A viability criterion for modified gravity with an extra force

Valerio Faraoni
Physics Department, Bishop's University
Sherbrooke, Quebec, Canada J1M 0C8

A recently proposed theory of modified gravity with an explicit “anomalous” coupling of the Ricci curvature to matter is discussed, and an inequality is derived which expresses a necessary and sufficient condition to avoid the notorious Dolgov-Kawasaki instability.

PACS numbers: 04.50.+h, 04.20.Cv, 95.35.+d

INTRODUCTION

Recently, modifications of gravity at cosmological scales have received much attention [1, 2, 3] in order to explain the cosmic acceleration discovered in 1998 using type Ia supernovae [4]. The alternative is to resort to a mysterious form of dark energy with exotic properties [5]:

The type Ia supernovae [4] scale has received much attention [1, 2, 3] in order to prevent such an inﬁnite future [7]. Modified gravity allows one to avoid such an approach.

Non-metric connection [17]. Here we focus on the metric part of the action does not depend on Γ^a_b_c but the matter part of the action does not depend on Γ^a_b_c [16]; and metric-affine gravity, in which also ℒ_m depends on the non-metric connection [17]. Here we focus on the metric approach.

\[
S_{EH} = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa} + \mathcal{L}_m \right),
\]

where \( f(R) \) is an (a priori) arbitrary function of \( R \), and the modiﬁcations are designed to affect cosmological scales and stay tiny at smaller scales in order not to violate the Solar System constraints [5]. The prototype \( f(R) = R - \mu^4/R \) (with \( \mu \sim H_0 \sim 10^{-35} \text{eV} \)) is now regarded as an unviable toy model at best because it is subject to a violent instability [9] and violates the experimental constraints [10].

In order to be viable, modiﬁed gravity theories must be free of short time scale instabilities and ghosts [9, 11, 12, 13], have a well-posed Cauchy problem [14], and have the correct cosmological dynamics including an early inﬂationary era followed by a radiation era, a matter era, and a late accelerated era (in many models there are problems with the exit from the radiation era [15]).

Modified \( f(R) \) gravity comes in three versions: metric formalism, in which the action (2) is varied with respect to the (inverse) metric tensor \( g^{ab} \); Palatini \( f(R) \) gravity, in which variation is with respect to both \( g^{ab} \) and an independent, non-metric, connection \( \Gamma^a_{bc} \) but the matter part of the action does not depend on \( \Gamma^a_{bc} \) [16]; and metric-affine gravity, in which also \( \mathcal{L}_m \) depends on the non-metric connection [17].

Recently, Bertolami, Böhmer, Harko, and Lobo (hereafter BBHL) [18] put a new twist on \( f(R) \) gravity by considering the action

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{f_1(R)}{2} + \frac{f_2(R)}{2} \left( 1 + \lambda f_2(R) \right) \mathcal{L}_m \right\},
\]

where \( f_1, f_2(R) \) are arbitrary functions of the Ricci curvature and \( \lambda \) is a small parameter (from now on we follow [18] and set \( \kappa = 8\pi G = 1 \)). The novelty consists of the coupling function \( f_2(R) \) which adds extra freedom and new features. The ﬁeld equations are

\[
\begin{align*}
&f_1'(R) R_{ab} - \frac{f_1(R)}{2} g_{ab} = \nabla_a \nabla_b f_1'(R) - g_{ab} \square f_1(R) \\
&- 2\lambda f_2'(R) \mathcal{L}_m R_{ab} + 2\lambda \left( \nabla_a \nabla_b - g_{ab} \square \right) (\mathcal{L}_m f_2'(R)) \\
&+ [1 + \lambda f_2(R)] T^{(m)}_{ab}.
\end{align*}
\]

where a prime denotes differentiation with respect to \( R \), \( \square \equiv g^{cd} \nabla_c \nabla_d \), and \( T^{(m)}_{ab} \) obeys [18]

\[
\nabla^b T^{(m)}_{ab} = \frac{\lambda f_2(R)}{1 + \lambda f_2(R)} \left( g_{ab} \mathcal{L}_m - T^{(m)}_{ab} \right) \nabla^b R.
\]

The BBHL theory contains intriguing phenomenology: all massive particles are subject to an extra force, similar to the one arising in scalar-tensor (ST) gravity following a conformal transformation to the Einstein frame [13, 20]. In Einstein frame ST gravity the extra force is due to an “anomalous” coupling of the matter Lagrangian to the Brans-Dicke-like scalar \( \phi \),

\[
S_{EF} = \int d^4x \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} \nabla^c \phi \nabla_c \phi - e^{-\alpha \phi} \mathcal{L}_m \right).
\]

There is, however, an important difference between Einstein frame ST gravity and BBHL theory: while in the former the units of time, length, and mass are not constant but scale with appropriate powers of the conformal factor of the conformal transformation deﬁning the Einstein frame (as explained in [13] and discussed extensively in [21]), in the latter there is no such scaling of
units. For this reason, the BBHL theory \(^{18}\) can not be reduced to "ordinary" ST or string gravity (in this respect, string theory has the same phenomenology of ST gravity—indeed, the low-energy limit of the bosonic string is a Brans-Dicke theory with Brans-Dicke parameter \(\omega = -1 \, ^{22}\)).

The extra force on massive particles generated by the \(\lambda f_2(R)\) coupling is always present and causes a deviation from geodesic paths; therefore, massive test particles simply do not exist. Due to this extra force, the acceleration law in the weak-field limit of BBHL theory assumes a form similar to the one of Modified Newtonian Dynamics (MOND) \(^{22}\), which was originally proposed to explain galactic rotation curves without dark matter. MOND has recently received a relativistic formulation in the rather complicated Tensor-Vector-Scalar (TeVeS) theory of \(^{24}\). The BBHL proposal exhibits MOND-like phenomenology but has a simpler formal structure than TeVeS: as shown below, it amounts to a ST theory with two coupling functions, one of which is the coupling of the scalar degree of freedom \(\phi = R\) to matter (unorthodox in ST gravity \(^{25}\)). Of course, in order to be viable, the BBHL theory must pass the tests mentioned above for \(f(R)\) gravity and it is not clear yet whether this is possible. In this paper we study one of these criteria, namely the stability of the theory with respect to local perturbations. In pure \(f(R)\) gravity, a fatal instability develops on time scales \(\sim 10^{-26} \, s\) \(^{9}\) when \(f''(R) < 0\). This instability, which we refer to as the "Dolgov-Kawasaki phenomenon", was discovered in the prototype model \(f(R) = R - \mu^2/R\) \(^9\) which is ruled out (and only cured by adding extra terms to \(f(R)\) \(^{11,12,26}\)) and then generalized to arbitrary \(f(R)\) models \(^{27}\). For the BBHL theory, the corresponding stability criterion turns out to be \(f''_1(R) + 2\lambda f''_2(R) \geq 0\) (see Sec. 3; see Refs. \(^{28}\) for other types of instabilities).

**EQUIVALENCE WITH AN ANOMALOUS ST THEORY**

It is well known that pure \(f(R)\) gravity \(^{11}\) is equivalent to a ST theory \(^{29}\); here we revisit this equivalence and generalize it to BBHL theory.

By introducing a new field \(\phi\), the action \(^3\) is written as

\[
S = \int d^4x\sqrt{-g} \left\{ f_1(\phi) \frac{\phi}{2} + \frac{1}{2} \frac{df_1}{d\phi}(R - \phi) + [1 + \lambda f_2(\phi)]L_m \right\}
\]

and, further introducing the field \(\psi(\phi) \equiv f'_1(\phi)\) (where now a prime denotes differentiation with respect to \(\phi\) \(^{37}\)), one can write

\[
S = \int d^4x\sqrt{-g} \left\{ \psi_4 \frac{2}{R} - V(\psi) + U(\psi)L_m \right\},
\]

where

\[
V(\psi) = \frac{\phi(\psi)f'_1(\phi)(\psi) - f_1(\phi)}{2}, \quad U(\psi) = 1 + \lambda f_2(\phi)\psi,
\]

with \(\phi(\psi)\) given by inverting \(\psi(\phi) \equiv f'_1(\phi)\). The actions \(^3\) and \(^5\) are equivalent when \(f''_1(R) \neq 0:\) in fact, by setting \(\phi = R\), eq. \(^5\) reduces trivially to eq. \(^3\). Vice-versa, variation of \(^3\) with respect to \(\phi\) yields

\[
(R - \phi) f''_1(\phi) + 2\lambda f''_2(\phi)L_m = 0.
\]

In vacuo \((L_m = 0)\), this equation yields \(\phi = R\) whenever \(f''_1(R) \neq 0\) \(^{29}\). In the presence of matter there seem to be other possibilities which are, however, excluded as follows. When \(L_m \neq 0\), the actions \(^3\) and \(^7\) are equivalent if \((R - \phi) f''_1(\phi) + 2\lambda f''_2(\phi)L_m \neq 0\). When eq. \(^11\) is satisfied, we have a pathological case which, upon integration of this equation, corresponds to

\[
\lambda f_2(\phi)L_m = \frac{f'_1(\phi)}{2}(\phi - R) - \frac{f_1(\phi)}{2}.
\]

But if eq. \(^12\) holds, then the action \(^7\) reduces to pure matter without the gravity sector and we dismiss this case. Then, the actions \(^3\) and \(^7\) are equivalent when \(f''_1(R) \neq 0\), as in pure \(f(R)\) gravity \(^{29}\). The action \(^3\) corresponds to a Brans-Dicke theory \(^{30}\) with a single scalar field, vanishing Brans-Dicke parameter \(\omega\), and an unorthodox coupling \(U(\psi)\) to matter. Actions of this kind have been contemplated before \(^{31,32,33}\), but little is known about them.

**BBHL THEORY AND INSTABILITIES**

The trace of the field equations is, in terms of \(R\),

\[
3 [f''_1(R) + 2\lambda L_m f''_2(R)]R + 3 [f''_1(R) + 2\lambda L_m f''_2(R)]
\]

\[
\nabla^\alpha R\nabla_\alpha + 12\lambda f''_2(R)\nabla^\alpha L_m\nabla_\alpha R + f'_1(R)R - 2f_1(R)
\]

\[
+ 2\lambda L_m f''_2(R)R = [1 + \lambda f_2(R)]T^{(m)} - 6\lambda f''_2(R)L_m,
\]

where \(T^{(m)} \equiv T^{(m)\alpha}_\alpha\). As customary in \(f(R)\) gravity, we parametrize the function \(f_1(R)\) as \(f_1(R) = R + \epsilon \phi(R)\), where \(\epsilon\) and \(\lambda\) must necessarily be small to respect the Solar System constraints \(^{34}\). Following \(^9\), we expand the spacetime quantities of interest as the sum of a background with constant curvature and a small perturbation: \(R = R_0 + R_1, T = T_0 + T_1, L_m = L_0 + L_1\), and the spacetime metric can locally be approximated by \(g_{ab} = \eta_{ab} + \eta_{ab}\), where \(\eta_{ab}\) is the Minkowski metric. There are really two approximations here. The first is an adiabatic expansion around a de Sitter space with constant curvature, which is justified on timescales much shorter than the Hubble time. The second is a local expansion
over small spacetime regions that are locally flat (hence the appearance of $\eta_{ab}$). Regions that are common in $f(R)$ gravity (e.g. \[ f(R) = R_0 + R_1 + \epsilon \varphi(R_0) + \epsilon \varphi''(R_0)R_1 + \ldots \] and the linearized version of the trace equation \[ \epsilon \varphi''(R_0)R_1 \] in the perturbations becomes

\[
\begin{align*}
3 [\epsilon \varphi''(R_0) + 2 \lambda f''(R_0)] & \nabla R_1 + [\epsilon \varphi''(R_0)R_0 - 1
- \epsilon \varphi'(R_0) + 2 \lambda f'(R_0)\nabla \lambda_0 + 2 \lambda \nabla_0 f''(R_0)R_0
- \lambda f''(R_0)R_0 + 2 \lambda f''(R_0)\nabla \lambda_0 + 2 \lambda \nabla_0 f''(R_0)R_0
= -2 \lambda f''(R_0)R_0 \nabla \lambda_1 + [1 + \lambda f_2(R_0)] T_1
- 6 \lambda f''(R_0) \nabla \lambda_1 ,
\end{align*}
\] (14)

where the zero order equation

\[
f'_1(R_0)R_0 - 2 f_1(R_0) + 2 \lambda \nabla_0 f'_2(R_0)R_0 = [1 + \lambda f_2(R_0)] T_0
\] (15)

has been used. Eq. (14) is further rewritten as

\[
\begin{align*}
\bar{R}_1 - \nabla^2 R_1 + m_{eff}^2 R_1 = & \left[3 \epsilon \varphi''(R_0) + 2 \lambda f''(R_0)\right] - 1 \left(2 \lambda f'_2(R_0)\nabla \lambda_1 - \left[1 + \lambda f_2(R_0)\right] T_1 + \lambda f'_2(R_0)\nabla \lambda_1 \right] (16)
\end{align*}
\]

where the effective mass $m_{eff}$ of the dynamical freedom $R_1$ is given by

\[
m_{eff}^2 = \left\{3 \epsilon \varphi''(R_0) + 2 \lambda f''(R_0)\right\}^{-1} \left[1 + \epsilon \varphi'(R_0) + \epsilon \varphi''(R_0)R_0 - 2 \lambda \nabla_0 \left[f'_2(R_0) + f''(R_0)R_0 + \lambda f'_2(R_0)T_0\right] .
\]

The dominant term on the right hand side is

\[
\left\{3 \epsilon \varphi''(R_0) + 2 \lambda f''(R_0)\right\}^{-1}
\]

and the effective mass squared must be non-negative for stability. Therefore, $\epsilon \varphi''(R) + 2 \lambda f''(R) \geq 0$ is the stability criterion for the BBHL theory against Dolgov-Kawasaki instabilities.

**OUTLOOKS**

The inequality $\epsilon \varphi''(R) + 2 \lambda f''(R) \geq 0$ generalizes the stability condition $f''(R) = \epsilon \varphi''(R) \geq 0$ found in pure $f(R)$ gravity \[ f(R) = R_0 + R_1 + \epsilon \varphi(R_0) + \epsilon \varphi''(R_0)R_1 + \ldots \]. The survival of BBHL theory \[ f(R) = R_0 + R_1 + \epsilon \varphi(R_0) + \epsilon \varphi''(R_0)R_1 + \ldots \] is subject to satisfying the other (physically independent) viability criteria mentioned above, which require a separate analysis and will be analyzed in future publications.

We thank Francisco Lobo for a discussion and the Natural Sciences and Engineering Research Council of Canada (NSERC) for financial support.
[54x41][15] D.N. Vollick, L. Amendola, D. Polarski, and S. Tsujikawa, arXiv:0709.4414.
[54x207][14] J. Cosmol. Astropart. Phys. Lett. 70, 043543 (2004); S. Nojiri and S.D. Odintsov, Phys. Lett. B 562, 147 (2003); L. Mersini, M. Bastero-Gil, and P. Kanti, Phys. Rev. D 64 043508 (2001); M. Bastero-Gil, P.H. Frampton, and L. Mersini, Phys. Rev. D 65 106002 (2002); P.H. Frampton, Phys. Lett. B 555, 139 (2003); E.O. Kahya and V.K. Onemli, Phys. Rev. D 76, 043512 (2007); L. Amendola and S. Tsujikawa, arXiv:0705.0396.
[72x197][7] R.R. Caldwell, M. Kamionkowski and N.N. Weinberg, Phys. Rev. D 57, 3349 (1998); L. Amendola, D. Polarski, and S. Tsujikawa, Class. Quantum Grav. 24, 2849 (2007); E.O. Kahya and V.K. Onemli, Phys. Rev. D 76, 043508 (2007); T. Koivisto and H. Kurki-Suonio, Class. Quantum Grav. 23, 2355 (2006); P. Wang, G.M. Kremer, D.S.M. Alves, and X.H. Meng, Gen. Rel. Grav. 38, 517 (2006); G. Allemandi, M. Capone, S. Capozziello, and M. Francaviglia, Gen. Rel. Grav. 38, 33 (2006); S. Nojiri and S.D. Odintsov, Int. J. Geom. Meth. Phys. 4, 115 (2007); E. Barausse, T.P. Sotiriou, and J.C. Miller, gr-qc/0703132; K. Uddin, J.E. Lidsey, and R. Tavakol, Class. Quantum Grav. 24, 3951 (2007); K. Kainulainen, J. Piilonen, V. Reijonen, and D. Sunhede, Phys. Rev. D 76, 024020 (2007); B. Li, J.D. Barrow, and D.F. Mota, arXiv:0707.2664.
[11] S. Nojiri and S.D. Odintsov, Phys. Lett. B 573, 1 (2003).
[72x552][7] J. Cosmol. Astropart. Phys. Lett. 91, 071301 (2003); S.M. Carroll, M. Hoffman and M. Trodden, Phys. Rev. D 68 023509 (2003).
[8] C.M. Will, Theory and Experiment in Gravitational Physics (Cambridge Univ. Press, Cambridge, 1993).
[9] A.D. Dolgov and M. Kawasaki, Phys. Lett. B 573, 1 (2003).
[10] A.L. Erickcek, T.L. Smith, and M. Kamionkowski, Phys. Rev. D 74, 121501(R) (2006); T. Chiba, T.L. Smith, and A.L. Erickcek, Phys. Rev. D 75, 124014 (2007).
[11] S. Nojiri and S.D. Odintsov, Phys. Rev. D 68, 123512 (2003); Gen. Rel. Gravit. 36, 1765 (2004).
[12] S. Nojiri, TPU Vestnik 44N7; 49: T. Multamaki and I. Vilja, Phys. Rev. D 73, 024018 (2006).
[13] I. Navarro and K. Van Acoleyen, J. Cosmol. Astropart. Phys. 03, 008 (2006); J. Phys. A 39, 6245 (2006); T. Clifton and J.D. Barrow, Phys. Rev. D 72, 123003 (2005); Class. Quantum Grav. 23, L1 (2006); A. Núñez and S. Solganiak, Phys. Lett. B 608, 189 (2005); hep-th/0403159; T. Chiba, J. Cosmol. Astropart. Phys. 0503, 008 (2005); A. De Felice, M. Hindmarsh, and M. Trodden, J. Cosmol. Astropart. Phys. 0608:005 (2006); G. Calcagni, B. de Carlos, and A. De Felice, Nucl. Phys. B 752, 404 (2006); J. Traschen and C.T. Hill, Phys. Rev. D 33, 3519 (1986); V. Müller, H.-J. Schmidt, and A.A. Starobinsky, Phys. Lett. B 202, 198 (1988); H.-J. Schmidt, Class. Quantum Grav. 5, 233 (1988); A. Battaglia Mayer and H.-J. Schmidt, Class. Quantum Grav. 10, 2441 (1993); M.R. Setare, Phys. Lett. B 644, 99 (2007).
[14] N. Lanahan-Tremblay and V. Faraoni, arXiv:0709.4414.
[15] L. Amendola, D. Polarski, and S. Tsujikawa, Phys. Rev. Lett. 98, 131302 (2007); L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, Phys. Rev. D 75, 083504 (2007); S. Capozziello, S. Nojiri, S.D. Odintsov, and A. Troisi, Phys. Lett. B 639, 135 (2006); S. Nojiri and S.D. Odintsov, Phys. Rev. D 74, 086005 (2006); A.W. Brookfield, C. van de Bruck, and L.M.H. Hall, Phys. Rev. D 74, 064028 (2006).
[16] D.N. Vollick, Phys. Rev. D 68, 063510 (2003); Class. Quantum Grav. 21, 3813 (2004); Class. Quantum Grav. 21, 951 (2004); Phys. Lett. B 584, 1 (2004); E.E. Flanagan, Phys. Rev. Lett. 92, 071101 (2004); T. Koivisto, Phys. Rev. D 73, 083517 (2006); Class. Quantum Grav. 23, 4289 (2006); T. Koivisto and H. Kurki-Suonio, Class. Quantum Grav. 23, 2355 (2006); P. Wang, G.M. Kremer, D.S.M. Alves, and X.H. Meng, Gen. Rel. Grav. 38, 11 (2004); Clifton and J.D. Barrow, Phys. Lett. B 594, 26 (2004); J.M. Cline, S. Jeon, and G.D. Moore, Phys. Rev. D 70, 043543 (2004); S. Nojiri and S.D. Odintsov, Phys. Lett. B 562, 147 (2003); L. Mersini, M. Bastero-Gil, and P. Kanti, Phys. Rev. D 64 043508 (2001); M. Bastero-Gil, P.H. Frampton, and L. Mersini, Phys. Rev. D 65 106002 (2002); P.H. Frampton, Phys. Lett. B 555, 139 (2003); E.O. Kahya and V.K. Onemli, Phys. Rev. D 76, 043512 (2007); L. Amendola and S. Tsujikawa, arXiv:0705.0396.