Discretized fast-slow systems near transcritical singularities

Maximilian Engel and Christian Kuehn

Zentrum Mathematik der TU München, Boltzmannstr. 3, D-85748 Garching bei München, Germany

E-mail: maximilian.engel@tum.de

Received 18 June 2018, revised 21 March 2019
Accepted for publication 3 April 2019
Published 30 May 2019

Abstract

We study a transcritical singularity in a fast-slow system given by the explicit Euler discretization of the corresponding continuous-time normal form. The analysis uses the blow-up method and direct trajectory-based estimates. We prove that the qualitative behaviour is preserved by a time-discretization with sufficiently small step size. This step size is fully quantified relative to the time scale separation. Our proof also yields the continuous-time results as a special case and provides more detailed calculations in the classical (or scaling) chart.

Keywords: transcritical bifurcation, slow manifolds, invariant manifolds, loss of normal hyperbolicity, discretization, blow-up method

Mathematics Subject Classification numbers: 34E15, 34E20, 37M99, 37G10, 34C45, 39A99

1. Introduction

We study the dynamics of the two-dimensional quadratic map

\[ p : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + h(x^2 - y^2 + \lambda \varepsilon) \\ y + \varepsilon h \end{pmatrix} \]  

(1.1)

for \( h, \varepsilon > 0 \). We interpret \( \varepsilon \) as a small time scale separation parameter between the fast variable \( x \) and the slow variable \( y \). The parameter \( h \) can be viewed as the step size for the explicit Euler discretization of the corresponding ordinary differential equation (ODE)

\[ \begin{align*}
  x' &= x^2 - y^2 + \lambda \varepsilon, \\
  y' &= \varepsilon,
\end{align*} \]

(1.2)
which represents the normal form of a fast-slow system exhibiting a transcritical singularity at the origin. The term transcritical refers to the fact that, if \(y\) is seen as a bifurcation parameter for the flow in the \(x\)-variable, a transcritical bifurcation occurs at the origin \((x, y) = (0, 0)\). The origin is singular since hyperbolicity of the dynamics breaks down at this point. The same holds for the map (1.1).

In the case of (1.2), Krupa and Szmolyan [16] analyze the dynamics around the origin by using the blow-up method for vector fields with singularities. The key idea to use the blow-up method [3, 4] for fast-slow systems goes back to Dumortier and Roussarie [5]. They observed that this technique may convert non-hyperbolic singularities at which fast and slow directions interact into partially hyperbolic problems. The method inserts a suitable manifold, e.g. a sphere, at the singularity and describes the extension of hyperbolic objects through a neighbourhood of the singularity via the partially hyperbolic dynamics on this manifold; see e.g. [19, chapter 7] for an introduction and [10, 15, 18, 20–23] for a list of a few, yet by no means exhaustive, list of different applications to planar fast-slow systems.

Krupa and Szmolyan also used the blow-up method for normal form of fold singularities in fast-slow systems [15]. For this case Nipp and Stoffer [26] transform the blow-up technique to the corresponding explicit Runge–Kutta, in particular Euler, discretization and prove the extension of slow manifolds for the discrete time system around the singularity. They treat the discretized dynamics in vicinity to the fold singularity as an application of a more general existence theory for invariant manifolds they develop in [26], describing these manifolds as perturbations of continuous-time reference manifolds by using Lipschitz-type estimates on the respective maps.

In this paper, we use instead a direct approach to analyze, how trajectories induced by (1.1) pass the singularity at the origin. The singularity is blown up to a sphere on which trajectories can be described directly via iterations of the map. This leads to the main result of the paper, theorem 3.1, which is the discrete-time extension of [16, theorem 2.1]. In this context, we only impose that \(h\) is bounded by \(\varepsilon\) and prove that there is no further restriction on the step size. Our theorem states explicitly, how for the cases \(\lambda < 1\), \(\lambda = 1\) and \(\lambda > 1\) in (1.1) attracting slow manifolds extend beyond the singularity at the origin. It gives estimates on the contraction rates of neighbourhoods of the manifolds. The case \(\lambda = 1\) corresponds with the problem of canard solutions [1, 5, 17] and is treated here in its simplest form, ignoring higher order terms which require additional analytical tools (see [7]). It should be noted that, by letting \(h \to 0\), our proof of theorem 3.1 can also be seen as a different way of proving [16, theorem 2.1] and our proof makes the results [16] for the scaling chart more explicit. Additionally, the blow-up method provides the insight that only in one chart around the singularity the preservation of stability behaviour is bound to the stability criteria of the Euler discretization derived from the Dahlquist test equation while in the other charts there is no such restriction.

This work is part of a broader effort to apply the blow-up method, which has so far mainly been used for flows, to fast-slow dynamical systems induced by maps. First, it is insightful to look at key examples that can be compared to continuous-time analogues, as in the case of the transcritical singularity, or in the case of the fold singularity as analyzed in [26]. In the future, one will also consider multiscale discrete-time problems, which have no correspondence to fast-slow flows so that no continuous-time reference manifolds are at hand and a direct analysis of the iterations of the maps, as explained in this paper, will most likely be helpful.

The remainder of the paper is structured as follows. In section 2, we summarize the results of Krupa and Szmolyan for transcritical singularities in continuous time [16]. In section 3, we discuss the problem in discrete time associated with (1.1). Our main result is theorem 3.1. The ingredients of the proof are developed in the following subsections. Section 3.1 introduces the blow-up method for the new discrete setting and, subsequently, the dynamics are analyzed.
in three different charts of the manifold that blows up the singularity, leading to the proof of theorem 3.1. In section 3.2, we describe how trajectories enter a neighbourhood of the origin via the entrance chart $K_1$ and leave this neighbourhood in the case of $\lambda < 1$. Section 3.3 builds the core of the proof: we analyze how trajectories pass the origin depending on $\lambda$ in the scaling chart $K_2$. In section 3.4, the exit dynamics through chart $K_3$ are described for the case $\lambda > 1$. Finally, in section 3.5 we combine the findings of the previous sections and give a separate argument for the case $\lambda = 1$ to finish the proof of theorem 3.1. We conclude with a short summary of our results and an outlook on future work in section 4.

2. Transcritical singularity in continuous time

We start with a brief review and notation for continuous-time fast-slow systems. Consider a system of singularly perturbed ordinary differential equations (ODEs) of the form

\[
\begin{align*}
\varepsilon \frac{dx}{dt} &= \varepsilon \dot{x} = f(x, y, \varepsilon), \\
\frac{dy}{dt} &= \dot{y} = g(x, y, \varepsilon),
\end{align*}
\]

(2.1)

where $f, g$ are $C^k$-functions with $k \geq 3$. Since $\varepsilon$ is a small parameter, the variables $x$ and $y$ are often called the fast variable(s) and the slow variable(s) respectively. The time variable in (2.1), denoted by $\tau$, is termed the slow time scale. The change of variables to the fast time scale $t := \tau / \varepsilon$ transforms the system (2.1) into the ODEs

\[
\begin{align*}
x' &= f(x, y, \varepsilon), \\
y' &= \varepsilon g(x, y, \varepsilon).
\end{align*}
\]

(2.2)

Both equations correspond with a respective limiting problem for $\varepsilon = 0$: the reduced problem (or slow subsystem) is given by

\[
\begin{align*}
0 &= f(x, y, 0), \\
\dot{y} &= g(x, y, 0),
\end{align*}
\]

(2.3)

and the layer problem (or fast subsystem) is

\[
\begin{align*}
x' &= f(x, y, 0), \\
y' &= 0.
\end{align*}
\]

(2.4)

We can understand the reduced problem (2.3) as a dynamical system on the critical manifold

\[
S_0 = \{ (x, y) \in \mathbb{R}^{n+m} : f(x, y, 0) = 0 \}.
\]

Observe that the manifold $S_0$ consists of equilibria of the layer problem (2.4). $S_0$ is called normally hyperbolic if the matrix $Df(p) \in \mathbb{R}^{m \times m}$ for all $p \in S_0$ has no spectrum on the imaginary axis. For a normally hyperbolic $S_0$ Fenichel Theory [8, 14, 19, 28] implies that for $\varepsilon$ sufficiently small, there is a locally invariant slow manifold $S_{\varepsilon}$ such that the restriction of (2.1) to $S_{\varepsilon}$ is a regular perturbation of the reduced problem (2.3). Furthermore, it follows from Fenichel’s perturbation results that $S_{\varepsilon}$ possess an invariant stable and unstable foliation, where the dynamics behave as a small perturbation of the layer problem (2.4).

A challenging phenomenon is the breakdown of normal hyperbolicity of $S_0$ such that Fenichel Theory cannot be applied. Typical examples of such a breakdown are found at bifurcation points $p \in S_0$, where the Jacobian $Df(p)$ has at least one eigenvalue with zero real
part. The simplest examples are folds or points of transversal self-intersection of \( S \) in planar systems (\( m = 1 = n \)). In the following we focus on the transcritical bifurcation in planar systems.

We briefly recall the main results for transcritical fast-slow singularities in the continuous-time setting from [16]. Without loss of generality, i.e. up to translation of coordinates, we may just assume that the transcritical point coincides with the origin. Any system of planar ODEs with a transcritical bifurcation at the origin, written on the fast time scale, can be brought [16] to the normal form

\[
\begin{align*}
x' &= x^2 - y^2 + \lambda \varepsilon + \mathcal{R}_1(x, y, \varepsilon), \\
y' &= \varepsilon (1 + \mathcal{R}_2(x, y, \varepsilon)),
\end{align*}
\]

(2.5)

where \( \lambda > 0 \) is a constant and

\[
\mathcal{R}_1(x, y, \varepsilon) = \mathcal{O}(x^3, x^2 y, xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2), \quad \mathcal{R}_2(x, y, \varepsilon) = \mathcal{O}(x, y, \varepsilon).
\]

The critical manifold \( S \) is the union of four branches. We denote them by \( S^+_a, S^-_a, S^+_e, S^-_e \) where a means attracting and \( r \) repelling with respect to the fast variables and + and − correspond to the sign of the \( y \)-variable, see also [16, figure 1]. We denote the corresponding slow manifolds for small \( \varepsilon > 0 \) by \( S^+_r, S^-_r, S^+_e, S^-_e \). We focus on the fate of \( S^+_e \), when it is continued through a neighbourhood of \((0, 0)\). For that purpose, we fix \( \rho > 0 \) and let \( J \) be a small open interval around 0 in \( \mathbb{R} \), potentially depending on \( \varepsilon \). Then one can define

\[
\Delta^\text{in} := \{(-\rho, y), y + \rho \in J\}, \quad \Delta^\text{out} = \{-\rho, y), y - \rho \in J\}, \quad \Delta^\text{out}_e = \{(\rho, y), y \in J\}.
\]

If \( \Pi_a \) and \( \Pi_r \) denote the transition maps from \( \Delta^\text{in} \) to \( \Delta^\text{out}_a \) and \( \Delta^\text{out}_e \) respectively, we can formulate the main result on the transcritical singularity [16, theorem 2.1].

**Theorem 2.1.** Fix \( \lambda \neq 1 \) in equation (2.5). There exists \( \varepsilon_0 > 0 \) such that the following assertions hold for \( \varepsilon \in (0, \varepsilon_0) \).

\[(T1)\] If \( \lambda > 1 \), then the manifold \( S^+_{a,e} \) passes through \( \Delta^\text{out}_e \) at a point \((\rho, h(\varepsilon))\) where \( h(\varepsilon) = \mathcal{O}(\sqrt{\varepsilon}) \). The section \( \Delta^\text{in} \) is mapped by \( \Pi_a \) to an interval containing \( S^+_{a,e} \cap \Delta^\text{out}_e \) of size \( \mathcal{O}(\varepsilon^{-C/\varepsilon}) \), where \( C \) is a positive constant.

\[(T2)\] If \( \lambda < 1 \), then the section \( \Delta^\text{in} \) (including the point \( \Delta^\text{in} \cap S^+_{a,e} \)) is mapped by \( \Pi_a \) to an interval about \( S^+_{a,e} \) of size \( \mathcal{O}(\varepsilon^{-C/\varepsilon}) \), where \( C \) is a positive constant.

Considering \( \varepsilon \) as a variable, one can write the problem in three variables

\[
\begin{align*}
x' &= x^2 - y^2 + \lambda \varepsilon + \mathcal{R}_1(x, y, \varepsilon), \\
y' &= \varepsilon (1 + \mathcal{R}_2(x, y, \varepsilon)), \\
\varepsilon' &= 0.
\end{align*}
\]

The total derivative of the above vector field \( X \) in \((x, y, \varepsilon)\) has only zero eigenvalues at the origin \((x, y, \varepsilon) = (0, 0, 0)\). In particular, the origin is a non-hyperbolic equilibrium. To gain hyperbolicity at the singularity one can use the blow-up method (see e.g. [3–5, 15, 16]), which maps the equilibrium point to an entire manifold, on which the dynamics can be desingularized. The shortest proof of theorem 2.1 [16] uses the quasi-homogeneous blow-up method

\[
\begin{align*}
x = r \xi, \quad y = r \eta, \quad \varepsilon = r^2 \tau,
\end{align*}
\]

where \((\xi, \eta, \tau, r) \in B := S^2 \times [0, r_0] \) for some \( r_0 > 0 \). The transformation can be formalized as a map \( \Phi : B \to \mathbb{R}^3 \), where \( r_0 \) is small enough such that the dynamics on \( \Phi(B) \) can be described by the normal form approximation. The map \( \Phi \) induces a vector field \( \tilde{X} \) on \( B \) by \( \Phi_*(\tilde{X}) = X \),
where $Φ_*$ is the pushforward induced by $Φ$. Note that, since $Φ(B)$ is a full neighbourhood of the origin, it suffices to study the vector field $X$, which is analyzed in three different charts. We will now provide a similar analysis for the discrete-time analogue.

### 3. Transcritical singularity in discrete time

We can now turn to the main part, i.e. we want to analyze the discrete-time problem obtained via an explicit Euler method. For that purpose, we first set the higher order terms in (2.6), represented by $R_1$ and $R_2$, to zero. We introduce the step size $h > 0$ of the Euler method as an additional variable and obtain a map $P : \mathbb{R}^4 \to \mathbb{R}^4$, whose iterations $P^n(z_0)$, for $n \in \mathbb{N}$ and $z_0 \in \mathbb{R}^4$, we are going to analyze close to the origin with $h, ε > 0$:

$$
P : \begin{pmatrix} x \\ y \\ ε \\ h \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{ε} \\ \tilde{h} \end{pmatrix} = \begin{pmatrix} x + h(x^2 - y^2 + λε) \\ y + εh \\ ε \\ h \end{pmatrix}.
$$

(3.1)

As in continuous time, the dynamical system induced by (3.1) has the critical manifold

$$S_0 = \{ (x, y, ε, h) \in \mathbb{R}^4 : x^2 = y^2 \}.$$

We split the set $S_0$ into the four branches

$$
S_a^- = \{ x = y < 0 \}, \quad S_a^+ = \{ -x = y > 0 \}, \\
S_r^- = \{ -x = y < 0 \}, \quad S_r^+ = \{ x = y > 0 \}.
$$

(3.2)

By linearization we see that these four branches are normally hyperbolic as long as for $(x, y) \in S_0 \setminus \{(0, 0)\}$ we have

$$|1 + 2hx| \neq 1.
$$

(3.3)

Since we want to restrict the analysis to a neighbourhood of the non-hyperbolic singularity at the origin $(x, y) = (0, 0)$, we always assume that $h$ is chosen small enough so that (3.3) holds and the stability properties along $S_0$ are analogous to the time-continuous case. For example, for a fixed initial condition with $x_0, y_0 < 0$, we have to ensure $1 + 2hx_0 > -1$ which yields the restriction $h < 1/|x_0|$, which then implies that $S_a^-$ is normally hyperbolic and locally attracting. Note in particular that the requirement $1 > |1 + 2hx|$ resembles the stability criteria of the Euler method derived from the Dahlquist test equation [2], as it will also occur in the detailed analysis of chart $K_1$ later on.

Due to normal hyperbolicity and according to [12, theorem 4.1], for $ε$ sufficiently small, there exist corresponding invariant slow manifolds $S_{a.c.h}^-, S_{a.c.h}^+, S_{r.c.h}^-, S_{r.c.h}^+$. However, the four branches of the critical manifold $S_0$ intersect at the origin $(x, y) = (0, 0)$, where we have $D_xP(0, 0) = 1$, i.e. we observe the loss of normal hyperbolicity as in the ODE case. We want to investigate where points around $S_{a.c.h}^-$ get mapped to by iterations of $P$ in order to find the continuation of $S_{a.c.h}^-$ beyond the singularity.

Similarly to the problem in continuous time, we fix some $\rho > 0$ and let $J$ be a small open interval around 0 in $\mathbb{R}$. We define

$$
Δ_ι^ι = \{ (-\rho, y) : y + \rho \in J \}, \quad Δ_ι^ι = \{ (-\rho, y) : y - \rho \in J \},
$$

$$
Δ_ε^ι = \{ (\rho, y) : y - \rho \in J \}, \quad Δ_ε^ι = \{ (\rho, y) : y \in J \}.
$$
where \( \varepsilon \) and \( h \) are fixed as prescribed by the map \( P \); see also figure 1. In contrast with flows, the intervals \( \Delta^{m}_{\varepsilon} \), \( \Delta^{ou}_{\varepsilon} \) are not necessarily hit by \( P'(-\rho,y) \) for fixed \( y \in \Delta^{m} \) and some \( n > 0 \). Notice that we used an abbreviated notation \( P'(x,y,\varepsilon,h) = P'(x,y) \) for the map \( P \) and also for points in \( \Delta^{m} \) just denoting them by their \( y \)-coordinate. We define the transition maps from \( \Delta^{m} \) to the vicinity of \( \Delta^{m}_{\varepsilon} \) and \( \Delta^{ou}_{\varepsilon} \) by

\[
\Pi_{n}(y) = P^{n}(\varepsilon)(-\rho,y), \quad \text{where } n^{*}(y) = \arg \min_{n \in \mathbb{N}} \text{dist}(P^{n}(-\rho,y), \Delta^{m}_{\varepsilon}), \quad y \in \Delta^{m},
\]

(3.4)

\[
\Pi_{c}(y) = P^{c}(\varepsilon)(-\rho,y), \quad \text{where } m^{*}(y) = \arg \min_{n \in \mathbb{N}} \text{dist}(P^{n}(-\rho,y), \Delta^{ou}_{\varepsilon}), \quad y \in \Delta^{m},
\]

(3.5)

\[
\Pi_{c}(y) = P^{c}(\varepsilon)(-\rho,y), \quad \text{where } k^{*}(y) = \arg \min_{n \in \mathbb{N}} \text{dist}(P^{n}(-\rho,y), \Delta^{ou}_{\varepsilon}), \quad y \in \Delta^{m}.
\]

(3.6)

We can formulate the main result on the transcritical singularity in discrete time (see figure 1 for an illustration):

**Theorem 3.1.** Fix \( \lambda \in \mathbb{R} \) and \( \rho > 0 \) in equation (3.1). There exists \( \varepsilon_{0} > 0 \) such that the following assertions hold for all \( \varepsilon \in [0, \varepsilon_{0}] \) and \( h > 0 \) such that \( h\rho < 1/8, h\varepsilon \ll \rho \) and \( 0 < h\rho^{3} < \varepsilon \), and any \( 0 < \lambda < \rho \).

\[\text{(T1)} \] If \( \lambda < 1 \), then the section \( \Delta^{m} \) (including the point \( \Delta^{m} \cap S^{+}_{\varepsilon,h} \)) is mapped by \( \Pi_{n} \) to a set about \( S^{+}_{\varepsilon,h} \) of \( y \)-width \( O \left( (1 - c)^{1/2} \right) \), where \( C \) is a positive constant, such that every point in \( \Pi_{n} \left( \Delta^{m} \right) \) is \( O(h\varepsilon) \)-close to \( \Delta^{ou}_{\varepsilon} \).

\[\text{(T2)} \] If \( \lambda > 1 \), then the manifold \( S_{\varepsilon,h}^{+} \) passes through \( \Delta^{ou}_{\varepsilon} \) at a point \( (\rho, k(\varepsilon)) \) where \( k(\varepsilon) = O(\varepsilon^{1/3}) \). The section \( \Delta^{m} \) is mapped by \( \Pi_{c} \) to a set about \( S^{+}_{\varepsilon,h} \) of \( y \)-width \( O \left( (1 - c)^{1/2} \right) \), where \( C \) is a positive constant, such that every point in \( \Pi_{c} \left( \Delta^{m} \right) \) is \( O(h(\varepsilon + \rho^{2})) \)-close to \( \Delta^{ou}_{\varepsilon} \).

\[\text{(T3)} \] If \( \lambda = 1 \), then the manifold \( S_{\varepsilon,h}^{+} = S^{+}_{\varepsilon} \) is connected to \( S^{+}_{\varepsilon} \) via a canard solution such that \( \Pi_{c} \left( \Delta^{m} \cap S_{\varepsilon,h}^{+} \right) \in S^{+}_{\varepsilon,h} \). The whole section \( \Delta^{m} \) is mapped by \( \Pi_{c} \) to a set about \( S^{+}_{\varepsilon,h} \) of \( y \)-width \( O \left( (1 - c^{2})^{1/2} \right) \), where \( C \) is a positive constant, such that every point in \( \Pi_{c} \left( \Delta^{m} \right) \) is \( O(h\varepsilon) \)-close to \( \Delta^{ou}_{\varepsilon} \).

**Remark 3.2.** Note that in the general case of the normal form (2.5) including higher order terms, the value of \( \lambda \) close to 1 giving a canard changes with the value of \( \varepsilon \). For continuous time, this phenomenon is studied in detail for canards in folds in [15] and discussed for the transcritical case in [16, remark 2.2 and section 3]. Using a Melnikov computation, one may show the existence of a function \( \lambda_{c}(\varepsilon^{1/2}) \) with \( \lambda_{c}(0) = 1 \) such that for \( \lambda = \lambda_{c}(\varepsilon^{1/2}) \) the slow manifold \( S_{\varepsilon}^{+} \) extends to \( S_{\varepsilon}^{+} \) for sufficiently small \( \varepsilon \). In our case, we have simply \( \lambda_{c}(\varepsilon^{1/2}) \equiv 1 \). A treatment of the more general problem including higher order terms and an expansion of \( \lambda_{c}(\varepsilon^{1/2}) \) for the discrete-time context requires a discrete Melnikov computation, which is more complicated. Therefore, we are going to treat the general canard problem in the separate study [7].

Note carefully that theorem 3.1 includes precise requirements for the relationship between three parameters, i.e. \( h\varepsilon \ll \rho \) for following the dynamics for meaningful time-lengths and \( 0 < h\rho^{3} < \varepsilon \) as well as \( h\rho < 1/8 \) as consequences of the proof (see lemmas 3.5, 3.8 and 3.9),
which means that the choice of step size for the Euler scheme is crucial. Furthermore, note that the stability criterion \( h \rho < 1 \) following from (3.3) is not only a direct consequence of \( h \rho < 1/8 \) but could also be obtained from the conditions \( h \epsilon < \rho \) and \( 0 < h \rho^3 < \epsilon \). Since we only work in the normal form, the parameter \( \rho \) does not have to be small and can, for example, be chosen equal to 1 such that the requirements read \( h < 1/8, h \epsilon \ll 1 \) and \( 0 < h < \epsilon \). Our aim is to prove theorem 3.1 using the blow-up method for the problem in four variables and to track individual trajectories inside the slow manifolds. The canard case \( \lambda = 1 \) will be proven without use of the blow-up method, since \( S_{a,c,h} = S_{a}^\perp \) and \( S_{r,c,h}^+ = S_r^+ \) are given explicitly in this case and can easily be analyzed directly.

Figure 1. The sketches illustrate theorem 3.1. For \( \lambda < 1 \) (figure (a)), \( \lambda > 1 \) (figure (b)) and \( \lambda = 1 \) (figure (c)), the figures show the extension of the slow manifold \( S_{a,c,h} \) (black curve) for system (3.1) around the non-hyperbolic singularity at the origin and the trajectory of a point \( y \in \Delta^\text{in} \) (dotted curve) to \( \Pi_i(y) \) near \( \Delta_i^{\text{out}} \) for \( i = a, e, c \). The dashed lines show the branches of the critical manifold \( S_c \).
3.1. Blow-up transformation

We conduct the quasihomogeneous blow-up transformation
\[ x = \bar{r}x, \quad y = \bar{r}y, \quad \varepsilon = \bar{r}^2 \bar{\varepsilon}, \quad h = \bar{h}/\bar{r}, \]
where \((\bar{x}, \bar{y}, \bar{\varepsilon}, \bar{h}, \bar{r}) \in B := S^2 \times (0, h_0) \times (0, \rho)\) for some \(h_0, \rho > 0\). The change of variables in \(h\) is chosen such that the map is desingularized in the relevant charts (see [26, section 13.3.1] for a similar approach). We exclude \(0\) from the domain of \(h\) since at \(h = 0\) every point is a neutral fixed point. Due to the transformation \(h = \bar{h}/\bar{r}\) we have to exclude \(0\) from the domain of \(\bar{r}\) as well.

The whole transformation can be formalised as a map \(\Phi : B \to \mathbb{R}^4\). The map \(\Phi\) induces a map \(\bar{\Phi}\) on \(B\) by \(\Phi \circ \bar{\Phi}^{-1} = P\). Analogously to the continuous time case, we are using the charts \(K_i, i = 1, 2, 3\), to describe the dynamics. The chart \(K_1\) focuses on the entry of trajectories for any value of \(\lambda\) and the exit of trajectories for \(\lambda < 1\), and is given by
\[
\begin{align*}
x &= -r_1, \quad y = r_1 y_1, \quad \varepsilon = r_1^2 \varepsilon_1, \quad h = h_1/r_1.
\end{align*}
\] (3.7)
In the scaling chart \(K_2\) the dynamics arbitrarily close to the origin are analyzed. It is given via the mapping
\[
\begin{align*}
x &= r_2 x_2, \quad y = r_2 y_2, \quad \varepsilon = r_2^2 \varepsilon_3, \quad h = h_2/r_2.
\end{align*}
\] (3.8)
The exit chart \(K_3\) plays a role for the dynamics emerging from a neighbourhood of the origin for \(\lambda > 1\) and is given by
\[
\begin{align*}
x &= r_3, \quad y = r_3 y_3, \quad \varepsilon = r_3^2 \varepsilon_3, \quad h = h_3/r_3.
\end{align*}
\] (3.9)
There are four relevant changes of coordinates between the charts. The map \(k_{12} : K_1 \to K_2\) is given by
\[
\begin{align*}
x_2 &= -\varepsilon_1^{-1/2}, \quad y_2 = \varepsilon_1^{-1/2} y_1, \quad r_2 = \varepsilon_1^{1/2} r_1, \quad h_2 = \varepsilon_1^{1/2} h_1,
\end{align*}
\] (3.10)
\(k_{31} : K_2 \to K_1\) is given by
\[
\begin{align*}
\varepsilon_1 &= x_3^{-2}, \quad y_1 = -x_3^{-1} y_2, \quad r_1 = -x_3 r_2, \quad h_1 = -x_3 h_2,
\end{align*}
\] (3.11)
\(k_{32} : K_3 \to K_2\) is given by
\[
\begin{align*}
x_2 &= \varepsilon_3^{-1/2}, \quad y_2 = \varepsilon_3^{-1/2} y_3, \quad r_2 = \varepsilon_3^{1/2} r_3, \quad h_2 = \varepsilon_3^{1/2} h_3,
\end{align*}
\] (3.12)
and \(k_{33} : K_2 \to K_3\) is given by
\[
\begin{align*}
\varepsilon_3 &= x_3^{-2}, \quad y_3 = x_3^{-1} y_2, \quad r_3 = x_3 r_2, \quad h_3 = x_3 h_2.
\end{align*}
\] (3.13)

3.2. Dynamics in the chart \(K_1\)

We choose \(\delta > 0\) small such that \(|\lambda \delta| \leq 1\), to be determined later in more detail which will also determine \(\varepsilon_0 = \rho^2 \delta\). Furthermore, we assume \(\nu := \rho h < \delta\) for fixed \(h \in (0, h_0]\). We are interested in trajectories entering \(B\) at \(\bar{r} = \rho\) which is best analyzed in the entering chart \(K_1\). At \(\bar{r} = \rho\) we have \(h_1 = \nu\). We investigate the dynamics within the domain
\[
D_1 := \{(r_1, y_1, \varepsilon_1, h_1) \in \mathbb{R}^4 : r_1 \in [0, \rho], \varepsilon_1 \in [0, 2\delta], h_1 \in [0, \nu]\}.
\]
Note that we have to bound \(h_1\) from below since for \(h_1 = 0\) everything is fixed and it is helpful to choose a uniform bound to get estimates on the contraction rates. A suitable choice
is $h_1 \geq \nu/2$. The proportionality $h = h_1/r_1$ implies that $r_1 \geq \rho/2$. Furthermore, we want to see what happens for $\varepsilon_1 = \delta$. Due to the invariant relation $\varepsilon_1 r_1 h_1 = \varepsilon h$, this implies taking $\varepsilon_1 \geq \delta/4$. These considerations lead to introducing the subdomain $D_1 \subset D_1$ which is given as

$$D_1 := \{(r_1, y_1, \varepsilon_1, h_1) \in \mathbb{R}^4 : r_1 \in [\rho/2, \rho], \varepsilon_1 \in [\delta/4, \delta], h_1 \in [\nu/2, \nu]\}.$$

We will later restrict $y_1$ to obtain a small neighbourhood of $\Delta^\delta$ as entering domain. To derive the blown-up map we calculate

$$\tilde{r}_1 = -\tilde{x} = -x - h(x^2 - y^2 + \lambda \varepsilon)$$

$$= r_1 - \frac{h_1}{r_1}(r_1^2 - r_1^2 y_1^2 + \lambda r_1^2 \varepsilon_1)$$

$$= r_1 (1 - h_1 (1 - y_1^2 + \lambda \varepsilon_1)).$$

Similarly, we can derive the maps for the other variables in chart $K_1$ leading to the following dynamics, desingularised due to $h = h_1/r_1$:

$$\tilde{r}_1 = r_1 (1 - h F_1(y_1, \varepsilon_1)),$$

$$\tilde{y}_1 = (y_1 + \varepsilon_1 h_1)(1 - h F_1(y_1, \varepsilon_1))^{-1},$$

$$\tilde{\varepsilon}_1 = \varepsilon_1 (1 - h F_1(y_1, \varepsilon_1))^{-2},$$

$$\tilde{h}_1 = h_1 (1 - h F_1(y_1, \varepsilon_1)), \quad (3.14)$$

where $F_1(y_1, \varepsilon_1) = 1 - y_1^2 + \lambda \varepsilon_1$. Now we have to analyze the dynamics of (3.14) in detail. For any $h_1 \in [0, \nu]$ system (3.14) has the fixed points.
\[ v_{a,1}^-(h_1) = (0, -1, 0, h_1), \quad v_{a,1}^+(h_1) = (0, 1, 0, h_1). \]

The points \( v_{a,1}^- \) and \( v_{a,1}^+ \) have a three-dimensional centre eigenspace and a one-dimensional eigenspace spanned by \((0, 1, 0, 0)\) with the eigenvalue \( \lambda_1 = 1 - 2h_1 \), which is stable as long as \( h_1 < 1 \). Note that the set

\[ \{w^m(h_1) := (0, 0, 0, h_1) : h_1 \in [0, \nu] \} \]

is an invariant set for system (3.14) within \( D_1 \). The points \( w^m(h_1) \) have two stable and two unstable eigenvalues

\[ \lambda_1 = 1 - 2h_1, \quad \lambda_2 = 1 - h_1, \quad \lambda_3 = (1 - h_1)^{-1}, \quad \lambda_4 = (1 - h_1)^{-2}, \]

such that again the stability depends on \( h_1 \) and is analogous to the time-continuous case, if \( h_1 < 1 \). The eigenvalues \( \lambda_1, \lambda_2 \) correspond with the \( h_1 \)- and \( r_1 \)-directions and \( \lambda_3, \lambda_4 \) with the \( y_1 \)-and \( \varepsilon_1 \)-directions. Moreover, we remark that we can re-interpret the stability conditions to obtain the same behaviour as in the continuous time case, such as

\[ 1 > |\lambda_1| = |1 - 2h_1|, \quad (3.15) \]

precisely as the stability criteria of the Euler method derived from the Dahlquist test equation [2] within each eigenspace of the continuous-time blow-up problem in chart \( K_1 \).

We observe that the two-dimensional planes

\[ S_{a,1}^\pm = \{(r_1, y_1, \varepsilon_1, h_1) \in D_1 : y_1 = \pm 1, \ \varepsilon_1 = 0\} \]

are invariant manifolds of \( D_1 \) only consisting of fixed points, attracting in the \( y_1 \)-direction and neutral in the other directions. One can extend these manifolds \( S_{a,1}^\pm \) to center-stable invariant manifolds \( M_{a,1}^\pm \) (see figure 2), which are given in \( D_1 \) by graphs \( y_1 = l_{\pm}(\varepsilon_1, h_1) \) for mappings \( l_{\pm} \). We can derive \( l_{\pm} \) from the discrete invariance equation

\[ l_{\pm}(\tilde{\varepsilon}_1, \tilde{h}_1) = \frac{l_{\pm}(\varepsilon_1, h_1) + \varepsilon_1 h_1}{1 - h_1 F_1(l_{\pm}(\varepsilon_1, h_1), \varepsilon_1)}. \quad (3.16) \]

Solving this equation allows us to make the following statement.

**Proposition 3.3.** *Equation (3.16) has the solutions*

\[ l_{-}(\varepsilon_1, h_1) = -1 + \frac{1 - \lambda}{2} \varepsilon_1 + \mathcal{O}(\varepsilon_1^2 h_1), \quad (3.17) \]

\[ l_{+}(\varepsilon_1, h_1) = 1 + \frac{1 + \lambda}{2} \varepsilon_1 + \mathcal{O}(\varepsilon_1^2 h_1) \quad (3.18) \]

*which characterize \( M_{a,1}^- \) and \( M_{a,1}^+ \) respectively. Furthermore, \( \varepsilon_1 \) is increasing on \( M_{a,1}^- \) and decreasing on \( M_{a,1}^+ \), whereas \( h_1, r_1 \) are decreasing on \( M_{a,1}^- \) and increasing on \( M_{a,1}^+ \).*

**Proof.** It is easy to derive that for \( l_{-}(\varepsilon_1, h_1) \) given by (3.17), we have

\[ F_1(l_{-}(\varepsilon_1, h_1), \varepsilon_1) = \varepsilon_1 + \mathcal{O}(\varepsilon_1^2) \quad (3.19) \]

Hence, we observe that

\[ \tilde{\varepsilon}_1 = \varepsilon_1(1 - h_1 F_1(l_{-}(\varepsilon_1, h_1), \varepsilon_1))^{-2} = \varepsilon_1 + \mathcal{O}(\varepsilon_1^2 h_1) \]
and

\[ \tilde{h}_1 = h_1 + \mathcal{O}(h_1^2 \varepsilon_1). \]

Therefore, we deduce that

\[
\frac{I_-(\varepsilon_1, h_1) + \varepsilon_1 h_1}{1 - h_1\hat{F}(I_-(\varepsilon_1, h_1), \varepsilon_1)} = (I_-(\varepsilon_1, h_1) + \varepsilon_1 h_1)(1 + h_1\varepsilon_1 + \mathcal{O}(\varepsilon_1^2 h_1))
= I_-(\varepsilon_1, h_1) - h_1\varepsilon_1 + \frac{1 - \lambda}{2} \varepsilon_1^2 h_1 + \varepsilon_1 h_1 + \mathcal{O}(\varepsilon_1^2 h_1)
= I_-(\varepsilon_1, h_1) + \mathcal{O}(\varepsilon_1^2 h_1) = -1 + \frac{1 - \lambda}{2} \varepsilon_1 + \mathcal{O}(\varepsilon_1^2 h_1) = I_-(\tilde{\varepsilon}_1, \tilde{h}_1),
\]

which shows the claim for \( I_-(\varepsilon_1, h_1) \). Since we can assume that \( h_1 \varepsilon_1 < 1 \), the dynamics on \( M_{a,1}^+ \) follow as stated. Similarly we can derive that for \( I_+(\varepsilon_1, h_1) \) given by \((3.18)\), we have

\[ F_1(I_+(\varepsilon_1, h_1)) = -\varepsilon_1 + \mathcal{O}(\varepsilon_1^2). \]

The statements then follow analogously to before. \( \square \)

For all trajectories, as explained above, we have to consider the entry region

\[ \Sigma_{1,-} := \{(r_1, y_1, \varepsilon_1, h_1) \in D_1 : r_1 = \rho, h_1 = \nu, \varepsilon_1 = \delta/4\}. \]

Before exiting \( \hat{D}_1 \) for the first time, the dynamics must reach the set

\[ \Sigma_{1,-}^{\text{out}} = \{(r_1, y_1, \varepsilon_1, h_1) \in \mathbb{R}^4 : \frac{\rho}{2} \leq r_1 \leq \frac{\rho}{2}(1 + \nu), \frac{\nu}{2} \leq h_1 \leq \frac{\nu}{2}(1 + \nu), \delta(1 - 2\nu) \leq \varepsilon_1 \leq \delta\}, \]

since \( F_1(y_1, \varepsilon_1) \leq 2 \). Next, we want to find a set \( R \subset \Sigma_{1,-}^{\text{out}} \) such that \( M_{a,1}^- \cap \Sigma_{1,-}^{\text{out}} \subset R \) and there is a well-defined map \( \Pi_{1,-} : R \to \Sigma_{1,-}^{\text{out}} \) that maps points in \( R \) along a trajectory of \((3.14)\) to a first entry point in \( \Sigma_{1,-}^{\text{out}} \). By proposition \( 3.3 \), this is feasible for \( R \) small enough such that trajectories through \( R \) stay sufficiently close to \( M_{a,1}^- \) in the first part of the passage in \( K_1 \).

We choose \( R \) to be the interval

\[ R_1 := \{(r_1, y_1, \varepsilon_1, h_1) \in \Sigma_{1,-}^{\text{in}} : -1 - \beta_1(\lambda) \leq y_1 \leq -1 + \beta_1(\lambda)\} \]

with, for example,

\[
\beta_1 := |\lambda - 1| \delta, \quad \beta_1(\lambda) := \begin{cases} \frac{\lambda}{15} \delta & \text{if } 0 < \lambda < 1 \\ \frac{2 \lambda - 1}{15} \delta & \text{otherwise}. \end{cases}
\]

Note that with these choices we have \( M_{a,1}^- \cap \Sigma_{1,-}^{\text{in}} \subset R_1 \) for \( \nu, \delta \) sufficiently small. Furthermore, these choices guarantee that the trajectories stay close to \( M_{a,1}^- \) such that \( F_1(y_1, \varepsilon_1) \) is positive, and, hence, we can formulate the following proposition (see figure 2).

**Proposition 3.4.** Trajectories in \( \hat{D}_1 \) starting in \( R_1 \) are increasing in \( \varepsilon_1 \) and decreasing in \( h_1, r_1 \). Hence, the transition map \( \Pi_{1,-} : R_1 \to \Sigma_{1,-}^{\text{out}} \) is well-defined.

**Proof.** It is enough to show that in this case \( F_1(y_1, \varepsilon_1) \) is positive. If \( \lambda \geq 1 \) or \( \lambda \leq 0 \), we observe that \( \beta_1(\lambda) = \frac{2 \lambda - 1}{15} \delta \) implies \( F_1 \geq \frac{\delta - \mathcal{O}(\delta^3)}{8} \). If \( 0 < \lambda < 1 \), we have

\[ 2375 \]
\[ F_1(y_1, \varepsilon_1) \geq 1 - \left( -1 - \frac{\lambda \delta}{16} \right)^2 + \lambda \varepsilon_1 \geq \frac{\lambda}{8} \delta - \left( \frac{\lambda}{16} \right)^2 \delta^2. \]

Together with the considerations above, we can conclude the claim. \(\square\)

We can make the following statement about the transition time from \(R_1\) to \(\Sigma^\omega_1\) which will be crucial for estimates on the contraction close to \(\mathcal{M}^\omega_1\). Define \(\gamma := 2 |\lambda - 1| + |\lambda|\) and assume without loss of generality that \(\nu < \frac{1}{2}\).

**Lemma 3.5.** The transition time \(N\) of system (3.14) from a point \(p = (\rho, y_1, \delta/4, \nu)\) in \(R_1\) to the point \(\Pi_{1,\nu}(p)\) in \(\Sigma^\omega_1\) satisfies
\[ N \geq \frac{1}{17\gamma \nu \delta}. \]

**Proof.** Let \((\varepsilon_1(n))_{n \in \mathbb{N}}\) denote the trajectory starting at \(\varepsilon_1(0) = \delta/4\) with
\[ \varepsilon_1(n + 1) = \varepsilon_1(n)(1 - h_1(n)F_1(y_1(n), \varepsilon_1(n)))^{-2}. \]

We can show by induction that for all \(n \in \mathbb{N}\) such that \(\varepsilon_1(n) \leq \delta\) we have
\[ \varepsilon_1(n) \leq \frac{\delta}{4} + n \left( 2\gamma \nu \delta^2 + f(\nu, \delta) \right), \quad (3.22) \]
where \(f(\nu, \delta) = \mathcal{O}(\nu \delta^3)\) does not depend on \(n\). In more detail, we observe that
\[
\begin{align*}
h_1(n)F_1(y_1(n), \varepsilon_1(n)) &\leq h_1(n) \left[ 1 - (-1 + |\lambda - 1| \delta)^2 + \lambda \varepsilon_1(n) \right] \\
&\leq \nu \left[ (2 |\lambda - 1| \delta + |\lambda| \delta) - (\lambda - 1)^2 \delta^2 \right] \\
&= \nu \gamma \delta - \nu (\lambda - 1)^2 \delta^2.
\end{align*}
\]
Hence, we conclude with a first order Taylor approximation that for some \(g(\nu, \delta) = \mathcal{O}(\nu \delta^2)\) we have
\[
\begin{align*}
\varepsilon_1(1) &\leq \frac{\delta}{4} (1 + \nu \gamma \delta + g(\nu, \delta))^2 = \frac{\delta}{4} + \nu \gamma \delta^2 + \frac{\delta}{4} (2g(\nu, \delta) + g(\nu, \delta)^2 + \nu^2 \gamma^2 \delta^2 + 2g(\nu, \delta) \nu \gamma \delta) \\
&\leq \frac{\delta}{4} + 2\gamma \nu \delta^2 + f(\nu, \delta),
\end{align*}
\]
where \(f(\nu, \delta) = \delta(2g(\nu, \delta) + \mathcal{O}(\nu^2 \delta^2)) = \mathcal{O}(\nu \delta^3)\). Similarly, the step from \(n\) to \(n + 1\) can be written as
\[
\begin{align*}
\varepsilon_1(n + 1) &\leq \varepsilon_1(n) \left( 1 + \nu \gamma \delta + g(\nu, \delta) \right)^2 \leq \varepsilon_1(n) + 2\gamma \nu \delta \varepsilon_1(n) + f(\nu, \delta), \\
&\leq \frac{\delta}{4} + n \left( 2\gamma \nu \delta^2 + f(\nu, \delta) \right) + 2\gamma \nu \delta^2 + f(\nu, \delta), \\
&= \frac{\delta}{4} + (n + 1) \left( 2\gamma \nu \delta^2 + f(\nu, \delta) \right).
\end{align*}
\]
This shows (3.22) for all \(n \in \mathbb{N}\) such that \(\varepsilon_1(n) \leq \delta\). We can rewrite the right hand side of (3.22), using a geometric series, as
\[ \frac{\delta}{4} + n \left( 2\gamma \nu \delta^2 + f(\nu, \delta) \right) = \frac{\delta}{1 - n \left( 8\gamma \nu \delta + \tilde{f}(\nu, \delta) \right)}, \]

where \( \tilde{f}(\nu, \delta) = O(\nu \delta^2) \). By definition of the transition time \( N \) we have \( \varepsilon_1(N) \geq \delta(1 - 2\nu) \). Hence, we deduce that

\[ \delta(1 - 2\nu) \leq \frac{\delta}{1 - N(8\gamma \nu \delta + \tilde{f}(\nu, \delta))}, \]

and therefore

\[ \delta \left(1 - 2\nu - \frac{1}{4}\right) \leq N \delta(1 - 2\nu)(8\gamma \nu \delta + \tilde{f}(\nu, \delta)). \]

Finally, for \( \delta \) sufficiently small and due to \( \nu < \frac{1}{8} \), this leads to

\[ N \geq \frac{\delta}{\delta(1 - 2\nu)(8\gamma \nu \delta + \tilde{f}(\nu, \delta))} \geq \frac{1}{17\gamma \nu \delta}, \]

which concludes the proof.

In addition to the first passage moving up the sphere, we already anticipate that for \( \lambda < 1 \) trajectories eventually re-enter \( K \) from \( K_2 \). With more precision to be added after the analysis in chart \( K_2 \), we define

\[ \Sigma_{1,+}^{\text{in}} := \{(r_1, y_1, \varepsilon_1, h_1) \in D_1 : |\varepsilon_1 - \delta| \text{ small}, |r_1 - \rho| \text{ small}, |h_1 - \nu| \text{ small}\}, \]

\[ \Sigma_{1,+}^{\text{out}} := \{(r_1, y_1, \varepsilon_1, h_1) \in D_1 : r_1 = \rho, h_1 = \nu, \varepsilon_1 = \delta/4, y_1 > 0\}, \]

and denote by \( \Pi_{1,+} : \Sigma_{1,+}^{\text{in}} \to \Sigma_{1,+}^{\text{out}} \) the map that sends points in \( \Sigma_{1,+}^{\text{in}} \) along a trajectory of (3.14) to the point of this trajectory, which is closest to \( \Sigma_{1,+}^{\text{out}} \). Note that \( \Pi_{1,+} \) is well-defined sufficiently close to \( M_{1,+}^\uparrow \) according to proposition 3.3. In more detail, for \( \beta_{1,+}^\uparrow \) and \( \beta_{1,+}^\downarrow \) to be determined more precisely in the analysis of chart \( K_2 \), there is

\[ R_2 := \left\{ (r_1, y_1, \varepsilon_1, h_1) \in \Sigma_{1,+}^{\text{in}} : 1 - \beta_{1,+}^\downarrow \leq y_1 \leq 1 + \beta_{1,+}^\uparrow \right\}, \]

(3.23)

such that \( M_{1,+}^\uparrow \cap \Sigma_{1,+}^{\text{in}} \subset R_2 \) and \( \Pi_{1,+} \) is well-defined on \( R_2 \); see also figure 2. Of course, a completely analogous result for the passage time as stated in proposition 3.5 also holds for the map \( \Pi_{1,+} \). We can use the lower bounds on the transition times to find the following lower bounds for the contraction rates of \( \Pi_{1,+}|_{R_1} \) and \( \Pi_{1,+}|_{R_2} \).

**Proposition 3.6.** There are constants \( K_1, K_2 > 0 \) such that for any \( c \) with \( 0 < c < \nu = \phi h \)

1. the map \( \Pi_{1,+}|_{R_1} \) is a contraction (in the \( y_1 \)-direction) with a rate stronger than

\[ K_1 (1 - c)^{\frac{1}{\phi h}} \]

2. the map \( \Pi_{1,+}|_{R_2} \) is a contraction (in the \( y_1 \)-direction) with a rate stronger than

\[ K_2 (1 - c)^{\frac{1}{\phi h}}. \]
Proof. The statement about $\Pi_{1,-}$ follows from lemma 3.5 and the fact that the stable eigenvalue at the fixed points in $S_{a,1}^+ \subset M_{a,1}$ is given by $1 - 2\epsilon_1 \leq 1 - \nu$, in combination with standard perturbation arguments.

The estimate for $\Pi_{1,+}$ uses the symmetry of system (3.14) with respect to the dynamics around $M_{a,1}^-$ and $M_{a,1}^+$; the transition time from $\Sigma_{a,1}^-$ to $\Sigma_{a,1}^+$ is of the same order as the transition time from $\Sigma_{a,1}^-$ to $\Sigma_{a,1}^+$, and the eigenvalues at $S_{a,1}^+$ are the same as at $S_{a,1}^-$. □

The graphs of the invariant manifolds, as given in (3.17) and (3.18), are up to leading order the same as in the continuous-time case. Yet, the transition time $N$, as estimated from below in lemma 3.5, is of order $O\left(\frac{1}{\rho\varepsilon_1}\right)$ compared to $O\left(\frac{1}{\varepsilon_1}\right)$ in continuous time (see [16, proof of theorem 2.1]). This can be seen from a lower bound analogue to (3.22) and a consequential analogous upper bound $N \leq C\frac{1}{\rho\varepsilon_1}$ for some $C > 0$, recalling that between $\Sigma_{a,1}^-$ and $\Sigma_{a,1}^+$ we have $h_1 \in [\frac{\rho}{2}, \nu], \varepsilon_1 \in [\frac{\rho}{2}, \delta]$.

Further note that, while the proof in the ODE case is immediate, as can be seen from [15, lemma 2.7] which [16, proof of theorem 2.1] refers to as analogous, the proof of lemma 3.5 requires additional estimates on the iterated maps. This analysis allows us to conclude proposition 3.6, giving explicit estimates on the contraction rates of the transition maps.

3.3. Dynamics in the scaling chart $K_2$

We turn to analyzing the dynamics in the scaling chart $K_2$ in order to understand the behaviour of trajectories past the origin. The chart $K_2$ covers the upper part of the sphere, where we can desingularize with respect to $\varepsilon$. Recall from (3.10) that the change of coordinates from $K_1$ to $K_2$ is given by $k_{12}: K_1 \to K_2$ with

$$x_2 = -\varepsilon_1^{-1/2}, \quad y_2 = \varepsilon_1^{-1/2} y_1, \quad r_2 = \varepsilon_1^{1/2} r_1, \quad h_2 = \varepsilon_1^{1/2} h_1.$$

It becomes clear from this transformation that the set of interest can be restricted to

$$D_2 := \left\{(x_2, y_2, r_2, h_2) \in \mathbb{R}^4 : \frac{\delta^{1/2}\rho}{2} \leq r_2 \leq \delta^{1/2}\rho, \frac{\delta^{1/2}\rho}{2} \leq h_2 \leq \delta^{1/2}\rho\right\}.$$

First of all, we need to make sure that $\kappa_{1,2}(\Pi_{1,-}(R_1)) \subset \Sigma_{a,2}^+$ for the entering set $\Sigma_{a,2}^+$. From the analysis in $K_1$ we derive that this is satisfied for

$$\Sigma_{a,2}^+ := \left\{(x_2, y_2, r_2, h_2) \in D_2 : -(\delta(1 - 2\nu))^{-1/2} \leq x_2 \leq -\delta^{-1/2}, \right.$$ 

$$\left. \delta^{-1/2}(-1 - \hat{\beta}_2(\lambda)) \leq y_2 \leq \delta^{-1/2}(-1 + \hat{\beta}_2)\right\},$$

where

$$\hat{\beta}_2 := |\lambda - 1|, \quad \hat{\beta}_2(\lambda) := \begin{cases} \frac{\lambda}{4}\delta & \text{if } 0 < \lambda < 1 \\ \frac{2\lambda - 1}{4}\delta & \text{otherwise}. \end{cases} \quad (3.25)$$

We derive the desingularized equations and thereby justify the choice of blow-up in $h$. Observe that $r_2 = r_2$ since $\varepsilon = \varepsilon$ and $\varepsilon = r_2^2$. Similarly, we have $h_2 = h_2$. Furthermore observe that

$$h_2 = h_2.$$
In addition to that, we obtain

\[
\tilde{x}_2 = \frac{\tilde{x}}{r_2} = \frac{r_2 x_2 + h_2 r_2^{-1}(r_2^2 x_2^2 - r_2^2 y_2^2 + \lambda r_2^2)}{r_2} = x_2 + h_2(x_2^2 - y_2^2 + \lambda).
\]

Hence, summarising, the dynamics in chart $K_2$ are given by iterating the map

\[
\begin{align*}
\tilde{x}_2 &= x_2 + h_2(x_2^2 - y_2^2 + \lambda), \\
\tilde{y}_2 &= y_2 + h_2, \\
\tilde{r}_2 &= r_2, \\
\tilde{h}_2 &= h_2.
\end{align*}
\]  

(3.26)

The transition areas from $K_2$ to another chart depend on $\lambda$. For $\lambda < 1$, we will return to chart $K_1$. Recall from (3.11) that the change of coordinates $k_{21} : K_2 \to K_1$ is given by

\[
\begin{align*}
\varepsilon_1 &= x_2^2, \\
y_1 &= -x_2^{-1}y_2, \\
r_1 &= -x_2 r_2, \\
h_1 &= -x_2 h_2.
\end{align*}
\]

We need to choose $\Sigma_{2,a}^\text{out}$ and the cuboid $R_2$ in $\Sigma_{1,e}^\text{in}$ in chart $K_1$ (see (3.23)) such that, firstly, trajectories starting in $\Sigma_{2,a}^\text{in}$ reach $\Sigma_{2,a}^\text{out}$ and, secondly, $k_{21}(\Sigma_{2,a}^\text{in}) \subset R_2$. It turns out (see proof of proposition 3.7 for the first criterion) that a suitable choice is given by

\[
\Sigma_{2,a}^\text{out} := \left\{ (x_2, y_2, r_2, h_2) \in D_2 : -\delta^{-1/2} - \frac{h_2}{2} \leq x_2 \leq -\delta^{-1/2} + \frac{h_2}{2}, \right. \\
\left. \delta^{-1/2}(1 - \tilde{\beta}_2^+) \leq y_2 \leq \delta^{-1/2}(1 + \tilde{\beta}_2^+) \right\},
\]  

(3.27)

where we define $\beta_2^+ := \frac{|\lambda + 1|}{2}\delta$ and $\tilde{\beta}_2^+ := \frac{|\lambda| + 1}{2}\delta$; see also figure 3. The second criterion is then satisfied by adapting $\Sigma_{2,e}^\text{in}$ in the $(\varepsilon_1, r_1, h_1)$-components accordingly via $k_{21}$ and choosing, for example, $\beta_1^+ := \frac{2|\lambda + 1|}{4}\delta$ and $\tilde{\beta}_1^+ := \frac{3|\lambda| + 1}{4}\delta$ in the definition of $R_2$.

For $\lambda > 1$, we set the area of exit as

\[
\Sigma_{2,e}^\text{out} := \{ (x_2, y_2, r_2, h_2) \in D_2 : \delta^{-1/2} \leq x_2 \leq \delta^{-1/2} + h_2(\lambda + \delta^{-1}), \\
0 \leq y_2 < \Omega(\delta^{-1/6}) \},
\]  

(3.28)

where $\Omega(\lambda) > 0$ is a constant for fixed $\lambda$; see also figure 3. In the situation of continuous time, the $y_2$-component in $\Sigma_{2,e}^\text{out}$ can be bounded by a constant independent from $\delta$, by using the Riccati equation [15, proposition 2.3]. As we do not have such a tool in the case of maps, we give an estimate from a first order expansion in $h$ of the iterated maps (see proof of proposition 3.7).

Recall from chart $K_1$ the attracting center manifolds $M_{a,1}^\pm$ (see figure 2). Similarly to [16], these manifolds correspond with the global manifolds $M_{a}^\pm$ on the blow-up manifold $B$. In chart $K_2$ we therefore have the attracting center manifolds $M_{a,2}^\pm = k_{21}(M_{a,1}^\pm)$ (with, reversely, $M_{a,1}^\pm = k_{21}(M_{a,2}^\pm)$), whose behaviour is described in the following.

Let us denote the sequence induced by iterating (3.26) for some initial condition $(x_0, y_0)$ as $(x_2(n), y_2(n))$ for $n \in \mathbb{N}$, and call such a sequence a trajectory. As for continuous time, the special case is the canard problem, i.e. when $\lambda = 1$. In this case, for any $\varepsilon = x_0 = y_0 = 0 \in \mathbb{R}$ the system of maps has the obvious solution $\gamma_2^\varepsilon(n)$ with $x_2(n) = y_2(n) = \varepsilon + nh_2$. For $\lambda \neq 1$ we
can make the following direct observations about the dynamics of the maps, where we recall that \( \nu := \rho h < \delta \), in particular \( \nu < \frac{1}{4} \).

**Proposition 3.7.** The following results hold:

(P1) If \( \lambda < 1 \), every trajectory starting in \( \Sigma_{2}^{\text{in}} \) passes through \( \Sigma_{2}^{\text{out}} \), and, hence, so does \( M_{4}^{-} \).

(P2) If \( \lambda > 1 \), every trajectory starting in \( \Sigma_{2}^{\text{in}} \) passes through \( \Sigma_{2}^{\text{out}} \), and, hence, so does \( M_{4}^{+} \).

The proof of this proposition is based on a couple of lemmas, which are shown in the following. We divide the diagonals \( \{x = y\} \) and \( \{x = -y\} \) into the subsets

\[
S_{a,2}^{-} := \{(x, y) \in \mathbb{R}^2 : y \leq 0, x = y\}, \quad S_{a,2}^{+} := \{(x, y) \in \mathbb{R}^2 : y > 0, x = y\}
\]

and

\[
S_{a,2}^{−} := \{(x, y) \in \mathbb{R}^2 : y \leq 0, x = -y\}, \quad S_{a,2}^{+} := \{(x, y) \in \mathbb{R}^2 : y > 0, x = y\}.
\]

Furthermore, we write as a shorthand \( x_{2,n} = x_{2}(n), y_{2,n} = y_{2}(n) \) for \( n \in \mathbb{N} \) and investigate the behaviour of the trajectories

\[
y_{2,n+1} = y_{2,n} + h_{2},
\]

\[
x_{2,n+1} = x_{2,n} + h_{2} \lambda + h_{2}(x_{2,n}^2 - y_{2,n}^2)
\]

for different values of \( \lambda \). In fact, even for \( \lambda < 1 \), there are subtle differences in the paths of trajectories (see figure 3).

**Lemma 3.8.** The following cases occur for \( \lambda < 1 \):

- Let \( 0 < \lambda < 1 \) and \( \delta \) be sufficiently small. Then any trajectory starting in \( \Sigma_{2}^{\text{in}} \) is strictly increasing in \( x \) as long as \( x_{2,n}, y_{2,n} < 0 \), and will be above the diagonal \( \{x = y\} \) at a certain point of time and stay there forever afterwards. In particular, if \( (x_{2,0}, y_{2,0}) \) in \( \Sigma_{2}^{\text{in}} \) with \( y_{2,0} < x_{2,0} < 0 \), there exists \( n^* \in \mathbb{N} \) such that \( x_{2,n}^{*} \leq y_{2,n}^{*} < 0 \) and

\[
\frac{n^* h_{2}}{n^* h_{2} + \frac{\delta}{\sqrt{2}}} \geq \lambda.
\]

- If \( \lambda \leq 0 \), any trajectory starting in \( \Sigma_{2}^{\text{in}} \) is strictly increasing in \( x \) as long as \( x_{2,n}, y_{2,n} < 0 \) and will be above the diagonal \( \{x = y\} \) at any point of time.

**Proof.** We start with the case \( 0 < \lambda < 1 \). Consider an initial condition \( (x_{2,0}, y_{2,0}) \) in \( \Sigma_{2}^{\text{in}} \) with \( y_{2,0} < x_{2,0} < 0 \), i.e. below \( S_{a,2}^{-} \). We obviously have \( x_{2,1} < x_{2,0} + \lambda h_{2} \). Furthermore, observe that

\[
x_{2,0}^2 - y_{2,0}^2 + \lambda \geq \delta^{-1} - (\delta^{-1/2}(-1 - \beta_{2}(\lambda)))^2 + \lambda
\]

\[
= \delta^{-1} - \delta^{-1} - \frac{1}{4\lambda} - \frac{1}{64} \lambda^2 \delta + \lambda
\]

\[
= \frac{3}{4} \lambda - \frac{1}{64} \lambda^2 \delta \geq \frac{1}{2} \lambda.
\]

Hence, \( x_{2,1} \geq x_{2,0} + \frac{1}{2} h_{2} \). Either we already have \( x_{2,1} \leq y_{2,1} < 0 \). If not, we can infer from the facts \( 0 > y_{2,1} > y_{2,0}, 0 > x_{2,1} > x_{2,0} \) and \( x_{2,1} - y_{2,1} < x_{2,0} - y_{2,0} \) that \( y_{2,1}^2 - x_{2,1}^2 < y_{2,0}^2 - x_{2,0}^2 \).

Hence, we have \( x_{2,0} + \lambda h_{2} < x_{2,2} < x_{2,0} + 2\lambda h_{2} \) and obviously \( y_{2,2} = y_{2,0} + 2h_{2} \). Therefore, we see inductively that for \( 0 > x_{2,n} > y_{2,n} \) both sequences are increasing and we either already
Thus, we can conclude that $x_{2,n+1} < y_{2,n+1} < 0$ or

$$x_{2,n+1} - y_{2,n+1} < x_{2,0} - y_{2,0} - (n+1)(1-\lambda)h_2 < \frac{\lambda}{8} \delta^{1/2} - (n+1)(1-\lambda)h_2$$

$$= \lambda \left( (n+1)h_2 + \frac{\delta^{1/2}}{8} \right) - (n+1)h_2.$$  

Thus, we can conclude that $x_{2,n^*} < y_{2,n^*} < 0$ for some $n^*$ such that $\lambda < \frac{n^* h_2}{n^* h_2 + \frac{\delta^{1/2}}{8}}$, if $\delta$ is chosen small enough such that $\frac{\delta^{1/2}}{8} > \lambda$. Namely, if there was $\hat{n} \in \mathbb{N}$ such that $0 > x_{2,\hat{n}} > y_{2,\hat{n}}$ for all $n < \hat{n}$ and $x_{2,\hat{n}} > 0$, we would have, for $h_2, \delta$ small enough, that $\frac{\delta^{1/2}}{8} > \lambda$ and would obtain
\[
\frac{(n-1)h_2}{\lambda} > \frac{\lambda h_2 + \frac{\delta}{1+\delta}}{\delta^{-1/2} + \frac{\delta}{1+\delta}} > \lambda,
\]

which is a contradiction, since this would imply \(x_{2,n-1} < y_{2,n-1}\) with the above.

Assume now that, at time \(n \in \mathbb{N}\), the trajectory is above the diagonal \(\{x = y\}\), i.e. \(x_{2,n} < y_{2,n}\). In particular, this covers the case when \(x_{2,n} \leq y_{2,n} < 0\), when the trajectory lies above \(S_{\lambda,2}\), which is relevant for the initial data. We have \(y_{2,n+1} = y_{2,n} + h_2\) and \(x_{2,n+1} \geq x_{2,n} + h_2\). If \(x_{2,n} = y_{2,n}\), then obviously \(x_{2,n+1} < y_{2,n+1}\). If \(x_{2,n} < y_{2,n}\), we observe that

\[
x_{2,n+1} > y_{2,n+1} \iff \frac{h_2(1-\lambda)}{y_{2,n} - x_{2,n}} \leq -h_2(x_{2,n} + y_{2,n}).
\]

(3.29)

Hence, since \(h_2 \mid x_{2,n} + y_{2,n} \mid < 2(1 - 2\nu)^{-1/2}\delta^{-1/2}\delta^{1/2}/\nu < 1\), we have \(x_{2,n+1} < y_{2,n+1}\) and the argument goes on inductively. We can also see that, for \(x_{2,n} \leq y_{2,n} < 0\), the trajectories stay close to \(S_{\lambda,2}\) since, if \(x_{2,n}^2 - y_{2,n}^2 > 1 - \lambda\), we have that \(y_{2,n+1} - x_{2,n+1} < y_{2,n} - x_{2,n}\). This concludes the proof of the first statement.

Next, we consider the case \(\lambda \leq 0\). Again, assume that at time \(n \in \mathbb{N}\) the trajectory is above the diagonal \(\{x = y\}\), i.e. \(x_{2,n} \leq y_{2,n}\). In particular, this covers the case when \(x_{2,n} \leq y_{2,n} < 0\) as relevant for the initial data. If \(x_{2,n} = y_{2,n}\), then obviously \(x_{2,n+1} < y_{2,n+1}\). If \(x_{2,n} < y_{2,n}\), we observe as before that (3.29) holds and that \(h_2 \mid x_{2,n} + y_{2,n} \mid < 1\). Hence, we have \(x_{2,n+1} < y_{2,n+1}\) and the argument goes on inductively. Therefore trajectories stay above the diagonal. Furthermore, observe that \(y_{2,0}^2 \leq (\delta^{-1/2} - 1 - (3\lambda - 1)/(4\delta))^2\) and therefore

\[
x_{2,n}^2 - y_{2,n}^2 + \lambda \geq \delta^{-1} - \delta^{-1} - \lambda + \frac{1}{2} \frac{(2\lambda - 1)^2}{16} \delta + \lambda = \frac{1}{2} \frac{(2\lambda - 1)^2}{16} \delta,
\]

which is greater than 0 for \(\delta\) small enough, depending on \(\lambda\). Hence, \(x_{2,1} > x_{2,0}\). We show that \(x_{2,n+1} > x_{2,n}\) as long as \(x_{2,n} < y_{2,n} < 0\) by proving that \(\xi_n := x_{2,n}^2 - y_{2,n}^2 + \lambda > 0\) implies \(x_{2,n+1}^2 - y_{2,n+1}^2 + \lambda > 0\). Assuming that \(x_{2,n} < y_{2,n} < 0\) and \(\xi_n > 0\) yields

\[
x_{2,n+1}^2 - y_{2,n+1}^2 + \lambda = (x_{2,n} + h_2\xi_n)^2 - (y_{2,n} + h_2)^2 + \lambda = \xi_n + h_2^2(\xi_n^2 - 1) + 2h_2(\mid y_{2,n} \mid - \xi_n \mid x_{2,n} \mid).
\]

From there, it is easy to observe that for \(h_2\) small enough the claim follows.

Although the argument is quite technical, the proof of the last lemma shows that the key steps in the scaling chart involve the sign of the nonlinear term for the \(x_2\)-variable. This idea can also be carried out in the case \(\lambda > 1\).

**Lemma 3.9.** Let \(\lambda > 1\) and \(\delta, \nu\) sufficiently small. Then all trajectories starting in \(\Sigma_{2,\lambda}^{in}\) are strictly increasing in \(x\) as long as \(x_{2,n} < y_{2,n} < 0\), will be below the diagonal \(\{x = y\}\) at a certain point of time and stay there forever afterwards. In particular, if \((x_{2,0}, y_{2,0})\) in \(\Sigma_{2,\lambda}^{in}\) with \(x_{2,0} < y_{2,0} < 0\), there is a \(n^* \in \mathbb{N}\) such that \(y_{2,n^*} \leq x_{2,n^*} < 0\) and

\[
n^* h_2 \geq \delta^{1/2}.
\]

**Proof.** We consider two cases, trajectories below and above the diagonal. First, we assume that, at time \(n \in \mathbb{N}\), the trajectory is below the diagonal \(\{x = y\}\) so that \(y_{2,n} \leq x_{2,n}\). In particular, this covers the case when \(y_{2,n} \leq x_{2,n} < 0\), i.e. the trajectory lies below \(S_{\lambda,2}\), which is
relevant for the initial data. If \( x_{2,n} = y_{2,n} \), then obviously \( y_{2,n+1} < x_{2,n+1} \). If \( y_{2,n} < x_{2,n} \), we observe similarly to (3.29) that

\[
y_{2,n+1} \geq x_{2,n+1} \quad \text{iff} \quad 1 + \frac{h_2(\lambda - 1)}{x_{2,n} - y_{2,n}} \leq -h_2(x_{2,n} + y_{2,n}).
\]

Hence, since \( h_2 |x_{2,n} + y_{2,n}| < (2(1 - 2\nu)^{-1/2} \delta^{-1/2} + \frac{2(\lambda - 1)^2}{16})\delta^{1/2} \nu < 1 \) due to \( \delta \lambda \leq 1 \) and \( \nu < \frac{1}{3} \), we have \( y_{2,n+1} < x_{2,n+1} \) and the argument goes on inductively.

Moreover, we check that the sequences are increasing in this case. We consider an initial condition \((x_{2,0}, y_{2,0})\) in \( \Sigma_{\text{out}}^n \) with \( y_{2,0} < x_{2,0} < 0 \), i.e. below \( S_{a,2} \). We obviously have \( x_{2,1} < x_{2,0} + \delta h_2 \). Furthermore, observe that

\[
x_{2,0}^2 - y_{2,0}^2 + \lambda \geq \delta^{-1} - (\delta^{-1/2}(-1 - \beta_2(\lambda)))^2 + \lambda
\]

\[
= \delta^{-1} - \delta^{-1} - \lambda + \frac{1}{2} - \frac{(2\lambda - 1)^2}{16} + \lambda
\]

\[
= \frac{1}{2} - \frac{(2\lambda - 1)^2}{16} \delta \geq 1
\]

if \( \delta \) is chosen sufficiently small in comparison to \( \lambda \). Hence, \( x_{2,1} \geq x_{2,0} + \frac{1}{2}h_2 \). Note in particular that if \( y_{2,0} = \delta^{-1/2}(-1 - \beta_2(\lambda)) =: y^* \), we have \( x_{2,0}^2 - y_{2,0}^2 + \lambda < \frac{1}{2} \). From here, it is easy to observe that, for \( \nu \) small enough, \( x_{2,1} - y_{2,1} < x_{2,0} - y^* \). This together with the fact that \( |y_{2,1} + x_{2,1}| < |y_{2,0} + x_{2,0}| \) yields \( x_{2,1}^2 - y_{2,1}^2 + \lambda > \frac{1}{4} \). Hence, we have \( x_{2,2} \geq \frac{1}{4} h_2 \) and obviously \( y_{2,2} = y_{2,0} + 2h_2 \). Therefore, we see inductively that for \( 0 > x_{2,n} > y_{2,n} \) both sequences are increasing.

As the second case, we consider a trajectory with initial condition \((x_{2,0}, y_{2,0})\) in \( \Sigma_{\text{out}}^n \) and \( x_{2,0} < y_{2,0} < 0 \), i.e. above \( S_{a,2} \). Either we already have \( y_{2,1} \leq x_{2,1} < 0 \). If not, we have \( x_{2,1} > x_{2,0} + \delta h_2 \) and obviously \( y_{2,1} = y_{2,0} + h_2 \). Therefore, we see inductively that for \( x_{2,n} < y_{2,n} < 0 \) both sequences are increasing and we either already have \( y_{2,n+1} \leq x_{2,n+1} < 0 \).
or
\[ x_{2,n+1} - y_{2,n+1} > x_{2,0} - y_{2,0} + (n + 1)(\lambda - 1)h_2 \]
\[ \geq \delta^{-1/2} - \delta^{-1/2}(1 - 2\nu)^{-1/2} - (\lambda - 1)\delta^{1/2} + (n + 1)(\lambda - 1)h_2. \]

Thus, we can conclude that \( y_{2,n^*} \leq x_{2,n^*} < 0 \) for some \( n^* \) such that \( h \geq \delta^{1/2} \left( 1 + \frac{(1 - 2\nu)^{-1/2} - 1}{\lambda - 1} \right). \)

Using \( (1 - 2\nu)^{-1/2} > 1 \), the claim follows.

Finally, we turn to the proof of proposition 3.7.

**Proof of proposition 3.7.** We distinguish three cases: (I) \( 0 < \lambda < 1 \), (II) \( \lambda \leq 0 \), and (III) \( \lambda > 1 \). The case distinction is going to allow us to apply each of the preliminary results obtained above.

**Case (I) \( 0 < \lambda < 1 \):** from lemma 3.8 we know that trajectories starting in \( \Sigma^m_2 \) will be above the diagonal \( \{ x = y \} \) at certain point of time and stay there forever afterwards. From this result and the fact that \( y_{2,n} \) and \( x_{2,n} \) are both strictly increasing uniformly as long as \( |y_{2,n}| \leq |x_{2,n}| \), we can conclude that any such trajectory reaches a point \( (x_{2,n}, y_{2,n}) \), with \( y_{2,n} > 0 \), such that \( x_{2,n}^2 > x_{2,n}^2 + \lambda \). We can conclude that there must be a minimal \( n^* \) such that \( y_{2,n^*}^2 > x_{2,n^*}^2 + \lambda \). Note that \( (x_{2,n^*}, y_{2,n^*}) \) lies between \( S_{2,n}^1 \) and \( S_{1,n}^2 \). As long as this is the case for \( n > n^* \), we have \( x_{2,n+1} < x_{2,n} \). Additionally, we observe that for \( \lambda > 2 \), we have \( y_{2,n+1} < y_{2,n} + x_{2,n} \). Hence, trajectories are rapidly approaching the vicinity of \( S_{2,n}^2 \). Similarly to (3.29), we find that for any such \( y_{2,n} > -x_{2,n} > 0 \)
\[ |x_{2,n+1}| \geq y_{2,n+1} \quad \text{iff} \quad 1 + \frac{h_2(1 + \lambda)}{y_{2,n} - |x_{2,n}|} \leq h_2(|x_{2,n}| + y_{2,n}). \]

Hence, since \( h_2 |x_{2,n}| + y_{2,n} < (2 + \delta + h_2)\nu < 1 \) before hitting \( \Sigma^m_{2,n} \), we have \( |x_{2,n+1}| < y_{2,n+1} \) and the argument goes on inductively before hitting \( \Sigma^m_{2,n} \). The fact that the trajectory will actually be located within \( \Sigma^m_{2,n} \) at a certain point of time can be inferred as follows: we observe from the above that \( (x_{2,n}, y_{2,n}) \) satisfies \( 0 \leq y_{2,n}^2 - x_{2,n}^2 \leq \lambda + 1 \) for large enough \( n \). First, we can conclude that \( x_{2,n} - x_{2,n+1} \leq h_2 \). Hence, there is an \( m \in \mathbb{N} \) such that \( x_{2,m} \in [-\delta^{-1/2} - \frac{h_2}{2}, -\delta^{-1/2} + \frac{h_2}{2}] \). Therefore we have \( y_{2,m} \geq \delta^{-1/2} - \frac{h_2}{2} \) and
\[ y_{2,m}^2 \leq \delta^{-1} + (\lambda + 1) - h_2\delta^{-1/2} + \frac{h_2^2}{4} \leq \delta^{-1} + 2\beta_2^2\delta^{-1} + \delta^{-1}(\beta_2^2)^2 = \left( \delta^{-1/2}(1 + \beta_2^2) \right)^2. \]

Figure 3(b) illustrates the behaviour of trajectories starting in \( \Sigma^m_{2,n} \) for \( 0 < \lambda < 1 \).

**Case (II) \( \lambda \leq 0 \):** we know from lemma 3.8 that any trajectory starting in \( \Sigma^m_{2,n} \) is strictly increasing in \( x \) as long as \( x_{2,n}, y_{2,n} < 0 \) and will be above the diagonal \( \{ x = y \} \) at any point of time. Analogously to before, we can conclude that there exists a minimal \( n^* \in \mathbb{N} \) such that \( y_{2,n^*} > 0 \) and \( y_{2,n^*} > x_{2,n^*} + \lambda \). Note that if \( (x_{2,n^*}, y_{2,n^*}) \) lies between \( S_{2,n}^1 \) and \( S_{1,n}^2 \), and stays in this region for all \( n > n^* \) before hitting \( \Sigma^m_{2,n^*} \), as for example for \( -1 \leq \lambda \leq 0 \), the arguments go exactly as before. Otherwise we observe, symmetrically to before, that \( (x_{2,n}, y_{2,n}) \) satisfies \( 0 \geq x_{2,n} - x_{2,n+1} \leq h_2 \) and...
conclude that there is an \( m \in \mathbb{N} \) such that \( x_{2,m} \in [-\delta^{-1/2} - \frac{h_2}{2}, -\delta^{-1/2} + \frac{h_2}{2}] \). Therefore we have \( y_{2,m} \leq \delta^{-1/2} + \frac{2h_2}{3} \) and for \( \delta \) sufficiently small depending on \( \lambda \)

\[
y_{2,m}^2 \geq \delta^{-1} - h_2\delta^{-1/2} + \frac{h_2^2}{4} + \lambda \geq \delta^{-1} + \lambda + \frac{(|\lambda| + 1)^2}{4} \delta - 1 = \left( \delta^{-1/2}(1 - \beta_1^+)^2 \right).
\]

Figure 3(a) illustrates the behaviour of trajectories starting in \( \Sigma_{2,a}^{\text{in}} \) for \( \lambda < 0 \).

Case (III) \( \lambda > 1 \): we can conclude from lemma 3.9 that trajectories starting in \( \Sigma_{2,a}^{\text{in}} \) will be below \( S_{2,a}^{\text{in}} \) at a certain point of time and stay below the diagonal \( \{x = y\} \) forever afterwards.

From that and the fact that \( y_{2,n} \) is strictly increasing for all time, we can conclude that any such trajectory will reach a point \( (x_{2,n}, y_{2,n}) \) with \( x_{2,n} > y_{2,n} > 0 \). Then the trajectory will increase its distance from the diagonal in each time step by

\[
h_2(\lambda - 1) + h_2(x_{2,n}^2 - y_{2,n}^2).
\]

Let us take the largest \( n \) such that \( x_{2,n} > 0 \geq y_{2,n} \). It is now obvious that there is an \( m \in \mathbb{N} \) such that \( \delta^{-1/2} \leq x_{2,m+n} \leq \delta^{-1/2} + h_2(\lambda + \delta^{-1}) \). We give an upper bound for \( y_{2,m+n} \) by expanding \( x_n \) up to \( h_2^3 \), which is the first order estimate in this case:

\[
(1 + 2\nu)\delta^{-1/2} \geq x_{2,m+n} > mh_2\lambda + \left( \sum_{k=1}^{m-1} k^2 \right)(\lambda^2 - 1)h_2^3
\]

\[
\geq y_{2,m+n} + \frac{1}{6}(\lambda^2 - 1)y_{2,m+n}(y_{2,m+n} - h_2)(2y_{2,m+n} - h_2)
\]

\[
\geq \lambda y_{2,m+n} + \frac{1}{6}(\lambda^2 - 1)y_{2,m+n}^3.
\]

Hence, we conclude that \( y_{2,m+n} = O(\delta^{-1/6}) \) and \( (x_{2,n+m}, y_{2,n+m}) \in \Sigma_{2,a}^{\text{out}} \). Figure 3(c) illustrates such a trajectory.

The consequences for \( M_{a,2}^- \) are immediate in each case. \( \square \)

Note that our way of proving proposition 3.7 differs significantly from the proof of the continuous-time analogue [16, proposition 2.1], which is based on the phase portraits of heteroclinic connections between the partially hyperbolic equilibria obtained from the blow-up (see [16, figure 3]), relying on the principle that trajectories of a planar flow cannot intersect.

In the discrete-time scenario, such arguments are not available. Hence, we follow the iterations of the (nonlinear) maps, using several analytical estimates, to predict the behaviour of trajectories, and, in particular, the continuation of the attracting center manifold \( M_{a,2}^- \). Note that this approach also differs from the proof of a corresponding result for the fold singularity in discretized time [26, theorem 13.1], where the attracting center manifold is shown to be a \( \varepsilon, \delta \)-perturbation of the reference manifold for the ODE. The proof is based on Lipschitz-type estimates on the maps which allow for application of general invariant manifold results [26, theorems 1.11 and 1.12]. Our proof focuses more directly on the iterations of the maps, independently from the continuous-time reference objects, and thereby gives a different perspective on the dynamical system under study.
3.4. Dynamics in the chart \( K_3 \)

We investigate the dynamics in the chart \( K_3 \) (3.9) for \( \lambda > 1 \). First, recall from (3.13) that the change of coordinates \( \varphi_2 : K_2 \to K_3 \) is given by
\[
\epsilon_3 = x_2^2, \quad y_3 = x_2^{-1} y_2, \quad r_3 = x_2 r_2, \quad h_3 = x_2 h_2.
\]
Symmetrically to the chart \( K_1 \), we define
\[
D_3 := \{(r_3, y_3, \epsilon_3, h_3) \in \mathbb{R}^4 : r_3 \in [0, \rho], \epsilon_3 \in [0, \delta], h_3 \in [0, \nu]\}
\]
and
\[
\hat{D}_3 := \{(r_3, y_3, \epsilon_3, h_3) \in \mathbb{R}^4 : r_3 \in [\rho/2, \rho], \epsilon_3 \in [\delta/4, \delta], h_3 \in [\nu/2, \nu]\}.
\]

Since we need to have \( k_{33} (\Sigma_3^{\text{out}}) \subset \Sigma_3^{\text{in}} \), a suitable choice is given by
\[
\Sigma_3^{\text{in}} := \{(r_3, y_3, \epsilon_3, h_3) \in D_3 : (\delta^{-1} + 4 \nu \delta^{-1})^{-1} \leq \epsilon_3 \leq \delta\}.
\]
Furthermore, we will simply set
\[
\Sigma_3^{\text{out}} := \{(r_3, y_3, \epsilon_3, h_3) \in D_3 : r_3 = \rho, h_3 = \nu, \epsilon_3 = \delta/4, \quad y_3 > 0\},
\]
and will end the analysis with the point of the trajectory which is closest to \( \Sigma_3^{\text{out}} \) (see figure 4). The dynamics, desingularized by choosing \( h = h_3/r_3 \), look as follows:
\[
\begin{align*}
\tilde{r}_3 &= r_3(1 + h_3 F_3(y_3, \epsilon_3)), \\
\tilde{y}_3 &= (y_3 + \epsilon_3 h_3) (1 + h_3 F_3(y_3, \epsilon_3))^{-1}, \\
\tilde{\epsilon}_3 &= \epsilon_3 (1 + h_3 F_3(y_3, \epsilon_3))^{-2}, \\
\tilde{h}_3 &= h_3 (1 + h_3 F_3(y_3, \epsilon_3)),
\end{align*}
\]
where \( F_3(y_3, \epsilon_3) = 1 - y_3^2 + \lambda \epsilon_3 \). For any \( h_3 \) system (3.30) has the fixed points
\[
v_{r,3}^- = (0, -1, 0, h_3), \quad v_{r,3}^+ = (0, 1, 0, h_3).
\]
The points \( v_{r,3}^- \) and \( v_{r,3}^+ \) have a three-dimensional centre eigenspace and a one-dimensional unstable eigenspace with the eigenvalue \( 1 + 2h_3 \). Hence, unlike the analogous case in the chart \( K_1 \), the stability does not depend on the size of \( h_3 \). The most relevant manifold for our problem is given by
\[
W := \{w^{\text{out}}(h_3) : (0, 0, 0, h_3) \quad h_3 \in [0, \nu]\},
\]
which is a line for system (3.30) within \( D_3 \). The points \( w^{\text{out}}(h_3) \) have two stable and two unstable eigenvalues
\[
\lambda_1 = (1 + h_3)^{-2}, \quad \lambda_2 = (1 + h_3)^{-1}, \quad \lambda_3 = 1 + h_3, \quad \lambda_4 = 1 + 2h_3,
\]
such that the stability corresponds to the time-continuous problem independently from \( h_3 \). Note that the chart \( K_3 \) differs in that respect from the chart \( K_1 \) where preservation of stability is bound to the stability criteria of the Euler method known from the Dahlquist test equation.

The eigenvalues \( \lambda_1, \lambda_2 \) correspond with the \( \epsilon_3 \)- and \( y_3 \)-directions and \( \lambda_3, \lambda_4 \) with the \( r_3 \)- and \( h_3 \)-directions. We extend the set \( W \) to the attracting invariant manifold \( M_{a,3} \) (see figure 4), which is given in \( D_3 \) by a graph \( y_3 = l_3(\epsilon_3, h_3) \). One can derive \( l_3 \) from the discrete invariance equation
\[
\lambda^2 = (1 + h_3)^{-2}, \quad \lambda_2 = (1 + h_3)^{-1}, \quad \lambda_3 = 1 + h_3, \quad \lambda_4 = 1 + 2h_3,
\]
such that the stability corresponds to the time-continuous problem independently from \( h_3 \). Note that the chart \( K_3 \) differs in that respect from the chart \( K_1 \) where preservation of stability is bound to the stability criteria of the Euler method known from the Dahlquist test equation.

The eigenvalues \( \lambda_1, \lambda_2 \) correspond with the \( \epsilon_3 \)- and \( y_3 \)-directions and \( \lambda_3, \lambda_4 \) with the \( r_3 \)- and \( h_3 \)-directions. We extend the set \( W \) to the attracting invariant manifold \( M_{a,3} \) (see figure 4), which is given in \( D_3 \) by a graph \( y_3 = l_3(\epsilon_3, h_3) \). One can derive \( l_3 \) from the discrete invariance equation
\[
\lambda^2 = (1 + h_3)^{-2}, \quad \lambda_2 = (1 + h_3)^{-1}, \quad \lambda_3 = 1 + h_3, \quad \lambda_4 = 1 + 2h_3,
\]
such that the stability corresponds to the time-continuous problem independently from \( h_3 \). Note that the chart \( K_3 \) differs in that respect from the chart \( K_1 \) where preservation of stability is bound to the stability criteria of the Euler method known from the Dahlquist test equation.

The eigenvalues \( \lambda_1, \lambda_2 \) correspond with the \( \epsilon_3 \)- and \( y_3 \)-directions and \( \lambda_3, \lambda_4 \) with the \( r_3 \)- and \( h_3 \)-directions. We extend the set \( W \) to the attracting invariant manifold \( M_{a,3} \) (see figure 4), which is given in \( D_3 \) by a graph \( y_3 = l_3(\epsilon_3, h_3) \). One can derive \( l_3 \) from the discrete invariance equation
\[
\lambda^2 = (1 + h_3)^{-2}, \quad \lambda_2 = (1 + h_3)^{-1}, \quad \lambda_3 = 1 + h_3, \quad \lambda_4 = 1 + 2h_3,
\]
$$l_3(\tilde{\epsilon}_3, \tilde{h}_3) = \frac{l_3(\epsilon_3, h_3) + \epsilon_3 h_3}{1 + \epsilon_3 F_3 l_3(\epsilon_3, h_3, \epsilon_3)}.$$  

(3.31)

Note that, analogously to the continuous-time case, where $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$ such that $\lambda_1 + \lambda_2 = \lambda_3$ (see [16, section 2.4], and [15, section 2.6] for more details), there is the resonance $\lambda_1 \lambda_2 = \lambda_3$, which makes the description of the dynamics close to $W$ and $M_{3,1}$ a delicate problem. However, the exiting behaviour can still be estimated by a relatively simple argument without a full analysis of the resonance as follows. Let $P_3$ denote the map given by (3.30) and $\pi_y$ the projection to the $y$-component.

**Proposition 3.10.** The transition map $\Pi_3$ from $\Sigma_3^\text{in}$ to the vicinity of $\Sigma_3^\text{out}$ given by

$$\Pi_3(z) = P_3^{\text{out}}(z) = \arg \min_{n \in \mathbb{N}} \text{dist}(P_3^m(z), \Sigma_3^\text{out}), \quad z \in \Sigma_3^\text{in},$$

is well-defined on $K_{23}(\Sigma_3^\text{out})$. Furthermore, for $z \in K_{23}(\Sigma_3^\text{out}) \subset \Sigma_3^\text{in}$ we have $\pi_y(\Pi_3(z)) = O(\delta^{1/3})$.

**Proof.** By the construction of $\Sigma_3^\text{out}$, proposition 3.7, and the fact that $y_3 = x_2^{-1} y_2$ we have $\pi_y(z) = O(\delta^{1/3})$ for any $z \in \Sigma_3^\text{in}$. Further note that $F_3$ is clearly positive as long as $y_3$ maintains some positive order of $\delta$. Since $\delta_3 h_3 = O(\delta^2)$ in $D_3$, we can immediately infer that

$$\pi_y(P_3^m(z)) = O\left(\delta^{1/3}\right) \quad \text{for all } n \leq m^*(z).$$

This implies both statements as $F_3$ stays positive along the trajectory.

\[\square\]

3.5. Connecting the charts and proof of the theorem

Finally, we can prove theorem 3.1 by combining the dynamics in $K_1$, $K_2$ and $K_3$ into a global picture.

**Proof of theorem 3.1.** We have proven the statements in charts $K_1, K_2, K_3$ for $\bar{\epsilon} = \delta$ with $\delta$ sufficiently small. Hence, we choose $\epsilon_0 = \rho^2 \delta_0$, where $\delta_0$ is the largest value of $\delta/4$ such that the statements hold. We did not use any further restrictions on $h$ apart from $\delta h < \delta$ and $\rho h < \frac{1}{8}$. Hence, it is enough to assume the latter, $\rho^3 h < \epsilon$ and, to guarantee sufficiently many time steps, $\epsilon h \ll \rho$.

As before, we distinguish several cases. First, we consider $\lambda < 1$. We define the map $\hat{\Pi}_a$ from $R_1 \subset \Sigma_1^\text{in}$ to the vicinity of $\Sigma_1^\text{out}$ by

$$\hat{\Pi}_a := \Pi_{1,+} \circ k_{21} \circ \Pi_2 \circ k_{12} \circ \Pi_{1,-},$$

where $\Pi_2 : \Sigma_2 \rightarrow \Sigma_2^\text{out}$ is the map well-defined by proposition 3.7. We have seen that

$$k_{12}(\Pi_{1,-}(R_1)) \subset \Sigma_2^\text{in} \text{ and } k_{21}(\Sigma_2^\text{out}) \subset \Sigma_2^\text{in}. $$

Hence, $\hat{\Pi}_a$ is indeed a well-defined map. In particular, we have seen from the analysis in the charts $K_1$ and $K_2$ (see propositions 3.3, 3.4 and 3.7) that $M_{a,1}^+$ equalling $M_{a,2}^+$ in $K_2$ is continued via $R_2$ to $\Sigma_2^\text{out}$ in the vicinity of $M_{a,1}^+$. We have that $\Pi_a = \Phi \circ \hat{\Pi}_a \circ \Phi^{-1}$, where $\Delta^\text{in} = \Phi(R_1)$ and $\Delta^\text{out} = \Phi(\Sigma_2^\text{out})$ is an interval about $S_a^+$ of the same size as $\Delta^\text{in}$. We observe with proposition 3.3 that $\Phi(M_{a,1}^+) \subset S_{a,\rho h}^+$, and, by the choices of $R_1$ and $R_2$, that...
\( \Delta^{\text{in}} \cap S_{\alpha \varepsilon \varepsilon}^- \) and \( \Pi_\varepsilon (\Delta^{\text{in}}) \cap S_{\alpha \varepsilon \varepsilon}^+ \) are nonempty. Summarizing, we can conclude that \( \Pi_\varepsilon \) maps \( \Delta^{\text{in}} \) including \( \Delta^{\text{in}} \cap S_{\alpha \varepsilon \varepsilon}^- \) to a set about \( S_{\alpha \varepsilon \varepsilon}^+ \). The distance between any point in \( \Pi_\varepsilon (\Delta^{\text{in}}) \) and \( \Delta^{\text{out}} \) is of order \( O(h\varepsilon) \) since for \( (x, y) \in \Delta^{\text{out}} \cap S_{\alpha \varepsilon \varepsilon}^+ \) it is bounded by \( h(x^2 - y^2 + \lambda \varepsilon) \) due to the definition of \( \Pi_\varepsilon \) and we have by (3.18) that

\[
|x^2 - y^2| = |x - y| |x + y| = O(\delta \rho)O(\rho) = O\left(\varepsilon \left(\frac{x}{\rho}\right)\right) O(\rho) = O(\varepsilon).
\]

Furthermore, proposition 3.6 says that \( \Pi_{1,-}|R_1 \) and \( \Pi_{1,+}|R_2 \) are contractions in the \( y_1 \) direction with rates of order at least \( O\left(\left(1 - c \frac{\varepsilon}{\rho}\right)^2\right) \) for some constant \( C > 0 \). Since \( \Pi_2 \) is also contracting and due to \( O(\delta) = O\left(\frac{\varepsilon}{\rho}\right) \), we obtain that \( \Pi_\varepsilon (\Delta^{\text{in}}) \) has \( y \)-width at most of order \( O\left((1 - c)^{\varepsilon/\rho}\right) \).

Let now \( \lambda > 1 \). We define the map \( \Pi_\varepsilon \) from \( R_1 \subset \Sigma_{\alpha \varepsilon \varepsilon}^{\text{in}} \) to the vicinity of \( \Sigma_{\alpha \varepsilon \varepsilon}^{\text{out}} \) by

\[
\Pi_\varepsilon := \Pi_3 \circ \kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_{1,-}.
\]

Again, we know that \( k_{12} (\Pi_{1,-}(R_1)) \subset \Sigma_{\alpha \varepsilon \varepsilon}^{\text{in}} \), and from proposition 3.10 that \( \Pi_3 \) is well-defined on \( k_{23} (\Sigma_{\alpha \varepsilon \varepsilon}^{\text{out}}) \subset \Sigma_{\alpha \varepsilon \varepsilon}^{\text{in}} \). Furthermore, the map \( \Pi_2 : \Sigma_{\alpha \varepsilon \varepsilon}^{\text{in}} \to \Sigma_{\alpha \varepsilon \varepsilon}^{\text{out}} \) is well-defined and keeps \( M_{\alpha \varepsilon \varepsilon}^3 \) invariant by proposition 3.7. Hence, \( \Pi_\varepsilon \) is indeed a well-defined map. We have that \( \Pi_\varepsilon = \Phi \circ \Pi_3 \circ \Phi^{-1} \), where again \( \Delta^{\text{in}} = \Phi(R_1) \) and \( \Delta^{\text{out}} = \Phi(\Sigma_{\alpha \varepsilon \varepsilon}^{\text{out}}) \) is an interval perpendicular to the \( x \)-axis. It follows immediately that \( S_{\alpha \varepsilon \varepsilon}^- \) is well-defined.

The fact that \( \Pi_\varepsilon (\Delta^{\text{in}}) \) has \( y \)-width \( O\left((1 - c)^{\varepsilon/\rho}\right) \) follows as for \( \lambda < 1 \). The distance between any point in \( \Pi_\varepsilon (\Delta^{\text{in}}) \) and \( \Delta^{\text{out}} \) is of order \( O(\rho \varepsilon / \rho^2) \) since for \( (x, y) \in \Delta^{\text{out}} \cap S_{\alpha \varepsilon \varepsilon}^- \) it is bounded by \( h(x^2 - y^2 + \lambda \varepsilon) \) due to the definition of \( \Pi_\varepsilon \).

This finishes the proof for \( \lambda \neq 1 \).

For \( \lambda = 1 \), we can show the statement directly from an analysis of (3.1), without using blow-up since we have refrained from including higher order terms in this study (see [7] for a comprehensive treatment of discrete-time canard phenomena) and the invariant manifolds can be determined explicitly, as opposed to \( \lambda \neq 1 \). In more detail, we can observe immediately that for fixed \( 0 < \varepsilon h \ll \rho \), the dynamical system induced by iterations of

\[
\begin{align*}
t_x &= x + h(x^2 - y^2) + h\varepsilon, \\
t_y &= y + h\varepsilon,
\end{align*}
\]

on \( \{(x, y) \in \mathbb{R}^2 : -\rho \leq x \leq \rho\} \), has the solution

\[
\gamma(n) = (x_n, y_n) = (-\rho + n\varepsilon, -\rho + n\varepsilon), \quad n \in \mathbb{N},
\]

which connects \( S_{\alpha \varepsilon \varepsilon}^- = S_{\varepsilon}^- \) and \( S_{\alpha \varepsilon \varepsilon}^+ = S_{\varepsilon}^+ \) in this case. Clearly, we have \( (x_0, y_0) = \Delta^{\text{in}} \cap S_{\alpha \varepsilon \varepsilon}^- \) and \( \Pi_\varepsilon (\Delta^{\text{in}} \cap S_{\alpha \varepsilon \varepsilon}^-) \in S_{\alpha \varepsilon \varepsilon}^+ \) in this case.

The linearization along the trajectory \( \gamma \) (3.32) is characterized by the variational equation

\[
v(n + 1) = \begin{pmatrix} 1 + 2x_n h & -2y_n h \\ 0 & 1 \end{pmatrix} v(n), \quad v(n) \in \mathbb{R}^2, \quad \text{for all } n \in \mathbb{N},
\]

where \( x_n = y_n = -\rho + n\varepsilon \). While the fixed point \( w = (1, 1)^T \) of (3.33) corresponds with the centre-direction along \( \gamma \), the solution of (3.33) starting at \( v(0) = (1, 0)^T \) corresponds with the transversal hyperbolic direction and can be explicitly solved to be

\[
\Delta^{\text{in}} \cap S_{\alpha \varepsilon \varepsilon}^-, \quad \Pi_\varepsilon (\Delta^{\text{in}}) \cap S_{\alpha \varepsilon \varepsilon}^+.\]
\[ v(n) = (v_1(n), v_2(n))^\top = (\prod_{k=0}^{N-1}(1 + 2n_k h), 0)^\top. \]

Let without loss of generality \( \frac{\rho}{\varepsilon} =: N \) be an even natural number such that \( x_0 = y_N = \rho \). Hence, the overall contraction and expansion in close vicinity to \( \gamma \) is given by

\[
\mu := v_1(N + 1) = \prod_{k=0}^{N/2-1} (1 - 2h(\rho - k\varepsilon)) \prod_{k=0}^{N/2-1} (1 + 2h(\rho - k\varepsilon)) = \prod_{k=0}^{N/2-1} (1 - (2h(\rho - k\varepsilon))^2).
\]

Hence, we can bound the contraction rate \( \mu \) by \( \mu \leq (1 - (\rho h)^2)^{N/4} \). Taking \( 0 < c < \rho h \) and inserting the definition of \( N \) proves the last claim of (T3).

Note that we have shown in the last part of the proof that the linearized contraction rates along the special solution \( \gamma \) on \( S^\gamma_a \) outweigh the linearized expansion rates on \( S^\gamma_r \). This is surprising since, in this particular case, the canard solution gains stability by the Euler discretization compared to the continuous-time analogue. In the latter case, denoting the special solution \( \tilde{\gamma}(t) = (x_t, y_t) = (-\rho + \varepsilon t, -\rho + \varepsilon t) \), the hyperbolic direction is characterized by \( d\tilde{v}_1(t)/dt = 2x_t \) and the overall contraction and expansion in close vicinity to \( \tilde{\gamma} \) is

\[
\tilde{\mu} = \exp \left( \int_0^{2\pi} -2x_t \, dx \right) = \exp \left( \int_{-\rho}^\rho -2x \, dx \right) = 1.
\]

Thus, in this example class the Euler method not only preserves the stability behaviour for trajectories close to the canard but even enhances stability as compared to the continuous-time case.

### 4. Summary and outlook

We have applied the blow-up method to the Euler discretization of a fast-slow system with a transcritical singularity at the origin. We have shown that the qualitative behaviour of the slow manifolds is preserved by the discretization for any choice of \( 0 < h < \varepsilon \) (setting \( \rho = 1 \)) for \( h, \varepsilon \) sufficiently small, where \( h \) denotes the time step size and \( \varepsilon \) the small time scaling parameter of the fast-slow system. The central part of the proof lies in the scaling chart \( K_2 \) of the manifold corresponding with the blown up singularity and is expressed in proposition 3.7. The proof of the proposition uses direct analysis of the map and, by that, can be seen of an alternative way of also showing the continuous analogue of the result when \( h \to 0 \). Furthermore, we are able to estimate transition times of trajectories by the analysis of the entering chart \( K_1 \) and give a bound for the \( y \)-component in the exiting chart \( K_3 \). In fact, our estimates provide a very fine control on individual trajectories, which is a potential advantage of the discrete-time framework for fast-slow systems.

We consider the work presented in this paper as a step towards a more comprehensive analysis of non-hyperbolic fixed points and non-hyperbolic submanifolds of fixed points in maps with multiple time scales. Whereas the normally hyperbolic theory for discrete-time multiple time scale systems is already quite well developed in [12, 26, 27], the geometric desingularisation of non-hyperbolic objects for maps still needs several extensions. For example, our problem (1.1) is based on an explicit Euler discretization, which is obviously the most straightforward scheme. We conjecture that one can use the more direct blow-up approach we use here for maps corresponding to ODEs also for other time-discretization schemes.

There are several reasons, why particular schemes should be checked: it is well-known from the area of geometric integration and the general theory of structure-preserving discretizations.
that only certain discrete-time schemes preserve relevant dynamical properties, e.g. adiabatic invariants for the Hamiltonian systems case [11] or certain asymptotic dynamics for the dissipative case [13]. For multiple time scale maps, Runge–Kutta methods have been studied from a geometric viewpoint [24, 25]. It remains to clarify more systematically, for which discretization the geometric blow-up approach can be applied and what the relation between the two small parameters \( 0 < h, \epsilon \ll 1 \) must be. In this context, an interesting problem are canard explosions in discrete-time [6, 9], which occur in this paper at \( \lambda = 1 \) and can be treated explicitly for the fixed value of \( \lambda \) since higher order terms are ignored. More generally, the small variation of the third parameter \( \lambda \) around the critical value, in our case \( \lambda = 1 \), is going to play a key role when higher order terms are regarded. Working out this case in more detail starting from the geometric approach by Krupa and Szmolyan for fast-slow ODEs [17] and using a discrete-time Melnikov method is the main object of the separate study [7].

**Acknowledgments**

CK and ME gratefully acknowledge support by the DFG via the SFB/TR109 Discretization in Geometry and Dynamics. We also thank several anonymous referees for their helpful comments.

**References**

[1] Benoît E, Callot J, Diener F and Diener M 1981 Chasse au canards Collect. Math. 32 37–76
[2] Dahlquist G G 1963 A special stability problem for linear multistep methods Nord. Tidskr. Inf. behandl. 3 27–43
[3] Dumortier F 1978 Singularities of Vector Fields (Monografias de Matemática [Mathematical Monographs] vol 32) (Rio de Janeiro: Instituto de Matemática Pura e Aplicada)
[4] Dumortier F 1993 Techniques in the theory of local bifurcations: blow-up, normal forms, nilpotent bifurcations, singular perturbations Bifurcations and Periodic Orbits of Vector Fields (Montreal, PQ, 1992) (NATO Advanced Science Institutes Series C, Mathematical and Physical Sciences vol 408) (Dordrecht: Kluwer) pp 19–73
[5] Dumortier F and Roussarie R 1996 Canard cycles and center manifolds Mem. Am. Math. Soc. 121 577
[6] El-Rabih A 2003 Canards solutions of difference equations with small step size J. Differ. Equ. Appl. 9 911–31
[7] Engel M, Kuehn C, Petrera M and Suris Y 2019 Discretized fast-slow systems with canard solutions (in preparation)
[8] Fenichel N 1979 Geometric singular perturbation theory for ordinary differential equations J. Differ. Equ. 31 53–98
[9] Fruchard A 1988 Canards discrets C. R. Acad. Sci. Paris Sér. I: Math. 307 41–6
[10] Gucwa I and Szmolyan P 2009 Geometric singular perturbation analysis of an autocatalator model Discrete Continuous Dyn. Syst. S 2 783–806
[11] Hairer E, Lubich C and Wanner G 2010 Geometric Numerical Integration (Springer Series in Computational Mathematics vol 31) (Structure-preserving algorithms for ordinary differential equations, reprint of the second (2006) edition) (Heidelberg: Springer)
[12] Hirsch M W, Pugh C C and Shub M 1977 Invariant Manifolds (Lecture Notes in Mathematics vol 583) (Berlin: Springer)
[13] Jin S 1999 Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations SIAM J. Sci. Comput. 21 441–54
[14] Jones C K R T 1995 Geometric singular perturbation theory Dynamical Systems (Montecatini Terme, 1994) (Lecture Notes in Mathematics vol 1609) (Berlin: Springer) pp 44–118
[15] Krupa M and Szmolyan P 2001 Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions SIAM J. Math. Anal. 33 286–314
[16] Krupa M and Szmolyan P 2001 Extending slow manifolds near transcritical and pitchfork singularities Nonlinearity 14 1473–91
[17] Krupa M and Szmolyan P 2001 Relaxation oscillation and canard explosion J. Differ. Equ. 174 312–68
[18] Kuehn C 2014 Normal hyperbolicity and unbounded critical manifolds Nonlinearity 27 1351–66
[19] Kuehn C 2015 Multiple Time Scale Dynamics (Applied Mathematical Sciences vol 191) (Berlin: Springer)
[20] Kuehn C 2016 A remark on geometric desingularization of a non-hyperbolic point using hyperbolic space J. Phys.: Conf. Ser. 727 012008
[21] Maesschalck P D and Dumortier F 2005 Time analysis and entry-exit relation near planar turning points J. Differ. Equ. Appl. 215 225–67
[22] Maesschalck P D and Dumortier F 2010 Singular perturbations and vanishing passage through a turning point J. Differ. Equ. 248 2294–328
[23] Maesschalck P D and Wechselberger M 2015 Neural excitability and singular bifurcations J. Math. Neurosci. 5 16
[24] Nipp K and Stoffer D 1995 Invariant manifolds and global error estimates of numerical integration schemes applied to stiff systems of singular perturbation type. I. RK-methods Numer. Math. 70 245–57
[25] Nipp K and Stoffer D 1996 Invariant manifolds and global error estimates of numerical integration schemes applied to stiff systems of singular perturbation type. II. Linear multistep methods Numer. Math. 74 305–23
[26] Nipp K and Stoffer D 2013 Invariant Manifolds in Discrete and Continuous Dynamical Systems (EMS Tracts in Mathematics vol 21) (Zürich: European Mathematical Society)
[27] Pötzsche C 2003 Slow and fast variables in non-autonomous difference equations J. Differ. Equ. Appl. 9 473–87 (dedicated to Professor George R Sell on the occasion of his 65th birthday)
[28] Wiggins S 1994 Normally Hyperbolic Invariant Manifolds in Dynamical Systems (Applied Mathematical Sciences vol 105) (with the assistance of György Haller and Igor Mezić) (New York: Springer)