MODULAR INVARIANT REPRESENTATIONS OF THE $N = 2$ SUPERCONFORMAL ALGEBRA

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Abstract. We compute the modular transformation formula of the characters for a certain family of (finitely or uncountably many) simple modules over the simple $N = 2$ vertex operator superalgebra of central charge $c_{p,p'} = 3 \left( 1 - \frac{2p'}{p} \right)$, where $(p, p')$ is a pair of coprime positive integers such that $p \geq 2$. When $p' = 1$, the formula coincides with that of the $N = 2$ unitary minimal series found by F. Ravanini and S.-K. Yang. In addition, we study the properties of the corresponding “modular $S$-matrix”, which is no longer a matrix if $p' \geq 2$.

1. Introduction

One of the most remarkable features in representation theory of vertex operator superalgebras (VOSAs) is the modular invariance property of the characters of their modules. The property is firstly established by Y. Zhu in [Zhu96] for rational, $C_2$-cofinite vertex operator algebras (VOAs) under some natural conditions. See [Miy04] and [VE13] for the generalization to irrational cases and to super cases, respectively. We note that all these previous works are based on the $C_2$-cofiniteness assumption which is introduced in [Zhu96] and is deeply related to the finiteness of the space of conformal blocks (in the sense of [VE13, Definition 4.2]). See [Zhu96], [Miy04], and [VE13] for more details.

In the present paper, we construct a “modular invariant” family of simple highest weight modules over the simple VOSA $L_{c_{p,p'}}$ associated with the $N = 2$ superconformal algebra of central charge

$$c_{p,p'} := 3 \left( 1 - \frac{2p'}{p} \right).$$

Here $(p, p')$ is a pair of coprime positive integers such that $p \geq 2$. We should note that the simple VOSA $L_{c_{p,p'}}$ is $C_2$-cofinite if and only if $p' = 1$ (see Corollary 2.3). When it is not $C_2$-cofinite, the dimension of the space of conformal blocks is not known to be finite (cf. [VE13 Theorem 1.3]). In fact, the space spanned by the character functions of simple $\frac{1}{2} \mathbb{Z}_{\geq 0}$-gradable $L_{c_{p,p'}}$-modules is not finite dimensional (see Remark 3.6). Therefore, we explain the precise meaning of the “modular invariance” in our case below.

For each pair $(p, p')$ as above, our family of simple $L_{c_{p,p'}}$-modules is divided into two classes, atypical modules and typical modules. All the atypical modules in this paper are obtained in either way:

(1) from the Kac-Wakimoto admissible highest weight $\widehat{sl}_2$-modules of level $k = -2 + \frac{2}{p'}$ (see [KW88]) by the Kazama-Suzuki coset construction (see [KSS9], [HT91], [FSST99], and [Sat16 §7]),
(2) from principal admissible \(\hat{sl}_{2|1}\)-modules of level \(\tilde{k} = -1 + \frac{p'}{p}\) (see [BHT98] and [KW16]) by the quantum Becchi-Rouet-Stora-Tyutin (BRST) reduction (see [KRW03] and [Ara05]).

See Appendix D.2 for the details of the second construction. Similarly to the atypical modules, the typical ones are also obtained in either way:

(1) from simple relaxed highest weight \(\hat{sl}_2\)-modules of level \(k = -2 + \frac{p}{p'}\) (see [CR13a] and Appendix [B]) by the Kazama-Suzuki coset construction,

(2) from typical highest weight \(\hat{sl}_{2|1}\)-modules (in the sense of [GK15]) of level \(\tilde{k} = -1 + \frac{p'}{p}\) by the quantum BRST reduction.

In response to the above two classes of modules, we introduce the following two families:

- characters of atypical modules indexed by a certain finite set \(\mathcal{S}_{p,p'}\) (see Definition 4.9 for the definition),
- characters of typical modules indexed by \(K_{p,p'} \times \mathbb{R}\), where \(K_{p,p'}\) is a finite set which parameterizes the Belavin-Polyakov-Zamolodchikov (BPZ) minimal series of central charge \(1 - \frac{6(p - p')^2}{pp'}\).

We note that the former characters can be written in terms of the Appell-Lerch sum (see [Zwe02] and [STT05]). See [KW16] and Lemma C.1 for the details.

Now we explain the modular transformation properties of the characters. To describe the whole picture of the modular invariance in super cases, we need to consider the following four types of formal characters:

(1) \(\text{ch}^{0,0}(M)\): Neveu-Schwarz twisted character,
(2) \(\text{ch}^{0,1}(M)\): Neveu-Schwarz twisted supercharacter,
(3) \(\text{ch}^{1,0}(M)\): Ramond twisted character,
(4) \(\text{ch}^{1,1}(M)\): Ramond twisted supercharacter.

For \(\lambda \in \mathcal{S}_{p,p'}\) and \((\mu, x) \in K_{p,p'} \times \mathbb{R}\), we denote the character functions of the corresponding highest weight modules by

\[
A^{\varepsilon, \varepsilon'}_\lambda \left( \frac{1}{\tau}, u, t \right) = \text{ch}^{\varepsilon, \varepsilon'}(\mathcal{L}_\lambda), \quad T^{\varepsilon, \varepsilon'}_{\mu, x} \left( \frac{1}{\tau}, u, t \right) = \text{ch}^{\varepsilon, \varepsilon'}(\mathcal{L}_{\mu, x}).
\]

Here \((\tau, u, t) \in \mathbb{H} \times \mathbb{C} \times \mathbb{C}\) stands for a certain coordinate of the Cartan subalgebra of the \(N = 2\) superconformal algebra. Then our “modular invariance” in this paper means the establishment of the following modular \(S\)-transformation

\[
A^{\varepsilon, \varepsilon'}_\lambda \left( \frac{1}{\tau}, u, t \right) = \sum_{\lambda' \in \mathcal{S}_{p,p'}} S^{\alpha, (\varepsilon, \varepsilon')}_{\lambda, \lambda'} A^{\varepsilon, \varepsilon'}_{\lambda'} \left( \frac{1}{\tau}, u, t \right) + \sum_{\mu' \in K_{p,p'}} \int_{\mathbb{R}} dx' S_{\lambda, (\mu', x')} T^{\varepsilon, \varepsilon'}_{\mu', x'} \left( \frac{1}{\tau}, u, t \right),
\]

\[
T^{\varepsilon, \varepsilon'}_{\mu, x} \left( \frac{1}{\tau}, u, t \right) = \sum_{\mu' \in K_{p,p'}} \int_{\mathbb{R}} dx' S_{(\mu, x), (\mu', x')} T^{\varepsilon, \varepsilon'}_{\mu', x'} \left( \frac{1}{\tau}, u, t \right)
\]

and the (rather trivial) modular \(T\)-transformation. See §3.3 and §4.4 for the details.

At last, we give some remarks on relationships between our result and the relevant previous works:
When \( p' = 1 \), the set \( K_{p,1} \) is empty and the index set \( \mathcal{I}_{p,1} \) bijectively corresponds to the \( N = 2 \) unitary minimal series of central charge

\[
c_{p,1} = 3 \left( 1 - \frac{2}{p} \right).
\]

Therefore, the space

\[
\text{span}_C \{ A_{\lambda}^{\varepsilon,\varepsilon'} | \varepsilon, \varepsilon' \in \{0, 1\}, \lambda \in \mathcal{I}_{p,1} \}
\]

forms a finite dimensional \( SL(2, \mathbb{Z}) \)-invariant space. Then our result recovers the modular transformation of the \( N = 2 \) minimal unitary characters which is firstly obtained by F. Ravanini and S.-K. Yang in [RY87] and proved by V. G. Kac and M. Wakimoto in [KW94a]. See Appendix D for the details.

In [STT05], A. M. Semikhatov, A. Taormina, and I. Yu. Tipunin studied the modular property of the characters of simple highest weight modules over the \( \hat{sl}_{2|1} \) of level \( k = -1 + \frac{p'}{p} \) for \( p' \geq 2 \). The \( N = 2 \) VOSA is obtained from the affine VOSA associated with \( \hat{sl}_{2|1} \) by the quantum BRST reduction (based on the result of T. Arakawa in [Ara05]) and the corresponding central charge is given by

\[
c = -3(2\bar{k} + 1) = c_{p,p'}.
\]

We can verify that the “admissible \( \hat{sl}_{2|1} \) representations” considered in [STT05] §B.3 correspond to principal admissible \( \hat{sl}_{2|1} \)-modules with spectral flow twists (cf. [KW16] §2). We note that the character formula for principal admissible \( \hat{sl}_{2|1} \)-modules is proved in [GK15] (see [KW16, Lemma 2.1] for the detail).

We should mention that the reduced version of the formula in [STT05, Theorem 4.1] is presented in [Gho03, (4.2.75)]. Then our result gives the alternative expression of [Gho03 (4.2.75)] purely in terms of the character functions.

By the Kazama-Suzuki coset construction (see [KS89], [HT91], [FST98], and [Sat16]), the modular invariant family of \( L_{c_{p,p'}} \)-modules in this paper can be regarded as the counterpart of that of \( sl_2 \)-modules at the Kac-Wakimoto admissible levels \( k = -2 + \frac{p'}{p} \) studied in [CR13a]. It is worth noting that the formal characters of “typical” \( \hat{sl}_2 \)-modules which are parameterized by \( K_{p,p'} \times \mathbb{R} \) are not convergent to functions defined in the upper half plane \( \mathbb{H} \). See Proposition 13.2 for the details.

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2. Preliminaries

2.1. Notation. For \((\tau, u) \in \mathbb{H} \times \mathbb{C}\), we set \((q, z) := (e^{2\pi i \tau}, e^{2\pi i u})\). We denote the eta function by
\[
\eta(\tau) := q^{1/24} \prod_{n > 0} (1 - q^n)
\]
and the theta functions by
\[
\vartheta_{\varepsilon, \varepsilon'}(u; \tau) := z^\frac{\tau}{2} q^{\frac{\tau}{24}} \prod_{n > 0} (1 - q^n) \left( 1 + (-1)^\varepsilon z q^{\frac{\tau}{24}} \right) \left( 1 + (-1)^{\varepsilon'} z^{-1} q^{\frac{\tau}{24}} \right)
\]
for \(\varepsilon, \varepsilon' \in \{0, 1\}\). We note that

\[
\vartheta_{\varepsilon, \varepsilon'}(u; \tau) = i^{-\varepsilon' \varepsilon} \sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{\tau}{2})^2 + 2\pi i (u+\frac{\varepsilon}{2})(n+\frac{\tau}{2})} = z^\frac{\tau}{2} q^{\frac{\tau}{24}} \vartheta_{0,0} \left( u + \frac{\varepsilon}{2} \tau + \frac{\varepsilon'}{2} \tau \right).
\]

By abuse of notation, we regard \(\eta(\tau)\) (resp. \(\vartheta_{\varepsilon, \varepsilon'}(u; \tau)\)) as a holomorphic function on \(\mathbb{H}\) (resp. \(\mathbb{C} \times \mathbb{H}\)) and also as a convergent series in \(q\) (resp. \(z\) and \(q\)).

2.2. The \(N = 2\) superconformal algebra. The Neveu-Schwarz sector of the \(N = 2\) superconformal algebra (firstly introduced in \([ABd+76]\)) is the Lie superalgebra

\[
\mathfrak{ns}_2 = \bigoplus_{n \in \mathbb{Z}} C L_n \oplus \bigoplus_{n \in \mathbb{Z}} C J_n \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} C G^+_r \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} C G^+_r \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} C C
\]

with the \(\mathbb{Z}_2\)-grading

\[
(\mathfrak{ns}_2)^0 = \bigoplus_{n \in \mathbb{Z}} C L_n \oplus \bigoplus_{n \in \mathbb{Z}} C J_n \oplus C C, \quad (\mathfrak{ns}_2)^1 = \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} C G^+_r \oplus \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} C G^+_r
\]

and with the following (anti-)commutation relations:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)C \delta_{n+m,0},
\]

\[
[L_n, J_m] = -mJ_{n+m}, \quad [L_n, G^\pm_r] = \left( \frac{n}{2} - r \right) G^\pm_{n+r},
\]

\[
[J_n, J_m] = \frac{n}{3} C \delta_{n+m,0}, \quad [J_n, G^\pm_r] = \pm G^\pm_{n+r},
\]

\[
[G^+_r, G^+_s] = 2L_{r+s} + (r-s)J_{r+s} + \frac{1}{3} \left( r^2 - \frac{1}{4} \right) C \delta_{r+s,0},
\]

\[
[G^+_r, G^-_s] = [G^-_r, G^+_s] = 0, \quad [\mathfrak{ns}_2, C] = \{0\},
\]

for \(n, m \in \mathbb{Z}\) and \(r, s \in \mathbb{Z} + \frac{1}{2}\).

2.3. The \(N = 2\) vertex operator superalgebra. Let \(\mathfrak{ns}_2 = (\mathfrak{ns}_2)^+ \oplus (\mathfrak{ns}_2)^0 \oplus (\mathfrak{ns}_2)^-\) be the triangular decomposition of \(\mathfrak{ns}_2\), where

\[
(\mathfrak{ns}_2)^+ := \bigoplus_{n > 0} C L_n \oplus \bigoplus_{n > 0} C J_n \oplus \bigoplus_{r > 0} C G^+_r \oplus \bigoplus_{r > 0} C G^-_r,
\]

\[
(\mathfrak{ns}_2)^- := \bigoplus_{n < 0} C L_n \oplus \bigoplus_{n < 0} C J_n \oplus \bigoplus_{r < 0} C G^+_r \oplus \bigoplus_{r < 0} C G^-_r,
\]

\[
(\mathfrak{ns}_2)^0 := \bigoplus_{n \geq 0} C L_0 \oplus \bigoplus_{n \leq 0} C J_0 \oplus C C,
\]

and set \((\mathfrak{ns}_2)^{>0} := (\mathfrak{ns}_2)^+ \oplus (\mathfrak{ns}_2)^0\).

For \((h, j, c) \in \mathbb{C}^3\), let \(\mathbb{C}_{h,j,c}\) be the \((1|0)\)-dimensional \((\mathfrak{ns}_2)^{>0}\)-module defined by \((\mathfrak{ns}_2)^+\). Then the induced module
Remark 2.2. The index set $\mathcal{L}_{h,j}$ is defined by $\mathcal{L}_{h,j}^{\mathfrak{ns}_2,0} \subset \mathcal{C}_{h,j}$ is the Verma module of $\mathfrak{ns}_2$. Denote by $L_{h,j}$ the simple quotient $\mathfrak{ns}_2$-module of $\mathcal{L}_{h,j}$.

We write $V_c = V_c(\mathfrak{ns}_2) \cong M_{0,0,c}/(U(\mathfrak{ns}_2)G^+ \cdot 0,0,c + U(\mathfrak{ns}_2)G^- \cdot 0,0,c)$ for the universal $N=2$ VOSA, where $(0,0,c)$ is the highest weight vector of $M_{0,0,c}$. When $c \neq 0$, we also write $L_c = L_c(\mathfrak{ns}_2) \cong \mathcal{L}_{0,0,c}$ for its simple quotient VOSA. See [Sat10 §2] for the details.

2.4. Classification of simple modules. Let $1 \leq r \leq p - 1$ and $0 \leq s \leq p' - 1$. We set $\mathcal{L}_{r,s;\lambda} := \mathcal{L}_{\Delta_{r,s} - \lambda \lambda', 0, \lambda', \lambda, p, p'}$, where $a := \frac{p'}{p}$ and $\Delta_{r,s} := \frac{(ar-s)^2-a^2}{4a}$. We also set $\lambda_{r,s} := \frac{r+a}{2} - \frac{\lambda}{a}$. Note that $\Delta_{r,s} = a\lambda_{r,s}(\lambda_{r,s} + 1)$.

The following classification is obtained by D. Adamović via the Kazama-Suzuki coset construction.

**Theorem 2.1** ([Ada99] Theorem 7.1 and 7.2). The complete representatives of the isomorphism classes of simple $\frac{1}{2}\mathbb{Z} \geq 0$-gradable $L_{c,p'}$-modules are given as follows:

$$\{\mathcal{L}_{r,0;\lambda, \theta} \mid 1 \leq r \leq p - 1, 0 \leq \theta \leq r - 1, \theta \in \mathbb{Z}\} \cup \{\mathcal{L}_{r,s;\lambda} \mid (r, s) \in K_{p,p'}, \lambda \in \mathbb{C}\},$$

where $K_{p,p'} := \{(m, n) \in \mathbb{Z}^2 \mid 1 \leq m \leq p - 1, 1 \leq n \leq p' - 1, mp + np \leq pp'\}$.

**Remark 2.2.** The index set $K_{p,p'}$ parameterizes the BPZ minimal series of central charge $1 - \frac{6(p-p')^2}{pp'}$ and we have $|K_{p,p'}| = \frac{(p-1)(p'-1)}{2}$.

**Corollary 2.3.** Let $c \in \mathbb{C} \setminus \{0\}$. Then the simple VOSA $L_c$ is $C_2$-cofinite if and only if $c = c_{p,1}$ for some $p \in \mathbb{Z}_{\geq 3}$.

**Proof.** First, the ‘if’ part follows from the regularity of the VOSA $L_{c_{p,1}}$ proved in [Ada01] Theorem 8.1 and the super-analog of [L99] Theorem 3.8.

Next, we verify the ‘only if’ part. We may assume that $c = c_{p,p'}$ for some pair of coprime integers $(p, p') \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$. In fact, otherwise, the simple quotient $L_c$ is isomorphic to the non $C_2$-cofinite VOSA $V_c$ by [GK07] Corollary 9.1.5 (ii). Here we assume that $p' \neq 1$. By the previous theorem and [KW91] Theorem 1.3, it follows that the Zhu algebra of $L_{c_{p,p'}}$ is infinite-dimensional. Then the infinite-dimensionality of (the even part of) the $C_2$-algebra of $L_{c_{p,p'}}$ follows from a slight generalization of [ALY14] Proposition 3.3 to the super case (see also [AM11] Introduction). This completes the proof of the ‘only if’ part.

In this paper, we introduce the notion of atypical and typical modules as follows.

**Definition 2.4.** We call a simple $L_{c_{p,p'}}$-module $\mathcal{L}_{r,s;\lambda}$ *typical* if $(r, s) \in K_{p,p'}$ and $\lambda \in \mathbb{C} \setminus S_{r,s}$, where $S_{r,s} := \{\lambda_{r,s}, \lambda_{p-r,p'-s}\} + \mathbb{Z}$. Otherwise, we call $\mathcal{L}_{r,s;\lambda}$ *atypical*.

2.5. Formal characters. Let $M$ be a weight $(\mathfrak{ns}_2, (\mathfrak{ns}_2)_0)$-module of central charge $c$, i.e., $(\mathfrak{ns}_2)_0$ acts semi-simply on $M$ and $C$ acts as the scalar $c$. Then the formal characters of $M$ is defined by

$$\text{ch}^{0,0}(M) := q^{-\frac{c}{2}} \sum_{\lambda \in (\mathfrak{ns}_2)_0^*} (\dim M_\lambda) e^\lambda,$$

$$\text{ch}^{0,1}(M) := q^{-\frac{c}{2}} \sum_{\lambda \in (\mathfrak{ns}_2)_0^*} (\text{sdim} M_\lambda) e^\lambda.$$
where \( M_\lambda := \{ v \in M \mid X.v = \langle \lambda, X \rangle v \text{ for any } X \in (\mathfrak{ns}_2)_0 \} \). We set \((q, z, w) := (e^{L_0}, e^{J_0}, e^{C^*})\), where \((L_0, J_0, C^*)\) is the dual basis of \((\mathfrak{ns}_2)_0^*\) with respect to the basis \((L_0, J_0, C)\). In what follows, we write \( \text{ch}^{0,\varepsilon}(M)(q, z, w) := \text{ch}^{0,\varepsilon}(M) \).

The following lemma is easily verified (see [Sat16 Lemma 5.2]).

**Lemma 2.5.** For the spectral flow twisted module \( M^\theta \) (see Appendix A for the definition) and \( \varepsilon \in \{0, 1\} \), we have

\[
\text{ch}^{0,\varepsilon}(M^\theta)(q, z, w) = q^{\frac{4\pi^2}{\eta(1)}} \varepsilon \text{ch}^{0,\varepsilon}(M)(q, z^q, w).
\]

Now we introduce the “half-twisted” characters as follows.

**Definition 2.6.** For \( \varepsilon \in \{0, 1\} \), the twisted character of \( M \) is defined by

\[
\text{ch}^{1,\varepsilon}(M)(q, z, w) := q^{\frac{\pi}{2}} z^{-\frac{\varepsilon}{2}} \text{ch}^{0,\varepsilon}(M)(q, z^{-\frac{1}{2}}, w).
\]

3. **Typical modular transformation law**

In this section, we derive the modular transformation formula for the character functions of typical modules. Throughout this section, we assume that \( p' \geq 2 \) and \( L_{r,s;\lambda} \) is typical.

3.1. **Character formula for typical modules.**

**Theorem 3.1.** We have an equality of formal series

\[
\text{ch}^{\varepsilon,\varepsilon'}(L_{r,s;\lambda})(q, z, w) = (-1)^{\varepsilon' - \varepsilon} q^{-a(\lambda + \frac{1}{24})^2} z^{a(\lambda + \frac{1}{24})} w^3(1 - 2a) \frac{\eta(1) \eta(\tau)}{\eta(\tau')^2} \chi_{r,s}(\tau),
\]

where

\[
\chi_{r,s}(\tau) := \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} \left( q^{p_r'(n + \frac{tp_r'-sp_r}{2pp'})^2} - q^{p_s'(n + \frac{tp_s'-sp_s}{2pp'})^2} \right).
\]

**Proof.** By [Sat16 Proposition 7.1], we have \( \Omega_\lambda^+(L_{\Delta_{r,s},a}, 2 + a - 1) \cong L_{r,s;\lambda} \) for any \( \lambda \in \mathbb{C} \), where \( L_{\Delta_{r,s},a}, 2 + a - 1 \) is the simple relaxed highest weight module over \( \mathfrak{sl}_2 \) (see Appendix B for the definition) and \( \Omega_\lambda^+ \) is the functor defined in [Sat16 Definition 4.1]. When \( \lambda \in \mathbb{C} \setminus S_{r,s} \), the character formula for \( L_{r,s;\lambda} \) is obtained by [CR13a Corollary 5] (see also Proposition B.2). Therefore we get the required character formula for \( \Omega_\lambda^+(L_{\Delta_{r,s},a}, 2 + a - 1) \) by [Sat16 Theorem 7.8]. By some computations, we also get the characters for \((\varepsilon, \varepsilon') \neq (0, 0)\). \( \square \)

**Remark 3.2.** (1) We can prove Theorem 3.1 by the character formula for certain \( \mathfrak{sl}_2 \)-modules of level \( \bar{k} = -1 + a = -1 + \frac{p'}{p} \) and the quantum BRST reduction. The detail is discussed in [KS].

(2) The convergent series \( \chi_{r,s}(\tau) \) is the normalized character of the corresponding BPZ minimal series. The modular transformation is given as follows:

\[
\chi_{r,s}(-\frac{1}{\tau}) = \sum_{(r',s') \in K_{p',p} \setminus \{ (r,s) \}} S_{(r,s),(r',s')} \chi_{r',s'}(\tau),
\]

\[
\chi_{r,s}(\tau + 1) = e^{2\pi i \left( \frac{tp_r'-sp_r}{2pp'} \right) \frac{1}{24}} \chi_{r,s}(\tau),
\]

where

\[
S_{(r,s),(r',s')} := \sqrt{\frac{8}{pp'}} (-1)^{(r+s)(r'+s')} \sin \left( \frac{\pi(p - p')rr'}{p} \right) \sin \left( \frac{\pi(p - p')ss'}{p'} \right).
\]
See [K11] Proposition 6.3 for the details.

The next corollary immediately follows from the previous character formula.

**Corollary 3.3.** For any \( \theta \in \mathbb{Z} \), the spectral flow twisted module \( (\mathcal{L}_{r,s;\lambda})^\theta \) is isomorphic to another typical module \( \mathcal{L}_{r,s;\lambda - \theta} \). In particular, the set of typical modules is closed under the spectral flow.

Now we introduce the corresponding character function.

**Definition 3.4.** Let \((r, s) \in K_{p, p'}\) and \(x \in \mathbb{C}\). Then a typical character function is defined as the following holomorphic function on \( \mathbb{H} \times \mathbb{C} \times \mathbb{C} : \)

\[
T_{r,s;x}^e(t, u, t') := (-1)^e q^{\frac{1}{24} (x - \frac{u}{2})^2 \frac{q^{2a}(x - \frac{u}{2})}{\eta(t)^2} (\frac{x - \frac{u}{2}}{2\pi}) (q, z, w),
\]

where \((q, z, w) = (e^{2\pi i}, e^{2\pi i u}, e^{2\pi i t})\).

The next lemma follows from Definition 2.4, Theorem 3.1, and Definition 3.4.

**Lemma 3.5.** As a function in \((q, z, w) = (e^{2\pi i}, e^{2\pi i u}, e^{2\pi i t})\),

\[
T_{r,s;x}^e(t, u, t) = \chi_{r,s}^e(\mathcal{L}_{r,s;ix})(q, z, w)
\]

holds if and only if \(x \in \mathbb{R} + i(\mathbb{R} \setminus S_{r,s})\).

**Remark 3.6.** Since the family \( \{T_{r,s;x}^e | x \in \mathbb{R} + i(\mathbb{R} \setminus S_{r,s})\} \) is linearly independent, the space spanned by the character functions of simple \( \frac{1}{2} \mathbb{Z}_{\geq 0} \)-gradable \( L_{c_{p, p'}} \)-modules is not finite dimensional.

### 3.2. Modular transformation of typical characters.

**Theorem 3.7.** For \((r, s) \in K_{p, p'}\) and \(x \in \mathbb{C}\), the following hold:

1. \[
T_{r,s;x}^e \left( -\frac{1}{T}, \frac{u}{T}, T - \frac{u^2}{2T^2} \right) = \sum_{(r', s') \in K_{p, p'}} \int_{\mathbb{R}} dx' S_{(r, s; x), (r', s'; x)}^{tt, (\varepsilon, \varepsilon')} T_{r', s'; x}^e(t, u, t),
\]

2. \[
T_{r,s;x}^e(1, T, 1 - \frac{u^2}{2T^2}) = e^{2\pi i} \left( a(x - \frac{u}{2})^2 + \frac{2\pi i}{yp} \right) \frac{1}{(r', s'); x}) T_{r', s'; x}^e(1^-\varepsilon'(1 - \varepsilon)(t, u, t),
\]

where

\[
S_{(r, s; x), (r', s'; x)}^{tt, (\varepsilon, \varepsilon')} := i^{-\varepsilon' \varepsilon} S_{(r, s), (r', s') \sqrt{2\pi a} e^{-4\pi i ax x'}, e^{-2\pi ia x'}} \frac{1}{\sqrt{2a}} \frac{\alpha_{ax} q_{ax x'} z_{q_{ax}}}{\sqrt{-iT}}.
\]

**Proof.** The \(S\)-transformation \([3.1]\) essentially follows from the Gaussian integral

\[
\frac{e^{2\pi i \eta^2}}{\sqrt{2a}} \frac{\alpha_{ax} q_{ax x'} z_{q_{ax}}}{\sqrt{-iT}} = \int_{\mathbb{R}} dx' q^{ax x'} e^{-4\pi i ax x'},
\]

where \((\hat{q}, \hat{x}) := (e^{-2\pi i}, e^{2\pi i})\). The \(T\)-transformation \([(3.2)]\) is clear. \(\square\)

### 3.3. Property of typical \(S\)-data.

The data \(S_{(r, s; x), (r', s'; x') \in K_{p, p'} \times \mathbb{R}} \) in the previous subsection satisfies the following conditions.

**Proposition 3.8.** For \((r, s; x), (r', s'; x') \in K_{p, p'} \times \mathbb{R})\), we have

\[
S_{(r, s; x), (r', s'; x')}^{tt, (\varepsilon, \varepsilon')} = S_{(r', s'; x'), (r, s; x)}^{tt, (\varepsilon, \varepsilon')}
\]

and

\[
\sum_{(r', s') \in K_{p, p'}} \int_{\mathbb{R}} dx'' S_{(r, s; x), (r', s'; x'')}^{tt, (\varepsilon, \varepsilon')} S_{(r', s'; x''), (r', s'; x')}^{tt, (\varepsilon, \varepsilon')}
\]
Note that we have \( \Omega \). It immediately follows from Corollary A.3.

**Proof.** The former is obvious. The latter follows from the fact that

\[
\sum_{(r'',s'') \in K p, p'} S_{(r,s),(r'',s'')} S_{(r'',s'),(r',s')} = \delta_{r,r'} \delta_{s,s'}
\]

and

\[
2a \int_{\mathbb{R}} dx'' e^{-4\pi i a (x - \frac{i \pi}{2}) (x'' - \frac{i \pi}{2})} e^{4\pi i a (x'' + \frac{i \pi}{2}) (x' + \frac{i \pi}{2})}
\]

\[
e^{-2\pi a c' (x + x')} \times 2a \int_{\mathbb{R}} e^{2\pi i (x' - x + i \varepsilon) \times 2a x''} dx''
\]

\[
e^{-4\pi a c' (x - \frac{i \pi}{2})} \delta(x' - x + i \varepsilon).
\]

For example, see [IK11, (9.10)] for the proof of the equality (3.3). □

4. **ATYPICAL MODULAR TRANSFORMATION LAW**

In this section, we present the modular transformation formula for the character functions of atypical modules (see Definition 2.4 for the definition of atypical modules). Its proof is given in Appendix C. Throughout this section, we assume that \( p' \geq 1 \).

4.1. **Parameterization of atypical modules.** First we assign a triple of certain integers to each atypical module as follows.

**Lemma 4.1.** Let \( M \) be a simple highest weight \( \mathfrak{sl}_2 \)-module. Then the following are equivalent:

1. \( M \) is an atypical module.
2. There exist \( 1 \leq r \leq p - 1, 0 \leq s \leq p' - 1 \), and \( \theta \in \mathbb{Z} \) such that \( M \cong \mathcal{L}(r, s)^{\theta} \), where \( \mathcal{L}(r, s) := \mathcal{L}_{r, s, \lambda_r, s} = \mathcal{L}_{a \lambda_r, 2a \lambda_r, c_p, p'} \).

**Proof.** It immediately follows from Corollary A.3 □

**Remark 4.2.** Note that we have \( \Omega^+_{\lambda_r, s, 0}(\mathcal{L}(r, s)) \cong \mathcal{L}(r, s)^{\theta} \), where \( L(r, s) \) is the Kac-Wakimoto admissible \( \mathfrak{sl}_2 \)-module of highest weight \( k - 2a \lambda_r, s \Lambda_0 + 2a \lambda_r, s \Lambda_1 \). Here \( k = -2 + a^{-1} = -2 + \frac{1}{p} \) and \( \Lambda_i \) stands for the \( i \)-th fundamental weight of \( \mathfrak{sl}_2 \).

Next we compute the set of equivalence class of all the modules of the form \( \mathcal{L}(r, s)^{\theta} \). When \( p' = 1 \), we have \( \mathcal{L}(r, 0)^{\theta} \cong \mathcal{L}(r'p, 0) \) and the periodic isomorphism \( \mathcal{L}(r, 0)^{\theta} \cong \mathcal{L}(r, 0) \) for any \( 1 \leq r \leq p - 1 \). Therefore, as well known, there are only finitely many inequivalent simple \( \frac{1}{2} \mathbb{Z}_{\geq 0} \)-gradable \( L_{c_p, 1} \)-modules. On the other hand, when \( p' \neq 1 \), we obtain by Corollary A.3 the following identification:

**Lemma 4.3.** Suppose that \( p' \geq 2 \). Then \( \mathcal{L}(r, s)^{\theta} \) is isomorphic to \( \mathcal{L}(r', s') \) if and only if one of the following holds:

1. \( (r, s; \theta) = (r, s; 0) \) and \( (r', s') = (r, s) \).
2. \( (r, s; \theta) = (r, 0; r) \) and \( (r', s') = (p - r, p' - 1) \).
3. \( (r, s; \theta) = (r', p' - 1; -p + r) \) and \( (r', s') = (p - r, 0) \).

As a consequence, if \( p' \geq 2 \), there exist infinitely many inequivalent atypical modules.
4.2. Character formula for atypical modules. In this subsection, we compute
the character formula for atypical modules and introduce the corresponding mero-
monic functions.

Let \((r, s)\) be a pair of integers such that \(1 \leq r \leq p - 1\) and \(0 \leq s \leq p' - 1\). We consider the meromorphic function
\[
\Psi_{p, p'; r, s}(u; \tau) := \sum_{n \in \mathbb{Z}} \left( q^{pp'} \left( n + \frac{r \epsilon' - s \epsilon}{2pp'} \right)^2 + q^{pp'} \left( n + \frac{2s \epsilon}{2pp'} \right)^2 \right)
\]
on \mathbb{C} \times \mathbb{H}, where its divisor is
\[
D_{p, r} := \{(m + pn\tau, \tau) \mid (m + (pn + r)\tau, \tau) \mid \tau \in \mathbb{H}, m, n \in \mathbb{Z}\}.
\]
Then we define the formal series \(\Phi_{p, p'; r, s}(z, q)\) as the expansion of \(\Psi_{p, p'; r, s}(u; \tau)\) with respect to the two variables \((z, q)\) with respect to the two variables \((z, q) = (e^{2\pi i u}, e^{2\pi i r})\) in the region
\[
\mathcal{A} := \{(u; \tau) \in \mathbb{C} \times \mathbb{H} \mid |q| < |z| < 1\} \subset (\mathbb{C} \times \mathbb{H}) \setminus D_{p, r}.
\]
That is, the formal series \(\Phi_{p, p'; r, s}(z, q)\) is given by
\[
\Phi_{p, p'; r, s}(z, q) = \sum_{n, m \geq 0} z^n q^m \varphi_{p, p'; r, s}(q) - \sum_{n, m < 0} z^n q^m \varphi_{p, p'; r, s}(q),
\]
where
\[
\varphi_{p, p'; r, s}(q) := q^{pp'} \left( n + \frac{r \epsilon' - s \epsilon}{2pp'} \right)^2 + q^{pp'} \left( n + \frac{2s \epsilon}{2pp'} \right)^2 + (pn + r)m.
\]

Theorem 4.4. For any \(\theta \in \mathbb{Z}\), we have an equality of formal series
\[
\text{ch}^{\epsilon, \epsilon'}(\mathcal{L}(r, s)\theta)(q, z, w) = (-1)^{\epsilon' - \epsilon} q^{-a(\lambda_r, s - \theta + \frac{\theta}{2})^2} z^{2a(\lambda_r, s - \theta + \frac{\theta}{2})} w^{3(1 - 2a)}
\]
\[
\times \frac{\partial_{\epsilon, \epsilon'}(u; \tau)}{\eta(\tau)^3} \Phi_{p, p'; r, s}(u_{\epsilon, \epsilon'} + \theta \tau, \tau),
\]
\[
\text{ch}^{0, 0}(\mathcal{L}(r, s))(q, z, w) = q^{-a\lambda_r, s} z^{2a\lambda_r, s} w^{3(1 - 2a)} \frac{\partial_{0, 0}(u; \tau)}{\eta(\tau)^3} \Phi_{p, p'; r, s}(-qz^{\frac{\theta}{2}}, q).
\]

Proof. By [Sat16, Theorem 7.13], we have
\[
\text{ch}^{0, 0}(\mathcal{L}(r, s))(q, z, w) = q^{-a\lambda_r, s} z^{2a\lambda_r, s} w^{3(1 - 2a)} \frac{\partial_{0, 0}(u; \tau)}{\eta(\tau)^3} \Phi_{p, p'; r, s}(-qz^{\frac{\theta}{2}}, q).
\]

By Lemma 2.5 and some computations, we also have the formula for \(\theta \neq 0\) and \((\epsilon, \epsilon') \neq (0, 0)\). \(\square\)

Definition 4.5. An atypical character function is defined as the following mero-
monic function on \(\mathbb{H} \times \mathbb{C} \times \mathbb{C}\):
\[
A_{r, s, \theta}(\tau, u, t) := (-1)^{\epsilon' - \epsilon} q^{-a(\lambda_r, s - \theta + \frac{\theta}{2})^2} z^{2a(\lambda_r, s - \theta + \frac{\theta}{2})} w^{3(1 - 2a)}
\]
\[
\times \frac{\partial_{\epsilon, \epsilon'}(u; \tau)}{\eta(\tau)^3} \Phi_{p, p'; r, s}(u_{\epsilon, \epsilon'} + \theta \tau, \tau),
\]
where \(u_{\epsilon, \epsilon'} := u + \frac{1 - \epsilon'}{2} r + \frac{1 - \epsilon}{2} \epsilon\).

The next lemma holds by definition.

Lemma 4.6. As a function in \((q, z, w) = (e^{2\pi i r}, e^{2\pi i u}, e^{2\pi i t})\),
\[
A_{r, s, \theta}(\tau, u, t) = \text{ch}^{\epsilon, \epsilon'}(\mathcal{L}(r, s)\theta)(q, z, w)
\]
holds if \(|q| < |z| \cdot |q|^{\theta + \frac{1 - \epsilon}{2}} < 1\).
Example 4.7. In the trivial case, i.e. \((p, p'; r, s) = (2, 1; 1, 0)\), we have
\[
\frac{\eta(z)^3}{\eta_{e,e}(u; \tau)} = (-1)^{e'} q^{-\frac{1}{2}(-\theta + e + 1)^2} \Psi_{2,1;1,0}(u,e' + \theta \tau; \tau).
\]
In particular, when \(\theta = 0\) and \((e, e') = (1, 1)\), it reproduces the Ramond denominator identity for \(\hat{g}_{1;1}^\theta\) (see [KW94a] Example 4.1):
\[
\prod_{i > 0} \frac{(1 - q^i)^2}{(1 - z q^{i-1})(1 - z^{-1} q^i)} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{(n+1)}{2}}}{1 - z q^n},
\]
where \(|q| < |z| < 1\).

4.3. Discrete spectra. In this subsection, we define the index set \(\mathcal{S}_{p,p'}\) as a finite subset of \(\{(r, s; \theta) \mid 1 \leq r \leq p - 1, 0 \leq s \leq p' - 1, \theta \in \mathbb{Z}\}\).

Let \((k, \ell)\) be the unique pair of integers such that \(p' = kp + \ell\) and \(1 \leq \ell \leq p - 1\). Since the pair \((p, p')\) is coprime, so is \((p, \ell)\). Then we obtain the following lemma by easy computations.

Lemma 4.8. For \(0 \leq m \leq p - 1\), let \((r_m, s_m)\) be the unique pair of integers such that \(0 \leq r_m \leq p - 1, 0 \leq s_m \leq \ell - 1\) and \(m = \ell r_m - ps_m\). Let \(\mathcal{S}_{p,p'}^{(m)}\) be the set of solutions \((r, s; \theta)\) of the equation
\[
2m + 2p' - p = 2(r - 2\theta)p' - (2s + 1)p
\]
under the condition
\[
1 \leq r \leq p - 1, 0 \leq s \leq p' - 1, \theta \in \mathbb{Z}.
\]
Then we have
\[
\mathcal{S}_{p,p'}^{(m)} = \left\{(r, s; \theta) = (2\theta + 1 + r_m, kr_m + s_m; \theta) \mid -\left\lfloor \frac{r_m}{2} \right\rfloor \leq \theta \leq \left\lfloor \frac{p - r_m}{2} \right\rfloor - 1 \right\}.
\]

Definition 4.9. We call
\[
\mathcal{S}_{p,p'} := \bigcup_{m=0}^{p-1} \mathcal{S}_{p,p'}^{(m)}
\]
the set of discrete spectra of central charge \(c = c_{p,p'}\).

Lemma 4.10. We have the following:
1. \(|\mathcal{S}_{p,p'}| = \frac{p(p-1)}{2}.
2. For \((r, s; \theta), (r', s'; \theta') \in \mathcal{S}_{p,p'}\), the two modules \(\mathcal{L}(r, s)\) and \(\mathcal{L}(r', s')\) are isomorphic if and only if \((r, s; \theta) = (r', s'; \theta')\).

Proof. (1) is easily verified by calculation. (2) directly follows from Lemma 4.3 \(\square\)

Example 4.11. We present some examples.
1. Since \((r_0, s_0) = (0, 0)\), we have
\[
\mathcal{S}_{p,p'}^{(0)} = \left\{(2i - 1, 0; i - 1) \mid 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor \right\}.
\]
2. If \((p, p') = (p, kp + 1)\), we have
\[
\mathcal{S}_{p,p}^{(2n-1)} = \left\{(2i, (2n-1)k; i - n) \mid 1 \leq i \leq \left\lfloor \frac{p - 1}{2} \right\rfloor \right\},
\]
\[
\mathcal{S}_{p,p}^{(2v)} = \left\{(2i - 1, 2vk; i - 1 - v) \mid 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor \right\}.
\]
for $1 \leq u \leq \left\lfloor \frac{u}{2} \right\rfloor$ and $1 \leq v \leq \left\lfloor \frac{v-1}{2} \right\rfloor$. In particular, when $k = 0$, the modules corresponding to $S_{p,1}$ coincide with the $N = 2$ unitary minimal series (see [Dö98] and [Sat16, Remark 7.12]).

4.4. Modular transformation of atypical characters. For $r, r', s, s', \theta, \theta' \in \mathbb{Z}$, we set
\[
S_{(r,s;\theta),(r',s';\theta')} : = i^{-\varepsilon'}(1-1-\varepsilon')s+1-\varepsilon'2 \sum_{p} \sin(\pi arp)e^{\pi arp}e^{\pi it(2\theta'-2\theta-1)}.
\]

In addition, if $(r', s') \in K_{p', p}$, we also set
\[
S_{at,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';\theta')} : = i^{-\varepsilon'}(1-1-\varepsilon')r'+r's'2 \sin(\pi arp')
\times \frac{\sin \left( \frac{\pi x'}{\delta} \right) e^{2\pi x} + \sin (2\pi \lambda_{r',1+s'} \pi x)}{\cosh (2\pi x_0) - \cos (2\pi \lambda_{r',s'})} e^{-4\pi \lambda_{r,s}(\theta+\frac{\pi}{2})}(x-\frac{\pi}{2})
\]

for $x \in \mathbb{R}$.

Now we state the main result of this paper.

**Theorem 4.12.** The following hold:
\[
A_{r,s;\theta}^{-\frac{1}{p},u} \left( -\frac{t}{\tau}, -\frac{u}{\tau} \right) = \sum_{(r', s'; \theta') \in \mathcal{A}_{p,p'}} S_{(r,s;\theta),(r',s';\theta')} A_{(r',s';\theta')}^{(\varepsilon,\varepsilon)}(\tau, u, t)
+ \sum_{(r'', s'') \in K_{p', p'}} \int_{\mathbb{R}} dx'' S_{at,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r'',s'';x'')} \mathcal{T}_{(r'',s'';x'')}^{(\varepsilon,\varepsilon)}(\tau, u, t),
\]
\[
A_{r,s;\theta}^{(\varepsilon,\varepsilon')}(\tau + 1, u, t) = e^{2\pi i [(ar+s-p')(\theta+\frac{1}{2})-a(\theta+\frac{1}{2})-\frac{1}{2}]} A_{r,s;\theta}^{(\varepsilon,\varepsilon'+(1-\varepsilon')(1-\varepsilon))}(u, \tau)
\]
for $1 \leq r \leq p-1, 0 \leq s \leq p'-1$, and $\theta \in \mathbb{Z}$.

See Appendix C for the proof of Theorem 4.12.

**Example 4.13.** When $(p, p') = (3, 2)$ and $(r, s; \theta) = (1, 0; 0)$, we have
\[
A_{1,0,0}^{(\varepsilon,\varepsilon') \left( -\frac{1}{\tau}, -\frac{u}{\tau}, -\frac{u^2}{6\tau} \right)} = \frac{e^{2\pi i \varepsilon'}}{\sqrt{3}} \left( A_{1,0,0}^{(\varepsilon,\varepsilon)}(\tau, u, t) - e^{\pi i \varepsilon} A_{1,1,-1}^{(\varepsilon,\varepsilon)}(\tau, u, t)
- e^{\pi i \varepsilon} A_{1,0,0}^{(\varepsilon,\varepsilon')}(\tau, u, t) + \int_{\mathbb{R}} e^{\frac{\pi i y}{2\cosh(2\pi y)}} T_{1,1,0}^{(\varepsilon,\varepsilon)}(\tau, u, t) dy \right).
\]

4.5. Property of atypical $S$-data. In this subsection, we verify the symmetricity and the unitarity of the data $S_{(r,s;\theta),(r',s';\theta')}^{(\varepsilon,\varepsilon')}$.

**Lemma 4.14.** Let $r, r' \in \mathbb{Z}_{\geq 0}$ such that $r, r' \leq p-1$ and $r - r'$ is even.

1. We have
\[
\sum_{j=0}^{p-1} \sin(\pi arj) \sin(\pi ar'j) = \frac{p}{2} \delta_{r,r'}.
\]
(2) Suppose that \( p \) is odd. Then we have

\[
\sum_{j=1}^{\frac{p-1}{2}} \sin(\pi ar(2j)) \sin(\pi ar'(2j)) = \frac{p}{4} \delta_{r,r'}.
\]

Proof. By using \( \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \) and straightforward calculation, we can verify both (1) and (2).

Lemma 4.15. Let \((r, s; \theta) \in \mathcal{S}_{p,p'}^{(m)} \) and \((r', s'; \theta') \in \mathcal{S}_{p,p'}^{(m')}. Then we have

\[
\sum_{(r'', s'', \theta'') \in \mathcal{S}_{p,p'}^{(m'')}} \sin(\pi ar'') \sin(\pi ar''') e^{\pi i a(r_m - r_{m'}) (r'' - 29'')} = \frac{p^2}{4} \delta_{r,r'} \delta_{s,s'} \delta_{\theta, \theta'}.
\]

Proof. Denote the left hand side by \( S \). It is rewritten as

\[
S = \sum_{m''=0}^{p-1} e^{\pi i a(r_m - r_{m'}) (r_{m''} + 1)} \sum_{(r'', s'', \theta'' \in \mathcal{S}_{p,p'}^{(m'')}} \sin(\pi ar'') \sin(\pi ar''').
\]

We set

\[
S_{\text{odd}} := \sum_{j=1}^{\frac{p-1}{2}} \sin(\pi ar(2j - 1)) \sin(\pi ar' (2j - 1)),
\]

\[
S_{\text{even}} := \sum_{j=1}^{\frac{p-1}{2}} \sin(\pi ar(2j)) \sin(\pi ar' (2j)).
\]

Then, by Lemma 4.14 we have

\[
\sum_{(r'', s'', \theta'' \in \mathcal{S}_{p,p'}^{(m'')}} \sin(\pi ar'') \sin(\pi ar''') = \begin{cases} S_{\text{odd}} & \text{if } r_{m''} \text{ is even}, \\ S_{\text{even}} & \text{if } r_{m''} \text{ is odd}. \end{cases}
\]

By easy computations, we have

\[
S_{\text{odd}} = \begin{cases} (-1)^p (r_m - r_{m'}) S_{\text{odd}} & \text{if } p \text{ is even}, \\ (-1)^p (r_m - r_{m'}) S_{\text{even}} & \text{if } p \text{ is odd} \end{cases}
\]

and

\[
S_{\text{even}} = \begin{cases} (-1)^p (r_m - r_{m'}) S_{\text{even}} & \text{if } p \text{ is even}, \\ (-1)^p (r_m - r_{m'}) S_{\text{odd}} & \text{if } p \text{ is odd}. \end{cases}
\]

First, we assume that \( p \) is odd. Since \( \{r_{m''} \mid 0 \leq m'' \leq p - 1\} = \{0, 1, \ldots, p - 1\} \), we have

\[
S = \sum_{\ell=1}^{p} e^{2\pi i a(r_m - r_{m'}) \ell} S_{\text{even}}. \]

Since \( r_m = r_{m'} \) holds if and only if \( m = m' \), combining with Lemma 4.14 (2), we obtain

\[
S = \frac{p^2}{4} \delta_{r,r'} \delta_{s,s'} \delta_{\theta, \theta'}.
\]

Next, we assume that \( p \) is even. Then \( p' \) is odd and it follows that \( S = 0 \) holds if \( r_m - r_{m'} \) is odd. In what follows, we assume that \( r_m - r_{m'} \) is even. Then we have

\[
S = \sum_{r_{m''} \text{ odd}} e^{\pi i a(r_m - r_{m'}) (r_{m''} + 1)} S_{\text{even}} + \sum_{r_{m''} \text{ even}} e^{\pi i a(r_m - r_{m'}) (r_{m''} + 1)} S_{\text{odd}}
\]

\[
= \frac{p}{2} \delta_{m,m'} \sum_{\ell=1}^{p-1} \sin(\pi ar_\ell) \sin(\pi ar'_\ell) = \frac{p^2}{4} \delta_{r,r'} \delta_{s,s'} \delta_{\theta, \theta'}.
\]
This completes the proof. □

**Proposition 4.16.** The modular $S$-matrix $S^{\alpha_0, (\lambda, \xi')}_{(r, s), (r', s'; \theta')}$ satisfies the following:

$$S^{\alpha_0, (\lambda, \xi')}_{(r, s), (r', s'; \theta')} = S^{\alpha_0, (\lambda', \xi')}_{(r, s), (r', s'; \theta')}$$

and

$$\sum_{(r''', s'''; \theta''') \in \mathcal{F}_{r, s; \theta; r, s; \theta}} S^{\alpha_0, (\lambda, \xi')}_{(r, s; \theta), (r'', s''; \theta'')} S^{\alpha_0, (\lambda', \xi')}_{(r'', s''; \theta''), (r', s'; \theta')} = \delta_{r, r'} \delta_{s, s'} \delta_{\theta, \theta'}.$$

**Proof.** Since the former equality is obvious, we verify the latter one. The left hand side is equal to

$$\epsilon^{2\pi i(1-\epsilon')}(\lambda_{r', \theta} - (\lambda_{r', \theta} - \theta')) \times \frac{4}{p^2} \times S.$$

Therefore it immediately follows from the Lemma [1.13]. □

5. **Conjecture on Verlinde coefficients**

In this section, we consider the structure constants of the “Verlinde ring” of the simple VOSA $L_{c, p, p'}$ in the spirit of [CR13b]. See [CR13b, ACTA §5], and references therein for the details. Throughout this section, we assume that $p' \geq 2$.

Let $(r, s; \theta) \in \mathcal{F}_{p, p'}$, $(r_j, s_j) \in K_{p, p'}$, and $x_j \in \mathbb{R} + i(\mathbb{R} \setminus S_{r_j, s_j})$ for $j \in \{1, 2\}$. Then we define the (atypical-typical-typical) *Verlinde coefficient* by

$$N^{\mu_2}_{\lambda, \mu_1} := \sum_{(r', s') \in K_{p, p'}} \int_\mathbb{R} dx' e^{2\pi i\theta(r', s', x')} S_{\lambda, \theta, r', s', x'} S_{\mu_1, \theta, r', s', x'} S_{\mu_2, \theta, r', s', x'},$$

where $\lambda := (r, s; \theta)$ and $\mu_j := (r_j, s_j; x_j)$ for $j \in \{1, 2\}$.

**Conjecture 5.1.** $N^{\mu_2}_{\lambda, \mu_1}$ can be expressed as a certain delta function with a non-negative integer coefficient. In addition, the integer coincides with the fusion rule of the corresponding simple $L_{c, p, p'}$-modules, i.e. the dimension of the space of intertwining operators of type

$$\left( \begin{array}{c} \mathcal{L}_{r_2, s_2; x_2} \\ \mathcal{L}(r, s) \end{array} \right) = \left( \begin{array}{c} \mathcal{L}_{r_1, s_1; x_1} \end{array} \right).$$

See [KW94b, Definition 1.6] for the definition of intertwining operators.

**Example 5.2.** When $(p, p') = (3, 2)$, we have

$$\mathcal{F}_{3, 2}^{(0)} = \{ \lambda_0 := (1, 0; 0) \}, \mathcal{F}_{3, 2}^{(1)} = \{ \lambda_1 := (1, 1; -1) \}, \mathcal{F}_{3, 2}^{(2)} = \{ \lambda_{-1} := (2, 0; 0) \}.$$

We also have $K_{3, 2} = \{ (1, 1) \}$ and $S_{1, 1} = \left\{ \frac{1}{3} \right\} + \frac{1}{2} \mathbb{Z}$. For $j \in \{1, 2\}$ and $x_j \in \mathbb{R} + i(\mathbb{R} \setminus S_{1, 1})$, we set $\mu_j := (1, 1; x_j)$. Then we obtain

$$N^{\mu_2}_{\lambda, \mu_1} = \int_\mathbb{R} dx' e^{2\pi i\epsilon'(x_2 - x_1 + \frac{4}{3})} = \delta(x_2 - x_1 + \frac{i\epsilon}{2})$$

for $\epsilon \in \{0, \pm 1\}$. In this case, the conjecture states that

$$\dim I(x_1, x_2, \epsilon) = \begin{cases} 1 & \text{if } x_2 = x_1 - \frac{i\epsilon}{2}, \\ 0 & \text{otherwise} \end{cases}.$$
holds for \((x_1, x_2, \epsilon)\) as above, where \(I(x_1, x_2, \epsilon)\) is the space of intertwining operators of type

\[
\begin{pmatrix}
\mathcal{L}_{-\frac{1}{8} + \frac{2}{5}(x_2)^2, \frac{4}{5}x_1, -1} \\
\mathcal{L}_{\frac{1}{5} + \frac{2}{5}(x_1)^2, \frac{4}{5}(x_1)^2, -1}
\end{pmatrix}.
\]

To the best of our knowledge, the fusion rule between the above modules has not ever appeared in the literature.

APPENDIX A. SPECTRAL FLOW AUTOMORPHISMS

A.1. Definition. We consider the Lie superalgebra automorphism \(U^\theta_{\mathfrak{n}s_2}\) of \(\mathfrak{n}s_2\) defined by

\[
U^\theta_{\mathfrak{n}s_2}(L_n) = L_n + \theta J_n + \frac{\partial^2}{\partial \tau^2} C \delta_{n,0}, \quad U^\theta_{\mathfrak{n}s_2}(J_n) = J_n + \frac{\partial}{\partial \tau} C \delta_{n,0},
\]

\[
U^\theta_{\mathfrak{n}s_2}(G^\pm_{r,\theta}) = G^\pm_{r,\theta}, \quad U^\theta_{\mathfrak{n}s_2}(C) = C.
\]

It is called the spectral flow automorphism of \(\mathfrak{n}s_2\).

Let \((U^\theta_{\mathfrak{n}s_2})^*: \mathfrak{n}s_2\text{-mod} \rightarrow \mathfrak{n}s_2\text{-mod}\) be the endofunctor induced by the pullback action with respect to \(U^\theta_{\mathfrak{n}s_2}\). For simplicity, we write \(M^\theta\) for \((U^\theta_{\mathfrak{n}s_2})^*(M)\). Since we have \(U^0_{\mathfrak{n}s_2} = \text{id}_{\mathfrak{n}s_2}\) and \(U^\theta_{\mathfrak{n}s_2} \circ U^{\theta'}_{\mathfrak{n}s_2} = U^{\theta + \theta'}_{\mathfrak{n}s_2}\) for any \(\theta, \theta' \in \mathbb{Z}\), the two functors \((U^\theta_{\mathfrak{n}s_2})^*\) and \((U^{-\theta}_{\mathfrak{n}s_2})^*\) are mutually inverse.

Note that we have the following lemma:

Lemma A.1 (Sat16 Lemma B.4). For any \(c \in \mathbb{C}\), the restriction of \((U^\theta_{\mathfrak{n}s_2})^*\) gives the categorical isomorphism \(L_c\text{-mod} \rightarrow L_c\text{-mod}; \ M \mapsto M^\theta\).

A.2. Spectral flow on irreducible highest weight modules. It is easy to verify the next lemma.

Lemma A.2. Let \((h, j, c) \in \mathbb{C}^3\). Then we have

\[
(L_{h,j,c})^1 \cong \begin{cases}
\mathcal{L}_{h+j+\frac{1}{2}, j+\frac{3}{2}, c} & \text{if } h = -\frac{1}{2}, \\
\mathcal{L}_{h+j+\frac{1}{2}, j+\frac{3}{2}, -1, c} & \text{if } h \neq -\frac{1}{2},
\end{cases}
\]

\[
(L_{h,j,c})^{-1} \cong \begin{cases}
\mathcal{L}_{h-j+\frac{1}{2}, j-\frac{3}{2}, c} & \text{if } h = \frac{1}{2}, \\
\mathcal{L}_{h-j+\frac{1}{2}, j-\frac{3}{2}, -1, c} & \text{if } h \neq \frac{1}{2}.
\end{cases}
\]

By the above lemma and some computations, we obtain the following.

Corollary A.3. Let \(1 \leq r \leq p - 1, \ 0 \leq s \leq p' - 1, \ \text{and } \theta \in \mathbb{Z}\). Then we have

\[
\mathcal{L}(r, s, \theta) \cong \begin{cases}
\mathcal{L}_{r, s+1, \lambda_{r, s+1} - \theta} & \text{if } s \neq p' - 1 \text{ and } \theta < 0, \\
\mathcal{L}(p - r, 0) & \text{if } s = p' - 1 \text{ and } \theta = -p + r, \\
\mathcal{L}_{r, s+1, \lambda_{r, s+1} - \theta} & \text{if } s = p' - 1 \text{ and } -p + r + \theta < 0, \\
\mathcal{L}_{r, s; \lambda_{r, s}, -\theta} & \text{if } s = 0 \text{ and } 0 \leq \theta < r, \\
\mathcal{L}(p - r, p' - 1) & \text{if } s = 0 \text{ and } \theta = r, \\
\mathcal{L}_{r, s; \lambda_{r, s}, -\theta} & \text{if } s \neq 0 \text{ and } 0 \leq \theta,
\end{cases}
\]

where \(\lambda_{m, n} := \frac{m+1}{2} - \frac{n}{m}\) for \(m, n \in \mathbb{Z}\).

APPENDIX B. RELAXED HIGHEST WEIGHT MODULES OVER \(\widehat{\mathfrak{s}l}_2\)

In this section, we construct the simple relaxed highest weight \(\widehat{\mathfrak{s}l}_2\)-module which appeared in the proof of Theorem 13.
B.1. Definition of relaxed verma module. Let \( \{E, H, F\} \subset \mathfrak{sl}_2 \) be the standard triple and \( U_0 := \{u \in U(\mathfrak{sl}_2) \mid [H, u] = 0\} \) the centralizer of the Cartan subalgebra \( h = \mathbb{C}H \). For \( (h, \lambda, k) \in \mathbb{C}^3 \), let \( C_{h, \lambda, k} \) be the 1-dimensional \( U_0 \)-module defined by \( \Omega.1 := 2(k+2)h \) and \( H.1 := 2\lambda \), where \( \Omega \in U(\mathfrak{sl}_2) \) is the Casimir element. We define the action of \( \mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K \) on the induced \( \mathfrak{sl}_2 \)-module \( \mathcal{R}_{h, \lambda, k} := \text{Ind}_{U_0}^{U(\mathfrak{sl}_2)} C_{h, \lambda, k} \) by \( X_n := X \otimes t^n \mapsto X\delta_{n,0}, K \mapsto k \text{id} \) for \( X \in \mathfrak{sl}_2 \) and \( n \geq 0 \). Then, we define the relaxed Verma module (firstly studied in \cite{Fut96} and \cite{FST98}) as the induced \( \mathfrak{sl}_2 \)-module \( \mathcal{R}_{h, \lambda, k} := \text{Ind}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\mathfrak{sl}_2 \oplus \mathbb{C}[t] \oplus \mathbb{C}K} \mathcal{R}_{h, \lambda, k} \). We write \( L_{h, \lambda, k} \) for its unique irreducible quotient module.

B.2. Construction of relaxed highest weight modules.

**Definition B.1.** Let \( X \in \{E, F\} \) and \( U_X \) the localization of \( U := U(\mathfrak{sl}_2) \) with respect to \( \mu \in \mathbb{C} \), we define a functor \( X^\mu : \mathfrak{sl}_2 \text{-mod} \to \tilde{\mathfrak{sl}}_2 \text{-mod} \) by \( X^\mu := \text{Res}_{U}^{U_X} \circ \text{Ad}(X^\mu)^* \circ \text{Ind}_{U_X}^{U_0} \), where \( \text{Ad}(X^\mu) \) is an \( \mathbb{C} \)-algebra automorphism of \( U_X \) defined by

\[
\text{Ad}(X^\mu)(-):= \sum_{i \geq 0}^{\mu}(i)! \text{ad}(X_0)^i(-)X_0^{-i}.
\]

Suppose that \( p' \neq 1 \). We define \( E_{r,s;\lambda} := F^{-\lambda+\lambda r,s} L(r, s), \) where \( \lambda \in \mathbb{C} \) and \( L(r, s) \) is the Kac-Wakimoto admissible \( \mathfrak{sl}_2 \)-module for \( (r, s) \in K_{p,p'} \) (see Remark 4.2 for the definition).

**Proposition B.2.** For any \( \lambda \in \mathbb{C} \), we have an equality of formal series

\[
(B.1) \quad \text{tr}_{E_{r,s;\lambda}} \left(q^{L_0 \text{Sug}} - \frac{\eta(q)}{\eta(q)^2} \right) = \frac{\sum_{n \in \mathbb{Z}} q^{\lambda+n} \chi_{r,s}(q)}{\eta(q)^2},
\]

where the 0-th mode Virasoro operator \( L_0^{\text{Sug}} \) on \( E_{r,s;\lambda} \) is given by the Sugawara construction and \( c \) stands for the corresponding central charge \( c_{p,p'} \).

**Proof.** By applying the exact functor \( F^{-\lambda+\lambda r,s} \) to the BGG resolution for \( L(r, s) \) \cite[Theorem A]{Mal91}, we obtain the relaxed Verma resolution for \( E_{r,s;\lambda} \). Thus the character formula follows. \( \square \)

When \( \lambda \in \mathbb{C} \setminus S_{r,s}, \) the character \( \text{(B.1)} \) coincides with \( \text{ch}(L_{\Delta_{r,s}, \lambda, k_{p,p'}}) \) given in \cite[Corollary 5]{CRT13}. Therefore \( E_{r,s;\lambda} \) is isomorphic to \( L_{\Delta_{r,s}, \lambda, k_{p,p'}} \) if \( \lambda \in \mathbb{C} \setminus S_{r,s} \).

**Appendix C. Proof of the atypical modular transformation law**

In this section, we compute the modular (S-)transformation of the atypical character function step by step.

**C.1. Expression in terms of Appell-Lerch sums.** For a positive integer \( K \), the level \( K \) Appell-Lerch sum is defined as

\[
A_K(u, v; \tau) := \sum_{n \in \mathbb{Z}}^{\frac{(-1)^n K^n y^n q^{K(n+1)}}{1 - q^n}}.
\]

See \cite{STT03} and \cite{ACT14} for the fundamental properties of this function.

We obtain the following by Definition 4.5.
Lemma C.1. We have
\[
\eta(t)^3 \frac{\eta(z,z')}{\eta(z,z') \tau} (u, t) = (-1)^{z' + 1} q^{\frac{1}{2} (a + s - p') + \frac{1}{2} (a - s + p') + 3 a (a + s - p') - 2 a (a + s - p') + \frac{1}{2} (a + s - p')}
\times \left( A_{2p'} \left( u, \tau, v, \theta, (p' - sp - pp') \tau; p \right) - q^{z' + 1} A_{2p'} \left( u, \tau, v, \theta - sp - pp' \tau; p \right) \right).
\]

C.2. Modular transformation law of Appell-Lerch sum. The following modular S-transformation property of the Appell-Lerch sum is a special case of [AC14 Corollary 3.4].

Proposition C.2.
\[
A_{2p'} \left( \frac{u}{\tau}, \frac{v}{\tau}, \frac{-p}{\tau} \right) = \frac{\tau}{p} e^{-2 \pi i (pu^2 - uv)/pr} \left[ A_{2p'} \left( \frac{u}{p}, \frac{v}{p}, \frac{\tau}{p} \right) + \theta \left( \sum_{m=0}^{p'-1} \frac{\theta \left( \sum_{n=0}^{2p'} \theta \left( \sum_{\ell=0}^{p'-1} A_{2p'} \left( u + n \tau, \frac{2p'n + m + p'}{p}, \frac{\tau}{p} \right) \right) \right) \right) \right],
\]
where \( f_m(\tau, v) := \int_{\mathbb{R}} \frac{x^2 + 2 \pi i x}{\cosh \pi x} \omega \) is the Mordell integral.

By Lemma C.1 and Proposition C.2 we see that the S-transformed character decomposes into the following two factors:
\[
A^{u, \tau}_{r, \tau, 0} \left( -\frac{1}{\tau}, \frac{u}{\tau}, \frac{-p}{\tau} \right) = (\text{discrete part}) + (\text{continuous part}).
\]

C.3. Computation of the discrete part. In this subsection, we compute the discrete part by rewriting all the objects in terms of the level 1 Appell-Lerch sum.

C.3.1. S-transformed side. In what follows, we write
\[
\tilde{u}_{x, z'} := u + \frac{1 - z'}{2} \tau - \frac{1 - z}{2} \equiv u_{x, z} \pmod{\mathbb{Z}}.
\]
By Lemma C.1 and Proposition C.2 the discrete part is given by
\[
(C.1) \quad A_{2p'} \left( \frac{\tilde{u}_{x, z'} - \theta}{p}, \frac{-p}{p} \right) = A_{2p'} \left( \frac{\tilde{u}_{x, z'} - \theta + r}{p}, \frac{p}{p} \right).
\]
We can rewrite this in terms of \( A_1 \) by the following lemma:

Lemma C.3.
\[
A_{2p'} \left( \frac{u}{p}, \frac{v}{p}, \frac{\tau}{p} \right) = \frac{1}{2p'} \sum_{n, m=0}^{p-1} z^{2m + 2p' - n} y^{m} q^{\frac{n(p' + m + p')}{2p'}} \times \sum_{\ell=0}^{2p' - 1} A_{1} \left( u + n \tau, \frac{2p'n + m + p'}{2p'}, \frac{\tau}{2p'} \right) \right).
\]

The proof is straightforward and we omit it.
Lemma C.3. The right hand side is also written in terms of $A$

Lemma C.4. Computation of the continuous part.
Then, by choosing appropriate $N$

Lemma C.6. Then the (untrasformed) atypical character is also rewritten in terms of $A_1$ as follows:

**Corollary C.4.** Let $N, M \in \mathbb{Z}$. We have

\[
\frac{\eta(\tau)^3}{\eta(\tau^3)} \vartheta_{e,e'}(u;\tau, u, t) = (-1)^{e+e'} \frac{(1-\varepsilon')}{2p'} \left( q \left( \theta + Np \right) \tau, u, (N - \varepsilon') \frac{pt}{2p'} \right) \]

\[
\times \sum_{\ell=0}^{2p'-1} \left[ A_1 \left( u, e, e' \left( \theta + Np \right) \tau, \frac{pt}{2p'} \right) \right.
\]

\[
- q^{(Mp+1)(2Np+Mp'-s-\frac{1}{2})} A_1 \left( u, e, e' \left( \theta + Np \right) \tau, \frac{pt}{2p'} \right) \],
\]

where $\bullet := 2(r-2\theta)p' - (2s+1)p$ and

\[
v_{\pm}(n) := \frac{2np'}{2p'} \pm \frac{sp'}{2p'} \theta - \frac{pt}{2p'} + \frac{\ell}{2p'}
\]

for $n \in \mathbb{Z}$.

**Proof.** Since we have $A_1(u + n\tau, v + n\tau; \tau) = (-y)^{-n} q^{-\frac{m}{2}} A_1(u, v, \tau)$, we only have to verify the equality for $N = M = 0$. It immediately follows from (C.2).

**C.3.3. Conclusion.**

**Proposition C.5.** We have

\[
(\text{discrete part}) = \frac{1}{p} (-1)^{(1-e')s+(1-e)s'} \sum_{(r',s';\theta') \in \mathcal{F}_{r',s',\theta'}^{p'}} \times \sin(\pi ar'v') p^{-ia(r-2\theta-1+e')(r'-2\theta'-1+e')} A_{r',s';\theta'}(u;\tau).
\]

**Proof.** The left hand side is equal to (C.3) and can be written in terms of $A_1$ by Lemma C.3. The right hand side is also written in terms of $A_1$ by Corollary C.4.

Then, by choosing appropriate $N$ and $M$, we obtain the formula.

**C.4. Computation of the continuous part.** In this subsection, we compute the continuous part.

**C.4.1. Computation of the theta part.** First we compute the theta functions in the continuous part. The following lemma is easily verified and we omit the proof.

**Lemma C.6.** We have

\[
\vartheta_{1,1} \left( u; \frac{\tau}{2p'} \right) = -i \sum_{\ell=0}^{2p'-1} e^{2\pi i \left( \ell + \frac{1}{2} \right) \left( \frac{u+\frac{1}{2}}{4p} \right)^2} \vartheta_{0,0} \left( 2pp' + \left( \ell + \frac{1}{2} \right) \tau; 2pp' \tau \right).
\]
Lemma C.10. We have

\[
\frac{2}{1 + e^{-2\pi(y + \frac{4\pi}{p}) - \frac{2\pi y}{p}}}
\]

Proof. By the previous lemma, we have

\[
\vartheta, \emptyset := \vartheta(\pi L) = 2\pi \sum_{\ell=0}^{2p'} q^{\pi i (\ell + \tau + \frac{2p'}{p}) \theta_0} \vartheta, \emptyset(\ell + \tau + \frac{2p'}{p}).
\]

Then the required formula follows.

Remark C.8. When \( p' = 1 \),

\[
\sum_{L=1}^{2p} (-1)^{Lm} \sin \left( \frac{\pi rL}{p} \right) \sum_{n \in \mathbb{Z}} q^{p(n + \frac{4\pi}{p})^2} = 0
\]

holds for \( 0 \leq m \leq 1 \). In particular, we get \( \vartheta, \emptyset(m) = 0 \).

C.4.2. Computation of the Mordell part. Second we compute the Mordell integral in the continuous part.

Lemma C.9. For \( 0 \leq m \leq 2p' - 1 \), we have

\[
\text{(Mordell)}_m := h \left( \frac{u - \theta - \frac{1-i}{2}}{p} - \frac{-rp' + sp - pm + pp'}{2pp'} - \left( \epsilon'p' - \frac{1}{2} \right) \frac{\tau}{2pp'} \right)
\]

\[
= 2p'e^{2\pi i (\theta + \frac{1-i}{2})(\epsilon' a - \frac{1}{2p})} e^{-\pi i \epsilon'(ar-s+m)} e^{-\pi i \left( \frac{-r\epsilon' \tau \tau - \frac{2p'}{p} - \mu}{2pp'} \right)}
\]

\[
\times q^{\frac{(2\epsilon' p')^2}{18p'^2}} z^{-\epsilon' a + \frac{1}{2p}} \int_{\mathbb{R} - i(\frac{1}{2} - \frac{1}{2p})} e^{2\pi [p' - m(\lambda^2 - \theta + \frac{1}{2})]} \frac{\sinh (2p' \pi y)}{q^{2\pi y}} 2\pi y dy.
\]

Proof. Since we have

\[
h(u - s\tau; \tau) = q^2 z^{-s} \int_{\mathbb{R} - is} \frac{e^{\pi i x^2 - 2\pi u x}}{\cosh \pi (x + is)} dx
\]

for \( s \in \mathbb{R} \), the equality follows from a direct computation.

Lemma C.10. We have

\[
e^{2p' \pi y} \sum_{m=0}^{2p'-1} \left( (-1)^{1 - \epsilon'} e^{- \frac{2\pi y}{p}} e^{-2\pi y} \right)^m = \frac{2 \sinh (2p' \pi y)}{1 + e^{-2\pi(y + \frac{4\pi}{p}) - \frac{2\pi y}{p}}}.
\]
Proof. Since
\[-\frac{i\varepsilon'}{2} + \left(n + \frac{1}{2} - \frac{L}{2p'}\right) i \notin \mathbb{R} - i \left(\frac{\varepsilon'}{2} - \frac{1}{4p'}\right)\]
for any \(n \in \mathbb{Z}\), the ratio in the left-hand side is not equal to 1. \qed

C.4.3. Conclusion. Finally we obtain the explicit form of the continuous part as follows.

**Proposition C.11.** We have

\[
(\text{continuous part}) = i^{1-\varepsilon'} \frac{2}{p} \sum_{(r',s') \in K_{p,p'}} (-1)^{r'+rs'} \sin(\pi r p')
\times \int_{\mathbb{R}} \sin \left(\frac{\pi x}{a}\right) e^{2\pi x'} + \sin \left(2\pi \lambda_{r',(1+s)s'}\right) \cosh(2\pi x') - \cos(2\pi \lambda_{r',s'})
\times e^{-4\pi a(\lambda_{r',s'} - \theta + \frac{\xi}{2})} \left(x' - \frac{i}{p}\right)^{2} \eta_{r',s'}(u;\tau) dx'.
\]

Proof. By Lemma C.10 and Proposition C.2, the continuous part is equal to \(C \times \partial_{\varepsilon',\varepsilon'}(u;\tau)\eta(\tau)^{-3}\), where

\[
C := i^{1-\varepsilon'} (-1)^{\varepsilon'} \frac{1}{p} \sum_{L=mp'} e^{2\pi i \frac{L}{p}} \sin \left(\frac{\pi r L}{p'}\right) \sum_{n \in \mathbb{Z}} \left(q^{n + \frac{1}{2p'}}\right)^{2\text{pp'} - 1} (\text{theta})_{m}(\text{Mordell})_{m}.
\]

By using Lemma C.10 and \(e^{2\pi i (\theta_{r',s'} - \theta)(a - \frac{1}{p'})} = e^{-2\pi i \theta_{r',s'}(a - \frac{1}{p'})} z a - \frac{1}{p'} q^{\frac{1-\varepsilon'}{2}}(\text{theta})_{\frac{L}{p'}}\), we have

\[
C = i^{1-\varepsilon'} (-1)^{\varepsilon'} \frac{2}{p} \sum_{L=mp'} e^{2\pi i \frac{L}{p}} \sin \left(\frac{\pi r L}{p'}\right) \sum_{n \in \mathbb{Z}} q^{n + \frac{1}{2p'}}^{2p' - 1} (\text{theta})_{m}(\text{Mordell})_{m}.
\]

\[
= i^{1-\varepsilon'} (-1)^{\varepsilon'} \frac{2}{p} \sum_{(r',s') \in K_{p,p'}} (-1)^{r'+rs'} e^{-\frac{\pi x}{a}} \sin(\pi r p')
\times \int_{\mathbb{R}} e^{-4\pi a(\lambda_{r',s'} - \theta + \frac{\xi}{2})} \left(x' - \frac{i}{p}\right)^{2} \eta_{r',s'}(\tau) d\tau
= i^{1-\varepsilon'} (-1)^{\varepsilon'} \frac{2}{p} \sum_{(r',s') \in K_{p,p'}} (-1)^{r'+rs'} \sin(\pi r p')
\times \int_{\mathbb{R}} e^{-2\pi i \theta_{r',s'}(a - \frac{1}{p'})} q^{2iay} z^{2ia\tau} q^{\frac{1-\varepsilon'}{2}}(\text{theta})_{\frac{L}{p'}} d\tau
= i^{1-\varepsilon'} (-1)^{\varepsilon'} \frac{2}{p} \sum_{(r',s') \in K_{p,p'}} (-1)^{r'+rs'} \sin(\pi r p')
\times \int_{\mathbb{R}} e^{2\pi Y} + \sin \left(2\pi \lambda_{r',(1+s)s'}\right) \cosh(2\pi Y) - \cos(2\pi \lambda_{r',s'})
\times (-1)^{\varepsilon'} q^{\frac{1-\varepsilon'}{2}}(\text{theta})_{\frac{L}{p'}} dY.
\]
In the last equality, we put \( Y := y - \frac{2z'}{\epsilon} \). Finally we shift the region of the integration by the residue theorem and obtain the required formula. 

\[ \square \]

### Appendix D. Comparison with the Results of Kac-Wakimoto

#### D.1. Modular transformation of the minimal unitary characters

In this subsection, we explain the relation between our result and the modular transformation properties for \( \mathcal{N} = 2 \) minimal unitary characters in [KW94a, Theorem 6.1]. We write \( A^{(p,0)} \) and \( A^{(p,\pm)} \) for the finite set \( A^+ \) and \( A^- \) defined in [KW94a, §6].

**Lemma D.1.** Let \( \varepsilon \in \{0, 1\} \) and \((j, k) \in A^{(p,\varepsilon)} \). Then we have

\[
\text{ch}_{j,k}^{p,\varepsilon}(\tau, u) = A_{j+k,0;k-\varepsilon}^{1-2\varepsilon,0}(\tau, u, 0), \quad \text{sch}_{j,k}^{p,\varepsilon}(\tau, u) = A_{j+k,0;k-\varepsilon}^{1-2\varepsilon,1}(\tau, u, 0),
\]

where \( \text{ch}_{j,k}^{p,\varepsilon}(\tau, u) \) and \( \text{sch}_{j,k}^{p,\varepsilon}(\tau, u) \) are defined in [KW94a, §6].

**Proof.** Since we have

\[
\Psi^{(\varepsilon)}(\tau, u) = \frac{\eta(\tau)^3}{\vartheta_{1-2\varepsilon,0}(u; \tau)}, \quad \Psi^{(\varepsilon)}(\tau, u) = \frac{\eta(\tau)^3}{(-1)^{1-2\varepsilon}\vartheta_{1-2\varepsilon,1}(u; \tau)}
\]

in [KW94a], we obtain

\[
\text{ch}_{j,k}^{p,\varepsilon}(\tau, u) = \left( \sum_{n,m \geq 0} - \sum_{n,m < 0} \right) (-1)^{n+m} z \vartheta_{1-2\varepsilon,0}(u; \tau) \eta(\tau)^3,
\]

\[
\text{sch}_{j,k}^{p,\varepsilon}(\tau, u) = (-1)^{1-2\varepsilon} \left( \sum_{n,m \geq 0} - \sum_{n,m < 0} \right) z \vartheta_{1-2\varepsilon,1}(u; \tau) \eta(\tau)^3.
\]

Our character formula gives

\[
A_{j+k,0;k-\varepsilon}^{1-2\varepsilon,\varepsilon'}(\tau, u, t)
= \text{ch}_{j+k,0;k-\varepsilon}^{1-2\varepsilon,\varepsilon'}(q, z, u)
= (-1)^{(1-2\varepsilon)\varepsilon'} q^{-\frac{\varepsilon'(j+k)^2}{4n}} \vartheta_{j+k,1-2\varepsilon,\varepsilon'}(u; \tau) \eta(\tau)^3 \Phi_{p,1;j+k,0}((-1)^{1-\varepsilon'} q^k, q)
\]

for \( \varepsilon' \in \{0, 1\} \). We can verify that these formulae coincide by some computations. 

\[ \square \]

Through the above identification, we obtain [KW94a, Theorem 6.1] as a special case of Theorem 1.12 for \( p' = 1 \) (see Remark C.8).

#### D.2. Non-unitary spectra of Kac-Wakimoto

In this subsection, we compare our spectra \( \mathcal{S}_{p,p'} \) with the set of highest weights considered in [KW16, §3]. In what follows, we identify the set of triples \( \mathcal{S}_{p,p'} \) with that of the highest weights of the corresponding \( \mathfrak{ns}_2 \)-modules.

Let \( \{\Lambda_0, \Lambda_1, \Lambda_2\} \) be the set of fundamental weights of the affine Lie superalgebra \( \hat{\mathfrak{sl}}_{2|1} \). We write \( \Lambda(s, i) \) for the integral weight \((p' - 1 - s)\Lambda_0 + s \Lambda_i\), where \( 0 \leq s \leq p' - 1 \) and \( i \in \{1, 2\} \). In [KW16, §3], V. Kac and M. Wakimoto considered a certain family of simple highest weight \( \mathfrak{ns}_2 \)-modules of central charge

\[
c = 3(1 - 2a) = 3 \left(1 - \frac{2p'}{p}\right)
\]
associated with the pair \((p, \Lambda(s, i))\) (see [KW16 Lemma 2.1]). The set of the corresponding \(s\)-highest weights \((h, j) = (h, j, 3(1 - 2a))\) is given as follows:

\[
S(a; s, i) := \{(h, j) = \Lambda_{k_1, k_2} \mid k_1, k_2 \in \mathbb{Z}_{\geq 0} \text{ and } k_1 + k_2 \leq p - 1\},
\]

where

\[
\Lambda_{k_1, k_2} := \left(a \left((k_1 + \frac{1}{2})(k_2 + \frac{1}{2}) - \frac{1}{4}\right) - s(k_1 + \frac{1}{2}), (-1)^i(a(k_2 - k_1) - s)\right).
\]

We note that they assume the integer \(p\) to be odd. See [KW16 §3] for the details.

**Example D.2.** Here we present two examples.

(1) If \((p, p') = (3, 2)\), we have

\[
S\left(\frac{2}{3}; 0, i\right) = \left\{(0, 0), \left(\frac{1}{3}, \pm\frac{2}{3}\right)\right\} = \mathcal{I}_{3, 2}
\]

for \(i \in \{1, 2\} \).

(2) If \((p, p') = (5, 2)\), we have

\[
S\left(\frac{2}{5}; 0, i\right) \cap \mathcal{I}_{5, 2} = \left\{(0, 0), \left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, 0\right), \left(\frac{7}{5}, \frac{2}{5}\right)\right\},
\]

\[
S\left(\frac{2}{5}; 1, i\right) \cap \mathcal{I}_{5, 2} = \left\{\left(\frac{1}{10}, \frac{(-1)^i}{5}\right)\right\},
\]

and

\[
\mathcal{I}_{5, 2} \subset S\left(\frac{2}{5}; 0, i\right) \cup S\left(\frac{2}{5}; 1, 1\right) \cup S\left(\frac{2}{5}; 1, 2\right)
\]

for \(i \in \{1, 2\} \).

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