Cardy-Verlinde formula for an axially symmetric dilaton-axion black hole

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Abstract: It is shown that the Bekenstein-Hawking entropy of an axially symmetric dilaton-axion black hole can be expressed as a Cardy-Verlinde formula. By utilizing the first order quantum correction in the Bekenstein-Hawking entropy we find the modified expressions for the Casimir energy and pure extensive energy. The first order correction to the Cardy-Verlinde formula in the context of axially symmetric dilaton-axion black hole are obtained with the use of modified Casimir and pure extensive energies.

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I. INTRODUCTION

The entropy $S_{CFT}$ of conformal field theory (CFT) in an arbitrary dimension $n$ has been related to its total energy $E$ and Casimir energy $E_C$ by a relation, named as the Cardy-Verlinde formula $S_{CFT} = \frac{2\pi R}{n} \sqrt{E_C(2E - E_C)}$ \[1\]. The entropy associated with the conformal field theory has been related to the Bekenstein-Hawking entropy for various black hole geometries with asymptotically anti-de Sitter (AdS) boundary \[2-10]\. Thus, one may naively expect that the entropy of all CFTs that have an AdS-dual description is given as the Cardy-Verlinde formula \[1\]. However, AdS black holes do not always satisfy the Cardy-Verlinde formula \[11\]. Recently, much interest has been developed in calculating the quantum corrections to the Bekenstein-Hawking entropy $S$ by using various techniques like radial null geodesics, Hamilton-Jacobi method and loop quantum gravity etc \[12-14]\. The leading-order correction is proportional to $\ln S$ which comes out to be the same with the use of above techniques. The leading order quantum correction to the classical Cardy-Verlinde formula has been studied by Carlip \[15\].

The thermodynamics of conformal field theories with gravity duals has been studied actively in literature with the remarkable resemblance of the relevant thermodynamic formulas \[1-10]\. It has been shown that the Cardy-Verlinde formula holds with a negative cosmological constant or a more general certain potential term for super-gravity scalars \[16\]. There it has been argued that the Cardy-Verlinde formula also holds for black hole geometry which are asymptotically flat instead of asymptotically AdS space. In the spirit of this Ref. \[16\], we discuss the entropy of dilaton-axion black hole which is asymptotically flat spacetime in terms of the Cardy-Verlinde formula. Here we consider the stationary axially-symmetric axion-dilaton black hole to study the Cardy-Verlinde formula and its first order correction. This black hole is a string theory inspired black hole in lower spacetime dimensions \[17, 18\]. The string theory inspired-models consist of two massless scalar fields namely dilaton and axion, in the low energy effective action in four dimension. The thermodynamics of axially-symmetric axion-dilaton black hole is investigated by various authors \[19\]. We shall demonstrate that the Cardy-Verlinde formula can be related with the Bekenstein-Hawking entropy of the stationary axially-symmetric axion-dilaton black hole. By employing the first order entropy correction to Bekenstein-Hawking entropy, we are able to find the leading order term of the Cardy-Verlinde formula.
The plan of the paper is: In the second section, we shall briefly discuss the thermodynamic quantities associated with the horizon of the stationary axially-symmetric dilaton-axion black hole. In third section, we will study the entropy of the axially-symmetric axion-dilaton black hole which can be represented by the Cardy-Verlinde formula. In the fourth section, we provide the leading order correction to the Cardy-Verlinde formula by using quantum corrected Bekenstein-Hawking entropy in the context of dilaton-axion black hole. Finally we shall conclude our results.

II. AXIALLY SYMMETRIC EINSTEIN-MAXWELL DILATON-AXION BLACK HOLE

In this section we shall consider the effective Lagrangian of the low-energy heterotic string theory in four dimensions given by [18, 20]

\[I = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R - 2g^{\mu\nu}\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2}e^{4\Phi}g^{\mu\nu}\nabla_\mu K_a \nabla_\nu K_a - e^{-2\Phi}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} - K_a F_{\mu\nu} \tilde{F}^{\mu\nu} \right), \]  

(1)

where the dual of electromagnetic field tensor \( F_{\mu\nu} \) is

\[ \tilde{F}^{\mu\nu} = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \]  

(2)

Here \( R \) is the Riemann curvature scalar, \( \epsilon_{\mu\nu\alpha\beta} \) is the Levi Civita symbol and \( g^{\mu\nu} \) is the metric tensor. Also \( \Phi \) and \( K_a \) are the massless dilaton field and the axion field respectively.

In the Boyer-Lindquist coordinates \((t, r, \theta, \varphi)\), the stationary axially-symmetric solution to the Einstein-Maxwell’s equations in the presence of the dilaton-axion is given by [18],

\[ ds^2 = \frac{-\Sigma - a^2 \sin^2 \theta}{\Delta} dt^2 - \frac{2a \sin^2 \theta}{\Delta} \left[ (r^2 - 2Dr + a^2) - \Sigma \right] dt d\varphi + \Delta dr^2 + \Delta d\theta^2 + \frac{\sin^2 \theta}{\Delta} \left[ (r^2 - 2Dr + a^2)^2 - \Sigma a^2 \sin^2 \theta \right] d\varphi^2, \]  

(3)

where

\[ \Delta = r^2 - 2Dr + a^2 \cos^2 \theta, \quad \Sigma = r^2 - 2mr + a^2, \]  

(4)

and

\[ e^{2\Phi} = \frac{W}{\Delta} = \frac{\omega}{\Delta} (r^2 + a^2 \cos^2 \theta), \quad \omega = e^{2\Phi_0}, \]  

(5)
\[
K_a = K_0 + \frac{2aD \cos \theta}{W},
\]
\[
A_t = \frac{1}{\Delta} (Qr - ga \cos \theta), \quad A_r = A_\theta = 0,
\]
\[
A_\varphi = \frac{1}{a \Delta} (-Q r a^2 \sin^2 \theta + g (r^2 + a^2) a \cos \theta).
\]
The mass \(M\), angular momentum \(J\), electric charge \(Q\), and magnetic charge \(P\), dilaton charge \(D\) of the black hole are given by

\[
M = m - D, \quad J = a(m - D), \quad Q = \sqrt{2\omega D(D - m)}, \quad P = g.
\]
The above results show that the stationary axis symmetric dilaton-axion black hole significantly differs from the the Kerr-Newmann black hole. The two horizons are the inner \(r_-\) and the outer one \(r_+\) of the black hole under consideration are

\[
r_\pm = M + D \pm \sqrt{(M + D)^2 - a^2}.
\]
Only \(r_+\) is the event horizon and one can associate thermodynamical quantities with it.

The Hawking temperature associated with the event horizon is

\[
T = \frac{\hbar}{4\pi} \left( \frac{r_+ - M - D}{r_+^2 - 2Dr_+ + a^2} \right).
\]
The angular velocity \(\Omega\) at the event horizon can be rewritten as

\[
\Omega = \frac{J/M}{r_+^2 - 2Dr_+ + a^2}.
\]
Here \(J\) is the angular momentum. The electrostatic potential can be given by

\[
\Phi = \frac{-2DM}{Q(r_+^2 - 2Dr_+ + a^2)}.
\]
The entropy associated with the event horizon of the dilaton-axion black hole is

\[
S = \frac{\pi}{\hbar} (r_+^2 - 2Dr_+ + a^2).
\]

**III. CARDY-VERLINDE FORMULA AND DILATON-AXION BLACK HOLE**

In this section, we introduce the Cardy-Verlinde formula which states that the entropy of a \((1+1)\)-dimensional CFT is given by

\[
S = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)},
\]
where \( c \) is the central charge and \( L_0 \) is the Virasoro generator. After appropriate identifications of \( c \) and \( L_0 \), the above Cardy formula, we obtain the generalized Cardy-Verlinde formula which takes the form

\[
S_{CFT} = \frac{2\pi R}{\sqrt{a_1 b_1}} \sqrt{E_C(2E - E_C)},
\]

where \( E \) is the total energy, \( E_C \) is the Casimir energy, \( a_1 \) and \( b_1 \) are arbitrary positive constants. Also \( R \) is the radius of the \( n + 1 \) dimensional spacetime, \( ds^2 = -dt^2 + R^2 d\Omega_n \).

The definition of Casimir energy is derived by the violation of the Euler relation as

\[
E_C = n(E + PV - TS - \Phi Q - J\Omega),
\]

where the pressure of the CFT is given by \( P = E/nV \). The total energy is the sum of two terms

\[
E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V).
\]

Here \( E_E \) is the purely extensive part of the total energy. The Casimir energy and the purely extensive part of the total energy are expressed as

\[
E_C = \frac{b_1}{2\pi R} S^{1-\frac{1}{n}},
\]

\[
E_E = \frac{a_1}{4\pi R} S^{1+\frac{1}{n}}.
\]

IV. ENTROPY OF AXIALLY SYMMETRIC AXION-DILATON BLACK HOLE AND CARDY-VERLINDE FORMULA

Using Eq. (12) with \( n = 2 \) and \( E = M \), we obtain

\[
E_C = 3M - 2TS - 2\Phi Q - 2\Omega J,
\]

\[
= 3M - \frac{1}{2}(r_+ - M - D) + \frac{4DM}{r_+^2 - 2Dr_+ + a^2} - \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)}.\]

From (16) we have

\[
2E - E_C = -M + 2TS + 2\Phi Q + 2\Omega J,
\]

\[
= -M + \frac{1}{2}(r_+ - M - D) - \frac{4DM}{r_+^2 - 2Dr_+ + a^2} + \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)}.\]
From (13) and (16), the extensive part of total energy becomes

\[ E_E = E - \frac{1}{2}E_C, \]
\[ = -\frac{1}{2}M + \frac{1}{4}(r_+ - M - D) - \frac{2DM}{r_+^2 - 2Dr_+ + a^2} + \frac{J^2}{M(r_+^2 - 2Dr_+ + a^2)}. \]  

(23)

Comparison of (14) and (16) yields

\[ R = \frac{b_1 S^{1/2}}{2\pi} \left[ 3M - 2TS - 2\Phi Q - 2\Omega J \right]^{-1}, \]
\[ = \frac{b_1}{2\pi} \sqrt{\frac{\pi}{2}} \frac{(r_+^2 - 2Dr_+ + a^2)}{3M - \frac{1}{2}(r_+ - M - D) + \frac{4DM}{r_+^2 - 2Dr_+ + a^2}}. \]  

(24)

Comparison of (15) and (18) yields

\[ R = \frac{a_1 S^{3/2}}{4\pi} \left[ -\frac{1}{2}M + TS + \Phi Q + \Omega J \right]^{-1}, \]
\[ = \frac{a_1}{4\pi} \left[ \frac{\pi}{2} (r_+^2 - 2Dr_+ + a^2) \right]^{3/2} \frac{3/2}{-\frac{1}{2}M + \frac{1}{4}(r_+ - M - D) - \frac{2DM}{r_+^2 - 2Dr_+ + a^2} + \frac{J^2}{M(r_+^2 - 2Dr_+ + a^2)}}. \]  

(25)

Combining the last two expressions (19) and (20), we obtain

\[ R = \frac{\sqrt{a_1 b_1 \pi}}{2\sqrt{2}} \frac{\pi}{\hbar} (r_+^2 - 2Dr_+ + a^2) \left[ 3M - \frac{1}{2}(r_+ - M - D) + \frac{4DM}{r_+^2 - 2Dr_+ + a^2} - \frac{2J^2}{M(r_+^2 - 2Dr_+ + a^2)} \right]^{-1} \]
\[ \times \left[ -\frac{1}{2}M + \frac{1}{4}(r_+ - M - D) - \frac{2DM}{r_+^2 - 2Dr_+ + a^2} + \frac{J^2}{M(r_+^2 - 2Dr_+ + a^2)} \right]^{-1}. \]  

(26)

Using (16), (17) and (21) in (11) yields

\[ S_{CFT} = \frac{\pi}{\hbar} (r_+^2 - 2Dr_+ + a^2) = S. \]  

(27)

V. LOGARITHMIC CORRECTION TO THE CARDY-VERLINDE FORMULA

In this section, we shall obtain the first order entropy correction by using corrected Bekenstein-Hawking entropy formula in the Cardy-Verlinde formula. The first order correction to the semi-classical Bekenstein-Hawking entropy \( S_0 \) is given by [21]

\[ S = S_0 - \frac{1}{2} \ln C. \]  

(28)

Here \( C \) is the heat capacity of the black hole evaluated at the event horizon. We suppose that \( C \simeq S = S_0 \) [21] so that the above equation (28) turns out

\[ S = S_0 - \frac{1}{2} \ln S_0. \]  

(29)
First we calculate the corrected Casimir energy and the corrected extensive part of the total energy by using first order corrected entropy (29) which admit

$$\tilde{E}_C = E_C + T \ln S_0,$$

(30)

$$\tilde{E}_E = E - \frac{1}{2} E_C - \frac{1}{2} T \ln S_0.$$  

(31)

By using modified Casimir energy (30) and the extensive part of the total energy (31) in the Cardy-Verlinde formula (16), we obtain the modified Cardy-Verlinde entropy relation

$$\tilde{S}_0 = \frac{2 \pi R}{\sqrt{a_1 b_1}} \sqrt{\tilde{E}_C (2E - \tilde{E}_C)}.$$  

(32)

Simplifying (32) we obtain

$$\tilde{S}_0 \simeq S_0 \left[ 1 + \frac{(E - E_C)}{E_C (2E - E_C)} T \ln S_0 \right].$$  

(33)

Finally using (33) in (29) yields the corrected entropy as

$$S \simeq \frac{2 \pi R}{\sqrt{a_1 b_1}} \sqrt{E_C (2E - E_C)} + \left[ \frac{2 \pi R}{\sqrt{a_1 b_1}} \frac{(E - E_C)}{E_C (2E - E_C)} - \frac{1}{2} \right] T \ln \left[ \frac{2 \pi R}{\sqrt{a_1 b_1}} \sqrt{E_C (2E - E_C)} \right].$$  

(34)

Hence the entropy correction to the semi-classical Bekenstein-Hawking entropy is obtained in terms of the modified Cardy-Verlinde formula which further investigates the AdS/CFT correspondence in terms of modified Cardy-Verlinde entropy formula. The first term corresponds to the usual CV formula while the second term relates to correction to Hawking entropy in terms of modified Cardy-Verlinde entropy formula.

VI. CONCLUSION

In this paper, we have shown that the Bekenstein-Hawking entropy of the axially-symmetric axion-dilaton black hole can also be expressed in the form of Cardy-Verlinde entropy formula which further investigates the AdS/CFT correspondence in terms of Cardy-Verlinde entropy formula. The axially symmetric dilaton axion black hole is asymptotically flat instead of AdS space. So our study indicates that the AdS/CFT correspondence still holds in the black hole geometries with asymptotically flat background. By using the logarithmic correction to the Bekenstein-Hawking entropy, we obtained the modified expressions for the Casimir and extensive energy relations. By utilizing modified expressions for Casimir
and extensive energy in the Cardy-Verlinde formula, we obtained the corrected $S_{CFT}$ relation which relates the entropy of a certain CFT to its total energy and Casimir energy. The second result of this paper is the entropy correction to the semi-classical Bekenstein-Hawking entropy in terms of the modified Cardy-Verlinde formula. The first term in (34) corresponds to the usual Cardy-Verlinde formula while the second term relates correction to Hawking entropy in terms of modified Cardy-Verlinde entropy formula.

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