Approximate Deadline-Scheduling with Precedence Constraints

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Abstract

We consider the classic problem of scheduling a set of \( n \) jobs non-preemptively on a single machine. Each job \( j \) has non-negative processing time, weight, and deadline, and a feasible schedule needs to be consistent with \textit{chain-like} precedence constraints. The goal is to compute a feasible schedule that minimizes the sum of penalties of late jobs. Lenstra and Rinnoy Kan [Annals of Disc. Math., 1977] in their seminal work introduced this problem and showed that it is strongly NP-hard, even when all processing times and weights are 1. We study the approximability of the problem and our main result is an \( O(\log k) \)-approximation algorithm for instances with \( k \) distinct job deadlines.

We also point out a surprising connection to a model for technology diffusion processes in networks that was recently proposed by Goldberg and Liu [SODA, 2013]. In an instance of such a problem one is given an undirected graph and a non-negative, integer threshold \( \theta(v) \) for each of its vertices \( v \). Vertices \( v \) in the graph are either \textit{active} or \textit{inactive}, and an inactive vertex \( v \) activates whenever it lies in component of size at least \( \theta(v) \) in the graph induced by itself and all active vertices. The goal is now to find a smallest cardinality seed set of active vertices that leads to the activation of the entire graph.

Goldberg and Liu showed that this problem has no \( o(\log(n)) \)-approximation algorithms unless NP has quasi-polynomial time algorithms, and the authors presented an \( O(rk \log(n)) \)-approximation algorithm, where \( r \) is the radius of the given network, and \( k \) is the number of distinct vertex thresholds. The open question is whether the dependence of the approximation guarantee on \( r \) and \( k \) is avoidable. We answer this question affirmatively for instances where the underlying graph is a spider. In such instances technology diffusion and precedence constrained scheduling problem with unit processing times and weights are equivalent problems.

1 Introduction

In an instance of the classic \textit{precedence-constrained single-machine deadline scheduling} problem we are given a set \( [n] := \{1, \ldots, n\} \) of jobs that need to be scheduled non-preemptively on a single machine. Each job \( j \in [n] \) has a non-negative deadline \( d_j \in \mathbb{N} \), a processing time \( p_j \in \mathbb{N} \) as well as a non-negative penalty \( w_j \in \mathbb{N} \). A feasible schedule has to be consistent with precedence constraints that are given implicitly by a directed acyclic graph \( G = ([n], E) \); i.e., job \( i \in [n] \) has to be processed before job \( j \) if \( G \) has a directed \( i,j \)-path. A feasible schedule incurs a penalty of \( w_j \) if job \( j \) is not completed before its deadline \( d_j \). Our goal is then to find a feasible schedule that minimizes the total penalty of late jobs. In the standard scheduling notation \[10\] the problem

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under consideration is succinctly encoded as $1|\text{prec}|\sum w_jU_j$, where $U_j$ is a binary variable that takes value 1 if job $j$ is late and 0 otherwise.

Single-machine scheduling with deadline constraints is a practically important and well-studied subfield of scheduling theory that we cannot adequately survey here. We refer the reader to Chapter 3 of [23] or Chapter 4 of [4], and focus here on the literature that directly relates to our problem. The decision version of the single-machine deadline scheduling problem without precedence constraints is part of Karp’s list of 21 NP-complete problems [17], and a fully-polynomial-time approximation scheme is known [8, 24]. The problem becomes strongly NP-complete in the presence of release dates as was shown by Lenstra et al. [21]. Lenstra and Rinnoy Kan [22] later proved that the above problem is strongly NP-hard even in the special case where each job has unit processing time and penalty, and the precedence digraph $G$ is a collection of vertex-disjoint directed paths.

Despite being classical, and well-motivated, little is known about the approximability of precedence-constrained deadline scheduling. This surprises, given that problems in this class were introduced in the late 70s, and early 80s, and that these are rather natural variants of Karp’s original 21 NP-hard problems. The sparsity of results to date suggests that the combination of precedence constraints and deadlines poses significant challenges. We seek to show, however, that these challenges can be overcome to achieve non-trivial approximations for these important scheduling problems. In this paper we focus on the generalization of the problem studied in [22], where jobs are allowed to have strongly NP-hard even in the special case where each job has unit processing time and penalty, and where we minimize the weighted sum of late jobs. Once more using scheduling notation, this problem is given by $1|\text{chains}|\sum w_jU_j$ (and hereafter referred to as $pDLS$). Our main result is the following.

**Theorem 1.1.** $pDLS$ has an efficient $O(\log k)$-approximation algorithm, where $k$ is the number of distinct job deadlines in the given instance.

We note that our algorithm finds a feasible schedule without late jobs if such a schedule exists.

In order to prove this result, we first introduce a novel, and rather subtle configuration-type LP. The LP treats each of the directed paths in the given precedence system independently. For each path, the LP has a variable for all nested collections of $k$ suffixes of jobs, and integral solutions set exactly one of these variables per path to 1. This determines which subset of jobs are executed after each of the $k$ distinct job deadlines. The LP then has constraints that limit the total processing time of jobs executed before each of the $k$ deadlines. While we can show that integral feasible solutions to our formulation naturally correspond to feasible schedules, the formulation’s integrality gap is large (see Appendix D for details. In order to reduce the gap, we strengthen the formulation using valid inequalities of Knapsack cover-type [1, 5, 14, 26] (see also [6, 19]).

The resulting formulation has an exponential number of variables and constraints, and it is not clear whether it can be solved efficiently. In the case of chain-like precedences, we are able to provide an alternate formulation that, instead of variables for nested collections of suffixes of jobs, has variables for job-suffixes only. Thereby, we reduce the number of variables to a polynomial of the input size, while increasing the number of constraints slightly. We do not know how to efficiently solve even this alternate LP. However, we are able to provide a relaxed separation oracle (in the sense of [5]) for its constraints, and can therefore use the Ellipsoid method [11] to obtain approximate solutions for the alternate LP of sufficient quality.

We are able to provide an efficiently computable map between solutions for the alternate LP, and those of the original exponential-sized formulation. Crucially, we are able to show that the latter solutions are structurally nice; i.e., no two nested families of job suffixes in its support cross! Such cross-free solutions to the original LP can then be rounded into high-quality schedules.

Several comments are in order. First, there is a significant body of research that investigates LP-based techniques for single-machine, precedence-constrained, minimum weighted completion-
time problems (e.g., see [13, 12, 25], and also [7] for a more comprehensive summary of LP-based algorithms for this problem). None of these LPs seem to be useful for the objective of minimizing the total penalty of late jobs. In particular, converting these LPs requires the introduction of so-called “big-M”-constraints that invariably yield formulations with large integrality gaps.

Second, using Knapsack-cover inequalities to strengthen an LP formulation for a given covering problem is not new. In the context of approximation algorithms, such inequalities were used by Carr et al. [5] in their work on the Knapsack problem and several generalizations. Subsequently, they also found application in the development of approximation algorithms for general covering and packing integer programs [19], in approximating column-restricted covering IPs [18, 6], as well as in the area of scheduling (without precedence constraints) [2]. Note that our strong formulations for pDLS use variables for (families of) suffixes of jobs in order to encode the chain-like dependencies between jobs. This leads to formulations that are not column-restricted, and they also do not fall into the framework of [19] (as, e.g., their dimension is not polynomial in the input size).

Third, it is not clear how what little work there has been on precedence-constrained deadline scheduling can be applied to the problem we study. The only directly relevant positive result we know of is that of Ibarra and Kim [15], who consider the single-machine scheduling problem in which \( n \) jobs need to be scheduled non-preemptively on a single machine while adhering to precedence constraints given by acyclic directed forests, with the goal to maximize the total profit of jobs completed before a common deadline \( T \). While the allowed constraints are strictly more general than the chain-like ones we study, this is more than outweighed by the fact that all jobs have a common deadline, which significantly reduces the complexity of the problem and renders it similar to the well-studied Knapsack problem. Indeed, we show in Appendix [B] that pDLS with forest precedences and a single deadline admits a pseudo-polynomial time algorithm as well. This implies that the decision version of pDLS is only weakly NP-complete in this special case. Given the strong NP-hardness of pDLS (as established in [22]), it is unclear how Ibarra and Kim’s results can be leveraged for our problem.

It is natural to ask whether the approximation bound provided in Theorem [1.1] can be improved. In Appendix [C] we provide an example demonstrating that this is unlikely if we use a path-independent rounding scheme (as in the proof of Theorem [1.1]). This example highlights that different paths can play vastly different roles in a solution, and be critical to ensuring that distinct necessary conditions are met. Thus, rounding paths independently can lead to many independent potential points of failure in the process, and significant boosting of success probabilities must occur if we are to avoid all failures simultaneously. This means, roughly speaking, that our analysis is tight and therefore our approximation factor cannot be improved without significant new techniques. Given the above, it is natural to look for dependent rounding schemes for solutions to our LP. Indeed, such an idea can be made to work for the special case of pDLS with two paths.

**Theorem 1.2.** pDLS with two paths admits a 2-approximation algorithm based on a correlated rounding scheme.

The proof of Theorem [1.2] is given in Appendix [C] and shows that the configurational LP used in the proof of Theorem [1.1] has an integrality gap of at most 2 for pDLS instances with two paths. This is accomplished using a randomized rounding scheme that samples families of suffix chains from the two paths in a correlated fashion instead of independently. The approach uses the fact that our instances have two paths, and extending it to general instances appears difficult.

We point out that the emphasis in Theorem [1.2] and its proof is on the techniques used rather than the approximation guarantee obtained. In fact, we provide a dynamic-programming-based exact algorithm for pDLS instances with a fixed number of chains (see Appendix [A] for details).
Theorem 1.3. \( \text{pDLS} \) can be solved exactly when the number of chains is fixed.

1.1 Deadline scheduling and technology diffusion

As we show now, the \textit{precedence-constrained single-machine deadline scheduling} problem is closely related to the \textit{technology diffusion} (TD) problem which was recently introduced by Goldberg and Liu \cite{GoldbergL20} in an effort to model dynamic processes arising in technology adaptation scenarios. In an instance of TD, we are given a graph \( G = (V, E) \), and thresholds \( \theta(v) \in \{\theta_1, \ldots, \theta_k\} \) for each \( v \in V \). We consider dynamic processes in which each vertex \( v \in V \) is either active or inactive, and where an inactive vertex \( v \) becomes active if, in the graph induced by it and the active vertices, \( v \) lies in a connected component of size at least \( \theta(v) \). The goal in TD is now to find a smallest seed set \( S \) of initially active vertices that eventually lead to the activation of the entire graph. Goldberg and Liu argued that it suffices (albeit at the expense of a constant factor loss in the approximation ratio) to consider the following connected abstraction of the problem: find a permutation \( \pi = (v_1, \ldots, v_n) \) of \( V \) such that the graph induced by \( v_1, \ldots, v_i \) is connected, for all \( i \), and such that
\[
S(\pi) = \{v_i : i < \theta(v_i)\}
\]
is as small as possible.

As Goldberg and Liu \cite{GoldbergL20} argue, TD has no \( o(\log(n)) \)-approximation algorithm unless NP has quasi-polynomial-time algorithms. The authors also presented an \( O(rk \log(n)) \)-approximation, where \( r \) is the diameter of the given graph, and \( k \) is the number of distinct thresholds used in the instance. Könenmann, Sadeghian, and Sanità \cite{KonemannSS19} recently improved upon this result by presenting a \( O(\min\{r, k\} \log(n)) \)-approximation algorithm. The immediate open question arising from \cite{GoldbergL20} and \cite{KonemannSS19} is whether the dependence of the approximation ratio on \( r \) and \( k \) is avoidable. As it turns out, our work here provides an affirmative answer for TD instances on \textit{spider} graphs (i.e., trees in which at most one vertex has degree larger than 2).

Theorem 1.4. TD is NP-hard on spiders. In these graphs, the problem also admits an \( O(\log(k)) \)-approximation.

The theorem follows from the fact that TD in spiders and pDLS with unit processing times, and penalties are equivalent. We sketch the proof. Given an instance of TD on spider \( G = (V, E) \), we create a job for each vertex \( v \in V \), and let \( d_v = n - \theta(v) + 1 \), and \( p_v = w_v = 1 \). We also create a dependence chain for each leg of the spider; i.e., the job for vertex \( v \) depends on all its descendants in the spider, rooted at its sole vertex of degree larger than 2. It is now an easy exercise to see that the TD instance has a seed set of size \( s \) iff the pDLS instance constructed has a schedule that makes \( s \) jobs late.

2 Notation

In the rest of the paper we will consider an instance of pDLS given by a collection \([n]\) of jobs. Each job \( j \) has non-negative processing time \( p_j \), penalty \( w_j \) and deadline \( d_j \). The precedence constraints on \([n]\) are induced by a collection of vertex-disjoint, directed paths \( P = \{P_1, \ldots, P_q\} \). In a feasible schedule job \( j \) has to precede job \( j' \) if there is a directed \( j, j' \)-path in one of the paths in \( P \); we will write \( j \preceq j' \) to indicate \( j \) has to precede \( j' \) from now on for ease of notation, and \( j < j' \) if we furthermore have \( j \neq j' \). We denote the set of distinct deadlines in our instance by \( D = \{D_1, \ldots, D_k\} \), with higher indices corresponding to later deadlines, that is, indexed such that \( D_i < D_{i'} \) whenever \( i < i' \). We use the notation \( i(j) \in [k] \) to denote the index that the deadline of job \( j \) has in the set \( D \), so we have that \( d_j = D_{i(j)} \) for all \( j \in [n] \). We say that a job is \textit{postponed}
or deferred past a certain deadline $D_i$ if the job is executed after $D_i$. Our goal is to find a feasible schedule that minimizes the total penalty of late jobs. Given a directed path $P$, we let $P_{j'} := \{ j' \in [n] : j \preceq j' \}$ be the suffix induced by job $j \in [n]$. We call a sequence $S = (S_1, S_2, \ldots, S_k)$ of suffixes of a given path $P \in \mathcal{P}$ a suffix chain if
\[ P \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_k; \]
while a suffix chain could have arbitrary length, we will only use suffix chains with length $k = |\mathcal{D}|$. Given two suffix chains $S$ and $S'$ with $k$ suffixes each, we say $S \preceq S'$ if $S_i \supseteq S'_i$ for all $i \in [k]$. If we have neither $S \preceq S'$ nor $S' \preceq S$, we say that $S$ and $S'$ cross. Given two suffix chains $S$ and $S'$, we obtain their join $S \lor S'$ by letting $(S \lor S')_i = S_i \cup S'_i$. Similarly, we let the meet of $S$ and $S'$ be obtained by letting $(S \land S')_i = S_i \cap S_i$.

### 3 An integer programming formulation

Our general approach will be to formulate the problem as an integer program, to solve its relaxation, and to randomly round the fractional solution into a feasible schedule of the desired quality. The IP will have a layered structure. For each deadline $D_i \in \mathcal{D}$, we want to decide which jobs in $[n]$ are to be postponed past deadline $D_i$. We start with the following two easy but crucial observations.

**Observation 3.1.** Consider a path $P \in \mathcal{P}$, and suppose that $j \in P$ is one of the jobs on this path. If $j$ is postponed past $D_i$ then so are all of $j$’s successors on $P$. Thus, we may assume w.l.o.g. that the collection of jobs of $P$ that are executed after time $D_i$ forms a suffix of $P$.

**Observation 3.2.** Consider a path $P \in \mathcal{P}$, and suppose that $j \in P$ is one of the jobs on this path. If $j$ is postponed past $D_i$, then it is also postponed past every earlier deadline $D_i < D_i$. Thus, we may assume w.l.o.g. that the collections $S_1, \ldots, S_k$ of jobs of $P$ that are executed after deadlines $D_1 < \cdots < D_k$, respectively, exhibit a chain structure, i.e. $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_k$.

Combining the above two observations, we see that for each path $P \in \mathcal{P}$, the collections of jobs postponed past each deadline form a suffix chain $S^P = S^P_1 \supseteq S^P_2 \supseteq \cdots \supseteq S^P_k$. In the following we let $S^P$ denote the collection of suffix chains for path $P$; we introduce a binary variable $x_S$ for each suffix chain $S \in S^P$ and each $P \in \mathcal{P}$. In an IP solution $x_S = 1$ for some $S \in S^P$ if for each $i \in [k]$ the set of jobs executed past deadline $D_i$ is precisely $S_i$. We now describe the constraints of the IP in detail.

**C1** At most one suffix chain of postponed jobs per path. Since a job can either be deferred or not, and there is no meaningful way to defer a job twice, we only want to choose at most one suffix chain per path $P \in \mathcal{P}$. Hence we obtain the constraint:
\[
\sum_{S \in S^P} x_S \leq 1 \quad \forall P \in \mathcal{P}. \tag{C1}
\]

**C2** Deferring sufficiently many jobs. In any feasible schedule, the total processing time of jobs scheduled before time $D_i$ must be at most $D_i$; conversely, the total processing time of jobs whose execution is deferred past time $D_i$ must be at least $\Gamma - D_i$, where $\Gamma = \sum_{j \in [n]} p_j$ is the total processing time of all jobs. This is captured by the following constraints:
\[
\sum_{P \in \mathcal{P}} \sum_{S \in S^P} p^i_S x_S \geq \Gamma - D_i \quad \forall i \in [k]
\]
where $p^i_S$ is the total processing time of the jobs contained in $S_i$. While the above constraints are certainly valid, in order to reduce the integrality gap of the formulation and successfully apply
our rounding scheme we need to strengthen them, as we now describe. To this end, suppose that we are given a chain

$$F^P = F_1^P \supseteq F_2^P \supseteq F_3^P \supseteq \ldots \supseteq F_k^P$$

of $k$ suffixes of deferred jobs for each path $P \in \mathcal{P}$, and let $F = \{ F_P \}_{P \in \mathcal{P}}$ be the family of these suffix chains. Suppose that we knew that we were looking for a schedule in which the jobs in $F_i^P$ are deferred past deadline $D_i$ for all $P \in \mathcal{P}$. For each $i \in [k]$, a feasible schedule must now defer jobs outside $\bigcup_{P \in \mathcal{P}} F_i^P$ of total processing time at least

$$\Theta^{i,F} := \max \left\{ (\Gamma - D_i) - \sum_{P \in \mathcal{P}} \sum_{j \in F_i^P} p_j, 0 \right\}. \tag{1}$$

We obtain the following valid inequality for any feasible schedule:

$$\sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}^P} p_S^{i,F} x_S \geq \Theta^{i,F} \quad \forall i \in [k], \forall F \in \mathcal{S}, \tag{C2}$$

where $\mathcal{S}$ is the collection of all families of suffix chains for $\mathcal{P}$ (including the empty family), and where $p_S^{i,F}$ is the minimum of $\Theta^{i,F}$ and the total processing time of jobs $j$ that are in $S_i$ but not in $F_i^P$; formally, for $F \in \mathcal{S}$, $i \in [k], P \in \mathcal{P}$, and $S \in \mathcal{S}^P$, we set

$$p_S^{i,F} := \min \left\{ \sum_{j \in S \cap \mathcal{F}_i^P} p_j, \Theta^{i,F} \right\}.$$

(C2) falls into the class of Knapsack Cover (KC) inequalities \cite{1, 5, 14, 26}, and the above capping of coefficients is typical for such inequalities.

All that remains to define the IP is to give the objective function. Consider a job $j$ on path $P \in \mathcal{P}$, and suppose that the IP solution $x$ picks suffix chain $S \in \mathcal{S}^P$. Job $j$ is late (i.e., its execution ends after time $d_j = D_{i(j)}$) if $j$ is contained in the suffix $S_{i(j)}$. We can therefore express the penalty of suffix chain $S$ succinctly as

$$w_S := \sum_{j \in P : j \in S_{i(j)}} w_j. \tag{2}$$

We can now state the canonical LP relaxation of the IP as follows

$$\min \left\{ \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}^P} w_S x_S : \text{(C1), (C2), } x \geq 0 \right\}. \tag{P}$$

For convenience we introduce auxiliary indicator variables $U_j$ for each job $j \in [n]$. $U_j$ takes value 1 if $j$’s execution ends after time $d_j$, and hence

$$U_j := \sum_{S \in \mathcal{S}^P : j \in S_{i(j)}} x_S, \tag{3}$$

where $P$ is the chain containing job $j$.

4 Rounding the relaxation

Our rounding scheme does not apply only to (suitable) feasible points for (P), but in fact allows us to round a much broader class of (not necessarily feasible) fractional points $(\hat{U}, \hat{x})$ to integral feasible solutions $(\hat{U}, \hat{x})$ of the corresponding IP, while only losing a factor of $O(\log k)$ in the objective value. As we will see in Section 5.2, being able to round this broader class of points is crucial for our algorithm. In order to formally describe the class of points we can round, we need to introduce the concept of canonical chain families. Informally, the canonical suffix chain for a path $P$ defers each job $j \in P$ as much as possible, subject to ensuring no job in $P$ is deferred past its deadline. The definition below makes this formal.
Definition 4.1. Given an instance of pDLS, we let $C_i^P$ be the longest suffix of path $P \in \mathcal{P}$ that consists only of jobs whose deadline is strictly greater than $D_i$. Jobs in $C_i^P$ may be scheduled to complete after $D_i$ without incurring a penalty. We call $$C^P := C_1^P \supseteq \ldots \supseteq C_k^P$$ the canonical suffix chain for path $P$, and let $C = \{C^P\}_{P \in \mathcal{P}}$ be the canonical suffix chain family.

Our general approach for rounding a solution $(U, x)$ to program $(P)$ is to split jobs into those with large $U_j$ values and those with small ones. While we can simply think of “rounding up” $U_j$ values when they are already large, we need to utilize the constraints (C1) and (C2) to see how to treat jobs with small $U_j$ values. As it turns out, in order to successfully round $(U, x)$ we need it to satisfy the KC-inequality for a single suffix chain family only. Naturally this family will depend on the set of jobs with large $U_j$ value. We can formalize the above as follows.

Consider any instance $\mathcal{I}$ of pDLS, and let $(U, x)$ be a solution to $(P)$. Define the set $L$ of jobs that are late to an extent of at least $1/(\gamma \log k)$ for a parameter $\gamma > 0$ (whose value we will make precise at a later point): $$L = \{ j : U_j \geq 1/(\gamma \log k) \}.$$ We now obtain a modified instance of pDLS, denoted $\mathcal{I}_L$, by increasing the deadline for the jobs in $L$ to $\Gamma$. Thus, jobs in $L$ can never be late in the modified instance $\mathcal{I}_L$. Note that since we do not modify the processing time of any job $j \in [n]$, we have that $p^i_F$ and $\Theta^i_F$ remain identical in $\mathcal{I}_L$ and $\mathcal{I}$ for all $i$, $F$, and $S$. Similarly, each job $j \in [n]$ has the same penalty $w_j$ in $\mathcal{I}$ and $\mathcal{I}_L$. Let $C$ be the canonical suffix chain family for $\mathcal{I}_L$. We are able to round a solution $(U, x)$ as long as it satisfies the following conditions:

(a) for each $P \in \mathcal{P}$, the set $\{ S \in S^P : x_S > 0 \}$ is cross-free

(b) $(U, x)$ is feasible for a relaxation (P') of $(P)$ that replaces the constraints (C2) by

$$\sum_{P \in \mathcal{P}} \sum_{S \in S^P} p^i_S x_S \geq \Theta^i_C \quad \forall i \in [k],$$

where $C$ is the canonical suffix chain family for the modified pDLS instance $\mathcal{I}_L$.

In the next section, we see how we can find solutions satisfying both of these conditions.

Suppose $(U, x)$ is a solution to $(P)$ that satisfies (a) and (b). Obtain $x^0$ by letting $x^0_S = x_S$ if $S$ makes at least one job $j \in [n]$ late in $\mathcal{I}_L$, and let $x^0_S = 0$ otherwise. Define $U^0 \leq U$ as in (3) (with $x^0$ in place of $x$), and note that $(U^0, x^0)$ satisfies (a) and (b). Let us now round $(U^0, x^0)$.

We focus on path $P \in \mathcal{P}$, and define the support of $(U^0, x^0)$ induced by $P$:

$$T^P := \{ S \in S^P : x^0_S > 0 \}.$$ As this set is cross-free by assumption (a), $T^P$ has a well-defined maximal element $S^*$ with $S \preceq S^*$ for all $S \in T^P$ (recall, $S \preceq S^*$ means $S$ defers no less jobs past every deadline $D_i$ than $S^*$ does). By definition, $S^*$ makes at least one job $j \in [n] \setminus L$ late. Since $S^*$ is maximal in $T^P$ it therefore follows that $j$ is late in all $S \in T^P$. Using the definition of $(U^0, x^0)$ as well as the fact that $j \notin L$ we obtain

$$\sum_{S \in T^P} x^0_S = \sum_{S \in S^P : j \in S(i)} x^0_S = U^0_j \leq U_j < \frac{1}{\gamma \log k}. \quad (4)$$

We let $(\bar{U}, \bar{x}) = \gamma \log k \cdot (U^0, x^0)$ and obtain the following lemma.

Lemma 4.2. $(\bar{U}, \bar{x})$ satisfies

$$\sum_{S \in S^P} \bar{x}_S \leq 1 \quad \forall i \in [k], \forall P \in \mathcal{P} \quad (C1)$$

$$\sum_{P \in \mathcal{P}} \sum_{S \in S^P} p^i_S \bar{x}_S \geq \gamma \log k \cdot \Theta^i_C \quad \forall i \in [k], \quad (C2)$$
where $C$ is the canonical suffix chain family defined for the modified instance $I_L$ of pDLS.

Proof. Observe that since $\bar{x} = \gamma \log k \cdot x$, we can view constraint (C1) as being precisely inequality (4) with both sides scaled up by a factor of $\gamma \log k$; similarly, we can also view constraint (C2) as constraint (C2') scaled up by this same factor. Thus, the lemma follows immediately from inequality (4) and the fact that $(U^0, x^0)$ is feasible for $(P')$.

We now randomly round $(\bar{U}, \bar{x})$ to an integral solution $(\hat{U}, \hat{x})$ as follows. For each $P \in P$, we independently select a single random suffix chain $S \in \mathcal{S}^P$ using marginals derived from $\bar{x}$, and set the corresponding $\hat{x}_S = 1$. In particular, we set $\hat{x}$ so that for all $P \in P$ and all $S \in \mathcal{S}^P$ we have

$$\Pr[\hat{x}_S = 1] = \begin{cases} \bar{x}_S & \text{if } S \in T^P \\ 1 - \sum_{S' \in T^P} \bar{x}_{S'} & \text{if } S = C^P. \end{cases}$$

Since $(\bar{U}, \bar{x})$ satisfies (C1), we can see that the above describes a valid randomized process. We run this process independently for each path $P \in P$ to obtain $\hat{x}$. A job $j \in [n] \setminus L$ is late if it is contained in level $i(j)$ of the suffix chain $S$ chosen for path $P$ by the above process. Thus, we set

$$\hat{U}_j := \sum_{S \in \mathcal{S}^P : j \in S_{i(j)}} \hat{x}_S,$$

We now claim that the expected value of $\hat{U}_j$ is precisely $\bar{U}_j$.

**Lemma 4.3.** For all $j \notin L$, $E[\hat{U}_j] = \bar{U}_j$.

Proof. Let $P$ be the path containing $j$, and consider a chain $S \in \mathcal{S}^P$ such that $j \in S_{i(j)}$. The probability for $\hat{x}_S$ to be 1 is precisely $\bar{x}_S$, and hence it immediately follows that

$$E[\hat{U}_j] = \sum_{S \in \mathcal{S}^P : j \in S_{i(j)}} \Pr[\hat{x}_S = 1] = \sum_{S \in \mathcal{S}^P : j \in S_{i(j)}} \bar{x}_S = \bar{U}_j.$$

The preceding lemma shows that the expected penalty of $(\bar{U}, \bar{x})$ in the modified instance $I_L$ is exactly $\sum_{j \in [n] \setminus L} w_j \bar{U}_j$. The following lemma shows that the schedule induced by $\hat{x}$ postpones at least $\Theta^{iC}$ jobs past deadline $D_i$ for all $i \in [k]$ with constant probability.

**Lemma 4.4.** With constant probability, we have

$$\sum_{P \in P} \sum_{S \in \mathcal{S}^P} p_i^{iC} \bar{x}_S \geq \Theta^{iC} \quad \forall i \in [k],$$

where $C$ is the canonical suffix chain family for the modified pDLS instance $I_L$. In particular, for $\gamma = 4$ the constraint holds with probability at least 0.7.

Proof. (of Lemma 4.4) Our proof relies on two bounds on random variables. Before proceeding with the proof itself, we begin by stating the two required bounds for the sake of completeness.

First, we need the following form of Bernstein inequality [16]. Let $X_1, \ldots, X_n$ be independent, nonnegative random variables uniformly bounded by some $M \geq 0$, i.e. such that $\Pr[X_i \leq M] = 1$ for all $i$. Then, if we let $X = \sum_i X_i$, we have that

$$\Pr[X \leq E[X] - \lambda] \leq \exp \left( -\frac{\lambda^2}{2(\text{Var}(X) + \lambda M/3)} \right),$$

for any $\lambda \geq 0$.

Second, we need the Bhatia-Davis Inequality [2], which states that for any random variable $X$ with support in the interval $[m, M]$, i.e. such that $\Pr[m \leq X \leq M] = 1$, we have

$$\text{Var}(X) \leq (M - E[X])(E[X] - m).$$
We now describe how we apply the above bounds to achieve the desired bound on the probability that (5) is satisfied. Fix some $i \in [k]$. Define random variables $\{X_P\}_{P \in \mathcal{P}}$ as

$$X_P := \sum_{S \in \mathcal{S}^P} p_{S}^{i,C} \hat{x}_S$$

where $C$ is the canonical chain family defined for the modified instance of pDLS; let $X = \sum_{P \in \mathcal{P}} X_P$ denote the sum of these random variables. We make the following observations on the random variables $X_P$:

- for $P, P' \in \mathcal{P}$, $P \neq P'$, we have that $X_P$ and $X_{P'}$ are independent since our rounding process made independent choices for the two paths;
- for each $P \in \mathcal{P}$ we know $X_P$ is nonnegative, since $p_{S}^{i,C}$ and $\hat{x}_S$ are always nonnegative;
- for each $P \in \mathcal{P}$ we can see that we always have

$$X_P = \sum_{S \in \mathcal{S}^P} p_{S}^{i,C} \hat{x}_S \leq \max_{S \in \mathcal{S}^P} p_{S}^{i,C} \leq \Theta^{i,C},$$

where the first inequality follows by constraint (C1), and the second inequality follows by the definition of $p_{S}^{i,C}$; and
- the expectation of $X = \sum_{P \in \mathcal{P}} X_P$ satisfies

$$E[X] = E\left[ \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}^P} p_{S}^{i,C} \hat{x}_S \right] = \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}^P} p_{S}^{i,C} \hat{x}_S \geq \gamma \log k \cdot \Theta^{i,C}.$$  

The inequality above follows by constraint (C2) in Lemma 4.2; the second equality above follows by observing that we always have $\Pr[\hat{x}_S = 1] = \bar{x}_S$ in our sum.

With the above observations in hand, we apply the Bhatia-Davis Inequality (7) to get that

$$\text{Var}(X) = \sum_{P \in \mathcal{P}} \text{Var}(X_P) \leq \sum_{P \in \mathcal{P}} (\Theta^{i,C} - E[X_P])(E[X_P] - 0) \leq \sum_{P \in \mathcal{P}} \Theta^{i,C} E[X_P] = \Theta^{i,C} E[X],$$

where the first equality follows by the fact that the variables $X_P$ are independent. Thus, applying the Bernstein inequality (6) with $M = \Theta^{i,C}$ and $\lambda = E[X] - \Theta^{i,C}$ gives us that

$$\Pr[X \leq \Theta^{i,C}] \leq \exp \left( -\frac{(E[X] - \Theta^{i,C})^2}{2(\Theta^{i,C} E[X] + (E[X] - \Theta^{i,C})\Theta^{i,C}/3)} \right) \leq \exp \left( -\frac{(E[X] - \Theta^{i,C})^2}{(4/3) (\Theta^{i,C} E[X])} \right) = \exp \left( -\frac{3}{4} \frac{(E[X] - \Theta^{i,C})^2}{(E[X])} \left( 1 - \frac{\Theta^{i,C}}{E[X]} \right)^2 \right).$$

The first inequality above follows from the previously mentioned application of the Bernstein Inequality, the second by observing $\Theta^{i,C} \geq 0$ always and gathering like terms in the denominator, and the equality follows by pulling a factor of $(E[X])^2$ out of the numerator. As noted earlier, however, we have that $E[X] \geq \gamma \log k \cdot \Theta^{i,C}$; taking $\gamma = 4$ and substituting this into the above gives us that, in fact, $\Pr[X \leq \Theta^{i,C}] \leq \frac{3}{10k}$. Since the constraint (5) holds for a given $i \in [k]$ if and only if $X \geq \Theta^{i,C}$, by taking a union bound we can see that the constraint holds for all $i \in [k]$ with probability at least 0.7.  

\[ \square \]
For each \( P \in \mathcal{P} \) let \( \hat{S}^P \) be the join of the suffix chain corresponding to solution \( \hat{x} \), and the canonical suffix chain \( C^P \); i.e., suppose that \( \hat{x}_S = 1 \) for \( S \in S^P \). Then
\[
\hat{S}^P = S \vee C^P.
\] (8)

Clearly, \( \hat{S}^P \) is a suffix chain for path \( P \). We use the following greedy algorithm to obtain a schedule.

```
for i = 1 to k do
    for all P ∈ \( \mathcal{P} \) do
        Schedule all jobs in \( P \setminus \hat{S}^P_i \) not already scheduled respecting the precedence constraints
    end for
end for

Schedule all remaining jobs respecting the precedence constraints
```

**Theorem 4.5.** The schedule produced by the above algorithm is feasible. Furthermore, if (5) holds, the schedule has cost at most \( \sum_{j \in \mathcal{L}} w_j \hat{U}_j \) in the instance \( I_L \).

**Proof.** We begin by noting that the schedule produced by the proposed algorithm respects the precedence constraints of all \( P \in \mathcal{P} \). This follows as \( \hat{S}^P \) is a suffix chain for all \( P \in \mathcal{P} \), and hence, the algorithm schedules the jobs in \( \hat{S}^P \setminus \hat{S}^P_i \) in iteration \( i \) for \( P \in \mathcal{P} \) in precedence order.

Next, we show that whenever (5) holds, we have that the penalty of the schedule produced by our algorithm is at most \( \sum_{j \in \mathcal{L}} w_j \hat{U}_j \) in the instance \( I_L \). Note that the total processing time of all jobs scheduled after iteration \( i \) by our algorithm is

\[
\sum_{P \in \mathcal{P}} \sum_{j \in \hat{S}^P} p_j \geq \sum_{P \in \mathcal{P}} \left( \sum_{S \in S^P} p_{S}^{i,C} \hat{x}_S + \sum_{j \in C^P} p_j \right) \geq \Theta^iC + \sum_{P \in \mathcal{P}} \sum_{j \in C^P} p_j \geq \Gamma - D_i,
\]

where \( \Gamma \) is the total processing time of all jobs. The first inequality above follows by the definitions of \( p_{S}^{i,C} \) and \( \hat{S} \); the second by our assumption that (5) holds; and the third by the definition of \( \Theta^iC \). This means, however, that all of the jobs scheduled during iterations 1, 2, \ldots, \( i \) of our algorithm will be completed by time \( D_i \). Now consider a job \( j \in P \setminus \mathcal{L} \) with deadline \( d_j = D_{i(j)} \). If \( j \) is late in the given schedule, then it must have been scheduled after iteration \( i \) of our algorithm is

\[
\hat{U}_j = \sum_{S \in S^P : j \in S_{i(j)}} \hat{x}_S = 1.
\]

Thus if we let \( \mathcal{L} \) be the set of jobs \( j \) that are late in the schedule produced by our algorithm, we can see that the penalty of that schedule in the modified instance \( I_L \) is

\[
\sum_{j \in \mathcal{L}} w_j \leq \sum_{j \notin \mathcal{L}} w_j \hat{U}_j,
\]

exactly as claimed.

**Corollary 4.6.** The schedule produced by the above algorithm is feasible and incurs penalty at most \( 8 \log k \cdot \sum_{j} w_j U_j \) in the original instance of the pDLS with constant probability.

**Proof.** (Proof of Corollary 4.6) Recall how we arrived at the integral solution \((\hat{U}, \hat{x})\). Given a solution \((U, x)\) for (\ref{eq:original}), we define the set \( \mathcal{L} \) of jobs \( j \) whose indicator variables \( U_j \) have value at least \( 1/(\gamma \log k) \). We then obtained a modified instance \( I_L \) of pDLS by increasing the deadlines
of jobs in $L$ to $\Gamma$. Assuming that $(U, x)$ satisfies constraint \( (C_2) \) for the canonical suffix chain family $C$ for $I_L$ we then generated a new solution $(\bar{U}, \bar{x})$ such that
\[
\sum_{j \notin L} w_j \bar{U}_j + \sum_{j \in L} w_j \leq \gamma \log k \sum_j w_j \bar{U}_j.
\]
We then rounded our solution $(\bar{U}, \bar{x})$ to produce the integral solution $(\hat{U}, \hat{x})$ to the modified instance.

Now, by combining Theorem 4.5 and Lemma 4.4, we can see that with probability at least 0.7 we get a feasible schedule whose cost is at most $\sum_{i \in L} w_j \hat{U}_j$ in the modified instance, while setting the parameter $\gamma = 4$. Consider how the cost of this schedule can change between the modified instance and the original instance: since our modification was precisely to set the deadline of every job $j \in L$ to $\Gamma$, we know that the cost of our schedule in the original setting can be at most $\sum_{i \in L} w_j \bar{U}_j + \sum_{j \in L} w_j$.

Now, by Lemma 4.3, we have that $E[\sum_{i \in L} w_j \hat{U}_j] = \sum_{i \notin L} w_j \bar{U}_j$. Thus, Markov’s Inequality gives us that $\sum_{i \notin L} w_j \bar{U}_j \leq 2 \sum_{i \in L} w_j \bar{U}_j$ with probability at least 1/2; taking a union bound over this and our probability of our rounding procedure producing a feasible schedule, we can conclude that with probability at least 0.2, we produce a feasible schedule the cost of which in the original instance is at most
\[
2 \sum_{i \notin L} w_j \bar{U}_j + \sum_{j \in L} w_j \leq 2 \cdot 4 \log k \cdot \sum_j w_j \bar{U}_j,
\]
effectively as claimed. \( \square \)

5 Solving LP (P)

While we have shown how we can express the pDLS problem as an IP in Section 4 and how we can round solutions to a weakened LP relaxation in Section 4, an important step in the process remains unspecified: how do we find a solution $(U, x)$ to round? This is especially problematic as the LP has an exponential (in $n$) number of both variables and constraints, and it is not clear how to solve such LPs in general. In this section, we show that (P) has a compact reformulation when precedences are chain-like.

5.1 An IP formulation with polynomial number of variables

Using the specific shape of precedences, we show how the important suffix-chain structure of postponed jobs can be captured more compactly. This allows us to reduce the number of variables drastically while slightly increasing the number of constraints. Roughly speaking, our new LP 

decouples decisions on job-postponement between the layers in $[k]$. The new IP has a binary variable $x^i_j$ for every job $j \in [n]$ and for all deadlines $D_i \in D$. In a solution $x^i_j = 1$ if job $j$ and all of its successors are executed after deadline $D_i$ and all of job $j$’s predecessors are executed before deadline $D_i$. We can see that our definition of the variables $x^i_j$ ensures the desired suffix structure on every path $P \in \mathcal{P}$ (see Observation 3.1); so we need only add constraints to ensure that the chosen suffixes for a path $P$ form a chain (in the sense of Observation 3.2).

At most one suffix of postponed jobs per path per layer. We want to choose at most one suffix of jobs to defer for each path $P \in \mathcal{P}$ on every layer $i \in [k]$. This yields the following constraint corresponding to (C1):
\[
\sum_{j \in P} x^i_j \leq 1, \quad \forall P \in \mathcal{P}, i \in [k]. \tag{D1}
\]
D2 Deferring sufficiently many jobs. As in our previous IP, our new formulation has a constraint for each suffix chain family $F \in S$ and each layer $i \in [k]$. As before, suppose that we were looking for a schedule that defers the jobs in $F_i$ past deadline $D_i$ for all $P \in P$, and for all $i \in [k]$. Then define $\Theta_{i,F}$ as in (1), and let

$$p_j^{i,F} := \min \left\{ \sum_{j' \leq j, j' \in F_i^p} p_{j'}, \Theta_{i,F} \right\},$$

for every job $j \in [n]$. The new constraint corresponding to (C2) is now:

$$\sum_{P \in P} \sum_{j \in P \setminus F_i^p} p_j^{i,F} x_j^i \geq \Theta_{i,F} \quad \forall i \in [k], \forall F \in S,$$

which enforces that the total (capped) weight of jobs deferred past $D_i$, beyond what is deferred by $F$ is sufficiently large.

D3 Chain structure of postponed suffixes. We need additional constraints to ensure our new IP chooses suffixes of jobs to defer that exhibit the chain structure that characterizes feasible schedules. Consider $i, i' \in [k]$, with $i < i'$, and let $j \in [n]$ be a job whose execution is postponed until after deadline $D_{i'}$. Then as previously observed, its execution must also be postponed until after deadline $D_i$, by the ordering of deadlines. We capture this with the following family of constraints:

$$\sum_{j' \leq j} x_j^{i+1} \leq \sum_{j' \leq j} x_j^{i'}, \quad \forall P \in P, \forall j \in P, \forall i \in [k-1].$$

We can now state the entire IP. As with (P), we introduce a binary variable $U_j$ that takes value 1 if job $j \in [n]$ is postponed past its deadline $d_j$. For a given job $j \in [n]$, we can define this variable in terms of the variables in our new IP as follows. Let $d_j = D_{i(j)}$. Then job $j$ is postponed if it is part of a chosen suffix for some layer $i \geq i(j)$. We can therefore set

$$U_j := \max_{i \geq i(j)} \sum_{j' \leq j} x_j^{i},$$

where the equality follows from (D3). The standard LP relaxation of the IP is:

$$\min \left\{ \sum_{j \in [n]} w_j U_j : (D1), (D2), (D3), x \geq 0 \right\}. $$

In the case of chain-like precedences, we are able to obtain an efficient, objective-value-preserving, and invertible map between fractional solutions to (P2) and fractional cross-free solutions to (P).

Theorem 5.1. Consider $\{x_S\}_{S \in S^P, P \in P}$, with $x \geq 0$, and let

$$\tilde{x}_j^i = \sum_{S \in S^P, S_i = P_{j+1}} x_S,$$

for all $i \in [k]$, $P \in P$, and $j \in P$. Then: (i) $\tilde{x}$ satisfies condition (D3); (ii) $\tilde{x}$ satisfies conditions (D1) if and only if $x$ satisfies conditions (C1); (iii) for any $F \in S$, $\tilde{x}$ satisfies (D2) for $F$ if and only if $x$ satisfies (C2) for family $F$. Furthermore, the objective value of $x$ in (P) equals that of $\tilde{x}$ in (P2).

Proof. We begin by showing that equation (9) implies that $\tilde{x}$ satisfies condition (D3). Fix $P \in P$, $j \in P$, and $i \in [k-1]$, and observe that

$$\sum_{j' : j' \leq j} \tilde{x}_j^{i+1} = \sum_{S \in S^P, S_{i+1} = P_{j+1}} \sum_{S_i = P_j} x_S = \sum_{S \in S^P, j \in S_{i+1}} x_S.$
If we apply the same transformation to \( \sum_{j' : j' \leq j} \tilde{x}_{j'} \), we see that condition (D3) is equivalent to
\[
\sum_{S \in \mathcal{S}^P : j \in S_{i+1}} x_S \leq \sum_{S \in \mathcal{S}^P : j \in S_i} x_S.
\] (10)

Since every \( S \in \mathcal{S}^P \) is a chain we have \( S_{i+1} \subseteq S_i \), and thus every summand on the left side of the inequality also appears on the right. Since \( x \geq \nu \), it follows that both inequality (10) and constraint (D3) hold.

To see that (ii) holds, we focus on path \( P \in \mathcal{P} \), and \( i \in [k] \), and observe that
\[
\sum_{j \in P} \tilde{x}_j = \sum_{j \in P} \sum_{S \in \mathcal{S}^P : S_i = P \geq j} x_S = \sum_{S \in \mathcal{S}^P} x_S.
\]
It immediately follows that \( \tilde{x} \) satisfies (D1) iff \( x \) satisfies (C1).

Next, we show that constraint (C2) holds for \( \tilde{x} \) if and only if constraint (D2) holds for \( x \).

Consider a chain family \( F \) on \( \mathcal{P} \), and a layer \( i \in [k] \). For \( P_i \in \mathcal{P} \) and \( j \in P_i \), we can see that for any \( S \in \mathcal{S}^P \) such that \( S_i = P \geq j \), we have
\[
p_{i,F}^{j,i,F} = \min \left\{ \sum_{j' : j' \leq j' \notin F_i^j} p_{j'}, \Theta_i^{j,F} \right\} = p_{S}^{i,F}.
\]

If we apply (9) followed by the above equality, we get that
\[
\sum_{P_i \in \mathcal{P}} \sum_{j \in P_i \setminus F_i^j} p_{i,F}^{j,i,F} x_j = \sum_{P_i \in \mathcal{P}} \sum_{j \in P_i \setminus F_i^j} p_{i,F}^{j,i,F} x_S = \sum_{P_i \in \mathcal{P}} \sum_{j \in P_i \setminus F_i^j} p_{i,F}^{j,i,F} x_S,
\]
where the final equality follows by combining the inner two summations and observing that \( p_{i,F}^{j,i,F} = 0 \) whenever \( F_i^j \supseteq S_i \). Thus we have that \( \tilde{x} \) satisfies the constraints in (D2) corresponding to \( F \) if and only if \( x \) satisfies the constraints in (C2) corresponding to \( F \).

Finally, fix some \( P_i \in \mathcal{P} \). Using (9) and the definition of \( \hat{U}_j \), we see that
\[
\sum_{j \in P_i} w_j \hat{U}_j = \sum_{j \in P_i} \sum_{j' : j' \leq j} w_j \tilde{x}_{j'} = \sum_{j \in P_i} \sum_{j' : j' \leq j} \sum_{S \in \mathcal{S}^P : S_i = P \geq j} w_j x_S = \sum_{j \in P_i} \sum_{S \in \mathcal{S}^P} w_j x_S,
\]
where the final equality follows simply by combining the two inner summations. Note, however, that if we change the order of the two summations on the right-hand side we obtain
\[
\sum_{j \in P_i} w_j \hat{U}_j = \sum_{S \in \mathcal{S}^P} \sum_{j \in S_i} w_j x_S = \sum_{S \in \mathcal{S}^P} w_S x_S.
\]

Summing the above over \( P_i \in \mathcal{P} \), it follows that \( \tilde{x} \) and \( x \) have the same objective values in (P2) and (P), respectively.

The above theorem immediately implies a natural algorithm for efficiently constructing a solution \( \tilde{x} \) to (P2) from a given solution \( x \) to (P). It fails to show how to perform the inverse operation, however. We now provide the missing piece.

**Theorem 5.2.** Given a solution \( \tilde{x} \) to (P2), we can efficiently construct a cross-free solution \( x \) to (P) that satisfies condition (9) from Theorem 5.1.

**Proof.** We begin by constructing the collection of suffix chains that lie in the support of our
claimed solution $x$ for (P2); we then describe the values $x$ associates with each of the suffix chains in this collection; and finally we argue that $x$ satisfies the claimed properties.

Consider a path $P ∈ P$, $i ∈ [k]$ and $α ∈ [0, 1]$. Let
\[
S_i(α) := \left\{ j ∈ P : ∑_{j' ≤ j} x_j' ≥ α \right\},
\]
and note that the non-negativity of $x$ implies that $S_i(α)$ is a suffix of $P$. Furthermore, constraint (D3) implies that
\[
S_1(α) ⊇ S_2(α) ⊇ ⋯ ⊇ S_k(α),
\]
and $S(α)$ is therefore a valid suffix chain for $P$. Furthermore, for $0 ≤ α ≤ α' ≤ 1$, and $i ∈ [k]$ one easily sees that $S_i(α) ⊇ S_i(α')$, and hence $S(α)$ and $S(α')$ do not cross. With this we now easily bound the number of distinct members of the family $\{S(α)\}_α$. Consider increasing $α$ continuously from 0 to 1, and count the number of times changes occur. For each $i ∈ [k]$, we know that $S_i(α)$ can only take on $|P| + 1$ different values. Since $α ≤ α'$ implies $S(α) ≤ S(α')$, it follows that $S_i(α)$ becomes smaller as $α$ increases. Thus, as we increase $α$ from 0 to 1, $S_i(α)$ can only change at most $|P|$ times. Since any time $S(α)$ changes, at least one $S_i(α)$ must change, we may conclude that $S(α)$ takes on at most $|P| + 1$ distinct values.

Now we show how to construct a solution $x$ for (P2) with support $\{S(α)\}_α$. For suffix chain $S ∈ S^P$ we let
\[
x_S := \sup\{α ∈ [0, 1] : S(α) = S\} − \inf\{α ∈ [0, 1] : S(α) = S\};
\]
if $S = S(α)$ for some $α ∈ [0, 1]$, and we let $x_S := 0$ otherwise.

Finally, we show that \cite{11} holds, i.e. that
\[
\tilde{x}_j = \sum_{S ∈ S^P : S_i = P_{≥ j}} x_S.
\]
Fix $i ∈ [k]$ and $j ∈ P$, and consider the sum on the right hand side of the above equality. Recall that we previously observed that $α ≤ α'$ implies $S(α) ≤ S(α')$; combining this with equation (12), we can easily see that the sum we care about will, in fact, be a telescoping sum that simplifies to
\[
\sum_{S ∈ S^P : S_i = P_{≥ j}} x_S = \sup\{α : S_i(α) = P_{≥ j}\} − \inf\{α : S_i(α) = P_{≥ j}\}.
\]
By our definition of $S_i(α)$, however, we can see that $j ∈ S_i(α)$ if and only if $∑_{j' ≤ j} x_{j'} ≥ α$. Further, $S_i(α)$ can only include elements strictly preceding $j$ in $P$ if $∑_{j' < j} x_{j'} ≥ α$. Thus, we may conclude that
\[
\sup\{α : S_i(α) = P_{≥ j}\} − \inf\{α : S_i(α) = P_{≥ j}\} = ∑_{j' ≤ j} \tilde{x}_{j'} − ∑_{j' < j} \tilde{x}_{j'} = \tilde{x}_j,
\]
effectively as required.

5.2 Solving the relaxation

The usual way to solve linear programs with an exponential number of constraints is to employ the ellipsoid method \cite{11}. The method famously allows us to reduce the problem of solving (P2) to that of efficiently separating an infeasible point from (P2). For a given candidate solution $(U, x)$ to (P2), it suffices to decide (in polynomial time) whether it is feasible, and if not, return a violated inequality. We do not know how to solve the separation problem for (P2), and it is in fact not known how to separate KC inequalities efficiently in general.

We will overcome this issue following the methodology proposed in \cite{5} and relying on a relaxed separation oracle. For this, we consider a relaxation (P2') of (P2) where we replace constraints
for a subset \( S' \subseteq S \) that initially contains only the canonical suffix chain family \( C \) for instance \( I \).

In an iteration, we apply the ellipsoid method to (P2'), and this generates a point \((\tilde{U}, \tilde{x})\) that is optimal and feasible for (P2'), but not necessarily feasible for (P2). Using the procedure developed in Section 4 we now attempt to map \((\tilde{U}, \tilde{x})\) to a cross-free solution to (P). Specifically, for parameter \( \gamma \) chosen there, we let

\[
L = \{ j : \tilde{U}_j \geq 1/(\gamma \log k) \},
\]

be the collection of jobs whose indicator variable is large; note that \( U \) remains constant under the correspondence between solutions to \( (P) \) and \( (P2) \) as given in Theorem 5.1, and so this is precisely the set \( L \) used in Section 4. As before, we imagine increasing the deadline for the jobs in \( L \) to \( \Gamma \) to produce a modified instance \( I_L \) of the pDLS.

We now check whether \((\tilde{U}, \tilde{x})\) violates (D2) for the canonical suffix chain family \( C \) of instance \( I_L \). In this case, we add \( C \) to \( S' \), and recurse. Otherwise, we know that we can apply the lifting operation of Theorem 5.2 to obtain a new candidate solution \((U, x)\) for \( (P) \) whose support is cross-free. By Theorem 5.1 we can see that, while \((U, x)\) may not be feasible for \( (P) \), it is feasible for the relaxation \( (P') \) used in our rounding procedure, and further has objective value no larger than the optimum of \( (P) \). Thus, applying the rounding procedure of Section 4 yields a solution to our instance of pDLS with penalty \( O(\gamma \log k) \) times the optimal with constant probability.

In other words, the process described above consists of applying the ellipsoid method for solving (P2') with a separation oracle that might fail in providing a violated inequality: we stop the algorithm at the first moment that our separation oracle fails, so as to guarantee that the number of iterations (and therefore the size of \( S' \)) is anyway polynomially bounded in the number of variables. The solution output at the end might be infeasible for (P2'), but (as discussed above) provides a lower bound on its optimal value and is feasible for the relaxation (P2'); this ensures we can lift it into a solution for (P) that has cross-free support and is feasible for the relaxation \( (P') \), which are precisely the properties required by our rounding procedure.

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A Constant number of paths

In this section, we present an algorithm for solving the pDLS problem when chain-like precedence constraints can be modeled by a constant number of paths, i.e., |P| is a constant. For each path \( P \in \mathcal{P} \), let \( V^P \) be a suffix of path \( P \), and let \( V \) be the vector formed by these suffixes. Further, for a given path \( P \) and any suffix \( V^P \neq \emptyset \), let \( \min(V^P) \) be the first job in suffix \( V^P \) according to the precedence order \( \preceq \) on jobs, i.e., the job we would have to schedule first among those in \( V^P \).

Now, we are ready to design a dynamic program for solving our problem. We define \( \text{OPT}(V, t) \) to be the smallest overall postponement cost we can incur on jobs in \( V \), when we schedule them beginning at time \( t \) while respecting precedence constraints. In order to compute the value of \( \text{OPT}(V, t) \), we need to consider how we can schedule the jobs in \( V \). In particular, consider the first job we choose to schedule. While we can choose this first job from any path \( P \) for which \( V^P \neq \emptyset \), the precedence constraints enforce that it must always be the earliest job in \( V^P \).

Define \( \text{cost}(j, t) \) as

\[
\text{cost}(j, t) = \begin{cases} w_j & \text{if } d_j < t + p_j; \\
0 & \text{otherwise,}
\end{cases}
\]

to capture the cost of scheduling job \( j \) at time \( t \). Then, we can express \( \text{OPT}(V, t) \) recursively as

\[
\text{OPT}(V, t) = \min_{P \in \mathcal{P}: V^P \neq \emptyset} \left( \text{cost}(j, t) + \text{OPT}(V \setminus j, t + p_j) \right),
\]

where \( j = \min(V^P) \) and we use \( V \setminus j \) to denote the vector \( V \) excluding job \( j \), i.e., we have

\[
(V \setminus j)^P = \begin{cases} V^P \setminus \{j\} & \text{if } j \in P; \\
V^P & \text{otherwise.}
\end{cases}
\]

Thus, if we abuse notation slightly and let \( \mathcal{P} \) denote the vector of suffix chains that includes every job on every path, and \( \emptyset \) denote the vector taking the empty suffix on every path, we can see that taking base cases of \( \text{OPT}(\emptyset, t) = 0 \) for all \( t \) and computing \( \text{OPT}(\mathcal{P}, 0) \) yields precisely the quantity we want to compute.

Running time: While it might appear that the above recursion describes a dynamic program that runs in pseudo-polynomial time, due to the second parameter, we note that the first parameter always fully determines the second. In particular, for any recursive call \( \text{OPT}(V, t) \) made while computing \( \text{OPT}(\mathcal{P}, 0) \), a simple induction shows that we always have that

\[
t = \sum_{P \in \mathcal{P}} \sum_{j \in P \setminus V^P} p_j.
\]

Thus, the number of values we need to compute is bounding by the number of possible vectors of suffix chains, which is \( \prod_{P \in \mathcal{P}} (|P| + 1) = O(n^{|P|}) \). Since our recurrence takes the best among at most \( |P| \) possiblities, we can immediately conclude that we can compute every value of \( \text{OPT}(V, t) \) of interest in time \( O(|P| n^{|P|}) \).

B Single Deadline

In this section, we present a pseudo-polytime algorithm for solving the pDLS problem in the case where all jobs face a single, common deadline. The algorithm we give addresses not just chain-like...
precedence constraints, but the more general case of precedence constraints forming a tree where jobs can have multiple predecessors (but still no more than one successor). This matches the setting of Ibarra and Kim [15], and highlights the critical role of multiple, distinct deadlines in the pDLS problem: Lenstra and Rinnoy Kan[22] proved that the pDLS problem is strongly NP-hard even when all jobs have unit processing time and deferral cost, and precedence constraints are chain-like; with only a single, common deadline, however, the existence of a pseudopolytime algorithm implies that the problem becomes only weakly NP-Hard, even in the more general case where precedence constraints form trees.

We solve our problem by designing an appropriate dynamic program. Before proceeding with the details, we begin by giving some intuition and making necessary definitions. Limiting all jobs to share a common deadline $D$ forces the problem to closely resemble the knapsack problem, and in fact our dynamic program follows in precisely that vein. With only a single deadline, our problem becomes one of deciding which jobs we choose to include before that deadline, and which we defer until after the deadline; in other words, we have a certain amount of time before the deadline occurs, and we need to decide what jobs we want to use that time on to minimize the cost of deferred jobs (or, equivalently, maximize the cost of on-time jobs). In the setting we consider, each job may have multiple predecessors but only a single successor. This implies that, if we consider any tree present in the precedence constraints, if we schedule a job $j$ from that tree we must also schedule every job in the subtree rooted at job $j$. Thus, our task is to find a subforest of the forest of precedence constraints. If we focus on one tree, we can consider whether or not to schedule the job at the root: if we choose to schedule the root job, then we must schedule the entire tree; if we choose not to schedule the root job, then we may decide independently how (and if) to schedule jobs from the subtrees rooted at each of its children. In effect, we choose to either schedule the entire tree, and then recurse on the remaining trees, or replace the tree with one tree for each of the root’s children, and then recurse on the modified forest.

In order to make the above intuition concrete, we need the following definitions. Let the collection $[n]$ of jobs be indexed according to a pre-order traversal of the precedence constraint forest, and for any job $j$, let $T(j)$ denote the subtree rooted $j$, i.e. all of $j$’s predecessors. Define $\text{next}(j) = \min\{j + 1, j + 1, \ldots, n + 1\} \setminus T(j)$. Note that this implies that every successor of $j$ will have a strictly earlier index $j' < j$, and the predecessors of $j$ will be precisely those jobs with indexes in the range $\{j + 1, j + 2, \ldots, \text{next}(j) - 1\}$. Further, let

$$W(j) = \sum_{j' \in T(j)} w_{j'}; \text{ and}$$

$$P(j) = \sum_{j' \in T(j)} p_{j'}$$

denote the total penalty and total processing of all jobs in $T(j)$. Recall that every job $j$ shares a single common deadline $d_j = D$.

We now describe our dynamic program. Let $\text{OPT}(j, t)$ represent the minimum penalty we can incur if we only have $t$ processing time in which to schedule jobs $\{j, j + 1, \ldots\}$. In order to compute this value, we need to decide whether or not to schedule job $j$ in the time we have left before the deadline. If we do schedule $j$, then we must schedule all of the jobs in $T(j)$. This uses up processing time $P(j)$ but incurs no penalty, and we may then make independent decisions about all remaining jobs. If we do not schedule job $j$, then we incur a penalty of $w_j$ but use no processing time, and then may make independent decisions about all remaining jobs. Thus, we may express $\text{OPT}(j, t)$ via the recurrence

$$\text{OPT}(j, t) = \min \{\text{OPT}(\text{next}(j), t - P(j)), w_j + \text{OPT}(j + 1, t)\},$$
with base cases of
\[
\text{OPT}(j,t) = \begin{cases} 
+\infty & \text{if } t < 0; \\
0 & \text{if } t \geq 0 \text{ and } j = n + 1.
\end{cases}
\]

From the above discussion, we can readily see that the value of $\text{OPT}(1,D)$ will be precisely the minimum penalty that can be occurred when scheduling all of the jobs in $[n]$.

**Running time:** Since we will only need to compute $\text{OPT}(j,t)$ for parameter settings $j \in [n+1]$ and $t \in [D]$, we can see that our dynamic program will require at most $O(n \cdot D)$ distinct values to be calculated. Since each calculation is simply a minimum of at most two options, we can compute the optimal value $rec_1, D$ in time $O(n \cdot D)$.

### C Correlated rounding

In this section, we consider whether we can replace our independent rounding scheme with a dependent one to improve our approximation ratio. We follow an approach similar to that of Carr et al. [5] for the knapsack problem. At a high level, the key idea in the rounding scheme of Carr et al. is to try and ensure that the possible solutions it might produce are as uniform in size as possible. In particular, they do this by ensuring that sets of items in each potential knapsack solution have size profiles that are as similar as possible. In our setting, this would correspond to trying to ensure that for every deadline $D_i$, the family of suffixes we defer on the set $P$ of paths has size that is as uniform as possible; unfortunately, we are much more constrained when selecting suffixes than we would be when choosing items for a knapsack.

Specifically, there are two key difficulties. First, since each solution can only use a single suffix from each path, and different paths can randomize over paths with very different size profiles, it may be impossible to ensure that every solution has a similar size profile. For example, if every path but one defers a negligible number of jobs past some deadline $D_i$, then we cannot make our solutions any more uniform in size than the distribution on that single critical path. Second, even if we can ensure that solutions defer sufficiently uniform total size of jobs past a given deadline $D_i$, we need a way to do this for all deadlines in $D$ simultaneously; that is, we need a way to ensure our approach makes consistent choices for every deadline $D_i$. This is problematic: even though we work with suffix chains and so know that, on average, the length of the particular suffix deferred on a given path decreases with the deadline $D_i$, the rate of this decrease could differ greatly between paths. For example, one path $P$ might defer much larger suffixes (on average) past some deadline $D_i$ than every other path in $P$; but if the size of suffixes deferred on $P$ past deadline $D_{i+1}$ becomes much smaller, while that of suffixes deferred on other paths remains relatively constant, the situation becomes exactly the reverse for deadline $D_{i+1}$. Since the relative sizes of suffixes deferred on each path can change quite dramatically between one deadline and the next, it becomes difficult to devise a scheme that ensures consistent choices for all of the deadlines in $D$.

It turns out that the first concern above is not difficult to deal with; it is the second concern that causes difficulties. When we only have two paths, however, we can overcome the second concern as well. In this section, we describe a correlated rounding scheme for two paths that provides a 2-approximation for the case of two paths. While we could achieve the same factor for two paths using a naive approach, we are hopeful the technique we describe here can be extended to more paths without the approximation factor increasing linearly with the number of paths (as the naive approach’s factor would).

We now describe our correlated rounding scheme for the case where we only have two paths, say $P = \{P_1, P_2\}$. We follow the general outline as the rounding procedure of Section 4 with two major changes. First, we adjust the definition of the set $L$, replacing the filtering parameter $\gamma \log k$
by 2. Second, we modify our rounding procedure to be correlated, rather than independent, for the two chains $P_1$ and $P_2$. We discuss the two changes in detail below.

Our first change in the rounding procedure is to adjust the filtering parameter that splits jobs based on whether their $x_j$ values are large or small. In particular, we now consider $x_j$ to be large only if it is at least $1/2$, rather than when it exceeds $1/\gamma \log k$ as in Section 4. This means the set $L$ of late jobs defining our modified instance $I_L$ is now $L = \{ j : U_j \geq 1/2 \}$.

Our second change will be to adjust the distribution used in our randomized rounding procedure. Before, we rounded the variables for each path $P \in \mathcal{P}$ independently; now, however, while our rounding scheme will continue to induce the same marginal probability distribution on each path, we modify our rounding process so that the random choices we make on the two paths are strongly dependent on each other. Retaining the same marginal distributions ensures the analysis of Section 4 remains valid up until Lemma 4.4 with only minor changes to accommodate the new definition of late jobs $L$. The major changes in our approach occur from that point on: we can replace the concentration result underlying that lemma with an averaging argument that leverages the dependence structure we have introduced into our rounding scheme (see Lemma C.4). This new argument allows us to prove a much stronger approximation guarantee than we obtained for independent rounding, which furthermore holds with certainty. Before detailing how we obtain our new guarantee, we will briefly review the initial steps of our rounding procedure. Since these initial steps only require that our new rounding procedure induces the same marginals on each path, we defer further details of the rounding scheme for now.

We begin by briefly recalling the overall structure of the rounding procedure from Section 4, suitably modified for our new definition of the set $L$ of late jobs (see that section for full details). We start with an instance $I$ of the pDLS, and a solution $(U, x)$ for our linear program $[P]$. Then, we define a modified instance $I_L$ of pDLS in which the set $L = \{ j : U_j \geq 1/2 \}$ of late jobs all have their deadlines changed to be $\Gamma$ (so they cannot be late in any schedule). Finally, we focus on a relaxation $(P')$ of $[P]$ in which the set of knapsack constraints $(C2)$ is reduced to just the ones corresponding to the canonical suffix chain family for the modified instance $I_L$.

We now describe the process we use to round our initial solution $(U, x)$. This process can be applied as long as $(U, x)$ both is feasible for the relaxed program $(P')$ and has support $\{ S \in \mathcal{S}^P : x_S > 0 \}$ on each path $P \in \mathcal{P}$ that is cross-free; recall that in Section 5 we saw how to produce exactly such a solution. Given a solution $(U, x)$ with these properties, we first modify it to produce a new solution $(\hat{U}, \hat{x})$ where each job $j \in L$ is no longer late, and each job $j \notin L$ is late to twice the extent it is in $(U, x)$. Formally, this means we set $\hat{U}_j = 2 \cdot U_j$ if $j \notin L$ and $\hat{U}_j = 0$ otherwise. We correspondingly set $\hat{x}_S = \begin{cases} 2 \cdot x_S & \text{if } S \text{ makes some job } j \notin L \text{ late;} \\ 0 & \text{otherwise.} \end{cases}$

Finally, we randomly round the fractional solution $(\hat{U}, \hat{x})$ to produce an integral solution $(\hat{U}, \hat{x})$, in such a way that $\Pr[\hat{x}_{S=1}] = \hat{x}_S$ for every path $P \in \mathcal{P}$ and every suffix chain $S \in \mathcal{S}^P$.

The same arguments as presented in Section 4 – with only minor adjustments – give us the following critical properties for $(\hat{U}, \hat{x})$ and $(\hat{U}, \hat{x})$. The two lemmas below are direct analogs of Lemmas 4.2 and 4.3 respectively. We state them without proof, as they follow from the same arguments as those given for their counterparts in Section 4, we refer the reader to that section for details.
Lemma C.1. \((\hat{U}, \hat{x})\) satisfies
\[
\sum_{S \in \mathcal{S}^P} \hat{x}_S \leq 1 \quad \forall i \in [k], \forall P \in \mathcal{P}
\]
where \(C\) is the canonical chain family defined for the modified instance \(\mathcal{I}_L\) of pDLS.

**Lemma C.2.** For all \(j \notin L\), \(E[\hat{U}_j] = \hat{U}_j\).

With the above two lemmas in hand, we are now ready to describe the rounding scheme we use to obtain \((\hat{U}, \hat{x})\) in detail, and prove our approximation guarantee.

We round our fractional solution \((\hat{U}, \hat{x})\) to the integral solution \((\hat{U}, \hat{x})\) as follows. For a given \(P \in \mathcal{P}\), for every \(\alpha \in [0, 1]\) we define the chain
\[
\mathcal{S}^P(\alpha) := \min\{S \in \text{supp}(\hat{x}) \cap \mathcal{S}^P : \sum_{S' \leq S} \hat{x}_{S'} > \alpha\},
\]
where \(\text{supp}(\hat{x}) = \bigcup_{P \in \mathcal{P}} \{S \in \mathcal{S}^P : \hat{x}_S > 0\}\). The minimum in the above definition is with respect to the partial order \(\preceq\) on suffix chains; recall that since we know \(\text{supp}(\hat{x})\) is cross-free, this is well-defined. If the set in the above definition is empty, i.e., we have that \(\sum_{S \in \mathcal{S}^P} \hat{x}_S \leq \alpha\), then we define \(\mathcal{S}^P(\alpha) = \mathcal{C}^P\), where \(\mathcal{C}\) is the canonical chain family for the modified instance \(\mathcal{I}_L\). We now round \(\hat{x}\) to \(\hat{x}\) as follows. Draw a single uniform random variable \(\alpha \sim \mathcal{U}[0, 1]\), and index the pair of paths in our instance as \(\mathcal{P} = \{P_1, P_2\}\). Our rounding procedure needs to choose one suffix chain for each of the paths; we use \(\alpha\) to correlate our choices in the following manner. For paths \(P_1\) and \(P_2\) we select the suffix chains \(S_1 = \mathcal{S}^{P_1}(\alpha)\) and \(S_2 = \mathcal{S}^{P_2}(1 - \alpha)\), respectively. At a high level, our goal is to use \(\alpha\) to correlate our choices on the two paths, pairing large suffix chains on one path with small suffix chains on the other, and vice versa. By balancing our choices on the two paths in this way, we ensure that the combined weight of the suffix chains we choose is always relatively large. We make this idea concrete by first defining the rounded solution \((\hat{U}, \hat{x})\) and then formalizing the above observation as a lemma.

As stated above, we want our rounded solution to schedule jobs on paths \(P_1\) and \(P_2\) according to the suffix chains \(S_1 = \mathcal{S}^{P_1}(\alpha)\) and \(S_2 = \mathcal{S}^{P_2}(1 - \alpha)\), respectively, where \(\alpha \sim \mathcal{U}[0, 1]\). Thus, we set \(\hat{x}_S^{P_1} \equiv \hat{x}_S^{P_2} = 1\), and set \(\hat{x}_S = 0\) for all other \(S \in \mathcal{S}^{P_1} \cup \mathcal{S}^{P_2}\). Correspondingly, for each job \(j\) such that \(j \notin L\), we set \(\hat{U}_j = 1\) if \(j \in S_1^{P_1}\) or \(j \in S_2^{P_2}\), respectively, depending on whether \(j \in P_1\) or \(j \in P_2\); for all other \(j \in [n]\) we set \(\hat{U}_j = 0\). The following lemma shows that the rounding scheme outlined above produces the same marginal probabilities for \((\hat{U}, \hat{x})\) as those in Section 4 and thereby establishing the validity of Lemmas C.1 and C.2.

**Lemma C.3.** When \((\hat{U}, \hat{x})\) is rounded to \((\hat{U}, \hat{x})\) as described above, we have that for all \(P \in \mathcal{P}\) and all \(S \in \mathcal{S}^P\),
\[
\Pr[\hat{x}_S = 1] = \begin{cases} 
\hat{x}_S & \text{if } S \neq \mathcal{C}^P; \\
1 - \sum_{S' \in \mathcal{S}^P \backslash \{\mathcal{C}^P\}} \hat{x}_{S'} & \text{if } S = \mathcal{C}^P,
\end{cases}
\]
where \(\mathcal{C}\) is the canonical suffix chain family for \(\mathcal{I}_L\).

**Proof.** First, observe that if we set \(\hat{x}_S = 1\), then we must have had that \(\hat{x}_S > 0\) (or \(S = \mathcal{C}^P\) for some \(P \in \mathcal{P}\) where \(\mathcal{C}\) is the canonical suffix chain family for \(\mathcal{I}_L\)); thus, for all other \(S\) we immediately have that \(\Pr[\hat{x}_S = 1] = 0 = \hat{x}_S\).

We begin by focusing on suffix chains \(S\) in the support of \(\hat{x}\). Recall that the fractional solution \((U, x)\) we began with had cross-free support, i.e., we had that \(\{S \in \mathcal{S}^P : x_S > 0\}\) was cross-free for all \(P \in \mathcal{P}\). Now, we defined \(\hat{x}\) so that for all \(S\) we have \(\hat{x}_S > 0\) implies \(x_S > 0\). Thus, we may conclude \((\hat{U}, \hat{x})\) has cross-free support as well. Fix some \(P \in \mathcal{P}\), and enumerate \(\mathcal{S}^P\) in sorted order as \(S_1 < S_2 < \cdots < S_m\), where \(m = |\text{supp}(\hat{x})| \leq |\mathcal{S}^P|\). For any \(\ell \in [m]\), we can compute \(\Pr[\hat{x}_{S_\ell}]\)
as follows. Recall that we set \( \hat{x}_{\ell'} = 1 \) if and only if we had that \( S_{\ell'} = S_{p_{1}}(\alpha) \) or \( S_{\ell'} = S_{p_{2}}(1 - \alpha) \) (for \( P = P_{1} \) or \( P = P_{2} \), respectively). Now, from our definition of \( S_{P}() \), we can see that these equalities hold, respectively, if and only if we have that

\[
\sum_{\ell' = 1}^{\ell - 1} x_{\ell'} \leq \alpha < \sum_{\ell' = 1}^{\ell} x_{\ell'} \quad \text{or} \quad \sum_{\ell' = 1}^{\ell - 1} x_{\ell'} \leq 1 - \alpha < \sum_{\ell' = 1}^{\ell} x_{\ell'}.
\]

Thus, the probability of selecting \( S_{\ell'} \) is precisely the probability of choosing an \( \alpha \) in one of the ranges above. Recalling that \( \alpha \sim U[0, 1] \), that \( \bar{x}_S \) is nonnegative for all \( S \), and that \( \sum_{S \in S_{P}} \bar{x}_S \leq 1 \) always (by Lemma C.2), we can conclude this probability is, in fact, the lengths of the intervals in question. Since both intervals have the same length, in either case we get that

\[
\Pr[\hat{x}_{\ell'} = 1] = \sum_{\ell' = 1}^{\ell} x_{\ell'} - \sum_{\ell' = 1}^{\ell - 1} x_{\ell'} = \bar{x}_{\ell'},
\]

exactly as desired.

Finally, for each \( P \in P \), we consider \( \Pr[\hat{x}_{C'P} = 1] \), where \( C \) is the canonical suffix chain family for \( \mathcal{I}_L \). Again, by the definition of our rounding process we can see this happens precisely when \( C_{p_{1}} = S_{p_{1}}(\alpha) \) or \( C_{p_{2}} = S_{p_{2}}(1 - \alpha) \). This two equalities holds, respectively, precisely when

\[
\sum_{S \in S_{p_{1}}} \bar{x}_S \leq \alpha \quad \text{or} \quad \sum_{S \in S_{p_{2}}} \bar{x}_S \leq 1 - \alpha.
\]

As in the previous case, however, the nonnegativity of \( \bar{x} \) and Lemma C.1 allow us to conclude that

\[
0 \leq \sum_{S \in S_{p_{1}}} \bar{x}_S, \sum_{S \in S_{p_{2}}} \bar{x}_S \leq 1.
\]

Thus, since \( \alpha \sim U[0, 1] \), we can conclude that in either case we have that

\[
\Pr[\hat{x}_{C'P} = 1] = 1 - \sum_{S \in S_{P}} \bar{x}_S
\]

exactly as claimed. \( \square \)

**Lemma C.4.** With probability 1, we have that

\[
\sum_{P \in P} \sum_{S \in S_{P}} p_{S}^{iC} \hat{x}_{S} \geq \Theta_{iC}^{C}
\]

for all \( i \in [k] \), where \( C \) is the canonical suffix chain family for \( \mathcal{I}_L \).

**Proof.** We obtain the desired bound by combining upper and lower bounds for the expected value of the sum on the left of inequality (14): \( \mathbb{E}[\sum_{P \in P} \sum_{S \in S_{P}} p_{S}^{iC} \hat{x}_{S}] \). Fix some \( i \in [k] \), and let \( M \) be the quantity we want to lower bound, i.e. the minimum value that the sum inside the expectation achieves. First, we show that the maximum value this sum ever achieves is at most \( M + \Theta_{iC}^{C} \); this immediate implies that the expected value of the sum is also at most \( M + \Theta_{iC}^{C} \). We then show a lower bound of \( 2\Theta_{iC}^{C} \) on the expected value of the sum. Chaining these two inequalities together immediately gives us that \( M \geq \Theta_{iC}^{C} \), exactly as desired.

We begin by proving our claimed upper bound on the sum \( \sum_{P \in P} \sum_{S \in S_{P}} p_{S}^{iC} x_{S} \). First, we simplify this sum by recalling details of our rounding procedure. For each path \( P \in P \) we set exactly one \( \hat{x}_S \) equal to 1, and set all others equal to 0, based on a uniform random variable \( \alpha \sim U[0, 1] \). Thus, our sum reduces to preciously the coefficients of the two variables in question; by the definition of our rounding procedure, we can see that we get that

\[
\sum_{P \in P} \sum_{S \in S_{P}} p_{S}^{iC} x_{S} \geq \Theta_{iC}^{C} = p_{S_{1}}^{iC} + p_{S_{2}}^{iC},
\]

where \( S_{1} = S_{p_{1}}(\alpha) \) and \( S_{2} = S_{p_{2}}(1 - \alpha) \).
The key to our upper bound is showing that for any two values of $\alpha$, the resulting values of the sum $p^{i,C}_{s_1} + p^{i,C}_{s_2}$ can differ by at most $\Theta^{i,C}$. To that end, fix $0 \leq \alpha \leq \alpha' \leq 1$; set $S^1 = S^{P_1}(\alpha)$ and $S^2 = S^{P_2}(1-\alpha)$ (as before); and set $S^1 = S^{P_1}(\alpha')$ and $S^2 = S^{P_2}(1-\alpha')$. Now, by our definition of $S^P(\cdot)$, we can immediately see that $\alpha \leq \alpha'$ implies that $S^1 \leq S^1$ and $S^2 \geq S^2$, and so $S^1 \supseteq S^1$ and $S^2 \supseteq S^2$. From the definition of $p^{i,C}_S$, however, we can then see that we have
\[
0 \leq p^{i,C}_{s_1} \leq \Theta^{i,C} \quad \text{and} \quad 0 \leq p^{i,C}_{s_2} \leq \Theta^{i,C}.
\]

The above immediately imply the slightly weaker pair of inequalities
\[
p^{i,C}_{s_1} \leq p^{i,C}_{s_1} \leq p^{i,C}_{s_1} + \Theta^{i,C} \quad \text{and} \quad p^{i,C}_{s_2} \leq p^{i,C}_{s_2} \leq p^{i,C}_{s_2} + \Theta^{i,C},
\]
which combine to imply that
\[
(p^{i,C}_{s_1} + p^{i,C}_{s_2}) \leq (p^{i,C}_{s_1} + p^{i,C}_{s_2}) + \Theta^{i,C} \quad \text{and} \quad (p^{i,C}_{s_1} + p^{i,C}_{s_2}) \leq (p^{i,C}_{s_1} + p^{i,C}_{s_2}) + \Theta^{i,C}.
\]

Recall that we chose $M = \min_0 \sum_{p \in P} \sum_{s \in S^P} p^{i,C}_s \hat{x}_s = \min_0 (p^{i,C}_{s_1} + p^{i,C}_{s_2})$. Thus, if we let $\alpha^*$ achieve this minimum value $M$, i.e. pick $\alpha^* \in \arg\min_0 (p^{i,C}_{s_1} + p^{i,C}_{s_2})$, we can see that applying the two inequalities above in the regions $[0, \alpha^*]$ and $[\alpha^*, 1]$, respectively, immediately gives us that
\[
E\left[\sum_{p \in P} \sum_{s \in S^P} p^{i,C}_s \hat{x}_s\right] = E[p^{i,C}_{s_1} + p^{i,C}_{s_2}] \leq E[M + \Theta^{i,C}] = M + \Theta^{i,C},
\]
exactly as claimed.

Our lower bound follows simply by combining Lemmas $C.2$ and $C.1$. In particular, we get that
\[
E\left[\sum_{p \in P} \sum_{s \in S^P} p^{i,C}_s \hat{x}_s\right] = \sum_{p \in P} \sum_{s \in S^P} p^{i,C}_s \Pr[\hat{x}_S = 1] = \sum_{p \in P} \sum_{s \in S^P} p^{i,C}_s \bar{x}_s \geq 2\Theta^{i,C},
\]
where the first equality follows since the $\hat{x}_s$ are all binary random variables, the second follows by Lemma $C.2$, and the inequality follows by Lemma $C.1$. Combining this with our previously found upper bound on this expected value, however, we can conclude that $M + \Theta^{i,C} \geq 2\Theta^{i,C}$ or equivalently $M \geq \Theta^{i,C}$, exactly as desired.

Finally, we observe that we can use the same procedure as outlined in Section 4 to produce a feasible schedule for $\mathcal{I}$ from the integral solution $(\hat{U}, \hat{x})$. As before, on each path $P \in P$ we defer all jobs that are in the suffix chain $S \in S^P$ such that $\hat{x}_S = 1$ or in the canonical suffix for $P$ in $\mathcal{I}_L$, i.e. all jobs in the join $S \wedge C^P$, and then running the algorithm outlined in that section to get a feasible schedule. We get the following results corresponding to Theorem 4.5 and Corollary 4.6 since their proofs follow from largely the same arguments as those given in Section 4; we only sketch the differences below.

**Theorem C.5.** Applying the process and algorithm from Section 4 to the solution $(\hat{U}, \hat{x})$ yields a feasible schedule with cost at most $\sum_{j \notin L} w_j \hat{U}_j$ in the instance $\mathcal{I}_L$.

**Proof.** The proof of this theorem follows from the exact same argument as that for Theorem 4.5. The only difference is the equivalent of constraint (5) for our current rounding scheme holds with certainty, due to Lemma $C.4$. Specifically, we have that
\[
\sum_{p \in P} \sum_{s \in S^P} p^{i,C}_s \bar{x}_s \geq \Theta^{i,C},
\]
holds with probability 1, and hence our upper bound on costs holds unconditionally, unlike its counterpart from Section 4. In all other aspects the theorem and its proof are identical to those found in Section 4, and we refer the reader to that section for further details.

**Corollary C.6.** The schedule produced by following the process and algorithm from Section 4 is feasible in the original instance $\mathcal{I}$ of pDLS, and incurs expected penalty of $2 \sum_{j} w_j U_j$. 

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Proof. The proof largely follows the same arguments as that of Corollary 4.6 and we refer the reader to that proof for full details. We sketch the main technical details below. Similarly to that section, since we defined \( L = \{ j : U_j \geq 1/2 \} \), and set \( \bar{U}_j = 2U_j \) for all \( j \notin L \), we get that
\[
\sum_{j \notin L} w_j \bar{U}_j + \sum_{j \in L} w_j \leq 2 \sum_{j} w_j U_j.
\]

Now, from Theorem C.5 we have the schedule we produce in the end has cost at most \( \sum_{j \notin L} w_j \hat{U}_j \) in the modified setting \( I_L \). Recall, however, that the only difference between \( I \) and \( I_L \) is that in the latter we increased the deadlines of all jobs in \( L \) to be \( \Gamma \). Thus, the cost of our schedule in \( I \) can increase versus the cost in \( I_L \) by at most the total penalty of all jobs in \( L \), i.e. by at most \( \sum_{i \in L} w_j \). Applying Lemma C.2, we can thus see that the expected cost of our schedule in \( I_L \) is at most
\[
E[\sum_{j \notin L} w_j \hat{U}_j + \sum_{i \in L} w_j] = \sum_{j \notin L} w_j \bar{U}_j + \sum_{j \in L} w_j \leq 2 \sum_{j} w_j U_j,
\]
exactly as claimed.

D LP gap example

In this section, we show that using the KC-inequalities (C2) in the LP relaxation (P) is critical to our approximation factor. In particular, we show that without utilizing KC-inequalities, the integrality gap of (P) would be \( \Omega(n/\log n) \). We do so by constructing a simple instance \( I \) of the pDLS, and considering the program (P) without the KC-inequality strengthening of constraint (C2). We will show that the LP relaxation of this weaker version of (P) admits a fractional solution whose value is a factor of \( \Omega(n/\log k) \) better than any integral solution. Of special note is the simplicity of the instance \( I \) we construct: the instance consists of a single path, all of the jobs on which have unit process time and unit weight.

In the rest of the section, we describe the instance \( I \) of pDLS and the fractional solution \((U, x)\) yielding our claimed integrality gap. As mentioned, our example consists of a single path \( P = 1 \preceq 2 \preceq \cdots \preceq n \), with \( w_j = p_j = 1 \) for all \( j \in [n] \). Each job has one of \( k = n/2 \) distinct deadlines, with the deadline of job \( j \) being given by
\[
d_j = \begin{cases} 
  j & \text{if } j \text{ is odd}; \\
  j - 1 & \text{if } j \text{ is even}.
\end{cases}
\]

Thus, we can see that deadline \( D_i = 2i - 1 \) for all \( i \in [k] \), and that \( i(j) = \lfloor \frac{j+1}{2} \rfloor \).

Since our example consists of a single path, there is only one feasible solution: simply schedule the jobs in the order they appear on that path. We chose our deadlines such that this schedule runs jobs with odd indices on time, and makes jobs with even indices late. This results in a total cost of \( n/2 \).

On the other hand, consider the following fractional solution to program (P) for the given example\(^1\). The support of our solution will be the set \( \{S^\ell\}_\ell \) of suffix chains indexed by \( \ell \in [k+1] \), where we define \( S^\ell \) by
\[
S^\ell_i = \begin{cases} 
  \{1, 2, \ldots, n\} & \text{if } i \leq \ell; \text{ and} \\
  \{2i + 1, 2i + 2, \ldots, n\} & \text{if } i > \ell.
\end{cases}
\]

\(^1\)in the interests of simplicity, for the rest of this discussion we omit any uses of the index \( P \) and use \( \leq \) in place of \( \preceq \) since, by construction, they are equivalent.
Our proposed solution then sets

\[
x_S = \begin{cases} 
\frac{1}{2k} & \text{if } S = S^0; \\
\frac{1}{2k} - \frac{1}{2(\ell+1)} & \text{if } S = S^\ell \text{ for } 1 \leq \ell < k; \\
\frac{1}{2k} & \text{if } S = S^k; \\
0 & \text{otherwise.}
\end{cases}
\]

Note that this solution has far lower cost than the integral one. In particular, for any \( j \in [n] \), the suffix chain \( S^j \) makes job \( j \) late if and only if \( j \in S^\ell_{i(j)} \). Whenever \( i(j) \leq \ell \), we have \( j \in S_j = [n] \); on the other hand, for \( i(j) > \ell \) we can see that the smallest element of \( S^\ell_{i(j)} \) is \( 2\left\lfloor \frac{i(j)+1}{2} \right\rfloor + 1 \geq j+1 \), since \( i(j) = \left\lfloor \frac{i(j)+1}{2} \right\rfloor \). So \( j \in S^\ell_{i(j)} \) if and only if \( i(j) \leq \ell \). Thus, we have that

\[
U_j = \sum_{S \in S : S \ni j} x_S = \sum_{\ell = i(j)}^k x_{S^\ell} = \sum_{\ell = i(j)}^{k-1} \left( \frac{1}{2\ell} - \frac{1}{2(\ell+1)} \right) + \frac{1}{2k} \leq \frac{1}{j}.
\]

The first inequality above follows by the definition of \( x_S \); the second by our definition of the support of \( x \) and our observations above on \( S^\ell_{i(j)} \); the third by our choice of \( x \); and the last by observing that we have a telescoping sum, and recalling that we have \( 2i(j) = 2\left\lfloor \frac{i(j)+1}{2} \right\rfloor \geq j \). Thus, we can see that the solution \((U, x)\) produces objective value

\[
\sum_{j \in [n]} w_j U_j \leq \sum_{j \in [n]} \frac{1}{j} = O(\log n).
\]

Thus, any program with integrality gap better than \( \Omega(n/\log n) \) must have constraints which this candidate solution \((U, x)\) violates. We next show that \((U, x)\) satisfies constraint (C1) from program \((P)\), as well as the version of constraint (C2) which does not use the KC-inequalities strengthening. This implies that the KC-inequalities are critical to achieving a small LP gap.

We begin by showing that \((U, x)\) satisfies constraint (C1) from the program \((P)\). First, note that we have defined \( x \) so that

\[
\sum_{S \subseteq S} x_{S} = \sum_{\ell = 0}^{\ell = k} x_{S^\ell} = \frac{1}{2} + \sum_{\ell = 1}^{\ell = k-1} \left( \frac{1}{2\ell} - \frac{1}{2(\ell+1)} \right) + \frac{1}{2k} = 1,
\]

and so constraint (C1) is satisfied.

Since our fractional solution \((U, x)\) satisfies (C1), we conclude constraint (C2) is crucial to bounding the integrality gap of \((P)\). Furthermore, we will now show that the “capping” of processing times \( p^i_{S} \) by \( \Theta^{i,F} \) is critical to achieving a good integrality gap for \((P)\). We do so by demonstrating that without the capping operation, \((U, x)\) would, in fact, satisfy constraint (C2), and hence show the integrality gap of \((P)\) is \( \Omega(n/\log n) \).

Consider constraint (C2) without the capping operation. Fix some suffix chain \( F \) on \( \mathcal{P} \). Using the previously given definition, for each \( i \in [k] \) we have that

\[
\Theta^{i,F} = \max\{n - D_i - |F_i|, 0\}.
\]

Furthermore, if we no longer enforce that the processing times we associate with suffixes be at most \( \Theta^{i,F} \) we get that

\[
p^i_{S} = \sum_{j \in S \setminus F_i} p_j = |S_i \setminus F_i|.
\]

Fix \( i \in [k] \). For constraint (C2) to hold in our setting, we need that

\[
\sum_{S \subseteq S : S \ni F_i} x_S \geq \Theta^{i,F}.
\]

We now show that the above always holds; we break our proof into three cases.

- Case: \( |F_i| \geq n - D_i \). Then we have that \( \Theta^{i,F} = 0 \), and the inequality holds trivially.
• Case: \(|F_i| = n - D_i - 1\). Then we have that \(\Theta^{i,F} = 1\); furthermore, since \(D_i = 2i - 1\), we can see that \(|F_i| = n - 2i\), and so \(F_i = \{2i+1, 2i+2, \ldots, n\}\). Recalling the definition of \(S^\ell_i\), however, we can see that \(S^\ell_i \supseteq F_i\) only if \(i \leq \ell\), and so \(S^\ell_i = [n]\). Thus, we get that
\[
\sum_{s \in S : S_i \supseteq F_i} p^{i,F}_s x_s = \sum_{\ell=1}^k p^i_{S^\ell_i} x_{S^\ell_i} = \vert[n] \setminus F_i\vert \left( \sum_{\ell=1}^{k-1} \left( \frac{1}{2\ell} - \frac{1}{2(\ell + 1)} \right) + \frac{1}{2k} \right) = 2i \cdot \frac{1}{2i} = 1.
\]
Thus the desired inequality holds in this case as well.

• Case: \(|F_i| \leq n - D_i - 2\). Then we have that \(S^\ell_i \supseteq F_i\) for all \(0 \leq \ell \leq k\). So can see that
\[
\sum_{s \in S : S_i \supseteq F_i} p^{i,F}_s x_s = \sum_{\ell=1}^{i-1} (n - 2i - |F_i|) x_{S^\ell_i} + \sum_{\ell=i}^k (n - |F_i|) x_{S^\ell_i}
= (n - |F_i|) \sum_{\ell=0}^{k-1} x_{S^\ell_i} - 2i \sum_{\ell=0}^{i-1} x_{S^\ell_i}
= (n - |F_i|) - 2i \left( 1 - \frac{1}{2i} \right)
= (n - |F_i|) - (2i - 1)
= n - D_i - |F_i|
= \Theta^{i,F},
\]

exactly as required.

In every case, we get that constraint (C2) holds. Since our choice of \(F\) was arbitrary, we may conclude that requiring \(p^{i,F}_s \leq \Theta^{i,F}\) is critical to ensuring a good integrality gap for (P).

### E Tightness of rounding scheme

Here, we give an example to show the limits of our current techniques when implemented with independent rounding. We construct an instance \(\mathcal{I}\) and a fractional solution \((U, x)\) for \(\mathcal{I}\) such that if we use the rounding procedure outlined in Section 4, while setting \(\gamma = O(1/\log^\varepsilon k)\) for some \(\varepsilon > 0\), then our probability of success will be \(o(1)\) in \(k\). In other words, if we want our rounding procedure to succeed with constant probability, we simply cannot replace our boosting factor of \(\gamma \log k\) with one that is \(O(\log^{1-\varepsilon} k)\). This shows that the result of Lemma 4.4 is tight, and so the approximation factor in Corollary 4.6 cannot be improved without significant new techniques.

We begin by constructing the instance \(\mathcal{I}\) of pDLS. Fix the number of deadlines \(k\), and let \(n\) be the number of jobs for some \(n\) divisible by \(k^2\). Each of our jobs will have unit runtime and unit weight, i.e. \(w_j = p_j = 1\) for all \(j \in [n]\). Our set of deadlines will be \(D = \{D_1, \ldots, D_k\}\) where
\[
D_i = \left( \frac{i}{k} \right) n - 1 \quad \text{for all } i \in [k].
\]

Our set of paths \(\mathcal{P}\) contains \(n/k\) identical paths of length \(k\); each path contains a single job with deadline \(D_i\) for each \(i \in [k]\), in increasing order along the path. Specifically, if one of our paths is \(P = \{j_1, j_2, \ldots, j_k\}\), with \(j_1 < \cdots < j_k\), then for all \(\ell \in [k]\) we have that \(i(j_i) = \ell\). In the following, we always index the jobs in a path \(P\) in this fashion for convenience, so that for any such path we always have that \(j_i < j_{i'}\) if and only if \(i < i'\), and that the deadline of job \(j_i\) is \(D_i\) for all \(i \in [k]\).

We now describe a fractional solution \((U, x)\) for \(\mathcal{I}\), parameterized by \(\gamma\). For every path \(P \in \mathcal{P}\), we will have only two suffix chains of \(P\) in the support of \(x\). One will be the canonical suffix chain \(C^P\) for \(P\), and the other will be a slight modification of the canonical suffix chain, which
we denote $\bar{C}^P$. Before defining these two suffix chains formally, we first note that we will place the majority of our solution’s weight on $C^P$, setting

$$x_S = \begin{cases} 
1 - \frac{1}{2\gamma \log k} & \text{if } S = C^P; \\
\frac{1}{2\gamma \log k} & \text{if } S = \bar{C}^P; \text{ and} \\
0 & \text{otherwise.}
\end{cases}$$

Since by definition the canonical suffix chain family never makes any job late, this solution ensures that $U_j \leq 1/2\gamma \log k$ for all $j \in [n]$. Thus, we can immediately conclude that for the solution $(U, x)$, we have

$$L = \{ j \in [n] : U_j \geq 1/\gamma \log k \} = \emptyset,$$

and so $I_L = I$. Thus, our rounding procedure works solely with the original instance $I$, and the canonical suffix chain family $C$ for $I$.

We now formally define the canonical suffix chain family $C$, the modification $\bar{C}$, and the KC-Inequalities for $I$ corresponding to $C$. First, by inspection we can see that for any path $P = \{ j_1, j_2, \ldots, j_k \} \in \mathcal{P}$, we have that

$$C^P = \{ j_2, \ldots, j_k \} \supseteq \{ j_3, \ldots, j_k \} \supseteq \cdots \supseteq \{ j_k \} \supseteq \emptyset,$$

i.e. we have that $C^P_i = \{ j_{i+1}, j_{i+2}, \ldots, j_k \}$ for all $i \in [k]$. Now, we define the modification $\bar{C}$ as follows. First, we partition the set $\mathcal{P}$ of paths into $k$ groups $\Pi_1, \Pi_2, \ldots, \Pi_k$, each containing exactly $n/k^2$ paths. Then, for any path $P \in \mathcal{P}$, we define the suffix chain $\bar{C}^P$ as

$$\bar{C}^P = \begin{cases} 
C^P_i & \text{if } P \notin \Pi_i; \text{ and} \\
C^P_{i-1} & \text{if } P \in \Pi_i,
\end{cases}$$

where we take $C^P_0$ to indicate the entire chain $P$. From the definitions of $C$, $\bar{C}$, and $x$ given above, we can compute that for each path $P = \{ j_1, \ldots, j_k \} \in \mathcal{P}$, and each $i \in [k]$, we have that

$$U_{j_i} = \begin{cases} 
1 - \frac{1}{2\gamma \log k} & \text{if } P \notin \Pi_i; \text{ and} \\
\frac{1}{2\gamma \log k} & \text{if } P \in \Pi_i; \text{ and} \\
0 & \text{otherwise.}
\end{cases}$$

Thus, as previously mentioned, we can see that no job is made late to an extent of $1/\gamma \log k$ or more, and so $I_L = I$.

**Lemma E.1.** For the described instance $I$ of pDLS, the constructed solution $(U, x)$ satisfies conditions (a) and (b) of the rounding procedure of Section 4, i.e. the solution has cross-free support and satisfies the reduced constraint (C2'), whenever we have $n/k^2 \geq 2\gamma \log k$, where $\gamma$ is the parameter for the rounding process.

**Proof.** We begin by showing that condition (a) is satisfied, i.e. the constructed $x$ has cross-free support. Fix some $P \in \mathcal{P}$. Now, we defined $x$ such that $x_S > 0$ if and only if $S \in \{ C^P, \bar{C}^P \}$ for all $S \in \mathcal{S}^P$. Recall, however, that for all $i$ we either have that $\bar{C}^P_i = C^P_i$, or have that $\bar{C}^P_i = C^P_{i-1} \supseteq C^P_i$. Thus, we may conclude that $\bar{C}^P \preceq C^P$ for all $P \in \mathcal{P}$, and hence the support of $x$ is cross-free.

Now, we show that condition (b) is satisfied, i.e. the KC-Inequalities corresponding to the canonical suffix chain family $C$ for the modified pDLS instance $I_L$ are satisfied. Recall, however, that we already saw that $U_j < 1/\gamma \log k$ for all $j \in [n]$, and so $I_L = I$; thus, we are actually interested in the KC-Inequalities corresponding to the canonical suffix chain family for the original pDLS instance $I$.

We begin by calculating the relevant constants for the KC-Inequalities. First, we compute $\Theta^{i, C}$. Recall that all of our jobs had unit processing times, and so we have $p_j = 1$ for all $j \in [n]$ and $\Gamma = n$. We claim that this gives us that $\Theta^{i, C} = 1$ for all $i \in [k]$. Fix some $i \in [k]$. Now, note that all of our paths are identical, and for each path $P = \{ j_1, \ldots, j_k \} \in \mathcal{P}$ we have that
$|C^P_i| = |\{j_{i+1}, \ldots, j_k\}| = k - i$. Thus, we can see that

$$\Theta_i^C = (1 - D_i) - |P| (k - i) = \left( n - \frac{i}{k} \cdot n + 1 \right) - \frac{n}{k} (k - i) = 1.$$  

Now, we consider the values of $P^i_C$ and $\tilde{P}^i_C$. First, we note that since $P^i_C$ denotes the (possibly capped) number of jobs suffix chain $S$ defers in addition to those deferred by the canonical suffix chain family $C$, we immediately can see that $P^i_C = 0$ always. Furthermore, since for each path $P \in P$, $\tilde{C}$ differs from $C$ only in that it defers a single additional job past deadline $D_i$ where $P \in \Pi_i$, we conclude that

$$p^i_C = \begin{cases} 1 & \text{if } P \in \Pi_i; \\ 0 & \text{otherwise.} \end{cases}$$

Combining the above, we can see that the KC-Inequality corresponding to the canonical suffix chain family $C$ holds whenever $n/k^2 \geq 2\log k$, exactly as claimed. To see this, we first compute that, for any $i \in [k]$ we have

$$\sum_{P \in P} \sum_{S \in S^P} p_S^i \cdot x_S = \sum_{P \in P} \sum_{P^i_C} \frac{1}{2\gamma \log k} = |\Pi_i| \cdot \frac{1}{2\gamma \log k} = \frac{n}{k^2} \cdot \frac{1}{2\gamma \log k},$$

where the first inequality follows by recalling that $x_S > 0$ only when $S \in \{C^P, \tilde{C}^P\}$, and $p^i_C = 0$; the second follows since $p^i_C = 1$ if and only if $P \in \Pi_i$ and is 0 otherwise; and the last since $|\Pi_i| = n/k^2$. Thus, since we already knew that $\Theta_i^C = 1$, we can substitute in our computed values and rearrange terms to get that the $(U, x)$ satisfies the KC-Inequalities corresponding to the canonical suffix chain family $C$ if and only if $n/k^2 \geq 2\gamma \log k$.

**Lemma E.2.** Let $\gamma$ be a function of $k$, such that $\gamma = O(1/\log^2 k)$ for some $\varepsilon > 0$. For the instance $I$ of pDLS described above, with $n = k^2 [2\gamma \log k]$, applying the rounding procedure of Section 4 to $(U, x)$ using $\log k$ as the boosting parameter has success probability that is $o(1)$ in $k$.

**Proof.** We begin by noting that the parameter settings outlined above are consistent with our example so far; in particular, we chose $n$ to be divisible by $k^2$, and furthermore such that $n/k^2 = \lceil 2\gamma \log k \rceil$ and so the condition for Lemma E.1 holds.

In order to calculate the probability of the rounding process succeeding, we begin by describing the conditions for it to succeed. Our claim essentially states that Lemma E.4 from Section 4 is tight, and we build on the analysis used to prove that lemma. Considering that lemma, we see that for each $i \in [k]$, one random variable $X_P$ is defined for each path $P \in P$ as

$$X_P := \sum_{S \in S^P} p_S^i \cdot x_S;$$

$\hat{x}$ is a random variable obtained by first modifying the solution $(U, x)$ to produce a new solution $(\hat{U}, \hat{x})$, and then using the value of $\hat{x}$ to define the marginal distribution for $\hat{x}$. While we refer the reader to Section 4 for the full details, we briefly describe the results of this process for our specific solution $(U, x)$ in the instance $I$. First, we note that from the definition of $(U, x)$ in this section and the method for producing $\hat{x}$, we will have that $\hat{x}^C_P = (\gamma \log k) \cdot (1/2\gamma \log k) = 1/2$ and $\hat{x}_S = 0$ for all other $S \in S^P \setminus \{\tilde{C}^P\}$. Second, we note that $p^i_C = 1$ if $P \in \Pi_i$ and equals 0 otherwise for all $P \in \Pi$ (see the proof of Lemma E.1 for details). Thus, if we consider the rounding process used to produce $\hat{x}$, we will see that for each $P \in \Pi_i$, $X_P$ will be a binary random variable which takes values 0 and 1 with equal probability; and for each $P \in \Pi \setminus \Pi_i$, $X_P = 0$ always. Furthermore, the random variables for each $P \in \Pi$ are independent.

Now, the rounding procedure succeeds for deadline $D_i$ if and only if the sum $X = \sum_{P \in \Pi} X_P$ of the above variables is at least $\Theta^i_C$. Now, for the setting $I$ we have that $\Theta^i_C = 1$ (see the proof
of Lemma E.1 for details). As we saw above, however, $X_P$ is identically 0 whenever $P \in \mathcal{P} \setminus \Pi_i$, so we conclude that the rounding process succeeds for deadline $D_i$ if and only if

$$X = \sum_{P \in \Pi_i} X_P \geq \Theta^i.C = 1.$$  

Recalling that each of the $X_P$ above is independently 1 with probability $1/2$ and 0 otherwise, we can see that

$$\Pr[X \geq 1] = 1 - \Pr[X = 0] = 1 - \prod_{P \in \Pi_i} \Pr[X_P = 0] = 1 - \left( \frac{1}{2} \right)^{|\Pi_i|} = 1 - \frac{1}{2^{n/k^2}},$$

and so the probability our rounding scheme succeeds for deadline $D_i$ is precisely $1 - 2^{-n/k^2}$.

Now, given the above, we want to compute the overall probability that our rounding scheme succeeds. In fact, this probability is precisely the probability that our rounding scheme succeeds for all of the deadlines in $\mathcal{D}$. Above, we saw that the probability our scheme succeeded for any single deadline was $1 - 2^{-n/k^2}$; we must be careful, however, because while our rounding scheme is independent for each path $P \in \mathcal{P}$, there is dependence in its behavior for a given path $P \in \mathcal{P}$ with respect to the different deadlines $D_i$. The key observation, however, is that the probability we succeed for deadline $D_i$ only depends on the random variables associated with paths $P \in \Pi_i$. Since $\Pi_1, \ldots, \Pi_k$ partition the set $\mathcal{P}$ of paths and the variables associated with each path $P \in \mathcal{P}$ are independent for every deadline, we have ensured by construction that the events that we succeed with respect to deadlines $D_i$ and $D_{i'}$ will be independent whenever $i \neq i'$. Thus, we conclude that the probability our rounding procedure succeeds for $(U, x)$ in the instance $I$ is precisely

$$\Pr[X \geq 1 \text{ for all } i \in [k]] = \prod_{i \in [k]} \Pr[X \geq 1 \text{ for } D_i] = \left( 1 - \frac{1}{2^{n/k^2}} \right)^k.$$  

Finally, we show that the probability we computed above is $o(1)$ when we have that $\gamma$ satisfies $\gamma = \Omega(1/\log^\varepsilon(k))$, some $\varepsilon > 0$, and $n = k^2[2\gamma \log k]$. First, we note that since $\gamma = O(1/\gamma \log^\varepsilon k)$, we must have that as $k$ goes to infinity, $\gamma \log k$ either converges to some fixed constant $c \geq 0$, or goes to infinity as well.

First, if $\gamma \log k \rightarrow c$, some $c \geq 0$, as $k \rightarrow \infty$, we immediately get that the success probability above is $o(1)$. This is because we can see that for all sufficiently large $k$, we have that $n/k^2 = [2\gamma \log k] \leq 2c + 1$. Thus, we can bound our probability of success as

$$\left( 1 - \frac{1}{2^{n/k^2}} \right)^k \leq \left( 1 - \frac{1}{2^{c+1}} \right)^k;$$

since $c > 0$ is a constant independent of $k$, we may conclude that our success probability goes to 0 as $k$ goes to infinity.

Second, we show that if $\gamma \log k \rightarrow \infty$ as $k \rightarrow \infty$, we again get that our success probability is $o(1)$. First, we note that this implies we must have that $n/k^2 \rightarrow \infty$ as $k \rightarrow \infty$; thus, we may conclude that for all sufficiently large $k$, we have that

$$\left( 1 - \frac{1}{2^{n/k^2}} \right)^k \leq \left( \frac{2}{e} \right)^{k \log^{n/k^2} k}.$$  

Thus, if we want to show that our success probability is $o(1)$, we need only show that the fraction $k/2^{n/k^2} \rightarrow \infty$ as $k \rightarrow \infty$. Note, however, that since $\gamma = O(1/\log^\varepsilon k)$, we know (again, for sufficiently large $k$) that $n/k^2 = [2\gamma \log k] < \lg^{1-c/2} k$, where $\lg k$ is the logarithm base 2 of $k$. Thus, we can conclude that when $k$ is sufficiently large we have that

$$\frac{k}{2^{n/k^2}} \geq \frac{k}{2^{\lg^{1-c/2} k}} = k \left( 1 - \frac{1}{\lg^{c/2} k} \right).$$  

Now, since $\frac{1}{\lg^{c/2} k} \rightarrow 0$ as $k \rightarrow \infty$, we may conclude from equation 16 that $\frac{k}{2^{n/k^2}} \rightarrow \infty$ as $k \rightarrow \infty$. Combining this with equation 15, however, we can see that our success probability
must converge to 0 as $k$ goes to infinity.

Thus, as we have shown above, in either case we get that our success probability goes to 0 as $k$ goes to infinity, i.e. our rounding process succeeds with probability $o(1)$ in $k$ exactly as claimed.