Quantum Geometric Contributions to the BKT Transition: Beyond Mean Field Theory

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We study quantum geometric contributions to the Berezinskii-Kosterlitz-Thouless (BKT) transition temperature, \( T_{\text{BKT}} \), in the presence of fluctuations beyond BCS theory. Because quantum geometric effects become progressively more important with stronger pairing attraction, a full understanding of 2D multi-orbital superconductivity requires the incorporation of preformed pairs. We find it is through the effective mass of these pairs that quantum geometry enters the theory. Increasing these geometric contributions tends to raise \( T_{\text{BKT}} \) which then competes with fluctuation effects which generally depress it. We quantify the magnitude of the geometric terms through the ratio of the pairing onset temperature \( T^* \) to \( T_{\text{BKT}} \). Both temperatures can be extracted from the same voltage-current measurements, thereby providing an important characterization of a given superconductor.

**Introduction**

The recent discovery of superconducting phases in twisted bilayer graphene (TBLG) at first magic angle has attracted much attention\(^1\)-\(^14\). The excitement surrounding this material is driven largely by the flatness of the energy bands, which effectively enhances the importance of electron-electron interactions. This stronger interaction effect is consistent with the observed high superconducting transition temperatures\(^2\) and has been speculated to place TBLG somewhere in the crossover between the BCS and the Bose-Einstein condensation (BEC) regimes\(^2,15,16\). Because of its two dimensionality (2D) this superconductivity is associated with a BKT instability, in which the transition temperature \( T_{\text{BKT}} \) is directly proportional to the superfluid phase stiffness\(^17\)-\(^19\). In a single flat band this stiffness vanishes; however, in multi-orbital band models, it was shown that the inclusion of quantum geometric effects may reinstate a finite transition temperature\(^20\)-\(^24\).

This physical picture of flat-band superconductivity has been established within BCS mean field (MF) theory, which is known to be problematic in 2D. Moreover, quantum geometric effects become most apparent outside the BCS regime, where non-condensed pairs, neglected in MF theory, play an important role in the phase stiffness.

In this paper we present a theory which addresses these shortcomings through studies of the interplay of preformed pairs with quantum geometric effects. We determine \( T_{\text{BKT}} \), in 2D superconductors using a simple two-band tight-binding model\(^25,26\) that captures some key ingredients in common with its TBLG counterpart, including potentially nontrivial band topology. The model has some formal similarities to a spin-orbit coupled Fermi gas Hamiltonian, where the nature of (albeit, three dimensional) pairing fluctuations within the BCS-BEC crossover is well studied\(^27\)-\(^31\). Built on the BCS-Leggett ground state\(^32\), our approach yields results for \( T_{\text{BKT}} \) that are consistent with the mean field literature at weak attraction, precisely where the MF theory is expected to work.

A major contribution of this paper is to establish the important competition: bosonic excitations lead to a decrease in the effective phase stiffness, whereas, geometric effects generally cause an increase. These latter become more appreciable as the bands become flatter. As a result, \( T_{\text{BKT}} \) remains substantial, even though it is reduced by beyond mean-field fluctuations. An important finding is that geometric contributions appear through the inverse pair mass, \( 1/M_B \) which necessarily depends on the fermionic excitation gap. Because \( M_B \) enters the excitation spectrum of the pairs, the effect of geometry must be present in a host of general characteristics beyond the superfluid stiffness including transport and thermodynamics\(^33\), persisting even into the pseudogap phase. Here “pseudogap phase” refers to the non-superconducting state with preformed pairs at \( T_{\text{BKT}} < T < T^* \). We reserve the term “normal state” for a non-interacting system without pairing.

To physically understand the relation between the pair mass and geometry, note that an increased magnitude of the quantum metric reflects an increased spatial extent of the normal state Wannier orbitals\(^34,35\). This increase leads to larger pairs, which have a bigger overlap, leading to higher pair mobility (smaller \( M_B \)). Nontrivial normal state band topology enhances these effects, which become most apparent in the so-called “isolated flat band limit”\(^21\), where the conventional contributions to the pair mobility are negligible. In analogy with earlier findings\(^20,21\) we demonstrate that a nontrivial band topology provides a lower bound for \( 1/M_B \) in this limit.

Finally, it is important to determine the size of the geometric contributions using experimentally accessible quantities. We find that the ratio of the pairing onset temperature, \( T^* \), and \( T_{\text{BKT}} \) allows quantification of the geometric contributions and characterization of a given 2D superconductor more generally. We demonstrate how both temperatures can be determined from the same voltage-current measurements\(^36\).

**Model**

Our tight-binding model\(^25,26\) is defined on a square lattice, which splits into two sublattices, \( \{A,B\} \), due to a staggered \( \pi \) magnetic flux. The flux is opposite for opposite spins with preserved time reversal symmetry. This symmetry and the absence of spin-orbit coupling reduces the four band pair-
ing problem, including sublattices and spin, to a two-band system with sub-lattices only and we henceforth drop the spin. Note that we only consider zero-center-of-mass momentum and spin singlet pairing.

As a result we have a simple normal state Hamiltonian in \( k \) space,

\[
H_\text{N}(k) = h_0(k) + h(k) \cdot s - \mu_F,
\]

written in the basis \((c_\alpha^+, c_\alpha)(k)\). Here \( s = (s_x, s_y, s_z) \) are Pauli matrices defined for the sublattice space, \( h_0 = -2t_\Gamma \cos(2k_x + k_y) + \cos(2k_x - k_y), h_z = -2t_\Sigma(\cos(k_x + k_y) - \cos(k_x - k_y)), h_x + i h_y = -2t e^{i(\phi - k_y)} \cos k_y + e^{i(\phi - k_y)} \cos k_x, \) with \( \phi = \pi/4 \), and \( \mu_F \) is the fermionic chemical potential. We set the lattice constant \( a_L = 1 \). Diagonalizing \( H_\text{N}(k) \) gives two energy bands, \( \xi_{\pm}(k) = h_0(k) \pm |h(k)| - \mu_F \), with a nonzero Chern number \( C = \mp 1 \).

**Theory** Our approach is based on a finite temperature formalism built on the BCS ground state, which can readily be extended to include stronger pairing correlations. This approach has been used to address pairing and pseudogap phenomena in Fermi gases and the cuprates as well as the effects of spin-orbit coupling on ultracold Fermi gases, and most recently to address the two dimensional BKT transition in some simple cases. In 2D, the natural energy scale parameter, \( n_B/M_B \), enters to describe \( T_{\text{BKT}} \), where \( n_B \equiv \) the areal density of the preformed pairs.

To determine \( n_B \) and \( M_B \) we begin with the pair susceptibility \( \chi(Q) \). We assume that \( \chi(Q) \) assumes a special form (involving one dressed and one bare Green’s function) such that the \( Q = 0 \) pole of the many body T-matrix \( t_{\text{pg}}(Q) \),

\[
t_{\text{pg}}(Q) = \frac{-U}{1 - U \chi(Q)},
\]

yields the usual BCS gap equation for the pairing gap \( \Delta_{\text{pg}} \) in the fermionic excitation energy spectrum, \( E_{\pm}(k) = \sqrt{\xi_{\pm}(k)^2 + \Delta_{\text{pg}}^2} \). This \( \Delta_{\text{pg}} \) is to be distinguished from the superconducting order parameter \( \Delta_{\text{sc}} \), which vanishes at any finite \( T \) in 2D. Here \( U > 0 \) is the strength of a local attractive Hubbard interaction. \( Q \equiv (i\Omega_m, q) \) with \( \Omega_m = 2\pi/MT \) the bosonic Matsubara frequency. Expressions for \( \chi(Q), t_{\text{pg}}(Q) \), and details of the following derivations can be found in the Supplemental Materials.

Within the pseudogap approximation \( t_{\text{pg}}(Q) \) is sharply peaked near \( Q = 0 \), close to an instability, so that

\[
\Delta_{\text{pg}}^2 = -T \sum_{Q \neq 0} t_{\text{pg}}(Q).
\]

Following Refs. 16, 37, and 40, for small \( Q \), we Taylor-expand \( t_{\text{pg}}^{-1}(Q) = Z^{-1} (i\Omega_m - q^2/(2M_B) + \mu_B \), where

\[
\frac{\mu_B}{Z} = -\frac{1}{U} + \frac{\chi(0)}{U} = -\frac{1}{U} + \sum_{k \in RBZ} \sum_{\alpha = \pm} \frac{\tanh(\beta E_{\alpha}/2)}{2 E_{\alpha}}.
\]

For brevity, we have suppressed the \( k \) dependence on the r.h.s. \( \"RBZ\" \) stands for reduced Brillouin zone. \( \mu_B \) is the bosonic pair chemical potential. When \( \mu_B \) is zero Eq. (4) can be recognized as the BCS gap equation, but for the present purposes we must include non-vanishing \( \mu_B \). Note that \( t_{\text{pg}}(Q) \) can be roughly viewed as a propagator for the preformed pairs with an energy \( E_B = q^2/(2M_B) - \mu_B \). Both expressions for \( Z \) and \( 1/M_B \) are obtained as functions of \( \{\Delta_{\text{pg}}, \mu_F\} \) from the Taylor expansion.

In 2D, with a simple parabolic pair dispersion, Eq. (3) yields

\[
n_B \equiv \sum_q f_B(q) = Z^{-1} \Delta_{\text{pg}}^2 = \frac{M_B^2}{2\pi^2} \ln(1 - e^{-\beta\mu_B}),
\]

where \( \beta = 1/T \), and \( f_B(x) = 1/(e^{\beta x} - 1) \). Then we have

\[
n_B/M_B = \Delta_{\text{pg}}^2/(M_B Z) = 2 \Delta_{\text{pg}}^2 (T_{\text{conv}} + T_{\text{geom}}),
\]

where we have split the contributions to the inverse pair mass into two terms: \( T_{\text{conv}} \) is the conventional contribution that only depends on the normal state dispersion while \( T_{\text{geom}} \) is the geometric contribution that carries information about the normal state wavefunction. Here we present an expression for \( T_{\text{geom}} \) (leaving \( T_{\text{conv}} \) to the SM).

\[
T_{\text{geom}} = \sum_{k \in RBZ} \sum_{\alpha = \pm} \frac{1}{4} \left[ 1 + \eta \xi_{\alpha} / E_{\alpha} \right] \times
\]

\[
\frac{n_F(\eta E_{\alpha}) - n_F(-\xi_{\alpha})}{n_F E_{\alpha} - \xi_{\alpha}} \left( -\alpha \right) \sum_{\mu = x,y} \partial_{\mu} \hat{h} \cdot \partial_{\mu} \hat{h},
\]

where \( n_F(x) = 1/(e^{\beta x} + 1) \) is the Fermi-Dirac distribution, and \( \hat{h}(k) \equiv h(k)/|h(k)| \). Interestingly, we see that \( T_{\text{geom}} \) contains both intra- and inter-band terms.

Quantum geometry enters into \( T_{\text{geom}} \), or equivalently \( n_B/M_B \), through the diagonal components of the quantum metric tensor, \( g_{\mu\nu}(k) \):

\[
g_{\mu\nu}(k) = \frac{1}{2} \partial_{\mu} \hat{h}(k) \cdot \partial_{\nu} \hat{h}(k),
\]

where \( \{\mu, \nu\} = \{x, y\} \), \( g_{\mu\nu} \) is a measure of the distance between two Bloch states in the projective normal state Hilbert space. In the BEC regime, where \( n_B = n/2 \), \( g_{\mu\nu} \) is directly connected to the inverse pair mass \( 1/M_B \). We stress that in contrast to other work, here \( 1/M_B \) depends on the self consistently determined pairing gap.

Finally, the electrons are subject to the number constraint \( n = \sum_{k \in RBZ} \sum_{\alpha = \pm} \left[ 1 - \frac{\xi_{\alpha}}{E_{\alpha}} \tanh(\beta E_{\alpha}/2) \right] \).

\[
n = \sum_{k \in RBZ} \sum_{\alpha = \pm} \left[ 1 - \frac{\xi_{\alpha}}{E_{\alpha}} \tanh(\beta E_{\alpha}/2) \right].
\]
Equations (4), (5), and (9) form a closed set that can be solved for $\Delta_{pg}$ and $\mu_F$, for given $(T, n, U)$, which also determines the important ratio $n_B/M_B$.

**BKT criterion** It was initially proposed in Ref. 45 based on experiments in Fermi gases, that the 2D BKT superconducting transition can be re-interpreted as a “quasi-condensation” of preformed Cooper pairs. The onset of quasi-condensation provides a normal state access to the BKT instability. Here the transition is approached from above, which is complementary to the superfluid phase stiffness based approach (from below). The quasi-condensation onset is quantified through the parameter $n_B/M_B$ which provides a natural 2D energy scale. More specifically, this approach to the BKT transition builds on a Monte-Carlo study of weakly interacting bosons\(^46\) where it was found that at the onset of quasi-condensation, i. e. $T = T_{BKT}$, one has:

$$\frac{n_B(T)}{M_B(T)} = \frac{D_B^{\text{crit}}}{2\pi} T. \quad (10)$$

Here $D_B^{\text{crit}}$ is the critical value of the phase space density, $D_B(T) \equiv n_B\lambda_B^2$ with $\lambda_B = \sqrt{2\pi/M_B T}$ the bosonic thermal de-Broglie wavelength (setting $h = k_B = 1$). This BKT criterion has been supported by experimental studies on atomic Bose gases\(^47–49\).

In general $D_B^{\text{crit}}$ depends on the non-universal boson-boson interaction strength $g_B$. In the most general case, $g_B$ is unknown for a fermionic superconductor where Cooper pairs are the emergent composite bosons. However, a small value of $g_B$ appears consistent with the BCS ground state, as the bosonic degrees of freedom enter this wavefunction in a quasi-ideal manner. Notably the dependence of $D_B^{\text{crit}}$ on $g_B$ is logarithmic and therefore weak\(^46\). Estimates for $D_B^{\text{crit}}$ for fermionic superfluids range from 4.9 to 6.45\(^45,50\). We choose $D_B^{\text{crit}} = 4.9$ that best fits the data on Fermi gases\(^50\).

**Isolated flat band limit** It is useful to arrive at some analytical insights on how $n_B/M_B$ depends on the normal state band topology. This can be done in the isolated flat band limit, corresponding to $W \ll U \ll E_p$ (which is also a BEC regime). In this limit, superconductivity is restricted to the lower flat band while the upper band is inactive, and Eq. (6) simplifies to

$$\frac{n_B}{M_B} \approx \Delta_{pg}^2 \sum_{k \in \text{RBZ}} \frac{\tanh(\beta E_-(k)/2)}{2E_-(k)} \frac{1}{2} \sum_{\mu = x,y} g_{\mu\mu}(k). \quad (11)$$

Using an inequality between the quantum metric tensor and the normal state band Berry curvature, one obtains

$$\frac{n_B}{M_B} \geq \Delta_{pg}^2 \frac{\tanh(\beta E_-/2) |C|}{4E_-} \pi, \quad (12)$$

which sets a lower bound for $n_B/M_B$ when $C \neq 0$, i.e. when the system is topologically nontrivial. Here $E_-$ is $k$ independent and $C = 1$ is the normal state conduction band Chern number. Interestingly, this lower bound is almost identical to the one derived for the MF superfluid phase stiffness in Ref. 21, provided one replaces $\Delta_{pg}$ with the MF superconducting order parameter.

**Numerical Results** In Fig. 1(a) we compare the calculated $T_{BKT}$ from our pairing fluctuation theory with that using the BCS MF superfluid phase stiffness $D_s$ for $F = 0.2$. Also plotted is the pairing onset temperature, $T^*$, approximated by the mean field transition temperature. In the weak-coupling BCS limit, all three temperatures converge. However, in the strong coupling regime, pairing fluctuations become important and our $T_{BKT}$ is significantly reduced relative to its MF counterpart, as a consequence of an additional bosonic excitation channel. Unlike the single band theory, where there is a more dramatic $T_{BKT}$ downturn near $U/t \approx 3$ (see below)\(^16\), in this multi-orbital model the geometric contribution prevents the expected strong decrease.

These features can be traced to the behavior of the pair mass, $M_B$, which is plotted along with $n_B$ in Fig. 1(b). In single band theories with conventional contributions only, due to a large suppression of pair hopping\(^51\) and an increase of pair-pair repulsion with pair density\(^52\), pairs tend to be localized near $U/t \approx 3$, corresponding to $M_B \rightarrow \infty$. The presence of geometric terms prevents this pair mass divergence. Figures 1(a) and (b) reveal that, while the small $U$ behavior of $T_{BKT}$ derives from variations in both $M_B$ and $n_B$, the behavior of $T_{BKT}$ in the BEC regime reflects that of $1/M_B$ only.

To see the importance of the geometric contributions more clearly, in Fig. 1(c) we present a decomposition of $T_{BKT}$ in terms of the conventional and geometric components, by separating the total $n_B/M_B$ into two terms, $(n_B/M_B)_{\text{conv}} = 2\Delta_{pg}^2 T_{\text{BKT}}$ and $(n_B/M_B)_{\text{geom}} = 2\Delta_{pg}^2 T_{\text{geom}}$. We then apply the BKT criterion in Eq. (10) to each of $(n_B/M_B)_{\text{conv}}$, $(n_B/M_B)_{\text{geom}}$ to arrive at the three curves in Fig. 1(c). Here we see that $T_{BKT}$ is almost completely geometric at $U/t \gtrsim 3$. The conventional contribution in Fig. 1(c) exhibits a dome-like dependence on $U$ with a maximum at $U \sim W$. Its contribution to $T_{BKT}$ in the pairing fluctuation theory falls precipitously to almost zero at

**FIG. 1. Behavior of calculated (a) $T_{BKT}$ (labeled “Present theory”) and (b) $\{n_B/n, M_B\}$. (c) decomposition of $T_{BKT}$ (“Tot”) into conventional (“Conv”) and geometric contributions (“Geom”) for topological bands, and (d) $T_{BKT}$ for a non-topological system, as a function of $U/t$, all with $F = 0.2$. In comparison, also plotted in (a,d) are $T^*$ and $T_{BKT}$ (“BCS MF”) calculated using the MF phase stiffness.**
$U/t \approx 3$ and remains extremely small at larger $U$, resulting from a cancellation between pair hopping and inter-pair repulsion effects\(^{41}\).

It is instructive to compare with a non-topological superconductor, as shown in Fig. 1(d). Our non-topological bands are constructed by adding a staggered on-site potential to the topologically nontrivial Hamiltonian $H_0$ in Eq. (1). For a meaningful comparison the trivial band structure is so chosen that both its conduction band width $W$ and band gap $E_g$ are comparable to the nontrivial $F = 0.2$ case. This insures that the conventional contributions to $T_{\text{BKT}}$, as well as the $U$ dependence of $\Delta_{\text{gg}}$ and $\mu_F$, are more or less the same in both cases. Comparison of $T_{\text{BKT}}$ in Fig. 1(d) and Fig. 1(a) at $U/t \gtrsim 4$, where the geometric component dominates, demonstrates that the geometric contribution to $T_{\text{BKT}}$ is significantly enhanced in the non-trivial case.

In Fig. 2(a) we present a comparison between the MF and present theory for a nearly flat conduction band, with $F \approx 0.01$. Just as in Fig. 1(a), pairing fluctuations suppress significantly the transition temperature relative to the mean field result. Also important is the absence of the conventional $T_{\text{BKT}}$ peak, seen in Fig. 1(a). There is a small residual feature at $U/t \approx 0.35t$ from the conventional term, which, however, is invisible in the plot. In this nearly flat band limit, $T_{\text{BKT}}$ is essentially purely geometric for the entire range of $U/t$ displayed. Notably, even a very small attraction ($U/t \approx 0.3$) puts the system in the BEC regime, where $n_b/n$ reaches 1/2\(^{41}\).

Also plotted in Fig. 2(a) are the pairing onset temperature $T^*$ (dot-dashed) along with the lower bound of $T_{\text{BKT}}$ in the isolated flat band limit (black dotted line), which is obtained by applying the BKT criterion in Eq. (10) to the r.h.s. of Eq. (12). Interestingly this bound is almost saturated by our calculated $T_{\text{BKT}}$ when $0.4 \lesssim U/t \lesssim 2$\(^{41}\).

Even with the reduction of $T_{\text{BKT}}$ relative to the BCS MF result, in the isolated flat band limit, $n_b/M_B$ is essentially equal to its BCS MF counterpart $D_s$ at $T = T_{\text{BKT}}$ and even for higher temperatures, provided $T \ll T^*$. This can be seen through the comparison in Fig. 2(b) between our $n_b/M_B$ in Eq. (11) and that of the MF $D_s$, where for clarity we have dropped the small but nonzero conventional term.

We turn finally to the physical implications of our calculations for a given 2D superconductor. We quantify the relative size of the geometric terms by use of the dimensionless ratio $T^*/T_{\text{BKT}}$ which is measurable in voltage current ($V-I$) experiments\(^{56}$ with consistency checks from STM data. As shown in Figure 3(a), $T^*/T_{\text{BKT}}$ increases monotonically with interaction strength $U$ for both the topological $F = 0.2$ and $F = 0.01$ cases, with an even more rapid increase as the system approaches the BEC regime. The fractional contribution of the geometric terms, $(n_b/M_B)^{\text{geom}}/(n_b/M_B)^{\text{tot}}$, is plotted in Fig. 3(b). Once in the BEC regime, $T_{\text{BKT}}$ is dominantly geometric.

To connect to experiments on TBLG, we present the experimental $V-I$ curves for an optimal example\(^2\), in the inset of 3(b). At $T = T_{\text{BKT}}$ the $V-I$ curve follows a power law, $V \propto R_N I_c (I/I_c)^{\alpha}$ with $\alpha = 3$; $I_c$ is the critical current and $R_N$ is the normal state resistance\(^{53-58}$). Importantly, when $T$ reaches $T^*$ the $V-I$ curve fully recovers its normal state Ohmic behavior, $V \propto R_N I$.

From the $V-I$ characteristics by Cao et al.\(^2\), we estimate $T^* \approx 4K$ and $T_{\text{BKT}} \approx 1K$, which yield $T^*/T_{\text{BKT}} = 4$. At this ratio, the normalized geometric contribution is about 70% and 50% for $F = 0.01$ and 0.2, respectively, in Fig. 3(b). This suggests that the system is in the intermediate BCS-BEC crossover regime, and has not yet passed into the BEC regime. However, we note that the $T^*/T_{\text{BKT}}$ ratio inferred from the $V-I$ measurements is somewhat variable in different experiments\(^{59-61}\). Further experiments are needed to firmly settle where magic angle TBLG is in the BCS-BEC spectrum. In future data, it would be useful to have a more continuous variation of the temperature scales to establish the Ohmic recovery point, $T^*$, more accurately.

In summary, we have established the quantum geometric contribution to superfluidity in a pair-fluctuation theory, where these contributions modify the pair mass. In general the quantum geometric contribution plays a dominant role in the strong coupling BEC regime. It restricts pairs from becoming infinitely heavy in a perfectly flat band, making them more mobile. We further show how to quantify the magnitude of the geometric contributions in a multi-orbital 2D superconductor in terms of the $T^*/T_{\text{BKT}}$ ratio. Our analysis was based on...
important experimental observations which have shown that the two temperature scales characterizing a 2D superconductor ($T_{BKT}$ and $T^*$) can be extracted from $V - J$ plots. We have ended this paper by presenting speculations on magic angle TBLG, concerning the size of the geometric terms and the location of this exotic superconductor within the BCS-BEC crossover. Despite our oversimplified band structure, our identification of these temperatures and their (measurable) ratio sets up a template which should be broadly useful in future to both theoretical and experimental communities.

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1. Y. Cao, V. Fatemi, A. Demir, S. Fang, S. L. Tomarken, J. Y. Luo, J. D. Sanchez-Yamagishi, K. Watanabe, T. Taniguchi, E. Kaxiras, *et al.*, Nature 556, 80 (2018).
2. Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Nature 556, 43 (2018).
3. R. Bistritzer and A. H. MacDonald, Proceedings of the National Academy of Sciences 108, 12233 (2011).
4. F. Wu, A. H. MacDonald, and I. Martin, Phys. Rev. Lett. 121, 257001 (2018).
5. I. F. Dodaro, S. A. Kivelson, Y. Schattner, and C. Wang, Phys. Rev. B 98, 075154 (2018).
6. J. Kang and O. Vafek, Phys. Rev. X 8, 031088 (2018).
7. M. Yankowitz, S. Chen, H. Polshyn, Y. Zhang, K. Watanabe, T. Taniguchi, D. Graf, A. F. Young, and C. R. Dean, Science 363, 1059 (2019).
8. N. F. Q. Yuan and L. Fu, Phys. Rev. B 98, 045103 (2018).
9. H. C. Po, L. Zou, A. Vishwanath, and T. Senthil, Phys. Rev. X 8, 031089 (2018).
10. C. Xu and L. Balents, Phys. Rev. Lett. 121, 087001 (2018).
11. B. Roy and V. Juricic, Phys. Rev. B 99, 121407 (2019).
12. H. Isobe, N. F. Q. Yuan, and L. Fu, Phys. Rev. X 8, 041041 (2018).
13. H. C. Po, L. Zou, T. Senthil, and A. Vishwanath, Phys. Rev. B 99, 195455 (2019).
14. G. Tarnopolsky, A. J. Kruchkov, and A. Vishwanath, Phys. Rev. Lett. 122, 106405 (2019).
15. Y. J. Uemura, Journal of Physics: Condensed Matter 16, S4515 (2004).
16. X. Wang, Q. Chen, and K. Levin, New Journal of Physics 22, 063050 (2020).
17. V. Berezinskii, Sov. Phys. JETP 34, 610 (1971).
18. J. M. Kosterlitz and D. J. Thouless, Journal of Physics C: Solid State Physics 6, 1181 (1973).
19. L. Benfatto, C. Castellani, and T. Giamarchi, Phys. Rev. Lett. 98, 117008 (2007).
20. S. Peotta and P. Törmä, Nature communications 6, 8944 (2015).
21. L. Liang, T. I. Vanhala, S. Peotta, T. Siro, A. Harju, and P. Törmä, Phys. Rev. B 95, 024515 (2017).
22. X. Hu, T. Hyart, D. I. Pikulin, and E. Rossi, Phys. Rev. Lett. 123, 237002 (2019).
23. A. Julku, T. J. Peltonen, L. Liang, T. T. Heikkilä, and P. Törmä, Phys. Rev. B 101, 060505 (2020).
24. F. Xie, Z. Song, B. Lian, and B. A. Bernevig, Phys. Rev. Lett. 124, 167002 (2020).
25. J. S. Hofmann, E. Berg, and D. Chowdhury, arXiv preprint arXiv:1912.08848 (2019).
26. T. Neupert, L. Santos, C. Chamion, and C. Mudry, Phys. Rev. Lett. 106, 236804 (2011).
27. L. He, X.-F. Huang, H. Hu, and X.-J. Liu, Phys. Rev. A 87, 053616 (2013).
28. J. Zhang, H. Hu, X.-J. Liu, and H. Pu, “Annual review of cold atoms and molecules,” (World Scientific, 2014) Chap. 2, pp. 81–143.
29. Z. Fu, L. Huang, Z. Meng, P. Wang, X.-J. Liu, H. Pu, H. Hu, and J. Zhang, Phys. Rev. A 87, 053619 (2013).
30. Z. Zheng, H. Pu, X. Zou, and G. Guo, Phys. Rev. A 90, 063623 (2014).
31. C.-T. Wu, B. M. Anderson, R. Boyack, and K. Levin, Phys. Rev. B 91, 220504 (2015).
32. A. J. Leggett, in Modern trends in the theory of condensed matter (Springer, 1980) pp. 13–27.
33. Q. Chen, I. Kosztin, and K. Levin, Phys. Rev. Lett. 85, 2801 (2000).
34. N. Marzari and D. Vanderbilt, Phys. Rev. B 56, 12847 (1997).
35. N. Marzari, A. A. Mostofi, J. R. Yates, I. Souza, and D. Vanderbilt, Rev. Mod. Phys. 84, 1419 (2012).
36. W. Zhao, Q. Wang, M. Liu, W. Zhang, Y. Wang, M. Chen, Y. Guo, K. He, X. Chen, Y. Wang, J. Wang, X. Xie, Q. Niu, L. Wang, X. Ma, J. K. Jain, M. Chan, and Q.-K. Xue, Solid State Communications 165, 59 (2013).
37. Q. Chen, J. Stajic, S. Tan, and K. Levin, Physics Reports 412, 1 (2005).
38. Q. Chen, I. Kosztin, B. Jankó, and K. Levin, Phys. Rev. B 59, 7083 (1999).
39. J. Maly, B. Jankó, and K. Levin, Physica C: Superconductivity 321, 113 (1999).
40. C.-T. Wu, B. M. Anderson, R. Boyack, and K. Levin, Phys. Rev. Lett. 115, 240401 (2015).
41. See Supplemental Material at [ ] for descriptions of the tight-binding model, derivations of the multi-orbital pairing fluctuation theory, discussions on connection between quantum geometry and non-condensed pair mass, equations used for BCS MF superfluid phase stiffness, additional numerical results, and further discussions on where the magic angle TBLG is in the BCS-BEC spectrum as well as on relation between our work and the literature.
42. J. P. Provost and G. Vallee, Communications in Mathematical Physics 76, 289 (1980).
43. M. Iskin, Phys. Rev. A 97, 033625 (2018).
44. M. Tovmasyan, S. Peotta, P. Törmä, and S. D. Huber, Phys. Rev. B 94, 245149 (2016).
45. M. G. Ries, A. N. Wenz, G. Zürn, L. Bayha, I. Boettcher, D. Kedar, P. A. Murthy, M. Neidig, T. Lompe, and S. Jochim, Phys. Rev. Lett. 114, 230401 (2015).
46. N. Prokof’ev and B. Svistunov, Phys. Rev. A 66, 043608 (2002).
47. J. V. Jose, 40 years of Berezinskii-Kosterlitz-Thouless theory (World Scientific, 2013).
48. S. Tung, G. Lamporesi, D. Lobser, L. Xia, and E. A. Cornell, Phys. Rev. Lett. 105, 230408 (2010).
49. P. Cladé, C. Ryu, A. Ramanathan, K. Helmerson, and W. D. Phillips, Phys. Rev. Lett. 102, 170401 (2009).
50. P. A. Murthy, I. Boettcher, L. Bayha, M. Holzmann, D. Kedar, M. Neidig, M. G. Ries, A. N. Wenz, G. Zürn, and S. Jochim,
Phys. Rev. Lett. 115, 010401 (2015).
51 P. Nozières and S. Schmitt-Rink, Journal of Low Temperature Physics 59, 195 (1985).
52 R. Micnas, J. Ranninger, and S. Robaszkiewicz, Rev. Mod. Phys. 62, 113 (1990).
53 B. Halperin and D. R. Nelson, Journal of low temperature physics 36, 599 (1979).
54 K. Epstein, A. M. Goldman, and A. M. Kadin, Phys. Rev. Lett. 47, 534 (1981).
55 D. J. Resnick, J. C. Garland, J. T. Boyd, S. Shoemaker, and R. S. Newrock, Phys. Rev. Lett. 47, 1542 (1981).
56 A. F. Hebard and A. T. Fiory, Phys. Rev. Lett. 50, 1603 (1983).
57 A. T. Fiory, A. F. Hebard, and W. I. Glaberson, Phys. Rev. B 28, 5075 (1983).
58 A. M. Kadin, K. Epstein, and A. M. Goldman, Phys. Rev. B 27, 6691 (1983).
59 X. Lu, P. Stepanov, W. Yang, M. A. Aamir, I. Das, C. Urgell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, Nature 574, 653 (2019).
60 P. Stepanov, I. Das, X. Lu, A. Fahimniya, K. Watanabe, T. Taniguchi, F. H. Koppens, J. Lischner, L. Levitov, and D. K. Efetov, Nature 583, 375 (2020).
61 Y. Cao, D. Rodan-Legrain, J. M. Park, F. N. Yuan, K. Watanabe, T. Taniguchi, R. M. Fernandes, L. Fu, and P. Jarillo-Herrero, arXiv preprint arXiv:2004.04148 (2020).
Supplemental Material for “Quantum Geometric Contribution to the BKT Transition: Beyond Mean Field Theory”

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I. TIGHT-BINDING MODEL

Our model is defined on a square lattice, with a kinetic energy contribution to the Hamiltonian, $H_K$, given by

$$H_K = \left\{ -t \sum_{\langle i,j \rangle} e^{i\phi_{ij}^c} c_{i\sigma}^\dagger c_{j\sigma}^c - t_2 \sum_{(i,j)_{2\sigma}} s_{(i,j)_{2\sigma}} c_{i\sigma}^\dagger c_{j\sigma}^c$$

$$- t_5 \sum_{(i,j)_{5\sigma}} c_{i\sigma}^\dagger c_{j\sigma}^c \right\} + \hbar \epsilon_c - \mu_F \sum_i n_i. \quad (S1)$$

Here $c_{i\sigma}^\dagger (c_{i\sigma})$ are electron creation (annihilation) operators at site $i$ for spin $\sigma$. $(t, t_2, t_5)$ are the magnitudes of the hopping integrals defined for the nearest neighbor (NN), second NN, and the fifth NN bond on the square lattice, respectively. $\mu_F$ is the fermionic chemical potential, and $n_i = \sum_{\sigma = \uparrow, \downarrow} c_{i\sigma}^\dagger c_{i\sigma}$ is the electron number at site $i$. The NN hopping amplitude is modulated by the phase $e^{i\phi_{ij}^c}$, where $\phi_{ij}^c = s_{ij} (\pi/4)$ if the hopping is along the direction of the arrows depicted in Fig. S1. $s_{ij} = +1 (-1)$ for spin $\uparrow (\downarrow)$. Because of $\phi_{ij}^c$ there is a net $\pm \pi$ flux through each square plaquette for given spin. This flux is staggered from one plaquette to the next (see Fig. S1), which breaks the original lattice translational symmetry and leads to two different sublattices $\{A, B\}$. However, time reversal symmetry is still preserved, because $\phi_{ij}^c$ are opposite for opposite spin $\sigma$ so that the total flux through each plaquette is zero. The sign of the second NN hopping amplitudes, $s_{(i,j)_{2\sigma}} = \pm$, is also staggered, as shown in Fig. S1.

Fourier transforming $H_K$ to $k$ space one finds the following block-diagonal Hamiltonian

$$H_K(k) = \left( \begin{array}{cc} H_{\uparrow}(k) & 0 \\ 0 & H_{\downarrow}(k) \end{array} \right), \quad (S2)$$

in the basis $\{ c_{A,\uparrow}^\dagger (k), c_{B,\uparrow}^\dagger (k), c_{A,\downarrow}^\dagger (k), c_{B,\downarrow}^\dagger (k) \}$. The diagonal block operating on the same spin is

$$H_{\sigma}(k) = h_0(k) + \hbar(k, \phi_{\sigma}) \cdot s - \mu_F, \quad (S3)$$

where $s = (s_x, s_y, s_z)$ are the three Pauli matrices defined for the sublattice space and

$$h_0(k) = -2t_0 \left[ \cos(2k_x + k_y) + \cos(2k_x - k_y) \right], \quad \text{(S4a)}$$

$$h_z(k) = -2t_2 \left[ \cos(k_x + k_y) - \cos(k_x - k_y) \right], \quad \text{(S4b)}$$

$$h_x(k, \phi_{\sigma}) + i \hbar y(k, \phi_{\sigma})$$

$$= -2t e^{i(\phi_{\sigma} - k_x)} \cos k_y - 2t e^{i(\phi_{\sigma} - k_x)} \cos k_x. \quad \text{(S4c)}$$

FIG. S1. (a) The tight binding model for $H_K$. $\{A, B\}$ denote two different sub-lattices, resulting from a staggered $\pi$ flux. The NN hopping amplitudes are $t e^{in\pi/4}$ for spin $\uparrow$ along the direction depicted by the arrows. Black dashed and blue dotted lines show the second NN bond with which the associated hopping amplitudes are $t_2$ and $-t_2$, respectively. There is also a uniform hopping between the fifth NN sites, which is not shown for clarity. (b) Fermi surfaces (FS), in blue, for the band flatness ratio $F = 0.2$ at electron density $n = 0.3$ per site. The regime bounded by the red dashed lines defines the reduced Brillouin zone (RBZ). (c) Corresponding band structure for $F = 0.2$. In the vertical axis, $\epsilon_k = h_0(k) \pm |\hbar(k)|$. (d) Band structure for $F = 0.01$. 

φσ = sσ(π/4). Diagonalizing $H_K(k)$ gives two energy bands, $ξ_±(k) = h_0(k) ± |h(k,φσ)| - \mu_F$, each of which are two-fold degenerate due to the spin. The two bands have a nonzerospin dependent Chern number $C_{ασ} = -αs_{σ}$, where $α = ±$.

Although $H_σ(k)$ depends on spin due to $φ_σ$, the final result of the time reversal invariant quantity, $n_σ/M_σ$ which determines the temperature, $T_{BKT}$, in our theory, is spin independent (see Sec. II). Therefore, in the main text we drop the spin and keep only the spin ↑ block Hamiltonian, i.e. $H_N ≡ H_τ$.

A. Non-topological model Hamiltonian

In Fig. 1(d) of the main text we also considered a topologically trivial band structure with zero Chern number. The corresponding trivial Hamiltonian is obtained from $H_K(k)$ by adding a staggered on-site potential term

$$H_K^{\text{trivial}}(k) = H_K(k) + m_z s_z ⊗ σ_0,$$

where $σ_0$ is the identity matrix in spin space. The resultant bands from $H_K^{\text{trivial}}(k)$ are trivial if $|m_z| > 4t_2$. Using $(t, t_2, t_3, m_z) = (1, 0.02, 0, -3)$ gives a two-band model with $W ≈ 1.2t$ and $E_g ≈ 5.8t$, corresponding to $F = 0.2$. $W$ and $E_g$ are comparable to those of the topological $F = 0.2$ band.

B. Attractive interaction

For the interaction we choose a local attractive Hubbard model

$$V = -U \sum_i n_{i,↑} n_{i,↓},$$

where $U > 0$. We do not discuss the possible origin of this attractive interaction in TBLG, which is not important for our purposes.

II. MULTI-ORBITAL BCS-BASED PAIRING FLUCTUATION THEORY

In the main text, we have sketched the derivation of our pairing fluctuation theory and outlined the main equations used. In this section we present the details. We first derive the expression for our pairing susceptibility and the corresponding many-body T-matrix. From the two we then obtain the two central quantities for our calculation of $T_{BKT}$, $n_B$ and $M_B$ of the preformed pairs.

A. Pairing susceptibility and many-body T-matrix $t_{pg}(Q)$

Our pairing fluctuation theory is one type of the many BCS-BEC crossover theories. The central assumption behind most of these theories is that even though the original BCS theory is a weak coupling one, the variational BCS ground state wavefunction has a wider applicability that goes beyond weak coupling. Our theoretical framework is designed such that the $T = 0$ ground state in this theory is identical to the BCS ground state and at the same time it includes pairing fluctuation effects at finite $T$. Therefore, to derive such a theory for our multi-orbital system, we first consider the corresponding BCS mean field problem.

Within the BCS mean field, the Cooper pairing instability can be derived from the pairing vertex function $Γ(Q)$. Assuming a local $s$-wave singlet pairing order parameter $Δ_{sc}(k) = Δ_σ iσ_y$, one can show that

$$\frac{1}{Γ(Q)} = \frac{1}{U} + \chi_0(Q),$$

$$\chi_0(Q) = \frac{T}{2} \sum_K \text{Tr}[G_0(K)iσ_y G_0(K - Q)(-iσ_y)].$$

$\chi_0(Q)$ is the bare pairing susceptibility. $K = (ω_n, k)$ with $ω_n = (2n + 1)πT$ is the fermionic Matsubara frequency. The summation over $k$ should be restricted to the Brillouin zone due to the unit cell doubling in real space. The trace is with respect to both sublattice and spin. $G_0(K)$ and $\tilde{G}_0(K)$ are the normal state electronic and hole Green’s function matrices, whose definitions are

$$G_0(K) = 1/(iω_n - H_K(k));$$

$$\tilde{G}_0(K) ≡ [G_0(-K)]^T.$$

$1/Γ(Q) = 0$ defines the BCS mean field $T_{c,BCS}$, which will be taken as an estimate for the pairing onset temperature $T^*$ in our theory, i.e., $T^* = T_{c,BCS}$.

Correspondingly, the mean field BCS gap equation for $Δ_{sc}$ is given by

$$-\frac{1}{U} + \frac{T}{2} \sum_K \text{Tr}[G(K)iσ_y \tilde{G}_0(K)(-iσ_y)] = 0,$$

where $G(K)$ is the electron Green’s function with the superconducting pairing self energy $Σ_{sc}(K)$ included

$$[G(K)]^{-1} = [G_0(K)]^{-1} - Σ_{sc}(K),$$

$$Σ_{sc}(K) = Δ^2_{sc} \tilde{G}_0(K).$$

The zero temperature solution of $Δ_{sc}$ to the above gap equation gives the BCS ground state.

Now we construct the pairing fluctuation theory. To account for the effects of scattering from non-condensed pairs on fermions, we include another pairing self energy, $Σ_{pg}$, into the dressed electronic Green’s function $G$

$$[G(K)]^{-1} = [G_0(K)]^{-1} - Σ_{pg}(K).$$

$Σ_{pg}$ results from scatterings of electrons from non-condensed pairs, to be distinguished from $Σ_{sc}$ which represents a true condensate. In three dimension (3D) we should include $Σ_{sc}$ as in the BCS mean field theory. In 2D and at finite temperature, which is what we focus on, $Σ_{sc} ≡ 0$ since there is no true long range superconducting order parameter.
\[ \Sigma_{pg}(K) \] is related to the many-body T-matrix \( t_{pg}(Q) \) by

\[ \Sigma_{pg}(K) = -T \sum_{Q \neq 0} t_{pg}(Q) \tilde{g}_0(K - Q). \quad (S15) \]

All pair scattering effects are encapsulated in \( t_{pg}(Q) \). Under the T-matrix approximation that has been widely used to understand BCS-BEC crossovers\(^3\)\(^4\),

\[ \frac{1}{t_{pg}(Q)} = -\frac{1}{U} + \chi(Q), \quad (S16) \]

where

\[ \chi(Q) = \frac{T}{2} \sum_K \text{Tr}[G(K)i\sigma_y \tilde{g}_0(K - Q)(-i\sigma_y)]. \quad (S17) \]

In the course of the developments of BCS-BEC crossover theories, there was a debate on whether the two Green’s functions used in the expression of \( \chi(Q) \) should be \( \tilde{g}_0 \tilde{G}_0 \) or \( \tilde{G} \tilde{G}_0 \). We choose the asymmetric form, \( \tilde{G} \tilde{G}_0 \), so that in 3D, when the superconducting transition is interpreted as a BEC of Cooper pairs, the ground state of this pairing fluctuation theory is given by the BCS wavefunction\(^7\). This is reflected in the pole structure of the T-matrix, determined by \( 1/t_{pg}(0) = 0 \) which yields the usual BCS gap equation for \( \Delta_{pg} \). It should be noted that the asymmetric form \( \tilde{G} \tilde{G}_0 \) can in fact be derived within the equation of motion approach\(^5\)\(^6\).

To proceed further, we note that for small pair chemical potential we may approximate \( t_{pg}(Q) \), noting that it is sharply peaked near \( Q = 0 \) so that Eq. (S15) can be written as

\[ \Sigma_{pg}(K) \approx \Delta_{pg}^2 \tilde{g}_0(K), \quad (S18) \]

\[ \Delta_{pg}^2 \equiv -T \sum_{Q \neq 0} t_{pg}(Q). \quad (S19) \]

We refer to this as the “pg approximation”, which (near the superconducting instability) is supported by numerical evidence\(^7\). Eq. (S18) is an analog to the BCS pairing self energy given in Eq. (S13). Just as in the BCS mean field theory, the above form of \( \Sigma_{pg}(K) \) leads to a pseudogap \( \Delta_{pg} \) in the fermionic excitation energy spectrum \( E_{\pm}(k) = \sqrt{\xi_{\pm}(k)^2 + \Delta_{pg}^2} \), which reflects the binding strength of non-condensed Cooper pairs.

For the Hamiltonian that is block diagonal in Eq. (S2), we can carry out the spin trace in Eq. (S17) and write

\[ \chi(Q) = \frac{1}{2} \left[ \chi_{\uparrow\downarrow}(Q) + \chi_{\downarrow\uparrow}(Q) \right], \quad (S20) \]

where

\[ \chi_{\sigma\sigma}(Q) = T \sum_K \text{Tr}[G_\sigma(K)\tilde{g}_{0,\sigma}(K - Q)]. \quad (S21) \]

\[ \sigma = \uparrow (\downarrow) \text{ if } \sigma = \downarrow (\uparrow). \quad G_\sigma \text{ and } \tilde{g}_{0,\sigma} \text{ are the spin } \sigma \text{ block of } G \text{ and the spin } \bar{\sigma} \text{ block of } \tilde{G}_0, \text{ respectively.} \]

Substituting the definitions of \( G_\sigma(K) \) and \( \tilde{g}_{0,\sigma}(K) \) into the expression of \( \chi_{\sigma\bar{\sigma}}(Q) \) and completing the fermionic Matsubara sum, one gets

\[ \chi_{\sigma\bar{\sigma}}(Q) = \sum_{k \in RBZ} \sum_{\{\alpha,\alpha',\sigma\}} \frac{1}{2} \left[ 1 + \eta \frac{\xi_\alpha(k)}{E_\alpha(k)} \right] \frac{n_F(\eta E_\alpha(k)) - n_F(-\xi_\alpha(k - q))}{n_F(\eta E_\alpha(k)) - n_F(-\xi_\alpha(k - q))} \text{Tr}[\hat{P}_{\alpha,\sigma}(k)\hat{P}_{\alpha',\bar{\sigma}}(k - q)], \quad (S22) \]

where \( n_F(x) = 1/(e^{\beta x} + 1) \) with \( \beta = 1/T \) is the Fermi-Dirac distribution function and \( \text{Tr}[\cdots] \) is with respect to the sublattice subspace.

\[ \hat{P}_{\alpha,\sigma}(k) \equiv \frac{1}{2} \left[ 1 + \alpha \hat{h}(k, \phi_\sigma) \cdot \hat{s} \right] \quad (S23) \]

is the projection operator defined for the normal state band with energy \( \xi_\alpha(k) \) and spin \( \sigma \). \( \hat{h}(k, \phi_\sigma) \equiv \hat{h}(k, \phi_\sigma)/|\hat{h}(k, \phi_\sigma)| \). Carrying out the trace in Eq. (S22) leads to

\[ \text{Tr}[\hat{P}_{\alpha,\sigma}(k)\hat{P}_{\alpha',\bar{\sigma}}(k - q)] = \frac{1}{2} \left[ 1 + \alpha \alpha' \hat{h}(k, \phi_\sigma) \cdot \hat{h}(k - q, \phi_\sigma) \right], \quad (S24) \]

\[ \chi(Q) \approx \chi(0) + b \Omega_m - c q^2, \quad (S25) \]

B. Small \( Q \) expansion of \( \chi(Q) \)

Within the “pg approximation” one can make the following small \( Q \) expansion for \( \chi(Q) \)\(^4\)\(^8\)\(^9\),

\[ \chi(Q) \approx \chi(0) + b \Omega_m - c q^2, \quad (S25) \]
where $\Omega_m = 2m\pi T$ is the bosonic Matsubara frequency, and

$$
\chi(0) = \sum_{k \in \text{RBZ}} \frac{1 - 2n_F(E_\alpha)}{2E_\alpha}, \quad (S26a)
$$

$$
b = - \sum_{k \in \text{RBZ}} \sum_{\{\alpha, \eta\} = \pm} \frac{\eta n_F(\eta E_\alpha) - n_F(-\xi_\alpha)}{2E_\alpha} \eta E_\alpha + \xi_\alpha,
$$

$$
c = -\frac{1}{2} \frac{\partial^2}{\partial q^2} \chi(Q) \bigg|_{Q=0} \equiv T_{\text{conv}} + T_{\text{geom}}. \quad (S26c)
$$

Here to determine the coefficient $c$, we use only the $q^2$ component of the $\chi(Q)$ expansion, since the system possesses a $C_4$ rotational symmetry.

For our later discussion on quantum geometry we have broken up $c$ into two separate terms,

$$
T_{\text{conv}} = \sum_{k \in \text{RBZ}} \sum_{\{\alpha, \eta\} = \pm} \frac{\eta}{4E_\alpha} \left\{ (\partial_x \xi_\alpha)^2 \left[ 2\frac{n_F(\eta E_\alpha) - n_F(-\xi_\alpha)}{(\eta E_\alpha + \xi_\alpha)^2} + \frac{\beta n_F(\xi_\alpha) n_F(-\xi_\alpha)}{\eta E_\alpha + \xi_\alpha} \right] - \partial_x^2 \xi_\alpha \frac{n_F(\eta E_\alpha) - n_F(-\xi_\alpha)}{\eta E_\alpha + \xi_\alpha} \right\},
$$

$$
T_{\text{geom}} = \sum_{k \in \text{RBZ}} \sum_{\{\alpha, \eta\} = \pm} \frac{1}{4} \left[ 1 + \eta \frac{\xi_\alpha}{E_\alpha} \right] \frac{n_F(\eta E_\alpha) - n_F(-\xi_\alpha)}{\eta E_\alpha + \xi_\alpha} (-\alpha') \frac{1}{2} \partial_x \hat{h} \cdot \partial_x \hat{h},
$$

(S27a)

(S27b)

where $\partial_x \equiv \partial_{k_x}$, and, for brevity, we have suppressed the $k$ dependence. The conventional term, $T_{\text{conv}}$, is derived from the $q_x$ derivative of the factors other than $\text{Tr}[\cdots]$ in Eq (S22); while the geometric term, $T_{\text{geom}}$, comes solely from that of the trace factor,

$$
\partial^2_{q_x} \text{Tr}[\hat{P}_{\alpha, \sigma}(k)\hat{P}_{\alpha', \sigma}(k-q)] \bigg|_{q=0} = (-\alpha') \frac{1}{2} \partial_{k_x} \hat{h}(k, \phi_\sigma) \cdot \partial_{k_x} \hat{h}(k, \phi_\sigma).
$$

(S28)

$T_{\text{geom}}$ depends on not only the normal state energy dispersion but also its wavefunctions, through the projection operators in the trace factor. This is in sharp contrast to $T_{\text{conv}}$. The scalar product, $\frac{1}{2} \partial_x \hat{h} \cdot \partial_x \hat{h}$, can be identified with the $xx-$component of the quantum metric tensor which will be defined and discussed in detail in Sec. III.

We note that although $\hat{h}(k, \phi_\sigma)$ depends on spin due to $\phi_\sigma$, $\frac{1}{2} \partial_x \hat{h} \cdot \partial_x \hat{h}$ does not because it is even in the sign of $\phi_\sigma$. As a result, $\chi(0), b, c$ are all spin independent. So are the characteristic parameters for the non-condensed bosons such as $n_B$ and $M_B$.

### C. $n_B$ and $M_B$

Next we calculate $n_B$ and $M_B$ from $\{\chi(0), b, c\}$. Substituting Eq. (S25) into Eq. (S16) leads to

$$
t_{\text{pg}}(Q) \approx \frac{Z}{i\Omega_m - q^2/(2M_B) + \mu_B}, \quad (S29)
$$

where

$$
\chi(0) = \sum_{k \in \text{RBZ}} \frac{1 - 2n_F(E_\alpha)}{2E_\alpha},
$$

and, for brevity, we have suppressed the $k$ dependence. The conventional term, $T_{\text{conv}}$, is derived from the $q_x$ derivative of the factors other than $\text{Tr}[\cdots]$ in Eq (S22); while the geometric term, $T_{\text{geom}}$, comes solely from that of the trace factor,

$$
\partial^2_{q_x} \text{Tr}[\hat{P}_{\alpha, \sigma}(k)\hat{P}_{\alpha', \sigma}(k-q)] \bigg|_{q=0} = (-\alpha') \frac{1}{2} \partial_{k_x} \hat{h}(k, \phi_\sigma) \cdot \partial_{k_x} \hat{h}(k, \phi_\sigma).
$$

(S27a)

(S27b)

The quantity $t_{\text{pg}}(Q)$ in Eq. (S29) can be interpreted as the propagator for non-condensed pairs with an energy dispersion $E_B = q^2/2M_B - \mu_B$, with $M_B$ the effective pair mass and $\mu_B$ the corresponding bosonic chemical potential. Then from Eqs. (S19) and (S29) one can relate the areal density of non-condensed pairs, $n_B$, to $\Delta_{\text{pg}}^2$ by

$$
n_B \equiv \sum_q f_B(E_B) = \frac{\Delta_{\text{pg}}^2}{Z} = \frac{M_B}{2\pi \beta} \left\{ -\ln[1 - e^{\beta \mu_B}] \right\}, \quad (S31)
$$

where $f_B(x) = 1/(e^{\beta x} - 1)$ is the Bose-Einstein distribution. To obtain the r.h.s. of the last equality we have neglected the upper bound in the $q$ summation which is associated with a lattice. This is consistent with the $\text{pg}$ approximation which implies, near the instability, a fast decrease of $t_{\text{pg}}(Q)$ at large $Q$.

Eqs. (S30) and (S31) combined together yield one independent nonlinear equation for two unknowns, $\Delta_{\text{pg}}$ and $\mu_B$, in terms of $\{T, n, U\}$. The other independent equation comes from the electron density constraint,$^4,8,9$

$$
n = \sum_{k \in \text{RBZ}} \sum_{\alpha = \pm} \left[ 1 - \frac{\xi_\alpha(k)}{E_\alpha(k)} \tanh \frac{\beta E_\alpha(k)}{2} \right]. \quad (S32)
$$

Solving the combined Eqs. (S30) to (S32) for given $\{T, n, U\}$ numerically we are able to compute $\Delta_{\text{pg}}$ and $\mu_B$, from which
show \( n_B/M_B \) is lower bounded by the nontrivial band topology.

The quantum metric tensor, \( g_{\alpha\sigma}^{\mu\nu}(k) \) with \( \{\mu, \nu\} = \{x, y\} \), is defined for each \( \alpha\sigma \) normal state band. It represents a distance in the projective Hilbert space between two states \( |\psi_{\alpha\sigma}(k)\rangle \) and \( |\psi_{\alpha\sigma}(k + dk)\rangle \):

\[
\Delta^2_{pg}(k) = 2 \left| g_{\alpha\sigma}^{\mu\nu}(k) dk_{\mu} dk_{\nu} + O((dk)^2) \right|^2.
\]

Here \( |\psi_{\alpha\sigma}(k)\rangle \) is an eigenstate of \( H_k(k) \) in Eq. (S2) with the quantum number \( \alpha = \pm \) and \( \sigma = \{+, \cdot\} \). Note that \( g_{\alpha\sigma}^{\mu\nu}(k) \) is independent of the arbitrary \( U(1) \) phase of \( |\psi_{\alpha\sigma}(k)\rangle \), and is therefore gauge invariant. By definition it is also positive definite.

The quantum metric tensor can be combined with the Berry curvature, \( F_{\mu\nu}^{\alpha\sigma} \), to define a quantum geometric tensor \( R_{\mu\nu}^{\alpha\sigma} \):

\[
R_{\mu\nu}^{\alpha\sigma} \equiv 2 \text{Tr}[\hat{P}_{\alpha\sigma} \partial_\mu \hat{P}_{\alpha\sigma} \partial_\nu \hat{P}_{\alpha\sigma}] = g_{\alpha\sigma}^{\mu\nu} + iF_{\mu\nu}^{\alpha\sigma}/2.
\]

Both \( g_{\alpha\sigma}^{\mu\nu} \) and \( F_{\mu\nu}^{\alpha\sigma} \) are real. Using the definition of \( \hat{P}_{\alpha\sigma} \) in Eq. (S23) one obtains

\[
F_{\mu\nu}^{\alpha\sigma}(k) = \alpha \epsilon_{\mu\nu} \hat{h}(k, \phi_\sigma) \times [\partial_\mu \hat{h}(k, \phi_\sigma) \times \partial_\nu \hat{h}(k, \phi_\sigma)],
\]

where \( \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \) is the Levi-Civita symbol. \( g_{\alpha\sigma}^{\mu\nu} \) is even under time reversal, and therefore independent of the spin \( \sigma \). In contrast, \( F_{\mu\nu}^{\alpha\sigma} \) is odd under time reversal, and therefore opposite for opposite spin. As a result, \( g_{\alpha\sigma}^{\mu\nu} \) in Eq. (S37a) is independent of \( \{\alpha\sigma\} \) for our model.

From its definition one can prove that \( R_{\mu\nu}^{\alpha\sigma} \) is positive definite, resulting an inequality between \( g_{\alpha\sigma}^{\mu\nu} \) and \( F_{\mu\nu}^{\alpha\sigma} \):

\[
g_{\alpha\sigma}^{\mu\nu} g_{\alpha\sigma}^{\nu\mu} \geq (F_{\mu\nu}^{\alpha\sigma})^2/4.
\]

The inequality implies

\[
\text{Tr}[g_{\alpha\sigma}^{\mu\nu}] \geq 2\sqrt{g_{\alpha\sigma}^{\mu\nu} g_{\alpha\sigma}^{\nu\mu}} \geq |F_{\mu\nu}^{\alpha\sigma}|.
\]

Here Tr is with respect to \( \{\mu, \nu\} \). Eq. (S38) shows that in general a nonzero Chern number, which necessarily implies a nonzero \( |F_{\mu\nu}^{\alpha\sigma}| \), enhances the magnitude of the quantum metric tensor. The physics behind this can be understood in terms of “Wannier obstruction”. Normal state Wannier functions \( |\psi_{\alpha\sigma}(R)\rangle \) can be constructed from the Bloch wavefunction \( |\psi_{\alpha\sigma}(k)\rangle \) in general not gauge invariant because of the \( U(1) \) phase ambiguity in defining \( |\psi_{\alpha\sigma}(k)\rangle \). Consequently, the spatial spread of \( |\psi_{\alpha\sigma}(R)\rangle \) contains both a gauge invariant and non-invariant part.

The enhancement of \( \text{Tr}[g_{\alpha\sigma}^{\mu\nu}] \) due to nontrivial band topology also affects the pairing state through \( n_B/M_B \). The latter reflects the degree of delocalization of the non-condensed pairs. Both a larger \( n_B \) and smaller \( M_B \) imply a larger overlap between individual pair wavefunctions, and therefore more
delocalized pairs. How delocalized the pairs are must be connected to how delocalized the normal state Wannier orbitals are. Therefore, it is not surprising that $g_{ij}^{\alpha\sigma}$, which provides a measure of how delocalized the normal states are, enters the expression of $n_B/M_B$ through $T_{\text{con}}$ in Eq. (S27b). However, $g_{ij}^{\alpha\sigma}$ appears in a complicated way because both the two normal bands can contribute, and because both intra- and inter-band processes matter. Interestingly, the inter- and intra-band contributions in Eq. (S27b) carry opposite signs; the former partially cancels the latter which is positive.

The above qualitative discussion suggests that in general, a nontrivial band topology enhances the quantum metric, which in turn increases $n_B/M_B$. This emerges most clearly in the isolated flat band limit, which was also heavily discussed in the literature addressing the superfluid phase stiffness $D_s$, where a lower bound for the mean field $D_s$ was found. In the following we show that a similar bound exists for $n_B/M_B$ in this limit.

**A. Isolated flat band limit**

The isolated flat band limit for the Hamiltonian in Sec. I is defined at $U$ such that $W \ll U \ll E_g$. This regime corresponds to a BEC superconductor. In this limit, superconductivity mainly occurs in the lower flat energy band while the upper one is inactive. As a consequence, all terms involving the upper energy band in the equations for $\{T_{\text{con}},T_{\text{geom}}\}$ drop out. Also, the lower flat band term in $T_{\text{geom}}$ can be neglected because the band is flat. The only remaining term comes from $T_{\text{geom}}$ which involves the lower flat band. Then from Eq. (S34), one finds

$$\frac{n_B}{M_B} = \Delta_{pg}^2 \sum_{\mathbf{k} \in \text{RBZ}} \frac{\tanh(\beta E_{\mathbf{k}}/2)}{2E_{\mathbf{k}}} g_{xx}(\mathbf{k}), \quad (S39)$$

where we have left the band dependence of $g_{ij}^{\alpha\sigma}$ unspecified since it is the same for different bands.

Interestingly, this expression for $n_B/M_B$ is almost identical to that of the BCS mean field $D_s$ in the same limit (see Eq. (S46) of Sec. IV and also Ref. 17). The only difference is that the gap parameter in $n_B/M_B$ is the pseudogap $\Delta_{pg}$ while that in $D_s$ is the BCS mean field superconducting order parameter.

Using Eq. (S38) and $g_{xx} = g_{yy}$, one can derive the following lower bound for $n_B/M_B$:

$$\frac{n_B}{M_B} \geq \frac{\Delta_{pg}^2}{4E_-} \sum_{\mathbf{k} \in \text{RBZ}} |F_{xx}(\mathbf{k})| / |C| / \pi. \quad (S40)$$

$E_-$ is $\mathbf{k}$ independent since the band is flat. We dropped the band dependence of the Berry curvature $F_{x\gamma}^{\alpha\sigma}(\mathbf{k})$ and also that of the Chern number $C_{\alpha\gamma}$, since their absolute values are the same for all bands. To obtain the last line we have used Eq. (S37b). This line clearly shows that $n_B/M_B$ is bounded below when the flat band has a nonzero Chern number, i.e. it is topologically nontrivial.

**IV. MEAN FIELD CALCULATION OF $D_s(T)$ AND $T_{\text{BKT}}$**

In Figs. (1) and (2) of the main text we have included the mean field results of $D_s$ and $T_{\text{BKT}}$ for comparison. This section gives a summary of the main equations used.

We start with the BCS mean field gap equation

$$\frac{1}{U} = \sum_{\mathbf{k} \in \text{RBZ}} \sum_{\alpha=\pm} \frac{1}{2E_{\mathbf{k}}(\alpha)} \tanh(\beta E_{\mathbf{k}}(\alpha)/2), \quad (S41)$$

where $E_\alpha \equiv \sqrt{\xi_{\alpha}^2 + \Delta_{sc}^2}$ with $\Delta_{sc}$ the BCS mean field superconducting gap. This equation is derived from Eq. (S11). The electron density equation is the same as in Eq. (S32). Solving the two equations for given $T$ and $U$ one obtains $\Delta_{sc}$ and $\mu_F$.

From the mean field $\Delta_{sc}$ and $\mu_F$ we calculate the mean field $D_s$ by (for derivations see Refs. 17 and 18)

$$D_s = \frac{1}{4} \sum_{\mathbf{k} \in \text{RBZ} \{i,j\} = \{1,2,3,4\}} \sum_{\alpha} \frac{n_F(\mathbf{E}_i) - n_F(\mathbf{E}_j)}{\mathbf{E}_i - \mathbf{E}_j} \left\{ \langle \Psi_i | \partial_x H_{\text{BdG}}[\Delta_{sc} = 0] | \Psi_j \rangle \langle \Psi_j | \partial_x H_{\text{BdG}} | \Psi_i \rangle - \langle \Psi_i | j^+_x | \Psi_j \rangle \langle \Psi_j | j_x | \Psi_i \rangle \right\}, \quad (S42)$$

where $\mathbf{E}_i = \pm E_\pm$ and $|\Psi_i\rangle$ are eigen-energies and eigenvectors of the following $4 \times 4$ mean field BdG Hamiltonian matrix

$$H_{\text{BdG}}(\mathbf{k}) = \begin{pmatrix} H_+^{\dagger}(\mathbf{k}) & \Delta_{sc}s_0 \\ -\Delta_{sc}s_0 & -H^T_+(-\mathbf{k}) \end{pmatrix}. \quad (S43)$$

In the curly brace in Eq. (S42), the first term is diamagnetic, while the second term is paramagnetic. $j_x(\mathbf{k}) = (\partial_x H_{\text{BdG}}(\mathbf{k}))\tau_z$ is the electric current operator, where $\tau_z$ is the $z$-component Pauli matrix defined for the Nambu space. Following Ref. 17 one can separate $D_s$ into the conventional and geometric contributions, $D_s = D_s^{\text{con}} + D_s^{\text{geom}}$. Their expressions are 17
\[ D_{\text{conv}}^{\text{conv}} = \frac{1}{4} \sum_{\mathbf{k} \in \text{RBZ}} \sum_{\alpha = \pm} \left[ -\frac{\beta}{2 \cosh^2(\beta E_{\alpha}(\mathbf{k})/2)} + \frac{\tanh(\beta E_{\alpha}(\mathbf{k})/2)}{E_{\alpha}(\mathbf{k})^2} \right] |\Delta_{\text{sc}}|^2 |\Delta_{\text{sc}}|_{\text{xx}}(\mathbf{k}), \quad (S44) \]

\[ D_{\text{geom}}^{\text{geom}} = \frac{1}{4} \sum_{\mathbf{k} \in \text{RBZ}} \sum_{\alpha = \pm} \left[ \frac{\tanh(\beta E_{\alpha}(\mathbf{k})/2)}{E_{\alpha}(\mathbf{k})} - \frac{\tanh(\beta E_{-\alpha}(\mathbf{k})/2)}{E_{-\alpha}(\mathbf{k})} \right] \frac{\xi_{-\alpha}(\mathbf{k}) - \xi_{\alpha}(\mathbf{k})}{\xi_{-\alpha}(\mathbf{k}) + \xi_{\alpha}(\mathbf{k})} |\Delta_{\text{sc}}|^2 |g_{xx}(\mathbf{k})|. \quad (S45) \]

The plummet of the pairing fluctuation theory \( T_{\text{BKT}} \) occurs near the point where \( \mu_F \) becomes negative. It is associated with a rapid decrease of a term in \( T_{\text{conv}} \) in Eq. (S27a), (the second one in the square bracket), which is \( \propto |\partial_{\xi_{\alpha}} n_F(\xi_{\alpha})| |\partial_{\xi} \xi_{\alpha}|^2 \). This term vanishes at \( T = 0 \) when \( \mu_F \) drops below the band bottom since \( |\partial_{\xi_{\alpha}} n_F(\xi_{\alpha})| |\partial_{\xi} \xi_{\alpha}|^2 = \delta(\xi_{\alpha}) |\partial_{\xi} \xi_{\alpha}|^2 \equiv 0 \) for any \( \mathbf{k} \). The remaining two terms in Eq. (S27a) cancel each other almost completely at \( T = 0 \) when \( \mu_F \) is negative, leading to the extremely small \( T_{\text{BKT}} \) at \( U/t \gtrsim 3 \). The near-complete cancellation does not occur when the electron density \( n \) is small so that the conduction band is much less than half-filled\(^9\), i.e., when the preformed pairs in the BEC regime are dilute. It suggests that the cancellation is a consequence of a competition between pair hopping and inter-site pair repulsion\(^9\), the latter of which originates from Pauli exclusion that prevents two pairs from occupying one site. The repulsion becomes more important as the density of the pairs, which is equal to \( n/2 \) in the BEC regime, increases, and it can severely restrict the motion of the pairs at high density\(^9\), leading to almost zero \( T_{\text{BKT}} \). This effect of the repulsion is naturally not included in the calculated mean field \( D_s \), even when the pair density is high and when \( U \) is very large\(^9\). To incorporate the inter-site pair repulsion effect into \( D_s \), one needs to include beyond mean field corrections\(^{21,22}\), in particular quantum fluctuation effects. On the other hand, numerical studies\(^{23–25}\) on a simple 2D attractive (single-orbital) Hubbard model on a square lattice do not seem to indicate a dramatic effect of the repulsion on \( T_{\text{BKT}} \). Of course, the numerical studies can be subject to finite size effects. At present, it is unclear if our calculated conventional \( n_B/M_B \) has overestimated the pair repulsion effect or not. Further studies are needed to resolve this issue.

The geometric contribution behaves similarly in the two theories. At small \( U \) it increases roughly linearly with \( U \) except where \( U \) is very small. At \( U/t \gtrsim 7 \), it begins to decrease, which comes from a cancellation between the inter- and intra-band contributions to \( T_{\text{geom}} \) in Eq. (S27b). The net result at large enough \( U \) is \( T_{\text{BKT}}^{\text{geom}} \propto (n_B/M_B)^{\text{geom}} \propto E_g^2/U \), as discussed in Sec. IID.

### V. ADDITIONAL NUMERICAL RESULTS

#### A. \( \mathcal{F} = 0.2 \)

![Fig. S2](image)

**FIG. S2.** Decomposition of the BCS MF \( T_{\text{BKT}} \) into the conventional (“Conv”) and geometric (“Geom”) contributions for the topological \( \mathcal{F} = 0.2 \) band. \( n = 0.3 \).

In Fig. 1 of the main text we have decomposed our pairing fluctuation theory \( T_{\text{BKT}} \) into the conventional and geometric contributions. Here we make the same decomposition for the corresponding BCS MF theory \( T_{\text{BKT}} \) in Fig. S2. Comparing the pairing fluctuation theory and MF results we see that the conventional term in both theories has a dome shape dependence on \( U \) with its maximum at \( U \sim W \). However, the decrease of the mean field \( T_{\text{BKT}} \) at large \( U \) is much slower and follows a \( t^2/U \) asymptote. In contrast, the corresponding pairing fluctuation result falls precipitously to almost zero at \( U/t \approx 3 \) and remains extremely small at larger \( U \).

#### B. \( \mathcal{F} = 0.01 \)

Fig. S3 illustrates some additional numerical results for the \( \mathcal{F} = 0.01 \) flat band. In Fig. S3(a) we present a zoomed view of the \( T_{\text{BKT}} \) results at small \( U \). One sees that there is a remnant \( T_{\text{BKT}} \) peak at \( U/t \approx 0.1 \), due to the small but still finite conventional contribution to \( n_B/M_B \). The latter comes from...
VI. FURTHER DISCUSSION

In this section we present some further discussion on the experimental $T^*/T_{\text{BKT}}$ ratio for magic angle TBLG, and also comment on the relation between our work and the literature.
A. $T^*/T_{BKT}$ and the BCS-BEC crossover in TBLG

As discussed in the main text, from the $V - I$ characteristics as measured by Cao et al.\cite{30}, we have extracted the ratio $T^*/T_{BKT}$ = 4 for TBLG. At this ratio, the corresponding $(n_B^*/M_B^*)_{\text{geom}}/(n_B^*/M_B^*)_{\text{tot}}$ calculated from our model is about 70% for $F = 0.01$, and about 50% for $F = 0.2$ (see Fig. (3b) of the main text). Which band flatness ratio is more appropriate for magic angle TBLG depends on one’s estimate of the effective bandwidth and bandgap $W$ and $E_g$. If we take $W \approx 3 \sim 5$meV, which is the energy range where the bare flat band density of states is appreciable, and $E_g \approx 20$meV\cite{30}, then $F \approx 0.15 \sim 0.25$.

At face value, this suggests that the $F = 0.2$ case is more relevant to magic angle TBLG. However, one should keep in mind that the estimated $W$ here only provides an upper bound, as the superconductivity in TBLG may be associated with a renormalized and therefore smaller effective band width $W$. In any case, the geometric contribution to $T_{BKT}$ is significant, ($\gtrsim 50\%$), and the system is in the BCS-BEC crossover regime, although it has not yet passed into the BEC.

We stress that the $T^*/T_{BKT}$ ratio inferred from the $V - I$ characteristics can be quite different in different experiments with different samples\cite{31-33}. For example, for one superconductor studied in Ref. 32, the ratio is only about 1.4, with $T_{BKT} = 710$mK and $T^* \approx 1$K. This puts the corresponding system in the BCS weak-coupling regime in Fig. (3a) of the main text, and consequently the corresponding geometric contribution to $T_{BKT}$ from Fig. (3b) of the main text is only about 10 $\sim 20\%$. However, one should also take note that the $T^*$ read off from all the existing $V - I$ curves is subject to uncertainty since none of the measurements provides a continuous sweep over closely separated temperature intervals.

When the $V - I$ measurements are not available, it appears that $T^*$ can be roughly estimated from dc transport. This is based on a temperature feature in the longitudinal resistivity $\rho(T)$, which corresponds to the point where $\rho(T)$ begins to drop below its normal state extrapolation\cite{34}. For example, in transport experiments on a TBLG sample with $T_{BKT} = 1$K in Ref. 30, this transport signature yields $T^* = 4 \sim 5$K, roughly consistent with the value obtained from $V - I$ measurements. While $T^*$ identified in this way is necessarily greater than or equal to $T_{BKT}$, depending on the carrier density and twist angle, it can be substantially larger. As seen from transport studies in Fig. I of Ref. 33, the $T^*/T_{BKT}$ ratio varies from a number close to 1 to a number much larger than 10 as the carrier density is tuned from one side of the superconducting dome to the other in a given sample\cite{35}.

One can speculate that this wide variation of $T^*/T_{BKT}$ obtained from transport, is unlikely to be due to disorder given that the measurements are on the same sample, though with different carrier density. Instead, variations in Coulomb screening, which crucially depends on the carrier density may play a key role. This is consistent with several recent experiments\cite{32,33,36-37}, where the importance of the Coulomb screening in the superconductivity has been emphasized.

Because of the sensitivity of the effective pairing interaction to band filling and Coulomb screening, determining whether superconducting magic angle TBLG is a weak-coupling or strong-coupling superconductor remains an open question\cite{38}.

To firmly settle the issue, further $V - I$ experiments over finely separated temperature intervals in order to establish the temperature for the Ohmic recovery are much needed. As in Ref. 39, for corroboration, these should ultimately be combined with STM measurements of the local pairing gap. STM experiments\cite{36,40-42} on magic angle TBLG to date tend to be limited to the normal state and have not yet reported signatures of the pairing gap or $T^*$.

B. Generalization and relation to the literature

Our rough comparison between theory and experiment is based on the assumption that the simple model we studied captures some essential features of the band structure of TBLG. While this sets up the general framework and identifies the issues, clearly, a calculation using a realistic band structure is ultimately needed. In our model, the band topology comes from a nonzero spin Chern number. On the other hand, (in the absence of complications arising from the effects of the hBN encapsulating substrate)\cite{43}, the relevant topology for the bare flat bands of TBLG was argued to be different and to correspond to a so called “fragile topology”\cite{44-48}. Whether this topology is associated with the normal state out of which the superconductivity emerges is still unclear\cite{49}. However, as demonstrated in a BCS mean field calculation\cite{44}, this fragile topology exhibits similar Wannier obstruction effects that can prevent the localization of Cooper pairs and hence enhance 2D superconductivity for a flat band system. Nevertheless, we expect most of our qualitative findings to survive in a more realistic band calculation with fragile topology included. The pairing fluctuation theory that we presented for our two band model can be easily generalized to a more-than-two-band structure, which is more relevant to TBLG. We leave
that for a future work.

We note that prior to our work there have been studies of the geometric contribution to the superfluid instability temperature that were associated with beyond-mean-field Gaussian pairing fluctuations. In a series of papers\textsuperscript{50}, M. Iskin called attention to the geometric contribution in 2D and 3D spin-orbit coupled Fermi gases. Notably, for this specific energy dispersion, the geometric contribution does not play a significant role and the conventional contribution dominates, due to the associated non-flat and unbounded band dispersion. It should be noted that within a Gaussian fluctuation theory, which is most appropriate for 3D superfluids, there does appear an inter-band geometric contribution to the inverse pair mass that is similar to our Eq. (S27b)\textsuperscript{50,51}.

Beyond mean field effects and quantum geometry have also been discussed in Ref. 17 in the context of dynamical mean field theory (DMFT). There it was similarly observed that the geometric contribution to the flat band phase stiffness survives, though reduced in magnitude. These DMFT calculations were shown to agree qualitatively with the results of strict mean field theory, not in the BCS regime but in the more strongly correlated BEC limit, where one might expect a mean field approach to be less appropriate. Finally, we note that there are other more analytical approaches which incorporate bosonic fluctuation effects on the superfluid phase stiffness across the entire BCS-BEC crossover\textsuperscript{27,52,53,54}. While the role of this additional “collective mode” bosonic branch is to degrade the superfluid phase stiffness, as we find here, these schemes have not addressed quantum geometric effects. Further investigations are needed to resolve these issues.

1 A. J. Leggett, in Modern trends in the theory of condensed matter (Springer, 1980) pp. 13–27.
2 J. Zhang, H. Hu, X.-J. Liu, and H. Pu, “Annual review of cold atoms and molecules,” (World Scientific, 2014) Chap. 2, pp. 81–143.
3 P. Nozières and S. Schmitt-Rink, Journal of Low Temperature Physics 59, 195 (1985).
4 Q. Chen, J. Stajic, S. Tan, and K. Levin, Physics Reports 412, 1 (2005).
5 L. P. Kadanoff and P. C. Martin, Phys. Rev. 124, 670 (1961).
6 Q. J. Chen, Generalization of BCS theory to short coherence length superconductors: A BCS–Bose-Einstein crossover scenario, Ph.D. thesis, University of Chicago (2000), arXiv:1801.06266.
7 J. Maly, B. Jankó, and K. Levin, Physica C: Superconductivity 321, 113 (1999).
8 C.-T. Wu, B. M. Anderson, R. Boyack, and K. Levin, Phys. Rev. Lett. 115, 240401 (2015).
9 X. Wang, Q. Chen, and K. Levin, New Journal of Physics 22, 063050 (2020).
10 L. He, X.-G. Huang, H. Hu, and X.-J. Liu, Phys. Rev. A 87, 053616 (2013).
11 J. P. Provost and G. Vallee, Communications in Mathematical Physics 76, 289 (1980).
12 J. Anandan and Y. Aharonov, Phys. Rev. Lett. 65, 1697 (1990).
13 S. Peotta and P. Törnä, Nature communications 6, 8944 (2015).
14 F. Xie, Z. Song, B. Lian, and B. A. Bernevig, Phys. Rev. Lett. 124, 167002 (2020).
15 N. Marzari and D. Vanderbilt, Phys. Rev. B 56, 12847 (1997).
16 N. Marzari, A. A. Mostofi, J. R. Yates, I. Souza, and D. Vanderbilt, Rev. Mod. Phys. 84, 1419 (2012).
17 L. Liang, T. I. Vanhala, S. Peotta, T. Siro, A. Harju, and P. Törnä, Phys. Rev. B 95, 024515 (2017).
18 A. Julku, T. J. Peltonen, L. Liang, T. T. Heikkilä, and P. Törnä, Phys. Rev. B 101, 060505 (2020).
19 R. Micas, J. Ranninger, and S. Robaszkiewicz, Rev. Mod. Phys. 62, 113 (1990).
20 P. J. H. Denteneer, G. An, and J. M. J. van Leeuwen, Phys. Rev. B 47, 6256 (1993).
21 N. Dupuis, Phys. Rev. B 70, 134502 (2004).
22 L. Benfatto, A. Toschi, and S. Caprara, Phys. Rev. B 69, 184510 (2004).
23 M. Keller, W. Metzner, and U. Schollwöck, Phys. Rev. Lett. 86, 4612 (2001).
24 A. Toschi, P. Barone, M. Capone, and C. Castellani, New Journal of Physics 7, 7 (2005).
25 T. Paiva, R. Scalettar, M. Randeria, and N. Trivedi, Phys. Rev. Lett. 104, 066406 (2010).
26 C. Rojas and J. V. José, Phys. Rev. B 54, 12361 (1996).
27 L. Capriotti, A. Cucoli, A. Fabini, V. Tognetti, and R. Vaia, Phys. Rev. Lett. 91, 247004 (2003).
28 J. S. Hofmann, E. Berg, and D. Chowdhury, arXiv preprint arXiv:1912.08848 (2019).
29 We note that the results presented in Fig. 1 of Ref. 28 are for electron density $n = 0.5$ per site; while our results are for $n = 0.3$ per site. However, we do not expect the qualitative $U$ dependence of $T_{AKT}$ and $T^*$ to change from $n = 0.5$ to $n = 0.3$.
30 Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Nature 556, 43 (2018).
31 X. Lu, P. Stepnov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, Nature 574, 653 (2019).
32 P. Stepnov, I. Das, X. Lu, A. Fakhim尼亚, K. Watanabe, T. Taniguchi, F. H. Koppens, J. Lischner, L. Levitov, and D. K. Efetov, Nature 583, 375 (2020).
33 Y. Cao, D. Rodan-Legrain, J. M. Park, F. N. Yuan, K. Watanabe, T. Taniguchi, R. M. Fernandez, L. Fu, and P. Jarillo-Herrero, arXiv preprint arXiv:2004.04148 (2020).
34 Similar dc transport signatures of $T^*$ have been observed previously in cuprates\textsuperscript{45}, although the pseudogap here can have a completely different origin from preformed Cooper pairs.
35 Here we ignore the intervening correlated insulating phase at half filling of the lower and upper flat band, and view the two superconducting domes flanking the insulating phase as one.
36 Y. Saito, J. Ge, K. Watanabe, T. Taniguchi, and A. F. Young, Nature Physics, 1 (2020).
37 X. Liu, Z. Wang, K. Watanabe, T. Taniguchi, O. Vafek, and J. Li, arXiv preprint arXiv:2003.11072 (2020).
38 This would also seem to imply that the possibility of a conventional phonon based pairing mechanism is unlikely since one does not expect the phonon spectrum, or its coupling to electrons, to change substantially for a given sample at a given twist angle.
39 W. Zhao, Q. Wang, M. Liu, W. Zhang, Y. Wang, M. Chen, Y. Guo, K. He, X. Chen, Y. Wang, J. Wang, X. Xie, Q. Niu, L. Wang, X. Ma, J. K. Jain, M. Chan, and Q.-K. Xue, Solid State Communications 165, 59 (2013).
As shown by recent experimental and theoretical studies, a coupling of the TBLG to the hBN substrate can break certain symmetry that leads to occurrence of Chern bands and the resulted anomalous Hall effect.

In principle, the normal state bands relevant to the superconductivity can be different from the bare band, due to renormalizations from interactions.