Signatures of Majorana fermions in hybrid normal–superconducting rings

Ph. Jacquod\(^1\) and M. Büttiker\(^3\)

\(^1\)Physics Department, University of Arizona, Tucson, AZ 85721, USA
\(^2\)College of Optical Sciences, University of Arizona, Tucson, AZ 85721
\(^3\)Theoretical Physics Department, University of Geneva, 1211 Geneva, Switzerland

(Dated: 26th June 2013)

We investigate persistent currents in metallic rings interrupted by a Coulomb blockaded topological superconducting segment. We show that the presence of Majorana bound states in the superconductor is reflected in the emergence of an \(h/e\) harmonics in the persistent current, whose sign is determined by the fermion parity of the superconductor. The Majorana bound states further render the current finite at zero flux, nevertheless the resulting peculiar symmetry of the persistent current is compatible with a free energy that is even in time-reversal symmetry breaking fields. These unique features of the persistent currents are robust against disorder and provide unambiguous signatures of the presence of Majorana fermions.

PACS numbers: 74.78.Na, 73.23.Ra, 74.45.+c, 03.65.Vf

Several recent experiments have reported features in transport\(^1–4\) and Josephson spectroscopy\(^5\) that have varying degrees of consistency with the presence of Majorana states\(^6\). The experimental setups are all based on the nanowire implementation of Kitaev’s chain\(^7\) proposed in Refs.\(^8\)–\(^10\). The presence of Majorana fermions in such systems manifests itself in a zero-bias peak in the tunneling conductance into the nanowire\(^10\)–\(^11\) and by a doubling of the periodicity of the Josephson current in the superconducting phase difference\(^11\)–\(^12\). These features were observed in Refs.\(^1\)–\(^5\). Other signatures of Majorana states in transport interferometry have also been theoretically investigated\(^11\)–\(^14\)–\(^16\).

Motivated by these experimental reports, a number of theoretical works have pointed out that zero-bias peaks in the tunneling conductance may also occur in the topologically trivial phase\(^17\)–\(^19\). Their observation is therefore not sufficient to demonstrate the presence of a Majorana state. Moreover, Ref.\(^20\) showed that a doubling of the periodicity of the Josephson current as reported in the AC Josephson setup of Ref.\(^5\) may also occur due to Landau-Zener transitions between standard Andreev bound states. Despite a slowly growing body of evidence in favor of the presence of Majorana states in nanowires, a smoking gun experiment is still missing. The consensus at this point is that only a zero-bias tunneling conductance peak quantized at \(2e^2/h\) would unambiguously reflect the presence of a Majorana state. The observation of such a quantized peak would however require ideal circumstances, in particular ultra-low temperatures beyond the reach of currently existing devices\(^21\).

In this manuscript, we propose an altogether new experiment to detect Majorana fermions in the nanowire platforms of Refs.\(^8\)–\(^10\). The system we investigate is sketched in Fig. 1. It consists of a normal metallic ring interrupted by a superconducting segment of length \(L \gg \xi\), much larger than the superconducting coherence length \(\xi\). The superconducting segment can be either

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{(Color online) Setup to detect signatures of Majorana fermions in persistent currents through metallic rings. A spin-orbit coupled metallic nanowire (orange) with induced superconductivity is embedded in a metallic ring pierced by a magnetic flux \(\phi\). A sufficiently strong Zeeman field \(B_2\) applied parallel to the axis of the nanowire creates two Majorana states, \(\xi_{1,2}\), localized at each end of the nanowire. In the presence of such states, individual electrons can be coherently transferred across the nanowire, even when the latter is longer than the superconducting coherence length, thereby generating a \(h/e\) harmonics in the persistent current. The nanowire is Coulomb blockaded and tunnel-coupled to the metallic ring (dark grey rectangles represent tunnel barriers). Its occupation number \(n_0\) can be externally tuned by a gate voltage \(V_g\), which fixes the fermion parity and allows to change the sign of the persistent current \(I(\phi) \propto (-1)^{n_0}\).}
\end{figure}

in a topologically trivial state, with a superconducting gap which allows the transfer of Cooper pairs only, or in a topologically nontrivial state, with Majorana subgap states at each of its ends allowing the coherent transfer of single quasiparticles\(^22\)–\(^24\). A persistent current is triggered by a magnetic flux \(\phi\) piercing the ring. In the former case, the current has even harmonics only \(\sim \sin(4\pi n\phi/\phi_0)\), with the flux quantum \(\phi_0 = h/e\) and
\[ n = 1, 2, \ldots \] In other words, only \( h/2e \) harmonics exist, because only Cooper pairs with charge \( 2e \) can be transferred through the superconductor. In the latter case, the transfer of a single electron generates odd harmonics \( \sim \sin(2\pi(2n + 1)\phi/\phi_0 - \chi) \). The presence of Majorana states breaks time-reversal symmetry, because the quasiparticle transfer amplitude \( \tau \) through the topological superconductor picks a relative minus sign when the direction of transfer is inverted. This follows directly from the fermionic anticommutation relations of Majorana creation and annihilation operators \( \gamma^+ \). Additionally, \( \tau \sim (-1)^n \) depends on the number \( n_0 \) of fermions on the superconductor. This leads to \( \chi = (-1)^n \phi/2 \) and, when the fermion parity is fixed, persistent current harmonics \( \sim (-1)^n \cos(2\pi(2n + 1)\phi/\phi_0) \) are obtained in the weak coupling limit, while the odd harmonics vanish identically when the fermion parity is not fixed and an average over \( n_0 \) is taken. Building up on Ref. [22, 23], we propose to fix the fermion parity via Coulomb blockade of the superconducting segment. The on-resonance persistent current then bears three unambiguous signatures of the presence of Majorana states: (i) the current develops a \( h/e \) harmonic, (ii) the current is finite at zero flux in the non-trivial phase but vanishes in the trivial phase, and (iii) the current changes sign when the fermion number parity on the superconductor is changed from \( n_0 = 2n \) (unoccupied Majorana states) to \( n_0 = 2n + 1 \) (occupied Majorana states). We finally show that despite the finite zero-field current, the system’s free energy is even in time-reversal symmetry breaking fields, \( \mathcal{F}(\phi, B_z) = \mathcal{F}(\phi, -B_z) \).

The origin of the \( h/e \) harmonics is the same as that of the fractional Josephson effect, however the effect is more robust for persistent currents than for the AC Josephson effect [3], the former in particular are immune to Landau-Zener transitions [20]. Measuring persistent currents in metallic rings is challenging but has been done by several groups [24, 25]. In persistent currents the emergence of a \( h/e \) harmonic may also occur when \( \xi \) increases and becomes comparable to \( L \) [25], which however can be monitored experimentally. Anomalous supercurrents (a non-zero persistent current at zero phase difference) in Josephson junctions formed of a mesoscopic conductor sandwiched between two ordinary superconductors are discussed in several theoretical works (See e.g. Refs. [30–32]). The anomalous supercurrent is a consequence of the spin-orbit interaction and Zeeman fields in the normal conductor. In contrast, in our work spin-orbit interaction and Zeeman fields exist only in the superconductor to the extent that they are needed to drive the latter into the topological phase.

We calculate the canonical persistent current using the effective Hamiltonian derived by Fu [22] for fixed-parity, Coulomb-blockaded topological superconductors,

\[
H = H_{\text{ring}} + \delta(f^\dagger f - 1/2) + \lambda_1 c_1^\dagger c_1 + \hbar c_c + h.c.
\]

\[
+ [\pi \lambda_2 (-1)^{n} f^\dagger f + h.c.]
\]

\[
\text{(1)}
\]

The superconductor is connected to a metallic ring with Hamiltonian \( H_{\text{ring}} = \sum_k \epsilon_k c_k^\dagger c_k \), pierced by a magnetic flux \( \phi = h/2e \), the fermionic operators \( c \) annihilate an electron in the ring, \( f \) is a fermion operator on the superconductor, combining Majorana operators and fermion parity isospin operators. The superconductor is Coulomb blocked and coupled to the ring via tunnel barrier, \( \lambda_{1,2} \ll t \), which allows to project the Hamiltonian into the subspace with only two superconductor charge states \( |n_0 \rangle \) and \( |n_0 + 1 \rangle \). The energy difference between these two states is \( \delta \), which can be tuned by an external gate voltage. Details of the construction of \( H \) are given in Ref. [22]. The above Hamiltonian is appropriate to calculate the \( h/e \) harmonics of the persistent current close to zero chemical potential in the ring.

In the limit when the superconductor-ring coupling is weak, and close to resonance between \( |n_0 \rangle \) and \( |n_0 + 1 \rangle \) (i.e. close to \( \delta = 0 \)) with \( M + n_0 \) electrons in total, \( H \) can be reduced to a \( 2 \times 2 \) Hamiltonian (see Ref. [34] for a similar treatment of a metallic quantum dot embedded in a metallic ring)

\[
H_{\text{red}} = \begin{pmatrix}
\epsilon_M & \lambda_1 + i\lambda_2 (-1)^{n_0} e^{-i\varphi} \\
\lambda_1 - i\lambda_2 (-1)^{n_0} e^{i\varphi} & \delta
\end{pmatrix}
\]

\[
\text{(2)}
\]

The total energy is given by the eigenvalues of \( H_{\text{red}} \) plus a constant sum over all energy levels in the ring \( \sum_{k=1}^{M-1} \epsilon_k \). The eigenvalues \( E_{\pm}(\varphi) \) of \( H_{\text{red}} \) are easily calculated, and one obtains the canonical persistent current \( I(\varphi) = -\partial_{\varphi} E_- \) as

\[
I(\varphi) = \frac{e}{\hbar} \frac{(-1)^n \lambda_1 \lambda_2 \cos \varphi}{\sqrt{(\epsilon_M - \delta)^2 + \lambda_1^2 + \lambda_2^2 + 2\lambda_1 \lambda_2 (-1)^n \sin \varphi}}
\]

\[
\text{(3)}
\]

The \( h/e \) harmonics of the current depends on the fermion parity, it is finite at zero flux and its magnitude is algebraically reduced away from resonance (\( \delta \neq \epsilon_M \)). The weak coupling condition means that \( \lambda \ll \delta \), with the energy level spacing \( \delta \) in the ring, so that unless \( \epsilon_M \) is anomalously close to zero, the \( \lambda \)-terms under the square root in the denominator are negligible at the degeneracy point between the two superconducting states, \( \delta = 0 \). Generically, min(\( |\epsilon_M| \)) \( \approx \delta \). Further specifying to a weakly disordered, \( N \)-site, one-dimensional tight-binding ring with nearest neighbor hopping, we obtain \( \lambda_1 = (\pi/2N)^{1/2} \lambda \), \( \delta \approx \pi t/N \) close to \( \epsilon_M = 0 \), so that \( I(\varphi) = (e/h)(-1)^n \lambda_1 \lambda_2 \cos \varphi/t \). In the weak coupling limit, the sign of the current is thus determined by the fermion parity, which can be tuned via the gate voltage. The change in sign is a signature of the Friedel sum rule [32]. In this weak coupling limit, the current does not depend on the length of the ring. Alternatively, one may tune the gate voltage and work at \( \epsilon_M = \delta \), in which case the current becomes (for \( \lambda_{1,2} = \lambda \))

\[
I(\varphi) = (e/h)(-1)^n \pi^{1/2} \lambda \cos \varphi/\sqrt{4N[1 + (-1)^n \sin \varphi]}
\]

In this case, the current decays as \( N^{-1/2} \) with the size
of the ring. One important consequence of Eq. (3) is
that upon disorder averaging, the \( I(n_0) = \varphi = \pi \) is tuned slightly away from resonance, \( \delta = 0 \). We checked, but do not show, that this anomalous even-odd effect generically disappears when \( \delta \) is tuned slightly away from resonance, when some disorder is added to the ring or when the latter becomes quasi-one-dimensional, with few transverse channels.

Data at stronger couplings \( \lambda_1 = \lambda_2 = 0.2t \) are shown in Fig. 4. The persistent currents acquire a more complicated harmonics content, which is qualitatively captured by Eq. (3). Some deviations are seen, which we attribute to the fact that more than a single level is coupled to the superconductor in this regime, while Eq. (3) assumes a single-level hybridization. For the same reason, the best fits with Eq. (3) shown have parameters that differ from the theoretical predictions. It is still remarkable that the single-level prediction of Eq. (3) captures the current symmetry \( I(n_0, \varphi) = -I(n_0 + 1, -\varphi) \), even when more than one ring level is coupled to the Majorana states.

Our numerics confirm our theoretical prediction that in clean, one-dimensional metallic rings interrupted by a Coulomb-blockaded superconducting segment carrying Majorana bound states, persistent currents have \( \hbar/e \) periodicity and finite magnitude even when no flux pierces the ring, and furthermore change parity with the fermion number parity on the superconductor. These predictions still hold, even when the metallic ring is disordered and
The persistent current is given by \[ I(\phi) = -\partial_y \mathcal{F} \] with the free energy \( \mathcal{F} \). In the absence of magnetization, the latter must be an even function of magnetic field, therefore one would expect \( I(\phi = 0) = 0 \). The finite zero-flux current predicted above comes about because creating a Majorana state requires to break time-reversal symmetry with a Zeeman magnetic field in the first place, and the free energy is even in the total field, \( \mathcal{F}(\phi, B_Z) = \mathcal{F}(-\phi, -B_Z) \). To show that this is the case, we specify to the standard nanowire Hamiltonian for Majorana bound states \[ \mathcal{H} = \left[ \frac{p^2}{2m} - \mu \right] \tau_z + u p_y \sigma_z \tau_z + B_Z \sigma_x + \Delta \tau_x . \] (4)

The wire is aligned in the \( y \)-direction, \( \sigma_\alpha \) and \( \tau_\alpha \) are Pauli matrices in spin and particle-hole space, respectively. Inverting \( B_Z \) is equivalent to space inversion \( \phi \rightarrow -\phi \), \( p_y \rightarrow -p_y \), \( \sigma_y \rightarrow -\sigma_y \), \( \sigma_{x,z} \rightarrow -\sigma_{x,z} \) and \( \tau_\alpha \rightarrow -\tau_\alpha \). Therefore, \( B_Z \rightarrow -B_Z \) is equivalent to interchanging the Majorana operators. This is equivalent to the substitution \( \lambda_{1}\phi \rightarrow \lambda_{2}\phi \) in our tight-binding formulation of Eq. \[ \mathcal{H} \], which can be absorbed by a relabelling the ring operators \( c_i \rightarrow c_{N-i+1} \) together with \( \phi \rightarrow -\phi \) (because the relabelling inverts the direction of counting sites along the ring). Thus the free energy is even, \( F(\phi, B_Z) = F(-\phi, -B_Z) \) and the persistent current is odd in the total magnetic field, as should be.

It is interesting to note that these symmetry considerations translate into an apparent violation of the Onsager reciprocity relation \( G(\phi) \neq G(-\phi) \) for the conductance of the system when it is connected to two external leads. Such an apparent violation has been reported in Ref. \[ 15 \], but its origin was not discussed. The above argument for the symmetry of the free energy can be applied to the transport problem, giving the true Onsager symmetry \( G(\phi, B_Z) = G(-\phi, -B_Z) \), containing both the Aharonov-Bohm flux and the time-reversal symmetry breaking field generating the Majorana states in the first place. Such reciprocity relations cannot be violated in two-terminal systems in the linear response regime. Simultaneously, the oscillating part of \( G(\phi) \) changes sign each time an electron is added on the superconductor, which agrees with the Friedel sum rule \[ 33 \], according to which the scattering phase jumps by \( \pi \) each time the energy of the scattering particle crosses a quasi-bound state of the scatterer. Topological superconductors therefore present a unique opportunity to verify this sum rule in the presence of superconductivity. In the topologically trivial regime, only Cooper pairs can be added, which result in unnoticeable scattering phase jumps of \( 2\pi \).

We thank C. Beenakker, B. Sothmann and M. Wimmer for interesting discussions. PJ thanks the Lorentz Institute, Leiden University and the Theoretical Physics Department, University of Geneva for their hospitality at the early stage of this project. Research in Geneva was supported by the Swiss NSF, the NCCR MaNEP and QSIT.

---

[1] V. Mourik, K. Zuo, S.M. Frolov, S.R. Plissard, E.P.A.M. Bakkers, and L.P. Kouwenhoven, Science 336, 1003 (2012).
[2] M.T. Deng, C.L. Yu, G.Y. Huang, M. Larson, P. Caroff, and H.Q. Xu, Nano Lett. 12, 6414 (2012).
[3] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nature Phys. 8, 887 (2012).
[4] A.D.K. Finck, D.J. Van Harlingen, P.K. Mohseni, K. Jung, and X. Li, Phys. Rev. Lett. 110, 126406 (2013).
[5] L.P. Roikinsson, X. Lui, and J.K. Furdyna, Nature Phys. 8, 795 (2012).
[6] For reviews on Majorana fermions in a condensed matter context, see: J. Alicea, Rep. Prog. Phys. 75, 076501 (2012); C.W.J. Beenakker, Annu. Rev. Con. Mat. Phys. 4, 113 (2013).
[7] A.Y. Kitaev, Phys. Usp. 44, 131 (2001).
[8] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[9] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
[10] K.T. Law, P.A. Lee, and T.K. Ng, Phys. Rev. Lett. 103, 237001 (2009).
[11] A. R. Akhmerov, J. P. Dahlhaus, F. Hassler, M. Wimmer, and C. W. J. Beenakker Phys. Rev. Lett. 106, 057001 (2011).
[12] L. Fu and C.L. Kane, Phys. Rev. B 79, 161408 (2009).
[13] F. Pietka, A. Romito, M. Dackheim, Y. Oreg, and F. von Oppen, New J. Phys. 15, 025001 (2013).
[14] L. Fu and C.L. Kane, Phys. Rev. lett. 102, 216403 (2009).
[15] C. Benjamin and J.K. Pachos, Phys. Rev. B 81, 085101 (2010).
[16] J. Li, G. Fleury, and M. B"uttiker, Phys. Rev. B 85, 125440 (2012).
[17] D. Bagrets and A. Altland, Phys. Rev. Lett. 109, 227005 (2012).
[18] J. Liu, A.C. Potter, K.T. Law, and P.A. Lee, Phys. Rev. Lett. 109, 267002 (2012).
[19] D.I. Pikulin, J.P. Dalhaus, M. Wimmer, H. Schomerus, and C.W.J. Beenakker, New J. Phys. 14, 125011 (2012).
[20] J.D. Sau, E. Berg, and B.I. Halperin, arXiv:1206.4596.
[21] M. Franz, Nature Nanotechnology 8, 149 (2013).
[22] J. Nilsson, A.R. Akhmerov, and C.W.J. Beenakker, Phys. Rev. Lett. 101, 120403 (2008).
[23] L. Fu, Phys. Rev. Lett. 104, 056402 (2010).
[24] L.P. Lévy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. 64, 2074 (1990).
[25] V. Chandrasekhar, R.A. Webb, M.J. Brady, M.B. Ketchen, W.J. Gallagher, and A. Kleinsasser, Phys. Rev. Lett. 67, 3578 (1991).
[26] D. Mailly, C. Chapelier, A. Benoit, Phys. Rev. Lett. 70, 2020 (1993).
[27] H. Bluhm, N.C. Koschnick, J.A. Bert, M.E. Huber, and K.A. Moler, Phys. Rev. Lett. 102, 136802 (2009).
[28] A.C. Bleszynski-Jayich, W.E. Shanks, B. Peaudecerf, E. Ginossar, F. von Oppen, L. Glazman, and J.G.E. Harris, Science 326, 272 (2009).
[29] M. Büttiker and T. M. Klapwijk, Phys. Rev. B 33, 5114 (1986).
[30] A.A. Reynoso, G. Usaj, C.A. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. Lett. 101, 107001 (2008).
[31] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, Phys. Rev. Lett. 103, 147004 (2009).
[32] A. Brunetti, A. Zazunov, A. Kundu, and R. Egger, arXiv:1305.3816.
[33] J. Friedel, Philos. Mag. 43, 153 (1952).
[34] M. Büttiker and C.A. Stafford, Phys. Rev. Lett. 76, 495 (1996).
[35] V. Shivamoggi, G. Refael, and J.E. Moore, Phys. Rev. B 82, 041405 (2010).
[36] K. Flensberg, Phys. Rev. B 82, 180516 (2010).