On the choice and goals of the project

The choice of topic came from a combination of interests and objectives. Because of my background in physics, I was set on having a project in mathematical physics, and gauge theory in particular interested me because of its relation to particle physics, an area in which I had worked before from the point of view of phenomenology. Among other projects considered by my advisor, Cristian Ortiz, and me, this one had as its strong points the heavy use of techniques from analysis and the timely occasion, with the original author of the main results, Karen Uhlenbeck, winning the Abel prize in 2019.

The subject being very technical and full of geometry and analysis suited me as I wished to learn the kind of techniques which could translate into future study and work in a myriad of possible areas. This has proven even more fitting, since throughout the development of the project I have come to appreciate the motivations for studying gauge theory in its many applications to mathematics itself, which include applications to low-dimensional topology via the study of Donaldson invariants and Floer theory.

The central goal was then to understand the setting and proof for Uhlenbeck’s compactness theorems from her paper Connections with $L^p$ Bounds on Curvature [Uhl82]. The main references followed were the original paper and the books [Weh04], [FU91] and [DK97].

Short introduction to gauge theory and Uhlenbeck compactness

Mathematically speaking, gauge theory can be seen as the study of principal $G$-bundles and connections. As is usual, we begin with a principal $G$-bundle $P \to M$, where $G$ is a compact Lie group and the base manifold $M$ is a compact, oriented Riemannian manifold. A connection on $P$ is a 1-form $A \in \Omega^1(P, g)$ which is $G$-equivariant and takes fixed values on vertical tangent vectors, and we write the space of connections as $\mathcal{A}(P)$. We also define the group $\mathcal{G}(P)$ of gauge transformations of $P$, which are the $G$-bundle automorphisms of the principal bundle, i.e., automorphisms of $P$ which cover the identity and are $G$-equivariant.

The gauge transformations act on the connections via pullback, and a typical object of interest is the moduli space of a class of connections given by the solutions to some PDE which is gauge invariant. A simple example is the class of flat connections; in the seminal paper [AB83], Atiyah and Bott studied Yang-Mills connections over Riemann surfaces; when the base manifold is four-dimensional, there is the notion of anti-self-dual connections (or instantons), which are automatically Yang-Mills, and these lead to the Donaldson invariants of 4-manifold topology; yet another important class of connections is given by solutions of
the Seiberg-Witten equations, and this leads to monopole Floer homology. This project focuses on Yang-Mills connections over a compact manifold.

Since $\mathcal{A}(P)$ and $\mathcal{G}(P)$ are infinite dimensional objects, we find ourselves within the realm of functional analysis. We would like to work with Banach spaces, and so the appropriate notions of Sobolev spaces of sections of vector and fiber bundles are introduced, and we seek to fit $\mathcal{A}(P)$ and $\mathcal{G}(P)$ into this framework.

If we define the associated vector bundle

$$\text{ad}(P) := \frac{P \times \mathfrak{g}}{G},$$

we may choose some reference connection $\check{A}$ and look at the space $\mathcal{A}(P)$ of connections on $P$ as

$$\mathcal{A}(P) \simeq \check{A} + \Omega^1(M, \text{ad}(P)).$$

We also define the auxiliary bundle

$$\text{Ad}(P) := \frac{P \times G}{G}$$

so that

$$\mathcal{G}(P) \simeq \Gamma(\text{Ad}(P)).$$

Then both the connections and gauge transformations may be seen as sections of some fiber bundle and can be extended to appropriate Sobolev spaces. We let $\mathcal{A}^{1,p}(P)$ be the $W^{1,p}$ Sobolev space of connections on $P$.

The first main result is as follows.$^1$

**Theorem (Uhlenbeck weak compactness).** A sequence of connections in $\mathcal{A}^{1,p}(P)$ with uniform $L^p$ bound on curvature is gauge equivalent to a sequence which has a $W^{1,p}$-weakly convergent subsequence.

If we impose further conditions on the connections, namely that they are Yang-Mills, then the result can be made stronger. A Yang-Mills connection satisfies the Yang-Mills equation,$^2$

$$d_A^* F_A = 0,$$

which is the Euler-Lagrange equation for the Yang-Mills functional

$$\mathcal{YM}(A) = \int_M |F_A|^2 \, dvol_g.$$ 

These expressions are invariant under gauge transformations.

**Theorem (Uhlenbeck strong compactness).** A sequence of Yang-Mills connections in $\mathcal{A}^{1,p}(P)$ with uniform $L^p$ bound on curvature is gauge equivalent to a sequence which has a subsequence converging uniformly to a smooth connection.

$^1$For the results stated here, there should be conditions on the Sobolev exponents considered, but these conditions are too technical in nature and not relevant in the present context.

$^2$For a manifold with boundary there is also a boundary condition.
The main technical difficulties in this work lie in the proof of an important lemma, called the *gauge fixing lemma* or *Uhlenbeck gauge theorem*. It is a local result which roughly states that for a connection with small curvature there is always a "good" gauge, which is the combination of the Coulomb (or Hodge) gauge condition and some bounds on the $L^p$ and $L^q$ norms of the connection for some $p$ and $q$. The proof follows the so called method of continuity or continuous induction, and makes use of Banach space calculus (especially the implicit function theorem), Sobolev embeddings, general results from functional analysis, and existence, uniqueness and regularity results for the Neumann problem.

References

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