i c_0 + c_{n-i-1,i} + c_{n-i,i-1} = 0 : 1 \leq i \leq n-1 \}$$

$$\text{BOT} = \{ c_{i,j} + c_{i-1,j} + c_{i,j-1} = 0 : 2 \leq i+j \leq n-1, i,j \geq 1 \}$$

However, we saw in Sections 3.7.4 and 3.7.5 that this has a solution when $n$ is even and when $n$ is odd. This implies the following theorem.

**Theorem 40.** There is a combinatorial $\mathcal{P}$-formula generated by $N_{\oplus}^n$ for every $n \in \mathbb{N}$.

Finally, we can finish the proof of Theorem 24.

**Proof of Theorem 24 (2).** By Lemma 39, the upper bound for $\text{Hom}_G(\mathcal{O}_{n,1})$ is $n$. Theorem 40 states that there is a combinatorial $\mathcal{P}$-formula of order exactly $n$ for every $n$. The formulas for $t = 1,2,\ldots,n$ are linearly independent for obvious reasons. Thus the upper bound is achieved. \square

As a direct consequence of this result, one can see that all invariants of $\mathcal{O}_{n,1}$ on the Wilson loop have the same value on two diagrams obtained from one another by changing the direction of a single arrow. The following corollary is immediate:

**Corollary 41.** Let $\mathcal{P}$ for $m \in \mathbb{N} \cup \{0\}$ denote the Gaussian parity. If $v \in \text{Hom}_G(\mathcal{O}_{n,1},\mathbb{Q})$ on the Wilson loop, then $v \circ I^n_{n,1}[\mathcal{P}]$ is Kauffman finite-type invariant of order $\leq n$ that is invariant under the virtualization move. Moreover, if $v \circ I^n_{n,1}[\mathcal{P}]$ is not the identically zero invariant, then $v \circ I^n_{n,1}[\mathcal{P}]$ is not of GPV finite-type.

### 4. Computational Results

#### 4.1. Very Brief Description of Mathematica Program.

Theorem 24 gives a method for determining combinatorial $\mathcal{P}$-formulae on $\mathcal{O}_{n,1}$. Also, Theorem 17 gives a method for finding some formulae on $\mathcal{O}_{n,n}$, but perhaps not a generating set. For the intermediate groups on the Wilson line, $\mathcal{O}_{n,k}$, $1 < k \leq n$, a Mathematica program was written to find formulae and dimensions. In this section, we give a very brief discussion of how the program works.

Fix a pair $n$ and $k$, $1 \leq k \leq n$. First one produces all signed arrow diagrams having between $k$ and $n$ arrows. For each diagram, one adds all of the ways in which you might add a 0 or a 1 to each arrow so that there are less than $k$ arrows signed 0. The corresponding embellished Gauss code is a string in Mathematica. These will play the role of the variables in a system of equations.

To reduce the number of variables, we use the simplification of [GPV]. We describe this simplification for our case. For diagrams having $n$ arrows, one only needs those which are signed $\oplus$ at every arrow. This follows from the fact that for the Q2 relations, a combinatorial formula $F$ must satisfy $\langle F, \mathcal{D} + \mathcal{D}' \rangle = 0$ if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by changing the sign of one arrow. Also, there is little point in producing variables whose Gauss diagrams have an isolated arrow labelled 0. The list of variables is ordered in some way to make an ordered basis of the free abelian group.

Secondly, one must produce the relations. This is done by first producing strings of Gauss codes corresponding to the drawn intervals in the relations $\Delta Q$. These strings are inserted into base diagrams corresponding to the intervals outside the drawn intervals. Each relation specifies an upper bound on the number of arrows in the base diagrams. Producing the base diagrams is therefore essentially the same as producing the variables of smaller degree (without the stated simplifications). Finally, one must take care to account for the signs of arrows in relations as in the previous paragraph. Note that some relations will be written more than once and that some relations may have Q1 relations in them.
Next, the relations are written as vectors corresponding to the ordered basis. At this time, all Q1 relations and duplicate relations are removed. The vectors correspond to rows of a matrix. Indeed, the matrix can be recognized in the short exact sequence:

\[ 0 \rightarrow \langle A^k, E_k, A_n, \Delta Q \rangle \rightarrow \mathbb{Z}[A^{(1,0)}] \rightarrow \mathcal{O}_{n,k} \rightarrow 0 \]

The above sequence is, modulo only the simplifications described above, the transpose of the inclusion of the kernel of the projection $\mathbb{Z}[A^{(1,0)}] \rightarrow \mathcal{O}_{n,k}$. The dual sequence is:

\[ 0 \rightarrow \text{Hom}_\mathbb{Z}(\mathcal{O}_{n,k}, \mathbb{Q}) \rightarrow \text{Hom}_\mathbb{Z}(\mathbb{Z}[A^{(1,0)}], \mathbb{Q}) \rightarrow \text{Hom}_\mathbb{Z}(\langle A^k, E_k, A_n, \Delta Q \rangle, \mathbb{Q}) \rightarrow 0 \]

Therefore, the kernel of the matrix of relations is identically the image of $\text{Hom}_\mathbb{Z}(\mathcal{O}_{n,k}, \mathbb{Q})$. *Mathematica* is used to find a basis for this kernel by using only integer arithmetic. The results are all combinatorial $\mathcal{P}$-formulae.

### 4.2. Tables and Formulae for Intermediate Groups on the Wilson Line

The table below gives the dimensions of the groups on the Wilson line as computed by the program for $n \leq 3$, $1 \leq k \leq n$. Note that the first column agrees identically with the theoretical result of Theorem 24 (1). As the theoretical result was obtained by a different technique, this supports our claim to the accuracy of the program.

| $\dim_{\mathbb{Z}}(\text{Hom}_\mathbb{Z}(\mathcal{O}_{n,k}, \mathbb{Q}))$ | $k = 1$ | $k = 2$ | $k = 3$ |
|-----------------|----------|----------|----------|
| $n = 1$         | 2        | *        | *        |
| $n = 2$         | 5        | 11       | *        |
| $n = 3$         | 9        | 46       | 66       |

Below is a generating set for the combinatorial $\mathcal{P}$-formulae on $O_2$ as found by the *Mathematica* program.

```
(1)
-1 ⊕ 0 1 -1 ⊕ 0 1 -1 ⊕ 1 0 +1 ⊕ 1 1

(2)
1 ⊕ 0 1 ⊕ 0 1 ⊕ 0 1

(3)
1 ⊕ 0 1 ⊕ 0 1 ⊕ 0 1

(4)
1 ⊕ 0 1 ⊕ 0 1 ⊕ 0 1
```
COMBINATORIAL FORMULAE FOR FINITE-TYPE INVARIANTS VIA PARITIES

(5) \[
\begin{align*}
1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 1 \\
+1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 1 \\
+1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 1 \\
+1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 1 \\
\end{align*}
\]

(6) \[
\begin{align*}
1 & \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 0 \\
+1 & \oplus 1 \oplus 1 \\
\end{align*}
\]

(7) \[
\begin{align*}
-1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 - 1 \oplus 1 \oplus 1 \\
+2 & \oplus 1 \oplus 0 - 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 0 \\
\end{align*}
\]

(8) \[
\begin{align*}
-1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 + 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 1 \\
-1 & \oplus 1 \oplus 1 - 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 0 \\
-1 & \oplus 1 \oplus 1 - 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 0 \\
\end{align*}
\]

(9) \[
\begin{align*}
-2 & \oplus 0 \oplus 1 - 2 \oplus 1 \oplus 1 - 2 \oplus 1 \oplus 0 - 1 \oplus 1 \oplus 0 \\
+1 & \oplus 1 \oplus 1 \\
\end{align*}
\]

(10) \[
\begin{align*}
1 & \oplus 1 \oplus 0 - 1 \oplus 0 \oplus 1 - 1 \oplus 1 \oplus 1 + 1 \oplus 1 \oplus 1 \\
-1 & \oplus 1 \oplus 0 + 1 \oplus 0 \oplus 1 \\
\end{align*}
\]
Micah Chrsiman: Monmouth University, West Long Branch, NJ, USA
e-mail: mchrisma at monmouth dot edu

Vassily Olegovich Manturov: People's Friendship University of Russia, Faculty of Sciences, 3 Ordjonikidze St., Moscow, 117923 and Moscow State University, Mechanical and Mathematical Dept., GSP-1, Leninskiye Gory, 1, MSU (main building) Moscow, 119991

e-mail vomanturov at yandex dot ru

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