Convolution estimates and the number of disjoint partitions

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Abstract

Let $X$ be a finite collection of sets. We count the number of ways a disjoint union of $n - 1$ subsets in $X$ is a set in $X$, and estimate this number from above by $|X|^{c(n)}$ where

$$c(n) = \left(1 - \frac{(n - 1)\ln(n - 1)}{n \ln n}\right)^{-1}.$$

This extends the recent result of Kane–Tao, corresponding to the case $n = 3$ where $c(3) \approx 1.725$, to an arbitrary finite number of disjoint $n - 1$ partitions.

1 Introduction

Let $\{0, 1\}^m$ be the Hamming cube of dimension $m \geq 1$. Set $1^m := (1, 1, \ldots, 1)$ to be the corner of $\{0, 1\}^m$. Take a finite number of functions $f_1, \ldots, f_n : \{0, 1\}^m \rightarrow \mathbb{R}$, and define the convolution at the corner $1^m$ as

$$f_1 * f_2 * \ldots * f_n(1^m) := \sum_{x_1, x_2, \ldots, x_n \in \{0, 1\}^m : x_1 + \ldots + x_n = 1^m} f_1(x_1) \cdots f_n(x_n).$$

Given $f : \{0, 1\}^m \rightarrow \mathbb{R}$ define its $L^p$ norm ($p \geq 1$) in a standard way

$$\|f\|_p := \left(\sum_{x \in \{0, 1\}^m} |f(x)|^p\right)^{1/p}.$$

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For $n \in \mathbb{N}$ we set
\[
p_n := \ln \frac{n^n}{(n-1)^{n-1}} \ln n.
\]

Our main result is the following theorem

**Theorem 1.** For any $n, m \geq 1$, and any $f_1, \ldots, f_n : \{0, 1\}^m \to \mathbb{R}$ we have
\[
f_1 \ast f_2 \ast \cdots \ast f_n(1^m) \leq \prod_{j=1}^{n} \|f_j\|_{p_n}.
\]

Moreover, for each fixed $n$ exponent $p_n$ is the best possible in the sense that it cannot be replaced by any larger number.

As an immediate application we obtain the following corollary (see Section 2.3 below).

**Corollary 2.** Let $X$ be a finite collection of sets. Then
\[
\left| \left\{ (A_1, \ldots, A_{n-1}, A) \in \underbrace{X \times \cdots \times X}_n : A = \bigsqcup_{j=1}^{n-1} A_j \right\} \right| \leq |X|^\frac{n}{p_n},
\]
where $\bigsqcup$ denotes the disjoint union, and $|X|$ denotes cardinality of the set.

The corollary extends a recent result of Kane–Tao [1], corresponding to the case $n = 3$ where $\frac{3}{p_3} \approx 1.725$, to an arbitrary finite number $n \geq 3$ disjoint partitions.

## 2 The proof of the theorem

Following [1] the proof goes by induction on the dimension of the cube $\{0, 1\}^m$. The case $m = 1$, which is the most difficult, is the main contribution of the current paper.

### 2.1 Basis: $m = 1$

In this case, set $f_j(0) = u_j$ and $f_j(1) = v_j$ for $j = 1, \ldots, n$. Then the inequality (1) takes the form
\[
\sum_{j=1}^{n} u_j \prod_{i=1, i \neq j}^{n} v_i \leq \prod_{j=1}^{n} (|u_j|^{p_n} + |v_j|^{p_n})^{1/p_n}.
\]

We do encourage the reader first to try to prove (3) in the case $n = 3$, or visit [1], to see what is the obstacle. For example, when $n = 3$ equality in (3) is attained at several points. Besides, direct differentiation of (3) reveals many “bad” critical points at which finding the values of (3) would require numerical computations [1]. The number of critical points together with equality cases increases as $n$ becomes larger, therefore one is forced
to come up with a different idea. We will overcome this obstacle by looking at (3) in dual coordinates.

Without loss of generality we can assume that \( u_j \) and \( v_j \) are nonnegative for \( j = 1, \ldots, n \). Moreover, we can assume that \( v_j \neq 0 \) for all \( j \) otherwise the inequality (3) is trivial.

Let us divide (3) by \( \prod_{j=1}^{n} v_j \). Denoting \( x_j := (u_j/v_j)^{p_n} \) we see that it is enough to prove the following lemma.

**Lemma 3.** For any \( n \geq 2 \) and all \( x_1, \ldots, x_n \geq 0 \) we have

\[
\left( \sum_{i=1}^{n} x_i^{1/p_n} \right)^{p_n} \leq \prod_{i=1}^{n} (1 + x_i),
\]

where \( p_n = \frac{\ln \frac{n^n}{(n-1)^{n-1}}}{\ln n} \).

**Proof.** For \( n = 2 \) the lemma is trivial. By induction on \( n \), monotonicity of the map

\[
p \rightarrow \left( \sum_{i=1}^{n} x_i^{1/p} \right)^{p},
\]

and the fact that \( p_n \) is decreasing, we can assume that all \( x_i \) are strictly positive. For convenience we set \( p := p_n \). Introducing new variables we rewrite (4) as follows

\[
p \ln \left( \sum x_i \right) \leq \sum \ln(1 + x_i^p).
\]

Concavity of the function \( \ln(x) \) provides us with a simple representation of the logarithmic function

\[
\ln(x) = \min_{b \in \mathbb{R}} (b + e^{-b} x - 1).
\]

Therefore we are left to show that for all \( x_i > 0 \) and all \( b_i \in \mathbb{R} \) we have

\[
B(x, b) := \sum (b_i + (1 + x_i^p) e^{-b_i} - 1) - p \ln \left( \sum x_i \right) \geq 0,
\]

where \( x = (x_1, \ldots, x_n) \) and \( b = (b_1, \ldots, b_n) \). Notice that given a vector \( b \in \mathbb{R}^n \), the infimum of \( B(x, b) \) in \( x \) cannot be reached at infinity because of the slow growth of the logarithmic function. Therefore, we look at critical points of \( B \) in \( x \)

\[
x_k^* = \frac{b_k}{\sum_i e^{b_i/p-1}} \quad \text{for} \quad k = 1, \ldots, n.
\]

Notice that \( \sum x_i^* = \left( \sum_i e^{b_i/p-1} \right)^{p-1} \). Therefore

\[
B(x^*, b) = \sum_k (b_k + e^{-b_k}) + 1 - (p - 1) \ln \left( \sum_k e^{b_k/p-1} \right).
\]
Setting \( r := p - 1 > 0 \), and introducing new variables again we are left to show that

\[
f(y) := 1 - n + \sum \ln y_i + \sum \frac{1}{y_i^r} - r \ln \left( \sum y_i \right) \geq 0
\]

for all \( y_i > 0 \). It is straightforward to check that \( f(y) \geq 0 \) on the diagonal, i.e., when \( y_1 = y_2 = \ldots = y_n \).

In general, we notice that critical points of \( f(y) \) satisfy the equation

\[
\frac{1}{y_i} - \frac{1}{y_i^{r+1}} = \frac{1}{y_j} - \frac{1}{y_j^{r+1}} = \frac{1}{\sum y_k}.
\]

Equation (5) gives the identity \( \sum y_i^{-r} = n - 1 \), and so at critical points (5) we are only left to show

\[
\sum \ln y_i - \ln \left( \sum y_i \right) \geq 0.
\]

Since the mapping

\[
s \to \frac{1}{s} - \frac{1}{s^{r+1}}, \quad s > 0,
\]

is increasing on \( (0, (1 + r)^{1/r}) \) and decreasing on the remaining part of the ray, we can assume without loss of generality that \( k \) numbers of \( x_i \) equal to \( u \geq (1 + r)^{1/r} \), and the remaining \( n - k \) numbers of \( x_i \) equal to \( v \leq (1 + r)^{1/r} \). Moreover, we can assume that \( 0 < k < n \) otherwise the statement is already proved. From (5), we have

\[
\frac{1}{u} - \frac{1}{u^{r+1}} = \frac{1}{v} - \frac{1}{v^{r+1}} = \frac{1}{ku + (n - k)v}.
\]

From the equality of the first and the third expressions in (7) it follows that

\[
v = \frac{u^{r+1}(1 - k) + ku}{(u^r - 1)(n - k)}.
\]

In order \( v \) to be positive we assume that the numerator of (8) is non negative. If we plug the expression for \( v \) from (8) into the first equality of (7) then after some simplifications we obtain the following equation in the variable \( z := u^r \geq 1 + r \)

\[
\frac{(z - 1)^r(n - k)^{r+1}}{(z(1 - k) + k)^r} = (n - 1)z - k.
\]

It follows from (7) that \( (ku + (n - k)v)^r = (\frac{z}{z-1})^r \), and so using (9) we obtain

\[
v^r = \frac{z(n - k)}{(n - 1)z - k}.
\]
Therefore at critical points \((6)\) simplifies as follows

\[
(n-1) \ln z + (n-k) \ln \frac{n-k}{(n-1)z-k} - r \ln \frac{z}{z-1} \geq 0. \tag{10}
\]

Now it is pretty straightforward to show that \((10)\) is non negative even under the assumption \(z \geq 1 + r\) for \(r = p - 1 = \frac{(n-1) \ln \frac{n}{n-1}}{\ln n}\). Indeed, notice that \(z > \frac{n}{n-1}\), and the map

\[k \to (n-k) \ln \frac{n-k}{(n-1)z-k}\]

is increasing on \([1, n-1]\). Therefore it is enough to check nonnegativity of \((10)\) when \(k = 1\), in which case the inequality follows again using \(z > \frac{n}{n-1}\), and the fact that the map

\[z \to \frac{\ln \left( 1 + \frac{1}{z(n-1)-1} \right)}{\ln \left( 1 + \frac{1}{1-1} \right)}\]

is increasing for \(z \geq \frac{n}{n-1}\). \(\square\)

**Remark** 4. Choice \(x_1 = \ldots = x_n = \frac{1}{n-1}\) gives equality in \((4)\), and this confirms the fact that \(p_n\) is the best possible in Theorem 1.

### 2.2 Inductive step

Inductive step is the same as in \([1]\) without any modifications. This is a standard argument for obtaining estimates on the Hamming cube (see for example \([2]\)). In order to make the paper self contained we decided to repeat the argument.

Suppose \((1)\) is true on the Hamming cube of dimension \(m\). Without loss of generality assume \(f_j \geq 0\), and set \(g_j := f_j^p\) for all \(j\). Define

\[B_n(y_1, \ldots, y_n) := y_1^{1/p_n} \cdots y_n^{1/p_n}.
\]

For \(x_j \in \{0, 1\}^{m+1}\), let \(x_j = (\bar{x}_j, x'_j)\) where \(\bar{x}_j\) is the vector consisting of the first \(m\) coordinates of \(x_j\), and number \(x'_j\) denotes the last \(m+1\) coordinate of \(x_j\). Set

\[\tilde{g}_j(x'_j) := \sum_{\bar{x}_j \in \{0, 1\}^m} g_j(\bar{x}_j, x'_j) \quad j = 1, \ldots, n.
\]
We have
\[
\sum_{x_j \in \{0,1\}^{m+1} : x_1 + \ldots + x_n = 1} B(g_1(x_1), \ldots, g_n(x_n)) = \sum_{x_j' \in \{0,1\} : x_1' + \ldots + x_n' = 1} \sum_{x_j \in \{0,1\}^m : x_1 + \ldots + x_n = 1} B(g_1(x_1), \ldots, g_n(x_n)) \text{ induction} \leq \sum_{x_j' \in \{0,1\} : x_1' + \ldots + x_n' = 1} B(\tilde{g}_1(x_1'), \ldots, \tilde{g}_n(x_n')) \text{ basis} \leq \sum_{x_j \in \{0,1\}^{m+1}} g_1(x_1), \ldots, \sum_{x_n \in \{0,1\}^{m+1}} g_n(x_n) \right).
\]

2.3 The proof of Corollary 2
Without loss of generality we may assume that all the sets $A$ in $X$ are subsets of $\{1, \ldots, m\}$ with some natural $m \geq 1$ (see [1]). For $j = 1, \ldots, n$ define functions
\[
f_j : \{0,1\}^m \to \{0,1\}
\]
as follows:
\[
f_1(a_1, \ldots, a_m) = \ldots = f_{n-1}(a_1, \ldots, a_m) = 1
\]
if the set $\{1 \leq i \leq m : a_i = 1\}$ lies in $X$, and $f_j = 0$ otherwise. Finally we define
\[
f_n(a_1, \ldots, a_m) = 1
\]
if the set $\{1 \leq i \leq m : a_i = 0\}$ lies in $X$, and $f_n = 0$ otherwise. Notice that in this case inequality (1) becomes (2).

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