A RIGIDITY THEOREM FOR CODIMENSION ONE SHRINKING
GRADIENT RICCI SOLITONS IN $\mathbb{R}^{n+1}$

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Abstract. We prove a splitting theorem for complete gradient Ricci soliton with nonnegative curvature and establish a rigidity theorem for codimension one complete shrinking gradient Ricci soliton in $\mathbb{R}^{n+1}$ with nonnegative Ricci curvature.

1. Introduction

A complete Riemannian metric $g$ on a smooth manifold $M^n$ is called a gradient Ricci soliton (GRS) (Hamilton [16], Perelman [23]) if there exists a smooth function $f$ on $M$ such that

(1.1) $\text{Ric} + \text{Hess} f = \frac{\lambda}{2} g,$

where $\lambda \in \mathbb{R}$. Below we assume that $\lambda = 1$, 0, or $-1$; these cases correspond to the GRS of shrinking, steady, or expanding type, respectively.

The classification of GRS, especially the noncompact shrinking GRS, has been a subject of interest to many people. By the work of Perelman [24], Ni and Wallach [22] and Cao, Chen, and Zhu [6], any 3-dimensional complete noncompact nonflat shrinking GRS must be the round cylinder $S^2 \times \mathbb{R}$ or its $\mathbb{Z}_2$ quotient. Naber [19] proved that four dimensional complete noncompact shrinking GRS with bounded nonnegative curvature operator are finite quotients of generalized cylinders $S^2 \times \mathbb{R}^2$ or $S^3 \times \mathbb{R}$. Note that there are several rigidity results for higher dimensional complete noncompact shrinking GRS under various geometric assumptions [22, 28, 25, 8, 18, 5].

On the other hand, Feldman, Ilmanen and Knopf [11] constructed $U(n)$–invariant shrinking Kähler GRS on the holomorphic line bundles $O(-k)$, $1 \leq k \leq n$, over $P^{n-1}$, $n \geq 2$. Their examples are cone-like at infinity, and have Euclidean volume growth, positive scalar curvature and quadratic curvature decay. However the Ricci curvature of these examples changes signs, more precisely the Ricci curvature is negative along the vertical (fiber) direction and positive along horizontal direction. We do not know whether there is any nontrivial (the universal cover does not split) example of complete noncompact nonflat shrinking GRS with nonnegative Ricci curvature.

The constant rank theorem is a powerful tool in the study of convex properties of solutions of nonlinear differential equations [4, 26, 2]. In this paper we first establish a constant rank theorem for Ricci tensor and for curvature operator of GRS (shrinking, steady, or expanding) and the corresponding splitting property of the GRS in section 3.

Theorem 1.1. Let $(M^n, g, f)$ be a GRS satisfying (1.1).
(I) If $g$ has nonnegative sectional curvature, then the rank of Ricci curvature is constant. Thus, either Ricci curvature is strictly positive or the universal covering $(\tilde{M}, \tilde{g}) = (N, h) \times \mathbb{R}^{n-k}$ splits isometrically and $(N, h)$ has strictly positive Ricci curvature;

(II) If $g$ has nonnegative curvature operator, then the rank of curvature operator is constant. Thus, either the curvature operator is strictly positive or the universal covering $(\tilde{M}, \tilde{g}) = (N, h) \times \mathbb{R}^{n-k}$ splits isometrically and $(N, h)$ has strictly positive curvature operator.

Note that since Ricci solitons satisfy Ricci flow, the splitting theorem can be obtained from the maximum principle for tensors in the parabolic setting. The maximum principle of this type was first proved by Hamilton for compact manifolds [15], we also refer [20] for the corresponding result for complete noncompact manifolds under certain growth condition on tensors.

With the help of Theorem [17] to classify shrinking GRS with nonnegative curvature operator, one only needs to consider GRS with positive curvature operator. Since compact GRS with positive curvature operator must be of constant curvature, therefore, to prove the rigidity of the complete shrinking GRS with nonnegative curvature operator one only needs to rule out the noncompact shrinking GRS with positive curvature operator. Note that in the Kähler case, Ni [21] proved that a complete shrinking GRS with positive bisectional curvature must be compact; therefore the GRS is isometric to complex projective space $\mathbb{P}^n$ by the Mori-Siu-Yau theorem.

In the second part of this paper, we consider codimension one shrinking GRS $(M^n, g)$ isometrically embedded in $\mathbb{R}^{n+1}$. If it has nonnegative Ricci curvature, then it has nonnegative curvature operator. By the classical convexity theorem of Sacksteder-van Heijenoort, $M$ is a convex hypersurface. The main result of this paper is the following.

**Theorem 1.2.** A complete codimension one shrinking GRS isometrically embedded in $\mathbb{R}^{n+1}$ with positive Ricci curvature must be compact. As a consequence, if $(M^n, g, f)$ is a complete shrinking GRS isometrically embedded in $\mathbb{R}^{n+1}$ with nonnegative Ricci curvature, then $(M, g)$ is a generalized cylinder $S^k \times \mathbb{R}^{n-k}$, $2 \leq k \leq n$.

The main ingredients of our proof are the estimate of the mean curvature and the eigenvalue estimate of a generalized Cheng-Yau operator associated to shrinking GRS. These results will be proved in section 4 and section 5 respectively.

**Acknowledgement:** P.L. is partially supported by a Simons grant. The work was done when Y.X. was supported by CRC Postdoctoral Fellowship at McGill University.

2. Preliminaries of Gradient Ricci Soliton

We collect some well known identities for gradient GRS below, they can be found in [16]. For a GRS satisfying (1.1), let $\{e_i\}$ is an orthonormal basis of $TM$, the Ricci curvature satisfies the following formula:

\[
\nabla R = 2 \text{Ric}(\nabla f),
\]

\[
\Delta_f R_{ij} = 2\lambda R_{ij} - 2\overset{\circ}{R}(\text{Ric})_{ij},
\]

where

\[
\Delta_f = \nabla^2_{ii} - \nabla_if
\]

and

\[
\overset{\circ}{R}(\text{Ric})_{ij} = \sum_k R(e_i, \text{Ric}(e_k), e_j, e_k).
\]
Moreover, after identifying $\Lambda^2 T_x M$ with $\mathfrak{so}(T_x M)$, the curvature operator is symmetric endomorphism $\mathcal{R} \in S^2(\mathfrak{so}(T_x M))$,

$$\mathcal{R}_{\alpha\beta} = R_{ijkl} \phi_i^j \phi_k^l, \quad \phi_\alpha = \phi_i^j e_i \wedge e_j \in \Lambda^2 T_x M.$$  

For a GRS satisfying (1.1), the curvature operator satisfies the following formula:

$$\Delta f \mathcal{R}_{\alpha\beta} = 2\lambda \mathcal{R}_{\alpha\beta} - \mathcal{R}_{\alpha\beta}^2 - \mathcal{R}_{\alpha\beta}^\sharp,$$

where

$$\langle \mathcal{R}^\sharp(\phi_\alpha), \phi_\alpha \rangle = \langle \text{ad} \circ (\mathcal{R} \wedge \mathcal{R}) \circ \text{ad}^\ast(\phi_\alpha), \phi_\alpha \rangle = \sum_{\beta, \gamma} \langle [\mathcal{R}(\phi_\beta), \mathcal{R}(\phi_\gamma)], \phi_\alpha \rangle \langle [\phi_\beta, \phi_\gamma], \phi_\alpha \rangle,$$

here $\text{ad} : \Lambda^2(\mathfrak{so}(T_x M)) \to \mathfrak{so}(T_x M), \phi \wedge \psi \mapsto \text{ad}(\phi \wedge \psi) = [\phi, \psi]$ is the adjoint representation.

The following identities involving the curvature and potential function are satisfied for shrinking GRS [16],

$$R + \Delta f = \frac{n}{2},$$

$$R + |\nabla f|^2 - f = C_0 (= 0)$$

for some constant $C_0$ (By adding a constant to $f$, we assume $C_0 = 0$ below).

The behavior of the potential function plays an important role in understanding the structure of shrinking GRS [24, 22, 6]. The following estimates are due to [23, 7, 17].

**Proposition 2.1.** Let $(M^n, g, f)$ be complete non-compact shrinking GRS satisfying (1.1). Let $x_0 \in M$ be the point such that $f(x_0) = \min_{x \in M} f(x)$. Then the potential function $f$ satisfies the estimates

$$\frac{1}{4}(r(x) - 5n)^2_+ \leq f(x) \leq \frac{1}{4}(r(x) + \sqrt{2n})^2,$$

$$|\nabla f| \leq \frac{1}{2}(r(x) + 2\sqrt{f(x_0)}),$$

where $r(x) = d(x, x_0)$ is the distance function and $a_+ := \max\{a, 0\}$. Consequently,

$$\int_M |u|e^{-f}d\mu < +\infty$$

for any continuous function $u$ on $M$ satisfying $|u(x)| \leq Ae^{\alpha r^2(x)}$ where $0 \leq \alpha < \frac{1}{4}$ and $A > 0$. In particular, the weighted volume of $M \int_M e^{-f}d\mu$ is finite.

Chen [9] proved that the scalar curvature of complete ancient solution of Ricci flow is always nonnegative. For complete non-flat shrinking GRS $(M^n, g, f)$ the asymptotic estimates for the potential function $f$ also controls the curvature growth rates. In particular, (2.6) and (2.7) imply that the scalar curvature grows at most quadratically,

$$0 \leq R(x) \leq \frac{1}{4}(r(x) + \sqrt{2n})^2.$$

When $(M, g, f)$ is assumed to have nonnegative Ricci curvature, by (2.1) we have

$$\langle \nabla R, \nabla f \rangle = 2 \text{Ric}(\nabla f, \nabla f) \geq 0,$$
thus scalar curvature $R$ is increasing along the gradient flow of potential $f$. It is established by Ni [21, Proposition 1.1] that there exists a $\delta_0 = \delta(M) \in (0, 1)$ such that
\begin{equation}
R \geq \delta_0 > 0.
\end{equation}
Combining $|\text{Ric}|^2 \leq R^2$ with (2.1), we conclude that the gradient of scalar curvature grows at most polynomial fast,
\begin{equation}
|\nabla R|^2 = |2\text{Ric}(\nabla f)|^2 \leq 4R^2|\nabla f|^2.
\end{equation}
By (2.5), (2.1), (2.6), together with the Bochner identity,
\begin{equation}
\Delta R = \Delta (f - |\nabla f|^2) \\
= \Delta f - 2(\nabla^2 f + \langle \nabla \Delta f, \nabla f \rangle + \text{Ric}(\nabla f, \nabla f)) \\
= \frac{n}{2} - R - 2|\text{Ric} - \frac{1}{2}g|^2 - 2(\nabla (\frac{n}{2} - R), \nabla f) - 2\text{Ric}(\nabla f, \nabla f) \\
= \frac{n}{2} - R - 2|\text{Ric} - \frac{1}{2}g|^2 + 2\text{Ric}(\nabla f, \nabla f).
\end{equation}
Hence,
\begin{equation}
|\Delta R| \leq \frac{n}{2} + |R| + 2|\text{Ric} - \frac{1}{2}g|^2 + 2|Ric||\nabla f|^2 \\
\leq n + 2R^2 + 2R|\nabla f|^2.
\end{equation}
We will need the following classification result for compact GRS which follows from the works of Hamilton [14, 15] (dimensions three and four) and Böhm and Wilking [3] (dimensions $\geq 5$).

**Theorem 2.2.** A compact GRS with positive curvature operator must be a space form.

3. Splitting Theorem of Gradient Ricci Soliton

In this section, we establish the constant rank Theorem 1.1 for GRS with nonnegative curvature via strong maximum principle. We will show that the distribution of the null space of the Ricci tensor is of constant dimension and is invariant under parallel translation. That would yield a splitting theorem for GRS. Similar conclusion also holds for the curvature operator.

3.1. Ricci curvature and constant ranking Theorem 1.1. Let $A = (a_{ij})_{n \times n}$ be a symmetric matrix. Define
\begin{equation}
\det(I + tA) = \sum_{l=0}^{n} \sigma_l(A)t^l.
\end{equation}
Note that $\sigma_l(A)$ is a smooth function of variables $a_{ij}$. When $A = \text{diag}[\lambda_1, \cdots, \lambda_n]$ is a diagonal matrix, then $\sigma_l(A)$ is the $l$-th elementary polynomial of $\lambda_1, \cdots, \lambda_n$.

If $A$ is any $n \times n$ symmetric matrix, we denote
\begin{equation}
\sigma^{ij}_l(A) := \frac{\partial \sigma_l(A)}{\partial a_{ij}}, \quad \sigma^{ij,kl}_l(A) := \frac{\partial^2 \sigma_l(A)}{\partial a_{ij} \partial a_{kl}}.
\end{equation}
In particular, we have
\begin{align*}
\sigma^{ij}_1(A) &= \delta_{ij}, \\
\sigma^{ij}_2(A) &= (\sum_{k=1}^{n} a_{kk})\delta_{ij} - a_{ij}.
\end{align*}
We also denote by \((A|i)\) the \((n-1) \times (n-1)\) matrix obtained from \(A\) by deleting the \(i\)-th row and \(i\)-th column, and by \((A|ij)\) the \((n-2) \times \(n-2)\) matrix obtained from \(A\) by deleting the \(i, j\)-th row and \(i, j\)-th column.

The following two propositions (See Proposition 3.1 and Proposition 3.2 below) are well known (e.g., see [2]).

**Proposition 3.1.** If \(A\) is a diagonal matrix. For any \(l, i, j\) we have
\[
\sigma_{l}^{ij}(A) = \begin{cases} 
\sigma_{l-1}(A|i), & \text{if } i = j; \\
0, & \text{otherwise.}
\end{cases}
\]
and
\[
\sigma_{l}^{ijk}(A) = \begin{cases} 
\sigma_{l-2}(A|ik), & \text{if } i = j, k = l, i \neq k; \\
-\sigma_{l-2}(A|ij), & \text{if } i = l, j = k, i \neq j; \\
0, & \text{otherwise.}
\end{cases}
\]

Let \((M^n, g, f)\) be a GRS with nonnegative Ricci curvature as in Theorem 1.1. In this case, we take \(A = \text{Ric}\). We may assume that \(r := \min_{x \in M} \text{rank Ric}(x) < n\); otherwise we have \(\text{Ric} > 0\). Let \(x_0 \in M\) be a point such that \(\text{rank Ric}(x_0) = r\). Pick a small open neighborhood \(\mathcal{O}\) of \(x_0\). We define function \(\phi\) on \(\mathcal{O}\) by
\[
\phi(x) = \sigma_{r+1}(\text{Ric}(x)).
\]

To prove Theorem 1.1, we first show that there is a positive constant \(C\) independent of \(\phi\) such that on \(\mathcal{O}\)
\[
\Delta \phi(x) \leq C(\phi(x) + |\nabla \phi(x)|).
\]

In the following we shall use the notations used in [2]. For any \(x \in \mathcal{O}\), let \(\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)\) be the eigenvalues of \(\text{Ric}(x)\). There is a positive constant \(C_0 > 0\) depending \(\mathcal{O}\), such that \(\lambda_1(x) \leq \lambda_2(x) \leq \cdots \lambda_{n-r}(x) \leq \frac{C_0}{1000n}\) and \(C_0 \leq \lambda_{n-r+1}(x) \leq \lambda_{n-r+2}(x) \leq \cdots \leq \lambda_n(x)\) for all \(x \in \mathcal{O}\). Let \(G = \{n-r+1, n-r+2, \cdots, n\}\) and \(B = \{1, \cdots, n-r\}\) be the “good” and “bad” sets of indices for eigenvalues of \(\text{Ric}\), respectively. Define diagonal matrix \(\Lambda_G = \text{diag} [0, \cdots, 0, \lambda_{n-r+1}, \lambda_{n-r+2}, \cdots, \lambda_n]\) and \(\Lambda_B = \text{diag} [\lambda_1, \cdots, \lambda_{n-r}, 0, \cdots, 0]\)

Use notation \(h = O(k)\) if \(|h(x)| \leq Ck(x)\) for \(x \in \mathcal{O}\) with some positive constant \(C\) under control. In particular, \(\lambda_i(\phi)\) for all \(i \in B\), and
\[
(\sum_{i \in B} \lambda_i)\sigma_r(\Lambda_G) = O(\phi).
\]

Based on Proposition 3.1 with the notation as above, we have

**Proposition 3.2.** Let \(A = \text{Ric}\) as above. Then we have that on \(\mathcal{O}\)
\[
\frac{\partial \sigma_{r+1}(A)}{\partial a_{ij}} = \begin{cases} 
\sigma_r(\Lambda_G) + O(\phi), & \text{if } i = j \in B; \\
O(\phi), & \text{otherwise.}
\end{cases}
\]
and
\[
\frac{\partial^2 \sigma_{r+1}(A)}{\partial a_{ij} \partial a_{kl}} = \begin{cases} 
\sigma_{r-1}(\Lambda_G|i) + O(\phi) = \frac{1}{\lambda_i} \sigma_r(\Lambda_G) + O(\phi), & \text{if } i = j \in G, k = l \in B; \\
\sigma_{r-1}(\Lambda_G|k) + O(\phi) = \frac{1}{\lambda_k} \sigma_r(\Lambda_G) + O(\phi), & \text{if } i = j \in B, k = l \in G; \\
\sigma_{r-1}(\Lambda_G) + O(\phi), & \text{if } i = j \in B, k = l \in B, i \neq k; \\
-\sigma_{r-1}(\Lambda_G|i) + O(\phi) = -\frac{1}{\lambda_i} \sigma_r(\Lambda_G) + O(\phi), & \text{if } i = l \in G, j = k \in B; \\
-\sigma_{r-1}(\Lambda_G|j) + O(\phi) = -\frac{1}{\lambda_j} \sigma_r(\Lambda_G) + O(\phi), & \text{if } i = l \in B, j = k \in G; \\
-\sigma_r(\Lambda_G) + O(\phi), & \text{if } i = l \in B, j = k \in B, i \neq j; \\
0, & \text{otherwise.}
\end{cases}
\]
From Proposition 3.2, we compute the first derivative
\( (3.4) \)
\[ \phi_\alpha = \sigma_{r+1}^{ij} R_{ij,\alpha} = \sigma_r(\Lambda_G) \sum_{i \in B} R_{ii,\alpha} + O(\phi), \]
and the second derivative
\[ \phi_{\alpha\beta} = \sigma_{r+1}^{ij} R_{ij,\alpha\beta} + \sigma_{r+1}^{kl} R_{ij,\alpha} R_{kl,\beta} \]
\[ = \sigma_r(\Lambda_G) \sum_{i \in B} R_{ii,\alpha\beta} + \sum_{i \in G} \sum_{k \in B} \frac{1}{\lambda_i} \sigma_r(\Lambda_G) R_{i\alpha,\alpha} R_{kk,\beta} + \sum_{i \in B} \sum_{k \in G} \frac{1}{\lambda_k} \sigma_r(\Lambda_G) R_{ii,\alpha} R_{kk,\beta} \]
\[ + \sum_{i,k \in B} \sigma_{r-1}(\Lambda_G) R_{ii,\alpha} R_{kk,\beta} - 2 \sum_{i \in G} \sum_{j \in B} \frac{1}{\lambda_i} \sigma_r(\Lambda_G) R_{ij,\alpha} R_{ji,\beta} \]
\( (3.5) \)
\[ - \sum_{i,j \in B} \sigma_{r-1}(\Lambda_G) R_{ij,\alpha} R_{ji,\beta} + O(\phi). \]

Take trace of (3.5) and using (3.4), we get
\[ \Delta \phi = \sigma_r(\Lambda_G) \sum_{i \in B} \Delta R_{ii} + 2 \sum_{i \in G} \sum_{j \in B} \frac{1}{\lambda_i} \phi_\alpha - 2 \sigma_r(\Lambda_G) \sum_{i \in G} \sum_{j \in B} \frac{1}{\lambda_i} |\nabla R_{ij}|^2 \]
\[ + \frac{\sigma_{r-1}(\Lambda_G)}{\sigma_r^2(\Lambda_G)} \phi_\alpha^2 - \sigma_{r-1}(\Lambda_G) \sum_{i,j \in B} |\nabla R_{ij}|^2 + O(\phi) \]
\( (3.6) \)
\[ = \sigma_r(\Lambda_G) \sum_{i \in B} \Delta R_{ii} - 2 \sigma_r(\Lambda_G) \sum_{i \in G} \sum_{j \in B} \frac{1}{\lambda_i} |\nabla R_{ij}|^2 - \sigma_{r-1}(\Lambda_G) \sum_{i,j \in B} |\nabla R_{ij}|^2 \]
\[ + O(\phi) + O(|\nabla \phi|). \]

By identity (2.2),
\[ \Delta \phi = \sigma_r(\Lambda_G) \sum_{i \in B} (\nabla \nabla \phi_R_{ii} + 2 \lambda R_{ii} - 2 \hat{R}(\text{Ric})_{ii}) \]
\( (3.7) \)
\[ - 2 \sigma_r(\Lambda_G) \sum_{i \in G} \sum_{j \in B} \frac{1}{\lambda_i} |\nabla R_{ij}|^2 - \sigma_{r-1}(\Lambda_G) \sum_{i,j \in B} |\nabla R_{ij}|^2 + O(\phi) + O(|\nabla \phi|). \]

To deal with the first term in the righthand side of (3.5), by (3.3) and (3.4) we have
\[ \sigma_r(\Lambda_G) \sum_{i \in B} (\nabla \nabla \phi_R_{ii} + 2 \lambda R_{ii}) = O(\phi) + O(|\nabla \phi|). \]

By the assumption of nonnegative sectional curvature, \( (3.9) \)
\[ \hat{R}(\text{Ric})_{ii} = \sum_{k} R(e_i, \text{Ric}(e_k), e_i, e_k) \geq \lambda_1 \sum_{k} R(e_i, e_k, e_i, e_k) \geq 0. \]
Combine (3.7), (3.8), and (3.9),
\[ \Delta \phi \leq C(\phi + |\nabla \phi|) - 2 \sigma_r(\Lambda_G) \sum_{i \in G} \sum_{j \in B} \frac{1}{\lambda_i} |\nabla R_{ij}|^2 - \sigma_{r-1}(\Lambda_G) \sum_{i,j \in B} |\nabla R_{ij}|^2. \]
Hence we have proved
\[ \Delta \phi \leq C(\phi + |\nabla \phi|). \]

\footnote{This is the only place that we need nonnegative sectional curvature.}
Since $\phi \geq 0$ on $\mathcal{O}$ and $\phi(x_0) = 0$, it follows from the strong maximum principle that $\phi \equiv 0$ on $\mathcal{O}$. We conclude that $\phi \equiv 0$ in $M$, i.e. rank Ric $\equiv r$.

Next we consider the null space of Ricci curvature null Ric. It follows from (3.10) that

\begin{equation}
2\sigma_r(\Lambda_G) \sum_{i,j \in B} \frac{1}{\lambda_i} |\nabla R_{ij}|^2 + \sigma_{r-1}(\Lambda_G) \sum_{i,j \in B} |\nabla R_{ij}|^2 \equiv 0.
\end{equation}

Hence for any $v \in \text{null Ric}$, $\nabla \text{Ric}(v) = 0$. On the other hand, for any section $v \in \text{null Ric}$ and for any index $k$ we have

$$0 = \nabla_k (R_{ij} v^i) = (\nabla_k R_{ij}) v^j + R_{ij} \nabla_k v^i,$$

thus $R_{ij} \nabla_k v^i = -(\nabla_k R_{ij}) v^j = 0$. This shows that $\nabla_k v \in \text{null Ric}$ and that null Ric is invariant under parallel translation.

Finally we show that the universal covering space of the GRS $(M,g)$ splits. Since the distribution null Ric is invariant under parallel translation, null Ric is involutive. Let $(\text{null Ric})^\perp$ be the distribution that generated by orthogonal complements of null Ric. For any sections $X, Y \in (\text{null Ric})^\perp$, $V \in \text{null Ric}$, then

$$g([X,Y], V) = g(\nabla X Y - \nabla Y X, V) = -g(Y, \nabla X V) + g(X, \nabla Y V) = 0.$$ 

Thus the distribution $(\text{null Ric})^\perp$ is also involutive. The classical deRham splitting theorem (see [11] Theorem 10.43) implies that $(M,g)$ locally splits.

Now consider the the universal covering space $(\tilde{M}, \tilde{g})$. We denote by $L$ the leaf of the integral manifold of null Ric, then $L$ is Ricci flat. By equation (1.1), on every leaf, Hess $f = \frac{1}{2} g$. Consequently, $L$ is isometric to $\mathbb{R}^{n-r}$. Hence $(\tilde{M}, \tilde{g}) = (N, h) \times \mathbb{R}^{n-r}$ splits isometrically along the null space of Ricci curvature, where $(N, h)$ has strictly positive Ricci curvature. We have finished the proof of Theorem L.1(II).

### 3.2. Curvature operator and constant rank Theorem L.1(II).

Similarly, we can establish the constant rank theorem Theorem L.1(II) for curvature operators. We may assume that $r := \min_{x \in M} \text{rank} \mathfrak{R}(x) < \frac{m(n-1)}{2}$. There is a point $x_0 \in M$ such that rank $\mathfrak{R}(x_0) = r$.

Pick an orthonormal basis $\{e_i\}$ around $x_0$. Let $\{\varphi_\alpha = \varphi_\alpha^j e_i \wedge e_j\}$ be the eigenvectors of curvature operator $\mathfrak{R}$, i.e. $\mathfrak{R}(\varphi_\alpha) = \lambda_\alpha \varphi_\alpha$. Define $\phi = \sigma_{r+1}(\mathfrak{R})$.

Below we adopt notations similar to the ones used in section 3.1. Using Proposition 3.2 and equation (2.3) and by a computation similar to the derivation of (3.7) we get

\begin{align}
\Delta \phi & \leq C(\phi + |\nabla \phi|) + \sigma_r(\Lambda_G) \sum_{\alpha \in B} \left( |\nabla \nabla f \mathfrak{R}_{\alpha \alpha} + 2\lambda_\alpha \mathfrak{R}_{\alpha \alpha} - \mathfrak{R}_{\alpha \alpha}^2 - \mathfrak{R}_{\alpha \alpha} \right) \\
& \quad - 2\sigma_r(\Lambda_G) \sum_{\alpha \in G, \beta \in B} \frac{1}{\lambda_\alpha} |\nabla \mathfrak{R}_{\alpha \beta}|^2 - \sigma_{r-1}(\Lambda_G) \sum_{\alpha, \beta \in B} |\nabla \mathfrak{R}_{\alpha \beta}|^2 \\
& \quad \leq C(\phi + |\nabla \phi|) - 2\sigma_r(\Lambda_G) \sum_{\alpha \in G, \beta \in B} \frac{1}{\lambda_\alpha} |\nabla \mathfrak{R}_{\alpha \beta}|^2 - \sigma_{r-1}(\Lambda_G) \sum_{\alpha, \beta \in B} |\nabla \mathfrak{R}_{\alpha \beta}|^2 \\
& \quad \leq C(\phi + |\nabla \phi|),
\end{align}

in some small neighborhood $\mathcal{O}$ of $x_0$. To get the second inequality above we have used the following

$$\langle \mathfrak{R}^2(\varphi_\alpha), \varphi_\alpha \rangle = \sum_{\alpha, \beta} \lambda_\beta \lambda_\gamma \langle [\varphi_\beta, \varphi_\gamma], \varphi_\alpha \rangle^2 \geq 0,$$
which follows from (2.4) and the assumption of the nonnegative curvature operator.

Since $\phi \geq 0$, and $\phi(x_0) = 0$, by applying the strong maximum principle to (3.12) we get $\phi \equiv 0$. We conclude that curvature operator $\mathcal{R}$ has constant rank.

By a similar proof as for the Ricci curvature case the null space of $\mathcal{R}$ is invariant under parallel translation. Moreover, it follows from (2.3) that null $\mathcal{R} \subset \text{null } \mathcal{R}^2$. By (2.3) we have $\langle [\varphi, \varphi]_1, \phi \rangle = 0$ for $\phi \in \text{null } \mathcal{R}$ and for any $\beta \neq \gamma$ with $\lambda_\beta > 0$ and $\lambda_\gamma > 0$. Since $\mathcal{R}$ is a self-adjoint operator,

$$\phi \in \text{image } \mathcal{R} \Leftrightarrow \langle \phi, \varphi_\alpha \rangle = 0, \quad \forall \alpha \text{ with } \lambda_\alpha = 0.$$ 

For any section $\phi, \omega \in \text{image } \mathcal{R}$, we have

$$\langle [\phi, \omega], \psi \rangle = \sum_{\beta, \gamma} \langle \phi, \varphi_\beta \rangle \langle \omega, \varphi_\gamma \rangle \langle [\varphi_\beta, \varphi_\gamma], \psi \rangle = 0, \quad \forall \psi \in \text{null } \mathcal{R},$$

hence $[\phi, \omega] \in \text{image } \mathcal{R}$. This implies that the image of $\mathcal{R}$ is a Lie subalgebra. Ambrose-Singer theorem ensures that the Lie algebra $\mathfrak{hol}(M, g)$ of Holonomy group is reduced to a lower dimension, so by deRham splitting theorem (see [1, Theorem 10.43]) the universal covering space $(\tilde{M}, \tilde{g})$ is a Riemannian product. Since one of the product factor is flat from our construction, $(\tilde{M}, \tilde{g}) = (N, h) \times \mathbb{R}^{n-m}$, $\frac{m(m-1)}{2} = r$, splits isometrically, where $(N, h)$ has strictly positive curvature operator. The proof of Theorem 1.1 is completed.

4. MEAN CURVATURE GROWTH ESTIMATE

In this section, we establish the following a priori interior estimate of the mean curvature for a convex hypersurface in $\mathbb{R}^{n+1}$. As a consequence, we can control the mean curvature growth for embedded codimension one GRS in $\mathbb{R}^{n+1}$, see (5.12) and (5.13).

Let $X: M^n \to \mathbb{R}^{n+1}$ be a hypersurface with induced metric $g$ and (outer) unit normal $\nu$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame filed on $M$, then

$$(4.1) \quad X_{ij} = -h_{ij}\nu, \quad 1 \leq i, j \leq n,$$

where $h = (h_{ij})$ is the second fundamental form. Let $\sigma_k = \sigma_k(h)$ be the $k$-th elementary symmetric function of the eigenvalues of $h$. In particular, $H = \sigma_1(h)$ and $R = 2\sigma_2(h)$ are the mean curvature and the scalar curvature respectively. If the scalar curvature of $M$ is positive, we take the unit normal $\nu$ such that $h$ lies in Garding’s $\Gamma_2$-cone. In particular, the differential operator

$$(4.2) \quad \square_h := \sigma_2^{ij}(h)\nabla^2_{ij}$$

is elliptic, where $\sigma_2^{ij}(h)$ is defined in (3.2).

**Theorem 4.1.** Let $X: (M^n, g) \to \mathbb{R}^{n+1}$ be a convex hypersurface with positive scalar curvature. If there exists a unit constant vector $a$ such that $\langle X, a \rangle$ is a nonnegative proper function, then we have the interior estimate

$$(4.3) \quad H(x) \leq C(n) \sup_{\{y \mid \langle X(y), a \rangle \leq 2\langle X(x), a \rangle\}} (1 + R^2(y) + \frac{1}{R(y)} + \frac{1}{R^2(y)}|\nabla R|^2(y) + \frac{1}{R(y)}|\Delta R|(y)).$$

We note that on a shrinking GRS, one can split out lines and reduce the GRS to be a convex hypersurface such that there exists automatically a vector $a$ such that $\langle X, a \rangle$ is a nonnegative proper function, see (5.9).

First of all, the following identity is well known (see, for example, (2.11) in [10]) and will be used to prove the theorem above.
Lemma 4.2. Let $X : M^n \to \mathbb{R}^{n+1}$ be a hypersurface with the second fundamental form $h$, then

$$\Box \sigma_1 := \sigma_{ij}^2 \sigma_{1,ij} = \Delta \sigma_2 + |\nabla h|^2 - |\nabla \sigma_1|^2 + 2\sigma_2 |h|^2 - (\sigma_1 \sigma_2 - 3\sigma_3) \sigma_1,$$

where $\sigma_{ij}^2 := \frac{\partial \sigma_1}{\partial n_{ij}}$ is defined in (3.2).

To prove Theorem 4.1, let $\phi(x) = r - \langle X(x), a \rangle$ be a cut off function with $r \geq 1$, we will apply second derivative test to the auxiliary function

$$\phi^2(x) \sigma_1(x)$$

in the domain $\Omega_r := \{ x \in M | \langle X(x), a \rangle \leq r \}$ to estimate $\sigma_1(x)$.

We may assume that $\phi^2 \sigma_1$ achieves its maximum at an interior point $\bar{x} \in \Omega_r$. Let $0 \leq \lambda_1(\bar{x}) \leq \lambda_2(\bar{x}) \leq \cdots \leq \lambda_n(\bar{x})$ be the principle curvature of $M$ at $x \in M$. Moreover, in a neighborhood of $\bar{x}$, we choose a local orthonormal frame $\{e_i\}$ such that $h_{ij}(\bar{x}) = \lambda_i(\bar{x}) \delta_{ij}$.

We consider three cases.

Case (I) : $\lambda_n(\bar{x}) \leq \max\{n^2, 100n\}$. Then $\sigma_1(\bar{x}) \leq C(n)$, and thus

$$\phi^2(\bar{x}) \sigma_1(\bar{x}) \leq C(n)r^2.$$

Case (II) : $\lambda_n(\bar{x}) > \max\{n^2, 100n\}$ and $\lambda_{n-1}(\bar{x}) \geq \lambda_n^{-\frac{1}{2}}(\bar{x})$, then the scalar curvature $R(\bar{x}) = \sum_{i \neq j} \lambda_i \lambda_j \geq \lambda_n \lambda_{n-1} \geq \lambda_n^{-\frac{1}{2}}(\bar{x})$ and $\sigma_1(\bar{x}) = \sum_{i=1}^{n} \lambda_i \leq n \lambda_n \leq nR^2(\bar{x})$. Hence

$$\phi^2(\bar{x}) \sigma_1(\bar{x}) \leq nR^2(\bar{x})r^2.$$

Case (III) : $\lambda_n(\bar{x}) > \max\{n^2, 100n\}$ and $\lambda_{n-1}(\bar{x}) < \lambda_n^{-\frac{1}{2}}(\bar{x})$. Then $\lambda_i(\bar{x}), i \neq n$ is much smaller than $\lambda_n(\bar{x})$. In this case, we have

$$\lambda_i < \frac{1}{n^2} \lambda_n, \quad i \neq n, \quad \text{and} \quad \lambda_n < \sigma_1 = \sum_{i=1}^{n} \lambda_i < (1 + \frac{1}{n^2}) \lambda_n.$$

To estimate $\phi^2(\bar{x}) \sigma_1(\bar{x})$ from above, we are left to consider case [Case (III)] Note that the function

$$\zeta := \ln (\phi^2 \sigma_1) = 2 \ln \phi + \ln \sigma_1$$

on $\Omega_r$ also achieve the maximum at $\bar{x}$. Apply the first and second derivative test, we get that at $\bar{x}$

$$0 = \zeta_i(\bar{x}) = 2 \frac{\phi_i}{\phi} + \frac{\sigma_1, i}{\sigma_1}, \quad \forall \ i = 1, \cdots, n.$$

and the matrix

$$0 \geq \zeta_{ij}(\bar{x})$$

$$= 2 \frac{\phi_{ij}}{\phi} - 2 \frac{\phi_i \phi_j}{\phi^2} + \frac{\sigma_1, ij}{\sigma_1} - \frac{\sigma_1, i \sigma_1, j}{\sigma_1^2}$$

$$= 2 \frac{\phi_{ij}}{\phi} + \frac{\sigma_1, ij}{\sigma_1} - \frac{3 \sigma_1, i \sigma_1, j}{2 \sigma_1^2},$$

where we used (4.8) in the last step.
Note that the positive scalar curvature on \( M \) implies that the operator \( \sigma_2^{-1}\nabla^2 \) is elliptic, see (4.2). Take the contraction of (4.9) with \( \sigma_2^{-1} \) and use (4.4), we get that at \( \bar{x} \)

\[
0 \geq \sigma_2^{ij} \zeta_{ij}.
\]

(4.10)

\[
\frac{\sigma_2^{ij} \phi_{ij}}{\phi} + \Delta \sigma_2 + |\nabla h|^2 - |\nabla \sigma_1|^2 + 2 \sigma_2 |h|^2 - (\sigma_1 \sigma_2 - 3 \sigma_3) \sigma_1 = - \frac{3 \sigma_2 \sigma_1 \sigma_{1,j}}{\sigma_1^2}.
\]

Below we will deal with the three terms in the right-hand side of (4.10) separately. All the related calculations are at point \( \bar{x} \).

For the first term, from (4.1), we have

\[
\sigma_2^{ij} \phi_{ij} = -\sigma_2^{ij} \langle X_{ij}, a \rangle = 2 \sigma_2 \langle \nu, a \rangle.
\]

To deal with the second term, note that

\[
|\nabla h|^2 - |\nabla \sigma_1|^2 = \sum_{i,j,k} h_{i,j,k}^2 - \sum_{i,j,k} h_{ii,k} h_{jj,k}
\]

(4.12)

\[
= \sum_{k} \sum_{i,j} h_{i,j,k}^2 - \sum_{k} \sum_{i,j \neq k} h_{ii,k} h_{jj,k}
\]

\[
= 2 \sum_{i \neq j} h_{ii,j}^2 + \sum_{i \neq j} h_{i,j,k}^2 - \sum_{k} \sum_{i \neq j} h_{ii,k} h_{jj,k},
\]

where we have used the Codazzi equation \( h_{ij,k} = h_{ik,j} \) to get the last equality. The term \( 2 \sum_{i \neq j} h_{ii,j}^2 \) in (4.12) will play a crucial role in our estimate. The term \( \sum_{i \neq j} h_{i,j,k}^2 \) can be discarded. However we need to control the negative term \( - \sum_{k} \sum_{i \neq j} h_{ii,k} h_{jj,k} \). In fact, by Gårding’s theory on hyperbolic polynomial (see [12, Lemma 3.2] or [13, Lemma 2.2]), we have

\[
- \sum_{i \neq j} h_{ii,k} h_{jj,k} \geq - \frac{1}{2} \frac{|\sigma_{2,k}|^2}{\sigma_2}, \forall k = 1, \ldots, n.
\]

Therefore

\[
|\nabla h|^2 - |\nabla \sigma_1|^2 \geq 2 \sum_{i \neq j} h_{ii,j}^2 - \frac{1}{2} \frac{|\nabla \sigma_2|^2}{\sigma_2}.
\]

We now handle the third term in (4.10). Since the principle curvatures \( \lambda_i, i \neq n \), are very small compared with \( \lambda_n \), thus \( \sigma_{2}^{n}= \sum_{i=1}^{n-1} \lambda_i \leq \sigma_1 \) is also very small. We may use (4.8) to substitute the partial derivative of mean curvature along the \( i = n \) direction, which in turn is bounded. We compute that for any \( \epsilon > 0 \)

\[
\frac{\sigma_2^{ij} \sigma_{1,i} \sigma_{1,j}}{\sigma_1^2} = \frac{\sigma_2^{n} \sigma_1^{2} \sigma_{1,n}^{2} + \sum_{i=1}^{n-1} \sigma_2^{ij} \sigma_1^{2} \sigma_{1,i}^{2}}{\sigma_1^2}
\]

\[
\leq \frac{4 \sigma_2^{n} \phi_{n}^{2}}{\phi_{n}^{2}} + \sigma_1 \sum_{i=1}^{n-1} \left( \frac{\sigma_{1,i}}{\sigma_{1}} \right)^2
\]

(4.15)

\[
= \frac{4 \sigma_2^{n} \phi_{n}^{2}}{\phi_{n}^{2}} + \sigma_1 \sum_{i=1}^{n-1} \left( \frac{h_{nn,i} + \sum_{j=1}^{n-1} h_{jj,i}^{2}}{\sigma_{1}} \right)^{2}
\]

\[
\leq \frac{4 \sigma_2^{n} \phi_{n}^{2}}{\phi_{n}^{2}} + \left( 1 + \frac{\epsilon}{\sigma_{1}} \right) \frac{\sum_{i=1}^{n-1} h_{nn,i}^{2}}{\sigma_{1}} + \left( 1 + \frac{\epsilon}{\sigma_{1}} \right) \frac{\sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} h_{jj,i}^{2} \right)^{2}}{\sigma_{1}}.
\]
To deal with the last term in (4.15), we consider the scalar curvature $\sigma_2$. For $i \leq n - 1$,

$$
\sigma_{2,i} = \sum_{j=1}^{n} \sigma_{2}^{jj} h_{jj,i} = \sigma_{2}^{nn} h_{nn,i} + \sum_{j=1}^{n-1} (\sigma_{1} - \lambda_j) h_{jj,i},
$$

where we have used Proposition 3.1 to get $\sigma_{2}^{ij} = \sigma_{1} - \lambda_j$ at $\bar{x}$. As a consequence, we get

$$
\left( \sum_{j=1}^{n-1} h_{jj,i} \right)^2 = \left( \frac{\sigma_{2,i}}{\sigma_1} - \frac{\sigma_{2}^{nn}}{\sigma_1} h_{nn,i} + \sum_{j=1}^{n-1} \frac{\lambda_j}{\sigma_1} h_{jj,i} \right)^2
$$

$$
\leq 3 \left( \frac{\sigma_{2,i}}{\sigma_1}^2 + \frac{\sigma_{2}^{nn}}{\sigma_1^2} h_{nn,i}^2 + (n - 1) \sum_{j=1}^{n-1} \frac{\lambda_j^2}{\sigma_1^2} h_{jj,i}^2 \right)
$$

$$
\leq 3 \left( \frac{\sigma_{2,i}}{\sigma_1^2} + \frac{\sigma_{2}^{nn}}{\sigma_1^2} h_{nn,i}^2 + \frac{2n}{\sigma_1^2} \sum_{j=1}^{n-1} h_{jj,i}^2 \right),
$$

where we have used $\lambda_j^2 \leq \lambda_n^{-1} < 2\sigma_1^{-1}$ for $j \leq n - 1$ to get the last inequality (see (4.7)).

Use equation (4.16) again, we get that for $i \leq n - 1$

$$
\sum_{j=1}^{n} h_{ii,i} = \frac{\sigma_{2,i}}{\sigma_2} - \frac{\sigma_{2}^{nn}}{\sigma_2^2} h_{nn,i} - \sum_{j=1,j\neq i}^{n-1} \frac{\sigma_{2}^{jj}}{\sigma_2^2} h_{jj,i}.
$$

It follows from (4.7) that $\sigma_1 \geq \sigma_{2}^{ii} = \sigma_1 - \lambda_i \geq \frac{n-1}{n} \sigma_1$ for $i \leq n - 1$. Then

$$
h_{ii,i}^2 \leq 3 \left( \frac{\sigma_{2,i}}{\sigma_1^2} + \frac{\sigma_{2}^{nn}}{\sigma_1^2} h_{nn,i}^2 + (n - 2) \sum_{j=1,j\neq i}^{n-1} \frac{\sigma_{2}^{jj}}{\sigma_1^2} h_{jj,i}^2 \right)
$$

$$
\leq 3 \left( \frac{n}{n-1} \right)^2 \left( \frac{\sigma_{2,i}}{\sigma_1^2} + \frac{\sigma_{2}^{nn}}{\sigma_1^2} h_{nn,i}^2 + (n - 2) \sum_{j=1,j\neq i}^{n-1} h_{jj,i}^2 \right)
$$

$$
\leq 6 \left( \frac{\sigma_{2,i}}{\sigma_1^2} + \frac{\sigma_{2}^{nn}}{\sigma_1^2} h_{nn,i}^2 + n \sum_{j=1,j\neq i}^{n-1} h_{jj,i}^2 \right).
$$

Combine (4.17) and (4.19) and use $\sigma_1 > n^2$, we have

$$
\left( \sum_{j=1}^{n-1} h_{jj,i} \right)^2 \leq 3 \left[ \frac{(\sigma_{2,i})^2}{\sigma_1^2} + \frac{(\sigma_{2}^{nn})^2}{\sigma_1^2} h_{nn,i}^2 + \frac{2n}{\sigma_1^2} \left( \sum_{j=1,j\neq i}^{n-1} h_{jj,i}^2 \right) \right]
$$

$$
+ 6 \left( \frac{\sigma_{2,i}}{\sigma_1^2} + \frac{\sigma_{2}^{nn}}{\sigma_1^2} h_{nn,i}^2 + n \sum_{j=1,j\neq i}^{n-1} h_{jj,i}^2 \right)
$$

$$
\leq 21 \left[ \frac{(\sigma_{2,i})^2}{\sigma_1^2} + \frac{(\sigma_{2}^{nn})^2}{\sigma_1^2} h_{nn,i}^2 + \frac{2n^2}{\sigma_1^2} \sum_{j=1,j\neq i}^{n-1} h_{jj,i}^2 \right].
$$
Plug (4.20) into (4.15), we have
\[
\frac{\sigma_{ij}^2}{\sigma_i^2} \leq 4 \frac{\sigma_{nn}^2}{\phi_i^2} + (1 + \epsilon) \sum_{i=1}^{n-1} h_{nn,i}^2 + 21 \left(1 + \frac{4}{\epsilon}\right) \sum_{i=1}^{n-1} \sum_{j=1,j\neq i}^{n} h_{jj,i}^2 \tag{4.21}
\]
where
\[
\delta := \epsilon + 21 \left(1 + \frac{4}{\epsilon}\right) \left(\frac{\sigma_{nn}^2}{\sigma_1^2}\right) + 21 (1 + \frac{4}{\epsilon}) \frac{2n^2}{\sigma_1^2}.
\]

Put (4.11), (4.14) and (4.21) into (4.10), we get
\[
0 \geq 4\sigma_2 \langle \nu, a \rangle \phi + \Delta \sigma_2 + \frac{1}{\sigma_1} \left(2 \sum_{i \neq j} h_{ii,i}^2 - \frac{1}{2} |\nabla \sigma_2|^2 \right) + \left(\frac{2|h|^2}{\sigma_1^2} - 1\right) \sigma_2 \sigma_1
\]
\[
+ 3\sigma_3 - \frac{3}{2} \left(\frac{4\sigma_{nn}^2}{\phi_i^2} + (1 + \delta) \frac{1}{\sigma_1} \sum_{i \neq j} h_{ii,i}^2 + 21 (1 + \frac{4}{\epsilon}) |\nabla \sigma_2|^2 \right) \tag{4.23}
\]
\[
\geq 4\sigma_2 \langle \nu, a \rangle \phi + \Delta \sigma_2 + \left(2 - \frac{3}{2} (1 + \delta)\right) \frac{1}{\sigma_1} \sum_{i \neq j} h_{ii,i}^2 - \frac{1}{2} |\nabla \sigma_2|^2
\]
\[
+ \left(\frac{2|h|^2}{\sigma_1^2} - 1\right) \sigma_2 \sigma_1 - \frac{6\sigma_{nn}^2 (\nu, a)^2}{\phi_i^2} - \frac{63}{2} (1 + \frac{4}{\epsilon}) |\nabla \sigma_2|^2 \frac{\sigma_1^2}{\sigma_2 \sigma_1},
\]
where we have used the Newton-MacLaurin inequality \(\sigma_1^2 \geq \frac{2n}{n+1} \sigma_2 \geq 2\sigma_2\).

By Case (III) assumption and (4.7) we have
\[
2 |h|^2 \frac{1}{\sigma_1^2} - 1 \geq \frac{2^n}{\sigma_1^2} - 1 \geq \frac{2n^2}{n^2 + 1} - 1 \geq \frac{1}{2} \tag{4.24}
\]
Again by (4.7) we have
\[
\sigma_{nn}^2 = \sum_{i=1}^{n-1} \lambda_i \leq n \lambda_1 < n \lambda_n^\frac{4}{3} < \frac{2n}{\sqrt{\sigma_1}},
\]
Take \(\epsilon = \frac{1}{10}\) in (4.22) and use \(\sigma_1 \geq 100n\), we get
\[
\delta \leq \frac{1}{10} + 100 \frac{4n^2}{\sigma_1^2} + 1000 \frac{2n^2}{\sigma_1} < \frac{1}{3} \tag{4.25}
\]
Hence it follows from (4.24), (4.25) and (4.23) that in Case III at point \(\bar{x}\)
\[
(\phi^2 \sigma_1) \leq \frac{2}{\sigma_2} \left[- 4\sigma_2 \langle \nu, a \rangle \phi + \frac{12n}{\sqrt{\sigma_1}} (\nu, a)^2 + \left( - \frac{\Delta \sigma_2}{\sigma_1} + 100 |\nabla \sigma_2|^2 \right) \phi^2 \right]. \tag{4.26}
\]

Combine the three cases (4.5), (4.6) and (4.26) all together and use \(H(x) = \sigma_1(x)\) and \(R(x) = 2\sigma_2(x)\), we have proved the interior estimate of the mean curvature
\[
\sup_{x \in \Omega_r} \phi^2(x) H(x) \leq C(n) \sup_{y \in \Omega_r} (1 + R^2(y)) + \frac{1}{R(y)} + \frac{1}{R^2(y)} |\nabla R|^2(y) + \frac{1}{R(y)} |\Delta R(y)| r^2. \tag{4.27}
\]
Given any \( x \in M \), we take \( r = 2(X(x),a) \), then \( \phi(x) = \langle X(x),a \rangle \). Hence by (4.27), we obtain (4.3). Theorem 4.1 is proved.

5. The Structure of Generalized Cylinder

In this section, we prove Theorem 1.2. First we define an operator which generalizes Cheng-Yau’s self-adjoint operator associated to a given Codazzi tensor [10].

**Proposition 5.1.** Let \((M^n,g)\) be a Riemannian manifold and let \( f \) be a smooth function on \( M \). Let \( \psi = \sum_{ij} \psi_{ij} \omega^i \omega^j \) be a symmetric \((2,0)\)-tensor on \( M \). Then the operator

\[
\Box_{\psi,f} := \psi_{ij} \nabla^2_{ij} - \psi_{ij} f_i \nabla_j
\]

is self-adjoint with respect the weighted measure \( e^{-f}d\mu \) if and only if the divergence of \( \sum_j \psi_{ij,j} = 0 \) for all \( i \). Here \( \nabla \) is the Levi-Civita connection and \( d\mu \) is the volume element associated with \( g = \sum_i \omega^2_i \).

In particular, given a symmetric Codazzi tensor \( h = h_{ij} \omega^i \omega^j \) with \( h_{ij,k} = h_{ik,j} \), define \( \sigma_2(h) \) and \( \sigma_{ij}^2(h) \) as in (3.1) and (3.2), then the tensor \( (\psi(h))_{ij} := (\sum_k h_{kk}) \delta_{ij} - h_{ij} = \sigma_2^2(h) \) is divergence free and the operator

\[
\Box_{\psi(h),f} = \sigma_2^2(h) \nabla^2_{ij} - \sigma_{ij}^2(h) f_i \nabla_j,
\]

is a self-adjoint operator with respect the weighted measure \( e^{-f}d\mu \).

**Proof.** For any compact supported \( C^2 \) functions \( u \) and \( v \) on \( M \), by Stoke’s theorem,

\[
\int_M \Box_{\psi,f} u \cdot ve^{-f} d\mu = \int_M (\psi_{ij} u_i v e^{-f})_j d\mu - \int_M \psi_{ij} u_i v_j e^{-f} d\mu - \int_M \psi_{ij,j} u_i v e^{-f} d\mu = -\int_M \psi_{ij} u_i v_j e^{-f} d\mu.
\]

Therefore the operator \( \Box_{\psi,f} \) is self-adjoint with respect to the measure \( e^{-f}d\mu \). \( \square \)

**Remark 1.** On a GRS satisfying (1.1), there is another natural intrinsic self-adjoint operator \( \Box_{\text{Ric}} = R_{ij} \nabla^2_{ij} \) with respect the weighted measure \( e^{-f}d\mu \). We believe that this operator should be useful.

We need the following proposition (analogue of [10 Proposition 2]).

**Proposition 5.2.** Let \((M^n,g)\) be a compact manifold with boundary and let \( f \) be a smooth function on \( M \). Suppose \( \psi = \sum_{ij} \psi_{ij} \omega^i \omega^j \) is a semipositive symmetric \((2,0)\)-tensor which is divergence free. Then the (possibly degenerate) elliptic operator \( \Box_{\psi,f} \) has the following property. For any \( C^2 \) positive function \( u \) and any non-negative \( C^2 \) function \( v \) satisfying \( v|_{\partial M} = 0 \), we have

\[
( -\int_M v \Box_{\psi,f} ve^{-f} d\mu ) \left( \int_M v^2 e^{-f} d\mu \right)^{-1} \geq \inf_M \frac{-\Box_{\psi,f} v}{u}.
\]

**Proof.** We only need to prove (5.2) assuming \( \Box_{\psi,f} \) is non-degenerate elliptic, as one may replace \( \Box_{\psi,f} \) by \( \Box_{\psi,f} + \epsilon \Delta_f \) and let \( \epsilon \to 0 \).

Let \( \lambda \) be the first (positive) eigenvalue and \( v_\lambda \) be the first eigenfunction of \( \Box_{\psi,f} \) over \( M \) with the zero boundary condition. Then it is well-known that the left hand of (5.2) is always not less than \( \lambda \) and \( v_\lambda \) is positive in the interior of \( M \).
Consider the function \( \frac{\lambda}{u} \). At the interior point where \( \frac{\lambda}{u} \) attains its maximum, we have
\[
0 = \nabla^2 \frac{\lambda}{u} = \frac{u \nabla v - v \nabla u}{u^2},
\]
(5.3)
\[
0 \geq \nabla^2 \frac{\lambda}{u} = \frac{u \nabla^2 v - v \nabla^2 u}{u^2}.
\]
(5.4)
Hence
\[
\lambda = -\Box_{\psi,f} v = -\frac{\psi_{ij} \nabla^2 v}{v} + \frac{\psi_{ij} f_i \nabla j v}{v} \geq -\frac{\psi_{ij} \nabla^2 v}{v} + \frac{\psi_{ij} f_i \nabla j v}{v}.
\]
This verifies (5.2).

Let \( X : M^n \to \mathbb{R}^{n+1} \) be a hypersurface with induced metric \( g \) and (outer) unit normal \( \nu \). Let \( h \) be the second fundamental form and let \( H \) be the mean curvature. Given a smooth function \( f \) on \( M \), we define two operators using a local orthonormal frame \( \{e_1, \ldots, e_n\} \)
\[
\Box_h := \sigma^i_{ij}(h) \nabla^2_{ij} \quad \text{and} \quad \Box_{h,f} := \Box_h - \sigma^i_{ij}(h) f_i \nabla_j,
\]
where \( \sigma^i_{ij}(h) \) are defined in (3.1) and (3.2). From Proposition (5.1), the operator \( \Box_{h,f} \) is a self-adjoint operator with respect to the weighted measure \( e^{-f} d\mu \).

We compute \( \Box_{h,f} \nu \). Note that
\[
X_{ij} = -h_{ij} \nu, \quad \nu_i = h_{id} e_i, \quad \nu_{ij} = h_{ij,d} e_i - h^2_{ij} \nu, \quad 1 \leq i, j \leq n.
\]
(5.5)
From (5.5), we have
\[
\Box_{h,f} \nu = \frac{\sigma^i_{ij}(h) (h_{ij,d} e_i - h^2_{ij} \nu) - \sigma^i_{ij}(h) f_i h_{ji} e_i}{2} = \frac{1}{2} \nabla R - (\sigma_1(h) \sigma_2(h) - 3 \sigma_3(h)) \nu - \text{Ric}(\nabla f).
\]
(5.6)
For a GRS satisfying (1.1), we have the following equation analogous to the one considered by Cheng-Yau in [10].

**Proposition 5.3.** Let \( (M^n, g, f) \) be an isometrically embedded hypersurface in \( \mathbb{R}^{n+1} \) with a GRS structure (1.1). Then
\[
\Box_{h,f} \nu = -(\sigma_1(h) \sigma_2(h) - 3 \sigma_3(h)) \nu.
\]
(5.7)
**Proof.** The proposition follows from (5.6) and (2.1).

Now we can prove Theorem 1.2. Since \( (M, g, f) \) is a gradient shrinking Ricci soliton with nonnegative Ricci curvature, then by the maximum principle, either \( (M, g, f) \) is flat or the scalar curvature is strictly positive (see (2.11)). Suppose \( (M, g) \) is an Euclidean hypersurface with nonnegative Ricci curvature and positive scalar curvature, then the second fundamental form \( h \) is semipositive. It follows that \( M \) is a convex hypersurface and the curvature operator is semipositive.

With the splitting Theorem 1.1 we can assume that the universal covering \( (\tilde{M}, \tilde{g}) = (N, h) \times \mathbb{R}^{n-k} \) split isometrically and \( (N, h) \) has strictly positive Ricci curvature. However, it is not illuminating to see that whether \( (N, h) \) or one of its quotient admit an isometric embedding in \( \mathbb{R}^{k+1} \). Alternatively, since \( (M, g) \) is Euclidean hypersurface, we can also establish an splitting theorem easily in an extrinsic way.
First of all, if $M$ is noncompact convex Euclidean hypersurface, then the Gauss image must lie in a closed hemisphere of $S^n$; moreover, if $(M, g)$ has positive section curvature, then Gauss image is an open convex subset of $S^n$ \cite{27}. Therefore, there is a unit vector $a \in S^n$ so that $\langle \nu, a \rangle \geq 0$ on $M$. If $\langle \nu, a \rangle = 0$ at one point, then we claim that $\langle \nu, a \rangle$ is identically zero. From (5.7) we get
\begin{equation}
\Box_{h,f} \langle \nu, a \rangle = -\langle \sigma_1(h)\sigma_2(h) - 3\sigma_3(h) \rangle \langle \nu, a \rangle. \tag{5.8}
\end{equation}
Note that the positive scalar curvature on $M$ implies that the operator $\Box_{h,f}$ is elliptic, see \cite{4.2}. Hence by applying the maximum principle we conclude that either $\langle \nu, a \rangle$ is everywhere positive or $\langle \nu, a \rangle \equiv 0$. In the later case, since $a$ is constant tangent vector and therefore parallel, we can split out one line globally along $a$. Continue by induction, we prove that $M^n = N \times \mathbb{R}^{n-k}$ for some $2 \leq k \leq n$, where $N$ does not contain any straight lines.

We note that on a codimension one shrinking GRS isometrically embedded in $\mathbb{R}^{n+1}$, we have the equation \cite{27.15}
\begin{equation}
\sigma_2(h) = \frac{1}{2}R = \frac{1}{2}(f - |\nabla f|^2). \tag{5.9}
\end{equation}
In the same way as did for the Ricci curvature in Theorem 1.1, we can deduce the constant rank theorem for the second fundamental form, and therefore the splitting theorem.

In the following we will show by contradiction argument that $N$ must be compact. With the splitting structure, let us assume $M (= N)$ is noncompact and there exists a vector $a$ such that $\langle \nu, a \rangle > 0$. In this case, $M$ is essentially a graph along $-a$, the set
\begin{equation}
\Omega_r = \{ x \in M | \langle X(x), -a \rangle \leq r \}
\end{equation}
is compact for all $r > 0$ and $\langle X(x), -a \rangle$ is asymptotic to the geodesic distance of $M$, see \cite{27.10, 10}.

Combine Proposition 5.2 and Proposition 5.3, we have that for any compact region $\Omega \subset M$ and for any nonnegative $C^2$ function $u$ with $u|_{\partial \Omega} = 0$
\begin{equation}
\min_{\Omega} \left( \sigma_1 \sigma_2 - 3\sigma_3 \right) = \min_{\Omega} \left( \Box_{h,f} \langle \nu, a \rangle \right) \tag{5.10}
\end{equation}
\begin{equation}
\leq \left( -\int_{\Omega} u \Box_{h,f} u e^{-f} \right) \left( \int_{\Omega} u^2 e^{-f} \right)^{-1}
= \left( \int_{\Omega} \sigma_{ij}^2 u_i u_j e^{-f} \right) \left( \int_{\Omega} u^2 e^{-f} \right)^{-1}.
\end{equation}

We apply (5.10) to $u(x) = r - \langle X(x), -a \rangle$ on $\Omega_r$ and get
\begin{equation}
\left( \int_{\Omega_r} \sigma_{ij}^2 u_i u_j e^{-f} \right) \left( \int_{\Omega_r} u^2 e^{-f} \right)^{-1}
= \left( \int_{\Omega_r} (H \delta_{ij} - h_{ij}) \langle e_i, a \rangle \langle e_j, a \rangle e^{-f} \right) \left( \int_{\Omega_r} (r - \langle X, -a \rangle)^2 e^{-f} \right)^{-1}
\leq \left( \int_{\Omega_r} H e^{-f} \right) \left( \frac{r^2}{4} \int_{\Omega_r} e^{-f} \right)^{-1}
= 4r^{-2} \left( \int_{\Omega_r^2} H e^{-f} \right) \left( \int_{\Omega_r^2} e^{-f} \right)^{-1}. \tag{5.11}
\end{equation}

Since $(M, g)$ is a shrinking GRS with positive Ricci curvature, combine the fact that the scalar curvature have a strictly lower bound \cite{2.11} with the scalar curvature growth
estimate (2.10), (2.12) and (2.14), (4.3) implies that
\[
H(x) \leq C(n)(1 + r(x))^4,
\]
where we used the fact that \(\langle X(x), -a \rangle\) is asymptotic to the intrinsic geodesic distance function \(r(x)\) of \(M\). In particular, the mean curvature is integrable with respect to the weighted measure \(e^{-f}d\mu\),
\[
\int_M He^{-f}d\mu < \infty.
\]
Combine with (5.10), (5.11) and (5.13), we have
\[
\min_{\Omega_r} \left( \sigma_1 \sigma_2 - 3\sigma_3 \right) \leq C(n)r^{-2}.
\]
Let \(r\) go to infinity, this implies that
\[
\inf_{M} \left( \sigma_1 \sigma_2 - 3\sigma_3 \right) = 0.
\]
On the other hand, since \(h\) is in the so-called Gårding cone \(\Gamma_2\), by Newton-MacLaurin inequality we have
\[
\sigma_1 \sigma_2 - 3\sigma_3 \geq C(n)\sigma_3^n > \delta,
\]
where the uniform positive lower bound of scalar curvature is ensured by (2.11). This contradicts with (5.15), and consequently, \(M\) must be compact.

Since \(M\) is compact hypersurface, then Ricci tensor has full rank at the elliptic point, i.e. there exist \(x_0 \in M\) such that \(\text{Ric}(x_0) > 0\). With Theorem 1.1, we conclude that Ricci tensor has constant rank; therefore the Ricci curvature and also the curvature operator are positive everywhere. Finally, by Theorem 2.2, the compact shrinking GRS with positive curvature operator has to be the round sphere or its metric quotient. Since \(M\) is an Euclidean hypersurface, then it must be a round sphere. Now the proof of Theorem 1.2 is complete.

References

[1] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 10. Springer-Verlag, Berlin, 1987.
[2] B. Bian and P. Guan, *A microscopic convexity principle for nonlinear partial differential equations*, Invent. Math. 177 (2009), no. 2, 307-335.
[3] C. Böhm, B. Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) 167 (2008), no. 3, 1079-1097.
[4] L. Caffarelli and A. Friedman, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. 52 (1985), no. 2, 431-456.
[5] M. Cai, *On shrinking gradient ricci soliton with nonnegative sectional curvature*, arXiv:1303.2728 [math.DG].
[6] H.D. Cao, B.L. Chen and X.P. Zhu, *Recent developments on Hamilton’s Ricci flow*, Surveys in Differential Geometry, vol. XII, 47-112, surv. differ.Geom., XII, Int. Press, Somerville, MA, 2008.
[7] H.D. Cao and D. Zhou, *On complete gradient shrinking Ricci solitons*, J. Differential Geom. 85 (2010), no. 2, 175-185.
[8] X. Cao, B. Wang and Z. Zhang, *On locally conformally flat gradient shrinking Ricci solitons*, Commun. Contemp. Math. 13 (2011), no. 2, 269-282.
[9] B.L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. 82 (2009), no. 2, 363-382.
[10] S.Y. Cheng and S.T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. 225 (1977), no. 3, 195-204.
[11] M. Feldman, T. Ilmanen and D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geom. 65 (2003), no. 2, 169-209.
[12] P. Guan, J. Li and Y. Li, *Hypersurfaces of prescribed curvature measure*, Duke Math. J. 161 (2012), no. 10, 1927-1942.

[13] P. Guan, C. Ren and Z. Wang, *Global C^2 estimates for convex solutions of curvature equations*, to appear in Comm. Pure & Appl. Math.

[14] R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255-306.

[15] R.S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. 24 (1986), no. 2, 153-179.

[16] R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7-136, Int. Press, Cambridge, MA, 1995.

[17] R. Haslhofer and R. Müller, *A compactness theorem for complete Ricci shrinkers*, Geom. Funct. Anal. 21 (2011), no. 5, 1091-1116.

[18] O. Munteanu and N. Sesum, *On gradient Ricci solitons*, J. Geom. Anal. 23 (2013), no. 2, 539-561.

[19] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. 645 (2010), 125-153.

[20] L. Ni, *Ricci flow and nonnegativity of sectional curvature*, Math. Res. Lett. 11 (2004), no. 5-6, 883-904.

[21] L. Ni, *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. 12 (2005), no. 5-6, 633-653.

[22] L. Ni and N. Wallach, *On a classification of gradient shrinking solitons*, Math. Res. Lett. 15 (2008), no. 5, 941-955.

[23] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 [math.DG]

[24] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109 [math.DG]

[25] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol. 14 (2010), no. 4, 2277-2300.

[26] I. Singer, B. Wong, S.T. Yau and Stephen Yau, *An estimate of gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4), 12 (1985), 319-333.

[27] H. Wu, *The spherical images of convex hypersurfaces*, J. Differential Geom. 9 (1974), 279-290.

[28] Z. Zhang, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. 242 (2009), no. 1, 189-200.

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