Equilibrium and eigenfunctions estimates in the semi-classical regime.

Brice Camus
Ruhr-Universität Bochum, Fakultät für Mathematik,
Universitätsstr. 150, D-44780 Bochum, Germany.
Email: brice.camus@univ-reims.fr

Abstract

We establish eigenfunctions estimates, in the semi-classical regime, for critical energy levels associated to an isolated singularity. For Schrödinger operators, the asymptotic repartition of eigenvectors is the same as in the regular case, excepted in dimension 1 where a concentration at the critical point occurs. This principle extends to pseudo-differential operators and the limit measure is the Liouville measure as long as the singularity remains integrable.

Key words: Quantum chaos, Schrödinger operators, Equilibria in classical mechanics.

1 Introduction.

The problem we consider here concerns the asymptotic behavior of eigenvectors of a self-adjoint operator and follows the works of Colin de Verdière [6] and Zelditch [13], on the basis of a result stated first by Schnirelman. We are more precisely interested in a proof of a microlocal concentration phenomena, in the semi-classical regime, as established in [1,7]. The adaptation to semi-classical analysis was done in [10], following a technic proposed by Voros [12]. We also mention [8] for a more general approach in the scattering setting.

Consider a quantum operator $P_h$, realized as a self-adjoint operator acting on a dense subset of $L^2(\mathbb{R}^n)$. A typical example, studied in section 2, is the Schrödinger operator $P_h = -h^2\Delta + V$ where the potential $V$ is smooth and

1 Work supported by the SFB/TR12 Symmetries & Universality in Mesoscopic Systems.
bounded from below. If the spectrum of \( P_h \) is discrete in \([E - \varepsilon, E + \varepsilon]\), a sufficient condition for this is given below, we can enumerate the eigenvalues in this interval as a sequence \( \lambda_j(h) \) with finite multiplicities. We note \( \psi_j^h \) the corresponding normalized eigenvectors, i.e.

\[
P_h \psi_j^h = \lambda_j(h) \psi_j^h, \quad ||\psi_j^h||_{L^2} = 1.
\]

Our objective is to establish eigenfunctions estimates:

\[
\begin{align*}
\nu_j(a) &= \langle \text{Op}_h^w(a) \psi_j^h, \psi_j^h \rangle, \\
\lambda_j(h) &\to E, \quad h \to 0^+,
\end{align*}
\]

where \( a \in S^0(\mathbb{R}^{2n}) \), so that by the Calderon-Vaillancourt Theorem:

\[
f \mapsto \text{Op}_h^w(a) f(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} a(\frac{x + y}{2}, \xi) e^{\frac{i}{h}(x-y,\xi)} f(y) dyd\xi,
\]

is bounded on \( L^2(\mathbb{R}^n) \). Note that the statement of the problem is local w.r.t. \( E \) and we are here interested in the case of \( E = E_c \) critical. Each \( \nu_j(a) \) measures the observable \( \text{Op}_h^w(a) \) in the state \( \psi_j^h \). Let \( \Phi_t = \exp(tH_p) \) be the Hamiltonian flow of the principal symbol \( p \) of \( P_h \). Interpreted as distributions, these measures are \( \Phi_t \)-invariant which easily follows from:

\[
\langle e^{-\frac{i}{h} P_h} \text{Op}_h^w(a) e^{\frac{i}{h} P_h} \psi_j^h, \psi_j^h \rangle = \langle \text{Op}_h^w(a) e^{\frac{i}{h} \lambda_j(h)} \psi_j^h, e^{\frac{i}{h} \lambda_j(h)} \psi_j^h \rangle = \nu_j(a).
\]

By Egorov’s Theorem \( e^{-\frac{i}{h} P_h} \text{Op}_h^w(a) e^{\frac{i}{h} P_h} \) is an operator of principal symbol \( (a \circ \Phi_t) \) and \( \nu_j \) is invariant under \( \Phi_t \), up to \( O(h) \).

We recall that \( E \) is regular if \( \nabla p \neq 0 \) on the energy surface:

\[
\Sigma_E = \{(x, \xi) \in T^*\mathbb{R}^n / p(x, \xi) = E \},
\]

and critical otherwise. When \( E_c \) is critical, \( \Sigma_{E_c} \) is not a smooth manifold. For \( E \) regular, \( \Sigma_E \) inherits a measure, invariant by \( \Phi_t \), given by:

\[
dLvol(z) = \frac{dz}{||\nabla p(z)||_{|\Sigma_E|}}, \quad z \in \Sigma_E,
\]

where \( dz \) is the Riemannian surface element. We note \( \mathcal{V}(E) \) the associated volume of \( \Sigma_E \) and we obtain a probability measure via:

\[
d\mu^E(z) = \frac{1}{\mathcal{V}(E)} dLvol(z).
\]

Note that if \( E_c \) is critical, \( d\mu^{E_c} \) has a sense if and only if \( 1 \in L^1(\Sigma_{E_c}, d\text{Lvol}) \).
Via a wave equation approach, substituting here the heat equation strategy of [6] on a compact manifold, the problem is related at the first order to the geometry of the energy surfaces. Accordingly, the integrability of dLvol has a strong effect on the asymptotic behavior of the sequence $\nu_j$.

The micro-local concentration near a singularity was studied in [7] in dimension one for a non-degenerate unstable equilibrium where they proved that $\nu_j$ converges to the Dirac mass at the equilibrium. We are first interested in the case of Schrödinger operators, but a generalization to pseudo-differential operators provides more examples. Finally, we recall that in Riemannian geometry, e.g. for compact surfaces of negative curvature, the question to know if the full sequence converges to the invariant measure (quantum unique ergodicity) is still an open problem.

Definitions.
We define now the objects used in sections 2&3. If $E_c$ is a critical energy level, we pick an $h$-dependant interval:

$$I(h) = [E_c - dh, E_c + dh], \ d > 0.$$ (1)

The associated counting function of eigenvalues is:

$$\Upsilon(h) = \sum_{\lambda_j(h)\in I(h)} \langle \psi^h_j, \psi^h_j \rangle = \# \{j / \lambda_j(h) \in I(h) \}.$$ (2)

For $A = Op^w_h(a)$ a pseudodifferential operator of order zero, whose principal symbol is $a \in S^0(\mathbb{R}^{2n})$ we put:

$$\Upsilon_a(h) = \sum_{\lambda_j(h)\in I(h)} \langle A\psi^h_j, \psi^h_j \rangle = \sum_{\lambda_j(h)\in I(h)} \nu_j(a).$$ (3)

Observe that $\Upsilon(h) = \Upsilon_1(h)$ since for every quantization used in this contribution the symbol of the identity is 1.

2 Schrödinger operators.

Let $p(x, \xi) = \xi^2 + V(x)$, where the potential $V \in C^\infty(\mathbb{R}^n)$ is real valued. To obtain a well defined spectral problem, we use:

$$(\mathcal{H}_1) \text{ There exists } C \in \mathbb{R} \text{ such that: } \liminf_{\infty} V > C.$$  \\

By a classical result, $P_h = -\hbar^2\Delta + V(x)$ is essentially self-adjoint. Note that $(\mathcal{H}_1)$ is always satisfied if $V$ goes to infinity at infinity. Let $J = [E_1, E_2]$, with $E_2 < \liminf_{\infty} V$. Since $p^{-1}(J)$ is compact, the spectrum $\sigma(P_h) \cap J$ is discrete
and consists in a sequence $\lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h)$ of eigenvalues of finite multiplicities, if $h$ is small enough. Next, we impose the singularity:

\((\mathcal{H}_2)\)  On $\Sigma_{E_c}$ the symbol $p$ has an isolated critical point $z_0 = (x_0, 0)$. This critical points can be degenerate but is associated to a local extremum of $V$

$$V(x) = E_c + V_{2k}(x) + O(||x - x_0||^{2k+1}), \ k \in \mathbb{N}^*,$$

where $V_{2k}$, homogeneous of degree $2k$, is definite positive or negative.

The case $k = 1$, i.e. a non-degenerate singularity in dimension $n$, is treated in [1] without any extremum condition. In Fig.1 the line in bold is the critical energy level attached to the top of a one dimensional symmetric degenerate double well. Observe the unstability near the recurrent critical point.

![Figure 1. Energies surfaces of $V(x) = -x^4 + x^6$.](image)

To simplify notations we write $z = (x, \xi) \in T^*\mathbb{R}^n$ and $z_0$ for a critical point. The first result concerns the statistical behavior of the sequence $\nu_j(a)$.

**Theorem 1** Assume (\(\mathcal{H}_1\)) and (\(\mathcal{H}_2\)) satisfied. If $n > 1$ we have

$$\lim_{h \to 0^+} \frac{\Upsilon_a(h)}{\Upsilon(h)} = \int a d\mu^{E_c},$$

but in dimension 1 we obtain:

$$\lim_{h \to 0^+} \frac{\Upsilon_a(h)}{\Upsilon(h)} = \langle \delta_{z_0}, a \rangle = a(x_0, 0).$$

These relations are statistical since when $h \to 0$ the number of eigenvalues in $I(h)$ tends to infinity, see Proposition 4. We define:

$$K(h) = \{ j \in \mathbb{N} \ / \ \lambda_j(h) \in I(h) \}.$$
Following the results of [1,6,13,10], for \( n > 1 \) Theorem 1 implies that if \( \Phi_t \) is ergodic on \( \Sigma_{E_c} \) there exists a density one subset \( L(h) \subset K(h) \) such that for all integer valued function \( h \to j(h) \in L(h) \) we have:

\[
\lim_{h \to 0} \langle \psi_j^h, \text{Op}_h(a)\psi_j^h \rangle = \int_{\Sigma_{E_c}} ad\mu_{E_c}.
\]

Here, density one simply means that:

\[
\lim_{h \to 0} \frac{\#L(h)}{\#K(h)} = 1.
\]

Since there is nothing new to prove, we refer to [1,10] for a precise study. More interesting, is the generalization of the result of [7]:

**Corollary 2** In dimension 1, assume that \( \Sigma_{E_c} \) is connected. Then, for the weak * topology, we have \( \nu_{j(h)} \to \delta(z_0) \), \( j(h) \in K(h) \), uniformly as \( h \to 0 \).

Hence, we obtain a concentration at \( z_0 \). Observe that \( \delta(z_0) \) is \( \Phi_t \)-invariant. If \( a \) is simply a function, the quantum probability satisfies:

\[
\lim_{h \to 0} \lambda_j^h(h) = E_c \Rightarrow \lim_{h \to 0} |\psi_j^h|^2(x) = \delta_{x_0}.
\]

The interpretation is as follows. In dimension one, the invariant measure on \( \Sigma_{E_c} \) has a singularity:

\[
dLvol(z) \sim c \frac{dz}{||z - z_0||}, \text{ near } z_0 = (x_0, 0).
\]

The measure is not integrable and the result has to be different. For \( n > 1 \), the singularity is integrable and an isolated critical point has no effect. This reinforce the universality of the Liouville measure in quantum ergodicity. But the case \( n = 1 \) is important since many problems, with symmetries, can be reduced to the study of such a singular Schrödinger equation. See e.g. [7] for an application in Riemannian-geometry.

**Preliminary remarks.**

The case of a local minimum of \( V \) is not really deep. Since \( \xi^2 \geq 0 \), \( z_0 \) is a local extremum of \( p \) and is an isolated point of \( \Sigma_{E_c} \). According to the results of [3], the contribution of a minimum is significative only if \( n = 1 \). We consider now the non-trivial case of a local maximum of \( V \), corresponding to an unstable equilibrium of the flow. Finally, since we use below the functional calculous, \( p \) has to be a symbol. But with \( (\mathcal{H}_1) \) we can eventually modify the potential \( V \) outside of a compact, without modifying the main results. Hence no extra assumption is required. Similar comments apply for section 3.

**Proof of Theorem 1.**
We use the semi-classical trace formula technic. This approach, analogous to
the trace of the heat operator of [6], uses the propagator $e^{\frac{t}{2}P_h}$ and a general-
ization of the Poisson summation formula for this operator. Let $\varphi \in S(\mathbb{R})$, a
Schwartz function. To approximate $\Upsilon$ we define:

$$\gamma(E_c, h, \varphi) = \sum_{|\lambda_j(h) - E_c| \leq \varepsilon} \varphi(\frac{\lambda_j(h) - E_c}{h}). \quad (4)$$

This object can be treated by mean of Fourier integral operators (F.I.O.), see
e.g. [9], and we refer to [1,2,3] for a detailed study of the trace. We recall that
the Tauberian approximation concerns expressions:

$$\Upsilon^\alpha_{E,h}(\varphi) = \sum_j \alpha_j(h) \varphi(\frac{\lambda_j(h) - E_c}{h}). \quad (5)$$

Under our assumptions, the behavior of $\Upsilon^\alpha_{E,h}$ determines the behavior of the
weighted counting functions:

$$N^\alpha_{E,d}(h) = \sum_{|\lambda_j(h) - E_c| \leq \varepsilon} \alpha_j(h).$$

Precisely, we will apply the results of section 6 of [1] for $\alpha_j(h) = 1$ or $\nu_j(a)$.
Strictly speaking, $\alpha_j(h) \geq 0$ is required but, by a standard result of pseudodif-
fferential calculus, we can modify the quantization to have $\nu_j(a) \geq 0$. This
does not change the main results, see Eq.(14) below.

To attain our objective, we can suppose that supp($\hat{\varphi}$) $\subset [-M, M]$, $M > 0$.
Let $\Theta \in C^\infty_0(]E_c - \varepsilon, E_c + \varepsilon[)$, such that $\Theta = 1$ in a neighborhood of $E_c$ and
$0 \leq \Theta \leq 1$ on $\mathbb{R}$. We localize the problem near $E_c$ by writing:

$$\gamma(E_c, h, \varphi) = \gamma_1(E_c, h, \varphi) + \gamma_2(E_c, h, \varphi),$$

with:

$$\gamma_2(E_c, h, \varphi) = \sum_{|\lambda_j(h) - E_c| \leq \varepsilon} \Theta(\lambda_j(h)) \varphi(\frac{\lambda_j(h) - E_c}{h})$$

$$= \text{Tr} \Theta(P_h) \varphi(\frac{P_h - E_c}{h}).$$

Where the last equality holds by support considerations. By a classical result,
see e.g. [2] Lemma 1, the term $\gamma_1 = \gamma - \gamma_2$ satisfies:

$$\gamma_1(E_c, h, \varphi) = \mathcal{O}(h^\infty), \text{ as } h \to 0. \quad (5)$$

The Fourier inversion formula for $\gamma_2$ and the previous estimate provide:

$$\gamma(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{itE_c}{h}} \varphi(t) \exp(-\frac{i}{h}tP_h) \Theta(P_h) dt + \mathcal{O}(h^\infty). \quad (6)$$
Next, with a function $\Psi \in C^\infty_0(T^*\mathbb{R}^n)$, with $\Psi = 1$ near $z_0$, we write:

$$\gamma_2(E_c, h, \varphi) = \gamma_{z_0}(E_c, h, \varphi) + \gamma_{\text{reg}}(E_c, h, \varphi),$$

where:

$$\gamma_{z_0}(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{it\varphi}{h}} \hat{\varphi}(t) \Psi^w(x, hD_x) \exp(-\frac{i}{\hbar} tP_h) \Theta(P_h) dt, \quad (7)$$

and $\gamma_{\text{reg}}$ is simply the difference. The micro-local term $\gamma_{z_0}$ contains the contribution of the singularity and the discussion below determines if this term is dominant. For finitely many critical point on $\Sigma_{E_c}$, we could repeat the procedure. For the convenience of the reader, we recall the contributions of an equilibrium to the trace formula.

**Proposition 3** Assume $(\mathcal{H}_1), (\mathcal{H}_2)$ and that $\hat{\varphi} \in C^\infty_0([-M, M])$, $M \leq M_0$. If $x_0$ is a local maximum of $V$ we have:

$$\gamma_{z_0}(E_c, h, \varphi) \sim h^{-n+\frac{\tilde{d}}{2} + \frac{n}{2k}} \sum_{m=0,1} \sum_{j,l \in \mathbb{N}^2} h^{\frac{j}{2} + \frac{l}{4k}} \log(h)^m \Lambda_{j,l,m}(\varphi).$$

If $\frac{n(k+1)}{2k} \in \mathbb{N}$ and $n$ is odd then the top-order term is:

$$C_{n,k} \log(h) h^{-n+\frac{\tilde{d}}{2} + \frac{n}{2k}} \int_{\mathbb{S}^{n-1}} |V_{2k}(\eta)|^{-\frac{\tilde{d}}{2k}} d\eta \int_{\mathbb{R}} |t|^\frac{n+\tilde{d}}{2k-1} \varphi(t) dt.$$

Otherwise the first non-zero coefficient are given by:

$$h^{-n+\frac{\tilde{d}}{2} + \frac{n}{2k}} \langle T_{n,k}, \varphi \rangle \int_{\mathbb{S}^{n-1}} |V_{2k}(\eta)|^{-\frac{\tilde{d}}{2k}} d\eta.$$

This result is the contribution of an equilibrium since the distributional coefficients have a non-discrete support, contrary to the Weyl-term supported in $t = 0$ and the contributions of closed orbits supported by the length spectrum. The distributions $T_{n,k}$ and the universal constants $C_{n,k} \neq 0$ depend only on $(n, k)$ and are explicitly determined in [4]. Mainly, we need the order w.r.t. $h$ of these contributions determined by the functions:

$$w_c(h) = h^{-n+\frac{\tilde{d}}{2} + \frac{n}{2k}} \log(h)^j, \quad j = 0 \text{ or } 1.$$ 

First, we give a natural application of Proposition 3.

**Proposition 4** The microlocal counting function satisfies:

$$\Upsilon(h) = \begin{cases} 2d\nu(E_c)(2\pi h)^{1-n} + o(h^{1-n}), & \text{if } n > 1, \\ \Lambda(\chi_{[-d,d]}) w_c(h) + o(w_c(h)), & \text{if } n = 1. \end{cases}$$
Here \( \Lambda \) is the first non-zero distribution of Prop. 3 and \( \chi_{[-d,d]} \) the characteristic function of \([−d,d,]\).

**Proof.** By construction we have:

\[
\gamma_{\text{reg}}(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{itE_c} \hat{\varphi}(t)(1 - \Psi^w(x, hD_x)) \exp(-\frac{i}{h}tP_h)\Theta(P_h) dt.
\]

By the standard calculus on F.I.O. and an easy application of the stationary phase method, as \( h \) tends to 0 we obtain:

\[
\gamma_{\text{reg}}(E_c, h, \varphi) \sim \frac{\hat{\varphi}(0)}{(2\pi h)^{n-1}} \text{Lvol}(\Sigma_{E_c} \cap \text{supp} (1 - \Psi)) + O(h^{2-n}).
\]

Hence \( \gamma_{\text{reg}} \) always contributes at the order \( h^{1-n} \).

**Case of** \( n > 1 \). We have \( w_c(h) = o(h^{1-n}) \) and :

\[
\text{Lvol}(\Sigma_{E_c} \cap \text{supp} (1 - \Psi)) \leq V(E_c) < \infty, \forall \Psi.
\]

It follows easily by shrinking the support of the cut-off \( \Psi \) that:

\[
\gamma(E_c, \varphi, h) = (2\pi h)^{1-n} \hat{\varphi}(0) V(E_c) + o(h^{1-n}).
\]  
(8)

Since the distributional factor is :

\[
\hat{\varphi}(0) = \int \varphi(t) dt = \langle 1, \varphi \rangle,
\]

replacing \( \varphi \) by \( \chi_{[-d,d]} \), via Theorem 6.3 of [1], provides:

\[
\Upsilon(h) = 2dV(E_c)(2\pi h)^{1-n} + o(h^{1-n}).
\]

**Case of** \( n = 1 \). Here the contribution of the critical point has a bigger order than the regular one. We obtain :

\[
\gamma(E_c, \varphi, h) = w_c(h) \Lambda(\varphi) + o(w_c(h)).
\]  
(9)

To apply the Tauberian argument of [1], we observe that our distribution \( \Lambda \in S'(\mathbb{R}) \) can be represented by an element of \( L^1_{\text{loc}}(\mathbb{R}) \) and hence can be extended as a linear form on \( C_0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). We obtain:

\[
\Upsilon(h) \sim \Lambda(\chi_{[-d,d]}w_c(h).
\]  
(10)

Which provides the desired result for \( n = 1 \). \( \blacksquare \)

As a matter of illustration, for \( n = k = 1 \), we have:

\[
\log(h)|V_{2k}(x_0)|^{-\frac{1}{2}} \int_{-d}^{d} dt = \frac{2d\log(h)}{|V_n(x_0)|^{\frac{1}{2}}}. \]  
(11)
Which is the result established for \( \Upsilon(h) \) in [1,7]. Observe that for \( n = 1, k = 1 \) is the only case where a logarithm occurs and \( \Upsilon(h) \) is slowly increasing w.r.t. \( k \) since for all \( k > 1 \):

\[
\Upsilon(h) \sim Ch^{\frac{1}{2}} \cdot |V^{(2k)}(x_0)|^{-\frac{1}{2k}}.
\] (12)

**Eigenfunctions estimates.**

We recall how to derive eigenfunctions estimates from the trace formula. First, to insert an observable \( A \) changes almost nothing. If \( \Pi \) is the spectral projector on \([E_c - \varepsilon, E + \varepsilon]\), computing the trace in the basis \( \psi^h_j \) and by cyclicity:

\[
\text{Tr} \left( \Pi A \varphi \left( \frac{P_h - E_c}{h} \right) \right) = \sum_{|\lambda_j(h) - E_c| \leq \varepsilon} \left\langle A \psi^h_j, \psi^h_j \right\rangle \varphi \left( \frac{\lambda_j(h) - E_c}{h} \right).
\]

Since \( A \) is a bounded operator, if \( \varphi \in \mathcal{S}(\mathbb{R}) \) we can again smooth the problem via an energy cut-off \( \Theta(P_h) \), with an error of order \( \mathcal{O}(h^\infty) \). Hence we can insert \( A = \text{Op}_h^w(a) \) in Eq.(7) and the results of Prop. 3 are the same after multiplication by \( a(z_0) \). Similarly, the regular contribution changes via:

\[
\frac{\hat{\varphi}(0)}{(2\pi h)^{1-n}} \int_{\Sigma E_c} a(z)(1 - \Psi(z))d\text{Lvol}(z).
\]

By evaluation of the trace, we have:

\[
\sum_{|\lambda_j(h) - E_c| \leq \varepsilon} \left\langle A \psi^h_j, \psi^h_j \right\rangle \varphi \left( \frac{\lambda_j(h) - E_c}{h} \right) \sim c_0(\varphi)w(h)m(a) + o(w(h)),
\]

where \( w(h) \) changes only if \( n = 1 \). By Theorem 6.3 of [1] we obtain:

\[
\sum_{\lambda_j(h) \in I(h)} \left\langle A \psi^h_j, \psi^h_j \right\rangle = m(a)w(h) + o(w(h)).
\]

In particular this implies that:

\[
\lim_{h \to 0^+} \frac{1}{\Upsilon(h)} \sum_{\lambda_j(h) \in I(h)} \left\langle A \psi^h_j, \psi^h_j \right\rangle = \lim_{h \to 0^+} \frac{\Upsilon_a(h)}{\Upsilon(h)} = \frac{m(a)}{m(1)}.
\] (13)

Substituting the correct expressions for these measures we obtain:

- for \( n > 1 \) : \( m \) is a constant multiple of the Liouville measure.
- for \( n = 1 \) : \( m \) is a multiple of the delta-Dirac distribution in \( z_0 \).

**Extraction of a subsequence.**

We chose \( a \geq 0 \) and modify the quantization. Different choices are possible: Friedrichs quantization \( \text{Op}^F \) as in [1] or the anti-Wick quantization \( \text{Op}^{AW} \) as in [10]. These quantization are positive, i.e.

\[
a \geq 0 \Rightarrow \left\langle f, \text{Op}^{AW}(a)f \right\rangle \geq 0, \forall f \in C_0^\infty(\mathbb{R}^n).
\]
Since \( \text{Op}_h^w(a) - \text{Op}_h^{AW}(a) \) is of order -1, we obtain:

\[
\langle \psi^h_j, (\text{Op}_h^w(a) - \text{Op}_h^{AW}(a))\psi^h_j \rangle = \mathcal{O}(h),
\]

and we can work with this positive operator. For \( n > 1 \), under the condition that \( \Phi_t \) is ergodic on \( \Sigma_{E_c} \), the extraction of a convergent subsequence of density one is the same as in [1,6,10] to which we refer for a detailed proof. For \( n = 1 \), if \( \Sigma_{E_c} \) is connected, there is only one probability measure invariant by \( \Phi_t \) and the full sequence converges to \( \delta_{z_0} \). Once the result is established for a positive symbol it can be extended by linearity to any \( a \in S^0(\mathbb{R}^{2n}) \).

### 3 Pseudo-differential operators.

The case of pseudo-differential operators provides more explicit examples. Let \( P_h = \text{Op}_h^w(p(x, \xi)) \), obtained by Weyl quantization, where the symbol \( p \) is real valued and smooth on \( T^*\mathbb{R}^n \). In general position, one can also consider \( h \)-dependent symbols \( \sum h^j p_j(x, \xi) \), see [10]. But, to simplify, we consider only the homogeneous case. As above we impose:

\begin{enumerate}
  \item[(A1)] There exists \( \varepsilon_0 > 0 \) such that \( p^{-1}([E_c - \varepsilon_0, E_c + \varepsilon_0]) \) is compact.
\end{enumerate}

As in section 2, \( \sigma(P_h) \cap [E_c - \varepsilon, E_c + \varepsilon] \) is discrete. A fortiori, \( (A1) \) insures that \( \Sigma_{E_c} \) is compact. Next, we chose an homogeneous singularity:

\begin{enumerate}
  \item[(A2)] On \( \Sigma_{E_c} \), \( p \) has a unique critical point \( z_0 = (x_0, \xi_0) \) and near \( z_0 \):

\[
p(z) = E_c + p_k(z) + \mathcal{O}(||(z - z_0)||^{k+1}), \quad k > 2,
\]

where \( p_k \) is homogeneous of degree \( k \) w.r.t. \( z - z_0 \).
\end{enumerate}

Strictly speaking, one could consider \( k = 2 \). But this case is precisely treated in [1]. The case of a critical point which is not an extremum is technical because the singularity is transferred on the blow up of \( z_0 \). To obtain a problem that can be explicitly solved, we consider the following hypothesis inspired by Hörmander’s real principal condition:

\begin{enumerate}
  \item[(A3)] We have \( \nabla p_k \neq 0 \) on the set \( C(p_k) = \{ \theta \in S^{2n-1} / p_k(\theta) = 0 \} \).
\end{enumerate}

For example \( p_3(x, \xi) = x^3 - \xi^3 \) is admissible and \( p(x, \xi) = x^3 - \xi^3 + x^4 + \xi^4 \) satisfies all our hypothesis for \( E_c = 0 \).

**Remark 5** With \( (A3) \), contrary to the case of a local extremum, \( z_0 \) is not an isolated point of \( \Sigma_{E_c} \) which imposes to study the classical dynamic in a neighborhood of \( z_0 \). The study of singularities like in \( (A3) \) is detailed in [11] chapter 4 to which we refer concerning the integrability of dLvol.
As in section 2, it is sufficient to study the local problem \( \gamma_{z_0} \) defined in Eq.(7). The contributions to the trace formula are:

**Proposition 6** Under \((A_1)\) to \((A_3)\), we have an asymptotic expansion:

\[
\gamma_{z_0}(E_c, \varphi, h) \sim h^{2n/\kappa} \sum_{m=0,1}^{\infty} \sum_{j=0}^{\infty} h^j \log(h)^m \Lambda_{j,m}(\varphi),
\]

where the logarithms only occur when \((2n+j)/k \in \mathbb{N}^*\) and \(\Lambda_{j,m} \in S'(\mathbb{R})\).

As concerns the leading term we obtain:

1. If \(k > 2n\) (non-integrable singularity on \(\Sigma_{E_c}\)) we have:

\[
\gamma_{z_0}(E_c, \varphi, h) \sim h^{2n/\kappa} \Lambda_{0,0}(\varphi) + \mathcal{O}(h^{2n+1/\kappa} \log(h)), \text{ as } h \to 0,
\]

where \(\Lambda_{0,0}\) is a universal distribution.

2. If the ratio \(2n/k \in \mathbb{N}\) we obtain logarithmic contributions:

\[
\gamma_{z_0}(E_c, \varphi, h) \sim h^{2n/\kappa} \log(h) \Lambda_{0,1}(\varphi) + \mathcal{O}(h^{2n/\kappa}), \text{ as } h \to 0,
\]

3. For \(2n > k\), \(2n/k \notin \mathbb{N}\) the result is as in (1) with a different distribution.

These results describe very precisely the singularity at \(z_0\). But this is not our purpose here and we refer to [5] for a detailed formulation of these contributions. For \(n = 1, k = 2\), the case (2) agrees with section 2 and allows to recover some results established in [1,7].

**Application to microlocal measures.**

The proof is exactly the same as in section 2. The main difference is that the singularity on \(\Sigma_{E_c}\) can be of arbitrary order. In our setting, according to Prop.6 the top order coefficient changes if and only if we have:

\[
\frac{2n}{k} - n < 1 - n \iff \frac{2n}{k} < 1.
\]

If \(k < 2n\) the singularity is integrable and contributes at a lower order compared to \(h^{1-n} \mathcal{V}(E_c)\). But if \(k \geq 2n\), which corresponds to a non-integrable singularity for \(d\mu^{E_c}\), the main term changes. To summarize, we obtain:

\[
\lim_{h \to 0} \frac{\Upsilon^a(h)}{\Upsilon(h)} = \begin{cases} 
\int a d\mu^{E_c}, & \text{for } k < 2n, \\
\frac{a(z_0)}{h}, & \text{for } k \geq 2n.
\end{cases}
\]

Contrary to section 2, observe that for \(k \geq 2n\) and if \(n > 1\) we do not obtain the convergence of the full sequence \(\nu_j(h), j(h) \in K(h)\), to the dirac-mass at the equilibrium. The obstruction is that an invariant probability measure can be supported by the closed orbits of \(\Sigma_{E_c}\).

**Comments.**
From these 2 families of example the conclusion is that the limiting measure changes only if \( \Sigma_{E_c} \) carries a measure such that \( 1 \not\in L^1_{loc}(\Sigma_{E_c}, dL\text{vol}) \). Interpreted as a quantum measurement, one obtain a very precise localization: if \( a = 0 \) around \( z_0 \) the limit is the Liouville-measure but if \( a(z_0) \neq 0 \) the limit strongly differs.

An interesting problem would be to study the repartition in presence of 2 equilibria \( z_1, z_2 \) on \( \Sigma_{E_c} \) of the same nature and with a non-integrable singularity. In this case any convex combination:

\[
\nu = a\delta(z_1) + (1 - a)\delta(z_2), \quad a \in [0, 1],
\]

provides an invariant probability measure. A natural question is to determine if the limiting measures are equally distributed between \( z_1 \) and \( z_2 \).

References

[1] R.Brummelhuis, T.Paul and A.Uribe, Spectral estimate near a critical level, Duke Mathematical Journal 78 (1995) no. 3, 477-530.

[2] B.Camus, A semi-classical trace formula at a totally degenerate critical level. Contributions of extremums, Commun. in Mathematical Physics 247 (2004) no.2, 513-526.

[3] B.Camus, Semi-classical spectral estimates for Schrödinger operators at a critical level. Case of a degenerate minimum of the potential. Submitted.

[4] B.Camus, Semi-classical spectral estimates for Schrödinger operators at a critical level. Case of a degenerate maximum of the potential, Journal of differential equations 226 (2006) no.1, 295-322.

[5] B.Camus, Spectral estimates for degenerate critical levels. Submitted.

[6] Y.Colin de Verdière, Ergodicité et fonctions propres du laplacien, Commun. in Mathematical Physics 102 (1985), 497-502.

[7] Y.Colin de Verdière and B.Parisse, Equilibre instable en régime semi-classique I. Concentration microlocale. Comm. Partial Differential Equations 19 (1994), no. 9-10, 1535-1563.

[8] P.Duclos and H.Hogreve, On the semiclassical localization of the quantum probability. J.Math. Phys. 34 (1993) no. 5, 1681-1691.

[9] J.J.Duistermaat and L.Hörmander, Fourier Integral Operators, Acta mathematica 128 (1972) no. 3-4, 183-269.

[10] B.Helffer, A.Martinez and D.Robert, Ergodicité et limite semi-classique. Commun. Math. Phys. 109 (1987) no. 2, 313-326.
[11] I.M. Guelfand and G.E. Chilov, Les distributions. Collection Universitaire de
Mathématiques, VIII Dunod, Paris (1962).

[12] A. Voros, Développements semi-classiques. Thèse, Université d’Orsay.

[13] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic
surfaces, Duke Mathematical Journal 55 (1987), 919-941.