Competing phases in spin ladders with ring exchange and frustration

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The ground state properties of spin-1/2 ladders are studied, emphasizing the role of frustration and ring exchange coupling. We present a unified field theory for ladders with general coupling constants and geometry. Rich phase diagrams can be deduced by using a renormalization group calculation for ladders with in-chain next nearest neighbor interactions and plaquette ring exchange coupling. In addition to established phases such as Haldane, rung singlet, and dimerized phases, we also observe a surprising instability towards an incommensurate phase for weak interchain couplings, which is characterized by an exotic coexistence of self-consistent ferromagnetic and anti-ferromagnetic order parameters.

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Spin ladders are heavily studied prototypical models exemplifying the role of quantum fluctuations in low dimensional quantum systems $[1,23]$. Already in their simplest form they show some of the most discussed quantum many-body properties, as for example a spin liquid or a Haldane gapped state with topological string order parameter. Theoretical studies have identified a number of remarkable ground state phases, but given the large variety of possible tuning parameters it is likely that this list is far from complete. At the same time experimental progress on novel materials $[26-28]$ as well as advancements in the field of optical lattices $[29-32]$ give renewed interest in spin ladders in their own right beyond the larger effort to gain a better insight into complex phases of frustrated two-dimensional (2D) models. This work now focuses on SU(2) invariant ladders in order to answer the important question which phases are accessible for a general choice of tunable couplings within the framework of an effective field theory. In particular, an incommensurate phase is postulated for a certain class of ladder systems in the weak coupling limit, which has so far received little attention.

The generalized SU(2) invariant spin-1/2 ladder is described by the Heisenberg Hamiltonian plus a ring exchange interaction

$$H = \sum_{<i,j>} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + K \sum_p (P_p + P_p^{-1}) ,$$  \hspace{1cm} (1)

where $J_{ij}$ take on values of antiferromagnetic nearest-neighbor coupling $J > 0$ and next-nearest neighbor (NNN) coupling $J_2$ in the two chains as well as two diagonal couplings $J_d$, $J_d'$ and a rung coupling $J_\perp$ between the chains as depicted in Fig. 1. The second term in Eq. (1) sums over the plaquettes of the system, where $P_p$ stands for the cyclic permutation of the spins on the four sites of the $p$-th plaquette. Such a ring exchange interaction arises from the higher order expansion of the Hubbard model in the strongly interacting limit $[33,34]$ and can also be written as a product of two-spin permutations $P_p = P_{p_1,p_2}P_{p_3,p_4}$, which in terms of Pauli matrices $\sigma$ reads, $P_{ij} = (1 + \sigma_i \cdot \sigma_j)/2$ $[35]$. The tunable parameters in Fig. (1) provide a generalization of previously studied ladder models. It has been established that weak interchain couplings give rise to four possible gapped quantum phases, which are characterized by correlations in form of singlets $[4,12]$: 

- **Rung singlet phase**: Singlet formation across the rungs for an antiferromagnetic rung coupling $J_\perp > 0$.
- **Haldane phase**: Singlet formation analogous to the so-called AKLT state $[36]$ in the spin-1 model with a topological string order parameter for $J_\perp < 0$.
- **Columnar dimerized phase**: Alternating singlet formation within each chain on the same bonds in both legs for $K < 0$.
- **Staggered dimerized phase**: Alternating singlet formation within each chain with a shift of one site between the two legs for $K > 0$.

The rung singlet and Haldane phases have short range correlations, while the dimer-dimer correlations in the...
dimerized phases are long range and break translational invariance. Those four phases with singlet formation are also dominant if one considers frustrating couplings \( J_d, J'_d, J_2 \), which normally enhance quantum fluctuations. Such frustrating diagonal coupling have been extensively studied in the form of a cross coupled ladder (CCL) for \( J_d = J'_d \) \(^{12, 18} \) or a diagonal ladder (DL) for \( J_d = 0 \) \(^{18-20} \). Both the DL and the CCL can be tuned from a rung singlet phase to a Haldane phase with increasing \( J_d/J_1 \). Interestingly, the columnar dimerized phase also appears for the DL for intermediate \( J_d/J_1 \) \(^{18} \), but such a phase is debated for the CCL since it would spontaneously break translational invariance \(^{14, 18} \). A frustrating in-chain next-nearest neighbor interaction is known to cause dimerization in single chains for \( J_2 > J_{2c} \approx 0.241167J \) \(^{37, 39} \) and therefore also stabilizes the corresponding dimer phases in ladder systems with next-nearest neighbor coupling \(^{21, 22} \).

We now want to answer the question if the enhanced entropy from competing phases may open the possibility for interesting additional phases. We therefore consider the interplay of all couplings in the full model in Fig. 1 using the framework of a renormalization group (RG) treatment. As we will see below there are two relevant operators which are responsible for the known four phases above, but for \( J_d \neq J'_d \) the enlarged unit cell allows an additional relevant operator which may dominate in parts of the phase diagram and gives rise to an interesting incommensurate phase.

Starting from the established effective field theory description of two decoupled chains, the spin operators acquire the following representation in the continuum limit \(^{40-44} \)

\[
S(x) \approx J(x) + (-1)^x \Omega \cdot n(x) \quad ,
\]

where \( \Omega \) is a non–universal constant \(^{44} \), and the lattice constant is set to unity. The uniform part of the spin operator is the sum of the chiral SU(2) currents of the Wess-Zumino-Witten (WZW) model, \( J = J_L + J_R \); the staggered part, is related to the fundamental field \( g \) of the WZW model via \( n \sim \text{tr} \sigma g \), while the trace of \( g \) gives the dimerization operator, \( \epsilon \sim \text{tr} g \), with \( \epsilon_j = (-1)^j S_j \cdot S_{j+1} \). Without interchain couplings the field theory can be written for each leg separately in the form

\[
H_0 = \frac{2\pi v}{3} \int dx \left[ : J_L \cdot J_L: + : J_R \cdot J_R: + \frac{3\lambda_n}{2} J_L \cdot J_R \right] \quad ,
\]

where the first two normal ordered terms are conformally invariant and the last part represents a backscattering marginal operator. Without any additional couplings between the two legs of the ladder this theory describes a spin liquid for each chain which is generally unstable to perturbations, however. The velocity \( v \approx \frac{\pi J_d}{J_2} - 1.65J_2 \) and the value of the marginal coupling \( \lambda_n \) can be tuned by the in-chain nearest neighbor coupling \( J_2 \) \(^{39, 41} \).

We denote the field theories for each chain with an additional index \( \eta = 1, 2 \). The allowed symmetries of the microscopic model dictate that up to four additional relevant or marginal operators can be generated by the coupling between the legs of the ladder

\[
\delta \mathcal{H} = 2\pi v \int dx \left( \lambda_n O_n + \lambda_c O_c + \lambda_e O_e + \lambda_b O_b \right) \quad .
\]

Here, the operator \( O_n = n_1 \cdot n_2 \) of scaling dimension 1 describes the coupling between the antiferromagnetic part of the spins, which drives the system into a rung singlet phase for \( \lambda_n > 0 \) and into a Haldane phase for \( \lambda_n < 0 \). An effective coupling of the dimerization is described by the relevant operator \( O_c = \epsilon_1 \epsilon_2 \) of scaling dimension 1, which may drive the system into a staggered \( \lambda_c > 0 \) or columnar \( \lambda_c < 0 \) dimerized phase. Interestingly, for \( J'_d \neq J_d \) the unit cell is enlarged which allows another relevant marginal coupling \( O_b = J_1 \cdot J_2, R + J_1 \cdot J_2, L \) of scaling dimension 2, which can tip the balance of which relevant operator dominates under renormalization, which in turn determines the phase of the ground state.

The corresponding RG equations of the bare coupling constants are determined up to second order according to the operator product expansion \(^{44, 48} \)

\[
d\lambda_k/dl = (2 - d_k) \lambda_k - \frac{\pi}{v} \sum_{i,j} C_{ijk} \lambda_i \lambda_j \quad ,
\]

with \( d_k \) the scaling dimension of each operator and the coefficients \( C_{ijk} \) are obtained by the operator product expansion \(^{44, 48} \) as discussed in the Appendix. For the operator content in Eq. 4, we arrive at the following RG flow (\( \lambda = d\lambda/dl \))

\[
\begin{align*}
\dot{\lambda}_a &= \lambda_a^2 + \frac{1}{2} \lambda_e^2 - \frac{1}{2} \lambda_n^2 \\
\dot{\lambda}_b &= \lambda_b^2 - \lambda_e \lambda_n + \lambda_n^2 \\
\dot{\lambda}_c &= \lambda_c + \frac{3}{2} \lambda_a \lambda_c - \frac{3}{2} \lambda_b \lambda_n - \frac{3}{2} \lambda_e^2 \\
\dot{\lambda}_n &= \lambda_n - \frac{1}{2} \lambda_a \lambda_n - \frac{1}{2} \lambda_b \lambda_e + \lambda_b \lambda_n - \lambda_c^2 \\
\dot{\lambda}_e &= \frac{1}{2} \lambda_c - \frac{1}{4} \lambda_a \lambda_c + \lambda_b \lambda_c + \frac{1}{2} \lambda_c \lambda_e - \lambda_c \lambda_n 
\end{align*}
\]

The bare coupling constants can be determined from the microscopic model using Eq. 2 as also discussed in the
Appendix

\[ \lambda_n = \frac{\Omega^2 J_{\perp} - J_d}{2\pi v}, \quad \lambda_c = \frac{3\Omega^2 K}{2\pi^2 2\pi v}, \quad \lambda_c = \Omega \frac{J_d - J'_d}{2\pi v}, \]
\[ \lambda_a = 1.723(J_2 - J_{2c}) + \left(2 - \Omega^2 + \frac{3\Omega^2}{\pi^2}(1 + \Omega^2)\right)\frac{K}{2\pi v}, \]
\[ \lambda_b = \frac{J_{\perp} + J_d}{2\pi v} + 4\left(1 - \frac{(1 + \Omega^2)^2}{\pi^2}\right)\frac{K}{2\pi v}. \]

where we have used \( J_d = J_d + J'_d \). As expected \( O_c \) is forbidden by symmetry for \( J'_d = J_d \) and stays zero under renormalization in Eqs. (6) in this case. The system remains approximately scale invariant above an energy scale \( \Lambda \) as long as the coupling constants are small. However, as the cutoff \( \Lambda(l) = \Lambda_0 e^{-l} \) is lowered typically one of the coupling constants effectively becomes of order unity under renormalization, which determines the dominant ground state correlations. In turn the value of \( \Lambda(l) \) at this breakdown point defines a new intrinsic energy scale, below which scale invariance is lost and no further renormalization is possible. Good agreement with numerically determined phase transitions has been achieved when using \( \Omega = 1 \) and integrating the RG equations up to a breakdown point where one coupling reaches \( \lambda_e = 1 \), which is the procedure we use in the following.

The operators with the smaller scaling dimension, i.e., \( O_n \) and \( O_c \), renormalize faster. They will determine the low energy physics in most of the cases in form of a rung singlet (\( \lambda_e > 0 \)), Haldane (\( \lambda_n < 0 \)), columnar dimer (\( \lambda_e < 0 \)), or staggered dimer phase (\( \lambda_c > 0 \)) as already discussed above. However, there are regions of the parameter space where the outcome of the renormalization procedure is less trivial to predict due to the competition of the frustrating couplings. In particular, if the bare couplings \( \lambda_e, \lambda_n \) are very small or vanish it is possible that other operators dominate. The relevant coupling \( \lambda_c \) is of particular interest in this respect since it may drive the phase into an incommensurate phase as discussed below. However, \( \lambda_e \) and \( \lambda_n \) will always be generated by higher order terms in the renormalization procedure \[14\] and may still dominate the low energy physics depending also on the flow of the marginal couplings.

To illustrate the interplay of different coupling constants we first consider the cross coupled ladder (CCL) with \( J_d = J_d' \) in the presence of in-chain frustration \( J_2 \) and ring exchange coupling \( K \). As shown in Fig. 2 the four known phases dominate the phase diagram since \( \lambda_e = 0 \) by symmetry in this case. For \( K = 0 \) there is a direct phase transition from rung singlet to Haldane phase at \( J_{\perp}/2J_d = 1 \) with no intermediate dimerized phase. The next nearest neighbor coupling \( J_2 \) does not change the basic topology of the phase diagram, but has a large effect on the range of the dimer phases. This is due to the fact that the in-chain frustration \( J_2 \) leads to a reduced starting value of \( \lambda_c \) and therefore makes the dimerization operator effectively more relevant. In fact, it is known that decoupled chains spontaneously dimerize for \( J_2 > J_{2c} \) \[37, 38\].

For comparison the phase diagram of the diagonal ladder (DL) with \( J'_d = 0 \) is shown in Fig. 3 for \( K = 0 \) and three values of the NNN interaction, \( J_2 = 0, 0.2J, J_{2c} \) as function of the two interchain couplings \( J_{\perp} \) and \( J_d \). The symmetry properties of this model are fundamentally different from the CCL, because the larger unit cell permits the presence of the operator \( O_c \), which plays an important role in generating the dimer operator \( O_e \) under the renormalization in Eq. (6). Interestingly, in contrast to the CCL, therefore an intermediate columnar dimer-
FIG. 4: Phase diagram of the diagonal ladder \((J_0' = 0)\) as a function of \(J_d\) and \(K\), for \(J_{\perp}/J_d = 1.1\) and \(J_2 = 0.2J\).

FIG. 5: Phase diagram as a function of \(-J_d'/J_d\) and \(K\), for \(J_{\perp} = 0\), \(J_2 = J_{2c}\), and \(J_d = 0.2J\).

Dimerized phase exists even without ring exchange, where the generated operator \(O_c\) dominates the correlations. This phase again becomes larger in the presence of in-chain frustration \(J_2\) for reasons mentioned above.

One interesting aspect of the phase diagram in Fig. 3 is the fact that there is no phase which is generated directly by the relevant operator \(O_c\) even in cases where its initial bare coupling constant \(\lambda_c\) may be largest. This invites interesting questions: First of all, is it ever possible to generate phases which are characterized by a dominant coupling \(\lambda_c\) in the extended parameter space? And secondly, if such a phase exist, what dominant correlation would be expected?

The first question is answered by finding suitable parameters for such a phase by taking advantage of canceling out the effect from competing dimer phases. We understand that the dimerized phase in Fig. 3 is created by \(O_c\), which grows beyond bounds after it is generated in second order by the initial bare coupling \(\lambda_c\). To suppress this generation it is possible to add a small ring exchange \(K\), which initially pushes the value of \(\lambda_c\) to be slightly positive in Eq. (6). The analogous reasoning applies for \(\lambda_n\), where a small positive bare value can be achieved by choosing \(J_{\perp}\) slightly larger than \(J_d\). Accordingly, we show the phase diagram for the diagonal ladder with \(J_{\perp} = 1.1J_d\) and \(J_2 = 0.2J\) as a function of \(K\) and \(J_d\). In this case there is indeed an extended region with this additional phase for \(K > 0\), which we describe as incommensurate for reasons that will be explained below. This incommensurate phase separates the staggered dimer from the columnar dimer phase.

To illustrate another example of possible parameters for this phase, we consider a general ratio of the diagonal couplings \(J_d'/J_d\) in Fig. 5 for \(J_{\perp} = 0\), \(J_2 = J_{2c}\) and \(J_d = 0.2J\). According to Eq. (6), the initial value of \(\lambda_c\) becomes strongest for \(J_d' = -J_d\), while the other initial couplings are zero for \(J_{\perp} = K = 0\) and \(J_2 = J_{2c}\). Indeed the intermediate incommensurate phase appears for small positive \(K\) and \(J_d' \approx -J_d\), again separating the staggered and columnar dimer phases.

To answer the question about the nature of the phase for strong \(O_c = J_1 \cdot n_2 - J_2 \cdot n_1\) it is possible to invoke a self-consistency argument, analogous to a chain mean field theory [49–51]. Clearly, the operator \(O_c\) causes an alternating magnetic order \(n\) in one chain to induce a collinear ferromagnetic order \(J\) in the other chain and vice versa. Assuming small finite order parameters we can write for the expectation values

\[
\langle J_2 \rangle = -\lambda_c \chi_0 \langle n_1 \rangle; \quad \langle n_1 \rangle = -\lambda_c \chi_1 \langle J_2 \rangle
\]

\[
\langle J_1 \rangle = \lambda_c \chi_0 \langle n_2 \rangle; \quad \langle n_2 \rangle = \lambda_c \chi_1 \langle J_1 \rangle
\]

where \(\chi_0\) is the uniform susceptibility and \(\chi_1\) is the alternating susceptibility. While \(\chi_0\) is finite [52], \(\chi_1\) diverges with \(1/T\) [50, 51], so there will always be a low enough energy scale at which \(\lambda_c^2 \chi_0 \chi_1 = 1\) results in a self-consistent order. Clearly, at this point both the alternating antiferromagnetic order and the collinear ferromagnetic order become finite in both chains. Since it is known that a small collinear ferromagnetic part effectively shifts the wavelength of the antiferromagnetic order [52, 53], we conclude that this behavior corresponds to an incommensurate state. Semi-classically \(n_{1(2)}\) is perpendicular to \(J_{1(2)}\), so therefore also \(n_1\) is perpendicular to \(n_2\) in this mean-field argument.

Incommensurate behavior due to frustration has been much discussed in the literature, not only for strongly frustrated chains [55] and coupled ladder systems [22, 56, 57], but more recently also for anisotropic triangular lattices [58, 60]. However, the incommensurate phase we have predicted in this paper arises for small frustration and interchain couplings. It is generated by a
complex combination of competing instabilities, which require only a small perturbation from the spin-liquid fixed point of decoupled chains. This opens the door for the search also in 2D systems for corresponding incommensurate phases near instabilities.

In conclusion, we have examined a general SU(2) invariant ladder model with focus on frustration and ring exchange. For small coupling strengths the renormalization flow to four known phases can be quantitatively examined. In addition, we predict that there is an instability to an incommensurate phase in parts of the parameter space. In contrast to other models with incommensurate behavior, this phase can be observed even for very small values of frustration and inter-chain couplings. While frustration is an important ingredient for enhanced quantum fluctuations in order to generate the phase, the underlying instability towards incommensurate order is already present in the field theory of any weakly coupled chains with a broken translational symmetry $J_\text{L} \neq J_\text{R}$. It therefore appears that an essential aspect for the observed incommensurate behavior is the enlarged unit cell, which in turn provides an important clue what choice of couplings may be promising to show corresponding phases in a 2D generalization of the model.

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APPENDIX

In this appendix additional information for the derivation of the renormalization group equations is given and the bosonization of the four spin interaction is discussed.

Field theory and renormalization group flow

For the derivation of the bosonization formulas and the operator product expansion (OPE) it is useful to consider an interacting spinful Fermion model as the underlying physical realization where only the spin channel will be considered in the low energy limit. For the half-filled Hubbard model the charge channel is gapped and the Heisenberg couplings considered in the paper corresponds to the spin channel.

The spin currents are therefore conveniently expressed in terms of left- and right-moving Fermion operators

$$J_\text{L}(z_\kappa) = \psi_\kappa \frac{\sigma^a_{\eta'\eta}}{2} \psi_{\eta'\eta} : (z_\kappa) ,$$

where $\sigma^a$ are the Pauli matrices with a summed over spin-index $\eta = \uparrow, \downarrow$, and $\kappa = \text{R, L}$ denotes the chirality. The chiral complex coordinates are $z_\text{L/R} = \pm ix + vt$. The dimerization and staggered magnetization operators are given by

$$\epsilon(z) \sim \frac{i}{2} \{ : \psi^\dagger_{R\eta} \psi_{L\eta} : (z) - : \psi^\dagger_{L\eta} \psi_{R\eta} : (z) \} ,$$

$$n^a(z) \sim \frac{1}{2} \sigma^a_{\eta'\eta} [ : \psi^\dagger_{R\eta} \psi_{L\eta'} : (z) + : \psi^\dagger_{L\eta} \psi_{R\eta'} : (z) ] ,$$

where $z$ implies a dependence on both chiral variables $z_\text{R}, z_\text{L}$ in this case.

The OPEs between $J_\text{a}^\text{a}$, $\epsilon$, and $n^b$ can be calculated using Wick’s theorem and the two point correlation function

$$\langle \langle \psi_{\kappa\eta}(z_\kappa) \psi^\dagger_{\kappa',\eta'} : (w_{\kappa'}) \rangle \rangle = \delta_{\kappa\kappa'} \delta_{\eta\eta'} \frac{\gamma}{z_\kappa - w_\kappa} ,$$

where we choose a normalization of $\gamma = 1/2\pi$. Keeping all relevant terms, the important OPEs are

$$J_\text{a}^\text{a}(z_\kappa) J_\text{a'}^{\text{a'}}(w_{\kappa'}) = \delta_{\kappa\kappa'} \left[ \frac{(\gamma^2/2) \delta_{ab}}{(z_\kappa - w_\kappa)^2} + i \epsilon_{abc}\gamma w_\kappa \frac{J_\beta^2}{z_\kappa - w_\kappa} \right] ,$$

$$J_\text{a}^\text{a}(z_\kappa) \epsilon(w) = i \kappa \gamma/2 z_\kappa - w_\kappa n^a(w) ,$$

$$J_\text{a}^\text{a}(z_\kappa) n^b(w) = i \frac{\gamma^2}{z_\kappa - w_\kappa} [ \epsilon_{abc} n^c(w) - \kappa \delta_{ab} \epsilon(w) ] ,$$

$$\epsilon(z) \epsilon(w) = -i \gamma |z - w| J_\text{L}(w) J_\text{L}(w)$$

$$n^a(z) n^b(w) = |z - w| \left[ J_\text{L}^2(w) \frac{1}{z_\text{L} - w_\text{L}} + J_\text{R}^2(w) \frac{1}{z_\text{R} - w_\text{R}} \right]$$

where, $\delta_{ab}$ is the Kronecker $\delta$-function, and $\epsilon_{abc}$ the Levi Civita symbol. Here

$$\hat{Q}^{ab} = \frac{1}{2} \sigma^a_{\eta'\eta} \sigma^b_{\tau'\tau} \psi_{R\eta}^\dagger \psi_{L\eta'} \psi_{L\tau}^\dagger \psi_{R\tau'} ,$$

denotes the zeroth order contraction between the fermionic fields. After freezing out the gapped charge degrees of freedom only the trace of this operator is relevant for the calculation of the renormalization group flow, which reads

$$\hat{Q}^{aa} = J_\text{L} \cdot J_\text{L} .$$

The evolution of the bare couplings is determined from the OPEs of the perturbing operators using

$$\frac{d\lambda_k}{dl} = (2 - d_k) \lambda_k - \frac{\pi}{v} \sum_{i,j} C_{ijk} \lambda_i \lambda_j ,$$

where $d_k$ is the scaling dimension of the operator, $v$ is the velocity, and $C_{ijk}$ is the coefficient extracted from the OPE $O_i(z) O_j(w) \sim O_k(w)$. Using Eqs. (12) and (13) we arrive at the renormalization group equations in the main text.
Four spin interactions

In this part we present the bosonization of the four-spin interactions. The ring exchange interaction is given by

$$H = K \sum_p \left( h_p + \frac{1}{4} \right), \quad (16)$$

where $p$ sums over the plaquettes of the system and the local energy operators are given by

$$h_p = S_{p_1} \cdot S_{p_2} + S_{p_3} \cdot S_{p_4} + S_{p_1} \cdot S_{p_4} + S_{p_2} \cdot S_{p_3} + \frac{1}{2} \left( \sum_{a,b} \delta_{ab} \epsilon(x) + \sum_{a,b} \epsilon(x) \right),$$

where due to the relative minus sign and the $\delta-$functions, relevant terms are eliminated, and only interchain leg parts we arrive at

$$H_L = 4K \sum_j \left( S_{1,j} \cdot S_{1,j+1} + S_{2,j} \cdot S_{2,j+1} \right), \quad (17)$$

$$H_R = 4K \sum_j \left( S_{1,j} \cdot S_{2,j} + S_{1,j+1} \cdot S_{2,j+1} \right) \quad (18)$$

The four-spin interactions are either on the same leg of the ladder or on the rungs of the ladder. The leg component $H_L$ and the rung $H_R$ are given by

$$H_L = 4K \sum_j \left( S_{1,j} \cdot S_{1,j+1} + S_{2,j} \cdot S_{2,j+1} \right);$$

$$H_R = 4K \sum_j \left( S_{1,j} \cdot S_{2,j} + S_{1,j+1} \cdot S_{2,j+1} \right).$$

Spin operators are substituted in the continuum with,

$$S_j / a \rightarrow S(x) \approx J(x) + (-1)\gamma \Omega n(x),$$

and the product of two spin operators on the same chain becomes with the help of Eq. (12),

$$aS^a(x)S^b(x + a) \approx -\frac{1}{2} a (2\gamma \delta_{ab} \epsilon(x) + (1 + a) \epsilon(x)).$$

Using this equation for the products of spin operators in the leg part we arrive at

$$H_L \approx 12a^3K \sum_j \left( S_{1,j} \cdot S_{2,j+1} + S_{2,j} \cdot S_{1,j+1} \right) \quad \text{(21)}$$

The rung part of the four spin interactions can be written as

$$H_R \approx 4a^3K \int dx S^a_{1}(x)S^b_{1}(x + a) \quad \text{(22)}$$

where due to the relative minus sign and the $\delta-$functions, relevant terms are eliminated, and only interchain marginal terms survive

$$H_R \approx -16aK \gamma^2 \left( 1 + a \Omega^2 \right)^2 \int dx \Omega_b(x). \quad \text{(23)}$$

Combining Eqs. (21) and (23) and carrying out a trivial calculation for the two spin interactions $H_2,$

$$H_2 \approx 4a^3K \int dx \left( (2 - a \Omega^2)O_a(x) + 4O_b(x) \right), \quad \text{(24)}$$

one arrives at the bosonized ring exchange interaction in the continuous limit

$$H \approx 4a^3K \int dx \left( 2 - a \Omega^2 + 3a \Omega^2 \left( 1 + a \Omega^2 \right) \Omega_a \right.$$

$$\left. + 4 \left( 1 - \frac{1}{2}a \Omega^2 \right) \Omega_b + \frac{36}{2+1} \Omega_e \right). \quad \text{(25)}$$

which is used to determine the corresponding bare coupling strengths in the main text.

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[1] T. Rice, Low Dimensional Magnets in High Magnetic Fields (C. Berthier, L. Levy, and G. Martinez, eds.), 595 of Lecture Notes in Physics, 139–160, Springer Berlin Heidelberg, 2002.

[2] E. Dagotto and T. M. Rice, Surprises on the Way from One- to Two-Dimensional Quantum Magnets: The Ladder Materials, Science 271, 618 (1996).

[3] H.-J. Mikeska and A. Kotakzhu, One-dimensional magnetism in Quantum Magnetism (U. Schollwöck, J. Richter, D. Farnell, and R. Bishop, eds.), 645 of Lecture Notes in Physics, 1, Springer Berlin Heidelberg, 2004.

[4] S. R. White, Equivalence of the antiferromagnetic Heisenberg ladder to a single $S =$1 chain, Phys. Rev. B 53, 52 (1996).

[5] D. G. Shelton, A. A. Nersesyan, and A. M. Tsvelik, Antiferromagnetic spin ladders: Crossover between spin $S =$1/2 and $S =$1 chains, Phys. Rev. B 53, 8521 (1996).

[6] Y. Nishiyama, N. Hatano, and M. Suzuki, Phase Transition and Hidden Orders of the Heisenberg Ladder Model in the Ground State, Journal of the Physical Society of Japan 64, 1967 (1995).

[7] M. Nakamura, T. Yamamoto, and K. Ide, Phase Diagram of the Spin-$\frac{1}{2}$ Bond-Alternating Two-Leg Ladder, Journal of the Physical Society of Japan 72, 1022 (2003).

[8] M. Müller, T. Vekua, and H.-J. Mikeska, Perturbation theories for the $S = \frac{1}{2}$ spin ladder with a four-spin ring exchange, Phys. Rev. B 66, 134423 (2002).

[9] V. Gritsev, B. Normand, and D. Baeriswyl, Phase diagram of the Heisenberg spin ladder with ring exchange, Phys. Rev. B 69, 094431 (2004).

[10] S. Capponi, P. Lecheminant, and M. Moliner, Quantum phase transitions in multileg spin ladders with ring exchange, Phys. Rev. B 88, 075132 (2013).

[11] K. Hiji and T. Sakai, Ground state phase diagram of an $S = \frac{1}{2}$ two-leg Heisenberg spin ladder system with negative four-spin interaction, Phys. Rev. B 88, 104403 (2013).

[12] E. H. Kim, G. Fáth, J. Sólyom, and D. J. Scalapino, Phase transitions between topologically distinct gapped phases in isotropic spin ladders, Phys. Rev. B 62, 14965 (2000).
[13] D. Allen, F. H. L. Essler, and A. A. Nersesyan, Fate of spinons in spontaneously dimerized spin-$\frac{1}{2}$ ladders, Phys. Rev. B 61, 8871 (2000).

[14] O. A. Starykh and L. Balents, Dimerized Phase and Transitions in a Spatially Anisotropic Square Lattice Antiferromagnet, Phys. Rev. Lett. 93, 127202 (2004).

[15] T. Hikihara and O. A. Starykh, Phase diagram of the frustrated spin ladder, Phys. Rev. B 81, 064432 (2010).

[16] G. Barcza, O. Legeza, R. M. Noack, and J. Sólyom, Dimerized phase in the cross-coupled antiferromagnetic spin ladder, Phys. Rev. B 86, 075133 (2012).

[17] H.-H. Hung, C.-D. Gong, Y.-C. Chen, and M.-F. Yang, Search for quantum dimer phases and transitions in a frustrated spin ladder, Phys. Rev. B 73, 224433 (2006).

[18] E. H. Kim, O. Legeza, and J. Sólyom, Topological order, dimerization, and dimeron deconfinement in frustrated spin ladders, Phys. Rev. B 77, 205121 (2008).

[19] G. Sierra, M. A. Martín-Delgado, S. R. White, D. J. Scalapino, and J. Dukelsky, Diagonal ladders: A class of models for strongly coupled electron systems, Phys. Rev. B 59, 7973 (1999).

[20] M. A. Martín-Delgado, J. Rodríguez-Laguna, and G. Sierra, Universality classes of diagonal quantum spin ladders, Phys. Rev. B 72, 104435 (2005).

[21] T. Vekua and A. Honecker, Quantum dimer phases in a frustrated spin ladder: Effective field theory approach and exact diagonalization, Phys. Rev. B 73, 214427 (2006).

[22] A. Lavareló, G. Roux, and N. Laflorencie, Melting of a frustration-induced dimer crystal and incommensurability in the $J_1$-$J_2$ two-leg ladder, Phys. Rev. B 84, 144407 (2011).

[23] A. Metavitsiadis, D. Sellmann, and S. Eggert, Spin-liquid versus dimer phases in an anisotropic $J_1$-$J_2$ frustrated square antiferromagnet, Phys. Rev. B 89, 241104 (2014).

[24] G.-H. Liu, H.-L. Wang, and G.-S. Tian, Existence of dimerized phases in frustrated spin ladder models, Phys. Rev. B 77, 214418 (2008).

[25] Y.-C. Li and H.-Q. Lin, Quantum phase diagram of the frustrated spin ladder with next-nearest-neighbor interactions, New Journal of Physics 14, 063019 (2012).

[26] R. Coldea, D. A. Tennant, R. A. Cowley, D. F. McMorrow, B. Dorner, and Z. Tylczynski, Quasi-1D $S = 1/2$ Antiferromagnet Cs$_2$CuCl$_4$ in a Magnetic Field, Phys. Rev. Lett. 79, 151 (1997).

[27] D. Schmidiger, P. Bouillot, T. Guidi, R. Biewel, C. Kollath, T. Giamarchi, and A. Zheludev, Spectrum of a Magnetized Strong-Leg Quantum Spin Ladder, Phys. Rev. Lett. 111, 107202 (2013).

[28] F. Casola, T. Shiroka, A. Feiguin, S. Wang, M. S. Grbić, M. Horvatić, S. Krámer, S. Mukhopadhyay, K. Conder, C. Berthier, H.-R. Ott, H. M. Ronnow, C. Rüegg, and J. Messot, Field-Induced Quantum Soliton Lattice in a Frustrated Two-Leg Spin-1/2 Ladder, Phys. Rev. Lett. 110, 187201 (2013).

[29] G. Sun, J. Jaramillo, L. Santos, and T. Vekua, Spin-orbit coupled fermions in ladder-like optical lattices at half filling, Phys. Rev. B 88, 165101 (2013).

[30] H. Zhang, Q. Guo, Z. Ma, and X. Chen, Spin model in the effective staggered magnetic field in optical lattices, Phys. Rev. A 87, 043625 (2013).

[31] J. Simon, W. S. Bakr, R. Ma, M. E. Tai, P. M. Preiss, and M. Greiner, Quantum simulation of antiferromagnetic spin chains in an optical lattice, Nature 472, 307 (2011).

[32] M. Nixon, E. Ronen, A. A. Friesem, and N. Davidson, Observing Geometric Frustration with Thousands of Coupled Lasers, Phys. Rev. Lett. 110, 184102 (2013).

[33] K. B. Lyons, P. A. Fleury, J. P. Remeika, A. S. Cooper, and T. J. Negran, Dynamics of spin fluctuations in lanthanum cuprate, Phys. Rev. B 37, 2353 (1988).

[34] C. J. Calzado and J.-P. Mailleux, Origin and evaluation of the four-spin operators in magnetic lattices, Phys. Rev. B 69, 094435 (2004).

[35] M. Roger, J. H. Hetherington, and J. M. Delrieu, Magnetism in solid $^3$He, Rev. Mod. Phys. 55, 1 (1983).

[36] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Valence bond ground states in isotropic quantum antiferromagnets, Comm. Mat. Phys. 115, 477 (1988).

[37] K. Nomura and K. Okamoto, Critical properties of $S = 1/2$ antiferromagnetic XXZ chain with next-nearest-neighbour interactions, J. Phys. A: Mathematical and General 27, 5773 (1994).

[38] G. Castilla, S. Chakravarty, and V. J. Emery, Quantum Magnetism of CuGeO$_3$, Phys. Rev. Lett. 75, 1823 (1995).

[39] S. Eggert, Numerical evidence for multiplicative logarithmic corrections from marginal operators, Phys. Rev. B 54, R9612 (1996).

[40] I. Affleck, Field Theory Methods and Quantum Critical Phenomena in Fields, Strings and Critical Phenomena (E. Brézin and J. Zinn-Justin, eds.), 563-640, North-Holland, Amsterdam, 1990. (proceedings of Les Houches Summer School, 1988).

[41] S. Eggert and I. Affleck, Magnetic impurities in half-integer-spin Heisenberg antiferromagnetic chains, Phys. Rev. B 46, 10866 (1992).

[42] I. Affleck, Exact critical exponents for quantum spin chains, non-linear-models and the quantum hall effect, Nucl. Phys. B 265, 490 (1986).

[43] M. A. Tsvelik, Quantum field theory in condensed matter physics. Cambridge university press, 2003.

[44] P. Di Francesco, P. Mathieu, and D. Senechal, Conformal field theory. Springer, 1997.

[45] S. Takayoshi and M. Sato, Coefficients of bosonized dimer operators in spin-$\frac{1}{2}$ XXZ chains and their applications, Phys. Rev. B 82, 214420 (2010).

[46] J. Cardy, Scaling and renormalization in statistical physics. Cambridge university press, 1996.

[47] D. Sénéchal, An introduction to bosonization in Theoretical Methods for Strongly Correlated Electrons (D. Sénéchal, A. M. S. Tremblay, and C. Bourbonnais, eds.), CRM Series in Mathematical Physics, ch. 4, 139–186, Springer, 2004.

[48] O. A. Starykh, A. Furuhashi, and L. Balents, Anisotropic pyrochlores and the global phase diagram of the checkerboard antiferromagnet, Phys. Rev. B 72, 094416 (2005).

[49] I. Affleck, M. P. Gelfand, and R. R. P. Singh, A plane of weakly coupled Heisenberg chains: theoretical arguments and numerical calculations, J. Phys. A: Mathematical and General 27, 7313 (1994).

[50] D. J. Scalapino, Y. Imry, and P. Pincus, Generalized Ginzburg-Landau theory of pseudo-one-dimensional systems, Phys. Rev. B 11, 2042 (1975).

[51] S. Eggert, I. Affleck, and M. D. P. Horton, Néel Order in Doped Quasi-One-Dimensional Antiferromagnets, Phys. Rev. Lett. 89, 047202 (2002).

[52] S. Eggert, I. Affleck, and M. Takahashi, Susceptibility of the spin 1/2 Heisenberg antiferromagnetic chain, Phys.
[53] I. Affleck and M. Oshikawa, Field-induced gap in Cu benzoate and other $S = \frac{1}{2}$ antiferromagnetic chains, Phys. Rev. B 60, 1038 (1999).

[54] S. Eggert and I. Affleck, Impurities in $S = 1/2$ Heisenberg Antiferromagnetic Chains: Consequences for Neutron Scattering and Knight Shift, Phys. Rev. Lett. 75, 934 (1995).

[55] S. R. White and I. Affleck, Dimerization and incommensurate spiral spin correlations in the zigzag spin chain: Analogies to the Kondo lattice, Phys. Rev. B 54, 9862 (1996).

[56] A. A. Nersesyan, A. O. Gogolin, and F. H. L. Eßler, Incommensurate Spin Correlations in Spin-1/2 Frustrated Two-Leg Heisenberg Ladders, Phys. Rev. Lett. 81, 910 (1998).

[57] S. Brehmer, A. K. Kolezhuk, H.-J. Mikeska, and U. Neugebauer, Elementary excitations in the gapped phase of a frustrated $S = 1/2$ spin ladder: from spinons to the Haldane triplet, J. Phys.: Cond. Mat. 10, 1103 (1998).

[58] S. V. Isakov, H.-C. Chien, J.-J. Wu, Y.-C. Chen, C.-H. Chung, K. Sengupta, and Y. B. Kim, Commensurate lock-in and incommensurate supersolid phases of hard-core bosons on anisotropic triangular lattices, Europhys. Lett. 87, 36002 (2009).

[59] A. Weichselbaum and S. R. White, Incommensurate correlations in the anisotropic triangular Heisenberg lattice, Phys. Rev. B 84, 245130 (2011).

[60] K. Harada, Numerical study of incommensurability of the spiral state on spin-$\frac{1}{2}$ spatially anisotropic triangular antiferromagnets using entanglement renormalization, Phys. Rev. B 86, 184421 (2012).

[61] X.-F. Zhang, S. Hu, A. Pelster, and S. Eggert, Quantum Domain Walls Induce Incommensurate Supersolid Phase on the Anisotropic Triangular Lattice, Phys. Rev. Lett. 117, 193201 (2016).